Quantum models of classical systems

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Abstract. Quantum statistical methods that are commonly used for the derivation of classical thermodynamic properties are extended to classical mechanical properties. The usual assumption that every real motion of a classical mechanical system is represented by a sharp trajectory is not testable and is replaced by a class of fuzzy models, the so-called maximum entropy (ME) packets. The fuzzier are the compared classical and quantum ME packets, the better seems to be the match between their dynamical trajectories. Classical and quantum models of a stiff rod will be constructed to illustrate the resulting unified quantum theory of thermodynamic and mechanical properties.

1. Introduction

There are some features of the classical world that seem to be incompatible with quantum mechanics:

**Realism** Properties such as position and momentum can be ascribed to a chair, say, independently of whether they are observed or not.

**Sharp Trajectories** By a common interpretation of classical mechanics, the real chair is even at a sharp point of its phase space at each time. Attempts to model this property by a quantum state with minimum uncertainty leads to coherent states that are pure.

**No Superpositions** The chair is never observed in a linear superposition of being, e.g., simultaneously in the kitchen as well as in the bedroom. However, pure states in quantum mechanics can be superposed in this manner.

**Robustness** Measurement of every classical observable can be done in such a way that the state of the observed system is arbitrarily weakly disturbed. However, pure quantum states are not disturbed only by measurements of very few very special observables.

Thus, attempts to solve the problem of Sharp Trajectories aggravate problems of Robustness and of No Superpositions.

There is a vast literature about the problems. Let us list examples of the most popular ideas: macroscopic systems do not obey quantum mechanics [1]; quantum decoherence theory [2]; only coarse-grained operators represent classical measurements [3]; Coleman-Hepp theory [4]; dynamical collapse theory [5, 6]. The list is incomplete.
Our theory is different. It rejects sharp trajectories and seeks quantum mechanical derivation of classical properties possessed by fuzzy mechanical states. The present paper is a short review of [7, 8] as well as of some new results.

2. Hypothesis of high entropy states
To motivate our approach, let us briefly recapitulate some ideas of statistical thermodynamics. Consider rarefied equilibrium gas in a vessel. There is a classical model $S_c$ of this gas offered by phenomenological thermodynamics, called “ideal gas”, and the properties of $S_c$ are examples of classical properties. They are described by thermodynamic quantities such as internal energy $E$, volume $\Omega$, pressure, entropy, temperature, specific heats, etc.

To obtain the values of such quantities from quantum mechanics, we need a quantum model $S_q$ of the gas. As $S_q$, we can choose a system of $N$ spin-zero point particles, each with mass $\mu$, in a deep potential well of volume $\Omega$ with Hamiltonian

$$H = \sum_{k=1}^{N} \frac{\hat{p}_k^2}{2\mu},$$

where $\hat{p}_k$ is the momentum of $k$-th particle in the rest system of $\Omega$. $H$ is then the operator of the internal energy and the classical internal $E$ energy is an average of $H$.

The most important assumption of the quantum model is the choice of state. It is the state that maximises the (von Neumann) entropy for fixed value $E$ of the average of the internal energy. It is called “Gibbs state”. All properties of the $S_c$ can then be calculated from $S_q$ as properties of the Gibbs states.

The main (heuristic) principle of our theory is a generalisation of this idea to all classical properties, including the mechanical ones. Thus, we state the following hypothesis:

**Assumption 1** Let a real system $S$ have a classical model $S_c$. Then, there is a quantum model $S_q$ of $S$ such that all properties of $S_c$ are selected properties of some high-entropy states of $S_q$.

An important reason for accepting this hypothesis is that it suggests ways in which all four problems mentioned in the introduction can be solved. Indeed, the Realism Problem could be approached as follows. Our theory of objective properties of quantum systems [9, 10] justifies the assumption that quantum states are objective. If classical properties are properties of some states of the quantum model, they will also be objective. The No-Superposition Problem is based of some properties of pure quantum states. But high-entropy states are not pure: they cannot be superposed. As for the robustness problem, we can use the fact that very many quantum states correspond to a single classical state. Even if quantum states may be disturbed by observation, the corresponding classical states need not be. Finally, there is no Sharp-Trajectories Problem for thermodynamics. All these points are just suggestions and must be more carefully studied on some mathematically well-defined models.

