LOCALLY RECOVERABLE CODES ON SURFACES

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Abstract. A linear error correcting code is a subspace of a finite dimensional space over a finite field with a fixed coordinate system. Such a code is said to be locally recoverable with locality $r$ if, for every coordinate, its value at a codeword can be deduced from the value of (certain) $r$ other coordinates of the codeword. These codes have found many recent applications, e.g., to distributed cloud storage. We will discuss the problem of constructing good locally recoverable codes and present some constructions using algebraic surfaces that improve previous constructions and sometimes provide codes that are optimal in a precise sense.

1. Introduction

Error correcting codes are used in the transmission of information through noisy channels that cause errors. Traditionally, the theory of error correcting codes has focused more on correction of errors, when part of the transmitted data is corrupted, but it has been long recognized that correcting erasures, when part of the transmitted message is simply missing, is also important. Motivated by recent applications to distributed cloud storage, a particular class of error correcting codes that efficiently correct erasures, the locally recoverable codes, has gained importance [GHSY12, TB14, PD14].

In a different vein, around 1980, Goppa introduced algebraic geometric codes. These attained importance when they were used to beat the asymptotic Gilbert-Varshamov bound. Algebraic geometric codes are constructed from algebraic varieties but it is the one-dimensional case of curves that has been used more extensively, with surfaces and higher dimensional objects being less studied. The standard reference on the subject is [TVN07].

There already exist several constructions of algebro-geometric locally recoverable codes [HMM18, BTV17, BHH+17, LMX19, MTT]. In particular, [BTV17] gives systematic way to produce optimal locally recoverable codes. As in previous optimal constructions, the codes obtained in the [BTV17] are short, i.e., have length bounded by the size of the field plus one. We present new systematic constructions using ruled surfaces and elliptic surfaces. We review carefully the algebro-geometric background in what follows, and present our construction and several examples derived from it. Some of these examples are optimal and provide the first long optimal locally recoverable error correcting codes for some values of the parameters. Ultimately, the codes we construct are obtained by evaluating functions on a curve sitting

2010 Mathematics Subject Classification. Primary 94B27, 14G50.
Key words and phrases. Error correcting codes, locally recoverable codes, algebraic surfaces.
on our surface and thus can be viewed as codes on curves. However, our proofs of the various properties these codes enjoy rely on the internal geometry of the ambient surface. This point of view guided our work throughout, so we have kept the perspective it affords.

1.1. Locally recoverable codes. Let $\mathbb{F}_q$ be the finite field of $q$ elements. A linear error correcting code is a subspace $C$ of $\mathbb{F}_q^n$ for some $n$, which is called the length of $C$. We denote by $k$ the dimension of $C$ as a $\mathbb{F}_q$-vector space and we denote by $d$ the minimum distance of $C$, defined as the minimum number of nonzero coordinates among the nonzero elements of $C$.

The code $C$ is said to be locally recoverable (LR) with locality $r$ if, for each $i = 1, \ldots, n$, there is a subset $J_i \subset \{1, \ldots, n\} - \{i\}$, $\#J_i = r$ (called the recovery set), such that, if we know the values $c_j$ for $j \in J_i$ of the coordinates of any $c \in C$, then we can recover $c_i$. Codes with small locality can be used in distributed storage systems as they can reconstruct data erasures with smaller storage overhead than traditional back-ups. It is desirable to have codes with small locality, large dimension (equivalently, high information rate $k/n$) and large minimum distance for these applications. However, these parameters are not independent. A basic constraint is that the parameters of $C$ satisfy

$$d \leq n - k - \lceil k/r \rceil + 2,$$

and $C$ is called an optimal LR code if equality holds. We write $d_{\text{opt}}$ for the right hand side of (1.1).

An explicit construction of optimal LR codes with $n \leq q$ is given in [TB14]. There are known upper bounds for the length of LR codes and some general existence theorems [GXY19]. One of the purposes of this paper is to explicitly construct longer optimal LR codes.

The LR codes we construct have the property that the sets $J_i \cup \{i\}$ form a partition of $\{1, \ldots, n\}$ but not every LR code has this property. The proof of (1.1) in general is quite complicated but for LR codes with this property we can give the following simple proof:

Note that the recovery map for any coordinate on inputs all equal to 0 is 0, since the zero vector is a codeword. Now take $b = \lceil k/r \rceil - 1$ so $br < k$ and choose $b$ disjoint sets of the form $J_i \cup \{i\}$ and set the $r$ coordinates indexed by each $J_i$ from this choice to 0. In addition, choose $k - 1 - br$ coordinates outside the union of the chosen $J_i \cup \{i\}$ and set them equal to 0 also. Thus, a total of $k - 1$ conditions are imposed and there exists a non-zero codeword satisfying them all as our code has dimension $k$. But this non-zero codeword also has zero $i$-th coordinates for all of the chosen $J_i \cup \{i\}$. This gives us $b$ additional zero coordinates. Hence the weight of this codeword is at most $n - (k - 1) - b = n - k - \lceil k/r \rceil + 2$.

1.2. Algebraic geometric codes. Let $X$ be a quasi-projective algebraic variety over a finite field $\mathbb{F}_q$. Concretely, this means that we select an open subset of affine or projective space where a collection of polynomials vanish. The function field of $X$ is the set of functions that can be expressed as quotients of polynomials in the coordinates of the ambient space.
modulo the equations defining $X$. Given a point $P$ on $X$ and an element $\sigma$ of the function field of $X$, if the denominator of $\sigma$ does not vanish at $P$, the function $\sigma$ can be evaluated at $P$ giving an element $\sigma(P)$ of $\mathbb{F}_q$.

Let $P_1, \ldots, P_n$ be a subset of the set $X(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of $X$ and $V$ a finite dimensional subspace of the function field of $X$. We assume that the evaluation, as above, of all elements of $V$ at all the points $P_1, \ldots, P_n$ is defined and we can consider the image $C$ of evaluation map, which is an error correcting code:

$$
ev_V: V \to (\mathbb{F}_q)^n$$

$$\sigma \mapsto (\sigma(P_1), \ldots, \sigma(P_n)).$$

The length of the code is $n$. The dimension $k$ of the code is

$$k = \dim_{\mathbb{F}_q}(\text{im } \ev_V) = \dim V - \dim_{\mathbb{F}_q}(\ker \ev_V)$$

which simplifies to $\dim V$ if $\ev_V$ is injective. The minimum distance $d$ is the smallest Hamming distance between elements of $C$. This is equal to $n$ minus the largest number of $\mathbb{F}_q$-points of $X$ vanishing on an element of $V \setminus \ker \ev_V$.

For $X$ a projective variety and $D$ a divisor on $X$, we denote by $\mathcal{L}(X, D)$ the Riemann-Roch space of functions $\sigma$ on $X$ such that either $\sigma = 0$ or $(\sigma) + D$ is an effective divisor, where $(\sigma)$ denotes the divisor of $\sigma$. The space $\mathcal{L}(X, D)$ is always finite dimensional and we denote its dimension by $\ell(X, D)$. We will typically define our vector space $V$ as above as a subspace of some $\mathcal{L}(X, D)$.

2. Baseline codes from high-dimensional varieties

Let $\mathbb{A}^m$ denote affine $m$-dimensional space over a finite field $\mathbb{F}_q$. In this section we construct locally recoverable codes, with local recoverability parameter $r$ from a projection morphism

$$\pi: \mathbb{A}^{r-1} \times \mathbb{A}^1 \to \mathbb{A}^1,$$

$$(x_1, \ldots, x_{r-1}; t) \mapsto t.$$

We shall impose the smallest possible amount of structure on our choice of points for evaluation. This will give us a baseline to assess the parameters of other constructions.

Let $M$ and $N$ denote positive integers. We shall use the space of functions

$$V_{M,N} := \left\{ a_0(t) + \sum_{i=1}^{r-1} a_i(t)x_i : \deg a_0 \leq M \text{ and } \deg a_i \leq N \text{ for } i = 1, \ldots, r-1 \right\}$$

to construct an evaluation code (so $V_{M,N}$ plays the rôle of the vector space $V$ from §1.2). We shall pick $r + 1$ points in each of $b$ fibers of $\pi$ as the set of points where we evaluate the above functions. Thus, the length of the resulting code will be $n = b(r + 1)$. The following lemma falls within the framework of [BHH+17, Proposition 4.2].
Lemma 2.1. Fix $t = t_0 \in \mathbb{F}_q$, and let $P_1, \ldots, P_{r+1}$ be \( \mathbb{F}_q \)-points in the fiber $\pi^{-1}(t_0)$, no $r$ of which lie on a hyperplane. Let $\sigma \in V_{M,N}$ be a function. Then the value of $\sigma(P_i)$ can be recovered from knowledge of the coordinates of $P_1, \ldots, P_{r+1}$ and the $r$ values $\sigma(P_1), \ldots, \sigma(P_{r+1})$.

