Derivations and automorphism groups of completed Witt Lie algebra

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Abstract. In this paper, a simple Lie algebra, referred to as the completed Witt Lie algebra, is introduced. Its derivation algebra and automorphism group are completely described. As a by-product, it is obtained that the first cohomology group of this Lie algebra with coefficients in its adjoint module is trivial. Furthermore, we completely determine the conjugate classes of this Lie algebra under its automorphism group, and also obtain that this Lie algebra does not contain any nonzero ad-locally finite element.

Key words: Completed Witt Lie algebra, Derivations, Automorphism groups.

§1. Introduction

The complex Witt algebra $W_1$, named after Ernst Witt, was first defined by E. Cartan [1] in 1909, which is the Lie algebra of meromorphic vector fields defined on the Riemann sphere that are holomorphic except at two fixed points. It is also the complexification of the Lie algebra of polynomial vector fields on a circle, and the Lie algebra of derivations of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$. Witt algebras occur in the study of conformal field theory. There are some related Lie algebras defined over finite fields, studied by Witt in the 1930s, that are also called Witt algebras.

The complex Witt algebras were the first examples of nonclassical infinite dimensional simple Lie algebra in 1937. It has been paying more and more attentions since it was found. As is known that the 1-dimensional central extension of the Witt algebra $W_1$ is the well-known Virasoro algebra, which plays important roles both in mathematics and physics. The classification of Harish-Chandra modules over the Virasoro algebras, high rank and higher rank Virasoro algebras are well developed in [10, 18, 17, 19].

The simple modules of the Witt algebra in prime characteristic were classified by Chang [2] in 1941. For a long time this was the only systematic knowledge of the representations of nonclassical simple Lie algebras. In the mid 1980s, Shen [13] obtained the “mixed product” realization over tensor product of vector spaces. In this way, a complete description of their graded and filtered simple representations was established in [13, 14, 15]. A simple description of the representations of the Witt algebra was given in [6].

Generalized Witt algebras were defined by Kaplansky [7] in the context of the classification problem of simple finite dimensional Lie algebras over fields of prime characteristic. Later on, generalizations of the simple Lie algebras of Witt type over a field of characteristic zero have been studied in [8, 11, 12, 28]. The structure spaces and some representations of these simple Lie algebras of generalized Witt type were presented in [22, 25]. The derivations, automorphisms and the second cohomology groups of such kind of algebras have been studied by several authors as indicated in the references (see e.g. [4, 23]). The bialgebras of generalized Witt type were also considered in [16, 26].
The classical one-sided Witt algebra $W^+_1$ is defined with basis $\{L_i \mid i \in \mathbb{Z}, i \geq -1\}$ and brackets
\[ [L_i, L_j] = (j - i)L_{i+j} \quad \text{for all } i, j \geq -1, \ i, j \in \mathbb{Z}. \]

Thus by realizing $L_i$ as $t^{i+1}\frac{d}{dt}$, one immediately observes that $W^+_1 = \text{Der} \mathbb{C}[t] = \mathbb{C}[t]\frac{d}{dt}$ is the derivation algebra of the polynomial algebra $\mathbb{C}[t]$. From this, one sees that it is very natural to consider the generalization of the Witt algebra by replacing the polynomial algebra $\mathbb{C}[t]$ by the power series algebra $\mathbb{C}[[t]]$. In this paper, by starting from the determination of the derivation algebra of $\mathbb{C}[[t]]$, we are able to define a Lie algebra $\mathcal{L}$, which we refer to as the completed Witt algebra, as follows: Let $L_i, i \geq -1, i \in \mathbb{Z}$ be symbols, denoted by $\mathcal{L}$ the $\mathbb{C}$-vector space consisting of elements $\sum_{i=-1}^{\infty} a_i L_i$ with $a_i \in \mathbb{C}$ such that $\sum_{i=-1}^{\infty} a_i L_i = \sum_{i=-1}^{\infty} b_i L_i$ if and only if $a_i = b_i$ for all $i \geq -1$. Then $\mathcal{L}$ is a Lie algebra under the brackets
\[ \left[ \sum_{i=-1}^{\infty} a_i L_i, \sum_{i=-1}^{\infty} b_i L_i \right] = \sum_{i=-1}^{\infty} \left( \sum_{j=-1}^{i} (2j - i)a_{i-j}b_j \right) L_i. \quad (1.1) \]

One will see from (2.3) that $\mathcal{L}$ is in fact the derivation algebra of $\mathbb{C}[[t]]$ and it is a simple Lie algebra (Theorem 2.4). One of the motivations in studying this Lie algebra also comes from a recent paper [29], where in order to determine a new quantization of the Witt algebra, it is necessary to consider its completion. Our first aim is to study the structure theory of this Lie algebra, such as the determination of the derivation algebra and the automorphism group of this Lie algebra. The main results will be summarized in Theorems 2.1, 4.1 and 5.9. As a byproduct, we completely determine the conjugate classes of $\mathcal{L}$ under the automorphism group, and also obtain that $\mathcal{L}$ does not contain any nonzero ad-locally finite element (Corollary 5.10). Furthermore, we shall also present some interesting examples of subalgebras of this Lie algebra in Section 3; in particular, the centerless Virasoro algebra and nongraded Virasoro algebra, Lie algebras of Block type can be all realized as subalgebras of this Lie algebra (Examples 3.2, 3.3 and 3.5). Our next aim will be the study of the representations of this Lie algebra.

