Determination of compact Lie groups with the Borsuk-Ulam property

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Abstract

A compact Lie group $G$ is said to have the Borsuk-Ulam property if the Borsuk-Ulam theorem holds for $G$-maps between representation spheres. It is well-known that an elementary abelian $p$-group $C_p^n$ ($p$ any prime) and an $n$-torus $T^n$, $n \geq 0$, have the Borsuk-Ulam property. In this paper, we shall discuss the classical question of which compact Lie groups have the Borsuk-Ulam property and in particular we shall show that every extension group of an $n$-torus by a cyclic group of prime order does not have the Borsuk-Ulam property. This leads us that the only compact Lie groups with the Borsuk-Ulam property are $C_p^n$ and $T^n$, which is a final answer to the question.

1 Introduction

The classical Borsuk-Ulam theorem [3] has been generalised from various aspects. From the viewpoint of equivariant topology or transformation group theory, the following theorem is well-known as a generalisation of the Borsuk-Ulam theorem, see for example [2], [8], [11], etc.

**Theorem** (Borsuk-Ulam type theorem). Let $\Gamma$ be an elementary abelian $p$-group $C_p^n$ ($p$ any prime) or an $n$-torus $T^n$. For any fixed-point-free orthogonal $\Gamma$-representations $V$ and $W$, i.e., $V^\Gamma = W^\Gamma = 0$, if there exists a $\Gamma$-map $f : S(V) \to S(W)$ between representation spheres, then the inequality $\dim V \leq \dim W$ holds.

We say that a compact Lie group $G$ has the Borsuk-Ulam property if such a Borsuk-Ulam type result holds, and such $G$ is called a **BU-group of type I** according to [12], that is, we call $G$ a BU-group of type I if the following property is satisfied: for any fixed-point-free orthogonal $G$-representations $V$ and $W$, if there exists a $G$-map $f : S(V) \to S(W)$ between representation spheres, then the inequality $\dim V \leq \dim W$ holds. We also call $G$ a **BU-group of type II** if the following property is satisfied: for any fixed-point-free orthogonal $G$-representations $V$ and $W$ with the same dimension, if there exists a $G$-map $f : S(V) \to S(W)$, then the degree of $f$ is $\deg f \neq 0$. It is also known that tori and elementary abelian $p$-groups are BU-groups of type II, see [9, Theorems 1 and 2]. It is natural and interesting to ask the following question:
Question. Which compact Lie groups have the Borsuk-Ulam property?

In previous studies of $G$-maps between spheres, several counterexamples of the Borsuk-Ulam property were sporadically found out; for example, $C_{pq}$, where $p$, $q$ are relatively prime positive integers, is not a BU-group of type II by [5, p.60] and not of type I by a result of [14]. As is observed in [10], $S^3 = S(\mathbb{H})$, $\mathbb{H}$ the skew field of quaternions, is not a BU-group of type I; in fact, the Hopf map $\pi: S^3 \to S^3/S^1 \cong S^2$ is considered as an $S^3$-map.

In the early 1990s, this question and related topics were systematically studied in [1] and [10]. Consequently, it has been shown that almost compact Lie groups are neither BU-groups of type I nor of type II. However, some unsolved cases were left until recently. In [12], we studied the remaining cases of finite groups and provided an answer to the above question in finite group case. In fact, we showed that the only finite BU-groups of type I [resp. II] are the elementary abelian $p$-groups (including the trivial group).

In this paper, we shall discuss the question in general compact Lie groups and provide a final answer to the question. As will be seen in the following sections, the question is reduced to the case of an extension of an $n$-torus by a cyclic group $C_p$ of prime order $p$, and then the following result will be shown.

Proposition 1.1. Let $G$ be any extension of an $n$-torus $T^n$ by a cyclic group $C_p$ of order $p$:
\[ 1 \to T^n \to G \to C_p \to 1, \quad n \geq 1. \]

Then $G$ is neither a BU-group of type I nor of type II.

As a consequence, we obtain a final answer to the question as follows:

Theorem 1.2. The following statements are equivalent.

1. A compact Lie group $G$ is a BU-group of type I.

2. A compact Lie group $G$ is a BU-group of type II.

3. A compact Lie group $G$ is isomorphic to an elementary abelian $p$-group $C_p^n$ or an $n$-torus $T^n$, $n \geq 0$.

This theorem is deeply related to some results of [10]. In [10], a compact Lie group $G$ is said to have the Borsuk-Ulam property in the strong sense $A$ (or the property $IIA$) if the following property is satisfied: for any $G$-map $f: S(V) \to S(W)$ with $\dim V = \dim W$ and $\dim V^G = \dim W^G$, if the degree $\deg f^G$ is prime to $|G/G_0|$, then $\deg f \neq 0$. Theorem 1.2 improves a result of [10] as follows:

Corollary 1.3. A compact Lie group $G$ has the Borsuk-Ulam property in the strong sense $A$ if and only if $G$ is isomorphic to an elementary abelian $p$-group $C_p^n$ or an $n$-torus $T^n$, $n \geq 0$.

Indeed, if $G$ has the Borsuk-Ulam property in the strong sense $A$, then $G$ is a BU-group of type II, since $\deg f^G = 1$ when $S(V)^G = S(W)^G = \emptyset$. Therefore $G$ is isomorphic to an elementary abelian $p$-group $C_p^n$ or an $n$-torus $T^n$, $n \geq 0$. On the other hand, the converse is already shown by [9, Theorems 1 and 2].

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Remark 1. Theorem 2 (b) of [10] is unfortunately incorrect. In particular, Lemma 1.2 of [10] does not hold for compact Lie groups with the property of type II A. Theorem [12] also gives a negative answer to the conjecture of [4] that \( G \) has the Borsuk-Ulam property if and only if \( G \cong T^n \times C_p^m \).

2 The property of BU-groups and reduction of cases

Since the main focus of this paper is to construct counterexamples of the Borsuk-Ulam property, we give the following definition.

Definition. A compact Lie group \( G \) is called an anti-BU-group of type I [resp. II] if \( G \) is not a BU-group of type I [resp. II].

In [12], we showed the fundamental property of BU-groups of each type. We restate it as follows:

Proposition 2.1 (([12])). Let \( G \) be a compact Lie group.

(1) If there exists a quotient group \( Q \) of \( G \) being an anti-BU-group of type I [resp. II], then \( G \) is an anti-BU-group of type I [resp. II].

(2) If there exists a closed subgroup \( H \) of finite index being an anti-BU-group of type I [resp. II], then \( G \) is an anti-BU-group of type I [resp. II].

