OCTONIONIC HERMITIAN MATRICES WITH NON-REAL EIGENVALUES

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ABSTRACT

We extend previous work on the eigenvalue problem for Hermitian octonionic matrices by discussing the case where the eigenvalues are not real, giving a complete treatment of the $2 \times 2$ case, and summarizing some preliminary results for the $3 \times 3$ case.

1. INTRODUCTION

In previous work [1,2; see also 3], we considered the real eigenvalue problem for $2 \times 2$ and $3 \times 3$ Hermitian matrices over the octonions $\mathbb{O}$. The $2 \times 2$ case corresponds closely to the standard, complex eigenvalue problem, since any $2 \times 2$ octonionic Hermitian matrix lies in a complex subalgebra $\mathbb{C} \subseteq \mathbb{O}$. The $3 \times 3$ case requires considerable care, resulting in some changes in the expected results. However, we also showed in [1] that there are octonionic Hermitian matrices which admit eigenvalues which are not real, and which it is the purpose of this paper to discuss.

We consider both the right eigenvalue problem

$$Av = v\lambda$$

and the left eigenvalue problem

$$Aw = \lambda w$$

where $A$ is a Hermitian octonionic matrix. The $2 \times 2$ case is reasonably straightforward, and can be completely solved. Although we argue that the right eigenvalue problem (1) is more fundamental, we obtain some intriguing results relating the sets of left and right eigenvectors, as well as the matrices whose eigenvectors they are.
We then briefly discuss our preliminary results in the $3 \times 3$ case. Although we have been able to obtain a 3rd-order characteristic equation for the (right) eigenvalues in this case, we have not been able to solve this equation, nor have we been able to extend our orthonormality results [1,2] from the real case.

Both of these cases have applications to physics. Three of the four superstring equations of motion can be written as (2 separate) $2 \times 2$ octonionic eigenvalue problems with real eigenvalue 0 [5,6,7]. The resulting equation is really the massless Dirac (Weyl) equation in 10 dimensions (and in momentum space), and this has recently been used in a model for dimensional reduction [8,9]. In this model each quaternionic subalgebra $\mathbb{H} \subset \mathbb{O}$ gives rise to a spectrum of (free) particles which corresponds exactly with the spins and helicities of a generation of leptons, including both massless and massive particles. Furthermore, in a natural sense there are precisely 3 such quaternionic subalgebras compatible with the dimensional reduction, which we have interpreted as generations.

Fundamental to this model is the use of the Lorentz group $SL(2,\mathbb{C}) \subset SL(2,\mathbb{O})$ to analyze the spin states of the resulting particles [10,11]. This is again an eigenvalue problem, this time for an octonionic self-adjoint operator. We discuss this operator eigenvalue problem below, showing that already at the quaternionic level it admits eigenvalues which are not real. This leads to spin states which are simultaneous eigenstates of all the spin operators, although not all the eigenvalues are real. This result could have implications for the interpretation of quantum mechanics [9].

The $3 \times 3$ case is of particular interest mathematically because it corresponds to the exceptional Jordan algebra, also known as the Albert algebra. There have been numerous attempts to use this algebra to describe quantum physics, which was in fact Jordan’s original motivation. More recently, Schray [11,12] has shown how to use the exceptional Jordan algebra to give an elegant description of the superparticle, which we have been attempting to extend to the superstring. Our dimensional reduction scheme extends naturally to this case [4], and we believe it is the natural language to describe the fundamental particles of nature.

The paper is organized as follows. In Section 2 we briefly review the properties of octonions. In Section 3 we consider $2 \times 2$ octonionic Hermitian matrices, and in Section 4 we discuss the $2 \times 2$ self-adjoint spin operators. In Section 5, we summarize our preliminary attempts to generalize these results to $3 \times 3$ octonionic Hermitian matrices. Along the way, we have need of several identities involving octonionic associators, which are closely related to the “3-Ψ’s rule” needed for supersymmetric theories; this is discussed in the Appendix. Finally, in Section 6 we discuss our results.

2. OCTONIONS

We summarize here only the essential properties of the octonions $\mathbb{O}$. For a more detailed introduction, see [1] or [13,14].

The octonions $\mathbb{O}$ are the nonassociative, noncommutative, normed division algebra over the reals. In terms of a natural basis, an octonion $a$ can be written

$$a = \sum_{q=1}^{8} a^q e_q$$  (3)
Figure 1: The representation of the octonionic multiplication table using the 7-point projective plane, where we have used the conventional names \{i, j, k, k\ell, j\ell, i\ell, \ell\} for \{e_2, \ldots, e_8\}. Each of the 7 oriented lines gives a quaternionic triple.

where the coefficients \(a^q\) are real, and where the basis vectors satisfy \(e_1 = 1\) and

\[
e^q_2 = -1 \quad (q = 2, \ldots, 8)
\] (4)

The multiplication table is conveniently encoded in the 7-point projective plane, shown in Figure 1. The product of any two imaginary units is given by the third unit on the unique line connecting them, with the sign determined by the relative orientation.

