We suggest a representation of Artin’s constant, which conjecturally describes the density of prime \( p \) such that “generic” \( g \in \mathbb{Z} \) is primitive modulo \( p \). Namely, \( A = \lim_{N \to \infty} R_k(N) \) for \( R_k(N) = \sum p^k \phi(p_i - 1) \), where the summation is over first \( N \) prime \( p_i, k \in \mathbb{Z}_+ \). The classical summation formula is as follows: \( A = \lim_{N \to \infty} \Sigma(N) \), where \( \Sigma(N) = \frac{1}{N} \sum_{i=1}^{N} \frac{\phi(p_i - 1)}{p_i - 1} \). The changes needed for arbitrary \( g \) are addressed in Theorem 1, a good exercise in basic analytic and algebraic number theory. The same procedure can be applied to other number-theoretic constants like \( A \) (see, e.g., [Ni]). In Theorem 2, we demonstrate how it works for the Stephens constant and for Artin’s constants of higher ranks (for the density of prime \( p \) such that a given set of “generic” integers generates \( \mathbb{Z}_p^* \)).

The following three features of this approach vs. the summation formulas are worth noticing.

1) The restricted summation suggested by P. Moore to make the \( \Sigma \)–formula matching the right heuristic density for arbitrary \( g \in \mathbb{Z} \) gives the desired answer in our approach only when \( g \) is not a pure (odd) power in \( \mathbb{Z} \). Otherwise, nontrivial rational multiplicative corrections occur; they are calculated in Theorem 1, (ii).

2) The \( p^k \)–terms in the denominator and numerator of \( R_k(N) \) do not influence the limit, which can be heuristically associated with switching from primitive roots in \( \mathbb{Z}_p \) to those in \((\mathbb{Z}/(p^{k+1}))^*\). The extra \( p^k \)–factors disappear (cancel) in the corresponding summation formula. When \( k = 0 \), our \( R \)–formula for \( A \) (without restricting the summation) follows from [Pi].

3) The \( R \)–formulas oscillate significantly around \( A \) (and the other constants). The magnitude of oscillations increases as \( k \) grows; see Figure 1. Representing \( R_k(N) = \sum_{i=1}^{N} w_i \phi(p_i - 1) \), the weights \( w_i \) change from \( O\left(\frac{\log N}{N^{k+1}}\right) \) for small \( i \) to \( O\left(\frac{\log N)^{k+1}}{N}\right) \) for \( i \sim N \). Thus, large \( p \)
receive greater weights in our approach, especially when \( k \) is large, which increases the range of oscillations.

1. Brief history. Artin’s primitive root conjecture states that given an integer \( g \), possibly negative but not a perfect square in \( \mathbb{Z} \), the number \( \mathcal{P}_N(g) \) of prime \( p \) among \( p_1 = 2, p_2 = 3, \ldots, p_N \) such that \( g \) is primitive modulo \( p \) approaches asymptotically \( A(g)N \) as \( N \to \infty \) for

\[
(1) \quad A(g) = A_h A_d, \quad A_d = \left( 1 - \mu(|d|) \prod_{p \mid d, p \mid h} \frac{1}{p - 2} \prod_{p \mid d, p \nmid h} \frac{1}{p^2 - p - 1} \right),
\]

\[
A_h = \prod_{p \mid h} \left( 1 - \frac{1}{p - 1} \right) \prod_{p \nmid h} \left( 1 - \frac{1}{p(p - 1)} \right), \quad \text{where}
\]

\[ d = \text{Discriminant} \left( \mathbb{Q}[\sqrt{g}] \right), \quad g = g^h_o \text{ for } g_o \in \mathbb{Z} \text{ and maximal } h \in \mathbb{N}. \]

Note that \( \mu(|d|) = 0 \) and, respectively, \( A(g) = A_h \) if and only if the discriminant \( d \) is not from \( 1 + 4\mathbb{Z} \). If \( h = 1 \) for such \( g \), then \( A(g) \) equals

\[
(2) \quad \text{Artin’s constant } = A = \prod_{\text{prime } p} \left( 1 - \frac{1}{p(p - 1)} \right).
\]

According to [St], Artin’s conjecture was finalized around 1965. In 1967, it was deduced by Hooley [Ho] from the generalized Riemann hypothesis for the fields \( K_m = \mathbb{Q}[\zeta_m, g^{1/m}] \) for squarefree \( m \). See [Mo1] for a comprehensive introduction (including some recent developments). See also [Mu, Le].