Finally, we would like to point out that since the algebras we are considering are infinite-dimensional with uncountable basis (and in fact it is impossible to find a basis of such an algebra), some techniques developed in [5] cannot be applied to our cases here. Also, since $\mathcal{L}$ does not contain any nonzero locally finite elements, some standard methods in determining automorphisms (such as those used in [23]) cannot be applied either. Thus it seems to us that it is necessary to develop some new techniques in dealing with the problems occurring in the present paper. This is also our another motivation to present this paper.

\section{Derivation algebra of $\mathbb{C}[[t]]$}

In this section, we shall determine the derivation algebra of the Laurent polynomial ring $\mathbb{C}[[t]] = \{ \sum_{i=0}^{\infty} a_i t^i \mid a_i \in \mathbb{C} \}$. Let us begin with the definition of derivations.

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A \textit{derivation} $D$ of $\mathcal{A}$ is a linear transformation on $\mathcal{A}$ such that
\[ D(xy) = D(x)y + xD(y) \quad \text{for } x, y \in \mathcal{A}. \quad (2.1) \]
The space Der $\mathcal{A}$ of derivations forms a Lie algebra with respect to the Lie bracket:
\[ [D_1, D_2] = D_1D_2 - D_2D_1 \quad \text{for } D_1, D_2 \in \text{Der} \mathcal{A}. \]
For any \( p(t) \in \mathbb{C}[[t]] \), one can define a derivation \( D = p(t) \frac{d}{dt} \in \text{Der} \mathbb{C}[[t]] \) as follows:

\[
D(a(t)) = p(t) \frac{d}{dt} a(t) \in \mathbb{C}[t] \quad \text{for any} \quad a(t) \in \mathbb{C}[[t]].
\]

We denote \( \mathbb{C}[[t]] \frac{d}{dt} = \{ p(t) \frac{d}{dt} | p(t) \in \mathbb{C}[[t]] \} \). The main result in this section is the following.

**Theorem 2.1** The derivation algebra of \( \mathbb{C}[[t]] \) is \( \text{Der} \mathbb{C}[[t]] = \mathbb{C}[[t]] \frac{d}{dt} \).

**Proof.** Let \( D \in \text{Der} \mathbb{C}[[t]] \). We shall prove that after a number of steps in each of which \( D \) is replaced by \( D - D' \) for some \( D' \in \mathbb{C}[[t]] \frac{d}{dt} \), the 0 derivation is obtained and thus proving that \( D \in \mathbb{C}[[t]] \frac{d}{dt} \). This will be done by two lemmas below. \( \square \)

**Lemma 2.2** Let \( D \in \text{Der} \mathbb{C}[[t]] \). By replacing \( D \) by \( D - D' \) for some \( D' \in \mathbb{C}[[t]] \frac{d}{dt} \), we can suppose \( D(t^i) = 0 \) for \( i \geq 0 \).

**Proof.** Let \( D \in \text{Der} \mathbb{C}[[t]] \). Take \( p(t) = D(t) \in \mathbb{C}[[t]] \). Re-denoting \( D - p(t) \frac{d}{dt} \) by \( D \), we have \( D(t) = 0 \). Using (2.1) and induction on \( i > 0 \), we have \( D(t^i) = 0 \). Obviously, we have \( D(t^0) = D(t^0t^0) = D(t^0) + D(t^0) \), and so \( D(t^0) = 0 \). \( \square \)

**Lemma 2.3** Suppose \( D \in \text{Der} \mathbb{C}[[t]] \) satisfying \( D(t^i) = 0 \) for \( i \geq 0 \). Then \( D = 0 \).

**Proof.** Let \( q(t) = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}[[t]] \). Assume \( D(q(t)) = \sum_{i=0}^{\infty} b_i t^i \). To compute \( b_{i_0} \) for any fixed \( i_0 \geq 0 \), we take a fixed \( N > i_0 \), and we have

\[
D(q(t)) = D\left( \sum_{i=0}^{N} a_i t^i + \sum_{i=N}^{\infty} a_i t^i \right) = D\left( \sum_{i=0}^{N} a_i t^i \right) + D\left( \sum_{i=N}^{\infty} a_i t^i \right) = D\left( \sum_{i=N}^{\infty} a_i t^i \right) = D\left( t^N \sum_{i=0}^{\infty} a_{i+N} t^i \right) = t^N D\left( \sum_{i=0}^{\infty} a_{i+N} t^i \right), \tag{2.2}
\]

where the third and last qualities follow from (2.1) and the fact that \( D(t^i) = 0 \) for \( i \in \mathbb{Z} \). Note that the right-hand side of (2.2), being a power series, does not contain the term \( t^{i_0} \) since \( N > i_0 \). Thus \( b_{i_0} = 0 \). Since \( i_0 \) is arbitrarily chosen, we obtain \( D(q(t)) = 0 \). \( \square \)

