A NOTE ON THE RANK OF A RESTRICTED LIE ALGEBRA

HAO CHANG

ABSTRACT. In this short note, we study the rank of a restricted Lie algebra $(\mathfrak{g}, [p])$ and give some applications, which concerns the dimensions of non-trivial irreducible modules. We also compute the rank of the restricted contact algebra $K(n)$.

1. Introduction

The theorem on the conjugacy of Cartan subalgebras in finite-dimensional Lie algebras over an algebraically closed field of characteristic 0 is a classical result in the Lie theory. However, it can not be extended to Lie algebras of prime characteristic.

Let $k$ be an algebraically closed field of characteristic $\text{char}(k) = p > 0$ and $(\mathfrak{g}, [p])$ be a restricted Lie algebra. By general theory, the Cartan subalgebras of $\mathfrak{g}$ are precisely the centralizers of maximal tori, see for instance [16, §2, Theorem 4.1]. There exist numerous examples of restricted Lie algebras that contain Cartan subalgebras of different dimensions. We are interested in the tori of maximal dimension, which play an important role in the classification theory ([1]). Premet’s theorem ensures that the invariance of the dimension of Cartan subalgebras with maximal toral part ([12, Theorem 1], see also [6, (3.5(1))]).

The dimension is the so-called rank of $\mathfrak{g}$, say $\text{rk}(\mathfrak{g})$.

The aim of this note is to study the rank of a restricted Lie algebra. We give some applications, which concerns the dimensions of non-trivial irreducible modules. We also compute the rank of the restricted contact algebra $K(n)$.

2. The rank of a Lie algebra

Throughout, we shall be working over an algebraically closed field $k$ of characteristic $\text{char}(k) = p > 0$. Given a restricted Lie algebra $(\mathfrak{g}, [p])$, we denote by $\mu(\mathfrak{g})$ and $\text{rk}(\mathfrak{g})$ the maximal dimension of all tori $t \subseteq \mathfrak{g}$ and the minimal dimension of all Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, respectively. Moreover, $\dim_k C_\mathfrak{g}(t) = \text{rk}(\mathfrak{g})$ for every torus $t$ of maximal dimension ([12 Theorem 1]). The following Lemma is well-known, see for example [17, Theorem 2.16] for maximal tori and [11, Lemma 1.7.2] for tori of maximal dimension.

Lemma 2.1. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra, $n \triangleleft \mathfrak{g}$ be a $p$-ideal. Then the following statements hold:

\begin{itemize}
  \item[1] $\mu(\mathfrak{g} / n) \geq \text{rk}(\mathfrak{g})$.
  \item[2] $\mu(\mathfrak{g} / n) = \text{rk}(\mathfrak{g})$ if and only if $n$ is a Cartan subalgebra.
\end{itemize}

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(1) If $t \subseteq g$ is a torus of maximal dimension, then $t \cap n$ and $t + n/n$ are tori of maximal dimension of $n$ and $g/n$, respectively.

(2) $\mu(g) = \mu(n) + \mu(g/n)$.

In fact, we can find some subtorus $t' \subseteq t$ such that $t = (t \cap n) \oplus t'$ (see for example [2, Page 1347]).

**Proposition 2.2.** Let $n \subseteq g$ be a $p$-ideal. Then the following statements hold:

1. $\text{rk}(g) \leq \text{rk}(n) + \text{rk}(g/n)$.
2. If $t \subseteq g$ is a torus of maximal dimension, then

$$\text{rk}(g) = \text{rk}(t + n) + \text{rk}(g/n) - \mu(g/n).$$

**Proof.** Let $t \subseteq g$ be a torus of maximal dimension. We write $t = (t \cap n) \oplus t'$ and observe that the short exact sequence

$$(0) \to n \to g \to g/n \to (0)$$

of $t$-modules splits. Consequently, $C_g(t) \cong C_n(t) \oplus C_{g/n}(t)$. Since $C_n(t) \subseteq C_n(t \cap n)$, Lemma 2.1 yields

$$\text{rk}(g) = \dim_k C_n(t) + \text{rk}(g/n) \leq \text{rk}(n) + \text{rk}(g/n). \tag{2.1}$$

