Loop Corrections in the Spectrum of 2D Hawking Radiation

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ABSTRACT

We determine the one-loop and the two-loop back-reaction corrections in the spectrum of the Hawking radiation for the CGHS model of 2d dilaton gravity by evaluating the Bogoliubov coefficients for a massless scalar field propagating on the corresponding backgrounds. Since the back-reaction can induce a small shift in the position of the classical horizon, we find that a positive shift leads to a non-Planckian late-time spectrum, while a null or a negative shift leads to a Planckian late-time spectrum in the leading-order stationary-point approximation. In the one-loop case there are no corrections to the classical Hawking temperature, while in the two-loop case the temperature is three times greater than the classical value. We argue that these results are consistent with the behaviour of the Hawking flux obtained from the operator quantization only for the times which are not too late, in accordance with the limits of validity of the semiclassical approximation.

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1. Introduction

Two-dimensional (2d) dilaton gravity models have turned out to be excellent toy models for understanding black hole formation and back-reaction of the Hawking radiation (for a review see [1]). The CGHS model [2] has been studied most extensively, due to its simplicity. In [2] it was shown that a thermal Hawking radiation can exist in the semiclassical limit, which was done by evaluating the Hawking flux from the trace anomaly. This result was later confirmed by a direct calculation of the Bogoliubov coefficients [3], which was the 2d analog of the famous Hawking calculation [4]. In a further development, explicit solutions which include the back-reaction were found, in one-loop [5, 6, 7] and two-loop [9] approximations.

In the one-loop case, the trace anomaly method [7], as well as the point-splitting method [8], yield the same Hawking flux as in the zero-loop case, which is a good indication that the one-loop back-reaction does not change the Planckian spectrum of the radiation. This result is surprising, because one expects that the back-reaction should change the zero-loop temperature for a small amount. This is especially puzzling in the case of the BPP solution [7], where the thermal Hawking flux gets turned off for late times (although discontinuously), leaving behind a static remnant geometry. Another puzzling feature of the BPP solution is that the corresponding semiclassical geometry has a horizon which is shifted for a small amount with respect to the classical horizon. The horizon shift also appears in the two-loop case, while the operator quantization yields an expected result for the Hawking flux, in the sense that the zero-loop flux changes under the two-loop back-reaction [9].

These results make interesting the calculation of the Bogoliubov coefficients, since one would like to see what kind of corrections these back-reaction effects would make in the spectrum of the Hawking radiation. Another interesting point to be examined is that one can evaluate the Hawking flux from the Bogoliubov coefficients and compare it to the Hawking flux obtained from the expectation value of the energy-momentum tensor operator. Therefore we are going to evaluate the Bogoliubov coefficients for a free scalar field propagating on the one-loop geometry of [7] and the two-loop geometry of [9].

In section two we briefly review the quantization of the CGHS model in the operator formalism of [8], and define the loop expansion. In section three we evaluate the Bogoliubov coefficients for the BPP background geometry and analyse the corresponding spectrum of the Hawking radiation. The same is done in section five, but in the case of the two-loop background geometry of ref. [9]. In section six we present our conclusions. In the appendix we give some theorems which are useful for obtaining
the asymptotic behaviour of the Bogoliubov coefficients.

2. Loop-expansion for 2d dilaton gravity

Our starting point is a classical theory described by the CGHS action \([2]\)

\[
S = \int_M d^2x \sqrt{-g} \left[ e^{-\phi} \left( R + (\nabla \phi)^2 + 4\lambda^2 \right) - \frac{1}{2} (\nabla f)^2 \right], \tag{2.1}
\]

where \(\phi\) is a dilaton scalar field, \(f\) is a matter scalar field, \(g\), \(R\) and \(\nabla\) are the determinant, the curvature scalar and the covariant derivative respectively, associated with a metric \(g_{\mu\nu}\) on a 2d manifold \(M\). The topology of \(M\) is that of \(\mathbb{R} \times \mathbb{R}\). The equations of motion can be solved in the conformal gauge \(ds^2 = -e^\rho dx^+ dx^- \) as

\[
e^{-\rho} = e^{-\phi} = -\lambda^2 x^+ x^- - F_+ - F_-, \quad f = f_+(x^+) + f_-(x^-), \tag{2.2}
\]

where

\[
F_\pm = a_\pm + b_\pm x^\pm + \int^{x^\pm} dy \int^y dz T_{\pm\pm}(z), \tag{2.3}
\]

and \(T_{\pm\pm}\) is the matter energy-momentum tensor

\[
T_{\pm\pm} = \frac{1}{2} \partial_\pm f \partial_\pm f. \tag{2.4}
\]

The residual conformal invariance is fixed by a gauge choice \(\rho = \phi\), and the independent integration constants are \(a_+ + a_-\) and \(b_\pm\).

The quantum theory can be obtained by quantizing the space of classical solutions defined by the equations (2.2) and (2.3) [3]. This is equivalent to a reduced phase space quantization of the action (2.1) [3]. In this way one obtains a quantum theory of a free massless scalar field \(f\) propagating on a flat fictitious background \(ds^2 = -dx^+ dx^-\). The dilaton and the conformal factor are given by the expression in (2.2), which is considered as an operator in the Heisenberg picture. The matter energy-momentum tensor operator is defined as

\[
T_{\pm\pm} = \frac{1}{2} : \partial_\pm f \partial_\pm f : , \tag{2.5}
\]

where the normal ordering in (2.5) is chosen to be with respect to the vacuum for the creation and annihilation operators \((a^+_k, a^-_k)\) defined by

\[
f_{\pm}(x^\pm) = \frac{1}{\sqrt{2\pi} \sqrt{2\omega_k}} \int_0^\infty \frac{dk}{\sqrt{2\omega_k}} \left[ a_{\pm k} e^{-ikx^\pm} + a_{\mp k}^+ e^{ikx^\pm} \right], \tag{2.6}
\]

where \(\omega_k = |k|\).
The physical Hilbert space of the model is the Fock space $\mathcal{F}(a_k)$ constructed from $a_k^\dagger$ acting on the vacuum $|0\rangle$. The model is unitary because the dynamics is generated by a free-field Hamiltonian

$$H = \int_{-\infty}^{\infty} dk \omega_k a_k^\dagger a_k + E_0 \ ,$$

which is a Hermitian operator acting on $\mathcal{F}$, where $E_0$ is the vacuum energy. Consequently the states at $t = \frac{1}{2}(x^+ + x^-) = \text{const.}$ surfaces are related by a unitary transformation

$$\Psi(t_2) = e^{-iH(t_2-t_1)}\Psi(t_1) \ .$$

The Heisenberg picture is defined as

$$\Psi_0 = e^{iHt}\Psi(t) \ , \ A(t) = e^{iHt}Ae^{-iHt} \ ,$$

which also serves to relate the covariant quantization expressions to the canonical quantization expressions. For example,

$$f(t, x) = e^{iHt}f(x)e^{-iHt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\omega_k}} \left[ a_k e^{i(kx - \omega_k t)} + a_k^\dagger e^{-i(kx - \omega_k t)} \right] \ ,$$

where $x = \frac{1}{2}(x^+ - x^-)$.

