THE TOPOLOGY OF PARABOLIC CHARACTER VARIETIES OF FREE GROUPS

INDRANIL BISWAS, CARLOS FLORENTINO, SEAN LAWTON, AND MARINA LOGARES

Abstract. Let $G$ be a complex affine algebraic reductive group, and let $K \subset G$ be a maximal compact subgroup. Fix $h := (h_1, \cdots, h_m) \in K^m$. For $n \geq 0$, let $X_{h,n}^G$ (respectively, $X_{h,n}^K$) be the space of equivalence classes of representations of the free group on $m + n$ generators in $G$ (respectively, $K$) such that for each $1 \leq i \leq m$, the image of the $i$-th free generator is conjugate to $h_i$. These spaces are parabolic analogues of character varieties of free groups. We prove that $X_{h,n}^K$ is a strong deformation retraction of $X_{h,n}^G$. In particular, $X_{h,n}^G$ and $X_{h,n}^K$ are homotopy equivalent. We also describe explicit examples relating $X_{h,n}^G$ to relative character varieties.

1. Introduction

Let $G$ be a complex affine algebraic reductive group. Fix a maximal compact subgroup $K$ of $G$.

Denote by $\text{Hom}(F_n, G)$ the space of representations in $G$ of the free group $F_n$ of rank $n$. The group $G$ acts on $\text{Hom}(F_n, G)$ by conjugation; the action of any $g \in G$ sends any $\rho \in \text{Hom}(F_n, G)$ to the homomorphism defined by $\gamma \mapsto g^{-1} \rho(\gamma) g$, where $\gamma \in F_n$. The GIT quotient space

$$X_n^G := \text{Hom}(F_n, G) \sslash G$$

is an affine algebraic variety, known as the $G$-character variety of $F_n$.

Similarly, we have the compact topological space

$$X_n^K := \text{Hom}(F_n, K) / K$$

that parametrizes the equivalence classes of homomorphisms from $F_n$ to $K$.

Whenever we will mention topology of $X_n^G$, we will mean the Euclidean topology of the underlying complex space.

In [FL1] it was shown that there is a strong deformation retraction from $X_n^G$ to $X_n^K$. In particular, these spaces have the same homotopy type. On the other hand, if $\pi$ is the fundamental group of a closed orientable surface, then it is known that the corresponding spaces $\text{Hom}(\pi, G) \sslash G$ and $\text{Hom}(\pi, K) / K$ do not have the same homotopy type; in fact, any connected component of $\text{Hom}(\pi, G) \sslash G$ differs cohomologically from the corresponding connected component of $\text{Hom}(\pi, K) / K$ [BF].

Our aim here is to consider the parabolic analogues of $X_n^G$ and $X_n^K$.

Take any integer $m \geq 0$, and fix $(h_1, \cdots, h_m) \in K^m$. Let $C_j^G$ (respectively, $C_j^K$) be the conjugacy class of $h_j$ in $G$ (respectively, $K$). For any $n \geq 0$, let $F_{m+n}$ be the free

2000 Mathematics Subject Classification. 14L30, 20E05, 14P25, 14L17.

Key words and phrases. Free group, representation space, conjugacy class, strong deformation retraction, complex algebraic group.
group on $m + n$ generators; the $i$-th generator of $F_{m+n}$ will be denoted by $e_i$. Define
\[
H^G_{h,n} := \{ \rho \in \text{Hom}(F_{m+n}, G) \mid \rho(e_j) \in C^G_j, \forall 1 \leq j \leq m \}
\]
and
\[
H^K_{h,n} := \{ \rho \in \text{Hom}(F_{m+n}, K) \mid \rho(e_j) \in C^K_j, \forall 1 \leq j \leq m \}.
\]
As before, the group $G$ (respectively, $K$) acts on $H^G_{h,n}$ (respectively, $H^K_{h,n}$) via the conjugation action of $G$ (respectively, $K$) on itself. Define
\[
X^G_{h,n} := H^G_{h,n} \// G \quad \text{and} \quad X^K_{h,n} := H^K_{h,n} / K.
\]
Note that if $m = 0$ these spaces reduce to the previous ones, $X^G_n$ and $X^K_n$, respectively.

We prove the following (see Corollary [10]):

**Theorem 1.** The quotient space $X^K_{h,n}$ is a strong deformation retraction of $X^G_{h,n}$. In particular, $X^G_{h,n}$ and $X^K_{h,n}$ are homotopy equivalent.

Take closed subgroups $K_1, \ldots, K_m$ of $K$. Let $G_i$ be the Zariski closure of $K_i$ in $G$. Theorem [1] is derived from the following more general result:

**Theorem 2.** Let \((\prod_{i=1}^m K/K_i) \times K^n) / K \) (respectively, \((\prod_{i=1}^m G/G_i) \times G^n) / G\) be the quotient for the diagonal left-translation action of $K$ (respectively, $G$) on the $m$ homogeneous spaces $K/K_i$ (respectively, $G/G_i$), and conjugation on the other $n$ factors of $K$ (respectively, $G$). Then \((\prod_{i=1}^m K/K_i) \times K^n) / K\) is a strong deformation retraction of \((\prod_{i=1}^m G/G_i) \times G^n) / G\).

Let $\Sigma$ be the complement of $p$ points of a compact surface of genus $g$, with $p \geq 1$. The fundamental group of $\Sigma$ is isomorphic to the free group $F_{m+n}$ of rank $m + n = 2g + p - 1$. By choosing a free generating set $\gamma = \{\gamma_1, \ldots, \gamma_{m+n}\}$ of $\pi_1(\Sigma, s_0)$, where $s_0$ is a base point, the space of representations of $\pi_1(\Sigma, s_0)$ into $G$ gets identified with $\text{Hom}(F_{m+n}, G)$ which is isomorphic to $G^{m+n}$, via the evaluation map
\[
ev_\gamma : \text{Hom}(\pi_1(\Sigma, s_0), G) \cong \text{Hom}(F_{m+n}, G) \rightarrow G^{m+n}, \rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_{m+n})) .
\]
Using this identification, we have
\[
X^G_{h,n} \subset \text{Hom}(\pi_1(\Sigma, s_0), G) / G .
\]

In the case of $m = 0$, by sending any flat connection on $\Sigma$ to its monodromy representation, the space $X^K_n$ (respectively, $X^G_n$) gets identified with the moduli space of flat principal $K$–bundles (respectively, completely reducible flat $G$–bundles) over $\Sigma$. For $m > 0$, we call $X^K_{h,n}$ and $X^G_{h,n}$ parabolic character varieties, because of their analogy with parabolic vector bundles.

As studied in [BG] and [La2], there is a natural “boundary” map associating the conjugacy classes around the punctures to a representation in $\text{Hom}(\pi_1(\Sigma, s_0), G)$. Relative character varieties are then defined as inverse images under the boundary map. In the last two sections, we consider various examples of parabolic character varieties and show how they relate to relative character varieties.
2. Deformation Retractions of Homogeneous Spaces

The base field is the complex numbers. Let $G$ be an affine algebraic reductive group. Fix a maximal compact subgroup $K \subset G$. Let $X$ be an affine $G$–variety. The action of $G$ on $X$ will be denoted by “$\cdot$”, meaning the action of $g \in G$ sends any $x \in X$ to $g \cdot x \in X$. The algebra of $G$-invariant regular functions on $X$ will be denoted by $\mathbb{C}[X]^G$. Let

$$X//G := \text{Spec}(\mathbb{C}[X]^G)$$

be the quotient, which is an affine variety $[\text{Do}]$. We call $X//G$ the affine quotient of $X$ by $G$. It satisfies the usual properties of good categorical quotients (see $[\text{Do}]$).