Let \( G_0 \) denote the identity component of a compact Lie group \( G \) and set \( F = G/G_0 \) a finite group. By [1, Theorem 2 and Proposition 2.2], connected compact Lie groups other than tori are anti-BU-groups of both types I and II. On the other hand, by [12], finite groups other than \( C_p^m \) are anti-BU-groups of both types I and II. Thus if \( G_0 \) is not a torus or if \( F \) is not an elementary abelian \( p \)-group, then \( G \) is an anti-BU-group of both types I and II by Proposition 2.1. Therefore it suffices to consider the following type of extension

\[
1 \to T^n \to G \to C_p^m \to 1 \quad (n \geq 1, m \geq 1).
\]

Furthermore, there exists a closed subgroup of index \( p^{m-1} \) of such \( G \); indeed, one can take \( \pi^{-1}(C_p) \) for some \( C_p^m \leq C_p^m \), where \( \pi : G \to C_p^m \) is the projection. Thus, by Proposition 2.1 \( G \) is reduced to the case of an extension

\[
1 \to T^n \to G \xrightarrow{\pi} C_p \to 1. \tag{2.1}
\]

In order to prove Proposition 2.1 a further reduction is needed. Here an \( n \)-torus \( T^n \) is described by

\[
T^n = \{t = (t_1, \ldots, t_n) \in \mathbb{C}^n \mid |t_i| = 1 \ (1 \leq i \leq n)\}.
\]

Let \( a \in C_p \) denote a (fixed) generator of \( C_p \). We here introduce a semi-direct product:

\[
\Gamma_p = T^p \rtimes C_p,
\]
where \( \rho : C_p = \langle a \rangle \to \text{Aut}(T^{p-1}) \) is the homomorphism defined by

\[
\rho(a)(t) = (t_{p-1}^{-1}, t_1 t_{p-1}^{-1}, \ldots, t_{p-2} t_{p-1}^{-1}) \tag{2.2}
\]

for \( t = (t_1, \ldots, t_{p-1}) \in T^{p-1} \). In fact, one can easily check that the \( p \)-fold composition of \( \rho(a) \) with itself is \( \text{id} \) and then \( \rho \) defines the conjugate action of \( C_p \) on \( T^n \):

\[ ata^{-1} = \rho(a)(t). \]

In particular, \( \Gamma_2 \) is isomorphic to \( \text{O}(2) \) the orthogonal group in dimension 2. The group \( \Gamma_p \) is a split extension of \( T^{p-1} \) by \( C_p \):

\[ 1 \to T^{p-1} \to \Gamma_p \to C_p \to 1, \]

and plays an important role in the proof of Proposition 1.1. The remainder of this section will be devoted to showing the following:

**Proposition 2.2.** For any extension (2.1), there exists a closed normal subgroup \( N \) such that \( G/N \) is isomorphic to \( S^1 \times C_p \) or \( \Gamma_p \).

We begin by observing the following fact.

**Lemma 2.3.** If \( C_p \) acts trivially on \( T^n \) for extension (2.1), then \( G \) is abelian.

**Proof.** Let \( \pi : G \to C_p \) be the projection and \( \alpha \in G \) an element such that \( \pi(\alpha) = a \). Let

\[ G = \coprod_{i=0}^{p-1} \alpha^i T^n \]

be a coset decomposition. Since the \( C_p \)-action on \( T^n \) is trivial, it follows that \( \alpha^i t \alpha^{-i} = t \) for any \( t \in T^n \) and \( 0 \leq i \leq p - 1 \). Thus, for any \( g = \alpha^i t, h = \alpha^j s \in G \), where \( t, s \in T^n \),

\[ gh = \alpha^i t \alpha^j s = \alpha^{i+j} ts = \alpha^j s \alpha^i t = hg. \]

Thus \( G \) is abelian. \( \square \)

By the following lemma, any extension (2.1) is reduced to a split extension.

**Lemma 2.4.** For any extension (2.1), there exists a closed normal subgroup \( N \) of \( G \) such that \( G/N \) is a split extension of \( T^l \) by \( C_p \):

\[ 1 \to T^l \to G/N \to C_p \to 1 \]

for some \( 1 \leq l \leq n \).
Proof. If $G$ is abelian, then $G \cong T^n \times C_p$ and $G$ itself is a split extension. Assume that $G$ is non-abelian. Then the $C_p$-action on $T^n$ is non-trivial by Lemma 2.3 and hence one can see that the fixed-point subgroup $(T^n)^{C_p}$ of $T^n$ coincides with the centre $Z(G)$ of $G$. Clearly for $\alpha \in G$ with $\pi(\alpha) = a$, it follows that $\alpha^p \in (T^n)^{C_p} = Z(G)$. Consider an extension

$$1 \to T^n/Z(G) \to G/Z(G) \xrightarrow{\pi} C_p \to 1.$$ 

Note that $T^n/Z(G) \cong T_1^l$ for some $1 \leq l \leq n$ since the $C_p$-action on $T^n$ is non-trivial. This extension splits, indeed, a splitting $s : C_p \to G/Z(G)$ is given by $s(a) = \overline{\alpha} \in G/Z(G)$, since $\overline{\alpha}^p = 1 \in G/Z(G)$ and $\overline{\alpha}(\overline{\alpha}) = a$. \hfill $\square$

By Lemma 2.3 it suffices to consider a semi-direct product

$$G_\sigma := T^n \rtimes_\sigma C_p,$$

where $\sigma : C_p \to \text{Aut}(T^n)$ is a homomorphism which gives the conjugate $C_p$-action on $T^n$. We introduce the following terminology. An $n$-torus $T^n$ with a homomorphism $\sigma : C_p \to \text{Aut}(T^n)$ is called a $C_p$-torus, denoted by $T^n_{\sigma}$. If a closed subgroup $H$ of $T^n_{\sigma}$ is $C_p$-invariant, then $H$ is called a $C_p$-subgroup. Clearly a $C_p$-subgroup $H$ is a closed normal subgroup of $G_\sigma$. In particular, if a subtorus $T$ of $T^n_{\sigma}$ is $C_p$-invariant, then $T$ is called a $C_p$-subtorus and denoted by $T_\sigma$.

As is well-known, $\text{Aut}(T^n)$ is naturally identified with $GL_n(\mathbb{Z})$. In fact, an automorphism $\phi \in \text{Aut}(T^n) = \text{Aut}(\mathbb{R}^n / \mathbb{Z}^n)$ corresponds to an isomorphism $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{\phi}(\mathbb{Z}^n) = \mathbb{Z}^n$, and this corresponds to an isomorphism $\tilde{\phi} := \tilde{\phi}|_{\mathbb{Z}^n} : \mathbb{Z}^n \to \mathbb{Z}^n$. For example, consider $\Gamma_p = T^{p-1} \rtimes_\rho C_p$ introduced before. Identifying $T^{p-1}$ with $\mathbb{R}^{p-1} / \mathbb{Z}^{p-1}$ by the exponential map

$$\exp([x_1, \ldots, x_{p-1}]) = (e^{2\pi \sqrt{-1} x_1}, \ldots, e^{2\pi \sqrt{-1} x_{p-1}}) \in T^{p-1}, \quad [x_1, \ldots, x_{p-1}] \in \mathbb{R}^{p-1} / \mathbb{Z}^{p-1},$$

one sees that $\rho(a) \in \text{Aut}(T^{p-1})$ is represented by the $(p-1) \times (p-1)$ matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \quad (2.3)$$

under the standard basis of $\mathbb{Z}^{p-1}$. Thus a $C_p$-torus $T^n_{\sigma}$ induces a $\mathbb{Z}[C_p]$-module $M_\sigma$ whose underling $\mathbb{Z}$-module is $\mathbb{Z}^n$ such that the action of $C_p$ on $M_\sigma$ is given by $a \cdot x = \sigma(a)x$ for $x \in M_\sigma$.

