VECTORS COHERENT STATES WITH AN UNBOUNDED INVERSE FRAME OPERATOR

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Abstract. We present a class of vector coherent states in the domain $D \times D \times \ldots \times D$ (n-copies), where $D$ is the complex unit disc, using a specific class of hermitian matrices. Further, as an example, we build vector coherent states in the unit disc by considering the unit disc as the homogeneous space of the group $SU(1, 1)$.

1. Introduction

As it is well-known, coherent states, CS has applications in several branches of quantum physics (for example, see [5]). CS can be defined in several ways [1]. In this article we take the following definition.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{\phi_m\}_{m=0}^{\infty}$ and $\mathbb{C}$ be the complex plane. For $z \in \mathcal{O}$, an open subset of $\mathbb{C}$, the states

$$|z\rangle = N(|z|) \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{\rho(m)}} \phi_m$$

are said to form a set of CS if

(a) The states $|z\rangle$ are normalized,

(b) The states $|z\rangle$ give a resolution of the identity, that is

$$\int \mathcal{O} |z\rangle W(|z|) \langle z | \, d\mu = I$$

where $N(|z|)$ is the normalization factor, $\{\rho(m)\}_{m=0}^{\infty}$ is a sequence of nonzero positive real numbers, $W(|z|)$ is a positive function called a weight function, $d\mu$ is an appropriately chosen measure and $I$ is the identity operator on $\mathcal{H}$.

In the literature, several interesting classes of CS were formed by changing the factors and parameters of (1.1) (for example, the deformed CS). In [7] we have extended the above definition to a class of vector coherent states, VCS by replacing the complex number $z$ by a matrix $Z$ and gave several examples using quaternion matrices. In this article we develop another class of VCS with a particular type of hermitian matrix.

2. Vector Coherent States with an Unbounded Frame Operator

In this section we present VCS in the domain $D \times D \times \ldots \times D$ (n-copies), where $D = \{z \in \mathbb{C} \mid |z| < 1\}$ is the complex unit disc, using a hermitian matrix.
2.1. Vector coherent states. For \( z_1, z_2, ..., z_{n-1} \in D \), let \( Z = (z_1, z_2, ..., z_{n-1}) \).
Define the \( n \times n \) hermitian matrix, \( Z \) as

\[
Z = I_n + \begin{pmatrix} 0 & Z \\ \overline{Z}^T & 0 \end{pmatrix}
\]

where \( \overline{Z}^T \) is the complex conjugate transpose of \( Z \). Let \( \{\chi^1, \chi^2, ..., \chi^n\} \) be the natural orthonormal basis of \( \mathbb{C}^n \). Form the Hilbert space \( \mathcal{H} = \mathbb{C}^n \otimes \mathcal{F} \), where \( \otimes \) is the tensor product. Let \( \psi_m = \frac{\delta_m}{m+1} \). The set, \( \{\Phi_{qm} = \chi^q \otimes \psi_m \mid q = 1 ... n, m = 0 ... \infty\} \), is a basis of \( \mathcal{F} \). With the above setup we form the set of VCS in \( L^2(D \times D \times ... \times D, d\mu) \), where we take

\[
d\mu = r_1r_2...r_{n-1}dr_1d\theta_1dr_2d\theta_2...dr_{n-1}d\theta_{n-1}
\]

by taking \( z_j = r_je^{i\theta_j}, j = 1, 2, ... n-1 \), as

\[
|Z, q\rangle = \mathcal{N}(\langle Z|) \sum_{m=0}^{\infty} R(m)Z^m \Phi_{qm}, \quad q = 1, 2, ..., n
\]

The number \( \mathcal{N} = N(\langle Z|) \) and the \( n \times n \) matrix \( R(m) \) have to be chosen suitably. First of all let us calculate \( Z^m \) by diagonalizing the matrix \( Z \). The eigenvalues of \( Z \) are \( 1, 1+a \) and \( 1-a \) with multiplicities \( n-2, 1 \) and \( 1 \) respectively, where \( a = \|Z\| = \sqrt{r_1^2 + r_2^2 + ... + r_{n-1}^2} \). A set of orthogonal normalized eigenvectors corresponding to the eigenvalue, 1 are

