Finite Element Modeling of Metasurfaces with Generalized Sheet Transition Conditions

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Abstract—A modeling of metasurfaces in the finite element method (FEM) based on generalized sheet transition conditions (GSTCs) is presented. The discontinuities in electromagnetic fields across a metasurface as represented by the GSTC are modeled by assigning nodes to both sides of the metasurface. The FEM-GSTC formulation in both 1D and 2D domains is derived and implemented. The method is extended to handle more general bianisotropic metasurfaces. The formulations are validated by several illustrative examples.

Index Terms—GSTC, FEM, Metasurface, Boundary condition, Susceptibility, Bianisotropy, Electromagnetic discontinuity.

I. INTRODUCTION

METASURFACES are electrically thin layers with embedded subwavelength-sized scatterers [1], [2]. They are two-dimensional (2D) reductions of three-dimensional (3D) metamaterials [3]–[5] and offer much richer functionalities than traditional frequency-selective surfaces [6]. The advantages of metasurfaces over metamaterials include lower loss, lighter weight, and easier fabrication. Applications of metasurfaces include polarization transformers [7], generalized refraction [8], broadband absorbers [9], spatial waveguides [10], remotely-controlled spatial processors [11], aberration free lens [12], [13], flat optical components [14], LED efficiency enhancers [15], and spatial isolators [16]. Metasurfaces achieve these functionalities by creating discontinuities in the electromagnetic fields. Such discontinuities can be modeled by generalized sheet transition conditions (GSTCs) [1], [17]. Therefore, it is important to develop numerical modeling of GSTCs for the study of metasurfaces.

The modeling of GSTCs in the finite difference method has been recently reported in [18]. In this paper, we present the modeling of GSTCs in the finite element method (FEM), which is one of the most widely used numerical methods to simulate electromagnetic boundary-value problems (BVP) [7], [19]. The FEM is particularly suited for solving practical engineering problems given its ability to model complex geometries with adaptive tetrahedral meshes. At present, commercial electromagnetic simulation softwares can model several boundary conditions, such as perfect electric conductor (PEC), perfect magnetic conductor (PMC), periodic boundary condition (PBC), standard impedance boundary condition (SIBC), radiation boundary condition (RBC), and perfectly matched layer (PML). Recently, a formulation to incorporate a generalized impedance boundary condition (GIBC) has also been proposed [20]. However, no commercial CAD tools have yet incorporated the modeling of GSTCs. Since the FEM is the computational method used in the most frequently used CAD tools and since metasurfaces have become increasingly prominent in electromagnetic engineering, there is clearly a need for the modeling of GSTCs in the FEM framework. It should be noted that physical metasurfaces have a finite subwavelength thickness. Simulating such structures directly would result in very dense meshes around the metasurfaces and hence compromise the simulation efficiency. By replacing a physical metasurface by an equivalent GSTC, the burden of mesh generation can be reduced significantly and the simulation efficiency can be enhanced considerably. This is particularly important in simulation scenarios where multiple metasurfaces are involved or when repetitive simulations are required for optimization [21].

The organization of the paper is as follows. Section II recalls the GSTC metasurface synthesis equations. This is followed by the FEM-GSTC formulation in 1D and 2D domains in Sections III and IV, respectively. Section V presents some simulation results, and Section VI extends the method to simulate bianisotropic metasurfaces. Conclusions are provided in Section VII.

II. METASURFACE SYNTHESIS EQUATIONS

For a metasurface placed perpendicular to the z direction of a cartesian coordinate system, GSTCs in their most general form are given as [17], [21]

\[ \begin{align*}
\hat{\vec{z}} \times \Delta \vec{H} &= j \omega \vec{P}_{||} - \hat{\vec{z}} \times \nabla || M_z, \\
\hat{\vec{z}} \times \Delta \vec{E} &= -j \omega \mu \vec{M}_{||} - \hat{\vec{z}} \times \nabla || \left( \frac{P_z}{\varepsilon} \right),
\end{align*} \]

