An exact sum rule for transversely polarized DIS

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Abstract

The Operator Product Expansion provides expressions for the $n^{th}$ moments of
$g_1(x)$ and $g_2(x)$ in terms of hadronic matrix elements of local operators for $n = \text{odd}$
integer. In some cases these matrix elements are expected to be small leading to
approximate sum rules for the $\text{odd}$ moments of $g_{1,2}(x)$. We have shown how, working
in a field-theoretic framework, one can derive expressions for the $\text{even}$ moments
of the valence parts of $g_{1,2}(x)$. These expressions cannot be written as matrix elements of
local operators and do not coincide with the analytic continuation to $n = \text{even}$ integer of the OPE results.

Just as for the OPE one can in some cases argue that the hadronic matrix elements should be small, leading to approximate sum rules for the moments of the valence parts of $g_{1,2}(x)$. But, most importantly, for the case $n = 2$ we have proved rigorously that the hadronic matrix element vanishes, yielding the exact ELT sum rule

$$\int_0^1 dx \, x \left[ g_1^V(x) + 2g_2^V(x) \right] = 0.$$

We have argued that the convergence properties of this sum rule are good and have discussed how it can be used to get information about $g_2(x)$ and as a further test of QCD.

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1 Introduction

The inelastic form factors $G_1$ and $G_2$, and their scaling versions $g_1$ and $g_2$, describing spin-dependent or polarized deep-inelastic scattering are attracting much attention at present with major experimental programmes in progress at CERN and SLAC and planned for HERA. For a comprehensive account see the review article [1] by Anselmino, Efremov and Leader (AEL). The theoretical and experimental status of $g_1$ and $g_2$ is rather different. There exists a simple partonic interpretation [2] of the scaling function $g_1(x)$ which is the only one of the two which survives in the strict Bjorken limit and, in that limit, completely controls the longitudinal polarization asymmetry. Longitudinal polarization dominates kinematically in this limit and is described in QCD as a leading (twist 2) effect. The function $g_1(x)$ is also the easier one to measure experimentally [3,4]. The main theoretical issue is the subtle effect whereby the triangle anomaly induces an anomalous gluon contribution in $g_1(x)$, in particular in its first moment [5,6].

$G_2$ and the corresponding dimensionless scaling function $g_2$ are more complicated. They describe the difference between the properties of a longitudinally and a transversely polarized hadron, and QCD twist 3 effects, for which there is no probabilistic interpretation, contribute significantly [7]. The transverse polarization effects are suppressed like $M/Q$ ($M$ is the hadron mass; recall that a massless particle is always longitudinally polarized). This makes the experimental studies more complicated as well. The first results from SMC and SLAC have just appeared [8] and it is hoped that the high intensity lepton beam and jet target will make possible the measurement of $g_2$ with high accuracy by the HERMES collaboration at HERA.

In this situation sum rules for $g_2$ are especially important. The Burkhardt-Cottingham superconvergence sum rule [9] is well known:

$$\int_0^1 g_2(x) dx = 0 \quad (1)$$

though it is not always realised that it does not follow from the Operator Product Expansion and that it may contradict the expected small-$x$ Regge behaviour [1,10].

The other sum rules that are often quoted are the Wandzura-Wilzcek sum rules [11]

$$\int_0^1 x^{n-1} \left[ \frac{n-1}{n} g_1(x) + g_2(x) \right] dx = 0 \quad n = 1, 3, 5 \ldots \quad (2)$$

which, as is discussed below, involve the neglect of twist-3 contributions and which assumes the validity of (1). If the sum rules in (2) are assumed to hold also for even values of $n$, one obtains the remarkable result

$$g_1(x) + g_2(x) = \int_x^1 \frac{g_1(x)}{x} dx. \quad (3)$$

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1 Further sum rules, for weak boson mediated DIS, based on the neglect of twist-3 contributions, have recently appeared [24].
The function $g_2(x)$ defined by (3) is often called $g_2^{WW}(x)$.

In [11] it is argued, on the basis of a model, that the twist 3 terms can be neglected. However, this argument is unreliable, since the selfsame model gives unacceptable results for $F_{1,2}(x)$ and $g_1(x)$.

We shall discuss the derivation of sum rules involving $g_2(x)$ from two different points of view. One is based upon the imposition of gauge invariance in a specific lepton-hadron reaction, namely polarized DIS, the second upon a study of the properties of the hadronic matrix elements involved in $g_{1,2}(x)$, without reference to any specific reaction.

In both these approaches we are able to produce new sum rules that do not follow from the operator product expansion. The OPE only makes statements about the odd moments of $g_{1,2}(x)$, corresponding to the fact one is essentially dealing with forward virtual Compton scattering, which, viewed in the t-channel, involves $p\bar{p} \rightarrow \gamma\gamma$, and thus only involves positive parity states.

