A combinatorial version of Sylvester’s four-point problem

Gregory S. Warrington
gwarring@cem.s.uvm.edu

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Abstract

J. J. Sylvester’s four-point problem asks for the probability that four points chosen uniformly at random in the plane have a triangle as their convex hull. Using a combinatorial classification of points in the plane due to Goodman and Pollack, we generalize Sylvester’s problem to one involving reduced expressions for the long word in \( S_n \). We conjecture an answer of \( 1/4 \) for this new version of the problem.

Remark 1. O. Angel and A. E. Holroyd [1] prove a more general result which implies our Conjecture 2.

1 Introduction

Fix \( n \geq 4 \). Pick uniformly at random a reduced expression \( w \) for the long word in \( S_n \). Then pick uniformly at random a 4-subset of \( \{1, \ldots, n\} \). This pair of choices determines a reduced expression \( v \) for the long word in \( S_4 \). What is the probability that

\[ v \in X := \{s_1s_2s_3s_1s_2, s_3s_2s_1s_2s_3s_2, s_2s_1s_2s_3s_2s_1, s_2s_3s_2s_1s_2s_3\}? \quad (1) \]

Conjecture 2. For any \( n \geq 4 \), the probability is \( 1/4 \).

For \( n = 4 \), the probability of \( 1/4 \) can be computed directly from the possible \( w \) listed in Figure 1. The conjecture can be checked for small values of \( n \) by computer:

- 5 points: \( \frac{960}{4768} = \frac{1}{4} \)
- 6 points: \( \frac{1,098,240}{292,864} = \frac{1}{4} \)
- 7 points: \( \frac{9,631,498,240}{1,100,742,656} = \frac{1}{4} \)
- 8 points: \( \frac{850,653,924,556,800}{48,608,795,688,960} = \frac{1}{4} \).

The computer check for \( n = 8 \) took several thousand hours on 3GHz CPUs.

Remark 3. The behavior of arbitrary reduced expressions for the long word in \( S_n \) (also known as sorting networks) has been considered in a number of contexts. Most notably, Angel et al. [2] consider several convergence questions. V. Reiner [11] computes the expected number of possible Yang-Baxter moves for such a reduced expression in the symmetric group while B. Tenner [14]...
performs the analogous calculation for the hyperoctahedral group. The above results all use connections to the theory of Young tableaux (see [4, 12]). Through the Goodman-Pollack correspondence [5], described here in Section 2, there are also connections to halving lines and k-sets of points in the plane (see, e.g., [8, 15]). The nature of the connections between Conjecture 2 and these other works is still unclear.

## 2 Motivation

Conjecture 2 combines a question of J.J. Sylvester with a combinatorial classification of points in the plane due to Goodman and Pollack.

In 1864, J.J. Sylvester [13] posed the following

**Problem 4.** Given four points chosen at random in the plane, find the probability that they form a reentrant (rather than convex) quadrilateral.

As was quickly realized at the time, the problem as stated is ill-posed; it does not specify how the plane is being modeled. Woolhouse (see [6]) realized that it sufficed to pick points in a closed, bounded region. The probability would be left invariant under the scaling necessary to model the plane. In fact, the exact probability is computable by integration for any convex region. Among convex regions, a triangle maximizes the probability at a value of 1/3 and a disk minimizes it at $35\pi^2/12 \approx 0.295$. The reader is referred to Pfeifer [10] for an excellent history of the problem.

Less than twenty years later, R. Perrin [9] considered the matter of combinatorially classifying collections of n points in general position in the plane. More recently, this work has been extended by Goodman and Pollack [5] in the form of allowable sequences (see also Knuth [7, §8]). There are several variations on Goodman and Pollack’s allowable sequences one can work with. Our setup is as follows.

For $n \geq 1$, let $\mathcal{P}_n$ denote the set of all possible n-tuples of points in the plane in general position. We consider the symmetric group $S_n$ on $\{1, 2, \ldots, n\}$ as generated by the adjacent transpositions $s_i = (i, i + 1)$ for $1 \leq i < n$. For any $w \in S_n$, a reduced expression for $w$ is a word in the $s_i$’s of minimum length among all those whose product equals $w$. Given $w \in S_n$, let $\mathcal{R}(w)$ to be the set of all possible reduced expressions for $w$. For the long word $w_0 = [n, n-1, \ldots, 2, 1]$, we now define a map $\phi : \mathcal{P}_n \to \mathcal{R}(w_0)$. The map $\phi$ depends on a fixed directed line $\ell$ in the plane which can be chosen arbitrarily. Denote by $\ell(\theta)$ the line $\ell$ rotated counterclockwise through $\theta$ radians about some point on $\ell$.

