A Novel Unified Framework for Solving Reachability, Viability and Invariance Problems

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Abstract

The level set method is a widely used tool for solving reachability and invariance problems. However, some shortcomings, such as the difficulties of handling dissipation function and constructing terminal conditions for solving the Hamilton-Jacobi partial differential equation, limit the application of the level set method in some problems with non-affine nonlinear systems and irregular target sets. This paper proposes a method that can effectively avoid the above tricky issues and thus has better generality. In the proposed method, the reachable or invariant sets with different time horizons are characterized by some non-zero sublevel sets of a value function. This value function is not obtained by solving a viscosity solution of the partial differential equation but by recursion and interpolation approximation. At the end of this paper, some examples are taken to illustrate the accuracy and generality of the proposed method.

Keywords: Reachability, Viability, Invariance, Optimal Control, Nonlinear system.

1 Introduction

Many strict definitions, such as controllability and observability, and many mature analytical methods have been proposed to analyze linear control systems’ behavior ([1]). These methods have been widely applied in engineering for solving stability and reliability problems. However, in nonlinear control systems, properties similar to those mentioned above are difficult to define and analyze ([2]).

Nonlinear control systems’ behavior can be studied using reachability and invariance analyses. In reachability and invariance analysis, it is first necessary to specify a time horizon and a target set in the state space, and then to compute the reachable and invariant sets of the target set ([3, 4]). Computing these sets, on the other hand, is a difficult task. Because of the variety of system states and control inputs, checking each state one at a time takes a long time ([5, 6]). As a result, simulation-based approaches are insufficient, and formal verification methods are required. Unfortunately, with the exception of a few solvable problems ([7]), computing these sets exactly is often intractable. Therefore, it’s customary to compute their approximate expressions.

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In recent decades, a variety of methods have been presented, which can be split into two categories: Lagrangian methods ([8]) and state space discretization methods. Lagrangian methods, which can handle high-dimensional problems but are mainly limited to linear dynamic systems, include ellipsoidal methods ([9, 10, 11]), polyhedral methods ([12, 13, 14]), and support vector machines ([8]). The state space discretization-based techniques require less of the form of the dynamical system and can solve non-linear problems. The level set method ([3, 5, 15, 16]) and the distance fields over grids (DFOG) method ([17, 18, 19]) are the two most common methods in this category, with the level set method being more widely applied in engineering.

More importantly, the level set method creatively characterizes the reachable and invariant sets in a unified form. In the level set method, the reachable or invariant set is represented as a zero level set of a value function, which is the viscosity solution of a Hamilton-Jacobi (HJ) partial differential equation (PDE) without running cost function ([3]). Different types of sets can be obtained by setting different terminal conditions and switching the MIN or MAX operator during the computation. On this basis, several mature toolboxes have been developed ([20, 21, 22, 23]), and solved a variety of practical engineering problems, such as flight control systems ([24, 25, 26]), ground traffic systems ([27, 28]), and air traffic management systems ([29]).

While the level set approach has evolved into a well-developed and dependable tool over time, it still has some limitations.

(1) The level set method entails the viscosity solution of an HJ PDE, which requires addressing an optimization problem known as the dissipation function. To our best knowledge, although this issue can be simplified in affine nonlinear systems, there is no practical approach for solving it in more general systems ([20, 29]). Therefore, the application of level set methods is mainly limited to affine nonlinear systems.

(2) The terminal condition of the HJ PDE is always set as a signed distance function of the target set. However, it is difficult to construct such a function for an irregular target set. In some studies, the irregular target set is replaced by a rectangle when constructing the required signed distance function ([25, 26]).

(3) In the level set method, saving the reachable or invariant sets for different time horizons requires saving the solutions of the HJ PDEs at different time points, which makes the storage space consumption proportional to the number of reachable or invariant sets to be saved.

To overcome these shortcomings, this paper proposes a novel unified framework for solving reachability and invariance problems. In this framework, the reachable and invariant sets can be described as some non-zero sublevel sets of some value functions, which are solutions of some HJ PDEs with running cost functions, and the HJ PDEs are solved by recursion and interpolation. Such a mechanism has the following advantages:

(1) Since there is no PDE solution involved, the computation of the dissipation function is effectively avoided. Therefore, the proposed method can be applied to non-affine nonlinear systems.

(2) As it is unnecessary to construct a signed distance function of the target set as the terminal condition of the HJ PDE, this method can easily handle irregular target sets.

(3) The reachable or invariant sets for different time horizons can be characterized by the solutions of the HJ PDEs at the same time point. Therefore, the required storage space is independent of the number of reachable or invariant sets to be saved.
The structure of this paper is as follows. Section 2 introduces reachable, viable, and invariant sets and briefly describes the level set method. Section 3 describes the proposed method in detail. In Section 4, a two-dimensional problem is taken as an example to analyze the accuracy of the proposed method. Section 5 takes an engineering problem as an example to illustrate the generality of our method. Finally, a brief conclusion is presented in Section 6.