\[
V_1 = \frac{1}{\sqrt{b_1}} \begin{pmatrix} 0 & -z_2 & z_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -z_3 & \bar{z}_2 & -z_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -z_4 & \bar{z}_3 & \bar{z}_2 & -z_1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -z_{n-1} & \bar{z}_{n-2} & \bar{z}_{n-3} & \cdots & \bar{z}_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & -z_n & \bar{z}_{n-1} & \bar{z}_{n-2} & \cdots & \bar{z}_{n-3} & \bar{z}_{n-2} & \bar{z}_{n-3} & \cdots & \bar{z}_{n-1} \end{pmatrix}
\]

where \( b_j = r_1^2 + r_2^2 + ... + r_{j+1}^2, \ j = 1, 2, ..., n-3 \). Eigenvectors corresponding to the eigenvalues \( 1+a \) and \( 1-a \) are

\[
V_{n-1} = \frac{1}{\sqrt{2a}} \begin{pmatrix} -a & \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_{n-1} \\ \bar{z}_1 & -z_1 & \bar{z}_2 & \cdots & \bar{z}_{n-2} \\ \bar{z}_2 & \bar{z}_1 & -z_2 & \cdots & \bar{z}_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{z}_{n-1} & \bar{z}_{n-2} & \bar{z}_{n-3} & \cdots & -z_{n-1} \end{pmatrix}
\]

\[
V_n = \frac{1}{\sqrt{2a}} \begin{pmatrix} a & \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_{n-1} \\ \bar{z}_1 & -z_1 & \bar{z}_2 & \cdots & \bar{z}_{n-2} \\ \bar{z}_2 & \bar{z}_1 & -z_2 & \cdots & \bar{z}_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{z}_{n-1} & \bar{z}_{n-2} & \bar{z}_{n-3} & \cdots & -z_{n-1} \end{pmatrix}
\]
respectively. The set \{V_1, V_2, ..., V_n\} is an orthonormal set with unit vectors. Form a matrix \(P\) by placing \(V_j\)s as columns, i.e.,

\[
P = \begin{pmatrix} V_1^T & V_2^T & \cdots & V_n^T \end{pmatrix}.
\]

Then \(P^T ZP = D\), a diagonal matrix with the eigenvalues over the diagonal. Thus we obtain \(Z^m = PD^mP^\dagger = (c_{lj})_{n \times n}\), where

\[
c_{11} = E_m, \quad c_{1l} = \frac{O_m s_{l-1}}{a}, \quad l = 2, 3, ..., n
\]

\[
c_{lj} = \frac{z_{j-1} z_{l-1}}{a^2} (E_m - 1), \quad l \neq j, l \neq 1, j \neq 1
\]

with

\[
(2.4) \quad E_m = \frac{1}{2} [(1 + a)^m + (1 - a)^m]
\]

(2.5) \quad \frac{1}{2}[ (1 + a)^m - (1 - a)^m].

Further notice that \((Z^m)^\dagger = Z^m\). Before we choose \(N\) and \(R(m)\) suitably, let us write \(R(m)\) tentatively as \(R(m) = (\alpha_{lj})\) where

\[
\alpha_{11} = R_m e^{im\theta}, \quad \alpha_{ll} = S_m e^{im\theta}, l \neq 1, \quad \alpha_{lj} = 0, l \neq j
\]

with

\[
\theta = \theta_1 + \theta_2 + \ldots + \theta_{n-1}.
\]

Consider

\[
|Z, 1\rangle \langle Z, 1| = N^2 \left| \sum_{m=0}^{\infty} R(m) Z^m \Phi_{1m} \right| \left| \sum_{k=0}^{\infty} R(k) Z^k \Phi_{1k} \right|
\]

\[
= N^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left| R(m) Z^m \Phi_{1m} \right| \left| R(k) Z^k \Phi_{1k} \right|
\]

\[
= N^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} R(m) Z^m \Omega_1 (Z^k)^\dagger (R(k))^\dagger \otimes | \psi_m \rangle \langle \psi_k |
\]

where \(\Omega_1 = \left( s_{lj}^{(1)} \right)_{n \times n}\) with \(s_{11}^{(1)} = 1\) and \(s_{lj}^{(1)} = 0\) for \(l \neq 1, j \neq 1\). The matrix multiplication yields,

\[
R(m) Z^m \Omega_1 (Z^k)^\dagger (R(k))^\dagger = \left( f_{lj}^{(1)} \right)_{n \times n}
\]
where