where \( \nabla || = \hat{x} \partial / \partial x + \hat{y} \partial / \partial y, \) \( \vec{P} \) and \( \vec{M} \) are the electric and magnetic polarization densities, respectively, and \( \varepsilon \) and \( \mu \) are the permittivity and permeability of the surrounding medium. Moreover, \( \Delta \vec{E} \) and \( \Delta \vec{H} \) represent respectively the differences between \( \vec{E} \) and \( \vec{H} \) on the two sides of the metasurface. For a problem of reflection and transmission, \( \Delta \psi = \psi^{tr} - (\psi^{ref} + \psi^{inc}) \) with the superscripts “tr,” “ref,” and “inc” denoting the transmitted, reflected, and incident fields, respectively. Throughout this work, the normal components of the polarization densities are assumed to be zero, i.e. \( P_z = M_z = 0 \) [18], [21]. The polarization densities can be expressed as

\[ \vec{P} = \varepsilon \varepsilon_{ee} \vec{E}_{av} + \sqrt{\varepsilon \mu} \chi_{em} \vec{H}_{av} \]

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\[ M = \bar{\chi} \bar{\chi}_{mm} \bar{H}_{av} + \sqrt{\epsilon \mu} \bar{\chi}_{me} \bar{E}_{av} \]  

(2b)

where \( \bar{\chi}_{ee}, \bar{\chi}_{mm}, \bar{\chi}_{em} \), and \( \bar{\chi}_{me} \) are the electric/magnetic (first e/m subscripts) susceptibility tensors describing the response to the electric/magnetic (second e/m subscripts) excitations, and the subscript “av” denotes the average of the fields on both sides of the metasurface. The computational domain discretization along with element numbers which material and field variations exist. The computational tensor elements with either \( \epsilon \) or \( \mu \) are applicable for a general bianisotropic metasurface.

Substituting (2) into (1) results in the following metasurface synthesis equations:

\[
\begin{align*}
\left(-\Delta H_y\frac{\partial}{\partial z}\right) &= j\omega \epsilon \left( \chi_{ee}_{xx} \chi_{ee}_{yy} + \chi_{ee}_{yx} \right) E_{x,av} + j\omega \sqrt{\epsilon \mu} \chi_{ee}_{xy} \left( H_{x,av} \right), \\
\left(\Delta E_y - \Delta E_x\right) &= j\omega \mu \left( \chi_{mm}_{xx} \chi_{mm}_{yy} + \chi_{mm}_{yx} \right) H_{x,av} + j\omega \sqrt{\epsilon \mu} \chi_{mm}_{xy} \left( E_{x,av} \right),
\end{align*}
\]

(3a, 3b)

which are applicable for a general bianisotropic metasurface. In the case of a mono-anisotropic metasurface, \( \bar{\chi}_{ee} = \bar{\chi}_{me} = 0 \). Throughout this paper, we have assumed, for simplicity but without loss of generality, that the cross-polarization terms (i.e., tensor elements with either \( xy \) or \( yx \) as the superscript) in all the four susceptibility tensors are zero.

### III. GSTCs in 1D FEM

Consider a 1D BVP where \( z \) is the only dimension along which material and field variations exist. The computational domain extends from \( z = 0 \) to \( z = L \) and a GSTC surface is located at \( z = z_m \), \( 0 < z_m < L \). Assuming \( E_z(z) \) and \( H_y(z) \) as the field quantities, the scalar wave equation for \( E_z(z) \) is given by

\[ \frac{d}{dz} \left[ \frac{1}{\mu_r(z)} \frac{dE_z}{dz} \right] + k_r^2 \epsilon_r(z) E_z(z) = 0. \]

(4)

The FEM domain discretization along with element numbers and global node numbers is shown in Fig. 1 where \( E_{z1} \) is located at \( z = 0 \) and \( E_{z5} \) is located at \( z = L \). Nodes 4 and 5 are placed on either side of the GSTC surface. The electric field in the element \( e \) can be expressed by interpolating the nodal electric field values using linear basis functions [19] as

\[ E^e_z(z) = \sum_{j=1}^{2} E^e_{zj} N_j^e(z), \]

(5)

where \( N_j^e(z) \) is the linear basis function. Using Galerkin’s method yields the following linear systems of equations for the unknown nodal values:

\[
\begin{bmatrix}
K_{11}^e & K_{12}^e & K_{13}^e & 0 \\
K_{21}^e & K_{22}^e + K_{13}^e & K_{33}^e & 0 \\
0 & K_{33}^e & K_{44}^e + K_{13}^e & K_{55}^e \\
0 & 0 & K_{55}^e & K_{66}^e + K_{13}^e \\
0 & 0 & 0 & K_{66}^e
\end{bmatrix}
\begin{bmatrix}
E_{x1}^e \\
E_{x2}^e \\
E_{x3}^e \\
E_{x4}^e \\
E_{x5}^e \\
E_{x6}^e
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{bmatrix}
\]