In our more general approach we obtain results also for the even moments of $g_{1,2}(x)$. But what is fascinating, and at first sight surprising, is that these sum rules involve only the valence contributions to the structure functions.

Amongst these the most interesting is the case $n = 2$, the so-called Efremov, Leader, Teryaev (ELT) sum rule, since it is exact and does not rely on any neglect of twist-3 contributions:

$$\int_0^1 dx \, x \left[ g_1^V(x) + 2g_2^V(x) \right] = 0$$

and which, as we shall discuss, can be tested experimentally.

In sections 2 and 3 we show how to derive these generalised sum rules first by appealing to gauge invariance in polarized DIS, second by a detailed study of the properties of various hadronic matrix elements. Section 4 discusses some aspects of the ELT sum rules and their generalization and in Section 5 we consider how the new sum rules might be tested experimentally.

## 2 Sum rules from gauge independence in DIS

Consider first the field-theoretic calculation of the antisymmetric part of the hadronic tensor $W_{\mu\nu}^{(A)}$ which controls polarized deep inelastic scattering, via the Feynman diagrams of Fig. 1.

Because we are dealing with twist 3 effects it is necessary to consider the quark-quark-gluon correlators:

$$b_A(x_1, x_2) = \int \frac{d\lambda_1}{2\pi} \frac{d\lambda_2}{2\pi} e^{i\lambda_1(x_1-x_2)+i\lambda_2 x_2} \tilde{b}_A(\lambda_1, \lambda_2)$$

where

$$\tilde{b}_A(\lambda_1, \lambda_2) = \frac{1}{2M} \langle P, S | \bar{\psi}(0) \not\! \! \! \! \! \gamma^\alpha S \cdot D(\lambda_1 n) \psi(\lambda_2 n) | P, S \rangle$$

and which, as we shall discuss, can be tested experimentally.
and
\[ b_V(x_1, x_2) = \int \frac{d\lambda_1}{2\pi} \frac{d\lambda_2}{2\pi} e^{i\lambda_1(x_1 - x_2) + i\lambda_2 x_2} \tilde{b}_V(\lambda_1, \lambda_2) \]  

(7)

where
\[ \tilde{b}_V(\lambda_1, \lambda_2) = -\frac{i}{2M} \epsilon^{\mu\nu\rho\sigma} S_\mu P_\nu n_\rho (\bar{\psi}(0) \not\! P \psi(\lambda_1 n)|P, S) \not\! D_\mu (\lambda_1 n) \psi(\lambda_2 n)|P, S) \]  

(8)

where \( D^\mu \) is the covariant derivative and \( n^\mu \) is a light-like gauge fixing vector: \( n^2 = 0, n \cdot A = 0, n \cdot P = 1. \) It also defines the transverse direction; for example, for the covariant spin vector \( S^\mu, \)
\[ S^\mu_T = S^\mu - (S \cdot n) P^\mu. \]  

(9)

The fractions \( x_1 \) and \( x_2 \) correspond to the fractions of the hadron momentum carried by the quarks. In these definitions the correlators \( b_A(x_1, x_2) \) and \( b_V(x_1, x_2) \) are real and dimensionless. They are related to the correlators used in AEL by \( b_V = iB^V/2 \) and \( b_A = -B^A/2. \) They have the symmetry properties
\[ b_V(x_1, x_2) = -b_V(x_2, x_1) \quad b_A(x_1, x_2) = b_A(x_2, x_1). \]  

(10)

In eqns. (5–8) we have suppressed the flavour label \( f \) on the quark fields.

Use of the equation of motion for the quark field of a given flavour leads to a very general relation between \( b_V, b_A \) and the quark-quark correlator function \( f_T(x) \) which directly gives the field-theoretic expression for the transverse combination of \( g_1 \) and \( g_2, \) namely
\[ g_T(x) \equiv g_1(x) + g_2(x) = \frac{1}{2} \sum_f Q_f^2 [f_T(x) + f_T(-x)] \]  

(11)

where
\[ f_T(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x} \tilde{f}_T(\lambda) \]  

(12)

and
\[ \tilde{f}_T(\lambda) = \frac{1}{2M} \langle P, S|\bar{\psi}(0)\gamma_5 S\psi(\lambda n)|P, S \rangle \]  

(13)

where, again, we suppress the flavour label.

