PSEUDO SYMPLECTIC GEOMETRY AS AN EXTENSION OF THE SYMPLECTIC GEOMETRY

NIK. TYURIN

Abstract. In [9] we've outlined how the Taubes result about nontriviality of the Seiberg — Witten invariants for symplectic manifolds can be generalized to a more general case, when a priori based manifold doesn't admit a symplectic structure. In the present paper we correct our result from [9] and discuss some properties which belong to pseudo symplectic 4-manifolds in the framework of the complex geometry.

§0. Introduction

In 1994 new invariants of smooth structures in 4-dimensional geometry were introduced by E. Witten in his seminal work [10]. These invariants are very close to the Donaldson invariants (and a relationship was established in [5]). But so as it's much more easy to compute the Seiberg — Witten invariants a number of unsolved problems have been solved in the framework of the Seiberg — Witten theory. Two first results proved in [10] have been proved earlier in the framework of the Donaldson theory, namely

1. that for a connected sum \( Y = X_1 \# X_2 \) where \( b_2^+(X_i) > 0 \) the invariants vanish (as well as for the Donaldson invariants);
2. and that for a Kahler surface \( S \) the Seiberg — Witten invariant for the canonical class \( K_S \) equals to \( \pm 1 \),
   (see [10]).

Based on this facts Witten reproved that a priori Kahler surface \( S \) isn’t diffeomorphic to its topomodel.

As in the Donaldson theory the invariants are defined if \( b_2^+ \) of based manifold bigger then 1 (for the Kahler case it just corresponds to \( p_g(S) > 0 \)).

But really the Seiberg — Witten invariants use to be slightly more usefull then the Donaldson one. We have in mind the following important result due to Taubes

**Theorem (Taubes, [6]).** Let \( X, \omega \) be symplectic manifold with \( b_2^+(X) > 1 \). Then the Seiberg — Witten invariant of the canonical class \( K_\omega \in H^2(X, \mathbb{Z}) \), associated to given symplectic form, equals to \( \pm 1 \).

In other words, for symplectic manifold the canonical class is a basic class.
But the inverse implication isn’t true: in [3] the counterexample was found. The construction is as follows. One can start with a symplectic manifold $X$ with $b_2^+(X) > 1$ and consider the connected sum of the following form

$$Y = X \# N,$$

where $b_1(N) = b_2^+(N) = 0$ but $\pi_1(N) \neq \{1\}$. Then $Y$ doesn’t admit symplectic structure but has nontrivial invariant (see [3]).

The idea to find a criterion for nontriviality of the invariants was quite reasonable, and the next step in this direction was done in [9]. In this paper we’ve extended the Taubes technique and modulo some remarks, represented in the present paper, the following result was established

**Theorem.** Let $X$ be pseudo symplectic manifold with $b_2^+(X) > 1$, so there exists a hermitian triple $(g, J, \omega)$ on $X$ with nontrivial image in $H^2(X, \mathbb{R})$ by the canonical map $\tau$. Then the invariant for $K_J \in H^2(X, \mathbb{Z})$ (which is, of course, the canonical class of $J$) is nontrivial.

Let us recall the definitions of pseudo symplectic manifolds and the canonical map $\tau$. For a smooth 4-manifold $X$ one can consider the space of all hermitian triples

$$\mathcal{M}_X = \{(g, J, \omega)\},$$

where the first element is a riemannian metric on $X$, the second is an almost complex structure, compatible with the first element, and the third is the corresponding 2-form (which is called an almost Kahler form) such that

$$\omega(u, v) = g(u, Jv).$$

It’s well known that this form $\omega$ is

1. self dual with respect to the conformal class $*_g$ and the orientation, chosen by $J$;
2. nondegenerated everywhere;
3. has the type $(1,1)$ with respect to $J$.

We have to emphrise that this $\omega$ hasn’t to be closed a priori. Because of this it is just *almost Kahler*.

The construction is pure local and it’s well known from the linear algebra that in a hermitian triple every element can be reconstructed from two others.

Further, there exists the canonical map

$$\tau : \mathcal{M}_X \to K^+ \subset H^2(X, \mathbb{R}),$$

where $K^+$ is the inner part of isotropic cone (so consists of 2-cohomology classes with positive squares), defined as follows. For a triple $(g, J, \omega)$ let us take the corresponding Hodge star operator

$$*_g : \Omega^i_X \to \Omega^{4-i}_X,$$
over $X$ defined by the conformal class of our riemannian metric $g$ and the orientation given by $J$. Then by the famous Hodge theorem ([2]) every form can be decomposed into three parts — harmonic part, exact and co-exact

$$\omega = \omega_H + d\rho_1 + d^*\rho_2,$$

(0.1)

where $\rho_1 \in \Omega^1_X, \rho_2 \in \Omega^3_X$.

Let us recall that $d^*$- operator on $\Omega^2$ is equal to the following double combination

$$d^* : \Omega^2 \rightarrow \Omega^1 \quad | \quad d^* = \ast_g d\ast_g$$

of the ordinary $d$ and two Hodge operators (in different dimensions).

But in our case $\omega$ is self dual, so one can apply $\ast_g$ to both sides of (0.1) and get

$$\ast_g \omega = \omega = \ast_g \omega_H + \ast_g d\rho_1 + d(\ast_g \rho_2).$$

From the uniqueness of the decomposition one imidiatelly gets that

$$\rho_1 = \ast_g \rho_2$$

and

$$\ast_g \omega_H = \omega_H,$$

hence one can rewrite the previous decomposition as follows

$$\omega = \omega_H + d^+ \rho_0, \quad \rho_0 \in \Omega^1_X \quad \text{where} \quad d^+ : \Omega^1_X \rightarrow \Omega^+_X,$$

(0.2)

and establish that the harmonic part $\omega_H$ is self dual too.

**Definition.** The image of the canonical map $\tau$ for given hermitian triple $(g, J, \omega)$ equals to the 2- cohomology class which is represented by the harmonic form $\omega_H$ defined in (0.1), (0.2)

$$\tau(g, J, \omega) = [\omega_H] \in H^2(X, \mathbb{R}).$$

(0.3)

As we’ve seen above $\omega_H$ is self dual so it has positive square

$$\int_X \omega_H \wedge \omega_H = \int_X |\omega_H|^2_g dm_g > 0$$

unless the case when $\omega_H$ is vanishing everywhere. Hence the image $\text{Im}\tau$ lies in $K^+$. 

**Definition.** One can called based manifold $X$ pseudo symplectic if there exists a hermitian triple with nontrivial image in $K^+$. 

And it was proved that the canonical class of pseudo symplectic manifold has the invariant equals to $\pm 1$ as well as in the original symplectic case.
Example. First of all it’s clear that every symplectic manifold is pseudo symplectic. For a given symplectic manifold $X$ endowed with a symplectic form $\omega$ one gets that for an appropriate riemannian metric $\omega$ is harmonic itself so $\omega_H = \omega$ and hence $\tau(g, J, \omega) = [\omega] \neq 0$. But we’d like to recall an example of pseudo symplectic manifold which doesn’t admit a symplectic structure. Namely, if one take the counterexample from [3] it isn’t too hard to establish that this connected sum $Y = X \sharp N$ (see [3], [9]) admits a hermitian triple with nontrivial image in $K^+$. Such triple on whole $Y$ can be constructed as an extension of the symplectic triple on the original symplectic manifold $X$ (we’ll discuss all details in the next section).

After all above is understood there is one absolutely reasonable question. It’s well known that in our setup the Seiberg — Witten invariant depends only on the choice of $Spin^C$ - structure so on the corresponding class $c \in H^2(X, \mathbb{Z})$ and doesn’t depend on the choice of riemannian metric and almost complex structure. But from the first viewpoint two hermitian triples $(g_i, J_i, \omega_i), i = 1, 2$ with the same canonical class $K = K_J$ can have rather different images in $K^+$. The first can be nontrivial whether the second is trivial.

In this paper we proof the following

Main Theorem. The nontriviality condition for image of $\tau$ is stable with respect to continuous deformations. It means that if $\tau(g_0, J_0, \omega_0) \neq [0] \in H^2(X, \mathbb{R})$ the same is true for every $(g, J, \omega)$ with $K_J = K_{J_0}$.

On the other hand we can reformulate the statement with respect to the action of the group

$$G = Aut(TX)$$

of all smooth fiberwise automorphisms of the tangent bundle over $X$. Namely there exists unique discrete invariant of triples — the corresponding canonical class $K_J \in H^2(X, \mathbb{Z})$. And it’s more or less clear that if two triples $(g_i, J_i, \omega_i), i = 1, 2$, have the same canonical class then there exists an element $u \in G$ which conjugates the first triple to the second. So as well one can say that the nontriviality of the image of $\tau$ is stable with respect to the $G$-action.

§1. PSEUDO SYMPLECTIC MANIFOLDS IN THE FRAMEWORK OF SEIBERG — WITTEN THEORY

In this section we’ll sketch the proof of the Taubes Theorem mentioned above and show in which point it uses symplectness of based manifolds.

