HOEFFDING’S INEQUALITY FOR NON-IRREDUCIBLE MARKOV MODELS

NIKOLA SANDRIĆ AND STJEPAN ŠEBEK

ABSTRACT. In this article, we establish Hoeffding’s inequality for bounded Lipschitz functions of a class of not necessarily irreducible Markov models. The result complements the existing literature on this topic where Hoeffding’s inequality for bounded measurable functions of a class of irreducible Markov models has been considered. Our approach is based on the assumption of uniform ergodicity of the underlying Markov model in $L^1$-Wasserstein space.

1. INTRODUCTION

Let $\{\xi_t\}_{t \in \mathbb{Z}_+}$ be a sequence of independent and bounded random variables (defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$), and let $S_t := \xi_0 + \cdots + \xi_t$ for $t \in \mathbb{Z}_+$. Assume that $\mathbb{P}(a_i \leq \xi_t \leq b_i)$ = 1 for some $\{a_i\}_{i \in \mathbb{Z}_+}, \{b_i\}_{i \in \mathbb{Z}_+} \subseteq \mathbb{R}$. The classical Chebyshev inequality then implies that for every $\varepsilon > 0$,

$$
\mathbb{P}\left( |S_{t-1} - E[S_{t-1}]| > \varepsilon t \right) \leq \frac{\sum_{i=0}^{t-1} (b_i - a_i)^2}{\varepsilon^2 t^2}.
$$

In his seminal work W. Hoeffding [Hoe63] has improved this result and showed that

$$
\mathbb{P}\left( |S_{t-1} - E[S_{t-1}]| > \varepsilon t \right) \leq 2 \exp \left\{ -2\varepsilon^2 t^2 / \sum_{i=0}^{t-1} (b_i - a_i)^2 \right\}.
$$

Hoeffding’s inequality has been widely applied in many problems arising in probability and statistics. However, the independence assumption limits its applicability in many situations. This, for instance, includes problems characterized by Markovian dependence, such as Markov chain Monte Carlo methods, time series analysis and reinforcement learning problems, see e.g. [FJS21], [OG02] and [Tan07]. Motivated by this, Hoeffding’s inequality has been extended to bounded measurable functions of a class of Markov models. However, to the best of our knowledge, in all the works on this topic (see the literature review part below) a common assumption is that the underlying Markov model is irreducible. In this article, we complement these results and discuss Hoeffding’s inequality for bounded Lipschitz functions of a class of not necessarily irreducible Markov models.

2. MAIN RESULT

Let $S$ be a Polish space, i.e. separable completely metrizable topological space. Denote the corresponding metric by $d$. We endow $(S, d)$ with its Borel $\sigma$-algebra $\mathcal{B}(S)$. Further, let $T = \mathbb{R}_+$ or $\mathbb{Z}_+$ be the time parameter set, and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \{\theta_t\}_{t \in T}, \{X_t\}_{t \in T}, \{\mathbb{P}_x\}_{x \in S})$, denoted by $\{X_t\}_{t \in T}$ in the sequel, be a time-homogeneous conservative strong Markov model with state space $(S, \mathcal{B}(S))$, in the sense of [BG68]. Recall that in the case when $T = \mathbb{Z}_+$, $\{X_t\}_{t \in T}$ is usually called a Markov chain, and in the case when $T = \mathbb{R}_+$, $\{X_t\}_{t \in T}$ is called a Markov process. In the latter case we also assume that $\{X_t\}_{t \in T}$ is progressively measurable (with respect to $\{\mathcal{F}_t\}_{t \in T}$), i.e. the map $(s, \omega) \mapsto X_s(\omega)$ from $[0, \infty) \times \Omega$ to $S$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}_s / \mathcal{B}(S)$ measurable for all $t \geq 0$. This will be in particular satisfied if $t \mapsto X_t(\omega)$ is right continuous for all $\omega \in \Omega$ (see [BG68, Exercise I.6.13]). Further, denote by $P^x(x, dy) := \mathbb{P}_x(X_t \in dy)$ the transition function of $\{X_t\}_{t \in T}$, and

2010 Mathematics Subject Classification. 60J05, 60J25.

Key words and phrases. Hoeffding’s inequality, Markov model, Wasserstein distance.
let $\mathcal{P}(S)$ be the class of all probability measures on $\mathcal{B}(S)$ having finite first moment. The $L^1$-Wasserstein distance on $\mathcal{P}(S)$ is defined by

$$\mathcal{W}(\mu_1, \mu_2) := \inf_{\Pi \in C(\mu_1, \mu_2)} \int_{S \times S} d(x, y) \Pi(dx, dy),$$

where $C(\mu_1, \mu_2)$ is the family of couplings of $\mu_1(dx)$ and $\mu_2(dy)$, i.e. $\Pi \in C(\mu_1, \mu_2)$ if, and only if, $\Pi(dx, dy)$ is a probability measure on $\mathcal{B}(S) \times \mathcal{B}(S)$ having $\mu_1(dx)$ and $\mu_2(dy)$ as its marginals. By Kantorovich-Rubinstein theorem it holds that

$$\mathcal{W}(\mu_1, \mu_2) = \sup_{\{f : \text{Lip}(f) \leq 1\}} |\mu_1(f) - \mu_2(f)|,$$

where the supremum is taken over all Lipschitz continuous functions $f : S \to \mathbb{R}$ with Lipschitz constant $\text{Lip}(f) \leq 1$ and, for a probability measure $\mu$ on $\mathcal{B}(S)$ and a measurable function $f : S \to \mathbb{R}$, the symbol $\mu(f)$ stands for $\int_S f(x) \mu(dx)$, whenever the integral is well defined.