Assumption 1 might work for thermodynamics, but what could be the high-entropy states for Newtonian mechanics?

3. Classical ME packets
Let us consider the classical mechanical model $S_c$ defined as a system with a single degree of freedom and Hamiltonian

$$H = \frac{p^2}{2\mu} + V(q).$$

The classical equations of motion are

$$\dot{q} = \frac{p}{\mu}, \quad \dot{p} = -\frac{dV}{dq}$$ (2)
and their solution is a sharp trajectory

\[ q = q(t), \quad p = p(t) \]

for every initial values \( q(0) \) and \( p(0) \).

Let us choose the corresponding quantum model \( S_q \) to be a system of one degree of freedom with position operator \( q \), momentum operator \( p \) and spin 0. Let the Hamiltonian be

\[ H = \frac{p^2}{2\mu} + V(q). \tag{3} \]

The Heisenberg equations of motion are

\[ \dot{q} = \frac{p}{\mu}, \quad \dot{p} = -\frac{dV}{dq}. \tag{4} \]

Then the time dependence of position and momentum averages \( Q = \langle q \rangle \) and \( P = \langle p \rangle \) in a state \( |\psi\rangle \) is

\[ \dot{Q} = \frac{P}{\mu}, \quad \dot{P} = -\left\langle \frac{dV}{dq} \right\rangle. \]

To evaluate the right-hand side of the second equation, let us expand the potential function in powers of \( q - Q \):

\[ V(q) = V(Q) + (q - Q) \frac{dV}{dQ} + \frac{1}{2}(q - Q)^2 \frac{d^2V}{dQ^2} + \ldots \]

so that

\[ \frac{dV}{dq} = \frac{dV}{dQ} + (q - Q) \frac{d^2V}{dQ^2} + \frac{1}{2}(q - Q)^2 \frac{d^3V}{dQ^3} + \ldots. \]

If we take the average of the last equation and use relations \( \langle (q - Q) \rangle = 0 \) and \( \langle (q - Q)^2 \rangle = \Delta Q^2 \), where \( \Delta Q \) is the variance of \( q \) in state \( |\psi\rangle \), we obtain

\[ \left\langle \frac{dV}{dq} \right\rangle = \frac{dV}{dQ} + \frac{1}{2}\Delta Q^2 \frac{d^3V}{dQ^3} + \ldots. \]

Let us assume that coordinate \( q \) and momentum \( p \) of \( S_c \) are obtained from the quantum model by formulas

\[ q = Q, \quad p = P. \]

Then, already for potentials of the third order, the quantum equations of motion for averages deviate from classical equation of motion for sharp trajectories. This deviation would be negligible for small \( \Delta Q \), that is, the spread of the wave packet \( |\psi\rangle \) over the space must be as small as possible. However, if the variance \( \Delta P \) is large, \( \Delta Q \) will quickly increase with time. This implies that the minimum-uncertainty wave packets may give the best approximation to classical sharp trajectories.

Let us stop here and ask: what is the reason for trying to get sharp trajectories from quantum mechanics? Clearly, it is the popularity of the specific form of classical realism mentioned in the Introduction: a real mechanical system possesses a sharp position and momentum at any instant of time. Let us call this assumption Sharp Trajectory Hypothesis (STH). There are many tacitly assumed consequences of STH, for example that a probability distributions on phase space is only an expression of insufficient knowledge of the real state.

However, there is no evidence supporting STH: indeed, as yet, any real observation of macroscopic bodies has been compatible with

\[ 2\Delta Q\Delta P \gg \hbar, \tag{5} \]

where \( \gg \) represents many orders of magnitude. This is well known but there can be two attitudes to Eq. (5):
(i) With improving techniques, the left-hand side of Eq. (5) will approach zero. This must be false if quantum mechanics holds true.

(ii) Sharp trajectory is just a handy model of a real, fuzzy, one. That is, it lies within a tube associated with the fuzzy trajectory. But then, a more realistic model of any Newtonian motion would be a probability distribution.

References [9, 8] assume the second attitude. For us, the most important consequence is that it is sufficient to approximate fuzzy Newtonian trajectories by quantum mechanics, where fuzzy trajectories are some probability distributions on the phase space of the system. Such a theory of classical properties can do without pure states. Of course, this probability distribution is not completely knowable and measurable: in any case, the sharp points do not exist [11, 12]. The fact that the points of the phase space do exist mathematically and must be used for mathematical description of a real state is only an unrealistic feature of Newtonian mechanics.