In this situation, one can show that there is a pair $(V, \iota)$, where $V$ is a $G$–module of finite (complex) dimension, and

$$\iota : X \hookrightarrow V$$

is a $G$–equivariant embedding. Fix a $K$-invariant Hermitian inner product $\langle -, - \rangle$ on $V$. For any $x \in X$, let

$$F_x : G \rightarrow \mathbb{R}$$

be the smooth function defined by $g \mapsto ||\iota(g \cdot x)||^2$. With respect to these choices of $V$, $\iota$ and $\langle -, - \rangle$, the Kempf–Ness set of $X$ is defined to be

$$(2.1) \quad \text{KN}_X := \{ x \in X \mid (dF_x)_e = 0 \},$$

where $e \in G$ is the identity element. This implies that $\text{KN}_X$ is a real algebraic subspace, and hence it is a CW-complex ($[\text{BCR}, \text{Hi}]$).

For any $x \in X$, let $K \cdot x$ denote the $K$-orbit passing through $x$. We recall a theorem of Kempf and Ness.

**Theorem 3** ([KN, Theorem 0.1, Theorem 0.2]). *All the critical points of $F_x$ are minima. Moreover, $F_x$ has a critical point if and only if $G \cdot x \subset X$ is closed. Also, if $x \in \text{KN}_X$, then $K \cdot x \subset \text{KN}_X$.*

A deformation retraction from a topological space $M$ to a subspace $N \subset M$ is a homotopy

$$\phi : [0, 1] \times M \rightarrow M$$

between the identity map of $M$, and a retraction $r_N : M \rightarrow N$. It is called a strong deformation retraction whenever it fixes the points in $N$, that is, $\phi(t, x) = x$ for all $x \in N$ and $t \in [0, 1]$ (see $[\text{Sp}]$). To abbreviate, we often express the above situation by saying that $N \hookrightarrow M$, i.e., the inclusion itself, is (strong) a deformation retraction.

**Remark 4.** If a group $J$ acts on $M$ and on $N$ such that there is a $J$–equivariant homotopy equivalence $\alpha : N \rightarrow M$ (i.e., a $J$-equivariant map with a homotopy inverse and whose corresponding homotopies are $J$-equivariant), then the induced map $\tilde{\alpha} : N/J \rightarrow M/J$ is again a homotopy equivalence. Indeed, a $J$–equivariant homotopy

$$[0, 1] \times M \rightarrow M$$

descends to a homotopy

$$[0, 1] \times (M/J) \rightarrow (M/J)$$

of quotient spaces.

We now recall a theorem of Neeman and Schwarz.
Theorem 5 ([Ne, Theorem 2.1], [Sc, Corollary 4.7, Corollary 5.3]). The composition
\[ \text{Kn}_X \hookrightarrow X \twoheadrightarrow X/G \]
is proper and induces a homeomorphism \( \text{Kn}_X/K \xrightarrow{\sim} X/G \). Moreover, there is a \( K \)-equivariant deformation retraction from \( X \) to \( \text{Kn}_X \). Therefore, there is a deformation retraction
\[ f : \text{Kn}_X/K \longrightarrow X/K. \]

The above theorem will be used in getting a deformation retraction between homogeneous spaces. We start with the following proposition which is basically an adaptation of some results in [FL2] to our situation.

Proposition 6. Suppose that we have the following commutative diagram of maps of CW complexes:
\[ \begin{array}{ccc}
Y & \xrightarrow{\alpha} & X \\
\downarrow f & & \downarrow \parallel \\
Z & \xrightarrow{\beta} & X
\end{array} \]
where \( f \) is an inclusion of a subcomplex, and the maps \( \alpha \) and \( \beta \) are homotopy equivalences. Then \( f \) is a strong deformation retraction.

Proof. For any nonnegative integer \( n \), let
\[ \alpha_n : \pi_n(Y) \longrightarrow \pi_n(X) \quad \text{and} \quad \beta_n : \pi_n(Z) \longrightarrow \pi_n(X) \]
be the homomorphisms induced by \( \alpha \) and \( \beta \) respectively. Since \( \alpha \) and \( \beta \) are homotopy equivalences, we know that \( \alpha_n \) and \( \beta_n \) are isomorphisms for all \( n \geq 0 \). Since
\[ \alpha = \beta \circ f, \]
it follows that \( f \) also induces isomorphisms on all homotopy groups. Then, from Whitehead’s Theorem (see [Ha, Theorem 4.5, page 346]) we conclude that \( Y \) is indeed a strong deformation retraction of \( Z \). \qed

Remark 7. Using the same notation as in Proposition 6, suppose that a group \( J \) acts on \( X, Y \) and \( Z \), in such a way that \( f \) is a \( J \)-equivariant inclusion of a subcomplex, and \( \alpha \) and \( \beta \) are \( J \)-equivariant homotopy equivalences. Then all maps in the diagram of Proposition 6 are \( J \)-equivariant, and by Remark 4 we obtain a commutative diagram of maps of quotient spaces
\[ \begin{array}{ccc}
Y/J & \xrightarrow{\tilde{\alpha}} & X/J \\
\downarrow \tilde{f} & & \downarrow \parallel \\
Z/J & \xrightarrow{\tilde{\beta}} & X/J
\end{array} \]
Thus, from Proposition 6 the induced map \( \tilde{f} : Y/J \longrightarrow Z/J \) is a strong deformation retraction, provided the quotient spaces \( Y/J \) and \( Z/J \) are CW complexes. Note however that \( f \) is not necessarily a \( J \)-equivariant strong deformation retract.

Recall that \( G \) is an affine algebraic reductive group, and \( K \) is a maximal compact subgroup of \( G \). Suppose that \( J \) is a closed subgroup of \( K \). Then the Zariski closure \( H \) of \( J \) (in \( G \)) is a reductive subgroup of \( G \) with
\[ J = H \cap K, \]
Consider the homogeneous space $G/H$. Since $H$ is reductive, we can regard $G/H$ as an affine quotient of the $H$–variety $G$, where $H$ acts by right translations on $G$. Note that $G/H = G//H$ is a smooth affine variety. Similarly, we have the quotient $K/J$, which is a compact manifold.

**Lemma 8.** There is a canonical inclusion of smooth manifolds $f : K/J \rightarrow G//H$.

**Proof.** Let $f : K/J \rightarrow G/H = G//H$ be the canonical map that sends any coset $kJ$, $k \in K$, to the coset $kH$. This $f$ is well-defined since $J \subset H$. Take any $k_1, k_2 \in K$ such that $k_1H = k_2H$. To prove the lemma it suffices to show that

$$k_1J = k_2J.$$  

(2.3)

To prove (2.3), take $h := k_2^{-1}k_1 \in H$. Let

$$\text{Lie}(H) = \text{Lie}(J) \oplus p$$

be the Cartan decomposition of the Lie algebra of $H$. Let

$$h = j \exp(p)$$

be the polar decomposition, where $j \in J$ and $p \in p$. Then,

$$j^{-1}k_2^{-1}k_1 = \exp(p).$$

(2.5)

The left–hand side of (2.5) is in the maximal compact subgroup $K$ of $G$, so by the uniqueness of polar decomposition in $G$, we have that

$$\exp(p) = e \text{ and } k_1 = k_2j.$$ 

Therefore, (2.3) holds.

**Lemma 9.** There is a $K \times K$-equivariant strong deformation retraction from $G$ to $K$.

**Proof.** The polar decomposition of $G$ is given by $K \times p$ and the diffeomorphism is given by $(k, p) \mapsto k\exp(p)$. Then $\phi_t(k\exp(p)) = k\exp(tp)$ is a strong deformation retraction. Consider $a, b \in K$. Then

$$a\phi_t(k\exp(p))b^{-1} = ak\exp(tp)b^{-1} = akb^{-1}\exp(t\text{Ad}_b(p)) = \phi_t(akb^{-1}\exp(\text{Ad}_b(p))) = \phi_t(ak\exp(p)b^{-1}).$$

Theorem 10. The inclusion $K/J \hookrightarrow G/H$ from Lemma 8 is a strong deformation retraction.

