Quantum Painlevé systems of type $A_l^{(1)}$

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Abstract

We propose quantum Painlevé systems of type $A_l^{(1)}$. These systems, for $l=1$ and $l \geq 2$, should be regarded as quantizations of the second Painlevé equation and the differential systems with the affine Weyl group symmetries of type $A_l^{(1)}$ studied by M. Noumi and Y. Yamada [13], respectively. These quantizations enjoy the affine Weyl group symmetries of type $A_l^{(1)}$ as well as the Lax representations. The quantized systems of type $A_1^{(1)}$ and type $A_l^{(1)}$ ($l=2n$) can be obtained as the continuous limits of the discrete systems constructed from the affine Weyl group symmetries of type $A_2^{(1)}$ and $A_{l+1}^{(1)}$, respectively.

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1 Introduction

The Painlevé equations $P_J$ ($J = I,...,VI$) were discovered by P. Painlevé and B. Gambier in the classification of second-order nonlinear ordinary differential equations without movable singular points in their solutions [17], [5]. It is established by K. Okamoto that the Painlevé equations $P_{II}, P_{III}, P_{IV}, P_{V}$, and $P_{VI}$ have the affine Weyl group symmetries of type $A^{(1)}_1, C^{(1)}_2, A^{(1)}_2, A^{(1)}_3$, and $D^{(1)}_4$, respectively, as the group of Bäcklund transformations [16]. He also revealed the Hamiltonian structures for the Painlevé equations completely, namely, we can consider the Painlevé equations as Hamiltonian systems with affine Weyl group symmetries. Therefore, a question naturally arises: Does there exist a quantization of Painlevé equations with affine Weyl group symmetries? What we mean by the quantization is the canonical quantization, that is, a Poisson bracket will be replaced with a commutator. The present paper aims to answer this problem affirmatively.

More on backgrounds. A. P. Veselov and A. B. Shabat studied the dressing chains [19], and noticed that they can be considered as higher analogues of $P_{IV}$ and $P_{V}$. Using the dressing chains, V. E. Adler introduced the symmetric form of the fourth Painlevé equation $P_{IV}$ [1]. Independently, using the invariant divisors of $P_{IV}$, M. Noumi and Y. Yamada introduced the symmetric form of the fourth Painlevé equation $P_{IV}$ [14]. The symmetric form of $P_{IV}$ enables us to clarify the structure of the affine Weyl group symmetry. Consequently, generalizing the Bäcklund transformations of the symmetric form of $P_{IV}$, M. Noumi and Y. Yamada succeeded in constructing a new representation of the Coxeter group associated with an arbitrary generalized Cartan matrix [12]. Moreover, they proposed nonlinear ordinary differential systems with the affine Weyl group symmetry of type $A^{(1)}_l$ ($l \geq 2$) [13]. These differential systems are equivalent to $P_{IV}$ and $P_{V}$ for $l = 2$ and $l = 3$, respectively, and they have polynomial Hamiltonians [13] and Lax representations [15].

Let us formulate our main theorem. For $l = 1$, let $\mathcal{K}_1$ be the skew field over $\mathbb{C}$ with generators $f_0, f_1, f_2, \alpha_0, \alpha_1, h$ and the fundamental relations

$$[f_1, f_0] = 2hf_2, \quad [f_0, f_2] = [f_2, f_1] = h, \quad (1.1)$$
$$[f_i, \alpha_j] = [h, \alpha_j] = 0, \quad (1.2)$$
$$[f_i, h] = 0, \quad (1.3)$$

and for each $l = 2, 3, \ldots$, let $\mathcal{K}_l$ be the skew field over $\mathbb{C}$ with generators $f_i, \alpha_i$ (0 $\leq i \leq l$), $h$, and the fundamental relations

$$[f_i, f_{i+1}] = h, \quad [f_i, f_j] = 0 \quad (j \neq i \pm 1), \quad [f_i, \alpha_j] = 0, \quad (1.4)$$
$$[\alpha_i, \alpha_j] = [h, f_j] = [h, \alpha_j] = 0 \quad (0 \leq i, j \leq l), \quad (1.5)$$

where the indices 0, 1, ..., $l$ are understood as elements of $\mathbb{Z}/(l + 1)\mathbb{Z}$. Then, the quantization problem of the Painlevé systems of type $A^{(1)}_l$ can be solved as follows:

**Theorem 1.1** Let the $\mathbb{C}$-derivation $\partial$ of $\mathcal{K}_l$ be defined as in Definition 2.3. The generators of $\mathcal{K}_l$ satisfy the following relations:
(0) For \( l = 1 \),
\[
\begin{align*}
\partial f_0 &= f_0 f_2 + f_2 f_0 + \alpha_0, & \partial f_1 &= -f_1 f_2 - f_2 f_1 + \alpha_1, & \partial f_2 &= f_1 - f_0, \\
\partial \alpha_i &= 0 \quad (i = 0, 1), & \partial h &= 0.
\end{align*}
\]

(1) For \( l = 2n \) \((n \geq 1)\),
\[
\begin{align*}
\partial f_i &= f_i \left( \sum_{1 \leq r \leq n} f_{i+2r-1} \right) - \left( \sum_{1 \leq r \leq n} f_{i+2r} \right) f_i + \alpha_i, \\
\partial \alpha_i &= 0 \quad (0 \leq i \leq l), & \partial h &= 0.
\end{align*}
\]

(2) For \( l = 2n + 1 \) \((n \geq 1)\),
\[
\begin{align*}
\partial f_i &= f_i \left( \sum_{1 \leq r \leq s \leq n} f_{i+2r-1} f_{i+2s} \right) - \left( \sum_{1 \leq r \leq s \leq n} f_{i+2r} f_{i+2s+1} \right) f_i \\
&\quad + \left( \frac{k \alpha_0}{2} - \sum_{1 \leq r \leq n} \alpha_{i+2r} \right) f_i + \alpha_i \sum_{1 \leq r \leq n} f_{i+2r}, \\
\partial \alpha_i &= 0 \quad (0 \leq i \leq l), & \partial h &= 0,
\end{align*}
\]

where \( k = \alpha_0 + \cdots + \alpha_l \).

**Theorem 1.2** The \( \mathbb{C} \)-derivation \( \partial \) commutes with the action of the extended affine Weyl group \( \tilde{W} = \langle s_0, \ldots, s_l, \pi \rangle \) of type \( A_1^{(1)} \) defined in Proposition 2.7.

In the classical case \( h = 0 \), when \( l = 1 \), the quantum system (1.6), (1.7) is nothing but the classical second Painlevé equation, and when \( l \geq 2 \), the quantum systems (1.8), (1.9) and (1.10), (1.11) are nothing but the classical systems proposed by M. Noumi and Y. Yamada. We call these systems the quantum Painlevé systems of type \( A_1^{(1)} \).

Apart from the above mentioned works on the Painlevé equations, the Painlevé equations can be formulated in the general theory of monodromy preserving deformation [7], [8]. As for the quantization of monodromy preserving deformation, only the cases of Poincaré rank 0 (namely, the regular singular case) and Poincaré rank 1 at the infinity are known. Let us briefly mention the works which are relevant to these cases.

The Schlesinger equations can be viewed as deformation equations that preserve the monodromy of the rational connection \( \partial/\partial z - \sum_{i=1}^{n} A_i/(z - z_i) \) \((A_i \in M_{l+1,l+1}(\mathbb{C}))\) with regular singularities. N. Reshetikhin introduced the generalized Knizhnik-Zamolodchikov equations and noticed that the original system of the Knizhnik-Zamolodchikov equations is a quantization of Schlesinger equations [13] (see also [3]). In the case of Poincaré rank 1 at the infinity where the rational connection is \( \partial/\partial z - [A + \sum_{i=1}^{n} B_i/(z - z_i)] \) \((A, B_i \in M_{l+1,l+1}(\mathbb{C}))\), a quantization is constructed [2], [4].

The classical differential systems for (1.6), (1.7), (1.8), (1.9) and (1.10), (1.11) describe monodromy preserving deformations of rational connections with irregular singularity of
Poincaré rank 3 \((l = 1)\), 2 \((l \geq 2)\) at \(z = \infty\). In Proposition 2.10, we establish that the quantum Painlevé systems of type \(A^{(1)}_l\) have the Lax representation.

This paper is organized as follows. In Section 2, we define the Hamiltonian of the quantum Painlevé systems of type \(A^{(1)}_l\). We will also redefine the quantum Painlevé systems of type \(A^{(1)}_l\) in terms of the Hamiltonian and establish the affine Weyl group symmetry. Moreover, we introduce a quantum canonical coordinate and rewrite the quantum Painlevé systems of type \(A^{(1)}_l\) into the Heisenberg equations and show that the quantum Painlevé systems of type \(A^{(1)}_l\) have Lax representations. In Section 3, we construct a discrete system from the action of the extended affine Weyl group of type \(A^{(1)}_l\) which is defined in Subsection 2.2 and take the continuous limit of the discrete system for \(l = 2, 2n + 1\). When \(l = 2\), we obtain the quantum second Painlevé equation as the continuous limit, and when \(l = 2n + 1\), we obtain the quantum Painlevé systems of type \(A^{(1)}_{2n}\). See Remark 3.5 for the discrete system whose continuous limit is the quantum Painlevé systems of type \(A^{(1)}_{2n+1}\).

