HOLOMORPHIC QUILLEN DETERMINANT LINE BUNDLES ON INTEGRAL
COMPACT KÄHLER MANIFOLDS

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Abstract. We show that any compact Kahler manifold with integral Kahler form, parametrizes
a natural holomorphic family of Cauchy-Riemann operators on the Riemann sphere such that the
Quillen determinant line bundle of this family is isomorphic to a sufficiently large tensor power of
the holomorphic line bundle determined by the integral Kahler form. We also establish a symplectic
version of the result. We conjecture that an equivariant version of our result is true.

1. Introduction

In geometric quantization, given a type (1,1) integral form ξ on a compact Kähler manifold,
there is a holomorphic line bundle L on the manifold with connection and curvature equal to ξ.
For positive integral (1,1) forms, we prove the following partial refinement.

Theorem 1. Any compact Kähler manifold M with integral Kähler form ω, parametrizes a natural
holomorphic family of Cauchy-Riemann operators \{∂_z : z ∈ M\} on CP^1 such that the Quillen
determinant line bundle det(∂) ∼= L⊗k as holomorphic line bundles, where L is the holomorphic
line bundle determined by ω, for some sufficiently large k.

The strategy of the proof is as follows. We first establish the theorem for complex projective
spaces CP^N for all integers N and for k = 1. This is achieved by viewing CP^N as the moduli space
of N-vortices on CP^1. The position of each N-vortex z determines a Cauchy-Riemann operator ∂_z
on CP^1, and the Quillen determinant line bundle (see [5]), det(∂) is isomorphic as a holomorphic
line bundle to the hyperplane bundle on CP^N, see [2]. The next step is to apply the Kodaira
embedding theorem to establish the theorem in general, see [3]. In §4 we prove a symplectic
analogue of our theorem, established using Gromov’s embedding theorem. In §5 we conjecture
that an equivariant version of our result is true.

2010 Mathematics Subject Classification. 58J52, 14D21, 53D50, 32Q15, 53D30.

Key words and phrases. holomorphic Quillen determinant line bundles, vortex moduli space, Kodaira embedding
theorem, Gromov’s embedding theorem.

Acknowledgments. The first author thanks the SYM project grant for support and the second author thanks the
Australian Research Council for support and the Harishchandra Research Institute for hospitality during the period
when this work was completed. Both authors thank M.S. Narasimhan for pointing out the error in the statement of
Kodaira’s embedding theorem in the previous version of the paper.
2. Moduli space of N-vortices on a sphere and determinant line bundle

2.1. **Vortex equations.** The vortex equations are as follows. Let Σ be a compact Riemann surface (in our case the sphere) and let ω = \( \frac{1}{2\pi} h^2 dz \wedge d\bar{z} \) be the purely imaginary volume form on it, (i.e. \( h \) is real). Let \( A \) be a unitary connection on a principal \( U(1) \)-bundle \( P \) i.e. \( A \) is a purely imaginary valued one form i.e. \( A = A^{(1,0)} + A^{(0,1)} \) such that \( A^{(1,0)} = -\bar{A}^{(0,1)} \). Let \( L \) be a complex line bundle associated to \( P \) by the defining representation. Let \( \Psi \) be a section of \( L \), i.e. \( \Psi \in \Gamma(\Sigma, L) \) and \( \bar{\Psi} \) be a section of its dual, \( \bar{L} \). There is a Hermitian metric \( H \) on \( L \), i.e. the inner product \( \langle \Psi_1, \Psi_2 \rangle_H = \Psi_1 H \bar{\Psi}_2 \) is a smooth function on \( \Sigma \). (Here \( H \) is real).

