Abstract

The matrix elements for $K \to \pi\pi l\nu$ decays are described by four form factors $F, G, H$ and $R$. We complete previous calculations by evaluating $R$ at next-to-leading order in the low-energy expansion. We then estimate higher order contributions using dispersion relations and determine the low-energy constants $L_1, L_2$ and $L_3$ from data on $K_e4$ decays and on elastic pion scattering. Finally, we present predictions for the slope of the form factor $G$ and for total decay rates.

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1 Introduction

In this article we analyze $K_{l4}$ decays,

$$K \rightarrow \pi \pi l \nu ; \; l = e, \mu \; ,$$

(1.1)
in the framework of chiral perturbation theory (CHPT). This method, also called "energy expansion" in the following, is based on an expansion of the Green functions in powers of the external momenta and of the light quark masses [1]-[5]. The matrix elements for $K_{l4}$ processes are described by four form factors $F, G, H$ and $R$. Their energy expansion reads

$$I = \frac{M_K}{F_\pi} \left\{ I^{(0)} + I^{(2)} + I^{(4)} + \cdots \right\} ; \; I = F, G, H, R \; ,$$

(1.2)

where $I^{(n)}$ is a quantity of order $E^n$. The predictions for the lowest order terms were first given by Weinberg [6]. The anomaly contribution $H^{(2)}$ has been determined in [7], whereas $F^{(2)}$ and $G^{(2)}$ have been evaluated in [8, 9]. For a calculation of $H^{(4)}$ see [10]. The experimental results [11] for $F, G$ turn out to be $30-50\%$ above the leading contributions. The missing piece must then come from higher orders. The expressions for $F^{(2)}, G^{(2)}$ involve the low-energy constants $L_1, \ldots, L_9$, which are not fixed by chiral symmetry alone and which must be determined phenomenologically at the present stage of our capability to solve QCD. This was done in [3], where the $L_i$’s were pinned down using experimental data (not related to $K_{l4}$ decays) and involving the large-$N_C$ prediction which states that, in the limit where the number of colors becomes large, certain combinations of the low-energy constants are suppressed. The decay (1.1) is the simplest process where this rule can be tested [8, 9]. In addition, it allows one to perform an independent determination of $L_1, L_2$ and $L_3$ and thus to check consistency with other data.

The aim of the present article is threefold. First, we fill the gap in the literature and evaluate also the next-to-leading order term $R^{(2)}$. [The amplitude $R$ is completely negligible in $K_{e4}$ decays, because its contribution to the rate is suppressed by the factor $m_l^2$. It must be retained, however, in the $K_{\mu4}$ channel ]. Second, we note that, because the strange quark mass is not very small on a typical hadronic scale, the corrections $I^{(2)}$ to the leading-order terms of the form factors are substantial. The determination of the $L_i$’s from $K_{l4}$ decays is therefore affected with substantial uncertainties if carried out using only the first two terms in the expansion (1.2), as was done in [8, 9]. Here we improve these calculations by estimating the size of higher order contributions to $F$. We use for this purpose the method developed in [12], which amounts to write a dispersive representation for the relevant partial wave amplitudes, fixing the corresponding subtraction constants with chiral perturbation theory. We are then able to reduce the uncertainties in the determination of $L_1, L_2$ and $L_3$ in a significant manner, even more so if data on elastic $\pi\pi$ scattering is considered in addition. Third, we predict the slope of the form factor $G$, and show that we may evaluate total decay rates for all channels in $K_{l4}$ decays within rather
small uncertainties, provided the leading $S$- and $P$-wave form factors have been determined experimentally e.g. from $K^+ \rightarrow \pi^+\pi^-\nu_e$ decays.

The plan of the paper is as follows. In section 2, we provide the necessary kinematics and the definition of the form factors. Section 3 contains the result of the one-loop calculation of these quantities. In section 4, we use dispersion relations to construct a $I = 0$ $S$-wave amplitude which has the correct phase to higher orders in the low-energy expansion. In section 5, we use this improved amplitude to determine the low-energy constants $L_1$, $L_2$ and $L_3$. Section 6 contains predictions, whereas a summary and concluding remarks are presented in section 7.

2 Kinematics and form factors

We discuss the decays

$$K^+(p) \rightarrow \pi^+(p_1) \pi^-(p_2) l^+(p_l) \nu_l(p_{\nu}) \ ,$$  
(2.1)

$$K^+(p) \rightarrow \pi^0(p_1) \pi^0(p_2) l^+(p_l) \nu_l(p_{\nu}) \ ,$$  
(2.2)

$$K^0(p) \rightarrow \pi^0(p_1) \pi^-(p_2) l^+(p_l) \nu_l(p_{\nu}) \ ; \ l = e, \mu \ ,$$  
(2.3)

and their charge conjugate modes. We do not consider isospin violating contributions and correspondingly set $m_u = m_d$, $\alpha_{\text{QED}} = 0$.

2.1 Kinematics

We start with the process (2.1). The full kinematics of this decay requires five variables. We will use the ones introduced by Cabibbo and Maksymowicz [13]. It is convenient to consider three reference frames, namely the $K^+$ rest system ($\Sigma_K$), the $\pi^+\pi^-$ center-of-mass system ($\Sigma_{2\pi}$) and the $l^+\nu_l$ center-of-mass system ($\Sigma_{l\nu}$). Then the variables are

1. $s_\pi$, the effective mass squared of the dipion system,
2. $s_l$, the effective mass squared of the dilepton system,
3. $\theta_\pi$, the angle of the $\pi^+$ in $\Sigma_{2\pi}$ with respect to the dipion line of flight in $\Sigma_K$,
4. $\theta_l$, the angle of the $l^+$ in $\Sigma_{l\nu}$ with respect to the dilepton line of flight in $\Sigma_K$; and
5. $\phi$, the angle between the plane formed by the pions in $\Sigma_K$ and the corresponding plane formed by the dileptons.

The angles $\theta_\pi$, $\theta_l$ and $\phi$ are displayed in Fig. 1.
The range of the variables is
\[
4M_\pi^2 \leq s_\pi = (p_1 + p_2)^2 \leq (M_K - m_l)^2 ,
\]
\[
m_l^2 \leq s_l = (p_l + p_\nu)^2 \leq (M_K - \sqrt{s_\pi})^2 ,
\]
\[
0 \leq \theta_\pi, \theta_l \leq \pi, 0 \leq \phi \leq 2\pi . \tag{2.4}
\]

It is useful to furthermore introduce the following combinations of four vectors
\[
P = p_1 + p_2, \quad Q = p_1 - p_2, \quad L = p_l + p_\nu, \quad N = p_l - p_\nu . \tag{2.5}
\]

Below we will also use the variables
\[
t = (p_1 - p)^2, \quad u = (p_2 - p)^2, \quad \nu = t - u . \tag{2.6}
\]

These are related to \(s_\pi, s_l\) and \(\theta_\pi\) by
\[
t + u = 2M_\pi^2 + M_K^2 + s_l - s_\pi ,
\]
\[
\nu = -2\sigma_\pi X \cos \theta_\pi , \tag{2.7}
\]
where
\[
\sigma_\pi = (1 - 4M_\pi^2/s_\pi)^{1/2} ,
\]
\[
X = \frac{1}{2} \lambda^{1/2}(M_K^2, s_\pi, s_l) ,
\]
\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz) . \tag{2.8}
\]

In addition we define
\[
\Sigma = M_K^2 + M_\pi^2 . \tag{2.9}
\]

### 2.2 Matrix elements and decay rates

The matrix element for \(K^+ \to \pi^+\pi^- l^+ \nu_l\) is
\[
T = \frac{G_F}{\sqrt{2}} V_{us}^* \bar{u}(p_\nu)\gamma\mu(1 - \gamma_5)\nu(p_l)(V^\mu - A^\mu) , \tag{2.10}
\]

where\footnote{In order to agree with the notation used by Pais and Treiman \[14\] and by Rosselet et al. \[11\], we have changed the previous convention \[8, 9\] in the definition of the anomaly form factor \(H\).}

\[
I_{\mu} = \langle \pi^+(p_1)\pi^-(p_2)_{\text{out}} | I_{\mu}^{A-i5}(0) | K^+(p) \rangle ; \quad I = V, A ,
\]

\[
V_{\mu} = -\frac{H}{M_K^3} \epsilon_{\mu\nu\rho\sigma} L^\nu P^\rho Q^\sigma ,
\]

\[
A_{\mu} = -i \frac{1}{M_K} [P_{\mu} F + Q_\mu G + L_{\mu} R] , \tag{2.11}
\]
and \( \epsilon_{0123} = 1 \). The matrix elements for the other channels (2.2, 2.3) may be obtained from (2.10, 2.11) by isospin symmetry, see below.