### 2 Quantum Painlevé systems of type \(A^{(1)}_l\)

For \(l = 1\), we can define the skew field \(K_1\) over \(\mathbb{C}\) with the generators

\[
f_0, f_1, f_2, \alpha_0, \alpha_1, h,
\]

and the following relations

\[
[f_1, f_0] = 2hf_2, \quad [f_0, f_2] = [f_2, f_1] = h, \quad (2.2)
\]
\[
[f_1, \alpha_j] = [h, \alpha_j] = 0, \quad (2.3)
\]
\[
[f_1, h] = 0, \quad (2.4)
\]

and for each \(l = 2, 3, \ldots\), we can define the skew field \(K_l\) over \(\mathbb{C}\) with the generators

\[
f_i, \alpha_i \quad (0 \leq i \leq l), h
\]

and the following relations

\[
[f_i, f_{i+1}] = h, \quad [f_i, f_j] = 0 \quad (j \neq i \pm 1), \quad [f_i, \alpha_j] = 0, \quad (2.6)
\]
\[
[\alpha_i, \alpha_j] = [h, f_j] = [h, \alpha_j] = 0 \quad (0 \leq i, j \leq l), \quad (2.7)
\]

where the indices 0, 1, \ldots, \(l\) are understood as elements of \(\mathbb{Z}/(l + 1)\mathbb{Z}\). We will also identify the generators \(\alpha_0, \ldots, \alpha_l\) with the simple roots of the affine root system of type \(A^{(1)}_l\). The associative algebra defined with the above relations is an Ore domain, and \(K_l\) is its quotient skew field (see, for example, [3] Chapter 1, Section 8).
2.1 Hamiltonian

Let us begin with the Hamiltonian which reproduces the quantum Painlevé systems of type $A_l^{(1)}$. In the classical case $\hbar = 0$, this Hamiltonian is nothing but the polynomial Hamiltonian for the classical Painlevé system of type $A_l^{(1)}$. Accordingly, we follow the notation of [13].

For each $i = 1, \ldots, l$, we denote by $\varpi_i$ the $i$-th fundamental weight of the finite root system of type $A_l$,

$$\varpi_i = \frac{1}{l+1} \{ (l+1-i) \sum_{r=1}^{i} r\alpha_r + i \sum_{r=i+1}^{l} (l+1-r)\alpha_r \}$$

$$= \sum_{r=1}^{l} (\min\{i, r\} - \frac{ir}{l+1})\alpha_r$$

(2.8)

and set $\varpi_0 = 0$.

Put $\Gamma = \mathbb{Z}/(l+1)\mathbb{Z}$. For each subset $C_{i,m} := \{i, i+1, \ldots, i+m-1\}$ ($m \in \mathbb{Z}_{>0}$, $m \leq l$) of $\Gamma$, we define $\chi(C_{i,m})$ by

$$\chi(C_{i,m}) := \varpi_i - \varpi_{i+1} + \cdots + (-1)^{m-1} \varpi_{i+m-1}.$$  

(2.9)

For each proper subset $C = \bigsqcup_i C_{i,m_i}$ (disjoint union) of $\Gamma$, we define $\chi(C)$ by

$$\chi(C) := \sum_i \chi(C_{i,m_i}),$$

(2.10)

where we assume that the intersection of $C_{i,m_{i+1}}$ and $C_{j,m_j}$ is empty for $i \neq j$. Then, we call each $C_{i,m_i}$ a connected component of $C$ with length $m_i$.

For each $d = 1, \ldots, l+1$, let $S_d$ be the set of the subset $K \subset \Gamma$ such that $|K| = d$, and the length of each connected component of $\Gamma \setminus K$ is even.

For $C_{i,m}$, we set

$$f_{C_{i,m}} = f_i f_{i+1} \cdots f_{i+m-1}.$$  

(2.11)

For each $K = \bigsqcup_i C_{i,m_i} \in S_d$ ($d = 1, \ldots, l$), we define $f_K$ by

$$f_K = \prod_i f_{C_{i,m_i}},$$

(2.12)

where $C_{i,m_i}$ is a connected component of $K$. Note that we do not define $f_K$ for $K \in S_{l+1}$.

**Definition 2.1** We define the Hamiltonian $H_0$ for the quantum Painlevé systems of type $A_l^{(1)}$ ([13], [10]) as follows:

(1) For even $l$:

$$H_0 = \begin{cases} f_0 f_1 f_2 + h f_1 + \sum_{K \in S_1} \chi(\Gamma \setminus K) f_K & (l = 2), \\ \sum_{K \in S_3} f_K + \sum_{K \in S_1} \chi(\Gamma \setminus K) f_K & (l = 2n, n \geq 2). \end{cases}$$

(2.13)
(2) For odd \(l\):

\[
H_0 = \begin{cases}
\frac{1}{2}(f_0f_1 + f_1f_0) + \alpha_1f_2 & (l = 1) \\
 f_0f_1f_2f_3 + hf_1f_2 + \sum_{\kappa \in S_2} \chi(\Gamma \backslash K)f_K + \left(\sum_{i=1}^{3}(-1)^{-i-1}\omega_i\right)^2 & (l = 3), \\
\sum_{\kappa \in S_4} f_K + \sum_{\kappa \in S_2} \chi(\Gamma \backslash K)f_K + \left(\sum_{i=1}^{l}(-1)^{-i-1}\omega_i\right)^2 & (l = 2n + 1, n \geq 2).
\end{cases}
\] (2.14)

Constant terms \(\left(\sum_{i=1}^{l}(-1)^{-i-1}\omega_i\right)^2\) in (2.14) are chosen so that \(H_0\) has the invariance under the affine Weyl group action (2.28) (see Proposition 2.7).

Example 2.2 The explicit forms \(H_0\) for \(l = 2, 3, 4, 5\) are as follows:

For \(l = 2\):

\[
H_0 = f_0f_1f_2 + hf_1 + \frac{1}{3}(\alpha_1 - \alpha_2)f_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2)f_1 - \frac{1}{3}(2\alpha_1 + \alpha_2)f_2.
\]

For \(l = 3\):

\[
H_0 = f_0f_1f_2f_3 + hf_1f_2 + \frac{1}{4}(\alpha_1 + 2\alpha_2 - \alpha_3)f_0f_1 + \frac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3)f_1f_2 \\
- \frac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3)f_2f_3 + \frac{1}{4}(\alpha_1 - 2\alpha_2 - \alpha_3)f_3f_0 + \frac{1}{4}(\alpha_1 + \alpha_3)^2.
\]

For \(l = 4\):

\[
H_0 = f_0f_1f_2 + f_1f_2f_3 + f_2f_3f_4 + f_3f_4f_0 + f_4f_0f_1 + \frac{1}{5}(2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4)f_0 \\
+ \frac{1}{5}(2\alpha_1 + 4\alpha_2 + \alpha_3 + 3\alpha_4)f_1 - \frac{1}{5}(3\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4)f_2 \\
+ \frac{1}{5}(2\alpha_1 - \alpha_2 + \alpha_3 + 3\alpha_4)f_3 - \frac{1}{5}(3\alpha_1 + \alpha_2 + 4\alpha_3 + 2\alpha_4)f_4.
\]

For \(l = 5\):

\[
H_0 = f_0f_1f_2f_3 + f_1f_2f_3f_4 + f_2f_3f_4f_5 + f_3f_4f_5f_0 + f_4f_5f_0f_1 + f_5f_0f_1f_2 \\
+ \frac{1}{3}(\alpha_1 + 2\alpha_2 + \alpha_4 - \alpha_5)f_0f_1 + \frac{1}{3}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 + 2\alpha_5)f_1f_2 \\
- \frac{1}{3}(2\alpha_1 + \alpha_2 - \alpha_4 + \alpha_5)f_2f_3 + \frac{1}{3}(\alpha_1 - \alpha_2 + \alpha_4 + 2\alpha_5)f_3f_4 \\
- \frac{1}{3}(2\alpha_1 + \alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5)f_4f_5 + \frac{1}{3}(\alpha_1 - \alpha_2 - 2\alpha_4 - \alpha_5)f_5f_0 \\
+ \frac{1}{3}(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5)f_0f_3 + \frac{1}{3}(\alpha_1 + 2\alpha_2 + \alpha_4 + 2\alpha_5)f_1f_4 \\
- \frac{1}{3}(2\alpha_1 + \alpha_2 + 2\alpha_4 + \alpha_5)f_2f_5 + \frac{1}{4}(\alpha_1 + \alpha_3 + \alpha_5)^2.
\]
Definition 2.3 \( \text{Put} \ k = \alpha_0 + \cdots + \alpha_l. \)

(1) For \( l = 1, 2n \), we define the \( \mathbb{C} \)-derivation \( \partial \) of \( K_l \) as follows:

\[
\partial f_i = \frac{1}{h}[H_0, f_i] + \delta_{i,0}k, \tag{2.15}
\]

\[
\partial \alpha_i = \frac{1}{h}[H_0, \alpha_i], \quad (0 \leq i \leq l), \quad \partial h = \frac{1}{h}[H_0, h]. \tag{2.16}
\]

(2) For \( l = 2n + 1 \), we define the \( \mathbb{C} \)-derivation \( \partial \) of \( K_l \) as follows:

\[
\partial f_i = \frac{1}{h}[H_0, f_i] - (-1)^i \frac{k}{2} f_i + \delta_{i,0}kx_0, \tag{2.17}
\]

\[
\partial \alpha_i = \frac{1}{h}[H_0, \alpha_i] \quad (0 \leq i \leq l), \quad \partial h = \frac{1}{h}[H_0, h], \tag{2.18}
\]

where \( x_0 = f_0 + f_2 + \cdots + f_{l-1} \).

We can check that \( \partial \) is the \( \mathbb{C} \)-derivation of \( K_l \) and that \( x_0 \) is a central element of \( K_l \) from the definition of \( K_l \).

Proof of Theorem 1.1 Case \( l = 1 \): It is straightforward to show that the right hand side of (2.15) equals to the right hand side of (1.6) from the definition (2.14).

