Accelerated expansion of a universe containing a self-interacting Bose–Einstein gas

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Received 15 June 2009, in final form 28 January 2010
Published 2 March 2010
Online at stacks.iop.org/CQG/27/065012

Abstract

Acceleration of the universe is obtained from a model of non-relativistic particles with a short-range attractive interaction, at low enough temperature to produce a Bose–Einstein condensate. Conditions are derived for negative-pressure behavior. In particular, we show that a phantom-accelerated regime at the beginning of the universe solves the horizon problem, consistently with nucleosynthesis.

PACS numbers: 98.80.c, 98.80.cq, 95.36.+x

(Some figures in this article are in colour only in the electronic version)

1. Introduction

There is observational evidence that the universe is undergoing an accelerated expansion driven by a cosmological constant or some form of energy that violates the strong energy condition ($\rho + 3p > 0$, $\rho$ and $p$ being the energy density and pressure, respectively, of the fluid conforming the universe) [1]. The observational data seem to admit models that even assume that the universe is dominated by a kind of fluid that violates the dominant energy condition ($\rho + p > 0$). This fluid is known in the literature as the phantom energy [2].

Many types of models have been proposed in recent years to explain the observed accelerated expansion: the cosmological constant, which presents a fine-tuning problem [3]; quintessence models [4] and quintessential inflation fields [5]; scalar-field models [6]; chameleon fields [7]; K-essence models [8]; modified gravity models [9]; the feedback of nonlinearities into evolution equations [10]; Chaplygin gases [11]; tachyons [12]; phantom dark energy and ghost condensates [2]; de Sitter vacua with flux compactification in string theory [13]; cyclic universe [14]. A more complete list and discussion can be found in [15]. Although all the models predict an accelerated expansion that is in good agreement with the observational data, few of them propose a microscopic understanding of dark energy.
Another stage of accelerated expansion has been proposed over the years in an attempt to explain the horizon, flatness and unwanted relic problems: inflation [16–18]. Most inflationary models assume that the expansion of the universe is dominated by a scalar field with a slow-roll potential, so that the scale factor evolves nearly exponentially with time. During the inflationary stage of the expansion the scale factor increases several orders of magnitude and the relic radiation energy density decreases dramatically. To allow nucleosynthesis to happen in the inflationary scenario, a reheating process should be introduced: the inflaton field decays into conventional fields by means of the potential. After reheating, the universe is dominated by radiation.

Given the lack of information about the nature of the dark energy, thermodynamics and holography may be helpful. As examples of this approach, Pollock and Singh in [19] study the thermodynamics of the de Sitter spacetime and quasi-de Sitter spacetime. One of the quasi-de Sitter models studied presents phantom-like behavior and the entropy associated with it is negative defined. In [20], an extension of these results is found for a more general phantom-type spacetime leading to negative entropies as well. In [21], a different approach is given and positive entropies are found for the phantom fluid by assuming that its temperature is negative defined. Babichev et al demonstrate in [22] that the mass of a black hole decreases due to the accretion of phantom energy. Making use of this result, in [23] the second law of thermodynamics (dS_{total}/dt > 0) is studied in a model of a universe dominated by a phantom fluid in the presence of black holes (the density of black holes being much smaller than the density of phantom dark energy), concluding that it is not preserved inside the event horizon.

Another approach applies local thermodynamics [24]. In this paper we use it to study a Bose–Einstein gas of self-interacting particles that produces acceleration applying statistical mechanics. Thus, our model has a detailed microscopical description. Bose–Einstein condensates have been widely studied in cosmology. Better candidates for cold dark matter (axions, WIMPs, scalar fields) can be found in the universe in the form of Bose–Einstein condensates [25]. Also, the quantum field equations describing the Bose–Einstein condensate allow us to test experimentally some of the predictions of semiclassical quantum gravity models of general relativity [26]. Here we demonstrate how accelerated expansion can be obtained by considering that the universe contains a Bose–Einstein gas of self-interacting particles. Specifically, the interaction between the particles of the gas gives accelerated solutions (with $\rho + 3p < 0$, and even phantom-like solutions with $\rho + p < 0$). It is well known in the literature that the presence of a self-interaction between elemental particles can cause an accelerated expansion of the universe [27]. In addition, the Bose–Einstein gas is a feasible option to describe both dark matter and dark energy, in contrast to radiation or ordinary non-relativistic matter. We also demonstrate how this model can be applied to describe the early stage of accelerated expansion, solving the horizon problem and respecting the nucleosynthesis scenario. In a future paper, we will demonstrate how present day accelerated expansion can be achieved in our model.

In section 2, we use expansions within statistical mechanics to obtain the equations of state of this gas. In section 3, we study the dynamics of a Friedman–Robertson–Walker (FRW) universe that contains the interacting Bose gas, making use of the results of the previous section. We obtain the parameter conditions leading to accelerated non-phantom and phantom solutions. In particular, we study the dynamics of a FRW containing the gas, and then with radiation. Then, we show how the horizon problem is solved for solutions with $\rho + p < 0$. We also investigate how the model matches the standard cosmology, respecting the nucleosynthesis scenario and producing adequate fluctuations. Finally, we study the stability of the model. In section 4, we summarize our findings.

We use units with $\hbar = k_B = c = 1$. 