The form factors \( F, G, R \) and \( H \) are analytic functions of the variables \( s_\pi, t \) and \( u \). The partial decay rate for (2.1) is given by

\[
d\Gamma = \frac{1}{2M_K(2\pi)^8} \sum_{\text{spins}} |T|^2 \delta^4(p - P - L) \frac{d^3p_1}{2p_1^0} \frac{d^3p_2}{2p_2^0} \frac{d^3p_3}{2p_3^0} \frac{d^3p_4}{2p_4^0} .
\] (2.12)

The quantity \( \sum_{\text{spins}} |T|^2 \) is a Lorentz invariant quadratic form in \( F, G, R \) and \( H \). All scalar products can be expressed in terms of the 5 independent variables \( s_\pi, s_t, \theta_\pi, \theta_t \) and \( \phi \), such that

\[
\sum_{\text{spins}} |T|^2 = 2G_F^2 |V_{us}|^2 M_K^{-2} J_5(s_\pi, s_t, \theta_\pi, \theta_t, \phi) .
\] (2.13)

Carrying out the integrations over the remaining \( 4 \cdot 3 - 5 = 7 \) variables in (2.12) gives [13]

\[
d\Gamma_5 = G_F^2 |V_{us}|^2 N(s_\pi, s_t) J_5(s_\pi, s_t, \theta_\pi, \theta_t, \phi) ds_\pi ds_t d(\cos \theta_\pi) d(\cos \theta_t) d\phi ,
\]

\[
N(s_\pi, s_t) = (1 - z_t) \sigma_\pi X/(2^{13} \pi^6 M_K^5) ,
\] (2.14)

where

\[
J_5 = 2(1 - z_t) \left[ I_1 + I_2 \cos 2\theta_t + I_3 \sin^2 \theta_t \cdot \cos 2\phi + I_4 \sin 2\theta_t \cdot \cos \phi + I_5 \sin \theta_t \cdot \cos \phi + I_6 \cos \theta_t \cdot \sin \phi + I_7 \sin \theta_t \cdot \sin \phi + I_8 \sin 2\theta_t \cdot \sin 2\phi \right] ,
\]

with

\[
I_1 = \frac{1}{4} \left\{ (1 + z_t)|F_1|^2 + \frac{1}{2}(3 + z_t) \left( |F_2|^2 + |F_3|^2 \right) \sin^2 \theta_\pi + 2z_t|F_4|^2 \right\} ,
\]

\[
I_2 = -\frac{1}{4}(1 - z_t) \left\{ |F_1|^2 - \frac{1}{2} \left( |F_2|^2 + |F_3|^2 \right) \sin^2 \theta_\pi \right\} ,
\]

\[
I_3 = -\frac{1}{4}(1 - z_t) \left\{ |F_2|^2 - |F_3|^2 \right\} \sin^2 \theta_\pi ,
\]

\[
I_4 = \frac{1}{2}(1 - z_t) \Re(F_1^* F_2) \sin \theta_\pi ,
\]

\[
I_5 = -\left\{ \Re(F_1^* F_3) + z_t \Re(F_4^* F_2) \right\} \sin \theta_\pi ,
\]

\[
I_6 = -\left\{ \Re(F_2^* F_3) \sin^2 \theta_\pi - z_t \Re(F_4^* F_1) \right\} ,
\]

\[
I_7 = -\left\{ \Im(F_1^* F_2) + z_t \Im(F_4^* F_3) \right\} \sin \theta_\pi ,
\]

\[
I_8 = \frac{1}{2}(1 - z_t) \Im(F_1^* F_3) \sin \theta_\pi ,
\]

\[
I_9 = -\frac{1}{2}(1 - z_t) \Im(F_2^* F_3) \sin^2 \theta_\pi ,
\] (2.15)
and
\[
F_1 = X \cdot F + \sigma_{\pi}(P L) \cos \theta_{\pi} \cdot G , \\
F_2 = \sigma_{\pi}(s_{\pi}s_l)^{1/2} G , \\
F_3 = \sigma_{\pi}X (s_{\pi}s_l)^{1/2} \frac{H}{M_K^2} , \\
F_4 = -(PL)F - s_lR - \sigma_{\pi}X \cos \theta_{\pi} \cdot G .
\]  
(2.16)

The definition of \(F_1, \ldots, F_4\) corresponds to the combinations used by Pais and Treiman [14] (the different sign in the terms \(\sim PL\) is due to our use of the metric \(\text{diag}(+- - -)\)). The form factors \(I_1, \ldots, I_9\) agree with the expressions given in [14].

### 2.3 Isospin decomposition

The \(K_{l4}\) decays (2.2) and (2.3) involve the same form factors as displayed in Eq. (2.11). We denote by \(A_{+-}, A_{00}\) and \(A_{0-}\) the current matrix elements of the processes (2.1)-(2.3). They are related by isospin symmetry
\[
A_{+-} = \frac{A_{0-}}{\sqrt{2}} - A_{00} .
\]  
(2.17)

This relation also holds for the individual form factors, which may be decomposed into a symmetric and an antisymmetric part under \(t \leftrightarrow u\) (\(p_1 \leftrightarrow p_2\)). Because of Bose symmetry and of the \(\Delta I = \frac{1}{2}\) rule of the relevant weak currents, one has
\[
(F, G, R, H)_{00} = -(F^+, G^-, R^+, H^-)_{+-} , \\
(F, G, R, H)_{0-} = \sqrt{2}(F^-, G^+, R^-, H^+)_{+-} ,
\]  
(2.18)

where
\[
F^\pm_{+-} = \frac{1}{2}[F(s_{\pi}, t, u) \pm F(s_{\pi}, u, t)] ,
\]  
(2.19)

and \(F(s_{\pi}, t, u)\) is defined in Eq. (2.11). \(G^\pm, R^\pm\) and \(H^\pm\) are defined similarly. The isospin relation for the decay rates is
\[
\Gamma(K^+ \rightarrow \pi^+\pi^-l^+\nu_l) = \frac{1}{2}\Gamma(K^0 \rightarrow \pi^0\pi^-l^+\nu_l) + 2\Gamma(K^+ \rightarrow \pi^0\pi^0l^+\nu_l) .
\]  
(2.20)

### 2.4 Partial wave expansion

The form factors may be written in a partial wave expansion in the variable \(\theta_{\pi}\). We consider a definite isospin \(\pi\pi\) state. Suppressing isospin indices, one has [14]
\[
F = \sum_{l=0}^{\infty} P_l(\cos \theta_{\pi}) f_l - \frac{\sigma_{\pi}PL}{X} \cos \theta_{\pi} G ,
\]  
(2.21)

\[\text{We use the Condon-Shortley phase conventions. Notice that we evaluate matrix elements and decay rates for } K^0 - \text{they differ from the corresponding } K_L-\text{quantities by a normalization factor.}\]
\[ G = \sum_{l=1}^{\infty} P'_{l}(\cos \theta_{\pi}) g_{l} , \]
\[ R = \sum_{l=0}^{\infty} P_{l}(\cos \theta_{\pi}) r_{l} + \frac{\sigma_{\pi} s_{\pi}}{X} \cos \theta_{\pi} G , \]
\[ H = \sum_{l=0}^{\infty} P'_{l}(\cos \theta_{\pi}) h_{l} , \]

(2.21)

where
\[ P'_{l}(z) = \frac{d}{dz} P_{l}(z) . \]

(2.22)

The partial wave amplitudes \( f_{l}, g_{l}, r_{l} \) and \( h_{l} \) depend on \( s_{\pi} \) and \( s_{I} \). Their phase coincides with the phase shifts \( \delta_{l}^{I} \) in elastic \( \pi\pi \) scattering (angular momentum \( l \), isospin \( I \)). More precisely, the quantities
\[ e^{-i\delta_{0}^{l} X_{2l}} , \]
\[ e^{-i\delta_{2l+1}^{l} X_{2l+1}} ; \quad l = 0, 1, \ldots ; \quad X = f, g, r, h , \]

(2.23)
are real in the physical region of \( K_{4} \) decay (in our overall phase convention). The form factors \( F_{1} \) and \( F_{4} \) therefore have a simple expansion,
\[ F_{1} = X \sum_{l} P_{l}(\cos \theta_{\pi}) f_{l} , \]
\[ F_{4} = - \sum_{l} P_{l}(\cos \theta_{\pi})(PLf_{l} + s_{l}r_{l}). \]

(2.24)

3 Theory

The theoretical predictions of \( K_{4} \) form factors have a long history which started in the sixties with the current algebra evaluation of \( F, G, R \) and \( H \). For an early review of the subject and for references to work prior to CHPT we refer the reader to [10] (see also [17]). Here we concentrate on the evaluation of the form factors in the framework of CHPT [8, 9, 18].

3.1 Tree level

The chiral representation of the form factors at leading order was originally given by Weinberg [3],
\[ F = G = \frac{M_{K}}{\sqrt{2} F_{\pi}} = 3.74 , \]
\[ R = \frac{M_{K}}{2 \sqrt{2} F_{\pi}} \left( \frac{s_{\pi} + \nu}{s_{I} - M_{K}^{2}} + 1 \right) , \]
\[ H = 0 . \]

(3.1)
The next-to-leading order corrections are displayed below, and the later sections contain an estimate of yet higher order contributions. Here we note that the total decay rates which follow from Eq. (3.1) are typically a factor of two (or more) below the data. As an example, consider the channel $K^+ \to \pi^+\pi^-e^+\nu_e$. Using (3.1), the total decay rate becomes $1297 \text{ sec}^{-1}$, whereas the experimental value is $3160\pm140 \text{ sec}^{-1}$.

3.2 The form factors at one-loop

In Ref. [8, 9], the form factors $F$, $G$ and $H$ have been evaluated in CHPT at order $p^4$ (see also [19]). We complement these works with the evaluation of $R$ at the same order. Below we display the result of our calculation, referring the reader to the above references and to available reviews [5] for the details of the methods used.

In order to make this article reasonably self-contained, we display the result of all four form factors. The result for $F$ may be written in the form

$$F(s_\pi, t, u) = \frac{M_K}{\sqrt{2}F_\pi} \left\{ 1 + \frac{1}{F^2_\pi} (U_F + P_F + C_F) + O(E^4) \right\} .$$

(3.2)

The contribution $U_F(s_\pi, t, u)$ denotes the unitarity correction generated by the one-loop graphs which appear at order $E^4$ in the low-energy expansion. It has the form

$$U_F(s_\pi, t, u) = \Delta_0(s_\pi) + A_F(t) + B(t, u) ,$$

(3.3)

with

$$\Delta_0(s_\pi) = \frac{1}{2} (2s_\pi - M^2_\pi) J^r_{\pi\pi}(s_\pi) + \frac{3s_\pi}{4} J^r_{KK}(s_\pi) + \frac{M^2_\pi}{2} J^r_{\eta\eta}(s_\pi) ,$$

$$A_F(t) = \frac{1}{16} \left[ (14M^2_K + 14M^2_\pi - 19t)J^r_{K\pi}(t) + (2M^2_K + 2M^2_\pi - 3t)J^r_{\eta K}(t) \right]$$

$$+ \frac{1}{8} \left[ (3M^2_K - 7M^2_\pi + 5t)K_{K\pi}(t) + (M^2_K - 5M^2_\pi + 3t)K_{\eta K}(t) \right]$$

$$- \frac{1}{4} \left[ 9(L_{K\pi}(t) + L_{\eta K}(t)) + (3M^2_K - 3M^2_\pi - 9t)(M^r_{K\pi}(t) + M^r_{\eta K}(t)) \right] ,$$

$$B(t, u) = -\frac{1}{2} (M^2_K + M^2_\pi - t)J^r_{KK}(t) - (t \leftrightarrow u) .$$

(3.4)

The loop integrals $J^r_{\pi\pi}(s_\pi), \ldots$ which occur in these expressions are listed in appendix A. The functions $J^r_{PQ}$ and $M^r_{PQ}$ depend on the scale $\mu$ at which the loops are renormalized. The scale drops out in the expression for the full amplitude (see below).

