Weighted theta functions for non-commutative graphs

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Grötschel, Lovász, and Schrijver generalized the Lovász $\vartheta$ function by allowing a weight for each vertex. We provide a similar generalization of Duan, Severini, and Winter’s $\tilde{\vartheta}$ on non-commutative graphs. While the classical theory involves a weight vector assigning a non-negative weight to each vertex, the non-commutative theory uses a positive semidefinite weight matrix. The classical theory is recovered in the case of diagonal weight matrices.

Most of Grötschel, Lovász, and Schrijver’s results generalize to non-commutative graphs. In particular, we generalize the inequality $\vartheta(G, w)\vartheta(G, x) \geq \langle w, x \rangle$ with some modification needed due to non-commutative graphs having a richer notion of complementation. Similar to the classical case, facets of the theta body correspond to cliques and if the theta body anti-blocker is finitely generated then it is equal to the non-commutative generalization of the clique polytope.

We propose two definitions for non-commutative perfect graphs, equivalent for classical graphs but inequivalent for non-commutative graphs.

I. INTRODUCTION

Lovász [22] introduced the $\vartheta$ function of a graph as an upper bound on the Shannon capacity – the independence number regularized under the strong graph product. The $\vartheta$ quantity is an upper bound on independence number, a lower bound on fractional chromatic number, and is multiplicative under the strong and the disjunctive graph products. It is a semidefinite program, hence efficiently computable both in theory and in practice. It is monotone under graph homomorphisms [8]; in fact its bound on independence and chromatic number follow from this.

Further insight into $\vartheta$ is gained by allowing vertices to be weighted [16, 18]. Weights are basically equivalent to duplicating vertices [18] except that weights don’t have to be whole numbers. Aside from only being defined for non-negative weights, the weighted $\vartheta$ of a graph resembles a norm on the weight vector: it scales linearly and is convex. In that language, $\vartheta$ of the complement graph is the dual norm. The set of weights $w$ for which $\vartheta(G, w) \leq 1$ is investigated in [16], where facets of this convex body are shown to correspond to clique constraints. This set is polyhedral if and only if the graph is perfect.

Lovász’s bound can be adapted to quantum channels via a suitable generalization of graphs where an operator subspace takes the place of the adjacency matrix [10]. These so called non-commutative graphs have since drawn interest in connection with quantum channels but also independently of any application. Several classical graph definitions and results carry over to non-commutative graphs, including homomorphisms [5, 25, 27, 30], chromatic numbers [17, 19, 27], Ramsey and Turán theorems [31, 32], asymptotic spectrum [20], a Haemers bound [15], and connectivity [6]. It can happen that there are multiple ways to generalize a particular concept: [3] presents two generalizations of $\vartheta$ distinct from the one in [10] (though possibly the same as each other).

The present work investigates a weighted version of the $\tilde{\vartheta}$ of [10], generalizing most of the results from [16]. We note that [3] defined a weighted version of their $\vartheta$ generalization, and investigated the corresponding theta body. It is not known whether it supports the sort of duality relations we find in this paper.

We will cover basic notation in section II, introduce our weighted $\tilde{\vartheta}$ in section III, prove a duality relation in section IV, extend this to $\mathcal{S}_0$-graphs in section V (with the core proof deferred to appendix A), and in section VI explore the geometry of the theta body for non-commutative graphs and its relation to perfect graphs. The presentation is self-contained, not requiring any background in quantum mechanics or quantum channels.

II. NOTATION AND BASIC DEFINITIONS

Hilbert spaces will be denoted by the symbols $\mathcal{A}, \mathcal{B}, \mathcal{Y}, \mathcal{Z}$. These will always be finite dimensional. Dual spaces are denoted $\mathcal{A}^*$, etc. Linear operators on $\mathcal{A}$ are denoted $\mathcal{L}(\mathcal{A})$; linear maps from $\mathcal{A}$ to $\mathcal{B}$ by $\mathcal{L}(\mathcal{A} \to \mathcal{B})$. The set of positive semidefinite operators on $\mathcal{A}$ is denoted $\text{Pos}(\mathcal{A})$, the set of Hermitian operators by $\text{Herm}(\mathcal{A})$. For operators $X, Y \in \mathcal{L}(\mathcal{A})$, $X \preceq Y$ means $Y - X \in \text{Pos}(\mathcal{A})$. For spaces $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$ we write the quotient
space as \( A/B = A \cap B^\perp \).

We will use Dirac’s bra-ket notation where \( |x\rangle \) is a vector, \( \langle x| \) is its dual, \( \langle x|y\rangle \) is an inner product, and \( \langle x|M|y\rangle \) is an inner product between \( \langle x| \) and \( M|y\rangle \) with \( M \) an operator. The adjoint of an operator is written \( M^\dagger \). Basis vectors are denoted \(|i\rangle\) or \(|j\rangle\) with \( i, j \in \{1, \ldots, n\} \).

For Hilbert spaces \( A \) and \( B \) of equal dimension, we choose an isomorphism \( A^\dagger \to B \), denoted \( |\Phi\rangle \). There is a canonical choice once an orthonormal basis has been chosen:

\[
|\Phi\rangle = \sum_i |i\rangle_A \otimes |i\rangle_B .
\]  

For an operator \( M \in \mathcal{L}(A) \) we define its vectorization \( |M\rangle = (M \otimes I)|\Phi\rangle \in A \otimes B \). We also use this isomorphism to define the transpose, taking it to move an operator from \( \mathcal{L}(A) \) to \( \mathcal{L}(B) \) so that \( (M \otimes I)|\Phi\rangle = (I \otimes M^\dagger)|\Phi\rangle \). Complex conjugate is defined similarly: \( \overline{M} = (M^\dagger)^T \) and \( (M^\dagger \otimes I)|\Phi\rangle = (I \otimes \overline{M})|\Phi\rangle \).

We take the definition of a non-commutative graph from [10].

**Definition 1.** A non-commutative graph is an operator subspace \( S \subseteq \mathcal{L}(A) \) satisfying \( S = S^\dagger \) and \( I \in S \).

In the above definition and throughout we use the shorthand notation \( S^\dagger = \text{span}\{x^\dagger : x \in S\} \), \( S + S' = \text{span}\{x + y : x \in S, y \in S'\} \), \( CI = \text{span}\{xi : x \in C\} \), etc.

Non-commutative graphs are analogous to adjacency matrices, but taken as subspaces rather than 0-1 matrices. Indeed, given any classical graph \( G \) we can define a corresponding non-commutative graph.

**Definition 2.** For a graph \( G \) define \( S_G = \text{span}\{|i\rangle\langle j| : i \sim_G j\} \).

Note that [27] breaks from the above definition, taking non-commutative graphs to be trace free, \( I \perp S \). That works better for graph homomorphisms, which require vertices to not be self adjacent. But for the present work the convention of definitions 1 and 2 is more appropriate.

We take the following definition from [3]. This is an extension to operators of the convex corners defined in [7, 12], which in turn are recovered by restricting to diagonal operators.

**Definition 3.** Let \( A \) be a Hilbert space. A convex corner in \( \mathcal{L}(A) \) is a non-empty closed convex subset \( C \subseteq \text{Pos}(A) \) such that

\[
A \in C \text{ and } 0 \leq B \leq A \implies B \in C.
\]

Condition (2) is called hereditary. For a general subset \( C \subseteq \text{Pos}(A) \) we denote by \( \text{her}(C) \) the hereditary closure, the smallest hereditary set containing \( C \):

\[
\text{her}(C) = \{A \in \text{Pos}(A) : \exists B \in C \text{ s.t. } A \leq B\}.
\]

The smallest convex corner containing a given subset \( C \subseteq \text{Pos}(A) \) of positive semidefinite operators is obtained by taking the hereditary closure of the closure of the convex hull,

\[
\text{her}(\text{conv}(C)).
\]

We then say this convex corner is generated by \( C \). We say a convex corner is finitely generated if it can be generated by a finite set.

Also from [3], and again an extension of a concept from [7, 12], we take the definition of anti-blockers.

**Definition 4.** Let \( A \) be a Hilbert space. The anti-blocker of \( C \subseteq \text{Pos}(A) \) is \( C^\bot = \{B \in \text{Pos}(A) : \text{Tr}(AB) \leq 1 \text{ for all } A \in C\} \).

The following theorem is from [2, lemma 2.2.10 and theorem 2.3.12].

**Lemma 5.** Let \( A \) be a Hilbert space. Convex corners satisfy the following basic facts.

(i) If \( C \subseteq \text{Pos}(A) \) is non-empty then \( C^\bot \) is a convex corner.
(ii) If \( \mathcal{B}, \mathcal{C} \subseteq \text{Pos}(\mathcal{A}) \) and \( \mathcal{B} \subseteq \mathcal{C} \) then \( \mathcal{C}^\perp \subseteq \mathcal{B}^\perp \).

(iii) A non-empty set \( \mathcal{C} \subseteq \text{Pos}(\mathcal{A}) \) satisfies \( \mathcal{C} = \mathcal{C}'' \) if and only if \( \mathcal{C} \) is a convex corner (the second anti-blocker theorem).

Following [3], we will occasionally refer to diagonal convex corners, analogous to definition 3 but restricted to diagonal matrices. Note this is equivalent to the classical notion of anti-blocker from [7, 12]. We will only be referencing diagonal convex corners and diagonal anti-blockers when comparing our theory to the previously established theory of classical (commutative) graphs.

We now construct a weighted version of \( \tilde{\vartheta} \) for classical graphs. Two quantities different from definition 7 are studied in [3]. These are all equivalent for classical graphs, they are in general different when generalized to non-commutative graphs. It is worth mentioning that for classical graphs there are many alternate forms for \( \tilde{\vartheta} \) [16, 18]. While these are all equivalent for classical graphs, they are in general different when generalized to non-commutative graphs. Two quantities different from definition 7 are studied in [3].

The following definition of weighted \( \tilde{\vartheta} \) for classical graphs is derived in a straightforward way from [18, section 6]. It is presented here for background and to motivate a similar definition for non-commutative graphs.

**Definition 6.** Let \( G \) be a graph with \( n \) vertices and \( |w\rangle \in \mathbb{R}^n \) be an entrywise positive vector. Let \( |r\rangle \) be the entrywise square root of \( |w\rangle \). Define

\[
\tilde{\vartheta}(G, w) = \min \{ \lambda : Y \geq |r\rangle \langle r|, Y_{ii} = \lambda, Y_{ij} = 0 \text{ for } i \neq j \}.
\]

The unweighted \( \tilde{\vartheta} \) is recovered by taking \( |w\rangle \) to be the all ones vector.

The non-weighted \( \tilde{\vartheta} \) has been defined for non-commutative graphs by [10], which presents the following equivalent definitions.

**Definition 7.** Let \( S \) be a non-commutative graph.

\[
\tilde{\vartheta}(S) = \max \{ \|T + I \otimes I\| : T \in S^+ \otimes \mathcal{L}(\mathcal{B}), T + I \otimes I \geq 0 \}
\]  
\[
= \max \{ \|\Phi T + I \otimes \rho(\Phi)\| : T \in S^+ \otimes \mathcal{L}(\mathcal{B}), T + I \otimes \rho \geq 0, \rho \geq 0, \text{Tr} \rho = 1 \}
\]  
\[
= \min \{ \|\text{Tr}_A Y\| : Y \in S \otimes \mathcal{L}(\mathcal{B}), Y \geq |\Phi\rangle \langle \Phi| \}
\]  
\[
= \min \{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y \leq \lambda I, Y \geq |\Phi\rangle \langle \Phi| \}
\]

If \( G \) is a graph and \( S_G = \text{span} \{ |i\rangle \langle j| : i \simeq_G j \} \) then \( \tilde{\vartheta}(S_G) = \tilde{\vartheta}(G) \).

It is worth mentioning that for classical graphs there are many alternate forms for \( \tilde{\vartheta} \) [16, 18]. While these are all equivalent for classical graphs, they are in general different when generalized to non-commutative graphs. Two quantities different from definition 7 are studied in [3].

We now construct a weighted version of \( \tilde{\vartheta}(S) \). Observing the similarity between (12) and definition 6, it seems reasonable that the weights should be absorbed into \( |\Phi\rangle \). The weights will be a positive semidefinite operator rather than an entrywise positive vector, which is common for quantum generalizations of classical concepts (cf. density operators vs. probability distributions).

**Definition 8.** Let \( S \) be a non-commutative graph and \( W \in \text{Pos}(\mathcal{A}) \). Let \( |W\rangle = (W \otimes I)|\Phi\rangle \in \mathcal{A} \otimes \mathcal{B} \) and \( |\sqrt{W}\rangle = (\sqrt{W} \otimes I)|\Phi\rangle \). Note that \( \text{Tr}_B \left( |\sqrt{W}\rangle \langle \sqrt{W}| \right) = W \) and \( \text{Tr}_A \left( |\sqrt{W}\rangle \langle \sqrt{W}| \right) = W^T \). Define

\[
\tilde{\vartheta}(S, W) = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y \leq \lambda I, Y \geq |\sqrt{W}\rangle \langle \sqrt{W}| \right\}.
\]
Theorem 9. For any classical graph $G$ and weight vector $|w\rangle$,
\[ \hat{\vartheta}(S_G, \text{diag}(w)) = \vartheta(G, w) \] (14)
where $S_G = \text{span}\{|i\langle j| : i \simeq_G j\}$ and $\text{diag}(w)$ is the diagonal matrix with $\text{diag}(w)_{ii} = w_i$.

Proof. Straightforward generalization of the proof of [10, corollary 12]. \qed

Theorem 9 invites a question: for classical graphs, does our $\tilde{\vartheta}$, by allowing non-diagonal weight matrices, provide additional information about the graph? In section V we will find the answer is no. For classical graphs, $\vartheta$ with non-diagonal weights is a function of $\tilde{\vartheta}$ with diagonal weights (Theorem 29).

Being a semidefinite program, (13) is efficiently computable. It is more computationally expensive than the classical $\vartheta$ because $\vartheta$ is of size $n^2 \times n^2$ rather than $n \times n$ where $n = \text{dim}(A) = |G|$. Experiments with the SCS solver [24, 29] show $n = 9$ takes one minute (i7-6820HQ CPU, circa 2015), with runtime scaling at about $O(n^6)$. Much insight can be gained through numerical experiments even with $n = 3, 4$. And, crucially, semidefinite programs have dual formulations.

Theorem 10. The dual of the semidefinite program (13) is
\[ \hat{\vartheta}(S, W) = \max \left\{ \left\langle \sqrt{W} | T + I \otimes \rho \right| \sqrt{W} \right\} : T \in S^+ \otimes \mathcal{L}(\mathcal{B}), T + I \otimes \rho \geq 0, \rho \geq 0, \text{Tr}\rho = 1 \} \] (15)

Proof. Set $M = |\sqrt{W}\rangle\langle \sqrt{W}|$ and rewrite (13),
\[ \hat{\vartheta}(S, W) = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y - \lambda I \leq 0, M - Y \leq 0 \right\} \] (16)
The Lagrangian is
\[ L(\lambda, Y; R, \rho) = \lambda + \langle \rho, \text{Tr}_A Y - \lambda I \rangle + \langle R, M - Y \rangle = \langle I \otimes \rho - R, Y \rangle + \lambda(1 - \text{Tr}\rho) + \langle R, M \rangle \] (17)

The dual program is then
\[ d^* = \max \left\{ \langle R, M \rangle : I \otimes \rho - R \in S^+ \otimes \mathcal{L}(\mathcal{B}), \text{Tr}\rho = 1, R \geq 0, \rho \geq 0 \right\} \] (19)
Defining $T = R - I \otimes \rho$ gives the right hand side of (15). The point $T = 0, \rho = I_{\mathcal{B}}/\dim(\mathcal{B})$ is in the relative interior of the feasible region so Slater’s condition holds and $d^* = \hat{\vartheta}(S, W)$. \qed

Theorem 11. Let $S \subseteq \mathcal{L}(\mathcal{A})$ be a non-commutative graph and $W \in \text{Pos}(\mathcal{A})$. Let $n = \text{dim}(\mathcal{A})$. We have the following alternate forms for $\hat{\vartheta}(S, W)$, where (23) requires $W$ to be non-singular.
\[ \hat{\vartheta}(S, W) = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y = \lambda I, Y \geq |\sqrt{W}\rangle\langle \sqrt{W}| \right\} \] (20)
\[ = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y = \lambda W^T, Y \geq |W\rangle\langle W| \right\} \] (21)
\[ = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y = \lambda W^T, Y \geq |W\rangle\langle W| \right\} \] (22)
\[ = \min \left\{ \lambda : Y \in S \otimes \mathcal{L}(\mathcal{B}), \text{Tr}_A Y = \lambda W^{-T}, Y \geq |\Phi\rangle\langle \Phi| \right\} \] (23)
\[ \hat{\vartheta}(S, W) = \max \left\{ \left\langle |\sqrt{W}\rangle \otimes I | T + I \otimes I \right| \left( |\sqrt{W}\rangle \otimes I \right) \right\} : T \in S^+ \otimes \mathcal{L}(\mathcal{B}), T + I \otimes I \geq 0, \} \] (24)
\[ = \max \left\{ \left\langle |\sqrt{T + I \otimes I} | (W \otimes I) \right| |\sqrt{T + I \otimes I} \right\} : T \in S^+ \otimes \mathcal{L}(\mathcal{B}), T + I \otimes I \geq 0, \} \] (25)
\[ = \max \left\{ n \left\langle |(\sqrt{W} \otimes I) Y (\sqrt{W} \otimes I) | \right\} : Y \in (S^+ + CI) \otimes \mathcal{L}(\mathcal{B}), Y \geq 0, \text{Tr}_A Y = I \} \] (26)
\[ = \max \left\{ n \left\langle |\sqrt{W} Y | \sqrt{W} \right\} : Y \in (S^+ + CI) \otimes \mathcal{L}(\mathcal{B}), Y \geq 0, \text{Tr}_A Y = I \} \] (27)
Proof. Let $n = \dim(A)$. Let $Y$ be feasible for (13) with value $\lambda$. Set $Y' = Y + n^{-1}I_A \otimes (\lambda I_B - \Tr_A Y)$. Then $\Tr_A Y' = \lambda I$. Since $I_A \in S$ we have $Y' \in S \otimes \mathcal{L}(B)$. Since $\Tr_A Y \leq \lambda I_B$, we have $Y' \geq Y$ and $Y'$ is feasible for (20). Therefore (20) $\leq$ (13).

Let $Y$ be feasible for (20). Define $Y' = (I_A \otimes \sqrt{W^T})Y(I_A \otimes \sqrt{W^T})^\dagger$. Then $\Tr_A Y' = \sqrt{W^T}(\Tr_A Y' \sqrt{W^T} = \lambda W^T$. And $Y' \geq (I_A \otimes \sqrt{W^T} ) |W\rangle \langle W| (I_A \otimes \sqrt{W^T})^\dagger = |W\rangle \langle W|$. So $Y'$ is feasible for (21), giving (21) $\leq$ (20).

Since (22) has a larger feasible region, (22) $\leq$ (21).

Let $Y$ be feasible for (22) and let $V$ be the inverse of $\sqrt{W^T}$, or the pseudoinverse if $W^T$ is singular. Set $Y' = (I_A \otimes V)Y(I_A \otimes V)^\dagger$. Then $\Tr_A Y' \leq \lambda V W^T V^\dagger \leq \lambda I_B$ (the last being equality if $W^T$ is not singular). And $Y' \geq (I_A \otimes V)|W\rangle \langle W|(I_A \otimes V)^\dagger = |W\rangle \langle W|$. So $Y'$ is feasible for (13) giving (13) $\leq$ (22). Therefore (13) $= (20) = (21) = (22)$.

