The heat equation for the Dirichlet fractional Laplacian with negative measures: Existence and nonexistence of nonnegative solutions

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Abstract

We establish conditions ensuring existence and nonexistence of nonnegative solutions for the heat equation generated by the Dirichlet fractional Laplacian perturbed by negative measure-valued potentials on bounded sets. The elaborated theory is supplied by some examples.

Key words: fractional Laplacian, heat equation, Dirichlet form.

1 Introduction

In this paper, we discuss the question of existence as well as nonexistence of nonnegative solutions for negatively perturbed Dirichlet fractional Laplacian on open bounded subsets of $\mathbb{R}^d$.

For every $0 < \alpha < \min(2, d)$ and every open bounded subset $\Omega \subset \mathbb{R}^d$, we designate by $L_0 := (-\Delta)^\alpha \big|_{\Omega}$ the fractional Laplacian with zero Dirichlet condition on $\Omega^c$ (as explained in the next section). We consider the associated perturbed heat (or parabolic) equation

$$
\begin{cases}
-\frac{\partial u}{\partial t} = L_0 u - u\mu, & \text{in } \Omega \times (0, T), \\
u(\cdot, t) = 0, & \text{on } \Omega^c, \forall 0 < t < T \leq \infty \\
u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
$$

(1.1)

where $u_0 \geq 0$ is a Borel measurable square integrable function on $\Omega$ and $\mu$ is a positive Radon measure on Borel subsets of $\mathbb{R}^d$ charging no sets having zero capacity. The meaning of a solution for the equation (1.1) will be explained in the next section.

We emphasize that the measure $\mu$ is not supposed to be in the (generalized) Kato class, so that the standard perturbation theory of Dirichlet forms does not help any more to

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decide whether a nonnegative solution occurs or not. Even for the special case of a Hardy potential
\[
\frac{d\mu(x)}{dx} = V(x) = \frac{c}{|x|^\alpha}, \quad x \neq 0, \ c > 0
\]  
(1.2)

only partial informations about the problem are established in the literature, to our best knowledge. Indeed, if \(0 \in \Omega\) one derives from the paper of Beldi–BelhajRhouna–BenAmor [BRB13], that for every \(0 < c \leq c^* := \frac{\Gamma(\frac{d+\alpha}{4})}{\Gamma(\frac{d-\alpha}{4})}\), a nonnegative solution exists. However, there is no answer for \(c > c^*\).

Our main task in this paper is to shed some light towards solving the problem by giving conditions ensuring existence as well as nonexistence of nonnegative solutions for (1.1). Focusing on nonnegative solutions is motivated, among other reasons, by the fact that they are physical solutions on one hand and on the other one by the significance of the above considered operator in physics and in other area of natural sciences. For instance, the case \(\alpha = 1/2\) corresponds to the nonrelativistic Schrödinger equation, whereas the general case models the so called Lévy motion or Lévy flight. For more about aspects of applications of the fractional Laplacian we refer the reader to [DSU08]. A prototype of application in biology is illustrated in [HWQ+12].

The inspiring point for us was the papers of Baras–Goldstein [BG84], Cabré–Martel [CM99] and Goldstein–Zhang [GZ03] where the problem was addressed and solved for the Dirichlet Laplacian (i.e. \(\alpha = 2\)). In the latter cited papers, the authors proved, in particular, that existence and nonexistence of nonnegative solutions in the case where the principal part of the equation is the Dirichlet Laplacian (or an elliptic operator) is related to the size of the bottom of the spectrum of the operator \(-\Delta|_\Omega - V\). However, there is a substantial difference between the Laplacian and the fractional Laplacian. Whereas it is known that the first one is local and therefore suitable for describing diffusions, the second one is nonlocal and commonly used for describing superdiffusions (Lévy flights). These differences are reflected in the way of computing for both operators (Green formula, integration by part, Leibniz formula....).

Nonetheless, we shall show that the method used in [CM99, GZ03] still apply in our setting. Especially, for the instantaneous blow-up part adequate generalization of the intermediate results to the nonlocal case are established (see in particular, formula (2.26) and Theorem (4.1)). These extensions make it possible to carry over the method to the nonlocal setting.

As a conclusion we approve that the used method provides a unified approach for handling the local as well as the nonlocal case.

2 Preliminaries on the fractional Laplacian and preparing results

To state our main results, it is convenient to introduce the following notations.

From now on we fix an open bounded subset \(\Omega \subset \mathbb{R}^d\) with Lipschitz boundary, a real number \(\alpha\) such that \(0 < \alpha < \min(2, d)\).
The Lebesgue spaces $L^2(\mathbb{R}^d, dx)$, resp. $L^2(\Omega, dx)$ will be denoted by $L^2$, resp. $L^2(\Omega)$. We shall write $\int \cdots$ as a shorthand for $\int_{\mathbb{R}} \cdots$.

The letters $C, c, C'$ will denote generic positive constants which may vary in value from line to line.

Consider the quadratic form $E$ defined in $L^2$ by

$$E(f,g) = \frac{1}{2} A(d, \alpha) \int \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} dy,$$

where

$$A(d, \alpha) = \frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha} \pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}.$$

Using Fourier transform $\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x) dx$, a straightforward computation yields the following identity (see [FLS08, Lemma 3.1])

$$\int |\xi|^\alpha |\hat{f}(\xi)|^2 d\xi = E[f], \quad \forall f \in W^{\alpha/2,2}(\mathbb{R}^d). \tag{2.3}$$

It is well known that $E$ is a transient Dirichlet form and is related (via Kato representation theorem) to the selfadjoint operator, commonly named the fractional Laplacian on $\mathbb{R}^d$, and which we shall denote by $(-\Delta)^{\alpha/2}$. We note that the domain of $(-\Delta)^{\alpha/2}$ is the fractional Sobolev space $W^{\alpha,2}(\mathbb{R}^d)$.