Conversely, a $\mathbb{Z}[C_p]$-module $M$ whose underling $\mathbb{Z}$-module is $\mathbb{Z}^n$ provides a homomorphism $\psi : C_p \to GL_n(\mathbb{Z})$ induced by the $C_p$-action on $M$. Then $\psi$ induces a homomorphism $\sigma : C_p \to \text{Aut}(T^n)$ via the exponential map. This provides a $C_p$-torus $T_M := T^n_{\sigma}$ and a semi-direct product $G_M := G_\sigma = T^n \rtimes_\sigma C_p$. Furthermore, a $\mathbb{Z}[C_p]$-submodule $L$ of
n provides a $C_p$-subtorus $T_L$ which is a closed normal subgroups of $G_M$. Note that, if rank$_\mathbb{Z}L = l$, then dim $T_L = l$ and $G_L$ is a split extension of $T_L$ by $C_p$:

$$1 \to T_L \to G_L \to C_p \to 1.$$ 

By integral representation theory, any $\mathbb{Z}[C_p]$-module $M$ whose underling $\mathbb{Z}$-module is $\mathbb{Z}^n$ decomposes into a direct sum of some indecomposable $\mathbb{Z}[C_p]$-modules (although the decomposition is not unique). The indecomposable $\mathbb{Z}[C_p]$-module is isomorphic to one of following three types, see [6, §74] for details.

1) $\mathbb{Z}$: the trivial $\mathbb{Z}[C_p]$-module.

2) $I$: a fractional ideal $I$ of the cyclotomic field $\mathbb{Q}(\xi_p)$, where $\xi_p = e^{2\pi \sqrt{-1}/p}$ a primitive $p$-th root of unity. Note that $I$ is regarded as a $\mathbb{Z}[C_p]$-module whose underling $\mathbb{Z}$-module is $\mathbb{Z}^{p-1}$ via the natural ring homomorphism $\mathbb{Z}[C_p] \to \mathbb{Z}[\xi_p]$. Furthermore, fractional ideals $I$ and $J$ are isomorphic as $\mathbb{Z}[C_p]$-modules if and only if these are in the same ideal class of $\mathbb{Q}(\xi_p)$. For example, the ideal class group of $\mathbb{Q}(\xi_{23})$ is of order 3, see [15], and so there are three isomorphism classes.

3) $(I, \nu) := I \oplus \mathbb{Z}y$ (rank$_\mathbb{Z}(I, \nu) = p$), where $I$ is a fractional ideal $I$ of $\mathbb{Q}(\xi_p)$ and $\nu$ is an element of $I$ such that $\nu \notin (\xi_p - 1)I$. The action of $a$ on $y$ is given by $ay = \nu + y \in (I, \nu)$. Note that the isomorphism class of $(I, \nu)$ depends only on the ideal class of $I$ and does not depend on the choice of $\nu$.

Proof of Proposition 2.2. Suppose that $G = G_\sigma$ for some $\sigma : C_p \to \text{Aut}(T^n)$. Let $M_\sigma$ be the $\mathbb{Z}[C_p]$-module corresponding to $T^n_\sigma \leq G_\sigma$. By the indecomposable decomposition of $M_\sigma$, one sees that there exists a $\mathbb{Z}[C_p]$-submodule $L$ such that $M_\sigma / L \cong \mathbb{Z}, I$ or $(I, \nu)$. A $\mathbb{Z}[C_p]$-module $I$ is a $\mathbb{Z}[C_p]$-submodule of $(I, \nu)$ and $(I, \nu)/I \cong \mathbb{Z}$ with the trivial $C_p$-action. Consequently, there exists a $\mathbb{Z}[C_p]$-submodule $L$ such that $M_\sigma / L \cong \mathbb{Z}$ or $I$ and such $L$ provides a $C_p$-subtorus $T_L$ which is a closed normal subgroup of $G_\sigma$. If $M_\sigma / L \cong \mathbb{Z}$, then it follows that $G_\sigma / T_L \cong G_Z \cong S^1 \times C_p$. Next suppose that $M_\sigma / L \cong I$. It then follows that $G_\sigma / T_L \cong G_I$. Let $I_0 = \mathbb{Z}[\xi_p]$ with $\mathbb{Z}$-basis

$$\{\xi_p, \xi_p^2, \ldots, \xi_p^{p-1}\}.$$ 

Since the action of the generator $a \in C_p$ on $I_0$ is given by the multiplication of $\xi_p$, the automorphism induced by the action of $a$ on $I_0$ is represented by the matrix (2.3). Hence it follows that $G_{I_0}$ is isomorphic to $\Gamma_p$. Since any ideal class is represented by an integral ideal of $\mathbb{Z}[\xi_p]$, one may assume that $I \subset I_0$ and $I$ is a $\mathbb{Z}[C_p]$-submodule of $I_0$, and $I$ and $I_0$ have the same $\mathbb{Z}$-rank $p - 1$. Therefore the inclusion $i : I \to I_0$ induces an $\mathbb{R}[C_p]$-isomorphism

$$\varphi := \mathbb{R} \otimes_\mathbb{Z} i : \mathbb{R} \otimes_\mathbb{Z} I \to \mathbb{R} \otimes_\mathbb{Z} I_0$$

such that $\varphi(I) \subset I_0$. Note that $\mathbb{R} \otimes_\mathbb{Z} I \cong \mathbb{R} \otimes_\mathbb{Z} I_0 \cong \mathbb{R}^{p-1}$ as $\mathbb{R}$-vector spaces. Then $\varphi$ induces a surjective $C_p$-homomorphism

$$\overline{\varphi} : T_I = \mathbb{R} \otimes_\mathbb{Z} I / I \cong T^{p-1} \to T_{I_0} = \mathbb{R} \otimes_\mathbb{Z} I_0 / I_0 \cong T^{p-1}$$
between $C_p$-tori. This also induces a surjective homomorphism $f : G_I \to G_{I_0}$ by setting $f(ta^k) = \varphi(t)a^k$ for $t \in T^{p-1}$ and $0 \leq k \leq p-1$. Since the kernel $\ker f = \ker \varphi$ is a finite $C_p$-subgroup, it follows that $\ker f$ is a finite normal subgroup of $G_I$. This implies that $G_I/\ker f \cong G_{I_0} \cong \Gamma_p$. Thus the proof of Proposition 2.2 is completed.

\[\square\]

### 3 The case of $\Gamma_p$

So far we have shown that Proposition 1.1 is reduced to the cases of $\Gamma_p$ and $S^1 \times C_p$. In this section, we shall show that $\Gamma_p$ for any prime $p$ is an anti-BU-group of both types I and II. First observe the following:

**Proposition 3.1.** If a compact Lie group $G$ is an anti-BU-group of type I, then $G$ is an anti-BU-group of type II.

**Proof.** Suppose that there exists a $G$-map $f : S(V) \to S(W)$ with

$$v := \dim V > w := \dim W$$

for some fixed-point-free representations. Then one can define a $G$-map

$$h : S(wV) \xrightarrow{\ast f} S(wW) \xrightarrow{\text{incl.}} S(vW),$$

where $\ast f$ denote the $w$-fold join of $f : S(V) \to S(W)$. Since

$$\dim wV = \dim vW = vw > \dim wW = w^2,$$

it follows that $\deg h = 0$ for dimensional reason. Thus $G$ is an anti-BU-group of type II.