Artin’s heuristic approach to this conjecture was based on the expectation that events “prime \( p \) does not split completely in \( K_q \) for prime \( q \)” are independent (subject to later qualitative and quantitative corrections). For instance, one can expect that \( A \) equals \( \lim_{N \to \infty} \mathcal{P}_N(g)/N \) if \( g \) is generic as far as primitive roots modulo prime \( p \) are concerned. It leads to the following heuristic summation formula for Artin’s constant:

\[
(3) \quad A = \lim_{N \to \infty} \Sigma(N), \quad \Sigma(N) = \frac{1}{N} \sum_{i=1}^{N} \frac{\phi(p_i - 1)}{p_i - 1},
\]

which can be checked unconditionally, without any reference to Artin’s conjecture. See [LL], [Mo1]. P. Moore extended it to arbitrary \( A(g) \) by switching to \( p \) in this summation such that \( g \) is a quadratic nonresidue modulo \( p \) and \( (p - 1, h) = 1 \) (heuristically, it makes sense); see below.
As Lehmers wrote, the convergence in (3) is “discouragingly slow” (they considered prime numbers $p < 1500000$). It remains very slow when prime numbers in much greater ranges are considered, generally, no better than the (conjectural) convergence of $\mathcal{P}_N(2)/N$ to $A = A(2)$; cf. Table “Artin’s constant estimates” from [Si] ($p < 10^{14}$).

2. Main Theorem. A refine version of this heuristic approach is from [Mo1] (for any integer $g$):

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varphi_g(p_i) = A(g) \left( \frac{\text{RH}}{\text{lim}_{N \to \infty} \mathcal{P}_N(g)/N} \right),$$

where

$$\varphi_g(p) \overset{\text{def}}{=} \begin{cases} 2 \frac{\phi(p - 1)}{p - 1} & \text{for } \left( \frac{g}{p} \right) = -1 \text{ and } (p - 1, h) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Heuristically, it is equally reasonable to expect that

$$A = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} \phi(p_i - 1)}{\sum_{i=1}^{N} (p_i - 1)} \quad \text{for sufficiently general } g.$$  

for sufficiently general $g$. Switching here to the restricted summation from (4), we come to the following theorem.

Theorem 1. (i) For arbitrary integers $k \geq 0$ and $g$ (possibly, negative),

$$A(g) = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} p_i^k \phi(p_i - 1)}{\sum_{i=1}^{N} p_i^k (p_i - 1)} ,$$

provided that $h = 1$, where the summation $\Sigma'$ is over prime $p_i$ such that $\left( \frac{g}{p_i} \right) = -1$ and $(p_i - 1, h) = 1$.

(ii) If $h > 1$, then the limit in the r.h.s. of (6) equals

$$\tilde{A}(g) = \tilde{A}_h \tilde{A}_d , \quad \text{where } \tilde{A}_h = \prod_{p \nmid h} \left( 1 - \frac{1}{p(p - 1)} \right),
$$

$$\tilde{A}_d = 1 \text{ if } d \mid h \text{ and } \tilde{A}_d = A_d \text{ otherwise.}$$

Sketch of proof. We follow [LL, Mo1], restricting ourselves with (5); see also [Pi]. Coupling the generalized Landau formula from [SZ]

$$\sum_{p \leq x} p^m = \frac{(1 + o(1)) x^{m+1}}{(1 + m) \log x}, \quad m \geq 0,$$
with the classical estimate
\[ \frac{\pi(x, d, 1)}{\pi(x)} = 1 + O(1/\log x), \]
where \( \pi(x, d, 1) \) is the number of prime numbers \( p \leq x \) in \( 1 + dN (\pi(x) = \pi(x, 1, 1)) \), one arrives at:
\[ \frac{\pi^{(m)}(x, d, 1)}{\pi^{(m)}(x)} = 1 + O(x^m/\log x), \]
where \( \pi^{(m)}(x, d, 1) = \sum_{p \leq x} p^m \) over prime \( p \in 1 + dN \). Then,
\[ \sum_{p \leq x} \phi(p-1) = \sum_{d \mid p-1} \mu(d) \frac{(p-1)}{d} = \sum_{d \leq x} \frac{\mu(d)}{d} (\pi^{(1)}(x, d, 1) - \pi(x, d, 1)). \]
Finally,
\[ \frac{\sum_{p \leq x} \phi(p-1)}{\sum_{p \leq x} (p-1)} \sim \sum_{d \leq x} \frac{\mu(d)}{d \phi(d)} \xrightarrow{x \to \infty} A. \]
We will omit the arguments (from the basic algebraic number theory) that give the rational corrections for arbitrary \( g \). □