Case \( l \geq 2 \): For each \( i = 0, \ldots, l \), we can define the \( \mathbb{C} \)-derivation \( \partial_i \) of \( K_l \) by

\[
\partial_i f_j = \delta_{ij}, \quad \partial_i \alpha_j = 0, \quad \partial_i h = 0. \tag{2.19}
\]

Then, for any \( \varphi \in K_l \) we have

\[
[f_i, \varphi] = h(\partial_{i+1} - \partial_{i-1})\varphi. \tag{2.20}
\]

Using the derivation \( \partial_i \), we compute \([H_0, f_i]\). For the cases where \( l = 2, 3 \), we can easily calculate \([H_0, f_i]\) from the definition (2.13), (2.14), and then from (2.15), and (2.17), we obtain the relations (1.8), (1.10), respectively. We now consider the case where \( l = 2n \) \((n \geq 2)\). For \( C \subset \Gamma \), we define \( S_d(C) \) by

\[
S_d(C) := \{ K \subset C \mid |K| = d, \ C \setminus K = \bigsqcup_{m_i: \text{even}} C_{i, m_i} \}. \tag{2.21}
\]

From the definition (2.13), we compute \( \frac{1}{h}[H_0, f_i] \) as follows:

\[
\frac{1}{h}[H_0, f_i] = (\partial_{i-1} - \partial_{i+1})H_0
\]

\[
= \sum_{K \in S_2(\Gamma \setminus \{i-1\})} f_K - \sum_{K \in S_2(\Gamma \setminus \{i+1\})} f_K + \chi(\Gamma \setminus \{i-1\}) - \chi(\Gamma \setminus \{i+1\})
\]

\[
= f_i \left( \sum_{r=1}^n f_{i+2r-1} \right) - \left( \sum_{r=1}^n f_{i+2r} \right) f_i + \alpha_i - \delta_{i,0}k.
\]

Thus, we obtain the relations (1.8). We can prove the case where \( l = 2n + 1 \) \((n \geq 2)\) in a similar way. \( \square \)
Remark 2.4 The relations in (1.6) and the \( l + 1 \) relations in (1.8) (as well as in (1.10)) are dependent among themselves. Namely, it holds that

\[
\partial (f_0 + f_1 + f_2^2) = k \quad (l = 1),
\]
\[
\partial \left( \sum_{r=0}^{l} f_r \right) = k \quad (l = 2n),
\]
\[
\partial \left( \sum_{r=0}^{n} f_{2r} \right) = \frac{k}{2} \sum_{r=0}^{n} f_{2r}, \quad \partial \left( \sum_{r=0}^{n} f_{2r+1} \right) = \frac{k}{2} \sum_{r=0}^{n} f_{2r+1} \quad (l = 2n + 1).
\]

2.2 Affine Weyl group symmetry

We will establish the affine Weyl group symmetry for the quantum Painlevé system of type \( A_i^{(1)} \), which generalizes the symmetry for the classical system. Let \( A = (a_{ij})_{i,j=0} \) be the generalized Cartan matrix of type \( A_i^{(1)} \)

\[
a_{ii} = 2, \quad a_{i,i \pm 1} = -1, \quad a_{ij} = 0 \quad (j \neq i, i \pm 1),
\]

and let \( U = (u_{ij})_{i,j=0} \) be the matrix defined by

\[
u_{i,i \pm 1} = \pm 1, \quad u_{ij} = 0 \quad (j \neq i \pm 1).
\]

**Proposition 2.5.** We can define automorphisms \( s_0, \ldots, s_l, \pi \in \text{Aut}_\mathbb{C} \mathcal{K}_i \) as follows:

1. For \( l = 1 \),

\[
s_0(f_0) = f_0, \quad s_0(f_1) = f_1 - f_2 \frac{\alpha_0}{f_0} - \frac{\alpha_0}{f_0} f_2 - \frac{\alpha_0^2}{f_0^2}, \quad s_0(f_2) = f_2 + \frac{\alpha_0}{f_0},
\]

\[
s_1(f_0) = f_0 + f_2 \frac{\alpha_1}{f_1} + \frac{\alpha_1}{f_1} f_2 - \frac{\alpha_1^2}{f_1^2}, \quad s_1(f_1) = f_1, \quad s_1(f_2) = f_2 - \frac{\alpha_1}{f_1},
\]

\[
s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_1) = \alpha_1 + 2\alpha_0, \quad s_1(\alpha_0) = \alpha_0 + 2\alpha_1, \quad s_1(\alpha_1) = -\alpha_1,
\]

\[
\pi(f_0) = f_1, \quad \pi(f_1) = f_0, \quad \pi(f_2) = -f_2, \quad \pi(\alpha_0) = \alpha_1, \quad \pi(\alpha_1) = \alpha_0,
\]

\[
s_0(h) = s_1(h) = \pi(h) = h.
\]

2. For \( l \geq 2 \),

\[
s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}, \quad s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(h) = h,
\]

\[
\pi(f_j) = f_{j+1}, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad \pi(h) = h, \quad (i, j \in \mathbb{Z}/(l + 1)\mathbb{Z}).
\]

**Theorem 2.6.** The automorphisms \( s_0, \ldots, s_l, \pi \) define a representation of the extended affine Weyl group \( \tilde{W} = \langle s_0, \ldots, s_l, \pi \rangle \) of type \( A_i^{(1)} \). Namely, they satisfy the commutation relations

\[
s_i^2 = 1, \quad (s_i s_j)^3 = 1 \quad (j = i \pm 1), \quad s_i s_j = s_j s_i \quad (j \neq i \pm 1),
\]

\[
\pi^{l+1} = 1, \quad \pi s_i = s_{i+1} \pi,
\]

where for \( l = 1 \) they satisfy the commutation relations except \( (s_i s_j)^3 = 1 \).
We can prove Proposition 2.5 and Theorem 2.6 by direct computations. Note that the extended affine Weyl group \( \tilde{W} = W \times \{1, \pi, \ldots, \pi^l\} \) is the extension of the ordinary affine Weyl group \( W = \langle s_0, \ldots, s_l \rangle \) by the cyclic group generated by the diagram rotation \( \pi \).

The automorphisms \( s_0, \ldots, s_l \) act on the Hamiltonian \( H_0 \) as follows. We shall deal with the action of the automorphism \( \pi \) on the Hamiltonian \( H_0 \) in the Appendix.

**Proposition 2.7** With respect to the action of \( W \), the Hamiltonian \( H_0 \) has the following:

1. For \( l = 1, 2n \),
   \[
   s_i(H_0) = H_0 + \delta_{i,0}k\frac{\alpha_0}{f_0} \quad (i = 0, \ldots, l).
   \] (2.31)

2. For \( l = 2n + 1 \),
   \[
   s_i(H_0) = H_0 + \delta_{i,0}k\frac{\alpha_0}{f_0}r_0 \quad (i = 0, \ldots, l).
   \] (2.32)

In particular, the Hamiltonian \( H_0 \) is invariant with respect to the action of the Weyl group \( W(A_l) = \langle s_1, \ldots, s_l \rangle \).

We prove this proposition through direct computations. For practical computations, it is convenient to use the Demazure operators \( \Delta_i \) \((i = 0, \ldots, l)\) defined by

\[
\Delta_i(\varphi) = \frac{1}{\alpha_i}(s_i(\varphi) - \varphi) \quad (\varphi \in K_i).
\] (2.33)

From the above definition we can easily show that \( \Delta_i \) \((i = 0, \ldots, l)\) satisfy the following relations:

\[
\begin{align*}
\Delta_i(\varphi \psi) &= \Delta_i(\varphi)\psi + s_i(\varphi)\Delta_i(\psi) \quad (\varphi, \psi \in K_i), \\
\Delta_i(\alpha_i) &= -2, \quad \Delta_i(\alpha_{i\pm 1}) = 1, \quad \Delta_i(\alpha_j) = 0 \quad (j \neq i, i \pm 1), \\
\Delta_i(f_i) &= 0, \quad \Delta_i(f_{i\pm 1}) = \pm \frac{1}{f_i}, \quad \Delta_i(f_j) = 0 \quad (j \neq i, i \pm 1).
\end{align*}
\] (2.34) (2.35) (2.36)

**Proof of Proposition 2.7** For the case where \( l = 1 \), we can easily calculate \( s_i(H_0) \) \((i = 0, 1)\) and obtain the formulas (2.31). For the cases where \( l = 2, 3 \), we can easily calculate \( \Delta_i(H_0) \) and obtain the formulas (2.31) and (2.32), respectively. In the case where \( l = 2n \) \((n \geq 2)\), we compute \( \Delta_i(H_0) \) as follows:

\[
\begin{align*}
\Delta_i \left( \sum_{K \in S_3} f_K + \sum_{K \in S_1} \chi(\Gamma \backslash K) f_K \right) \\
= \Delta_i(f_{i-1}f_if_{i+1}) + \Delta_i \left( f_if_{i+1} \sum_{r=1}^{n-1} f_{i+2r} \right) + \Delta_i \left( f_{i+1} \sum_{1 \leq r \leq s \leq n-1} f_{i+2r}f_{i+2s+1} \right)
\end{align*}
\]

