GRAPHS WITH PRESCRIBED LOCAL NEIGHBORHOODS OF THEIR UNIVERSAL COVERINGS.

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Abstract. Given a collection of $n$ rooted trees with depth $h$, we give a necessary and sufficient condition for this collection to be the collection of $h$-depth universal covering neighborhoods at each vertex.

1. Reconstruction of a graph with its universal covering.

Let $G = (V, E)$ be a finite, connected graph. A graph $G' = (V', E')$ is a covering of $G$ if there is a surjective map $\iota : V' \to V$ which is a local isomorphism: for every $x \in V'$, $\iota$ induces a bijection between the edges incident to $x$ and the edges incident to $\iota(x)$. The universal covering of $G$, denoted by $T_G$, is the unique (up to isomorphism) covering which is a tree. Note that $T_G$ can be infinite if $G$ is not itself a tree. For instance, the universal covering of any $d$-regular graph is the infinite $d$-regular tree $T_d$. Note that $T_G$ is also the universal covering of any covering of $G$.

Let $h$ be a positive integer. Given any vertex $x$ of $G$, its $h$-depth universal covering neighborhood is the unlabeled ball of radius $h$ in $T_G$ around any antecedent of $x$ by $\iota$. One can easily see that this ball does not depend — up to isomorphism — on the chosen antecedent $x$.

Question 1.1. Let $t = (t_1, ..., t_n)$ be a collection of $n$ unlabeled rooted trees with maximal depth $h$. Is it the collection of $h$-depth universal covering neighborhoods of some (simple) graph $G$?

If this is the case, we call the $n$-tuple $t$ a graphical $h$-neighborhood and we say that $G$ is a realization of $t$.

Date: October 2, 2018.
1.1. **Notation.** From now on, we will adopt the term “tree” instead of “unlabeled, rooted tree”, unless explicitly stated otherwise. If $G$ is any graph and $x$ is a vertex of $G$, we will note $\deg_G(x)$ the number of neighbors of $x$ in $G$. In a directed graph $G$, we will note $\deg^+_G(x)$ and $\deg^-_G(x)$ the in and out degree of the vertices. Generally, the root of a (rooted) graph will be noted $\bullet$ and if $k$ is an integer and $g$ a rooted graph, $(g)_k$ denotes the ball $B_g(\bullet, k)$ of radius $k$ around the root of $g$. The set of all (unlabeled, rooted) trees with depth $h$ will be noted $\mathcal{T}_h$.

1.2. **Related work.** Graph reconstruction problems ask the following question: given any property $\mathcal{P}$ about graphs, how can we ensure that there is a graph (or digraph, or multigraph) having this property $\mathcal{P}$? What is the number of graphs that have this property? Can we determine the properties $\mathcal{P}$ that have a single graph realization?

Reconstructing a graph (or bipartite graph, or digraph) only by the list of its degree has been a well-known and studied problem since the seminal works of Erdős, Gallai and many others. In fact, question 1.1 had been settled long time ago for $h = 1$ by the celebrated Erdős-Gallai theorem. Suppose that $t = (t_1,...,t_n)$ is an $n$-tuple of 1-depth trees. A 1-depth tree $t_i$ is just a root with some leaves, say $d_i$ leaves; thus, a 1-depth neighborhood can be identified with a $n$-tuple of integers $(d_1,...,d_n)$.

![Figure 2](image-url) Is there a graph on 8 vertices with this 1-depth neighborhood? The associated degree sequence if $d = (3,1,2,3,5,2,3,1)$. According to Theorem 1, yes.

Finding a graph $G$ with $t$ as 1-depth neighborhood boils down to finding a graph $G$ with degree sequence $d$ — such sequences are called *graphical*. All integer sequences are not graphical; the Erdős-Gallai theorem gives one necessary and sufficient condition for an integer sequence to be graphical.