The imaginary part of $F^{-2}_\pi \Delta_0(s_\pi)$ contains the $I = 0$, $S$-wave $\pi\pi$ phase shift

$$\delta^0_0(s_\pi) = \frac{1}{32\pi F^2_\pi} (2s_\pi - M^2_\pi) \sigma_\pi + O(E^4) ,$$

(3.5)

3If not stated otherwise, we use $F_\pi = 93.2$ MeV, $|V_{us}| = 0.22$ and $(M_\pi, M_K) = (139.6, 493.6)$ MeV, $(135, 493.6)$ MeV and $(137, 497.7)$ Mev for the decays (2.1), (2.2) and (2.3), respectively.
as well as contributions from $K\bar{K}$ and $\eta\eta$ intermediate states. The functions $A_F(t)$ and $B(t, u)$ are real in the physical region.

The contribution $P_F(s, t, u)$ is a polynomial in $s, t, u$ obtained from the tree graphs at order $E^4$. We find

$$P_F(s, t, u) = \sum_{i=1}^{9} p_{i,F}(s, t, u) L_i^r,$$

where

$$p_{1,F} = 32(s_{\pi} - 2M^2_{\pi}),$$
$$p_{2,F} = 8(M^2_K + s_{\pi} - s_l),$$
$$p_{3,F} = 4(M^2_K - 3M^2_{\pi} + 2s_{\pi} - t),$$
$$p_{4,F} = 32M^2_{\pi},$$
$$p_{5,F} = 4M^2_{\pi},$$
$$p_{9,F} = 2s_l.$$  \hspace{1cm} (3.7)

The remaining coefficients $p_{i,F}$ are zero. The quantities $L_i^r$ denote the renormalized coupling constants which parametrize the effective lagrangian at order $E^4$ \[3\]. Their scale dependence is

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{\Gamma_i}{16\pi^2} \ln \frac{\mu_1}{\mu_2}.$$  \hspace{1cm} (3.8)

Observable quantities are independent of the scale $\mu$, once the loop contributions are included. The coefficients $\Gamma_i$ are displayed in table 1, together with the value \[3\] of the couplings $L_i^r$ at $\mu = M_{\rho}$.

The contributions $C_F$ contain logarithmic terms, independent of $s_{\pi}, t$ and $u$:

$$C_F = \frac{1}{256\pi^2} \left[ 5M^2_{\pi} \ln \frac{M^2_{\pi}}{\mu^2} - 2M^2_K \ln \frac{M^2_K}{\mu^2} - 3M^2_{\eta} \ln \frac{M^2_{\eta}}{\mu^2} \right].$$  \hspace{1cm} (3.9)

The corresponding decomposition of the form factor $G$ is

$$G(s, t; u) = \frac{M_K}{\sqrt{2}F_{\pi}} \left\{ 1 + \frac{1}{F^2_{\pi}} (U_G + P_G + C_G) + O(E^4) \right\},$$

$$U_G(s, t; u) = \Delta_1(s_{\pi}) + A_G(t) + B(t, u).$$  \hspace{1cm} (3.10)

with

$$\Delta_1(s_{\pi}) = 2s_{\pi} \left\{ M_{\pi\pi}(s_{\pi}) + \frac{1}{2} M_{K\pi}(s_{\pi}) \right\},$$

$$A_G(t) = \frac{1}{16} \left[ (2M^2_K + 2M^2_{\pi} + 3t) J_{K\pi}(t) - (2M^2_K + 2M^2_{\pi} - 3t) J_{\eta K}(t) \right]$$

$$+ \frac{1}{8} \left[ (3M^2_K + 7M^2_{\pi} - 5t) K_{K\pi}(t) + (M^2_K + 5M^2_{\pi} - 3t) K_{\eta K}(t) \right]$$

$$- \frac{3}{4} \left[ L_K(t) + L_{\eta K}(t) - (M^2_K - M^2_{\pi} + t)(M_{K\pi}(t) + M_{\eta K}(t)) \right].$$  \hspace{1cm} (3.11)
Table 1: Phenomenological values and source for the renormalized coupling constants $L_i^r(M_\rho)$ according to Ref. [3]. The quantities $\Gamma_i$ in the fourth column determine the scale dependence of the $L_i^r(\mu)$ according to Eq. (3.8). $L_{11}^r$ and $L_{12}^r$ are not directly accessible to experiment.

| $i$ | $L_i^r(M_\rho) \times 10^3$ | source | $\Gamma_i$ |
|-----|----------------------------|--------|----------|
| 1   | 0.7 ± 0.5                  | $\pi\pi$ $D$-waves, Zweig rule | 3/32   |
| 2   | 1.3 ± 0.7                  | $\pi\pi$ $D$-waves      | 3/16   |
| 3   | −4.4 ± 2.5                 | $\pi\pi$ $D$-waves, Zweig rule | 0    |
| 4   | −0.3 ± 0.5                 | Zweig rule               | 1/8    |
| 5   | 1.4 ± 0.5                  | $F_K : F_\pi$            | 3/8    |
| 6   | −0.2 ± 0.3                 | Zweig rule               | 11/144 |
| 7   | −0.4 ± 0.2                 | Gell-Mann-Okubo,$L_5$, $L_8$ | 0    |
| 8   | 0.9 ± 0.3                  | $M_{K^0} - M_{K^+}$, $L_5$, $L_8$ | 5/48  |
|     |                            | $(2m_s - m_u - m_d):(m_d - m_u)$ |        |
| 9   | 6.9 ± 0.7                  | $< r^2 >_{\pi m}$       | 1/4    |
| 10  | −5.5 ± 0.7                 | $\pi \rightarrow e\nu\gamma$ | −1/4  |
| 11  |                            |                         | −1/8   |
| 12  |                            |                         | 5/24   |

The imaginary part of $F_{\pi}^{-2}\Delta_1(s_\pi)$ contains the $I = 1$, $P$-wave phase shift

$$\delta_1^I(s_\pi) = (96\pi F_{\pi}^2)^{-1}s_\pi \sigma_\pi^3 + O(E^4). \quad (3.12)$$

as well as contributions from $K\bar{K}$ intermediate states. The function $A_G$ is real in the physical region. The polynomials

$$P_G = \sum_{i=1}^{9} p_{i,G}(s_\pi, t, u)L_i^r \quad (3.13)$$

are

$$
\begin{align*}
p_{2,G} &= 8(t - u) , \\
p_{3,G} &= 4(t - M_K^2 - M_\pi^2) , \\
p_{5,G} &= 4M_\pi^2 , \\
p_{9,G} &= 2s_l .
\end{align*}
$$

(3.14)

The remaining $p_{i,G}$ vanish. The logarithms contained in $C_G$ are

$$C_G = -C_F. \quad (3.15)$$

The form factor $R$ contains a pole part $Z(s_\pi, t, u)/(s_l - M_K^2)$ and a regular piece $Q$. [Since the axial current acts as an interpolating field for a kaon, the residue of...
the pole part is related to the $KK \to \pi\pi$ amplitude in the standard manner. We write

\[
R = \frac{M_K}{2\sqrt{2}F_\pi} \left\{ \frac{Z}{s_I - M^2_K} + Q + O(E^4) \right\},
\]

\[
I = B_I + \frac{1}{F^2_\pi}(U_I + P_I + C_I), \quad I = Z, Q. \quad (3.16)
\]

The Born terms $B_I$ are

\[
B_Z = s_\pi + \nu,
\]

\[
B_Q = 1. \quad (3.17)
\]

For the loop corrections $U_I, P_I$ and $C_I$ we find for the residue $Z$

\[
U_Z = s_\pi \Delta_0(s_\pi) + \nu \Delta_1(s_\pi) - \frac{4}{9} M^2_K M^2_\pi J^r_{\eta\eta}(s_\pi)
\]

\[
+ \frac{1}{32} \left[ 11(s_\pi - \nu)^2 - 20\Sigma(s_\pi - \nu) + 12\Sigma^2 \right] J^r_{K\pi}(t)
\]

\[
+ \frac{1}{96} \left[ 3(s_\pi - \nu) - 2\Sigma \right]^2 J^r_{\eta\eta}(t)
\]

\[
+ \frac{1}{4} (s_\pi + \nu)^2 J^r_{K\pi}(u)
\]

\[
+ \frac{1}{4} (M^2_K - M^2_\pi) \left[ 5(s_\pi - \nu) - 6\Sigma \right] K_{K\pi}(t)
\]

\[
+ \frac{1}{4} (M^2_K - M^2_\pi) \left[ 3(s_\pi - \nu) - 2\Sigma \right] K_{\eta\eta}(t)
\]

\[
+ \frac{3}{8} \left[ 2s_\pi(\nu + 4\Sigma) - 3s^2_\pi + \nu^2 - 16M^2_\pi M^2_K \right] \left[ M^r_{K\pi}(t) + M^r_{\eta\eta}(t) \right]
\]

\[
- \frac{3}{4} (3s_\pi + \nu - 2\Sigma)(L_{\eta\eta}(t) + L_{K\pi}(t)), \quad (3.18)
\]

and

\[
P_Z(s_\pi, t, u) = \sum_{i=1}^9 p_{i,Z}(s_\pi, t, u) L^r_i, \quad (3.19)
\]

with

\[
p_{1,Z} = 32(s_\pi - 2M^2_K)(s_\pi - 2M^2_\pi),
\]

\[
p_{2,Z} = 8(s^2_\pi + \nu^2),
\]

\[
p_{3,Z} = -2 \left[ (\nu + 4\Sigma)s_\pi - 5s^2_\pi - \nu^2 - 16M^2_\pi M^2_K \right],
\]

\[
p_{4,Z} = 32 \left[ \Sigma s_\pi - 4M^2_K M^2_\pi \right],
\]

\[
p_{5,Z} = 4 \left[ (s_\pi + \nu)\Sigma - 8M^2_K M^2_\pi \right],
\]

\[
p_{6,Z} = 128M^2_K M^2_\pi,
\]

\[
p_{8,Z} = 64M^2_K M^2_\pi. \quad (3.20)
\]
The remaining $p_{i,Z}$ vanish. Finally, the logarithms in $C_Z$ are

$$C_Z = -\frac{M_K^2 - M_\pi^2}{128\pi^2} \left[ 3M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - 2M_K^2 \ln \frac{M_K^2}{\mu^2} - M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \right]. \quad (3.21)$$