If we realize \( L_i \) as \( t^{i+1} \frac{d}{dt} \), then the Lie algebra \( \hat{L} \) defined in (1.4) is simply

\[
\hat{L} = \text{Der} \mathbb{C}[[t]] = \mathbb{C}[[t]] \frac{d}{dt}. \tag{2.3}
\]

**Theorem 2.4** The Lie algebra \( \hat{L} \) is simple.

**Proof.** Let \( I \) be any nonzero ideal of \( \mathbb{C}[[t]] \), and let \( x = \sum_{i=i_0}^{\infty} a_i L_i \in I \) be a nonzero element, where \( i_0 \geq -1 \) is the smallest integer such that \( a_{i_0} \neq 0 \). Applying \( \text{ad} L_{-1} \) to \( x \) several times and re-denoting the result by \( x \) if necessary, we can suppose \( i_0 = -1 \). By rescaling \( x \), we can suppose \( a_{-1} = 1 \). Now let \( y = \sum_{i=-1}^{\infty} b_i L_i \in \hat{L} \) be any element. We can always find some \( z = \sum_{i=-1}^{\infty} c_i L_i \in \hat{L} \) for some \( c_i \in \mathbb{C} \) such that \( y = [x, z] \), which is equivalent to

\[
b_i = \sum_{j=-1}^{i+1} (2j-i) a_{i-j} c_j = c_{i+1} + \sum_{j=-1}^{i} (2j-i) a_{i-j} c_j \quad \text{for} \quad i \geq -1. \tag{2.4}
\]
Regarding (2.4) as a system of linear equation on $c_i$, $i \geq -1$, we see that there exists a unique solution for $c_i$, $i \geq -1$, such that $y = [x, z]$ holds. This implies $y \in I$, i.e., $I = \hat{L}$. □

§3. Some interesting examples of subalgebras of $\hat{L}$

We present some interesting examples of subalgebras of $\hat{L}$ below. One may expect that their structure and representation theories are of interests and worth further studies.

**Example 3.1** (1) Let $B_1 = \text{Span}\{L_n = 2 \sin nt \frac{d}{dt} \mid n \in \mathbb{Z}\}$. Then $L_i = -L_{-i}$ and

$$[L_i, L_j] = (j - i)L_{i+j} + (i + j)L_{i-j}. \quad (3.1)$$

(2) Let $B_2 = \text{Span}\{L_n = 2 \sin nt \frac{d}{dt}, M_n = 2 \cos nt \frac{d}{dt} \mid n \in \mathbb{Z}\}$. Then $L_i = -L_{-i}$, $M_i = M_{-i}$, and we have (3.1) and

$$[L_i, M_j] = (j - i)M_{i+j} - (i + j)M_{i-j}, \quad [M_i, M_j] = (i - j)L_{i+j} + (i + j)L_{i-j}.$$ 

**Example 3.2** Let $B_3 = \text{Span}\{L_n = \exp nt \frac{d}{dt} \mid n \in \mathbb{Z}\}$. Then $[L_i, L_j] = (j - i)L_{i+j}$.

**Example 3.3** Let $B_4 = \text{Span}\{L_{m,n} = \sin mt \exp nt \frac{d}{dt}, M_{m,n} = \cos mt \exp nt \frac{d}{dt} \mid m, n \in \mathbb{Z}\}$. Then $L_{m,n} = -L_{-m,n}$, $M_{m,n} = M_{-m,n}$, and

$$[L_{m,n}, L_{k,l}] = \frac{k - m}{2}L_{m+k,n+l} + \frac{k + m}{2}L_{m-k,n+l} + \frac{n - l}{2}M_{m+k,n+l} + \frac{l - n}{2}M_{m-k,n+l},$$

$$[L_{m,n}, M_{k,l}] = \frac{k - m}{2}M_{m+k,n+l} - \frac{k + m}{2}M_{m-k,n+l} + \frac{l - n}{2}L_{m+k,n+l} + \frac{l - n}{2}L_{m-k,n+l},$$

$$[M_{m,n}, M_{k,l}] = \frac{m - k}{2}M_{m+k,n+l} + \frac{k + m}{2}M_{m-k,n+l} + \frac{l - n}{2}L_{m+k,n+l} + \frac{l - n}{2}L_{m-k,n+l}.$$ 

**Example 3.4** Let $B_5 = \text{Span}\{L_{m,n} = t^n \exp nt \frac{d}{dt} \mid m, n \in \mathbb{Z}\}$. Then

$$[L_{m,n}, L_{k,l}] = (k - m)L_{m+k,n+l} + (l - n)L_{m+k,n+l-1}.$$ 

Thus $B_5$ is in fact the centerless nongraded Virasoro defined in [23] (4.5)].

