An Upper Bound on Computation for the Anharmonic Oscillator

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Abstract

For a quantum system with energy $E$, there is a limitation in quantum computation which is identified by the minimum time needed for the state to evolve to an orthogonal state. In this paper, we will compute the minimum time of orthogonalization (i.e. quantum speed limit) for a simple anharmonic oscillator and find an upper bound on the rate of computations. We will also investigate the growth rate of complexity for the anharmonic oscillator by treating the anharmonic terms perturbatively. More precisely, we will compute the maximum rate of change of complexity and show that for even order perturbations, the rate of complexity increases while for the odd order terms it has a decreasing behavior.

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1 Introduction

It is well-known that the laws of physics can in principle determine the ultimate power of work done by computers; for example, the amount of information that a computer can process and also the speed of processing, as the two fundamental subjects, can be addressed theoretically by the laws of physics. In this way, the energy and the number of degrees of freedom that the system could achieve, can control the speed limit and the amount of memory limitations, respectively. It is worth mentioning that quantum mechanics imposes a limit on the speed of the evolution of quantum systems. In other words, the minimal time a system needs to evolve from an initial state to a final orthogonal state is defined as the quantum speed limit [1]. Actually, this limitation on the speed of processing can be translated into the maximum number of distinct states that a given system passes per unit of time. The states are orthogonal and in this sense, one can say that an upper bound on the speed of processing is enforced by the quantum mechanics.

The connection between theoretical physics and information theory has a rich history dating back to the study of black hole physics [2]. Nowadays, it is believed that black holes might be considered as a computational device that compresses a given definite amount of energy. Therefore, in this approach, the black hole can indeed perform with a certain rate of operations [3]. There is in fact a concrete relation between the black hole physics and the quantum information theory. Bekenstein [4] argued that black holes set a theoretical maximum on information storage which means memory is bounded.

Theoretically, Lloyd’s bound [5] defines an upper limit for the information processing, for a quantum system with a given average energy $E$, the upper limit on the number of operations is given by $\frac{2E}{\pi \hbar}$; this is the Lloyd’s bound.
Entanglement entropy and complexity are two significant concepts in theoretical physics that might be considered as a bridge between the fundamental laws of physics and information theory [6, 7]. Essentially, the entanglement entropy is a measure of entanglement between the degrees of freedom of two subsystems while the complexity measures the difficulty of doing a physical task, e.g., it quantifies the difficulty of evolving a reference quantum state into another (target) state. The computational complexity of an operation in computer science is defined by the minimum number of logical gates that are needed for solving a problem [8]. On the other hand, in quantum mechanics, quantum complexity is a criterion that shows us how difficult a task is. It is defined by the number of elementary unitary operations which are required to build up a desired final state from a given reference state [9].

It is also worth mentioning that black holes can, in principle, set fundamental limits on density, entropy and computational complexity [10]. Also, the action of the interior of the black hole can set an upper limit on information processing, as we mentioned above. The action of the interior of the black hole changes with a rate which is proportional to $\frac{2E}{\hbar}$. Further investigation of this bound seems to be important; In Ref. [11], we have disturbed the system by both a magnetic and an electric field and have studied this bound. Our main goal in this article is to examine Lloyd’s bound for a quantum system, anharmonic oscillator, with perturbation.

The layout of this paper is as follows. In Section 2, we will calculate the orthogonalization time for a simple anharmonic oscillator. In Section 3, we will explore the Lloyd’s bound for this system. The concluding remarks are given in section 4. Finally, we close the paper with an appendix where by explicit mathematical calculation we show that the growth rate of complexity for an anharmonic oscillator with even order perturbation, increases while for odd order the rate has a decreasing behavior.

