MATTER OF RESOLUTION: FROM QUASICLASSICS TO FINE TUNING

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Recently, there appeared results of lattice measurements in Yang-Mills theories which indicate non-trivial dependences on the lattice spacing of many observables. In particular, volume occupied by fermionic zero modes shrinks to zero in the continuum limit. These results are in apparent disagreement with quasiclassical models which assume that all the non-perturbative effects develop on the scale of $\Lambda_{QCD}$. We emphasize that this kind of contradictions might be superficial since the results in point depend in fact on the measurements procedure. The lattice simulations correspond to measurements with high resolution while the quasiclassical picture assumes poor resolution. We will argue that the general trend is that what looks quasiclassical in measurements with poor resolution becomes fine tuned in measurements with fine resolution. The main emphasis is on the topological fermionic modes and we argue that shrinking of the volume occupied by the modes could have been predicted theoretically.

1 Introduction

Some time ago it was demonstrated\(^1\) that confining fields are soft, $A_\mu \sim \Lambda_{QCD}$. Namely, both confining and topological properties survive if one removes from the vacuum fields large quantum fluctuations, $A_\mu \sim 1/a$, where $a$ is the lattice spacing.

More recently, however, there appeared measurements on original quantum field configurations $\{A_\mu^a(x)\}$, without cooling, which demonstrate unexpected dependences on the lattice spacing, or the ultraviolet cut off. In particular, the so called center vortices which appear to be responsible for confinement, for references see\(^2\) possess ultraviolet divergent non-Abelian action:\(^3\)

$$S_{vort} \approx 0.54 \frac{(Area)_{vort}}{a^2},$$

where $a$ is the lattice spacing. On the other hand, the area of the vortices is in physical units:

$$(Area)_{vort} \sim \Lambda_{QCD}^2 V_{tot},$$

where $V_{tot}$ is the volume of the lattice and references can be found in\(^2\). Thus, according to measurements\(^1\) and\(^2\) the confining fields are not soft but rather fine tuned, exhibiting dependences both on the UV and IR scales.

Most recently, it was also found that the volume occupied by low-lying fermionic modes shrinks to zero in the continuum limit.\(^4\) In more detail, the volume is defined in terms of the so called Inverse Participation Ratio (IPR), see, e.g.\(^5\). One finds eigenfunctions of the Dirac equation,

$$D_\mu \gamma_\mu \psi_n = \lambda_n \psi_n,$$

where the covariant derivatives $D_\mu$ are constructed on quantum vacuum configurations of the gauge fields $\{A_\mu^a(x)\}$. Furthermore, one introduces $\rho_n(x) = (const) |\phi_n|^2$ nor-
malized as
\[ \int d^4 x \rho_n(x) = 1 . \] (4)

The IPR is defined then as
\[ (IPR)_n \equiv V_{tot} \int d^4 x \rho_n^2(x) \equiv \frac{V_{tot}}{(V_{mode})_n} , \] (5)

where \( V_{tot} \) is the total volume and \((V_{mode})_n\) is the volume occupied by the mode, by definition. The observation is that for low-lying modes one has:
\[ \lim_{a \to 0} V_{mode} \sim (a \cdot \Lambda_{QCD})^r V_{tot} , \] (6)

where \( a \) is the lattice spacing and \( r \) is a positive number of order unit. In other words, the localization volume apparently shrinks to zero in the continuum limit \( a \to 0 \). \(^a\)

All this accumulating evidence in favor of non-trivial dependences on the lattice spacing came unexpected because the common belief is that all non-perturbative effects in Yang-Mills theories are controlled by the scale of \( \Lambda_{QCD} \). In particular, the fermionic zero modes are a classical application of the instanton model, beginning with the seminal paper\(^7\).

In an attempt to appreciate the result (6), let us first emphasize that by measuring (IPR) and repeating or averaging over such measurements we do not determine any matrix element. Therefore, answers to such questions as ‘what is the volume occupied by zero modes’ can well depend on the measuring procedure itself. The latest results on the low-lying fermionic modes\(^4\) correspond to measurements with high resolution. Indeed, fixing a particular vacuum configuration \( \{A^\mu_a(x)\} \) via Monte Carlo simulations is equivalent to measuring all the fields with resolution \( a \). On the other hand, the classical instanton calculus, see, e.g.,\(^7\), applies to the case when one specifies initial and final states and no measurements are performed in between. In other words, quasiclassical picture corresponds to measurements with poor resolution.

