ON THE PHONON–INDUCED SUPERCONDUCTIVITY OF DISORDERED ALLOYS

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A model of alloy is considered which includes both quenched disorder in the electron subsystem ("alloy" subsystem) and electron-phonon interaction. For given approximate solution for the alloy part of the problem, which is assumed to be conserving in Baym’s sense, we construct the generating functional and derive the Eliashberg-type equations which are valid to the lowest order in the adiabatic parameter. The renormalization of bare electron-phonon interaction vertices by disorder is taken into account consistently with the approximation for the alloy self-energy. For the case of exact configurational averaging the same set of equations is established within the usual $T$-matrix approach. We demonstrate that for any conserving approximation for the alloy part of the self-energy the Anderson’s theorem holds in the case of isotropic singlet pairing provided disorder renormalizations of the electron-phonon interaction vertices are neglected. Taking account of the disorder renormalization of the electron-phonon interaction we analyze general equations qualitatively and present the expressions for $T_c$ for the case of weak and intermediate electron-phonon coupling. Disorder renormalizations of the logarithmic corrections to the effective coupling, which arise when the effective interaction kernel for the Cooper channel has the second energy scale, as well as the renormalization of the dilute paramagnetic impurity suppression are discussed.

Keywords: Electron- Phonon Interaction, Quenched Disorder, Superconductivity, Generating Functional Approach

1. Introduction

A problem of how disorder influences a superconductive transition temperature $T_c$, first studied in pioneering papers by Anderson and by Abrikosov and Gor’kov, draws a great deal of physicists attention. However, even if a usual mechanism of the superconductivity owing to electron-phonon interaction (EPI) and isotropic $s$-type pairing are concerned — it is this subject present paper deals with — one cannot regard the problem as being solved completely. The situation where disorder scattering is weak or the concentration of impurities (both magnetic and nonmagnetic) is small was studied rigorously.

As for strong disordering, in early work this problem was considered within the framework of semiphenomenological approaches which were called for explaining qualitatively empirical rules (e.g., Miedema rules) determining superconductive transition temperature in alloys. In a pioneering paper by Anderson et al. the...
a mechanism of universal \( T_c \)-degradation was proposed which was based on the 
enhancement of screened Coulomb pseudopotential due to disordering effects in 
compounds with A-15 structure. Further a great number of works appeared touching 
terrelations between the phonon-induced superconductivity and Anderson’s 
localization, and other effects of strong disorder, see papers, \cite{16,17,18,19,20} 
and also a recent review by Belitz and Kirkpatrick \cite{21} where mapping on the non-linear matrix 
\( \sigma \)-model was used when treating the Anderson’s localization and related problems.

Recently a new impetus has appeared for studying the influence of disorder on 
\( T_c \). This impetus comes mainly from discussing mechanisms of superconductivity in 
novel Cu-oxide based compounds (see, e.g., papers by Abrikosov \cite{22} where 
the influence of non-magnetic impurities on \( T_c \) is discussed in the case of strongly 
quasimomentum-dependent singlet order parameter). These substances are sup-
posed to be strongly correlated systems which may be described in the simplest 
way by an “alloy analogy” within some approaches, as for instance Hubbard-III ap-
proximation \cite{23} or static approximation within functional integral approach \cite{24,25,26,27} 
where this analogy emerges in a rather natural way. In particular, the renormaliza-
tion of the phonon-induced \( T_c \) was discussed in Ref. \cite{28} where the use was made of 
Hubbard-III approximation when treating the effects of strong electron on-site repul-
sion, however only a part of contributions was taken into account. Although the 
approximations of Hubbard-III type are considered now as not being quite adequate, 
from the formal point of view they have many similarities with modern approaches 
to highly correlated systems, which are based on the \( d \to \infty \) limit: \cite{29,30,31,32,33} the 
electron self-energy possesses strong energy dependence in the normal phase, while 
the quasimomentum dependence is absent.

Thus, long time after creating BCS theory the classical problem of the disorder 
influence on \( T_c \) attracts attention of researchers.

At the same time, existence of large number of works and approximations has 
lead to that it is difficult to compare results of various approaches between each 
other. Even the order parameter is understand in different ways. Therefore it is 
instructive to study the problem in a general form by which we mean the follow-
ing. Given an approximate self-energy expression for the averaged electron Green’s 
function in the normal phase and without electron-phonon interaction, we search 
for the equations determining the order parameter in the superconductive phase for 
the electron-phonon pairing mechanism, which are consistent (in some still unde-
termined sense) with the approximation.

In the present paper we study systematically the question of how strong quenched 
substitutional disorder (components of an alloy are assumed to be nonmagnetic) in-
fluences \( T_c \), leaving aside an interesting in its own right and fascinating problem of 
an interplay between superconductivity and localization phenomenon (the latter is 
known may occure in disordered systems \cite{34}). We shall demonstrate that for a given 
approximation describing normal state of an alloy in the absence of EPI and satisfy-
ing natural requirement of being thermodynamically consistent in Baym-Kadanoff 
sense \cite{35,36} (later on rather the terms “\( \phi \)-derivable” or “conserving” will be used in-
terchangeably when referring to such approximations), and with suitable meaning of the term “consistent”, the problem can be solved in an unique way. We propose a procedure which enables constructing the Eliashberg-type equations, which are valid to the lowest order in the adiabatic parameter and properly take into account the disorder renormalization of EPI vertices. The structure of these equations is the same for all $\phi$-derivable approximations and such approximations, as Virtual Crystal, Diffusional, Averaged $T$–Matrix and Coherent Potential approximations, etc., can be discussed within an unified scheme. However, the linearized version of the equations is so complicated that further simplifications are required and we establish also what kind of simplifications leads to the results previously obtained within some (in fact, within the most of already mentioned) approximations widely accepted in treating the “alloy” problems.

We should note that the problem in a similar form has been considered by Belitz with the use of exact eigenfunction representation. However, in this approach the very definition of the order parameter differs from the traditional one, which exploits quasimomentum or coordinate representation, and establishing relations between these approaches is, in general, a very difficult problem. The approach being proposed in this paper seems to be more directly related with observable characteristics, example of which is the quasimomentum dependence of the superconductive gap. In this connection, the traditional and often not specially discussed supposition, that the order parameter is constant over the Fermi surface in dirty metals and, therefore, the type of averaging procedure is not crucially important, does not, generally speaking, hold (see Ref. 22).

The paper is organized as follows. In Section 2 we discuss the model. The generating functional is constructed and the equations of the strong coupling theory are then derived. An alternative approach with the use of standard $T$-matrix formulation for the configurational averaging is given in Section 3 where we discuss also the relationships between the two approaches. In Section 4 we demonstrate that for the case of the isotropic singlet pairing owing to electron-phonon interaction the Anderson’s theorem may be valid when formulated analogously to that for the BCS model, and we explicitly specify the conditions for the theorem to hold (all earlier treatments appear to be incomplete). In Section 5 reduced isotropic equations are derived in which the anomalous self-energy contribution coming from disorder scattering (“disorder” contribution to the anomalous part of the self-energy) is eliminated exactly. Section 6 deals with qualitative analysis of the reduced equations within certain approximations for the isotropic contributions of electron-phonon vertices and general expressions for $T_c$ are presented there for the case of weak and intermediate electron-phonon coupling. We establish relations between effective electron-phonon coupling and generating function for arbitrary thermodynamically consistent single-site approximation for the alloy self-energy and write down explicit expressions for renormalized electron-phonon coupling within a number of widely known single-site approximations in Section 7. In Appendix A we give a derivation of Ward-type identities. Appendix B deals with renormalizations of the
$T_c$-suppression by dilute paramagnetic impurities and in Appendix C some useful relations between generating functions for “disorder” self-energy and for vertex corrections are established within arbitrary single-site approximation.

2. The Model, Generating Functional and Eliashberg-type Equations

To begin with, we consider the model of alloy with substitutional disorder described by the Frölich-type model Hamiltonian

$$H = H_e + H_{e-ph} + H_{ph},$$

(2.1)

where $H_e$ is Hamiltonian of an electron subsystem, which in the Wannier representation has the form

$$H_e = \sum_{ij,\sigma} h_{ij} c_{i\sigma}^\dagger c_{j\sigma},$$

(2.2)

$$h_{ij} = t_{ij} - \mu \delta_{ij} + \xi_{ij} + \varepsilon_i \delta_{ij},$$

(2.3)

and $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) is electron creation (annihilation) operator in the Wannier state with spin $\sigma$ on site $i$; $t_{ij}$, non-random part of transfer integral; $\xi_{ij}$, random part of transfer integral and $\langle \xi_{ij} \rangle = 0$ ($\langle \ldots \rangle$ means configurational average with certain probability distribution); $\varepsilon_i$, on-site electron energy; $\delta_{ij}$, usual Kronecker delta symbol and $\mu$, the chemical potential.

In the case of substitutional disorder of a general type both $\xi_{ij}$ and $\varepsilon_i$ are random variables which depend on kinds of atoms placed at the corresponding lattice sites and particular cases of alloy disorder may be obtained by appropriate choosing the quantities $\xi_{ij}$ and $\varepsilon_i$ to be random or not. The so called diagonal disorder emerges when one sets all the $\xi_{ij}$ to be identically zero and quantities $\varepsilon_i$ are random and depend on the sort of atoms at the corresponding lattice sites. If, on the contrary, $\xi_{ij}$ are random, the Hamiltonian (2.3) describes the case of off-diagonal alloy disorder.

To complete the description of a disorder type one should fix distributions for the random quantities entering the Hamiltonian (2.2). For the case of random binary alloy model with diagonal disorder Eq. (2.3) takes the form

$$h_{ij} = t_{ij} - \mu \delta_{ij} + \varepsilon_A \delta_{ij} + V \eta_i \delta_{ij},$$

(2.4)

$$\text{Prob}(\eta_i = 1) = c, \quad \text{Prob}(\eta_i = 0) = 1 - c,$$

(2.5)

with $c$ being the concentration of B-type component of the alloy in the host which is assumed to be of A-type, and $V = \varepsilon_B - \varepsilon_A$ is the scattering potential of B-type atoms measured relatively to the host potential. We will use this specific model of alloy in Sections 6 and 7 and in Appendices B and C. In the remaining part of the paper a discussion is rather general and does not rely upon any specific alloy model.

The Hamiltonian of the phonon subsystem is of the following form

$$H_{ph} = \sum_{\mathbf{q}, \omega} \omega_{\mathbf{q}} b_{\mathbf{q}}^\dagger b_{\mathbf{q}},$$

(2.6)
with $b_{q,s}^\dagger$ ($b_{q,s}$) being phonon creation (annihilation) operator with quasimomentum $q$ and the branch index $s$, $\omega_{q,s}$ being the phonon frequencies and we assume that $\omega_{q,s}$ are non-random functions despite they are renormalized by all the interactions in our model.

Finally, the Hamiltonian of the electron-phonon interaction reads

$$H_{e-ph} = \sum_{ij,\sigma} \sum_{l,s} M_{l;ij}^{(s)} c_{i\sigma}^\dagger c_{j\sigma} \frac{1}{\sqrt{N}} \sum_{q} \exp(-iqr) \left( b_{q,s}^\dagger + b_{-q,s} \right),$$

where $M_{l;ij}^{(s)}$ is matrix element of the electron-phonon interaction (EPI) and supposed to be independent of disorder.

Deriving explicit expressions for $M_{l;ij}^{(s)}$ requires some care since because the scattering by impurities is elastic only in the coordinate frame moving with crystall lattice as was pointed out by Blount. We will not write down the expressions for $M_{l;ij}^{(s)}$ here; corresponding details and the results of calculations are given in Refs. 39, 40, 41, 42.

As is known, one should treat $\omega_{q,s}$ as completely renormalized by all the relevant interactions when considering the phenomenon of superconductivity in such model, and effects of EPI on electron spectrum should be taken into account in the lowest (in fact, second) order in $M_{l;ij}^{(s)}$ for each disorder configuration. This actually corresponds to the lowest order of the perturbation theory in adiabatic parameter as was demonstrated by Migdal. Of course the two simplifying approximations about non-random character of $M_{l;ij}^{(s)}$ and $\omega_{q,s}$ cannot be proven rigorously if the real situation is kept in mind, and therefore they have a model character within the framework of the Frölich-type model. However, as we will see in what follows, even with these approximations the model remains complicated and interesting enough for studying.

In the Gorkov-Nambu representation, and (2.7) take the form

$$H_e = \sum_{ij} h_{ij} C_i^\dagger \tau_3 C_j,$$

$$H_{e-ph} = \sum_{ij} \sum_{l,s} M_{l;ij}^{(s)} C_i^\dagger \tau_3 C_j \frac{1}{\sqrt{N}} \sum_{q} \exp(-iqr) \left( b_{q,s}^\dagger + b_{-q,s} \right),$$

where we have introduced Nambu row- and column- spinors

$$C_i^\dagger = \left( c_{i\uparrow}^\dagger, c_{i\downarrow} \right), \quad C_i = \left( c_{i\uparrow}, c_{i\downarrow}^\dagger \right)$$

and

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are usual Pauli matrices.
To derive equations of strong coupling theory, or so-called Eliashberg-type equations, we use the following trick. We construct a functional $W[G, D]$ of Luttinger-Ward type. As is known, the functional derivative of such a functional with respect to one-particle Green’s function $G$ give the self-energy $\Sigma[G, D]$ (or mass operator), as functionals of $G$ and $D$, and, along with Dyson’s equation for Green’s function $G$, this procedure leads to a set of self-consistent equations on $\Sigma[G, D]$ and $G$. Within the Fröhlich-type model being used here, “bare” phonon Green’s function $D$ is fully determined by $H_{e-ph}$ Eq. (2.6), and coincides with the renormalized one.

We suppose also that the functional $W^e[G]$ determining certain approximate self-energy $\Sigma^e[G]$ for the alloy part of the problem, Eqs. (2.2,2.3,2.8) (i.e., without EPI), is known and what remains is to find the contribution of EPI to $W[G, D]$ up to the lowest order in EPI coupling with account of those renormalizations of EPI vertices by disorder which are consistent with given approximate $\Sigma^e[G]$ (or, which is the same, with the approximation to $W^e[G]$).

More formally, we seek for the set of equations of the following general form

\[
W[G, D] = W^e[G] + W^{e-ph}[G, D], \tag{2.12}
\]

\[
\Sigma[G, D] = \Sigma^e[G] + \Sigma^{e-ph}[G, D], \tag{2.13}
\]

\[
\Sigma^e[G] = \frac{\delta W^e[G]}{\delta G^T}, \tag{2.14}
\]

\[
\Sigma^{e-ph}[G, D] = \frac{\delta W^{e-ph}[G, D]}{\delta G^T}, \tag{2.15}
\]

\[
G = G_0 + G_0 \Sigma[G, D], \tag{2.16}
\]

where $W^e[G]$ and $W^{e-ph}[G, D]$ are the contributions to the functional $W[G, D]$ which are due to disorder scattering and EPI respectively; $\Sigma^e[G]$ and $\Sigma^{e-ph}[G, D]$, the corresponding contributions to the self-energy $\Sigma[G, D]$, and in (2.14) $G_0$ is “bare” one-particle Green’s function determined by non-random part of $H_e$ Eq. (2.2).

To construct the functional $W^{e-ph}[G, D]$ we note that to lowest order in EPI

\[
\Pi[G, D] = 2 \frac{\delta W[G, D]}{\delta D^T} = 2 \frac{\delta W^{e-ph}[G, D]}{\delta D^T}, \tag{2.17}
\]

where $\Pi[G, D]$ is the polarization operator, and

\[
\frac{\delta \Pi[G, D]}{\delta D^T} = 0, \tag{2.18}
\]

so that $\Pi[G, D]$ is a functional of $G$ only, $\Pi = \Pi[G]$. The functional $W^{e-ph}[G, D]$ may then be written as

\[
W^{e-ph}[G, D] = \frac{1}{2} \operatorname{Sp} D \Pi[G], \tag{2.19}
\]

in some formal notation.
Now we may give the definition of the renormalizations of EPI which are consistent with the approximation for $\Sigma^e[G]$. We will call the contributions to $\Sigma^{e-\text{ph}}[G,D]$ consistent with the approximation to $\Sigma^e[G]$ if this contributions are derived from the functional (2.19), where the polarization operator $\Pi[G]$ is consistent with the approximation for $\Sigma^e[G]$ in Boym-Kadanoff sense.

To obtain explicit expression we divide $W^{e-\text{ph}}[G,D]$ into two parts

$$W^{e-\text{ph}}[D,G] = W^H[D,G] + W^F[D,G], \quad (2.20)$$

where $W^H[G,D]$ and $W^F[G,D]$ are Hartree- and Fock-type contributions to the generating functional respectively.

Consider Fock-type contribution first. Contributions to the polarization operator

$$\Pi^F[G] = 2 \frac{\delta W^F[D,G]}{\delta D^T} \quad (2.21)$$

are shown in Fig. 1. Restoring the generating functional $W^F[G,D]$ with the use of Eq. (2.19) yields

$$W^F[G,D] = -\frac{T}{2} \sum_{ip_s,ip_m} D_{\gamma\gamma'}(ip_s - ip_m) M^\gamma_{\alpha\alpha'} \left\{ G_{\alpha\beta}(ip_m) G_{\beta'\alpha'}(ip_s) + G_{\alpha\mu}(ip_m) G_{\mu'\alpha'}(ip_s) \Gamma_{\mu\nu\mu'\nu'}(ip_m,ip_s) G_{\beta'\nu'}(ip_s) G_{\nu\beta}(ip_m) \right\} M^\gamma_{\beta'\beta} \quad (2.22)$$

and we depicted the functional $W^F[G,D]$ in Fig. 2. Here and in what follows

Fig. 1. Fock-type contributions to the polarization operator. Black circle denotes “bare” electron-phonon three-leg vertex; bold line, the renormalized electron Green’s function; the square labeled with $\Gamma$, full disorder four-leg vertex. Arrows points the direction of index arrangement.

Fig. 2. Fock-type contributions to the generating functional. Bold wavy line denotes “bare” phonon Green’s function and meaning of the other graphical elements is the same as in Fig. 1.
implicit summation over doubly repeated Greek indices is implied,

\[ M_{\alpha\alpha'}^{\gamma\gamma'} = \left( \Gamma_{\gamma\gamma'} \right)_{\alpha\alpha'} \]  

(2.23)

is bare EPI vertex, and we have introduced multy-indices encapsulating site \textit{and} spinor indices (e.g. \( \alpha, \beta, \) etc. in (2.23)) or site \textit{and} branch indices (\( \gamma, \gamma' \) in (2.23)).

The quantity \( \Gamma_{\alpha\beta;\beta'\alpha'}(ip_m, ip_s) \) is full disorder four-leg vertex which obeys the Bethe-Salpeter equation

\[
\Gamma_{\alpha\beta;\beta'\alpha'}(ip_s, ip_m) = U_{\alpha\beta;\beta'\alpha'}(ip_s, ip_m) + 
+ U_{\mu\nu;\mu'\nu'}(ip_s, ip_m) G_{\mu\nu}(ip_s) G_{\nu'\mu'}(ip_m) \Gamma_{\nu'\beta;\beta'\nu'}(ip_s, ip_m),
\]

(2.24)

and \( U_{\mu\nu;\nu'\mu'}(ip_s, ip_m) \) is irreducible disorder four-leg vertex generated by the approximation used for \( \Sigma'[G] \) (in fact this vertex is the second variational derivative of the functional \( W'[G] \) with respect to \( G \)).

Apart from the standard Fock-type contribution, there is still a contribution of Hartree-type. This contribution is usually neglected following Muttalib and Anderson. However, as was demonstrated in a series of papers by Belitz et al., this contribution is important and do play role in final expression for \( T_c \).

To take Hartree-type contribution into account we introduce, in an analogy with Fock case, a polarization operator \( \Pi^H[G] \) as

\[
\Pi^H[G] = \frac{\delta W^H[D, G]}{\delta D^I} 
\]

(2.25)

(see Fig.3 for corresponding contributions). We then have

Fig. 3. Hartree-type contributions to the polarization operator.

\[
W^H[G, D] = \frac{T}{2} \sum_{ip_s, ip_m} D_{\gamma\gamma'}(0) M_{\alpha\alpha'}^{\gamma\gamma'} \left\{ G_{\alpha'\alpha}(ip_m) G_{\beta\beta'}(ip_s) + 
+ G_{\alpha'\mu}(ip_m) G_{\nu\alpha}(ip_m) \Gamma_{\mu\nu;\mu'\nu'}(ip_m, ip_s) G_{\beta\beta'}(ip_s) G_{\nu'\beta'}(ip_s) \right\} M_{\beta'\beta}, 
\]

(2.26)

and the functional \( W^H[G, D] \) is shown in Fig. 4.