For an arbitrary test function \( \sigma(x) \) one finds [12]
\[ \int dxdy \{[\sigma(x) + \sigma(y)] b_A(x, y) + [\sigma(x) - \sigma(y)] b_V(x, y)\} = -2 \int dx\sigma(x) x f_T(x). \]  

(14)

Further, demanding that the results for \( W^{(A)}_{\mu\nu} \) be independent of the gauge fixing vector \( n^\mu \) leads to a relation between \( b_A \) and the quark-quark correlator function \( h_L(x) \) which gives the field-theoretic formula for \( g_1(x), \) namely
\[ g_1(x) = \frac{1}{2} \sum_f Q_f^2 [h_L(x) + h_L(-x)] \]  

(15)

where
\[ h_L(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x} \tilde{h}_L(\lambda) \]  

(16)
and
\[ h_L(\lambda) = \frac{1}{2Mn \cdot \vec{S}} \langle P, S| \bar{\psi}(0) \not\gamma_5 \psi(\lambda n)| P, S \rangle. \]  

(17)

One finds [12]
\[ \int dxdy \left[ \frac{\sigma(x) - \sigma(y)}{x - y} \right] b_A(x, y) = \int dx \sigma(x) \left[ f_T(x) - h_L(x) \right]. \]  

(18)

Note that in eqns. (14) and (18) the range of integration is \(|x| \leq 1, |y| \leq 1 \) and \(|x-y| \leq 1\).

For the longitudinal case it is possible to associate \( h_L \), for each flavour, with a polarized quark or antiquark number density:
\[ \Delta q(x) = h_L(x) \quad \Delta \bar{q}(x) = h_L(-x) \]  

(19)

but such a connection is not possible for the transverse spin case.

In (18) let us now choose \( \sigma(x) = x^{n-1} \), with \( n \) odd. The integral \(-1 \leq x \leq 1\) on the RHS of (18) can then be converted into an integral \( 0 \leq x \leq 1 \), leading via (11) and (15) to [12]
\[ \int_0^1 dx x^{n-1} g_2(x) = \frac{1}{2} \sum_f Q_f^2 \int dxdy \left[ \frac{n-1}{2} (x^{n-2} + y^{n-2}) + \phi_{n-1}(x, y) \right] b_A(x, y) \quad (n \text{ odd}) \]  

(20)

where
\[ \phi_n(x, y) \equiv \frac{x^n - y^n}{x - y} - \frac{n}{2} \left( x^{n-1} + y^{n-1} \right). \]  

(21)

Note that
\[ \phi_n(x, y) = 0 \quad \text{if} \quad x = y. \]  

(22)

Now let us choose \( \sigma(x) = x^{n-2} \) in (14) with \( n \) odd. By analogous arguments (14) becomes
\[ \int_0^1 dx x^{n-1} \left[ \frac{n-1}{n} g_1(x) + g_2(x) \right] = -\frac{1}{4} \sum_f Q_f^2 \int dxdy \left\{ (x^{n-2} + y^{n-2}) b_A(x, y) + (x^{n-2} - y^{n-2}) b_V(x, y) \right\} \]  

(n odd).

(23)

It follows that [12],
\[ \int_0^1 dx x^{n-1} \left[ \frac{n-1}{n} g_1(x) + g_2(x) \right] = \frac{1}{4(n+1)} \sum_f Q_f^2 \int dxdy \left\{ \phi_{n-1}(x, y) b_A(x, y) - \frac{n-1}{2} (x^{n-2} - y^{n-2}) b_V(x, y) \right\} \]  

\[ n = 1, 3, 5 \ldots \]  

(24)
The set of sum rules (24) is perfectly equivalent to what one obtains from the Operator Product Expansion for \( n = 3, 5, 7 \cdots \). The OPE, however, says nothing about the case \( n = 1 \). Indeed (24) may not be valid for \( n = 1 \) because the integrals could diverge.

We see that the LHS of (24) is just the LHS of the Wandzura-Wilczek sum rule (2). The WW sum rule was originally derived from the Operator Product Expansion by neglecting twist 3 operators on the RHS and by assuming that the operator product result can be continued smoothly to \( n = 1 \), where, of course, the WW sum rule just reduces to the Burkhardt-Cottingham sum rule (1).

The quark-quark-gluon correlators in our approach combine the contributions related to terms of twist 2 and 3 in the operator product expansion. However, it is possible to separate them. Note that the matrix elements, containing the covariant derivative, are actually not gauge invariant. This is because the derivative and the quark field it is acting on are taken at different points on the light cone. One can easily pass to a gauge-invariant form by shifting the gluon field to the point of the quark field and express the compensating term \( A^\mu(2) - A^\mu(3) \) in terms of the gluon field strength (the latter is possible because an axial gauge is used). This contradicts earlier statements [13] that transverse momentum and gluon field are combined together in a gauge-invariant way.

If we neglect the contribution coming from \( G^{\mu\nu} \) but keep the transverse momentum contribution embedded in the covariant derivative, we obtain a result of the form [14]

\[
b_A(x_1, x_2) = \phi_A(x_1) \delta(x_1 - x_2) \quad b_V(x_1, x_2) = 0.
\]

From this follows, via (14),

\[
f_T(x) = -\frac{\phi_A(x)}{x}
\]

and, via (18),

\[
h_L(x) - f_T(x) = \frac{d}{dx}\phi_A(x)
\]

yielding

\[
\frac{df_T}{dx} = -\frac{h_L(x)}{x}.
\]

Integrating and using (11) and (15) then yields once again the Wandzura-Wilczek relation given in eqn. (3), and which, as mentioned, was originally “derived” from the OPE by neglecting twist 3 contributions. The above discussion shows that the twist 2 contributions do take account of the transverse motion of the quark [15,16].