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\[
+ \Delta_i \left( \sum_{r=1}^{n-1} f_{i+2r+1} f_{i-1} \right) + \Delta_i \left( \sum_{1 \leq r \leq s \leq n-1} f_{i+2r} f_{i+2s+1} f_{i-1} \right) \\
+ \sum_{K \in S_i} \Delta_i (\chi(\Gamma \setminus K)) f_K + s_i (\chi(\Gamma \setminus \{i+1\})) \frac{1}{f_i} - s_i (\chi(\Gamma \setminus \{i-1\})) \frac{1}{f_i} \\
= (-f_{i+1} + f_{i-1} - \frac{\alpha_i}{f_i}) + \sum_{r=1}^{n-1} f_{i+2r} + \frac{1}{f_i} \sum_{1 \leq r \leq s \leq n-1} f_{i+2r} f_{i+2s+1} - \sum_{r=1}^{n-1} f_{i+2r+1} \\
- \sum_{1 \leq r \leq s \leq n-1} f_{i+2r} f_{i+2s+1} \frac{1}{f_i} + (\chi(\Gamma \setminus \{i+1\}) + \alpha_i) \frac{1}{f_i} - (\chi(\Gamma \setminus \{i-1\}) - \alpha_i) \frac{1}{f_i} \\
+ \sum_{r=1}^{2n} (-1)^{r-1} f_{i+r} \\
= \frac{\alpha_i}{f_i} + (\omega_{i-1} - 2\omega_i + \omega_{i+1}) \frac{1}{f_i} = \frac{\alpha_i}{f_i} + (\alpha_i + \delta_{i,0} k) \frac{1}{f_i} = \delta_{i,0} k \frac{1}{f_i}.
\]

Consequently, we obtain the formula (2.31). We can prove the case where \( l = 2n + 1 \) \((n \geq 2)\) in a similar way. \(\square\)

**Proof of Theorem 1.2.** This theorem immediately follows from the definition (2.15), (2.17), and Proposition 2.7. \(\square\)

### 2.3 Quantum canonical coordinate and Heisenberg equation

In the same manner as in the classical case, we introduce a quantum canonical coordinate for the quantum Painlevé system of type \( A_l^{(1)} \). We discuss the cases of \( A_1^{(1)}, A_{2n}^{(1)} \) and \( A_{2n+1}^{(1)} \) separately.

**Case \( A_1^{(1)} \):** Let a new quantum coordinate system be defined by

\[
(q; p; x) = (f_1; f_2; f_0 + f_1 + f_2^2).
\]

It is easy to show that

\[
[p, q] = h, \quad [p, x] = [q, x] = 0.
\]

\( H_0 \) can be rewritten as a non-commutative polynomial \( H = H(q; p; x) \) in the quantum canonical coordinate \((q; p; x)\). Then, we see that the quantum Painlevé system of type \( A_1^{(1)} \) is equivalent to the Heisenberg equations

\[
\partial q = \frac{1}{h} [H, q], \quad \partial p = \frac{1}{h} [H, p], \quad \partial x = k.
\]

**Case \( A_{2n}^{(1)} \):** We define a new quantum coordinate system

\[
(q; p; x) = (q_1, \ldots, q_n; p_1, \ldots, p_n; x),
\]

(2.37)
using the following formulas

\[ q_i = f_{2i}, \quad p_i = \sum_{r=1}^{i} f_{2r-1} \quad (i = 1, \ldots, n), \] (2.41)

\[ x = f_0 + f_1 + \cdots + f_l. \] (2.42)

The inverse of this coordinate transformation is given by

\[ f_0 = x - \sum_{r=1}^{n} q_r - p_n, \quad f_1 = p_1, \quad f_2 = q_1, \] (2.43)

\[ f_{2i-1} = p_i - p_{i-1}, \quad f_{2i} = q_i \quad (i = 2, \ldots, n). \] (2.44)

It is easy to show that

\[ [p_i, q_j] = \hbar \delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = [p_i, x] = [q_i, x] = 0, \] (2.45)

for \( i, j = 1, \ldots, n \). By (2.43) and (2.44), \( H_0 \) can be rewritten as a non-commutative polynomial \( H = H(q; p; x) \) in the quantum canonical coordinate \((q; p; x)\). Then, we see that the quantum Painlevé system of type \( A_l^{(1)} \) (1.8) is equivalent to the Heisenberg equations

\[ \partial q_i = \frac{1}{\hbar} [H, q_i], \quad \partial p_i = \frac{1}{\hbar} [H, p_i], \quad \partial x = k, \] (2.46)

where \( i = 1, \ldots, n \).

Case \( A_{2n+1}^{(1)} \): Note that from (2.24), by putting

\[ \tilde{f}_{2r} = x_0 f_{2r}, \quad \tilde{f}_{2r+1} = x_0^{-1} f_{2r+1} \quad (r = 0, 1, \ldots, n), \] (2.47)

we obtain

\[ \partial \tilde{q}_i = \frac{1}{\hbar} [H_0, \tilde{q}_i] + \delta_{i,0} x_0^2 \quad (i = 0, 1, \ldots, 2n + 1). \] (2.48)

We introduce a new coordinate system

\[ (q; p; x) = (q_1, \ldots, q_n; p_1, \ldots, q_n; x_0, x_1) \] (2.49)

by the following formulas:

\[ q_i = x_0 f_{2i}, \quad p_i = x_0^{-1} \sum_{r=1}^{i} f_{2r-1} \quad (i = 1, \ldots, n), \] (2.50)

\[ x_0 = f_0 + f_2 + \cdots + f_{2n}, \quad x_1 = f_1 + f_3 + \cdots + f_{2n+1}. \] (2.51)

The inverse of this coordinate transformation is given by

\[ f_0 = x_0 - x_0^{-1} \sum_{r=1}^{n} q_r, \quad f_1 = x_0 p_1, \quad f_2 = x_0^{-1} q_1, \] (2.52)
\[ f_{2i-1} = x_0(p_i - p_{i-1}), \quad f_{2i} = x_0^{-1} q_i \quad (i = 2, \ldots, n). \] (2.53)

It holds that
\[ [p_i, q_j] = h\delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0, \quad (2.54) \]
\[ [p_i, x_0] = [q_i, x_0] = [p_i, x_1] = [q_i, x_1] = [x_0, x_1] = 0, \quad (2.55) \]

for \( i, j = 1, \ldots, n \). By (2.52) and (2.53), \( H_0 \) can be rewritten as a non-commutative polynomial \( H = H(q; p; x) \) in the quantum canonical coordinate \((q; p; x)\). Then, we see that the quantum Painlevé system of type \( A_{l}^{(1)} \) is equivalent to the Heisenberg equations
\[ \partial q_i = \frac{1}{h}[H, q_i], \quad \partial p_i = \frac{1}{h}[H, p_i], \quad \partial x_0 = \frac{k}{2} x_0, \quad \partial x_1 = \frac{k}{2} x_1, \] (2.56)

where \( i = 1, \ldots, n \).

The above results can be summarized as follows.

**Theorem 2.8** (1) The quantum Painlevé system of type \( A_{1}^{(1)} \) and of type \( A_{2n}^{(1)} \) is equivalent to the Heisenberg equations:
\[ \partial q_i = \frac{1}{h}[H, q_i], \quad \partial p_i = \frac{1}{h}[H, p_i], \quad \partial x = k, \] (2.57)

where \( i = 1 \) (\( l = 1 \)) and \( i = 1, \ldots, n \) (\( l \geq 2 \)).

(2) The quantum Painlevé system of type \( A_{2n+1}^{(1)} \) is equivalent to the Heisenberg equations:
\[ \partial q_i = \frac{1}{h}[H, q_i], \quad \partial p_i = \frac{1}{h}[H, p_i], \quad \partial x_0 = \frac{k}{2} x_0, \quad \partial x_1 = \frac{k}{2} x_1, \] (2.58)

where \( i = 1, \ldots, n \).

### 2.4 Lax representation

The classical Painlevé systems of type \( A_{l}^{(1)} \) arise from the compatibility condition of the linear problem [15], namely, we have the Lax representations for those systems. In this subsection, we show that the quantum Painlevé systems of type \( A_{l}^{(1)} \) also have Lax representations.

Let \( \mathcal{A}_1 \) be the skew field over \( \mathbb{C} \) with the generators
\[ f_0, f_1, f_2, u_1, u_2, \epsilon_0, \epsilon_1, h, t \] (2.59)

and the following relations
\[ [f_1, f_0] = 2hf_2, \quad [f_0, f_2] = [f_2, f_1] = h, \] (2.60)
\[ [u_1, f_0] = [f_0, u_2] = [u_1, f_1] = [f_1, u_2] = h, \] (2.61)
[u_1, f_2] = [u_2, f_2] = [f_i, e_j] = [h, e_j] = [t, e_j] = 0,  \hspace{1cm} (2.62)
[f_i, h] = [f_i, t] = 0, \hspace{1cm} (2.63)

and let \( \mathcal{A}_l \) \( (l \geq 2) \) be the skew field over \( \mathbb{C} \) with the generators

\[ f_i, u_i, e_i, \quad (0 \leq i \leq l), t, h, \] \hspace{1cm} (2.64)

and the following relations

\[ [f_i, f_{i+1}] = [f_i, u_{i+1}] = [u_i, f_i] = h, \] \hspace{1cm} (2.65)
\[ [f_i, f_j] = 0 \quad (j \neq i \pm 1), \quad [f_i, e_j] = [u_i, e_j] = [t, e_j] = [h, e_j] = 0, \] \hspace{1cm} (2.66)
\[ [f_i, t] = [u_i, t] = [f_i, h] = [u_i, h] = [t, h] = 0, \] \hspace{1cm} (2.67)
\[ f_i - f_{i+1} = u_i - u_{i+2}, \] \hspace{1cm} (2.68)
\[ f_0 + f_1 + \cdots + f_l = t, \] \hspace{1cm} (2.69)

with indices understood as elements in \( \mathbb{Z}/(l + 1)\mathbb{Z} \).

