On the Mass Spectrum of the SU(2) Higgs Model in 2+1 Dimensions

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Abstract

We calculate the masses of the low-lying states with quantum numbers $J^{PC}=0^{++},1^{-+}$ in the Higgs and confinement regions of the three-dimensional SU(2) Higgs model, which plays an important rôle in the description of the thermodynamic properties of the standard model at finite temperatures. We extract the masses from correlation functions of gauge-invariant operators which are calculated by means of a lattice Monte Carlo simulation. The projection properties of our lattice operators onto the lowest states are greatly improved by the use of smearing techniques. We also consider cross correlations between various operators with the same quantum numbers. From these the mass eigenstates are determined by means of a variational calculation. In the symmetric phase, we find that some of the ground state masses are about 30\% lighter than those reported from previous simulations. We also obtain the masses of the first few excited states in the symmetric phase. Remarkable among these is the occurrence of a $0^{++}$ state composed almost entirely of gauge degrees of freedom. The mass of this state, as well as that of its first excitations, is nearly identical to the corresponding glueball states in three-dimensional SU(2) pure gauge theory, indicating an approximate decoupling of the pure gauge sector from the Higgs sector of the model. We perform a detailed study of finite size effects and extrapolate the lattice mass spectrum to the continuum.


1 Introduction

The study of three-dimensional field theories has attracted a lot of attention over the past few years. While some models are investigated for field theoretic reasons or because they are more easily accessible than their four-dimensional homologues, others have an immediate physical meaning in the context of four-dimensional field theory at finite temperature. It has been known for a long time that for temperatures much higher than any mass scale of a given theory its non-static Matsubara modes may be integrated out perturbatively to yield a three-dimensional effective theory for the zero modes [1]. This effective theory describes the static long-range physics of the underlying four-dimensional finite-temperature theory, and moreover contains all the infrared divergences and non-perturbative phenomena that spoil a purely perturbative treatment of the latter.

In particular, the three-dimensional SU(2) Higgs model represents an effective high-temperature theory for the standard electroweak model, after neglecting the U(1) sector and fermions in a first approximation. Since it was realised that the baryon asymmetry of the universe could have been generated during a first-order electroweak phase transition [2], a lot of effort has been spent to determine the order and dynamics of this transition in detail. The perturbative procedure of dimensionally reducing the four-dimensional SU(2) Higgs model at finite temperature to a three-dimensional effective theory has been carried out in great detail in ref. [3], and the corresponding relations between the three-dimensional and four-dimensional parameters and temperature may be found there. Bearing these relations in mind, we shall stay entirely within the framework of the 2+1 dimensional SU(2) Higgs model in this paper.

There are already several analytical [4]-[7] and numerical [8]-[10] studies of the three-dimensional SU(2) Higgs model. While the main motivation for these studies was the phase transition itself, there also emerged the problem of understanding the structure of the symmetric phase. Due to infrared divergences in vector loops, straightforward perturbation theory breaks down in the symmetric phase, and until recently not much was known about the particle spectrum and the effective interactions in this parameter region of the theory. In a recent analytic calculation, the loop expansion was reorganised by resumming masses and vertices, which led to a set of gauge independent gap equations for the vector boson and the Higgs masses, defined on their respective mass shells. The solutions of these equations predict a non-vanishing vector boson mass and scalar vacuum expectation value in the “symmetric phase”, which thus would be interpreted as another Higgs phase, just with different parameters. On the other hand, lattice simula-
tions in four dimensions at finite temperature [11] as well as in three dimensions [9, 10] have reported vector boson masses about four times larger than predicted by the gap equations [1]. The picture conjectured from the lattice results is one of a symmetric phase with confining behaviour (in the sense of QCD) and a particle spectrum consisting of bound states. Similar conclusions may be drawn from analytic investigations of truncated renormalisation group equations which indicate strong coupling effects in the symmetric phase [7]. The picture of a QCD-like symmetric electroweak phase has also been employed for a model calculation of bound state masses [5]. One possible explanation of the large discrepancy between the two approaches is a breakdown of the resummed loop expansion of [4] in higher orders. In principle, however, it is also conceivable that an extremely low-lying state might not have been visible on the lattice sizes that have been investigated. Moreover, all simulations have emphasised the difficulty of measuring correlation functions in the symmetric phase due to the extremely low signal-to-noise ratio.

The purpose of the present paper is to shed more light on the situation in the symmetric phase by employing new techniques which allow a more reliable lattice calculation of the mass spectrum. The masses are extracted from correlation functions of gauge invariant operators. In order to improve the sensitivity to low-lying or bound states we construct a large set of non-local operators by employing a “blocking” technique similar to the one which has proved to be useful in pure gauge calculations [12, 13]. As we shall see, this procedure turns out to be very effective in enhancing the projection of our operators onto the lowest states. Moreover, it reduces the statistical errors significantly, yielding more accurate results for the masses. We also measure cross correlations between different operators. Diagonalisation of the corresponding correlation matrix then unmixes the superposition of the ground state and the excited states. This procedure further improves the signal for the lowest states and, more importantly, enables us to estimate the masses of the first few excited states. It also allows us to extract information about the overlap any individual operator has with a given state, and hence the coupling between the different states. We then perform a study of finite-size effects and an extrapolation to the continuum limit for two points in parameter space, one located in the symmetric and one in the Higgs phase.

The paper is organised as follows. In section[2] the lattice action and the basic operators used in mass calculations are discussed. The details and more technical aspects of

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1 For a detailed comparison of the lattice and analytic approaches in three and four dimensions in the context of the electroweak phase transition, see [5].
our simulation are described in section 3. In section 4 we present our results, analysing in detail the reliability of our mass estimates of the ground state, and including the extrapolation to the continuum limit. Finally, section 5 contains our conclusions.

2 Action and basic operators

The action of the SU(2) Higgs model in 2+1 dimensions and its general properties in the continuum and on the lattice have been discussed previously in the literature [4]-[10]. In order to fix the notation and to give all equations used in this paper we list some of these general aspects here.

The continuum action is given by

\[ S = \int d^3 x \text{Tr} \left[ \frac{1}{2} W_{\mu\nu} W^{\mu\nu} + (D_\mu \varphi)^\dagger D_\mu \varphi + \mu_3^2 \varphi^\dagger \varphi + 2\lambda_3 (\varphi^\dagger \varphi)^2 \right], \]

(1)

where all fields are in a $2 \times 2$ matrix notation,

\[ \varphi = \frac{1}{2}(\sigma + i \vec{\pi} \cdot \vec{\tau}), \quad D_\mu \varphi = (\partial_\mu - ig_3 W_\mu)\varphi, \quad W_\mu = \frac{1}{2} \vec{\tau} \cdot \vec{W}_\mu. \]

(2)

The gauge coupling $g_3$ and the scalar coupling $\lambda_3$ have mass dimension 1/2 and 1, respectively. The action (1) can be parametrised by two dimensionless parameters, which may be chosen to be $\lambda_3/g_3^2$ and $\mu_3^2/g_3^4$. Fixing these parameters determines the physical properties of the theory. The corresponding lattice action may be defined as

\[ S[U, \phi] = \beta_G \sum_p \left( 1 - \frac{1}{2} \text{Tr} U_p \right) + \sum_x \left\{ - \beta_H \sum_{\mu=1}^3 \frac{1}{2} \text{Tr} \left( \phi^\dagger(x) U_\mu(x) \phi(x + \hat{\mu}) \right) \right. \\
+ \frac{1}{2} \text{Tr} \left( \phi^\dagger(x) \phi(x) \right) + \beta_R \left[ \frac{1}{2} \text{Tr} \left( \phi^\dagger(x) \phi(x) \right) - 1 \right]^2 \right\}. \]

