RELATIVE HYPERBOLICITY AND ARTIN GROUPS

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Abstract. Let $G = \langle a_1, \ldots, a_n \mid a_i a_j a_i \ldots = a_j a_i a_j \ldots, i < j \rangle$ be an Artin group and let $m_{ij} = m_{ji}$ be the length of each of the sides of the defining relation involving $a_i$ and $a_j$. We show if all $m_{ij} \geq 7$ then $G$ is relatively hyperbolic in the sense of Farb with respect to the collection of its two-generator subgroups $\langle a_i, a_j \rangle$ for which $m_{ij} < \infty$.

1. Introduction

The notion of a word-hyperbolic group introduced by Gromov [18] has played a pivotal role in the development of geometric group theory for the last fifteen years. In [18, 19] Gromov also suggested a way of generalizing this notion to that of a group relatively hyperbolic with respect to a collection of subgroups called parabolic subgroups. Several researchers proposed ways of making these ideas precise. First, Farb [15] defined relative hyperbolicity by requiring that the Cayley graph of a group with cosets of the parabolic subgroups “coned-off” is a hyperbolic metric space. Later Bowditch [5] gave a rigorous interpretation of Gromov’s original approach which mimics the case of a geometrically finite Kleinian group. Recently Yaman [31] showed that the Gromov-Bowditch version of relative hyperbolicity can be characterized in terms of convergence group actions. Bowditch had earlier obtained such a characterization for word-hyperbolic groups [6]. Szczepański [29] investigated the relationship between different versions of relative hyperbolicity. He proved that relative hyperbolicity in the sense of Gromov-Bowditch-Yaman implies that of Farb but not vice versa. Bowditch [5] observed that relative hyperbolicity in the sense of Farb together with what Farb termed the “bounded coset penetration property” implies relative hyperbolicity in the sense of Gromov-Bowditch-Yaman. A number of interesting results regarding relatively hyperbolic groups are obtained in [7, 8, 9, 17, 12, 13, 16, 21, 23, 30].

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In particular, Szczepański [30] recently provided a way of constructing groups relatively hyperbolic in the sense of Farb by mimicking the hyperbolization of polyhedra construction.

In this paper we study the class of Artin groups, that is, groups given by a presentation of the form:

\[(\dagger) \quad G = \langle a_1, \ldots, a_n \mid u_{ij} = u_{ji}, \text{ for } 1 \leq i < j \leq n \rangle\]

where for \(i \neq j\)

\[u_{ij} := a_i a_j a_i \ldots \underbrace{a_i a_i \ldots a_i}_{m_{ij} \text{ terms}}\]

and where \(m_{ij} = m_{ji}\) for each \(i < j\). We allow \(m_{ij} = \infty\) in which case the relation \(u_{ij} = u_{ji}\) is omitted from presentation \((\dagger)\).

In the theory of Artin groups a subgroup of an Artin group \(G\) generated by a subset of \(\{a_1, \ldots, a_n\}\) is called a \emph{parabolic subgroup} (see, for example, [27]). It follows from the results of Appel-Schupp [2] that if in \((\dagger)\) all \(m_{ij} \geq 4\) then for any \(i < j\) the two-generator parabolic subgroup \(G_{ij}:= \langle a_i, a_j \rangle \leq G\) is itself an Artin group with the presentation

\[G_{ij} = \langle a_i, a_j \mid u_{ij} = u_{ji} \rangle.\]

Although we will not use the fact, it is easy to see that for an even \(m_{ij} = 2k\) the group \(G_{ij}\) is isomorphic to the Baumslag-Solitar group \(B(k,k) = \langle x, y \mid y^{-1}x^k y = x^k \rangle\). Similarly, if \(m_{ij} = 2k + 1\) is odd, then \(G_{ij}\) is isomorphic to the torus-knot group \(\langle x, y \mid x^{2k+1} = y^2 \rangle\).

An Artin group is said to be of \emph{extra large type} if all \(m_{ij} \geq 4\) in presentation \((\dagger)\). Even before the general theory of hyperbolic groups Appel and Schupp [2] proved theorems about Artin groups of extra large type by showing that they were “relatively small cancellation” with respect to the collection of subgroups \(G_{ij}\). Our main result here is:

**Theorem A.** Let \(G\) be an Artin group given by presentation \((\dagger)\) above. Assume that for all \(i < j\) we have \(m_{ij} \geq 7\). Then \(G\) is relatively hyperbolic in the sense of Farb with respect to the collection of subgroups \(\{G_{ij} \mid m_{ij} < \infty\}\), where \(G_{ij} = \langle a_i, a_j \rangle \leq G\).

Thus Theorem A asserts that \(G\) is relatively hyperbolic with respect to the family of non-free two-generator parabolic subgroups. This provides additional justification for using the term “parabolic subgroup” in the context of Artin groups.