There are good reasons to believe that BC sum rule will fail because the expected Regge behaviour for \( g_2(x) \) as \( x \to 0 \) might make the integral over \( g_2(x) \) diverge [10].

Contrary to the Operator Product approach, one can certainly choose \( \sigma(x) = x^{n-1} \) with \( n \) even in (18) and \( \sigma(x) = x^{n-2} \) with \( n \) even in (14), to obtain a totally new set of sum rules, which, however, do not involve \( g_1(x) \) or \( g_2(x) \) as such, but a part of them, \( g_1^V(x) \) and \( g_2^V(x) \) which can be regarded as the valence contribution to them. For \( g_1(x) \), which has a simple partonic interpretation, this is straightforward. For \( g_2(x) \), which does
not have a partonic interpretation it is not clear what $g_2^V(x)$ means physically. However, it is a well defined object, which can be measured, and thus sum rules involving it are of physical importance.

The difference between $n$ odd and $n$ even appears in the following way. The LH sides of (20) and (23) originally involve integrals of the form, for example,

$$\frac{1}{2} \sum f Q_f \int_{-1}^{1} dx x^{n-1} h_L(x).$$

Because $n$ was odd this could be written

$$\frac{1}{2} \sum f Q_f \int_{0}^{1} dx x^{n-1} [h_L(x) + h_L(-x)] = \int_{0}^{1} dx x^{n-1} g_1(x).$$

For $n$ even the last step will lead to expressions of the form

$$\frac{1}{2} \sum f Q_f \int_{0}^{1} dx x^{n-1} [\Delta q_f(x) - \Delta \bar{q}_f(x)] = \int_{0}^{1} dx x^{n-1} g_2^V(x).$$

We shall define $g_2^V(x)$ by [see(11)]

$$g_2^V(x) = -g_1^V(x) + \frac{1}{2} \sum f Q_f^2 [f_T(x) - f_T(-x)].$$

Then the sum rules (20), (23) and (24) hold also for even $n$ with $g_1(x) \to g_1^V(x)$ and $g_2(x) \to g_2^V(x)$.

Of particular interest is the case $n = 2$, because the contribution of the twist 3 correlators on the RHS of (24) vanishes when $n = 2$. Thus one has

$$\int_{0}^{1} dx x \left[ g_1^V(x) + 2g_2^V(x) \right] = 0.$$

This so-called Efremov, Leader, Teryaev (ELT) sum rule was incorrectly stated in AEL [1] where the label “$V$” was not indicated.

We shall return to discuss certain aspects of the ELT sum rule, the possibility of testing it physically, its convergence properties and whether or not it can be generalised, after first discussing a quite different approach to the sum rule.

3 Sum rules from properties of hadronic matrix elements

The derivation of the sum rules in Section 2 is a little unsatisfactory in that it appeals to a particular lepton-hadron reaction to derive properties inherent to the nucleon. The
following derivation deals only with nucleon matrix elements. The sum rules can be derived from a careful study of the structure and gauge properties of the matrix elements and use of the equation of motion of the quark field. In this approach [1] one sees very clearly why sum rules like the Burkhardt-Cottingham one may fail because of the non-invertability of certain Fourier transforms.

Consider first the forward matrix element of the bilocal operator

$$\bar{\psi}(0)\gamma^\mu\gamma_5\psi(x)$$

on the light-cone $x^2 = 0$. Its most general form is

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\psi(x)\rangle_{P,S} = A_1 S^\mu + (x \cdot S) A_2 P^\mu + (x \cdot S) A_3 x^\mu$$

(33)

where $\langle \cdots \rangle$ is short for $\langle P, S | \cdots | P, S \rangle$. The scalar functions $A_{1,2,3}$ are functions only of $x \cdot P$.

From (33) we deduce

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\partial^\nu\psi(x)\rangle_{P,S} = A'_1 S^\mu P^\nu + A'_2 P^\mu S^\nu +$$

$$+ (x \cdot S) [A'_2 P^\mu P^\nu + A'_3 x^\mu P^\nu + A_3 g^{\mu\nu}]$$

(34)

where

$$A' \equiv \frac{dA(x \cdot P)}{d(x \cdot P)}.$$ (35)

We now put $x^\mu = \lambda n^\mu$. Then

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\psi(\lambda n)\rangle_{P,S} = A_1 S^\mu + \lambda (n \cdot S) [A_2 P^\mu + \lambda A_3 n^\mu]$$