(3)

Due to the super-renormalisability of the theory (1), the only parameter receiving ultraviolet renormalisation is the scalar mass parameter $\mu_3^2/g_3^4$, whose corrections have been determined at the two-loop level in perturbation theory using the \( \overline{MS} \) scheme. The corresponding two-loop calculation in lattice perturbation theory was carried out in [14]. Requiring that the renormalised mass parameters be the same in both regularisation schemes, a relation between the parameters labelling the continuum and lattice theories has been established [14],

\[ \beta_G = \frac{4}{a g_3^2}, \]

(4)
\[ \beta_R = \frac{\lambda_3 \beta_H^2}{g_3^2 \beta_G}, \quad (5) \]
\[ \mu_3^2 \frac{g_3}{g_3^4} = \frac{\beta_H^2}{8} \left( \frac{1}{\beta_H} - 3 - \frac{2 \beta_H \lambda_3}{\beta_G g_3^2} \right) + \frac{3 \Sigma \beta_G}{32 \pi} \left( 1 + \frac{4 \lambda_3}{g_3^2} \right) \]
\[ + \frac{1}{16\pi^2} \left[ \left( 16 + 9 \frac{\lambda_3}{g_3^2} - 12 \left( \frac{\lambda_3}{g_3^2} \right)^2 \right) \left( \ln \frac{3 \beta_G}{2} + \zeta \right) + 5.0 + 5.2 \frac{\lambda_3}{g_3^2} \right], \quad (6) \]
with the numerical constants \( \Sigma = 3.17591 \) and \( \zeta = 0.09 \).

A Monte Carlo simulation of any quantity in the theory (3) is carried out for a given set of bare parameters \( \beta_G, \beta_H, \beta_R \). In order to establish contact with the desired continuum physics one first has to perform an infinite volume limit, i.e., do simulations on lattices much larger than the largest correlation length of the theory such that the results do not show any dependence on the lattice size. Secondly, one has to perform a continuum limit \( a \to 0 \), i.e., to simulate at different values of \( \beta_G \) and extrapolate to \( \beta_G \to \infty \). This limit has to be taken in such a way that the renormalised quantities parametrizing the theory remain constant. The corresponding ‘lines of constant physics’ in the space of the lattice parameters are given by equations (4)-(6).

The actions (1) and (3) have an \( SU(2)_{\text{local}} \times SU(2)_{\text{global}} \) symmetry. Physical states are described by gauge invariant operators. After decomposing \( \phi(x) \) as
\[ \phi(x) = \rho(x)\alpha(x), \quad \rho^2(x) = \frac{1}{2} \text{Tr} \left( \phi^\dagger(x)\phi(x) \right), \quad \rho(x) \geq 0, \quad \alpha(x) \in SU(2), \quad (7) \]
one may define the gauge-invariant composite field
\[ V_\mu(x) = \alpha^\dagger(x)U_\mu(x)\alpha(x + \hat{\mu}). \quad (8) \]

While \( \rho(x) \) and \( V_\mu(x) \) are invariant under local transformations, they transform under the diagonal global \( SU(2)_{\text{diag}} \) subgroup, customarily termed weak isospin, as
\[ \rho'(x) = \rho(x), \quad V'_\mu(x) = \Lambda V_\mu(x)\Lambda^{-1}, \quad \Lambda \in SU(2)_{\text{diag}}, \quad (9) \]
i.e., the lowest excitation of \( \rho(x) \) describes the isoscalar Higgs boson while the matrix-valued \( V_\mu(x) \) transforms as an isovector. A single field representing the spin-one, isospin-one \( W \) boson may be obtained from the composite link variable by taking the trace with an insertion of a Pauli matrix \( \tau^a \). Taking the trace without \( \tau^a \)-insertion produces another spin zero isoscalar operator. A third \( 0^{++} \) isoscalar operator is given by the plaquette. Thus we consider the following set of basic operators for the description of the low-lying states,
\[ 0^{++}: \; R(x) \equiv \frac{1}{2} \text{Tr} \left( \phi^\dagger(x)\phi(x) \right), \]
\[ 0^{++} : L(x) \equiv \sum_{\mu=1}^{2} \frac{1}{2} \text{Tr} \left( \alpha^\dagger(x)U_\mu(x)\alpha(x + \hat{\mu}) \right), \]

\[ 0^{++} : P(x) \equiv U_1(x)U_2(x + \hat{1})U_1^\dagger(x + \hat{2})U_2^\dagger(x), \]

\[ 1^{--} : V_\mu^a(x) \equiv \frac{1}{2} \text{Tr} \left( \tau^a \alpha^\dagger(x)U_\mu(x)\alpha(x + \hat{\mu}) \right). \] (10)

The plaquette \( P \) is particularly interesting because it consists of gauge degrees of freedom only, and in the three-dimensional pure gauge theory it is the simplest operator one can use to describe the \( 0^{++} \) glueball \([13]\). In the theory with scalars one expects a mixing of this operator with the other \( 0^{++} \) operators due to the coupling of gauge and scalar degrees of freedom.

The phase structure of the three-dimensional model with lattice action (3) has not been fully mapped out by numerical simulations as yet. However, from analyticity considerations \([13]\) and numerical studies of the four-dimensional model \([15]\) one expects the following qualitative picture: the three-dimensional parameter space spanned by \( \beta_G, \beta_H, \beta_R \) is divided into Higgs and confinement-like (or symmetric) regions by a surface of first-order phase transitions which is crossed by changing \( \beta_H \) for fixed \( \beta_G, \beta_R \). At sufficiently large values of \( \beta_R \) and small values of \( \beta_G \) this surface is expected to terminate so that the two regions are analytically connected. In this region, there is no phase transition but just a crossover as \( \beta_H \) is varied. Numerically, however, this region in the phase diagram has so far not been accessed in the three-dimensional theory. The continuum limit is represented by a single point in the phase diagram, \( \beta_G \to \infty, \beta_R \to 0, \beta_H \to 1/3 \). In order to describe different continuum theories, the continuum limit has to be taken along different paths in the parameter space, as specified by equations (4)-(6).

The term confinement-like is chosen to distinguish the behaviour of the theory in this region of parameter space from the confinement realised in the three-dimensional pure gauge theory. There the potential between static charges rises linearly with distance, without any bound. In the Higgs model one expects a flattening of the potential at some large distance, due to pair creation of scalars breaking the string between the static charges, just as fermions break the string in QCD. From the analytic connectedness of the Higgs and the confinement regions it follows that for every state in the Higgs region, there is a corresponding one in the confinement region. In particular, the same operators (10) may be used to describe physical states in both regimes. The global isospin symmetry is realised in the Higgs as well in the confinement region, so one expects low-lying states with the same quantum numbers in both regions.
3 The Simulation

The purpose of this paper is a closer investigation of the mass spectrum in the confinement-like phase, by calculating correlations of operators of the type in eq. (10).

In this section, we describe the details of our calculation, including the simulation algorithm, the construction of “blocked” or “smeared” operators for the Higgs and vector bosons and the way in which these are used to compute matrix correlators. We conclude this section with details about the statistical analysis and the fitting procedure employed to obtain our final mass estimates.