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2. Short-range interactive Bose gas

The thermodynamic potential $\Omega$ of a Bose gas, described by single-particle quantum states with energy levels $\epsilon_k$ associated with momentum $k$ [28], is given by

$$\Omega = T \sum_k \log(1 - e^{(\mu - \epsilon_k)/T}),$$

where $T$ is the temperature, and $\mu$ the chemical potential. It leads to the average occupation number

$$\bar{n}_k = \frac{1}{e^{(\epsilon_k - \mu)/T} - 1},$$

where $\epsilon_k$ contains the kinetic energy of a particle with momentum $k$ and mass $m$; considering phase space, the number of particles associated with a given state is

$$dN_k = g V \left(\frac{2\pi}{\hbar}\right)^3 \bar{n}_k d^3k,$$

where for spin-zero particles the degeneracy factor is $g = 1$ and $V$ is the volume. The total number of particles is

$$N = \int dN_k. \tag{4}$$

Consistency of equation (4) requires macroscopic occupation of the lowest state at low enough temperature $T$. In the free case, this starts at the critical temperature $T_c$ such that

$$N/V = \frac{gV(T_c,m)^{3/2}}{\sqrt{2\pi}^2} \int_0^\infty dz \frac{z^{1/2}}{e^{z} - 1},$$

or equivalently

$$T_c = 3.31 \left(\frac{N}{V}\right)^{2/3} \frac{1}{g^{2/3}m}.$$ \tag{5}

The chemical potential remains constant and zero for $T < T_c$; the latter condition is required for the condensate phase to be present. In addition, a non-relativistic description is assured if $m \gg T_c$, or equivalently,

$$m \gg \left(\frac{N}{V}\right)^{1/3}. \tag{6}$$

A short-range two-particle interaction $V(x)$ modifies free-particle behavior. We consider temperatures below the critical temperature, $T \leq T_c$. Near $T_c$ and $T < T_c$, the single-particle energy is given (to first order) by [31]

$$\epsilon_k = \frac{k^2}{2m} + 2v'0 n_{\epsilon > 0}, \tag{8}$$

where the second term is the potential energy of $N_{\epsilon > 0}$ non-condensate particles associated with $V(x)$, $n_{\epsilon > 0} = N_{\epsilon > 0}/V$, and

$$v'_0 = \int d^3x V(x). \tag{9}$$

For equation (8), the chemical potential is

$$\mu = 2n_{\epsilon} v'_0, \tag{10}$$

instead of the $\mu = 0$ value for a free gas with $T < T_c$. The value of $T_c$ in equation (5) and the constancy of $\mu$ remain valid also in this case.
A similar constant potential term is approximated near total condensation $T \sim 0$ [28] with $n_c = N_c/V$, $N_c$ being the condensate particle number, for which

$$\epsilon_k = \frac{k^2}{2m} + v_0' n_c,$$

and

$$\mu = n_c v_0'.$$

The presence of a factor 2 within equation (10) expresses the exchange effect, which accounts for the interaction of different states, while these corrections are absent to this order for the condensate. We use the limits in equation (8) for $T$ near $T_c$ and equation (11) for $T$ near 0 to construct an interpolated potential term. This takes into account the potential particle exchange between the condensate and non-condensate components, so an appropriate average is

$$v_0 = v_0' \left[ \left( N_c^2 + 2(N - N_c)^2 + 2N_c (N - N_c) \right) / N^2 \right] (N = N_c + N_{>0}).$$

One way to derive the energy corresponding to $\Omega_1$ is using the thermodynamic identity

$$E = -\mu \frac{\partial \Omega}{\partial \mu} - T \frac{\partial \Omega}{\partial T} + \Omega.$$

Another way to obtain it in the interactive case is to substitute the single-particle distribution into the Bose distribution in equation (3). Then,

$$E = \frac{gVm^{3/2}}{\sqrt{2\pi}^2} \int d\epsilon \epsilon^{3/2} \frac{1}{e^{(\epsilon - \mu)/T} - 1} + \frac{1}{2V} \sum_i v_0,$$

$$\approx \frac{gVm^{3/2}}{\sqrt{2\pi}^2} T^{3/2} \int_0^\infty dz z^{3/2} e^{-z} - 1 + \frac{v_0}{2V} N^2,$$

where the second term in $E$ sums over pairs of interactions averaged over volume from equation (9), and equation (14) takes the thermodynamic limit.

The pressure can be obtained from the definition $p = -\left( \frac{\partial E}{\partial V} \right)_{N,S}$ and is

$$p = \frac{2}{3} E + \frac{1}{6} v_0 N^2 / V^2.$$

We obtain the entropy of the gas $S$ in terms of the energy $E$, number of particles $N$ and volume $V$ from the well-known thermodynamic relation

$$TS = E + pV - N \mu,$$

where the temperature is $T = \left( \frac{\partial E}{\partial S} \right)_{V,N}$. In our case, equation (17) reads

$$\left( \frac{\partial E}{\partial S} \right)_{V,N} S = \frac{5}{3} \left( E - \frac{1}{2} v_0 N^2 / V \right),$$

which can be integrated to

$$S = C V^\frac{2}{5} \left( E - \frac{1}{2} v_0 N^2 / V \right)^{\frac{7}{5}},$$

where $C = \frac{1}{5} (128g)^{2/5} m^{3/5}$ is set from the interactive-gas entropy dependence on the kinetic energy, which is similar to that of the non-interactive gas entropy in [28].
3. Cosmology of the interacting Bose gas