Putting everything together one obtains for \( W^{\varepsilon-ph}[G, D] \) the expression

\[
W^{\varepsilon-ph}[G, D] = \frac{T}{2} \sum_{ip_s, ip_m} D_{\gamma\gamma'}(ip_s - ip_m) M_{\alpha\alpha'}^{\gamma\gamma'} \left\{ G_{\alpha'\alpha}(ip_m) G_{\beta\beta'}(ip_s) + 
+ G_{\alpha'\mu}(ip_m) G_{\nu\alpha}(ip_m) \Gamma_{\mu\nu;\mu'\nu'}(ip_m, ip_s) G_{\beta\beta'}(ip_s) G_{\nu'\beta'}(ip_s) \right\} M_{\beta'\beta} + \frac{T}{2} \sum_{ip_s, ip_m} D_{\gamma\gamma'}(ip_s - ip_m) M_{\alpha\alpha'}^{\gamma\gamma'} G_{\alpha'\beta'}(ip_m) G_{\beta\alpha}(ip_s) M_{\beta'\beta}, 
\]

(2.27)
and we have introduced renormalized by the disorder EPI vertex:

\[ M^\gamma_{\alpha\alpha'}(ip_s, ip_m) = M^\gamma_{\alpha\alpha'} + \]
\[ + \Gamma_{\alpha\beta;\beta'\alpha'}(ip_s, ip_m)G_{\beta\beta'}(ip_s)G_{\mu\nu'}(ip_m)M^\gamma_{\mu\nu'}, \]
\[ (2.28) \]
\[ M^\gamma_{\alpha\alpha'}(ip_s, ip_m) = M^\gamma_{\alpha\alpha'} + \]
\[ U_{\alpha\beta;\beta'\alpha'}(ip_s, ip_m)G_{\beta\beta'}(ip_s)G_{\mu\nu'}(ip_m)M^\gamma_{\mu\nu'}(ip_s, ip_m). \]
\[ (2.29) \]

The use of (2.12–2.16) then yields the following set of coupled equations:

\[ G_{\alpha\beta}(ip_s) = G_{\alpha\beta}^0(ip_s) + G_{\alpha\mu}^0(ip_s)\Sigma_{\mu\nu}(ip_s)G_{\nu\beta}(ip_s), \]
\[ (2.30) \]
\[ \{ G_0^{-1}(ip_s) \}_{ij} = ip_s\tau_0\delta_{ij} - (t_{ij} - \mu_0\delta_{ij})\tau_3, \]
\[ (2.31) \]
\[ \Sigma_{\mu\nu}(ip_s) = \Sigma_{\mu\nu}^e(ip_s) + \Sigma_{\mu\nu}^{ph}(ip_s), \]
\[ (2.32) \]
\[ \Sigma_{\mu\nu}^e(ip_s) = \frac{\delta W^e[G]}{\delta G_{\mu\nu}(ip_s)}, \]
\[ (2.33) \]
\[ \Sigma_{\mu\nu}^{ph}(ip_s) = \Sigma_{\mu\nu}^H(ip_s) + \Sigma_{\mu\nu}^F(ip_s), \]
\[ (2.34) \]

\[ \Sigma_{\mu\nu}^F(ip_s) = -T \sum_{ip_m} D_{\gamma\rho}(ip_s - ip_m)\left\{ M^\gamma_{\rho\alpha} G_{\alpha\beta}(ip_m)M^\gamma_{\beta\nu'} + \right\} \]
\[ + \Gamma_{\mu\nu';\beta'\alpha'}(ip_s, ip_m)G_{\nu\alpha'}(ip_s)M^\gamma_{\alpha\beta}(ip_m)G_{\beta'\nu'}(ip_m)M^\gamma_{\beta'\nu'}, \]
\[ + M^\gamma_{\mu\nu}(G_{\mu\nu'}(ip_s)G_{\alpha\beta'}(ip_m)M^\gamma_{\beta\alpha}(ip_m)G_{\beta'\nu'}(ip_s)\Gamma_{\alpha\beta\nu'}(ip_m, ip_s) + \right\} \]
\[ + \Theta_{\mu\nu';\beta'\nu'}(ip_s, ip_m, ip_s)\times \]
\[ \times G_{\mu\alpha'}(ip_s)M^\gamma_{\alpha\beta}(ip_m)G_{\beta'\nu}(ip_m)M^\gamma_{\beta'\nu'}(ip_s) \}, \]
\[ (2.35) \]

\[ \Sigma_{\mu\nu}^H(ip_s) = T \sum_{ip_m} D_{\alpha\gamma}(0)\left\{ M^\gamma_{\beta\alpha} G_{\alpha\beta}(ip_m)M^\gamma_{\beta\nu'} + \right\} \]
\[ + \Gamma_{\mu\nu';\beta'\alpha'}(ip_s, ip_m)G_{\beta\alpha'}(ip_m)M^\gamma_{\alpha\beta}(ip_m)G_{\beta'\nu'}(ip_m)M^\gamma_{\beta'\nu'}, \]
\[ + M^\gamma_{\mu\nu}(G_{\mu\nu'}(ip_s)G_{\alpha\beta'}(ip_m)M^\gamma_{\beta\alpha}(ip_m)G_{\beta'\nu'}(ip_s)\Gamma_{\alpha\beta\nu'}(ip_m, ip_s) + \right\} \]
\[ + \Theta_{\mu\nu';\beta'\nu'}(ip_s, ip_m, ip_s)\times \]
\[ \times G_{\mu\alpha'}(ip_s)M^\gamma_{\alpha\beta}(ip_m)G_{\beta'\nu}(ip_m)M^\gamma_{\beta'\nu'}(ip_s) \}, \]
\[ (2.36) \]
where $\partial \Gamma$ denotes the functional derivative of four-leg vertex

$$
\partial \Gamma_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) = \frac{\delta \Gamma_{\nu'\nu;\alpha\beta}(i_{p_s}, i_{p_m})}{\delta G_{\mu\nu}(i_{p_s})}
$$

(2.37)

Note, that in this expression and in similar expressions in what follows the derivative is rather a partial variational derivative. A possibility to use partial variational derivatives stems from the fact that both fermionic loops and "crossing" propagation lines are absent in disorder vertices for the case of quenched disorder. Therefore, in the right hand side of the expression (2.37) the arguments $i_{p_s}$ and $i_{p_m}$ label two independent functional variables and should be formally considered as being distinct. Latter on we will exploit this fact without mentioning explicitly.

The use of the Bethe-Salpeter equation (2.24) and definition (2.37) gives the equation on $\partial \Gamma$:

$$
\partial \Gamma_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) = U_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) + 
+ U_{\mu\nu;\alpha\beta;\xi\eta}(i_{p_s}, i_{p_m}, i_{p_s}) G_{\xi\eta}(i_{p_s}) G_{\mu\nu}(i_{p_m}) 
+ \Gamma_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}) 
+ \delta U_{\mu'\mu;\alpha\beta}(i_{p_s}, i_{p_m})/\delta G_{\nu\nu'}(i_{p_s})
$$

(2.38)

and

$$
\partial \Gamma_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) = \delta U_{\nu'\mu;\alpha\beta}(i_{p_s}, i_{p_m})/\delta G_{\nu\nu'}(i_{p_s})
$$

(2.39)

is irreducible six-vertex due to disorder scattering. Using Eq. (2.24) once again we may express $\partial \Gamma$ through $\Gamma$, $U_4$ and $U_6$ only and after straightforward but lengthy manipulations we arrive at the result

$$
\partial \Gamma_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) = U_{\mu\nu;\alpha\beta;\mu'\nu'}(i_{p_s}, i_{p_m}, i_{p_s}) + 
+ \Gamma_{\nu'\mu;\alpha\beta}(i_{p_s}, i_{p_m}) G_{\nu\nu'}(i_{p_s}) G_{\mu\nu}(i_{p_m}) 
+ \delta U_{\mu'\mu;\alpha\beta}(i_{p_s}, i_{p_m})/\delta G_{\nu\nu'}(i_{p_s}) 
+ \delta U_{\mu'\mu;\alpha\beta}(i_{p_s}, i_{p_m})/\delta G_{\nu\nu'}(i_{p_s})
$$

(2.40)

Then substitution of (2.40) into (2.35 2.36) with the use of (2.28 2.29) leads to final expression for $\Sigma^{\nu\nu'}(G, D)$ (see Figs. 5 and 6)

$$
\Sigma^{\nu\nu'}(i_{p_s}) = \Sigma^{H}(i_{p_s}) + \Sigma^{F}(i_{p_s}),
$$

(2.41)

$$
\begin{align*}
\Sigma^{F}(i_{p_s}) &= -T \sum_{i_{p_m}} D_{\nu\nu'}(i_{p_s} - i_{p_m}) \\
&\left\{ M^{\gamma}_{\mu\nu}(i_{p_s}, i_{p_m}) G_{\alpha'\beta}(i_{p_m}) M^{\gamma}_{\beta'\nu'}(i_{p_m}, i_{p_s}) + 
+ U_{\mu\nu;\xi\eta}(i_{p_s}, i_{p_m}, i_{p_s}) G_{\mu\nu}(i_{p_s}) M^{\gamma}_{\alpha\beta}(i_{p_s}, i_{p_m}) G_{\alpha'\beta}(i_{p_m}) G_{\beta'\nu'}(i_{p_s}) \right\}
\end{align*}
$$

(2.42)
\[
\Sigma^{H}_{\mu \nu}(i \delta s) = T \sum_{i \delta m} D_{\gamma \gamma'}(0) \left\{ M_{\mu \nu} G_{\alpha \beta}(i \delta m) M_{\beta \alpha}' + 
+ \Gamma_{\mu \nu; \delta \delta}(i \delta s, i \delta m) G_{\delta \beta}(i \delta m) M_{\beta \beta}' G_{\beta \delta}(i \delta m) G_{\delta \nu}(i \delta s) M_{\nu \nu}' + 
+ M_{\mu \nu} G_{\mu \nu'}(i \delta s) G_{\delta \beta}(i \delta m) M_{\beta \beta}' G_{\beta \delta}(i \delta m) G_{\delta \nu'}(i \delta s) M_{\nu \nu'} + 
+ \Gamma_{\mu \nu; \delta \delta'}(i \delta s, i \delta m) G_{\delta \nu'}(i \delta s) G_{\delta \beta}(i \delta m) M_{\beta \beta}' G_{\beta \delta}(i \delta m) G_{\delta \nu}(i \delta s) M_{\nu \nu} \right\} 
\times G_{\mu' \xi}(i \delta s) M_{\xi \xi'} G_{\delta' \alpha'}(i \delta m) G_{\alpha \mu'}(i \delta m) M_{\gamma \gamma'} G_{\xi \nu'}(i \delta s) + 
+ U_{\mu \nu; \alpha \beta; \nu' \nu'}(i \delta s, i \delta m, i \delta s) \times 
\times G_{\mu \nu} G_{\mu \nu'}(i \delta s) M_{\alpha \beta}' G_{\alpha \mu'}(i \delta m) M_{\beta \beta'} G_{\beta \delta}(i \delta m) G_{\delta \nu}(i \delta s) M_{\nu \nu'} + 
+ U_{\mu \nu; \alpha \beta; \nu' \nu'}(i \delta s, i \delta m, i \delta s) \times 
\times G_{\mu \nu} M_{\alpha \beta}' G_{\alpha \mu'}(i \delta m) M_{\beta \beta'} G_{\beta \delta}(i \delta m) G_{\delta \nu}(i \delta s) M_{\nu \nu'} + 
+ U_{\mu \nu; \alpha \beta; \nu' \nu'}(i \delta s, i \delta m, i \delta s) \times 
\times G_{\mu \nu} M_{\alpha \beta}' G_{\alpha \mu'}(i \delta m) M_{\beta \beta'} G_{\beta \delta}(i \delta m) G_{\delta \nu}(i \delta s) M_{\nu \nu'} + \right\}, \tag{2.43}
\]

The formulas (2.30, 2.33) and (2.41, 2.43) are those which constitute closed set of self-consistent equations in the case of EPI renormalizations taken into account consistently with the approximate \( \Sigma^{c}[G] \) given.

To obtain the equations for the superconductive transition temperature \( T_{c} \) one should linearize the equation for \( \Sigma[G, D] \) in anomalous part which is assumed to be small near \( T_{c} \). This is formally achieved by expanding \( \Sigma[G] \) in a functional series in \( \delta G, \delta G \) being anomalous part of \( G \), and then using Dyson’s equation to express...
Fig. 6. Hartree-type contributions to the electron-phonon self-energy.
\( \delta G \) back through \( \delta \Sigma \), where \( \delta \Sigma \) is the anomalous contribution to \( \Sigma[G, D] \). We have

\[
\Sigma[G_n + \delta G, D] = \Sigma[G_n, D] + \frac{\delta \Sigma[G, D]}{\delta G} \bigg|_{G=G_n} \delta G = \\
\Sigma[G_n, D] + \frac{\delta \Sigma[G, D]}{\delta G} \bigg|_{G=G_n} G_n \delta \Sigma G_n = \Sigma_n + \delta \Sigma
\]  

(2.44)

(subscript “n” means that the corresponding quantity is taken in the normal state of superconductor) and equating contributions of order \( O(\delta \Sigma) \) in left- and right-hand sides of Eq. (2.44) we obtain formal equation

\[
\delta \Sigma = \frac{\delta \Sigma[G, D]}{\delta G} \bigg|_{G=G_n} G_n \delta \Sigma G_n,
\]  

(2.45)

and \( T_c \) may be found from solubility condition to this equations. Note, that the solubility condition may be written as equality of some formal determinant to zero, but we omit explicit formal expression of such a form. The diagrammatic representation of (2.45) in the case under discussion contains ninety seven terms and is too cumbersome to be written down here.

Hence, normal part of \( \Sigma[G, D] \) near \( T_c \) is still determined by Eqs. (2.32,2.33) and (2.41–2.43) with one-particle Green function \( G \) taken in normal phase of superconductor, and we have formal equation (2.45), from which the expression for \( T_c \) may be derived after suitably fixing approximation for \( \Sigma^e[G] \).

In the conclusion of this Section some remarks are to the point.

Obviously, the approximation to \( \Sigma[G, D] \) given by (2.32,2.33,2.41–2.43) is conserving in a whole by construction.

The expression for \( W^{e-ph}[G, D] \) (2.27) does not actually depend on particular alloy model since the assumptions on the alloy model enter into this expression only implicitly through irreducible four-leg vertex, so that the form of \( W^{e-ph}[G, D] \) is independent of the disorder type. We will demonstrate the correctness of the statement above in subsequent Section for the substitutional disorder of a general type carrying out configurational averaging in a formally exact way.

The equations Eqs. (2.32,2.33) and (2.41–2.43) enable also to consider “inconsistent” renormalizations of EPI vertices. In this case we assume that \( \Sigma^e[G] \) is calculated within certain conserving approximation and renormalizations of EPI are taken into account within another, but still conserving approximation for \( \Sigma^e[G] \). This happens, for instance, when discussing the so-called Anderson’s theorem where one assumes that disorder does not influence the EPI at all. The last, in turn, implies that the approximation for \( \Sigma^e[G] \) within which we renormalize EPI vertices is \( \Sigma^e[G] = 0 \), and therefore irreducible four-leg disorder vertex and of course higher vertices vanishes identically.

3. T–Matrix Approach to the Superconductivity of Disordered Alloys

In this Section we derive the equations (2.32,2.33) and (2.41–2.43) for the case of formally exact configurational (disorder) averaging. Making use of multiple scattering
theory, or so called $T$–matrix approach\cite{6,7,8,9,10}, we establish relations with the approach of the previous Section and demonstrate that the Eqs. (2.30–2.33) and (2.41–2.43) are actually the case for general type of substitutional disorder when the disorder averaging is carried out exactly. However, if the matter is a specific approximation within certain model of substitutional alloy, it is more convenient to formulate the approximation in terms of generating functional, so that the approaches of Section 2 and of this Section complement each other.

For the first time the application of $T$–matrix approach to the problem of alloy superconductivity has been given in Refs.\cite{6,7} where the model of a binary alloy was considered with local (single-site) BCS-type interaction and, hence, with purely local superconductive pairing. In these papers the equation for the self-energy was derived within the framework of Coherent Potential Approximation (CPA) and resulting BCS coupling renormalized by disorder was presented in Ref.\cite{7}.

The results of this Section are a straightforward generalization of the approach of Refs.\cite{6,7,8,9,10} to the case of non-local superconductive pairing owing to EPI with bare EPI vertices being independent of disorder.

For particular alloy configuration, consider one-particle Matsubara Green’s function $g(\tau)$ with matrix elements

$$
\{ g(\tau) \}_{\alpha\beta} = -< T_{\tau} C_{\alpha}(\tau) C_{\beta}^d(0) >_{H^*},
$$

where $< \ldots >_{H^*}$ means usual Gibbs averaging, with $H$ being the Hamiltonian (2.1).

After transforming to Matsubara frequencies and taking EPI into account one has for $g^{-1}(ip_s)$ the expression

$$
g^{-1}(ip_s) = ip_s \tau_0 - H \tau_3 - V_{e-ph}(ip_s),
$$

with $H$ being the matrix with elements (2.3); $V_{e-ph}(ip_s)$ being the effective interaction due to EPI in this disorder configuration, possessing functional dependence on $g(ip_s)$.

To derive equations required it is sufficient to know the functional $V_{e-ph}(ip_s)$ to the lowest order in EPI of the self-consistent perturbation theory. To this order the expression for $V_{e-ph}(ip_s)$ has the form

$$
V_{e-ph}(ip_s) = V^H(ip_s) + V^F(ip_s),
$$

$$
V^H_{\alpha\beta}(ip_s) = T \sum_{ip_m} D_{\gamma\gamma'}(ip_s) M_{\alpha\beta}^\gamma g_{\alpha'\beta'}(ip_m) M_{\beta'\alpha'}^{\gamma'},
$$

$$
V^F_{\alpha\beta}(ip_s) = -T \sum_{ip_m} D_{\gamma\gamma'}(ip_s - ip_m) M_{\alpha\beta}^\gamma g_{\alpha'\beta'}(ip_m) M_{\beta'\alpha'}^{\gamma'},
$$

where $V^H(ip_s)$ and $V^F(ip_s)$ are Hartree and Fock contributions to the effective potential respectively.
For the configurational averaging to be carried out we introduce $T$-matrix. In
the usual manner one has

\[ T(ip_s) = (V(ip_s) - \Sigma(ip_s)) + (V(ip_s) - \Sigma(ip_s)) G(ip_s) T(ip_s), \]
\[ T(ip_s) = (V(ip_s) - \Sigma(ip_s)) + T(ip_s) G(ip_s) (V(ip_s) - \Sigma(ip_s)), \]

where

\[ V(ip_s) = V^e + V^{e-ph}(ip_s), \]

\[ g(ip_s) = G(ip_s) + G(ip_s) T(ip_s), \]
\[ < g(ip_s) > = G(ip_s) + G(ip_s) < T(ip_s) > G(ip_s), \]
\[ G(ip_s) = G_0(ip_s) + G_0(ip_s) \Sigma(ip_s) G(ip_s). \]

Here $< \ldots >$ stands for configurational averaging with particular probability dis-
tribution and $G_0(ip_s)$ is completely determined by non-random part of $H$. For
self-consistency $\Sigma(ip_s)$ is fixed by the condition

\[ < T(ip_s) > = 0, \]

and, hence, from (3.9,3.10) and (3.12) it follows

\[ < g(ip_s) > = G(ip_s). \]

To carry out configurational averaging we must use two apparently different
schemes of averaging: one for the alloy part of the problem and another for the
effective scattering owing to EPI (the latter scheme is reminiscent of Virtual Crystal
Approximation (VCA) for ordinary “alloy” problems). To this end we decompose
$\Sigma(ip_s)$ according to natural partitioning of $V(ip_s)$ into two parts owing to disorder
scattering and EPI respectively

\[ \Sigma(ip_s) = \Sigma^e(ip_s) + \Sigma^{e-ph}(ip_s), \]

and then rewrite Eqs. (3.6-3.7) for $T$-matrix in the following form

\[ T(ip_s) = T_1(ip_s) + T_2(ip_s), \]
\[ T_1(ip_s) = V_1(ip_s) + V_1(ip_s) G(ip_s) T(ip_s), \]
\[ T_2(ip_s) = V_2(ip_s) + V_2(ip_s) G(ip_s) T(ip_s), \]

where

\[ V_1(ip_s) = V^e - \Sigma^e(ip_s), \]
\[ V_2(ip_s) = V^{e-ph}(ip_s) - \Sigma^{e-ph}(ip_s). \]
After introducing partial $T$-matrices $T^c_e$ and $T^{e-\text{ph}}$ by

$$
T^c_e(i_p) = V_1(i_p) + V_1(i_p)G(i_p)T^c_e(i_p),
$$

$$
T^{e-\text{ph}}(i_p) = V_2(i_p) + V_2(i_p)G(i_p)T^{e-\text{ph}}(i_p),
$$
equations (3.16,3.17) for $T_1$ and $T_2$ transform as

$$
T_1(i_p) = T^c_e(i_p) + T^c_e(i_p)G(i_p)T_2(i_p),
$$

$$
T_2(i_p) = T^{e-\text{ph}}(i_p) + T^{e-\text{ph}}(i_p)G(i_p)T_1(i_p).
$$

Bearing in mind the lowest order in EPI let us find the full $T$-matrix equations (3.16,3.17) for $T^c_e(i_p)$ up to the first order in $V_2$. In this case Eq. (3.21) becomes

$$
T^{e-\text{ph}}(i_p) = V_2(i_p).
$$

To this order in $V_2$ the result for the full $T$-matrix reads

$$
T(i_p) = T^c_e(i_p) + V_2(i_p)G(i_p)T^c_e(i_p) + T^c_e(i_p)G(i_p)V_2(i_p)G(i_p)T^c_e(i_p).
$$

Fixing up to now arbitrary quantity $\Sigma^c_e(i_p)$ by the condition

$$
< T^c_e(i_p) >= 0,
$$
we obtain from (3.25) the equation for $\Sigma^{e-\text{ph}}(i_p)$:

$$
\Sigma^{e-\text{ph}}(i_p) = < V^{e-\text{ph}}(i_p) > + < V^{e-\text{ph}}(i_p)G(i_p)T^c_e(i_p) > + < T^c_e(i_p)G(i_p)V^{e-\text{ph}}(i_p) > + < T^c_e(i_p)G(i_p)V_2(i_p)G(i_p)T^c_e(i_p) >,
$$

where the use was made of Eqs. (3.16,3.17) and (3.26).

