A Global Trace Formula for Reductive Lie Algebras and the Harish-Chandra Transform on the Space of Characteristic Polynomials

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Abstract

In this paper an integral transform between spaces of nonstandard test functions on the affine space of dimension \( n \) is constructed. The integral transform satisfies a summation formula of Poisson type, which is derived from an analogue of the Arthur-Selberg trace formula for the Lie algebra of \( n \times n \) matrices. This paper also contains a proof of a trace formula for a general reductive Lie algebra defined over the field of rational numbers.

Introduction

The classical summation formula of Poisson

\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)
\]

has been related to the modularity of theta functions and the functional equation of zeta functions since the days of Jacobi and Riemann.

In his seminal paper \[1859\] Riemann derived the functional equation of the zeta function which now bears his name from the modular identity of Jacobi’s theta function, which is a consequence of the Poisson summation formula.

In his thesis \[Ta50\] Tate recast Riemann’s argument over the adeles and derived the functional equation of a large class of \( L \)-functions directly from his adelic Poisson summation formula. Tate’s method was later generalized to the standard \( L \)-functions of higher rank general linear groups and central simple algebras by Godement and Jacquet in \[GJ72\]. In their argument the group \( GL(n) \) is embedded into its Lie algebra \( gl(n) \), and the functional equation of standard \( L \)-functions of \( GL(n) \) is derived from the adelic Poisson summation formula for the vector space \( gl(n) \).

Recently in the work of Braverman and Kazhdan \[BK00\], Ngô \[Ng12\], Sakellaridis \[Sa12\] and Lafforgue \[La13\], the argument of Godement and Jacquet was partially generalized to more general automorphic \( L \)-functions:

Let \( G \) be a reductive group and let \( \rho \) be a representation of the dual group \( \hat{G} \) satisfying some mild assumptions, then the group \( G \) embeds into an algebraic monoid \( M_\rho \) which depends on \( \rho \), and the functional equation of the automorphic \( L \)-functions \( L(s, \pi, \rho) \) would follow from a conjectural summation formula of Poisson type for the monoid \( M_\rho \). In general \( M_\rho \) is nonlinear and possibly singular, and the conjectural Poisson summation formula would involve a nonstandard Fourier transform and spaces of nonstandard test functions.
The purpose of this paper is to construct a toy example of such a Poisson summation formula for the space $\mathcal{A}_n$ of characteristic polynomials of $n \times n$ matrices:

(6.4.4) Proposition There exist two nonstandard Schwartz spaces $S_0(\mathcal{A}_n(\mathbb{A}))$ and $S_1(\mathcal{A}_n(\mathbb{A}))$ on $\mathcal{A}_n(\mathbb{A})$ such that $S_0(\mathcal{A}_n(\mathbb{A}))$ is dense in $L^2(\mathcal{A}_n(\mathbb{A}))$ and an invertible integral transform $\mathcal{H}$ from $S_0(\mathcal{A}_n(\mathbb{A}))$ to $S_1(\mathcal{A}_n(\mathbb{A}))$ such that:

For each point $X$ in $\mathcal{A}_n(\mathbb{Q})$ there exists a nonzero constant $a(X)$ such that for all $\varphi$ in $S_0(\mathcal{A}_n(\mathbb{A}))$

$$
\sum_{X \in \mathcal{A}_n(\mathbb{Q})} a(X) \cdot \varphi(X) = \sum_{X \in \mathcal{A}_n(\mathbb{Q})} a(X) \cdot \mathcal{H}\varphi(X),
$$

the summand for a polynomial $X$ with repeated roots needs to be interpreted appropriately.

The Poisson summation formula for $\mathcal{A}_n$ is deduced from an analogue of the Arthur-Selberg trace formula for the Lie algebra $\mathfrak{gl}(n)$. In this paper such a trace formula is established for a general reductive Lie algebra $\mathfrak{g}$. One possible problem for future investigation is the Poisson summation formula for the affine space $\mathcal{A}_G$ defined as the adjoint quotient of $\mathfrak{g}$ for a general reductive group $G$. Such a Poisson summation formula will likely require a stable trace formula for $\mathfrak{g}$.

The organization of this paper is as follows:

In the first three chapters an analogue of the refined noninvariant version of the Arthur-Selberg trace formula for a general reductive Lie algebra $\mathfrak{g}$ is established. The arguments are parallel to the original arguments of Arthur, and the readers who are familiar with the trace formula for reductive groups should feel free to skip to Proposition (3.4.1).

In the fourth chapter the refined noninvariant trace formula for $\mathfrak{g}$ is converted to an analogue of the invariant version of the Arthur-Selberg trace formula.

In the fifth chapter the invariant trace formula for $\mathfrak{g}$ is lifted to an identity between vector-valued distributions that naturally generalize the orbital integrals on $\mathfrak{g}(\mathbb{A})$.

In the sixth chapter the spaces $S_0(\mathcal{A}_n(\mathbb{A}))$ and $S_1(\mathcal{A}_n(\mathbb{A}))$ and the integral transform $\mathcal{H}$ are constructed, and the Poisson summation formula is deduced from the trace formula for $\mathfrak{gl}(n)$.

The fifth chapter and the sixth chapter are independent of each other and could be read in either order.

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1 Preliminaries

1.1 Notations and definitions

(1.1.1) Let $G$ be a connected reductive group defined over the field of rational numbers $\mathbb{Q}$, let $\mathfrak{g}$ be its Lie algebra equipped with the adjoint action of $G$ from the right. More generally an algebraic group will be denoted by a capital roman letter, its Lie algebra by the corresponding lowercase fraktur letter, except for $\mathfrak{a}$ which is reserved for a Euclidean vector space.

Denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ at a place $v$. If $S$ is a finite set of places, denote by $\mathbb{Q}_S$ the direct product of $\mathbb{Q}_v$ for all $v$ in $S$. Denote by $\mathbb{A}$ the ring of adeles of $\mathbb{Q}$. Define $S$-local and global norms by

$$\forall x \in \mathbb{Q}_S \left| x \right|_S = \prod_{v \in S} \left| x \right|_v, \quad \forall x \in \mathbb{A} \left| x \right|_\mathbb{A} = \lim_S \left| x \right|_S.$$  

The local norms are normalized in such a way that

$$\forall x \in \mathbb{Q} \setminus \{0\} \left| x \right|_\mathbb{A} = 1.$$  

The groups $G(\mathbb{Q}_v)$, $G(\mathbb{Q}_S)$ and $G(\mathbb{A})$ of $\mathbb{Q}_v$, $\mathbb{Q}_S$ and $\mathbb{A}$-valued points of $G$ are locally compact with respect to the analytic topology. The group $G(\mathbb{Q})$ of $\mathbb{Q}$-valued points of $G$ is a discrete subgroup of $G(\mathbb{A})$ with respect to the analytic topology.

(1.1.2) Fix a minimal parabolic subgroup $P_0$ of $G$. Fix a Levi subgroup $M_0$ of $P_0$ with split component $A_0$. A parabolic subgroup $P$ of $G$ is said to be standard if $P$ contains $P_0$. Denote by $N_P$ the unipotent radical of $P$, by $M_P$ the unique Levi subgroup of $P$ containing $M_0$, by $A_P$ the split component of $M_P$. Such a Levi subgroup $M_P$ is said to be standard. Denote by $\overline{P}$ the parabolic subgroup group opposite to $P$, by $\overline{N}_P$ its unipotent radical. To simplify notations the standard Levi, split and unipotent components of $P_i$ will be denoted by $M_i, A_i$ and $N_i$ where $i$ is a natural number.

Let $M$ and $L$ be Levi subgroups of $G$ such that $M$ is contained in $L$, denote by $\mathcal{F}^L(M)$ the set of parabolic subgroups of $L$ that contain $M$, by $\mathcal{P}^L(M)$ the set of parabolic subgroups of $L$ whose Levi component is $M$, by $\mathcal{L}^L(M)$ the set of Levi subgroups of $L$ that contain $M$. To simplify notations denote by $\mathcal{F}(M)$ the set $\mathcal{F}^G(M)$, by $\mathcal{F}^L$ the set $\mathcal{F}^L(M_0)$, by $\mathcal{F}$ the set $\mathcal{F}^G(M_0)$. Similar notations apply to $\mathcal{P}$ and $\mathcal{L}$.

Let $P$ be a parabolic subgroup of $G$, denote by $X(M_P)$ the group of rational characters of $M_P$

$$X(M_P) = \text{Hom}_{\text{Grp}/\mathbb{Q}}(M_P, \text{GL}(1, \mathbb{Q})).$$  

Let $a_P$ denote the real vector space

$$a_P = \text{Hom}_{\mathbb{Z}}(X(M_P), \mathbb{R}),$$  

let $a_P^\ast$ denote the dual space

$$a_P^\ast = X(M_P) \otimes_{\mathbb{Z}} \mathbb{R}.$$  

Let

$$\Phi_P, \Delta_P \subset a_P^\ast, \quad \Phi_P^\vee, \Delta_P^\vee \subset a_P.$$
denote respectively the set of roots, simple roots, coroots, simple coroots of $A_P$ in $\mathfrak{g}$ and $\mathfrak{n}_P$. The quadruple $(X(M_0), \Phi_0, X(M_0)^*, \Phi_0^*)$ is called the root datum of $G$.

Let $P_1$ and $P_2$ be parabolic subgroups of $G$ with $P_1$ contained in $P_2$, denote

\begin{equation}
N_1^2 = N_1 \cap M_2, \quad \overline{N}_1^2 = \overline{N}_1 \cap M_2.
\end{equation}

Let $\Delta_1^2$ and $\Delta_1^{2,\vee}$ be the set of simple roots and coroots of $A_1$ in $\mathfrak{n}_1^2$. There are canonical splittings

\begin{equation}
a_1 = a_1^2 \oplus a_2, \quad a_1^* = a_1^{2*} \oplus a_2^*.
\end{equation}

The sets $\Delta_1^2$ and $\Delta_1^{2,\vee}$ form bases of $a_1^{2*}$ and $a_1^2$. The respective dual bases are called the coweights and weights and denoted by $\hat{\Delta}_1^2$ and $\hat{\Delta}_1^{2,\vee}$.

Let $W^G_0$ be the Weyl group of the pair $(G, A_0)$. Let $M_1$ and $M_2$ be two Levi subgroups of $G$, define the Weyl set $W(a_1, a_2)$ to be the set of linear isomorphisms from $a_1$ to $a_2$ obtained by restricting the action of elements of the Weyl group. The group $W^G_0$ operates on the root datum of $G$, hence on $a_0$ and $a_0^*$. Fix Euclidean inner products on $a_0$ and $a_0^*$ which are compatible with each other and the underlying root datum, hence invariant under the Weyl group action. The inner product on $a_0$ induces an inner product on $a_1^2$.

Let $\tau^2_1$ denote the characteristic function on $a_0$ of the points that are positive with respect to every element of $\Delta_1^2$, let $\hat{\tau}^2_1$ denote the characteristic function on $a_0$ of the set of points that are positive with respect to every element of $\hat{\Delta}_1^{2,\vee}$:

\begin{equation}
\tau^2_1 = I_{\{H \in a_0: a(H) > 0, \forall a \in \Delta_1^2\}}, \quad \hat{\tau}^2_1 = I_{\{H \in a_0: a(H) > 0, \forall a \in \hat{\Delta}_1^{2,\vee}\}}.
\end{equation}

To simplify notations denote by $\tau_1$ the set $\tau^2_1$, by $\hat{\tau}_1$ the set $\hat{\tau}^2_1$.

\section{1.3} Define $G(\mathbb{A})^1$ to be the subgroup of $G(\mathbb{A})$ consisting of elements $g$ of $G(\mathbb{Q})$ such that

\begin{equation}
\forall \chi \in X(G) \quad |\chi(g)|_\mathbb{A} = 1.
\end{equation}

Fix an admissible maximal compact subgroup

\begin{equation}
K = \prod_v K_v
\end{equation}

of $G(\mathbb{A})$ such that the Iwasawa decomposition

\begin{equation}
G(\mathbb{A}) = P(\mathbb{A})K = M_P(\mathbb{A})^1 \exp(a_P) N_P(\mathbb{A})K
\end{equation}

holds. Denote by

\begin{equation}
H_P : G(\mathbb{A}) \rightarrow a_P
\end{equation}

the natural projection.

The Tamagawa measure on $G(\mathbb{A})$ is the measure induced from the choice of a basis of rational 1-forms on $G$, which is well-defined by the product formula \textbf{[1.1.2]}. The Euclidean vector space $a_G$ has a translation invariant measure, which without loss of generality assigns the coweight lattice

\begin{equation}
\text{Hom}_\mathbb{Z}(X(G), \mathbb{Z}) \subset a_G
\end{equation}

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covolume one.

The various measures are compatible under the Iwasawa decomposition in the sense that

\[(1.1.3.5) \int_{G(A)} f(g) \, dg = \int_{M^p(A)} \int_{A^p} \int_{N^p(A)} \int_{K} f(ma_nk) \, dk \, dn \, dmadm \]

\[= \int_{M^p(A)} \int_{A^p} \int_{N^p(A)} \int_{K} f(ma_nk)e^{-2\rho_p(H_0(a))} \, dk \, dn \, dmadm \]

where \(dn\) is the Tamagawa measure on \(N^p(A)\) which could also be characterized by assigning \(N^p(Q)\) covolume one in \(N^p(A)\), the point \(\rho_p\) in \(a_0^p\) is the Weyl vector defined as the half sum of the roots of \(\mathfrak{p}\). The choices of measures on \(G(A)\) and \(a_G\) determine a Haar measure on \(G(A)^1\), hence a measure on the automorphic quotient \(G(Q) \backslash G(A)^1\).

Let \(T'\) be a point in \(a_0\), let \(\omega\) be a compact subset of \(N_0(A)M_0(A)^1\). The Siegel set \(\mathcal{S}(T', \omega)\) is the subset of \(G(A)^1\) defined by

\[(1.1.3.6) \mathcal{S}(T', \omega) = \left\{ x = pak : p \in \omega, a \in \exp(a_0), k \in K, \beta(H_0(a) - T') \geq 0 \forall \beta \in \Delta_0 \right\} .

A Siegel set is said to be a Siegel domain if it contains a fundamental domain for \(G(Q) \backslash G(A)^1\):

\[(1.1.3.7) G(A)^1 = G(Q) \mathcal{S}(T', \omega).

Let \(T\) be a sufficiently positive point in \(a_0\), such a \(T\) is said to be a truncation parameter. Let \(\mathcal{S}(T', \omega)\) be a Siegel domain. The truncated Siegel domain \(\mathcal{S}^T(T', \omega)\) is defined by

\[(1.1.3.8) \mathcal{S}^T(T', \omega) = \left\{ x \in \mathcal{S}(T', \omega) : \varpi(H_0(x) - T) \leq 0 \forall \varpi \in \hat{\Delta}_0 \right\} .

**Proposition** (Borel, Harish-Chandra)

There exist a point \(T'\) and a compact set \(\omega\) such that \(\mathcal{S}(T', \omega)\) is a Siegel domain.

**Proof** See §13 of [Bo69].

The Siegel domain \(\mathcal{S}(T', \omega)\), therefore \(G(Q) \backslash G(A)^1\), has finite volume. The truncated Siegel domain \(\mathcal{S}^T(T', \omega)\) is compact and exhausts \(\mathcal{S}(T', \omega)\) as \(T\) approaches infinity. Fix a pair \((T', \omega)\) such that \(\mathcal{S}(T', \omega)\) is a Siegel domain, denote by \(F^G(x, T)\) the characteristic function on \(G(Q) \backslash G(A)^1\) of \(\mathcal{S}^T(T', \omega)\). There are analogues \(F^P(x, T)\) on \(P(Q) \backslash G(A)^1\).

\[(1.1.4) \quad \text{The space } \mathcal{S}(g(A)) \text{ of Schwartz functions on } g(A) \text{ is defined as the tensor product}

\[(1.1.4.1) \bigotimes_p^\text{res} C_c^\infty(g(Q_p)) \otimes \mathcal{S}(g(\mathbb{R}))

\text{restricted at all but finitely many finite primes with respect to the unit vector } \mathbb{I}_{g(\mathbb{Z}_p)} \text{ in } C_c^\infty(g(Q_p)), \text{ equipped with the final topology with respect to}

\[(1.1.4.2) \quad \mathcal{S}(g(A)) = \lim_{S} \mathcal{S}(g(Q_S)).\]
Fix a nondegenerate $G$-invariant rational bilinear form $\langle \ , \ \rangle$ on $\mathfrak{g}$, fix a global additive unitary character $\psi$ on $\mathbb{A}$

$$\psi : \mathbb{A} \to \text{U}(1)$$

such that

$$\forall x \in \mathbb{Q} \quad \psi(x) = 1,$$

and $\psi(\langle \ , \ \rangle)$ identifies $\mathbb{Q}$ and $\mathbb{A}/\mathbb{Q}$ as Pontryagin duals of each other. The Fourier transform on $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ is defined by

$$\hat{f}(X) = \int_{\mathfrak{g}(\mathbb{A})} f(Y)\psi(\langle X, Y \rangle) \, dY.$$

The global Fourier transform on $\mathfrak{g}(\mathbb{A})$ factorizes as the tensor product of the local Fourier transforms on $\mathfrak{g}(\mathbb{Q}_v)$ with respect to compatible choices of $\psi_v(\langle \ , \ \rangle)$. Denote by $\check{\psi}$ the inverse Fourier transform.

**Proposition** (Poisson summation formula, Tate)

*For every Schwartz function $f$ on $\mathfrak{g}(\mathbb{A})$,*

$$\sum_{X \in \mathfrak{g}(\mathbb{Q})} f(X) = \sum_{X \in \mathfrak{g}(\mathbb{Q})} \hat{f}(X),$$

*the sums are the absolutely convergent.*

**Proof** See §4.2 of [Ta50].

(1.1.5)

**Define an equivalence relation $\sim$ on $\mathfrak{g}(\mathbb{Q})$ by**

$$X \sim Y \quad \text{if} \quad \exists g \in G(\mathbb{Q}) \, Y_{\text{ss}} = X_{\text{ss}} \cdot \text{ad}(g).$$

In general $\sim$ is weaker than conjugacy by $G(\mathbb{Q})$. A typical equivalence class will be denoted by $\sigma$.

Let $D$ be the discriminant function on $\mathfrak{g}(\mathbb{Q})$. Let $X$ be an element of $\mathfrak{g}(\mathbb{Q})$, define $D(X)$ to be the coefficient of the characteristic polynomial of $\text{ad}(X)$, acting on $\mathfrak{g}(\mathbb{Q})$ as a linear endomorphism, in degree $r$, the absolute rank of $G$:

$$r = \text{rank}_{\mathbb{Q}}(G \otimes_{\mathbb{Q}} \mathbb{Q}).$$

Alternatively $D$ could be defined as the product of the roots of $\mathfrak{g}$ over an algebraically closed field. Denote by $D^M$ the discriminant function on $\mathfrak{m}$ for a Levi subgroup $M$ of $G$.

Let $X$ be a semisimple element of $\mathfrak{g}(\mathbb{Q})$. Then $X$ is said to be regular if $D(X)$ does not vanish. Denote by $\mathfrak{g}_{\text{reg, ss}}$ the locus of regular semisimple points on $\mathfrak{g}$. If an equivalence class $\sigma$ contains a regular semisimple element, the set $\sigma$ is a $G(\mathbb{Q})$-orbit consisting of regular semisimple elements. Such an $\sigma$ is said to be regular; otherwise $\sigma$ is said to be singular. A semisimple element $X$ is said to be $\mathbb{Q}$-elliptic if it is stabilized under the adjoint action by a maximal torus that is anisotropic over $\mathbb{Q}$ modulo the center of $G$.

Denote by $G_X$ the centralizer of $X$ in $G$, let $G_X^0$ be the connected component of the identity of $G_X$, let $\pi_0(G_X)$ be the group of connected components of $G_X$. There is an exact sequence

$$1 \to G_X^0 \to G_X \to \pi_0(G_X) \to 1.$$
Let $S$ be a finite set of places of $\mathbb{Q}$, denote by $G_S$ the base change $G \otimes \mathbb{Q} \mathbb{Q}_S$ of $G$. Let $v$ be a place of $\mathbb{Q}$, denote by $G_v$ the base change $G \otimes \mathbb{Q} \mathbb{Q}_v$ of $G$. Similar notations apply to the Lie algebra $\mathfrak{g}$. The underlying topological groups of $G(\mathbb{Q}_S)$ and $G_S(\mathbb{Q}_S)$ are the same, however

$$(1.1.6) \quad X(G) = \text{Hom}_{\text{Grp}/\mathbb{Q}}(G, \text{GL}(1, \mathbb{Q})), \quad X(G_S) = \text{Hom}_{\text{Grp}/\mathbb{Q}_S}(G_S, \text{GL}(1, \mathbb{Q}_S))$$

are in general different.

All the constructions above generalize to the local and $S$-local settings with similar caveats:

- Fix a minimal parabolic subgroup $P_{S,0}$ of $G_S$ contained in $P_{0,S}$, fix a Levi subgroup $M_{S,0}$ contained in $M_{0,S}$ with split component $A_{S,0}$ containing $A_{0,S}$. The choice of $P_{S,0}$ is equivalent to a choice of a minimal parabolic $P_{v,0}$ of $G_v$ for each $v$ in $S$, similarly for $M_{S,0}$ and $A_{S,0}$.

- Let $M_S$ and $L_S$ be Levi subgroups of $G_S$ such that $M_S$ is contained in $L_S$, denote by $\mathcal{F}^{L_S}(M_S)$, $\mathcal{P}^{L_S}(M_S)$ and $\mathcal{L}^{L_S}(M_S)$ the analogous sets of parabolic and Levi subgroups. The Levi subgroups $M_S$ and $L_S$ determine local Levi subgroups $M_v$ and $L_v$ for each $v$ in $S$. There are bijections

$$(1.1.6.2) \quad \mathcal{F}^{L_S}(M_S) = \prod_{v \in S} \mathcal{F}^{L_v}(M_v), \quad \mathcal{P}^{L_S}(M_S) = \prod_{v \in S} \mathcal{P}^{L_v}(M_v), \quad \mathcal{L}^{L_S}(M_S) = \prod_{v \in S} \mathcal{L}^{L_v}(M_v).$$

If $M_S$ and $L_S$ are the base change of Levi subgroups $M$ and $L$ of $G$ from $\mathbb{Q}$ to $\mathbb{Q}_S$, there are diagonal inclusions

$$(1.1.6.3) \quad \mathcal{F}^L(M) \subset \mathcal{F}^{L_S}(M_S), \quad \mathcal{P}^L(M) \subset \mathcal{P}^{L_S}(M_S), \quad \mathcal{L}^L(M) \subset \mathcal{L}^{L_S}(M_S).$$

- Let $P_S$ be a parabolic subgroup of $G_S$, define real vector spaces

$$(1.1.6.4) \quad a_{P_S} = \text{Hom}_\mathbb{Z}(X(M_{P_S}), \mathbb{R}), \quad a_{P_S}^* = X(M_{P_S}) \otimes \mathbb{Z} \mathbb{R}.$$

If $P_S$ is the base change of a parabolic subgroup $P$ of $G$ from $\mathbb{Q}$ to $\mathbb{Q}_S$, there are diagonal inclusions

$$(1.1.6.5) \quad a_P \subset a_{P_S} = \bigoplus_{v \in S} a_{P_v}, \quad a_P^* \subset a_{P_S}^* = \bigoplus_{v \in S} a_{P_v}^*.$$

- A maximal torus $T_S$ of $G_S$ is equivalent to the choice of a maximal torus $T_v$ of $G_v$ for each $v$ in $S$. The associated Cartan subalgebra $t_S$, which is a free $\mathbb{Q}_S$-module, is equal as an abelian group to the direct sum of the $\mathbb{Q}_v$-vector spaces $t_v$ for all $v$ in $S$.

A maximal torus $T_S$ is said to be elliptic in $G_S$ if it is anisotropic modulo $A_{G_S}$ over $\mathbb{Q}_S$. A maximal torus $T_S$ is elliptic in $G_S$ if and only if $T_v$ is elliptic in $G_v$ for each $v$ in $S$. If this is the case, $T_S(\mathbb{Q}_S)$ is compact modulo $A_{G_S}(\mathbb{Q}_S)$ in the analytic topology.

Denote by $\mathcal{T}_{\text{ell}}(G_S)$ the set of conjugacy classes of elliptic maximal tori of $G_S$.

- Let $W_{S,0}^{G_S}$ be the Weyl group of the pair $(G_S, A_{S,0})$, there is a bijection

$$(1.1.6.6) \quad W_{S,0}^{G_S} = \prod_{v \in S} W_{v,0}^{G_v},$$

the linear representation of $W_{S,0}^{G_S}$ on $a_{S,0}$ is the direct sum of the local representations.

Denote by $W(G_S, T_S)$ the Weyl group of the pair $(G_S(\mathbb{Q}_S), T_S(\mathbb{Q}_S))$. There is a bijection

$$(1.1.6.7) \quad W(G_S, T_S) = \prod_{v \in S} W(G_v, T_v).$$
The Schwartz space $S(\mathfrak{g}_S(\mathbb{Q}_S))$, the Fourier transform $\wedge$, the discriminant function $D^{G_S}$, and the regular semisimple locus $\mathfrak{g}_{S,\text{reg.ss}}(\mathbb{Q}_S)$ are unchanged under base change from $\mathbb{Q}$ to $\mathbb{Q}_S$.

**Proposition**  (Weyl integration formula)

If $f_S$ is a Schwartz function on $\mathfrak{g}_S(\mathbb{Q}_S)$, then

$$
(1.1.6.8) \quad \int_{\mathbb{Q}_S(\mathbb{Q}_S)} f_S(X) \, dX = \sum_{M_S \in \mathcal{L}^{G_S}} |W_{S,0}^{M_S}||W^{G_S}_{S,0}|^{-1} \sum_{T_S \in T_{\text{ad}}(M_S)} |W(M_S, T_S)|^{-1} \times
$$

$$
\times \int_{1_S(\mathbb{Q}_S)} |D^{G_S}(X)|_S \int_{A_{MS}(\mathbb{Q}_S) \backslash G_S(\mathbb{Q}_S)} f_S(X \cdot \text{ad}(x)) \, dx \, dX.
$$

**Proof**  For the $p$-adic case see §7.11 of [Ko05]. For the real case see Lemma 2 on page 35 of [Va77] and the references therein. □

**Definition**  Let $f_S$ be a Schwartz function on $\mathfrak{g}_S(\mathbb{Q}_S)$, let $X$ be an element of $\mathfrak{g}_{S,\text{reg.ss}}(\mathbb{Q}_S)$. Define the $S$-local orbital integral $I_G^X(X, f_S)$ to be the invariant distribution on $\mathfrak{g}_S(\mathbb{Q}_S)$ such that

$$
(1.1.6.9) \quad I_G^X(X, f_S) = |D^G(X)|_S^{1/2} \int_{G^0_{S,X}(\mathbb{Q}_S) \backslash G_S(\mathbb{Q}_S)} f_S(X \cdot \text{ad}(x)) \, dx
$$

where $G^0_{S,X}$ denotes the connected component of the identity of the stabilizer subgroup of $X$ in $G_S$.

## 2 The non-invariant trace formula

In this chapter a preliminary version of the trace formula for the reductive Lie algebra $\mathfrak{g}$ established in Chaudouard [Ch02] is recalled.

### 2.1 A motivating example

**Definition**  Let $f$ be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $\sigma$ be a $\sim$ equivalence class on $\mathfrak{g}(\mathbb{Q})$. Define the kernel functions $K(x, f)$ and $K_{\sigma}(x, f)$ by

$$
(2.1.1) \quad \forall x \in G(\mathbb{Q}) \backslash G(\mathbb{A}) \quad K(x, f) = \sum_{X \in \mathfrak{g}(\mathbb{Q})} f(X \cdot \text{ad}(x)), \quad K_{\sigma}(x, f) = \sum_{X \in \sigma} f(X \cdot \text{ad}(x)).
$$

**Remark**  By the Poisson summation formula [1.1.4.6], the function $K(x, f)$ satisfies the functional equation

$$
(2.1.2.1) \quad K(x, f) = K(x, f^*).
$$

**Proposition**  Let $f$ be a Schwartz function on $\mathfrak{g}(\mathbb{A})$. If $G$ is anisotropic over $\mathbb{Q}$, then

$$
(2.1.3.1) \quad \lim_{S} \sum_{\sigma \in \mathfrak{g}(\mathbb{Q}) / \sim} \text{Vol}(G^0_{X_{\sigma}(\mathbb{Q}) \backslash G^0_{X_{\sigma}(\mathbb{A})}}) \cdot I_{G}^{X}(X_{\sigma}, f_{S}) + \sum_{\sigma \in \mathfrak{g}(\mathbb{Q}) / \sim} \text{Vol}(G^0_{X_{\sigma}(\mathbb{Q}) \backslash G^0_{X_{\sigma}(\mathbb{A})}}) \cdot I_{\sigma}(f)
$$

$$
= \lim_{S} \sum_{\sigma \in \mathfrak{g}(\mathbb{Q}) / \sim} \text{Vol}(G^0_{X_{\sigma}(\mathbb{Q}) \backslash G^0_{X_{\sigma}(\mathbb{A})}}) \cdot I_{G}^{X}(X_{\sigma}, f^*_{S}) + \sum_{\sigma \in \mathfrak{g}(\mathbb{Q}) / \sim} \text{Vol}(G^0_{X_{\sigma}(\mathbb{Q}) \backslash G^0_{X_{\sigma}(\mathbb{A})}}) \cdot I_{\sigma}(f^*),
$$
for each class $\mathfrak{o}$ let $X_\mathfrak{o}$ be an element of $\mathfrak{o}$, if $\mathfrak{o}$ is singular define

$$I_\mathfrak{o}(f) = \int_{G(X_\mathfrak{o}(A)) \setminus G(A)} f(X_\mathfrak{o} \cdot \text{ad}(x)) \ dx.$$  \hfill (2.1.3.2)

(2.1.4) **Proof** The function $K(x, f)$ is continuous. By assumption $G$ is anisotropic, so $G(Q) \setminus G(A)$ is compact. Therefore $K(x, f)$ is absolutely integrable, hence

$$\int_{G(Q) \setminus G(A)} K(x, f) \ dx = \int_{G(Q) \setminus G(A)} \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} K_\mathfrak{o}(x, f) \ dx$$

$$= \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} \int_{G(Q) \setminus G(A)} K_\mathfrak{o}(x, f) \ dx$$

$$= \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} \text{Vol}(G_{X_\mathfrak{o}}^0(Q) \setminus G_{X_\mathfrak{o}}^0(A)) \int_{G(X_\mathfrak{o}(A)) \setminus G(A)} f(X_\mathfrak{o} \cdot \text{ad}(x)) \ dx$$

$$\lim_S \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} \text{Vol}(G_{X_\mathfrak{o}}^0(Q) \setminus G_{X_\mathfrak{o}}^0(A)) \cdot I_G^G(X_\mathfrak{o}, f_S)$$

$$+ \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} \text{Vol}(G_{X_\mathfrak{o}}^0(Q) \setminus G_{X_\mathfrak{o}}^0(A)) \cdot I_\mathfrak{o}(f^\prime),$$

the equality (2.1.4.2) follows from

$$\forall X_\mathfrak{o} \in \mathfrak{g}(Q) \exists S \forall S' \supset S \ |D(X_\mathfrak{o})|_{S'} = 1$$

by the product formula (1.1.1.2). The proposition follows from the functional equation (2.1.2.1). \hfill \Box

2.2 The non-invariant trace formula of Chaudouard

(2.2.1) **Definition** Let $f$ be a Schwartz function on $\mathfrak{g}(A)$, let $\mathfrak{o}$ be a $\sim$ equivalence class on $\mathfrak{g}(Q)$, let $T$ be a truncation parameter. Define the **truncated kernel function** $k_\mathfrak{o}^T(\cdot, f)$ on $G(Q) \setminus G(A)$ by

$$\forall x \in G(Q) \setminus G(A) \quad k_\mathfrak{o}^T(x, f) = \sum_{P \in \mathcal{F}} (-1)^{\dim(A_P / A_G)} \sum_{\delta \in \mathfrak{p}(Q) \setminus G(Q)} \hat{\tau}_P(H_0(\delta x) - T) K_{P, \mathfrak{o}}(\delta x)$$

$$\text{where}$$

$$K_{P, \mathfrak{o}}(x, f) = \sum_{X \in \mathfrak{m}(Q) \cap \mathfrak{o}} \int_{P(A)} f((X + N) \cdot \text{ad}(x)) \ dN.$$  \hfill (2.2.1.2)

Define distributions $J_\mathfrak{o}^T$ and $J^T$ on $\mathfrak{g}(A)$ by

$$\forall f \in \mathcal{S}(\mathfrak{g}(A)) \quad J_\mathfrak{o}^T(f) = \int_{G(Q) \setminus G(A)} k_\mathfrak{o}^T(x, f) \ dx,$$

$$J^T(f) = \int_{G(Q) \setminus G(A)} \sum_{\mathfrak{o} \in \mathfrak{g}(Q) / \sim} k_\mathfrak{o}^T(x, f) \ dx.$$
(2.2.2) Proposition Let $f$ be a Schwartz function on $g(\mathbb{A})$, let $T$ be a truncation parameter, then

\[(2.2.2.1) \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{o \in g(\mathbb{Q})/\sim} |k^T_o(x, f)| \, dx < \infty.\]

(2.2.3) Proof This is Théorème 3.1 of [Ch02]. The following is a sketch of the argument, for the details please refer to loc. cit.

The proof depends on the following lemma.

Lemma (Combinatorial lemma of Langlands)
Let $P_1$ and $P_3$ be parabolic subgroups of $G$ such that $P_1$ is contained in $P_3$, then

\[(2.2.3.1) \quad \sum_{P_5 \in F, P_1 \subset P_2 \subset P_3} (-1)^{\dim(A_2/A_3)} r_1^2(H) r_1^3(H) = \begin{cases} 1 & \text{if } P_1 = P_3, \\ 0 & \text{otherwise}. \end{cases}\]

Proof See § 6 of [Ar78]. \qed

By (2.2.3.1)

\[(2.2.3.2) \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{o \in g(\mathbb{Q})/\sim} |k^T_o(x, f)| \, dx \leq \sum_{P_1, P_4 \in F} \int_{P_1(\mathbb{Q}) \backslash G(\mathbb{A})} \sum_{P_1 \subset P_2 \subset P_4} \sum_{o \in g(\mathbb{Q})/\sim} \int_{n_4(\mathbb{A})} f((X + Y + N) \cdot \text{ad}(x)) \, dN \times \sum_{P_3 \in F, P_1 \subset P_2 \subset P_3} \sum_{o \in g(\mathbb{Q})/\sim} (-1)^{\dim(A_2/A_3)} K_{P_3, o}(x, f) \, dx \times \sum_{P_5 \in F, P_4 \subset P_5} \sum_{P_1 \subset P_2 \subset P_3} (-1)^{\dim(A_3/A_4)} \sigma_4^4(H_0(x) - T) \times \sum_{P_5 \in F, P_4 \subset P_5} (-1)^{\dim(A_4/A_5)} r_1^5(H) r_5^3(H)\]

where

\[(2.2.3.3) \quad \sigma_4^4(H) = \sum_{P_5 \in F, P_4 \subset P_5} (-1)^{\dim(A_3/A_4)} r_1^5(H) r_5^3(H).\]

The second factor of the integrand of the right hand side of (2.2.3.2) satisfies the inequality

\[(2.2.3.4) \quad \sum_{P_1 \in F, P_1 \subset P_3 \subset P_4} (-1)^{\dim(A_3/A_4)} K_{P_3, o}(x, f) \leq \sum_{P_2 \in F, P_1 \subset P_2 \subset P_3} \sum_{P_3 \in F, P_2 \subset P_3} (-1)^{\dim(A_3/A_4)} \sum_{X \in \mathfrak{n}_2(\mathbb{Q}) \cap o} \int_{n_3(\mathbb{A})} f((X + Y + N) \cdot \text{ad}(x)) \, dN \times \sum_{Y \in \mathfrak{n}_2(\mathbb{Q})} \int_{n_3(\mathbb{A})} f((X + Y + N) \cdot \text{ad}(x)) \, dN + \sum_{X \in \mathfrak{n}_4(\mathbb{Q}) \cap o} \int_{n_3(\mathbb{A})} f((X + N) \cdot \text{ad}(x)) \, dN + \]

\[(2.2.3.5) \quad = \left| \sum_{X \in \mathfrak{m}_4(\mathbb{Q}) \cap o} \int_{n_3(\mathbb{A})} f((X + N) \cdot \text{ad}(x)) \, dN \right| + \]
where $m^Q(Q)'$ denotes the set of points of $m^Q(Q)$ not contained in any proper parabolic subalgebra of $q$. Similar notation applies to $n$. The equality (2.2.3.5) follows from the Poisson summation formula applied to $n^3(Q)$ as a lattice in $n^3(A)$.

By the inclusion-exclusion principle applied to (2.2.3.4), the last expression in (2.2.3.4) reduces to a majorant of the form

$$\prod_{\alpha \in \Delta^4} \int_0^{\infty} (1 + t_\alpha)^{p_\alpha} e^{-q_\alpha t_\alpha} \, dt_\alpha$$

for some natural numbers $p_\alpha$ and $q_\alpha$, which is finite. $\square$

**Definition** Let $P_2$ be a standard parabolic subgroup of $G$, let $T$ be a truncation parameter in $a_0$. Define the geometric gamma function $\Gamma'_{P_2}(,T)$ on $a_0$ by

$$\forall H \in a_0 \quad \Gamma'_{P_2}(H,T) = \sum_{P_3 \in F \atop P_2 \subseteq P_3} (-1)^{\dim(A_3/A_G)} \tau_3(H) \tau_3(H - T).$$

**Remark** For each parabolic subgroup $P_1$ of $G$ the geometric gamma functions satisfy the identity

$$\hat{\tau}_1(H - T) = \sum_{P_2 \in F \atop P_1 \subseteq P_2} (-1)^{\dim(A_2/A_G)} \tau_2(H) \Gamma'_{P_2}(H,T).$$

For a proof see page 13 of [Ar81].

**Proposition** Let $f$ be a Schwartz function on $\mathfrak{g}(A)$, let $o$ be a $\sim$ equivalence class, let $T$ be a truncation parameter. Then $J^T_o(f)$ and $J^T(f)$ are polynomials in $T$ of degree at most $\dim(A_0/A_G)$.

**Proof** This is Théorème 4.2 of [Ch02]. The following is a sketch of the argument for $J^T_o(f)$, the argument for $J^T(f)$ is the same. For the details please refer to loc. cit.
Fix a point $T'$ in $\mathfrak{a}_0$ that is sufficiently positive and assume that $T$ dominates $T'$. By (2.2.5.1)

\begin{equation}
(2.2.7.1) \quad J_0^T(f) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \left( \sum_{P_1 \in F} (-1)^{\dim(A_1/A_G)} \times \right. \\
\times \sum_{\delta \in P_1 \backslash G(\mathbb{Q})} \tau_1(H_0(\delta x) - T) \cdot K_{P_1, \delta}(\delta x) \left. \right) dx \\
= \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \left( \sum_{P_1, P_2 \in F} (-1)^{\dim(A_1/A_2)} \times \right. \\
\times \sum_{\delta \in P_1 \backslash G(\mathbb{Q})} \tau_1^2(H_{P_1}(\delta x) - T') \Gamma_{P_2}'(H_{P_2}(\delta x) - T', T - T') \times \\
\times \sum_{X \in m_1(\mathbb{Q}) \cap \mathfrak{a}_0} \int_{\mathfrak{n}_1(\mathbb{A})} f((X + N) \cdot \text{ad}(\delta x)) \ dN \left. \right) dx \\
= \sum_{P_2 \in F} \int_{M_2(\mathbb{Q}) \backslash M_2(\mathbb{A})} \left( \sum_{P_1 \in F} \left( (-1)^{\dim(A_1/A_2)} \times \right. \right. \\
\times \sum_{\delta \in P_1 \cap M_2(\mathbb{Q}) \backslash M_2(\mathbb{Q})} \tau_1^2(H_{P_1}(\delta x) - T') \sum_{X \in m_1(\mathbb{Q}) \cap \mathfrak{a}_0} \int_{\mathfrak{n}_1(\mathbb{A})} f((X + N) \cdot \text{ad}(\delta x) + N' \cdot \text{ad}(k)) \ dN \left. \right) dx \times \\
\times \int_{\mathfrak{a}_0^{G}} \Gamma_{P_2}'(H - T', T - T') \ dH \\
= \sum_{P_2 \in F} J_0^{M_2; T'}(f_{P_2}) \int_{\mathfrak{a}_0^{G}} \Gamma_{P_2}'(H, T - T') \ dH
\end{equation}

where $J_0^{M}$ is defined to be the sum of $J_0^{M}$ over all the $M(\mathbb{Q}) \sim$ equivalence classes $\mathfrak{o}'$ contained in $\mathfrak{o}$ and $f_P$ is defined by

\begin{equation}
(2.2.7.2) \quad f_P(X) = \int_{K} \int_{\mathfrak{n}_P(\mathbb{A})} f((X + N) \cdot \text{ad}(k)) \ dNdk.
\end{equation}

Because $\int \Gamma_{P}'(H, T - T') \ dH$ is a polynomial in $T$ which is homogeneous of degree $\dim(A_P/A_G)$, Proposition (2.2.6) follows by induction.