For the proof of (2), we note that

$$\text{rk}(t + n) = \dim_k C_{t + n}(t) = \dim_k C_n(t) + \dim_k t' = \dim_k C_n(t) + \mu(g/n),$$

so that (2.1) implies

$$\text{rk}(g) = \dim_k C_n(t) + \text{rk}(g/n) = \text{rk}(t + n) + \text{rk}(g/n) - \mu(g/n), \tag{2.2}$$

as desired. \qed

A restricted Lie algebra $(g, [p])$ is called $p$-nilpotent if $g$ consists of only $p$-nilpotent elements. In particular, $g$ is $p$-nilpotent if and only if $\mu(g) = 0$.

**Corollary 2.3.** Let $n \subseteq g$ be a $p$-ideal. If $g/n$ is $p$-nilpotent, then $\mu(g) = \mu(n)$ and $\text{rk}(g) = \text{rk}(n) + \dim_k g/n$.

**Proof.** By assumption, we know $\mu(g/n) = 0$ and $\dim_k g/n = \text{rk}(g/n)$, so that our assertions follow from Lemma 2.1(2) and Proposition 2.2(2). \qed

Let $(g, [p])$ be a restricted Lie algebra. We denote by

$$\text{Tor}(g) := \{t \subseteq g; \dim_k t = \mu(g)\}$$

the toral variety of $g$ (see [6, 2]). Moreover, the variety $\text{Tor}(g)$ is irreducible ([1, (1.6)]). Let $t \in \text{Tor}(g)$ be a torus of maximal dimension. We denote by $S(g, t)$ the toral stabilizer of $g$ relative to $t$ (see [6] Page 4194) for details. In particular, $S(g, t)$ is a subgroup of $\text{Aut}_p(t) \cong \text{GL}_{\mu(g)}(\mathbb{F}_p)$.

**Theorem 2.4.** Let $V$ be a restricted $g$-module of Lie algebra $(g, [p])$, and $t \in \text{Tor}(g)$ a torus of maximal dimension. Suppose that $S(g, t) \cong \text{GL}_{\mu(g)}(\mathbb{F}_p)$.

1. If $\dim_k V < p^{\mu(g)}-1$, then the trivial $g$-module $k$ is the only composition factor of $V$ (i.e. $V = k^{\dim_k V}$ in the Grothendieck group).
(ii) Moreover, if $g = g^{[p]} + [g, g]$, then $V \cong k^\dim V$.

Proof. Let

$$V = \bigoplus_{\lambda \in \Lambda_V} V_{\lambda} \quad (2.3)$$

relative to $t$. According to [2, Corollary 6.2(2)], we know that all weight spaces belonging to a non-zero weights have the same dimension. Let

$$I := \{ x \in g; \; x \cdot V = (0) \}$$

be the annihilator of $V$ in $g$. By assumption, this is a $p$-ideal that contains $t$, it follows from Lemma 2.1(2) that $\mu(g/I) = 0$, so that $g/I$ is $p$-nilpotent. We obtain (i).

(ii) A classical result by Hochschild (see [8, Theorem 2.1]) states that

$$\text{Ext}^1_{U_0(g)}(k, k) \cong H^1_{\ast}(g, k) \cong (g/(g^{[p]} + [g, g]))^* = (0). \quad (2.4)$$

Now (i) implies the assertion. □

Corollary 2.5. Let $h \subseteq g$ be a $p$-subalgebra of the restricted Lie algebra $(g, [p])$. Suppose that $\mu(h) = \mu(g)$. If $t \in \text{Tor}(h)$ such that

(a) $S(h, t) \cong \text{GL}_{\mu(h)}(F_p)$, and

(b) $\dim_k g - \dim_k h < p^\mu(g) - 1$,

then $\text{rk}(g) = \text{rk}(h) + \dim_k g - \dim_k h$.

