FIRST ORDER $p$-ADIC DEFORMATIONS OF WEIGHT ONE NEWFORMS

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Abstract. This article studies the first-order $p$-adic deformations of classical weight one newforms, relating their Fourier coefficients to the $p$-adic logarithms of algebraic numbers in the field cut out by the associated projective Galois representation.

Contents

Introduction 1

Part A. The regular setting 2
  1. The general case 3
  2. CM forms 10
  3. RM forms 17

Part B. The irregular setting 19
  4. Generalised eigenspaces 20
  5. The general case 22
  6. CM forms 25
  7. RM forms 28

References 30

Introduction

Let $g$ be a classical cuspidal newform of weight one, level $N$ and nebentypus character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Q}_p^\times$, with Fourier expansion

$$g(q) = \sum_{n=1}^{\infty} a_n q^n.$$ 

The $p$-stabilisations of $g$ attached to a rational prime $p \nmid N$ are the eigenforms of level $Np$ defined by

$$g_\alpha(q) := g(q) - \beta \cdot g(q^p), \quad g_\beta(q) := g(q) - \alpha \cdot g(q^p),$$

where $\alpha$ and $\beta$ are the (not necessarily distinct) roots of the Hecke polynomial

$$x^2 - a_p x + \chi(p) = (x - \alpha)(x - \beta).$$

The forms $g_\alpha$ and $g_\beta$ are eigenvectors for the Atkin $U_p$ operator, with eigenvalues $\alpha$ and $\beta$ respectively. Since $\alpha$ and $\beta$ are roots of unity, these eigenforms are both ordinary at $p$.

An important feature of classical weight one forms is that they are associated to odd, irreducible, two-dimensional Artin representations, via a construction of Deligne-Serre. Let $\varrho_g : G_\mathbb{Q} \to \text{GL}_2(\mathbb{C})$ denote this Galois representation, and write $V_g$ for the underlying representation space.
A fundamental result of Hida asserts the existence of a $p$-adic family of ordinary eigenforms specialising to $g_\alpha$ (or to $g_\beta$) in weight one. Bellaiche and Dimitrov [BD] later established the uniqueness of this Hida family, under the hypothesis that $g$ is regular at $p$, i.e., that $\alpha \neq \beta$, or equivalently, that the Frobenius element at $p$ acts on $V_g$ with distinct eigenvalues. In the intriguing special case where $g$ is the theta series of a character of a real quadratic field $F$ in which the prime $p$ is split, the result of Bellaiche-Dimitrov further asserts that the unique ordinary first-order infinitesimal $p$-adic deformation of $g$ is an overconvergent (but not classical) modular form of weight one. In [DLR2], the Fourier coefficients of this non-classical form were expressed as $p$-adic logarithms of algebraic numbers in a ring class field of $F$, suggesting that a closer examination of such deformations could have some relevance to explicit class field theory for real quadratic fields.

The primary purpose of this note is to extend the results of [DLR2] to general weight one eigenforms.

Part A considers the regular setting where $\alpha \neq \beta$, in which the results exhibit a close analogy to those of [DLR2].

Part B takes up the case where $g$ is irregular at $p$. Here the results are more fragmentary and less definitive. Let $\tilde{S}^1_p(N,\chi)[[g]]$ denote the generalised eigenspace attached to the system of Hecke eigenvalues of an irregular weight one form $g \in S_1(N,\chi)$. The main conjecture of the second part asserts that $\tilde{S}^1_p(N,\chi)[[g]]$ is always four dimensional, with a two-dimensional subspace consisting of classical forms. Under this conjecture, an explicit description of the elements of the generalised eigenspace in terms of their $q$-expansions is provided. The resulting concrete description of the generalised eigenspace that emerges from Part [B] is an indispensable ingredient in the extension of the “elliptic Stark conjectures” of [DLR1] to the irregular setting that will be presented in [DLR3].

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Part A. The regular setting

Let $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$ denote the Iwasawa algebra, and let

$$\mathcal{W} := \text{Hom}_{\text{cts}}(1 + p\mathbb{Z}_p, \mathbb{C}_p^\times) = \text{Hom}_{\text{alg}}(\Lambda, \mathbb{C}_p)$$

denote the associated weight space. For each $k \in \mathbb{Z}_p$, write $\nu_k \in \mathcal{W}$ for the “weight $k$” homomorphism sending the group-like element $a \in 1 + p\mathbb{Z}_p$ to $a^{k-1}$. The rule $\lambda(k) := \nu_k(\lambda)$ realises elements of $\Lambda$ as analytic functions on $\mathbb{Z}_p$. The spectrum $\tilde{\mathcal{W}} := \text{Hom}_{\text{alg}}(\tilde{\Lambda}, \mathbb{C}_p)$ of a finite flat extension $\tilde{\Lambda}$ of $\Lambda$ is equipped with a “weight map”

$$w : \tilde{\mathcal{W}} \longrightarrow \mathcal{W}$$

of finite degree. A $\mathbb{Q}_p$-valued point $x \in \tilde{\mathcal{W}}$ is said to be of weight $k$ if $w(x) = \nu_k$, and is said to be étale over $\mathcal{W}$ if the inclusion $\Lambda \subset \tilde{\Lambda}$ induces an isomorphism between $\Lambda$ and the completion of $\tilde{\Lambda}$ at the kernel of $x$, denoted $\tilde{\Lambda}_x$. An element of this completion thus gives rise to an analytic function of $k \in \mathbb{Z}_p$ in a natural way.
A Hida family is a formal q-series
\[ g := \sum a_n q^n \in \tilde{\Lambda}[[q]] \]
with coefficients in a finite flat extension \( \tilde{\Lambda} \) of \( \Lambda \), specialising to a classical ordinary eigenform of weight \( k \) at almost all points \( x \) of \( \tilde{W} \) of weight \( k \in \mathbb{Z}_{\geq 2} \). Two Hida families \( g_1 \in \tilde{\Lambda}_1[[q]] \) and \( g_2 \in \tilde{\Lambda}_2[[q]] \) are regarded as equal if the \( \Lambda \)-algebras \( \tilde{\Lambda}_1 \) and \( \tilde{\Lambda}_2 \) can be embedded in a common extension \( \tilde{\Lambda} \) in such a way that \( g_1 \) and \( g_2 \) are identified. A well known theorem of Hida and Wiles asserts the existence of a Hida family specialising to \( g_\alpha \) in weight one. The following uniqueness result for this Hida family plays an important role in our study.

**Theorem** (Bellaiche, Dimitrov). Assume that the weight one form \( g \) is regular at \( p \), and let \( x_\alpha \) and \( x_\beta \) denote the distinct points on \( \tilde{W} \) attached to \( g_\alpha \) and \( g_\beta \) respectively. Then

(a) The curve \( \tilde{W} \) is smooth at \( x_\alpha \) and \( x_\beta \), and in particular there are unique Hida families \( g_\alpha, g_\beta \in \tilde{\Lambda}[[q]] \) specialising to \( g_\alpha \) and \( g_\beta \) at \( x_\alpha \) and \( x_\beta \) respectively.

(b) The weight map \( w : \tilde{W} \rightarrow W \) is furthermore étale at \( x_\alpha \) if and only if

(†) \( g \) is not the theta series of a character of a real quadratic \( K \) in which \( p \) splits.

The setting where \( g \) is regular at \( p \) but \( w \) is not étale at \( x_\alpha \) has been treated in [DLR2], and the remainder of Part A will therefore focus on the scenarios where (†) is satisfied. In that case, after viewing elements of the completion \( \tilde{\Lambda}_{x_\alpha} \) of \( \tilde{\Lambda} \) at \( x_\alpha \) as analytic functions of the “weight variable” \( k \), one may consider the canonical q-series
\[ g'_\alpha := \left( \frac{d}{dk} g_\alpha \right)_{k=1} \]
representing the first-order infinitesimal ordinary deformation of \( g \) at the weight one point \( x_\alpha \), along this canonical “weight direction”. The q-series \( g'_\alpha \) is analogous to the overconvergent generalised eigenform considered in [DLR2], with the following differences:

(a) While the overconvergent generalised eigenform of [DLR2] is a (non-classical, but overconvergent) modular form of weight one, such an interpretation is not available for the q-series \( g'_\alpha \), which should rather be viewed as the first order term of a “modular form of weight 1 + \varepsilon”.

(b) In the non-étale setting of [DLR2], the absence of a natural local coordinate with respect to which the derivative would be computed meant that the overconvergent generalised eigenform of loc.cit. could only be meaningfully defined up to scaling by a non-zero multiplicative factor. This ambiguity is not present in the definition of \( g'_\alpha \), whose Fourier coefficients are therefore entirely well-defined.

The main results of Part A give explicit formulae for these Fourier coefficients: they are stated in Theorems 1.10, 2.1, 2.3, and 3.1 below.

1. The general case

The goal of this section is to describe a general formula for the Fourier coefficients of \( g'_\alpha \).

The Artin representation \( V_g \) can be realised as a two-dimensional \( L \)-vector space, where \( L \) is a finite extension of \( \mathbb{Q} \), contained in a cyclotomic field. Let \( W_g = \text{hom}(V_g, V_g) \) denote the adjoint equipped with its usual conjugation action of \( G_\mathbb{Q} \), denoted
\[ \sigma \cdot w := g_y(\sigma) w g_y(\sigma)^{-1}, \quad \sigma \in G_\mathbb{Q}, \quad w \in W_g. \]
Let \( H \subset H_g \) denote the finite Galois extensions of \( \mathbb{Q} \) cut out by the representations \( W_g \) and \( V_g \) respectively, and write \( G := \text{Gal}(H/\mathbb{Q}) \).

For notational simplicity, the following assumption is made in the rest of this paper:
Assumption 1.1. The prime \( p \) splits completely in the field \( L \) of coefficients of the Artin representation \( V_g \).

This assumption amounts to a simple congruence condition on \( p \). The choice of an embedding of \( L \) into \( \mathbb{Q}_p \), which is fixed henceforth, will allow us, when it is convenient, to view \( V_g \) and \( W_g \) as representations of \( G \) with coefficients in \( \mathbb{Q}_p \), and the weight one form \( g \) as a modular form with Fourier coefficients in \( \mathbb{Q}_p \) rather than in \( L \). The \( \mathbb{Q}_p \)-vector spaces \( V_g \) and \( W_g \) are thus equipped with natural \( G \)-stable \( L \)-rational structures, denoted \( V^L_g \) and \( W^L_g \) respectively.

An embedding of \( \overline{\mathbb{Q}} \) into \( \mathbb{Q}_p \) is fixed once and for all, determining a prime \( \wp \) of \( H \) and of \( H_g \) above \( p \), and an associated Frobenius element \( \tau_p \) in \( \text{Gal}(H_g/\mathbb{Q}) \) and in \( G \). Let \( G_\wp \subset G \) be the decomposition subgroup generated by \( \tau_p \).

The representations \( V_g \) and \( W_g \) admit the following decompositions as \( \tau_p \)-modules:

\[
V_g = V^\alpha_g \oplus V^\beta_g, \quad W_g = W^\alpha_g \oplus W^\beta_g, \quad V^\alpha_g \oplus W^\beta_g, \quad W^\alpha_g \oplus V^\beta_g,
\]

where \( V^\alpha_g \) and \( V^\beta_g \) denote the \( \alpha \) and \( \beta \)-eigenspaces for the action of \( \tau_p \) on \( V_g \), and

\[
W^{\xi\eta}_g := \text{hom}(V^\xi_g, V^\eta_g), \quad \text{for } \xi, \eta \in \{\alpha, \beta\}
\]

is a \( G_\wp \)-stable line, on which \( \tau_p \) acts with eigenvalue \( \eta/\xi \). Let

\[
W^\text{ord}_g := \text{hom}(V^\alpha_g, V^\beta_g) = W^\beta_g \oplus W^\alpha_g.
\]

Of course, \( W^\text{ord}_g \) is stable under the action of \( G_\wp \) but not under the action of \( G \).

We propose to give a general formula for the \( \ell \)-th Fourier coefficient of \( \eta_I^g \) as the trace of a certain explicit endomorphism of \( V_g \), which is constructed via a series of lemmas. In the lemma below, we let \( G \) act on \( \mathcal{O}_H^\times \otimes W_g \) diagonally on both factors in the tensor product.

Lemma 1.2. The \( \mathbb{Q}_p \)-vector space \( (\mathcal{O}_H^\times \otimes W_g)^G \) of \( G \)-invariant vectors is one-dimensional.

Proof. Let \( G_\infty \) be the subgroup of \( G \) generated by a complex conjugation \( c \), which has order two, since \( V_g \) is odd. By Dirichlet’s unit theorem, the global unit group \( \mathcal{O}_H^\times \otimes \mathbb{Q}_p \) is isomorphic to \( \text{Ind}_{G_\infty}(\mathbb{Q}_p) - \mathbb{Q}_p \) as a \( \mathbb{Q}_p[G] \)-module. Let \( W^0_g \) denote the three-dimensional representation of \( G_\infty \) consisting of trace zero endomorphisms of \( V_g \). As a representation of \( G \), we have \( W_g = W^0_g \oplus \mathbb{Q}_p \), and \( W^0_g \) does not contain the trivial representation as a constituent. By Frobenius reciprocity,

\[
\dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W_g)^G) = \dim_{\mathbb{Q}_p}((\mathcal{O}_H^\times \otimes W^0_g)^G) = \dim_{\mathbb{Q}_p}((W^0_g)^{c-1}) = 1.
\]

The result follows. \( \square \)

Assume that the field \( L \) of coefficients is large enough so that the semisimple ring \( L[G] \) becomes isomorphic to a direct sum of matrix algebras over \( L \). The \( L[G] \)-module \( \mathcal{O}_H^\times \otimes L \) decomposes as a direct sum of \( V \)-isotypic components,

\[
\mathcal{O}_H^\times \otimes L = \bigoplus V \mathcal{O}_H^\times[V],
\]

where \( V \) runs over the irreducible representations of \( G \), and \( \mathcal{O}_H^\times[V] \) denotes the largest subrepresentation of \( \mathcal{O}_H^\times \otimes L \) which is isomorphic to a direct sum of copies of \( V \) as an \( L[G] \)-module. For a general, not necessarily irreducible, representation \( W \), the module \( \mathcal{O}_H^\times[W] \) is defined as the direct sum of the \( \mathcal{O}_H^\times[V] \) as \( V \) ranges over the irreducible constituents of \( W \). Because \( W_g \) (viewed, for now, as a representation with coefficients in \( L \)) is self-dual, Lemma 1.2 can be recast as the assertion that \( \mathcal{O}_H^\times[W_g] \) is isomorphic to a single irreducible subrepresentation of \( W_g \). More precisely:
In the case of “exotic weight one forms” where \( \varrho_g \) has non-dihedral projective image (isomorphic to \( A_4, S_4 \) or \( A_5 \)), then
\[
\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[W_g^0] \simeq W_g^0,
\]
and hence is three-dimensional.

If \( \varrho_g \) is induced from a character \( \psi_g \) of an imaginary quadratic field \( K \), then
\[
W_g = L \oplus L(\chi_K) \oplus V_\psi,
\]
where \( \chi_K \) is the odd quadratic Dirichlet character associated to \( K \) and \( V_\psi \) is the two-dimensional representation of \( G \) induced from the ring class character \( \psi = \psi_g/\psi_g' \) which cuts out the abelian extension \( H \) of \( K \). The representation \( V_\psi \) is irreducible if and only if \( \psi \) is non-quadratic, and in that case,
\[
\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[V_\psi] \simeq V_\psi.
\]

In the special case where \( \psi \) is quadratic, the representation \( V_\psi \) further decomposes as the direct sum of one-dimensional representations attached to an even and an odd quadratic Dirichlet character, denoted \( \chi_1 \) and \( \chi_2 \) respectively. That special case, in which \( V_g \) is also induced from a character of the real quadratic field cut out by \( \chi_1 \), is thus subsumed under \( \text{(1)} \) below.

If \( \varrho_g \) is induced from a character \( \psi_g \) of a real quadratic field \( F \), then
\[
W_g = L \oplus L(\chi_F) \oplus V_\psi, \quad V_\psi := \text{Ind}_{F}^{\mathbb{Q}}(\psi), \quad \psi := \psi_g/\psi_g',
\]
and one always has
\[
\mathcal{O}_H^\times[W_g] = \mathcal{O}_H^\times[\chi_F] \simeq L(\chi_F),
\]
i.e., \( \mathcal{O}_H^\times[W_g] \) is generated by a fundamental unit of \( F \).

