Noether conservation laws in classical mechanics

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Abstract. In Lagrangian mechanics, Noether conservation laws including the energy one are obtained similarly to those in field theory. In Hamiltonian mechanics, Noether conservation laws are issued from the invariance of the Poincaré–Cartan integral invariant under one-parameter groups of diffeomorphisms of a configuration space. Lagrangian and Hamiltonian conservation laws need not be equivalent.

Classical non-relativistic mechanics can be formulated as a particular field theory whose configuration space is a fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ [5, 6, 7, 10]. This configuration space is equipped with bundle coordinates $(t, q^i)$ where $t$ is the Cartesian coordinate on $\mathbb{R}$ possessing transition functions $t' = t + \text{const.}$, i.e., time reparametrizations are not considered. Of course, a fibre bundle $Q \to \mathbb{R}$ is trivial. However, it need not admit a preferable trivialization and, moreover, its different trivializations correspond to different non-relativistic reference frames. Therefore, we do not fix a trivialization of $Q \to \mathbb{R}$ fixed, i.e., bundle coordinates $q^i$ possess arbitrary time-dependent transition functions $q^i \to q'^i(t, q^j)$.

We obtain Noether conservation laws in Lagrangian and Hamiltonian non-relativistic mechanics similarly to those in field theory (see [5] for a detailed exposition).

1 Lagrangian conservation laws

Following general formalism of dynamical systems on fibre bundles (see, e.g., [3, 11]), the velocity phase space of non-relativistic mechanics on a configuration space $Q$ is the first order jet manifold $J^1Q$ of sections of $Q \to \mathbb{R}$. It is equipped with the adapted coordinates $(t, q^i, \dot{q}^i)$. There is the canonical imbedding of $J^1Q \to TQ$ onto the subbundle of the tangent bundle $TQ$ of $Q$ given by the condition $dt|v = 1, v \in TQ$. Due to this imbedding, any connection $\Gamma : Q \to J^1Y$ on a fibre bundle $Q \to \mathbb{R}$ is represented by a vector field

\[ \Gamma = \partial_t + \Gamma^i(t, q^j)\partial_i, \quad dt|\Gamma = 1, \]
on $Q$, and *vice versa*. Furthermore, one can associate to any connection $\Gamma$ (1) an atlas of the fibre bundle $Q \to \mathbb{R}$ with time-independent transition functions such that $\Gamma = \partial_t$ with respect to the corresponding bundle coordinates. In particular, there is one-to-one correspondence between the Ehresmann connections represented by complete vector fields (1) and the trivializations of $Q \to \mathbb{R}$ [5, 6]. From the physical viewpoint, any connection $\Gamma$ (1) defines a non-relativistic reference frame such that $q^i_t - \Gamma^i$ are relative velocities with respect to this reference frame [5, 8, 10].

A Lagrangian of non-relativistic mechanics is defined as a density

$$L = \mathcal{L}(t, q^i, q^i_t)dt$$

(2)
on the velocity phase space $J^1Q$. Its variation $\delta L$ is the second order Euler–Lagrange operator

$$\mathcal{E}_L = (\partial_i - d_t \partial^i_t) \mathcal{L} \theta^i \wedge dt,$$

(3)where $d_t = \partial_t + q^i_t \partial_i + q^i_{tt} \partial^i_t$ is the total derivative and $\theta^i = dq^i - q^i_t dt$ are contact forms. Its kernel $\ker \mathcal{E}_L \subset J^2Q$ defines the Lagrange equation

$$\mathcal{E}_i = (\partial_i - d_t \partial^i_t) \mathcal{L} = 0.$$

(4)

Here, $J^2Q$ is the second order jet manifold of sections of $Q \to \mathbb{R}$ equipped with the adapted coordinates $(t, q^i, q^i_t, q^i_{tt})$.

There are different Lagrangians whose variations provide the same Euler–Lagrange operator. They make up an affine space modelled over the vector space of variationally trivial Lagrangians $L_0$, i.e., $\delta L_0 = 0$. One can show that a first order Lagrangian $L_0$ is variationally trivial iff

$$L_0 = h_0(\varphi) = (\varphi_t + q^i_t \varphi_i) dt,$$

(5)where $\varphi = \varphi_t dt + \varphi_i dq^i$ is a closed one-form on $Q$ and

$$h_0(dt) = dt, \quad h_0(\theta^i) = 0$$
is the horizontal operator acting on semibasic exterior forms on $J^1Q \to Q$.