The nonpole part $Q$ receives the following one-loop contributions:

$$U_Q = \Delta_0(s_\pi) + \frac{M_K^2 - s_l}{32} \left\{ 11J_K^*(t) + 8J_K^*(u) + 3J_{\eta K}^*(t) \right\}$$

$$- \frac{1}{8} \left( 5(s_\pi - \nu) + 5(M_K^2 - s_l) - 6\Sigma \right) K_K(t)$$

$$- \frac{1}{8} \left( 3(s_\pi - \nu) + 3(M_K^2 - s_l) - 2\Sigma \right) K_{\eta K}(t)$$

$$- \frac{9}{4} \left( L_{\eta K}(t) + L_K(t) \right)$$

$$+ \frac{3}{8} \left( 4(\nu + 2M_\pi^2) - 3(M_K^2 - s_l) \right) (M_{K\pi}^*(t) + M_{\eta K}^*(t)), \quad (3.22)$$

and

$$P_Q(s_\pi, t, u) = \sum_{i=1}^{9} p_{i,Q}(s_\pi, t, u) L_i^r, \quad (3.23)$$

with

$$p_{1,Q} = 32(s_\pi - 2M_\pi^2),$$

$$p_{2,Q} = 8(M_K^2 - s_l),$$

$$p_{3,Q} = 2(4(s_\pi - 2M_\pi^2) + M_K^2 - s_l),$$

$$p_{4,Q} = 32M_\pi^2,$$

$$p_{5,Q} = 4\Sigma,$$

$$p_{9,Q} = 2 \left[ (s_\pi + \nu) - (M_K^2 - s_l) \right]. \quad (3.24)$$

The remaining $p_{i,Q}$ vanish. Finally, the logarithms in $C_Q$ are

$$C_Q = \frac{1}{128\pi^2} \left[ 5M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - 2M_K^2 \ln \frac{M_K^2}{\mu^2} - 3M_\eta^2 \ln \frac{M_\eta^2}{\mu^2} \right]. \quad (3.25)$$

The first nonvanishing contribution in the chiral expansion of the form factor $H$ is due to the chiral anomaly [21]. The prediction is [7]

$$H = -\sqrt{2} \frac{M_K^3}{8\pi^2 F_\pi^3} = -2.66, \quad (3.26)$$

in excellent agreement with the experimental value [11] $H = -2.68 \pm 0.68$. The next-to-leading order corrections to $H$ have also been calculated [10]. If the new
low-energy parameters are estimated using the vector-mesons only, these corrections are small.

The results for $F, G$ and $R$ must satisfy two nontrivial constraints: i) Unitarity requires that $F, G$ and $R$ contain, in the physical region $4M_{\pi}^2 \leq s_{\pi} \leq (M_K - m_{l})^2$, imaginary parts governed by $S$- and $P$-wave $\pi\pi$ scattering [these imaginary parts are contained in the functions $\Delta_0(s_{\pi}), \Delta_1(s_{\pi})$]. ii) The scale dependence of the low-energy constants $L_i$ must be compensated by the scale dependence of $U_{F,G,Z,Q}$ and $C_{F,G,Z,Q}$ for all values of $s_{\pi}, t, u, M_{\pi}^2, M_K^2$. [Since we work at order $E^4$ in the chiral expansion, the meson masses appearing in the above expressions satisfy the Gell-Mann-Okubo mass formula.] We have checked that these constraints are satisfied.

It is one of the aims of this article to determine the low-energy constants $L_1, L_2$ and $L_3$ from experimental data on $K^+ \to \pi^+\pi^-e^+\nu_e$ decays and on $\pi\pi$ threshold parameters. In Ref. [8, 9], the above one-loop expressions have been used for this purpose. Because the one-loop contributions are rather large, the values of the $L_i$’s so extracted suffer from substantial uncertainties. In the following section, we therefore first estimate the effects from higher orders in the chiral expansion, using then this improved representation for the form factors in a comparison with the data.

4 Beyond one-loop

To investigate the importance of higher order terms, we employ the method developed in Ref. [12]. It amounts to writing a dispersive representation of the partial wave amplitudes, fixing the subtraction constants using chiral perturbation theory. Here, we estimate the higher order terms in the $S$-wave projection of the amplitude $F_1$,

$$f(s_{\pi}, s_l) = (4\pi X)^{-1} \int d\Omega F_1(s_{\pi}, t, s_l),$$

(4.1)

because this form factor plays a decisive role in the determination of $L_1, L_2$ and $L_3$, and it is influenced by $S$-wave $\pi\pi$ scattering which is known [22, 23] to produce substantial corrections.

4.1 Analytic properties of $f(s_{\pi}, s_l)$

Only the crossing even part

$$F_1^+ = XF^+ + \sigma_{\pi}(PL)\cos\theta_{\pi} \cdot G^-$$

(4.2)

contributes in the projection $[4.1]$. The partial wave $f$ has the following analytic properties:

1. At fixed $s_l$, it is analytic in the complex $s_{\pi}$-plane, cut along the real axis for $\text{Re } s_{\pi} \geq 4M_{\pi}^2$ and $\text{Re } s_{\pi} \leq 0$.

2. In the interval $0 \leq s_{\pi} \leq 4M_{\pi}^2$, it is real.
3. In $4M^2_\pi \leq s_\pi \leq 16M^2_\pi$, its phase coincides with the isospin zero $S$-wave phase $\delta_0^0$ in elastic $\pi\pi$ scattering,

$$f_+ = e^{2i\delta_0^0} f_-, \quad f_\pm = f(s_\pi \pm i\epsilon, s_\pi).$$  \hspace{1cm} (4.3)

The proof of these properties is standard \[24\]. Here we only note that the presence of the cut for $s_\pi \leq 0$ follows from the relations

$$t = M^2_\pi + \frac{M^2_K + s_l - s_\pi}{2} - \sigma_\pi X \cos \theta_\pi,$$

$$t(\cos \theta_\pi = -1, s_\pi < 0) \geq (M_K + M_\pi)^2.$$  \hspace{1cm} (4.4)

Since $F^+$ and $G^-$ have cuts at $t \geq (M_K + M_\pi)^2$ [see e.g. Eqs. 3.2–3.11], the claim is proven.

### 4.2 Unitarization

We introduce the Omnès-function

$$\Omega(s_\pi) = \exp \left[ \frac{s_\pi}{\pi} \int_{4M^2_\pi}^{A^2} \frac{ds}{s} \frac{\delta_0^0(s)}{s - s_\pi} \right],$$  \hspace{1cm} (4.5)

where $\Lambda$ will be chosen of the order of $1$ GeV below. According to (4.3), multiplication by $\Omega^{-1}$ removes the cut in $f$ for $4M^2_\pi \leq s_\pi \leq 16M^2_\pi$. Consider now

$$f = f_L + f_R,$$  \hspace{1cm} (4.6)

where $f_L(f_R)$ has only the left-hand (right-hand) cut, and introduce

$$v = \Omega^{-1}(f - f_L).$$  \hspace{1cm} (4.7)

Then $v$ has only a right-hand cut, and we may represent it in a dispersive way,

$$v = v_0 + v_1 s_\pi + \frac{s^2_\pi}{\pi} \int_4^{A^2} \frac{ds}{s^2} \frac{\text{Im} \Omega^{-1}(f - f_L)}{s - s_\pi}.$$ \hspace{1cm} (4.8)

We expect the contributions from the integral beyond 1GeV$^2$ to be small. Furthermore, $\Omega^{-1} f$ is approximately real between $16M^2_\pi$ and 1GeV$^2$, as a result of which one has

$$v = v_0 + v_1 s_\pi - \frac{s^2_\pi}{\pi} \int_{4M^2_\pi}^{A^2} \frac{ds}{s^2} \frac{f_L \text{Im} \Omega^{-1}}{s - s_\pi}.$$  \hspace{1cm} (4.9)

For given $v_0, v_1, f_L$ and $\Omega$, the form factor $f$ is finally obtained from

$$f = f_L + \Omega v.$$  \hspace{1cm} (4.10)

The behaviour of $f_L$ at $s_\pi \to 0$ is governed by the large $|t|$-behaviour of $F^+$ and $G^-$, see (4.4). Therefore, instead of using CHPT to model $f_L$, we approximate the
left-hand cut by resonance exchange. To pin down the subtraction constants $v_0$ and $v_1$, we require that the threshold expansion of $f$ and $f_{\text{CHPT}}$ agree up to and including terms of order $O(E^2)$. For a specific choice of $f_L$, this fixes $v_0, v_1$ in terms of the quantities which occur in the one-loop representation of $F^+$ and $G^-$. In the case where $f_L = 0$, $f$ has then a particularly simple form at $s_I = 0$.

\[
\begin{align*}
  f(s, s_l = 0)_{|f_L = 0} & = \Omega(v_0 + v_1 s) \\
v_0 & = \frac{M_K}{\sqrt{2} F_\pi} \left\{ 1.05 + \frac{1}{F_\pi^2} \left[ -64M_\pi^2 L_1^r + 8M_K^2 L_2^r \\
                           + 2(M_K^2 - 8M_\pi^2)L_3 + \frac{2}{3}(M_K^2 - 4M_\pi^2)(4L_2^r + L_3) \right] \right\} \\
v_1 & = \frac{M_K}{\sqrt{2} F_\pi} \left\{ 0.38 + \frac{1}{F_\pi^2} \left[ 32L_1^r + 8L_2^r + 10L_3 \\
                           - \frac{2}{3} \frac{M_K^2 - 4M_\pi^2}{4M_\pi^2}(4L_2^r + L_3) \right] \right\} .
\end{align*}
\]

We relegate the details of the calculation of $f_L, v_0$ and of $v_1$ to appendix C.

