A DYADIC DECOMPOSITION APPROACH TO A FINITELY DEGENERATE HYPERBOLIC PROBLEM

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To the memory of Stefano Benvenuti

Abstract. We use the Littlewood-Paley decomposition technique to obtain a $C^\infty$-well-posedness result for a weakly hyperbolic equation with a finite order of degeneration.

1. Introduction

Let us consider the following operator

\[ L = \partial_t^2 - \sum_{j,k=1}^n \partial_{x_j}(a_{jk}(t,x)\partial_{x_k}) + \sum_{j=1}^n b_j(t,x)\partial_{x_j} + c(t,x), \]

where $a_{jk} = a_{kj} \in B^\infty([0,T] \times \mathbb{R}^n)$, $b_j, c \in C^0([0,T], B^\infty(\mathbb{R}^n))$ and $B^\infty$ is the space of the infinitely differentiable functions which are bounded with bounded derivatives. We assume that $L$ is weakly hyperbolic, i.e.

\[ a(t,x,\xi) = \sum_{j,k=1}^n a_{jk}(t,x)\xi_j\xi_k / |\xi|^2 \geq 0 \]

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We suppose also that the operator $L$ has a finite order degeneration in the points in which $a(t,x,\xi) = 0$, i.e. there exists a positive integer $k$ such that

\[ \sum_{j=0}^k |\partial_t^j a(t,x,\xi)| \neq 0 \]

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We are interested in the $C^\infty$-well-posedness in the Cauchy problem for the operator $L$ with respect to the hyperplane $\{t = 0\}$. It is well-known that a usual hypothesis to such kind of results concerns a relation between the terms of first and second order of the operator, the so-called Levi condition. Our Levi condition reads as follows: that there exist $C_0 > 0$, $\gamma \geq 0$ such that

\[ |b(t,x,\xi)| = \sum_{j=1}^n b_j(t,x)\xi_j / |\xi| \leq C_0 a(t,x,\xi)^\gamma \]

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. 

\[ 1 \]
The $C^\infty$-well-posedness for $L$ satisfying the conditions (2), (3) and (4) with

$$\gamma + \frac{1}{k} \geq \frac{1}{2}$$

has been proved in [6] under the supplementary hypothesis that all the coefficients of the operator $L$ depend only on $t$. A well-known example in [13] shows that the condition (5) is sharp. Let us remark that if $k = 2$ then condition (5) reduces to $\gamma \geq 0$, i.e. no Levi condition is requested: it is the case of the effective hyperbolicity studied in [12]; on the other hand, if $k$ approaches to infinity then condition (5) approaches to $\gamma \geq 1/2$: some special results in this case were obtained in [11]. The result of [6] has been extended in [5] to the case that the coefficient $c$ depends also on the variable $x$. The case of operators $L$ having only time-dependent coefficients in the principal part, while all the other ones depend on $t$ and $x$ has been considered in [9], but in this case the $C^\infty$-well-posedness has been obtained replacing the condition (5) by the more restrictive one

$$\gamma + \frac{1}{2(k - 1)} \geq \frac{1}{2}.$$

Recently, in [1], it has been considered the case in which the principal part of the operator $L$ has the following structure:

$$\partial_t^2 - \alpha(t) \sum_{j,k=1}^n \partial_x^j (\beta_{jk}(x) \partial_x^j).$$

In this situation the hypotheses (2), (3), (4) and (5) are sufficient to prove the $C^\infty$-well-posedness. We remark that the result in [1] improves that one in [9] only in the case of one space variable.

The proofs of all the above quoted results are based on the use of the so-called approximate energies, which origin goes back to the papers of Colombini, De Giorgi, Jannelli and Spagnolo [3], [7] (a survey on the matter can be found in [4]). In particular in [6] the technique is very similar to that one in [7]. In [5] and [9] some convolution weights are introduced to treat the coefficients depending on $x$, while in [1] the approximate energies method is used together with some pseudodifferential theory arguments.

