A simple trace formula for arithmetic groups

by

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Introduction

Let $G$ denote a semisimple Lie group and $\Gamma$ an arithmetic subgroup. Suppose first that $\Gamma$ is cocompact, i.e. the quotient manifold $G/\Gamma$ is compact. Then the Selberg trace formula is the equality

$$J_{\text{geom}}(f) = J_{\text{spectral}}(f),$$

where $J_{\text{geom}}(f)$, the geometric side, is a sum of Orbital integrals and the spectral side is the trace of $f$ on $L^2(G/\Gamma)$.

In the case that $\Gamma$ is not cocompact in $G$, the trace formula as developed by J. Arthur, gives a similar identity, but now both sides are more complicated. The geometric side is an alternating sum of contributions belonging to the classes of parabolic subgroups. This formula is set up and refined by Arthur to fit the needs of number theorists who primarily want to compare the trace formulas for two different groups. For geometric applications one would want a trace formula whose geometric side consists of orbital integrals just as in the cocompact case. In this paper we show that such a formula can be deducted from Arthur’s work by plugging in special test functions.
For groups of rational rank equal to one we show how the geometric side also can be computed further.

1 A simple trace formula

We derive a simple version of Arthur’s trace formula by inserting functions with certain restrictive properties which guarantee the vanishing of the parabolic terms in the trace formula.

1.1 Let $G$ be a semi simple simply connected linear algebraic group over $\mathbb{Q}$ and write $G$ for the group of real points. Let $\mathbb{A}$ denote the adele-ring over $\mathbb{Q}$. On the locally compact group $G(\mathbb{A})$ we will fix a Haar measure. Let $\mathbb{A}_{\text{fin}}$ denote the subring of finite adeles then $\mathbb{A} \cong \mathbb{A}_{\text{fin}} \times \mathbb{R}$, hence $G(\mathbb{A}) \cong G(\mathbb{A}_{\text{fin}}) \times G(\mathbb{R})$. We will distribute the Haar measure onto the factors.

According to Harish-Chandra, every Haar-measure on $G$ comes from a scalar multiple $B$ of the Killing form and this form fixes Haar-measures on the closed subgroups on $G$. We will keep this form at our disposal and will only fix it in later sections.

1.2 We will consider $G$ as a subgroup of some $GL_m$. Let $\Gamma$ be a congruence subgroup, i.e. $\Gamma$ is a rational lattice which contains a principal congruence subgroup $\Gamma(N) = \{g \in (G(\mathbb{Q}) \cap GL_m(\mathbb{Z}))/g \equiv 1 \mod(N)\}$ for some natural number $N$.

Invariantly stated the congruence property is pronounced as follows: $\Gamma \subset G(\mathbb{Q})$ is a congruence subgroup if there is a compact open subgroup $K_{\Gamma} \subset G(\mathbb{A}_{\text{fin}})$, where $\mathbb{A}_{\text{fin}}$ is the ring of finite adeles over $\mathbb{Q}$, such that $\Gamma = G(\mathbb{Q}) \cap K_{\Gamma}$. By strong approximation there is a canonical bijection $\Psi : \Gamma \setminus G = \Gamma \setminus G(\mathbb{R}) \longrightarrow G(\mathbb{Q}) \setminus G(\mathbb{A})/K_{\Gamma}$

given by $\Gamma g \mapsto G(\mathbb{Q})gK_{\Gamma}$. We will further assume $\Gamma$ to be weakly neat, i.e. $\Gamma$ is torsion free and for any $\gamma \in \Gamma$ the adjoint morphism $\text{Ad}(\gamma)$ on the $\mathfrak{g}$ has no root of unity other then 1 as an eigenvalue. Note that any arithmetic $\Gamma$ has a weakly neat subgroup of finite index.

1.3 Let $\mathcal{P} \neq G$ denote a parabolic subgroup defined over $\mathbb{Q}$ and let $\mathcal{P}_1 \subset \mathcal{P}$ be a minimal parabolic defined over $\mathbb{R}$. Let $\mathcal{P} = LN$ and $\mathcal{P}_1 = L_1N_1$ denote
Levi decompositions and write \( \mathcal{A} \) resp. \( \mathcal{A}_1 \) for the split components where we assume \( \mathcal{A} \subset \mathcal{A}_1 \). Note that we have \( P = \mathcal{P}(\mathbb{R}) = \mathcal{L}(\mathbb{R})^1 \mathcal{A}(\mathbb{R})^0 \mathcal{N}(\mathbb{R}) = MAN \), where \( \mathcal{L}(\mathbb{R})^1 \) is the subgroup of all \( m \in \mathcal{L}(\mathbb{R}) \) such that \( \chi(m) \) has absolute value 1 for all rational characters \( \chi \) of \( \mathcal{L} \).

1.4 Fix a maximal compact subgroup \( K \subset G = \mathcal{G}(\mathbb{R}) \) such that the Lie algebra of \( K \) is orthogonal to the Lie algebra of \( \mathcal{A}_1(\mathbb{R}) \). Choose a maximal compact open subgroup \( K_{\text{max}} = \prod_p K_p \) of \( G(\mathbb{A}_{\text{fin}}) \) such that \( G(\mathbb{A}_{\text{fin}}) = K_{\text{max}} P(\mathbb{A}_{\text{fin}}) \), this is achieved by assuming \( K_p \) to be a good maximal compact subgroup for all \( p \). We assume \( K_\Gamma \subset K_{\text{max}} \).

