THE BORWEIN CONJECTURES OVER ARITHMETIC PROGRESSIONS

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Abstract. We obtain asymptotic formulas for sums of coefficients over arithmetic progressions of polynomials related to the Borwein conjectures. Let \( a_i \) denote the coefficient of \( q^i \) in the polynomial \( \prod_{j=1}^{n} \prod_{k=1}^{p-1} (1 - q^{p^j - k})^i \), where \( p \) is an odd prime, and \( n, s \) are positive integers. In this note, we prove that

\[
\left| \sum_{i \equiv b \mod 2pn} a_i - \frac{(p-1)p^{sn-1}}{2n} \right| \leq p^{sn/2},
\]

if \( b \) is divisible by \( p \), and

\[
\left| \sum_{i \equiv b \mod 2pn} a_i + \frac{p^{sn-1}}{2n} \right| \leq p^{sn/2},
\]

if \( b \) is not divisible by \( p \). This improves a recent result of Goswami and Pantangi [6].

1. Introduction

Let \( p \) and \( s \) be two positive integers. For a positive integer \( n \), let the sequence \( (a_i) \) be defined by

\[
\prod_{j=1}^{n} \prod_{k=1}^{p-1} (1 - q^{p^j - k})^i = \sum_{i=0}^{sn(p-1)p/2} a_i q^i.
\]  

(1.1)

In 1990, Peter Borwein discovered some intriguing sign patterns of the coefficients \( a_i \) for three different cases \((p, s) = (3, 1), (3, 2), (5, 1)\). They have three repeating sign patterns + − −, + + − and + − − − respectively. Equivalently, the sign of \( a_i \) is determined by \( i \mod p \). These conjectures were formalized by Andrews in 1995 [1], which are stated as follows.

Conjecture 1.1 (First Borwein conjecture). For the polynomials \( A_n(q), B_n(q) \) and \( C_n(q) \) defined by

\[
\prod_{j=1}^{n} (1 - q^{3j-2})(1 - q^{3j-1}) = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3),
\]
each has non-negative coefficients.

Conjecture 1.2 (Second Borwein conjecture). For the polynomials \( \alpha_n(q), \beta_n(q) \) and \( \gamma_n(q) \) defined by

\[
\prod_{j=1}^{n} (1 - q^{3j-2})^2(1 - q^{3j-1})^2 = \alpha_n(q^3) - q\beta_n(q^3) - q^2\gamma_n(q^3),
\]
each has non-negative coefficients

Conjecture 1.3 (Third Borwein conjecture). For the polynomials \( \nu_n(q), \phi_n(q), \chi_n(q), \psi_n(q) \) and \( \omega_n(q) \) defined by

\[
\prod_{j=1}^{n} (1 - q^{5j-4})(1 - q^{5j-3})(1 - q^{5j-2})(1 - q^{5j-1}) = \nu_n(q^5) - q\phi_n(q^5) - q^2\chi_n(q^5) - q^3\psi_n(q^5) - q^4\omega_n(q^5),
\]
each has non-negative coefficients.

All these conjectures had been open for many years. In a recent paper [2], Wang gave an analytic proof of the first Borwein conjecture using the saddle point method and a formula discovered by Andrews [1, Theorem 4.1] for the polynomials \( A_n(q), B_n(q) \) and \( C_n(q) \). It is not clear if his method can be applied to other conjectures. Even for the first Borwein conjecture, an algebraic proof would be very interesting.
Instead of evaluating $a_i$ directly, it is natural to consider the Borwein conjectures on average over arithmetic progressions. Let $d$ be an integer divisible by $p$ and $b$ be an integer with $0 \leq b \leq d - 1$. If we define

$$S_{d,b} := \sum_{i=b \text{ mod } d} a_i,$$

then the positivity (negativity) part of the Borwein conjectures follows from the positivity (negativity, respectively) of $S_{d,b}$ for sufficiently large $d$, say $d \geq s n^2 (p-1)p/2$. Please note that here $S_{d,b}$ should be $S_{p,s,n,d,b}$. For notational simplicity the subscripts $p$, $s$ and $n$ are omitted when there is no confusion.