Having formula (2.3) in hands one can explicitly evaluate the fractional Laplacian:

$$(-\Delta)^{\alpha/2} f(x) = F^{-1}\left( |\cdot|^\alpha \hat{f}(\cdot) \right)(x), \quad \forall f \in W^{\alpha,2}(\mathbb{R}^d). \tag{2.4}$$

From the definition of correspondence between the quadratic form and the operator we observe that the expression of the operator $(-\Delta)^{\alpha/2}$ is given by

$$(-\Delta)^{\alpha/2} f(x) = A(d, \alpha) \int \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy, \quad \forall f \in W^{\alpha,2}(\mathbb{R}^d). \tag{2.5}$$

The latter identity suggests to extend the definition of the fractional Laplacian for functions which are not in the domains as follows:

$$(-\Delta)^{\alpha/2} f(x) = A(d, \alpha) \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^d, |y - x| > \epsilon\}} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} dy, \quad \forall f \in W^{\alpha,2}(\mathbb{R}^d) \tag{2.6}$$

provided the limit exists and is finite.

Using Fourier transform one derives the following expression:

$$\int (-\Delta)^{\alpha/2} f(x) \cdot g(x) dx = A(d, \alpha) \int \int \frac{f(x)g(y)}{|x - y|^{d+\alpha}} dx dy, \quad \forall f, g \in W^{\alpha/2}(\mathbb{R}^d). \tag{2.7}$$

We shall say that a property holds quasi everywhere (q.e. for short) if it holds up to a set having zero capacity, where the capacity is the one induced by the Dirichlet form $E$. 


Set \( L_0 := (-\Delta)^{\alpha/2}|_\Omega \) the localization of \((-\Delta)^{\alpha/2} \) on \( \Omega \), i.e., the operator which Dirichlet form in \( L^2(\Omega, dx) \) is given by

\[
D(\mathcal{E}_\Omega) = W^{\alpha/2,2}_0(\Omega): = \{ f \in W^{\alpha/2,2}(\mathbb{R}^d): f = 0 \text{ q.e. on } \Omega^c \}
\]

\[
\mathcal{E}_\Omega(f,g) = \mathcal{E}(f,g) = \frac{1}{2} A(d,\alpha) \int_{\Omega} \int_{\Omega} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dx \, dy + \int_{\Omega} f(x)g(x)\kappa_\Omega(x) \, dx,
\]

where

\[
\kappa_\Omega(x) := A(d,\alpha) \int_{\Omega^c} \frac{1}{|x - y|^{d+\alpha}} \, dy.
\]

The Dirichlet form \( \mathcal{E}_\Omega \) coincides with the closure of \( \mathcal{E} \) restricted to \( C^\infty_c(\Omega) \), and is therefore regular and furthermore transient.

Formula (2.7) suggests to introduce the following local space, that will play a crucial role in the notion of solution

\[
W_{\text{loc}}(\Omega) := \{ f: f \in L^2_{\text{loc}}(\Omega), \int_{\Omega} \int \frac{|f(x)\phi(y)|}{|x - y|^{d+\alpha}} \, dx \, dy < \infty, \forall \phi \in C_c^\infty(\mathbb{R}^d) \}. \tag{2.9}
\]

The space \( W_{\text{loc}}(\Omega) \) deserves some comments that we collect in the following remark.

**Remark 2.1.** Other types of function spaces are introduced as being local spaces by Frank–Lenz–Wingert [RLW] and Felsinger–Kassmann–Voigt [FKV], with the notation \( H^\ast_{\text{loc}}(\Omega, \mathbb{R}^d) \). However, we would like to emphasize that the space \( W_{\text{loc}}(\Omega) \) is strictly larger than the the space \( H^\ast_{\text{loc}}(\Omega, \mathbb{R}^d) \) even for smooth domains. Indeed, assume that \( \Omega \) is a bounded smooth domain containing zero and \( \varphi_0 \) the ground state of the Dirichlet fractional Laplacian on \( \Omega \) with critical Hardy potential

\[
V_\ast(x) = \frac{c^*}{|x|^{\alpha}}, \quad x \neq 0, \text{ where } c^* := \frac{2\alpha\Gamma^2(\frac{d+\alpha}{4})}{\Gamma^2(\frac{d-\alpha}{4})}. \tag{2.10}
\]

Then by [BRB13, Theorem 4.2], the following sharp estimate holds true

\[
\varphi_0 \sim | \cdot |^{-\frac{d+\alpha}{2}} \delta^\frac{\alpha}{2}, \text{ where } \delta(x) = \text{dist}(x, \Omega^c). \tag{2.11}
\]

On the other side it is proved in [RLW, Theorem 3.2] and in [FKV, Remark 2.2] that in this situation \( W^{\alpha/2,2}_0(\Omega) = H^\ast_{\text{loc}}(\Omega, \mathbb{R}^d) \). Thus if \( \varphi_0 \) were in \( W^{\alpha/2,2}_0(\Omega) \) we would get the Hardy inequality

\[
\int \frac{\varphi_0^2}{|x|^{\alpha}} \, dx \leq \frac{1}{c^*} \mathcal{E}_\Omega[\varphi_0] < \infty. \tag{2.12}
\]

However, thanks to the estimate (2.11) we conclude that the left-hand-side of the latter inequality is infinite!
Let us prove that in fact \( \varphi_0 \in W_{\text{loc}} \). Let \( \varphi \in C^\infty_\text{c}(\mathbb{R}^d) \) and set \( w(x) := |x|^{-\frac{d+\alpha}{2}} \), \( x \neq 0 \). Since \( \Omega \) is bounded, thanks to the sharp estimate (2.11), we get
\[
\int_\Omega \int \frac{|w(x)\varphi(y)|}{|x-y|^{d+\alpha}} \, dx \, dy \leq C \int_\Omega |\varphi(y)| \left( \frac{1}{\Omega} \int_\Omega \frac{V_s(x)w(x)}{|x-y|^{d+\alpha}} \, dx \right) + C \int_\Omega \int_\Omega \frac{|w(x)\varphi(y)|}{|x-y|^{d+\alpha}} \, dx \, dy. \tag{2.13}
\]
Obviously the second integral is finite.