1.5 Consider the function \( f = f_{\text{fin}} \otimes f_\infty \in C_c(\mathcal{G}(\mathbb{A})) \) defined by \( f_{\text{fin}} = \frac{1}{\text{vol}(K_\Gamma)} 1_{K_\Gamma} \), the characteristic function of the compact open subgroup divided by the volume. On the function \( f_\infty : G \to \mathbb{C} \) we put the following restriction: At first we insist that \( f_\infty \) has compact support and is \( j \)-times continuously differentiable for some \( j \in \mathbb{N} \) which is assumed to be large enough \( [1] \). We further insist that for any \( \mathcal{Q} \)-parabolic \( P = MAN \neq G \) we have

\[
f_\infty(x^{-1}qx) = 0
\]

for any \( x \in G \) and \( q \in MN \).

This implies that \( f(\text{man}) = 0 \) for \( \text{man} \in MAN \) if \( a = 1 \). One might call these elements \( P \)-singular. Elements which are not \( P \)-singular will be called \( P \)-\textbf{regular}. So we insist that \( f_\infty \) vanishes on all \( P \)-singular elements for all nontrivial \( \mathcal{Q} \)-parabolic subgroups. We say for short that \( f \) \textbf{vanishes on \( \mathcal{Q} \)-parabolically singular elements}.

1.6 An element of \( G \) is called \textbf{regular} if its centralizer is a torus. Regular elements are semi simple and the set \( G' \) of regular elements is dense in \( G \). An element which is not regular will be called \textbf{singular}. It is not hard to show that regular elements are \( P \)-regular for any nontrivial parabolic \( P \) in \( G \). This implies that if \( f_\infty \) vanishes on singular elements then it already vanishes on parabolically singular elements.

1.7 Given \( f \) we define \( K(x, y) \) by

\[
K(x, y) = \sum_{\gamma \in \mathcal{G}(\mathbb{Q})} f(x^{-1} \gamma y).
\]

Since \( f \) has compact support the sum is locally finite and so \( K(x, y) \) inherits the smoothness from \( f \).
1.8 Fix a $\mathbb{Q}$-parabolic $\mathcal{P}$. As in [1], p.923 define

$$K_{\mathcal{P},o}(x,y) = \sum_{L(\mathbb{Q}) \cap o(x,y)} \int_{N(\mathbb{A})} f(x^{-1} \gamma ny) \, d\gamma.$$ 

To define the geometric side, Arthur fixes some functions $\hat{\tau}_p(H(\delta x) - T)$. Here the only thing we need to know is that this factor equals 1 for the trivial parabolic $P = G$. Then Arthur defines

$$k^T_o(x,f) = \sum_{P(-1)} (-1)^{\dim A_p} \sum_{\delta \in \mathcal{P}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{Q})} K_{\mathcal{P},o}(\delta x, \delta x) \hat{\tau}_p(H(\delta x) - T),$$

and

$$J_{\text{geom}}(f) = \int_{\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A})} \sum_{o} k^T_o(x,f) \, dx.$$ 

We will show

**Lemma 1.9** Suppose $\mathcal{P} \neq \mathcal{G}$ and $f$ satisfies 1.5, then $K_{\mathcal{P},o}(x,x) = 0$ for any $x \in \mathcal{G}(\mathbb{A})$.

**Proof:** Let $\gamma \in L(\mathbb{Q}) \subset L(\mathbb{A})^1$ and $n \in N(\mathbb{A})$. Then $\gamma n \in \mathcal{P}(\mathbb{A})^1$. Now let $q \in \mathcal{P}(\mathbb{A})^1$ be arbitrary and $x \in \mathcal{G}(\mathbb{A})$, we will show that $f(x^{-1}qx) = 0$. Assume therefore $q = q_{\text{fin}}q_{\infty}$ with $x^{-1}q_{\text{fin}}x \in \text{supp}(f_{\text{fin}}) = K_T$, then it follows that $q_{\text{fin}} \in xK_Tx^{-1} \cap \mathcal{P}(\mathbb{A})$, a compact subgroup of $\mathcal{P}(\mathbb{A})$. Any continuous quasicharacter with values in $]0, \infty[$ will therefore be trivial on $q_{\text{fin}}$, hence $q_{\text{fin}} \in \mathcal{P}(\mathbb{A})^1$. Since $q$ already was in $\mathcal{P}(\mathbb{A})^1$ it follows $q_{\infty} \in \mathcal{P}(\mathbb{A})^1 \cap \mathcal{P}(\mathbb{R}) = L'_Q(\mathbb{R})N_Q(\mathbb{R}) = MN$ but this implies $f_{\infty}(q_{\infty}) = 0$ as claimed.

Q.E.D.

1.10 For $g \in C_c(G)$ and $y \in G$ define the **orbital integral** as

$$O_y(g) := \int_{G_y \setminus G} g(x^{-1}yx) \, dx.$$ 

Where $G_y$ denotes the centralizer of $y$ in $G$. (Recall our conventions on Haar-measures.) It is known that the integral always converges.

1.11 Let $R$ denote the representation of $\mathcal{G}(\mathbb{A})$ on $L^2(\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A}))$ and let
From the lemma it follows that along the diagonal the kernel $K$ coincides with the modified kernel as in [1]. So it follows that the integral over $G(\mathbb{Q}) \backslash G(\mathbb{A})$ of the diagonal $K(x,x)$ exists.

