Global existence and exponential decay of solutions for generalized coupled non-degenerate Kirchhoff system with a time varying delay term

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Abstract
The paper studies a system of nonlinear viscoelastic Kirchhoff system with a time varying delay and general coupling terms. We prove the global existence of solutions in a bounded domain using the energy and Faedo–Galerkin methods with respect to the condition on the parameters in the coupling terms together with the weight condition as regards the delay terms in the feedback and the delay speed. Furthermore, we construct some convex function properties, and we prove the uniform stability estimate.

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1 Introduction
The Kirchhoff equation belongs to the famous wave equation’s models describing the transverse vibration of a string fixed in its ends. It has been introduced in 1876 by Kirchhoff [8] and it is more general than the D’Alembert equation. In one dimensional space it takes the following form:

\[
\frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{\rho h} + \frac{E}{2L\rho} \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}
\]

where the function \( u(x,t) \) is the vertical displacement at the space coordinate \( x \), varying in the segment \([0,L]\) and over time \( t > 0 \), \( \rho \) is the mass density, \( h \) is the area of the cross section of the string, \( P_0 \) is the initial tension on the string, \( L \) is the length of the string and \( E \) is the Young modulus of the material. The nonlinear coefficient

\[
C(t) = \int_0^L \left| \frac{\partial u}{\partial x}(x,t) \right|^2 dx
\]
is obtained by the variation of the tension during the deformation of the string. When we do not have an initial tension (i.e. $P_0 = 0$), we call that a degenerate case as opposed to the non-degenerate case.

In this paper, we are interested in studying, in $A = \Omega \times (0, \infty)$, the following coupled viscoelastic Kirchhoff system:

\[
\begin{cases}
|u_t|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu_1 \Delta u(x,t-t(t)) + f_1(u,v) = 0 & \text{in } \Omega \times (0, +\infty[,} \\
|v_t|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t h_2(t-s)\Delta v(s)\,ds - \mu_2 \Delta v(x,t-t(t)) + f_2(u,v) = 0 & \text{in } \Omega \times (0, +\infty[,} \\
u(x,t) = v(x,t) = 0 & \text{on } \Gamma \times (0, +\infty[,} \\
(u(x,0), v(x,0)) = (u_0(x), v_0(x)), & (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) \text{ in } \Omega, \\
(u_t(x,t-t(0)), v_t(x,t-t(0))) = (f_1(x,t-t(0)), g_0(x,t-t(0))) & \text{in } \Omega \times (0, t(0)[,} 
\end{cases}
\]

in which $\Omega$ is an $n$ dimensional bounded domain of $\mathbb{R}^n$ and we have a smooth boundary $\Gamma$, $l > 0$, $\mu_1$ and $\mu_2$ are positive real constants, $h_1$ and $h_2$ are positive functions with exponential decay, and $t(t)$ is a positive time varying delay. In addition the initial condition $(u_0, v_0, u_1, v_1, f_0, g_0)$ will be specified in their function space later. $M$ is a smooth function defined by

\[
M : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad \text{ with } a, b > 0, \text{ and } \gamma \geq 1. \quad f_1 \text{ and } f_2 \text{ are two functions taking a particular form that we will make precise later.}
\]

The problem (1.2) is a description of axially moving viscoelastic strings composed of two different materials (like the wires of electricity) that are nonhomogeneous and which will be of influence on its moving, specially on the acceleration. From the mathematical point of view, this influence is represented by $|w_t|^l w''$, where $|w_t|^l$ is the material density, varying the velocity. A lot of work has been published with this term, for example see [11] and [14], where we find different results about the global existence and nonexistence of solutions and the decay of energy.

In recent years, the study of wave equations with delay has become an active area and with different forms of delay (constant [7], switching [5], varying in time [12], distributed [6]). The delay appears in modeling of a lot of domains, like the physical, chemical, biological and engineering domains. It is introduced when we have a time lag between an action on a system and a response of the system to this action. Furthermore, a delay can be small enough in feedback yet can destabilize a system [10], or improve the performance of the system [17].

In the absence of delay, Cavalcanti et al. [3] studied the following viscoelastic wave equations with strong damping:

\[
|u_t|^l u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu \Delta u(x,t) = 0, \quad \text{in } \Omega \times (0, +\infty[.}
\]
They used the Fadeo–Galerkin method to prove the global existence of a solution; also an explicit decay rate of the energy has been given provided \( m > 0 \).

In the other hand, in the same case and for \( l = 0 \), Raslan et al. [16] and El-Sayed et al. [4] have studied coupled equal width wave equations with strong damping, as they were looking for the new exact solution.

The problem treated in [2] has the following form:

\[
\begin{align*}
\partial_t^2 u - \Delta u + \mu_1 \sigma(t) g_1(u_t(x,t)) + \mu_2 \sigma(t) g_2(u_t(x,t - \tau(t))) &= 0, \quad \text{in } \Omega \times ]0, +\infty[.
\end{align*}
\]

Under the assumptions set on \( g_1, g_2, \sigma \) and \( \tau \), the authors have gotten the global existence of a solution and the decay rate of the energy.

Recently, Mezouar and Boulaaras [13] have studied the viscoelastic non-degenerate Kirchhoff equation with varying delay term in the internal feedback.

In the present paper, we extend our recently published paper in [13] for a coupled system (1.2). The famous technique of using the presence of a delay in the PDE problem is to set a new variable defined by a velocity dependent on the delay, which will give us a new problem equivalent to our studied problem; but the last one is a coupled system without delay. After this, we can prove the existence of global solutions in suitable Sobolev spaces by combining the energy method with the Fadeo–Galerkin procedure and under the choice of a suitable Lyapunov functional, we establish an exponential decay result.

The outline of the paper is as follows: In the second section, some hypotheses related to the problem are given and we state our main result. Then in the third section, the global existence of weak solutions is proven. Finally, in the fourth section, we give the uniform energy decay.

### 1.1 Preliminaries and assumptions

Similar to that [12], we present the new variables

\[
\begin{align*}
\varphi_1(x, \rho, t) &= u_t(x, t - \rho \tau(t)), \quad x \in \Omega, \rho \in (0,1), t > 0, \\
\varphi_2(x, \rho, t) &= v_t(x, t - \rho \tau(t)), \quad x \in \Omega, \rho \in (0,1), t > 0.
\end{align*}
\]

Then we have

\[
\begin{align*}
\tau(t) \varphi_1'(x, \rho, t) + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} \varphi_1(x, \rho, t) &= 0, \quad \text{in } \Omega \times (0,1) \times (0, +\infty). \quad (1.3)
\end{align*}
\]

In the same way, we have

\[
\begin{align*}
\tau(t) \varphi_2'(x, \rho, t) + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} \varphi_2(x, \rho, t) &= 0, \quad \text{in } \Omega \times (0,1) \times (0, +\infty). \quad (1.4)
\end{align*}
\]
Therefore, problem (1.2) is equivalent to

\[
\begin{align*}
|u_1|^2u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t h_1(t-s)\Delta u(s)\,ds - \mu_1 \Delta z_1(x,t) + f_1(u,v) &= 0 \\
&\text{in } \Omega \times (0, +\infty], \\
|v_1|^2v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t h_2(t-s)\Delta v(s)\,ds - \mu_2 \Delta z_2(x,t) + f_2(u,v) &= 0 \\
&\text{in } \Omega \times (0, +\infty], \\
\tau(t)z_1'(x,\rho, t) + (1 - \rho \tau'(t))\frac{\partial}{\partial \rho} z_1(x,\rho, t) &= 0, \quad \text{in } \Omega \times (0, 1) \times (0, +\infty], \\
\tau(t)z_2'(x,\rho, t) + (1 - \rho \tau'(t))\frac{\partial}{\partial \rho} z_2(x,\rho, t) &= 0, \quad \text{in } \Omega \times (0, 1) \times (0, +\infty], \\
u(x,t) = v(x,t) = 0, \quad \text{on } \partial \Omega \times [0, +\infty], \\
(z_1(x,0,t), z_2(x,0,t)) &= (u_0(x), v_0(x)), \quad \text{on } \Omega \times (0, +\infty], \\
u(x,0) = (u_0(x), v_0(x)) \quad \text{in } \Omega, \\
(z_1(x,\rho, 0), z_2(x,\rho, 0)) &= (h_0(x,\rho \tau(0)), g_0(x,\rho \tau(0))), \quad \text{in } \Omega \times (0, 1].
\end{align*}
\tag{1.5}
\]

Throughout this work and for simplifying our formulas, we will adopt the notation $z_i, u$ and $v$ instead of $z_i(x,\rho, t), u(x,t)$ and $v(x,t)$, except if that makes things inconvenient.

In order to demonstrate the main result in this paper, a few assumptions are needed.

(A-1) Consider that $0 < l \leq \gamma$ verifies

\[
\begin{cases}
\gamma \leq \frac{2}{n+2} & \text{in the case } n > 2, \\
\gamma < \infty & \text{in the case } n \leq 2.
\end{cases}
\]

(A-2) As regards the relaxation functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$, we see that they are bounded $C^1$ functions such that

\[a - \int_0^\infty h_i(s)\,ds \geq k > 0.\]

We assume also that there exist some positive constants $\xi_i$ verifying

\[h_i'(t) \leq -\xi_i h_i(t)\]

for $i = 1, 2$.

(A-3) We have $\tau \in C^2([0, T], [\tau_0, \tau_1])$ a positive function, where

\[\tau'(t) \leq d < 1, \quad \forall t \in [0, T].\]

(A-4) $f_1(u,v) = \alpha u + b_1|\nu|^{p+1}|u|^{p-1}u$ and $f_2(u,v) = \alpha u + b_2|u|^{p+1}|\nu|^{p-1}v$ where $\alpha > 0$, $b_1 = (p+1)(p+q)$, $b_2 = (q+1)(p+q)$ such that $p$ and $q$ are conjugate (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), $p, q < \gamma - \frac{1}{2}$ and satisfy

\[2 \leq p, q \leq \begin{cases} 
\sqrt{\frac{n}{2(n-2)}} & \text{if } n > 2, \\
+\infty & \text{if } n \leq 2.
\end{cases}\]
The energy related to the system solution of (1.5) is defined as follows:

\[
E(t) = \frac{1}{t + 2} \left( \|u_t\|_{L^2}^2 + \|v_t\|_{L^2}^2 \right) + \frac{b}{2(\gamma + 1)} \left( \|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\
\quad + \frac{1}{2} \left( a - \int_0^t \eta_1(s) \, ds \right) \|\nabla u\|^2 + \frac{1}{2} \left( a - \int_0^t \eta_2(s) \, ds \right) \|\nabla v\|^2 \\
\quad + \frac{1}{2} \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\
\quad + \frac{1}{2} \eta_1 \phi \nabla u(t) + \frac{1}{2} \eta_2 \phi \nabla v(t) + \xi \tau(t) \int_0^1 \left( \|\nabla z_1\|^2 + \|\nabla z_2\|^2 \right) \, d\rho \\
\quad + \alpha \int_{\Omega} uv \, dx + (p + q) \int_{\Omega} |u|^{p+1} |v|^{q+1} \, dx,
\]

(1.6)

where \(\xi\) is a positive constant such that

\[
\max\{\mu_1, \mu_2\} \frac{2(1-d)}{2^{(1-d)}} < \xi
\]

(1.7)

and

\[
(h_1 \circ w)(t) = \int_0^t h_1(t-s) \|w(\cdot, t) - w(\cdot, s)\|^2 \, ds, \quad \text{for } i = 1, 2.
\]

Lemma 1.1 (Sobolev–Poincaré’s inequality) Let \(q\) be a number with

\[2 \leq q < +\infty \quad (n = 1, 2) \quad \text{or} \quad 2 \leq q \leq 2n/(n-2) \quad (n \geq 3).
\]

Then there exists a constant \(C_s = C_s(\Omega, q)\) such that

\[
\|u\|_q \leq C_s \|\nabla u\|_2 \quad \text{for } u \in H^1_0(\Omega).
\]

We present the following lemma.