In the partial wave $f$, the effects of the final-state interactions are substantial, because they are related to the $I = 0, S$-wave $\pi\pi$ phase shift. On the other hand, for the leading partial wave in $G^+ = ge^{i\delta_p} + \cdots$, these effects are reduced, because the phase $\delta_p$ is small at low energies. We find it more difficult to assess an estimate for the higher order corrections in this case – we come back to this point in the following section.

## 5 Determination of $L_1, L_2$ and $L_3$

Here we determine the low-energy constants $L_1^r, L_2^r$ and $L_3$ from experimental data on $K^+ \to \pi^+\pi^-e^+\nu_e$ decays and on $\pi\pi \to \pi\pi$ threshold parameters, using the improved $S$-wave amplitude $f$ set up above. We are aware that our results will not be the last word: future kaon facilities like DAΦNE [20] will allow a more refined comparison of the chiral representation with the data. Nevertheless, we believe that it is instructive to see what one has to expect from higher order contributions. A comparison with earlier work [9] will be provided at the end of this section.

### 5.1 The data

Experimentally the study of $K_{l4}$ decays is dominated by the work of Rosselet et al. [11] which measures $K^+ \to \pi^+\pi^-e^+\nu_e$ with good statistics. The total decay rate, the absolute value of the form factors $F, G$ and of $H$ and the difference of the phases $\delta^0_0 - \delta^1_1$ were determined by use of

\[
\begin{align*}
  F & = f_s e^{i\delta_s} + f_p e^{i\delta_p} \cos \theta_s + D - \text{wave} \\
  G & = g e^{i\delta_p} + D - \text{wave} \\
  H & = h e^{i\delta_h} + D - \text{wave} .
\end{align*}
\]
The form factor $f_p$ was found to be compatible with zero and hence set equal to zero when the final value for $g$ was derived. No $s_\pi, s_l$ dependence of the ratios $g/f_s$ and $h/f_s$ was seen. Parametrizing $f_s$ in the form

\begin{align}
  f_s &= f_s(0)(1 + \lambda_f q^2) , \\
  q^2 &= (s_\pi - 4M_\pi^2)/4M_\pi^2 , \\
\end{align}

then gives

\begin{align}
  g &= g(0)(1 + \lambda_g q^2) , \\
  h &= h(0)(1 + \lambda_h q^2) ,
\end{align}

(5.3)

with $\lambda_f = \lambda_g = \lambda_h$. Rosselet et al. found \[11\]

\begin{align}
  f_s(0) &= 5.59 \pm 0.14 , \\
  g(0) &= 4.77 \pm 0.27 , \\
  h(0) &= -2.68 \pm 0.68 , \\
  \lambda_f &= 0.08 \pm 0.02 ,
\end{align}

(5.4)

where we have used $|V_{us}| = 0.22$ in transcribing their results. Notice that the experimental numbers (5.4) have been obtained in \[11\] under assumptions which are in conflict with our theoretical formulae, like absence of higher waves, $s_l$ independence of the form factors and equality of the slopes, $\lambda_f = \lambda_g = \lambda_h$. It would of course be desirable to analyze forthcoming data without any additional assumptions.

The total decay rate is \[11\]

$$
\Gamma_{K^e} = (3.26 \pm 0.15) \times 10^3 \text{sec}^{-1} .
$$

(5.5)

In \[9\] it has been observed that, using as input the central values (5.4), one obtains $\Gamma_{K^e} = 2.94 \times 10^3 \text{sec}^{-1}$, which disagrees with (5.5). On the other hand, the value (5.5) was used in Ref. \[11\] to normalize the form factors in (5.4). We do not understand the origin of this contradiction.

The $\pi\pi$ threshold data used below are taken from Ref. \[25\]. We display them in table 2, column 6.

### 5.2 The fits

In the following, we perform various fits to $f_s(0), \lambda_f, g(0)$ and to the $\pi\pi$ threshold parameters listed in table 2. We introduce for this purpose the quantities

\begin{align}
  \tilde{f}(s_\pi, s_l) &= (4\pi X)^{-1} \int d\Omega F_1(s_\pi, t, s_l) = |f(s_\pi, s_l)| , \\
  \tilde{g}(s_\pi, s_l) &= \frac{3}{8\pi} \int d\Omega \sin^2 \theta_\pi G(s_\pi, t, s_l) ,
\end{align}

(5.6)
Table 2: Results of fits with one-loop and unitarized form factors, respectively. The errors quoted for the $L_i$’s are statistical only. The $L_i$ are given in units of $10^{-3}$ at the scale $\mu = M_\rho$, the scattering lengths $a_l^i$ and the slopes $b_l^i$ in appropriate powers of $M_{\pi^+}$.

|       | $K_{e4}$ data alone | $K_{e4}$ and $\pi\pi$ data | experiment |
|-------|---------------------|----------------------------|------------|
|       | one-loop | unitarized | one-loop | unitarized |          |
| $L_1^r$ | 0.65 ± 0.27 | 0.36 ± 0.26 | 0.60 ± 0.24 | 0.37 ± 0.23 |          |
| $L_2^r$ | 1.63 ± 0.28 | 1.35 ± 0.27 | 1.50 ± 0.23 | 1.35 ± 0.23 |          |
| $L_3$ | −3.4 ± 1.0 | −3.4 ± 1.0 | −3.3 ± 0.86 | −3.5 ± 0.85 |          |
| $a_0^0$ | 0.20 | 0.20 | 0.20 | 0.20 | 0.26 ± 0.05 |
| $b_0^0$ | 0.26 | 0.25 | 0.26 | 0.25 | 0.25 ± 0.03 |
| −$10 \ a_0^0$ | 0.40 | 0.41 | 0.40 | 0.41 | 0.28 ± 0.12 |
| −$10 \ b_0^0$ | 0.67 | 0.72 | 0.68 | 0.72 | 0.82 ± 0.08 |
| $10a_1^1$ | 0.36 | 0.37 | 0.36 | 0.37 | 0.38 ± 0.02 |
| $10b_1^1$ | 0.44 | 0.47 | 0.43 | 0.48 |          |
| $10^2a_0^0$ | 0.22 | 0.18 | 0.21 | 0.18 | 0.17 ± 0.03 |
| $10^2a_2^2$ | 0.39 | 0.21 | 0.37 | 0.20 | 0.13 ± 0.3 |
| $\chi^2/N_{DOF}$ | 0/0 | 0/0 | 8.8/7 | 4.9/7 |          |

where the factor $3/2 \sin^2 \theta$ appears because $G$ is expanded in derivatives of Legendre polynomials. Below, we confront $[f_s(0), g(0)]$ with $[\bar{f}(4M_{\pi^+}^2, s_l)]$, $\bar{g}(4M_{\pi^+}^2, s_l)$, which depend on $s_l$. Furthermore, we compare the slope $\lambda_f$ with

$$\lambda_f(s_{\pi}, s_l) = \frac{\bar{f}(s_{\pi}, s_l) - \bar{f}(4M_{\pi^+}^2, s_l)}{\bar{f}(4M_{\pi^+}^2, s_l)} \frac{4M_{\pi^+}^2}{s_{\pi} - 4M_{\pi^+}^2},$$

which depends on both $s_{\pi}$ and $s_l$. Below we use these dependences to estimate systematic uncertainties in the determination of the low-energy couplings. [In future high statistics experiments, the $s_l$-dependence of the form factors will presumably be resolved. It will be easy to adapt the procedure to this case.]

We have used MINUIT [26] to perform the fits. The results for the choice $s_l = 0, s_{\pi} = 4.4M_{\pi^+}^2$ are given in table 2. In the columns denoted by ”one-loop”, we have evaluated $\bar{f}, \bar{g}$ and $\lambda_f$ from the one-loop representation given above. In the fit with the unitarized form factor (columns 3 and 5), we have evaluated $\bar{f}$ from Eqs. (4.10[27]), inserting in the Omnès function the parametrization of the $\pi\pi$ S-wave phase shift proposed by Schenk [27, solution B]. For the form factor $G$, we have again used the one-loop representation. The statistical errors quoted for the $L_i$’s are the ones generated by the procedure MINOS in MINUIT and correspond to an increase of $\chi^2$ by one unit.

A few remarks are in order at this place.

---

4We always use for $L_4^i, \ldots, L_9^i$ the values quoted in table 1.
1. It is seen that the overall description of the $\pi\pi$ scattering data is better using the unitarized form factors, in particular so for the $D$-wave scattering lengths.

2. The errors quoted do not give account of the fact that the simultaneous determination of the three constants produces a strong correlation between them. To illustrate this point we note that, while the values of the $L_i$'s in column 4 and 5 are apparently consistent with each other within one error bar, the $\chi^2$ in column 5 increases from 4.9 to 30.7 if the $L_i$'s from column 4 are used in the evaluation of $\chi^2$ in column 5. (For a discussion about the interpretation of the errors see [26]).

