Boltzmann scaling of spontaneous Hall current and nonequilibrium spin-polarization

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We extend the semiclassical Boltzmann formalism for the anomalous Hall effect (AHE) in non-degenerate multiband electron systems to the spin Hall effect (SHE) and unconventional Edelstein effect (UEE, cannot be accounted for by the conventional Boltzmann equation, unlike the conventional Edelstein effect). This extension is confirmed by extending the Kohn-Luttinger density-matrix transport theory in the weak disorder-potential regime. By performing Kubo linear response calculations in a prototypical multiband model, the Boltzmann scaling for the AHE/SHE and UEE is found to be practically valid only if the disorder-broadening of bands is quite smaller than the minimal intrinsic energy-scale around the Fermi level. Discussions on this criterion in various multiband systems are also presented. A qualitative phase diagram is proposed to show the influences of changing independently the impurity density and strength of disorder potential on the AHE/SHE and UEE.

I. INTRODUCTION

Disorder effects on nonequilibrium properties of Bloch electrons is a basic issue in condensed matter physics. Many of them can be discussed within the relaxation time approximation of the conventional semiclassical Boltzmann equation [1]. However, some transport phenomena related to the spin-orbit coupling, such as the spin Hall effect (SHE) and anomalous Hall effect (AHE) [2, 3], contain intriguing disorder-induced effects that cannot be treated by the conventional Boltzmann equation [2, 3]. Another spin-orbit-induced nonequilibrium phenomenon is the Edelstein effect – nonequilibrium spin-polarization driven by external electric fields [3]. The conventional Edelstein effect is described by the conventional Boltzmann equation [6]. While the unconventional Edelstein effect (UEE), in which a nonequilibrium spin-polarization arises in the direction perpendicular to that in the conventional Edelstein effect [7, 8], is related to aforementioned intriguing disorder effects [9]. In the presence of exchange coupling to a local magnetization, the conventional and unconventional Edelstein effects give rise to the fieldlike and dampinglike spin-orbit torques on the magnetization, respectively [3, 4].

Those intriguing effects due to static disorder, including the skew scattering [3], side-jump [3, 4, 12] and scattering off pairs of impurities [12, 13], have been incorporated into the generalized semiclassical Boltzmann formalism by semiclassical or semi-phenomenological arguments. For the AHE, the generalized Boltzmann formalism formulated in the weak disorder-potential regime has its root in the Kohn-Luttinger density-matrix transport approach to electrical conductivities [1, 4, 11, 14, 15]. However, in the case of the SHE and UEE, such a necessary identification is still absent. In the present paper it will be provided by extending the Kohn-Luttinger approach to the SHE and UEE [16].

The Boltzmann formalism yields the Boltzmann scaling [13]

$$\alpha = c_{sk} + (c_{in} + c_{AQ}) \rho_{yy}$$

(1)

in the presence of one type of static disorder. Here $\alpha$ can represent the anomalous Hall ratio ($\sigma_{xy}/\sigma_{yy}$, $\sigma_{xy}$ is the Hall conductivity, $\sigma_{yy}$ is the longitudinal conductivity), spin Hall ratio ($\sigma_{xy}/\sigma_{yy}$, $\sigma_{xy}^2$ is the spin Hall conductivity) and the UEE-efficiency per current (e.g., $\chi_{yy}/\sigma_{yy}$, $\chi_{yy}$ is the UEE response coefficient). $c_{sk}$ stands for the longitudinal resistivity and $\rho_{yy} \gg \rho_{xy}$ is assumed. $c_{in}$ comes from the intrinsic contribution, whereas $c_{sk}$ and $c_{AQ}$ come from the skew scattering and anomalous quantum (called side-jump in Refs. [2, 3]) contributions, respectively. These nomenclatures are explained in Sec. III. A well-defined scaling relation exists only if the scaling parameters remain constant as the scaling variables change. In the Boltzmann framework $c_{sk}$, $c_{in}$ and $c_{AQ}$ remain constant when the impurity density is changed. Thus the so tuned $\rho_{yy}$ plays the role of a scaling variable, and $c_s$’s scaling parameters. The multivariable Boltzmann scaling for the AHE in the presence of more than one type of disorder has also been proposed [17] via an approach equivalent to the Boltzmann formalism [9, 12, 13]. The Boltzmann scaling and its multivariable generalization have played the central role in understanding measurements and analyzing numerical results in the field of AHE/SHE [17, 19, 22].

However, theoretically the regime of validity of Boltzmann scaling remains unclear. This is the second topic in the present paper. The Boltzmann formalism is intuitively anticipated to work well only if the disorder-broadening $h/\tau$ of bands is quite smaller than the minimal intrinsic energy scale $\Delta$ of the band structure around the Fermi level. $\Delta$ is usually the minimal interband splitting around the Fermi level and depends on the position of the latter. Although some previous researches on the intrinsic AHE/SHE support this idea [24, 26], some other work suggest that the Boltzmann scaling is valid up to $h/\tau \lesssim \eta F$ [27, 28] or $h/\tau \lesssim 0.1 \eta F$ [29]. This situation has caused confusion in understanding experimental results [32]. Focusing on the case of short-range weak disorder-potential ($DV_0 \lesssim 0.1$ in practice, $D$ is the typical density of states around the Fermi level, $V_0$ is the Fourier component of the disorder potential $V (r)$ at zero wavevector), we find that the Boltzmann scaling is practically or ap-
proximately valid if $\frac{\hbar}{\tau} < \frac{A}{\pi}$, i.e., $\left(\frac{\hbar}{2\pi}\right)^2 < 0.1$. This is obtained in a prototypical multiple conduction-band model and found to be applicable in various other systems. Moreover, a qualitative phase diagram is proposed to show the influences of changing independently the impurity density and the strength of disorder potential on the AHE/SHE and UEE.

The present paper is organized as follows. The Boltzmann formulations are outlined in Sec. II, whereas the regime of validity of the Boltzmann scaling for AHE/SHE and UEE is analyzed in Sec. III. Section V summarizes the paper. Appendices A and B include necessary discussions on the semiclassical Boltzmann formalism, whereas some calculation details are given in Appendix C.

II. KOHN-LUTTINGER DERIVATION OF THE BOLTZMANN TRANSPORT

In the Boltzmann formalism of linear response, the average value of an observable $A$ (quantum mechanically, Hermitian operator $\hat{A}$) in the presence of a dc weak uniform electric field $\mathbf{E}$ is given by $A = \sum_j f_j A_j$, with the index $l$ denoting the carrier state. In the present paper we consider non-degenerate multiband carrier systems. The semiclassical distribution function $f_j$ is governed by the generalized semiclassical Boltzmann equation in nonequilibrium steady-states in the presence of elastic carrier-impurity scattering. $A_j$ can be written as

$$A_j = A_j^0 + \delta^{\text{in}} A_j + \delta^{\text{s}} A_j,$$

where $\delta^{\text{in}} A_j = -\hbar e \mathbf{E} \cdot \sum_{\nu, \nu' \neq \nu} 2 \text{Im} \langle u_{\nu} | \hat{\mathbf{V}} | u_{\nu'} \rangle \delta_{kk'} A_{\nu \nu'} / d_{\nu \nu'}^2$ and

$$\delta^{\text{s}} A_j = \sum_{\nu, \nu' \neq \nu} \frac{\langle V_{\nu} \hat{V}_{\nu'} \rangle A_{\nu \nu'} | d_{\nu \nu'}^2 | + 2 \text{Re} \sum_{\nu, \nu' \neq \nu} \langle V_{\nu} \hat{V}_{\nu'} \rangle A_{\nu \nu'} | d_{\nu \nu'}^2 |.$$  