Let \( U_g^\times \) be any generator of the one-dimensional \( \mathbb{Q}_p \)-vector space \( (\mathcal{O}_H^\times \otimes W_g)^G \) and let
\[
U_g := (\log_p \otimes \text{id})(U_g^\times) \in H_p \otimes W_g
\]
be the image of this vector under the linear map
\[
\log_p \otimes \text{id} : \mathcal{O}_H^\times \otimes W_g \rightarrow H_p \otimes W_g,
\]
where \( \log_p \) is the \( p \)-adic logarithm on the \( p \)-adic completion \( H_p \) of \( H \) at \( p \).

**Lemma 1.3.** There exists a non-zero endomorphism \( A \in H_p \otimes W_g \) satisfying the following conditions:

(a) \( \text{Trace}(AU_g) = 0 \).

(b) \( A \) belongs to \( H_p \otimes W_g^{\text{ord}} \), i.e., \( A(V_g^\alpha) = 0 \).

This endomorphism is unique up to scaling.

**Proof.** The space \( H_p \otimes W_g \) is four-dimensional over \( H_p \) and the conditions in Lemma 1.3 amount to three linear conditions on \( A \). More precisely, choose a \( \tau_p \)-eigenbasis \( (u_\alpha, v_\beta) \) for \( V_g \) for which
\[
\tau_p u_\alpha = \alpha u_\alpha, \quad \tau_p v_\beta = \beta v_\beta.
\]
Relative to this basis, the endomorphism \( U_g \) is represented by a matrix of the form
\[
U_g := \begin{pmatrix}
\log_p(u_1) & \log_p(u_\beta/\alpha) \\
\log_p(u_\alpha/\beta) & -\log_p(u_1)
\end{pmatrix},
\]
where \( u_1, u_{\alpha/\beta}, \) and \( u_{\beta/\alpha} \) are generators of \( \mathcal{O}_H^\times[W_g] \) which (when non-zero) are eigenvectors for \( \tau_p \), satisfying
\[
\tau_p(u_1) = u_1, \quad \tau_p(u_{\beta/\alpha}) = (\beta/\alpha)u_{\beta/\alpha}, \quad \tau_p(u_{\alpha/\beta}) = (\alpha/\beta)u_{\alpha/\beta}.
\]
The endomorphism $A$ satisfies condition (b) above if and only if the matrix representing it in the basis $(v_\alpha, v_\beta)$ is of the form

$$A : \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}, \quad x, y \in H_\psi,$$

and condition (a) implies the further linear relation

$$\log_\psi(u_{\alpha/\beta}) \cdot x - \log_\psi(u_1) \cdot y = 0. \tag{6}$$

The injectivity of $\log_\psi : O^*_H \otimes \mathbb{L} \rightarrow H_\psi$, which follows from the linear independence over $\overline{\mathbb{Q}}$ of logarithms of algebraic numbers, implies that the coefficients $\log_\psi(u_{\alpha/\beta})$ and $\log_\psi(u_1)$ in (6) vanish simultaneously if and only if

$$u_{\alpha/\beta} = u_1 = 0$$

in $O^*_H \otimes \mathbb{L}$, i.e., if and only if $O^*_H[W_g]$ is one-dimensional over $\mathbb{L}$ and generated by $u_{\beta/\alpha}$. This immediately rules out (2) and (3) as scenarios for the structure of $O^*_H[W_g]$, leaving only (4). Hence, $V_\psi$ is induced from a character of a real quadratic field $F$. In that case, the lines spanned by $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ are interchanged under the action of any reflection in $G$, and hence the condition $u_{\alpha/\beta} = 0$ implies that $u_{\beta/\alpha} = 0$ as well, thus forcing the vanishing of the full $O^*_H[W_g]$. This contradiction to Lemma 1.2 shows that (6) imposes a non-trivial linear condition on $x$ and $y$, and therefore that $A$ is unique up to scaling. \hfill \Box

**Lemma 1.4.** Let $A$ be any element of $H_\psi \otimes W_g$ satisfying the conditions in Lemma 1.3. Then the following are equivalent:

(a) $\text{Trace}(A) \neq 0$;

(b) The representation $\varrho_g$ is not induced from a character of a real quadratic field in which the prime $p$ splits.

**Proof.** The vanishing of $\text{Trace}(A)$ is equivalent to the vanishing of the entry $y$ in (6), and hence to the vanishing of $\log_\psi(u_{\alpha/\beta})$, and therefore of $u_{\alpha/\beta}$ and $u_{\beta/\alpha}$ as well. This implies that $O^*_H[W_g]$ is one-dimensional and generated by $u_1$. As in the proof of Lemma 1.3, this rules out (2) and (3), leaving only (4) as a possibility, i.e., $V_\psi$ is necessarily induced from a character of a real quadratic field $F$. Furthermore, $\tau_\psi$ fixes the group $O^*_H[W_g]$ generated by the fundamental unit of $F$, which occurs precisely when $p$ splits in $F$. The lemma follows. \hfill \Box

Assume from now on that the equivalent conditions of Lemma 1.4 hold. One can then define $A_g \in H_\psi \otimes W_g$ to be the unique $H_\psi^*$-multiple of $A$ satisfying

$$\text{Trace}(A_g) = 1.$$

As in lemma 1.2, $H_\psi \otimes W_g$ is endowed with the diagonal action of $G_\psi$ which acts on both $H_\psi$ and on $W_g$ in a natural way. Given $A \in H_\psi \otimes W_g$ and $\sigma \in G_\psi$, let us write $\sigma A$ for the image of $A$ by the action of $\sigma$ on the first factor $H_\psi$, and $\sigma \cdot A_g$ for the image of $A$ by the action of $\sigma$ by conjugation on the second factor $W_g$.

**Lemma 1.5.** The endomorphism $A_g$ belongs to the space $(H_\psi \otimes W_g)^{G_\psi}$ of $G_\psi$-invariants for the diagonal action of $G_\psi$ on $H_\psi \otimes W_g$, i.e.,

$$\tau_\psi A_g = \tau_\psi^{-1} \cdot A_g.$$

**Proof.** Relative to the $\overline{\mathbb{Q}}_p$-basis for $V_\psi$ used in the proof of Lemma 1.3, the endomorphism $A_g$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & \log_\psi(u_1) \\ 0 & \log_\psi(u_{\alpha/\beta}) \end{pmatrix}.$$
The lemma follows immediately from this in light of the fact that conjugation by \( \rho_g(\tau) \) preserves the diagonal entries in such a matrix representation while multiplying its upper right hand entry by \( \alpha/\beta \), whereas \( \tau \) acts on the upper right-hand entry of the above matrix as multiplication by \( \beta/\alpha \).

The matrix \( A_g \) gives rise to a \( G \)-equivariant homomorphism \( \Phi_g : H^\times \to H_\wp \otimes W_g \) by setting

\[
\Phi_g(x) = \sum_{\sigma \in G} \log_\wp(\sigma x) \cdot (\sigma^{-1} \cdot A_g),
\]

where, just as above, the group \( G \) acts on \( H_\wp \otimes W_g \) trivially on the first factor and through the usual conjugation action induced by \( \rho_g \) on the second factor.

**Lemma 1.6.** The homomorphism \( \Phi_g \) takes values in \( W_g \).

**Proof.** For any \( x \in (H \otimes \mathbb{Q}_p)^\times \) we have

\[
\tau_v \Phi_g(x) = \sum_{\sigma \in G} \log_\wp(\tau_v \sigma x) \cdot (\sigma^{-1} \cdot \tau_v A_g) = \sum_{\sigma \in G} \log_\wp(\tau_v \sigma x) \cdot (\sigma^{-1} \cdot \tau_v^{-1} \cdot A_g) = \sum_{\sigma \in G} \log_\wp(\tau_v \sigma x) \cdot ((\tau_v \sigma)^{-1} \cdot A_g) = \Phi_g(x),
\]

where Lemma 1.5 has been used to derive the second equation.

By a slight abuse of notation, we shall continue to denote with the same symbol the homomorphism \( \Phi_g : (H \otimes \mathbb{Q}_p)^\times \to H_\wp \otimes W_g \) obtained from (7) by extending scalars. Note that \( H_\wp^\times \) embeds naturally in \( (H \otimes \mathbb{Q}_p)^\times \).

**Lemma 1.7.** The homomorphism \( \Phi_g \) vanishes on \( \mathcal{O}_H^\times \otimes \mathbb{Q}_p \) and \( \Phi_g(H_\wp^\times) \subseteq H_\wp \otimes W_g^{\text{ord}} \).

**Proof.** Picking \( u \in \mathcal{O}_H^\times \) and an arbitrary \( B \in W_g \), set

\[
U_g^\times := \sum_{\sigma \in G} \sigma u \otimes (\sigma \cdot B) \in (\mathcal{O}_H^\times \otimes W_g)^G, \quad U_g := (\log_\wp \otimes \text{id})(U_g)
\]
as in the statement of Lemma 1.3. Note that \( U_g^\times \) is either trivial or a generator of the one-dimensional space \( (\mathcal{O}_H^\times \otimes W_g)^G \). We have

\[
\text{Trace}(\Phi_g(u) \cdot B) = \text{Trace}\left( \sum_{\sigma \in G} \log_\wp(\sigma u) \cdot (\sigma^{-1} \cdot A_g) \cdot B \right)
\]

\[
= \text{Trace}\left( A_g \cdot \left( \sum_{\sigma \in G} \log_\wp(\sigma u) \cdot (\sigma \cdot B) \right) \right)
\]

\[
= \text{Trace}\left( A_g \cdot (\log_\wp \otimes \text{id})(U_g^\times) \right) = \text{Trace}\left( A_g \cdot U_g \right).
\]

It follows from Property (a) satisfied by \( A \) (and hence \( A_g \) in particular) in Lemma 1.3 that

\[
\text{Trace}(\Phi_g(u) \cdot B) = 0, \quad \text{for all } B \in H_\wp \otimes W_g.
\]

The first assertion in the lemma follows from the non-degeneracy of the \( H_\wp \)-valued trace pairing on \( H_\wp \otimes W_g \). The second assertion follows from Property (b) satisfied by \( A \) and by \( A_g \) in Lemma 1.3.
Let now \( \ell \nmid \mathcal{P} \) be a rational prime, and let \( \lambda \) be a prime of \( H \) above \( \ell \). Let \( u(\lambda) \in \mathcal{O}_H[1/\lambda]^{\times} \otimes \mathbb{Q} \) be a \( \lambda \)-unit of \( H \) satisfying
\[
\text{Norm}^H(u(\lambda)) = \ell.
\]
This condition makes \( u(\lambda) \) well-defined up to the addition of elements in \( \mathcal{O}_H^{\times} \otimes \mathbb{Q} \), and hence the element
\[
A_g(\lambda) := \Phi_g(u(\lambda)) = \sum_{\sigma \in G} \log_\wp(\sigma^g u(\lambda)) \cdot (\sigma^{-1} \cdot A_g)
\]
is well-defined, by Lemma 1.7. Although \( A_g(\lambda) \) only belongs to \( H_p \otimes W_g \) a priori, we have:

**Lemma 1.8.** The trace of the endomorphism \( A_g(\lambda) \) is equal to \( \log_\wp(\ell) \).

**Proof.** Since the trace of \( A_g \) and its conjugates are all equal to 1, we have
\[
\text{Trace}(A_g(\lambda)) = \sum_{\sigma \in G} \log_\wp(\sigma^g u(\lambda)) \cdot \text{Trace}(\sigma^{-1} \cdot A_g)
\]
\[
= \sum_{\sigma \in G} \log_\wp(\sigma^g u(\lambda))
\]
\[
= \log_\wp \left( \text{Norm}^H(u(\lambda)) \right).
\]
The latter expression is equal to \( \log_\wp(\ell) \), by (8). \( \square \)

**Remark 1.9.** Although \( A_g(\lambda) \) belongs to \( W_g \) by Lemma 1.6, the entries of the matrix representing \( A_g(\lambda) \) relative to an \( L \)-basis for \( V^L \) are \( L \)-linear combinations of products of \( \wp \)-adic logarithms of units and \( \ell \)-units in \( H \), and in particular \( A_g(\lambda) \) need not lie in \( W^L_g \). (In fact, it never does, since its trace is not algebraic.)

In addition to the invariant \( A_g(\lambda) \), the choice of the prime \( \lambda \) of \( H \) above \( \ell \) also determines a well-defined Frobenius element \( \tau_\lambda \) in \( G = \text{Gal}(H/\mathbb{Q}) \), and even in \( \text{Gal}(H_g/\mathbb{Q}) \), since \( \text{Gal}(H_g/H) \) lies in the center of this group.

We are now ready to state the main theorem of this section:

**Theorem 1.10.** For all rational primes \( \ell \nmid \mathcal{P} \),
\[
a_\ell(g_\alpha') = \text{Trace}(g_\alpha(\tau_\lambda)A_g(\lambda)).
\]

**Remark 1.11.** This invariant does not depend on the choice of a prime \( \lambda \) of \( H \) above \( \ell \), since replacing \( \lambda \) by another such prime has the effect of conjugating the endomorphisms \( g_\alpha(\tau_\lambda) \) and \( A_g(\lambda) \) by the same element of \( \text{Aut}(V_g) \).

**Proof of Theorem 1.10.** Let \( \mathbb{Q}[\varepsilon] \) denote the ring of dual numbers over \( \mathbb{Q}_p \), with \( \varepsilon^2 = 0 \), and let
\[
\tilde{g}_g : G_Q \longrightarrow \text{GL}_2(\mathbb{Q}_p[\varepsilon])
\]
be the unique first order \( \alpha \)-ordinary deformation of \( g_g \) satisfying
\[
\det \tilde{g}_g = \chi_g(1 + \log_\wp \chi_{\text{cyc}} \cdot \varepsilon).
\]
This representation may be written as
\[
\tilde{g}_g = (1 + \varepsilon \cdot \kappa_g) \cdot g_g \quad \text{for some} \quad \kappa_g : G_Q \longrightarrow W_g.
\]
The multiplicativity of \( \tilde{g}_g \) implies that the function \( \kappa_g \) is a 1-cocycle on \( G_Q \) with values in \( W_g \), whose class in \( H^1(Q, W_g) \) (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of \( \tilde{g}_g \). Furthermore,
\[
a_\ell(g_\alpha) + \varepsilon \cdot a_\ell(g_\alpha') = \text{Trace}(\tilde{g}_g(\tau_\lambda)) = a_\ell(g) + \varepsilon \cdot \text{Trace}(\kappa_g(\tau_\lambda)g_g(\tau_\lambda)),
\]
and hence
\[
a_\ell(g_\alpha) = \text{Trace}(g_g(\tau_\lambda)\kappa_g(\tau_\lambda)).
\]
To make $\kappa_g(\tau_\lambda)$ explicit, observe that the inflation-restriction sequence combined with global class field theory for $H$ gives rise to a series of identifications

$$H^1(Q, W_g) \xrightarrow{\text{res}} \text{Hom}(G_H, W_g)^G = \text{Hom}_{G}(\left( \frac{O_H \otimes Q_p}{\mathcal{O}_H^{\times}} \right)^{\times}, W_g).$$

Under this identification, the class $\kappa_g$ can be viewed as an element of the space

$$H^1_{\text{ord}}(Q, W_g) = \left\{ \Phi \in \text{Hom}_{G}(\left( \frac{O_H \otimes Q_p}{\mathcal{O}_H^{\times}} \right)^{\times}, W_g) \mid \text{ such that } \Phi(H^0_{\text{ord}}) \subset W_g^{\text{ord}} \right\}.$$

But the homomorphism $\Phi_g$ of (11) belongs to the same one-dimensional space, by Lemma 1.6 and 1.7. By global class field theory, the endomorphism $\kappa_g(\tau_\lambda)$ is therefore a $Q_p^{\times}$-multiple of $\Phi_g(u_\lambda(\lambda)) = A_g(\lambda)$. The fact that these endomorphisms are actually equal now follows by comparing their traces and noting that

$$\text{Trace}(\kappa_g(\tau_\lambda)) = \log_p \chi_{\text{cyc}}(\ell) = \log_p(\ell),$$

while

$$\text{Trace}(A_g(\lambda)) = \log_p(\ell),$$

by Lemma 1.8, Theorem 1.10 follows. \qedhere

**Corollary 1.12.** If the rational prime $\ell \not| Np$ splits completely in $H/Q$, then

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

**Proof.** The hypothesis implies that $g_\ell(\tau_\lambda)$ is a scalar, and hence that $g_\ell(\tau_\lambda) = \frac{1}{2} a_\ell(g)$. It follows that

$$\text{Trace}(g_\ell(\tau_\lambda)A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \text{Trace}(\lambda_A_g(\lambda)) = (1/2) \cdot a_\ell(g) \cdot \log_p(\ell).$$

The corollary now follows from Theorem 1.10. \qed

**Example 1.13.** Let $\chi$ be a Dirichlet character of conductor 171 with order 3 at 9 and 2 at 19. Then $S(171, \chi)$ is a $Q(\chi)$-vector space of dimension 2. It is spanned by an eigenform

$$g = q + \zeta q^2 + \zeta^3 q^3 - \zeta^2 q^4 + (\zeta^2 - 1)q^6 + \cdots$$

defined over $L := Q(\zeta)$, with $\zeta$ a primitive 12th root of unity, and its Galois conjugate. (See [BL] for all weight one eigenforms of level at most 1500.) The associated projective representation $g_\ell$ has $A_4$-image and factors through the field

$$H = Q(a), a^4 + 10a^3 + 45a^2 + 81a + 81 = 0.$$  

Let $p = 13$, which splits completely in $L$. The representation $g_\ell$ is regular at 13, with eigenvalues $\alpha = \zeta$ and $\beta = -\zeta^3$. We computed the first order deformations through each of $g_{a\ell}$ and $g_{3\ell}$ to precision $13^{10}$, and $q$-adic precision $q^{37,000}$, using methods based upon the algorithms in [La].