\[ f_{11}^{(1)} = R_m R_k E_m E_k e^{i \theta (m-k)} \]
\[ f_{ll}^{(1)} = S_m S_k O_m O_k \frac{r_l^2 - r_l^2}{a^2} e^{i \theta (m-k)} , \quad l = 2, 3, ..., n \]
\[ f_{1l}^{(1)} = R_k E_k S_m O_m \frac{r_{l-1}}{a} e^{i (m \theta - k \theta + \theta_{l-1})} , \quad l = 2, 3, ..., n \]
\[ f_{11}^{(1)} = R_k E_k S_m O_m \frac{r_l - 1}{a} e^{i (m \theta - k \theta - \theta_{l-1})} , \quad l = 2, 3, ..., n \]
\[ f_{lj}^{(1)} = S_m S_k O_m O_k r_{l-1}^2 e^{i (m \theta - k \theta + \theta_{j-1} - \theta_{l-1})} , \quad j, l = 2, 3, ..., n; j \neq l. \]

Thus we obtain

\[ |Z, 1 \rangle \langle Z, 1 | = N^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (f_{lj}^{(1)})_{n \times n} \otimes | \psi_m \rangle \langle \psi_k | . \]

Similarly, in order to obtain \(|Z, 2 \rangle \langle Z, 2 |\) we calculate

\[ R(m) Z^m \Omega_2 (Z^k)^\dagger (R(k))^\dagger = (f_{lj}^{(2)})_{n \times n} \]

where

\[ f_{11}^{(2)} = R_m R_k O_m O_k \frac{r_l^2}{a^2} e^{i \theta (m-k)} \]
\[ f_{22}^{(2)} = S_m S_k \frac{(r_1^2 E_m + a^2 - r_l^2)}{a^2} (r_1^2 E_k + a^2 - r_1^2) e^{i \theta (m-k)} \]
\[ f_{ll}^{(2)} = S_k S_m \frac{(E_m - 1)(E_k - 1)}{a^4} r_1^2 r_{l-1}^2 e^{i \theta (m-k)} , \quad l = 3, 4, ..., n \]
\[ f_{lj}^{(2)} = A_{ljmk}^{(2)} e^{i (m \theta - k \theta - \theta_{l-1})} , \quad j = 1, l = 2, 3, ..., n \]
\[ f_{lj}^{(2)} = B_{ljmk}^{(2)} e^{i (m \theta - k \theta + \theta_{j-1})} , \quad l = 1, j = 2, 3, ..., n; j \neq l. \]
\[ f_{lj}^{(2)} = C_{ljmk}^{(2)} e^{i (m \theta - k \theta + \theta_{j-1} - \theta_{l-1})} , \quad \text{otherwise} \]

Where \(A_{ljmk}^{(2)}, B_{ljmk}^{(2)} \) and \(C_{ljmk}^{(2)} \) are some functions of \(m, k\) and \(r_l, r_j\), exact expressions are irrelevant for our purpose. Thus we obtain

\[ |Z, 2 \rangle \langle Z, 2 | = N^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (f_{lj}^{(2)})_{n \times n} \otimes | \psi_m \rangle \langle \psi_k | . \]

In general, in order to get \(|Z, q \rangle \langle Z, q |\), we calculate

\[ R(m) Z^m \Omega_q (Z^k)^\dagger (R(k))^\dagger = (f_{lj}^{(q)})_{n \times n} \]
where
\[
\begin{align*}
    f_{11}^{(q)} &= R_m R_k O_m O_k \frac{r_{q-1}^2 - a^2}{a^2} e^{i \theta (m-k)} \\
    f_{22}^{(q)} &= S_m S_k \frac{r_{q-1}^2 E_m + a^2 - r_{q-1}^2}{a^2} (r_{q-1}^2 E_k + a^2 - r_{q-1}^2) e^{i \theta (m-k)} \\
    f_{ll}^{(q)} &= S_l S_m \frac{(E_m - 1)(E_k - 1)}{a^4} r_{q-1}^2 r_{l-1}^2 e^{i \theta (m-k)}, \quad l \neq q, l = 2, 4, ..., n \\
    f_{ij}^{(q)} &= A_{jlmk} e^{i (m \theta - k \theta - \theta_{l-1})}, \quad j = 1, l = 2, 3, ..., n \\
    f_{ij}^{(q)} &= B_{jlmk} e^{i (m \theta - k \theta + \theta_{j-1})}, \quad l = 1, j = 2, 3, ..., n; j \neq l \\
    f_{ij}^{(q)} &= C_{jlmk} e^{i (m \theta - k \theta + \theta_{j-1} - \theta_{l-1})}, \quad \text{otherwise}
\end{align*}
\]