(2.2.8) Proposition Let $f$ be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, let $T$ be a truncation parameter. For every positive $\epsilon$

\begin{equation}
(2.2.8.1) \quad \left| J^T(f) - \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} F^G(x, T) \sum_{X \in \mathfrak{g}(\mathbb{Q})} f(X \cdot \text{ad}(x)) \ dx \right| = O(e^{-\epsilon\|T\|})
\end{equation}
where \( \| \| \) denotes the Euclidean norm on \( a_0 \), as \( T \) approaches infinity such that \( T \) is uniformly bounded away from the walls of the positive chamber.

(2.2.9) **Proof** This is a corollary of the proof of Lemma 3.2.2.1.

(2.2.10) **Proposition** (Non-invariant trace formula of Chaudouard) Let \( f \) be a Schwartz function on \( g(\mathbb{A}) \), then

\[
\sum_{o \in g(\mathbb{Q})/\sim} J_o^T(f) = \sum_{o \in g(\mathbb{Q})/\sim} J_o^T(f^-)
\]

holds as an equality between polynomials in \( T \).

(2.2.11) **Proof** By (2.1.2.1)

\[
\int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} F^G(x,T) \sum_{X \in g(\mathbb{Q})} f(X \cdot \text{ad}(x)) \, dx = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} F^G(x,T)K(x,f) \, dx = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} F^G(x,T)K(x,f^-) \, dx.
\]

By (2.2.8.1) the difference between \( J^T(f) \) and \( J^T(f^-) \) converges to zero as \( T \) approaches infinity. By Proposition (2.2.6) \( J^T(f) \) and \( J^T(f^-) \) are both polynomials in \( T \), hence equal to each other.

(2.2.12) **Lemma** There exists a unique point \( T_0 \) in \( a_0^G \) such that

\[
\forall s \in W_0^G \quad H_0(w_s^{-1}) = T_0 - s^{-1}T_0
\]

where \( w_s \) denotes a representative of \( s \) in \( G(\mathbb{Q}) \).

(2.2.13) **Proof** See Lemma 1.1 of [Ar81].

(2.2.14) **Definition** Define distributions \( J \) and \( J_o \) on \( g(\mathbb{A}) \) by

\[
\forall f \in S(g(\mathbb{A})) \quad J(f) = J^{T_0}(f), \quad J_o(f) = J_o^{T_0}(f).
\]

(2.2.15) **Remark** The coefficients of the polynomials \( J^T(f) \) and \( J_o^T(f) \) in positive degrees depend only on the orbital integrals of \( f_P \) along proper Levi subalgebras of \( g \). Therefore the constant terms \( J(f) \) and \( J_o(f) \) contain the essential information. The choice of \( T_0 \) implies that the distributions \( J \) and \( J_o \) are independent of the choice of the minimal parabolic subgroup \( P_0 \).

(2.2.16) **Proposition** Let \( f \) be a Schwartz function on \( g(\mathbb{A}) \). If \( o \) is the conjugacy class of a regular semisimple element \( X \) which is contained in a standard parabolic \( p \) and elliptic in its Levi component \( m \), then

\[
J_o(f) = |\pi_0(G_X)|^{-1} \text{Vol}(M_X^0(\mathbb{Q}) \backslash M_X^0(\mathbb{A})) \int_{G_X^0(\mathbb{A}) \backslash G(\mathbb{A})} f(X \cdot \text{ad}(x)) \, v_M(x) \, dx
\]

where the weight factor \( v_M(x) \) is defined to be the volume of the convex hull of

\[
\left\{ -H_P(x) : P \in \mathcal{P}(\mathbb{M}) \right\} \subset a_M^G.
\]
\textbf{(2.2.17) Proof} This is a corollary of Theorem 3.3.15.1. See also (5.4) of [Ch02].

\textbf{(2.2.18) Proposition} Let $f$ be a Schwartz function on $\mathfrak{g}(A)$, let $\sigma$ be a $\sim$ equivalence class, then

\begin{equation}
(2.2.18.1) \forall x \in G(A)^1 \quad J_\sigma(f \circ \text{ad}(x)) = \sum_{P \in F} |W_0^M| |W_0^G|^{-1} J_\sigma^M(f_{P,x})
\end{equation}

where $J_\sigma^M$ is defined to be the sum of $J_{\sigma'}^M$ over all the $M(Q) \sim$ equivalence classes $\sigma'$ contained in $\sigma$, and the function $f_{P,x}$ on $\mathfrak{m}_P(A)$ is defined as

\begin{equation}
(2.2.18.2) f_{P,x}(X) = \int_K \int_{\mathfrak{m}_P(A)} f((X + N) \cdot \text{ad}(k)) \cdot v'_P(kx) dNdk
\end{equation}

where the weight factor $v'_P(x)$ is defined by

\begin{equation}
(2.2.18.3) v'_P(x) = \int_{\delta_P^G} \Gamma_P(^H, -H_P(x)) \ dH.
\end{equation}

\textbf{(2.2.19) Proof} By definition

\begin{equation}
(2.2.19.1) J_\sigma^T(f \circ \text{ad}(x)) = \int_{G(Q) \backslash G(A)^1} \left( \sum_{P_1 \in F \text{ standard}} \sum_{P_1 \subset P_2} \sum_{\sigma_1} \hat{\tau}_1(H_0(\delta yx) - T) \cdot K_{P_1,\sigma}(\delta yx) \ d\delta y \right).
\end{equation}

By (2.2.5.1)

\begin{equation}
(2.2.19.2) \hat{\tau}_1(H_P(\delta yx) - T) = \sum_{P_2 \in F, P_1 \subset P_2} (-1)^{\dim(A_2/A_G)} \hat{\tau}_1^2(H_P(\delta y) - T) \times \Gamma_{P_2}(H_P(\delta y) - T, -H_P(kx))
\end{equation}

where $k$ is a $K$ component of $\delta y$ under the decomposition of $G(A)$ as $P_1(A)K$, the point $H P_1(kx)$ is independent of the choice of the element $k$. Hence the right hand side of (2.2.19.1) is equal to

\begin{equation}
(2.2.19.3) \sum_{P_2 \subset F} \sum_{\text{standard}} \int_{M_2(Q) \backslash M_2(A)^1} \left( \sum_{P_1 \subset P_2} \sum_{\sigma_1} \hat{\tau}_1^2(H_P(\delta y) - T) \times \right.
\end{equation}

\begin{equation}
\left. \sum_{X \in \mathfrak{m}_1(Q) \cdot \sigma_0} \int_{n_1^2(A)} \left( \int_{K_n^2(A)} f((X + N) \cdot \text{ad}(\delta yx) + N' \cdot \text{ad}(kx)) \right. \times \right.
\end{equation}

\begin{equation}
\left. \left( \int_{\delta_P^G} \Gamma_{P_2}(H - T, -H_{P_2}(kx)) \ dH \right) \ dN' \ dN \right) \ d\delta y.
\end{equation}

Substituting $T_0$ for $T$, the expression (2.2.19.3) becomes

\begin{equation}
(2.2.19.4) \sum_{P_2 \subset F} J_\sigma^M(f_{P_2,x}) = \sum_{P_2 \subset F} |W_0^{M_2}| |W_0^G|^{-1} J_\sigma^M(f_{P_2,x})
\end{equation}

since the distribution $J_\sigma^M$ is independent of the choice of the minimal parabolic subgroup $P_0$. \hfill \Box
(2.2.20) Remark  The identities (2.2.16.1) and (2.2.18.1) are the first steps towards the refined trace formula (3.4.1.1) and the invariant trace formula (4.2.1.1) for a reductive Lie algebra.

3  Refined expansions

In this chapter the distribution $J_0$ is decomposed as a linear combination of weighted orbital integrals following the methods of Arthur [Ar85] [Ar86].

3.1  Weighted orbital integrals

(3.1.1) Definition  Let $M$ be a standard Levi subgroup of $G$. A collection of complex-valued functions

$$\left\{ c_P \in C^\infty(i\mathfrak{a}^*_M) : P \in \mathcal{P}(M) \right\}$$

is said to be a $(G,M)$-family if for each pair of adjacent parabolic subgroups $P$ and $P'$ in $\mathcal{P}(M)$, the functions $c_P$ and $c_{P'}$ agree on the hyperplane spanned by the common wall of the positive chambers of $i\mathfrak{a}^*_M$ defined by $P$ and $P'$.

Let $(c_P)$ be a $(G,M)$-family, define the function $c_M$ on the complement of the coroot hyperplanes in $i\mathfrak{a}^*_M$ by

$$\forall \lambda \in i\mathfrak{a}^*_M \quad c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}$$

where

$$\theta_P(\lambda) = \text{Vol}(i\mathfrak{a}^*_M/Z(\Delta_P^\vee))^{-1} \prod_{\alpha \in \Delta_P} \langle \alpha^*, \lambda \rangle.$$

The function $c_M$ extends smoothly over $i\mathfrak{a}^*_M$. Denote by $c_M$ its value at the origin of $i\mathfrak{a}^*_M$.

(3.1.2) Definition  Let $Q$ be a parabolic subgroup in $\mathcal{F}(M)$ with Levi component $L$, let $(c_P)$ be a $(G,M)$-family. Let

$$i^G_Q : \mathcal{P}^L(M) \to \mathcal{P}^G(M)$$

be the map that sends a parabolic subgroup $P$ in $\mathcal{P}^L(M)$ to the unique parabolic subgroup in $\mathcal{P}^G(M)$ that is contained in $Q$ whose intersection with $L$ is $P$. Denote by $(c_P^G)$ the $(L,M)$-family

$$\left\{ c_{P}^{G} : P \in \mathcal{P}^L(M) \right\}.$$

Let

$$j^L_M : i\mathfrak{a}^*_L \to i\mathfrak{a}^*_M$$

be the natural inclusion map. Denote by the $(c_P)$ the $(G,L)$-family

$$\left\{ j^L_M(c_P') : P \in \mathcal{P}^G(L) \right\}.$$
where $P'$ is a parabolic subgroup in $P^G(M)$ contained in $P$, and the function $j_{M}^{L_{*}}(c_{P'})$ is independent of the choice of $P'$ in $P^G(M)$.

Let $(c_P)$ and $(d_P)$ be two $(G, M)$-families, denote by $((cd)_{P})$ the product of $(c_P)$ and $(d_P)$, which is a $(G, M)$-family. For each parabolic subgroup $Q$ in $\mathcal{F}(M)$ there exists a function $c_{Q}'$ on $ia_{M}^{*}$ such that

$$\forall \lambda \in ia_{M}^{*} \quad (cd)_{M}(\lambda) = \sum_{Q \in \mathcal{F}(M)} c_{Q}'(\lambda)d_{M}^{Q}(\lambda).$$

(3.1.2.5)

Denote by $c_{Q}'$ the value $c_{Q}'(0)$.

(3.1.3) Definition Let $M$ be a standard Levi subgroup of $G$. A collection of points

$$\mathcal{Y}_{M} = \left\{ Y_{P} \in a_{M} : P \in \mathcal{P}(M) \right\}$$

is said to be a $(G, M)$-orthogonal set if for each pair of adjacent parabolic subgroups $P$ and $P'$ in $\mathcal{P}(M)$, the vector

$$Y_{P} - Y_{P'} \in a_{M}$$

is orthogonal to the hyperplane spanned by the common wall of the positive chambers defined by $P$ and $P'$.

Let $\alpha_{P'}$ be the unique coroot that is positive for $P$ and negative for $P'$. A $(G, M)$-orthogonal set $\mathcal{Y}_{M}$ is positive if

$$\exists t > 0 \quad Y_{P} - Y_{P'} = t\alpha_{P'}.$$

(3.1.3.3)

(3.1.4) Remark Let $\mathcal{Y}_{M}$ be a positive $(G, M)$-orthogonal set, then the collection of functions

$$\left\{ v_{P}(\mathcal{Y}_{M})(\lambda) = e^{\langle \lambda, Y_{P} \rangle} : P \in \mathcal{P}(M) \right\}$$

forms a $(G, M)$-family. The associated constant $v_{M}(\mathcal{Y}_{M})$ as in (3.1.2) is equal to the volume of the convex hull of $\mathcal{Y}_{M}$ in $a_{M}$.

(3.1.5) Definition Let $M$ be a standard Levi subgroup of $G$, let $x$ be an element of $G(Q_{S})$.

The collection of points

$$\mathcal{Y}_{M}(x) = \left\{ -H_{P}(x) : P \in \mathcal{P}(M) \right\}$$

forms a positive $(G, M)$-orthogonal set. Define the weight factor $v_{M}(x)$ to be the associated constant

$$v_{M}(x) = v_{M}(\mathcal{Y}_{M}(x)).$$

(3.1.5.2)

Let $X$ be a point in $m(Q_{S})$ such that $G^{0}_{X}(Q_{S})$ is contained in $M(Q_{S})$, define the weighted orbital integral $J_{M}^{G}(X)$ to be the distribution on $g(Q_{S})$ such that

$$\forall f \in S(g(Q_{S})) \quad J_{M}^{G}(X, f) = |D^{G}(X)|^{1/2}_{S} \int_{G^{0}_{X}(Q_{S}) \backslash G(Q_{S})} f(X \cdot ad(x)) v_{M}(x) dx.$$
For a general element $X$ in $\mathfrak{m}(\mathbb{Q}_S)$ define the weighted orbital integral $J^G_M(X, f)$ to be the distribution on $g(\mathbb{Q}_S)$ such that

\[(3.1.5.4) \quad \forall f \in \mathcal{S}(g(\mathbb{Q}_S)) \quad J^G_M(X, f) = \lim_{A \to 0} \sum_{L \in \mathcal{L}(M)} r^L_M(\exp(X_{\text{nil}}), \exp(A))J^G_L(X + A, f)\]

where $A$ is a sequence of $\mathbb{Q}_S$-points of $\text{Lie}(A_M)$ such that

\[(3.1.5.5) \quad C^0_{X + A}(\mathbb{Q}_S) \subset M(\mathbb{Q}_S)\]

and $r^L_P(x, a)$ is an auxiliary $(L, M)$-family constructed by Arthur in §5 of [Ar88a], see also §2.4 of [HW13] and §III.2 of [Wa95].

(3.1.6) Lemma Let $M$ be a standard Levi subgroup of $G$. Let $X$ be an element of $\mathfrak{m}(\mathbb{Q}_S)$, let $x$ be an element of $G(\mathbb{Q}_S)$. Let $f$ be a Schwartz function on $\mathcal{S}(g(\mathbb{Q}_S))$. Then

\[(3.1.6.1) \quad J^G_M(X, f \circ \text{ad}(x)) = \sum_{P \in \mathcal{F}(M)} J^M_P(X, f_{P,x})\]

where $f_{P,x}$ is the Schwartz function on $\mathfrak{m}_P(\mathbb{Q}_S)$ defined as

\[(3.1.6.2) \quad f_{P,x}(X) = \int_K \int_{\mathfrak{a}_P(\mathbb{A})} f((X + N) \cdot \text{ad}(k)) \nu'_P(kx) dN dk\]

where for each $y$ in $G(\mathbb{Q}_S)$ and for each parabolic subgroup $Q$ in $\mathcal{F}(M)$ the constant $\nu'_Q(y)$ is the constant $\nu'_Q$ intervening in (3.1.5.1) associated to the $(G, M)$-orthogonal set $\mathcal{Y}_M(x)$ defined in (3.1.5.1).

(3.1.7) Proof See III.3.(f) of [Wa95].

3.2 Nilpotent orbits

(3.2.1) Definition Let $\mathfrak{g}_{\text{nil}}$ be the $\sim$ equivalence class of the origin in $\mathfrak{g}(\mathbb{Q})$, hence the nilpotent locus of $\mathfrak{g}$. Let $J_{\text{nil}}$ and $J^T_{\text{nil}}$ denote respectively $J_{\g_{\text{nil}}}$ and $J^T_{\g_{\text{nil}}}$. Let $J^G_{\text{nil}}$ denote $J_{\text{nil}}$ for convenience in inductive arguments involving Levi subgroups.

(3.2.2) Lemma There exists a continuous seminorm $\| \|$ on $\mathcal{S}(\mathfrak{g}(\mathbb{A}))$ such that for every truncation parameter $T$ in $\mathfrak{a}_0$

\[(3.2.2.1) \quad \forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A})) \quad \left| J^T_{\text{nil}}(f) - \int_{G(\mathbb{Q}) \setminus G(\mathbb{A})^T} F^G(x, T) \sum_{X \in \mathfrak{g}_{\text{nil}}(\mathbb{Q})} f(X \cdot \text{ad}(x)) \, dx \right| \leq \|f\| e^{-\frac{d(T)}{2}}\]

where $d(T)$ denotes the distance from $T$ to the root hyperplanes.

(3.2.3) Proof The argument is valid for a general class $\mathfrak{o}$ and the sum of all the classes $\mathfrak{o}$, hence contains (2.2.8.1) as a special case. The necessary estimates are established by Chaudouard in the proof of Proposition 4.4 of [Ch02]. The following is a sketch of the argument, for the details please refer to loc. cit. See also Theorem 3.1 of [Ar85].
Following the first part of the proof of (2.2.2.1), by the combinatorial lemma of Langlands, expand \( J_{\text{nil}}^f (f) \) as a sum indexed by triples of nested standard parabolic subgroups whose leading term corresponding to \((G,G,G)\) is

\[
(3.2.3.1) \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} F^G(x,T) \sum_{X \in \mathcal{B}_{\text{nil}}(\mathbb{Q})} f(X \cdot \text{ad}(x)) \, dx.
\]

Hence the left hand side of the inequality in (3.2.2.1) is bounded by

\[
(3.2.3.2) \quad \sum_{P_1,P_2,P_3 \in \mathcal{F} \atop \text{standard}} \int_{P_1(\mathbb{Q}) \backslash G(\mathbb{A})^1} F^{P_1}(x,T) a_1^3 \left( H_0(x) - T \right) \sum_{X \in m_1^2(\mathbb{Q}) \cap \varnothing} \left| \sum_{Y \in \mathcal{P}_2(\mathbb{Q})} \Phi_X(x,Y) \right| \, dx
\]

where \( \Phi \) denotes the partial Fourier transform of \( f \)

\[
(3.2.3.3) \quad \Phi_X(x,Y) = \int_{n_2(\mathbb{A})} f((X + N) \cdot \text{ad}(x)) \cdot \psi((N,Y)) \, dN.
\]

Changing variables each summand of (3.2.3.2) is bounded by

\[
(3.2.3.4) \quad \sup_{y \in \Gamma} \left( \int_{A_1^0(\mathbb{R})} \delta_0^2(a_1a)^{-1} \sigma_1^3(H_{P_1}(a_1) - T) \times \right.
\]

\[
\left. \times \sum_{X \in m_1^2(\mathbb{Q}) \cap \varnothing} \left| \sum_{Y \in \mathcal{P}_2(\mathbb{Q})} \Phi_{X \cdot \text{ad}(a_1a)}(y,Y \cdot \text{ad}(a_1a)) \right| \, dada_1 \right),
\]

where \( \Gamma \) is a fixed compact subset of \( M_0(\mathbb{A})^1 \) which is independant of the truncation parameter \( T \), the subset \( A_{0,T',T}(\mathbb{R}) \) of \( A_0^1(\mathbb{R}) \) is the \( T',T \)-truncated part as in (1.1.3.8), and \( \delta_0^2 \) is the modulus function of \( P_2 \).

Let \( n \) be a natural number, let \( \mathcal{D} \) be an invariant differential operator on \( \mathfrak{g}(\mathbb{R}) \) of degree \( n \), denote by \( \Phi^\mathcal{D} \) the partial Fourier transform of \( \mathcal{D} f \) where \( \mathcal{D} \) operates on \( f \) via its archimedean component \( f_\infty \), then

\[
(3.2.3.5) \quad \left| \Phi_{X \cdot \text{ad}(a_1a)}(y,Y \cdot \text{ad}(a_1a)) \right| = C(y) \cdot \left| Y \cdot \text{ad}(a_1a) \right|^{-n} \cdot \left| \Phi_{X \cdot \text{ad}(a_1a)}(y,Y \cdot \text{ad}(a_1a)) \right|
\]

for some constant \( C(y) \) that depends continuously on \( y \). For a Schwartz function \( f \) on \( \mathfrak{g}(\mathbb{A}) \) define \( N(f) \) to be the smallest natural number \( N \) such that \( f \) is supported on \( \frac{1}{N} \mathbb{Z} \times \mathbb{R} \) where \( \mathbb{Z} \) denotes the profinite completion of \( \mathbb{Z} \). Let the natural number \( n \) be large enough so that

\[
(3.2.3.6) \quad \sum_{Y \in \mathcal{P}_2(\mathbb{Q})} \left| Y \cdot \text{ad}(a_1a) \right|^{-n} \leq C \prod_{\alpha \in \Delta_3^0} e^{-k_\alpha \sigma(H_0(a_1a))}
\]

for some constant \( C \) and natural numbers \( k_\alpha \). Define the seminorm \( \| \cdot \| \) by

\[
(3.2.3.7) \quad \| f \| = \sup_{X \in \mathfrak{g}(\mathbb{A})} |\mathcal{D} f(X)|.
\]

A Schwartz function on the \( \mathbb{A} \)-valued points of a rational vector space is bounded by a product of Schwartz functions on each coordinate, hence for \( Z \) in \( p_2(\mathbb{A}) \)

\[
(3.2.3.8) \quad |\mathcal{D} f(Z \cdot \text{ad}(y))| \leq \left( \prod_{\mu \in \Phi_0 - \Phi_2} \phi_\mu(Z_\mu) \right) \phi_{n_2}(Z_{n_2})
\]
where \( Z_\mu \) and \( Z_{n_2} \) are the components of \( Z \) on the weight space of \( \mu \) and \( n_2 \), and \( \phi_\bullet \) are positive Schwartz functions. If \( a_0 \) is an element of \( A_0(\mathbb{R}) \), denote by \( \Psi(a_0) \) for the sum

\[
(3.2.3.9) \quad \Psi(a_0) = \sum_{X \in m_1^2(\mathbb{N}(j)/\mathbb{Z})) \cap \alpha} \left( \prod_{\mu \in \Phi_0 - \Phi_2} \phi_\mu(\mu(a_0)^{-1}X_\mu) \right).
\]

Then \((3.2.3.4)\) is bounded by

\[
(3.2.3.10) \quad \sup_{y \in \Gamma} C(y) \int_{\mathbb{A}_1^1(\mathbb{R})} \int_{\mathbb{A}_0^1(\mathbb{R})} \left( \delta^2_\alpha(a_1)^{-1}a_1^2(H_{\mu_1}(a_1) - T) \times \Psi(a_1) \cdot C \prod_{\alpha \in \Delta^2} e^{-\alpha(H_0(a_1))} \right) da_1,
\]

which only depends on the Schwartz function \( f \) via its componentwise bounds \( \phi_\bullet \) and the lattice \( \mathbb{N}(j)/\mathbb{Z}) \), hence is proportional to the seminorm \( \| \| \) defined by

\[
(3.2.3.11) \quad \forall f \in \mathcal{S}(g(\mathbb{A})) \quad \| f \| = \sup_{y \in \Gamma} C(y) \cdot C \cdot \| f \|' \cdot N(f)^n.
\]

The constant \( \sup_{y \in \Gamma} C(y) \cdot C \) is independent of both the Schwartz function \( f \) and the truncation parameter \( T \).

Substituting the definition of the seminorm \( \| \| \), the majorant \((3.2.3.10)\) reduces to

\[
(3.2.3.12) \quad \| f \| \cdot \text{Vol}(\mathbb{A}_0^1(\mathbb{R})) \prod_{\alpha \in \Delta^1} \left( e^{a(T)} \int_0^\infty p(t) e^{-a} \, dt \right)
\]

where \( p(t) \) is a polynomial. Because

\[
(3.2.3.13) \quad \prod_{\alpha \in \Delta^1} e^{-a(T)} \int_0^\infty p(t) e^{-a} \, dt \leq e^{-d(T)},
\]

and \( \text{Vol}(\mathbb{A}_0^1(\mathbb{R})) \) is of polynomial growth in \( T \), the majorant \((3.2.3.12)\) reduces to

\[
(3.2.3.14) \quad \| f \| e^{-d(T)/2}.
\]

\(\square\)

\((3.2.4)\) \textbf{Definition} \ Let \( v \) be a place of \( \mathbb{Q} \), let \( \beta_v \) be a bump function on \( \mathbb{Q}_v \), let \( \nu \) be a \( G \)-orbit contained in \( g_{01} \), let \( \{ p_1, \ldots, p_l \} \) be a collection of polynomials rational with rational coefficients cutting out the Zariski closure \( \mathfrak{7} \). Let \( f \) be a Schwartz function on \( g(\mathbb{A}) \), let \( \epsilon \) be a positive real number. Define the \textit{truncated function} \( f_{\nu,v}^\epsilon \) on \( g(\mathbb{A}) \) as in \([\text{Ar85}]\) by

\[
(3.2.4.1) \quad \forall X \in g(\mathbb{A}) \quad f_{\nu,v}^\epsilon(X) = f(X)\beta_v(\epsilon^{-1}|p_1(X)|_v)\ldots\beta_v(\epsilon^{-1}|p_l(X)|_v).
\]

\((3.2.5)\) \textbf{Lemma} \ Let the place \( v \), the bump function \( \beta_v \), the orbit \( \nu \) and the polynomials \( \{ p_1, \ldots, p_l \} \) be as in \((3.2.4)\), then there exists a natural number \( m \) and another seminorm \( \| \|_1 \) for which the inequality \((3.2.2.7)\) holds such that

\[
(3.2.5.1) \quad \forall \epsilon \text{ such that } 0 < \epsilon < 1 \forall f \in \mathcal{S}(g(\mathbb{A})) \quad \| f_{\nu,v}^\epsilon \| \leq \epsilon^{-ml} \| f \|_1.
\]
(3.2.6) Proof It is enough to consider the case when there is a single polynomial $p$. There are
two cases:

- If $v$ is the archimedean place then
  \[(3.2.6.1)\]
  \[N(f^ε_{v,v}) = N(f),\]
  so by properties of the derivative
  \[(3.2.6.2)\]
  \[\|f^ε_{v,v}\|' ≤ ε^{-m}C\|f\|'.\]
  Define $\|f\|'_1$ to be $C\|f\|$.

- If $v$ is finite then
  \[(3.2.6.3)\]
  \[\|f^ε_{v,v}\|' = \|f\|',\]
  so
  \[(3.2.6.4)\]
  \[N(f^ε_{v,v}) ≤ ε^{-m}N(f)\]
  since $N(f)$ only depends on the support of $f$ finite and the assignment
  \[(3.2.6.5)\]
  \[f \mapsto f^ε_{v,v} = f \prod β_v(ε^{-1}|p|_v)\]
  shrinks the support of $f$ by a factor of $ε^m$, up to a multiplicative constant. Define $\|f\|'$ to be $\|f\|$.

Hence (3.2.5.1) follows since
\[(3.2.6.6)\]
\[\|f\| = \|f\|'N(f).\]

(3.2.7) Lemma Let the place $v$, the bump function $β_v$, the orbit $ν$ and the polynomials $\{p_1, \ldots, p_l\}$
be as in (3.2.4), then there exists a positive real number $r$ such that
\[(3.2.7.1)\]
\[∀ε > 0 \ ∀f ∈ S(g(𝔸)) \int_{G(𝔸)} \ F^G(x,T) \sum_{X ∈ g(ℚ) - Γ(ℚ)} |f^ε_{v,v}(X \cdot \text{ad}(x))| \ dx ≤ \|f\|'ε^r(1 + |T|)^{d_0}\]
where $d_0$ is $\text{dim}(𝔸_0/𝔸_G)$, the split rank of $G$.

(3.2.8) Proof This is Lemma 4.1 of [Ar85].

(3.2.9) Proposition Let $T$ be a truncation parameter. For each nilpotent orbit $ν$ there exists
a distribution $J^T_ν$ on $g(𝔸)$ such that for each Schwartz function $f$ on $g(𝔸)$, the expression $J^T_ν(f)$ is
a polynomial in $T$ of degree at most $d_0$, and
\[(3.2.9.1)\]
\[J^T_ν(f) = \sum_{ν ∈ \text{null orbit}} J^T_ν(f).\]
There exists a continuous seminorm $\|\|$ on $S(g(𝔸))$ and a positive real number $ε$ such that
\[(3.2.9.2)\]
\[\|J^T_ν(f) - \int_{G(ℚ) \setminus G(𝔸)^1} \ F^G(x,T) \sum_{X ∈ ν(ℚ)} f(X \cdot \text{ad}(x)) \ dx\| ≤ \|f\|e^{-εd(T)}.\]
(3.2.10) **Proof** Define the polynomials recursively by the formula

\[(3.2.10.1)\]

\[
J^T_ν(f) = \lim_{\epsilon \to 0} J^T_\text{nil}(f^\epsilon_{ν,v})
\]

where \(J^T_\nu\) denotes the sum of \(J^T_{ν'}\) for those orbits \(ν'\) contained in \(ν\).

The limit of \(J^T_\text{nil}(f^\epsilon_{ν,v})\) as \(\epsilon\) approaches 0 exists because

\[(3.2.10.2)\]

\[
\sum_{X ∈ \nu(Q)} f(X · ad(x)) = \sum_{X ∈ \nu(Q)} f^\epsilon_{ν,v}(X · ad(x))
\]

implies that

\[(3.2.10.3)\]

\[
\left| J^T_\text{nil}(f^\epsilon_{ν,v}) - \int_{G(Q) \backslash G(A)} F^G(x, T) \sum_{X ∈ \nu(Q)} f(X · ad(x)) \, dx \right|
\]

\[
≤ \left| J^T_\text{nil}(f^\epsilon_{ν,v}) - \int_{G(Q) \backslash G(A)} F^G(x, T) \sum_{X ∈ g_\text{nil}(Q)} f^\epsilon_{ν,v}(X · ad(x)) \, dx \right|
\]

\[
+ \int_{G(Q) \backslash G(A)} F^G(x, T) \sum_{X ∈ g_\text{nil}(Q) \backslash G(Q)} \left| f^\epsilon_{ν,v}(X · ad(x)) \right| \, dx.
\]

By \((3.2.2.1)\) the first summand of the right hand side of \((3.2.10.3)\) is bounded by

\[(3.2.10.4)\]

\[
\|f^\epsilon_{ν,v}\|e^{-\frac{d(T)}{2}} \leq e^{-l\|f\|_1 \cdot e^{-\frac{d(T)}{2}}}
\]

which follows from \((3.2.5.1)\).

The second summand of the right hand side of \((3.2.10.3)\) is bounded by summing over the complement of \(G(Q)\) in \(g(Q)\) instead of in \(g_\text{nil}(Q)\), hence by \((3.2.7.1)\) is bounded by

\[(3.2.10.5)\]

\[
\|f\|_1 \cdot \epsilon^r \cdot (1 + |T|)^d_0.
\]

Therefore the right hand side of \((3.2.10.3)\) is bounded by

\[(3.2.10.6)\]

\[
\|f\|_1 \left( e^{-l\|f\|_1 \cdot e^{-\frac{d(T)}{2}}} + \epsilon^r (1 + |T|)^d_0 \right).
\]

It suffices to take \(δ^n\) as \(\epsilon\) with \(δ\) bounded strictly between 0 and 1 and \(n\) a sequence natural numbers approaching infinity. Then the majorant \((3.2.10.6)\) satisfies the inequality

\[(3.2.10.7)\]

\[
\|f\|_1 \left( e^{|\log(δ)|l_m \cdot \frac{d(T)}{2}} + δ^{\tau n} (1 + |T|)^d_0 \right) \leq \|f\| \cdot δ^{\tau n} \cdot (1 + |T|)^d_0
\]

provided \(d(T)\) is bounded below by \(C|\log(δ)|n\) for some constant \(C\) and \(||\|\) is another continuous seminorm with the same properties.

Therefore for every natural number \(n\) and every \(T\) in the open subcone

\[(3.2.10.8)\]

\[
\left\{ T : d(T) > C|\log(δ)|(n + 1) \right\}
\]

of the positive chamber, the following inequality holds

\[(3.2.10.9)\]

\[
|J^T_\text{nil}(f^δ_{ν,v}) - J^T_\text{nil}(f^{δ_{n+1}}_{ν,v})| \leq 2\|f\|(1 + |T|)^d_0 δ^{\tau n}.
\]

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The left hand side is a polynomial in $T$ of degree at most $d_0$, hence the polynomial extrapolation lemma in Lemma 5.2 of [Ar82] applies:

There exists a constant $A$ such that for all $T$,

\begin{equation}
|J^T_{\text{nil}}(f^{\delta_n}_{\nu,v}) - J^T_{\text{nil}}(f^{\delta_{n+1}}_{\nu,v})| \leq A \cdot \|f\| \cdot (1 + |T|)^{d_0} \cdot (|\log(\delta)|(n + 1))^{d_0} \cdot \delta^{rn}.
\end{equation}

Since

\begin{equation}
\sum_{n=0}^{\infty} (|\log(\delta)|(n + 1))^{d_0} \delta^{rn} < \infty,
\end{equation}

by telescoping the series

\begin{equation}
\sum_{n=0}^{\infty} J^T_{\text{nil}}(f^{\delta_n}_{\nu,v}) - J^T_{\text{nil}}(f^{\delta_{n+1}}_{\nu,v})
\end{equation}

the sequence $J^T_{\text{nil}}(f^{\delta_n}_{\nu,v})$ has a limit as $n$ approaches infinity. The limit is a polynomial in $T$ of degree at most $d_0$, denoted by $J^T_{\nu}(f)$.

By construction

\begin{equation}
\left| J^T_{\nu}(f) - \int_{G(Q)\backslash G(A)^1} F^G(x,T) \sum_{X \in \pi(Q)} f(X \cdot \text{ad}(x)) \, dx \right|
\end{equation}

\begin{equation}
\leq \left| J^T_{\text{nil}}(f^{\delta_n}_{\nu,v}) - J^T_{\nu}(f) \right| + \left| J^T_{\text{nil}}(f^{\delta_n}_{\nu,v}) - \int_{G(Q)\backslash G(A)^1} F^G(x,T) \sum_{X \in \pi(Q)} f(X \cdot \text{ad}(x)) \, dx \right|
\end{equation}

\begin{equation}
\leq \sum_{n=0}^{\infty} A\|f\|(1 + |T|)^{d_0}(|\log(\delta)|(n + 1))^{d_0} \delta^{rn} + \|f\|\delta^{rn}(1 + |T|)^{d_0}
\end{equation}

where (3.2.10.14) holds for a fixed $\delta$, a sufficiently positive $T$ and $n$ the largest natural number such that

\begin{equation}
d(T) \geq C|\log(\delta)|n.
\end{equation}

Therefore it is possible choose a new seminorm $\|\|$ and a new constant $\epsilon > 0$ such that

\begin{equation}
\left| J^T_{\nu}(f) - \int_{G(Q)\backslash G(A)^1} F^G(x,T) \sum_{X \in \pi(Q)} f(X \cdot \text{ad}(x)) \, dx \right| \leq \|f\|e^{-\epsilon d(T)}.
\end{equation}

The corresponding statements for $\nu$ instead of $\nu'$ follow recursively by setting

\begin{equation}
J^T_{\nu}(f) = \sum_{\nu' \subseteq \nu \text{ orbit}} J^T_{\nu'}(f).
\end{equation}

(3.2.11) Proposition Let $S$ be a finite set of places of $Q$ containing the archimedean places, let $f$ be a Schwartz function on $g(A)$ such that $f_p$ is the characteristic function of the standard lattice $g(Z_p)$ whenever $p$ is not contained in $S$. Let $M$ be a standard Levi subgroup of $G$. Denote by $m_{\text{nil}}(Q)_{M,S}$ the set of $M(Q_S)$-conjugacy classes in $m_{\text{nil}}(Q)$. For each nilpotent conjugacy class $\nu$ in $m_{\text{nil}}(Q)_{M,S}$ here exists a constant $a^M(S,\nu)$ such that

\begin{equation}
J^G_{\text{nil}}(f) = \sum_{M \in \mathcal{L}} |W^M_0||W^G_0|^{-1} \sum_{\nu \in m_{\text{nil}}(Q)_{M,S}} a^M(S,\nu)J^G_M(\nu, f_S).
\end{equation}
(3.2.12) **Remark** By Lemma 7.1 of [Ar85], the sets $m_{nil}(Q)_{M,S}$ and $m_{nil}(Q_S)/M(Q_S)^1$ are in natural bijection, hence the expression $J^G_M(\nu, f_S)$ makes sense.

(3.2.13) **Proof** The argument is based on the following two lemmas:

**Lemma** For every $x$ in $G(A)^1$

\[(3.2.13.1) \quad J^G_{nil}(f \circ \text{ad}(x)) = \sum_{Q \in \mathcal{F}} |W_0^M||W_0^G|^{-1} J^M_{nil}(f_{Q,x}),\]

where the function $f_{Q,x}$ on $m(Q)$ is defined as in (2.2.18.2).

**Proof** This is a special case of (2.2.18.1). \hfill \Box

**Lemma** Let $M$ be a standard Levi subgroup. For every $x$ in $G(Q_S)^1$ and $\nu$ in $m_{nil}(Q_S)/M(Q_S)^1$

\[(3.2.13.2) \quad J^G_M(\nu, f_S \circ \text{ad}(x)) = \sum_{Q \in \mathcal{F}(M)} J^M_M(\nu, f_{S,Q,x}).\]

**Proof** This is a special case of (3.1.6.1). \hfill \Box

Argue by induction. Assume that the constants $a^L(S, \nu)$ and the identities

\[(3.2.13.3) \quad J^L_{nil}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^L|^{-1} \sum_{\nu \in m_{nil}(Q)_{M,S}} a^M(S, \nu) J^L_M(\nu, f_S)\]

are known for every proper Levi subgroup $L$ of $G$.

Define the distribution $T^G$ on $g(Q_S)$ by

\[(3.2.13.4) \quad \forall f_S \in S(g(Q_S)) \quad T^G(f_S) = J^G_{nil}(f_S \otimes \bigotimes_{p \notin S} I_{g(Z_p)}) - \sum_{M \in \mathcal{L}, M \neq G} |W_0^M||W_0^G|^{-1} \sum_{\nu \in m_{nil}(Q)_{M,S}} a^M(S, \nu) J^G_M(\nu, f_S).\]

The distribution $T^G$ is supported on $g_{nil}(Q_S)$. By (3.2.13.1) and (3.2.13.2)

\[(3.2.13.5) \quad T^G(f_S \circ \text{ad}(x)) - T^G(f_S) = \left( \sum_{Q \in \mathcal{F}} |W_0^M||W_0^G|^{-1} J^M_{nil}(f_S \otimes \bigotimes_{p \notin S} I_{g(Z_p)} f_{Q,x}) \right) \]

\[- \left( \sum_{M \in \mathcal{L}, Q \in \mathcal{F}(M)} |W_0^M||W_0^G|^{-1} \times \right.\]

\[\left. \sum_{\nu \in m_{nil}(Q)_{M,S}} a^M(S, \nu) J^M_M(\nu, f_{S,Q,x}) \right) \]

\[- J^G_{nil}(f_S \otimes \bigotimes_{p \notin S} I_{g(Z_p)}) - \sum_{M \in \mathcal{L}, M \neq G} |W_0^M||W_0^G|^{-1} \times \]
Stratify the nilpotent locus which vanishes by the inductive hypothesis that $T^\nu$ is invariant applied to the Levi subgroup $M_Q$ with $Q$ a proper parabolic subgroup of $G$. Therefore $T^G$ is invariant under the action of $G(\mathbb{Q}_S)^1$.

Construct the constants $a^G(S,\nu)$ subject to the new identity

$$T^G(f_S) = \sum_{\nu \in \mathcal{G}_G} a^G(S,\nu)J^G_G(\nu, f_S).$$

Stratify the nilpotent locus $\mathfrak{g}_{\text{nil}}(\mathbb{Q}_S)$ equivariantly by codimension. More precisely define for each natural number $d$ the open set

$$\mathfrak{g}_{\text{nil},d}(\mathbb{Q}_S) = \bigcup_{\nu \in \mathcal{G}_G \cap \nu(\mathbb{Q}_S), \text{ codim}(\nu) \leq d} \nu(\mathbb{Q}_S).$$

Denote by $T^G_d$ the distribution obtained by restricting $T^G$ to $\mathfrak{g}_{\text{nil},d}(\mathbb{Q}_S)$.