3. The low-energy constants $\bar{l}_1, \bar{l}_2$ which occur in $SU(2)_L \times SU(2)_R$ analyses may be evaluated from a given set of $L^r_1, L^r_2$ and $L_3$ [3]. Their value changes in a significant way by using the unitarized amplitude instead of the one-loop formulae: the values for $(\bar{l}_1, \bar{l}_2)$ in column 4 and 5 are $(-0.5 \pm 0.88, 6.4 \pm 0.44)$ and $(-1.7 \pm 0.85, 6.0 \pm 0.4)$, respectively.

4. $L^r_1, L^r_2$ and $L_3$ are related to $\pi\pi$ phase-shifts through sum rules [23, 29]. In principle, one should take these constraints into account as well. We do not consider them here, because we find it very difficult to assess a reliable error for the integrals over the total $\pi\pi$ cross sections which occur in those relations.

The statistical error in the data is only one source of the uncertainty in the low-energy constants, which are in addition affected by the ambiguities in the estimate of the higher order corrections. These systematic uncertainties have several sources:

i) Higher order corrections to $\bar{g}$ have not been taken into account.

ii) The determination of the contribution from the left-hand cut is not unique.

iii) The quantities $\bar{f}$ and $\bar{g}$ depend on $s_t$, and $\bar{\lambda}_f$ is a function of both $s_t$ and $s_\pi$.

iv) The Omnès function depends on the elastic $\pi\pi$ phase shift and on the cutoff $\Lambda$ used.

We have considered carefully these effects. As for the first point, we have evaluated the higher orders in $\bar{g}$ in two ways:

- We define the quantity [9]

$$\Delta \bar{g} = \frac{(g(0) - \bar{g}^{(2)})^2}{\bar{g}^{(2)}},$$

where $\bar{g}^{(2)}$ is the CHPT prediction at leading-order. We then add $\Delta \bar{g}$ in quadrature to the experimental error in $g(0)$ and redo the fit. This generates slightly larger errors than before. To illustrate, the entries $(0.23, 0.23, 0.85)$ in column 5 in table 2 become $(0.29, 0.28, 1.1)$.

5We thank B. Moussallam for pointing this out to us.
Table 3: Fits made with different theoretical input to determine the systematic uncertainties in \( L_1^r, L_2^r \) and \( L_3 \). \( \Delta L_1 \) shows the difference to the value of \( L_1^r \) displayed in the second row. \( L_i^r \) are given in units of \( 10^{-3} \) at the scale \( \mu = M_{\rho} \). The fits include \( K_{e4} \) and \( \pi\pi \) data.

| Column 5th table 2 | \( L_1^r \) | \( \Delta L_1 \) | \( L_2^r \) | \( \Delta L_2 \) | \( L_3 \) | \( \Delta L_3 \) |
|---------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| full propagators    |             |             |             |             |             |             |
| \( f_L = 0 \)       | 0.34        | -0.03       | 1.08        | -0.27       | -2.8        | -0.7        |
| \( s_\pi = 5.6M_\pi^2, s_l = 0 \) | 0.42        | 0.05        | 1.30        | -0.05       | -3.5        | 0.0         |
| \( s_\pi = 4.4M_\pi^2, s_l = M_K^2/10 \) | 0.38        | 0.01        | 1.35        | 0.00        | -3.5        | 0.0         |
| \( \pi\pi \) phase from [27, solution A] | 0.33        | -0.04       | 1.26        | -0.09       | -3.3        | -0.2        |
| \( \pi\pi \) phase from [27, solution C] | 0.46        | 0.09        | 1.36        | 0.01        | -3.5        | 0.0         |

- The main contribution to the one-loop correction in \( \bar{g} \) stems from \( L_3 \). On the other hand, the low-energy constants \( L_i^r \) are saturated by resonance exchanges whose contribution is evaluated in the limit of large resonance mass [30]. We find it therefore reasonable to estimate higher orders by using the complete resonance propagators. In order to evaluate the uncertainty induced by this correction, we made the fit including the full propagators in \( \bar{g} \) and in \( \bar{f} \). This changes the central values for the \( L_i^r \)'s, see table 3.

Concerning the effect of the left-hand cut, we estimate its uncertainty by dropping this piece altogether. The results of the fit in this case are again given in table 3. It is seen that the \( L_i^r \)'s depend rather weakly on the presence of the left-hand cut. Figures 2 and 3 illustrate the effect of \( f_L \) in more detail. In Fig. 2 is shown the form factor \( \bar{f} \) with and without the left-hand cut (dashed and dash-dotted line, respectively). In both cases, \( L_1^r, L_2^r, L_3 \) from (5.9) have been used. At large values of \( s_\pi \), the difference between the two evaluations of \( \bar{f} \) is not negligible for the following reason. At threshold, the value of the form factor and of its slope cannot change much by construction–these quantities are constrained in our procedure by the low-energy expansion at one-loop order. On the other hand, the quadratic piece is unconstrained and receives contributions from \( f_L \) through the dispersive integral. The corresponding change in \( \bar{f} \) is less than 10\% at \( s_\pi = 10M_\rho^2 \). In Fig. 3, we display the differential decay rate \( d\Gamma/ds_\pi \), obtained by using the form factor with and without \( f_L \). The figure illustrates that the contribution from the quadratic piece in \( s_\pi \) to observables is strongly suppressed by phase space and will therefore be difficult to observe.

The dependence of the fits on \( s_l, s_\pi \) and on the \( \pi\pi \) phase shift used in the Omnès function \( \Omega \) is illustrated in the last four rows in table 3. Furthermore, changing the cutoff \( \Lambda = 1\text{GeV} \) in \( \Omega \) to \( \Lambda = 0.8\text{GeV} \) induces small changes in the \( L_i^r \)'s only. We conclude that a global fit to all the available data is rather stable against the
systematic uncertainties considered here.

To finally give the best determinations of $L^r_1$, $L^r_2$ and $L_3$, we take the central values from the global fit displayed in table 2, column 5. For the corresponding errors, we take the ones generated by using the theoretical error bars for the higher orders in $\bar{g}$, and find in this manner

$$
\begin{align*}
10^3L^r_1(M_\rho) &= 0.4 \pm 0.3 , \\
10^3L^r_2(M_\rho) &= 1.35 \pm 0.3 , \\
10^3L_3(M_\rho) &= -3.5 \pm 1.1 .
\end{align*}
$$

(5.9)

For $SU(2)_L \times SU(2)_R$ analyses it is useful to know the corresponding values for the constants $\bar{l}_1$ and $\bar{l}_2$,

$$
\begin{align*}
\bar{l}_1 &= -1.7 \pm 1.0 , \\
\bar{l}_2 &= 6.1 \pm 0.5 .
\end{align*}
$$

(5.10)

The value and uncertainties in these couplings play a decisive role in a planned experiment [31] to measure the lifetime of $\pi^+\pi^-$ atoms, which will provide a completely independent measurement of the $\pi\pi$ scattering lengths $|a_0^0 - a_0^2|$.

One motivation for the analysis in [8, 9] was to test the large $N_C$ prediction $L^r_2 = 2L^r_1$. The above result shows that a small non-zero value is preferred. To obtain a clean error analysis, we have repeated the fitting procedure using the variables

$$
\begin{align*}
X_1 &= L^r_2 - 2L^r_1 - L_3 , \\
X_2 &= L^r_2 , \\
X_3 &= (L^r_2 - 2L^r_1)/L_3 .
\end{align*}
$$

We performed a fit to $K_{e4}$ and $\pi\pi$ data, including the theoretical error in $G$ as discussed above, and found

$$
\begin{align*}
X_1 &= (4.8 \pm 0.8) \cdot 10^{-3} , \\
X_3 &= -0.17^{+0.12}_{-0.22} .
\end{align*}
$$

(5.11)

The result is that the large $N_C$ prediction works remarkably well.

### 5.3 Comparison with earlier work

It is of interest to compare the present procedure to determine the low-energy constants $L^r_1$, $L^r_2$ and $L_3$ with the method used in [9]. There are two main differences:

1. The definition of the slope $\lambda_f$ and of the threshold value of the form factors $f_s, g$ chosen in [9] differs from the one used here. These quantities have of course a unique meaning in principle – on the other hand, one may wish to approximate a particular experimental situation. The procedure used in [9] was adapted to Ref. [11], whereas a slight variation of the method proposed here may be useful once the $s_l$-dependence of the form factors has experimentally been resolved.
2. Higher order corrections are estimated in [9] in a rather crude manner. In the present approach, the final-state interactions in the $I = 0, S$-wave amplitude are instead taken into account, and higher order terms in $\bar{g}$ are estimated with resonance exchange.

The main effect of these differences can be described as follows. The different slope and form factors used in [9] lead to slightly different central values for $L_1^r, L_2^r$ and $L_3^r$ at one-loop order, whereas the errors turn out to be very similar in both cases. The higher order estimates in [9] lead to the same central values with larger error bars, whereas the unitarization performed in the present work leads to different central values with slightly smaller error bars than before, see columns 2/3 and 4/5 in table 2. This effect can be easily understood by considering the simplified expression (4.11), which shows how the Omnès function affects the influence of the $L_i$'s and hence their value in the fit.

5.4 Improvements

As we mentioned at the beginning of this section, there is room for improvement in the above treatment, both on the theoretical and on the experimental side. Concerning the latter, one should determine in future experiments the form factors $f_s$ and $g$ without additional assumptions [11] which are in contradiction with the chiral representation. It remains to be seen whether this can be achieved by comparing the data directly with a modified chiral representation. In the latter, the full $S$- and $P$-wave parts of $F_1$ and $F_2$ could be inserted, using the chiral representation solely to describe the small background effects due to higher partial waves $l \geq 2$. To be more precise, one would take for $R$ and $H$ the one-loop chiral representation, whereas for $G$ one writes

$$G = g(s_\pi, s_t)e^{i\delta_p} + \Delta G,$$

$$\Delta G = G_{\text{CHPT}} - \frac{3}{8\pi} \int d\Omega \sin^2 \theta_\pi G_{\text{CHPT}}, \quad \text{(5.12)}$$

and similarly for $F$. The unknown amplitudes $g(s_\pi, s_t), f_s(s_\pi, s_t)$ and the phases $\delta_p, \delta_s$ would then be determined from the data. We have checked that, if the errors in the form factors determined in this manner can be reduced by e.g. a factor 3 with respect to the ones shown in (5.4), one could pin down particular combinations of $L_1^r, L_2^r$ and $L_3^r$ to considerably better precision than was shown above. This is true independently of an eventual improvement in the theoretical determination of the higher order corrections in the form factor $G$ – which is a theoretical challenge in any case.