In order to construct a $\Gamma_p$-map $f : S(V) \to S(W)$ for some representations $V, W$ with $\dim V > \dim W$, we use the following fact from equivariant obstruction theory.

**Proposition 3.2.** Let $G$ be a compact Lie group and $W$ a $G$-representation. Let $X$ be a finite $G$-CW complex and $A$ a $G$-subcomplex of $X$ (possibly empty). Suppose that there exists a $G$-map $f_A : A \to S(W)$. Let $Y = X \setminus A$. If

$$\dim Y^H/W_G(H) \leq \dim S(W)^H$$

(3.1)

for any isotropy subgroup $H$ of $Y$, then there exists a $G$-map $f : X \to S(W)$ extending $f_A$. Here $W_G(H)$ denotes $N_G(H)/H$, and $N_G(H)$ is the normaliser of $H$ in $G$.

**Proof.** This is a consequence of equivariant obstruction theory [7, Chapter II, 3]. indeed, since $S(W)^H$ is $(\dim S(W)^H - 1)$-connected, there are no obstructions to an extension of the $G$-map $f_A$.

\[\square\]
We first consider the case of $p = 2$. Then $\Gamma_2$ is isomorphic to the orthogonal group $O(2)$ in dimension 2. The orthogonal group $O(2)$ has the orthogonal 2-dimensional irreducible $O(2)$-representations $U_k^t$, $k \in \mathbb{Z}$, whose underlying space is $\mathbb{R}^2 \cong \mathbb{C}$ and $t \in S^1$ acts by $t \cdot z = t^k z$, $z \in \mathbb{C}$, and the generator $a \in C_2$ acts by $a \cdot z = \bar{z}$ the complex conjugate. There are 1-dimensional $O(2)$-representations $\mathbb{R}$ and $V_1^t$, where $\mathbb{R}$ is the trivial representation and $V_1^t$ is given by the lift of the non-trivial 1-dimensional $C_2$-representation, i.e., $S^1$ acts trivially on $V_1^t$ and $a = -1$ acts by $a \cdot x = -x$, $x \in V_1^t$. The following shows that $\Gamma_2$ is an anti-BU-group of type I, and hence of type II.

**Lemma 3.3.** There exists a $\Gamma_2$-map $f : S(U_1^t \oplus U_1^t) \to S(U_2^t \oplus V_1^t)$.

*Proof.* Set $V = U_1^t \oplus U_1^t$ and $W = U_2^t \oplus V_1^t$. Then $S(V)$ has two isotropy types (1) and (C2). In fact, the set $\text{Iso}(S(U_1^t))$ of isotropy subgroups of $U_1^t$ consists of the subgroups $\langle ta \rangle$, $t \in S^1$. Note that $\langle ta \rangle$ is conjugate to $C_2$. For any $x = (z, w) \in S(U_1^t \oplus U_1^t)$, the isotropy subgroup $G_x$ of $x$ is equal to $G_z \cap G_w$ for $z, w \in U_1^t$. This deduces that the set of conjugacy classes of isotropy subgroups of $S(V)$ is

$$\text{Iso}(S(V))/\Gamma_2 = \{(1), (C_2)\}.$$ 

Let $K = W_{\Gamma_2}(C_2)$. Since the normaliser $N_{\Gamma_2}(C_2)$ is $Z \times C_2$, where $Z = \{ \pm 1 \}$ is the centre of $\Gamma_2$, hence $K \cong Z$. Then $S(V)^{C_2}$ is a free $K$-sphere of dimension 1. Indeed, $V^{C_2} \cong \mathbb{R}^2$ and $K$ acts antipodally on $S(V)^{C_2}$. On the other hand, $S(W)^{C_2} = S(U_2^t)^{C_2} \cong S^0$ has the trivial $K$-action. Take a constant map

$$f_{C_2} : S(V)^{C_2} \to S(W)^{C_2},$$

which is $K$-equivariant. Taking the $\Gamma_2$-orbits of $S(V)^{C_2}$ and $S(W)^{C_2}$, one obtains a (well-defined) $\Gamma_2$-map

$$f_{(C_2)} : \Gamma_2 S(V)^{C_2} \to \Gamma_2 S(W)^{C_2}$$

which is defined by $f_{(C_2)}(gx) = gf_{C_2}(x)$ for $g \in \Gamma_2$, $x \in S(V)^{C_2}$.

One sees that $S(V) \times \Gamma_2 S(V)^{C_2}$ is a free $\Gamma_2$-space and $\dim S(V)/\Gamma_2 = 2 = \dim S(W)$. By Proposition 3.2, there exists a $\Gamma_2$-map $f : S(V) \to S(W)$. \qed

A similar argument is valid for $\Gamma_p$, where $p$ is an odd prime. We shall summarise facts on $\Gamma_p$ here.

**Lemma 3.4.** Let $\Gamma_p = T^{p-1} \rtimes_p C_p$ as before.

1. For any $t \in T^{p-1}$, the order of $ta$ is $p$ and $\langle ta \rangle$ is conjugate to $C_p = \langle a \rangle$.

2. The centre $Z(\Gamma_p)$ of $\Gamma_p$ is

$$Z(\Gamma_p) = \langle (\xi_p, \xi_p^2, \ldots, \xi_p^{p-1}) \rangle \leq T^{p-1}.$$

3. The normaliser of $C_p$ is $N_{\Gamma_p}(C_p) = Z(\Gamma_p) \times C_p$. 

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More generally, \( \rho T \) is given by the induced representations of non-trivial irreducible \( \rho \)-representations and the lifts of irreducible \( \rho \)-representations. In the following, we prove this statement.

**Proof.** (1) The automorphism \( \rho(a) \in \text{Aut}(T^{p-1}) \) is represented by the matrix \( A \) in (2.1). More generally, \( \rho(a^i), 1 \leq i \leq p-1 \), is represented by

\[
A^i = \begin{pmatrix}
0 & \cdots & 0 & -1 & 1 & 0 \\
\vdots & \ddots & \ddots & & \ddots & \ddots \\
0 & \cdots & 0 & -1 & 0 & 1 \\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & -1 & 0 & \cdots & 0
\end{pmatrix}
\]

(3.2)

where the \((p-i)\)-th column is \( t(-1, \ldots, -1) \), and \( A^p = I \).

Let \( X \in \text{GL}_{p-1}(\mathbb{Z}) \) and \( \phi \in \text{Aut}(T^{p-1}) \) the automorphism induced by \( X \). Set \( t^X = \phi(t) \) for \( t \in T^{p-1} \). When \( X = (x_{ij}) \), one sees that

\[
t^X = \left( t^{x_{11}}_1 x_1, \ldots, t^{x_{11}}_{p-1} x_{p-1}, \ldots, t^{x_{11}}_1 x_1, \ldots, t^{x_{11}}_{p-1} x_{p-1} \right),
\]

and \( t^X + Y = t^X t^Y \) for matrices \( X, Y \in \text{GL}_{p-1}(\mathbb{Z}) \).