The pair \((A, \Psi)\) will be said to satisfy the vortex equations if

\[
\begin{align*}
(1) & \quad F(A) = \frac{1}{2} (1 - |\Psi|^2_H) \omega, \\
(2) & \quad \partial_A \Psi = 0,
\end{align*}
\]

where \( F(A) \) is the curvature of the connection \( A \) and \( d_A = \partial_A + \bar{\partial}_A \) is the decomposition of the covariant derivative operator into \((1, 0)\) and \((0, 1)\) pieces. Let \( S \) be the space of solutions to (1) and (2). There is a gauge group \( G \) acting on the space of \((A, \Psi)\) which leaves the equations invariant. We take the group \( G \) to be abelian and locally it looks like Maps(\( \Sigma, U(1) \)). If \( g \) is an \( U(1) \) gauge transformation then \((A_1, \Psi_1) \) and \((A_2, \Psi_2)\) are gauge equivalent if \( A_2 = g^{-1}dg + A_1 \) and \( \Psi_2 = g^{-1}\Psi_1 \). Taking the quotient by the gauge group of \( S \) gives the moduli space of solutions to these equations and is denoted by \( \mathcal{M} \).

\[ i \int_{\Sigma} F(A) = 4\pi N \] where \( N \) is an integer called the vortex number.

There is a theorem by Taubes (cf.\[11\]) and Bradlow \[2\], which says that the moduli space of vortices is parametrised by the zeroes of \( \Psi \). Thus the moduli space is the symmetric product, \( \text{Sym}^N(\Sigma) \). When the Riemann surface is \( \mathbb{C}P^1 \), the moduli space is complex projective space, \( \text{Sym}^N(\mathbb{C}P^1) = \mathbb{C}P^N \).

2.2. **The metric and symplectic form.** Let \( \mathcal{A} \) be the space of all unitary connections on \( P \) and \( \Gamma(\Sigma, L) \) be sections of \( L \). Let \( \mathcal{C} = \mathcal{A} \times \Gamma(\Sigma, L) \) be the affine space on which equations (1) and (2) are imposed. Let \( p = (A, \Psi) \in \mathcal{C}, X = (\alpha_1, \beta), Y = (\alpha_2, \eta) \in T_p \mathcal{C} \equiv \Omega^1(\Sigma, i\mathbb{R}) \times \Gamma(\Sigma, L) \) i.e. \( \alpha_i = \alpha_i^{(0,1)} + \alpha_i^{(1,0)} \) such that \( \alpha_i^{(0,1)} = -\bar{\alpha_i}^{(1,0)} \), \( i = 1, 2 \). On \( \mathcal{C} \) one can define a metric

\[
\mathcal{G}(X, Y) = \int_{\Sigma} \ast \alpha_1 \wedge \alpha_2 + i \int_{\Sigma} \text{Re} < \beta, \eta >_H \omega
\]

and an almost complex structure \( \mathcal{I} = \begin{bmatrix} * & 0 \\ 0 & i \end{bmatrix} : T_p \mathcal{C} \to T_p \mathcal{C} \) where \( * : \Omega^1 \to \Omega^1 \) is the Hodge star operator on \( M \) such that \( *\alpha^{1,0} = -i\alpha^{1,0} \) and \( *\alpha^{0,1} = i\alpha^{0,1} \) (i.e. it makes \( A^{0,1} \) the holomorphic coordinate on \( \mathcal{A} \)).
It is easy to check that \( G \) is positive definite. In fact, if \( \alpha_1 = \alpha^{(1,0)} + \alpha^{(0,1)} = adz - \overline{a}d\overline{z} = i(\dot{A}_1 dx + A_2 dy) \) is an imaginary valued 1-form, \( *\alpha_1 = -i(adz + \overline{a}d\overline{z}) \) and

\[
G(X, X) = \int_\Sigma 4|a|^2 dx \wedge dy + \int_\Sigma |\beta|^2_H h^2 dx \wedge dy
\]

where \( \omega = \frac{1}{2} h^2 dz \wedge d\overline{z} = -ih^2 dx \wedge dy. \)

We define

\[
\Omega(X, Y) = \frac{1}{2} \int_\Sigma \alpha_1 \wedge \alpha_2 + \frac{i}{2} \int_\Sigma \text{Re} < i\beta, \eta > H \omega
\]

such that \( G(X, Y) = 2\Omega(X, Y) \). It is closed, since it is constant. In fact, \( \Omega \) is the real Kähler form on the affine space, \([14]\) page 93. It is positive, since it comes from a positive definite metric.