Here $|l\rangle = |k\rangle |u_{\nu}\rangle$ is the Bloch state, $l = (\eta, \mathbf{k})$ with $\eta$ the band index and $\mathbf{k}$ the momentum. $\langle \ldots \rangle$ represents the average over disorder configurations, $d_{\nu \nu'} \equiv \epsilon_{\nu} - \epsilon_{\nu'}$, $d_{\nu \nu'}^2 \equiv d_{\nu \nu'} \pm i \hbar s$ with $s \to 0^+$. In the case of $\hat{A} = \hat{\mathbf{V}}$, $\delta^{\text{in}} \mathbf{V}_l$ and $\delta^{\text{s}} \mathbf{V}_l$ coincide with the Berry-curvature anomalous velocity [4] and the side-jump velocity [4], respectively. Both of them have microscopic derivations [8]. While, in other cases $\delta^{\text{in}} A_j$ and $\delta^{\text{s}} A_j$ were only added into the Boltzmann formalism semi-phenomenologically [4].

In the present paper we give the microscopic derivation to Eq. (2) in the case of $\hat{A}$ other than $\hat{\mathbf{V}}$. Because this justification is obtained by resorting to the Kohn-Luttinger density-matrix approach [14], we provide it in Appendix A in order not to introduce too many notations in the main text. From that derivation one can see that, Eq. (2) accounts for the off-diagonal response of the out-of-equilibrium single-particle density-matrix in the band-eigenstate representation [31].

III. REGIME OF VALIDITY OF BOLTZMANN SCALING

A. Two-conduction-band model calculation

We consider the 2D Hamiltonian

$$\hat{H}_0 = \frac{\hbar^2 \mathbf{k}^2}{2m} + \alpha_R \hat{\mathbf{\sigma}} \cdot (\mathbf{k} \times \hat{\mathbf{z}}) - \epsilon_I \hat{\sigma}_z,$$

where $m$ is the effective mass of conduction electron, $\mathbf{k} = k (\cos \phi, \sin \phi)$ the 2D wavevector, $\hat{\mathbf{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the Pauli matrices for electron spin. In different qualitative realizations of this Hamiltonian, $\alpha_R > 0$ and $\epsilon_I > 0$ have different physical interpretations. In ultrathin ferromagnets embedded between two asymmetric interfaces [32], $\alpha_R$ describes the Rashba coupling due to the gating field. In gaped transition-metal dichalcogenides [3, 33, 34], $\alpha_R$ is the Rashba spin-orbit coupling coefficient, $\epsilon_I$ is the exchange coupling. In gated systems with multiple-Fermi-surfaces and avoided band-anticrossing point [2]. In this case $\epsilon_I$ plays the role of the spin-orbit coupling that lifts the accidental degeneracy of band dispersions with the velocity $\alpha_R / \hbar$ [27, 28]. Although this special 2D model breaks both the inversion and time reversal symmetries, some generic qualitative insights can still be acquired, which apply to various ferromagnetic [2, 27, 28, 35], and nonmagnetic [30] materials possessing multiple-Fermi-surfaces.

For any energy $\epsilon > \epsilon_I$ there are two iso-energy rings corresponding to the two subbands $\eta = \pm$: $k^2_\eta (\epsilon) = \frac{2m}{\hbar^2} (\epsilon - \eta \Delta_\eta (\epsilon))$ where $\epsilon_R = m (\Delta_\eta)^2$ and $\Delta_\eta (\epsilon) = \sqrt{\epsilon^2 + \epsilon_R^2 + 2 \epsilon \eta \epsilon_R - \eta^2 R}$. We focus on the case where both subbands are partially occupied in our analytic treatment, whereas the regime $\epsilon_F < \epsilon_I$ will also be addressed later (Sec. III. B). The qualitative insights obtained in the former case can also be applied to the latter one. Randomly distributed identical $\delta$-scatterers are assumed. For the simplest assumption of scalar disorder, the pathological properties of model 1 in the case of both subbands partially occupied, e.g., the vanishing AHE/SHE and UEE irrespective of the impurity density under the noncrossing approximation in the case of weak disorder potential [2, 3, 4, 23], make it inconvenient to extract general insights. Fortunately, one can get around this inconvenience by just assuming another type of short-range disorder $\tilde{V} = V_0 \hat{\sigma}_z$. Although such a kind of disorder has its root in realistic considerations as detailed in Refs. 4, 37, 38, we just regard it as an approach that gets around the pathological properties of the Rashba model and makes the model a prototypical one from which general qualitative insights can be extracted. In the following we will focus on the UEE in this model, whereas the considerations on the AHE and SHE are completely similar.
We will only focus on the aspects of the model that can be meaningful for general multiband systems.

1. Boltzmann calculation

In model \(|\{\mathbf{S}\}|\), due to the UEE there is a nonequilibrium spin density \((\delta \mathbf{S})_i\parallel\) parallel to the driving electric field. The Boltzmann theory yields \((\delta \mathbf{S})_i\parallel = \delta^{ii} \mathbf{S} + \delta^{AQ} \mathbf{S} + \delta^{sk} \mathbf{S} \). Here \(\delta^{ii} \mathbf{S}\) is the intrinsic contribution, \(\delta^{AQ} \mathbf{S}\) is termed the anomalous quantum contribution which arises from disorder but turns out to be independent of the impurity density, \(\delta^{sk} \mathbf{S}\) is the skew scattering contribution inversely proportional to the impurity density. The anomalous/spin Hall current within the Boltzmann framework can also be parsed in the same way. Two necessary notes on the Boltzmann calculation are in order.

First, in the weak disorder-potential regime \(\delta^{sk} \mathbf{S}\) is dominated by the contribution from \(o(V^3)\) non-Gaussian disorder correlation \(\langle V^3 \rangle \sim n_{im} V_0^3\). Here \(n_{im}\) is the impurity density, \(\langle \cdot \rangle\) is the connected part of disorder correlation. The transport time of the \(o(V^3)\) skew scattering is of scale \(DV_0\tau\) (Appendix C). The higher-order skew scattering is usually negligible compared to the \(o(V^3)\) one in the weak disorder-potential regime, because, e.g., the transport time of the \(o(V^3)\) skew scattering \([4, 11]\) is of scale \((DV_0)^2 \tau \ll DV_0\tau\) (Appendix C).

Second, the effect of scattering off pairs of impurities enters into \(\delta^{AQ} \mathbf{S}\) via both the noncrossing-diagram and crossing-diagram parts of \(\omega_{ji}\) in \(o(V^4)\) (\(\omega_{ji}\) is the semiclassical scattering rate \([4, 8]\), see Appendix B). We only address the noncrossing-diagram part in the concrete calculation. For the purpose of this paper, the quantitative difference due to the inclusion of the crossing part \([42]\) is unimportant.