On the other hand by \([\text{BRB}13, \text{Lemma2.1}]\), one has
\[
w(x) = c \int \frac{V_s(y)w(y)}{|x-y|^{d-\alpha}} \, dy := K_{d,\alpha}(|\cdot|w)(x), \ x \neq 0. \tag{2.14}\]
Thereby, as \( \Omega \) is bounded we obtain
\[
\int_\Omega |\varphi(y)| \left( \frac{1}{\Omega} \int_\Omega \frac{V_s(x)w(x)}{|x-y|^{d+\alpha}} \, dx \right) \, dy \leq C \int_\Omega |\varphi(y)| \left( \frac{1}{\Omega} \int_\Omega K_{d,\alpha}(V_s w)(x) \, dx \right)
= C \int_\Omega |\varphi(y)| \, w(y) \, dy < \infty. \tag{2.15}
\]

As we are going to integrate functions in some Lebesgue equivalence classes against the measure \( \mu \), we should be sure that such computation is consistent. The following lemma ensure it

**Lemma 2.1.** Every element from \( W_{\text{loc}}(\Omega) \) has a quasi continuous representative on \( \Omega \).

**Proof.** Pick a function \( f \in W_{\text{loc}}(\Omega) \). Assume first that \( \text{supp}(f) \subset \Omega \) and that \( f \in L^\infty \). Then a straightforward computation yields that \( f \in W^{\alpha/2}_0(\Omega) \) and it is known that it has then a quasi continuous representative (q.c. for short). Assume that \( \text{supp}(f) \subset \Omega \).

Then by the first step \( f_k := f \wedge k \) is q.c. and \( f_k \uparrow f \). Thus \( f \) is q.c. Finally for general \( f \in W_{\text{loc}}(\Omega) \), let \( \Omega_k \subset \Omega, \Omega_k \uparrow \Omega \) and \( f_k = f \mathbf{1}_{\Omega_k} \). Then \( f_k \) fulfills conditions of the second step and hence q.c. As \( f_k \uparrow f \) on \( \Omega \), we conclude that \( f \) is q.c. as well. \( \square \)

From now on we assume that elements from the space \( W_{\text{loc}}(\Omega) \) has been chosen so as to be q.c. As by assumptions the measure \( \mu \) does not charge sets having zero capacity we obtain that if \( f = g - \text{q.e.} \) then \( f = g - \mu \text{ a.e.} \). Thus the integration against \( \mu \) is consistent.

**Lemma 2.2.** Let \( f \in W_{\text{loc}}(\Omega) \) be such that \( \text{supp}(f) \subset \Omega \). Then for every \( \varphi \in C^\infty_\text{c}(\mathbb{R}^d) \), we have
\[
\int \int \frac{f(x)\varphi(y)}{|x-y|^{d+\alpha}} \, dx \, dy = \int f(x)(-\Delta)^{\frac{d}{2}} \varphi(x) \, dx. \tag{2.16}
\]
Proof. Let \((f_k) \subset C_c^\infty(\mathbb{R}^d)\) s.t. \(\text{supp}(f_k) \subset \Omega\), \(f_k \rightarrow f\) a.e. and \(|f_k| \uparrow |f|\). Then by Formula (2.7),
\[
\int \int \frac{f_k(x)\phi(y)}{|x-y|^{d+\alpha}} \, dx \, dy = \int f_k(x)(-\Delta)_\alpha^2 \phi(x) \, dx. \tag{2.17}
\]
As \(f \in W_{\text{loc}}(\Omega)\), using the dominated convergence theorem we derive
\[
\int f_k(x)(-\Delta)_\alpha^2 \phi(x) \, dx \rightarrow \int \int \frac{f(x)\phi(y)}{|x-y|^{d+\alpha}} \, dx \, dy. \tag{2.18}
\]
Fatou lemma now yields that \(\int |f(x)(-\Delta)_\alpha^2 \phi(x)| \, dx < \infty\). Thus using the dominated convergence theorem once again yields
\[
\int f_k(x)(-\Delta)_\alpha^2 \phi(x) \, dx \rightarrow \int f(x)(-\Delta)_\alpha^2 \phi(x) \, dx = \int \int \frac{f(x)\phi(y)}{|x-y|^{d+\alpha}} \, dx \, dy, \tag{2.19}
\]
which finishes the proof. \(\square\)

We give the notion of solution for the heat equation (1.1).

**Definition 2.1.**
A Borel measurable function \(u \in L^2_{\text{loc}}((0,T) \times \Omega)\) is a solution of problem (1.1) if
1. \(u(t,\cdot) = 0\) a.e. on \(\Omega^c\).
2. \(u \in L^1_{\text{loc}}((0,T) \times \Omega, dt \otimes d\mu)\).
3. \(u \in L^1_{\text{loc}}([0,T], W_{\text{loc}}(\Omega))\) and for every \(0 \leq t_1 < t_2 < T\) and every \(\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)\), the following identity holds true
\[
\int_{\Omega} u\phi|_{t_1}^{t_2} \, dx + \int_{t_1}^{t_2} \int_{\Omega} u(t,x)(-\phi_t(t,x) + (-\Delta)_\alpha^2 \phi(t,x)) \, dx \, dt \leq \int_{t_1}^{t_2} \int_{\Omega} u(t,x)\phi(t,x) \, d\mu(x) \, dt. \tag{2.20}
\]

Let us emphasize, that regarding Remark 2.1 the solution may not be an energy solution or even locally an energy solution.

In the sequel we designate by \(\mathcal{C}_\mu\) any core of the symmetric operator \(L_0|_{C_c^\infty(\Omega)} - \mu\) in \(L^2(\Omega)\) and denote by
\[
\lambda_0^\mu := \inf_{\phi \in \mathcal{C}_\mu \setminus \{0\}} \frac{\mathcal{E}_\Omega[\phi]}{\int \phi^2 \, d\mu}. \tag{2.21}
\]
Being a Radon measure, it is known that there is a sequence of Kato measures \(\mu_k\) such that \(\mu_k \uparrow \mu\). From now on we fix definitively such a sequence. We denote by \((P_k)\) the heat equation corresponding to the Dirichlet fractional Laplacian perturbed by the measure \(\mu_k\) instead of the measure \(\mu\). The standard theory of quadratic forms implies the existence of a unique nonnegative energy solution given by \(u_k = e^{-tL_k}u_0\), \(t > 0\) for the problem \((P_k)\),
where $L_k$ is the selfadjoint operator associated to the closed quadratic form $E_{\Omega} - \mu_k$. Furthermore, the solution $u_k$ lies in the space $W^{\alpha/2,2}_0(\Omega) \cap L^\infty$ in its spacial variable, is continuous in its time variable and satisfies Duhamel’s formula:

$$u_k(t,x) = e^{-tL_0}u_0(x) + \int_0^t \int_{\Omega} p_{t-s}u_k(s,x) d\mu_k(y) ds,$$  \hspace{1cm} (2.22)

where $p_t$ is the heat kernel of $e^{-tL_0}$.