**Theorem 1** (Erdős, Gallai, [EG60]). Let $d = (d_1,...,d_n)$ be a $n$-tuple of integers. Rearrange them in decreasing order $d(1) \geq ... \geq d(n)$. Then, $d$ is graphical if and only if it satisfies the two following conditions:

\[
d_1 + \cdots + d_n \text{ is even,} \tag{1.1}
\]

and the “Erdős-Gallai condition”

\[
\forall k \in [n], \quad d_1 + \cdots + d_k \leq k(k-1) + \sum_{i=k+1}^n d_i \wedge k. \tag{1.2}
\]

A short and constructive proof is available at [TV03]. In fact, the Erdős-Gallai condition is not the only sufficient and necessary condition for an integer sequence to be graphical; there are some other (equivalent) conditions, notably listed in [SH91]. The corresponding realization problem for *digraphs* had also been solved quite early; see the interesting note [Ber14] for a complete history and presentation of the many variants.

**Theorem 2.** Let $d^\pm = (d^+_1,d^-_i)_{i \in [n]}$ be a $2n$-tuple of integers. We order the first component by decreasing order $d^+_1 \geq ... \geq d^+_n$. Then, $d^\pm$ is the sequence of oriented in and out degrees of some digraph $G$ if and only if it satisfies the two following conditions:

\[
\sum_{i=1}^n d^+_i = \sum_{i=1}^n d^-_i, \tag{1.3}
\]
and the “directed Erdős-Gallai condition”:

\[ \forall k \in [n], \quad \sum_{i=1}^{k} d^+_i \leq \sum_{i=1}^{k} d^-_i \wedge (k - 1) + \sum_{i=k+1}^{n} k \wedge d^-_i \tag{1.4} \]

where the couples \((d^+_i, d^-_i)\) are sorted in lexicographic order.

This settled our question for \(h = 1\). The case \(h = 2\) had recently been solved by [BGE’15, BD17]; a 2-depth neighborhood is called a neighborhood degree list (NDL). In [BD17], the authors not only settle Question 1.1 and give a sufficient and necessary condition for a NDL to be graphical, but they also characterize those NDL that are “unigraphical”, meaning that they have a unique graphical realization — we do not adress this problem, but we solve Question 1.1 for arbitrary depths \(h\).

For \(h = 1\), the number of labeled graphs with a given degree sequence is asymptotically known in many asymptotic regimes, see notably [Jan14], [Bol01, Theorem 2.16] and references therein. For general \(h\), this question has been recently addressed in [BC15] in the regime where the maximal degree is uniformly bounded. The motivation came from the Benjamini-Schramm topology of rooted graphs.

In this paper, we only deal with universal covering neighborhoods, thus ignoring the eventual cycles in the \(h\)-neighborhood of a vertex. If a \(h\)-depth neighborhood is graphical, then it might as well have very different realizations, for instance ones that are \(h\)-locally tree-like, or others having many short cycles. When the same question is addressed with graph \(h\)-neighborhoods, Question 1.1 becomes much more arduous; a similar problem in graph reconstruction, the famous Kelly-Ulam reconstruction problem, was asked during the 1940s and still remains opened.

1.3. Definitions and statement of the main result. Fix some \(h\)-depth neighborhood \(t = (t_1, ..., t_n)\). The associated degree sequence \(d = (d_1, ..., d_n)\) is the sequence of degrees of the root \(\bullet\) of the tree \(t_i\), that is \(d_i = \deg_{t_i}(\bullet)\). An obvious necessary condition for \(t\) to be graphical is that \(d\) is itself a graphical sequence, hence satisfying (1.1)-(1.2). From now on, we will assume that \(d_1 + ... + d_n = 2m\) where \(m\) is an integer.

Let \(t\) be a tree with depth at most \(h\) and root \(\bullet\). Let \(e\) be an edge incident with the root, say \(e = (\bullet, x)\). The tree \(t \setminus e\) has exactly two connected components. The connected component containing the root is \(r'\) and the other one is \(s\); we root \(s\) at \(x\). We erase from \(r'\) all the vertices which were at depth exactly \(h\) in \(t\), and we keep the same root; this yields a new rooted tree \(r\) — see Figure 3. The type of the edge \(e\) is defined as the couple of rooted trees \((r, s)\) and we will denote it by \(\tau(e)\).