We consider a flat Friedmann–Robertson–Walker (FRW) universe with line element
\[
ds^2 = -dt^2 + a(t)^2 [dr^2 + r^2 d\Omega_\theta^2],
\]
where \(a(t)\) is the scale factor, and \(t, r\) and \(\Omega_\theta\) are the time, the radius and solid-angle comoving coordinates of the metric, respectively. The Einstein equations \([17]\) read
\[
H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_P^2} \rho, \tag{20}
\]
\[
\dot{\rho} + 3H(\rho + p) = 0, \tag{21}
\]
where \(m_P\) is the Planck mass, and \(\rho\) and \(p\) are the total energy density and total pressure of the fluid conforming the universe, respectively. For a universe containing \(n\) fluids, the total energy density and pressure are \(\rho = \sum_{j=0}^n \rho_j\) and \(p = \sum_{j=0}^n p_j\), where \(\rho_j\) and \(p_j\) are the energy density and pressure of each component \(j\). If there is no interaction between the different components, we can generalize equation (21) to a set of \(n\) independent equations
\[
\dot{\rho}_j + 3H(\rho_j + p_j) = 0. \tag{22}
\]
Combining equations (20) and (21), we find the acceleration parameter
\[
\frac{\ddot{a}}{a} = -\frac{4\pi}{3m_P^2} (\rho + 3p). \tag{23}
\]
One concludes from this equation that the universe is undergoing an accelerated expansion if the fluid violates the strong energy condition \(\rho + 3p > 0\). Additionally, we can consider the dominant energy condition \(\rho + p > 0\) that assures that the speed of sound in the fluid be lower than \(c\).

3.1. Friedmann–Robertson–Walker metric containing the interacting Bose gas

The pressure of the gas, from equation (19), is
\[
p = \frac{2}{3} \rho + \frac{1}{6} \nu_0 \frac{N^2}{V^2}. \tag{24}
\]
In an expanding FRW universe the volume changes with time as \(V = V_i (a(t)/a_i)^3\), where \(V_i\) and \(a_i\) denote the volume and scale factor at an arbitrary reference instant \(t_i\), respectively. Assuming a FRW universe model that expands adiabatically, the above equation reads
\[
p = \frac{2}{3} \rho + \frac{1}{6} \nu_0 n_i^2 \left(\frac{a_i}{a(t)}\right)^6, \tag{25}
\]
where \(n_i\) is the number density of particles (i.e. \(n_i = N / V_i\), where \(N\) is defined in equation (4)) at \(t_i\). Using the equation of state (25) in equation (21), we obtain the evolution of the energy density
\[
\rho = \rho_{ci} \left(\frac{a_i}{a(t)}\right)^5 + \frac{1}{2} \nu_0 n_i^2 \left(\frac{a_i}{a(t)}\right)^6, \tag{26}
\]
where \(\rho_{ci}\) denotes the energy density of the condensate at the reference instant \(t_i\).

The Bose-gas equality \(T = (5/3) E_c / S \) \([28]\) for the condensate energy contribution \(E_c = \rho_{ci} V\) confirms that once the condition \(T < T_c\) is satisfied at some point, it is always satisfied, as both \(T\) (equation (26)) and \(T_c\) (equation (6)) decrease as \(a^{-2}\). This also means the non-relativistic condition in equation (7) can be consistently maintained throughout.
From equation (23), the universe acceleration containing the gas is

\[
\ddot{a} = -\frac{4\pi}{3m_p^2} \left[ 3\rho_{ci} \left( \frac{a_i}{a(t)} \right)^5 + 2\nu_0 n_i^2 \left( \frac{a_i}{a(t)} \right)^6 \right].
\]  

(27)

It is possible to obtain accelerated solutions to the Einstein equations if the interaction term is negative \((\nu_0 < 0)\), i.e. if the interaction is attractive. From this point on, we assume \(\nu_0 < 0\). When the acceleration given by equation (27) is positive at \(a = a_i\) (i.e. \(-2\nu_0 n_i^2 > 3\rho_{ci}\)), the model goes through a transitory stage of accelerated expansion that ends when the interaction term (which scales as \(a^{-6}\)) becomes negligible as compared to the condensate term (which scales as \(a^{-5}\)). We evaluate the scale factor for which the accelerated stage ends (i.e. \(\ddot{a}(a/a_i) = 0\)). This is at

\[
a/a_i = -\frac{2\nu_0 n_i^2}{3\rho_{ci}}.
\]  

(28)

General relativity (GR) requires that the total energy density be positive, i.e. \(-\nu_0 n_i^2 < 2(\rho_{ri} + \rho_{ci})\). Using this condition in equation (28), we conclude that the accelerated expansion stage ends at a scale factor \(a/a_i\) always lower than \(4/3\). The transitory acceleration stage predicted by this model is hence more suitable to describe the acceleration era of the early universe than the present accelerated era. To describe the present era of expansion with this model, the interaction parameter \(\nu_0\) evolves with time, or if the number density of the gas particles \(n_i\) is not conserved with the expansion.

We now explore the conditions under which the transitory accelerated stage predicted by this model can describe the acceleration of the early universe, and in particular, solve the horizon problem.