From now on we shall write

$$e^{-tL_0}(v\mu)(t,x) := \int_{\Omega} p_t(x,y)v(y) d\mu(y).$$  \hspace{1cm} (2.23)

We list the relevant properties of the sequence $(u_k)$.

**Lemma 2.3.**

i) The sequence $(u_k)$ is increasing.

ii) If problem (1.1) has a nonnegative solution $u$, then $u_k \leq u$, $\forall k$. Moreover $\lim_{k \to \infty} u_k$ is a nonnegative solution of problem (1.1) as well.

**Proof.**

i) By Duhamel’s formula, one has

$$u_{k+1}(t) - u_k(t) = e^{-tL_{k+1}}u_0 - e^{-tL_k}u_0 = \int_0^t e^{-sL_k}e^{-sL_{k+1}}(u_0\mu_{k+1} - u_0\mu_k)(y) ds \geq 0.$$  \hspace{1cm} (2.24)

ii) The domination inequality $u_k \leq u$ follows once again from Duhamel’s formula. Let us prove that the limit of the sequence $(u_k)$ is a nonnegative solution. Set $u_\infty := \lim_{n \to \infty} u_k$. Then $u_\infty \leq u$ and therefore $u_\infty \in W_{loc}(\Omega)$ as well. Thus $u_\infty \in L^1_{loc}([0,T), W_{loc}(\Omega))$. Being solutions of the heat equation with potentials $\mu_k$, the $u_k$’s satisfy: for every $0 \leq t_1 < t_2$ and every $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$,

$$\int_\Omega u_k \phi dx + \int_{t_1}^{t_2} \int_\Omega u_k(t,x)(-\phi(t,x) + (-\Delta)^{\frac{\alpha}{2}} \phi(t,x)) dx dt = \int_{t_1}^{t_2} \int_\Omega u_k(t,x) \phi(t,x) d\mu_k(x) dt.$$  \hspace{1cm} (2.25)

By dominated convergence theorem we conclude that $u_\infty$ satisfies equation (2.20). The other properties required for $u_\infty$ to be a solution are obviously satisfied. \hfill \square

The following lemma is crucial for the development of the paper. It is inspired from the ‘gradient’ case.

**Lemma 2.4.** Let $u_k$ be the nonnegative solution of the approximate problem $(P_k)$ and $\phi \in W^{\alpha/2,2}_0(\Omega) \cap L^\infty$ having support in $\Omega$. Then $\frac{\partial^2}{u_k} \in W^{\alpha/2,2}_0(\Omega)$ and

$$E_{\Omega}(u_k, \frac{\partial^2}{u_k}) \leq E_{\Omega}[\phi].$$  \hspace{1cm} (2.26)
Proof. It suffices to give the proof for positive \( \phi \). Let \( \phi \geq 0 \) and \( u_k \) be as specified in the lemma. As \( \mathcal{E}_\Omega \) is a Dirichlet form, to prove the first part it suffices to prove that \( \frac{\phi}{u_k} \in W^{\alpha/2,2}_0(\Omega) \cap L^\infty \).

Clearly for every compact subset \( K \subset \Omega \) there are constants \( \kappa_k, \kappa'_k \) such that \( \kappa'_k \geq u_k \geq \kappa_k \) on \( K \) and then \( \frac{\phi}{u_k} \in L^\infty \). To show that the latter function has finite energy we shall proceed directly.

An elementary computation yields
\[
\left( \frac{\phi(x)}{u_k(x)} - \frac{\phi(y)}{u_k(y)} \right)^2 = \frac{\phi^2(x)}{u_k^2(x)} - 2 \frac{\phi(x)\phi(y)}{u_k(x)u_k(y)} + \frac{\phi^2(y)}{u_k^2(y)} \\
\leq \frac{\phi^2(x)}{\kappa_k^2} - 2 \frac{\phi(x)\phi(y)}{(\kappa'_k)^2} + \frac{\phi^2(y)}{\kappa_k^2} \\
\leq C \left( 2(\phi(x) - \phi(y))^2 \right). \tag{2.27}
\]

Thereby we derive
\[
\mathcal{E}_\Omega[\frac{\phi}{u_k}] = \frac{1}{2} A(d, \alpha) \int \int_\Omega \frac{(\frac{\phi}{u_k}(x) - \frac{\phi}{u_k}(y))^2}{|x-y|^{d+\alpha}} \, dx \, dy + \int \Omega \frac{\phi^2}{u_k^2}(x)\kappa^\alpha \Omega(x) \, dx < \infty. \tag{2.28}
\]

Hence \( \frac{\phi}{u_k} \in W^{\alpha/2,2}_0(\Omega) \).

We proceed now to prove inequality (2.26). An elementary computation yields
\[
(u_k(x) - u_k(y))(\frac{\phi^2(x)}{u_k(x)} - \frac{\phi^2(y)}{u_k(y)}) = \phi^2(x) + \phi^2(y) \\
- \frac{u_k(x)}{u_k(y)}\phi^2(y) - \frac{u_k(y)}{u_k(x)}\phi^2(x) \\
= \phi^2(x) + \phi^2(y) - \frac{u_k^2(x)\phi^2(y) + u_k^2(y)\phi^2(x)}{u_k(x)u_k(y)} \\
\leq (\phi(x) - \phi(y))^2. \tag{2.29}
\]

Thus
\[
\mathcal{E}_\Omega(u_k, \frac{\phi^2}{u_k}) = \frac{1}{2} A(d, \alpha) \int \int \frac{(u_k(x) - u_k(y))(\frac{\phi^2}{u_k}(x) - \frac{\phi^2}{u_k}(y))}{|x-y|^{d+\alpha}} \, dx \, dy \leq \mathcal{E}_\Omega[\phi], \tag{2.30}
\]

which was to be proved.

By the end of this section, we give a technical result dealing about the comparability of the ground state of the operator \( L_0 \) that will be needed in the proof of the nonexistence part.