Since we do not actually use it, we refer to Farb’s paper [15] for the precise definition of the “bounded coset penetration property.” But
it is easy to see that if two subgroups $H_1, H_2$ of a group $G$ have infinite intersection then $G$ does not satisfy the bounded coset penetration property with respect to the collection $(H_1, H_2)$ (or with respect to any larger collection of subgroups). Thus for an Artin group $G$ as in Theorem A if there are distinct $i, j, k$ such that $m_{ij}$ and $m_{ik}$ are finite then $G$ does not have the bounded coset penetration property with respect to the collection of subgroups $G_{ij}$. This raises the interesting question of finding different conditions which ensure good group-theoretic properties in the presence of Farb’s definition of relative hyperbolicity. It seems plausible that most Artin groups satisfy some sort of a “nested” version of the bounded coset penetration property, especially since many of these groups are known to be CAT(0) and biautomatic (see for example, [3, 4, 10, 25, 28]).

2. Artin groups and small cancellation theory

In this section we explain how one can apply small cancellation theory to study Artin groups, even though the given finite presentation \([1]\) does not have good small cancellation properties. In order to do this we assume that the reader is familiar with the basics of standard small cancellation theory. For definitions and terminology see Lyndon and Schupp [22].

Let $R_{ij}$ be the symmetrized set of words in $F(a_i, a_j)$ generated by the defining relator, $u_{ij}u_{ji}^{-1}$, and its inverse. The cancellation between two noninverse elements of $R_{ij}$ can be almost half of each of the words but not an entire half. Consider, for example, a product such as $$(a_1a_2a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1})(a_2a_1^{-1}a_2^{-1}a_1^{-1}a_2^{-1}a_1)$$

It is not difficult to verify that the set $R_{ij}$ satisfies the “flat” small cancellation condition $C(4) - T(4)$. To exploit this fact we must use the right normal form for elements of $F(a_1, \ldots, a_n)$.

Convention 2.1. For the remainder of this paper we denote $F := F(a_1, \ldots, a_n)$. In the free group $F$ every nontrivial reduced word $w$ has a unique normal form with exponents

$$w = a_{h_1}^{n_1} \cdots a_{h_s}^{n_s}$$

where each $h_t \neq h_{t+1}$ and each $n_t \neq 0$. The integer $s$ is the syllable length of $w$ and is written $||w||$. The subwords $a_{h_i}^{n_i}$ are called the syllables of $w$. 
Notation 2.2. For each $i < j$ such that $m_{ij} < \infty$ let $R_{ij}$ be the set of all nontrivial cyclically reduced words in $F(a_i, a_j)$ which are equal to the identity in the group $G_{ij}$.

By introducing “strips” to study the fine geometry of $C(4)$-$T(4)$ diagrams, Appel and Schupp [2] proved:

Proposition 2.3. Suppose $2 \leq m_{ij} < \infty$ and let $w$ be a word from $R_{ij}$. Then $||w|| \geq 2m_{ij}$.

Notation 2.4. To deal with the Artin group $G$ we now switch to the following infinite presentation of $G$:

\[
G = \langle a_1, \ldots, a_n \mid R \rangle.
\]

where

\[
R = \bigcup_{m_{ij} < \infty} R_{ij}.
\]

Remark 2.5. The point of shifting to the infinite presentation (‡) is that it allows a strong use of minimality. In considering van Kampen diagrams for a word $w = 1$ in $G$, it suffices to consider minimal diagrams, that is, diagrams with as few as regions as possible. In a minimal diagram $\Delta$ over (‡) distinct regions labelled by relators from the same $R_{ij}$ cannot have even a vertex in common since they could then be combined into a single region, contradicting minimality. Also, any common edge shared by regions labelled from some $R_{ij}$ and $R_{il}$ where $j \neq l$ must be labelled by a power of the generator $a_i$, that is, the edge label has syllable length one. If $G$ is of extra large type, Proposition [2.3] will show that minimal diagrams for the presentation (‡) have “hyperbolic” $C(8)$-geometry.

Notation 2.6. As in [22], if $\Delta$ is a map and $D$ is a region of $\Delta$, then $i(D)$ will denote the interior degree of $D$, that is, the number of $\Delta$-interior edges of $D$, where each edge occurring twice in a boundary cycle of $D$ is counted twice in $i(D)$. Recall also that a region $D$ of $\Delta$ is said to be a boundary region if $\partial D \cap \partial \Delta \neq \emptyset$. Note that a boundary region need not contain a boundary edge of $\Delta$ as the intersection may contain only vertices of $\partial \Delta$. We will say that a boundary region $D$ is a simple boundary region of $\Delta$ if the intersection $\partial D \cap \partial \Delta \neq \emptyset$ is connected and if the edges of $\partial D \cap \partial \Delta$ are consecutively traversed in some boundary cycle of $\Delta$.

Lemma 4.1 in Ch. V of [22] asserts that in a $(6, 3)$-map the boundary of every region is a simple closed curve. In particular this means that if
$D$ is a simple boundary region of a $(6,3)$-map $\Delta$ then the intersection $\partial D \cap \partial \Delta$ is consecutively traversed in some boundary cycle of $D$.