**Definition 2.9** Let \( \mathcal{A}_l[z] \) be the polynomial ring, and we define the elements \( L, B \in M_{l+1,l+1}(\mathcal{A}_l[z]) \) as follows:

1. For \( l = 1 \),

\[ L = \begin{bmatrix}
\epsilon_1 + zf_2 & f_1 + z \\
zf_0 + z^2 & \epsilon_0 - zf_2
\end{bmatrix}, \quad B = \begin{bmatrix}
u_1 & 1 \\
z & u_2
\end{bmatrix}. \] \hspace{1cm} (2.70)

2. For \( l \geq 2 \),

\[ L = \begin{bmatrix}
\epsilon_1 & f_1 & 1 \\
\epsilon_2 & f_2 & 1 \\
\vdots & \vdots & \vdots \\
z & \epsilon_1 & f_l \\
zf_0 & z & \epsilon_0
\end{bmatrix}, \quad B = \begin{bmatrix}
u_1 & 1 \\
u_2 & 1 \\
\vdots & \vdots \\
z & u_l & 1 \\
z & u_0
\end{bmatrix}. \] \hspace{1cm} (2.71)

Let \( \partial_z \) be the \( \mathcal{A}_l \)-derivation of \( \mathcal{A}_l[z] \) that maps \( z \) to 1.

**Proposition 2.10** For any \( \mathbb{C} \)-derivation \( \partial_t \) of \( \mathcal{A}_l[z] \) that maps \( t \) to 1 and \( z \) to 0 such that

\[ [z\partial_z + L, \partial_t + B] = 0, \] \hspace{1cm} (2.72)

the following formulas hold: for \( l = 1, 2n \),

\[ \partial_t f_i = \partial f_i, \quad \partial_t \alpha_i = 0, \quad \partial_t h = 0, \] \hspace{1cm} (2.73)

and for \( l = 2n + 1 \),

\[ \partial_t f_i = \frac{2}{l} \partial f_i, \quad \partial_t \alpha_i = 0, \quad \partial_t h = 0, \] \hspace{1cm} (2.74)

where \( \alpha_0 = 1 - \epsilon_1 + \epsilon_0, \alpha_i = \epsilon_i - \epsilon_{i+1} \) \( (1 \leq i \leq l) \), and \( k = 1 \). Namely, \( \partial_t \) defines the quantum Painlevé system for \( f_i \).
Remark 2.11 The condition (2.72) determines a $C$-derivation $\partial_t$ of $A_l$ up to the action on $u_i$. Examples of such a derivation can be constructed with the Hamiltonian $H_0$.

Proof. When $l = 1$, the condition (2.72) is equivalent to the following system of equations
\begin{align}
\partial_t \epsilon_i &= 0, \\
\partial_t f_2 &= f_1 - f_0, \quad (2.75) \\
\partial_t f_0 &= f_0 f_2 + f_2 f_0 + \epsilon_2 - \epsilon_1 + 1, \quad (2.76) \\
\partial_t f_1 &= -f_1 f_2 - f_2 f_1 + \epsilon_1 - \epsilon_2. \quad (2.77)
\end{align}

Hence, we have the formulas (2.73).

When $l \geq 2$, the condition (2.72) is equivalent to the following system of equations
\begin{align}
\partial_t \epsilon_i &= \epsilon_i u_i - u_i \epsilon_i, \quad (2.78) \\
f_i - f_{i+1} &= u_i - u_{i+1}, \quad (2.79) \\
\partial_t f_i &= -u_i f_i + f_i u_{i+1} + \alpha_i, \quad (2.80)
\end{align}

with indices understood as elements in $\mathbb{Z}/(l + 1)\mathbb{Z}$. From the equations (2.78), we have $\partial_t \alpha_i = 0$, and the equations (2.79) are the defining relations (2.68). Moreover, one can eliminate variables $u_i$ from the right hand side of the equation (2.80) by using (2.65), (2.66), (2.67), (2.68), and (2.69). Then, one obtains the equations (2.73) and (2.74). We explain the procedure in detail. Note that inserting (2.65) into (2.80), we have
\begin{align}
\partial_t f_i &= f_i (-u_i + u_{i+1}) - h + \alpha_i. \quad (2.81)
\end{align}

Case $l = 2n$: From (2.79) we get
\begin{align}
\sum_{r=1}^{n} (f_{i+2r-1} - f_{i+2r}) &= \sum_{r=1}^{n} (u_{i+2r-1} - u_{i+2r+1}) = -u_i + u_{i+1}. \quad (2.82)
\end{align}

From (2.81) and (2.82) we obtain
\begin{align}
\partial_t f_i &= f_i \sum_{r=1}^{n} (f_{i+2r-1} - f_{i+2r}) - h + \alpha_i \\
&= f_i \left( \sum_{r=1}^{n} f_{i+2r-1} \right) - \left( \sum_{r=1}^{n} f_{i+2r} \right) f_i + \alpha_i.
\end{align}

Thus, we have (2.73).

Case $l = 2n + 1$: From (2.79) we have
\begin{align}
\sum_{r=0}^{n} (f_{2r} - f_{2r+1}) &= \sum_{r=0}^{n} (u_{2r} - u_{2r+2}) = 0, \quad (2.83)
\end{align}
hence, we have
\begin{align}
\sum_{r=0}^{n} f_{2r} &= \sum_{r=0}^{n} f_{2r+1} = \frac{t}{2}, \quad (2.84)
\end{align}
From (2.80) we get

\[ \sum_{r=0}^{n} \partial_t f_{i+2r} = \sum_{r=0}^{n} (f_{i+2r}(-u_{i+2r} + u_{i+2r+1}) + \alpha_{i+2r} - h). \]  

(2.85)

For each \( r = 1, \ldots, n \), we define \( B_r \in A_l \) by the following relation:

\[ u_{i+2r} - u_{i+2r+1} = u_i - u_{i+1} + B_r. \]  

(2.86)

Then, from (2.79) we have

\[ B_r = \sum_{k=1}^{r} f_{i+2k-1} - \sum_{k=r+1}^{n+1} f_{i+2k-1} - \sum_{k=1}^{r-1} f_{i+2k} + \sum_{k=r+1}^{n} f_{i+2k}. \]  

(2.87)

From (2.84) and (2.85) we have

\[ -u_i + u_{i+1} = 2t \left\{ \frac{1}{2} + \sum_{r=1}^{n} f_{i+2r} B_r - \sum_{r=0}^{n} \alpha_{i+2r} + (n+1)h \right\}. \]  

(2.88)

Inserting (2.88) into (2.81), we have

\[ t^2 \frac{\partial_t f_i}{2} = f_i \left\{ \frac{1}{2} + \sum_{r=1}^{n} f_{i+2r} B_r - \sum_{r=0}^{n} \alpha_{i+2r} + (n+1)h \right\} + t^2 (-h + \alpha_i). \]  

(2.89)

Substituting (2.87) for \( B_r \), we have

\[ t \frac{\partial_t f_i}{2} = f_i \left\{ \frac{1}{2} + \sum_{r=1}^{n} f_{i+2r} \left( \sum_{k=1}^{r} f_{i+2k-1} - \sum_{k=r+1}^{n+1} f_{i+2k-1} - \sum_{k=1}^{r-1} f_{i+2k} + \sum_{k=r+1}^{n} f_{i+2k} \right) \right. \\
\left. - \sum_{r=0}^{n} \alpha_{i+2r} + (n+1)h \right\} + t^2 (-h + \alpha_i). \]  

(2.90)

Since

\[ \sum_{r=1}^{n} f_{i+2r} \sum_{k=1}^{r} f_{i+2k-1} = \sum_{r=1}^{n} \sum_{k=1}^{r} f_{i+2k-1} f_{i+2r} - nh, \]  

(2.91)

and

\[ \sum_{r=1}^{n} f_{i+2r} \left( - \sum_{k=1}^{r-1} f_{i+2k} + \sum_{k=r+1}^{n} f_{i+2k} \right) = 0, \]  

(2.92)

we obtain

\[ t \frac{\partial_t f_i}{2} = f_i \left\{ \frac{1}{2} + \sum_{r=1}^{n} \sum_{k=1}^{r} f_{i+2k-1} f_{i+2r} - \sum_{r=1}^{n} \sum_{k=r+1}^{n+1} f_{i+2r} f_{i+2k-1} - \sum_{r=0}^{n} \alpha_{i+2r} + h \right\} \]
Thus, we have (2.74). □

As in the classical case [11], from the viewpoint of the Lax representation, the origin of the affine Weyl group symmetry for the quantum Painlevé systems can be explained as follows.

Let \( G_i(z), \Lambda(z) \) be matrices of \( M_{l+1,l+1}(A_l[z,z^{-1}]) \) defined by

\[
G_0(z) = 1 + \frac{\alpha_0}{f_0} z^{-1} E_{1,l+1}, \quad G_i(z) = 1 + \frac{\alpha_i}{f_i} E_{i+1,i} \quad (1 \leq i \leq l),
\]

\[
\Lambda(z) = \sum_{i=1}^{l} E_{i,i+1} + z E_{1,l+1},
\]

where \( E_{ij} \) is the matrix unit with 1 at the \((i,j)\) entry and 0 for other entries.

We define the action of \( w = s_0, \ldots, s_l, \pi \) on \( A_l \) as follows:

\[
z \partial_z + w(L) = G_w(z) (z \partial_z + L) (G_w(z))^{-1},
\]
\[
\partial_t + w(B) = G_w(z) (\partial_t + B) (G_w(z))^{-1},
\]

where \( G_w = G_i \) for \( w = s_0, \ldots, s_l \) and \( G_\pi = \Lambda \). Then, from the definition, the action of \( s_0, \ldots, s_l, \pi \) on \( A_l \) commutes with the derivation \( \partial_t \) that satisfies (2.72). It can be seen that the action of \( s_0, \ldots, s_l, \pi \) for \( f_i, \alpha_i \) is nothing but the action (2.28) of \( s_0, \ldots, s_l, \pi \) on \( K_l \).