First off all in absolutely general almost complex situation for a hermitian triple $(g, J, \omega)$ one has the following objects:

1. the decomposition on self dual and anti self dual forms represents as

$$\Lambda^+ = (\Lambda^{2,0} \oplus \Lambda^{0,2})_\mathbb{R} \oplus \mathbb{R} < \omega >,$$

$$\Lambda^- = (\mathbb{R} < \omega >)^\perp \subset \Lambda^{1,1};$$

(1.1)

2. spinor bundles correspond to $Spin^C$ - structure $c_0 = -K_\omega \in H^2(X, \mathbb{Z})$ have the form

$$W^+ = \Lambda^{0,0} \oplus \Lambda^{0,2} = I \oplus K^{-1},$$

$$W^- = \Lambda^{0,1}$$

(1.2)
and our self dual form $\omega$ (as the section of $\Lambda^+ \cong adW^+$) acts on $W^+$ as diagonal operator with eigenvalues $2i$ and $-2i$ on the direct summands $K^{-1}$ and $I$ respectively;

3. the gauge group is

$$G = Aut_h(detW^+) = Aut_h(K^{-1});$$

(1.3)

4. the configuration space for the Seiberg — Witten system is represented by space of triples $(a, \alpha, \beta)$ where $a$ is a hermitian connection on $K^{-1}$, $\alpha$ is a section of $I$ so is a complex valued function and $\beta$ is a section of $K^{-1}$;

5. the Seiberg — Witten system in this setup reads as follows

$$D_a(\alpha \oplus \beta) = \partial a \alpha + \partial^* a \beta = 0$$

$$iF^+_a = (|\beta|^2 - |\alpha|^2)\omega + \alpha \beta + \beta \bar{\alpha};$$

(1.4)

and let us recall that in this case the moduli space of solutions is zero dimensional. (For all details see [4], [6], [9], [10].)

The first step. In any case there exists unique up to gauge transformations hermitian connection $a_0 \in \mathcal{A}_h(K^{-1})$ such that the projection of the corresponding connection (on whole $W^+ = I \oplus K^{-1}$ ) to the first direct summand is equal to ordinary $d$

$$\nabla_{a_0}|_I = d \text{ on } I.$$  

(1.5)

So if one take spinor filed of the form $(s \oplus 0) \in I \oplus K^{-1}$ where $s$ a constant function then

$$\nabla_{a_0}(s \oplus 0) = b \in \Gamma(K^{-1} \otimes T^*X)$$

(1.6)

and $b$ essentially is the torsion of our almost complex structure $J$. So in the case of integrable complex structure one has $b = 0$ and hence $D_{a_0}(s \oplus 0)$ automatically equals to zero. But in the symplectic case Taubes observes that despite of nontriviality of $\nabla_{a_0}(s \oplus 0) = b$ the corresponding Dirac operator $D_{a_0}$ vanishes on $s \oplus 0$ if and only if the form $\omega$ is closed (so is a symplectic form).

Taubes arguments on this step are extremly usefull and deep exercises in the theory. Namely he acts by the self dual form $\omega$ on the spinor field $s \oplus 0$

$$\omega(s \oplus 0) = -2i \cdot (s \oplus 0)$$

and then applies the corresponding covariant derivative to the both sides of the previous equality

$$\nabla_{a_0}(\omega(s \oplus 0)) = -2ib.$$  

But on the right side we have by the Liebnitz rule

$$\nabla_{a_0}(\omega(s \oplus 0)) = C \cdot d^* \omega \otimes (s \oplus 0) + \omega(b) = C \cdot d^* \omega \otimes (s \oplus 0) + 2ib$$

(where $C$ is an integer number) hence using Clifford multiplication one gets

$$D_{a_0}(s \oplus 0) = \frac{iC \cdot s}{4}(d^* \omega)^{0,1}$$

(1.7)

where the last term $(d^* \omega)^{0,1}$ can be equal to zero if and only if $d^* \omega$ is equal to zero so if and only if our form $\omega$ is closed and hence symplectic.
The second step. After it was established that the spinor field \( s \oplus 0 \) is harmonic with respect to the Dirac operator \( D_{a_0} \), Taubes considers some special perturbation of the original Seiberg — Witten system namely

\[
D_{a_0} (\alpha \oplus \beta) = 0
\]

\[
i F_a^+ = i F_{a_0}^+ + (|\beta|^2 - |\alpha|^2 + 1) \omega + \alpha \overline{\beta} + \overline{\alpha} \beta. \tag{1.4'}
\]

It’s clear that the triple \( (a = a_0, \alpha = 1, \beta = 0) \) is a solution of this perturbed system. But the point is that one can derive the Seiberg — Witten invariant from this perturbed system as well as from the original one. It’s so since \textit{there are no reducible solutions} for the perturbed system. Really for such solution (with trivial spinor part) one has

\[
i F_a^+ = i F_{a_0}^+ + \omega
\]

but these two curvature forms are cohomologically the same so

\[
\omega = d^+ \rho
\]

where \( \rho \) is a real 1- form. But so as we are now in the symplectic category it’s impossibly hence there are no reducible solutions.

The third step. Then Taubes imposed an additional term to the right side of the second equation

\[
i F_a^+ = i F_{a_0}^+ + (|\beta|^2 - |\alpha|^2 + 1) \omega + \alpha \overline{\beta} + \overline{\alpha} \beta + \frac{4r}{1 + r |\alpha|^2} (\overline{\alpha} < b, \nabla_\xi \alpha > + \alpha < b, \nabla_\xi \alpha >). \tag{1.8}
\]

Here \( < ., . > \) is the \( \mathbb{C} \)-bilinear extension to \( T^*_X \) of the metric inner product, \( b \) is the above-defined section of \( K^{-1} \otimes T^* X \), and \( \nabla_\xi = d + i \xi \) is a covariant derivative of complex-valued functions on \( X \), where the pure imaginary form \( \xi \) is the difference \( \xi = a - a_0 \) of two Hermitian connections from \( A_h(K^{-1}) \) (see [6], [4]). Then it was established that for sufficiently large \( r >> 0 \) there exists unique up to gauge transformations solution with multiplicity one for this perturbed system and hence the invariant equals to \( \pm 1 \) (see [6], [4]).

We recalled more or less rigouresly just the first and the second steps because only on these steps Taubes uses \textit{closedness} of the self dual 2- form \( \omega \).

So to generalize this important result one has to avoid in these steps contradictions and to impose a condition on hermitian triples.

The first step. We’ve seen that in absolutely general situation (in the framework of the almost complex geometry)

\[
D_{a_0} (1 \oplus 0) = \frac{i C}{4} (d^* \omega)^{0,1} \in \Omega^{0,1}_X = \Gamma(W^-)
\]

supposing that \( s = 1 \). Then one can deform the connection \( a_0 \) such that the corresponding covariant derivative has the form

\[
\nabla_{a_1} = \nabla_{a_0} + \left( \begin{array}{cc} \frac{C}{4i} & 0 \\ 0 & \frac{C}{4i} \end{array} \right) \otimes d^* \omega \in A_h(W^+). \tag{1.9}
\]
It corresponds to such hermitian connection \( a_1 \in \mathcal{A}_h(\det W^+) \) that
\[
a_1 = a_0 + \frac{C}{4t} d^* \omega \tag{1.10}
\]
and it’s clear so as the last term is a pure imaginary 1-form that such deformation is well defined.

Then we can use this new connection \( a_1 \) instead of the original one in absolutely general case to establish that
\[
D_{a_1} (1 \oplus 0) = 0
\]
always (direct substitution of (1.9) to (1.7)).

**The second step.** So on this level one can use a familiar perturbation of the original system which differs from the Taubes perturbation just in the second equation
\[
iF^+ = iF^+ + (|\beta|^2 - |\alpha|^2 + 1) \omega + \alpha \bar{\beta} + \bar{\alpha} \beta. \tag{1.8'}
\]
Therefore again we have ”primitive” solution which is now \((a = a_1, \alpha = 1, \beta = 0)\) and one has to establish non existence of reducible solutions for our perturbed system.

At this point our definition of pseudo symplectic triples has to be imposed. Namely let us suppose that a reducible solution exists for our choosen triple \((g, J, \omega)\). Then we get that
\[
iF^+ - iF^+ = \omega
\]
but again two curvature forms in the left side are cohomologically the same so in this case it should be a real 1-form \( \rho \) such that
\[
\omega = d^+ \rho. \tag{1.9}
\]
But such \( \rho \) can exist if and only if the corresponding harmonic part \( \omega_H \) in the Hodge decomposition (0.1)
\[
\omega = \omega_H + d\rho_1 + d^* \rho_2 = \omega_H + d^+ \rho_0
\]
is trivial. So if our hermitian triple \((g, J, \omega)\) has nontrivial image \( \tau(g, J, \omega) \neq [0] \) in \( H^2(X, \mathbb{R}) \) (see the Introduction) there are no reducible solutions for our perturbed system. Then consider the following version of the SW - system, combined form (1.8) and (1.8’):
\[
D_{a_1}(\alpha + \beta) = 0, \quad iF^+ = iF^+ + (|\beta|^2 - |\alpha|^2 + 1) \omega + \alpha \bar{\beta} + \bar{\alpha} \beta + 4r
\]
\[
\left( \frac{1}{1 + r|\alpha|^2} \right) \left( \alpha < b, \nabla_\xi \alpha > + \alpha + < b, \nabla_\xi \alpha > \right), \tag{1.8’’}
\]
where all the notation are explained above in (1.8). So it differs from the Taubes \( r > 0 \) perturbation (see [4], [6]) just by the self- dual form \( iF^+ - iF^+ \) in the second equation. Now we claim that our basic solution \((a_1, 1 \oplus 0)\) satisfies the both equations
from (1.8”). Really, to check this one could test only the fifth term of the second equation right side. So it’s sufficient to check that
\[ < b, a_1 - a_0 > = 0. \]

But by the construction above the difference \( a_1 - a_0 \) is proportional to \( d^*\omega \). And moreover, the same construction from the Step One gives us that our \( b \) is related to this real 1-form. Namely
\[ b \in \Gamma(\Lambda^{0,2} \otimes T^*X) \mapsto b' \in \Gamma(\Lambda^{1,2}) \mapsto \rho_b \in \Gamma(T^*_C), \]

where the last map is defined by the operator \( \Lambda \), adjoint to the wedge product by \( \omega \). By the construction \( \rho_b \) is proportional to \((d^*\omega)^{0,1}\) (see the Step One), so using local computation one gets that
\[ < b, \rho_b > \in \Gamma(\Lambda^{0,2}) \]

is trivial for every section of \( K^{-1} \otimes T^*X \) and hence it is trivial for our \( b \).