Proof. Thanks to Theorem 2.4 the torus $t$ of $h$ acts trivially on $g/h$. Hence, $g_{\alpha} = h_{\alpha}$ for every root $\alpha$ and

$$\dim_k g - \dim_k h = \dim_k C_g(t) - \dim_k C_h(t) = \text{rk}(g) - \text{rk}(h).$$

□

Corollary 2.6. Let $h \subseteq g$ be a simple $p$-subalgebra of the restricted Lie algebra $(g, [p])$. If $t \in \text{Tor}(h)$ such that

(a) $S(h, t) \cong \text{GL}_{\mu(h)}(F_p)$, and

(b) $\dim_k g - \dim_k h < p^\mu(h) - 1$,

then $h$ is an ideal in $g$.

Proof. Now apply Theorem 2.4 to the $h$-module $g/h$. Note that $h$ is simple, it follows that the module $g/h$ is trivial. □

Example. Let $W(n)$ be the restricted Jacobson-Witt algebra. Suppose that $W(n) \subseteq g$ has codimension $< p^n - 1$. Note that $W(n)$ is simple, hence is an ideal of $g$ (Corollary 2.6). We have the short exact sequences of restricted Lie algebras:

$$(0) \to W(n) \to g \to g/W(n) \to (0).$$

Note that $W(n)$ is centerless with all derivations being inner (see [16, §4, Theorem 8.5]), it follows from a direct computation that

$$g = C_g(W(n)) \oplus W(n).$$
Corollary 2.7. Suppose that $p > 3$. Let $\mathfrak{g}$ be a non-classical restricted simple Lie algebra and $V$ be a restricted $\mathfrak{g}$-module. If $\dim_k V < p^{\mu(\mathfrak{g})} - 1$, then $V$ is a trivial module.

Proof. According to the classification of restricted simple Lie algebras (cf. [15]), we know that $\mathfrak{g}$ should be a Lie algebra of Cartan type or Melikian algebra. Thanks to [2, Theorem 5.3, 5.5], we have $S(\mathfrak{g}, t) \cong GL_{\mu(\mathfrak{g})}(\mathbb{F}_p)$. Since $\mathfrak{g}$ is simple, now Theorem 2.4 implies that $V$ must be trivial. □

Remark. In [9, Lemma 3.6], Jantzen gave a computation of the lower bound of non-trivial irreducible modules for the restricted simple Lie algebras of Cartan type.

Example. The simple modules for the Witt algebra $W(1)$ were determined in [3]. There are precisely $p$ isomorphism classes of simple restricted $W(1)$-modules. These modules are represented by $L(i)$; $0 \leq i \leq p - 1$ and the dimensions are given by $\dim_k L(0) = 1$, $\dim_k L(p - 1) = p - 1$ and $\dim_k L(i) = p$; $1 \leq i \leq p - 2$.

More generally, if a simple Lie algebra $S$ possesses a faithful irreducible representation of dimension $\leq p - 1$, then $S$ is either classical or the Witt algebra $W(1)$ (see [14]).

3. Rank of Contact algebra $K(n)$

In this section, we assume that $p \geq 3$. The reader is referred to [16, §4] for basic facts concerning the Lie algebras of Cartan type. We denote by $W(n)$, $S(n)$, $H(n)$; $n = 2r$ and $K(n); n = 2r + 1$ the restricted simple Lie algebras of Cartan type. We recall the following results due to Demushkin [4, 5], as corrected in [15, §7.5].

Lemma 3.1. (1) $\mu(W(n)) = \text{rk}(W(n)) = n$.
(2) $\mu(S(n)) = n - 1$ and $\text{rk}(S(n)) = (n - 1)(p - 1)$.
(3) $\mu(H(n)) = r$ and $\text{rk}(H(n)) = p^r - 2$.
(4) $\mu(K(n)) = r + 1$.