Given a physical state $\Psi_0$, one can associate an effective metric to $\Psi(t) = e^{-iHt}\Psi_0$ via

$$e^{\rho_{\text{eff}}(t,x)} = e^{\phi_{\text{eff}}(t,x)} = \langle \Psi(t) | e^{\phi(x)} | \Psi(t) \rangle = \langle \Psi_0 | e^{\phi(t,x)} | \Psi_0 \rangle \ ,$$

where $e^{\rho_{\text{eff}}}$ is the effective conformal factor. The geometry which is generated by $e^{\rho_{\text{eff}}}$ via $ds^2 = -e^{\rho_{\text{eff}}} dx^+ dx^-$ makes sense only in the regions of $M$ where the quantum fluctuations are small [6, 8]. The condition for this is

$$\sqrt{|\langle e^{2\phi}\rangle - \langle e^\phi \rangle^2|} \ll \langle e^\phi \rangle \ ,$$

which defines the limits of validity of the semiclassical approximation. At least perturbatively, the region defined by (2.12) roughly coincides with the weak-coupling region [6, 8].

The effective conformal factor can be calculated perturbatively by using an expansion in powers of the energy-momentum tensor, which is equivalent to the expansion in matter loops [6, 8]

$$\langle (\lambda^2 x^+ x^- - F)^{-1} \rangle = e^{\phi_0} \sum_{n=0}^{\infty} e^{n\phi_0} \langle \delta F^n \rangle \ .$$

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where \( F_0 \) is a c-number function, 
\[ e^{-\phi_0} = -\lambda^2 x^+ x^- - F_0 \]
and \( \delta F = F - F_0 \). A convenient choice for \( F_0 \) is
\[ F_0 = \langle F_+ \rangle + \langle F_- \rangle , \tag{2.14} \]
since then the lowest order metric is a one-loop semiclassical metric
\[ e^{-\phi_0} = -\lambda^2 x^+ x^- - \langle F_+ \rangle - \langle F_- \rangle . \tag{2.15} \]

The state \( \Psi_0 \) is chosen such that it is as close as possible to the classical matter distribution \( f_0(x^+) \) which creates the black hole. The corresponding classical metric is described by
\[ e^{-\rho} = \frac{M(x^+)}{x^+} - \lambda^2 x^+ \Delta(x^+) - \lambda^2 x^+ x^- \tag{2.16} \]
where
\[ M(x^+) = \lambda \int_{-\infty}^{x^+} dy \, y T^0_{++}(y) \quad , \quad \lambda^2 \Delta = \int_{-\infty}^{x^+} dy \, T^0_{++}(y) \tag{2.17} \]
and \( T^0_{++} = \frac{1}{2} \partial_+ f_0 \partial_+ f_0 \). The geometry is that of a black hole of the mass
\[ M = \lim_{x^+ \to +\infty} M(x^+) , \tag{2.18} \]
and the horizon is at
\[ x^- = -\Delta = - \lim_{x^+ \to +\infty} \Delta(x^+) . \tag{2.19} \]
In the limit of a shock-wave matter distribution, for which
\[ T^0_{++} = a \delta(x^+ - x^+_0) , \tag{2.20} \]
we have
\[ M(x^+) = \lambda a x^+_0 \theta(x^+ - x^+_0) \quad , \quad \Delta(x^+) = \frac{a}{\lambda^2} . \tag{2.21} \]
The asymptotically flat coordinates \((\eta^+, \eta^-)\) at the past null infinity \( \mathcal{I}^- \) are given by
\[ \lambda x^+ = e^{\lambda \eta^+} \quad , \quad x^- = -\Delta e^{-\lambda \eta^-} , \tag{2.22} \]
while the asymptotically flat coordinates \((\sigma^+, \sigma^-)\) at the future null infinity \( \mathcal{I}^+ \) satisfy
\[ \lambda x^+ = e^{\lambda \sigma^+} \quad , \quad \lambda (x^- + \Delta) = -e^{-\lambda \sigma^-} . \tag{2.23} \]

Note that a conformal change of coordinates \( x^\pm \to \xi^\pm \) defines a new set of creation and annihilation operators \((b^\dagger_k, b_k)\) through
\[ f^\pm = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2\omega_k}} \left[ b_{\mp k} e^{-ik\xi^\pm} + b^\dagger_{\pm k} e^{ik\xi^\pm} \right] . \tag{2.24} \]
The old and the new creation and annihilation operators are related by a Bogoliubov transformation

\[ a_k = S^{-1} b_k S = \int_{-\infty}^{\infty} dq (b_q \alpha_{qk} + b^*_q \beta_{qk}), \]

and the new vacuum is given by \(|0_\xi\rangle = S |0\rangle\). Since initially the geometry should be as close as possible to the dilaton vacuum, we take for \(\Psi^0\) a coherent state

\[ \Psi^0 = e^{A} |0^+_\eta\rangle \otimes |0^-_\eta\rangle, \]

where \(|0^\pm\rangle = |0^\pm_\eta\rangle\otimes |0^-_\eta\rangle\) is the vacuum for the “in” coordinates (2.22), while

\[ A = \int_0^\infty dk [f_0(k)a^\dagger_{-k} - f_0^*(k)a_{-k}], \]

and \(f_0(k)\) are the Fourier modes of \(f_0(x^+).\) The choices (2.26) and (2.27) complete the initial set up of the quantization method of [8].

3. One-loop spectrum

A one-loop effective metric can be obtained from the expression (2.15), which was calculated in [8]. This gives

\[ e^{-\rho^0} = e^{-\phi^0} = C + b_\pm x^\pm - \lambda^2 x^+ x^- - \frac{\kappa}{4} \log |\lambda^2 x^+ x^-| - \frac{1}{2} \int_{-\infty}^{x^+} dy^+ (x^+ - y^+) \left( \frac{\partial f_0}{\partial y^+} \right)^2. \]

The expression (3.1) can be also obtained as a solution of the equations of motion of an effective one-loop action

\[ S_{eff} = S_0 - \frac{\kappa}{4} \int d^2 x \sqrt{-g} R \square^{-1} R - \kappa \int d^2 x \sqrt{-g} (R \phi - (\nabla \phi)^2), \]

where \(S_0\) is the CGHS action (2.1) [8]. By choosing \(C = -\frac{1}{4} \kappa [1 - \log(\kappa/4)]\) one can obtain a consistent semiclassical geometry [8]. In the case of the shock-wave matter this geometry is well defined in the \(x^+ > 0, x^- < 0\) quadrant. For \(x^+ < x^+_0\)

\[ e^{-\rho^0} = e^{-\phi^0} = C - \lambda^2 x^+ x^- - \frac{\kappa}{4} \log(-\lambda^2 x^+ x^-), \]

which is defined for \(\sigma \geq \sigma_{cr}\), where \(\sigma = \log(-\lambda^2 x^+ x^-)\) is the static coordinate. At \(\sigma = \sigma_{cr}\) there is a singularity, and this line is interpreted as the boundary of the strong-coupling region.