The open set $\mathfrak{g}_{\text{nil},0}(\mathbb{Q}_S)$ is the regular nilpotent orbit. Since $T^G_0$ is invariant it is equal to a multiple of $J^G_G(\nu_{\text{reg}}, \cdot)$, where $\nu_{\text{reg}}$ denotes the regular nilpotent orbit. Define $a^G(S,\nu_{\text{reg}})$ to be the constant of proportionality.

The other constants $a^G(S,\nu)$ are constructed by induction on the codimension $d$ which ranges among $0, 1, 2, \ldots, \text{dim}(\mathfrak{g}_{\text{nil}})$. Let $T^G, d$ be the distribution on the complement of $\mathfrak{g}_{\text{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\text{nil}}(\mathbb{Q}_S)$ defined by

$$T^G, d(f_S) = T^G(f_S) - \sum_{\nu \in \mathcal{G}_G \cap \text{codim}(\nu) < d} a^G(S,\nu)J^G_G(\nu, f_S).$$

Denote by $T^G_d$ its restriction to the complement of $\mathfrak{g}_{\text{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\text{nil},d}(\mathbb{Q}_S)$.

Since the complement of $\mathfrak{g}_{\text{nil},d-1}(\mathbb{Q}_S)$ in $\mathfrak{g}_{\text{nil},d}(\mathbb{Q}_S)$ has an open partition by the nilpotent orbits of codimension $d$, and the distribution $T^G_d$ is invariant, there exist constants $a^G(S,\nu)$, one for each $\nu$ of codimension $d$, such that

$$T^G_d(f_S) = \sum_{\nu \in \mathcal{G}_G \cap \text{codim}(\nu) = d} a^G(S,\nu)J^G_G(\nu, f_S).$$

The constants $a^G(S,\nu)$ are required to satisfy (3.2.13.6), which is equivalent to

$$T^G(f_S) = T^G_0(f_S) + T^G_1(f_S) + T^G_2(f_S) + \cdots + T^G_{\text{dim}(\mathfrak{g}_{\text{nil}})}(f_S).$$
Let $v \in S$ be a place of $Q_S$, let $\nu$ be a nilpotent orbit, define the function $f^\epsilon_{\tilde{S},\nu,v}$ by the same formula (3.2.14.1) as for $f^\epsilon_{\nu,v}$. Let $\nu_d$ denote the complement of $\mathfrak{g}_{\text{nil},d-1}$ in $\mathfrak{g}_{\text{nil},d}$, the union of the nilpotent orbits $\nu$ of codimension $d$. Then the expression $f^\epsilon_{\tilde{S},\nu_d,v}$ makes sense, and

$$T_d^{G,S}(f_S) = \lim_{\epsilon \to 0} T^G(f^\epsilon_{\tilde{S},\nu_d,v}) - \lim_{\epsilon \to 0} T^G(f^\epsilon_{\tilde{S},\nu_{d+1},v}).$$

Therefore

$$T_d^{G,0}(f_S) + T_1^{G,1}(f_S) + T_2^{G,2}(f_S) + \cdots + T_d^{G,\dim(\mathfrak{g}_{\text{nil}})}(f_S)$$

$$= \lim_{\epsilon \to 0} \left( T^G(f^\epsilon_{\tilde{S},\nu_0,v}) - T^G(f^\epsilon_{\tilde{S},\nu_1,v}) + T^G(f^\epsilon_{\tilde{S},\nu_1,v}) - T^G(f^\epsilon_{\tilde{S},\nu_2,v}) + T^G(f^\epsilon_{\tilde{S},\nu_2,v}) - \cdots \right.$$  

$$\left. \cdots - T^G(f^\epsilon_{\tilde{S},\nu_{\dim(\mathfrak{g}_{\text{nil}})},v}) + T^G(f^\epsilon_{\tilde{S},\nu_{\dim(\mathfrak{g}_{\text{nil}})},v}) - T^G(f^\epsilon_{\tilde{S},\nu_{\dim(\mathfrak{g}_{\text{nil}})+1},v}) \right)$$

$$= T^G(f_S).$$

(3.2.14) **Remark** If $\nu$ is the orbit consisting of the origin, the coefficient $a^M(S,\nu)$ is independent of $S$ and equal to the corresponding Tamagawa number

$$a^M(S,0) = \text{Vol}(M(Q)\backslash M(A)^1).$$

3.3 General orbits

(3.3.1) **Definition** Let $\mathfrak{g}$ be a $\sim$ equivalence class on $\mathfrak{g}(Q)$. Let $P_1$ be a parabolic subgroup of $G$, let $M_1$ be the Levi component of $P_1$ which is standard. Fix a semisimple element $\Sigma$ in $\mathfrak{g}$ such that $\Sigma$ is contained in the Levi subalgebra $\mathfrak{m}_1$, but not in any proper parabolic subalgebra of $\mathfrak{p}_1$. The group $P_{1,\Sigma}^0$ is a minimal parabolic subgroup of $G_{\Sigma}^0$ with minimal Levi component $M_{1,\Sigma}^0$. Denote by $F_{\Sigma}$ the set of parabolic subgroups of $G_{\Sigma}^0$ containing $M_{1,\Sigma}^0$. A parabolic subgroup $Q$ in $F_{\Sigma}$ is said to be standard if $Q$ contains $P_{1,\Sigma}^0$. Fix a maximal compact subgroup $K_{\Sigma}$ of $G_{\Sigma}^0(A)$ that is admissible with respect to $M_{1,\Sigma}^0$ such that for each parabolic subgroup $Q$ in $F_{\Sigma}$ there is the associated function

$$H_Q : G_{\Sigma}^0(A) \rightarrow \mathfrak{a}_Q.$$ 

There is a unique point $T_{\Sigma,1}$ in $\mathfrak{a}_1$ modulo $\mathfrak{a}_{G_{\Sigma}^0}$ defined in the same manner as $T_0$ in $\mathfrak{a}_G^0$ that satisfies an identity analogous to (2.2.12.1). Let $L$ be a Levi subgroup of $G_{\Sigma}^0$ containing $M_{1,\Sigma}^0$, denote by $W^1_L$ the Weyl group of $L$ with respect to the split torus $A_1$.

Let $\pi_{\Sigma}$ be the surjection from $F(M_1)$ onto $F_{\Sigma}$ defined by

$$\forall P \in F(M_1) \quad \pi_{\Sigma}(P) = P_{\Sigma}^0.$$ 

Let $Q$ be a parabolic subgroup in $F_{\Sigma}$. Let $F_{\Sigma}(M_1)$ be the inverse image $\pi_{\Sigma}^{-1}(Q)$. Define subsets $\tilde{F}_Q(M_1)$ and $\tilde{F}_Q(M_1)$ of $F(M_1)$ by

$$\tilde{F}_Q(M_1) = \left\{ P \in F(M_1) : \pi_{\Sigma}(P) = Q, \mathfrak{a}_P = \mathfrak{a}_Q \right\}$$

$$\tilde{F}_Q(M_1) = \left\{ P \in F(M_1) : \pi_{\Sigma}(P) \supset Q \right\}.$$
There are inclusions

\[(3.3.1.4) \quad \forall Q \in \mathcal{F}^\Sigma, \quad \hat{\mathcal{F}}_Q(M_1) \subset \mathcal{F}_Q(M_1) \subset \hat{\mathcal{F}}_Q(M_1) \subset \mathcal{F}(M_1). \]

**3.3.2 Definition** Let \( P_1, M_1, \alpha, \Sigma \) be as in \((3.3.1)\). Let \( Q \) be a parabolic subgroup in \( \mathcal{F}^\Sigma \). Let \( \mathcal{Y} \) be a collection of points \( \{ Y_P \in a_0 : P \in \hat{\mathcal{F}}_Q(M_1) \} \) satisfying the compatibility conditions defining a \((G, M_1)\)-orthogonal set in \((3.1.3)\) namely that for each pair of adjacent parabolic subgroups \( P \) and \( P' \) in \( \hat{\mathcal{F}}_Q(M_1) \) the difference between \( Y_P \) and \( Y_{P'} \) is orthogonal to the common wall of the positive chambers defined by \( P \) and \( P' \).

The collection \( \mathcal{Y} \) has a unique extension from \( \mathcal{F}_Q(M_1) \) to \( \hat{\mathcal{F}}_Q(M_1) \). Let \( Q' \) be a parabolic subgroup in \( \mathcal{F}^\Sigma \) containing \( Q \), denote by \( \mathcal{Y}_{Q'} \) the collection of points \( \{ Y_P : P \in \mathcal{F}_{Q'}(M_1) \} \).

**3.3.2.1**

\[ \mathcal{Y}_{Q'} = \left\{ Y_P : P \in \mathcal{F}_{Q'}(M_1) \right\}. \]

Let \( Q_3 \) be a parabolic subgroup in \( \mathcal{F}^\Sigma \). Define the gamma function \( \Gamma'_{Q_3}(\cdot, \mathcal{Y}_{Q_3}) \) on \( a_1 \) by

\[(3.3.2.3) \quad \forall H \in \mathcal{F}_{Q_1}(M_1) \quad \Gamma'_{Q_3}(H, \mathcal{Y}_{Q_3}) = \sum_{Q_4 \in \mathcal{F}^\Sigma} \tau^{(H)}_3 \left( P \right) \left( \sum_{P \in \mathcal{F}_{Q_4}(M_1)} (-1)^{\dim(A_P/A_C)} \hat{\tau}_P(H - Y_P) \right). \]

**3.3.3 Remark** The function \( \Gamma'_{Q_3}(\cdot, \mathcal{Y}_{Q_3}) \) factorizes through the projection from \( a_1 \) onto \( a_3^G \) and depends continuously on \( \mathcal{Y} \). For each parabolic subgroup \( Q_2 \) in \( \mathcal{F}^\Sigma \),

\[(3.3.3.1) \quad \sum_{P \in \mathcal{F}_{Q_2}(M_1)} (-1)^{\dim(A_P/A_C)} \hat{\tau}_P(H - Y_P) = \sum_{Q_3 \in \mathcal{F}^\Sigma} \sum_{Q_3 \supset Q_2} (-1)^{\dim(A_3/A_C)} \hat{\tau}_3(H) \Gamma'_{Q_3}(H, \mathcal{Y}_{Q_3}). \]

See \( \S 4 \) of \[Ar86\].

**3.3.4 Lemma** Let \( Q \) be a parabolic subgroup in \( \mathcal{F}^\Sigma \), let \( \mathcal{Y} \) be a collection of points as in \((3.3.2.2)\). The function \( \Gamma'_{Q}(\cdot, \mathcal{Y}_{Q}) \) is compactly supported as a function on \( a_3^G \).

Let \((c_p)\) be the \((G, M_1)\)-family associated with a \((G, M_1)\)-orthogonal set that contains \( \mathcal{Y}_Q \) as in \((3.1.4.1)\). Let \( c'_p \) be the functions intervening in \((3.1.2.3)\). Let \( \Gamma'^{\sim}_{Q}(\cdot, \mathcal{Y}_{Q}) \) denote the Fourier transform of \( \Gamma'_{Q}(\cdot, \mathcal{Y}_{Q}) \). Then

\[(3.3.4.1) \quad \forall \lambda \in i a_3^{G^*}, \quad \Gamma'^{\sim}_{Q}(\lambda, \mathcal{Y}_{Q}) = \sum_{P \in \mathcal{F}_Q(M_1)} c'_p(\lambda). \]

**3.3.5 Proof** See Lemma 4.1 of \[Ar86\]. \( \square \)

**3.3.6 Remark** The main ingredient of the proof is the following generalization of \((2.2.3.1)\):

Let \( Q \) be a parabolic subgroup in \( \mathcal{F}^\Sigma \), let \( P \) be a parabolic subgroup in \( \hat{\mathcal{F}}_{Q}(M_1) \), then

\[(3.3.6.1) \quad \sum_{P' \in \mathcal{F}_{Q}(M_1)} (-1)^{\dim(A_P/A_{C})} \tau^{(P')}_{Q}(H) \hat{\tau}^P_{P'}(H) = \begin{cases} 1 & \text{if } P \in \mathcal{F}_{Q}(M_1) \text{ and } H \in a_{P}, \\ 0 & \text{otherwise}. \end{cases} \]

This is Lemma 4.2 of \[Ar86\].
Decompose the sum defining Lemma (Global semisimple descent)

\[
\begin{align*}
(3.3.7.1) \quad J_\phi(f) &= |\pi_0(G_\Sigma)|^{-1} \int_{G^0_\Sigma(A) \backslash G(A)} \left( \sum_{Q \in \mathcal{F}^0} |W_1^{MQ}| |W_1^{G_\Sigma}|^{-1} J_{1Q}^{MQ} (\Phi_{Q,x}^{T_0-T_\Sigma,1}) \right) dx
\end{align*}
\]

where for a truncation parameter \( T \) in \( a_0 \), the function \( \Phi_{Q,x}^{T} \) on \( m_\Sigma(A) \) is defined by

\[
(3.3.7.2) \quad \forall X \in m_\Sigma(A) \quad \Phi_{Q,x}^{T}(X) = \int_{K_\Sigma} \int_{a_\Sigma(A)} f\left((\Sigma + (X+N) \cdot \text{ad}(k)) \cdot \text{ad}(x)\right) v'_Q(kx,T) d\text{Ndk}
\]

where the weight factor \( v'_Q \) is defined as

\[
(3.3.7.3) \quad v'_Q(kx,T) = \int_{a_\Sigma^0} \Gamma'_Q(H,Y_Q^T(k,x)) \ dH
\]

where \( Y^T(k,x) \) is the collection of points defined by

\[
(3.3.7.4) \quad \forall P \in \mathcal{F}_Q(M_1) \quad Y^T_P(k,x) = -H_P(kx) - T_\Sigma + T
\]

where \( T_\Sigma \) is a truncation parameter in \( a_1 \) such that \( T_\Sigma - T_{\Sigma,1} \) is the projection of \( T - T_0 \).

**Proof** This argument follows the proof of Lemma 6.2 of [Ar86].

Let \( T \) be a truncation parameter in \( a_0 \). Define a second truncated kernel function \( j''_\phi( , f) \) on \( G(Q) \backslash G(A)^{1} \) by

\[
(3.3.8.1) \quad \forall x \in G(Q) \backslash G(A)^1 \quad j''_\phi(x, f) = \sum_{P \in \mathcal{F}} (-1)^{\dim(A_P/A_C)} \sum_{\delta \in P(Q) \backslash G(Q)} \hat{\tau}_P(H_0(\delta x) - T) j_P(\delta x)
\]

where

\[
(3.3.8.2) \quad J_P(\phi, x, f) = \sum_{X \in m_{P}(Q) \cap a} \sum_{\eta \in N_{P,x} \backslash N_{P}(Q)} \int_{m_{P,x} \backslash (\Sigma)} f((X+N) \cdot \text{ad}(\eta x)) \ d\text{N}.
\]

**Lemma** The function \( j''_\phi( , f) \) is integrable on \( G(Q) \backslash G(A)^1 \), and

\[
(3.3.8.3) \quad J''_\phi(f) = \int_{G(Q) \backslash G(A)} j''_\phi(x, f) \ dx.
\]

**Proof** Following the proof of (2.2.2.1), by the combinatorial lemma of Langlands (2.2.3.1), it suffices to prove that

\[
(3.3.8.4) \quad \sum_{P_2 \in \mathcal{F}_{\text{standard}}} \left| \int_{P_2(Q) \backslash G(A)} F_{P_2}(x,T) \sigma^5_{P_2}(H_0(x) - T) \right| \sum_{P_4 \in \mathcal{F}} (-1)^{\dim(A_4/A_C)} J_{P_4,\phi}(x, f) \ dx < \infty.
\]

Decompose the sum defining \( J_{P_4,\phi}(x, f) \) over the set of parabolic subgroups contained in \( P_4 \),

\[
(3.3.8.5) \quad J_{P_4,\phi}(x, f) = \sum_{P_3 \in \mathcal{F}} \sum_{P_2 \in \mathcal{P}_2 \cap \mathcal{P}_3} \sum_{X \in m_3(Q) \cap a} \sum_{Y \in m_3(Q)} \ldots
\]
where (3.3.8.6) follows from Corollary 2.4 of [Ch02] which states that

\[ \eta \in N_{4,(X+Y)_{ss}}(Q) \backslash N_{4}(Q) \]

\[ \sum_{\eta \in N_{4,(X+Y)_{ss}}(Q) \backslash N_{4}(Q)} \int_{n_{4,(X+Y)_{ss}}(A)} f((X + Y + N) \cdot \ad(\eta x)) \, dN \]

(3.3.8.6)

\[ = \sum_{P_{2} \in F} \sum_{P_{2} \subseteq P_{3} \subseteq P_{4}} \sum_{X \in \mathfrak{m}_{3}(Q) \cap F} \sum_{Z \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\eta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\delta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\delta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} f((X + Z + N) \cdot \ad(\delta^{\perp})) \cdot \ad(\delta \eta x) \, dN \]

\[ \times \int_{n_{4,(X+Z)_{ss}}(A)} f((X + Z + N) \cdot \ad(\eta x)) \, dN \]

(3.3.8.7)

\[ = \sum_{P_{2} \in F} \sum_{P_{2} \subseteq P_{3} \subseteq P_{4}} \sum_{X \in \mathfrak{m}_{3}(Q) \cap F} \sum_{Z \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\eta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\delta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\delta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} \sum_{\delta \in N_{3, X_{ss}}(Q) \backslash N_{3}(Q)} f((X + N) \cdot \ad(\eta x)) \cdot \psi((N, Z)) \, dN \]

where (3.3.8.6) follows from Corollary 2.4 of [Ch02] which states that

(3.3.8.8)

\[ \sum_{Z \in N_{X_{ss}}(Q)} \sum_{\delta \in N_{X_{ss}}(Q) \backslash N(Q)} f((X + Z) \cdot \ad(\delta)) = \sum_{Y \in n(Q)} f(X + Y) \]

and (3.3.8.7) follows from the Poisson summation formula. The right hand side of (3.3.8.7) is independent of \( P_{4} \), hence the alternating sum in \( P_{4} \) cancels by the inclusion-exclusion principle, therefore the same estimates in the proof of (2.2.2.11) implies the integrability of \( j_{0}^{T}(f, f) \).

For the integral representation of \( j_{0}^{T}(f) \), consider the \((P_{2}, P_{3})\) summand of the integral of \( j_{0}^{T}(f, f) \) over \( G(Q) \backslash G(A)^{1} \):

(3.3.8.9)

\[ \int_{P_{2}(Q) \backslash G(A)^{1}} F^{P_{2}}(x, T) \sigma_{2}^{5}(H_{0}(x) - T) \sum_{P_{4} \in F} (-1)^{\dim(A_{4}/A_{G})} J_{P_{4}, \delta}(x, f) \, dx \]

\[ = \int_{M_{2}(Q) \backslash M_{2}(A) \backslash G(A)^{1}} F^{P_{2}}(x, T) \sigma_{2}^{5}(H_{0}(x) - T) \sum_{P_{4} \in F} (-1)^{\dim(A_{4}/A_{G})} \sum_{X \in \mathfrak{m}_{4}(Q) \cap \mathfrak{m}_{3}(Q)} \sum_{\eta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} f((X + N) \cdot \ad(\eta n_{2}x)) \cdot n_{2} \, dN \, dN \, dN \, dN \]

(3.3.8.10)

\[ = \int_{M_{2}(Q) \backslash M_{2}(A) \backslash G(A)^{1}} F^{P_{2}}(x, T) \sigma_{2}^{5}(H_{0}(x) - T) \sum_{P_{4} \in F} (-1)^{\dim(A_{4}/A_{G})} \sum_{X \in \mathfrak{m}_{4}(Q) \cap \mathfrak{m}_{3}(Q)} \sum_{\eta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} \sum_{\delta \in N_{4, X_{ss}}(Q) \backslash N_{4}(Q)} f((X + N) \cdot \ad(\eta n_{2}x)) \cdot n_{2} \, dN \, dN \, dN \, dN \]
\[ \int_{N_2(Q)\backslash N_2(A)} f((X + N) \cdot \text{ad}(n x)) \, dN \, d\nu_2 \] 

(3.3.8.11) \[ \int_{M_2(Q)\backslash M_2(A)} F^{P_2}(x, T) \sigma_2^5(H_0(x) - T) \sum_{P_2 \in F} (-1)^{\dim(A_4/A_G)} \sum_{X \in \mathfrak{m}_4(Q) \cap \sigma} (3.3.8.10) \]

\[ \sum_{P_2 \supset P_4 \subset P_5} (3.3.8.11) \]

\[ \times (\int_{N_2(Q)\backslash N_2(A)} \int_{\mathfrak{n}_4(A)} f((X + N) \cdot \text{ad}(n x)) \, dN \, d\nu_2) \, dx. \]

The equality (3.3.8.10) holds since \( N_4(Q) \backslash N_4(A) \) has volume 1, and the equality (3.3.8.11) follows from Corollary 2.5 of \([Ch02]\) which is the integral analogue of (3.3.8.8). Reversing the combinatorial manipulations to the right-hand side of (3.3.8.11) skipping the step (3.3.8.10).

\[ \int_{M_2(Q)\backslash M_2(A)} F^{P_2}(x, T) \sigma_2^5(H_0(x) - T) \sum_{P_2 \in F} (-1)^{\dim(A_4/A_G)} \sum_{X \in \mathfrak{m}_4(Q) \cap \sigma} (3.3.8.12) \]

\[ \sum_{P_2 \supset P_4 \subset P_5} (3.3.8.11) \]

\[ \times (\int_{N_2(Q)\backslash N_2(A)} \int_{\mathfrak{n}_4(A)} f((X + N) \cdot \text{ad}(n x)) \, dN \, d\nu_2) \, dx. \]

\[ = \int_{P_2(Q)\backslash G(A)} F^{P_2}(x, T) \sigma_2^5(H_0(x) - T) \sum_{P_2 \supset P_4 \subset P_5} (3.3.8.11) \]

which is the \((P_2, P_5)\) summand of the integral of \( k_0^T(, f) \) over \( G(Q)\backslash G(A)^{1} \), hence \( J_{\sigma}^T(f) \). 

By (3.3.8.3) it is enough to consider the function \( J_{P, \sigma}(, f) \) instead of \( K_{P, \sigma}(, f) \) for a parabolic subgroup \( P \) of \( G \) containing \( P_1 \) for the proof of (3.3.7.1). Every element \( X \) in \( \mathfrak{m}_P(Q) \cap \sigma \) is conjugate under the adjoint action of \( G(Q) \) to the sum of \( \Sigma \) and \( N_\Sigma \) for some \( N_\Sigma \) in \( \mathfrak{g}_{\Sigma, \text{nil}}(Q) \). More precisely

(3.3.8.13) \[ \exists P'_1 \in \mathcal{F}, \ P'_1 \text{ standard, } P'_1 \supset P \ \exists s \in W(a_{P_1}, a_{P_1}) \ \exists \mu \in M_0^P(Q) \]

\[ \exists N_\Sigma \in \mathfrak{m}_P(Q) \cdot \text{ad}(w_s) \cap \mathfrak{g}_{\Sigma, \text{nil}}(Q) \]

\[ X = (\Sigma + N_\Sigma) \cdot \text{ad}(w_s^{-1} \mu) \]

where

- the element \( w_s \) in \( G(Q) \) is a representative of \( s \);
- the double coset

(3.3.8.14) \[ [s] \in W_0^{M_P} \backslash W(a_{P_1}, a_{P_1}) / W_1^{G_{\Sigma}} \]

is uniquely determined;
- for a fixed choice of the element \( s \), the coset

(3.3.8.15) \[ [\mu] \in M_P(Q) \cap w_s G_{\Sigma}(Q) w_s^{-1} \backslash M_P(Q) \]

is uniquely determined;

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for a fixed choice of the representative \( w_s \) and the element \( \mu \), the element \( N_{\Sigma} \) is uniquely determined.

Let \( W(a_1; M_P^+, G_{\Sigma}^{0, +}) \) be the subset of the Weyl group of \( G \) defined by

\[
(3.3.8.16) W(a_1; M_P^+, G_{\Sigma}^{0, +}) = \bigcup_{\substack{P' \in \mathcal{F} \\
\text{standard} \ P' \subset P}} \left\{ s \in W(a_{P_1}, a_{P'_1}) : \forall \alpha \in \Delta_P^+, s^{-1}\alpha > 0, \forall \beta \in \Delta_{\Sigma} G_{\Sigma}^{0, +} s\beta > 0 \right\}.
\]

The map

\[
(3.3.8.17) \left( W(a_1; M_P^+, G_{\Sigma}^{0, +}) \right) \times \left( \mathcal{M}_P(Q) \cap w_s G_{\Sigma}^{0, +}(Q) w_s^{-1} \right) \mathcal{M}_P(Q) \times \left( \mathcal{m}_P(Q) \cdot \text{ad}(w_s) \cap \mathcal{g}_{\Sigma, \text{nil}}(Q) \right)
\]

\[
\rightarrow \; \mathcal{m}_P(Q) \cap \eta
\]

defined by

\[
(3.3.8.18) \; (s, \mu, N_{\Sigma}) \mapsto (\Sigma + N_{\Sigma}) \cdot \text{ad}(w_s^{-1} \mu)
\]

is surjective and the group \( \pi_0(G_{\Sigma}) \) operates simply transitively on each fiber. Hence

\[
(3.3.8.19) \; \forall x \in G(Q) \backslash G(A)^1
\]

\[
J_{P, \sigma}(x, f) = \sum_{s \in W(a_1; M_P^+, G_{\Sigma}^{0, +})} \sum_{\mu \in \mathcal{M}_P(Q) \cap w_s G_{\Sigma}^{0, +}(Q) w_s^{-1} \mathcal{M}_P(Q)}
\]

\[
\times \sum_{N_{\Sigma} \in \mathcal{m}_P(Q) \cdot \text{ad}(w_s) \cap \mathcal{g}_{\Sigma, \text{nil}}(Q)}
\]

\[
\times \left| \pi_0(G_{\Sigma}) \right|^{-1} \int_{\eta \in \mathcal{N}_{P, \Sigma, \text{ad}(w_s^{-1} \mu)}(Q) \backslash \mathcal{N}_P(Q)} \left( (\Sigma + N_{\Sigma}) \cdot \text{ad}(w_s^{-1} \mu) + N \right) \cdot \text{ad}(\eta x) \right) \, dN
\]

\[
(3.3.8.20) = \left| \pi_0(G_{\Sigma}) \right|^{-1} \sum_{s \in W(a_1; M_P^+, G_{\Sigma}^{0, +})} \sum_{N_{\Sigma} \in \mathcal{m}_P(Q) \cdot \text{ad}(w_s) \cap \mathcal{g}_{\Sigma, \text{nil}}(Q)}
\]

\[
\times \int_{\pi \in \mathcal{P}(Q) \cap w_s G_{\Sigma}^{0, +}(Q) w_s^{-1} \mathcal{P}(Q)} \left( \Sigma + N_{\Sigma} + N \right) \cdot \text{ad}(w_s^{-1} \pi x) \right) \, dN
\]

where (3.3.8.20) follows from the change of variables

\[
(3.3.8.21) \quad \mathcal{M}_P(Q) \cap w_s G_{\Sigma}^{0, +}(Q) w_s^{-1} \mathcal{M}_P(Q) \times \mathcal{N}_{P, \Sigma, \text{ad}(w_s^{-1} \mu)}(Q) \backslash \mathcal{N}_P(Q)
\]

\[
\rightarrow \; \mathcal{P}(Q) \cap w_s G_{\Sigma}^{0, +}(Q) w_s^{-1} \mathcal{P}(Q).
\]
Substitute (3.3.8.19) into the formula (3.3.8.5),

(3.3.8.22) \[ J^T_s(f) = |\pi_0(G_\Sigma)|^{-1} \int_{G(Q) \backslash G(A)} \left( \sum_{P \in \mathcal{F}} \sum_{\delta \in \mathcal{P}(Q) \backslash G(Q)} (-1)^{\dim(A_P / A_G)} \right) \times \left( \sum_{s \in W(a_1; M_P^+, G_\Sigma^0)} \sum_{N_\Sigma \in m_P \cdot \ad(w_s) \cap g_\Sigma} \int_{n_\Sigma Q} f((\Sigma + N_\Sigma + N) \cdot \ad(\xi)) \, dN \times \right. \\
\left. \times \hat{\tau}_P(H_p(w_\Sigma x) - T) \right) \, dx \]

(3.3.8.23) \[ = |\pi_0(G_\Sigma)|^{-1} \int_{G(Q) \backslash G(A)} \left( \sum_{Q \in \mathcal{F}^Q} \sum_{\xi \in Q \backslash G(Q)} (-1)^{\dim(A_P / A_G)} \right) \times \left( \sum_{N_\Sigma \in m_Q \cdot \ad(w_s) \cap g_\Sigma} \int_{n_\Sigma Q} f((\Sigma + N_\Sigma + N) \cdot \ad(\xi)) \, dN \times \right. \\
\left. \times \hat{\tau}_P(H_p(w_\Sigma x) - T) \right) \, dx \]

\[ = |\pi_0(G_\Sigma)|^{-1} \int_{G(Q) \backslash G(A)} \left( \sum_{Q \in \mathcal{F}^Q} \sum_{\xi \in Q \backslash G(Q)} (-1)^{\dim(A_P / A_G)} \hat{\tau}_P(H_p(w_\Sigma x) - T) \right) \, dx. \]

where in (3.3.8.23) \( \xi \) denotes the product \( w^{-1}_s \pi \delta \) and \( Q \) denotes the standard parabolic subgroup of \( G_\Sigma^0 \) defined by

(3.3.8.24) \[ Q = w^{-1}_s P w_s \cap G_\Sigma^0 \]

with Levi decomposition

(3.3.8.25) \[ m_Q = m_P \cdot \ad(w_s) \cap g_\Sigma, \quad n_Q = n_P \cdot \ad(w_s) \cap g_\Sigma. \]

The assignment

(3.3.8.26) \[ (P, s) \mapsto P' = w^{-1}_s P w_s \]

defines a bijection

(3.3.8.27) \[ \left\{ P \in \mathcal{F}, \ s \in W(a_1; M_P^+, G_\Sigma^0) : P \text{ standard, } w^{-1}_s P w_s \cap G_\Sigma^0 = Q \right\} \overset{\sim}{\longrightarrow} \mathcal{F}_Q(M_1), \]
hence

\begin{equation}
J^T_\delta(f) = |\pi_0(G_\Sigma)|^{-1} \int_{G(Q)\backslash G(A)^1} \left( \sum_{Q \in F^\Sigma} \sum_{\xi \in Q(Q)\backslash G(Q)} \right.
\times \sum_{N_\Sigma \in m_{Q,\text{nil}}(Q)} \int_{n_\delta(Q)} f((\Sigma + N_\Sigma + N) \cdot \text{ad}(\xi x)) \, dN \times
\times \sum_{P' \in F_Q(M_1)} (-1)^{\text{dim}(A_{P'}/A_{G})} \tau_{P'}(H_{P'}(\xi x) - s^{-1}(T - T_0) - T_0) \Bigg) \, dx
\end{equation}

where $s$ denotes the second component of the inverse image of $P'$ under the map defined in \ref{3.3.8.26}.

Let $T_\Sigma$ be a truncation parameter for the triple $(G_\Sigma(A)^0, M_1, \Sigma(A), K_\Sigma)$ in $\mathfrak{a}_1$ such that $T_\Sigma - T_\Sigma, 1$ is the projection of $T - T_0$. By \ref{3.3.3.1}

\begin{equation}
\sum_{P \in F_Q(M_1)} (-1)^{\text{dim}(A_{P}/A_{\Sigma})} \tau_P(H_P(\delta x y) - s^{-1}(T - T_0) - T_0)
= \sum_{P \in F_Q(M_1)} (-1)^{\text{dim}(A_{P}/A_{G})} \tau_P((H_Q(\delta x) - T_\Sigma) - Y_P^T(\delta x, y))
= \sum_{Q' \in F^\Sigma, Q' \supseteq Q} (-1)^{\text{dim}(A_{Q}/A_{Q'})} \tau_{Q'}^Q(H_Q(\delta x) - T_\Sigma) \times \Gamma_{Q'}(H_{Q'}(\delta x) - T_S, \gamma_{Q'}^T(\delta x, y))
\end{equation}

where the family $\gamma_{Q'}^T(\delta x, y)$ is defined by

\begin{equation}
\forall P \in F_Q(M_1) \quad Y_P^T(\delta x, y) = -H_P(ky) + s^{-1}(T - T_0) - T_\Sigma + T_0,
\end{equation}

where $k$ is the $K_\Sigma$ component of $\delta x$ under the Iwasawa decomposition with respect to $P_\Sigma$ and $s$ is the second component of the inverse image of $P$ under the map defined in \ref{3.3.8.26}. On the right hand side of \ref{3.3.8.28} make the change of variables

\begin{equation}
Q(Q) \backslash G(Q) \times G(Q) \backslash G(A)^1 \xrightarrow{\sim} Q(Q) \backslash G_\Sigma^0(Q) \times G_\Sigma^0(Q) \backslash G_\Sigma^0(A) \cap G(A)^1 \times G_\Sigma^0(A) \backslash G(A).
\end{equation}

Then

\begin{equation}
J^T_\delta(f) = |\pi_0(G_\Sigma)|^{-1} \int_{G(B)^1} \left( \sum_{Q \in F^\Sigma} \sum_{Q' \supseteq Q} \sum_{\delta \in Q(Q)\backslash G_\Sigma^0(Q)} \right.
\times (-1)^{\text{dim}(A_{Q}/A_{Q'})} \sum_{N_\Sigma \in m_{Q,\text{nil}}(Q)} \int_{n_\delta(Q)} f((\Sigma + N_\Sigma + N) \cdot \text{ad}(\delta x y)) \, dN \times
\times \tau_{Q'}^Q(H_Q(\delta x) - T_\Sigma) \Gamma_{Q'}(H_{Q'}(\delta x) - T_S, \gamma_{Q'}^T(\delta x, y)) \Bigg) \, dxdy
\end{equation}

\begin{equation}
= |\pi_0(G_\Sigma)|^{-1} \int_{G_\Sigma^0(B)^1, G(A)^1} \left( \sum_{Q' \in F^\Sigma} \int_{K_\Sigma} \int_{A_{Q'}(\mathbb{R}) \cap G(A)^1} \int_{M_\Sigma(Q)\backslash M_{Q'}(A)^1} \right.
\end{equation}
\[
\times \left( \sum_{Q \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \sum_{Q'' \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \right.
\]
\[
\times \int_{n_{Q}(A) \cap M_{Q'}(A)} (-1)^{\dim(A_{Q}/A_{Q'})} \Phi_{Q',a,k,y}^{T}(N_{\Sigma} + N) \cdot \text{ad}(\mu m)) \, dN \times
\]
\[
\times \int_{n_{Q}(A) \cap M_{Q'}(A)} \phi_{Q'}(H_{Q}(\mu m) - T_{\Sigma}) \, dmdak \, dy
\]
\[
= |\pi_{0}(G_{\Sigma})|^{-1} \int_{G_{0}^{g}(A) \cap G(A)} \sum_{Q' \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \sum_{Q'' \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \right.
\]
\[
\times \left( \int_{K_{\Sigma}} \int_{A_{Q}(\mathbb{R}) \cap G(A)^{1}} j_{\text{nil}(\Phi_{Q',y})}^{M_{Q'}(T_{\Sigma})} \, d\mu_{m} \right) \, dy.
\]
where on the right hand side of \((3.3.8.33)\) \(\Phi_{Q',a,k,y}^{T}(m)\) denotes the function on \(m_{Q'}(A)\) defined by
\[(3.3.8.34) \forall X \in m_{Q'}(A)
\[
\Phi_{Q',a,k,y}^{T}(X) = \left( \int_{m_{Q'}(A)} f((\Sigma + X + N) \cdot \text{ad}(k y)) \, dN \right) \Gamma_{Q'}^{T}(H_{Q'}(a) - T_{\Sigma}, \gamma_{Q'}^{T}(k, y)),
\]
and the equality \((3.3.8.33)\) follows from the changes of variables
\[(3.3.8.35) \quad \frac{Q(Q) \backslash G_{0}^{g}(Q) \times G_{0}^{g}(Q) \backslash G_{0}^{g}(A) \cap G(A)^{1}}{Q(Q) \cap M_{Q'}(Q) \backslash M_{Q'}(Q) \times Q'(Q) \backslash G_{0}^{g}(A) \cap G(A)^{1}} \sim
\]
and
\[(3.3.8.36) \quad n_{Q}(A) \sim n_{Q}(A) \cap m_{Q'}(A) \times n_{Q'}(A)
\]
and the Iwasawa decomposition for \(G_{0}^{g}(A) \cap G(A)^{1}\).

Evaluate \(T\) at \(T_{0}\) and \(T_{\Sigma}\) at \(T_{\Sigma,1}\), and apply \((3.3.4.1)\), then
\[(3.3.8.37) \quad J_{0}(f) = |\pi_{0}(G_{\Sigma})|^{-1} \int_{G_{0}^{g}(A) \cap G(A)} \sum_{Q'' \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \sum_{Q'' \in F^g} \sum_{\mu \in \mathcal{Q}(Q) \cap M_{Q'}(Q)} \sum_{N \Sigma \in n_{Q',nil}(Q)} \right.
\]
\[
\times \left( \int_{K_{\Sigma}} \int_{A_{Q}(\mathbb{R}) \cap G(A)^{1}} j_{\text{nil}(\Phi_{Q',y})}^{M_{Q'}(T_{\Sigma})} \, d\mu_{m} \right) \, dy.
\]
By the defining property \((2.2.12.1)\) of the points \(T_{0}\) and \(T_{\Sigma,1}\),
\[(3.3.8.38) \quad j_{\text{nil}(\Phi_{Q',y})}^{M_{Q'}(T_{0}-T_{\Sigma,1})} = j_{\text{nil}(\Phi_{Q'',y})}^{M_{Q''}(T_{0}-T_{\Sigma,1})}
\]
whenever \(Q'\) and \(Q''\) are parabolic subgroups in \(F^g\) conjugate under the Weyl group of \(G_{0}^{g}\), hence
\[(3.3.8.39) \quad J_{0}(f) = |\pi_{0}(G_{\Sigma})|^{-1} \int_{G_{0}^{g}(A) \cap G(A)} \left( \sum_{Q \in F^g} |W_{1}^{M_{Q}}||W_{1}^{G_{0}^{g}}|^{-1} j_{\text{nil}(\Phi_{Q,x})}^{M_{Q}(T_{0}-T_{\Sigma,1})} \right) \, dx.
\]
**Definition** Fix a finite set of places $S$. Let $M$ be a standard Levi subgroup of $G$, let $\Xi$ be a semisimple element of $m(\mathbb{Q})$, let $\nu$ be a nilpotent element of $m_{\Xi}(\mathbb{Q}_{S})$ defined up to the adjoint action of $M_{\Xi}(\mathbb{Q}_{S})$. The element

$$X = \Xi + \nu \in m(\mathbb{Q}_{S})$$

is well-defined modulo $M(\mathbb{Q}_{S})$. Denote by $D_{G}$ the discriminant function on $g$.

Let $T$ be a point in $a_{M}$, let $x$ be an element of $G(\mathbb{Q}_{S})$. Let $(v_{P}(x,T))$ be the $(G,M)$-family defined by

$$\forall \lambda \in i_{\mathbb{A}}^{*}M v_{P}(x,T)(\lambda) = v_{P}(x)(\lambda) \times e^{(\lambda,T)}.$$ 

Let $Q$ be a parabolic subgroup of $G_{\Xi}^{0}$ containing $M_{\Xi}^{0}$. Define the weight factor $v'_{Q}$ by

$$v'_{Q}(x,T) = \sum_{P \in \mathcal{F}_{Q}(M)} v_{P}(x,T)$$

where the set $\mathcal{F}_{Q}(M)$ is defined with respect to the semisimple element $\Xi$. Let $T_{\Xi,M}$ be the point in $a_{M_{\Xi}^{0}}$ modulo $a_{G_{\Xi}^{0}}$ defined in the same manner as $T_{0}$ in $a_{0}^{G}$ that satisfies an identity analogous to (2.2.12.1).

**Lemma** Let $L$ be a Levi subgroup in $L(M)$, let $T$ be a point in $a_{M}$, then

$$\forall x \in G(\mathbb{Q}_{S}) \forall y \in G_{\Xi}^{0}(\mathbb{Q}_{S}) \quad v_{L}(yx) = \sum_{Q \in \mathcal{F}_{\Xi}(M_{\Xi}^{0})} v_{L}^{Q}(y)v'_{Q}(kx,T)$$

where $v_{L}(x)$ is the weight factor defined in (3.1.5.2) and $k$ is the $K_{\Xi}$-component of $y$ under the Iwasawa decomposition of $G_{\Xi}^{0}(\mathbb{Q}_{S})$ with respect to the parabolic subgroup $Q_{\Xi}^{0}$.

