Unimodular Constraint on global scale Invariance

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Abstract: We study a model with global scale invariance within the framework of unimodular gravity. The global scale invariant gravitational action which follows the unimodular general coordinate transformations is considered without invoking any scalar field. This is generalization of conformal theory described in the Ref. [1]. The possible solutions for the gravitational potential under static linear field approximation are discussed. The new modified solution has additional corrections to the Schwarzschild solution which describe the galactic rotational curve. A comparative study of unimodular theory with conformal theory is also presented. Furthermore, the cosmological solution is studied and it is shown that the unimodular constraint preserve the de Sitter solution explaining the dark energy of the universe.

1 Introduction

Surprisingly, we live in the era of late time acceleration of universe [2,3]. The acceleration in expansion of universe offers the possibility of new physics of the unknown component “dark energy”. The dark matter, whose physical nature is only partially known, is also one of the interesting subjects of research in cosmology. Dark matter was introduced by Fritz Zwicky in year 1933 [4,5] to account for evidence of missing mass in Coma cluster. The more interesting issue to notice is that dark matter and dark energy dominate the energy density of the universe. The observed dark energy and dark matter contribute approximately 72.8% and 22.7% respectively to the total energy content of the universe. There are number of scalar field theories such as Quintessence, K-essence, Chaplygin gas model and theory of modified gravity such as \( f(R) \), DGP model, etc. to describe the dark side of universe. In the recent literature [6–8], authors discuss the absence of the dark matter in the vicinity of solar system. The dark matter might be internal property of the space and hence might be explained by the modified theory of gravity [1,9–14]. In this paper, such a theory of modified gravity is considered which has global scale invariance along with the unimodular constraint to explain the galactic rotational curve and acceleration of universe expansion. The scale symmetry prevents any dimensionful parameter in the action and hence might give a possible explanation for the cosmological constant problem [15–21]. This symmetry is broken if the theory contains any dimensionful parameters, such as particle masses, cosmological constant, gravitational constant etc.. Although scale invariance is generally believed to be anomalous, it is possible to maintain this symmetry in the full quantum theory if the symmetry is broken by a soft mechanism [18,21]. It has been shown [20,23] that scale invariance has its other advantages in explaining the dark sector of the universe. The local scale invariance explains the current era of the universe [22], favoring the \( \Lambda CDM \) model. In [22,23] the symmetry breaking mechanism of local scale invariance generates the dark energy and vector field which acts as dark matter field. The global scale transformation is given by

\[
x^\mu \rightarrow \Lambda x^\mu
\]

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where $\Lambda$ is a constant parameter. The above transformations make the action like $\int R^2 \sqrt{-g} d^4x$ or $\int \phi^2 R \sqrt{-g} d^4x$ invariant. The theories of higher order invariants in the action were initially started by Weyl in 1919 and by Eddington in 1923. One of the advantage of $f(R)$ theory is it describes the early universe [24–26]. Further, it also explains the late time acceleration [14, 27] in the expansion of universe and might be an alternative for the dark matter [1]. The theory follows the principle of covariance similar as the Einstein-Hilbert action. It modifies the Einstein equation as well as the gravitational potential. The only quadratic terms of the curvature scalar in the action preserves the global scale invariance. It is reasonable to take only quadratic terms of the curvature scalar in the action, since the solution could include the Schwarzschild solution and the corrections in addition [1]. Using the Gauss-Bonnet identity, we may write such global scale invariant action as [28]

$$-\alpha \int (R_{\mu\nu} R_{\mu\nu} + \gamma R^2) \sqrt{-g} d^4x$$ (1)

In this paper we generalize the conformal theory [1] by imposing the unimodular constraint. Conformal theory is one of the special case of (1), where $\gamma = -1/3$ [1]. In subsection (1.1), the field equation of metric is derived. In section 2, the corresponding equation for the unimodular theory is given. The field equation is solved under static linear field approximation and corresponding galactic rotational curves are discussed in section 2. The cosmological solution with unimodular constraint is discussed in section 3. The last section 4 contains the discussion and conclusions.

### 1.1 Field Equation

The variation of action (1) gives the field equation as

$$W^1_{\mu\nu} + W^2_{\mu\nu} = 0 .$$ (2)

Here $W^1_{\mu\nu}$ and $W^2_{\mu\nu}$ are the terms corresponding to the variation of $R^2$ and $R_{\mu\rho} R^{\rho\sigma}$ and these are given as

$$W^1_{\mu\nu} = -\frac{\gamma}{2} R^2 g_{\mu\nu} + 2\gamma \left[ g^\alpha\beta R_{\alpha\beta ; \mu
u} - R_{;\mu,\nu} \right] + 2\gamma RR_{\mu\nu},$$

$$W^2_{\mu\nu} = -\frac{1}{2} R_{\mu\rho} R^{\rho\sigma} g_{\mu\nu} + 2R_{\mu\rho} R_{\nu\sigma} g^{\rho\sigma} - 2(R^{\alpha\beta})_{;\mu,\nu} + (R_{\mu\nu})^{;\mu} + (R^{\alpha\beta})_{;\beta,\alpha} g_{\mu\nu}$$ (3)

respectively.