**Lemma 2.5.** Set \( \varphi_0 > 0 \) the normalized ground state of the operator \( L_0 \) and \( h(t, x) := e^{-tL_0}u_0(x) \) for every \( t > 0 \) and every \( x \in \Omega \). Then
\[
h(t, \cdot) \sim \varphi_0 \text{ for every fixed } t > 0. \tag{2.31}
\]
Proof. By a result due to Kaleta–Kulczycki [KK10], the operator $e^{-tL_0}$ is intrinsically ultracontractive. Hence $p_t(x,y) \leq c_t \phi_0(x) \varphi_0(y)$, which leads to $h(t,x) \leq c_t \phi_0(x)$.

The reversed inequality follows from the intrinsic ultracontractivity as well (see [DS84, p.13]).

3 Existence of nonnegative solutions

Theorem 3.1. Assume that $\lambda_0^\mu > -\infty$. Then the heat equation (1.1) has at least one nonnegative solution.

The substance of Theorem 3.1 may be established by generalizing the results corresponding to the absolutely continuous case (see [Sto87, Proposition] [Voi86, Proposition5.7]) to the case of measures. However, for the convenience of the reader we shall give an adapted proof.

Proof. We follow the local case [CM99, GZ03]. Let $u_n$ be the solution of the approximate problem $(P_n)$. Then, since $u_n$ lies in the domain of the operator $L_n$, we obtain

$$
\frac{d}{dt} \|u_n\|_{L^2}^2 = -2(L_n u_n, u_n) \leq -2\lambda_0^\mu \int_\Omega u_n^2(x,t) dx.
$$

The latter inequality is an immediate consequence of the finiteness of $\lambda_0^\mu$. Hence by Gronwall’s lemma we achieve the upper estimate

$$
\|u_n\|_{L^2} \leq \|u_0\|_{L^2(\Omega)} e^{-\lambda_0^\mu t}, \forall t > 0.
$$

Thus the sequence $(u_n)$ increases to a positive finite function $u$ for every $t > 0$ and a.e. $x \in \mathbb{R}^d$. Furthermore $u = 0$ on $\Omega^c$.

We are in position now to prove that $u$ solves the heat equation (1.1). Indeed, having Duhamel’s formula in hands for the $u_n$’s, we conclude by monotone convergence theorem that

$$
u(t,x) = e^{-tL_0} u_0(x) + \int_0^t \int_\Omega p_{t-s}(x,y) u(s,y) d\mu(y) ds,
$$

from which follows by the properties of the heat kernel $p_t$, namely $p_t(x,y) > 0$ on $\Omega \times \Omega$, that $u \in L^1_{loc}((0,T) \times \Omega, dt \otimes d\mu)$ and hence satisfies Duhamel’s formula

$$
u(t,x) = e^{-tL_0} u_0(x) + \int_0^t e^{-(t-s)L_0} (u \mu)(s,x) ds.
$$

Now utilizing the equation fulfilled by the $u_n$’s being solutions of the $P_n$’s we obtain by dominated convergence theorem, for every $0 \leq t_1 < t_2$ and every $\phi \in C^\infty_c([0,T) \times \mathbb{R}^d)$,

$$
\int_\Omega \phi_{t_1}^t dx + \int_{t_1}^{t_2} \int_\Omega u(t,x) (-\phi_t(t,x) + (-\Delta)^{\frac{\mu}{2}} \phi(t,x)) dx dt
= \int_{t_1}^{t_2} \int_\Omega u(t,x) \phi(t,x) d\mu(x) dt.
$$

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The latter identity together with formula (2.16) for the $u_n$ leads to
\[
\int_{t_1}^{t_2} E_\Omega(u_n, \phi) \to \int_{t_1}^{t_2} \int_{\Omega} u(t, x)(-\Delta)^{\alpha/2} \phi(t, x) \, dx \, dt.
\] (3.6)

Thus by dominated convergence theorem once again we conclude that
\[
\int_{t_1}^{t_2} E_\Omega(u, \phi) = \int_{t_1}^{t_2} \int_{\Omega} u(t, x)(-\Delta)^{\alpha/2} \phi(t, x) \, dx \, dt.
\] (3.7)

Therefore
\[
u \in L^1_{\text{loc}}([0, T), W^\alpha_{\text{loc}}).
\]

Remark 3.1. 1. The mapping
\[T_t : t \mapsto u(t), \ t > 0,\]
defines an $L^2$-semigroup which is exponentially bounded, i.e.:\[
\|T_t\|_{L^2(\Omega)} \leq e^{-\lambda_0^\alpha t}, \ \forall \ t > 0.
\] (3.8)
By the standard theory of semigroups, there is a unique selfadjoint operator associated to $T_t$.

2. The following equivalence holds true
\[
\lambda_0^\alpha > -\infty \iff \sup_n \|u_n(t)\|_{L^2} < \infty, \ \forall \ t > 0.
\] (3.9)

4 Nonexistence and blow up

In order to prove the nonexistence part we are going first, to establish an estimate for the integral $\int \ln u$, whenever $u$ is a nonnegative solution. Such estimate has an independent interest and is involved to derive regularity properties for the solutions. Its use in our context is inspired from the one corresponding to the Dirichlet-Laplacian (see [CM99, GZ03]).

Theorem 4.1. Assume that $u$ is a nonnegative solution of the heat equation (1.1). Then for all $0 < t_1 < t_2 < T$, and all $\Phi \in C^\infty_c(\Omega)$, we have
\[
\int_{\Omega} \Phi^2 \, d\mu - E_\Omega[\Phi] \leq \frac{1}{t_2 - t_1} \int_{\Omega} \ln \left(\frac{u(t_2)}{u(t_1)}\right) \Phi^2 \, dx.
\] (4.1)

Proof. Pick a function $\Phi \in C^\infty_c(\Omega)$. As $u_n$ is a solution of the heat equation and $\frac{\Phi^2}{u_n} \in W^{\alpha/2}_0(\Omega)$ (see Lemma 2.4), using the energy estimate of Lemma 2.4 we achieve
\[
\int_{\Omega} \Phi^2 \, d\mu = \int_{\Omega} (\partial_t u_n) \frac{\Phi^2}{u_n} \, dx + \int_{\Omega} (-\Delta)^{\alpha/2} (u_n) \frac{\Phi^2}{u_n} \, dx
\]
\[
= \int_{\Omega} (\partial_t u_n) \frac{\Phi^2}{u_n} \, dx + E_\Omega(u_n, \frac{\Phi^2}{u_n})
\]
\[
= \frac{d}{dt} \int_{\Omega} \ln(u_n) \Phi^2 \, dx + E_\Omega(u_n, \frac{\Phi^2}{u_n})
\]
\[
\leq \frac{d}{dt} \int_{\Omega} \ln(u_n) \Phi^2 \, dx + E_\Omega[\Phi].
\] (4.2)
Hence
\[ \int_{\Omega} \Phi^2 \, d\mu_n - \mathcal{E}_{\Omega}[\Phi] \leq \frac{d}{dt} \int_{\Omega} (\ln u_n)\Phi^2 \, dx. \quad (4.3) \]