We now state the main result of this article.

**Theorem 1.** Let $f : S \to \mathbb{R}$ be bounded and Lipschitz continuous, and let $S_{t-1} := \int_{(0,t)} f(X_s) \tau(dt)$ for $t \in \mathbb{T}$. Here, $\tau(dt)$ stands for the counting measure when $\mathbb{T} = \mathbb{Z}_+$ and the Lebesgue measure when $\mathbb{T} = \mathbb{R}_+$. Assume that $\{X_t\}_{t \in \mathbb{T}}$ admits an invariant probability measure $\pi(dx)$ (i.e. a measure satisfying $\int_S P_t(x, dy)\pi(dx) = \pi(dy)$ for all $t \in \mathbb{T}$) such that

1. $\gamma := \sup_{x \in S} \int_{\mathbb{T}} \mathcal{W}(P^t(x, \cdot), \pi(\cdot))\tau(dt) < \infty.$

Then for any $\varepsilon > 0$,

2. $P_x(\{|S_{t-1} - \pi(f)| > \varepsilon t\}) \leq \begin{cases} 2 \exp \left\{ -\frac{(\varepsilon t - 2 \text{Lip}(f)\varepsilon^2)}{8(\text{Lip}(f)\varepsilon + \|f\|_\infty)^2} \right\}, & \mathbb{T} = \mathbb{Z}_+, \\
2 \exp \left\{ -\frac{(\varepsilon t - 2 \text{Lip}(f)\varepsilon^2)}{8(\text{Lip}(f)\varepsilon + \|f\|_\infty)(\varepsilon t + 1)} \right\}, & \mathbb{T} = \mathbb{R}_+. \end{cases}$

According to [But14, Theorems 2.1 and 2.4] the relation in eq. (1) will hold if

(i) the metric $d$ is bounded (without loss of generality by 1)

(ii) there is $\rho \in (0, 1)$ such that for all $x, y \in S$ and all $t$ large enough,

$$\mathcal{W}(P^t(x, \cdot), P^t(y, \cdot)) \leq (1 - \rho)d(x, y).$$

(iii) there are $\kappa \in \mathbb{R}$, measurable and bounded $V : S \to \mathbb{R}_+$ and concave, differentiable and increasing to infinity function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(0) = 0$, such that

3. $\mathbb{E}_x[V(X_t)] - V(x) \leq \kappa t - \int_{(0,t)} \mathbb{E}_x[\phi \circ V(X_s)]\tau(ds)$

(iv) there is $\epsilon \in (0, 1)$ such that

$$\int_{[1,\infty)} \left( \phi \circ \Phi^{-1}(t) \right)^{\epsilon-1} \tau(dt) < \infty,$$

where $\Phi(u) := \int_0^u 1/\phi(v) dv$.

Examples satisfying conditions (i)-(iv) are given in Section 5.
Hoeffding’s inequality is a key tool in the analysis of many problems arising in both probability and statistics. As already mentioned above, it was originally proved by W. Hoeffding [Hoe63] in the context of independent and bounded random variables. However, many applied problems require an extension of the result to the case where certain dependence of the components is involved, in particular Markovian dependence (see e.g. [FJS21], [OG02] and [Tan07]). Therefore, variants of the Hoeffding’s inequality in the context of different types of Markov models have been studied recently. There are two main approaches to this problem: (i) based on spectral methods (see [CLLM12], [FJS21], [LP04], [Lez98] [Mia14] and [Rao19]) and (ii) based on Foster-Lyapunov inequality (see [AB15], [Bou09], [CL19], [DMoH11], [GO02] and [LL21]). A common assumption in all these works is that the underlying Markov model is irreducible. Hence, according to eq. (Hoeffding’s inequality) given in [Hoeffding’s inequality] and 2. Recall, a set \( \mathcal{E}(\mathbb{S}) \) is uniformly bounded, which is the main step in the proof of \([LL21, Theorems 1 and 2]\). It has been shown that the solution to the corresponding stochastic Poisson equation (see eq. (4)) is uniformly bounded, which is the main step in the proof of \([LL21, Theorems 1 and 2]\). Recall, a set \( \mathcal{C}(\mathbb{S}) \) is called an atom for \( \{X_t\}_{t \in \mathbb{T}} \) if \( \mathcal{P}(x, B) = \mathcal{P}(y, B) \) for all \( x, y \in \mathcal{C} \) and \( B \in \mathcal{B}(\mathbb{S}) \). Let us remark here that this result can be slightly generalized, i.e. the conclusions of \([LL21, Proposition 1 and Theorem 3] \), and then also the main results (Hoeffding’s inequality) given in \([LL21, Theorems 1 and 2] \), remain valid by assuming eq. (4) with \( \mathcal{C} \) being a petite set for \( \{X_t\}_{t \in \mathbb{T}} \). Namely, under these assumptions in \([GM96, Theorems 2.3 and 3.2] \) it has been shown that the solution to the corresponding stochastic Poisson equation (see eq. (4)) is petite for \( \{X_t\}_{t \in \mathbb{T}} \).