(36)

where now $A_i = A_i(\lambda)$, and

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\partial^\nu\psi(\lambda n)\rangle_{P,S} = A'_1 S^\mu P^\nu + A'_2 P^\mu S^\nu +$$

$$+ \lambda \{A_3 n^\mu S^\nu + (n \cdot S) [A'_2 P^\mu P^\nu + \lambda A'_3 n^\mu P^\nu + A_3 g^{\mu\nu}]\}.$$ (37)

We assume that all scalar functions are such that $\lambda A(\lambda) \to 0$ as $\lambda \to 0$ for all terms occurring in (36) and (37). Then at $\lambda = 0$ we have the simple structures

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\psi(0)\rangle_{P,S} = A_1(0) S^\mu$$

(38)

and

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma^\mu\gamma_5\partial^\nu\psi(0)\rangle_{P,S} = A'_1(0) S^\mu P^\nu + A_2(0) P^\mu S^\nu.$$ (39)

We shall also require, from (37)

$$\frac{1}{M}\langle\bar{\psi}(0)\gamma_5\partial\psi(\lambda n)\rangle_{P,S} = -\lambda (n \cdot S) [M^2 A'_2 + 5 A_3 + \lambda A'_3]$$

(40)
so that at $\lambda = 0$

$$\frac{1}{M} \langle \bar{\psi}(0) \gamma_5 \partial \psi(0) \rangle_{P,S} = 0. \quad (41)$$

Finally note, from (37), that

$$\frac{1}{M} \langle \bar{\psi}(0) \gamma_\mu \gamma_5 n \cdot \partial \psi(\lambda n) \rangle_{P,S} = A_1^\mu S^\mu + (n \cdot S) [(A_2 + \lambda A_2') P^\mu + \lambda (2A_3 + \lambda A_3') n^\mu]$$

$$= \frac{1}{M} \frac{d}{d\lambda} \langle \bar{\psi}(0) \gamma_\mu \gamma_5 \psi(\lambda n) \rangle_{P,S}. \quad (42)$$

Consider now the gluonic matrix element

$$\frac{1}{M} \langle \bar{\psi}(0) \gamma_\mu \gamma_5 g A^\nu(\lambda n) \psi(\lambda n) \rangle_{P,S}$$

with $x = \lambda n$. Its most general form is

$$\lambda (S \cdot n) \left[ B_1 P^\mu P^\nu + \lambda B_2 P^\mu n^\nu + \lambda B_3 n^\mu P^\nu + \lambda^2 B_4 n^\mu n^\nu \right]$$

$$+ B_5 S^\mu P^\nu + B_6 P^\mu S^\nu + \lambda B_7 S^\mu n^\nu + \lambda B_8 n^\mu S^\nu. \quad (43)$$

The gauge condition $\gamma_\mu A^\mu = 0$ implies that

$$B_5 = 0, \quad \lambda B_1 = -B_6, \quad \lambda B_3 = -B_8$$

so that

$$\frac{1}{M} \langle \bar{\psi}(0) \gamma_\mu \gamma_5 g A^\nu(x) \psi(x) \rangle_{P,S}$$

$$= \lambda B_1 [(S \cdot n) P^\mu P^\nu - P^\mu S^\nu] + \lambda (S \cdot n) [B_2 P^\mu n^\nu + \lambda B_4 n^\mu n^\nu]$$

$$+ \lambda^2 B_3 [(S \cdot n) n^\mu P^\nu - n^\mu S^\nu] + \lambda B_7 S^\mu n^\nu. \quad (45)$$

Notice the crucial feature, that the imposition of the gauge condition, together with the assumptions about the vanishing of products like $\lambda B(\lambda)$ as $\lambda \to 0$, leads to the vanishing of (45) at $\lambda = 0$, i.e.

$$\langle \bar{\psi}(0) \gamma_\mu \gamma_5 g A^\nu(0) \psi(0) \rangle_{P,S} = 0. \quad (46)$$

This result will be crucial for deriving the Efremov-Leader-Teryaev sum rule.

Let us now relate some of the above coefficients to the functions occurring in the discussion of $g_1$ and $g_2$. From (17) and (36) we have

$$\tilde{h}_L(\lambda) = \frac{1}{2} [A_1(\lambda) + \lambda A_2(\lambda)]. \quad (47)$$

From (13) and (36)

$$\tilde{f}_T(\lambda) = \frac{1}{2} A_1(\lambda). \quad (48)$$

Then from (15) and (16), if the Fourier transforms can be inverted,

$$\int_0^1 dx g_1(x) = \frac{Q^2}{2} \tilde{h}_L(0) = \frac{Q^2}{4} A_1(0) \quad \text{by (47).} \quad (49)$$
Similarly, from (11) and (12)

\[ \int_0^1 dx \left[ g_1(x) + g_2(x) \right] = \frac{Q_2^2}{2} \tilde{f}_T(0) = \frac{Q_2^2}{4} A_1(0) \quad \text{by (48).} \]  

Eqns. (49) and (50) imply the Burkhardt-Cottingham sum rule

\[ \int_0^1 dx g_2(x) = 0. \]  

As is discussed in ref. [10] the above derivation may fail because of the non-invertability of the Fourier transforms. We turn now to the Efremov-Leader-Teryaev sum rule.