(6a, 6b)

The matrix elements are calculated by using

\[
K_{ij} = \int_{z_m}^{z_m} \left[ \frac{1}{\mu_r} \frac{dN_i^e}{dz} - k_i^2 \epsilon_r N_j^e \right] \frac{dE_r}{dz} dz,
\]

(7)

where \( \epsilon_r \) and \( \mu_r \) are the relative permittivity and relative permeability of the medium within the element. The elements of the right-hand-side vectors in (6) are given by

\[
\begin{align*}
b_1 &= -\frac{1}{\mu_r} \frac{dE_x}{dz} \bigg|_{z=0}, \\
b_2 &= \frac{1}{\mu_r} \frac{dE_x}{dz} \bigg|_{z=z_m}, \\
b_3 &= \frac{1}{\mu_r} \frac{dE_x}{dz} \bigg|_{z=z_m}, \\
b_4 &= \frac{1}{\mu_r} \frac{dE_x}{dz} \bigg|_{z=L}. \end{align*}
\]

(8a, 8b, 8c, 8d)

The \( b_1 \) and \( b_4 \) can be evaluated by applying the first-order ABC [19]. Such an ABC will excite the computational domain with a plane wave propagating along the +z direction and absorb all the outgoing waves. For the case of an incident wave represented by \( e^{-jkz} \), \( b_1 \) and \( b_4 \) are given by

\[
\begin{align*}
b_1 &= -\frac{jk}{\mu_r} E_{x1} + \frac{2jk}{\mu_r}, \\
b_4 &= \frac{1}{\mu_r} E_{x8}. \end{align*}
\]

(9a, 9b)

The \( b_4 \) and \( b_8 \) can be used to incorporate the metasurface synthesis equations [3]. From the Maxwell-Faraday equation,

\[
\begin{align*}
b_4 &= -j \omega \mu_r H_y \bigg|_{z=z_m}, \\
b_5 &= j \omega \mu_r H_y \bigg|_{z=z_m}. \end{align*}
\]

(10a, 10b)

For a mono-isotropic metasurface, equation (3) reduces to

\[
\begin{align*}
E_x|_{z=z_m} - E_x|_{z=z_m} &= -j \omega \mu_r \nabla_{x} \nabla_{x} \left( H_y|_{z=z_m} + H_y|_{z=z_m} \right), \\
H_y|_{z=z_m} - H_y|_{z=z_m} &= -j \omega \mu_r \nabla_{x} \nabla_{x} \left( E_x|_{z=z_m} + E_x|_{z=z_m} \right). \end{align*}
\]

(11a, 11b)

Inverting the above equations yields the magnetic field components on either side of the GSTC surface in terms of the electric field components as

\[
\begin{bmatrix}
j \omega \mu_r H_y |_{z=m} \\
-j \omega \mu_r H_y |_{z=m}
\end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} E_x |_{z=m} \\
E_x |_{z=m}\end{bmatrix}
\]

(12)
with $A$ and $B$ given by
\[
A = \frac{k^2 \chi_{ee} \chi_{mm}^{yy} - 4}{4 \chi_{mm}^{yy}}, \quad (13a)
\]
\[
B = \frac{k^2 \chi_{ee} \chi_{mm}^{yy} + 4}{4 \chi_{mm}^{yy}}. \quad (13b)
\]

With these, $b_4$ and $b_5$ can be written in terms of the electric field values adjacent to the GSTC surface as
\[
b_4 = \frac{B}{\mu_r} E_{x4} + \frac{A}{\mu_r} E_{x4}, \quad (14a)
\]
\[
b_5 = \frac{A}{\mu_r} E_{x4} + \frac{B}{\mu_r} E_{x4}. \quad (14b)
\]

The above two equations incorporate GSTC in 1D FEM. In the code, the locations of the nodes corresponding to $E_{x4}$ and $E_{x5}$ can be at $z = z_m$. However, $E_{x4}$ should be used to calculate the field solution in element 3 and $E_{x5}$ should be used to calculate the field solution in element 4.