Using examples, we will argue that the general trend is that quasiclassical fields measured with high resolution appear fine tuned.

Our main emphasis is on topological fermionic modes, section 3. We argue that in fact the result (6) could have been predicted theoretically and suggest bounds on the value of \( r \):
\[ 1 \leq r \leq 3 . \] (7)

Section 2 is a kind of introduction where we reiterate well-known facts about monopole clusters. The reason is that similar structures arise in other cases which we will consider. In Section 4 we emphasize the short-distance facet of the lattice strings mentioned above, see Eqs (1) and (2). In Section 5 we discuss briefly possible connection between strings and fermionic modes.

\(^a\) Independent evidence in favor of shrinking volume of topologically nontrivial gauge-field configurations was obtained in Ref\(^6\).
2 A case study: monopole clusters

As a kind of introduction let us describe briefly a case where the phenomenon which we call transition from quasiclassics to fine tuning is well known and explicit. We mean description of the phase transition to confinement in compact U(1) theory in terms of percolating cluster. In more general terms, we will recapitulate a few points, in a non-systematic manner, from so called polymer approach to field theory, see, e.g.,.

2.1 Percolating cluster

In all the cases which we will consider fine tuned configurations are in fact percolating clusters of trajectories or surfaces. Let us recall, therefore, the simplest model for percolating cluster. Assume that probability of a given link to belong to a ‘trajectory’ is equal to \( p \) and is independent on other links. Then the probability to find a connected trajectory of length \( L \) is proportional to

\[
W(L) \sim p^{L/a} N_L \approx \exp\left(\frac{L}{a} \ln p + c_{\text{geom}}\right)
\]

where \( N_L \) is the number of different trajectories of same length \( L \) and \( c_{\text{geom}} \) is the number of choices, on a given lattice to continue the trajectory. For example, for hyper-cubic lattice in 4d

\[
c_{\text{geom}} = \ln 7,
\]

if we concentrate on closed trajectories.

If we start with very small \( p \) and gradually increase it, we encounter a phase transition to percolation at

\[
p_{cr} = 1/7.
\]

At this point, infinite-length trajectory is allowed and an infinite, or percolating cluster emerges. One of the central points is that even at \( (p - p_{cr}) \equiv \epsilon \geq 0 \) the percolating cluster is dilute as far as \( \epsilon \) is small:

\[
\theta_{\text{link}} \sim (p - p_{cr})^\alpha,
\]

where \( \theta_{\text{link}} \) is the probability of a given link to belong to the infinite cluster and the critical exponent is positive, \( \alpha > 0 \).

2.2 Compact U(1)

The percolation picture can readily be mapped onto a free field theory (in Euclidean space), see, e.g.. As an example close to our subject consider theory with action

\[
S = \frac{1}{4e^2} \int d^4xF_{\mu\nu}^2,
\]
where \( F_{\mu\nu} \) is the electromagnetic field. Then there exists a classical solution, Dirac monopole with magnetic field \( \mathbf{H} \sim (1/e)r/r^3 \). The corresponding self energy diverges in the ultraviolet:

\[
M_{\text{mon}} = \frac{\text{const}}{e^2a},
\]

where \( a \) is the lattice spacing and the constant is known explicitly for a given regularization. Then the action associated with the monopole trajectory is

\[
S_{\text{mon}} = \frac{\text{const} L}{e^2a},
\]

and the factor \( \exp(-S_{\text{mon}}) \) can be replaced by the factor \( p^{L/a} \) in Eq \((8)\) if we choose \( p = \exp(-\text{const}/e^2) \), where the constant is the same as in Eq. \((12)\).

Next steps are the same as in the preceding subsection. The phase transition to percolation of monopoles is simultaneously phase transition to confinement of external electric charges.