Using estimates of exponential sums, Zaharescu \cite{Zaharescu} first studied $S_{d,b}$ for a large classes of $d$. He proved the following theorem.

**Theorem 1.4 (Zaharescu).** Let $p, q$ be two distinct odd primes with $q \leq n$, and let $b$ be an integer with $0 \leq b \leq pq - 1$. Then

$$\left| S_{pq,b} - \frac{(p-1)p^{n-1}}{q} \right| \leq \frac{(p-1)(q-1)p^{[n/q]-1}2^s q (p-1) (n-[n/q])}{q},$$

if $b$ is divisible by $p$, and

$$\left| S_{pq,b} + \frac{p^{sn-1}}{q} \right| \leq \frac{(p-1)(q-1)p^{[n/q]-1}2^s q (p-1) (n-[n/q])}{q},$$

if $b$ is not divisible by $p$, where $[x]$ denote the greatest integers bounded by $x$.

For instance, when $(p, s) = (3, 1)$, Zaharescu’s bound gives

$$\left| S_{3q,b} - \frac{2 \cdot 3^{n-1}}{q} \right| \leq \frac{2(q-1)3^{[n/q]-1}2^s q (n-[n/q])}{q},$$

for $b$ divisible by 3. Note that to insure this bound is nontrivial, $q$ must be a prime bounded by $n$. Thus a new question naturally arises.

**Problem 1.5.** For larger $d$, give a reasonable bound for $S_{d,b}$.

In the case $(p, s) = (3, 1)$, Li \cite{Li} removed the condition that $q$ is a prime and in fact obtained an estimate with a very small error bound. He showed that

**Theorem 1.6 (Li).** Let $p = 3$, $s = 1$, and $b$ be an integer with $0 \leq b \leq 3n - 1$. Then

$$\left| S_{3n,b} - \frac{2 \cdot 3^{n-1}}{n} \right| \leq 2^n,$$

if $b$ is divisible by 3, and

$$\left| S_{3n,b} + \frac{3^{n-1}}{n} \right| \leq 2^n,$$

if $b$ is not divisible by 3.

Goswami and Pantangi \cite{GoswamiPantangi} generalized this bound to general cases $(p, s)$ and $d = pn$ following Li’s argument and Li-Wan’s sieving argument. They proved the following theorem.

**Theorem 1.7 (Goswami and Pantangi).** Let $p$ be an odd prime and $b$ be an integer with $0 \leq b \leq pn - 1$. Then

$$\left| S_{pn,b} - \frac{(p-1)p^{sn-1}}{n} \right| \leq p^{sn/2},$$

if $b$ is divisible by $p$, and

$$\left| S_{pn,b} + \frac{p^{sn-1}}{n} \right| \leq p^{sn/2},$$

if $b$ is divisible by $p$.

In this note, we improve the result of Goswami and Pantangi to arithmetic progressions with a larger common difference of $2pm$. We proved the following result:
Theorem 1.8. Let $p$ be an odd prime and $b$ be an integer with $0 \leq b \leq 2pn - 1$. Then

$$\left| S_{2pn,b} - \frac{(p-1)p^{n-1}}{2n} \right| \leq p^{n}/2,$$

if $b$ is divisible by $p$, and

$$\left| S_{2pn,b} + \frac{p^{n-1}}{2n} \right| \leq p^{n}/2,$$

if $b$ is not divisible by $p$.

Notation. The congruence notion $a \equiv b \mod n$ means $a - b$ is divisible by $n$. We use $\text{ord}(\chi)$ to denote the order of the character $\chi$ and $|E|$ to denote the cardinality of the set $E$. If $S$ is a statement, we use $1_S$ to denote the indicator function of $S$, thus $1_S = 1$ when $S$ is true and $1_S = 0$ when $S$ is false.