For \(x^+ > x^+_0\) one obtains an evaporating black hole solution

\[ e^{-\rho^0} = e^{-\phi^0} = C + \frac{M}{\lambda} - \lambda^2 x^+ (x^- + \Delta) - \frac{\kappa}{4} \log(-\lambda^2 x^+ x^-). \]
The corresponding Hawking radiation flux at $I^+$ is determined in the operator formalism by evaluating

$$\langle \Psi_0 | T_{--}(\xi^-) | \Psi_0 \rangle$$

(3.5)

where $T_{--}(\xi^-)$ is normal ordered with respect to the asymptotically flat coordinates $\xi^\pm$ of the metric (3.4) at $I^+$. Since the $\xi^\pm$ coordinates are the same as the “out” coordinates (2.23) for the classical black hole solution, one obtains by using the point-splitting regularization

$$\langle T_{--}(\xi^-) \rangle = \frac{\lambda^2}{48} \left[ 1 - (1 + \lambda \Delta e^{\lambda \sigma^-})^{-2} \right]$$

(3.6)

The expression (3.6) gives a late-time flux which corresponds to a thermal Hawking radiation, with the temperature $T = \lambda/2\pi$ [2, 3]. The Hawking radiation shrinks the apparent horizon of the solution (3.4), so that the apparent horizon line meets the curvature singularity in a finite proper time, at

$$x^+_i = \frac{1}{\lambda^2 \Delta} \left( -\kappa/4 + e^{1+\frac{4}{3}(C+M/\lambda)} \right) \quad x^-_i = \frac{-\Delta}{1 - \frac{\kappa}{4} e^{-1-\frac{1}{3}(C+M/\lambda)}}$$

(3.7)

The curvature singularity then becomes naked for $x^+ > x^+_i$. However, a static solution (3.3) of the form

$$e^{-\rho_0} = e^{-\phi_0} = \dot{C} - \lambda^2 x^+(x^- + \Delta) - \frac{\kappa}{4} \log(-\lambda^2 x^+(x^- + \Delta))$$

(3.8)

can be continuously matched to (3.4) along $x^- = x^-_i$ if $\dot{C} = -\frac{1}{4} \kappa [1 - \log(\kappa/4)]$. A small negative energy shock-wave emanates from that point, and for $x^- > x^-_i$ the Hawking radiation stops, while the static geometry (3.8) has a null ADM mass. There is again a critical line $\tilde{\sigma} = \tilde{\sigma}_{cr}$, corresponding to a singularity of the geometry (3.8). Note that the scalar curvature of (3.8) is bounded at $x^- = x^-_i$, and the singularity comes from the pathological behaviour of $e^{-\phi_0}$, which becomes ill-defined for $x^- > x^-_i$. This singularity can be interpreted as the boundary of the region where higher-order corrections become important. The spatial geometry of the remnant (3.8) is that of a semi-infinite throat, extending to the strong coupling region.

Note that the form of the Hawking flux (3.6) is necessary but not a sufficient condition for the Hawking radiation to have a Planckian spectrum. In order to determine this, one has to evaluate the Bogoliubov coefficients. Since the “in” and the “out” coordinates are exactly the same as in the classical black hole background case, one can conclude that the one-loop Bogoliubov coefficients are the same as the classical Bogoliubov coefficients calculated in [4]. Consequently, a Planckian one-loop spectrum should be obtained in the late-time approximation. However, by a closer
examination of the one-loop geometry one notices that the one-loop horizon is given by the line \( x^- = x_i^- \). This means that the classical horizon \( x^- = -\Delta \) has undergone a small shift due to the one-loop backreaction. The shift \(-\Delta - x_i^-\) is very small, of the order of \( \Delta e^{-M/\lambda} \). Still, as we are going to show, this shift will have a non-trivial consequence for the spectrum of the Hawking radiation.

The Bogoliubov coefficients can be calculated by using the formalism of [3]. The “in” plain-wave basis is given by

\[
u_\omega = \frac{1}{\sqrt{2\omega}} e^{-i\omega \eta^-}, \quad (3.9)
\]

while the “out” basis is given by

\[
u_\omega = \frac{1}{\sqrt{2\omega}} e^{-i\omega \sigma^-} \theta(\eta_i - \eta^-), \quad (3.10)
\]

where \( \eta_i \) is the position of the one-loop horizon, given by

\[
\lambda \eta_i = \log \left[ 1 - \frac{k}{4} e^{-\frac{1}{2}(M/\lambda + C - 1)} \right]. \quad (3.11)
\]

In the semiclassical approximation, when \( M >> \lambda \), the shift is an exponentially small quantity. Still, it produces a change in the classical “out” basis, for which \( \eta_i = 0 \). Following [3], one obtains

\[
\alpha_{\omega\omega'} = -\frac{i}{\pi} \int_{-\infty}^{\eta_i} d\eta^- v_\omega \partial_{\eta^-} u_{\omega'}^*, \quad \beta_{\omega\omega'} = \frac{i}{\pi} \int_{-\infty}^{\eta_i} d\eta^- v_\omega \partial_{\eta^-} u_{\omega'}, \quad (3.12)
\]

so that

\[
\alpha_{\omega\omega'} = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- e^{-i\omega \sigma^- + i\omega' \eta^-}, \quad (3.13)
\]

while

\[
\beta_{\omega\omega'} = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- e^{-i\omega \sigma^- - i\omega' \eta^-} = -i\alpha_{-\omega\omega'}. \quad (3.14)
\]