### 3.2. Friedmann–Robertson–Walker metric containing the interacting Bose gas and radiation

To address this application of the accelerated stage, we add to the model a radiation term with density \(\rho_r = \rho_{ri} (a_i/a(t))^4\) and pressure \(p_r = \rho_r/3\). This term is introduced to fit the standard cosmology, as indeed it eventually dominates the expansion, allowing for nucleosynthesis to happen.

The total energy density and pressure of the universe are

\[
\rho = \rho_{ri} \left( \frac{a_i}{a(t)} \right)^4 + \rho_{ci} \left( \frac{a_i}{a(t)} \right)^5 + \frac{1}{2} \nu_0 n_i^2 \left( \frac{a_i}{a(t)} \right)^6,
\]  

(29)

\[
p = \frac{1}{3} \rho_{ri} \left( \frac{a_i}{a(t)} \right)^4 + \frac{2}{3} \rho_{ci} \left( \frac{a_i}{a(t)} \right)^5 + \frac{1}{2} \nu_0 n_i^2 \left( \frac{a_i}{a(t)} \right)^6.
\]  

(30)

The free parameters in this model are the density of the radiation \(\rho_{ri}\), the energy density of the condensate \(\rho_{ci}\) and the interaction term \(\nu_0 n_i^2\). The parameters are constrained by the condition \(-\nu_0 n_i^2 < 2(\rho_{ri} + \rho_{ci})\), as the total energy density entering the Einstein equations is required to be positive, i.e. \(\rho(a_i) > 0\). The reference term \(a_i\) can be interpreted as the scale factor at which our model starts to be valid.

The acceleration reads

\[
\ddot{a} = -\frac{4\pi}{3m_p^2} \left[ 2\rho_{ri} \left( \frac{a_i}{a(t)} \right)^4 + 3\rho_{ci} \left( \frac{a_i}{a(t)} \right)^5 + 2\nu_0 n_i^2 \left( \frac{a_i}{a(t)} \right)^6 \right].
\]  

(31)

1 In this case, the second term on the right-hand side of equation (25) can be interpreted as an effective viscous pressure. The dynamics of a universe containing a viscous fluid has been widely studied and its presence allows the expansion to accelerate under certain conditions [32, 33].
Similarly to the model in subsection 3.1, it is possible to obtain accelerated solutions to the Einstein equations if \(\nu_0 < 0\). The acceleration eventually becomes negative, as the interaction term scales as \((a/a)^6\), while the contributions of the radiation and the condensate scale as \((a/a)^4\) and \((a/a)^5\), respectively. When the latter terms dominate, the expansion becomes decelerated.

The left and right panels of figure 1 show the evolution of the total density and acceleration, respectively, for three possible types of solutions, which are characterized by the acceleration at \(a_i\). These are as follows:

(i) **Decelerated solutions.** The acceleration given by equation (31) is negative (or zero) at the scale factor \(a = a_i\) when \(\rho(a_i) + 3p(a_i) > 0\), i.e. \(-\nu_0 n_i^2 \leq (3/2)\rho_{ci} + \rho_{ri}\). Then, the universe maintains a decelerated expansion from \(a = a_i\) on.

(ii) **Non-phantom accelerated solutions.** These solutions are obtained when the free parameters fulfill the conditions \((3/2)\rho_{ci} + \rho_{ri} < -\nu_0 n_i^2 < (5/3)\rho_{ci} + (4/3)\rho_{ri}\), where the lower limit is given by the positiveness of the acceleration in equation (31) at the scale factor \(a = a_i\), and the upper limit by the condition \(\rho + p > 0\).

(iii) **Phantom-accelerated solutions.** The acceleration given by equation (31) at \(a_i\) is positive, and the condition \(\rho + p > 0\) is violated at the scale \(a = a_i\), if \(-\nu_0 n_i^2 > (5/3)\rho_{ci} + (4/3)\rho_{ri}\). In this case, the total density grows with the scale factor as \(d\rho/da = -3(\rho + p)/a > 0\).

First, the universe goes through a super-exponential acceleration phase similar to that in the phantom-energy models. When the interaction term is diluted for large values of \(a\), the condition \(\rho + p < 0\) is no longer valid and the energy density becomes a decreasing
function of $a$. Consequently, the energy density presents a maximum $\rho_{\text{max}}$. The scale factor $a_*$ for which $\rho(a_*) = \rho_{\text{max}}$ is

$$
a_* = \frac{-6\upsilon_0 n_i^2}{5\rho_i + \sqrt{25\rho_i^2 - 48\upsilon_0 n_i^2 \rho_i}}. \tag{32}
$$

The plausibility and stability of phantom models have been widely studied [34]. The big-rip problem (infinite expansion of the universe in a finite-time span) in some phantom-energy models that explain the current universe acceleration [2] is not present in our model because the phantom-accelerated stage is transitory and ends before the scale factor diverges.

The scale factor $a$ at which the accelerated expansion stage ends in this scenario is always lower than $\sqrt{2}a_i$ (see appendix A for details).

### 3.3. Solving the horizon problem through the super-exponential expansion regime

Next, after briefly reviewing the horizon problem, we obtain the parameter conditions which solve the horizon problem and are compatible with the nucleosynthesis scenario. Also, the spectrum and evolution of the initial density perturbations is addressed later.