The approximate energies have been coupled with the Littlewood-Paley decomposition technique in [8], where some energy inequalities and some well-posedness results for strictly hyperbolic operators with non-Lipschitz-continuous coefficients have been obtained. This approach has been successfully used also in the case of strictly hyperbolic operators with oscillating coefficients in [10].

In this note we use the approximate energies together with the Littlewood-Paley decomposition technique to improve slightly some of
the recalled results of $C^\infty$-well-posedness for weakly hyperbolic operators with a finite order of degeneration. In particular we will consider operators having the principal part of the form

$$\partial_t^2 - \alpha(t) \sum_{j,k=1}^n \partial_{x_j} (\beta_{jk}(t,x) \partial_{x_k}),$$

where

$$\sum_{j,k=1}^n \beta_{jk}(t,x) \xi_j \xi_k / |\xi|^2 \geq \lambda_0 > 0$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. The proof of this result is inspired by that one of [6], where the use of Fourier transform is replaced by the localization in the phase space given by the dyadic decomposition of Littlewood-Paley.

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2. Main results

The main result of the paper is contained in the following theorem.

**Theorem 1.** Let $L$ be the operator defined in (1). Suppose that the hypotheses (2), (3), (4) and (5) hold. Suppose moreover that there exist $\alpha, \beta_{jk} \in B^\infty$ such that $a_{jk}(t,x) = \alpha(t) \beta_{jk}(t,x)$ for all $j,k = 1,\ldots,n$ and for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and there exist $\Lambda_0, \lambda_0 > 0$ such that

$$\Lambda_0 \geq \sum_{j,k=1}^n \beta_{jk}(t,x) \xi_j \xi_k / |\xi|^2 \geq \lambda_0 > 0$$

for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

Then there exist $\delta > 0$ such that

$$\sup_{0 \leq t \leq T} \{ \| u(t,\cdot) \|_{H^{m+1-\delta}} + \| \partial_t u(t,\cdot) \|_{H^{m-\delta}} \} \leq C_m(\| u(0,\cdot) \|_{H^{m+1}} + \| \partial_t u(0,\cdot) \|_{H^m} + \int_0^T \| Lu(t,\cdot) \|_{H^m} dt)$$

for all $m \in \mathbb{R}$ and for all $u \in C^2([0,T], H^\infty(\mathbb{R}^n))$.

The usual consequence of the energy inequality (7) is stated in the following corollary.

**Corollary 2.** Let $L$ be the operator defined in (1). Suppose that the hypotheses of the Theorem 1 hold.

Then the Cauchy problem for $L$ is $C^\infty$-well-posed.
3. Proofs

We divide the proof of the Theorem 1 into several steps.

a) the dyadic decomposition

We collect in this step some of the well-known facts on the Littlewood-Paley decomposition, referring to [2] and [8] for the details. Let \( \phi_0 \in C_0^\infty(\mathbb{R}^n) \), \( 0 \leq \phi_0(\xi) \leq 1 \), \( \phi_0(\xi) = 1 \) if \( |\xi| \leq 1 \), \( \phi_0(\xi) = 0 \) if \( |\xi| \geq 2 \), \( \phi_0 \) radial and decreasing in \( |\xi| \). We set \( \phi_0(\xi) = \phi_0(2\xi) \) and, if \( \nu \) is an integer greater or equal than 1, \( \phi_0(2^{-\nu}y) \). Let \( w \) be a function in \( H^\infty(\mathbb{R}^n) \); we define

\[
\phi_0(D_x) w(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\xi} \hat{\phi}_0(\xi) \hat{w}(\xi) \, d\xi
\]

We have that for all \( m \in \mathbb{R} \) there exists \( K_m > 0 \) such that

\[
\frac{1}{K_m} \sum_{\nu=1}^\infty \| w_\nu \|_{L^2}^2 2^m \leq \| w \|_{H^m} \leq K_m \sum_{\nu=1}^\infty \| w_\nu \|_{L^2}^2 2^m.
\]