6 Predictions

Having determined the constants $L_1^r, L_2^r$ and $L_3$, there are several predictions which we can make. Whereas the slope $\lambda_g$ was assumed to coincide with the slope $\lambda_f$ in
the final analysis of the data in Ref. [11], these two quantities may differ in the chiral representation. Furthermore, our amplitudes allow us to evaluate partial and total decay rates. In this section, we consider the slope $\lambda_g$ and the total rates.

6.1 The slope $\lambda_g$

We consider the form factor $\tilde{g}$ introduced in (5.6) and determine its slope $\lambda_g$

$$\begin{align*}
g(s_\pi, s_l) = \tilde{g}(4M_\pi^2, s_l)(1 + \lambda_g(s_l)q^2 + O(q^4))
\end{align*}$$

(6.1)

from the one-loop expression for $G$. The result is $\lambda_g(0) = 0.08$. As the slope is a one-loop effect, higher order corrections may affect its value substantially. For this reason, we have evaluated $\lambda_g$ also from the modified form factor obtained by using the complete resonance propagators (and the corresponding $L_i$'s), compare the discussion above. The change is $\Delta \lambda_g = 0.025$. We believe this to be a generous error estimate and obtain in this manner

$$\lambda_g(0) = 0.08 \pm 0.025 .$$

(6.2)

The central value indeed agrees with the slope $\lambda_f$ in (5.4).

6.2 Total rates

Once the leading partial waves $\bar{f}$ and $\bar{g}$ are known from e.g. $K^+ \rightarrow \pi^+\pi^-e^+\nu_e$ decays, the chiral representation allows one to predict the remaining rates within rather small uncertainties. We illustrate the procedure for $K^+ \rightarrow \pi^0\pi^0e^+\nu_e$. According to Eq. (2.18), the relevant amplitude is determined by $F^+, G^-, R^+$ and $H^-$. The contribution from $H$ is kinematically strongly suppressed and completely negligible in all total rates, whereas the contribution from $R$ is negligible in the electron modes. Using the chiral representation of the amplitudes $F^+$ and $G^-$, we find that the rate is reproduced to about 1%, if one neglects $G^-$ altogether and uses only the leading partial wave in the remaining amplitude, $F^+ \approx -X\bar{f}$. From the measured [11] form factor $\bar{f} = 5.59(1 + 0.08q^2)$ we then find $\Gamma_{K^+ \rightarrow \pi^0\pi^0e^+\nu_e} = 1625\text{sec}^{-1}$. Finally, we estimate the error from

$$\begin{align*}
\Delta \Gamma &= \left\{ \left[ \Gamma(f_s(0) + \Delta f_s, \lambda_f) - \Gamma(f_s(0), \lambda_f) \right]^2 + \\
&\quad \left[ \Gamma(f_s(0), \lambda_f + \Delta \lambda_f) - \Gamma(f_s(0), \lambda_f) \right]^2 \right\}^{1/2} = 90\text{sec}^{-1},
\end{align*}$$

(6.3)

where $\Delta f_s = 0.14, \Delta \lambda_f = 0.02$. The final result for the rate is shown in the row "final prediction" in table 5, where we have also listed the tree and the one-loop result, together with the experimental data. The evaluation of the remaining rates is done in a similar manner – see table 4 for the simplifications used and table 5 for the corresponding predictions.
Table 4: Approximations used to evaluate the total rates in table 5. Use of $\bar{f} = \bar{f}_{\text{CHPT}}, \bar{g} = \bar{g}_{\text{CHPT}}$ reproduces the one-loop results in table 5 to about 1%.

a) $K^+$ decays

|      | $\pi^0\pi^0e^+\nu_e$          | $\pi^+\pi^-\mu^+\nu_\mu$ | $\pi^0\pi^0\mu^+\nu_\mu$ |
|------|-------------------------------|----------------------------|---------------------------|
| $F_1$ | $-X\bar{f}$                  | $X\bar{f} + \sigma_\pi(PL)\cos\theta_\pi\bar{g}$ | $-X\bar{f}$              |
| $F_2$ | 0                             | $\sigma_\pi(s_\pi s_t)^{1/2}\bar{g}$           | 0                         |
| $F_3$ | 0                             | 0                                         | 0                         |
| $F_4$ | $(PL)\bar{f}$                | $\{(PL)\bar{f} + s_tR_{\text{CHPT}} + \sigma_\pi X \cos\theta_\pi\bar{g}\}$ | $\{(PL)\bar{f} + s_tR_{\text{CHPT}}\}$ |

b) $K^0$ decays. Shown are the amplitudes divided by $\sqrt{2}$.

|      | $\pi^0\pi^-e^+\nu_e$        | $\pi^0\pi^0\mu^+\nu_\mu$ |
|------|-------------------------------|---------------------------|
| $F_1$ | $XF_{\text{CHPT}}^- + \sigma_\pi(PL)\cos\theta_\pi\bar{g}$ | $XF_{\text{CHPT}}^- + \sigma_\pi(PL)\cos\theta_\pi\bar{g}$ |
| $F_2$ | $\sigma_\pi(s_\pi s_t)^{1/2}\bar{g}$ | $\sigma_\pi(s_\pi s_t)^{1/2}\bar{g}$ |
| $F_3$ | 0                             | 0                         |
| $F_4$ | $\{(PL)F_{\text{CHPT}}^- + \sigma_\pi X \cos\theta_\pi\bar{g}\}$ | $\{(PL)F_{\text{CHPT}}^- + s_tR_{\text{CHPT}} + \sigma_\pi X \cos\theta_\pi\bar{g}\}$ |

We have assessed an uncertainty due to contributions from $F^-_{\text{CHPT}}, R_{\text{CHPT}}$ in the following manner. i) We have checked that the results barely change by using the tree level expression for $R_{\text{CHPT}}$ instead of its one-loop representation. We conclude from this that the uncertainties in $R_{\text{CHPT}}$ do not matter. ii) The uncertainty from $F^-_{\text{CHPT}}$ is taken into account by adding to $\Delta\Gamma$ in quadrature the change obtained by evaluating $F^-_{\text{CHPT}}$ at $L_3 = -3.5 + 1.1 = -2.4$. iii) In $K^0$ decays, we have also added in quadrature the difference generated by evaluating the rate with $M_\pi = 135$ MeV.

The decay $K^0 \to \pi^0\pi^-e^+\nu_e$ has recently been measured [33] with considerably higher statistics than before [32]. We display the result for the rate in the first column of table 5b. The quoted errors correspond to the errors in the branching ratio [33] and do not include the uncertainty in the total decay rate quoted by the PDG [32]. Notice that the value for $L_3$ determined in [33] should be multiplied with $-1$ [34].

7 Summary and conclusion

1. The matrix elements for $K_{l4}$ decays depend on four form factors $F, G, H$ and $R$. This article contains the full expressions for these at order $E^4$ in CHPT, thus completing already published calculations [7, 8, 9] of $F, G, H$ at this order.
Table 5: Total decay rates in sec$^{-1}$. To evaluate the rates at one-loop accuracy, we have used $L_1^r$, $L_2^r$ and $L_3^r$ from (5.9). The remaining low-energy constants are from table 1. The final predictions are evaluated with the amplitudes shown in table 4, using $\bar{f} = 5.59(1 + 0.08q^2)$, $\bar{g} = 4.77(1 + 0.08q^2)$. For the evaluation of the uncertainties in the rates see text.

a) $K^+$ decays

|                | $\pi^+\pi^- e^+\nu_e$ | $\pi^0\pi^0 e^+\nu_e$ | $\pi^+\pi^- \mu^+\nu_\mu$ | $\pi^0\pi^0 \mu^+\nu_\mu$ |
|----------------|-------------------------|-------------------------|-----------------------------|-----------------------------|
| tree           | 1297                    | 683                     | 155                         | 102                         |
| one-loop       | 2447                    | 1301                    | 288                         | 189                         |
| final prediction | input                  | 1625                    | 333                         | 225                         |
|                | $\pm90$                 | $\pm15$                 | $\pm11$                     |                             |
| experiment     | 3160                    | 1700                    | 1130                        |                             |
| $[32]$         | $\pm140$                | $\pm320$                | $\pm730$                    |                             |

b) $K^0$ decays

|                | $\pi^0\pi^- e^+\nu_e$ | $\pi^0\pi^- \mu^+\nu_\mu$ |
|----------------|-------------------------|-----------------------------|
| tree           | 561                     | 55                          |
| one-loop       | 953                     | 94                          |
| final prediction | 917                    | 88                          |
|                | $\pm170$                | $\pm22$                    |
| experiment     | 998                     | $\pm39 \pm43$              |
| $[33]$         |                          |                             |

2. We have estimated higher order terms in the $S$-wave amplitude of the form factor $F$ by use of a dispersive representation, determining the subtraction constants in the standard manner $[12]$ from CHPT. This procedure puts earlier attempts $[9]$ to estimate these corrections on a more firm basis.

3. Using the improved $S$-wave amplitude, we have determined $L_1^r$, $L_2^r$ and $L_3^r$ from $K^+ \to \pi^+\pi^- e^+\nu_e$ decays and $\pi\pi$ threshold data. Unitarizing the amplitude affects the related $SU(2) \times SU(2)$ constant $\bar{l}_1$ in a significant manner. As a result of this, the $D$-wave scattering lengths are in better agreement with the values given by Petersen $[25]$ than was the case before $[9]$. All in all, a remarkably good agreement with $K_{e4}$ and $\pi\pi$ data is obtained.