We integrate between \( t_1 \) and \( t_2 \), and pass to the limit to obtain inequality (4.1), which finishes the proof. \( \square \)

**Theorem 4.2.** Assume that \( \lambda_0^{(1-\epsilon)}\mu = -\infty \) for some \( \epsilon > 0 \). Then the heat equation (1.1) has no nonnegative solutions. Moreover all nonnegative solutions blow up completely and instantaneously, i.e.: \( \lim_{n \to \infty} u_n(t, x) = \infty \) for every \( t > 0 \) and every \( x \in \Omega \).

**Remark 4.1.** Let us emphasize that there is a small gap between Theorem 3.1 and Theorem 4.2. Namely, the existence of a positive solution does not yield in general the finiteness of the bottom of the associated quadratic form. However, if the mapping
\[ t \mapsto u(t), \ t > 0 \]
satisfies the semigroup property and \( \|u(t)\|_{L^2} \) is exponentially bounded, then \( \lambda_0^n > -\infty \).

**Proof.** Assume that a nonnegative solution \( u \) exists. Relying on Lemma 2.3 we may and shall assume that \( u \) is the increasing limit of the \( u_n \)'s.

Let \( 0 < \rho \in C_c(\Omega) \) be such that \( \ln \rho \in L^p(\Omega) \) for any \( p > 1 \).

**Claim:** There exists at most one point \( t_1 \in (0, T) \) such that \( u(\cdot, t_1)\rho \in L^1(\Omega) \). Indeed, suppose that the contrary holds true. Then there exist \( t_1, t_2 \in (0, T) \) such that \( t_2 > t_1 \) and for \( i = 1, 2 \) \( u(\cdot, t_i)\rho \in L^1(\Omega) \).

For a small \( \eta > 0 \), set
\[ \Omega_\eta := \{ x \in \Omega : \delta(x) \geq \eta \}. \quad (4.4) \]

Note that \( u(x, t_i)\rho(x) \geq c > 0 \) when \( x \in \Omega_\eta \) for some \( c \) depending on \( t_i \) and \( \eta \). Thus using Jensen’s inequality twice we first get that \( \ln(u(\cdot, t_i)\rho) \in L^1(\Omega_\eta) \) and then
\[ \int_{\Omega_\eta} |\ln ((u(x, t_i)\rho(x)))|^p \, dx < \infty. \quad (4.5) \]

Now we decompose the set \( \Omega \setminus \Omega_\eta \) into:
\[ \Omega \setminus \Omega_\eta = S_1 \cup S_2 \quad (4.6) \]
\[ = \{ x \in \Omega_\eta^c : u(x, t_i)\rho(x) \geq m \} \cup \{ x \in \Omega_\eta^c : u(x, t_i)\rho(x) < m \}. \quad (4.7) \]

Observing that \( \ln^p s \) is a concave function of \( s \), for \( s \geq m \) with \( m \) sufficiently large and applying Jensen’s inequality on \( S_1 \), we derive that, for any \( p > 1 \),
\[ \int_{S_1} |\ln ((u(x, t_i)\rho(x)))|^p \, dx \leq \ln^p \left( \int_{S_1} u(x, t_i)\rho(x) \, dx \right) < \infty. \quad (4.8) \]
For $x \in S_2$, we have
\begin{align*}
m > u(x, t_i) \rho(x) &= \left[ \int_{\Omega} p_{t_i}(x, y) u_0(y) dy \right. \\
&\quad + \int_0^{t_i} \int_{\Omega} p_{t_i-s}(x, y) u(y, s) d\mu(y) ds \right] \rho(x) \\
&\geq \int_{\Omega} p_{t_i}(x, y) u_0(y) dy \rho(x) \\
&:= h(x, t_i) \rho(x), \ i = 1, 2. \tag{4.11}
\end{align*}

Thus
\begin{align*}
\ln m \geq \ln \left( u(x, t_i) \rho(x) \right) &= \ln \left( \frac{u(x, t_i) \rho(x)}{h(x, t_i) \rho(x)} \right) + \ln \left( h(x, t_i) \rho(x) \right) \\
&\geq \ln \left( h(x, t_i) \rho(x) \right), \tag{4.12}
\end{align*}

leading to the estimate
\begin{align*}
|\ln(u(x, t_i) \rho(x))| \leq |\ln m| + |\ln(h(x, t_i) \rho(x))|, \ i = 1, 2. \tag{4.13}
\end{align*}

As $\Omega$ has Lipschitz boundary it is known that (see [CKSI10])
\begin{align*}
\varphi_0 \geq C\delta^{\alpha/2}. \tag{4.14}
\end{align*}

Thus by Jensen’s inequality together with Lemma 2.5 we get that
\begin{align*}
\int_{S_2} |\ln[h(x, t_i) \rho(x)]| \rho dx < \infty. \tag{4.15}
\end{align*}

Therefore $u(\cdot, t_i) \in L^p(\Omega), \ i = 1, 2.$

Now we conclude that
\begin{align*}
\ln \frac{u(\cdot, t_2)}{u(\cdot, t_1)} &= \ln \left( u(\cdot, t_2) \rho(.) \right) - \ln \left( u(\cdot, t_1) \rho(.) \right) \\
&\in L^p(\Omega), \forall p > 1. \tag{4.16}
\end{align*}

On the other hand it is well known that, being in the space $L^p(\Omega)$ for $p > d/\alpha$, the function $\ln \frac{u(\cdot, t_2)}{u(\cdot, t_1)}$ is in fact in the Kato class and whence it satisfies the following: For any $r > 0$, there exists $C(r) > 0$ such that
\begin{align*}
\frac{1}{t_2 - t_1} \int_{\Omega} \ln \frac{u(x, t_2)}{u(x, t_1)} \Phi^2(x) dx \leq r\mathcal{E}(\Phi) + C(r) \int_{\Omega} \Phi^2(x) dx, \ \forall \Phi \in W_0^{\alpha/2}(\Omega). \tag{4.17}
\end{align*}