Concrete calculations of \(\delta S_{yy} = \lambda_{yy} E_y\) have been presented in Appendix C and Ref. \([23]\). Here we write down the UEE efficiency \(\alpha = \lambda_{yy}/\sigma_{yy}\) in a form

\[
\alpha = -e\alpha R D_0 \left[ \frac{1}{mV_0} \frac{f_{sk}}{f_L} \frac{f_{sk}^{UEE}}{f_L} (C_1, 0) \right] + f_{in+AQ} (C_1, D_2, 0) \rho_{yy},
\]

which is convenient to be compared with the corresponding result Eq. (5) obtained in the Kubo-Streda formula. Here \(D_0 = \frac{m}{2\pi e}\), and other notations are described in the next subsection. What is important is that \(f_{sk}^{UEE} (C_1, 0)\), \(f_L (C_1, 0)\) and \(f_{in+AQ} (C_1, D_2, 0)\) are all independent of both the impurity density and disorder potential. Thus when tuning \(\rho_{yy}\) via changing the impurity density, Eq. (5) is just the Boltzmann scaling \([11]\).

2. Kubo calculation

The linear response to a dc uniform electric field in the single-particle picture with static disorder can be found by the Kubo-Streda formula \([43]\). In the weak disorder-potential regime not far away from the weak disorder-potential limit, one can apply the standard ladder approximation and consider the conventional Mercedes star diagrams for the \(o(V^3)\) skew scattering \([44]\), leading to \(\lambda_{yy} = \lambda_{yy}^b + \lambda_{yy}^f + \lambda_{yy}^{sk} \) at the zero-temperature limit with

\[
\lambda_{yy}^b = -e\alpha R D_0 \tau \chi_D I_2,
\]

\[
\lambda_{yy}^f = -e\alpha R D_0 f_{in+AQ} (I_1, \tau I_2, I_2),
\]

\[
\lambda_{yy}^{sk} = -e\alpha R D_0 f \frac{mV_0}{\pi^2} \chi_{UEE} (I_1, I_2),
\]

and \(\sigma_{yy} = \frac{e^2}{\pi^2 R} f \epsilon f_L (I_1, I_2^2)\). \(\lambda_{yy}^b\) is the bubble contribution, \(\lambda_{yy}^f\) is the skew scattering contribution, \(\lambda_{yy}^{sk}\) is the skew scattering correction, \(\lambda_{yy} = \lambda_{yy}^b + \lambda_{yy}^f + \lambda_{yy}^{sk}\) with \(\lambda_{yy}^f\) the ladder vertex correction. \(\chi_{UEE}\) is the Fermi sea term \([43]\). The so-called Fermi sea term \([43]\) \(\lambda_{yy}^{ll}\) equals zero in the present case \([23]\). Thus the UEE efficiency reads

\[
\alpha = -e\alpha R D_0 \left[ \frac{1}{mV_0} \frac{f_{sk}}{f_L} \frac{f_{sk}^{UEE}}{f_L} (C_1, 0) \right] + f_{in+AQ} (I_1, \tau I_2, I_2) \rho_{yy} + f_{in+AQ} (I_1, \tau I_2, I_2) \rho_{yy}. \tag{6}
\]

The \(f's\) depend on disorder via their arguments. The expressions of \(f's\) are given in Appendix C, with

\[
I_1 = \sum_{\eta} \left[ \frac{\sin^2 \eta_\theta + 1 + \frac{\cos^2 \eta_\theta}{(2\Delta_{\eta} (\epsilon_F) \tau)^2}}{1 + 2\Delta_{\eta} (\epsilon_F) \tau^2} \right] \frac{D_\eta (\epsilon_F)}{4D_0},
\]

\[
I_2 = -i \frac{\tau}{\hbar} \sum_{\eta} \frac{\epsilon_I}{1 + 2\Delta_{\eta} (\epsilon_F) \tau^2} \frac{D_\eta (\epsilon_F)}{D_0}. \tag{7}
\]

Here \(\cos \eta_\theta = \frac{\epsilon_J}{\Delta_{\eta} (\epsilon_F)}\), \(\sin \eta_\theta = \frac{\alpha_{sk} (\epsilon_F)}{\Delta_{\eta} (\epsilon_F)}\). The results for Hall conductivities are also presented in Appendix C.

\[
\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.pdf}
\caption{Behaviors of \(I_1\) and \(I_2\) as \(h/\tau\) increases.}
\end{figure}
\end{figure}

\(I_{1,2}\) depend on the parameter \(2\Delta_{\eta} (\epsilon_F) \tau\) which measures the competition between the intrinsic energy scales and the disorder-broadening of bands around the Fermi level. When \(2\Delta_{\eta} (\epsilon_F) > h/\tau\), the topology of Fermi surfaces remains unchanged and the multiband structure around the Fermi level survives, so the Boltzmann theory is applicable. However, when \(2\Delta_{\eta} (\epsilon_F) < h/\tau\), the
intrinsically multiband structure around the Fermi level collapses owing to the large disorder-broadening. This case cannot be described by the Boltzmann theory. More accurately, we take \( \left( \frac{h}{2\Delta_+ (\epsilon_F) / \tau} \right)^2 \lesssim 0.1 \), i.e., \( \frac{h}{\tau} \lesssim \frac{2\Delta_+ (\epsilon_F)}{\pi} \), as the practical criterion for the validity of Boltzmann theory. Because \( \Delta_+ < \Delta_- \), \( 2\Delta_+ \) is the minimal intrinsic energy scale around the Fermi level. Thus the criterion can be refined to be \( \frac{h}{\tau} \lesssim \frac{2\Delta_+ (\epsilon_F)}{\pi} \).

As shown in Fig. 1, \( I_1 \) is quite robust against increasing \( h/\tau \) and \( I_2 \ll I_1 \) when \( h/\tau \) is smaller than \( \Delta_+ (\epsilon_F) \). When \( h/\tau \ll \Delta_+ (\epsilon_F) \), Eq. 7 yields \( I_1 = C_1 + D_1 \left( \frac{h}{\tau} \right)^2 \)

and \( I_2 = D_2 \frac{h}{\tau} \). Here \( C_1 \), \( D_1 \), and \( D_2 \) are disorder-independent quantities. Thus \( \chi_{yy}^b \sim \tau I_2 \) is nearly constant when \( h/\tau \ll \frac{2\Delta_-}{\pi} \), and

\[
(c_{in} + c_{AQ}) \propto f_{in+AQ} \left( I_1, \tau I_2 \right) \simeq f_{in+AQ} (C_1, D_2, 0),
\]

where \( f_{in+AQ} (C_1, D_2, 0) \) just corresponds to the Boltzmann value of \( (c_{in} + c_{AQ}) \). Thus the scaling parameter \( c_{in} + c_{AQ} \) is well-defined up to \( h/\tau \lesssim \frac{2\Delta_-}{\pi} \). For the skew scattering,

\[
c_{sk} \propto \frac{mV_0}{h^2} \frac{f_{sk}^{UEE} (I_1, I_2)}{f_L (I_1, I_2)} = \frac{mV_0}{h^2} \frac{f_{sk}^{UEE} (C_1, 0)}{f_L (C_1, 0)}
\]

is also expected to be insensitive to the increasing impurity density when \( h/\tau \ll \frac{2\Delta_-}{\pi} \), since the corresponding Boltzmann value is just \( \frac{mV_0}{h^2} \frac{f_{sk}^{UEE} (C_1, 0)}{f_L (C_1, 0)} \).