4. $K_{e4}$ decays may be used to test the large-$N_C$ prediction $L_2^r = 2L_1^r$ $[8, 9]$. Using the improved representation of the amplitudes, we have confirmed the earlier $[9]$ finding: The large-$N_C$ rule works at the one standard deviation level for this combination of the constants.
5. The above determination of $L_1^r$, $L_2^r$ and $L_3$ will presumably be even more reliable, once high statistics data from kaon facilities like DAΦNE \[20\] will become available.

6. We also predict the slope $\lambda_g$ of the form factor $G$ and total decay rates, see Eq. (5.2) and table 5.

7. We have made some effort to find out whether any of the $K_{l4}$ decays could serve to determine some of the other low-energy constants which occur in the amplitude. We believe that it will be very difficult to pin down any of these (in particular $L_4^r$) to better precision than already known, because the higher order corrections tend to wash out their effect.

8. Finally, we would like to recall that the determination of the low-energy constants from $K_{l4}$ decays or the prediction of the total rates is not the only issue: these decays are in addition the only known source for a precise determination of the isoscalar $\pi\pi$ $S$-wave phase shift near threshold. The possibilities to determine those in future high statistics experiments are presently under investigation \[33\].

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A Loop integrals

In this appendix we define the loop integrals used in the text. We consider a loop with two masses, $M$ and $m$. All needed functions can be given in terms of the subtracted scalar integral $ar{J}(t) = J(t) - J(0)$ evaluated in four dimensions,

$$J(t) = -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{((p+k)^2 - M^2)(p^2 - m^2)}, \quad (A.1)$$

with $t = k^2$. The functions used in the text are then:

$$\bar{J}(t) = -\frac{1}{16\pi^2} \int_0^1 dx \log \frac{M^2 - tx(1-x) - \Delta x}{M^2 - \Delta x},$$

$$J'(t) = \bar{J}(t) - 2k,$$

$$M'(t) = \frac{1}{12t} \left\{ t - 2\Sigma \right\} \bar{J}(t) + \frac{\Delta^2}{3t^2} \bar{J}(t) + \frac{1}{288\pi^2} \frac{k}{6} - \frac{1}{96\pi^2t} \left\{ \Sigma + 2 \frac{M^2m^2}{\Delta} \log \frac{m^2}{M^2} \right\},$$

$$L(t) = \frac{\Delta^2}{4t} \bar{J}(t),$$

$$K(t) = \frac{\Delta}{2t} \bar{J}(t),$$

$$\Delta = M^2 - m^2,$$

$$\Sigma = M^2 + m^2,$$

$$\lambda = \lambda(t, M^2, m^2) = (t + \Delta)^2 - 4tM^2. \quad (A.2)$$

In the text these are used with subscripts,

$$\bar{J}_{ij}(t) = \bar{J}(t) \quad \text{with} \quad M = M_i, m = M_j, \quad (A.3)$$

and similarly for the other symbols. The subtraction point dependent part is contained in the constant $k$

$$k = \frac{1}{32\pi^2} \frac{M^2 \log \left( \frac{M^2}{\mu^2} \right) - m^2 \log \left( \frac{m^2}{\mu^2} \right)}{M^2 - m^2}, \quad (A.4)$$

where $\mu$ is the subtraction scale.

B Resonance contribution to the form factors

Below we display the contributions to the form factors $F$ and $G$ from resonance exchange (spin less than or equal to one, see also \[36, 37, 30\]). We quote them for...
\[ K^+ \rightarrow \pi^+\pi^-\ell^+\nu_\ell. \] The others can be derived using isospin relations (2.18). These contributions have been used both to provide a reasonable approximation of the left-hand cut, and to estimate higher order corrections in \( \bar{g} \).

**VECTORS**

- **t-channel**

\[
F_V^t = \frac{M_K G_V}{2\sqrt{2} F_\pi^3} \frac{1}{M_V^2 - t} \times \\
\left[ F_V(t - u + 2s_\ell) + G_V(t - u - 3s_\pi - s_\ell + M_K^2 + 8M_\pi^2) \right],
\]

\[
G_V^t = \frac{M_K G_V}{2\sqrt{2} F_\pi^3} \frac{1}{M_V^2 - t} \times \\
\left[ F_V(M_K^2 + s_\ell - s_\pi) + G_V(t - u + M_K^2 + s_\pi - s_\ell) \right], \tag{B.1}
\]

- **s_\pi-channel**

\[
F_\pi^s = -\frac{M_K G_V}{2\sqrt{2} F_\pi^3} \frac{1}{M_V^2 - s_\pi} [F_V - 2G_V] (t - u),
\]

\[
G_\pi^s = \frac{M_K G_V}{2\sqrt{2} F_\pi^3} \frac{1}{M_V^2 - s_\pi} \times \\
\left[ (F_V - 2G_V)(s_\ell - M_K^2) + (F_V + 2G_V)s_\pi \right], \tag{B.2}
\]

**SCALARS**

- **t-channel**

\[
F_S^t = \sqrt{\frac{2M_K}{F_\pi^3}} \frac{1}{M_S^2 - t} \times \\
\left[ c_d(M_K^2 + M_\pi^2 - t) - c_dc_m(M_K^2 + M_\pi^2) \right],
\]

\[
G_S^t = -F_S^t \tag{B.3}
\]

- **s_\pi-channel**

\[
F_\pi^s = -\frac{2\sqrt{2} M_K}{F_\pi^3} \frac{1}{M_S^2 - s_\pi} \times \\
\left[ c_d^2(2M_\pi^2 - s_\pi) - 2c_d c_m M_\pi^2 \right],
\]

\[
G_\pi^s = 0 \tag{B.4}
\]

The values used for the couplings \( G_V, F_V \) and \( c_d, c_m \) are \[30\]

\[
F_V = 154 \text{ MeV},
\]

\[
G_V = 69 \text{ MeV},
\]

\[
c_d = 32 \text{ MeV},
\]

\[
c_m = 42 \text{ MeV}, \tag{B.5}
\]

27
while for the masses we used

\[ M_V = 770 \text{ MeV}, \]
\[ M_S = 985 \text{ MeV}. \]  

(B.6)

C Evaluation of \( f_L, v_0 \) and \( v_1 \)

\( f_L \) is calculated from

\[
f_L = \frac{1}{4\pi} \int d\Omega \left\{ (F_V^+ + F_S^+) + \frac{\sigma_{PL}}{X} \cos \theta (G_V^- + G_S^-) \right\} \quad \text{(C.1)}
\]

Only the \( t \)-channel contributes to \( f_L \). The \( s \)-channel has only singularities on the right-hand cut. The quantities in Eq. (C.1) are defined as:

\[
F_{V,S}^t = \frac{1}{2} \left( F_{V,S}^t + F_{V,S}^t (t \leftrightarrow u) \right), \\
G_{V,S}^- = \frac{1}{2} \left( G_{V,S}^- - G_{V,S}^- (t \leftrightarrow u) \right), 
\]

(C.2)

analogously to (2.19), see appendix B for \( G_t^{V,S}, F_t^{V,S} \). To evaluate \( v_0 \) and \( v_1 \), we impose that the unitarized amplitude \( f = f_L + \Omega v \) matches the chiral one-loop representation \( f_{\text{CHPT}} \) at the threshold \( s_\pi = 4M_\pi^2 \). We write

\[
f_{\text{CHPT}} = \frac{M_K}{\sqrt{2}F_\pi} (f_{\text{CHPT}}^{(0)} + f_{\text{CHPT}}^{(2)} + O(E^4)),
\]

\[
f_{\text{CHPT}}^{(0)} = 1, 
\]

(C.3)

with obvious notation, and have

\[
f_{\text{CHPT}}^{(0)} + f_{\text{CHPT}}^{(2)} = \frac{\sqrt{2}F_\pi}{M_K} f_L + (1 + \Delta) \left( v_0^{(0)} + v_0^{(2)} + v_1^{(0)} s_\pi + O(E^4) \right),
\]

(C.4)

where

\[
\Delta = \frac{s_\pi}{\pi} \int_{4M_\pi^2}^{\Lambda^2} \frac{ds \delta_0^0(s)}{s - s_\pi}, \\
\delta_0^0 = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \sqrt{1 - 4M_\pi^2/s},
\]

(C.5)

and where \( \delta_0^0 \) is a quantity of order \( E^2 \). The quantity \( 1 + \Delta \) is the expansion of \( \Omega \) in CHPT to the required order. The \( v_i^{(k)} \) are obtained by equating the threshold expansion of the left-and right-hand side in (C.4).
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Figure captions

Fig. 1 Kinematic variables for $K_{l4}$ decays. The angle $\theta_\pi$ is defined in $\Sigma_{2\pi}$, $\theta_l$ in $\Sigma_{l\nu}$ and $\phi$ in $\Sigma_K$.

Fig. 2 The partial wave amplitude $\bar{f}(s_\pi, s_l = 0)$. The dashed line shows $\bar{f}$, evaluated according to Eqs. (4.10,C.1), with $L_1^r, L_2^r$ and $L_3$ from (3.9). The dash-dotted line is evaluated with $\bar{f}_L = 0$ according to (4.11), using the same $L_i$’s, whereas the solid line displays $f_s = 5.59(1 + 0.08q^2)$.

Fig. 3 Differential rate $d\Gamma/ds_\pi$ for $K^+ \rightarrow \pi^+\pi^-e^+\nu_e$ decays in arbitrary units. The evaluation is done with $F_1^- = F_2 = F_3 = F_4^- = 0$, and $F_1^+ = X \bar{f}(s_\pi, s_l = 0), F_4^+ = -PL/X F_1^+$. The input for the dashed (dash-dotted) line is the same as for the dashed (dash-dotted) line in Fig. 2.
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