Consider first eqn. (14) which followed from the equations of motion. Choosing \( \sigma(x) = \delta(x - z) \) and then integrating over \( z \), using (10), (5) and (12) there results:

\[ \tilde{b}_A(0,0) = -i \left. \frac{d \tilde{f}_T}{d\lambda} \right|_{\lambda=0} \]  

where we have taken the quark mass to be zero for simplicity and where we have taken, on the basis of (12),

\[ x f_T(x) = i \int \frac{d\lambda}{2\pi} e^{i\lambda x} \tilde{f}_T d\lambda(\lambda). \]  

Now because of (46), from (6)

\[ \tilde{b}_A(0,0) = \frac{i}{2M} \langle \bar{\psi}(0) \not\!\gamma_5 (S_T \cdot \partial) \psi(0) \rangle_{P,S} \]  

so that via (39)

\[ \tilde{b}_A(0,0) = -i A_2(0). \]  

Use of this and (48) in (52) yields

\[ A_2(0) = \left. \frac{d}{d\lambda} A_1(\lambda) \right|_{\lambda=0} = A'_1(0). \]  

Now by arguments similar to those that lead to (53), we have

\[ \int_{-1}^1 dx x h_L(x) = \left. i \frac{d\tilde{h}}{d\lambda} \right|_{\lambda=0} \]  

\[ = \frac{i}{2} [A_1'(0) + A_2(0)] \quad \text{by (47)} \]  

\[ = i A'_1(0) \quad \text{by (55).} \]  

Similarly we have, using (53) and (48)

\[ 2 \int_{-1}^1 dx x f_T(x) = 2i \left. \frac{d \tilde{f}_T}{d\lambda} \right|_{\lambda=0} \]  

\[ = i A'_1(0). \]
Subtracting (56) from (57) and repeating the kind of argument that led to eqn. (30) we obtain, once again, the ELT sum rule

$$\int_0^1 dx \left[ g_1^V(x) + 2g_2^V(x) \right] = 0. \quad (58)$$

4 Discussion of the ELT sum rule and a generalization

We discuss here firstly the question of the convergence of the ELT sum rule (32), then consider an analogous sum rule involving the complete functions $g_{1,2}(x)$ and not just their valence parts and then comment upon an implication for the concept of handedness of jets.

As mentioned earlier the Burkhardt-Cottingham sum rule (1) may well diverge because of a possible $1/x^2$ growth of $g_2(x)$ as $x \to 0$. It is important to note that such a singular behaviour will not spoil the convergence of the ELT sum rule (32), since the singularity will cancel out in the subtraction in (31).

Consider now the question of the analogue of (32) for the complete functions $g_{1,2}(x)$. In contrast to the Operator Product Expansion, the sum rules (14) hold for $\sigma(x) = x^{n-1}$ with $n$ odd or even and the sum rules (18) hold for $\sigma(x) = x^{n-2}$ with $n$ odd or even. For $n$ odd and $\geq 3$ they reproduce the OPE results for the moments of $g_{1,2}(x)$. For $n$ even they produce new sum rules for the moments of the valence parts of $g_{1,2}(x)$. However, it is possible to consider sum rules for $n$ even from a different point of view, namely from the analytic continuation in $n$ of the results for $n$ odd. Hence we wish to begin with (20) and (23) and analytically continue in $n$. As written the RH sides of (20) and (23) do not have a unique analytic continuation since $x$ and $y$ can be negative so that terms of the form $x^n$ and $y^n$ effectively reproduce factors of $(-1)^n$ which grow exponentially in the imaginary $n$ direction and spoil the uniqueness of the analytic continuation. However, starting with $n$ odd we can rewrite all the integrals in (20) and (23) in such a way that $0 \leq x \leq 1$ and $0 \leq y \leq 1$ after which the analytic continuation is unique. We shall not give the detailed results for arbitrary $n$, but for $n = 2$ we find

$$\int_0^1 dx \left[ g_1(x) + 2g_2(x) \right] = \sum_f Q_f^2 \int_0^1 dy \int_0^{1-y} dx \left[ \frac{x-y}{x+y} b_A(x,-y) - b_V(x,-y) \right] \quad (59)$$

The matrix elements on the RHS of (59) are not zero and cannot be expressed as a finite series of matrix elements of local operators. However they are of twist-3 and are proportional to the square root of the product of the probability to find a gluon and the probability to find a $q\bar{q}$ pair in the nucleon. The latter was estimated to be small from the study of QCD sum rules by Shuryak and Vainshtein [17]. So it may be that the RHS of (59) is negligible, corresponding to the Wandzura-Wilczek sum rules (2) continued to $n = 2$. Together with the Burkhardt-Cottingham sum rule (1), this means that $g_2^{WW}(x)$...
should intersect the experimental $g_2(x)$ at least twice in the interval $0 < x < 1$ which seems compatible with the present SLAC data [8].

