One-Sided Repeated-Root Two-Dimensional Cyclic and Constacyclic Codes

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Abstract

In this paper, we study some repeated-root two-dimensional cyclic and constacyclic codes over a finite field $\mathbb{F} = \mathbb{F}_q$. We obtain the generator matrices and generator polynomials of these codes and their duals. We also investigate when such codes are self-dual. Moreover, we prove that if there exists an asymptotically good family of one-sided repeated-root two-dimensional cyclic or constacyclic codes, then there exists an asymptotically good family of simple root two-dimensional cyclic or constacyclic codes with parameters at least as good as the first family. Furthermore, we show that several of the main results of the papers Rajabi and Khayyarnesh (2018) \cite{20} and Sepasdar and Khayyarnesh (2016) \cite{23} are not accurate and find other conditions needed for them to hold.

Keywords: Two-dimensional cyclic codes, Two-dimensional constacyclic codes, Self-dual codes, Asymptotically good family of codes.

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1 Introduction

Two-dimensional (2D, for short) cyclic codes which have a long history, see for example \cite{12, 13}, still gain attention, see \cite{9, 11, 22, 23} and the references therein. As mentioned in \cite{9}, these codes are special cases of quasi-cyclic

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codes which form an important and well-studied class of codes (see, for example, [1, 3, 7, 18, 21, 25] and their references). Also constacyclic codes which are a generalization of cyclic codes are investigated over finite fields and some other types of rings, see [3, 24] and their references. In [20], 2D constacyclic codes were introduced and studied as a generalization of 2D cyclic codes.

As in one-dimensional cyclic codes, it is more common to consider the simple root case, that is, when both dimensions of the code is coprime to the size of the base field, mainly because in this case the structure of the code can be characterized by the set of its zeros. In the one-dimensional case, it is known that some optimal repeated-root cyclic codes exist (see [26]). Such results caused several authors to study one-dimensional repeated-root cyclic codes, see [6, 8, 14, 16, 26] and the references therein. In this paper, we consider two-dimensional cyclic codes in which at least one dimension is coprime to the size of the base field but the other dimension is arbitrary. We indeed state our results for two-dimensional constacyclic codes which are generalizations of two-dimensional cyclic codes.

We recall the definition of 2D constacyclic codes ([20]). We always assume that \( p \) is a prime number, \( \mathbb{F} = \mathbb{F}_q \) is a finite field with \( q = p^r \) elements and \( \lambda \) and \( \delta \) are units in \( \mathbb{F} \). Consider

\[
\tau_\lambda : \mathbb{F}^n \longrightarrow \mathbb{F}^n
\]

\[
(d_0, d_1, \ldots, d_{n-1}) \longmapsto (\lambda d_{n-1}, d_0, \ldots, d_{n-2}), \quad \text{where } d_j \in \mathbb{F}
\]

and

\[
\Upsilon_\delta : (\mathbb{F}^n)^m \longrightarrow (\mathbb{F}^n)^m
\]

\[
(a_0, a_1, \ldots, a_{m-1}) \longmapsto (\delta a_{m-1}, a_0, \ldots, a_{m-2}), \quad \text{where } a_j \in \mathbb{F}^n.
\]

Assume that \( a = (a_0, a_1, \ldots, a_{m-1}) \) is an element of \( \mathbb{F}^{nm} \), where \( a_j = (a_{j0}, a_{j1}, \ldots, a_{jn-1}) \in \mathbb{F}^n \). For any \( i, j, 0 \leq j \leq m - 1 \) and \( 0 \leq i \leq n - 1 \), define

\[
\Theta_{\delta,\lambda}^{j,i}(a) = \Upsilon_{\delta}^{j}(\tau_{\lambda}^{i}(a_0), \tau_{\lambda}^{i}(a_1), \ldots, \tau_{\lambda}^{i}(a_{m-1})).
\]

A 2D linear code \( D \) of length \( nm \) is called \((\lambda, \delta)-\text{constacyclic code}\) over \( \mathbb{F} \), if \( \Theta_{\delta,\lambda}^{j,i}(D) = D \) for any \( 0 \leq j \leq m - 1 \) and \( 0 \leq i \leq n - 1 \). In \( \mathbb{F}^{nm} \simeq M_{m \times n}(\mathbb{F}) \), any \( nm \)-array \( (a_0, a_1, \ldots, a_{m-1}) \) corresponds to a polynomial in \( \mathbb{F}[x, y] \) with \( x \)-degree less than \( n \) and \( y \)-degree less than \( m \), say \( a(x, y) = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} a_{ji} x^i y^j \). With this correspondence, any \((\lambda, \delta)\)-constacyclic code
of length \(nm\) over \(F\) is identified with an ideal of the quotient ring \(S = F[x, y]/(x^n - \lambda, y^m - \delta)\).

In Theorem 4.1 of [23] which is its main theorem, a set of generators for 2D cyclic codes of length \(s2^k\) over fields of odd characteristic is presented. In [20], this theorem is used to find a generating set for certain 2D constacyclic codes of length \((2p^s)2^k\) and their duals over fields of odd characteristic. Here in Section 2, we present a counterexample to [23, Theorem 4.1] and find the other conditions needed for this theorem to hold. We show that because of the inaccuracy in this theorem, several results of [20] are also erroneous. Then in Section 3, we study the algebraic structure of constacyclic codes of length \(nm\) where at least one of \(n\) or \(m\) is relatively prime to \(p\). Moreover, we show how we can find the dimension and a generator matrix of such codes. In Section 4, we study the dual of such constacyclic codes and present a method to find the parity check matrix of these codes. We also investigate when such a code is self-dual. Finally in Section 5, we prove that if there exists an asymptotically good family of such (repeated-root) 2D constacyclic codes, then there exists an asymptotically good family of simple root 2D constacyclic codes with parameters at least as good as the first family.

2 A counterexample and a correction to some previous results

First, by presenting a counterexample to the main theorem (Theorem 4.1) of [23], we show that this theorem is not accurate. Consequently, as all of the main theorems of [20] use this incorrect result, they are not correct either. First we recall the statement of [23, Theorem 4.1]. We note that in [23], the authors work over a finite field \(F\) with \(\text{char} F \neq 2\) and consider 2D cyclic codes of length \(n = 2^k s\) which correspond to the ideals of the ring \(F[x, y]/(x^s - 1, y^{2^k} - 1)\).

**Incorrect Theorem 2.1 ([23, Theorem 4.1]).** Suppose that \(C\) is a 2D cyclic code of length \(n = 2^k s\). Then the corresponding ideal \(I\) has the generating
set of polynomials as follows:

\[
I = \left< p_1(x) \left( \sum_{i=0}^{2k-1} y^i \right), p_2(x) \left( \sum_{i=0}^{2k-1} (-1)^i y^i \right), p_3(x) \left( \sum_{i=0}^{2k-1} (-1)^i y^{2i} \right), p_4(x) \left( \sum_{i=0}^{2k-2-1} (-1)^i y^{4i} \right), \ldots, p_{k+1}(x) \left( \sum_{i=0}^{2k-1} (-1)^i y^{2k-1}\right) \right>. \quad (2.1)
\]

The counterexample presented here is the case that \( k = 2 \). Note that in the case \( k = 2 \), this theorem indeed says that every ideal \( I \) of the ring \( \mathbb{F}[x, y] \) which contains \( x^s - 1 \) and \( y^4 - 1 \) can be generated using \( x^s - 1, y^4 - 1 \) and three polynomials of the form \( p_1(x)(1 + y + y^2 + y^3), p_2(x)(1 - y + y^2 - y^3), p_3(x)(1 - y^2) \).

**Example 2.2.** Suppose that \( \mathbb{F} = \mathbb{F}_5 \) and \( I = \langle x^s - 1, y + 2 \rangle \) for an integer \( s \geq 1 \). Note that \( y^4 - 1 = (y + 1)(y - 1)(y + 2)(y - 2) \), so that \( y^4 - 1 \in I \). If \( I = \langle x^s - 1, y^4 - 1, p_1(x)(1 + y + y^2 + y^3), p_2(x)(1 - y + y^2 - y^3), p_3(x)(1 - y^2) \rangle \), then as \( p_3(x)(1 - y^2) \in \langle x^s - 1, y + 2 \rangle \) and by evaluating at \( y = -2 \) we have \( 2p_3(x) \in \langle x^s - 1 \rangle \). Hence \( p_3(x)(1 - y^2) \) can be dropped from the aforementioned generating set of \( I \). Thus it follows that

\[
y + 2 \in \langle x^s - 1, y^4 - 1, p_1(x)(1 + y + y^2 + y^3), p_2(x)(1 - y + y^2 - y^3) \rangle.
\]

Now by evaluating the above relation at \( x = 1 \), we get that \( y + 2 \in \langle y^4 - 1, 1 + y + y^2 + y^3, 1 - y + y^2 - y^3 \rangle = \langle 1 + y^2 \rangle \) in \( \mathbb{F}[y] \), which is a contradiction. Therefore, \( I \) has not a generating set as in Theorem 2.1 and that theorem is not correct.

In the proof stated for Theorem 2.1 in [23], the problem arises from the fact that \( I_3, I_4, \ldots, I_{k+1} \) (see pages 104, 109 or 111 of [23]) are not always ideals as assumed in [23]. For instance, in the Example 2.2 we have

\[
y + 2 \in I_3 = \{ g(x, y) \in \mathbb{F}[x, y]/\langle x^s - 1, y^2 - 1 \rangle | g(x, y)(1 - y^2) \in I \}
\]

but \((1 + 2y)(1 - y^2) \notin I \) (that is, \( 1 + 2y \notin I_3 \)), although in \( \mathbb{F}[x, y]/\langle x^s - 1, y^2 - 1 \rangle \), it holds that \((y + 2)y = 1 + 2y \). Next we investigate for which \( \mathbb{F}, k, s \) Theorem 2.1 does hold.