In (5,6), only the leading powers of \( p_i \) matter; for instance, one can take here the ratio \( \sum_p \phi(p^{k+1})/\sum_p \phi(p^{k+1}) \), the heuristic probability for \( g \) being a primitive root modulo \( p^{k+1} \) over all prime \( p \), which leads to the same Artin constant.

3. Further examples. Let us apply the same procedure to Stephens’ constant \( S(a, b) \) and the higher rank Artin constants. The former is defined for a given pair \( a, b \in \mathbb{Q}^* \) such that \( a^r b^s = 1 \implies r = 0, s = 0 \) for \( r, s \in \mathbb{Z} \); it describes the density of prime \( p \) such that \( b = a^m \mod p \) for some \( m \in \mathbb{Z} \). Modulo the generalized Riemann hypothesis, it equals
\[ C_{ab} S \overset{\text{def}}{=} \prod_{i=1}^{\infty} \left( 1 - \frac{p_i}{p_i^2 - 1} \right), \]
where the factors \( C_{ab} \) are rational [S, MS]. These factors were calculated explicitly in [MS] under the condition that the group \( \mathbb{Q}^*/\langle a, b, -1 \rangle \)
is torsion free. Assuming that $a, b$ are “random”, the heuristic probability $PS(p)$ that $b = a^m \mod p$ for some $m$ can be readily calculated:

$PS(p) = S(p) / (p - 1)^2$, where $S(p) \equiv \sum_{d|p-1} d\phi(d)$

$= \prod_{j=1}^{m} \frac{q_j^{2k_j+1} + 1}{q_j + 1}$ for prime factorization $p - 1 = \prod_{j=1}^{m} q_j^{k_j}$.  

We naturally omit the prime numbers that divide the numerators or denominators of $a, b$.

The rank $r$ Artin constant $A_r(g_1, \ldots, g_r)$ describes the heuristic density of prime $p$ such that a given set of nonzero integers $\{g_1, \ldots, g_r\}$ (or rationals) generates $\mathbb{Z}_p^*$. See [CP]. Its “generic” value (modulo the generalized Riemann hypothesis) is as follows:

$A_r \equiv \prod_{i=1}^{\infty} \left( 1 - \frac{1}{p_i^r (p_i - 1)} \right)$.

The corresponding probability at $p$ for “random” $g_1, \ldots, g_r$ equals

$PA_r(p) = A_r(p) / (p - 1)^r$ for $A_r(p) \equiv \prod_{j=1}^{m} (q_j^{r_{k_j}} - q_j^{r(k_j - 1)})$

in terms of the prime factorization $p - 1 = \prod_{j=1}^{m} q_j^{k_j}$.

The summation (unconditional) limiting formulas are as follows:

$S = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} PS(p_i), \ A_r = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} PA_r(p_i)$.  

The proof of the following theorem is similar to that of Theorem 1.

**Theorem 2.** For an arbitrary integer $k \geq 0$,

$S = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} p_i^k S(p_i)}{\sum_{i=1}^{N} p_i^k (p_i - 1)^2}$,  

$A_r = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} p_i^k A_r(p_i)}{\sum_{i=1}^{N} p_i^k (p_i - 1)^r}$,  

where the summation is over consecutive prime numbers $p_i$.  

\[\square\]
4. Numerical aspects. For the constants $C$ considered above, we plot $\frac{\Sigma(N)}{C} - 1$, shown blue-thin, for the classical summation $\Sigma(N)$ and $\frac{R(N)}{C} - 1$, which are red-thick, for our ratio approximations $R(N)$; for instance, $\Sigma(N)$ is from (3) for Artin’s $A$. The range is $N \leq 1000M$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Deviation from $A$ as $k = 0, 1$ for $1000M$ primes.}
\end{figure}
Figure 1 compares the stabilization of (3), blue-thin, to the Artin constant and the stabilization of (5), red-thick, as \( k = 0, 1 \). The function \( \Sigma(N) - A \) remains positive in the range \( N \leq 1000M \); \( R_k(N) \) oscillate around Artin’s constant \( A \approx 0.37395581361920228805 \) (the zero level of this graph). The amplitude of oscillations become larger for \( R_{k=1} \) vs. \( R_{k=0} \), however the graphs are very much similar.