![Figure 3. Construction of the type \(\tau(e) = (r, s)\) of edge \(e\) in some tree \(t\).](image-url)
If $\tau = (r, s)$ is a type, its opposite type $\tau^{-1}$ is defined as $(s, r)$. A type is an element of $\mathcal{T}_{h-1} \times \mathcal{T}_{h-1}$. The set of all types induced by the edges in $t$ is noted $\text{types}(t)$. It can be decomposed into the disjoint union of three sets

$$\text{types}(t) = \Delta \cup A \cup B$$

where

- $\Delta$ is the set of “diagonal” types $\tau = (r, r)$ for some $r \in \mathcal{T}_{h-1}$;
- $A \cup B$ is the set of types $\tau = (r, s)$ with $r \neq s$, and the sets $A, B$ are chosen such that if $\tau \in A$, then $\tau^{-1} \in B$.

If $\tau \in \text{types}(t)$, we define

- the $\tau$-degree of any index $i \in [n]$ as the number of edges in $t_i$ incident to the root and whose type is $\tau$. We will denote it by $d^\tau_i$;
- the $\tau$-number $N_\tau$ as the total number of edges in $t$ with type $\tau$, that is

$$N_\tau = \sum_{i \in [n]} d^\tau_i.$$ 

It should be clear that if $i \in [n]$ is a vertex, then $\sum_{\tau \in \text{types}(t)} d^\tau_i = d_i$.

**Theorem 3.** Let $t$ be a $h$-depth neighborhood; it is graphical if and only if it satisfies the following conditions:

- for every $\tau \in \Delta$, the integer sequence $(d^\tau_i)_{i \in [n]}$ is graphical;
- for every $\tau \in A$, the integer double sequence $(d^\tau_i, d^{\tau^{-1}}_i)_{i \in [n]}$ is digraphical.

Using classical characterizations of graphical and digraphical sequences given earlier in Theorems 1 and 2, this result can be detailed:

**Theorem 4.** Let $t$ be a $h$-depth neighborhood; it is graphical if and only if it satisfies the following conditions:

- for every $\tau \in \Delta$, the integer $N_\tau$ is even and for every $k \in [n]$ we have

$$\sum_{i=1}^{k} d^\tau_i \leq k(k-1) + \sum_{i=k+1}^{n} d_i \wedge k,$$

(1.5)

- for every $\tau \in A$, we have $N_\tau = N_{\tau^{-1}}$ and for every $k \in [n]$, we have

$$\sum_{i=1}^{k} d^\tau_i \leq \sum_{i=1}^{k} d^{\tau^{-1}}_i + \sum_{i=k+1}^{n} d^{\tau^{-1}}_i \wedge k,$$

(1.6)

where indices correspond to lexicographic reordering.

Note that those conditions together imply that $(d_1, ..., d_n)$ is itself a graphical sequence (sum over all the types $\tau$), which is a necessary, but clearly non sufficient condition.

### 2. Proof of Theorem 3.

We assume without loss of generality $h \geq 2$. The conditions are easily seen to be necessary, for if $t$ is graphical and $\tau$ is a type, then

- either $\tau \in \Delta$ and the graph induced in $G$ by keeping only the edges $e$ such that $\tau(e) = \tau$ has $(d^\tau_i)_{i \in [n]}$ has its degree sequence,
• either \( \tau \notin \Delta \); in this case either \( \tau \in A \) or \( \tau^{-1} \in A \), so without loss of generality we can assume that \( \tau \in A \). The graph induced by edges such that \( \tau(e) = \tau \) can be oriented: if \( e = (i, j) \in G_\tau \), then one vertex \( k \in \{i, j\} \) satisfies \( \tau(e) = \tau \) in \( t_k \). We orient the edge \((i, j)\) from \( k \) to the other vertex. This yields a digraph \( \vec{G}_\tau \) with oriented bi-degree sequence \((d^+_\tau(i), d^-_\tau(i))_{i \in [n]}\), so the second condition of Theorem 3 is met.