Suppose $W$ is non-singular and let $Y$ be feasible for (22). Then $Y' = (I \otimes W^{-T}) Y (I \otimes W^{-T})^\dagger$ is feasible for (23). Conversely, if $Y'$ is feasible for (23) then $Y = (I \otimes W^T) Y'(I \otimes W^T)^\dagger$ is feasible for (22). Therefore (23) $= (22)$.

Let $T$ be feasible for (24) with value $\lambda$. Let $|\psi\rangle$ be the normalized vector achieving

$$\lambda = \langle \psi | (\sqrt{W} \otimes I)(T + I \otimes I)(\sqrt{W} \otimes I) | \psi \rangle. \quad (28)$$

Let $\rho$ be such that $(I \otimes \sqrt{\rho})|\Phi\rangle = |\psi\rangle$. This requires that $|\psi\rangle$, seen as an operator $\mathcal{L}(B^\dagger \to A)$ (under the isomorphism between $A$ and $B^\dagger$ induced by $|\Phi\rangle$), is positive semidefinite. This is always achievable because (24) is invariant under unitary transform on the $B$ side of $T$. Note that $\langle \psi | \psi \rangle = 1$ gives $\Tr \rho = 1$.

Define $T' = (I \otimes \sqrt{\rho})^\dagger T (I \otimes \sqrt{\rho})$. Then $T' \in S^\perp \otimes \mathcal{L}(B)$ and

$$\lambda = \langle \Phi | (\sqrt{W} \otimes \sqrt{\rho})(T + I \otimes I)(\sqrt{W} \otimes \sqrt{\rho}) | \Phi \rangle \quad (29)$$

$$= \langle \sqrt{W} | (I \otimes \sqrt{\rho})(T + I \otimes I)(I \otimes \sqrt{\rho}) \sqrt{W} \rangle \quad (30)$$

$$= \langle \sqrt{W} | T' + I \otimes \rho \sqrt{W} \rangle \quad (31)$$

So $T'$ is feasible for (15) with value $\lambda$, giving (15) $\geq$ (24).

To show (24) $\geq$ (15) run this proof in reverse, starting with $T'$ being feasible for (15). There are a couple bumps in this road. First, $|\psi\rangle$ may not be the eigenvector for the largest eigenvalue in (28). This is not a significant issue because we only seek $24 \geq \lambda$. The second issue is that finding $T$ satisfying $T' = (I \otimes \sqrt{\rho})^\dagger T (I \otimes \sqrt{\rho})$ requires that the null space of $T'$ contains the null space of $I \otimes \rho$. And indeed this is the case. Let $P$ be the projector onto the null space of $\rho$. Then

$$T' + I \otimes \rho \geq 0 \implies (I \otimes P)^\dagger (T' + I \otimes \rho)(I \otimes P) \geq 0 \quad (32)$$

$$\implies (I \otimes P)^\dagger T'(I \otimes P) \geq 0 \quad (33)$$

Since $I \in S$ and $T' \in S^\perp \otimes \mathcal{L}(A)$, we have $\Tr_A T' = 0$. Therefore

$$0 = \Tr \left((I \otimes P)^\dagger (T'(I \otimes P)\right) \quad (34)$$

$$= \Tr \left( (I \otimes P)^\dagger T'(I \otimes P) \right) \quad (35)$$

Any positive semidefinite operator with vanishing trace vanishes so $(I \otimes P)^\dagger T'(I \otimes P) = 0$; the null space of $T'$ contains the null space of $\rho$.

The equivalence of (24) and (25) follows directly from the relation $\|\sqrt{A} B \sqrt{A}\| = \|\sqrt{B} A \sqrt{B}\|$, valid for all $A, B \geq 0$.

Any solution to (24) can be transformed into a solution to (26) of the same value by taking $Y = n^{-1}(T + I \otimes I)$. Conversely, if $Y$ is a solution to (26) we can take $T = nY - I \otimes I$. Note that $\Tr_A T = 0$ so $Y \in (S^\perp + CI) \otimes \mathcal{L}(B)$ implies $T \in S^\perp \otimes \mathcal{L}(B)$. Therefore (26) $= (24)$.

Similarly, any solution to (15) can be transformed into a solution to (27) by defining $Y = n^{-1}(T + I \otimes \rho)$. Any solution to (27) can be transformed into a solution to (15) by defining $\rho = \Tr_A Y$ and $T = nY - I \otimes \rho$. We have $\Tr_A T = n\Tr_A Y - n\rho = 0$ so $T \in (CI)^\perp \otimes \mathcal{L}(B)$. Since $Y \in (S^\perp + CI) \otimes \mathcal{L}(B)$ we have $T \in S^\perp \otimes \mathcal{L}(B)$. Therefore (27) $= (15)$. \qed
The weighted \( \tilde{\vartheta} \) satisfies all the same basic properties of the classical \( \vartheta \).

**Theorem 12.** For matrices \( W, X \in \text{Pos}(A), W' \in \text{Pos}(A') \), scalar \( \alpha \geq 0 \), and non-commutative graphs \( S \subseteq \mathcal{L}(A), S' \subseteq \mathcal{L}(A') \) the following basic properties hold:

(i) \( S \subseteq S' \implies \tilde{\vartheta}(S, W) \geq \tilde{\vartheta}(S', W) \)

(ii) \( W \leq X \implies \tilde{\vartheta}(S, W) \leq \tilde{\vartheta}(S, X) \)

(iii) \( \tilde{\vartheta}(S, \alpha W) = \alpha \tilde{\vartheta}(S, W) \)

(iv) \( \tilde{\vartheta}(S, W + X) \leq \tilde{\vartheta}(S, W) + \tilde{\vartheta}(S, X) \)

(v) \( \tilde{\vartheta}(\mathcal{C}I, W) = \text{dim}(A)\text{Tr}W \)

(vi) \( \tilde{\vartheta}(\mathcal{L}(A), W) = \|W\| \)

(vii) \( \|W\| \leq \tilde{\vartheta}(S, W) \leq \text{dim}(A)\text{Tr}W \)

(viii) \( \tilde{\vartheta}(S \otimes S', W \otimes W') = \tilde{\vartheta}(S \ast S', W \otimes W') = \tilde{\vartheta}(S, W) \tilde{\vartheta}(S', W') \) where \( S \ast S' = (S/\mathcal{C}I) \otimes \mathcal{L}(A') + \mathcal{L}(A) \otimes (S'/\mathcal{C}I) + \mathbb{C} \otimes I \).

Note that \( S \otimes S' \) is analogous to the strong product for classical graphs and \( S \ast S' \) is analogous to the disjunctive product.

**Proof.** (i): In (13) the feasible set for \( \tilde{\vartheta}(S, W) \) is contained in the feasible set for \( \tilde{\vartheta}(S', W) \).

(ii): The objective function in (25) is monotone in \( W \).

(iii): The objective function in (25) is linear in \( W \).

(iv): The objective function in (25) is subadditive in \( W \).

(v): Consider (20). The conditions \( Y \in S \otimes \mathcal{L}(B) \) and \( \text{Tr}_A Y = \lambda I \) force \( Y = \lambda n^{-1}I \otimes I \) where \( n = \text{dim}(A) \).

The operator norm of \( \sqrt{W} \sqrt{\mathcal{V}} \) is \( \langle \sqrt{W} \mathcal{V} \rangle = (\Phi \mathcal{W} \otimes I \Phi) = \text{Tr}W \). So \( Y \geq \sqrt{W} \sqrt{\mathcal{V}} \iff \lambda n^{-1} \geq \text{Tr}W \).

(vi): Since \( CI \subseteq S \subseteq \mathcal{L}(A) \) this follows from (i), (v), and (vi).

(vii): Suppose \( S \subseteq \mathcal{L}(A) \) and \( S' \subseteq \mathcal{L}(A') \). Let \( Y, \lambda \) and \( Y', \lambda' \) be optimal for (20) so \( \lambda = \tilde{\vartheta}(S, W) \) and \( \lambda' = \tilde{\vartheta}(S', X) \). Then \( Y \otimes Y' \in \mathcal{L}(A \otimes A' \otimes B \otimes B') \) is feasible for (20) for \( \tilde{\vartheta}(S \otimes S', W \otimes X) \) with value \( \lambda \lambda' \), giving \( \tilde{\vartheta}(S \otimes S', W \otimes X) \leq \tilde{\vartheta}(S, W) \tilde{\vartheta}(S', X) \).

Since \( S \otimes S' \subseteq S \ast S' \), (i) gives \( \tilde{\vartheta}(S \ast S', W \otimes X) \leq \tilde{\vartheta}(S \otimes S', W \otimes X) \).

Let \( Y \) and \( Y' \) be optimal for (26). Then \( Y \in (S/\mathcal{C}I) \otimes \mathcal{L}(B) = (S/\mathcal{C}I)^\perp \otimes \mathcal{L}(B) \). Similarly, \( Y' \in (S'/\mathcal{C}I)^\perp \otimes \mathcal{L}(B') \).

\[
Y \otimes Y' \in (S/\mathcal{C}I)^\perp \otimes (S'/\mathcal{C}I)^\perp \otimes \mathcal{L}(B \otimes B')
\]

\[
= (S/\mathcal{C}I \otimes \mathcal{L}(A') + \mathcal{L}(A) \otimes S'/\mathcal{C}I)^\perp \otimes \mathcal{L}(B \otimes B')
\]

\[
= ((S \ast S')^\perp + \mathcal{C}I \otimes I) \otimes \mathcal{L}(B \otimes B').
\]

And \( \text{Tr}_{\mathcal{A} \otimes \mathcal{A}'}(Y \otimes Y') = (\text{Tr}_A Y)(\text{Tr}_A Y') = I_{B \otimes B'} \) so \( Y \otimes Y' \) is feasible for (26) for \( \tilde{\vartheta}(S \otimes S', W \otimes W) \). Since operator norm is multiplicative under tensor product and \( \sqrt{W} \otimes \sqrt{X} = \sqrt{W} \otimes \sqrt{X} \), the value of this solution is \( \tilde{\vartheta}(S, W) \tilde{\vartheta}(S', X) \). Therefore \( \tilde{\vartheta}(S \otimes S', W \otimes W) \geq \tilde{\vartheta}(S, W) \tilde{\vartheta}(S', X) \).

\[\Box\]

**Theorem 13.** \( \tilde{\vartheta}(S, W) \) is uniformly continuous in \( W \). In fact, for \( W, X \in \text{Pos}(A) \) we have \( |\tilde{\vartheta}(S, W) - \tilde{\vartheta}(S, X)| \leq n \|W - X\|_{\text{Tr}} \), where \( n = \text{dim}(A) \).

**Proof.** Let \( W, X \geq 0 \). Using theorems 12(ii), 12(iv) and 12(viii), we have

\[
\tilde{\vartheta}(S, W) = \tilde{\vartheta}(S, X + (W - X)) \leq \tilde{\vartheta}(S, X + |W - X|)
\]

\[
\leq \tilde{\vartheta}(S, X) + \tilde{\vartheta}(S, |W - X|)
\]

\[
\leq \tilde{\vartheta}(S, X) + n \|W - X\|_{\text{Tr}}.
\]

Therefore \( \tilde{\vartheta}(S, W) - \tilde{\vartheta}(S, X) \leq n \|W - X\|_{\text{Tr}} \). Similar logic with \( W \) and \( X \) swapped yields \( \tilde{\vartheta}(S, X) - \tilde{\vartheta}(S, W) \leq n \|W - X\|_{\text{Tr}} \).

\[\Box\]
For classical graphs there is a formulation of $\vartheta(G, w)$ which makes clear that $\{w \geq 0 : \vartheta(G, w) \leq 1\}$ is in fact a spectrahedral shadow [18, section 29]. As a side note, this form tends to be more powerful for adding extra constraints [13].

$$\vartheta(G, w) = \min \left\{ \lambda : \left[ \frac{\lambda}{|w|} \right] \preceq Z \right\} \geq 0, \text{diag}(Z) = w, Z_{ij} = 0 \text{ for } i \neq j, i \sim j \right\} \quad (43)$$

Something similar can be done for non-commutative graphs. Note this optimization is over $(n^2+1) \times (n^2+1)$ matrices where $n = \text{dim}(A)$.

**Theorem 14.**

$$\tilde{\vartheta}(S, W) = \min \left\{ \lambda : \left[ \frac{\lambda}{|W|} \right] \preceq Z \right\} \geq 0, Z \in S \otimes \mathcal{L}(B), \text{Tr}_A Z = W^T \right\}. \quad (44)$$

**Proof.** Taking (21) from theorem 11 and defining $Z = \lambda^{-1} Y$ gives

$$\tilde{\vartheta}(S, W) = \min \{ \lambda : Z \in S \otimes \mathcal{L}(B), \text{Tr}_A Z = W^T, Z - \lambda^{-1}|W| \preceq 0 \}. \quad (45)$$

But $Z - \lambda^{-1}|W| \preceq 0$ is the Schur complement of the block matrix in (44), so its positive semidefiniteness is equivalent to the positive semidefiniteness of that block matrix. \qed

Fundamental to the theory of weighted $\vartheta$ for classical graphs is the *theta body*.

**Definition 15.** The *theta body* of a graph $G$, denoted $\text{TH}(G)$, is a set of entrywise non-negative vectors given by the following equivalent definitions.

$$\text{TH}(G) = \{ x \geq 0 : \vartheta(G, x) \leq 1 \} = \{ x \geq 0 : \langle y | x \rangle \leq 1 \text{ for all } y \geq 0, \vartheta(G, y) \leq 1 \}. \quad (46) (47)$$

We define this for non-commutative graphs by extending (47) rather than (46) because graph complement is more subtle for non-commutative graphs. (Complements of non-commutative graphs will be explored in the following sections.) Note that [3], having defined a different theta, defines a different theta body, which we will not be investigating here.

**Definition 16.** The *theta body* for a non-commutative graph $S$ is

$$\text{TH}(S) = \{ X \in \text{Pos}(A) : \text{Tr}(XW) \leq 1 \text{ for all } W \in \text{Pos}(A), \tilde{\vartheta}(S, W) \leq 1 \}. \quad (48)$$

Though we define $\text{TH}(S)$ above, we will generally be more interested in its anti-blocker, $\text{TH}^\ddagger(S)$. This in fact has a simpler definition as we shall now see.

**Theorem 17.** The theta body of $S$ satisfies the following basic properties.

(i) $\text{TH}(S)$ is a convex corner.

(ii) $\text{TH}^\ddagger(S) = \{ W \in \text{Pos}(A) : \tilde{\vartheta}(S, W) \leq 1 \}$.  

(iii) $\tilde{\vartheta}(S, W) = \max\{\text{Tr}(XW) : X \in \text{TH}(S)\}$.  

**Proof.**  

(i): $\text{TH}(S) = \{ W \in \text{Pos}(A) : \tilde{\vartheta}(S, W) \leq 1 \}$. By theorem 5(i), the anti-blocker of any non-empty subset of $\text{Pos}(A)$ is a convex corner.

(ii): Define $\mathcal{C} = \{ W \in \text{Pos}(A) : \tilde{\vartheta}(S, W) \leq 1 \}$. From the monotonicity and convexity of $\tilde{\vartheta}$ (theorems 12(ii) and 12(iv)) it follows that $\mathcal{C}$ is a convex corner. And $\mathcal{C}^\ddagger = \text{TH}(S)$ follows directly from definition 16 and the definition of anti-blocker. By the second anti-blocker theorem (theorem 5(iii)), $\mathcal{C} = \mathcal{C}^{\ddagger\ddagger} = \text{TH}^\ddagger(S)$.

(iii): For $W \in \text{Pos}(A)$ we have

$$\max\{\text{Tr}(XW) : X \in \text{TH}(S)\} \leq 1 \iff W \in \text{TH}^\ddagger(S) \iff \tilde{\vartheta}(S, W) \leq 1. \quad (49) \quad (50)$$

The first implication follows from the definition of anti-blocker, the second from (ii). Then (iii) follows from linearity of $\tilde{\vartheta}$. \qed
Though we’ve formed the definitions in terms of $\text{TH}(S)$ for historical reasons, we will generally find more use for $\text{TH}^\#(S)$. The following theorem shows that $\text{TH}^\#(S)$ is a spectrahedral shadow.

**Theorem 18.**

$$\text{TH}^\#(S) = \left\{ W \in \text{Pos}(A) : \frac{1}{|W|} \left\langle \frac{W}{Z} \right\rangle \geq 0, Z \in S \otimes \mathcal{L}(B), \text{Tr}_A Z = W^T \right\}. \tag{51}$$

**Proof.** Follows from theorem 14 and theorem 17(ii). \qed

It is worth noting that not only is $\text{TH}^\#(S)$ a convex corner, it is in fact available as an SDP subroutine. That is, the variables and constraints from (51) can be part of a larger SDP. For example, we could find the $W \in \text{TH}^\#(S)$ that maximizes the inner product $\text{Tr}(WX)$ for some given $X$, the subject of section IV. Or the same but constraining $W$ to be block diagonal. Or something more exotic such as the entropy of this convex corner. Entropy of convex corners is discussed in [2, section 2.4.3] and computing entropy via SDP in [11]. Such numerical investigations have been instrumental in discovering many of the theorems in this paper.

**IV. DUALITY**

Classical $\vartheta$ functions satisfy an interesting duality relation [16]. For $w, x \geq 0$ (entrywise positive vectors),

$$\vartheta(G, w) \vartheta(G^\perp, x) \geq \langle w | x \rangle \tag{52}$$

and for every $w$ there is some $x$ that saturates this inequality. Equivalently,

$$\vartheta(G, x) = \max\{ \langle w | x \rangle : \vartheta(G, w) \leq 1 \}. \tag{53}$$

And in terms of theta bodies,

$$\text{TH}(G) = \text{TH}^\#(G) \tag{54}$$

where $\flat$ is the classical anti-blocker, $\text{TH}^\#(G) = \{ w \geq 0 : \langle w | x \rangle \leq 1 \text{ for all } x \in \text{TH}(G) \}$

The goal of this section is an analogous theorem for non-commutative graphs:

**Theorem 19.** Let $S \in \mathcal{L}(A)$ be a non-commutative graph and $n = \text{dim}(A)$. For any $W, X \in \text{Pos}(A)$,

$$\tilde{\vartheta}(S, W) \tilde{\vartheta}(S^\perp + CI, X) \geq n \text{Tr}(WX) \tag{55}$$

and for every $W$ there is some $X$ that saturates this inequality. Equivalently,

$$\tilde{\vartheta}(S^\perp + CI, X) = \max\{ n \text{Tr}(WX) : W \geq 0, \tilde{\vartheta}(S, W) \leq 1 \}. \tag{56}$$

In terms of theta bodies,

$$\text{TH}^\#(S^\perp + CI) = n^{-1} \text{TH}(S) \tag{57}$$

Notice the extra factor $n$ compared to (52). Although our weighted thetas match the classical definition for classical graphs (theorem 9), our graph complement is less dense and therefore yields a larger $\tilde{\vartheta}$. Specifically, the classical complement would be $\text{span}\{ S^\perp, |i\rangle\langle i| : i \in \{1, \ldots, n\} \}$ whereas we use here $\text{span}\{ S^\perp, I \}$. In section V we will investigate this further, showing that the diagonal elements added (or, in this case, not added) into the graph complement are responsible for this scaling factor.

Before proving theorem 19 we record a simple lemma.

**Lemma 20.** For $M \geq 0$, the following are equivalent:

(i) $M \geq |x\rangle\langle x|$

(ii) There is some $|h\rangle$ such that $\langle h | h \rangle \leq 1$ and $\sqrt{M} |h\rangle \langle h| \sqrt{M} = |x\rangle\langle x|$. 
Proof. The second implies the first because $I \geq |h\rangle\langle h| \implies M \geq \sqrt{M}|h\rangle\langle h|\sqrt{M}$. On the other hand if the first is true then take $|h| = L|x|$ where $L$ is the pseudo-inverse of $\sqrt{M}$. Then $\sqrt{M}|h\rangle = P|x|$ where $P$ is the projector onto the support of $M$. But $M \geq |x\rangle\langle x|$ requires $|x\rangle$ to be in the support of $M$ so $P|x| = |x\rangle$.