We shall need the following “basic fact” of small cancellation theory [22]:

**Proposition 2.7.** Let $\Delta$ be a $(6,3)$-map without vertices of degree one and with no simple boundary regions of interior degree zero. Then at least one of the following occurs:

(a) The map $\Delta$ contains at least two simple boundary regions of interior degree one.
(b) The map $\Delta$ has at least three simple boundary regions of interior degree at most three.

Proposition 2.3 and Proposition 2.7 together imply [2]:

**Proposition 2.8.** Suppose $G$ is given by $(\dagger)$ where all $m_{ij} \geq 4$. Then:

1. Every minimal diagram over the infinite presentation $(\ddagger)$ satisfies $C(8)$ and every interior edge in such a diagram is labeled by a power of some generator $a_i$.
2. If $w$ is a nontrivial freely reduced word representing $1$ in $G$ then $w$ contains a subword $v$ such that $v$ is also a subword of some $r \in R_{ij}$ with $r = vu$, $||u|| \leq 3$ and $||v|| \geq 2m_{ij} - 3$.
3. If $m_{ij}, m_{ik} < \infty$ for some $k \neq j$ then $G_{ij} \cap G_{ik} = \langle a_i \rangle$. Moreover, if $\{i, j\} \cap \{t, k\} = \emptyset$ then $G_{ij} \cap G_{tk} = \{1\}$ in $G$.

**Convention 2.9.** For the remainder of this section we assume that $G$ is given by presentation $(\ddagger)$ where all $m_{ij} \geq 4$.

**Definition 2.10.** Let $w$ be a freely reduced word in $F(a_1, \ldots, a_n)$. We say that $w$ is *Artin-reduced* if $w$ does not contain a subword $v$ such that $v$ is also a subword of some $r \in R_{ij}$ with $r = vu$, $||u|| \leq 3$.

Similarly, a word $w$ is *strongly Artin-reduced* if $w$ does not contain a subword $v$ such that $v$ is also a subword of some $r \in R_{ij}$ with $r = vu$, $||u|| \leq 4$.

An equality diagram for a pair of Artin-reduced words has the standard structure, consisting of “islands” in which each region “cuts through” the diagram from “top to bottom” as described precisely in the following lemma.

**Lemma 2.11.** Suppose $\Delta$ is a minimal diagram over $(\ddagger)$ such that the boundary cycle of $\Delta$ corresponds to the relation $u =_{G} v$ where both $u$ and $v$ are Artin-reduced. Then $\Delta$ is a union of several (possibly none) linearly arranged disk components connected by (possibly degenerate) arcs, as shown in Figure 1. Moreover, each of these disk components
corresponds to an equality \( u' =_G v' \), where \( u' \) is a subword of \( u \) and \( v' \) is a subword of \( v \), and the component has the form shown in Figure 2 satisfying the following properties:

1. Each region \( Q_s \) corresponds to a relator from \((\mathcal{R})_\alpha\).
2. Each interior edge \( p_s \) is labeled by a single syllable \( a_{i_s}^{n_s} \) and has one endpoint on the “top” part of \( \partial \Delta \) corresponding to \( u \) and its other endpoint on the “bottom” part of \( \partial \Delta \) corresponding to \( v \).
3. Each region \( Q_s \) has at least one boundary edge contained in the top part of \( \partial \Delta \) and at least one boundary edge contained in the bottom part of \( \partial \Delta \).

\[ \begin{array}{c}
\text{Figure 1. Equality diagram}
\end{array} \]

Proof. By removing interior vertices of degree two and combining edge labels, we follow the usual convention that \( \Delta \) has no interior vertices of degree two. Since \( u \) and \( v \) are Artin-reduced, it is clear that \( \Delta \) is indeed a union of linearly arranged disk components, each corresponding to an equality diagram for \( u' =_G v' \) for some subwords \( u' \) of \( u \) and \( v' \) of \( v \), as shown in Figure 1. Also, each of the disk components is a minimal diagram over \((\mathcal{R})_\alpha\).

Recall that by Proposition 2.8 in a minimal diagram over \((\mathcal{R})_\alpha\) no two regions corresponding to relators from the same \( \mathcal{R}_{ij} \) have a common edge. Moreover, if \( r \in \mathcal{R}_{ij}, r' \in \mathcal{R}_{il} \), where \( l \neq j \), have a nontrivial common subword, this subword must be a power of \( a_i \). Thus every interior edge of a minimal diagram over \((\mathcal{R})_\alpha\) is labeled by some \( a_{i_s}^{n_s} \).

Consider an individual disk component \( \Delta_0 \) corresponding to an equality \( u' =_G v' \), where \( u' \) is a subword of \( u \) and \( v' \) is a subword of \( v \).