**Proof** This is Corollary 8.4 of [Ar88a].

**Lemma** (Local semisimple descent)

Let $f_{S}$ be a Schwartz function on $g(\mathbb{Q}_{S})$, let $X$ and $\nu$ be as in (3.3.9), then

$$J_{M}^{G}(X,f_{S}) = |D^{G}(\Xi)|^{1/2}_{S} \int_{G_{\Xi}^{0}(\mathbb{Q}_{S}) \backslash G(\mathbb{Q}_{S})} \left( \sum_{Q \in \mathcal{F}_{\Xi}(M_{\Xi}^{0})} J_{M_{\Xi}^{0}}^{Q}(\nu,\Phi_{S,Q,x}^{T_{0}-T_{\Xi,M}}) \right) dx$$

where $\Phi_{S,Q,x}^{T}$ is the function on $m_{Q}(\mathbb{Q}_{S})$ defined by

$$\Phi_{S,Q,x}^{T}(Y) = \int_{K_{\Xi}} \int_{m_{Q}(\mathbb{Q}_{S})} f_{S} \left( (\Xi + (Y + N) \cdot \text{ad}(k)) \cdot \text{ad}(x) \right) v'_{Q}(kx,T) dNdk$$

where the weight factor $v'_{Q}(x,T)$ is defined as in (3.3.9.3).
(3.3.13) **Proof** This argument follows Corollary 8.7 of [Ar88a], see also the discussion in §2.6 of [HW13].

Since \( T_0 - T_{\Xi,M} \) lies in \( \mathfrak{a}_M \), the identity (3.3.10.1) is valid for the weight factor \( v'_Q(x, T_0 - T_{\Xi,M}) \), hence

\[
(3.3.13.1) \quad J^G_M(X, f_S) = \lim_{A \to 0} \left( \left| D^G(X + A) \right|_{S}^{1/2} \int_{G^0_{\Xi}(Q_S) \setminus G(Q_S)} \int_{G^0_{X+\mathfrak{a}}(Q_S) \setminus G^0_{X+\mathfrak{a}}(Q_S)} \times f_S((X + A) \cdot \text{ad}(y x)) \times \\
\times \left( \sum_{L \in \mathcal{L}(M)} r^L_M \left( \exp(X_{nil}), \exp(A) v_L(y x) \right) dy dx \right) \right)
\]

\[
= \lim_{A \to 0} \left( \left| D^G(X + A) \right|_{S}^{1/2} \int_{G^0_\Xi(Q_S) \setminus G(Q_S)} \int_{M^0_X(Q_S) \setminus M^0_\Xi(Q_S)} \times f_S((X + A) \cdot \text{ad}(y x)) \times \\
\times \left( \sum_{Q \in \mathcal{F}^\Xi(M^0_\Xi)} \sum_{L \in \mathcal{L}^M(Q^0_\Xi)} r^L_M \left( \exp(X_{nil}), \exp(A) \right) \right) \times \\
v^Q_L(y) v'_Q(k x, T_0 - T_{\Xi,M}) \right) dy dx
\]

(3.3.13.2) \[
= \lim_{A \to 0} \left( \left| D^G(X + A) \right|_{S}^{1/2} \left| D^{G_\Xi}(X + A) \right|_{S}^{1/2} \int_{G^0_\Xi(Q_S) \setminus G(Q_S)} \left( \sum_{Q \in \mathcal{F}^\Xi(M^0_\Xi)} \sum_{L \in \mathcal{L}^M(Q^0_\Xi)} r^L_M \left( \exp(X_{nil}), \exp(A) \right) \right) \times \\
J^M_Q(X_{nil} + A, \Phi^{T_0 - T_{\Xi,M}}_{S,Q,y}) \right) dy \\
= \left| D^G(\Xi) \right|_{S}^{1/2} \int_{G^0_\Xi(Q_S) \setminus G(Q_S)} \left( \sum_{Q \in \mathcal{F}^\Xi(M^0_\Xi)} \sum_{Q \in \mathcal{F}^\Xi(M^0_\Xi)} J^M_Q(X_{nil}, \Phi^{T_0 - T_{\Xi,M}}_{S,Q,y}) \right) dy
\]

where the equality (3.3.13.2) follows from the change of variables

(3.3.13.3) \[
M^0_X(Q_S) \setminus G^0_\Xi(Q_S) \sim M_X(Q_S) \setminus M^0_\Xi(Q_S) \times N_Q(Q_S) \times K_\Xi
\]

for each parabolic subgroup \( Q \) in \( \mathcal{F}^\Xi(M^0_\Xi) \).

□

(3.3.14) **Definition** Let \( \equiv \) be the equivalence relation on \( \mathfrak{m}(Q) \cap \mathfrak{o} \) defined by

\[
(3.3.14.1) \quad \forall X \in \mathfrak{m}(Q) \cap \mathfrak{o} \quad \forall Y \in \mathfrak{m}(Q) \cap \mathfrak{o} \quad X \equiv Y \quad \text{if} \quad \exists \delta \in \mathfrak{m}(Q) \exists \eta \in M^0_{X_{ss}}(Q_S) \quad X_{ss} = Y_{ss} \cdot \text{ad}(\delta) \quad X_{nil} = Y_{nil} \cdot \text{ad}(\delta \eta).
\]

Let \( (\mathfrak{m}(Q) \cap \mathfrak{o})_{M,S} \) denote the collection of \( \equiv \) equivalence classes in \( \mathfrak{m}(Q) \cap \mathfrak{o} \). By abuse of notation denote the \( \equiv \) equivalence class of an element \( X \) in \( \mathfrak{m}(Q) \cap \mathfrak{o} \) by the same symbol \( X \).
(3.3.15) Proposition Let $\sigma$ be a $\sim$ equivalence class on $g(\mathbb{Q})$. Let $S_\sigma$ be a sufficiently large finite set of places of $\mathbb{Q}$. Then for each standard Levi subgroup $M$ of $G$ and for each $\equiv$ equivalence class $X$ in $m(\mathbb{Q}) \cap \sigma$ there exists a constant $a^M(S_\sigma, X)$ such that

$$J^G_\sigma(f) = \sum_{M \in \mathcal{L}} \sum_{X \in (m(\mathbb{Q}) \cap \sigma)_{M,S_\sigma}} a^M(S_\sigma, X) J^G_M(X, f_{S_\sigma}).$$

(3.3.16) Proof Retain the notations of (3.3.7.1) and (3.3.12.1). Let $\Xi$ be equal to $\Sigma$, hence

$$X = \Sigma + \nu \in m(\mathbb{Q}_S)/ad(M(\mathbb{Q}_S)),$$

and

$$T_0 - T_{\Xi,M} = T_0 - T_{\Sigma,1}.$$

Without loss of generality assume that the finite set $S_\sigma$ is large enough such that

$$\forall p \notin S_\sigma \quad \bullet \Sigma \in g(\mathbb{Z}_p);
\bullet \|D(\Sigma)_p = 1;
\bullet g(\mathbb{Z}_p) \cdot ad(K_p) = g(\mathbb{Z}_p);
\bullet K_p \cap G^0(\mathbb{Q}_p) = K_{\Sigma,p} is hyperspecial in $G^0(\mathbb{Q}_p)$;
\bullet \forall x_p \in G(\mathbb{Q}_p) (\Sigma + g(\mathbb{m}_nil(\mathbb{Q}_p))) \cdot ad(x_p) \cap g(\mathbb{Z}_p) \neq \emptyset
\Rightarrow x_p \in G^0(\mathbb{Q}_p)K_p.$$

The last condition in (3.3.16.3) is satisfied for large enough $S_\sigma$ by Lemma 6.1 of [Ar86]. Since

$$\left( H \to \Gamma_Q(H - T_{\Sigma,1}, Y^T_Q(k, x)) \right)^\wedge = v_Q(kx, T_0 - T_{\Sigma,1}(\lambda))$$

the conditions (3.3.16.3) imply that

$$\Phi_{Q,x}^{T_0 - T_{\Sigma,1}} = \Phi_{S_\sigma,Q,x_{S_\sigma}}^{T_0 - T_{\Sigma,1}} \otimes \bigotimes_{p \notin S_\sigma} \mathbb{I}_{G^0(\mathbb{Q}_p)K_p}(x) \cdot \mathbb{I}_{m_Q(\mathbb{Z}_p)}.$$

By (3.3.7.1) and (3.2.11.1),

$$J^G_\sigma(f) = |\pi_0(G_\Sigma)|^{-1} \int_{G^0(\mathbb{Q}_S) \setminus G(\mathbb{Q}_S)} \left( \sum_{Q \in F_{\Sigma}} \sum_{L \in \mathcal{L}^{Q}(M_{1,\Sigma})} \left| W^G_{L} \right| \left| W^G_{Q} \right|^{-1} \times \sum_{\nu \in (h_{nil}(\mathbb{Q}))_{L,S_\sigma}} a^L(S_\sigma, \nu) J^{M_Q}(\nu, \phi^{T_0 - T_{\Sigma,1}}_{Q,\sigma,\nu}) \right) dx$$

$$= |\pi_0(G_\Sigma)|^{-1} \sum_{M \in \mathcal{L}_{Q}(M_{1})} \left| W^G_{M} \right| \left| W^G_{1} \right|^{-1} \times \sum_{\nu \in (m_{\Sigma,nil}(\mathbb{Q}))_{M_{\Sigma},S_\sigma}} a^{M_{\Sigma}}(S_\sigma, \nu) J^{M_{\Sigma}}(\Sigma + \nu, f_{S_\sigma}).$$
where on the right hand side of (3.3.16.7) the symbol \( \hat{\mathcal{L}}_\Sigma(M_1) \) denotes the set
\[
\hat{\mathcal{L}}_\Sigma(M_1) = \left\{ M \in \mathcal{L}(M_1) : A_M = A_{M_0} \right\}.
\]

The equality (3.3.16.7) follows from (3.3.12.1) and the bijection
\[
\pi_\Sigma : \hat{\mathcal{L}}_\Sigma(M_1) \sim \rightarrow \mathcal{L}^{G_\Sigma}_G(M^0_{1,\Sigma})
\]
defined by
\[
\forall M \in \hat{\mathcal{L}}_\Sigma(M_1) \quad \pi_\Sigma(M) = M^0_\Sigma = L
\]
and the equation
\[
|D(\Sigma)|_{S_0} = 1
\]
which follows from (3.3.16.3).

Define the constant \( a^M(S_0, X) \) by
\[
a^M(S_0, X) = |\pi_0(M_\Sigma)|^{-1} \sum_{\nu \in (m_\Sigma, nii)_{M^0_{1,\Sigma}, S_0}} a^{M^0_{1,\Sigma}}(S_0, \nu) \times \sum_{X \in (m(Q) \cap o)_{M, S_0}} a^M(S_0, X)J^G_M(X, f_{S_0}).
\]

The set
\[
\left\{ (M, \sigma) : M \in \mathcal{L}, \sigma \text{ is a semisimple } M(Q)\text{-orbit in } m(Q), \Sigma \text{ is conjugate to a point in } \sigma \text{ under } G(Q). \right\}
\]

admits a natural action by \( W^G_0 \) which preserves \( a^M(S_0, X) \) and \( J^G_M(X, f_{S_0}) \), where each pair \( (M, \sigma) \) has stabilizer \( W^M_0 \). Denote by \( W^{G_\Sigma}_1 \) the quotient of the normalizer of \( A_1 \) in \( G_\Sigma \) by \( M^0_{1,\Sigma} \). There is an exact sequence
\[
1 \rightarrow W^{G_\Sigma}_1 \rightarrow W^G_1 \rightarrow \pi_0(G_\Sigma) \rightarrow 1.
\]

Similar notations apply to \( M_\Sigma \). The subset of (3.3.16.14)
\[
\left\{ (M, \sigma) : M \in \mathcal{L}(M_1) \text{ and } \sigma = \Sigma \cdot \text{ad}(M(Q)) \right\}
\]
admits a natural action of \( W_{1G}^G \), where each pair \((M, \sigma)\) has stabilizer \( W_{1M}^G \). The quotients
\[
\text{(3.3.16.17)} \quad \left\{ (M, \sigma) : M \in \mathcal{L}, \sigma \text{ is a semisimple } M(\mathbb{Q})\text{-orbit in } m(\mathbb{Q}), \Sigma \text{ is conjugate to a point in } \sigma \text{ under } G(\mathbb{Q}) \right\} / W_0^G
\]
and
\[
\text{(3.3.16.18)} \quad \left\{ (M, \sigma) : M \in \mathcal{L}(M_1) \text{ and } \sigma = \Sigma \cdot \text{ad}(M(\mathbb{Q})) \right\} / W_{1G}^G
\]
are in bijection, therefore
\[
\text{(3.3.16.19)} \quad J_0^G(f) = \sum_{M \in \mathcal{L}(M_1)} |W_{M\Sigma}^G||W_{1G}^G|^{-1} \sum_{X \in (m(\mathbb{Q})\cap \sigma)_{M,S_0}} a^M(S_0, X)J_M^G(X, f_{S_0})
\]
\[
= \sum_{M \in \mathcal{L}} \frac{|W_0^M|}{|W_G^0|} \left( \frac{|W_{M\Sigma}^G|}{|W_{1G}^G|} \right)^{-1} |W_{1M}^G||W_{1G}^G|^{-1} \times \sum_{X \in (m(\mathbb{Q})\cap \sigma)_{M,S_0}} a^M(S_0, X)J_M^G(X, f_{S_0})
\]
\[
= \sum_{M \in \mathcal{L}} |W_0^M||W_G^0|^{-1} \sum_{X \in (m(\mathbb{Q})\cap \sigma)_{M,S_0}} a^M(S_0, X)J_M^G(X, f_{S_0}).
\]

**Remark** For a semisimple element \( X \) in \( \sigma \),
\[
\text{(3.3.17.1)} \quad a^M(S_0, X) = |\pi_0(MX)|^{-1} \text{Vol}(M^0_X(\mathbb{Q}) \setminus M^0_X(\mathbb{A}))
\]
if \( X \) is \( \mathbb{Q} \)-elliptic in \( m(\mathbb{Q}) \), and vanishes otherwise. If \( X \) is in addition assumed to be regular, the identity \((3.3.15.1)\) reduces to \((2.2.16.1)\).

### 3.4 The refined trace formula

**Proposition** (Refined trace formula)

For each sufficiently large finite set \( S \) of places of \( \mathbb{Q} \), for each \(~\) equivalence class \( \sigma \) in \( g(\mathbb{Q}) \), for each \( \equiv \) equivalence class \( X \) in \( (m(\mathbb{Q}) \cap \sigma)_{M,S} \), there exists a constant \( a^M(S, X) \) such that
\[
\text{(3.4.1.1)} \quad \forall f \in S(g(\mathbb{A})) \quad \lim_{S} \sum_{\sigma \in g(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_G^0|^{-1} \sum_{X \in (m(\mathbb{Q})\cap \sigma)_{M,S}} a^M(S, X)J_M^G(X, f_{S})
\]
\[
= \lim_{S} \sum_{\sigma \in g(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_G^0|^{-1} \sum_{X \in (m(\mathbb{Q})\cap \sigma)_{M,S}} a^M(S, X)J_M^G(X, f_{S}).
\]

**Proof** By \((2.2.10.1)\) it is enough to show that the left hand side of \((3.4.1.1)\) is equal to
\[
\text{(3.4.2.1)} \quad J(f) = \sum_{\sigma \in g(\mathbb{Q})/\sim} J_\sigma(f).
\]
For each individual class \( \sigma \) by \((3.3.15.1)\)
\[
\text{(3.4.2.2)} \quad J_\sigma(f) = \lim_{S} \sum_{M} |W_0^M||W_G^0|^{-1} \sum_{X} a^M(S, X)J_M^G(X, f_{S})
\]
where the limit stabilizes as \( S \) grows large enough.
Lemma. Let $\Gamma$ be a compact subset of $\mathfrak{g}(A)$. Then there are only finitely many classes $\mathfrak{o}$ such that $\mathfrak{o} \cdot \text{ad}(G(A))$ intersects $\Gamma$.

Proof. The lemma follows from Corollary A.2 of [Ar86] which states that there exists a compact set $G(\Gamma)$ contained in $G(A)^1$ such that

$$\forall x \in G(A)^1 - G(Q)G(\Gamma) \quad \mathfrak{g}(Q)' \cdot \text{ad}(x) \cap \Gamma = \emptyset$$

where $\mathfrak{g}(Q)'$ denotes the set of points of $\mathfrak{g}(Q)$ not contained in any proper parabolic subalgebra. This is established by a reduction theory argument.

To prove the lemma first consider a class $\mathfrak{o}$ contained in $\mathfrak{g}(Q)'$. There exists a point $X$ in $\mathfrak{o}$ such that $X \cdot \text{ad}(G(\Gamma))$ is contained in $\Gamma$, hence there are only finitely many such $X$, hence finitely many such $\mathfrak{o}$.

Next for an arbitrary class $\mathfrak{o}$ there is some Levi subalgebra $\mathfrak{m}$ such that $\mathfrak{o}$ intersects $\mathfrak{m}(Q)'$. Then by the Iwasawa decomposition

$$\mathfrak{o} \cdot \text{ad}(G(A)^1) = \mathfrak{o} \cdot \text{ad}(M(A)^1) \cdot \text{ad}(N(Q) \setminus N(A)) \cdot \text{ad}(K)$$

the lemma follows by induction.

The Schwartz function $f$ is compactly supported if its component at infinity $f_\infty$ is compactly supported. For such an $f$ the limit on the left hand side of (3.4.1.1) stabilizes for every $S$ such that

$$S \supset \bigcup_{\mathfrak{o} \in \mathfrak{g}(Q)/\sim} S_\mathfrak{o}$$

where the union is supported on a finite set of $\mathfrak{o}$. Hence for a compactly supported $f$ the left hand side of (3.4.1.1) is equal to $J(f)$.

However the function spaces $C^\infty_c(\mathfrak{g}(\mathbb{R}))$ and hence $C^\infty_c(\mathfrak{g}(Q_S))$ are dense in $\mathcal{S}(\mathfrak{g}(\mathbb{R}))$ and $\mathcal{S}(\mathfrak{g}(Q_S))$ respectively, therefore $C^\infty_c(\mathfrak{g}(A))$ is dense in $\mathcal{S}(\mathfrak{g}(A))$ which by (1.1.4.2) is defined as

$$\mathcal{S}(\mathfrak{g}(A)) = \lim_{S \to \infty} \mathcal{S}(\mathfrak{g}(Q_S))$$

equipped with the final topology.

Since $J(f)$ extends continuously to all Schwartz functions, the limit on the left hand side of (3.4.1.1) exists, and the interchangeability of the operations of taking the limit as $S$ approaches infinity and taking the sum over all classes $\mathfrak{o}$ extends from $C^\infty_c(\mathfrak{g}(A))$ to $\mathcal{S}(\mathfrak{g}(A))$. This establishes the identity (3.4.1.1) for all Schwartz functions $f$ on $\mathfrak{g}(A)$.

4 The trace formula in invariant form

In this chapter the refined trace formula (3.4.1.1) is transformed into an identity between invariant distributions on $\mathfrak{g}(A)$.

4.1 Invariant weighted orbital integrals

(4.1.1) Definition. Let $M$ be a standard Levi subgroup of $G$, let $X$ be an element of $\mathfrak{m}(Q_S)$. Define the invariant weighted orbital integral $I^G_M(X, \ )$ to be the distribution on $\mathfrak{g}(Q_S)$ such that

$$\forall f_S \in \mathcal{S}(\mathfrak{g}(Q_S))$$
\[ I_M^G(X, fs) = J_M^G(X, fs) - \sum_{L \in \mathcal{L}(M)} \sum_{L' \in \mathcal{L}_S} |W_{S,0}^{L_S}||W_{S,0}^{L_S}|^{-1} \times \]
\[ \times \sum_{T_S \in \mathcal{T}(L'_S)} |W(L'_S, T_S)|^{-1} \times \]
\[ \times \int_{t_S(Q_S)} J^G_L(Y, fs) |I^L_M(X, )^*(Y)| |D^L(Y)|_S^{1/2} dY \]

where \( I_M^G(X, ) \) denotes the standard orbital integral on \( m(Q_S) \) defined by (1.1.6.9).

(4.1.2) Remark  The invariant weighted orbital integral \( I_M^G(X, ) \) is well-defined.

- The distribution \( I_M^G(X, ) \) does not depend on the choice of the Fourier transforms on \( g(Q_S) \) and its Levi subalgebras \( l(Q_S) \) as long as these are compatible. See [Wa95] Lemme VI.5 for the \( p \)-adic case, the same argument also works for the real and \( S \)-local cases.

- The integral in (4.1.1) converges since the distribution \( I_M^L(X, )^* \) is represented by a smooth function supported on \( g_{\text{reg}, ss}(Q_S) \) such that the function

\[ Y \mapsto \left( I^L_M(X, )^*(Y) \right) |D^L(Y)|_S^{1/2} \]

is locally

\[ O \left( \max \{1, - \log(|D^G(Y)|_S^N) \} \right) \]

on \( t(Q_S) \) for some natural number \( N \) and tempered at infinity, and the function

\[ Y \mapsto J^G_L(Y, f^* s) \]

is locally

\[ O \left( \max \{1, - \log(|D^G(Y)|_S^M) \} \right) \]

on \( t(Q_S) \) for some natural number \( M \) and rapidly decreasing at infinity. For the \( p \)-adic case see Lemma VI.3(iv) and Corollaire III.6 of [Wa95]. For the real case see Proposition 9 on page 108 of [Va77] and Corollary 7.4 of [Ar76].

(4.1.3) Proposition  Let \( M \) be a standard Levi subgroup of \( G \), let \( X \) be an element of \( m(Q) \). Then \( I_M^G(X, ) \) is an invariant distribution on \( g(Q_S) \), i.e.

\[ \forall x \in G(Q_S) \forall fs \in \mathcal{S}(g(Q_S)) \quad I^G_M(X, f_S \circ \text{ad}(x)) = I^G_M(X, f_S). \]

(4.1.4) Proof  For the \( p \)-adic case see Proposition VI.1 of [Wa95], the same argument also works for the \( S \)-local case.  \( \square \)
(4.1.5) **Remark** To quote Waldspurger from §Introduction of [Wa95]:

"On dispose de deux ensembles de distributions invariantes sur $G$:

1. les intégrales orbitales associées aux éléments semi-simples de $G$;

2. les caractères de représentations tempérées irréductibles de $G$.

(......)

Remplaçons $G$ par son algèbre de Lie $\mathfrak{g}$ et considérons l’espace de distributions invariantes par l’action adjointe de $G$. L’ensemble (1) a un analogue évident: les intégrales orbitales associées aux éléments semi-simples de $\mathfrak{g}$. Le seul but de cet article est de fournir un support un peu consistant à l’idée, d’ailleurs banale, que l’analogue de (2) est l’ensemble des transformées de Fourier des intégrales orbitales précédents (invariantes)."

The duality between the orbital integrals and the Fourier transforms of the invariant weighted orbital integrals is embodied in the global and local invariant trace formulae on the Lie algebra $\mathfrak{g}$.

### 4.2 The global and local invariant trace formulae

**Proposition** (Global invariant trace formula)

*For each sufficiently large finite set $S$ of places of $\mathbb{Q}$, for each $\sim$ equivalence class $\mathfrak{o}$ in $\mathfrak{g}(\mathbb{Q})$, for each $\equiv$ equivalence class $X$ in $(\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{M,S}$, there exists a constant $a^M(S,X)$ such that*

\[
\forall f \in \mathcal{S}(\mathfrak{g}(\mathbb{A})) \quad \lim_{S} \sum_{\mathfrak{o} \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{X \in (\mathfrak{g}(\mathbb{Q}) \cap \mathfrak{o})_{G,S}} a^G(S,X)I^G_M(X,f_S) = \lim_{S} \sum_{\mathfrak{o} \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{X \in (\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{M,S}} a^M(S,X)I^G_M(X,f^S).
\]

**Proof** Argue by induction. Assume that for each proper Levi subgroup $L$ of $G$, the identity (4.2.1.1) is valid when $G$ is replaced by $L$.

By (4.1.1.1)

\[
\lim_{S} \sum_{\mathfrak{o} \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{X \in (\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{M,S}} a^M(S,X)I^G_M(X,f^S) = \lim_{S} \sum_{\mathfrak{o} \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{X \in (\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o})_{M,S}} a^M(S,X) \left( J^G_M(X,f^S) - \sum_{L \in \mathcal{L}(M) \setminus G} \right.
\]

\[
\times \left( \sum_{L_S \in \mathcal{L}^S} |W_{S,0}^{L_S}| |W_{S,0}^{L_S}|^{-1} \sum_{T_S \in \mathcal{T}_{\mathcal{LL}}(L_S)} |W(L_S,T_S)|^{-1} \times \right.
\]

\[
\times \int_{t_S(\mathbb{Q})} J^G_L(Y,f_S) \left( I^H_M(X, \gamma(Y)) |D^L(Y)|_{S}^{1/2} dY \right)
\]

(4.2.2)
By the inductive hypothesis (4.2.1.1) applied to the proper Levi subgroup $L$ of $G$

$$\lim_{S} \sum_{L \in \mathcal{L}} \left( \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_{0}^{M}||W_{0}^{G}|^{-1} \sum_{X \in (m(Q)\cap \sigma)_{M,S}} a^{M}(S, X) \times \right.$$

$$\times \sum_{L'_{S} \in \mathcal{L}_{S}} |W_{S,0}^{L_{S}}||W_{S,0}^{G}|^{-1} \sum_{T_{S} \in T_{\mathcal{L}}(L'_{S})} |W(L'_{S}, T_{S})|^{-1} \times$$

$$\times \int_{t_{s}(Q_{S})} J_{L}^{G}(Y, f_{S}) \left( I_{M}^{L}(X, )^{\ast}(Y) \right) |D^{L}(Y)|_{S}^{1/2} dY \right).$$

By (3.4.1.1) the minuend on the right hand side of (4.2.2.2) is equal to

(4.2.2.3) \[ \lim_{S} \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_{0}^{M}||W_{0}^{G}|^{-1} \sum_{X \in (m(Q)\cap \sigma)_{M,S}} a^{M}(S, X) J_{M}^{G}(X, f_{S}). \]

The subtrahend on the right hand side of (4.2.2.2) is equal to

(4.2.2.4) \[ \sum_{L \in \mathcal{L}} |W_{0}^{L}||W_{0}^{G}|^{-1} \times \]

$$\times \left( \lim_{S} \sum_{L'_{S} \in \mathcal{L}_{S}} |W_{S,0}^{L_{S}}||W_{S,0}^{G}|^{-1} \sum_{T_{S} \in T_{\mathcal{L}}(L'_{S})} |W(L'_{S}, T_{S})|^{-1} \int_{t_{s}(Q_{S})} J_{L}^{G}(Y, f_{S}) \times \right. \]

$$\left. \times \left( \lim_{S} \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_{0}^{M}||W_{0}^{L}|^{-1} \sum_{X \in (m(Q)\cap \sigma)_{M,S}} a^{M}(S, X) \left( I_{M}^{L}(X, )^{\ast}(Y) \right) \right) \times \right.$$

$$\times \left| D^{L}(Y)|_{S}^{1/2} dY \right).$$

By the inductive hypothesis (4.2.1.1) applied to the proper Levi subgroup $L$ of $G$

(4.2.2.5) \[ \lim_{S} \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_{0}^{M}||W_{0}^{L}|^{-1} \sum_{X \in (m(Q)\cap \sigma)_{M,S}} a^{M}(S, X) \left( I_{M}^{L}(X, )^{\ast}(Y) \right) \]

$$= \lim_{S} \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{X \in (m(Q)\cap \sigma)_{L,S}} a^{L}(S, X) \left( I_{L}^{L}(X, )^{\ast}(Y) \right)$$

as an equality between tempered distributions on $I(A)$ in the variable $Y$, the expression (4.2.2.4) is equal to

(4.2.2.6) \[ \sum_{L \in \mathcal{L}} \left( \sum_{L'_{S} \in \mathcal{L}_{S}} |W_{S,0}^{L_{S}}||W_{S,0}^{G}|^{-1} \sum_{T_{S} \in T_{\mathcal{L}}(L'_{S})} |W(L'_{S}, T_{S})|^{-1} \int_{t_{s}(Q_{S})} J_{L}^{G}(Y, f_{S}) \times \right. \]

$$\times \left( \lim_{S} \sum_{\sigma \in \mathcal{I}(Q)/\sim} \sum_{X \in (m(Q)\cap \sigma)_{L,S}} a^{L}(S, X) \left( I_{L}^{L}(X, )^{\ast}(Y) \right) \right) |D^{L}(Y)|_{S}^{1/2} dY \right).$$
by the Weyl integration formula (1.1.6.8).
Therefore the right hand side of (4.2.2.2) is equal to

\[
\lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} \left| W_{0,0}^M \right| \left| W_{0}^G \right|^{-1} \sum_{X \in (\mathfrak{m}(Q) \cap \mathfrak{o})_M, S} a^M(S, X) J_{L}^{G}(X, f_{S})
\]

which is the left hand side of (4.2.1.1).

(4.2.3) Proposition  (Local invariant trace formula of Waldspurger)
Let \( v \) be a place of \( Q \), let \( f_{v} \) and \( g_{v} \) be Schwartz functions on \( \mathfrak{g}_{\nu}(Q_{\nu}) \), then

\[
\sum_{M_{\nu} \in \mathcal{L}^{G_{\nu}}} \left| W_{0,v,0}^M \right| \left| W_{0,v}^G \right|^{-1} \sum_{T_{\nu} \in T_{\mathfrak{d}}(M_{\nu})} \left| W(M_{\nu}, T_{\nu}) \right|^{-1} \times 
\int_{t_{v}(Q_{v})} (-1)^{\dim(A_{M_{\nu}}/A_{G_{\nu}})} J_{M_{\nu}}^{G_{\nu}}(X_{v}, f_{v}^*) I_{G_{\nu}}^{G_{v}}(X_{v}, g_{v}) \ dX_{v} 
\]

\[
= \sum_{M_{\nu} \in \mathcal{L}^{G_{\nu}}} \left| W_{0,v,0}^M \right| \left| W_{0,v}^G \right|^{-1} \sum_{T_{\nu} \in T_{\mathfrak{d}}(M_{\nu})} \left| W(M_{\nu}, T_{\nu}) \right|^{-1} \times 
\int_{t_{v}(Q_{v})} (-1)^{\dim(A_{M_{\nu}}/A_{G_{\nu}})} J_{M_{\nu}}^{G_{\nu}}(X_{v}, g_{v}^*) I_{G_{\nu}}^{G_{v}}(X_{v}, f_{v}) \ dX_{v}. 
\]

(4.2.4) Proof  For the \( p \)-adic case see Théorème VII.1 of \([Wa93]\). The argument only uses combinatorial properties of \((G_{\nu}, M_{\nu})\)-families and standard results from harmonic analysis on a \( Q_{\nu}\)-vector space, hence works equally well in the real case.

(4.2.5) Corollary  (Global invariant trace formula II)
For each Levi subgroup \( M \) in \( \mathcal{L} \) choose a parabolic subgroup \( ^{M}P \) in \( \mathcal{P}(M) \). For each sufficiently large finite set \( S \) of places of \( Q \), for each \( \sim \) equivalence class \( \mathfrak{o} \) in \( \mathfrak{g}(Q) \), for each \( \equiv \) equivalence class \( X \) in \( (\mathfrak{m}(Q) \cap \mathfrak{o})_M, S \), there exists a constant \( a^M(S, X) \) such that

\[
\lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} \left| W_{0}^M \right| \left| W_{0}^G \right|^{-1} (-1)^{\dim(A_{M}/A_{G})} \sum_{X \in (\mathfrak{m}(Q) \cap \mathfrak{o})_M, S} a^M(S, X) J_{M}^{M}(X, f_{S, MP}) 
\]

\[
= \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} \left| W_{0}^M \right| \left| W_{0}^G \right|^{-1} (-1)^{\dim(A_{M}/A_{G})} \sum_{X \in (\mathfrak{m}(Q) \cap \mathfrak{o})_M, S} a^M(S, X) J_{M}^{G}(X, f^* S) 
\]

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where \( f_{S,MP} \) is the Schwartz function on \( \mathfrak{m}(Q_S) \) is defined by

\[
(4.2.6.2) \quad \forall X \in \mathfrak{m}(Q_S) \quad f_{S,MP}(X) = \int_{K_S} \int_{\mathfrak{a}(Q_S)} f_S((X + N) \cdot \text{ad}(k)) \, dN dk
\]

where \( \mathfrak{n} \) is the Lie algebra of the unipotent radical of \( M^P \), and \( I_M^G(X, \cdot) \) is the invariant distribution on \( \mathfrak{g}(Q_S) \) defined by

\[
(4.2.6.3) \quad \forall f_S \in S(\mathfrak{g}(Q_S)) \quad I_M^G(X, f_S) = \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(A_M/A_L)} I_M^L(X, f_{S,LP}).
\]

(4.2.6) **Proof**  Denote by \( f_{MP} \) the Schwartz function on \( \mathfrak{m}(A) \) whose \( S \)-local component \( f_{MP,S} \)

\[
(4.2.6.1) \quad \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{X \in (\mathfrak{m}(Q) \cap \phi)_{M,S}} a^M(S, X) I_M^L(X, f_{S,MP})
\]

\[
= \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)} \left( \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{X \in (\mathfrak{t}(Q) \cap \phi)_{L,S}} a^L(S, X) I_M^L(X, f_{L,P,S}) \right)
\]

\[
(4.2.6.2) = \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)} \left( \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_0^L|^{-1} \times \right.
\]

\[
\left. \sum_{X \in (\mathfrak{m}(Q) \cap \phi)_{M,S}} a^M(S, X) I_M^L(X, (f_{L,P})^* S) \right)
\]

\[
(4.2.6.3) = \sum_{L \in \mathcal{L}} |W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)} \left( \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_0^L|^{-1} \times \right.
\]

\[
\left. \sum_{X \in (\mathfrak{m}(Q) \cap \phi)_{M,S}} a^M(S, X) I_M^L(X, f_{S,LP}) \right)
\]

where the equality \((4.2.6.2)\) follows from the identity \((4.2.1.1)\) applied to each \( L \), and the equality \((4.2.6.3)\) follows from the fact that the operator

\[
(4.2.6.4) \quad f_S \mapsto f_{S,MP}
\]

intertwines the Fourier transforms on \( \mathfrak{g}(Q_S) \) and \( \mathfrak{m}(Q_S) \). See §13.13 of [Ko05] and (5.1.3.1).

Rearrange the double sum over \( L \) and \( M \), the right hand side of \((4.2.6.3)\) is equal to

\[
(4.2.6.5) \quad \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{X \in (\mathfrak{m}(Q) \cap \phi)_{M,S}} a^M(S, X) \times
\]

\[
\times \left( \sum_{L \in \mathcal{L}(M)} (-1)^{\dim(A_M/A_L)} I_M^L(X, f_{S,LP}) \right)
\]

\[
= \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{X \in (\mathfrak{m}(Q) \cap \phi)_{M,S}} a^M(S, X) I_M^G(X, f_{S,L,P}).
\]
(4.2.7) Corollary (Local invariant trace formula II)
Let \( v \) be a place of \( \mathbb{Q} \). For each Levi subgroup \( M_v \) in \( \mathcal{L}^G_v \) choose a parabolic subgroup \( M_v P_v \) in \( \mathcal{P}^G_v(M_v) \). Let \( f_v \) and \( g_v \) be Schwartz functions on \( \mathfrak{g}_v(\mathbb{Q}_v) \), then

\[
(4.2.7.1) \sum_{M_v \in \mathcal{L}^G_v} |W_{M_v,0}^M||W_{G_v,0}^G|^{-1} \sum_{T_v \in T_{\mathrm{ell}}(M_v)} |W(M_v, T_v)|^{-1} \int_{t_v(Q_v)} I_{M_v}^G(X_v, f_v) I_{G_v}^G(X_v, g_v) \, dX_v
= \sum_{M_v \in \mathcal{L}^G_v} |W_{M_v,0}^M||W_{G_v,0}^G|^{-1} \sum_{T_v \in T_{\mathrm{ell}}(M_v)} |W(M_v, T_v)|^{-1} \int_{t_v(Q_v)} I_{M_v}^G(X_v, g_v) I_{G_v}^G(X_v, f_v) \, dX_v
\]

where \( I_{M_v}^G(X_v, ) \) is the invariant distribution on \( \mathfrak{g}_v(Q_v) \) defined by

\[
(4.2.7.2) \forall f_v \in \mathcal{S}(\mathfrak{g}_v(Q_v)) \quad I_{M_v}^G(X_v, f_v) = \sum_{L_v \in \mathcal{L}^G_v(M_v)} (-1)^{\dim(A_{M_v}/A_{L_v})} I_{L_v}^G(X_v, f_v) .
\]

(4.2.8) Proof The left hand side of (4.2.7.1) is equal to

\[
(4.2.8.1) \sum_{M_v \in \mathcal{L}^G_v} |W_{M_v,0}^M||W_{G_v,0}^G|^{-1} \sum_{T_v \in T_{\mathrm{ell}}(M_v)} |W(M_v, T_v)|^{-1} \times \int_{t_v(Q_v)} \sum_{L_v \in \mathcal{L}^G_v(M_v)} (-1)^{\dim(A_{M_v}/A_{L_v})} I_{L_v}^G(X_v, (f_v)_{L_v P_v}) I_{G_v}^G(X_v, g_v) \, dX_v
= \sum_{L_v \in \mathcal{L}^G_v} |W_{L_v,0}^L||W_{G_v,0}^G|^{-1} \left( \sum_{M_v \in \mathcal{L}^G_v} |W_{M_v,0}^M||W_{L_v,0}^L|^{-1} \sum_{T_v \in T_{\mathrm{ell}}(M_v)} |W(M_v, T_v)|^{-1} \times \int_{t_v(Q_v)} (-1)^{\dim(A_{M_v}/A_{L_v})} I_{L_v}^G(X_v, (f_v)_{L_v P_v}) I_{G_v}^G(X_v, g_v) \, dX_v \right).
\]

Hence (4.2.7.1) follows from the identity (4.2.3.1) applied to each \( L_v \) and the fact that

\[
(4.2.8.2) \forall X_v \in t_{v, \text{reg}, \text{ss}}(\mathbb{Q}_v) \quad I_{L_v}^G(X_v, g_v, (f_v)_{L_v P_v}) = I_{G_v}^G(X_v, g_v).
\]

See §13.12 of [Ko05] and (5.1.2.1).

5 The trace formula for polyhedron-valued orbital integrals

In this chapter the invariant trace formulæ (4.2.5.1) and (4.2.7.1) are lifted to identities between vector-valued distributions on \( \mathfrak{g}(\mathbb{A}) \).

5.1 Parabolic descent and parabolic induction

(5.1.1) Definition Let \( P \) be a parabolic subgroup in \( \mathcal{F} \) with Levi component \( M \) and unipotent radical \( N \). Let \( f_S \) be a Schwartz function on \( \mathfrak{g}(\mathbb{Q}_S) \). Define \( f_{S,P} \) to be the Schwartz function on \( \mathfrak{m}(\mathbb{Q}_S) \) by parabolic descent along \( P \)

\[
(5.1.1.1) \forall X \in \mathfrak{m}(\mathbb{Q}_S) \quad f_{S,P}(X) = \int_{K_S} \int_{\mathfrak{n}(\mathbb{Q}_S)} f_S((X + N) \cdot \text{ad}(k)) \, dN dk.
\]

For each parabolic subgroup \( P_S \) in \( \mathcal{F}^{G_S} \) define the Schwartz function \( f_{S,P_S} \) on \( \mathfrak{m}_S(\mathbb{Q}_S) \) analogously.
(5.1.2) Remark Parabolic descent preserves orbital integrals, i.e.
\[(5.1.2.1) \forall X \in m_{\text{reg,ss}}(Q_S) \quad I_G^M(X, f_S) = I_M^M(X, f_{S,P}).\]

See §13.12 of [Ko05]. Hence the orbital integrals of $f_{S,P}$ on $m_{\text{reg,ss}}(Q_S)$ is independent of the choice of the parabolic subgroup $P$ in $P(M)$. In this case denote $f_{S,P}$ by the alternative notation $f_{S,M}$ such that
\[(5.1.2.2) \forall X \in m_{\text{reg,ss}}(Q_S) \quad I_M^M(X, f_{S,M}) = I_M^M(X, f_{S,P}).\]

is well-defined.

(5.1.3) Remark With compatible choices of Fourier transforms on $g(Q_S)$ and $m(Q_S)$, parabolic descent intertwines the Fourier transforms, i.e.
\[(5.1.3.1) (f_S^*)_P = (f_{S,P})^*.\]

See §13.13 of [Ko05].