Introducing in a standard fashion\textsuperscript{14} full four- and six-leg vertices

$$
\Gamma_{\mu\mu';\nu\nu'}(i_p, i_m) = < T^c_e(i_p)T^c_e(i_p) >,
$$

$$
\Gamma_{\alpha\alpha';\beta\beta';\gamma\gamma'}(i_p, i_p, i_p) = < T^c_{\alpha\alpha'}(i_p)T^c_{\beta\beta'}(i_p)T^c_{\gamma\gamma'}(i_p) >,
$$
writing down indices (the summation over doubly repeated Greek indices is implicitly meant here and in what follows) and then substituting (3.4,3.7) and (3.9,3.10) into (3.27) we obtain

$$
\Sigma^{e-\text{ph}}(i_p) + \Gamma_{\mu\mu';\nu\nu'}(i_p)(i_p)G_{\mu'\alpha}(i_p)\Sigma^{e-\text{ph}}(i_p)G_{\beta\nu}(i_p) = \Sigma^{(1)}_{\mu\nu}(i_p) + \Sigma^{(2)}_{\mu\nu}(i_p),
$$

where

$$
\Sigma^{(1)}_{\mu\nu}(i_p) = - T \sum_{i_p, i_m} D_{\gamma\gamma'}(i_p, i_m) \left\{ M_{\mu\alpha}G_{\alpha\beta}(i_m)M_{\beta\nu} \right\},
$$
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\[ \Gamma_{\mu \nu; \beta \gamma} (ip_i, ip_m) G_{\mu \alpha} (ip_i) M_{\alpha \beta} (ip_m) G_{\gamma \nu} (ip_m) M_{\nu \mu} + \]
\[ M_{\mu \nu} G_{\mu \alpha} (ip_i) G_{\alpha \beta} (ip_m) M_{\beta \nu} G_{\gamma \nu} (ip_m) \Gamma_{\alpha \alpha; \nu \nu} (ip_i, ip_m) + \]
\[ + \Gamma_{\mu \nu; \beta \nu; \nu \mu} (ip_i, ip_m, ip_s) \times \]
\[ \times G_{\mu \alpha} (ip_s) M_{\alpha \alpha} G_{\alpha \beta} (ip_m) G_{\beta \nu} (ip_m) M_{\nu \mu} G_{\gamma \beta} (ip_m) \}
\]
\[ (3.31) \]

and
\[ \Sigma^{(2)}_{\mu \nu} (ip_s) = T \sum_{ip_m} D_{\gamma \gamma} (0) \left\{ M_{\beta \alpha} G_{\alpha \beta} (ip_m) M_{\mu \nu} + \right. \]
\[ \left. \Gamma_{\mu \nu; \beta \gamma} (ip_i, ip_m) G_{\beta \alpha} (ip_m) M_{\alpha \beta} (ip_i) G_{\gamma \nu} (ip_m) M_{\nu \mu} + \right. \]
\[ \left. M_{\mu \nu} G_{\mu \alpha} (ip_i) G_{\alpha \beta} (ip_m) M_{\beta \nu} G_{\gamma \nu} (ip_m) \Gamma_{\alpha \alpha; \nu \nu} (ip_i, ip_m) + \right. \]
\[ \left. + \Gamma_{\mu \nu; \beta \nu; \nu \mu} (ip_i, ip_m, ip_s) \times \right. \]
\[ \times G_{\mu \alpha} (ip_s) M_{\alpha \alpha} G_{\alpha \beta} (ip_m) G_{\beta \nu} (ip_m) M_{\nu \mu} G_{\gamma \beta} (ip_m) \} \}
\[ (3.32) \]

The equations (3.30–3.32) are essentially a generalization of corresponding equations of Refs [16, 17] to the case with EPI-induced superconductive pairing and when configurational averaging is carried out exactly, and they may be used to discuss the phenomenon of the superconductivity of alloys on the grounds of certain approximate solution for \( T^e \) [18] and \( \Sigma^e (ip_s) \) [19]. In particular, for the case of diagonal alloy disorder the replacing \( T^e \) with \( t \) in (3.30–3.32), where \( t \) is the single-site \( T \)-matrix,
\[ t(ip_s) = (\varepsilon_i \tau_3 - \Sigma^e (ip_s)) + (\varepsilon_i \tau_3 - \Sigma^e (ip_s)) G_{ii} (ip_s) t(ip_s), \]
\[ (3.33) \]
retaining single-site EPI contributions only and using CPA expression for \( \Sigma^e \), one may obtain the equations of Refs [16, 17] with the exception that superconductive pairing is due to EPI, not BCS interaction.

However we prefer the equations for the self-energy \( \Sigma(ip_s) \) in the form, which does not rely upon \( T \)-matrix and gives a possibility to analyze arbitrary conserving single-site approximation for \( \Sigma(ip_s) \) within an unified scheme. Therefore we rederive the expressions (2.41–2.43) for \( \Sigma^{e-\text{ph}} (ip_s) \) from (3.30–3.32) to demonstrate that the two expressions are in fact equivalent in the case of exact configurational averaging.

In a standard way, the irreducible disorder four-leg vertex, \( U_4 \), is introduced by the Bethe-Salpeter equation [16, 17]
\[ \Gamma_{\mu \nu; \beta \gamma} (ip_i, ip_m) = U_{\mu \nu; \beta \gamma} (ip_i, ip_m) \]
\[ + U_{\mu \alpha; \beta \nu} (ip_i, ip_m) G_{\alpha \alpha} (ip_s) G_{\nu \beta} (ip_m) \Gamma_{\alpha \alpha; \nu \nu} (ip_s, ip_m) \]
\[ (3.34) \]
(note the inverse order of introducing of full and irreducible vertices in this and in the previous Sections). Multiplying (3.30) by the combination \( GU_G \) (where indices must be identified explicitly) we can express \( < T^e G \Sigma^{e-\text{ph}} G T^e > \) in the left-hand side of (3.30) through \( U_4, \Gamma_4, \Gamma_0 \) and \( G \). Substituting the result obtained back into (3.30) and collecting terms we then arrive to the result which reads
\[ \Sigma^{e-\text{ph}}_{\mu \nu} (ip_s) = \Sigma^H_{\mu \nu} (ip_s) + \Sigma^F_{\mu \nu} (ip_s), \]
\[ (3.35) \]
\[ \Sigma^F T_{\mu\nu}(i\rho_s) = -T \sum_{i\rho_m} D_{\gamma\gamma'}(i\rho_s - i\rho_m) \left\{ M_{\mu\alpha}^\gamma G_{\alpha\beta}(i\rho_m) M_{\beta\nu}' + \Gamma_{\mu\nu';\beta\beta'}(i\rho_s, i\rho_m) G_{\mu\alpha}(i\rho_s) M_{\alpha\beta}^\gamma G_{\beta\nu'}(i\rho_m) M_{\nu\nu'}' + M_{\mu\nu'}^\gamma G_{\mu\alpha}(i\rho_m) G_{\alpha\beta}(i\rho_m) M_{\beta\nu}^\gamma G_{\nu\nu'}(i\rho_s) G_{\gamma\beta}(i\rho_m) \right\} \]

\[ \Sigma^H T_{\mu\nu}(i\rho_s) = T \sum_{i\rho_m} D_{\gamma\gamma'}(0) \left\{ M_{\mu\alpha}^\gamma G_{\alpha\beta}(i\rho_m) M_{\beta\nu}' + \Gamma_{\mu\nu';\beta\beta'}(i\rho_s, i\rho_m) G_{\beta\alpha}(i\rho_m) M_{\nu\nu'}^\gamma G_{\gamma\beta}(i\rho_m) G_{\beta\nu'}(i\rho_s) M_{\nu\nu'}' + \right\} T_{\mu\nu}, \tag{3.36} \]

and

\[ \Theta_{\mu\nu';\beta\beta'}(i\rho_s, i\rho_m) = \Gamma_{\mu\nu';\beta\beta'}(i\rho_s, i\rho_m, i\rho_s) - \sum_{i\rho_m} U_{\mu\nu',\xi}(i\rho_s, i\rho_s) G_{\xi\xi'}(i\rho_s) \Gamma_{\xi\gamma':\delta\delta'}(i\rho_s, i\rho_m) \]

\[ U_{\mu\nu',\xi}(i\rho_s, i\rho_s) G_{\xi\xi'}(i\rho_s) \Gamma_{\xi\gamma':\delta\delta'}(i\rho_s, i\rho_m) \Gamma_{\delta\delta':\nu\nu'}(i\rho_s, i\rho_m, i\rho_s). \tag{3.37} \]

Here the expression (3.38) is nothing but a variational derivative \( \partial \Gamma \) of \( \Gamma \) with respect to \( G \). Indeed, the variation of the \( T \)-matrix (3.28) with respect to \( G \) gives

\[ \delta T^e / \delta G = -T^e \delta \left( (T^e)^{-1} \right) / \delta G T^e \]

\[ = -T^e \left\{ \delta \left( (V^e - \Sigma^e)^{-1} - G \right) / \delta G \right\} T^e \]

\[ = T^e \left\{ - (V^e - \Sigma^e)^{-1} \delta \Sigma^e / \delta G (V^e - \Sigma^e)^{-1} + \delta G / \delta G \right\} T^e \]

\[ = -(1 + T^e G) \delta \Sigma^e / \delta G (1 + G T^e) + T^e \delta G / \delta G T^e, \tag{3.39} \]

and

\[ \frac{\delta G_{\alpha\beta}}{\delta G_{\gamma\delta}} = \delta_{\alpha\gamma} \delta_{\beta\delta}. \tag{3.40} \]

Introducing

\[ \frac{\delta \Sigma^e_{\alpha\beta}(i\rho_s)}{\delta G_{\gamma\delta}(i\rho_s)} = U_{\alpha\gamma,\beta\delta}(i\rho_s, i\rho_s), \tag{3.41} \]

explicitly writing down indices in (3.39) and then substituting (3.39) into the following expression

\[ \partial \Gamma_{\mu\nu';\beta\beta'}(i\rho_s, i\rho_m, i\rho_s) = \frac{\delta T^e_{\mu\nu'}(i\rho_s)}{\delta G_{\nu\nu'}(i\rho_s)} T^e_{\beta\beta'}(i\rho_m), \tag{3.42} \]
one obtains

\[
\partial T_{\mu\mu';\delta\xi_{\nu'}(ip_s,ip_m,ip_s)} = \Gamma_{\mu\mu';\delta\xi_{\nu'}(ip_s,ip_m,ip_s)} - U_{\mu\mu';\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}\Gamma_{\xi_{\nu'}(ip_m)} - U_{\mu\mu';\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}\Gamma_{\xi_{\nu'}(ip_m)} - U_{\mu\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}G_{\xi_{\nu'}(ip_m)}
\]

\[
\Gamma_{\delta\xi_{\nu'}(ip_s,ip_m,ip_s)} = \Gamma_{\delta\xi_{\nu'}(ip_s,ip_m,ip_s)} - U_{\mu\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}\Gamma_{\xi_{\nu'}(ip_m)} - U_{\mu\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}\Gamma_{\xi_{\nu'}(ip_m)} - U_{\mu\xi_{\nu'}(ip_s,ip_m)}G_{\xi_{\nu'}(ip_s)}G_{\xi_{\nu'}(ip_m)}
\]

where we have used (3.41) and definitions (3.28) and (3.29). The four-leg vertex (3.41) is completely determined by the Bethe-Salpeter equation and, as shown in Appendix A, the definition (3.41) is in fact the Ward's identity.

Comparing Eqs. (3.43) and (3.38) one sees that they coincide with each other and, consequently, (3.35–3.37) and (2.34–2.36) are identical. In Appendix A we will demonstrate that Eq. (3.26) determining the self-energy \(\Sigma_e\) can be represented in the form (2.33). Simple repetition of the transformations of the previous Section (Eqs. (2.34–2.43)) then leads to Eqs. (2.41–2.43). This completes the demonstration that two sets of equations (2.33,2.41–2.43) and (3.26,3.30–3.32) are equivalent for the case of exact configurational averaging.

Note that the case of exact configurational averaging we dealt with in this Section may be considered as an extreme case of conserving approximation in the sense that \(\Sigma_e[G]\) coincides with the exact value, and the condition of \(\phi\)-derivability of the approximation

\[
\delta \Sigma_e^{\alpha\beta}(ip_s) = \frac{\delta \Sigma_e^{\alpha\beta}(ip_s)}{\delta G^{\beta\alpha}(ip_s)},
\]

can be readily verified (see Appendix A). Therefore, as for conserving approximations, formal results which will be obtained in the rest of the paper are valid for this case too, and we will not discuss it separately in the following.

Concluding this Section let us stress some important points.

In contrast to Refs. 6, 7, 8, 9, 10, where site-diagonal contributions to the effective scattering potential (which come from BCS-like interaction) have been treated on an equal footing with disorder contributions, that is within the same approximation (in fact CPA), we rather apply the same Virtual Crystal type averaging procedure to both site-diagonal and site-off-diagonal EPI contributions to \(V(ip_s)\). The decomposition of \(\Sigma[G, D]\), Eq. (3.14), and, hence, of the full \(T\)-matrix, Eqs. (3.15–3.17), gives a possibility to carry out two different averaging procedures. The very content of the Migdal-Eliashberg theory, based on the addiabaticity theorem, dictates such the way of partitioning the self-energy \(\Sigma[G, D]\) into the parts owing to disorder scattering and EPI, as well as it dictates the virtual crystal type averaging procedure for the EPI contributions. Then the Eliashberg-type equations acquire the character of effective field equations with two effective fields \(\Sigma_e[G]\) and \(\Sigma_e^{\text{ph}}[G, D]\) coupled via Dyson’s equation for the one-particle electron Green’s function \(G\).

Suppose for a moment that we have decomposed the full self-energy into, for example, three parts owing to disorder scattering, site-diagonal and site-off-diagonal EPI contributions, respectively. Then, if we use CPA-like, or even exact, averaging
scheme for site-diagonal EPI contributions, this will give results which are formally beyond the accuracy of the Migdal-Eliashberg theory as they contain, besides the lowest order contribution, higher order EPI contributions to $\Sigma^{c-\text{ph}}[G, D]$.

Another important feature is that the decomposition (3.8) of $V(i\rho_s)$ and $\Sigma[G, D]$, Eq. (3.14), is the only possible for both the Ward’s identity (3.41) and $\phi$-derivability criterion (3.44) for $\Sigma^c[G]$ to hold. Rather different approaches of this Section and of Section 2 are then in tight analogy with each other: the form of the equations are the same and detailed structure of underlying quantities may be established using the connections between disorder vertices and the $T$-matrix.

4. Ward’s Identities and the Anderson’s Theorem

As a first application of the approach being developed in this paper we consider the so-called Anderson’s theorem which concerns the influence of nonmagnetic disorder on the superconductive transition temperature $T_c$.

This theorem was established within BCS model and Virtual Crystal Approximation (VCA) for dilute nonmagnetic impurities and for concentrated nonmagnetic impurities taken into account within Coherent Potential Approximation (CPA). For the latter case the theorem was discussed on more general grounds.

For isotropic superconductor where pairing owing to EPI is primarily of $s$-character there is a standard consideration in the literature of how dilute nonmagnetic impurities (essentially the case of weak alloy disordering) influence $T_c$.

For the case where disorder scattering strength is arbitrary, which was treated by using Average $T$-matrix Approximation (the concentration of the impurities is small none the less), and in the absence of EPI renormalizations this question was discussed in connection with one-particle state density variations near the Fermi energy. It was demonstrated that, while usual cancellation of anomalous disorder-scattering contributions holds in the equation for the gap function, the asymmetry of the normal-phase one-particle state density near the chemical potential leads to non-zero (in fact, non-constant) spectral shifts. In this case, equations for the gap function and for the spectral weight renormalization function $Z(i\rho_s)$ no longer decouple which results in the disorder contributions to the gap function via explicit dependence of the corresponding equation on $Z(i\rho_s)$. Then, as was concluded, in the presence of any structure of the density of states (DOS) near the Fermi level, the Anderson’s theorem breaks down, which seems to be in contrast with the situation for the BCS model. This point needs further clarification.

Further we shall see that the account of the Fermi-surface energy dependence and its electron-hole symmetry is not directly related to the Anderson’s theorem. It is strong $k$-dependence of the superconducting order parameter ($\Delta_k(i\rho_s)$ is not constant over the Fermi surface) only which may lead to the suppression of superconductivity by non-magnetic impurities in the case of weak disorder. However from the point of view of the standard considerations this situation does not correspond to $s$-pairing (basis function in the expansion in partial waves for an anisotropic Fermi surface is unity for $s$-channel!). So, the term “$s$-pairing” will be understand
hereafter as follows: the gap function $\Delta_{\mathbf{k}}$ has the same point symmetry as the “bare” electronic spectrum $\epsilon_{\mathbf{k}}$, that is $\Delta_{\mathbf{k}} = \Delta(\epsilon_{\mathbf{k}})$. One may expect that this holds true for all conventional superconductors.

Within the Bogolyubov’s formulation of the BCS-like model, that is for the model with disorder-independent attractive four-fermionic interaction which is almost of infinite range and has a separable kernel, and after adding disorder-dependent two-fermionic contributions, the Anderson’s theorem may be understood basing on qualitative considerations. It states that the expression for $T_c$ has usual BCS form with the normal state one-particle spectral density renormalized by the disorder, replacing bare one. Indeed, by analogy with the case where disorder is absent, one can construct an approximate Hamiltonian of effective-field type, which gives asymptotically exact (in the thermodynamical limit) solutions for the thermodynamical quantities and for correlation functions as well. For fixed disorder configuration the effective field (in fact, the superconductive gap) which enters the approximate Hamiltonian, is spatially homogeneous and can be determined through the self-consistency conditions containing anomalous averages of fermionic operators forming Cooper pairs in this disorder configuration. The anomalous averages are then expressed in terms of the one-particle spectral density which is non-negative and is the same for both spin-up and spin-down electron states because, as was pointed out by Anderson, the states of electrons forming Cooper pair are time-reversal images of each other. Hence, the effective field may depend on disorder via the corresponding dependence of the spectral density, but the “disorder” fluctuations of the effective field are strongly suppressed in the thermodynamical limit because of infinite range of the interaction, so that the effective field can be safely replaced by its configurationally averaged value in the self-consistency equation. Residual disorder averaging touches only the spectral density; the self-consistency equations become similar to those of usual BCS theory; however they contain the renormalized spectral density instead of bare one. The statement of the Anderson’s theorem then follows. Note, that for this particular model the conclusion takes place irrespectively of the strength and type of quenched substitutional disorder and also of whether renormalized and/or bare state densities possess any structure.

The situation for the model (2.1) is somewhat different. In this case effective EPI-induced electron-electron interaction is rather of short or intermediate range than of (infinitely) long range, “disorder” fluctuations of the electron-phonon part of the self-energy are not small in general and lead to renormalizations of “bare” EPI vertices. But if one neglects completely these renormalizations the statement may be established which is quite analogous to the Anderson’s theorem for the BCS-like model.

In this Section we demonstrate that, for superconductive alloy with isotropic $s$-type pairing owing to EPI and for arbitrary conserving approximation for the disorder self-energy $\Sigma^c[G]$, quenched substitutional disorder influences superconductive transition temperature $T_c$ only through normal-state one-particle spectral density renormalized by all the interactions in the system, provided disorder does
not change “bare” EPI vertices. Surely, this makes the theorem to be rather a formal statement because the effects of EPI-vertex renormalizations, as we will see, are not small except for the case of weak alloy disordering (dilute alloys). Note, also, the Anderson’s theorem is not a statement of mathematical regour but holds to certain approximations usually accepted in conventional theory of superconductivity.

The most general form for spatial Fourier transforms of $\Sigma^{e}(ip_s)$ and $\Sigma^{e-ph}(ip_s)$ is

$$\Sigma^{e}_{k}(ip_s) = ip_s[1 - \gamma_{k}(ip_s)]\tau_0 + \chi^{e}_{k}(ip_s)\tau_3 + \phi^{e}_{k}(ip_s)\tau_1, \quad (4.1)$$

$$\Sigma^{e-ph}_{k}(ip_s) = ip_s\gamma_{k}(ip_s)[1 - Z_{k}(ip_s)]\tau_0 + \chi^{e-ph}_{k}(ip_s)\tau_3 + \phi^{e-ph}_{k}(ip_s)\tau_1, \quad (4.2)$$

where only $\tau_1$ contribution to the anomalous part is retained (in the absence of external magnetic fields we can always make $\tau_2$-contributions vanish by suitable gauge fixing because usual gauge invariance under phase transformations is preserved within $\phi$-derivable approximations) and

$$\gamma_{k}(ip_s) = 1 - \frac{\Sigma^{e}_{k}(ip_s) - \Sigma^{e}_{-k}(-ip_s)}{2ip_s}. \quad (4.3)$$

The representation (4.2) for $\Sigma^{e-ph}_{k}(ip_s)$ differs from the standard one in that the disorder contribution have been picked out explicitly into the factor $\gamma_{k}(ip_s)$ (which is nothing but the disorder self-energy correction).