Note that the mapping of percolation theory into the free field theory can be completed by evaluating path integral for propagator of a particle with the classical action \((13)\). The propagating mass, \( m_{\text{phys}} \) turns to be:

\[
m_{\text{phys}}^2 \approx \frac{\text{const}'}{a^2} \left( \frac{\text{const}}{e^2} - \ln 7 \right).
\]

The central point is that only fine tuned theories with

\[
e^2 - e^2_{\text{crit}} \equiv \epsilon \sim a^2m^2
\]

are interesting. Indeed, only in this case \( m^2 \) can be kept independent of the lattice spacing with \( a \to 0 \).

2.3 Fine tuning and short distances

Keeping \( m^2 \) in Eq \((14)\) finite corresponds to fine tuning between energy and entropy. Both quantities are UV divergent while their difference is not. Indeed, Eq \((14)\) is an example of a general relation:

\[
\text{(Free energy)} = \text{(Energy)} - \text{(Entropy)}.
\]

Only free energy is ‘physical’ and it does not depend on the lattice spacing. One can wonder what is then use to follow fine tuning between the energy and entropy if none of these quantities is of direct physical meaning.

The answer to this question is that by observing fine tuning, as cancellation between two UV divergent quantities, we check that the monopoles are indeed point-like. The polymer approach to field theory which we are in fact highlighting now turns relevant to the monopole physics because monopoles are defined at short distances, in terms of violations of Bianchi identities:

\[
j_{\mu} \equiv \partial_{\nu} \tilde{F}_{\mu\nu}.
\]

‘Compactness’ of U(1) is equivalent to postulating that the Dirac string does not cost action.
where $j_\nu$ is the monopole current. Clearly enough, Bianchi identities can be violated only on singular fields.

Thus, in measurement with resolution of order lattice spacing $a$ percolating monopole cluster is very dilute, see (10) and there is fine tuning, see (14). In the classical picture, on the other hand, the same phenomenon of the monopole condensation is described as

$$<\phi_M> \neq 0,$$  

(18)

where $\phi_M$ is magnetically charged field. In the classical picture $<\phi_M>\neq 0$ and is the same everywhere. This picture corresponds to smearing of the dilute cluster seen at short distances.

3 Topological fine tuning of fermionic modes

3.1 Tunneling time: QM case

Rather recently, there was an extensive discussion, how long does it take for a particle to traverse a (quantum-mechanical) barrier, for review see (11). Below, we will argue that the results obtained are a kind of analogy to what we are observing now with localization of fermions in YM.

In more detail, the tunneling time is defined in the following way. Consider the famous two-slit experiment. But put now a barrier on one of the two trajectories, so that classically light traveling along this trajectory cannot reach the detector. Quantum mechanically, however, some light goes through and one does observe the interference pattern on the screen. By analyzing the interference pattern, furthermore, one can deduce the time $\Delta t$ which the particle spent ‘under the barrier’: The result is:

$$\Delta t = 0.$$  

(19)

No matter how strange the result might look at first sight, it was predicted within QM. However, the interpretation is not of much interest for us here.

Note that the information on small probability of tunneling is not lost. The point is that the intensity of light penetrating the barrier is small. But, by observing the interference one picks up only the particles which did go through the barrier. And it is on these, selected trajectories that one observes instantaneous transition (19).

Two remarks are now in order. There is no contradiction, of course, between the quasiclassical calculation of the probability of the barrier transition and observation (19). To verify the quasiclassical calculation one needs measurements with poor resolution, averaging over long time intensity of light penetrating the barrier. The instantaneous transition (19) is observed if the resolution is fine: by observing the interference we pick up the particle which went through the barrier.

The second point to mention, is that the result (19) allows for a nice, although might be too naive interpretation. Namely, one can say that particle under the barrier travels in ‘imaginary time’. And then it is only natural that by measuring time, which is real, we cannot detect how long the particle lived in the imaginary time.

3.2 Lessons?

In case of YM theory, there are barrier transitions described by instantons and in the next section we will establish analogy between the QM example just described and measure-
ments in the YM case. Here we will emphasize that the problem we are confronting now is not calculation of quantum corrections to the probability of the instanton transition.