Octonionic conjugation is given by reversing the sign of the imaginary basis units

\[
\overline{a} = a^1e_1 - \sum_{q=2}^{8} a^q e_q
\] (5)

Conjugation is an antiautomorphism, since it satisfies

\[
\overline{ab} = \overline{b}\overline{a}
\]

The real and imaginary parts of an octonion \(a\) are given by

\[
\text{Re}(a) = \frac{1}{2}(a + \overline{a}) \quad \text{Im}(a) = \frac{1}{2}(a - \overline{a})
\] (6)

The inner product on \(\mathbb{O}\) is the one inherited from \(\mathbb{R}^8\), namely

\[
a \cdot b = \sum_q a^q b^q
\] (7)

which can be rewritten as

\[
a \cdot b = \frac{1}{2}(a\overline{b} + b\overline{a}) = \frac{1}{2} (\overline{b}a + \overline{a}b)
\] (8)
and which satisfies the identities
\[ a \cdot (xb) = b \cdot (xa) \]  
\[ (ax) \cdot (bx) = |x|^2 a \cdot b \]  
for any \( a, b, x \in \mathbb{O} \). The norm of an octonion is just
\[ |a| = \sqrt{a\bar{a}} = \sqrt{a \cdot a} \]  
which satisfies the defining property of a normed division algebra, namely
\[ |ab| = |a||b| \]  
The associator of three octonions is
\[ [a, b, c] = (ab)c - a(bc) \]  
which is totally antisymmetric in its arguments, has no real part, and changes sign if any one of its arguments is replaced by its octonionic conjugate. Although the associator does not vanish in general, the octonions do satisfy a weak form of associativity known as alternativity, namely
\[ [b, a, a] = 0 = [b, a, \bar{a}] \]  
The underlying reason for alternativity is Artin’s Theorem [15,16], which states that that any two octonions lie in a quaternionic subalgebra of \( \mathbb{O} \), so that any product containing only two octonionic directions is associative. We will also have use for the associator identity
\[ [a, b, c]d + a[b, c, d] = [ab, c, d] - [a, bc, d] + [a, b, cd] \]  
for any \( a, b, c, d \in \mathbb{O} \), which is proved by writing out all the terms.

3. 2 \times 2 OCTONIONIC HERMITIAN MATRICES

The general \( 2 \times 2 \) octonionic Hermitian matrix can be written
\[ A = \begin{pmatrix} p & a \\ \bar{a} & m \end{pmatrix} \]  
with \( p, m \in \mathbb{R} \) and \( a \in \mathbb{O} \), and satisfies its characteristic equation
\[ A^2 - (\text{tr} A) A + (\det A) I = 0 \]  
where \( \text{tr} A \) denotes the trace of \( A \), and where there is no difficulty with commutativity and associativity in defining the determinant of \( A \) as usual via
\[ \det A = pm - |a|^2 \]  
since the components of \( A \) lie in a complex subalgebra \( \mathbb{C} \subset \mathbb{O} \). If \( a = 0 \) the eigenvalue problem is trivial, so we assume \( a \neq 0 \). We also set
\[ v = \begin{pmatrix} x \\ y \end{pmatrix} \]  
with \( x, y \in \mathbb{O} \).
a) Left Eigenvalue Problem

As pointed out in [1], even quaternionic Hermitian matrices can admit left eigenvalues which are not real, as is shown by the following example:

$$
\begin{pmatrix}
1 & -i \\
i & 1
\end{pmatrix}
\begin{pmatrix}
1 + j \\
k + i
\end{pmatrix}
= (1 + j)
\begin{pmatrix}
k \\
1
\end{pmatrix}
$$

(20)

Direct computation allows us to determine which Hermitian matrices $A$ admit left eigenvalues which are not real. Inserting (19) in (2) leads to

$$
(\lambda - p)x = ay \\
(\lambda - m)y = \overline{\pi}x
$$

(21)

which in turn leads to

$$
\frac{\overline{\pi}(\lambda - p)x}{|a|^2} = y = \frac{(\overline{\lambda} - m)(\overline{\pi}x)}{|\lambda - m|^2}
$$

(22)

Assuming without loss of generality that $x \neq 0$ and taking the norm of both sides yields

$$
|a|^2 = |\lambda - p||\lambda - m|
$$

(23)

resulting in

$$
\frac{\overline{\pi}(\lambda - p)x}{|\lambda - p|} = \frac{(\overline{\lambda} - m)(\overline{\pi}x)}{|\lambda - m|}
$$

(24)

This equation splits into two independent parts, the terms (in the numerator) which involve the imaginary part of $\lambda$, which is nonzero by assumption, 3 and those which don’t. Looking first at the latter leads to

$$
p = m
$$

(25)

which in turn reduces (24) to

$$
\overline{\pi}(\lambda x) = \overline{\lambda}(\overline{\pi}x)
$$

(26)

which forces $a$ to be purely imaginary (and $\lambda \cdot a = 0$), but which puts no conditions on $x$.

Denoting the $2 \times 2$ identity matrix by $I$ and setting

$$
J(\hat{r}) = \begin{pmatrix}
0 & -\hat{r} \\
\hat{r} & 0
\end{pmatrix}
$$

(27)

for any pure imaginary unit octonion $\hat{r}$, and noting that this latter condition can be written as $\hat{r}^2 = -1$, we have

**Lemma 1:** The set of $2 \times 2$ Hermitian matrices $A$ for which left eigenvalues exist which are not real is

$$
A := \{ A: A = pI + qJ(\hat{r}); \quad p, q \in \mathbb{R}, q \neq 0, \hat{r}^2 = -1 \}
$$

(28)

The set $A$ has some remarkable properties, which will be further discussed below. Without loss of generality, we can take $\hat{r} = i$, so that $A$ takes the form

$$
A = \begin{pmatrix}
p & -iq \\
iq & p
\end{pmatrix}
$$

(29)

3 We can in fact assume without loss of generality that $\text{Re}(\lambda) = 0$ by replacing $A$ with $A - \text{Re}(\lambda)I$. 