2. Reduction to a subset-sum type problem

As in [4], we first reduce the problem to a subset-sum type problem over the (additive) group of integers modulo $2pn$. The starting point is the following equality

$$(1 - q^1) = -q^1(1 - q^{-1}),$$

which allows us to write the polynomial (2.1) as

$$\prod_{j=1}^{n} \prod_{k=1}^{p-1} (1 - q^{pj-k})^s = \prod_{j=1}^{n} \prod_{k=1=-(p+1)/2}^{p-1} ((-1)^s q^s(pj-k)(1 - q^{-(pj-k)})^s) \prod_{k=1}^{(p-1)/2} (1 - q^{pj-k})^s$$

$$= (-1)^{sn(p-1)/2} q^{sn(p-1)(2pn+1-p)/8} \prod_{j=-(n-1)}^{n} \prod_{k=1}^{(p-1)/2} (1 - q^{pj-k})^s.$$  

Let $b_i$ denote the coefficient $q^i$ in the Laurent polynomial $\prod_{j=-(n-1)}^{n} \prod_{k=1}^{(p-1)/2} (1 - q^{pj-k})^s$. Then the above equation implies $a_i = (-1)^{sn(p-1)/2} b_{i-sn(p-1)(2pn+1-p)/8}$. In particular, we have

$$S_{2pn,b} = \sum_{i=b \mod 2pn} a_i = (-1)^{sn(p-1)/2} \sum_{i=b-sn(p-1)(2pn+1-p)/8 \mod 2pn} b_i. \quad (2.1)$$

Thus to prove Theorem 1.8 it suffices to consider the sum $\sum_{i=b \mod 2pn} b_i$.

Let $D$ denote set $\{pj-k : -(n-1) \leq j \leq n, 1 \leq k \leq (p-1)/2\}$. Given integers $0 \leq m_i \leq |D|$, $1 \leq i \leq s$ and $0 \leq b \leq 2pn - 1$, we define $N(m_1, m_2, \ldots, m_s; b)$ to be cardinality of the set

$$N_D(m_1, m_2, \ldots, m_s, b) := \# \{(V_1, V_2, \ldots, V_s) : V_i \subset D \mid |V_i| = m_i, 1 \leq i \leq s, \sum_{i=1}^{s} \sum_{x \in V_i} x = b \mod 2pn\}.$$  

That is, $N_D(m_1, m_2, \ldots, m_s, b)$ is the number of ordered $s$-tuples of subsets of $D$ with prescribed cardinalities $m_i$ which sum to $b$. In the subset-sum problem, we count the number of subsets (equivalently, 1-tuples of subsets) with prescribed cardinality which sum to a given element. Thus this problem can be viewed as a variant of the subset-sum problem. We also define $N_D(b)$ to be the alternating sum of $N_D(m_1, m_2, \ldots, m_s, b)$

$${\sum_{0 \leq m_i \leq |D|, 1 \leq i \leq s}} (-1)^{\sum_{i=1}^{s} m_i} N_D(m_1, m_2, \ldots, m_s, b). \quad (2.2)$$

From the definitions of $b_i$ and $N_D(b)$, it is not hard to see that

$$N_D(b) = \sum_{i=b \mod 2pn} b_i. \quad (2.3)$$

The problem is now reduced to counting $N_D(b)$ and thus to counting $N_D(m_1, m_2, \ldots, m_s, b)$, which can be viewed as a subset-sum type problem over the group of integers modulo $2pn$. 

3. Li-Wan sieve and some combinatorial formulas

For the purpose of the proof, we briefly introduce the Li-Wan sieve [5] and present some combinatorial formulas.

Let $A$ be a finite set and let $A^m$ be the $m$-th fold Cartesian product of $A$. Let $X$ be a subset of $A^m$. Let $\mathcal{X}$ denote the elements in $X$ with distinct coordinates

$$\mathcal{X} = \{ (x_1, x_2, \ldots, x_m) \in X : x_i \neq x_j : \forall \ i \neq j \}.$$ 

Let $S_m$ be the symmetric group on the set $\{1, 2, \ldots, m\}$. Given a permutation $\tau \in S_m$, we can write it as a disjoint cycle product $\tau = C_1C_2 \cdots C_{\ell(\tau)}$, where $\ell(\tau)$ denote the number of disjoint cycles of $\tau$. We define the signature of $\tau$ to be $\text{sign}(\tau) = (-1)^{k-\ell(\tau)}$. We also define the set $X_\tau$ to be

$$X_\tau = \{ (x_1, x_2, \ldots, x_m) \in X : x_i \text{ are equal for } i \in C_j, 1 \leq j \leq \ell(\tau) \}.$$ 