3 Continuous limit

3.1 Discrete system

We construct a quantum discrete system with affine Weyl group symmetry of type \( A_l^{(1)} \) in the same way as in [12]. We introduce the shift operators \( T_i \) \((1 \leq i \leq l+1)\) by

\[
T_1 = \pi s_l s_{l-1} \cdots s_1,
\]
Then, we have the following relations:

\[ T_i T_j = T_j T_i, \quad T_1 \cdots T_{l+1} = 1, \]
\[ T_i(\alpha_{i-1}) = \alpha_{i-1} + k, \quad T_i(\alpha_i) = \alpha_i - k, \quad T_i(\alpha_j) = \alpha_j \quad (j \neq i, i-1). \]

We consider the shift operator \( T_i \) as the time evolution operator. Since \( T_j = \pi^{j-1} T_1 \pi^{l-j} \) for \( j = 2, \ldots, l+1 \), in the following we take \( T_1 \) without loss of generality. The quantum discrete system with \( T_1 \) as its time evolution operator has the form

\[ f_i[n+1] = G_i[n] \quad (0 \leq i \leq l), \quad (3.1) \]

where \( f_i[n] \) stands for \( T_1^n(f_i) \), \( G_i[n] \) is a rational function of \( f_j[n], \alpha_j[n] \) for each \( i = 0, 1, \ldots, l \).

For example, when \( l = 2 \), the quantum discrete system is written as follows:

\[
\begin{cases}
    f_0[n+1] = f_1[n] + \frac{\alpha_0[n]}{f_0[n]} - \frac{\alpha_2[n] + \alpha_0[n]}{f_2[n]} - \frac{\alpha_0[n]}{f_0[n]}, \\
    f_1[n+1] = f_2[n] - \frac{\alpha_0[n]}{f_0[n]}, \\
    f_2[n+1] = f_0[n] + \frac{\alpha_2[n] + \alpha_0[n]}{f_2[n]} - \frac{\alpha_0[n]}{f_0[n]}. \\
\end{cases} \quad (3.2)
\]

For each \( l \geq 2 \), the quantum discrete system has the affine Weyl group symmetry of type \( A_{l-1}^{(1)} \), because automorphisms \( s_0 s_1 s_0, s_2, \ldots, s_l \) of \( K_l \) commute with \( T_1 \), and \( s_0 s_1 s_0, s_2, \ldots, s_l \) define a representation of the affine Weyl group of type \( A_{l-1}^{(1)} \). Therefore, if one can take an appropriate continuous limit of this discrete system, one would obtain a continuous system with affine Weyl group symmetry of type \( A_{l-1}^{(1)} \) in \( K_l \).

Indeed, we can take an appropriate continuous limit if \( l \) is either 2 or \( 2n + 1 \) (\( n = 1, 2, \ldots \)). We shall see how to take a continuous limit in the next subsection.

### 3.2 How to take a continuous limit

When \( l = 2 \), we obtain the quantum second Painlevé equation as the continuous limit, and when \( l = 2n + 1 \), we obtain the quantum Painlevé system of type \( A_{2n}^{(1)} \) as the continuous limit as we shall see below.

Informally, we consider the continuous limit as follows: First, we introduce the parameter \( \epsilon \) called the lattice parameter, and then we introduce the continuous time variable \( t \).
such that $t = n\epsilon$, where $n$ is the discrete time variable. Second, for a function of $n$ we set

$$f[n] = y_0 + y_1\epsilon + y_2\frac{\epsilon^2}{2} + \cdots,$$

where $y_i$ is a function of $t$. Third, assuming

$$f[n + 1] = f[n] + \epsilon\frac{df[n]}{dt} + \frac{\epsilon^2}{2}\frac{d^2f[n]}{dt^2} + \cdots,$$

and comparing the above equation with $f[n + 1] = G[n]$, where $G[n]$ is a function of $f[n]$, we obtain the derivative $dy_i/dt$. The differential equations for $dy_i/dt$ (i = 0, 1, ...) are its continuous limit.

Now we will take the continuous limit for the case where $l = 2n + 1$. We can define the skew field $\mathcal{F}_l$ over $\mathbb{C}$ with the generators $\varphi_i$, $\beta_i$ (0 ≤ $i$ ≤ $l$), $t$, $h'$ and the following relations

$$[\varphi_i, \varphi_{i+1}] = h', \quad [\varphi_i, \varphi_j] = 0 \quad (j \neq i \pm 1),
[\varphi_i, \beta_j] = 0, \quad [\beta_i, \beta_j] = 0,
[\varphi_i, t] = [\beta_i, t] = [\varphi_i, h'] = [\beta_i, h'] = [t, h'] = 0,
\varphi_0 + \varphi_2 + \cdots + \varphi_{2n} = 0, \quad \varphi_1 + \varphi_3 + \cdots + \varphi_{2n+1} = 0,
\beta_0 + \beta_1 + \cdots + \beta_l = 1,$$

where the indices 0, 1, ..., $l$ are understood as elements of $\mathbb{Z}/(l + 1)\mathbb{Z}$. Also, since $f_0 + f_2 + \cdots + f_{2n}, f_1 + f_3 + \cdots + f_{2n+1} \in \mathcal{K}_l$ are central elements in $\mathcal{K}_l$ and invariants of the action of the affine Weyl group $W$, we put these elements as a constant in this quantum discrete system. In particular, we set $f_0 + f_2 + \cdots + f_{2n} := 1, f_1 + f_3 + \cdots + f_{2n+1} := 1$.

**Lemma 3.1** Let $\mathcal{F}_l(\epsilon)$ be the quotient skew field of the polynomial ring $\mathcal{F}_l[\epsilon]$ with coefficients in $\mathcal{F}_l$. We can define the homomorphism $\Psi : \mathcal{K}_l \to \mathcal{F}_l(\epsilon)$ as follows:

$$\Psi(f_0) = 1 + \epsilon\varphi_0, \quad \Psi(f_1) = 1 + \epsilon\varphi_1, \quad \Psi(f_i) = \epsilon\varphi_i \quad (2 \leq i \leq l),
\Psi(\alpha_0) = -1 + \epsilon t + \epsilon^2\beta_0, \quad \Psi(\alpha_1) = 1 - \epsilon t + \epsilon^2\beta_1, \quad \Psi(\alpha_i) = \epsilon^2\beta_i \quad (2 \leq i \leq l),
\Psi(h) = \epsilon^2 h'.$$

**Proof.** One can show that $\Psi$ preserves the defining relations by using the definition of $\mathcal{K}_l$ and $\mathcal{F}_l$. \[\square\]

We introduce the elements $\psi_i$ and $\gamma_i$ of $\mathcal{F}_l$ (0 ≤ $i$ ≤ $l - 1$) by

$$\psi_0 = \varphi_0 + \varphi_1 + t, \quad \psi_i = \varphi_{i+1} \quad (1 \leq i \leq l - 1),
\gamma_0 = \beta_0 + \beta_1, \quad \gamma_i = \beta_{i+1} \quad (1 \leq i \leq l - 1).$$

We denote $s_0s_1s_0, s_2, \ldots, s_l$ by $r_0, r_1, \ldots, r_{l-1}$, respectively.
We can define the action of the subgroup $\tilde{W}' = \langle T_1, r_0, \ldots, r_{l-1} \rangle$ of $\tilde{W}$ on $F_l(\epsilon)$ as follows:

$$T_1(\epsilon) = \epsilon, \quad T_1(t) = t + \epsilon, \quad T_1(\beta_i) = \beta_i \quad (i = 0, \ldots, l),$$

$$T_1(\psi_i) = \frac{\Psi T_1(f_{i+1})}{\epsilon} \quad (i = 1, \ldots, l - 1),$$

and for each $i = 0, \ldots, l - 1$,

$$r_i(\epsilon) = \epsilon, \quad r_i(t) = t,$$

$$r_i(\gamma_j) = \gamma_j - \gamma_i a_{ij} \quad (j = 0, \ldots, l - 1),$$

$$r_i(\psi_j) = \psi_j + \frac{\Psi r_i(f_{j+1})}{\epsilon} \quad (j = 1, \ldots, l - 1),$$

$$r_0(\beta_0) = -\beta_1, \quad r_l(\beta_0) = \beta_0 + \beta_l, \quad r_i(\beta_0) = \beta_0 \quad (1 \leq i \leq l - 1),$$

where $A = (a_{ij})_{i,j=0}^{l-1}$ is the generalized Cartan matrix of type $A_{l-1}^{(1)}$. Then the homomorphism $\Psi$ is an $\tilde{W}'$-intertwiner.

We have the next lemma from the action of $\langle T_1, r_0, \ldots, r_{l-1} \rangle$ on $F_l(\epsilon)$.

**Lemma 3.2** Let $F_l[[\epsilon]]$ be the ring of the formal power series with coefficients in $F_l$. Then, we can define the action of $\tilde{W}'$ on $F_l[[\epsilon]]$ by the relations (3.5), (3.6). Furthermore, for $\phi \in F_l[[\epsilon]]$, we have

$$T_1(\phi) = \phi + \epsilon \hat{\phi}, \quad \text{for some } \hat{\phi} \in F_l[[\epsilon]].$$

**Proof.** We can directly compute $r_i(\psi_j)$ and obtain

$$r_i(\psi_j) = \psi_j + \frac{\gamma_j}{\psi_i} u_{ij} + \epsilon \phi, \quad \text{for some } \phi \in F_l[[\epsilon]],$$

where $U = (u_{ij})_{i,j=0}^{l-1}$ as in (2.26).