Lemma 1.2 [15] For \(h, \varphi \in C^1\)-real functions, we have

\[
\frac{d}{dt} \left[ (h \circ \varphi)(t) - \left( \int_0^t h(s) \, ds \right) \|\varphi\|^2 \right] \\
= (h^2 - 2) \int_{\Omega} \int_0^t h(t-s) \varphi(t) \varphi_s(t) \, dx \, ds \quad \forall t \geq 0.
\]

(1.8)

Lemma 1.3 Let \((u, v, z_1, z_2)\) be a solution of the problem (1.5). Then the energy functional defined by (1.6) satisfies

\[
E(t) \leq -\beta \left( \|\nabla z_1(\cdot, 1, t)\|^2 + \|\nabla z_2(\cdot, 1, t)\|^2 \right) + \lambda \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\
\quad + \frac{1}{2} \left[ (h_1 \circ \nabla u)(t) + h_2 \circ \nabla v)(t) \right],
\]

(1.9)

where \(\lambda = \xi + \frac{\mu}{2}, \beta = \xi (1-d) - \frac{\mu}{2}\) and \(\mu = \max\{\mu_1, \mu_2\}\) are positive.
Proof. After the multiplication of the first equation in (1.5) by \( u_t \) followed by integration of the result by parts over \( \Omega \), we get

\[
\frac{d}{dt} \left[ \frac{1}{l+2} \left\| u_t \right\|^2_s + \frac{b}{2(\gamma + 1)} \left\| \nabla u \right\|^2_{s+1} + \frac{1}{2} d \left\| \nabla u_t \right\|^2 + \frac{1}{2} \left\| \nabla u_t \right\|^2 \right]
- \int_\Omega \int_0^t h_1(t-s) \nabla u(s) \nabla u_t(t) \, ds \, dx
+ \mu_1 \int_\Omega \nabla u_t \nabla z_1(x,1,t) \, dx + \alpha \int_\Omega v u_t \, dx + b_1 \int_\Omega |v|^{p+1} |u|^{p-1} u u_t \, dx = 0. \tag{1.10}
\]

Using (1.8) and (1.10) leads to

\[
\frac{d}{dt} \left[ \frac{1}{l+2} \left\| v_t \right\|^2_s + \frac{b}{2(\gamma + 1)} \left\| \nabla v \right\|^2_{s+1} + \frac{1}{2} \left( a - \int_0^t h_1(s) \, ds \right) \left\| \nabla v \right\|^2 \right]
+ \frac{1}{2} \left\| \nabla v_t \right\|^2 + \frac{1}{2} (h_1 \circ \nabla u)(t)
+ \frac{1}{2} h_1(t) \left\| \nabla v \right\|^2 - \frac{1}{2} (h'_1 \circ \nabla u)(t)
+ \mu_1 \int_\Omega \nabla u_t \nabla z_1(x,1,t) \, dx
+ \alpha \int_\Omega v u_t \, dx + b_1 \int_\Omega |v|^{p+1} |u|^{p-1} u u_t \, dx = 0. \tag{1.11}
\]

Similarly by multiplying the second equation in (1.5) by \( v_t \), integrating over \( \Omega \) and using integration by parts, we get

\[
\frac{d}{dt} \left[ \frac{1}{l+2} \left\| v_t \right\|^2_s + \frac{b}{2(\gamma + 1)} \left\| \nabla v \right\|^2_{s+1} + \frac{1}{2} \left( a - \int_0^t h_2(s) \, ds \right) \left\| \nabla v \right\|^2 \right]
+ \frac{1}{2} \left\| \nabla v_t \right\|^2 + \frac{1}{2} (h_2 \circ \nabla v)(t)
+ \frac{1}{2} h_2(t) \left\| \nabla v \right\|^2 - \frac{1}{2} (h'_2 \circ \nabla v)(t)
+ \mu_2 \int_\Omega \nabla v_t \nabla z_2(x,1,t) \, dx + \alpha \int_\Omega u v_t \, dx
+ b_2 \int_\Omega |u|^{p+1} |v|^{p-1} v v_t \, dx = 0. \tag{1.12}
\]

Multiplying the third equation in (1.5) by \( \xi \Delta z_1 \) and integrating the result over \( \Omega \times (0,1) \), we obtain

\[
\xi \tau(t) \int_0^1 \Delta z_1 \, d\rho \, dx = -\xi \int_{\Omega \times (0,1)} (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1 \Delta z_1 \, d\rho \, dx.
\]

Consequently,

\[
\frac{d}{dt} \left( \xi \tau(t) \int_0^1 \left\| \nabla z_1 \right\|^2 \, d\rho \right) = \xi \int_{\Omega \times (0,1)} \left[ \tau'(t) \left\| \nabla z_1 \right\|^2 - \rho \tau'(t) \frac{\partial}{\partial \rho} \left\| \nabla z_1 \right\|^2 \right] \, d\rho \, dx
= -\xi \int_0^1 \frac{\partial}{\partial \rho} \left( (1 - \rho \tau'(t)) \left\| \nabla z_1 \right\|^2 \right) \, d\rho
= \xi \left[ \left\| \nabla u_t \right\|^2 - \xi (1 - \tau'(t)) \left\| \nabla z_1(x,1,t) \right\|^2 \right]. \tag{1.13}
\]
Similarly we get
\[
\frac{d}{dt} \left( \xi \tau(t) \int_0^1 \| \nabla z_2 \|^2 \, d\rho \right) = \xi \left[ \| \nabla v_1 \|^2 - (1 - \tau'(t)) \| \nabla z_2(x, 1, t) \|^2 \right]. \tag{1.14}
\]

Combining (1.11)–(1.14), taking the derivation of energy leads to
\[
E'(t) = \xi \left[ \| \nabla u_t \|^2 + \| \nabla v_t \|^2 - (1 - \tau'(t)) \left( \| \nabla z_1(x, 1, t) \|^2 + \| \nabla z_2(x, 1, t) \|^2 \right) \right]
- \frac{1}{2} \left[ h_1(t) \| \nabla u_t \|^2 + h_2(t) \| \nabla v_t \|^2 \right]
+ \frac{1}{2} \left[ (h'_1 \circ \nabla u)(t) + (h'_2 \circ \nabla v)(t) \right] - \mu_1 \int_\Omega \nabla u_t(x, t) \nabla z_1(x, 1, t) \, dx
- \mu_2 \int_\Omega \nabla v_t(x, t) \nabla z_2(x, 1, t) \, dx.
\]

From (A3), we find the following bound:
\[
E'(t) \leq - \left( \xi(1 - d) - \frac{\mu_1}{2} \right) \int_\Omega \| \nabla z_1(x, 1, t) \|^2 \, dx
- \frac{1}{2} h_1(t) \| \nabla u_t(t) \|^2 + \left( \xi + \frac{\mu_1}{2} \right) \| \nabla u_t(t) \|^2
- \frac{1}{2} h_2(t) \| \nabla v(t) \|^2
+ \left( \xi + \frac{\mu_2}{2} \right) \| \nabla v_1(t) \|^2 \tag{1.15}
+ \frac{1}{2} \left[ (h'_1 \circ \nabla u)(t) + (h'_2 \circ \nabla v)(t) \right].
\]

Using (1.7), we complete the proof of the lemma. \qed

2 Global existence

**Theorem 2.1** Let \((u_0, v_0) \in (H^2(\Omega) \cap H^1_0(\Omega))^2, (u_1, v_1) \in (H^2_0(\Omega))^2\) and \((f_0, g_0) \in (H^1_0(\Omega), H^1(0, 1))^2\) satisfy the compatibility condition
\[
(f_0(\cdot, 0), g_0(\cdot, 0)) = (u_1, v_1).
\]

Assume that (A1)–(A3) hold. Then the problem (1.2) admits a weak solution such that \(u, v \in L^\infty(0, \infty; H^2(\Omega) \cap H^1_0(\Omega)), u_t, v_t \in L^\infty(0, \infty; H^1_0(\Omega)), \) and \(u_{tt}, v_{tt} \in L^2(0, \infty, H^1_0(\Omega)). \)

**Proof** As in the previous assumptions in [2] for the initial conditions \(u_0, v_0 \in H^2(\Omega) \cap H^1_0(\Omega), u_1, v_1 \in H^1_0(\Omega), f_0, g_0 \in H^1_0(\Omega), H^1(0, 1))\) and the basic functions, we introduce the approximate solutions \((u^k, v^k, z_1^k, z_2^k), k = 1, 2, 3, \ldots, \) in the form
\[
u^k(t) = \sum_{j=1}^k b^k(t) w^j, \quad v^k(t) = \sum_{j=1}^k d^k(t) \phi^j.
\]

Similarly, we get
\[
E'(t) = \xi \left[ \| \nabla u_t \|^2 + \| \nabla v_t \|^2 - (1 - \tau'(t)) \left( \| \nabla z_1(x, 1, t) \|^2 + \| \nabla z_2(x, 1, t) \|^2 \right) \right]
- \frac{1}{2} \left[ h_1(t) \| \nabla u_t \|^2 + h_2(t) \| \nabla v_t \|^2 \right]
+ \frac{1}{2} \left[ (h'_1 \circ \nabla u)(t) + (h'_2 \circ \nabla v)(t) \right] - \mu_1 \int_\Omega \nabla u_t(x, t) \nabla z_1(x, 1, t) \, dx
- \mu_2 \int_\Omega \nabla v_t(x, t) \nabla z_2(x, 1, t) \, dx.
\]

From (A3), we find the following bound:
\[
E'(t) \leq - \left( \xi(1 - d) - \frac{\mu_1}{2} \right) \int_\Omega \| \nabla z_1(x, 1, t) \|^2 \, dx
- \frac{1}{2} h_1(t) \| \nabla u_t(t) \|^2 + \left( \xi + \frac{\mu_1}{2} \right) \| \nabla u_t(t) \|^2
- \frac{1}{2} h_2(t) \| \nabla v(t) \|^2
+ \left( \xi + \frac{\mu_2}{2} \right) \| \nabla v_1(t) \|^2 \tag{1.15}
+ \frac{1}{2} \left[ (h'_1 \circ \nabla u)(t) + (h'_2 \circ \nabla v)(t) \right].
\]

Using (1.7), we complete the proof of the lemma. \qed
where $a^i, b^i, c^i$ and $d^i$ ($j = 1, 2, \ldots, k$) are determined by the following ordinary differential equations:

\[
\begin{cases}
\left| |u^k_t|^i u^k_{tt}, w_j\right| + M\left(\|\nabla u^k(t)\|^2\right)\left(\nabla u^k, \nabla w_j\right) + \left(\nabla u^k, \nabla w_j\right) \\
- \int_0^t h_1(t-s)\left(\nabla u^k(s), \nabla w_j\right) ds + \mu_1\left(\nabla z^j_1(1, 1), \nabla w_j\right) + (f_1(u^k, v^k), w_j) = 0, \\
1 \leq j \leq k, \\
\left| |v^k_t|^i v^k_{tt}, w_j\right| + M\left(\|\nabla v^k(t)\|^2\right)\left(\nabla v^k, \nabla w_j\right) + \left(\nabla v^k, \nabla w_j\right) \\
- \int_0^t h_2(t-s)\left(\nabla v^k(s), \nabla w_j\right) ds + \mu_2\left(\nabla z^j_2(1, 1), \nabla w_j\right) + (f_2(u^k, v^k), w_j) = 0, \\
1 \leq j \leq k, \\
z^j_1(x, 0, t) = u^k_t(x, t), \\
z^j_2(x, 0, t) = v^k_t(x, t),
\end{cases}
\]

\begin{align}
\tag{2.1}
& u^k(0) = u^k_0 = \sum_{j=1}^k (u_0, w_j) w_j \to u_0, \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega) \text{ as } k \to +\infty, \\
& v^k(0) = v^k_0 = \sum_{j=1}^k (v_0, w_j) w_j \to v_0, \quad \text{in } H^2(\Omega) \cap H^1_0(\Omega) \text{ as } k \to +\infty, \\
& u^k_t(0) = u^k_t = \sum_{j=1}^k (u_1, w_j) w_j \to u_1, \quad \text{in } H^1_0(\Omega) \text{ as } k \to +\infty, \\
& v^k_t(0) = v^k_t = \sum_{j=1}^k (v_1, w_j) w_j \to v_1, \quad \text{in } H^1_0(\Omega) \text{ as } k \to +\infty.
\end{align}

Also

\begin{align}
& \left(\tau(t) + \frac{\sqrt{\varepsilon}}{2t^2}\right) z^j_1 + (1 - \rho \tau'(t)) \frac{\sqrt{\varepsilon}}{2t^2} z^j_1, \phi^i_j = 0, \quad 1 \leq j \leq k, \\
& \left(\tau(t) + \frac{\sqrt{\varepsilon}}{2t^2}\right) z^j_2 + (1 - \rho \tau'(t)) \frac{\sqrt{\varepsilon}}{2t^2} z^j_2, \phi^i_j = 0, \quad 1 \leq j \leq k.
\end{align}

\begin{align}
& z^j_1(\rho, 0) = \sum_{j=1}^k (f_0, \phi^i_j) \phi^i_j \to f_0, \quad \text{in } H^1(\Omega, H^1(0, 1)) \text{ as } k \to +\infty, \\
& z^j_2(\rho, 0) = \sum_{j=1}^k (g_0, \phi^i_j) \phi^i_j \to g_0, \quad \text{in } H^1(\Omega, H^1(0, 1)) \text{ as } k \to +\infty.
\end{align}

Noting that \( \frac{1}{2(\nu+1)} + \frac{1}{2(\nu+1)} + \frac{1}{2} = 1 \), by applying the generalized Hölder inequality, we find

\[ \left( |u^k_t|^i u^k_{tt}, w_j \right) = \int_\Omega |u^k_t|^i u^k_{tt} w_j dx \leq \left( \int_\Omega |u^k_t|^{2(\nu+1)} dx \right)^\frac{\nu}{2(\nu+1)} \|u^k_t\|_{2(\nu+1)} \|w_j\|_2. \]

Since (A1) holds, according to the Sobolev embedding the nonlinear terms \(||u^k_t|^i u^k_{tt}, w_j||_2 \) and \(||v^k_t|^i v^k_{tt}, w_j||_2 \) in (2.1) make sense (see [2]).