Having inequality (4.11) in hands, we achieve
\begin{align*}
\int_{\Omega} \Phi^2(x) d\mu(x) - \mathcal{E}(\Phi) \leq r\mathcal{E}(\Phi) + C(r) \int_{\Omega} \Phi^2(x), \ \forall \Phi \in \mathcal{C}_\mu. \tag{4.18}
\end{align*}

Therefore for every $\Phi \in \mathcal{C}_\mu$ such that $\int_{\Omega} \Phi^2 dx = 1$, we have
\begin{align*}
\frac{-C(r)}{1 + r} \leq \mathcal{E}[\Phi] - (1 + r)^{-1} \int_{\Omega} \Phi^2(x) d\mu(x). \tag{4.19}
\end{align*}
Whence
\[ \lambda_0^{(1+r)^{-1}} > -\infty, \forall r > 0, \] \hspace{1cm} (4.20)
which contradicts the assumption of the theorem and the claim is finally proved.

Given \( x \in \Omega \) and \( t \in (0, T) \), we take \( \rho = \rho(y) = p_\ast(x, y) \). Owing to the sharp estimate of Lemma 2.5 together with the lower bound (4.14), we conclude that \( \ln \rho \in L^p(\Omega) \) as was the case for \( h \).

On the other hand from the properties of the heat kernel for the Dirichlet fractional Laplacian, we have \( \rho(y) > 0 \) for any \( y \in \Omega \).

If there is no \( s \in (0, t_2] \) such that \( \rho(., s) \in L^1(\Omega) \), then by using Duhamel’s principle, we have
\[
\begin{align*}
\int_\Omega p_{\frac{t}{2}}(x, y)u(y, s)dy &= \int_\Omega \rho(y)u(y, s)dy = \infty.
\end{align*}
\] \hspace{1cm} (4.21)

In case \( s \in (0, \frac{t}{2}] \) is the only point such that \( \rho(., s) \in L^1(\Omega) \), making use of Duhamel’s principle once again, we obtain
\[
\begin{align*}
\int_\Omega p_{\frac{t}{2}}(x, y)u(y, s)dy &= \int_\Omega \rho(y)u(y, s)dy = \infty.
\end{align*}
\] \hspace{1cm} (4.22)

From the intrinsic ultracontractivity property for the semigroup \( e^{-tL_0} \), we derive that for very \( x \in \Omega \), every \( r > 0 \) such that \( B_r(x) \subset \Omega \) and every \( z \in B_r(x) \), and every small \( \gamma > 0 \), there is a constant \( = c(t, \gamma, r) \) such that
\[
p_{\frac{t + \gamma r}{2}}(z, y) \geq C p_{\frac{t}{2}}(x, y), \forall y \in \Omega.
\] \hspace{1cm} (4.23)

Indeed
\[
\begin{align*}
p_{\frac{t + \gamma r}{2}}(z, y) & \sim \varphi_0(z)\varphi_0(y) \geq c \frac{\inf_{B_r(x)} \varphi_0}{\sup_{B_r(x)} \varphi_0} \varphi_0(x)\varphi_0(y) \\
& \sim \varphi_{\frac{t}{2}}(x, y).
\end{align*}
\] \hspace{1cm} (4.24)

Making use of the latter claim we achieve
\[
\begin{align*}
u(z, \frac{t + s + \gamma r}{2}) & \geq e^{-(\frac{t + \gamma r}{2})L_0}u(z, \frac{s}{2}) = \int_\Omega p_{\frac{t + \gamma r}{2}}(z, y)u(y, \frac{s}{2})dy \\
& \geq c \int_\Omega p_{\frac{t}{2}}(x, y)u(y, \frac{s}{2})dy \geq \int_\Omega \rho(y)u(y, \frac{s}{2})dy = \infty.
\end{align*}
\] \hspace{1cm} (4.25)
By the semigroup property (or Duhamel’s formula), we have
\[
    u(x, t) \geq \int_{\Omega} p_{t-\frac{s}{2}}(x, y) u(y, \frac{s}{2}) dy
    = \int_{\Omega} \int_{\Omega} p_{t-\frac{s}{2}+\gamma}(x, z) p_{\frac{s}{2}}(z, y) u(y, \frac{s}{2}) dz dy
    = \int_{\Omega} p_{t-\frac{s}{2}+\gamma}(x, z) \left[ \int_{\Omega} p_{\frac{s}{2}}(z, y) u(y, \frac{s}{2}) dy \right] dz
    = \int_{\Omega} p_{t-\frac{s}{2}+\gamma}(x, z) u(z, \frac{t+s+\gamma r}{2}) dz
    \geq \int_{B_r(x)} p_{t-\frac{s}{2}+\gamma}(x, z) u(z, \frac{t+s+\gamma r}{2}) dz = \infty.
\]
(4.26)

Since \((x, t)\) is arbitrary, this proves the blow-up an the proof is finished. \(\square\)

5 Examples

In this section we provide some examples that support the already developed theory. We begin by a positive criteria ensuring existence of nonnegative solutions.

For every Borel measurable functions on \(\mathbb{R}^d\), we set
\[
    J(f, g) := A(d, \alpha) \int \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \, dx \, dy,
\]
provided \(\int \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \, dx \, dy < \infty\).

Remark 5.1. If \(f\) and \(w\) are two functions such that \(wf \in W^{\alpha/2}(\mathbb{R}^d)\) then \(J(w, wf^2)\) is well defined in the sense indicated above. Indeed, an elementary computation yields
\[
    (w(x) - w(y))(w(x)f^2(x) - w(y)f^2(y)) \leq (w(x)f(x) - w(y)f(y))^2,
\]
yielding that \(J(w, wf^2)\) makes sense and
\[
    J(w, wf^2) \leq E[w].
\]
(5.3)

Proposition 5.1. Assume that there is a real constant \(c\) and a quasi continuous function \(w > 0, \text{q.e.}\) such that
\[
    J(w, f) - \int f w \, d\mu \geq c \int f w \, dx,
\]
(5.4)

for every positive function \(f\) such that \(J(w, f)\) makes sense. Then \(\lambda_0^\mu \geq c\).