For any \( t \in T^{p-1} \), since \( at = \rho(a)(t)a = t^A a \), it follows that

\[
(ta)^p = t^{I + A + \cdots + A^{p-1}} a^p = t^O = 1.
\]

Therefore the order of \( ta \) is \( p \). Next we show that \( \langle ta \rangle \) is conjugate to \( C_p \). Indeed, we set

\[
s_k = t^{(p-k)/p}_1 \cdots t_k^{(p-k)/p} t_{k+1}^{(p-k)/p} \cdots t_{p-1}^{(p-k)/p} \in S^1
\]

for \( 1 \leq k \leq p-1 \), and \( s = (s_1, \ldots, s_{p-1}) \in T^{p-1} \). Then \( s a s^{-1} = s(s^A)^{-1} a \), and

\[
s(s^A)^{-1} = (s_1 s_{p-1}, s_2 s_{p-1}^{-1} s_{p-1}, \ldots, s_{p-1} s_{p-2} s_{p-1}).
\]

By a direct computation, one can see that \( s(s^A)^{-1} = t \) and \( s a s^{-1} = ta \). Thus \( \langle ta \rangle \) is conjugate to \( C_p \).

(2) Since \( ata^{-1} = t \) if and only if \( t^A = t \), it follows that

\[
t_{p-1}^{-1} = t_1, \quad t_1 t_{p-1}^{-1} = t_2, \ldots, t_{p-2} t_{p-1}^{-1} = t_{p-1}.
\]

These equations imply that \( t_k = t_k^i, 1 \leq k \leq p-1 \), and \( t_1^p = 1 \). Thus

\[
Z(\Gamma_p) = \langle (\xi_p, \xi_p^2, \ldots, \xi_p^{p-1}) \rangle.
\]

(3) Clearly \( C_p \leq N_{\Gamma_p}(C_p) \). If \( t \in N_{\Gamma_p}(C_p) \) for \( t \in T^{p-1} \), then \( t^{-1} at = a^k \) for some \( 1 \leq k \leq p-1 \), and this implies that \( t^{-1} ata^{-1} = t^{-1} t^A = a^{k-1} \in T^{p-1} \cap C_p = \{1\} \), and so \( k = 1 \) and \( t^{-1} t^A = 1 \). Therefore \( t \in Z(\Gamma_p) \). Thus the desired result holds. 

The irreducible unitary \( \Gamma_p \)-representations are obtained from the argument of (13). Consequently, these are given by the induced representations of non-trivial irreducible \( T^{p-1} \)-representations and the lifts of irreducible \( C_p \)-representations. In the following, we
only consider specific representations below. For any \( k \in \mathbb{Z} \setminus \{0\} \), an irreducible unitary \( \Gamma_p \)-representation \( U_k \) is given by

\[
U_k = \text{Ind}_{\Gamma_p}^{\Gamma_p} \overline{U}_k,
\]

where \( \overline{U}_k \) is a 1-dimensional unitary \( T^{p-1} \)-representation on which \( t = (t_1, \ldots, t_{p-1}) \in T^{p-1} \) acts by \( t \cdot z = t_1^i z \) for \( z \in \overline{U}_k \). Regarding \( U_k \) as a direct sum \( \oplus_{i=0}^{p-1} a^i \overline{U}_k \), one sees that \( a \) acts by permutation of components: \( a \cdot a^i \overline{U}_k = a^{i+1} \overline{U}_k \), and \( t \) acts on \( a^i \overline{U}_k \) by

\[
t \cdot w_i = a^i (t^{A^{-i}}) z_i \in a^i \overline{U}_k
\]

for \( w_i = a^i z_i \in a^i \overline{U}_k \), \( z_i \in \overline{U}_k \), where \( (t^{A^{-i}})_1 \) denotes the first component of \( t^{A^{-i}} \in T^{p-1} \).

More concretely, it follows from the matrix \((3.2)\) that, for any \( w = (w_0, w_1, \ldots, w_{p-1}) \in \oplus_{i=0}^{p-1} a^i \overline{U}_k \),

\[
t \cdot w = (t^k z_0, at_1^{-k} z_1, \ldots, a^{p-2} t_2^{-k} z_{p-2}, a^{p-1} t_{p-1}^{-k} z_{p-1}).
\]  \hspace{1cm} (3.3)

For any \( k \in \mathbb{Z}/p \), an irreducible unitary \( \Gamma_p \)-representation \( V_k \) is given by the lift of a 1-dimensional unitary \( C_p \)-representation \( \overline{V}_k \) on which \( a \) acts by \( a \cdot z = \xi^k z \) for \( z \in \overline{V}_k \). In particular, \( T^{p-1} \) acts trivially on \( V_k \) and also \( V_0 \) is the trivial \( \Gamma_p \)-representation.

In general, for an arbitrary unitary \( G \)-representation \( V \), the kernel \( \text{Ker} V \) of \( V \) is defined by the kernel of the representation homomorphism \( \varphi : G \to \text{U}(n) \) of \( V \), or equivalently, \( \text{Ker} V \) is the closed subgroup consisting of elements \( g \in G \) trivially acting on \( V \). We note the following:

**Lemma 3.5.** (1) For any \( k \geq 1 \), \( \text{Ker} U_k = \text{Ker} U_{-k} = \mathbb{Z}_k^{p-1} \leq T^{p-1} \), where \( \mathbb{Z}_k = \langle \xi_k \rangle \leq S^1 \) and for any \( k \in \mathbb{Z}/p \setminus \{0\} \), \( \text{Ker} V_k = T^{p-1} \).

(2) The centre \( Z(\Gamma_p) \) is a subgroup of \( \text{Ker} U_p \).

**Proof.** (1) By formula \((3.3)\), the first result is verified. The second result is trivial by definition of \( V_k \).

(2) Since \( Z(\Gamma_p) = \langle (\xi_p, \ldots, \xi_{p-1}) \rangle \) by Lemma 3.4, this is clear by (1). \( \square \)

**Remark 2.** The \( C_p \)-homomorphism \( \varphi : T^{p-1} \to T^{p-1} \), \( f(t) = t^k \) induces the isomorphism \( \Gamma_p/\mathbb{Z}_k^{p-1} \cong \Gamma_p \). Since \( \text{Ker} U_k = \mathbb{Z}_k^{p-1} \), the fixed-point representation \( U_k^{Z_k^{p-1}} \) is regarded as a \( \Gamma_p \)-representation and then \( U_k^{Z_k^{p-1}} \cong U_1 \) as \( \Gamma_p \)-representations. Conversely, \( U_k \) is regarded as the lift of \( U_1 \) by the projection \( q : \Gamma_p \to \Gamma_p/\mathbb{Z}_k^{p-1} \cong \Gamma_p \).

Next we summarise some facts on the isotropy subgroups of \( S(U_1) \) and \( S(V_1) \).