The definitions of the intrinsic, anomalous quantum and skew scattering contributions are introduced in the last subsection in the Boltzmann framework. Given that the Boltzmann scaling holds practically in the regime \( h/\tau \ll \frac{2\Delta_-}{\pi} \), above definitions of these contributions also remain valid in practice in this regime. Therefore, in the case of finite but weak disorder potential, the Boltzmann scaling can be valid even if the impurity density is not dilute in experiments. When the impurity density increases further so that \( h/\tau > \frac{2\Delta_-}{\pi} \), apparent \( n_{im} \)-dependence of \( (c_{in} + c_{AQ}) \) is anticipated as in Fig. 2(d), thus the Boltzmann scaling no longer work well. In this case the conventional definitions \[3] of the intrinsic, anomalous quantum and skew scattering contributions, which are in fact born in the Boltzmann regime, are no longer suitable. All these points can be read out from Fig. 4.

B. General ideas based on model (4)

1. Multiple intrinsic energy scales near the Fermi level

In complicated multiband systems there exist multiple intrinsic energy scales around the Fermi level, e.g., \( 2\Delta_+ (\epsilon_F) \) and \( 2\Delta_- (\epsilon_F) \) in the case of both subbands partially occupied in model (4). The behaviors of \( c_{sk}, c_{in} \) and \( c_{AQ} \) are predominantly dictated by the smallest intrinsic energy scale \( 2\Delta_+ \). As a specific example, one can assume \( 2\Delta_- \ll 2\Delta_+ \), then \( I_2 \) and the \( \tau \)-dependent part of \( I_1 \) are dictated by \( 2\Delta_+ / \tau \). This understanding accounts well for the numerical finding in the intrinsic AHE of a multi-d-orbital tight-binding model \[24\]. In Ref. \[24\] the minimal intrinsic energy scale around the Fermi level is about \( \Delta = 0.417 \) Ry (1 Ry = 13.6 eV), thus the Boltzmann scaling for the intrinsic contribution is anticipated to be valid up to \( \gamma = \frac{h}{\tau} \simeq \frac{\Delta_-}{\pi} = 0.066 \) Ry according to our arguments. This is in exact agreement with the numerical results presented in Ref. \[24\]. In the transition
energy scales exist only slightly away from the Fermi level. This case occurs also in model (4), as shown in Fig. 3(b): the band-anticrossing region is located slightly away from the Fermi level. When the Fermi surface is smeared by increasing disorder, the dominant intrinsic energy scale changes from $2\Delta \ell$ to $2\ell$. The change of the dominant intrinsic energy scale may induce complicated behaviors of the AHE/SHE and UEE that need case by case analysis, because the magnitude of these effects may be different for different dominant intrinsic energy scales. In the case of Fig. 3(b), because the intrinsic Hall current takes the largest value in the narrow band-anticrossing region, it is expected to increase first as the band-anticrossing region is involved when increasing disorder. After reaching a maximum value the Hall current begins to decrease as the disorder density increases further, because the multiband structure around the Fermi level finally collapses. This observation accounts for the non-monotonic behavior of the intrinsic SHE with respect to increasing $\gamma$ suggested by tight-binding calculations in transition metal Ta [29]. Thermal smearing of Fermi surface has similar influences if the dominant intrinsic energy scale is very small ($< 26$ meV). Shitade et al. [46] once showed the non-monotonic intrinsic anomalous Hall conductivity with respect to increasing temperatures in the 2D massive Dirac model.

2. Multiple extrinsic energy scales

As we have mentioned, there are other extrinsic energy scales than $\frac{h}{\gamma}$ in the case of weak disorder potential, such as $\Delta \sim \frac{h}{\tau |DV_0|}$. One can roughly estimate that the crossover between the skew scattering and intrinsic-plus-anomalous-quantum $(in + AQ)$ regimes occurs at $\frac{h}{\tau |DV_0|} \sim \Delta$. In the very narrow ($\ell < \ell_R$) resonant window of model (4), the skew-scattering-to-intrinsic crossover was estimated [28] to occur at $\frac{h}{\tau} \sim \frac{2|V_0|}{\pi |DV_0|} \ell \sim 2\pi D|V_0|\epsilon_f$, consistent with our idea. However, out of the resonant region, we do not find a general and rigorous theoretical criterion for the crossover. From Fig. 2, one can see that the naively expected criterion $\frac{h}{\tau} \sim 2\pi D|V_0|\Delta_+ \sim \epsilon_F$ is only qualitatively useful, and it is likely that other intrinsic energy scales also affects the crossover.

The $o(V^4)$ skew scattering is linked to the extrinsic energy scale $\frac{h}{\tau |DV_0|^2}$ and thus is expected to decay significantly at $\frac{h}{\tau} \sim (DV_0)^2 \Delta$. Thus the $o(V^4)$ skew scattering is much smaller in magnitude and decays much faster than the $o(V^3)$ one in the case of weak disorder potential. Although it is much larger than the $in + AQ$ contribution in the limit of dilute impurities, it is, meanwhile, overwhelmed by the $o(V^3)$ skew scattering. Thus we neglect the $o(V^4)$ skew scattering.

In this paper we only consider zero-range static impurities, for which the transport time and quantum lifetime of electrons are not much different. This makes the qualitative analysis of the time scale of skew scattering reliable. If the charged impurities dominate, especially

metal Pt it was found that [20] the minimal interband splitting around the Fermi level is $\Delta = 0.035$ Ry, thus the constant behavior of the intrinsic spin Hall conductivity is anticipated to be valid up to $\gamma \simeq \frac{\Delta}{\pi} = 5.5 \times 10^{-3}$ Ry, in exact agreement with the tight-binding numerical results presented in Ref. [20].

The position of Fermi level dictates which intrinsic energy scales are relevant to determining the results presented in Ref. [26].

FIG. 3. Some cases of the Fermi-level position in model (4) addressed in the qualitative discussions in Sec. III. B.
in 2D high-mobility semiconductor heterojunctions, the ratio of the transport time and lifetime can be very large [47]. In this case one should be cautious when making qualitative conclusions about the skew scattering [48].

C. Qualitative phase diagram

In the last subsection we have discussed the possible rich behaviors in the Boltzmann regime. Now we assume the simplest case where only one dominant intrinsic energy scale $\Delta$ is present around the Fermi level and other intrinsic energy scales exist far away. We adopt the qualitative criterion $\frac{b}{\Delta} \approx 2\pi V_0 \frac{\Delta}{\pi}$ for the crossover from the skew scattering regime to the $i_n + AQ$ regime. Then we give the phase diagram in Fig. 4 for the AHE/SHE and UEE in the case of weak disorder potential.

Figure 4 reveals that, in analyzing disorder effects on the AHE/SHE and UEE, the conventional discussion based only on the dichotomy between the weak scattering and strong scattering limits is not complete. Moreover, the usually used term “weak disorder regime” is not clearly defined. Instead, the strength of the disorder potential and the impurity density should be considered independently. And thus one should distinguish the “weak disorder-potential regime” and “dilute impurity regime”. In the present paper we have focused on the weak disorder-potential regime, whereas we comment on the dilute-impurity and strong disorder-potential case in the last paragraph of this section. The x-axis label of Fig. 4 measures the strength of the disorder potential, whereas the y-axis label measures the impurity density. The sk-to-$i_n + AQ$ crossover is represented qualitatively by the red dashed curve, whereas the green curve is the boundary of the Boltzmann and non-Boltzmann regimes. We expect that this qualitative phase diagram provides a necessary clarification of the way of thinking about disorder effects on the AHE/SHE and UEE.