2 Orthogonality Time in a Quartic Perturbed Anharmonic Oscillator

In the field of information processing and quantum measurements, the evolution of a quantum state to an orthogonal state and how fast the state can evolve are two important issues that have attracted a considerable attention. As a matter of fact, the orthogonal states can be distinguished from each other, therefore, a transition from a state to an orthogonal, might be considered as an elementary step of a computational process. In this regard, the maximal speed of evolution of a quantum system can be identified with the minimal time that a system needs to evolve from an initial state to another state orthogonal to it. This time is known as the quantum speed limit time. For a system of total energy $E$, the Margolus-Levitin formula [12] gives us the minimum time $\tau_\perp$, which it takes for $|\psi_0\rangle$ to evolve into an orthogonal state. In other words, for a given quantum state, say $|\Psi\rangle$, the orthogonality time which is needed to evolve to the orthogonal state, $|\Psi_{\tau_\perp}\rangle$ can be obtained via the following relation

$$S(\tau_\perp) \equiv \langle \Psi | \Psi_{\tau_\perp} \rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{-iE_n\tau_\perp} = 0. \quad (1)$$

which yields [11]

$$\tau_\perp = \frac{\pi}{2E}.$$

The above formula sets a fundamental limit on quantum computation. In this paper, we examine the minimum time of evolution for a simple harmonic oscillator with odd and even terms of perturbation. The corresponding Hamiltonian in one dimension is given by

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2 + \lambda_0x^{2k-1} + \lambda_e x^{2k}, \quad (2)$$
where $0 \leq \lambda_o, \lambda_e \ll 1$ and $k$ is a positive integer number. To do so, first let us study a quartic anharmonic oscillator that perturbed by the external fields and the generalization to all has been done in appendix.

The effect of the week external electric field on the charged anharmonic oscillator can be studied by adding the term like $H_p = qE x + \lambda x^4$ which leading to the following Hamiltonian\(^1\)

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 + qE x + \lambda x^4; \quad (3)$$

where $m$ and $q$ stand for the mass and charge of system, respectively. The lowest nonvanishing order of the energy shift is then given by

$$E_n = \omega(n + \frac{1}{2}) + \frac{3\lambda}{4m^2 \omega^2}[2n^2 + 2n + 1] - 2m \frac{\omega_e^2}{\omega^2} \quad (4)$$

where $n$ is positive integer or zero and $\lambda$ is small positive real parameter, also $\omega_e$ is defined by

$$\frac{qE}{2m} = \omega_e \quad (5)$$

\(1\)Note that through this paper we have used the natural units $\hbar = c = 1$.

Figure 1: The orthogonality time for generic coefficient (dashed line) and the case that we have chosen in the paper (solid line), for $N = 20$ (left panel) and $N = 100$ (right panel). In both figures we set $m = 1, \lambda = 3 \times 10^{-3}$ and $\omega_e = 3 \times 10^{-1}$.

In order to compute the average energy in the state $| \Psi \rangle$, let us suppose that our system passes through $N$ mutually orthogonal states in time $\tau$, consequently, the eigenfunction is displayed as follows

$$| \Psi \rangle = \sum_{n=0}^{N-1} c_n | n \rangle \quad (6)$$

For a quartic anharmonic oscillator, the expectation value of energy is given by

$$E = \langle \Psi | H | \Psi \rangle = \sum_{n=0}^{N-1} |c_n|^2 \left[ \omega \left(n + \frac{1}{2}\right) + \frac{3\lambda}{4m^2 \omega^2}[2n^2 + 2n + 1] - 2m \frac{\omega_e^2}{\omega^2} \right]. \quad (7)$$
To make calculations easier, one can start by considering the set of evolutions that pass through an exact cycle of $N$ which are supposed to be mutually orthogonal states at a constant rate in time $\tau$; In this case and for large $N$, one has $c_n = \sqrt{\frac{1}{N}}$, therefore, one obtain

$$E = \langle \Psi | H | \Psi \rangle$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left( \omega [n + \frac{1}{2}] + \frac{3 \lambda}{4m^2 \omega^2} [2n^2 + 2n + 1] - 2m \frac{\omega^2}{\omega^2} \right)$$

$$= \frac{N \omega}{2} + \frac{\lambda}{4m^2 \omega^2} + \frac{N^2 \lambda}{2m^2 \omega^2} - 2m \frac{\omega^2}{\omega^2}$$

(8)

It is worth mentioning that, for sufficiently large $N$, in figure.(1), we have plotted the resultant orthogonality time for generic four random coefficient for $N = 20$ and $N = 100$. As the figure shows for large $N$, the curves coincide to what we have considered in the paper.

Moreover, making use the relation given in appendix and also $\omega^2 = \frac{\lambda}{2m^3}$, one can show that

$$\tau_\perp \geq \frac{\pi}{N \omega + m \omega^2 + 2mN^2 \omega^2 - 4m \omega^2}.$$  

(9)

According to our definition of the time which takes for a given quantum system of energy $E$ to go from one state to an orthogonal state, the relation (9) shifts the bound to upper bound of the rate of complexity.