Let \( x_1 \) be the common initial vertex of the paths labeled \( u' \) and \( v' \) in the boundary of \( \Delta_0 \). Similarly, let \( x_2 \) be their common terminal
vertex. If \( \Delta_0 \) has only one region, Lemma 2.11 certainly holds. Thus we now suppose that \( \Delta_0 \) has at least two regions. Since \( \Delta_0 \) is minimal over presentation (\[\text{[1]}\]), by Proposition 2.8 \( \Delta_0 \) is a \( C(8) \)-diagram. Hence we can apply part Proposition 2.7 to \( \Delta_0 \). Since \( u' \) and \( v' \) are Artin-reduced, case (b) of Proposition 2.7 cannot occur. Thus case (a) takes place and there are two distinct boundary regions \( Q \) and \( Q' \) in \( \Delta_0 \) each with interior degree one and with a single syllable written on its interior edge, as shown in Figure 3. Denote those edges by \( q \) and \( q' \) accordingly.
It is clear that \( x_1 \) belongs to one of these regions and \( x_2 \) belongs to the other. Suppose \( x_1 \) is in \( Q \) and \( x_2 \) is in \( Q' \). If \( \Delta_0 \) consists of two or three regions, the statement of the lemma obviously holds. Otherwise let \( \Delta' \) be obtained from \( \Delta_0 \) by first removing the regions \( Q, Q' \), and, if their removal creates any vertices of degree one, then successively removing vertices of degree one and their incident edges until no such vertices remain. If \( \Delta' \) has fewer than two regions then the statement of the lemma holds.

Otherwise the diagram \( \Delta' \) contains at least two regions and we can apply Proposition \ref{prop:two-vertices} to it. Again, clearly case (b) of Proposition \ref{prop:two-vertices} cannot occur and case (a) takes place. Let \( S, S' \) be two distinct boundary regions in \( \Delta' \), each of interior degree one and thus with a single syllable written on the interior edge. Since \( u \) and \( v \) are weakly Artin-reduced, one of these regions, say \( S \), contains the edge \( q \) and the other region \( S' \) contains \( q' \).

We claim that each of \( S, S' \) has an edge in common with both the top and the bottom portions of the boundary of \( \Delta_0 \). If this is not the case then either \( u' \) or \( v' \) contains a subword of \( r_{ij}^* \) missing at most two letters, contradicting the assumption that \( u' \) and \( v' \) are Artin-reduced. Hence we can remove \( S \) and \( S' \) from \( \Delta' \) and repeatedly apply the same argument to the remaining diagram.

This yields the statement of Lemma \ref{lem:two-regions}.

\[ \square \]

3. Coned-off Cayley graphs and relative hyperbolicity

Recall that a \textit{geodesic path} in a metric space \((X, d)\) is an isometric embedding \( \gamma : [a, b] \to X \), where \([a, b] \subseteq \mathbb{R} \). In this situation we call the set \( \gamma([a, b]) \) a \textit{geodesic segment} from \( x = \gamma(a) \) to \( y = \gamma(b) \) in \( X \) and denote it by \([x, y]\). A metric space \((X, d)\) is said to be \textit{geodesic} if for any \( x, y \in X \) there exists a geodesic segment \([x, y]\) in \( X \). The general notion of a “hyperbolic metric space” is, of course, due to Gromov \cite{Gromov}:

\textbf{Definition 3.1 (Hyperbolic Metric Space).} A geodesic metric space \((X, d)\) is said to be \textit{hyperbolic} if there is a number \( \delta \geq 0 \) such that for any three points \( x, y, z \in X \), for any geodesic segments \([x, y]\), \([x, z]\) and \([y, z]\) and for any point \( p \in [x, y] \) there is a point \( q \in [x, z] \cup [y, z] \) such that

\[ d(p, q) \leq \delta. \]

We refer the reader to \cite{Drutu, Hruska, Papadopoulos, Gromov} for the background material on hyperbolic spaces and hyperbolic groups.

We need to recall Farb’s definitions of coned-off Cayley graphs and relative hyperbolicity \cite{Farb}:
Definition 3.2 (Coned-off Cayley Graph). Let $G$ be a group with a finite generating set $A$ and let $H_1, \ldots, H_t$ be a family of subgroups of $G$. Let $\Gamma = \Gamma_A(G)$ be the Cayley graph of $G$ with respect to $A$. The coned-off Cayley graph $X = \hat{\Gamma}_A(G; H_1, \ldots, H_t)$ is obtained from $\Gamma$ as follows: for each coset $gH_i$ (where $g \in G$, $1 \leq i \leq t$) we add a new vertex $v(gH_i)$ to $\Gamma$ and for every element $g' \in gH_i$ add an edge of length $1/2$ from $v(gH_i)$ to $g'$.

The vertices $v(gH_i)$ are referred to as cone vertices and the edges from $v(gH_i)$ to elements of $gH_i$ are called cone edges.