---

Fig. 7. Standard contribution to the normal electron-phonon self-energy without disorder renormalizations.

Fig. 8. Full contribution to the anomalous self-energy. Dashed circle represents the anomalous self-energy; dashed square, the irreducible disorder four-leg vertex. All the other graphical elements are the same as before.

Linearized Eliashberg-type equations without disorder renormalization of EPI
is the full anomalous contribution to \( \Sigma_k \) related to anomalous contribution to \( \Sigma \), where

\[
\chi^\varepsilon_{-ph}(ip_s) = -T_c \sum_{q,ip_m} \lambda_{kq}(ip_s - ip_m) \Psi_q(ip_m) \varepsilon_q(ip_m), \tag{4.4}
\]

\[
\phi_k(ip_s) = T_c \sum_{q,ip_m} \lambda_{kq}(ip_s - ip_m) \Psi_q(ip_m) \phi_q(ip_m) + \sum_q U_{kq}^{\uparrow\downarrow}(ip_s) \Psi_q(ip_s) \phi_q(ip_s), \tag{4.6}
\]

where

\[
\phi_k(ip_s) = \phi_k^e(ip_s) + \phi_k^{\varepsilon_{-ph}}(ip_s) \tag{4.7}
\]

is the full anomalous contribution to \( \Sigma_k(ip_s) \), \( U_{kq}^{\uparrow\downarrow}(ip_s) \) is proper irreducible four-leg vertex related to anomalous contribution to \( \Sigma^e(ip_s) \), and

\[
\Psi_k(ip_s) = \left[ (p_s \gamma_k(ip_s) Z_k(ip_s))^2 + (\varepsilon_k(ip_s))^2 \right]^{-1}, \tag{4.8}
\]

\[
\varepsilon_k(ip_s) = t_k + \chi^\varepsilon_k(ip_s) + \chi^\varepsilon_{-ph}(ip_s), \tag{4.9}
\]

with \( t_k \) being bare electron dispersion defined by non-random part of \( H \), Eqs. (2.2, 2.3). Quantities \( \gamma_k(ip_s) \) and \( \chi^\varepsilon_k(ip_s) \) describe disorder scattering and satisfy formal equation:

\[
ip_s[1 - \gamma_k(ip_s)]\tau_0 + \chi^\varepsilon_k(ip_s)\tau_3 = \left. \frac{\delta W^e[G]}{\delta G_k^e(ip_s)} \right|_{T = T_c} \tag{4.10}
\]

Explicit form of this equation depends on the approximation used for \( W^e[G] \) or \( \Sigma^e[G] \) and is not important for further consideration. Note that we have also included Hartree contributions to the renormalization of the chemical potential.

At last, the expression for \( \lambda_{kq}(i\omega_n) \) reads

\[
\lambda_{kq}(i\omega_n) = \int_0^\infty d\Omega \alpha^2 F_{kq}(\Omega) \frac{\Omega}{\omega_n^2 + \Omega^2}. \tag{4.11}
\]

Here \( \alpha^2 F_{kq}(\Omega) \) coincides with the usual Eliashberg function up to a factor of dimension energy and has the form

\[
\alpha^2 F_{kq}(\Omega) = \sum_s \left| M^{(s)}_{kq} \right|^2 \left\{ -\frac{1}{\pi} \text{Im} D^{(s)}_{k-q}(\Omega + i0) \right\}. \tag{4.12}
\]

Setting in (A.10) of Appendix A

\[
G' = G_{ij,\uparrow}(ip_s), \quad G = G_{ji,\downarrow}(-ip_s), \tag{4.13}
\]

\[
\Sigma' = \Sigma^e_{ij,\uparrow}(ip_s), \quad \Sigma = \Sigma^e_{ji,\downarrow}(-ip_s), \tag{4.14}
\]
where all quantities are taken in the normal state, and Fourier transforming one then obtains

\[ \Sigma_{k \uparrow}^e (ip_s) - \Sigma_{k \downarrow}^e (-ip_s) = \sum_{q} U_{kq}^{\uparrow \downarrow} (ip_s) (G_{q \uparrow}^e (ip_s) - G_{-q \downarrow}^e (-ip_s)), \]  

(4.15)

or, with account of the definition (4.3) for \( \gamma_k (ip_s) \),

\[ \gamma_k (ip_s) = 1 + \sum_{q} U_{kq}^{\uparrow \downarrow} (ip_s) \Psi_q (ip_s) Z_q (ip_s) \gamma_q (ip_s), \]  

(4.16)

where the use was made of Eqs. (4.1, 4.2, 4.8, 4.9) and of the expression

\[ G_{k \sigma}^{-1} = ip_s - t_k - \Sigma_{k \sigma}^e (ip_s) - \Sigma_{-k \sigma}^{e-p_h} (ip_s), \]  

(4.17)

and it is essential that the irreducible four-leg vertices which enter Eqs. (4.6) and (4.16) are the same.

Introducing in a standard fashion normal-state one-particle spectral density

\[ A_{k \sigma}^{\uparrow \downarrow} (\epsilon) = -\frac{1}{\pi} \text{Im} G_{k \sigma}^{\uparrow \downarrow} (\epsilon + i0) \]  

(4.18)

and noting that

\[ A_{k \uparrow}^{\uparrow \downarrow} (\epsilon) = A_{-k \downarrow}^{\uparrow \downarrow} (\epsilon) \]  

(4.19)

(this is, in fact, a consequence of time reversal symmetry of the problem), we can write

\[ \gamma_k (ip_s) Z_k (ip_s) \Psi_k (ip_s) = \overline{\Psi}_k (ip_s), \]  

(4.20)

\[ \overline{\Psi}_k (ip_s) = \int d\epsilon \frac{A_{k \sigma}^{\uparrow \downarrow} (\epsilon)}{p_s^2 + \epsilon^2} \]  

(4.21)

and

\[ \overline{Y}_k (ip_s) = \bar{e}_k (ip_s) \overline{\Psi}_k (ip_s) = \int d\epsilon \frac{\epsilon A_{k \sigma}^{\uparrow \downarrow} (\epsilon)}{p_s^2 + \epsilon^2}. \]  

(4.22)

Here we fix the zero of the energy self-consistently at the true chemical potential.

The system of the equations (4.4–4.6) then reads

\[ ip_s \gamma_k (ip_s) [1 - Z_k (ip_s)] = -T_c \sum_{q, ip_m} \lambda_{kq} (ip_s - ip_m) \overline{\Psi}_q (ip_m) ip_s, \]  

(4.23)

\[ \chi_k^{e-p_h} (ip_s) = -T_c \sum_{q, ip_m} \lambda_{kq} (ip_s - ip_m) \overline{Y}_q (ip_s) \]  

(4.24)

and

\[ \gamma_k (ip_s) Z_k (ip_s) \Delta_k (ip_s) = T_c \sum_{q, ip_m} \lambda_{kq} (ip_s - ip_m) \overline{T}_q (ip_m) \Delta_q (ip_m) + \sum_{q} U_{kq}^{\uparrow \downarrow} (ip_s) \Psi_q (ip_s) Z_q (ip_s) \gamma_q (ip_s) \Delta_q (ip_s), \]  

(4.25)
where
\[
\Delta_k(i_p) = \frac{\phi_k(i_p)}{Z_k(i_p)\gamma_k(i_p)}
\] (4.26)
is, as usual, the gap function.

Solving (4.23) for \(Z_k(i_p)\):
\[
Z_k(i_p) = 1 + \frac{1}{i_p} \gamma_k(i_p) T_c \sum_{q,i_p,m} \lambda_{kq}(i_p - i_p_m) \overline{\Psi}(i_p_m)i_p_m \] (4.27)
and inserting the result into (4.25) we rewrite the equation for \(\Delta_k(i_p)\) in the form
\[
\sum_q Q_{kq}(i_p) \gamma_q(i_p) \Delta_q(i_p) = T_c \sum_{q,i_p,m} \lambda_{kq}(i_p - i_p_m) \overline{\Psi}(i_p_m)i_p_m - \Delta_k(i_p) T_c \sum_{q,i_p,m} \lambda_{kq}(i_p - i_p_m) \overline{\Psi}(i_p_m)i_p_m,
\] (4.28)
where
\[
Q_{kq}(i_p) = \delta_{kq}(i_p) - U_{kq}(i_p) Z_q(i_p) \Psi_q(i_p)
\] (4.29)
and up to now no approximations has been made.

Usual way to proceed further is to introduce harmonics which constitute orthogonal and full set at some “bare” Fermi surface, that is so called Fermi surface harmonics. We have
\[
\sum_k F_J(k) F_J'(k) \delta(\epsilon - \epsilon_k^*) = N_0(\epsilon) \delta_{JJ'},
\] (4.30)
\[
N_0(\epsilon) = \sum_k \delta(\epsilon - \epsilon_k^*).
\] (4.31)
The expansion over Fermi surface harmonics for various \(k\)-dependent quantities reads
\[
C_{k} = \sum_J \int d\epsilon \delta(\epsilon - \epsilon_k^*) F_J(k) C_J(\epsilon),
\] (4.32)
\[
K_{kq} = \sum_{J,J'} \int d\epsilon d\epsilon' \delta(\epsilon - \epsilon_k^*) \delta(\epsilon' - \epsilon_q^*) F_J(k) F_{J'}(q) K_{JJ'}(\epsilon\epsilon'),
\] (4.33)
\[
K_{kq} = \sum_J \int d\epsilon \delta(\epsilon - \epsilon_k^*) F_J(k) K_{J,\epsilon}(\epsilon),
\] (4.34)
\[
A_k B_k = \sum_J C_{JLL'} A_{J,LL'} B_{L'}(\epsilon),
\] (4.35)
\[
\sum_k A_k B_k = \int d\epsilon N_0(\epsilon) \sum_L A_{L}(\epsilon) B_{L}(\epsilon),
\] (4.36)
where coefficients $C_{JLL'}$, being Clebsch-Gordan coefficients for the harmonics, obey the property
\[ C_{0LL'} = C_{L0L'} = C_{LL'0} = \delta_{LL'}, \tag{4.37} \]

$\varepsilon_k^*$ is “bare” dispersion and we choose $\varepsilon_k^* = t_k$, and $t_k$ is bare electron dispersion defined by nonrandom part of $H^e$, Eq. (2.2).

With the use of (4.30–4.36) the equation (4.28) can be represented as

\[
\sum_{J'} \int d\epsilon' N_0(\epsilon') Q_{J'J}(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is, im) \sum_{LL'} C_{J'LL'} \psi_{L}(\epsilon'; ip_i, is, im) =
\]

\[
= T_c \sum_{ip_i, im} \sum_{J'} \int d\epsilon' N_0(\epsilon') \lambda_{J'J}(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is, im) \times
\]

\[
\times \sum_{LL'} C_{J'LL'} \psi_{L}(\epsilon'; ip_i, is, im) \Delta_{L'}(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is, im) =
\]

\[
= T_c \sum_{ip_i, im} \sum_{J'} \int d\epsilon' N_0(\epsilon') \lambda_{J'J}(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is, im) \times
\]

\[
\times \sum_{LL'} C_{J'LL'} \psi_{L}(\epsilon'; ip_i, is, im) \Delta_{L'}(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is, im) \Delta(\epsilon'\epsilon'\epsilon\epsilon'\epsilon\epsilon'; ip_i, is).
\tag{4.38} \]

Since $s$-type pairing dominates for the case of isotropic superconductor and, moreover, $\Delta_L(\epsilon; ip_i)$ is almost independent of $\epsilon$ in a narrow energy region $|\epsilon| < \omega_D$ the following approximation takes place

\[ \Delta_L(\epsilon; ip_i) \rightarrow \Delta_0(\epsilon^*; ip_i) = \Delta(ip_i). \tag{4.39} \]

After substituting (4.33) into (4.38) and using (4.37) we get

\[
\sum_{q} Q_0.q(\epsilon^*; ip_i) \gamma_q(\epsilon^*; ip_i) \Delta(\epsilon^*; ip_i) = T_c \sum_{q; ip_m} \lambda_{0,q}(\epsilon^*; ip_i, ip_m) \psi_q(ip_m) \Delta(ip_m) -
\]

\[
- \frac{\Delta(ip_i)}{ip_i} T_c \sum_{q; ip_m} \lambda_{0,q}(\epsilon^*; ip_i, ip_m) \psi_q(ip_m) ip_m \tag{4.40} \]

The Ward’s identity (4.16) can be rewritten as

\[ \sum_{q} Q_{kq}(ip_i) \gamma_q(ip_i) = 1, \tag{4.41} \]

where $Q_{kq}(ip_i)$ is defined by (4.29). After carrying out expansion over Fermi surface harmonics, the expression (4.41) becomes

\[ \sum_{q} Q_{J,Lq}(\epsilon, ip_i) \gamma_q(ip_i) = \delta_{J0}, \tag{4.42} \]

and thus

\[ \sum_{q} Q_0.q(\epsilon, ip_i) \gamma_q(ip_i) = 1. \tag{4.43} \]
The use of (4.43) in the left-hand side of (4.40) immediately yields

\[
\Delta(i_p) \left\{ 1 + \frac{1}{i_p s} T_c \sum_{\mathbf{q},i_p m} \lambda_0,\mathbf{q}(\epsilon^*; i_p s - i_p m) \overline{\Psi}(i_p m)\overline{\Psi}(i_p m) \right\} =
\]

\[
= T_c \sum_{\mathbf{q},i_p m} \lambda_0,\mathbf{q}(\epsilon^*; i_p s - i_p m) \overline{\Psi}(i_p m) \Delta(i_p m). \tag{4.44}
\]

This final expression clearly shows that disorder contributions enter the gap-function equation only through \( \overline{\Psi}_k(i_p s) \) and, thus, through renormalized normal-state one-particle spectral density \( A_k(\epsilon) \) (Eq. (4.15)). Disorder contributions to anomalous part of the self-energy cancel the factor \( \gamma_k(i_p s) \) in the left hand side of (4.43) owing to exact Ward’s identity (4.16) and we may conclude that it is a type of Ward’s cancellations which leads to the Anderson’s theorem.

5. Reduced Equations for the Isotropic superconductor. The Abrikosov’s Identity

In this section we will derive a set of linearized Eliashberg-type equations within isotropic approximation for the gap function \( \Delta_k(i_p s) \), that is for the case where \( \mathbf{k} \)-dependence of \( \Delta_k(i_p s) \) is rather weak and may be completely neglected in the narrow energy interval \( \pm \omega_D \) near the true chemical potential. Analogously to the case where disorder contributions to EPI matrix elements have been neglected, we will demonstrate that, owing to exact Ward’s identity, anomalous contributions to the self-energy, which come from disorder scattering, may be eliminated from the full equation for the gap function.

With the use of (4.1–4.3) linearized Eliashberg-type equations can be formally written as

\[
i_p s \gamma_k(i_p s) [1 - Z_k(i_p s)] = -T_c \sum_{\mathbf{q},i_p m} L^{(z)}_{\mathbf{kq}}(i_p s, i_p m), \tag{5.1}
\]

\[
\chi_k^{\epsilon-p\hbar}(i_p s) = -T_c \sum_{\mathbf{q},i_p m} L^{(x)}_{\mathbf{kq}}(i_p s, i_p m), \tag{5.2}
\]

\[
\phi_k(i_p s) = T_c \sum_{\mathbf{q},i_p m} L^{(1)}_{\mathbf{kq}}(i_p s, i_p m) \Psi(q)(i_p m) + \sum_{\mathbf{q}} L^{(2)}_{\mathbf{kq}}(i_p s, i_p m) \Psi(q)(i_p s) + \sum_{\mathbf{q}} L^{(2)}_{\mathbf{kq}}(i_p s, i_p m) \Psi(q)(i_p s) \phi(q)(i_p s), \tag{5.3}
\]

where we have denoted contributions to \( Z_k(i_p s) \) and \( \chi_k(i_p s) \) as \( L^{(z)}_{\mathbf{kq}}(i_p s, i_p m) \) and \( L^{(x)}_{\mathbf{kq}}(i_p s, i_p m) \) in Eqs. (5.1) and (5.2) respectively. Analogously, in (5.3) \( L^{(1)}_{\mathbf{kq}}(i_p s, i_p m) \) means the contributions where anomalous part \( \phi_k(i_p s) \) is summed over both quasimomentum \( \mathbf{q} \) and frequency \( i_p m \), and \( L^{(2)}_{\mathbf{kq}}(i_p s, i_p m) \), the contributions which contain \( \phi_k(i_p s) \) summed over the quasimomentum only. Note, by the way, that while
both Hartree- and Fock-type diagrams contribute to $L^{(2)}_{\mathbf{kq}}(i_p, i_p m)$, only Fock-type diagrams give non-zero contribution to $L^{(1)}_{\mathbf{kq}}(i_p, i_p m)$. The last term in (5.3) is anomalous disorder scattering contribution to $\phi_k(i_p s)$, and we have also explicitly written down sums over internal quasimomentum and frequency.

Making use of Eq. (4.19) we can express (5.3) in terms of the one-particle spectral density. A little manipulation yields

$$Z_k(i_p s) \gamma_k(i_p s) \Delta_k(i_p s) = T_c \sum_{\mathbf{q}, i_p m} L^{(2)}_{\mathbf{kq}}(i_p, i_p m) \Psi_{\mathbf{q}}(i_p m) \Delta_{\mathbf{q}}(i_p m) +$$

$$+ T_c \sum_{\mathbf{q}, i_p m} L^{(2)}_{\mathbf{kq}}(i_p, i_p m) \Psi_{\mathbf{q}}(i_p s) \Delta_{\mathbf{q}}(i_p s) + \sum_{\mathbf{q}} U^{\uparrow \downarrow}_{\mathbf{kq}}(i_p s) \Psi_{\mathbf{q}}(i_p s) \Delta_{\mathbf{q}}(i_p s),$$

(5.4)

and

$$\gamma_k(i_p s) = \sum_{\mathbf{q}} U^{\uparrow \downarrow}_{\mathbf{kq}}(i_p s) \Psi_{\mathbf{q}}(i_p s).$$

(5.5)

Again, the gap function $\Delta_k(i_p s)$ has been introduced, see Eq. (4.26).

Formal solution to the equation (5.1) for $Z_k(i_p s)$ has the form

$$Z_k(i_p s) = 1 + \frac{1}{i_p s} \gamma_k(i_p s) T_c \sum_{\mathbf{q}, i_p m} L^{(z)}_{\mathbf{kq}}(i_p, i_p m).$$

(5.6)

The substitution of (5.6) and (4.16) into (5.4) gives

$$\Delta_k(i_p s) \left\{ 1 + \frac{1}{i_p s} T_c \sum_{\mathbf{q}, i_p m} L^{(z)}_{\mathbf{kq}}(i_p, i_p m) \right\} +$$

$$+ \Delta_k(i_p s) \sum_{\mathbf{q}} U^{\uparrow \downarrow}_{\mathbf{kq}}(i_p s) \Psi_{\mathbf{q}}(i_p s) =$$

$$= T_c \sum_{\mathbf{q}, i_p m} \left\{ L^{(1)}_{\mathbf{kq}}(i_p, i_p m) \Psi_{\mathbf{q}}(i_p m) \Delta_{\mathbf{q}}(i_p m) +$$

$$+ L^{(2)}_{\mathbf{kq}}(i_p, i_p m) \Psi_{\mathbf{q}}(i_p s) \Delta_{\mathbf{q}}(i_p s) \right\} + \sum_{\mathbf{q}} U^{\uparrow \downarrow}_{\mathbf{kq}}(i_p s) \Psi_{\mathbf{q}}(i_p s) \Delta_{\mathbf{q}}(i_p s),$$

(5.7)

and, after expanding over Fermi surface harmonics (4.30) and using (4.32–4.36), one obtains

$$\sum_{\mathbf{LL}'} \Delta_{\mathbf{L}}(\epsilon; i_p s) \left\{ \delta_{\mathbf{LJ}} + C_{\mathbf{JJ}LL'} \frac{1}{i_p s} T_c \sum_{\mathbf{q}, i_p m} L^{(z)}_{\mathbf{L'}q}(\epsilon; i_p, i_p m) \right\} +$$

$$+ \sum_{\mathbf{LL}'} C_{\mathbf{JJ}LL'} \Delta_{\mathbf{L}}(\epsilon; i_p s) \sum_{\mathbf{q}} U^{\uparrow \downarrow}_{\mathbf{L'}q}(\epsilon; i_p s) \Psi_{\mathbf{q}}(i_p s) =$$

$$= T_c \sum_{i_p m} \sum_{j'} \sum_{\mathbf{LL}'} \int d\epsilon' N_0(\epsilon') C_{\mathbf{j}'LL'} \times$$
Eliashberg-type equations we then can write for the isotropic superconductor the following set of linearized equations explicitly.

renormalized EPI vertices involved in the equation for auxiliary self-energy \( \tilde{\Sigma}(i\epsilon) \) in the previous Section and the remaining explicit dependence on disorder is due to the disorder-scattering contribution to the self-energy can be eliminated owing to the picking out of isotropic contributions is meant implicitly. We see that the anomalous behavior is given by the expression

\[
\Delta(i\epsilon) = T_c \left\{ \sum_{q\cdot p_m} L_{0,q}^{(1)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) + \right. \\
+ T_c \sum_{q\cdot p_m} L_{0,q}^{(2)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) \left. \right\}.
\]

As in the previous section, a weak dependence of \( \Delta_j(\epsilon; i\epsilon) \) on \( \epsilon \) within the region \( |\epsilon| \leq \omega_D \) near \( E_F \) is supposed and we assume that the pairing is mainly of \( s \)-type, that is the harmonics with \( J = 0 \) give dominant contribution into (5.8). Substituting (1.39) into (5.8) and recalling (1.32) we then have

\[
\Delta(i\epsilon) = T_c \sum_{q\cdot p_m} \left\{ L_{0,q}^{(1)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) + \right. \\
+ T_c \sum_{q\cdot p_m} L_{0,q}^{(2)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) \left. \right\},
\]

and this equation does not contain anomalous contribution due to disorder scattering explicitly.