A Noether conservation law results from the invariance of a Lagrangian $L$ under a local one-parameter group of bundle automorphisms of a configuration bundle $Q \to \mathbb{R}$ [1, 2, 5, 9]. We agree to call them gauge transformations by analogy with field theory. The infinitesimal generator of a one-parameter gauge transformation group is a projectable vector field

$$u = u^i \partial_i + u^i(t, q^i) \partial_i$$

(6)
on $Q$, where $u^t = 0, 1$ because time reparametrizations are not considered. If $u^t = 0$, we have a vertical vector field $u = u^i \partial_i$ which takes its values into the vertical cotangent bundle $VQ$ of $Q \rightarrow \mathbb{R}$. If $u^t = 1$, a vector field $u$ (6) is a connection on the configuration bundle $Q \rightarrow \mathbb{R}$. Connections $\Gamma (2)$ on $Q \rightarrow \mathbb{R}$ make up an affine space modelled over the vector space of vertical vector fields on $Q \rightarrow \mathbb{R}$, i.e., the sum of a connection and a vertical vector field is a connection, while the difference of two connections is a vertical vector field on $Q \rightarrow \mathbb{R}$.

The canonical jet prolongation of a vector field $u$ (6) onto the velocity phase space $J^1Q$ reads

$$J^1u = u^t \partial_t + u^i \partial_i + d_t u^i \partial_i^t.$$  

(7)

A Lagrangian $L (2)$ is invariant under a one-parameter group of of gauge transformations generated by a vector field $u$ (6) iff its Lie derivative

$$L_{J^1u}L = J^1u]dL + d(u^i L) = (J^1u]dL)dt = (u^t \partial_t + u^i \partial_i + d_t u^i \partial_i^t)\mathcal{L}dt$$

(8)

along $J^1u$ (7) vanishes. The first variational formula provides the canonical decomposition

$$L_{J^1u}L = (u^i - u^t q_i^t)\mathcal{E}_i dt + d_t (u]H_L) dt,$$

(9)

of the Lie derivative (8), where

$$H_L = L + \partial^i L \theta^i$$

(10)

is the Poincaré–Cartan form. For instance, if $L = L_0$ is a variationally trivial Lagrangian (5), the first variational formula (9) gives the equality

$$L_{J^1u}L_0 = d_t (u] \varphi) = d_t (u^t \varphi_t + u^i \varphi_i).$$

(11)

If $L_{J^1u}L = 0$, we have the Noether conservation law

$$0 \approx -d_t \xi_u$$

(12)

of the symmetry function

$$\xi_u = -u]H_L = (u^t q_i^t - u^i)\partial_i^t \mathcal{L} - u^i \mathcal{L}$$

(13)

on the shell (4).

Since $u^t = 0, 1$, there are the following two types of Lagrangian conservation laws (12). Let $u = v^i \partial_i$ be a vertical vector field on $Q \rightarrow \mathbb{R}$. If the Lie derivative $L_{J^1u}L$ vanishes, we obtain the Noether conservation law

$$d_t (v^i \partial_i^t \mathcal{L}) \approx 0$$

(14)
of the momentum
\[ \nabla_v = -v^i \partial_i \mathcal{L} \] (15)
along a vector field \( u \). In particular, let \( u(q) \neq 0 \) at a point \( q \in Q \). Then, there exists an open neighbourhood \( U \) of \( q \) provided with coordinates \( q^i \) such that \( u = \partial / \partial q^1 \). A glance at the expression (8) shows that the Lie derivative \( \mathcal{L} \) on \( U \) vanishes if a Lagrangian \( \mathcal{L}(t, q^i, q^j_t) \) is independent of the coordinate \( q^1 \).