Thus, instead of a sharp point of the phase space a distribution on the phase space can be considered as the real state of a mechanical system. It is determined by preparation similarly as in quantum theory. In this way, we preserve the realism (for more details on realism, see [10]) as such but change the form of it as expressed by STH.

This opens the problem to application of Bayesian methods, see, e.g., [13]. These methods recommend maximising entropy in the cases of missing knowledge. Let us define a fuzzy state called maximum-entropy packet (ME packet) as a phase-space distribution maximising entropy for given averages and variances of mechanical state coordinates.

More precisely, for the classical model $S_c$, we consider the states described by distribution function $\rho(q,p)$ on the phase space spanned by $q$ and $p$. The function $\rho(q,p)$ is dimensionless and normalised by
\[
\int \frac{dq}{v} \frac{dp}{v} \rho = 1 ,
\]
where $v$ is an auxiliary phase-space volume to make $\rho$ dimensionless. The entropy of $\rho(q,p)$ can be defined by
\[
S := -\int \frac{dq}{v} \frac{dp}{v} \rho \ln \rho .
\]
The value of entropy will depend on $v$ but most other results will not. Classical mechanics does not offer any idea of how to fix $v$. We shall get its value from quantum mechanics.

**Definition 1** ME packet is the distribution function $\rho$ that maximizes the entropy subject to the conditions:
\[
\langle q \rangle = Q , \quad \langle q^2 \rangle = \Delta Q^2 + Q^2 ,
\]
and
\[
\langle p \rangle = P , \quad \langle p^2 \rangle = \Delta P^2 + P^2 ,
\]
where $Q$, $P$, $\Delta Q$ and $\Delta P$ are given values.

We have used the abbreviation
\[
\langle x \rangle = \int \frac{dq}{v} \frac{dp}{v} x \rho .
\]

The explicit form of $\rho$ can be found using the Lagrange-multiplier and partition-function method [7]:

**Theorem 1** The distribution function of the classical ME packet for a one-degree-of-freedom system with given averages and variances $Q$, $\Delta Q$ of coordinate and $P$, $\Delta P$ of momentum, is
\[
\rho[Q, P, \Delta Q, \Delta P](q,p) = \left( \frac{v}{2\pi} \right) \frac{1}{\Delta Q \Delta P} \exp \left[ -\frac{(q-Q)^2}{2\Delta Q^2} - \frac{(p-P)^2}{2\Delta P^2} \right] .
\]
In this way, to describe the mechanical degrees of freedom, we need twice as many variables as the standard mechanics. The doubling of state coordinates is due to the necessity to define a fuzzy distribution rather than a sharp trajectory.

The model can be generalised to any number of degrees of freedom. Also, the ME packet could be defined by different pairs of conjugate variables. It seems plausible that our main results would then remain valid.

4. Quantum ME packets

**Definition 2** State $\mathcal{T}$ of quantum model $S_q$ with one degree of freedom that maximizes von Neumann entropy

$$S = -\text{tr}(\mathcal{T} \ln \mathcal{T})$$

under the conditions

$$\text{tr}[\mathcal{T}_q] = Q, \quad \text{tr}[\mathcal{T}_q^2] = Q^2 + \Delta Q^2,$$

$$\text{tr}[\mathcal{T}_p] = P, \quad \text{tr}[\mathcal{T}_p^2] = P^2 + \Delta P^2,$$

where $Q$, $P$, $\Delta Q$ and $\Delta P$ are given numbers, is called quantum ME packet.

The following theorem can be proved by the method of Lagrange multipliers and partition function, but the proof is non-trivial because of non-commuting factors [7]:

**Theorem 2** The state operator of the ME packet of a one-degree-of-freedom system with given averages and variances $Q$, $P$, $\Delta Q$ and $\Delta P$ is

$$\mathcal{T}[Q, P, \Delta Q, \Delta P] = \frac{2}{\sqrt{\nu^2 - 1}} \exp\left(-\frac{\nu}{2} \ln \frac{\nu + 1}{\nu - 1} K\right),$$

where

$$K = \frac{(q - Q)^2}{2\Delta Q^2} + \frac{(p - P)^2}{2\Delta P^2}$$

and

$$\nu = \frac{2\Delta P \Delta Q}{\hbar}.$$  \hspace{1cm} (10)