The predictions made from Theorem 1.10 for $a_\ell(g'_\alpha)$ depend upon the conjugacy class of the Frobenius at $\ell$ in $G(H/Q)$. For all primes $\ell < 37,000$ which split completely in $H$, such as $\ell = 109, 179, 449, 467, 521, \ldots$, we verified that

$$a_\ell(g'_\alpha) = (1/2) \cdot a_\ell(g) \cdot \log_{13}(\ell) \pmod{13^{10}},$$

as asserted by Corollary 1.12.
2. CM forms

This section focuses on the case where $g = \theta_{\psi_g}$ is the CM theta series attached to a character $\psi_g : G_K \rightarrow \mathbb{L}^\times$ of a quadratic imaginary field $K$. The main theorems are Theorems 2.1 and 2.3 below, which will be derived in two independent ways, both “from first principles” and by specialising Theorem 1.10.

As in the previous section, the choice of an embedding of $L$ into $\mathbb{Q}_p$ allows us to view $\psi_g$ as a $\mathbb{Q}_p^\times$-valued character, and the weight one form $g$ as a modular form with coefficients in $\mathbb{Q}_p$.

For a character $\psi : G_K \rightarrow \mathbb{L}^\times$, the notation $\psi'$ will be used to designate the composition of $\psi$ with conjugation by the non-trivial element in $\text{Gal}(K/Q)$:

$$\psi'(\tau) = \psi(\tau \sigma \tau^{-1}),$$

where $\tau$ is any element of $G_K$ which acts non-trivially on $K$.

The Artin representation $\varrho_g$ is induced from $\psi_g$ and its restriction to $G_K$ is the direct sum $\psi_g \otimes \psi'_g$ of two characters of $K$, which are distinct by the irreducibility of $\varrho_g$ resulting from the fact that $g$ is a cusp form. In this case, the field $H$ is the ring class field of $K$ which is cut out by the non-trivial ring class character $\psi := \psi_g/\psi'_g$. The Galois group $G := \text{Gal}(H/K)$ is a generalised dihedral group containing $Z := \text{Gal}(H/K)$ as its abelian normal subgroup of index two.

The case of CM forms can be further subdivided into two sub-cases, depending on whether $p$ is split or inert in $K$.

2.1. The case where $p$ splits in $K$. Write $p\mathcal{O}_K = p\mathfrak{p} \mathfrak{p}'$, and fix a prime $\nu$ of $\mathbb{Q}$ above $p$. The roots of the $p$-th Hecke polynomial of $g$ are

$$\alpha = \psi_g(\mathfrak{p}), \quad \beta = \psi_g(\mathfrak{p}').$$

This case is notable in that the Hida family $\mathfrak{g}$ passing through $\varrho_\alpha$ can be written down explicitly as a family of theta series. Its weight $k$ specialisation $\mathfrak{g}_k$ is the theta-series attached to the character $\psi_g \Psi^{k-1}$, where $\Psi$ is a CM Hecke character of weight $(1,0)$ which is unramified at $p$. For all rational primes $\ell \nmid Np$, the $\ell$-th Fourier coefficient of $\mathfrak{g}_k$ is given by

$$a_\ell(\mathfrak{g}_k) = \begin{cases} 
\psi_g(\lambda)\Psi^{k-1}(\lambda') + \psi'_g(\lambda')\Psi^{k-1}(\lambda) & \text{if } \ell = \lambda \lambda' \text{ splits in } K; \\
0 & \text{if } \ell \text{ is inert in } K.
\end{cases}$$

(11)

Letting $h$ be the class number of $K$ and $t$ the cardinality of the unit group $\mathcal{O}_K^\times$, the character $\Psi$ satisfies

$$\Psi(\lambda)^ht = u_\lambda^t, \quad \text{where } (u_\lambda) := \lambda^h,$$

for any prime ideal $\lambda$ of $\mathcal{O}_K$ whose norm is the rational prime $\ell = \lambda \lambda'$. Let $u_\lambda'$ denote the conjugate of $u_\lambda$ in $K/\mathbb{Q}$. It follows that

$$\frac{d}{dk}\Psi^{k-1}(\lambda)_{k=1} = \log_p(u(\lambda)), \quad \text{where } u(\lambda) := u_\lambda \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q},$$

and likewise

$$\frac{d}{dk}\Psi^{k-1}(\lambda')_{k=1} = \log_p(u(\lambda'))', \quad \text{where } u(\lambda')' := u_\lambda' \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q}.$$  

In light of (11), we have obtained:

**Theorem 2.1.** For all rational primes $\ell$ that do not divide $Np$,

$$a_\ell(\mathfrak{g}_k) = \begin{cases} 
(\psi_g(\lambda) \log_p(u(\lambda')) + \psi'_g(\lambda') \log_p(u(\lambda))) & \text{if } \ell = \lambda \lambda' \text{ splits in } K; \\
0 & \text{if } \ell \text{ is inert in } K.
\end{cases}$$

(12)

where $u(\lambda)'$ denotes the conjugate of $u(\lambda)$ in $K/\mathbb{Q}$. It follows that

$$\frac{d}{dk}\Psi^{k-1}(\lambda)_{k=1} = \log_p(u(\lambda)), \quad \text{where } u(\lambda) := u_\lambda \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q},$$

and likewise

$$\frac{d}{dk}\Psi^{k-1}(\lambda')_{k=1} = \log_p(u(\lambda'))', \quad \text{where } u(\lambda')' := u_\lambda' \otimes \frac{1}{h} \in \mathcal{O}_H[1/\ell]^\times \otimes \mathbb{Q}.$$
Thus, the prime fourier coefficients of \( g_{\nu} \) are supported at the primes \( \ell \) which are split in \( K \), where they are (algebraic multiples of) the \( p \)-adic logarithms of \( \ell \)-units in this quadratic field. This general pattern will persist in the other settings to be described below, with the notable feature that the fourier coefficients of \( g_{\nu} \) will be more complicated expressions involving, in general, the \( p \)-adic logarithms of units and \( \ell \)-units in the full ring class field \( H \).

The reader will note Theorem 2.1 is consistent with Theorem 1.10 and could also have been deduced from it. More precisely, choose a basis of \( V_{\nu} \) consisting of eigenvectors for the action of \( G_{\nu} \) (and hence also, of \( \tau_{\nu} \)) which are interchanged by some element \( \tau \in K_{Q} - G_{\nu} \). Relative to such a basis, the endomorphisms \( U_{\nu} \) and \( A_{\nu} \) are represented by the following matrices, in which \( u_{\psi} \) and \( \tau u_{\psi} \) are generators of the spaces of \( \psi \) and \( \psi^{-1} \)-isotypic vectors in the group of elliptic units in \( O_{H}^{\times} \otimes L \):

\[
U_{\nu} : \begin{pmatrix} 0 & \log_{p}(u_{\psi}) \\ \log_{p}(u_{\psi}) & 0 \end{pmatrix}, \quad A_{\nu} : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It follows that, if \( \ell = \lambda \lambda' \) is split in \( K \), then \( A_{\nu}(\lambda) \) is represented by the matrix

\[
A_{\nu}(\lambda) : \begin{pmatrix} \log_{p}(u(\lambda')) & 0 \\ 0 & \log_{p}(u(\lambda)) \end{pmatrix},
\]

while \( A_{\nu}(\lambda) = \frac{1}{2} \log_{p}(\ell) \) is the scalar matrix with trace equal to \( \log_{p}(\ell) \) if \( \ell \) is inert in \( K \).

2.2. The case where \( p \) is inert in \( K \). We now turn to the more interesting case where \( p \) is inert in \( K \). Let \( \sigma_{p} := \tau_{p}^{2} \) denote the frobenius element in \( G_{\nu} \) attached to the prime \( \nu \) of \( H \) (which is well-defined modulo the inertia subgroup at \( \nu \)). Note that the prime \( p \) splits completely in \( H/K \), since the image of \( \tau_{p} \) in \( G \) is a reflection in this generalised dihedral group. The image of \( \sigma_{p} \) in \( \operatorname{Gal}(H_{p}/K) \) therefore belongs to the subgroup \( \operatorname{Gal}(H_{p}/H) \) whose image under \( \varphi \) consists of scalar matrices. Similar notations and remarks apply to any rational prime \( \nu \) which is inert in \( K/Q \).

Relative to an eigenbasis \((e_{1}, e_{2})\) for the action of \( G_{\nu} \) on \( V_{p} \), the Galois representation \( \varphi_{\nu} \) takes the form

\[
\varphi_{\nu}(\sigma) = \begin{pmatrix} \psi_{\nu}(\sigma) & 0 \\ 0 & \psi_{\nu}(\sigma) \end{pmatrix} \text{ for } \sigma \in G_{\nu}.
\]

The homomorphisms \( \psi_{\nu}, \psi'_{\nu} : G_{\nu} \rightarrow \mathbb{Q}_{p}^{\times} \) factor through \( \operatorname{Gal}(H_{\nu}/K) \) and satisfy

\[
\psi_{\nu}(\sigma \tau^{-1}) = \psi'_{\nu}(\sigma), \quad \text{for all } \tau \in K_{Q} - G_{\nu}, \quad \sigma \in G_{\nu}.
\]

It follows that \( \varphi_{\nu}(\tau) \) interchanges the lines spanned by \( e_{1} \) and \( e_{2} \), for any element \( \tau \in K_{Q} - G_{\nu} \). The restriction of \( \varphi_{\nu} \) to \( G_{Q} - G_{\nu} \) can therefore be described in matrix form by

\[
\varphi_{\tau}(\tau) = \begin{pmatrix} 0 & \eta_{\nu}(\tau) \\ \eta'_{\nu}(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_{Q} - G_{\nu},
\]

where \( \eta_{\nu} \) and \( \eta'_{\nu} \) are \( L \)-valued functions on \( G_{Q} - G_{\nu} \) that satisfy

\[
\eta_{\nu}(\tau_{1}) \eta'_{\nu}(\tau_{2}) = \psi_{\nu}(\tau_{1} \tau_{2}) = \psi'_{\nu}(\tau_{2} \tau_{1}), \quad \text{for all } \tau_{1}, \tau_{2} \in G_{Q} - G_{\nu},
\]

as well as the relations

\[
\eta_{\nu}(\sigma \tau) = \psi_{\nu}(\sigma) \eta_{\nu}(\tau), \quad \eta_{\nu}(\sigma \tau) = \psi'_{\nu}(\sigma) \eta'_{\nu}(\tau), \quad \text{for all } \sigma \in G_{\nu}, \quad \tau \in G_{Q} - G_{\nu}.
\]

After re-scaling \( e_{1} \) and \( e_{2} \) if necessary, we may assume that \( \tau_{p} \in G_{Q} - G_{\nu} \) is sent to the matrix

\[
\varphi_{\tau}(\tau_{p}) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}, \quad \text{with } \zeta^{2} = -\chi_{\nu}(p).
\]
The eigenvalues of $\varrho_g(\tau_p)$ are equal to $\alpha := \zeta$ and $\beta := -\zeta$, and hence $g$ is always regular in this setting.

Let
\[
\tilde{\varrho}_g : G_{\mathbb{Q}} \rightarrow \text{GL}(\tilde{V}_g)
\]
denote the first-order infinitesimal deformation of $\varrho_g$ attached to the Hida family $g$ passing through a choice of $p$-stabilization $g_0$ of $g$, where $\alpha \in \{\zeta, -\zeta\}$. The module $\tilde{V}_g$ is free of rank two over the ring $\mathbb{Q}_p[\varepsilon]/(\varepsilon^2) = \mathbb{Q}_p[[T]]/(T^2)$ arising from the mod $T^2$ reduction of the representation $\varrho_g$ attached to $g$. Choose any $\mathbb{Q}_p[\varepsilon]$-basis $(\tilde{e}_1, \tilde{e}_2)$ of $\tilde{V}_g$ lifting $(e_1, e_2)$, and note that the restriction of $\tilde{\varrho}_g$ to $G_K$ is given by:
\[
\tilde{\varrho}_g(\sigma) = (\psi_g(\sigma), (1 + \kappa(\sigma) \cdot \varepsilon) \cdot \psi_g'(\sigma), (1 + \kappa'(\sigma) \cdot \varepsilon) \cdot \psi_g''(\sigma)), \quad \text{for all } \sigma \in G_K.
\]
In this expression,
\begin{enumerate}
\item The functions $\kappa$ and $\kappa'$ are continuous homomorphisms from $G_K$ to $\mathbb{Q}_p$, i.e., elements of $H^1(K, \mathbb{Q}_p)$, which are interchanged by conjugation by the involution in $\text{Gal}(K/\mathbb{Q})$: \\
$\kappa(\sigma \tau \tau^{-1}) = \kappa'(\sigma)$, $\tau \in G_{\mathbb{Q}} - G_K$, $\sigma \in G_K$.
\item The functions $\kappa_{\psi}, \kappa_{\psi}' : G_K \rightarrow \mathbb{Q}_p$ are one-cocycles with values in $\mathbb{Q}_p(\psi)$, and give rise to well defined classes \\
$\kappa_{\psi} \in H^1(K, \mathbb{Q}_p(\psi))$, $\kappa_{\psi}' \in H^1(K, \mathbb{Q}_p(\psi^{-1}))$,
\end{enumerate}
which also satisfy
\[
\kappa_{\psi}(\tau \sigma \tau^{-1}) = \kappa_{\psi}'(\sigma), \quad \tau \in G_{\mathbb{Q}} - G_K, \quad \sigma \in G_K.
\]
For each rational prime $\ell \mid Np$, the $\ell$-th Fourier coefficient $a_\ell(g'_\alpha)$ is given by
\[
a_\ell(g'_\alpha) = \frac{d}{dk}\text{Trace}(\varrho_g(\tau_\lambda))_{k=1}
\]
Observe that the spaces $H^1(K, \mathbb{Q}_p)$ and $H^1(K, \mathbb{Q}_p(\psi))$ are of dimensions two and one respectively over $\mathbb{Q}_p$, since $\psi \neq 1$. More precisely, restriction to the inertia group at $p$ combined with local class field theory induces an isomorphism
\[
H^1(K, \mathbb{Q}_p) = \text{hom}(\mathcal{O}_K^\times, \mathbb{Q}_p) = \mathbb{Q}_p \log_p(z) \oplus \mathbb{Q}_p \log_p(z').
\]
Let $\mathcal{O}_H^\times, \psi$ denote the (one-dimensional) $\psi$-isotypic component of $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$ on which $\text{Gal}(H/K)$ acts through the character $\psi$, and denote by $\varphi$ the prime of $H$ above $p$ arising from our chosen embedding of $\mathbb{Q}$ into $\mathbb{Q}_p$. Restriction to the inertia group at $\varphi$ in $G_H$ likewise gives rise to an identification
\[
H^1(K, \mathbb{Q}_p(\psi)) = \text{hom}(\mathcal{O}_H^\times/\mathcal{O}_H^\times, \mathbb{Q}_p) = \mathbb{Q}_p \cdot (\log_p(u') \log_p(z) - \log_p(u') \log_p(z')).
\]
In the above equation, $u'$ is to be understood as the natural image in $\mathcal{O}_H^\times = \mathcal{O}_K^\times$ of an element of the form
\[
\sum_{\sigma \in \mathbb{Z}} \psi^{-1}(\sigma) u^\sigma \in (\mathcal{O}_H^\times \otimes L)^\psi,
\]
where $u$ is an $L[G]$-module generator of $\mathcal{O}_H^\times \otimes L$, and $u'$ is the image of $u_\psi$ under the conjugation action $K_p \rightarrow K_p$. Note that replacing $u$ by $\lambda u$ for some $\lambda \in L[G]$ has the effect of multiplying both $u_\psi$ and $u'$ by $\psi(\lambda)$ in $\mathbb{Q}_p$, so that the $\mathbb{Q}_p$-line spanned by the right-hand side of (21) is independent of the choice of $u \in \mathcal{O}_H^\times$.