Proof. Write $\sigma = a_0(t) + \sum_{i=1}^{r-1} a_i(t)x_i$. Let $a_i = a_i(t_0)$ for $i = 1, \ldots, r+1$. Then we have the matrix equation

\[
\begin{pmatrix}
1 & x_1(P_1) & \cdots & x_{r-1}(P_1) \\
1 & x_1(P_2) & \cdots & x_{r-1}(P_2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_1(P_{r+1}) & \cdots & x_{r-1}(P_{r+1})
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_r
\end{pmatrix}
= \begin{pmatrix}
\sigma(P_1) \\
\sigma(P_2) \\
\vdots \\
\sigma(P_{r+1})
\end{pmatrix}.
\]

(2.1)

Since no $r$ of the points $P_1, \ldots, P_{r+1}$ lie on a hyperplane, the $r \times r$ matrix in (2.1) is invertible, and hence we may compute $a_0, \ldots, a_r$ from knowledge of the coordinates of $P_1, \ldots, \hat{P}_i, \ldots, P_{r+1}$ and the $r$ values $\sigma(P_1), \ldots, \sigma(P_{i-1}), \sigma(P_{i+1}), \ldots, \sigma(P_{r+1})$. We conclude that

$\sigma(P_i) = a_0 + a_1x_1(P_i) + \cdots + a_rx_{r-1}(P_i)$. \qed

To construct what we will call a baseline code, let

$$\{t_1, \ldots, t_b\} \subseteq \mathbb{A}^1(\mathbb{F}_q)$$

be $b$ distinct points on the target $\mathbb{A}^1$ of the morphism $\pi$, and for each $t_i$, choose $r+1$ points $P_{i,1}, \ldots, P_{i,r+1}$ on the fiber $\pi^{-1}(t_i)$, no $r$ of which lie on a hyperplane.

Proposition 2.2. Suppose that $b - M, b - N \geq 1$. The baseline code

$$C = \{(\sigma(P_{i,j}))_{1 \leq i \leq b, 1 \leq j \leq r+1} : \sigma \in V_{M,N}\}.$$

has local recoverability $r$ and its parameters satisfy

$$n = b(r+1),$$

$$k = (M+1) + (r - 1)(N + 1),$$

$$d \leq (r + 1)(b - (N + 1)) - (M - N) - \left\lfloor \frac{M - N}{r} \right\rfloor + 2,$$

$$d \geq \min\{(b - M)(r + 1), 2(b - N)\}.$$
the corresponding fibers of \( \pi \), the function \( \sigma \) defines a hyperplane. The hypothesis that no \( r \) points on a fiber lie on a hyperplane ensures that \( \sigma \) takes on a nonzero value on at least two points in each of the \( (b - N) \) fibers. Hence, \( d \geq \min\{(b - M)(r + 1), 2(b - N)\} \), as claimed.

Local recoverability of \( C \) follows from Lemma 2.1.

**Remark 2.3.** The proof of Proposition 2.2 shows that if \( \min\{(b - M)(r + 1), 2(b - N)\} = (b - M)(r + 1) \), then in fact \( d = (b - M)(r + 1) \). In addition, if \( M + N > b \) and \( 2N > b \) then it is always possible to construct a function \( \sigma \) whose associated code word has weight exactly \( 2(b - N) \). So under the conditions (2.2), the lower bound for \( d \) in Proposition 2.2 is in fact sharp.

**Example 2.4.** We specialize to the case where \( r = 3 \), \( M = b - 1 \) and \( N = b - 2 \). Then the upper and lower bounds for \( d \) meet and we have \( d = 4 \). This gives optimally recoverable codes with parameters

\[
(n, k, d, r) = (4b, 3b - 2, 4, 3).
\]

Note that the information rate \( k/n \) is approximately 75%, and since \( b \leq q \), one can construct codes with \( n = 4q \) and high information rate that are optimal locally recoverable. In particular, over any \( \mathbb{F}_q \) with \( q \geq 9 \), we can construct a code with parameters \( (n, k, d, r) = (32, 22, 4, 3) \).

**Example 2.5.** If we now take \( b \leq q, r \) arbitrary and \( M = N = b - 1 \), then the upper and lower bounds of Proposition 2.2 also coincide and the code has distance \( d = 2 \).

The last two examples are the only cases where the upper and lower bounds of Proposition 2.2 coincide and a baseline code with no additional properties is optimal (see Remark 2.3). To see this, let \( \delta := M - N \); we consider two cases:

- \( \min\{(b - M)(r + 1), 2(b - N)\} = (b - M)(r + 1) \): Then
  \[
  (b - M)(r + 1) = (r + 1) (b - (N + 1)) - (M - N) - \left\lceil \frac{M - N}{r} \right\rceil + 2,
  \]
  from which one can conclude that
  \[
  (r + 1)(\delta - 1) - \delta - \left\lceil \frac{\delta}{r} \right\rceil + 2 = 0. \tag{2.3}
  \]
  Write \( \left\lceil \frac{\delta}{r} \right\rceil = \frac{\delta}{r} + \epsilon, 0 \leq \epsilon < 1 \), then
  \[
  \delta(r - 1/r) = r - 1 + \epsilon. \tag{2.4}
  \]
  This implies in particular that \( \delta > 0 \). If \( \delta \geq 2 \) then, since \( r \geq 3 \), we have
  \[
  \delta(r - 1/r) \geq 2r - 1 > r - 1 + \epsilon
  \]
  and hence \( \delta = 1 \), since it is an integer.
If $\delta = 1$ then the hypothesis $2(b - N) \geq (b - M)(r + 1)$ gives

$$b \leq M + \frac{2}{r-1} \leq M + 1 \text{ (whenever } r \geq 3),$$

from which we conclude that $b - M = 1$, and hence that $b - N = 2$. It follows that

$$(r + 1) = (b - M)(r + 1) = d = \min\{(b - M)(r + 1), 2(b - N)\} = \min\{(r + 1), 4\},$$

whence $r + 1 \leq 4$. Since we want codes with $r \geq 3$, we must have $r = 3$ and $d = 4$.

- $\min\{(b - M)(r + 1), 2(b - N)\} = 2(b - N)$:

  Then

  $$b = N + 1 + \frac{\delta}{r-1} + \frac{1}{r-1} \left\lceil \frac{\delta}{r} \right\rceil. \tag{2.5}$$

  On the other hand, $\min\{(b - M)(r + 1), 2(b - N)\} = 2(b - N)$ gives

  $$b \geq \frac{M(r + 1) - 2N}{r - 1}. \tag{2.6}$$

Substituting the value for $b$ obtained in (2.5) into the inequality (2.6) we get

$$\delta \left(1 + \frac{1}{r-1}\right) \leq 1 + \frac{1}{r-1} \left\lceil \frac{\delta}{r} \right\rceil.$$

The latter implies that $\delta \leq 1$. If $\delta = 1$ then $M = N + 1$ and thus

$$2(b - N) \leq (b - (N + 1))(r + 1) \Rightarrow b \geq N + 1 + \frac{2}{r - 1}.$$

We also have

$$b = N + 1 + \frac{1}{r-1} \left\lceil \frac{1}{r} \right\rceil + \frac{1}{r-1} \text{ (by (2.5))}$$

$$= N + 1 + \frac{2}{r - 1} \leq N + 2.$$

The distance is thus given by $d = 2(b - N) \leq 4$ and by our analysis, the inequality $2(b - N) \leq (b - (N + 1))(r + 1)$ is sharp, so $(b - M)(r + 1) \leq 4$, which forces $r \leq 3$. Finally, since we assumed $r \geq 3$, we conclude that in fact $r = 3$ and $d = 4$.

If $\delta = 0$ then (2.5) gives $b = N + 1$, which implies that $d = 2$. If $\delta \leq -1$, we get $b \leq N$ which is not possible.