2 Preliminaries

2.1 Statement of Problem

Consider a continuous time control system with fully observable state:

\[ \dot{s} = f(s, u) \]  

where \( s \in \mathbb{R}^n \) is the system state, \( u \in \mathcal{U} \) is the control input (\( \mathcal{U} \) is a closed set). The function \( f(., .) : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n \) is Lipschitz continuous and bounded. Let \( \mathcal{W} \) denote the set of Lebesgue measurable functions from the time interval \([0, \infty)\) to \( \mathcal{U} \). Then, given the initial state \( s_0 \) at time \( t_0 \) and \( u(.) \in \mathcal{W} \), the evolution of system (1) in time interval \([t_0, t_1]\) can be expressed as a continuous trajectory \( \phi_{t_1}^{t_0}(., s_0, u) : [t_0, t_1] \rightarrow \mathbb{R}^n \) and \( \phi_{t_1}^{t_0}(t_0, s_0, u) = s_0 \). Given a target set \( K \) and a time horizon \( T \), four important definitions can be proposed:

Definition 1. Maximal reachable set

\[ R_{\text{max}}(K, T) = \{ s_0 | \exists u(.) \in \mathcal{W}, \exists t \in [0, T] \phi_{t}^{T}(t, s_0, u) \in K \} \]  

Definition 2. Minimal reachable set

\[ R_{\text{min}}(K, T) = \{ s_0 | \forall u(.) \in \mathcal{W}, \exists t \in [0, T] \phi_{t}^{T}(t, s_0, u) \in K \} \]  

Definition 3. Maximal invariant set

\[ I_{\text{max}}(K, T) = \{ s_0 | \exists u(.) \in \mathcal{W}, \forall t \in [0, T] \phi_{t}^{T}(t, s_0, u) \in K \} \]  

Definition 4. Minimal invariant set

\[ I_{\text{min}}(K, T) = \{ s_0 | \forall u(.) \in \mathcal{W}, \forall t \in [0, T] \phi_{t}^{T}(t, s_0, u) \in K \} \]  

The keys to solve the reachability and invariance problems are to compute the above-mentioned sets.

2.2 Level set Method

In the level set method, two value functions \( V_1 \) and \( V_2 \) are characterized as viscosity solutions to the following Hamilton-Jacobi (HJ) partial differential equations (PDEs):

\[
\begin{align*}
\frac{\partial V_1}{\partial t}(s, t) + \min_{u \in \mathcal{U}} \left[ 0, \max_{s \in \mathcal{U}} \frac{\partial V_1}{\partial s}(s, t) f(s, u) \right] &= 0 \\
\text{s.t. } V_1(s, T) &= l(s) \\
\frac{\partial V_2}{\partial t}(s, t) + \min_{u \in \mathcal{U}} \left[ 0, \min_{s \in \mathcal{U}} \frac{\partial V_2}{\partial s}(s, t) f(s, u) \right] &= 0 \\
\text{s.t. } V_2(s, T) &= l(s)
\end{align*}
\]  

(6)
where \(l(.) : \mathbb{R}^n \rightarrow \mathbb{R}\) is a Lipschitz continuous function and satisfies:

\[
K = \{ s \in \mathbb{R}^n | l(s) \geq 0 \}
\]

A common choice for the function \(l(.)\) is the signed distance to the target set \(K\). Then, the maximal and minimal invariant sets are denoted as zero super-level sets of two value functions, i.e.:

\[
I_{\text{max}}(K,T) = \{ s_0 | V_1(s_0,0) \geq 0 \}
\]

\[
I_{\text{min}}(K,T) = \{ s_0 | V_2(s_0,0) \geq 0 \}
\]

Some literature point out that the invariance and reachability problems are duals of one another ([3]), i.e.

\[
R_{\text{max}}(K,T) = \overline{\mathbb{R}^n} \left[ I_{\text{min}}(\overline{\mathbb{R}^n} - K,T) \right]
\]

\[
R_{\text{min}}(K,T) = \overline{\mathbb{R}^n} \left[ I_{\text{max}}(\overline{\mathbb{R}^n} - K,T) \right]
\]

The user of the level set toolboxes is required to provide the following parameters: a computational domain, the number of grid nodes in each dimension of the computational domain, the terminal condition of the HJ PDE, the Hamiltonian, and the dissipation function.

**Remark 1.** In the level set method, for \(T_1, \ldots, T_M \in [0, \infty)\), the invariant sets of these time horizons are characterized by the zero super-level sets of the solutions of Eq. (6) at time points \(T_1, \ldots, T_M\), i.e.

\[
I_{\text{max}}(K,T_i) = \{ s_0 | V_1(s_0,T_i) \geq 0 \}
\]

\[
I_{\text{min}}(K,T_i) = \{ s_0 | V_2(s_0,T_i) \geq 0 \}
\]

... 

\[
I_{\text{max}}(K,T_M) = \{ s_0 | V_1(s_0,T_M) \geq 0 \}
\]

\[
I_{\text{min}}(K,T_M) = \{ s_0 | V_2(s_0,T_M) \geq 0 \}
\]

and the solutions of Eq. (6) at different time are generally different from each other. This means that the storage space required to save these invariant sets is proportional to \(M\).

## 3 Recursive Value Function Method

In this section, we propose a method that can avoid the above-mentioned difficulties, namely recursive value function method. According to Eq. (9), the maximal and minimal invariant sets can be obtained by computing the minimal and maximal reachable sets, respectively. To avoid repetitive descriptions, this paper focuses on the computation of the maximal and minimal reachable sets.