Figure 2: The rate of normalized complexification of the anharmonic oscillator for different values of mass for a fixed $N = 100$. Left panel: $\omega \lambda = 8 \times 10^{-3}$ and $\omega e = 65 \times 10^{-2}$. Right panel: $\omega \lambda = 10^{-2}$ and $\omega e = 65 \times 10^{-2}$.

3 Upper Bound of Complexification

Lloyd’s bound defines a fundamental upper limit on the speed of computations for a classical computer [5]. The information process is defined by successive application of logical gates which are needed to transform a given initial state to a final state. In other words, a logical gate refers

$\omega e \leq \sqrt{\frac{\lambda(\frac{1}{2} + N^2) + m^2 N \omega^3}{2m^3}}$

$^{2}$We are dealing with large $N$, therefore, the following limit on the electric field leads to non-negative energy
Figure 3: The rate of normalized complexification of the anharmonic oscillator. When the anharmonic parameter induction reaches a definite critical value given by Eq.(13), the behavior of this rate changes. Note that we set $N = 100$, $m = 1$. Note we have defined $\dot{\tilde{\mathcal{C}}}_\text{norm} = \pi \dot{\mathcal{C}} / N \omega$.

to the actual physical device that performs a logical operation and performing any operation takes some time $t$. If one defines a Hamiltonian action $H_g$ to a task which is done by given gate, then the operation is given by $U(t) = e^{-i H_g t}$. The unitary evolution takes the initial state $|0\rangle$ to desired final state $U(t)|0\rangle$ after time $t$. Consequently, a sequential application of $n$ gates, are given by [13]

$$|0\rangle \rightarrow U(t)|0\rangle, \quad U(t) = T \prod_i U(t_{i+1}, t_i),$$

where $T$ is the time ordered operator and $U(t_{i+1}, t_i)$ stand for an orthogonalizing gate. In the present case the rate of complexification gets a strong bound as follows

$$\dot{\mathcal{C}} \leq \frac{1}{\tau_\perp},$$

where $\tau_\perp$ is the orthogonalization time of the system. Therefore, in our case the rate of complexification is given by

$$\dot{\tilde{\mathcal{C}}} \leq \frac{1}{\pi} \left( N \omega + m \frac{\omega^2}{\omega^2} + 2m N^2 \frac{\omega^2}{\omega^2} - 4m \frac{\omega^2}{\omega^2} \right)$$

In figure (3) we have plotted the rate of complexity for different values of $\omega_\lambda$ and $\omega_e$. As it is shown from figure (2) and (3), there is a critical value for anharmonic parameter which beyond that the rate of complexity changes its behavior drastically which is given by

$$\omega^{\text{cri}}_\lambda = \frac{2\omega_e}{\sqrt{1 + 2N^2}}.$$  

The results indicate that the normalized complexification of the charged anharmonic oscillator saturates to a definite value for large $\omega$. However, there is a critical value for anharmonic parameter which defines the behavior of this saturation.

Figure (4) shows that the electric field can completely neutralize the additive effect of the anharmonic parameter, so that with the electric field and anharmonic parameter we do not see any change in the rate of complexity namely one has $\Delta \tilde{\mathcal{C}} = 0$. In this case, Eq.(13) is fully satisfied.

If we want to make a simple comparison with a charged harmonic oscillator in the presence of electric and magnetic fields, we can easily see that for charged anharmonic oscillator, like the
\[ \tilde{c} = 16 \times 10^{-3} \]
\[ \tilde{c} = 17 \times 10^{-3} \]
\[ \tilde{c} = 18 \times 10^{-3} \]
\[ \tilde{c} = 19 \times 10^{-3} \]
\[ \tilde{c} = 2 \times 10^{-2} \]

\[ \omega \approx \lambda \]

Figure 4: The effect of anharmonic parameter and electric field on the rate of complexity, where \( \tilde{\Delta C} = \frac{\pi - N \omega}{m} \). Note that we set \( N = 100 \), \( \tilde{\omega}_\lambda \equiv \frac{\omega}{\omega} \) and \( \tilde{\omega}_e \equiv \frac{\omega}{\omega} \)

Charged harmonic oscillator, the electric field has a decreasing effect on the rate of complexity. Instead, as can be clearly seen in figure (5), the anharmonic parameter, like the magnetic field in the charged harmonic oscillator, plays an effective role in increasing the rate of complexity [11].