From (2.23) we obtain

\[
\alpha_{\omega\omega'}^\pm = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- e^{i\pi \log[\lambda\Delta (e^{-\lambda \eta^- - 1})] \mp i\omega' \eta^-}, \quad (3.15)
\]

where \( \alpha^+ = \alpha \) and \( \alpha^- = \beta \). The integral in (3.15) can be rewritten as

\[
I_1 = \int_0^{x_i} dx (1 - x)^a x^{-1-a+b} \quad (3.16)
\]

where \( a = i\omega / \lambda, \ b = \pm i\omega' / \lambda, \ x_i = e^{\lambda \eta}. \) The integral (3.16) is the incomplete Beta function \( B(1+a, b-a, x_i) \) [14], so that

\[
\alpha_{\omega\omega'}^\pm = \frac{1}{2\pi \lambda} \sqrt{\frac{\omega'}{\omega}} (\lambda \Delta)^{i\omega / \lambda} B \left( 1 + i\omega / \lambda, -i(\omega + \omega') / \lambda, x_i \right). \quad (3.17)
\]
The incomplete Beta function can be evaluated for \( x_i \) close to one via the expansion

\[
B(1 + a, b - a; x) = B(1 + a, b - a) - \frac{x^{b-a}}{1 + a} \sum_{n=0}^{\infty} \frac{(1 + b)_n}{(2 + a)_n} (1 - x)^{n+1+a},
\]

(3.18)

where \((c)_n = \frac{\Gamma(c+n)}{\Gamma(c)}\). When \( x_i = 1 \), the formula (3.18) gives the expression for the Bogoliubov coefficients obtained in [3].

The number of particles of a given energy emitted at \( I^+ \) is given by

\[
N_{\omega} = \int_{0}^{\infty} |\beta_{\omega\omega'}|^2 d\omega',
\]

(3.19)

which is divergent when (3.17) is used. This divergence is the artefact of the plain-wave basis we are using, and can be avoided by using a normalisable basis [4]. A convenient basis is formed by the wave-packets

\[
v_{j\epsilon}(n) = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\omega e^{2\pi in\omega} v_{\omega},
\]

(3.20)

which are centered around \( \sigma^- = \frac{2\pi n}{\epsilon} \), where \( n, j \in \mathbb{N} \) [3]. In the late-time approximation, i.e. when \( n \) is large, the main contribution in the \( \beta \)-coefficient integral comes from the vicinity of the horizon, for \( \lambda\eta^- = O(\exp(-2\pi n/\epsilon)) \), so that one can use the approximation \( e^{-\lambda\eta^-} - 1 \approx -\lambda\eta^- \), which gives

\[
\beta_{\omega,\omega'} \approx \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- e^{\frac{\epsilon}{\lambda^2} \log[-\lambda^2 \Delta\eta^-] - i\omega' \eta^-}.
\]

This is also equivalent to large \( \omega' \) asymptotics of (3.17), since the contribution from small \( \eta^- \) is equivalent to \( \log \omega' \approx 2\pi n/\epsilon \).

The expression (3.21) can be evaluated by using the integral

\[
J_1 = \int_{\epsilon_i}^{\infty} dx x^a e^{-bx} = b^{-1-a} \Gamma(1 + a, b\epsilon_i),
\]

(3.22)

where \( \Gamma(\alpha, x) \) is the incomplete gamma-function, \( a = i\omega/\lambda, b = \pm i\omega'/\lambda \) and \( \epsilon_i = -\lambda\eta_i \). The following expansion

\[
\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha + n)} ,
\]

(3.23)

can be used to approximate the incomplete gamma function for small \( x \) [10], while for large values of \( x \) one can use [10]

\[
\Gamma(\alpha, x) = x^{\alpha-1} e^{-x} \left(1 + \frac{\alpha - 1}{x} + O(x^{-2})\right),
\]

(3.24)
By using the approximation (3.21) and the formula (3.22) we obtain

\[
\alpha_{\omega}\approx \frac{1}{2\pi\lambda} \sqrt{\frac{\omega'}{\omega}} (\lambda\Delta)^{i\omega/\lambda}(\pm i\omega'/\lambda)^{-1-\omega'/\lambda} \Gamma(1+i\omega/\lambda, \mp i\omega') .
\] (3.25)

Note that the approximation (3.25) follows from the exact formula (3.17) when \(\omega'\) is large, in accordance with the expected late-time asymptotics. This can be seen from (3.18) by approximating each term in (3.18) by its leading large-\(b\) expression. In this way one obtains the first two terms of (3.25) when expanded via (3.23).

From (3.25) and (3.23) we obtain

\[
\left|\frac{\alpha}{\beta}\right|^2 \approx e^{2\omega \pi/\lambda} \left( \frac{\Gamma(1+i\omega/\lambda) - i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots}{\Gamma(1+i\omega/\lambda) + i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots} \right)^2 .
\] (3.26)

One can see that the expression (3.26) gives in the limit \(\eta \to 0\) a Planckian spectrum with the temperature \(T = \lambda/2\pi\). However, the \(\eta\) corrections do not correspond to a Planckian spectrum with a shifted temperature \(T + \delta T\), as one would naively expect. This can be also seen by applying the large \(\omega'\) formula (3.24) to (3.25), so that

\[
\left|\frac{\alpha}{\beta}\right|^2 \approx e^{4\omega \pi/\lambda} \left( \frac{\Gamma(1+i\omega/\lambda) - i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots}{\Gamma(1+i\omega/\lambda) + i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots} \right)^2 ,
\] (3.27)

where \(T\) is independent of \(\omega'\). Note that when \(\omega' \to \infty\) the expression (3.27) tends to one, which in the thermal case would mean that \(T \to \infty\). This indicates that the corresponding Hawking flux will diverge. We can calculate the Hawking flux by calculating \(N_\omega\), since

\[
\langle T \rangle = \int_0^\infty d\omega \omega N_\omega .
\] (3.28)

\(N_\omega\) can be calculated by going into the basis (3.20), so that

\[
\beta_{H}(n) = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\nu e^{2\pi i \nu} \beta_{H} ,
\] (3.29)

where \(\omega = j\epsilon\). By applying the expansion (3.23) to \(\beta_{H}\), we obtain in the large-\(n\) limit

\[
\left|\beta_{H}(n)\right|^2 = \frac{1}{\pi \lambda} \left( \epsilon^{2\pi\omega/\lambda} - 1 \right) \frac{1}{\omega'} \frac{\sin^2 \epsilon z/2}{z^2} \left( \frac{\Gamma(1+i\omega/\lambda) - i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots}{\Gamma(1+i\omega/\lambda) + i \exp(-\pi\omega/2\lambda) \sin \left( \frac{\pi\omega}{4\lambda}\right) \zeta + \cdots} \right) + \cdots
\] (3.30)

where c.c. denotes the complex conjugate of the previous term, while \(z = -\frac{1}{\epsilon} \log(\omega'/\lambda^2 \Delta) + \frac{2\pi n}{\epsilon}\) and \(z_1 = \frac{1}{\epsilon} \log(-\eta\lambda^2 \Delta) + \frac{2\pi n}{\epsilon}\). In order to derive (3.30) one needs

\[
\int_{j\epsilon}^{(j+1)\epsilon} d\nu e^{i\alpha \nu} = \frac{2}{\alpha} \sin \frac{\alpha \epsilon}{2} e^{i(j+\frac{1}{2})\epsilon a} .
\] (3.31)
\[ |\Gamma(1 + ix)|^2 = \frac{\pi x}{\sinh \pi x} \quad . \tag{3.32} \]