Thus we obtain
\[
(2.8) \quad |Z, q\rangle \langle Z, q| = \mathcal{N}^2 \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (f_{ij}^{(q)})_{n \times n} \otimes |\psi_m \rangle \langle \psi_k|.
\]

Now, when we compute
\[
\begin{align*}
    \sum_{q=1}^{n} \int_{D \times \ldots \times D} |Z, q\rangle \langle Z, q | \ d\mu,
\end{align*}
\]
all the off-diagonal terms vanish with one of the $\theta_1, \theta_2, ..., \theta_{n-1}$ integrals and only in the diagonals terms with $m = k$ survive. Thus we obtain
\[
\begin{align*}
    \sum_{q=1}^{n} \int_{D \times \ldots \times D} |Z, q\rangle \langle Z, q | \ d\mu &= (D_{11})_{n \times n}
\end{align*}
\]
where
\[
\begin{align*}
    D_{11} &= \sum_{q=1}^{n} f_{11}^{(q)} = \sum_{m=0}^{\infty} \left( \int_{D \times \ldots \times D} \mathcal{N}^2 R_m^2 (E_m^2 + O_m^2) \ d\mu \right) \otimes |\psi_m \rangle \langle \psi_m | \\
    D_{ll} &= \sum_{q=1}^{n} f_{ll}^{(q)} \\
    &= \sum_{m=0}^{\infty} \left( \int_{D \times \ldots \times D} \mathcal{N}^2 S_m^2 r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2 \ d\mu \right) \otimes |\psi_m \rangle \langle \psi_m |, \quad l = 2, ..., n \\
    D_{ij} &= 0 \quad \text{otherwise}.
\end{align*}
\]
Choose
\[
(2.9) \quad \mathcal{N} = \frac{\alpha}{\sqrt{(2\pi)^{n-1}}}.
\]

Then
\[
\begin{align*}
    D_{11} &= \sum_{m=0}^{\infty} \left( \int_{0}^{1} \ldots \int_{0}^{1} a^2 R_m^2 (E_m^2 + O_m^2) dr_1 \ldots dr_{n-1} \right) \otimes |\psi_m \rangle \langle \psi_m | \\
    D_{ll} &= \sum_{m=0}^{\infty} \left( \int_{0}^{1} \ldots \int_{0}^{1} S_m^2 (r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2) dr_1 \ldots dr_{n-1} \right) \otimes |\psi_m \rangle \langle \psi_m |, \quad l = 2, 3, ..., n.
\end{align*}
\]
The following integrals are finite and positive

\[ N_1 m = \int_0^1 \cdots \int_0^1 a^2 R_m^2 (E_m^2 + O_m^2) dr_1 \cdots dr_{n-1} \]

\[ N_2 m = \int_0^1 \cdots \int_0^1 S_m^2 (r_{l-1}^2 (E_m^2 + O_m^2) + a^2 - r_{l-1}^2) dr_1 \cdots dr_{n-1} \quad \text{(for any)} \ l = 2, 3, \ldots, n. \]

Further \( N_1 m = (n-1)N_2 m - \frac{1}{4} (n-2). \) Now choose

\[ (2.10) \quad R_m = \frac{1}{\sqrt{N_1 m}} \quad \text{and} \quad S_m = \frac{1}{\sqrt{N_2 m}}. \]

Then we obtain

\[ \sum_{q=1}^n \int_{D \times \cdots \times D} |Z, q \rangle \langle Z, q | \ d\mu = \mathbb{1}_n \otimes \sum_{m=1}^{\infty} |\psi_m \rangle \langle \psi_m |. \]

Now let

\[ (2.11) \quad T = \mathbb{1}_n \otimes \sum_{m=1}^{\infty} |\psi_m \rangle \langle \psi_m |, \]

which is a bounded invertible operator with \( \text{Ker} T = \{0\} \), for

\[ \text{Ker} T = \{ \Phi \in \tilde{H} | T \Phi = 0 \}. \]