If we introduce an auxiliary quantity \( \tilde{\Sigma}(i\epsilon) \) by the expression

\[
\tilde{\Sigma}(i\epsilon) = i\epsilon [1 - \tilde{Z}(i\epsilon)] \tau_0 + \chi(i\epsilon) \tau_3 + \tilde{Z}(i\epsilon) \Delta(i\epsilon) \tau_1,
\]

we then can write for the isotropic superconductor the following set of linearized Eliashberg-type equations

\[
ip_s[1 - \tilde{Z}(i\epsilon)] = -T_c \sum_{q\cdot p_m} L_{0,q}^{(1)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m),
\]

\[
\chi(i\epsilon) = -T_c \sum_{q\cdot p_m} L_{0,q}^{(1)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m),
\]

\[
\tilde{Z}(i\epsilon) \Delta(i\epsilon) = T_c \sum_{q\cdot p_m} L_{0,q}^{(2)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) + \right. \\
+ T_c \sum_{q\cdot p_m} L_{0,q}^{(2)}(i\epsilon; i\epsilon; i\epsilon) \mathbf{\Pi}_q(i\epsilon_m) \Delta(i\epsilon_m) \left. \right\}.
\]

Being considered as equations on auxiliary self-energy \( \tilde{\Sigma}(i\epsilon) \) these equations may be depicted in a form formally coinciding with that shown in Figs. 5 and 6, except that picking out of isotropic contributions is meant implicitly. We see that the anomalous disorder-scattering contribution to the self-energy can be eliminated owing to the Ward’s identity for the disorder part of the self-energy similarly to the case studied in the previous Section and the remaining explicit dependence on disorder is due to renormalized EPI vertices involved in the equation for auxiliary self-energy \( \tilde{\Sigma}(i\epsilon) \).
The fact of the Ward’s cancellations of anomalous contributions coming from disorder scattering and disorder self-energy contributions to the gap function manifests itself also in the so-called Abrikosov’s identity. To establish this identity, we consider the linearized $\phi_\mathbf{k}(i\mathbf{p}_s)$ contribution,

$$\phi_\mathbf{k}(i\mathbf{p}_s) = \sum_\mathbf{q} U^{\uparrow\downarrow}_{\mathbf{kq}}(i\mathbf{p}_s) \Psi_{\mathbf{q}}(i\mathbf{p}_s) Z\mathbf{q}(i\mathbf{p}_s) \gamma\mathbf{q}(i\mathbf{p}_s) \Delta\mathbf{q}(i\mathbf{p}_s),$$  \hspace{1cm} (5.14)

to the function $\phi_\mathbf{k}(i\mathbf{p}_s)$. Bearing in mind isotropic $s$-type pairing and making use of the expansion over harmonics (4.30) one obtains

$$\phi_\mathbf{k}(i\mathbf{p}_s) = \sum_\mathbf{q} U^{\uparrow\downarrow}_{\mathbf{kq}}(i\mathbf{p}_s) \Psi_{\mathbf{q}}(i\mathbf{p}_s) Z\mathbf{q}(i\mathbf{p}_s) \gamma\mathbf{q}(i\mathbf{p}_s) \Delta(i\mathbf{p}_s),$$  \hspace{1cm} (5.15)

where we have retained a residual $\mathbf{k}$-dependence of the irreducible four-leg vertex. Recalling (4.16) we have

$$\phi_\mathbf{k}(i\mathbf{p}_s) \Delta(i\mathbf{p}_s) = \gamma\mathbf{k}(i\mathbf{p}_s) - 1.$$  \hspace{1cm} (5.16)

Substituting in this expression the definition (4.3) for $\gamma\mathbf{k}(i\mathbf{p}_s)$ and using the time reversal symmetry we arrive at the Abrikosov’s identity

$$\phi_\mathbf{k}(i\mathbf{p}_s) \Delta(i\mathbf{p}_s) = -i \text{Im}\Sigma\mathbf{k}(i\mathbf{p}_s) / i\mathbf{p}_s$$  \hspace{1cm} (5.17)

in the form proposed by Lustfield, so that this identity is not specific for BCS-like models but holds also for the case where pairing is due to EPI and irrespectively of the strength and type of quenched substitutional disorder.

6. Qualitative Analysis of the Reduced Isotropic Equations

General formal equations (2.33, 2.41–2.43) from which one may obtain the explicit expression for the superconductive transition temperature $T_c$, are too cumbersome to work with directly. However, if certain approximations are made when calculating isotropic contributions to $\hat{\Sigma}(i\mathbf{p}_s)$, qualitative analysis of the reduced equations (5.11–5.13) becomes possible. Confining ourselves hereafter by the binary alloy model (2.3) introduced in Section 2 we present here general formulas for $T_c$ in the case of weak and intermediate EPI coupling and for the renormalization of the EPI vertices by disorder scattering.

To pick out isotropic contributions to $\hat{\Sigma}(i\mathbf{p}_s)$, let us consider terms of order $M$ of the self-consistent perturbational series in $V$ contributing to $\Sigma(i\mathbf{p}_s)$. The isotropic part resulting from a particular $M$th-order diagram can be represented in terms of the functions $G_J(\varepsilon; i\mathbf{p}_s)$ and $\lambda_{JJ'}(\varepsilon'; i\omega_n)$ which are the coefficients of the expansions over Fermi surface harmonics of the one-particle Green’s function $G_\mathbf{k}(i\mathbf{p}_s)$ and of the function $\lambda_{\mathbf{kq}}(i\omega_n)$ (defined by (4.11)) respectively.
There arise also combinations of the form

$$N_0^{-1}(\epsilon) \sum_{k,p,q} F_{J_1}(k) F_{J_2}(p) F_{J_3}(q) F_{J_3}(k - p + q)$$

$$\delta(\epsilon - t_k) \delta(\epsilon_1 - t_p) \delta(\epsilon_2 - t_q) \delta(\epsilon_3 - t_k - p + q)$$

and the like containing more complex combinations of $\delta$-functions and Fermi surface harmonics $F_J(k)$. Quite convincing check of the statement made is to write down the expression for the isotropic part of $\Sigma(\epsilon)$ within simplest alloy approximation, that is VCA, but corresponding details are mainly of auxiliary character and we do not present them here.

Our further approximations are formulated as follows.

Firstly, we suppose that approximation adopted for $\Sigma^e_k(ip_s)$ is such as not to disturb drastically the $k$-dependence of $G_k^e(ip_s)$ comparatively with that of the bare one-particle Green’s function, at least for the narrow energy region $|\epsilon| < \omega_D$ near $E_F$, what implies that $\Sigma^e_k(ip_s)$ is almost $k$-independent in this region. This holds exactly for all energies within arbitrary single-site approximation to $\Sigma^e_k(ip_s)$ and approximately for $\Sigma^{e-ph}_k(ip_s)$, to a good extent however. More exactly, we may suppose that $\Sigma^e_k(ip_s) \approx \Sigma(t_k; ip_s)$, with $t_k$ being “bare” electron spectrum, and in particular we may have $\Sigma^e_k(ip_s) = \Sigma(ip_s)$ within an arbitrary single-site approximation. It is also assumed that dominant contribution to $\lambda_{J',J}(\epsilon'; i\omega_n)$ results from harmonics with $J = J' = 0$. Hence, we have the following substitutions:

$$G_J(\epsilon; ip_s) \rightarrow G_0(\epsilon; ip_s) \delta_{J0}, \quad \text{ (6.2)}$$

$$\lambda_{J',J}(\epsilon'; i\omega_n) \rightarrow \lambda_{00}(\epsilon'; i\omega_n) \delta_{J0} \delta_{J'0}. \quad \text{ (6.3)}$$

In this situation combination (6.1) simplifies

$$N_0^{-1}(\epsilon) \sum_{k,p,q} \delta(\epsilon - t_k) \delta(\epsilon_1 - t_p) \delta(\epsilon_2 - t_q) \delta(\epsilon_3 - t_k - p + q), \quad \text{ (6.4)}$$

and so do the like.

Secondly, the dependence of $\lambda_{00}(\epsilon'\epsilon; i\omega_n)$ on $\epsilon$ and $\epsilon'$ is supposed to be sufficiently weak in the region $|\epsilon|, |\epsilon'| < \omega_D$ and therefore may be completely neglected. This leads to the replacement

$$\lambda_{00}(\epsilon'\epsilon; i\omega_n) \rightarrow \lambda_{00}(\epsilon^*\epsilon; i\omega_n) = \lambda(i\omega_n), \quad \text{ (6.5)}$$

where $\epsilon^*$ is some point within the energy interval specified.

Thirdly, we decouple expressions of the type (6.4) and the like in a simple manner

$$N_0^{-1}(\epsilon) \sum_{k,p,q} \delta(\epsilon - t_k) \delta(\epsilon_1 - t_p) \delta(\epsilon_2 - t_q) \delta(\epsilon_3 - t_k - p + q) \rightarrow$$

$$N_0^{-1}(\epsilon) \sum_{k,p,q} \delta(\epsilon - t_k) \delta(\epsilon_1 - t_p) \delta(\epsilon_2 - t_q) \delta(\epsilon_3 - t_k - p + q) \rightarrow$$

$$N_0^{-1}(\epsilon) \sum_{k,p,q} \delta(\epsilon - t_k) \delta(\epsilon_1 - t_p) \delta(\epsilon_2 - t_q) \delta(\epsilon_3 - t_k - p + q) \sum_{p'} \delta(\epsilon_3 - t_{k+q-p'}) \bigg|_{p'}$$

$$= N_0(\epsilon_1) N_0(\epsilon_2) N_0(\epsilon_3). \quad \text{ (6.6)}$$
In these approximations one-particle Green’s function enters the analytical expressions for diagrams in the combination

$$\int d\epsilon N_0(\epsilon) G_0(\epsilon; i_p s) = \sum_k G_k(i_p s) = \frac{1}{N} \sum_i G_{ii}(i_p s) = G(i_p s) \quad (6.7)$$

where the use was made of (4.30–4.36). What happens when accepting these simplifications is that the vertices become purely local quantities. They contain only site-diagonal contributions, so that these approximations may be justified for the case where site-off-diagonal contributions carrying all remaining \(k\)-dependence of the vertices are not important. The latter holds true, in particular, when the localization of electrons by disorder is absent or at least weak for the energies within the energy interval of interest.

In any respect, these simplifications are fully transparent for single-site approximations to the alloy self-energy where all irreducible vertices are purely local quantities and as for more complex non-single-site approximations we suppose the validity of them, of course with the remarks above in mind. Note also, that the other way of introducing the approximations proposed is to use the limit of infinite spatial dimensionality \(d = \infty\), which has become very popular last time when discussing problems of the theory of strongly correlated systems.

The problem of the renormalization of EPI by disorder may then be formulated in terms of the Holstein model in the limit \(d = \infty\) and with the disorder contributions added. This problem appears to be a single-site problem and the \(k\)-dependence of both the self-energy and the vertex corrections is absent from the very beginning.

And at last fourthly, the expression (6.7) can be represented as

$$G(i_p s) = -\sum_k \int \frac{d\epsilon A_k(\epsilon)}{p^2_s + \epsilon^2} \{i_p \tau_0 + \epsilon \tau_3 + \Delta(i_p s) \tau_1\}$$

$$= -\int \frac{d\epsilon N(\epsilon)}{p^2_s + \epsilon^2} \{i_p \tau_0 + \epsilon \tau_3 + \Delta(i_p s) \tau_1\} \quad (6.8)$$

where we have expanded \(G(i_p s)\) up to the first order in \(\Delta(i_p s)\) bearing in mind equation for \(T_c\) and introduced renormalized one-particle state density

$$N(\epsilon) = \sum_k A_k(\epsilon) \quad (6.9)$$

and \(A_k(\epsilon)\) is defined by (4.18,4.19). Choosing then \(E_F = 0\) and assuming weak dependence of \(N(\epsilon)\) on \(\epsilon\) in the interval \(|\epsilon| < \omega_D\) we can write

$$G(i_p s) = -i\pi N(0) \text{Sign}(p_s) \tau_0 - \frac{\Delta(i_p s)}{|p_s|} \pi N(0) \tau_1. \quad (6.10)$$

It is this simple frequency dependence of normal part of \(G(i_p s)\) which enables, along with approximations (5.2, 6.6), to analyze qualitatively the equations (5.11, 5.13).
Key idea of such analysis is this. After using approximations (6.2–6.6) and (6.10), decoupling of dependence on frequency and disorder potential $V$ and concentration $c$ proves to be possible order by order in the perturbational expansion. The equations (5.11–5.13) can then be represented as

$$T_c \sum_{i, p_m} \sum_{\alpha \beta} f_{\alpha \beta}(c, \pi N(0) \lambda N(0) \lambda (ip_s - ip_m) \zeta_\alpha(ip_s) \zeta_\beta(ip_m), \pi N(0) \lambda \Delta(ip_s)(ip_m)),$$

(6.11)

where $ip_s$-dependence of $\zeta_\alpha(ip_s)$ is known and $\zeta_\alpha(ip_s)$ possesses one of the following simple forms

$$1, \ \text{Sign}(p_s), \ \frac{\text{Sign}(p_s)}{|p_s|}, \ \frac{\Delta(p_s)}{|p_s|},$$

(6.12)

and the factors $f_{\alpha \beta}(c, \pi N(0)V)$ in (6.11) are wholly determined by the approximation used to calculate $\Sigma^e(ip_s)$, and $\alpha, \beta = 1, \ldots, 4$.

We consider this possibility in detail for Fock contributions to the self-energy first.

Characterize diagrams of order $M$ by three numbers $N, L, Q$, with $N, L$ and $Q$ being the numbers of successive disorder scattering processes before, between and after two successive EPI processes respectively (as we shall see shortly only a parity of $N, L$ and $Q$ is of importance). Obviously, the number of full electron lines in the $M$'th-order diagram equals $M + 1$, the number of $\tau_3$ Pauli matrices is $M + 2$ and $N + L + Q = M$. We also refer to the subset of graphs with triple $NLQ$ fixed as to $NLQ$-family.

Within the approximations (6.2–6.6) and (6.10) the contribution of $M$'th-order self-energy diagram reads (see Fig. 9)

$$B_{NLQ}^{(D)} V^M T_c \sum_{i, p_m} \lambda (ip_s - ip_m) \tau_3 (G(ip_s) \tau_3)^N (G(ip_m) \tau_3)^{L+1} (G(ip_s) \tau_3)^Q$$

(6.13)

where $B_{NLQ}^{(D)}(c)$ is a product of the renormalized cumulants for the diagram $(D)$ given.
To proceed with the calculation of normal part of $\tilde{\Sigma}(ip_s)$ we note that diagrams with $M$ being even or odd contribute to $\tilde{Z}(ip_s)$ or $\chi(ip_s)$ respectively. Indeed, graphs with even (odd) $M$ contain even (odd) number of $\tau_3$-matrices, normal contribution to $G(ip_s)$ is proportional to unity $\tau_0$-matrix (see expression (6.10)), therefore the contribution of such graphs to $\tilde{\Sigma}(ip_s)$ is proportional to $\tau_0$ ($\tau_3$)-matrix.

In turn, we further classify graphs contributing to $\tilde{Z}(ip_s)$ with the parity of $N$, $L$ and $Q$. There are only four combinations for the decomposition of $M$ even on $N$, $L$ and $Q$:

$$eee, \quad oeo, \quad eoo, \quad ooe,$$

and $e$ ($o$) means even (odd).

Consider for example $eee$-type contributions. After inserting the normal part of $G(ip_s)$, which has the form $-i\pi N(0)\text{Sign}(p_s)$, general expression (6.13) reduces to

$$B^{(D)}_{N L Q}(c)(-i\tilde{V})^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m)\tau_3^{M+2}\text{Sign}^{N+Q}(p_s)\text{Sign}^{L+1}(p_m), \quad (6.15)$$

where

$$\tilde{V} = \pi N(0)V, \quad \tilde{\lambda}(i\omega_n) = N(0)\lambda(i\omega_n).$$

(6.16)

Then, for $eee$-type contribution one has

$$-iB^{(D)}_{N L Q}(c)(-i\tilde{V})^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m)\text{Sign}(p_m). \quad (6.17)$$

Analogously, for other parity combinations, we have

$$-iB^{(D)}_{N L Q}(c)(-i\tilde{V})^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m)\text{Sign}(p_m), \quad (6.18)$$

$$-iB^{(D)}_{N L Q}(c)(-i\tilde{V})^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m)\text{Sign}(p_m), \quad (ooe, ooe) \quad (6.19)$$

and, as we see, the precise form of frequency dependence for the expressions under sums is completely determined by the parity of the numbers $N$, $L$ and $Q$.

Putting everything together over all orders in $V$ we then have the contribution of the Fock diagrams to the equation on $\tilde{Z}(ip_s)$ of the following form

$$-i \left\{ f_{eee}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m)\text{Sign}(ip_m) -$$

$$-i \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m)\text{Sign}(ip_s), \quad (6.20)$$

and the coefficient functions $f_{\alpha\beta\gamma}(c, \tilde{V})$ are

$$f_{\alpha\beta\gamma}(c, \tilde{V}) = \sum_{N L Q = \{\alpha\beta\gamma\}} \sum_{\{D\}} B^{(D)}_{N L Q}(c)(-i\tilde{V})^{N+L+Q}. \quad (6.21)$$
Here $B_{N_{LQ}}^{(D)}(c)$ is a product of cumulants for a particular graph $(D)$, inner sum runs over a set $\{D\}$ of the Fock-type graphs with triple $NLQ$ fixed, that is over the $NLQ$-family, and outer sum is over all $NLQ$-families with definite parity $\{\alpha/\beta/\gamma\}$ of $NLQ$ triple.

Similarly one obtains the Fock contribution to $\chi(ip_s)$:

$$-i \left\{ f_{ooe}(c, \tilde{V}) + f_{oee}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) -$$

$$-i \left\{ f_{eco}(c, \tilde{V}) + f_{ece}(c, \tilde{V}) \right\} \text{Sign}(ip_s) \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \text{Sign}(ip_m). \quad (6.22)$$

Consider the anomalous contributions to $\tilde{\Sigma}(ip_s)$. A particular $M$'th-order graph for the normal part of $\tilde{\Sigma}(ip_s)$ generates $M + 1$ graphs contributing by successively replacing each normal state electron line with anomalous part of $(6.10)$ which is proportional to $\tau_1$-matrix. These graphs may be further partitioned into two complementary sets. The first set contains graphs where anomalous part of $G(ip_s)$ is between two successive EPI processes; the second, all the graphs remained.

For the first set we have

$$B_{N_{LQ}}^{(D)}(c) \left( -i \tilde{V} \right)^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m) \text{Sign}^{N+Q}(ip_s) \times$$

$$\times \text{Sign}^L(p_m) \frac{\Delta(ip_m)}{|p_m|} \left\{ \sum_{S=1}^{L+1} \tau_3^{N+S-1} \tau_1 \tau_3^{L+Q+1-S} \right\}, \quad (6.23)$$

and the expression of the form

$$B_{N_{LQ}}^{(D)}(c) \left( -i \tilde{V} \right)^M \pi T_c \sum_{ip_s} \tilde{\lambda}(ip_s - ip_m) \text{Sign}^{N+Q-1}(ip_s) \text{Sign}^{L+1}(p_m) \frac{\Delta(ip_s)}{|ip_s|} \times$$

$$\times \left\{ \sum_{S=1}^{N} \tau_3^{S-1} \tau_1 \tau_3^{N+L+Q+1-S} + \sum_{S=1}^{Q} \tau_3^{N+L+S} \tau_1 \tau_3^{Q-S} \right\} \quad (6.24)$$

for the second set.

