COMPACT CLOSED CATEGORIES AND Γ-CATEGORIES
(with an appendix by Andrés Joyal)

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ABSTRACT: In this paper we extend the notion of compact closed categories to coherently commutative monoidal categories. We construct a model category of (permutative) compact closed categories and a model category of coherently compact closed categories which are manifestations of the aforementioned extension. We show that the Segal’s nerve functor is a Quillen equivalence between the two model categories. The construction of a model category of coherently compact closed categories leads to a proof of the one dimensional cobordism hypothesis based on a purely homotopical algebra argument.

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1. Introduction

An abelian group is a commutative monoid having the special property that each element has a two-sided inverse. A Picard groupoid is a symmetric monoidal category having the special property that each object has a two-sided inverse up to isomorphism. A compact closed category is a symmetric monoidal category having the special property that each object has a left (and therefore a right) dual. The archetype example of compact closed categories is the category of finite dimensional vector spaces. Some other prominent examples of compact closed categories include the category of finitely generated projective modules over a commutative ring and the category of finite dimensional representations of a compact group. The category of abelian groups can be characterized as a reflective localization of the category of commutative monoids, namely the localization functor has a right adjoint which is the fully faithful inclusion of the full subcategory of abelian groups (local-objects). The generator of this localization is the inclusion map $i : \mathbb{N} \to \mathbb{Z}$. Similarly, the category of (permutative) Picard groupoids [GM97] is a reflective localization of the category of permutative (or strict symmetric monoidal) categories $\text{Perm}$. In this paper we obtain a category of (permutative) compact closed categories as a reflective localization of the category of permutative categories $\text{Perm}$. The generator of this localization is the inclusion $i^\otimes : F^\otimes(\ast) \to F^cc(\ast)$ of the free permutative category on one generator into the free compact closed category on one generator. The main objective of this paper is to compare the aforementioned category with a (model) category of coherently compact closed categories obtained by a left Bousfield localization of a (model) category of coherently commutative monoidal categories [Sha20] with respect to the ($\Gamma$-categories representation of) same generator $i^\otimes$. The main result of this paper may be regarded as a coherence theorem for (coherently) compact closed categories.

The classical cobordism hypothesis [BD95], [Lur09] informally states that the (framed) 1-Bordism category namely the category whose objects are framed 0-dimensional manifolds and morphisms are (diffeomorphism classes of) framed 1-dimensional manifolds with boundary, is a model for the free compact closed category on one generator. To a purely algebraic problem, the cobordism hypothesis provides an answer which is rooted in differential topology. In this paper we seek an answer to the same algebraic problem within homotopic algebra. This paper is a first in a series of papers aimed at developing a theory for compact closed ($\infty, n$)-categories and thereby writing a rigorous homotopic algebra based proof of the cobordism hypothesis. In this paper we construct a model category whose fibrant objects can be described as categories equipped with a coherently commutative multiplication structure wherein each object has a dual. This model category is intended to be a prototype for subsequently constructing model categories whose fibrant objects are models for $(n + k, n)$-categories equipped with a coherently commutative monoidal structure and which are fully dualizable.

The category of compact closed categories and symmetric monoidal functors has a subcategory, denoted $\text{ccPerm}$, whose objects are compact closed permutative categories namely those compact closed categories whose underlying category has a strict symmetric monoidal (or permutative) structure and whose morphisms are strict symmetric monoidal functors namely those symmetric monoidal functors that preserve the symmetric monoidal structure strictly. This category has more morphisms than another category of compact closed categories considered in [KL80] wherein the morphisms are those strict symmetric monoidal functors that preserve the dualizability data strictly. As indicated above, the category $\text{ccPerm}$ is a reflective subcategory of the category of permutative categories $\text{Perm}$. We use this fact to construct a model category structure on $\text{ccPerm}$ by transferring the natural model category structure on $\text{Perm}$, see [Sha20]. Normalized coherently commutative monoidal categories were introduced in the paper [Seg74] where they were...
called Γ-categories. These (normalized) objects have also been referred to in the literature as special Γ-categories. A model category \( \Gamma \text{Cat}^{\otimes} \) whose fibrant objects are coherently commutative monoidal categories was constructed in [Sha20]. Unlike a symmetric monoidal category, higher coherence data is specified as a part of the definition of a coherently commutative monoidal category. Moreover, in the latter, a tensor product of two objects is unique only up to a contractible space of choices. In this paper we extend the notion of compact closed categories to the more generalized setting of coherently commutative monoidal categories. More precisely, we define a notion of coherently commutative monoidal categories wherein each object has a dual. We call these objects coherently compact closed categories. We construct another model category structure on the (functor) category \( \Gamma \text{Cat}^{\otimes} \) whose fibrant objects are coherently compact closed categories. This model category, denoted \( \Gamma \text{Cat}^{cc} \), is a (left) Bousfield localization of the model category of coherently commutative monoidal categories \( \Gamma \text{Cat}^{\otimes} \). We go on to show that the Segal’s Nerve functor [Sha20] (restricted to \( \text{ccPerm} \)) is a left Quillen functor of a Quillen equivalence between the aforementioned model category structure on \( \text{ccPerm} \) and \( \Gamma \text{Cat}^{cc} \) and thereby justify our claim that \( \Gamma \text{Cat}^{cc} \) is a model category of coherently compact closed categories.

The Barrat-Priddy-Quillen theorem was reformulated in the language of Γ-spaces in [Seg74]. In the same paper, Segal constructed a functor from the category of Γ-spaces \( \Gamma S \) to the category of (connective) spectra. This functor maps the unit of the symmetric monoidal structure on \( \Gamma S \), namely the free Γ-space on one generator \( \Gamma^1 \), to the sphere spectrum. In section 2 of the same paper, Segal constructed a Γ-space, which he denoted \( B\Sigma \), which can also be described as (nerve of) the (categorical) Segal’s nerve [Sha20] of the (skeletal) permutative category of finite sets and bijections, denoted \( K(\Lambda) \). The reformulated theorem states that the spectrum associated to the Γ-space \( B\Sigma \) is stably equivalent to the sphere spectrum. In other words, the reformulation states that the Γ-space \( \Gamma^1 \) is equivalent to \( B\Sigma \) in the stable model category of Γ-spaces constructed in [Sch99]. A stronger version of this theorem called the (special) Barrat-Priddy-Quillen theorem appeared in the paper [dBM17]. This theorem states that the two Γ-spaces in context are also unstable equivalent i.e. they are equivalent in a model category of special Γ-spaces. In this paper we formulate and prove a homotopical algebra version of the one-dimensional cobordism hypothesis which is a reformulation of the classical cobordism hypothesis along the lines of the aforementioned Segal’s reformulation of the Barrat-Priddy-Quillen theorem. Informally, the reformulated theorem states that a Γ-categories representation of the 1-Bordism category \( \Gamma \text{Cob}^{\otimes}(1) \) is equivalent to the (representable) Γ-category \( \Gamma^1 \) in the model category \( \Gamma \text{Cat}^{cc} \). A proof of this theorem follows readily from the construction of the model category \( \Gamma \text{Cat}^{cc} \).

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2. Compact closed permutative categories

A compact closed category is a symmetric monoidal category wherein each object has the special property of having a left (and hence a right) dual. The category of all (small) symmetric monoidal categories has a subcategory \( \text{Perm} \) which inherits a model category structure from the natural model category \( \text{Cat} \). The objects of \( \text{Perm} \) are permutative categories (or strict symmetric monoidal categories) which are those symmetric monoidal categories whose tensor product is strictly associative and strictly unital. The morphisms of \( \text{Perm} \) are those symmetric monoidal functors which preserve the symmetric monoidal structure strictly. In this section we will construct a model category structure on a subcategory of the category of permutative categories \( \text{Perm} \) whose objects are compact closed categories. We begin by recalling the definition of a compact closed category from [KL80]:

**Definition 2.1.** A compact closed category is a symmetric monoidal category \( C \) in which each object \( c \in \text{Ob}(C) \) can be assigned a triple \((c^\bullet, \eta_c, \epsilon_c)\) where \( c^\bullet \) is an object of \( C \) (called right dual of \( c \)) and \( \eta_c : 1_C \to c^\bullet \otimes c \) and \( \epsilon_c : c \otimes c^\bullet \to 1_C \) are two maps in \( C \) such that the following two maps are identities:

\[
\text{(1)} \quad c^\bullet \cong 1_C \otimes c^\bullet \otimes id \quad c^\bullet \otimes id \quad c^\bullet \otimes 1_C \cong c^\bullet \\
\text{(2)} \quad c \cong c \otimes 1_C \quad id \otimes \eta_c \quad c \otimes c^\bullet \otimes id \quad c \otimes 1_C \cong c
\]

**Remark 1.** The symmetric monoidal structure ensures that the right dual is also a left dual and therefore we will just call \( c^\bullet \) as the dual of \( c \).