3. Codes from ruled surfaces: affine intimations

3.1. Tamo-Barg codes. We present the construction of Tamo and Barg [TB14] of optimal LR codes of length at most $q$ from the perspective of the last section, which we believe is new. We retain the notation of the previous section.
Let \( g(x) \in \mathbb{F}_q[x] \) be a polynomial of degree \( r + 1 \), viewed as a morphism \( g: \mathbb{A}^1 \to \mathbb{A}^1 \). Choose distinct \( t_1, \ldots, t_b \in \mathbb{F}_q \) such that the fiber \( g^{-1}(t_i) \) consists of \( r + 1 \) distinct elements \( u_{i,1}, \ldots, u_{i,r+1} \) of \( \mathbb{F}_q \), for \( i = 1, \ldots, b \). Note that the \( u_{i,j} \) are therefore \( n = b(r + 1) \) distinct elements of \( \mathbb{F}_q \). We define the points \( P_{i,j} = (u_{i,j}, u_{i,j}^2, \ldots, u_{i,j}^{r-1}) \in \mathbb{A}^{r-1}(\mathbb{F}_q) \), and we consider the projection map

\[
\pi: \mathbb{A}^{r-1} \times \mathbb{A}^1 \to \mathbb{A}^1, \\
(x_1, \ldots, x_{r-1}; t) \mapsto t.
\]

For a fixed \( i \), the fiber above \( t_i \) is an affine space \( \mathbb{A}^{r-1} \) containing the points \( P_{i,j} \) for \( j = 1, \ldots, r+1 \). Moreover, by their construction, these points lie on an affine rational normal curve, i.e., they lie on the image of the map

\[
h: \mathbb{A}^1 \to \mathbb{A}^{r-1}, \\
x \mapsto (x, x^2, \ldots, x^{r-1}).
\]

This guarantees that no \( r \) of them lie on a hyperplane. As in §2 we take the space of functions \( V_{M,N} \), but specialize to the case where \( M = N \), and build a code \( C \). Lemma 2.1 guarantees that \( C \) has local recoverability \( r \). Put differently, the fact that the points \( P_{i,j} \) lie on rational normal curves implies that the \( r \times r \) matrix in (2.1) is a Vandermonde matrix, thus invertible.

The parameters \( n, k, \) and \( r \) for the code \( C \) are as before. However, in this special situation, we get a better lower bound for the minimal distance \( d \) as follows. Note that

\[
\sigma(P_{i,j}) = a_0(g(u_{i,j})) + \sum_{\ell=1}^{r-1} a_{\ell}(g(u_{i,j}))u_{i,j}^{\ell}
\]

is the value at \( x = u_{i,j} \) of a polynomial of degree at most \( N(r + 1) + r - 1 \) in \( x \). This degree is an upper bound on the number of its zeros and thus \( d \geq n - (N(r + 1) + r - 1) \). On the other hand, as in the previous section, the upper bound (1.1) for \( d \) when \( M = N \) is

\[
(r + 1)(b - (N + 1)) - (M - N) - \left\lceil \frac{M - N}{r} \right\rceil + 2 = n - (N(r + 1) + r - 1),
\]

showing that these codes are optimal LR codes.

As mentioned above, these codes have \( n \leq q \). To achieve \( n \) near \( q \) one needs to choose the polynomial \( g(x) \) in such a way that the preimage of many values of \( t \in \mathbb{F}_q \) under \( g \) consists of \( r + 1 \) elements of \( \mathbb{F}_q \). One such choice is \( g(x) = x^{r+1} \) if \( (r + 1) \mid (q - 1) \). For other choices and a full discussion, see [TB14].

3.2. Ruled surfaces perspective. An algebraic surface \( S \) over a field \( k \) is called a ruled surface if it is endowed with a morphism \( \pi: S \to B \) to a base algebraic curve \( B \) such that for all but finitely many \( b \in B(\bar{k}) \), the fiber \( \pi^{-1}(b) \) is a smooth rational curve, where \( \bar{k} \) is a fixed algebraic closure of \( k \). There is a ruled surface operating behind the scenes in our recasting of the Tamo–Barg codes [TB14], which we now describe.
Using the notation of §3.1, we let \( S = \mathbb{A}_x^1 \times \mathbb{A}_t^1 \), which maps to \( \mathbb{A}_{r-1}^r \times \mathbb{A}_t^1 \) via
\[
h \times \text{id}: (x,t) \mapsto (x, x^2, \ldots, x^{r-1}; t).
\]
The variety \( S \) fits into the commutative diagram
\[
\begin{array}{ccc}
S & \xrightarrow{h \times \text{id}} & \mathbb{A}_{r-1}^r \times \mathbb{A}_t^1 \\
\downarrow{\pi'} & & \downarrow{\pi} \\
\mathbb{A}_x^1 & \xrightarrow{g} & \mathbb{A}_t^1
\end{array}
\]
where the map \( \pi' \): \( S \to \mathbb{A}_x^1 \) is projection onto the first coordinate. The variety \( S \) is our ruled surface, and the code constructed in §3.1 can be described as an evaluation code on \( S \), as follows. Given \( t_1, \ldots, t_b \) outside the branch locus of the morphism \( g: \mathbb{A}_x^1 \to \mathbb{A}_t^1 \), i.e., such that the fiber \( g^{-1}(t_i) \) consists of \( b \) distinct points \( u_{i,1}, \ldots, u_{i,r+1} \) in \( \mathbb{A}_x^1(\mathbb{F}_q) \), we set
\[
P_{i,j} = (u_{i,j}, t_i) \in V(\mathbb{F}_q) \quad \text{for } 1 \leq i \leq b, 1 \leq j \leq r + 1,
\]
so that the recovery set for the point \( P_{i,j} \) is
\[
J_{i,j} := \{ P_{i,k} : 1 \leq k \leq r + 1, k \neq j \}.
\]
Then, letting
\[
V_N = \left\{ a_0(t) + \sum_{i=1}^{r-1} a_i(t)x^i : \deg a_i \leq N \text{ for } i = 0, \ldots, r - 1 \right\}
\]
the Tamo–Barg codes are of the form
\[
C = \{ (\sigma(P_{i,j}))_{1 \leq i \leq b, 1 \leq j \leq r+1} : \sigma \in V_N \}.
\]

3.3. Recasting and extending Barg–Tamo–Vlăduţ codes. Just as §§3.1–3.2 give a reinterpretation of the construction of [TB14], in this section we reinterpret the construction of [BTV17] but here we go further and, aided by our geometric point of view, obtain better codes by a judicious choice of the space of functions to evaluate. Some of the codes we obtain are optimal.

In broad terms, we consider a curve \( C \) in the surface \( S = \mathbb{A}_x^1 \times \mathbb{A}_t^1 \) and embed \( S \) (and consequently \( C \)) in \( \mathbb{A}_{r-1}^r \times \mathbb{A}_t^1 \) as above by \( (x,t) \mapsto (x, x^2, \ldots, x^{r-1}, t) \). We choose \( C \) so that the projection in the \( t \) coordinate has degree \( r + 1 \) and choose the values of \( t \in \mathbb{F}_q \) to be those for which their preimage consists of \( r + 1 \) rational points. Then, just as before, we can evaluate these points on a space of polynomials such as \( V_{M,N} \) above to get an LR code with locality \( r \).

\(^1\)Keen readers will immediately note that \( S = \mathbb{A}_x^2 \). We prefer to use the product \( \mathbb{A}_x^1 \times \mathbb{A}_t^1 \) because, as we shall see in §4, the correct projective compactification of \( S \) to work with is \( \mathbb{P}^1 \times \mathbb{P}^1 \), and not \( \mathbb{P}^2 \).
In §3.2 all the points in $S$ used for the Tamo–Barg evaluation code lied on the curve $g(x) = t$. In this section, we instead consider the curve

$$C : \quad x^{r+1} = t^2 + 1,$$

(3.1)

which is a cyclic cover of $\mathbb{A}^1_{\mathbb{F}_q}$ via the map $(x, t) \mapsto t$. In order to have many fibers of cardinality $r + 1$ over $\mathbb{F}_q$, we take $q \equiv 1 \mod r + 1$. Fix a positive integer $d$. The space of functions we use to define the code consists of functions of the form

$$\sigma = a_0(t) + a_1(t)x + \cdots + a_{r-1}(t)x^{r-1},$$

(3.2)

where the $a_j(t)$ vary in the vector space defined by the inequalities

$$\deg a_j \leq \frac{n - \vartheta}{r + 1} - \epsilon_j$$

and

$$\epsilon_j = \begin{cases} 
0 & \text{if } j = 0, \\
1 & \text{if } 0 < j \leq (r + 1)/2, \\
2 & \text{otherwise.}
\end{cases}$$

The local recoverability with locality $r$ of the resulting code follows, since for fixed $t$, with $r + 1$ distinct values for $x$, the matrix determining the missing value is a Vandermonde matrix. The inequalities defining the space of functions to be evaluated ensure that the minimum distance of this code is at least $d$, because $x$ has a pole of order 2 at infinity and $t$ has a pole of order $r + 1$ at infinity.