Instantons determine the matrix element of transition between two topologically distinct vacua. Keeping the exponential alone the probability of the tunneling is proportional:

$$P_{\text{tunneling}} \sim \exp \left( - \frac{8\pi^2}{g^2} g^2 V_{\text{tot}} \right),$$  \hspace{1cm} (20)

where $g^2$ is the (classical) coupling. The quantum correction to this result on one loop level was fully calculated$^7$ and the result is

$$P_{\text{tunneling}} \sim \exp \left( - \frac{8\pi^2}{g^2 (\rho^2)} d^4 x d\rho \rho^5 \right),$$  \hspace{1cm} (21)

where $\rho$ is the instanton size. Eq (21) completes calculation of quantum corrections of the probability of the tunneling. However, the standard instanton trajectory dominates (21) provided that we observe only initial and final states which are separated by a very long 'time',

$$T \gg \rho \sim \Lambda_{\text{QCD}}^{-1}$$

There are no measurements in between. Now, we will consider measurements with high resolution. The matrix element (21) does not change, as far as the measuring procedure does not induce topologically non-trivial transitions. Which we assume to be the case since measurements in YM case make simply manifest zero-point fluctuations.

The problem which we will address now is how to predict properties of the 'instanton' trajectory provided that the measurements are made on scale $a$ and

$$a \ll \Lambda_{\text{QCD}}^{-1}.$$ 

This is a problem different from evaluating quantum corrections to (20). Indeed, prediction (19) is not a result of quantum corrections to the standard quasiclassical calculation of the probability of the barrier transition but an answer to a different question.

3.3 Protected matrix elements in the YM case

Quantum-mechanical examples suggest two tools to translate the quasiclassical picture into the picture obtained with high resolution. First, one can use matrix elements which are protected against the ultraviolet noise. These should not depend on the resolution. Second, we arrive at the principle that tunneling time cannot be measured, even if the measurements are made with high resolution. Combining the two observations we shall be able to derive fine tuning of the fermionic modes.

In case of the Yang-Mills theory, one measures two matrix element related to the low-lying fermionic modes. These are the topological susceptibility (Witten relation),

$$<Q^2_{\text{top}}> \sim \Lambda_{\text{QCD}}^4 V_{\text{tot}},$$  \hspace{1cm} (22)

and density of the near-zero modes (Banks-Casher relation):

$$<\bar{q}q>_{\text{quenched}} = - \pi \rho (\lambda_n \to 0) \sim \Lambda_{\text{QCD}}^3.$$  \hspace{1cm} (23)
Both matrix elements (22) and (23) can be estimated within quasiclassical, instanton picture and measured on soft-field configurations which obtained via cooling, for review see, e.g.,12. Since perturbation theory does not contribute to (22), (23) one expects that these relations remains true in measurements with high resolution. And, indeed, in both cases there is no dependence on the lattice spacing, see second paper in Ref.4.

3.4 Tunneling volume: YM case

Imagine that we decide to design an experiment in the YM case which is analogous to measuring tunneling time in QM. Then, naturally, we would proceed in the following way.

Barrier transitions in YM theories are well known, that is, instantons. Thus, we would like to observe an instanton trajectory, with good resolution and measure time (in 4d case, 4d volume in fact) it takes. Well, it is not so simple since measurements on the trajectory introduce fields of order

\[(A^a_\mu)_{\text{quant}} \sim 1/a ,\]

while the quasiclassical fields in the physical vacuum are of order

\[(A^a_\mu)_{\text{class}} \sim \Lambda_{\text{QCD}} \ll a^{-1} .\]

Thus, we have a huge noise. However, presumably, this noise is topologically trivial. Then, an ingenious way to suppress the noise is to concentrate on zero fermionic modes. Indeed, the index theorem guarantees

\[n^+ - n^- = \Delta Q_{\text{top}} .\]

To derive this equation one integrates over the whole volume. However, in reality the integral is saturated by the regions which dominate \(|\psi_0|^2\). Thus, to measure the volume

\[n^+ - n^- = \Delta Q_{\text{top}} .\]

There exist also ‘unprotected’ matrix elements relevant to the study of the topological properties of the vacuum. Consider correlator of topological densities. Perturbatively, the correlator is easy to calculate:

\[< G\tilde{G}(x), G\tilde{G}(0) > = - \frac{144}{\pi^2 x^8} ,\]

and at short distances this is a valid approximation. From Eq (24) alone one concludes that local value of topological charge density is of order

\[G\tilde{G}(x) \sim \frac{12}{\pi^2 (\Delta x)^4} ,\]

where \(\Delta x\) is the resolution. Thus, distribution of the density of topological charge is sensitive to the resolution.