Thus our \( (a_1, 1 \oplus 0) \) is a solution of (1.8”) for every \( r \geq 0 \). Then one can repeat all Taubes calculation to establish that for \( r > 0 \) there exits only one solution of the system (1.8”) and this gives us that the Seiberg — Witten invariant is non trivial for \( \text{Spin}^C \) - structure \( -K_J \). We do not discuss the question of the multiplicity of this solution, because rather then in the symplectic case one can’t claim that it equals to one.

Let us continue with the example of pseudo symplectic but non symplectic manifolds mentioned in the Introduction. So \( X \) is a symplectic manifold with \( b^+_2 > 1 \) and symplectic form \( \omega \). Consider the following connected sum
\[ Y = X \# N \]

where \( N \) is a manifold with \( b_1(N) = b^+_2(N) = 0 \). It was shown in [3] that such \( Y \) has the same invariants as the original \( X \) but doesn’t admit symplectic structure if the fundamental group \( \pi_1(N) \) admits nontrivial finite quotient. To establish that there are no symplectic structures on \( Y \) it’s sufficiently to consider the universal covering of \( Y \) which is a smooth manifold of the following form
\[ \tilde{Y} = X \# X \# \ldots \# X \# N' \]

and here we have \( d \) copies of \( X \) where \( d \) is the corresponding order of the quotient. Since any symplectic structure on \( Y \) induces the corresponding symplectic structure on \( \tilde{Y} \) combing this with the fact that every connected sum where summands have positive \( b^+_2 \) has trivial Seiberg — Witten invariants one gets that \( Y \) doesn’t admit any symplectic structure.

But for our aims such examples are quite appropriate. Let us formulate the result as follows
Lemma. Let $X$ be symplectic 4-manifold endowed with symplectic form $\omega_0$. Then for every 4-manifold $N$ such that $b_1(N) = b_2^+(N) = 0$ the connected sum

$$Y = X \# N$$

is a pseudo symplectic manifold.

To prove this Lemma one has to construct a pseudo symplectic structure on $Y$ so to find a hermitian triple $(g, J, \omega)$ with nontrivial image in $H^2(Y, \mathbb{R})$.

Let us start with the given symplectic structure on $X$ represented by a triple $(g_0, J_0, \omega_0)$ where the third element is our given symplectic form and we choose some compatible riemannian metric $g_0$ to form such triple. Then let us extend arbitrary this original triple to the whole $Y$, getting some triple $(g_1, J_1, \omega_1) \in \mathcal{M}_Y$.

First of all the corresponding canonical class $K_{J_1} \in H^2(Y, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus H^2(N, \mathbb{Z})$ has the following form

$$K_{J_1} = K_{J_0} + \sum_{1}^{b_2(N)} c_i$$

(1.10)

where $c_i$ is the basis in $H^2(N, \mathbb{Z})$ in which the intersection form $Q_N$ has standard form $(-1, ..., -1)$. Obviously we can extend all the elements if such choice is done. Really, let us realize our almost complex structure $J_0$ as a section of the corresponding spinor bundle $W^+_X \rightarrow X$ with the Chern classes $c_1(W^+_X) = K_{J_0}, c_2(W^+_X) = 0$. Then it’s clear that over $N$ the sum $\sum_{1}^{b_2(N)} c_i$ is a Spin$^C$-structure (because it is a characteristic element of $Q_N$) and one can take the corresponding spinor bundle $W^+_N$ over $N$ with the Chern classes $c_1(W^+_N) = \sum_{1}^{b_2(N)} c_i, c_2(W^+_N) = 1$ induced by some appropriate riemannian metric over $N$. Then over the whole $Y$ we get the spinor bundle $W^+_Y$ with the Chern classes $c_1(W^+_Y) = K_{J_1}, c_2(W^+_Y) = 0$ (it’s easy to check it directly). So one can extend the chosen section of $W^+_X$ corresponded to our given almost complex structure and then get a nonvanishing section of $W^+_Y$ which would induce an almost complex structure on the whole $Y$.

So we claim now to deduce that the triple $(g_1, J_1, \omega_1)$ has nontrivial image $\tau(g_1, J_1, \omega_1)$ in $H^2(Y, \mathbb{R})$. For this let us take the harmonic with respect to $g_1$ 2-form $\omega_H$ such that

$$[\omega_H] = [\omega_0] \in H^2(Y, \mathbb{R}).$$

Then consider the following integral

$$\int_Y \omega_1 \wedge \omega_H.$$  (1.11)

Since both 2-forms are self dual with respect to $g_1$, this expression equals to the inner product so if it is nontrivial then the projection of $\omega_1$ to the ray $\mathbb{R} < \omega_H >$ in the harmonic space is nontrivial and hence $\tau(g_1, J_1, \omega_1)$ has to be nontrivial too.
As usual for connected sums let’s divide the integral (1.11) into three parts
\[ \int_Y \omega_1 \wedge \omega_H = \int_{X \setminus B_1} \omega_1 \wedge \omega_H + \int_{\text{Neck}} \omega_1 \wedge \omega_H + \int_{N \setminus B_2} \omega_1 \wedge \omega_H \]
where \( B_i \) are small balls using for the glueing procedure.

Then since over punctured \( X \setminus B_1 \) these two 2- forms are very closed to the original symplectic form we get that
\[ \int_{X \setminus B_1} = 2VolX + o(r), \]
where \( r \) is the radius of the balls. So as the Neck is conformal flat one gets
\[ \int_{\text{Neck}} \omega_1 \wedge \omega_H = o(r), \]
and since there are no self dual harmonic form over \( N \) the same expression can be obtained for the third summand
\[ \int_{N \setminus B_2} \omega_1 \wedge \omega_H = o(r). \]

Taking all together we see that
\[ \int_Y \omega_1 \wedge \omega_H = 2VolX + o(r) \quad (1.12) \]
so shrinking the Neck (standard trick, see f.e. [1]) we get that this integral is nontrivial and positive.

The Lemma proved above gives us a number of examples of pseudo symplectic but non symplectic 4- manifolds.

This story represents the motivation to introduce the definition of pseudo symplectic manifolds. Now we’d like to repeat the natural conjecture which was stated above: we know that the invariants doesn’t depend of the choice of riemannian metric and depends only on the corresponding class in \( H^2(X, \mathbb{Z}) \). Is the same true for nontriviality of the image of our canonical map \( \tau \)?

The answer (included in our Main Theorem) is positive and in the following sections we prove this statement.

§2. THE STRUCTURE OF THE SPACE \( M_X \) FOR A GIVEN \( X \)

In this section we’ll describe the structure of the hermitian triple space for a given smooth 4 - manifold \( X \).

First of all let’s consider the discrete invariants, which divides all the \( M_X \) into discrete set of subspaces.

The most generic division of \( M_X \) corresponds to the orientation. For an orientable \( X \) there are two choices of orientation. And if one fix an almost complex
structure then the corresponding orientation is fixed. So one can divide $\mathcal{M}_X$ into two pieces

$$\mathcal{M}_X = \mathcal{M}_X^+ \cup \mathcal{M}_X^-,$$

(2.1)

where triples from the first subspace are compatible with one orientation and triples from the second one are compatible with the other. There are not any preferences in the choice of the orientations but let us assign $+$ and $-$ arbitrary to distinguish the two orientations.

The next (and the last) discrete data are the labelling of $\mathcal{M}_X$ by the corresponding canonical classes. Namely let’s consider the following obvious map

$$Can : \mathcal{M}_X \to K_X \subset H^2(X, \mathbb{Z})$$

(2.2)

defining by

$$Can(g, J, \omega) = K_J \in H^2(X, \mathbb{Z})$$

where $K_J$ is the canonical class of $J$. There are two approaches to define this canonical class. First of all, the tangent bundle $TX$ together with our chosen almost complex structure $J$ can be regarded as rank two complex bundle $T_{C,0}X$, so one can take the first Chern class of this bundle and

$$K_J = -c_1(T_{C,0}X).$$

The second approach is to consider the direct sum

$$\Lambda^2(T^*X) = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are real rank-3 bundles on self dual (respectively anti self dual) 2-forms with respect to the conformal class $*_g$ and the orientation defined by $J$. It’s well known that there is trivial real subbundle $\mathbb{R} < \omega > \subset \Lambda^+$ and in the presence of the riemannian metric $g$ one gets the orthogonal to $\mathbb{R} < \omega >$ subbundle $(\mathbb{R} < \omega >)^\perp \subset \Lambda^+$. Moreover, the vector multiplication by our form $\omega$ defines a complex structure on $(\mathbb{R} < \omega >)^\perp$ and one can take the first Chern class of this line complex bundle and it is our canonical class again.