Every vector \( \Phi \) in \( \tilde{H} \) can be written as

\[ \Phi = \left( \sum_{k=1}^{\infty} \alpha_{k1} \phi_k, \sum_{k=1}^{\infty} \alpha_{k2} \phi_k, \ldots, \sum_{k=1}^{\infty} \alpha_{kn} \phi_k \right)^T \]

Thus \( T \Phi = 0 \) gives

\[ \text{diag} \left( \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} | \phi_m \rangle \langle \phi_m |, \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} | \phi_m \rangle \langle \phi_m |, \ldots, \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} | \phi_m \rangle \langle \phi_m | \right) \cdot \]

\[ \left( \sum_{k=1}^{\infty} \alpha_{k1} \phi_k, \sum_{k=1}^{\infty} \alpha_{k2} \phi_k, \ldots, \sum_{k=1}^{\infty} \alpha_{kn} \phi_k \right)^T = (0, 0, \ldots, 0) \]

Hence, for each \( i \), we get

\[ \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} \alpha_{mi} \phi_m = 0. \]

With the fact that \( \{ \phi_m \}_{m=0}^{\infty} \) is an orthonormal basis, for each \( m \) and for each \( i \), we have \( \alpha_{mi} = 0. \) Thus we have \( \text{Ker} T = \{0\}. \) The inverse of \( T \) exists and the domain of \( T^{-1} \), \( \text{D}(T^{-1}) \) is dense in \( \tilde{H} \). For \( \phi \in \text{D}(T^{-1}) \), consider

\[ T(T^{-1} \phi) = \left[ \sum_{q=1}^n \int_D |Z, q \rangle \langle Z, q | \ d\mu \right] T^{-1} \phi. \]

That is

\[ \phi = \sum_{q=1}^n \int_D |Z, q \rangle \langle Z, q | T^{-1} \phi \rangle d\mu. \]
Thus, for the vectors in $D(T^{-1})$ we have a proper decomposition in the above sense. Now let $\phi \in \tilde{H}$, then there exists a sequence $\{\eta_m\}_{m=0}^\infty \subset D(T^{-1})$ such that $\eta_m \rightarrow \phi$ as $m \rightarrow \infty$ in $\tilde{H}$. Further for each $\eta_m$ we have,

$$\eta_m = \sum_{q=1}^n \int_D |Z,q \rangle \langle Z,q | T^{-1} \eta_m \rangle d\mu.$$ 

Now by taking limit both sides as $m \rightarrow \infty$, we can have

$$\phi = \lim_{m \rightarrow \infty} \left( \sum_{q=1}^n \int_D |Z,q \rangle \langle Z,q | T^{-1} \eta_m \rangle d\mu \right).$$

In this regard, we have a weak decomposition for the vectors in $\tilde{H} - D(T^{-1})$. Even though we do not have a complete resolution of the identity the CS form a frame with the frame operator $T$ (see [1]). The inverse operator, $T^{-1}$ is unbounded.

3. Example: Vector coherent states with $SU(1,1)$

In this section we build vector coherent states on $D$ by considering it as the homogeneous space of the group $SU(1,1)$. In order to introduce the concept we need the following preliminaries.

The non-compact group $SU(1,1)$ is defined as,

$$SU(1,1) = \left\{ g | g = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \det g = |\alpha|^2 - |\beta|^2 = 1 \right\},$$

and its maximal compact subgroup $K$ is given by

$$K = \left\{ k | k = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} : 0 \leq \phi \leq 2\pi \right\}.$$ 

The Cartan decomposition of an arbitrary $g \in SU(1,1)$ is,

$$g = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} = |\alpha| \begin{pmatrix} 1 & z \\ \frac{\bar{z}}{\bar{\alpha}} & \frac{\bar{\alpha}}{\alpha} \end{pmatrix} \begin{pmatrix} 1 & \bar{z} \\ 0 & 1 \end{pmatrix}$$

where $z = \beta \alpha^{-1}$ and $|\alpha| = (1 - |z|^2)^{-\frac{1}{2}}$. Thus the coset space $SU(1,1)/K$ can be identified with the unit disc. Further it is known that the measure

$$d\nu(z,\bar{z}) = \frac{1}{2\pi i (1 - |z|^2)^2} \frac{dz \wedge d\bar{z}}{1 - |z|^2}$$