Moreover, Eq (29) implies that there can be no accumulation of topological charge on a finite 4d volume. Thus, structure of the density (of topological charge) distribution, imposed by perturbation theory, looks as lumps of size of order \(a^4\) of large charge density, \(\sim 1/a^4\) with alternating sign. If one follows the chain of lumps of the topological charge of the same sign, the emerging structure seems to be 3d percolating volume. It is worth emphasizing that this ‘3d’ volume does not know anything about \(\Lambda_{\text{QCD}}\) and has such a fractal dimension that ‘3d’ volume fills in a finite part of the 4d volume.

To summarize, gross features of the topological charge density distribution could well be explained by perturbation theory. For further conclusions it would be very useful to have a ‘reference point’ and perform similar measurements of the density distribution in a trivial case of an Abiène gauge field.
occupied by the zero mode seems to be a smart way to determine how long the instanton (barrier) transition takes place!

Moreover, there is a general relation\[^{13}\]

\[
\langle \bar{G} \tilde{G}(x), \bar{G} \tilde{G}(0) \rangle > 0 ,
\]

which follows from unitarity alone and which is satisfied, of course, by the perturbative contribution. Note that for instantons the sign is opposite,

\[
\langle \bar{G} \tilde{G}(x), \bar{G} \tilde{G}(0) \rangle_{\text{instanton}} < 0 ,
\]

as far as the points \((0, x)\) are within one instanton. Thus, trying to separate instanton trajectory from the perturbative noise we are trying to pick up a ‘non-unitary’ contribution. This is an analogy to the ‘imaginary vs real’ time in case of the barrier transition in QM.

Since this point is crucial let us reiterate the argument in terms of dispersion relations. Introduce to this end the function \(f(Q^2)\):

\[
\int d^4x \exp(iqx) \langle 0|T\{\bar{G} \tilde{G}(x), \bar{G} \tilde{G}(0)\}|0 \rangle \equiv f(Q^2) ,
\]

where \(Q^2 = -q^2 > 0\). Then

\[
f(Q^2 = 0) \sim < Q^2_{\text{top}} > > 0 .
\]

In the language of the dispersion relations, the positive sign of \(f(Q^2 = 0)\) can be ensured only by introducing a proper subtraction constant in dispersion relations\[^{16}\]. Indeed,

\[
\text{Im} f(Q^2 < 0) < 0 ,
\]

and the dispersive contribution to \(f(Q^2 = 0)\) would be negative.

Thus, by measuring the volume occupied by zero modes we perform measurements on the ‘size’ of a subtraction constant. Naturally we reveal its local nature, manifested by the vanishing of the volume occupied by zero modes (which echo the shape of the nontrivial topological gauge field configuration).

Note that the low-lying modes appear fine tuned (see second paper in Ref.\[^{4}\]) in the sense that their volume tends to zero while the energy stays stable. With the realization that non-trivial topological transitions correspond to a subtraction constant we get an explanation of this phenomenon of the topological fine tuning.\[^{6}\]

### 3.5 Dimensionality of the "subtraction volume"

An open question is, to which manifold shrink the topological modes. Trying to get an answer, notice first that in the quenched approximation the fermionic modes are delocalized, i.e.,

\[
V_{\text{mode}} \sim V_{\text{tot}} .
\]

\[^{6}\text{One could expect that there exists also a more general mechanism for IPR growing with } a \to 0. \text{ It is interesting to note that removing the topological modes seemingly removes all the non-trivial values of IPR.}\]

\[^{16}\text{Thus there is no other mechanism for fine tuning, but the topological one.}\]
Indeed, if there were localized modes, then there existed mobility edge, \( \lambda_{mob} \) separating localized and delocalized modes. Furthermore, there exists an obvious inequality:

\[
\lambda_{mob}^2 \leq m^2.
\] (34)

Indeed, by mass we understand the lowest eigenvalue which corresponds to the ordinary plane wave, propagating through the whole volume. The eigenfunction corresponding to the mass is not localized. Localized states correspond to a kind of a bound state since the particle moves within a finite volume.