It follows from (20) and (21) that the total deformation space of $\varrho_g$ (before imposing any ordinarity hypotheses, or restrictions on the determinant) is three dimensional.

Let $v^+_g := e_1 + e_2$ and $v^-_g := e_1 - e_2$ be the eigenvectors for $\tau_p$ acting on $\tilde{V}_g$, with eigenvalues $\zeta$ and $-\zeta$ respectively. Let $\kappa_p$ and $\kappa_{\psi, \varphi}$ denote the restrictions $\kappa$ and $\kappa_{\psi}$ to the inertia groups
at \( p \) and \( \varphi \) in \( G_H \) and \( G_K \) respectively. Both can be viewed as characters of \( K^\times_p = H^\times_p \) after identifying the abelianisations of \( G_{K_p} \) and \( G_{H_p} \) with a quotient of \( K^\times_p \) via local class field theory.

**Lemma 2.2.** The following are equivalent:

(a) The inertia group at \( \varphi \) acts as the identity on some lift \( \tilde{v}_g^+ \) of \( v_g^+ \) to \( \tilde{V}_g \);

(b) The inertia group at \( \varphi \) acts as the identity on all lifts \( \tilde{v}_g^+ \) of \( v_g^+ \) to \( \tilde{V}_g \);

(c) The restrictions \( \kappa_p \) and \( \kappa_{\psi,p} \) satisfy

\[
\kappa_p(x) = -\kappa_{\psi,p}(x), \quad \text{for all } x \in \mathcal{O}_K^\times.
\]

Similar statements hold when \( v_g^+ \) is replaced by \( v_g^- \), where the conclusion is that \( \kappa_p = \kappa_{\psi,p} \).

**Proof.** The equivalence of the first two conditions follows from the fact that \( \varepsilon \tilde{V}_g \simeq V_g \) is unramified at \( p \) and hence that inertia acts as the identity on the kernel of the natural map \( \tilde{V}_g \rightarrow V_g \). To check the third, note that the inertia group \( I_p \) at \( p \) is contained in \( G_K \), since \( K \) is unramified at \( p \) and observe that any \( \sigma \in I_p \) sends \( \tilde{e}_1 + \tilde{e}_2 \) to

\[
\tilde{g}_\sigma(\tilde{e}_1 + \tilde{e}_2) = \tilde{e}_1 + \tilde{e}_2 + \varepsilon \cdot (\kappa(\sigma)\tilde{e}_1 + \kappa'(\sigma)\tilde{e}_2 + \kappa_\psi(\sigma)\tilde{e}_1 + \kappa'(\sigma)\tilde{e}_2) = \tilde{e}_1 + \tilde{e}_2 + \varepsilon \cdot ((\kappa(\sigma) + \kappa_\psi(\sigma))\tilde{e}_1 + (\kappa'(\sigma) + \kappa'(\sigma))\tilde{e}_2).
\]

The lemma follows. \( \square \)

A lift \( \tilde{g}_\sigma \) of \( g_\sigma \) is ordinary relative to the space spanned by \( v_g^+ \) if and only if it satisfies the equivalent conditions of Lemma 2.2. This lemma merely spells out the proof of the Bellaiche-Dimitrov theorem on the one-dimensionality of the tangent space of the eigencurve at the point associated to \( g_\alpha \). More precisely, the general ordinary first-order deformation of \( g_\sigma \) is completely determined by the pair \( (\kappa_p, \kappa_{\psi,p}) \), which depends on a single linear parameter \( \mu \in \bar{Q}_p \) and is given by the rule

\[
(22) \quad \kappa_p(z) = \mu(\log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z) - \log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z'))
\]

\[
(23) \quad \kappa_{\psi,p}(z) = \pm \mu(\log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z) - \log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z'))
\]

where the sign in the second formula depends on whether one is working with the ordinary deformation of \( g_\alpha \) or \( g_\beta \).

Let us now make use of the fact that

\[
\det(\tilde{g}_\sigma) = 1 + \varepsilon \log_{\varphi} \chi_{\text{cyc}} = 1 + \varepsilon \log_{\varphi}(zz').
\]

Since \( \det(\tilde{g}_\sigma) = 1 + \varepsilon(\kappa + \kappa') \), this condition implies that

\[
\mu = \frac{1}{\log_{\varphi}(u_\psi') - \log_{\varphi}(u_\psi)},
\]

and hence that \( \kappa_p \) and \( \kappa_{\psi,p} \) are given by

\[
(24) \quad \kappa_p(z) = \frac{\log_{\varphi}(u_\psi') \cdot \log_{\varphi}(z) - \log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z')}{\log_{\varphi}(u_\psi') - \log_{\varphi}(u_\psi)}
\]

\[
(25) \quad \kappa_{\psi,p}(z) = \pm \frac{\log_{\varphi}(u_\psi') \cdot \log_{\varphi}(z) - \log_{\varphi}(u_\psi) \cdot \log_{\varphi}(z')}{\log_{\varphi}(u_\psi') - \log_{\varphi}(u_\psi)}
\]

Equations (24) and (25) give a completely explicit description of the first order deformation \( \tilde{g}_\alpha \) and \( \tilde{g}_\beta \), from which the Fourier coefficients of \( \tilde{g}_\alpha' \) and \( \tilde{g}_\beta' \) shall be readily calculated.

The formula for the \( \ell \)-th Fourier coefficient of \( \tilde{g}_\alpha' \) involves the unit \( u_\psi \) above as well as certain \( \ell \)-units in \( \mathcal{O}_H[1/\ell] \) \( \otimes L \) whose definition depends on whether or not the prime \( \ell \) is split or inert in \( K/\mathbb{Q} \).
If \( \ell = \lambda \lambda' \) splits in \( K/Q \), let \( u(\lambda) \) and \( u(\lambda') \) denote, as before, the \( \ell \)-units in \( O_K[1/\ell]^\times \otimes \mathbb{Q} \) of norm \( \ell \) with prime factorisation \( \lambda \) and \( \lambda' \) respectively. Set

\[
\psi_g(\lambda) := u(\lambda) \otimes \psi_g(\lambda) + u(\lambda') \otimes \psi_g(\lambda'), \quad \psi_g(\lambda') := u(\lambda') \otimes \psi_g(\lambda') + u(\lambda) \otimes \psi_g(\lambda').
\]

In other words, \( \psi_g(\lambda) \) is the unique element of \( O_K[1/\ell]^\times \otimes L \) whose prime factorisation is equal to \( \psi_g(\lambda) \cdot \lambda + \psi_g(\lambda') \cdot \lambda' \). Note that, if \( \ell \) splits completely in \( H/Q \), \( \ell \otimes \psi_g(\lambda) \) is equal to a scalar \( \zeta \), then \( \psi_g(\lambda) = \psi_g(\lambda') = \ell \otimes \zeta \), but that otherwise \( \psi_g(\lambda) \) and \( \ell \) generate the \( L \)-vector space \( O_K[1/\ell]^\times \otimes L \) of \( \ell \)-units of \( K \) (tensored with \( L \)).

If \( \ell \) is inert in \( K/Q \), choose a prime \( \lambda \) of \( H \) lying above \( \ell \), and let \( u(\lambda) \in O_H[1/\lambda]^\times \otimes \mathbb{Q} \) be any \( \lambda \)-unit of \( H \) satisfying \( \text{ord}_\lambda(u(\lambda)) = 1 \), which is well defined up to units in \( O_H^\times \). Define the elements

\[
\begin{align*}
\psi(u(\lambda)) &= \sum_{\sigma \in \mathbb{Z}} \psi^{-1}(\sigma) \otimes \sigma u(\lambda) \in L \otimes O_H[1/\ell]^\times, \\
\psi^\prime(u(\lambda)) &= \tau_\psi \psi(u(\lambda)) \in L \otimes O_H[1/\ell]^\times.
\end{align*}
\]

Thus \( \psi(u(\lambda)) \) lies in the \( \psi \)-component \( O_H[1/\ell]^\times[\psi] \) and is well-defined up to the addition of multiples of \( \psi(u(\lambda)) \), where

\[
\begin{align*}
u(\psi) &= \sum_{\sigma \in \mathbb{Z}} \psi^{-1}(\sigma) \otimes \sigma u \in L \otimes O_H[1/\ell]^\times,
\end{align*}
\]

for any unit \( u \in O_H^\times \), while \( \psi^\prime(u(\lambda)) \) lies in the \( \psi^{-1} \) component and is well-defined up to the addition of multiples of \( \psi^\prime(u(\lambda)) \), where

\[
\begin{align*}
\psi^\prime(u(\lambda)) &= \tau_\psi \psi(u(\lambda)).
\end{align*}
\]

Recall the function \( \psi \) defined in \([14]\), with values in the roots of unity of \( L \). The main result of this section is:

**Theorem 2.3.** Let \( \ell \nmid Np \) be a rational prime.

(a) If \( \ell = \lambda \lambda' \) splits in \( K/Q \), then

\[
a_\ell(\psi(\lambda)) = a_\ell(\psi(\lambda')) = \frac{\log_\psi(\psi^\prime(u(\lambda)) \cdot \log_\psi(u(\lambda)) \cdot \log_\psi(u(\lambda')))}{\log_\psi(\psi(u(\lambda)) \cdot \log_\psi(u(\lambda)) \cdot \log_\psi(u(\lambda'))).}
\]

(b) If \( \ell \) remains inert in \( K/Q \), then

\[
a_\ell(\psi(\lambda)) = \psi^\prime(\tau_\lambda) \frac{\log_\psi(\psi^\prime(u(\lambda)) \cdot \log_\psi(u(\lambda)) \cdot \log_\psi(u(\lambda')))}{\log_\psi(\psi(u(\lambda)) \cdot \log_\psi(u(\lambda)) \cdot \log_\psi(u(\lambda'))).}
\]

**Proof.** Let us first compute first the Fourier coefficients at primes \( \ell \nmid Np \) that split as \( \ell = \lambda \lambda' \) in \( K \). Let \( \sigma_\lambda \) and \( \sigma_{\lambda'} \) be the Frobenius elements associated to \( \lambda \) and \( \lambda' \) respectively. They are well-defined elements in the Galois group of any abelian extension of \( K \) in which \( \ell \) is unramified.

It follows from \([19]\) and the matrix expression for \( \tilde{\psi}(g) \) given in \([18]\) that

\[
\begin{align*}
a_\ell(\psi(\lambda)) &= \psi^\prime(\psi(\lambda)) \psi(\lambda) \kappa(\lambda' \kappa(\lambda') \\
&= \psi^\prime(\psi(\lambda)) \psi(\lambda) \kappa(\psi(\lambda)) \psi^\prime(\psi(\lambda)).
\end{align*}
\]

Equation \([20]\) then follows from the formula for \( \psi^\prime(z) \) given in \([24]\). We now turn now to the computation of the Fourier coefficients of \( \psi^\prime \) at primes \( \ell \nmid Np \) that remain inert in \( K \). Let \( \tau_\lambda \) denote the Frobenius element in \( \text{Gal}(M/Q) \) associated to the choice of a prime ideal \( \lambda \) above \( \ell \) in \( Q \), and let \( \sigma_\lambda := \tau_\lambda^\psi \) denote the associated Frobenius element in \( \text{Gal}(M/K) \).
Since \( \tau_\lambda \) belongs to \( G_{\mathbb{Q}} - G_K \), it follows from (14) that the matrix \( \tilde{g}_g(\tau_\lambda) \) is of the form
\[
\tilde{g}_g(\tau_\lambda) = \begin{pmatrix}
\eta_g(\tau_\lambda)(1 + r_\ell \cdot \varepsilon) & \eta_g(\tau_\lambda)(1 + s_\ell \cdot \varepsilon) \\
\eta_g'(\tau_\lambda)(1 + r_\ell \cdot \varepsilon) & \eta_g'(\tau_\lambda)(1 + s_\ell \cdot \varepsilon)
\end{pmatrix},
\]
for suitable scalars \( r_\ell, s_\ell, t_\ell, \) and \( u_\ell \in \mathbb{Q}_p \). Since
\[a_\ell(g_a) + a_\ell(g'_a)\varepsilon = \text{Trace}(\tilde{g}_g(\tau_\lambda)) = (r_\ell + u_\ell) \cdot \varepsilon,
\]
it follows that
\[(27) \quad a_\ell(g'_a) = r_\ell + u_\ell.\]

In order to compute this trace, we observe that it arises in the upper right-hand and lower left-hand entries of the matrix
\[
\tilde{g}_g(\sigma_\lambda)^2 = \begin{pmatrix}
\psi_g(\sigma_\lambda)(1 + (s_\ell + t_\ell) \cdot \varepsilon) & \psi_g(\sigma_\lambda)(r_\ell + u_\ell) \cdot \varepsilon \\
\psi_g'(\sigma_\lambda) \cdot \kappa_{\psi}(\sigma_\lambda) \cdot \varepsilon & \psi_g'(\sigma_\lambda) \cdot \kappa_{\psi}^{-1}(\sigma_\lambda) \cdot \varepsilon
\end{pmatrix}.
\]

On the other hand, since \( \sigma_\lambda \) belongs to \( G_K \) it follows from (13) that
\[(29) \quad \tilde{g}_g(\sigma_\lambda) = \begin{pmatrix}
\psi_g(\sigma_\lambda) \cdot (1 + \kappa_{\psi}(\sigma_\lambda) \cdot \varepsilon) & \psi_g'(\sigma_\lambda) \cdot \kappa_{\psi}(\sigma_\lambda) \cdot \varepsilon \\
\psi_g'(\sigma_\lambda) \cdot \kappa_{\psi}(\sigma_\lambda) \cdot \varepsilon & \psi_g'(\sigma_\lambda) \cdot (1 + \kappa^{-1}(\sigma_\lambda) \cdot \varepsilon).
\end{pmatrix}
\]

By comparing upper-right entries in the matrices in (28) and (29) and invoking (27) together with the relation \( \psi_g'(\sigma_\lambda) \eta_g(\tau_\lambda)^{-1} = \eta_g'(\tau_\lambda) \) arising from (15), we deduce that
\[a_\ell(g'_a) = \eta_g'(\tau_\lambda) \kappa_{\psi}(\sigma_\lambda).
\]

It is worth noting that each of the expressions \( \eta_g'(\tau_\lambda) \) and \( \kappa_{\psi}(\sigma_\lambda) \) depend on the choice of a prime \( \lambda \) of \( H \) above \( \ell \) that was made to define \( \tau_\lambda \) and \( \sigma_\lambda \), since changing this prime replaces \( \tau_\lambda \) and \( \sigma_\lambda \) by their conjugates \( \sigma_\lambda \tau_\lambda \sigma^{-1} \) and \( \sigma_\lambda \lambda^{-1} \) by some \( \sigma \in G_K \). More precisely, by (16) and the cocycle property of \( \kappa_{\psi} \),
\[\eta_g'(\sigma \tau_\lambda \sigma^{-1}) = \psi^{-1}(\sigma) \eta_g'(\tau_\lambda), \quad \kappa_{\psi}(\sigma \sigma^{-1}) = \psi(\sigma) \kappa_{\psi}(\sigma_\lambda).
\]

In particular, the product \( \eta_g'(\tau_\lambda) \kappa_{\psi}(\sigma_\lambda) \) is independent of the choice of a prime above \( \ell \), as it should be. Note that \( \eta_g'(\tau_\lambda) \) is a simple root of unity belonging to the image of \( \psi_g \), while \( \kappa_{\psi}(\sigma_\lambda) \) represents the interesting “transcendental” contribution to the Fourier coefficient \( a_\ell(g'_a) \).