**Notation 2.2.** Unless specified otherwise, in this paper a compact closed category will mean a permutative category which is compact closed.

**Notation 2.3.** The category of all (small) compact closed categories and strict symmetric monoidal functors is denoted by \( \text{ccPerm} \).

We recall that a compact closed category \( C \) is a closed symmetric monoidal category wherein the internal hom object, between two objects \( c_1, c_2 \in C \) is defined as follows:

\[ [c_1, c_2]_C := c_1 \otimes c_2. \]

Let \( \mathcal{T} = \{-1, 1\} \) be a set with. We define a category whose objects are all finite collections \( k = (k_1, k_2, \ldots, k_r) \) of subsets of \( \mathcal{T} \), i.e. each \( k_i \subseteq \mathcal{T} \) for \( 1 \leq i \leq r \).

Let \( l = (l_1, l_2, \ldots, l_s) \) be another object. A morphism \( F : k \to l \) in the desired category is a commutative triangle:

![Diagram](image-url)

where the top arrow is a bijection and \( i_k \) and \( i_l \) are the obvious inclusion maps. This defines a permutative groupoid which we denote by \( \mathcal{L}(1)^{fr} \).

**Remark 2.** The above permutative groupoid is isomorphic to \( \mathcal{L}(2) \), see [Sha20].
The set $T$ is equipped with a bijection $in : T \rightarrow T$ which changes the sign i.e. $in(1) = -1$ and $in(-1) = 1$. Each subset map $k_i \subseteq T$ uniquely determines another subset by the composite map $k_i \subseteq T \xrightarrow{in} T$ which we denote by $k_i^\bullet$.

**Notation 2.4.** For each object $k = (k_1, k_2, \ldots, k_r)$ in the above permutative category, we denote the object $(k_r^\bullet, k_{r-1}^\bullet, \ldots, k_1^\bullet)$ by $k^\bullet$.

For each object $k$ in the above permutative category, we specify two edges
\[(3) \quad \eta_k : \emptyset \rightarrow (k^\bullet, k) \quad \text{and} \quad \epsilon_k : (k, k^\bullet) \rightarrow \emptyset\]
Now we enhance the underlying graph of the above category $L(1)^{fr}$ denoted by $G = U(L(1)^{fr})$. The enhanced graph, denoted $G^E$, has a set of edges consisting of all finite collections $(f_1, f_2, \ldots, f_r)$, where each $f_i$ is either an arrow of $L(1)^{fr}$ or one of the above two maps (3). We now consider the free category $F(G^E)$ generated by this enhanced graph. Each arrow $f$ of $L(1)^{fr}$ has a representative $f^\bullet$.

We want to construct a quotient category from this free category by imposing the following relations:

(i) For each pair $f, g$ of composable arrows in $L(1)^{fr}$ we impose the relation $(g \circ f) = (g^\bullet \circ (f^\bullet))$.

(ii) For each object $k \in L(1)^{fr}$ we impose the two relations (1) and (2).

(iii) For each pair of arrows $f : k \rightarrow m, g : l \rightarrow n$ in the free category $F(G^E)$, we impose relations such that the following two diagram commutes:

\[
\begin{array}{ccc}
(k, l) & \xrightarrow{f} & (l, k) \\
\downarrow^{(f,g)} & & \downarrow^{(g,f)} \\
(m, n) & \xrightarrow{\epsilon_k} & (n, m)
\end{array}
\]
where the horizontal arrows are the symmetry isomorphisms in the permutative category $L(1)^{fr}$.

(iv) For each pair of objects $l, k \in L(1)^{fr}$ we impose relations such that the following diagrams commute:

\[
\begin{array}{ccc}
(\emptyset, \emptyset) & \xrightarrow{(\eta_l, \eta_k)} & (l^\bullet, l, k^\bullet, k) \\
\downarrow^{\eta_k(l, k)} & & \downarrow^{\eta_k(l, k)} \\
\emptyset & \xrightarrow{\epsilon_k(l, k)} & (k^\bullet, l^\bullet, l, k)
\end{array}
\quad \quad \begin{array}{ccc}
(l, l^\bullet, k, k^\bullet) & \xrightarrow{(\epsilon_l, \epsilon_k)} & (\emptyset, \emptyset) \\
\downarrow^{\epsilon_k(l, k)} & & \downarrow^{\epsilon_k(l, k)} \\
(l, k, k^\bullet, l^\bullet) & \xrightarrow{\epsilon_k(l, k)} & \emptyset
\end{array}
\]
This quotient category is a permutative compact closed category with permutative structure being concatenation.

**Notation 2.5.** We will denote the above constructed (permutative) compact closed category by $\mathcal{C}ob^{fr}(1)$. The object $(1)$ in $\mathcal{C}ob^{fr}(1)$ will be called the *generator*.

**Remark 3.** The symmetry isomorphisms in the permutative category $\mathcal{C}ob^{fr}(1)$ are inherited from the permutative category $L(1)^{fr}$.

**Remark 4.** The permutative category $\mathcal{C}ob^{fr}(1)$ is equivalent to the symmetric monoidal category whose objects are 0-dimensional framed manifolds and morphisms are (boundary preserving) diffeomorphism classes of 1-dimensional framed manifolds with boundary. In other words $\mathcal{C}ob^{fr}(1)$ is an algebraic model for the framed 1-Bordism category.
Definition 2.6. We will refer to $\mathsf{Cob}^{fr}(1)$ either as the free compact closed category on one generator or as the algebraic 1-Bordism category.

Remark 5. The free compact closed category over one generator $\mathsf{Cob}^{fr}(1)$ is equipped with an inclusion (strict) symmetric monoidal functor $i: L(1) \to \mathsf{Cob}^{fr}(1)$.

Proposition 2.7. A permutative category is compact closed if and only if it is a $\{i\}$-local object.

Proof. For any compact closed category $C$, an argument similar to the proof of proposition [A.4] shows that the following map, which is the evaluation map on the generator, is an equivalence of groupoids:

$$J \left( [i, C]^{str}_{\otimes} \right): J \left( \mathsf{Cob}^{fr}(1), C|^{str}_{\otimes} \right) \to J \left( [L(1), C]^{str}_{\otimes} \right) \cong J(C)$$

where $J$ is the right adjoint to the inclusion map $i: \mathsf{Gpd} \to \mathsf{Cat}$. Thus each permutative compact closed category is an $\{i\}$-local object. Conversely, let us assume that $C$ is an $\{i\}$-local object. We recall that for any category $D$ we have the following equality of object sets: $\text{Ob}(D) = \text{Ob}(J(D))$. By assumption the functor $J \left( [i, C]^{str}_{\otimes} \right)$ is an equivalence of groupoids which now implies that each object of $C$ is isomorphic to some object in the image of $J \left( [i, C]^{str}_{\otimes} \right)$. Now the result follows from the observation that each object in the image of $J \left( [i, C]^{str}_{\otimes} \right)$ has a dual. $\square$

Definition 2.8. A map of permutative categories $F: C \to D$ will be called a compact closed equivalence of permutative categories if it is a $\{i\}$-local equivalence.

Remark 6. A strict symmetric monoidal functor $F: C \to D$ between cofibrant permutative categories is a compact closed equivalence if the following functor is an equivalence of groupoids:

$$J([F, E]^{str}_{\otimes}): J([D, E]^{str}_{\otimes}) \to J([C, E]^{str}_{\otimes})$$

for each permutative compact closed category $E$.

Corollary 2.9. The category of all (small) compact closed permutative categories $\mathsf{ccPerm}$ is complete.

