Chaotic dynamics of charged particles in the field of a finite non-uniform wave packet

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A generalization of the Chirikov-Taylor model is introduced to study the dynamics of a charged particle in the field of an electrostatic wave packet with an arbitrary but finite number of harmonics. The dependence of both the edge of chaos (dissipative regime) and the deterministic diffusion on the wave packet width is predicted theoretically and confirmed numerically, including the case of relativistic particles. Diverse properties of the standard maps are shown to be non-universal in the framework of the wave-particle interaction, because these maps correspond to an infinite number of waves.

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During the past quarter of century the Hamiltonian [1] and dissipative [2] versions of the so-called standard map (SM) have been widely studied as basic models of the dynamics of charged particles in the field of an electrostatic wave packet (see, e.g., refs. [3-4]):

$$\ddot{x} + \gamma \dot{x} = -\frac{e}{m_e} \sum_{n=-N}^{N} E_n \sin (k_n x - \omega_n t) ,$$  \hspace{1cm} (1)

where $E_n$, $k_n$, and $\omega_n$ are the amplitudes, wave numbers, and frequencies, respectively, of the $(2N+1)$ plane waves, and $\gamma$, $e$ and $m_e$ are the damping coefficient, the charge and the mass of the particle, respectively. Specifically, the SM describes the particular case of an infinite set of waves having the same amplitudes, same wave numbers, and integer frequencies. While its diffusion properties ($\gamma \equiv 0$) have been shown to be non-universal in the sense that it does not generalize to a wave spectrum with uncorrelated phases [5], its assumption of an infinite and uniform amplitude distribution also seems quite restrictive since it does not permit one to study the sensitivity of the dynamics to changes in the wave packet
width. Physically, this sensitivity yields changes in the properties and structures of the phase space, as can be appreciated when comparing, e.g., the cases of two waves [6-9] and infinite waves [4]. In this Letter a generalized model of the wave packet structure is introduced to take into account such a finite-size effect on the particle dynamics. Specifically, it is assumed that \( k_n = k_0, \omega_n = \omega_0 + n\Delta \omega, E_n = E_n(m) \equiv E_0 \text{sech} [n\pi K(1 - m)/K(m)] \), where \( K(m) \) is the complete elliptic integral of the first kind, i.e., a sech distribution is assumed for the amplitudes such that the effective width is controlled by a single parameter: the elliptic parameter \( m \). This specific form of \( E_n(m) \) is motivated by the following properties:

(i) \( E_n(m = 0) = E_0 \delta_{n0}, \) with \( \delta_{n0} \) being the Kronecker delta, i.e., one recovers the (non-chaotic) limiting case of a single plane wave. (ii) \( E_n(m = 1) = E_0, \forall n, \) i.e., one recovers the limiting case described by the SM. (iii) For any \( m \in [0, 1) \), one may define an effective number of harmonics forming the wave packet as follows. Let us choose quite freely a real number \( \zeta \in (0, 1) \) such that \( N_{\text{eff}} \) is the largest integer satisfying \( E_{N_{\text{eff}}} / E_0 \geq \zeta \), then \( E_n / E_0 < \zeta, \forall n > N_{\text{eff}}, \) i.e., \( N_{\text{eff}} = N_{\text{eff}}(m) \equiv \lceil K(m) \cosh^{-1} (1/\zeta) / (\pi K(1 - m)) \rceil + 1 \) where the brackets stand for the integer part. Thus, for this choice of the wave packet structure, one has \( \sum_{n = -N}^{N} E_n \sin (k_n x - \omega_n t) = \sin (k_0 x - \omega_0 t) \sum_{n = -N_{\text{eff}}}^{N_{\text{eff}}} E_n(m) \cos (n\Delta \omega t) \) and, after extending the summation from \( -\infty \) to \( \infty \), eq. (1) transforms into the form

\[
\ddot{x} + \gamma \dot{x} = -\frac{eE_0}{m_e} \sin (k_0 x - \omega_0 t) D(t; T, m),
\]

where \( T \equiv 2\pi/\Delta \omega \) is the characteristic period of the field and \( D(t; T, m) \equiv 2K(m) \text{dn} [2K(m)t/T; m] / \pi \) [10], with \( \text{dn} \) being the Jacobian elliptic function of parameter \( m \). Note that this approximation does not imply a loss of generality at variance with the case of the SM. In a reference frame moving with the main wave, eq. (2) transforms into the equation