IV. GSTCs in 2D FEM

The 2D computational domain considered in this work is shown in Fig. 2 where a finite-sized GSTC surface is located at $z = z_m$. Assuming a TE polarization ($E_x, E_z, H_y$), the wave equation for $H_y(x, z)$ reads
\[
\frac{\partial}{\partial x} \left[ \frac{1}{\epsilon_r(x, z)} \frac{\partial H_y}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \frac{1}{\epsilon_r(x, z)} \frac{\partial H_y}{\partial z} \right] + k_0^2 \mu_r(x, z) H_y = 0. \quad (15)
\]

Applying Galerkin’s method to (15) results in a system of equations for nodal values of $H_y$, which can be written as
\[
\begin{bmatrix}
K_{11} & K_{12} & \cdots & K_{1N} \\
K_{21} & K_{22} & \cdots & K_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{N1} & K_{N2} & \cdots & K_{NN}
\end{bmatrix}
\begin{bmatrix}
H_{y1} \\
H_{y2} \\
\vdots \\
H_{yN}
\end{bmatrix}
= \begin{bmatrix}
g_1 \\
g_2 \\
\vdots \\
g_N
\end{bmatrix}, \quad (16)
\]

where $N$ is the total number of nodes in the computational domain. The calculation of the elements of the stiffness matrix, $K_{ij}$, is same as that of 2D FEM without any GSTC surfaces [19]. The elements $g_j$ are non-zero only if the corresponding node (i.e. node $j$) lies on the boundary or if the node lies on a surface (or curve) where there is a field discontinuity [19].

Similar to the 1D case, a GSTC surface is modeled by placing nodes both above and below the GSTC surface as shown in Fig. 3. It should be noted that the nodes above and below the GSTC surface do not necessarily need to have different locations. But the nodes above the GSTC surface should be used to calculate the field inside the triangular elements which are located above the GSTC surface and the same applies for the nodes below the GSTC surface. Consider Fig. 3 where

Fig. 2. 2D FEM computational domain.

Fig. 3. Double nodes around the GSTC surface.
where the coefficients $C$ and $D$ are given by

$$
C = \frac{k^2 \chi_{xx} \chi_{yy}}{4 \chi_{ee}}, \quad (20a)
$$

$$
D = \frac{k^2 \chi_{xx} \chi_{yy}}{4 \chi_{ee}}, \quad (20b)
$$

In the segment $ab$, $E_x \big|_{z^m}$ will be denoted as $E^{ab}_x(\xi)$ and in the segment $bc$, $E_x \big|_{z^m}$ will be denoted as $E^{bc}_x(\xi)$. Following this notation, we obtain

$$
\begin{align*}
\frac{j \omega \xi}{\epsilon_r} E^{ab}_x(\xi) & = \frac{C_p}{\epsilon_r} H^{ab}_y(\xi) + \frac{D_p}{\epsilon_r} H^{de}_y(\xi) \quad (21a) \\
\frac{j \omega \xi}{\epsilon_r} E^{bc}_x(\xi) & = \frac{C_r}{\epsilon_r} H^{bc}_y(\xi) + \frac{D_r}{\epsilon_r} H^{ef}_y(\xi) \quad (21b)
\end{align*}
$$

where it has been assumed that the edges $ab$ and $bc$ are short enough such that the coefficients $C$ and $D$ are approximately constant on these segments. Therefore, $C_p, D_p, C_r,$ and $D_r$ are the values of the coefficients $C$ and $D$ at the edge midpoints $p$ and $r$, respectively, as shown in Fig. 3. Since linear finite elements are used, linear interpolation can be used to obtain the values of $H_y$ at the edges:

$$
\begin{align*}
H^{ab}_y(\xi) & = H_{ya} \left( 1 - \frac{\xi}{l_{ab}} \right) + H_{yb} \frac{\xi}{l_{ab}}, \quad 0 \leq \xi \leq l_{ab}, \quad (22a) \\
H^{de}_y(\xi) & = H_{yd} \left( 1 - \frac{\xi}{l_{ab}} \right) + H_{ye} \frac{\xi}{l_{ab}}, \quad 0 \leq \xi \leq l_{ab}, \quad (22b) \\
H^{bc}_y(\xi) & = H_{yb} \left( 1 - \frac{\xi}{l_{bc}} \right) + H_{yc} \frac{\xi}{l_{bc}}, \quad 0 \leq \xi \leq l_{bc}, \quad (22c) \\
H^{ef}_y(\xi) & = H_{ye} \left( 1 - \frac{\xi}{l_{bc}} \right) + H_{yf} \frac{\xi}{l_{bc}}, \quad 0 \leq \xi \leq l_{bc}. \quad (22d)
\end{align*}
$$