The space of functions at which we evaluate points of the curve has dimension, for $r$ odd,

$$k = \frac{r}{r + 1} (n - \vartheta) - \sum_{i=0}^{r-1} \epsilon_j + r = \frac{r}{r + 1} (n - \vartheta) + \frac{5 - r}{2}.$$  

Note that the upper bound (1.1) for the distance of this code is

$$n - k - \left\lfloor \frac{k}{r} \right\rfloor + 2 = n - \frac{r}{r + 1} (n - \vartheta) + \frac{r - 5}{2} = \left\lfloor \frac{1}{r + 1} (n - \vartheta) + \frac{5 - r}{2r} \right\rfloor + 2$$

$$= \vartheta + \frac{r - 5}{2} + 2.$$

The last equality holds for $r \geq 5$ whereas, for $r = 3$, we just get $\vartheta$. So the codes constructed this way are optimal for $r = 3$; for $r > 3$, these codes are further from the optimal bound the larger $r$ gets.

For $r$ even, a similar calculation gives $\vartheta + r/2$ as the upper bound for the distance when $r > 2$ and $\vartheta$ when $r = 2$. So the codes constructed are optimal for $r = 2$; for $r > 3$, these codes are further from the optimal bound the larger $r$ gets.

We note again the similarity with the Tamo-Barg codes discussed above, which uses a space of functions of the same form as (3.2) but with $\deg a_j \leq k/r - 1$ and a curve of the
form \( g(x) = t \) for a polynomial \( g(x) \) in place of \( C \). The length of their codes is at most \( q \), whereas the codes above can be longer if the curve \( C \) in (3.1) has more than \( q \) affine points.

4. Codes on ruled surfaces: \( \mathbb{P}^1 \times \mathbb{P}^1 \)

In this section we add one more layer of geometry to the codes we constructed in §3 by considering codes on the ruled surface \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), which is a projective compactification of the surface \( \mathbb{A}_x^1 \times \mathbb{A}_t^1 \). This extra layer of geometry affords important conceptual insights: a lower bound for the minimum distance of a code can be interpreted as an intersection number of two curves in \( S \), and good lower bounds for a minimum distance can be achieved by forcing curves to intersect with high multiplicity at the point \((\infty, \infty) \in S\).

We begin with a toy model for our code, that is far from optimal, but which helps set ideas and notation. We let \( S := \mathbb{P}^1(x:y) \times \mathbb{P}^1(s:t) \), where \((x:y)\) and \((s:t)\) are respective homogeneous coordinates for the factors of \( S \).

4.1. A coarse construction. Let \( r \) be a positive odd integer, let \( b \leq q \) be a positive integer, and set \( n = b(r + 1) \). Choose an integer \( \delta \) such that

\[
N := \frac{n - \delta}{r + 1}
\]

is an integer, as well as \( \alpha \in \mathbb{Z}_{>0} \) such that \( \alpha \mid (r + 1) \). Consider a curve of the form

\[
C : g(x, y; s, t) = 0
\]

in \( S \), where \( g \) is a bi-homogeneous polynomial of the bi-degree \((r + 1, \alpha)\). In other words, every monomial of \( g \) has total degree \( r + 1 \) in the variables \( x \) and \( y \), and total degree \( \alpha \) in the variables \( s \) and \( t \). We say that \( C \) is of type \((r + 1, \alpha)\). Our code will be an evaluation code on the \( \mathbb{F}_q \)-vector space of functions of the form

\[
\sigma = a_0(s, t)y^{r-1} + a_1(s, t)y^{r-2}x + \cdots a_{r-1}(s, t)x^{r-1},
\]

(4.1)

where the \( a_i(s, t) \) are homogeneous polynomials of degree \( N \) in \( s \) and \( t \). We write \( V_{r-1,N} \) for this vector space. Each function \( \sigma \in V_{r-1,N} \) defines itself a curve in \( X \) given by \( \sigma = 0 \). We write \((\sigma)\) for this curve; it is a curve of type \((r - 1, N)\).

Write \( p : S \to \mathbb{P}^1(s:t) \) for the projection onto the second factor. To construct our code, we pick \( b \) points \((s_i : t_i) \in \mathbb{P}^1(s:t)(\mathbb{F}_q)\) such that the fiber \( p^{-1}((s_i : t_i)) \cap C \) consists of \( r + 1 \) distinct points

\[
(u_{i,1} : v_{i,1}), \ldots, (u_{i,r+1} : v_{i,r+1}) \in \mathbb{P}^1(x:y)(\mathbb{F}_q)
\]

and set

\[
P_{i,j} = ((u_{i,j} : v_{i,j}), (s_i : t_i)) \in S(\mathbb{F}_q).
\]

---

2The notation \((\sigma)\) is the usual notation in algebraic geometry for the divisor of zeroes of a global section of a line bundle; see §1.2.
Proposition 4.1. The code

\[ C := \{ (\sigma(P_{i,j}))_{1 \leq i \leq b, 1 \leq j \leq r+1} : \sigma \in V_{r-1,N} \} \]

has parameters satisfying

\[ \begin{align*}
    n &= b(r + 1) \\
    k &= r(N + 1) = \frac{r}{r + 1} \cdot (n - \mathfrak{d}) + r \\
    d &\leq \mathfrak{d} - r + 1 \\
    d &\geq \mathfrak{d} - \alpha(r - 1)
\end{align*} \]

Proof. The parameter \( k \) is simply the dimension of the \( \mathbb{F}_q \)-vector space \( V_{r-1,N} \). The upper bound for the distance is the bound (1.1):

\[ d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2 = n - \frac{r}{r + 1} \cdot (n - \mathfrak{d}) - r - \frac{1}{r + 1} \cdot (n - \mathfrak{d}) - 1 + 2 = \mathfrak{d} - r + 1. \]

We have used here the divisibility relation \( (r + 1) \mid (n - \mathfrak{d}) \). For the lower bound on the distance, we note that the largest number of zeros in a code word in \( C \) is bounded above by

\[ \max_{\sigma \in V_{r-1,N}} \# (C \cap (\sigma)) , \]

i.e., the largest number of intersection points between \( C \) and the curve \( (\sigma) \subset S \) given by \( \sigma = 0 \), as \( \sigma \) varies over the vector space \( V_{r-1,N} \). The intersection theory of \( S \) shows that this number is independent of \( \sigma \): indeed, the intersection of divisors on \( S \) of type \((a, b)\) and \((a', b')\) is \( ab' + a'b \) [Har77, V, Example 1.4.3]. Since \( C \) is a curve of type \((r + 1, \alpha)\) and \((\sigma)\) is a curve of type \((r - 1, N)\), we have

\[ \# (C \cap (\sigma)) = N(r + 1) + \alpha(r - 1) = n - \mathfrak{d} + \alpha(r - 1). \]

Hence, the lowest weight for a code word in \( C \) is

\[ d \geq n - \# (C \cap (\sigma)) = \mathfrak{d} - \alpha(r - 1), \]

as claimed. \( \square \)

Remark 4.2. The codes in the above proposition have locality \( r \). However, we defer the discussion of locality until after we refine the code in the next section.

Remark 4.3. The upper and lower bounds for \( d \) in Proposition 4.1 meet if and only if \( \alpha = 1 \); this is precisely the habitat for the Tamo–Barg codes. In the notation of §3.1, the affine curve \( g(x) = t \) lies in the open set \( \mathbb{A}^1_x \times \mathbb{A}^1_y = \{ y, s \neq 0 \} \) of \( S \); its projective closure
in $S$ is given by $y^{r+1}g(x/y) = t$, which is a curve of type $(r + 1, 1)$ in the notation of this section.

4.2. **Refining the construction.** In this section, we show that one can narrow the gap between the upper and lower bounds for $d$ in Proposition 4.1 by

1. choosing $C$ judiciously,
2. using a particular proper subspace $V \subset V_{N,r-1}$ for the evaluation code,
3. using only points $P_{i,j} = ((u_{i,j} : v_{i,j}), (s_i : t_i))$ with $u_{i,j} = s_i = 1$.