\[
\frac{d^2 \xi}{d\tau^2} + \sin \xi = -\delta - \eta \frac{d\xi}{d\tau} - [D(\tau; \alpha, m) - 1] \sin \xi, \tag{3}
\]

where \( \xi \equiv k_0 x - \omega_0 t, \Omega_0 \equiv (e k_0 E_0/m_e)^{1/2}, \tau \equiv \Omega_0 t, \delta \equiv \gamma \omega_0 \Omega_0^{-2}, \eta \equiv \gamma \Omega_0^{-1}, \) and \( \alpha \equiv \Omega_0 T \) are all dimensionless variables and parameters. It is worth noting that \( \alpha \equiv \sqrt{k} \), where \( k \) is the stochasticity parameter of the SM [9]. The parameters \( k_0, \omega_0, E_0, \) and \( \Omega_0 \) are held constant throughout. Physically, eq. (3) represents a damped pendulum subjected to a periodic string of finite pulses having an effective width and an amplitude controlled by \( m \). Thus, model (2) permits one to study the structural stability of the system under changes in the width of the wave packet by solely varying the parameter \( m \) (and hence \( N_{\text{eff}} \)) between
0 and 1. In particular, the case of a single plane wave \((m = 0)\) is described by a purely damped pendulum, while the case of an infinity of plane waves \((\text{SM}: m = 1)\) is described by a delta-kicked rotator. The dependence of the chaotic dynamics’ features on the effective number of harmonics will be considered here.

**Dissipative Regime:** Consider first the case of weak dissipation \((0 < \gamma \ll 1)\). Since \(D(\tau; \alpha, m) - 1 = (2/E_0) \sum_{n=1}^{\infty} E_n(m) \cos[2n\pi\tau/\alpha]\), eq. (3) may be regarded as a perturbed pendulum \((0 < \delta, \eta \ll 1)\) for \(m \in [0, 1)\) and then one can apply Melnikov’s method (MM) \([11-13,3]\) to obtain an analytical estimate of the edge of chaos in the parameter space. The application of MM to eq. (3) gives the Melnikov function \(M^\pm(\tau_0) = -D^\pm + (16\pi^3/E_0) \sum_{n=1}^{\infty} n^2E_n(m)b_n(\alpha) \sin(2n\pi\tau_0/\alpha)\), with \(D^\pm = 8\eta \pm 2\pi\delta, b_n(\alpha) \equiv \alpha^{-2} \csch(n\pi^2/\alpha)\), and where the positive (negative) sign refers to the top (bottom) homoclinic orbit of the underlying conservative pendulum. It is well known that the simple zeros of the Melnikov function imply transversal intersections of stable and unstable manifolds (i.e., a homoclinic bifurcation occurs), giving rise to Smale horseshoes and hence hyperbolic invariant sets \([12]\). From \(M^\pm(\tau_0)\) one sees that a homoclinic bifurcation (signifying the possibility of chaotic behavior) is guaranteed for trajectories whose initial conditions are sufficiently close to the separatrix of the underlying conservative pendulum if \(|2\eta \pm \pi\delta/2| < U(m, \alpha) \equiv (4\pi^3/E_0) \sum_{n=1}^{\infty} n^2E_n(m)b_n(\alpha)\), where \(U(m, \alpha)\) is the chaotic threshold function. It is straightforward to obtain the following properties: \((i)\) \(U(m, \alpha)\) increases, as a function of \(m\), as \(m\) is increased, i.e., the possibility of chaos increases as the spectral width is increased; \((ii)\) \(U(m, \alpha \to 0, \infty) = 0\), i.e., for any spectral width, the possibility of chaos diminishes when the small-amplitude frequency of the non-perturbed equivalent pendulum \((d^2\xi/dt^2 + \Omega_0^2 \sin \xi = 0)\) is much higher or much lower than the characteristic spectral frequency; \((iii)\) \(U(m, \alpha)\) presents a maximum, as a function of the parameter \(\alpha\), at \(\alpha_{\text{max}} \equiv \alpha_{\text{max}}(m), \forall m \in (0, 1)\), such that \(\alpha_{\text{max}}(m)\) is a monotonously increasing function and exhibits the asymptotic behavior \(\alpha_{\text{max}}(m \to 1) \sim \ln (1 - m)^{-1/2}\); and \((iv)\) \(U(m \to 1, \alpha) = 4\pi^3 \sum_{n=1}^{\infty} n^2b_n(\alpha) > U(m, \alpha), \forall m \in [0, 1)\), i.e., it is expected that the limiting case \(m = 1\) be maximal with respect to the extension of dissipative chaos in parameter space. Note that properties \((iii)\) and \((iv)\) mean that the dissipative SM is non-universal in the framework of the wave-particle interaction, because this map corresponds to an infinite set of waves having the same amplitudes, and hence it does not present the aforementioned maximum. Extensive Lyapunov exponent (LE) calculations of eq. (3) are coherent with
properties (i)-(iv). One typically finds that the maximal LE, $\lambda^+$, presents a maximum as a function of $\alpha$ at $\alpha_{\text{max}}^* \equiv \alpha_{\text{max}}^*(m)$ and that both $\lambda^+(\alpha_{\text{max}}^*)$ and $\alpha_{\text{max}}^*$ increase with the spectral width in accordance with the MM-based predictions (It is worth mentioning that one cannot expect too good a quantitative agreement between $\alpha_{\text{max}}(m)$ and $\alpha_{\text{max}}^*(m)$ because MM is a perturbative method generally related to transient chaos, while LE provides information concerning solely steady chaos [12]). Figure 1 shows an illustrative example for three increasing values of the spectral width.