The \( \omega' \) integral over the first term in (3.30) gives the standard thermal contribution. However, the \( \omega' \) integral over the second and higher order in \( \eta_i \) terms in (3.30) are all divergent. This is caused by the fact that we have taken only a finite number of terms in the expansion (3.23), which is a good approximation only for a sufficiently small \( x = \omega'\epsilon_i \), or equivalently not too big \( \omega' (\omega' < \epsilon_i^{-1}) \). For big \( \omega' (\omega' > \epsilon_i^{-1}) \), the asymptotic expansion (3.24) is better. Consequently, we have

\[
N_\omega(n) = \int_0^{\epsilon_i^{-1}} d\omega' |\beta_{\omega\omega'}|^2 + \int_{\epsilon_i^{-1}}^{\infty} d\omega' |\beta_{\omega\omega'}|^2 \\
\approx \int_0^{\epsilon_i^{-1}} d\omega' |\beta^s_{\omega\omega'}|^2 + \int_{\epsilon_i^{-1}}^{\infty} d\omega' |\beta^l_{\omega\omega'}|^2 \quad . \tag{3.33}
\]

where \( \beta^s \) and \( \beta^l \) stand for small and large \( \omega' \) approximations, respectively. The first integral in (3.33) has the integrand given by (3.30), and because of the cut-off in \( \omega' \), it is finite. In the second integral we use the approximation (3.24), so that

\[
\beta^l_{\omega\omega'} = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} (\lambda\Delta)^{\omega/\lambda} \frac{x_i^{-\omega/\lambda}}{-i\omega'} \exp(i(\omega/\lambda) \log \epsilon_i) \quad . \tag{3.34}
\]

This gives a logarithmic divergent contribution. However, when \( \epsilon_i = 0 \), the large \( \omega' \) asymptotics changes by a phase-factor \( \exp[-i(\omega/\lambda) \log(\omega'/\lambda)] \), which gives a finite \( N_\omega(n) \).

Note that the one-loop flux is also given by the expectation value (3.6), which is finite. From the Bogoliubov coefficients analysis it follows that a finite flux is possible only if the shift \( \epsilon_i \leq 0 \), while the one-loop geometry seems to suggest that \( \epsilon_i > 0 \). The way out of this paradox is provided by the fact that the line \( x^- = x_i^- \) gets very close to the strong-coupling region. Therefore the one-loop geometry is not a good approximation there, and hence our assumption that the effective horizon was given by the line \( x^- = x_i^- \) was not a good assumption.

4. Two-loop spectrum

A two-loop metric can be obtained by truncating the expansion (2.13) after \( n = 2 \). Outside of the region occupied by the matter pulse centered around \( x_0^+ \), the metric can be written as [9]

\[
ds^2 = -e^{\phi_2} dx^+ dx^- ;
\]

\[
e^{\phi_2} = e^{\phi_0} \left[ 1 + e^{2\phi_0} (C_- + C_+(x^+ - x_0^+))^2 \theta(x^+ - x_0^+) \right] \quad , \tag{4.1}
\]
where $e^{\phi_0}$ is the one-loop dilaton solution

$$e^{-\phi_0} = C - a(x^+ - x_0^+)\theta(x^+ - x_0^+) - \lambda^2 x^+ x^- - \frac{\lambda}{4} \log |\lambda^2 x^+ x^-| , \quad (4.2)$$

whith $a = \lambda^2 \Delta$ and $C_{\pm}$ are integration constants, analogous to the constant $C$ of the one-loop solution. The difference now is that $C_+$ depends on the matter pulse profile, and it is negative for very narrow pulses (those which are shorter than a critical length defined by the momentum cut-off [9]) and positive otherwise. The constant $C_-$ is matter independent, and its value is regularization dependent. Consistent semiclassical geometries appear for $C_+ \geq 0$ [9].

As in the one-loop case, the relevant quadrant is $x^+ \geq 0, x^- \leq 0$. For $x^+ < x_0^+$ the solution (4.1) is static, and it can be written as

$$e^{\phi_2} = e^{\phi_0} \left[ 1 - e^{2\phi_0} \alpha^2 \right] , \quad e^{-\phi_0} = C + e^\sigma - \frac{k}{4} \sigma , \quad (4.3)$$

where $C_- = -\alpha^2$ and $\sigma = \log(-\lambda^2 x^+ x^-)$ is the static coordinate. The solution (4.3) describes a two-loop corrected dilaton vacuum. The corresponding scalar curvature diverges at the line

$$e^{-\phi_0} - \alpha = 0 , \quad (4.4)$$

which corresponds to the one-loop singularity line with $C$ replaced by $C - \alpha$. The semiclassical geometry will be defined for $\sigma \geq \sigma_c$, where $C + e^{\sigma_c} - \frac{k}{4} \sigma_c = \alpha$. The curvature singularity will be absent for $C \geq \alpha - \frac{k}{4}(1 - \log \frac{k}{4})$. Therefore if we want to avoid a naked singularity at two loops, the one-loop $C$ has to be increased to $C = \alpha - \frac{k}{4}(1 - \log \frac{k}{4})$. Note that in the case when $C_- = \alpha^2$, there is no curvature singularity, and the only singularity comes from the one-loop critical line determined by the second formula in (4.3).

For $x^+ > x_0^+$ the solution (4.1) becomes

$$e^{\phi_2} = e^{\phi_0} \left[ 1 - e^{2\phi_0} (\alpha^2 - C_+(x^+ - x_0^+)^2) \right] , \quad (4.5)$$

where

$$e^{-\phi_0} = C + \frac{M}{\lambda} - \lambda^2 x^+(x^- + \Delta) - \frac{\lambda}{4} \log(-\lambda^2 x^+ x^-) . \quad (4.6)$$

It describes a two-loop corrected evaporating black hole geometry. The curvature singularity line is given by

$$e^{-2\phi_0} - \alpha^2 + C_+(x^+ - x_0^+)^2 = 0 . \quad (4.7)$$

When $C_+ = 0$, the equation (4.7) becomes a shock wave singularity equation (4.4), and it describes a singularity line of the one-loop metric with a smaller ADM mass.
The apparent horizon line is given by the equation $\partial_+ \phi_2 = 0$, which can be rewritten as

$$x^- + \Delta + \frac{k}{4\lambda^2 x^+} = \frac{-2C_+ (x^+ - x_0^+)}{e^{-2\phi_0} - 3\alpha^2 + 3C_+ (x^+ - x_0^+)^2} . \quad (4.8)$$