It’s easy to describe the image of $Can$ in $H^2(X, \mathbb{Z})$. Let us fix an orientation on $X$, then for any $K_0 \in \text{Im}Can \subset H^2(X, \mathbb{Z})$ one has

$$K_0 = w_2(X)(\text{mod } 2), \quad K_0^2 = 2\chi + 3\sigma,$$

(2.3)

where $w_2(X)$ is the second Stiefel-Whitney class, $\chi$ is the euler characteristic (and these elements are independent of the choice of orientation) and $\sigma$ is the signature, corresponding to the chosen orientation.

Further, for any $K_0 \in \text{Im}Can$ one has that $-K_0 \in \text{Im}CanK$. Namely for any triple $(g, J, \omega)$ with $K_0$ we have $(g, -J, -\omega)$ with $-K_0$. So the space $\mathcal{M}_X$ has 

**standard real structure** $\Theta_X$ without real points. For any point $(g, J, \omega) \in \mathcal{M}_X$ we have the conjugated point

$$\Theta_X(g, J, \omega) = (g, -J, -\omega) \in \mathcal{M}_X$$

(2.4)
and obviously our real structure $\Theta_X$ preserves both $M^\pm_X$ and induces just the multiplication by $-1$ on $K_X \subset H^2(X, \mathbb{Z})$.

Hence our space $M_X$ is labelled by the discrete image of the map $Can$ and we have

$$M_X/\Theta_X = \bigcup_i M_{X_i}^K,$$

where $M_{X_i}^K = Can^{-1}(K_i)$.

To describe the structure of the space $M_X$ we consider a triple $(g, J, \omega)$ with fixed canonical class as the corresponding pair $g, J$. Then there are two natural projections: to the first and the second components

$$\pi_1(g, J) = g, \quad \pi_2(g, J) = J. \quad (2.5)$$

The images lie in the homogeneous space of all riemannian metrics $\mathcal{H}$, compatible with the given smooth structure, and in the space of all almost complex structures with the fixed canonical class. Now we want to describe the fibers of these two projections. But for this at a moment we’ll use not the space $M_{X_i}^K$ but the space of pairs $(\ast g, J)$ where $\ast g$ is the conformal class of a riemannian metric. Let us denote this space as $\tilde{M}_{X_i}^K$, so we have the following natural fibration

$$\text{con} : M_{X_i}^K \to \tilde{M}_{X_i}^K,$$

which is a principal $e^\mathbb{R}$- bundle.

To describe the projections above it’s very useful to translate all constructions to the language of projectivization. In this setup the conformal class $\ast g$ of a metric $g$ corresponds to the following object. Over each point $p \in X$ one has in

$$\mathbb{C}P^3 = \mathbb{P}(T^C_p X)$$

non-degenerated quadric $Q_p \subset \mathbb{C}P^3$ together with standard real structure

$$\Theta_p : \mathbb{C}P^3 \to \mathbb{C}P^3, \quad \Theta_p^2 = id$$

such that $Q_p$ is real but without real points. Then any almost complex structure $J$ compatible with this conformal class corresponds to a pair of projective lines $l^1_p, l^2_p$ on the quadric. It’s well known that there are two families of projective lines on a nondegenerated quadric. And our two lines lie in the same family because

$$\Theta_p(l^1_p) = l^2_p,$$

and if $l^1_p, l^2_p$ will be in two different families then the intersection point would be real. Our nondegenerated quadric $Q_p$ is the direct product of two projective lines

$$Q_p = \mathbb{P}^+ \times \mathbb{P}^-$$

(since the orientation is chosen by the fixing of $K_i$ one can distinguish these two projective lines by the signs). So by the definition $l^i_p$ is represented by points in $\mathbb{P}^+$ (for details see [10]).

In this language the description of $\pi_i$ is quite obvious.
The first projection. Now we’re ready to describe a fiber of
\[ \pi_1 : \mathcal{M}^{K_i} \to \mathcal{H}, \]
where \( \mathcal{H} \) is the homogeneous space of all riemannian metrics compatible with the given smooth structure.

Let us fix a riemannian metric \( g \) and a canonical class \( K_i \). Then one gets the following lifting of the corresponding projective bundle \( \mathbb{P}^+ \). Since our \( X \) is orientable (and the corresponding orientation is fixed by our canonical class) the lifting is defined by the choice of the first Chern class (which is called the choice of \( \text{Spin}^c \)-structure). So if we choose \( K_i \) as the first Chern class we immediately get the corresponding vector bundle \( W^+ \) which is called spinor bundle. The topological type of the bundle is defined by our chosen \( K_i \)
\[ c_1(W^+) = K_i, \quad c_2(W^+) = 0. \]
Then one can consider the space of smooth everywhere nonvanishing sections:
\[ \Gamma^*(W^+) = \{ \phi \in \Gamma(W^+) \mid (\phi_0)|_0 = \emptyset \}. \quad (2.6) \]
Then any section \( \phi \in \Gamma^*(W^+) \) defines an almost complex structure as follows. There is natural pairing \( \phi \to (\phi \otimes \phi_0) \) and in the composition with well-known isomorphism
\[ adW^+ = \Lambda^+ \]
one gets the corresponding self dual 2-form \( \omega_\phi \) which is nondegenerated everywhere. So from \( g \) and \( \omega_\phi \) one can reconstruct the second element in hermitian triple and it is our almost complex structure (see, f.e., [8]).

But it’s clear that two nonvanishing sections define the same complex structure if and only if they are gauge equivalent with respect to the gauge group
\[ \mathcal{G}_0 = \text{Aut}_h(\text{det}W^+) \]
of fiberwise transformations of the determinant line bundle \( \text{det}W^+ \). So a fiber of \( \pi_1 \) above is naturally isomorphic to
\[ \pi_1^{-1} = \Gamma^*(W^+)/\mathcal{G}_0. \quad (2.7) \]

Equivalently one can describe the first projection in terms of the space of nonvanishing self dual 2-forms. Again, since we have chosen orientation and riemannian metric \( g \), we can decompose the space \( \Omega^2_X \) of all smooth 2-forms into self dual and anti self dual parts
\[ \Omega^2_X = \Omega^+ \oplus \Omega^- . \]
Then let us derive the subset \( \Omega^+_\times \subset \Omega^+ \) of nonvanishing everywhere self dual forms. Let us recall that for self dual 2-forms there are just two possibilities: to have rank 0 or rank 4 (full rank). So the subset \( \Omega^+\times \) is defined by just one condition: to be nonvanishing. Then the subset \( \Omega^+\times \) consists of a number of disconnected components, labelled as above by the corresponding canonical classes. In this sense the present
picture is more universal then the picture above (with the spinor bundle) because now all the components of $\pi_1^{-1}(g)$ are defined on the whole space $M_X$. So the corresponding connected component of $\Omega^+_X$ is the fiber of $\pi^{-1} \in M^X_k$.

On the other hand we can realize the situation globally using the following global construction. For our fixed riemannian metric $g$ we consider pointwise all self dual 2-forms which correspond to almost Kahler forms. Namely over each point $p$ we have a point $\omega_p$ in the fiber $\Lambda^+_p$. But one has to unify all these forms so that each $\omega_p$ has the norm equals to $\sqrt{2}$. Hence we have 2-sphere $S^3_p \subset \Lambda^+_p$ consists of locally compatible forms. After globalization one gets the following fibration

$$s : S \rightarrow X$$

with a fiber $s^{-1}(p) = S^2$. It’s clear that

a) $S$ is smooth compact orientable 6-dimensional manifold;

b) the topological type of $S$ is fixed by the topological type of $X$;

c) every global everywhere nonvanishing self dual 2-form $\omega$ defines smooth inclusion

$$i_\omega X \hookrightarrow S$$

such that the image $i_\omega(X)$ intersects all the fibers transversally;

d) the 4-homology classes of the images $[i_\omega_i] \in H_4(S, \mathbb{Z})$ are in one-to-one correspondence with the canonical classes defined by $\omega_i$.

Moreover since every 2-sphere fiber is endowed with the natural riemannian metric one gets the corresponding riemannian metric $G_g$ on whole $S$. So:

e) the projection of $G_{i_\omega(X)}$ is isomorphic to $g$ on $X$.

And we can define an almost complex structure $J_g$ on $S$ as follows. Over each point $t \in S$ there is ortogonal decomposition of the tangent space induced by our metric $G_g$

$$T_S(t) = TS^2 \oplus TX$$

and we take the standard complex structure induced by the vector multiplication by $\omega$ on the first summand and the tautological almost complex structure defined by $g$ and $\omega$ on the second summand.

Really the space $S$ is well known in differential geometry as twistor space but originally it was defined for conformal classes instead of riemannian metrics themselves. But the correspondence is absolutely clear and will be used further in this paper.

**The second projection.** Now let’s consider the second projection

$$\pi_2 : M^K_k \rightarrow K_k$$

to the space of almost complex structure with the same canonical classes.

For this we’d like to present the procedure to reconstruct riemannian metrics compatible with a given almost complex structure from some additional data.