In other words, $X_\tau$ is the set of elements in $X$ fixed under the action of $\tau$ defined by $\tau \circ (x_i)_{x_i \leq m} := (x_{\tau(i)})_{1 \leq \tau \leq m}$. The Li-Wan sieve gives a formula for calculating sums over $\mathcal{X}$ via sums over $\mathcal{X}_\tau$.

**Theorem 3.1** ([5], Theorem 2.6). Let $f : X \to \mathbb{C}$ be a complex-valued function defined over $X$. Then we have

$$\sum_{x \in \mathcal{X}} f(x) = \sum_{\tau \in S_m} \text{sign}(\tau) \sum_{x \in X_\tau} f(x).$$ 

A permutation $\tau \in S_m$ is said to be of type $(c_1, c_2, \ldots, c_m)$ if it has $c_i$ cycles of length $i$, $1 \leq i \leq m$. Let $N(c_1, c_2, \ldots, c_m)$ denote the number of permutations of type $(c_1, \ldots, c_m)$. It is well-known [7] that

$$N(c_1, c_2, \ldots, c_m) = \frac{m!}{1^{c_1}c_1!2^{c_2}c_2! \cdots m^{c_m}c_m!}.$$ 

(3.1)

If we define an $m$-variate polynomial $Z_m$ via

$$Z_m(t_1, t_2, \ldots, t_m) = \frac{1}{m!} \sum_{i_1, \ldots, i_m = m} N(c_1, c_2, \ldots, c_m) t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m},$$

then it follows from (3.1) that $Z_m$ satisfies the generating function

$$\sum_{m \geq 0} Z_m(t_1, t_2, \ldots, t_m) u^m = \exp(t_1u + t_2 \frac{u}{2} + t_3 \frac{u^3}{3} + \cdots).$$ 

(3.2)

We give some combinatorial lemmas that will be used later.

**Lemma 3.2** ([4], Lemma 2.3). If $t_i = a$ for $d \mid i$ and $t_i = 0$ otherwise, then we have

$$Z_m(t_1, t_2, \ldots, t_m) = Z_m(0, \ldots, 0, a, 0, \ldots, 0, a, \ldots) = [u^m] (1 - u^d)^{-a/d}.$$

**Lemma 3.3.** Let $B$ be a finite set of complex numbers. If $t_i = \sum_{b \in B} b \bar{z} a$ for $d \mid i$ and $t_i = 0$ otherwise. Then we have

$$Z_m(t_1, t_2, \ldots, t_m) = Z_m(0, \ldots, 0, \sum_{d_1 \mid d} b^{\bar{z} a} + \sum_{d_1 \mid d} b^{\bar{z} a} \ldots ) = [u^m] \prod_{b \in B} (1 - bu^d)^{-a/d}.$$

Proof. Substituting the values of $t_i$ into (3.2), we see that

$$Z_m(t_1, t_2, \ldots, t_m) = [u^m] \exp(\sum_{i=1}^{\infty} \frac{\sum_{b \in B} b^{\bar{z} a} d}{d^i}) = [u^m] \exp(-\frac{a}{d} \sum_{b \in B} \log(1 - bu^d)) = [u^m] \prod_{b \in B} (1 - bu^d)^{-a/d}.$$ 

$\Box$
4. Proof of the main result

Now we prove Theorem \[\text{1.8}\]. In view of \[\text{2.1}\] and \[\text{2.3}\], we have

\[
S_{2pn,b} = (-1)^{sn(p-1)/2} N_D(b - sn(p-1)(2pn + 1 - p)/8).
\]  \hspace{1cm} (4.1)