**Proposition 2.3.** Fix a field \( \mathbb{F} \) with an odd characteristic and consider integers \( s > 1 \) and \( k > 0 \). Then every 2D cyclic code of length \( 2^k s \) over \( \mathbb{F} \) as an ideal of \( R = \mathbb{F}[x, y]/\langle x^s - 1, y^{2k} - 1 \rangle \) has a generating set of the form (2.1), if and only if both of the following conditions hold:
(i) \(x^{2^k-1} + 1\) is irreducible in \(\mathbb{F}[x]\);

(ii) \(\gcd(x^s - 1, x^{2^k} - 1)|x^2 - 1\) in \(\mathbb{F}[x]\).

Proof. \((\Rightarrow)\): Note that in \((2.1)\), the polynomials \(p_i(x)\) can be chosen to be monic factors of \(x^s - 1\), since the ring \(\mathbb{F}[x]/\langle x^s - 1 \rangle\) is a principal ideal ring. Assume that in \(\mathbb{F}[x]\) the number of monic factors of \(x^s - 1\) is \(a\). Then the number of ideals of \(R\) is at most \(a^{k+1}\). On the other hand, if \(y^{2^k} - 1 = f_1 \cdots f_t\) is an irreducible factorization of \(y^{2^k} - 1\), then by the Chinese Remainder Theorem (CRT)

\[
R \cong \left( \frac{\mathbb{F}[y]/\langle y^{2^k} - 1 \rangle}{\langle x^s - 1 \rangle} \right) \cong \left( \bigoplus_{i=1}^{t} \frac{\mathbb{F}[y]/\langle f_i \rangle}{\langle x^s - 1 \rangle} \right) \cong \bigoplus_{i=1}^{t} K_i[x], \tag{2.2}
\]

where \(K_i = \mathbb{F}[y]/\langle f_i \rangle\) is an extension field of \(\mathbb{F}\). Because in every field we have

\[
y^{2^k} - 1 = (y - 1)(y + 1)(y^2 + 1) \cdots (y^{2^{k-1}} + 1), \tag{2.3}
\]

it follows that \(t \geq k + 1\). Suppose \(x^s - 1\) has \(b_i\) monic factors in the extension field \(K_i\). Then we have \(b_i \geq a\) for all \(i\). Thus the number of ideals of \(R\) is \(\prod_{i=1}^{t} b_i \geq a^t \geq a^{k+1}\). Consequently, \(a^{k+1} = \prod_{i=1}^{t} b_i\) and hence \(t = k + 1\) and \(b_i = a\) for all \(i\). From \(t = k + 1\) it follows that, for each \(1 \leq i \leq k - 1\) the polynomial \(y^{2^i} + 1\) is irreducible over \(\mathbb{F}\), in particular, \(x^{2^i-1} + 1\) is irreducible in \(\mathbb{F}[x]\). Also we can assume that \(f_1 = y - 1\) and \(f_i = y^{2^{i-2}} + 1\), for \(i \geq 2\). From \(b_i = a\) we deduce that the number of irreducible factors of \(x^s - 1\) is the same in \(\mathbb{F}[x]\) and \(K_i[x]\). Now if condition \((\dagger)\) does not hold, then an irreducible factor of \(x^{2^k} - 1\), say \(x^{2^i} + 1\) for some \(1 \leq i \leq k - 1\), divides \(x^s - 1\). Since in \(K_{i+2}\) we have \(y^{2^i} + 1 = 0\), in \(K_{i+2}[x]\) we have \(x - y/x^{2^i} + 1\) and \(x - y\) is a monic factor of \(x^s - 1\) over \(K_{i+2}[x]\) but not over \(\mathbb{F}[x]\). Hence \(b_{i+2} > a\), a contradiction from which \((\dagger)\) follows.

\((\Leftarrow)\): By \((\dagger)\) it follows that \(x^{2^i} + 1\) is irreducible over \(\mathbb{F}\) for all \(i \leq k - 1\). So \((2.3)\) is an irreducible factorization and we deduce that \(t = k + 1\). If for some \(i \geq 1\), a factor of \(x^{2^i} + 1\) in \(K_i[x]\), divides \(x^s - 1\), then as \(x^{2^i} + 1\) is irreducible in \(\mathbb{F}[x]\), we must have \(x^{2^i} + 1|x^s - 1\), which contradicts \((\dagger)\). Thus the factors of \(x^s - 1\) over \(\mathbb{F}\) and over \(K_i\) are the same. So \(b_i = a\) for all \(i\) and hence \(R\) has \(a^{k+1}\) ideals.

Therefore, to show that every ideal is as in \((2.1)\), we just need to show that the number of ideals of \(R\) which have the claimed form is \(a^{k+1}\). As we can choose all \(p_i(x)\) from the set of monic factors of \(x^s - 1\) in \(a^{k+1}\) ways, it suffices to prove that for different choices of \(p_i(x)\), different ideals are generated by Equation \((2.1)\).
To see this, let $f_1 = y - 1$, $f_i = y^{2^{i-2}} + 1$ and $f_j = (y^{2^j} - 1)/f_i$ and note that $f_1 = \sum_{i=0}^{2^{j-1}} y^i$ and $f_{j+1} = -\sum_{i=0}^{2^{j-1}} (-1)^i y^{2^j}$ for all $0 \leq j \leq k$. Thus under the isomorphism (2.2), $f_j$ is mapped to the vector with all entries zero except for the $i$-th entry which is a unit in $K_i[x]/(x^s - 1)$. Thus an ideal $I$ satisfying (2.1) is mapped to the ideal generated by the vector with the only nonzero entry at the $i$-th position and equal to $p_i(x)$. Hence if $I$ and $I'$ both satisfy (2.1) with $p_i(x)$ and $p'_i(x)$, respectively, where $p_i$ and $p'_i$ are monic factors of $x^s - 1$, then $I = I' \iff p_i(x) = p'_i(x)$ for each $1 \leq i \leq k + 1$.

Using known results on irreducibility and factors of $x^t \pm 1$ over finite fields, we can restate Proposition 2.3 as follows.

**Theorem 2.4.** Fix a field $\mathbb{F}$ with an odd characteristic and consider integers $s > 1$ and $k > 0$. Then every 2D cyclic code of length $2^k s$ over $\mathbb{F}$ as an ideal of $R = \mathbb{F}[x, y]/(x^s - 1, y^{2^k} - 1)$ has a generating set of the form (2.1), if and only if one of the following conditions holds:

(i) $k = 1$;

(ii) $k = 2$, $q \equiv 3 \mod 4$ and $4 \nmid s$.

**Proof.** It is clear that if $k = 1$, then both conditions of Proposition 2.3 hold. If $k > 2$, then it follows from [15, Theorem 3.75], that $x^{2^k - 1} + 1$ is not irreducible and Proposition 2.3 does not hold. If $k = 2$, then using [15, Theorem 3.75], we see that condition (i) of 2.3 holds if and only if $q \equiv 3 \mod 4$. Also by [15, Corollary 3.7], $\gcd(x^s - 1, x^{2^k} - 1) = x^{\gcd(s, 2^k)} - 1$ and hence the condition (ii) holds if and only if 4 $\nmid s$. Consequently, the result follows Proposition 2.3.

At the end we mention that Theorems 3.2, 4.1 and 5.7 of [20] are not correct either, because they are based on Theorem 2.1. Indeed, the first part of [20, Theorem 4.1] (about the generating set of the code) is just a special case of Theorem 2.1 and Example 2.2 serves as a counterexample to that theorem, too. Also if in Example 2.2, we replace $x^s - 1$ with $x^{2^p s} + 1$, then the same argument shows that the image of the ideal $I$ in the ring $\mathbb{F}[x, y]/(x^{2^p s} + 1, y^{2^k} - 1)$ has no generating set as mentioned in [20, Theorem 3.2]. For Theorem 5.7 of [20], note that it is proved in [20], that the ring in Theorem 5.7 is isomorphic to a ring as in Theorems 3.2 or 4.1. Thus by this isomorphism, counterexamples to the latter theorems map to counterexamples of Theorem 5.7. It should be mentioned that Theorem 2.1 or an argument quite similar to its proof and also the isomorphism defined in [20, Section 5], can be
applied in conditions of Theorems 3.2, 4.1 and 5.7 of [20] to see exactly when they are correct. But here in the next section, we more generally present generating sets for all constacyclic 2D codes which are repeated root in at most one direction.

3 One-sided repeated-root 2D constacyclic codes

In this paper, we deal with 2D constacyclic codes which are either simple root or have repeated roots in at most one direction. We call such codes one-sided repeated-root codes, as defined below.

Definition 3.1. We call a two-dimensional \((\lambda, \delta)\)-constacyclic code \(D\) of length \(nm\) over \(\mathbb{F}_{p^r}\), one-sided repeated root, if either \(\gcd(n, p) = 1\) or \(\gcd(m, p) = 1\).

First, we fix some notations.

Notation 3.2. From now on, we assume that \(n, m\) are two integers, such that \(\gcd(n, p) = 1\), \(m = m'p^s\) and \(\gcd(m', p) = 1\). Also we assume that \(\lambda, \delta\) are non-zero elements of \(\mathbb{F}_p\). We let \(S = \mathbb{F}[x, y]/\langle x^n - \lambda, y^m - \delta \rangle\). Moreover, we assume that \(x^n - \lambda = \prod_{j=1}^{\eta} f_j(x)\), where \(f_j(x), 1 \leq j \leq \eta\), are monic irreducible coprime polynomials in \(\mathbb{F}[x]\). Also we set \(d_j = \deg f_j, K_j = \mathbb{F}[x]/\langle f_j(x) \rangle \cong \mathbb{F}_{q^{d_j}}\) and \(S_j = K_j[y]/\langle y^m - \delta \rangle\). We consider elements of \(S\) as those elements of \(\mathbb{F}[x, y]\) whose \(x\)-degree and \(y\)-degree is less than \(n\) and \(m\), respectively. A similar notation holds for elements of \(S_j\) and \(K_j\).