4 Remarks and Conclusions

The Margolus-Levitin theorem [12] sets a limit on quantum computation by identifying the minimum time for a quantum system with an energy \( E \) to go from one state to an orthogonal state. In this paper, we have calculated the minimal time of orthogonalization for a simple anharmonic oscillator and have observed that the \( \langle E \rangle \) plays a crucial role in this process. On the other hand, there is a limitation or an upper bound on the rate of computations. This bound known as Lloyd’s bound and it deals with the minimal time to perform a task done by a physical device.

The minimal time is controlled by the energy \( E = \langle H \rangle \), which means that the energy of a system sets a limitation of computation. The upper limit on the number of operations for a quantum system with a given average energy \( E \), is given by \( \frac{2E}{\pi \hbar} \). The operations done on a state can be implemented quantum mechanically, in fact, there are inputs and outputs in any step of the computation which are classical states. These states are supposed to be orthogonal and have no superpositions with each other.

On the other hand in computer science, the complexity of an operation is a criterion that shows us how difficult the task is. In this way the complexity is related to the number of elementary unitary operations which is required to build up a desired state from a given reference state [9]. And in this sense, the complexity is affect by the energy of the system. The processing of information might be given by the time rate of change of the complexity. In this way, Lloyd’s bound defines the upper bound as \( \frac{d}{dt} \mathcal{C} \leq \frac{2E}{\pi \hbar} \).

In our previous work, we studied the complexity of a charged harmonic oscillator. We considered the harmonic oscillator in the presence of both magnetic and electric fields and found the minimum time needed for any state of our system to evolve into an orthogonal state. We observed that the time of orthogonality for small and large values of magnetic field behaves differently, and also, the rate of complexity increases/decreases by turning on the magnetic/electric field.

The main goal of this paper has been to examine the rate of complexification for a simple
Figure 5: contour plot $\Delta \hat{C}$ in terms of $\tilde{\omega}_\lambda$ and $\tilde{\omega}_e$. Note that we set $N = 100$. It is clear that the rate of complexity increases by $\tilde{\omega}_\lambda$ while it decreases by increasing of $\tilde{\omega}_e$.

anharmonic oscillator when evolving from an initial reference state to an orthogonal final state. The numerical computation showed that for those perturbation up to all orders, that have odd power of $x$, the rate of complexity decreases whereas the even power increases this rate and leads to more complexification.

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5 Appendix

5.1 Eigenvalue for perturbation theory

In this appendix, we find the energy change of the simple anharmonic oscillator whose Hamiltonian is given by (2), for the even as well as the odd perturbations. It is shown that the growth rate of complexity for an anharmonic oscillator for the even order perturbation, increases while for the odd order the rate has a decreasing behavior. The first-order change in the odd perturbation vanishes, on the other hand, the second-order correction to the energy becomes

$$E_{n_{\text{odd}}}^2 = -\lambda_0^2 \sum_{m \neq n} \frac{|\langle m | x^{2k-1} | n \rangle|^2}{E_{mn}}$$

which is negative, while one gets a positive number for even perturbation

$$E_{n_{\text{even}}}^1 = \lambda_e \langle n | x^{2k} | n \rangle,$$
where the position operator $x$ is given by
\[
x = \sqrt{\frac{1}{2m\omega}}(a + a^\dagger).
\] (16)

Not let us calculate the first-order corrections for a harmonic oscillator with applied perturbation say as $\lambda_1 x, \lambda_2 x^2, \lambda_3 x^3, \ldots$, which leads to
\[
\langle m | (a + a^\dagger) | n \rangle = \sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}
\]
\[
\langle m | (a + a^\dagger)^2 | n \rangle = 2n\delta_{m,n} + \delta_{m,n} + \sqrt{n-1}\sqrt{n}\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\delta_{m,n+2}
\]
\[
\langle m | (a + a^\dagger)^3 | n \rangle = 3n^{3/2}\delta_{m,n-1} + 3\sqrt{n+1}\delta_{m,n+1} + \sqrt{n-2}\sqrt{n-1}\sqrt{n}\delta_{m,n-3}
\]
\[
+ 3\sqrt{n+1}\delta_{m,n+1} + \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\delta_{m,n+3}
\] (17)