3.1 Simulation algorithm and parameters

Our Monte Carlo simulation is performed using the lattice action in eq. (3), containing the bare parameters $\beta_G$, $\beta_H$ and $\beta_R$.

For the update of the gauge variables we use a combination of the standard heatbath and over-relaxation algorithms for SU(2) \cite{17, 18}. The scalar degrees of freedom are updated using the algorithm proposed in \cite{19}, which uses the four real components of the scalar field $\phi(x)$. Thus, no separate updates of the radial and angular parts $\rho(x), \alpha(x)$ are required, leading to a simple implementation of the algorithm. As explained in \cite{19}, over-relaxation (reflection) steps in the update of the scalar field can be easily incorporated, provided the Higgs self-coupling $\beta_R$ is not too large, which would lead to a poor acceptance rate. In our simulation, where $\beta_R = O(10^{-4})$, we achieved acceptance rates of well over 90%. Higher values of $\beta_R$ could be simulated, for instance, by using the reflection algorithm described in \cite{20}.

In our simulation, a “compound” sweep consists of a combination of heatbath (HB) and several reflection (REF) updates of the gauge and scalar fields,

$$1 \text{HB}\{U\} + 1 \text{HB}\{\phi\} + n_{\text{OR}} \text{REF}\{U\} + n_{\text{ref}} \text{REF}\{\phi\}. \quad (11)$$

In accordance with ref. \cite{19}, we chose $n_{\text{OR}}$ to be roughly equal to the inverse scalar mass in order to achieve maximum decorrelation. With this choice we found that the average integrated autocorrelation time estimated using the scalar mass was close to one, in agreement with \cite{19}.

Our simulations were performed for inverse gauge couplings $\beta_G = 7, 9$ and 12. We restricted our attention to one point in the symmetric and one point in the Higgs region of parameter space chosen sufficiently away from the phase transition, so that the system
does not tunnel between the phases. In order to compare our results directly with those of a previous calculation of the lightest scalar and vector masses [9], we work at the same fixed value of $\frac{\lambda_3}{g_3^2}$,

$$\frac{\lambda_3}{g_3^2} = \frac{\beta_R \beta_G}{\beta_H^2} = 0.0239,$$

which in the context of the four-dimensional theory corresponds to a Higgs mass at tree-level and zero temperature of $M_H \simeq 35 \text{ GeV}$.

In the symmetric phase of the model we initially chose $\beta_G = 12, \beta_H = 0.3411$. The value of $\beta_R$ was then fixed by the relations (4)-(6). The same relations determine the continuum scalar mass parameter in the $\overline{MS}$ scheme as $\mu_3^2/g_3^4 = 0.089$. Our point in the Higgs phase of the model was fixed to be $\beta_G = 12, \beta_H = 0.3418$, which corresponds to $\mu_3^2/g_3^4 = -0.020$ in the continuum. At $\beta_G = 7$ and 9 the corresponding values of $\beta_H$ and $\beta_R$ were chosen according to the "lines of constant physics", eqs. (4)-(6), using the constraint eq. (12).

At $\beta_G = 12$, Monte Carlo runs were performed on lattice sizes ranging from $10^2 \cdot 12$ up to $40^3$ in order to analyse finite-size effects in detail. This is of special importance in the symmetric phase of the model, where we are particularly interested in the possible occurrence of very light states.

For all our observables, statistics were gathered from about 30 000 compound sweeps. In a few cases, statistics were increased to a total of 75 000 sweeps.

### 3.2 Constructing improved operators

The main difficulty encountered in recent attempts to compute the mass spectrum in the symmetric phase of the SU(2) Higgs model [11, 9], was the low signal-to-noise ratio in the computation of the correlation function

$$C(t) \equiv \sum_{x,x'} e^{i\mathbf{p} \cdot (x - x')} \langle \varphi^\dagger(x,t) \varphi(x',0) \rangle_c = \sum_{n>0} |\langle 0|\varphi(0)|n \rangle|^2 e^{-aE_n t}$$

$$\equiv \sum_{n>0} |c_n|^2 e^{-aE_n t} t \to \infty \simeq |c_1|^2 e^{-aE_1 t},$$

where $\varphi(x,t)$ denotes any one of the operators in eqs. (10), and $E_n > E_{n-1}$ is implied. For our numerical calculation of the masses we use the zero momentum timeslice averages of the original operators, i.e., $p = 0$ in the above expression.

One important goal of our simulations is to investigate the possible existence of very low-lying states in the symmetric phase of the model, such as predicted by the
analytic approach in [4]. If an operator has a bad projection onto the lightest state one must be able to follow the signal to sufficiently large $t$ before the ground state dominates $C(t)$. A poor signal-to-noise ratio of the correlation function will then hamper any effort to establish the existence of such a state. The problem is further exacerbated in the symmetric phase, where, due to the confining behaviour of the theory, the particle spectrum may consist of bound states, having a larger spatial extension than their point-like counterparts in the Higgs region. Previous experience with calculations of the glueball spectrum in pure gauge theory shows that conventional local operators have indeed a bad projection onto bound states in confining theories [12, 13]. The situation could be considerably improved by constructing “blocked” or “smeared”, non-local operators [12, 21], which are of similar extended structure as the bound states they are supposed to project on. Similar techniques, which preserve gauge invariance, have been developed and successfully applied in simulations of lattice QCD [22].

Here we are applying and extending these ideas in order to construct non-local versions of the operators defined in eq. (10). Some of these techniques were applied to the four-dimensional SU(2) Higgs model in ref. [23].

Following [12], we construct composite (“blocked”) link variables $U_{\mu}^{(n)}(x)$ of blocking level $n$ according to

$$U_{\mu}^{(n)}(x) = U_{\mu}^{(n-1)}(x)U_{\mu}^{(n-1)}(x + \hat{\mu}) + \sum_{\nu=1, \nu\neq \mu}^{+2} U_{\nu}^{(n-1)}(x)U_{\mu}^{(n-1)}(x + \hat{\nu})U_{\mu}^{(n-1)}(x + \hat{\mu} + \hat{\nu})U_{\nu}^{(n-1)\dagger}(x + 2\hat{\mu}).$$

(15)

The links $U_{\mu}^{(n)}(x)$ are twice as long as those at the lower blocking level $n - 1$. We shall refer to this as “link blocking” in the following. It seems natural to design a similar procedure for the scalar fields $\phi(x)$. A “site-blocked” scalar field $\phi^{(n)}(x)$ at blocking level $n$ can be constructed iteratively from a field at a given lattice site and its covariant connection with the four nearest neighbours,

$$\phi^{(n)}(x) = \phi^{(n-1)}(x) + \sum_{\mu=1}^{2} [U_{\mu}^{(n-1)}(x)\phi^{(n-1)}(x + \hat{\mu}) + U_{\mu}^{(n-1)\dagger}(x - \hat{\mu})\phi^{(n-1)}(x - \hat{\mu})].$$

(16)

Clearly both blocking procedures can be iterated, thereby quickly increasing the number of links and sites contributing to a given composite variable.