**Dissipationless Regime:** When dissipation is negligible ($\gamma = 0$), system (3) is generated by the Hamiltonian $H(\xi, p_\xi, \tau) = p_\xi^2/2 - D(\tau; \alpha, m) \cos \xi, p_\xi \equiv k_0 x / \Omega_0$, and MM provides an estimate of the width of the stochastic layer generated around the unperturbed separatrix [14]: $d(m, \alpha) = U(m, \alpha)/2 \equiv (2\pi^3/E_0) \sum_{n=1}^{\infty} n^2 E_n(m) b_n(\alpha)$. Thus, the aforementioned properties of the chaotic threshold function hold for the width of the stochastic layer and hence one expects an intensification of the deterministic diffusion as the spectral width is increased. Figures 2(b, d, f) provide an illustrative sequence for three increasing values of $m$ at $\alpha \simeq \alpha_{\text{max}}^*(m)$, respectively. Moreover, one typically finds the gradual disappearance of the invariant curves region inside the separatrix cell (cf. figs. 2(c, b, d, f)). Numerical simulations also confirm that the stochastic layer exhibits a maximal width as a function of $\alpha$ at a value close to $\alpha_{\text{max}}^*(m)$. This can be appreciated in the sequence of figs. 2(a, c, e).

Numerical simulations also indicate that another difference with respect to the SM is that the phase space of the Hamiltonian $H(\xi, p_\xi, \tau)$ is bounded by Kolmogorov-Arnold-Moser (KAM) tori at any values of $\alpha$ and $m \in (0, 1)$, i.e., global stochasticity is not possible for any arbitrary but finite spectral width (notice that, contrary to the SM, the phase space of the Hamiltonian $H(\xi, p_\xi, \tau)$ is not periodic in $p_\xi$). One thus concludes that, in the context of the wave-particle interaction, such a transition to global stochasticity is a peculiarity of the SM.

**Relativistic Regime:** Similarly to the case of the relativistic standard map (RSM) [15,16], the relativistic generalization of model (2) ($\gamma \equiv 0$) is necessary if the acceleration of particles is sufficiently large. The relativistic equations corresponding to model (2),

\[
\dot{x} = pc^2 (m_{e0} c^4 + p^2 c^2)^{-1/2}, \quad \dot{p} = -e E_0 \sin (k_0 x - \omega_0 t) D(t; T, m),
\]

where $p$, $m_{e0}$, and $c$ are the momentum and the rest mass of the particle, and the light velocity, respectively, may be
conveniently transformed into the dimensionless form

\[ \frac{d\xi}{d\tau} = \frac{p_{\xi}}{\sqrt{1 + \epsilon p_{\xi}^2}} - \Gamma, \]

\[ \frac{dp_{\xi}}{d\tau} = -\sin \xi D(\tau; \alpha, m), \]