**Proof.** In (2.1), set $X = G$ equipped with the right–translation action of $H$. By the Kempf–Ness construction, we obtain a commutative diagram of natural inclusions

$$K \hookrightarrow G \xrightarrow{\phi} KN_G \hookrightarrow G.$$ 

(2.6)

As a special case of Lemma 14 below, $K \hookrightarrow KN_G$. 

and $J$ is a maximal compact subgroup of $H$. 

Consider the homogeneous space $G/H$. Since $H$ is reductive, we can regard $G/H$ as an affine quotient of the $H$–variety $G$, where $H$ acts by right translations on $G$. Note that $G/H = G//H$ is a smooth affine variety. Similarly, we have the quotient $K/J$, which is a compact manifold.

**Lemma 8.** There is a canonical inclusion of smooth manifolds $f : K/J \rightarrow G//H$.

**Proof.** Let $f : K/J \rightarrow G/H = G//H$ be the canonical map that sends any coset $kJ$, $k \in K$, to the coset $kH$. This $f$ is well-defined since $J \subset H$. Take any $k_1, k_2 \in K$ such that $k_1H = k_2H$. To prove the lemma it suffices to show that

$$k_1J = k_2J.$$  

(2.3)

To prove (2.3), take $h := k_2^{-1}k_1 \in H$. Let

$$\text{Lie}(H) = \text{Lie}(J) \oplus p$$

be the Cartan decomposition of the Lie algebra of $H$. Let

$$h = j \exp(p)$$

be the polar decomposition, where $j \in J$ and $p \in p$. Then,

$$j^{-1}k_2^{-1}k_1 = \exp(p).$$

(2.5)

The left–hand side of (2.5) is in the maximal compact subgroup $K$ of $G$, so by the uniqueness of polar decomposition in $G$, we have that

$$\exp(p) = e \text{ and } k_1 = k_2j.$$ 

Therefore, (2.3) holds.

**Lemma 9.** There is a $K \times K$-equivariant strong deformation retraction from $G$ to $K$.

**Proof.** The polar decomposition of $G$ is given by $K \times p$ and the diffeomorphism is given by $(k, p) \mapsto k\exp(p)$. Then $\phi_t(k\exp(p)) = k\exp(tp)$ is a strong deformation retraction. Consider $a, b \in K$. Then

$$a\phi_t(k\exp(p))b^{-1} = ak\exp(tp)b^{-1} = akb^{-1}\exp(t\text{Ad}_b(p)) = \phi_t(akb^{-1}\exp(\text{Ad}_b(p))) = \phi_t(ak\exp(p)b^{-1}).$$

Theorem 10. The inclusion $K/J \hookrightarrow G/H$ from Lemma 8 is a strong deformation retraction.

**Proof.** In (2.1), set $X = G$ equipped with the right–translation action of $H$. By the Kempf–Ness construction, we obtain a commutative diagram of natural inclusions

$$K \hookrightarrow G \xrightarrow{\phi} KN_G \hookrightarrow G.$$ 

(2.6)

As a special case of Lemma 14 below, $K \hookrightarrow KN_G$. 

and $J$ is a maximal compact subgroup of $H$. 

Consider the homogeneous space $G/H$. Since $H$ is reductive, we can regard $G/H$ as an affine quotient of the $H$–variety $G$, where $H$ acts by right translations on $G$. Note that $G/H = G//H$ is a smooth affine variety. Similarly, we have the quotient $K/J$, which is a compact manifold.

**Lemma 8.** There is a canonical inclusion of smooth manifolds $f : K/J \rightarrow G//H$.

**Proof.** Let $f : K/J \rightarrow G/H = G//H$ be the canonical map that sends any coset $kJ$, $k \in K$, to the coset $kH$. This $f$ is well-defined since $J \subset H$. Take any $k_1, k_2 \in K$ such that $k_1H = k_2H$. To prove the lemma it suffices to show that

$$k_1J = k_2J.$$  

(2.3)

To prove (2.3), take $h := k_2^{-1}k_1 \in H$. Let

$$\text{Lie}(H) = \text{Lie}(J) \oplus p$$

be the Cartan decomposition of the Lie algebra of $H$. Let

$$h = j \exp(p)$$

be the polar decomposition, where $j \in J$ and $p \in p$. Then,

$$j^{-1}k_2^{-1}k_1 = \exp(p).$$

(2.5)

The left–hand side of (2.5) is in the maximal compact subgroup $K$ of $G$, so by the uniqueness of polar decomposition in $G$, we have that

$$\exp(p) = e \text{ and } k_1 = k_2j.$$ 

Therefore, (2.3) holds.

**Lemma 9.** There is a $K \times K$-equivariant strong deformation retraction from $G$ to $K$.

**Proof.** The polar decomposition of $G$ is given by $K \times p$ and the diffeomorphism is given by $(k, p) \mapsto k\exp(p)$. Then $\phi_t(k\exp(p)) = k\exp(tp)$ is a strong deformation retraction. Consider $a, b \in K$. Then

$$a\phi_t(k\exp(p))b^{-1} = ak\exp(tp)b^{-1} = akb^{-1}\exp(t\text{Ad}_b(p)) = \phi_t(akb^{-1}\exp(\text{Ad}_b(p))) = \phi_t(ak\exp(p)b^{-1}).$$

Theorem 10. The inclusion $K/J \hookrightarrow G/H$ from Lemma 8 is a strong deformation retraction.

**Proof.** In (2.1), set $X = G$ equipped with the right–translation action of $H$. By the Kempf–Ness construction, we obtain a commutative diagram of natural inclusions

$$K \hookrightarrow G \xrightarrow{\phi} KN_G \hookrightarrow G.$$ 

(2.6)

As a special case of Lemma 14 below, $K \hookrightarrow KN_G$.
As semi-algebraic spaces, $K$ can be taken to be a sub-complex of the CW-complex $\text{KN}_G$, as can the quotients $K/J \hookrightarrow \text{KN}_G/J$ (BCR, HI, Sc).

From Theorem 3 we know that $\text{KN}_G$ is closed under the right-translation action of $J := H \cap K$. So all the maps in (2.6) are $J$-equivariant.

The top map in (2.6) induces a right $J$-equivariant strong deformation retraction by Lemma 9. The bottom map likewise induces a $J$-equivariant strong deformation retraction by Theorem 5. Therefore, by Proposition 6 and Remark 7, the inclusion $	ilde{\varphi} : K/J \rightarrow \text{KN}_G/J$

given by $\varphi$ in (2.6) is a strong deformation retraction. Finally, by the homeomorphism $\text{KN}_G/J \cong G/H$ in Theorem 5, we obtain a strong deformation retraction of $G/H$ to $K/J$.

Remark 11. Fix an element $k \in K$. Let $G \cdot k := \{ gkg^{-1} \mid g \in G \}$ be its orbit under the conjugation action of $G$. Let $G_k := \{ g \in G \mid gkg^{-1} = k \}$ be the centralizer. Since $k$ is semisimple, it follows that $G_k$ is a complex reductive subgroup of $G$ [Hu1, § 2.2, p. 26, Theorem]. Sending any $g \in G$ to $gkg^{-1} \in G \cdot k$, we get an identification of $G/G_k$ with $G \cdot k$. This identification takes the left-translation action of $G$ on $G/G_k$ to the adjoint action of $G$ on $G \cdot k$. Moreover, this mapping $G/G_k \rightarrow G \cdot k$ defines an isomorphism of quasi-projective varieties [Hu2, p. 83].

Similarly, the $K$–orbit $K \cdot k$ gets identified with $K/K_k$, where $K_k$ is the centralizer of $k$ inside $K$. Note that $K_k = G_k \cap K$.

Corollary 12. Fix any $k \in K$. The inclusion of orbits $K \cdot k \subset G \cdot k$ is a strong deformation retraction.