First of all let us recall that our standard real structure $\Theta_p$ on $\mathbb{C}P^3_p$ induces quaternionic real structures on our projective lines $\mathbb{P}^\pm_p$ which parametrize the two families on the quadric $Q_p$. 
Suppose that we already fixed an almost complex structure $J$ and got over each point $p \in X$ two projective lines $l^i_p$. The same picture exists over each point and one can see that $J$ defines an inclusion

$$
\rho_J : \mathbb{P}^- \hookrightarrow \mathbb{C}P^3
$$

of projective bundle (global!) $\mathbb{P}^- \to X$ to our projective bundle $\mathbb{C}P^3 \to X$ so the image is a projective subbundle of $\mathbb{C}P^3$. Over each point

$$
\rho_J(\mathbb{P}^-)|_p = l^1_p.
$$

(2.9)

Then we can reconstruct a quadric $Q_p$ from the corresponding quaternionic real structure $\theta_p$ on $\mathbb{P}^-_p$ by the following procedure. Together with the identification (2.9) above one gets the corresponding quaternionic real structure

$$
\theta'_p = (\rho_J)_*(\theta_p)
$$

on $l^1_p$. We define for each point $s^1 \in l^1_p$ a point $s^2 \in l^2_p$ as

$$
\Theta_p(\theta'_p(s^1)) = s^2.
$$

Then if we take all projective lines of the form $<s^1, s^2>$ we’ll get a nondegenerated quadric which is of course our $Q_p$. An extra datum for the reconstruction procedure is the following: over each point we have a global pintwise quaternionic structure on $\mathbb{P}^-$. So the space of all conformal classes (we have to emphrise that we get just a conformal class) compatible with a given almost complex structure $J$ can be regarded as the space of all global sections of the corresponding principal $PGL(2, \mathbb{C})$-bundle. After local consideration we again insure that $Aut(TX)$ acts transitively on $\mathcal{M}_X^K$. But it’s clear that for every global $\theta$ there is nontrivial stabilizer in $PGL(2, \mathbb{C})$.

Let us note that really the picture of the reconstruction is more universal in the following sense. For a fixed quaternionic real structure on the projective bundle $\mathbb{P}^-$ one can get different metrics using different almost complex structures with different (ad hoc) canonical classes. So the unique fixed additional datum corresponds to the subset in the whole $\mathcal{M}_X$.

The existance of the stabilizers gives us the following conclusion. The space $\mathcal{M}_X$ has no structure of a principal bundle but is a homogeneous space. This is easy to see from the consideration of any of our two projections. So let’s now consider the fibration descried above

$$
Can : \mathcal{M}_X \to K_X.
$$

Then we have the action of $Aut(TX)$ on each fiber. But for every point $(g, J, \omega) \in \mathcal{M}_X$ there is nontrivial stabilizer which is the space of sections of a principal $U(2)$-bundle. We’d like to postpone more precise description to a future since our goal now is just to study the pseudo symplecticity condition.
§3. Inclusions into twistor space

We can describe a fiber of the first projection $\pi_1$ as it was outlined above using smooth inclusions and turning to an analogy with complete linear systems in the algebraic geometry.

Let us fix a conformal class $*_g$ and consider the corresponding fiber of $\pi_1$. Then over each point we have $\mathbb{P}^+_p$ projective lines so that our quadric $Q_p$ is the direct product of two projective lines parametrizing two families of projective lines on $Q_p$. Let us globalize this picture over whole $X$ to get a projective bundle $\mathbb{P}^+ \to X$. The total space of this bundle is twistor space of the conformal class which we’ll denote as $Y$. The topological type of this $Y$ is fixed by the topological type of $X$ and moreover it has a smooth structure and an almost complex structure $J_g$ both defined by the smooth structure on $X$ and our conformal class. We have

$$\pi : Y \to X$$

(3.1)

such that $\pi^{-1}(p) = \mathbb{CP}^1$ which is our projective line above. So for any compatible almost complex structure $J$ on $X$ one has an inclusion

$$i_J : X \to Y$$

(3.2)

such that

1. $i_J$ is smooth inclusion;
2. the projection $d\pi(J_g|_{i_J(X)}) = J$ by the definition;
3. the corresponding 4- homological class $[i_J(X)] \in H^4(Y, \mathbb{Z})$ depends only on the corresponding canonical class $K_J$.

Here we need some computation to find what it the homological class $[i_J(X)]$ in $H_4(Y, \mathbb{Z})$. For this one can note that topologically our twistor space is isomorphic to the projectivization $\mathbb{P}(W^+)$ of the spinor bundle $W^+$. Then we use the standard knowledgements about projectivizations of complex bundles over $X$. Namely if $E$ is a complex rank 2 bundle over $X$ then the projectivization $\mathbb{P}(W^+)$ has the following cohomology ring

$$H^2(X, \mathbb{Z}) \oplus \mathbb{Z}[H]\setminus(c_2(E) + c_1(E) \cdot H + H^2).$$

Of course, this generator $H$ is defined modulo twisting by complex line bundles. So in our case with $E = W^+$ we choose $W^+ = I \oplus K_J^{-1}$ and thus we get that in this case the homology class $[i_J(X)]$ is Poincare dual to the class $[H]$. Moreover, for any other $J_1$ we have

$$[i_{J_1}(X)]^* = [H] + \frac{1}{2}(\pi^*(K_{J_1} - K_J)).$$

For example, for the almost complex structure $-J = J^{-1}$ with canonical class $-K_J$ we get

$$[i_{-J}(X)] = [H] - \pi^*(K_J),$$

such that

$$[i_{-J}(X)] \cap [i_J(X)] = [0] \in H_2(Y, \mathbb{Z}).$$
It can be illustrated as follows. We have an involution $\sigma$

$$\sigma : Y \to Y$$

which is represented by quaternionic real structures on the fibers

$$[J] \in \pi^{-1}(pt) \mapsto [-J].$$

Each fiber hasn’t real points, and obviously the intersection

$$i_J(X) \cap i_{-J}(X) = \emptyset$$

is trivial for each $J$.

One can compare this picture with the construction of $S$ above. Of course, the ambient spaces are the same, and one can reduce the problem from conformal classes to metrics by fixing some appropriate nondegenerated positive (with respect to the orientation) 4-form and then for every conformal class there is unique riemannian metric from this class which has this fixed 4-form as volume form. So one can represent a fiber of the projection $\pi_1$ as follows. Let one has the twistor space $Y, J_g$ with almost complex structure defined by our conformal class. Then

$$\pi_1^{-1}(\ast_g) = \{X' \subset Y | X' satisfies : \}$$

1. $X'$ is a smooth representation of the class $([H])^* \in H_4(Y, \mathbb{Z})$
2. $X'$ intersects all the fibers transversally.

Due to this twistor picture one can prove the following

**Proposition 1.** Let $(g_0, J, \omega) \in M^K_X$ has nontrivial image $\tau(g_0, J, \omega) \in K^+$ so

$$\tau(g_0, J, \omega) \neq [0].$$

Then the same is true for any $(g_0, J_1, \omega_1)$.

To prove this Proposition one can consider for a fixed metric $g_0$ the corresponding twistor space

$$\pi : Y \to X$$

with the almost complex structure $J_{g_0}$ defined by the conformal class of $g_0$. But moreover using the standard metric of $\mathbb{CP}^1 = \pi^{-1}(p)$, given metric $g_0$ and a connection on the $PU(2)$-bundle corresponding to the Levi-Civita connection of the metric $g_0$ one gets as well a riemannian metric $G_{g_0}$ on the whole $Y$ (it is much more clear in the setup of the construction of $S$ (2.8) above — all the necessary definitions were given there). So our riemannian metric $g_0$ defines a hermitian triple $(G_{g_0}, J_{g_0}, \Omega_{g_0})$ on $Y$ and we have the following inclusion

$$h : \mathcal{H}_X \to \mathcal{M}_Y$$

and it’s easy to see that the image lies in the following component

$$\text{Im} h \subset \mathcal{M}_Y^{K_{g_0}} \subset \mathcal{M}_Y.$$
Further, let us consider the second element $J$ of the original triple. It corresponds to an inclusion of $X$ to $Y$ as described above.

It’s obvious that
\[
\begin{align*}
  d\pi(J g_0 | i_J(X)) &= J \\
  d\pi(G g_0 | i_J(X)) &= g_0 \\
  d\pi(\Omega g_0 | i_J(X)) &= \omega,
\end{align*}
\]
and the triviality or nontriviality of $\tau(g_0, J, \omega)$ can be related to the question of the triviality or nontriviality of the corresponding class
\[
\bar{\tau}(G g_0, J g_0, \Omega g_0) \in H^2(Y, \mathbb{R}),
\]
restricted on the submanifold $i_J(X) \subset Y$. Here the map $\bar{\tau} : \mathcal{M}_Y \to H^2(Y, \mathbb{R})$ has the same definition as the map $\tau$ above (but really there is no $K^+$ in $H^2(Y, \mathbb{R})$ so the analogy isn’t absolutely direct).

The relation between the following two integrals
\[
\int_X \omega_H^2, \quad \text{and} \quad \int_{i_J(X)} \Omega_{g_0}^2_H,
\]
where ”H” denotes the corresponding harmonic components, defined by (0.1), can be derived as follows. These two integral don’t coincide a priori, because of the ”vertical” component of the wedge square in the second one. But using our involution $\sigma$ one can “kill” this vertical component, namely
\[
\int_X \omega_H^2 = \frac{1}{2} \left\{ \int_{i_J(X)} \Omega_{g_0}^2_H - \int_{\sigma(i_J(X))} \Omega_{g_0}^2_H \right\}. \quad (3.6)
\]
But since $\sigma(i_J(X)) = i_{-J}(X)$ and because of the harmonicity of $\Omega_{g_0}_H \ (3.6)$ can be reduced to the equality
\[
\int_X \omega_H^2 = \langle [\Omega_{g_0}_H], \pi^*(K_J) >_\mathbb{R}, \quad (3.6')
\]
where the right side is a topological constant. Thus one can repeat all the discussion above to an arbitrary $Ji$ with canonical class $K_{Ji}$, getting the same equality $(3.6')$. This means that the expression in the left side of $(3.6')$ depends only on canonical class of our almost complex structure.