Non-local blocked operators are now constructed from the basic ones (14) by replacing the scalar and link variables with composite ones at a desired blocking level. Note that the blocking steps are constructed in a way which preserves the gauge invariance of
the original operators. The basic operators $R$ and $P$ contain only site and link variables, respectively. By applying the corresponding blocking procedure to these operators, we get $N$ operators of different spatial extension, where $N$ denotes the maximal blocking level. These we write as $R^{(n)}(x)$ and $P^{(n)}$, with $n = 0, \ldots, N$. For the operators $L$ and $V$ both site and link blocking can be applied, so from each of them we construct a set of $N \times N$ operators, denoted by $L^{(nm)}_{\mu}(x)$ and $V^{(nm)}_{\mu}(x)$, $n, m = 0, \ldots, N$, where the first upper index stands for site and the second for link blocking.

3.3 Cross correlations

The blocking procedure described in the previous subsection is designed to yield an optimal operator for a given set of quantum numbers. In an attempt to further separate the excitations from the ground state we can also utilise the information contained in our non-optimal operators by considering cross correlations between different operators in the same channel.

For a given set of quantum numbers we construct a set of, say, $N$ lattice operators, $\phi_i : i = 1, \ldots, N$, with those quantum numbers. We normalise these operators so that $\langle \phi_i \phi_i \rangle = 1$, and we impose the same normalisation on all the operators we discuss below. To find the energy of the lightest state we use a variational criterion. That is to say, we find the linear combination of the $\phi_i$ that maximises

$$\langle \phi_1 \phi \rangle = \langle \phi e^{-H} \phi \rangle .$$

Call this operator $\Phi_1$. In the limit where the basis $\{\phi_i\}$ is complete, this procedure becomes exact. That is to say, if the lightest state is $|1\rangle$ and the corresponding energy is $E_1$, then

$$\Phi_1|vac\rangle = |1\rangle ,$$

and

$$e^{-aE_1} = \langle \Phi_1 \phi \rangle .$$

We can find higher excited states by a simple extension of this procedure. Let the first excitation be $|2\rangle$ and let the corresponding energy be $E_2$. We consider the subspace $\{\phi_i\}'$ of $\{\phi_i\}$ that is orthogonal to $\Phi_1$, i.e., such that $\langle \Phi_1 \phi \rangle = 0$. We apply the same variational criterion as above, but restricted to this subspace. This gives us an operator $\Phi_2$. In the limit where our original basis becomes complete, we have

$$\Phi_2|vac\rangle = |2\rangle ,$$
We can continue this procedure obtaining operators $\Phi_3, \Phi_4, \ldots$ from which we can obtain the energies of higher excited states.

In our case our basis is finite, and we can obtain at most $N$ operators $\Phi$. With such a limited basis eq. (19) provides at best an estimate for $aE_1$. We improve upon this estimate by calculating the correlation function $\langle \Phi_1^\dagger(t)\Phi_1(0) \rangle$ for all $t$. If we define an effective energy by

$$e^{-aE_{\text{eff}}(t)} = \langle \Phi_1^\dagger(t)\Phi_1(0) \rangle,$$

(22)

then we know that $E_{\text{eff}}(t)$ will approach $E_1$ from above as $t$ increases. The more effective our variational procedure, the smaller the value of $t$ at which this occurs (for a basis that is complete we would find $E_{\text{eff}}(t) = E_1$ for all $t$). So we can estimate $aE_1$ from the value of $aE_{\text{eff}}(t)$ on its ‘plateau’. In practice, what we actually do is to fit the correlation function to an exponential in $t$ for $t$ large enough (as described below). From the exponent we then obtain our estimate for $aE_1$. From the coefficient of the exponential we obtain the normalised projection of our operator onto the lightest state, i.e. $|\langle 1 | \Phi_1 | \text{vac} \rangle|^2$. If we have a good basis of operators then this projection will be close to one. In practice this is always the case in the scalar channel, where the projection is often consistent with unity. In the vector channel the projection tends to be $\sim 0.8$.

We follow the same procedure for excited states, extracting $aE_i$ by fitting an exponential to $\langle \Phi_i(t)\Phi_i(0) \rangle$ for large enough $t$. One must be more careful here than with the ground state because, with a finite basis, the operator $\Phi_i$ will have some projection onto all states, not just onto $|i\rangle$. So as $t \to \infty$ its correlation function will ultimately vary as $\sim \exp(-E_1 t)$ and not as $\sim \exp(-E_i t)$. Thus, by fitting an exponential at larger $t$ we may underestimate the value of $aE_i$. In practice this is not a problem where the operators are very good. For example if the projection of $\Phi_1$ onto the lightest state is $1 - \epsilon$, then the projection of $\Phi_2$ onto this lightest state is $\leq \epsilon$. If $\epsilon$ is as small as it is in our calculations, then this potential contamination of $E_2$ by $E_1$ is insignificant. The same argument can be used for higher excited states. In general, where we quote a mass without qualifications, we are confident, by examining the relevant projections, that our mass estimate is not significantly contaminated by admixtures of any of the lighter states that we list.

In practice our lattice is finite and so in the above we replace $e^{-Et}$ by $e^{-Et} + e^{-E(T-t)}$ where $T$ is the length of the lattice in the $t$-direction.

The procedure we follow to obtain the $\Phi_i$ is standard [24]. Define the $N \times N$ corre-
lation matrix $C(t)$ by

$$C_{ij}(t) = \langle \phi_i^\dagger(t) \phi_j(0) \rangle .$$  \hfill (23)$$

Let the eigenvectors of the matrix $C^{-1}(0)C(a)$ be $v^i; i = 1, \ldots, N$. Then

$$\Phi_i = c_i \sum_{k=1}^{N} v^i_k \phi_k \equiv \sum_{k=1}^{N} a_{ik} \phi_k ,$$  \hfill (24)$$

where the constant $c_i$ is chosen so that $\Phi_i$ is normalised to unity.

We would like to emphasise that there are many possible variations on the above variational procedure. For example we could apply it to $t = 2a$ rather than to $t = a$. As a check we have performed such an alternative calculation. We further remark that, in practice, the best of our original $\phi_i$ operators is already so good that the calculation of the ground state in each channel is not greatly improved by going to the $\Phi_i$ operators. It is if we wish to obtain the excited states that this analysis becomes indispensable.

In our actual calculations, we typically compute a $9 \times 9$ matrix of correlators in the $0^{++}$ channel, which consists of the three operators $R$, $P$ and $L$, each taken at three different blocking levels. In the $1^{--}$ channel, where only operators of type $V$ are known, three different blocking levels are used to compute a $3 \times 3$ correlation matrix.

### 3.4 Fits and error analysis

All our mass estimates are obtained from measured correlation functions of operators $\Phi_i$ in the diagonalised basis defined in the preceding subsection. The ansatz we use for the asymptotic behaviour of the correlation function on a finite lattice, for large $T$, is

$$\tilde{C}_i(t) \equiv \langle \Phi_i^\dagger(t) \Phi_i(0) \rangle = A_i \left( e^{-aM_i t} + e^{-aM_i(T-t)} \right) ,$$  \hfill (25)$$

where $i$ labels the operator, and $T$ denotes the extent of the lattice in the time direction. This expression would be exact for all $t$ if the basis of operators was complete. To monitor deviations from this behaviour we define an effective mass according to

$$aM_{\text{eff}}(t) = \text{arcosh} \left\{ \frac{C(t+1) + C(t-1)}{2C(t)} \right\} ,$$  \hfill (26)$$

where $C(t)$ denotes either $\tilde{C}_i(t)$ or $C_{ii}(t)$. As one readily sees, this definition has the desired property that $aM_{\text{eff}}(t) = aM_i$ for those $t$ where $C(t)$ is accurately given by (25).