### 2 Unimodular Gravity

Unimodular gravity was introduced in [29,30] and has been reviewd in [31]. The theory is subclass of general theory of relativity but with a constraint in addition, i.e., the determinant of the metric is not dynamical: $g_{\mu\nu} \delta g^{\mu\nu} = 0$. The motivation of the unimodular gravity is to solve the cosmological constant problem as we don’t have any such term in the action. Further, in the reference [32], authors discuss dynamics of expansion of universe with the unimodular theory of gravity taking the dynamical part of determinant of metric as a separate scalar field. However, in this paper, any scalar field is not considered. The condition $g_{\mu\nu} \delta g^{\mu\nu} = 0$ modifies the Einstein equation as
following [30][31],

\[ R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = \kappa \left( T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right), \tag{4} \]

where \( \kappa \) is coupling constant, \( T_{\mu\nu} \) is energy momentum tensor of source field and \( T \) is its trace. The Eq. 4 is traceless part of the Einstein equation. The variation of action 1 gives following field equation

\[ W_{\alpha\beta} = -\frac{1}{2} g_{\alpha\beta} \left( \gamma R^2 + R_{\mu\rho} R^{\rho\sigma} \right) + 2\gamma \left[ g^{\mu\nu} (R)_{;\mu\nu} g_{\alpha\beta} - (R)_{;\alpha\beta} \right] + 2R_{\alpha\nu} R_{\beta\rho} g^{\mu\nu} \]

\[ - 2 (R_{\alpha\nu})_{;\alpha} g_{\mu\beta} + (R_{\mu\nu})_{;\lambda} g_{\nu\alpha} g_{\alpha\beta} + (R_{\nu\mu})_{;\nu} g_{\alpha} g_{\alpha\beta} + 2\gamma R R_{\alpha\beta} = 0. \tag{5} \]

The same procedure of constraint of unimodular gravity over the action given in Eq. 1 gives the following field equation,

\[ W^{uni}_{\alpha\beta} = -\frac{1}{2} g_{\alpha\beta} \left( \gamma R^2 + R_{\mu\rho} R^{\rho\sigma} \right) + 2\gamma \left[ g^{\mu\nu} (R)_{;\mu\nu} g_{\alpha\beta} - (R)_{;\alpha\beta} \right] + 2R_{\alpha\nu} R_{\beta\rho} g^{\mu\nu} \]

\[ - 2 (R_{\alpha\nu})_{;\alpha} g_{\mu\beta} + (R_{\mu\nu})_{;\lambda} g_{\nu\alpha} g_{\alpha\beta} + (R_{\nu\mu})_{;\nu} g_{\alpha} g_{\alpha\beta} + 2\gamma R R_{\alpha\beta} - \frac{W}{4} g_{\alpha\beta} = 0, \tag{6} \]

where \( W = W_{\alpha\beta}^{\alpha\beta} \) is trace of tensor \( W_{\alpha\beta} \). The Eq. 6 is the traceless part of Eq. 5. Here \( g_{\mu\nu} \delta g^{\mu\nu} = 0 \) is used, i.e., the action does not have any constant term.

### 2.1 Vacuum Solution for the Conformal Theory

For \( \gamma = -1/3, 1/B \) of \( W^{rr} \) component of Eq. 6 gives the following equation

\[ \frac{B'B'''}{6} - \frac{B'^2}{12} - \frac{BB'''}{3r} + \frac{3B'}{3r} - \frac{BB''}{3r^2} - \frac{B^2}{3r^3} + \frac{2BB'}{3r^3} - \frac{B^2}{3r^3} + \frac{1}{3r^4} = 0, \tag{7} \]

where the metric is given by,

\[ ds^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega. \tag{8} \]

The exact vacuum of Eq. 7 may be written as [1]

\[ B(r) = 1 - \frac{C_1(2 - 3C_1C_2)}{r} - 3C_1C_2 + C_2r - C_3r^2, \tag{9} \]

where, \( C_1, C_2 \) and \( C_3 \) are constants. Now in next subsection we generalize this for general \( \gamma \).