4. PROOF OF THEOREM 1

In this section, we prove Theorem 1. We follow and adapt the approach from \([GO02]\). By Kantorovich-Rubenstein theorem we have that

\[
\left| \mathbb{E}_x[f(X_\tau) - \pi(f)] \right| \leq \text{Lip}(f) \mathcal{H}(\mathcal{P}(x, \cdot), \pi(\cdot)).
\]

Hence, according to eq. (1) it follows that

\[
\hat{f}(x) := \int_{\mathbb{T}} \mathbb{E}_x[f(X_\tau) - \pi(f)] \tau(\text{d}t)
\]

is well defined and bounded. Furthermore, it clearly solves the stochastic Poisson equation

\[
\mathbb{E}_x[\hat{f}(X_\tau)] - \hat{f}(x) = - \int_{(0,t)} \mathbb{E}_x[f(X_s) - \pi(f)] \tau(\text{d}s),
\]

which in turn implies that

\[
M_t := \hat{f}(X_t) - \hat{f}(X_0) + \int_{(0,t)} (f(X_s) - \pi(f)) \tau(\text{d}s), \quad t \in \mathbb{T},
\]

for all \( t \in \mathbb{T} \), hence the solution is \( \mathcal{C}(\mathbb{S}) \) for the corresponding stochastic Poisson equation.
is a bounded martingale. Namely, for \( s < t \) it follows that
\[
E_x \left[ M_s \mid P_s \right] = M_s + E_x \left[ \hat{f}(X_{t-s}) - \hat{f}(X_s) + \int_{(0,t-s)} \left( E_x \left[ f(X_u) \right] - \pi(f) \right) \tau(du) \right] = M_s.
\]

By employing Markov inequality, for any \( \varepsilon > 0 \) and \( \theta \geq 0 \) it follows that
\[
P_x \left( S_{t-1} - \pi(f)t > t\varepsilon \right) \leq e^{-\beta \varepsilon} E_x \left[ e^{\theta (S_{t-1} - \pi(f)t)} \right] = e^{-\beta \varepsilon} E_x \left[ e^{\theta (M_t - \hat{f}(X_{t}) + \hat{f}(X_0))} \right] \leq e^{-\theta \varepsilon + 2\theta \|f\|_\infty^2} E_x \left[ \exp \left\{ \theta \left( \sum_{s=1}^{[t]} (M_s - M_{s-1}) + M_t - M_{[t]} \right) \right\} \right].
\]

Observe that when \( T = Z_+ \), then \( t = [t] \). Further, it clearly holds that
\[
\|M_s - M_{s-1}\| \leq 2\|\hat{f}\|_\infty + 2\|f\|_\infty \quad \text{and} \quad \|M_t - M_{[t]}\| \leq 2\|\hat{f}\|_\infty + 2\|f\|_\infty,
\]
and from the proof of [DGL96, Lemma 8.1] it then follows that
\[
E_x \left[ e^{\theta (M_s - M_{s-1}) \mid P_s} \right] \leq e^{2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)} \quad \text{and} \quad E_x \left[ e^{\theta (M_t - M_{[t]} \mid P_{[t]})} \right] \leq e^{2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)}.
\]

Thus,
\[
E_x \left[ \exp \left\{ \theta \left( \sum_{s=1}^{[t]} (M_s - M_{s-1}) + M_t - M_{[t]} \right) \right\} \right] \leq \begin{cases} e^{2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2}, & T = Z_+, \\
 e^{2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2(t+1)}, & T = R_+.
\end{cases}
\]

We then have
\[
P_x \left( S_{t-1} - \pi(f)t > t\varepsilon \right) \leq \begin{cases} e^{-\theta \varepsilon + 2\theta \|f\|_\infty^2 + 2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2}, & T = Z_+, \\
 e^{-\theta \varepsilon + 2\theta \|f\|_\infty^2 + 2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2(t+1)}, & T = R_+.
\end{cases}
\]

Analogously we conclude that
\[
P_x \left( S_{t-1} - \pi(f)t < -t\varepsilon \right) \leq \begin{cases} e^{-\theta \varepsilon + 2\theta \|f\|_\infty^2 + 2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2}, & T = Z_+, \\
 e^{-\theta \varepsilon + 2\theta \|f\|_\infty^2 + 2\theta^2 (\|f\|_\infty + \|f\|_\infty^2)^2(t+1)}, & T = R_+.
\end{cases}
\]

Finally, using \( \|\hat{f}\|_\infty \leq \text{Lip}(f) \gamma \) (recall that \( \gamma = \sup_{x \in S} \int_{t} \mathcal{W} \left( \mathcal{P}^t(x, \cdot), \pi(\cdot) \right) \tau(\cdot) \right) \text{ and optimizing over } \theta \text{ we obtain eq. (2).}

5. EXAMPLES

In this section, we discuss several examples of non-irreducible Markov models satisfying conditions of Theorem 1.