Note that every \((\lambda, \delta)\)-constacyclic code of length \(nm\) over \(\mathbb{F}\) is an ideal of \(S\). As the following remark shows, \(S\) is the direct sum of \(S_j\)’s. By using this structure, we determine the ideals of \(S\).

Remark 3.3. Using the CRT we see that

\[
S = \mathbb{F}[x, y]/\langle x^n - \lambda, y^m - \delta \rangle \cong \prod_{j=1}^{\eta} \mathbb{F}[x]/\langle f_j(x) \rangle \cong \prod_{j=1}^{\eta} S_j.
\]

This isomorphism is \(\psi : S \to \bigoplus_{j=1}^{\eta} S_j\) with

\[
\psi(h(x, y)) = (\psi_1(h(x, y)), \ldots, \psi_\eta(h(x, y))),
\]

where \(\psi_j : S \to S_j\) is defined with \(\psi_j(h(x, y)) = h(x, y) \mod f_j(x)\).
Now, we can determine the general form of ideals of \( S \). If \( C \) is a \((\lambda, \delta)\)-constacyclic code over \( F \), then \( C \) is an ideal of \( S \). Hence \( \psi(C) = \bigoplus_{j=1}^{\eta} C_j \), where \( C_j, 1 \leq j \leq \eta \), is an ideal of \( S_j = \frac{K_i[y]}{(y^m - \delta)} \). In fact, \( C_j \) is a \( \delta \)-constacyclic code over \( K_j \). Thus as an ideal of \( S_j \), \( C_j = \langle g_j(x, y) \rangle \), where \( g_j(x, y) \in K_j[y] \) is the generator polynomial for \( C_j \), that is, the unique monic polynomial of minimum \( y \)-degree in \( C_j \) which divides \( y^m - \delta \) in \( K_j[y] \). In particular, \( C_j = 0 \) if and only if \( g_j(x, y) = y^m - \delta \).

**Theorem 3.4.** Let \( C \) be a \((\lambda, \delta)\)-constacyclic code over \( F \). Then there exist unique polynomials \( g_j(x, y) \) such that \( g_j(x, y) \mid y^m - \delta \) in \( K_j[y] \), \( g_j(x, y) \) is monic when considered as a polynomial in \( y \) and as an ideal of \( S \),

\[
C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), g_2(x, y) \prod_{i \neq 2} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle.
\]

Moreover, \( \dim(C) = mn - \sum_{j=1}^{\eta} d_j t_j \), where \( t_j = \deg_y g_j \).

**Proof.** Suppose that \( \psi(C) = \bigoplus_{j=1}^{\eta} \ll g_j(x, y) \gg \). Set

\[
D = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), g_2(x, y) \prod_{i \neq 2} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle.
\]

For any \( l \neq j \), we have \( \psi(lg_j(x, y) \prod_{i \neq j} f_i(x)) = 0 \) (because \( \psi(lf_i(x)) = 0 \)). Also, \( \psi(lg_i(x, y) \prod_{i \neq l} f_i(x)) = g_l(x, y) \prod_{i \neq l} f_i(x) \) \( \bmod f_l \) in \( S_l \). Since \( \prod_{i \neq l} f_i(x) \) is a unit in \( K_l[y] \), so \( \psi(D) = \bigoplus_{j=1}^{\eta} \ll g_j(x, y) \gg = \psi(C) \). Thus \( C = D \).

Now, let \( C = \langle g'_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g'_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle \), for some \( g'_j(x, y) \) such that \( g'_j(x, y) \mid y^m - \delta \) in \( K_j[y] \) and \( g'_j(x, y) \) is monic when considered as a polynomial in \( y \). We claim that \( g'_j(x, y) = g_j(x, y) \). For any \( j \), we have \( \psi_j(C) = \langle g'_j(x, y) \rangle = \langle g_j(x, y) \rangle \). Since \( g_j \) and \( g'_j \) are monic and divide \( y^m - \delta \), so \( g'_j(x, y) = g_j(x, y) \).

Let \( C_j = \langle g_j(x, y) \rangle \). Thus

\[
|C| = |\psi(C)| = |C_1||C_2| \cdots |C_\eta| = (q^d_1)^{m-t_1}(q^d_2)^{m-t_2} \cdots (q^d_\eta)^{m-t_\eta} = q^{m-n} \sum_{j=1}^{\eta} d_j t_j = q^{mn - \sum_{j=1}^{\eta} d_j t_j}.
\]

Consequently, \( \dim(C) = mn - \sum_{j=1}^{\eta} d_j t_j \). \( \square \)

The above theorem introduces a generating set for an ideal \( C \) of \( S \) and shows that this generating set is unique. In the sequel, by
\[
C = \ll g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \gg.
\]
we mean that all \(g_j(x, y)\) satisfy the conditions of Theorem 3.4 and
\[
C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle.
\]

**Remark 3.5.** With the notations of Theorem 3.4, the polynomial \(\sum_{j=1}^{\eta} g_j(x, y) \prod_{i \neq j} f_i(x)\) generates \(C\). To see this, assume that \(D = \langle \sum_{j=1}^{\eta} g_j(x, y) \prod_{i \neq j} f_i(x) \rangle\). Note that each \(\psi_l\) is a ring isomorphisms and \(\psi_l(g_j(x, y) \prod_{i \neq j} f_i(x)) = 0\), if \(l \neq j\). It follows that \(\psi_l(C) = \psi_l(D)\) for all \(l\) and hence \(C = D\). Note that \(S\) is a principal ideal ring (PIR).

Let \(C = \langle h_1(x, y), h_2(x, y), \ldots, h_r(x, y) \rangle\), as an ideal of \(S\), where \(\deg_y h_j < m\). We show how to find the unique generator polynomials for \(C\) as in the previous theorem. For this, we compute \(\psi_j(h_1(x, y))\), for each \(j, l, 1 \leq l \leq r, 1 \leq j \leq \eta\). Then using Euclidean algorithm, we calculate the greatest common divisor of \(\psi_j(h_1(x, y))\) for all \(l\). Assume that
\[
g_j(x, y) = \gcd(\psi_j(h_1(x, y)), \ldots, \psi_j(h_r(x, y)), y^m - \delta) \text{ in } K_j[y].
\]
Here we assume that \(\gcd\) returns a monic polynomial and also \(\gcd(0, f) = f\).

Thus \(g_j(x, y) \mid y^m - \delta\) and is monic as a polynomial in \(y\) and also \(\psi_j(C) = \langle g_j(x, y) \rangle\). So
\[
C = \ll g_1(x, y) \prod_{i \neq 1} f_i(x), g_2(x, y) \prod_{i \neq 2} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \gg.
\]

**Example 3.6.** Suppose that \(S = \frac{\mathbb{F}_2[x, y]}{(x^2 + 1, y^2 + 1)}\). We have
\[
\begin{align*}
x^3 - 1 &= (x + 1)(x^2 + x + 1) \quad \text{in } \mathbb{F}_2[x] \\
y^3 - 1 &= (y + 1)(y^2 + y + 1) \quad \text{in } \frac{\mathbb{F}_2[x]}{(x + 1)}[y] \\
y^3 - 1 &= (y + 1)(y + x)(y + x + 1) \quad \text{in } \frac{\mathbb{F}_2[x]}{(x^2 + x + 1)}[y].
\end{align*}
\]

If \(C\) is an ideal of \(S\), then \(C = \ll g_1(x, y)(x^2 + x + 1), g_2(x, y)(x + 1) \gg\), where \(g_1(x, y) \mid y^3 - 1\) in \(\frac{\mathbb{F}_2[x]}{(x + 1)}[y]\) and \(g_2(x, y) \mid y^3 - 1\) in \(\frac{\mathbb{F}_2[x]}{(x^2 + x + 1)}[y]\). The number of such codes is \(2^2 \times 2^3 = 32\). Consider the code \(C = \langle h_1(x, y), h_2(x, y), h_3(x, y) \rangle\), where
\[
\begin{align*}
h_1(x, y) &= y^2x^2 + y^2x + x + 1, \\
h_2(x, y) &= y^2x + yx + x \quad \text{and} \\
h_3(x, y) &= y^2x^2 + y^2 + yx^2 + y + x^2 + 1.
\end{align*}
\]
We have

\[ \psi_1(h_1) = 0, \quad \psi_2(h_1) = y^2 x^2 + y^2 x + x^2 = x(y + x)(y^2 + y + x), \]
\[ \psi_1(h_2) = y^2 + y + 1, \quad \psi_2(h_2) = y^2 x + y x + x = (y + x)(y + x + 1), \]
\[ \psi_1(h_3) = 0 \text{ and } \psi_2(h_3) = y^2 x + y x + x = x(y + x)(y + x + 1). \]

So

\[ \gcd(\psi_1(h_1), \psi_1(h_2), \psi_1(h_3), y^3 - 1) = y^2 + y + 1 \text{ in } \mathbb{F}_2[x]/(x + 1)[y]. \]
\[ \gcd(\psi_2(h_1), \psi_2(h_2), \psi_2(h_3), y^3 - 1) = y + x \text{ in } \mathbb{F}_2[x]/(x^2 + x + 1)[y]. \]

Therefore, \( C = \langle 1 + y + y^2 \rangle(x^2 + x + 1), (y + x)(x + 1) \rangle. \) Note that here \( f_1 = x + 1, f_2 = x^2 + x + 1, g_1 = 1 + y + y^2 \) and \( g_2 = y + x \) and hence we get \( \dim C = 9 - 4 = 5. \)

**Corollary 3.7.** Consider a \((\lambda, \delta)\)-constacyclic code \( C \) over \( \mathbb{F} \) such as

\[ C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), g_2(x, y) \prod_{1 \leq i \leq 2} f_i(x), \ldots, g_\eta(x, y) \prod_{1 \leq i \leq \eta} f_i(x) \rangle. \]

Then the following set is a basis for \( C \) over \( \mathbb{F} \).

\[ \Delta = \bigcup_{j=1}^\eta \{ x^ry^j g_j(x, y) \prod_{i \neq j} f_i(x) \mid 0 \leq r < d_j, 0 \leq l < m - t_j \}, \]

where \( t_j = \deg_y g_j. \) Note that, if \( g_j = y^m - \delta \), then the set \( \{ x^ry^j g_j(x, y) \prod_{i \neq j} f_i(x) \mid 0 \leq l < m - t_j \} \) is empty for all \( r \).