Proof. The group $K_k$ is compact because it is closed in $K$. Its Zariski closure in $G$ is $G_k$. In view of the canonical inclusion $f : K/K_k \rightarrow G/G_k$ in Lemma 8, we can apply Theorem 10 with $J = K_k$ and $H = G_k$. Therefore, $K \cdot k$ is a strong deformation retraction of $G \cdot k$.

Note that the retraction in Theorem 10 is more general than that in Corollary 12 since not every closed subgroup $J \subset K$ is the stabilizer of some $k \in K$.

3. PARABOLIC CHARACTER VARIETIES OF A FREE GROUP

Let $m \geq 0$ be an integer, and let $K_i$ for $1 \leq i \leq m$ be closed subgroups of the compact group $K$. For each $1 \leq i \leq m$, let $G_i \subset G$ be the Zariski closure of $K_i$; note that $G_i$ is reductive, and $\text{Lie}(G_i) = \text{Lie}(K_i) \oplus \sqrt{-1} \cdot \text{Lie}(K_i)$. Define

\begin{equation}
\mathcal{X}_{m,n} := \left( \prod_{i=1}^{m} G/G_i \right) \times G^n
\end{equation}

and

\begin{equation}
\mathcal{Y}_{m,n} := \left( \prod_{i=1}^{m} K/K_i \right) \times K^n.
\end{equation}
Since \( G = G/G_j \) if \( K_j = \{ e \} \), the parameter \( n \) could be removed; but it is retained because it will be useful for defining parabolic character varieties later on. As usual, the zero-fold Cartesian product \( X^0 \) consists of only the empty tuple. So \( X_{0,n} = G^n \) and \( Y_{0,n} = K^n \).

Let \( K \) act as left–translations on the homogeneous spaces \( K/K_j \) and \( G/G_j \) for \( 1 \leq j \leq m \); the group \( K \) acts on \( K \) and \( G \) through inner automorphisms. These actions provide an action of \( K \) on both \( Y_{m,n} \) and \( X_{m,n} \). Clearly, the map

\[
(3.3) \quad f : Y_{m,n} \hookrightarrow X_{m,n}
\]

(see (3.1) and (3.2)) is \( K \)-equivariant, and it is injective by Lemma 8.

The \( K \)-equivariant strong deformation retraction in \[FL1\], and the strong deformation retraction in Theorem 10 together prove the following proposition.

**Proposition 13.** For any \( m, n \geq 0 \), there is a strong deformation retraction from \( X_{m,n} \) to \( Y_{m,n} \).

The group \((\prod_{i=1}^m G_i) \times G\) acts on \( G^m \times G^n \) as follows:

\[
(h_1, \ldots, h_m, g) \cdot (f_1, \ldots, f_m, g_1, \ldots, g_n) = (gf_1h_1^{-1}, \ldots, f_mh_m^{-1}, gg_1g_1^{-1}, \ldots, gg_ng_n^{-1}).
\]

It is easy to see that the two GIT quotients \((G/G_1 \times \cdots \times G/G_m \times G^n)/G \) and \((G^m \times G^n)/((\prod_{i=1}^m G_i) \times G)\) are isomorphic; likewise for the compact quotients \((K^m \times K^n)/((\prod_{i=1}^m K_i) \times K)\) and \( Y_{m,n}/K \).

One can apply the Kempf-Ness construction to the affine \((\prod_{i=1}^m G_i) \times G\)-variety \( G^m \times G^n \).

**Lemma 14.** \( K^m \times K^n \) is a subcomplex of \( KN_{G^m \times G^n} \).

**Proof.** The fact that it is a subset follows from the computation in the proof of Proposition A.1 in \[FL2\]. Since both sets are algebraic, \( K^m \times K^n \) may be taken to be a subcomplex by \[Hi, BCR\]. \( \square \)

Therefore, from the above lemma we deduce the following:

**Theorem 15.** There is a strong deformation retraction from \( X_{m,n}/G \) to \( Y_{m,n}/K \).

**Proof.** The proof is analogous to that of Theorem 10. We have a commutative diagram of \((\prod_{i=1}^m K_i) \times K\)-equivariant inclusions

\[
K^m \times K^n \xrightarrow{\varphi} G^m \times G^n \xrightarrow{\beta} G^m \times G^n \xrightarrow{\alpha} KN_{G^m \times G^n} \xrightarrow{\varphi} G^m \times G^n.
\]

The bottom map corresponds to a \((\prod_{i=1}^m K_i) \times K\)-equivariant strong deformation retraction by Theorem 5, the top map corresponds to a \((\prod_{i=1}^m K_i) \times K\)-equivariant strong deformation retraction by Lemma 9 and Lemma 14 gives the inclusion of complexes \( \varphi \) that descends to a map on quotients that are themselves complexes \[Hi, BCR, Sc\].

Then, by Proposition 6 and Remark 7 one obtains a strong deformation retraction from

\[
X_{m,n}/G \cong (G^m \times G^n)/(\prod_{i=1}^m G_i \times G) \cong KN_{G^m \times G^n}/(K \times \prod_{i=1}^m K_i)
\]
to the compact quotient
\[(K^m \times K^n)/\left(\prod_{i=1}^{m} K_i \times K\right) \cong \mathcal{Y}_{m,n}/K.\]

Finally, we apply Theorem 15 to the parabolic character varieties. Fix an element
\[h := (h_1, \ldots, h_m) \in K^m.\]

For \(1 \leq j \leq m\), let
\[C^K_j := K \cdot h_j = \{kh_jk^{-1} \mid k \in K\}\]
be the conjugation orbits in \(K\) and \(G\) respectively. Let \(n \geq 0\) be an integer. Define
\[H^G_{h,n} := \{(g_1, \ldots, g_{m+n}) \in G^{m+n} \mid g_j \in C^G_j, j = 1, \ldots, m\} = C^G_1 \times \cdots \times C^G_m \times G^n.\]
Consider the diagonal action of \(G\) on \(G^{m+n}\) constructed using the adjoint action of \(G\) on itself. This action clearly preserves the subset \(H^G_{h,n} \subset G^{m+n}\). The restriction of this action to the subgroup \(K \subset G\) preserves the subset
\[H^K_{h,n} := \{(g_1, \ldots, g_{m+n}) \in K^{m+n} \mid g_j \in C^K_j, j = 1, \ldots, m\} = C^K_1 \times \cdots \times C^K_m \times K^n\]
of \(H^G_{h,n}\).

We have the affine algebraic quotient
\[X^G_{h,n} := H^G_{h,n}/G\]
and the compact quotient
\[X^K_{h,n} := H^K_{h,n}/K,\]
which are called the parabolic character varieties.

**Corollary 16.** Take integers \(m \geq 0\) and \(n \geq 0\). For any \(h \in K^m\), there is a strong deformation retraction of \(X^G_{h,n}\) onto \(X^K_{h,n}\).