Estimates for the masses $aM_i$ and amplitudes $A_i$ are obtained from correlated fits of $\tilde{C}_i(t)$ to eq. (25) over a finite interval $[t_1, t_2]$. Our choice of the fitting interval is guided by
the plateaux observed in the effective masses (26), and is constrained by the requirement that a reasonable $\chi^2$/dof should be obtained.

Our individual measurements of $\tilde{C}_i(t)$ are accumulated in bins of typically 500 measurements each. Statistical errors on $aM_i$ and $A_i$ are obtained from a jackknife analysis of the fits to the average of $\tilde{C}_i(t)$ in each jackknifed bin.

It has been known for some time that correlated fits may amplify hidden systematic errors in the data [25]. Therefore we repeated all our fits using an uncorrelated covariance matrix. The difference between the results obtained using either correlated or uncorrelated fits are quoted as a (symmetric) systematic error on our mass estimates. In most cases we found the systematic error arising from this procedure much smaller than the statistical error. For the final extrapolation of masses and mass ratios to the continuum limit, statistical and systematic errors are added in quadrature before the extrapolation is performed.

4 Results

In this section, we present our main results. We start with a discussion of the effects of the blocking procedure and the diagonalisation of operators in subsections 4.1 and 4.2, using our data at $\beta_G = 12$ on the largest lattices we investigated in the confinement phase ($40^3$), and in the Higgs phase ($20^3$). The main results on the spectrum, which were obtained using diagonalised operators at all three values of $\beta_G$, are presented in subsections 4.3 and 4.4. Finally, in subsection 4.6 we give our mass estimates extrapolated to the continuum limit.

4.1 The effects of the blocking procedure

A priori nothing is known about the projection properties of the individual operators in our set $\{\phi_i\}$. The candidates with the best projection onto the lowest states have to be determined from actual simulations. A criterion to judge the performance of an operator is its effective mass at time separation one, where the lowest value indicates the least contamination from excited states. Figure 1 illustrates the effect of the blocking procedure for the purely scalar/gauge operators $R/P$ (cf. eqs. (10)) in the Higgs phase and the confinement phase, respectively. In the Higgs phase, nothing is gained by blocking the $R$ operator, while in the confinement phase four iterations are necessary before it reaches its optimal projection. For the plaquette $P$, three blocking steps are required
to get to the minimal effective mass in either phase, but the improvement is far more pronounced in the confinement phase.

Figure 2 shows effective mass plots for the operators with the best projection of each basic type in the $0^{++}$ channel, again for the Higgs and confinement phases, respectively. In both regimes the operator with the best projection onto the ground state is of the type $R$, with nearly 100% overlap at the optimal blocking level. The ground state could in principle also be extracted from the correlation function of the best candidate of the type $L$ at large time separations. However, its projection is much worse, five to six lattice spacings are needed until excitations have died away, and a mass calculated from this correlation function would be much less accurate. Of particular interest is the behaviour of the plaquette correlations. While they are dominated by noise in the Higgs phase, they suggest a separate plateau in the effective mass plot in the confinement phase. Up to those time separations for which we have a good signal, there seems to be no tendency for this operator to mix with the other $0^{++}$ operators. We shall return to this observation below.

Finally, Figure 3 shows the result of the blocking procedure on the effective masses of the vector boson. In the Higgs phase blocking slightly improves the projection of the operator $V$, but it is not difficult to extract a mass also from the unblocked one. In the confinement phase the situation is rather different. The unblocked operator does not give
any signal beyond noise, and the figure displays how even for the best blocked candidates excitations die out only very slowly.

In particular, the figure illustrates how one might easily extract too large a mass for the vector boson, if one only used a non-optimal operator such as the one symbolised by the triangles. The effective masses produced by the different operators do not seem to merge at a common ground state up to the distances to which we can follow the signal. An indication that we do really see the ground state is the fact that further iterations in the blocking procedures for either links or scalars, beyond the level of the operator $V^{55}$, again result in a worse projection. Figure 3 also displays a nice side effect of the blocking procedure. Since the improved operators have a better projection onto low mass states the corresponding correlation functions fall less steeply than those of the unblocked ones, hence the signal-to-noise ratio is improved, leading to considerably smaller statistical errors.

In summary, we find that blocking has little or no effect in the Higgs phase, where the original local operators exhibit a rather good projection onto the ground state in each channel. In the confinement phase, on the contrary, blocking turns out to be necessary in order to obtain any useful signal at all. This is particularly pronounced in the $1^{--}$ channel. It was also demonstrated by using a large set of operators, that in the confinement phase for time separations up to ten timeslices one is typically still rather far away from
the asymptotic region where all excited states have died out. This implies that the latter are rather light compared to the ones in the Higgs phase, as we shall see more explicitly in the next subsections. This is precisely what spoils an easy mass measurement in the confinement phase. We have increased blocking levels on each kind of operator until we could explicitly identify the ones with the best projection onto the lowest states. Thus we can be sure that we have found the optimal operators that can be constructed from (10) by means of the blocking technique described in subsection 3.2.

4.2 Correlations of eigenstates

Now we discuss the correlations of the eigenstates of the matrix correlators $C_{ij}(t)$ introduced in subsection 3.3. Consider first the 1−− channel. Employing a basis composed of the three operators used in Figure 3 we obtain, after diagonalisation, the three sets of effective $W$ masses shown in Figure 4 for the Higgs and confinement phases. Comparing with Figure 3, we conclude that this procedure has only slightly improved the projection onto the ground state. However, it has clearly separated off the excitations. Even though it is not always possible to identify extended plateaux for these excited states, one can nevertheless conclude from the comparison between Higgs and confinement phases that the gap between the ground state and the excitations is much larger in the former. Clearly, in order to obtain more quantitative information about the exci-
tations, one would need to choose a larger basis of operators. We shall not pursue this possibility here, since in this channel our main interest is in the ground state.

Next, consider the three lowest states in the $0^{++}$ channel shown in Figure 5. In the Higgs region, the situation is rather simple with an isolated Higgs ground state and a large gap to excitations. Because the excited states are much higher in mass, their correlation functions fall rapidly, one loses the signal after a few timeslices, and it is difficult to identify well-defined excited states. Here one would also need to increase the basis of operators and to reduce the lattice spacing, in order to improve the situation. In contrast, in the confinement phase the diagonalisation has isolated three distinct states which were mixed previously. In Table 1 the coefficients $a_{ij}$ (cf. eq. (24)) with which the individual operators contribute to various eigenstates are shown. The labelling is such that $\Phi_1$ denotes our best operator for the ground state, $\Phi_2$ the one for the first excited state, etc.

According to this analysis the ground state in the confinement phase consists predominantly of $R$- and $L$-contributions. The next state has contributions from all types of operators, with a dominance by $R$ and $L$. As in the spin-one case, the gap between the lowest and first excited states is much smaller than in the Higgs phase. The separate plateau of the plaquette operators survives diagonalisation, representing a rather definite state. Table 1 shows that the plaquette operators indeed have practically no overlap with
the ground state $\Phi_1$. Conversely, the other $0^{++}$ operators do not contribute to the state $\Phi_3$, which thus appears to be an object composed exclusively of gauge degrees of freedom and very little mixing with operators containing scalars. In the pure gauge theory this object would correspond to a glueball. It seems natural to interpret this state in the Higgs model as a $0^{++}$ “$W$-ball”, composed almost entirely of gauge bosons. As shown in Table I, the basis of eigenstates contains two more states $\Phi_6, \Phi_9$ with almost exclusively plaquette contributions, thus appearing to be excitations of the state $\Phi_3$. Details of the spectrum of excited states will be presented in subsection 4.4.