Proof. The category $\mathsf{Perm}$ is locally presentable therefore it is sufficient to show that the limit of a diagram in $\mathsf{ccPerm}$ is a compact closed category. Let $F: D \to \mathsf{ccPerm}$ be such a diagram. Now we have the following commutative square in $\mathsf{Cat}$ for:

$$\begin{array}{ccc}
\mathsf{Cob}^{fr}(1), \lim F|^{str}_{\otimes} & \xrightarrow{[i, \lim F]^{str}_{\otimes}} & [L(1), \lim F]^{str}_{\otimes} \\
\cong & & \cong \\
\lim [\mathsf{Cob}^{fr}(1), F]^{str}_{\otimes} & \xrightarrow{H} & \lim [L(1), F]^{str}_{\otimes}
\end{array}$$

where the limit in the above diagram is taken in $\mathsf{Perm}$ and $H = \varprojlim \tau$ for $\tau: [\mathsf{Cob}^{fr}(1), F(-)]^{str}_{\otimes} \Rightarrow [L(1), F(-)]^{str}_{\otimes}$. The lower horizontal arrow $H$ is an equivalence of categories because $\tau$ is a natural equivalence. Now the two out of three property of weak-equivalences in model categories implies that the top horizontal arrow $[i, \lim F]^{str}_{\otimes}$ is an equivalence of categories. Since both $\mathsf{Cob}^{fr}(1)$ and $L(1)$ are cofibrant, proposition [2.7] implies that $\lim F$ is a compact closed permutative category. $\square$

In light of the above corollary, the formal criterion for the existence of an adjoint $\mathsf{Mac}^{-1}$ Thm. X.2] implies that the forgetful functor $U: \mathsf{ccPerm} \to \mathsf{Perm}$ has a left adjoint $F^{cc}: \mathsf{Perm} \to \mathsf{ccPerm}$. 
Proposition 2.10. The category $\text{ccPerm}$ is cocomplete.

Proof. It is sufficient to show that $\text{ccPerm}$ has all (small) coproducts and coequalizers for all pairs of arrows. Let $\{C_j\}_{j \in J}$ be a collection of objects of $\text{ccPerm}$. Then the coproduct of the above collection denoted by $\bigvee_{j \in J} C_j$ exists in $\text{Perm}$. Moreover it is equipped with a (unit) map $\eta : \bigvee_{j \in J} C_j \to \mathcal{F}^{cc}(\bigvee_{j \in J} C_j)$ such that it induces the following bijection for each compact closed permutative category $D$:

$$\text{Perm}(\eta, D) : \text{ccPerm}(\mathcal{F}^{cc}(\bigvee_{j \in J} C_j), D) \cong \text{Perm}(\bigvee_{j \in J} C_j, D).$$

The aforementioned coproduct provides a collection of maps $\{\iota_j\}_{j \in J}$ defined as follows:

$$\iota_j : C_j \xrightarrow{i} \bigvee_{j \in J} C_j \xrightarrow{\eta} \mathcal{F}^{cc}(\bigvee_{j \in J} C_j)$$

These arrows induce the following bijection of hom sets which show that $\mathcal{F}^{cc}(\bigvee_{j \in J} C_j)$ is a coproduct of the collection $\{C_j\}_{j \in J}$:

$$\text{ccPerm}(\iota_j, D) : \text{ccPerm}(\mathcal{F}^{cc}(\bigvee_{j \in J} C_j), D) \cong \text{Perm}(\bigvee_{j \in J} C_j, D) \cong \prod_{j \in J} \text{ccPerm}(C_j, D).$$

However $\mathcal{F}^{cc}$ is a left adjoint and therefore it preserves coproducts which implies that

$$\mathcal{F}^{cc}(\bigvee_{j \in J} C_j) \cong \bigvee_{j \in J} \mathcal{F}^{cc}(C_j).$$

Since each $C_j$ is an object of $\text{ccPerm}$ therefore $\mathcal{F}^{cc}(C_j) \cong C_j$. This implies the following chain of isomorphisms:

$$\mathcal{F}^{cc}(\bigvee_{j \in J} C_j) \cong \bigvee_{j \in J} \mathcal{F}^{cc}(C_j) \cong \bigvee_{j \in J} C_j.$$

Thus $\bigvee_{j \in J} C_j$ is a coproduct of the collection $\{C_j\}_{j \in J}$ in $\text{ccPerm}$. A similar argument shows that if $H : D \to E$ is a coequalizer in $\text{Perm}$ of a pair of arrows $F, G : C \to D$ in $\text{ccPerm}$ then it is also a coequalizer of the pair $F, G$ in $\text{ccPerm}$. \qed

Theorem 2.11. There is a model category structure on $\text{ccPerm}$ wherein a strict symmetric monoidal functor $F : C \to D$ is a

1. weak equivalence if $F$ is a weak-equivalence in the natural model category $\text{Perm}$.
2. fibration if $F$ is a fibration in the natural model category $\text{Perm}$.
3. cofibration if it has the left lifting property with respect to all strict symmetric monoidal functors which are simultaneously a weak-equivalences and fibrations.

Proof. We will use theorem [C2] to prove this theorem by applying it to the adjunction $(\mathcal{F}^{cc}, U)$. Let $F : C \to D$ be a strict symmetric monoidal functor which is a cofibration and which has the left lifting property with respect to all fibrations. We can factorize $F$ into an acyclic cofibration $i$ followed by a fibration $p$ in the natural model category $\text{Perm}$ which gives us the following (outer) commutative diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{i} & E \\
\downarrow F & & \downarrow p \\
D & \xleftarrow{\sim} & D
\end{array}
$$
Since \( i \) is a weak-equivalence therefore \( E \) is a compact closed category therefore the above diagram is in \( \text{ccPerm} \). Further, \( p \) is a fibration therefore, by assumption, there exists a (dotted) lifting arrow \( L \) which makes the entire diagram commutative. Now the two out of six property of model categories tells us that \( F \) is a weak-equivalence.

\[ \square \]

**Remark 7.** A strict symmetric monoidal functor in \( \text{ccPerm} \) which is a cofibration in \( \text{Perm} \) is a cofibration in the above model category structure.

**Proposition 2.12.** The adjunction \((F^{cc}, U)\) is a Quillen adjunction.

The next proposition is an easy consequence of the construction of the model category structure on \( \text{ccPerm} \):

**Proposition 2.13.** The forgetful functor \( U \) preserves fibrations and acyclic fibrations.

**Remark 8.** The right adjoint of the adjunction \((F^{cc}, U)\) is fully faithful therefore for each compact closed category \( C \) the counit map \( \epsilon_C : F^{cc}(U(C)) \to C \) is an isomorphism.

Next, we provide a characterization of cofibrations in the model category of compact closed permutative categories:

**Theorem 2.14.** A strict symmetric monoidal functor \( F : C \to D \) is a cofibration in the model category of compact closed permutative categories if and only if it is a cofibration in the natural model category \( \text{Perm} \).

**Proof.** One side of the proof is obvious. Let us assume that \( F \) is a cofibration in the model category of compact closed permutative categories, then \( F \) has the left lifting property with respect to all acyclic fibrations (in \( \text{Perm} \)) between compact closed permutative categories. We define a functor \( E : \text{mon} \to \text{Perm} \), where \( \text{mon} \) is the category of monoids and monoid homomorphisms, which assigns to each monoid \( M \) its category of translations \( EM \). We recall that \( EM \) is the permutative category whose object monoid is \( M \) and there is a unique arrow between any pair of objects. We observe that \( EM \) is a (permutative) Picard groupoid for each monoid \( M \). We further observe that for each surjective homomorphism of monoids \( f : M \to N \), the strict symmetric monoidal functor \( E(f) : EM \to EN \) is an acyclic fibration between (permutative) Picard groupoids (hence compact closed permutative categories). By assumption, \( F \) has the left lifting property with respect to all strict symmetric monoidal functors in the set:

\[ \{ E(f) : EM \to EN; f \text{ is surjective} \} \]

This implies that \( \text{Ob}(F) \) has the left lifting property with respect to all surjective monoid homomorphisms. Now the result follows from [Sha20, lem 3.8].

\[ \square \]

The rest of this section is dedicated to describing some special properties of the left adjoint functor \( F^{cc} \):

**Lemma 2.15.** The left Quillen functor \( F^{cc} \) preserves fibrations and acyclic fibrations between compact closed categories.