5
Let us find the general solution of the left eigenvalue problem for these matrices. Taking $A$ as in (29) and $v$ as in (19) we can rewrite (2) as

$$\frac{\lambda - p}{q} x = -iy \quad \frac{\lambda - p}{q} y = ix$$  \hspace{1cm} (30)

Taking the norm of both sides immediately yields

$$|x|^2 = |y|^2$$  \hspace{1cm} (31)

and we can normalize both of these to 1 without loss of generality. We thus obtain

$$\frac{\lambda - p}{q} = -(iy)x = (ix)y$$

$$= -[i, y, x] - i(yx) = [i, x, y] + i(xy)$$  \hspace{1cm} (32)

But since

$$[z, y, x] = -[z, y, x] = [z, x, y] = -[z, x, y]$$

for any $z$, the two associators cancel, and we are left with

$$x \cdot y = 0$$  \hspace{1cm} (34)

Thus, $x$ and $y$ correspond to orthonormal vectors in $\mathbb{O}$ thought of as $\mathbb{R}^8$. This argument is fully reversible; any suitably normalized $x$ and $y$ which are orthogonal yield an eigenvector of $A$. We have therefore shown that all matrices in $\mathbb{A}$ have the same left eigenvectors:

**Lemma 2:** The set of left eigenvectors for any matrix $A \in \mathbb{A}$ is given by

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : |x|^2 = |y|^2; \ x \cdot y = 0 \right\}$$  \hspace{1cm} (35)

The left eigenvalue is given in each case by (30). Furthermore, left multiplication by an arbitrary octonion preserves the set $\mathcal{V}$, so that matrices in $\mathbb{A}$ have the property that left multiplication of left eigenvectors yields another left eigenvector (albeit with a different eigenvalue).  \hspace{1cm} (4) It follows from (30) and (31) that

$$|\lambda - p| = q$$  \hspace{1cm} (36)

Inserting this into either of (30), multiplying both sides by $i$, and using the identities (9) and (10) then shows that (34) forces

$$\lambda \cdot i = 0$$  \hspace{1cm} (37)

However, these are the only restrictions on $\lambda$, since (30) can be used to construct eigenvectors having any eigenvalue satisfying these two conditions.

**b) Right Eigenvalue Problem**

As discussed in [1], the right eigenvalues of quaternionic Hermitian matrices must be real, which is a strong argument in favor of (1) over (2). However, as pointed out in [1],

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4 Direct computation shows that, other than real matrices, the matrices in $\mathbb{A}$ are the only $2 \times 2$ Hermitian matrices with this property.
there do exist octonionic Hermitian matrices which admit right eigenvalues which are not real, as is shown by the following example:

\[
\begin{pmatrix}
1 & -i \\
i & 1
\end{pmatrix}
\begin{pmatrix}
j \\
\ell
\end{pmatrix}
= \begin{pmatrix}
j - i\ell \\
\ell + k
\end{pmatrix} (1 + k\ell)
\]

(38)

Proceeding as we did for left eigenvectors, we can determine which matrices \( A \) admit right eigenvalues which are not real. Inserting (19) into (1) leads to

\[x(\lambda - p) = ay \quad y(\lambda - m) = \overline{a}x\]

(39)

which in turn leads to

\[
\frac{\overline{a}(x(\lambda - p))}{|a|^2} = y = \frac{(\overline{a}x)(\overline{\lambda} - m)}{|\lambda - m|^2}
\]

(40)

Taking the norm of both sides (and assuming \( x \neq 0 \)) again yields (23), resulting in

\[
\frac{\overline{a}(x(\lambda - p))}{|\lambda - p|} = \frac{(\overline{a}x)(\overline{\lambda} - m)}{|\lambda - m|}
\]

(41)

Just as for the left eigenvector problem, this equation splits into two independent parts, the terms (in the numerator) which involve the imaginary part of \( \lambda \), which is nonzero by assumption, and those which don’t.\(^3\) Looking first at the latter again forces \( p = m \), which in turn forces \( |y| = |x| \). The remaining condition is now

\[
\overline{a}(x\lambda) = (\overline{a}x)\overline{\lambda}
\]

(42)

so that \( \overline{a} \), \( \text{Im}(\lambda) \), and \( x \) antiassociate. In particular, this forces both \( a \) and \( x \) to be pure imaginary, as well as

\[
\lambda \cdot a = 0
\]

(43)

\[
\lambda \cdot x = 0 = a \cdot x
\]

(44)

with corresponding identities also holding for \( y \).\(^5\) We conclude that the necessary and sufficient condition for matrices to admit right eigenvalues which are not real is that \( A \in \mathbb{A} \):

**Lemma 3:** The set of \( 2 \times 2 \) Hermitian matrices \( A \) for which right eigenvalues exist which are not real is \( \mathbb{A} \) as defined in (28).

Thus, all \( 2 \times 2 \) Hermitian matrices which admit right eigenvalues which are not real also admit left eigenvalues which are not real, and vice versa!

**Corollary:** A \( 2 \times 2 \) octonionic Hermitian matrix admits right eigenvalues which are not real if and only if it admits left eigenvalues which are not real.