4.3 Mass spectrum and finite-size analysis

Now we proceed to presenting our complete set of results for the spectrum of the SU(2) Higgs model in three dimensions. We perform an analysis of finite-size effects and finally extrapolate our results to the continuum limit.

In Tables 2 and 3 we summarise the results on all lattices and for all values of $\beta_G, \beta_H$ used in our calculation. All masses quoted in this section have been obtained by fitting the correlation functions to the functional form in eq. (25). As has been demonstrated in subsection 4.2, our signals for the lowest states show quite pronounced plateaux. The situation is more difficult for the excited states. In Tables 2 and 3 we only record masses
Table 1: Coefficients $a_{ij}$ as defined in eq. (24) of the operators used in the simulation for the three lowest $0^{++}$ states in the confinement phase ($\beta_G = 12$, $\beta_H = 0.3411$, $L^2 \cdot T = 40^3$). In the header, we also introduce the labelling for scalar states used below.

for which we could identify a plateau of at least three timeslices extension in an effective mass plot. Those cases where the statistical errors of the correlation function were large, or where the overlap $a_{ij}$ of the diagonalised operators onto the desired state was small, are marked by an asterisk.

Table 2: Mass estimates in the $0^{++}$ and in the $1^{--}$ channels in the Higgs phase. The first error is statistical, the second is an estimate of systematic effects.

We investigate finite volume effects in detail for $\beta_G = 12$. Numerically, the infinite volume limit is reached when the change in a mass with increasing lattice size is smaller than the statistical errors. In order to avoid additional finite size studies for the smaller
values of $\beta_G$, we take the required spatial length corresponding to the large volume limit of the lattice at $\beta_G = 12$ in units of the Higgs mass, $M_{H_1}L$, and scale it down to the lower $\beta_G$-values. This way we ensure that the simulations at the smaller values of $\beta_G$ are done in the same physical volume as for $\beta_G = 12$. Strictly speaking, this procedure is only valid if the considered range of values for $\beta_G$ is in the scaling region, an assumption which turns out to be satisfied rather well, as we shall see a posteriori. After infinite-volume masses have been determined for different $\beta_G$-values they can be extrapolated to $\beta_G \to \infty$.

The large-volume limit of the Higgs phase is rather easy to reach. Table 3 gives the Higgs and $W$ boson masses in lattice units as measured on lattices with spatial lengths $L = 16, L = 20$ at $\beta_G = 12$. It is seen that for both states the masses on the two lattices are compatible within the statistical errors.

Again the situation is much more difficult in the confinement phase, as is illustrated in Figure 6. There are strong finite size effects for the lightest scalar state, which are only under control for lattices larger than $L = 32 (1/L = 0.031)$. We estimate that the infinite-volume limit for the scalar ground state in the confinement phase is reached for $aM_{H_1}L \approx 10$. The vector boson mass, on the other hand, shows only little dependence on the volume. The $W$-ball is just getting close to the large volume limit on a lattice with

| $\beta_G$ | $\beta_H$ | $L^2 \cdot T$ | $aM_{H_1}$ | $aM_{H_2}$ | $aM_{H_3}$ | $aM_W$ |
|-----------|-----------|---------------|-------------|-------------|-------------|--------|
| 12        | 0.3411    | $40^3$        | 0.2903(15)(12) | 0.514(4)(1) | 0.563(5)(2) | 0.447(8)(3) |
| 32        |           | $28.85(25)(15)$ | 0.440(9)(4) | 0.527(13)(13) | 0.442(4)(5) |
| 26        |           | $28.13(22)(5)$ | 0.334(10)(4) | 0.544(12)(2) | 0.443(9)(2) |
| 20        |           | $22.66(6)$ | 0.3068(3) | 0.509(11)(10) | 0.423(10)(4) |
| 16         |           | $17.39(32)(19)$ | 0.247(12)(5) | 0.540(7)(3) | 0.422(8)(4) |
| 10         |           | $12.12(4)(1)$ | 0.136(8)(1)$^*$ |  | 0.469(11)(1) |

Table 3: Mass estimates in the $0^{++}$ and in the $1^{--}$ channels in the symmetric phase. The first error is statistical, the second is an estimate of systematic effects.
$L = 40 \ (1/L = 0.025)$. The most pronounced finite size effects of all states investigated are displayed by the intermediate state $\Phi_2$. We conclude that the ground state masses in both channels have reached the infinite volume limit, while for the excitations it would be desirable to go to larger lattices. In order to get an estimate for the situation in larger volumes it is instructive to consider $\beta_G = 7$, where a given lattice size (here we consider $L = 30$) corresponds to a larger volume in physical units than at $\beta_G = 12$. Since the mass of the lowest state $\Phi_1$ is free of finite size effects at $L = 40$ the ratio of this mass at the two $\beta_G$-values may be used to scale lattice size and masses according to

$$La(\beta_G = 12) = 30a(\beta_G = 7) \frac{aM_1(\beta_G = 7, L = 30)}{aM_1(\beta_G = 12, L = 40)} = 52.9a(\beta_G = 12),$$

$$aM_i(\beta_G = 12, L = 52.9) = aM_i(\beta_G = 7, L = 30) \frac{aM_i(\beta_G = 12, L = 40)}{aM_1(\beta_G = 7, L = 30)}.$$

(27)

The result of this scaling is shown as the open data points in Figure 6. Now the state $\Phi_2$ also seems to have approached the large volume limit. However, since it is very close to the $W$-ball at large volumes one expects some mixing between these states. Comparing our data for the coefficients $a_{ij}$ from the lattices with $L \geq 26$ we find growing admixtures of plaquette operators to $\Phi_2$ with increasing volume, while the composition of the state $\Phi_3$ remains rather unchanged by the approaching $\Phi_2$. In order to be absolutely sure that $\Phi_2$ really represents an isolated state further investigations are required on larger lattices or at different parameter values, where $\Phi_2$ and $\Phi_3$ might be more clearly separated.

In ref. [9] it was stated that at $\beta_G = 12$ the results for the lowest $0^{++}$ state were practically indistinguishable on lattices of size $30^3$ and $20^3$. In contrast to this, we find a rather strong dependence of $aM_{H1}$ in this range of lattice sizes. In particular, our results on $32^3$ and $20^3$ are clearly incompatible. We ascribe this to a presumably incomplete isolation of the ground state in ref. [9]. In addition, we observe that on the $20^3$ lattice our vector boson mass in the confinement phase is about 35%, and the scalar ground state about 25% lower than those reported in [9].

We conclude that the construction of improved operators is an indispensable tool in the study of the mass spectrum of our model in the confinement phase. In view of this, it would be very interesting to apply this technique to mass calculations in the vicinity of the phase transition, and at higher values of $\lambda_3/g_3^2$ corresponding to more realistic zero temperature Higgs masses.