There is a regime where the Boltzmann scaling holds and the $i_n + AQ$ contribution dominates, thus the constant behavior of $(c_{i_n} + c_{AQ})$ with varying $n_{im}$ is possible in experiments. Larger $D V_0$ shrinks the range of $n_{im}$ in which the constant behavior of $(c_{i_n} + c_{AQ})$ may exist.

In order to address the regime $\frac{b}{\Delta} > \Delta$, one may try to employ the equivalence $c_{i_n} = \chi_{yy} + \chi_{yy}^{11}$, $c_{AQ} = \chi_{yy}^{\text{ver}}$ and $c_{sk} = \chi_{yy}^{sk} \rho_{yy}$ in the Boltzmann regime to “continue” $c_{i_n}$, $c_{AQ}$ and $c_{sk}$ out of the Boltzmann regime, similar to the analytical continuation in complex analysis. Along this route, according to Eq. (6) one can view $\lambda_{yy} + \chi_{yy}^{11}$, $\chi_{yy}^{\text{ver}}$ and $\chi_{yy}^{sk}$ as the intrinsic, anomalous quantum and skew scattering contributions, respectively, when $\frac{b}{\Delta} > \Delta$ in the weak disorder-potential regime [49]. This “continuation” has already been widely employed in discussing the intrinsic AHE/SHE, e.g., in Refs. [24, 25, 40]. In the weak disorder-potential regime the continuation for the anomalous quantum and skew scattering contributions is also feasible. As shown in Fig. 2(d), when $\frac{b}{\Delta} \geq 2\Delta$ the skew scattering contributions of $\chi_{yy}^{sk}$, one cannot observe the well-defined Boltzmann scaling or $\tau$-independent $(c_{i_n} + c_{AQ})$. This situation is represented by the regime above the green curve in Fig. 4 and most relevant in the case of very small $\Delta$ [27, 28, 40, 40] or very high impurity density [20].

In the case of $DV_0 \gtrsim 0.1$ in Fig. 4, the skew scattering always dominates over the $i_n + AQ$ contribution in the Boltzmann regime. When $\frac{n_{im}}{b} \Delta$ increases into the non-Boltzmann regime $\hbar/\tau > \Delta$, the $i_n + AQ$ contribution gradually dominates over the skew scattering, but meanwhile the semiclassical Boltzmann formalism already breaks down and one cannot observe $n_{im}$-independent constant $(c_{i_n} + c_{AQ})$.

Before ending this section, we mention that Luttinger and Kohn also designed a transport formalism in the dilute impurity limit without limiting the strength of disorder potential, based on a impurity-density expansion [50]. The Boltzmann equation for free electrons ($\epsilon_k = \frac{\hbar^2 k^2}{2m}$) was produced, from which the nonequilibrium distribution function of leading order $o\left(n_{im}^{-1}\right)$ and sub-leading order $o\left(n_{im}^0\right)$ can be obtained [50]. Thus it is anticipated that when the impurity density is low but finite, the Boltzmann formalism is still valid if the disorder potential is not too strong. This is consistent with the trend of our qualitative phase diagram in the larger-$DV_0$ part in Fig. 4. Accordingly we speculate that the rich transport physics, such as the crossover from the skew scattering to $i_n + AQ$ regime, in the case of strong disorder-potential mainly occurs out of the Boltzmann regime. Nevertheless, a comprehensive picture for the anomalous quantum contribution in the Boltzmann formalism in the strong disorder-potential and dilute impurity case is still absent. This issue calls for more future attention.

FIG. 4. A qualitative phase diagram for the AHE/SHE and UEE in the presence of static impurities in the weak disorder-potential regime. The localization effect is not included in our research. The Boltzmann scaling works well in the regime below the green curve, and the skew scattering (sk) dominates in the regime below the red dashed curve. In the brown regime, the intrinsic-plus-anomalous-quantum ($i_n + AQ$) contribution still dominates, but cannot be well described by the semiclassical Boltzmann formalism. Here we assume $V_0 > 0$. 

\[DV_0\]

\[\text{AQ, sk, and in + AQ, but beyond Boltzmann}\]
IV. SUMMARY

In summary, we have extended the semiclassical Boltzmann formalism for the AHE to SHE and UEE, and confirmed this semiclassical formalism by extending the Kohn-Luttinger density-matrix transport approach in the weak disorder-potential regime to the linear response of spin current and spin density. Then we investigated the regime of validity of the Boltzmann scaling for the AHE/SHE and UEE, by performing Kubo linear response calculations in a simple but prototypical multiband independent-carrier (electron or hole) model. It is found that the Boltzmann scaling is practically valid provided that the disorder-broadening of bands is quite small than the minimal intrinsic energy-scale around the Fermi level. We also illustrated that the qualitative insights acquired in the prototypical model system indeed account for the behaviors of the AHE/SHE in various realistic systems. Moreover, we proposed a qualitative phase diagram showing the influences of changing independently the impurity density and the strength of disorder potential on the AHE/SHE and UEE.

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Appendix A: Justification of Eq. (2)

We only address the part of the Kohn-Luttinger density-matrix approach that is necessary for confirming our Eq. (2). The original Kohn-Luttinger approach deals with the response of electric current to the external electric field. Here we just extend it to other observables such as spin current and spin density. This extension only concerns the response of off-diagonal elements of the single-particle density matrix to the external weak dc uniform electric field.

1. Preliminaries: Kohn-Luttinger single-particle formulation and linear response

We introduce the notation $\hat{A}$ to mean the representation of operator $A$ in the second-quantized formalism. For a single-carrier operator, i.e., $\hat{A} = \sum_i \hat{A}_i$ where $\hat{A}_i$ depends only on the dynamical variables of the $i$-th carrier, one has $\hat{A} = \sum_{nn'} A_{nn'} a_n^\dagger a_{n'}$, where $A_{nn'}$ are the matrix elements in the $n$ representation of single-carrier space, $a_n^\dagger (a_n)$ is the creation (annihilation) operator on the single-carrier eigenstate $|n\rangle$. The expectation value of $\hat{A}$ is given by $\langle A \rangle = Tr \left( \hat{n}_t \hat{A} \right)$, where $Tr$ denotes the trace operation in the occupation-number space, and the many-particle density matrix $\hat{\rho}_T$ in the occupation-number representation is governed by the quantum Liouville equation $i\hbar \frac{\partial}{\partial t} \hat{\rho}_T = [\hat{H}_T, \hat{\rho}_T]$.