(4)

where \( \epsilon \equiv (\Omega_0 / k_0)^2 / c^2 \) is the relativistic parameter, \( p_{\xi} \equiv k_0 p / (m e_0 \Omega_0) \) is the dimensionless momentum, and \( \Gamma \equiv \omega_0 / \Omega_0 \). To also characterize the relativistic dynamics in the coordinate-velocity phase space, \( (\xi, d\xi/d\tau) \), eq. (4) is rewritten as a second-order differential equation:

\[ \frac{d^2 \xi}{d\tau^2} = -D(\tau; \alpha, m) \sin \xi \left[ 1 - \epsilon (d\xi/d\tau + \Gamma)^2 \right]^{3/2}. \]

A relativistic effect is the reduction of the system symmetry: eq. (4) presents two mirror symmetries with respect to \( \xi = 0 \) and \( p_{\xi} = \Gamma \), respectively, for non-relativistic particles, while it solely presents the former symmetry for relativistic particles when \( \Gamma > 0 \). Another relativistic effect is the modification of the fixed points existing in the classical (Newtonian) regime: \( (\xi, p_{\xi}) = \{ (0, \Gamma / \sqrt{1 - \epsilon \Gamma^2}), (\pm \pi, \Gamma / \sqrt{1 - \epsilon \Gamma^2}) \} \). Two limiting cases may be distinguished: \( \epsilon \ll \Gamma^{-2} \) and \( \epsilon \lesssim \Gamma^{-2} \). Physically, the resonance condition \( \epsilon = \Gamma^{-2} \) means that the phase velocity of the main wave \( (\omega_0 / k_0) \) is equal to the velocity of light. In the former case, when the phase velocity differs significantly from the velocity of light, it is found numerically that the stochastic motion is restricted to the neighborhood of the fixed points, such that the corresponding chaotic layers are bounded by KAM tori at any values of \( \alpha \) and \( m \in (0, 1) \) (see figs. 3 and 4). For relativistic particles just beyond the Newtonian regime \( (\epsilon \gtrsim 0) \), it is found numerically that the extension of the stochasticity regions in the coordinate-momentum phase space for relativistic and nonrelativistic particles, respectively, is approximately the same, while it diminishes relatively in the coordinate-velocity phase space for relativistic particles, as can be appreciated in the examples shown in figs. 4 and 3, respectively. Clearly, this shrinkage is a consequence of the relativistic constraint \( |d\xi/d\tau| \leq \epsilon^{-1/2} - \Gamma \) (i.e., \( |\dot{x}| \leq c \)). Note that a proper comparison of the structures appearing in the two phase spaces would require taking the initial momentum as \( p_{\xi}(0) \equiv [(d\xi/d\tau)(0) + \Gamma] \left\{ 1 - \epsilon [(d\xi/d\tau)(0) + \Gamma]^2 \right\}^{-1/2} \), where \( (d\xi/d\tau)(0) \) is the corresponding initial velocity. A preliminary quantitative estimate of such a relativistic effect in the coordinate-velocity phase space can be obtained from the equation giving the first-order relativistic correction:

\[ \frac{d^2 \xi}{d\tau^2} + \sin \xi = -[D(\tau; \alpha, m) - 1] \sin \xi + \frac{3}{2} \epsilon D(\tau; \alpha, m) (d\xi/d\tau + \Gamma)^2 \sin \xi + O(\epsilon^2). \]

Similarly to eq. (3), this
equation may be considered as a perturbed pendulum for \( m \in [0, 1) \) and, after applying MM to it, one straightforwardly obtains the following expression for the width of the stochastic layer: 
\[
d_R(m, \alpha, \epsilon) = \frac{U(m, \alpha, \epsilon)}{2} \equiv \frac{2\pi^3}{E_0} \sum_{n=1}^{\infty} n^2 E_n(m) \left[ b_n(\alpha) - \epsilon r_n(\alpha) \right],
\]
where \( r_n(\alpha) \equiv [2\alpha^{-2} + n^2 \pi^2 \alpha^{-4}/2] \csc \left( \frac{n\pi^2}{\alpha} \right) \) and \( U_R(m, \alpha, \epsilon) \) is the first-order relativistic threshold function. One obtains that, for fixed \( m, \alpha \), the relativistic width \( d_R \) decreases as \( \epsilon \) is increased, and that this decrease is ever more noticeable as \( m \) is increased. Numerical results confirm these two predictions, as in the example shown in fig. 3. Proceeding similarly to obtain a first-order relativistic correction in the coordinate-momentum phase space by expanding the square root in eq. (4), the MM yields a null correction to the Newtonian width because the corresponding integrals vanish. Numerical results confirm this prediction, as in the example shown in fig. 4. As the phase velocity approaches the velocity of light \( (\epsilon \lesssim \Gamma^{-2}) \), the momentum of the fixed points increases continuously, and the KAM torus limiting the stochastic region from above expands into the higher-momentum region (see fig. 5(a)). When the resonance condition \( \epsilon = \Gamma^{-2} \) is met exactly, the particles can be accelerated to very high energies, which is indicated by the vertical traces in fig. 5(b). This mechanism of particle acceleration [17,18] is also observed in the RSM for the case \( \omega_0 = 2\pi l/T \), with \( l \) being an integer [15,16]. Notice that, far from the resonance condition, the analysis of the RSM indicated [15,16] a relativity-induced decrease of the stochasticity in the coordinate-momentum phase space. However, the above findings clearly show that such a property does not hold for finite wave packets.