**Theorem 1.12** Let the function $f$ on $G(\mathbb{A})$ satisfy [L3] then the geometric side of the trace formula $J_{\text{geom}}(f)$ equals:

$$
\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} K(x,x) \, dx = \sum_{[\gamma]} \text{vol}(\Gamma_{\gamma} \backslash G(\mathbb{R})) \cdot O_{\gamma}(f_{\infty}),
$$

where the sum on the right hand side runs over the set of all conjugacy classes $[\gamma]$ in the group $\Gamma$.

**Proof:** Consider the bijection $\Psi : \Gamma \backslash G \to G(\mathbb{Q}) \backslash G(\mathbb{A})/K_{\Gamma}$ and let $\Psi_{*}$ denote the unitary map

$$
\Psi_{*} : L^2(\Gamma \backslash G) \to L^2(\mathcal{G}(\mathbb{Q}) \backslash G(\mathbb{A})/K_{\Gamma}) \quad \text{by} \quad \sqrt{\text{vol}(K_{\Gamma})}^{-1}
$$

further let

$$
\Phi_{*} : L^1(\Gamma \backslash G) \to L^1(\mathcal{G}(\mathbb{Q}) \backslash G(\mathbb{A})/K_{\Gamma}) \quad \text{by} \quad \text{vol}(K_{\Gamma})^{-1}
$$

At first note that by construction

$$
\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \Phi_{*}\varphi(x)dx = \int_{\Gamma \backslash G} \varphi(x)dx.
$$

Next we have $\Psi_{*}R(f_{\infty}) = R(f)\Psi_{*}$ and this gives $\Psi_{*1}\Psi_{*2}K = K_{\infty}$, where $R(f_{\infty})$ is the convolution by $f_{\infty}$ on $L^2(\Gamma \backslash G)$ and $K_{\infty}$ its kernel. The indices at $\Psi_{*}$ indicate that it is applied to each argument separately. Let $K(x) = K(x,x)$ and $K_{\infty}(x) = K_{\infty}(x,x)$ as a function in one argument. Clearly $(\Psi_{*1}\Psi_{*2}K)(x,x) = \Phi_{*}K(x)$ and thus

$$
\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} K(x,x) \, dx = \int_{\Gamma \backslash G} K_{\infty}(x,x) \, dx.
$$

Arthur showed absolute convergence in [1]. The rest is the usual calculation expressing the integral of the diagonal as a sum of orbital integrals.

Q.E.D.
1.13 Now we consider the spectral side. We will now assume that the rank of $\mathcal{G}$ over $\mathbb{Q}$ equals 1. Then there is, up to conjugation, only one nontrivial $\mathbb{Q}$-parabolic $\mathcal{P}$ in $\mathcal{G}$. By Arthur’s kernel identity we get that

$$\int_{\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})} K(x, x) dx$$

equals the sum of

$$\sum_{\pi \in \hat{\mathcal{G}}(\mathbb{A})} m_{\pi} \text{tr } \pi(f)$$

and

$$\sum_{\chi} \frac{1}{4\pi i} \int_{i_a^*} \sum_{\phi \in B_{\mathcal{P}, \chi}} \left( \wedge^T E(\cdot, I_\mathcal{P}(\lambda, f)\phi, \lambda), \wedge^T E(\cdot, \phi, \lambda) \right) d\lambda,$$

where $m_{\pi}$ is the multiplicity of $\pi$ in $L^2(\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}))_{\text{disc}}$, the discrete part, further the sum $\sum_{\chi}$ runs over all cuspidal representations of $\mathcal{L}(\mathbb{A})$ and $B_{\mathcal{P}, \chi}$ is a orthonormal basis of the $\chi$-isotype and $\wedge^T E$ is the truncated Eisenstein series as in [2]. Using the fact that the above is independent of the cutoff parameter $T$, by the Maass-Selberg relations ([19], [2]) and the usual Fourier analysis [11] the latter summand is seen to be

$$\frac{1}{4\pi i} \int_{i_a^*} \text{tr} (1 + M(\lambda)^{-1} M'(\lambda) I_\lambda(f)) d\lambda - \frac{1}{4} \text{tr} (M(0) I_0(f)),$$

where $M(\lambda)$ is the intertwining operator of [2] and $I_\lambda$ is the induced representation on the space of which $M(\lambda)$ acts. For the convenience of the reader we will recall the definition of $I_\lambda$ and $M(\lambda)$ next.

1.14 We have the decomposition $\mathcal{G}(\mathbb{A}) = \mathcal{N}(\mathbb{A}) \mathcal{L}(\mathbb{A})^1 A(\mathbb{A})^0 K_{\text{max}} K$. In this decomposition any $y \in \mathcal{G}(\mathbb{A})$ writes as $y = nm \exp(Y)k fk$ where $Y \in a$ is uniquely determined. We define a map $H_P : \mathcal{G}(\mathbb{A}) \rightarrow a$ mapping $y$ to $Y$.