We now prove the sufficiency. We suppose that \( t \) is a \( h \)-depth neighborhood satisfying the assumptions in Theorem 3 and we build a graph \( G \) which is a realization of \( t \). We first fix some type \( \tau \).

• We suppose in the first time that \( \tau \in A \), in particular \( \tau = (r, s) \) with \( r \neq s \). As \((d^+_\tau(i), d^-_\tau(i))_{i \in [n]}\) is digraphical, there is some digraph \( \vec{G}_\tau \) on \( n \) vertices such that \( \deg_{\vec{G}_\tau}(i) = d^+_\tau(i) \) and \( \deg_{\vec{G}_\tau}(i) = d^-_\tau(i) \) for every vertex \( i \in [n] \). We now define a (non-directed) multigraph \( G_\tau \) by simply forgetting the directions of edges in \( \vec{G}_\tau \)—indeed, this multigraph will be proven to be simple in Lemma 2.1.

• Else, if \( \tau \in \Delta \), then by assumption \((d^+_\tau(i))_{i \in [n]}\) is graphical and there is a simple graph \( G_\tau \) such that \( \deg_{G_\tau}(i) = d^+_\tau(i) \).

We now “glue together” the graphs \( G_\tau \) to get our realization of \( t \), namely \( G \). Formally, if \( E(G_\tau) \) denotes the set of edges in \( G_\tau \), then \( G = ([n], E) \) with the edge set \( E \) being defined as

\[
E := \bigcup_{\tau \in \Delta \cup A} E(G_\tau). \tag{2.1}
\]

The following lemma is the crucial ingredient of the proof of Theorem 3.

**Lemma 2.1.** \( G \) is a simple graph.

**Proof.** Suppose that \( G_\tau \) contains a double edge, for instance \((x, y)\). We are going to prove the two following facts:

1. first, this double edge can not arise from two distinct \( G_\tau \). In other words, if \((x, y) \in G_\tau \), then \((x, y) \notin G_{\tau'} \) for every \( \tau' \neq \tau \);
2. then we check that for every \( \tau \in A \), the multigraph \( G_\tau \) contains no double edge.

Together, those two facts imply that \( G \) is simple: indeed, if there is a double edge, then it can only belong to a single \( G_\tau \); but if \( \tau \in A \), \( G_\tau \) cannot contain any double edge, and if \( \tau \in \Delta \) then \( G_\tau \) is simple by construction, hence the conclusion.

Suppose that there is a double edge between vertices \( i \) and \( j \), one belonging to \( G(\tau) \) and the other to \( G(\tau') \) for two types \( \tau = (r, s) \) and \( \tau' = (r', s') \). We prove that \( \tau = \tau' \). As manipulating unlabeled rooted trees is quite inconvenient, we will work with two labeled rooted trees \( T_i, T_j \) in the equivalence classes of \( t_i, t_j \), and the same with \( R, R', S, S' \) which are representatives of the equivalence classes of \( r, r', s, s' \). We are going to prove that \( R \simeq R' \) and \( S \simeq S' \) (as rooted labeled trees), hence proving \( r = r' \) and \( s = s' \) as needed. The following arguments are illustrated in Figure 4.

• The presence of an edge between \( i \) and \( j \) in \( G_\tau \) has the following consequence: there is an edge \( e \) in \( T_i \), adjacent with the root, such that \( T_i \setminus e \) has two connected components, one isomorphic with \( S \) and the other having its ball of radius \( h \) isomorphic with \( R \). On the other hand, as \((i, j) \in G(\tau')\), there is an edge \( e' \) such that \( T_i \setminus e' \) has one component isomorphic with \( S' \) and the ball of radius \( h - 1 \) of the other is isomorphic with \( R' \).