Proof of theorem 19. Fix $X \geq 0$ and consider the optimization in (56). For brevity, define the set $F = \{ Y \in Y \otimes \mathcal{L}(\mathcal{B}) : Y \geq 0, \text{Tr} A Y = I \}$. Using (20),

$$\max \{ \text{Tr}(W X) : W \geq 0, \tilde{\vartheta}(S, W) \leq 1 \} = \max \left\{ \text{Tr}(W X) : W \geq 0, Y \in F, Y \geq |\sqrt{W}\rangle\langle \sqrt{W}| \right\} = \max \left\{ \langle \sqrt{W}|X \otimes I|\sqrt{W}\rangle : W \geq 0, Y \in F, Y \geq |\sqrt{W}\rangle\langle \sqrt{W}| \right\}$$

Using lemma 20,

$$(60) = \max \left\{ \langle \sqrt{W}|X \otimes I|\sqrt{W}\rangle : W \geq 0, Y \in F, \langle h|h\rangle \leq 1, \sqrt{Y}|h\rangle = |\sqrt{W}| \right\} = \max \left\{ \langle h|\sqrt{Y}(X \otimes I)|\sqrt{h}\rangle : \langle h|h\rangle \leq 1, Y \in F \right\}$$

For the equality in (62) we make use of unitary freedom on $\mathcal{L}(\mathcal{B})$. That is, for any feasible solution to (62), to get a feasible solution to the prior equation we need that $\sqrt{Y}|h\rangle = |\sqrt{W}|$ for some $W$. This requires that $\sqrt{Y}|h\rangle$, viewed as an operator $\mathcal{L}(\mathcal{B}^\dagger \rightarrow \mathcal{A})$ (under the isomorphism between $\mathcal{A}$ and $\mathcal{B}^\dagger$ induced by $|\Phi\rangle$), is positive semidefinite. That can be achieved by applying some unitary on the $\mathcal{B}$ side of $Y$, which is allowed because the condition $Y \in F$ allows that unitary degree of freedom. Continuing,

$$(62) = \max \left\{ \|\sqrt{Y}(X \otimes I)\sqrt{Y}\| : Y \in F \right\} = \max \left\{ \|Y(X \otimes I)(X \otimes I)^\dagger\| : Y \in F \right\}$$

Using (26) from theorem 11 and the fact that $Y \in S \otimes \mathcal{L}(\mathcal{B}) \iff Y \in ((S^\perp + CI)^\perp + CI) \otimes \mathcal{L}(\mathcal{B})$ we have

$$n^{-1}\tilde{\vartheta}(S^\perp + CI, X).$$

Therefore (56) holds.

As for (57), we have

$$\text{TH}(S^\perp + CI) = \{ X \geq 0 : \tilde{\vartheta}(S^\perp + CI, X) \leq 1 \}$$

$$= \{ X \geq 0 : \max \{ n \text{Tr}(W X) : W \geq 0, \tilde{\vartheta}(S, W) \leq 1 \} \leq 1 \}$$

$$= \{ X \geq 0 : n \text{Tr}(W X) \leq 1 : \forall W \geq 0, \tilde{\vartheta}(S, W) \leq 1 \}$$

$$= \{ n^{-1}X \geq 0 : \text{Tr}(W X) \leq 1 : \forall W \geq 0, \tilde{\vartheta}(S, W) \leq 1 \}$$

Taking the anti-blocker of both sides and applying the second anti-blocker theorem yields (57).

V. GRAPHS WITH A BLOCK STRUCTURE

The main result of the previous section, theorem 19, is not a true generalization of the classical $\vartheta$ duality relation (52). It has two problems: an extra factor of $n$ and a different type of graph complement. The purpose of this section is to remedy both of these problems, which as we will see are related. If $G$ is a classical graph, $S_G = \text{span}\{ |i\rangle\langle j| : i \approx_G j \}$ is the corresponding non-commutative graph. The non-commutative graph for $\overline{G}$ is then $S_{\overline{G}} = S_{G^\perp} + \mathcal{D}$ where $\mathcal{D}$ is the space of diagonal matrices. This differs from the inverse $S^\perp + CI$.
used in theorem 19. In general, how are we to know whether $S^\perp + CI$ or $S^\perp + D$ is the appropriate complement? Or something different from either of these?

The answer comes from [9], the extended arXiv version of [10]. The non-commutative graph associated with a classical graph has a special linear algebraic structure. We take the following definition from [9] but omit discussion of the $S_0$-valued inner product, which we will not need.

**Definition 21.** For a Hilbert space $A$ and a $C^*$-algebra $S_0 \subseteq \mathcal{L}(A)$, a non-commutative graph $S$ is said to be an $S_0$-graph if $S$ is an $S_0$ bimodule, i.e., $S_0 \subseteq S$ and $S_0SS_0 = S$.

The space $S_0$ can be thought of as the “vertices” of $S$. For our purposes $S_0$ will be significant in a number of ways. We will use it to define the graph complement, taking $S^c = S^\perp + S_0$. We will find that convex corners associated with $\vartheta$ or with cliques end up taking maximal values in $S_0\perp$, the commutant of $S_0$. We will find the shape of $S_0$ to factor into a generalized version of theorem 19, taking the place of the spurious factor of $n$ appearing in that theorem.

Note that if $S$ is an $S_0$-graph and $T_0$ is a subalgebra of $S_0$ then $S$ is also a $T_0$-graph. In particular, any $S_0$-graph is also a $CI$-graph. But when speaking of $S$ as being an $S_0$-graph or a $CI$-graph, the corresponding complement graphs will be different.

If $G$ is a classical graph and $S_G = \text{span}\{|i\langle j| : i \in G, j \rangle\}$ then $S_G$ is a $D$-graph with $D = \{|i\langle i| : i \in V(G)\}$. And $S_G^\perp = S_G^\perp + D$ is also a $D$ graph. In general we will take the complement of an $S_0$-graph to be $S^\perp + S_0$.

**Theorem 22.** If $S$ is an $S_0$-graph, then $S^c = S^\perp + S_0$ is also an $S_0$-graph.

**Proof.** Clearly $S_0 \in S^c$. It remains to show $S_0S^cS_0 = S^c$. Suppose $X \in S^\perp$ and $K,L \in S_0$. We will show $KXL \in S^\perp$. For any $Y \in S$ we have

\[
\text{Tr}(Y^\dagger KXL) = \text{Tr}(LY^\dagger KX) = \text{Tr}((K^\dagger YL^\dagger)^\dagger X). \tag{70}
\]

But $K^\dagger, L^\dagger \in S_0$ and $S_0SS_0 = S$ so $K^\dagger YL^\dagger \in S$. Since $X \in S^\perp$, the trace vanishes and we have $KXL \in S^\perp$. Therefore $S_0S^\perp S_0 \subseteq S^\perp$. And $I \in S_0$ so $S^\perp = IS^\perp I \subseteq S_0S^\perp S_0$. Thus we have $S^\perp = S_0S^\perp S_0 and

\[
S_0S^cS_0 = S_0(S^\perp + S_0)S_0 \tag{72}
\]

\[
= S_0S^\perp S_0 + S_0S_0S_0 \tag{73}
\]

\[
= S^\perp + S_0. \tag{74}
\]

\[\square\]

Going forward, we will be using a particular decomposition of the $S_0$ space. The structure theorem for a finite dimensional $C^*$-algebra $S_0 \subseteq \mathcal{L}(A)$ gives [9]

\[
S_0 = \bigoplus_{i=1}^r \mathcal{L}(A_i) \otimes I_{Y_i} \quad \text{with} \quad \mathcal{A} = \bigoplus_{i=1}^r A_i \otimes Y_i. \tag{75}
\]

For a $C^*$-algebra $S_0 \subseteq \mathcal{L}(A)$, we denote by $S_0'$ its commutant: the set of operators in $\mathcal{L}(A)$ commuting with every element of $S_0$. This is also a $C^*$-algebra, and takes the form

\[
S_0' = \bigoplus_{i=1}^r I_{A_i} \otimes \mathcal{L}(Y_i). \tag{76}
\]

The commutant will play a significant role as we move forward: we will show the maximal values of TH$^2(S)$ are in $S_0'$. Classical graphs have the peculiar property that $S_0' = S_0$.

The duality relations we will find for $S_0$ graphs have a more complicated structure, requiring the use of various projection and scaling operators.
Lemma 25. Let $S$ be an $S_0$-graph, with $S_0$ decomposed as in (75). Let $P_i$ be the projector onto $A_i \otimes Y_i$. For $W \in \mathcal{L}(A)$ define
\[
\Delta(W) = \sum_i \dim(A_i)^{-1} I_{A_i} \otimes \text{Tr}_{A_i}(P_i WP_i) \quad (77)
\]
\[
\Psi(W) = \sum_i \dim(Y_i)^{-1} I_{A_i} \otimes \text{Tr}_{A_i}(P_i WP_i) \quad (78)
\]
\[
D = \sum_i \dim(A_i)^{-1} \dim(Y_i)P_i. \quad (79)
\]

Lemma 24. The entities from definition 23 satisfy the following basic facts.

(i) $\Psi$ and $\Delta$ are completely positive superoperators.
(ii) $\Delta(W)$ is the projector onto the subspace $S_0'$, and in particular if $W \in S_0'$ then $\Delta(W) = W$.
(iii) If $X \in S_0'$ then $\Psi(XWX^\dagger) = X\Psi(W)X^\dagger$ and $\Delta(XWX^\dagger) = X\Delta(W)X^\dagger$.
(iv) $\Delta(W) = \Psi(\sqrt{D}W\sqrt{D})$ and $\Psi(W) = \Delta(D^{-1/2}WD^{-1/2})$.
(v) $S_0'^{\perp}$ is in the null space of $\Psi$.

Proof. (i): Conjugation by $P_i$, partial trace, and direct product with a positive semidefinite operator are all completely positive operations; the composition and sum of these operations is also completely positive.

(ii): That $\Delta(W) = W$ for all $W \in S_0'$ is clear by inspection. On the other hand, suppose $W \in S_0'^{\perp}$. Since $I_{A_i} \otimes M \in S_0'$ for any $1 \leq i \leq r$ and $M \in \mathcal{L}(Y_i)$ (taking the other terms of the direct sum in (76) to be zero), we have
\[
0 = \text{Tr}((I_{A_i} \otimes M)P_i WP_i) = \text{Tr}(MP_i WP_i). \quad (80)
\]
Therefore all terms of (77) vanish and $\Delta(W) = 0$.

(iii): Let $X \in S_0'$. Then $X = \oplus_i (I_{A_i} \otimes X_i)$ with $X_i \in \mathcal{L}(Y_i)$. We have
\[
\Psi(XWX^\dagger) = \sum_i \dim(Y_i)^{-1} I_{A_i} \otimes \text{Tr}_{A_i}(P_i XWX^\dagger P_i). \quad (82)
\]
\[
= \sum_i \dim(Y_i)^{-1} I_{A_i} \otimes X_i \text{Tr}_{A_i}(P_i WP_i)X_i \quad (83)
\]
\[
= X \left( \sum_i \dim(Y_i)^{-1} I_{A_i} \otimes \text{Tr}_{A_i}(P_i WP_i) \right) X \quad (84)
\]
\[
= X \Psi(W)X. \quad (85)
\]

The derivation for $\Delta(XWX^\dagger) = X\Delta(W)X^\dagger$ is analogous.

(iv): Since $\Delta$ and $\Psi$ differ only by scaling of the blocks, we have $\Delta(W) = \sqrt{D}\Psi(W)\sqrt{D}$. And $\sqrt{D} \in S_0'$ so by (iii) can be moved inside the parentheses, giving $\Delta(W) = \Psi(\sqrt{D}W\sqrt{D})$. Since $D$ is invertible, we also have $\Psi(W) = D^{-1/2}\Delta(W)D^{-1/2} = \Delta(D^{-1/2}WD^{-1/2})$.

(v): Let $W \in S_0'^{\perp}$. Since $\Delta$ is the projector onto $S_0'$ we have $\Delta(W) = 0$. But as shown above, $\Psi(W) = D^{-1/2}\Delta(W)D^{-1/2}$, so this vanishes as well. \hfill \Box

The following lemma explores the effect on $\hat{\vartheta}$ of removing the vertex set $S_0$ from $S$ and replacing it with the “thin” vertex set $\mathcal{C}I$. Though it is of little interest on its own, this is the core technical lemma from which we build the rest of the results of this section.

Lemma 25. Let $S$ be an $S_0$-graph and take $\Psi$ from definition 23. Then
\[
\hat{\vartheta}(S/S_0 + \mathcal{C}I,W) = \hat{\vartheta}(S, n\Psi(W)). \quad (86)
\]
In particular, $\hat{\vartheta}(S/S_0 + \mathcal{C}I,W)$ is insensitive to any component of $W$ perpendicular to $S_0'$, the commutant of $S_0$. 


Proof. The proof details are tedious and are deferred to appendix A. Here we tie together the results from that appendix.

If $W$ is not singular we have by theorem 64 $\tilde{\vartheta}(S/S_0 + CI, W) \leq \tilde{\vartheta}(S, n\Psi(W))$ and by theorem 67 $\tilde{\vartheta}(S/S_0 + CI, W) \geq \tilde{\vartheta}(S, n\Psi(W))$. This is extended to singular $W$ by continuity of $\vartheta$, theorem 13.

By theorem 24(v), $\Psi(W)$ is insensitive to any component of $W$ perpendicular to $S_0$. Since $\tilde{\vartheta}(S, \Psi(W))$ is insensitive to this component, $\tilde{\vartheta}(S/S_0 + CI, W)$ is as well. □

### Corollary 26.
Let $G$ be a classical graph. Let $S = CI + \text{span}\{[i|j] : i \sim_G j\}$. Then for any $W \in \text{Pos}(A)$,

$$\tilde{\vartheta}(S, W) = |V(G)|\vartheta(G, \text{diag}(W)).$$  \hspace{1cm} (87)

In terms of proof complexity, lemma 25 is the main technical result of this paper. But this theorem tells us something about $\tilde{\vartheta}(S, \Psi(W))$ whereas what we really want is insight about $\tilde{\vartheta}(S, W)$. That is the goal of the remainder of this section. First we need a couple lemmas about convex corners.

### Lemma 27.
Every convex corner $C$ has non-empty relative interior. And there is some $M \in \text{relint}(C)$ such that $PCP = C$ where $P$ is the projector onto the support of $M$.

**Proof.** Let $M \in C \subseteq L(A)$ have the largest rank possible. If $M$ is full rank we are done, as $M/2$ is in the interior of $C$. Otherwise, let $P$ be the projector onto the support of $M$. Note that $PXP = X$ for all $X \in C$ because otherwise $(M + X)/2 \in C$ would contradict $M$ having the largest possible rank. Then $\{X \in L(A) : PXP = X\}$ defines a subspace containing $C$ and $M/2$ is in the interior of $C$ relative to this subspace. □

### Lemma 28.
Let $C \subseteq \text{Pos}(A)$ be a convex corner and $D \subseteq L(A)$ be a subspace with $D^\perp = D$. Suppose the convex corner has no structure in directions perpendicular to $D$,

$$W \in C, E \in D^\perp, W + E \geq 0 \implies W + E \in C$$  \hspace{1cm} (88)

Then $C^\parallel = \text{her}(C^\parallel \cap D)$ where her is the hereditary closure (3). In other words, the maximal elements of $C^\parallel$ are all in $D$ (assuming $C^\parallel$ is bounded).

**Proof.** Let $X \in C^\parallel$. We will show there is a $Y \in C^\parallel \cap D$ such that $Y \geq X$. Since $X \in C^\parallel$ we have $\max\{\text{Tr}(X^\dagger W) : W \in C\} \leq 1$, or using (88), $p^* \leq 1$ where

$$p^* = \max\{\text{Tr}((W + E)^\dagger X) : W \in C, E \in D^\perp, W + E \geq 0\}.  \hspace{1cm} (89)$$

This is a convex optimization problem on the convex domain $G = \{(W, E) : W \in C, E \in D^\perp\}$. We will find the dual by the recipe in [4, chapter 5]. The Lagrangian is

$$L(W, E, Q) = \text{Tr}((W + E)^\dagger X) + \text{Tr}((W + E)^\dagger Q)$$

\hspace{1cm} (90)

$$= \text{Tr}(W^\dagger (X + Q)) + \text{Tr}(E^\dagger (X + Q))$$  \hspace{1cm} (91)

on the domain $(W, E) \in G, Q \geq 0$. For a given $Q$ the Lagrange dual is the maximization of $L$ over the domain of the primal,

$$g(Q) = \max\{L(W, E, Q) : (W, E) \in G\}$$

\hspace{1cm} (92)

$$= \max\{\text{Tr}(W^\dagger (X + Q)) + \text{Tr}(E^\dagger (X + Q)) : (W, E) \in G\}$$  \hspace{1cm} (93)

$$= \begin{cases} 
\max\{\text{Tr}(W^\dagger (X + Q)) : W \in C\} & \text{if } X + Q \in D \\
\infty & \text{otherwise.} 
\end{cases}$$  \hspace{1cm} (94)

The dual of (89) is the minimization of this over $Q \geq 0$,

$$d^* = \min\{g(Q) : Q \geq 0\}$$

\hspace{1cm} (95)

$$= \min\{\max\{\text{Tr}(W^\dagger (X + Q)) : W \in C\} : X + Q \in D, Q \geq 0\}$$  \hspace{1cm} (96)

$$= \min\{\max\{\text{Tr}(W^\dagger Y) : W \in C\} : Y \in D, Y \geq X\}.  \hspace{1cm} (97)$$

Since $C$ has non-empty relative interior by lemma 27, (89) is strictly feasible and Slater’s condition applies, giving $d^* = p^*$. But $p^* \leq 1$ so $d^* \leq 1$ and there is a feasible solution to (97) with $\max\{\text{Tr}(W^\dagger Y) : W \in C\} \leq 1$.

In other words $Y \in C^\parallel$. Since $Y \in C^\parallel \cap D$ and $Y \geq X$, we have shown that any $X \in C^\parallel$ is dominated by an element of $C^\parallel \cap D$. □
We are now in position to leverage lemma 25 into a statement about how the structure of the vertex set $S_0$ affects $\vartheta(S,W)$.

**Theorem 29.** Let $S$ be an $S_0$-graph and $S^c = S^\perp + S_0$ its complement. Take $\Psi$ from definition 23. Then

\[
\text{TH}(S) = \{ W \geq 0 : \Psi(W) \in \text{TH}^\sharp(S^c) \} \tag{98}
\]
\[
\text{TH}^\sharp(S) = \text{her}(\text{TH}^\sharp(S) \cap S_0') \tag{99}
\]

where $S_0'$ is the commutant of $S_0$. Equivalently, for any $W \in \text{Pos}(A)$,

\[
\tilde{\vartheta}(S,W) = \max\{ \text{Tr}(WZ) : Z \geq 0, \tilde{\vartheta}(S^c, \Psi(Z)) \leq 1 \} \tag{100}
\]
\[
= \min\{ \tilde{\vartheta}(S,X) : X \geq W, X \in S_0' \}. \tag{101}
\]

The maximal elements of $\text{TH}^\sharp(S)$ are all in $S_0'$. In particular, for classical graphs the maximal elements of $\text{TH}^\sharp(S)$ are diagonal.

**Proof.** Let $n = \dim(A)$. Theorem 19 gives

\[
\tilde{\vartheta}(S,W) = n \max\{ \text{Tr}(WZ) : Z \geq 0, \tilde{\vartheta}(S^\perp + CI, Z) \leq 1 \} \tag{102}
\]
\[
= \max\{ \text{Tr}(WZ) : Z \geq 0, \tilde{\vartheta}(S^\perp + CI, n^{-1}Z) \leq 1 \} \tag{103}
\]

By lemma 25,

\[
\tilde{\vartheta}(S^\perp + CI, n^{-1}Z) = \tilde{\vartheta}(S^c, \Psi(Z)). \tag{104}
\]

So relation (100) is proved.