In the quenched approximation chiral fermions cannot acquire mass by symmetry considerations, \( m^2 = 0 \) and Eq (34) immediately implies that fermionic modes cannot be localized.

This means in turn that the modes extend through the whole volume and occupy at least a \( d = 1 \) manifold:

\[
V_{mode} \sim V_{tot} (a \cdot \Lambda_{QCD})^r, \quad r \leq 3.
\]

In this way we come to the upper bound in Eq (7). The lower bound on the value of \( r \), see Eq (7) follows from the observation that the anti-unitary sign in (30) corresponds to one of the coordinates, ‘time’ having extra factor of \( \sqrt{-1} \). The extension of the instanton in the direction of this coordinate cannot be measured.

Consider now the unquenched case. A non-vanishing mobility edge is still not allowed. Indeed, let us start with the Banks-Casher relation, see Eq. (23) which is true in the unquenched case as well. We will also assume that the density \( \rho(0) \) is not vanishing. Then it remains true that the near-zero modes constrained by this relation are delocalized. Indeed, recall that near-zero modes become zero modes in the limit of the infinite volume, according to this relation. If these ‘near-zero’ modes were localized in a finite volume, they would not be sensitive to the total volume which is much larger than the localization volume. The rest of argumentation leading to (7) remains unchanged compared to the quenched case.

3.6 Summary on the fermionic modes

Let us summarize briefly the issue of the topological fermionic modes.

Low-lying modes fill in the window in the perturbative spectrum:

\[
0 \leq \lambda_n \leq \frac{\pi}{L},
\] (35)

where \( L \) is the lattice size. The low values of the eigenvalues is a reflection of the topological nature of the modes. Their shape follows the shape of topologically non-trivial gluonic fields.

Furthermore, we argued that topological transitions are in fact ‘anti-unitary’. Therefore measuring the volume occupied by the topological fermionic modes is like performing measurements on a subtraction constant which is local. That is why performing measurements with better and better resolution we observe that the topological modes shrink to a vanishing 4d volume. This is a mechanism of topological fine-tuning: the volume of a state tends to zero while its ‘energy’ remains small.

Thus, with vanishing lattice spacing, \( a \to 0 \) the topological modes become a kind of percolating cluster. Predicting of the fractal dimension of this cluster remains a challenge. At this moment, we can only suggest bounds (7).
4 Lattice strings

On the theoretical side, there is no direct relation between the topological fermionic modes, or topologically non-trivial gluonic fields and confinement. On the other hand, the lattice data suggest strongly that deconfinement phase transition and restoration of chiral symmetry happen at the same temperature and one is invited to speculate that both phenomena originate from the same vacuum field configurations.

As for the confining fields, there are good reasons to believe that they are represented by the center vortices, for review see [2]. In measurements with fine resolution vortices appear fine tuned, see Eqs (1), (2) and then we reserve the word ‘strings’ for the vortices since they do look like thin 2d surfaces. As far as one thinks about chiral symmetry breaking as triggered by instantons, confining fields look unrelated to the breaking of chiral symmetry. However, now that we are beginning to understand that in measurements with fine resolution fermionic modes are also becoming a kind of a cluster, a close connection between strings and topologically non-trivial fields is no longer ruled out.

We will outline the latest data [19] on this connection in the next section. In this section we will emphasize connection between lattice strings and strings discussed within continuum theory, see, e.g., [20]. This connection is revealed through measurements with high resolution.

4.1 ’t Hooft loop

There are two external probes of the vacuum state in non-Abelian case, that is heavy quarks and heavy monopoles. The heavy-quark potential is related to the vacuum expectation value of the Wilson line:

\[ \langle W \rangle \sim \exp(-V_{QQ}(R)T) , \]

while the heavy-monopole potential is related to the ’t Hooft loop:

\[ \langle H \rangle \sim \exp(-V_{M\bar{M}}(R)T) . \]

Note that heavy monopoles are defined not in terms of an Abelian subgroup but rather in terms of the center group. We will concentrate on the \( S(2) \) case and then the center group is \( Z_2 \).