Intuitively, our construction guarantees that the point $(\infty, \infty) := ((0 : 1), (0 : 1)) \in S(\mathbb{F}_q)$ lies in the intersection $C \cap (\sigma)$ for all $\sigma \in V$ with high multiplicity. This allows us to certify the code $C$ has minimum distance $d = 0$.

Consider the curve $C : s^\alpha x^{r+1} - (t^\alpha + s^\alpha)y^{r+1} = 0$, which is a particular curve of type $(r + 1, \alpha)$ in $S$. We shall use functions of the form (4.1), but we constrain the degree in $t$ of the polynomials $a_i(s,t)$, as follows:

$$\deg_t a_i(1, t) \leq N - \left\lceil \frac{\alpha i}{r + 1} \right\rceil.$$ 

In other words, setting

$$\epsilon_i := \left\lceil \frac{\alpha i}{r + 1} \right\rceil,$$

we assume that for each $0 \leq i \leq r - 1$,

$$a_i(s,t) = s^{\epsilon_i} \cdot a'_i(s,t)$$

for a homogeneous polynomial $a'_i(s,t)$. When this is the case, the vector space of functions

$$V := \{ \sigma \in V_{r-1,N} : \sigma = a_0(s,t)y^{r-1} + s^{\epsilon_1} \cdot a_1(s,t)y^{r-2}x + \cdots + s^{\epsilon_{r-1}}a_{r-1}(s,t)x^{r-1} \}$$

has dimension

$$k = r(N + 1) - \sum_{i=0}^{r-1} \epsilon_i. \quad (4.2)$$

We pick $b$ points $(1 : t_i) \in \mathbb{P}^1_{(s,t)}(\mathbb{F}_q)$ such that the fiber $p^{-1}((1 : t_i)) \cap C$ consists of $r + 1$ distinct points

$$(u_{i,1} : 1), \ldots, (u_{i,r+1} : 1) \in \mathbb{P}^1_{(x,y)}(\mathbb{F}_q).$$

Put

$$P_{i,j} = ((u_{i,j} : 1), (1 : t_i)) \in S(\mathbb{F}_q).$$

**Proposition 4.4.** The code

$$C := \{ (\sigma(P_{i,j}))_{1 \leq i \leq b, 1 \leq j \leq r+1} : \sigma \in V \}$$
has locality \( r \) and its parameters satisfy
\[
    n = b(r + 1),
    \quad k = r(N + 1) + 2\alpha - \frac{(\alpha + 1)(r + 1)}{2},
    \quad d \leq \delta + \frac{(\alpha - 1)(r - 3)}{2} - \left\lfloor \frac{2\alpha}{r} - \frac{(\alpha + 1)(r + 1)}{2r} \right\rfloor,
    \quad d \geq \delta.
\]

In particular, the code \( C \) is an optimal LR code if \( \alpha = 1 \) or \( r = 3 \).

**Proof.** By (4.2), to establish the claim on \( k = \dim_{\mathbb{F}_q} V \), it suffices to show that
\[
    \sum_{i=0}^{r-1} \epsilon_i = \frac{(\alpha + 1)(r + 1)}{2} - 2\alpha.
\]

The sequence of integers \( \epsilon_0, \ldots, \epsilon_{r-1} \) has the form
\[
    0, 1,\ldots, 1, 2,\ldots, 2, 3,\ldots, 3,\ldots, \alpha - 1,\ldots, \alpha - 1, \alpha,\ldots, \alpha.
\]

Hence
\[
    \sum_{i=0}^{r-1} \epsilon_i = \sum_{l=1}^{\alpha-1} l \cdot \frac{r + 1}{\alpha} + \alpha \left( \frac{r + 1}{2} - 2 \right)
    = \frac{(\alpha - 1)\alpha}{2} \cdot \frac{r + 1}{\alpha} + (r + 1) - 2\alpha
    = (\alpha - 1) \cdot \frac{r + 1}{2} + (r + 1) - 2\alpha
    = \frac{(\alpha + 1)(r + 1)}{2} - 2\alpha.
\]

For the lower bound on the distance, note that the largest number of zeros in a code word in \( C \) is bounded above by
\[
    \max_{\sigma \in V} \# (C \cap (\sigma)),
\]
just as in Proposition 4.1. We have already seen that
\[
    C \cdot (\sigma) = \alpha(r - 1) + n - \delta.
\]

However, for every \( \sigma \in V \), the curves \( C \) and \( (\sigma) \) intersect at point \( (\infty, \infty) \in S(\mathbb{F}_q) \). We claim this happens with multiplicity at least \( \alpha(r - 1) \), and hence
\[
    \max_{\sigma \in V} \# (C \cap (\sigma)) \leq C \cdot (\sigma) - \alpha(r - 1) = n - \delta,
\]
from which we deduce that
\[
    d \geq n - \max_{\sigma \in V} \# (C \cap (\sigma)) \geq \delta.
\]
To establish the claim on the multiplicity of $C$ and $(\sigma)$ at $(\infty, \infty)$, note that the point $(\infty, \infty)$ is the origin of the affine patch $\mathbb{A}^2_{(y,s)}$ of $S$. In this patch, an affine equation for $C$ is

$$C : s^\alpha = (1 + s^\alpha)y^{r+1},$$

which is in fact singular at the origin (this only helps increase the multiplicity of the intersection with the curve $(\sigma)$). In the complete local ring of $C$ at the origin, the quantity $1 + s^\alpha$ has an $\alpha$-th root. More precisely, let

$$A = k[y, s]/(s^\alpha - (1 + s^\alpha)y^{r+1})$$

be the affine coordinate ring of $C$, and let $m = (y, s)$ be the maximal ideal corresponding to the origin. Then in the completed local ring $\hat{A}_m$, the binomial expansion shows that

$$w = (1 + s^\alpha)^{1/\alpha} = 1 + \left(\frac{1}{\alpha}\right)s^\alpha + \left(\frac{1}{\alpha}\right)^2s^{2\alpha} + \left(\frac{1}{\alpha}\right)^3s^{3\alpha} + \ldots.$$

Let $\zeta$ denote an $\alpha$-th root of unity in an algebraic closure of $\mathbb{F}_q$. Geometrically, $C$ has $\alpha$ branches at the origin:

$$s = wy^{(r+1)/\alpha}, s = \zeta wy^{(r+1)/\alpha}, \ldots, s = \zeta^{\alpha-1}wy^{(r+1)/\alpha},$$

For each one of these branches, $y$ is a uniformizer for the ideal $m$, and $s$ has valuation $(r + 1)/\alpha$ with respect to this uniformizer.\(^3\) For $\sigma \in V$, a local equation for $(\sigma)$ in the affine patch $\mathbb{A}^2_{(y,s)}$ is

$$a_0(s, 1)y^{r-1} + s^{\epsilon_1} \cdot a_1(s, 1)y^{r-2} + \cdots + s^{\epsilon_{r-1}}a_{r-1}(s, 1) = 0.$$

The monomial $s^{\epsilon_i}y^{r-i}$ has $m$-adic valuation

$$\left\lfloor \frac{\alpha i}{r+1} \right\rfloor \cdot \frac{r + 1}{\alpha} + r - 1 - i.$$

As $i$ ranges through $0, \ldots, r - 1$, the smallest value of this quantity is $r - 1$. Hence, on each branch of $C$ the minimal $m$-adic valuation of $\sigma \in V$ is $r - 1$, and therefore $C$ and $(\sigma)$ intersect at $(\infty, \infty)$ with multiplicity $\geq \alpha(r - 1)$. This concludes the proof of the lower bound for $d$.

\(^3\)By this we mean: let $B = \overline{\mathbb{F}}_q[y, s]/(s - \zeta^iws^{(r+1)/\alpha})$ be the geometric local coordinate ring of one of the branches of $C$. Then the $m$-adic completion $\hat{B}_m$ at the maximal ideal $m = (y, s)$ corresponding to the origin is a local discrete valuation ring. Hence the ideal $m\hat{B}_m$ is principal [AM69, Proposition 9.2]. The equation of the branch shows that $y$ is a generator for this ideal, and that $s \in m^{(r+1)\alpha} \setminus m^{(r+1)\alpha - 1}$, which is to say that $s$ has $m$-adic valuation $(r + 1)/\alpha$. 