\textbf{A. First estimate.}

Since the sequences \( u^k_t, v^k_t, u^k_t, v^k_t, z^j_1(\rho, 0) \) and \( z^j_2(\rho, 0) \) converge and from Lemma 1.3 with employing Gronwall’s lemma, we find \( C_1 > 0 \) independent of \( k \) such that

\[ E^k(t) + \beta \int_0^t \left( \|\nabla z^j_1(x, 1, s)\|^2 + \|\nabla z^j_2(x, 1, s)\|^2 \right) ds \leq C_1, \tag{2.9} \]
Differentiating (2.6) with respect to \( t \), we get

\[
\text{over } j \text{ from 1 to } k, \text{we have}
\]

Noting \((2.6)\) and the estimate \((2.9)\) yields

\[
\begin{align*}
u^k, \nu^h & \text{ are bounded in } L^\infty_{\text{loc}}(0, \infty, H^1_0(\Omega)), \\
u^k_t, \nu^h_t & \text{ are bounded in } L^\infty_{\text{loc}}(0, \infty, H^1_0(\Omega)), \\
z^1_t(x, \rho, t), z^2_t(x, \rho, t) & \text{ are bounded in } L^\infty_{\text{loc}}(0, \infty, L^1(0, 1, H^1_0(\Omega))).
\end{align*}
\]

**B. The second estimate.**

By multiplying the first side of equation (respectively, the second equation) in (2.1) by \( a^j_{tt} \) (respectively, by \( b^j_{tt} \)), by summing \( j \) from 1 to \( k \), then

\[
\begin{align*}
\int_{\Omega} |u^k_t|^2 |u^k_{tt}|^2 \, dx + \int_{\Omega} M(||\nabla u^k||^2) \nabla u^k \nabla u^k_{tt} \, dx + ||\nabla u^k_{tt}||^2 \\
= \int_{\Omega} h_1(t-s) \nabla u^k(s) \nabla u^k_{tt}(t) \, dx \, ds \\
- \mu_1 \int_{\Omega} \nabla u^k \nabla (z^1_t(x, 1, t)) \, dx - \int_{\Omega} f_1(u^k, \nu^h) u^k_{tt}(t) \, dx, \\
\int_{\Omega} |v^k_t|^2 |v^k_{tt}|^2 \, dx + \int_{\Omega} M(||\nabla \nu^h||^2) \nabla \nu^h \nabla v^k_{tt} \, dx + ||\nabla v^k_{tt}||^2 \\
= \int_{\Omega} h_2(t-s) \nabla \nu^h(s) \nabla v^k_{tt}(t) \, dx \, ds \\
- \mu_2 \int_{\Omega} \nabla \nu^h \nabla (z^2_t(x, 1, t)) \, dx - \int_{\Omega} f_2(u^k, \nu^h) v^k_{tt}(t) \, dx.
\end{align*}
\]

Differentiating (2.6) with respect to \( t \), we get

\[
\begin{align*}
\left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) \frac{\partial \alpha}{\partial t} z^1 + \frac{\partial (t)}{(1-\rho^t(t))} \frac{\partial^2 z^1}{\partial \rho^t \partial t} + \frac{\partial^2 \alpha}{\partial \rho^t \partial t^2} z^1(t, \phi') = 0, \\
\left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) \frac{\partial \alpha}{\partial t} z^2 + \frac{\partial (t)}{(1-\rho^t(t))} \frac{\partial^2 z^2}{\partial \rho^t \partial t} + \frac{\partial^2 \alpha}{\partial \rho^t \partial t^2} z^2(t, \phi') = 0.
\end{align*}
\]

Multiplying the first equation by \( c^j_{tt} \) (respectively the second equation by \( d^j_{tt} \)), summing over \( j \) from 1 to \( k \), we have

\[
\begin{align*}
\frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial \alpha}{\partial t} z^1||^2 + \frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial z^1}{\partial t}||^2 + \frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial^2 z^1}{\partial \rho^t \partial t}||^2 = 0, \\
\frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial \alpha}{\partial t} z^2||^2 + \frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial z^2}{\partial t}||^2 + \frac{1}{2} \left( \frac{d}{dt} \frac{\partial (t)}{(1-\rho^t(t))} \right) ||\frac{\partial^2 z^2}{\partial \rho^t \partial t}||^2 = 0.
\end{align*}
\]
Integrating over $(0, 1)$ with respect to $\rho$, we obtain

\[
\left\{ \begin{array}{l}
\frac{1}{2} \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \left( \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho \right) \\
+ \frac{1}{2} \left\| \frac{\partial}{\partial \tau} \zeta^1 (x, 1, t) \right\|^2 - \frac{1}{2} \left\| \zeta^1_t (x, t) \right\|^2 = 0,
\end{array} \right. \tag{2.14}
\]

Summing (2.13), (2.14) and as $M(r) \geq a$, we get

\[
\left\{ \begin{array}{l}
\int_{\Omega} \left\| \zeta^1 \right\|^2 d\xi + \left\| \nabla \zeta^1 \right\|^2 \leq a \int_{\Omega} \zeta^1_t \zeta^1 dx + \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} \zeta^1_t \zeta^1 dx \right) + \frac{1}{2} \left\| \zeta^1_t \right\|^2 (x, 1, t) \left\| \right. \\
+ \int_{\Omega} \zeta^1_t (x, t) \zeta^1 dx - \int_{\Omega} \int_{\Omega} \left. \zeta^1 \right| \zeta^1_t \zeta^1 dx ds - \mu_1 \int_{\Omega} \nabla \zeta^1 \nabla \zeta^1 dx,
\end{array} \right. \tag{2.15}
\]

We estimate the right hand side of (2.15) as follows:

From the integration by parts, we have

\[- \int_{\Omega} f_1(\zeta^1, \zeta^1) \zeta^1 dx = \alpha \int_{\Omega} \zeta^1 \zeta^1 dx - b_1 \int_{\Omega} \left\| \zeta^1 \right\|^{q+1} \zeta^1 \zeta^1 dx.
\]

Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Sobolev–Poincaré inequalities, we obtain

\[
\alpha \int_{\Omega} \zeta^1 \zeta^1 dx \leq \frac{\alpha}{2} \left( \left\| \zeta^1 \right\|^2 + \left\| \zeta^1 \right\|^2 \right) \leq \frac{C_2 \alpha}{2} \left( \left\| \nabla \zeta^1 \right\|^2 + \left\| \zeta^1 \right\|^2 \right). \tag{2.16}
\]

On the other hand, by recalling (A-4) and Lemma 1.1 and using Young’s inequality, we get

\[
\left| \int_{\Omega} \left\| \zeta^1 \right\|^{q+1} \left\| \zeta^1 \right\|^{q+1} \zeta^1 \zeta^1 dx \right| \leq \frac{1}{2} \int_{\Omega} \left\| \zeta^1 \right\|^{2(q+1)} \zeta^1 dx + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \int_{\Omega} \left\| \zeta^1 \right\|^{2(q+1)} dx + \frac{1}{8\eta} \int_{\Omega} \left\| \zeta^1 \right\|^{2q} dx + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \left\| \zeta^1 \right\|^{2(q+1)} + \frac{1}{8\eta} \left\| \zeta^1 \right\|^{2q} + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \left\| \zeta^1 \right\|^{4(q+1)} \left\| \nabla \zeta^1 \right\|^2 + \frac{C_2^2 \eta}{8\eta} \left\| \zeta^1 \right\|^2 + \frac{C_2^2}{2} \left\| \zeta^1 \right\|^2 \tag{2.17}
\]

Integrating over $(0, 1)$ with respect to $\rho$, we obtain

\[
\left\{ \begin{array}{l}
\frac{1}{2} \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \left( \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho \right) \\
+ \frac{1}{2} \left\| \frac{\partial}{\partial \tau} \zeta^1 (x, 1, t) \right\|^2 - \frac{1}{2} \left\| \zeta^1_t (x, t) \right\|^2 = 0,
\end{array} \right. \tag{2.14}
\]

Summing (2.13), (2.14) and as $M(r) \geq a$, we get

\[
\left\{ \begin{array}{l}
\int_{\Omega} \left\| \zeta^1 \right\|^2 d\xi + \left\| \nabla \zeta^1 \right\|^2 \leq a \int_{\Omega} \zeta^1_t \zeta^1 dx + \frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} \zeta^1_t \zeta^1 dx \right) + \frac{1}{2} \left\| \zeta^1_t \right\|^2 (x, 1, t) \left\| \right. \\
+ \int_{\Omega} \zeta^1_t (x, t) \zeta^1 dx - \int_{\Omega} \int_{\Omega} \left. \zeta^1 \right| \zeta^1_t \zeta^1 dx ds - \mu_1 \int_{\Omega} \nabla \zeta^1 \nabla \zeta^1 dx,
\end{array} \right. \tag{2.15}
\]

We estimate the right hand side of (2.15) as follows:

From the integration by parts, we have

\[- \int_{\Omega} f_1(\zeta^1, \zeta^1) \zeta^1 dx = \alpha \int_{\Omega} \zeta^1 \zeta^1 dx - b_1 \int_{\Omega} \left\| \zeta^1 \right\|^{q+1} \zeta^1 \zeta^1 dx.
\]

Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Sobolev–Poincaré inequalities, we obtain

\[
\alpha \int_{\Omega} \zeta^1 \zeta^1 dx \leq \frac{\alpha}{2} \left( \left\| \zeta^1 \right\|^2 + \left\| \zeta^1 \right\|^2 \right) \leq \frac{C_2 \alpha}{2} \left( \left\| \nabla \zeta^1 \right\|^2 + \left\| \zeta^1 \right\|^2 \right). \tag{2.16}
\]

On the other hand, by recalling (A-4) and Lemma 1.1 and using Young’s inequality, we get

\[
\left| \int_{\Omega} \left\| \zeta^1 \right\|^{q+1} \left\| \zeta^1 \right\|^{q+1} \zeta^1 \zeta^1 dx \right| \leq \frac{1}{2} \int_{\Omega} \left\| \zeta^1 \right\|^{2(q+1)} \zeta^1 dx + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \int_{\Omega} \left\| \zeta^1 \right\|^{2(q+1)} dx + \frac{1}{8\eta} \int_{\Omega} \left\| \zeta^1 \right\|^{2q} dx + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \left\| \zeta^1 \right\|^{2(q+1)} + \frac{1}{8\eta} \left\| \zeta^1 \right\|^{2q} + \frac{1}{2} \left\| \zeta^1 \right\|^2 \\
\leq \frac{\eta}{2} \left\| \zeta^1 \right\|^{4(q+1)} \left\| \nabla \zeta^1 \right\|^2 + \frac{C_2^2 \eta}{8\eta} \left\| \zeta^1 \right\|^2 + \frac{C_2^2}{2} \left\| \zeta^1 \right\|^2 \tag{2.17}
\]