In the confining theory [18],

\[ \lim_{R \to \infty} V_{QQ} = \sigma R ; \lim_{R \to \infty} V_{M\bar{M}} = const . \]

Generically, confinement of color is ensured by condensation of the ‘magnetic degrees of freedom’. The same condensation explains also screening of the magnetic charges and flattening of \( V_{M\bar{M}} \) at large distances.

What are magnetic degrees of freedom is a dynamical question. Assume that the Wilson line is determined in terms of strings living in 5d space with non-trivial geometry which can be open on the Wilson line, see, e.g., [20]. Then by magnetic degrees of freedom it is natural to understand strings which can be open on the ’t Hooft loop. Such objects are indeed commonly introduced in the continuum theory, see, e.g., [21].
4.2 $Z_2$ gauge theory

To make the guess on magnetic, or dual strings more transparent consider analogy to $Z_2$ gauge theory. In this theory, see, e.g. [22], one integrates over link variables $Z_\mu(x) = \pm 1$. The corresponding plaquettes take values $P = \pm 1$ as well. The action is defined as proportional to the total area of the negative plaquettes:

$$S_{Z_2} = \beta A_{neg}.$$  \hspace{1cm} (39)

For the Wilson loop in this theory one can derive:

$$\langle W \rangle_{Z_2} \sim \Sigma_{AC} \exp(-\beta^* A_C),$$ \hspace{1cm} (40)

where the sum is taken over all the surfaces span on the Wilson contour $C$ and $A_C$ is the area of a surface while the constant $\beta^*$ is defined in terms of the original constant $\beta$ in the following way:

$$\beta^* = -\ln \tanh \frac{\beta}{2}.$$ \hspace{1cm} (41)

Similar representation exists for the 't Hooft line:

$$\langle H \rangle_{Z_2} \sim \Sigma_{AH} \exp(-\beta A_H).$$ \hspace{1cm} (42)

Naively, the representations (40) and (41) imply the area law both for the Wilson and 't Hooft lines. And this would be in contradiction with the confinement criteria mentioned above.

The resolution of the paradox is that one of the representations (40), (41) is actually formal for any value of the constant $\beta$. 'Formal' means that the sum over the surfaces is in fact divergent since the entropy factor for the surfaces overweighs the suppression due to the action. Consider, for example, $\beta$ small. Then $\beta^*$ is large and the sum (40) can be approximated by the surface with smallest area. And this implies the area law for the Wilson line. The sum (41) for the 't Hooft line is, however, divergent. If the constant $\beta$ and its dual are equal to each other,

$$\beta = \beta^*,$$

then there is a phase transition.

Physics-wise, this divergence implies that the corresponding surfaces can form an infinite percolating cluster. Generically, the logic here is the same as was used when we considered percolating monopole clusters. The surfaces which percolate in the vacuum are closed surfaces which can be open on the 't Hooft line.

In the non-Abelian case, there are arguments that string representation is valid for the Wilson and 't Hooft lines in terms of strings living in higher-dimensional spaces with non-trivial geometry, for review and references see [23]. Following the example of the $Z_2$ gauge theory, one can expect that the strings which can be open on the 't Hooft loop are actually condensed and there is percolating cluster of such strings in the vacuum state of Yang-Mills theories.\footnote{To our knowledge this point, or speculation has not been made in the continuum-theory literature on the magnetic (dual) strings.}
4.3 Finding strings

The problem of finding such strings in the vacuum seems to be a formidable task, even if they exist. It is amusing, therefore, that phenomenologically this problem has been (approximately) solved, for references see rather long time ago and without reference to the theoretical developments mentioned above. These strings are center vortices.

Algorithmically, the center vortices are defined in terms of a \( Z_2 \) projection. Namely, one replaces the original field configurations by the closest configuration of the \( Z_2 \)-fields,

\[
\{ A_\mu^a \} \rightarrow \{ Z_\mu(x) \}
\]

and the closeness of the two sets of the fields is defined in terms of the norm of the fields, summed up through the whole lattice. The center vortices are then defined in terms of negative plaquettes, evaluated in the \( Z_2 \) projection.