Let $\mathcal{H}_0^0$ be the space of functions

$$\varphi : N(\mathbb{A}) \mathcal{L}(\mathbb{Q}) A(\mathbb{R})^0 \backslash \mathcal{G}(\mathbb{A}) \rightarrow \mathbb{C}$$

such that

- for any $x \in \mathcal{G}(\mathbb{A})$ the function $m \mapsto \varphi(mx)$, $m \in \mathcal{L}(\mathbb{A})$ is $\mathfrak{z}_{\mathcal{L}(\mathbb{R})}$-finite, where $\mathfrak{z}_{\mathcal{L}(\mathbb{R})}$ is the center of the universal enveloping algebra of $l = \text{Lie}(\mathcal{L}(\mathbb{R}))$,
• $\varphi$ is right $K_{\text{max}}K$-finite,

$$
\| \varphi \| ^2 = \int_{K_{\text{max}}K} \int_{\mathcal{A}(\mathbb{R})^n \mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})} | \varphi(mk) |^2 dmdk < \infty.
$$

The last point defines an obvious scalar product on $\mathcal{H}_{P}^0$. Let $\mathcal{H}_P$ be the Hilbert space completion of $\mathcal{H}_{P}^0$. For $\lambda \in \mathfrak{a}$, $\varphi \in \mathcal{H}_P$ and $x, y \in \mathcal{G}(\mathbb{A})$ put

$$(I_{\lambda}(y)\varphi)(x) = \varphi(xy)e^{\lambda+\rho,H_P(xy)-H_P(x)}.$$

We will write $I_{\lambda}$ for the resulting $\mathcal{G}(\mathbb{A})$-representation. Note that this is just the induced representation from $\mathcal{P}(\mathbb{A})$ to $\mathcal{G}(\mathbb{A})$ of $L^2(\mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})^1) \otimes \lambda \otimes 1$.

1.15 Let $w \in K_{\text{max}}K \cap \mathcal{G}(\mathbb{Q})$ be a representative for the nontrivial element of the Weyl group $W(g,a)$. For $\lambda \in \mathfrak{a}$ with $\text{Re}(\lambda) >> 0$ we define an operator $M(\lambda)$ on $\mathcal{H}_P$ by

$$M(\lambda)\varphi(x) = \int_{N(\mathbb{A})} \varphi(wnx)e^{\lambda+\varphi,H_P(wnx)+\langle \lambda-\rho,H_P(x) \rangle} dn.$$

Langlands has shown that this operator extends meromorphically in $\lambda$ and satisfies

$$M(-\lambda) = M(\lambda)^{-1}, \quad M(\lambda)^* = M(\bar{\lambda}).$$

These equations especially imply that $M(\lambda)$ is unitary if $\lambda$ is purely imaginary.

Let $\text{Pr} : \mathcal{H}_P \to \mathcal{H}_P^{K_P}$ be the projection to the space of $K_P$-fixed vectors. In our considerations only the operator $M(\lambda)^\Gamma := \text{Pr} M(\lambda) \text{Pr}$ will occur.

1.16 Since $\mathcal{G}$ has $\mathbb{Q}$-rank one it follows that $\mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})^1$ is compact. So $L^2(\mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})^1)$ decomposes as $\mathcal{L}(\mathbb{A})^1$-representation

$$L^2(\mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})^1) = \bigoplus N(\chi)\chi,$$

where the sum runs over the unitary dual of $\mathcal{L}(\mathbb{A})^1$ and the multiplicities $N(\chi)$ are finite. Therefore

$$I_{\chi} = \text{Ind}_{\mathcal{P}(\mathbb{A})}^{\mathcal{G}(\mathbb{A})}(L^2(\mathcal{L}(\mathbb{Q}) \backslash \mathcal{L}(\mathbb{A})^1) \otimes \lambda \otimes 1)$$

$$= \bigoplus N(\chi)\text{Ind}_{\mathcal{P}(\mathbb{A})}^{\mathcal{G}(\mathbb{A})}(\chi \otimes \lambda \otimes 1).$$
Write $\pi_{\chi,\lambda}$ for $\text{Ind}_{P(A)}^{G(A)}(\chi \otimes \lambda \otimes 1)$. We consider $\chi \otimes \lambda = \chi_\lambda$ as an irreducible admissible representation of $\mathcal{L}(A) = \mathcal{L}(A)^1 \mathcal{A}(\mathbb{R})^0$. Since any irreducible admissible representation of $\mathcal{L}(A)$ can be written as a tensor product of local representations we get

$$\chi_\lambda = \bigotimes_p \chi_{\lambda,p} \otimes \chi_{\lambda,\infty}.$$ 

For example, $\chi_{\lambda,\infty}$ is a representation of $\mathcal{L}(\mathbb{R}) = MA$. For later use we introduce the notation $\chi_\infty := \chi_{\lambda,\infty}|_M$, which does not depend on $\lambda$. Accordingly we get

$$\pi_{\chi_\lambda} = \bigotimes_p \pi_{\chi_{\lambda,p}} \otimes \pi_{\chi_{\lambda,\infty}} = \pi_{\chi_{\lambda,\text{fin}}} \otimes \pi_{\chi_{\lambda,\infty}}.$$ 

Note that with this notation we have $\pi_{\chi_{\lambda,\infty}} = \pi_{\chi_{\infty,\lambda}}$. For the space of $K_\Gamma$-fixed vectors we get

$$(\pi_{\chi_\lambda})^{K_\Gamma} = (\pi_{\chi_{\lambda,\text{fin}}})^{K_\Gamma} \otimes \pi_{\chi_{\lambda,\infty}}.$$ 

The space $(\pi_{\chi_{\lambda,\text{fin}}})^{K_\Gamma}$ is finite dimensional, its dimension $N_\Gamma(\chi)$ does not depend on $\lambda$. As a $G$-representation

$$I^{K_\Gamma}_\chi = \bigoplus_\chi N(\chi) N_\Gamma(\chi) \pi_{\chi_{\lambda,\infty}}.$$ 

The representation $\pi_{\chi_{\lambda,\infty}}$ equals the direct Hilbert sum of its $K$-isotypes

$$\pi_{\chi_{\lambda,\infty}} = \bigoplus_{\tau \in K} \pi_{\chi_{\lambda,\infty}}(\tau),$$

and each isotype is finite dimensional. The Weyl-group $W = W(A, G)$ acts on the unitary dual $\hat{M}$ of the group $M = \mathcal{L}(\mathbb{R})^1$. Write $O$ for an orbit in $\hat{M}$, then

$$\mathcal{H}^{K_\Gamma}_P = \bigoplus_O \bigoplus_\tau \pi_O(\tau),$$

with $\pi_O(\tau) = \bigoplus_{\chi_{\lambda,\infty} \in O} \pi_{\chi_{\lambda,\infty}}(\tau)$ (here any $\lambda$ will give the same space, but the representation on it varies.) The spaces $\pi_O(\tau)$ are finite dimensional and stable under $M(\lambda)$ for any $\lambda$.