By the description of \( \kappa_{\psi}(\sigma_\lambda) \) arising from local and global class field theory, we conclude from (25) that
\[(30) \quad a_\ell(g'_a) = \eta_g'(\tau_\lambda) \frac{\log_v(u'_\psi) \log_{\psi}(u_\psi(\lambda)) - \log_{\psi}(u_\psi) \log_{\psi}(u'_\psi(\lambda))}{\log_v(u'_\psi) - \log_{\psi}(u_\psi)},
\]
as was to be shown.

A more efficient (but somewhat less transparent) route to the proof of Theorem (2.3) is to specialise Theorem (1.10) to this setting. Relative to a basis of the form \( (v, \tau v) \) for \( V_g \), where \( v \) spans a \( G_K \)-stable subspace of \( V_g \) on which \( G_K \) acts via \( \psi_g \), the matrix for \( U_g \) is proportional to one of the form
\[U_g : \begin{pmatrix} 0 & \log_v(u_\psi) \\ \log_{\psi}((\tau v)^{-1} u_\psi) & 0 \end{pmatrix},\]
and the ordinary condition implies that the matrix representing \( A_g \) is proportional to a matrix of the form
\[A : \begin{pmatrix} x & -x \\ y & -y \end{pmatrix}.
\]

The relations \( \text{Trace}(A_g U_g) = 0 \) and \( \text{Trace}(A_g) = 1 \) show that \( A_g \) is represented by the matrix
\[A_g : \begin{pmatrix} 1 & \log_v(u_\psi) - \log_{\psi}(u_\psi) \\ \log_{\psi}(u_\psi) & -\log_v(u_\psi) \end{pmatrix}.
\]
and Theorem 2.3 is readily deduced from the general formula for the Fourier coefficients of \( g'_\chi \) given in Theorem 1.10. The details are left to the reader.

2.3. Numerical examples. We begin with an illustration of Theorem 2.3 in which the image of \( g_\chi \) is isomorphic to the symmetric group \( S_3 \).

Example 2.4. Let \( \chi \) be the quadratic character of conductor 23 and 

\[
g = q - q^2 - q^3 + q^6 + q^8 + \cdots \in S_1(23, \chi)
\]

be the theta series attached to the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{-23}) \). The Hilbert class field \( H \) of \( K \) is 

\[
H = \mathbb{Q}(\alpha) \quad \text{where} \quad \alpha^6 - 6\alpha^4 + 9\alpha^2 + 23 = 0.
\]

Write \( \text{Gal}(H/K) = \langle \sigma \rangle \). The smallest prime which is inert in \( K \) is \( p = 5 \). The deformations \( g'_1 \) and \( g'_{-1} \) were computed to a 5-adic precision of 540, and \( g'_0 \) to 40 digits of 5-adic precision, confirming Part (a) of Theorem 2.3.

Consider the inert prime \( \ell = 7 \) in \( K \). Let \( u(7) = (2\alpha^4 - 7\alpha^2 + 5)/9 \), a root of \( x^3 - x^2 + 2x - 7 = 0 \). Taking \( \omega \) a primitive cube root of unity we have

\[
\begin{align*}
\log_5(u_\psi(7)) &= \log_5(u(7)) + \omega \log_5(u(7)^\sigma) + \omega^2 \log_5(u(7)^{\sigma^2}) \\
\log_5(u'_\psi(7)) &= \log_5(u(7)) + \omega^2 \log_5(u(7)^\sigma) + \omega \log_5(u(7)^{\sigma^2}).
\end{align*}
\]

Let \( u = (\alpha^2 - 1)/3 \) be the elliptic unit in \( H \), a root of \( x^3 - x^2 + 1 = 0 \). Then likewise we have

\[
\begin{align*}
\log_5(u_\psi) &= \log_5(u) + \omega \log_5(u^{\sigma}) + \omega^2 \log_5(u^{\sigma^2}) \\
\log_5(u'_\psi) &= \log_5(u) + \omega^2 \log_5(u^{\sigma}) + \omega \log_5(u^{\sigma^2}).
\end{align*}
\]

Now

\[
a_7(g'_1) = -a_7(g'_{-1}) = 4083079847610157092272537548 \cdot 5 \mod 5^{40}
\]

and one checks to 40 digits of 5-adic precision that

\[
a_7(g'_1) = \frac{\log_5(u_\psi(7)) \log_5(u'_\psi(7)) - \log_5(u'_\psi(7)) \log_5(u_\psi)}{\log_5(u_\psi) - \log_5(u'_\psi)}
\]

as predicted by part (b) of Theorem 2.3.

Consider next the prime \( \ell = 13 \), which splits in \( K \) and factors as \( (l) = \lambda \lambda' \), where \( \lambda^3 = (u_\lambda) \) is a principal ideal generated by \( u_\lambda = -6\alpha^3 + 18\alpha - 37 \), a root of \( x^2 + 74x + 2197 \). After setting \( u(\lambda) = u_\lambda \otimes \frac{1}{3} \), we let

\[
\begin{align*}
\log_5(u_\psi(\lambda)) &= \left( \omega \log_5(u(\lambda)) + \omega^2 \log_5(u'(\lambda)) \right) \\
\log_5(u'_\psi(\lambda)) &= \frac{1}{3} \left( \omega^2 \log_5(u(\lambda)) + \omega \log_5(u'(\lambda)) \right).
\end{align*}
\]

We have

\[
a_{13}(g'_1) = a_{13}(g'_{-1}) = -638894131680830198852008592 \cdot 5 \mod 5^{40}
\]

and one sees that

\[
a_{13}(g'_{\pm 1}) = \frac{\log_5(u'(\psi)) \log_5(u_\psi(\lambda)) - \log_5(u(\psi)) \log_5(u'_\psi(\lambda))}{\log_5(u'_\psi(\lambda)) - \log_5(u_\psi(\lambda))}
\]

to 40 digits of 5-adic precision, confirming Part (a) of Theorem 2.3.

The experiment below focuses on the case where \( \psi_2 \) is a quartic ring class character, so that \( g_2 \) has image isomorphic to the dihedral group of order 8. The associated ring class character \( \psi = \psi_2/\psi_2' = \psi_2^2 \) of \( K \) is quadratic, i.e., a genus character which cuts out a biquadratic extension \( H \) of \( \mathbb{Q} \) containing \( K \). Let \( F \) denote the unique real quadratic subfield of \( H \), and let \( K' \) the unique imaginary quadratic subfield of \( H \) which is distinct from \( F \). The unit \( u_\psi \) is a power of the fundamental unit of \( F \). Observe that the prime \( p \) is necessarily inert in \( K'/\mathbb{Q}, \)


since otherwise $\varrho_g$ would be induced from a character of the real quadratic field $F$ in which $p$ splits. It follows that $u'_\psi = u^{-1}_\psi$, so that by Theorem 2.3

$$a_\ell(g'_\alpha) = \frac{-\log_p u_\psi \cdot (\log_p(u_g(\ell)) + \log_p(u'_g(\ell)))}{-2\log_p(u_\psi)} = \frac{1}{2} \cdot (\log_p(u_g(\ell)) + \log_p(u'_g(\ell))).$$

It follows from the definition of $u_g(\ell)$ that

$$a_\ell(g'_\alpha) = \text{trace}(\varrho_g(\lambda)) \cdot \log_p(\ell).$$

In particular, we obtain

$$a_\ell(g'_\alpha) = \begin{cases} \log_p(\ell) & \text{if } \psi_g(\lambda) = 1, \\ 0 & \text{if } \psi_g(\lambda) = \pm i, \\ -\log_p(\ell) & \text{if } \psi_g(\lambda) = -1, \end{cases}$$

in perfect agreement with the experiments below.

\textbf{Example 2.5.} Let $\chi = \chi_3\chi_{13}$ where $\chi_3$ and $\chi_{13}$ are the quadratic characters of conductors 3 and 13, respectively. The space $S_1(39, \chi)$ is one dimensional and spanned by the form $g = q - q^3 - q^7 + q^9 + \cdots$. The representation $\varrho_g$ has projective image $D_4$ and is induced from characters of two imaginary quadratic fields and one real quadratic field. In particular, it is induced from the quadratic character $\psi_g$ of the Hilbert class field

$$H = \mathbb{Q}(\sqrt{-39}, a), \quad a^4 + 4a^2 - 48 = 0$$

of $\mathbb{Q}(\sqrt{-39})$ (and also ramified characters of ray class fields of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{13}))$. Let $p = 7$, which is inert in $\mathbb{Q}(\sqrt{-39})$. We computed $g'_\pm 1$ to 20-digits of 7-adic precision (and to $q$-adic precision $q^{100}$).

First consider the case of $\ell = \lambda \nu$ split in $\mathbb{Q}(\sqrt{-39})$. Then one observes to 20-digits of 7-adic precision and all such $\ell < 900$ that both $a_\ell(g'_1)$ and $a_\ell(g'_{-1})$ satisfy (31). Next we consider the case that $\ell$ is inert in $\mathbb{Q}(\sqrt{-39})$. Here one observes numerically that the Fourier coefficients are zero when $\ell$ is inert in $\mathbb{Q}(\sqrt{-3})$. When $\ell$ is split in $\mathbb{Q}(\sqrt{-3})$ the Fourier coefficients of the two stabilisations are opposite in sign and both equal to the $p$-adic logarithm of a fundamental $\ell$ unit of norm 1 in $\mathbb{Q}(\sqrt{-3})$. (Observe that $p$ is split in $\mathbb{Q}(\sqrt{-3})$ and our numerical observations are consistent in this example with both our theorems for $p$ split and $p$ inert in the CM case.)

3. RM forms

Consider now the case where $g$ is the theta series attached to a character

$$\psi_g : G_K \to \mathbb{C}$$

of mixed signature of a real quadratic field $K$. As before, assume for simplicity that the field $L$ may be embedded into $\mathbb{Q}_p$ and fix one such embedding. We also continue to denote $H_\varphi$ the abelian extensions of $K$ which is cut out by $\varrho_g$, and let $H$ be the ring class field of $K$ cut out by the non-trivial ring class character $\psi := \psi_g/\psi_g'$. Since $\psi_g$ has mixed signature, it follows that $\psi$ is totally odd and thus $H$ is totally imaginary. As before, write $G := \text{Gal}(H/\mathbb{Q})$ and $Z := \text{Gal}(H/K)$.

As explained in the introduction, the case where $p$ splits in $K$ was already dealt with in \cite{DLR2}, so in the section we only consider the case where $p$ is inert in $K/\mathbb{Q}$. The prime $p$ then splits completely in $H/K$ and we fix a prime $\varphi$ of $H$ above $p$. This choice determines an embedding $H \to H_\varphi = K_\varphi$ and we write $z \mapsto z'$ for the conjugation action of $\text{Gal}(K_\varphi/\mathbb{Q}_p)$.

Let $u_K \in \mathcal{O}_K^\times$ denote the fundamental unit of $\mathcal{O}_K$ of norm 1, which we regard as an element of $K_\varphi^\times = H_\varphi$ through the above embedding, and let $u'_K = u_K^{-1}$ denote its algebraic conjugate.
Let \((v_1, v_2)\) be a basis for \(V_g\) consisting of eigenvectors for the action of \(G_K\), and which are interchanged by the Frobenius element \(\tau_p\). Just as in the previous section, relative to this basis the Galois representation \(\varrho_g\) takes the form

\[
\varrho_g(\sigma) = \begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi_g(\sigma) \end{pmatrix}
\]

for \(\sigma \in G_K\), where \(\psi_g\) is induced from a character of a real quadratic field, but of no imaginary quadratic field.

\[
\varrho_g(\tau) = \begin{pmatrix} 0 & \eta_g(\tau) \\ \eta_g(\tau)^{-1} & 0 \end{pmatrix}
\]

for \(\tau \in G_{\mathbb{Q}} - G_K\), where \(\eta_g\) and \(\eta_g^{\prime}\) are functions taking values in the group of roots of unity in \(L^\times\). The element \(U_g \in (H_\nu \otimes W_g)\) of [5] is thus represented the matrix

\[
U_g : \begin{pmatrix} \log_p(u_K) & 0 \\ 0 & \log_p(u'_K) \end{pmatrix}
\]

and hence the endomorphism \(A_g\) of Lemma [13] is represented by the particularly simple matrix

\[
A_g : \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]

It follows that, if \(\ell = \lambda \lambda'\) is split in \(K/\mathbb{Q}\), we have

\[
A_g(\lambda) : \frac{1}{2} \begin{pmatrix} \log_p(\ell) & 0 \\ 0 & \log_p(\ell) \end{pmatrix}
\]

while if \(\ell\) is inert in \(K/\mathbb{Q}\) and \(\lambda\) is a prime of \(H\) lying above \(\ell\),

\[
A_g(\lambda) : \frac{1}{2} \begin{pmatrix} \log_p(\ell) & -\log_p(u'_\psi(\lambda)) \\ -\log_p(u'_\psi(\lambda)) & \log_p(\ell) \end{pmatrix}
\]

**Theorem 3.1.** For all rational primes \(\ell \nmid Np\),

(a) If \(\ell\) is split in \(K/\mathbb{Q}\), then

\[
a_\ell(g'_\alpha) = \frac{1}{2} a_\ell(g) \cdot \log_p(\ell).
\]

(b) If \(\ell\) is inert in \(K/\mathbb{Q}\), then

\[
a_\ell(g'_\alpha) = - (\eta_g(\lambda) \log_p(u'_\psi(\lambda)) + \eta_g(\lambda') \log_p(u_\psi(\lambda))).
\]

**Proof.** This follows directly from Theorem [111] in light of [433] and [434]. \(\square\)

**Remark 3.2.** As already remarked in [DLR2], Theorem 3.1 above (and also Theorem 7.4 of Part B) display a striking analogy with Theorem 1.1. of [DL4] concerning the Fourier expansions of mock modular forms whose shadows are weight one theta series attached to characters of imaginary quadratic fields. The underlying philosophy is that the \(p\)-adic deformations considered in this paper behave somewhat like mock modular forms of weight one, “with \(\infty\) replaced by \(p\)”. This explains why the analogy remains compelling when the quadratic imaginary fields of [DL3] are replaced by real quadratic fields in which \(p\) is inert (these fields being “imaginary” from a \(p\)-adic perspective).

We illustrate Theorem 3.1 on the form of smallest level whose associated Artin representation is induced from a character of a real quadratic field, but of no imaginary quadratic field. The projective image in this example is the dihedral group \(D_8\) of order 8:

**Example 3.3.** Let \(\chi = \chi_5\chi_{29}\) where \(\chi_5\) and \(\chi_{29}\) are quadratic and quartic characters of conductor 5 and 29, respectively. Then \(S_4(145, \chi)\) is one dimensional and spanned by the eigenform

\[g = q + iq^4 - iq^5 + iq^9 + (-i - 1)q^{11} - q^{16} + (-i - 1)q^{19} + \cdots.\]

The form \(g\) is induced from a quartic character of a ray class group of \(K = \mathbb{Q}(\sqrt{5})\) (see [DLR1] Example 4.1) for a further discussion on this form). The relevant ring class field \(H\) is

\[H = \mathbb{Q}(\alpha)\] where \(\alpha^8 - 2\alpha^7 + 4\alpha^6 - 26\alpha^5 + 94\alpha^4 - 212\alpha^3 + 761\alpha^2 - 700\alpha + 980 = 0.\]
Write $\text{Gal}(H/K) = \langle \psi \rangle$. Take $p = 13$, and note that $\chi(p) = 1$ and so $\alpha = i$ and $\beta = -i$. We compute $g'_1$ to 10 digits of 13-adic precision (and $q$-adic precision $q^{28,000}$).

Consider first the prime $\ell = 7$ which is inert in $K$. We take the 7-unit $u(7) \in H$ to satisfy $x^4 + 13x^3 + 38x^2 + 5x + 343 = 0$ and define

$$\log_{13}(u(7, \pm i)) := \log_{13}(u(7)) \mp i \log_{13}(u(7)^\psi) \mp \log_{13}(u(7)^{\psi^2}) \pm i \log_{13}(u(7)^{\psi^3})$$

and so $\log_{13}(u(7, i)) + \log_{13}(u(7, -i)) = \log_{13}(v)$ where $v = u(7)/u(7)^2 \in H$. Then one checks that to 10 digits of 13-adic precision

$$a_7(g'_1) = -\frac{1}{6} \cdot \log_{13}(v)$$

which is in line with Theorem 3.1. Next we take the prime $\ell = 11$ which is split in $K$. Then to 10-digits of 13-adic precision

$$a_{11}(g'_1) = -\frac{i + 1}{2} \cdot \log_{13}(11)$$

exactly as predicted by Theorem 3.1.