Integrating over $(0, 1)$ with respect to $\rho$, we obtain

\[
\left\{ \begin{array}{l}
\frac{1}{2} \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \left( \int_0^1 \frac{\sigma(t)}{(1-p t^{\eta})} \left\| \frac{\partial}{\partial \tau} \zeta^1 \right\|^2 d\rho \right) \\
+ \frac{1}{2} \left\| \frac{\partial}{\partial \tau} \zeta^1 (x, 1, t) \right\|^2 - \frac{1}{2} \left\| \zeta^1_t (x, t) \right\|^2 = 0,
\end{array} \right. \tag{2.14}
\]
Hence from summing (2.16) and (2.17) we deduce that
\[
- \left| \int_{\Omega} f_1(u^k, v^k) u^k_\alpha(t) \, dx \right| \leq \frac{C^2_2}{2} \left( \| \nabla v^k \|^2 + \| \nabla u^k_\alpha \|^2 \right) \\
+ \frac{b_1 \eta}{2} |\Omega|^{\frac{q+1}{q(\alpha+1)}} \| \nabla v^k \|^2 \| \nabla u^k_\alpha \|^2 \\
+ \frac{b_1 C^2_1}{8\eta} \| \nabla u^k \|^2 \| \nabla v^k \|^2 + \frac{b_1 C^2_1}{2} \| \nabla u^k_\alpha \|^2. \tag{2.18}
\]

Similarly
\[
- \left| \int_{\Omega} f_2(u^k, v^k) v^k_\alpha(t) \, dx \right| \leq \frac{C^2_2}{2} \left( \| \nabla v^k \|^2 + \| \nabla u^k_\alpha \|^2 \right) \\
+ \frac{b_2 \eta}{2} |\Omega|^{\frac{q}{q(p+1)}} \| \nabla u^k \|^2 \left( \nabla v^k \right)^p \\
+ \frac{b_2 C^2_2}{8\eta} \| \nabla v^k \|^2 \left( \nabla u^k \right)^2 + \frac{b_2 C^2_1}{2} \| \nabla v^k \|^2. \tag{2.19}
\]

Also by Young’s inequality, we get
\[
\begin{align*}
| \int_{\Omega} a \nabla u^k \cdot \nabla u^k_\alpha \, dx | \leq \eta \| \nabla u^k_\alpha \|^2 + \frac{C_2}{4\eta} \| \nabla u^k \|^2, \\
| \int_{\Omega} a \nabla v^k \cdot \nabla v^k_\alpha \, dx | \leq \eta \| \nabla v^k_\alpha \|^2 + \frac{C_2}{4\eta} \| \nabla v^k \|^2. \tag{2.20}
\end{align*}
\]

We have
\[
\left| \int_{0}^{t} h_1(t - s) \left| \int_{\Omega} \nabla u^k(s) \cdot \nabla u^k_\alpha(t) \, dx \right| \, ds \right|
\leq \eta \left\| \nabla u^k_\alpha \right\|^2 + \frac{1}{4\eta} \int_{\Omega} \left( \int_{0}^{t} h_1(t - s) \left| \nabla u^k(s) \right| \, ds \right)^2 \, dx
\leq \eta \left\| \nabla u^k_\alpha \right\|^2 + \frac{1}{4\eta} \int_{0}^{t} h_1(s) \, ds \int_{\Omega} \left( \int_{0}^{t} h_1(t - s) \left| \nabla u^k(s) \right| \, ds \right) \, dx
\leq \eta \left\| \nabla u^k_\alpha \right\|^2 + \frac{a - k}{4\eta} \int_{0}^{t} h_1(t - s) \left\| \nabla u^k(s) \right\|^2 \, ds
\leq \eta \left\| \nabla u^k_\alpha \right\|^2 + \frac{(a - k) h_1(0)}{4\eta} \int_{0}^{t} \left\| \nabla u^k(s) \right\|^2 \, ds. \tag{2.21}
\]

Similarly
\[
\left| \int_{0}^{t} h_2(t - s) \left| \int_{\Omega} \nabla v^k(s) \cdot \nabla v^k_\alpha(t) \, dx \right| \, ds \right|
\leq \eta \left\| \nabla v^k_\alpha \right\|^2 + \frac{(a - k) h_2(0)}{4\eta} \int_{0}^{t} \left\| \nabla v^k(s) \right\|^2 \, ds \tag{2.22}
\]

and
\[
\begin{align*}
| \mu_1 \int_{\Omega} \nabla u^k \cdot \nabla z^k_1(x, 1, t) \, dx | \leq \eta \mu_1^2 \left\| \nabla u^k \right\|^2 + \frac{1}{4\eta} \left\| \nabla z^k_1(x, 1, t) \right\|^2,
| \mu_2 \int_{\Omega} \nabla v^k \cdot \nabla z^k_2(x, 1, t) \, dx | \leq \eta \mu_2^2 \left\| \nabla v^k \right\|^2 + \frac{1}{4\eta} \left\| \nabla z^k_2(x, 1, t) \right\|^2. \tag{2.23}
\end{align*}
\]
Taking into account (2.18)–(2.23) into (2.15) yields

\[
\begin{align*}
\int_{\Omega} |u_{m}^{i}| |u_{n}^{j}|^{2} \, dx + \| \nabla u_{m}^{i} \|^{2} &+ \| \nabla u_{n}^{j} \|^{2} + \frac{1}{2} \int_{0}^{T} \left( \int_{0}^{1} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \| \frac{\partial}{\partial \tau} z_{1}^{i} \|^{2} \, d\rho \right) + \frac{1}{2} \| \frac{\partial}{\partial \tau} z_{1}^{j}(x, 1, t) \|^{2} \\
&\leq (\eta (\mu_{2}^{2} + 2) + \frac{C^{2}}{2}) \| \nabla u_{m}^{i} \|^{2} + \frac{1}{2} \| \tau_{0} z_{1}^{i}(x, 1, t) \|^{2} \\
&\quad + \| \nabla u_{n}^{j} \|^{2} + \| \nabla u_{m}^{i} \|^{2} + \| \nabla u_{n}^{j} \|^{2} + \frac{b_{1} C_{1}^{2}}{20} \| \nabla u_{m}^{i} \|^{2} \\
&\quad + \frac{b_{1} C_{1}^{2}}{20} \| \nabla u_{n}^{j} \|^{2} + \frac{b_{1} C_{1}^{2}}{20} \| \nabla u_{m}^{i} \|^{2} \\
&\quad + \frac{1}{2} \| \tau_{0} z_{1}^{i}(x, 1, t) \|^{2} \leq C_{2} + \frac{1}{20} (\alpha - \kappa_{1}) \tau_{1}(0) C_{1} T,
\end{align*}
\]

where $C_{2}$ is a positive constant that depends on $\eta, \alpha, a, C_{i}, |\Omega|, b_{1}, b_{2}, p, q, C_{1}$ for $i = 1, 2$. 

Integrating (2.24) over $(0, t)$ we obtain

\[
\begin{align*}
\int_{0}^{t} \int_{\Omega} |u_{m}^{i}| |u_{n}^{j}|^{2} \, dx \, dt + (1 - (\eta (\mu_{2}^{2} + 2) + \frac{(1+b_{2}) C_{1}^{2}}{2})) \int_{0}^{T} \| \nabla u_{m}^{i} \|^{2} \, ds \\
&\quad + \frac{1}{2} \int_{0}^{T} \left( \int_{0}^{1} \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \| \frac{\partial}{\partial \tau} z_{1}^{i} \|^{2} \, d\rho \right) + \frac{1}{2} \| \tau_{0} z_{1}^{i}(x, 1, t) \|^{2} \, dt \leq (C_{2} + \frac{1}{20} (\alpha - \kappa_{1}) \tau_{1}(0) C_{1} T) T,
\end{align*}
\]

For a suitable $\eta > 0$ such that $1 - (\eta (\mu_{2}^{2} + 2) + \frac{(1+b_{2}) C_{1}^{2}}{2}) > 0$ for $i = 1, 2$, we obtain the second estimate

\[
\begin{align*}
\int_{0}^{t} \left( \| \nabla u_{m}^{i} \|^{2} + \| \nabla u_{n}^{j} \|^{2} \right) \, ds \\
&\quad + \int_{0}^{1} \frac{\tau(t)}{1 - \rho \tau'(t)} \left( \| \frac{\partial}{\partial \tau} z_{1}^{i} \|^{2} + \| \frac{\partial}{\partial \tau} z_{1}^{j} \|^{2} \right) \, d\rho \leq C_{3}.
\end{align*}
\]

We observe from the estimate (2.9) and (2.25) that there exist subsequences $(u^{m})$ of $(u^{i})$ and $(v^{m})$ of $(v^{j})$ such that

\[
(u^{m}, v^{m}) \rightharpoonup (u, v) \quad \text{weakly star in } L^{\infty}(0, T, H_{0}^{1}(\Omega)),
\]

\[
(\cdot, \nabla) \quad \text{weakly in } L^{2}(0, T, H^{-1}(\Omega)).
\]
\[(u_t^m, v_t^m) \rightharpoonup (u_t, v_t) \text{ weakly star in } L^\infty(0, T, H^1_0(\Omega)), \quad (2.27)\]

\[(u_{tt}^m, v_{tt}^m) \rightharpoonup (u_{tt}, v_{tt}) \text{ weakly in } L^2(0, T, H^1_0(\Omega)), \quad (2.28)\]

\[(z_1^m, z_2^m) \rightharpoonup (z_1, z_2) \text{ weakly star in } L^\infty(0, T, L^2(\Omega, L^2(0,1))), \quad (2.29)\]

\[\left(\frac{\partial}{\partial t} z_1^m, \frac{\partial}{\partial t} z_2^m\right) \rightharpoonup \left(\frac{\partial}{\partial t} z_1, \frac{\partial}{\partial t} z_2\right) \text{ weakly star in } L^\infty(0, T, L^2(\Omega \times (0,1))). \quad (2.30)\]

In the following, we will treat the nonlinear term. From the first estimate (2.9) and Lemma 1.1, we deduce

\[
\|u_t^m\|_{L^2(0,T,L^2(\Omega))}^2 \leq C_4^2(1 + \|\nabla u_t^m\|_{L^2(0,T,L^2(\Omega))}^2),
\]

where \(C_4\) depends only on \(C_s, C_1, T, l\).

On the other hand, from the Aubin–Lions theorem (see Lions [9]), we deduce that there exists a subsequence of \((u_t^m)\), still denoted by \((u_t^m)\), such that

\[u_t^m \to u_t \text{ strongly in } L^2(0,T,L^2(\Omega)), \quad (2.31)\]

which implies

\[u_t^m \to u_t \text{ almost everywhere in } \mathcal{A}. \quad (2.32)\]

Hence

\[|u_t^m| |u_t^m| \to |u_t| |u_t| \text{ almost everywhere in } \mathcal{A}, \quad (2.33)\]

where \(\mathcal{A} = \Omega \times (0, T)\). Thus, using (2.31), (2.33) and the Lions lemma, we derive

\[|u_t^m| |u_t^m| \to |u_t| |u_t| \text{ weakly in } L^2(0,T,L^2(\Omega)); \quad (2.34)\]

similarly

\[|v_t^m| |v_t^m| \to |v_t| |v_t| \text{ weakly in } L^2(0,T,L^2(\Omega)) \quad (2.35)\]

and

\[(z_1^m, z_2^m) \to (z_1, z_2) \text{ strongly in } L^2(0,T,L^2(\Omega)), \quad (2.36)\]

which implies \((z_1^m, z_2^m) \to (z_1, z_2)\) almost everywhere in \(\mathcal{A}\).

The sequences \((u_t^m)\) and \((v_t^m)\) satisfy

\[f_1(u^m, v^m) \to f_1(u, v) \text{ strongly in } L^\infty(0, T, L^2(\Omega)) \quad (2.37)\]
and

\[ f_2(u^m, v^m) \to f_2(u, v) \quad \text{strongly in } L^\infty(0, T, L^2(\Omega)); \]  

we have

\[ \|f_1(u^m, v^m) - f_1(u, v)\|^2 = \int_\Omega |\nabla|^{q+1} u^k||u^k|^p u^k - |v|^{q+1}|u|^p u|^2 \, dx. \]

As we add and subtract \(|v^k|^{q+1}|u|^p u\) to the previous formula, we obtain

\[ \|f_1(u^m, v^m) - f_1(u, v)\|^2 \leq 4C \left[ \int_\Omega |v^k|^{2(q+1)} |u^k|^p u^k - |u|^p u|^2 \, dx \right. \]

\[ \left. + \int_\Omega |u|^{2(q+1)} |v^k - v|^{2} \right] \left( |v^k|^{2q} + |v|^2 \right) \, dx. \]  

We use the following elementary inequalities:

\[ |a^k - b^k| \leq C|a - b|(|a|^{k-1} + |b|^{k-1}), \]

\[ |a^k a - |b|^k b| \leq C|a - b|(\|a\|^k + |b|^k), \]

and

\[ (a + b)^2 \leq 2(a^2 + b^2), \]

for some constant \(C\), \(\forall k \geq 1\) and \(\forall a, b \in \mathbb{R}\). Hence (2.38) becomes

\[ \|f_1(u^m, v^m) - f_1(u, v)\|^2 \leq 4C \left[ \int_\Omega |v^k|^{2(q+1)} |u^k|^p u^k - |u|^p u|^2 \, dx \right. \]

\[ \left. + \int_\Omega |u|^{2(q+1)} |v^k - v|^{2} \right] \left( |v^k|^{2q} + |v|^2 \right) \, dx. \]  

The typical term in the above formula can be estimated as follows.