The expectation value of a single-carrier operator $\hat{A}$ can be expressed in terms of $\hat{A}$ and a single-carrier operator $\hat{\rho}_T$:

$$\langle A \rangle = \sum_{nn'} A_{nn'} \langle \hat{\rho}_T \rangle_{nn'} = Tr \left( \hat{A} \hat{\rho}_T \right),$$

for $(\hat{\rho}_T)_{nn'} \equiv Tr \left( \hat{\rho}_T a_n^\dagger a_{n'} \right).$ (A1)

Here $tr$ denotes the trace operation in the single-carrier Hilbert space. Kohn-Luttinger noticed that [14], when the total Hamiltonian is also a single-carrier operator $\hat{H}_T = \sum_{mm',nn'} (\hat{H}_T)_{mm',nn'} a_n^\dagger a_{n'}$, the equation of motion for $(\hat{\rho}_T)_{nn'}$ reads (Schroedinger picture) $i\hbar \frac{\partial}{\partial t} (\hat{\rho}_T)_{nn'} = [\hat{H}_T, \hat{\rho}_T]_{nn'}$. The $n$ representation in the single-carrier Hilbert space is arbitrary thus

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}_T = [\hat{H}_T, \hat{\rho}_T],$$

with the operators acting on the single-carrier space. $\hat{\rho}_T$ satisfies $(\hat{\rho}_T)_{nn} = \langle N_n \rangle \geq 0$ and $tr \hat{\rho}_T = N_c$ with $N_n = a_n^\dagger a_n$ and $N = \sum_n N_n$. Although normalized to the carrier number $N_c$ instead of 1, $\hat{\rho}_T$ is often referred to as the single-particle density matrix, the diagonal elements of which represent the average occupation numbers of single-particle eigenstates rather than occupation probability. This character implies that $\hat{\rho}_T$ can be regarded as a quantum-statistical generalization of the single-particle density function described by the classical Boltzmann equation, and thus the equation of motion for $\hat{\rho}_T$ may reduce to a Boltzmann-type transport equation for diagonal elements of $\hat{\rho}_T$. This idea motivates one to split the quantum Liouville equation into diagonal and off-diagonal parts in the following.

The single-carrier Hamiltonian reads $\hat{H}_T = \hat{H}_0 + \hat{H}' + \hat{H}_F$, where $\hat{H}_0$ is the single-particle free Hamiltonian, $\hat{H}' = \lambda \hat{V}$ with $\lambda$ a dimensionless parameter and $\hat{V}$ the potential produced by randomly distributed static impurities, and the field term $\hat{H}_F = \hat{H}_1 e^{i\omega t}$ with $\hat{H}_1 = -e \hat{E} \cdot \hat{r}$ arises from the electric field adiabatically switched-on from the remote past $t = -\infty$. The infinitesimal positive $s$ in $\hat{H}_F$ can be taken to be the same as the $s$ which appears as a regularization factor in the T-matrix theory of the semiclassical Boltzmann formalism (see the main text). This is because the physical situation is obtained by taking the limit $s \to 0^+$. We remind that a similar note on the infinitesimal positive $s$ has appeared in the
The Kohn-Luttinger theory starts from Eq. (A2) in the linear response regime where \( \hat{\rho}_T = \hat{\rho} + \hat{\rho}_F \). Here \( \hat{\rho} \) is the equilibrium value of the single-particle density matrix, \( \hat{\rho}_F \) is linear in the electric field and satisfies \( \hat{\rho}_F(t \to -\infty) = 0 \). Kohn and Luttinger proceeded by employing the ansatz \( \hat{\rho}_F = f e^{\omega t} \), where \( f = \hat{\rho}_F(t = 0) \) is independent of time. Then Eq. (A2) reduces to

\[
d_\mu^l f_\nu^l = \sum_{\nu'} (f_{\nu'}^l H_{\nu'\nu} - H_{\nu'\nu} f_{\nu'}^l) + C_{\nu'}^l
\]

in the band-epsilon-state representation of \( \hat{H}_0 \). Here \( C_{\nu'}^l \equiv [\hat{\rho}, \hat{H}_1]_{\nu'} \) reads \( C_{\nu'}^l = i e E \cdot [\partial_k + \partial_{k'}] \rho_{\nu'} + [\mathbf{J}, \rho_{\nu'}] \) for \( l \neq \nu' \), and \( C_{\nu'}^l \equiv i e E \cdot [\partial_k \rho_{\nu'} + [\mathbf{J}, \rho_{\nu'}]] \), where \( [\mathbf{J}, \rho_{\nu'}] \equiv \sum_{\nu''} (J_{\nu''\nu'} \rho_{\nu''} - \rho_{\nu''} J_{\nu''\nu'}) \). Here \( \mathbf{v}_{\nu'} = i e \frac{\partial}{\partial k} f_{\nu'}^l + i \mathbf{J}_{\nu'} \) and \( \mathbf{J}_{\nu'} \equiv \delta_{kk'} (u_l | \partial_k | u_{\nu'}) \) are used.

The linear response of an observable \( A \) is thus

\[
\delta A = \text{tr} \left\{ \hat{\delta} \hat{A} \right\} = \sum_i \langle f_i^l \rangle A_{ll'} + \sum_{\nu'} \langle f_{\nu'}^l \rangle A_{\nu'l'}, \tag{A4}
\]

Hereafter the notation \( \sum_{\nu'} \) means that all the index equalities should be avoided in the summation.

Equation (A3) can be split into

\[
d_\mu^l f_\nu^l = \sum_{\nu'} \left( f_{\nu'}^l H_{\nu'\nu} - H_{\nu'\nu} f_{\nu'}^l \right) + \left( f_l - f_\nu^l \right) H_{l\nu}^l + C_{\nu'}^l
\]

for \( l \neq \nu' \), and

\[
- i h s f_l^l = \sum_{\nu' \neq l} \left( f_{\nu'}^l H_{l\nu'}^l - H_{l\nu'}^l f_{\nu'}^l \right) + C_l^l
\]

for \( l = \nu' \).

Here \( H_{l\nu}^l \) is the first-order energy correction in the bare quantum mechanical perturbation theory, which has been absorbed into \( H_0 \), thus \( H_{l\nu}^l = 0 \) hereafter. In the case of weak disorder-potential, an iterative analysis of Eqs. (A5) and (A6) in terms of the parameter \( \lambda \) is possible. Assuming \( s f_l^l \to 0 \) when \( s \to 0^+ \) is equivalent to assuming that \( f_l^l \) starts from the order of \( \lambda^{-2} \). Then an order-by-order analysis with respect to the disorder potential follows:

\[
\begin{align*}
\delta f_l &= f_l^{(-2)} + f_l^{(-1)} + f_l^{(0)} + \ldots, \\
\delta f_{\nu'} &= f_{\nu'}^{(-1)} + f_{\nu'}^{(0)} + f_{\nu'}^{(1)} + \ldots \quad (l \neq \nu'), \\
C_{\nu'} &= C_{\nu'}^{(0)} + C_{\nu'}^{(1)} + C_{\nu'}^{(2)} + \ldots.
\end{align*}
\]

The superscript means the order of \( \lambda \). The iterative solutions are not repeated here. Then one gets the expressions for the off-diagonal elements in terms of the diagonal ones, e.g.,

\[
f_{l\nu'}^{(-1)} = \frac{f_l^{(-2)} - f_{\nu'}^{(-2)}}{d_{l\nu'}} H_{l\nu'}^l,
\]

and a transport equation which only concerns the diagonal elements. These equations are microscopic equations, and the required macroscopic equations are obtained from them by disorder-averaging. In so doing, Kohn and Luttinger assumed that \( f_l \) does not contain any physically important, rapidly varying exponential factors, thus in the thermodynamic limit

\[
\langle f_l^{(-2)} f_{\nu'}^{(-2)} \rangle = \langle f_l^{(-2)} \rangle \langle f_{\nu'}^{(-2)} \rangle = \langle H_{l\nu'}^l H_{l\nu'}^l \rangle f_l^{(-2)}
\]

Therefore, one can see that, only when this assumption is true, the semiclassical distribution function and thus the Boltzmann formalism can be defined. The validity of this vital assumption has been confirmed by subsequent researches, but beyond the scope of our study.