**Part B. The irregular setting**

Denote by $S_k(Np, \chi)$ (resp. by $S_k^{(p)}(N, \chi)$) the space of classical (resp. $p$-adic overconvergent) modular forms of weight $k$, level $Np$ (resp. tame level $N$) and character $\chi$, with coefficients in $\mathbb{Q}_p$. The Hecke algebra $\mathbb{T}$ of level $Np$ generated over $\mathbb{Q}$ by the operators $T_\ell$ with $\ell \nmid Np$ and $U_\ell$ with $\ell \mid Np$ acts naturally on the spaces $S_k(Np, \chi)$ and $S_k^{(p)}(N, \chi)$.

As in the introduction, let $g \in S_1(N, \chi)$ be a newform and let $g_\alpha \in S_1(Np, \chi)$ be a $p$-stabilisation of $g$. The eigenform $g_\alpha$ gives rise to a ring homomorphism $\varphi_{g_\alpha} : \mathbb{T} \rightarrow L$ to the field $L$ generated by the fourier coefficients of $g_\alpha$, satisfying

$$\varphi_{g_\alpha}(T_\ell) = a_\ell(g_\alpha) \text{ if } \ell \nmid Np, \quad \varphi_{g_\alpha}(U_\ell) = \begin{cases} a_\ell(g_\alpha) & \text{if } \ell \mid N; \\ \alpha & \text{if } \ell = p. \end{cases}$$

For any ideal $I$ of a ring $R$ and any $R$-module $M$, denote by $M[I]$ the $I$-torsion in $M$. Let $I_{g_\alpha} \triangleleft \mathbb{T}$ be the kernel of $\varphi_{g_\alpha}$, and set

$$S_1(Np, \chi)[g_\alpha] := S_1(Np, \chi)[I_{g_\alpha}], \quad S_1(Np, \chi)[[g_\alpha]] := S_1(Np, \chi)[I_{g_\alpha}^2].$$

Our main object of study is the subspace

$$S_1(N, \chi)[[g_\alpha]] := S_1^{(p)}(N, \chi)[I_{g_\alpha}].$$

of the space of overconvergent $p$-adic modular forms of weight one, which is contained in the generalised eigenspace attached to $I_{g_\alpha}$. An element of $S_1^{(p)}(N, \chi)[[g_\alpha]]$ is called an overconvergent generalised eigenform attached to $g_\alpha$, and it is said to be classical if it belongs to $S_1(Np, \chi)[[g_\alpha]]$. The theorem of Bellaïche and Dimitrov stated in the opening paragraphs of Part A implies that the natural inclusion

$$S_1(Np, \chi)[[g_\alpha]] \hookrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]]$$

is an isomorphism, i.e., every overconvergent generalised eigenform is classical, except possibly in the following cases:

(a) $g$ is the theta series attached to a finite order character of a real quadratic field in which the prime $p$ splits, or

(b) $g$ is irregular at $p$, i.e., $\alpha = \beta$. 

The study of \( S_1^{(p)}(N, \chi)[[g_\alpha]] \) in scenario (a) was carried out in [DLR2] when \( \alpha \neq \beta \). The main result of loc.cit. is the description of a basis \((g_\alpha, V_\alpha^0)\) for \( S_1^{(p)}(N, \chi)[[g_\alpha]] \) which is canonical up to scaling, and an expression for the fourier coefficients of the non-classical \( g_\alpha^b \) (or rather, of their ratios) in terms of \( p \)-adic logarithms of certain algebraic numbers.

Assume henceforth that \( g \) is not regular at \( p \), i.e., that \( \alpha = \beta \). In that case, the form \( g \) admits a unique \( p \)-stabilisation \( g_\ell = g_\beta \). The Hecke operators \( T_\ell \) for \( \ell \nmid Np \) and \( U_\ell \) for \( \ell \mid N \) act semisimply (i.e., as scalars) on the two-dimensional vector space

\[
S_1(Np, \chi)[[g_\alpha]] = \mathbb{Q}_{p}g_\alpha \oplus \mathbb{Q}_{p}g', \quad g'(q) := g(q^p),
\]

but the Hecke operator \( U_p \) acts non-semisimply via the formulæ

\[
U_p g_\alpha = \alpha g_\alpha, \quad U_p g' = g_\alpha + \alpha g'.
\]

Because

\[
(a_1(g_\alpha), a_p(g_\alpha)) = (1, \alpha), \quad (a_1(g'), a_p(g')) = (0, 1),
\]

the classical subspace \( S_1(Np, \chi)[[g_\alpha]] \) has a natural linear complement in \( S_1^{(p)}(N, \chi)[[g_\alpha]] \), consisting of the generalised eigenforms \( \tilde{g} \) whose \( q \)-expansions satisfy

\[
(36) \quad a_1(\tilde{g}) = a_p(\tilde{g}) = 0.
\]

A modular form satisfying (36) is said to be normalised, and the space of normalised generalised eigenforms is denoted \( S_1^{(p)}(N, \chi)[[g_\alpha]]^0 \). The main goal of Part B is to study this space and give an explicit description of its elements in terms of their fourier expansions. The idoneous fourier coefficients will be expressed as determinants of \( 2 \times 2 \) matrices whose entries are \( p \)-adic logarithms of algebraic numbers the number field \( H \) cut out by the projective Galois representation attached to \( g \) (cf. Theorems [38] [91] and [94]).

### 4. Generalised eigenspaces

We begin by recalling some of the notations that were already introduced in Part A. Let

\[
g_\beta : G_\mathbb{Q} \longrightarrow \text{Aut}_{\mathbb{Q}_p}(V_g) \cong \text{GL}_2(\mathbb{Q}_p)
\]

be the odd, two-dimensional Artin representation associated to \( g \) by Deligne and Serre (but viewed as having \( p \)-adic rather than complex coefficients; as in Part A, we assume for simplicity that the image of \( g_\beta \) can be embedded in \( \text{GL}_2(\mathbb{Q}_p) \) and not just in \( \text{GL}_2(\mathbb{Q}_p) \)).

The four-dimensional \( \mathbb{Q}_p \)-vector space \( W_g := \text{Ad}(V_g) := \text{End}(V_g) \) of endomorphisms of \( V_g \) is endowed with the conjugation action of \( G_\mathbb{Q} \),

\[
\sigma \cdot M := g_\beta(\sigma) \circ M \circ g_\beta(\sigma)^{-1}, \quad \text{for any } \sigma \in G_\mathbb{Q}, \quad M \in W_g.
\]

Let \( H \) be the field cut out by this Artin representation. The action of \( G_\mathbb{Q} \) on \( W_g \) factors through a faithful action of the finite quotient \( G := \text{Gal}(H/\mathbb{Q}) \). Let \( W_g^o := \text{Ad}^0(V_g) \) denote the three-dimensional \( G_\mathbb{Q} \)-submodule of \( W_g \) consisting of trace zero endomorphisms. The exact sequence

\[
0 \longrightarrow W^o_g \longrightarrow W_g \longrightarrow \mathbb{Q}_p \longrightarrow 0
\]

of \( G \)-modules admits a canonical \( G \)-equivariant splitting

\[
p : W_g \longrightarrow W^o_g, \quad p(A) := A - 1/2 \cdot \text{Tr}(A).
\]

Because the action of \( G_\mathbb{Q} \) on \( V_g \) also factors through a finite quotient, the field \( L \subset \mathbb{Q}_p \) generated by the traces of \( g_\beta \) is a finite extension of \( \mathbb{Q} \), and \( g_\beta \) maps the semisimple algebra \( L[G_\mathbb{Q}] \) to a central simple algebra of rank 4 over \( L \). By eventually enlarging \( L \), it can be assumed that \( g_\beta(L[G_\mathbb{Q}]) \cong \mathbb{M}_2(L) \), and therefore that \( g_\beta \) is realised on a two-dimensional \( L \)-vector space \( V_g^L \) equipped with an identification \( \iota : V_g^L \otimes_L \mathbb{Q}_p \longrightarrow V_g \). The spaces

\[
W^L_g := \text{Ad}(V_g^L), \quad W^{oL}_g := \text{Ad}^0(V_g^L)
\]
likewise correspond to $G$-stable $L$-rational structures on $W_g$ and $W_g^\circ$ respectively, equipped with identifications

$$
\iota : W_g^L \otimes_L \mathbb{Q}_p \to W_g, \quad \iota : W_g^\circ \otimes_L \mathbb{Q}_p \to W_g^\circ.
$$

The spaces $W_g$ and $W_g^\circ$ (as well as $W_g^L$ and $W_g^\circ L$) are equipped with the Lie bracket $[\ , \ ]$ and with a symmetric non-degenerate pairing $\langle \ , \ \rangle$ defined by the usual rules

$$
[A, B] := AB - BA, \quad \langle A, B \rangle := \text{Tr}(AB),
$$

which are compatible with the $G$-action in the sense that

$$
[\sigma \cdot A, \sigma \cdot B] = \sigma \cdot [A, B], \quad \langle \sigma \cdot A, \sigma \cdot B \rangle = \langle A, B \rangle, \quad \text{for all } \sigma \in G.
$$

These operations can be combined to define a $G$-invariant determinant function---i.e., a non-zero, alternating trilinear form---on $W_g^\circ$ and on $W_g^\circ L$ by setting

$$
\det(A, B, C) := \langle [A, B], C \rangle.
$$

The rule described in \cite{Ma} gives rise to natural identifications

$$
S_1(Np, \chi)[g_\alpha] \simeq \text{Hom}(\mathbb{T}/I_{g_\alpha}, \mathbb{Q}_p), \quad S_1^{(p)}(N, \chi)[[g_\alpha]] \simeq \text{Hom}(\mathbb{T}/I_{g_\alpha}^2, \mathbb{Q}_p),
$$

and hence the dual of the short exact sequence

$$
0 \to I_{g_\alpha}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha} \to 0
$$

can be identified with

$$
0 \longrightarrow S_1(Np, \chi)[g_\alpha] \longrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]] \longrightarrow S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \longrightarrow 0.
$$

In particular, one has the isomorphism

$$
S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \simeq \text{Hom}(I_{g_\alpha}/I_{g_\alpha}^2, \mathbb{Q}_p).
$$

Let $\mathbb{Q}_p[\varepsilon] = \mathbb{Q}_p[x]/(x^2)$ denote the ring of dual numbers. Given $g^\flat \in S_1^{(p)}(N, \chi)[[g_\alpha]]_0$, the modular form $\tilde{g} := g_\alpha + \varepsilon \cdot g^\flat$ is an eigenform for $\mathbb{T}$ with coefficients in $\mathbb{Q}_p[\varepsilon]$. Its associated Galois representation

$$
\tilde{g}_5 : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathbb{Q}_p[\varepsilon])
$$

satisfies

(i) $\tilde{g}_5 = g_5 \pmod{\varepsilon}$ and $\det(\tilde{g}_5) = \chi$,

(ii) for every prime number $\ell \nmid Np$, the trace of an arithmetic Frobenius $\tau_\ell$ at $\ell$ is

$$
\text{Tr}(\tilde{g}_5(\tau_\ell)) = a_\ell(g_\alpha) + \varepsilon \cdot a_\ell(g^\flat).
$$

Conjecture 4.1. Assume that $g$ is irregular at $p$. Then the assignment $g^\flat \mapsto \tilde{g}_5$ gives rise to a canonical isomorphism between $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ and the space $\text{Def}^0(\tilde{g}_5)$ of isomorphism classes of deformations of $\tilde{g}_5$ to the ring of dual numbers, with constant determinant.

We now derive some consequences of this conjecture.

Proposition 4.2. Assume Conjecture 4.1. If $g$ is irregular at $p$, then the space $S_1^{(p)}(N, \chi)[[g_\alpha]]_0$ is two-dimensional over $\mathbb{Q}_p$.

Proof. Since any $\tilde{g} \in \text{Def}^0(\tilde{g}_5)$ has constant determinant, it may be written as

$$
\tilde{g} = (1 + \varepsilon \cdot c) \cdot g_5 \quad \text{for some} \quad c = c(\tilde{g}) : G_{\mathbb{Q}} \longrightarrow W_g^\circ.
$$

The multiplicativity of $\tilde{g}$ implies that the function $c$ is a 1-cocycle of $G_{\mathbb{Q}}$ with values in $W_g^\circ$, whose class in $H^1(\mathbb{Q}, W_g^\circ)$ (which shall be denoted with the same symbol, by a slight abuse of notation) depends only on the isomorphism class of $\tilde{g}$. The assignment $\tilde{g} \mapsto c(\tilde{g})$ realises an isomorphism (cf. for instance \cite{Ma}, §1.2)

$$
\text{Def}^0(\tilde{g}_5) \longrightarrow H^1(\mathbb{Q}, W_g^\circ).
$$
Under Conjecture 4.1 this yields an isomorphism

\[ S_1^{(p)}(N, \chi)[[g_0]]_0 \sim \to H^1(Q, W_g^0), \quad g^0 \mapsto c_g^0. \]

The inflation-restriction sequence combined with global class field theory for \( H \) now gives rise to a series of identifications

\[ H^1(Q, W_g^0) \overset{\text{res}_H}{\to} \text{hom}(G_H, W_g^0)^G = \text{hom}_G \left( \frac{(O_H \otimes \mathbb{Z}_p)\times}{O_H^\times \otimes \mathbb{Z}_p}, W_g^0 \right) = \text{hom}_G \left( H_p^\times, W_g^0 \right) = \ker \left( \text{hom}_G(H_p, W_g^0) \overset{\text{res}_U}{\to} \text{hom}_G(U, W_g^0) \right), \]

where \( U \) denotes the natural image of \( O_H^\times \otimes \mathbb{Z}_p \) in \( H_p := H \otimes \mathbb{Q}_p \) under the \( p \)-adic logarithm map

\[ \log_p : H_p^\times \to H_p. \]

As representations for \( G \), the space \( H_p \) is isomorphic to the regular representation

\[ H_p \simeq \text{Ind}_G^1 \mathbb{Q}_p, \]

while \( U \), by the Dirichlet unit theorem, is induced from the trivial representation of the subgroup \( G_{\infty} \subset G \) generated by a complex conjugation:

\[ U \simeq \text{Ind}_{G_{\infty}}^G \mathbb{Q}_p. \]

Complex conjugation acts on \( W_g^0 \) with eigenvalues 1, \(-1\) and \(-1\), and hence by Frobenius reciprocity,

\[ \dim_{\mathbb{Q}_p} \text{hom}_G(H_p, W_g^0) = 3, \quad \dim_{\mathbb{Q}_p} \text{hom}_G(U, W_g^0) = 1. \]

It follows from (41) that \( H^1(Q, W_g^0) \) is two-dimensional over \( \mathbb{Q}_p \). Proposition 4.2 follows.

For any \( \ell \nmid Np \), the \( \ell \)-th Fourier coefficient of \( g^0 \) is given in terms of the associated cocycle \( c_g^0 \) by the rule

\[ a_\ell(g^0) = \text{Tr}(c_\ell(\sigma_\lambda)\theta_g(\sigma_\lambda)) \]

where \( \lambda | \ell \) is any prime above \( \ell \) and \( \sigma_\lambda \) denotes the arithmetic Frobenius associated to it. Note that the right-hand side of (43) does not depend on the choice of \( \lambda \).

Our next goal is to parametrise the elements of (41) explicitly, and then to derive concrete formulae for the Fourier expansions of the associated modular forms in \( S_1^{(p)}(N, \chi)[[g_0]] \) via (40) and (43). After treating the general case in Section 5, Sections 6 and 7 focus on the special features of the scenarios where \( W_g^0 \) is reducible, i.e.,

(i) the CM case where \( V_g \) is induced from a character of an imaginary quadratic field;
(ii) the RM case where \( V_g \) is induced from a character of a real quadratic field.

## 5. The General Case

The Galois representation \( W_g^0 \) is irreducible if and only if \( G := \text{Gal}(H/\mathbb{Q}) \) is isomorphic to \( A_4 \), \( S_4 \), or \( A_5 \). Otherwise, the representation \( g_0 \) has dihedral projective image and \( G \) is isomorphic to a dihedral group.