### 2.2 Vacuum Solution for Unimodular Gravity

In this subsection, we solve for the gravitational potential with unimodular gravity considering the line element [9]. \(-(1/B)\) of \( t-t \) component, \(-B\) of \( r-r \) component and \( 1/r^2 \) of \( \theta-\theta \) component
of field Eq. 6 are given by

\[
(1 + 2\gamma) \frac{B'^2}{4} - \gamma \frac{B^2}{r^2} + (1 + 4\gamma) \frac{B'B''}{r} - \frac{(1 + 2\gamma)}{r^4} - (1 + 4\gamma) \frac{B'^2}{r^4} + (2 + 6\gamma) \frac{B'}{r^4} - \frac{B'B''}{2} - (2 + 3\gamma) \frac{BB''}{r} - \gamma \frac{BB''}{2} - (1 + \gamma) \frac{BB''}{2} + 2(1 + 3\gamma) \frac{B'}{r^3} = 0 ,
\]

(10)

\[
- (1 + 2\gamma) \frac{B'^2}{4} + \gamma \frac{B'^2}{r^2} - (1 + 4\gamma) \frac{B'B''}{r} + \frac{(1 + 2\gamma)}{r^4} - (7 + 20\gamma) \frac{B^2}{r^4} + (6 + 18\gamma) \frac{B'}{r^4} + (2 + 4\gamma) \frac{BB'}{r^3} - \gamma \frac{B'B''}{2} - (2 + 5\gamma) \frac{BB''}{r} + (4 + 13\gamma) \frac{BB''}{2} - (1 + 3\gamma) \frac{BB''}{2} - 2(1 + 3\gamma) \frac{B'}{r^3} = 0
\]

(11)

respectively. Now, considering linear approximation, i.e., \( B(r) \approx 1 + \phi(r) \), we have following three equations

\[
- \frac{2\gamma}{r^4} + \frac{2\gamma}{r^3} - \frac{\gamma}{r^2} - (2 + 3\gamma) \frac{\phi''}{r} - \frac{(1 + \gamma)}{2} \phi''' = 0 ,
\]

(13)

\[
- \frac{(8 + 22\gamma)}{r^4} - \frac{2\gamma}{r^3} + \frac{(4 + 13\gamma)}{r^2} - (2 + 5\gamma) \frac{\phi''}{r} - \frac{(1 + 3\gamma)}{2} \phi''' = 0
\]

(14)

and

\[
- \frac{(4 + 10\gamma)}{r^4} - \frac{2\gamma}{r^3} + \frac{(2 + 7\gamma)}{r^2} - \frac{\gamma}{r} \phi'' - \frac{\gamma}{2} \phi''' = 0
\]

(15)

respectively. The solution of Eq. 13 is given by

\[
\phi = C_1 r^{\frac{2\gamma}{1 + \gamma}} + \frac{C_2}{r} + C_3 r + C_4 r^2 .
\]

(16)

Plugging this solution, either in Eq. 14 or 15, we get the same constraint over the constants which is given as follows

\[
(1 + 3\gamma) \left[ -(6\gamma^3 + \gamma^2 - 5\gamma - 2)C_1 + 2(1 + \gamma) r^{1 + \frac{2\gamma}{1 + \gamma}} C_3 \right] = 0 .
\]

(17)
Now, we have different solution for allowed values of $\gamma$ and other constants. The constraint Eq. 17 gives one of the case where $\gamma = -1/3$. For this value we get

$$\phi = \frac{C_2}{r} + (C_1 + C_3)r + C_4r^2,$$

(18)

which is same solution as in Eq. 9 for the conformal theory. However, for $\gamma \neq -1/3$, we have $C_3 = 0$ and

$$6\gamma^3 + \gamma^2 - 5\gamma - 2 = 0,$$

(19)

which implies

$$\gamma = \frac{2}{3}, \gamma = -\frac{1}{2} \text{ and } \gamma = 1.$$

(20)

For these values of $\gamma$ the solutions are given by

$$\phi = C_1r^4 + \frac{C_2}{r} + C_4r^2,$$

(21)

$$= C_2r^2 + (C_1 + C_4)r^2;$$

(22)

$$= \frac{(C_1 + C_2)}{r} + C_4r^2$$

(23)

respectively. The solution 22 or 23 with the data of galactic rotational curve 33,34 for the Milky Way galaxy is plotted in Fig 1. For the large scale, data is taken from the simulation II given in the table (3) of the Ref. 33 and for small scale the data is taken from the table (2) of the Ref. 34.