Definition 3.3 (Relative Hyperbolicity). Let $G$ be a finitely generated group and let $H_1, \ldots, H_t$ be a family of subgroups of $G$. We say that $G$ is relatively hyperbolic in the sense of Farb with respect to the $t$-tuple of subgroups $(H_1, \ldots, H_t)$ if for some finite generating set $A$ of $G$ the coned-off Cayley graph $X = \hat{\Gamma}_A(G; H_1, \ldots, H_t)$ is a hyperbolic metric space.

Recall that a finitely generated group $G$ is hyperbolic if for some (and hence for any) finite generating set $A$ of $G$ the Cayley graph $\Gamma(G, A)$ is a hyperbolic metric space. Thus $G$ is hyperbolic if and only if it is relatively hyperbolic with respect to the trivial subgroup $H = \{1\}$.

It is shown in [15] that $G$ being relatively hyperbolic with respect to the given collection of subgroups does not depend on the choice of the generating set $A$.

4. Geodesics in the coned-off Cayley graph of an Artin group

Let $G$ be given by presentation where $m_{ij} \geq 7$ for all $i \neq j$. Let $\Gamma = \Gamma_A(G)$ where $A = \{a_1, \ldots, a_n\}$. Let $X = \hat{\Gamma}_A(G; \{G_{ij} | m_{ij} < \infty\})$ be the coned-off Cayley graph of $G$. If $e$ is an edge of a graph, we will denote by $o(e)$ the initial vertex of $e$ and by $t(e)$ the terminal vertex of $e$. These notations and conventions will be fixed for the remainder of the paper.

Convention 4.1. Let $g \in G$ be an arbitrary element and let $\alpha$ be a geodesic in $X$ from $1 \in G$ to $g$. The goal of this section is to show that $\alpha$ is 4-close in $X$ to a path $\gamma$ from 1 to $g$ in $\Gamma$ whose label is strongly Artin-reduced. As an intermediate step in constructing this path $\gamma$ we will first define an auxiliary path $\beta$. Unless specified otherwise, we shall fix $g$ and $\alpha$ for the remainder of this section.
4.1. **Construction of the path** $\beta$. The path $\alpha$ can be subdivided as a concatenation of paths

$$\alpha = \alpha_1 \ldots \alpha_t$$

where $t \geq 1$ and where each $\alpha_k$ is either a path in $\Gamma$ from a vertex of $\Gamma$ to a vertex of $\Gamma$, in which case $\alpha_k$ is referred to as a $\Gamma$-block, or $\alpha_k = e_k e'_k$, where $e_k$ and $e'_k$ are cone-edges separated by a cone-vertex, in which case $\alpha_k$ is referred to as a cone-block. Thus $t(e_k) = o(e'_k)$ is a cone-vertex of $X$ and $t(e'_k) = o(e_k) g_k$ for some $g_k \in G_{ij} - \{1\}$. Moreover, we can assume that the $\Gamma$-blocks are chosen maximally, so that no two such blocks are consecutive in $\alpha$. For each cone-block $\alpha_k$ choose a freely reduced word $v_k$ in $G_{ij}$ of the smallest possible syllable length representing $g_k$. Note that if $g_k$ belongs to two distinct $G_{ij}$-subgroups and thus is a power of the generator then the pair $\{i, j\}$ may not be uniquely defined by $g_k$, but in that case the word $v_k$ of syllable length one will be uniquely determined by $g_k$.

**Remark 4.2.** The choice of the word $v_k$ for the cone-block $\alpha_k$ as a word in the corresponding $G_{ij}$ which is “geodesic” with respect to syllable length is a crucial feature of our argument.

If $\alpha_k$ is a $\Gamma$-block then the label $v_k$ of $\alpha_k$ is an $A$-geodesic word.

**Notation 4.3.** The word

$$v = v_1 \ldots v_t$$

certainly represents the element $g$. We denote the path in $\Gamma$ from 1 to $g$ with label $v$ by $\beta$. Thus

$$\beta = \beta_1 \ldots \beta_t$$

where $\beta_k$ is the path in $\Gamma$ with label $v_k$ from the initial point of $\alpha_k$ to the terminal point of $\alpha_k$.

The following lemma is a straightforward consequence of the definitions:

**Lemma 4.4.** Let $\alpha$ and $\beta$ be as above. Then:

1. The label $v_k$ of each $\Gamma$-block $\alpha_k$ is a $\Gamma$-geodesic word.
2. If $v'$ is a subword of the label $v_k$ of some $\Gamma$-block such that $v'$ is a word in some $G_{ij}$ then $|v'| \leq 1$.
3. Suppose that $\alpha_{k+1}$ is a $\Gamma$-block and that $\alpha_k$ is a cone-block with $v_k \in G_{ij}$. Then the first syllable of $v_{k+1}$ is not a power of either $a_i$ or $a_j$. 


Suppose that $\alpha_k$ is a $\Gamma$-block and that $\alpha_{k+1}$ is a cone-block with $v_{k+1} \in G_{ij}$. Then the last syllable of $v_k$ is not a power of either $a_i$ or $a_j$.