on $D$ is invariant under the action of $SU(1,1)$. In polar coordinates

$$Z = \begin{pmatrix} 1 & z \\ \frac{\bar{z}}{\bar{\alpha}} & \frac{\bar{\alpha}}{\alpha} \end{pmatrix}$$

$$d\nu(z,\bar{z}) = d\nu(r,\theta) = \frac{rdrd\theta}{\pi(1 - r^2)^2}$$

where $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$. Let $\{\chi^1, \chi^2\}$ be the natural basis of $\mathbb{C}^2$. Now, as before, form the Hilbert space $\tilde{H} = \mathbb{C}^2 \otimes \tilde{H}$ and take $\{\Phi_{qm} = \chi^q \otimes \psi_m | q = \}$
1, 2; m = 0, 1, ..., ∞} as a basis of it. With the above set up we form the set of coherent states in \( L^2(D, d\mu) \) as,

\[
|Z, q\rangle = \mathcal{N}(|Z|) \sum_{m=0}^{\infty} R_m Z^m \Phi_{qm}, \quad q = 1, 2
\]

by suitably choosing the number \( \mathcal{N} = \mathcal{N}(|Z|) \) and a \( 2 \times 2 \) matrix \( R(m) \).

As we did before through diagonalization we obtain

\[
Z^m = \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{i\theta} & E_m \end{pmatrix}
\]

with

\[
E_m = \frac{1}{2}[(1 + r)^m + (1 - r)^m] \\
O_m = \frac{1}{2}[(1 + r)^m - (1 - r)^m].
\]

Take

\[
R(m) = \begin{pmatrix} R_m e^{im\theta} & 0 \\ 0 & R_m e^{im\theta} \end{pmatrix}.
\]

Now with the above choices, matrix multiplication yields

\[
|Z, 1\rangle = \mathcal{N} \sum_{m=0}^{\infty} \begin{pmatrix} R_m e^{im\theta} E_m \psi_m \\ R_m e^{im\theta} O_m \psi_m \end{pmatrix}
\]

\[
|Z, 2\rangle = \mathcal{N} \sum_{m=0}^{\infty} \begin{pmatrix} R_m e^{im\theta} O_m \psi_m \\ R_m e^{im\theta} E_m \psi_m \end{pmatrix}
\]

First let us calculate the following finite integrals

\[
\int_0^1 E_m^2 r dr = \int_0^1 \frac{1}{4}(1 + r)^{2m} + \frac{1}{2}(1 - r^2)^m + \frac{1}{4}(1 - r)^{2m} r dr
\]

\[
= \frac{1}{4} \left( \sum_{j=0}^{2m} \binom{2m}{j} \frac{1}{j+2} + \sum_{j=0}^{2m} \binom{2m}{j} \frac{(-1)^j}{j+2} \right) + \frac{1}{4(m+1)}
\]

\[
= \frac{1}{4} \frac{2^m + 1 + m}{(2m + 1)(m + 1)}
\]

and

\[
\int_0^1 O_m^2 r dr = \int_0^1 \left( \frac{1}{4}(1 + r)^{2m} - \frac{1}{2}(1 - r^2)^m + \frac{1}{4}(1 - r)^{2m} r dr
\]

\[
= \frac{1}{4} \left( \sum_{j=0}^{2m} \binom{2m}{j} \frac{1}{j+2} + \sum_{j=0}^{2m} \binom{2m}{j} \frac{(-1)^j}{j+2} \right) - \frac{1}{4(m+1)}
\]

\[
= \frac{m}{2} \frac{4^m - 1}{(2m + 1)(m + 1)}.
\]

With the above two integrals we can also have

\[
\int_0^1 (E_m^2 + O_m^2) r dr = \frac{2 \times 4^m m + 1}{2(2m + 1)(m + 1)}.
\]
Similarly we get
\[\langle Z, 1 | Z, 1 \rangle = \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) \frac{r^2 d\theta dr}{\pi(1+r^2)^2}\]
\[\langle Z, 2 | Z, 2 \rangle = \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) \frac{r^2 d\theta dr}{\pi(1+r^2)^2}.\]

Now let us choose
\[
N = \frac{\sqrt{6}}{\pi} (1 + r^2),
\]
\[
R_m = \sqrt{\frac{(2m+1)(m+1)}{2 \times 4^m m + 1}}.
\]