**Proof.** Let \( p : X \to Y \) be a fibration in the natural model category \( \text{Perm} \). Let \( i : C \to D \) be an acyclic cofibration in \( \text{ccPerm} \). In order to show that \( F^{cc}(p) \) is a fibration in \( \text{ccPerm} \), we need to
show that whenever we have the following (outer) commutative diagram, there is a (dotted) lifting arrow $L$ which makes the entire diagram commutative:

\[
\begin{array}{ccc}
C & \xrightarrow{F} & F^{\text{cc}}(X) \\
\downarrow & & \downarrow \ \\
D & \xrightarrow{L} & F^{\text{cc}}(Y)
\end{array}
\]

This follows from remark [8] which implies that $F^{\text{cc}}(X) \cong X$ and $F^{\text{cc}}(Y) \cong Y$. A similar argument shows that $F^{\text{cc}}$ preserves acyclic fibrations.

The next property of the left adjoint functor $F^{\text{cc}}$ is that it preserves weak-equivalences between compact closed categories:

**Proposition 2.16.** The left adjoint functor $F^{\text{cc}}$ preserves weak-equivalences between compact closed categories.

**Proof.** Let $F : C \to D$ be a weak-equivalence in $\text{Perm}$ where $C$ and $D$ are both compact closed categories. By definition of weak-equivalences in $\text{ccPerm}$, $F$ is also a weak-equivalence in $\text{ccPerm}$. We want to show that $F^{\text{cc}}(F)$ is a weak-equivalence in $\text{ccPerm}$. We observe that the following diagram commutes

\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\downarrow \ \\
F^{\text{cc}}U(C) & \xrightarrow{F^{\text{cc}}U(F)} & F^{\text{cc}}U(D) \\
\downarrow \ \\
F^{\text{cc}}(C) & \xrightarrow{F^{\text{cc}}(F)} & F^{\text{cc}}(D)
\end{array}
\]

where $\epsilon$ is the counit of the adjunction $(F^{\text{cc}}, U)$ which is a (natural) isomorphism by remark [8]. Now the result follows from the two out of three property of weak-equivalences in a model category. \qed
3. Coherently compact closed categories

In this section we will construct another model category structure on the category of $\Gamma$-categories $\Gamma\text{Cat} = [\Gamma^{\text{op}}, \text{Cat}]$. The main result of this section is that the Segal’s nerve functor $L$ is a right Quillen functor of a Quillen equivalence between the model category of coherently compact closed categories constructed in this section and the model category of compact closed permutative categories constructed in the previous section. We construct the desired model category by carrying out a left Bousfield localization of the model category of coherently commutative monoidal categories constructed in [Sha20], which we denote by $\Gamma\text{Cat}^{\otimes}$. We begin by briefly recalling that the Segal’s nerve functor $K: \text{Perm} \to \Gamma\text{Cat}$ constructed in [Sha20]:

**Definition 3.1.** For each $n \in \mathbb{N}$ we will now define a permutative groupoid $L(n)$. The objects of this groupoid are finite sequences of subsets of $n$. We will denote an object of this groupoid by $(S_1, S_2, \ldots, S_r)$, where $S_1, \ldots, S_r$ are subsets of $n$. A morphism $(S_1, S_2, \ldots, S_r) \to (T_1, T_2, \ldots, T_k)$ is an isomorphism of finite sets $F: S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r \to T_1 \sqcup T_2 \sqcup \cdots \sqcup T_k$ such that the following diagram commutes

\[
\begin{array}{ccc}
S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r & \xrightarrow{F} & T_1 \sqcup T_2 \sqcup \cdots \sqcup T_k \\
\downarrow & & \downarrow \\
\Delta n & \xrightarrow{\sim} & \Delta n
\end{array}
\]

where the diagonal maps are the unique inclusions of the coproducts into $n$.

The (classical) Segal’s nerve functor is now defined in degree $n$ as follows:

\[K(C)(n^+) := [L(n), C]^{\text{str}}\]

where $C$ is a permutative category. The functor $K: \text{Perm} \to \Gamma\text{Cat}$ has a left adjoint, denoted $L$, see [Sha20]. The Segal’s nerve of the free compact closed category on one generator $\text{Cob}^{fr}(1)$ constructed in the previous section, denoted by $K(\text{Cob}^{fr}(1))$, is equipped with an inclusion map

\[j: \Gamma^1 \to K(\text{Cob}^{fr}(1))\]

which is determined by the generator of $\text{Cob}^{fr}(1)$.

**Remark 9.** The coherently commutative monoidal category $K(\text{Cob}^{fr}(1))$ is equivalent to the Segal’s nerve of the framed 1-Bordism (symmetric monoidal) category described in remark 1.

**Definition 3.2.** A coherently commutative monoidal category $X$ is called a coherently compact closed category if it is a $\{j\}$-local object.

**Remark 10.** The Segal nerve $K(C)$ of a compact closed permutative category $C$ is a coherently compact closed category.

**Definition 3.3.** We will refer to the coherently compact closed category $K(\text{Cob}^{fr}(1))$ as the Segal’s nerve of the algebraic 1-Bordism category.

**Definition 3.4.** A $\{j\}$-local equivalence will be called a compact closed equivalence of $\Gamma$-categories.

**Proposition 3.5.** A morphism of $\Gamma$-categories $F: X \to Y$ is a $\{j\}$-local equivalence if and only if for each compact closed permutative category $Z$ we have the following homotopy equivalence of function complexes:

\[\text{Map}_{\Gamma\text{Cat}^{\otimes}}^h(F, K(Z)) : \text{Map}_{\Gamma\text{Cat}^{\otimes}}^h(Y, K(Z)) \to \text{Map}_{\Gamma\text{Cat}^{\otimes}}^h(X, K(Z))\]
Proof. The proposition follows from the observation that for each coherently compact closed category $W$, the unit map $\eta_W : W \to \mathcal{K}(W)$ is a strict equivalence of $\Gamma$-categories. □

Theorem 3.6. There is a left-proper, combinatorial model category structure on the category $\Gamma\text{-Cat}$ wherein a map is a

1. cofibration if it is a strict cofibration of $\Gamma$-categories, namely a cofibration in the strict model category of $\Gamma$-categories.
2. weak-equivalence if it is a compact closed equivalence of $\Gamma$-categories.
3. a fibration if it has the right lifting property with respect to maps which are simultaneously cofibrations and weak-equivalences.

The fibrant objects of this model category are coherently compact closed categories.

Proof. The model category structure is obtained by carrying out a left Bousfield localization of the natural model category structure on $\text{Perm}$ with respect to $\{j\}$, this follows from [Bar07, Thm. 2.11]. The same theorem implies that the model category is combinatorial and left-proper. □

Notation 3.7. We denote the above model category by $\Gamma\text{-Cat}^{cc}$ and refer to it as the model category of coherently compact closed categories.

In the paper [Sha20] an adjoint pair $(L, K)$ was described whose right adjoint $K$ is known as the Segal’s Nerve functor. This adjoint pair is a Quillen equivalence between the natural model category of permutative categories $\text{Perm}$ and the model category of coherently commutative monoidal categories $\Gamma\text{-Cat}^{cc}$.

Lemma 3.8. The adjoint pair $(L, K)$ is a Quillen pair between the natural model category of permutative categories $\text{Perm}$ and the model category $\Gamma\text{-Cat}^{cc}$.

Proof. We recall from above that the model category $\Gamma\text{-Cat}^{cc}$ is a left Bousfield localization of the model category of coherently commutative monoidal categories $\Gamma\text{-Cat}^{\otimes}$ therefore it has the same cofibrations as $\Gamma\text{-Cat}^{\otimes}$. Since the adjoint pair in context is a Quillen pair between $\text{Perm}$ and $\Gamma\text{-Cat}^{\otimes}$ therefore the left adjoint $L$ preserves cofibrations between the two model categories in the context of the theorem. The fibrations between fibrant objects in $\Gamma\text{-Cat}^{cc}$ are strict fibrations of $\Gamma$-categories which are preserved by $K$. Now [Joy08] tells us that the adjoint pair $(L, K)$ is a Quillen pair between $\text{Perm}$ and $\Gamma\text{-Cat}^{cc}$. □

Lemma 3.9. The Segal’s nerve functor $K$ preserves compact closed equivalences of permutative categories.

