ON THE POSITIVITY OF A CERTAIN FUNCTION RELATED WITH THE DIGAMMA FUNCTION

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Abstract. It is proved that

\[
\left( \frac{x^n}{1 - e^{-x}} \right)^{(n)} > 0
\]

for all \( x \in (\log 2, \infty) \) and \( n \in \mathbb{N} \), which improves the result of [Al-Musallam and Bustoz in Ramanujan J. 11 (2006) 399-402].

1. Introduction

A function \( f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R} \) is completely monotonic if it is infinitely differentiable and

\[
(-1)^n f^{(n)}(x) \geq 0
\]

for all \( x \in (a, b) \) and \( n \in \mathbb{N} \). A function \( f(-x) \) is called absolutely monotonic on \((-b, -a)\) if and only if \( f(x) \) is completely monotonic on \((a, b)\). Absolutely monotonic functions were pioneeringly introduced by Bernstein. Bernstein himself, and later Widder independently, discovered that a necessary and sufficient condition for \( f \) to be completely monotonic on \((0, \infty)\) is that

\[
f(x) = \mathcal{L}(\mu)(x) = \int e^{-xt} \, d\mu(t),
\]

where \( \mu \) is a positive measure on \([0, \infty)\) and the integral converges for all positive \( x \). (These and other classical results on absolutely/completely monotonic functions can be found in [8, Chapter IV] and [3].) As it was remarked in [2], by Bernstein’s theorem, it is easy to see that the absolute value of the digamma function, \( \psi = \Gamma'/\Gamma \), and the absolute value of its derivatives (the polygamma functions) are completely monotonic functions on \((0, \infty)\). Indeed,

\[
(-1)^{n+1} \psi^{(n)}(x) = \mathcal{L} \left( \frac{t^n}{1 - e^{-t}} \right)(x)
\]

for all \( x \in (0, \infty) \) and \( n \in \mathbb{N} \).

In [4] Clark and Ismail introduced the functions

\[
F_m(x) = x^m \psi(x), \quad G_m(x) = -x^m \psi(x).
\]

They proved that \( F_m^{(m+1)} \) is completely monotonic on \((0, \infty)\) for \( m \in \mathbb{N} \setminus \{0\} \) [4 Theorem 1.2] and that \( G_m^{(m)} \) is completely monotonic on \((0, \infty)\) for \( m = 1, 2, \ldots, 16 \).

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Theorem 1.3]. Afterwards, they wrote: “We believe Theorem 1.3 \( G'(m) \) is completely monotonic on \((0, \infty)\) is true for all \( m \) \[\ldots\]”. However, Alzer, Berg, and Kommandos \[2, Theorem 1.1\] proved that there exists an integer \( m_0 \) such that for all \( m \geq m_0 \) the function \( G'(m) \) is not completely monotonic. From this and the relation \[4 (2.4)\]

\[
G'(m)(x) = L\left(\frac{t^m}{1 - e^{-t}}\right)(x),
\]

it follows that the following conjecture of Clark and Ismail \[4, Conjecture 1.4\] is false:

**Conjecture.**

\[
\left(\frac{x^n}{1 - e^{-x}}\right)^{(n)} > 0
\]

for all \( x \in (0, \infty) \) and \( n \in \mathbb{N} \).

By showing that \(2\) holds for \( n = 1, 2, \ldots, 16 \) (and using \(1\)), Clark and Ismail proved that \( G'(n) \) is (strictly) completely monotonic on \((0, \infty)\) for these values of \( n \). Regardless of the fact that the conjecture is not true, the inequality \(2\) is of interest in its own right. It remains an open problem to determine the smallest positive number \( a \) (positive integer \( n_0 \)) such that \(2\) remains positive for all \( x \in (a, \infty) \) and \( n \in \mathbb{N} \ (x \in (0, \infty) \) and \( n > n_0 \) with \( n \in \mathbb{N} \). (This open problem was also placed in \[2, Section 4\].) In \[1\] Theorem 2.1, Al-Musallam and Bustoz proved that \(2\) holds for all \( x \in (2 \log 2, \infty) \) and \( n \in \mathbb{N} \). (This was also proved independently in \[2, p. 112\] using the same idea: an inequality proved by Szegö \[8, Theorem 17a, p. 168\].) Our main theorem, which improves the result in \[1\], reads as follows:

**Theorem.** \(2\) holds for all \( x \in (\log 2, \infty) \) and \( n \in \mathbb{N} \).

As in \[1, Theorem 3.1\], now using the above theorem, the next result follows. (The details are left to the reader.)

**Corollary.**

\[
\left(\frac{x^{n+\alpha}}{1 - e^{-x}}\right)^{(n)} > 0
\]

for all \( \alpha \in (0, \infty) \), \( x \in (\log 2, \infty) \), and \( n \in \mathbb{N} \).