**Proof.** If \(m = 0\) the result follows from \([FL1]\). Now assume \(m \geq 1\). Let \(h := (h_1, \ldots, h_m) \in K^m\). The centralizer of \(h_j\) in \(G\) (respectively, \(K\)) will be denoted by \(G_j\) (respectively, \(K_j\)); so \(G_j\) is the Zariski closure of \(K_j\) in \(G\), and \(\text{Lie}(G_j) = \text{Lie}(K_j) \oplus \sqrt{-1} \cdot \text{Lie}(K_j)\). As noted before in Remark \([1]\) \(G/G_j\) (respectively, \(K/K_j\)) is identified with the orbit \(G \cdot h_j\) (respectively, \(K \cdot h_j\)) for the adjoint action. These give identifications
\[H^G_{h,n} \cong \mathcal{X}_{m,n}\]
and
\[H^K_{h,n} \cong \mathcal{Y}_{m,n},\]
where the space \(\mathcal{Y}_{m,n}\) (respectively, \(\mathcal{X}_{m,n}\)) is constructed as in \((3.2)\) (respectively, \((3.1)\)) using the subgroups \(K_1, \ldots, K_m\) (respectively, \(G_1, \ldots, G_m\)). Under these identifications, the diagonal conjugation action of \(K\) on \(H^K_{h,n}\) becomes the diagonal action for the left–translation action on the homogeneous spaces \(K/K_j\) and the conjugation action on the factors \(K\). A similar statement holds for the actions of \(G\) on \(\mathcal{X}_{m,n}\) and \(H^G_{h,n}\). Therefore, the strong deformation retraction of \(X^G_{h,n} := H^G_{h,n}/G\) to \(X^K_{h,n} := H^K_{h,n}/K\) is a direct consequence of Theorem 15. \(\square\)
4. The generic case of parabolic character varieties

In this section we discuss some examples, and the generic case, that is the case when \( h = (x_1, x_2, \cdots, x_m) \in K^m \) has a regular component.

4.1. Elementary examples. First we consider the case where \( m = 0 \). In this case, as there is no parabolic data (no conjugation classes), we get back the deformation retraction from the \( G \)-character variety to the \( K \)-character variety of the free group of rank \( n \):

\[
X^K_n = \text{Hom}(F_n, K)/K \hookrightarrow X^G_n = \text{Hom}(F_n, G)\!/G,
\]

that was obtained in [FL1]. Note in particular, for \( n = 1 \), that we have the inclusion

\[
K/K \hookrightarrow G/G.
\]

For a rather trivial example, let us consider \( m = 1 \) and \( n = 0 \). In this case the vector \( h \in K^m \) is given by a single element \( x \in K \). Then, the parabolic representation spaces are single orbits

\[
H^K_{x,0} = K \cdot x
\]

and

\[
H^G_{x,0} = G \cdot x.
\]

Since the \( K \)-action (respectively, the \( G \)-action) is transitive on these orbits, the parabolic character varieties consist of a single point:

\[
X^G_{x,0} = H^G_{x,0}/G = H^K_{x,0}/K = X^K_{x,0}.
\]

4.2. The case \( m = 1 \). Let us consider arbitrary \( n \), a compact group \( K \), and again \( m = 1 \), so we are dealing with only one conjugacy class. Since every element \( x \in K \) lies in a maximal torus of \( K \), and all maximal tori of \( K \) are conjugate, it is no loss of generality to assume that \( x \in T \), for some fixed maximal torus \( T \subset K \).

Let us now consider the “generic” situation: assume that \( x \in T \) is regular, that is \( x \) is not fixed by any non-trivial element of the Weyl group. Such regular elements form a dense set inside \( T \), and moreover it is known that, for regular \( x \), the centralizer of \( x \) in \( K \) is just the torus: \( K_x = T \).

In this situation, we have:

\[
H^K_{x,n} = (K \cdot x) \times K^n
\]

and the parabolic character variety can be described as follows. Consider the action of \( T \subset K \) on \( K^n \) by simultaneous conjugation and the corresponding quotient space \( K^n/T \).

**Proposition 17.** When \( x \in T \subset K \) is regular, there is a natural isomorphism of quotient spaces

\[
X^K_{x,n} = H^K_{x,n}/K \sim K^n/T.
\]

**Proof.** We will describe the isomorphism explicitly. Consider the map:

\[
\eta : K^n/T \longrightarrow H^K_{x,n}/K \quad [(k_1, \cdots, k_n)] \longmapsto [(x, k_1, \cdots, k_n)].
\]

The map \( \eta \) is well defined, since if \( (k_1, \cdots, k_n) \) and \( (h_1, \cdots, h_n) \) represent the same class in \( K^n/T \), then there is \( t \in T \) such that \( tk_jt^{-1} = h_j \) for all \( 1 \leq j \leq n \) and because \( x = txt^{-1} \), the elements \( (x, h_1, \cdots, h_n) \) and \( (x, k_1, \cdots, k_n) \) in \( H^K_{x,n} \) represents the same
class. The map $\eta$ is surjective: if $(y, h_1, \cdots, h_n) \in H^K_{x,n} = (K \cdot x) \times K^n$ is arbitrary, since $y \in K \cdot x$, there exists $k \in K$ such that $x = kyk^{-1}$, so
$$\eta(kh_1k^{-1}, \cdots, kh_nk^{-1}) = [(x, kh_1k^{-1}, \cdots, kh_nk^{-1})] = [(y, h_1, \cdots, h_n)].$$
Finally, $\eta$ is also injective, because if $[(x, k_1, \cdots, k_n)] = [(x, h_1, \cdots, h_n)]$, then there is an element $k \in K$ such that $kxk^{-1} = x$ and $kk_jk^{-1} = h_j$ for all $1 \leq j \leq n$. But since $K_x = T$ we conclude that $k \in T$ and so $[(k_1, \cdots, k_n)] = [(h_1, \cdots, h_n)]$. \hfill $\square$

The same argument allows us to show that we have a natural isomorphism
$$X^G_{x,n} = H^G_{x,n}/G \overset{\sim}{\longrightarrow} G^m/T$$
whenever we have $G_x = T$ and $x \in T \subset T \subset G$, for the maximal torus $T$ of $G$ which is the complexification of $T$.

4.3. The generic case when $m > 1$. The case when $m > 1$ is not so easy to describe in general, but the “generic” case can also be written explicitly as a quotient by an abelian group of the $m - 1$ parabolic character variety. In fact, we have:

Theorem 18. Let $x_1 \in T \subset K$ be a regular element. Let $h = (x_1, x_2, \cdots, x_m) \in K^m$ and $h' = (x_2, \cdots, x_m) \in K^{m-1}$. Then, there are natural isomorphisms of quotient spaces
$$X^K_{h,n} = H^K_{h,n}/K \overset{\sim}{\longrightarrow} H^K_{h',n}/T$$
and
$$X^G_{h,n} = H^G_{h,n}/G \overset{\sim}{\longrightarrow} H^G_{h',n}/T.$$  

Proof. As in the proof of Proposition 17 we can construct explicit isomorphisms:
$$\eta: H^K_{h',n}/T \longrightarrow H^K_{h,n}/K$$
$$[(x_2, \cdots, x_n, k_1, \cdots, k_n)] \longmapsto [(x_1, x_2, \cdots, x_n, k_1, \cdots, k_n)],$$
and an analogous map for $X^G_{h,n} = H^G_{h,n}/G$. The proof that $\eta$ is well defined and bijective is the same as before. \hfill $\square$

Remark 19. The above theorem remains true if any one of $x_i$ in $(x_1, \cdots, x_m)$ is regular.

5. Relative versus Parabolic Character Varieties

Let $b \geq 1$ be an integer and $\Sigma$ be a genus $g$ surface with $b$ points removed. Then there is a presentation of the fundamental group of $\Sigma$ as

$$\pi_1(\Sigma) \cong \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_p \mid \prod_{i=1}^g [\alpha_i, \beta_i] \prod_{j=1}^b \gamma_j = 1 \rangle,$$
where each $\gamma_j$ corresponds to a loop around a puncture. Note that $\pi_1(\Sigma)$ is a free group of rank $2g + b - 1$. As discussed in [BG] and [La2], there is a boundary map
$$\partial: \text{Hom}(\pi_1(\Sigma), G)/G \to (G/G)^b$$
given by
$$\partial([\rho]) = ([\rho(\gamma_1)], \cdots, [\rho(\gamma_p)]),$$
where $[\rho]$ denotes the equivalence class of $\rho \in \text{Hom}(\pi_1(\Sigma), G)$. 

In general, $G//G$ is an irreducible affine variety isomorphic to $T//W$ where $T$ is a maximal torus and $W$ is the Weyl group (see [[St]]. For a given $b$ in the image of $\partial$, the affine variety $X^b_{rel} := \partial^{-1}(b)$ is called a relative character variety.