Example 1 (Deterministic SDE). Consider the following one-dimensional (deterministic) SDE:
\[
dX_t = -|X_t|^\alpha dt \\
X_0 = x \in [1, 2]
\]

with \( \alpha \in [1, 2] \). The SDE is well posed and it admits a unique strong solution which is a conservative strong Markov process with continuous sample paths on \( S = [1, 2] \) (endowed with the standard Euclidean metric \( d(x, y) = |x - y| \) and Borel \( \sigma \)-algebra \( \mathcal{B}([-1, 1]) \)). When \( \alpha = 1 \) the solution is given by \( X_t = xe^{-t} \), and when \( \alpha \in (1, 2) \) it is given by
\[
X_t = \frac{x}{((\alpha - 1)|x|^{-1}t + 1)^{(\alpha - 1)}}.
\]
Furthermore, it clearly holds that $\mathcal{P}^t(x, dy) = \delta_{X_t}(dy)$, and the unique invariant probability measure of $\{X_t\}_{t \in \mathbb{R}_+}$ is $\delta_0(dy)$. Here, $\delta_x(dy)$ stands for the Dirac delta measure at $x \in \mathbb{S}$. We now have that

$$
\mathcal{W}(\mathcal{P}^t(x, \cdot), \delta_0(\cdot)) = |X_t| \leq \begin{cases} 
e^{-t}, & \alpha = 1, \\ \frac{1}{((\alpha - 1)t + 1)/\alpha - t}, & \alpha \in (1, 2). \end{cases}
$$

Thus,

$$
\sup_{x \in \mathbb{S}} \int_0^{\infty} \mathcal{W}(\mathcal{P}^t(x, \cdot), \delta_0(\cdot)) dt < \infty,
$$

which shows that the condition in eq. (1) is satisfied and we can apply Theorem 1 with any Lipschitz function $f : [-1, 1] \to \mathbb{R}$. Observe also that $\{X_t\}_{t \in \mathbb{R}_+}$ is not irreducible and $\mathcal{P}^t(x, dy)$ cannot converge to $\delta_0(dy)$, as $t \to \infty$, in the total variation distance.

We now give two examples of discrete-time Markov models satisfying conditions (i)-(iv) from Section 2. We first consider an autoregressive model of order one (see e.g. [MT09]).

**Example 2 (Autoregressive model).** Let $X_0$ and $\{\xi_t\}_{t \geq 1}$ be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $X_0$ is independent of $\{\xi_t\}_{t \geq 1}$, $\{\xi_t\}_{t \geq 1}$ is an i.i.d. sequence, $\mathbb{P}(X_0 \in [0, 1]) = 1$ and $\mathbb{P}(\xi_0 = 0) = \mathbb{P}(\xi_1 = 1/2) = 1/2$. Define

$$
X_{t+1} := \frac{1}{2} X_t + \xi_{t+1}.
$$

Clearly, $\{X_t\}_{t \in \mathbb{Z}_+}$ is a Markov chain on $\mathbb{S} = [0, 1]$ (endowed with the standard Euclidean metric $d(x, y) = |x - y|$) and Borel $\sigma$-algebra $\mathcal{B}([0, 1])$ with transition function $\mathcal{P}(x, dy) = \mathcal{P}(\xi_t + x/2 \in dy)$. Observe that $\{X_t\}_{t \in \mathbb{Z}_+}$ is not irreducible. Namely, for $x \in [0, 1] \cap \mathbb{Q}$ it holds that $\mathcal{P}^t(x, [0, 1] \cap \mathbb{Q}) = 0$ for all $t \geq 1$, and analogously for $x \in [0, 1] \cap \mathbb{Q}^c$ it holds that $\mathcal{P}^t(x, [0, 1] \cap \mathbb{Q}) = 0$ for all $t \geq 1$. Next, a straightforward computation shows that

$$
\mathcal{W}(\mathcal{P}(x, \cdot), \mathcal{P}(y, \cdot)) \leq \frac{1}{2} d(x, y),
$$

and since

$$
\mathcal{W}(\mathcal{P}^2(x, \cdot), \mathcal{P}^2(y, \cdot)) \leq \inf_{\Pi \in C(\mathcal{P}(x, \cdot), \mathcal{P}(y, \cdot))} \int_{\mathbb{S} \times \mathbb{S}} \mathcal{W}(\mathcal{P}(u, \cdot), \mathcal{P}(v, \cdot)) \Pi(du, dv)
$$

$$
\leq \frac{1}{2} \inf_{\Pi \in C(\mathcal{P}(x, \cdot), \mathcal{P}(y, \cdot))} \int_{\mathbb{S} \times \mathbb{S}} d(\Pi)(du, dv)
$$

$$
= \frac{1}{2} \mathcal{W}(\mathcal{P}(x, \cdot), \mathcal{P}(y, \cdot)),
$$

we conclude that

$$
\mathcal{W}(\mathcal{P}^t(x, \cdot), \mathcal{P}^t(y, \cdot)) \leq \frac{1}{2^t} d(x, y).
$$