The method used in Section 3 to derive sum rules for the first and second moments of $g_{1,2}(x)$ highlights an interesting aspect of the Burkhardt-Cottingham sum rule. The assumption that all the scalar functions $A(\lambda)$ are well behaved as $\lambda \to 0$, as implied by the assumed behaviour $\lambda A(\lambda) \to 0$ as $\lambda \to 0$ means, as can be seen from (36), that the first moments of the longitudinal $g_L(x) \equiv g_1(x)$ and the transverse $g_T(x) \equiv g_1(x) + g_2(x)$ depend on the matrix element of the axial vector current which is proportional to just the single vectorial structure $S^\mu$. There is no reference to any direction which could differentiate longitudinal from transverse, so the first moments of $g_L(x)$ and $g_T(x)$ coincide. This seems very similar to the “naive” derivation of the BC sum rule from rotational invariance [2] as well as to the early QCD derivation [18]. (It would be interesting to understand analogously the physical meaning of (32) written as $\int_0^1 dx x g_L^V(x) = 2 \int_0^1 dx x g_T^V(x)$.)

An analogous situation arises for the new spin-dependent variable handedness (H) introduced in [19], which allows the study of the polarization of a quark or gluon which has fragmented into a jet. H is given as a product of the quark polarization times the analyzing power $A$ of the fragmentation reaction. The analyzing power is described by light-cone functions analogous to $h_L(x)$ and $f_T(x)$. As discussed in [19] longitudinal and transverse analyzing powers coincide in the case of particle decay as a consequence of rotational invariance, but in the “decay” of the jet the light-cone vector $n^\mu$ “remembers” the jet direction resulting in a difference between longitudinal and transverse analyzing powers. But by the same reasoning as above, the first moment of the longitudinal and transverse analyzing powers should coincide. The integration variable in this case is $z$, the fraction of the parton’s momentum carried by a pair (or triple) of particles used to define the jet.

Let us now consider how the new sum rules can be used to learn about $g_2(x)$ and to test QCD.

5 Phenomenological tests of the ELT sum rule

The general field theoretic expression for $g_2(x)$ in terms of hadronic matrix elements of operators is given in (11), (12) and (13). As mentioned earlier, despite appearances to the contrary, $g_2(x)$ does not have any simple probabilistic parton model interpretation even though only quark operators appear in the matrix element (13). Nonetheless it is given by a sum over contributions coming from quark operators of definite flavour $f$ (the flavour label was suppressed in Section 2), so that the contribution of a given flavour of quark or antiquark to $g_2(x)$ is meaningful.

Moreover, since the flavour label is clearly irrelevant in the derivation, it must be true that (32) holds for the contribution to $g_2(x)$ of each flavour. Hence one has, for each flavour $f$,

$$\int_0^1 dx x [g_{1,f}^V(x) + 2g_{2,f}^V(x)] = 0. \quad (60)$$
Information about the contributions of a given flavour to $g_2(x)$ can be obtained by studying reactions with different targets and by studying non-purely-electromagnetic DIS, for example charge changing DIS involving $W^\pm$ exchange or, at large $Q^2$, interference between $\gamma$ and $Z^0$ exchange. There is also the possibility of focussing on specific flavours by looking at semi-inclusive DIS.

There thus appear to be several possibilities to learn about $g_{2,f}(x)$.

1. Assuming, as usual, that the contributions from sea quarks are the same in protons and neutrons, we can derive a kind of analogue of the Bjorken sum rule. For, then, from (32) or (60)

$$0 = \int_0^1 dx \{ g_p^1(x) + 2g_p^2(x) - g_n^1(x) - 2g_n^2(x) \}$$

$$= \int_0^1 dx \left\{ \frac{1}{6} [\Delta u_V(x) - \Delta d_V(x)] + 2g_{2,u}^V - 2g_{2,d}^V(x) \right\}. \quad (61)$$

Hence we have the interesting new sum rule

$$\int_0^1 dx \{ g_p^2(x) - g_n^2(x) \} = -\frac{1}{12} \int_0^1 dx [\Delta u_V(x) - \Delta d_V(x)] . \quad (62)$$

2. In unpolarized semi-inclusive DIS it is claimed that the study of meson production

$$\ell + N \rightarrow \ell' + M + X$$

where $M = \pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0$ etc. allows one to identify the contribution of a given $q_f$ or $\bar{q}_f$ to the unpolarized structure functions and it is proposed to use the same approach, but with a longitudinally polarized target at CERN [20] to identify the individual $\Delta q_f(x)$ and $\Delta \bar{q}_f(x)$ contributions to $g_1(x)$.