Turning to the eigenvectors, inserting \( p = m \) into (39) leads to

\[
\overline{a}(ay) = \overline{y}(\overline{a}x)
\]

(45)

\(^5\) This also implies that \( a\lambda \cdot x = 0 \), that is, \( x \) (and \( y \)) must be orthogonal to the quaternionic subalgebra generated by \( \lambda \) and \( a \).
and inserting the conditions on \( a, x, \) and \( y \) now leads to
\[
x \cdot y = 0 \tag{46}
\]
just as for left eigenvectors. All right eigenvectors with non-real eigenvalues are hence in \( \mathbb{V} \), although the converse is false (since right eigenvectors have no real part). Furthermore, not all of the remaining elements of \( \mathbb{V} \) will be eigenvectors for any given matrix \( A \) (since right eigenvectors have no “quaternionic” part).

Putting all of this together, typical solutions of the (right) eigenvalue problem for \( A \) as in (29) can thus be written as
\[
v = n \begin{pmatrix} j \\ k \end{pmatrix} \quad \lambda_v = p + q s \\
w = n \begin{pmatrix} k \\ j \end{pmatrix} \quad \lambda_w = p - q s \tag{47}
\]
where \( p, q, n \in \mathbb{R} \) and where
\[
s = \cos \theta + k \ell \sin \theta \tag{48}
\]
The example given in (38) is a special case of the first of (47) with \( p = q = n = 1 \) and \( \theta = \pi/2 \).

c) Characteristic Equation

We now derive a generalized characteristic equation which is satisfied by right eigenvalues of \( 2 \times 2 \) octonionic Hermitian matrices. \(^6\) Along the way, we also rederive some of the results of the previous subsection.

Multiplying the first of (39) on the left by \( \lambda \) and the second of (39) on the right by \( \lambda \) and subtracting leads to
\[
y \left( \lambda^2 - \lambda (\text{tr} A) + (\text{det} A) \right) = [a, y, \lambda] \tag{49}
\]
which can be solved for the characteristic equation in the form
\[
\lambda^2 - \lambda (\text{tr} A) + (\text{det} A) = \frac{\overline{y} [\overline{x}, x, \lambda]}{|y|^2} \tag{50}
\]
Using (15), we have
\[
\overline{y} [\overline{x}, x, \lambda] = [\overline{y} \overline{x}, x, \lambda] - [\overline{y}, \overline{x}, \lambda] + [\overline{y}, \overline{x}, \xi \lambda] - [\overline{y}, \overline{x}, x] \lambda \tag{51}
\]
But (39) immediately implies
\[
[a y, x, \lambda] = 0 = [y, \overline{x}, x, \lambda] \tag{52}
\]
so that the first 2 terms on the right-hand-side of (51) vanish. Using (39) again brings (51) to the form
\[
\overline{y} [\overline{x}, x, \lambda] = [\overline{y}, \overline{x}, xp + ay] - [\overline{y}, \overline{x}, x] \lambda = [\overline{y}, \overline{x}, x] (p - \lambda) \tag{53}
\]
and inserting this into (50) leads finally to the generalized characteristic equation for \( \lambda \), namely
\[
\lambda^2 - \lambda (\text{tr} A) + (\text{det} A) = [\overline{x}, y, \lambda] \frac{(\lambda - p)}{|y|^2} = [a, y, x] \frac{(\lambda - m)}{|x|^2} \tag{54}
\]
\(^6\) Analogous results hold for left eigenvalues, but they are much less elegant.
where the final equality follows by symmetry.

If the associator \([a, x, y]\) vanishes, then \(\lambda\) satisfies the ordinary characteristic equation, and hence is real (since \(A\) is complex Hermitian). Otherwise, comparing real and imaginary parts of the last two terms in (54) provides an alternate derivation of \(|y| = |x|\), and we recover \(p = m\) as expected. Furthermore, since the left-hand-side of (54) lies in a complex subalgebra of \(\mathcal{O}\), so does the right-hand-side, and it is then straightforward to solve for \(\lambda\) by considering its real and imaginary parts. The generalized characteristic equation (54) then yields the following equation for \(\lambda\)

\[
(\text{Re}(\lambda))^2 - \text{Re}(\lambda)(\text{tr} A) + (\text{det} A) = (\text{Im}(\lambda))^2 < 0
\] (55)

together with the requirement that

\[
\frac{[\pi, x, y]}{|x||y|} = 2 \text{Im}(\lambda)
\] (56)

The explicit form of the eigenvalues given in (47) and (48) verifies that there are no further restrictions on \(\lambda\) other than (43) and (55). Furthermore, having shown in the previous subsection that \(a\) and \(x\) (and therefore also \(y\)) are pure imaginary, (56) yields an alternate derivation that \(\lambda\) is orthogonal to \(a\), which is (43), as well as to \(x\) and \(y\), which is (44).

It follows directly from the generalized characteristic equation (55) that

\[
|\text{Im}(\lambda)| \leq |a|
\] (57)

but this can be made more precise. Given only that \(m = p\), it is straightforward to rewrite (55) as (compare (36))

\[
|\lambda - p|^2 = |a|^2
\] (58)

and it is intriguing that this seems to be almost the condition for the vanishing of the determinant of \(Q = A - \lambda I\). But if \(\lambda \not\in \mathbb{R}\), \(Q\) is not Hermitian, and hence has no well-defined determinant. However, \(QQ^\dagger\) is Hermitian, and \(\det(QQ^\dagger) = 0\) does indeed reduce to (58) for \(A\) of the form (29), that is with \(p = m\) and \(\text{Re}(a) = 0\), provided that in addition (43) is assumed to hold.