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Next, we compute an upper bound for $d$ using (1.1):

\[ d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2 \]

\[ = n - r(N + 1) - 2\alpha + \frac{(\alpha + 1)(r + 1)}{2} - (N + 1) - \left\lceil \frac{2\alpha}{r} - \frac{(\alpha + 1)(r + 1)}{2r} \right\rceil + 2 \]

\[ = d - (r + 1) - 2\alpha + \frac{(\alpha + 1)(r + 1)}{2} - \left\lceil \frac{2\alpha}{r} - \frac{(\alpha + 1)(r + 1)}{2r} \right\rceil + 2 \]

\[ = d + \frac{(\alpha - 1)(r - 3)}{2} - \left\lceil \frac{2\alpha}{r} - \frac{(\alpha + 1)(r + 1)}{2r} \right\rceil. \]

Finally, we discuss the locality of the code $C$. Since all points $P_{i,j}$ used to construct $C$ have $u_{i,j} = s_i = 1$, the set $\{P_{i,j}\}$ lies entirely in the affine patch $\mathbb{A}^1 \times \mathbb{A}^1$ of $S$. Proceeding as in §3.2, we map this affine patch to $\mathbb{A}^{r-1} \times \mathbb{A}_t$ via

\[(x, t) \mapsto (x, x^2, \ldots, x^{r-1}; t).\]

The image of the points $\{P_{i,j}\}$ lie on a rational normal curve, so no $r$ of them lie on a hyperplane, and hence Lemma 2.1 shows the code $C$ has locality $r$. \qed

5. Codes on Hirzebruch surfaces

The ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$ is an example of a Hirzebruch surface, which are ruled surfaces determined by a non-negative integer $m$. After recalling some of the geometry of these surfaces, we adapt the construction of codes in §4 to the setting of Hirzebruch surfaces.

5.1. Hirzebruch surfaces $\mathbb{F}(m, 0)$. Let $m \in \mathbb{Z}_{\geq 0}$; we let two copies of the multiplicative group $\mathbb{G}_m \times \mathbb{G}_m$ act on the product of two punctured affine planes $\mathbb{A}^2 \setminus \{(0, 0)\} \times \mathbb{A}^2 \setminus \{(0, 0)\}$ via

\[(\lambda, 1) : (u, v; s, t) \mapsto (u, \lambda^{-m}v; \lambda s, \lambda t) \]

\[(1, \mu) : (u, v; s, t) \mapsto (\mu u, \mu v; s, t).\]

The Hirzebruch surface $S = F(m, 0)$ is the quotient

\[\mathbb{A}^2 \setminus \{(0, 0)\} \times \mathbb{A}^2 \setminus \{(0, 0)\} / \mathbb{G}_m \times \mathbb{G}_m.\]

Such surfaces are endowed with a natural fibration $p: S \to \mathbb{P}^1_{(s, t)}$ given by

\([(u : v), (s : t)] \mapsto (s : t). \quad (5.1)\]

**Lemma 5.1.** Let $S = F(m, 0)$ be as above. The following hold:

1. The Picard group $\text{Pic}(S)$ is isomorphic to $\mathbb{Z}^2$, generated by the classes of the curves $A = \{as + bt = 0\}$ and $B = \{v = 0\}$, which are, respectively, a fiber of (5.1) and the so-called negative section of $S$. 

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(2) The intersection pairing on $\text{Pic}(S)$ is determined by
\[ A^2 = 0, \quad A \cdot B = 1 \quad \text{and} \quad B^2 = -m. \]

(3) Let $M = mA + B \in \text{Pic}(S)$. The canonical divisor $K_S$ is linearly equivalent to $(m - 2)A - 2M$.

(4) For non-negative integers $\alpha, \beta$ satisfying $\alpha \geq m\beta - 1$, the Riemann–Roch space $L(S, \alpha A + \beta B)$ has dimension
\[ \ell(S, \alpha A + \beta B) = (\alpha + 1)(\beta + 1) - m \frac{\beta(\beta + 1)}{2}. \]

Proof. For (1), (2) and (3) see [Rei97], Sections B.2.9 and B.2.7. The Riemann–Roch theorem for surfaces gives the Euler characteristic of the class $\alpha A + \beta B$:
\[ \frac{(\alpha A + \beta B) \cdot (\alpha A + \beta B - K_S)}{2} + 1 = (\alpha + 1)(\beta + 1) - n \frac{\beta(\beta + 1)}{2}. \]

By, e.g., [CH17, Thm. 2.1.], the conditions $\beta \geq 0$ and $\alpha \geq m\beta - 1$ guarantee that this Euler characteristic coincides with the dimension of the Riemann–Roch space $L(S, \alpha A + \beta B)$.

\[ \square \]

Remark 5.2. The morphism $\phi: X \to \bar{X} \subset \mathbb{P}^m$ defined by the sections generating the projectivized Riemann-Roch space $|M|$ is the natural resolution of the cone over the rational normal curve of degree $n$. The map $\phi$ contracts $B$ to the vertex of the cone (see [Rei97, B 2.9]).

5.2. Riemann-Roch spaces for codes. In this section, we give an explicit description of the elements of the Riemann–Roch spaces $V_{\alpha, \beta} := L(S, \alpha A + \beta B)$ appearing in Lemma 5.1. We assume throughout that $\alpha$ and $\beta$ are non-negative integers.

Lemma 5.3. Let $\alpha = \varepsilon + m\beta$ with $\varepsilon \geq 0$. The elements of $V_{\alpha, \beta}$ have the form
\[ \sigma = a_0(s, t)u^\beta + a_1(s, t)u^{\beta-1}v + \cdots + a_\beta(s, t)v^\beta \]
where $a_i(s, t)$ is a homogeneous polynomial of degree $\varepsilon + im$ for $i = 0, \ldots, \beta$. We have
\[ \dim V_{\alpha, \beta} = (\alpha + 1)(\beta + 1) - m \frac{\beta(\beta + 1)}{2}. \]

Proof. Let $\sigma$ be as in the statement of the lemma. First, we show that $\sigma \in V_{\alpha, \beta}$. Since $A$ and $B$ generate $\text{Pic}(S)$, there are $\alpha'$ and $\beta'$ such that $(\sigma) = \alpha'A + \beta'B$ as classes in $\text{Pic}(S)$. To determine $\alpha'$ and $\beta'$ we use the intersection pairing on $\text{Pic}(S)$.

Since $A$ is a curve defined by fixing the ratio $s/t$, we have that
\[ (\sigma) \cdot A = \beta. \]

On the other, since $B = \{v = 0\}$, we see that
\[ (\sigma) \cdot B = \varepsilon. \]
We obtain the system of equations
\[
\beta = (\sigma) \cdot A = \alpha' \cdot A^2 + \beta' A \cdot B = \beta', \\
\varepsilon = (\sigma) \cdot B = \alpha' A \cdot B + \beta' B^2 = \alpha' - m\beta'.
\]
Thus \(\beta' = \beta\) and \(\alpha' = \varepsilon + m\beta' = \alpha\) as claimed. Note that the condition \(a_i(s, t)\) homogeneous of degree \(\varepsilon + im\) ensures that the monomials are invariant under the action \((\lambda, 1) \in \mathbb{G}_m \times \mathbb{G}_m\).

The subspace of \(V_{\alpha, \beta}\) generated by elements of the form \(\text{[5.2]}\) has dimension
\[
k = (\varepsilon + 1) + (\varepsilon + 1 + m) + \cdots + (\varepsilon + 1 + \beta m)
= \sum_{i=0}^{\beta} (\varepsilon + 1) + im
= (\beta + 1)(\varepsilon + 1) + m\frac{\beta(\beta + 1)}{2}
= (\alpha + 1)(\beta + 1) - m\frac{\beta(\beta + 1)}{2}
\]
and hence must be equal to the entire vector space, by Lemma \(\text{[5.1]}(4)\). \(\square\)

5.3. **Construction of codes.** Consider the curves \(C_1\) and \(C_2\) in \(S\) with affine models given by
\[
C_1 : u^{r+1} = t^{\alpha} + 1 \quad \text{and} \quad C_2 : v^{r+1} = t^{\alpha} + 1.
\]
The projective closure of these curves in \(S\) is respectively given by:
\[
s^{\alpha} u^{r+1} - (t^{\alpha} + s^{\alpha} v^{r+1}) = 0 \quad \text{and} \quad s^{\alpha+m(r+1)} u^{r+1} - (t^{\alpha} + s^{\alpha}) u^{r+1} = 0. \tag{5.3}
\]
The left hand sides of the above equations are elements of the vector space \(V_{\alpha+m(r+1),(r+1)}\).