We note that the best way to calculate \( A \) and similar constants is based on the known product formulas in terms of \( \zeta(n) \) for integers \( n > 1 \).

Figure 2 shows the convergence of (4) and (6) to \( A(g = 5) \) for \( k = 0 \); notice that the blue-thin curve remains beyond the red-thick one in this range.

The last two plots show the graphs for the Stephen constant \( S \approx 0.57595996889294543964 \), Figure 3, and the rank 3 Artin constant \( A_3 \approx 0.85654044485354217443 \), Figure 4; here \( k = 0 \). The convergence rate and other features of these four graphs are similar to those for \( A \) and \( A(5) \). There is striking (qualitative) similarity of these two figures, including the oscillations, although the convergence rate in Figure 4 (for \( A_3 \)) is significantly (almost 10 times) greater than in Figure 3.
Figure 3. Deviation from $S$ for 1000M primes.

Figure 4. Deviation from $A_{[t=3]}$ (rank=3) for 1000M primes.
Qualitatively, the behavior of $R(N)$ for large $N$ can be evaluated following [Pi]. For instance, the functions $|R(N)/A − 1|$ in Figure 1 must be no greater than $C_m(\log N)^{-m}$ for any fixed $m > 0$ and proper constant $C_m$ (depending on $k$) as $N >> 0$. Generally speaking, $C_m$ can be estimated in terms of (the order of) $N$, but we will not discuss it. Indeed, the graphs of $R(N)/A − 1$, red-thick from Figure 1, look like $O((\log N)^{-4})$ in the range $N < 1000M$.

The nature of oscillations of the functions $R(N)$ around the corresponding constants remains unclear.

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References

[CP] L. Cangelmi, and F. Pappalardi, *On the r-ranking Artin conjecture II*, J. Number Theory 75 (1999), 120–132.

[Ho] C. Hooley, *Artins conjecture for primitive roots*, J. Reine Angew. Math. 225 (1967), 209-220.

[LL] D.H. Lehmer, and E. Lehmer, *Heuristics, anyone?*, in “Studies in Mathematical Analysis and Related Topics”, 202-210, Stanford Univ. Press, Stanford, CA, 1962.

[Le] H.W. Lenstra, jr., *On Artins conjecture and Euclid’s algorithm in global fields*, Invent. Math. 42 (1977), 202-224.

[Mo1] P. Moree, *Artins primitive root conjecture*, Preprint arXiv:math/0412262 [math.NT], 2004.

[Mo2] P. Moree, *Asymptotically exact heuristics for (near) primitive roots*, J. Number Theory 83:1 (2000), 155–181.

[Mo3] P. Moree, *Asymptotically exact heuristics for (near) primitive roots. II*, Japan. J. Math. (N.S.) 29:2 (2003), 143–157.

[MS] P. Moree, and P. Stevenhagen, *A two variable Artin conjecture*, J. Number Theory 85 (2000), 291–304.

[Mu] R. Murty, *Artins conjecture for primitive roots*, Math. Intelligencer 10:4 (1988), 59-67.

[Ni] G. Niklasch, *Some number-theoretical constants arising as products of rational functions of p over primes*, Preprint 2002; http://www.gn-50uma.de/alula/essays/Moore/Moore.en.shtml.
[Pi] S.S. Pillai, On the sum function connected with primitive roots, Proc. Indian Acad. Sci., Sect. A. 13 (1941), 526–529. 1, 3, 9

[Si] T. Oliveira e Silva, Least primitive root of prime numbers, Preprint 2004; http://www.ieeta.pt/~tos/p-roots.html . 3

[SZ] T. Salát, and S. Znám, On the sums of prime powers, Acta Fac. Rer. Univ. Com. Math. 21 (1968), 21–25. 3

[S] P.J. Stephens, Prime divisors of second order linear recurrences, J. Number Theory 8:3 (1976), 313-345. 4

[St] P. Stevenhagen, The correction factor in Artins primitive root conjecture, J. Théor. Nombres Bordeaux 15 (2003), 383-391. 2

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