Thus, condition (ii) from Section 2 holds with $\rho \leq 1/2$. Conditions (iii) and (iv) trivially hold by taking $\kappa = 1$, $\mathcal{V}(x) \equiv 1$ and $\phi(t) = t$. Hence, we can apply Theorem 1 to $\{X_t\}_{t \in \mathbb{Z}_+}$ and any Lipschitz function $f : [0, 1] \to \mathbb{R}$. Observe also that $\text{Leb}(dy)$ (on $\mathcal{B}([0, 1])$) is the (unique) invariant probability measure for $\{X_t\}_{t \in \mathbb{Z}_+}$, which is singular with respect to $\mathcal{P}(x, dy)$ for any $t \in \mathbb{Z}_+$ and $x \in [0, 1]$. Hence, $\mathcal{P}^t(x, dy)$ cannot converge to $\text{Leb}(dy)$, as $t \to \infty$, in the total variation distance. Let us remark here that from [But14, Theorem 2.1] follows that for any $\epsilon \in (0, 1)$ there are $c_1(\epsilon), c_2(\epsilon) > 0$, such that

$$
\mathcal{W}(\mathcal{P}^t(x, \cdot), \text{Leb}(\cdot)) \leq c_1(\epsilon) e^{-c_2(\epsilon)t}.
$$

We now discuss a simple symmetric random walk on torus.
Example 3 (Random walk on torus). Let $Y_0$ and $\{\xi_i\}_{i \geq 1}$ be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $Y_0$ is independent of $\{\xi_i\}_{i \geq 1}$, $\{\xi_i\}_{i \geq 1}$ is an i.i.d. sequence and $\mathbb{P}(\xi_i = -1) = \mathbb{P}(\xi_i = 1) = 1/2$. Define

$$Y_{i+1} := Y_i + \xi_{i+1}.$$ 

Clearly, $\{Y_i\}_{i \in \mathbb{Z}_+}$ is a Markov model on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Denote the corresponding transition function by $P_Y(x, dy)$. Next, for $x \in \mathbb{R}$ let

$$[x] := \{y \in \mathbb{R} : x - y \in 2\pi \mathbb{Z}\}, \quad \text{and} \quad S^1 := \{[x] : x \in \mathbb{R}\}.$$

Clearly, $S^1$ is obtained by identifying the opposite faces of $[0, 2\pi]$. The corresponding Borel $\sigma$-algebra is denoted by $\mathcal{B}(S^1)$, which can be identified with the sub-$\sigma$-algebra of $\mathcal{B}(\mathbb{R})$ of sets of the form $\bigcup_{k \in \mathbb{Z}} \{x + k : x \in B\}$ for $B \in \mathcal{B}([0, 2\pi])$. The covering map $\mathbb{R} \ni x \mapsto [x] \in S^1$ is denoted by $\Pi(x)$. The projection of $\{Y_i\}_{i \in \mathbb{Z}_+}$, with respect to $\Pi(x)$, on the torus $S^1$, denoted by $\{X_i\}_{i \in \mathbb{Z}_+}$, is a Markov model on $(S^1, \mathcal{B}(S^1))$ with transition kernel given by

$$P^t(x, B) = P^t_Y(z, \Pi^{-1}(B))$$

for $x \in S^1$, $B \in \mathcal{B}(S^1)$ and $z \in \Pi^{-1}([x])$. Denote by $d(x, y)$ the arc-length metric on $S^1$. It is evident that $\mathcal{B}(S^1)$ is generated by this metric. It is also clear that $\{X_i\}_{i \in \mathbb{Z}_+}$ is not irreducible. For example, for $x = [1]$ it holds that $P^t(x, \Pi([k + 2\ell \pi : k, l \in \mathbb{Z}])) = 0$ for all $t \geq 1$, and analogously for $x = [\sqrt{2}]$ it holds that $P^t(x, \Pi([k + 2\ell \pi : k, l \in \mathbb{Z}])) = 0$ for all $t \geq 1$. A straightforward computation shows that