Thus we need to estimate the quantity \(N_D(b)\) and thus to estimate the quantity \(N_D(m_1, \ldots, m_s, b)\). As in \[\text{4}\], we use character sums to estimate it. Let \(G = \mathbb{Z}/2pn\mathbb{Z}\) be the cyclic group of integers modulo \(2pn\). Let \(X = D^m\) denote the \(m\)-th fold Cartesian product of \(D = \{pj - k : -(n-1) \leq j \leq n, 1 \leq k \leq (p-1)/2\}\) and \(X_i\) denote the set of elements in \(X_i\) with distinct coordinates. For an ordered \(k\)-tuple \(x = (x_1, x_2, \ldots, x_m) \in D^m\), where \(m\) is a positive integer, let \(s(x) := \sum_{i=1}^m x_m\) denote the sum of its coordinates. Using the fact that \(\frac{1}{|G|} \sum_{\chi \in G} \chi(x) = 1\) if \(x = 0\) and is 0 otherwise, we can express \(N_D(m_1, m_2, \ldots, m_2, b)\) as

\[
N_D(m_1, m_2, \ldots, m_s, b) = \frac{1}{m_1! m_2! \cdots m_s!} \sum_{(x_1, x_2, \ldots, x_s) \in \chi_1 \times \chi_2 \times \cdots \chi_s} \frac{1}{G} \sum_{\chi \in G} \chi(s(x_1) + s(x_2) + \cdots + s(x_s) - b)
\]

\[
= \frac{1}{|G|} \sum_{\chi \in G} \chi(b) \sum_{(x_1, x_2, \ldots, x_s) \in \chi_1 \times \chi_2 \times \cdots \chi_s} \prod_{i=1}^s \left( \frac{1}{m_i!} \chi(s(x_i)) \right).
\]

Thus we need to evaluate the character sums of the form

\[
S_m(\chi) := \frac{1}{m!} \sum_{x \in X} \chi(s(x)) = \frac{1}{m!} \sum_{(x_1, x_2, \ldots, x_m) \in X} \chi(x_1) \chi(x_2) \cdots \chi(x_m),
\]

where \(X = D^m\) and \(X_{\tau}\) consists of elements in \(X\) fixed by \(\tau\).

Evaluating \(S_m(\chi)\) a distinct coordinate counting problem that can be handled by the Li-Wan sieve. Applying Theorem \[\text{3.1}\] we can write \(S_m(\chi)\) as

\[
S_m(\chi) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \text{sign}(\tau) \sum_{(x_1, x_2, \ldots, x_m) \in X_\tau} \chi(x_1) \chi(x_2) \cdots \chi(x_m).
\]

Let \(\tau = C_1 \cdots C_j\) be a disjoint cycle product of \(\tau\). Then from the definition of \(X_\tau\), we have

\[
\sum_{(x_1, x_2, \ldots, x_m) \in X_\tau} \chi(x_1) \chi(x_2) \cdots \chi(x_m) = \prod_{i=1}^j \left( \sum_{x \in D} \chi^{\ell(C_i)}(x) \right),
\]

(4.5)

where \(\ell(C_i)\) denotes the length of the cycle \(C_i\), \(1 \leq i \leq j\). Thus we have to determine character sums over the set \(D = \{pj - k : -(n-1) \leq j \leq n, 1 \leq k \leq (p-1)/2\}\).

Let \([D]\) denote the image of \(D\) under the quotient map \(q : \mathbb{Z} \to G\) that sends \(a\) to \(a + 2pn\mathbb{Z}\). We observe that \([D]\) is a disjoint union of translations of the subgroup \(pG\), where \(pG = \{pg : g \in G\}\). Precisely, we have \([D] = \bigcup_{k=1}^{(p-1)/2} (pG - k)\). Thus

\[
\sum_{x \in D} \chi(x) = \sum_{x \in [D]} \chi(x) = \sum_{k=1}^{(p-1)/2} \sum_{x \in pG} \chi(x - k) = \left( \sum_{k=1}^{(p-1)/2} \chi(k) \right) \sum_{x \in pG} \chi(x)
\]

The sum \(\sum_{x \in pG} \chi(x)\) vanishes, unless \(\chi\) is a trivial character on \(pG\) for which \(\text{ord}(\chi) = 1\) or \(p\). This implies that