(5.1.4) Definition Let $P$ be a parabolic subgroup in $F$ with Levi component $M$ and unipotent radical $N$. Let $X$ be an element in $m(Q_S)$. Define $\text{Ind}^G_{M,P}(X)$ to be the $\text{ad}(G(Q_S))$-invariant subset of $g(Q_S)$ by parabolic induction along $P$
\[(5.1.4.1) \text{Ind}^G_{M,P}(X) = \left( (X \cdot \text{ad}(M) + n) \cdot \text{ad}(G) \right)_{\text{reg}}(Q_S)\]

where the subscript reg denotes the regular locus, the set of points with minimal dimensional isotropy group in $G$.

(5.1.5) Remark With a fixed choice of $M$, parabolic induction along $P$ is independent of the choice of the parabolic subgroup $P$ in $P(M)$, hence adopt the alternative notation $\text{Ind}^G_M(X)$. For a proof see Satz 2.6 of [Bo81]. If the inducing Levi subgroup $M$ is understood tacitly, denote $\text{Ind}^G_M(X)$ by the alternative notation $X^G$.

(5.1.6) Remark If $X$ is regular semisimple then $\text{Ind}^G_M(X)$ is equal to the $\text{ad}(G(Q_S))$-orbit of $X$. In general $\text{Ind}^G_M(X)$ is a finite union of $\text{ad}(G(Q_S))$-orbits which are geometrically conjugate to each other, but $\text{Ind}^G_M(X)$ is still called the induced orbit of $X$. See page 255 of [Ar88a] and §2.1 of [Bo81]. If $L$ is a Levi subgroup in $L(M)$, define the invariant weighted orbital integral along the induced orbit $X^L$ by
\[(5.1.6.1) \forall f_S \in g(Q_S) \quad I_L^G(X^L, f_S) = \sum_{\Omega \leq X^L} I_G^G(\Omega, f_S).\]

(5.1.7) Remark Parabolic induction is transitive in nested chains of Levi subgroups, i.e.
\[(5.1.7.1) \forall L \in L^G(M) \quad \text{Ind}^G_L(X) = \text{Ind}^G_M(\text{Ind}^G_L(X)).\]

See §2.3 of [Bo81].
(5.1.8) Remark Parabolic induction is compatible with Jordan decomposition, i.e.

\[ \text{Ind}_M^G(X) = \left( X_{ss} + \text{Ind}_M^G(X_{nill}) \right) \cdot \text{ad}(G(\mathbb{Q}_S)). \]

See §2.4 of [Bo81]. Also see Lemma 2 of [Ho13].

(5.1.9) Lemma (Descent and splitting of \((G, M)\)-families)

Let \( M \) be a Levi subgroup in \( L \). There exist a function \( \text{d}_M^G(\ ,\ ) \)

\[ \text{d}_M^G : \mathcal{L}(M) \times \mathcal{L}(M) \to \mathbb{R}, \]

and a partially defined map \( s(\ ,\ ) \)

\[ s : \mathcal{L}(M) \times \mathcal{L}(M) \to \mathcal{F}(M) \times \mathcal{F}(M) \]

whose domain contains the pairs \((L_1, L_2)\) for which \( \text{d}_M^G(L_1, L_2) \) is nonzero, such that

- if \((L_1, L_2)\) is contained in the domain of \( s \), then
  \[ s(L_1, L_2) \in \mathcal{P}(L_1) \times \mathcal{P}(L_2); \]

- if \((c_p)\) is a \((G, M)\)-family and \( L \) is a Levi subgroup in \( \mathcal{L}(M) \), then
  \[ c_L = \sum_{L' \in \mathcal{L}(M)} \text{d}_M^G(L, L') c_M^{Q'} \]

  where \( Q' \) denotes the second component of \( s(L, L') \);

- if \((c_p)\) and \((d_p)\) are \((G, M)\)-families, then
  \[ (cd)_M = \sum_{L_1, L_2 \in \mathcal{L}(M)} \text{d}_M^G(L_1, L_2) c_M^{Q_1} d_M^{Q_2} \]

  where

  \[ (Q_1, Q_2) = s(L_1, L_2). \]

Analogous results hold for the groups \( G_v \) and \( G_S \).

(5.1.10) Proof See §7 of [Ar88b]. \( \Box \)

(5.1.11) Remark The constant \( \text{d}_M^G(L_1, L_2) \) is defined to be the volume in \( \mathfrak{a}_M^G \) of the image of a fundamental parallelotope in the direct sum of \( \mathfrak{a}_{L_1}^M \) and \( \mathfrak{a}_{L_2}^M \) under the natural map

\[ \mathfrak{a}_{L_1}^M \oplus \mathfrak{a}_{L_2}^M \to \mathfrak{a}_M^G \]

if \((5.1.11.1)\) is an isomorphism, and zero otherwise. In the former case \( \text{d}_M^G(L_1, L_2) \) is equal to the volume in \( \mathfrak{a}_{L_2}^G \) of the image of a fundamental parallelotope in \( \mathfrak{a}_{L_1}^M \) under the natural isomorphism

\[ \mathfrak{a}_{L_1}^M \to \mathfrak{a}_{L_2}^G. \]
Remark 5.1.12 The map $s$ depends on the choice of a vector $\xi$ in general position in $a_M^G$. Let $L_1$ and $L_2$ be Levi subgroups in $L(M)$ such that (5.1.11) is an isomorphism, then

$$\exists \xi_1 \in a_{L_1}^G, \exists \xi_2 \in a_{L_2}^G \quad \xi = \frac{\xi_1}{2} - \frac{\xi_2}{2}.$$  

Define $s(L_1, L_2)$ to be $(Q_1, Q_2)$ where $Q_i$ is the parabolic subgroup in $P(L_i)$ whose corresponding positive chamber in $a_{L_i}^G$ contains the vector $\xi_i$ where the index $i$ is 1 or 2.

Lemma 5.1.13 (Descent and splitting of invariant weighted orbital integrals) Let $M$ be a Levi subgroup in $L$, let $X$ be an element of $m(Q_S)$. Let $f_S$ be a Schwartz function on $g(Q_S)$. Let $\xi$ be a vector in general position in $a_M^G$, let $d_{L(G)}^G: L_{G}(M) \times L_{G}(M) \rightarrow \mathbb{R}$ be defined as in Remark (5.1.11) and Remark (5.1.12) with respect to $\xi$.

- Let $L$ be a Levi subgroup in $L^G(M)$, then
  $$I^G_L(X, f_S) = \sum_{L' \in L(M)} d^G_M(L, L') I^1_{M}(X, f_{S, Q'})$$
  where $Q'$ denotes the second component of $s(L, L')$.

- Let $S$ be the set $\{v_1, v_2\}$. Let $f_S$ be of the form $f_{v_1} \otimes f_{v_2}$ where $f_{v_i}$ is a Schwartz function on $g(Q_{v_i})$ where the index $i$ is 1 or 2. Then
  $$I^G_M(X, f_S) = \sum_{L_1, L_2 \in L(M)} d^G_M(L_1, L_2) I^1_{M}(X, f_{v_1, Q_1}) I^2_{M}(X, f_{v_2, Q_2})$$
  where
  $$(Q_1, Q_2) = s(L_1, L_2).$$

Local identities analogous to (5.1.13.2) also hold for $G_v$.

Proof 5.1.14 This is the Lie algebra analogue of the main results of §8 and §9 of [Ar88b].

Corollary 5.1.15 (Descent and splitting of orbital integrals) Let $X$ be an element of $g(Q_S)$. Let $f_S$ be a Schwartz function on $g(Q_S)$.

- Let $L$ be a Levi subgroup in $L$ such that $X$ is contained in $l(Q_S)$, then
  $$I^G_L(X^G, f_S) = I^L(X, f_{S,L}).$$

- Let $S$ be the set $\{v_1, v_2\}$. Let $f_S$ be of the form $f_{v_1} \otimes f_{v_2}$ where $f_{v_i}$ is a Schwartz function on $g(Q_{v_i})$ where the index $i$ is 1 or 2. Then
  $$I^G_V(X, f_{v_1} \otimes f_{v_2}) = I^G_V(X, f_{v_1}) I^G_V(X, f_{v_2}).$$

Local identities analogous to (5.1.15.1) also hold for $G_v$. 

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(5.1.16) **Proof** The identity (5.1.15.1) follows from (5.1.13.2) where the only nonzero summand on the right hand side corresponds to the Levi subgroup $L$ in $L(L)$. The identity (5.1.15.2) follows from (5.1.13.3) where the only summand on the right hand side corresponds to the pair $(G, G)$ in $L(G) \times L(G)$.

(5.1.17) **Remark** If $X$ is a regular semisimple element of $m(Q)$ then the local analogue of (5.1.13.2) becomes

\[(5.1.17.1) \quad I_{M_v}^G(x, f_v) = \sum_{L'_v \in L(G)} d_{M_v}^{L_v}(L_v, L'_v) I_{M_v}^{L'_v}(x, f_v, Q'_v).\]

Repeatedly applying the identities (5.1.13.3) and (5.1.17.1) reduces a regular semisimple invariant weighted orbital integral $I_{M_v}^G(x)$ that appears generically in the global invariant trace formula (4.2.1.1) to a linear combination of products of local invariant weighted orbital integrals $I_{M'_v}^{G_v}(x, f_v)$ where $M'_v$ is contained in $M_v$ and $X$ is elliptic in $m'_v(Q_v)$.

### 5.2 Orbital integrohedra

(5.2.1) **Definition** Let $\mathbb{E}^n$ denote the $n$-dimensional Euclidean space. A polytope in $\mathbb{E}^n$ that is closed with nonempty interior is said to be **proper**. Let $P$ and $Q$ be proper convex polytopes in $\mathbb{E}^n$. Then $P$ and $Q$ are said to be **translational scissors congruent** if there exist convex polytopes $P_1, P_2, \ldots, P_l$ and $Q_1, Q_2, \ldots, Q_l$ in $\mathbb{E}^n$ such that

\[(5.2.1.1) \quad P = \bigcup_{i=1}^l P_i \quad Q = \bigcup_{i=1}^l Q_i\]

and $P_i$ is a translation of $Q_i$ for each index $i$ among $1, 2, \ldots, l$. Define the **scissors group** of $\mathbb{E}^n$, denoted by $S(\mathbb{E}^n)$, to be the quotient of the free abelian group generated by the proper convex polytopes in $\mathbb{E}^n$ modulo translational scissors congruence.

(5.2.2) **Definition** A flag $\Phi$ of linear subspaces in $\mathbb{E}^n$ is said to be **strict of length** $l$ if

\[(5.2.2.1) \quad \Phi = V_0 \supset V_1 \supset \cdots \supset V_l\]

where $V_i$ has codimension $i$. Let $\Phi$ be a strict flag of length $l$. A **rigging** $r$ of $\Phi$ is a collection

\[(5.2.2.2) \quad r = \{r_1, r_2, \ldots, r_l\}\]

where $r_i$ is a real linear functional on $V_{i-1}$ with kernel $V_i$ for each $i$ among $1, 2, \ldots, l$. Two riggings $r$ and $r'$ are **equivalent** if $r_i$ and $r_i'$ are positive multiples of each other for each $i$ among $1, 2, \ldots, l$. Denote by $\mathcal{Rig}(\Phi)$ the collection of equivalence classes of riggings of $\Phi$. A **rigged flag** $\Phi^r$ is defined to be a strict flag $\Phi$ together with a choice of an element $r$ in $\mathcal{Rig}(\Phi)$.

An **orientation** of $\mathbb{E}^n$ is an ordered basis of $\mathbb{E}^n$ defined up to a linear transformation with positive determinant. The product of an orientation of $\mathbb{E}^n$ with a translation invariant measure on $\mathbb{E}^n$ is equal to a volume form on $\mathbb{E}^n$. Fix an ordered basis $\mathcal{B}$

\[(5.2.2.3) \quad \mathcal{B} = (b_1, b_2, \ldots, b_n)\]
of \( \mathbb{E}^n \) such that
\[
(5.2.2.4) \quad \mathcal{B}' = (b_{l+1}, b_{l+2}, \ldots, b_n)
\]
is an ordered basis of \( V_l \). Let \( \mathbf{r} \) be a rigging of \( \Phi \), choose vectors
\[
(5.2.2.5) \quad c_1 \in V_0, \quad c_2 \in V_1, \quad \ldots, \quad c_l \in V_{l-1}
\]
such that
\[
(5.2.2.6) \quad \forall i = 1, 2, \ldots, l \quad r_i(c_i) > 0.
\]
Denote by \( \mathcal{B}^{\mathbf{r}} \) the ordered basis
\[
(5.2.2.7) \quad \mathcal{B}^{\mathbf{r}} = (c_1, c_2, \ldots, c_l, b_{l+1}, b_{l+2}, \ldots, b_n)
\]
of \( \mathbb{E}^n \). Define the *sign* of \( \mathbf{r} \) by
\[
(5.2.2.8) \quad \text{sign}(\mathbf{r}) = \begin{cases} 
1 & \text{if } \mathcal{B} \text{ and } \mathcal{B}^{\mathbf{r}} \text{ define the same orientation of } \mathbb{E}^n, \\
-1 & \text{if } \mathcal{B} \text{ and } \mathcal{B}^{\mathbf{r}} \text{ define opposite orientations of } \mathbb{E}^n.
\end{cases}
\]
Let \( \Phi^{\mathbf{r}} \) be a rigged flag of length \( l \) in \( \mathbb{E}^n \). Let \( P \) be a proper convex polytope in \( \mathbb{E}^n \). Then the \( \Phi^{\mathbf{r}} \)-boundary \( \partial_{\Phi^{\mathbf{r}}} P \) of \( P \) is defined by
\[
(5.2.2.9) \quad \partial_{\Phi^{\mathbf{r}}} P = r_1^{\min}(r_2^{\min}(\ldots(r_{l-1}^{\min}(r_l^{\min}(P)) \ldots))
\]
where for each subset \( S \) of \( V_{l-1} \)
\[
(5.2.2.10) \quad r_i^{\min}(S) = \begin{cases} 
v_i + r_i^{-1}\left(\min_{s \in S} r_i(s)\right) & \text{if } v_i \text{ is a vector in } V_{l-1} \text{ such that} \\
v_i + r_i^{-1}\left(\min_{s \in S} r_i(s)\right) & \text{is a subset of } V_i \text{ with nonempty interior,} \\
\emptyset & \text{if no such } v_i \text{ exists,}
\end{cases}
\]
which is a subset of \( V_i \) defined up to translation. Fix a translation invariant measure on the Euclidean space \( V_l \). This determines the volume of the convex polytope \( \partial_{\Phi^{\mathbf{r}}} P \). Then define the *Hadwiger invariant* \( \text{Had}_\Phi \) of \( P \) with respect to \( \Phi \) by
\[
(5.2.2.11) \quad \text{Had}_\Phi(P) = \sum_{\mathbf{r} \in \mathcal{R}_{\Phi}} \text{sign}(\mathbf{r}) \text{Vol}(\partial_{\Phi^{\mathbf{r}}} P).
\]

**Remark** Each Hadwiger invariant defines a real-valued additive function on \( \mathfrak{S}(\mathbb{E}^n) \).

**Lemma** Let \( P \) and \( Q \) be proper convex polytopes in \( \mathbb{E}^n \). Then \( P \) are \( Q \) are translational scissors congruent if and only if for every \( l \) among \( 0, 1, 2, \ldots, n \), for each strict flag \( \Phi \) of length \( l \),
\[
(5.2.4.1) \quad \text{Had}_\Phi(P) = \text{Had}_\Phi(Q).
\]
(5.2.5) Proof See Corollary 2 in §4 of [Mo93a]. ☐

(5.2.6) Lemma Let \((H_\Phi)\) be a collection of real numbers indexed by the set of all strict flags \(\Phi\) in \(\mathbb{E}^n\) which vanishes for all but finitely many \(\Phi\). Then there exists an element \([P]\) in the scissors group \(S(\mathbb{E}^n)\) such that

\[
\forall \text{ strict flag } \Phi \text{ in } \mathbb{E}^n \quad H_\Phi = \text{Had}_\Phi([P])
\]

if and only if for every \(l\) among \(0, 1, 2, \ldots, n\), for each strict flag of length \(l\), for every \(i\) among \(1, 2, \ldots, l - 1\),

\[
\sum_{\Phi' \in F_s(\Phi, i)} H_{\Phi'} = 0
\]

where \(F_s(\Phi, i)\) is the set defined by

\[
F_s(\Phi, i) = \left\{ \Phi' \text{ strict flag in } \mathbb{E}^n : \Phi' = V_0 \supset V_1 \supset \cdots \supset V_i \supset U_i \supset V_{i+1} \supset \cdots \supset V_l \right\},
\]

and

\[
\sum_{\Phi' \in F_s(\Phi, l)} H_{\Phi'} \wedge dU_l = 0
\]

where

\[
\Phi' = V_0 \supset V_1 \supset \cdots \supset V_{l-1} \supset U_l
\]

and \(dU_l\) is the element of \(\wedge^{n-l} \mathbb{E}^{n,*}\) defined as the product of the fixed orientation and translation invariant measure on \(U_l\).

(5.2.7) Proof See Corollary 3 in §4 of [Mo93a]. ☐

(5.2.8) Remark The scissors group \(S(\mathbb{E}^n)\) has the structure of a real vector space.

(5.2.9) Definition Let \(M\) be a Levi subgroup in \(\mathcal{L}\). Let \(\mathcal{Y}_M\) and \(\mathcal{Z}_M\) be positive \((G, M)\)-orthogonal sets. Then \(\mathcal{Y}_M\) and \(\mathcal{Z}_M\) are said to be \(\textit{scissors congruent as } (G, M)\)-orthogonal sets if the convex hull of \(\mathcal{Y}_M\) in \(a_M^G\) is translational scissors congruent to the convex hull of \(\mathcal{Z}_M\) in \(a_M^G\). Define the \(\textit{scissors group} of a_M^G\), denoted by \(S(a_M^G)\), to be the quotient of the free abelian group generated by the positive \((G, M)\)-orthogonal sets modulo translational scissors congruence in \(a_M^G\).

(5.2.10) Remark The Hadwiger invariants of \(\mathcal{Y}_M\) are supported on the strict flags \(\Phi\) in \(a_M^G\) of the form

\[
\Phi = a_M^0 \supset a_M^1 \supset \cdots \supset a_M^l
\]

where

\[
L^0 \supset L^1 \supset \cdots \supset L^l
\]
is a nested chain of Levi subgroups in $\mathcal{L}(M)$. Denote the collection of such $\Phi$ by $\mathcal{F}_s(a_M^G)$.

A rigging $r$ of $\Phi$ is equivalent to a nested chain of parabolic subgroups

\[(5.2.10.3)\quad r = Q^0 \supset Q^1 \supset \cdots \supset Q^l\]

such that $Q^l$ is a parabolic subgroup in $\mathcal{P}(L^i)$ for each $i$ among $0, 1, 2, \ldots, l$.

Fix a vector $\xi$ in general position in $a_M^G$. Then $\xi$ defines a total order on $\Delta_M^G$ for each Levi subgroup $L$ in $\mathcal{L}(M)$, which induces a consistent choice of signs for all rigged flags $\Phi^r$ with $\Phi$ in $\mathcal{F}_s(a_M^G)$. More precisely choose an element $s$ of the Weyl group that stabilizes $M$ such that the parabolic subgroup $sQ^l$ is standard. The nested chain

\[(5.2.10.4)\quad sQ^0 \supset sQ^1 \supset \cdots \supset sQ^l\]

determines a sequence of positive roots $\alpha^i$ where $i$ ranges among $1, 2, \ldots, l$. Let $\sigma$ be the permutation on $l$ letters such that $\alpha^{\sigma(i)}$ is strictly increasing with respect to the total order determined by $\xi$. Then the sign of the rigging $r$ is equal to

\[(5.2.10.5)\quad \text{sign}(r) = \text{sign}(\text{Det}(s))\text{sign}(\sigma).\]

Each subspace $a_M^G$ of $a_M^G$ is equipped with the translation invariant measure determined by the covéntage lattice. This choice of orientations and measures determines the numerical values of the Hadwiger invariants $\text{Had}_{\Phi}(\mathcal{Y}_M)$.

\subsection{5.2.11 Definition} Let $M$ be a Levi subgroup in $\mathcal{L}$, let $X$ be an element of $m(Q_S)$. Define the scissors-congruence-valued orbital integral, or orbital integrohedron $I_M^G(X, f_S)$ to be the vector-valued distribution on $g(Q_S)$ taking values in $\bigoplus_{\Phi \in \mathcal{F}_s(a_M^G)} \mathbb{C}$ such that

\[(5.2.11.1)\quad \forall f_S \in S(g(Q_S)) \quad I_M^G(X, f_S) = \left( \sum_{r \in \mathcal{R}_{ig}(\Phi)} \text{sign}(r) I_M^{ij}(X, f_{S,Q^l}) \right)_{\Phi \in \mathcal{F}_s(a_M^G)}\]

where $\Phi$, $r$, $L^i$ and $Q^l$ are related as in \[(5.2.10.1)\], \[(5.2.10.2)\] and \[(5.2.10.3)\].

For each Levi subgroup $M_S$ in $\mathcal{L}_M^G$ define $I_{M_S}^G(X, f_S)$ by the analogous formula.

\subsection{5.2.12 Lemma} Let $M$ be a Levi subgroup in $\mathcal{L}$, let $X$ be an element of $m(Q_S)$, let $f_S$ be a real-valued Schwartz function on $g(Q_S)$. Then $I_M^G(X, f_S)$ defines a unique element of $\mathcal{S}(a_M^G)$.

\subsection{5.2.13 Proof} The Schwartz function $f_S$ is real-valued, so $I_M^G(X, f_S)$ is a collection of real numbers indexed by the strict flags $\Phi$ in $\mathcal{F}_s(a_M^G)$. It suffices to verify \[(5.2.6.3)\] and \[(5.2.6.5)\].

The left hand side of \[(5.2.13.1)\] is equal to

\[(5.2.13.1)\quad \sum_{\Phi' \in \mathcal{F}_s(\Phi, i)} \sum_{r' \in \mathcal{R}_{ig}(\Phi')} \text{sign}(r') I_M^{ij'}(X, f_{S,Q^l}) = \sum_{Q^{j'}_0 \supset Q^{j'}_1 \supset \cdots \supset Q^{j'}_l} \sum_{\forall j=0,1,\ldots,l} \sum_{\forall j'=0,1,\ldots,l} \text{sign}(r') I_M^{ij'}(X, f_{S,Q^{j'}})\]

where $\Phi$, $L^i$ and $\Phi'$, $r'$, $L'^i$, $Q^{j'}$ are related as in \[(5.2.10.1)\], \[(5.2.10.2)\] and \[(5.2.10.3)\]. The summands of the right hand side of \[(5.2.13.1)\] with a fixed minimal term $Q^{j'}$ are in $(1, 1)$ correspondence with the sequences of roots $(\alpha^j)$ that are positive with respect to $Q^{j'}$ such that

\[(5.2.13.2)\quad \alpha^1, \alpha^2, \ldots, \alpha^{i-1}, \alpha^{i+2}, \alpha^{i+3}, \ldots, \alpha^l\]
are determined by Φ. Hence the summand

\[ I_M^{L'}(X, f_S, Q') \]

appears twice on the right hand side of (5.2.13.1) with opposite signs, so (5.2.13.1) vanishes. Hence \( \Pi_M^L(X, f_S) \) satisfies (5.2.6.3).

The left hand side of (5.2.6.5) is equal to

\[
\sum_{\Phi' \in F_s(\Phi, l)} \sum_{r' \in R_{\Theta}(\Phi')} \text{sign}(r') I_M^{L'}(X, f_S, Q') \wedge da_M^{L'}
\]

\[
= \sum_{Q_0 \supset Q_1 \supset \cdots \supset Q_{l-1} \supset Q'} \text{sign}(r') I_M^{L'}(X, f_S, Q') \wedge da_M^{L'}
\]

where Φ, Q′ and Φ′, r′, L′, Q′ are related as in (5.2.10.1), (5.2.10.2) and (5.2.10.3). It suffices to show that (5.2.13.4) vanishes as a differential form on \( a_{M}^{L_{l-1}} \). Let L be a Levi subgroup in \( L_{l-1}(M) \) such that

\[ \dim(A_M/A_L) = 1, \]

(5.2.13.5)

denote by \( da_L^{L_{l-1}} \) the differential form on \( a_{M}^{L_{l-1}} \) defined by pulling back the volume form on \( a_{L}^{L_{l-1}} \) along the natural projection

\[ a_{M}^{L_{l-1}} \rightarrow a_{L}^{L_{l-1}}. \]

Then the orthogonal projection of (5.2.13.4) onto the one dimensional subspace

\[ \text{span}(da_L^{L_{l-1}}) \subset \bigwedge^{\dim(A_M/A_G)-l} a_{M}^{L_{l-1}}, \]

(5.2.13.7)
is equal to

\[
\sum_{Q_0 \supset Q_1 \supset \cdots \supset Q_{l-1} \supset Q'} \text{sign}(r') d_M^{L_{l-1}}(L, L':) I_M^{L'}(X, f_S, Q') \wedge da_L^{L_{l-1}}
\]

(5.2.13.8)

by the definition of the constant \( d_M^{L_{l-1}}(L, L') \) in Remark (5.1.11). The summation in (5.2.13.8) is taken over the set

\[
\{(L'', Q'' \cap L_{l-1}) : L'' \in \mathcal{L}^{L_{l-1}}(M), \dim(a_{L''}^{G}) = l, Q'' \in P(L''), Q'' \subset Q_{l-1}\}
\]

(5.2.13.9)
\[ \subset \mathcal{L}^{L_{l-1}}(M) \times \mathcal{F}^{L_{l-1}}(M) \]

Let \( \xi^{l-1} \) be the projection of the vector \( \xi \) in \( a_M^{G} \) used to define the orientation onto \( a_{M}^{L_{l-1}} \). Then \( \xi^{l-1} \) determines a partial map

\[ s^{l-1} : \mathcal{L}^{L_{l-1}}(M) \times \mathcal{L}^{L_{l-1}}(M) \rightarrow \mathcal{F}^{L_{l-1}}(M) \times \mathcal{F}^{L_{l-1}}(M) \]

(5.2.13.10)
as in Remark (5.1.12) which is positive in the sense that

\[ \text{sign}(r) = 1 \]

(5.2.13.11)
if the rigging \( \mathbf{r} \) corresponds to the element \((L', Q^{l'}_+)\) in the set (5.2.13.9) where \( Q^{l'}_+ \) denotes the second component of \( s^{l-1}(L, L') \). Let \( Q^{l'}_- \) denote the opposite parabolic of \( Q^{l'}_+ \), then (5.2.13.9) is equal to the disjoint union

\[
(5.2.13.12) \quad \left\{ (L', Q^{l'}_+) : L' \in L^{L^{-1}}(M) \right\} \coprod \left\{ (L', Q^{l'}_-) : L' \in L^{L^{-1}}(M) \right\},
\]

hence (5.2.13.8) is equal to

\[
(5.2.13.13) \quad \left( \sum_{L' \in L^{L^{-1}}(M)} d_{M}^{L^{-1}}(L, L') f_{M}^{L'}(X, f_{S, Q^{l'}}) \right) - \left( \sum_{L' \in L^{L^{-1}}(M)} d_{M}^{L^{-1}}(L, L') f_{M}^{L'}(X, f_{S, Q^{l'-}}) \right) \wedge d_{l}^{L^{-1}} \left( \right)
\]

\[
(5.2.13.14) \quad = \left( \left( H_{L}^{L^{-1}}(X, f_{S, Q^{l'}}) - H_{L}^{L^{-1}}(X, f_{S, Q^{l'-}}) \right) \wedge d_{l}^{L^{-1}} \left( \right) \right) = 0
\]

where the equality (5.2.13.14) follows from the \( S \)-local version of (5.1.13.2) and the fact that \( Q^{l'}_- \) is the second component of \( s^{L^{-1}}(L, L') \) where \( s^{L^{-1}}(, , ) \) is the partial map determined as in Remark (5.1.12) by the vector \(-\ell^{L^{-1}}\). Hence (5.2.13.8) vanishes for every \( L \), so (5.2.13.4) vanishes as a differential form on \( a_{M}^{G} \). Hence \( I_{M}^{G}(X, f_{S}) \) satisfies (5.2.6.5).

(5.2.14) **Remark** In §23 of [Ko05] Kottwitz defined weight factors and weighted orbital integrals taking values in the complexified \( K \)-group of the toric variety of the fan of the root hyperplanes in \( a_{M}^{G+} \). In §4 of [Mo93b] Morelli proved that this \( K \)-group is the additive group of translational scissors congruent classes of positive \((G, M)\)-orthogonal sets whose vertices are contained in the coweight lattice in \( a_{M}^{G} \).

(5.2.15) **Definition** Let \( M \) be a Levi subgroup in \( L \). Define the **total scissors ring** of \( a_{M}^{G} \), denoted by \( S(a_{L(M)}^{G}) \), to be the direct sum

\[
(5.2.15.1) \quad S(a_{L(M)}^{G}) = \bigoplus_{L \in L^{G}(M)} S(a_{L}^{G}).
\]

Define a bilinear product \( \boxtimes \) on \( S(a_{L(M)}^{G}) \) by

\[
(5.2.15.2) \quad \forall L_{1}, L_{2} \in L^{G}(M) \quad \forall \mathcal{Y}_{L_{i}} \in S(a_{L_{i}}^{G})
\]

where \( i = 1, 2 \)

\[
[\mathcal{Y}_{L_{1}}] \boxtimes [\mathcal{Y}_{L_{2}}] = \begin{cases} j^{*}(\mathcal{Y}_{L_{1}} \times \mathcal{Y}_{L_{2}}) & \text{if the natural map} \\ j : a_{L_{1} \cap L_{2}}^{G} \longrightarrow a_{L_{1}}^{G} \oplus a_{L_{2}}^{G} \text{is an isomorphism,} \\ 0 & \text{otherwise,} \end{cases}
\]
where the \((G \times G, L_1 \times L_2)\)-family \(Y_{L_1} \times Y_{L_2}\) is well-defined up to translational scissors congruence
in \(a^G_{L_1} \oplus a^G_{L_2}\), and \(j^*\) is the homomorphism
\[(5.2.15.3) \quad j^*: \mathbb{S}(a^G_{L_1} \oplus a^G_{L_2}) \rightarrow \mathbb{S}(a^G_{L_1 \cap L_2})\]
induced by \(j\).

\[\textbf{(5.2.16)} \quad \textbf{Remark} \quad \text{The total scissors ring } \mathbb{S}(a^G_{\mathcal{L}(M)}) \text{ is graded by} \]
\[(5.2.16.1) \quad \forall n \in \mathbb{N} \quad \left(\mathbb{S}(a^G_{\mathcal{L}(M)})\right)_n = \bigoplus_{L \in \mathcal{L}^G(M), \dim(a^G_L) = n} \mathbb{S}(a^G_L).
\]
The grading \[(5.2.16.1)\] has a refinement into the \(\mathcal{L}^G(M)\)-grading defined by
\[(5.2.16.2) \quad \forall L \in \mathcal{L}^G(M) \quad \left(\mathbb{S}(a^G_{\mathcal{L}(M)})\right)_L = \mathbb{S}(a^G_L)
\]where the monoid structure on \(\mathcal{L}^G(M)\) is defined by intersection.
Each Levi subgroup \(L\) in \(\mathcal{L}^G(M)\) defines a homogeneous ideal
\[(5.2.16.3) \quad \bigoplus_{L' \in \mathcal{L}^G(M), \dim(a^G_{L'}) \cap \mathcal{L} > \mathcal{L}} \mathbb{S}(a^G_{L'}) \subset \mathbb{S}(a^G_{\mathcal{L}(M)}),
\]and taking the quotient of \(\mathbb{S}(a^G_{\mathcal{L}(M)})\) modulo \[(5.2.16.3)\] defines the homomorphism
\[(5.2.16.4) \quad q^L_M: \mathbb{S}(a^G_{\mathcal{L}(M)}) \rightarrow \mathbb{S}(a^G_{\mathcal{L}(L)}).
\]The map \(q^G_M\) is the quotient modulo the augmentation ideal onto the graded component \((\mathbb{S}(a^G_{\mathcal{L}(M)})\big)_G\), which is a copy of \(\mathbb{R}\).

\[\textbf{(5.2.17)} \quad \textbf{Definition} \quad \text{Let } M \text{ be a Levi subgroup in } \mathcal{L}, \text{ let } X \text{ be an element of } \mathfrak{m}(Q_S). \text{ Define the total orbital integrohedron } T^G_M(X, ) \text{ to be the } \mathbb{S}(a^G_{\mathcal{L}(M)}) \otimes \mathbb{C}\text{-valued distribution on } \mathfrak{g}(Q_S) \text{ such that}
\]
\[(5.2.17.1) \quad \forall f_S \in \mathbb{S}(\mathfrak{g}(Q_S)) \quad T^G_M(X, f_S) = \left((-1)^{\dim(a_L \otimes A^G)} W^L_0 \| W^G_L \|^{-1} \mathcal{E}^G_L(X^L, f_S)\right)_{L \in \mathcal{L}^G(M)}
\]For each Levi subgroup \(M_S\) in \(\mathcal{L}^G\) define \(T^G_{M_S}(X, )\) by the analogous formula.

\[\textbf{(5.2.18)} \quad \textbf{Lemma} \quad \text{(Induction and splitting of orbital integrohedra)}
\]Let \(M\) be a Levi subgroup in \(\mathcal{L}\), let \(X\) be an element of \(\mathfrak{m}(Q_S)\), let \(f_S\) be a Schwartz function on \(\mathfrak{g}(Q_S)\).

- Let \(L\) be a Levi subgroup in \(\mathcal{L}(M)\), then
  \[(5.2.18.1) \quad T^G_L(X^L, f_S) = q^L_M\left(T^G_M(X, f_S)\right).
\]
- Let \(S\) be the set \(\{v_1, v_2\}\), let \(f_S\) be of the form \(f_{v_1} \otimes f_{v_2}\) where \(f_{v_i}\) is a Schwartz function on \(\mathfrak{g}(Q_{v_i})\) where the index \(i\) is 1 or 2, then
  \[(5.2.18.2) \quad T^G_M(X, f_{v_1} \otimes f_{v_2}) = T^G_M(X, f_{v_1}) \otimes T^G_M(X, f_{v_2}).
\]Local identities analogous to \[(5.2.18.1)\] also hold for \(G_v\).
The splitting identity (5.2.18.2) follows by comparing the Hadwiger invariants of the two sides of (5.2.18.2). Retain the notations of (5.2.11.1). Let \( L \) be a Levi subgroup in \( \mathcal{L}(M) \), let \( \Phi \) be a strict flag in \( \mathcal{F}_x(a_1^G) \). It suffices to show that the \( \Phi \)-components of the Hadwiger invariants of the \( L \)-components of the two sides of (5.2.18.2) are equal.

The \( L \)-component of the right hand side of (5.2.18.2) is

\[
(5.2.19.1) \sum_{L_1, L_2 \in \mathcal{L}(L)} (-1)^{\dim(L_1/L_2)} |W_0^{L_1}||W_0^{L_2}||W_0^G|^{-1} \eta^G_{L_1}(X^{L_1}, f_{v_1}) \otimes \eta^G_{L_2}(X^{L_2}, f_{v_2}),
\]

whose Hadwiger invariant has \( \Phi \)-component equal to

\[
(5.2.19.2) \sum_{L_1, L_2 \in \mathcal{L}(L)} (-1)^{\dim(L_1/L_2)} |W_0^{L_1}||W_0^{L_2}||W_0^G|^{-1} d^L_{L_1}(L_1, L_2) \times
\]
\[
\left( \sum_{r_1 \in \text{Rig}(\Phi_1)} \text{sign}(r_1) I^L_{1L_1}(X^{L_1}, f_{v_1}, Q^L_1) \right) \left( \sum_{r_2 \in \text{Rig}(\Phi_2)} \text{sign}(r_2) I^L_{1L_2}(X^{L_2}, f_{v_2}, Q^L_2) \right)
\]

where \( \Phi_i \) denotes the strict flag in \( a_i^G \) induced by \( \Phi \), and \( \Phi_i, r_i, L_i \) and \( Q^L_i \) are related as in (5.2.10.1), (5.2.10.2) and (5.2.10.3) where the index \( i \) is 1 or 2. Combine \( r_1 \) and \( r_2 \) into a rigging \( r \) of \( \Phi \), the expression (5.2.19.2) becomes

\[
(5.2.19.3) \sum_{L_1, L_2 \in \mathcal{L}(L)} (-1)^{\dim(L_1/L_2)} |W_0^{L_1}||W_0^{L_2}||W_0^G|^{-1} \sum_{Q^L_1 \in \mathcal{P}(L_1)} \sum_{Q^L_2 \in \mathcal{P}(L_2)} \text{sign}(r) \times
\]
\[
\times d^L_{L_1}(L_1, L_2) I^L_{1L_1}(X^{L_1}, f_{v_1}, Q^L_1) I^L_{1L_2}(X^{L_2}, f_{v_2}, Q^L_2)
\]

\[
= \sum_{L_1, L_2 \in \mathcal{L}(L)} (-1)^{\dim(L_1/L_2)} |W_0^{L_1}||W_0^{L_2}||W_0^G|^{-1} \sum_{Q^L_1 \in \mathcal{P}(L_1)} \sum_{Q^L_2 \in \mathcal{P}(L_2)} \text{sign}(r) \times
\]
\[
\times d^L_{L_1}(L_1, L_2) \sum_{L_0 \in \mathcal{L}(L_0)} d^L_{L_1}(L_1, L_0) I^L_{L_1}(X^{L_1}, f_{v_1}, Q^L_1) \sum_{L_0 \in \mathcal{L}(L_0)} d^L_{L_2}(L_2, L_0) I^L_{L_2}(X^{L_2}, f_{v_2}, Q^L_2)
\]

by (5.1.13.2), where \( Q^L_i \circ \) is the second component of \( s_{Q^L_i}(L_i, L_0^L) \) for the choice of a collection of partial maps \( (s_{Q^L_i})^L_i \) such that

\[
(5.2.19.4) \forall Q^L_i \in \mathcal{P}(L^L_i) \quad Q_i^L \subset Q^L_i
\]

where the index \( i \) is 1 or 2. The choice of such a collection is equivalent to the choice of a collection of vectors \( (\xi_{Q^L_i})^L_i \) such that

\[
(5.2.19.5) \forall Q^L_i \in \mathcal{P}(L^L_i) \quad \xi_{Q^L_i} \in a^L_i
\]

in general position and positive with respect to \( Q^L_i \). By definition

\[
(5.2.19.6) d^L_{L_1}(L_1, L_2) d^L_{L_1}(L_1, L_0^L) d^L_{L_2}(L_2, L_0^L) = d^L_{L_1}(L_0^L, L_0^L),
\]
hence the right hand side of (5.2.19.3) is equal to

\[(5.2.19.7)\]

\[
(-1)^{\dim(A_L/A_G)} \sum_{r \in R_{L^0}(\Phi)} \text{sign}(r) \times \sum_{L_1, L_2 \in \mathcal{L}(L)} \sum_{L_1^0 \in \mathcal{L}^{L_1}_L(L)} \sum_{L_2^0 \in \mathcal{L}^{L_2}_L(L)} |W_0^{L_1}| |W_0^{G_{L_1}}|^{-1} |W_0^{L_2}| |W_0^{G_{L_2}}|^{-1} \times
\]

\[
\times d_{L_1^0}^L(L_1, L_2^0) I_{L_1}^L(X^L, f_{v_1,Q_1^0}) I_{L_2}^L(X^L, f_{v_2,Q_2^0})
\]

where \((Q_1^{L_0}, Q_2^{L_0})\) is the image of \((L_1^0, L_2^0)\) under the partial map \(s_{Q_1^L, Q_2^L}\) determined by the vector

\[(5.2.19.8)\]

\[
\frac{\xi_{Q_1^L}}{2} - \frac{\xi_{Q_2^L}}{2} \in a_{L_1^0}^L = a_{L_1^0}^L \oplus a_{L_2^0}^L.
\]

For each fixed \(L\) and fixed \(\Phi\) with minimal term \(L^l\), the map

\[(5.2.19.9)\]

\[
\left\{(L_1, L_2, L_1^0, L_2^0) : L_1, L_2, L_1^0, L_2^0 \in \mathcal{L}(L), L_1^0 \in \mathcal{L}^{L_1^0}(L), L_2^0 \in \mathcal{L}^{L_2^0}(L) \right\}
\]

\[\rightarrow \left\{(L_1^0, L_2^0) : L_1^0 \in \mathcal{L}^{L_1^0}(L), L_2^0 \in \mathcal{L}^{L_2^0}(L) \right\}
\]

defined by

\[(5.2.19.10)\]

\[
(L_1, L_2, L_1^0, L_2^0) \mapsto (L_1^0, L_2^0)
\]

is surjective. Fix Levi subgroups \(L_1\) and \(L_2\) in \(\mathcal{L}(L)\) and parabolic subgroups \(P_1 \in \mathcal{P}(L_1)\) and \(P_2 \in \mathcal{P}(L_2)\), and let \(P_{1,2}\) be the parabolic subgroup in \(\mathcal{P}(L)\) whose unipotent radical \(N_{P_{1,2}}\) contains \(N_{P_1}\) and \(N_{P_2}\), then the map \((5.2.19.10)\) restricts to a bijection between Levi subgroups \(L_1^0\) and \(L_2^0\) that are standard with respect to \(P_1\) and \(P_2\) or \(P_{1,2}\). Hence each fiber of the map \((5.2.19.10)\) containing the pair \((L_1, L_2)\) has cardinality

\[(5.2.19.11)\]

\[
\frac{|W_0^G|/|W_0^L|}{(|W_0^{G_{L_1}}|/|W_0^{L_1}|)(|W_0^{G_{L_2}}|/|W_0^{L_2}|)},
\]

hence \((5.2.19.7)\) is equal to

\[(5.2.19.12)\]

\[
(-1)^{\dim(A_L/A_G)} |W_0^L|^{|W_0^G|^{-1}} \sum_{r \in R_{L^0}(\Phi)} \text{sign}(r) \times \sum_{L_1^0, L_2^0 \in \mathcal{L}^{L^l}(L)} \sum_{L_1^0 \in \mathcal{L}^{L_1^0}(L)} \sum_{L_2^0 \in \mathcal{L}^{L_2^0}(L)} |W_0^{L_1}| |W_0^{G_{L_1}}|^{-1} |W_0^{L_2}| |W_0^{G_{L_2}}|^{-1} \times
\]

\[
\times d_{L_1^0}^L(L_1, L_2^0) I_{L_1}^L(X^L, f_{v_1,Q_1^0}) I_{L_2}^L(X^L, f_{v_2,Q_2^0})
\]

\[= (-1)^{\dim(A_L/A_G)} |W_0^L|^{|W_0^G|^{-1}} \sum_{r \in R_{L^0}(\Phi)} \text{sign}(r) I_{L^l}(X^L, f_{v_1,Q^l_1} \oplus f_{v_2,Q^l_2})\]

by \((5.1.13.3)\) where \(\Phi, r, L^l\) and \(Q^l\) are related as in \((5.2.10.1)\), \((5.2.10.2)\) and \((5.2.10.3)\), which is equal to the \(\Phi\)-component of the Hadwiger invariant of the \(L\)-component of the left hand side of \((5.2.18.2)\).
5.3 The global and local trace formulae for orbital integrohedra

(5.3.1) Definition Let $M$ be a Levi subgroup in $L$. Let $\mathcal{M}(\mathfrak{a}_M^G)$ denote the vector space of translation invariant complex measures on $\mathfrak{a}_M^G$. Define the total measure coalgebra of $\mathfrak{a}_M^G$, denoted by $\mathcal{M}(\mathfrak{a}_M^G)$, to be the direct sum

\begin{equation}
\mathcal{M}(\mathfrak{a}_M^G) = \bigoplus_{L_1,L_2 \in \mathcal{L}(G)} \mathcal{M}(\mathfrak{a}_{L_1}^{L_2}).
\end{equation}

Define a bilinear coproduct $\Delta$ on $\mathcal{M}(\mathfrak{a}_M^G)$ by

\begin{equation}
\forall L_1,L_2 \in \mathcal{L}(G) \forall \mu \in \mathcal{M}(\mathfrak{a}_{L_1}^{L_2}) \quad \Delta(\mu) = \sum_{L' \in \mathcal{L}(L_1)} j_*(\mu)
\end{equation}

where $j_*$ is the pushforward

\begin{equation}
j_* : \mathcal{M}(\mathfrak{a}_{L_1}^{L_2}) \longrightarrow \mathcal{M}(\mathfrak{a}_{L_1}^{L_3} \otimes \mathfrak{a}_{L_1}^{L_2}) = \mathcal{M}(\mathfrak{a}_{L_1}^{L_3}) \otimes \mathcal{M}(\mathfrak{a}_{L_1}^{L_2})
\end{equation}
on measures induced by the natural isomorphism $j$ from $\mathfrak{a}_L^G$ to the direct sum of $\mathfrak{a}_{L_1}^{L_3}$ and $\mathfrak{a}_{L_1}^{L_2}$.