**Proof.** Let \( D \) be the \( \mathbb{F} \)-subspace of \( S \), generated by \( \Delta \). We show that \( D = C \). Since \( C \) is an ideal of \( S \), \( \Delta \subseteq C \). So \( D \subseteq C \). We will show that the elements of \( \Delta \) are linearly independent over \( \mathbb{F} \). Suppose that

\[ B = \sum_{j=1}^\eta \left( \sum_{r=0}^{m-t_j-1} a_{rj}(x)y^r \right) g_j(x, y) \prod_{i \neq j} f_i(x) = 0 \]

in \( S \), where \( \deg a_{rj}(x) < d_j \). Hence \( \psi(B) = 0 \). So for any \( j, 1 \leq j \leq \eta, \psi_j(B) \in \langle y^m - \delta \rangle. \) Thus

\[ (\sum_{r=0}^{m-t_j-1} a_{rj}(x)y^rg_j(x, y))\psi_j(\prod_{i \neq j} f_i(x)) = 0 \]

in \( S \). Since \( \prod_{i \neq j} f_i(x) \) is a unit in \( K_j \), \( \sum_{r=0}^{m-t_j-1} a_{rj}(x)y^rg_j(x, y) = 0 \) in \( S \). Since the \( y \)-degree of \( \sum_{r=0}^{m-t_j-1} a_{rj}(x)y^rg_j(x, y) \) is less than \( m \), \( \sum_{r=0}^{m-t_j-1} a_{rj}(x)y^rg_j(x, y) = 0 \) in \( K_j[y] \). Thus \( (\sum_{r=0}^{m-t_j-1} a_{rj}(x)y^r) = 0 \) in
$K_j[y]$. Hence $a_{rj}(x) = 0$ in $K_j$ for all $r$. Since $\deg a_{rj} < \deg f_j$, $a_{rj}(x) = 0$ in $F[x]$. So all coefficients of $a_{rj}(x)$ are zero and hence $\Delta$ is an independent set. Now,

$$| D | = (q^{d_1})^{m-t_1}(q^{d_2})^{m-t_2} \cdots (q^{d_n})^{m-t_n} = q^{\sum_{j=1}^n d_j t_j} = | C |.$$ 

Therefore, $C = D$. 

The following example shows how we can find a generator matrix for a given code using Corollary 3.7.

**Example 3.8.** With the notations of Example 3.6, consider $C = \ll (1 + y + y^2)(x^2 + x + 1), (y + x)(x + 1) \gg$. We have $| C | = 2^5 = 32$. Also, the generator matrix of $C$ is of the following form.

$$G = \begin{bmatrix}
(1 + y + y^2)(x^2 + x + 1) \\
(y + x)(x + 1) \\
x(y + x)(x + 1) \\
y(y + x)(x + 1) \\
xy(y + x)(x + 1)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}.$$ 

Note that every row of $G$ as a constacyclic codeword is seen as a $3 \times 3$ array. For example, the array form of the second row is 

$$1 \quad x \quad x^2$$

$$1 \quad 0 \quad 1 \quad 1$$

$$y \quad 1 \quad 1 \quad 0$$

$$y^2 \quad 0 \quad 0 \quad 0$$

and its cyclic shift corresponding to multiplication by $x$ is the third row.

# 4 The dual of one-sided repeated-root 2D constacyclic codes

For any $(\lambda, \delta)$-constacyclic code $C \subseteq F^{nm}$, let

$$C^\perp = \{ u \in F^{nm} \mid u \cdot w = 0 \text{ for any } w \in C \}$$

be the dual of the code $C$, where $u \cdot w$ is the Euclidean inner product of $u$ and $w$ in $F^{nm}$. By [20, Proposition 2.2], $C^\perp$ is a $(\lambda^{-1}, \delta^{-1})$-constacyclic code over $F$. In this section, we shall determine the unique generating set of the dual of $C$ as an ideal of $T = \frac{F[x,y]}{<x^n-\lambda^{-1}, y^m-\delta^{-1}>}$. 

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Proof. (i): Suppose that $a \in S$ have similar meanings. Note that for $f, g \in \mathbb{F}[x, y]$, we have $x^n - \lambda = \prod_{i=1}^n f_i(x)$ for some $u \in \mathbb{F}$.

Similar to Remark 3.3 we can show that

$$T \simeq \bigoplus_{i=1}^n \frac{\mathbb{F}[x]}{(f_i(x))} [y],$$

where $f_i(x) = f^*(x)$.

Proposition 4.1 ([20, Proposition 2.1]). Assume that $\lambda_1$ and $\lambda_2$ are non-zero elements of $\mathbb{F}$ and $f(x, y) = f_0(x) + \cdots + f_{m-1}(x)y^{m-1}$, $g(x, y) = g_0(x) + \cdots + g_{m-1}(x)y^{m-1} \in \mathbb{F}[x, y]$, where $f_i(x) = \sum_{j=0}^{m-1} a_{ji}(x)x^j$ and $g_i(x) = \sum_{j=0}^{m-1} b_{ji}(x)x^j$ for $i = 0, 1, \ldots, m - 1$. Then $f(x, y)g(x, y) = 0$ in $\mathbb{F}[x, y]$ iff $(a_0, a_1, \ldots, a_{m-1})$ is orthogonal to $(b_{n-1}, \ldots, b_0)$ and all its $(\lambda_1^{-1}, \lambda_2^{-1})$-constacyclic shifts, where $a_i = (a_0, \ldots, a_{n-1}, i)$, $b_i = (a_{n-1}, \ldots, a_0)$ for $0 \leq i \leq m - 1$.

Suppose that $f(x, y) \in \mathbb{F}[x, y]$ has $x$- and $y$-degree less than $n$ and $m$, respectively. Then we define $f^\circ(x, y) = x^{-n}y^{-m-1}f(x, y)$. If $f \in S$ (resp. in $T$), we consider $f^\circ$ as an element of $T$ (resp. $S$). With this definition, Proposition 4.1 says that $\text{ann}(g) = (g^\circ)$$^1$ for any $g \in S$. It is easy to see that $f^\circ(x, y) = f(x, y)$ and if $f, g \in S$, then and $(f + g)^\circ = f^\circ + g^\circ$. Now for any polynomial $f(x, y) \in \mathbb{F}[x, y]$ define $f^*(x, y) = x^{\deg f}f(x, y)$.

Then $f^\circ(x, y) = uf^\circ(x, y)$ for some $u \in T$ and

$$\langle f^\circ(x, y) \rangle = \langle f^\circ(x, y) \rangle.$$

Note that for $f, g \in S$ by $(fg)^\circ$ we mean $h^\circ$ where $h$ is a polynomial with $x$-degree less than $n$ and $y$-degree less than $m$ such that we have $h = fg$ in $S$. Also for $A \subseteq S$, by $A^\circ$ we mean $\{a^\circ | a \in A\}$. The notions $A^\circ$ and $(fg)^*$ have similar meanings.

Lemma 4.2. Suppose that $f, g \in S$ and let $I$ be an ideal of $S$. Then

(i) $(fg)^\circ = uf^\circ g^\circ$ for some unit $u \in T$;

(ii) $(fg)^* = uf^* g^*$ for some unit $u \in T$.

(iii) $I^\circ = \langle I^\circ \rangle$.

Proof. (i): Suppose that $fg = h$ in $S$. Then in $\mathbb{F}[x, y]$ there exist polynomials $a$ and $b$ such that $fg = h + a(x^n - \lambda) + b(y^m - \delta)$. Since $x$- and $y$-degrees
of \( fg \) are at most \( 2n - 2 \) and \( 2m - 2 \), respectively, we can assume that \( \deg_x(a) \leq n - 2 \), \( \deg_y(a) \leq 2m - 2 \), \( \deg_x(b) \leq 2n - 2 \) and \( \deg_y(b) \leq m - 2 \). Now in \( \mathbb{F}[x, y] \) we have

\[
f^\circ g^\circ = x^{2n-2}y^{2m-2}f\left(\frac{1}{x}, \frac{1}{y}\right)g\left(\frac{1}{x}, \frac{1}{y}\right) = x^{2n-2}y^{2m-2}h\left(\frac{1}{x}, \frac{1}{y}\right) + x^{n-2}y^{2m-2}a\left(\frac{1}{x}, \frac{1}{y}\right)(1 - \lambda x^n) + x^{2n-2}y^m b\left(\frac{1}{x}, \frac{1}{y}\right)(1 - \delta y^m) = x^{n-1}y^{m-1}h^\circ - \lambda a' (x, y)(x^n - \lambda^{-1}) - \delta b' (x, y)(y^m - \delta^{-1}),
\]

for some \( a', b' \in \mathbb{F}[x, y] \). Therefore in \( \mathcal{T} \) we have

\[
f^\circ g^\circ = x^{n-1}y^{m-1}h^\circ = u(fg)^\circ,
\]

for some unit \( u \in \mathcal{T} \).