In the case of weak disorder-potential, the off-diagonal response only concerns the lowest nonzero order of \( \langle f_{\nu'}^l \rangle \) (see below), while the analysis of diagonal response has to go to higher orders in the perturbation expansion of \( f_l \). In these higher-order contributions some trivial renormalization effects appear, only giving rise to negligible higher-order contributions in the weak disorder-potential limit to AHE/SHE and UBE. The qualitatively and quantitatively important part of the diagonal response of density matrix in the weak disorder-potential regime is just the generalized semiclassical Boltzmann equation.

\section{Off-diagonal response}

After disorder average, assuming \( \langle H_{l\nu'}^l \rangle = 0 \) one has

\[
\begin{align*}
\sum_{\nu'} & \langle f_{\nu'}^l \rangle A_{l\nu'} = \sum_{\nu'} \langle f_{\nu'}^{(0)} \rangle A_{l\nu'} = \sum_{\nu'} C_{\nu'}^{(0)} A_{l\nu'} + \sum_{\nu' \neq l} \left( f_l^{(-2)} - f_{\nu'}^{(-2)} \right) \left( f_{\nu'}^{(-2)} - f_{\nu'}^{(-2)} \right) \langle H_{l\nu'}^l H_{l\nu'}^l \rangle A_{l\nu'}.
\end{align*}
\]

Due to \( C_{\nu'}^{(0)} = i e E \cdot J_{\nu'} (\rho_{\nu'} - \rho_{\nu'}) \) and \( v_{\nu'} \delta_{kk'} = -\frac{i}{\hbar} d_{l\nu'} J_{l\nu'}^l \) for \( l \neq \nu' \), we have

\[
\sum_{\nu'} C_{\nu'}^{(0)} A_{l\nu'} = 2 e \sum_{\nu'} \rho_{\nu'} \text{Im} E \cdot J_{\nu'} A_{l\nu'} = -2 e \sum_{\nu'} \rho_{\nu'} \text{Im} E \cdot \text{Im} \langle u_l | v_{\nu'} \rangle A_{l\nu'}
\]

\[
= \sum_{\nu'} f_l^l \delta_{\nu l} A_{l\nu'}.
\]
where \( \rho_t = f_0^t \).

Besides, by interchanging the indices \( l, l' \) and \( l'' \) here and there and some simple algebra, we find

\[
\sum_{l', l''} \langle H_{l',l'}^{(2)} H_{l''}^{(2)} \rangle \left( \frac{f_{l'}^{(-2)} - f_{l'}^{(-2)}}{d_{l',l'}^{(2)}} - \frac{f_{l''}^{(-2)} - f_{l''}^{(-2)}}{d_{l''}^{(2)}} \right) A_{l'l''} = \sum_{l', l''} \langle f_{l'}^{(-2)} \rangle \left[ \frac{(H_{l'}^{(2)} H_{l''}^{(2)}) A_{l'l''}}{d_{l',l'}^{(2)} d_{l''}^{(2)}} + \text{c.c.} \right] + \sum_{l', l''} \langle f_{l'}^{(-2)} \rangle \langle H_{l'}^{(2)} H_{l''}^{(2)} A_{l'l''} \rangle \left( \frac{1}{d_{l',l'}^{(2)}} - \frac{1}{d_{l''}^{(2)}} \right) \frac{1}{d_{l',l''}^{(2)}} = \sum_{l'} \langle f_{l'}^{(-2)} \rangle \delta^{s_j} A_{l'},
\]

(A9)

where \( \delta^{s_j} A_{l'} \) coincides with Eq. 9.

Summarizing the contents of this subsection, we proved in the weak disorder-potential regime and linear response regime

\[
\sum_{l'} \langle f_{l'}^{(-2)} \rangle A_{l'l'} = \sum_{l'} f_0^{l'} \delta^{\eta_{l'}} A_{l'} + \sum_{l'} \langle f_{l'}^{(-2)} \rangle \delta^{s_j} A_{l'}.\tag{A10}
\]

Here \( \langle f_{l'}^{(-2)} \rangle \) is just the conventional nonequilibrium distribution function obtained in the lowest Born approximation. This equation confirms what was obtained phenomenologically previously in the Boltzmann formalism 2.

**Appendix B: Scattering off pairs of impurities**

In the modern semiclassical Boltzmann theory developed in studying the AHE, the anti-symmetric part \( \omega_{l'l''}^A \equiv \frac{1}{2} \left( \omega_{l'l''}^{(4)} - \omega_{l'l'}^{(4)} \right) \) of the fourth-order scattering rate \( \omega_{l'l''}^{(4)} \) was calculated only within the noncrossing approximation, giving rise to the intrinsic-shear-scattering-induced anomalous quantum contribution [4]. Here we show that the crossing part also contributes to the anomalous quantum contribution. Both the noncrossing and crossing contributions arise from scattering off pairs of impurity centers [42].

Starting from [41]

\[
\omega_{l'l''}^A = -\frac{2\pi}{\hbar} \delta (d_{l''}) \sum_{l', l''} \left[ \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle \frac{1}{d_{l'',l''}^{(2)}} \right]
\]

+ \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle \frac{1}{d_{l'',l''}^{(2)}}

+ \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle \frac{1}{d_{l'',l''}^{(2)}}

we get

\[
\omega_{l'l''}^A = -\frac{(2\pi)^2}{\hbar^2} \delta (d_{l''}) \sum_{l', l''} \delta (d_{l''}) \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle

+ \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle + \text{Im} \left\langle V_{l''} V_{l''} V_{l''} V_{l''} \right\rangle

\]

where the \( l, l' \) and \( l'' \) states are on-shell, whereas the \( l''' \) state may be off-shell. When taking the average over disorder configurations, there exist some different possibilities: the non-Gaussian contribution from \( \omega (V_{l'''}) \) disorder correlation [39–41], the Gaussian non-crossing [13] and crossing [42] contributions. For the noncrossing Gaussian contribution, one can find that an interband off-shell scattering is contained in each term. While for crossing Gaussian contribution, the interband off-shell scattering may be present or not. To be more specific, we present the expressions for \( \omega_{l'l''}^A \) in the smooth scalar disorder-potential limit. In this limit, the two momenta linked by the disorder potential are close to each other thus

\[
V_{l''} = V_{kk} \left\{ \delta_{\eta_{l''}} + \left( k'_{\mu} - k_{\mu} \right) J_{\mu}^{l''} \right\}

+ \frac{1}{2} \left( k'_{\mu} - k_{\mu} \right) \left( k'_{\nu} - k_{\nu} \right) J_{\mu\nu}^{l''} \right\}

+ \cdots,
\]

where \( J_{\mu}^{l''} \) and \( J_{\mu\nu}^{l''} \) are the Gaussian non-crossing and crossing contributions. For the crossing coherent-skew-scattering-induced anomalous quantum contribution [13, 18]. The Einstein summation convention is used hereafter for the indices \( \mu, \nu \). For real elastic process there must be \( \eta = \eta' \) in the smooth scalar disorder-potential limit in non-degenerate multiband system. When taking the disorder average, we only consider Gaussian disorder. The noncrossing part contributes

\[
\omega_{l'l''}^{\delta_{l'l''}} = -\frac{2\pi n_{im} V_{l''}^2}{\hbar} \left( k \times k' \right) \left( \delta_{\eta_{l''}} - \delta_{\eta'} \right) \delta (d_{l''})