Define the cap product

\begin{equation}
\sim : \mathcal{M}(\mathfrak{a}_M^G) \otimes \mathcal{S}(\mathfrak{a}_M^G) \longrightarrow \mathcal{M}(\mathfrak{a}_M^G)
\end{equation}
to be the composition

\begin{equation}
\mathcal{M}(\mathfrak{a}_M^G) \otimes \mathcal{S}(\mathfrak{a}_M^G) \xrightarrow{\Delta \otimes 1} \mathcal{M}(\mathfrak{a}_M^G) \otimes \mathcal{M}(\mathfrak{a}_M^G) \otimes \mathcal{S}(\mathfrak{a}_M^G) \xrightarrow{1 \otimes (, , )} \mathcal{M}(\mathfrak{a}_M^G)
\end{equation}
where $(, , )$ denotes the bilinear pairing between $\mathcal{M}(\mathfrak{a}_M^G)$ and $\mathcal{S}(\mathfrak{a}_M^G)$ such that

\begin{equation}
\forall L_1,L_2,L_3 \in \mathcal{L}(G) \forall \mu_L \in \mathcal{M}(\mathfrak{a}_{L_1}^{L_3}) \quad \langle \mu_{L_1}, Y_{L_3} \rangle = \begin{cases} 
const & \text{the volume of the convex hull of } Y_{L_3} \\
& \text{with respect to the complex measure } \mu_{L_1} \text{ if } L_1 = L_3 \\
& \text{and } L_2 = G, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

Define a linear endomorphism $b_M$ on $\mathcal{M}(\mathfrak{a}_M^G)$ by

\begin{equation}
\forall L_1,L_2 \in \mathcal{L}(G) \forall \mu \in \mathcal{M}(\mathfrak{a}_{L_1}^{L_2}) \quad \mu^{b_M} = \sum_{L' \in \mathcal{L}(G)} \frac{|W_{L_1}^L||W_{L_2}^L|}{|W_0^L||W_0^L|} j_{\ast}^{-1}(\mu)
\end{equation}

where $j_*$ is the pushforward

\begin{equation}
j_* : \mathcal{M}(\mathfrak{a}_M^G) \longrightarrow \mathcal{M}(\mathfrak{a}_{L_1}^{L_2})
\end{equation}
on measures induced by the natural isomorphism $j$ from $\mathfrak{a}_M^G$ to $\mathfrak{a}_{L_1}^{L_2}$. Denote $b_{M_0}$ by $b_0$.

Denote by $b^\ast$ the bilinear product between $\mathcal{M}(\mathfrak{a}_M^G)$ and $\mathcal{S}(\mathfrak{a}_M^G)$ defined by the composition

\begin{equation}
\mathcal{M}(\mathfrak{a}_M^G) \otimes \mathcal{S}(\mathfrak{a}_M^G) \xrightarrow{\sim} \mathcal{M}(\mathfrak{a}_M^G) \xrightarrow{b_M} \mathcal{M}(\mathfrak{a}_M^G).
\end{equation}
(5.3.2) **Remark** Each Levi subgroup $L$ in $\mathcal{L}^G(M)$ defines a subcoalgebra

$$j^L_M: M(\mathfrak{a}^G_{\mathcal{L}(L)}) \rightarrow M(\mathfrak{a}^G_{\mathcal{L}(M)}).$$

The maps $j^L_M$ and $q^L_M$ form a pair of adjoint linear transformations with respect to the bilinear pairings $\langle \, , \rangle$ between $M(\mathfrak{a}^G_{\mathcal{L}(M)})$ and $S(\mathfrak{a}^G_{\mathcal{L}(M)})$ and between $M(\mathfrak{a}^G_{\mathcal{L}(L)})$ and $S(\mathfrak{a}^G_{\mathcal{L}(L)})$.

(5.3.3) **Remark** Let $\overline{\mathfrak{z}}^G$ be the linear endomorphism on $M(\mathfrak{a}^G_{\mathcal{L}(M)})$ defined by

$$\forall L_1, L_2 \in \mathcal{L}^G(M) \forall \mu \in M(\mathfrak{a}^G_{\mathcal{L}(L_1)}) \quad \mu^{\overline{\mathfrak{z}}^G} = \sum_{L' \in \mathcal{L}(L_1)} j_*(\mu)$$

where $j_*$ is the pushforward

$$j_*: M(\mathfrak{a}^G_{\mathcal{L}(L_1)}) \rightarrow M(\mathfrak{a}^G_{\mathcal{L}(L_2)})$$

on measures induced by the natural isomorphism $j$ from $\mathfrak{a}^G_{\mathcal{L}(L_1)}$ to $\mathfrak{a}^G_{\mathcal{L}(L_2)}$. Let $\overline{\rho}^G$ be the bilinear product between $M(\mathfrak{a}^G_{\mathcal{L}(M)})$ and $S(\mathfrak{a}^G_{\mathcal{L}(M)})$ defined by the composition

$$M(\mathfrak{a}^G_{\mathcal{L}(M)}) \otimes S(\mathfrak{a}^G_{\mathcal{L}(M)}) \xrightarrow{\overline{\rho}^G} M(\mathfrak{a}^G_{\mathcal{L}(M)}) \xrightarrow{\hat{\rho}^G} M(\mathfrak{a}^G_{\mathcal{L}(M)}).$$

Then the $\overline{\rho}^G$-cap product defines the structure of a right $S(\mathfrak{a}^G_{\mathcal{L}(M)})$-module on $M(\mathfrak{a}^G_{\mathcal{L}(M)})$, i.e.

$$\forall [Y], [Z] \in S(\mathfrak{a}^G_{\mathcal{L}(M)}) \forall \mu \in M(\mathfrak{a}^G_{\mathcal{L}(M)}) \quad \left([Y] \cap [Z]\right) \overline{\rho}^G \mu = [Y] \overline{\rho}^G \left([Z] \overline{\rho}^G \mu\right).$$

(5.3.4) **Definition** Let $M$ be a Levi subgroup in $\mathcal{L}$, let $X$ be an element of $\mathfrak{m}(Q)_{M,S}$. Define a complex measure $\mathfrak{a}^G_M(S, X)$ on $\mathfrak{a}^G_M$ by

$$\mathfrak{a}^G_M(S, X) = (-1)^{\dim(A_M/A_G)} a^M(S, X) \mu^G_M$$

where $a^M(S, X)$ is the constant defined as in (5.3.16.12) and $\mu^G_M$ is the translation invariant measure on $\mathfrak{a}^G_M$ which assigns the coweight lattice covolume one.

(5.3.5) **Theorem** (Global trace formula for orbital integrohedra)

Let $f$ be a Schwartzs function on $g(A)$, then

$$\lim_{S} \sum_{s \in g(Q)/\sim} \sum_{M \in \mathcal{L}} \sum_{X \in (M(Q)/\sigma)_{M,S}} \mathfrak{a}^G_M(S, X) b^s j^M_M(X, f_{S,M}) = \lim_{S} \sum_{s \in g(Q)/\sim} \sum_{M \in \mathcal{L}} \sum_{X \in (M(Q)/\sigma)_{M,S}} \mathfrak{a}^G_M(S, X) b^s \mathcal{I}^G_M(X, f^s).$$
(5.3.6) Proof The right hand side of (5.3.5.1) is equal to

\[(5.3.6.1) \quad \lim_S \sum_{\phi \in \mathcal{Q}/\sim} \sum_{M \in \mathcal{L}} \sum_{X \in (\mathcal{M}(Q) \cap \phi)_{M,S}} \left( \sum_{L \in \mathcal{L}(M)} (\sum (-1)^{\dim(A_M/A_G)} a^M(S, X) \mu^L_{M} \times \right.
\]
\[ \times \left( \mu^G_{L}, (-1)^{\dim(A_L/A_G)} |W_0^L| |W_0^G|^{-1} \right) \right)_{b_0}
\]
\[ = \lim_S \sum_{\phi \in \mathcal{Q}/\sim} \sum_{M \in \mathcal{L}} \sum_{X \in (\mathcal{M}(Q) \cap \phi)_{M,S}} \sum_{L \in \mathcal{L}(M)} (\sum (-1)^{\dim(A_M/A_L)} |W_0^L| |W_0^G|^{-1} \times \right.
\]
\[ \times \left( a^M(S, X) I^G_L(X, f^S) \right) \times \left( \mu^G_{M} \right)_{b_0}
\]
\[ = \lim_S \sum_{\phi \in \mathcal{Q}/\sim} \sum_{M \in \mathcal{L}} \sum_{X \in (\mathcal{M}(Q) \cap \phi)_{M,S}} \sum_{L \in \mathcal{L}(M)} \left( (-1)^{\dim(A_M/A_L)} |W_0^L| |W_0^G|^{-1} a^M(S, X) \times \right.
\]
\[ \times \left( \sum_{L^0 \in \mathcal{L}(M_0)} \frac{|W_0^M| |W_0^G|}{|W_0^L| |W_0^G|} \left( \sum_{L \in \mathcal{L}(M)} \frac{|W_0^M| |W_0^G|}{|W_0^L| |W_0^G|} \right) \right) \]}

where the equality (5.3.6.2) follows from (5.113.2) and \(Q'\) is the second component of \(s(L, L')\) for the choice of a partial map \(s\). By the identity (7.1) in [Ar88b] which states that

\[(5.3.6.4) \quad \forall M' \in \mathcal{L}_2^M \quad d^G_M(L_1, L_2) = \sum_{L' \in \mathcal{L}(M')} d^G_M(M', L_1) d^G_M(L', L_2),
\]

the right hand side of (5.3.6.3) is equal to

\[(5.3.6.5) \quad \sum_{L' \in \mathcal{L}} \left( \lim_S \sum_{\phi \in \mathcal{Q}/\sim} \sum_{M \in \mathcal{L}'} \sum_{X \in (\mathcal{M}(Q) \cap \phi)_{M,S}} \sum_{L^0 \in \mathcal{L}(M_0)} \left( (-1)^{\dim(A_M/A_G)} |W_0^M| |W_0^G|^{-1} a^M(S, X) I^G_L(X, f^S) \times \right.
\]
\[ \times \left( \sum_{L^0 \in \mathcal{L}(M_0)} \frac{|W_0^M| |W_0^G|}{|W_0^L| |W_0^G|} \left( \sum_{L \in \mathcal{L}(M)} \frac{|W_0^M| |W_0^G|}{|W_0^L| |W_0^G|} \right) \right) \right)
\]
\[ = \lim_S \sum_{\phi \in \mathcal{Q}/\sim} \sum_{L' \in \mathcal{L}} \sum_{X \in (\mathcal{M}(Q) \cap \phi)_{L', S}} \left( (-1)^{\dim(A_M/A_G)} a^L(S, X) I^L_L(X, f_{S, L'}) \right) \times \left( \mu^G_{L'} \right)_{b_0}
\]

which is equal to the left hand side of (5.3.5.1).  \(\square\)
(5.3.7) **Definition** Let \( \sigma \) be a \( \sim \) equivalence class on \( \mathfrak{g}(Q) \). Let \( L \) be a Levi subgroup in \( \mathcal{L} \). By (5.1.8.2), if \( X \) and \( Y \) are elements in \( I(Q) \cap \sigma \) that are \( \equiv \) equivalent in \( I(Q) \), then \( X^G \cap \sigma \) and \( Y^G \cap \sigma \) are \( \equiv \) equivalent in \( \mathfrak{g}(Q) \), hence parabolic induction defines a map

\[
(\_)^G : (I(Q) \cap \sigma)_{L,S} \to (\mathfrak{g}(Q) \cap \sigma)_{G,S}.
\]

Let \( X \) be a semisimple element in \( \sigma \). A pair \((M,\nu)\) where \( M \) is a Levi subgroup in \( \mathcal{L} \) and \( \nu \) is a nilpotent conjugacy class in \( m_X(Q_S) \) is said to be a **rigid nilpotent orbit** of \( X \) if \( \nu \) is not contained in the class parabolically induced from some nilpotent conjugacy class \( \nu' \) in \( m'_X(Q_S) \) where \( M' \) is a Levi subgroup in \( \mathcal{L}^M \). Denote by \( N_{rig,S}(X) \) the collection of \( G_X(Q_S) \)-conjugacy classes of rigid nilpotent orbits of \( X \).

(5.3.8) **Remark** Assume that \( S \) does not contain the archimedean place. By the theory of Shalika germs, see \S 17 of [Ko05], for a fixed Schwartz function \( f_S \) the orbital integral \( I^G_X(X,f_S) \) is locally constant on \( \mathfrak{g}_{reg,ss}(Q_S) \) as a function in \( X \). If \( X_0 \) is a singular point, there exists a finite collection of germs \( \Gamma(X,\nu) \) indexed by the rigid classes \((M,\nu)\) in \( N_{rig,S}(X_0) \) such that

\[
I^G_X(X,f_S) = \sum_{(M,\nu)\in N_{rig,S}(X_0)} c_{M,\nu} \Gamma(X,\nu)
\]

for all \( X \) sufficiently close to \( X_0 \). The coefficients are equal to the orbital integrals, i.e.

\[
c_{M,\nu} = I^G_X((X_0 + \nu)^G,f_S).
\]

Similar remarks apply to the orbital integrohedra \( T^G_M(X,f_S) \). See \S 9 of [Ar88a] and Proposition VI.4 of [Wa95].

(5.3.9) **Definition** Define the total measure \( \alpha^G_M(S,X) \) to be the element of \( M(\mathfrak{a}^G_M) \) by

\[
\alpha^G_M(S,X) = \left( a^G_L(S,X^L) \right)_{L \in \mathcal{L}^G(M)}
\]

where \( X^L \) is the element of \( I(Q)_{L,S} \) defined by the induced class \( X^L \).

(5.3.10) **Corollary** (Global trace formula for orbital integrohedra II)

*Let \( f \) be a Schwartz function on \( \mathfrak{g}(\mathbb{A}) \), then*

\[
\lim_S \sum_{X \in \mathfrak{b}_{ss}(Q)/ad(G(Q))} \sum_{(M,\nu)\in N_{rig,S}(X)} \alpha^G_M(S,X + \nu) \overset{\text{by}}{\to} I^M_M(X + \nu, f_{S,M})
\]

\[
= \lim_S \sum_{X \in \mathfrak{b}_{ss}(Q)/ad(G(Q))} \sum_{(M,\nu)\in N_{rig,S}(X)} \alpha^G_M(S,X + \nu) \overset{\text{by}}{\to} T^G_M(X + \nu, f^S).
\]

(5.3.11) **Proof** The left hand side of (5.3.10.1) is equal to

\[
\lim_S \sum_{X \in \mathfrak{b}_{ss}(Q)/ad(G(Q))} \sum_{(M,\nu)\in N_{rig,S}(X)} \sum_{L \in \mathcal{L}(M)} a^G_L(S, (X + \nu)^L) \overset{\text{by}}{\to} I^M_M(X + \nu, f_{S,M})
\]

\[
= \lim_S \sum_{X \in \mathfrak{b}_{ss}(Q)/ad(G(Q))} \sum_{(M,\nu)\in N_{rig,S}(X)} \sum_{L \in \mathcal{L}(M)} a^G_L(S, (X + \nu)^L) \overset{\text{by}}{\to} I^L_L((X + \nu)^L, f_{S,L})
\]

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by \((5.1.15.1)\). Since each \(\equiv\) equivalence class in \((l(Q) \cap o)_{L,S}\) is induced from a unique rigid class in \(N_{rig,S}(X)\) for a semisimple element \(X\) in \(g_{ss}(Q)\) which is uniquely determined modulo conjugacy by \(G(Q)\), the right hand side of \((5.3.11.1)\) is equal to

\[
(5.3.11.2) \quad \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{L \subseteq \mathcal{L}} \sum_{Y \in ((l(Q) \cap o)_{L,S})} a^G_{L}(S, Y) \overset{\text{by}}{\sim} I^G_{L}(Y, f_S, L)
\]

\[
= \lim_S \sum_{\phi \in \mathfrak{g}(Q)/\sim} \sum_{L \subseteq \mathcal{L}} \sum_{Y \in ((l(Q) \cap o)_{L,S})} a^G_{L}(S, Y) \overset{\text{by}}{\sim} I^G_{L}(Y, f^* S)
\]

by the identity \((5.3.5.1)\). The right hand side of \((5.3.11.2)\) is equal to

\[
(5.3.11.3) \quad \lim_S \sum_{X \in g_{ss}(Q)/\text{ad}(G(Q))} \sum_{(M, \nu) \in N_{rig,S}(X)} \sum_{L \subseteq \mathcal{L}(M)} a^G_{L}(S, (X + \nu)^L) \overset{\text{by}}{\sim} I^G_{L}(X + \nu, f^* S)
\]

\[
= \lim_S \sum_{X \in g_{ss}(Q)/\text{ad}(G(Q))} \sum_{(M, \nu) \in N_{rig,S}(X)} \sum_{L \subseteq \mathcal{L}(M)} a^G_{M}(S, X + \nu) \overset{\text{by}}{\sim} I^G_{M}(X + \nu, f^* S)
\]

which is the right hand side of \((5.3.10.1)\). The equality \((5.3.11.4)\) follows from \((5.2.18.1)\).

(5.3.12) **Proposition** (Local trace formula for orbital integrohedra)

Let \(v\) be a place of \(Q\), let \(f_v\) and \(g_v\) be Schwartz functions on \(g_v(Q_v)\), then

\[
(5.3.12.1) \quad \sum_{M_v \in \mathcal{L}^{G_v}} |W^{M_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} \sum_{T_v \in \mathcal{T}_{\text{all}}(M_v)} |W(M_v, T_v)|^{-1} \int_{l_v(Q_v)} I^G_{M_v}(X_v, f_v) I^{G_v}_{G_v}(X_v, g_v) dX_v
\]

\[
= \sum_{M_v \in \mathcal{L}^{G_v}} |W^{M_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} \sum_{T_v \in \mathcal{T}_{\text{all}}(M_v)} |W(M_v, T_v)|^{-1} \int_{l_v(Q_v)} I^G_{M_v}(X_v, g_v) I^{G_v}_{G_v}(X_v, f_v) dX_v.
\]

(5.3.13) **Proof** Retain the notations of \((5.2.11.1)\). Let \(L_v\) be a Levi subgroup in \(\mathcal{L}(M_v)\), let \(\Phi\) be a strict flag in \(F_s(a^G_{L_v})\). It suffices to show that the \(\Phi\)-components of the Hadwiger invariants of the \(L_v\)-component of the two sides of \((5.3.12.1)\) are equal.

The \(L_v\)-component of the left hand side of \((5.3.12.1)\) is

\[
(5.3.13.1) \quad \sum_{M_v \in \mathcal{L}^{L_v}} |W^{M_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} \sum_{T_v \in \mathcal{T}_{\text{all}}(M_v)} |W(M_v, T_v)|^{-1} \times
\]

\[
\int_{l_v(Q_v)} (-1)^{\dim(A_{L_v}/A_{G_v})} |W^{L_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} I^{L_v}_{L_v}(X_v, f_v) I^{G_v}_{G_v}(X_v, g_v) dX_v,
\]

whose Hadwiger invariant has \(\Phi\)-component equal to

\[
(5.3.13.2) \quad \sum_{M_v \in \mathcal{L}^{L_v}} |W^{M_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} \sum_{T_v \in \mathcal{T}_{\text{all}}(M_v)} |W(M_v, T_v)|^{-1} (-1)^{\dim(A_{L_v}/A_{G_v})} |W^{L_v}_{v,0}| |W^{G_v}_{v,0}|^{-1} \times
\]

\[
\int_{l_v(Q_v)} \sum_{r \in \mathcal{R}_{ig}(\Phi)} \text{sign}(r) I^{L_v}_{L_v}(X_v, (f_v)^{-1} Q_v) I^{G_v}_{G_v}(X_v, g_v) dX_v
\]

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such that

\[ \text{For a regular semisimple element} \]

where \( \Phi, r, L^\ell, \text{and } Q^t \) are related as in (5.2.10.1), (5.2.10.2) and (5.2.10.3). By (5.1.13.2) the right hand side of (5.3.13.2) is equal to

\[
(5.3.13.3) \sum_{r \in \mathcal{R}ig(\Phi)} \text{sign}(r)(-1)^{\dim(A_{L^\ell}/A_{G_v})} \left| W_{v,0}^{L^\ell} \right| \left| W_{v,0}^{G_v} \right|^{-1} \times \\
\times \sum_{M_v \in L^{L^\ell}} \left| W_{v,0}^{M_v} \right| \left| W_{v,0}^{G_v} \right|^{-1} \sum_{T_v \in T_{\text{fill}}(M_v)} \left| W(M_v, T_v) \right|^{-1} \times \\
\times \int_{t_v(Q_v)} (-1)^{\dim(A_{L^\ell}/A_{L_v^\ell})} I_{L_v^\ell}^L (X_v, (f_v^{-})Q_v^t) I_{G_v^t}^G (X_v, g_v) \, dX_v
\]

where \( Q^t \) is the second component of \( s_{Q_v^t} (L_v, L_v^0) \) for the choice of a collection of partial maps \( s_{Q_v^t} \) such that

\[
(5.3.13.4) \quad \forall Q^t_v \in \mathcal{P}(L_v^\ell) \quad Q^t_v \subset Q^t_v.
\]

For a regular semisimple element \( X_v \) in \( \mathcal{C}_v(Q_v) \), the induced orbit \( X_v^{G_v} \) is equal to \( X_v \cdot \text{ad}(G_v(Q_v)) \), hence by (5.1.15.1) the right hand side of (5.3.13.3) is equal to

\[
(5.3.13.5) \sum_{r \in \mathcal{R}ig(\Phi)} \text{sign}(r) \sum_{L^t_v \in L^{L^\ell}_v(M_v)} \left| W_{v,0}^{L^t_v} \right| \left| W_{v,0}^{G_v} \right|^{-2} \times \\
\times \sum_{M_v \in L^{L^t_v}} \left| W_{v,0}^{M_v} \right| \left| W_{v,0}^{L^t_v} \right|^{-1} \sum_{T_v \in T_{\text{fill}}(M_v)} \left| W(M_v, T_v) \right|^{-1} \times \\
\times \int_{t_v(Q_v)} (-1)^{\dim(A_{M_v}/A_{L^t_v})} I_{M_v^t}^L (X_v, (f_v^{-})Q_v^t) I_{G_v^t}^L (X_v, g_v) \, dX_v
\]

\[
= \sum_{r \in \mathcal{R}ig(\Phi)} \text{sign}(r) \sum_{L^t_v \in L^{L^\ell}_v(M_v)} \left| W_{v,0}^{L^t_v} \right| \left| W_{v,0}^{G_v} \right|^{-2} \times \\
\times \sum_{M_v \in L^{L^t_v}} \left| W_{v,0}^{M_v} \right| \left| W_{v,0}^{L^t_v} \right|^{-1} \sum_{T_v \in T_{\text{fill}}(M_v)} \left| W(M_v, T_v) \right|^{-1} \times \\
\times \int_{t_v(Q_v)} (-1)^{\dim(A_{M_v}/A_{L^t_v})} I_{M_v^t}^L (X_v, (f_v^{-})Q_v^t) I_{G_v^t}^L (X_v, g_v) \, dX_v
\]
\[
\times \sum_{M_v \in \mathcal{L}^0} |W_{v,0}^M|^{-1} \sum_{T_v \in \mathcal{T}(M_v)} |W(M_v, T_v)|^{-1} \times \\
\sum_{M_v \in \mathcal{L}^0} |W_{v,0}^L|^{-1} \sum_{T_v \in \mathcal{T}(M_v)} |W(M_v, T_v)|^{-1} \times \\
\int_{\lambda_v(Q_v)} (-1)^{\dim(\Lambda_{M_v}/\Lambda_{L_v})} I_{M_v}^{L_v}(X_v, (g_v, \Phi_v)^{-1}) I_{L_v}^{L_v}(X_v, f_v, \Phi_v) \, dX_v
\]

by (4.2.3.1). The right hand side of (5.3.13.5) is equal to

\[
(5.3.13.6) \quad \sum_{M_v \in \mathcal{L}^0} |W_{v,0}^M|^{-1} \sum_{T_v \in \mathcal{T}(M_v)} |W(M_v, T_v)|^{-1} (-1)^{\dim(\Lambda_{M_v}/\Lambda_{G_v})} |W_{v,0}^L|^{-1} |W_{v,0}^G|^{-1} \times \\
\sum_{T_v \in \mathcal{T}(M_v)} \sum_{r \in R(\Phi)} \text{sign}(r) I_{M_v}^{L_v}(X_v, (g_v, \Phi_v)^{-1}) I_{G_v}^{L_v}(X_v, f_v) \, dX_v,
\]

which is equal to the \(\Phi\)-component of the Hadwiger invariant of the \(L_v\)-component of the right hand side of (5.3.12.1).

\section{The Harish-Chandra transform on the space of characteristic polynomials}

In this chapter an integral transform on the space of characteristic polynomials satisfying a summation formula of Poisson type is constructed.

\subsection{Notations and definitions}

\begin{enumerate}
\item In this chapter \(G\) denotes the general linear group \(\text{GL}(n, \mathbb{Q})\) for some natural number \(n\), with the standard choice of the minimal Levi subgroup \(M_0\) and the Borel subgroup \(B\) to be the subgroup consisting of the diagonal matrices and the upper triangular matrices respectively.
\item Let \(A_G\) denote the affine space of characteristic polynomials of \(n \times n\) matrices over \(\mathbb{Q}\), i.e.
\end{enumerate}

\begin{equation}
A_G = \text{gl}(n, \mathbb{Q})/\!\!/\text{GL}(n, \mathbb{Q})
\end{equation}

where \(\!\!/\) denotes the affine quotient and \(\text{GL}(n, \mathbb{Q})\) acts on \(\text{gl}(n, \mathbb{Q})\) from the right by conjugation. The discriminant function \(D\) on \(\text{gl}(n, \mathbb{Q})\) descends to a polynomial on \(A_G\), denote by \(A_{G,\text{reg}}\) the open subset where \(D\) does not vanish.

Let \(M\) be a standard Levi subgroup of \(G\). Denote by \(A_M\) the affine quotient of \(m\) by the adjoint action of \(M\), then \(A_M\) is an affine space and there exists a partition \((n_1, n_2, \ldots, n_r)\) of \(n\) such that

\begin{equation}
A_M = A_{G_1} \times A_{G_2} \times \cdots \times A_{G_r}.
\end{equation}

The embedding of \(m\) into \(g\) induces a map

\begin{equation}
\pi_M : A_M \to A_G
\end{equation}

which is finite of degree \(|W_0^M|^{-1}|W_0^G|\) and étale over \(A_{G,\text{reg}}\).
(6.1.3) Let \( v \) be a place of \( \mathbb{Q} \). Equip \( A_G(\mathbb{Q}_v) \) with the measure

\[
|D(X_v)|_v^{-1/2}dX_v
\]

where \( dX_v \) denotes the standard translation invariant measure on the affine space \( A_G(\mathbb{Q}_v) \). This is equal to the pushforward of the translation invariant measure on \( m_0(\mathbb{Q}_v) \) along the Chevalley morphism

(6.1.3.2) \[
\pi_{M_0} : m_0 = A_{M_0} \longrightarrow m_0//W^G_0 = A_G.
\]

The complement of \( A_{G,\text{reg}}(\mathbb{Q}_v) \) in \( A_G(\mathbb{Q}_v) \) is a null set.

For each standard Levi subgroup \( M \) of \( G \) equip \( A_M(\mathbb{Q}_v) \) with the product measure

(6.1.3.3) \[
\prod_{i=1}^r |D^{G_i}(X_i,v)|_v^{-1/2}dX_i,v
\]

where \( G_1, G_2, \ldots, G_r \) are related to \( M \) as in (6.1.2.1).

Denote by \( A_{M,\text{reg}}(\mathbb{Q}_v)_{\text{ell}} \) the subset of \( A_{M,\text{reg}}(\mathbb{Q}_v) \) consisting of the images of the \( \mathbb{Q}_v \)-elliptic elements in \( m(\mathbb{Q}_v) \). Then

(6.1.3.4) \[
A_{G,\text{reg}}(\mathbb{Q}_v) = \prod_{M \in \mathcal{L}} \pi_M(A_{M,\text{reg}}(\mathbb{Q}_v)_{\text{ell}}).
\]

This decomposition is compatible with the measures on \( A_G(\mathbb{Q}_v) \) and \( A_M(\mathbb{Q}_v) \).

The local measures induce \( S \)-local and global measures on \( A_G(\mathbb{Q}_S) \) and \( A_G(A) \).

(6.1.4) The arguments in this chapter only require a weaker version of the results from the previous chapter which does not involve scissors-congruence-valued orbital integrals:

Definition Let \( S \) be a finite set of places of \( \mathbb{Q} \), let \( M \) be a Levi subgroup in \( \mathcal{L} \), let \( X \) be an element of \( m(\mathbb{Q}_S) \). Define the vector-valued orbital integral \( I^G_{M}(X,\cdot) \) to be the vector-valued distribution on \( g(\mathbb{Q}_S) \) taking values in \( \bigoplus_{Q \in \mathcal{F}(M)} \mathbb{C} \) such that

(6.1.4.1) \[
\forall f_S \in \mathcal{S}(g(\mathbb{Q}_S)) \quad I^G_{M}(X,f_S) = \left(-1\right)^{\dim(A_M/A_{MQ})} I^{MQ}_{M}(X,f_{S,Q}) \quad \forall Q \in \mathcal{F}(M).
\]

Lemma (Induction and splitting) Let \( S \) be a finite set of places of \( \mathbb{Q} \), let \( M \) be a Levi subgroup in \( \mathcal{L} \), let \( X \) be an element of \( m(\mathbb{Q}_S) \), let \( f_S \) be a Schwartz function on \( g(\mathbb{Q}_S) \).

- If \( L \) is a Levi subgroup in \( \mathcal{L}(M) \), then \( I^G_{L}(X,f_S) \) is completely determined by \( I^G_{M}(X,f_S) \).
- If \( S \) is the set \( \{v_1,v_2,\ldots,v_r\} \) for some natural number \( r \) and

(6.1.4.2) \[
\forall i = 1, 2, \ldots, r \quad \exists f_{v_i} \in \mathcal{S}(g(\mathbb{Q}_{v_i})) \quad f_S = f_{v_1} \otimes f_{v_2} \otimes \cdots \otimes f_{v_r},
\]

then \( I^G_{M}(X,f_S) \) is completely determined by \( I^G_{M}(X_{v_i},f_{v_i}) \) where \( i \) ranges among \( 1, 2, \ldots, r \).
**Proposition** (Local trace formula)

Let $v$ be a place of $\mathbb{Q}$, let $f_v$ and $g_v$ be Schwartz functions on $\mathfrak{g}(\mathbb{Q}_v)$, then

\[
(6.1.4.3) \quad \sum_{M \in \mathcal{L}} |W(M)^{\ell}||W_0^G|^{-1} \sum_{T_v \in T_{\mathfrak{g}(M)}} |W(M, T_v)|^{-1} \int_{G_{\mathbb{Q}_v}} T_M^G(X_v, f_v) I^G_M(X_v, g_v) \, dX_v \\
= \sum_{M \in \mathcal{L}} |W(M)^{\ell}||W_0^G|^{-1} \sum_{T_v \in T_{\mathfrak{g}(M)}} |W(M, T_v)|^{-1} \int_{G_{\mathbb{Q}_v}} T_M^G(X_v, g_v) I^G_M(X_v, f_v) \, dX_v.
\]

**Proposition** (Global trace formula)

Let $f$ be a Schwartz function on $\mathfrak{g}(\mathbb{A})$, then

\[
(6.1.4.4) \quad \lim_S \sum_{\phi \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} \frac{|W(M)^{\ell}||W_0^G|^{-1}}{\phi(M)^2} \sum_{X \in (m(\mathbb{Q})\cap \mathfrak{a})_{M,S}} a_M^{G}(S, X) I_M^G(X, f_{S,M}) \\
= \lim_S \sum_{\phi \in \mathfrak{g}(\mathbb{Q})/\sim} \sum_{M \in \mathcal{L}} \frac{|W(M)^{\ell}||W_0^G|^{-1}}{\phi(M)^2} \sum_{X \in (m(\mathbb{Q})\cap \mathfrak{a})_{M,S}} a_M^{G}(S, X) \operatorname{Tr}\left(T_M^G(X, f^*)\right)
\]

where the coefficients $a_M^{G}(S, X)$ are defined by

\[
(6.1.4.5) \quad a_M^{G}(S, X) = (-1)^{\dim(M/A_G)} a_M(S, X)
\]

where $a_M(S, X)$ are defined as in (3.3.16.12), and $\operatorname{Tr}$ denotes the linear functional on $\bigoplus_{Q \in \mathcal{F}(M)} \mathbb{C}$ such that

\[
(6.1.4.6) \quad \forall (c_Q)_{Q \in \mathcal{F}(M)} \in \bigoplus_{Q \in \mathcal{F}(M)} \mathbb{C} \quad \operatorname{Tr}\left((c_Q)_{Q \in \mathcal{F}(M)}\right) = \sum_{Q \in \mathcal{F}(M)} |P(Q)|^{-1} c_Q.
\]

**Proof** See (5.1.13.2), (5.1.13.3), (4.2.7.1) and (4.2.5.1).

\[\square\]

### 6.2 The local and global Schwartz spaces

**Definition** Let $v$ be a place of $\mathbb{Q}$. Let $X_v$ be an element of $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v)$. Define the maximal orbital integral $I_{\text{max}}^G(X_v)$ to be the vector-valued distribution on $\mathfrak{g}(\mathbb{Q}_v)$ such that

\[
(6.2.1.1) \quad \forall f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v)) \quad I_{\text{max}}^G(X_v, f_v) = I_{\text{max}}^{G[\tilde{X}_v, \text{ell}]}(\tilde{X}_v, f_v)
\]

where $\tilde{X}_v$ is a regular semisimple element in $\mathfrak{g}(\mathbb{Q}_v)$ lifting $X_v$, and $M[\tilde{X}_v, \text{ell}]$ is a standard Levi subgroup of $G$ such that $m[\tilde{X}_v, \text{ell}](\mathbb{Q}_v)$ contains $X_v$ as a $\mathbb{Q}_v$-elliptic element.

**Definition** Let $v$ be a place of $\mathbb{Q}$. Define the local Schwartz space $\mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v))$ to be the space of complex-valued functions on $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v)$ such that

\[
(6.2.2.1) \quad \varphi_v \in \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \iff \exists f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v)) \forall X_v \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \quad \varphi_v(X_v) = I_G^G(X_v, f_v).
\]

Define the local Schwartz space $\mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v))$ to be the space of vector-valued functions on $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v)$ such that

\[
(6.2.2.2) \quad \varphi_v \in \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)) \iff \exists f_v \in \mathcal{S}(\mathfrak{g}(\mathbb{Q}_v)) \forall X_v \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \quad \varphi_v(X_v) = I_{\text{max}}^G(X_v, f_v).
\]

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(6.2.3) **Definition** Let $v$ be a place of $\mathbb{Q}$. If $v$ is $p$-adic, denote by $\Lambda_0$ the standard lattice

\[
\Lambda_0 = \text{gl}(n, \mathbb{Z}) / (\mathbb{Z}_p) \subset \text{gl}(n, \mathbb{Q}) / (\mathbb{Q}_p)
\]

and denote by $\mathbb{I}_{\Lambda_0}$ its characteristic function. If $v$ is archimedean, denote by $E$ the Gaussian function on $g(\mathbb{R})$ which is self-dual with respect to the unitary Fourier transform, i.e.

\[
\forall M \in g(\mathbb{R}) \quad E(M) = e^{-\pi \text{Tr}(M^T M)}
\]

where $M^T$ denotes the transpose of $M$ and $\text{Tr}$ denotes the trace of an $n \times n$ matrix.