**Lemma 3.6.** (1) Any subgroup \( K \) conjugate to \( C_p \) is a maximal isotropy subgroup of \( S(U_1) \).

(2) Any isotropy subgroup \( K \) of \( S(U_1) \) not conjugate to \( C_p \) is a subgroup of \( T^{p-1} \).

(3) For any \( x \in S(V_k) \), \( k \in \mathbb{Z}_p \setminus \{0\} \), the isotropy subgroup \( (\Gamma_p)_x \) is \( T^{p-1} \).
Proof. (1) We may assume that \( K = C_p \), since the set \( \text{Iso}(S(U_1)) \) of isotropy subgroups is closed under conjugation. By definition of the \( C_p \)-action on \( U_1 \), we have

\[
U_1^{C_p} = \{(z, a z \ldots, a^{p-1} z) \in \oplus_{i=0}^{p-1} a \overline{U}_k | z \in \mathbb{C} \} \cong \mathbb{C}.
\]

Let \( H = (\Gamma_p)_{u} \) be the isotropy subgroup at \( u \in S(U_1)^{C_p} \). Clearly \( H \geq C_p \) and \( H \) forms an extension

\[
1 \rightarrow H \cap T^{p-1} \rightarrow H \rightarrow C_p \rightarrow 1.
\]

Since \( S(U_1)^H \neq \emptyset \) and \( S(U_1)^H \subset S(U_1)^{C_p} \), it follows that \( S(U_1)^H = S(U_1)^{C_p} \cong S(\mathbb{C}) \). For any \( t \in H \cap T^{p-1} \) and \( u = (z, a z \ldots, a^{p-1} z) \in S(U_1)^H \), one sees

\[
t \cdot u = (t_1 z, a_1 t_2 z, \ldots, a^{p-2} t_{p-2} t_{p-1} z, a^{p-1} t_{p-1}^{-1} z) \in S(U_1)^H
\]

by formula (3.3). Since \( t \cdot u = u \) and \( z \neq 0 \), it follows that \( t = 1 \) and hence \( H \cap T^{p-1} = 1 \). Thus \( H = C_p \) and \( C_p \) is a maximal isotropy subgroup.

(2) Suppose that \( H \) is an isotropy subgroup of \( S(U_1) \). If \( \pi(H) = C_p \), where \( \pi : \Gamma_p \rightarrow C_p \) is the projection, then there exists an element \( t a \in H \) for some \( t \in T^{p-1} \). Since \( C'_p = \langle t a \rangle \leq H \) is is a maximal isotropy subgroup by Lemma 3.4. The maximality implies that \( H = C'_p \) and \( H \) must be conjugate to \( C_p \). This contradicts that \( H \) is not conjugate to \( C_p \) by assumption. It thus follows that \( \pi(H) = 1 \) and so \( H \) is a subgroup of \( T^{p-1} \).

(3) Since \( \text{Ker} V_k = T^{p-1} \) and \( C_p \) acts freely on \( S(V_k) \), it follows that \( (\Gamma_p)_x = T^{p-1} \). \( \square \)

Remark 3. Any isotropy subgroup of \( S(U_1) \) included in \( T^{p-1} \) is isomorphic to an \( m \)-torus \( T^m \) for some \( 0 \leq m < p - 1 \).

The proof of Proposition 3.1 is finished by the next lemma.

Lemma 3.7. For any odd prime \( p \), there exists a \( \Gamma_p \)-map

\[
f : S(U_1 \oplus U_1) \rightarrow S(U_p \oplus (p - 1)V_1).
\]

Proof. Set \( V = U_1 \oplus U_1 \) and \( W = U_p \oplus (p - 1)V_1 \). Observing that the isotropy subgroup of \((u_1, u_2) \in V\) is

\[
(\Gamma_p)_{(u_1, u_2)} = (\Gamma_p)_{u_1} \cap (\Gamma_p)_{u_2},
\]

one can see that any subgroup \( K \) conjugate to \( C_p \) is a maximal isotropy subgroup of \( S(V) \) by Lemma 3.4. Set \( K := W_{\Gamma_p}(C_p) \), which is isomorphic to \( Z(\Gamma_p) \) by Lemma 3.4. By formula (3.3), it follows that \( S(V)^{C_p} \cong S^3 \) is a free \( K \)-sphere of dimension 3, and also \( S(W)^{C_p} \cong S^1 \) has the trivial \( K \)-action, since \( Z(\Gamma_p) \leq \text{Ker} W \). Take a constant map \( f_{C_p} : S(V)^{C_p} \rightarrow S(W)^{C_p} \). Then \( f_{C_p} \) is \( K \)-equivariant and so one can obtain a \( \Gamma_p \)-map

\[
f_{(\Gamma_p)} : \Gamma_p S(V)^{C_p} \rightarrow \Gamma_p S(W)^{C_p}.
\]

Since any isotropy subgroup \( H \) of \( Y := S(V) \setminus \Gamma_p S(V)^{C_p} \) is not conjugate to \( C_p \), it follows from Lemma 3.4 that \( H \) is a closed subgroup of \( T^{p-1} \). Then for any \( H \in \text{Iso}(Y) \), we shall verify the condition of Proposition 3.2.

\[
\text{dim } Y^H / W_{\Gamma_p}(H) \leq \text{dim } S(W)^H.
\]
Assertion 1. It holds that $\dim_{\mathbb{R}} U_1^H \leq \dim_{\mathbb{R}} U_p^H$ for any non-trivial closed subgroup $H \leq T^{p-1}$.

Indeed, by the isomorphism $\varphi : \Gamma_p/\mathbb{Z}_{p-1} \to \Gamma_p$ defined by $\varphi(ta^i) = t^p a^i$, it follows that $U_p^{\mathbb{Z}_{p-1}}$ is isomorphic to $U_1$, see Remark 2. Then one sees

$$\dim_{\mathbb{R}} U_p^H = \dim_{\mathbb{R}} U_p^{\mathbb{Z}_{p-1}} = \dim_{\mathbb{R}}(U_p^{\mathbb{Z}_{p-1}}/\mathbb{Z}_{p-1}) = \dim_{\mathbb{R}} U_1^H \geq \dim_{\mathbb{R}} U_1^H,$$

where $H^p = \{t^p \in H \mid t \in H\} \leq H$. Thus Assertion 1 holds.

Remark 4. If $H$ is a subtorus of $T^{p-1}$, then $H^p = H$ and $\dim_{\mathbb{R}} U_1^H = \dim_{\mathbb{R}} U_p^H$, see Remark 2.

Since $\dim_{\mathbb{R}} U_1^H \leq 2p - 2$ for $H \neq 1$, one sees that

$$\dim_{\mathbb{R}} V^H \leq \dim_{\mathbb{R}} U_1^H + \dim_{\mathbb{R}} U_1^H \leq 2p - 2 + \dim_{\mathbb{R}} U_p^H = \dim_{\mathbb{R}} W^H.$$ 

This inequality shows that the inequality (3.3) holds for $H \neq 1$. When $H = 1$, it follows that

$$\dim Y/\Gamma_p = \dim S(V)/\Gamma_p = 3p \leq \dim S(W) = 4p - 3,$$

since $p \geq 3$. Thus there exists a $\Gamma_p$-map $f$ extending $f_{(C_p)}$ by Proposition 3.2.