### 3.1 Characterizations of minimal and maximal reachable sets

Consider a case where the control input \(u(.)\) is chosen to avoid the trajectory from reaching the target set or delay the time when the trajectory first touches the target set. With such a control input, the initial state of a trajectory that can touch the target set within the time horizon \(T\) must be an element of the minimal reachable set. Thus, we can construct the following value function:
Then the minimal reachable set is the $T$–sublevel set of function $W_1(\cdot)$, i.e.:

$$R_{\text{min}}(K,T) = \{ s_0 | W_1(s_0) \leq T \}$$

Then the maximal reachable set can be characterized by:

$$R_{\text{max}}(K,T) = \{ s_0 | W_2(s_0) \leq T \}$$

In order to transform Eqs. (11) and (13) to cost-to-go functions to construct the recursive formulas, we need to build a modified dynamic system:

$$\dot{s} = f(s,u) = \begin{cases}
    f(s,u), & s \notin K \\
    0, & s \in K
\end{cases}$$

and a modified running cost function:

$$\bar{c}(s) = \begin{cases}
    1, & s \notin K \\
    0, & s \in K
\end{cases}$$

Given the state $s_0$ at time $t_0$ and $u(\cdot) \in \mathcal{U}$, the evolution of the modified system (15) in time interval $[t_0, t_f]$ can also be expressed as a continuous trajectory $\phi_{t_f}^{t_0}(\cdot, s_0, u) : [t_0, t_f] \rightarrow \mathbb{R}^n$.

**Remark 2.** As long as the trajectory $\phi_{t_f}^{t_0}(\cdot, s_0, u)$ evolves outside the target set $K$, it is the same as traject-
tory $\phi^{t_0}_t(., s_0, u)$ and the modified running cost is identically equal to 1. When the trajectory $\bar{\phi}^{t_1}_t(., s_0, u)$ touches the border of $K$, then it stays on the border under the dynamics (15) and the running cost is identically equal to 0.

Then, two cost-to-go functions can be defined:

$$\bar{W}_1(s_0, \bar{T}) = \begin{cases} 
\max_{u(.)} \int_0^\bar{T} \bar{c}(s(t)) \, dt \\
\text{s.t. } \dot{s}(t) = \bar{f}(s(t), u(t)) \forall t \in [0, \bar{T}] \\
s(0) = s_0 \\
u(t) \in U \forall t \in [0, \bar{T}]
\end{cases}$$

(17)

$$\bar{W}_2(s_0, \bar{T}) = \begin{cases} 
\min_{u(.)} \int_0^\bar{T} \bar{c}(s(t)) \, dt \\
\text{s.t. } \dot{s}(t) = \bar{f}(s(t), u(t)) \forall t \in [0, \bar{T}] \\
s(0) = s_0 \\
u(t) \in U \forall t \in [0, \bar{T}]
\end{cases}$$

(18)

Then, we may deduce the following results:

**Theorem 1.** If $0 < \tau < \bar{T}$, then $\bar{W}_1(s, \bar{T}) = W_1(s)$ holds for any $s \in \{s_0 | \bar{W}_1(s_0, \bar{T}) \leq \tau\}$.

**Proof.** Case 1: $s \in K$. According to Eq. (11), when $s \in K$, $W_1(s) = 0$, and the trajectory of the modified system (15) initialized from $s$ always stays at $s$. Therefore,

$$\bar{W}_1(s, \bar{T}) = \int_0^{\bar{T}} 0 \, dt = 0$$

(19)

This means that

$$s \in K \implies s \in \{s_0 | \bar{W}_1(s_0, \bar{T}) \leq \tau\}$$

(20)

and

$$\bar{W}_1(s, \bar{T}) = W_1(s)$$

(21)

Case 2: $s \notin K$. In this case, $\bar{W}_1(s, \bar{T})$ satisfies the following inequality:

$$\bar{W}_1(s, \bar{T}) \leq \int_0^{\bar{T}} \max_{s' \in \mathbb{R}^n} \bar{c}(s') \, dt = \int_0^{\bar{T}} 1 \, dt = \bar{T}$$

(22)

Moreover, $\bar{W}_1(s, \bar{T}) = \bar{T}$ if and only if the trajectory of the modified system (15) initialized from $s$ always evolves outside the target set $K$ on the time interval $[0, \bar{T}]$ under any $u(.) \in \mathcal{U}$. Thus,

$$\bar{W}_1(s, \bar{T}) \leq \tau < \bar{T} \implies \forall u(.) \in \mathcal{U} \exists t_f \in [0, \bar{T}) \phi^{t_f}_0(t_f, s, u) \in K$$

(23)

Again, according to Remark 2, $\phi^{t_f}_0(., s, u)$ is the same as $\phi^{t_f}_0(., s, u)$ as long as it evolves outside the target set, therefore,

$$\bar{W}_1(s, \bar{T}) \leq \tau < \bar{T} \implies \forall u(.) \in \mathcal{U} \exists t_f \in [0, \bar{T}) \phi^{t_f}_0(t_f, s, u) \in K$$

(24)
The preceding equation belongs to the second case in Eq. (11). Consequently,

\[
W_1(s, \bar{T}) \leq \tau < \bar{T} \implies \begin{cases}
\max_{u(.) \in U} \left( \int_0^{t_f} 1dt + \int_{t_f}^{T} 0dt \right) \\
\text{s.t.} \quad \dot{s}(t) = f(s(t), u(t)) \quad \forall t \in [0, t_f] \\
\quad s(t) \in K \quad \forall t \in [t_f, \bar{T}] \\
\quad s(0) = s_0 \\\n\quad u(t) \in U \quad \forall t \in [0, t_f] \\
\quad s(t) \notin K \quad \forall t \in [0, t_f]
\end{cases} = W_1(s_0)
\]

To sum up, if \( \tau < \bar{T} \), \( W_1(s, \bar{T}) = W_1(s) \) holds for any \( s \in \{s_0|W_1(s_0, \bar{T}) \leq \tau\} \).