Define the *basic function* $\phi_{0,v}$ to be the element of $\mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v))$ such that

\[
(6.2.3.2) \quad \forall X_v \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \quad \phi_{0,v}(X_v) = \begin{cases} 
I_v^G(X_v, \mathbb{I}_{\Lambda_0}) & \text{if } v \text{ is } p\text{-adic}, \\
I_v^G(X_v, E) & \text{if } v \text{ is archimedean}.
\end{cases}
\]

Define the *basic function* $\phi_{1,v}$ to be the element of $\mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v))$ such that

\[
(6.2.3.3) \quad \forall X_v \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \quad \phi_{1,v}(X_v) = \begin{cases} 
I_v^G(X_v, \mathbb{I}_{\Lambda_0}) & \text{if } v \text{ is } p\text{-adic}, \\
I_v^G(X_v, E) & \text{if } v \text{ is archimedean}.
\end{cases}
\]

(6.2.4) **Remark** If $v$ is a $p$-adic place, then a complex-valued function $\varphi_v$ on $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v)$ belongs to $\mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v))$ if and only if the following conditions are satisfied:

- $\varphi_v$ is a locally constant function on $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v)$;
- after extending by zero to $\mathcal{A}_G(\mathbb{Q}_v)$, $\varphi_v$ is compactly supported;
- for each singular point $Z_v$ in the complement of $\mathcal{A}_{G,\text{reg}}(\mathbb{Q}_S)$ in $\mathcal{A}_G(\mathbb{Q}_v)$, there exists an open neighborhood $U_{\varphi_v, Z_v}$ of $Z_v$ in $\mathcal{A}_G(\mathbb{Q}_v)$ such that

\[
(6.2.4.1) \quad \varphi_v|_{U_{\varphi_v, Z_v}} \in \text{span} \left\{ \Gamma_v^G(\cdot, \nu)|_{U_{\varphi_v, Z_v}} : \nu \in g_{\tilde{Z}_v}(\mathbb{Q}_v) \text{ nilpotent orbit} \right\}
\]

where $\varphi|_U$ denotes the restriction of the function $\varphi$ to the open subset $U$, the point $\tilde{Z}_v$ is a regular semisimple element in $g(\mathbb{Q}_v)$ lifting $Z_v$, and $\Gamma_v^G(\cdot, \nu)$ denotes the Shalika germ at the nilpotent orbit $\nu$, see §17 of [Ko05].

**Proof** Away from the singular locus the function $\varphi_v$ is a linear combination of the characteristic functions of small compact open sets, and these all have lifts in $\mathcal{S}(g(\mathbb{Q}_v))$.

Near the singular locus the theory of Shalika germs implies that the asymptotic behavior of $\varphi_v$ near the singular point $Z_v$ is necessary. It remains to show that all the Shalika germs appear in $\mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v))$, in other words there is no nontrivial linear relation among the possible asymptotes of $I_v^G(X_v, f_v)$ as $f_v$ ranges over $\mathcal{S}(g(\mathbb{Q}_v))$. The Shalika germ associated to the nilpotent orbit $\nu$ is homogeneous of degree equal to the codimension of $\nu$ in the nilpotent locus, hence the only possible linear relations are among the Shalika germs associated to the nilpotent orbits of the same dimension. But such nilpotent orbits are separated, so there are Schwartz functions $f_v$ on $g(\mathbb{Q}_v)$ that vanish on all but one of the nilpotent orbits of a given dimension, hence these too are linearly independent. \qed
(6.2.5) **Remark** Let \( v \) be a \( p \)-adic place. It is conjectured by Jacquet for a general reductive Lie algebra and proven by Waldspurger for \( \text{gl}(n) \) that a Schwartz function \( f_v \) on \( g(Q_v) \) has the property that

\[
\exists \lambda \in \mathbb{C} \forall X_v \in A_{G,\text{reg}}(Q_v) \quad I_G^v(X_v, f_v) = \lambda \cdot \phi_{0,v}(X_v)
\]

if and only if \( I_G^v(X_v, f_v) \) and \( I_G^v(X_v, f_v^*) \) are both supported on the subset of \( A_G(Q_v) \) consisting of the characteristic polynomials with coefficients in \( \mathbb{Z}_v \).

(6.2.6) **Definition** Define the *global Schwartz spaces* \( S_0(A_G(\mathbb{A})) \) and \( S_1(A_G(\mathbb{A})) \) to be the tensor products of local Schwartz spaces as defined in (6.2.2.1) and (6.2.2.2)

\[
S_0(A_G(\mathbb{A})) = \bigotimes_v S_0(A_{G}(Q_v)) \quad S_1(A_G(\mathbb{A})) = \bigotimes_v S_1(A_{G}(Q_v))
\]

restricted with respect to the basic functions \( \phi_{0,v} \) and \( \phi_{1,v} \) as defined in (6.2.3.2) and (6.2.3.3).

6.3 **The Harish-Chandra transform**

(6.3.1) **Definition** Let \( v \) be a place of \( \mathbb{Q} \). For an element \( \varphi_v \) in \( S_0(A_G(Q_v)) \), choose a Schwartz function \( f_v \) on \( g(Q_v) \) such that

\[
\forall X_v \in A_{G,\text{reg}}(Q_v) \quad \varphi_v(X_v) = I_G^v(X_v, f_v).
\]

Let \( f_v^* \) be the Fourier transform of \( f_v \), denote by \( H_v(\varphi_v) \) the element in \( S_1(A_G(Q_v)) \) such that

\[
\forall X_v \in A_{G,\text{reg}}(Q_v) \quad H_v(\varphi_v)(X_v) = T_{\text{max}}^G(X_v, f_v^*).
\]

Define the *local Harish-Chandra transform* to be the linear operator

\[
H_v : \quad S_0(A_G(Q_v)) \rightarrow S_1(A_G(Q_v))
\]

defined by

\[
H_v : \quad \varphi_v \mapsto H_v(\varphi_v).
\]

(6.3.2) **Definition** Let \( v \) be a place of \( \mathbb{Q} \). If \( X_v \) is an element of \( A_{G,\text{reg}}(\mathbb{Q}) \), then the orbital integral \( T_{\text{max}}^G(X_v, \cdot) \) is a vector-valued tempered distribution on \( g(Q_v) \). Denote by \( T_{\text{max}}^G(X_v, \cdot)^* \) its componentwise Fourier transform, which is represented by a conjugation invariant function on \( g(Q_v) \), hence descends to a function on \( A_G(Q_v) \) denoted by

\[
Y_v \mapsto \left( T_{\text{max}}^G(X_v, \cdot)^* \right)(Y_v).
\]

Define the *local Harish-Chandra kernel function* \( K_v(\cdot, \cdot) \) to be the vector-valued bivariate function on \( A_{G,\text{reg}}(Q_v) \) such that

\[
\forall X_v, Y_v \in A_{G,\text{reg}}(Q_v) \quad K_v(X_v, Y_v) = |D(Y_v)|_v^{1/2} \left( T_{\text{max}}^G(X_v, \cdot)^* \right)(Y_v).
\]

(6.3.3) **Remark** In the case when \( v \) is a \( p \)-adic place, each component of the vector-valued kernel function \( K_v \) is denoted by \( I_M^G(X_v, Y_v) \) for some Levi subgroup M by Waldspurger in [Wa95].
(6.3.4) Lemma  Let \( v \) be a place of \( \mathbb{Q} \). The local Harish-Chandra transform \( \mathcal{H}_v \) is an integral operator with integral kernel \( \mathcal{K}_v \), i.e.

\[
(6.3.4.1) \quad \forall \varphi_v \in \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \forall X_v \in \mathcal{A}_G(\mathbb{Q}_v)
\]

\[
\mathcal{H}_v(\varphi_v)(X_v) = \int_{\mathcal{A}_G(\mathbb{Q}_v)} \mathcal{K}_v(X_v, Y_v) \varphi_v(Y_v) \abs{D(Y_v)}^{1/2} dY_v.
\]

In particular the operator \( \mathcal{H}_v \) is well-defined.

(6.3.5) Proof  Every \( X_v \) in \( \mathcal{A}_G(\mathbb{Q}_v) \) has a \( \delta \)-sequence in \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \). More precisely there exists a sequence of functions \( \delta_{X_v, 1}, \delta_{X_v, 2}, \delta_{X_v, 3}, \ldots \) in \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \) such that

\[
(6.3.5.1) \quad \lim_{i \to \infty} \delta_{X_v, i} = \delta_{X_v}
\]

as distributions on \( \mathcal{A}_G(\mathbb{Q}_v) \), where \( \delta_{X_v} \) denotes the Dirac distribution at \( X_v \). For each natural number \( i \) choose a Schwartz function \( \tilde{\delta}_{X_v, i} \) on \( g(\mathbb{Q}_v) \) such that

\[
(6.3.5.2) \quad \forall Y_v \in \mathcal{A}_G(\mathbb{Q}_v) \quad \tilde{\delta}_{X_v, i}(Y_v) = I_G^G(Y_v, \tilde{\delta}_{X_v, i}).
\]

Let \( f_v \) be a Schwartz function on \( g(\mathbb{Q}_v) \) such that

\[
(6.3.5.3) \quad \forall Y_v \in \mathcal{A}_G(\mathbb{Q}_v) \quad \varphi_v(Y_v) = I_G^G(Y_v, f_v).
\]

By the local trace formula \([6.1.4.3]\)

\[
(6.3.5.4) \quad \int_{\mathcal{A}_G(\mathbb{Q}_v)} T_{\text{max}}^G(Y_v, f_v) I_G^G(Y_v, \tilde{\delta}_{X_v, i}) \abs{D(Y_v)}^{1/2} dY_v
\]

\[
= \int_{\mathcal{A}_G(\mathbb{Q}_v)} T_{\text{max}}^G(Y_v, \tilde{\delta}_{X_v, i}) I_G^G(Y_v, f_v) \abs{D(Y_v)}^{1/2} dY_v,
\]

hence

\[
(6.3.5.5) \quad \mathcal{H}_v(\varphi_v)(X_v) = \int_{\mathcal{A}_G(\mathbb{Q}_v)} \mathcal{H}_v(\varphi_v)(Y_v) \delta_{X_v}(Y_v) \abs{D(Y_v)}^{1/2} dY_v
\]

\[
= \lim_{i \to \infty} \int_{\mathcal{A}_G(\mathbb{Q}_v)} \mathcal{H}_v(\varphi_v)(Y_v) \delta_{X_v, i}(Y_v) \abs{D(Y_v)}^{1/2} dY_v
\]

\[
= \lim_{i \to \infty} \int_{\mathcal{A}_G(\mathbb{Q}_v)} T_{\text{max}}^G(Y_v, \tilde{\delta}_{X_v, i}) \varphi_v(Y_v) \abs{D(Y_v)}^{1/2} dY_v.
\]

Hence \( \mathcal{H}_v \) is an integral operator with integral kernel

\[
(6.3.5.6) \quad \lim_{i \to \infty} T_{\text{max}}^G(Y_v, \tilde{\delta}_{X_v, i}) = \abs{D(Y_v)}^{1/2} \left( T_{\text{max}}^G(X_v, \cdot) \right)(Y_v)
\]

which is independent of the choice of the sequence \( (\tilde{\delta}_{X_v, i})_{i=1}^\infty \) and equal to the local Harish-Chandra kernel \( \mathcal{K}_v(X_v, Y_v) \). \hfill \Box

(6.3.6) Remark  The local Harish-Chandra transform preserves the basic functions, i.e.

\[
(6.3.6.1) \quad \mathcal{H}_v : \phi_{0,v} \mapsto \phi_{1,v}.
\]
(6.3.7) Definition  For a finite set \( S \) of places of \( \mathbb{Q} \), define the \( S \)-local Harish-Chandra transform \( \mathcal{H}_S \) to be the tensor product

\[
\mathcal{H}_S = \bigotimes_{v \in S} \mathcal{H}_v : \bigotimes_{v \in S} \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \rightarrow \bigotimes_{v \in S} \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)).
\]

By (6.3.6.1) the limit of \( \mathcal{H}_S \) as \( S \) approaches infinity defines a linear operator

\[
\lim_S \mathcal{H}_S : \bigotimes_v \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \rightarrow \bigotimes_v \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)),
\]

which is defined to be the global Harish-Chandra transform

\[
\mathcal{H} : \mathcal{S}_0(\mathcal{A}_G(\mathbb{A})) \rightarrow \mathcal{S}_1(\mathcal{A}_G(\mathbb{A})).
\]

6.4 The Poisson summation formula

(6.4.1) Definition  Let \( Z \) be a singular point in the complement of \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \) in \( \mathcal{A}_G(\mathbb{Q}) \). Denote by \( \mathfrak{o}_Z \) the \( \sim \) equivalence class in \( \mathfrak{g}(\mathbb{Q}) \) which is the fiber of \( Z \). Let \( S \) be a finite set of places of \( \mathbb{Q} \), let \( \varphi_S \) be an element of \( \bigotimes_{v \in S} \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)) \), let \( f_S \) be a Schwartz function on \( \mathfrak{g}(\mathbb{Q}_S) \) such that

\[
\forall X \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_S) \quad \varphi_S(X) = I^G(X, f_S)
\]

By (6.1.5.1), if \( Z' \) is a \( \equiv \) equivalence class in \( (\mathfrak{m}(\mathbb{Q}) \cap \mathfrak{o}_Z)_{\mathbf{M},\mathbf{S}} \), then \( I^G(Z', f_S) \) is determined by \( \varphi_S(X) \) where \( X \) ranges over the elements of \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_S) \) close to \( Z' \). Denote

\[
\varphi_S(Z') = I^G(Z', f_S).
\]

(6.4.2) Definition  Let \( \mathbf{M} \) be a Levi subgroup in \( \mathcal{L} \), let \( X \) be an element of \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \) contained in \( \pi_{\mathbf{M}}(\mathcal{A}_M(\mathbb{Q})) \). Let \( v \) be a place of \( \mathbb{Q} \), let \( \varphi_v \) be an element of \( \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)) \). By the induction lemma in (6.1.4) if \( f_v \) is a Schwartz function on \( \mathfrak{g}(\mathbb{Q}_v) \) such that

\[
\varphi_v(X) = T^G_{\mathbf{M}[X_{\text{reg}}]}(\tilde{X}_v, f_v)
\]

where \( \tilde{X}_v \) is a regular semisimple element of \( \mathfrak{m}(\mathbb{Q}_v) \) lifting \( X \), then \( T^G_{\mathbf{M}}(\tilde{X}_v, f_v) \) is determined by \( \varphi_v(X) \). Denote

\[
q^M(\varphi_v(X)) = T^G_{\mathbf{M}}(\tilde{X}_v, f_v).
\]

Let \( S \) be a finite set of places of \( \mathbb{Q} \), let \( \varphi_S \) be an element of \( \bigotimes_{v \in S} \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)) \). By the splitting lemma in (6.1.4) if \( \varphi_S \) is of the form \( \bigotimes_{v \in S} \varphi_v \) where for each \( v \) in \( S \) the local factor \( \varphi_v \) lies in \( \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)) \) and \( f_v \) is a Schwartz function on \( \mathfrak{g}(\mathbb{Q}_v) \) such that

\[
q^M(\varphi_v(X)) = T^G_{\mathbf{M}}(\tilde{X}, f_v)
\]

where \( \tilde{X} \) is a regular semisimple element of \( \mathfrak{m}(\mathbb{Q}) \) lifting \( X \), then \( T^G_{\mathbf{M}}(\tilde{X}, \bigotimes_{v \in S} f_v) \) is determined by \( \bigotimes_{v \in S} \varphi_v(X) \). Denote

\[
q^M(\varphi_S(X)) = T^G_{\mathbf{M}}(\tilde{X}, \bigotimes_{v \in S} f_v)
\]

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and extend the definition to \( q^M(\varphi_S(X)) \) for all \( \varphi_S \) in \( \bigotimes_{v \in S} S_1(A_G(Q_v)) \) by linearity.

Let \( Z \) be a singular point in the complement of \( A_{G,\text{reg}}(Q) \) in \( A_G(Q) \). Denote by \( \sigma_Z \) the \( \sim \) equivalence class in \( g(Q) \) which is the fiber of \( Z \). Let \( f_S \) be a Schwartz function on \( g(Q_S) \) such that

\[
\forall Y \in A_{G,\text{reg}}(Q_S) \quad q^M(\varphi_S(Y)) = T^G_M(\tilde{Y}, f_S)
\]

where \( \tilde{Y} \) is a regular semisimple element of \( m(Q) \) lifting \( Y \). By (3.1.5.4), if \( Z' \) is a \( \equiv \) equivalence class in \( (m(Q) \cap \sigma_Z)_{M,S} \), then \( T^G_M(Z', f_S) \) is determined by \( q^M(\varphi_S(Y)) \) where \( Y \) ranges over the elements of \( A_{G,\text{reg}}(Q_S) \) close to \( Z' \). Denote

\[
q^M(\varphi_S(Z')) = T^G_M(Z', f_S).
\]

(6.4.3) Definition Let \( X \) be an element of \( A_{G,\text{reg}}(Q) \), let \( \tilde{X} \) be a regular semisimple element of \( g(Q) \) lifting \( X \), let \( M[\tilde{X}, \text{ell}] \) be a Levi subgroup in \( \mathcal{L} \) such that \( m[\tilde{X}, \text{ell}] \) contains \( \tilde{X} \) as a \( \text{Q-elliptic} \) element. Then by (3.3.17.1) the constant \( a^{M[\tilde{X}, \text{ell}]}(S, \tilde{X}) \) as defined in (3.3.16.12) is independent of the finite set of places \( S \) and determined by \( X \). Denote

\[
a(X) = |W_0^{M[\tilde{X}, \text{ell}]}||W_0^G|^{-1}a^{G}_{M[\tilde{X}, \text{ell}]}(S, X)
\]

where \( a^{G}_{M}(S, X) \) is defined as in (6.4.4.5) and denote

\[
q^\text{ell}(\varphi_S(X)) = q^{M[\tilde{X}, \text{ell}]}(\varphi_S(X))
\]

where \( q^M(\varphi_S(X)) \) is defined as in (6.4.2.6).

(6.4.4) Proposition (Poisson summation formula)

Let \( \varphi \) be an element of \( S_0(A_G(A)) \), then

\[
\sum_{X \in A_{G,\text{reg}}(Q)} a(X)\varphi(X) + \sum_{Z \in A_G(Q) - A_{G,\text{reg}}(Q)} \lim_S\left( \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{Z' \in (m(Q) \cap \sigma_Z)_{M,S}} a^{G}_{M}(S, X)\varphi_S(Z') \right)
\]

\[
= \sum_{X \in A_{G,\text{reg}}(Q)} a(X)\text{Tr}\left(q^{\text{ell}}(H(\varphi)(X))\right) + \sum_{Z \in A_G(Q) - A_{G,\text{reg}}(Q)} \lim_S\left( \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \times \right.
\]

\[
\times \sum_{Z' \in (m(Q) \cap \sigma_Z)_{M,S}} a^{G}_{M}(S, X)\text{Tr}\left(q^{\text{ell}}(H_S(\varphi_S)(Z'))\right)\right).
\]

(6.4.5) Proof Follows from the global trace formula (6.4.4.4).

(6.4.6) Remark The Poisson summation formula (6.4.4.1) has the general form

\[
\sum_{X \in A_{G,\text{reg}}(Q)} a(X)\varphi(X) + \sum_{Z \in A_G(Q) - A_{G,\text{reg}}(Q)} (-) = \sum_{X \in A_{G,\text{reg}}(Q)} a(X)\text{Tr}\left(q^{\text{ell}}(H(\varphi)(X))\right) + \sum_{Z \in A_G(Q) - A_{G,\text{reg}}(Q)} (-).
\]

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Corollary Let \( v \) and \( w \) be places of \( \mathbb{Q} \). Let \( \varphi \) be an element of \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{A})) \) such that \( \varphi_v \) vanishes on a neighborhood of the complement of \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \) in \( \mathcal{A}_G(\mathbb{Q}_v) \) and \( \mathcal{H}_w(\varphi_w) \) vanishes on a neighborhood of the complement of \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_w) \) in \( \mathcal{A}_G(\mathbb{Q}_w) \). Then

\[
\sum_{X \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q})} a(X) \varphi(X) = \sum_{X \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q})} a(X) \text{Tr}\left(q^\text{ell}(\mathcal{H}(\varphi)(X))\right).
\]

Proof Follows from (6.4.6.1).

Corollary Let \( v \) be a place of \( \mathbb{Q} \). The local Harish-Chandra transform \( \mathcal{H}_v \) is bijective.

Proof The following argument is suggested by Sakellaridis.

The local Harish-Chandra transform \( \mathcal{H}_v \) is surjective by definition. For injectivity assume for contradiction that there exist an element \( \varphi_v \) in \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{A})) \) and a point \( X_v \) in \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \) such that

\[
\forall Y_v \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \quad \mathcal{H}_v(\varphi_v)(Y_v) = 0
\]

and

\[
\varphi_v(X_v) \neq 0.
\]

Since \( \varphi_v \) is smooth on \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_v) \), without loss of generality \( X_v \) lies in the dense subset \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \).

Let \( \varphi \) be an element of \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{A})) \) whose local component at \( v \) is equal to \( \varphi_v \), then by the Poisson summation formula (6.4.4.1)

\[
\sum_{X \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q})} a(X) \varphi(X) + \sum_{Z \in \mathcal{A}_G(\mathbb{Q}) - \mathcal{A}_{G,\text{reg}}(\mathbb{Q})} \left( \text{contribution from the singular locus} \right) = 0
\]

since \( \mathcal{H}(\varphi) \) vanishes identically at the place \( v \). There are two cases:

- If \( v \) is \( p \)-adic, then there exists an integer \( N \) such that \( X_v \) lies in the image of \( \mathfrak{g}_{\text{reg,ss}}(N^{-1}\mathbb{Z}) \) in \( \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \) under the natural projection.

At a finite place \( w \) distinct from \( v \), denote by \( \Lambda_{N,w} \) the lattice

\[
\Lambda_{N,w} = \mathfrak{g}(N^{-1}\mathbb{Z}_w) \subseteq \mathfrak{g}(\mathbb{Q}_w).
\]

Let \( \varphi_w \) be the element of \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_w)) \) such that

\[
\forall Y_w \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_w) \quad \varphi_w(Y_w) = I_G^\mathfrak{g}(Y_w, \Pi_{\Lambda_{N,w}})
\]

where \( \Pi_{\Lambda_{N,w}} \) denotes the characteristic function of \( \Lambda_{N,w} \).

At infinity, the set of rational points where \( \bigotimes_{w < \infty} \varphi_w \) is nonzero is contained in the discrete subset

\[
\pi_{\infty}(\mathfrak{g}_{\text{reg,ss}}(N^{-1}\mathbb{Z})) \subseteq \mathcal{A}_{G,\text{reg}}(\mathbb{R})
\]

where \( \pi_{\infty} \) denotes the natural projection from \( \mathfrak{g}(\mathbb{R}) \) to \( \mathcal{A}_G(\mathbb{R}) \). Choose \( \varphi_{\infty} \) to be a bump function supported away from the complement of \( \mathcal{A}_{G,\text{reg}}(\mathbb{R}) \) in \( \mathcal{A}_G(\mathbb{R}) \) such that \( X_v \) is the only point contained in

\[
\text{supp}(\varphi_{\infty}) \cap \pi_{\infty}(\mathfrak{g}_{\text{reg,ss}}(N^{-1}\mathbb{Z}))
\]
Choose \( \varphi \) to be \( \bigotimes_w \varphi_w \), then the left hand side of (6.4.10.3) is equal to

\[
(6.4.10.8) \quad a(X_v)\varphi(X_v) = \pm |\pi_0(M_{X_v})|^{-1} \text{Vol}(T_{X_v}(\mathbb{Q}) \backslash T_{X_v}(\mathbb{A})^1) \lim_{S} \left( \prod_{w \in S} \varphi_w(X_v) \right)
\]

for some Levi subgroup \( M \) in \( \mathcal{L} \) and some torus \( T_{X_v} \), which is nonzero.

- If \( v \) is archimedean, then choose a regular semisimple element \( \tilde{X}_\infty \) of \( g(\mathbb{Q}) \) lifting \( X_\infty \) and choose a Schwartz function \( f_\infty \) on \( g(\mathbb{A}) \) such that

\[
(6.4.10.9) \quad \forall Y_\infty \in \mathcal{A}_{G,\text{reg}}(\mathbb{R}) \quad \varphi_\infty(Y_\infty) = I_G^G(Y_\infty, f_\infty).
\]

If \( N \) is an integer, denote by \( \Lambda_{N,\infty} \) the lattice

\[
(6.4.10.10) \quad \Lambda_{N,\infty} = g(N \mathbb{Z}) \subset g(\mathbb{R}).
\]

Since \( f_\infty \) is a Schwartz function on \( g(\mathbb{R}) \), the quantity \( \varphi_\infty(Y_\infty) \) is rapidly decreasing as \( Y_\infty \) approaches infinity in \( \mathcal{A}_{G,\text{reg}}(\mathbb{R}) \) in such a way that \( |D(Y_\infty)| \) is uniformly bounded below, hence for every positive real numbers \( \epsilon \) and \( r \) there exists a natural number \( N_\epsilon \) such that

\[
(6.4.10.11) \quad \forall N \geq N_\epsilon \quad \sum_{\substack{Y \in \pi_\infty(X_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty, |D(Y)| \geq 1}} |\varphi_\infty(Y)| < \epsilon N^{-r}
\]

where \( \pi_\infty \) denotes the natural projection from \( g(\mathbb{R}) \) to \( \mathcal{A}_G(\mathbb{R}) \). Choose

\[
(6.4.10.12) \quad \epsilon = \frac{1}{2(n!)^m} |a(X_\infty)\varphi_\infty(X_\infty)|
\]

where \( n \) is the rank of \( G \) and choose \( r \) to be \( 2\text{dim}(g) \). Let \( N \) be a natural number greater than \( N_\epsilon \) such that

\[
(6.4.10.13) \quad \forall Y \in \pi_\infty(\tilde{X}_\infty + \Lambda_{N,\infty}) \quad D(Y) \neq 0.
\]

At a finite place \( w \), denote by \( \Lambda_{N,w} \) the lattice

\[
(6.4.10.14) \quad \Lambda_{N,w} = g(N \mathbb{Z}_w) \subset g(\mathbb{Q}_w).
\]

Let \( \varphi_w \) be the element of \( \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_w)) \) such that

\[
(6.4.10.15) \quad \forall Y_w \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}_w) \quad \varphi_w(Y_w) = I_G^G(Y_w, \mathbb{1}_{\tilde{X}_\infty + \Lambda_{N,w}})
\]

where \( \mathbb{1}_{\tilde{X}_\infty + \Lambda_{N,w}} \) denotes the characteristic function of the translation of \( \Lambda_{N,w} \) by \( \tilde{X}_\infty \).

Choose \( \varphi \) to be \( \bigotimes_w \varphi_w \), then the left hand side of (6.4.10.3) is equal to

\[
(6.4.10.16) \quad a(X_\infty) \prod_{w < \infty} \varphi_w(X_\infty) \cdot \varphi_\infty(X_\infty) + \sum_{\substack{Y \in \pi_\infty(\tilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty, |D(Y)| \geq 1}} a(Y) \prod_{w < \infty} \varphi_w(Y) \cdot \varphi_\infty(Y)
\]

\[
= C \cdot \left( a(X_\infty)\varphi_\infty(X_\infty) + \sum_{\substack{Y \in \pi_\infty(\tilde{X}_\infty + \Lambda_{N,\infty}) \\ Y \neq X_\infty, |D(Y)| \geq 1}} \left( a(Y) \prod_{w < \infty} \varphi_w(Y) \right) \varphi_\infty(Y) \right)
\]

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where $C$ is a nonzero constant. Let $Y$ be an element of $\mathcal{A}_{G,\text{reg}}(\mathbb{Q})$. By definition
\begin{equation}
(6.4.10.17) \quad a(Y) = \pm |\pi_0(M_Y)|^{-1} \text{Vol}(T_Y(\mathbb{Q}) \backslash T_Y(\mathbb{A})^1)
\end{equation}
where $M$ is a Levi subgroup in $\mathcal{L}$ and $T_Y$ is a maximal torus in $G$. By the Main theorem in §5 of [On63],
\begin{equation}
(6.4.10.18) \quad \text{Vol}(T_Y(\mathbb{Q}) \backslash T_Y(\mathbb{A})^1) \leq |H^1_{\text{Gal}}(\mathbb{Q}, \hat{T}_Y)|
\end{equation}
where $H^1_{\text{Gal}}$ denotes the first Galois cohomology group and $\hat{T}_Y$ denotes the Galois module of algebraic characters of $T_Y$. Let $F$ be the splitting field of $T_Y$, denote by $\Gamma$ the Galois group of $F$ over $\mathbb{Q}$. Then by the inflation-restriction exact sequence
\begin{equation}
(6.4.10.19) \quad 0 \longrightarrow H^1(\Gamma, \hat{T}_Y) \longrightarrow H^1_{\text{Gal}}(\mathbb{Q}, \hat{T}_Y) \longrightarrow H^1_{\text{Gal}}(F, \hat{T}_Y)^\Gamma \longrightarrow \text{Hom}(\text{Gal}(F/F), \mathbb{Z}^n)^\Gamma
\end{equation}
and the compactness of $\text{Gal}(\overline{F}/F)$ which implies that the only continuous homomorphism from $\text{Gal}(\overline{F}/F)$ to $\mathbb{Z}^n$ is the trivial homomorphism,
\begin{equation}
(6.4.10.20) \quad |H^1(\Gamma, \hat{T}_Y)| = |H^1_{\text{Gal}}(\mathbb{Q}, \hat{T}_Y)|.
\end{equation}
The group $H^1(\Gamma, \hat{T}_Y)$ is annihilated by the order of $\Gamma$ which is bounded by the factorial of $n$ since the $\hat{T}_Y$ splits over the splitting field of the characteristic polynomial represented by $Y$ which is of degree $n$. By the bar resolution, the group $H^1(\Gamma, \hat{T}_Y)$ is a subquotient of $\bigoplus_{g \in \Gamma} \mathbb{Z}^n$ which is generated by at most $n \cdot n!$ elements, hence by (6.4.10.17), (6.4.10.18) and (6.4.10.20)
\begin{equation}
(6.4.10.21) \quad \forall Y \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \quad |a(Y)| \leq (n!)^{n-n!}.
\end{equation}
As $N$ approaches infinity, either $\prod_{\varphi w < \infty} \varphi_w(Y)$ vanishes or
\begin{equation}
(6.4.10.22) \quad N^{-\dim(g)} \leq \prod_{\varphi w < \infty} \varphi_w(Y) \leq 1,
\end{equation}
and $\prod_{\varphi w < \infty} \varphi_w(X_\infty)$ is nonzero. Hence without loss of generality $N$ is large enough so that
\begin{equation}
(6.4.10.23) \quad \forall Y \in \mathcal{A}_{G,\text{reg}}(\mathbb{Q}) \quad \frac{\prod_{\varphi w < \infty} \varphi_w(Y)}{\prod_{\varphi w < \infty} \varphi_w(X_\infty)} \leq N^r.
\end{equation}
By (6.4.10.21) and (6.4.10.23),
\begin{equation}
(6.4.10.24) \quad \sum_{\begin{subarray}{c} Y \in \pi_\infty(X_\infty + A_{N,\text{reg}}) \\ Y \neq X_\infty, |D(Y)| \geq 1 \end{subarray}} \left| a(Y) \prod_{\varphi w < \infty} \varphi_w(Y) \prod_{\varphi w < \infty} \varphi_w(X_\infty) \varphi_\infty(Y) \right| \leq N^r(n!)^{n-n!} \sum_{\begin{subarray}{c} Y \in \pi_\infty(X_\infty + A_{N,\text{reg}}) \\ Y \neq X_\infty, |D(Y)| \geq 1 \end{subarray}} |\varphi_\infty(Y)| < N^r(n!)^{n-n!} \epsilon N^{-r}
\end{equation}
\begin{equation}
(6.4.10.25) \quad = (n!)^{n-n!} \frac{1}{2(n!)^{n-n!}} |a(X_\infty)\varphi_\infty(X_\infty)|
\end{equation}
\begin{equation}
(6.4.10.26) \quad = \frac{1}{2} |a(X_\infty)\varphi_\infty(X_\infty)|
\end{equation}
where the inequality (6.4.10.25) follows from (6.4.10.11) and the equality (6.4.10.26) follows from (6.4.10.12). Hence the right hand side of (6.4.10.26) is nonzero.

In either case the left hand side of (6.4.10.3) is nonzero, which is a contradiction. \qed
(6.4.11) Remark Injectivity of $H_v$ is analogous to the classical result of Harish-Chandra on density of regular semisimple orbital integrals which states that for a $p$-adic reductive Lie algebra $\mathfrak{g}$, if $f$ is a Schwartz function on $\mathfrak{g}$ such that all the regular semisimple orbital integrals of $f$ vanish, then $D(f)$ vanishes for every invariant distribution $D$ on $\mathfrak{g}$. See §27 of [Ko05].

A Appendix: the local basic functions for $GL(2)$

In this appendix the local basic functions $\phi_{0,v}$ (6.2.3.2) and $\phi_{1,v}$ (6.2.3.3) are calculated for the group $GL(2, \mathbb{Q}_v)$ where $v$ is an odd rational prime. Similar computations could be found in [Ev98].

A.1 Notations and definitions

(A.1.1) In this appendix $v$ denotes an odd rational prime $p$, and $| |$ denotes the $p$-adic absolute value. Let $G$ denote the group $GL(2, \mathbb{Q}_p)$, let $M$ denote the minimal Levi subgroup of $G$ consisting of the diagonal matrices, then $M$ is the Levi component of the Borel subgroup $B$ consisting of the upper triangular matrices. The groups $G$ and $M$ are the only standard Levi subgroups of $G$.

Let $g'$ denote the Lie algebra $sl(2, \mathbb{Q}_p)$, equipped with the adjoint action of $G$ from the right. Let $m'$ denote the intersection of $m$ and $g'$. Denote by $A$ the affine quotient $g'//G$ which is identified with the affine line $\mathbb{L}$ via the negative determinant map

$$-\det : g' \rightarrow \mathbb{L} = A.$$

Remark Let $z$ denote the center of $g$ consisting of the diagonal matrices on which $G$ operates trivially. Since

$$g = g' \oplus z$$

as representations of $G$, for computing the local basic functions and the local Harish-Chandra transforms it suffices to consider $g'$ instead of $g$.

(A.1.2) The discriminant function $D$ on $A$ is equal to $4X$ where $X$ denotes the coordinate function on $A$. Let $\eta$ be an element of $A_{reg}(\mathbb{Q}_p)$ which is identified with the subset of units $\mathbb{Q}_p^\times$ of $\mathbb{Q}_p$. Then $\eta$ is said to be

- *split* if $\eta$ has a square root in $\mathbb{Q}_p$;
- *unramified elliptic* if $\eta$ does not have a square root in $\mathbb{Q}_p$ but the $p$-adic valuation of $\eta$ is even;
- *ramified elliptic* if the $p$-adic valuation of $\eta$ is odd.

The image of $m'_{reg,ss}(\mathbb{Q}_p)$ in $A_{reg}(\mathbb{Q}_p)$ is equal to the subset of split elements.

(A.1.3) The algebraic differential form

$$d_\eta X = \frac{dx \wedge dy}{y} = \frac{dy \wedge dz}{2x} = \frac{dz \wedge dx}{z}$$

on the orbit

$$g'_\eta = \left\{ \left( \begin{array}{cc} x & y \\ z & -x \end{array} \right) \in g' : x^2 + yz = \eta \right\}$$

where $\eta$ lies in $A_{reg}$ is invariant under the action of $G$.  

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Lemma  Let $f$ be a Schwartz function on $\mathfrak{g}'(\mathbb{Q}_p)$, then

$$\forall \eta \in \mathcal{A}_{\text{reg}}(\mathbb{Q}_p) \quad I_G^G(\eta, f) = \frac{1}{1 + p^{-1}} \int_{\mathfrak{g}'_p(\mathbb{Q}_p)} f(X) \, |d_\eta X|. \quad (A.1.3.3)$$

Proof  There exists a multiplicative constant $\lambda$ such that

$$\forall f \in S(\mathfrak{g}'(\mathbb{Q}_p)) \forall \eta \in \mathcal{A}_{\text{reg}}(\mathbb{Q}_p) \quad I_G^G(\eta, f) = \lambda \cdot \int_{\mathfrak{g}'_p(\mathbb{Q}_p)} f(X) \, |d_\eta X|, \quad (A.1.3.4)$$

hence the constant $\lambda$ is determined by

$$I_G^G(1, \mathbb{I}_{\mathfrak{g}'(\mathbb{Q}_p)}) = \lambda \cdot \int_{\mathfrak{g}'_p(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Q}_p)}(X) \, |d_1 X|. \quad (A.1.3.5)$$

By parabolic descent along the Borel subgroup $B$, the left hand side of $(A.1.3.5)$ is equal to

$$I_M^M(1, (\mathbb{I}_{\mathfrak{g}'(\mathbb{Q}_p)})_B) = I_M^M(1, \mathbb{I}_{m'(\mathbb{Q}_p)}) \quad (A.1.3.6)$$

which is equal to one. Hence by $(A.1.3.5)$ the reciprocal of $\lambda$ is equal to

$$\int_{\mathfrak{g}'_p(\mathbb{Q}_p)} \mathbb{I}_{\mathfrak{g}'(\mathbb{Q}_p)} \left( \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) \left| \frac{dx \wedge dy}{y} \right|$$

$$= \int_{\mathbb{Q}_p} \left( \int_{\{ y \in \mathbb{Q}_p: z = \frac{1}{y} \} \subset \mathbb{Q}_p} \frac{dy}{|y|} \right) dx$$

$$= \int_{|1+x|<1} \left( \int_{|1-x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx + \int_{|1-x|<1} \left( \int_{|1-x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx +$$

$$+ \int_{|1+x|=1} \left( \int_{|1-x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx$$

$$= \sum_{i=1}^{\infty} \left( \int_{|1+x|=p^{-i}} dx \right) \left( \int_{p^{-i} \leq |y| \leq 1} \frac{dy}{|y|} \right) + \sum_{j=1}^{\infty} \left( \int_{|1-x|=p^{-j}} dx \right) \left( \int_{p^{-j} \leq |y| \leq 1} \frac{dy}{|y|} \right) +$$

$$+ \left( \int_{|1+x|=1} dx \right) \left( \int_{|y|=1} \frac{dy}{|y|} \right)$$

$$= \sum_{i=1}^{\infty} \left( \frac{1}{p^i} - \frac{1}{p^{i+1}} \right) \left( (i+1)(p-1) \right) + \sum_{j=1}^{\infty} \left( \frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \left( (j+1)(p-1) \right) +$$

$$+ \left( \frac{p-2}{p} \right) \left( \frac{p-1}{p} \right)$$

$$= \frac{p-1}{p} \left( 2 \cdot \left( \frac{1}{p} + \frac{1}{p-1} \right) + \frac{p-2}{p} \right)$$

$$= 1 + p^{-1}. \quad \square$$

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A.2 The local basic function $\phi_{0,v}$

(A.2.1) Lemma Let $\eta$ be an element of $\mathcal{A}_{reg}(\mathbb{Q}_p)$, then

$$
\phi_{0,v}(\eta) = \begin{cases} 
1 & \text{if } \eta \text{ is split and contained in } \mathbb{Z}_p - \{0\}, \\
1 - \frac{2|\eta|^{1/2}}{1+p} & \text{if } \eta \text{ is unramified elliptic and contained in } \mathbb{Z}_p - \{0\}, \\
1 - \frac{p^{-1/2}|\eta|}{1+p} & \text{if } \eta \text{ is ramified elliptic and contained in } \mathbb{Z}_p - \{0\}, \\
0 & \text{if } \eta \text{ does not lie in } \mathbb{Z}_p - \{0\}.
\end{cases}
$$

(A.2.2) Proof If $\eta$ does not lie in $\mathbb{Z}_p - \{0\}$, then the orbit $g_{\eta}'(\mathbb{Q}_p)$ is disjoint from the support of the characteristic function $\mathbb{I}_{g_{\eta}'(\mathbb{Z}_p)}$, hence $\phi_{0,v}(\eta)$ vanishes.

If $\eta$ is split and contained in $\mathbb{Z}_p - \{0\}$, then by (A.1.3.6) the value $\phi_{0,v}(\eta)$ is equal to one.

If $\eta$ is unramified elliptic and contained in $\mathbb{Z}_p - \{0\}$, then by (A.1.3.3)

$$
\phi_{0,v}(\eta) = \frac{1}{1 + p^{-1}} \int_{g_{\eta}'(\mathbb{Q}_p)} \mathbb{I}_{g_{\eta}'(\mathbb{Z}_p)}(X) \frac{d\eta X}{|x|} = \frac{1}{1 + p^{-1}} \int_{g_{\eta}'(\mathbb{Q}_p)} \mathbb{I}_{g_{\eta}'(\mathbb{Z}_p)} \left( \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right) \frac{dx \wedge dy}{y} = \frac{1}{1 + p^{-1}} \int_{|x| \leq 1} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx.
$$

Decompose the integral on the right hand side of (A.2.2.2) over two disjoint regions:

- If $|x^2|$ is strictly less than $|\eta|$, then $|\eta - x^2|$ is equal to $|\eta|$, hence

$$
\int_{|x^2| < |\eta|} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx = \left( \int_{|x^2| < |\eta|} dx \right) \left( \int_{|\eta| \leq |y| \leq 1} \frac{dy}{|y|} \right) = \left( p^{-\left(\frac{\text{val}(\eta)}{2} + 1\right)} \right) \left( \frac{\text{val}(\eta) + 1)(p - 1)}{p} \right)
$$

where $\text{val}(\eta)$ denotes the $p$-adic valuation of $\eta$.