After doing some algebra one also finds
\[
\langle n | x^2 | n \rangle = \frac{2n + 1}{2m\omega}
\]
\[
\langle n | x^4 | n \rangle = \frac{3(2n^2 + 2n + 1)}{4m^2\omega^2}
\]
\[
\langle n | x^6 | n \rangle = \frac{5(2n + 1)(2n^2 + 2n + 3)}{8m^3\omega^3}
\]
\[
\langle n | x^8 | n \rangle = \frac{35(2n^4 + 4n^3 + 10n^2 + 8n + 3)}{16m^4\omega^4}
\]
\[
\langle n | x^{10} | n \rangle = \frac{63(2n + 1)(2n^4 + 4n^3 + 18n^2 + 16n + 15)}{32m^5\omega^5}
\] (18)

We can also get energy corrections up to the second level
\[
\sum_{m \neq n} \frac{|\langle m | x | n \rangle|^2}{E_{mn}} = \frac{1}{(2m\omega)\omega}
\]
\[
\sum_{m \neq n} \frac{|\langle m | x^3 | n \rangle|^2}{E_{mn}} = \frac{30n^2 + 30n + 11}{(2m\omega)^3\omega}
\]
\[
\sum_{m \neq n} \frac{|\langle m | x^5 | n \rangle|^2}{E_{mn}} = \frac{630n^4 + 1260n^3 + 2030n^2 + 1400n + 449}{(2m\omega)^5\omega}
\]
\[
\sum_{m \neq n} \frac{|\langle m | x^7 | n \rangle|^2}{E_{mn}} = \frac{3(4004n^6 + 12012n^5 + 42350n^4 + 64680n^3 + 81788n^2 + 51450n + 14793)}{(2m\omega)^7\omega}
\] (19)

References

[1] P. Pfeifer, How fast can a quantum state change with time? Phys. Rev. Lett. 70, 3365 (1993). https://doi.org/10.1103/PhysRevLett.70.3365
[2] S. W. Hawking, “Black hole explosions”, Nature 248, 30 (1974). doi:10.1038/248030a0

[3] L. B. Levitin, “Physical limitations of rate, depth, and minimum energy in information processing”, Int. J. Theor. Phys. 21 299 (1982). https://doi.org/10.1007/BF01857732

[4] J. D. Bekenstein, “Black Holes and Entropy”, Phys. Rev. D 7, 2333 (1973). https://doi.org/10.1103/PhysRevD.7.2333

[5] S. Lloyd, “Ultimate physical limits to computation”, Nature 406 1047 (2000), https://doi.org/10.1038/35023282. arXiv: quant-ph/9908043.

[6] M. Van Raamsdonk, “Evaporating Firewalls,” JHEP 1411, 038 (2014) doi:10.1007/JHEP11(2014)038 [arXiv:1307.1796 [hep-th]].

[7] L. Susskind, “Entanglement is not enough,” Fortsch. Phys. 64, 49 (2016) doi:10.1002/prop.201500095 [arXiv:1411.0690 [hep-th]].

[8] Sanjeev Arora, Boaz Barak, “Computational Complexity, A Modern Approach”, Cambridge University Press (2009).

[9] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entanglement Entropy: An Overview,” J. Phys. A 42, 504008 (2009) doi:10.1088/1751-8113/42/50/504008 [arXiv:0905.0932 [hep-th]].

[10] S. Lloyd, Y. J. Ng, “Black hole computers”. Sci. Am. 291, 52–61 (2004).

[11] R. Pirmoradian, M. Tanhayi, “On the Complexity of a Charged Quantum Oscillator,” Journal of the Korean Physical Society, 77, 2 (2020) 102-106, doi:10.3938/jkps.77.102 arXiv:1911.08886 [hep-th].

[12] N. Margolus and L. B. Levitin, The Maximum speed of dynamical evolution,” Physica D 120, 188 (1998). DOI: 10.1016/S0167-2789(98)00054-2, arXiv: quant-ph/9710043 [quant-ph].

[13] W. Cottrell and M. Montero, “Complexity is simple!,” JHEP 1802, 039 (2018) doi:10.1007/JHEP02(2018)039 [arXiv:1710.01175 [hep-th]].