**Example 3.5** (1) Let $B_6 = \text{Span}\{L_{m,n} = t^n(t + a)^m \frac{d}{dt} \mid m, n \in \mathbb{Z}\}$ ($a \neq 0$ is fixed). Then

$$[L_{m,n}, L_{k,l}] = (k - m)L_{m+k-1,n+l} + (l - n)L_{m+k,n+l-1}.$$ 

(2) Let $B_7 = \text{Span}\{L_{m,n} = (\sin t)^m(\cos t)^n \frac{d}{dt} \mid m, n \in \mathbb{Z}\}$. Then

$$[L_{m,n}, L_{k,l}] = (k - m)L_{m+k-1,n+l+1} - (l - n)L_{m+k+1,n+l-1}.$$ 

Thus $B_6, B_7$ are in fact Lie algebras of Block type studied in [3, 9, 20, 21, 27, 30], etc.
§4. Derivations of $\hat{\mathcal{L}}$

In this section, we aim to determine all the derivations of the completed Witt Lie algebra $\hat{\mathcal{L}}$ with the Lie bracket defined by equation (1.1). As in (2.1), a linear transformation $D : \hat{\mathcal{L}} \rightarrow \hat{\mathcal{L}}$ is a derivation of $\hat{\mathcal{L}}$, if it satisfies $D([x, y]) = [D(x), y] + [x, D(y)]$ for $x, y \in \hat{\mathcal{L}}$. If, in addition, there exists some $x \in \hat{\mathcal{L}}$ such that $D(y) = ad x(y) = [x, y]$ for $y \in \hat{\mathcal{L}}$, then $D = ad x$ is called an inner derivation. We denote by $\text{Der} \hat{\mathcal{L}}$ (resp., $\text{ad} \hat{\mathcal{L}}$) the set of all derivations (resp., inner derivations). Then the main result in this section is the following.

**Theorem 4.1** The derivation algebra of $\hat{\mathcal{L}}$ is $\text{Der} \hat{\mathcal{L}} = ad \hat{\mathcal{L}}$. In particular, the first cohomology group of $\hat{\mathcal{L}}$ with coefficients in its adjoint module is trivial, namely, $H^1(\hat{\mathcal{L}}, \hat{\mathcal{L}}) = 0$.

**Proof.** Let $D \in \text{Der} \hat{\mathcal{L}}$. Again we shall prove that after a number of steps in each of which $D$ is replaced by $D - D'$ for some $D' \in \text{ad} \hat{\mathcal{L}}$, the 0 derivation is obtained and thus proving that $D \in \text{ad} \hat{\mathcal{L}}$. This will be done by three lemmas below. \hfill $\square$

**Lemma 4.2** Let $D \in \text{Der} \hat{\mathcal{L}}$, by replacing $D$ by $D - D'$ for some $D' \in \text{ad} \hat{\mathcal{L}}$, we can suppose $D(L_i) = a_i L_i$ for some $a_i \in \mathbb{C}$ with $i \geq -1$ and $a_0 = 0$.

**Proof.** Let $D \in \text{Der} \hat{\mathcal{L}}$. Denote $D(L_0) = \sum_{i=-1}^{\infty} a_i L_i$. Taking $v = \sum_{i \neq 0} (-\frac{a_i}{i}) L_i \in \hat{\mathcal{L}}$, one has

$$(D - \text{ad} v)(L_0) = D(L_0) - [v, L_0] = \sum_{i=-1}^{\infty} a_i L_i - \sum_{i \neq 0} (-\frac{a_i}{i}) [L_i, L_0] = a_0 L_0.$$ 

Re-denoting $D - \text{ad} v$ by $D$, we have $D(L_0) = a_0 L_0$. Suppose that $D(L_i) = \sum_{j=1}^{\infty} a_{ij} L_j$ ($i \neq 0$).

Applying $D$ to $[L_0, L_i] = i L_i$, we have $i a_0 L_i + \sum_{j \neq i} a_{ij} i L_j = i \sum_{j \neq i} a_{ij} L_j$, which implies $a_0 = 0$ since $i \neq 0$, and $a_{ij} = 0$ for $j \neq i$. Namely, $D(L_0) = 0$ and $D(L_i) = a_i L_i$, $i \neq 0$. Let $a_i = a_{ii}$ for $i \geq -1$, then we get $D(L_i) = a_i L_i$ with $a_0 = 0$. \hfill $\square$

**Lemma 4.3** Let $D \in \text{Der} \hat{\mathcal{L}}$, by replacing $D$ by $D - D'$ for some $D' \in \text{ad} \hat{\mathcal{L}}$, we can suppose $D(L_i) = 0$ for all $i \geq -1$.