**Remark** For any measure space $(\Omega, \mu)$ and any Hilbert space $V$ we write $L^2(\Omega, V)$ for the space of all $V$-valued square integrable functions on $\Omega$. 

We now come to the discussion of $\pi_{\chi,\lambda}(f) = Pr_\Gamma \otimes \pi_{\chi,\lambda}(f_{\infty})$; recall [1.16] for notations. Using the compact model of induction we see that $\pi_{\chi,\lambda}(f_{\infty})$ can be written as an integral operator on the space

$$L^2(K, \chi_{\infty}) := \{ \varphi \in L^2(K, V_{\chi_{\infty}}) | \varphi(mk) = \chi_{\infty}(m)f(k), m \in K \cap M \},$$

with kernel

$$k_{f,\lambda}(k, k') = \int_{M \backslash K} f_{\infty}(m^{-1}mk'a^{\rho+\lambda}\chi_{\infty}(m)dm)da$$

$$= \int_{\mathbb{R}} \tilde{f}(k, k', \exp(tH))e^{\lambda(H)t}dt,$$

for a compactly supported function $\tilde{f}$ on $K \times K \times A$. Thus we may view the kernel $k_{f,\lambda}$ pointwise as a Paley-Wiener function in $\lambda$.

**Lemma 1.18** For any irreducible admissible representation $\chi_\lambda$ of $\mathcal{L}(A)$ it holds

$$\int_{i\mathbb{R}_0} \text{tr} \pi_{\chi_\lambda,\lambda}(f) d\lambda = 0.$$

**Proof:** Since $\pi_{\chi_\lambda}(f) = Pr_\Gamma \otimes \pi_{\chi_\lambda,\infty}(f_{\infty})$ it remains to show

$$\int_{i\mathbb{R}_0} \text{tr} \pi_{\chi_\lambda,\infty}(f_{\infty}) d\lambda = 0.$$

Let $V_{\chi_{\infty}} = \bigoplus_{\tau \in K_M} V_{\chi_{\infty}}(\tau)$ be the decomposition of $V_{\chi_{\infty}}$ into $K_M = K \cap M$-isotypes. To this corresponds the decomposition

$$L^2(K, \chi_{\infty}) = \bigoplus_{\tau \in K_M} L^2(K, \chi_{\infty}, \tau),$$

where

$$L^2(K, \chi_{\infty}, \tau) = \{ \varphi \in L^2(K, V_{\chi_{\infty}}(\tau)) | \varphi(mk) = \chi_{\infty}(m)f(k), m \in K_M \}.$$

Let $Pr_\tau$ denote the projection $L^2(K, \chi_{\infty}) \to L^2(K, \chi_{\infty}, \tau)$. Since the operator $\pi_{\chi_\lambda,\infty}(f_{\infty})$ is given by the smooth kernel $k_f(k, k')$ it follows that the operator $\pi_{\chi_\lambda,\infty}(f) Pr_\tau$ is given by the kernel

$$k_{f,\tau}(k, k') = \int_{K_M} k_f(k, k_M k') \text{tr} \tau(k_M) dk_M.$$
The latter is smooth, so it follows
\[ \text{tr}(\pi_{\chi, \infty}(f_\infty)Pr) = \int_K \text{tr}(k_{f, \tau}(k, k))dk \]
\[ = \int_\mathbb{R} \int_K \text{tr}\tilde{f}_\tau(k, k, \exp(tH))e^{t\lambda(H)}dt, \]
where \( \tilde{f}_\tau(k, k', a) \) is defined to be
\[ \int_{K_M} \int_{L(\mathbb{R}) \times N(\mathbb{R})} f_\infty(k^{-1}mank'Mk')e^{t\lambda(H)}\chi_\infty(m)\text{tr}\tau(k_M)dmdndk_M. \]
It follows
\[ \int_{ia_0} \int_{ia_0} \int_\mathbb{R} \int_K \text{tr}\tilde{f}_\tau(k, k, \exp(tH))e^{t\lambda(H)}d\lambda \]
\[ = \frac{1}{2\pi} \sum_\tau \int_K \text{tr}\tilde{f}_\tau(k, k, 1)dk \]
by the Fourier-inversion theorem. Since \( f_\infty \) vanishes on \( P \)-singular elements it follows that \( \tilde{f}_\tau(k, k, 1) = 0 \) for any \( k \in K, \tau \in \hat{K}_M \).
Q.E.D.