In the shock-wave limit ($C_+ = 0$) the intersection point of the apparent horizon and the curvature singularity line is given by the one-loop expressions (3.7). The change in the intersection point when $C_+ \neq 0$ can be evaluated perturbatively [9], and it is given by

$$\delta x_i^- = -\frac{C_+ (x_i^+ - x_0^+)^2}{2\alpha a x_i^+} x_i^- + O(\beta^4)$$

$$\delta x_i^+ = -\frac{2\lambda^2 C_+ (x_i^+ - x_0^+)^2}{\kappa\alpha a} x_i^- x_i^+ + O(\beta^4) . \quad (4.9)$$

The line $x^- = x_h = x_i^- + \delta x_i^-$ can be considered as the horizon of the two-loop semiclassical geometry. Clearly the quantum corrections have shifted the position of the horizon, which will affect the Hawking radiation. In the region $x^- > x_h$ a naked singularity will appear, unless we impose an appropriate boundary condition. In the shock-wave case, one can impose a static solution (4.3) for $x^- > x_h$

$$\dot{e}^{\phi_2} = e^{\phi_0} \left[ 1 - e^{2\phi_0 \alpha^2} \right] , \quad e^{-\phi_0} = \dot{C} - \lambda^2 x^+ (x^- + \Delta) - \frac{\kappa}{4} \log(-\lambda^2 x^+ (x^- + \Delta)) ,$$

which can be continuously matched to (4.5) at $x^- = x_h$ if

$$\dot{C} = -\frac{\kappa}{4} (1 - \log \frac{\kappa}{4}) - \alpha . \quad (4.11)$$

However, in contrast to the one-loop case, the metric (4.10) has a curvature singularity at $\sigma = \sigma_{cr}$, and the naked singularity is not removed. When $C_+ < 0$, there is no static two-loop dilaton vacuum solution for $x^- > x_h$ which can be continuously matched to (4.3) and the naked singularity remains. However, when $C_+ > 0$, the naked singularity lays in the strong-coupling region, and therefore it can be ignored since it lays outside of the region of validity of the metric (4.1). Also note that in the case when $C_- > 0$, the solution with $C_+ > 0$ does not have a curvature singularity. As a consequence of these properties the Hawking flux will be free of singularities only for $C_+ > 0$.

The asymptotically flat coordinates $(\sigma^+, \sigma^-)$ at $I^+_R$ are given by

$$\lambda x^+ = e^{\lambda \sigma^+} , \quad \lambda(x^- + \Delta) = -e^{-\lambda \tilde{\sigma}} , \quad (4.12)$$

where

$$\tilde{\sigma} = \sigma^- - \frac{C_+}{2\lambda^3 e^{2\lambda \tilde{\sigma}}} . \quad (4.13)$$
This can be written as

\[ \sigma^- = \frac{1}{\lambda} \log[-\lambda(\Delta + x^-)] + \frac{C_+}{2\lambda^5} (\Delta + x^-)^{-2}, \quad (4.14) \]

so that a non-zero correction appears at two loops. As a direct consequence, the Hawking flux at \( I^+ \) will not have a zero-loop form (3.6), which can be seen by evaluating

\[ \langle T_{-\epsilon} \rangle |_{I^+} = -1 \]

so that a non-zero correction appears at two loops. As a direct consequence, the Hawking flux at \( I^+ \) will not have a zero-loop form (3.6), which can be seen by evaluating

\[ \langle T_{-\epsilon} \rangle |_{I^+} = -1/48 \left( \eta'' \eta' - \frac{3}{2} \left( \frac{\eta''}{\eta'} \right)^2 \right), \quad (4.15) \]

where the primes denote \( d/d\sigma^- \). One then obtains

\[ \mathcal{T} = \langle T_{-\epsilon} \rangle |_{I^+} = \frac{\lambda^2 y^4[y^4(\Delta y + \frac{1}{2}\Delta^2) - C_+ P_1(y) + C^2_+ P_2(y)]}{(y - \Delta)^2(C_+ + y^2)^4}, \quad (4.16) \]

where \( y = x^- + \Delta, P_2 = -2y^2 + 3\Delta y - 3/2\Delta^2 \) and \( P_4 = y^2(-4y^2 + 10\Delta y - 5\Delta^2) \).

From (4.16) one can see that \( \mathcal{T}(y) \) does not diverge for \( C_+ \geq 0 \). In that case \( \mathcal{T}(y) \) goes to zero for late times \( (y \to 0) \), indicating that higher-order loop corrections can turn off the Hawking radiation. This also means that the spectrum of the Hawking radiation can not be thermal for very late times.

Relation (4.14) clearly shows that the two-loop “out” coordinates are not the same as the analogous zero and one loop coordinates, which indicates that the spectrum of the radiation will change. This can be confirmed by evaluating the Bogoliubov coefficients in the late-time approximation. The two-loop Bogoliubov coefficients can be evaluated from the formulas (3.13-14), where now

\[ \lambda \sigma^- = -\log[\lambda \Delta(e^{-\lambda \eta^-} - 1)] + \frac{C_+}{2\lambda^5 \Delta^2} (e^{-\lambda \eta^-} - 1)^{-2} \]

and

\[ \lambda \eta_i = \log \left[ 1 - \frac{k}{4} e^{-\frac{i}{\lambda}(M/\lambda+C)-1} \right] + C_+ \frac{(x_i^- - x_i^+)^2}{2\alpha \sigma x_i^+}, \quad (4.17) \]

so that

\[ \alpha_{\omega,\omega}^\pm = \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- \exp \left( \frac{i}{\lambda} \log[\lambda \Delta(e^{-\lambda \eta^-} - 1)] - \frac{i\omega C_+}{2\lambda^5 \Delta^2} (e^{-\lambda \eta^-} - 1)^{-2} \pm i\omega' \eta^- \right). \quad (4.18) \]

For late times, we can use the approximation \( e^{-\lambda \eta^-} - 1 \approx -\lambda \eta^- \), so that

\[ \alpha_{\omega,\omega}^\pm \approx \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\eta_i} d\eta^- \exp \left( \frac{i}{\lambda} \log(-\lambda^2 \Delta \eta^-) - \frac{i\omega C_+}{2\lambda^5 \Delta^2} (\lambda \eta^-)^{-2} \pm i\omega' \eta^- \right). \quad (4.19) \]