Noting that \(\frac{1}{q} + \frac{1}{2q} + \frac{1}{2} = 1\), by applying the generalized Hölder inequality, we find

\[ \int_\Omega |v^k|^{2(q+1)} |u^k - u|^2 |u^k|^{2p} \, dx \]

\[ \leq \left( \int_\Omega |v^k|^{4q+1} \, dx \right)^{\frac{3}{2}} \left( \int_\Omega |u^k - u|^{4q} \, dx \right)^{\frac{1}{2q}} \left( \int_\Omega |u^k|^{4q^2} \, dx \right)^{\frac{1}{2p}}. \]  

Recalling (A4), Lemma 1.1 and (2.9), we get

\[ \int_\Omega |v^k|^{2(q+1)} |u^k - u|^2 |u^k|^{2p} \, dx \leq C \|\nabla(u^k - u)\|^2. \]  

(2.41)
Hence (2.39) yields

\[
\|f_i(u^{m}, v^m) - f_i(u, v)\|^2 \leq C\left[\|\nabla (u^k - u)\|^2 + \|\nabla (v^k - v)\|^2\right]. \tag{2.42}
\]

As \((u^m), (v^m)\) are Cauchy sequences in \(L^\infty(0, T, H^1_0(\Omega))\) (we prove it as in [1]) then we deduce (2.36). Similarly we get the convergence (2.37).

By multiplying (2.1) and (2.6) by \(\theta(t) \in D(0, T)\) and by integrating over \((0, T)\), it follows that

\[
\begin{cases}
-\frac{1}{\mu} \int_0^T (|u_1^k(t)|^2 u_1^k(t), \omega^j(t)) dt + \int_0^T M(|\nabla u_1^k(t)|^2)(\nabla u_1^k(t), \nabla \omega^j(t)) dt \\
+ \int_0^T (\nabla u_1^k, \nabla \omega^j(t)) dt - \int_0^T h_1(t - s)(\nabla u_1^k(s), \nabla \omega^j(t)) ds dt \\
+ \mu_1 \int_0^T (\nabla u_1^k(1, 1), \nabla \omega^j(t)) dt + \int_0^T (f_1(u_1^k, v_1^k), \omega^j(t)) dt = 0,
\end{cases}
\tag{2.43}
\]

for all \(j = 1, \ldots, k\).

The convergence of (2.26)–(2.30), (2.34), (2.36) and (2.37) is sufficient to pass to the limit in (2.43). This completes the proof of the theorem. \(\square\)

3 Exponential decay rate

In order to make precise the asymptotic behavior of our solutions, we introduce some functionality to determine a suitable Lyapunov functional equivalent to \(E\).

**Theorem 3.1** Assume that (A1)–(A3) hold. Then for every \(t_0 > 0\) there exist positive constants \(K\) and \(c\) such that the energy defined by (1.6) obeys the following decay:

\[
E(t) \leq Ke^{-c\tau}, \quad \forall t \geq t_0. \tag{3.1}
\]

**Lemma 3.2** Along a solution of the problem (1.5) the functional

\[
I(t) = \tau(t) \int_0^1 e^{-2\tau^2(t)} \left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right) d\rho \tag{3.2}
\]

satisfies the following estimates:

\[
|I(t)| \leq \frac{1}{\xi} E(t), \tag{3.3}
\]

\[
I'(t) \leq -2(\tau(t))e^{-2\tau^2} \int_0^1 \left(\|\nabla z_1\|^2 + \|\nabla z_2\|^2\right) d\rho + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + (1 - d)e^{-2\tau^2} \left(\|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2\right). \tag{3.4}
\]
Proof (i) A direct derivation of (3.2) gives

\[ I'(t) = \int_0^1 \left[ T^\nu(t)^{\nu(t)} (\| \nabla z_1 \|^2 + \| \nabla z_2 \|^2) + \tau(t) e^{\nu(t)} \left( \| \nabla z'_1 \|^2 + \| \nabla z'_2 \|^2 \right) \right] d\rho. \]

Recalling (1.3)–(1.4)

\[ I'(t) = \int_0^1 \left[ T^\nu(t)^{\nu(t)} (\| \nabla z_1 \|^2 + \| \nabla z_2 \|^2) - \tau(t) e^{\nu(t)} (1 - \rho \tau(t)) \left( \| \nabla z_1 \|^2 + \| \nabla z_2 \|^2 \right) \right] d\rho \]

\[ = -\int_0^1 \left[ \frac{\partial}{\partial \rho} \left( e^{\nu(t)} (1 - \rho \tau(t)) \left( \| \nabla z_1 \|^2 + \| \nabla z_2 \|^2 \right) \right) \right] d\rho \]

\[ = \| \nabla u \|^2 + \| \nabla v \|^2 - e^{\nu(t)} (1 - \tau(t)) \left( \| \nabla z_1(x, 1, t) \|^2 + \| \nabla z_2(x, 1, t) \|^2 \right) - 2I(t). \]

Because the exponential function $e^{\nu(t)}$ decreases on $(0, 1) \times (\tau_0, \tau_1)$ and from (A3), we get the results of this lemma. \qed

**Lemma 3.3** Along a solution of the problem (1.5) the functional

\[ \phi(t) = \frac{1}{l+1} \int_{\Omega} (|u|^l u u + |v|^l v v) \, dx + \int_{\Omega} \nabla u_t \nabla u \, dx + \int_{\Omega} \nabla v_t \nabla v \, dx \]

verifies the estimates

\[ |\phi(t)| \leq \frac{1}{l+2} \left[ \| u \|_{l+2}^2 + \| v \|_{l+2}^2 \right] + \left( \frac{\mu_1^2}{4\eta} \| \nabla z_1(x, 1, t) \|^2 + \frac{\mu_2^2}{4\eta} \| \nabla z_2(x, 1, t) \|^2 \right) \]

\[ + \frac{1}{2} \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right) \quad (3.5) \]

and

\[ \phi'(t) \leq \frac{1}{l+1} \left( \| u \|_{l+2}^2 + \| v \|_{l+2}^2 \right) \]

\[ + \left( \eta (a - k + 1) - k \left( \frac{b_1 + b_2}{2} + \alpha \right) \right) \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right) \]

\[ + \frac{1}{4\eta} \left[ (h_1 \nabla u)(t) + (h_2 \nabla v)(t) \right] + \frac{\mu_1^2}{4\eta} \| \nabla z_1(x, 1, t) \|^2 + \frac{\mu_2^2}{4\eta} \| \nabla z_2(x, 1, t) \|^2 \]

\[ + \| \nabla u_t \|^2 + \| \nabla v_t \|^2. \quad (3.6) \]

**Proof** (i) Applying Young's inequality, Sobolev–Poincaré’s inequality and $L^{l+2} \hookrightarrow L^2$, we find

\[ |\phi(t)| \leq \frac{1}{l+2} \| u \|_{l+2}^2 + \left( \frac{\mu_1^2}{4\eta} \right) \| \nabla z_1(x, 1, t) \|^2 \]

\[ + \frac{1}{2} \| \nabla u \|^2 + \frac{1}{2} \| \nabla u \|^2 + \frac{1}{2} \| \nabla v \|^2 + \frac{1}{2} \| \nabla v \|^2. \]
\[ \frac{1}{l + 1} \| u_t \|_{L^2}^2 + \frac{(l + 1)^{-1}}{l + 2} \| u_{tt} \|_{L^2}^2 + \frac{1}{l + 2} \| v_t \|_{L^2}^2 + \frac{(l + 1)^{-1}}{l + 2} \| v_{tt} \|_{L^2}^2 \]
\[ + \frac{1}{2} \| \nabla u_t \|_2^2 + \frac{1}{2} \| \nabla u \|_2^2 + \frac{1}{2} \| \nabla v \|_2^2 + \frac{1}{2} \| \nabla v_t \|_2^2 \]
\[ \leq \frac{1}{l + 1} (\| u_t \|_{L^2}^2 + \| v_t \|_{L^2}^2) + \left( \frac{(l + 1)^{-1}}{l + 2} \| \nabla u \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \right) \]
\[ + \frac{1}{2} \left( \| \nabla u_t \|_2^2 + \| \nabla v_t \|_2^2 \right). \]

(ii) Taking a direct derivation of (3.2) and replacing \( |u_t|^{q+1} |v_t|^{q+1} \) from the first and seconde equations of (1.5) give

\[ \phi'(t) = \frac{1}{l + 1} \int _\Omega \left( |u_t|^q |u_t|^q \right) u \, dx + \frac{1}{l + 1} \int _\Omega |u_t|^{q+2} \, dx \]
\[ + \frac{1}{l + 1} \int _\Omega \left( |v_t|^q |v_t|^q \right) v \, dx + \frac{1}{l + 1} \int _\Omega |v_t|^{q+2} \, dx \]
\[ + \int _\Omega \nabla u_t \cdot \nabla u \, dx + \int _\Omega \nabla u \cdot \nabla u_t \, dx + \int _\Omega \nabla v_t \cdot \nabla v \, dx + \int _\Omega \nabla v \cdot \nabla v_t \, dx \]
\[ = \int _\Omega \left( |u_t|^q |u_t|^q \right) u \, dx + \frac{1}{l + 1} \| u_t \|_{L^2}^2 + \int _\Omega \left( |v_t|^q |v_t|^q \right) v \, dx + \frac{1}{l + 1} \| v_t \|_{L^2}^2 \]
\[ - \int _\Omega \Delta u_t u \, dx + \| \nabla u_t \|_2^2 - \int _\Omega \Delta v_t v \, dx + \| \nabla v_t \|_2^2 \]
\[ = \frac{1}{l + 1} \left( \| u_t \|_{L^2}^2 + \| v_t \|_{L^2}^2 \right) + \int _\Omega -f_1(u, v) + M(\| \nabla u \|_2^2) \Delta u \]
\[ - \int _\Omega h_1(t - s) \Delta u(s) ds + \mu_1 \Delta z_1(x, t) \int _\Omega u \, dx \]
\[ + \int _\Omega -f_2(u, v) + M(\| \nabla v \|_2^2) \Delta v - \int _\Omega h_2(t - s) \Delta v(s) ds + \mu_2 \Delta z_2(x, t) \int _\Omega v \, dx \]
\[ + \| \nabla u_t \|_2^2 + \| \nabla v_t \|_2^2 \]
\[ = \frac{1}{l + 1} \left( \| u_t \|_{L^2}^2 + \| v_t \|_{L^2}^2 \right) - M(\| \nabla u \|_2^2) \| \nabla u \|_2^2 \]
\[ + \int _\Omega \nabla u(t) \int _0^t (h_1(t - s) \Delta u(s) ds - \mu_1 \int _\Omega \nabla z_1(x, t) \nabla u \, dx \]
\[ - M(\| \nabla v \|_2^2) \| \nabla v \|_2^2 + \int _\Omega \nabla v(t) \int _0^t (h_2(t - s) \nabla v(s) ds - \mu_2 \int _\Omega \nabla z_2(x, t) \nabla v 
\[ - (b_1 + b_2) \int _\Omega |v|^{q+1} |u|^{q+1} dx - 2s \int _\Omega uv \, dx. \] (3.7)
Similarly,

\[
\int_0^t \nabla v(t) \int_0^t h_2(t-s) \nabla v(s) \, ds \, dx \leq (1+\eta) \|\nabla v(t)\|^2 + \frac{1}{4\eta} \int_0^t h_1(t-s) \|\nabla u(s) - \nabla u(t)\|^2 \, ds
\]

and from (A4)

\[
-(b_1 + b_2) \int_0^t |v|^{q+1} |u|^{p+1} \, dx - 2\alpha \int_\Omega uv \, dx \leq \left(\frac{b_1 + b_2}{2} + \alpha\right) C_4 \left(\|\nabla v\|^2 + \|\nabla u\|^2\right).
\]