The irregularity assumption implies that the prime \( p \) splits completely in \( H \), and \( H \) can therefore be viewed as a subfield of \( \mathbb{Q}_p \) after fixing an embedding \( H \hookrightarrow \mathbb{Q}_p \) once and for all. This amounts to choosing a prime \( \varphi \) of \( H \) above \( p \). Let \( \log \varphi : H_p^\times \to \mathbb{Q}_p \) denote the associated \( \varphi \)-adic logarithm map, which factors through \( \log_p \).
The Dirichlet unit theorem implies (via the second equation in (42)) that
\[ \dim_L(O_H^\times \otimes W^{\sigma L}_g)^G = 1. \]

In particular, for all \( u \in \mathcal{O}_H^\times \) and all \( w \in W^{\sigma L}_g \), the element
\[ \xi(u, w) := \frac{1}{\# G} \times \sum_{\sigma \in G} (\sigma u) \otimes (\sigma \cdot w) \in (O_H^\times \otimes W^{\sigma L}_g)^G \]
only depends on the choices of \( u \) and \( w \) up to scaling by a (possibly zero) factor in \( L \). As \( u \) varies over \( \mathcal{O}_H^\times \) and \( w \) over \( W^{\sigma L}_g \), the elements
\[ \xi_p(u, w) := (\log_p \otimes \text{id}) \xi(u, w) = \frac{1}{\# G} \times \sum_{\sigma \in G} \log_p(\sigma u) \cdot (\sigma \cdot w) \in W^{\sigma}_g \]
therefore lie in a one-dimensional \( L \)-vector subspace of \( W^{\sigma}_g \). Choose a generator \( w(1) \) for this space. The coordinates of \( w(1) \) relative to a basis \( (e_1, e_2, e_3) \) for \( W^{\sigma L}_g \) are \( \wp \)-adic logarithms of units in \( \mathcal{O}_H \), namely, we can write
\[ w(1) = \log_p(u_1) e_1 + \log_p(u_2) e_2 + \log_p(u_3) e_3, \]
for appropriate \( u_i \in (O_H^\times) \otimes \mathbb{Z} \). Let \( \ell \nmid \mathfrak{N}_p \) be a rational prime. For any prime \( \lambda \) of \( H \) above \( \ell \), let \( \tilde{u}_\lambda \) be a generator of the principal ideal \( \lambda^h \), where \( h \) is the class number of \( H \), and set
\[ u_\lambda := \tilde{u}_\lambda \otimes h^{-1} \in (O_H[1/\ell]^\times) \otimes \mathbb{Z} / L. \]
Let
\[ \tilde{w}_\lambda := g_\ell(\lambda) \in W^{\sigma L}_g, \quad w_\lambda := p(\tilde{w}_\lambda) \in W^{\sigma L}_g \]
be the endomorphisms of \( V_\ell \) arising from the image of \( \sigma_\lambda \) under \( g_\ell \). The element \( u_\lambda \) is well-defined up to multiplication by elements of \( \mathcal{O}_H^\times \), and hence the elements
\[ \xi(u_\lambda, w_\lambda) := \frac{1}{\# G} \times \sum_{\sigma \in G} (\sigma u_\lambda) \otimes (\sigma \cdot w_\lambda) \in (O_H[1/\ell]^\times \otimes W^{\sigma L}_g)^G, \]
\[ w(\ell) = \xi_p(u_\lambda, w_\lambda) := \frac{1}{\# G} \times \sum_{\sigma \in G} \log_p(\sigma u_\lambda) \cdot (\sigma \cdot w_\lambda) \in W^{\sigma}_g \]
are defined up to translation by elements of the one-dimensional \( L \)-vector spaces \( (O_H^\times \otimes W^{\sigma L}_g)^G \) and \( L \cdot w(1) \) respectively. Furthermore, the image of \( w(\ell) \) in the quotient \( W^{\sigma}_g / (L \cdot w(1)) \) does not depend on the choice of the prime \( \lambda \) of \( H \) above \( \ell \) that was made to define it. The Lie bracket
\[ \mathfrak{m}(\ell) := [w(1), w(\ell)] \in W^{\sigma}_g \]
is thus independent of the choices that were made in defining \( w(\ell) \).

**Remark 5.1.** The coordinates of \( w(\ell) \) relative to a basis \( (e_1, e_2, e_3) \) for \( W^{\sigma L}_g \) are \( \wp \)-adic logarithms of \( \ell \)-units in \( H \), i.e., one can write
\[ w(\ell) = \log_p(v_1) e_1 + \log_p(v_2) e_2 + \log_p(v_3) e_3, \]
with \( v_i \in (O_H[1/\ell]^\times)_L \) for \( i = 1, 2, 3 \). A direct computation shows that
\[ \dim_{\wp}(W^{\sigma}_g)_{\sigma_\lambda=1} = \begin{cases} 1 & \text{if } g \text{ is regular at } \ell; \\ 3 & \text{if } g \text{ is irregular at } \ell. \end{cases} \]

It follows that for all regular primes \( \ell \),
\[ \dim_L(O_H[1/\ell]^\times \otimes W^{\sigma L}_g)^G = 2, \]
and therefore that the element $\xi(v, w)$ attached to any pair $(v, w) \in \mathcal{O}^{1}[\ell]^{\times} \times W_{g}^{oL}$ as in (48) is well-defined up to scaling by $L$ and up to translation by elements of the one-dimensional space $(\mathcal{O}_{1}^{o} \otimes W_{g}^{oL})^{G}$. In particular, the associated vector $\mathfrak{M}(\ell)$ lies in a canonical one-dimensional subspace of $W_{g}^{o}$, namely, the orthogonal complement in $W_{g}^{o}$ of

$$(\log \wp \otimes \text{Id})(\mathcal{O}_{H}^{1}[\ell]^{\times} \otimes W_{g}^{oL})^{G} \subset W_{g}^{o}.$$ 

If the basis $(e_{1}, e_{2}, e_{3})$ for $W_{g}^{oL}$ in (46) and (49) is taken to be the standard basis

$$e_{1} = \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right), \quad e_{2} = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right), \quad e_{3} = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right),$$

then

$$\mathfrak{M}(\ell) = \det \left( \begin{array}{ccc} u_{2} & u_{3} \\ v_{2} & v_{3} \end{array} \right) \cdot e_{1} + 2 \det \left( \begin{array}{ccc} u_{1} & u_{2} \\ v_{1} & v_{2} \end{array} \right) \cdot e_{2} - 2 \det \left( \begin{array}{ccc} u_{1} & u_{3} \\ v_{1} & v_{3} \end{array} \right) \cdot e_{3}.$$ 

Remark 5.2. Observe that if the prime $\ell$ is irregular for $g$, the vector $\tilde{w}_{\lambda}$ is a scalar endomorphism in $W_{g}$ and hence $w_{\lambda} = w(\ell) = \mathfrak{M}(\ell) = 0$.

Our main result is

**Theorem 5.3.** Assume Conjecture 4.1. For all $w \in W_{g}^{o}$, there exists an overconvergent generalised eigenform $g_{w}^{p} \in S_{1}^{(p)}(N, \chi)[[g_{a}]]_{0}$ satisfying

$$a_{\ell}(g_{w}^{p}) = \langle w, \mathfrak{M}(\ell) \rangle = \det(w, w(1), w(\ell)),$$

for all primes $\ell \nmid Np$. The assignment $w \mapsto g_{w}^{p}$ induces an isomorphism between $W_{g}^{o}/U$ and $S_{1}^{(p)}(N, \chi)[[g_{a}]]_{0}$.

Proof. The semi-local field $H_{p} = H \otimes_{\mathbb{Q}} \mathbb{Q}_{p} = \bigoplus_{p \nmid \ell} \mathbb{Q}_{p}$ is naturally identified with the set of vectors $h = (h_{p})_{p \nmid \ell}$ of entries $h_{p} \in \mathbb{Q}_{p}$, indexed by the primes of $H$ above $p$. The function which to $w \in W_{g}^{o}$ associates the linear transformation

$$\hat{\varphi}_{w} : H_{p} \longrightarrow W_{g}^{o}, \quad \hat{\varphi}_{w}(h) = \frac{1}{\# G} \times \sum_{\sigma \in G} (\sigma^{-1} h)_{p} \cdot (\sigma \cdot w)$$

identifies $W_{g}^{o}$ with $\text{hom}_{G}(H_{p}, W_{g}^{o})$. The linear function $\hat{\varphi}_{w}$ is trivial on $U := \log_{p}(\mathcal{O}_{H}^{o}) \subset H_{p}$ if and only if, for all $u \in \mathcal{O}_{H}^{o}$ and all $w' \in W_{g}^{o}$,

$$\langle \hat{\varphi}_{w}(\log_{p}(u)), w' \rangle = 0.$$ 

But

$$\langle \hat{\varphi}_{w}(\log_{p}(u)), w' \rangle = \frac{1}{\# G} \times \left( \sum_{\sigma \in G} \log_{p}(\sigma^{-1}(u)) \cdot (\sigma \cdot w), w' \right)$$

$$= \frac{1}{\# G} \times \sum_{\sigma \in G} \log_{p}(\sigma^{-1}(u)) \cdot \langle \sigma \cdot w, w' \rangle$$

$$= \frac{1}{\# G} \times \sum_{\sigma \in G} \log_{p}(\sigma^{-1}(u)) \cdot \langle w, \sigma^{-1} \cdot w' \rangle$$

$$= \langle w, \xi_{p}(u, w') \rangle,$$

and hence $\hat{\varphi}_{w}$ is trivial on $U = \log_{p}(\mathcal{O}_{H}^{o})$ if and only if $w$ is orthogonal in $W_{g}^{o}$ to the line spanned by $w(1)$. It follows that the $G$-equivariant linear function

$$\varphi_{w} := \hat{\varphi}_{w,w(1)} : H_{p} \longrightarrow W_{g}^{o}$$

factors through $H_{p}/U$. The assignment $w \mapsto \varphi_{w}$ identifies $W_{g}^{o}/(L \cdot w(1))$ with $\text{hom}_{G}(H_{p}/U, W_{g}^{o})$, and gives an explicit description of the latter space.
Let $\tilde{g}_w = g + \varepsilon g_w$ be the eigenform with coefficients in $\mathbb{Q}_p[\varepsilon]$ which is attached to the cocycle $\varphi_w \in \text{hom}_G(H_p/U, W_g^\circ) = H^1(G, W_g^\circ)$. Equation (43) with $g' = \tilde{g}_w$ (and hence $c_{g'} = \varphi_w$) combined with (47) shows that the $\ell$-th the fourier coefficient of $\tilde{g}_w$ at a prime $\ell \nmid Np$ is equal to

\begin{equation}
 a_\ell(g_w) = \text{Tr}(\varphi_w(\sigma_\lambda) \cdot g(y)) = \langle \varphi_w(\sigma_\lambda), \tilde{w}_\lambda \rangle = \langle \varphi_w(\sigma_\lambda), w_\lambda \rangle.
\end{equation}

Class field theory for $H$ implies that

$\varphi_w(\sigma_\lambda) = \frac{1}{\# G} \sum_{\sigma \in G} \log_p(\sigma^{-1}u_\lambda) \cdot \sigma \cdot [w, w(1)]$.

Hence

\begin{equation}
 a_\ell(g_w) = \frac{1}{\# G} \times \left( \sum_{\sigma \in G} \log_p(\sigma^{-1}u_\lambda) \cdot \sigma \cdot [w, w(1)], w_\lambda \right)
\end{equation}

\begin{equation}
 = \frac{1}{\# G} \times \left( \sum_{\sigma \in G} \log_p(\sigma^{-1}u_\lambda) \cdot [\sigma \cdot [w, w(1)], w_\lambda \rangle
\end{equation}

\begin{equation}
 = \frac{1}{\# G} \times \left( \sum_{\sigma \in G} \log_p(\sigma^{-1}u_\lambda) \cdot ([w, w(1)], \sigma^{-1} \cdot w_\lambda)
\end{equation}

\begin{equation}
 = \langle [w, w(1)], w(\ell) \rangle = \det(w, w(1), w(\ell)) = \langle w, \mathfrak{M}(\ell) \rangle.
\end{equation}

The theorem follows. $\square$

If $w$ in a vector in $W_g^oL$, Theorem 5.3 shows that the associated overconvergent generalised eigenform $g_w^o$ has fourier coefficients which are $L$-rational linear combinations of determinants of $2 \times 2$ matrices whose entries are the $\varphi$-adic logarithms of algebraic numbers in $H$. In the CM and RM cases to be discussed below, the representation $W_g^o$ is reducible and decomposes further into non-trivial irreducible representations. In that case the choice of an $L$-basis for $W^oL$ which is compatible with this decomposition leads to canonical elements of $S_1^{(p)}(N, \chi)[[g_w]]$, which can sometimes be re-scaled so that their fourier expansions admit even simpler expressions, as will be described in the next two sections.

6. CM FORMS

Assume that $g$ is the theta series attached to a character of a quadratic imaginary field $K$, i.e., that

$\psi_L^g = \text{Ind}_K^G \psi_g$,

where $\psi_g : \text{Gal}(\bar{K}/K) \to L^X$ is a finite order character. Let $\psi'_g$ denote the character deduced from $\psi_g$ by composing it with the involution in $\text{Gal}(\bar{K}/Q)$. The irreducibility assumption on $V_g^L$ implies that the characters $\psi_g$ and $\psi'_g$ are distinct, and therefore the representations $V_g^L$ and $V_g$ decompose canonically as a direct sum of two $G_K$-stable one-dimensional subspaces

$V_g^L = \mathcal{L}_{\psi_g}^L \oplus \mathcal{L}_{\psi'_g}^L$, \quad V_g = \mathcal{L}_{\psi_g} \oplus \mathcal{L}_{\psi'_g}$

on which $G_K$ acts via the characters $\psi_g$ and $\psi'_g$ respectively. The representations $W_g^L$ and $V_g$ also decompose as direct sums of four $G_K$-stable lines

$W_g^L = \left( \text{hom}(\mathcal{L}_{\psi_g}^L, \mathcal{L}_{\psi_g}^L) \oplus \text{hom}(\mathcal{L}_{\psi'_g}^L, \mathcal{L}_{\psi'_g}^L) \right) \oplus \left( \text{hom}(\mathcal{L}_{\psi_g}^L, \mathcal{L}_{\psi'_g}^L) \oplus \text{hom}(\mathcal{L}_{\psi'_g}^L, \mathcal{L}_{\psi_g}^L) \right)$,

$W_g = \left( \text{hom}(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi_g}) \oplus \text{hom}(\mathcal{L}_{\psi'_g}, \mathcal{L}_{\psi'_g}) \right) \oplus \left( \text{hom}(\mathcal{L}_{\psi_g}, \mathcal{L}_{\psi'_g}) \oplus \text{hom}(\mathcal{L}_{\psi'_g}, \mathcal{L}_{\psi_g}) \right)$. 

The direct summands in parentheses are also stable under $G_\mathbb{Q}$ and are isomorphic to the induced representations $\text{Ind}_{H}^{\mathcal{O}_{H}} \chi$ and $\text{Ind}_{H}^{\mathcal{O}_{H}} \psi$ respectively, where $\psi := \psi_{g}/\psi_{g}'$, is the ring class character of $K$ associated to $\psi_{g}$. It follows that

$$W_{g}^{oL} = L(\chi_{K}) \oplus Y_{g}^{L}, \quad W_{g}^{o} = \mathcal{O}_{p}(\chi_{K}) \oplus Y_{g}, \quad Y_{g}^{L} := \text{Ind}_{K}^{G} \psi, \quad Y_{g} := Y_{g}^{L} \otimes L \mathbb{Q}_{p}.$$ 

It will be convenient to choose a basis $(e_{1}, e_{2}) \in L_{\psi_{g}} \times L_{\psi_{g}}$ for $V_{g}^{L}$, and denote by $e_{11}, e_{12}, e_{21}, e_{22}$ the resulting basis of $W_{g}^{L}$, where $e_{ij}$ is the elementary matrix whose $(i', j')$-entry is $\delta_{i'} \delta_{j'}$. Relative to the identification of $W_{g}^{oL}$ with the space of $2 \times 2$ matrices of trace zero with entries in $L$ in this basis, the representation $L(\chi_{K}) = L \cdot (e_{11} - e_{22})$ is identified with the space of diagonal matrices of trace 0, while $Y_{g}^{L} = L \cdot e_{12} \oplus L \cdot e_{21}$ is identified with the space of off-diagonal matrices in $M_{2}(L)$. Fix an element $\tau \in G_{\mathbb{Q}} = G_{K}$ once and for all. By eventually re-scaling $e_{1}$ and $e_{2}$, it can (and shall, henceforth) be assumed that $\varrho_{g}(\tau)$ is represented by the matrix

$$
\begin{pmatrix}
0 & t \\
t & 0
\end{pmatrix}
$$

in this basis, where $-t^{2} := \chi(\tau)$.