Let \( u = \Gamma = \partial_t + \Gamma^i \partial_i \) be a connection. The corresponding symmetry function (13) is the energy function
\[ \nabla_\Gamma = (q^i_t - \Gamma^i) \partial_i \mathcal{L} - \mathcal{L} \] (16)
[2, 5, 9]. If the Lie derivative \( \mathcal{L} \) vanishes, we have the energy conservation law
\[ d_t((q^i_t - \Gamma^i) \partial_i \mathcal{L} - \mathcal{L}) \approx 0. \] (17)
As was mentioned above, one can always choose bundle coordinates on \( Q \to \mathbb{R} \) such that \( \Gamma^i = 0 \). A glance at the expression (8) shows that the Lie derivative \( \mathcal{L} \) vanishes if a Lagrangian \( L \) written with respect to these coordinates is independent of time. Then, the energy conservation law (17) takes the form
\[ d_t(q^i_t \partial_i \mathcal{L} - \mathcal{L}) \approx 0. \]

We observe that there are different energy functions \( \nabla_\Gamma \) (16) corresponding to different connections \( \Gamma \) on \( Q \to \mathbb{R} \). Moreover, if an energy function \( \nabla_\Gamma \) is conserved and the momentum \( \nabla_v \) (15) is so, the energy function
\[ \nabla_{\Gamma + v} = \nabla_\Gamma + \nabla_v \]
is also conserved. Therefore, the problem is to select an energy function describing a true physical energy. Simple examples show that it may be an energy function \( \nabla_\Gamma \) where a connection \( \Gamma \) takes its values into the kernel of the Legendre map (19). The problem is that, if a Lagrangian \( L \) is degenerate, such a connection fails to be unique.

**Example.** Let us consider a one-dimensional motion of a point particle subject to friction. It is described by the dynamic equation
\[ q_{tt} = -kq_t, \quad k > 0. \]
This equation is equivalent to the Lagrange equation of the Lagrangian
\[ L = \frac{1}{2} \exp[kt]q^2_t dt. \] (18)
Let us consider the vector field 
\[ \Gamma = \partial_t - \frac{k}{2} q \partial_q. \]

Its jet prolongation (7) reads
\[ J^1 \Gamma = \partial_t - \frac{k}{2} q \partial_q - \frac{k}{2} q_t \partial^t_q. \]

It is readily observed that the Lie derivative of the Lagrangian (18) along \( J^1 \Gamma \) vanishes. Then, the energy function
\[ \Sigma_\Gamma = \frac{1}{2} \exp[kt]q_t(q_t + kq) \]
is conserved.

It however may happen that the Lie derivative of a Lagrangian does not vanish, but a conservation law takes place as follows.

(i) Every Lagrangian \( L \) (2) yields the Legendre map
\[ \hat{L} : J^1 Q \rightarrow V^* Q, \quad p_i \circ \hat{L} = \partial^t_i L, \]
of the velocity phase space \( J^1 Q \) to the vertical cotangent bundle \( V^* Q \) of \( Q \rightarrow \mathbb{R} \) equipped with the holonomic coordinates \((t, q^i, p_i)\). This bundle plays a role of the momentum phase space of non-relativistic mechanics (see next Section). A Lagrangian \( L \) is called regular if the Legendre map (19) is of maximal rank, i.e., a local diffeomorphism. Otherwise, it is said to be degenerate. Let \( L \) be a degenerate Lagrangian and \( \nu = v^i \partial_i \) a vertical vector field on \( Q \rightarrow \mathbb{R} \) which belongs to the annihilator \( \text{Ann}(\hat{L}(J^1 Q)) \subset V Q \) of \( \hat{L}(J^1 Q) \), i.e.,
\[ v^i(t, q^j)\partial^t_i \mathcal{L}(t, q^j, q^j_t) = 0. \]

In this case, the first variational formula (9) takes the form
\[ \mathbf{L}_{J^1 \nu} L = u^i \mathcal{E}_i dt, \]
i.e., the Lie derivative \( \mathbf{L}_{J^1 \nu} L \) vanishes on the shell (4). Let now \( u \) be a vector field such that \( \mathbf{L}_{J^1 \nu} L = 0 \), and let us consider the sum \( u + \nu \). The Lie derivative \( \mathbf{L}_{J^1 (u + \nu)} L \) does not vanish, but it vanishes on-shell. Applying to it the first variational formula (4), we recover the Noether conservation law (12).