Equation (4.19) can be rewritten as

\[ \alpha_{\omega,\omega}^\pm \approx \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} (\lambda \Delta)^{i\omega/\lambda} \int_{\epsilon_i}^{\infty} dy^a y^a e^{-by - cy^2} \approx \frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} (\lambda \Delta)^{i\omega/\lambda} J_2(a,b,c) \quad (4.20) \]
where \( a = i\omega/\lambda \), \( b = \pm i\omega'/\lambda \), \( \epsilon_i = -\lambda \eta \), and \( c = i\omega C_+/2\lambda^5 \Delta^2 \). Since we are interested in the large-\( b \) asymptotics, we can analyze \( J_2 \) by using the method of stationary points. We can rewrite \( J_2 \) as

\[
J_2 = \int_{\epsilon_i}^\infty dy y^a e^{-f(y)}, \tag{4.21}
\]

where \( f(y) = by + cy^2 \). \( f(y) \) has a stationary point at \( y = (2c/b)^{\frac{1}{3}} \), which for a sufficiently large \( b \) will lay outside of the interval \( [\epsilon_i, \infty) \). Hence the main contribution to \( J_2 \) will come from the vicinity of the point \( y = \epsilon_i \), so that as \( b \to \infty \)

\[
J_2 \approx \frac{1}{b} (\epsilon_i)^a \exp(-b\epsilon_i - c\epsilon_i^{-2}) \tag{4.22}
\]

Note that the argument we have used to derive (4.22) clearly holds for \( a, b, c \) real and positive. When \( a, b, c \) are complex, an analogous argument applies (see [11] and the appendix), and (4.22) is still valid. We also show in the appendix that the same method can be applied to the exact expression (4.18), and the result (4.22) is again obtained when \( b \) is large.

The expression (4.22) has the same asymptotics as the incomplete gamma function (3.22), and therefore we will encounter the same problem as in the one-loop case. Since \( N_\omega(n) \) will then diverge, the corresponding Hawking flux will also diverge for late times. On the other hand, the Hawking flux from the expectation value (4.16) is finite for late times, and we have the same discrepancy as in the one-loop case. Since the line \( x^\pm = x_i^\pm + \delta x_i^\pm \), which is suggested by the semiclassical geometry as the horizon, gets close to the border of the strong-coupling region where the spacetime geometry is not well-defined, we can invoke the same argument as in the one-loop case, and resolve the discrepancy by taking \( \epsilon_i \leq 0 \). In that case the stationary-point method gives thermal Bogoliubov coefficients (see the appendix), and we obtain a Planckian late-time spectrum in the leading-order approximation with the temperature \( T = 3\lambda/2\pi \). This increase in the temperature is an expected back-reaction effect. However, an unexpected feature is that the increase is not perturbative in \( C_+ \).

A related problem is that the Hawking flux (4.16) deviates only slightly for small enough \( C_+ \) from the zero-loop flux which corresponds to the temperature \( T = \lambda/2\pi \). Note that for big enough \( C_+ \) there is a late time \( y_0 \) for which \( T(y_0) \) coincides with the thermal flux corresponding to the temperature \( T = 3\lambda/2\pi \). In this case one can argue that there is an agreement with the Bogoliubov coefficients calculation when \( y \approx y_0 \), while for the times \( y > y_0 \), the discrepancy can be justified by the argument that the background geometry is not well-defined (strong-coupling region) and hence the Bogoliubov coefficients are not well defined there. Still, there is a discrepancy in the weak-coupling region for small \( C_+ \), and a possible explanation is that the approx-
imation (A.15) is not good enough, and that the $a_2$ and higher-order terms should have been taken into account.

5. Conclusions

The results of our analysis indicate that small back-reaction corrections in the background geometry can induce large changes in the Hawking radiation spectrum. We have seen that a small positive shift in the position of the classical horizon gives a non-Planckian late-time Hawking radiation spectrum whose flux diverges. On the other hand, a small negative or a null shift gives a Planckian late-time spectrum. Consistency considerations then select a null or a negative shift as an effective horizon shift, contrary to what is suggested by the semiclassical geometry. This is explained by the breakdown of the semiclassical approximation for late times near the classical horizon.

In the one-loop case the Bogoliubov coefficients give a late-time Planckian spectrum which is consistent with the operator quantization Hawking flux only for times $x^- < x_i^-$. After that the BPP solution gives a zero flux, which indicates that one enters a non-thermal regime. It is plausible that the higher-order stationary point contributions to the Bogoliubov coefficients may account for this. In the two-loop case the relation between the late-time spectrum and the operator quantization flux is similar but slightly more complex. The consistency again requires that the time is not too late, and the new requirement is that the constant $C_+$ is not too small. It is likely that this restriction is caused by the low-order approximation we are using to calculate the two-loop Bogoliubov coefficients. Including the higher-order terms in the stationary-point approximation may also account for a non-Planckian nature of the very late time spectrum, which is suggested by the behaviour of the operator quantization Hawking flux $\mathcal{T}$. However, a further investigation is necessary in order to clarify these issues.

Note that when $C_+ < 0$, there is a total disagreement between the flux from the Bogoliubov coefficients and the operator quantization flux $\mathcal{T}$. In that case $\mathcal{T}$ diverges to minus infinity for late times. This cannot be reconciled neither with the positive horizon shift case (the corresponding flux diverges to plus infinity), nor with the negative horizon shift case (the corresponding flux is finite and positive). However, $C_+ < 0$ solution can be considered as an unphysical solution, since it requires a matter pulse whose width is shorter than the 2d analog of the Planck length $\mathcal{L}$. 

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Appendix

In this appendix we give some relevant theorems from [11], and use them to obtain the asymptotic expressions for the relevant integrals.

In the one-loop case, the relevant integral is

\[ I_1(b) = \int_0^{x_i} d\tau \tau^{b-a-1}(1 - \tau)^a \]  \hfill (A.1)

where \( x_i = \exp(-\epsilon_i) \), \( a = i\omega/\lambda \), \( b = \pm i\omega'/\lambda \). This integral can be rewritten as

\[ I_1(b) = x_i^{b-a} \int_0^\infty dt e^{-bt}(e^t - x_i)^a \]  \hfill (A.2)

where \( \tau = x_i e^{-t} \). Now we use a theorem from [11], page 108

Theorem (1): Let \( I(z) = \int_0^\infty dt e^{-zt} q(t) \), such that \( q(t) \) is real or complex, \( |q^{(s)}(t)| \leq A_s e^{\sigma|t|} \) \( (\sigma \in \mathbb{R}) \) and \( q(t) \) holomorphic in the sector \( S = \{ \alpha_1 \leq \text{Arg} t \leq \alpha_2 \} \) \( (\alpha_1 < 0, \alpha_2 > 0) \). Then as \( z \to \infty \)

\[ I(z) \approx \sum_{s=0}^{\infty} \frac{q^{(s)}(0)}{z^{s+1}} , \]  \hfill (A.3)

in the sector \( S_\pi = \{ -\alpha_1 - \pi/2 \leq \text{Arg} z \leq -\alpha_2 + \pi/2 \} \).