Generalisation to any number of degrees of freedom is easy. It is amusing to observe how the forms of Eqs. (8) and (9) approach each other in the limit $\Delta P \Delta Q \to \infty$. Indeed,

$$\lim_{\nu \to \infty} \frac{\nu}{2} \ln \frac{\nu + 1}{\nu - 1} = 1.$$

The entropy of state (9) can be shown [7] to be an increasing function of $\nu \in (1, \infty)$ diverging for $\nu \to \infty$. For $\nu = 1$ (minimum quantum uncertainty), $\mathcal{T}$ is a pure state with wave function

$$\psi(q) = \left(\frac{1}{\pi \Delta Q^2}\right)^{1/4} \exp\left[-\frac{1}{4\Delta Q^2}(q - Q)^2 + \frac{iPq}{\hbar}\right].$$

This is just a Gaussian wave packet and the entropy is zero. Thus, quantum ME packets are generalization of Gaussian wave packets.
5. Comparing classical and quantum evolutions

Let us consider the time evolution of the averages and variances for ME packet (8) with initial data \( Q, P, \Delta Q, \Delta P \) at \( t = 0 \) and let us define the classical trajectory of the classical model \( S_c \) by the quadruple \( Q_c(t), P_c(t), \Delta Q_c(t), \Delta P_c(t) \). Let \( Q_q(t), P_q(t), \Delta Q_q(t), \Delta P_q(t) \) be an analogous trajectory for the quantum model \( S_q \) starting in state (9). Each of the two trajectories is described by four real functions so that they can be compared.

Let us first study the special case of at most quadratic potential:

\[
V(q) = V_0 + V_1 q + \frac{1}{2} V_2 q^2 ,
\]

where \( V_k \) are constants with suitable dimensions. If \( V_1 = V_2 = 0 \), we have a free particle, if \( V_2 = 0 \), it is a particle in a homogeneous force field and if \( V_2 \neq 0 \), it is a harmonic or anti-harmonic oscillator. In these cases, exact solutions can be found:

\[
Q_c(t) = Q_q(t) = f_0(t) + Q f_1(t) + P f_2(t) ,
\]

\[
\Delta Q_c(t) = \Delta Q_q(t) = \sqrt{f_1^2(t) \Delta Q^2 + f_2^2(t) \Delta P^2} ,
\]

\[
P_c(t) = P_q(t) = g_0(t) + Q g_1(t) + P g_2(t) ,
\]

\[
\Delta P_c(t) = \Delta P_q(t) = \sqrt{g_1^2(t) \Delta Q^2 + g_2^2(t) \Delta P^2} .
\]

If \( V_2 \neq 0 \), the functions are

\[
f_0(t) = -\frac{V_1}{V_2} (1 - \cos \omega t) , \quad f_1(t) = \cos \omega t , \quad f_2(t) = \frac{1}{\xi} \sin \omega t ,
\]

\[
g_0(t) = -\frac{V_1}{V_2} \sin \omega t , \quad g_1(t) = -\xi \sin \omega t , \quad g_2(t) = \cos \omega t ,
\]

where

\[
\xi = \sqrt{\mu V_2} , \quad \omega = \sqrt{\frac{V_2}{\mu}} .
\]

If \( V_2 = 0 \), we obtain

\[
f_0(t) = -\frac{V_1}{2 \mu} t^2 , \quad f_1(t) = 1 , \quad f_2(t) = \frac{t}{\mu} ,
\]

\[
g_0(t) = -V_1 t , \quad g_1(t) = 0 , \quad g_2(t) = 1 .
\]

Hence, for at-most-quadratic potentials, classical and quantum trajectories coincide. However, already for a third order potentials, there are non-trivial quantum corrections. For example, if \( V = V_3 q^3 / 6 \), one can use Heisenberg equations of motion (4) to calculate the 9-th time derivative of \( p \) where there is finally a term,

\[
\frac{d^9 p}{dt^9} = -\frac{125}{4} \frac{V_3^5}{\mu^4} q^4 ,
\]

that has a quantum correction, namely

\[
\langle q^6_q \rangle = \langle q^6_c \rangle + 9 \Delta Q^6 \nu^{-1} - 3 \Delta Q^6 \nu^{-3} ,
\]

where \( \nu \) is defined by Eq. (10) and

\[
\langle q^6_c \rangle = Q^6 + 15 Q^4 \Delta Q^2 + 45 Q^2 \Delta Q^4 + 15 \Delta Q^6 .
\]