**d) Decompositions**

One of the main results of \([1]\) was that if \(v, w\) are (normalized) eigenvectors of the 2 \(\times\) 2 octonionic Hermitian matrix \(A\) corresponding to different real eigenvalues \(\lambda_v, \lambda_w\), then \(A\) can be expanded as

\[
A = \lambda_v vv^\dagger + \lambda_w ww^\dagger
\] (59)

Furthermore, \(v\) and \(w\) are automatically orthogonal in the generalized sense

\[
(vv^\dagger) w = 0
\] (60)

We now ask whether a decomposition analogous to (59) exists when the eigenvalues are not real. Consider first the case of left eigenvalues, so that \(A \in \mathbb{A}\) and \(v \in \mathbb{V}\). If \(v\) is given by (19) with \(v^\dagger v = 1\), let

\[
w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = \begin{pmatrix} y \\ x \end{pmatrix} \in \mathbb{V}
\] (61)

\footnote{Using the (square root of the) determinant of the Hermitian square \(QQ^\dagger\) is just Dieudonné’s prescription \([16]\) for the determinant of a 2 \(\times\) 2 quaternionic matrix \(Q\); this is also briefly discussed in \([17]\).}
This leads to $w^\dagger w = 1$, and furthermore
\[ v^\dagger w = \bar{x} \cdot \bar{y} = (y \cdot x) = 0 \] (62)
by the definition of $\mathbb{V}$. Provided
\[ [i, x, y] = 0 \] (63)
so that the entire problem is quaternionic and hence associative, it turns out that (59) does hold, \(^8\) where $\lambda_v = \lambda$ is obtained by solving (either equation in) (30) and $\lambda_w$ is obtained from $\lambda_v$ by interchanging $x$ and $y$. We illustrate this result by returning to the example (20), for which we obtain the decomposition
\[
\begin{pmatrix}
1 & -i \\
0 & 1
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & j \\
k & 1
\end{pmatrix} \bar{v} + \frac{1}{2} \begin{pmatrix}
1 & -j \\
k & 1
\end{pmatrix} \bar{w}
\] (64)
where the factor of 2 is due to the normalization of the eigenvectors.

The above construction fails if $[i, x, y] \neq 0$. Remarkably, a similar construction still works for right eigenvalues! Direct computation establishes:

**Theorem 1:** For any $A \in \mathbb{A}$ and (normalized) $v \in \mathbb{V}$ such that $Av = v\lambda_v$.

Then $A$ can be expanded as
\[ A = \lambda_v (vv^\dagger) + \lambda_w (ww^\dagger) \] (65)
where $w$ is defined by (61) and satisfies $Aw = w\lambda_w$.

As before, we can assume without loss of generality that $A$ is given by (29) and that $v$ and $w$ are given by (47). Returning to our example (38) yields the explicit decomposition
\[
\begin{pmatrix}
1 & i \\
-i & 1
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & j \\
k & 1
\end{pmatrix} \bar{v} + \frac{1}{2} \begin{pmatrix}
1 & -j \\
k & 1
\end{pmatrix} \bar{w}
\] (66)

While it is true that
\[ (vv^\dagger) w = v (v^\dagger w) \] (67)
for any $v, w$ related by (61) (but not necessarily in $\mathbb{V}$), the decomposition (65) is surprising because the eigenvalues $\lambda_v, \lambda_w$ do not commute or associate with the remaining terms. Specifically, although (67) is zero here, we have
\[ (\lambda_v (vv^\dagger)) w \neq 0 \] (68)

Remarkably, there is another decomposition theorem which does not have this problem. Direct computation establishes:

**Theorem 2:** Given $A \in \mathbb{A}$ and (normalized) $v \in \mathbb{V}$ such that $Av = v\lambda_v$. Then $A$ can be expanded as
\[ A = (v\lambda_v) v^\dagger + (w\lambda_w) w^\dagger \] (69)
where $w$ is defined by (61) and satisfies $Aw = w\lambda_w$.

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\(^8\) No parentheses are needed because of the assumed associativity.
The decomposition (69) is less surprising than (65) when one realises that orthogonality in the form
\[
( (v\lambda) v^\dagger ) w = (v\lambda) (v^\dagger w) = 0
\]
holds for any \( \lambda \in \mathbb{O} \) and \( v, w \in \mathbb{V} \) satisfying (61). Furthermore, since
\[
( (v\lambda) v^\dagger ) v = (v\lambda) (v^\dagger v)
\]
for any \( v \) and \( \lambda \) (see the Appendix), one can use the decomposition (69) to construct \( 2 \times 2 \) octonionic matrices with arbitrary octonionic (right) eigenvalues. However, such matrices will not in general be Hermitian; this requires the imaginary parts of the eigenvalues to be equal and opposite, as well as some further restrictions on the eigenvalues (compare (47)).

4. SPIN

In standard quantum theory, the infinitesimal generators \( r_\alpha \) (with \( \alpha = x, y, z \)) of angular momentum or spin are just the anti-Hermitian matrices obtained by multiplying the Pauli matrices by the imaginary complex unit (which for us is \( \ell \), not \( i \)) and dividing by 2. Explicitly, we have
\[
\begin{align*}
r_x &= \frac{1}{2} \begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix} \\
r_y &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
r_z &= \frac{1}{2} \begin{pmatrix} \ell & 0 \\ 0 & -\ell \end{pmatrix}
\end{align*}
\]
where we have set \( \hbar = 1 \). One then normally multiplies by \( -\ell \) to obtain a description of the Lie algebra \( su(2) \) in terms of Hermitian matrices.