- (iii): This follows from part (ii) and (i).
- (iii): It is easy to see that if \( d = \deg_x(f), d' = \deg_y(f) \), then \( f^\circ = (x^{n-d-1}y^{m-d-1}f^\circ) \) and hence \( I^\circ \subseteq I^\circ \). Also from (4.1), it follows that \( I^\circ \subseteq \langle I^* \rangle \). Thus it suffices to show that \( I^\circ \) is an ideal. It is closed under addition because \( (f + g)^\circ = f^\circ + g^\circ \). Now assume that \( f^\circ \in I^* \), that is \( f \in I \) and \( g \in \mathcal{T} \). By part (ii), we have \( (gf^\circ)^\circ = u(g^\circ f) \in I \) for some unit \( u \in \mathcal{S} \) and hence \( gf^\circ = (gf^\circ)^\circ \in I^\circ \).

Using Proposition 4.1.1 and Lemma 4.2 we have the following lemma that shows the relationship between the elements of \( \text{ann}(C) \) and \( C^\perp \).

**Lemma 4.3.** Let \( C \) be a \((\lambda, \delta)\)-constacyclic code over \( \mathbb{F} \). Then

\[
C^\perp = \text{ann}^\circ(C) = \langle \text{ann}^*(C) \rangle.
\]

**Proof.** By Lemma 4.2 \( \text{ann}^\circ(C) = \langle \text{ann}^*(C) \rangle \). Also as \( (fg)^\circ = 0 \) in \( \mathcal{T} \) if and only if \( fg = 0 \) in \( \mathcal{S} \), it follows that \( \text{ann}^\circ(C) = \text{ann}(C^\circ) \). Note that Proposition 4.1.1 indeed says that \( \text{ann}(g) = \langle g^\circ \rangle \). Now from

\[
C^\perp = \bigcap_{g \in C} \langle g \rangle^\perp = \bigcap_{g \in C} \text{ann}(g^\circ) = \text{ann}(C^\circ),
\]

the claim follows.

Let \( C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle \) be a \((\lambda, \delta)\)-constacyclic code over \( \mathbb{F} \). Assume that

\[
h_i(x, y) = \frac{y^m - \delta}{g_i(x, y)} \quad (4.2)
\]
in $K_1[y]$. If $g_i(x, y) = 0$, we assume that $h_i(x, y) = 1$. Suppose that $h_i^x(x, y)$ is the monic polynomial in $\mathbb{F}[x, y]$ such that

$$h_i^x(x, y) = \frac{h_i^*(x, y)}{h_i(x, 0)} \quad (4.3)$$

in $\frac{\mathbb{F}[x]}{(f_j^*(x))}[y]$. With this notations, we have the following theorem that gives the generating set of the dual of the code $C$.

**Theorem 4.4.** Let $C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle$ be a $(\lambda, \delta)$-constacyclic code over $\mathbb{F}$. Then

$$C^\perp = \langle h_1^x(x, y) \prod_{i \neq 1} f_i^x(x), h_2^x(x, y) \prod_{i \neq 2} f_i^x(x), \ldots, h_\eta^x(x, y) \prod_{i \neq \eta} f_i^x(x) \rangle.$$

**Proof.** We can see that $h_j^x(x, y), 0 \leq j \leq \eta$, is a monic polynomial, by (4.3), and divides $y^m - \delta^{-1}$ in $\frac{\mathbb{F}[x]}{(f_j^*(x))}[y]$, by (4.2). Let $D = \langle h_1^x(x, y) \prod_{i \neq 1} f_i^x(x), \ldots, h_\eta^x(x, y) \prod_{i \neq \eta} f_i^x(x) \rangle$. For $l \neq j$, we have $a(x, y) = (g_j(x, y) \prod_{i \neq j} f_i(x)) (h_l(x, y) \prod_{i \neq l} f_i(x)) = 0$ in $\mathcal{S}$, because, $x^\eta - \lambda | a(x, y)$ in $\mathbb{F}[x, y]$. For $l = j$, we have $h_l(x, y) g_l(x, y) = 0$. So $h_l(x, y) \prod_{i \neq l} f_i(x) \in \text{ann}(C)$. Hence by Lemma 4.3, $(h_l(x, y) \prod_{i \neq l} f_i(x))^* \in C^\perp$ for any $l, 1 \leq l \leq \eta$. So by Lemma 4.2, $uv h_i^x(x, y) \prod_{i \neq l} f_i^x(x) \in C^\perp$ for some unit $u \in \mathcal{T}$ and hence $v h_i^x(x, y) \prod_{i \neq l} f_i^x(x) \in C^\perp$ for some unit $v \in \mathcal{T}$. Consequently, $D \subseteq C^\perp$. Because $h(x, 0) \neq 0 \neq f_i(0)$, we have $\deg h_i^x = \deg h_i^* = \deg h_i$ and $\deg f_i^x = \deg f_i$. So

$$\dim(D) = mn - \sum_{j=1}^{\eta} d_j(m - t_j) = \sum_{j=1}^{\eta} d_j t_j = \dim(C^\perp).$$

Hence $C^\perp = D$. \qed

Now we can find the parity check matrix of the code $C$, as the following example shows.

**Example 4.5.** Suppose that $\mathcal{S} = \frac{\mathbb{F}_2[x, y]}{(x^3 - 1, y^4 - 1)}$ and $C$ is the code defined in
Example 3.8 that is, \(C = \ll (1 + y + y^2)(x^2 + x + 1), (y + x)(x + 1) \gg\). Hence

\[
h_1(x, y) = \frac{y^3 - 1}{y^2 + y + 1} = y + 1 \quad \text{in } \mathbb{F}_2[x]/(x + 1)[y]
\]

\[
h_1^x(x, y) = \frac{y(\frac{1}{y} + 1)}{0 + 1} = 1 + y
\]

\[
h_2(x, y) = \frac{y^3 - 1}{y + x} = y^2 + xy + x + 1 \quad \text{in } \mathbb{F}_2[x]/(x^2 + x + 1)[y]
\]

\[
h_2^x(x, y) = \frac{y^2(x(\frac{1}{y} + 1) + \frac{1}{x} + 1)}{x + 1} = (y + x)(y + 1)
\]

We have \(C^\perp = \ll (1 + y)(x^2 + x + 1), (y + 1)(y + x + 1)(x + 1) \gg\). Also, the parity check matrix of \(C\) has the following form.

\[
H = \begin{bmatrix}
(1 + y)(x^2 + x + 1) \\
y(1 + y)(x^2 + x + 1) \\
(y^2 + xy + x + 1)(x + 1) \\
x(y^2 + xy + x + 1)(x + 1)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]

Every two columns of \(H\) are linearly independent and there are 3 linearly dependent columns in \(H\). So \(d_{\text{min}}(C) = 3\).

Next we study when \(C\) is self-dual, that is, \(C = C^\perp\). To see why it is important to study and find self-dual codes see for example [19, Section 3]. Note that if \(C\) is self-dual, then it is both \((\lambda, \delta)\)-constacyclic and \((\lambda^{-1}, \delta^{-1})\)-constacyclic.

**Lemma 4.6.** Suppose that \(C\) is both \((\lambda, \delta_1)\)-constacyclic and \((\lambda', \delta_2)\)-constacyclic, where \(\delta_1 \neq \delta_2\) (but we may have \(\lambda' = \lambda\)). Then there exists a (one-dimensional) \(\lambda\)-constacyclic code \(C_0\) of length \(n\) such that

\[C = C_0^m = \{(c_0, \ldots, c_{m-1}) | \forall 0 \leq i < m: \ c_i \in C_0\}.
\]

Also in this case, \(C\) is \((\lambda, \delta)\)-constacyclic for every \(\delta \in \mathbb{F}\). A similar result holds for \((\lambda_1, \delta)\)- and \((\lambda_2, \delta')\)-constacyclic codes.

**Proof.** Let \(C_0\) be the set of \(c \in \mathbb{F}^n\) such that \(c\) appears as the \(i\)-th entry of a codeword of \(C\). Note that since \(C\) is \(\mathbb{F}\)-linear and constacyclic, \(C_0\) is independent of the choice of \(i\). Also it is routine to see that \(C_0\) is a \(\lambda\)-constacyclic code of length \(n\) and \(C \subseteq C_0^m\). Conversely, if \(c \in C_0\), say \(u = (c_0, \ldots, c_{m-1}) \in C\) with \(c = c_{m-1},\) then as \(C\) is both \((\lambda, \delta_1)\)- and
\((\lambda', \delta_2)\)-constacyclic, \(v = (\delta_1 c, \ldots, c_{m-2})\) and \(w = (\delta_2 c, \ldots, c_{m-2})\) are in \(C\) and hence \(v - w = ((\delta_1 - \delta_2)c, 0, \ldots, 0)\) is in \(C\). Thus by \(F\)-linearity and being constacyclic, we see that \((0, \ldots, 0, c, 0, \ldots, 0)\) is in \(C\), where the position of \(c\) is arbitrary. Again as \(c \in C_0\) was chosen arbitrarily and by linearity, we deduce that \(C_0^m \subseteq C\).

**Proposition 4.7.** Suppose that \(C\) is a \((\lambda, \delta)\)-constacyclic code of length \(nm\) and \(\delta^2 \neq 1\). Then \(C\) is self-dual if and only if \(C = C_0^m\) for some self-dual \(\lambda\)-constacyclic code \(C_0\) of length \(n\). A similar result holds when instead of \(\delta^2 \neq 1\) we assume \(\lambda^2 \neq 1\).

**Proof.** By Proposition 2.2, \(C = C^\perp\) is a \((\lambda^{-1}, \delta^{-1})\)-constacyclic code and as \(\delta^{-1} \neq \delta\), we can apply 4.6 to see that \(C = C_0^m\). It is routine to check that \(C_0^m\) is self-dual if and only if \(C_0\) is self-dual.