Up to this point we have shown:

**Theorem 1.19** Let the function \( f \) on \( \mathcal{G}(\mathfrak{a}) \) satisfy \([1.5]\) then the sum
\[ \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma)\mathcal{O}_\gamma(f_\infty) \]
equals
\[ \sum_{\pi \in \hat{G}} N_\Gamma(\pi)\text{tr}(f_\infty) - \frac{1}{4} \text{tr}(M(0)I_0(f)) \]
\[ + \frac{1}{4\pi i} \int_{ia_0^*} \text{tr}((M(\lambda)^{-1}M'(\lambda))I_\lambda(f))d\lambda. \]
2 The continuous contribution

In this section we will give the continuous contribution

\[ \frac{1}{4\pi i} \int_{\alpha_0} \text{tr}((M(\lambda)^{-1}M'(\lambda))I_\lambda(f))d\lambda \]

a different shape. This section is more general than the rest of the paper since we can take for \( f_\infty \) an arbitrary element of \( C^j_c(G) \).

2.1 For \( r > 0 \) and \( a \in \mathbb{C} \) let \( B_r(a) \) be the closed disk around \( a \) of radius \( r \). Let \( g \) be a meromorphic function on \( \mathbb{C} \) with poles \( a_1, a_2, \ldots \). We say that \( g \) is essentially of moderate growth if there is a natural number \( N \), a constant \( C > 0 \) and a sequence of positive real numbers \( r_n \), tending to zero such that the disks \( B_{r_n}(a_n) \) are pairwise disjoint and that on the domain \( D = \mathbb{C} - \bigcup_n B_{r_n}(a_n) \) it holds \( |g(z)| \leq C|z|^N \). In that case the constant \( N \) is called the growth exponent.

Lemma 2.2 Let \( f \) be a meromorphic function of finite order and let \( g = f'/f \) be its logarithmic derivative. Then \( g \) is essentially of moderate growth with growth exponent equals the order of \( f \) plus two.

Proof: This is a direct consequence of Hadamard’s factorization theorem applied to \( f \).

Q.E.D.

2.3 Recall the definition of \( M(\lambda) \). The integral over \( \mathcal{N}(\mathbb{A}) \) may be written as a product of an integral over \( \mathcal{N}(\mathbb{A}_{\text{fin}}) \) and an integral over \( \mathcal{N}(\mathbb{R}) \). Thus \( M(\lambda) = M_{\text{fin}}(\lambda) \otimes M_\infty(\lambda) \), and so \( M(\lambda)^{-1}M'(\lambda) = M_{\text{fin}}(\lambda)^{-1}M'_{\text{fin}}(\lambda) + M_\infty(\lambda)^{-1}M'_\infty(\lambda) \). According to [3], Theorem 2.1 we write \( M_{\text{fin}}(\lambda)|_{\pi_\mathcal{O}(\tau)} = r_{\text{fin}}(\lambda)R_{\text{fin}}(\lambda) \) with a scalar-valued meromorphic function \( r_{\text{fin}}(\lambda) \). Property \( (R_6) \) in Theorem 2.1 of loc. cit. implies that \( R_{\text{fin}}(\lambda) \) is of finite order, the order being independent on \( \mathcal{O} \) and \( \tau \), hence the same holds for \( \det(R_{\text{fin}}(\lambda)) \). Further p. 39 of loc. cit. shows that \( r_{\text{fin}} \) is of finite order independent on \( \mathcal{O} \) and \( \tau \).

Lemma 2.4 The function \( \lambda \mapsto \text{tr}M_{\text{fin}}(\lambda)^{-1}M'_{\text{fin}}(\lambda)|_{\pi_\mathcal{O}(\tau)} \) is of essentially moderate growth with growth exponent independent of \( \mathcal{O} \) and \( \tau \).
Proof: Let $\psi(\lambda) := \det(M_{\text{fin}}(\lambda)|_{\pi_{O}(\tau)})$ then $\text{tr}M_{\text{fin}}(\lambda)^{-1}M'_{\text{fin}}(\lambda)|_{\pi_{O}(\tau)}$ equals $\psi'/\psi(\lambda)$, so by Lemma 2.3 it suffices to show that the order of $\psi$ is independent on $O$ and $\tau$. This is clear by the above.

\[ \text{Q.E.D.} \]

Lemma 2.5 The matrix valued function $M_{\infty}(\lambda)^{-1}M'_{\infty}(\lambda)|_{\pi_{O}(\tau)}$ is essentially of moderate growth with growth exponent independent on $O$ and $\tau$.

Proof: For the length of this proof write $M(\lambda)$ for $M(\lambda)|_{\pi_{O}(\tau)}$ and $M_{\infty}(\lambda)$ for $M_{\infty}(\lambda)|_{\pi_{O}(\tau)}$. In [29], p. 514, it is shown that $M(\lambda)$ is a matrix valued meromorphic function of order $\dim G/K + 2$. Above we showed that $M_{\text{fin}}$ is of finite order independent of $O$ and $\tau$. Together the same follows for $M_{\infty}(\lambda)$. According to Theorem 2.1 in [3] we have $M_{\infty}(\lambda) = r_{\infty}(\lambda)R_{\infty}(\lambda)$ and hence $M_{\infty}(\lambda)^{-1}M'_{\infty}(\lambda) = r'_{\infty}/r_{\infty}(\lambda) + R_{\infty}(\lambda)^{-1}R'_{\infty}(\lambda)$. By formula (3.5) of [3] it follows that the normalizing factor $r_{\infty}$ satisfies the same growth conditions as $M_{\infty}(\lambda)$. Now Lemma 2.2 applies to the first summand. The second summand is rational by p. 29 in [3] and the proof on page 37 of [3] implies that the degree of $R_{\infty}(\lambda)$ and $R_{\infty}(\lambda)^{-1}$ and hence of $R(\lambda)^{-1}R'(\lambda)$ does only depend on $G$.

\[ \text{Q.E.D.} \]

2.6 We want to give the integral over $ia^{*}_0$ in Theorem 1.19 a different shape. To this end recall that the kernel $k_{f,\lambda}$ is a Paley-Wiener function in the argument $\lambda$.