With these choices we have
\[
\langle Z, 1 | Z, 1 \rangle = \langle Z, 2 | Z, 2 \rangle = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 \frac{6}{\pi^2} \frac{(2m+1)(m+1)}{2 \times 4^m m + 1} (E_m^2 + O_m^2) r^2 d\theta dr
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} \frac{6}{\pi^2}
\]
\[
= 1.
\]

Let us turn our attention to the resolution of identity with a calculation of
\[
| Z, 1 \rangle \langle Z, 1 | = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} N^2 | R_m Z^m \phi_{1m} \rangle \langle R_k Z^k \phi_{1k} |
\]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} N^2 R_m Z^m | \phi_{1m} \rangle \langle \phi_{1k} | R_k Z^k T
\]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} N^2 \begin{pmatrix} R_m e^{im\theta} & 0 \\ R_m e^{im\theta} & 0 \end{pmatrix} \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{-i\theta} & E_m \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_m & O_m e^{i\theta} \\ O_m e^{-i\theta} & E_m \end{pmatrix} \begin{pmatrix} R_m e^{-im\theta} & 0 \\ 0 & R_m e^{-im\theta} \end{pmatrix}
\]
\[
= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} N^2 \begin{pmatrix} R_m R_k E_m E_k e^{i\theta(m-k)} & R_m R_k E_m O_k e^{i\theta(m+k+1)} \\ R_m R_k O_m E_k e^{-i\theta(m+k+1)} & R_m R_k O_m O_k e^{i\theta(m-k)} \end{pmatrix}
\]

Thus we get
\[
\int_D W | Z, 1 \rangle \langle Z, 1 | d\mu = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]
where
\[
A = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 N^2 R_m^2 E_m^2 W \frac{r^2 d\theta dr}{\pi(1+r^2)^2} | \psi_m \rangle \langle \psi_m |
\]
\[
B = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 N^2 R_m^2 O_m^2 W \frac{r^2 d\theta dr}{\pi(1+r^2)^2} | \psi_m \rangle \langle \psi_m |.
\]

Similarly we get
\[
\int_D W | Z, 2 \rangle \langle Z, 2 | d\mu = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}.
\]
With all these we have
\[ (3.9) \quad \int_D W \big| Z, 1 \big\rangle \big\langle Z, 1 \big| \, d\mu + \int_D W \big| Z, 2 \big\rangle \big\langle Z, 2 \big| \, d\mu = \begin{pmatrix} A + B & 0 \\ 0 & A + B \end{pmatrix}. \]

Now let us calculate \( A + B \)
\[
A + B = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) W \frac{r \, d\theta \, dr}{\pi (1 + r^2)^2} \otimes | \psi_m \rangle \langle \psi_m |.
\]
In order to get
\[
\int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) W \frac{r \, d\theta \, dr}{\pi (1 + r^2)^2} = 1
\]
let us choose
\[ W = \frac{\pi^2}{6}. \]

With this choice we have
\[
\int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) W \frac{r \, d\theta \, dr}{\pi (1 + r^2)^2}
\]
\[ \quad = \int_0^{2\pi} \int_0^1 \frac{6(1 + r^2)^2 (2m + 1)(m + 1)}{\pi^2} \frac{(E_m^2 + O_m^2)^2}{2 \times 4^m m + 1} \frac{r \, d\theta \, dr}{(1 + r^2)^2}
\]
\[ \quad = \frac{2 \times 4^m m + 1}{2(2m + 1)(m + 1) \pi^2} \frac{6 (2m + 1)(m + 1) \pi^2}{2 \times 4^m m + 1} \frac{2\pi}{6}
\]
\[ \quad = 1
\]
Now we have
\[ (3.10) \quad A + B = \sum_{m=0}^{\infty} | \psi_m \rangle \langle \psi_m | \]
which yields
\[ \int_D W \big| Z, 1 \big\rangle \big\langle Z, 1 \big| \, d\mu + \int_D W \big| Z, 2 \big\rangle \big\langle Z, 2 \big| \, d\mu = I_2 \otimes \sum_{m=0}^{\infty} | \psi_m \rangle \langle \psi_m |.
\]

Let
\[ T = I_2 \otimes \sum_{m=0}^{\infty} | \psi_m \rangle \langle \psi_m |.
\]
Thus we have an operator \( T \) as in equation (2.11) with \( n = 2 \). The decomposition of any vector \( \phi \) in \( \tilde{H} \) follows by replacing \( n = 2 \) in the discussion which we have right after equation (2.11).