For $i = 1, \ldots, l - 1$, one can prove inductively that the following formulas hold:

$$\Psi \pi s_i \cdots s_{i+2} f_i = \begin{cases} 
1 + \epsilon (\varphi_i + \varphi_{i+2} + \ldots + \varphi_l - \varphi_0 - t) + \epsilon^2 \phi & \text{for even } i \\
1 + \epsilon (\varphi_{i+1} + \varphi_{i+2} + \ldots - \varphi_l + \varphi_0) + \epsilon^2 \phi & \text{for odd } i
\end{cases}$$

where $\phi$ is some element of $F_l[[\epsilon]]$. Applying these formulas to

$$T_1(f_{i+1}) = \pi s_i \cdots s_{i+2} f_{i+1} + \pi s_i \cdots s_{i+1} \left( \frac{\alpha_i}{f_i} \right)$$

for $i = 1, \ldots, l - 1$, we obtain that $T_1(\psi_i) = \psi_i + \epsilon \phi$ for some $\phi \in F_l[[\epsilon]]$. As a result, we obtain the formulas (3.7).

From Lemma 3.2, we can define the following $\mathbb{C}$-derivation $\partial$ of $F_l$. 

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Definition 3.3  We define the $C$-derivation $\partial$ of $F_i$ by

$$\partial \phi = \frac{T_i(\phi) - \phi}{\epsilon} \bigg|_{\epsilon=0},$$

(3.11)

for $\phi \in F_i$.

From Lemma 3.2 it holds that $r_i (\epsilon F_i[[\epsilon]]) \subset \epsilon F_i[[\epsilon]]$, hence the actions of $r_0, \ldots, r_{l-1}$ on $F_i = F_i[[\epsilon]]/\epsilon F_i[[\epsilon]]$, are induced. Then we have the next Theorem.

Theorem 3.4 (1) The $C$-derivation $\partial$ of $F_i$ acts on $\psi_i$ $(i = 0, \ldots, l - 1)$ as follows:

$$\partial \psi_i = \psi_i \left( \sum_{1 \leq r \leq n} \psi_{i+2r-1} \right) - \left( \sum_{1 \leq r \leq n} \psi_{i+2r} \right) \psi_i + \gamma_i.$$  

(3.12)

(2) The action of $r_0, \ldots, r_{l-1}$ on $F_i$ commutes with the derivation $\partial$.

Proof. (1) For $i = 1, \ldots, l - 1$, we define the elements $a_i, b_i \in F_i$ by

$$\Psi \pi s_1 \cdots s_{i+1}(f_i) = 1 + \epsilon a_i + \epsilon^2 b_i + \epsilon^3 \phi$$

for some $\phi \in F_i[[\epsilon]].$  

(3.13)

Then, we have

$$\Psi \pi s_1 \cdots s_{i+1}(f_i) = \Psi \pi s_1 \cdots s_{i+2}(f_i - \frac{\alpha_{i+1}}{f_{i+1}})$$

$$= \epsilon \varphi_{i+1} - \left\{ -1 + \epsilon t + \epsilon^2 (\beta_{i+2} + \cdots + \beta_l + \beta_0) \right\}$$

$$\times \left\{ 1 - \epsilon a_{i+1} - \epsilon^2 (b_{i+1} - a_{i+1}^2) \right\} + \epsilon^3 \phi'$$

$$= 1 + \epsilon a_i + \epsilon^2 (a_{i+1} + t) + \epsilon^3 (\beta_{i+1} + a_{i+1}^2 - a_i^2 + t(a_{i+1} - a_i)) + \epsilon^3 \phi''$$

where $\phi'$ and $\phi''$ are some elements of $F_i[[\epsilon]].$ Consequently, by using (3.9) and (3.10), we have

$$\Psi T_1(f_{i+1}) = \Psi (\pi s_1 \cdots s_{i+2}(f_{i+1}) + \pi s_1 \cdots s_{i+1}(\frac{\alpha_{i+1}}{f_{i+1}}))$$

$$= 1 + \epsilon a_{i+1} + \epsilon^2 b_{i+1} + \left\{ 1 - \epsilon t + \epsilon^2 (\beta_{i+1} + \cdots + \beta_l + \beta_0) \right\}$$

$$\times \left\{ 1 - \epsilon a_i + \epsilon^2 (a_{i+1} + t) + \epsilon^3 (\beta_{i+1} + a_{i+1}^2 - a_i^2 + t(a_{i+1} - a_i)) \right\}$$

$$= \epsilon(a_{i+1} + a_i + t) + \epsilon^2 (\beta_{i+1} + a_{i+1}^2 - a_i^2 + t(a_{i+1} - a_i)) + \epsilon^3 \phi''$$

where $\phi''$ and $\phi'''$ are some elements of $F_i[[\epsilon]].$ From (3.9), for even $i$, we have

$$\partial \psi_i = \beta_{i+1} + a_{i+1}^2 - a_i^2 + t(a_{i+1} - a_i)$$

$$= \gamma_i + (\varphi_{i+2} - \varphi_{i+3} + \cdots - \varphi_l + \varphi_0)^2 - (\varphi_{i+1} - \varphi_{i+2} + \cdots + \varphi_l - \varphi_0 - t)^2$$

$$+ t(t - \varphi_{i+1} - 2(\varphi_{i+2} + \cdots + \varphi_l - \varphi_0))$$

$$= \gamma_i - \varphi_{i+1}(\varphi_{i+2} + \cdots + \varphi_l - \varphi_0) + t\varphi_{i+1}$$

$$- (\varphi_{i+2} + \cdots + \varphi_l - \varphi_0)\varphi_{i+1} - \varphi_{i+1}^2$$
\[ = \gamma_i + t\psi_i - \psi_i \sum_{r=1}^n \psi_{i+2r} - \sum_{r=1}^n \psi_{i+2r} - \psi_i^2 \]
\[ = \gamma_i + \psi_i \sum_{r=1}^n \psi_{i+2r-1} - \sum_{r=1}^n \psi_{i+2r} \psi_i. \]

For odd \(i\), one can compute it in a similar way. Hence, the formulas (3.12) hold.

(2) The action of \(r_0, \ldots, r_{l-1}\) commutes with the action of \(T_1\). Hence, from the definition of the derivation \(\partial\), the action of \(r_0, \ldots, r_{l-1}\) commutes with \(\partial\). \(\Box\)

Hence, the derivation \(\partial\) of \(F_{2n}\) defines the quantum Painlevé system of type \(A_{2n}^{(1)}\). However, it is not clear that there is some connection between the derivation \(\partial\) of \(K_{2n}\) defined by the Hamiltonian \(H_0\) in Section 2 and the derivation \(\partial\) of \(F_{2n}\) defined by the time evolution \(T_1\) in this section.

**Remark 3.5** In the classical case, a continuous limit of the discrete system constructed from a representation of \(W(A_2^{(1)}) \times W(A_1^{(1)}) (l \geq 2)\) in [10] is the classical Painlevé system of type \(A_l^{(1)}\). If one can quantize the representation of \(W(A_2^{(1)}) \times W(A_1^{(1)})\), then one would obtain the quantum discrete system whose continuous limit is the quantum Painlevé system of type \(A_l^{(1)}\).

An appropriate continuous limit of the quantum discrete system for \(l = 2\) is the quantum second Painlevé equation. We can take the limit in the similar way as in the case where \(l = 2n + 1\), though how to set \(y_i\) in (3.3) is not exactly the same. Let \(F_2\) be the skew field over \(\mathbb{C}\) with the generators \(\psi, \varphi_0, \varphi_1, \beta_0, \beta_1, \beta_2, t, h'\) and the following relations:

\([\psi, \varphi_i] = \frac{h'}{2} (i = 0, 1), \quad [\varphi_0, \varphi_1] = 0, \quad \beta_i, t, h'\) are central, \quad \beta_0 + \beta_1 + \beta_2 = 1.

Since \(f_0 + f_1 + f_2\) is central in \(K_2\) and the invariant of the action of \(W\), we put \(f_0 + f_1 + f_2 = 2\).