The effective velocity of star may be written as

$$v^2 = \frac{f}{2} \left(r \frac{\partial \phi}{\partial r}\right),$$

(24)

where $f = 9 \times 10^6$ to make velocity unit as (100Km/sec). For the best fit, the values of constants are

$$C_2 = -7.39 \times 10^{-6} \text{ Kpc and } (C_1 + C_4) = 1.67 \times 10^{-10} \text{ Kpc}^{-2} \text{ for solution 22,}$$

$$C_1 + C_2 = -7.39 \times 10^{-6} \text{ Kpc and } C_4 = 1.67 \times 10^{-10} \text{ Kpc}^{-2} \text{ for solution 23.}$$

(25)

The plot is shown by the dotted line. The further plot of the solution 21 with solid line is shown in Fig. 1. The values of constants for this case are as following

$$C_1 = -5.16 \times 10^{-14} \text{ Kpc}^{-4}, \ C_2 = -7.22 \times 10^{-6} \text{ Kpc} \text{ and } C_4 = 3.78 \times 10^{-10} \text{ Kpc}^{-2}.$$ 

(26)

The plot for the conformal theory is also shown with dashed line, where the gravitational potential is given by the Eq. 9 and for the best fit the values of the constants are given by

$$C_1 = 2.6526 \times 10^{-6} \text{ Kpc, } \ C_2 = 5.0460 \times 10^{-8} \text{ Kpc}^{-1} \text{ and } C_3 = 4.1366 \times 10^{-10} \text{ Kpc}^{-2}.$$ 

(27)

The values of $\chi^2_{\text{min}}$ per degree of freedom for the best fit for the solution 9, 21 and 22 are given by 3.19, 5.54 and 6.15 respectively. However, the solution 21 gives the best fit for the scale
Figure 1: The variation of velocity with distance $r$. The dotted curve is plot of solution given by (22) or (23) whereas solid curve is for the solution (24). The dashed curve is for the solution (9). Data for Milky Way Galaxy is shown with the error bar.

> $15 \text{Kpc}$ as shown in the Fig. (1). For the large scale $> 15 \text{Kpc}$, we find $\chi^2_{\text{min}}$ for the conformal theory; Eq. (9) as 2.25 whereas for the case of unimodular gravity; Eq. (22) and (21) it is as 1.09 and 0.77 respectively. Hence for the large scale, the theory of unimodular gravity describes the galactic rotational curve with the best fit.

3 Cosmological Solution

It is known to us that Gauss-Bonnet action explains acceleration in the expansion of the universe $[35–37]$. Further, in the modified theory of gravity $f(R) = R^2$, we have exact de-Sitter solution $[14]$ for the vacuum. In this section, we test it explicitly as now the action has the unimodular constraint in addition. For the FRW metric $[-1, a^2, a^2, a^2]$, where $a$ is scale factor of the universe, Eq. (8) gives the same equation for $0 - 0$ and $i - j$ components and it is given by

$$-(6 + 18\gamma) \left( \frac{a'}{a} \right)^4 + (9 + 27\gamma) \frac{a'' a'''}{a^3} - (3 + 9\gamma) \left( \frac{a''}{a} \right)^2 + (1 + 3\gamma) \frac{a''}{a^2} - (1 + 3\gamma) \frac{a'''}{a} = 0. \ (28)$$
The Eq. (28) may be written as independent of the parameter $\gamma$ as

$$-6 \left( \frac{a'}{a} \right)^4 + 9 \frac{a''a'''}{a^3} - 3 \left( \frac{a''}{a} \right)^2 + \frac{a'}{a^2} - \frac{a''''}{a} = 0.$$  \hfill (29)

Looking over Eq. (29), one may conclude for the exact de-Sitter solution,

$$a = a_0 e^{H_0 t},$$  \hfill (30)

which explain the acceleration in the expansion of universe, where $H_0$ is Hubble constant. Hence, the de Sitter solution satisfies both the conformal theory and the theory of unimodular gravity.

### 4 Discussion and Conclusions

A scale invariant model of higher order invariant in the action is presented. The unimodular constraint on the theory is also considered. Scale invariance allows only quadratic terms of curvature scalar in the action, whereas consideration of unimodular theory in addition constrain on the values of the parameter of the resulting theory. It is shown that for the parameter $\gamma = -1/2$ and 1, the solution of the gravitational potential includes the Schwarzschild solution as well as the term corresponding to the integration constant. The solution for this case explains the galactic rotational curve, but the corresponding gravitational field increases as distance increases whereas for $\gamma = -2/3$, the solution has one more term proportional to $r^4$ so that the velocity or corresponding gravitational field decreases after $\sim 42$ Kpc. Furthermore, the solution of conformal theory is recovered for $\gamma = -1/3$. The conformal solution has a lighter bump at $\sim 30$ Kpc. Hence, the unimodular theory of gravity has good behavior for the large scale rather than that of conformal theory. The proper scale invariant matter source term in the action might describe the rotational curve for the low range. We will proceed it further in the future publication. The theory is interesting as it does not require the dark matter which has not been observed in the solar neighborhood so far. Furthermore, the de Sitter solution is also obtained for the considered theory explaining the dynamics of current era.

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