**Proof.** Suppose $D \in \text{Der} \hat{\mathcal{L}}$ satisfies Lemma 4.2. Re-denoting $D - \text{ad} L_0$ by $D$, we obtain $D(L_0) = D(L_1) = 0$. Applying $D$ to $[L_i, L_1] = (1 - i) L_{i+1}$, we obtain $a_i (1 - i) L_{i+1} = (1 - i) a_{i+1} L_{i+1}$, which implies

$$a_i = a_{i+1} \text{ if } i \neq 1.$$ \hfill (4.1)

So $a_{-1} = a_0 = 0$. Applying $D$ to $[L_{-1}, L_2] = 3 L_1$, we get $[L_{-1}, a_2 L_2] = 0$, i.e. $3 a_2 L_1 = 0$, and thus $a_2 = 0$. Then the result follows from (4.1). \hfill $\square$

**Lemma 4.4** Suppose $D \in \text{Der} \hat{\mathcal{L}}$ satisfying $D(L_i) = 0$ for $i \geq -1$. Then $D = 0$.

**Proof.** Let $x = \sum_{i=-1}^{\infty} a_i L_i \in \hat{\mathcal{L}}$. Assume $D(x) = \sum_{i=-1}^{\infty} b_i L_i$. To compute $b_{i_0}$ for a fixed $i_0 \geq -1$, we take a fixed $N > i_0 + 1$, and we have

$$D(x) = D\left(\sum_{i=-1}^{\infty} a_i L_i\right) = D\left(\sum_{i=-1}^{N-2} a_i L_i + \sum_{i=N-1}^{\infty} a_i L_i\right) = D\left(\sum_{i=N-1}^{\infty} a_i L_i\right)$$

$$= D\left(\sum_{i=-1}^{N-1} a_i L_{i+N}\right) = D\left(L_N, \sum_{i \neq N} \frac{a_{i+N}}{i-N} L_i\right) + a_2 L_{2N}$$

$$= \left[L_N, D\left(\sum_{i \neq N \neq \infty} \frac{a_{i+N}}{i-N} L_i\right)\right].$$
If we write \( D(\sum_{i \neq N} a_i N_L i) = \sum_{i=-1}^\infty c_i L_i \), then we obtain \( \sum_{i=-1}^\infty b_i L_i = [L_N; \sum_{i=-1}^\infty c_i L_i] = \sum_{i=-1}^\infty c_i (i - N) L_{i+N} \), which in particular implies \( b_i = 0 \). Since \( i_0 \) is arbitrarily chosen, we obtain \( D(x) = 0 \). 

\[ \square \]

§5. Automorphism Groups of \( \hat{L} \)

In this section we compute the automorphism group \( \text{Aut} \hat{L} \) of the completed Witt algebra \( \hat{L} \). This will be done by several lemmas. We remark that since \( \hat{L} \) does not contain any nonzero locally finite elements (Corollary 5.10), some standard methods such as those used in [24] cannot be applied in our case here.

Lemma 5.1 For any \( \sigma \in \text{Aut} \hat{L} \), write \( \sigma(L_0) = \sum_{i=-1}^\infty a_i L_i \). Then one has \( (a_{-1}, a_0) \neq (0, 0) \).

Proof. If not the case, then \( \sigma(L_0) = \sum_{i=1}^\infty a_i L_i \). Write \( \sigma(L_{-1}) = \sum_{j=j_0}^\infty b_j L_j \), where \( j_0 \) is the smallest index such that \( b_{j_0} \neq 0 \). Applying \( \sigma \) to \( [L_0, L_{-1}] = -L_{-1} \), one has \( [\sigma(L_0), \sigma(L_{-1})] = -\sigma(L_{-1}) \), namely, \( \sum_{i=-1}^\infty a_i L_i; \sum_{j=j_0}^\infty b_j L_j = -\sum_{j=j_0}^\infty b_j L_j \). Note that the left-hand side does not contain the term \( L_{j_0} \), which implies that \( b_{j_0} = 0 \), a contradiction with the assumption that \( b_{j_0} \neq 0 \). 

For convenience, we introduce the following filtration of \( \hat{L} \):

\[ \hat{L} = \hat{L}_{-1} \supset \hat{L}_0 \supset \hat{L}_1 \supset \cdots \text{ with } \hat{L}_i = \left\{ \sum_{j=i}^\infty a_j L_j \mid a_j \in \mathbb{C} \right\}, \quad (5.1) \]

which satisfies

\[ \bigcap_{i=-1}^\infty \hat{L}_i = 0, \quad [\hat{L}_i, \hat{L}_j] \subset \hat{L}_{i+j} \text{ for } i, j \geq -1. \quad (5.2) \]

Remark 5.2 For any \( x = \sum_{i=-1}^\infty a_i L_i \in \hat{L} \), if we need to determine the coefficient \( a_{i_0} \) of \( L_{i_0} \) in \( x \) for a fixed \( i_0 \), we can always do the computation under modulo \( \hat{L}_N \) for a fixed \( N \gg 0 \). In this way, we can simplify the computation by reducing a sum with infinite terms to a sum with finite terms.