Let now $M$ be odd. Pauli matrices enter such diagrams in combinations

$$(\tau_3)^o \tau_1 (\tau_3)^e = i\tau_2, \quad (6.25)$$

$$(\tau_3)^e \tau_1 (\tau_3)^o = -i\tau_2 \quad (6.26)$$

(here $e$ ($o$) means even (odd) power respectively), leading formally to non-zero $\tau_2$ contributions to anomalous part of $\tilde{\Sigma}(ip_s)$. However full contribution of such graphs vanishes identically owing to usual gauge symmetry under phase transformations. This may be established on rather general grounds for the equations (2.30–2.33, 2.41–2.43) without any computational simplifications accepted when evaluating contribution of a particular graph provided the approximation to $\Sigma_k^\tau(ip_s)$ used possesses a property of being conserving. But the last is precisely what we deal with.
Now we check that within computational simplifications \(6.2\)–\(6.6,6.10\) vanishing of \(\tau_2\)-contributions is preserved. Consider a particular graph of asymmetric \(NLQ\)-family, \(N \neq Q\). There is always a graph in the \(QLN\)-family which is symmetric to initially given one by reflection with respect to a vertical line since full contribution to \(\tilde{\Sigma}(i\hspace{0.1em}p_s)\) possesses this symmetry. Both graphs generate \(M + 1\) contributions to the anomalous part but with reversed order of Pauli matrices relatively to each other. Therefore whole contribution of two asymmetric \(NLQ\)- and \(QLN\)- families vanishes identically (recall Eqs. \((6.27,6.28)\)). Analogously, this vanishing takes place for the symmetric \(NLQ\)-family, \(N = Q\), but within this family, because graphs with the symmetry specified belong now to the same family: anomalous graphs of the first set cancel within the set and so do graphs of the second set. Thus we conclude that such contributions vanish identically order by order in the perturbation series.

For even \(M\), after partitioning on \(NLQ\) we have for the anomalous part of \(\tilde{\Sigma}(i\hspace{0.1em}p_s)\) the contribution which for the first set reads

\[
B^{(D)}_{NLQ}(c) \left(-i\hat{V}\right) M \pi T_c \sum_{i\hspace{0.1em}p_s} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) \frac{\Delta(i\hspace{0.1em}p_m)}{|p_m|}, \quad (\text{eee}) \quad (6.27)
\]

\[
- \left(-i\hat{V}\right) M \pi T_c \sum_{i\hspace{0.1em}p_s} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) \frac{\Delta(i\hspace{0.1em}p_s)}{|p_s|}, \quad (\text{oeo}) \quad (6.28)
\]

and for the second set

\[
B^{(D)}_{NLQ}(c) \left(-i\hat{V}\right) M \pi T_c \sum_{i\hspace{0.1em}p_s} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) \frac{2\Delta(i\hspace{0.1em}p_s)}{|p_s|} \text{Sign}(p_s)\text{Sign}(p_m) \quad (\text{eee}) \quad (6.29)
\]

\[
B^{(D)}_{NLQ}(c) \left(-i\hat{V}\right) M \pi T_c \sum_{i\hspace{0.1em}p_s} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) \frac{\Delta(i\hspace{0.1em}p_s)}{|p_s|}, \quad (\text{eoo}, \text{ooe}) \quad (6.30)
\]

All other parity combinations from \((6.14)\) give zero contributions to the anomalous self-energy.

Again, summing anomalous contributions over all orders of perturbation expansion we obtain the contribution of the Fock diagrams to the anomalous part of the self-energy \(\tilde{\Sigma}(i\hspace{0.1em}p_s)\):

\[
\left\{ f_{\text{eee}}(c, \hat{V}) - f_{\text{oeo}}(c, \hat{V}) \right\} \pi T_c \sum_{i\hspace{0.1em}p_m} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) \frac{\Delta(i\hspace{0.1em}p_m)}{|p_m|} + \\
+ \left\{ f_{\text{eoo}}(c, \hat{V}) + f_{\text{ooe}}(c, \hat{V}) \right\} \frac{\Delta(i\hspace{0.1em}p_s)}{|p_s|} \pi T_c \sum_{i\hspace{0.1em}p_m} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m) + \\
+ 2f_{\text{eee}}(c, \hat{V}) \frac{\Delta(i\hspace{0.1em}p_s)}{|p_s|} \pi T_c \sum_{i\hspace{0.1em}p_m} \tilde{\lambda}(i\hspace{0.1em}p_s - i\hspace{0.1em}p_m)\text{Sign}(i\hspace{0.1em}p_s)\text{Sign}(i\hspace{0.1em}p_m), \quad (6.31)
\]

where functions \(f_{\alpha\beta\gamma}\) are defined by Eq.\((1.21)\).

Now we briefly consider evaluation of the Hartree-type contributions to the self-energy \(\tilde{\Sigma}(i\hspace{0.1em}p_s)\). The evaluation of these contributions follow closely same roots as above, and minor modifications are only necessary.
General Hartree-type diagram is shown in Fig. 10.

We characterize Hartree-type diagrams by three numbers $N$, $L$ and $Q$, where $N$ and $Q$ have the same meaning as before and $L$ is the number of successive disorder scattering processes on a fermionic loop. Analytical expression for Hartree-type contributions has the form

$$-B_{NLQ}^{(D)} V^M T_c \sum_{ip_m} \lambda(0) \tau_3 (G(ip_m)\tau_3)^N \text{Sp} \left \{ (G(ip_m)\tau_3)^{L+1} \right \} (G(ip_3)\tau_3)^Q, \quad (6.32)$$

$B_{NLQ}^{(D)}(c)$ is the product of the cumulants for particular diagram $(D)$ with the triple $NLQ$ fixed, and diagram $(D)$ belongs now to the set $\{D\}$ of Hartree-type diagrams. Then making the approximation (6.10) we have the contributions to the equations on $\tilde{Z}(ip_s)$ and $\chi(ip_s)$ of the form

$$2i \left \{ f_{oo}(c,\tilde{V}) + f_{ooe}(c,\tilde{V}) \right \} \tilde{\lambda}(0) U \text{Sign}(ip_s), \quad (6.33)$$

$$2i \left \{ f_{o}(c,\tilde{V}) + f_{oo}(c,\tilde{V}) \right \} \tilde{\lambda}(0) U \quad (6.34)$$

and to the anomalous part of the self-energy:

$$-2 \left \{ f_{oo}(c,\tilde{V}) + f_{ooe}(c,\tilde{V}) \right \} \tilde{\lambda}(0) U \frac{\Delta(ip_s)}{|p_s|}. \quad (6.35)$$

Here

$$U = \pi T_c \sum_{|p_s| \leq W} 1, \quad (6.36)$$

and we have introduced the cut-off $W$, with $W$ being of order of the bandwidth. Note also that the functions $f_{\alpha\beta\gamma}$ in the expressions (6.33) and (6.35) are defined by Eq.(6.21) where inner sum runs over Hartree set of diagrams. However the results of the summation for both Hartree- and Fock-type diagrams appear to be the same.
as may be demonstrated by straightforward manipulations using the method of generating functions. Moreover, it is not even necessary to calculate the coefficient functions (7.21) for Hartree-type diagrams because Hartree-type diagrams do not explicitly contribute to the final $T_c$ expression within the approximations used in this Section.

Finally, taking into account (6.31) and (6.33–6.35) one obtains the following set of the equations:

$$i\rho_s[1 - \tilde{Z}(\rho_s)] = 2\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \lambda(0)U\text{Sign}(i\rho_s)$$

$$-i\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m)\text{Sign}(ip_m)$$

$$-i\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m)\text{Sign}(ip_s) \quad (6.37)$$

$$\chi(\rho_s) = 2i\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \lambda(0)U$$

$$-i\left\{ f_{ooe}(c, \tilde{V}) + f_{eoo}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m)$$

$$-i\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \text{Sign}(ip_s)\pi T_c \sum_{ip_m} \lambda(ip_s - ip_m) \text{Sign}(ip_m) \quad (6.38)$$

$$\tilde{Z}(\rho_s)\Delta(\rho_s) = -2\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \lambda(0)U \frac{\Delta(\rho_s)}{|\rho_s|}$$

$$\left\{ f_{eoo}(c, \tilde{V}) - f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m) \frac{\Delta(ip_m)}{|p_m|} +$$

$$\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \frac{\Delta(ip_s)}{|\rho_s|} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m) +$$

$$2f_{eoo}(c, \tilde{V}) \frac{\Delta(ip_s)}{|\rho_s|} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m) \text{Sign}(ip_s) \text{Sign}(ip_m) \quad (6.39)$$

Having been solved, the Eq.(6.37) yields for $\tilde{Z}(\rho_s)$ the expression

$$\tilde{Z}(\rho_s) = 1 - 2\left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \lambda(0)U \frac{1}{|\rho_s|}$$

$$+ \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \frac{1}{|\rho_s|} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m) \text{Sign}(ip_m) \text{Sign}(ip_s)$$

$$+ \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \frac{1}{|\rho_s|} \pi T_c \sum_{ip_m} \lambda(ip_s - ip_m). \quad (6.40)$$

Substitution of (6.40) into (6.39) then leads to the equation for the gap function:

$$\Delta(\rho_s) = \pi T_c \sum_{ip_m} \lambda_{eff}(ip_s - ip_m) \frac{\Delta(ip_m)}{|p_m|}$$
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\[
\frac{\Delta(p_s)}{|p_s|} \pi T_c \sum_{i p_m} \lambda^{eff}(i p_s - i p_m) \text{Sign}(i p_s) \text{Sign}(i p_m), \tag{6.41}
\]

where

\[
\lambda^{eff}(i \omega_n) = \kappa(c, \tilde{V}) N(0) \lambda(i \omega_n)
= \kappa(c, \tilde{V}) \lambda(i \omega_n), \tag{6.42}
\]

\[
\kappa(c, \tilde{V}) = f_{ee}(c, \tilde{V}) - f_{eo}(c, \tilde{V}), \tag{6.43}
\]

with \( \lambda^{eff}(i \omega_n) \) being EPI kernel renormalized by disorder scattering and \( \kappa(c, \tilde{V}) \) being the renormalization factor depending on the approximation used for disorder self-energy. As we have stated earlier Hartree-type contributions do not explicitly appear in the equation for the gap function (6.41), which is the result of compensation between Hartree contributions to the function \( \tilde{Z}(i p_s) \) and those of to the anomalous self-energy \( \phi(i p_s) \). Of course, such compensation does not take place in general, and is rather due to the character of the computational simplifications used, of which the local approximation (e.g., the limit \( d = \infty \)) is principal.

Comparing (6.41) with the standard result for \( \Delta(i p_s) \) which reads

\[
\Delta(i p_s) = \pi T_c \sum_{i p_m} \frac{\lambda(i p_s - i p_m) \Delta(i p_m)}{|p_m|} - \pi T_c \sum_{i p_m} \frac{\lambda(i p_s - i p_m) \text{Sign}(i p_s) \text{Sign}(i p_m)}{|p_m|}, \tag{6.44}
\]

we see that the two equations differ in the only respect: the first contains \( \lambda^{eff}(i \omega_n) \) and the second, \( \lambda(i \omega_n) \); the form of the equation remains intact. Then using square-well approximation \( \lambda_{00}(i p_s - i p_m) \) in the first term of Eq.(6.41) \( \lambda_{00}(i p_s - i p_m) \rightarrow \lambda \theta(\omega_1 - |p_s| \theta(\omega_1 - |p_m|) \tag{6.45} \)

and a slightly different version in the contribution which comes from the expression for \( \tilde{Z}(i p_s) \)

\[
\lambda_{00}(i p_s - i p_m) \rightarrow \lambda \theta(\omega_1 - |p_s - p_m|) \tag{6.46}
\]

we immediately obtain for \( T_c \)

\[
T_c = 1.13 \omega_D \exp \left\{ - \frac{1 + \lambda^{eff}}{\lambda^{eff}} \right\}, \tag{6.47}
\]

where

\[
\lambda^{eff} = \kappa(c, \tilde{V}) N(0) \lambda_b, \quad \lambda_b = 2 \int_0^{\infty} d\Omega \Omega^2 F_{00}(\epsilon^* \epsilon^*; \Omega) \frac{\Omega}{\Omega} \tag{6.48}
\]

The Eq.(6.41) enables one to consider more general situation where, along with EPI mechanism, another pairing mechanism exists which will be called, for definiteness, “excitonic”. Here we treat only the case of weak coupling in both EPI and
“excitonic” channels. The corresponding interaction may originate from electron-electron interaction, but should not constitute its main part. For instance, one may think of the Weber model\textsuperscript{57} where strong Mott-Hubbard correlations in the d-system and pairing via crystal-field excitations are considered.

In this situation “excitonic” channel leads to the appearance of the second scale in the effective interaction kernel and to the logarithmic corrections to $\lambda_{\text{eff}}$. Within square-well model for the interaction kernel, $\lambda_{00}(i_{p}s - i_{p}m)$ may be approximated as

$$
\lambda_{00}(i_{p}s - i_{p}m) \rightarrow \lambda_{1}\Theta(\omega_{1} - |p_{s}|)\Theta(\omega_{1} - |p_{m}|) + \lambda_{2}\Theta(\omega_{2} - |p_{s}|)\Theta(\omega_{2} - |p_{m}|),
$$

(6.49)

where $\omega_{1}$ and $\omega_{2}$ ($\omega_{1} < \omega_{2}$) are characteristic phonon and exciton energy scales, $\lambda_{1}$ and $\lambda_{2}$ are corresponding coupling parameters of dimension energy. Substituting (6.49) into (6.41) we obtain

$$
T_{c} = 1.13\omega_{1}\exp\left\{-\frac{1}{\lambda_{\text{eff}}}\right\}
$$

(6.50)

with $\lambda_{\text{eff}}$ defined by an expression of the form

$$
\lambda_{\text{eff}} = \tilde{\lambda}_{1} + \frac{\tilde{\lambda}_{2}}{1 - \lambda_{2}\ln|\omega_{2}/\omega_{1}|}
$$

(6.51)

and

$$
\tilde{\lambda}_{1} = \kappa(c, \tilde{V})N(0)\lambda_{1}, \quad \tilde{\lambda}_{2} = \kappa(c, \tilde{V})N(0)\lambda_{2}.
$$

(6.52)

The expression (6.51) retains its usual form, but up to a renormalization of bare couplings by disorder scattering, which is absorbed in the factor $\kappa(c, \tilde{V})$.

Let us consider now disorder renormalizations of dilute paramagnetic impurity contributions to the gap function and $T_{c}$ equations. In this case the equation (6.41) has to be changed to

$$
\Delta(i_{p}s) = \pi T_{c}\sum_{i_{p}m_{s}}\lambda^{\text{eff}}(i_{p}s - i_{p}m_{s})\frac{\Delta(i_{p}m_{s})}{|p_{m}|} - \frac{\Delta(i_{p}s)}{|p_{s}|}\pi T_{c}\sum_{i_{p}m_{s}}\lambda^{\text{eff}}(i_{p}s - i_{p}m_{s})\text{Sign}(i_{p}s)\text{Sign}(i_{p}m_{s}) - 2\gamma^{\text{eff}}_{P}\frac{\Delta(i_{p}s)}{|p_{s}|},
$$

(6.53)

where

$$
\gamma_{P} = c_{P}V_{P}^{2}S(S + 1), \quad \bar{\gamma}_{P} = N(0)\gamma_{P},
$$

(6.54)

$$
\gamma^{\text{eff}}_{P} = \kappa(c, \tilde{V})\bar{\gamma}_{P},
$$

(6.55)

with $c_{P}$ being the concentration of the paramagnetic impurities, $V_{P}$ being paramagnetic impurity scattering potential, $S$ being the impurity spin, and some details of the derivation of (6.53) are given in Appendix B. We see that, analogously to the situations without paramagnetic impurities, effects of non-magnetic disorder lead to the replacement of bare couplings by the renormalized ones in the gap function.
equation and the renormalization factor is the same for all bare couplings. The gap
function equation preserves its form (compare with the case where non-magnetic
disorder is absent), and the results for the critical concentration of the paramagnetic
impurities and for the $T_c$ suppression may be obtained by standard means. Thus
we have the equation determining the critical concentration $n_c$ of the paramagnetic
impurities
\[ \gamma_c = (1 + \lambda^{eff}) T_{c0}/26, \]
where $T_{c0}$ is the critical temperature with $\gamma^{eff} = 0$, $\gamma_c = n_c V^2 \pi S(S + 1)$, and an
approximate expression
\[ \ln \left( \frac{T_{c0}}{T_c} \right) = \Psi'(\alpha + 1/2) - \Psi(1/2), \]
gives the $T_c$ suppression by dilute paramagnetic impurities,
\[ \alpha = \frac{\gamma^{eff}}{\pi T_c(1 + \lambda^{eff})} = 0.14 \frac{\gamma^{eff}}{\gamma_c} \frac{T_{c0}}{T_c} \]
is pair-breaking parameter and $\Psi(z)$, the digamma function.

7. Calculations of the Renormalization Factor within Conserving Single-site Approximations to the Alloy Self-energy.

In this section we demonstrate that within an arbitrary conserving single-site ap-
proximation determining the self-energy $\Sigma^\nu(ip_s)$ in the normal phase of a super-
conductor the renormalization factor $\kappa(c, \tilde{V})$ (6.43) can be expressed in terms of
suitably defined generating function for $\Sigma^\nu(ip_s)$. To this end we recall that the co-
efficient $B_{NLQ}^{(D)}(c)$ in (6.21) is nothing but a product of all the cumulants resulting
from disorder-vertex contributions for the diagram $(D)$ given, and these cumulants
are the same which enter $\Sigma^\nu(ip_s)$, for the EPI renormalizations are consistent with
the approximation to $\Sigma^\nu(ip_s)$ by construction.

Let us introduce $Q$-function, generating function connected with the number of
successive disorder scattering processes, by the following expression
\[ Q(z_1, z_2, z_3) = \sum_{NLQ \{ D \}} \sum B_{NLQ}^{(D)}(c) z_1^N z_2^L z_3^Q. \]

Here $B_{NLQ}^{(D)}(c)$ is the product of the cumulants for the Fock-type graph $(D)$ having
the triple $NLQ$ fixed, inner sum runs over all graphs of $NLQ$-family and outer sum
over all positive integers $NLQ$. Clearly, the functions $f_{\alpha\beta\gamma}(c, \tilde{V})$ can be expressed
through $Q(z_1, z_2, z_3)$ by picking out even and odd parts of the latter with respect
to each of its three arguments, that is
\[ f_{\alpha\beta\gamma}(z_1, z_2, z_3) = \frac{1}{2^{24}} \sum_{\eta_1, \eta_2, \eta_3} \eta_1^\alpha \eta_2^\beta \eta_3^\gamma Q(\eta_1 z_1, \eta_2 z_2, \eta_3 z_3), \]
where $\eta_i = \pm 1$, $(i = 1, 2, 3)$, and

$$f_{\alpha\beta\gamma}(c, \tilde{V}) = f_{\alpha\beta\gamma}(z_1, z_2, z_3) |_{z_1 = z_2 = z_3 = -i \tilde{V}}. \quad (7.3)$$

According to its definition the function $Q(z_1, z_2, z_3)$ is the product of analogous functions for vertex corrections

$$Q(z_1, z_2, z_3) = Q_3(z_1, z_2)Q_3(z_2, z_3) \{1 + Q_6(z_1, z_2, z_3)\}, \quad (7.4)$$

where we have introduced $Q$-functions for irreducible three- and six-leg vertices.

$$Q_3(z_1, z_2) = 1 + Q_3(z_1, z_2)Q_4(z_1, z_2), \quad (7.5)$$

$$Q_3(z_1, z_2) = (1 - Q_4(z_1, z_2))^{-1}. \quad (7.6)$$

Now we find the relations between the disorder self-energy $\Sigma^e(i\nu_s)$ and generating functions introduced above. From Eqs. (C.4, C.10, C.12) of Appendix C we have

$$\Sigma^e[G] = VP_2(VG), \quad (7.7)$$

$$U[G', G''] = V^2P_4(VG, VG'), \quad (7.8)$$

$$U[G, G', G''] = V^3P_6(VG, VG', VG''), \quad (7.9)$$

where

$$P_2(z) = \sum_{n=0}^{\infty} q_n(c)z^n, \quad (7.10)$$

and $q_n(c)$ are renormalized cumulants for the single-site approximation being used. (Note that we refer to the generating functions introduced in Appendix C as to generating $P$-functions.)

Generating functions for four- and six-leg vertices can be expressed as

$$P_4(z_1, z_2) = \frac{1}{2\pi i} \int_C dz \frac{P_2(z)}{(z - z_1)(z - z_2)}, \quad (7.11)$$

$$P_6(z_1, z_2, z_3) = \frac{1}{2\pi i} \int_C dz \frac{P_2(z)}{(z - z_1)(z - z_2)(z - z_3)}, \quad (7.12)$$

where contours $C$ in the integrals encircle only the peculiarities of the denominator of integrands. Noting that the irreducible vertex of $2n$’th order has $n$ potential lines more than full electron lines and using (7.8, 7.9) one obtains the relations:

$$Q_4(z_1, z_2) = z_1z_2P_4(z_1, z_2), \quad (7.13)$$

$$Q_6(z_1, z_2, z_3) = z_1z_2z_3P_6(z_1, z_2, z_3). \quad (7.14)$$
Finally, we write down the expression for $\kappa(c, \tilde{V})$ once more for reader convinence:

$$\kappa(c, \tilde{V}) = f_{eeo}(c, \tilde{V}) - f_{eo}(c, \tilde{V}).$$  \hfill (7.15)

So, we see that the knowledge of $P$-generating function for $\Sigma^c[G]$ is sufficient to calculate renormalization factor $\kappa(c, \tilde{V})$.