**Lemma 1.** \( \Omega \) is a symplectic form on the vortex moduli space.

**Proof.** Let \( \zeta \in \Omega(M, i\mathbb{R}) \) be the Lie algebra of the gauge group (the gauge group element being \( g = e^{\zeta} \)); note that \( \zeta \) is purely imaginary. It generates a vector field \( X_\zeta \) on \( \mathbb{C} \) as follows:

\[
X_\zeta(A, \Psi) = (d\zeta, -\zeta \Psi) \in T_p \mathbb{C}
\]

where \( p = (A, \Psi) \in \mathbb{C}. \)

We show next that \( X_\zeta \) is Hamiltonian. Let us define \( H_\zeta : \mathbb{C} \to \mathbb{C} \) as follows:

\[
H_\zeta(A, \Psi) = \frac{1}{2} [\int_\Sigma \zeta \cdot (F(A) - \frac{1}{2}(1 - |\Psi|^2_H)\omega]
\]

to be the Hamiltonian for the gauge group action. Then for \( X = (\alpha, \eta) \in T_p \mathbb{C}, \)

\[
dH_\zeta(X) = \frac{1}{2} \int_\Sigma \zeta d\alpha + \frac{1}{4} \int_M \zeta(\Psi H\bar{\eta} + \bar{\Psi} H\eta)\omega
\]

\[
= -\frac{1}{2} \int_\Sigma (d\zeta) \wedge \alpha - \frac{1}{4} \int_\Sigma \left[ (-\zeta \Psi) H\bar{\eta} - (\bar{\zeta} \Psi) H\eta \right] \omega
\]

\[
= \Omega(X_\zeta, X),
\]

where we use that \( \bar{\zeta} = -\zeta. \)

Thus we can define the moment map \( \mu : \mathbb{C} \to \Omega^2(\Sigma, i\mathbb{R}) = G^\ast \) (the dual of the Lie algebra of the gauge group) to be

\[
\mu(A, \Psi) = \frac{1}{2} [F(A) - \frac{1}{2}(1 - |\Psi|^2_H)\omega].
\]

Thus equation (1) is \( \mu = 0 \) and the form descends as a symplectic form to \( \mu^{-1}(0)/G \). This follows from Marsden and Weinstein, \([7]\), who proved the symplectic reduction even for infinite-dimensional case. \( \square \)
The moduli space of vortices, $\text{Sym}^N(\Sigma)$ is a smooth Kähler manifold with the Manton-Nasir Kähler form $\omega_{MN}$, (which is equivalent to our metric).

This metric is given by (6, eqn (2.14)).

This is the same as the descendent of our metric:

$$G(X, X) = \int_{\Sigma} 4|a|^2 \, dx \wedge dy + \int_{\Sigma} |\beta|^2 H^2 \, dx \wedge dy$$

$$= \int_{\Sigma} (A_1 \dot{A}_1 + A_2 \dot{A}_2) \, dx \wedge dy + \int_{\Sigma} |\beta|^2 H^2 \, dx \wedge dy$$

The complex structure defined by us is exactly the usual complex structure of the vortex moduli space i.e. on $\text{Sym}^N(\Sigma)$. This can be proved using Ruback’s argument mentioned in the appendix of [10], or in the mathscinet review of [6]. (Recall it takes $\alpha \to i\alpha$ and $\beta \to i\beta$ which is the case in Ruback’s argument).

Now we restrict our attention to the case when the Riemann surface is $\mathbb{C}P^1$ of radius $R$. The Manton-Nasir Kähler form on the moduli space of $N$ vortices on $\mathbb{C}P^1$ of radius $R$ is given by, [6], [9],

$$\omega_{MN} = \frac{i}{2} \sum_{r,s=1}^N \left( \frac{4R^2 \delta_{rs}}{(1 + |z_r|^2)^2} + 2 \frac{\partial \bar{b}_s}{\partial z_r} \right) dz_r \wedge d\bar{z}_s$$

This real 2-form is exactly the symplectic reduction of $\Omega$ on the affine space $\mathcal{C}$ of $(A, \Psi)$ to the vortex moduli space, when the number of vortices is $N$ and the Riemann surface is $\mathbb{C}P^1$ of radius $R$. Thus $\Omega = \omega_{MN}$. (This statement is true even for a compact Riemann surface of genus $g > 0$.)