So the proof of the Proposition 1 is completed.

But one could note that we got even more, then the statement of Proposition 1. Namely the formula $(3.6')$ ensures that the riemannian norm of the corresponding harmonic component $(\omega)_H$ is constant under the continuous variations of the second element in our given hermitian triple.

Really, since all $(\omega)_H$ are self dual, then
\[
\| (\omega)_H \|_{g_0} = \sqrt{\int_X \omega_H^2} = \text{const.}
\]
Thus for a given metric $g_0$ we have a correspondence $K_{Ji} \to \mathbb{R}_{\geq 0}$ between canonical classes and a set or non negative real numbers.
§4. SEMITWISTOR SPACES

Now we know that nontriviality of $\tau$ is stable with respect to deformations of the second elements of the triples. The next step is to establish the same stability with respect to deformations of the first elements.

For this we’ll study a new object: an analog of the twistor space, which depends on a fixed almost complex structure (instead of a riemannian metric). Above we got twistor space as total space for all (pointwise) almost complex structures compatible with a given riemannian metric. If one fixed vice versa an almost complex structure $J$ one can consider all riemannian metrics pointwise compatible with $J$. Let us input some linear algebra to establish what is a fiber in this case.

For $\mathbb{R}^4$ with given standard complex structure

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

we are interested in such positive defined bilinear forms which are preserved by $J$. Every form can be represented by a symmetric matrix $A_Q$. Direct calculations give us the following answer

$$A_Q = \begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and

$$ac - b^2 > 0, \quad a + c > 0,$$

for positivity.

The set $L^+$ of points satisfied the two conditions above is the inner half part of quadratic cone $ac - b^2 = 0$ which belongs to the condition $a + c > 0$. Points on the boundary cone $ac - b^2 = 0$ correspond to degenerated metrics.

In turn to the conformal classes (of compatible riemannian metrics) we get a bundle

$$w : W \to X$$

where $w^{-1}(pt)$ is topologically a two-dimensional disc. Let us call the total space of this bundle $W$ semitwistor space corresponding to the given $J$. A reason to call it so is as follows: let us take together with $L^+$ the other part

$$L^- = \{(a, b, c) \mid ac - b^2 > 0 \text{ but } a + c < 0\}.$$

This part corresponds to negative defined metrics, but nevertheless after ”projectivization” we get a sphere $S^2_p$ with marked equator $S^1_p \subset S^2_p$ containing conformal classes of degenerated metrics. Globalizing this picture over whole $X$ one gets a bundle on 2-spheres

$$t : T \to X,$$

where $t^{-1}(p) = S^2_p$, and this bundle, at least topologically, corresponds to twistor bundle $\pi : \mathbb{P}^1 \to X$ above.
But now we’d like to consider the following construction. Let us realise the disc, which belongs to the set of all conformal classes, compatible with a given almost complex structure \( J \) as the **Lobachevsky plane**. Really, on \( \mathcal{L}^+ \) we have the natural norm
\[
v = (a, b, c) \mapsto |v| = ac - b^2
\]
and one can consider all vectors with the unit norm. They form hyperbolic surface inside \( \mathcal{L}^+ \) which we’ll denote as \( \text{Lob} \). It’s clear that \( \text{Lob} \) is naturally isomorphic to the Lobachevsky plane. So one can regard
\[
w : W \rightarrow X
\]
as fibration on the Lobachevsky planes. So as on \( \text{Lob} \) there exists special riemannian metric one can define the universal conformal class \( *_G \) on the whole \( W \) in the same way as it was defined for the universal almost complex structure on twistor space. Unfortunately an analog for Levi- Civita connection doesn’t exist. To avoid this one can choose a compatible riemannian metric \( g_0 \) to identify all the fibers For each \( s \in W \) the corresponding to \( g_0 \) Levi- Civita connection defines the decomposition
\[
TW_s = TX_{w(s)} \times T(\text{Lob})_s,
\]
and on the second component we have standard riemannian metric and on the first component one has tautological conformal class on \( X \) corresponding to this point \( s \). So we have the universal conformal class \( *_J \) on our semitwistor space \( W \) and hence one can define an almost complex structure on \( W \). Namely, this universal conformal class defines over each point the orthogonal decomposition of the tangent space so one can define the direct product of two complex structure - the given on \( TX \) and the standard on \( T(\text{Lob}) \).

So for every almost complex structure \( J \) on \( X \), equipped with an additional tool - a compatible riemannian metric \( g_0 \), we defined a 6- dimensional open manifold \( W \) with boundary \( S \) with universal pair \( *_J \) and \( J_J \). But another problem is that \( *_J \) doesn’t admit a nondegenerated extension to the boundary \( S \) and the same happens with our universal almost complex structure \( J_J \).

Hence we have an analogy with the twistor case but now the situation is much more complicated. First of all while a twistor space is compact (and is smooth 6- dimensional manifold), a semitwistor space is not compact and is represented by an open almost complex 6- dimensional riemannian manifold. The boundary can be described as \( S^1 \)- bundle over \( X \) and is denoted as
\[
\partial W = S \rightarrow X.
\]

Now the description of all **globally defined** conformal classes compatible with our given \( J \) is the following. For each \( *_g \) on based manifold \( X \) one has the corresponding smooth inclusion
\[
i_{*_g}(X) \rightarrow W
\]
such that \( i_{*_g}(X) \) is a 4- submanifold which intersects the fibers transversally. Again we have
\[
dw(*_J|_{i_{*_g}(X)}) = *_g
\]
\[
dw(J_J|_{i_{*_g}(X)}) = J.
\]

(4.5)
Now we want to repeat the argument, which took place in the previous section to establish nontriviality of our canonical map $\tau$ for all metrics compatible with our given almost complex structure $J$. But in the present case one has to use two additional arguments and one additional tool.

First of all, the construction of semitwistor spaces above deals with *conformal classes* instead of riemannian metrics. So we have to fix a nondegenerated 4-form on $X$, denoted as $d\mu$ (the volume form) and then get a one-to-one correspondence between conformal classes and riemannian metrics with the same volume form (so this fixing defines a section of fibration

$$\tilde{W} \to W, \quad i_{d\mu}(W) \hookrightarrow \tilde{W} \quad (4.6)$$

where $\tilde{W}$ is the space of all riemannian metrics pointwise compatible with $J$).

Then one gets on $W$ the *universal riemannian metric* $G$ instead of the universal conformal class $\ast G$ and universal 2-form $\Omega$.

Further, the second point is that $W$ isn’t compact and one has to impose the following argument to establish stability of our canonical map $\tau$ with respect to changing of the first element in hermitian triple. Namely instead of ordinary cohomology one has to work with cohomology with compact support. Then since our universal riemannian metric $G$ descends to zero on the boundary we can repeat all arguments on the Hodge decomposition of $\Omega$ in terms of the cohomology with compact support and get the following statement:

**Proposition 2.** Let one fix a hermitian triple $(g, J, \omega)$ with nontrivial image in $K^+ \subset H^2(X, \mathbb{R})$. Then for every metric $g_1$ compatible with $J$ and with the same volume form $d\mu$ as for the original one the image of $(g_1, J, \omega_1)$ is nontrivial.

An additional tool, which one needs, is an analog of the involution $\sigma : Y \to Y$, used in the previous section. Really, we again have an involution of our semitwistor space $W$ due to the chosen riemannian metric $g_0$. Namely, over each point of $X$ we have the corresponding to $g_0$ framed point. So the involution can be defined as the rotation of each fiber, centred in this framed points

$$\sigma_{g_0} : W \to W; \quad \sigma_{g_0}^2 = id.$$

Then we can repeat with slight modifications all our arguments from the previous section. First of all, we have

$$\int_X (\omega_1)_H^2 = \frac{1}{2} \left\{ \int_{i_{g_1}(X)} (\Omega_J)_H^2 + \int_{\sigma_{g_0}(i_{g_1}(X))} (\Omega_J)_H^2 \right\} \quad (4.7)$$

where $\omega_1$ corresponds to the fixed almost complex structure $J$ an a riemannian metric $g_1$, and the sign ”+” in the right side of (4.7) reflects the difference between ”classical” involution $\sigma$ for twistor spaces and ”temporar” involution $\sigma_{g_0}$ (which has the fixed point, for example, when $\sigma$ hasn’t real points at all).

It remains just to use all tricks from the previous section. Namely, one can rearrange (4.7) in ”cohomological” style: for this it’s sufficient to note that since our fibers are simply connected then $[i_{g_1}(X)]$ and $\sigma_{g_0}(i_{g_1}(X))$ are homologically the same.
and since we’ve fixed the volume form these two submanifolds are sufficiently “far” from the boundary. Our harmonic 2-form \((\Omega_J)_H\) descends to the boundary since our universal metric \(G_J\) does, so it defines a compactly supported cohomological class \([(\Omega)_H] \in H^2_c(W, \mathbb{R})\), thus
\[
\int_X (\omega_1)_H^2 = <[(\Omega)_H]^2; [i_{g_1}(X)] > \in \mathbb{R}.
\]
(4.8)

These arguments can be exploited for any riemannian metric with the fixed volume form, which gives us the statement of Proposition 2.