In the above equations, it is assumed that $l_{ab} = l_{de}$ and $l_{bc} = l_{ef}$, which is equivalent of having an identical mesh just above and below the GSTC surface. Substituting (22) into (21) and followed by further substitution into the integrals in (18), an expression for $g_b$ can be obtained as

$$
g_b = \frac{C_{p} l_{ab}}{6 \epsilon_r} H_{ya} + \frac{D_{p} l_{ab}}{6 \epsilon_r} H_{yd} + \left( \frac{C_{p} l_{ab}}{3 \epsilon_r} + \frac{C_{r} l_{bc}}{3 \epsilon_r} \right) H_{yb} + \left( \frac{D_{p} l_{ab}}{3 \epsilon_r} + \frac{D_{r} l_{bc}}{3 \epsilon_r} \right) H_{ye}.
$$

By following the same procedure, an expression for $g_c$ can be obtained as

$$
g_c = \frac{C_{p} l_{de}}{6 \epsilon_r} H_{yd} + \frac{D_{p} l_{de}}{6 \epsilon_r} H_{ya} + \left( \frac{C_{p} l_{de}}{3 \epsilon_r} + \frac{C_{r} l_{ef}}{3 \epsilon_r} \right) H_{yb} + \left( \frac{D_{p} l_{de}}{3 \epsilon_r} + \frac{D_{r} l_{ef}}{3 \epsilon_r} \right) H_{ye}.
$$

The evaluation of $g_b$ and $g_c$ for every node above and below the GSTC surface completes the incorporation of GSTC into 2D FEM.

V. SIMULATION RESULTS

This section presents simulation examples to validate FEM-GSTC in both 1D and 2D domains. The metasurface susceptibilities are synthesized by using (11). For vacuum on either side of the metasurface, the susceptibilities are obtained as

$$
\chi^{xx}_{ee} = \frac{2}{j \omega \mu_o} \left[ H^{inc}_{y} + H^{ref}_{y} - H^{tr}_{y} \right], \quad (25a)
$$

$$
\chi^{yy}_{mm} = \frac{2}{j \omega \epsilon_o} \left[ E^{inc}_{x} + E^{ref}_{x} - E^{tr}_{x} \right]. \quad (25b)
$$

The simulation frequency is 5 GHz. For both 1D and 2D codes, a simple first-order analytical ABC is used, although the use of the second-order ABC or PML would result in more accurate results.

A. 1D Example

For validating 1D FEM-GSTC, a fully absorbing metasurface is simulated, for which the reflected and transmitted fields are zero in (25). The susceptibilities for such a metasurface are thus found as $\chi^{xx}_{ee} = \chi^{yy}_{mm} = 2 / j \omega \epsilon_o$, whose negative imaginary nature indicate dissipation.

The total length of the computational domain is 20 $\lambda$. The metasurface is located at $z_m = 10 \lambda$. A plane wave with the electric field of magnitude 1 V/m is incident on the metasurface from the left. The simulation results are plotted in Fig. 4 where it can be seen that the transmitted field to the right of the metasurface is zero, as specified. On the left side of the metasurface, only the incident wave is present, corresponding to unity-magnitude quadrature real and imaginary phasor parts. If a reflected wave were present, one would observe a partly standing-wave pattern with varying field magnitude.

1 This may be easily verified by setting $\epsilon_o = 1 + \chi^{ee}_{ee,im}$. We have then $k_z = n k_0 = \sqrt{\epsilon_o k_0} = \sqrt{1 + j \chi^{ee}_{ee,im} k_0} = \left( 1 + j \chi^{ee}_{ee,im}/2 \right) k_0$. So a negative $\chi^{ee}_{ee,im}$ implies a decaying wave along $+z$ direction.
B. 2D Examples

The dimension of the computational domain used for the two 2D FEM-GSTC examples is $26\lambda \times 26\lambda$. The first-order ABC is used on all of the four boundaries of the rectangular computational domain. A finite-sized GSTC surface is located at $z_m = 13\lambda$ with a dimension along the $x$-axis of $20\lambda$. A plane wave multiplied by a Gaussian profile (Gaussian variation along the $x$ direction) is incident on the metasurface from below.