We study this form on the moduli space of $N$ vortices on the sphere of radius $R$. In the following we refer to Romao, [9], where $\omega_{MN}$ is denoted by $\omega_{\text{sam}}$. If $\frac{1}{2\pi} \Omega$ is integral, then $\frac{1}{2\pi} [\Omega] = \ell [\omega_{FS}]$, for some $\ell \in \mathbb{Z}$ since $H^2(\mathbb{C}P^N, \mathbb{Z})$ is generated by $[\omega_{FS}]$ (the cohomology class of the Fubini-Study Kähler form.)

By [9], we can fine tune the volume of the sphere such that $\ell = 1$. Namely, we take $R^2 = \frac{1}{2} + N$ and $\kappa = 2k, k \in \mathbb{Z}$. This satisfies the constraints that $\kappa$ and $\kappa(R^2 - N)$ is an integer. Note that Bradlow criterion for existence of solution holds, since area of the sphere is greater than $4\pi N$, [9] equation (18), [2].

Then $[\frac{1}{2\pi} \Omega] = [\frac{1}{2\pi} \omega_{MN}] = [\omega_{FS}]$.

2.3. Quillen Determinant Line bundle and the hyperplane bundle on $\mathbb{C}P^N$. We denote the Quillen bundle $\mathcal{P} = \det(\bar{\partial}_A)$ which is well defined on $\mathcal{C} = A \times \Gamma(L)$ (over every $(A, \Psi)$ the fiber is that of $\det(\bar{\partial} + A^{0,1})$). Following Biswas and Raghavendra’s work on stable triples, [11], we give $\mathcal{P}$ a modified Quillen metric, namely, we multiply the Quillen metric $e^{-\zeta_A(0)}$ by the factor $e^{-\frac{1}{4\pi} \int_{\Sigma} |\Psi|^2 \bar{\partial}_i \omega}$, where recall $\zeta_A(s)$ is the zeta-function corresponding to the Laplacian of the $\bar{\partial} + A^{0,1}$ operator. We calculate the curvature for this modified metric on the affine space. The factor
\( e^{-\zeta_A(0)} \) contributes \( \frac{i}{2\pi} \left( -\frac{1}{2} \int \alpha_1 \wedge \alpha_2 \right) \) to the curvature, and the factor \( e^{-\frac{i}{4\pi} \int_M |\Psi|^2 \omega} \) contributes \( \frac{i}{2\pi} \left( -\frac{1}{4} \int \Sigma (\beta H \bar{\eta} - \bar{\beta} H \eta) \omega \right) \) to the curvature.

**Lemma 2.** The curvature \( \Omega_{\text{det}} \) of \( \mathcal{P} \) with the modified Quillen metric is indeed \( \frac{i}{2\pi} \Omega \) on the affine space \( \mathcal{C} \). \( \Omega \) is the real \((1,1)\) given above which is positive.

**Proof.** Quillen, [8], constructs the determinant line bundle on the affine space \( \mathcal{A}^{0,1} \).

We consider the space of unitary connections, i.e \( \mathcal{A}^{0,1} = -\mathcal{A}^{1,0} \). The complex structure on \( \mathcal{C} \) defined by \( I = \begin{bmatrix} * & 0 \\ 0 & i \end{bmatrix} \) makes \( (\mathcal{A}^{0,1}, \Psi) \) the holomorphic variables on \( \mathcal{C} \). This is because \( *\alpha^{0,1} = i\alpha^{0,1} \).

Thus the holomorphic coordinate \( w \) in [8] corresponds to \( \mathcal{A}^{0,1} \) in our notation.

By Quillen’s computation, the curvature two form is \( \overline{\partial} \partial |\sigma||^2 \) where \( \log |\sigma||^2 = -\zeta_A(0) \) where \( \zeta_A \) is the zeta-function corresponding to the Laplacian of \( \overline{\partial} A \).