Consider the convex corner

\[
\text{TH}^\sharp(S) = \{ W \in \text{Pos}(A) : \tilde{\vartheta}(S,W) \leq 1 \} \tag{105}
\]
\[
= \{ W \in \text{Pos}(A) : \text{Tr}(WZ) \leq 1 \text{ for all } Z \geq 0, \tilde{\vartheta}(S^c, \Psi(Z)) \leq 1 \} \tag{106}
\]
\[
= \{ Z \in \text{Pos}(A) : \tilde{\vartheta}(S^c, \Psi(Z)) \leq 1 \}^\sharp \tag{107}
\]
\[
= \{ Z \in \text{Pos}(A) : \Psi(Z) \in \text{TH}^\sharp(S^c) \}^\sharp \tag{108}
\]

where (105) comes from theorem 17(ii), (106) follows from (100), and (107) from the definition of anti-blocker. Applying the second anti-blocker theorem (theorem 5(iii)) to the last line then gives (98). This requires the r.h.s. to be a convex corner. Convexity holds because $\Psi$ is linear and $\text{TH}^\sharp(S^c)$ is convex. Hereditarity holds because $\Psi$ is completely positive, giving $Z' \leq Z \implies \Psi(Z') \leq \Psi(Z)$.

By theorem 24(v), $S_0'^\perp$ is in the null space of $\Psi$, so considering (98) it is clear $\text{TH}(S)$ meets the conditions for lemma 28 with $D = S_0$, giving (99). So the maximal elements of $\text{TH}^\sharp(S)$ are all in $S_0'$. From this and the linearity of $\tilde{\vartheta}$, (101) follows.

\[\square\]

From (101) we see that for classical graphs we gain no new information by using non-diagonal weights: $\tilde{\vartheta}(S,W)$ is a function of $\tilde{\vartheta}(S,X)$ with $X$ diagonal, which by theorem 9 corresponds to the classical quantity $\vartheta(G, \text{diag}(X))$. This leads to a correspondence between the theta bodies of classical and non-commutative graphs.

**Theorem 30.** Let $G$ be a classical graph and $S = \{|i\langle j| : i \simeq_G j \}$. Then

\[
\text{TH}^\sharp(S) = \text{her}(\text{diag}(w) : w \in \text{TH}^\sharp(G)) \tag{109}
\]
\[
\text{TH}^\sharp(G) = \{ \text{diag}(W) : W \in \text{TH}^\sharp(S) \} \tag{110}
\]

where $\text{diag}(w)$ is the diagonal matrix with entries from the vector $w$ and $\text{diag}(W)$ is the vector with entries taked from the diagonal of $W$. 

Proof. Let \( D \) be the space of diagonal matrices. Starting from (99) and using the correspondence between \( \hat{\vartheta} \) and \( \vartheta \) for classical graphs and diagonal weights,

\[
\text{TH}^\sharp(S) = \text{her}(\text{TH}^\sharp(S) \cap D) = \text{her}\{W \in D : W \geq 0, \hat{\vartheta}(S, W) \leq 1\} = \text{her}\{\text{diag}(w) : w \geq 0, \vartheta(G, w) \leq 1\} = \text{her}\{\text{diag}(w) : w \in \text{TH}^\sharp(G)\}.
\]

And

\[
\text{TH}^\flat(G) = \{w : w \geq 0, \vartheta(G, w) \leq 1\} = \{\text{diag}(W) : W \geq 0, W \in D, \hat{\vartheta}(S, W) \leq 1\} = \{\text{diag}(W) : W \in D \cap \text{TH}^\flat(S)\} = \{\text{diag}(W) : W \in \text{TH}^\flat(S)\}
\]

where the last equality follows from the fact that by theorem 29 the maximal elements of \( \text{TH}^\sharp(S) \) are all in \( D \).

\[\Box\]

Applying (100) with \( S = S_0 \), so that \( S^c = L(A) \), gives the following corollary.

**Corollary 31.** Let \( S_0 \subseteq L(A) \) be a C*-algebra and \( W \in \text{Pos}(A) \). Let \( D \) be the set of diagonal matrices. Then

\[
\hat{\vartheta}(S_0, W) = \max\{\text{Tr}(WZ) : Z \geq 0, ||\Psi(Z)|| \leq 1\}
\]

\[
\hat{\vartheta}(D, W) = \max\{\text{Tr}(WZ) : Z \geq 0, Z_{ii} \leq 1 \text{ for } i \in \{1, \ldots, n\}\}
\]

The second line is equivalent to \( \gamma_2^*(W) \), the dual of the factorization norm from [21] (but in our case restricted to positive semidefinite operators).

We now present a duality relation that generalizes both the classical inequality (52) as well as the duality relation from the previous chapter, theorem 19. This will make clear that the extra factor of \( n \) in theorem 19 was due to use of the graph complement \( S^\perp + CI \); for \( S_0 = CI \) the scaling factor in the below theorem takes the form \( D = nI \). When \( S_0 = D \), this theorem reduces to the classical \( \vartheta \) duality relation (52). In this case \( S'_0 = S_0 = D \), so equality is achievable for diagonal weights.

**Theorem 32.** Let \( S \) be an \( S_0 \)-graph, and take \( D \) from definition 23. Then for any \( W, V \in \text{Pos}(A) \),

\[
\hat{\vartheta}(S, V)\hat{\vartheta}(S^\perp + S_0, W) \geq \text{Tr}(V\sqrt{D}W\sqrt{D}).
\]

And if one of \( V \in S'_0 \) or \( W \in S'_0 \), the other can be chosen (also in \( S'_0 \)) to achieve equality.

**Proof.** Define \( S^c = S^\perp + S_0 \). For now assume \( V \in S'_0 \). Taking (100) from theorem 29,

\[
\hat{\vartheta}(S, V) = \max\{\text{Tr}(VZ) : Z \in \text{TH}^\sharp(S^c)\} = \max\{\text{Tr}(V\sqrt{D}W\sqrt{D}) : \Psi(\sqrt{D}W\sqrt{D}) \in \text{TH}^\sharp(S^c)\} = \max\{\text{Tr}(V\sqrt{D}\Delta(W)\sqrt{D}) : \Delta(W) \in \text{TH}^\sharp(S^c)\} = \max\{\text{Tr}(V\sqrt{D}W\sqrt{D}) : W \in \text{TH}^\sharp(S^c)\}
\]

where the last equality uses the fact that \( V \in S'_0 \implies \sqrt{D}V\sqrt{D} \in S'_0 \), so the trace can only see the \( S'_0 \) projection of \( W \). Since all instances of \( W \) are projected onto \( S'_0 \), we can equivalently just restrict this variable to \( S'_0 \), giving

\[
\hat{\vartheta}(S, V) = \max\{\text{Tr}(V\sqrt{D}W\sqrt{D}) : W \in S'_0, W \in \text{TH}^\sharp(S^c)\}
\]
Considering that \( W \in \text{TH}^\sharp(S^c) \iff W \geq 0, \tilde{\vartheta}(S^c, W) \leq 1 \), by linearity of \( \tilde{\vartheta} \) this can be written
\[
\tilde{\vartheta}(S, V) = \max\{\text{Tr}(V\sqrt{D}W\sqrt{D})/\tilde{\vartheta}(S^c, W) : W \geq 0, W \in S'_0\}.
\] (127)

Therefore, under our assumption \( V \in S'_0 \), (121) holds for all \( W \geq 0, W \in S'_0 \) and there is some \( W \geq 0, W \in S'_0 \) achieving equality.

Now drop the \( V, W \in S'_0 \) assumptions. By (101) of theorem 29 there are \( V', W' \in S'_0 \) such that \( V' \geq V, W' \geq W, \tilde{\vartheta}(S, V') = \tilde{\vartheta}(S, V) \) and \( \tilde{\vartheta}(S^c, W') = \tilde{\vartheta}(S^c, W) \). Then
\[
\tilde{\vartheta}(S, V)\tilde{\vartheta}(S^c, W) = \tilde{\vartheta}(S, V')\tilde{\vartheta}(S^c, W')
\] (128)
\[
\geq \text{Tr}(V'\sqrt{D}W'\sqrt{D})
\] (129)
\[
\geq \text{Tr}(V\sqrt{D}W\sqrt{D}).
\] (130)

\( \square \)

At this point it will be instructive to explore this bound through some simple examples.

**Example 33.** Let \( S = S_0 = CI \subseteq \mathcal{L}(A) \). Then \( S^c = S^1 + S_0 = \mathcal{L}(A) \) and \( D = nI \). We have \( \tilde{\vartheta}(S, V) = n\text{Tr}(V) \) and \( \tilde{\vartheta}(S^c, W) = \|W\| \). Theorem 32 then says \( n\text{Tr}(V)\|W\| \geq \text{Tr}(V\sqrt{n}W\sqrt{n}) = n\text{Tr}(VW) \). And for every \( V \in S'_0 = \mathcal{L}(A) \) there is \( W \in S'_0 \) giving equality.

**Example 34.** Let \( S = S_0 = \mathcal{L}(A) \). Then \( S^c = S^1 + S_0 = \mathcal{L}(A) \) and \( D = n^{-1}I \). We have \( \tilde{\vartheta}(S, V) = \|V\| \) and \( \tilde{\vartheta}(S^c, W) = \|W\| \). Theorem 32 gives \( \|V\|\|W\| \geq \text{Tr}(Vn^{-1/2}Wn^{-1/2}) = n^{-1}\text{Tr}(VW) \). And for every \( V \in S'_0 = CI \) there is \( W \in S'_0 \) giving equality, i.e., \( \|V\|^2 = n^{-1}\text{Tr}(I) \).

**Example 35.** Let \( S = S_0 \) for some arbitrary \( C^* \)-algebra \( S_0 \subseteq \mathcal{L}(A) \). Then \( S^c = S^1 + S_0 = \mathcal{L}(A) \) and \( \tilde{\vartheta}(S^c, W) = \|W\| \). Theorem 32 gives \( \tilde{\vartheta}(S, V)\|W\| \geq \text{Tr}(V\sqrt{D}W\sqrt{D}) \). If \( V \in S'_0 \) then \( \tilde{\vartheta}(S, V) = \max\{\text{Tr}(V\sqrt{D}W\sqrt{D})/\|W\| : W \in S'_0\} = \text{Tr}(V) \), since the max is obtained with \( W = I \).

In the case of \( S_0 \) being the diagonal matrices, we have \( S'_0 = S_0, D = I \) and \( \tilde{\vartheta}(S, V) = \text{Tr}(V) \). In this case \( S \) is the classical empty graph.

Having proved a duality relation for \( \tilde{\vartheta} \) on an \( S_0 \) graph versus its \( S_0 \)-complement, we now translate it into a statement about the corresponding theta bodies. Whereas classical graphs satisfy \( \text{TH}(G) = \text{TH}(\overline{G}) \) (with \( \overline{b} \) being the classical anti-blocker), for the case of non-commutative graphs we need a special type of anti-blocker that bakes in the \( D \) scaling seen in theorem 32.

Henceforth we will be using the notation \( \Psi(C) = \{\Psi(X) : X \in C\} \) for any convex corner \( C \). Note that since \( \Psi \) is convex (in fact, linear) and completely positive, \( \Psi(C) \) is a convex corner whenever \( C \) is. We similarly extend \( \Delta \) to operate on convex corners.

**Definition 36.** For a convex corner \( C \), define
\[
C^\circ = \{W \geq 0 : \Psi(W) \in C\} \tag{131}
\]

The following two lemmas will be needed for proving some useful properties of \( C^\circ \).

**Lemma 37.** Let \( C \) be a convex corner such that all maximal elements are in \( S'_0 \). Then
\[
\Delta(C) = C \cap S'_0 \tag{132}
\]

**Proof.** Clearly \( \Delta(C) \supseteq C \cap S'_0 \) because \( \Delta(W) = W \) whenever \( W \in S'_0 \).

Suppose \( W \in \Delta(C) \). Then there is \( X \in C \) with \( W = \Delta(X) \). Since maximal elements of \( C \) are in \( S'_0 \), there is some \( Y \in C \cap S'_0 \) such that \( Y \geq X \). But \( \Delta \) is completely positive, so \( W = \Delta(X) \leq \Delta(Y) = Y \). Because \( C \) is hereditary, \( W \in C \). Being in the image of \( \Delta \), we have \( W \in S'_0 \). Therefore \( \Delta(C) \subseteq C \cap S'_0 \). \( \square \)

**Lemma 38.** Let \( C \subseteq \mathcal{L}(A) \) be a convex corner and \( Z \in \mathcal{L}(A) \) invertible. Then \( (ZCZ^1)^\sharp = Z^{-1}C^\sharp Z^{-1} \).

\[
\]
Proof. 

\[(ZCZ^\dagger)^2 = \{ W \geq 0 : \text{Tr}(W^\dagger X) \leq 1 \text{ for all } X \in ZCZ^\dagger \}\]  
\[= \{ W \geq 0 : \text{Tr}(W^\dagger ZXZ^\dagger) \leq 1 \text{ for all } X \in C \}\]  
\[= \{ W \geq 0 : \text{Tr}((Z^\dagger WZ)^\dagger X) \leq 1 \text{ for all } X \in C \}\]  
\[= \{ W \geq 0 : Z^\dagger WZ \in C^\dagger \}\]  
\[= Z^{-1}C^\dagger Z^{-1}. \]  

Lemma 39. For a convex corner \( C \), the following hold.

(i) The maximal elements of \( C^\circ \) are in \( S_0' \).
(ii) \( C^\circ = \text{her}(\Psi(C^\dagger)) \), if the maximal elements of \( C \) are all in \( S_0' \).
(iii) \( C^{\circ\circ} = \text{her}(C \cap S_0') \).

Proof. (i): Since \( S_0'^\perp \) is in the null space of \( \Psi \), lemma 28 applies.
(ii): Let \( B = \{ W \geq 0 : \Psi(W) \in C \} \) so that \( C^\circ = B^\sharp \). Then

\[\Delta(B) = \{ \Delta(W) : W \geq 0, \Psi(W) \in C \}\]  
\[= \{ \Delta(W) : W \geq 0, D^{-1/2}\Delta(W)D^{-1/2} \in C \}\]  
\[= \{ \Delta(W) : W \geq 0, \Delta(W) \in \sqrt{D}C\sqrt{D} \}\]  
\[= \sqrt{D}C\sqrt{D} \cap S_0' \]  
\[= \sqrt{D}(C \cap S_0')\sqrt{D} \]  

where the second line uses theorems 24(iii) and 24(iv) and the fourth line uses that the image of \( W \geq 0 \) under \( \Delta \) is \( \text{Pos}(A) \cap S_0' \). Since the maximal elements of \( C^\circ \) are all in \( S_0' \), and the rest follows from the hereditary condition, we need only consider \( C^\circ \cap S_0' \).

\[C^\circ \cap S_0' = \{ X \in S_0' : X \geq 0, \text{Tr}(XW) \leq 1 \text{ for all } W \in B \}\]  
\[= \{ X \in S_0' : X \geq 0, \text{Tr}(\Delta(X)W) \leq 1 \text{ for all } W \in B \}\]  
\[= \{ X \in S_0' : X \geq 0, \text{Tr}(X\Delta(W)) \leq 1 \text{ for all } W \in B \}\]  

where the second line uses \( \Delta(X) = X \) when \( X \in S_0' \) and the third line follows because \( \Delta \) is a projection (onto \( S_0' \)) and hence is self-adjoint (as a superoperator). Now, \( \{ \Delta(W) : W \in B \} \) is equivalent to \( \{ W : W \in \Delta(B) \} \), so

\[C^\circ \cap S_0' = \{ X \in S_0' : X \geq 0, \text{Tr}(XW) \leq 1 \text{ for all } W \in \Delta(B) \}\]  
\[= S_0' \cap (\Delta(B))^\sharp. \]  

Since \( S_0'^\perp \) is in the null space of \( \Delta \), lemma 28 applies and the maximal elements of \( (\Delta(B))^\sharp \) are all in \( S_0' \). Apply lemma 37 to the right hand side to give \( C^\circ \cap S_0' = \Delta((\Delta(B))^\sharp). \) Substituting (142) gives

\[C^\circ \cap S_0' = \Delta((\sqrt{D}(C \cap S_0')\sqrt{D})^\sharp) \]  
\[= \Delta(D^{-1/2}(C \cap S_0')^\sharp D^{-1/2}) \]  
\[= \Delta(D^{-1/2}C^\dagger D^{-1/2}) \]  
\[= \Psi(C^\dagger) \]  

where the second line uses lemma 38, and the third line uses \( C^\dagger = (C \cap S_0')^\dagger \) because the maximal elements of \( C \) are in \( S_0' \) and the anti-blocker only cares about the maximal elements. The last line uses theorem 24(iv).

Since, by (i), all maximal elements of \( C^\circ \) are in \( S_0' \), we have

\[C^\circ = \text{her}(C^\circ \cap S_0') \]  
\[= \text{her}(\Psi(C^\dagger)) \]
(iii): The maximal elements of $C^\circ$ are in $S'_0$ so we can apply (ii).

$$C^{oo} = \text{her}(\Psi(C^{oo}))$$

$$= \text{her}(\Psi(\{W \geq 0 : \Psi(W) \in C\}^\sharp))$$

$$= \text{her}(\Psi(\{W \geq 0 : \Psi(W) \in C\}))$$

where the second line uses definition 36 and the third line uses the second anti-blocker theorem. That requires the argument to be a convex corner, which it is: convexity follows from linearity of $\Psi$ and convexity of $C$ whereas hereditarity follows from hereditarity of $C$ and $\Psi$ being positive semidefinite. Moving the first $\Psi$ into the set notation, we have

$$C^{oo} = \text{her}(\{\Psi(W) : W \geq 0, \Psi(W) \in C\})$$

$$= \text{her}(C \cap S'_0)$$

where the second line holds because the image of $\Psi$ is $S'_0$.

Finally, we apply the $\diamond$ anti-blocker to give a duality relation between theta bodies. When $S_0 = D$ (the diagonal matrices), this reproduces the classical result $\text{TH}(\overline{G}) = \text{TH}^\flat(G)$ When $S_0 = CI$ this reproduces the theta body relation from theorem 19.

**Theorem 40.** Let $S$ be an $S_0$-graph and $S^c = S^\perp + S_0$ its complement graph. Then

$$\text{TH}^\sharp(S^c) = \text{TH}^{\diamond\circ}(S)$$

**Proof.** Starting from (98),

$$\text{TH}(S) = \{Z \geq 0 : \Psi(Z) \in \text{TH}^\sharp(S^c)\}$$

$$\text{TH}^\sharp(S) = \{Z \geq 0 : \Psi(Z) \in \text{TH}^\sharp(S^c)\}^\sharp$$

$$= \text{TH}^{\diamond\circ}(S^c)$$

Applying this to $S^c$, which is also an $S_0$ graph, gives $\text{TH}^\sharp(S^c) = \text{TH}^{\diamond\circ}(S^c) = \text{TH}^{\diamond\circ}(S)$.