Thus, (3.6) is valid. \(\square\)

**Lemma 3.4** Along a solution of the problem (1.5) the functional

\[
\psi(t) = \int_\Omega \left(\Delta u_t - \frac{1}{l+1} |u_t|^{l+1} u_t \right) \int_0^t h_1(t-s) (u(t)-u(s)) \, ds \, dx + \int_\Omega \left(\Delta v_t - \frac{1}{l+1} |v_t|^{l+1} v_t \right) \int_0^t h_2(t-s) (v(t)-v(s)) \, ds \, dx
\]

satisfies the estimates

\[
|\psi(t)| \leq \frac{1}{2} \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) + \frac{1}{2} (a-k) \left(1 + \frac{(l+1)^{-1}}{(l+2)} (a-k)^{l+2} \left\{(h_1 o \nabla u)(t) + (h_2 o \nabla v)(t)\right\}
\]

\[
+ \frac{(l+1)^{-1}}{(l+2)} (a-k)^{l+2} \left\{2^{l+1} \|\nabla u\|^{2(l+1)} + \|\nabla v\|^{2(l+1)}\right\}
\]
\[ + \frac{1}{l+2} \left( \|u_t\|_{l^{1/2}}^{l^{1/2}} + \|v_t\|_{l^{1/2}}^{l^{1/2}} \right) \]  

(3.9)

and

\[
\psi'(t) \leq \delta \left[ (a - k) + \frac{(l + 1)^{-1}}{(l + 2)} (h_1(0))^{l^{1/2}} c_s^{l^{1/2}} 2^{2(l+1)} + b_2 \frac{c_s^{4(q+1)}}{2} \right. \\
+ \left. \frac{c_s^{4q}}{2} b_1 \right] M(\|\nabla u\|^{2}) \|\nabla u\|^{2} \\
+ \left( 2\delta (a - k)^{2} + \frac{\alpha c_s^{2}}{2} \right) \|\nabla u\|^{2} + \left( \frac{M(\|\nabla u\|^{2})}{4\delta} \right) \\
+ \left( 2\delta + \frac{1}{3\delta} + \frac{\alpha c_s^{2}}{2} \right) (a - k) \right) (h_1 \circ \nabla u)(t) \\
- \frac{h_1(0)}{4\delta} \left( 1 + \frac{(l + 1)^{-1}}{(l + 2)} (h_1(0))^{l^{1/2}} c_s^{l^{1/2}} \right) \left( h_1' \circ \nabla u \right)(t) \\
+ \left( \delta - \int_{0}^{t} h_1(s) ds \right) \|\nabla u_t\|^{2} + \mu_1^{2} \delta \|\nabla z_1(x, 1, t)\|^{2} \\
+ \frac{1}{l+1} \left( 1 - \int_{0}^{t} h_1(s) ds \right) \|u_t\|_{l^{1/2}}^{l^{1/2}} \\
+ \delta \left[ (a - k) + \frac{(l + 1)^{-1}}{(l + 2)} (h_2(0))^{l^{1/2}} c_s^{l^{1/2}} 2^{2(l+1)} + b_1 \frac{c_s^{4(q+1)}}{2} \right. \\
+ \left. \frac{c_s^{4q}}{2} b_2 \right] M(\|\nabla v\|^{2}) \|\nabla v\|^{2} \\
+ \left( 2\delta (a - k)^{2} + \frac{\alpha c_s^{2}}{2} \right) \|\nabla v\|^{2} + \left( \frac{M(\|\nabla v\|^{2})}{4\delta} \right) \\
+ \left( 2\delta + \frac{1}{3\delta} + \frac{\alpha c_s^{2}}{2} \right) (a - k) \right) (h_2 \circ \nabla v)(t) \\
- \frac{h_2(0)}{4\delta} \left( 1 + \frac{(l + 1)^{-1}}{(l + 2)} (h_2(0))^{l^{1/2}} c_s^{l^{1/2}} \right) \left( h_2' \circ \nabla v \right)(t) \\
+ \left( \delta - \int_{0}^{t} h_2(s) ds \right) \|\nabla v_t\|^{2} + \mu_2^{2} \delta \|\nabla z_2(x, 1, t)\|^{2} \\
+ \frac{1}{l+1} \left( 1 - \int_{0}^{t} h_2(s) ds \right) \|v_t\|_{l^{1/2}}^{l^{1/2}}, \tag{3.10} \]

where \( \delta > 0 \) and \( c_s \) is the Sobolev embedding constant.

**Proof** We have

\[
\psi(t) = - \int_{\Omega} \nabla u_t \int_{0}^{t} h_1(t - s) \left( \nabla u(t) - \nabla u(s) \right) ds dx \\
- \int_{\Omega} \frac{1}{l+1} |u_t|^{l} |u_t| \int_{0}^{t} h_1(t - s) (u(t) - u(s)) ds dx
\]
\[- \int_{\Omega} \nabla v_t \int_0^t h_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx \]
\[- \int_{\Omega} \frac{1}{l+1} |v_t|^l v_t \int_0^t h_2(t-s)(v(t) - v(s)) \, ds \, dx.\]

We use Young’s inequality with the conjugate exponents \( q' = \frac{l+2}{l+1} \) and \( q = l+2 \), then the second term in the right hand side can be estimated as

\[
\left| - \int_{\Omega} \frac{1}{l+1} |u_t|^l u_t \int_0^t h_1(t-s)(u(t) - u(s)) \, ds \, dx \right|
\leq \frac{1}{l+1} \left| \int_{\Omega} (|u_t|^l u_t) \left( \int_0^t h_1(t-s)(u(t) - u(s)) \, ds \right) \, dx \right|
\leq \frac{1}{l+1} \left[ \frac{1}{q'} \int_{\Omega} |u_t|^l u_t \, dx + \frac{1}{q} \int_{\Omega} \left( \int_0^t h_1(t-s)(u(t) - u(s)) \, ds \right) ^q \, dx \right]
\leq \frac{1}{l+1} \left[ \frac{1}{q'} \int_{\Omega} (|u_t|^{l+1}) \, dx + \frac{1}{q} \int_{\Omega} \left( \int_0^t h_1(t-s)(u(t) - u(s)) \, ds \right) ^q \, dx \right]
\leq \frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} \int_{\Omega} \left[ \int_0^t (h_1(t-s))^{\frac{l+1}{l+2}} \left( (h_1(t-s))^{\frac{1}{l+2}} |u(t) - u(s)| \right) \, ds \right] ^{l+2} \, dx. \quad (3.11)

We get by using Hölder’s inequality

\[
\int_{\Omega} \left[ \int_0^t (h_1(t-s))^{\frac{l+1}{l+2}} \left( (h_1(t-s))^{\frac{1}{l+2}} |u(t) - u(s)| \right) \, ds \right] ^{l+2} \, dx
\leq \int_{\Omega} \left[ \left( \int_0^t \left( (h_1(t-s))^{\frac{l+1}{l+2}} \right) ^{\frac{1}{l+2}} |u(t) - u(s)| \, ds \right) ^{\frac{1}{l+2}} \right] ^{l+2} \, dx
\leq \int_{\Omega} \left[ \left( \int_0^t h_1(t-s) \, ds \right) ^{\frac{l+1}{l+2}} \left( \int_0^t h_1(t-s) |u(t) - u(s)| ^{l+2} \, ds \right) ^{\frac{1}{l+2}} \right] ^{l+2} \, dx
\leq \left( \int_0^t h_1(t-s) \, ds \right) ^{\frac{l+1}{l+2}} \int_0^t h_1(t-s) \|u(t) - u(s)\|_{l+2}^{l+2} \, ds
\leq (a-k)^{l+1} c_s^{l+2} \int_0^t \sqrt{h_1(t-s)} \sqrt{h_1(t-s)} \|\nabla u(t) - \nabla u(s)\|^{l+1} \|\nabla u(t) - \nabla u(s)\| \, ds
\leq (a-k)^{l+1} c_s^{l+2} \left( \frac{1}{2} \int_0^t h_1(t-s) \|\nabla u(t) - \nabla u(s)\|^{2l+2} \, ds \right)
\leq (a-k)^{l+1} c_s^{l+2} \left( \frac{1}{2} \int_0^t h_1(t-s) 2\|\nabla u(t)\|^{2l+2} \, ds + \frac{1}{2} (h_1 o \nabla u)(t) \right)
\leq (a-k)^{l+1} c_s^{l+2} \left( 2^{2l+1}(a-k) \|\nabla u(t)\|^{2l+1} + \frac{1}{2} (h_1 o \nabla u)(t) \right). \quad (3.12)
Combining (3.12) with (3.11) we obtain

\[
\left| -\int_{\Omega} \frac{1}{l+1} |u_t|^l u_t \int_0^t h_1(t-s) (u(t) - u(s)) \, ds \, dx \right| \\
\leq \frac{1}{l+2} \|u_t\|_{l+2}^2 \\
+ \frac{(l+1)^{l+1}}{l+2} \left[ (a-k)^{l+1} c_1^{l+2} \left( 2^{2i+1} (a-k) \|\nabla u(t)\|^{2i+1} + \frac{1}{2} (h_1 \circ \nabla u)(t) \right) \right]. \tag{3.13}
\]

In the same way, we get

\[
\left| -\int_{\Omega} \nabla u_t \int_0^t h_1(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \right| \\
\leq \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \int_{\Omega} \left( \int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)| \, ds \right)^2 \, dx \\
\leq \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} (a-k) (h_1 \circ \nabla u)(t). \tag{3.14}
\]

Similarly

\[
\left| -\int_{\Omega} \frac{1}{l+1} |\nabla v_t|^l \nabla v_t \int_0^t h_2(t-s) (v(t) - v(s)) \, ds \, dx \right| \leq \frac{1}{l+2} \|v_t\|_{l+2}^2 \\
+ \frac{(l+1)^{l+1}}{l+2} \left[ (a-k)^{l+1} c_1^{l+2} \left( 2^{2i+1} (a-k) \|\nabla v(t)\|^{2i+1} + \frac{1}{2} (h_2 \circ \nabla v)(t) \right) \right]. \tag{3.15}
\]

Combining (3.13), (3.14) and (3.15), we deduce (i).

(ii) We use the Leibnitz formula and the first and second equations of (1.5) to find

\[
\psi'(t) = \int_{\Omega} \left( \Delta u_t - |u_t|^l u_t \right) \int_0^t h_1(t-s) (u(t) - u(s)) \, ds \, dx \\
+ \int_{\Omega} \left( \Delta u_t - \frac{1}{l+1} u_t |u_t|^l \right) \left( \int_0^t h_1(t-s) (u(t) - u(s)) \, ds \right) \, dx \\
+ \int_{\Omega} \left( \Delta v_t - |v_t|^l v_t \right) \int_0^t h_2(t-s) (v(t) - v(s)) \, ds \, dx \\
+ \int_{\Omega} \left( \Delta v_t - \frac{1}{l+1} |v_t|^l v_t \right) \left( \int_0^t h_2(t-s) (v(t) - v(s)) \, ds \right) \, dx \\
= \int_{\Omega} f_1(u,v) \int_0^t h_1(t-s) (u(t) - u(s)) \, ds \, dx \\
+ \int_{\Omega} M(\|\nabla u\|^2) \nabla u(t) \int_0^t h_1(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \int_{\Omega} \int_0^t h_1(t-s) \nabla u(s) \, ds \int_0^t h_1(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
+ \mu_1 \int_{\Omega} \nabla z_1(x,1,t) \int_0^t h_1(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \int_{\Omega} \nabla u_t \int_0^t h_1(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx
\]
- \frac{1}{l+1} \int_{\Omega} |u_t|^4 u_t \int_0^t h'_1(t-s)(u(t) - u(s)) \, ds \, dx \\
- \|\nabla u_t\|^2 \int_0^t h_1(s) \, ds - \frac{1}{l+1}\|u_t\|^{l+2}_2 \int_0^t h_1(s) \, ds \\
+ \int_{\Omega} f_2(u, v) \int_0^t h_2(t-s)(u(t) - u(s)) \, ds \, dx \\
+ \int_{\Omega} M(\|\nabla v\|^2) \nabla v(t) \int_0^t h_3(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx \\
- \int_{\Omega} \int_0^t h_2(t-s)\nabla v(s) \, ds \int_0^t h_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx \\
+ \mu_2 \int_{\Omega} \nabla z_2(x, 1, t) \int_0^t h_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx \\
- \int_{\Omega} \nabla v_t \int_0^t h'_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx \\
- \frac{1}{l+1} \int_{\Omega} |v_t|^4 v_t \int_0^t h'_2(t-s)(v(t) - v(s)) \, ds \, dx \\
- \|\nabla v_t\|^2 \int_0^t h_2(s) \, ds - \frac{1}{l+1}\|v_t\|^{l+2}_2 \int_0^t h_2(s) \, ds \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 - \|\nabla u_t\|^2 \int_0^t h_1(s) \, ds - \frac{1}{l+1}\|u_t\|^{l+2}_2 \int_0^t h_1(s) \, ds \\
- \|\nabla v_t\|^2 \int_0^t h_2(s) \, ds - \frac{1}{l+1}\|v_t\|^{l+2}_2 \int_0^t h_2(s) \, ds, \tag{3.16}