The first example considers a generalized refracting metasurface with no reflection [21]. The metasurface transforms a normally incident plane wave to a plane wave propagating at $\pi/4$ radians to the metasurface normal. The susceptibilities are synthesized using (25) and explicitly given, plotted and interpreted in [?]. These monoisotropic susceptibilities essentially correspond to a phase-gradient metasurface with loss (and hence negative imaginary susceptibility). The simulation results are plotted in Figs. 5 and 6. From these figures, the expected refraction at $\pi/4$ radians is clearly observed. In Fig. 5, it can be seen that the metasurface does not create any reflections, as specified. Note that the slight standing-wave pattern observed at the top right corner of Fig. 6 is due to the reflections from the absorbing boundary of the computational domain, where the first-order ABC was used. If the second-order ABC or a PML is used, these reflections can be reduced. Since the goal of this work is to implement GSTC in FEM, this issue is not further studied in this case. In Fig. 6 weak scattering can be seen at the left end of the Gaussian beam. This could be due to the fact that the susceptibilities were synthesized for the case of plane waves on either side of the metasurface, whereas in the simulation, a plane wave modulated by a transverse Gaussian profile is used. Such a wave is not an exact solution to the vector wave equation. The COMSOL simulation results for a generalized refraction metasurface are reported in [18]. In the COMSOL simulations, the metasurface was represented as a thin slab of a subwavelength thickness. The COMSOL simulation results in [18] showed unspecified refracted beams. Similar to the FDTD-GSTC [18], the FEM-GSTC does not result in these spurious refracted beams.

The second example is of a fully absorbing metasurface, which is simply the bi-dimensional counterpart of Sec. V.A. The simulation results are shown in Figs. 7 and 8. As specified, zero transmission and reflection can be verified in these figures. The COMSOL simulations for the same problem are reported in [18], where the results showed a partial transmission of the incident beam.

VI. FEM-GSTC FOR BIANISOTROPIC METASURFACES

The FEM-GSTC described in the previous sections can be extended to model bianisotropic metasurfaces. In this section, we consider bianisotropic metasurfaces where the off-diagonal terms of all the four susceptibility tensors are zero, i.e. $\chi_{xy}^{ee} = \chi_{yx}^{ee} = \chi_{xy}^{om} = \chi_{yx}^{om} = 0$. The FEM-GSTC formulation can then be applied as follows.
\( \chi_{ee}^{x} = 0, \chi_{em}^{x} = \chi_{me}^{y} = 0, \chi_{mm}^{x} = \chi_{mm}^{y} = 0, \) and \( \chi_{me}^{y} = \chi_{me}^{x} = 0 \) in (3). In this case, equation (3) becomes

\[
E_{x}\big|_{z_{m}} = -\frac{j\omega \mu \chi_{mm}^{y}}{2} \left( H_{y}\big|_{z_{m}} + H_{y}\big|_{z_{m}} \right)
\]

\[
H_{y}\big|_{z_{m}} - H_{y}\big|_{z_{m}} = -\frac{j\omega \sqrt{\mu \varepsilon \chi_{me}^{x}}}{2} \left( E_{x}\big|_{z_{m}} + E_{x}\big|_{z_{m}} \right)
\]

\[
E_{y}\big|_{z_{m}} - E_{y}\big|_{z_{m}} = -\frac{j\omega \sqrt{\mu \varepsilon \chi_{me}^{y}}}{2} \left( E_{x}\big|_{z_{m}} + E_{x}\big|_{z_{m}} \right)
\]

\[
H_{x}\big|_{z_{m}} - H_{x}\big|_{z_{m}} = -\frac{j\omega \sqrt{\mu \varepsilon \chi_{me}^{x}}}{2} \left( E_{y}\big|_{z_{m}} + E_{y}\big|_{z_{m}} \right)
\]

It may be observed from the coupling between field components in these equations that the diagonal elements of the \( \chi_{em}^{x} \) and \( \chi_{me}^{y} \) tensors result in a gyrotropic (chiral) metasurface.