Relative character varieties are in one-to-one correspondence with framed $G$-local systems over $\Sigma$ as described by Fock and Goncharov in [FG] Section 2, page 39, which in turn are studied in recent work by Biquard, García-Prada and Mundet i Riera in relation with the moduli spaces of parabolic $G$-Higgs bundles [BGM]. This latter work aims at generalizing Simpson’s correspondence [[Si]] between $GL(\ell, \mathbb{C})$-local systems on $\Sigma$ with fixed monodromy in $U(\ell)$ around the punctures and parabolic Higgs bundles, to any reductive Lie group $G$ and any fixed monodromy in $G$. (This was done earlier for classical groups in [[BS]], and it was done for finite order monodromies around punctures in [[Bi]].)

The parabolic character varieties we consider in this paper can also be obtained in a similar way. Indeed, let $F_{m+n}$ be the free group with generators $e_1, \ldots, e_{m+n}$ as before, and consider the map
\[
\partial_{par} : \text{Hom}(F_{m+n}, G)//G \cong G^{m+n}//G \to (G//G)^m
\]
given by
\[
\partial_{par}([\rho]) = ([\rho(e_1)], \ldots, [\rho(e_m)]).
\]
Let $q : G^m \to (G//G)^m$ be the canonical quotient map. For a given $h = (h_1, \ldots, h_m) \in G^m$, the parabolic character variety $X^G_{h,n} \subset \text{Hom}(F_{m+n}, G)//G$ is obtained by sending the first $m$ generators $e_1, \ldots, e_m$ to the conjugacy classes of $h_1, \ldots, h_m$. So, we have the identification
\[
X^G_{h,n} = \partial_{par}^{-1}(q(h)).
\]
Note that we required the parabolic data $h \in K^m$ for the deformation retraction in Theorem 1 to work. However, we can consider here the more general situation $h \in G^m$. Let $Z(G)$ be the center of $G$.

**Proposition 20.** For $n + m \geq 2$, the parabolic character variety $X^G_{h,n}$ is an affine variety such that
\[
\dim X^G_{h,n} \geq (n + m - 1) \dim G + \dim Z(G) - m \rank G
\]
with equality holding for almost every choice of $h$.

**Proof.** Since $n + m \geq 2$, and only $Z(G)$ acts trivially on $G^{n+m}$, we conclude $d_1 := \dim G^{n+m}//G = (n + m - 1) \dim G + \dim Z(G)$. Since $G//G \cong T//W$, we have $d_2 := \dim (G//G)^m = m \rank G$. The map $\partial_{par}$ is always surjective, hence $\dim X^G_{h,n} = \partial_{par}^{-1}(q(h))) \geq d_1 - d_2$ for all points $q(h) \in (G//G)^m$ and with equality holding on an open dense subset $U$ of $(G//G)^m$ [[St]. Section 6, Theorem 7]. Then equality holds for almost all $h$ since the projection $q : G^m \to (G//G)^m$ is surjective and continuous, and thus $q^{-1}(U)$ is also (Zariski) open and non-empty, and hence dense. □

We now relate the relative character varieties to the parabolic character varieties. Note, however, that relative character varieties depend on the homeomorphism class of the underlying surface. On the other hand, character varieties and parabolic character varieties only depend on the free group $F_{m+n}$. Because of this, we need a concrete identification of the surface group $\pi_1(\Sigma)$ with a free group, that works for any genus $g$ surface $\Sigma$ with $b$ points removed.
Using the presentation of $\pi_1(\Sigma)$ as in (5.1), and supposing we have $b > m \geq 0$, $n, g \geq 0$ and $m + n = 2g + b - 1$, define the following isomorphism of free groups:

$$
\varphi : F_{m+n} \rightarrow \pi_1(\Sigma)
$$

given by

$$
\varphi(e_i) := \begin{cases} 
\gamma_i, & i = 1, \ldots, b - 1 \\
\alpha_{1+i-b}, & i = b, \ldots, g + b - 1 \\
\beta_{1+i-b-g}, & i = b + g, \ldots, 2g + b - 1.
\end{cases}
$$

One easily checks that $\varphi$ is an isomorphism because

$$
\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{b-1}\}
$$
is a sequence of free generators of $\pi_1(\Sigma)$, and $\gamma_b$ is expressed as

$$
\gamma_b = \left(\prod_{i=1}^{g}[\alpha_i, \beta_i]^{b-1} \prod_{j=1}^{b} \gamma_j \right)^{-1}.
$$

Suppose that $\rho \in \text{Hom}(\pi_1(\Sigma), G)$. By composing with $\varphi$, we obtain a representation $\rho \circ \varphi \in \text{Hom}(F_{m+n}, G)$, and this defines an isomorphism

$$
\varphi^* : \text{Hom}(\pi_1(\Sigma), G)/G \rightarrow \text{Hom}(F_{m+n}, G)/G
$$
since this composition is equivariant with respect to the conjugation action on the representation spaces.

**Proposition 21.** Let $b > m \geq 0$. There is a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}(\pi_1(\Sigma), G)/G & \overset{\partial_{par}}{\longrightarrow} & (G/G)^n \\
\varphi^* \downarrow & & \downarrow \pi \\
\text{Hom}(F_{m+n}, G)/G & \overset{\partial_{par}}{\longrightarrow} & (G/G)^m
\end{array}
$$

where $\pi$ is the projection onto the first $m$ factors.

**Proof.** Let $\rho \in \text{Hom}(\pi_1(\Sigma), G)$. The boundary of $\Sigma$ corresponds to the loops $\gamma_1, \ldots, \gamma_b$. Since $b > m$, commutativity follows from the computations

$$
\pi(\partial([\rho])) = \pi([\rho(\gamma_1)], \ldots, [\rho(\gamma_b)]) = ([\rho(\gamma_1)], \ldots, [\rho(\gamma_m)])
$$

and

$$
\partial_{par}([\varphi^*(\rho)]) = ([\varphi^*(\rho)(e_1)], \ldots, [\varphi^*(\rho)(e_m)]) = ([\rho(\gamma_1)], \ldots, [\rho(\gamma_m)]).
$$

This completes the proof. \(\square\)

**Corollary 22.** Let $h = (h_1, \ldots, h_m) \in G^m$ and let $b \in \pi^{-1}(q(h))$. Then, with the notation as above, $X_{rel}^b \subset X_{h,n}^G$ as affine sub-varieties. In fact,

$$
X_{h,n}^G = \bigcup_{b \in \pi^{-1}(q(h))} X_{rel}^b.
$$

**Proof.** From the definitions, and from commutativity of the diagram one can write

$$
X_{h,n}^G = \partial_{par}^{-1}(h) = \partial^{-1}(\pi^{-1}(q(h))) = \bigcup_{b \in \pi^{-1}(q(h))} \partial^{-1}(b) = \bigcup_{b \in \pi^{-1}(q(h))} X_{rel}^b,
$$
completing the proof. \(\square\)
5.1. Examples. In the following examples we consider $g = 0$. Thus $m + n = b - 1$, and $\Sigma$ is a sphere with $b = m + n + 1$ punctures. In particular, our restriction $b > m$ is automatically satisfied. Now, the identification of $F_{m+n}$ with $\pi_1(\Sigma)$ under equation (5.4) becomes simply $\varphi(e_i) := \gamma_i, i = 1, \ldots, m + n$, and such that the last loop verifies $\gamma_b = (\gamma_1 \cdots \gamma_{b-1})^{-1}$. We will also restrict to the case $n = 0$, so that $m = b - 1$ becomes the rank of these free groups.