(ii) The Euler-Lagrange operator \( \mathcal{E}_L \) (3) is invariant under a one-parameter group of gauge transformations generated by a vector field \( u \) iff its Lie derivative \( \mathbf{L}_{J^2 u} \mathcal{E}_L \) along the
jet prolongation $J^2u$ of $u$ onto the second order jet manifold $J^2Q$ vanishes. There is the relation
\[ L_{J^2u}E_L = \delta(L_{J^1u}L) = E_{L_{J^1u}L}, \] (20)
i.e., the Lie derivative $L_{J^2u}E_L$ is the Euler–Lagrange operator associated to the Lagrangian $L_{J^1u}L$ [3]. It follows that $L_{J^2u}E_L = 0$ iff the Lie derivative $L_{J^1u}L$ is a variationally trivial Lagrangian (5), i.e.,
\[ L_{J^1u}L = h_0(\varphi), \] (21)
where $\varphi$ is a closed form on $Q$. Substituting this expression into the first variational formula (9), we obtain the weak equality
\[ h_0(\varphi) \approx dt(u]H_L) \] (22)
on the shell (4). If $\varphi = d\sigma$ is an exact form on $Q$, this equality is brought into the weak conservation law
\[ 0 \approx dt(u]H_L - \sigma) \] (23)
due to the relation $dt \circ h_0 = h_0 \circ d$. Let $L' = L + L_0$ be another Lagrangian whose variation is the Euler–Lagrange operator $E_L$. Due to the equality (11), we come to the same conservation law (23). Note that, if $L_{J^1u}L = h_0(\varphi)$, the Lie derivative of the Poincaré–Cartan form $H_L$ (10) is
\[ L_{J^1u}H_L = J^1u]dH_L + d(J^1u]H_L) = \varphi, \] (24)
and vice versa. This fact follows at once from the equality
\[ L_{J^1u}H_L = L_{J^1u}L + \partial^i(J^1u]d\mathcal{L})\theta^i. \] (25)

**Example.** Let us consider a one-dimensional free motion of a point particle described by the Lagrangian
\[ L = \frac{1}{2}q^2dt. \] (26)
The vector field $u = vt\partial_q$, $v =$const., is the infinitesimal generator of a one-parameter group of the Galilei transformations. Its jet prolongation (7) reads
\[ J^1u = vt\partial_q + v\partial^q_t. \]
The Lie derivative of the free motion Lagrangian (26) along $J^1u$ is
\[ L_{J^1u}L = vq_t = dq(vq). \]
Then, the equality (23) shows that $(q_t - q)$ is a constant of motion.
2 Hamiltonian conservation laws

As was mentioned above, the momentum phase space of non-relativistic mechanics on a configuration space $Q$ is the vertical cotangent bundle $V^*Q$ of $Q \to \mathbb{R}$ equipped with holonomic coordinates $(t,q^i,p_i)$. The cotangent bundle $T^*Q$ of $Q \to \mathbb{R}$ coordinated by $(t,q^i,p,p_i)$ plays a role of the homogeneous momentum phase space. It is provided with the canonical Liouville form $\Xi = p dt + p_i dq^i$ and the canonical symplectic form $\Omega = d\Xi$.

There are three equivalent description of the dynamics of non-relativistic Hamiltonian mechanics.

(i) There is the trivial affine bundle

$$\zeta : T^*Q \to V^*Q. \tag{27}$$

Its section

$$h : V^*Q \to T^*Q, \quad p \circ h = -\mathcal{H}(t,q^i,p_i) \tag{28}$$

is a Hamiltonian of (time-dependent) non-relativistic mechanics. The pull-back $h^*\Xi$ of $\Xi$ onto $V^*Q$ by means of a section $h$ (28) is the well-known Poincaré–Cartan integral invariant

$$H = p_i dq^i - \mathcal{H} dt. \tag{29}$$

We agree to call it a Hamiltonian form. There exists a unique vector field $\gamma_H$ on $V^*Q$ such that

$$dt |_{\gamma_H} = 1, \quad \gamma_H |_{dH} = 0, \quad \gamma_H = \partial_t + \partial^i \partial q^i - \partial_i \mathcal{H} \partial^i. \tag{30}$$