We will formulate a general remark on Paley-Wiener functions. For a natural number $n$ let $C^{n}_{c}(\mathbb{R})$ denote the space of $n$-times continuously differentiable compactly supported functions on $\mathbb{R}$. By a Paley-Wiener function of order $n$ we mean a function $h$ which is the Fourier transform of some $g \in C^{n}_{c}(\mathbb{R})$. Since it better fits into our applications we will change coordinates from $z$ to $iz$. So a Paley-Wiener function $h$ will be of the form

$$ h(z) = \int_{-\infty}^{\infty} g(t)e^{zt}dt $$

for some $g \in C^{n}_{c}(\mathbb{R})$.

Proposition 2.7 Let $h$ be a Paley-Wiener function of order $n$ and fix $a \in \mathbb{C}$. There is a unique decomposition

$$ h = h_{a}^{+,n} + h_{a}^{-,n} $$
such that the functions \( h_\alpha^\pm, n \) are holomorphic in \( \mathbb{C} \setminus \{a\} \), both have at most a pole of order \( < n \) at \( a \). Further for some \( C > 0 \) the following estimates hold:

\[
|h_\alpha^+, n(z)| \leq \frac{C}{|z - a|^n} \quad \text{for} \quad \text{Re}(z) \leq 0, \quad z \neq a,
\]

\[
|h_\alpha^-, n(z)| \leq \frac{C}{|z - a|^n} \quad \text{for} \quad \text{Re}(z) \geq 0, \quad z \neq a.
\]

**Proof:** Let us show uniqueness first. Suppose we are given two decompositions \( h = h^+ + h^- = h^+_1 + h^-_1 \) of the above type then \( \tilde{h} = h^+ - h^-_1 \) satisfies \( |\tilde{h}(z)| \leq \frac{2C}{|z - a|^n} \) for all \( z \neq a \). Therefore the entire function \( (z - a)^n \tilde{h}(z) \) is bounded, hence constant. But this function vanishes at \( a \) by the pole order condition, whence the claim.

For the existence assume

\[
h(z) = \int_{-\infty}^{\infty} g(t)e^{zt}dt
\]

for some \( g \in C^\infty_c(\mathbb{R}) \). Now define

\[
h_\alpha^+, n(z) := (\frac{1}{z - a})^n \int_0^{\infty} (g(t)e^{at})^n e^{(z-a)t} dt - \frac{c(g)}{(z-a)^n}
\]

and

\[
h_\alpha^-, n(z) := (\frac{1}{z - a})^n \int_{-\infty}^{0} (g(t)e^{at})^n e^{(z-a)t} dt + \frac{c(g)}{(z-a)^n},
\]

where \( c(g) = \int_{0}^{\infty} (g(t)e^{at})^n dt \). Partial integration shows that \( h = h_\alpha^+, n + h_\alpha^-, n \), the rest is clear.

Q.E.D.

**2.8** Note that if \( g \) vanishes at \( t = 0 \) to order \( j + 1 \) and \( n \leq j \), then

\[
h_\alpha^+, n = h_\alpha^+, n-1 = \ldots = h_\alpha^+, 1
\]

and this further equals

\[
h^\pm(z) := \int_0^{\infty} g(\pm t)e^{\pm tz} dt.
\]

In this case we say that \( h \) is **orthogonal to polynomials of degree \( \leq j \).**
If $a = 0$, we will generally drop the index, so $h_0^{±,n} = h^{±,n}$.

Finally note that, by the formula given above one sees that if $g$ depends differentiably or holomorphically on some parameter then the same holds for $h^{±}$.

2.9 Fix some $n \leq j$, but still large and denote by $k_{f,λ,a}^{±,n}$ the kernels we get by applying this construction to $k_{f,λ}$ as a function in $λ$. Write $T_{f,λ,a}^{±,n}$ for the corresponding operator at infinity and $I_{λ,a}^{±,n}(f)$ for the global operator $Pr_Γ \otimes T_{f,λ,a}^{±,n}$. Suppose $a \in a^*$ has negative real part and does not coincide with a pole of $M(λ)^{-1}M'(λ)$. We get that

$$ \frac{1}{4πi} \int_{ia_0^+} \text{tr} M(λ)^{-1}M'(λ)I_λ(f) dλ $$

equals

$$ \frac{1}{4πi} \int_{ia_0^-} \text{tr} M(λ)^{-1}M'(λ)I_{λ,a}^{+,n}(f) dλ $$

$$ + \frac{1}{4πi} \int_{ia_0^+} \text{tr} M(λ)^{-1}M'(λ)I_{λ,a}^{-,n}(f) dλ. $$

We move the integration paths to the left and the right resp. to get the residues plus a term which tends to zero according to the Lemmas 2.4 and 2.5. The above becomes

$$ \frac{1}{2} \sum_{\Re λ < 0} \text{tr} R_λ I_{λ,a}^{+,n}(f) $$

$$ + \frac{1}{2} \text{res}_{λ = a} \text{tr} M(λ)^{-1}M'(λ)I_{λ,a}^{+,n}(f) $$

$$ - \frac{1}{2} \sum_{\Re λ > 0} \text{tr} R_λ I_{λ,a}^{-,n}(f), $$

where $R_λ_0 := \text{res}_{λ = λ_0} M(λ)^{-1}M'(λ)$.