Proof. Let $F : C \to D$ be a compact closed equivalence of permutative categories. It follows from [Sha20 Cor. 6.19] that for each permutative category $C$, the counit map $\epsilon_C : \mathcal{L}(C) \to C$ is a weak-equivalence in $\text{Perm}$ i.e. the underlying functor of $\epsilon_C$ is an equivalence of categories. We consider the following diagram of function complexes for each compact closed permutative category.
Let $\eta : X \to K\mathcal{L}(X)$ be the counit map because the adjunction $(\mathcal{L}, K)$ is a Quillen equivalence. Since $X$ is a coherently compact closed category by assumption, it follows from [Hir02, Thm. 17.7.7] that the top horizontal arrow and the upper two vertical arrows are homotopy equivalences of simplicial sets. Now the two out of three property of weak equivalences in a model category implies that the lower horizontal map, namely $\Map^h_{\Gamma\text{-Cat}^{cc}}(\mathcal{L}(F), \mathcal{K}(Z))$ is a homotopy equivalence of simplicial sets. Now proposition 3.5 and [Hir02, Thm. 17.7.7] together imply that $\mathcal{K}(F)$ is a weak-equivalence in $\Gamma\text{-Cat}^{cc}$.

Now we want to give a characterization of fibrant objects in the model category $\Gamma\text{-Cat}^{cc}$:

**Lemma 3.10.** A coherently commutative monoidal category $X$ is a coherently compact closed category if and only if $\mathcal{L}(X)$ is a compact closed permutative category.

**Proof.** If $\mathcal{L}(X)$ is a compact closed permutative category then $\mathcal{K}\mathcal{L}(X)$ is a coherently compact closed category because the adjunction $(\mathcal{L}, \mathcal{K})$ is a Quillen equivalence. Since $X$ is a coherently commutative monoidal category by assumption, it follows from [Sha20, lemma 6.15] that the counit map $\eta_X : X \to K\mathcal{L}(X)$ is a strict equivalence of $\Gamma$-categories. The natural equivalence $\eta : \mathcal{K} \Rightarrow K\mathcal{L}$ constructed in the proof of [Sha20, Cor. 6.19] gives us a strict equivalence of $\Gamma$-categories $\eta(X) : \mathcal{K}\mathcal{L}(X) \to K\mathcal{L}(X)$. This implies that $K\mathcal{L}(X)$ is a coherently compact closed category. Now the counit strict equivalence $\eta_X : X \to K\mathcal{L}(X)$ implies that $X$ is a coherently compact closed category.

Conversely, let us assume that $X$ is a coherently compact closed category. The unit map $\eta_X : X \to K\mathcal{L}X$ is a strict equivalence of $\Gamma$-categories therefore $K\mathcal{L}X$ is also a coherently compact closed category. This gives us the following homotopy equivalence of simplicial sets:

$$\Map^h_{\Gamma\text{-Cat}^{cc}}(j, K\mathcal{L}(X)) : \Map^h_{\Gamma\text{-Cat}^{cc}}(K\mathcal{L}(X)) \to \Map^h_{\Gamma\text{-Cat}^{cc}}(\Gamma^1, K\mathcal{L}(X))$$

Let $q : Q \Rightarrow id$ be the cofibrant replacement functor obtained by the chosen functorial factorizations of the model category $\Gamma\text{-Cat}^{cc}$. We consider the following commutative diagram:
where the bottom horizontal map $K = \mathcal{M}ap^h_{\text{Perm}}(\mathcal{L}(Q(j)), \mathcal{K}(X))$. The two vertical isomorphisms follow from [Hir02 Prop. 7.4.16]. The top two vertical arrows in the above diagram are homotopy equivalences of simplicial sets. The two out of three property of weak-equivalences implies that the bottom arrow is a homotopy equivalence of simplicial sets. It follows from [Sha20 Thm. 6.17] that the left Quillen functor $\mathcal{L}$ preserves weak-equivalences therefore the map $\mathcal{M}ap^h_{\text{Perm}}(\mathcal{L}(j)), \mathcal{L}(X))$ is a homotopy equivalence of simplicial sets. It follows from [Sha20 Cor. 6.19] that the counit $\epsilon$ of the Quillen equivalence $(\mathcal{L}, K)$ is a natural weak-equivalence which gives us the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}ap^h_{\text{Perm}}(\mathcal{L}(\mathcal{K})\mathcal{C}ob^h(1)), \mathcal{L}(X)) & \xrightarrow{\mathcal{M}ap^h_{\text{Perm}}(\mathcal{L}(j)), \mathcal{K}(X))} & \mathcal{M}ap^h_{\text{Perm}}(\mathcal{L}(\Gamma^1), \mathcal{L}(X)) \\
\mathcal{M}ap^h_{\text{Perm}}(\mathcal{C}ob^h(1)), \mathcal{L}(X)) & \xrightarrow{\mathcal{M}ap^h_{\text{Perm}}(i), \mathcal{L}(X))} & \\
\end{array}
$$

means that the map $\mathcal{M}ap^h_{\text{Perm}}(i, \mathcal{L}(X))$ is a homotopy equivalence of simplicial sets, where $i : \mathcal{L}(\Gamma^1) \cong \mathcal{L}(1) \to \mathcal{C}ob^h(1)$. Thus we have shown that $\mathcal{L}(X)$ is a compact closed permutative category.

**Notation 3.11.** We denote the left and right adjoint functors of the following composite adjunction

$$
\mathcal{F}^{cc} \mathcal{L} : \Gamma \mathcal{C}at \cong \mathcal{C}c \mathcal{P}erm : \mathcal{K}U
$$

by $\mathcal{L}^{cc}$ and $\mathcal{K}^{cc}$ respectively.

**Theorem 3.12.** The adjoint pair $(\mathcal{L}^{cc}, \mathcal{K}^{cc})$ is a Quillen equivalence between the natural model category of compact closed permutative categories $\mathcal{C}c \mathcal{P}erm$ and the model category of coherently commutative monoidal categories $\Gamma \mathcal{C}at^{cc}$.

**Proof.** Let $X$ be a cofibrant $\Gamma$-category, we will show that the composite map:

$$
X \xrightarrow{\eta_X} \mathcal{K}^{cc}\mathcal{L}(X) \xrightarrow{\mathcal{K}^{cc}(r_{\mathcal{L}(X)})} \mathcal{K}^{cc}(\mathcal{R}(\mathcal{L}^{cc}(X)))
$$

is a compact closed equivalence of $\Gamma$-categories, where $r_{\mathcal{L}(X)} : \mathcal{L}^{cc}(X) \to \mathcal{R}(\mathcal{L}^{cc}(X))$ is a fibrant replacement of $\mathcal{L}^{cc}(X)$ in $\Gamma \mathcal{C}at^{cc}$. The above composite map is the following map in $\Gamma \mathcal{C}at$:

$$
X \xrightarrow{\eta_X} \mathcal{K}\mathcal{L}(X) \xrightarrow{H} \mathcal{K}U\mathcal{F}^{cc}\mathcal{L}(X) \xrightarrow{\mathcal{K}(r_{\mathcal{F}^{cc}\mathcal{L}(X)})} \mathcal{K}U(\mathcal{F}^{cc}\mathcal{L}(X))
$$

where $H$ is the composite $\mathcal{K}\mathcal{L}(X) \xrightarrow{\mathcal{K}(\eta_{\mathcal{L}(X)})} \mathcal{K}U\mathcal{F}^{cc}\mathcal{L}(X)$ in which $\eta$ is the unit of the adjunction $(\mathcal{F}^{cc}, U)$.

It follows from [Sha20 Cor. 6.19] that the map $\eta_X$ is a weak-equivalence in $\Gamma \mathcal{C}at^{cc}$ therefore it is a weak-equivalence in $\Gamma \mathcal{C}at^{co}$. Since $X$ is cofibrant therefore $\eta_{\mathcal{L}(X)}$ is a compact closed equivalence of permutative categories. By lemma [3.9] the map $H = \mathcal{K}(\eta_{\mathcal{L}(X)})$ is a weak-equivalence in $\Gamma \mathcal{C}at^{cc}$. The map $r_{\mathcal{F}^{cc}\mathcal{L}(X)}$ is a weak-equivalence in $\text{Perm}$ therefore $\mathcal{K}(r_{\mathcal{F}^{cc}\mathcal{L}(X)})$ is also a weak-equivalence in $\Gamma \mathcal{C}at^{cc}$.