The fibration \(p : S \rightarrow \mathbb{P}^1_{(s,t)}\) in \(\text{[5.1]}\) gives \(S\) the structure of a ruled surface. To construct evaluation codes using \(C_1\) or \(C_2\), pick \(b\) points \((s_i : t_i) \in \mathbb{P}^1_{(s,t)}(\mathbb{F}_q)\) such that the fiber \(p^{-1}((s_i : t_i)) \cap C_k\) consists of \(r+1\) distinct points
\[
(u_{i,1} : v_{i,1}), \ldots, (u_{i,r+1} : v_{i,r+1}).
\]
Put
\[
P_{i,j} = ((u_{i,j} : v_{i,j}), (s_i : t_i)) \in S(\mathbb{F}_q),
\]
so that there are \(n = b(r + 1)\) points of the form \(P_{i,j}\) in total. Choose an integer \(d\) such that
\[
N := \frac{n - d}{r + 1}
\]
is an integer, and set \(\varepsilon = N\) and \(\beta = r - 1\). We shall use the vector space
\[
V_{\varepsilon+m\beta, \beta} = V_{N+m(r-1),r-1}
\]
to construct our evaluation codes.
Proposition 5.4. The code

\[ C := \{ (\sigma(P_{i,j}))_{1 \leq i \leq b, 1 \leq j \leq r+1} : \sigma \in V_{N+n(r-1),r-1} \}, \]
constructed using either \( C_1 \) or \( C_2 \), has locality \( r \) and its parameters satisfy

\[ n = b(r + 1) \]
\[ k = (N + 1)r + m \frac{r(r - 1)}{2} \]
\[ d \leq \vartheta - (r - 1) - m \frac{(r^2 - 1)}{2} \]
\[ d \geq \vartheta - (r - 1)(\alpha + m(r + 1)). \]

Proof. By Lemma 5.2, we have

\[ k = \dim V_{N+m(r-1),r-1} = r(N + 1) + m \frac{r(r - 1)}{2}. \]

Next, if \( r \) is odd or \( m \) even, we have

\[ \left\lceil \frac{k}{r} \right\rceil = N + 1 + m \frac{(r - 1)}{2}. \]

Otherwise,

\[ \left\lceil \frac{k}{r} \right\rceil = N + 1 + m \frac{(r - 1)}{2} + \frac{1}{2} \geq N + 1 + m \frac{(r - 1)}{2}. \]

Hence, an upper bound for \( d \) using (1.1) is

\[ d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2 \]
\[ \leq n - r(N + 1) - m \frac{r(r - 1)}{2} - (N + 1) - m \frac{(r - 1)}{2} + 2 \]
\[ = n - (n - \vartheta) - (r + 1) - m \frac{r(r - 1)}{2} - m \frac{(r - 1)}{2} + 2 \]
\[ = \vartheta - (r - 1) - m \frac{(r^2 - 1)}{2}. \]

As in the proof of Proposition 4.4, a lower bound for the minimum distance of \( C \) is

\[ d \geq n - \max_{\sigma \in V} \# (C_k \cap (\sigma)) \]
\[ \geq n - C_k \cdot (\sigma) \text{ for any } \sigma \in V_{N+m(r-1),r-1} \]

Since the equation for either \( C_1 \) or \( C_2 \) is an element of \( V_{\alpha+m(r+1),(r+1)} \), we may use Lemma 5.1(2) to compute

\[ C_k \cdot (\sigma) = ((\alpha + m(r + 1))A + (r + 1)B) \cdot ((N + m(r - 1))A + (r - 1)B) \]
\[ = (r - 1)(\alpha + m(r + 1)) + (N + m(r - 1))(r + 1) - m(r^2 - 1) \]
\[ = (r - 1)(\alpha + m(r + 1)) + n - \vartheta, \]
and hence
\[ d \geq 0 - (r - 1)(\alpha + m(r + 1)). \]
as claimed. Finally, the locality is \( r \) by the same argument as in the end of the proof of Proposition 4.4.

When \( n = 0 \), we have \( S = \mathbb{F}(0,0) = \mathbb{P}^1 \times \mathbb{P}^1 \). In this case, the bounds on the distance for \( C \) coincide with the bounds of Proposition 4.1, as one would expect.

**Remark 5.5.** The upper and lower bounds for the minimum distance in Proposition 5.4 meet when
\[ 1 + m \frac{(r + 1)}{2} = \alpha + m(r + 1). \]
Since \( \alpha, m \) and \( r \) are nonnegative, we must have \( m = 0 \) (i.e., \( S = \mathbb{P}^1 \times \mathbb{P}^1 \)) and \( \alpha = 1 \).

6. **Locally recoverable codes from elliptic surfaces**

6.1. **Elliptic surfaces.** The definitions of this section hold over an arbitrary field \( k \).

An algebraic surface \( E \) is called an **elliptic surface** if it is endowed with a morphism \( \pi : E \to B \) to a base algebraic curve \( B \) such that

i) for all but finitely many \( t \in B(\bar{k}) \), the fiber \( \pi^{-1}(t) \) is a genus one curve, where \( \bar{k} \) is a fixed algebraic closure of \( k \).

ii) there is a section \( \sigma \) to \( \pi \), i.e., a morphism \( \sigma : B \to E \) such that \( \pi \circ \sigma = \text{id}_B \).

The morphism \( \pi \) is called an elliptic fibration. Condition ii) implies that all but finitely many fibers of \( \pi \) are indeed elliptic curves.

Let \( \pi : E \to B \) be an elliptic fibration. A section \( P : B \to E \) is, by definition, a regular map such that \( \pi \circ P \) is the identity on \( B \). We denote by \( \mathcal{O} \) the zero section and by abuse of notation also the zero element of any fiber. The set of sections of the fibration \( \pi \) in the above sense can be made into an abelian group with identity \( \mathcal{O} \) (in the same way one defines the group law on an elliptic curve). This group is called the Mordell-Weil group of \( E \) and it is finitely generated by the Néron-Severi-Mordell-Weil theorem.

We also have that \( E \) has a Weierstrass equation
\[
y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6
\]
where \( a_i \in k(B) \). We consider the divisor \( D = n \cdot \infty + m \cdot \mathcal{O} \), where \( \infty \) is the “fiber above \( \infty \)”, and \( \mathcal{O} \) is the zero section. A function on \( E \) whose polar divisor is bounded by \( D \) is of the form
\[
\sum_{2i \leq m} \alpha_i x^i + \sum_{2i+3 \leq m} \beta_i x^i y,
\]
where \( \alpha_i \) and \( \beta_i \) are functions in the Riemann Roch space \( \mathcal{L}(B,n \cdot \infty) \).

Each fiber \( E \) is embedded in \( \mathbb{P}^{n-1} \) by the linear system \( |n\mathcal{O}| \) (where \( \mathcal{O} \) is the identity of \( E \)).
6.2. **General code construction.** Let $\pi : \mathcal{E} \to B$ be an elliptic fibration. We denote by $\mathcal{O}$ the zero section and by abuse of notation also the zero element of any fiber. We denote by $E_t = \pi^{-1}(t)$ the fiber above $t$ and by $E_t[2]$ its subgroup of elements of order at most 2.

**Lemma 6.1.** Assume that for each $t$ in a subset of $B(\mathbb{F}_q)$ such that the fiber $E_t$ over $t$ is an elliptic curve, we are given $\Gamma_t \subset E_t(\mathbb{F}_q) - E_t[2]$ all of same cardinality $r + 1$ for some integer $r$ with the property that $\sum_{P \in \Gamma_t} P \in E_t[2]$ in the group law of $E_t$.

Let $\Gamma = \bigcup_t \Gamma_t$ and $V$ a finite dimensional $\mathbb{F}_q$-vector space of functions on $\mathcal{E}$ such that the restriction of any element of $V$ to a fiber above any $t$ is in the Riemann-Roch space $\mathcal{L}(E_t, r\mathcal{O})$. We form a code $C$ by evaluating the functions on $V$ on the points of $\Gamma$. The code $C$ is locally recoverable with locality $r$.