**VI. A SANDWICH THEOREM AND PERFECT GRAPHS**

For classical graphs the theta body is sandwiched between the vertex packing polytope $\text{vp}(G)$, defined as the convex hull of indicator functions of independent sets, and the fractional vertex packing polytope, equal to $\text{vp}^\flat(\overline{G})$: [16]

$$\text{vp}(G) \subseteq \text{TH}(G) \subseteq \text{vp}^\flat(\overline{G}).$$

Or, considering the complement graph and making use of the fact that $\text{TH}(\overline{G}) = \text{TH}^\flat(G)$,

$$\text{vp}(\overline{G}) \subseteq \text{TH}^\flat(G) \subseteq \text{vp}^\flat(\overline{G}).$$

We will generalize this latter statement to non-commutative graphs. Note that $\text{vp}(\overline{G})$ is the polytope of indicator functions of cliques of $G$. Given that

$$\alpha(G) = \max \left\{ \sum w_i : w \in \text{vp}(G) \right\}$$

$$\vartheta(G) = \max \left\{ \sum w_i : w \in \text{TH}(G) \right\}$$

$$\chi^*(\overline{G}) = \max \left\{ \sum w_i : w \in \text{vp}^\flat(\overline{G}) \right\}$$

(where $\alpha$ is independence number and $\chi^*$ is fractional chromatic number), (163) can be seen as a more granular version of the well known sandwich inequality $\alpha(G) \leq \vartheta(G) \leq \chi^*(\overline{G})$.

Also from [16] comes the marvelous result that the following are equivalent:
1. $G$ is a perfect graph.
2. $\text{TH}(G)$ is a polytope.
3. $\text{TH}(G) = \text{vp}(G)$.
4. $\text{TH}(G) = \text{vp}^S(G)$.

We will show this partially generalizes to non-commutative graphs. There is currently no definition of “perfect” for non-commutative graphs, so we may take the non-commutative generalization of $\text{vp}(G) = \text{vp}^S(G)$ to be the definition. Since we are using an operator generalization of convex corners and anti-blockers, there are (at least) two possible analogues of “polytope”: we could consider convex corners generated by finitely many vertices, or those bounded by finitely many inequalities. We will find that if $\text{TH}^S(S)$ is finitely generated then it is equal to the non-commutative generalization of $\text{vp}(G)$, but that the converse is not true. This can be considered a first dipping of the toes into the theory of non-commutative perfect graphs, with many questions remaining open.

The following definition from [3] gives the non-commutative generalization of $\text{vp}(G)$ that we will be using. They also define $S$-abelian and $S$-clique projectors, the former being analogous to independent sets and the latter being a looser definition of clique, but we will not be using those here.

**Definition 41.** Let $S \subseteq \mathcal{L}(A)$ be a non-commutative graph. A projector $P \in \mathcal{L}(A)$ is called $S$-full if $P \mathcal{L}(A)P \subseteq S$. We define the convex corner generated by these projections,

$$\text{fp}(S) = \operatorname{her}(\operatorname{conv}\{P : P \text{ an } S\text{-full projection}\})$$

(168)

**Theorem 42.** Let $S \subseteq \mathcal{L}(A)$ be an $S_0$-graph. If $P$ is a maximal $S$-full projection then $P \in S_0'$. Consequently, the maximal elements of $\text{fp}(S)$ are in $S_0'$.

**Proof.** Let $P$ be a maximal $S$-full projection. Then $P = \sum_{i=1}^m |\psi_i\rangle\langle\psi_i|$ for some collection of normalized vectors $\{|\psi_i\rangle\}$, and $|\psi_i\rangle\langle\psi_j| \in S$ for all $i, j \in \{1, \ldots, m\}$. By the definition of an $S_0$ graph, $S_0S = SS_0 = S$. Therefore, for any $X \in S_0$ we have $X|\psi_i\rangle\langle\psi_j| \in S$ and $|\psi_i\rangle\langle\psi_j|X^\dagger \in S$. It must be that $X|\psi_i\rangle \in \text{span}\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$ for all $X \in S_0$ and all $i$. Otherwise, $\text{span}\{X|\psi_i\rangle, |\psi_1\rangle, \ldots, |\psi_m\rangle\}$ would define a larger $S$-full space, in contradiction to $P$ being maximal.

Consider a unitary $U \in S_0$. By the above reasoning, $U|\psi_i\rangle$ is in the support of $P$ for all $i \in \{1, \ldots, m\}$. Then $UPU^\dagger = \sum_{i} U|\psi_i\rangle\langle\psi_i|U^\dagger$ has support contained in the support of $P$. Since $UPU^\dagger$ is also a projector, it must be that $U^\dagger U = P$. Equivalently, $UP = PU$. Since $P$ commutes with all unitary $U \in S_0$, it must also commute with the algebra generated by those unitaries, which is all of $S_0$. Therefore $P \in S_0'$.

Since each maximal $S$-full projection is in $S_0'$, so must be the closure of their convex hull,

$$\operatorname{conv}\{P : P \text{ a maximal } S\text{-full projection}\} \subseteq S_0'.$$

(169)

Every maximal element of $\text{fp}(S)$ must be a member of this, because every $S$-full projection is dominated by some maximal $S$-full projection.$\square$

We are now ready to present our sandwich theorem, the generalization of (164) to $S_0$-graphs.

**Theorem 43.**

$$\text{fp}(S) \subseteq \text{TH}^S(S) \subseteq \text{fp}^S(S^c)$$

(170)

**Proof.** Let $P$ be an $S$-full projector. Since $Y = |P\rangle\langle P|$ is feasible for (13) with $\lambda = 1$, we have $\hat{\theta}(S, P) \leq 1$ so $P \in \text{TH}^S(S)$. Since $\text{TH}^S(S)$ is a convex corner and contains all $S$-full projectors, it must also contain the convex corner generated by the $S$-full projectors. Therefore $\text{fp}(S) \subseteq \text{TH}^S(S)$.

Applying the above results to the graph $S^c$, we have

$$\text{fp}(S^c) \subseteq \text{TH}^S(S^c)$$

(171)

$$\{W \geq 0 : \Psi(W) \in \text{fp}(S^c)\} \subseteq \{W \geq 0 : \Psi(W) \in \text{TH}^S(S^c)\}$$

(172)

$$\{W \geq 0 : \Psi(W) \in \text{fp}(S^c)\}^* \supseteq \{W \geq 0 : \Psi(W) \in \text{TH}^S(S^c)\}^*$$

(173)

$$\text{fp}^S(S^c) \supseteq \text{TH}^S(S^c)$$

(174)

$$= \text{TH}^S(S)$$

(175)
where the third line uses the fact that taking the anti-blocker of both sides reverses the inclusion order (theorem 5(ii)) and the last uses the duality relation of theorem 40. □

It’s worth noting this sandwich theorem is symmetric under graph complement. Taking the ◦ anti-blocker of (170) and considering that this reverses inclusion order and that \( \text{TH}^\#(S) = \text{TH}^\#(S^c) \), we get

\[
\text{fp}^\#(S) \supseteq \text{TH}^\#(S^c) \supseteq \text{fp}(S^c).
\]

This is equivalent to (170) applied to the complement graph.

It is also worth noting that the right inclusion of (170) was derived from the left inclusion without using any special properties of \( \text{fp}(S) \). Any convex corner contained in \( \text{TH}^\#(S) \) would give a similar sandwich theorem. This opens the possibility of tightening (170) by finding some convex corner larger than \( \text{fp}(S) \) but still contained in \( \text{TH}^\#(S) \).

Since a classical graph is perfect if and only if \( \text{vp}(G) = \text{vp}^\#(G) \), we take the non-commutative analogue to be one possible definition of perfect for non-commutative graphs.

**Definition 44.** An \( S_0 \)-graph \( S \) is \( \text{fp} \)-perfect if \( \text{fp}(S) = \text{fp}^\#(S) \).

Note that, by theorem 43, \( \text{fp} \)-perfect graphs necessarily also satisfy \( \text{TH}^\#(S) = \text{fp}(S) \).

**Theorem 45.** For a graph \( G \) define \( S = \text{span}\{\langle i|j \rangle : i \simeq_G j \} \) and \( S_0 = \text{span}\{|i\langle i \rangle : i \in V(G)\} \). Then \( S \) is \( \text{fp} \)-perfect if and only if \( G \) is perfect.

**Proof.** The graph \( G \) is perfect if and only if \( \text{vp}(G) = \text{vp}^\#(G) \) where \( \text{vp} \) is the vertex packing polytope, the convex hull of incidence vectors of cliques, and \( \text{vp}^\#(G) = \text{vp}^\#(G) \cap S_0' \) is the diagonal anti-blocker [16]. Note \( \text{vp}(G) \) is the convex hull of incidence vectors of cliques,

\[
\text{vp}(G) = \text{conv}\left\{\sum_{i \in X} |i\langle i \rangle : X \subseteq V(G) \text{ is a clique of } G \right\}. \tag{176}
\]

By [3, theorem 3.5, corollary 3.7],

\[
\Delta(\text{fp}(S)) = \text{fp}(S) \cap S_0' = \text{vp}(G) \tag{177}
\]

\[
\Delta(\text{fp}^\#(S)) = \text{fp}^\#(S) \cap S_0' = \text{vp}^\#(G) \tag{178}
\]

And by [3, lemma 3.6], if \( A \) is a diagonal convex corner and \( B \) is a convex corner such that \( A = D \cap B = \Delta(B) \), where \( D = S_0 = S_0' \) are the diagonal operators, then \( A^\# = D \cap B^\# = \Delta(B) \). Taking \( B = \{W \geq 0 : \Delta(W) \in \text{fp}(S^c)\} \), we have \( D \cap B = \Delta(B) = D \cap \text{fp}(S^c) \). By (177), \( D \cap \text{fp}(S^c) = \text{vp}(G) \), so taking \( A = \text{vp}(G) \) gives \( A^\# = \text{vp}^\#(G) = \Delta(B^\#) \). By definition 36, \( B^\# = \text{fp}^\#(S') \). So

\[
\Delta(\text{fp}^\#(S^c)) = \text{fp}^\#(G). \tag{179}
\]

Suppose \( S \) is \( \text{fp} \)-perfect. Then

\[
\text{fp}(S) = \text{fp}^\#(S^c) \implies \Delta(\text{fp}(S)) = \Delta(\text{fp}^\#(S^c)) \tag{180}
\]

\[
\implies \text{vp}(G) = \text{vp}^\#(G), \tag{181}
\]

where the second implication follows from (177) and (179). So \( G \) is perfect.

On the other hand, suppose \( G \) is perfect so \( \text{vp}(G) = \text{vp}^\#(G) \). Then

\[
\text{vp}(G) = \text{vp}^\#(G) = \text{fp}(S) \cap S_0' = \Delta(\text{fp}^\#(S^c)) \tag{182}
\]

\[
\implies \text{fp}(S) \supseteq \Delta(\text{fp}^\#(S^c)) \tag{183}
\]

\[
\implies \text{her}(\text{fp}(S)) \supseteq \text{her}(\Delta(\text{fp}^\#(S^c))) \tag{184}
\]

\[
\implies \text{fp}(S) \supseteq \text{fp}^\#(S^c), \tag{185}
\]

where the first implication follows from (177)-(178) and the last from \( \text{fp}(S) = \text{her}(\text{fp}(S)) \) and theorem 39(ii) (since \( \Delta \) and \( \Psi \) are equivalent when \( S_0 = D \)). By theorem 43, \( \text{fp}(S) \subseteq \text{fp}^\#(S^c) \) so in fact \( \text{fp}(S) = \text{fp}^\#(S^c) \) and \( S \) is \( \text{fp} \)-perfect. □

For classical graphs the weak perfect graph theorem states that a graph is perfect if and only if its complement is perfect. This holds also for \( \text{fp} \)-perfect graphs.
Theorem 46. If an \( S_0 \)-graph \( S \) is fp-perfect then \( S^c \) is fp-perfect.

Proof. Suppose \( S \) is fp-perfect. Starting from the definition of fp-perfect and applying theorem 39(iii) we have

\[
\begin{align*}
\text{fp}(S) &= \text{fp}^\circ(S^c) \\
\text{fp}(S)^\circ &= \text{fp}^\circ\circ(S^c) \\
&= \text{her}(\text{fp}(S^c) \cap S'_0) \\
&= \text{fp}(S^c)
\end{align*}
\]

where the last equality follows from the fact that \( \text{fp}(S^c) \) is generated by elements of \( S'_0 \) (theorem 42).

We give an example of an fp-perfect graph which will also later serve as an important counterexample. This \( S_0 \)-graph is not a classical graph, though its complement is the classical empty graph (or, it would be aside from the fact we take \( S_0 = CI \) rather than the algebra of diagonal matrices).

Example 47. The graph

\[
S = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}
\]

is an fp-perfect \( S_0 \)-graph with \( S_0 = CI \).

Proof. Clearly no \( S \)-full projector can be rank-2 because \( S \neq L(A) \). Rank-1 projectors have trace 1, so to be in \( (190) \) they must have \( a = 1/2 \). In fact, any such projector is easily verified to be \( S \)-full. We have then

\[
\begin{align*}
\text{fp}(S) &= \text{her} \left( \text{conv} \left\{ \begin{pmatrix} 1/2 & 1 \\ \phi & 1 \end{pmatrix} : \phi \in \mathbb{C}, |\phi| = 1 \right\} \right) \\
&= \left\{ M \in L(\mathbb{C}^2) : M \geq 0, M_{11} \leq 1/2, M_{22} \leq 1/2 \right\}.
\end{align*}
\]

The complement graph is

\[
S^c = S^\perp + CI = \left\{ M \in L(\mathbb{C}^2) : M \text{ is diagonal} \right\}.
\]

Again there are no rank-2 \( S^c \)-full projectors. And clearly the only rank-1 projectors in \( S^c \) are the projectors onto the two basis vectors. We have then

\[
\begin{align*}
\text{fp}(S^c) &= \text{her} \left( \text{conv} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right) \\
&= \left\{ M \in L(\mathbb{C}^2) : M \geq 0, M_{11} \leq 1, M_{22} \leq 1 \right\}.
\end{align*}
\]

Since the anti-blocker only cares about the extreme points of \( \text{fp}(S^c) \), the anti-blocker is

\[
\begin{align*}
\text{fp}^\sharp(S^c) &= \left\{ M \in L(\mathbb{C}^2) : M \geq 0, M_{11} \leq 1, M_{22} \leq 1 \right\} \\
&= 2\text{fp}(S).
\end{align*}
\]

With \( S_0 = CI \subseteq L(\mathbb{C}^2) \) we have \( \Psi(W) = W/2 \) so

\[
\begin{align*}
\text{fp}^\circ(S^c) &= \left\{ W \geq 0 : \Psi(W) \in \text{fp}(S^c) \right\}^\sharp \\
&= \left\{ W \geq 0 : W/2 \in \text{fp}(S^c) \right\}^\sharp \\
&= (2\text{fp}(S^c))^\sharp \\
&= 1/2 \text{fp}^\sharp(S^c) \\
&= \text{fp}(S)
\end{align*}
\]
For classical graphs, [16] showed that a graph \( G \) is perfect if and only if \( \text{TH}(G) \) is a polytope, and that facets of \( \text{TH}(G) \) correspond to cliques of \( G \). The remainder of this section will be devoted to exploring to which extent this generalizes to \( S_\theta \)-graphs. We are using convex corners on operator spaces, which are more complicated than the diagonal convex corners which \( \text{TH}(G) \) lives in. Diagonal convex corners are polyhedral if they are finitely generated if they are defined by finitely many inequalities. This is not true for the non-diagonal convex corners we are using. For example, \( \text{her}(\{I\}) \), the set of bounded semidefinite operators, is a finitely generated convex corner but is not polyhedral. Before proceeding we will need some groundwork on the geometry of convex corners and anti-blockers.

**Definition 48.** Let \( C \subseteq \text{Herm}(A) \) be a nonempty closed convex set and \( T \subseteq C \) a subset. A supporting hyperplane of \( C \) at \( T \) is an affine subspace \( \{X \in \text{Herm}(A) : \text{Tr}(XY) = \alpha\} \), with \( Y \in \text{Herm}(A) \) and \( \alpha \in \mathbb{R} \), such that \( \text{Tr}(XY) \leq \alpha \) for all \( X \in C \) and \( \text{Tr}(XY) = \alpha \) for all \( X \in T \).

A vertex \( X \in C \) is an element on the boundary of \( C \) such that the intersection of all supporting hyperplanes of \( C \) at \( X \) is an affine subspace of dimension 0. In other words, a supporting hyperplane can be “wiggled” a small amount in every direction, pivoting on \( X \).

A facet \( F \subseteq C \) is a subset of the boundary of \( C \), of affine codimension 1 (affine dimension \( \dim(L(A)) - 1 \)), defined by a single supporting hyperplane: \( F = \{X \in C : \text{Tr}(XY) = \alpha\} \). Note that facets are necessarily convex.

Many authors only require that vertices are not part of any line segment in \( C \). We take the stricter definition from [14, definition 2.3]. The corners of a cube are vertices, the boundary points of a closed ball are not. Also, the boundary points of a closed ball are not facets: they are defined by a single supporting hyperplane but are not affine codimension 1.

We will need the following facts regarding the geometry of convex corners.

**Theorem 49.** Let \( C \subseteq L(A) \) be a convex corner. Then the following hold.

\( (i) \) If \( \dim(A) > 1 \) and \( C \) has a facet then \( C \) is necessarily of full dimension, containing some positive definite element.

\( (ii) \) For \( \dim(A) > 1 \), supporting hyperplanes defining facets of \( C \) do not pass through the origin. Therefore, facets always take the form

\[
F = \{X \in C : \text{Tr}(XY) = 1\}. \tag{202}
\]

\( (iii) \) If (202) is a facet then \( Y \) is a vertex of \( C^G \).

\( (iv) \) Non-zero vertices of \( C \) are maximal in the sense that if \( X \) is a vertex and \( Y \geq X, Y \neq X \) then \( Y \not\in C \).

\( (v) \) Suppose \( C \) is finitely generated and \( \{X_i : i \in \{0, \ldots, m\}\} \) is a minimal set of generators, so \( C = \text{her}(\text{conv}\{X_i : i \in \{0, \ldots, m\}\}) \). Then each \( X_i \) is a vertex of \( C \) and

\[
F = \{Z \in C^G : \text{Tr}(X_iZ) = 1\} \tag{203}
\]

is a facet of \( C^G \).

**Proof.** Proofs are in appendix B.

Intriguingly, many of these facts are not true for diagonal convex corners. That is to say, the usage of the Loewner order for the hereditarity condition, i.e. \( X \in C, 0 \leq Y \leq X \implies Y \in C \), imposes stricter requirements on the existence of vertices and facets for non-commutative convex corners as compared to diagonal convex corners, in which hereditarity is defined by elementwise less-than of vectors.

For example, consider the diagonal convex corner \( \{(x, y) \subseteq \mathbb{R}^2 : x, y \geq 0, x, y \leq 1\} \). The edge \( y = 0 \) forms a facet whose supporting hyperplane passes through the origin, in violation of theorem 49(ii). The point \( x = 1, y = 0 \) is a vertex but is not maximal, in violation of theorem 49(iv). The analogous non-commutative convex corner is \( C = \{X \in \text{Pos}(\mathbb{R}^2) : \|X\| \leq 1\} \). Here too we have that \( \text{diag}(0,1) \) is not a maximal element, but in this case it is also not a vertex: the tangent cone of \( C \) at \( \text{diag}(0,1) \) contains the line \( \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \).

We are ready to show that facets of \( \text{TH}(S) \) correspond to \( S \)-full projectors. The derivation follows in spirit that of [16], but using the language of compatible matrices from [18, chapter 29]. The following lemma states that if we take optimal solutions for \( \tilde{\vartheta} \) for a non-commutative graph and its complement that saturate the duality relation of theorem 19, then the matrices associated with the Schur complement form of the \( \vartheta \) SDP.
(theorem 14) will be orthogonal to each other. From this we can read off some useful relations among the blocks of these matrices. When TH(S) has a facet, there are multiple linearly independent instances of these relations, forcing the matrix from theorem 14 into a particular form: it must be rank-1. Such a solution must necessarily correspond to an $S$-full projector.

**Lemma 50.** Let $S$ be a non-commutative graph. Suppose $W, V \in \text{Pos}(A)$ saturate the inequality of theorem 19 so $\tilde{\vartheta}(S, V)\tilde{\vartheta}(S^\perp + CI, W) = n \text{Tr}(VW)$.