In sum, a generalized model of the dynamics of a charged particle in the field of an electrostatic wave packet with an arbitrary but finite number of harmonics has been presented. The dependence of both the chaotic threshold (dissipative regime) and the deterministic diffusion on the spectral width was predicted theoretically and confirmed numerically, including the case of relativistic particles. Several properties of the SM and the RSM were shown to be non-universal in the framework of the wave-particle interaction, because these maps correspond to an infinite number of waves.

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A. Figure Captions

Figure 1. Maximal LE vs parameter $\alpha$ for three $m$ values: $1-10^{-4}$, $1-10^{-6}$, and $1-10^{-8}$, which correspond to $N_{\text{eff}} = 5$, 7, and 8, respectively, for $\zeta = 0.05$. Vertical arrows indicate the position of the respective maxima. System parameters: $\delta = \eta = 0.1$. 
Figure 2. Trajectories in the stroboscopic (for $\tau = n\alpha$ with $n = 0, \ldots, 400$) Poincaré section ($p_\xi$ vs $\xi/\pi$) for the system (3) with no dissipation ($\delta = \eta = 0$), and (a) $m = 0.5, \alpha = 1.4$, (b) $m = 0.95, \alpha = 5.95 \simeq \alpha_{\text{max}}(m = 0.95)$, (c) $m = 0.5, \alpha = 5.3 \simeq \alpha_{\text{max}}(m = 0.5)$, (d) $m = 1 - 10^{-6}, \alpha = 11.7 \simeq \alpha_{\text{max}}(m = 1 - 10^{-6})$, (e) $m = 0.5, \alpha = 7.9$, and (f) $m = 1 - 10^{-14}, \alpha = 23.5 \simeq \alpha_{\text{max}}(m = 1 - 10^{-14})$. For fixed $\zeta = 0.05$, one has $N_{\text{eff}} = 2, 3, 7, 14$ for $m = 0.5, 0.95, 1 - 10^{-6}, 1 - 10^{-14}$, respectively.

Figure 3. Trajectories in the stroboscopic (for $\tau = n\alpha$ with $n = 0, \ldots, 400$) Poincaré section ($d\xi/d\tau$ vs $\xi/\pi$) for the relativistic system (see text), and (a) $m = 0.6, \epsilon = 0$, (b) $m = 0.6, \epsilon = 1/9$, (c) $m = 0.9, \epsilon = 0$, and (d) $m = 0.9, \epsilon = 1/9$. System parameters: $\alpha = 5.3, \Gamma = 0$.

Figure 4. Trajectories in the stroboscopic (for $\tau = n\alpha$ with $n = 0, \ldots, 400$) Poincaré section ($p_\xi$ vs $\xi/\pi$) for the relativistic system (see text), and (a) $m = 0.6, \epsilon = 0$, (b) $m = 0.6, \epsilon = 1/9$, (c) $m = 0.9, \epsilon = 0$, and (d) $m = 0.9, \epsilon = 1/9$. The initial conditions correspond to those used in fig. 4. System parameters: $\alpha = 5.3, \Gamma = 0$.

Figure 5. Trajectories in the stroboscopic (for $\tau = n\alpha$ with $n = 0, \ldots, 400$) Poincaré section ($p_\xi$ vs $\xi/\pi$) for the relativistic system (see text), and (a) $\epsilon = 0.8$, and (b) $\epsilon = 1$. System parameters: $m = 0.5, \alpha = 5.3, \Gamma = 1$. 