+ \sum_{l', l''} \left( \delta_{\eta_{l'}} \delta (d_{l''}) \left( \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} \right) \right)

\times \text{Im} \left\langle J_{\mu}^{l''} \right\rangle \left( k \times k' \right) \left( \delta_{\eta_{l'}} - \delta_{\eta'} \right) \left( \delta_{\eta_{l''}} - \delta_{\eta'} \right)

\]

Thus an interband off-shell scattering \( \left( \eta'' \neq \eta \right) \) is unavoidable in each term of the noncrossing intrinsic-shear-scattering-induced anomalous quantum contribution [13, 18]. For the crossing coherent-skew-scattering-induced anomalous quantum contribution [42], we get

\[
\omega_{l'l''}^{\delta_{l'l''}} = -\frac{2\pi n_{im} V_{l''}^2}{\hbar} \left( \delta_{\eta_{l''}} - \delta_{\eta'} \right) \delta (d_{l''}) \sum_{l', l''} \left( \delta_{\eta_{l'}} \delta (d_{l''}) \left( \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} + \delta_{k_{l'}k_{l''}} \right) \right)

\times \text{Im} \left\langle J_{\mu}^{l''} \right\rangle \left( k \times k' \right) \left( \delta_{\eta_{l'}} - \delta_{\eta'} \right) \left( \delta_{\eta_{l''}} - \delta_{\eta'} \right)

\]

which contains both intraband \( \left( \eta'' = \eta \right) \) and interband \( \left( \eta'' \neq \eta \right) \) terms. \( \Omega_{l''} \) is the momentum-space Berry curvature. In fact, this expression was already obtained by Luttinger sixty years ago [13].
Appendix C: Calculation details

1. Boltzmann calculation

For identical pointlike scalar impurities in model [4] the conventional skew scattering from $o(V^3)$ non-Gaussian disorder vanishes due to $\omega_{0l}^\alpha = 0$ [11], then the skew scattering induced by the $o(V^4)$ non-Gaussian correlation $\langle V^4 \rangle = n_{im} V_0^4$ plays an important role in the dilute limit [29] [11]. The nonequilibrium distribution function responsible for this $o(V^4)$ skew scattering reads [11] $g_{\eta}^{sk}(\epsilon) = (\partial_\epsilon f^0) (\hat{\mathbf{z}} \times e \mathbf{E}) \cdot \mathbf{v}_\eta$ with

$$
\tau_{\eta}^{sk}(\epsilon) = -\tau \left( \frac{m V_0}{\hbar^2} \right)^2 \frac{\eta \epsilon \tau_{\eta}^{sk}(\epsilon) (\epsilon^2 + 2\epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}}{2\pi \Delta^2 (\epsilon) (\epsilon^2 + \epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}},
$$

then the corresponding nonequilibrium spin-polarization $\delta_{sk}^{\mathbf{S}} = \sum_i g_i^{sk} \mathbf{S}_i^0$ takes the form

$$
\delta_{sk}^{\mathbf{S}} = -\epsilon \alpha R D_0 n_{im} \frac{\epsilon \tau_{\eta}^{sk}(\epsilon) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}}{\Delta (\epsilon^2 + \epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}} E_y \hat{y}.
$$

Here $\tau \left( \frac{m V_0}{\hbar^2} \right)^2 = 2\pi \hbar D_0 / n_{im}$ is used. From this particular model case, one can extract a generic information that the skew scattering contribution due to the $o(V^4)$ non-Gaussian disorder correlation is characterized by a relaxation time of scale $\tau \left( \frac{m V_0}{\hbar^2} \right)^2$.

However, in usual case the $o(V^3)$ skew scattering is nonzero and dominates the skew scattering contribution. The peculiar property of model [11] with scalar short-range disorder in the case of both subbands partially occupied thus makes it inconvenient to extract general insights.

As for the disorder model chosen in the main text, we obtain ($l = (\eta, k) = (\eta, \epsilon, \phi)$)

$$
\omega_0^\eta_l = -\frac{1}{\tau} \frac{m V_0}{\hbar^2} \frac{\eta \epsilon \phi}{2D_0 \Delta \eta (\epsilon) \Delta \eta (\epsilon)} \delta (d_{ll})
$$

for the $o(V^3)$ skew scattering contribution. Substituting $\omega_0^\eta_l$ into the Boltzmann equation and following the general recipe given in Ref. [41], we get

$$
\tau_{\eta}^{sk}(\epsilon) = -\tau \left( \frac{m V_0 \eta \epsilon R (\epsilon^2 + 2\epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}}{\Delta (\epsilon^2 + \epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}} \right).
$$

Thus $\delta_{sk}^{\mathbf{S}} = \sum_i g_i^{sk} \mathbf{S}_i^0$ is given by

$$
\delta_{sk}^{\mathbf{S}} = -\epsilon \alpha R D_0 \frac{m V_0}{\hbar^2} \frac{\epsilon \tau_{\eta}^{sk}(\epsilon) (\epsilon^2 + 2\epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}}{\Delta (\epsilon^2 + \epsilon R) \ln \frac{k_- (\epsilon)}{k_+ (\epsilon)}} E_y \hat{y}.
$$

2. Kubo calculation

Some expressions needed in Sec. III A. 2. are presented here:

$$
\begin{align*}
&f_{in+AQ} \left( I_1, \tau_1 I_2, I_2^2 \right) = \frac{2\tau I_2}{(1 + I_1)^2 + (i I_2)^2}, \\
&f_{sk}^{UHE} \left( I_1, I_2^2 \right) = \left( \frac{1 + I_1}{(1 + I_1)^2 + (i I_2)^2} - 1 \right) \\
&\quad \times \left( 2 - \frac{1 + I_1}{(1 + I_1)^2 + (i I_2)^2} - 1 \right) \\
&- \left( \frac{i I_2}{(1 + I_1)^2 + (i I_2)^2} \right)^2,
\end{align*}
$$

and $f_L \left( I_1, I_2^2 \right) \equiv 1 + \frac{\sigma_\eta}{\eta \epsilon R} - 2 \frac{\sigma_\eta}{\eta \epsilon R} \frac{1 - I_2^2 - (i I_2)^2}{(1 + I_1)^2 + (i I_2)^2}$.

For the anomalous Hall conductivity, we get $\sigma_{xy} = \frac{e^2}{\eta \epsilon R \tau} \tau I_2$, $\sigma_{\tau 1} = \frac{e^2}{\eta \epsilon R} f_{in+AQ} \left( I_1, \tau_1 I_2, I_2^2 \right)$ and $\sigma_{\tau k} = -e^2 / \eta \epsilon R f_{sk}^{\text{AHE}} \left( I_1, I_2^2 \right)$, where

$$
\begin{align*}
&f_{sk}^{\text{AHE}} \left( I_1, I_2^2 \right) = \left( 1 - I_1^2 - (i I_2)^2 \right) \left( 1 + I_1^2 + (i I_2)^2 \right) \\
&- \left( \frac{2 I_2}{(1 + I_1)^2 + (i I_2)^2} \right)^2.
\end{align*}
$$

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