If $\alpha_k, \alpha_{k+1}$ are cone-blocks and $v_k \in G_{ij}, v_{k+1} \in G_{st}$ then $\{i, j\} \neq \{s, t\}$.

The paths $\alpha$ and $\beta$ are 2-Hausdorff close in $X$.

Lemma 4.5. Suppose that $\alpha_{k-1}, \alpha_k, \alpha_{k+1}$ are cone-blocks with $v_{k-1} \in G_{ij}, v_k \in G_{js}, v_{k+1} \in G_{sq}$. If the first syllable of $v_k$ is a power of $a_j$ and the last syllable of $v_k$ is a power of $a_i$ then $||v_k|| \geq 3$.

Proof. If $||v_k|| = 2$ then we can replace the path $\alpha_{k-1}\alpha_k\alpha_{k+1}$ of length 3 in $X$ by a path of length 2 in $X$ consisting of two pairs of cone-edges: the first corresponding to $G_{i,j}$ and the second corresponding to $G_{sq}$. This contradicts the assumption that $\alpha$ is a geodesic in $X$. \qed

4.2. Construction of the path $\gamma$. Note that the word $v$ is not necessarily reduced. Indeed, it is possible that $\alpha_k, \alpha_{k+1}$ are two consecutive cone blocks such that the last syllable of $v_k$ and the first syllable of $v_{k+1}$ are powers of the same generator. (The above lemmas imply that this is the only way in which $v$ may fail to be freely reduced). We will modify $v$ and $\beta$ to rectify this problem by “condensing syllables” from left to right as follows.

We define a sequence of words $u_1, \ldots, u_t$ inductively.

Put $u_1 := v_1$. Suppose $1 \leq k < t$ and $u_1, \ldots, u_k$ are already defined. We have two cases to consider:

(a) Suppose $\alpha_k$ is a cone-block with $v_k \in G_{ij}$.

Let $u_k = z_k s_k$ where $s_k$ is the last syllable of $u_k$.

If $\alpha_{k+1}$ is also a cone-block, let $v_{k+1} = s_{k+1} y_{k+1}$ where $s_{k+1}$ is the first syllable of $v_{k+1}$. If the last syllable of $v_k$ and the first syllable of $v_{k+1}$ are not powers of the same generator, then let $u_{k+1} = v_{k+1}$. If the last syllable of $v_k$ and the first syllable of $v_{k+1}$ are powers of the same generator, then let $u_k'$ be the reduced form of $u_k s_{k+1}$ and redefine $u_k$ to be $u_k'$. Set $u_{k+1} := y_{k+1}$.

If $\alpha_{k+1}$ is a $\Gamma$-block, put $u_{k+1} := v_{k+1}$.

(b) If $\alpha_k$ is a $\Gamma$-block, put $u_{k+1} := v_{k+1}$.

Notation 4.6. We set

\[ u := u_1 \ldots u_t, \]

and denote by $\gamma$ the path in $\Gamma$ from 1 to $g$ with label $u$. Also, denote by $\gamma_k$ the subpath of $\gamma$ corresponding to $u_k$, so that

\[ \gamma = \gamma_1 \ldots \gamma_t. \]
We summarize the relevant properties of $u$ and $\gamma$:

**Lemma 4.7.** The following hold:

1. The word $u$ as in (♠) is freely reduced.
2. Suppose $\alpha_k$ is a cone-block with $u_k \in G_{ij}$. Then for any word $w$ in $G_{ij}$ representing the same element as $u_k$ we have $\|u_k\| \leq \|w\| + 1$.
3. For any $k < t$ we have $\|u_k u_{k+1}\| = \|u_k\| + \|u_{k+1}\|$.
4. For any $\Gamma$-block $\alpha_k$ we have $\gamma_k = \beta_k = \alpha_k$ and $u_k = v_k$.
5. The paths $\gamma$ and $\beta$ are $2$-Hausdorff close in $X$.

**Lemma 4.8.** Let $w$ be a reduced word in some $G_{ij}$, $m_{ij} < \infty$ such that $\|w\| \geq 4$. Suppose $w$ is a subword of $u$ and let $w'$ be obtained from $w$ by removing the first and the last syllables of $w$. Then $w'$ is a subword of some $u_k$ such that $\alpha_k$ is a cone-block of $\alpha$.