**Lemma 3.6** We can define the homomorphism \(\Psi : K_2 \to F_2(\epsilon)\) as follows:

\[ \Psi(f_0) = 1 + \epsilon \psi + \epsilon^2 \varphi_0, \quad \Psi(f_1) = 1 - \epsilon \psi + \epsilon^2 \varphi_1, \quad \Psi(f_2) = -\epsilon^2 (\varphi_0 + \varphi_1), \]
\[ \Psi(\alpha_0) = -1 + \epsilon^2 t + \epsilon^3 \beta_0, \quad \Psi(\alpha_1) = 1 - \epsilon^2 t + \epsilon^3 \beta_1, \quad \Psi(\alpha_2) = \epsilon^3 \beta_2, \]
\[ \Psi(h) = \epsilon^3 h'. \]

**Proof.** One can show that \(\Psi\) preserves the defining relations by using the definition of \(K_2\) and \(F_2\). \(\Box\)
We can define the action of the subgroup \( \tilde{W}' = \langle T_1, r_0, r_1 \rangle \) of \( \tilde{W} \) on \( F_2(\epsilon) \) as follows:

\[
T_1(\epsilon) = \epsilon, \quad T_1(t) = t + \epsilon, \quad T_1(\beta_i) = \beta_i \quad (i = 0, 1, 2),
\]
\[
T_1(\psi) = \psi + \epsilon(2(\varphi_0 + \varphi_1) - \psi^2 + t),
\]
\[
T_1(\varphi_0) = \frac{\Psi T_1(f_0) - 1 - \epsilon T_1(\psi)}{\epsilon^2}, \quad T_1(\varphi_1) = \frac{\Psi T_1(f_1) - 1 + \epsilon T_1(\psi)}{\epsilon^2},
\]

and for each \( i = 0, 1, \)

\[
r_i(\epsilon) = \epsilon, \quad r_i(t) = t,
\]
\[
r_0(\psi) = \psi - \frac{\beta_0 + \beta_1}{\varphi_0 + \varphi_1 + t - \psi^2}, \quad r_1(\psi) = \psi - \frac{\beta_2}{\varphi_0 + \varphi_1},
\]
\[
r_i(\varphi_0) = \frac{\Psi r_i(f_0) - 1 - \epsilon r_i(\psi)}{\epsilon^2}, \quad r_i(\varphi_1) = \frac{\Psi r_i(f_1) - 1 + \epsilon r_i(\psi)}{\epsilon^2},
\]
\[
r_0(\beta_0) = -\beta_1, \quad r_0(\beta_1) = -\beta_0, \quad r_0(\beta_2) = 2 - \beta_2,
\]
\[
r_1(\beta_0) = \beta_0 + \beta_2, \quad r_1(\beta_1) = \beta_1 + \beta_2, \quad r_1(\beta_2) = -\beta_2.
\]

Then, the homomorphism \( \Psi \) is an \( \tilde{W}' \)-intertwiner, and we can define the action of \( \tilde{W}' \) on \( F_2[[\epsilon]] \) by the above relations. Furthermore, we have

\[
T_1(\varphi_0) = \varphi_0 + \epsilon(\psi \varphi_1 + \varphi_1 \psi + \psi^3 - t \psi + \beta_0 - \beta_2) + \epsilon^2 \phi, \tag{3.14}
\]
\[
T_1(\varphi_1) = \varphi_1 + \epsilon(\psi \varphi_0 + \varphi_0 \psi - \psi^3 + t \psi - \beta_0) + \epsilon^2 \phi, \tag{3.15}
\]
\[
r_0(\varphi_0) = \frac{1}{\varphi_0 + \varphi_1 + t - \psi^2} \psi (\beta_0 + \beta_1)
\]
\[
+ \frac{1}{\varphi_0 + \varphi_1 + t - \psi^2} (\beta_0 + \psi \varphi_1 - \varphi_0 \psi) \frac{\beta_0 + \beta_1}{\varphi_0 + \varphi_1 + t - \psi^2} \epsilon \phi', \tag{3.16}
\]
\[
r_0(\varphi_1) = \frac{1}{\varphi_0 + \varphi_1 + t - \psi^2} \psi
\]
\[
+ \frac{1}{\varphi_0 + \varphi_1 + t - \psi^2} (\beta_1 + \psi \varphi_0 - \varphi_1 \psi) \frac{\beta_0 + \beta_1}{\varphi_0 + \varphi_1 + t - \psi^2} \epsilon \phi'', \tag{3.17}
\]

where \( \phi, \phi', \) and \( \phi'' \) are some elements of \( F_2[[\epsilon]] \). Hence, we can define the \( \mathbb{C} \)-derivation \( \partial \) of \( F_2 \) by

\[
\partial \phi = \left. \frac{T_1(\phi) - \phi}{\epsilon} \right|_{\epsilon=0}, \tag{3.18}
\]

for \( \phi \in F_2 \), and on \( F_2 \), the actions of \( r_0, r_1 \) are induced. Then we have the next theorem.

**Theorem 3.7** (1) The \( \mathbb{C} \)-derivation \( \partial \) of \( F_2 \) acts on \( \psi, \varphi_0 \) and \( \varphi_1 \) as follows:

\[
\partial \psi = 2(\varphi_0 + \varphi_1) - \psi^2 + t,
\]
\[
\partial \varphi_0 = \psi \varphi_1 + \varphi_1 \psi + \psi^3 - t \psi + \beta_0 - \beta_2, \quad \partial \varphi_1 = \psi \varphi_0 + \varphi_0 \psi - \psi^3 + t \psi - \beta_0.
\]

Therefore, we have

\[
\partial^2 \psi = 2\psi^3 - 2t \psi - 2\beta_2 + 1. \tag{3.19}
\]

(2) The actions of \( r_0, r_1 \) on \( F_2 \) commute with the derivation \( \partial \).
In the classical case \( h' = 0 \), the equation (3.19) is nothing but the classical second Painlevé equation \( P_{II} \). We call the equation (3.19) the quantum second Painlevé equation.

Also, putting \( f_0 = - (\varphi_0 + \varphi_1), f_1 = \varphi_0 + \varphi_1 + t - \psi^2, f_2 = \psi \), (3.19) reduces to (1.6). However, in the same as \( l = 2n \) case, it is not clear that there is some connection between the derivation \( \partial \) of \( K_1 \) defined by the Hamiltonian \( H_0 \) in Section 2 and the derivation \( \partial \) of \( \mathcal{F}_1 \) defined by the time evolution \( T_1 \) in this section.

A Appendix: Properties of the Hamiltonians \( H_j \)

In the following, we introduce a family of Hamiltonians \( H_1, \ldots, H_l \), by the diagram rotation. Namely,

\[
H_j := \pi(H_{j-1}). \tag{A.1}
\]

In the classical case, these Hamiltonians have some remarkable properties in relation with the action of \( W \).

**Proposition A.1** With respect to the action of the affine Weyl group \( W \) (2.24) and (2.28), the Hamiltonians have the following:

1. For \( l = 1, 2n \),
   \[
s_i(H_j) = H_j + \delta_{ij}k \frac{\alpha_j}{f_j} \quad (i, j = 0, \ldots, l). \tag{A.2}
   \]

2. For \( l = 2n + 1 \),
   \[
s_i(H_j) = H_j + \delta_{ij}k \frac{\alpha_j}{f_j} x_j \quad (i, j = 0, \ldots, l), \tag{A.3}
   \]

where the index of \( x_j \) (2.51) is regarded as in \( \mathbb{Z}/2\mathbb{Z} \). In particular, the Hamiltonian \( H_j \) is invariant with respect to the action of the Weyl group \( W(A_l) = \langle s_0, \ldots, s_{j-1}, s_{j+1}, \ldots, s_l \rangle \).

**Proof.** This proposition is a generalization of Proposition 2.7. From (2.31), (2.32), by using the definition of \( H_j \) (A.1) and the relation \( \pi s_i = s_{i+1} \pi \), we obtain the formulas (A.2), (A.3) respectively. \( \square \)

**Proposition A.2** (1) In the case of \( A_{2n}^{(1)} \), for each \( j = 0, \ldots, 2n \), one has

\[
H_{j+1} - H_j = k \sum_{r=1}^{n} f_{j+2r} - \frac{nk}{2n + 1} x,
\]

where \( x = f_0 + f_1 + \cdots + f_{2n} \) (2.42).

(2) In the case of \( A_{2n+1}^{(1)} \), for each \( j = 0, \ldots, 2n + 1 \), one has

\[
H_{j+1} - H_j = k \sum_{1 \leq r \leq s \leq n} f_{j+2r} f_{j+2s+1} - \frac{nk}{2n + 1} \sum_{K \in S_2} f_K + (-1)^j \frac{k}{4} \sum_{i=0}^{l} (-1)^i \alpha_i. \tag{A.5}
\]
Proof. For $l = 2, 3$, we can prove through direct computations. For $l = 2n$ ($l = 2n, n \geq 2$), we have

$$H_0 = \sum_{K \in S_3} f_K + \sum_{i=0}^{2n} \chi(\Gamma \setminus \{i\}) f_i,$$

$$H_1 = \sum_{K \in S_3} f_K + \sum_{i=0}^{2n} \pi(\chi(\Gamma \setminus \{i\})) f_i.$$

Hence,

$$H_1 - H_0 = \sum_{i=0}^{2n} \pi(\chi(\Gamma \setminus \{i-1\}) - \chi(\Gamma \setminus \{i\})) f_i. \quad (A.6)$$

Computing $\pi(\chi(\Gamma \setminus \{i-1\}) - \chi(\Gamma \setminus \{i\}))$ from the definition, we obtain

$$\pi(\chi(\Gamma \setminus \{i-1\}) - \chi(\Gamma \setminus \{i\})) = \begin{cases} 
- \frac{n k}{2n + 1} & (i = 0 \text{ or } i = \text{odd}) \\
\frac{2n + 1}{(n + 1)k} & (i \neq 0, i = \text{even}) \\
\frac{2n + 1}{k} & (i \neq 0, i = \text{even}) 
\end{cases}. \quad (A.7)$$

Therefore, we obtain

$$H_1 - H_0 = \sum_{i=0}^{2n} \frac{-nk}{2n + 1} f_i + \sum_{r=1}^{n} k f_{2r} = k \sum_{r=1}^{n} f_{j+2r} - \frac{nk}{2n + 1} x. \quad (A.8)$$

Similarly we can show the formula in the case where $l = 2n + 1$ ($n \geq 2$). \qed

In the classical case, we have $\tau$-functions $\tau_0, \ldots, \tau_l$ such that $h_j = k(\log \tau_j)'$, where $h_j$ is the classical Hamiltonian corresponding to quantum Hamiltonian $H_j$. The affine Weyl group symmetry lifts to the level of $\tau$-functions. In fact, Proposition A.1 and Proposition A.2 illustrate how to lift the affine Weyl group symmetry to the level of $\tau$-functions.

Unfortunately, the formulation of $\tau$-functions in the quantum case is not completed yet and we hope to report on this in a near future.

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