Let $Z$ be optimal for (44) for $\lambda = \tilde{\vartheta}(S, V)$ and $Z'$ be optimal for $\lambda' = \tilde{\vartheta}(S^\perp + CI, W)$. Then

\begin{align*}
\lambda|W\rangle &= nZ'|V\rangle \\
\lambda'|V\rangle &= nZ|W\rangle \\
|V\rangle\langle W| &= nZZ'.
\end{align*}

**Proof.** Consider the block matrices from (44) for $S$ and $S^\perp + CI$, with the second conjugated by a diagonal matrix to change the sign of and scale the second row and column.

\begin{align*}
M &= \begin{pmatrix}
\lambda & \langle V | \\
|V\rangle & Z
\end{pmatrix} \\
M' &= \begin{pmatrix}
1 & 0 \\
0 & -nI
\end{pmatrix} \begin{pmatrix}
\lambda' & \langle W | \\
|W\rangle & Z'
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & -nI
\end{pmatrix}
\end{align*}

Their product is

\begin{align*}
MM' &= \begin{pmatrix}
\lambda \lambda' - n\langle V | W\rangle & -\lambda n\langle W | + n^2\langle V | Z' \\
\lambda' \langle V | - nZ \langle W | & -n\langle V | W\rangle + n^2\langle W | Z'
\end{pmatrix}
\end{align*}

We will show that $M$ and $M'$ are orthogonal under the Hilbert-Schmidt inner product, i.e., $\text{Tr}(MM') = 0$. The upper-left block vanishes: $\langle V | W\rangle = \text{Tr}(VW)$ and by (204), $\lambda \lambda' = n \text{Tr}(VW)$. What remains of the trace is the lower-right block,

\[\text{Tr}(MM') = -n \text{Tr}(|V\rangle \langle W|) + n^2 \text{Tr}(ZZ')\]

\[= -n \text{Tr}(VW) + n^2 \text{Tr}(ZZ').\]

Since $Z \in S \otimes L(B)$ and $Z' \in (S^\perp + CI) \otimes L(B)$, the Hilbert-Schmidt inner product of $Z$ and $Z'$ can only see the projection of these variables onto $CI \otimes L(B)$. So we have

\[\text{Tr}(ZZ') = \text{Tr}\left((n^{-1}I \otimes \text{Tr}_A Z)(n^{-1}I \otimes \text{Tr}_A Z')\right)\]

\[= n^{-1} \text{Tr}(\text{Tr}_A Z(\text{Tr}_A Z'))\]

\[= n^{-1} \text{Tr}(V^T W^T)\]

\[= n^{-1} \text{Tr}(VW),\]

where the third line comes from the condition $\text{Tr}_A Z = V^T$, $\text{Tr}_A Z' = W^T$ of (44). Therefore $\text{Tr}(MM') = 0$ and the $M$ and $M'$ operators are orthogonal. Positive semidefinite orthogonal operators have orthogonal supports, so in fact $MM' = 0$. The blocks of $MM'$ listed in (211) being zero gives (205)-(207).

---

1 We use the complement $S^\perp + CI$ regardless of whether $S_0 = CI$. It is possible to make a more complicated version of this lemma using the complement $S^\perp + S_0$, but this will not be necessary.
Theorem 51. If \( S \subseteq \mathcal{L}(A) \) is a non-commutative graph and

\[
\mathcal{F} = \{ X \in \text{TH}(S) : \text{Tr}(XV) = 1 \}
\]

is a facet of \( \text{TH}(S) \) then \( V \) is a maximal \( S \)-full projector. Note that by theorem 49(ii), all facets of a convex corner take this form.

Proof. Define \( S^c = S^\perp + CI \) and let \( n = \dim(A) \). By theorem 49(iii), \( V \) is a vertex of \( \text{TH}^F(S) \). By theorem 49(iv) vertices are maximal, so \( \tilde{\vartheta}(S,V) = 1 \).

Being a facet, \( \mathcal{F} \) has \( \dim(\mathcal{L}(A)) = n^2 \) elements spanning a \( \dim(\mathcal{L}(A)) - 1 = n^2 - 1 \) dimensional affine subspace. Label these elements \( X_1, \ldots, X_{n^2} \).

Since each \( X_i \in \mathcal{F} \), we have

\[
\text{Tr}(X_i V) = 1 \quad \text{(219)}
\]

\[
\tilde{\vartheta}(S^c, n^{-1}X_i) = 1 \quad \text{(220)}
\]

By lemma 50 there is \( Z \) optimum for (44) for \( \tilde{\vartheta}(S,V) \) with\[
\langle V | V \rangle = nZ | W_i \rangle.
\]

Since the \( X_i \) span an \( n^2 - 1 \) dimensional affine subspace of \( \mathcal{L}(A) \), so do the \( W_i \). And the \( | W_i \rangle \) span an \( n^2 - 1 \) dimensional affine subspace of \( A \otimes B \). Their differences then span a codimension 1 subspace of \( A \otimes B \). Since\[
Z(| W_i \rangle - | W_j \rangle) = n^{-1}(| V \rangle - | V \rangle) = 0,
\]

we have that \( Z \) is rank 1. Considering (223) and \( \langle V | W_i \rangle = \text{Tr}(W_i V) = n^{-1} \), we have\[
Z = | V \rangle \langle V |.
\]

This gives \( \text{Tr}_A Z = (V^T)^2 \) but by (44) we have \( \text{Tr}_A Z = V^T \). Therefore \( V \) is a projector. For any \( M \in \mathcal{L}(A) \) we have\[
VMV^\dagger = \text{Tr}_B((I \otimes M^T)|V\rangle\langle V|)
\]

\[
= \text{Tr}_B((I \otimes M^T)Z)
\]

\[
\in S \quad \text{(227)}
\]

\[
\in S \quad \text{(228)}
\]

\[
\in S \quad \text{(229)}
\]

where the last relation follows from \( Z \in S \otimes \mathcal{L}(B) \), a requirement of (44). Therefore \( V \) is an \( S \)-full projector. Since \( V \) is a vertex of \( \text{TH}^F(S) \), it is maximal in \( \text{TH}^F(S) \). Therefore \( V \) is a maximal \( S \)-full projector.

After having shown that facets of \( \text{TH}(G) \) correspond to cliques of \( G \), [16] is able to immediately state that \( \text{TH}(G) \) being a polytope implies \( G \) is perfect: \( \text{TH}(G) \) being defined by clique constraints forces \( \text{TH}(G) = \text{vp}^\flat(G) \). Since \( \text{TH}^F(G) = \text{TH}(G) \) must also be polyhedral we similarly have \( \text{TH}(G) = \text{vp}(G) \), giving \( \text{vp}(G) = \text{vp}^\flat(G) \) (i.e. \( G \) is perfect). For non-commutative graphs the situation is not so simple. What we will be able to show is that if \( \text{TH}^F(S) \) is finitely generated, then it is equal to \( \text{fp}(S) \). But the converse does not necessarily hold. And \( \text{TH}^F(S) = \text{fp}(S) \) does not necessarily mean that \( S \) is fp-perfect.

Theorem 52. If \( \text{TH}^F(S) \) is finitely generated then it is equal to \( \text{fp}(S) \).
Proof. Suppose $\text{TH}^f(S)$ is finitely generated and let $\{X_i : i \in \{0, \ldots, m\}\}$ be a minimal set of generators, so $C = \text{her}(\text{conv}\{X_i : i \in \{0, \ldots, m\}\})$. By theorem 49(v), for each $X_i$,

$$F_i = \{Z \geq 0 : \text{Tr}(X_i Z) = 1\}$$

(230)

is a facet of $\text{TH}^f(S) = \text{TH}(S)$. By theorem 51 these facets correspond to $S$-full projectors. Specifically, each $X_i$ is an $S$-full projector. All $S$-full projectors are in $\text{fp}(S)$, so we have $\text{TH}^f(S)$ being generated by elements of $\text{fp}(S)$; therefore $\text{TH}^f(S) \subseteq \text{fp}(S)$. But by theorem 43, $\text{fp}(S) \subseteq \text{TH}^f(S)$. Therefore $\text{fp}(S) = \text{TH}^f(S)$.

This invites a second possible definition of perfection for non-commutative graphs.

Definition 53. A non-commutative graph $S$ is fg-perfect if $\text{TH}^f(S)$ is finitely generated.

Theorem 54. For a graph $G$ define $S = \text{span}\{|i\rangle\langle j| : i \simeq_G j\}$ and $S_0 = \text{span}\{|i\rangle \langle i| : i \in V(G)\}$. Then $S$ is fg-perfect if and only if $G$ is perfect.

Proof. If $G$ is perfect then $\text{TH}^f(G) = \text{TH}(G)$ is a polytope [16], and hence is finitely generated. By theorem 30, $\text{TH}^f(S) = \text{her}(\text{diag}(w) : w \in \text{TH}^f(G))$. If $w_1, \ldots, w_n$ are generators of $\text{TH}^f(G)$ then $\text{diag}(w_1), \ldots, \text{diag}(w_n)$ generate $\text{TH}^f(S)$. So $S$ is fg-perfect.

On the other hand, suppose $S$ is fg-perfect. By theorem 30, $\text{TH}^f(G) = \{\text{diag}(W) : W \in \text{TH}^f(S)\}$. If $W_1, \ldots, W_n$ generate $\text{TH}^f(S)$ then $\text{diag}(W_1), \ldots, \text{diag}(W_n)$ generate $\text{TH}^f(G)$. Being finitely generated, $\text{TH}^f(G)$ is a polytope. Then by [16] it is perfect.

We close this section with a series of examples exploring the relation between fp-perfect graphs and fg-perfect graphs.

Example 55. It is not necessarily the case that $S$ being fg-perfect implies $S^c$ is fg-perfect. Consider $S = \mathcal{L}(A)$, $S_0 = \mathbb{C}I$. $S$ is fg-perfect but $S^c$ is not.

Proof. We have $\text{TH}^f(S) = \text{TH}^f(\mathcal{L}(A)) = \{X \geq 0 : \|X\| \leq 1\}$ which is generated by a single element: $\text{TH}^f(S) = \text{her}(|I\rangle\langle I|)$. On the other hand, $\text{TH}^f(S^c) = \{X \geq 0 : \text{dim}(A)\text{Tr}(X) \leq 1\}$ which is not finitely generated: $\text{dim}(A)^{-1}P$ is maximal in $\text{TH}^f(S^c)$ for any rank-1 projector $P$.

Example 56. fg-perfect does not imply fp-perfect. Again consider $S = \mathcal{L}(A)$, $S_0 = \mathbb{C}I$. $S$ is fg-perfect but is not fp-perfect.

Proof. That $S$ is fg-perfect was shown in the previous example. $\text{fp}(S) = \text{her}(|I\rangle\langle I|)$ because $I$ is an $S$-full projector. On the other hand, there are no $S$-full projectors of $S^c = \mathbb{C}I$, so $\text{fp}(S^c) = \{0\}$ and $\text{fp}^o(S^c) = \text{Pos}(A)$. $\text{fp}(S) \neq \text{fp}^o(S^c)$ so $S$ is not fp-perfect.

Example 57. fp-perfect does not imply fg-perfect. A counterexample is the graph from example 47. This is not fg-perfect, though its complement is.

Proof. In example 47 this graph was shown to be fp-perfect, so $\text{TH}^f(S) = \text{fp}(S)$. By (191), $\text{fp}(S)$ is not finitely generated: any rank-1 projector of the form $\left\{\frac{1}{2} \begin{pmatrix} 1 & \phi \\ \phi^* & 1 \end{pmatrix} : \phi \in \mathbb{C}, |\phi| = 1\right\}$ is maximal in $S$.

Example 58. Maximal S-full projectors are not necessarily vertices of $\text{fp}(S)$, though they are maximal elements.

Proof. Consider the graph from example 47. Take the maximal S-full projector $X = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$, corresponding to $\phi = 1$ in (191). Notice that $\text{fp}(S)$ has a tangent line at this point given by $\phi = 1 \pm i\varepsilon$. Any supporting hyperplane of $\text{fp}(S)$ at $X$ must contain this line. Therefore the intersection of all supporting hyperplanes is not dimension 0, and $X$ is not a vertex. In fact, by the symmetry of (191), all maximal S-full projectors are not vertices.
VII. CONCLUSION AND OPEN QUESTIONS

Broadly speaking, there were three interrelated topics explored in this paper: the weighed theta function, the geometry of convex corners, and perfect graphs. While the basic theory of weighted theta functions is essentially complete, the theory of perfect graphs stands as little more than an invitation to the topic.

There are numerous open questions surrounding all aspects of this work. For the weighted theta function, the biggest question is whether there is any application. In terms of what may be generalized to non-commutative graphs, the most tantalizing application is the result of [1] that independence number approaches \( \vartheta \) when activated through the strong product:

\[
\sup_H \alpha(G \boxtimes H) = \vartheta(G \boxtimes H) = 1. \tag{231}
\]

The first step of the proof uses \( \alpha(G \boxtimes G) = n \) or, rather, the weighted version of this. This is trivial for classical graphs since \( \{(v, v) : v \in V(G)\} \) is an independent set. The analogous statement for non-commutative graphs seems not so easy. Possibly the rest of the proof of (231) could be made to work.

The entropy of the theta body is investigated in [23]. Entropy of non-commutative convex corners is explored in [2, section 2.4.3]. Is there application for the entropy of the theta body for non-commutative graphs? Using the techniques of [11], such a quantity can be estimated to arbitrary precision via an SDP.

An alternate definition of weighted theta was presented in [3]. Does this support a duality relation like theorem 19 or theorem 32? Are maximal elements of the corresponding theta body anti-blocker in \( S_0' \) like we have in theorem 29?

While we have made much progress on understanding the geometry of vertices and facets for non-commutative convex corners, a few questions remain open. Is it true that vertices of a convex corner always correspond to facets of the anti-blocker? Is a convex corner defined by finitely many inequalities if and only if its anti-blocker is finitely generated? If a convex corner is generated by its vertices, does this mean it is finitely generated?

As for perfect graphs, we have offered nothing beyond a couple definitions and some examples. The most pressing question is whether there is some analogue to the strong perfect graph theorem. That is to say, can fp- or fg-perfection be characterized by forbidden subgraphs? If not, is there any other characterization?

It seems unsatisfactory that \( S = CI, S_0 = CI \) is not fp-perfect. Can this be remedied by finding a tighter version of the sandwich theorem (theorem 43)? For instance, perhaps we should replace fp(\( S \)), the non-commutative analogue of the clique polytope, with a convex corner generated by entanglement assisted cliques (however those may be defined). Expanding the definition of perfect graphs by tightening the sandwich theorem has some precedent, e.g., circular-perfect graphs [26].

Source code in the Julia language for computing weighted thetas on non-commutative graphs is available [28].

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Appendix A: Proof of main theorem

The goal of this appendix is a constructive proof of lemma 25, showing how to convert a feasible solution for \( \vartheta(S/S_0 + CI, W) \) into a feasible solution for \( \vartheta(S, n\Psi(W)) \), and vice versa.

Lemma 59. Let \( W \in B \otimes Z \) be Hermitian and suppose \( W \) and \( \text{Tr}_B W \) are invertible. Then

\[
I_A \otimes W^{-1} \geq |\Phi\rangle\langle\Phi| \otimes (\text{Tr}_B W)^{-1}. \tag{A1}
\]
Proof. Consider

\[ P = (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes (\text{Tr}_B W)^{-1}) (I_A \otimes \sqrt{W}). \]  

(A2)

This is a projector since it is Hermitian and (as depicted in fig. 1),

\[ P^2 = (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes (\text{Tr}_B W)^{-1}) (I_A \otimes \sqrt{W}) (I_A \otimes \sqrt{W}) \]

(A3)

\[ = (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes I_Z)(\text{Tr}_B W)^{-1}(|\Phi\rangle\langle\Phi| \otimes I_Z)(I_A \otimes \sqrt{W})(I_A \otimes \sqrt{W}) \]

(A4)

\[ = (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes I_Z)(\text{Tr}_B W)^{-1}(\text{Tr}_B W)^{-1}(|\Phi\rangle\langle\Phi| \otimes I_Z)(I_A \otimes \sqrt{W}) \]

(A5)

\[ = (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes I_Z)(\text{Tr}_B W)^{-1}(|\Phi\rangle\langle\Phi| \otimes I_Z)(I_A \otimes \sqrt{W}) \]

(A6)

\[ = P. \]  

(A7)

As a projector, we have \( P \leq I_A \otimes I_B \otimes Z \), so

\[ (I_A \otimes \sqrt{W}) (|\Phi\rangle\langle\Phi| \otimes (\text{Tr}_B W)^{-1}) (I_A \otimes \sqrt{W}) \leq I_A \otimes I_B \otimes I_Z \]  

(A8)

\[ |\Phi\rangle\langle\Phi| \otimes (\text{Tr}_B W)^{-1} \leq I_A \otimes W^{-1}. \]  

(A9)

\[ \square \]

Lemma 60. Let \( G \) be a subgroup of the unitary operators on \( A \). Define

\[ P = \int UdU \]  

(A10)

\[ \Omega(\rho) = \int U\rho U^\dagger dU \]  

(A11)

where integration is with respect to the Haar measure on \( G \). Then for \( \rho \in \mathcal{L}(A), \rho \geq 0 \),

\[ P\rho P^\dagger \leq \Omega(\rho). \]  

(A12)
Proof. We first show \( P_Ω(ρ)P = P_ρP \). Since we are integrating over a group, change of variables \( U' = UW \) and \( V' = W^†V \) gives

\[
P_Ω(ρ)P = \iiint (UWρW^†V)dUdVdW \tag{A13}
\]
\[
= \iiint (U'ρV')dU'dV'dW \tag{A14}
\]
\[
= P_ρP^\dagger. \tag{A15}
\]

\( P \) commutes with \( Ω(ρ) \). Indeed, taking \( V' = UV \),

\[
P_Ω(ρ) = \iint (UVρV^†)dVdU \tag{A16}
\]
\[
= \iint (V'ρV'^†U)dV'dU \tag{A17}
\]
\[
= Ω(ρ)P. \tag{A18}
\]

Since \( P = P^\dagger = P^2 \), it is a projector. A projector that commutes with a positive semidefinite operator shrinks it, \( P_Ω(ρ)P^\dagger \leq Ω(ρ) \). Since \( P_Ω(ρ)P^\dagger = P_ρP^\dagger \), we are done. \hfill \Box

Lemma 61. Let \( G \) be a subgroup of the unitary operators on \( A \) and define

\[
T = (U ⊗ U)dU. \tag{A19}
\]

If \( Y ∈ L(A ⊗ B) \), \( Y \geq 0 \) then

\[
\text{Tr}_A(TYT^\dagger) \leq \int (U(\text{Tr}_A Y)U^†)dU \tag{A20}
\]

where \( U \) is defined such that \( (I ⊗ U)|Φ⟩ = (U^† ⊗ I)|Φ⟩ \).

Proof. The operators \( U ⊗ U \) form a group, \( (U ⊗ U)(V ⊗ V) = UV ⊗ UV = UV ⊗ U^TV \), so lemma 60 applies, giving

\[
TYT^\dagger \leq \int ((U ⊗ U)Y(U ⊗ U)^†)dU. \tag{A21}
\]

Taking the partial trace,

\[
\text{Tr}_A(TYT^\dagger) \leq \int \text{Tr}_A (U ⊗ U)Y(U ⊗ U)^†dU. \tag{A22}
\]
\[
≤ \int \text{Tr}_A ((I ⊗ U)Y(I ⊗ U)^†)dU. \tag{A23}
\]
\[
≤ \int (U(\text{Tr}_A Y)U^†)dU. \tag{A24}
\]

\hfill \Box

For a \( C^\ast \)-algebra \( S_0 ⊆ A \) we have, by the structure theorem for finite dimensional \( C^\ast \)-algebras, the decomposition (75). As explained in section II, the SDPs associated with \( \tilde{ν} \) involve operators in the augmented space \( L(A ⊗ B) \), with the vector \( |Φ⟩ \) giving an isomorphism \( A^\dagger \rightarrow B \). Under this isomorphism, we can decompose the \( B \) space in a way matching (75) for \( A \),

\[
B = \bigoplus_{i=1}^r B_i ⊗ Z_i. \tag{A25}
\]

This decomposition will be assumed for the remainder of this section.
Lemma 62. Let $S_0 \subseteq \mathcal{L}(A)$ be a $C^*$-algebra. Let $Y \in \mathcal{L}(A \otimes B)$ be Hermitian and $Y' = TYT'$ where

$$T = \int (U \otimes U) dU$$

(A26)

with integration being over unitaries in $S_0$ under the Haar measure. Then

$$Y' = \sum_{ij} |\Phi_{A_i \otimes B_j}\rangle \langle \Phi_{A_i \otimes B_j}| \otimes Q_{ij}$$

(A27)

for some $Q_{ij} \in \mathcal{L}(Y_j \otimes Z_j \rightarrow Y_i \otimes Z_i)$.