Proof. From the very definition of \(\lambda_0^\mu\) one has
\[
    \lambda_0^\mu = \inf \left\{ \frac{E[wf] - \int f^2 w^2 \, d\mu}{\int f^2 w^2 \, dx}, \, f : \, wf \in \mathcal{C}^\mu \setminus \{0\} \right\}.
\]
(5.5)
As for every $f \in C^\mu$ such that $wf \in C^\mu$, $J(w, wf^2)$ makes sense we plug the function $wf^2$ in inequality (5.4) and obtain

$$J(w, wf^2) - \int f^2 w^2 \, d\mu \geq c \int f^2 w^2 \, dx,$$

(5.6)

Having inequality (5.3) in hands we achieve

$$\mathcal{E}_\Omega[wf] - \int f^2 w^2 \, d\mu \geq c \int f^2 w^2 \, dx, \quad \forall f: wf \in C^\mu,$$

(5.7)

which finishes the proof.

\[ \square \]

**Remark 5.2.**

1. Observing that

$$- \infty < \lambda^\mu_0 \iff \inf \left\{ \frac{\mathcal{E}_\Omega[f] + \beta \int f^2 \, dx - \int f^2 \, d\mu}{\int f^2 \, dx}, \quad f: \in C^\mu \setminus \{0\} \right\} > -\infty,$$

(5.8)

for some, and hence every $\beta \geq 0$, we deduce that condition (5.3) is fulfilled if in particular the operator

$$(-\Delta)^{\alpha/2} + \beta - \mu,$$

(5.9)

has a superharmonic function for some $\beta \geq 0$.

2. Owing to the first part of the latter observation together with [BBA12, Theorem 3.6], the converse of Proposition 5.1 is true as well.

**Example 5.1.** Let $w > 0 - a.e.$ be such that $w$ is superharmonic, i.e. $w$ defines a tempered distribution and

$$(-\Delta w)^{\alpha/2} \geq 0,$$

in the sense of distributions.

(5.10)

Choose the measure $\mu = \frac{(-\Delta w)^{\alpha/2}}{w}$. Then $w$ is harmonic with respect to $(-\Delta)^{\alpha/2} - \mu$. Hence according to Remark 5.2–1., the heat equation related to $L_0 - \mu$ has a nonnegative solution.

Finally we give some examples where nonexistence of nonnegative solutions occurs.

**Lemma 5.1.** Assume that there is $\lambda > 1$ and a sequence of balls $B_k \subset \Omega$ such that their Lebesgue volumes $|B_k| \downarrow 0$ and a sequence $(\phi_k) \subset C_\mu$ with $\text{Supp} \phi_k \subset B_k$, $\int \phi_k^2(x) \, dx = 1$, $\forall k$ such that

$$\int \phi_k^2(x) \, d\mu \geq \lambda \mathcal{E}_\Omega[\phi_k], \quad \forall k.$$

(5.11)

Then the heat equation (1.1), has no nonnegative solution.
Proof. As a consequence of the condition, there is \( \lambda' > 1 \) and \( \epsilon \in (0, 1) \) such that

\[
(1 - \epsilon) \int \phi_k^2(x) \, d\mu \geq \lambda' \mathcal{E}_\Omega[\phi_k], \quad \forall \, k.
\]  (5.12)

Thus

\[
- \mathcal{E}_{B_k}[\phi_k] + (1 - \epsilon) \int \phi_k^2(x) \, d\mu \geq (\lambda' - 1) \mathcal{E}_\Omega[\phi_k] \geq c |B_k|^{-\alpha/d}, \quad \forall \, k.
\]  (5.13)

Hence \( \lambda_0^{(1-\epsilon)\mu}(B_k) \leq -c |B_k|^{-\alpha/d} \to -\infty \), as \( k \to \infty \). Now observing that

\[
\lambda_0^{(1-\epsilon)\mu}(B_k) \geq \lambda_0^{(1-\epsilon)\mu},
\]  (5.14)

yields the result. \( \square \)

**Example 5.2.** Hardy potential with interior singularity.

Let \( \alpha < \min(2, d) \), and \( \Omega \subset \mathbb{R}^d \) an open bounded subset with Lipschitz boundary and containing 0. Set \( V_\epsilon(x) = \frac{\alpha}{|x|^{d-\alpha}}, \ x \neq 0 \) and \( c \geq 0 \). If \( 0 \leq c \leq c^* := \frac{2^{d-1}\Gamma(\frac{d+\alpha}{2})}{\Gamma^2(\frac{d-\alpha}{2})} \), then the heat equation associated to \( L_{V_\epsilon} \) has a nonnegative solution (owing to Hardy’s inequality). However if \( c > c^* \) then the heat equation has no nonnegative solution. Indeed, owing to the sharpness of the Hardy’s inequality

\[
\int_{\Omega} \frac{f^2(x)}{|x|^\alpha} \, dx \leq \frac{1}{c^*} \mathcal{E}_\Omega[f], \quad \forall \, f \in W^{\alpha/2,2}_0(\Omega),
\]  (5.15)

there is \( \lambda > 1 \) and a sequence \( (\phi_k) \subset \mathcal{C}_V \) such that

\[
\int \phi_k^2(x) V(x) \, dx \geq \lambda \mathcal{E}_\Omega[\phi_k], \quad \forall \, k.
\]  (5.16)

Now an elementary computation shows that the sequence \( \psi_k \) defined by \( \psi_k(x) = \phi_k(kx) \) and \( B_k := B_{R/k} \) fulfills the conditions of Lemma 5.1.

**Example 5.3.** Hardy potential with boundary singularity.

Assume that the following Hardy’s inequality holds true

\[
\int \frac{f^2(x)}{\delta^\alpha(x)} \, dx \leq \frac{1}{\kappa} \mathcal{E}_\Omega[f], \quad \forall \, f \in W^{\alpha/2,2}_0(\Omega),
\]  (5.17)

with sharp constant \( 1/\kappa^* \). Take \( V_\kappa = \frac{\kappa}{\delta^\alpha(x)}, \ \kappa \geq 0 \). Arguing as in Example 5.2, we conclude that for \( \kappa > \kappa^* \), the related heat equation has no nonnegative solution, whereas it has for \( \kappa \leq \kappa^* \).

According to [CS03, Corollary 2.4], inequality (5.17) is satisfied if \( d \geq 2 \) and \( \alpha \neq 1 \).
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