**Theorem 2.** If 0 < \( \tau < \bar{T} \), then \( W_2(s, \bar{T}) = W_2(s) \) holds for any \( s \in \{s_0|W_2(s_0, \bar{T}) \leq \tau\} \).

**Proof.** Case 1: \( s \in K \). In this case, the trajectory initialized from \( s \) touches the target set at time 0 regardless of the choice of the control input, and obviously \( W_2(s) = 0 \). The trajectory of the modified system (15) initialized from \( s \) always stays at \( s \). Therefore,

\[
W_2(s, \bar{T}) = \int_0^{\bar{T}} 0dt = 0
\]

This means that

\[
s \in K \implies s \in \{s_0|W_2(s_0, \bar{T}) \leq \tau\}
\]

and

\[
W_2(s, \bar{T}) = W_2(s)
\]

Case 2: \( s \notin K \). In this case, \( W_2(s, \bar{T}) \) satisfies the following inequality:

\[
W_2(s, \bar{T}) \leq \int_0^{\bar{T}} \max_{s' \in \mathbb{R}^n} c(s') dt = \int_0^{\bar{T}} 1dt = \bar{T}
\]

Moreover, \( W_2(s, \bar{T}) = \bar{T} \) if and only if there exists a \( u(.) \in U \) and a \( t_f \in [0, \bar{T}] \) such that the trajectory of the modified system (15) initialized from \( s \) can touch the target set \( K \) at time \( t_f \). Thus,

\[
W_2(s, \bar{T}) \leq \tau < \bar{T} \implies \exists u(.) \in U \ \exists t_f \in [0, \bar{T}] \ \phi^T_0(t_f, s, u) \in K
\]

Again, according to Remark 2, \( \tilde{\phi}^T_0(., s, u) \) is the same as \( \phi^T_0(., s, u) \) as long as it evolves outside the target set, therefore,

\[
W_2(s, \bar{T}) \leq \tau < \bar{T} \implies \exists u(.) \in U \ \exists t_f \in [0, \bar{T}] \ \phi^T_0(t_f, s, u) \in K
\]
The preceding equation belongs to the first case in Eq. (13). Consequently,

\[
W_2(s_0, T) = \begin{cases} 
\min_{u(.) \in \mathcal{U}} & \left( \int_0^{t_f} 1 \, dt + \int_{t_f}^T 0 \, dt \right) \\
\text{s.t.} & \dot{s}(t) = f(s(t), u(t)) \text{ and } s(t) \notin K \forall t \in [0, t_f] \\
& s(t) \in K \forall t \in [t_f, T] \\
& s(0) = s_0 \\
& u(t) \in \mathcal{U} \forall t \in [0, t_f] \\
& s(t) \notin K \forall t \in [0, t_f] \end{cases} = W_2(s_0) \tag{32}
\]

To sum up, if \( \tau < \bar{T} \), \( W_2(s, \bar{T}) = W_2(s) \) holds for any \( s \in \{ s_0 \mid W_2(s_0, \bar{T}) \leq \tau \} \).

Theorem 1 indicates that, given the modified value functions \( W_1(., \bar{T}) \), due to the equivalence between \( W_1(., \bar{T}) \) and \( W_1(., \bar{T}) \) in the region \( \{ s_0 \mid W_1(s_0, T) \leq \tau \} \) for any \( \tau \in (0, \bar{T}) \), for any \( T < \bar{T} \), the minimal reachable set \( R_{\text{min}}(K, T) \) can be characterized by:

\[
R_{\text{min}}(K, T) = \{ s_0 \mid W_1(s_0) \leq T \} = \{ s_0 \mid W_1(s_0, \bar{T}) \leq T \} \tag{33}
\]

Similarly, for any \( T < \bar{T} \), the maximal reachable set \( R_{\text{max}}(K, T) \) can be characterized by:

\[
R_{\text{max}}(K, T) = \{ s_0 \mid W_2(s_0) \leq T \} = \{ s_0 \mid W_2(s_0, \bar{T}) \leq T \} \tag{34}
\]

**Remark 3.** *In our method, for \( T_1, ..., T_M \in [0, \infty) \), if \( \bar{T} > \max(T_1, ..., T_M) \), then the reachable sets of these time horizons can be represented as \( T_i - \text{sublevel sets} \ (i = 1, ..., M) \) of \( W_1(., \bar{T}) \) and \( W_2(., \bar{T}) \), i.e.*

\[
\begin{align*}
R_{\text{min}}(K, T_1) &= \{ s_0 \mid W_1(s_0, \bar{T}) \leq T_1 \} \\
R_{\text{max}}(K, T_1) &= \{ s_0 \mid W_2(s_0, \bar{T}) \leq T_1 \} \\
& \vdots \\
R_{\text{min}}(K, T_M) &= \{ s_0 \mid W_1(s_0, \bar{T}) \leq T_M \} \\
R_{\text{max}}(K, T_1) &= \{ s_0 \mid W_2(s_0, \bar{T}) \leq T_M \} 
\end{align*} \tag{35}
\]