Proof. Suppose $M \subseteq \mathcal{L}(A)$ satisfies $UMU^\dagger = M$ for all unitary $U \in S_0$. Then $M$ commutes with $U$. Also $M$ must commute with the algebra generated by these unitaries, which is all of $S_0$. Since the commutator of $M$ takes the form (76), we have $M \in \bigoplus I_{A_i} \otimes \mathcal{L}(Y_i)$. And $(M \otimes I)|\Phi\rangle = \sum_i |\Phi_{A_i \otimes B_j}\rangle \otimes |\psi_i\rangle$ with $|\psi_i\rangle \in Y_i \otimes Z_i$.

Any vector $|M\rangle \in A \otimes B$ can be written as $(M \otimes I)|\Phi\rangle$ for some $M \in \mathcal{L}(A)$. If $(U \otimes U)|M\rangle = |M\rangle$ for all unitary $U \in S_0$ then $UMU^\dagger = M$ and by the above reasoning

$$|M\rangle = \sum_i |\Phi_{A_i \otimes B_j}\rangle \otimes |\psi_i\rangle$$

(A28)

for some $|\psi_i\rangle \in Y_i \otimes Z_i$.

Because of the twirling, $Y'$ satisfies $(U \otimes U)Y' = Y'$. Extending (A28) by linearity gives $Y' = \sum_i |\Phi_{A_i \otimes B_j}\rangle \otimes R_i$ with $R_i \in Y_i \otimes Z_i \otimes A^\dagger \otimes B^\dagger$. Any Hermitian operator matching this form must take the form (A27).

Theorem 63. Let $S$ be an $S_0$-graph, with $S_0$ decomposed as in (75). Let $P_i$ be the projector onto $A_i \otimes Y_i$. Suppose $X > 0$ commutes with all of $S_0$. Note that $X > 0$ requires $X$ to be invertible. Set $\lambda = \vartheta(S, X)$. Then there are $Y'$ and $Q_{ij}$ such that

$$Y' \in S \otimes \mathcal{L}(B)$$

(A29)

$$Q_{ij} \in \mathcal{L}(Y_j \otimes Z_j \rightarrow Y_i \otimes Z_i)$$

(A30)

$$Y' = \sum_{ij} |\Phi_{A_i \otimes B_j}\rangle \langle \Phi_{A_i \otimes B_j}| \otimes Q_{ij}$$

(A31)

$$Y' \geq |\Phi\rangle \langle \Phi|$$

(A32)

$$I_{B_i} \otimes \text{Tr}_{Y_i} Q_{ii} = \lambda (P_i X P_i)^{-T}$$

(A33)

where $|\Phi_{A_i \otimes B_j}\rangle$ is defined similar to $|\Phi\rangle$ but on the space $A_i \otimes B_i$ rather than $A \otimes B$.

Proof. Let $Y$ be optimal for (23) (of theorem 11) for $\vartheta(S, X)$,

$$Y \in S \otimes \mathcal{L}(B),$$

(A34)

$$\text{Tr}_A Y \leq \lambda X^{-T},$$

(A35)

$$Y \geq |\Phi\rangle \langle \Phi|.$$  

(A36)

Define the twirling operator

$$T = \int (U \otimes U) dU$$

(A37)

where we integrate over unitaries in $S_0$ under the Haar measure and set $Y' = TYT^\dagger$.

Since $USU^\dagger \subseteq S_0 SS_0 \subseteq S$, we have

$$Y' \in S \otimes \mathcal{L}(B)$$

(A38)

By lemma 62, $Y'$ takes the form (A31). Since $T|\Phi\rangle = |\Phi\rangle$ we have

$$Y' \geq |\Phi\rangle \langle \Phi|.$$  

(A39)
By lemma 61 we have

\[ \text{Tr}_A Y' \leq \int \mathcal{U}(\text{Tr}_A Y') \mathcal{U}^\dagger dU \quad (A40) \]
\[ \leq \lambda \int \mathcal{U} X^{-T} \mathcal{U}^\dagger dU \quad (A41) \]
\[ \leq \lambda \int X^{-T} \mathcal{U} \mathcal{U}^\dagger dU \quad (A42) \]
\[ \leq \lambda X^{-T} \quad (A43) \]

where we use the fact that \( X \) (and thus \( X^{-1} \)) commutes with \( U \in S_0 \), therefore \( X^{-T} \) commutes with \( U \).

Considering (A31) we have

\[ \text{Tr}_A Y' = \sum_i \text{Tr}_A(P_i Y') \quad (A44) \]
\[ = \sum_i \text{Tr}_A([\Phi_{A_i} \otimes B_i] \langle \Phi_{A_i} \otimes B_i \rangle) \otimes \text{Tr}_Y Q_{ii} \quad (A45) \]
\[ = \sum_i I_{B_i} \otimes \text{Tr}_Y Q_{ii} \quad (A46) \]

Projecting onto \( P_i^T \in \mathcal{L}(\mathcal{B}) \) (for a given \( i \)) and applying (A43) gives

\[ I_{B_i} \otimes \text{Tr}_Y Q_{ii} \leq \lambda P_i X^{-T} P_i. \quad (A47) \]

Since \( P_i \in S_0 \) and \( X \) commutes with all of \( S_0 \) we have \( P_i X^{-T} P_i = (P_i X P_i)^{-T} \) so

\[ I_{B_i} \otimes \text{Tr}_Y Q_{ii} \leq \lambda (P_i X P_i)^{-T}. \quad (A48) \]

We are done aside from (A48) being an inequality rather than an equality. This can be fixed by adding to each \( Q_{ii} \) a term of the form \( I_{Y_i} \otimes \sigma_i \) with \( \sigma_i \geq 0 \). Conditions (A32) and (A29) still hold, the former because we’re adding a positive semidefinite term and the latter because \( [\Phi_{A_i} \otimes B_i] \langle \Phi_{A_i} \otimes B_i \rangle \otimes I_{Y_i} \otimes \sigma_i \in S_0 \otimes \mathcal{L}(\mathcal{B}) \subseteq S \otimes \mathcal{L}(\mathcal{B}) \).

**Theorem 64.** Let \( S \) be an \( S_0 \)-graph and take \( \Psi \) from definition 23. Let \( S' = S/S_0 + CI \). Let \( W \in \text{Pos}(\mathcal{A}) \) be non-singular. Then

\[ \hat{\vartheta}(S', W) \leq \hat{\vartheta}(S, n\Psi(W)). \quad (A49) \]

**Proof.** With \( S_0 \) decomposed as in (75), let \( P_i \) be the projector onto \( A_i \otimes Y_i \). Set \( n_i = \text{Tr} P_i, n = \sum_i n_i = \text{dim}(\mathcal{A}), \) and \( W_i = P_i W P_i \).

Define \( X = n\Psi(W) \). The image of \( \Psi \) is \( S'_0 \) so \( X \) commutes with all of \( S_0 \) and theorem 63 applies. Let \( Y' \) and \( Q_{ij} \) satisfy (A29)-(A33) with \( \lambda = \hat{\vartheta}(S, X) \). Substituting the definition of \( X \) into (A33) gives

\[ I_{B_i} \otimes \text{Tr}_Y Q_{ii} = \lambda \left( P_i X P_i \right)^{-T} \quad (A50) \]
\[ = \lambda \left( n \text{dim}(\mathcal{Y}_i)^{-1} I_{A_i} \otimes \text{Tr}_{A_i} W_i \right)^{-T} \quad (A51) \]
\[ = \lambda n^{-1} \text{dim}(\mathcal{Y}_i) I_{B_i} \otimes (\text{Tr}_{A_i} W_i)^{-T} \quad (A52) \]
\[ \text{Tr}_Y Q_{ii} = \lambda n^{-1} \text{dim}(\mathcal{Y}_i) (\text{Tr}_{A_i} W_i)^{-T}. \quad (A53) \]

We will construct a feasible solution for \( \hat{\vartheta}(S', W) \). Let \( Y_{ij}' = P_i Y' P_j \). Because \( Y' \) takes the form (A31), we have \( Y_{ij}' = [\Phi_{A_i} \otimes B_i] \langle \Phi_{A_j} \otimes B_j \rangle \otimes Q_{ij} \). We will get \( Y' \) into the space \( S' \otimes \mathcal{L}(\mathcal{B}) \) by adding correction terms that are positive semidefinite. We first adjust the diagonal blocks \( Y_{ii}' \) to be perpendicular to trace-free elements of \( P_i S_0 P_i \), i.e., trace free operators from \( \mathcal{L}(\mathcal{A}_i) \otimes I_{Y_i} \). Consider the adjustment term

\[ Z_i = \lambda n^{-1} I_{Y_i} \otimes \left( I_{A_i} \otimes W_i^{-T} - [\Phi_{A_i} \otimes B_i] \langle \Phi_{A_i} \otimes B_i \rangle \otimes (\text{Tr}_{A_i} W_i)^{-T} \right) \quad (A54) \]
which, by lemma 59, is positive semidefinite (note \((\Tr_A W_i)^{-T} = (\Tr_B W_i^T)^{-1}\), and we are applying the lemma to \(W_i^T\)). Define

\[ Y'' = Y' + \sum_i Z_i \geq Y'. \]  

(A55)

Since \(Z_i \in S_0 \otimes \L(B) \subseteq S \otimes \L(B)\), we have \(Y'' \in S \otimes \L(B)\). For any \(R \in P_i S_0 P_i\) we have, by the structure of \(S_0\), \(R = R' \otimes I_{Y_i}\) for some \(R' \in \L(A_i)\). Then

\[
\Tr_A(RY'') = \Tr_A(\otimes Y_i)((R' \otimes I_{Y_i})(Y'_{ii} + Z_i))
\]

(A56)

\[
= \Tr_A(\otimes Y_i)((R' \otimes I_{Y_i})Y'_{ii} + \Tr_A,\otimes Y_i)((R' \otimes I_{Y_i})Z_i)
\]

(A57)

\[
= \Tr_A,(R'\otimes\Phi_{A,\otimes B},(\Phi_{A,\otimes B})) \otimes Y_i, Q_{ii} + \Tr_A,\otimes Y_i((R' \otimes I_{Y_i})Z_i)
\]

(A58)

\[
= R'T \otimes \Tr_{Y_i} Q_{ii} + \lambda n^{-1} \Tr(I_{Y_i}) (\Tr(R')W_i^{-T} - R'T \otimes (\Tr_A W_i)^{-T})
\]

(A59)

\[
= R'T \otimes (\Tr_{Y_i} Q_{ii} - \lambda n^{-1} \dim(Y_i) (\Tr_A W_i)^{-T}) + \lambda n^{-1} \dim(Y_i) \Tr(R')W_i^{-T}.
\]

(A60)

Substituting in (A53), the first term vanishes, leaving

\[
\Tr_A(RY'') = \lambda n^{-1} \dim(Y_i) \Tr(R'W_i^{-T}) = \lambda n^{-1} \Tr(R)W_i^{-T}
\]

(A61)

(A62)

In particular, \(\Tr_A(P_i Y'') = \lambda n^{-1} n_i W_i^{-T}\) and for trace-free \(R\), \(\Tr_A(RY'') = 0\). So

\[ Y'' \in (S/S_0 + \Span\{P_i\}) \otimes B. \]  

(A63)

Now define \(Y''' = (I \otimes \sqrt{WT}) Y'' (I \otimes \sqrt{WT})\), giving

\[
\Tr_A Y''' = \lambda n^{-1} n_i \sqrt{WT} W_i^{-T} \sqrt{WT}.
\]  

(A64)

Now, \(\sqrt{WT} W_i^{-T} \sqrt{WT}\) is a projector because it is Hermitian and equal to its square,

\[
\left(\sqrt{WT} W_i^{-T} \sqrt{WT}\right)^2 = \sqrt{WT} W_i^{-T} W^T W_i^{-T} \sqrt{WT}
\]

(A65)

\[
= \sqrt{WT} W_i^{-T} W^T W_i^{-T} \sqrt{WT}
\]

(A66)

\[
= \sqrt{WT} W_i^{-T} \sqrt{WT}
\]

(A67)

where the second line uses that the support of \(W_i^{-T}\) is \(P_i\). Being a projector, \(\sqrt{WT} W_i^{-T} \sqrt{WT} \leq I\) and we have

\[
\Tr_A Y''' = \lambda n^{-1} n_i \sqrt{WT} W_i^{-T} \sqrt{WT}
\]

\[
\leq \lambda n^{-1} n_i I_B.
\]  

(A68)

(A69)

We can add to that to get equality. Let \(\sigma_i\) be the slack in the (A69) inequality and set

\[
Y''' = Y'' + \sum_i \dim(A_i) I_{A_i} \otimes \sigma_i,
\]  

(A70)

giving

\[
\Tr_A Y''' = \lambda n^{-1} n_i I_B.
\]  

(A71)

We have \(Y''' \in S' \otimes \L(B)\). Indeed, since \(Y'' \in (S/S_0 + \Span\{P_i\}) \otimes B\) by (A63), and since we’ve only multiplied by \(\sqrt{WT}\) on the \(B\) side and added terms in \((S/S_0 + \Span\{P_i\}) \otimes B\), we have \(Y''' \in (S/S_0 + \Span\{P_i\}) \otimes B\). But in fact \(Y'''\) is perpendicular to anything in \(\Span\{P_i\}/C I\). For, suppose \(R \in \Span\{P_i\}/C I\). Then \(R = \alpha_1 P_i + \cdots + \alpha_n P_n\) and

\[
\Tr_A(RY'''') = \sum_i \alpha_i \lambda n^{-1} n_i I_B
\]

(A72)

\[
= \lambda n^{-1} I_B \sum_i \alpha_i n_i
\]  

(A73)

\[
= \lambda n^{-1} I_B \Tr(R) = 0.
\]

(A74)
Therefore \(Y''' \in (S/S_0 + CI) \otimes \mathcal{L}(B) = S' \otimes \mathcal{L}(B)\). Substituting \(R = I_A\) gives

\[
\text{Tr}_A(Y''') = \lambda I_B.
\] (A75)

And finally,

\[
Y''' \geq (I \otimes \sqrt{W}^T) Y'(I \otimes \sqrt{W}^T) = |\sqrt{W}\rangle\langle \sqrt{W}|.
\] (A78)

Therefore, \(Y'''\) is feasible for definition 8 for \(\tilde{\vartheta}(S', W)\) with value \(\lambda\).

We now work toward the reverse inequality, \(\tilde{\vartheta}(S', W) \geq \tilde{\vartheta}(S, n\Psi(W))\).

**Lemma 65.** Let \(U_{\alpha \beta} \in \mathcal{L}(A)\) be the generalized Pauli operators

\[
U_{\alpha \beta} = \sum_j \omega^{\beta j} |j + \alpha\rangle\langle j|
\] (A79)

where \(\omega\) is a primitive root of unity of order \(\dim(A)\) and the indices \(\alpha\) and \(\beta\) range from 0 to \(\dim(A) - 1\). Then for any \(M \in \mathcal{L}(A)\) we have

\[
\sum_{\alpha \beta} U_{\alpha \beta} M U_{\alpha \beta}^\dagger = \dim(A) \text{Tr}(M) I.
\] (A80)

**Proof.** The generalized Paulis form a basis of \(\mathcal{L}(A)\), so we can write \(M\) in this basis,

\[
M = \sum_{\alpha' \beta'} M_{\alpha' \beta'} U_{\alpha' \beta'}.
\] (A81)

The generalized Paulis satisfy the following braiding relation:

\[
U_{\alpha \beta} U_{\alpha' \beta'} = \omega^{\alpha' \beta - \alpha \beta'} U_{\alpha' \beta'} U_{\alpha \beta}.
\] (A82)

Using this braiding relation, we can evaluate the sum,

\[
\sum_{\alpha \beta} U_{\alpha \beta} M U_{\alpha \beta}^\dagger = \sum_{\alpha' \beta'} M_{\alpha' \beta'} \sum_{\alpha \beta} U_{\alpha \beta} U_{\alpha' \beta'} U_{\alpha' \beta'}^\dagger U_{\alpha \beta}^\dagger
\] (A83)

\[
= \sum_{\alpha' \beta'} M_{\alpha' \beta'} \omega^{\alpha' \beta - \alpha \beta'} U_{\alpha' \beta'}^\dagger U_{\alpha \beta}^\dagger
\] (A84)

\[
= \sum_{\alpha' \beta'} M_{\alpha' \beta'} \omega^{\alpha' \beta - \alpha \beta'} U_{\alpha' \beta'}
\] (A85)

\[
= \sum_{\alpha \beta} M_{00} U_{00}
\] (A86)

\[
= \dim(A)^2 M_{00} I
\] (A87)

where (A86) follows because \(\sum_{\alpha} \omega^{\alpha 0'} = 0\) unless \(0' = 0\), and similarly for the sum over \(\beta\). Since \(M_{00} = \dim(A)^{-1} \text{Tr} M\), we have the desired result. \(\square\)

The following corollary follows by linearity.

**Corollary 66.** Let \(U_{\alpha \beta} \in \mathcal{L}(A)\) be as in lemma 65. For any \(M \in \mathcal{L}(A) \otimes \mathcal{L}(B)\) we have

\[
\sum_{\alpha \beta} (U_{\alpha \beta} \otimes I_B) M (U_{\alpha \beta}^\dagger \otimes I_B) = \dim(A) I_A \otimes \text{Tr}_A M.
\] (A88)
**Theorem 67.** Let $S$ be an $S_0$-graph and take $\Psi$ from definition 23. Let $S' = S/S_0 + CI$. For $W \in \text{Pos}(A)$, 
\[ \tilde{\vartheta}(S', W) \geq \tilde{\vartheta}(S, n\Psi(W)). \]  

(A89)

**Proof.** With $S_0$ decomposed as in (75), let $P_i$ be the projector onto $A_i \otimes \mathcal{Y}_i$. Set $n_i = \text{Tr} P_i$, $n = \sum_i n_i = \text{dim}(\mathcal{A})$, and $W_i = P_i WP_i$.

Define $X = n\Psi(W)$. Let $Y$ be optimal for (21) for $\tilde{\vartheta}(S', W)$,
\[ Y \in S' \otimes \mathcal{L}(\mathcal{B}), \]
\[ \text{Tr}_A Y = \lambda W^T, \]
\[ Y \geq |W\rangle\langle W|, \]

(A90) (A91) (A92)

with $\lambda = \tilde{\vartheta}(S', W)$.

We will construct a feasible solution for $\tilde{\vartheta}(S, n\Psi(W))$. For each $A_i$ let $U_{i\alpha\beta} \in \mathcal{L}(A_i)$ be the generalized Pauli operators
\[ U_{i\alpha\beta} = \sum_j \omega^{\beta j} |j + \alpha \rangle\langle j|, \]

(A93)

where $\omega$ is a primitive root of unity of order $\text{dim}(A_i)$. The indices $\alpha$ and $\beta$ range from 0 to $\text{dim}(A_i) - 1$ so there are $\text{dim}(A_i)^2$ such operators.