In the case of the integral (A.2), \( q(t) = (e^t - x_i)^a \), and the conditions of the theorem (1) are satisfied, so that as \( b \to \infty \)

\[ I_1(b) \approx x_i^{b-a} \left( \frac{1 - x_i^a}{b} \right) + O(b^{-2}) \]  \hfill (A.4)

Note that the expression (A.4) has the same large-\( b \) asymptotics as (3.22) for \( x_i \) close to one.

When \( x_i = 1 \), then a generalization of the theorem (1) applies [11], page 114

Theorem (2): Let \( q(t) \) be holomorphic in \( S \) and \( q(t) = O(e^{\sigma|t|}) \) as \( t \to \infty \). Let

\[ q(t) \approx \sum_{s=0}^{\infty} a_s t^{(s+\nu-\mu)/\mu} \]

for \( t \to 0^+ \) and \( t \in S \), where \( \mu > 0 \) and \( \text{Re} \nu > 0 \). Then

\[ I(z) \approx \sum_{s=0}^{\infty} \Gamma \left( \frac{s + \nu}{\mu} \right) \frac{a_s}{z^{(s+\nu)/\mu}} \]  \hfill (A.5)
as \( z \to \infty \) in the sector \( S_\pi \).

In our case, \( q(t) = (e^t - 1)^a \approx \sum_{s=0}^{\infty} a_s t^{s+a} \), so that \( \mu = 1 \) and \( \nu = 1+a = 1+i\omega/\lambda \), and hence we can use the theorem (2). Therefore, as \( b \to \infty \)

\[
I_1(b) \approx \frac{\Gamma(1+a)}{b^{1+a}} \left(1 + O(b^{-1})\right), \quad (A.6)
\]

whose leading-order term is (3.22) for \( \epsilon_i = 0 \).

In the two-loop case, we have

\[
I_2(b) = \int_0^{x_i} d\tau \, \tau^{b-a-1}(1-\tau)^a \exp \left[-c(\tau^{-1} - 1)^{-2}\right]
\]

\[
= x_i^{b-a} \int_0^{\infty} dt e^{-bt} (e^t - x_i)^a \exp \left[-c(x_i^{-1} e^t - 1)^{-2}\right] . \quad (A.7)
\]

When \( x_i < 1 \), then

\[
q(t) = (e^t - x_i)^a \exp \left[-c(x_i^{-1} e^t - 1)^{-2}\right] \approx \sum_{s=0}^{\infty} a_s t^s
\]

so that the theorem (1) applies, and hence

\[
I_2(b) \approx x_i^{b-a} \left(1 - x_i\right)^a b \exp \left[-c \left(\frac{x_i}{1-x_i}\right)^2\right] + O(b^{-2}) . \quad (A.8)
\]

It is easy to see that the large-\( b \) asymptotics of the expressions (A.8) and (4.22) coincide for \( x_i \) close to one.

When \( x_i = 1 \), then \( q(t) \approx t^a \exp(-ct^{-2}) \) as \( t \to 0^+ \), so that neither theorem (1) nor theorem (2) can be used. One can then use a theorem from [11], pg 127

Theorem (3): Let

\[
I(z) = \int_a^b dt \, e^{-zp(t)} q(t) = \int_{C_{ab}} dt \, e^{-zp(t)} q(t)
\]

such that

(i) \( p(t) \) and \( q(t) \) are holomorphic and single-valued in a domain \( T \)

(ii) curve \( C_{ab} \in T \)

(iii) \( p'(t) \) has a simple zero \( t_0 \in C_{ab} \)

(iv) \( \theta = \text{Arg} \, z \in [\theta_1, \theta_2] \), \( |z| \geq Z, \theta_2 - \theta_1 < \pi, Z > 0, I(z) \) converges absolutely and uniformly at \( a \) and \( b \).

(v) \( \text{Re} \{e^{i\theta}(p(t) - p(t_0))\} \) is positive on \( C_{ab} \), except at \( t_0 \), and is bounded away from zero uniformly with respect to \( \theta \) as \( t \to a \) or \( b \) along \( C_{ab} \).
Then as \( z \to \infty \)
\[
I(z) \approx 2e^{-zp(t)} \sum_{s=0}^{\infty} \frac{a_{2s}}{z^{s+1/2}} \, , \tag{A.9}
\]
for \( \theta_1 \leq \text{Arg } z \leq \theta_2 \), where

\[
a_0 = \frac{q}{\sqrt{2p''}} \, , \quad a_2 = \left[ 2q'' - \frac{2p'''q'}{p''} + \left( \frac{5p''^2}{6p''^2 - \frac{p'''}{2p''}} \right) q \right] (2p'')^{-3/2} \, , \tag{A.10}
\]
and so on.

In the case of the integral (A.7) when \( x_i = 1 \) one can take

\[
p(t) = t + \frac{c}{b}(e^t - 1)^{-2} \, , \quad q(t) = (e^t - 1)^a \, . \tag{A.11}
\]

However, theorem (3) cannot be applied directly, since the role of \( z \) is played by \( b \), and in the choice (A.11) \( p(t) \) depends on \( b \). In order to obtain a proper \( p(t) \), note that if we rewrite \( I_2 \) as

\[
I_2(b) = \int_0^\infty dt \, e^{-f(t)} \tag{A.12}
\]
where \( f(t) = -a \log(e^t - 1) + bt + c(e^t - 1)^{-2} \), then a main contribution to \( I_2 \) comes from the stationary point \( f'(t_0) = 0 \), which for large \( b \) is given as \( t_0 \approx (2c/b)^{1/3} \). Since then \( t_0 \) is close to zero, one can approximate \( f(t) \) with its small \( t \) asymptotics, which is the late-time approximation we used in sections 3 and 4. Therefore as \( b \to \infty \)

\[
bp(t) \approx bt + ct^{-2} \, , \quad q(t) \approx t^a \, , \tag{A.13}
\]
and we make a rescaling \( t = \tau(c/b)^{1/3} \), so that

\[
bp(t) \approx z\tilde{p}(\tau) = z(\tau + \tau^{-2}) \, ,
\]
where \( z = (b^2c)^{1/3} \). Consequently

\[
I_2 \approx (c/b)^{(a+1)/3} \int_0^\infty d\tau \, e^{-z(\tau + \tau^{-2})} \tau^a \, , \tag{A.14}
\]
and the theorem (3) can be now applied. We then obtain in the leading-order approximation \( (a_0 \text{ term}) \)

\[
\left| \frac{\alpha_{\omega \nu'}}{\beta_{\omega \nu'}} \right|^2 \approx |(-1)^{a/3}|^2 = \exp(2\omega \pi /3\lambda) \, , \tag{A.15}
\]
so that the leading-order two-loop Hawking temperature is given by \( T = 3\lambda/2\pi \).
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