Under the isomorphisms $\text{Hom}(F_m, G) \bowtie G \cong G^m \bowtie G \cong \text{Hom}(\pi_1(\Sigma), G) \bowtie G$, we can then write the diagram of Proposition 21 in the following way:

$$
\begin{array}{c}
G^m \bowtie G \\
\downarrow \partial_{\text{par}} \\
(G/\!\!/G)^{m+1} \\
\downarrow \pi \\
(G/\!\!/G)^m
\end{array}
$$

where $\pi$ is the projection onto the first $m$ factors, and according to (5.2) and (5.3), we can write for a given $(g_1, \ldots, g_m) \in G^m$

\[ \partial_{\text{par}}([g_1, \ldots, g_m]) = ([g_1], \ldots, [g_m]), \]

and

\[ \partial([g_1, \ldots, g_m]) = ([g_1], \ldots, [g_m], [(g_1 \cdots g_m)^{-1}]). \]

We concentrate on the lower rank simple groups $G = \text{SL}(\ell, \mathbb{C})$, as these allow for explicit descriptions. For $G = \text{SL}(\ell, \mathbb{C})$, a generating set for the the coordinate ring $\mathbb{C}[G/\!\!/G] = \mathbb{C}[G]^G$ is given by the coefficients of the characteristic polynomial of a generic $X \in \text{SL}(\ell, \mathbb{C})$. In particular, we have $G/\!\!/G \cong \mathbb{C}^{\ell-1}$.

As the $m = 1$ cases were described in Section 4.2 we consider now $m > 1$.

Example 23. Let $G = \text{SL}(2, \mathbb{C})$. Then $G/\!\!/G$ can be naturally identified with $\mathbb{C}$ by $[g] \mapsto \text{tr}(g)$, for $[g] \in G/\!\!/G$. Now let $m = 2$. Then

$$
\text{Hom}(F_2, G) \bowtie G = \text{SL}(2, \mathbb{C})^2 \bowtie \text{SL}(2, \mathbb{C}) \cong \mathbb{C}^3
$$

by the Fricke-Klein-Vogt Theorem [FK, Vö] and the boundary map is an isomorphism from $G^2 \bowtie G$ to $\mathbb{C}^3$ (see [BG, Theorem 2.1.1]), which is explicitly given by

\[ \partial([g_1, g_2]) = (\text{tr}(g_1), \text{tr}(g_2), \text{tr}(g_1g_2)), \]

since $\text{tr}(g_1g_2)^{-1} = \text{tr}(g_1g_2)$ in $\text{SL}(2, \mathbb{C})$. Then the diagram for this example is

$$
\begin{array}{c}
G^2 \bowtie G \cong \mathbb{C}^3 \\
\downarrow \partial_{\text{par}} \\
(G/\!\!/G)^3 \cong \mathbb{C}^3 \\
\downarrow \pi \\
(G/\!\!/G)^2 \cong \mathbb{C}^2
\end{array}
$$

where $\partial_{\text{par}}([g_1, g_2]) = (\text{tr}(g_1), \text{tr}(g_2))$. Thus, taking $K = \text{SU}(2)$ and $h = (h_1, h_2) \in \text{SU}(2)^2$ we have,

$$
X_{h,0}^G = \partial_{\text{par}}^{-1}(\text{tr}(h_1), \text{tr}(h_2)) = \partial^{-1}(\pi^{-1}(\text{tr}(h_1), \text{tr}(h_2))) = \partial^{-1}(\mathbb{C}) \cong \mathbb{C},
$$

where we used the isomorphism $\partial : G^2 \bowtie G \overset{\sim}{\longrightarrow} \mathbb{C}^3$ and the obvious fact that $\pi^{-1}(z, w) \cong \mathbb{C}$ for any $z, w \in \mathbb{C}$.
Example 24. Let again $G = \text{SL}(2, \mathbb{C})$, but set $m = 3$. In this case, $\text{Hom}(F_3, G)/G$ can be identified with a cubic hyper-surface in $\mathbb{C}^7$ (a branched double cover over $\mathbb{C}^6$) in the following way (see [BG], Section 3.B for details).

First identify $\rho \in \text{Hom}(F_3, G)$ with $(\rho(e_1), \rho(e_2), \rho(e_3)) = (g_1, g_2, g_3) \in G^3$ and define $a = \text{tr}(g_1), b = \text{tr}(g_2), c = \text{tr}(g_3), d = \text{tr}(g_1 g_2 g_3), x = \text{tr}(g_1 g_2), y = \text{tr}(g_2 g_3)$ and $z = \text{tr}(g_3 g_1)$. With these seven traces as coordinates of $\mathbb{C}^7$, the quotient $G^3/G = \text{Hom}(F_3, G)/G$ becomes the subvariety of $\mathbb{C}^7$, where the following cubic polynomial vanishes

$$x^2 + y^2 + z^2 + xyz - (ab + cd)x - (ad + bc)y - (ac + bd)z - 4 + a^2 + b^2 + c^2 + d^2 - abcd.$$ 

Denote this polynomial by $p(a, b, c, d, x, y, z)$. Again, identifying $G/G$ with $\mathbb{C}$ using the trace, we have the commutative diagram

$$\mathbb{C}^7 \supset G^3/G \xrightarrow{\partial} (G/G) \cong \mathbb{C}^4 \xleftarrow{\partial_{\text{par}}} (G/G)^3 \cong \mathbb{C}^3$$

As in the previous example, $\partial_{\text{par}}(g_1, g_2, g_3) = (a, b, c)$, but now $\partial(g_1, g_2, g_3) = (a, b, c, d)$ since in $F_3$ we have $e_0 = (e_1 e_2 e_3)^{-1}$, and $\text{tr}(g_1 g_2 g_3)^{-1} = \text{tr}(g_1 g_2 g_3)$ in $\text{SL}(2, \mathbb{C})$. The relative character variety results from $\text{tr}$ fixing $b = (a, b, c, d) \in \mathbb{C}^4$ (corresponding to the four boundaries), so

$$X_{\text{rel}}^b = \{(x, y, z) \in \mathbb{C}^3 : p(a, b, c, d, x, y, z) = 0\},$$

which is manifestly a cubic surface in $\mathbb{C}^3$. Similarly, the parabolic character variety results from fixing conjugacy classes in only the first three boundaries, so, with $h = (h_1, h_2, h_3) \in K^3$, where $K = \text{SU}(2)$ and $a = \text{tr}(h_1), b = \text{tr}(h_2)$ and $c = \text{tr}(h_3)$, we obtain

$$X_{h,0}^G = \{(x, y, z, d) \in \mathbb{C}^4 : p(a, b, c, d, x, y, z) = 0\}$$

which is a cubic 3-fold in $\mathbb{C}^4$, defined over $\mathbb{R}$ (since $a, b, c \in [-2, 2]$ are traces of $\text{SU}(2)$ matrices).

Example 25. In this example, let $G = \text{SL}(3, \mathbb{C})$, and $m = 2$. As shown in [La1], $\text{SL}(3, \mathbb{C})^2//\text{SL}(3, \mathbb{C})$ is isomorphic to an degree 6 hyper-surface in $\mathbb{C}^9$ which maps onto $\mathbb{C}^8$ generically two-to-one. The relative character varieties in this case (for both the three-holed sphere and the one-holed torus) were studied in [La3]. The diagram in this case is this:

$$\mathbb{C}^9 \supset \text{SL}(3, \mathbb{C})^2//\text{SL}(3, \mathbb{C}) \xrightarrow{\partial} (\text{SL}(3, \mathbb{C})//\text{SL}(3, \mathbb{C}))^3 \cong \mathbb{C}^6 \xleftarrow{\partial_{\text{par}}} (\text{SL}(3, \mathbb{C})//\text{SL}(3, \mathbb{C}))^2 \cong \mathbb{C}^4$$

The isomorphism $\text{SL}(3, \mathbb{C})//\text{SL}(3, \mathbb{C}) \cong \mathbb{C}^2$ is given by $[g] \mapsto (\text{tr}g_1, \text{tr}g_1^{-1})$, and the map that embeds $\text{SL}(3, \mathbb{C})^2//\text{SL}(3, \mathbb{C})$ in $\mathbb{C}^9$ is given by

$$\{(g_1, g_2) \mapsto (t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9),$$

where $t_1 = \text{tr}(g_1), t_2 = \text{tr}(g_1^{-1}), t_3 = \text{tr}(g_2), t_4 = \text{tr}(g_2^{-1}), t_5 = \text{tr}(g_1 g_2), t_6 = \text{tr}((g_1 g_2)^{-1}), t_7 = \text{tr}(g_1^{-1} g_2), t_8 = \text{tr}(g_1 g_2^{-1}), t_9 = \text{tr}(g_1 g_2 g_1^{-1} g_2^{-1})$. 