For any \( x = \sum_{i=-1}^\infty b_i L_i \in \hat{L}_1 \), we can define a linear transformation \( \exp^{ad}x \) on \( \hat{L} \) by

\[ \exp^{ad}x(y) = \sum_{k=0}^\infty \frac{1}{k!} (\text{ad } x)^k(y) \text{ for } y \in \hat{L}. \quad (5.3) \]

Note that for any \( y \in \hat{L} \), the right-hand side of (5.3) well defines an element in \( \hat{L} \) by observing the following: for any fixed \( N \), since \( x \in \hat{L}_1 \), by (5.2), we have \( (\text{ad } x)^k(y) \in \hat{L}_N \) when \( k > N \), thus Remark 5.2 can be applied.

Lemma 5.3 If \( x \in \hat{L}_1 \), then \( \exp^{ad}x \in \text{Aut} \hat{L} \).

Proof. Obviously, \( \exp^{ad}x \) is a bijection, since one can immediately check that \( \exp^{ad}(-x) \exp^{ad}x \) is the identity transformation. Thus it remains to prove

\[ [\exp^{ad}x(y), \exp^{ad}x(z)] = \exp^{ad}x([y, z]) \text{ for } y, z \in \hat{L}. \quad (5.4) \]
By \((5.2)\) and Remark \(5.2\), we can suppose \(x, y, z\) and the right-hand side of \((5.3)\) all contain only finite terms. In this case, \((5.4)\) is obvious.

Similarly, we can prove that \(\exp^{ad} L_0 \in \text{Aut} \hat{L}\). Denote

\[
\Gamma = \text{the subgroup of Aut} \hat{L} \text{ generated by } \exp^{a_0 \text{ad} L_0} \text{ and } \exp^{a \text{ad} x} \text{ for } a_0, x \in \hat{L}_1. \tag{5.5}
\]

**Proposition 5.4** The automorphism group of \(\hat{L}\) is \(\text{Aut} \hat{L} = \Gamma\).

**Proof.** The proof will be done by several lemmas below.

**Lemma 5.5** With notions as in Lemma \(5.1\), there must exist \(x \in \hat{L}_1\) such that \(\sigma(L_0) = \exp^{ad} (a_0 L_0 + a_{-1} L_{-1})\). Thus by re-denoting \(\exp^{ad} x \sigma\) by \(\sigma\), we can now suppose \(\sigma(L_0) = a_{-1} L_{-1} + a_0 L_0\).

**Proof.** We need to prove that there exists \(x = \sum_{i=1}^{\infty} b_i L_i \in \hat{L}_1\) for some \(b_i \in \mathbb{C}\) such that

\[
\sum_{i=-1}^\infty a_i L_i = \sigma(L_0) = \exp^{ad} (a_0 L_0 + a_{-1} L_{-1}) = \sum_{i=0}^{\infty} \frac{1}{i!} (\text{ad} x)^i (a_0 L_0 + a_{-1} L_{-1})
\]

\[
= (a_0 L_0 + a_{-1} L_{-1}) + [x, a_0 L_0 + a_{-1} L_{-1}] + \frac{1}{2!} [x, [x, [a_0 L_0 + a_{-1} L_{-1}]]] + \cdots \tag{5.6}
\]

By comparing the coefficients of \(L_j\) for \(j \geq -1\), we obtain the following system of equations on unknown variables \(b_i, i \geq 1:\)

\[
\begin{align*}
a_0 &= a_0 - 2a_{-1} b_1, \\
a_1 &= -a_0 b_1 - 3a_{-1} b_2 + a_{-1} b_1^2, \\
a_2 &= -2a_0 b_2 - 4a_{-1} b_3 + 2a_{-1} b_1 b_2, \\
a_3 &= -3a_0 b_3 - 5a_{-1} b_4 + a_{-1} b_1 b_3 - \frac{1}{2} a_0 b_1 b_2 + \frac{3}{2} a_{-1} b_2^2 + \frac{1}{3} a_{-1} b_1^2 b_2, \\
&\quad \ldots
\end{align*}
\]

\(5.7\)

We see that no matter whether \(a_{-1} \neq 0\) or \(a_{-1} = 0, a_0 \neq 0\), the system \((5.7)\) has a unique solution for \(b_i, i \geq 1\), which can be expressed in terms of \(a_j\) with \(-1 \leq j < i\). Hence we can choose \(b_i, i \geq 1\) such that \((5.6)\) holds. The lemma follows.

The following Lemma will play a key role in proving our main result.

**Lemma 5.6** For any \(x = \sum_{i=-1}^{\infty} a_i L_i \in \hat{L}\), suppose \(\sigma(x) = \sum_{i=-1}^{\infty} a_i^\prime L_i\). Then for every \(i_0\), the coefficient \(a_i^\prime\) of \(L_{i_0}\) in \(\sigma(x)\) is the same as that in \(\sum_{i=-1}^{N-1} a_i \sigma(L_i)\) for \(N \gg 0\).