Cosmic microwave background (CMB) photons from different regions of the sky are in thermal equilibrium at temperatures with relative fluctuations up to $10^{-5}$. The most natural explanation for this phenomenon is that photons from different regions had been thermalized by being in causal contact previously. However, the dynamics of a FRW universe containing only radiation and/or non-relativistic matter (dust) cannot describe such a situation as there was no time for those regions to interact before the photons were emitted. In other words, the distance light could travel from the instant when the photons were emitted, $t_{\text{dec}}$, until the photons were released at the instant $t_0$ is much smaller than the horizon distance today ($t_0$), i.e.

$$
a_0 \int_{t_{\text{dec}}}^{t_0} \frac{dt'}{a(t')} = a_0 \int_{a(t_{\text{dec}})}^{a(t_0)} \frac{da'}{a^2 H(a')} \ll a_0 \int_{t_{\text{dec}}}^{t_0} \frac{dt'}{a(t')}, \tag{33}
$$

where $a_0$ represents the scale factor today. This problem is known in the literature as the horizon problem [17, 18].

An alternative description of this problem considers the Hubble radius $1/H$, where $H$ is defined by equation (20). The Hubble radius is closely related to the horizon [35]. A point in space is no longer in causal contact with an observer when its distance to it, $\lambda$, becomes larger than $1/H$. In radiation and dust-dominated universes, $1/H$ increases with a slope with respect to $a$ larger than one. This implies that a point $A$ in space for which $\lambda_A = 1/H_0$ (in causal contact with the observer for the first time today $t_0$) was never before in causal contact with the observer (i.e. $\lambda_A > 1/H(a)$ for any $a < a_0$).

In both the accelerated and phantom-accelerated cases of the FRW model containing radiation and the Bose gas exposed in the previous subsection, there is an initial period for which different length scales $\lambda$ are in causal contact, leaving the Hubble radius and re-entering it later in time, when the universe is no longer accelerated. This process is illustrated in figure 2. For the accelerated region, $d(1/H)/(da)$ is lower than 1 but positive until the expansion becomes decelerated. For the super-exponential accelerated region, $d(1/H)/(da)$ is negative until the expansion becomes decelerated. This allows for a large range of length scales to leave the causal contact region and to reenter it later. Finally, we also note that for the decelerated region this phenomenon is not manifested, as $d(1/H)/(da)$ is always larger than 1.
Figure 2. Two different length scales that evolve with the scale factor as \( \lambda = \lambda_i(a(t)/a_i) \) (black solid lines). The Hubble radius, \( 1/H \), is plotted for different choices of the parameters: (i) decelerated universe with the parameters \( \rho_{ri} = 0.5\rho_i, \rho_{ci} = 1.5\rho_i \) and \( (\nu_0n_i^2)/2 = -\rho_i \) (solid line); (ii) accelerated solution with the parameters \( \rho_{ri} = 0.5\rho_i, \rho_{ci} = 5\rho_i \) and \( (\nu_0n_i^2)/2 = -4.5\rho_i \) (dotted line); (iii) super-exponential accelerated solution with the parameters \( \rho_{ri} = 3\rho_i, \rho_{ci} = 5\rho_i \) and \( (\nu_0n_i^2)/2 = -7\rho_i \) (dashed line). For this plot, \( \rho_i = \rho_{ri} + \rho_{ci} + (\nu_0n_i^2)/2 \).

The particle horizon tends to zero when \( a \) decreases, as for the radiation or the non-relativistic matter FRW models.

It is possible to solve the horizon problem if we consider phantom-accelerated values for the free parameters of our model; specifically, we choose the scale factor \( a_i \) to be small enough so that the Hubble radius is big enough. In this scenario, choosing \( 1/H_i \gg 1/H(a_*) (a_*/a_i) \) defined by equation (32), the distance that light can travel before the photons were released can be large enough to solve the horizon problem because

\[
\int_{a_i}^{a_*} \frac{da'}{a^2 H(a')} \gg \int_{t_0}^{t_{dec}} dt' a(t').
\] (34)

Then, we ensure that there be a wide range of scales that leave the region of causal contact. Figure 3 illustrates an example of a phantom regime for which \( 1/H_i \gg 1/H(a_*) \). This implies that the energy density increases from \( \rho_i \) until it reaches its maximum \( \rho_{max} \approx 10^{60}\rho_i \).

We evaluate the range of the free parameters that solve the horizon problem. It is plausible to assume that \( \rho_{max} \) cannot be larger than the Planck scale, \( m_P \) (GR is valid for \( \rho < m_P^4 \)). We first define the new set of parameters

\[
X_r^* = \frac{\rho_{ri}}{\rho_{max}} \left( \frac{a_i}{a_*} \right)^4, \quad X_c^* = \frac{\rho_{ci}}{\rho_{max}} \left( \frac{a_i}{a_*} \right)^5, \\
X_v^* = \frac{1}{2} \frac{\nu_0n_i^2}{\rho_{max}} \left( \frac{a_i}{a_*} \right)^6.
\] (35)
Figure 3. $1/H$ as a function of $a/a_i$ in a log–log plot for a set of parameters that allow $1/H = 10^{30}/H(a_i)$. The solid lines represent two different length scales that evolve with the scale factor as $\lambda = \lambda_i(a(t)/a_i)$.