As with Example 23, \( e_3 \) is homotopic to \((e_1e_2)^{-1}\), but unlike in Example 23 the trace is not invariant under inversion, although this does not pose a problem since in this case we need both the boundary loop and its inverse for the boundary map.

Fix the six invariants \( t_1, t_2, t_3, t_4, t_5, t_6 \) corresponding to the three boundaries; call those fixed values \( a_1, a_2, a_3, a_4, a_5, a_6 \) respectively.

We have

\[
\partial^{-1}(a_1, \ldots, a_6) = \{ (a_1, \ldots, a_6, t_7, t_8, t_9) \in \mathbb{C}^9 \mid p(a_1, \ldots, a_6, t_7, t_8, t_9) = 0 \}
\]

for \( p \) a polynomial of degree 6 in \( \mathbb{C}^9 \). Note that after fixing the 6 coordinates \( p \) has degree 3 in the other 3 variables. Thus, \( \partial^{-1}(a_1, \ldots, a_6) \) is isomorphic to a cubic hyper-surface in \( \mathbb{C}^3 \); as shown in [La2] its smooth points form a complex symplectic surface. It likewise follows that

\[
\partial_{\text{par}}^{-1}(a_1, a_2, a_3, a_4) = \{ (a_1, \ldots, a_4, t_5, t_6, t_7, t_8, t_9) \in \mathbb{C}^9 \mid p(a_1, \ldots, a_4, t_5, \ldots, t_9) = 0 \},
\]

is isomorphic to a quartic hyper-surface in \( \mathbb{C}^5 \) since the degree of \( p \) in the last 5 variables is 4.

Acknowledgments.

The first, third, and fourth authors thank Instituto Superior Técnico for hospitality; their visit to IST was funded by the FCT project PTDC/MAT/099275/2008. The second author is partially supported by FCT project PTDC/MAT/120411/2010. The third author additionally wishes to thank the GEAR network for funding his visit to the Institut Henri Poincaré. The fourth author is grateful to Michel Brion for fruitful discussions. Finally, we also thank Oscar García-Prada for communicating to us the work in [BGM]. Lastly, we thank the referee for useful suggestions which led us to a clarification of some of proofs.

References

[BG] R. L. Benedetto, W. M. Goldman: The topology of the relative character variety of a quadruply-punctured sphere, Experiment. Math. 8 (1999), 85–103.

[BGM] O. Biquard, O. García-Prada, I. Mundet i Riera: Parabolic Higgs bundles and representations of the fundamental group of a punctured surface, (in preparation).

[BF] I. Biswas and C. Florentino: The topology of moduli spaces of group representations: the case of compact surface, Bull. Sci. Math. 135 (2011), 395–399.

[Bi] I. Biswas: Parabolic principal Higgs bundles, Jour. Ramanujan Math. Soc. 23 (2008), 311–325.

[BS] I. Biswas and M. Stemmler: Hermitian-Einstein connections on polystable orthogonal and symplectic parabolic Higgs bundles, Jour. Geom. Phy. 62 (2012), 1358–1365.

[BCR] Bochnak, Jacek and Coste, Michel and Roy, Marie-Françoise: Real algebraic geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 36, Springer-Verlag, Berlin, 1998, x+430.

[CG] N. Chriss and V. Ginzburg: Representation theory and complex geometry, Birkhäuser, Boston, 1997.

[Do] I. Dolgachev: Lectures on invariant theory, London Mathematical Society Lecture Note Series, Vol. 296, Cambridge University Press, Cambridge, 2003.

[FL1] C. Florentino and S. Lawton: The topology of moduli spaces of free group representations, Math. Ann. 345 (2009), 453–489.

[FL2] C. Florentino and S. Lawton: Character Varieties and the moduli of quiver representations, In the tradition of Ahlfors-Bers, Papers from the 5th Ahlfors-Bers Colloquium held at Rice University, Contemporary Mathematics, American Mathematical Society (to appear).
I. Biswas, C. Florentino, S. Lawton, and M. Logares

[FG] V. Fock and A. Goncharov: Moduli space of local systems and higher Teichmüller theory, Publ. Math. Inst. Hautes Études Sci. 103 (2006), 1–211.

[FK] R. Fricke and F. Klein: Über die theorie der automorphen modulgruppen, Kgl. Ges. d. W. Nachrichten, Math-Phys. Klasse (1896), 91–93.

[Ha] A. Hatcher: Algebraic topology, Cambridge University Press, Cambridge, 2002.

[Hi] Hironaka, Heisuke: Triangulations of algebraic sets Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), p. 165-185. Amer. Math. Soc., Providence, RI, 1975.

[Hu1] J. E. Humphreys: Conjugacy classes in semisimple algebraic groups, Math. Surveys and Monographs, Vol. 43, American Mathematical Society, Providence, RI, 1995.

[La1] S. Lawton: Generators, relations and symmetries in pairs of $3 \times 3$ unimodular matrices, Jour. Alg. 313 (2007), 782–801.

[La2] S. Lawton: Poisson geometry of $\text{SL}(3, \mathbb{C})$-character varieties relative to a surface with boundary, Trans. Amer. Math. Soc. 361 (2009), 2397–2429.

[La3] S. Lawton: Obtaining the one-holed torus from pants: duality in an $\text{SL}(3, \mathbb{C})$-character variety, Pacific Jour. Math. 242 (2009), 131–142.

[Ne] A. Neeman: The topology of quotient varieties, Ann. of Math. 122 (1985), 419–459.

[Re] R. Ree: Commutators in semi-simple algebraic groups, Proc. Amer. Math. Soc. 15 (1964), 457–460.

[Sc] G. W. Schwarz: The topology of algebraic quotients, Topological methods in algebraic transformation groups, (New Brunswick, NJ, 1988), 135–151, Progr. Math., Vol. 80, Birkhauser, Boston, 1989.

[Sh] I. R. Shafarevich: Basic Algebraic Geometry 1, Springer-Verlag, Berlin Heiderberg, 1994.

[S1] C. T. Simpson: Harmonic bundles on noncompact curves, Jour. Amer. Math. Soc. 15 (1990), 713–770.

[Si] C. T. Simpson: Products of matrices, Differential Geometry, Global Analysis, and Topology, Proc. Canad. Math. Soc. Conference, 12, Amer. Math. Soc., Providence (1992), 157–185.

[Sp] E. H. Spanier: Algebraic Topology, Springer-Verlag, New York, 1966.

[St] R. Steinberg: Regular elements of semisimple algebraic groups, Publ. Math. Inst. Hautes Études Sci. 25 (1965), 49–80.

[Vo] M. Vogt: Sur les invariants fondamentaux des equations differentielles lineaires du second ordre, Ann. Sci. Ecol. Norm. Supér. Troi. 6 (1889).

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Departamento Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

E-mail address: cfloren@math.ist.utl.pt

Department of Mathematics, The University of Texas-Pan American, 1201 West University Drive Edinburg, TX 78539, USA

E-mail address: lawtonsd@utpa.edu

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Nicolás Cabrera 15, Campus Cantoblanco UAM, 28049 Madrid, Spain

E-mail address: marina.logares@icmat.es