Remark 5. In case of $p = 2$, it still follows that there exists a $\Gamma_2$-map

$$f : S(U_1 \oplus U_1) \to S(U_2 \oplus V_1).$$

Indeed, by Lemma 3.3, there exits a $\Gamma_2$-map $f : S(U_1' \oplus U_1') \to S(U_2' \oplus V_1')$ between orthogonal representation spheres. By complexification, one sees that $U_k = \mathbb{C} \otimes U_k'$ and $V_k = \mathbb{C} \otimes V_k'$, and $f$ induces a $\Gamma_2$-map $f_{\mathbb{C}} : S(U_1' \oplus U_1') \to S(U_2' \oplus V_1')$.

4 The case of $S^1 \times C_p$

In this section, we shall show that $G = S^1 \times C_p$ is an anti-BU-group of types I and II, and complete the proof of Proposition 2.1. In this case, since the obstruction to extension of a $G$-map may appear, the proof is more complicated. As the first step, using an argument similar to that in [12], we shall show the following result.

Proposition 4.1. The group $G = S^1 \times C_p$ is an anti-BU-group of type II. Namely, there exists a $G$-map $f : S(V) \to S(W)$ with $\deg f = 0$ for some fixed-point-free $G$-representations $V$ and $W$ with the same dimension.

Let $G = S^1 \times C_p$ and $a \in C_p$ be a generator of $C_p$ as before. The irreducible unitary $G$-representations are given as follows. Let $V_{1,0}$ denote the lift of the 1-dimensional unitary $S^1$-representation $U^t_1$ on which $t \in S^1$ acts by $t \cdot z = tz$ for $z \in V_{1,0}$. Let $V_{0,1}$ denote the lift of the 1-dimensional unitary $C_p$-representation $U_1$ on which $a$ acts by $a \cdot z = \xi_p z$ for $z \in V_{0,1}$. Every irreducible unitary $G$-representation is given by $V_{k,l} := V_{1,0}^k \otimes V_{0,1}^l$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}/p$. We consider $G$-representations $V_{1,0} \oplus V_{1,1}$ and $V_{0,0} \oplus V_{0,1}$.
**Lemma 4.2.** For $V = V_{1,0} \oplus V_{1,1}$ and $W = V_{p,0} \oplus V_{0,1}$, there exists a $G$-map $h : S(V) \to S(W)$.

**Proof.** Note that $\text{Ker} V_{1,0} = C_p = \langle a \rangle$ and $\text{Ker} V_{1,1} = C_p' := \langle \xi_p^{-1}a \rangle$. Therefore $\text{Iso}(S(V))$ consists of $C_p$, $C_p'$ and 1. Similarly, $\text{Ker} V_{p,0} = \mathbb{Z}_p \times C_p$, where $\mathbb{Z}_p = \langle \xi_p \rangle \leq S^1$ and $\text{Ker} V_{0,1} = S^1$, and $\text{Iso}(S(W))$ consists of $\mathbb{Z}_p \times C_p$, $S^1$ and $\mathbb{Z}_p$. A $G$-map

$$h_{C_p} : S(V)^{C_p} = S(V_{1,0}) \to S(W)$$

is defined by $h_{C_p}(z) = (z^p, 0)$ for $z \in S(V_{1,0})$, and also a $G$-map

$$h_{C_p'} : S(V)^{C_p'} = S(V_{1,1}) \to S(W)$$

is defined by $h_{C_p'}(w) = (w^p, 0)$ for $w \in S(V_{1,1})$. Using these maps, we obtain a $G$-map

$$h^{>1} : S(V)^{>1} \to S(W),$$

where $S(V)^{>1}$ is the singular set:

$$S(V)^{>1} := \{ x \in S(V) \mid G_x \neq 1 \} = S(V)^C \coprod S(V)^{C_p'}.$$

Since $G$ acts freely on $S(V) \setminus S(V)^{>1}$ and $\dim(S(V) \setminus S(V)^{>1})/G = 2 < \dim S(W) = 3$, there exists a $G$-map $h : S(V) \to S(W)$ extending $h^{>1}$ by Proposition 3.2. 

Next we shall show $\deg h = 0$ for any $G$-map $h$ as above and finish the proof of Proposition 4.1. To do that, we use the Euler classes of an oriented orthogonal representation $V$, i.e., the $G$-action on $V$ is orientation preserving under a given orientation on $V$. A unitary $G$-representation $V$ has a canonical orientation given by the complex structure of $V$ and oriented as an orthogonal $G$-representation. Generally, the Euler class

$$e_G(V) \in H^n(BG, R)$$

of an oriented orthogonal representation $V$ of dimension $n$ is defined to be the Euler class of the associated vector bundle $\pi : EG \times_G V \to BG$ over the classifying space $BG$. In case of a unitary $G$-representation $V$, the Euler class $e_G(V)$ coincides with the top Chern class of the associated complex vector bundle. Although the coefficient ring $R$ can be taken to be $\mathbb{F}_p$, $\mathbb{Z}$ or $\mathbb{Q}$, etc, we here take $R = \mathbb{Q}$ as coefficients, because the cohomology ring of $BG$ becomes simpler and it is sufficient for our purpose. A key result is the following special case of the result of [11], see also [12].

**Proposition 4.3 ([11], [12]).** Let $G$ be a compact Lie group and $V, W$ fixed-point-free, oriented orthogonal $G$-representations with the same dimension $n$. Suppose that there exists a $G$-map $h : S(V) \to S(W)$. Then

$$e_G(W) = (\deg h)e_G(V) \in H^n(BG; R).$$

In particular, under $R = \mathbb{Q}$, if $e_G(V) \neq 0$ and $e_G(W) = 0$, then $\deg h = 0$. 

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Now we return to the case of $G = S^1 \times C_p$. The next lemma shows Proposition 4.4.

**Lemma 4.4.** Let $V = V_{10} \oplus V_{11}$ and $W = V_{p,0} \oplus V_{0,1}$. For any $G$-map $h : S(V) \to S(W)$, it follows that $\deg h = 0$.

**Proof.** Since $BG \cong BS^1 \times BC_p$ and $BC_p$ is $\mathbb{Q}$-acyclic, it follows that

$$\Res_{S^1} = i^* : H^*(BG; \mathbb{Q}) \to H^*(BS^1; \mathbb{Q})$$

is a graded ring isomorphism, where $i : S^1 \to G$ is the natural inclusion. Since $H^*(BS^1; \mathbb{Q}) \cong \mathbb{Q}[c]$ as graded rings, where $c$ is the first Chern class of the canonical complex line bundle over $BS^1 \cong \mathbb{C}P^\infty$, one obtains $H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[c]$ as graded rings. Clearly $\Res_{S^1} V_{1,0}$ and $\Res_{S^1} V_{1,1}$ are isomorphic to the standard $S^1$-representation $\overline{\mathbb{U}}$ whose associated complex vector bundle is isomorphic to the canonical one over $BS^1$, hence $e_G(V) = e_G(V_{1,0})e_G(V_{1,1}) = c^2 \neq 0$. On the other hand, $\Res_{S^1} V_{0,1}$ is the trivial $S^1$-representation, hence $e_G(V_{0,1}) = 0$. This implies that $e_G(W) = e_G(V_{p,0})e_G(V_{0,1}) = 0$. Thus $\deg h = 0$ by Proposition 4.3. $\square$

The second step is to construct a $G$-map $f : S(V) \to S(W)$ for some fixed-point-free representations $V, W$ with $\dim V > \dim W$. We shall prove this using a $G$-map $h$ of degree 0. The goal is to prove the following result.