$X^*$ has been defined to be positive when the interaction is attractive. These parameters are the contribution of each energy component at $a_*$ to $\rho_{\text{max}}$. We also define

$$X(a) = \frac{\rho(a)}{\rho_{\text{max}}}$$

(37)

as the total energy density rescaled to $\rho_{\text{max}}$. Its evolution in terms of the scale factor can be expressed as

$$X(a) = X_r^* \left( \frac{a_*}{a} \right)^4 + X_c^* \left( \frac{a_*}{a} \right)^5 - X_v^* \left( \frac{a_*}{a} \right)^6.$$  

(38)

The rescaled parameters are related because $X(a_*) = 1$, that is,

$$X_r^* + X_c^* - X_v^* = 1.$$  

(39)

In addition, the derivative of the total density with respect to $a$ evaluated at $a_*$ is equal to zero. This condition reads

$$-4X_r^* - 5X_c^* + 6X_v^* = 0.$$  

(40)

Combining conditions (39) and (40), we relate the parameters in the form

$$X_r^* = 5 - X_v^*, \quad X_c^* = 2X_v^* - 4,$$  

(41)

and, then, the only free parameter to consider is the interaction $X_v^*$. Given equation (41) and because $X_r^* > 0$, $X_c^* < 5$. Additionally, $X_v^* > 0$, which implies that $X_v^* > 2$. The range of parameters that can solve the horizon problem in our model are then given by equation (41) with $2 < X_v^* < 5$. Figure 4 shows the energy density as a function of the scale factor, for this range of parameters. The energy density reaches the null value at the scale

$$\frac{a_*}{a_*} = \frac{X_v^*}{X_v^* - 2 + \sqrt{X_v^* + 4}}.$$  

(42)
and the horizon problem is solved, and equation (33) is satisfied because the energy density increases from $\rho(a) \approx 0$ at a scale factor $a \gtrsim a_{\infty}$. Note that if we consider $a_{\infty}$ as the initial scale, $1/H(a_{\infty})$ diverges, and so does the horizon. This divergence is also found in the inflationary de Sitter models, for which the horizon diverges when $a(t_e) \to 0$.

3.4. Fluctuations

One of the main predictions of inflation is the presence of scale-invariant fluctuations which form the seed of later large-scale structure formation. In the conventional description for a nearly de Sitter universe, the constancy of the fluctuation amplitude is determined by the constant horizon size (which in turn is set by the Hawking temperature $T_H = H/(2\pi)$); this occurs at the horizon scale $k = H a$, where $k$ is the Fourier-transformed comoving coordinate.

We have shown that the universe can be initially causally connected in our model; we shall also argue that the properties of a Bose gas allow for a scale-invariant perturbations regime: we show that thermal fluctuations originating in the non-condensate contribution have the appropriate properties for the standard cosmology.

During the phantom-accelerated regime, the expansion is dominated first by the interaction term in equation (29) and the non-condensate contribution; the latter then dominates as $a$ grows (see figure 1). In general, the mean square density fluctuation, for a given volume (and therefore, constant particle number or entropy), is given by [29]

$$\langle (E - \bar{E})^2 \rangle = T^2 \left( \frac{\partial E}{\partial T} \right)_{N,V} = T^2 N c_v,$$

(43)

where $c_v$ is the specific heat per particle (constant in our case).

While this calculation represents a useful benchmark for later comparison, we actually need the contribution of the fluctuations at a given mode $k$. This may be obtained from the approximate large-distance non-condensate wavefunction component $\delta \Phi_k$ [30]:

$$\delta \Omega = \frac{1}{2mT} \rho_m V \sum_{k_{\text{phys}}} k^2_{\text{phys}} |\delta \Phi_k|^2,$$

(44)
where $\rho_m$ is the non-condensate mass density, and we used the physical $k_{\text{phys}}$; this term contributes to the kinetic term in equation (29). Then, one obtains the expression for the statistical average:

$$\langle |\delta \Phi_k|^2 \rangle = \frac{T m^2}{V \rho_m k_{\text{phys}}^2}. \quad (45)$$

By considering the phase-space factor $V d^3 k_{\text{phys}} = d^3 k$, one finds that the $k$ contribution to the total fluctuation in equation (43) is proportional to $T$. While $T$ is $a$-dependent, the ratio in the rms density contrast is not, nor is $\xi_{\text{rms}} = \delta \rho_{\text{rms}}/(\rho + p)$, as one can use the relation $p_c = (2/3) \rho_c$ for the non-condensate part. We hence obtain that the gauge-invariant quantity contribution $\zeta_{\text{rms}}$ is constant beyond the horizon. This leads to Gaussian scale-invariant fluctuations, since their origin is thermodynamic, and they all have the same $k$-power dependence for their amplitude.

Once the fluctuation mode $k$ crosses outside the Hubble radius, it evolves as a classical perturbation [17]; the fluctuation evolution of the universe in our model becomes equivalent to the standard universe. Thus, we find that it is possible to obtain scale-invariant fluctuations, assuming the non-condensate fluctuations dominate, even as the horizon size $1/H$ is not constant, but decreases with $a$ (or equivalently with $t$).

### 3.5. Nucleosynthesis

We find the condition under which our model solves the horizon problem consistently with standard cosmological requirements, and specifically with nucleosynthesis [17]. Nucleosynthesis requires that the universe be dominated by radiation at times of around 1 s in order for the light elements to be produced.