In this section, we would like to compute the rank of $K(n)$. Let $A(n) := k[X_1, \ldots, X_n]/(X_1^{p^n}, \ldots, X_n^{p^n})$ ($n \geq 1$) be the truncated polynomial ring. Define

$$K''(n) := \{ D \in W(n); D \omega_K \in A(n)\omega_K \},$$

where $n = 2r + 1$, $\omega_K$ is the Cartan differential form of type $K$ (cf. [16, §4.1]). By definition, the derived algebra $K(n) := [K''(n), K''(n)]$ is simple and restricted, and we call it contact algebra (cf. [16, Page 174]). According to [16, Page 174, Theorem 5.5], we know that $K(n) = K''(n)$ provided $n + 3 \neq 0 \text{ mod } (p)$. Following [16, §4.5], we endow the truncated polynomial ring $A(n)$ with the
structure of a restricted Lie algebra; the corresponding Lie bracket \( \langle , \rangle \) is called contact bracket. More precisely, define the map (see [16 Page 169])

\[
D_K : A(n) \to W(n).
\]

(3.1)

It turns out that the image of \( D_K \) is exactly \( K''(n) \).

First, we consider the case \( n + 3 \neq 0 \mod (p) \), so that \( K(n) = K''(n) \). Let

\[
t := \oplus _{i=1} ^{r} kx_i x_{r+i} \oplus k(1 + x_n) \subseteq K(n).
\]

According to [2 page 1357], \( t \) is a torus of maximal dimension in \( K''(n) \). Then \( t \) is also a torus of maximal dimension of the centralizer \( c := C_{K''(n)}(1 + x_n) \). By [15 (7.5.15)], there is an isomorphism \( \phi : c \to P(2r) \) of restricted Lie algebras such that \( \phi(1 + x_n) = 1 \), where \( P(2r) \) is the restricted Possion algebra with toral center (cf. [15 Page 403]).

Also, there is a short exact sequence of restricted Lie algebras:

\[
(0) \to k \to P(2r) \xrightarrow{D_H} H''(2r) \to (0).
\]

(3.2)

Here \( H''(2r) \) is the annihilator of the Hamiltonian form (see [16 Page 163]). In particular, \( H''(2r) \) is a \( p \)-subalgebra of \( W(2r) \). Let \( H'(2r) \) denote the image of \( P(2r) \) under \( D_H \) and the derived algebra \( H'(2r)^{(1)} = H(2r) \) is the Hamiltonian algebra. We obtain a short exact sequence of restricted Lie algebras:

\[
(0) \to k \to P(2r) \xrightarrow{D_H} H'(2r) \to (0).
\]

(3.3)

**Proposition 3.2.** Keep the notations as above. Then the following statements hold:

1. \( \text{rk}(H'(2r)) = p^r - 1 \).
2. \( \text{rk}(P(2r)) = p^r \).
3. \( \text{rk}(K''(n)) = p^r \).

**Proof.** Note that \( \mu(P(2r)) = r + 1 \) (cf. [15 Page 403]). Lemma 2.1(2) and (3.3) ensure that \( \mu(H'(2r)) = r \). We also have \( \text{dim}_k H'(2r) = \text{dim}_k P(2r) - 1 = p^{2r} - 1 \). Since \( \text{dim}_k H(2r) = p^{2r} - 2 \) (see [16 Page 166]) and \( \mu(H(2r)) = \mu(H'(2r)) = r \) (see [15 Corollary 7.5.9(2)]), we have \( H(2r) \) is a codimension 1 restricted ideal of \( H'(2r) \). Now Corollary 2.3 implies that \( \text{rk}(H'(2r)) = \text{rk}(H(2r)) + 1 = p^r - 1 \) (Lemma 3.1(3)). Furthermore, the exact sequence (3.3) and Proposition 2.2 imply that

\[
\text{rk}(P(2r)) = r + 1 + \text{rk}(H'(2r)) - \mu(H'(2r)) = p^r.
\]

Note the centralizer \( c := C_{K''(n)}(1 + x_n) \) is isomorphic to the restricted Possion algebra \( P(2r) \) and \( \text{rk}(K''(n)) = \text{rk}(c) \). Now (3) follows from (2). \( \square \)

**Remark.** It should be notice that the following facts:

1. \( H(2r) \subseteq H'(2r), \mu(H(2r)) = \mu(H'(2r)) = r \),
2. \( S(H(2r), t) \cong \text{GL}_r(\mathbb{F}_p) \).