**Example.** It is easy to obtain from \[5, (1), p. 11\], for \( n \in \mathbb{N} \setminus \{0\} \), the power series

\[
\left(\frac{x^n}{1 - e^{-x}}\right)^{(n)} = \frac{n!}{2} + \sum_{j=2}^{\infty} \frac{(j + n - 1)!}{j!} B_j x^{j-1}
\]

valid in the disk \(|x| < 2\pi\) which extends to the nearest singularities \( x = \pm 2\pi i \) of \( x/(e^x - 1) \). (The coefficients \( B_j \) are the Bernoulli numbers. The odd Bernoulli numbers are all zero after the first, but it is a highly complex task to determine the
even Bernoulli numbers.) Let us imagine that we are questioned about the sign of the following sum:

\[ S_n = \frac{(n + 0)!}{0! \cdot 1!} B_0 + \frac{(n + 1)!}{1! \cdot 2!} B_2 + \frac{(n + 3)!}{3! \cdot 4!} B_4 + \frac{(n + 5)!}{4! \cdot 5!} B_6 + \cdots \]

\[ = n! + \sum_{j=1}^{\infty} \frac{(2j + n - 1)!}{(2j - 1)!} \frac{B_{2j}}{(2j)!}. \]

(Recall that \( B_0 = 1 \).) Note that

\[ \left( \frac{x^n}{1 - e^{-x}} \right)^{(n)} \bigg|_{x=1} = S_n - \frac{n!}{2}. \]

Since \( \log 2 \approx 0.693147 < 1 < 2\pi \), our main results gives

\[ S_n > n!/2 \]

\( n \in \mathbb{N} \setminus \{0\} \). It is worth pointing out that from the results obtained in [1, 2], it is not possible to conclude this because \( 2 \log 2 \approx 1.38629 > 1 \). Now it only remains to check that \( S_n \) converges, which follows from

\[ \lim_{j \to \infty} 2j \sqrt{\frac{(2j + n - 1)!}{(2j - 1)!} \frac{|B_{2j}|}{(2j)!}} = \frac{1}{2\pi} < 1. \]

In [2] the relation of the function given in (2) with a function of Hardy and Littlewood was extensively explored.

2. PROOF OF THE THEOREM

Set

\[ f_n(x) = \frac{d^n}{dx^n} \left( \frac{x^n}{1 - e^{-x}} \right). \]

If \( c > 0 \) is arbitrary and fixed, the series

\[ \frac{1}{1 - e^{-x}} = \sum_{j=0}^{\infty} e^{-jx}, \]

converges uniformly on \([c, \infty)\). We then write \( f_n \) in the form

\[ f_n(x) = \sum_{j=0}^{\infty} \frac{d^n}{dx^n} \left( e^{-jx}x^n \right). \]

Recall that [6] (5), p. 188] \( n!L_n(x) = e^x(d^n/dx^n)(e^{-x}x^n) \), \( L_n \) being the Laguerre polynomial of degree \( n \), and so

\[ n!e^{-jx}L_n(jx) = \frac{d^n}{dx^n} \left( e^{-jx}x^n \right). \]

Hence

\[ f_n(x) = n! \sum_{j=0}^{\infty} L_n(jx)e^{-jx} \]
on \([c, \infty)\). There is a well-known connection between the Laguerre and Hermite polynomials due to Feldheim \([6, (33), p. 195]\):

\[
\int_0^\infty e^{-t^2} H_n^2(t) \cos(2^{1/2} y t) \, dt = \sqrt{\pi} 2^{n-1} n! L_n(y^2),
\]

\(H_n\) being the Hermite polynomial of degree \(n\). Write

\[y^2 = j x.\]

From the above expressions, we have

\[
f_n(x) = \frac{1}{\sqrt{\pi} 2^{n-1}} \sum_{j=0}^{\infty} \int_0^\infty g_j(t) \, dt,
\]

where

\[g_j(t) = e^{-t^2} e^{-j x} H_n^2(t) \cos(\sqrt{2} j x t)\]

for all \(x \in [c, \infty)\). (Recall that it is not true that uniform convergence is sufficient to allow the interchanging of the sum and integral when the integral is over an infinity interval.) However, the function \(g_j\) is integrable and

\[
\sum_{j=0}^{\infty} \int_0^\infty |g_j(t)| \, dt < 0,
\]

for all \(x \in [c, \infty)\). These conditions allow the interchanging of the above sum and integral, see, for instance, \([7, Corollary 17.4.7]\). Indeed, since

\[|g_j(t)| < e^{-t^2} e^{-j x} H_n^2(t),\]

we see at once that

\[
\int_0^\infty |g_j(t)| \, dt < e^{-j x} \int_0^\infty e^{-t^2} H_n^2(t) \, dt
\]

\[< e^{-j x} \int_{-\infty}^\infty e^{-t^2} H_n^2(t) \, dt \leq \sqrt{\pi} 2^n n! e^{-j c} < \infty,
\]

and the integrability of \(g_j\) is guaranteed. Moreover,

\[
\sum_{j=0}^{\infty} \int_0^\infty |g_j(t)| \, dt < \sqrt{\pi} 2^n n! \sum_{j=0}^{\infty} e^{-j x}
\]

\[\leq \sqrt{\pi} 2^n n! \frac{e^c}{e^c - 1} < \infty.
\]

Consequently, we can interchange the sum and integral to obtain

\[
\sqrt{\pi} 2^{n-1} f_n(x) = \int_0^\infty e^{-t^2} H_n^2(t) \sum_{j=0}^{\infty} e^{-j x} \cos(\sqrt{2} j x t) \, dt.
\]

Finally, note that

\[
\sum_{j=0}^{\infty} e^{-j x} \cos(\sqrt{2} j x t) > 1 - \sum_{j=1}^{\infty} e^{-j x} = 1 - \frac{1}{e^x - 1} = g(x).
\]

Thus \(g(x) \geq 0\) if and only if \(x > \log 2\). This completes the proof.
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