The finite volume effects that we have analysed so far are to do with the size of the spatial volume. There are, however, additional finite volume effects which have to do with the finite extent in time of the lattice. Of course the eigenvalues of the transfer
Figure 6: Finite volume study for the confinement phase at $\beta_G = 12$. Squares, circles and diamonds represent the three lowest $0^{++}$ states, whereas triangles denote the lowest $1^{--}$ state. Open symbols indicate the data extrapolated from $\beta_G = 7$ according to eq. (27).
matrix and (lattice) Hamiltonian, $H$, are not altered by varying $T$. However, what may change is the relationship between those eigenvalues and the exponents in the decays of our calculated correlation functions. For example, the fact that the values of our masses are with respect to the vacuum energy arises because our expectation values contain the partition function, $Z = \text{Tr}\{e^{-HT}\}$, as a normalisation factor and this factor will normally be dominated by the vacuum energy. If $T$ is sufficiently small, however, then $Z$ may receive significant contributions from excited states and the masses we calculate from our correlation functions may be shifted by the corresponding shift in the ‘effective’ vacuum energy. Exactly what the effect of this is going to be is a complicated matter since, on the one hand, similar contributions occur in the numerator of the correlation function and this may lead to a partial cancellation of this correction. On the other hand our scalar masses involve the subtraction of a vacuum expectation value of the operator, and this will also be affected. Nonetheless, although we cannot easily estimate where such effects may be important, we note that since the leading correction to $Z$ is $O(e^{-aM_{H_1}T})$, we need to be concerned once $aM_{H_1}$ is small.

To obtain a quantitative control over this potential problem, we have taken our $10^2$ spatial lattice at $\beta_G = 12$ (because it has the smallest value of $aM_{H_1}$ and we have repeated the calculations, with the same basis of operators (which, unusually, happened to be 6 in this case), for $T = 20$ and $T = 12$. We have extracted masses in the same way as on the $T = 30$ lattice and have found $aM_{H_1} = 0.125(3)$ for $T = 20$ and $aM_{H_1} = 0.111(3)$ for $T = 12$. Thus there are no finite-$T$ effects within these small errors down to $T = 20$ and even at $T = 12$, where $e^{-aM_{H_1}T} \simeq 0.24$, the shift in the extracted mass is only $\sim 10\%$. At $T = 20$, $aM_{H_1}T \sim 2.5$ and this gives us a benchmark value for judging when we should be safe from such corrections. We find no significant effects, within errors, for our other states. Of course, these effects may be somewhat different in the different phases, and, to the extent that scaling is violated, at different $\beta_G$. However the volumes that we use for extracting our final masses have values of $aM_{H_1}T$ so much larger than the above benchmark value that we saw no reason to repeat this analysis in those other cases.

### 4.4 Higher excitations

The diagonalisation procedure also enables us to compute masses of more highly excited states, which were not mentioned in Table 3. These, however, are determined with much less accuracy, since the variational basis for these states is smaller. We nevertheless find it instructive to give a qualitative discussion of that part of the spectrum. Since the gap between the ground state and excited states in the Higgs phase is rather large for both
the Higgs and the vector boson (as can be seen, e.g. in Figures 4 and 5), we restrict our discussion to the symmetric phase.

In Table 4 we present the results for those states where we felt confident enough to quote a mass estimate. For the $W$-ball, the correlation functions of the first and second excited states were those where plaquette contributions were clearly dominant (see e.g. Table 1 for the overlaps of operators $\Phi_6$ and $\Phi_9$ at $\beta_G = 12$, $L^2 \cdot T = 40^3$).

It is instructive to compare the mass estimates for the $W$-ball and its excitations with the glueball spectrum in the pure gauge theory. In Table 5 the masses in lattice units of these states are compared with those of the corresponding $0^{++}$ glueball and its first two excitations at $\beta_G = 12$ in three-dimensional pure $SU(2)$ gauge theory [13, 26]. The striking agreement between these states in the two theories indicates a remarkably complete decoupling of the pure gauge sector from the Higgs part in the $SU(2)$ Higgs model.

| $\beta_G$ | $\beta_H$ | $L^2 \cdot T$ | $aM_W$   | $aM_{H3}^*$ | $aM_{H3}^{**}$ |
|-----------|-----------|--------------|----------|-------------|--------------|
| 12        | 0.3411    | $40^3$       | 0.682(24)| 0.840(24)   | 1.02(2)      |
|           | 32        | 0.622(19)    | 0.804(10)| 0.983(16)   |
|           | 26        | 0.636(23)    | 0.773(12)| 0.974(30)   |
| 9         | 0.3438    | $26^3$       | 0.854(34)|             |
| 7         | 0.3467    | $30^3$       | 1.193(17)|             |
|           | 20        | 1.219(21)    |          |             |

Table 4: Mass estimates for the excitations of the vector boson and the $W$-ball in the symmetric phase.

|          | $aM$    | $aM^*$   | $aM^{**}$ |
|----------|---------|----------|-----------|
| $SU(2)$ pure gauge | 0.563(5) | 0.805(8) | 0.982(14) |
| $SU(2)$ Higgs        | 0.563(5) | 0.840(24)| 1.02(2)   |

Table 5: Comparison of $0^{++}$ glueball and $W$-ball and their first two excitations at $\beta_G = 12$. 
The existence of a separate $W$-ball which does not mix at all with other $0^{++}$ states is rather unexpected in view of the coupling between scalar and link variables in the tree-level action, and this suppression of mixing must be of dynamical origin. It would be interesting to see whether this isolation of the pure gauge sector persists also in the $1^{--}$ channel.

4.5 How certain is the ground state?

Measuring and diagonalising the correlation matrices provided us with valuable insight into the excitation spectrum of the theory. What can we say about the existence of very light states? In the $0^{++}$ channel we have a nearly complete projection onto the lowest state, and in the $1^{--}$ case the projection onto the lowest state looks quite acceptable as well. In the last section it was demonstrated that our operator basis includes the optimal operators which can be obtained from the operator types (10) by means of the blocking techniques (15),(16). Although we do find the lowest $1^{--}$ state to be about 30–40% smaller than that quoted in [9], our lowest masses are still much larger than the Higgs and vector boson masses predicted by the gap equations [4].

Of course, we cannot strictly rule out the existence of a lighter state which may only show up at distances larger than those up to which we have a good signal. If there were such states, however, they would have to have a rather poor overlap with our operator basis. This can be made more quantitative as follows. All effective masses presented so far were obtained from (26) under the assumption that the corresponding correlation function is dominated by a single lowest state. Let us now assume that there is one lighter state in each channel such that our measured correlation functions would correspond to a superposition of two states,

$$\tilde{C}_i(t) \simeq A_i \left( e^{-aM_i t} + e^{-aM_i (T-t)} \right) + A_0 \left( e^{-am t} + e^{-am (T-t)} \right),$$

(28)

where $am$ corresponds to the supposed very light mass and $aM_i$ is of the size of the mass we extracted assuming a single exponential correlation function as in eq. (25). Fixing the assumed light mass $am$ to values motivated by the study in [4], we try to fit our data for the low states by the correlation function (28). Some results are presented in Table 6.