- If $|x^2|$ is greater than or equal to $|\eta|$, then $|\eta - x^2|$ is equal to $|x^2|$, hence

$$
\int_{|\eta| \leq |x^2| \leq 1} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx = \int_{|\eta| \leq |x^2| \leq 1} \left( \int_{|x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) dx = \int_{|\eta| \leq |x^2| \leq 1} (2\text{val}(x) + 1) \left( \frac{p - 1}{p} \right) dx
$$

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\[ \frac{p - 1}{p} \left( 1 - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} \right) + 2 \int_{|\eta| \leq |x^2| \leq 1} \text{val}(x) \, dx \]

\[ = \frac{p - 1}{p} \left( 1 - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} \right) + 2 \cdot \sum_{i=0}^{\text{val}(\eta)} i \left( \frac{1}{p^i} - \frac{1}{p^{i+1}} \right) \]

\[ = \frac{p - 1}{p} \left( 1 - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} \right) + 2 \cdot \left( - \left( \frac{\text{val}(\eta)}{2} + 1 \right) p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} + \frac{1 - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)}}{p - 1} \right). \]

Hence the right hand side of (A.2.2.2) is equal to

\[ \frac{1}{1 + p^{-1}} \cdot \frac{p - 1}{p} \left( \left( \text{val}(\eta) + 1 \right) p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} - \left( \text{val}(\eta) + 2 \right) p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} + \right. \]

\[ + \left. \left( 1 - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} \right) \left( 1 + \frac{2}{p - 1} \right) \right) \]

\[ = \frac{p - 1}{p + 1} \left( - p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} - \left( \frac{p + 1}{p - 1} \right) p^{-\left( \frac{\text{val}(\eta)}{2} + 1 \right)} + \frac{p + 1}{p - 1} \right) \]

\[ = 1 - \frac{2}{p + 1} p^{-\text{val}(\eta)/2}. \]

If \( \eta \) is ramified elliptic and contained in \( \mathbb{Z}_p - \{0\} \), then by (A.1.3.3)

\[ (A.2.2.6) \quad \phi_{0,v}(\eta) = \frac{1}{1 + p^{-1}} \int_{|x| \leq 1} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) \, dx \]

Decompose the integral on the right hand side of (A.2.2.6) over two disjoint regions:

- If \( |x^2| \) is less than \( |\eta| \), then \( |\eta - x^2| \) is equal to \( |\eta| \), hence

\[ (A.2.2.7) \quad \int_{|x^2| < |\eta|} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) \, dx = \left( \int_{|x^2| < |\eta|} \, dx \right) \left( \int_{|\eta| \leq |y| \leq 1} \frac{dy}{|y|} \right) \]

\[ = \left( p^{-\left( \frac{\text{val}(\eta) + 1}{2} \right)} \right) \left( \frac{\text{val}(\eta) + 1)(p - 1)}{p} \right). \]

- If \( |x^2| \) is greater than \( |\eta| \), then \( |\eta - x^2| \) is equal to \( |x^2| \), hence

\[ (A.2.2.8) \quad \int_{|\eta| \leq |x^2| \leq 1} \left( \int_{|\eta - x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) \, dx \]

\[ = \int_{|\eta| \leq |x^2| \leq 1} \left( \int_{|x^2| \leq |y| \leq 1} \frac{dy}{|y|} \right) \, dx \]

\[ = \int_{|\eta| \leq |x^2| \leq 1} \left( 2 \text{val}(x) + 1 \left( \frac{p - 1}{p} \right) \right) \, dx \]
\[
\begin{align*}
&= \frac{p-1}{p} \left( 1 - p^{-\left(\frac{\val(\eta)+1}{2}\right)} \right) + 2 \int_{|\eta| \leq |x^2| \leq 1} \val(x) \, dx \\
&= \frac{p-1}{p} \left( 1 - p^{-\left(\frac{\val(\eta)+1}{2}\right)} \right) + 2 \sum_{i=0}^{\val(\eta)-1} \left( \frac{1}{p^i} - \frac{1}{p^{i+1}} \right) \\
&= \frac{p-1}{p} \left( 1 - p^{-\left(\frac{\val(\eta)+1}{2}\right)} \right) + 2 \left( -\left(\frac{\val(\eta)+1}{2}\right) p^{-\left(\frac{\val(\eta)+1}{2}\right)} + \frac{1-p^{-\left(\frac{\val(\eta)+1}{2}\right)}}{p-1} \right).
\end{align*}
\]

Hence the right hand side of (A.2.2.6) is equal to
\[
\begin{align*}
(A.2.2.9) \quad &\frac{1}{1+p^{-1}} \cdot \frac{p-1}{p} \left( \left(\frac{\val(\eta)+1}{2}\right) p^{-\left(\frac{\val(\eta)+1}{2}\right)} - \left(\frac{\val(\eta)+1}{2}\right) p^{-\left(\frac{\val(\eta)+1}{2}\right)} + \\
&\quad + \left(1 - p^{-\left(\frac{\val(\eta)+1}{2}\right)}\right) \left(1 + \frac{2}{p-1}\right) \right) \\
&= \frac{p-1}{p+1} \left( 1 - p^{-\left(\frac{\val(\eta)+1}{2}\right)} \right) \left( \frac{p+1}{p-1} \right) \\
&= 1 - p^{-1/2} p^{-\left(\frac{\val(\eta)}{2}\right)}.
\end{align*}
\]

\[\square\]

A.3 The local basic function \(\phi_{1,v}\)

(A.3.1) **Definition** The weighted orbital integral \(J^G_M(\tilde{\eta}, )\) where \(\tilde{\eta}\) is an element of \(q_{reg,ss}(\mathbb{Q}_p)\) lifting \(\eta\) is defined is \(\eta\) is split. For such an \(\eta\) choose a square root \(\sqrt{\eta}\) of \(\eta\) in \(\mathbb{Q}_p\) and fix the lift \(\tilde{\eta}\) to be
\[
(A.3.1.1) \quad \tilde{\eta} = \begin{pmatrix} \sqrt{\eta} \\ -\sqrt{\eta} \end{pmatrix}.
\]

(A.3.2) **Remark** With respect to the choice of \(B\) as the Borel subgroup and \(G(\mathbb{Z}_p)\) as the maximal compact subgroup, the weight factor \(v( )\) appearing in the definition of \(J^G_M(\tilde{\eta}, )\) is the function on \(G(\mathbb{Q}_p)\) such that \(v( )\) is invariant under left translation by \(M(\mathbb{Q}_p)\) and right translation by \(G(\mathbb{Z}_p)\) and
\[
(A.3.2.1) \quad \forall u \in \mathbb{Q}_p, \quad v \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \begin{cases} 0 & \text{if } u \text{ is contained in } \mathbb{Z}_p, \\
-\val(u) & \text{otherwise}. \end{cases}
\]

See §12.1 of [Ko05].

(A.3.3) **Lemma** Let \(\eta\) be a split element of \(A_{reg}(\mathbb{Q}_p)\), then
\[
(A.3.3.1) \quad J^G_M(\tilde{\eta}, \underline{\mathfrak{g}}'(\mathbb{Z}_p)) = \begin{cases} \frac{\val(\eta)}{2} - \frac{1}{p-1} + \frac{|\eta|^{1/2}}{p-1} & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\
0 & \text{otherwise}. \end{cases}
\]

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(A.3.4) Proof By definition

\[(A.3.4.1) \quad J_M^G(\tilde{\eta}, \mathbb{I}_{g'(Z_p)}) = |D(\eta)|^{1/2} \int_{M(Q_p) \backslash G(Q_p)} \mathbb{I}_{g'(Z_p)}(\tilde{\eta} \cdot \text{ad}(g)) \, v(g) \, dg\]

\[(A.3.4.2) = |\eta|^{1/2} \int_{Q_p} \mathbb{I}_{g'(Z_p)} \left( \begin{array}{cc} \sqrt{\tilde{\eta}} & 2u\sqrt{\eta} \\ -\sqrt{\tilde{\eta}} & \end{array} \right) \left( \begin{array}{cc} 1 & u \\ 1 & \end{array} \right) \, v \left( \begin{array}{cc} 1 & u \\ 1 & \end{array} \right) \, du \]

\[(A.3.4.3) = |\eta|^{1/2} \int_{1 < |u| \leq 2\sqrt{\eta}} -\text{val}(u) \, du \]

\[(A.3.4.4) = |\eta|^{1/2} \sum_{i=1}^{\text{val}(\eta)} i \left( \frac{1}{p^{-i}} - \frac{1}{p^{-i+1}} \right) \]

\[(A.3.4.5) = |\eta|^{1/2} \left( \frac{\text{val}(\eta)}{2} - \frac{p^{\text{val}(\eta)}}{p - 1} \right)\]

where the equality (A.3.4.2) follows from the Iwasawa decomposition. \[\square\]

(A.3.5) Lemma Let \( \eta \) be a split element of \( A_{\text{reg}}(\mathbb{Q}_p) \), then

\[(A.3.5.1) \quad J_M^G(\tilde{\eta}, \mathbb{I}_{g'(Z_p)}) = \begin{cases} \frac{\text{val}(\eta)}{2} + \frac{|\eta|^{1/2}}{p - 1} - \frac{2p + 1}{p^2 - 1} & \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\ \frac{|\eta|^{-1}}{1 - p^{-2}} - \frac{|\eta|^{-1/2}}{1 - p^{-1}} & \text{otherwise}. \end{cases} \]

(A.3.6) Proof By definition

\[(A.3.6.1) \quad J_M^G(\tilde{\eta}, \mathbb{I}_{g'(Z_p)}) = J_M^G(\tilde{\eta}, \mathbb{I}_{g'(Z_p)}) - \int_{m'(Q_p)} J_M^G(X, \mathbb{I}_{g'(Z_p)}) \psi'(\text{Tr}(\tilde{\eta}^T X)) \, dX \]

\[(A.3.6.2) = J_M^G(\tilde{\eta}, \mathbb{I}_{g'(Z_p)}) - \int_{Q_p} J_M^G \left( \begin{array}{cc} x & \\ -x & \end{array} \right), \mathbb{I}_{g'(Z_p)} \right) \psi(2\sqrt{\eta}x) \, dx \]

where \( \psi \) denotes the additive character of \( \mathbb{Q}_p \) involved in the definition of the Fourier transform on a \( p \)-adic vector space, the superscript \( T \) denotes the transpose of a \( 2 \times 2 \) matrix and \( \text{Tr} \) denotes the trace of a \( 2 \times 2 \) matrix.

The Fourier transform of \( \text{val}(\ ) \mathbb{I}_{Z_p}(\ ) \) on \( \mathbb{Q}_p \) is the function such that

\[(A.3.6.3) \forall x \in \mathbb{Q}_p, \quad \left( \text{val}(\ ) \mathbb{I}_{Z_p}(\ ) \right)^*(x) = \left( \sum_{i=1}^{\infty} \mathbb{I}_{p^i Z_p}(\ ) \right)^*(x)\]
\[= \sum_{i=1}^{\infty} \frac{1}{p^i} \mathbb{I}_{p^{-i}}(x)\]

\[= \sum_{i=\max\{1,-\text{val}(x)\}}^{\infty} \frac{1}{p^i} \mathbb{I}_{p^{-i}}(x)\]

\[= \frac{p^{-\max\{1,-\text{val}(x)\}}}{1 - p^{-1}}\]

\[= \min\left\{ \frac{p^{-1}}{1 - p^{-1}}, \frac{|x|^{-1}}{1 - p^{-1}} \right\}.\]

The Fourier transform of \(| \mathbb{I}_{z_p}(x) |\) on \(Q_p\) is the function such that

\[\forall x \in Q_p \quad (| \mathbb{I}_{z_p}(x) |)^{\hat{}}(x) = \left( \mathbb{I}_{z_p}(x) + \sum_{i=1}^{\infty} \left( \frac{1}{p^i} - \frac{1}{p^{i-1}} \right) \mathbb{I}_{p^{i-1}}(x) \right)^{\hat{}}(x)\]

\[= \mathbb{I}_{z_p}(x) + \sum_{i=\max\{1,-\text{val}(x)\}}^{\infty} \left( \frac{1}{(-p)^{2i}} + \frac{1}{(-p)^{2i-1}} \right) \mathbb{I}_{p^{-i}}(x)\]

\[= \mathbb{I}_{z_p}(x) + \frac{(-p)^{-2\max\{1,-\text{val}(x)\}-1}}{1 + p^{-1}}\]

\[= \mathbb{I}_{z_p}(x) - \min\left\{ \frac{p^{-1}}{1 + p^{-1}}, \frac{|p|x^{-2}}{1 + p^{-1}} \right\}.\]

Hence by (A.3.3.1) and (A.3.6.1) the invariant weighted orbital integral \(I_{z_p}(\eta, \mathbb{I}_{z_p})\) is equal to

\[\left( \frac{\text{val}(\eta)}{2} - \frac{1}{p - 1} + \frac{|\eta|^{1/2}}{p - 1} \right) \mathbb{I}_{z_p - \{0\}}(\eta)\]

\[-\left( \left( \frac{\text{val}(\eta)}{2} - \frac{1}{p - 1} + \frac{|\eta|^{1/2}}{p - 1} \right) \mathbb{I}_{z_p}(\eta) \right)^{\hat{}}(2\sqrt{\eta})\]

\[= \left( \frac{\text{val}(\eta)}{2} - \frac{1}{p - 1} + \frac{|\eta|^{1/2}}{p - 1} \right) \mathbb{I}_{z_p - \{0\}}(\eta) + \frac{1}{p - 1} \mathbb{I}_{z_p}(\eta)\]

\[-\left( \text{val}(\eta) \mathbb{I}_{z_p}(\eta) \right)^{\hat{}}(2\sqrt{\eta}) - \frac{1}{p - 1} \left( | \mathbb{I}_{z_p}(\eta) |^{\hat{}}(2\sqrt{\eta})\right)\]

\[= \begin{cases} 
\frac{\text{val}(\eta)}{2} + \frac{|\eta|^{1/2}}{p - 1} \\
\frac{-1}{1 - p^{-1}} - \frac{1}{p - 1} \left( 1 - \frac{p^{-1}}{1 + p^{-1}} \right) \quad \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\
\frac{-|2\sqrt{\eta}|^{-1}}{1 - p^{-1}} - \frac{1}{p - 1} \left( -p\frac{|2\sqrt{\eta}|^{-2}}{1 + p^{-1}} \right) \quad \text{otherwise},
\end{cases}\]
\[
\begin{align*}
\text{val}(\eta) + |\eta|^{1/2} - \frac{1}{p-1}\left(\frac{2+p^{-1}}{1+p^{-1}}\right) & \quad \text{if } \eta \text{ lies in } \mathbb{Z}_p - \{0\}, \\
-\frac{|\eta|^{1/2}}{1-p^{-1}} + \frac{|\eta|^{-1}}{(1-p^{-1})(1+p^{-1})} & \quad \text{otherwise,}
\end{align*}
\]

where the equality \((A.3.6.5)\) follows from \((A.3.6.2)\) and \((A.3.6.3)\).

\((A.3.7)\) Remark If \(\eta\) is a split element of \(A_{\text{reg}}(\mathbb{Q}_p)\), then \(\phi_{1,v}(\eta)\) has three components indexed by the parabolic subgroups \(B, \overline{B}\) and \(G\), and

\[(A.3.7.1)\]

\[\phi_{1,v}(\eta)_B = \phi_{1,v}(\eta)_{\overline{B}} = 1 \quad \phi_{1,v}(\eta)_G = -I_{M}^G(\overline{\eta}, I_{\mathbb{Z}_p}).\]

If \(\eta\) is an elliptic element of \(A_{\text{reg}}(\mathbb{Q}_p)\), then \(\phi_{1,v}(\eta)\) has one component corresponding to the parabolic subgroup \(G\) which is equal to \(\phi_{0,v}(\eta)\).

**List of symbols**

- \(G, g\) a connected reductive group defined over \(\mathbb{Q}\) and its Lie algebra
- \(v, S\) a place of \(\mathbb{Q}\) and a finite set of places of \(\mathbb{Q}\)
- \(Q_v, Q_S, A\) the completions of \(\mathbb{Q}\) at \(v\) and \(S\) and the ring of adeles of \(\mathbb{Q}\)
- \(|v|, |S|, |A|\) the norms on \(Q_v, Q_S\) and \(A\)
- \(G(Q_v), G(Q_S), G(A), G(\mathbb{Q})\) the topological groups of \(Q_v, Q_S, A\) and \(\mathbb{Q}\)-valued points of \(G\)
- \(P_0, M_0, A_0\) a minimal parabolic subgroup, a minimal Levi subgroup and a maximal split torus
- \(P, M_P, N_P, A_P\) a parabolic subgroup, its Levi component, its unipotent radical and its split component
- \(P_i, M_i, N_i, A_i\) a parabolic subgroup, its Levi component, its unipotent radical and its split component
- \(\overline{P}, N_P\) the parabolic subgroup opposite to \(P\) and its unipotent radical
- \(\mathcal{F}, P, \mathcal{L}\) finite sets of parabolic subgroups and Levi subgroups
- \(X(M_P), X(M_P)^*\) the group of rational characters of \(M_P\) and its dual
- \(a_P, a_P^*\) a Euclidean space and its dual space
- \(\Phi_P, \Delta_P, \Phi_P^\vee, \Delta_P^\vee\) the sets of roots, simple roots, coroots and simple coroots of \(P\)
- \(\Delta_1^2, \Delta_1^2, \Delta_1^2, \Delta_1^2\) sets of simple roots and coroots, coweights and weights
- \(a_1^2, a_2^2\) subquotients of \(a\) and \(a^*\)
- \(W_0^G, W(a_1, a_2)\) the Weyl group of \(G\) and the Weyl set from \(a_1\) to \(a_2\)
- \(\tau_1^2, \tau_1^2\) characteristic functions of positive cones in \(a\)
- \(G(\mathbb{A})^1\) the subgroup of \(G(\mathbb{A})\) of elements of norm one
- \(K, K_v\) global and local admissible maximal compact subgroups
- \(H_P\) a height function on \(G(\mathbb{A})\)
- \(\rho_P\) the Weyl vector in \(a^*\)
- \(T^0, T\) a fixed negative point and a truncation parameter in \(a\)
- \(\omega\) a compact subset of \(N_0(\mathbb{A})M_0(\mathbb{A})^1\)
- \(\mathcal{G}(T', \omega), \mathcal{G}^T(T', \omega)\) a Siegel domain and a truncated Siegel domain
$F^G(x, T), F^P(x, T)$ characteristic functions of a truncated Siegel domain

$S, C_c^\infty$ spaces of Schwartz functions and compactly supported smooth functions

$[g(z_p)]$ the characteristic function of the points in $g(\mathbb{Q}_p)$ with $p$-adically integral coordinates

$\langle , \rangle, \psi, \bar{\psi}_v$ a nondegenerate invariant bilinear form on $g$, a additive character on $A$ and its $\mathbb{Q}_v$-component

$\land, \lor$ the Fourier transform and the inverse Fourier transform

$\sim, o$ an equivalence relation on $g(\mathbb{Q})$ and an equivalence class

$X_{ss}, X_{nil}$ the semisimple and nilpotent components of $X$ under the Jordan decomposition

$D, D^M$ the discriminant functions on $g$ and $m$

$\mathfrak{g}_{\text{reg, ss}}$ the locus of regular semisimple elements of $g$

$G_X, G^0_X, \pi_0(G_X)$ the centralizer of $X$ in $G$, its identity component and its group of connected components

$G_S, G_v$ the base changes of $G$ to $\mathbb{Q}_S$ and $\mathbb{Q}_v$

$P_{S,0}, M_{S,0}, A_{S,0}$ a pair of minimal parabolic subgroups, minimal Levi subgroups

$P_{r,v,0}, M_{r,v,0}, A_{r,v,0}$ and maximal split tori of $G_S$ and $G_v$

$T_S, T_v, t_S, t_v$ maximal tori of $G_S$ and $G_v$ and the associated Cartan subalgebras

$\mathcal{T}_o(G_S)$ the set of conjugacy classes of elliptic maximal tori of $G_S$

$W(G_S, T_S)$ the Weyl group of the pair $(G_S(Q_S), T_S(Q_S))$

$I^G_C(X, )$ the standard normalized orbital integral

$K(x, f), K_o(x, f)$ the kernel functions of Chaudouard

$I_o(f)$ a variant of the standard orbital integral in the anisotropic case

$K_{P, o}(x, f), k^T_o(, f)$ a variant of the kernel function and the truncated kernel function of Chaudouard

$J^T, J^T$ distributions on $g(A)$

$\sigma^T_o( )$ a combinatorial function on $a$

$m^Q_{\mathbb{P}}, m^Q_{\mathbb{P}}(Q), n_2^3$ the quotient of $m_Q$ by $m_{\mathbb{P}}$, a subset of $m^Q_{\mathbb{P}}(Q)$ and the quotient of $n_2$ by $n_3$

$\Gamma^p_o(, T)$ the geometric gamma' function

$J^M_o(, T)$ a sum of distributions on $m(A)$

$f_P$ the parabolic descent of $f$ along the parabolic subgroup $P$

$\| \|$ the Euclidean norm on $a$

$T_0$ a distinguished point in $a$

$s, s$ a representative of the element $s$ of the Weyl group in $G(Q)$

$\mathcal{J}, J_o$ distributions on $g(A)$

$\nu_M(x)$ the weight factor

$f_{P,x}, v^P_{P}(X)$ a noninvariant version of the parabolic descent of $f$ along $P$ and a variant of the weight factor $\nu_M(x)$

$\theta_P$ a polynomial on $i\mathbb{A}_M^*$

$(c_P), c_M$ a $(G, M)$-family and its associated constant

$(c_P^Q), (c_P)$ the $(L, M)$-family and the $(G, L)$-family associated to the $(G, M)$-family $(c_P)$

$c_Q$ the function or constant associated to the $(G, M)$-family $(c_P)$

$\mathcal{Y}_M, Y_P$ a $(G, M)$-orthogonal set and an element of $\mathcal{Y}_M$

$(v^P_{P}(\mathcal{Y}_M))$ the $(G, M)$-family associated to the $(G, M)$-orthogonal set $\mathcal{Y}_M$
\( \mathcal{Y}_M(x), v_M(x) \) a positive \((G, M)\)-orthogonal and its associated constant which is the weight factor \( J^D_G(X) \) the weighted orbital integral \( r^c_G(x, a) \) an auxiliary \((L, M)\)-family \( f_{F,P,x}, v_Q(y) \) an \( S \)-local noninvariant parabolic descent of \( f \) along \( P \) and its associated weight factor \( g_{\text{nil}}, J_{\text{nil}}, J^T_{\text{nil}} \) the nilpotent locus and the distributions supported on \( g_{\text{nil}} \) \( \| \|, d(T) \) a continuous seminorm on \( S(g(\mathbb{A})) \) and the distance from \( T \) to the root hyperplanes \( \Phi_X(x, Y) \) a partial Fourier transform of \( f \) \( \Gamma, A^{0,T}_Q(\mathbb{R}), \delta_0^2 \) a compact subset of \( M_0(\mathbb{A})^1 \), a compact subset of \( A^1_Q(\mathbb{R}) \) and the modulus function of \( P_2 \) \( D, \Phi^D \) an invariant differential operator on \( g(\mathbb{R}) \) and its action on \( \Phi \) via the Fourier transform \( N(f) \) a natural number determined by a Schwartz function \( f \) \( \hat{Z} \) the profinite completion of \( Z \) \( \| \|' \) a seminorm on \( S(g(\mathbb{A})) \) \( Z_\mu, Z_\nu \) the components of \( Z \) on the weight space of the weight \( \mu \) and on \( \nu \) \( \phi_\bullet, \Psi(a_0) \) positive Schwartz functions and the upper bounded determined by the collection of \( \phi_\bullet \) \( \beta_\nu \) a bump function on \( Q_\nu \) \( \nu, \{p_1, \ldots, p_l\} \) a nilpotent orbit and a collection of polynomials vanishing on \( \nu \) \( f_{\nu,x}^c \) a function obtained by truncating \( f \) around \( \nu \) \( \| \|_1 \) a seminorm on \( S(g(\mathbb{A})) \) \( J^F_G \) a distribution supported on \( \mathcal{V} \) \( \| \| \) a seminorm on \( S(g(\mathbb{A})) \) \( J^F_P \) a distribution supported on \( \mathcal{V} \) \( m_{\text{nil}}(Q)_{M,S} \) the set of \( M(Q_S)\)-conjugacy classes in \( m_{\text{nil}}(Q) \) \( a^M(S, \nu) \) a global coefficient that depends on \( S \) \( T^G \) a distribution supported on \( g_{\text{nil}}(Q_S) \) \( g_{\text{nil},d}(Q_S) \) an open subset of \( g_{\text{nil}}(Q_S) \) \( T^G_d, T^{G,d}_d \) \( T^{G,d}_d \) distributions constructed from \( T^G \) and \( g_{\text{nil},d}(Q_S) \) \( \Sigma \) a semisimple element in \( \sigma \) \( P^0_{1,\Sigma}, M^0_{1,\Sigma}, K_\Sigma \) a minimal parabolic subgroup and a minimal Levi subgroup of \( G^0_\Sigma \) and a maximal compact subgroup of \( G^0_\Sigma(\mathbb{A}) \) \( T_{\Sigma,1} \) a distinguished point in \( \mathfrak{a}^{G_\Sigma}_1 \) \( \mathcal{F}^\Sigma, \mathcal{F}_Q, \mathcal{F}_Q \) finite sets of parabolic subgroups \( \pi_\Sigma \) a surjection from \( \mathcal{F} \) to \( \mathcal{F}^\Sigma \) \( \mathcal{Y}, \mathcal{Y}_Q \) variants of a \((G, M)\)-orthogonal set \( \Gamma^D_G(\mathcal{Y}_Q), \Gamma^D_G(\mathcal{Y}_Q) \) a gamma' function and its Fourier transform \( \Phi^D_{Q,x}, v'_Q, Y^D(k, x) \) a variant of the parabolic descent of \( f \) along \( Q \) and a weight factor and a \((G, M)\)-orthogonal set involved in its definition \( T_\Sigma \) a truncation parameter in \( \mathfrak{a}_1 \) \( J^D_0(, f), J_{P,0}(x, f) \) variants of a truncated kernel function and its summand \( \Xi \) a semisimple element of \( m(\mathbb{Q}) \) \( (v_P(x, T)), v_Q \) a \((G, M)\)-family and a weight factor

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\[ T_{\Xi\gamma,\eta} \quad \text{a distinguished point in } \mathfrak{a}_{M\mathfrak{g}}^{G_0} \quad \text{page 35} \]
\[ \Phi^T_{S,\mathfrak{g},\eta} \quad \text{an } S\text{-local variant of the parabolic descent of } f_S \text{ along } \mathfrak{g} \quad \text{page 35} \]
\[ \equiv \quad \text{an equivalence relation on } \mathfrak{m}(\mathfrak{g}) \cap \mathfrak{g} \quad \text{page 36} \]
\[ (\mathfrak{m}(\mathfrak{g}) \cap \mathfrak{g})_{M,S} \quad \text{the collection of } \equiv \text{ equivalence classes in } \mathfrak{m}(\mathfrak{g}) \cap \mathfrak{g} \quad \text{page 36} \]
\[ \mathfrak{S}_\gamma \quad \text{a sufficiently large finite set of places of } \mathfrak{g} \quad \text{page 37} \]
\[ a^M(S,\mathfrak{g}) \quad \text{a global coefficient that depends on } S \quad \text{page 37} \]
\[ \mathcal{L}_\mathcal{G}(M_1) \quad \text{a finite set of Levi subgroups} \quad \text{page 38} \]
\[ a^M(S,\mathfrak{g}) \quad \text{a global coefficient that depends on } S \quad \text{page 39} \]
\[ \Gamma, \mathcal{G}_\gamma, \mathfrak{a}(\mathfrak{q})' \quad \text{a compact subset of } \mathfrak{g}(\mathfrak{a}), \text{ a compact subset of } \mathfrak{g}(\mathfrak{a})^1 \text{ and a subset of } \mathfrak{g}(\mathfrak{q}) \quad \text{page 40} \]
\[ I_{\mathcal{L},\mathfrak{a}}^G(X,\gamma), I_{\mathcal{L},\mathfrak{a}}^J(X,\gamma) \quad \text{the invariant weighted orbital integral and its Fourier transform} \quad \text{page 40} \]
\[ I_{\mathfrak{a},\mathfrak{g}}^G(X,\gamma) \quad \text{a parabolic subgroup in } \mathcal{P}(\mathfrak{M}) \quad \text{page 41} \]
\[ I_{\mathfrak{a},\mathfrak{g}}^G(X,\gamma) \quad \text{an alternating sum of invariant weighted orbital integrals} \quad \text{page 42} \]
\[ I_{\mathfrak{a},\mathfrak{g}}^G(X,\gamma) \quad \text{an alternating sum of local invariant weighted orbital integrals} \quad \text{page 43} \]
\[ J_{\mathfrak{a},\mathfrak{g}}^G(X,\gamma) \quad \text{the parabolic descent of } f_S \text{ along } \mathfrak{g} \quad \text{page 44} \]
\[ \text{Ind}_{\mathfrak{a},\mathfrak{g}}^G(X,\gamma) \quad \text{the orbit in } \mathfrak{g}(\mathfrak{q}) \text{ parabolically induced from } X \text{ along } \mathfrak{g} \quad \text{page 45} \]
\[ a^G_{M}(\mathfrak{a},\mathfrak{g}) \quad \text{a bivariate function on } \mathcal{L}(\mathfrak{M}), \text{ a vector in general position in } \mathfrak{a} \text{ and the partial map determined by } \xi \quad \text{page 46} \]
\[ E_\gamma, S(E_\gamma) \quad \text{the } \gamma\text{-dimensional Euclidean space and its scissors group} \quad \text{page 50} \]
\[ \Phi, V_\gamma, r, \mathcal{I}_\mathcal{R}(\Phi) \quad \text{a strict flag in } E_\gamma, \text{ the } i\text{th subspace of } \Phi, \text{ a rigging of } \Phi \text{ and the collection of equivalence classes of riggings of } \Phi \quad \text{page 51} \]
\[ B, B_\xi, \text{sign}(\xi) \quad \text{two ordered bases of } E_\gamma \text{ and the sign of } \xi \quad \text{page 52} \]
\[ \Phi^*, \partial_\Phi(P), r^\mathfrak{g}_{\text{min}}(\mathfrak{p}) \quad \text{a rigged flag, the } \Phi^\mathfrak{g}\text{-boundary of the polytope } P \text{ and an auxiliary function} \quad \text{page 53} \]
\[ \text{Had}_\Phi(P) \quad \text{the Hadwiger invariant of the polytope } P \quad \text{page 54} \]
\[ (H_{\Phi}, F_\Phi, (\Phi, (i) \text{ the scissors group of positive } (G, \mathfrak{M})\text{-orthogonal sets and a collection of strict flags in } \mathfrak{g}) \quad \text{page 55} \]
\[ \text{S}(a^G_{\mathfrak{M}}), F_S(a^G_{\mathfrak{M}}) \quad \text{the scissors group of positive } (G, \mathfrak{M})\text{-orthogonal sets and a collection of strict flags in } a^G_{\mathfrak{M}} \quad \text{page 56} \]
\[ \mathbb{E}_{\gamma}^\mathfrak{a}(\mathbb{E}_{\gamma}) \quad \text{the scissors-congruence-valued orbital integral} \quad \text{page 57} \]
\[ d\mathfrak{a}^{L_{\gamma}-1} \quad \text{a differential form on } \mathfrak{a}_{\mathfrak{M}}^{L_{\gamma}-1} \quad \text{page 58} \]
\[ S(a^G_{\mathfrak{L}(\mathfrak{M})}) \quad \text{the total scissors ring of } a^G_{\mathfrak{M}} \quad \text{page 59} \]
\[ \otimes \gamma, j^\gamma \quad \text{a bilinear product on the total scissors ring and two auxiliary maps} \quad \text{page 59} \]
\[ (\mathfrak{S}(a^G_{\mathfrak{L}(\mathfrak{M})})), \mathfrak{S}(a^G_{\mathfrak{L}(\mathfrak{M})}) \quad \text{the graded components of the total scissors ring with respect to two different gradings} \quad \text{page 60} \]
\[ d^G_{\mathfrak{M}} \quad \text{a homomorphism from } S(a^G_{\mathfrak{L}(\mathfrak{M})}) \text{ to } S(a^G_{\mathfrak{L}(\mathfrak{L})}) \quad \text{page 61} \]
\[ \hat{\mathcal{L}}_{\mathfrak{G}}^G(X,\gamma) \quad \text{the total orbital integrohedron} \quad \text{page 62} \]
\[ M(a^G_{\mathfrak{L}(\mathfrak{M})}), M(a^G_{\mathfrak{L}(\mathfrak{L})}) \quad \text{the total measure coalgebra of } a^G_{\mathfrak{M}} \text{ and its components} \quad \text{page 63} \]
\[ \Delta, j_\gamma, (\gamma, \gamma) \quad \text{a bilinear coproduct on the total measure coalgebra, two auxiliary maps, a bilinear pairing between the total measure coalgebra and the total scissors ring and the cap product} \quad \text{page 64} \]
\[ b^G_{\mathfrak{M}}, j_\gamma, b^G_{\mathfrak{M}} \quad \text{a linear endomorphism of the total measure coalgebra, two auxiliary maps and a variant of the cap product} \quad \text{page 65} \]
\[ j^G_{\mathfrak{M}} \quad \text{a homomorphism from } M(a^G_{\mathfrak{L}(\mathfrak{L})}) \text{ to } M(a^G_{\mathfrak{L}(\mathfrak{M})}) \quad \text{page 66} \]
$\mathfrak{g}_M, j, j^*, \tilde{\mathfrak{g}}_M$ a linear endomorphism of the total measure coalgebra, two auxiliary maps and a variant of the cap product

\[ a^G_M(S, X), \mu^G_M \] a measure-valued global coefficient that depends on $S$ and a measure on $a^G_M$

\[ \mathcal{N}_{\text{rig}, S}(X), \Gamma( , \nu) \] the collection of $G_X(\mathbb{Q}_S)$-conjugacy classes of rigid nilpotent orbits of $X$ and a Shalika germ for a rigid nilpotent orbit

\[ \alpha^G_M(S, X) \] a total measure-valued global coefficient that depends on $S$

\[ \mathfrak{g}_m(\mathbb{Q}) \] the subset of semisimple elements of $\mathfrak{g}(\mathbb{Q})$

\[ G, M_0, B \] the general linear group $GL(n, \mathbb{Q})$ and the standard choice of the minimal Levi subgroup and the Borel subgroup

\[ \mathcal{A}_G, D, \mathcal{A}_{G, \text{reg}}, \tau_M \] the space of characteristic polynomials, the discriminant function, the nonvanishing locus of $D$ and the natural map from $\mathcal{A}_M$ to $\mathcal{A}_G$

\[ |D(X_v)|^{1/2} dX_v \] a measure on $\mathcal{A}_G(\mathbb{Q}_v)$

\[ \mathcal{A}_{M, \text{reg}}(\mathbb{Q}_v) \] the subset of $\mathcal{A}_{M, \text{reg}}(\mathbb{Q}_v)$ consisting of the $\mathbb{Q}_v$-elliptic elements

\[ T^G_{\text{max}}(X_v, \ ) \] the vector-valued orbital integral

\[ q^G_M(S, X), \text{Tr} \] a signed global coefficient that depends on $S$ and a linear functional

\[ T^G_{\text{max}}(X_v, ) \] the maximal orbital integral

\[ \tilde{X}_v, M[X_v] \] a lift of $X_v$ in $\mathfrak{g}(\mathbb{Q}_v)$ and a standard Levi subgroup whose Lie algebra contains $\tilde{X}_v$

\[ \mathcal{S}_0(\mathcal{A}_G(\mathbb{Q}_v)), \mathcal{S}_1(\mathcal{A}_G(\mathbb{Q}_v)) \] the local Schwartz spaces

\[ \Lambda_0, \mathbb{E}, M^T, \text{Tr} \] the standard lattice in $\mathfrak{g}(\mathbb{Q}_p)$, its characteristic function, the standard Gaussian on $\mathfrak{g}(\mathbb{R})$, the transpose of the matrix $M$ and the matrix trace

\[ \phi_{0,v}, \phi_{1,v} \] the local basic functions

\[ Z_v, U_{\varphi_v, Z_v}, \Gamma^G_v( , \nu) \] a singular point, an open neighborhood of $Z_v$ and the Shalika germ at a nilpotent orbit $\nu$

\[ \mathcal{S}_0(\mathcal{A}_G(\mathbb{A})), \mathcal{S}_1(\mathcal{A}_G(\mathbb{A})) \] the global Schwartz spaces

\[ \mathcal{H}_v \] the local Harish-Chandra transform

\[ T^G_{\text{max}}(X_v, ) \] the function representing the Fourier transform of the maximal orbital integral

\[ \mathcal{K}_v( , ) \] the local Harish-Chandra kernel

\[ \tilde{\mathfrak{g}}_M( , ) \] Waldspurger’s notation for the local Harish-Chandra kernel

\[ \delta_\chi_v(, \delta_{\chi_v,i}), \tilde{\delta}_{\chi_v,i} \] the Dirac distribution at $\chi_v$, a $\delta$-sequence converging to $\delta_{\chi_v}$ and a lift of $\delta_{\chi_v,i}$ on $\mathfrak{g}(\mathbb{Q}_v)$

\[ \mathcal{H}_S, \mathcal{H} \] the $S$-local and global Harish-Chandra transforms

\[ Z, \sigma_Z, Z', \varphi_S(Z') \] a singular point in $\mathcal{A}_G(\mathbb{Q})$, its fiber in $\mathfrak{g}(\mathbb{Q})$, a $\equiv$ equivalence class in $(\mathfrak{m}(\mathbb{Q}) \cap \sigma_Z)_M^{S,S}$ and the value determined by the nearby regular values $\varphi_S(X)$

\[ q^M(\varphi_v(X)), q^M(\varphi_{S}(X)) \] the value determined by $\varphi_v(X)$ via descent of invariant weighted orbital integrals, and the $S$-local value determined by the local values $q^M(\varphi_v(X))$ via splitting of invariant weighted orbital integrals

\[ q^M(\varphi_{S}(Z')) \] the value determined by the nearby regular values $q^M(\varphi_{S}(X))$

\[ a(X), q^{\text{ell}}(\varphi_S(X)) \] the global coefficient for a regular $X$, and the $\mathbb{Q}$-elliptic value determined by $\varphi_S(X)$ for a regular $X$
Λ_{N,w}, I_{Λ_{N,w}} a lattice in \( g(\mathbb{Q}_w) \) for a finite place \( w \) and its characteristic function page 73

\( \pi_\infty \) the natural projection from \( g(\mathbb{R}) \) to \( A_G(\mathbb{R}) \) page 73

\( T_{X_v} \) the centralizer of \( X_v \) which is a torus page 74

\( \Lambda_{N,\infty}, \Lambda_{N,w} \) a lattice in \( g(\mathbb{R}) \), a lattice in \( g(\mathbb{Q}_w) \) for a finite place \( w \), and the characteristic function of the translation of \( \Lambda_{N,w} \) by \( \tilde{X}_\infty \) page 74

\( T_Y, \hat{T}_Y, F, \Gamma \) the centralizer of \( Y \) which is a maximal torus, its group of algebraic characters, the splitting field of \( T_Y \) and its Galois group page 75

\( \mathcal{D} \) an invariant distribution on a \( p \)-adic reductive Lie algebra page 76

\( v, p, \mid \mid \) an odd rational prime and the \( p \)-adic absolute value page 76

\( G, M, B \) the group \( GL(2) \) and the standard choice of the minimal Levi subgroup and the Borel subgroup page 76

\( g, g', m', \mathfrak{g} \) the Lie algebras \( gl(2), sl(2), m \cap g' \) and the center of \( g \) page 76

\( A, L, D, A_{\text{reg}}, \mathbb{Q}_p \times \) the affine quotient of \( g' \) by \( G \), the affine line, the discriminant function on \( A \), the nonvanishing locus of \( D \) and the group of units of \( \mathbb{Q}_p \) page 76

\( \eta, g'_\eta, d_\eta \mathcal{X} \) a point of \( A_{\text{reg}} \), the fiber of \( \eta \) in \( g' \) and a differential form on \( g'_\eta \) page 76

\( \text{val}(\eta) \) the \( p \)-adic valuation of \( \eta \) page 78

\( \tilde{\eta} \) a lift of a split \( \eta \) in \( g'_{\text{reg}, ss}(\mathbb{Q}_p) \) page 80

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