**Proof.** Given a function $f$ and codeword $c = (f(P))_{P \in \Gamma}$ and suppose we need to recover $f(P_0)$. We have that $P_0 \in \Gamma_t$ for some $t$. Now, the restriction of $f$ to $E_t$ is a rational function $f_t$ on $E_t$, which is an element of the Riemann-Roch space $\mathcal{L}(E_t, r\mathcal{O})$. We claim that $f_t(P_0)$ can be uniquely recovered from the values of $f_t(P), P \in \Gamma_t - \{P_0\}$. If there are two such functions with the same values, their difference vanishes at $\Gamma_t - \{P_0\}$ but has a pole of order at most $r$ at $\mathcal{O}$, that would imply $\sum_{P \in \Gamma_t - \{P_0\}} P = \mathcal{O}$ and thus $P_0 \in E_t[2]$, which contradicts our hypothesis. This shows that the map $L(r\mathcal{O}) \to \mathbb{F}_q^r, h \mapsto (h(P))_{P \in \Gamma_t - \{P_0\}}$ is injective. As these spaces have the same dimension by Riemann-Roch, it is also surjective. \qed

A natural example is to take sections $P_i, i = 1, \ldots, r$ of the elliptic fibration $\pi : \mathcal{E} \to B$. If we let $P_{r+1} = - \sum_{i=1}^r P_i$ and $\Gamma_t = \{P_1(t), \ldots, P_{r+1}(t)\}$, we are in the above situation.

We can also use an irreducible curve $C$ in $\mathcal{E}$. Then we have a map $C \to B$ and we assume that it has degree $r + 1$ and take as $\Gamma_t$ the fibers of this map above points that split completely. To ensure that the points of $\Gamma_t$ add to zero we need to check the algebraic point defined by $C$ has trace zero. Often the following lemma is useful.

**Lemma 6.2.** Let $\pi : \mathcal{E} \to B$ be an elliptic surface with finite Mordell-Weil group. Let $C$ be an irreducible curve in $\mathcal{E}$ such that the map $C \to B$ is separable of degree $r + 1$. If, for one $t \in B$ with $\pi^{-1}(t)$ an elliptic curve and whose preimage $\Gamma_t = (\pi|_C)^{-1}(t)$ in $C$ has $r + 1$ distinct points we have that $\sum_{P \in \Gamma_t} P = \mathcal{O}$, then for all other such $t$, we also have $\sum_{P \in \Gamma_t} P = \mathcal{O}$.

**Proof.** We can base change $\pi : \mathcal{E} \to B$ to $\pi' : \mathcal{E}' \to C$ via $C \to B$ and $C$ itself pulls back to a section $s$ of $\pi'$ and we can then take the $C \to B$ trace of this section to get a section of $\pi$. Concretely, this section consists of adding the points on $(\pi|_C)^{-1}(t)$ and viewing that as a function of $t \in B$. By the assumption on the Mordell-Weil group, this section is of finite order. It is known that, for sections of finite order, the specialization map to a smooth fiber is injective. By assumption, for one such fiber, the specialization of $s$ is zero. It follows that $s$ itself is zero. \qed

Here are some explicit examples.
**Example 6.3.** Take $\mathcal{E}$ the Legendre family $y^2 = x(x-1)(x-t)$ and consider the curve $C: (u^2+t+1)^2 = u(u-1)(u-t)$ of genus 1 embedded in $\mathcal{E}$ by taking $x = u, y = u^2+t+1$, so $r = 3$. Lemma 6.3 applies with $t = -1$. If $\Gamma$ has $n$ points and $d < n, 4|\delta(n-d)$, we consider functions of the form $f = a(t) + b(t)x + c(t)y$ with $\deg a \leq (n-d)/4, \deg b, \deg c < (n-d)/4$ and these restrict to $C$ as a function of degree at most $n-d$, so the minimum distance is at least $d$. The dimension is $k = 3(n-d)/4 + 1$ and it follows that $d = n - k - \lceil k/3 \rceil + 2$, i.e., the code is optimal, but typically not as long as the optimal codes from the previous sections.

**Example 6.4.** Let $\mathcal{E}$ be the elliptic surface $y^2 = x^3 + x - t^2 - 1$ over $\mathbb{F}_q$ and $C$ the curve given by $x = y^2$ inside $\mathcal{E}$, which is $y^4 = t^2 + 1$. The elliptic surface has trivial Mordell Weil group over $\mathbb{F}_q(t)$ so the multisection corresponding to $C$ automatically has trace zero. This leads to the same family of codes corresponding to the case $r = 5$ of subsection 3.3 by considering evaluation on functions of the form $f = a_0(t) + a_1(t)x + a_2(t)y + a_3(t)x^2 + a_4(t)xy$.

**Example 6.5.** We can also recover the case $r = 3$ of subsection 3.3 by taking $\mathcal{E}$ to be the elliptic surface $y^2 + xy = x^3 + t^2 + 2$ over $\mathbb{F}_q$ and $C$ the curve given by $x^2 = y = u$ inside $\mathcal{E}$, which is $u^4 = t^2 + 2$ and evaluation on functions of the form $f = a_0(t) + a_1(t)x + a_2(t)y$. We can take, for $q = 5, 13$ respectively, sets of size $b = 2, 4$ and get codes of length $n = 8, 16$.

Yet another example is a variant of the examples constructed by Ulmer [14] leading to the following theorem.

**Theorem 6.6.** For every odd prime (power) $p$ and integer $d \leq 2(p+1)(p-2), (p+1)|d$, there exists a locally recoverable code $C$ over $\mathbb{F}_{p^2}$ of recoverability $p$, length $n = 2(p+1)(p-2)$, minimal distance $d$ and dimension

$$k = \frac{p(n-d)}{p+1} - \frac{p-1}{2}.$$ 

**Proof.** Consider the surface $\mathcal{E}: y^2 = x(x+1)(x+t^2+1)$ over $\mathbb{F}_{p^2}$, $p$ odd and the curve $C$ defined by $u^{p+1} = t^2 + 1$. Then $C$ embeds in $\mathcal{E}$ by taking $x = u, y = u(u+1)^{(p+1)/2}$. The points on $C$ on the fiber above $t = b$ are of the form $(c, c(c+1)^{(p+1)/2})$ for each $c$ satisfying $c^{p+1} = b^2 + 1$. The function $y(x+1)^{(p-1)/2} - (x + b^2 + 1)$ has degree $p+2$ and vanishes on all these points and on the point $(-b^2 - 1, 0)$ of order 2. So Lemma 6.1 applies once we exclude the points on $C$ with $c = 0, c^{p+1} = 1$. Each allowed value of $c$ gives two values of $b$ since $c^{p+1} - 1 \in \mathbb{F}_p$ so has square roots in $\mathbb{F}_{p^2}$. So we have $n = 2(p+1)(p-2)$ points in $C$ we can use to form $\Gamma$.

To construct a code we consider the following vector space, where $x_i = x^{(i+1)/2}, i$ odd and $x_i = yx^{(i-2)/2}, i$ even, $i > 0$.

$$V = \left\{ a_0(t) + \sum_{i=1}^{p-1} a_i(t)x_i : \deg a_i \leq N_i \text{ for } i = 0, \ldots, p-1 \right\}$$

where $N_0 = \frac{n-d}{p+1}$,
\[ N_i = \frac{n-d}{p+1} - 1, \ \text{i odd}, \quad \text{(6.1)} \]
\[ N_i = \frac{n-d}{p+1} - 2, \ \text{i even}, \ i > 0. \]

chosen so that the elements of \( V \) restrict to functions of degree \( n-d \) on \( C \) and the codewords have weight at least \( d \). The dimension \( k \) satisfies
\[ k = \sum_{i=0}^{p-1} (N_i + 1) \]
and the result follows. \( \square \)

**Remark 6.7.** Note that, in the above theorem \( d_{opt} = n - k - \lfloor k/p \rfloor + 2 = d + (p + 3)/2 \).

**Acknowledgements**

The authors would like to thank the following institutions for providing the opportunity for them to meet and/or for financial support: IMPA, BIRS-Oaxaca, MPIM, IHP, University of Canterbury and M. Stoll’s Rational Points workshop series. Cecília Salgado was partially supported by FAPERJ grant E-26/203.205/2016, the Serrapilheira Institute (grant number Serra-1709-17759), Cnpq grant PQ2 310070/2017-1 and the Capes-Humboldt program. Anthony Várilly-Alvarado was partially supported by NSF grants DMS-1352291 and DMS-1902274. José Felipe Voloch was partially supported by the Simons Foundation (grant #234591) and the Marsden Fund Council administered by the Royal Society of New Zealand. He would also like to thank A. Dimakis for a helpful conversation.

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