The Quillen curvature form is \( \overline{\partial} \partial |\sigma||^2 = \partial \overline{\partial}[-\log |\sigma||^2] = \partial \overline{\partial} \zeta_A'(0) \). Now

\[
\frac{\partial^2 \zeta_A'(0)}{\partial w \partial \bar{w}} dw \wedge d\bar{w} = \frac{i}{2\pi} \int_{\Sigma} \delta A^{0,1} \wedge \delta A^{0,1},
\]

that is,

\[
\left( \frac{\partial^2 \zeta_A'(0)}{\partial w \partial \bar{w}} dw \wedge d\bar{w} \right)(\alpha_1^{0,1}, \alpha_2^{0,1}) = \frac{i}{2\pi} \int_{\Sigma} \frac{1}{2} [\alpha_1^{0,1} \wedge \alpha_2^{0,1} - \alpha_2^{0,1} \wedge \alpha_1^{0,1}]
\]

\[
= \frac{i}{2\pi} \int_{\Sigma} \frac{-1}{2} [\alpha_1 \wedge \alpha_1]
\]

where recall we have used the fact that \( \alpha_i^{0,1} = -\alpha_i^{1,0} \) This precisely corresponds to \( i/2\pi \) times the first term in our Kähler form.

Next, the term \( e^{-\frac{i}{4\pi} \int_M |\Psi|^2 \omega} \) contributes to the curvature as:

\[
-\frac{i}{4\pi} \tau = -\frac{i}{4\pi} \int_{\Sigma} <\delta \Psi \wedge \delta \bar{\Psi}>_H \omega
\]

where \( -\frac{i}{4\pi} \tau \) is a two-form on the affine space \( \Gamma(\Sigma, L) \). Details of this calculation can be found in [4]. Here \( \tau(\beta, \eta) = \frac{1}{2} \int_{\Sigma} (\beta H \bar{\eta} - \bar{\eta} H \beta) \omega \).

Thus the contribution is

\[
-\frac{i}{4\pi} \tau(\beta, \eta) = \frac{i}{2\pi} \left[ -\frac{1}{4} \int_M (\beta H \bar{\eta} - \bar{\eta} H \beta) \omega \right],
\]

which is precisely corresponds to \( i/2\pi \) times the second term in our Kähler form. \( \square \)

To define it on \( \mathcal{C}/\mathcal{G} \), we repeat the argument in [3], [4] with \( B = 0 \).
Lemma 3. \( \mathcal{P} \) descends to a well defined line bundle (denoted by the same symbol) on \( \mathcal{C}/\mathcal{G} \).

Proof. Let \( D = \bar{\partial} + A^{(0,1)} \) and let \( D_g = g(\bar{\partial} + A^{(0,1)})g^{-1} \) (the gauge transformed operator), then the gauge transformed Laplacian of \( D \), namely \( \Delta_g = g\Delta g^{-1} \). Thus there is an isomorphism of eigenspaces, namely, \( s \to gs \).

Let \( K^a(\Delta) \) be the direct sum of eigenspaces of the operator \( \Delta \) of eigenvalues < \( a \), over the open subset \( U^a = \{A^{(1,0)} | a \notin \text{Spec} \Delta \} \) of the affine space \( \mathcal{C} \). The determinant line bundle is defined using the exact sequence

\[
0 \to \text{Ker} D \to K^a(\Delta) \to D(K^a(\Delta)) \to \text{Coker} D \to 0
\]

Thus one identifies \( \wedge^\text{top} (\text{Ker} D)^* \otimes \wedge^\text{top} (\text{Coker} D) \) with \( \wedge^\text{top} (K^a(\Delta))^* \otimes \wedge^\text{top} (D(K^a(\Delta))) \) (see [8], for more details) and there is an isomorphism of the fibers as \( D \to D_g \). Thus one has

\[
\wedge^\text{top} (K^a(\Delta))^* \otimes \wedge^\text{top} (D(K^a(\Delta))) \equiv \wedge^\text{top} (K^a(\Delta_g))^* \otimes \wedge^\text{top} (D(K^a(\Delta_g))).
\]

\( U^a_g = g \cdot U^a \) where \( U^a_g \) is the open set formed out of gauge transformation of \( U^a \), namely, \( U^a_g = \{A_g, | a \notin \text{Spec}(g\Delta g^{-1}) \} \). But \( A_g \in U^a_g \) imples \( a \notin \text{Spec}(\Delta) \) and thus \( A_g \in U^a \). Hence \( U^a \subset U^a_g \). Similarly, \( U^a_g \subset U^a \). Thus \( U^a = U^a_g \).