As discussed in [8,9], however, in the octonionic setting care must be taken with this last step, and we define instead the operators
\[
L_\alpha \psi = -(r_\alpha \psi)\ell
\]
where \( \psi \) is a 2-component octonionic column (representing a Majorana-Weyl spinor in 10 spacetime dimensions). The operators \( L_\alpha \) are self-adjoint with respect to the inner product
\[
(\psi, \chi) = \pi(\psi^\dagger \chi)
\]
where the map
\[
\pi(q) = \frac{1}{2} (q + \ell q \ell)
\]
projects \( \mathbb{O} \) to a preferred complex subalgebra \( \mathbb{C} \subset \mathbb{O} \).

Spin eigenstates are obtained as usual as the eigenvectors of \( L_z \) with eigenvalues \( \pm \frac{1}{2} \). Particular attention is paid in [8,9] to the eigenstates
\[
\Psi^+ = \begin{pmatrix} 1 \\ k \end{pmatrix} \quad \Psi^- = \begin{pmatrix} -k \\ 1 \end{pmatrix}
\]
which were proposed as representing particles \(^9\) at rest with spin \( \pm \frac{1}{2} \), respectively. Note that \( \Psi^+ \) and \( \Psi^- \) are orthogonal with respect to the above inner product, that is
\[
(\Psi^+, \Psi^-) = 0
\]

We will therefore focus on these eigenstates, which have some extraordinary properties.

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\(^9\) These two papers used different conventions to distinguish particles from antiparticles; we adopt the conventions used in [9].
Consider now the remaining spin operators $L_x$, $L_y$ acting on these eigenstates. We have

$$L_x \Psi^+ = \frac{1}{2} \left( -\frac{k}{1} \right) = \Psi^+ \left( -\frac{k}{2} \right)$$

(77)

and

$$L_y \Psi^+ = \frac{1}{2} \left( -\frac{k\ell}{\ell} \right) = \Psi^+ \left( -\frac{k\ell}{2} \right)$$

(78)

with similar results holding for $\Psi^-$. This illustrates the fact that this quaternionic self-adjoint operator eigenvalue problem admits eigenvalues which are not real. More importantly, as claimed in [8,9], it shows that $\Psi^+$ is a simultaneous eigenvector of the 3 self-adjoint spin operators $L_x$, $L_y$, $L_z$!

This result could have significant implications for quantum mechanics. In this formulation, the inability to completely measure the spin state of a particle, because the spin operators fail to commute, is thus ultimately due to the fact that the eigenvalues don’t commute. Explicitly, we have

$$4L_x(L_y \Psi^+) = 2L_x(-\Psi^+ k\ell) = 2r_x(\Psi^+ k\ell) \ell = -2r_x(\Psi^+ \ell) k\ell = +(2L_x \Psi^+) k\ell$$

$$= -\Psi^+ k\ell \ell = +\Psi^+ \ell$$

(79)

$$4L_y(L_x \Psi^+) = -\Psi^+ k\ell k = -\Psi^+ \ell$$

(80)

which yields the usual commutation relation in the form

$$[L_x, L_y] \Psi^+ = \frac{1}{2} \Psi^+ \ell = L_z \Psi^+ \ell$$

Consider now the more general eigenstate

$$L_z \left( \Psi^+ e^{i\theta} \right) = \left( \Psi^+ e^{i\theta} \right) \left( \frac{1}{2} \right)$$

(81)

and note that

$$L_x \left( \Psi^+ e^{i\theta} \right) = \left( \Psi^+ e^{i\theta} \right) \left( -\frac{k\ell e^{2i\theta}}{2} \right)$$

$$L_y \left( \Psi^+ e^{i\theta} \right) = \left( \Psi^+ e^{i\theta} \right) \left( -\frac{k\ell e^{2i\theta}}{2} \right)$$

(82)

so that the non-real eigenvalues depend on the phase. It is intriguing to speculate on whether the value of the non-real eigenvalues, which determine the phase, can be used to specify (but not measure) the actual direction of the spin, and whether this might shed some insight on basic properties of quantum mechanics such as Bell’s inequality.

### 5. 3 × 3 OCTONIONIC HERMITIAN MATRICES

The general $3 \times 3$ octonionic Hermitian matrix can be written

$$\mathcal{A} = \begin{pmatrix} p & a & b \\ \bar{a} & m & c \\ b & c & n \end{pmatrix}$$

(83)

with $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$. Remarkably, $\mathcal{A}$ satisfies the (natural generalization of the) usual characteristic equation for $3 \times 3$ matrices [18]. Equally remarkably, the real eigenvalues do not satisfy this equation, and, as shown originally by Ogievetsky [19], $\mathcal{A}$ has 6, rather
than 3, real eigenvalues. As shown in [1] (see also [3]), the eigenvalues naturally belong to 2 distinct families, each containing 3 real eigenvalues. Furthermore, within each family, the corresponding eigenvectors are orthogonal in the sense of (60), and lead to a natural decomposition of \( \mathcal{A} \) along the lines of (59).

In our previous work [1], we derived a generalized characteristic equation satisfied by the real eigenvalues. In this section, we summarize our preliminary efforts to generalize this equation to non-real eigenvalues. Since the calculations are somewhat involved, and since we have so far been unable to solve the resulting equations, we give here only the final results of those calculations, the full details of which are available online together with some examples and further discussion [20].