Moreover, for all \( \nu \geq 1 \), we obtain

\[
2^{\nu-1} \| w_\nu \|_{L^2} \leq \| \nabla_x w_\nu \|_{L^2} = (\sum_j \| \partial_{x_j} w_\nu \|_{L^2}^2)^{1/2} \leq 2^{\nu+1} \| w_\nu \|_{L^2}.
\]

b) the estimate for the microlocalized approximate energy

Let \( \varepsilon \) be a positive real number less or equal than 1. Let \( u(t,x) \) be a function in \( C^1([0,T], H^\infty(\mathbb{R}^n)) \). We set \( u_\nu(t,x) = \phi_0(D_x) u(t,x) \). We introduce the microlocalized approximate energy

\[
E_{\nu,\varepsilon} u(t) = \| \partial_t u_\nu(t,\cdot) \|_{L^2}^2 + \sum_{j,k=1}^n \langle \partial_{x_j} u_\nu(t,\cdot) + \delta_{jk} \varepsilon \partial_{x_k} u_\nu(t,\cdot), \partial_{x_j} u_\nu(t,\cdot) \rangle_{L^2},
\]

where \( \delta_{jk} \) is Kroneker’s symbol. An easy computation gives

\[
\frac{d}{dt} E_{\nu,\varepsilon} u(t) = 2 \text{Re} \sum_j \langle \varepsilon \partial_{x_j} u_\nu, \partial_{x_j} \partial_t u_\nu \rangle + \sum_{j,k} \langle \partial_t a_{jk} \partial_{x_k} u_\nu, \partial_{x_j} u_\nu \rangle \\
-2 \text{Re} \sum_j \langle b_j \partial_{x_j} u_\nu, \partial_t u_\nu \rangle - 2 \text{Re} \langle cu_\nu, \partial_t u_\nu \rangle \\
+ 2 \text{Re} \sum_{j,k} \langle [\phi_0, a_{jk}] \partial_{x_k} u, \partial_t u_\nu \rangle + 2 \text{Re} \sum_j \langle [\phi_0, b_j] \partial_{x_j} u, \partial_t u_\nu \rangle + 2 \text{Re} \langle (Lu)_\nu, \partial_t u_\nu \rangle.
\]
We have
\[ 2\left| \sum_j \langle \varepsilon \partial_{x_j} u, \partial_{x_j} \partial_t u \rangle \right| \leq \frac{\varepsilon 2^\nu}{(\alpha(t) + \varepsilon)^{\frac{1}{2}}} \left( (\alpha(t) + \varepsilon) \left( \sum_j \| \partial_{x_j} u \|_{L^2}^2 \right) + n \| \partial_t u \|_{L^2}^2 \right) \]
\[ \leq C_1 \frac{\varepsilon 2^\nu}{(\alpha(t) + \varepsilon)^{\frac{1}{2}}} E_{\nu, \varepsilon} u(t), \]
where \( C_1 > 0 \) depends only on \( \lambda_0 \) and \( n \); similarly
\[ \sum_{j,k} |\langle \partial_t a_{jk} \partial_{x_k} u, \partial_{x_j} u \rangle| \leq C_2 \frac{|\alpha'(t)|}{\alpha(t) + \varepsilon} E_{\nu, \varepsilon} u(t), \]
where \( C_2 > 0 \) depends only on \( \Lambda_0 \), \( \lambda_0 \) and \( \sup |\partial_t \beta_{jk}| \). From (4) we deduce that
\[ 2\left| \sum_j \langle b_j \partial_{x_j} u, \partial_t u \rangle \right| \leq C_3 \frac{1}{\varepsilon 2^\nu} E_{\nu, \varepsilon} u(t), \]
where \( C_3 > 0 \) depends only on \( C_0 \) and \( \Lambda_0 \), and finally,
\[ 2|\langle cu, \partial_t u \rangle| \leq C_4 \frac{1}{\varepsilon 2^\nu} E_{\nu, \varepsilon} u(t), \]
where again \( C_4 > 0 \) depends only on \( \sup |c| \). We choose now \( \varepsilon = \varepsilon_\nu = 2^{-\frac{d}{2 + 2^\nu}} \). We remark that with this choice \( \varepsilon 2^\nu = 2^{\frac{d}{2 + 2^\nu}} \geq 1 \). We obtain
\[ \frac{d}{dt} E_{\nu, \varepsilon} u(t) \]
\[ \leq \tilde{C} \left( \frac{\varepsilon_\nu 2^\nu}{(\alpha(t) + \varepsilon_\nu)^{\frac{1}{2}}} + \frac{|\alpha'(t)|}{\alpha(t) + \varepsilon_\nu} + \frac{1}{(\alpha(t) + \varepsilon_\nu)^{\frac{1}{2}}} + 1 \right) E_{\nu, \varepsilon} u(t) \]
\[ + 2\text{Re} \sum_{j,k} \langle \partial_{x_j} ([\varphi_{\nu}, a_{jk} \partial_{x_k} u], \partial_{x_j} \partial_t u) + 2\text{Re} \sum_j \langle [\varphi_{\nu}, b_j] \partial_{x_j} u, \partial_t u \rangle \]
\[ + 2\text{Re} \langle [\varphi_{\nu}, c] u, \partial_t u \rangle + 2\text{Re} \langle (Lu)_{\nu}, \partial_t u \rangle, \]
where \( \tilde{C} \) depends only on \( a_{jk}, b_j \) and \( c \).