Consider a 1D metasurface problem similar to that in Section III. For a BVP with variations only along the \( z \) direction, there are two independent field modes: \( \{ E_{x}, H_{y} \} \) and \( \{ E_{y}, H_{x} \} \). A bianisotropic metasurface induces a coupling between \( \{ E_{x}, H_{y} \} \) and \( \{ E_{y}, H_{x} \} \) field modes, as seen in (26). Therefore, simulating a bianisotropic metasurface will require a simultaneous processing of the \( x \) and \( y \) field components. The same principle would apply to a metasurface with off-diagonal components in \( \chi_{ee}^{x} \) and \( \chi_{mm}^{x} \). Thus both \( E_{x} \) and \( E_{y} \) need to be assigned to the FEM nodes. The domain discretization and node assignment for the case of 8 elements are shown in Figs. 4 and 7 for \( E_{x} \) and \( E_{y} \), respectively. The solution vector is \( [E_{x1} \cdots E_{x8} \ E_{y1} \cdots E_{y8}]^{T} \). As in

Section III, a system of equations can be written for the \( E_{y} \) components as

\[
\begin{bmatrix}
K_{11}^{e} & K_{12}^{e} & 0 & 0 \\
K_{21}^{e} & K_{22}^{e} + K_{11}^{e} & 0 & 0 \\
0 & K_{21}^{e} & K_{22}^{e} + K_{11}^{e} & 0 \\
0 & 0 & K_{21}^{e} & K_{22}^{e} + K_{11}^{e}
\end{bmatrix}
\begin{bmatrix}
E_{y1} \\
E_{y2} \\
E_{y3} \\
E_{y4}
\end{bmatrix}
= \begin{bmatrix}
b_{9} \\
b_{10} \\
b_{11} \\
b_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{11}^{m} & K_{12}^{m} & 0 & 0 \\
K_{21}^{m} & K_{22}^{m} + K_{11}^{m} & 0 & 0 \\
0 & K_{21}^{m} & K_{22}^{m} + K_{11}^{m} & 0 \\
0 & 0 & K_{21}^{m} & K_{22}^{m} + K_{11}^{m}
\end{bmatrix}
\begin{bmatrix}
E_{y5} \\
E_{y6} \\
E_{y7} \\
E_{y8}
\end{bmatrix}
= \begin{bmatrix}
b_{13} \\
b_{14} \\
b_{15} \\
b_{16}
\end{bmatrix}
\]

The stiffness matrix elements are given by equation (1). The elements of the right-hand-side vectors in (27) are given by

\[
b_{9} = -\frac{1}{\mu_{r}} \frac{dE_{y}}{dz} \big|_{z=0},
\]

\[
b_{10} = \frac{1}{\mu_{r}} \frac{dE_{y}}{dz} \big|_{z=z_{m}},
\]

\[
b_{11} = -\frac{1}{\mu_{r}} \frac{dE_{y}}{dz} \big|_{z=z_{m}^{+}},
\]

\[
b_{12} = \frac{1}{\mu_{r}} \frac{dE_{y}}{dz} \big|_{z=L}.
\]

Similar to \( b_{1} \) and \( b_{8} \), the elements \( b_{9} \) and \( b_{16} \) can be evaluated with the first-order ABC. The other elements \( b_{4}, b_{5}, b_{12}, \) and \( b_{13} \) can be evaluated by using Maxwell’s equations and GSTCs in (26). From the Maxwell-Faraday equation, \( b_{4}, b_{5}, b_{12}, \) and \( b_{13} \) can be converted to expressions in terms of \( H_{x} \) and \( H_{y} \) on either side of the metasurface. This is followed by using (26) to express \( H_{y}\big|_{z_{m}}, \), and \( H_{y}\big|_{z_{m}^{+}}, H_{x}\big|_{z_{m}}, \) and \( H_{x}\big|_{z_{m}^{+}} \), in terms of \( E_{x}\big|_{z_{m}}, E_{y}\big|_{z_{m}}, E_{x}\big|_{z_{m}^{+}}, \) and \( E_{x}\big|_{z_{m}^{+}} \). The final expressions for \( b_{4}, b_{5}, b_{12}, \) and \( b_{13} \) are