$$\mathcal{H}(P(x, t), P(y, t)) \leq \frac{1}{2}d(x, y),$$

and similarly as in eq. (6) we conclude that

$$\mathcal{H}(P^t(x, \cdot), P^t(y, \cdot)) \leq \frac{1}{2t}d(x, y),$$

which is exactly condition (ii) from Section 2 (with $\rho \leq 1/2$). As in Example 2, conditions (iii) and (iv) trivially hold by taking $\kappa = 1$, $\mathcal{W}(x) \equiv 1$ and $\phi(t) = t$. Hence, we can apply Theorem 1 to $\{X_i\}_{i \in \mathbb{Z}_+}$ and any Lipschitz function $f : S^1 \to \mathbb{R}$. Similarly as in the previous example, Leb(dy) (on $\mathcal{B}(S^1)$) is the (unique) invariant probability measure for $\{X_i\}_{i \in \mathbb{Z}_+}$, which is singular with respect to $P^t(x, dy)$ for any $t \in \mathbb{Z}_+$ and $x \in S^1$. Hence, $P^t(x, dy)$ cannot converge to Leb(dy), as $t \to \infty$, in the total variation distance. From [But14, Theorem 2.1] it follows that for any $\epsilon \in (0, 1)$ there are $c_1(\epsilon), c_2(\epsilon) > 0$, such that

$$\mathcal{H}(P^t(x, \cdot), \text{Leb}(\cdot)) \leq c_1(\epsilon)e^{-c_2(\epsilon)t}.$$ 

\[\square\]

At the end, we remark that one of typical ways of obtaining Markov models from a given Markov model is through a random time-change method. Recall, a subordinator $\{S_t\}_{t \in T_S}$ is a non-decreasing right-continuous (in the case when $T_S = \mathbb{R}_+$) stochastic process on $\mathbb{R}_+$ with stationary and independent increments. If $T_S = \mathbb{Z}_+$, $\{S_t\}_{t \in T_S}$ is a random walk; and if $T_S = \mathbb{R}_+$, it is a Lévy process. Let now $\{X_t\}_{t \in T}$ be a Markov model with transition kernel $P^t(x, dy)$, and let $\{S_t\}_{t \in T_S}$ be a subordinator independent of $\{X_t\}_{t \in T}$. If $T = \mathbb{Z}_+$, we assume that $\{S_t\}_{t \in T_S}$ takes values in $\mathbb{Z}_+$. The process $X_S^t := X_S$, obtained from $\{X_t\}_{t \in T}$ by a random time change through $\{S_t\}_{t \in T_S}$, is referred to as the subordinate process $\{X_t\}_{t \in T_S}$ with subordinator $\{S_t\}_{t \in T_S}$. It is easy to see that $\{X_S^t\}_{t \in T_S}$ is again a Markov model with transition kernel

$$P_S^t(x, dy) = \int_{T_S} P^t(x, dy) \mu_t(ds),$$
where $\mu_t(dx) = \mathbb{P}(S_t \in dx)$. It is also elementary to check that if $\pi(dx)$ is an invariant probability measure for $\{X_t\}_{t \in \mathbb{T}}$, then it is also invariant for the subordinate process $\{X^S_t\}_{t \in \mathbb{T}}$. Furthermore, in [APS, Proposition 1.1] it has been shown that if $W(\mathcal{P}^t(x, \cdot), \pi(\cdot)) \leq c(x) r(t)$ for some Borel measurable $c : S \to \mathbb{R}_+$ and $r : \mathbb{T} \to \mathbb{R}_+$, then

$$W(\mathcal{P}^t(x, \cdot), \pi(\cdot)) \leq c(x)E[r(S_t)].$$

Let us now apply this method to Markov models from Examples 1 to 3. Assume first that $\mathbb{T}_S = \mathbb{Z}_+$. In particular, this means that $\{S_t\}_{t \in \mathbb{T}_S}$ is given as $S_t = S_{t-1} + \xi_t$, where $S_0 = 0$ and $\{\xi_t\}_{t \geq 1}$ is a sequence of i.i.d. non-negative integer-valued random variables. Assume additionally that $\mathbb{P}(\xi_t = 0) = 0$. This procedure is sometimes referred to as discrete subordination and it was introduced in [BSC12]. Then, in order to apply Theorem 1 to $\{X^S_t\}_{t \in \mathbb{Z}_+}$, it suffices to show that $\sum_{t \in \mathbb{Z}_+} E[r(S_t)] < \infty$. Observe that in the case of Example 1 we have that $c(x) = 1$ and

$$r(t) = \begin{cases} e^{-t}, & \alpha = 1, \\ \frac{1}{(a-1)t^{a-1}}, & \alpha \in (1, 2), \end{cases}$$

while in Examples 2 and 3,

$$\sum_{t \in \mathbb{Z}_+} E[r(S_t)] = \begin{cases} \sum_{t \in \mathbb{Z}_+} \left(E[e^{-\xi_t}]\right)^t, & \text{Example 1 with } \alpha = 1, \\ \sum_{t \in \mathbb{Z}_+} \frac{E[e^{-c(\xi_t)^t}]}{((a-1)t^{a-1})^{1/(a-1)}}, & \text{Example 1 with } \alpha \in (1, 2), \\ \sum_{t \in \mathbb{Z}_+} e^{-\xi_t}, & \text{Example 1 with } \alpha = 1, \\ \sum_{t \in \mathbb{Z}_+} \frac{1}{((a-1)t^{a-1})^{1/(a-1)}}, & \text{Examples 2 and 3}, \\ \sum_{t \in \mathbb{Z}_+} e^{-c(\xi_t)^t}, & \text{Examples 2 and 3}. \end{cases}$$

Thus, we can apply Theorem 1 to $\{X^S_t\}_{t \in \mathbb{Z}_+}$ and any Lipschitz function $f : [0, 1] \to \mathbb{R}$, $f : [-1, 1] \to \mathbb{R}$ and, respectively, $f : \mathbb{S}^1 \to \mathbb{R}$. Observe also that in all three cases $\{X^S_t\}_{t \in \mathbb{Z}_+}$ is not irreducible and the corresponding transition function cannot converge to the invariant probability measure in the total variation distance.