2.10 We say that a function $f \in C^j_ξ(G)$ is orthogonal to polynomials of degree $\leq j$ if the operator valued function $λ \mapsto π_ξ(λ)$ satisfies this condition for any $ξ \in M$. In that case 2.8 immediately gives that the above equals

$$ \frac{1}{2} \sum_{\Re λ < 0} \text{tr} R_λ I_{λ}^{+,n}(F) - \frac{1}{2} \sum_{\Re λ > 0} \text{tr} R_λ I_{λ}^{-}(F). $$
Furthermore $f$ is called even if $\lambda \mapsto \pi_{\xi,\lambda}(f)$ is an even function for any $\xi \in \mathcal{M}$. In that case the functional equation gives $\text{tr}R_{\lambda}I^{-}_{\lambda}(f) = \text{tr}R_{\lambda}J^{+}_{\lambda}(f)$ and so for $f$ even and orthogonal to polynomials of degree $\leq j$ we end up with the simple expression
\[ \sum \text{Re} \lambda < 0 \text{tr}R_{\lambda}I^{-}_{\lambda}(f). \]

2.11 Recall the definition of $M(\lambda)$ as an integral over $\mathcal{N}(\mathbb{A})$. This integral can be written as a product of an integral over $\mathcal{N}(A_{\text{fin}})$ and an integral over $\mathcal{N}(R)$ giving a decomposition $M(\lambda) = M_{\text{fin}}(\lambda) \otimes M_{\infty}(\lambda)$. The second factor, restricted to the contribution of a single $\chi \in \mathcal{L}(\mathbb{A})^{1}$, coincides with the Knapp-Stein intertwining operator at the infinite place and can separately be continued to a meromorphic function on the plane $\mathbb{C}$. Therefore the first also extends meromorphically and we get $M(\lambda)^{-1}M'(\lambda) = M_{\text{fin}}(\lambda)^{-1}M'_{\text{fin}}(\lambda) + M_{\infty}(\lambda)^{-1}M'_{\infty}(\lambda)$ and so $R_{\lambda} = R_{\text{fin,}\lambda} \otimes 1 + 1 \otimes R_{\infty,\lambda}$ which implies that $\text{tr}R_{\lambda} \pi^{+}_{\lambda,\infty}(f)$ equals
\[ \sum \chi N(\chi) \text{tr}(R^{\Gamma}_{\text{fin,}\lambda}) \pi^{+}_{\chi,\infty}(f) + \sum \chi N(\chi) N_{\Gamma}(\chi) \text{tr}R_{\infty,\lambda} \pi^{+}_{\chi,\infty}(f). \]

Lemma 2.12 The number $N_{\Gamma}(\chi,\lambda) := \text{tr}(R^{\Gamma}_{\text{fin,}\lambda})$ is an integer.

Proof: Let $A(\lambda)$ be the finite dimensional matrix $M_{\text{fin,}\lambda}(\lambda)^{\Gamma}$, then
\[ \text{tr}(R^{\Gamma}_{\text{fin,}\lambda}) = \text{tr}(\text{res}_{\lambda=\lambda_{0}}A(\lambda)^{-1}A'(\lambda)) = \text{res}_{\lambda=\lambda_{0}} \text{tr}A(\lambda)^{-1}A'(\lambda) = \text{res}_{\lambda=\lambda_{0}} \text{tr} \frac{\partial}{\partial \lambda} \log A(\lambda) = \text{res}_{\lambda=\lambda_{0}} \frac{\partial}{\partial \lambda} \log \det A(\lambda) \]

Let $c(\lambda) = \det(A(\lambda))$ then the last expression becomes $\text{res}_{\lambda=\lambda_{0}} \frac{c'(\lambda)}{c(\lambda)}$, which is an integer.

Q.E.D.

2.13 Let $\hat{f}_{\infty}$ be the trace of the Fourier-transform of $f_{\infty}$, i.e. for an admissible representation $\pi$ of $G$ of finite length let $\hat{f}_{\infty}(\pi) := \text{tr} \pi(f_{\infty})$. The
function $\lambda \mapsto \hat{f}_\infty(\pi_{\lambda,\infty})$ is a Paley-Wiener function and so it decomposes as above where we write $\hat{f}_\infty(\pi_{\lambda,\infty}) = \hat{f}_\infty^{+,n}(\chi,\lambda) + \hat{f}_\infty^{-,n}(\chi,\lambda)$. Since the trace equals the integral over the kernel it follows $\text{tr} \pi_{\lambda}^{+,n}(f_\infty) = \hat{f}_\infty^{+,n}(\chi,\lambda)$. We can summarize the results of this section in

**Proposition 2.14** Let $f_\infty$ in $C^j_c(G)$ be even and orthogonal to polynomials of degree $\leq j$, then

$$\frac{1}{4\pi i} \int_{a_0} \text{tr}((M(\lambda)^{-1}M'(\lambda))I_A(f))d\lambda$$

equals

$$\sum_{\text{Re} \lambda < 0} \text{tr} R_\lambda I_{\lambda}^+(f),$$

and this can be written as

$$+ \sum_{\chi} N(\chi) \left( \sum_{\text{Re} (\lambda) < 0} N_{\Gamma(\chi,\lambda)} \hat{f}_\infty^+(\chi,\lambda) + N(\chi) \text{tr} R_{\Gamma(\chi,\lambda)} \pi_{\chi,\infty}^+(f_\infty) \right).$$

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