The amplitude $A_0$ is consistent with zero in all cases. Adding two standard deviations to $A_0$ we get a bound at 90% CL for the ratios of the amplitudes. The square root of this ratio, which is given in the last column of the table, may serve as an estimate of the maximal matrix element that a lower state has with the corresponding eigenstate of our
\[ a_m = 0.1 \]

| \( aM_i \) | \( A_i \) | \( A_0 \) | \( \sqrt{(A_0 + 2\sigma)/A_i} \) |
|-----|-----|-----|-----------------|
| \( \Phi_1 \) | 0.287(4) | 1.810(18) | -0.016(17) | 0.10 |
| \( \Phi_2 \) | 0.515(8) | 0.341(2) | 0.0004(17) | 0.11 |
| \( \Phi_3 \) | 0.567(8) | 1.031(8) | 0.003(5) | 0.11 |
| \( \Phi_{W1} \) | 0.449(14) | 0.881(52) | 0.0004(25) | 0.078 |

| \( aM_i \) | \( A_i \) | \( A_0 \) | \( \sqrt{(A_0 + 2\sigma)/A_i} \) |
|-----|-----|-----|-----------------|
| \( \Phi_1 \) | 0.288(3) | 1.805(12) | -0.0095(97) | 0.074 |
| \( \Phi_2 \) | 0.515(7) | 0.3411(3) | 0.0003(12) | 0.089 |
| \( \Phi_3 \) | 0.566(7) | 1.031(8) | 0.0020(34) | 0.092 |
| \( \Phi_{W1} \) | 0.448(12) | 0.880(50) | 0.0002(15) | 0.06 |

Table 6: Results of double exponential fits with a fixed assumed light mass to the lowest states in the confinement phase (\( \beta_G = 12, \beta_H = 0.3411, L^2 \cdot T = 40^3 \)).

basis. This suggests that it is rather unlikely that significantly lighter states have been missed.

A potential source of systematic errors in the reported values of ground state masses is the residual contamination of the correlation function by higher excitations. Of course, the blocking procedure in conjunction with our variational technique is designed to optimise the projection onto the ground state. In the vector channel, however, the plateaux set in at larger values of \( t \), thus showing that the ground state does not dominate the correlation function at very early timeslices.

In order to quantify this systematic error, we performed a double exponential fit similarly to eq. (28). Here, however, the mass \( a_m \) was fixed to the mass estimate for the first excited state in either the scalar or the vector channel.

Extending the fitting interval to earlier timeslices, we found that the double exponential fit does not change at all the mass of the lowest \( 0^{++} \) state, thus confirming that a nearly perfect projection has been achieved. In the vector channel, the double exponential fit gave slightly lower results for the mass of the ground state. We found that the mass decreased by about 5%, but that none of the mass estimates using a double exponential fit were incompatible with the result using a single exponential.
We conclude that practically all contamination from higher states has been eliminated in the scalar channel, while higher excitations might lead to an uncertainty of about 5% in the mass of the vector channel.

4.6 The continuum limit

Our next task is to extrapolate the lattice spectrum to the continuum by taking the limit \( \beta_G \to \infty \). The continuum limit is performed only for the lowest states where our results are accurate enough. We have taken our results at all three \( \beta_G \)-values at the largest respective lattice sizes, which, as Figure 6 shows, have reached, or are close to, the infinite volume limit.

Figure 7: Continuum limit in the Higgs (left) and confinement (right) regions. Squares, circles and diamonds represent the three lowest 0\(^{++}\) states, whereas triangles denote the lowest 1\(^{--}\) state. Open symbols indicate the data extrapolated to \( 1/\beta_G = 0 \).

In the confinement phase, the dimensionless combinations \( aM \beta_G/4 = M/g^2_3 \) were extrapolated linearly in \( 1/\beta_G \) for the three lowest 0\(^{++}\) and the lowest 1\(^{--}\) states. In the Higgs phase, only the lowest scalar and vector states were extrapolated, since higher excited states could not be clearly identified at all three \( \beta_G \)-values. In Figure 7 the data at the three \( \beta_G \)-values are shown together with the extrapolated results. In addition, we extrapolated the dimensionless ratio \( aM_{H1}/aM_W \) linearly in \( 1/\beta_G \) in both phases. Table 7 shows a summary of the continuum values of the individual masses and the mass ratio for both phases.
Table 7: Continuum values of the three lowest scalar and the lowest vector states, as well as the ratio \( M_{H1}/M_W \) in the Higgs and confinement phases. Since the extrapolation of \( M_{H2}/g_3^2 \) was performed using only the data at \( \beta_G = 12, 7 \), we cannot quote \( \chi^2/\text{dof} \), the error is a subjective estimate.

| \( \lambda_3/g_3^2 = 0.0239 \) | \( M_{H1}/g_3^2 \) | \( M_{H2}/g_3^2 \) | \( M_{H3}/g_3^2 \) | \( M_W/g_3^2 \) | \( M_{H1}/M_W \) |
|---------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Higgs, \( \mu_3^2/g_3^4 = -0.020 \) | mass             | 0.547(12)       | –               | 1.91(3)         | 0.282(8)        |
|                           | \( \chi^2/\text{dof} \) | 0.80            | –               | 1.25            | 1.70            |
| Confinement, \( \mu_3^2/g_3^4 = 0.089 \) | mass             | 0.839(15)       | 1.47(4)         | 1.60(4)         | 1.27(6)         |
|                           | \( \chi^2/\text{dof} \) | 0.74            | 0.42            | 0.06            | 0.58            |

5 Summary and Conclusions

We have presented results for the mass spectrum of the continuum SU(2) Higgs model in three dimensions at selected points in the symmetric and broken phases of the model. In order to get reliable mass estimates, the use of improved lattice operators turned out to be crucial. This is of particular importance for the investigation of the possibility of very low-lying states of the kind predicted by the analytic approach presented in [4].

Using our particular blocking procedure, we were able to increase the projection onto the ground state dramatically. In most cases in the scalar channel, we achieved projections of essentially 100%, whereas in the vector channel values for the overlap ranged between 75–95%. Undoubtedly, with a more refined smearing or blocking procedure, one could improve the signal for the ground state in the \( 1^{--} \) channel even further. We wish to emphasise the importance of a high projection onto the desired state, since otherwise the possible misidentification of plateaux in the effective masses is a source of large systematic errors which are difficult to quantify. Due to our use of the blocking procedure, we observe quantitative differences in the masses of the lightest scalar and vector states on specific lattices in the symmetric phase compared to ref. [9]. Furthermore, we observe strong finite-size effects in the ground state of the \( 0^{++} \) channel in the symmetric phase.

Within the framework of our calculation we find no evidence for very small masses in the scalar and vector channels in the symmetric phase, as predicted by [4]. We wish to point out, however, that we considered correlations of gauge-invariant composite operators, whereas the correlators of elementary fields used in the analytic approach in [4] are gauge-dependent. There are indications from the numerical work reported in [27] that
correlations of these gauge-dependent operators indeed exhibit a signal corresponding to a very low effective mass of the gauge boson. This needs to be better understood.

Our computation of masses of excited states confirms the existence of a dense spectrum of states in the confinement phase of the model. This appears to be consistent with the picture that bound states constitute the particle content of the symmetric phase. A surprising result of our calculation is the existence of states that are composed almost entirely out of gauge degrees of freedom. This “W-ball” and its excitations are almost identical in mass to their gluonic counterparts in the pure SU(2) gauge theory. We are thus led to conclude that the pure gauge sector in the SU(2) Higgs model approximately decouples from the scalar degrees of freedom, a phenomenon which must be of dynamical origin.

We have shown in this paper that by using various refined calculational tools in lattice simulations of the SU(2) Higgs model, detailed information of the mass spectrum in the symmetric phase can be gained. This is important for the development of effective theories of the symmetric phase, which will serve to analyse the nature of the phase transition at very large Higgs masses [14, 28], and to describe the thermodynamics of the electroweak plasma in the high temperature symmetric phase in the early universe.

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