- \(\sum_{x \in D} \chi(x) = 0\) if \(\text{ord}(\chi) \neq 1, p\);
- \(\sum_{x \in D} \chi(x) = (\sum_{k=1}^{(p-1)/2} \chi(k)) |pG| = (\sum_{k=1}^{(p-1)/2} \chi(k)) |G|/p\) if \(\text{ord}(\chi) = 1\) or \(p\). Note that in the case \(\text{ord}(\chi) = 1\), the formula can be further simplified as \(\sum_{x \in D} \chi(x) = |D|\).
Now suppose that the order of the character $\chi$ is $e$. From the above discussion, we see that for $p \nmid e$, \( \sum_{x \in D} \chi^i(x) = |D| \) if $e \mid i$ and $\sum_{x \in D} \chi^i(x) = 0$ otherwise; for $p \mid e$, $\sum_{x \in D} \chi^i(x) = (\sum_{k=1}^{(p-1)/2} S^k \chi^i(k))|G|/p$ if $p \mid i$ and $\sum_{x \in D} \chi^i(x) = 0$ otherwise. Thus we have two cases.

**Case 1:** $p \nmid e$. In this case, we have $\sum_{x \in D} \chi^i(x) = |D|1_{e|i}$. Then according to (2.4) and (4.5), we deduce that

\[
S_m(\chi) = \frac{1}{m!} \sum_{\tau \in S_m} \text{sign}(\tau) \prod_j \left( \sum_{x \in D} \chi^i(C_j)(x) \right)
\]

\[
= \frac{1}{m!} \sum_{\sum_{i \in \mathbb{Z}} (-1)^m \sum_{i=1}^m \chi(i) \prod_{k=1}^{(p-1)/2} \chi(k)^{G/p, 0 \leq j \leq \frac{p}{p-1}} \sum_{k=1}^{(p-1)/2} \chi(k)^{G/p}}
\]

\[
= (-1)^m [u^m] (1 - u^e)^{|D|/e}.
\]

The last step is due to Lemma 3.2.

**Case 2:** $p \mid e$. In this case, we have $\sum_{x \in D} \chi^i(x) = (\sum_{k=1}^{(p-1)/2} \chi^i(k))|G|/p 1_{e|i}$. A similar calculation as in Case 1 shows that the sum $S_m(\chi)$ equals

\[
S_m(\chi) = (-1)^m \sum_{k=1}^{(p-1)/2} \chi(k)^{G/p, 0 \leq j \leq \frac{p}{p-1}} \sum_{k=1}^{(p-1)/2} \chi(k)^{G/p}
\]

\[
= (-1)^m [u^m] \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{e/p})^{G/p},
\]

where we used Lemma 3.3.

To sum up, we have

\[
S_m(\chi) = \begin{cases} 
(-1)^m [u^m] (1 - u^{|\text{ord}(\chi)|})^{G/p}, & \text{if } p \nmid \text{ord}(\chi); \\
(-1)^m [u^m] \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{\text{ord}(\chi)/p})^{G/p}, & \text{if } p \mid \text{ord}(\chi).
\end{cases}
\]

Now we are ready to estimate $N_D(b)$. In view of (2.2) and (4.2), we have

\[
N_D(b) = \sum_{0 \leq m_s \leq |D|, 1 \leq i \leq s} (-1)^{\sum_{i=1}^s m_s} N_D(m_1, m_2, \ldots, m_s, b)
\]

\[
= \frac{1}{|G|} \sum_{0 \leq m_s \leq |D|} \sum_{1 \leq i \leq s} (-1)^{\sum_{i=1}^s m_s} \sum_{\chi \in G} \chi(b) \prod_{i=1}^s (1 - \chi(k)u^{\text{ord}(\chi)})^{G/p}.
\]