The expression (7.15) for $\kappa(c, \tilde{V})$ is not quite convenient to work with directly and may be transformed further. To this end, we introduce, in tight analogy with the Bethe-Salpeter equation (2.24), the generating function $\Gamma_4(z_1, z_2)$ by

$$\Gamma_4(z_1, z_2) = P_4(z_1, z_2) + z_1 z_2 P_4(z_1, z_2) \Gamma_4(z_1, z_2).$$ \hfill (7.16)

Then from (7.5, 7.6, 7.13) and (7.16), it follows

$$Q_3(z_1, z_2) = 1 + z_1 z_2 \Gamma_4(z_1, z_2),$$ \hfill (7.17)

and

$$Q_3(z_1, z_2) = \frac{\Gamma_4(z_1, z_2)}{P_4(z_1, z_2)}.$$ \hfill (7.18)

Using (7.14) we can rewrite expression (7.3) for $Q(z_1, z_2, z_3)$ as

$$Q(z_1, z_2, z_3) = 1 + z_1 z_2 \Gamma_4(z_1, z_2) + z_2 z_3 \Gamma_4(z_2, z_3) + z_1 z_2 z_3 \delta Q(z_1, z_2, z_3),$$ \hfill (7.19)

where

$$\delta Q(z_1, z_2, z_3) = z_2 \Gamma_4(z_1, z_2) \Gamma_4(z_2, z_3) + Q_3(z_1, z_2) P_6(z_1, z_2, z_3) Q_3(z_2, z_3).$$ \hfill (7.20)

To proceed further we use the Ward-type identity which relates $P$-functions for vertices of different order. From (C.13, C.14) of Appendix C the function $P_6(z_1, z_2, z_3)$ can be expressed as

$$P_6(z_1, z_2, z_3) = \frac{P_4(z_1, z_2) - P_4(z_3, z_2)}{z_1 - z_3}.$$ \hfill (7.21)

In its turn, from (7.16) one has

$$P_4(z_1, z_2) = (\Gamma_4^{-1}(z_1, z_2) + z_1 z_2)^{-1}$$ \hfill (7.22)

and using then (7.16, 7.22) one can obtain

$$P_4(z_1, z_2) - P_4(z_3, z_2) = \frac{P_4(z_1, z_2) P_4(z_3, z_2)}{\Gamma_4(z_1, z_2) \Gamma_4(z_3, z_2)} \times (\partial \Gamma(z_1, z_2, z_3) - z_2 \Gamma_4(z_1, z_2) \Gamma_4(z_3, z_2))(z_1 - z_2),$$ \hfill (7.23)

where

$$\partial \Gamma(z_1, z_2, z_3) = \frac{\Gamma_4(z_1, z_2) - \Gamma_4(z_3, z_2)}{z_1 - z_3}.$$ \hfill (7.24)
Substituting now the expression (7.23) into (7.21) one has
\[ P_0(z_1, z_2, z_3) = \frac{P_4(z_1, z_2)P_4(z_3, z_2)}{\Gamma_4(z_1, z_2)\Gamma_4(z_3, z_2)} \times \]
\[ (\partial \Gamma(z_1, z_2, z_3) - z_2\Gamma_4(z_1, z_2)\Gamma_4(z_3, z_2)) \]. \quad (7.25)

Recalling (7.18) and Eq. (7.20) for \( \delta Q(z_1, z_2, z_3) \), we can write
\[ \delta Q(z_1, z_2, z_3) = \partial \Gamma(z_1, z_2, z_3), \]
so that \( Q(z_1, z_2, z_3) \) becomes
\[ Q(z_1, z_2, z_3) = 1 + z_1z_2\Gamma_4(z_1, z_2) + z_2z_3\Gamma_4(z_2, z_3) + z_1z_2z_3\partial \Gamma(z_1, z_2, z_3). \quad (7.27) \]

As for the combination \( f_{eee}(z_1, z_2, z_3) - f_{ooe}(z_1, z_2, z_3) \), a little algebra with the use of (7.2) yields
\[ \frac{1}{4} \{ Q(z_1, z_2, -z_3) + Q(-z_1, -z_2, z_3) + Q(z_1, -z_2, -z_3) + Q(-z_1, z_2, z_3) \}. \quad (7.28) \]

Noting the symmetry \( Q(z_1, z_2, z_3) = Q(z_3, z_2, z_1) \) one has
\[ \kappa(c, \tilde{V}) = \frac{1}{2} \left\{ Q(z, -z, -z) + Q(-z, z, z) \right\} \bigg|_{z = -i\tilde{V}} \quad (7.29) \]
\[ = \frac{1}{2} \left\{ Q(z, -z, -z) + Q(z, z, -z) \right\} \bigg|_{z = -i\tilde{V}} \quad (7.30) \]
\[ = \frac{1}{2} \left\{ Q(z_1, z_2, z_2) + Q(z_1, z_1, z_2) \right\} \bigg|_{z_1 = -z_2 = -i\tilde{V}}. \quad (7.31) \]

Substitution of (7.27) into (7.31) leads
\[ \kappa(c, \tilde{V}) = 1 + \frac{1}{2(z_1 - z_2)^2} \left\{ z_1^2(z_1 - z_2)\Gamma_4(z_1, z_1) - z_2^2\Gamma_4(z_2, z_2) + 2z_1z_2(z_1 - z_2)^2\Gamma_4(z_1, z_2) \right\} \bigg|_{z_1 = z_2 = -i\tilde{V}} \quad (7.32) \]
and on a line \( z_1 = -z_2 \) this expression further reduces to
\[ \kappa(c, \tilde{V}) = 1 + \frac{z_1^2\Gamma_4(z_1, z_1) - 2z_1^2z_2^2\Gamma_4(z_1, z_2) + z_2^4\Gamma_4(z_2, z_2)}{(z_1 - z_2)^2} \bigg|_{z_1 = -z_2 = -i\tilde{V}}, \quad (7.33) \]
\[ \kappa(c, \tilde{V}) = 1 + \tilde{V}^2 \text{Im}_{z_1}\text{Im}_{z_2}\Gamma_4(z_1, z_2) \bigg|_{z_1 = z_2 = i\tilde{V}}, \quad (7.34) \]
where \( \text{Im}_z \) means picking out “imaginary” part with respect to a complex variable \( z \), which may be written as
\[ \text{Im}_z f(\ldots, z, \ldots) = \frac{1}{2i} \left( f(\ldots, z, \ldots) - f(\ldots, z^*, \ldots) \right) \quad (7.35) \]
and points stand for variables other than \( z \), and we retain explicit dependence on \( z_1 \) and \( z_2 \) in (7.33,7.34) for further use when analyzing the renormalizations within CPA.
Now everything is ready to evaluate $\kappa(c, \tilde{V})$ within certain widely known single site approximations. To be concrete we consider three such approximations: VCA, ATA and CPA.

### 7.1. Virtual Crystal Approximation and EPI Renormalizations.

For the case of VCA the self-energy reads

$$\Sigma_{VCA}^{e}(ip_s) = cV + c(1 - c)VG(ip_s)$$  (7.36)

and cumulants $q_n(c)$ are

$$q_0(c) = c, \quad q_1(c) = c(1 - c), \quad q_n(c) = 0 \quad n > 1.$$  (7.37)

The generating function $P_2(c)$ within this approximation has the form

$$P_2(z) = c + c(1 - c)z$$  (7.38)

and the use of (7.11) gives

$$P_4(z_1, z_2) = c(1 - c).$$  (7.39)

Then, from (7.16) and within the VCA accuracy, we have

$$\Gamma_4(z_1, z_2) = P_4(z_1, z_2) = c(1 - c)$$  (7.40)

and from (7.33, 7.34) it follows

$$\kappa(c, \tilde{V}) = 1.$$  (7.41)

So, there is no disorder contributions to the renormalization of EPI coupling, except for usual ones through one particle state density.

### 7.2. Renormalizations of EPI within Average T-matrix Approximation.

For ATA approximation the expression for $\Sigma_{ATA}^{e}(ip_s)$ has the form

$$\Sigma_{ATA}^{e}(ip_s) = \frac{cV}{1 - (1 - c)VG(ip_s)}.$$  (7.42)

General expression for $q_n(c)$ within ATA reads

$$q_n(c) = c(1 - c)^n, \quad n \geq 1$$  (7.43)

and the generating function $P_2(c)$ is

$$P_2(z) = c(1 - c)z^{-1}.$$  (7.44)

Then, similarly to the previous case, we obtain

$$P_4(z_1, z_2) = c(1 - c)[1 - (1 - c)z_1]^{-1}[1 - (1 - c)z_2]^{-1}$$  (7.45)
and to required accuracy
\[ \Gamma_4(z_1, z_2) = P_4(z_1, z_2), \]
so that in this case the result for \( \kappa(c, \tilde{V}) \) reads
\[ \kappa(c, \tilde{V}) = 1 + \frac{c(1-c)c^3\tilde{V}^4}{1 + (1-c)^2\tilde{V}^2} \]  \( (7.47) \)

Let us consider now the so called split band limit, \( V \to \infty \). Supposing that the chemical potential lies in the lower A-subband, that is
\[ \lim_{\tilde{V} \to \infty} N_B(0) = 0. \]  \( (7.48) \)

Introducing projected one particle state densities for A- and B-type atoms
\[ N(\epsilon) = (1 - c)N_A(\epsilon) + cN_B(\epsilon), \]
we find in this limit that
\[ \kappa(c, \infty) = \lim_{\tilde{V} \to \infty} \kappa(c, \tilde{V}) = (1 - c)^{-1} \]  \( (7.50) \)
and effective EPI coupling becomes
\[ \lambda_{eff} = \kappa(c, \infty)N(0)\lambda = \lambda N_A(0). \]  \( (7.51) \)

Such a limiting behavior is fully transparent and might be expected on physical grounds. Indeed, suppose \( \tilde{V} \) is so large that electrons can hop and interact with phonons only on A-type atoms (the probability of finding an electron on B-type atom is exponentially small in this case, so that one may completely neglect this process). Then two effects arise. The first is the renormalization of the electron state density for A-type atoms by both B-type impurities and EPI. The second is the renormalization of chemical potential which is due to the fact that B-type atoms are no longer accessible for the electron motion. The electrons can now propagate in those states which belong to the renormalized band for A-type atoms and the pairing occurs for these states only. The temperature \( T_c \) depends solely on the properties of A-band states but not on the (macrosscopical) size of superconductive cluster and this is the reason why \( \lambda_{eff} \) obeys the limiting expression of the type \( (7.51) \) where the dependence on the concentration of B-type impurities and on the strength of disorder scattering enters implicitly through the renormalized electron state density \( N_A(0) \) and, of course, through the renormalized chemical potential. However the self-consistent version of ATA used here fails in describing normal state properties of an alloy (see Ref. 37 and references therein for critique on this approximation), so that more complex approximations have to be considered.

7.3. Coherent Potential Approximation and EPI Renormalizations.
For CPA the equation for a self-energy $\Sigma_{CPA}(ip_s)$ reads
\[< t(ip_s) > = 0 \quad (7.52)\]
where
\[t(ip_s) = (V\eta_i - \Sigma_{CPA}) + (V\eta_i - \Sigma_{CPA})G_{ii}(ip_s)t(ip_s). \quad (7.53)\]
We may introduce $P_2(z)$ generating function by
\[< t(z) > = 0, \quad (7.54)\]
\[t(z) = (\eta - P_2(z)) + (\eta - P_2(z))zt(z), \quad (7.55)\]
where $\eta$ is a random variable with the distribution
\[\text{Prob}\{\eta = 1\} = c, \quad \text{Prob}\{\eta = 0\} = 1 - c. \quad (7.56)\]
To calculate $P_4(z_1, z_2)$ we use the Ward-type identity (7.11). Then a kind of transformations used in Appendix A, Eq.(A.5), leads
\[P_4(z_1, z_2) = \frac{< t(z_1)t(z_2) >}{1 + z_1z_2 < t(z_1)t(z_2) >}, \quad (7.57)\]
and from Bethe-Salpeter type equation (7.16) one has
\[\Gamma_4(z_1, z_2) = < t(z_1)t(z_2) >. \quad (7.58)\]
Using (7.34) we can write $\kappa(c, \tilde{V})$ in the form
\[\kappa(c, \tilde{V}) = 1 + \tilde{V}^2 < (\text{Im}(\tilde{V}))^2 >. \quad (7.59)\]
A more familiar form arises if one uses general expression for $\kappa(c, \tilde{V})$ in the form (7.33). To this end we rewrite the expression (7.33) in the form
\[\kappa(c, \tilde{V}) = \frac{< (z_1 + z_2^2t(z_1) - z_2 - z_2^2t(z_2))^2 >}{(z_1 - z_2)^2} |z_1 = -z_2 = -i\tilde{V}, \quad (7.60)\]
where the use was made of Eqs.(7.53) and (7.57), and finally
\[\kappa(c, \tilde{V}) = \frac{< |\text{Im}(z + z^2t(z))|^2 >}{(\text{Im}z)^2} |z = -i\tilde{V}. \quad (7.61)\]
Within the approximation (6.10) one has
\[\text{Re}G(+i0) = 0, \quad \text{Im}G(+i0) = -i\pi N(0), \quad (7.62)\]
with $G(E + i0)$ being analytical continuation of $G(ip_s)$ onto the upper half-plane of the complex energy plane, and the expression (7.61) becomes
\[\kappa(c, \tilde{V}) = \frac{< N_4^2(0) >}{< N_4(0) >^2}, \quad (7.63)\]
where in the course of transformations we have introduced projected electron state densities as

\[ N_{\eta}(E) = -\frac{1}{\pi} \text{Im} F_{\eta}(E), \quad F_{\eta}(E) = G(E) + G(E)t_{\eta}(E + i0)G(E), \]  

with \( t(E + i0) \) being analytical continuation of \( t(i\phi) \), Eq.(7.53), onto the upper half-plane and \( G(E) \) being on-site one-particle Green’s function.

As one may see the formula (7.63) is exactly the same as in Refs.5,7 and obeys the expected limiting behavior Eq.(7.51) (the reasons for this are the same as in the case of ATA).

8. Conclusions

In this paper we have studied the electron system which includes both quenched substitutional disorder and electron-phonon interaction. We have developed a rather general approach for constructing Eliashberg-type equations describing superconductive transition in such systems and have analyzed a linearized version of the equations in a number of widely known “alloy” approximations to find explicit expressions for \( T_c \). The examples are Eq.(7.63) for CPA, which coincide with that given in Refs.5,7, and also Eq.(7.47) for ATA (along with general expression (6.47) for \( T_c \) and Eq.(6.43) for the renormalization factor \( \kappa(c,\tilde{V}) \) of course).

Analogously, one may derive rather simple and compact expressions for \( T_c \) in any other “alloy” approximation for normal phase and within more general models of quenched disorder including off-diagonal one. The structure of the equations (2.30–2.33,2.41–2.43) remains intact irrespectively of the type of quenched substitutional disorder as well as the approximation for the “alloy” self-energy used (obviously, the latter must obey \( \phi \)-derivability criterion, Eq.(3.44)).

We want also mark a general conclusion on extreme importance of EPI vertex renormalization by alloy disordering and on the Anderson’s theorem for arbitrary type of quenched disorder treated within conserving approximations. As for results (6.50–6.52) and (6.55–6.58) concerning the influence of disordering on logarithmic corrections to the effective EPI coupling and on dilute paramagnetic impurity suppression of \( T_c \) respectively, to our knowledge they have been presented in such a general form for the first time here.

It would be interesting enough to generalize the approach presented to the case of non-singlet types of pairing, which seems to be quite possible. In this case, as one knows, the Anderson’s theorem does not hold and the effects of the disorder influence on \( T_c \) are much stronger.

Let us discuss briefly the applicability of the results for \( T_c \) obtained to the “alloy analogy” for the Hubbard model. Seemingly, they are not valid literally because of very different structure of diagrams for quenched disorder and for effective disorder within static approximation for the Hubbard model, where disorder is not quenched, but annealed. In this case the “concentrations” of “alloy” components themselves are functionally dependent on one-particle Green’s function and should be varied.
too. At a diagrammatic level of discussion the difference among two types of the
disorder is the appearance of graphs with loops contributing for the case of annealed
disorder, while such graphs are absent for quenched disorder. But even if one
neglects this difference (which may be quite reasonable, at least for not too strong
on-site repulsion when distributions for fluctuating fields are weakly renormalized
by electrons) there is still another reason for why the present results are not directly
applicable to the “alloy analogy” for the Hubbard model. The matter is that in
this case fluctuating fields contain both charge and spin components, so that the
effective disorder is by no means nonmagnetic.

One can also see that simple approximations of a type used by us earlier are
rather dangerous since a type of Ward’s cancellations may take place just as in the
case of Anderson’s theorem and careless generalization of the present approach to
the “alloy analogy” within functional integral treatments of the Hubbard model is
required.

Let us stress once more the difference between our treatment and the approach
by Belitz. Belitz, following ideas due to Anderson, used the superconducting order parameter $\Delta_\nu \propto C^{\dagger}_\nu C_{\tau\downarrow}$ where $\nu$ and $\tau$ are the states connected by the
time reversal symmetry for a particular disorder configuration. At the same time
we use $k$-dependent order parameter in a translationally invariant effective medium.
Therefore both approaches are not equivalent from formal point of view: diagrams
of the same form in $\nu$- and $k$-representation need not to give the same contri-
bution and, hence, the same result for $T_c$. In particular, in the approach by Belitz
the Hartree-type diagrams do not contribute to the anomalous self-energy unlike
present approach.

To establish the equivalence one has to solve very difficult, yet not investigated,
problem whether the random quantity $\Delta_\nu$ is a self-averaging one. On the other
hand, the order parameter used here seems to be directly related to the quasiparticle
spectrum in the superconducting phase and consequently to observables. To our
opinion, treatment of such important problems as, for instance, order parameter
anisotropy in high $T_c$ superconductors is more adequate within our approach. In
any case, the order parameter used here is physically satisfactory, provided the
quasimomentum remains well defined, that is, far from the localization transition.
As for the essentially strong disorder (near the Anderson’s transition), both the
approaches apparently do not work and the choice of the order parameter in the
intermediate region may be done empirically.

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Appendix A The Ward identities

In this Appendix we present the derivation of Ward’s identities (3.41) and (4.15). The identities of this type was established in Ref. 60 within the CPA in connection with the calculation of transport properties of binary alloys. The relations between self-energy $\Sigma_e[G]$, as given by CPA, and irreducible four-leg vertex presented in Ref. 60 was also studied in Ref. 61 with the use of diagrammatic technique.

To begin with, we establish the connection between variation of $\Sigma[G]$ caused by finite change in $G$. Such an identity was derived within certain approximation and used when discussing the problem of localization by disorder.

Given arbitrary but finite variation in one-particle Green’s function $G$

$$G' = G + \Delta G,$$  \hspace{1cm}  (A.1)

we consider

$$< \Delta T> = < T^{\text{ee}} - T^e >,$$  \hspace{1cm}  (A.2)

where $T^e$ is given by

$$T^{\text{ee}}(ip_s) = (V^e - \Sigma^e(ip_s)) + (V^e - \Sigma^e(ip_s))G(ip_s)T^e(ip_s),$$  \hspace{1cm}  (A.3)

$T^{\text{ee}}$ is defined by similar expression, but with $G$ and $\Sigma$ replaced by $G'$ and $\Sigma'$ respectively,

$$\Sigma' = \Sigma^e + \Delta \Sigma,$$  \hspace{1cm}  (A.4)

and we impose the self-consistency condition $< T > = < T' > = 0$ for $\Sigma$ to acquire proper, albeit implicit, functional dependence on $G$.

The following chain of equalities holds:

$$0 = < \Delta T^e > = < T^{\text{ee}}[G', \Sigma'] - T^{\text{ee}}[G, \Sigma] >$$  

$$= - < T^{\text{ee}} (T^{\text{ee}})^{-1} - (T^e)^{-1} > T^e >$$  

$$= - < T^{\text{ee}} \left((V^e - \Sigma^e)^{-1} - (V^e - \Sigma^e)^{-1} - (G' - G)\right) > T^e >$$  

$$= - < (1 + T^e G') \Delta \Sigma (1 + G T^e) - T^e \Delta G T^e >$$  

$$= < T^{\text{ee}} \Delta G T^e > - < (1 + T^{\text{ee}} G') \Delta \Sigma (1 + G T^e) > - \Delta \Sigma.$$  \hspace{1cm}  (A.5)
In explicit index notation the last line of Eq. (A.5) reads

\[
\Delta \Sigma_{\mu\nu} + \Gamma_{\mu\alpha;\beta\nu}[G', G]G_{\alpha\alpha'}\Delta \Sigma_{\alpha'\beta'}G_{\beta'\beta} = \Gamma_{\mu\alpha;\beta\nu}[G', G]\Delta G_{\alpha\beta},
\] (A.6)

where, analogously to (3.28), we have introduced full disorder four-leg vertex

\[
\Gamma_{\mu\alpha;\beta\nu}[G', G] = <T^F_{\mu\alpha}T^F_{\beta\nu}>. (A.7)
\]

The Bethe-Salpeter equation gives the irreducible four-leg vertex

\[
\Gamma_{\mu\alpha;\beta\nu}[G', G] = U_{\mu\alpha;\beta\nu}[G', G] + U_{\mu'\nu';\nu'\nu}[G', G]G'_{\mu'\alpha'}G_{\beta'\nu'}\Gamma_{\alpha'\alpha;\beta'\beta}[G', G]. (A.8)
\]

Multiplying (A.6) by the combination \(G'U[\cdot, \cdot]G\) and using the Bethe-Salpeter equation (A.8) we can write

\[
\Gamma_{\alpha\alpha'\beta'}[G', G]G'_{\alpha'\mu}G_{\beta'\beta}\Delta \Sigma_{\mu\nu} = U_{\alpha\alpha'\beta'}[G', G]G'_{\alpha'\mu}G_{\beta'\beta}\Gamma_{\alpha'\beta'\gamma}[G', G]\Delta G_{\gamma\gamma}. (A.9)
\]

After substituting this expression into (A.6) and using Eq. (A.8) once again one obtains the Ward’s identity

\[
\Delta \Sigma_{\mu\nu} = U_{\mu\alpha;\beta\nu}[G', G]\Delta G_{\alpha\beta}. (A.10)
\]

Putting now \(G' = G + \Delta G\), where \(\Delta G\) tends to zero we immediately have

\[
\frac{\delta \Sigma_{\mu\nu}}{\delta G_{\alpha\beta}} = U_{\mu\alpha;\beta\nu}[G, G]. (A.11)
\]

By the definition (A.7) the vertex function \(\Gamma[\cdot, \cdot]\) has the property

\[
\Gamma_{\mu\alpha;\beta\nu}[G', G] = \Gamma_{\beta\nu;\mu\alpha}[G, G] (A.12)
\]

which induces corresponding property for irreducible four-leg vertex

\[
U_{\mu\alpha;\beta\nu}[G', G] = U_{\beta\nu;\mu\alpha}[G, G']. (A.13)
\]

Then the use of (A.11) leads to the condition of \(\phi\)-derivability, and the Eqs. (3.26) which implicitly determine the self-energy \(\Sigma^e[G]\) can be represented in the form stated in Eq. (2.33) with some functional \(W^e[G]\).