We suggest that the same method, but using a transversely polarized target, will allow the identification of the contributions $g_{2,f}(x)$ to $g_2(x)$ coming from a given flavour quark or antiquark.

Hence, in principle, the valence contribution to $g_2(x)$ of a given flavour, $g_{2,f}^V(x)$, can be measured.

3. A simpler method is to assume dominance of the $u$ and $d$ contributions and to study

$$\ell + N \rightarrow \ell' + JET + X$$

using a transversely polarized target and with identification of the charge of the jet ($\pm$). If the differences of cross-sections when the transverse spin is reversed, $\Delta_T d\sigma^{JET+}$ and $\Delta_T d\sigma^{JET-}$, are measured then $[\Delta_T d\sigma^{JET+} - \Delta_T d\sigma^{JET-}]$ will involve the combinations [22]

$$(g_{2,u} + g_{2,d}) - (g_{2,d} + g_{2,u}) = g_{2,u}^V - g_{2,d}^V. \quad (63)$$
It would seem possible to carry out such a measurement in the upgraded SMC experiment HMC with a forward magnetic spectrometer or in the HERMES experiment at HERA which uses a polarized gas jet target.

4. In charge changing DIS mediated by $W^\pm$ bosons the coupling to quarks and antiquarks is of opposite sign. If the cross-section differences under reversal of the transverse nucleon polarization can be measured for

$$\mu^+ N \to \bar{\nu}_\mu + X$$

and for

$$\mu^- N \to \nu_\mu + X$$

then, for the difference of these one has [21]

$$\Delta_T d\sigma^{\mu^+ \to \bar{\nu}_\mu} - \Delta_T d\sigma^{\mu^- \to \nu_\mu} \propto g^{W^+}_2 - g^{W^-}_2. \quad (64)$$

The precise relation between cross-sections and scaling functions is given in ref. [1]. However the expression for $g^W_2(x)$ given there, which was taken from ref. [23] is incorrect. In fact $g^W_2(x)$ is given in terms of the function $f_T(x)$ as occurs in (11). The only difference is in the coupling constants involved. Hence the combination occurring in (64) can be expressed in terms of the purely electromagnetic $g^V_2(x)$ valence parts discussed above:

$$g^{W^+}_2(x) - g^{W^-}_2(x) = 18g^V_{2,d}(x) - \frac{9}{2}g^V_{2,u}(x). \quad (65)$$

For an isoscalar target $A_0$ one then has

$$\left[ g^{W^+}_2(x) - g^{W^-}_2(x) \right]^{A_0}_{\text{per nucleon}} = \frac{25}{4} \left[ g^V_{2,u}(x) + g^V_{2,d}(x) \right]. \quad (66)$$

In principle one could combine (66) and (62) to study the individual $u$ and $d$ valence contributions to $g^V_2(x)$.

5. If an asymmetry measurement with transversely polarized target can be done at sufficiently large $Q^2$ so that $\gamma-Z^0$ interference is important, then

$$g^{\gamma Z}_2(x) = 2 \sum_f \left( \frac{g^V_f}{Q_f} \right) g_{2,f}(x) \quad (67)$$

where $g^V_u = \frac{1}{2} - \frac{4}{3} \sin^2 \theta_W$, $g^V_d = -\frac{1}{2} + \frac{2}{3} \sin^2 \theta_W$, $Q_f$ is the charge and $g_{2,f}(x)$ is the flavour-$f$ contribution to the pure electromagnetic $g^V_2(x)$. Measurement of $g^{\gamma Z}_2(x)$ thus provides further information about the flavour-$f$ contributions to $g^V_2(x)$.
6 Conclusions

The Operator Product Expansion provides expressions for the $n^{th}$ moments of $g_1(x)$ and $g_2(x)$ in terms of hadronic matrix elements of local operators for $n = \text{odd integer}$. In some cases these matrix elements are expected to be small leading to approximate sum rules for the odd moments of $g_{1,2}(x)$. We have shown how, working in a field-theoretic framework, one can derive expressions for the even moments of the valence parts of $g_{1,2}(x)$. These expressions cannot be written as matrix elements of local operators and do not coincide with the analytic continuation to $n = \text{even integer}$ of the OPE results.

Just as for the OPE one can in some cases argue that the hadronic matrix elements should be small, leading to approximate sum rules for the moments of the valence parts of $g_{1,2}(x)$. But, most importantly, for the case $n = 2$ we have proved rigorously that the hadronic matrix element vanishes, yielding the exact ELT sum rule

$$\int_0^1 dx \left[ g_1^V(x) + 2g_2^V(x) \right] = 0.$$ 

We have argued that the convergence properties of this sum rule are good and have discussed how it can be used to get information about $g_2(x)$ and as a further test of QCD.

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Figure 1: Simplest Feynman diagrams contributing to DIS at twist 2 and twist 3 level. (Crossed diagrams are not shown.)