It defines the first order Hamilton equation

$$d_t q^i = \partial^i \mathcal{H}, \quad d_t p_i = -\partial_i \mathcal{H} \tag{31}$$

on $V^*Q$, where $d_t = \partial_t + q^i_t \partial_i + p_i \partial^i$ is the total derivative written with respect to the adapted coordinates $(t,q^i,p_i,q^i_t,p_i)$ on the jet manifold $J^1V^*Q$ of the fibre bundle $V^*Q \to \mathbb{R}$.

(ii) Let us take the pull-back of a Hamiltonian form $H$ (29) onto $J^1V^*Q$, and let us consider the Lagrangian

$$L_H = h_0(H) = (p_i q^i_t - \mathcal{H}(t,q^i,p_j)) dt \tag{32}$$
on $J^1V^*Q$. It is readily observed that the Lagrange equation of $L_H$ is exactly the Hamilton equation (31).

(iii) Let us consider the pull-back $\zeta^*H$ of the Hamiltonian form $H$ (29) onto $T^*Q$. Then

$$\textbf{H} = \partial_t \mid (\Xi - \zeta^*H) = p + \mathcal{H}$$

(33)
is a function on $T^*X$. Let us regard it as a Hamiltonian of an autonomous Hamiltonian system on the symplectic manifold $(T^*Q, \Omega)$ provided with the corresponding Poisson bracket

$$\{f, f'\}_T = \partial^p f \partial_t f' + \partial^i f \partial_i f' - \partial_t f \partial^p f' - \partial_i f \partial^i f', \quad \partial^p = \partial / \partial p.$$ Then, the relation

$$\zeta^*(\mathbb{L}_u f) = \{\textbf{H}, \zeta^*f\}_T$$

(34)
holds for any smooth real function $f \in C^\infty(V^*Q)$. For instance, $f$ is a first integral of motion iff the bracket $\{\textbf{H}, \zeta^*f\}_T$ vanishes.

Turn now to Noether conservation laws. Let $u$ be a projectable vector field (6) on $Q \rightarrow \mathbb{R}$ treated as the generator of a one-parameter group of gauge transformations. Its canonical lift onto $V^*Q$ reads

$$\tilde{u} = u^i \partial_i + u^t \partial_t - p_j \partial_j u^i.$$ (35)
The Lie derivative of the Hamiltonian form $H$ (29) along $\tilde{u}$ (35) reads

$$\mathbb{L}_{\tilde{u}}H = \tilde{u} \mid dH + d(\tilde{u} \mid H) = (\partial_t(p_i u^i - u^t \mathcal{H}) - u^t \partial_i \mathcal{H} + \partial_i u^i p_j \partial^j \mathcal{H})dt.$$ (36)

Let $J^1\tilde{u}$ be the jet prolongation of $\tilde{u}$ onto $J^1V^*Q$. A simple computation shows that the pull-back of the Lie derivative (36) onto $J^1V^*Q$ obeys the relation

$$\mathbb{L}_{\tilde{u}}H = \mathbb{L}_{J^1\tilde{u}}L_H.$$ (37)

In particular, the Hamiltonian form $H$ (29) is invariant under a one-parameter group of gauge transformations iff the Lagrangian $L_H$ (32) is so.

Regarding the Hamilton equation (31) as the Lagrange equation of the Lagrangian $L_H$ (32), we have the condition (21) of its invariance under a one-parameter group of gauge transformations generated by a vector field $u$ (6), i.e.,