Define the projection and twirling operation
\[ K = n \sum_{i\alpha\beta} n_i^{-1} V_{i\alpha\beta} \otimes V_{i\alpha\beta}^T. \]

(A94)

where
\[ V_{i\alpha\beta} = (U_{i\alpha\beta} \otimes I_Y) P_i. \]

(A95)

Define
\[ Y' = KYK^\dagger. \]

(A96)

Since $Y \in S' \otimes \mathcal{L}(\mathcal{B}) \subseteq S \otimes \mathcal{L}(\mathcal{B})$ and $K \in S_0 \otimes \mathcal{L}(\mathcal{B})$ we have $Y' \in S_0 SS_0 \otimes \mathcal{L}(\mathcal{B}) = S \otimes \mathcal{L}(\mathcal{B})$. Consider now its partial trace,
\[ \text{Tr}_A Y' = n^2 \sum_{ii'\alpha\beta'\beta'} n_i^{-1} n_i'^{-1} \text{Tr} \left( (V_{i\alpha\beta} \otimes V_{i\alpha\beta}^T) Y (V_{i'\alpha'\beta'}^T \otimes V_{i'\alpha'\beta'}^T) \right). \]

(A97)

\[ = n^2 \sum_{ii'\alpha\beta'\beta'} n_i^{-1} n_i'^{-1} V_{i\alpha\beta} \text{Tr}_A \left( Y (V_{i'\alpha'\beta'}^T V_{i\alpha\beta} \otimes I_B) \right) V_{i'\alpha'\beta'}^T. \]

(A98)

Consider the partial trace $\text{Tr}_A (Y (V_{i'\alpha'\beta'}^T V_{i\alpha\beta} \otimes I_B))$. We have $Y \in S' \otimes \mathcal{L}(\mathcal{B}) = (S/S_0 + CI) \otimes \mathcal{L}(\mathcal{B})$ and $V_{i'\alpha'\beta'}^T V_{i\alpha\beta} \otimes I_B \in S_0 \otimes \mathcal{L}(\mathcal{B})$. These spaces only overlap on $CI \otimes \mathcal{L}(\mathcal{B})$ so we only need to consider the component of $V_{i'\alpha'\beta'}^T V_{i\alpha\beta}$ that lies in $CI$. The projection of this factor onto $CI$ is
\[ n^{-1} I_A \text{Tr}(V_{i'\alpha'\beta'}^T V_{i\alpha\beta}) = n^{-1} I_A \text{Tr}(P_i) \delta_{ii'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \]

(A99)

where $\delta$ is the Kronecker delta. Continuing,
\[ \text{Tr}_A Y' = n \sum_{i\alpha\beta} n_i^{-2} \text{Tr}(P_i) V_{i\alpha\beta}^T \text{Tr}_A (Y) V_{i\alpha\beta}^T \]

(A100)

\[ = \lambda n \sum_{i\alpha\beta} n_i^{-1} V_{i\alpha\beta} W^T V_{i\alpha\beta}^T \]

(A101)

\[ = \lambda n \sum_i n_i^{-1} \sum_{\alpha\beta} ((U_{i\alpha\beta} \otimes I_Y) W_i (U_{i\alpha\beta} \otimes I_Y)^\dagger)^T. \]

(A102)
Applying corollary 66,

\[ \text{Tr}_A Y' = \lambda n \sum_i n^{-1}(\dim(A_i)I_{A_i} \otimes \text{Tr}_A W_i) \]

\[ = \lambda n \sum_i \dim(Y_i)^{-1}I_{A_i} \otimes \text{Tr}_A W_i \]

\[ = \lambda X, \]

as desired.

Conjugating both sides of (A92) with \( K \) yields

\[ Y' \geq K|W\rangle\langle W|K^\dagger. \]

We have

\[ K|W\rangle = n \sum_{i\alpha\beta} n_i^{-1}(V_{i\alpha\beta} \otimes V_{i\alpha\beta}^T)|W\rangle. \]

\[ = n \sum_{i\alpha\beta} n_i^{-1}(V_{i\alpha\beta}W \otimes V_{i\alpha\beta}^T)|\Phi\rangle. \]

\[ = n \sum_{i\alpha\beta} n_i^{-1}(V_{i\alpha\beta}WV_{i\alpha\beta}^\dagger \otimes I_B)|\Phi\rangle. \]

\[ = n \sum_{i\alpha}\dim(Y_i)^{-1}(I_{A_i} \otimes \text{Tr}_A W_i \otimes I_B)|\Phi\rangle. \]

\[ = (X \otimes I_B)|\Phi\rangle. \]

\[ = |X\rangle. \]

Therefore \( Y' \geq |X\rangle\langle X| \) and \( Y' \) satisfies (21) for \( \tilde{\vartheta}(S, X) \) with value \( \lambda = \tilde{\vartheta}(S', W) \).

**Appendix B: Proofs for convex corner geometry**

**Proof of theorem 49(i).** Suppose \( \dim(A) > 1 \) and that \( C \) has a facet. By lemma 27 there is some \( X_0 \in C \) in the relative interior of \( C \), and \( PCP = C \) where \( P \) is the projector onto the support of \( X_0 \). It must be the case that \( X_0 > 0 \), i.e., \( X_0 \) is not just in the relative interior but in fact in the interior. Otherwise the projector \( P \) is not full rank, and \( C \) lies in a subspace \( P\mathcal{L}(A)P \) of dimension \( \text{rank}(X_0)^2 \), strictly smaller than \( \dim(\mathcal{L}(A)) - 1 \), the required dimension of a facet.

**Proof of theorem 49(ii).** Suppose the supporting hyperplane defining a facet passes through the origin. That is, suppose the facet is of the form

\[ \mathcal{F} = \{ X \in \mathcal{C} : \text{Tr}(XY) = 0 \}. \]

for some \( Y \), with \( \text{Tr}(XY) \leq 0 \) for all \( X \in \mathcal{C} \).

By theorem 49(i) there is some \( X_0 \in \mathcal{C}, X_0 > 0 \). Then for any \( Z \geq 0 \) there is some \( \epsilon > 0 \) such that \( 0 \leq \epsilon Z \leq X_0 \), and hence by hereditarity \( \epsilon Z \in \mathcal{C} \). It must be the case that \( Y \leq 0 \). Otherwise there is some \( Z \geq 0 \) such that \( \text{Tr}(ZY) > 0 \) (e.g. the projector onto the positive eigenvalues of \( Y \) ). Then there is some \( \epsilon > 0 \) such that \( \epsilon Z \in \mathcal{C} \), and \( \text{Tr}(\epsilon ZY) > 0 \), contradicting that \( \text{Tr}(XY) \leq 0 \) for all \( X \in \mathcal{C} \).

Let \( P \) be the projector onto the support of \( Y \). The condition \( \text{Tr}(XY) = 0 \) for all \( X \in \mathcal{F} \), combined with \( Y \leq 0 \), requires \( PXP = 0 \) for all \( X \in \mathcal{F} \). But this is inconsistent with \( \mathcal{F} \) having affine dimension \( \dim(\mathcal{L}(A)) - 1 \). If \( P \neq 0 \), \( \dim(\mathcal{F}) \leq (\dim(A) - \text{rank}(P))^2 < \dim(\mathcal{L}(A)) - 1 \) (assuming \( \dim(A) > 1 \)). If \( P = 0 \) then \( Y = 0 \) and \( \dim(\mathcal{F}) = \dim(\mathcal{C}) = \dim(\mathcal{L}(A)) > \dim(\mathcal{L}(A)) - 1 \). In either case \( \dim(\mathcal{F}) = \dim(\mathcal{L}(A)) - 1 \) is not possible. So it is not possible for the hyperplane to pass through the origin.

A hyperplane not passing through the origin is of the form \( \{ X \in \text{Herm}(A) : \text{Tr}(XY) = \alpha \} \) for some \( \alpha \neq 0 \). Since supporting hyperplanes satisfy \( \text{Tr}(XY) \leq \alpha \) for all \( X \in \mathcal{C} \), and since \( 0 \in \mathcal{C} \), it is not possible that \( \alpha < 0 \). Therefore \( \alpha > 0 \). We can then rescale, defining \( Y' = Y/\alpha \) to get \( \mathcal{F} = \{ X \in \text{Herm}(A) : \text{Tr}(XY') = 1 \} \).
Proof of theorem 49(iii). Let \(X_0 \in \text{relint}(\mathcal{F})\). Note first that
\[
X_0 + \epsilon Y \notin \mathcal{C}
\] (B2)
for all \(\epsilon > 0\) because \(Y\) forms a supporting hyperplane, and adding \(\epsilon Y\) puts the point on the wrong side of that hyperplane: \(\text{Tr}((X_0 + \epsilon Y) Y) = 1 + \epsilon \text{Tr}(Y^2) > 1\) unless \(Y = 0\), which in turn is forbidden by \(\text{Tr}(X Y) = 1\) for \(X \in \mathcal{F}\).

We first show \(Y \geq 0\). The hypothesis \(\text{Tr}(X_0 Y) = 1\) forbids \(Y \leq 0\), so \(Y\) has at least one positive eigenvalue. If \(Y\) also has at least one negative eigenvalue then there is some \(Z > 0\) such that \(\text{Tr}(Z Y) = 0\). Since \(X_0 \in \text{relint}(\mathcal{F})\) and \(\text{Tr}(Z Y) = 0\) (so \(Z\) runs parallel to the hyperplane), \(Z\) can be chosen small enough that \(X_0 + Z \in \mathcal{F}\). Since \(Z > 0\) there is some \(\epsilon > 0\) such that \(-Z \leq \epsilon Y \leq Z\). Now,
\[
X_0 - Z \in \mathcal{F} \implies X_0 - Z \geq 0
\]
\[
\implies X_0 + \epsilon Y \geq 0
\] (B3)
Also, since \(\epsilon Y \leq Z\),
\[
X_0 + Z \in \mathcal{F} \implies X_0 + Z \in \mathcal{C} = \mathcal{C}^{\#}
\]
\[
\implies \text{Tr}((X_0 + Z)X) \leq 1 \text{ for all } X \in \mathcal{C}^{\#}
\]
\[
\implies \text{Tr}((X_0 + \epsilon Y)X) \leq 1 \text{ for all } X \in \mathcal{C}^{\#}
\] (B4)
By the second anti-blocker theorem, \(\mathcal{C} = \mathcal{C}^{\#} = \{W \geq 0 : \text{Tr}(W X) \leq 1 \text{ for all } X \in \mathcal{C}^{\#}\}\) so (B3)-(B4) imply \(X_0 + \epsilon Y \in \mathcal{C}\). This violates (B2). Therefore \(Y \geq 0\).

Since \(Y \geq 0\) and \(\text{Tr}(X Y) \leq 1\) for all \(X \in \mathcal{C}\) (because \(Y\) is a supporting hyperplane of \(\mathcal{C}\)), we have \(Y \in \mathcal{C}^{\#}\).

Every element \(X \in \mathcal{F}\) defines a supporting hyperplane of \(\mathcal{C}^{\#}\) at \(Y\), because we have \(\text{Tr}(X Y) = 1\) from (202) and \(\text{Tr}(X Y') \leq 1\) for all \(Y' \in \mathcal{C}^{\#}\) due to \(X \in \mathcal{C} = \mathcal{C}^{\#}\).

Since \(\mathcal{F}\) has affine dimension \(\dim(\mathcal{L}(A)) - 1\) and that affine space doesn’t pass through the origin (by theorem 49(ii)), we have linearly independent \(X_1, \ldots, X_{\dim(\mathcal{L}(A))} \in \mathcal{F}\). These are all supporting hyperplanes of \(\mathcal{C}^{\#}\) at \(Y\), and their intersection is the single point \(Y\), so \(Y\) is a vertex. \qed

Proof of theorem 49(iv). Let \(X \neq 0\) be a vertex of a convex corner \(\mathcal{C}\). Suppose \(X\) is not maximal. Then there is \(Y \geq 0, Y \neq 0\) such that \(X + Y \in \mathcal{C}\). We will show this leads to the local shape of \(\mathcal{C}\) at \(X\) being flat or curved, not a sharp point as would be required for \(X\) to be a vertex.

Let \(R_X, R_Y\) be such that \(R_X R_X^\dagger = X, R_Y R_Y^\dagger = Y\), and \(R_X R_Y^\dagger + R_Y R_X^\dagger \neq 0\). It suffices to take \(R_X = \sqrt{X}\) and \(R_Y = \sqrt{Y}\) unless \(\text{Tr}(X Y) = 0\), in which case it suffices to take \(R_Y = \sqrt{Y} U\) where \(U\) is a unitary causing the support of \(R_Y\) to overlap with the support of \(R_X\).

For \(0 < \epsilon < 1\) let
\[
Z = X + \epsilon^2(Y - X)
\] (B5)
\[
W = \epsilon \sqrt{1 - \epsilon^2} \left( R_{X} R_{Y}^\dagger + R_{Y} R_{X}^\dagger \right)
\] (B6)
Then
\[
Z \pm W = (1 - \epsilon^2)X + \epsilon^2 Y \pm \epsilon \sqrt{1 - \epsilon^2} \left( R_{X} R_{Y}^\dagger + R_{Y} R_{X}^\dagger \right)
\] (B7)
\[
\geq 0
\] (B8)
\[
(X + Y) - (Z \pm W) = Y + \epsilon^2(X - Y) \mp W
\] (B9)
\[
= (1 - \epsilon^2) Y + \epsilon^2 X \mp \epsilon \sqrt{1 - \epsilon^2} \left( R_{X} R_{Y}^\dagger + R_{Y} R_{X}^\dagger \right)
\] (B10)
\[
\geq 0
\] (B11)
\[
\text{so } 0 \leq Z \pm W \leq X + Y.
\]
Since \(X + Y \in \mathcal{C}\), by hereditarity we have \(Z \pm W \in \mathcal{C}\).
Let $V, \alpha$ define a supporting hyperplane of $\mathcal{C}$ at $X$ that is not parallel to $W$ (note that the direction of $W$ does not depend on $\epsilon$),

$$\text{Tr}(V X) = \alpha$$
$$\text{Tr}(V T) \leq \alpha \text{ for all } T \in \mathcal{C}$$
$$\text{Tr}(V W) \neq 0.$$

If $X$ is a vertex then such a hyperplane must exist: if all hyperplanes are parallel to $W$ then their intersection cannot be dimension 0. Now consider the points $Z \pm W \in \mathcal{C}$,

$$Z \pm W \in \mathcal{C} \implies \text{Tr}(V(Z \pm W)) \leq \alpha$$
$$\implies \text{Tr}(V(Z - X \pm W)) \leq 0$$

As $\epsilon \to 0$, $Z - X$ scales as $O(\epsilon^2)$ whereas $W$ scales as $O(\epsilon)$. So for sufficiently small $\epsilon$ the $\text{Tr}(V W)$ term dominates, requiring $\pm \text{Tr}(V W) \leq 0$ in contradiction of (B16). So all supporting hyperplanes of $\mathcal{C}$ at $X$ must be parallel to $W$ and $X$ cannot be a vertex. $\square$

**Proof of theorem 49(v)**. By the symmetry of the problem we need only prove that $X_0$ is a vertex and forms a facet of $\mathcal{C}^t$. Consider the semidefinite program

$$p^* = \min \{ \lambda : \text{Tr}(X_0 Y) = 1, \ \text{Tr}(X_i Y) \leq \lambda \text{ for } i \in \{1, \ldots, m\}, \ Y \geq 0 \}. \quad \text{(B19)}$$

After showing this is feasible with $\lambda < 1$ we will show that (203) is a facet and contains $Y$. Intuitively, the conditions of (B19) mandate that $Y$ is a supporting hyperplane of $\mathcal{C}^t$ at $X_0$, and remains a supporting hyperplane if “wiggled” in a direction perpendicular to $X_0$. We will return to this at the end.

The Lagrangian of this SDP is

$$L(\lambda, Y; \mu, b_i) = \lambda + \mu(1 - \text{Tr}(X_0 Y)) + \sum_{i=1}^{m} b_i(\text{Tr}(X_i Y) - \lambda)$$
$$= \mu + \text{Tr} \left( Y \left( \sum_{i=1}^{m} b_i X_i - \mu X_0 \right) \right) + \lambda \left( 1 - \sum_{i=1}^{m} b_i \right). \quad \text{(B20)}$$

The dual program is

$$d^* = \max \left\{ \mu : \sum_{i=1}^{m} b_i X_i \geq \mu X_0 \right\} \quad \text{(B22)}$$
$$\sum_{i=1}^{m} b_i = 1, b_i \geq 0 \}. \quad \text{(B23)}$$

This cannot be feasible for $\mu \geq 1$, since that would imply $X_0 \leq \sum b_i X_i$, meaning $X_0$ is dominated by a convex combination of the other $X_i$, in contradiction to the generators being a minimal set. Therefore $d^* < 1$. Slater’s condition is satisfied, with $Y = I/\text{Tr}(X_0)$ and $\lambda$ large being a feasible point in the relative interior of (B19). Therefore $p^* = d^*$, giving $p^* < 1$.

Take $Y$ to be feasible for (B19) with $\lambda < 1$. Let $K_0 = 0$ and let $\{K_j : j \in \{1, \ldots, \dim(\mathcal{L}(A)) - 1\}\}$ be a Hermitian basis of $\mathcal{L}(A)/X_0$ (the space perpendicular to $X_0$ under the Hilbert-Schmidt inner product), with $\|K_j\| \leq 1$. With $\epsilon > 0$ small (we shall later see how small), define

$$Y_j = (1 + \epsilon \text{Tr}(X_0))^{-1}(Y + \epsilon(I + K_j)) \quad \text{for } j \in \{0, \ldots, \dim(\mathcal{L}(A)) - 1\}. \quad \text{(B24)}$$

Since $K_j$ is orthogonal to $X_0$ and $\text{Tr}(X_0 Y) = 1$, we have

$$\text{Tr}(X_0 Y_j) = (1 + \epsilon \text{Tr}(X_0))^{-1}\text{Tr}(X_0(Y + \epsilon(I + K_j)))$$
$$= 1. \quad \text{(B25)}$$

$$Y_j \geq 0. \quad \text{(B24)}$$

$$\text{Tr}(X_0 Y_j) = (1 + \epsilon \text{Tr}(X_0))^{-1}\text{Tr}(X_0(Y + \epsilon(I + K_j)))$$
$$= 1. \quad \text{(B25)}$$

$$Y_j \geq 0. \quad \text{(B24)}$$
As for the rest of the $X_i$,

$$\text{Tr}(X_i Y_j) = (1 + \epsilon \text{Tr}(X_0))^{-1} \text{Tr}(X_i (Y + \epsilon (I + K_j)))$$  \hfill (B27)

$$\leq (1 + \epsilon \text{Tr}(X_0))^{-1} (\lambda + \epsilon \text{Tr}(X_i (I + K_j)))$$ \hfill (B28)

$$\leq \lambda + \epsilon \text{Tr}(X_i (I + K_j))$$ \hfill (B29)

$$< 1$$ \hfill (B30)

where the last line uses $\lambda < 1$ and $\epsilon$ sufficiently small.

Since $\{X_i : i \in \{0, \ldots, m\}\}$ generate $C$, each element of $C$ is dominated by a convex combination of these generators. We have then

$$\text{Tr}(X Y_j) \leq 1 \text{ for all } X \in C$$ \hfill (B31)

By (B24) and (B31), each $Y_j \in C^\natural$. By (B26), the $Y_j$ are on the surface of $C^\natural$. Since the $K_j$ (and hence the $Y_j$) span an affine space of dimension $\dim(L(A)) - 1$, they define a facet of $C^\natural$. And by (B26), $F_0 = \{Z \in \text{Herm}(A) : \text{Tr}(X_0 Z) = 1\}$ is the supporting hyperplane defining this facet.

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