But again we get even more than this statement. Really, we’ve got (with (4.8)) that for any two riemannian metric \(g_0, g_1\), compatible with a given almost complex structure, with the same volume forms \(d\mu_{g_0} = d\mu_{g_1}\) we have
\[
\|(\omega_0)_H\|_{g_0} = \|(\omega_1)_H\|_{g_1}.
\]

It is a marvel, because in this case we vary the norm as well as riemannian metric, but the value doesn’t depend on such variation.

§5. From metric to conformal class

As we’ve seen in the previous section the nontriviality condition for \(\tau\) is stable with respect to changing of the second element in hermitian triples. Now we’d like to consider the next step.

**Proposition 3.** Let for \((g, J, \omega)\) the image \(\tau(g, J, \omega) \in K^+_X\) is nontrivial. Then the same is true for any triple of the shape \((e^f \cdot g, J, e^f \cdot \omega)\) where \(f : X \to \mathbb{R}\) is a smooth real function
\[
\tau(e^f \cdot g, J, e^f \cdot \omega) \neq [0].
\]

**Proof.** Let \((g, J, \omega)\) is a triple with nontrivial image
\[
\tau(g, J, \omega) = [\omega_H] \in H^2(X, \mathbb{R}) \quad [\omega_H] \neq [0]
\]
where \(\omega_H\) is the corresponding harmonic 2-form with respect to \(*_g\) and the orientation. Consider the following integral
\[
\int_X e^f \omega \wedge \omega_H.
\]
(5.1)

Since both forms appearing in this integral are self dual this expression equals to projection of the first form \(e^f \omega\) to the harmonic ray \(\mathbb{R} < \omega_H >\) modulo the norm of \(\omega_H\). So to prove the Proposition 3 it’s sufficient to show that this integral isn’t equal to zero.

For this one can observe that if
\[
(\omega, \omega_H) \in C^\infty(X \to \mathbb{R})
\]
is non negative real function then the integral above has to be strictly positive. So now our claim is to establish that the inner product \((\omega, \omega_H)\), induced by our
original riemannian metric $g$, is non negative everywhere. (And let’s recall that $\omega \wedge \omega_H = (\omega \wedge \omega_H)$ everywhere because of the self duality.)

It’s clear that since
\[
\int_X \omega \wedge \omega_H = \int_X \omega_H \wedge \omega_H > 0 \quad (5.3)
\]
(by the definition), the function
\[
s = (\omega, \omega_H)
\]
has positive integral over $X$ with respect to the volume form $d\mu_g$. So if there exist points with negative volumes of $s$ then it should impose the following picture. There is smooth 3-dimensional submanifold $B \subset X$ which is
\[
B = s^{-1}(0)
\]
and such that $X \setminus B = U^+ \cup U^-$ where
\[
U^\pm = s^{-1}(\mathbb{R}^\pm).
\]

First of all, it’s easy to see that $\omega_H$ is equal to zero form on $B$
\[
\omega_H|_B = 0.
\]

Really, let us consider smooth inclusion
\[
i : B \hookrightarrow X
\]
which gives us riemannian metric $i^*g$, nondegenerated everywhere 2-form $i^*\omega$ and inherited orientation on $B$. So one can use again the Hodge theorem over $B$ and it’s clear that the harmonic part of $i^*\omega$ should be equal to $i^*\omega_H$:
\[
i^*\omega = i^*\omega_H + d\rho_1 + d^*\rho_2 \quad (5.4)
\]
over $B$ where $\rho_1 \in \Omega^1_B, \rho_2 \in \Omega^3_B$. Since we supposed that $(\omega, \omega_H) = 0$ over $B$, we immediately get
\[
\int_B (i^*\omega_H, i^*\omega_H) d\mu_{i^*g} = \int_B (i^*\omega, i^*\omega_H) d\mu_{i^*g} = 0 \quad (5.5)
\]
so $B$ has to be a subset of zero set for $\omega_H$
\[
\omega_H|_B = 0. \quad (5.6)
\]

These arguments can be used to establish what we need. Namely, let $B_1$ is a smooth 3-dimensional submanifold of $X$ such that
\[
B_1 \subset U^-
\]
and 

\[ i_1 : B_1 \hookrightarrow X \]

is the corresponding smooth inclusion. Then with respect to \( i_1^* g \) and the inherited orientation over \( B_1 \) one has the same decomposition as (5.4)

\[ i_1^* \omega = i_1^* \omega_H + d\rho_3 + d^* \rho_4. \quad (5.4') \]

And the point is that

\[ \int_{B_1} (i_1^* \omega_H, i_1^* \omega_H) d\mu_{i_1^* g} = \int_{B_1} (i_1^* \omega, i_1^* \omega_H) d\mu_{i_1^* g} \quad (5.5') \]

has to be negative so we get the contradiction.

From all above one can conclude that:

1. \( (\omega, \omega_H) \geq 0 \) everywhere and moreover
2. zerosets of \( (\omega, \omega_H) \) and \( (\omega_H, \omega_H) \) coincide.

Because of this for any smooth function \( f \in C^\infty(X \to \mathbb{R}) \) we get

\[ \int_X e^t \omega \wedge \omega_H = \int_X e^t (\omega, \omega_H) d\mu_g \geq \int_X e^t (\omega, \omega_H) d\mu_g = e^t [\omega_H]^2 > 0 \quad (5.7) \]

where \( t \) is the minimum of our smooth function \( f \) on our compact smooth manifold \( X \).

Hence we get the statement of the Proposition 3.

§ 6. GOOD PATH, JOINING TWO TRIPLES

Only one problem remains — namely, how one can join two triples from a component \( \mathcal{M}_X^{K_i} \subset \mathcal{M}_X \) by an appropriate path, covered by our Propositions 1, 2, 3 which would give us the statement of the Main Theorem. Good path, joining two triples \( (g_0, J_0, \omega_0) \) and \( (g_1, J_1, \omega_1) \) can be constructed using the following

"Junction" Lemma. Let \( (g_0, J_0, \omega_0) \) and \( (g_1, J_1, \omega_1) \) be a pair of hermitian triples over \( X \) with the same canonical class \( K_{J_i} = K \in H^2(X, \mathbb{Z}) \). Then there exist an almost complex structure \( J_{\text{junct}} \) with the same canonical class together with two riemannian metrics \( g_p, g_q \) such that

1) \( g_p \) is compatible simultaneously with \( J_0 \) and \( J_{\text{junct}} \) and
2) \( g_q \) is compatible simultaniously with \( J_{\text{junct}} \) and \( J_1 \).

So we’ll construct a good path in \( \mathcal{M}_X^K \) joining two given triple such that this path consists of five "linear" pathes — the first one is induced by changing of riemannian metric with the fixed almost complex structure \( J_0 \), the second is induced by changing of almost complex structures with the fixed riemannian metric \( g_p \), the third is induced by changing of riemannian metric with the fixed almost complex
structure $J_{junct}$, the fourth is induced by changing of almost complex structures with the fixed riehmannian metric $g_q$. Of course, one has to impose the last "line" — changing of riemannian metrics from $g_q$ to $g_1$ with fixed almost complex structure $J_1$.

The main step in this procedure is to construct an appropriate almost complex structure $J_{junct}$ so we begin with the explanation of this point.

We work locally over a point $x \in X$. So we have in $\mathbb{CP}^3 = \mathbb{P}(T^C_x X)$ two pairs of projective lines $l_i, \Theta_x(l_i), i = 0, 1$, which correspond to our given almost complex structures $J_0, J_1$. It’s clear that in general there is no a riehmannian metric (i.e.: a nondegenerated real quadric) in general which contains all four projective lines. Let’s consider what we have in $\mathbb{CP}^5 = \mathbb{P}(\Lambda^2 T^*_x \mathbb{C} X)$. There are two pairs of points on the Grassmanian $Gr \subset \mathbb{CP}^5$ and our four projective lines lie on the same quadric in $\mathbb{CP}^3$ if and only if the corresponding four points in $\mathbb{CP}^5$ lie on the same real 2-plane (which is $\mathbb{P}(\Lambda^+_{2})$ of course).

So given these four points $l_0, \Theta_x(l_0), l_1, \Theta_x(l_1)$ our aim is to construct such pair of points $l_{junct}, \Theta_x(l_{junct})$ that:

the span $\langle l_i, \Theta_x(l_i), l_{junct}, \Theta_x(l_{junct}) \rangle$ is equal to projective 2-plane $\mathbb{P}^2_i, i = 0, 1$. The point is that these two planes are automatically real (so $\Theta_x(\mathbb{P}^2_i) = \mathbb{P}^2_i$) and one can choose $g_p$ and $g_q$ such that $\mathbb{P}^2_0 \subset \mathbb{CP}^5$ is the projectivization of $\Lambda_{g_p}^+$ and $\mathbb{P}^2_1$ is the projectivization of $\Lambda_{g_q}^+$.

So the remaining part of the proof is just an exercise in classical projective geometry. To construct two conjugated points $l_{junct}, \Theta_x(l_{junct})$ one can use the following procedure. Let us take the set of all projective 2-planes which contain the projective line $< l_0, l_1 >$. Then if we find such a 2-plane containing this line that it has two real points $r_0, r_1$ (such points that

$$\Theta_x(r_i) = r_i$$

our problem will be solved. Really, let’s suppose that we’ve found such projective 2-plane $\pi$ that

$$l_0, l_1, r_0, r_1 \in \pi$$

where $l_0, l_1$ our given points and $r_0, r_1$ some distinct real points. Then there exists conjugated 2-plane $\Theta_x(\pi)$ which contains the points $\Theta_x(l_0), \Theta_x(l_1), r_0, r_1$. So one can construct the following two points

$$l'_{junct} = < l_0, r_0 > \cap < l_1, r_1 > \in \pi$$

$$\Theta_x(l'_{junct}) = < \Theta_x(l_0), r_0 > \cap < \Theta_x(l_1), r_1 > \in \Theta_x(\pi).$$

(6.1)

It’s easy to see that projective lines $< l_0, \Theta_x(l_0) >$ and $< l'_{junct}, \Theta_x(l'_{junct}) >$ lie in the same projective 2-plane (which isn’t our $\pi$!) and the same is true for projective lines $< l_1, \Theta_x(l_1) >$ and $< l'_{junct}, \Theta_x(l'_{junct}) >$.