Next we have to show that for each compact closed permutative category $C$, the following composite map is a weak-equivalence in $\mathcal{C}c \mathcal{P}erm$:

$$
\mathcal{L}^{cc}(Q(\mathcal{K}^{cc}(C))) \xrightarrow{\mathcal{L}^{cc}(\eta_{\mathcal{K}^{cc}(C)})} \mathcal{L}^{cc}\mathcal{K}^{cc}(C) \xrightarrow{\epsilon_{\mathcal{K}^{cc}(C)}} C
$$

where $\eta_{\mathcal{K}^{cc}(C)} : Q(\mathcal{K}^{cc}(C)) \to \mathcal{K}^{cc}(C)$ is a cofibrant replacement of $\mathcal{K}^{cc}(C)$ and therefore a strict equivalence of $\Gamma$-categories between compactly closed categories. It follows from [Sha20]...
Thm. 6.17] and proposition 2.16 that \( \eta_{Kcc(C)} \) is mapped to a weak-equivalence in \( ccPerm \) by \( Lcc \). The map \( \epsilon_{cc} \) is the following composite:

\[
F^{cc}(\mathcal{L}K(U(C))) \xrightarrow{F^{cc}\tau_{U(C)}} F^{cc}U(C) \cong C,
\]

where \( \tau_{U(C)} \) is the counit map of the Quillen equivalence \((\mathcal{L}, K)\). For each compact closed category \( D \), this counit map \( \tau_D \) is a weak-equivalence between compact closed categories and therefore it mapped to a weak-equivalence in \( ccPerm \) by \( F^{cc} \), see proposition 2.16. The isomorphism on the right is the counit of the adjunction \((F^{cc}, U)\).

\( \square \)

The morphism \( j \) defined in (4) can be factorized as a cofibration followed by an acyclic fibration in the model category \( \Gamma Cat^{cc} \) as follows:

\[
\begin{array}{ccc}
\Gamma\mathcal{C}ob^{fr}(1) & \xrightarrow{j_c} & \mathcal{K}(\mathcal{C}ob^{fr}(1)) \\
\downarrow p & & \downarrow \\
\Gamma^{1} & \xrightarrow{j} & \mathcal{K}(\mathcal{C}ob^{fr}(1))
\end{array}
\]

Remark 11. The coherently compact closed category \( \Gamma\mathcal{C}ob^{fr}(1) \) is a (cofibrant) \( \Gamma \)-categories representation of the (algebraic) 1-Bordism category. However, in light of remark (11), we regard it as a (cofibrant) \( \Gamma \)-categories representation of the (framed) 1-Bordism category.

Definition 3.13. We will refer to the coherently compact closed \( \Gamma\mathcal{C}ob^{fr}(1) \) as the \( \Gamma \)-categories representation of the (algebraic) 1-Bordism category.

Since the map \( j_c \) is a weak-equivalence in \( \Gamma Cat^{cc} \) therefore \( j_c \) is an acyclic cofibration. This observation allows us to formulate a corollary of theorem 3.6 which we refer to as a homotopical algebra version of the one dimensional cobordism hypothesis:

Corollary 3.14. The 1-Bordism category \( \Gamma\mathcal{C}ob^{fr}(1) \) is a fibrant approximation of the unit of the symmetric monoidal structure on \( \Gamma Cat \) namely the \( \Gamma \)-category \( \Gamma^{1} \).

The proof of the above corollary follows immediately by the construction of the model category \( \Gamma Cat^{cc} \).

For any coherently compact closed category \( X \) the mapping space \( \mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma\mathcal{C}ob^{fr}(1), X) \) is the space of all coherently commutative monoidal functors between the two coherently commutative monoidal categories \( \Gamma\mathcal{C}ob^{fr}(1) \) and \( X \). Since \( \Gamma\mathcal{C}ob^{fr}(1) \) is also a coherently compact closed category therefore \( \mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma\mathcal{C}ob^{fr}(1), X) \) is a Kan complex hence \( \mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma\mathcal{C}ob^{fr}(1), X) = J\mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma\mathcal{C}ob^{fr}(1), X) \). By definition of the map \( j_c \) one may regard \( \mathcal{M}ap_{\Gamma Cat^{cc}}(j_c, X) \) to be an evaluation map on the generator of the 1-Bordism category. An algebraic analog of the (classical) cobordism hypothesis for coherently compact closed categories can be stated as follows: The following simplicial map is a homotopy equivalence of Kan complexes:

\[
(5) \quad \mathcal{M}ap_{\Gamma Cat^{cc}}(j_c, X) : \mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma\mathcal{C}ob^{fr}(1), X) \rightarrow J(\mathcal{M}ap_{\Gamma Cat^{cc}}(\Gamma^{1}, X)) \xrightarrow{\cong} JN(X(1^{+})).
\]

In light of remark (11) the aforementioned algebraic analog is equivalent to the (classical) cobordism hypothesis for coherently compact closed categories. In other words the coherently compact closed
category $\Gamma\text{Cob}^{fr}(1)$ can be replaced in \([15]\) with a cofibrant replacement of the Segal’s nerve of the (framed) 1-Bordism category. The following proposition, along with the above discussion, is a justification of our claim that corollary 3.14 is a homotopical algebra version of the 1-dimensional cobordism hypothesis:

**Proposition 3.15.** The following statements are equivalent:

1. The inclusion map $j_c : \Gamma^1 \to \Gamma\text{Cob}^{fr}(1)$ is an acyclic cofibration in $\Gamma\text{Cat}^{cc}$.
2. For any coherently compact closed category $X$, the following map is an equivalence of function spaces:

$$\text{Map}_{\Gamma\text{Cat}^\otimes}(j_c, X) : \text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X) \to J(\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma^1, X)) \cong JN(X(1^+))$$

**Proof.** We begin by observing that since $\Gamma\text{Cob}^{fr}(1)$ is a coherently compact closed category therefore the mapping space $\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X)$ is a Kan complex therefore $\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X) = J(\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X))$. Moreover, the model category $\Gamma\text{Cat}^\otimes$ is enriched over the model category of quasi-categories. It follows from [Sha, Appendix B] that $J(\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X)) = \text{Map}^h_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X)$ and $JN(X(1^+)) \cong J(\text{Map}_{\Gamma\text{Cat}^\otimes}(\Gamma^1, X)) = \text{Map}^h_{\Gamma\text{Cat}^\otimes}(\Gamma^1, X)$. In order to show that (2) $\Rightarrow$ (1), we only need to show that the map induced on the (homotopy) function complexes:

$$\text{Map}^h_{\Gamma\text{Cat}^\otimes}(j_c, X) : \text{Map}^h_{\Gamma\text{Cat}^\otimes}(\Gamma\text{Cob}^{fr}(1), X) \to \text{Map}^h_{\Gamma\text{Cat}^\otimes}(\Gamma^1, X)$$

is a homotopy equivalence of Kan complexes. We recall that $j_c$ is a cofibration by construction. This homotopy equivalence is now obvious from the statements above and the fact that $X$ is fibrant in $\Gamma\text{Cat}^\otimes$.

(1) $\Rightarrow$ (2). If $j_c$ is weak-equivalence in $\Gamma\text{Cat}^{cc}$ then $\text{Map}^h_{\Gamma\text{Cat}^\otimes}(j_c, X)$ is a weak-equivalence. By the above statements the simplicial map $J(\text{Map}_{\Gamma\text{Cat}^\otimes}(j_c, X))$ is also a weak-equivalence. □
APPENDIX A. ASPECTS OF DUALITY

by André Joyal

The results of this appendix are folklore and where possible, we will provide a reference.