*This means that the storage space required to save these reachable sets is independent of \( M \).*

### 3.2 Recursive Formulas of the Modified Value Functions

Value functions \( W_1(., .) \) and \( W_2(., .) \) are cost-to-go functions, which, according to Bellman's principle of optimality [30], can be expressed as solutions of the following HJ equations with operating cost function \( \bar{c}(.) \):

\[
\begin{align*}
\frac{\partial W_1}{\partial t}(s, t) &= \max_{u(.) \in \mathcal{U}} \left[ \frac{\partial W_1}{\partial s}(s, t) \bar{f}(s, u) + \bar{c}(s) \right] \\
& \text{s.t.} \ W_1(s, 0) = 0 \\
\frac{\partial W_2}{\partial t}(s, t) &= \min_{u(.) \in \mathcal{U}} \left[ \frac{\partial W_2}{\partial s}(s, t) \bar{f}(s, u) + \bar{c}(s) \right] \\
& \text{s.t.} \ W_2(s, 0) = 0 \tag{36}
\end{align*}
\]

According to Eq. (33) and Eq. (34), The crucial point in characterizing the minimal and maximal
reachable sets is to compute the modified value functions. However, $W_1(.,.)$ and $W_2(.,.)$ are often discontinuous, which makes it impossible to obtain the viscosity solutions to the HJ equations in Eq. (36). In the current research, we use a recursive and interpolation-based approach to approximate these two functions.

Denote the discrete form of system (15) as

$$s(t + \Delta t) = F(s(t), u(t))$$

(37)

The preceding equation can be yielded by the Euler’s method or the Runge-Kutta method. If $\Delta t$ is small enough, the recursive formula of Eq. (36) is

$$\begin{align*}
W_1(s, (k + 1)\Delta t) &= \max_{u \in U} \left[ W_1(F(s, u), k\Delta t) + \bar{c}(s)\Delta t \right] \\
\text{s.t.} \quad &W_1(s, 0) = 0
\end{align*}$$

(38)

$$\begin{align*}
W_2(s, (k + 1)\Delta t) &= \min_{u \in U} \left[ W_2(F(s, u), k\Delta t) + \bar{c}(s)\Delta t \right] \\
\text{s.t.} \quad &W_2(s, 0) = 0
\end{align*}$$

3.3 Approximation of the Modified Value Functions

Typically, computing the analytical forms of $W_1(.,k\Delta t)$ and $W_2(.,k\Delta t)$ for any $k$ is difficult. In the current study, a rectangular subset of the state space is designated as the computational domain, which is divided into a Cartesian grid structure. The values of $W_1(.,k\Delta t)$ or $W_2(.,k\Delta t)$ at each grid point are stored in an array with the same dimensions as the dynamic system. As an example, consider a two-dimensional system with $s = [x, y]^T$. The function $W_1(.,k\Delta t)$ or $W_2(.,k\Delta t)$ may be represented by the bilinear interpolation of the aforementioned array. When updating the value of a grid point $s$ near the boundary of the computational domain, its transferred state $F(s, u)$ may be outside the computational domain. The linear extrapolation is applied to evaluate the value of function at the transferred state. As shown in Fig. 1, the value of the unknown function $W(.)$ at point $s = [x, y]^T$, marked by the green point, is to be estimated. It is easy to find the four nearest grid points to $s$. Denote these four grid points as $s_{11}, s_{12}, s_{21},$ and $s_{22}$. The value of the function $W(.)$ at $s_{ij}$ is denoted as $v_{ij}$, which is stored in a two-dimensional array in advance, and denote the coordinate of $s_{ij}$ by $[x_{ij}, y_{ij}]^T$. Then $W(s)$ is approximated by

$$W(s) = \frac{v_{11}A_{22} + v_{12}A_{21} + v_{21}A_{12} + v_{22}A_{11}}{A}$$

(39)

where

$$\begin{align*}
A_{ij} &= \Delta x_i \Delta y_j \forall i, j \in \{1, 2\} \\
\Delta x_1 &= x - x_{11} = x - x_{12}, \quad \Delta x_2 = x_{21} - x = x_{22} - x \\
\Delta y_1 &= y - y_{11} = y - y_{21}, \quad \Delta y_2 = y_{12} - y = y_{22} - y
\end{align*}$$

(40)

Regardless of whether $s$ is in the computational domain or not, the estimation of $W$ is as in Eqs. (39) and (40). $\Delta x_1, \Delta x_2, \Delta y_1, \Delta y_2$ are positive when $s$ lies in the computational domain, as shown in Fig. 1(a), and some of these values are negative when $s$ is outside the computational, as shown in Fig. 1(b).
Given all the techniques introduced above, the complete algorithm of the proposed method is described in Algorithm 1.