$$\mathbb{L}_{J^1\tilde{u}}L_H = \mathbb{L}_{\tilde{u}}H = h_0(\phi),$$
where $\phi$ is a closed form on $V^*Q$. Moreover, since the left-hand side of this relation is independent of jet coordinates $q^t_i$ and $p^ti$, this condition takes the form
\[ \mathbf{L}_{J^1_u}L_H = \mathbf{L}_u H = \partial_t f(t)dt, \] (38)
where $f$ is a function of time only. Then, applying the first variational formula (8) to the Lagrangian $L_H$ (32), one can obtain Noether conservation laws in Hamiltonian mechanics [5, 7, 10]. This first variational formula reads
\[ \mathbf{L}_{J^1_u}L_H = -[(u^i - u^i q^t_i)(p^ti + \partial_i H) + (p^j \partial_i u^j + u^j p^ti)(q^t_i - \partial^i H) + d_t(u^i H - u^i p^ti)]dt. \] (39)
If the Lie derivative $\mathbf{L}_{J^1_u}L_H$ obeys the relation (38), we obtain the conservation law
\[ 0 \approx -d_t(u^i H - u^i p^t_i + f) \] (40)
on the shell (31). If the Lie derivative (39) vanishes, this conservation law takes the form
\[ 0 \approx -d_t(u^i H - u^i p^t_i). \] (41)
We agree to call
\[ T_u = u^i H - u^i p^t_i \] (42)
a symmetry function. In next Section, we will relate it to a symmetry function in Lagrangian mechanics.

Equivalently, the conservation law (41) results from the equality
\[ \mathbf{L}_u H = -\gamma_H]dT_u \]
and from the fact that $d_t f = \gamma_H]df$ on-shell for any function $f$ on $V^*Q$. Then, it follows from the relation (34) that a conserved symmetry function $T_u$ is a first integral.

In particular, let $u = v^i \partial_i$ be a vertical vector field on $Q \rightarrow \mathbb{R}$. If the Lie derivative $\mathbf{L}_{J^1_u}L_H$ vanishes, we obtain the Noether conservation law
\[ d_t(v^i p^t_i) \approx 0 \] (43)
of the momentum
\[ T_v = -v^i p^t_i \] (44)
along a vector field $u$. Let $u(q) \neq 0$ at a point $q \in Q$. As was mentioned above, there exists an open neighbourhood $U$ of $q$ provided with coordinates $q^t_i$ such that $u = \partial / \partial q^t_i$. 9
A glance at the expression (36) shows that the Lie derivative $L_{J_1}L_H$ on $U$ vanishes iff $\mathcal{H}(t, q^i, p'_j)$ is independent of the coordinate $q^i$.

Let $u = \Gamma = \partial_t + \Gamma^i \partial_i$ be a connection. The corresponding symmetry function (42) is the energy function

$$T_\Gamma = \mathcal{H}_\Gamma = \mathcal{H} - p_i \Gamma^i.$$ (45)

If the Lie derivative $L_{J_1}L_H$ vanishes, we have the energy conservation law

$$d_t \mathcal{H}_\Gamma \approx 0.$$ (46)

As was mentioned above, one can always choose bundle coordinates on $Q \to \mathbb{R}$ such that $\Gamma_i = 0$ and $\mathcal{H} = \mathcal{H}_\Gamma$. A glance at the expression (36) shows that the Lie derivative $L_{J_1}L_H$ vanishes iff the energy $\mathcal{H}_\Gamma$ written with respect to these coordinates is independent of time.

### 3 Relations between Lagrangian and Hamiltonian conservation laws

Lagrangian and Hamiltonian formulations of non-relativistic mechanics fail to be equivalent. The relationship between Lagrangian and Hamiltonian mechanics [5, 7] is a particular case of that between Lagrangian and Hamiltonian formulations of field theory [3, 4].

As was mentioned above, every Lagrangian $L$ (2) on the velocity phase space $J^1 Q$ yields the Legendre map $\hat{L}$ (19). Conversely, any Hamiltonian form $H$ (29) on the momentum phase space $V^* Q$ yields the momentum map

$$\hat{H} : V^* Q \to J^1 Q, \quad q_i^t \circ \hat{H} = \partial^i \mathcal{H}.$$ (47)

Given a Lagrangian $L$ on $J^1 Q$, a Hamiltonian form $H$ on $V^* Q$ is said to be associated with $L$ if $H$ satisfies the relations

$$\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L},$$
$$\mathcal{H} = p_i \partial^i \mathcal{H} - \hat{H}^* \mathcal{L},$$ (48) (49)

where $\hat{H}^* \mathcal{L} = \mathcal{L}(t, q^i, \partial^j \mathcal{H})$ is the pull-back of $\mathcal{L}$ onto $V^* Q$. An associated Hamiltonian form need not exists or it is not necessarily unique. Here, we will restrict our consideration to the cases of hyperregular Lagrangians (see [5] for the case of semiregular Lagrangians).