There is ample evidence that central vortices defined in this way are relevant, or responsible for the confinement. However, there are many other ways to introduce \( Z_2 \) projections. Phenomenologically, only those projections are successful which find a string of negative plaquettes introduced by hand on the lattice. We are now in position to argue that the theoretical meaning of this phenomenological observation is that such strings can be open on the 't Hooft loop. To our mind, it is quite a remarkable success of the theory.

From our perspective, it is crucial that the lattice strings are fine tuned. Indeed, observations and imply that

\[
(String\ tension) = (Bare\ tension) - (Entropy) \sim \Lambda_{QCD}^2 ,
\]

while the bare tension and entropy are ultraviolet divergent. This is another example of the relation which we discussed in connection with the monopole clusters. Eq suggests existence of a quasiclassical description of the strings as well.

4.4 Summary on the lattice strings

Lattice strings are fine tuned. It is well known, see, e.g., that fine tuning is the only way to realize strings (in Euclidean space) quantum mechanically. However, there is no consistent string theory in 4d. The quasiclassical image of the same strings probably is produced by theories with extra dimensions and non-trivial geometry. In these notes, we gave only one example of possible relation between the lattice strings and strings of the continuum theory. Namely, originally lattice strings are defined in terms of projected fields and the definition is far from being transparent. Now, one can appreciate theoretically, which projections are indeed physical.

5 Bringing parts 3 and 4 together

In two preceding sections we considered topologically non-trivial gluon fields and confining field configurations. Both, when measured with fine resolution appear to be percolating clusters, or submanifolds of the 4d space.

\[a\]Namely, the center vortices are defined on the dual lattice as unification of all the plaquettes orthogonal to the negative projected plaquettes on the original lattice, for details see.

\[h\]The 't Hooft loop is the trajectory of end points of the Dirac string. The Dirac string, in turn pierces negative plaquettes.
It is tempting to speculate that the two clusters finally merge with each other. One can check this hypothesis through direct measurements. Namely one measures correlation between intensity of topological fermionic modes and lattice strings. The correlation turns to be positive and strong.

In more detail, the vortices live on the dual lattice and one considers the correlator of the points on the dual lattice, $P_i$, which belong to the strings with the value of $\rho(x)$ averaged over the vertices of the 4d hypercube, $H$, dual to $P_i$, where $\rho_\lambda(x)$ is the density of the topological modes introduced in Eq (4). Thus, the correlator studied is defined as:

$$C_\lambda = \frac{\Sigma_{P_i} \Sigma_{x \in H} (V \rho_\lambda(x) - <V \rho_\lambda(x)>)}{\Sigma_{P_i} \Sigma_{x \in H} I}$$

(45)

It was found that the correlator is positive and large numerically for the topological modes. Also, the larger eigenvalue $\lambda$ is, the smaller is the correlator.

Moreover, the correlator depends on the number of the vortex plaquettes, $N_{vort}$ attached to the point $P_i$. For low-lying modes, roughly speaking one gets:

$$C_\lambda \sim 0 \approx 0.1 N_{vort}$$

(46)

where $3 \leq N_{vort} \leq 10$.

These results are in agreement with the hypothesis that the vortices are related to the chiral symmetry breaking. Further efforts are needed, however, to clarify the connection between the lattice strings and the chiral symmetry breaking.

6 Conclusions

We argued that quasiclassical and fine-tuned description can be dual to each other and apply to measurements with poor and fine resolutions, respectively. While the quasiclassical picture is more traditional and, therefore, intuitive the fine-tuned picture directly addresses physics of short distances. A remote analogy is that finding fine-tuned objects with ultraviolet divergent action relevant to non-perturbative physics is similar to finding point-like quarks of the perturbative physics.

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\[\text{Indirectly it has been known for some time that vortices and zero modes are related to each other. Namely, by removing vortices one eliminates in fact topological fermionic modes. However, by this procedure one actually affects 3d volumes, not only strings. Now, there are direct measurements available on the correlation of the modes with vortices and 3d volumes. The correlation turns positive in both cases.}\]
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