\[
\begin{bmatrix}
b_{4} \\
b_{5} \\
b_{12} \\
b_{13}
\end{bmatrix} = \begin{bmatrix}
-j\omega \mu \varepsilon H_{y}\big|_{z_{m}} \\
j\omega \varepsilon H_{y}\big|_{z_{m}} \\
j\omega \mu H_{x}\big|_{z_{m}} \\
-j\omega \mu H_{x}\big|_{z_{m}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{1} & A_{2} & A_{3} & A_{4} \\
A_{2} & A_{3} & -A_{4} & -A_{3} \\
A_{5} & A_{6} & A_{7} & A_{8} \\
-A_{6} & -A_{3} & A_{7} & A_{8}
\end{bmatrix}
\begin{bmatrix}
E_{x}\big|_{z_{m}} \\
E_{x}\big|_{z_{m}^{+}} \\
E_{y}\big|_{z_{m}} \\
E_{y}\big|_{z_{m}^{+}}
\end{bmatrix}
= \begin{bmatrix}
A_{1} & A_{2} & A_{3} & A_{4} \\
A_{2} & A_{3} & -A_{4} & -A_{3} \\
A_{5} & A_{6} & A_{7} & A_{8} \\
-A_{6} & -A_{3} & A_{7} & A_{8}
\end{bmatrix}
\begin{bmatrix}
E_{x}\big|_{z_{m}} \\
E_{x}\big|_{z_{m}^{+}} \\
E_{y}\big|_{z_{m}} \\
E_{y}\big|_{z_{m}^{+}}
\end{bmatrix}
\]

\[
A_{1} = k^{2} \chi_{mm}^{y} \left( \chi_{ee}^{x} \chi_{mm}^{x} - \chi_{mm}^{x} \chi_{ee}^{x} \chi_{mm}^{y} \right) + 4 \chi_{mm}^{x}
\]

\[
A_{2} = k^{2} \chi_{mm}^{y} \left( \chi_{ee}^{x} \chi_{mm}^{x} - \chi_{mm}^{x} \chi_{ee}^{x} \chi_{mm}^{y} \right) - 4 \chi_{mm}^{x}
\]
The metasurface is illuminated with two plane waves, $E^{\text{inc}}_x = e^{-jk_0z}$ and $E^{\text{inc}}_y = e^{-jk_0z}$. The simulation parameters are same as those in Section V. The metasurface is located at $z_m = 10\lambda$. The real parts of $E_x$ and $E_y$ are shown in Fig. 10 where it can be observed that on the right side of the

![Fig. 10. FEM-GSTC simulation results for a bianisotropic metasurface.](image-url)

metasurface, there is only an $E_y$ component. This is due to the fact that the $x$ polarized wave incident on the metasurface from its left side is transformed to a $y$ polarized wave and the $y$ polarized wave incident on the metasurface from its left side is completely absorbed. It can also be seen that for $z < 10\lambda$, there are no reflections.

VII. CONCLUSION

In this paper, we presented the FEM modeling of metasurfaces based on GSTCs, where the discontinuities in electromagnetic fields across a metasurface were modeled by assigning nodes to both sides of the metasurface. We derived the FEM-GSTC formulation in both 1D and 2D domains and extended it to handle more general bianisotropic metasurfaces. We also presented several illustrative examples to validate the FEM-GSTC formulation. Future work includes extension of the method to 3D problems and curved metasurfaces. In 3D problems, the susceptibility tensor elements will be functions of both $x$ and $y$. The extension of 2D FEM-GSTC to 3D is straightforward. In 2D, the line integrals along the element edges were used to calculate the right-hand-side of the FEM system of equations. In 3D, surface integrals on the tetrahedral faces should be used to obtain the right-hand-side of the FEM system of equations. Even though this paper assumed zero off-diagonal elements in the susceptibility tensors, the formulation can be easily extended to non-zero off-diagonal elements. In such a case, the matrix elements in (29) will be more involved. It should be noted that in this paper, the GSTC surface is handled by placing field nodes on either side of the GSTC surface. A similar procedure is used in the FDFD formulation in [18]. However, the advantage of FEM-GSTC is its flexibility in placing these nodes conforming to the metasurface geometry. The rectangular Yee cells of FDFD
limits this flexibility. Hence FEM-GSTC is more efficient for simulating arbitrarily shaped metasurfaces or in general arbitrarily shaped spatial electromagnetic field discontinuity.

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