On \( U^a \) one defines the equivalence class of the fiber

\[
\wedge^\text{top} (K^a(\Delta))^* \otimes \wedge^\text{top} (D(K^a(\Delta))) \equiv \wedge^\text{top} (K^a(\Delta_g))^* \otimes \wedge^\text{top} (D(K^a(\Delta_g))).
\]

\( \square \)

When the form \( \frac{1}{2\pi} \Omega \) is integral, this construction holds for a compact Riemann surface of genus \( g \).

Thus \( \mathcal{P} \) descends to the quotient of the affine space \( \mathcal{C} \) modulo the gauge group and further restricts to a line bundle on the vortex moduli space. We consider this in the case when the Riemann surface is \( \mathbb{C}P^1 \) of radius \( R \) where \( R^2 = \frac{1}{2} + N \). We denote this line bundle again by \( \mathcal{P} \) and its curvature by \( \Omega_{\text{det}} = \frac{i}{2\pi} \Omega \) such that \( \Omega \) coincides with the usual Kähler form on the vortex moduli space, namely \( \omega_{MN} \), which is cohomologous to \( 2\pi \omega_{FS} \). The Chern class of \( \mathcal{P} \) is \( \frac{1}{2\pi}[\Omega] \) which is integral and \( \Omega \) is positive.

The first Chern class of \( \mathcal{P} \), namely \( \frac{1}{2\pi}[\Omega] \), is equal to the first Chern class of the hyperplane line bundle on \( \mathbb{C}P^N \), namely \([\omega_{FS}]\).

Since the Picard variety of \( \mathbb{C}P^N \) is trivial, the hyperplane bundle is uniquely defined as a holomorphic line bundle by its first Chern class and so is the determinant bundle. Since the first Chern classes agree, they are the equivalent as holomorphic line bundles.

Thus we have the following proposition (where we denote \( \mathcal{P} = \det(\bar{\partial}) \)).
Proposition 4. The determinant line bundle $\mathcal{P} = \det(\overline{\partial})$ of the family of Cauchy-Riemann operators on $\mathbb{C}P^1$, as a line bundle on $\mathbb{C}P^N$ is equivalent, as holomorphic line bundles, to the hyperplane line bundle on $\mathbb{C}P^N$, if the radius $R$ of $\mathbb{C}P^1$ satisfies $R^2 = \frac{1}{2} + N$.

3. KODAIRA EMBEDDING THEOREM AND APPLICATION

Here we recall the Kodaira embedding theorem (cf. [5, 13]) in a form that is suitable for our application.

Theorem 2 (Kodaira embedding theorem). Let $M$ be a compact Kähler manifold with integral Kähler form $\omega$. Let $L \to M$ be the line bundle with connection $\nabla$ and curvature $\omega$. Then there is a positive integer $k_0$ (which we will assume is minimal) such that for all $k \geq k_0$ the natural map

$$\phi_k : M \hookrightarrow \mathbb{P}(H^0(M, \mathcal{O}(L^\otimes k))^*)$$

is a holomorphic embedding. In particular, one has the equality of cohomology classes $[\phi_k^*(\Omega_k)] = k[\omega]$, where $\Omega_k$ denotes the Kähler form of the Fubini-Study metric on $\mathbb{P}(H^0(M, \mathcal{O}(L^\otimes k))^*)$.

Here $H^0(M, \mathcal{O}(L^\otimes k))$ denotes the finite dimensional vector space of all holomorphic sections of the line bundle $L^\otimes k$, which has a natural $L^2$-metric using the Kähler structure on $M$, which in turn induces the Fubini-Study Kähler form $\Omega_k$ on $\mathbb{P}(H^0(M, \mathcal{O}(L^\otimes k))^*)$.