Inserting
\[
v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\]
(84)
into the (right) eigenvalue problem (1) leads, after considerable manipulation [20], to the generalized characteristic equation for the eigenvalues \( \lambda \) in the form
\[
z \left( \lambda^3 - (\text{tr} \mathcal{A}) \lambda^2 + \sigma(\mathcal{A}) \lambda - \det \mathcal{A} \right) = b \left( a(cz) \right) + \overline{c} \left( \overline{b} z \right) - \left( b(ac) + (\overline{c} \overline{a}) \overline{b} \right) z
\]
\[
+ b \left[ a, y, \lambda \right] + \overline{c} \left[ b, x, \lambda \right] \left( \lambda - m \right) + \left[ c, x, \lambda \right] \left( \lambda - p \right)
\]
(85)
The term in parentheses on the left-hand-side is just the usual characteristic equation; in particular, we have
\[
\text{tr} \mathcal{A} = p + m + n
\]
\[
\sigma(\mathcal{A}) = pm + pn + mn - |a|^2 - |b|^2 - |c|^2
\]
\[
\det \mathcal{A} = pmn + b(ac) + \overline{b} \overline{ac} - n|a|^2 - m|b|^2 - p|c|^2
\]
(86)

If \( \lambda \) is real, all the associators on the right-hand-side vanish, and we recover the generalized characteristic equation given in [1]. As shown there, the requirement in that case that the right-hand-side be a real multiple of \( z \) (since the left-hand-side is) then constrains \( z \), resulting in precisely 2 values for that real multiple, and reducing (85) to 2 cubic equations, one for each family of real eigenvalues.

While we find the form of (85) attractive, as there are no extraneous terms involving both \( z \) and \( \lambda \), we have so far been unable to further simplify (85) when \( \lambda \) is not real. (It is straightforward to eliminate one of \( x, y \) from this equation, but not both.)

Another possible approach to finding the eigenvalues relies on the associator identity
\[
[v^\dagger, v, \lambda] := (v^\dagger v) \lambda - v^\dagger (v \lambda) \equiv 0
\]
(87)
which follows for any octonionic vector \( v \) and \( \lambda \in \mathbb{O} \) by alternativity, and which is further discussed in the Appendix. If \( v \) is a normalized right eigenvector of \( \mathcal{A} \) with eigenvalue \( \lambda \), then
\[
v^\dagger (\mathcal{A} v) = v^\dagger (v \lambda) = (v^\dagger v) \lambda = \lambda
\]
(88)
which yields an equation for $\lambda$ in terms of $A$ and the components of $v$. After some rearrangement, further details of which can again be found in [20], one obtains

$$\text{Re}(\lambda) = \frac{x \cdot (ay) + z \cdot (bx) + p|x|^2}{|x|^2}$$

(and similar expressions obtained by cyclic permutation) and

$$\text{Im}(\lambda) = [x, a, y] + [z, b, x] + [y, c, z]$$

We had hoped to use these various expressions to impose conditions on $A$ which would in turn enable us to solve for $\lambda$, but have not yet found a way to do so.

6. DISCUSSION

As pointed out in [1], the orthonormality relation (60) is equivalent to assuming that

$$vv^\dagger + \ldots + w w^\dagger = I$$

If we define a matrix $U$ whose columns are just $v, \ldots, w$, then this statement is equivalent to

$$UU^\dagger = I$$

Furthermore, the eigenvalue equation (1) can now be rewritten in the form

$$AU = UD$$

where $D$ is a diagonal matrix whose entries are the eigenvalues. Decompositions of the form (69) now take the form

$$A = (UD)U^\dagger$$

and multiplication of (93) on the right by $U^\dagger$ shows that

$$(AU)U^\dagger = (UD)U^\dagger = A = A(UU^\dagger)$$

Thus, just as in [1], decompositions of the form (69) can be viewed as the assertion of associativity

$$(AU)U^\dagger = A(UU^\dagger)$$

For non-real eigenvalues, we know of no way to express decompositions of the form (59) in similar language, which leads us to suspect that (69) is more fundamental. We further conjecture that the correct notion of orthogonality is (70), not (60), to which it of course reduces if the eigenvalues are real. In any case, it is intriguing that this notion of orthogonality can be written as

$$((Ax) v^\dagger) w = 0$$

which explicitly involves $A$.

Putting these ideas together, it would be natural to conjecture that all eigenvectors of a $3 \times 3$ octonionic Hermitian matrix come in families of 3, which form a decomposition in the sense that (96) is satisfied, and which are orthogonal in the sense of (70). However, based on examples, this conjecture appears to be false [20].

There is, however, another intriguing possibility. The examples considered in [20] suggest that the eigenvectors of $3 \times 3$ octonionic Hermitian matrices may come in sets of 6 (or more), rather than in sets of 3. This would fit nicely with our recent result with Okubo [21] that,
for real eigenvalues, it takes all 6 eigenvectors in order to decompose an arbitrary vector into a linear combination of eigenvectors, despite the fact that only 3 eigenvectors are needed to decompose the original matrix.

We therefore conjecture that, for any 3×3 octonionic Hermitian matrix, (96) should hold when suitably rewritten for a set consisting of $n$ eigenvectors, where $n$ presumably divides 24, the number of (real) independent eigenvectors with real eigenvalues. Whether or in what form orthogonality would hold in such a context is an interesting open question.