Thus to see when a constacyclic code \(C\) is self-dual, it remains to consider the case that \(\lambda^2 = \delta^2 = 1\). So assume that \(\lambda^2 = \delta^2 = 1\) and \(C\) is a \((\lambda, \delta)\)-constacyclic code of length \(nm\). Also suppose that \(f^i_j = f_i\) for all of the irreducible factors \(f_i\) of \(x^n - \lambda\). This is because in this case, \(\prod_{i \neq j} f^i_j = \prod_{i \neq j} f_i\) for all \(j\) and hence according to 4.4 \(C\) is self-dual if and only if \(g_j(x, y) = h^j_2(x, y)\) for all \(j\). Note that since \(\delta = \pm 1\), we have \(y^m - \delta = (y^m' - \delta)^p\).

Let in \(K_j[y]\), \(y^m' - \delta = \prod_{l=1}^{t_j} h_{jl}(x, y)\), where \(h_{jl}(x, y)\), \(1 \leq l \leq t_j\), are monic irreducible coprime polynomials in \(K_j[y]\). Assume that \(h_{jl}(x, y) = h^j_{2l}(x, y)\) for \(1 \leq l \leq a_j\) and \(h^j_{2l} \neq h_{jl}\) for \(a_j < l\). Since \((y^m' - \delta)^s = y^m - \delta - 1 = y^m - \delta\), so for each \(1 \leq l \leq t_j\), we have \(h^j_{2l} = h_{jl}\) for some \(1 \leq l' \leq t_j\). Thus we can suppose that

\[
y^m' - \delta = \prod_{l=1}^{a_j} h_{jl}(x, y) \prod_{l=a_j+1}^{b_j} h_{jl}(x, y) \prod_{l=a_j+1}^{b_j} h^j_{2l}(x, y). \tag{4.4}
\]

**Theorem 4.8.** Let \(p = 2\), \(s > 0\), \(f^i_j(x) = f_i(x)\) for all \(i\), and

\[
C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), g_2(x, y) \prod_{i \neq 2} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle
\]

be a \((\lambda, \delta)\)-constacyclic code of length \(n(2^s m')\) over \(F\), where \(\lambda^2 = \delta^2 = 1\). The code \(C\) is self-dual if and only if for every \(j\),

\[
g_j(x, y) = \prod_{l=1}^{a_j} h^j_{2l}(x, y) \prod_{l=a_j+1}^{b_j} h^{\alpha_{jl}}_{jl}(x, y) \prod_{l=a_j+1}^{b_j} (h^j_{2l})^{2^s - \alpha_{jl}}(x, y), \tag{4.5}
\]

for some \(\alpha_{jl}\), \(0 \leq \alpha_{jl} \leq 2^s\).
Proof. Assume that \( C = C^\perp \). So for any \( j \), we have \( g_j(x, y) = h_j^+(x, y) \) or equivalently \( g_j^+(x, y) = h_j(x, y) \). This means,
\[
g_j(x, y) = \frac{y^m - \delta}{h_j(x, y)} = \frac{y^m - \delta}{g_j^+(x, y)}.
\]
If
\[
g_j(x, y) = \prod_{l=1}^{a_j} h_j^{\gamma_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} h_j^{\alpha_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} (h_j^\#)^{\beta_{jl}}(x, y),
\]
then
\[
g_j^+(x, y) = \prod_{l=1}^{a_j} h_j^{\gamma_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} (h_j^\#)^{\alpha_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} h_j^{\beta_{jl}}(x, y).
\]
Hence
\[
y^m - \delta = g_j(x, y) g_j^+(x, y)
\]
\[
= \prod_{l=1}^{a_j} h_j^{2\gamma_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} h_j^{\alpha_{jl} + \beta_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} (h_j^\#)^{\alpha_{jl} + \beta_{jl}}(x, y).
\]
So \( 2\gamma_{jl} = \alpha_{jl} + \beta_{jl} = 2^s \). Therefore, \( \gamma_{jl} = 2^{s-1} \) and \( \alpha_{jl} = 2^s - \beta_{jl} \). Conversely, suppose that \( g_j(x, y) \) is of the form (4.5). So
\[
h_j(x, y) = \frac{y^m - \delta}{g_j(x, y)} = \prod_{l=1}^{a_j} h_j^{2^{s-1}}(x, y) \prod_{l=a_j+1}^{b_j} h_j^{2^s - \alpha_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} (h_j^\#)^{\alpha_{jl}}(x, y).
\]
Hence
\[
h_j^+(x, y) = \prod_{l=1}^{a_j} h_j^{2^{s-1}}(x, y) \prod_{l=a_j+1}^{b_j} (h_j^\#)^{2^s - \alpha_{jl}}(x, y) \prod_{l=a_j+1}^{b_j} h_j^{\alpha_{jl}}(x, y) = g_j(x, y).
\]
Thus \( C = C^\perp \). \( \blacksquare \)

Example 4.9. Suppose that \( S = \frac{\mathbb{F}_2[x, y]}{(x-1, y^{x-1}-1)} \). We have
\[
x^3 - 1 \quad = \quad (x + 1)(x^2 + x + 1) \quad \text{in} \quad \mathbb{F}_2[x]
\]
\[
y^{12} - 1 \quad = \quad (y + 1)^4(y^2 + y + 1)^4 \quad \text{in} \quad \frac{\mathbb{F}_2[x]}{(x+1)}[y]
\]
\[
y^{12} - 1 \quad = \quad (y + 1)^4(y + x)^4(y + x + 1)^4 \quad \text{in} \quad \frac{\mathbb{F}_2[x]}{(x^2 + x + 1)}[y].
\]

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Let $C = \ll (y^2 + y + 1)^2 (y+1)^2 (x^2 + x + 1), (y+1)^2 (y+x)^3 (y+x+1)(x+1) \gg$ be a 2D constacyclic code of length 36 over $\mathbb{F}_2$. By Theorem 4.8, $C$ is a self-dual code. But the code

$D = \ll (y^2 + y + 1)(y+1)^2 (x^2 + x + 1), (y+1)^2 (y+x)^3 (y+x+1)(x+1) \gg$ is not self-dual. We have

$D^\perp = \ll (y^2 + y + 1)^3 (y+1)^2 (x^2 + x + 1), (y+1)^2 (y+x)^3 (y+x+1)^2 (x+1) \gg .$

Note that $(y+x)^\# = y + x + 1$.

The following theorem shows that, for an odd prime number $p$ the existence of self-dual one-sided repeated root 2D constacyclic codes depends on the factorization of $y^{m'} - \delta$.

**Theorem 4.10.** Assume that $f^\sharp_i(x) = f_i(x)$, for all $i$ and $\lambda^2 = \delta^2 = 1$. Let $p$ be an odd prime number or $s = 0$. There exists a self-dual 2D $(\lambda, \delta)$-constacyclic code of length $nm = n(p^s m')$ over $\mathbb{F}$ if and only if in (4.4), $a_j = 0$ for all $j$. In this case, a code

$C = \ll g_1(x,y) \prod_{i \neq 1} f_i(x), g_2(x,y) \prod_{i \neq 2} f_i(x), \ldots, g_\eta(x,y) \prod_{i \neq \eta} f_i(x) \gg$

is self-dual if and only if

$g_j(x,y) = \prod_{l=1}^{b_j} h_{jl}^{\alpha_{jl}}(x,y) \prod_{l=1}^{b_j} (h_{jl}^{\sharp})^{p^s - \alpha_{jl}}(x,y),$ for some $\alpha_{jl}, 0 \leq \alpha_{jl} \leq p^s$.

**Proof.** Assume that $p$ is an odd prime number and

$y^{m'} - \delta = \prod_{l=1}^{b_j} h_{jl}(x,y) \prod_{l=1}^{b_j} h_{jl}^{\sharp}(x,y).$

Consider the polynomials $g_j(x,y) = \prod_{l=1}^{b_j} h_{jl}^{\alpha_{jl}}(x,y) \prod_{l=1}^{b_j} (h_{jl}^{\sharp})^{p^s - \alpha_{jl}}(x,y),$ for $j, 1 \leq j \leq \eta$. Thus

$h_j(x,y) = \frac{y^{m'} - \delta}{g_j(x,y)} = \prod_{l=1}^{b_j} h_{jl}^{p^s - \alpha_{jl}}(x,y) \prod_{l=1}^{b_j} (h_{jl}^{\sharp})^{\alpha_{jl}}(x,y).$

Hence

$h_{jl}^{\sharp}(x,y) = \prod_{l=1}^{b_j} (h_{jl}^{\sharp})^{p^s - \alpha_{jl}}(x,y) \prod_{l=1}^{b_j} h_{jl}^{\alpha_{jl}}(x,y) = g_j(x,y).$
\[ C = g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \] is a self-dual code. Conversely, suppose that \( y^m - \delta \) is of the form (4.4) with \( a_j \neq 0 \) for some \( j \) and \( C = g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \) is a self-dual code. For any \( j \), we have

\[ g_j(x, y) = \prod_{l=1}^{a_j} h_\gamma^{jl}(x, y) \prod_{l=a_j+1}^{b_j} h_\gamma^{jl}(x, y) \prod_{l=a_j+1}^{b_j} (h_\gamma^{jl})^{\beta jl}(x, y), \]

for some \( \alpha jl, \gamma jl \) and \( \beta jl \), \( 0 \leq \alpha jl, \gamma jl, \beta jl \leq p^s \). Now, similar to the proof Theorem 4.8, we have \( \alpha jl + \beta jl = 2 \gamma jl = p^s \). Since \( p \) is an odd prime number or \( p^s = 1 \), this is impossible. So \( a_j = 0 \) and \( g_j(x, y) \) has the claimed form for all \( j \).