**Algorithm 1** Computation of the maximal and minimal reachable sets

1: **Input:** Dynamic system (1), admissible control set $\mathcal{U}$, time horizons $T_1, \ldots, T_M$, target set $K$, number of time steps $m$, a small positive real number $\epsilon$, computational domain $\Omega = [x_l, x_u] \times [y_l, y_u]$, number of grid points $N_x \times N_y$;
2: Construct the modified system in discretized form (37);
3: Construct the modified running cost (16);
4: $T_{\text{max}} \leftarrow \max(T_1, \ldots, T_M)$, $\bar{T} \leftarrow T_{\text{max}} + \epsilon$;
5: $\Delta t \leftarrow \frac{\bar{T}}{m}$, $\Delta x = \frac{x_u - x_l}{N_x - 1}$, $\Delta y = \frac{y_u - y_l}{N_y - 1}$;
6: Let $\mathcal{W}$ and $\mathcal{W}'$ be two $N_x \times N_y$ arrays and initialize them to 0;
7: for $k \leftarrow 1, \ldots, m$ do
8: Construct the bilinear interpolation function $\mathcal{W}(\cdot)$ using $\mathcal{W}$;
9: for $i_x \leftarrow 0, \ldots, N_x - 1$ do
10: for $i_y \leftarrow 0, \ldots, N_y - 1$ do
11: $s_0 \leftarrow [x_l + i_x \Delta x, y_l + i_y \Delta y]^T$;
12: $\mathcal{W}'[i_x][i_y] \leftarrow \begin{cases} \max_{u \in \mathcal{U}} \left[ \bar{c}(s_0) \Delta t + \mathcal{W}(F(s_0, u)) \right], & \text{for computation of } \mathcal{W}_1(\cdot, \bar{T}) \\ \min_{u \in \mathcal{U}} \left[ \bar{c}(s_0) \Delta t + \mathcal{W}(F(s_0, u)) \right], & \text{for computation of } \mathcal{W}_2(\cdot, \bar{T}) \end{cases}$
13: end for
14: end for
15: Copy $\mathcal{W}'$ to $\mathcal{W}$;
16: Return $\{s | \mathcal{W}(s) \leq T_1\}, \ldots, \{s | \mathcal{W}(s) \leq T_M\}$;

### 3.4 Complexity of the algorithm

Suppose the number of grid points in each dimension of the computational domain is $N$, then the total number of grid points is $N^n$. In the level set method, for each grid point, the left and right derivatives in
n dimensions need to be computed. Therefore, at each time step, the time consumed to traverse all grid points is proportional to $nN^n$. According to the CFL condition (time step size that does not satisfy the CFL condition can lead to numerical instability), the time step size in the level set method is proportional to the grid size, which is inversely proportional to $N$. Thus, given the time horizon, the required number of time steps is proportional to $N$. Finally, the time complexity of the level set method is $O(nN^{n+1})$.

In our method, for each grid point, the time consumption of the multilinear interpolation is proportional to the number of vertices of the $n$-dimensional cube. Therefore, at each time step, the time consumed to traverse all grid points is proportional to $2^nN^n$. Since the time step size of our method is not determined by the CFL condition but is given by the user, the number of time steps is independent of the number of grids. Finally, the time complexity of our method is $O(2^nN^n)$.

The space complexities of our method and the level set method are the same, both require two $n$-dimensional arrays to be stored in memory during the computation. Therefore, the space complexities are $O(N^n)$.

4 Two-dimensional system example

Consider the simple control problem below:

$$\dot{s} = \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = f(s, u) = \begin{bmatrix} u \\ -x \end{bmatrix}$$  \hspace{1cm} (41)

In the preceding equation, $s = [x, y]^T$ is the system state, $u \in \mathcal{U} = [-1, 1]$ is the control input. The target set is $K = \{[x, y]^T | -1 < y < 1\}$, and the time horizons are $T_1 = 0.5$, $T_2 = 1$, $T_3 = 1.5$, and $T_4 = 2$. The task is to compute the maximal invariant sets $\mathcal{I}_{\text{max}}(K, T_1)$, $\mathcal{I}_{\text{max}}(K, T_2)$, $\mathcal{I}_{\text{max}}(K, T_3)$, and $\mathcal{I}_{\text{max}}(K, T_4)$. This problem can be computed analytically and thereby compared with the results of our method and the level set method. According to the duality described by Eq. (9), the minimal reachable sets of the complement of $K$ can be used to characterize the maximal invariant sets.

4.1 Analytical solution

The analytical expression of the value function constructed in Eq. (11) with $\mathcal{C}_{\mathbb{R}^n}K$ as the target set is

$$W_1(s) = \begin{cases} 
0, & \text{if } y \leq 1 \lor y \geq 1 \\
-x - \sqrt{-2 + x^2 + 2y}, & \text{if } -1 < y < 1 \land x < 0 \land -2 + x^2 + 2y \geq 0 \\
x - \sqrt{-2 + x^2 - 2y}, & \text{if } -1 < y < 1 \land x \geq 0 \land -2 + x^2 - 2y \geq 0 \\
\infty, & \text{if } -1 < y < 1 \land x < 0 \land -2 + x^2 + 2y < 0 \\
\infty, & \text{if } -1 < y < 1 \land x \geq 0 \land -2 + x^2 - 2y < 0
\end{cases}$$  \hspace{1cm} (42)

where "\land" and "\lor" are the logical operators "AND" and "OR", respectively. The analytical expressions
of the minimal reachable sets of $\mathbb{C}_{\mathbb{R}^n}K$ can be obtained from the above equation:

$$
\mathcal{R}_{\text{min}}(\mathbb{C}_{\mathbb{R}^n}K, T) = \{[x, y]^T | y \leq -1 \lor y \geq 1\} \cup \\
\{[x, y]^T | -T \leq x < 0 \land -2 + x^2 + 2y \geq 0\} \cup \\
\{[x, y]^T | x \leq -T \land y \geq xT + 1 + \frac{1}{2}T^2\} \cup \\
\{[x, y]^T | 0 \leq x \leq T \land -2 + x^2 - 2y \geq 0\} \cup \\
\{[x, y]^T | x \geq T \land y \leq xT - 1 - \frac{1}{2}T^2\} \cup \\
\{[x, y]^T | y \leq -1 \lor y \geq 1\}
$$