where

\begin{align*}
I_1 &= \int_{\Omega} M(\|\nabla u\|^2) \nabla u(t) \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
+ \int_{\Omega} M(\|\nabla v\|^2) \nabla v(t) \int_0^t h_3(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx, \\
I_2 &= -\int_{\Omega} f_2^1 \int_0^t h'_1(t-s)(\nabla u(s)) \, ds \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \int_{\Omega} \int_0^t h_2(t-s)\nabla v(s) \, ds \int_0^t h_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx, \\
I_3 &= \mu_1 \int_{\Omega} \nabla z_2(x, 1, t) \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
+ \mu_2 \int_{\Omega} \nabla z_2(x, 1, t) \int_0^t h'_2(t-s)(\nabla v(t) - \nabla v(s)) \, ds \, dx, \\
I_4 &= -\int_{\Omega} \nabla u_t \int_0^t h'_1(t-s)(\nabla u(s)) \, ds \, dx \\
- \int_{\Omega} \nabla v_t \int_0^t h'_2(t-s)(\nabla v(s)) \, ds \, dx, \\
I_5 &= -\frac{1}{l+1} \int_{\Omega} |u_t|^4 u_t \int_0^t h'_1(t-s)(u(t) - u(s)) \, ds \, dx \\
- \frac{1}{l+1} \int_{\Omega} |v_t|^4 v_t \int_0^t h'_2(t-s)(v(t) - v(s)) \, ds \, dx, \\
I_6 &= \int_{\Omega} f_2(u, v) \int_0^t h_1(t-s)(u(t) - u(s)) \, ds \, dx \\
+ \int_{\Omega} f_2(u, v) \int_0^t h'_2(t-s)(u(t) - u(s)) \, ds \, dx.
\end{align*}

and

Next we will estimate $I_1, \ldots, I_6$. 
For $I_1$, by applying Hölder's and Young's inequalities, we obtain

$$
|I_1| \leq M(\|\nabla u\|^2) \int_\Omega |\nabla u(t)| \left( \int_0^t h_1(s) ds \right) \frac{1}{2} \left( \int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)|^2 ds \right) \frac{1}{2} dx
+ M(\|\nabla v\|^2) \int_\Omega |\nabla v(t)| \left( \int_0^t h_2(s) ds \right) \frac{1}{2} \left( \int_0^t h_2(t-s) |\nabla v(t) - \nabla v(s)|^2 ds \right) \frac{1}{2} dx
\leq M(\|\nabla u\|^2) \left[ \delta \int_\Omega |\nabla u(t)|^2 \int_0^t h_1(s) ds dx \right]
+ \frac{1}{4\delta} \int_\Omega \int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx
+ M(\|\nabla v\|^2) \left[ \delta \int_\Omega |\nabla v(t)|^2 \int_0^t h_2(s) ds dx \right]
+ \frac{1}{4\delta} \int_\Omega \int_0^t h_2(t-s) |\nabla v(t) - \nabla v(s)|^2 ds dx
\leq M(\|\nabla u\|^2) \left( \delta(a-k) \|\nabla u(t)\|^2 + \frac{1}{4\delta} (h_1 o \nabla u)(t) \right)
+ M(\|\nabla v\|^2) \left( \delta(a-k) \|\nabla v(t)\|^2 + \frac{1}{4\delta} (h_2 o \nabla v)(t) \right). \quad (3.17)
$$

Similarly,

$$
|I_2| \leq \delta \int_\Omega \left( \int_0^t h_1(t-s) |\nabla u(s)| ds \right)^2 dx
+ \frac{1}{4\delta} \int_\Omega \left( \int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)| ds \right)^2 dx
+ \delta \int_\Omega \left( \int_0^t h_2(t-s) |\nabla v(s)| ds \right)^2 dx
+ \frac{1}{4\delta} \int_\Omega \left( \int_0^t h_2(t-s) |\nabla v(t) - \nabla v(s)| ds \right)^2 dx
\leq \delta \int_\Omega \left( \int_0^t h_1(t-s) (|\nabla u(s)| + |\nabla u(t)|) ds \right)^2 dx
+ \frac{1}{4\delta} \left( \int_0^t h_1(s) ds \right)(h_1 o \nabla u)(t)
+ \delta \int_\Omega \left( \int_0^t h_2(t-s) (|\nabla v(s)| + |\nabla v(t)|) ds \right)^2 dx
+ \frac{1}{4\delta} \left( \int_0^t h_2(s) ds \right)(h_2 o \nabla v)(t)
\leq 2\delta \|\nabla u(t)\|^2 \left( \int_0^t h_1(t) ds \right)^2 dx
+ \left( 2\delta + \frac{1}{4\delta} \right) \left( \int_0^t h_1(s) ds \right)(h_1 o \nabla u)(t)
+ 2\delta \|\nabla v(t)\|^2 \left( \int_0^t h_2(t) ds \right)^2 dx
+ \left( 2\delta + \frac{1}{4\delta} \right) \left( \int_0^t h_2(s) ds \right)(h_2 o \nabla v)(t)
\leq 2\delta \|\nabla u(t)\|^2 (a-k)^2 + \left( 2\delta + \frac{1}{4\delta} \right) (a-k)(h_1 o \nabla u)(t)
+ 2\delta \|\nabla v(t)\|^2 (a-k)^2
+ \left( 2\delta + \frac{1}{4\delta} \right) (a-k)(h_2 o \nabla v)(t), \quad (3.18)
$$
\[ |I_3| \leq \delta \left( \mu^2 \| \nabla z_1(x_1, t) \|^2 + \mu^2 \| \nabla z_2(x_1, t) \|^2 \right) \]
\[
+ \frac{(a-k)}{4\delta} (h_1 o \nabla u)(t) + \frac{(a-k)}{4\delta} (h_2 o \nabla v)(t),
\]
\[
|I_4| \leq \delta \int \nabla u_i^2 \, dx + \frac{1}{4\delta} \int \left( \int_0^t |h_1'(t-s)\nabla u(t) - \nabla u(s)| \, ds \right)^2 \, dx
\]
\[
+ \delta \int \nabla v_i^2 \, dx + \frac{1}{4\delta} \int \left( \int_0^t |h_2'(t-s)\nabla v(t) - \nabla v(s)| \, ds \right)^2 \, dx
\]
\[
\leq \delta \| \nabla u_i \|^2 + \frac{1}{4\delta} \int_0^t (-h_1'(t-s)) \, ds \int_0^t (-h_1^2) \, ds \, dx
\]
\[
+ \delta \| \nabla v_i \|^2 + \frac{1}{4\delta} \int_0^t (-h_2'(t-s)) \, ds \int_0^t (-h_2^2) \, ds \, dx
\]
\[
\leq \delta \| \nabla u_i \|^2 - \frac{h_1(0)}{4\delta} (h_1' o \nabla u)(t) + \delta \| \nabla v_i \|^2 - \frac{h_2(0)}{4\delta} (h_2' o \nabla v)(t),
\]
and using the fact that \( l \leq \gamma \)

\[
|I_5| \leq \frac{1}{l+2} \left( \| u_i \|_{t_i}^{l+2} + \| v_i \|_{t_i}^{l+2} \right) + \frac{(l+1)^{-1}}{l+2} \left[ (h_1(0))^{l+1} \int_0^t (-h_1'(t-s)) \| u(t) - u(s) \|_{t_i}^{l+2} \, ds \right]
\]
\[
+ (h_2(0))^{l+1} \int_0^t (-h_2'(t-s)) \| v(t) - v(s) \|_{t_i}^{l+2} \, ds \right]
\]
\[
\leq \frac{1}{l+2} \left( \| u_i \|_{t_i}^{l+2} + \| u_i \|_{t_i}^{l+2} \right)
\]
\[
+ \frac{(l+1)^{-1}}{l+2} c_2 \left[ (h_1(0))^{l+1} \int_0^t (-h_1'(t-s)) \| \nabla u(t) - \nabla u(s) \|_{t_i}^{l+2} \, ds \right]
\]
\[
+ (h_2(0))^{l+1} \int_0^t (-h_2'(t-s)) \| \nabla v(t) - \nabla v(s) \|_{t_i}^{l+2} \, ds \right]
\]
\[
\leq \frac{1}{l+2} \left( \| u_i \|_{t_i}^{l+2} + \| u_i \|_{t_i}^{l+2} \right)
\]
\[
+ \frac{(l+1)^{-1}}{l+2} c_2 \left[ (h_1(0))^{l+1} \left[ \delta 2^{(l+1)}h_1(0) \| \nabla u(t) \|_{2^{(l+1)}} - \frac{1}{4\delta} (h_1' o \nabla u)(t) \right] \right]
\]
\[
+ (h_2(0))^{l+1} \left[ \delta 2^{(l+1)}h_2(0) \| \nabla v(t) \|_{2^{(l+1)}} - \frac{1}{4\delta} (h_2' o \nabla v)(t) \right] \right]
\]
\[
\leq \frac{1}{l+2} \left( \| u_i \|_{t_i}^{l+2} + \| u_i \|_{t_i}^{l+2} \right)
\]
\[
+ \frac{(l+1)^{-1}}{l+2} c_2 \left[ (h_1(0))^{l+1} \left[ \delta 2^{(l+1)}h_1(0)M \| \nabla u(t) \|_{2^{(l+1)}} \| \nabla u(t) \|^2 \right]
\]
\[
- \frac{1}{4\delta} (h_1' o \nabla u)(t) \right]
\]
\[
+ (h_2(0))^{l+1} \left[ \delta 2^{(l+1)}h_2(0)M \| \nabla v(t) \|_{2^{(l+1)}} \| \nabla v(t) \|^2 \right]
\]
\[
- \frac{1}{4\delta} (h_2' o \nabla v)(t) \right].
\]
For $I_6$, we have

\[
I_6 = \alpha \int_{\Omega} v(t) \int_{0}^{t} h_1(t-s)(u(t)-u(s)) \, ds \, dx + \alpha \int_{\Omega} u(t) \int_{0}^{t} h_2(t-s)(v(t)-v(s)) \, ds \, dx
\]
\[
+ b_1 \int_{\Omega} |v|^{p+1} |u|^{p-1} u \int_{0}^{t} h_1(t-s)(u(t)-u(s)) \, ds \, dx
\]
\[
+ b_2 \int_{\Omega} |u|^{p+1} |v|^{q-1} v \int_{0}^{t} h_2(t-s)(v(t)-v(s)) \, ds \, dx
\]
\[
= I_6^0 + b_1 I_6^1 + b_2 I_6^2, \quad (3.21)
\]

and

\[
|I_6^0| \leq \frac{\alpha \delta^2}{2} \left( \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + (a - k) \left( (h_1 o \nabla u)(t) + (h_2 o \nabla v)(t) \right) \right), \quad (3.22)
\]

By using the Young and Hölder inequalities and Lemma 1.1, we find

\[
I_6^{11} = \frac{1}{2} \int_{\Omega} |v|^{2(q+1)} \int_{0}^{t} |h_1(t-s)(u(t)-u(s))| \, ds \, dx
\]
\[
\leq \frac{\delta}{2} \int_{\Omega} |v|^{2(q+1)} \, dx + \frac{1}{8\delta} \int_{\Omega} \left[ \int_{0}^{t} h_1(t-s) |u(t)-u(s)| \, ds \right]^2 \, dx
\]
\[
\leq \frac{\delta}{2} \|\nabla v\|^{2(q+1)} + \frac{1}{8\delta} (a - k) (h_1 o \nabla u)(t)
\]
\[
\leq \frac{\delta}{2} M \|\nabla v\|^2 \|\nabla u\|^2 + \frac{1}{8\delta} (a - k) (h_1 o \nabla u)(t). \quad (3.23)
\]

Also by following a similar technique to above, we get

\[
|I_6^{12}| \leq \frac{\delta^{q+1}}{2} M \|\nabla u\|^2 \|\nabla u\|^2 + \frac{1}{8\delta} (a - k) (h_1 o \nabla u)(t). \quad (3.24)
\]

Hence

\[
|I_6^1| \leq \frac{\delta^{q+1}}{2} M \|\nabla v\|^2 \|\nabla u\|^2 + \frac{\delta^{q+1}}{2} M \|\nabla u\|^2 \|\nabla u\|^2
\]
\[
+ \frac{1}{4\delta} (a - k) (h_1 o \nabla u)(t). \quad (3.25)
\]

Similarly

\[
|I_6^2| \leq \frac{\delta^{q+1}}{2} M \|\nabla u\|^2 \|\nabla u\|^2 + \frac{\delta^{q+1}}{2} M \|\nabla v\|^2 \|\nabla v\|^2
\]
\[
+ \frac{1}{4\delta} (a - k) (h_2 o \nabla v)(t). \quad (3.26)
\]
Summing (3.22), (3.26) and (3.27), we get
\[
I_{6} \leq \left( b_{2} c_{1}^{4(p+1)} + c_{1}^{4q} \right) M_{2} \left( \| \nabla u \|^{2} \right) + b_{1} M \left( \| \nabla v \|^{2} \right)
\]
\[
+ \left( b_{2} c_{1}^{4q} \right) M_{2} \left( \| \nabla v \|^{2} \right)
\]
\[
+ \left( \alpha c_{1}^{2} + \frac{1}{4\delta} \right) (a - k) \left( (2\alpha \nabla u)(t) + (2\nabla v)(t) \right) + \frac{\alpha c_{1}^{2}}{2} \left( \| \nabla u \|^{2} + \| \nabla v \|^{2} \right).
\]
Combining (3.16) and (3.17)–(3.28), we complete the proof.