Pick $P \in \mathcal{P}_n$. For each pair of distinct points $i, j \in P$, there is a unique angle $\theta' \in [0, \pi)$ for which $\ell(\theta')$ is orthogonal to the line through $i$ and $j$. Running over all unordered pairs of points, these $\theta'$ define a sequence

$$0 \leq \theta_1 < \theta_2 < \cdots < \theta_{\binom{n}{2}} < \pi. \tag{2}$$

We now label the points according to their projections onto $\ell$. (In the probability-zero case that $\theta_1 = 0$, we instead project onto $\ell(-\epsilon)$ for some sufficiently small $\epsilon$.) For each angle $\theta$ not occurring in the sequence (2), the projection of $P$ onto $\ell(\theta)$ determines a permutation $\pi_\theta$ with respect to the initial labeling. Write $\pi_{\theta_0} = [1, 2, \ldots, n]$. We set $\phi(P) := w = w_1 w_2 \cdots w_{\binom{n}{2}}$ where $w_k = s_i$ if $\pi_{\theta_{k+1}} \pi_{\theta_k}^{-1} s_i$ for $\delta = (\theta_{k+1} - \theta_k)/2$.

We illustrate $\phi(\mathcal{P}_4)$ in Figure 1. Each expression $w \in \mathcal{R}(w_0)$ is written as a string diagram (see, e.g., [3]). To clarify conventions, we note that the first reduced expression in the first row of the figure is $s_2 s_3 s_2 s_1 s_2 s_3$. Below each $w$ is a configuration of points $P$ for which $\phi(P) = w$ (assuming
$\ell$ is horizontal and directed to the right). It is worth pointing out that Goodman and Pollack [7] show that $\phi$ is not surjective for $n \geq 5$.

Figure 1: Reduced expressions for $w_0 \in S_4$.

The reentrant configurations in the first two columns of Figure 1 correspond to the reduced expressions comprising the set $X$ in (4).

3 Combinatorial Version

We now rephrase Sylvester’s problem in terms of the elements of $R(w_0)$. Fix a closed, bounded region $R$ in the plane along with a directed line $\ell$. The uniform distribution on $R$ induces a probability distribution $f_n$ on $\phi(P_n)$ for each $n$. We can think of picking four points at random from $R$ as picking one of the 16 elements of $R([4, 3, 2, 1])$ according to $f_n$. The distribution $f_n$ is displayed for several different shapes in Figure 2. For brevity in the figure, we have identified the two rows of Figure 1. (Entries in the same column of Figure 1 correspond to choosing the opposite direction for the line $\ell$ and will therefore appear with equal frequencies in $f_n$.)

Figure 2: Monte Carlo simulations for three regions (1000 trials).

For the general, combinatorial version of this problem, we dispense with the geometry of our region $R$. We start with the uniform distribution on $R(w_0)$ rather than the geometrically induced
distribution $f_n$.

**Problem 5.** For $n \geq 4$ and $w_0 \in S_n$, pick $w \in \mathcal{R}(w_0)$ uniformly at random and a four-subset \{a, b, c, d\} $\subseteq \{1, 2, \ldots, n\}$ uniformly at random. The pair of choices induces a reduced expression $v$ for the long word in $S_{\{a,b,c,d\}} \cong S_4$. Find the probability that $v \in X$.

An example of the procedure of Problem 5 is illustrated in Figure 3.

![Figure 3: Two-stage combinatorial version illustrated for n = 7.](image)

**Example 6.** R. Stanley [12] showed that $|\mathcal{R}(w_0)|$ equals the number of standard Young tableaux of staircase shape $(n-1, n-2, \ldots, 1)$. Applying the Frame-Robinson-Thrall hook formula for the number of such tableaux, we find that there are 768 such reduced expressions when $n = 5$. They fall into three classes according to how many of the $5 \choose 4$ four-tuples lead to $v$ being in $X$. Figure 4 displays the size of each class, a representative, and a point configuration realizing one of the reduced expressions in the class. (Recall that some of the expressions are not realizable; the point configurations are for intuition only.)

![Figure 4: Computation of probability for n = 5.](image)

From this data, we can calculate

$$\Pr(v \text{ reentrant}) = \frac{40 \cdot 4 + 400 \cdot 2 + 328 \cdot 0}{5 \choose 4} \left(40 + 400 + 328\right) = \frac{1}{4}. \quad (3)$$

Similar computations can be made on computer for $n \in \{6, 7, 8\}$.

**Remark 7.** As can be seen in Figure 1, the reduced expressions fall into four natural classes. In terms of points in the plane, these are generated by the direction of rotation (clockwise or counterclockwise) and the choice of direction for the directed line $\ell$. A natural assumption is that Conjecture 2 holds for the three “convex” classes as well. This is not true. For example, with $n = 5$, the number of pairs $(w, v)$ falling into the each of the four classes is, respectively, 960, 980, 972 and 928.
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