Hence, we can also use Corollary 2.5 to compute the rank of \( H'(2r) \).
Theorem 3.3. Let $K(n)$ be the restricted contact algebra. Then

$$\text{rk}(K(n)) = \begin{cases} p^r, & n + 3 \neq 0 \mod (p) \\ p^r - 1, & n + 3 = 0 \mod (p) \end{cases} \quad (3.4)$$

Proof. Thanks to Proposition 3.2, we only need consider the case $n + 3 = 0 \mod (p)$. We know that $\dim_k K(n) = \dim_k K''(n) - 1 = \dim_k A(n) - 1 = p^{2r+1} - 1$ (cf.[16 §4.5]). Since $K(n)$ is a restricted ideal of $K''(n)$ and the quotient is $p$-nilpotent, our assertion immediately follows from Corollary 2.3. □

Corollary 3.4. The contact algebra $K(n)$ affords no self-centralizing torus except $p = n = 3$.

Proof. Suppose that there exists a self-centralizing torus. Thanks to [6, Corollary 3.8], we know that every torus of $K(n)$ of maximal dimension is self-centralizing. It follows from Lemma 3.1(4) and Theorem 3.3 that

$$r + 1 = p^r; \ n + 3 \neq 0 \mod (p)$$

or

$$r + 1 = p^r - 1; \ n + 3 = 0 \mod (p).$$

By a direct computation, this holds only when $n = p = 3$. □

Corollary 3.5.

$$\dim \text{Tor}(K(n)) = p^n - p^r$$

Proof. Thanks to [7 (1.6)] and [5 (3.5)], the dimension of the variety $\text{Tor}(K(n))$ is $\dim_k K(n) - \text{rk}(K(n))$. Now our assertion follows from Theorem 3.3. □

Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with connected automorphism group $G := \text{Aut}_p(\mathfrak{g})^\circ$. A torus $t \in \text{Tor}(\mathfrak{g})$ is called generic if the orbit $G.t$ is a dense subset of the variety $\text{Tor}(\mathfrak{g})$.

Remarks. (1). It is known that the contact algebra $K(n)$ affords no generic torus ([2, Theorem 6.6]). In his doctoral dissertation [10, Page 48], the author give a direct proof of the fact. It is not true since he used the incorrect fact that $K(n)$ admits a self-centralizing torus (see Corollary 3.4). For convenience, we consider the case $n + 3 \neq 0 \mod (p)$. As before, we know that $K(n) = K''(n)$. Let $G := \text{Aut}_p(K(n))^\circ$ be the connected automorphism group. Accoring to [11, Proposition 3.2] and Corollary 3.5, we know that

$$\dim G = \dim_k K(n) - (2r+1) = p^{2r+1} - (2r+1) > p^{2r+1} - p^r = \dim \text{Tor}(K(n)).$$

Hence, we can not get the nonexistence of generic tori by directly comparing the dimensions of $G$ and $\text{Tor}(K(n))$ (By assumption, we do not consider the case $r = 1$ and $p = 3$).

(2). Suppose that $p > 3$. Let $(\mathfrak{g}, [p])$ be a non-classical restricted simple Lie algebra and $t \in \text{Tor}(\mathfrak{g})$ a torus of maximal dimension. According to [2, Theorem 5.3, 5.5 and Corollary 6.2], we have the following equality:

$$\dim_k \mathfrak{g} = \text{rk}(\mathfrak{g}) + (p^{\mu(\mathfrak{g})} - 1) \dim_k \mathfrak{g}_\alpha.$$
where $g_\alpha$ is some root space. By a direct check (see Lemma 3.1, Theorem 3.3 and [13, Lemma 4.1] for the Melikian algebra), we obtain:

$$\dim_k g_\alpha = \max\{n \in \mathbb{N};\ n(p^{\mu(g)} - 1) < \dim_k g\}.$$ 

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Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, 24098 Kiel, Germany

E-mail address: chang@math.uni-kiel.de