Let now $\mathbb{T}_S = \mathbb{R}_+$. In this case, the Laplace transform of $\{S_t\}_{t \in \mathbb{R}_+}$ takes the form $\mathbb{E}[e^{-uS_t}] = e^{-\psi(u)}$. The characteristic (Laplace) exponent $\psi : (0, \infty) \to (0, \infty)$ is a Bernstein function, i.e. it is of class $C^\infty$ and $(-1)^n \psi^{(n)}(u) \geq 0$ for all $n \in \mathbb{Z}_+$. It is well known that every Bernstein function admits a unique (Lévy-Khintchine) representation

$$\psi(u) = bu + \int_{(0, \infty)} (1 - e^{-uy}) \nu(dy),$$

where $b \geq 0$ is the drift parameter and $\nu(dy)$ is a Lévy measure, i.e. a Borel measure on $\mathfrak{B}((0, \infty))$ satisfying $\int_{(0, \infty)} (1 \wedge y) \nu(dy) < \infty$. For additional reading on Bernstein functions we refer the reader to the monograph [SSV12]. Let now $\{S_t\}_{t \in \mathbb{R}_+}$ be the Poisson process (with parameter $\lambda > 0$) as the simplest (non-trivial) continuous-time subordinator. Observe that in this case $b = 0$ and $\nu(dy) = \lambda \delta_y(dy)$. We then have

$$\int_0^\infty \mathbb{E}[r(S_t)] \, dt = \begin{cases} \int_0^\infty \sum_{n \in \mathbb{Z}_+} \frac{e^{-n(\lambda t)^n}}{n!} \frac{1}{(a-1)n^{1/(a-1)}} \, dt, & \text{Example 1 with } \alpha = 1, \\ \int_0^\infty \sum_{n \in \mathbb{Z}_+} \frac{e^{-n(\lambda t)^n}}{n!} \frac{1}{((a-1)n^{1/(a-1)})^{1/(a-1)}} \, dt, & \text{Example 1 with } \alpha \in (1, 2), \\ \int_0^\infty \sum_{n \in \mathbb{Z}_+} \frac{1}{((a-1)n^{1/(a-1)})^{1/(a-1)}}, & \text{Examples 2 and 3}, \end{cases}$$

$$= \begin{cases} \frac{1}{\lambda} \sum_{n \in \mathbb{Z}_+} \frac{1}{((a-1)n^{1/(a-1)})^{1/(a-1)}}, & \text{Example 1 with } \alpha = 1, \\ \frac{1}{\lambda} \sum_{n \in \mathbb{Z}_+} \frac{1}{((a-1)n^{1/(a-1)})^{1/(a-1)}}, & \text{Examples 2 and 3}. \end{cases}$$
Thus, we can again apply Theorem 1 to \( \{ X^S_t \}_{t \in \mathbb{R}_+} \) and any Lipschitz function \( f : [-1, 1] \to \mathbb{R}, \)
\( f : [0, 1] \to \mathbb{R} \) and, respectively, \( f : \mathbb{S}^1 \to \mathbb{R} \). In all three cases \( \{ X^S_t \}_{t \in \mathbb{R}_+} \) is not irreducible
and the corresponding transition function cannot converge to the invariant probability measure
in the total variation distance.

So far we have considered subordinators taking values in \( \mathbb{Z}_+ \) only. However, the Markov
model from Example 1 can be subordinated by more general subordinators. In order to apply
Theorem 1 to such processes we again need to guarantee that
\[
\int_0^\infty \mathbb{E}[r(S_t)] \, dt < \infty. 
\]
Note that for \( \alpha \in [1, 2) \) we have that \( r(t) \leq c(1 + t)^{-\beta} \) for some \( c > 0 \) and \( \beta > 1 \) (possibly depending on \( \alpha \)). From [DSS17, Theorem 1.1 and Lemma 3.1] we know that if
\[
\liminf_{s \to \infty} \frac{\psi(u)}{\log u} > 0 \quad \text{and} \quad \liminf_{s \to 0} \frac{\psi(\rho u)}{\psi(u)} > 1
\]
for some \( \rho > 1 \), then
\[
\mathbb{E}[r(S_t)] \leq c \left( 1 \wedge \psi^{-1}(1/u) \right)^{1/(\alpha-1)}.
\]
Hence, eq. (7) holds if eq. (8) and
\[
\int_0^\infty \left( 1 \wedge \psi^{-1}(1/u) \right)^{1/(\alpha-1)} \, dt < \infty
\]
hold true. Typical examples of such characteristic exponents (subordinators) are given by \( \psi(u) = u^\gamma \) for \( \gamma \in (0, 1) \) (\( \gamma \)-stable subordinator) and \( \psi(u) = \log(1+u) \) (geometric \( 1 \)-stable subordinator).