We consider the scale factor $a_{\text{eq}}$ at which the radiation and Bose-gas densities are equal, i.e.

$$X_r^* \left(\frac{a}{a_{\text{eq}}}\right)^4 X^*_c \left(\frac{a}{a_{\text{eq}}}\right)^5 X^*_v \left(\frac{a}{a_{\text{eq}}}\right)^6 = 0. \quad (46)$$

From the scale $a_{\text{eq}}$ on, the radiation term will dominate the expansion of the universe. We evaluate the time span $t_{\text{eq}}$ which corresponds to this scale factor. If $t_{\text{eq}} < 1$ s, then the nucleosynthesis scenario will be preserved by the model. The time span $t_{\text{eq}}$ is found by integration of equation (20),

$$t_{\text{eq}} = \sqrt{\frac{3}{8 \pi}} \frac{m_p}{\rho_{\text{max}}} \int_{a_{\text{eq}}}^{a_{\text{eq}}} d\left(\frac{a}{a_{\text{eq}}}\right) \sqrt{\frac{X^*_r}{X^*_c}}. \quad (47)$$

If there is no radiation in our model, the parameters read $X^*_r = 0$, $X^*_c = 6$, $X^*_v = 5$. In this particular case, $t_{\text{eq}}$ tends to infinity. These parameters, although they solve the horizon problem, are in contradiction with the nucleosynthesis scenario.

Depending on the value of $\rho_{\text{max}}$ it is possible to numerically evaluate how much radiation density is initially needed (i.e. how large $X^*_r$ must be) to allow for $t_{\text{eq}} < 1$ s. This minimal radiation amount decreases its value as $\rho_{\text{max}}$ increases, as can be seen from equation (47). Assuming $\rho_{\text{max}} = m_p^4$ and that $X^*_r \approx 10^{-17}$, $X^*_c \approx 6 - 2 \times 10^{-17}$, $X^*_v \approx 5 - 10^{-17}$, then equation (47) reads $t_{\text{eq}} \approx 10^{42} m_p^4 \approx 0.1$ s. In this limit, nucleosynthesis will be allowed for any $X^*_r \geq 10^{-17}$. If we consider that $\rho_{\text{max}}$ is lower than $m_p^4$, the minimal amount of radiation needed (evaluated from equation (47) with $t_{\text{eq}} = 1$ s) is larger than $10^{-17}$.

Consideration of the Bose-gas particles’ mass requires the contribution $m_n$ in the energy density. This term scales as $a^{-3}$, and eventually becomes the dominant contribution. To ensure
that this term does not dominate the expansion before nucleosynthesis, we have to impose the condition \( mn_i \ll \rho_{ri} \). The latter condition, together with the non-relativistic gas condition given by equation (7), implies \( \rho_{ri} \ll \rho_{ci}, (v_0 n_i^2)/2 \). In terms of the rescaled parameters, this is \( X_r^* \approx 0, X_i^* \approx 2 \) and \( X_c^* \approx 3 \).

We can conclude that nucleosynthesis does not bound the free parameters of the model strongly. If the radiation term does not differ from the other two terms for more than ten orders of magnitude, nucleosynthesis will occur. In such a scenario, our model starts with an accelerated era that explains the horizon problem, later evolves to a radiation-dominated universe in which light elements can be created and, eventually, the expansion is dominated by non-relativistic matter. In other words, our model of universe behaves as the standard cosmology, besides the initial stage of expansion. As we mentioned before, late time acceleration can also be obtained in our model if we consider that the number of particles of the gas or the interaction term is not constant. This problem will be addressed in a future paper.

3.6. Stability of the interacting Bose gas in the cosmological frame

A confined Bose-gas condensate with an attractive interaction is stable under certain conditions [36], unlike the case of a Bose condensate with an attractive interaction within an infinite volume, which always collapses. When the gas particles are confined in a given volume (e.g., by a harmonic trap), finite-size effects due to the confining potential generate an additional kinetic-energy contribution that compensates the interaction energy, avoiding the collapse, as long as the confined particle number in the potential is lower than a critical value \( N_{cr} \). This number can be computed by equating the exterior potential plus kinetic energy terms with the interaction term [36], which translates to

\[
N_{cr} \lesssim 4\pi \frac{a_{ho}}{m |v_0|},
\]

where \( a_{ho} \) is the physical trap characteristic length scale, and \( m \) is the particle mass.

We approach the collapse problem in our model applying dimensional analysis. In the cosmological frame, we assign \( a_{ho} \) to the Hubble radius \( 1/H \), as the latter represents the region of space in causal contact. In our case, \( a_{ho} \) grows with time, as the volume expands [37]. Thus, if the Hubble volume \((1/H)^3\) contains less than \( N_{cr} \) particles, the collapse can be avoided. So is the case for any smaller volume with a characteristic length \( a_{hol} \) lower than \( 1/H \), as such a volume will contain less than \( N_{cr} \) particles.