• The same holds with \( T_j \).
It is clear that \( \deg_{R}(\bullet) + 1 = \deg_{T}(\bullet) = \deg_{R}(\bullet) \). The same is true with \( S, S' \); we have just proven that \((R)_{1} \simeq (R')_{1} \) and \((S)_{1} \simeq (S')_{1} \). We are now going to prove that if \((S)_{k} \simeq (S')_{k} \) and \((R)_{k} \simeq (S')_{k} \) for some \( k < h - 1 \), then this is also true with \( k + 1 \).

First, the ball \((T_{i})_{k+1} \) can be decomposed in two ways shown in Figure 4:

\[
(T_{i})_{k+1} = e \cup (S)_{k} \cup (R)_{k+1} \quad \text{and} \quad (T_{i})_{k+1} = e' \cup (S')_{k} \cup (R')_{k+1}
\]

but as \((S)_{k} \simeq (S')_{k} \), we can erase both branches pending at \( e \) and \( e' \), to get \((R)_{k+1} \simeq (R')_{k+1} \). The same idea applies to \( T_{j} \), to show that \((S')_{k+1} \simeq (S')_{k+1} \), hence closing the recurrence. We have proven that \((S)_{h-1} \simeq (S')_{h-1} \) and \((R)_{h-1} \simeq (R')_{h-1} \), thus \( r = r' \) and \( s = s' \) as needed. We thus have proven the first point exposed earlier.

![Figure 4. An illustration of the proof of Lemma 2.1. The green parts represent \((S)_{k} \) and \((S')_{k} \) which are isomorphic (as recurrence hypothesis) and the dark red parts are representing \((R)_{k} \) and \((R')_{k} \), which are isomorphic too; hence, the light pink parts are also isomorphic, thus proving \((R)_{k+1} \simeq (R')_{k+1} \). A similar procedure applies to \( T_{j} \).](attachment:figure4.png)

We now check the second point, i.e. that for every \( \tau \in A \), the multi-graph \( G_{\tau} \) is indeed a simple graph. The proof runs along the same lines: suppose that there is a double directed edge between \( i \) and \( j \) in \( G_{\tau} \). This can only happen if \((i, j) \) and \((j, i) \) are both directed edges in \( G_{\tau} \). We suppose that \( \tau = (r, s) \), and with a recurrence we prove that \( r = s \), hence \( \tau \in \Delta \) which had been discarded since \( \Delta \cap A = \emptyset \).

To do this, first check that \( \deg_{\tau}(\bullet) = \deg_{s}(\bullet) \), then suppose that for some \( k < h \), we have \((r)_{k} = (s)_{k} \) and prove that \((r)_{k+1} = (s)_{k+1} \). This step uses the exact same procedure as before. \( \square \)

We now check that \( G \) solves our problem.

**Lemma 2.2.** \( G \) is a realization of \( t \).

*Proof.* We want to show that the \( h \)-neighborhood of any vertex \( i \) in the universal cover of \( G \) matches \( t_{i} \). We show by strong recurrence that for \( k \leq h \), if \( \bar{t}_{i} \) denotes the \( h \)-neighborhood of \( i \) in the universal cover, then for every \( i \in [n] \) we have \((t_{i})_{k} = (\bar{t}_{i})_{k} \). It is clear by our construction of \( G \) that \( \deg_{G}(i) = \sum d_{i}^{e} = d_{i} \), hence \((t_{i})_{1} = (\bar{t}_{i})_{1} \). Now suppose that \((t_{i})_{k} = (\bar{t}_{i})_{k} \) for some \( k < h \). If \( N_{G}(i) \) is the set of neighbors of \( i \) in \( G \), then for every \( j \in N_{G}(i) \) we have \((t_{j})_{k} = (\bar{t}_{j})_{k} \) by the recurrence hypothesis. This readily implies that \((t_{i})_{k+1} = (\bar{t}_{i})_{k+1} \), hence the lemma is proven. \( \square \)
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