Proof of Theorem 3.1 Now, for \( M, \varepsilon_{1} > 0 \), we introduce the following functional:
\[
F(t) = ME(t) + I(t) + \psi(t) + \varepsilon_{1}\phi(t).
\]
Firstly we prove that \( F(t) \) is equivalent to \( E(t) \); for this we show that \( F(t) \) verifies the following boundedness:
\[
\kappa_{1}E(t) \leq F(t) \leq \kappa_{2}E(t)
\]
for some positive constants \( \kappa_{1}, \kappa_{2} \).

We recall (3.3), (3.5), and (3.9) and, using the fact that \( l \leq r \), we get
\[
\left| I(t) + \psi(t) + \varepsilon_{1}\phi(t) \right|
\]
\[
\leq \frac{\varepsilon_{1} + 1}{l + 2} \left( \| u \|_{l_{2}}^{l_{r}} + \| v \|_{l_{2}}^{l_{r}} \right) + \frac{\varepsilon_{1} + 1}{2} \left( \| \nabla u \|^{2} + \| \nabla v \|^{2} \right)
\]
\[
+ \left( \frac{\varepsilon_{1}c}{2} + \frac{l + 1}{l + 2} \right) c_{1}^{l_{r}^{2}} \left( \| \nabla u \|^{2(l_{r} + 1)} + \| \nabla v \|^{2(l_{r} + 1)} \right)
\]
\[
+ \frac{a - k}{2} \left( 1 + \frac{l + 1}{l + 2} \right) \left( (2\alpha \nabla u)(t) + (2\nabla v)(t) \right) + \frac{1}{\xi} E(t)
\]
\[
\leq \kappa E(t),
\]
where \( \kappa > 0 \) depending the \( \varepsilon_{1}, a, b, l, c, c_{1}, k, \xi \). For the choice of \( M = \kappa + \epsilon \) with \( \epsilon > 0 \), we get \( F(t) \sim E(t) \).

By recalling (1.9), (3.4), (3.6), (3.10) and (A2), we deduce that
\[
F' \leq \left( \mu^{2} + \frac{\mu^{2}}{4\eta} - (1 - d)e^{-\tau_{1}} - M_{2} \right) \left( \| \nabla Z_{1}(x, 1, t) \|^{2} + \| \nabla Z_{2}(x, 1, t) \|^{2} \right)
\]
\[
- 2\tau(t)e^{-\tau_{1}} \int_{0}^{1} \left( \| \nabla Z_{1} \|^{2} + \| \nabla Z_{2} \|^{2} \right) d\rho
\]
\[
- (\varepsilon_{1} \left[ k - \eta(a - k) + \frac{b_{1} + b_{2}}{2} + \alpha \right]) c_{1}^{l_{r}^{2}}
\]
\[
- 2\delta(a - k)^{2} - \delta \left[ \left( \frac{l + 1}{l + 2} \right) (h_{3}c_{1}^{l_{r}^{2}}2^{l_{r}}) + \omega \right] M_{2}\left( \| \nabla u \|^{2} + \| \nabla v \|^{2} \right)
\]
\[
- \frac{1}{l + 1} \left( h_{0} - 1 - \varepsilon_{1} \right) \left( \| u \|_{l_{2}}^{l_{r}^{2}} + \| v \|_{l_{2}}^{l_{r}^{2}} \right)
\]
\[-(h_0 - \delta - M\lambda - 1 - \varepsilon_1)(\|\nabla u_t\|^2 + \|\nabla v_t\|^2)
\]
\[+ \left[ \frac{M}{2} - \frac{h_1}{4\delta} \left(1 + \frac{(l + 1)^{-1}}{(l + 2)}h_1^i \right) \right] - \frac{1}{\zeta} \frac{\varepsilon_1}{4\eta} M_0 + \frac{M_0}{4\delta}
\]
\[+ \left(2\delta + \frac{1}{3\delta} + \frac{ac^2}{2}\right)(a - k) \right) \left( (h'_1 \sigma u)(t) + (h'_2 \sigma v)(t) \right), \quad \forall t \geq t_0 > 0,
\]

where \(M_0 = \max \{M\|\nabla u\|^2, M\|\nabla v\|^2\}, h_0 = \min \{\int_{t_0}^0 h_1(s) \, ds, \int_{t_0}^0 h_2(s) \, ds\}, h_1 = \min \{h_1(0), h_2(0)\}, h_2 = \max \{h_1(0), h_2(0)\}, \omega = \max \{b_1\frac{\delta m}{q}, b_2\frac{\delta m}{q} + \frac{2\delta}{2}, b_1\} \) and \(\xi = \max \{\zeta_1, \zeta_1\} \).

Let \(\varepsilon > 0\) be sufficiently small so \(M\) is fixed, we take \(h_0 - M\lambda - 1 > \varepsilon_1\) and \(\delta\) small enough such that

\[a_3 = h_0 - 1 - \varepsilon_1 > 0 \quad \text{and} \quad a_4 = h_0 - \delta - M\lambda - 1 - \varepsilon_1.
\]

Further, we choose \(\eta\) small enough such that

\[a_1 = \mu^2 \delta + \varepsilon_1 \frac{\mu^2}{4\eta} - (1 - d)e^{-\tau_1} - M\beta > 0,
\]
\[a_2 = \varepsilon_1 \left[ k - \eta(a - k + 1) - \left(\frac{b_1 + b_2}{2} + \alpha \right) \right] - 2\delta(a - k)^2
\]
\[- \delta \left( a - k \right) + \frac{(l + 1)^{-1}}{(l + 2)}(h_2 c_1)^{(l+2)(l+1)} + \omega \right) M_0 > 0,
\]

and

\[a_5 = \frac{M}{2} - \frac{h_1}{4\delta} \left(1 + \frac{(l + 1)^{-1}}{(l + 2)} h_1^i \right) - \frac{1}{\zeta} \frac{\varepsilon_1}{4\eta} M_0 + \frac{M_0}{4\delta}
\]
\[+ \left(2\delta + \frac{1}{3\delta} + \frac{ac^2}{2}\right)(a - k) < 0.
\]

Thus

\[F'(t) \leq -a_3 \frac{1}{l + 2} \left( \|u_t\|^2_{l_1} + \|v_t\|^2_{l_1} \right) - a_2 \left( \|\nabla u\|^2 + \|\nabla v\|^2 \right)
\]
\[- 2\tau(t)e^{-\tau_1} \int_0^1 \left( \|\nabla z_1\|^2 + \|\nabla z_2\|^2 \right) \, d\rho
\]
\[+ a_1 \left( \|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2 \right) - a_4 \left( \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right)
\]
\[+ a_5 \left( (h'_1 \sigma u)(t) + (h'_2 \sigma v)(t) \right)
\]
\[\leq -mE(t) - cE'(t), \quad (3.31)
\]

where \(m = \min \{\frac{2e^{-\tau_1}}{2}, 2\frac{a_3}{a_1}, a_4\}\) and \(c = \min \{\frac{a_1}{2}, \frac{a_4}{2}, -2a_5\} \).

Let \(L(t) = F(t) + cE(t) \sim E(t)\). From (3.31), we get

\[L'(t) \leq -c'L(t), \quad \forall t \geq t_0, \quad (3.32)
\]

for some \(c' > 0\). A simple integration over \((t_0, t)\) yields

\[L(t) \leq L(t_0)e^{-c'(t-t_0)}, \quad \forall t \geq t_0, \quad (3.33)
\]

Thanks to the equivalence between \(L\) and \(E\), we obtain (3.1). \(\square\)
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References
1. Agre, K., Rammaha, M.A.: Systems of nonlinear wave equations with damping and source terms. Differ. Integral Equ. 19(11), 1235–1270 (2006)
2. Benaissa, A., Benaissa, A., Messaoudi, S.A.: Global existence and energy decay of solutions for a wave equation with a time varying delay term in weakly nonlinear internal feedback. J. Math. Phys. 53, 123514 (2012)
3. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ferreira, J.: Existence and uniform decay for a non-linear viscoelastic equation with strong damping. Math. Methods Appl. Sci. 24, 1043–1053 (2001)
4. El-Sayed, M.F., Moatimid, G.M., Moussa, M.H.M., Al-Khawli, M.A., El-Shiekh, R.M.: New exact solutions for coupled equal width wave equation and (2 + 1)-dimensional Nizhnik–Novikov–Veselov system using modified Kudryashov method. Int. J. Adv. Appl. Math. Mech. 2(1), 19–25 (2014)
5. Fragnelli, G., Pignotti, C.: Stability of solutions to nonlinear wave equations with switching time delay. Dyn. Partial Differ. Equ. 13(1), 31–51 (2016)
6. Guesmia, A., Tatar, N.: Some well-posedness and stability for abstract hyperbolic equation with infinite memory and distributed time delay. Commun. Pure Appl. Anal. 14(2), 457–491 (2015)
7. Kirane, M., Said-Houari, B.: Existence and asymptotic stability of a viscoelastic wave equation with a delay. Z. Angew. Math. Phys. 62, 1065–1082 (2011)
8. Kirchhoff, G.: Vorlesungen über Mechanik. Teubner, Leipzig (1883)
9. Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris (1969) (in French)
10. Logemann, H., Rebarber, R., Weiss, G.: Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop. SIAM J. Control Optim. 34(2), 572–600 (1996)
11. Mezouar, N., Abdelli, M., Rachah, A.: Existence of global solutions and decay estimates for a viscoelastic Petrovsky equation with a delay term in the non-linear internal feedback. Electron. J. Differ. Equ. 2017, 58 (2017)
12. Mezouar, N., Boulaaras, S.: Global existence of solutions to a viscoelastic non-degenerate Kirchhoff equation. Appl. Anal. (2018). In press. https://doi.org/10.1080/00036811.2018.1544621
13. Mezouar, N., Boulaaras, S.: Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation. Bull. Malays. Math. Sci. Soc. 43, 725–755 (2020)
14. Mezouar, N., Piskin, E.: Decay rate and blow up solutions for coupled quasilinear system. Bol. Soc. Mat. Mex. (2019). https://doi.org/10.1007/s40590-019-00243-3
15. Park, J.Y., Kang, J.R.: Global existence and uniform decay for a nonlinear viscoelastic equation with damping. Acta Appl. Math. 110, 1393–1406 (2010)
16. Raslan, K.R., El-Danaf, T.S., Ali, K.K.: New exact solutions of coupled generalized regularized long wave equations. J. Egypt. Math. Soc. 25, 400–405 (2017)
17. Xu, G.Q., Yang, S.P., Li, L.K.: Stabilization of wave systems with input delay in the boundary control. ESAIM Control Optim. Calc. Var. 20(4), 779–785 (2016)