c) the estimate for the total energy
We remark that from [6, Lemma 1 and 2] we have that there exists \( C > 0 \) depending only on the function \( \alpha \) such that
\[ h(\nu, t) \]
\[ = \tilde{C} \int_0^t \left( \frac{\varepsilon_\nu 2^\nu}{(\alpha(s) + \varepsilon_\nu)^{\frac{1}{2}}} + \frac{|\alpha'(s)|}{\alpha(s) + \varepsilon_\nu} + \frac{1}{(\alpha(s) + \varepsilon_\nu)^{\frac{1}{2}}} + 1 \right) ds \]
\[ \leq C \nu \]
for all \( t \in [0, T] \) and for all \( \nu \geq 1 \). We will need also the following result; the proof is very similar to those ones for the quoted lemmas from [6]: we let it to the reader.
Lemma 3. There exists $C > 0$ depending only on the function $\alpha$ such that

$$|h(\nu, t) - h(\nu + 1, t)| \leq C$$

for all $t \in [0, T]$ and for all $\nu \geq 1$.

Now we define the total energy

$$\tilde{E}(t) = \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} E_{\nu, \xi, \nu} u(t),$$

where $\sigma$ is a positive constant to be fixed. Our goal is to prove that

$$\tilde{E}'(t) \leq \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \| (Lu)_{\nu} \|_{L^2}^2,$$

and from this the inequality (1) will follow. From (9) we obtain that

$$\tilde{E}'(t) \leq \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \sum_{j,k} \langle [\varphi_{\nu}, a_{jk}] \partial_{x_k} u, \partial_{x_j} \partial_t u_{\nu} \rangle$$

$$\quad + \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \sum_{j} \langle [\varphi_{\nu}, b_j] \partial_{x_j} u, \partial_t u_{\nu} \rangle$$

$$\quad + \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \langle [\varphi_{\nu}, c] u, \partial_t u_{\nu} \rangle$$

$$\quad + \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \langle (Lu)_{\nu}, \partial_t u_{\nu} \rangle.$$

We start to estimate

$$A = \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \sum_{j,k} \langle [\varphi_{\nu}, a_{jk}] \partial_{x_k} u, \partial_{x_j} \partial_t u_{\nu} \rangle.$$