One can see from these equations that the quantum correction becomes negligible in the limit \( \Delta Q \to \infty \). In general, one can show:
Theorem 3 For general polynomial potential, the classical and quantum trajectories satisfy:

\[ \lim_{\Delta Q \to \infty, \Delta P \to \infty} \frac{Q_q(t) - Q_c(t)}{Q_c(t)} = 0 \quad \lim_{\Delta Q \to \infty, \Delta P \to \infty} \frac{P_q(t) - P_c(t)}{P_c(t)} = 0, \]

and

\[ \lim_{\Delta Q \to \infty, \Delta P \to \infty} \frac{\Delta Q_q(t) - \Delta Q_c(t)}{\Delta Q_c(t)} = 0 \quad \lim_{\Delta Q \to \infty, \Delta P \to \infty} \frac{\Delta P_q(t) - \Delta P_c(t)}{\Delta P_c(t)} = 0, \]

for all \( t \) for which the formulas make sense.

The proof is based on the calculation of averages of polynomials in \( q \) and \( p \) described in [7] and will be published elsewhere.

Hence, the fuzzier the compared ME packets are, the better their dynamical evolutions match each other. In other words, the classical limit for mechanical degrees of freedom is

\[ \Delta Q \to \infty, \quad \Delta P \to \infty. \] (11)

This seems to contradict the usual belief that the classical physics is best approximated by minimum-uncertainty (\( \nu = 1 \)) quantum states. But the explanation of this paradox is simple. The two answers to the question which states best approximate classical physics are different because the questions asked are, in fact, different: the first one compares two fuzzy states, the second one compares a quantum state with a sharp classical trajectory.

The conjecture that the classical limit is given by equation (11) seems to be plausible, but it has yet to be shown for non-polynomial potential function, such as the Coulomb potential (Kepler orbits).

6. Mechanical and thermostatic properties unified

We have tried to prove that classical mechanical properties of an object can be obtained from its quantum model as properties of high-entropy quantum states. However, this also holds for classical thermostatic properties, such as internal energy, temperature, entropy, specific heats etc., which suggests that the quantum theory of classical properties can be based on a single principle. In the present section, we try to show in more detail how such “unified” theory could look like.

We use a very simple model so that its quantum equations are exactly solvable and we can concentrate on conceptual questions. As a real object \( S \), consider a thin stiff rod of mass \( M \) and length \( L \) extended and moving freely in one space dimension. Let its classical model \( S_c \) be a one-dimensional continuum. Its (classical) state is determined by the values of 5 quantities: internal energy \( E_{\text{int}} \), average \( X \) and variance \( \Delta X \) of its centre-of-mass coordinate as well as average \( P \) and variance \( \Delta P \) of its total momentum.

Let the quantum model \( S_q \) be a chain of \( N + 1 \) particles, each of mass \( \mu \). We denote the position operator of the \( n \)-th particle by \( x_n \) and that of its momentum by \( p_n \), \( n = 1, \ldots N + 1 \). Let the Hamiltonian of the quantum model be

\[ H = \frac{1}{2\mu} \sum_{n=1}^{N+1} p_n^2 + \frac{\kappa^2}{2} \sum_{n=2}^{N+1} (x_n - x_{n-1} - \xi)^2. \]

The potential represents nearest-neighbour elastic forces, \( \kappa \) being the oscillator strength and \( \xi \) the equilibrium inter-particle distance. The algebra of observables of \( S_q \) is generated by \( x_n \) and \( p_n \), a set of \( 2N + 2 \) operators. We assume that \( N \approx 10^{23} \). This implies that the quantum state contains much more information than the classical one.
A linear (in fact, Fourier) transformation of variables $x_n$ and $p_n$ to normal modes $u_n$ and $q_n$ diagonalizes the Hamiltonian \([9, 8]\). Moreover, it becomes the sum of the total momentum part and the internal energy part $E_{\text{int}}$ (see \([9, 8]\)):

$$H = \frac{1}{2M}P^2 + E_{\text{int}},$$

where $M = (N + 1)\mu$ is the total mass of the chain, $P$ its total momentum,

$$E_{\text{int}} = \frac{1}{2\mu} \sum_{m=1}^{N} q_m^2 + \frac{\mu}{2} \sum_{m=1}^{N} \omega_m^2 u_m^2$$

and $\omega_m$, $m = 1, \ldots, N$ are “phonon” frequencies:

$$\omega_m = \frac{2\kappa}{\sqrt{\mu}} \sin \left( \frac{\pi}{2(N + 1)} m \right).$$

The mechanical evolution thus decouples from the thermodynamics.