4. Remarks and Discussions

In this section we will discuss some other possibilities of our choices over the construction and resulting difficulties.

- We have chosen a basis of the Hilbert space in an unusual way as \( \psi_m = \frac{\phi_m}{m + 1} \). We explain the convenience of this choice using the \( SU(1,1) \) example.
Instead of this choice if we take \( \phi_m \), in equations (3.1) and (3.2) we will have \( \phi_m \) instead of \( \psi_m \), which will change equations (3.3) and (3.4) as

\[
\langle Z, 1 | Z, 1 \rangle = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) \frac{r dr dr}{\pi (1 + r^2)^2}.
\]

\[
\langle Z, 2 | Z, 2 \rangle = \sum_{m=0}^{\infty} \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) \frac{r dr dr}{\pi (1 + r^2)^2}.
\]

Further, in equation (3.9) \( \psi_m \) will be replaced by \( \phi_m \), i.e,

\[
A + B = \sum_{m=0}^{\infty} | \phi_m \rangle \langle \phi_m | \int_0^{2\pi} \int_0^1 N^2 R_m^2 (E_m^2 + O_m^2) W \frac{r dr dr}{\pi (1 + r^2)^2}.
\]

So in (4.3) by choosing

\[
N = \left( \frac{1 + r^2}{\sqrt{2}} \right), \quad R_m = \frac{1}{\sqrt{N_m}}, \quad \text{and} \quad W = 1
\]

we can have the resolution of identity. But these choices will make the series in (4.1) and (4.2) diverge. In order to have the convergence in (4.1) and (4.2), at least we have to have

\[
R_m^2 \cdot \frac{2 \times 4^m m + 1}{2(2m + 1)(m + 1)} \sim \frac{1}{m^p}
\]

with \( p > 1 \). If we make such a choice for \( R_m \), our moment problem will be

\[
\int_0^1 (E_m^2 + O_m^2) W r dr = \frac{2 \times 4^m m + 1}{2(2m + 1)(m + 1)} m^p.
\]

We have experienced difficulty in solving this moment problem by taking \( W \) as a function of \( r \) only, but it can be solved if we take \( W = f(r, m) \). Generally, as a weight function, dependence of \( W \) on \( m \) is not allowed. In this regard, we made the choice \( \psi_m = \frac{2}{m + r} \). Notice also that \( \psi_m \) can be chosen in many ways in a similar fashion.

- We have obtained proper decomposition for the vectors in our intended space up to a dense subset. The effort of getting the decomposition for the remaining nowhere dense set is limited by the unboundedness of the inverse operator \( T^{-1} \). In this case, we have an approximation for the decomposition.

- The \( SU(1, 1) \) example presented here can be considered as a preliminary step of constructing vector coherent states on classical domains. The construction directly depends on the explicit form of the elements of the classical groups. As we know, many of the matrix groups consist elements with an unpleasant explicit form in terms of calculations. In this regard, construction of vector coherent states on other classical domains may need little more effort.
References

[1] Ali, S.T., Antoine, J-P. and Gazeau, J-P., Coherent States, Wavelets and their Generalizations, Springer-Verlag, New York, 2000.

[2] Brif, C., Vourdas, A. and Mann, A., Analytic representation based on $SU(1, 1)$ coherent states and their applications, J.Phys.A:Math.Gen. pp 5873-5885, vol.29 (1996).

[3] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G., Higher Transcendental functions, McGraw-Hill, New York, 1953.

[4] Gilmore, R., Lie Groups, Lie Algebras, and Some of their applications, John Wiley & Sons, New York (1974).

[5] Klauder, J.R, Skagerstam, B.S., Coherent States, Applications in Physics and Mathematical Physics, World Scientific, Singapore, (1985).

[6] Perelemov, A.M., Generalized coherent states and their applications, Springer-Verlag, Berlin, 1986.

[7] Kengatharam Thirulogasanthar, Ali Twareque,S., A class of vector coherent states defined over matrix domains, preprint, math-ph/0305036.

[8] Vourdas, A. and Wunsche, A., Resolution of the identity in terms of line integrals of $SU(1, 1)$ Coherent States, J.Phys.A:Math.Gen., pp 9341-9352, vol.31 (1998).

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