We use the technique of \[8, Lemma 4.4\]. We set $\psi_{\mu} = \varphi_{\mu-1} + \varphi_{\mu} + \varphi_{\mu+1}$ ($\varphi_{-1} \equiv 0$); using also \[8\] we have

$$A \leq \sum_{\nu, \mu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \sum_{j,k} \langle [\varphi_{\nu}, a_{jk}] \psi_{\mu}, \partial_{x_k} u_{\mu}, \partial_{x_j} \partial_t u_{\nu} \rangle$$

$$\leq \sum_{\nu, \mu=0}^{\infty} e^{-\frac{(h(\nu, t) - h(\mu, \xi))}{2}} n^2 \nu^2 \| [\varphi_{\nu}, a] \psi_{\mu} \|_{L^2}$$

$$\cdot e^{-\frac{h(\nu, \xi)}{2} - \sigma t} \| \nabla u_{\mu} \|_{L^2} e^{-\frac{h(\nu, t)}{2} - \sigma t} \| \partial_t u_{\nu} \|_{L^2}.$$
where \( \| \cdot \|_\mathcal{L} \) is the norm of bounded linear operators from \( (L^2(\mathbb{R}^n))^n \) into itself. We remark now that

\[
\| [\varphi_\nu, a] \psi_\mu \|_\mathcal{L} \leq \alpha(t) \| [\varphi_\nu, \beta] \psi_\mu \|_\mathcal{L} \\
\leq (\alpha(t) + \varepsilon_\mu) \| [\varphi_\nu, \beta] \psi_\mu \|_\mathcal{L} \\
\leq c(\alpha(t) + \varepsilon_\mu)^{\frac{1}{2}} \| [\varphi_\nu, \beta] \psi_\mu \|_\mathcal{L},
\]

where \( c \) depends only on \( \alpha \). Finally

\[
(\alpha(t) + \varepsilon_\mu)^{\frac{1}{2}} \| \nabla u_\mu \|_{L^2} \leq \ell' \sum_{j,k} (a_{jk} + \delta_{jk} \varepsilon_\mu) \partial_{x_k} u_\mu, \partial_{x_j} u_\mu |^{\frac{1}{2}},
\]

where \( \ell' \) depends only on \( \beta_{jk} \). We deduce that

\[
A \leq c'' \sum_{\nu, \mu=0}^\infty e^{-\frac{(h(\nu, t) - h(\mu, t))}{2}} 2^{\nu} \| [\varphi_\nu, a] \psi_\mu \|_\mathcal{L} \\
\cdot e^{-\frac{\alpha(t)}{2} - \sigma t} \sum_{j, k} (a_{jk} + \delta_{jk} \varepsilon_\mu) \partial_{x_k} u_\mu, \partial_{x_j} u_\mu |^{\frac{1}{2}} e^{-\frac{h(\nu, t)}{2} - \sigma t} \| \partial_t u_\nu \|_{L^2}.
\]

The idea is now to use Schur’s lemma. We introduce

\[
k_{\nu\mu}(t) = e^{-\frac{(h(\nu, t) - h(\mu, t))}{2}} 2^{\nu} \| [\varphi_\nu, a] \psi_\mu \|_\mathcal{L}.
\]

We have to consider

\[
\sup_{\nu} \sum_{\mu} |k_{\nu\mu}(t)|, \quad \sup_{\mu} \sum_{\nu} |k_{\nu\mu}(t)|.
\]

To this end the following lemma will be useful.

**Lemma 4.** Let \( |\nu - \mu| \leq 2 \).

Then there exists \( C \) depending only on \( \beta \) such that

\[
\| [\varphi_\nu, \beta] \psi_\mu \|_\mathcal{L} \leq C 2^{-\nu}.
\]

Let \( |\nu - \mu| \geq 3 \).

Then for all \( N > 0 \) there exists \( C_N > 0 \) depending only on \( N \) and \( \beta \) such that

\[
\| [\varphi_\nu, \beta] \psi_\mu \|_\mathcal{L} \leq C_N 2^{-N \max\{\nu, \mu\}}.
\]

**Proof.** It is easy to prove (15) (see [8, Prop. 3.6]). The inequality (16) can be proved taking into account the fact that the supports of \( \varphi_\nu \) and \( \psi_\mu \) are disjoint and using the asymptotic formula of \( [\varphi_\nu, \beta] \psi_\mu \) (see [8, Prop. 4.5]).