For instance, for a compact Riemann surface of genus greater than 2, the minimal choice of $k_0$ is equal to 3.

Proof of Theorem 2. By Proposition 4, we know that Theorem 1 is true for complex projective space $\mathbb{C}P^N$ for all positive integers $N$. By the Kodaira embedding theorem, Theorem 2 and with the minimal choice of $k_0 = N$, Theorem 1 is true in the general case. □

4. SYMPLECTIC VARIANT AND APPLICATION

The symplectic variant of the Kodaira embedding theorem is due to Gromov and we quote a version in [12], Remark following Theorem B.

Theorem 3 (Gromov’s embedding theorem). Let $M$ be a compact symplectic manifold with integral symplectic form $\omega$. Let $L \to M$ be the line bundle with connection $\nabla$ and curvature $\omega$. Then there is a positive integer $k_0$ (which we will assume is minimal) such that for all $k \geq k_0$ there is a symplectic embedding

$$\phi_k : M \hookrightarrow \mathbb{C}P^k.$$

In particular, $\phi_k^*(\Omega_k) = \omega$, where $\Omega_k$ denotes the symplectic form of the Fubini-Study metric on $\mathbb{C}P^k$. 

Our next result is the symplectic analogue of Theorem 1.

**Theorem 4.** Any compact symplectic manifold $M$ with integral symplectic form $\omega$, parametrizes a natural smooth family of Cauchy-Riemann operators $\{\overline{\partial} z : z \in M\}$ on $\mathbb{C}P^1$ such that the determinant line bundle $\det(\overline{\partial}) \cong \mathcal{L}$ as complex line bundles, where $\mathcal{L}$ is the prequantum line bundle determined by $\omega$.

*Proof.* By Proposition 4, we know that Theorem 4 is true for complex projective space $\mathbb{C}P^N$ for all positive integers $N$. By Gromov’s embedding theorem, Theorem 3, and with the minimal choice of $k_0 = N$, Theorem 1 is true in the general case. □

**Remark:** The determinant line bundle $\mathcal{P}$ with the modified metric generalises to the $N$-vortex moduli space for compact Riemann surface of genus $g$, with its curvature exactly $i/2\pi \omega_{MN}, \omega_{MN}$ the Kähler form on the moduli space, when the latter satisfies an integerality condition. However, it is difficult to determine exactly what this line bundle is when the moduli space is thought of as a symmetric product of the Riemann surface. The Kodaira embedding theorem gives us a strategy for finding the Hilbert space of this theory. Embed the moduli space, which is a Kähler manifold with $\omega_{MN}$, into $\mathbb{C}P^{k_0}$ for $k_0$ high enough and the hyperplane bundle on $\mathbb{C}P^{k_0}$ when restricted to the Kähler manifold, is equivalent to $\mathcal{P}^k$, for some $k$ large enough. The holomorphic sections of this hyperplane bundle determine the Hilbert space of the theory.

5. **Conjecture**

We conjecture that equivariant versions of our results are true. More precisely,

**Conjecture.** Let $M$ be a compact Kähler $G$-manifold $M$ with integral Kähler form $\omega$, where $G$ is a compact Lie group. That is, $\omega$ is a $G$-invariant Kähler form on the $G$-manifold $M$ and all structures are $G$-invariant. Let $\mathcal{L}$ denote the holomorphic $G$-line bundle determined by $\omega$. Then $M$ parametrizes a natural holomorphic $G$-equivariant family of Cauchy-Riemann operators $\{\overline{\partial} z : z \in M\}$ on a compact Kähler $G$-manifold $Z$ such that the Quillen determinant line bundle $\det(\overline{\partial}) \cong \mathcal{L}^\otimes k$ as holomorphic $G$-line bundles, for some sufficiently large $k$.

There is an analogous conjecture in the equivariant symplectic case. If the conjectures are valid, then they would apply to coadjoint integral maximal orbits and thus to representation theory.

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