It is sufficient to apply the above condition for initial values of the parameters in our model, as the interaction energy term (proportional to \( n_i^2 \)) dilutes faster with the expansion than the condensate term in equation (29), and eventually becomes negligible. With the association of \( N_{cr} \) with the ground-state (or condensate) particle density \( n_c \), equation (48) reads

\[
n_i^2 |v_0| \lesssim \frac{\rho_{ri} + \rho_{ci}}{\frac{1}{2} + \frac{3m_P^2 m}{2(4\pi)^2 n_i^2}},
\]

where we used \( a_{ho} \sim 1/H \) and having in mind that \( m < m_P \), it is convenient to explore two extreme cases.
(i) $n_i \gg n_c$. Only a small amount of the total particles are in the ground state. As the temperature of the gas is $T = T_c \left( \frac{N-n_c}{N} \right)^{2/3}$, we have $T \sim T_c$. If $m$ is such that the second term in the denominator in equation (49) satisfies
\[
\frac{3m_p^2 m c_n}{2(4\pi)^2 n_i^2} \ll 1,
\]
then equation (49) reads $n_i^2 |v_0| \lesssim 2(\rho_{ri} + \rho_{ci})$, which is consistent with the condition required for the total energy density entering the Einstein equation to be positive. In this case, it is possible to obtain phantom-accelerated, non-phantom accelerated and decelerated solutions that do not collapse.

(ii) $n_i = n_c$. All the particles of the gas are in the ground state (i.e. the temperature of the gas is $T = 0$) and consequently,
\[
\frac{3m_p^2 m c_n}{2(4\pi)^2 n_i^2} = \frac{3m_p m}{2(4\pi)^2 n_i} \gg 1
\]
\[
\Rightarrow \rho_{ri} + \rho_{ci} \gg \frac{\rho_{ri} + \rho_{ci}}{1 + \frac{3m_p^2 m c_n}{2(4\pi)^2 n_i^2}} \gtrsim n_i^2 |v_0|.
\]
In this case $n_i^2 |v_0| \ll \rho_{ri} + \rho_{ci}$. The condition of $\rho > 0$ is compatible with the latter condition. But conditions for the accelerated and phantom kind solutions in section 3.2 ($\rho_{ri} + \rho_{ci} < -3\nu_0 n_i^2 < (5/3)\rho_{ri} + (4/3)\rho_{ci}$ and $-3\nu_0 n_i^2 > (5/3)\rho_{ri} + (4/3)\rho_{ci}$, respectively) are in contradiction with the collapse condition in this case. Consequently, models for which $n_i = n_c$ and that do not present collapse are suitable to be used in cosmology (as $\rho > 0$) but always lead to a decelerated expansion of the universe.

The above conditions are further relaxed in a cosmological context, as an attractive interaction ultimately accelerates the expansion, and temperature effects will contribute to counter the attraction. To summarize this subsection, stable solutions of the Bose–Einstein gas are possible, while accelerated ones favor the case $n_i \gg n_c$.

4. Conclusions

A Bose–Einstein gas with self-interacting particles leads to accelerated expansion of the universe. The interaction between the particles of the gas naturally gives solutions of the type $\rho + 3p < 0$, if the interaction is attractive. Phantom-accelerated solutions with $\rho + p < 0$ can also be obtained, if we consider a model of a universe that contains the Bose gas plus radiation and in which the interaction term initially dominates the expansion.

As such a term, responsible for the accelerated behavior, decreases faster with the expansion than the condensate and radiation terms, the accelerated stage eventually ends. This fact makes the model more suitable to describe the acceleration of the early universe than the present accelerated era.

Our model solves the horizon problem when we consider phantom-accelerated solutions. In addition, nucleosynthesis is feasible by starting with a large-enough radiation term.

Acknowledgments

The authors would like to acknowledge Rocío Jauregui for her useful comments. GI also acknowledges support from the “Programa de Becas Postdoctorales de la UNAM”, and JB acknowledges support from the Helen program for CERN work sojourns.
Appendix.

In this appendix we calculate a higher limit for the scale factor $a$ for which the accelerated stage of the FRW model of the interacting gas plus radiation of section 2 ends. Assuming that the acceleration of the expansion given by equation (31) is positive at the scale $a_i$, we thus evaluate the scale factor $a$ for which $\ddot{a}(a) = 0$. This is at

$$a = \frac{-3\rho_{ri} + \sqrt{9\rho_{ri}^2 - 16\rho_{ri}v_0 n_i^2}}{4\rho_{ri}}.$$  \hspace{1cm} (A.1)

By definition, $\rho_{ri} > 0$. We define the new variables $y = \frac{\rho_{ci}}{\rho_{ri}} > 0$, $z = -\frac{v_0 n_i^2 / 2}{\rho_{ri}} > 0$, \hspace{1cm} (A.2)

and express equation (A.1) in their terms:

$$a = \frac{-3y + \sqrt{9y^2 + 32z}}{4}.$$  \hspace{1cm} (A.3)

GR requires the total energy density to be positive ($-v_0 n_i^2 < \rho_{ri} + \rho_{ci}$), which reads in terms of the new variables $z < 1 + y$. Making use of the latter condition and of equation (A.3), we get

$$a < \frac{-3y}{4} + \frac{\sqrt{9y^2 + 32(y + 1)}}{4}. \hspace{1cm} (A.4)$$

The right-hand side term of inequality (A.4) is a decreasing function of $y$. Its maximum value in the limit $y \to 0$ is $\sqrt{2}$. Consequently, we conclude that the acceleration stage in the radiation plus interacting Bose gas scenario ends at $a/a_i$ always lower than $\sqrt{2}$.

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