A Lagrangian $L$ is said to be hyperregular if the Legendre map $\hat{L}$ is a bundle isomorphism of $J^1 Q \to Q$ onto $V^* Q \to Q$. In this case, there exists a unique associated
Hamiltonian form $H$ such that: (i) the momentum map $\tilde{H} = \tilde{L}^{-1}$ is the inverse bundle isomorphism, (ii) the Hamiltonian form $H$ is the pull-back $H = \tilde{H}^*H_L$ of the Poincaré–Cartan form $H_L$, and (iii) the Poincaré–Cartan form $H_L$ is the pull-back $H_L = \tilde{L}^*H$ of $H$. The property (i) takes the coordinate form

\[
p_i = \partial^k_i\mathcal{L}(t, q^i, \partial^k\mathcal{H}(t, q^k, p_k)), \quad q^k_i = \partial^k\mathcal{H}(t, q^i, \partial^j\mathcal{L}(t, q^j, q^k_i)).
\] (50)

The property (ii) recovers the relation (49). The property (iii) leads to the coordinate equality

\[
q^i_i\partial_i^j\mathcal{L}(t, q^i, q^j_i) - \mathcal{L}(t, q^i, q^j_i) = \mathcal{H}(t, q^i, \partial^j\mathcal{L}(t, q^k, q_i^k)).
\] (51)

Furthermore, one can show that, if $\tilde{s}(t) = (q^i(t), p_i(t))$ is a solution of the Hamilton equation (19) of $H$, then $s(t) = (q^i(t))$ is a solution of the Lagrange equation (4) of $L$. Conversely, if $s(t)$ is a solution of the Lagrange equation (4) of $L$, then

\[
\tilde{s}(t) = (\tilde{L} \circ s)(t) = (q^i(t), p_i(t)) = \partial^j\mathcal{L}(t, q^j(t), \partial t q^j(t))
\] (52)

is a solution of the Hamilton equation (19) of $H$. Thus, it seems that Lagrangian and Hamiltonian formalisms in the case of hyperregular Lagrangians are completely equivalent. However, it appears that the relation between Lagrangian and Hamiltonian conservation laws is more intricate.

Let $u$ be a projectable vector field (6) on $Q$, and let $J^1u$ (7) and $\bar{u}$ (35) be its prolongations onto $J^1Q$ and $V^*Q$, respectively. The key point is that, though the Hamiltonian form $H$ and the Poincaré–Cartan form $H_L$ are the pull-back of each other, their Lie derivatives $L_{\bar{u}}H$ and $L_{J^1u}H_L$ fail to be so because the vector fields $J^1u$ and $\bar{u}$ are not transformed into each other by morphisms $\tilde{L}$ and $\tilde{H}$. At the same time, using the formulas (49) – (51), one can obtain the relations

\[
\tilde{H}^*\xi_u = \tau_u, \quad \tilde{L}\tau_u = \xi_u
\]

between the symmetry functions $\xi_u$ (13) and $\tau_u$ (42). It follows that the symmetry function $\xi_u$ is constant on a solution $s$ of the Lagrange equation, i.e., $\partial s^*\xi_u = 0$ iff the symmetry function $\tau_u$ is constant on the corresponding solution $\tilde{s}$ (52) of the Hamilton equation, and vice versa.

In general case, there is no one-to-one correspondence between solutions of the Lagrange and Hamilton equations. For instance, it may happen that different solutions $s$ and $s'$ of the Lagrange equation of a Lagrangian $L$ are associated to solutions $\tilde{s}$ and $\tilde{s}'$ of the Hamilton equations of different Hamiltonian forms $H$ and $H'$ associated with $L$ such that we have different symmetry functions constant on $\tilde{s}$ and $\tilde{s}'$ [5].
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