A.1. On certain monoidal transformations. Let $\mathcal{C}$ be a symmetric monoidal category. If $C$ is a dualisable object in $\mathcal{C}$, with dual object $C^*$, let us denote by $\eta_C : I \rightarrow C \otimes C^*$ and $\epsilon_C : C^* \otimes C \rightarrow I$ the unit and counit of the duality. If $D$ is another dualisable object, then the dual $f^* : D^* \rightarrow C^*$ of a morphism $f : C \rightarrow D$ is defined to be the composite

$$
D^* \xrightarrow{D^* \otimes \eta_C} D^* \otimes C \otimes C^* \xrightarrow{D^* \otimes f \otimes C^*} D^* \otimes D \otimes C^* \xrightarrow{\epsilon_D \otimes C^*} C^*
$$

Let us say that a morphism $g : C^* \rightarrow D^*$ respects the morphism $f : C \rightarrow D$ if the following diagrams commutes

$$
\begin{array}{ccc}
C^* \otimes C & \xrightarrow{\epsilon_C} & I \\
g \otimes f & \downarrow & \downarrow g \\
D^* \otimes D & \xrightarrow{\epsilon_D} & I \\
\end{array}
\quad
\begin{array}{ccc}
I & \xrightarrow{\eta_C} & C \otimes C^* \\
I & \xrightarrow{f \otimes g} & I \\
I & \xrightarrow{\eta_D} & D \otimes D^* \\
\end{array}
$$

Lemma A.1. If a morphism $g : C^* \rightarrow D^*$ respects a morphism $f : C \rightarrow D$, then $g \circ f^* = 1_{D^*}$ and $f^* \circ g = 1_{C^*}$. Hence the morphisms $f$ and $g$ are invertible.

Proof. Let us compute $g \circ f^*$. The following diagram commutes by naturality:

$$
\begin{array}{ccc}
D^* & \xrightarrow{D^* \otimes \eta_C} & D^* \otimes C \otimes C^* \\
& \xrightarrow{D^* \otimes f \otimes C^*} & D^* \otimes D \otimes C^* \\
& \xrightarrow{\epsilon_D \otimes C^*} & C^* \\
\end{array}
\quad
\begin{array}{ccc}
D^* \otimes C \otimes D^* & \xrightarrow{D^* \otimes \eta_D} & D^* \otimes D \otimes D^* \\
& \xrightarrow{\epsilon_D \otimes D^*} & D^* \\
\end{array}
$$

It follows that the morphism $g \circ f^*$ is the composite

$$
D^* \xrightarrow{D^* \otimes \eta_C} D^* \otimes C \otimes C^* \xrightarrow{D^* \otimes f \otimes C^*} D^* \otimes D \otimes C^* \xrightarrow{\epsilon_D \otimes C^*} C^*
$$

But we have $(f \otimes g) \eta_C = \eta_D$, since $g$ respects $f$. Hence the morphism $g \circ f^*$ is the composite

$$
D^* \xrightarrow{D^* \otimes \eta_D} D^* \otimes D \otimes D^* \xrightarrow{\epsilon_D \otimes D^*} D^*
$$

But $(\epsilon_D \otimes D^*)(D^* \otimes \eta_D) = 1_{D^*}$, by the duality between $D$ and $D^*$. This shows that $g \circ f^* = 1_{D^*}$. The proof that $f^* \circ g = 1_{C^*}$ is similar.

Lemma A.2. Let $\alpha : F \rightarrow G$ be a monoidal natural transformation between symmetric monoidal functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ between symmetric monoidal categories $\mathcal{C} = (\mathcal{C}, \otimes, I)$ and $\mathcal{D} = (\mathcal{D}, \otimes, J)$. If the monoidal category $\mathcal{C}$ is compact closed, then $\alpha$ is invertible.

Proof. Let us show that the map $\alpha_C : F(C) \rightarrow G(C)$ is invertible for every object $C \in \mathcal{C}$. The object $C$ has a dual $C^*$, since the category $\mathcal{C}$ is compact closed. Let $\eta_C : I \rightarrow C \otimes C^*$ and $\epsilon_C : C^* \otimes C \rightarrow I$ be the unit and counit of the duality. The object $F(C)$ has then a dual $F(C)^* := F(C^*)$. The unit $\eta_{F(C)} : J \rightarrow F(C) \otimes F(C^*)$ is defined to be the composite

$$
\begin{array}{cccc}
J & \xrightarrow{\cong} & F(I) & \xrightarrow{F(\eta_C)} \\
& & F(C \otimes C^*) & \xrightarrow{\cong} \\
& & F(C) \otimes F(C^*) & \\
\end{array}
$$
and the counit $\epsilon_{F(C)} : F(C^\ast) \otimes F(C) \to J$ is defined to be the composite

$$F(C^\ast) \otimes F(C) \xrightarrow{\sim} F(C^\ast \otimes C) \xrightarrow{F(\epsilon_C)} F(I) \xrightarrow{\sim} J$$

Similarly, the object $G(C)$ has a dual $G(C)^\ast := G(C^\ast)$. The unit $\eta_{G(C)} : J \to G(C) \otimes G(C^\ast)$ is defined to be the composite

$$J \xrightarrow{\sim} G(I) \xrightarrow{G(\eta_C)} G(C \otimes C^\ast) \xrightarrow{\sim} G(C) \otimes G(C^\ast)$$

and the counit $\epsilon_{G(C)} : G(C^\ast) \otimes G(C) \to J$ is defined to be the composite

$$G(C^\ast) \otimes G(C) \xrightarrow{\sim} G(C^\ast \otimes C) \xrightarrow{G(\epsilon_C)} G(I) \xrightarrow{\sim} J$$

Let us show that the morphism $\alpha_{C^\ast} : F(C^\ast) \to G(C^\ast)$ respects the morphism $\alpha_C : F(C) \to G(C)$. The following diagram commutes, since the natural transformation $\alpha : F \to G$ is monoidal

$$\begin{array}{ccc}
J & \xrightarrow{\sim} & F(I) \\
\downarrow & & \downarrow F(\eta_C) \\
J & \xrightarrow{\sim} & G(I) \end{array}$$

Hence the following square commutes

$$\begin{array}{ccc}
J & \xrightarrow{\eta_{F(C)}} & F(C) \otimes F(C^\ast) \\
\downarrow & & \downarrow \alpha_C \otimes \alpha_{C^\ast} \\
J & \xrightarrow{\eta_{G(C)}} & G(C) \otimes G(C^\ast) \\
\end{array}$$

Similarly, the following square commutes

$$\begin{array}{ccc}
F(C^\ast) \otimes F(C) & \xrightarrow{\epsilon_{F(C)}} & J \\
\downarrow \alpha_{C^\ast} \otimes \alpha_C & & \downarrow \\
G(D^\ast) \otimes G(D) & \xrightarrow{\epsilon_{G(D)}} & J \\
\end{array}$$

This shows that the morphism $\alpha_{C^\ast} : F(C^\ast) \to G(C^\ast)$ respects the morphism $\alpha_C : F(C) \to G(C)$. It then follows by Lemma A.1 that $\alpha_C$ is invertible. \qed

If $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories, let us denote by $\text{Hom}_{SM}(\mathcal{C}, \mathcal{D})$ the category of symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$. The category $\text{Hom}_{SM}(\mathcal{C}, \mathcal{D})$ is symmetric monoidal.

A different proof of the following proposition appears in [DM18, Prop. 1.13]

**Proposition A.3.** If $\mathcal{C}$ is a compact closed symmetric monoidal category, then the symmetric monoidal category $\text{Hom}_{SM}(\mathcal{C}, \mathcal{D})$ is a groupoid for every symmetric monoidal category $\mathcal{D}$.

**Proof.** Let $\alpha : F \to G$ be a morphism in the category $\text{Hom}_{SM}(\mathcal{C}, \mathcal{D})$. The natural transformation $\alpha$ is invertible by Lemma A.2. Its inverse $\alpha^{-1} : G \to F$ is monoidal (by a general result). Thus, $\text{Hom}_{SM}(\mathcal{C}, \mathcal{D})$ is a groupoid. \qed
A.2. **On the compact closed symmetric monoidal category free on one generator.** Let me denote by $B$ the compact closed symmetric monoidal category freely generated by one object $U \in B$. By definition, for every compact closed symmetric monoidal category $C$ and every object $C \in C$ there exists a symmetric monoidal functor $F : B \to C$ such that $F(U) = C$, and the functor $F$ is unique up to unique isomorphism: if $G : B \to C$ is another functor such that $G(U) = C$, then there exists a unique monoidal natural isomorphism $\alpha : F \to G$ such that $\alpha_U = 1_C$.

If $C$ is a category, then the subcategory of invertible morphisms of $C$ is a groupoid called the **core** of $C$. I will denote the core of $C$ by $C^{\text{cor}}$. The core of a symmetric monoidal category $C$ is a symmetric monoidal subcategory of $C$.