**Proof.** Suppose first that the occurrence of $w$ in $u$ overlaps some $u_k = v_k$ corresponding to a $\Gamma$-block $\alpha_k$ of $\alpha$. Then some syllable $s$ of $w$ is a subword of this $u_k$. Recall that no two $\Gamma$-blocks are adjacent in $\alpha$. Part 2 of Lemma 4.4 implies that neither of the syllables preceding or following $s$ of $w$ can overlap $u_k$. Let $u_k = asb$. Among the words $a, b$ choose the one of largest syllable length. By symmetry we may assume that $\|a\| \geq \|b\|$. Hence $\|a\| \geq 2$ since $\|w\| \geq 4$. Since $\|a\| > 0$ then $\alpha_{k-1}$ is a cone-block. Also, since $\|a\| \geq 2$ and $w$ is a word in $G_{ij}$, we conclude that both $s$ (which is the first syllable of $u_k$) and $u_{k-1}$ are words in $G_{ij}$ (this is true even if $\|u_{k-1}\| = 1$). This is impossible by Lemma 4.4.

Thus $w$ does not overlap any $u_k$ corresponding to $\Gamma$-blocks $\alpha_k$.

Assume now that there is some $u_k$ contained in $w$ such that $w$ also overlaps $u_{k-1}$ and $u_{k+1}$. Each of $\alpha_{k-1}$, $\alpha_k$ and $\alpha_{k+1}$ is a cone-block. If $\|u_k\| = 1$ then either the part of $w$ following $u_k$ or the part of $w$ preceding $u_k$ has syllable length at least two. Since $w$ is a word in $G_{ij}$ this yield a contradiction with part 3 of Lemma 4.4 and the definition of $u$. Hence $\|u_k\| \geq 2$. If either the either the part of $w$ following $u_k$ or the part of $w$ preceding $u_k$ has syllable length at least two, then either both $v_{k-1}, v_k$ or both $v_k, v_{k+1}$ belong to $G_{ij}$. Again, this is impossible by part 5 of Lemma 4.4. The statement of Lemma 4.8 now follows.

Suppose now that there is no $u_k$ contained in $w$ such that $w$ also overlaps $u_{k-1}$ and $u_{k+1}$. Thus $w$ overlaps at most two of the words $u_k$. If $w$ is contained in a single $u_k$, the statement of Lemma 4.8 obviously holds. Assume now that $w$ is a subword of $u_k u_{k+1}$ and that $w$ overlaps both of these words. If both overlaps have syllable length at least two, then $g_k, g_{k+1} \in G_{ij}$, contrary to part 5 of Lemma 4.4. If one of
the overlaps has syllable length one, then all but the first or the last syllable of $w$ is contained in either $u_k$ or $u_{k+1}$ and the statement of Lemma 4.8 holds.

Proposition 4.9. The word $u$ is strongly Artin reduced and the paths $\alpha$ and $\gamma$ are 4-Hausdorff close in $X$.

Proof. The paths $\alpha$ and $\gamma$ are 4-close since each is 2-close to the path $\beta$ by part 6 of Lemma 4.4 and by part 5 of Lemma 4.7.

We now show that $u$ is strongly Artin reduced. If not, there is a nontrivial relator $r \in R_{ij}$ for some $m_{ij} < \infty$ and a subword $w$ of $u$ such that $r = wy$ and $||y|| \leq 4$. Thus $w = y^{-1}$ in $G$.

It is now that we use the assumption that all $m_{ij} \geq 7$. By Proposition 2.3 this condition implies that $||r|| \geq 14$ and hence $||w|| \geq 10$. Let $w = s_1w's_2$ where $s_1, s_2$ are respectively the first and the last syllables of $w$. Then $||w'|| \geq 8$ and by Lemma 4.8 $w'$ is a subword of some $u_k$ such that $\alpha_k$ is a cone-block of $\alpha$.

Moreover, if we write $u_k$ as $u_k = zw'z'$ then $||u_k|| = ||z|| + ||w'|| + ||z'||$. Using the relator $r$, we have

$$w' = G s_1^{-1} y^{-1} s_2^{-1}$$

and hence $u_k = G z s_1^{-1} y^{-1} s_2^{-1} z'$. But

$$||z s_1^{-1} y^{-1} s_2^{-1} z'|| \leq ||z|| + 1 + 4 + 1 + ||z'|| < ||z|| + ||w'|| - 1 + ||z'|| = ||u_k|| - 1,$$

where the last inequality holds since $||w'|| \geq 8$. Hence there exists a word representing the same element as $u_k$ but with syllable length less than $||u_k|| - 1$.

This contradicts part 2 of Lemma 4.7. □

5. Proof of the Main Result

We recall the following useful fact due to Papasoglu [26]:

Proposition 5.1 (Thin bigons criterion). Let $C$ be a connected graph with the simplicial metric $d$. Then $C$ is hyperbolic if and only if there is some $\delta > 0$ such that for any points $x, y \in C$ (possibly inside edges) any two geodesic paths from $x$ to $y$ in $C$ are $\delta$-Hausdorff close.

Remark 5.2. The above result was stated in [26] only for Cayley graphs of finitely generated groups. However, it is easy to see that Papasoglu’s proof [26] does not use the Cayley graph assumption and works for any connected graph with the simplicial metric. This was noted, for example, in [24].