As the state of $S_p$, the tensor product $T_{\text{therm}} \otimes T_{\text{mech}}$ can, therefore, be chosen. Next, we apply the unifying principle: $T_{\text{therm}}$ is the maximum entropy quantum state for a given value $E$ of the averages of $E_{\text{int}}$ and $T_{\text{mech}}$ is the maximum entropy state for given averages of $X$ and $P$ and variances $\Delta X$ and $\Delta P$. It then follows that $T_{\text{therm}}$ is the Gibbs state and $T_{\text{mech}}$ is an ME packet.

The phonons of species $m$ form statistically independent subsystems. Hence, the Gibbs state factorizes and the factors are Gibbs states $T_m$ of the species:

$$T_m = \sum_{r=0}^{\infty} |r\rangle P^m_r \langle r|,$$

where $r$ is the number of phonons of species $m$,

$$P^m_r = \left( 1 - e^{-\lambda \omega_m} \right) e^{-\lambda \omega_m r}$$

and $\lambda$ is the Lagrange multiplier of the variational problem for the conditional maximum of entropy. The variational principle couples the value of $\lambda$ with the value of the energy average $E$. As $\lambda$ is interpreted as $1/kT$, $k$ being the Boltzmann constant and $T$ the temperature, the relation between internal energy and temperature results. The average number $\langle r\rangle$ of photons in state $T_m$ is the Bose distribution.

All properties of the classical model (such as the temperature and length of the rod, its dynamical trajectory etc.) have been obtained, in a good approximation, from the quantum one (\([9, 8]\)). For example the length $L$ of the rod is the average of a natural rod-length operator $x_{N+1} - x_1$. The calculation in \([9, 8]\) yields

$$L = N\xi,$$

This is independent of the parameter $E$ of the Gibbs state. Hence, the model describes a rigid rod. The relative variances of the internal energy and length are indirectly proportional to $N$. 

8
7. Conclusion and outlook
Our results suggest that there is a unified theory for both thermostatic and mechanical properties. It is based on the assumption that the states of quantum system that exhibit classical properties are some states with high entropy.

The fuzzier are the compared mechanical states, the better is the match between classical and quantum mechanical trajectories, if the \( \nu \)-Conjecture holds true. This would confirm the feeling that quantum mechanics is more accurate and finer than Newtonian mechanics. We hope to be able to prove this statement for non-polynomial potentials later.

The paper suggests promising ideas of how all four conceptual problems can be solved. More detailed models that would describe such solutions for some simple cases ought to be constructed.

The project is in its beginnings. Only extremely simple models have been studied. Also, a generalisation of the idea to classical electro- and magnetostatic properties, as well as a generalisation to the relativistic classical electrodynamics is missing as yet.

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References
[1] Leggett A J 2002 J. Phys.: Condens. Matter 14 R415
[2] Schlosshauer M 2004 Rev. Mod. Phys. 76 1267
[3] Kampen van N G Physica 1995 194 542
[4] Hepp K 1972 Helvetica Phys. Acta 45 237
[5] Pearle P 1989 Phys. Rev. A 39 2277
[6] Ghirardi G C 2002 The Stanford Encyclopedia of Philosophy http://plato.stanford.edu/entries/qm-collapse
[7] Hájíček P 2009 Found. Phys. 39 1072
[8] Hájíček P 2013 Entropy 15 789
[9] Hájíček P and Tolar J 2009 Found. Phys. 39 411
[10] Hájíček P 2013 J. Phys.: Conf. Ser. 442 012043
[11] Exner F 1922 Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften (Deuticke: Leipzig)
[12] Born M 1955 Phys. Blätter 11 49
[13] Jaynes E T 2003 Probability Theory. The Logic of Science (Cambridge University Press: Cambridge, UK)