ACKNOWLEDGEMENTS

Financial support through Alexander von Humboldt Foundation (No. HRV 1151902 HFST-E)
and Croatian Science Foundation under project 8958 (for N. Sandrić), and the Croatian Science Foundation under project 4197 (for S. Šebek) is gratefully acknowledged.

REFERENCES

[AB15] R. Adamczak and W. Bednorz. Exponential concentration inequalities for additive functionals of Markov chains. ESAIM Probab. Stat., 19:440–481, 2015.

[APS] A. Arapostathis, G. Pang, and N. Sandrić. Subexponential upper and lower bounds in Wasserstein distance for Markov processes. Preprint (2020), arXiv:1907.05250.

[BG68] R. M. Blumenthal and R. K. Getoor. Markov processes and potential theory. Academic Press, New York-London, 1968.

[Bou09] T. R. Boucher. A Hoeffding inequality for Markov chains using a generalized inverse. Statist. Probab. Lett., 79(8):1105–1107, 2009.

[BSC12] A. Bendikov and L. Saloff-Coste. Random walks on groups and discrete subordination. Math. Nachr., 285(5-6):580–605, 2012.

[But14] O. Butkovsky. Subgeometric rates of convergence of Markov processes in the Wasserstein metric. Ann. Appl. Probab., 24(2):526–552, 2014.

[CL19] M. C. H. Choi and E. Li. A Hoeffding’s inequality for uniformly ergodic diffusion process. Statist. Probab. Lett., 150:23–28, 2019.

[CLLM12] K.-M. Chung, H. Lam, Z. Liu, and M. Mitzenmacher. Chernoff-Hoeffding bounds for Markov chains: generalized and simplified. In 29th International Symposium on Theoretical Aspects of Computer Science, volume 14 of LIPIcs. Leibniz Int. Proc. Inform., pages 124–135. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2012.

[DGL96] L. Devroye, L. Györfi, and G. Lugosi. A probabilistic theory of pattern recognition. Springer-Verlag, New York, 1996.

[DMOvH11] R. Douc, E. Moulines, J. Olsson, and R. van Handel. Consistency of the maximum likelihood estimator for general hidden Markov models. Ann. Statist., 39(1):474–513, 2011.

[DSS17] C.-S. Deng, R. L. Schilling, and Y.-H. Song. Subgeometric rates of convergence for Markov processes under subordination. Adv. in Appl. Probab., 49(1):162–181, 2017.
Hoeffding’s inequality for non-irreducible Markov models

[FJS21] J. Fan, B. Jiang, and Q. Sun. Hoeffding’s inequality for general Markov chains and its applications to statistical learning. *J. Mach. Learn. Res.*, 22:Paper No. 139, 35, 2021.

[GM96] P. W. Glynn and S. P. Meyn. A Liapounov bound for solutions of the Poisson equation. *Ann. Probab.*, 24(2):916–931, 1996.

[GO02] P. W. Glynn and D. Ormoneit. Hoeffding’s inequality for uniformly ergodic Markov chains. *Statist. Probab. Lett.*, 56(2):143–146, 2002.

[Hoe63] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963.

[Lez98] P. Lezaud. Chernoff-type bound for finite Markov chains. *Ann. Appl. Probab.*, 8(3):849–867, 1998.

[LL21] Y. Liu and J. Liu. Hoeffding’s inequality for Markov processes via solution of Poisson’s equation. *Front. Math. China*, 16(2):543–558, 2021.

[LP04] C. A. León and F. Perron. Optimal Hoeffding bounds for discrete reversible Markov chains. *Ann. Appl. Probab.*, 14(2):958–970, 2004.

[Mia14] B. Miasojedow. Hoeffding’s inequalities for geometrically ergodic Markov chains on general state space. *Statist. Probab. Lett.*, 87:115–120, 2014.

[MT09] S. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition, 2009.

[OG02] D. Ormoneit and P. W. Glynn. Kernel-based reinforcement learning in average-cost problems. *IEEE Trans. Automat. Control*, 47(10):1624–1636, 2002.

[Rao19] S. Rao. A Hoeffding inequality for Markov chains. *Electron. Commun. Probab.*, 24:Paper No. 14, 11, 2019.

[SSV12] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions*. Walter de Gruyter & Co., Berlin, second edition, 2012.

[Tan07] Y. Tang. A Hoeffding-type inequality for ergodic time series. *J. Theoret. Probab.*, 20(2):167–176, 2007.

(Nikola Sandrić) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, ZAGREB, CROATIA
*Email address: nikola.sandric@math.hr*

(Stjepan Šebek) DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF ELECTRICAL ENGINEERING AND COMPUTING, UNIVERSITY OF ZAGREB, ZAGREB, CROATIA
*Email address: stjepan.sebek@fer.hr*