Let $Z := \text{Gal}(H/K)$ be the maximal abelian normal subgroup of the dihedral group $G = \text{Gal}(H/\mathbb{Q})$. Note that every element in $G - Z$ (such as the image of $\tau$ in $G$) is an involution, and that $Z$ operates transitively on $G - Z$ by either left or right multiplication.

The field $H$ through which $W_{g}^{o}$ factors is the ring class field of $K$ attached to the character $\psi$. The group $\mathcal{O}_{H}^{\times} \otimes \mathbb{Q}$ of units of $H$ is isomorphic to the regular representation of $Z$ minus the trivial representation, and a finite index subgroup of $\mathcal{O}_{H}^{\times}$ can be constructed explicitly from the elliptic units arising in the theory of complex multiplication. Let

$$e_{\psi} := \frac{1}{\#Z} \sum_{\sigma \in Z} \psi^{-1}(\sigma) \sigma$$

be the idempotent in the group ring of $Z$ giving rise to the projection onto the $\psi$-isotypic component for the action of $Z$. Choose a unit $u \in \mathcal{O}_{H}^{\times}$ and let

$$u_{\psi} := e_{\psi} u, \quad \tau u_{\psi} = e_{\psi}(\tau u)$$

be elements of $\mathcal{O}_{H}^{\times} \otimes L$ on which $Z$ acts via the characters $\psi$ and $\psi' = \psi^{-1}$ respectively. With these choices, we can let

$$w(1) = \begin{pmatrix}
0 & \log_{\psi}(u_{\psi}) \\
\log_{\psi}(\tau u_{\psi}) & 0
\end{pmatrix}.$$ 

The description of the canonical vectors $w(\ell), \mathcal{U}(\ell) \in W_{g}^{o}$ attached to a rational prime $\ell \nmid Np$ depends in an essential way on whether $\ell$ is split or inert in $K/\mathbb{Q}$.

If $\ell = \lambda \lambda'$ is split in $K$ and $\ell$ is regular for $g$, i.e., $\psi_{g}(\sigma_{\lambda}) \neq \psi_{g}(\lambda')$, then the natural map

$$(\mathcal{O}_{K}[1/\ell]^{\times} \otimes W_{g}^{oL})^{G} \subset \left( \frac{\mathcal{O}_{H}[1/\ell]^{\times}}{\mathcal{O}_{H}^{\times}} \otimes W_{g}^{oL} \right)^{G}$$

is an isomorphism of $L$-vector spaces.

Let $\bar{u}_{\lambda}$ be a generator of $\lambda^{h}$ where $h$ is the class number of $K$, and set

$$u_{\lambda} := \bar{u}_{\lambda} \otimes h^{-1}.$$ 

Since

$$\bar{w}_{\lambda} = \begin{pmatrix}
\psi_{g}(\lambda) & 0 \\
0 & \psi_{g}(\lambda')
\end{pmatrix}, \quad w_{\lambda} = \frac{\psi_{g}(\lambda) - \psi_{g}(\lambda')}{2} \times \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},$$

a direct calculation shows that

$$w(\ell) = \log_{\psi}(u_{\lambda}/u_{\lambda}') \times \frac{(\psi_{g}(\lambda) - \psi_{g}(\lambda'))}{2} \times \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.$$
It follows that

\begin{equation}
\mathfrak{M}(\ell) = \log_\wp(u_{\lambda}/u'_{\lambda}) \times (\psi g(\lambda) - \psi g(\lambda')) \times \begin{pmatrix}
0 & -\log_\wp(u_{\psi}) \\
\log_\wp(u'_{\psi}) & 0
\end{pmatrix}.
\end{equation}

If \( \ell \) is inert in \( K \) then \( \ell \) is always regular for \( g \) since \( g_g(\tau_{\ell}) \) has trace 0 and hence has distinct eigenvalues. The prime \( \ell \) splits completely in \( H/K \), and hence the group \( (\mathcal{O}_H[1/\ell]^\times) \otimes L \) is isomorphic to two copies of the regular representation of \( \mathbb{Z} \) minus a trivial representation. The choice of a prime \( \lambda \) of \( H \) above \( \ell \) determines a matrix (and not just a conjugacy class)

\[ \bar{w}_{\lambda} = w_{\lambda} = \varrho(\sigma_{\lambda}) = \begin{pmatrix}
0 & b_{\lambda} \\
c_{\lambda} & 0
\end{pmatrix} \]

with entries in \( L \). Let \( u_{\lambda} \) be an element of \( (\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times) \otimes L \) whose prime factorisation is given by

\begin{equation}
(u_{\lambda}) = b_{\lambda} \lambda + c_{\lambda}(\tau_{\lambda}).
\end{equation}

This \( \ell \)-unit is only well defined by \( 53 \) up to translation by \( \mathcal{O}_H^\times \otimes L \), and the defining equation \( 53 \) of course depends crucially on the choice of the prime \( \lambda \) above \( \ell \). However, the \( \psi \)-isotypic projection

\begin{equation}
u_{\psi}(\ell) := e_\psi u_{\lambda}
\end{equation}

is independent of this choice. A direct calculation shows that

\[ w(\ell) = \frac{1}{2} \times \begin{pmatrix}
0 & \log_\wp(u_{\psi}(\ell)) \\
\log_\wp(\tau_{\psi}(\ell)) & 0
\end{pmatrix}.\]

It follows that

\begin{equation}
\mathfrak{M}(\ell) = \begin{pmatrix}
R_\psi(\ell) & 0 \\
0 & -R_\psi(\ell)
\end{pmatrix},
\end{equation}

where

\[ R_\psi(\ell) = \det \begin{pmatrix}
\log_\wp(u_{\psi}) & \log_\wp(\tau_{\psi}) \\
\log_\wp(u_{\psi}(\ell)) & \log_\wp(\tau_{\psi}(\ell))
\end{pmatrix} \]

is an \( \ell \)-unit regulator attached to \( \psi \), which is independent of the choice of prime \( \lambda \) of \( H \) above \( \ell \). The function \( \ell \mapsto R_\psi(\ell) \) does depend on the choice of the unit \( u \), but only up to scaling by \( L^\times \).

**Theorem 6.1.** The space \( S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \) has a canonical basis \( (g_1^\alpha, g_2^\alpha) \) which is characterised by the properties:

(i) The Fourier coefficients \( a_\ell(g_1^\alpha) \) are 0 for all primes \( \ell \nmid Np \) that are inert in \( K \). If \( \ell = \lambda\lambda' \) is split in \( K \), then

\[ a_\ell(g_1^\alpha) = (\psi g(\lambda) - \psi g(\lambda')) \times \log_\wp(u_{\lambda}/u'_{\lambda}) \]

is a simple algebraic multiple of the \( p \)-adic logarithm of the fundamental \( \ell \)-unit of norm 1 in \( K \).

(ii) The Fourier coefficients \( a_\ell(g_2^\alpha) \) are 0 at all the primes \( \ell \mid Np \) that are split in \( K \). If \( \ell \) is inert in \( K \), then

\[ a_\ell(g_2^\alpha) = R_\psi(\ell). \]

**Proof.** This follows directly from the calculation of the matrices \( \mathfrak{M}(\ell) \) in \( 52 \) and \( 55 \) in light of Theorem 5.3. \( \Box \)
Example 6.2. Let \( \chi \) be the quadratic character of conductor 59. The space \( S(59, \chi) \) is one dimensional and spanned by the theta series
\[
g = q - q^3 + q^4 - q^5 - q^7 - q^{12} + q^{15} + q^{16} + 2q^{17} - \cdots.
\]
Here \( K = \mathbb{Q}(\sqrt{-59}) \) and the ring class field attached to \( \psi \) is
\[
H = K(\alpha) \quad \text{where} \quad \alpha^3 - 3\alpha + 46\sqrt{-59} = 0.
\]
The inert primes \( \ell \) in \( K \) are 2, 3, 13, 23, \cdots and the unit and first few \( \ell \)-units are
\[
u = \frac{1}{612} (13\alpha^2 - 7\sqrt{-59}\alpha - 26), \quad u_2 = \frac{1}{612} (-5\alpha^2 - 13\sqrt{-59}\alpha - 194)
\]
\[
u_1 = \frac{1}{300} (5\alpha^2 + 13\sqrt{-59}\alpha - 112), \quad u_3 = \frac{1}{612} (13\alpha^2 - 7\sqrt{-59}\alpha - 1250)
\]
\[
u_2 = \frac{1}{200} (-\alpha^2 + 11\sqrt{-59}\alpha + 138).
\]

Let \( p = 17 \), an irregular prime for \( g \). We computed a basis of \( q \)-expansions for the generalised eigenspace modulo \( p^{20} \) and \( q^{30000} \). One observes that it contains the classical space spanned by the forms \( g_u(\psi) \) and \( q(\psi) \) and in addition a complementary space of dimension two. This space is canonically spanned by two normalised generalised eigenforms
\[
\tilde{\gamma}_1 = q^3 + \cdots + 0 \cdot q^3 + \cdots \quad \text{and} \quad \tilde{\gamma}_2 = q^2 + 0 \cdot q^3 + \cdots + 0 \cdot q^8 + \cdots.
\]
Note that the natural scaling of the forms output by our algorithm is with leading Fourier coefficients equal to 1. By Theorem 6.1 one expects that for \( \ell \) inert in \( K \), or \( \ell \) split in \( K \) but irregular, we have \( a_\ell(\tilde{\gamma}_1) = 0 \); and for \( \ell \) split in \( K \) we have that
\[
a_\ell(\tilde{\gamma}_1) = \frac{\log_p(u_\ell)}{\log_p(u_3)}
\]
where \( u_\ell \) is a fundamental \( \ell \)-unit in \( K \) (the logarithm of this is well-defined up to sign). We checked this to 20-digits of 17-adic precision for primes \( \ell < 1000 \). Further, one expects that
\[
a_\ell(\tilde{\gamma}_2) = \frac{R_{\psi}(\ell)}{R_{\psi}(2)} \quad \text{for} \quad \ell \quad \text{inert in} \quad K, \quad \text{and} \quad a_\ell(\tilde{\gamma}_2) = 0 \quad \text{for} \quad \ell \quad \text{split in} \quad K.
\]
We checked this for all split primes \( \ell < 30,000 \) and for the inert primes \( \ell = 2, 3, 11 \) and 23, constructing \( R_\psi(\ell) \) using the unit \( u \) and \( \ell \)-unit \( u_\ell \) above.

7. RM forms

We now turn to the RM setting where \( F \) is a real quadratic field and
\[
V_g = \text{Ind}_F^Q \psi_g,
\]
where \( \psi_g : \text{Gal}(\bar{F}/F) \to L^\times \) is a finite order character of mixed signature. Letting \( \psi'_g \) denote the character deduced from \( \psi_g \) by composing it with the involution in \( \text{Gal}(F/\mathbb{Q}) \), the ratio \( \psi := \psi_g/\psi'_g \) is a totally odd \( L \)-valued ring class character of \( F \).

As before, let \( H \) denote the ring class field of \( F \) which is fixed by the kernel of \( \psi \), and set \( Z := \text{Gal}(H/F) \) and \( G := \text{Gal}(H/\mathbb{Q}) \). Just as in the previous section,
\[
W_g^0 = \chi_K \oplus Y_g, \quad Y_g := \text{Ind}_K^Q \psi,
\]
and we can set
\[
w(1) = \begin{pmatrix} \log_p(u_F) & 0 \\ 0 & -\log_p(u_F) \end{pmatrix},
\]
where \( u_F \) is a fundamental unit of \( F \).

If \( \ell \) is split in \( K/\mathbb{Q} \), it is easy to see that the vector \( w(\ell) \) is proportional to \( w(1) \), and hence that
\[
2\mathcal{M}(\ell) = 0.
\]
If \( \ell \) is inert in \( K \), let \( U_\ell \) and \( U_p(\ell) \) denote the subspaces \((\mathcal{O}_H^\times \otimes Y_g)^G_{\mathbb{Q}} \) and \((\mathcal{O}_H[1/\ell]^\times \otimes Y_g)^G_{\mathbb{Q}} \). The dimensions of these spaces are 0 and 1 respectively. Choose a prime \( \lambda \) of \( H \) above \( \ell \), and let \( u_\lambda \) and \( u_\psi(\ell) \) be the elements of \( \mathcal{O}_H[1/\ell]^\times \) determined by the relations

\[
(u_\lambda) = b_\lambda \lambda + c_\lambda \tau \lambda, \quad u_\psi(\ell) = e_\psi(u_\lambda), \quad u_\psi'(\ell) = \tau u_\psi(\ell),
\]

where

\[
\varrho_g(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}.
\]

The \( \varrho \)-adic logarithms

\[
\log_\varrho(u_\psi(\ell)), \quad \log_\varrho(u_\psi'(\ell))
\]

are well-defined invariants of \( \ell \) and \( g \) which do not depend on the choice of a prime \( \lambda \) lying above \( \ell \), and

\[
w(\ell) = \frac{1}{2} \begin{pmatrix} 0 & \log_\varrho(u_\psi(\ell)) \\ \log_\varrho(u_\psi'(\ell)) & 0 \end{pmatrix}.
\]

It follows that

\[
\mathfrak{M}(\ell) = \log_\varrho(u_F) \times \begin{pmatrix} 0 & \log_\varrho(u_\psi(\ell)) \\ -\log_\varrho(u_\psi'(\ell)) & 0 \end{pmatrix}.
\]

**Theorem 7.1.** The space \( S_1^{(p)}(N, \chi)[[g_\alpha]]_0 \) has a canonical basis \((g_1^\flat, g_2^\flat)\) which is characterised by the properties:

(i) The Fourier coefficients of \( g_1^\flat \) and \( g_2^\flat \) are 0 at all primes \( \ell \nmid Np \) that are split in \( F \).

(ii) If \( \ell \) is inert in \( F \), then

\[
a_\ell(g_1^\flat) = \log_\varrho(u_\psi(\ell)), \quad a_\ell(g_2^\flat) = \log_\varrho(u_\psi'(\ell)).
\]

**Proof.** This follows directly from Theorem 5.3 in light of equations (56) and (57). \( \Box \)

**Example 7.2.** Let \( \chi_8 \) and \( \chi_7 \) denote the quadratic characters of conductors 8 and 7, respectively, and define \( \chi := \chi_8 \chi_7 \). Then \( S_1(56, \chi) \) is one-dimensional and spanned by the form

\[
g = q - q^2 + q^4 - q^7 - q^8 - q^9 + q^{14} + q^{16} + q^{18} + 2q^{23} - \cdots.
\]

We take \( p = 23 \), an irregular prime for \( g \), and compute a basis for the generalised eigenspace modulo \((p^{15}, q^{3000})\). The two dimensional space complementary to the classical space has a natural basis

\[
\tilde{g}_1^\flat = q^3 + \cdots + 0 \cdot q^p + \cdots \quad \text{and} \quad \tilde{g}_2^\flat = q^2 + 0 \cdot q^3 + \cdots + 0 \cdot q^p + \cdots.
\]

Take

\[
g_1^\flat := \frac{1}{2} \cdot \log_\varrho(u_2) \cdot \tilde{g}_1^\flat \quad \text{and} \quad g_2^\flat := \log_\varrho(u_3) \cdot \tilde{g}_2^\flat.
\]

Here \( u_\ell, \ell = 2 \) and 3, denotes a fundamental \( \ell \)-unit of norm 1 in \( \mathbb{Q}(\sqrt{-7}) \) and \( \mathbb{Q}(\sqrt{-56}) \), respectively. One finds that the coefficients at primes \( \ell \) which are split in \( \mathbb{Q}(\sqrt{8}) \) of both forms \( g_1^\flat \) and \( g_2^\flat \) are zero. At inert primes the coefficients of \( g_1^\flat \) are the logarithms of fundamental \( \ell \)-units of norm 1 in \( \mathbb{Q}(\sqrt{-7}) \), and those of \( g_2^\flat \) are the logarithms of fundamental \( \ell \)-units of norm 1 in \( \mathbb{Q}(\sqrt{-56}) \) (such logarithms are well-defined up to sign; interestingly, the forms \( g_j^\flat \) single out a consistent choice of signs).
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