But two constructed points aren’t what we need because a priori these points don’t lie on our Grassmanian $Gr$ so don’t represent any lines in our $\mathbb{CP}^3$. So one has to construct the intersection

$$< l'_{junct}, \Theta_x(l'_{junct}) > \cap Gr = \{l_{junct}, \Theta_x(l_{junct})\}$$

(6.2)
and these two points would be our $l_{junct}$ and $\Theta_x(l_{junct})$. It’s not hard to see that this intersection is a pair of points indeed because our quadric $Gr \subset \mathbb{CP}^5$ is real with respect to $\Theta_x$ and our projective line $<l'_{junct}, \Theta_x(l'_{junct})>$ is real but without real points (by the construction).

So it remains to explain why there exists such appropriate projective 2-plane $\pi$ which contains given two points $l_0, l_1$ and a pair of distinct real points.

We construct $\pi$ as follows:

1. consider the span

$$<l_0, l_1, \Theta_x(l_0), \Theta_x(l_1) > = \mathbb{CP}^3 \subset \mathbb{CP}^5;$$

2. our standard real structure $\Theta_x$ restricted on $\mathbb{CP}^3$ gives us a standard real structure (which we denote as $\Theta'_x$) on $\mathbb{CP}^3$;

3. since $\Theta'_x$ is a standard real structure there exists invariant with respect to $\Theta'_x$ subset $\mathbb{RP}^3 \subset \mathbb{CP}^3$;

4. consider a 2-plane $\pi$, containing the projective line $<l_0, l_1>$ and lieing in our $\mathbb{CP}^3$;

5. since every two projective planes in $\mathbb{CP}^3$ have non trivial intersection (which is a projective line) we get the projective line

$$\rho = \pi \cap \Theta'_x(\pi);$$

6. this projective line $\rho$ is invariant with respect to the $\Theta'_x$- action (i.e. is a real projective line)

$$\Theta'_x(\rho) = \rho;$$

7. every real projective line in $\mathbb{CP}^3$ has at least two real points.

One can see that really every 2-plane, contained in the span (6.3), satisfies the required property (i.e. has two distinct real points).

We have to check only step 2 and step 7 from the list above. The other steps are rather obvious.

First of all, let us recall that there exist two types of real structure on projective spaces. The first one is standard real structure and in appropriate homogeneous coordinates in $\mathbb{CP}^n$ it has the form

$$\Theta_s(z_0 : \ldots : z_n) = (\overline{z}_0 : \ldots : \overline{z}_n).$$

Evedently, it has real points. The second type exists only on $\mathbb{CP}^n$ with $n$ odd. In this odd dimensional case we have quaternionic real structure:

$$\Theta_q(z_0 : z_1 : \ldots : z_{n-1} : z_n) = (-\overline{z}_1 : \overline{z}_0 : \ldots : -\overline{z}_n : \overline{z}_{n-1}).$$

This quaternionic structure has no real point (it can be checked directly).

**Step 2.** Since our $\mathbb{CP}^3$ is preserved by $\Theta_x$ (the four points are transformed ones to the others) we get well defined real structure $\Theta'_x$ on $\mathbb{CP}^3$. We have to establish that this restricted $\Theta'_x$ is a standard real structure. For this it is sufficient to show that there exists a real point (because there are only two possibilities — standard and quaternionic). So consider the projective line $<l_0, \Theta_x(l_0) > \subset \mathbb{CP}^3 \subset \mathbb{CP}^5$. This line is preserved by the real structure $\Theta_x$. Hence this reduction of the step 2 together with the step 7 will be proved if we prove the following
**Statement.** Let a projective line $\rho \in \mathbb{CP}^n$ be preserved by a fixed standard real structure $\Theta_s$. Then there exist at least two real points on this projective line.

Proof. Let us fix appropriate homogeneous coordinates on $\mathbb{CP}^n$ such that our given real structure $\Theta_s$ has the form (??) in this coordinate system. Then consider a system of homogeneous equations in the same coordinates, corresponding to our projective line $\rho$:

$$
\sum_{i=0}^{n} \alpha_j^iz_i = 0, \quad (6.5)
$$

$\alpha_j^i \in \mathbb{C}, j = 1, \ldots, n - 1$. Because of the $\Theta_s$- invariance we get that the points of $\rho$ satisfy the following system

$$
\sum_{i=0}^{n} \alpha_j^iz_i = 0 \quad (6.5')
$$

as well as the original one. Then it is easy to see that the points

$$
r_0 = (Rez_0 : \ldots : Rez_0), \quad r_1 = (Imz_0 : \ldots : Imz_0)
$$

should satisfy both systems. So we’ve found two real points.

Of course, there exist not only two real points on $\rho$. There is a real projective line

$$
\rho_\mathbb{R} \cong \mathbb{R}\mathbb{P}^1, \quad \rho_\mathbb{R} \subset \rho
$$

containing real points, but for our aim it is sufficient to find only two distinct real points.

With the Statement in hand we prove the step 2 and the step 7, applying this result to the case $n = 5$ and the case $n = 3$ respectively.

It remains to note that one can globalize this local picture over whole $X$ (using patching arguments) and then deduce that such good path always exists.

So now we have a good “partially linear” path joining two given hermitian triples, which were chosen arbitrary, and then applying Propositions 1, 2 and 3 along this path one gets the statement of our Main Theorem.

§7. Additional remarks

First of all it’s necessary to say that the Main Theorem above gives us new invariants of smooth structures in 4-geometry.

Really, it’s clear that for two diffeomorphic 4- manifolds $X_1, X_2$ with some diffeomorphism

$$
\phi : X_1 \rightarrow X_2
$$

if $K_2 \in H^2(X_2, \mathbb{Z})$ has nontrivial generalized image in $H^2(X_2, \mathbb{R})$ (so if for every triple $(g, J, \omega)$ with $K_J = K_2$ the image $\tau(g, J, \omega) \neq [0]$) the corresponding class $\phi^*K_2 \in H^2(X_1, \mathbb{Z})$ should be ”nontrivial” in this sense too.

Let us define the following subset in $H^2(X, \mathbb{Z})$ for a smooth compact orientable 4- manifold $X$. A class $K_i \in H^2(X, \mathbb{Z})$ is called $PS$-class if the generalized image is nontrivial. The subset $N_{PS} \subset H^2(X, \mathbb{Z})$ is preserved by the group $Diff^+X$ (as
the basic classes in the gauge theories). Of course, the generalization of the Taubes result ([9]) gives us the following inclusion

\[ N_{PS} \subset \{ \text{the set of basic classes} \} \]

but we’d like to emphasize that our definition of the PS-classes works in the case \( b_2^+ = 1 \) as well as in the ”classical” for the gauge theories case \( b_2^+ > 1 \) (of course, if \( b_2^+ = 0 \) the subset \( N_{PS} \) is empty).

Moreover, with remarks from the ends of Sections 3 and 4 in hand one can prove the following

**Theorem.** Let \((g_1, J_1, \omega_1)\) and \((g_2, J_2, \omega_2)\) a pair of hermitian triples over \(X\) s.t.

\[ K_{J_1} = K_{J_2}, \quad d\mu_{g_1} = d\mu_{g_2}. \]

Then

\[ \| (\omega_1)_H \|_{g_1} = \| (\omega_2)_H \|_{g_2}. \]

This allows us to define a version of the PS- invariants, defined above. Namely, consider the following real nonegative function

\[ \tau_\mathbb{R} : \mathcal{M} \to \mathbb{R}_{\geq 0} \]

s.t.

\[ (g, J, \omega) \mapsto \| (\omega)_H \|_g. \]

Obviously this function is invariant under the \( Diff^+(X) \)-action.

But one could try to derive some more think invariants of smooth structures from the construction of the canonical map \( \tau \). Let us consider the complete image \( \text{Im}\tau \subset K^+ \subset H^2(X, \mathbb{R}) \). It’s clear again that

1. \( \text{Im}\tau \) is a ”subcone” in \( K^+ \). The point is that if some point \( k \in K^+ \) is realized as the image of a triple \((g, J, \omega)\) then for every positive real number \( r > 0 \) the point \( r \cdot K \in K^+ \) is realized too;

2. ”subcone” \( \text{Im}\tau \) is a connected subset and the canonical map \( \tau \) is a continous map;

3. it’s quite natural to projectivize this ”subcone” and then the topology of the porjective ”manifold” reflects properties of the underling smooth structure. Namely if there are two smooth structures on the same topological based manifold then two corresponding ”projectivized subcones” should be topologically isomorphic one to other.

But now it’s the time to finish this article hoping that the new notion of pseudo symplectic manifolds inspired by the considerations in the framework of Seiberg — Witten theory will lead to new results in the smooth classification of 4- manifolds.

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MPI, BONN

E-mail address: tyurin@tyurin.mccme.ru jtyurin@mpim-bonn.mpg.de