Using (4.3) and the above results for $S_m(\chi)$, we conclude that

\[
N_D(b) = \frac{1}{|G|} \sum_{\chi \in G \cap p(\text{ord}(\chi))} \chi(b) \sum_{0 \leq m_s \leq |D|} \prod_{i=1}^s (u^{m_i}) \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{\text{ord}(\chi)})^{G/p}
\]

\[
+ \frac{1}{|G|} \sum_{\chi \in G \cap p(\text{ord}(\chi))} \chi(b) \prod_{i=1}^s \left( \sum_{m_i = 0}^{[D]} (u^{m_i}) \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{\text{ord}(\chi)})^{G/p} \right)
\]

\[
= \frac{1}{|G|} \sum_{\chi \in G \cap p(\text{ord}(\chi))} \chi(b) \prod_{i=1}^s \left( \sum_{m_i = 0}^{[D]} (u^{m_i}) \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{\text{ord}(\chi)})^{G/p} \right)
\]

\[
+ \frac{1}{|G|} \sum_{\chi \in G \cap p(\text{ord}(\chi))} \chi(b) \prod_{i=1}^s \left( \sum_{m_i = 0}^{[D]} (u^{m_i}) \prod_{k=1}^{(p-1)/2} (1 - \chi(k)u^{\text{ord}(\chi)})^{G/p} \right)
\]
From the equality (1) the definition of
\[ P = \sum_{\chi \in \hat{G}^{p}} (1 - \chi(\bar{a})) \frac{x^{\chi(\bar{a})}}{\eta^{\chi(\bar{a})}} \]
where we used the fact that
\[ \{ \chi \in \hat{G}^{p} : \text{ord} \chi \} = 1 \]
Then we compute the argument of
\[ P = \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a})) \frac{\chi^{G_{\bar{a}}}}{\eta^{G_{\bar{a}}}} \]
Combining the results of the modulus of argument of \( P \)
\[ N_{D}(b) = \frac{1}{2pn} \sum_{\chi \in \hat{G}^{p} : \text{ord} \chi = p} \chi(b) \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a})) \frac{\chi^{G_{\bar{a}}}}{\eta^{G_{\bar{a}}}} + O\left( \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a})) \frac{\chi^{G_{\bar{a}}}}{\eta^{G_{\bar{a}}}} \right). \]
Note that the implied constant in the big \( O \) can be 1.
Since \( G = \mathbb{Z}/2pn\mathbb{Z} \), we have \(|G| = 2pn\). A substitution into the above equation yields
\[ N_{D}(b) = \frac{1}{2pn} \sum_{\chi \in \hat{G}^{p} : \text{ord} \chi = p} \chi(b) \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2\pi n} + O\left( \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{\pi n} \right). \] (4.6)
For a character \( \chi \) of order \( p \), we consider the following product of \( \chi \):
\[ P(\chi) := \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2}. \]
To determine the value of \( P(\chi) \), we evaluate its modulus and argument separately.
We first compute the modulus of \( P(\chi) \). By definition, we have
\[ |P(\chi)|^{2} = \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2} \prod_{k=1}^{(p-1)/2} (1 - \chi(-\bar{a}))^{2} \]
\[ = \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2} \prod_{k=1}^{(p-1)/2} (1 - \chi(p - \bar{a}))^{2} = \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2} = p^{2}. \]
where we used the fact that \( \{ \chi(\bar{a}), 1 \leq k \leq p-1 \} \) gives a complete list of primitive \( p \)-roots of unity.
This gives \( |P(\chi)| = p \). Consequently, we have \(|P(\chi)|^{2} = p^{2}\). Thus the error term in (4.6) is \( O(p^{n/2}) \).
Then we compute the argument of \( P(\chi) \). To this end, we need the explicit form of \( \chi \). Since \( \chi \) is a character of order \( p \), it must be of the form \( \chi(\bar{a}) = e^{2\pi i a/p} \) for some integer \( a \) with \( 1 \leq a \leq p-1 \). From the definition of \( P(\chi) \), we have
\[ \arg(P(\chi)) = \sum_{k=1}^{(p-1)/2} \arg((1 - \chi(\bar{a}))^{2}) \mod 2\pi. \]
From the equality \((1 - e^{i\theta})^{2} = 2(1 - \cos(\theta))e^{i(\theta + \pi)} \), we have \( \arg((1 - \chi(\bar{a}))^{2}) = \arg((1 - e^{2\pi i a/p})^{2}) = 2\pi a/p + \pi \mod 2\pi \). This implies that
\[ \arg(P(\chi)) = \sum_{k=1}^{(p-1)/2} (2\pi a/p + \pi) \mod 2\pi = \pi \left( \frac{p^{2} - 1}{4p} - a + \frac{p - 1}{2} \right) \mod 2\pi, \]
Combining the results of the modulus of argument of \( P(\chi) \) together, we conclude that
\[ P(\chi) = \prod_{k=1}^{(p-1)/2} (1 - \chi(\bar{a}))^{2} = pe^{\pi i \left( \frac{p^{2} - 1}{4p} - a + \frac{p - 1}{2} \right)}, \]
for \( \chi \) defined by \( \chi(\bar{a}) = e^{2\pi i a/p} \). Substituting this into (4.6), we see that
\[ N_{D}(b) = \frac{p^{n}}{2pn} \sum_{a=1}^{p-1} e^{2\pi i a/p} e^{\pi i n \left( \frac{p^{2} - 1}{4p} - a + \frac{p - 1}{2} \right)} + O(p^{n/2}). \] (4.7)
The above sum is a geometric series with common ratio of $e^{\pi i (\frac{2b}{p} + sn \frac{p^2 - 1}{4p})}$. Thus when it is 1, that is, $\frac{2b}{p} + sn \frac{p^2 - 1}{4p} \equiv 0 \pmod{2}$, we have