**Proposition 4.5.** The group $G = S^1 \times C_p$ is an anti-BU-group of type I. Namely, there exists a $G$-map $f : S(V) \to S(W)$ for some fixed-point-free representations $V, W$ with $\dim V > \dim W$.

In order to prove this, we again use equivariant obstruction theory [7, Chapter II, 3]. In this case the obstruction may appear; however, the computation of the obstruction class is not easy in general. In order to avoid this difficulty, we use the existence of a $G$-map of degree 0. This idea is based on an argument of [14].

First recall the equivariant primary obstruction class. Let $G$ be a compact Lie group. Let $X$ a finite $G$-CW complex and $Y$ an $n$-simple and $(n - 1)$-connected $G$-space, $n \geq 1$. Let $X^{>1}$ be the singular set of $X$ and suppose that there exists a $G$-map $g : X^{>1} \to Y$. Let $X_{(m)}$ be an $m$-skeleton relative to $X^{>1}$, i.e., $X_{(m)}$ is the union of free $i$-cells $G \times e^i$ of $X \smallsetminus X^{>1}$ for $i \leq m$ with $X^{>1}$. Since $Y$ is $(n - 1)$-connected, there exists a $G$-map $f_n : X_{(m)} \to Y$ extending $g$, since the obstructions to extension vanish. In this situation, the equivariant primary obstruction $\gamma(g)$ is defined in the equivariant cohomology group $\mathcal{H}_G^{n+1}(X, X^{>1}; \pi_n(Y))$ and there exists a $G$-map $f_n : X_{(n+1)} \to Y$ extending $g$ if and only if $\gamma(g) = 0$.

**Lemma 4.6.** Let $Z$ be another $n$-simple and $(n - 1)$-connected $G$-space.

1. For any $G$-map $h : Y \to Z$, it follows that $\gamma(h \circ g) = h_\#(\gamma(g))$, where

   $$h_\# : \mathcal{H}_G^{n+1}(X, X^{>1}; \pi_n(Y)) \to \mathcal{H}_G^{n+1}(X, X^{>1}; \pi_n(Z))$$

   is the homomorphism induced by $h$. 

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(2) If there exists a $G$-map $h : Y \to Z$ such that $h_* = 0 : \pi_n(Y) \to \pi_n(Z)$, then there exists a $G$-map $f_{n+1} : X_{(n+1)} \to Z$ extending $h \circ g$.

**Proof.** (1) From [7] Chapter II, 3], the equivariant primary obstruction $\gamma(f)$ is represented by a cocycle

$$c^{n+1}(g) : C_{n+1}(X_{(n+1)}, X_n) \xleftarrow{\partial} \pi_{n+1}(X_{(n+1)}, X_n) \xrightarrow{\delta} \pi_n(X_n) \xrightarrow{f_*} \pi_n(Y),$$

where $\rho$ is the Hurewicz homomorphism

$$\rho : \pi_{n+1}(X_{(n+1)}, X_n) \to H_{n+1}(X_{(n+1)}, X_n; \mathbb{Z}) = C_{n+1}(X_{(n+1)}, X_n).$$

Since an extension $f'_n : X_n \to Z$ of $h \circ g$ is given by $f'_n = h \circ f_n$, the obstruction class $\gamma(h \circ g)$ is represented by

$$c^{n+1}(h \circ g) : C^{n+1}(X_{(n+1)}, X_n) \xleftarrow{\partial} \pi_{n+1}(X_{(n+1)}, X_n) \xrightarrow{\delta} \pi_n(X_n) \xrightarrow{(h \circ f_n)_*} \pi_n(Z).$$

Clearly $c^{n+1}(h \circ g) = h_*(c^{n+1}(g))$ and thus $\gamma(h \circ g) = h_\#(\gamma(g))$.

(2) Since $h_* = 0$, the obstruction class $\gamma(h \circ g) = h_\#(\gamma(g))$ vanishes. Therefore there exists a $G$-map $f_{n+1} : X_{(n+1)} \to Z$ extending $h \circ g$. \qed

**Proof of Proposition 4.3.** Consider $G$-representations

$$V = 2V_{1,0} \oplus 2V_{1,1}, \quad U = V_{1,0} \oplus V_{p,0} \oplus V_{1,1}, \quad W = 2V_{p,0} \oplus V_{0,1}.$$  

Observing that

$$V^{C_p} = 2V_{1,0}, \quad U^{C_p} = V_{1,0} \oplus V_{p,0}, \quad V^{C_p} = 2V_{1,1}, \quad U^{C_p} = V_{p,0} \oplus V_{1,1},$$

one can define $G$-maps

$$f^{C_p} : S(V)^{C_p} \to S(U)^{C_p}, (z, w) \mapsto (z, w^p)/\|z, w^p\|$$

and

$$f^{C_p} : S(V)^{C_p} \to S(U)^{C_p}, (z, w) \mapsto (z^p, w)/\|z^p, w\|.$$  

Therefore, there exists a $G$-map $g : S(V)^{>1} \to S(U)$, where $S(V)^{>1} = S(V)^{C_p} \coprod S(V)^{C_p}$. By Lemmas 4.2 and 4.4, there exists a $G$-map

$$h : S(V_{1,0} \oplus V_{1,1}) \to S(V_{p,0} \oplus V_{0,1})$$

of degree 0. We define a $G$-map $\tilde{h} : S(U) \to S(W)$ by

$$\tilde{h} = h \ast id : S(U) \cong S(V_{1,0} \oplus V_{1,1}) \ast S(V_{p,0}) \to S(V_{p,0} \oplus V_{0,1}) \ast S(V_{p,0}) \cong S(W)$$

where $\ast$ means join. Then $\tilde{h}_* = 0$ on $\pi_5(S(U))$, since $\deg(h \ast id) = 0$. Since $S(U)$ and $S(W)$ are 4-connected, it follows from Lemma 1.6 that there exists a $G$-map $f : S(V)^{0} \to S(W)$ extending $\tilde{h} \circ g$. Since $\dim S(V)/G = 6$, it follows that $S(V)^{0}$ coincides with the whole space $S(V)$. Thus there exists a $G$-map $f : S(V) \to S(W)$. \qed
Remark 6. In case of \( p = 2 \), by an argument similar to that in Lemma 3.3, one can see that there exists an \( S^1 \times C_2 \)-map between orthogonal representation spheres as a counterexample of the Borsuk-Ulam property of type I.

Proof of Theorem 1.2. It is already known that statement (3) implies (2), see for example [9]. Statement (2) implies (1) by Proposition 3.1. The discussion so far shows that if \( G \) is neither \( C'_p \) nor \( T^n \), then \( G \) is an anti-BU-group of both types I and II. In particular, statement (1) implies (3).

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