**Example 4.11.** Suppose that \( S = \frac{F_{9}[x, y]}{(x^2 - 1, y^2 + 1)} \). We have

\[ x^2 - 1 = (x - 1)(x + 1) \text{ and } y^2 + 1 = (y + \alpha)(y + \alpha^{-1}), \]

where \( \alpha^2 = -1 \) in \( F_9 \). Since

\[ (x + 1)^2 = x + 1 \text{ and } (x - 1)^2 = x - 1, \]

we can use Theorem 4.10 to determine self-dual codes over \( F_9 \). By Theorem 4.10,

\[ D_1 = (y + \alpha)(x - 1), (y + \alpha^{-1})(x + 1) \]

and

\[ D_2 = (y + \alpha^{-1})(x - 1), (y + \alpha)(x + 1) \]

are self-dual codes. Note that we also have \( S \cong \frac{F_{9}[x, y]}{(x^2 + 1, y^2 + 1)} \), but if we view \( S \) as this ring, then since \( (x + \alpha)^2 \neq (x + \alpha) \), we can not use Theorem 4.10.

### 5 Asymptotic badness of one-sided repeated-root 2D constacyclic codes

Theorem 3.4 enables us to state and prove 2D versions of some known results on repeated root (one dimensional) cyclic and constacyclic codes. One such result is [6, Theorem 4], which states that repeated-root cyclic codes cannot be asymptotically better than simple root cyclic codes. In this section, similar to [6], we relate the minimum distance of a one-sided repeated-root 2D constacyclic code to some simple-root 2D constacyclic codes and from this, we deduce that one-sided repeated-root \((\lambda, \delta)\)-constacyclic codes can not be
asymptotically better than simple-root 2D constacyclic codes. It should be mentioned that the ideas and techniques used here are quite similar to those used in sections III and IV of [4] and hence we do not state most of the proofs here.

Note that for any $\delta \in \mathbb{F}$, there exists a $\delta_0 \in \mathbb{F}$ with the property that $\delta = \delta_0^n$ (see, for example [4, Lemma 3.1]). Let $C = \langle g_1(x,y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x,y) \prod_{i \neq \eta} f_i(x) \rangle$ be a $(\lambda, \delta)$-constacyclic code over $\mathbb{F}$. Recall that $C$ is an ideal of $\mathbb{F}[x,y]$ where $\gcd(n, p) = 1$, $m = m \cdot p^s$ with $\gcd(m', p) = 1$ and $x^n = \lambda = \prod_{i=1}^\eta f_i(x)$ where $f_i(x), 1 \leq i \leq \eta$, are monic irreducible coprime polynomials in $\mathbb{F}[x]$ and $\deg f_i = d_i$. Suppose that in $K[y]$ we have $y^{m'} - \delta_0 = \prod_{j=1}^{t_j} h_{jl}(x,y)$, where $h_{jl}(x,y), 1 \leq l \leq t_j$, are monic irreducible coprime polynomials in $K[y]$. Then as $g_j(x,y)$ is a divisor of $(y^{m'} - \delta_0)p^s = y^m - \delta$ in $K[y]$, we have $g_j(x,y) = \prod_{l=1}^{t_j} h_{jl}^{\alpha_{jl}}(x,y)$ for some $\alpha_{jl}$, $0 \leq \alpha_{jl} \leq p^s - 1$. For $0 \leq t \leq p^s - 1$, let $g_{jt}(x,y)$ be the product of those irreducible factors $h_{jl}(x,y)$ such that $\alpha_{jl} > t$ in $g_j(x,y)$. So

$$\bar{C}_t = \langle \bar{g}_{1t}(x,y) \prod_{i \neq 1} f_i(x), \bar{g}_{2t}(x,y) \prod_{i \neq 2} f_i(x), \ldots, \bar{g}_{\eta t}(x,y) \prod_{i \neq \eta} f_i(x) \rangle$$

is an ideal of $\mathbb{S} = \mathbb{F}[x,y]_{(x^n, \lambda, y^{m'} - \delta_0)}$ and a simple root 2D constacyclic code. 

Note that, if $t' \geq t$, then $\bar{C}_{t'} \supseteq \bar{C}_t$.

**Example 5.1.** Suppose that $S = \mathbb{F}_2[x,y]_{(x^2 + y^2 - 1)}$ (see Example 4.9). Let $C = \langle (y^2 + y + 1)(x^2 + x + 1), (y + 1)(y + x)^3(x + 1) \rangle$ be a 2D constacyclic code of length 36 over $\mathbb{F}_2$. So

$$\bar{C}_0 = \langle (y^2 + y + 1)(x^2 + x + 1), (y + 1)(y + x)(x + 1) \rangle$$

$$\bar{C}_1 = \langle (x^2 + x + 1), (y + x)(x + 1) \rangle$$

$$\bar{C}_2 = \langle (x^2 + x + 1), (y + x)(x + 1) \rangle$$

$$\bar{C}_3 = \langle (x^2 + x + 1), (x + 1) \rangle$$

are ideals of $\mathbb{F}_2[x,y]_{(x^2 + y^2 - 1)}$, that is, simple-root 2D constacyclic codes of length 9 over $\mathbb{F}_2$. 

We start by the following lemma.

**Lemma 5.2.** Let $C = \langle g_1(x,y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x,y) \prod_{i \neq \eta} f_i(x) \rangle$ be a $(\lambda, \delta)$-constacyclic code over $\mathbb{F}$ and $\bar{v}_t(x,y)$ be a non-zero polynomial in $C_t$. Then the polynomial

$$\bar{v}_t(x,y) = (y^{m'} - \delta_0)^t \bar{v}_t^{p^s}(x,y) \mod (y^m - \delta, x^n - \lambda) \quad (5.1)$$
is a non-zero polynomial in $C$.

**Proof.** Since $\bar{v}(x,y)$ is a non-zero polynomial in $\bar{C}_t$, so

$$\bar{v}(x,y) = \sum_{j=1}^{\eta} \left( A_j(x,y)\bar{g}_{jt}(x,y) \prod_{i \neq j} f_i(x) \right),$$

where $A_j(x,y) \in \bar{S}$.

When $j \neq l$, we have

$$(\prod_{i \neq l} f_i(x)) (\prod_{i \neq j} f_i(x)) = 0 \mod (y^m - \delta, x^n - \lambda).$$

Hence

$$\bar{v}(x,y) = \sum_{j=1}^{\eta} B_j(x,y)\bar{g}_{jt}(x,y) \prod_{i \neq j} f_i(x) \mod (y^m - \delta, x^n - \lambda).$$

Thus

$$\bar{v}^p_t(x,y) \in \ll g_{t1}^p(x,y) (\prod_{i \neq 1} f_i(x))^p, \ldots, g_{tq}^p(x,y) (\prod_{i \neq q} f_i(x))^p \gg.$$

Now, completely similar to the first part of [6, Lemma 1], this lemma can be proved. \hfill \Box

In what follows, by $w(c)$ and $d_{\text{min}}(C)$ we mean the Hamming weight of a codeword $c$ and the minimum Hamming distance of a code $C$. Let $P_t = w((y^m - \delta)^t)$. By Lemma 5.2 and using the techniques used in the second part of the proof of [6, Lemma 1], we have the following lemma.

**Lemma 5.3.** Let $C$ be a $(\lambda, \delta)$-constacyclic code over $F$. Then $d_{\text{min}}(C) \leq P_t d_{\text{min}}(\bar{C}_t)$, for any $0 \leq t \leq p^s - 1$.

Consider the set

$$T = \{ t < p^s \mid P_t \leq P_{t'} \text{ for any } t' \in \{ t+1, \ldots, p^s - 1 \} \}. \tag{5.2}$$

Note that $p^s - 1 \in T$ and for any $0 \leq t \leq p^s - 1$ set

$$\bar{t} = \min \{ t' \in T \mid t' \geq t \}. \tag{5.3}$$

**Lemma 5.4.** Let $C$ be a $(\lambda, \delta)$-constacyclic code of length $nm$ over $F$. Assume that $c(x,y) \in C$ and $c(x,y) = (y^{m'} - \delta_0)^{t} v(x,y)$, where $y^{m'} - \delta_0 \nmid v(x,y)$ in $F[x,y]$. Then the polynomial

$$\bar{c}(x,y) = (y^{m'} - \delta_0)^{t} v^p(x,y) \mod (y^m - \delta, x^n - \lambda),$$

where $\bar{v}(x,y) = v(x,y) \mod (y^m - \delta, x^n - \lambda)$, is a non-zero polynomial of $C$ and satisfies $w(\bar{c}(x,y)) \leq w(c(x,y))$. 

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Proof. The proof is similar to the proof of [6, Lemma 2], but with some minor modifications. To show how these modifications can be accounted (in this and also other omitted proofs), we state the complete proof here. Since \( c(x, y) \in C, \) \( c(x, y) = \sum_{j=1}^{\eta} A_j(x, y)g_j(x, y) \prod_{i \neq j} f_i(x) \), for some \( A_j(x, y) \in \mathcal{S} \). So in \( \mathcal{S} \),

\[
\sum_{j=1}^{\eta} A_j(x, y)g_j(x, y) \prod_{i \neq j} f_i(x) = (y^{m'} - \delta_0)^t v(x, y).
\]

Thus in \( \mathbb{F}[x, y] \),

\[
\sum_{j=1}^{\eta} A_j(x, y)g_j(x, y) \prod_{i \neq j} f_i(x) = (y^{m'} - \delta_0)^t v(x, y) + (y^m - \delta) D + (x^n - \lambda) E,
\]

for some \( D, E \in \mathbb{F}[x, y] \). Now, in \( K_j[y] \), we have

\[
A_j(x, y)g_j(x, y) \prod_{i \neq j} f_i(x) = (y^{m'} - \delta_0)^t v(x, y) + (y^m - \delta) D.
\]

If for some \( 0 \leq l \leq t_j \), \( \alpha_{jl} > t \), then \( h_{jl}^{t+\alpha_{jl}-t}(x, y) \mid g_j(x, y) \) and \( h_{jl}^{t+\alpha_{jl}-t}(x, y) \mid y^m - \delta \). Thus in \( K_j[y] \),

\[
h_{jl}^{t+\alpha_{jl}-t}(x, y) \mid (y^{m'} - \delta_0)^t v(x, y).
\]