$$N_D(b) = \frac{p^m}{2pn} e^{\pi i \frac{p^2 - 1}{2pn}} (p - 1) + O(p^{n/2}) = (-1)^{sn(p - 1)/2} \frac{(p - 1)p^m}{2pn} + O(p^{n/2}).$$

Otherwise we have

$$N_D(b) = \frac{p^m}{2pn} e^{\pi i (\frac{2b}{p} + sn \frac{p^2 - 1}{4p})} - e^{\pi i sn \frac{p^2 - 1}{4p}} + O(p^{n/2})$$

$$= \frac{p^m}{2pn} e^{\pi i \frac{2b}{p} + sn \frac{p^2 - 1}{4p}} - 1 + \frac{1}{2} e^{\pi i \frac{2b}{p} + sn \frac{p^2 - 1}{4p}} + O(p^{n/2})$$

$$= - (-1)^{sn(p - 1)/2} \frac{p^m}{2pn} + O(p^{n/2}).$$

where we used $p^2 - 1 \equiv 0 \pmod{8}$ for odd primes $p$.

In summary, we have

$$N_D(b) = \begin{cases} (-1)^{sn(p - 1)/2} \frac{(p - 1)p^m}{2pn} + O(p^{n/2}), & \text{if } \frac{2b}{p} + sn \frac{p^2 - 1}{4p} \equiv 0 \pmod{2}; \\ -(-1)^{sn(p - 1)/2} \frac{p^m}{2pn} + O(p^{n/2}), & \text{otherwise}. \end{cases} \quad (4.8)$$

It then follows from (4.11) and (4.8) that $S_{2pn, b} = \frac{(p - 1)p^m}{2pn} + O(p^{n/2})$ if

$$\frac{2b}{p} - \frac{sn(p - 1)(2pn + 1 - p)}{8} + sn \frac{p^2 - 1}{4p} \equiv 0 \pmod{2}, \quad (4.9)$$

and $S_{2pn, b} = - \frac{p^m}{2pn} + O(p^{n/2})$ otherwise. A direct simplification shows that the condition (4.9) is equivalent to

$$\frac{2b}{p} - sn \left\{ \frac{n - 1}{2} \right\} (p - 1) \equiv 0 \pmod{2},$$

which is equivalent to $p \mid b$, since $\frac{n - 1}{2}$ is always an integer and $p - 1$ is an even number. Therefore we conclude that

$$S_{2pn, b} = \begin{cases} \frac{(p - 1)p^{n - 1}}{2pn} + O(p^{n/2}), & \text{if } p \mid b; \\ - \frac{p^m}{2pn} + O(p^{n/2}), & \text{if } p \nmid b. \end{cases}$$

This completes the proof. \qed

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