(43)

4.2 Results of the proposed method

We numerically solved this problem using the method described in Algorithm 1 on a computational domain $\Omega = [-2, 2] \times [-2, 2]$ with grid points $201 \times 201$. The modified system is

$$
\dot{s} = \bar{f}(s, u) = \begin{cases} 
  f(s, u), & s \notin \mathbb{C}_{\mathbb{R}^2}K \\
  0, & s \in \mathbb{C}_{\mathbb{R}^2}K
\end{cases}
$$

(44)

and the modified running cost function is

$$
\bar{c}(s) = \begin{cases} 
  1, & s \notin \mathbb{C}_{\mathbb{R}^2}K \\
  0, & s \in \mathbb{C}_{\mathbb{R}^2}K
\end{cases}
$$

(45)

Since $\max(T_1, T_2, T_3, T_4) = 2$, $\bar{T}$ is specified as 2.16 and the time step size $\Delta t$ is set as 0.02. The computational results of our method are shown in Fig. 2. It can be seen that in the region where the function $\bar{W}_1(., \bar{T})$ takes values less than $\bar{T}$, the surfaces representing $W_1(., \bar{T})$ and $\bar{W}_1(., \bar{T})$ almost overlap, and the invariant sets computed by the proposed method almost coincide with the analytic solutions.
Figure 2: The results of solving the two-dimensional example using our method. In the left subplot, the grey surface represents the analytical solution of $W_1(s)$, and the orange surface represents $\overline{W}_1(s, \bar{T})$ derived from our method. In the four subplots on the right, the orange areas indicate the invariant sets computed by our method, and the analytical solutions of the invariant sets are outlined by the black curves.

Fig. 2 not only displays the results of the proposed method but also visually depicts how the results are stored. All four invariant sets are represented as the complements of the sublevel sets of function $\overline{W}_1(s, \bar{T})$, i.e.

$$I_{\text{max}}(K, T_1) = \mathbb{C}_\mathbb{R}^2 (\{s | \overline{W}_1(s, \bar{T}) \leq T_1\})$$

$$...$$

$$I_{\text{max}}(K, T_4) = \mathbb{C}_\mathbb{R}^2 (\{s | \overline{W}_1(s, \bar{T}) \leq T_4\})$$

This means that only $\overline{W}_1(s, \bar{T})$ needs to be saved to save these four invariant sets. This is different from the level set method, in which the four invariant sets are represented as zero super-level sets of $V_1(s, ..)$ at different time if the terminal condition of the HJ equation about $V_1(s, ..)$ is set to $V_1(s, T_4) = l(s)$, i.e.

$$I_{\text{max}}(K, T_1) = \{s | V_1(s, T_4 - T_1) \geq 0\}$$

$$...$$

$$I_{\text{max}}(K, T_4) = \{s | V_1(s, 0) \geq 0\}$$

Since $V_1(s, ..)$ at different time are usually different from each other, the level set method needs to save these four invariant sets by saving $V_1(s, ..)$ at four time points, which requiring four times more storage space than our method for the same number of grids. The Fig. 3 shows how the level set method saves the invariant sets.
4.3 Convergence

This subsection analyzes the variation of computational error with grid size and time step size. The solver settings of Algorithm 1 are listed in Table 1.

| Parameter                           | Setting                                                                 |
|-------------------------------------|-------------------------------------------------------------------------|
| Computational domain $\Omega$      | $[-2, 2] \times [-2, 2]$                                                |
| Grid points $N_x \times N_y$       | $51 \times 51, 101 \times 101, 151 \times 151, 201 \times 201, 251 \times 251$ |
| $\bar{T}$                          | 2.16                                                                    |
| Number of time steps $h$            | 216, 108, 72, 54                                                        |
| Time step size $\Delta t = \frac{T}{h}$ | 0.01, 0.02, 0.03, 0.04                                                  |

We use the Jaccard index ([31]) to quantify the errors between the numerical and analytical solutions. Specifically, the relative volume error between sets $A$ and $B$ is

$$e(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|}$$  \hspace{1cm} (48)

Take set $I_{\text{max}}(K, T_k)$ as an example, Fig. 4 shows the relative volume errors under different time step sizes against the number of grid points per dimension. The level set method is also involved in the comparison. It can be seen that the computational accuracy of the proposed method is not sensitive to the time step size and is not significantly different from that of the level set method.
5 Aircraft ground motion example

Consider a simplified aircraft ground motion dynamics system:

\[
\dot{s} = \begin{bmatrix}
\dot{v}_x \\
\dot{v}_y \\
\dot{r}
\end{bmatrix} = \begin{bmatrix}
rv_y + \frac{F_y}{m} \\
-rv_x + \frac{F_x}{m} \\
\frac{M_z}{I_z}
\end{bmatrix}
\]  