Note also that for any conserving approximation for \(\Sigma^e[G]\) the condition of \(\phi\)-derivability holds automatically and the identity of the type (A.10) may be established along the lines of Appendix of Ref. and making use of the fact that complete families of self-energy terms contribute to \(\Sigma^e[G]\) in this case.

Formal identity (A.10) may be used to derive various relations between alloy self-energy \(\Sigma^e(i\rho_s)\) and one particle Green’s function \(G(i\rho_s)\). For example, substituting

\[
\Sigma' = \Sigma^e_\phi(i\rho_s), \quad \Sigma = \Sigma^e_\phi(i\rho_m)
\] (A.14)
and
\[ G' = G'_\sigma(i\epsilon_s), \quad G = G_\sigma(i\epsilon_m) \] (A.15)
into (A.10) we obtain
\[ \Sigma_{ij\sigma}(i\epsilon_s) - \Sigma_{ij\sigma}(i\epsilon_m) = U_{ij\sigma'}(i\epsilon_s, i\epsilon_m) \{ G_{i\sigma'}(i\epsilon_s) - G_{i\sigma'}(i\epsilon_m) \} \] (A.16)
which after Fourier transforming and using
\[ \text{Im} \Sigma_{k\sigma}(E) = \frac{1}{2i} \left\{ \Sigma_{k\sigma}(E + i\delta) - \Sigma_{k\sigma}(E - i\delta) \right\}, \quad \delta \to +0, \] (A.17)
\[ \text{Im} G_{k\sigma}(E) = \frac{1}{2i} \left\{ G_{k\sigma}(E + i\delta) - G_{k\sigma}(E - i\delta) \right\}, \quad \delta \to +0, \] (A.18)
takes the form
\[ \text{Im} \Sigma_{k\sigma}(E) = \sum_q U_{q\sigma\sigma'}(E^+, E^-) \text{Im} G_{k\sigma}(E), \quad E^\pm = E \pm i0. \] (A.19)

This is nothing but an analogue of the unitarity condition for alloy type problems we deal with in the present paper.

Appendix B Disorder-Induced Renormalization of Paramagnetic Impurity \( T_C \)-Suppression

In this Appendix we treat the problem of how \( T_C \)-suppression by dilute paramagnetic impurities is renormalized by non-magnetic disorder. Using the approach of Section 3 we derive the expression for the self-energy correction which is due to the scattering on the paramagnetic impurities, and then we apply the analysis of Section 5 and 6.

Three assumptions are made. Firstly, paramagnetic impurities are distributed randomly with the concentration \( c_P \) over the host and independently of the non-magnetic ones. Secondly, the disorder due to paramagnetic impurities is purely site-diagonal. Thirdly, both the concentration of paramagnetic impurities and the scattering intensity \( V_P \) on them are small, so that the relevant small parameter is \( c_P V_P^2 \).

The contribution to the Hamiltonian (2.1) due to paramagnetic impurities reads
\[ H_P = \sum_i V_P^i \left\{ S_i^z \mathbf{C}_i^\dagger \tau_0 \mathbf{C}_i + S_i^+ \mathbf{C}_i^\dagger \tau_- \mathbf{C}_i + S_i^- \mathbf{C}_i^\dagger \tau_+ \mathbf{C}_i \right\}, \] (B.1)
where \( \mathbf{C}_i^\dagger \) and \( \mathbf{C}_i \) are row and column spinors (2.10) on a site \( i \), \( S_i^z \) is the \( z \)-component of the impurity spin operator of spin \( S \), the matrices \( \tau_+ \) and \( \tau_- \) are \( 1/2(\tau_1 + i\tau_2) \) and \( 1/2(\tau_1 - i\tau_2) \) respectively, \( S_i^+ \) and \( S_i^- \) have similar definitions, and
\[ V_P^i = V_P \zeta_i, \] (B.2)
\[ \text{Prob}\{\zeta_i = 0\} = (1 - c_P), \quad \text{Prob}\{\zeta_i = 1\} = c_P, \]
with $\zeta_i$ being independent random variables. Up to the lowest order in $c_p V_P^2$, one has the correction to the effective potential $V(p_s)$ Eq. (3.3),

$$\delta V_{\alpha\beta}(p_s) = c_p V_P^2 S(S + 1) g_{\alpha\beta}(p_s) \delta_{\alpha\beta}, \quad (B.3)$$

and we have neglected uninteresting contribution of the first order in $V_P$ which gives a shift of the chemical potential.

The expression (B.3) can be written in the form analogous to the effective scattering potential caused by EPI

$$\delta V_{\alpha\beta}(p_s) = S_{\gamma\gamma'} K_{\alpha\alpha'}^{\gamma}(p_s) g_{\alpha\beta}(p_s) K_{\beta\beta'}^{\gamma'}, \quad (B.4)$$

$$\{K\}_{i \pm j} = \delta_{i \pm j} \tau_0, \quad (B.5)$$

$$S_{\gamma\gamma'} = S_{\gamma\gamma'} = \delta_{\gamma\gamma'} c_p V_P^2 S(S + 1) = \delta_{\gamma\gamma'} c_p V_P^2 S(S + 1). \quad (B.6)$$

The procedure of Section 3 then yields the correction to the full self-energy $\Sigma[G, D]$, with the form

$$\delta \Sigma_{\mu\nu}(p_s) = S_{\gamma\gamma'}\times \{ R_{\mu\alpha'}^{\gamma}(p_s, p_s) G_{\alpha'\beta'}^{\gamma}(p_s) R_{\beta\beta'}^{\gamma'}(p_s, p_s) + U_{\mu\nu;\beta\beta'}^{\gamma}(p_s, p_s, p_s) \
\times G_{\mu'\alpha}^{\gamma'}(p_s) R_{\alpha\alpha'}^{\gamma}(p_s, p_s) G_{\alpha\beta}^{\gamma}(p_s) G_{\beta\beta'}^{\gamma'}(p_s, p_s) G_{\beta\beta'}^{\gamma'}(p_s) \}, \quad (B.7)$$

In this expression $R_{\alpha\beta}^{\gamma}(p_s, p_s)$, is renormalized “paramagnetic impurity” vertex which obeys the equation

$$R_{\alpha\alpha'}^{\gamma}(p_s, p_m) = K_{\alpha\alpha'}^{\gamma} + \Gamma_{\alpha\beta;\beta'\alpha'}^{\gamma}(p_s, p_m) G_{\beta\beta'}^{\gamma}(p_s) G_{\beta'\beta'}^{\gamma'}(p_m) K_{\mu\mu'}^{\gamma'}, \quad (B.8)$$

$$R_{\alpha\alpha'}^{\gamma}(p_s, p_m) = K_{\alpha\alpha'}^{\gamma} + U_{\alpha\beta;\beta'\alpha'}^{\gamma}(p_s, p_m) G_{\beta\beta'}^{\gamma}(p_s) G_{\beta'\beta'}^{\gamma'}(p_m) R_{\mu\mu'}^{\gamma'}(p_s, p_m). \quad (B.9)$$

The expression (B.7) may be depicted in a form similar to that for $\Sigma^{\gamma\gamma'}[G, D]$ (see Fig. 5) with graphical representation of the combination $c_p V_P^2 S(S + 1)$ replacing phonon lines and with the “paramagnetic impurity” three-leg vertices replacing EPI vertices.

Further manipulations repeat literally the analysis of Section 5 and, after adopting binary alloy model for non-magnetic disorder, we apply the qualitative analysis of Section 6 to the reduced equations obtained.

Introducing

$$\gamma_P = c_p V_P^2 S(S + 1), \quad \gamma_P = N(0) \gamma_P, \quad (B.10)$$

one can write the expression for the correction to $\tilde{Z}(p_s)$ as

$$\delta \tilde{Z}(p_s) = \pi \gamma_P \left\{ f_{eoe}(c, \tilde{V}) + f_{oeo}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) + f_{oee}(c, \tilde{V}) \right\} \frac{1}{|p_s|} \quad (B.11)$$

and for the contribution to $\chi(p_s)$ one has the following form

$$\delta \chi(p_s) = -i \pi \gamma_P \left\{ f_{ooe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) + f_{eoe}(c, \tilde{V}) + f_{oee}(c, \tilde{V}) \right\} \quad (B.12)$$
which has no energy dependence (within the approximations of Section 6, of course) and leads to an uninteresting shift of chemical potential only.

The contribution of paramagnetic impurities to the anomalous part of the self-energy has the form

\[
\delta \left( \tilde{Z}(ip_s) \Delta(ip_s) \right) = -\pi \gamma_p \left\{ f_{eoo}(c, \tilde{V}) - f_{ooe}(c, \tilde{V}) - f_{ooe}(c, \tilde{V}) - 3f_{ooe}(c, \tilde{V}) \right\} \frac{\Delta(ip_s)}{|p_s|} \tag{B.13}
\]

and the correction to the gap function becomes

\[
\delta \Delta(ip_s) = -2\gamma_p^{eff} \frac{\Delta(ip_s)}{|p_s|}. \tag{B.14}
\]

In Eq. (B.14) we have introduced

\[
\gamma_p^{eff} = \kappa(c, \tilde{V}) \gamma_p, \tag{B.15}
\]

where \(\kappa(c, \tilde{V})\) is defined by Eq. (B.44). So we see that, up to the factor \(\kappa(c, \tilde{V})\), the contribution of the dilute paramagnetic impurities to the equations for the self-energy has the same form as in the case without alloy disordering.

Full equations on \(\tilde{Z}(ip_s)\), \(\chi(ip_s)\) and \(\Delta(ip_s)\) which take account of the paramagnetic impurity contributions read

\[
\text{Re} \begin{align*}
\text{ii} \left[ 1 - \tilde{Z}(ip_s) \right] &= 2i \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \tilde{\lambda}(0) \mathcal{U} \text{sign}(ip_s) \\
&-i \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \text{sign}(ip_m) \\
&-i \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \text{sign}(ip_m) \\
&-i \pi \gamma_p \left\{ f_{eoo}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) + f_{ooe}(c, \tilde{V}) \right\} \text{sign}(p_s), \tag{B.16}
\end{align*}
\]

\[
\chi(ip_s) = 2i \left\{ f_{eoe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) \right\} \tilde{\lambda}(0) \mathcal{U} \\
-1 \left\{ f_{eoe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \\
-1 \left\{ f_{eoe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) \right\} \text{sign}(ip_s) \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \text{sign}(ip_m) \\
-1 \pi \gamma_p \left\{ f_{eoe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) + f_{eoe}(c, \tilde{V}) \right\} \text{sign}(ip_m), \tag{B.17}
\]

\[
\text{Re} \begin{align*}
\tilde{Z}(ip_s) \Delta(ip_s) &= -2 \left\{ f_{eoe}(c, \tilde{V}) + f_{eeo}(c, \tilde{V}) \right\} \tilde{\lambda}(0) \mathcal{U} \frac{\Delta(ip_s)}{|p_s|} \\
&+ \left\{ f_{eoe}(c, \tilde{V}) - f_{eeo}(c, \tilde{V}) \right\} \pi T_c \sum_{ip_m} \tilde{\lambda}(ip_s - ip_m) \frac{\Delta(ip_m)}{|p_m|}
\end{align*}
\]
\[
\begin{align*}
&+ \left\{ f_{eo}(c, \tilde{V}) + f_{oe}(c, \tilde{V}) \right\} \frac{\Delta(i_{ps})}{|p_s|} \pi T_c \sum_{i_{pm}} \hat{\lambda}(i_{ps} - i_{pm}) \\
&+ 2f_{oeo}(c, \tilde{V}) \pi T_c \sum_{i_{pm}} \hat{\lambda}(i_{ps} - i_{pm}) \text{Sign}(i_{ps}) \text{Sign}(i_{pm}) \\
&- \pi \tilde{\gamma}_p \left\{ f_{eot}(c, \tilde{V}) - f_{eoe}(c, \tilde{V}) - f_{oeo}(c, \tilde{V}) - 3f_{roe}(c, \tilde{V}) \right\} \frac{\Delta(i_{ps})}{|p_s|}. \quad (B.18)
\end{align*}
\]

Solving the equation (B.16) on \( \tilde{Z}(i_{ps}) \) and inserting into (B.18) one finds

\[
\Delta(i_{ps}) = \pi T_c \sum_{i_{pm}} \lambda^{\text{eff}}(i_{ps} - i_{pm}) \frac{\Delta(i_{pm})}{|p_m|} \\
- \Delta(i_{ps}) \pi T_c \sum_{i_{pm}} \lambda^{\text{eff}}(i_{ps} - i_{pm}) \text{Sign}(i_{ps}) \text{Sign}(i_{pm}) - 2\tilde{\gamma}_p \frac{\Delta(i_{ps})}{|p_s|}, \quad (B.19)
\]

— the equation on the gap function, from which results for the critical concentration and for \( T_c \)-suppression follow by standard consideration.

**Appendix C  Generating Functions for Single-site Approximations**

In this Appendix we establish relations between suitably introduced generating function for arbitrary conserving single-site approximation for disorder self-energy \( \Sigma^e[G] \) and generating functions for irreducible disorder four-leg vertex and for higher vertices. The Ward’s type identities for generating functions will be also derived.

Generating functional for arbitrary single site approximation has the following analytical form

\[
W^e[G] = \sum_i \sum_{i_{ps}} \sum_{n=1}^{\infty} q_{n-1}(c) (VG_{ii}(i_{ps}))^n \quad (C.1)
\]

and outer sum runs over lattice sites. For the corresponding self-energy one has the expression

\[
\Sigma^e[G] = V \sum_{n=0}^{\infty} q_n(c) (VG_{ii}(i_{ps}))^n, \quad (C.2)
\]

where \( q_n(c) \) is \( n \)-th order cumulant renormalized by the multiple occupancy corrections within approximation of interest. Particular choice of these cumulants determines wholly the approximation for the disorder self-energy. We introduce the generating function:

\[
P_2(z) = \sum_{n=0}^{\infty} q_n(c) z^n \quad (C.3)
\]

The expression for \( \Sigma^e[G] \) can then be written in the form

\[
\Sigma^e[G] = VP_2(VG). \quad (C.4)
\]

Note also that generating functions we use here are connected with the number of Green’s functions in the corresponding contributions to the self-energy \( \Sigma^e[G] \).
Diagrams for the irreducible disorder four-leg vertex are generated by varying corresponding contributions to the self-energy and the expression for $U_4[G_1, G_2]$ reads

$$U[G_1, G_2] = V^2 \sum_{n=2}^{\infty} q_{n-1}(c) \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \delta_{n,m+l+2} (VG_1)^m (VG_2)^n$$

$$= V^2 \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} q_{m+l+1}(c) (VG_1)^m (VG_2)^n,$$

where we have used the fact that for quenched disorder the self-energy $\Sigma^e[G]$ does not contain fermionic loops and therefore the disorder four-leg vertex may be considered as some functional of two independent functional variables. Generally speaking, we may think of disorder $2m$-leg vertex as a functional of $m$ independent functional variables by the same reason.

Analogously to (C.3) we introduce the generating function

$$P_4(z_1, z_2) = \sum_{n=2}^{\infty} q_{n-1}(c) \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \delta_{n,m+l+2} z^m z^n$$

$$= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} q_{m+l+1}(c) z^m z^n,$$

and hence

$$U[G_1, G_2] = V^2 P_4(VG_1, VG_2).$$

In general, for $2m$-leg disorder vertex within arbitrary single-site approximation one can readily obtain

$$U[G_1, \ldots, G_m] = V^m \sum_{n=m}^{\infty} q_{n-1}(c) \prod_{j=1}^{m} (VG_j)^{n_j}$$

$$= V^m \prod_{j=1}^{m} q_{n_1+\cdots+n_m-1}(c) \prod_{j=1}^{m} (VG_j)^{n_j},$$

so that

$$P_2m(z_1, \ldots, z_m) = \sum_{n=m}^{\infty} q_{n-1}(c) \prod_{j=1}^{m} z_j^{n_j}$$

$$= \prod_{j=1}^{m} q_{n_1+\cdots+n_m-1}(c) \prod_{j=1}^{m} z_j^{n_j},$$

and then

$$U[G_1, \ldots, G_m] = V^m P_{2m}(VG_1, \ldots, VG_m).$$

We now express cumulant of the $n$-th order in terms of Cauchi integral

$$q_n(c) = \frac{1}{2\pi i} \int_C \frac{P_2(z)}{z^{n+1}}.$$
The substitution of \( q_n(c) \) in the form (C.11) into (C.9) leads

\[
\begin{align*}
P_{2m}(z_1, \ldots, z_m) &= \frac{1}{2\pi i} \int_C dz P_2(z) \sum_{n_1=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{1}{z^{n_1+n_2+\cdots+n_m+m}} \prod_{j=1}^{m} z_j^{n_j} \\
&= \frac{1}{2\pi i} \int_C dz P_2(z) \prod_{j=1}^{m} \left\{ \sum_{n_j=0}^{\infty} \left( \frac{z_j}{z} \right)^{n_j} \right\} \\
&= \frac{1}{2\pi i} \int_C dz P_2(z) \prod_{j=1}^{m} \frac{1}{z-z_j} , \quad (C.12)
\end{align*}
\]

where the contour of integration encircles singularities of the denominator only, and \( P_{2m}(z_1, \ldots, z_m) \) possess the property

\[
P_{2m}(z_1, \ldots, z_m) = P_{2m}(zp_1, \ldots, zp_m), \quad (C.13)
\]

with \( \mathcal{P} \) being arbitrary permutation of indices \( \{1 \ldots m\} \).

The expression (C.12) relates generating function \( P_2(z) \) with the corresponding generating functions for 2\( m \)-leg vertices. In particular, we have

\[
P_4(z_1, z_2) = \frac{1}{2\pi i} \int_C dz \frac{P_2(z)}{(z-z_1)(z-z_2)} \quad (C.14)
\]

and

\[
P_6(z_1, z_2, z_3) = \frac{1}{2\pi i} \int_C dz \frac{P_2(z)}{(z-z_1)(z-z_2)(z-z_3)} . \quad (C.15)
\]

For \( m \geq 2 \) rewriting the product in (C.12) as

\[
\prod_{j=1}^{m} \frac{1}{z-z_j} = \frac{1}{z-z_1} \frac{1}{z-z_2} \prod_{j=3}^{m} \frac{1}{z-z_j} = \frac{1}{z_2-z_1} \left\{ \frac{1}{z-z_2} - \frac{1}{z-z_1} \right\} \prod_{j=3}^{m} \frac{1}{z-z_j} \quad (C.16)
\]

and substituting this expression into (C.12) we obtain the analogue of the Ward’s identity which connects the generating functions for vertices of different order

\[
P_{2m}(z_1, \ldots, z_m) = \frac{P_{2(m-1)}(z_2, z_3, \ldots, z_m) - P_{2(m-1)}(z_1, z_3, \ldots, z_m)}{z_2-z_1} . \quad (C.17)
\]

This equation gives

\[
P_4(z_1, z_2) = \frac{P_2(z_2) - P_2(z_1)}{z_2-z_1} \quad (C.18)
\]

and

\[
P_6(z_1, z_2, z_3) = \frac{P_4(z_2, z_3) - P_4(z_1, z_3)}{z_2-z_1} \quad (C.19)
\]

for \( m = 2 \) and \( m = 3 \) respectively.
Yet another way to establish the relation (C.18) is to use the Ward’s identity derived in Appendix A. Indeed, noting that within arbitrary single-site approximation irreducible four-leg vertex is a purely local quantity we rewrite (A.7) as

$$V \{ P_2(VG_2) - P_2(VG_1) \} = VP_4(VG_2, VG_1) \{ VG_2 - VG_1 \},$$

where the use was made of (C.4) and (C.7). Then substitutions

$$VG_1 \to z_1, \quad VG_2 \to z_2$$

immediately lead to the expression (C.18).