APPENDIX

In deriving the foregoing results, we have made use of various associator identities involving octonionic vectors, such as (71) and (87). In this appendix, we derive several such identities, including these two. As we then show, an important application of these identities is to give a particularly elegant derivation of the so-called “3-Ψ’s rule” needed for supersymmetry.

a) Vector Associators

Let $U, V, W$ be arbitrary octonionic vectors, i.e. $1 \times n$ octonionic matrices. Define the vector associator via

$$[U, V, W] := (UV^\dagger)W - U(V^\dagger W)$$

Then direct computation using alternativity establishes

$$[W, V, V] \equiv 0$$

which was used in (71). Setting $W = V$ yields

$$[V, V, V] \equiv 0$$

Furthermore, essentially the same argument also establishes a similar formula when $W$ is replaced by an octonionic scalar $\lambda$, so that

$$[\lambda, V, V] = 0$$

where we have implicitly defined yet another associator, namely

$$[\lambda, V, W] := (\lambda V^\dagger)W - \lambda(V^\dagger W)$$

An interesting consequence of this result is the Hermitian conjugate relation

$$V^\dagger (V\overline{\lambda}) = (V^\dagger V)\overline{\lambda}$$

which is equivalent to (87).

We can polarize (99) to obtain

$$[U, V, W] + [U, W, V] \equiv 0$$

a special case of which is

$$[V, V, W] + [V, W, V] \equiv 0$$

obtained by setting $U = V$. A further special case of (104) is

$$[U, V, W] + [U, W, V] + [V, W, U] + [V, U, W] + [W, U, V] + [W, V, U] \equiv 0$$
obtained by adding cyclic permutations of (104), or alternatively, without requiring (99), by repeated polarization of (100).

b) The 3-Ψ’s Rule

An essential ingredient in the construction of the Green-Schwarz superstring [22,23] is the spinor identity

$$e^{klm} \gamma^\mu \Psi_k \overline{\Psi} \gamma^\mu \Psi_m = 0 \quad (107)$$

for anticommuting spinors $\Psi_k$, $\Psi_l$, $\Psi_m$, where $e^{klm}$ indicates total antisymmetrization. This identity can be viewed as a special case of a Fierz rearrangement. An analogous identity holds for commuting spinors $\Psi$, namely

$$\gamma^\mu \overline{\Psi} \gamma^\mu \Psi = 0 \quad (108)$$

To our knowledge, Schray [11,12] was the first to formally refer to (107) as the 3-Ψ’s rule; we extend this usage to the commuting case (108).

It is well-known that the 3-Ψ’s rule holds for Majorana spinors in 3 dimensions, Majorana or Weyl spinors in 4 dimensions, Weyl spinors in 6 dimensions, and Majorana-Weyl spinors in 10 dimensions. Thus, the Green-Schwarz superstring exists only in those cases [22,23]. As was shown by Fairlie and Manogue [6], the 3-Ψ’s rule in all these cases is equivalent to an identity on the $\gamma$-matrices, which holds automatically for the natural representation of the $\gamma$-matrices in terms of the 4 division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, corresponding precisely to the above 4 types of spinors. Manogue and Sudbery [7] then showed how to rewrite these spinor expressions in terms of $2 \times 2$ matrices over the appropriate division algebra, thus eliminating the $\gamma$-matrices completely.

In the commuting case, the (unpolarized) 3-Ψ’s rule can be written in terms of a 2-component octonionic “vector” (really a spinor) $V$ as [12]

$$(\overline{V} V) V = 0 \quad (109)$$

where

$$\overline{A} := A - \text{tr} A \quad (110)$$

corresponding to time reversal. It is straightforward to check that this equation holds by alternativity. This can also be seen by using the identity

$$\text{tr} (VV^\dagger) = \text{tr} (V^\dagger V) = V^\dagger V \in \mathbb{R} \quad (111)$$

to write

$$(\overline{V} V) V = (VV^\dagger - V^\dagger V) V = (VV^\dagger)V - V(V^\dagger V) = [V, V, V] \quad (112)$$

and we see that the 3-Ψ’s rule for commuting spinors is just (100).

Equivalently, we can rewrite the (polarized) 3-Ψ’s rule in terms of 2-component “vectors” $U$, $V$, $W$ as

$$(\overline{UV} + \overline{VU}) W + (\overline{VW} + \overline{WV}) U + (\overline{WU} + \overline{UW}) V = 0 \quad (113)$$

Using the identity

$$\text{tr} (UV^\dagger + VU^\dagger) = \text{tr} (V^\dagger U + U^\dagger V) = V^\dagger U + U^\dagger V \quad (114)$$

we have

$$\overline{UV} + \overline{VU} = (UV^\dagger + VU^\dagger) - (U^\dagger V + V^\dagger U) \quad (115)$$
where the last term is Hermitian and hence real. We thus have
\[(\tilde{\Psi}^\dagger UV + VU^\dagger)W = (UV^\dagger + VU^\dagger)W - W(U^\dagger V + V^\dagger U)\] (116)

Using this 3 times shows that the 3-Ψ’s rule (113) is precisely the same as (106).

An analogous argument can be given for anticommuting spinors; this is essentially the approach used in [12]. Combining these results, the 3-Ψ’s rule can be written without γ-matrices in terms of 2-component octonionic spinors \(\psi_\alpha\) as

\[\{\psi_1, \psi_2, \psi_3\} \pm \{\psi_1, \psi_3, \psi_2\} + \{\psi_2, \psi_3, \psi_1\} \pm \{\psi_2, \psi_1, \psi_3\} + \{\psi_3, \psi_1, \psi_2\} \pm \{\psi_3, \psi_2, \psi_1\} \equiv 0\] (117)

for both commuting (+) or anticommuting (−) spinors. Both of these identities follow from the identity (100) applied to 2-component octonionic vectors, which is a special case of the more general identity (99) which holds for octonionic vectors of arbitrary rank.

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