We denote now by \( \tilde{k}_{\nu\mu}(t) \) the value \( k_{\nu\mu}(t) \chi_{\{|\nu - \mu| \leq 2\}}(\nu, \mu) \), where \( \chi_\Omega \) is the characteristic function of the set \( \Omega \). Then

\[
\sum_{\mu} |\tilde{k}_{\nu\mu}(t)| = \sum_{j=\nu-2}^{\nu+2} |k_{\nu j}(t)|.
\]
From (11) and (15) we have that 
\[ \sum |\tilde{k}_{\nu\mu}(t)| \leq C \] for all \( t \in [0, T] \), where \( C \) does not depend on \( \nu \). Similarly \( \sum |\tilde{k}_{\nu\mu}(t)| \leq C \) for all \( t \in [0, T] \) and for all \( \mu \). We denote by \( k^*_{\nu\mu}(t) \) the value \( k_{\nu\mu}(t) \chi_{\{|\nu-\mu|\geq 3\}}(\nu, \mu) \).

From (10) we have that 
\[ |h(\nu, t) - h(\mu, t)| \leq C_1(\nu + \mu) \] for all \( \nu, \mu \) and for all \( t \in [0, T] \). Consequently using (16) we deduce 
\[ |k^*_{\nu\mu}(t)| \leq C_N e^{C_1(\nu + \mu) + \nu - N \cdot \max\{\nu, \mu\}}. \]

Easily we have 
\[ \sum_{\mu} |k^*_{\nu\mu}(t)| \leq C, \] where \( C \) does not depend on \( \nu \); similarly in the remaining case. We obtain 
\[ A \leq C \left( \sum_{\nu=0}^{\infty} e^{-\frac{h(\nu, t)}{2} - \sigma \cdot 2} \right) \sum_{j,k} \langle (a_{jk} + \delta_{jk} \varepsilon \nu) \partial_x j u_\nu, \partial_x k u_\nu \rangle \right) \frac{1}{2} 
\cdot \left( \sum_{\nu=0}^{\infty} e^{-\frac{h(\nu, t)}{2} - \sigma \cdot 2} \| \partial_t u_\nu \|_{L^2}^2 \right) \frac{1}{2}, \]

so that there exists \( C > 0 \) which does not depend on \( \sigma \) such that 
\[ A \leq C \tilde{E}(t). \]

Let us consider briefly the next term in (14). We set 
\[ B = \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \sum_{j} \langle [\varphi_\nu, b_j] \partial_x j u, \partial_t u_\nu \rangle. \]

As before 
\[ B \leq \sum_{\nu, \mu=0}^{\infty} e^{-\frac{(h(\nu, t) - h(\mu, t))}{2}} 2 \| [\varphi_\nu, b] \psi_\mu \|_{L^2} \cdot e^{-\frac{h(\mu, t)}{2} - \sigma \cdot 2} \| \nabla u_\mu \|_{L^2} e^{-\frac{h(\nu, t)}{2} - \sigma \cdot 2} \| \partial_t u_\nu \|_{L^2}, \]

\[ \leq c \sum_{\nu, \mu=0}^{\infty} e^{-\frac{(h(\nu, t) - h(\mu, t))}{2}} \| [\varphi_\nu, b] \psi_\mu \|_{L^2} \cdot e^{-\frac{h(\mu, t)}{2} - \sigma \cdot 2} \| \nabla u_\mu \|_{L^2} \cdot e^{-\frac{h(\nu, t)}{2} - \sigma \cdot 2} \| \partial_t u_\nu \|_{L^2}. \]

A computation similar to the previous one gives 
\[ B \leq C \tilde{E}(t). \]

The estimate of the other terms from (14) is straightforward. We finally obtain 
\[ \tilde{E}'(t) \leq (-2\sigma + C) \tilde{E}(t) + \sum_{\nu=0}^{\infty} e^{-h(\nu, t) - 2\sigma t} \| (Lu)_\nu \|_{L^2}^2, \]
where $C$ depends on $a_{jk} , b_j , c$ but does not depend on $\sigma$. We choose $\sigma > C/2$ and (13) follows. From this the inequality (7) is obtained with a standard argument (see [8, p. 695]). The proof is complete.

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