**Proof.** Let \(i_0\) be fixed. Let \(L_{i_0+2}^i := \sigma^{-1}(L_{i_0+2}) = \sum_{j=j_0}^{\infty} b_j L_j\), where \(j_0 \geq -1\) is the smallest integer such that \(b_{j_0} \neq 0\). For any \(N \geq \max\{i_0, 2j_0 + 1\}\), we want to prove there exists \(y = \sum_{i=N-j_0}^{\infty} c_i L_i \in \hat{L}\) for some \(c_i \in \mathbb{C}\) such that

\[
\sum_{i=N}^{\infty} a_i L_i = [y, L_{i_0+2}^i], \tag{5.8}
\]

which is equivalent to

\[
a_i = \sum_{j=N-j_0}^{i-j_0} (i - 2j) c_j b_{i-j} = (2j_0 - i) c_{i-j_0} b_{j_0} + \sum_{j=N-j_0}^{i-j_0-1} (i - 2j) c_j b_{i-j} \quad \text{for } i \geq N. \tag{5.9}
\]
Regarding (5.9) as a system of linear equations on \( c_{i-j_0}, \ i \geq N \), and noting that the coefficient 
\((2j_0 - i)b_{j_0}\) of \( c_{i-j_0}\) is always nonzero since \( b_{j_0} \neq 0 \) and \( i \geq N > 2j_0 \), we obtain that there exists a unique solution of \( c_i, \ i \geq N - j_0 \) such that (5.8) holds.

Now using (5.8), we have

\[
\sigma(x) = \sigma \left( \sum_{i=1}^{N-1} a_i L_i + \sum_{i=N}^{\infty} a_i L_i \right) = \sum_{i=1}^{N-1} a_i \sigma(L_i) + \sigma \left( \sum_{i=N}^{\infty} a_i L_i \right) = \sum_{i=1}^{N-1} a_i \sigma(L_i) + \sigma(\sum_{i=N}^{\infty} a_i L_i) + \sigma(y, L_{i_0} + 2).
\]

It is easy to see that the smallest index in the second term is equal or greater than \( i_0 + 1 \). Hence, the coefficient \( a'_{i_0} \) of \( L_{i_0} \) in \( \sigma(x) \) is the same as that in \( \sum_{i=1}^{N-1} a_i \sigma(L_i) \).

\[ \Box \]

**Lemma 5.7** We have \( a_{-1} = 0 \) and \( a_0 = 1 \).

**Proof.** Write \( \sigma(L_i) = \sum_{j=1}^{\infty} b_{i,j} L_j \) for \( i \geq -1 \). Applying \( \sigma \) to \( iL_i = [L_0, L_i] \) and comparing the coefficients of \( L_j \) in both sides, we obtain

\[ ib_{i,j} = (j + 2)a_{-1}b_{i,j+1} + ja_0b_{i,j} \quad \text{for} \quad i, j \geq -1. \]  

(5.10)

First assume \( a_{-1} \neq 0 \). From (5.10), we inductively obtain

\[ b_{i,j} = \left( \frac{a_0}{a_{-1}} \right)^{j+1} b_{i,-1} \quad \text{and} \quad b_{i,-1} \neq 0 \quad \text{for} \quad i, j \geq -1, \]  

(5.11)

where in general, the binomial coefficient \( \binom{a}{j} \) is defined to be \( \frac{a(a-1)\cdots(a-j+1)}{j!} \) if \( 0 \leq j \in \mathbb{Z} \), or zero otherwise, and where, the second equation follows from the first and the fact that \( \sigma(L_i) \neq 0 \). Note that (5.11) has meaning even if \( a_0 = 0 \) as we can rewrite (5.11) so that \( a_0 \) does not appear in the denominator. Applying \( \sigma \) to \( (j-i) L_{i+j} = [L_i, L_j] \), and comparing the coefficients of \( L_p \) in both sides, we obtain

\[ (j-i)b_{i+1,j,p} = \sum_{k=-1}^{p+1} (p - 2k)b_{i,k}b_{i,j,p-k} \quad \text{for} \quad i, j, p \geq -1. \]  

(5.12)

Taking \( p = -1 \) and using (5.11), we obtain

\[ a_{-1}b_{i+1,j,-1} = b_{i,-1}b_{j,-1} \quad \text{for} \quad i, j \geq -1 \quad \text{with} \quad i \neq j. \]  

(5.13)

Taking \( j = -1 \), we see that \( b_{-1,-1} \neq 0 \). From this, one immediately solves

\[ b_{i,-1} = \left( \frac{a_{-1}}{b_{-1,-1}} \right)^{i+1} b_{-1,-1} \quad \text{for} \quad i \geq -1. \]  

(5.14)

Now take \( x = \sum_{i=-1}^{\infty} \frac{1}{b_{i,-1}} L_i \in \hat{\mathcal{C}} \). Let us compute the coefficient, denoted by \( \lambda \), of \( L_{-1} \) in \( \sigma(x) \). By Lemma 5.6, \( \lambda \) is equal to the coefficient of \( L_{-1} \) in \( \sum_{i=1}^{N} \frac{1}{b_{i,-1}} \sigma(L_i) \) for all \( N \gg 0 \), i.e., \( \lambda = N \) for all \( N \gg 0 \), which is impossible since \( \lambda \) is a constant number.