We are now ready to prove Theorem $\Lambda$. 
Theorem A. Let $G$ be an Artin group given by presentation $\{a_1, \ldots, a_n\}$. Suppose that for all $i < j$ we have $m_{ij} \geq 7$.

Then $G$ is relatively hyperbolic in the sense of Farb with respect to the collection of subgroups $\{G_{ij}|m_{ij} < \infty\}$.

Proof. Put $A = \{a_1, \ldots, a_n\}$. Let $G = \Gamma_A(G)$ and let

$$X = \Gamma_A(G; \{G_{ij}|m_{ij} < \infty\})$$

be the coned-off Cayley graph of $G$. We need to show that $X$ is hyperbolic.

Note that if we barycentrically subdivide each of the $\Gamma$-edges of $X$ to obtain the graph $X'$, then twice the metric of $X$ coincides with the simplicial metric for $X'$. Hence to establish Theorem A by using Papasoglu’s criterion it suffices to show that geodesic bigons in $X$ (with endpoints possibly inside edges) are uniformly thin. Moreover, the definition of $X$ implies that it is enough to prove the following:

Claim. There exists a constant $K > 0$ with the following property. Let $\alpha_1, \alpha_2$ be geodesics in $X$ from $h_1 \in G$ to $g_1 \in G$ and from $h_2 \in G$ to $g_2 \in G$ respectively such that:

1. Either $h_1^{-1}h_2 \in G_{ij}$ for some $m_{ij} < \infty$ or $d_\Gamma(h_1, h_2) \leq 1$.
2. Either $g_2^{-1}g_1 \in G_{sl}$ for some $m_{sl} < \infty$ or $d_\Gamma(g_1, g_2) \leq 1$.

Then $\alpha_1$ and $\alpha_2$ are $K$-Hausdorff close in $w$.

To verify the Claim, let $a, b \in G$ be such that $h_1 = h_2a$ and $g_1 = g_2b$. By Proposition 4.9 for $l = 1, 2$ there is a path $\gamma_l$ in $\Gamma$ from $h_l$ to $g_l$ such that the label $U_l$ of $\gamma_l$ is a strongly Artin-reduced word and such that the paths $\gamma_l$ and $\alpha_l$ are 4-close in $X$, as shown in Figure 4.

If $d_\Gamma(h_1, h_2) \leq 1$ then $a$ is either trivial or a single letter of $A \cup A^{-1}$. Let $W_1$ be the freely reduced form of the word $aU_1$. Since $U_1$ is strongly Artin-reduced, the word $W_1$ is Artin-reduced.

Assume now that $d_\Gamma(h_1, h_2) \geq 2$, so that $a \in G_{ij}$ for some $m_{ij} < \infty$. If $a$ belongs to the cyclic subgroup generated by one of the letters of $A$ (in which case $G_{ij}$ may not be uniquely determined by $a$), say $a = a_i^q$, we let $W_1$ be the freely reduced from of the word $a_i^qU_1$. Again, the word $W_1$ is Artin-reduced because $U_1$ is strongly Artin-reduced. Assume now that $a$ is not a power of the generator, so that $a \in G_{ij}$ for a unique pair $\{i, j\}$. Let $U_1 = Y_1Z_1$ where $Y_1$ is the maximal initial segment of $U_1$ which is a word in $G_{ij}$. Let $Y_1'$ be the reduced word of smallest possible syllable length in $G_{ij}$ representing the element $aY_1$. Then $Y_1'$ and $Z_1$ are strongly Artin-reduced and the word $W_1 = Y_1'Z_1$ is Artin-reduced.

In either of the above cases, using the cone-vertex $v(h_1G_{ij})$ it is easy to see that the path $\gamma_1$ is 2-close in $X$ to the path $\lambda_1$ in $\Gamma$ from $h_2$ to $g_1$ labeled by $W_1$. 

Similarly, by considering the product $U_2 b$ we can find an Artin-reduced word $W_2$ and a path $\lambda_2$ from $h_2$ to $g_1$ in $\Gamma$ with label $W_2$ such that $\gamma_2$ and $\lambda_2$ are 2-close in $X$.

Now consider an equality diagram for the equality $W_1 =_G W_2$ of smallest area over the infinite presentation $[3]$ of $G$. Applying Lemma 2.11 to this diagram and using cone vertices we see that $\lambda_1$ and $\lambda_2$ are 2-close in $X$. Adding the distances between the paths considered in Figure 4 we see that $\alpha_1$ and $\alpha_2$ are 14-close in $X$ and the Claim is verified. \hfill \Box

Remark 5.3. The value $K = 14$ obtained in the above proof of Theorem A using Papasoglu’s criterion does not depend on the particular group $G$. Thus all the coned-off Cayley graphs $X$ of the groups $G$ covered by Theorem A are all $\delta$-hyperbolic for some $\delta > 0$ which is independent of $X$ and hence of the choice of $G$. So the relative hyperbolicity of $G$ with respect to $G_{ij}$’s is in a sense uniform over the class of Artin groups covered by the Theorem.

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