(49)

where \( s = [v_x, v_y, r]^T \) is system state. \( v_x \) and \( v_y \) are the longitudinal and lateral velocities, respectively, \( r \) is the yaw rate, \( F_x \) and \( F_y \) are the longitudinal and lateral resultant forces, \( M_z \) is the resultant moment. The expressions for \( F_x, F_y, M_z \) are as follows:

\[
\begin{align*}
F_x &= -D \cos \beta - Y \sin \beta - Q_n - Q_{ml} - Q_{mr} + P \\
F_y &= Y \cos \beta - D \sin \beta - F_n - F_{ml} - F_{mr} \\
M_z &= n + (F_{ml} + F_{mr}) a_m - F_n a_n + (Q_{ml} + Q_{mr}) \frac{b_w}{2}
\end{align*}
\]

(50)

where \( D = \frac{1}{2} \rho v^2 SC_Y \beta \) is the drag, \( Y = \frac{1}{2} \rho c^2 SC_D \) is the aerodynamic lateral force, \( F_n, F_{ml}, F_{mr} \) are the ground lateral forces, \( Q_n, Q_{ml}, Q_{mr} \) are the ground longitudinal forces. See Figure 5 for the specific meanings of these variables.
Assume that the ground friction coefficient $\mu = 0.04$, then these variables can be derived by solving the force and moment balance equations ([32]). It is worth pointing out that the expression of the lateral force of the nose-wheel is:

$$F_n = -\mu_d \sin \{\mu_w \arctan (\mu_b \tan \beta_n) - \mu_e \arctan (\mu_b \tan \beta_n - \arctan (\mu_b \tan \beta_n))\} P_n \tag{51}$$

and

$$\beta_n = \arctan \left( \frac{v + ra_n}{u \cos \theta_W + (v + ra_n) \sin \theta_W} \right) \tag{52}$$

where $\theta_W \in [-0.15, 0.15]$ is the nose-wheel deflection and is considered as the control input. See literatures ([33, 34]) for the meaning of each symbol in Eq. (51) and Eq. (52). In summary, the aircraft ground motion is a highly nonlinear system, that it cannot be reduced into an affine nonlinear form. Moreover, in this example, we let the target set $K$ be a set with irregular shape, which is regarded as a combination of some cubic cells, and whether a cell is contained in the target set is determined by its center, see Figure. 6. Due to these factors, this example cannot be solved by level set method.
The parameters of the reference aircraft are summarized in Table 2.

### Table 2: Numerical values representative of the reference aircraft

| Parameter | Value  | Parameter | Value  |
|-----------|--------|-----------|--------|
| $m$       | 104915.9 | $I_z$     | 10504308.1 |
| $S$       | 249.9   | $\bar{b}$ | 12.1 |
| $\rho$    | 1.293   | $b_w$     | 23.8 |
| $a_n$     | 17.9    | $\bar{a}_m$ | 2.3 |
| $b_W$     | 6.9     | $\mu_d$   | 0.1014 |
| $\mu_b$  | -10.11  | $\mu_c$   | 1.438 |
| $\mu_e$  | -0.8507 | $C_{D0}$  | 0.061 |
| $C_{Y\beta}$ | -1.4 | $C_{n\beta}$ | 0.2 |
| $C_{nr}$  | -1.5    | $H$       | 9.47 |
| $C_{L0}$  | -0.053  |           |        |

The control objective is to steer the state towards the target set within time horizon $T = 2$. In other words, the maximal reachable set $R_{\text{max}}(K,T)$ needs to be computed. Table 3 summarizes the parameters of Algorithm 1.

### Table 3: Solver settings for Algorithm 1

| Parameter | Setting |
|-----------|---------|
| Computational domain $\Omega$ | $[30,37] \times [-20,12] \times [-0.3,0.3]$ |
| Grids | $101 \times 101 \times 101$ |
| $T$ | 2.1 |
| Number of time steps $h$ | 210 |
| Time step size $\Delta t = \frac{T}{h}$ | 0.01 |

Figure 7 depicts the maximal reachable set.
6 Conclusions

This paper proposes a unified framework for dealing with reachability and invariance problems. In this framework, the reachable or invariant sets with different time horizons are characterized by a family of non-zero sublevel sets of the solution of an HJ PDE with a running cost function, which is approximated by recursion and interpolation. This mechanism avoids the computation of the dissipation function and can reduce storage space consumption compared to the level set method. It also avoids the construction of the signed distance function of the target set, which acts as the terminal condition of the HJ PDE, such that it can handle the irregular target sets.

The suggested method, like many others, suffers from the exponential escalation in memory and computational cost as the system’s dimension grows ([15, 35]). To overcome this problem, some ideas are worth taking into account, such as decomposing a high-dimensional system into a number of lower-dimensional subsystems based on dependencies ([5, 36, 37]) or differences in the rate of change ([38]) between state variables, and solving these sub-problems sequentially.

Improving the interpolation algorithm to reduce the required quantity of grids might be another effective approach to overcome the curse of dimensionality. Furthermore, in our method, different running cost functions can be constructed for real problems, such as fuel consumption, distance traveled per unit of time. Then some more generalized reachability or invariance problems can be discussed. These will be considered in our future work.

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