This proves \( a_{-1} = 0 \). Thus \( a_0 \neq 0 \). Now (5.10) shows that \( b_{i,j} = 0 \) if \( j \neq \frac{i}{a_0} \). However, for each \( i_0 \geq -1 \), there exists at least one integer \( j_0 \geq -1 \) such that \( b_{i_0,j_0} \neq 0 \). This shows that \( j_0 \) must be \( \frac{i_0}{a_0} \), thus \( \frac{i_0}{a_0} \geq -1 \) is an integer for all \( i_0 \geq -1 \). In particular, \( a_0 = 1 \).  

\[ \Box \]
Denote $b_i = b_{i,i}$. Then \( (5.10) \) implies
\[
\sigma(L_i) = b_i L_i \quad \text{for} \quad i \geq -1 \quad \text{with} \quad b_0 = 1.
\]
Applying $\sigma$ to $(j-i)L_{i+j} = [L_i, L_j]$, we can easily solve that $b_i = b^j$ for $i \geq -1$, where $b = b_1$.

**Lemma 5.8** For any $\sigma \in \text{Aut} \hat{L}$, by re-denoting $\exp^{a_0 \text{ad} L_0} \sigma$ by $\sigma$ for some $a_0 \in \mathbb{C}$, we can suppose $\sigma(L_i) = L_i$ for $i \geq -1$. Furthermore $\sigma = 1$.

**Proof.** Taking $a_0 = -\ln b \in \mathbb{C}$, one can easily check
\[
\exp^{a_0 \text{ad} L_0} \sigma(L_1) = \exp^{a_0 \text{ad} L_0} (b L_1) = L_1.
\]
Re-denoting $\exp^{a_0 \text{ad} L_0} \sigma$ by $\sigma$, one can assume that $\sigma(L_1) = L_1$, i.e., $b = 1$. Thus, $\sigma(L_i) = L_i$ for $i \geq -1$. Now using Lemma 5.6, we obtain that $\sigma(x) = x$ for any $x = \sum_{i=-1}^{\infty} a_i L_i \in \hat{L}$, i.e., $\sigma = 1$. This completes the proof of of Lemma 5.8.

From Lemmas 5.5 and 5.8, we in fact have the following.

**Theorem 5.9** Let $\hat{L}_1$ be defined as in $(5.1)$. Then the automorphism group of $\hat{L}$ is $\text{Aut} \hat{L} = \{ \exp^{x \text{ad} L_0} \exp^{a_0 \text{ad} L_0} \mid a_0 \in \mathbb{C}, x \in \hat{L}_1 \}$.

As the application of the above proof, we can obtain Corollary 5.10. First we give a concepts. An element $s \in \hat{L}$ is called ad-locally finite if for any given $v \in \hat{L}$, the subspace $\text{Span}\{(ad s)^m \cdot v \mid m \in \mathbb{Z}_+ \}$ of $\hat{L}$ is finite-dimensional. Two elements $y$ and $z$ in $\hat{L}$ are said to be $\text{(Aut} \hat{L})$-conjugate or conjugate under the automorphism group of $\hat{L}$ if $y \in (\text{Aut} \hat{L})z$.

**Corollary 5.10** (1) Two elements $y$ and $z$ in $\hat{L}$ are conjugate under the automorphism group of $\hat{L}$ if and only if there exists some $i \geq -1$ such that $y, z \in \hat{L}_i \hat{L}_{i+1}$ and moreover in case $i = 0$, $y - z \in \hat{L}_1$.

(2) The Lie algebra $\hat{L}$ does not contain any nonzero ad-locally finite element.

**Proof.** Let $y = \sum_{i=-1}^{\infty} a_i L_i$ be any nonzero element in $\hat{L}$ such that $i_0 \geq -1$ is the smallest integer with $a_{i_0} \neq 0$. As in the proof of Lemma 5.5 we can choose some $\sigma = \exp^{x \text{ad} L_0} \in \text{Aut} \hat{L}$ with $x \in \hat{L}_1$ such that $\sigma(x) = a_{i_0} L_{i_0}$. Furthermore, $a_{i_0} L_{i_0} = \sigma_0(L_{i_0})$ for some $\sigma_0 \in \text{Aut} \hat{L}$ if and only if $a_{i_0} = 1$, or $i_0 \neq 1$ and $\sigma_0 = \exp^{x \text{ad} L_0}$ for some $\lambda = \frac{\ln a_{i_0}}{i_0}$. From this, we obtain (1).

Now suppose $y$ is a nonzero ad-locally finite element. Then $L_{i_0} = \frac{1}{a_{i_0}} \sigma(y)$ is also an ad-locally finite element. But for $z = \sum_{i=-1}^{\infty} L_i$, one can easily check that $(ad L_{i_0})^k(z) = \sum_{j=\max\{k(i_0-1),-1\}}^{\infty} (j-2i_0)^k L_j$ for $k = 0, 1, \ldots$, are linear independent, which is a contradiction. This completes the proof of the corollary.

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