So \( h_{jl}(x, y) \mid v(x, y) \). Hence \( \bar{g}_{jl}(x, y) \mid v(x, y) \) in \( K_j[y] \) for each \( j \). Thus \( \bar{v}(x, y) \in \bar{C}_t \). Since \( y^{m'} - \delta_0 \not\mid v(x, y) \) and \( \deg_x v(x, y) < n \), \( \bar{v}(x, y) \neq 0 \) in \( \bar{S} \). Since \( t \geq t \), \( \bar{C}_t \subseteq \bar{C}_t \). Hence by Lemma 5.2

\[
0 \neq \bar{c}_t(x, y) = (y^{m'} - \delta_0)^t \bar{v}^{\rho t}(x, y) \in C. \tag{5.4}
\]

There exist polynomials \( v_i \in \mathbb{F}[x, y] \) and \( v_{ij} \in \mathbb{F}[y] \) such that

\[
v(x, y) = \sum_{i=0}^{m'-1} y^i v_i(x, y^{m'}) \quad \text{and} \quad v_i(x, y^{m'}) = \sum_{j=0}^{n-1} x^j v_{ij}(y^{m'}).\]

Hence

\[
v(x, y) = \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} x^j y^i v_{ij}(y^{m'}),
\]

and

\[
\bar{v}(x, y) = \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} x^j y^i v_{ij}(\delta_0) \mod (y^{m'} - \delta_0, x^n - \lambda).
\]
Let \( N_\nu = w(\bar{v}(x,y)) \), which according to the above formula equals the number of \( v_{ij} \)'s with \( v_{ij}(\delta_0) \neq 0 \). Now we have

\[
w(c) = w((y^{m'} - \delta_0)^t v(x,y)) = w((y^{m'} - \delta_0)^t \sum_{i=0}^{m'-1} y^i v_i(x, y^{m'}))
\]

\[
= w\left( \sum_{i=0}^{m'-1} y^i (y^{m'} - \delta_0)^t v_i(x, y^{m'}) \right) = \sum_{i=0}^{m'-1} w((y^{m'} - \delta_0)^t v_i(x, y^{m'}))
\]

\[
= \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} w(x^j (y^{m'} - \delta_0)^t v_{ij}(y^{m'}))
\]

\[
= \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} \sum_{j=0}^{n-1} w((y^{m'} - \delta_0)^t v_{ij}(y^{m'})) = \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} w((y - \delta_0)^t v_{ij}(y)).
\]

If \( v_{ij}(y) \neq 0 \), since \((y - \delta_0)^t v_{ij}(y)\) can be written as \( \sum_{i=1}^{\nu} b_i (y - \delta_0)^t \) with \( b_i \in \mathbb{F} \) and by \([17, \text{Theorem 6.1}]\), we have

\[
w((y - \delta_0)^t v_{ij}(y)) \geq \sum_{i=0}^{m'-1} \sum_{j=0}^{n-1} \min_{\nu' > t} w((y - \delta_0)^{t'}) = P_t.
\]

Thus \( w(c(x,y)) \geq P_t N_\nu \), where \( N_\nu \) is the number of nonzero \( v_{ij}(y) \). Also \( w(\bar{c}_i(x,y)) \leq P_t w(\bar{v}^{\delta'}(x,y)) = P_t w(\bar{v}(x,y)) \) by \([5.4]\) and so \( w(\bar{c}_i(x,y)) \leq \bar{t} N_\nu \). If \( v_{ij}(\delta_0) \neq 0 \), then \( v_{ij}(y) \neq 0 \). Hence \( N_\nu \geq N_\nu \). Therefore,

\[
w(\bar{c}_i(x,y)) \leq P_t N_\nu \leq P_t N_\nu \leq w(c(x,y)).
\]

Hence \( w(\bar{c}_i(x,y)) \leq w(c(x,y)) \). \( \square \)

Using Lemmas \([5.2][5.4]\) instead of Lemmas 1 and 2 of \([6]\), one can prove the following result similar to \([6, \text{Theorem 1}]\).

**Proposition 5.5.** Let \( C \) be a \((\lambda, \delta)\)-constacyclic code of length \( nm \) over \( \mathbb{F} \). Then \( d_{\text{min}}(C) = P_t d_{\text{min}}(C_{\bar{t}}) \), for some \( \bar{t} \in T \).

Also the next theorem can be proved mutatis mutandis to \([6, \text{Theorem 2}]\).
Theorem 5.6. Let $C$ be a one-sided repeated-root $(\lambda, \delta)$-constacyclic code of length $nm$ over $\mathbb{F}$. Then there exists a simple-root $(\lambda, \delta)$-constacyclic code $\hat{C}$ over $\mathbb{F}$ of length $nm'$ with both rate and relative minimum distance at least as large as the corresponding values for $C$.

Note that, since the code $\hat{C}$ is of length $nm'$ and $C$ is of length $nm' p^s$, Theorem 5.6 does not mean that simple root 2D constacyclic codes are better than one-sided repeated-root 2D constacyclic codes.

Let $C = \langle g_1(x, y) \prod_{i \neq 1} f_i(x), \ldots, g_\eta(x, y) \prod_{i \neq \eta} f_i(x) \rangle$ be a $(\lambda, \delta)$-constacyclic code of length $nm$ over $\mathbb{F}$. Then $\dim(C) = mn - \sum_{j=0}^{\eta} d_j t_j$, where $d_j = \deg f_j$ and $t_j = \deg y g_j$. In what follows, we assume that $C$ and $\langle g_j(x, y) \rangle$ (as an ideal of $K_j[y]/\langle y^{m-\delta} \rangle$), are of the rate $r$ and $r_j$ respectively.

So we have the following lemma.

Lemma 5.7. For any $0 < R < 1$, if $R < r$, then there exists an $l$ ($1 \leq l \leq \eta$) such that $R < r_l$.

Proof. Note that,

$$r = \frac{mn - \sum_{j=0}^{\eta} d_j t_j}{mn} = \frac{\sum_{j=0}^{\eta} d_j (m - t_j)}{mn} = \frac{1}{n} \sum_{j=0}^{\eta} \frac{m - t_j}{m} d_j = \frac{1}{n} \sum_{j=0}^{\eta} r_j d_j.$$ 

Suppose that for any $j$, $R \geq r_j$. So $\sum_{j=0}^{\eta} d_j R \geq \sum_{j=0}^{\eta} r_j d_j$. Hence $nR \geq \sum_{j=0}^{\eta} r_j d_j$. Thus $R \geq r$ which is impossible. \[\square\]

Now, one can prove the counterparts of Lemma 3 and Theorems 3 and 4 of [6], with completely similar arguments. Here we just mention the statements.

Lemma 5.8. Let $C$ be a $(\lambda, \delta)$-constacyclic code of length $nm$ over $\mathbb{F}$ and rate $r$. For any $0 < R < 1$, there exists a constant $\gamma(R)$ such that if $r > R$, then $\lim_{i \to \infty} \frac{d_{\min}(C_i)}{m_i n_i} = 0$.

Proposition 5.9. Any sequence of $(\lambda, \delta)$-constacyclic codes $C_i$ over $\mathbb{F}$ of length $n_i m_i$, where $m_i = m_i' p^{s_i}$ and $\gcd(n_i, p) = \gcd(m_i', p) = 1$, with rates $r_i > R > 0$ such that $\liminf_{i \to \infty} s_i = \infty$, satisfies

$$\lim_{i \to \infty} \frac{d_{\min}(C_i)}{m_i n_i} = 0.$$ 

Theorem 5.10. If there exist a sequence of $(\lambda, \delta)$-constacyclic codes $C_i$ over $\mathbb{F}$ of length $n_i m_i$, where $m_i = m_i' p^{s_i}$ and $\gcd(n_i, p) = \gcd(m_i', p) = 1$, with rates $r_i > R > 0$ such that
\[ \lim_{i \to \infty} m_i n_i = \infty, \]

and

\[ \lim_{i \to \infty} \frac{d_{\text{min}}(\mathcal{C}_i)}{m_i n_i} = \Delta > 0, \]

then there exists a sequence of \((\lambda, \delta)\)-constacyclic codes \(\hat{\mathcal{C}}_i\) over \(\mathbb{F}\) of length \(\hat{n}_i \hat{m}_i\) and \(\gcd(\hat{n}_i, p) = \gcd(\hat{m}_i, p) = 1\), and of the rate \(\hat{r}_i > R > 0\) such that

\[ \lim_{i \to \infty} \hat{n}_i \hat{m}_i = \infty \]

and

\[ \lim_{i \to \infty} \frac{d_{\text{min}}(\hat{\mathcal{C}}_i)}{\hat{n}_i \hat{m}_i} \geq \Delta > 0. \]

This theorem shows that if there exists an asymptotically good family of one-sided repeated-root 2D constacyclic codes, then there exists an asymptotically good family of simple-root 2D constacyclic codes with similar or better parameters.

6 Conclusion

In this paper, we studied the algebraic structure of one-sided repeated 2D cyclic and constacyclic codes, found their generator and parity check matrices and also their duals. Moreover, we showed that if there exists an asymptotically good family of one-sided repeated-root 2D cyclic or constacyclic codes, then there exists an asymptotically good family of simple-root 2D constacyclic codes with similar or better parameters. We also mentioned some needed corrections to some known previous results on such codes.

Although our work suggests that one should not look for asymptotically good families of codes in this class of codes, but we may have some optimal codes among one-sided repeated-root cyclic or constacyclic codes. We leave their study to a future work. Another problem which remains to be studied in future works is to find self-dual codes when the assumption made here (that is, \(f_i^2 = f_i\)) does not hold.

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