I will use the following construction in the proof of the next proposition. Let me denote by $J$ the groupoid freely generated by one isomorphism $i : 0 \to 1$. If $C$ is a category then an object of the category $C^J$ is an isomorphism $f$ in $C$. The source and target functors $s, t : C^J \to C$ are connected by a natural isomorphism $h : s \to t$ defined by putting $h(f) = f : s(f) \to t(f)$. The category $C^J$ is symmetric monoidal if $C$ is symmetric monoidal. Moreover, the source and target functors $s, t : C^J \to C$ and the natural transformation $h : s \to t$ are symmetric monoidal. The category $C^J$ is compact closed if $C$ is compact closed, since the functor $s : C^J \to C$ is an equivalence of symmetric monoidal categories.

**Proposition A.4.** Let $B$ the compact closed symmetric monoidal category freely generated by one object $U \in B$. If $C$ is a compact closed symmetric monoidal category, then the evaluation functor

$$e_U : \text{Hom}_{\text{SM}}(B, C) \to C$$

defined by putting $e_U(F) := F(U)$ takes its values in the core of $C$. Moreover, the induced functor

$$e_U' : \text{Hom}_{\text{SM}}(B, C) \to C^{\text{cor}}$$

is an equivalence of symmetric monoidal categories.

**Proof.** The category $\text{Hom}_{\text{SM}}(B, C)$ is a groupoid by Proposition A.3. Hence the functor $e_U$ takes its values in the core of $C$. Let us show that the induced functor $e_U'$ is an equivalence of categories. For every object $C \in C$ there exists a symmetric monoidal functor $F : B \to C$ such that $F(U) = C$, since $C$ is compact closed and $B$ is compact closed and freely generated by the object $U \in B$. We then have $e_U'(F) := e_U(F) := F(U) = C$. We have proved that the functor $e_U'$ is surjective on objects. Let us show that the functor $e_U'$ is fully faithful. If $F, G : B \to C$ are symmetric monoidal functors, let us show that for every isomorphism $f : F(U) \to G(U)$ there exists a unique monoidal natural isomorphism $\alpha : F \to G$ such that $\alpha_U = f$. We shall first prove the existence of $\alpha$. The symmetric monoidal category $C^J$ is compact closed, since the symmetric monoidal category $C$ is compact closed by hypothesis. The isomorphism $f$ is an object in $C^J$. By the freeness of $B$, there exists a symmetric monoidal functor

$$H : B \to C^J$$

such that $H(U) = f$. We have $sH(U) = s(f) = F(U)$, since $f : F(U) \to G(U)$. The functor $sH : B \to C$ is symmetric monoidal, since the functors $H$ and $s$ are. Thus, there exists a unique monoidal natural isomorphism $\rho : F \to sH$ such that $\rho_U = 1_{F(U)}$. Similarly, if $t : C^J \to C$ is the target functor, then $tH(U) = t(f) = G(U)$. Thus, there exists a unique monoidal natural isomorphism $\lambda : tH \to G$ such that $\lambda_U = 1_{G(U)}$. If $h : s \to t$ is the canonical isomorphism, then the composite $\alpha := \lambda h \rho$ is a monoidal natural isomorphism $\alpha : F \to G$

$$\begin{array}{ccc}
F & \xrightarrow{\rho} & sH \\
\xrightarrow{h \circ H} & \xrightarrow{\lambda} & tH \\
\xrightarrow{\lambda} & & \xrightarrow{G}
\end{array}$$
We have $\alpha_U = f$, since $\rho_U = 1_{F(U)}$, $(h \circ H)_U = h(H(U)) = h(f) = f$ and $\lambda_U = 1_{G(U)}$. The existence of $\alpha : F \to G$ is proved. Let us show that $\alpha$ is unique. Let $\beta : F \to G$ a monoidal natural isomorphism such that $\beta_U = f$. Then $\gamma := \beta^{-1}\alpha : F \to F$ is a monoidal natural isomorphism such that $\gamma_U = 1_U$. It follows that $\gamma = 1_F$, since $\mathcal{B}$ is freely generated by the object $U \in \mathcal{B}$. We have proved that the functor $\epsilon'_U : \text{Hom}_{\text{SM}}(\mathcal{B}, \mathcal{C}) \to \mathcal{C}^{\text{cor}}$ is fully faithful. It is thus an equivalence of categories, since it is surjective on objects. It is also an equivalence of symmetric monoidal categories, since it is a symmetric monoidal functor. \qed
Appendix B. Localization in model categories

We begin by recalling the notion of a left Bousfield localization:

**Definition B.1.** Let $\mathcal{M}$ be a model category and let $S$ be a class of maps in $\mathcal{M}$. The left Bousfield localization of $\mathcal{M}$ with respect to $S$ is a model category structure $L_S\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that

1. The class of cofibrations of $L_S\mathcal{M}$ is the same as the class of cofibrations of $\mathcal{M}$.
2. A map $f : A \to B$ is a weak equivalence in $L_S\mathcal{M}$ if it is an $S$-local equivalence, namely, for every fibrant $S$-local object $X$, the induced map on homotopy function complexes

   $$f^* : \text{Map}^h_{\mathcal{M}}(B, X) \to \text{Map}^h_{\mathcal{M}}(A, X)$$

   is a weak homotopy equivalence of simplicial sets. Recall that an object $X$ is called fibrant $S$-local if $X$ is fibrant in $\mathcal{M}$ and for every element $g : K \to L$ of the set $S$, the induced map on homotopy function complexes

   $$g^* : \text{Map}^h_{\mathcal{M}}(L, X) \to \text{Map}^h_{\mathcal{M}}(K, X)$$

   is a weak homotopy equivalence of simplicial sets.

We recall the following theorem which will be the main tool in the construction of the desired model category. This theorem first appeared in an unpublished work [?] but a proof was later provided by Barwick in [Bar07].

**Theorem B.2.** [Bar07, Theorem 2.11] If $\mathcal{M}$ is a combinatorial model category and $S$ is a small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ along any set representing $S$ exists and satisfies the following conditions.

1. The model category $L_S\mathcal{M}$ is left proper and combinatorial.
2. As a category, $L_S\mathcal{M}$ is simply $\mathcal{M}$.
3. The cofibrations of $L_S\mathcal{M}$ are exactly those of $\mathcal{M}$.
4. The fibrant objects of $L_S\mathcal{M}$ are the fibrant $S$-local objects $Z$ of $\mathcal{M}$.
5. The weak equivalences of $L_S\mathcal{M}$ are the $S$-local equivalences.
Appendix C. Transfer model structure on locally presentable categories

In this section we will show that one can always transfer a model category structure on a locally presentable category $C$ along an adjunction $F : D \rightleftarrows C : G$, where $D$ is a cofibrantly generated model category.

This result is a special case of the following theorem:

**Theorem C.1.** [GS07, Theorem 3.6] Let $F : D \rightleftarrows C : G$ be an adjoint pair and suppose $D$ is a cofibrantly generated model category and $C$ is both complete and cocomplete. Let $I$ and $J$ be chosen sets of generating cofibrations and acyclic cofibrations of $D$, respectively. Define a morphism $f : X \to Y$ in $C$ to be a weak equivalence or a fibration if $G(f)$ is a weak equivalence or fibration in $D$. Suppose further that

1. The right adjoint $G : C \to D$ commutes with sequential colimits; and
2. Every cofibration in $C$ with the LLP with respect to all fibrations is a weak equivalence.

Then $C$ becomes a cofibrantly generated model category. Furthermore the collections \( \{ F(i) \mid i \in I \} \) and \( \{ F(j) \mid j \in J \} \) generate the cofibrations and the acyclic cofibrations of $C$ respectively.

Now we provide a statement of this special case which will be useful throughout this paper:

**Theorem C.2.** Let $F : D \rightleftarrows C : G$ be an adjoint pair and suppose $D$ is a combinatorial model category and $C$ is a locally presentable category. Then there exists a cofibrantly generated model category structure on $C$ in which a map $f$ is

1. a weak equivalence if $G(f)$ is a weak equivalence in $D$
2. a fibration if the map $G(f)$ is a fibration in $D$
3. a cofibration if it has the left lifting property with respect to maps which are both fibrations and weak equivalences.

Let $I_D$ and $J_D$ be a chosen class of generating cofibrations and generating acyclic cofibrations of $D$ respectively. Then the collections $F(I_D) = \{ F(i) \mid i \in I_D \}$ and $F(J_D) = \{ F(j) \mid j \in J_D \}$ generate the cofibrations and the acyclic cofibrations of $C$ respectively.
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