Pronounced minimum of the thermodynamic Casimir forces of O(n) symmetric film systems: analytic theory

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Abstract

Thermodynamic Casimir forces of film systems in the O(n) universality classes with Dirichlet boundary conditions are studied below bulk criticality. Substantial progress is achieved in resolving the long-standing problem of describing analytically the pronounced minimum of the scaling function observed experimentally in $^4$He films ($n = 2$) by R. Garcia and M.H.W. Chan, Phys. Rev. Lett. 83, 1187 (1999) and in Monte Carlo simulations for the three-dimensional Ising model ($n = 1$) by O. Vasilyev et al., EPL 80, 60009 (2007). Our finite-size renormalization-group approach yields excellent agreement with the depth and the position of the minimum for $n = 1$ and semiquantitative agreement with the minimum for $n = 2$. Our theory also predicts a pronounced minimum for the $n = 3$ Heisenberg universality class.

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Thermodynamic Casimir forces occur in a large variety of confined condensed matter systems [1] and have attracted the interest of many theoretical and experimental researchers over the past decades until very recently [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Of particular interest are O(n) symmetric film systems where both long-range Goldstone and critical fluctuations are the physical origin of such Casimir forces. One of the most prominent systems is superfluid $^4$He ($n = 2$) where the Casimir force causes a surprising and as yet unexplained effect close to the superfluid transition:
a pronounced minimum of the Casimir force scaling function as observed experimentally by a thinning of liquid $^4$He films [4]. Although this effect has been confirmed by Monte Carlo (MC) simulations for the XY model ($n = 2$) [7, 9, 13] there is still no theoretical understanding and analytic description based on a coherent theory without adjustments of parameters. Additional information comes from more recent MC data for the three-dimensional Ising model ($n = 1$) with free boundary conditions (BC) [9] and from a numerical analysis of the O($n$) $\varphi^4$ model in the large - $n$ limit [16] in which cases similar minima were found. This calls for a general theoretical explanation of the minima that is not specific to the superfluid transition and the XY universality class and is largely unrelated to the existence of and the crossover to a Goldstone regime at low temperatures. It is the goal of this Letter to provide such an explanation and to make quantitative predictions for the O($n$) universality classes with general finite $n \geq 1$.

An early renormalization-group (RG) description of the thermodynamic Casimir effect in $d = 4 - \varepsilon$ dimensions [3] covers only the region above bulk criticality where this effect is quite small and where no indication of the large minimum below bulk criticality of $^4$He is recognizable. Subsequent theoretical work is based on mean field (MF) theory [8, 10, 15] which, however, is not capable of making a prediction of the depth of the minimum because of the strong dependence on an undetermined nonuniversal parameter. A RG improved version of MF theory presented in [8] yields a minimum that is roughly five times deeper than the experimentally measured minimum. Furthermore the position of the MF minimum differs considerably from the experimentally observed position [10]. Recently an analytic RG calculation of the minimum of O($n$) symmetric film systems with periodic BC [12, 17] was found to be in agreement with MC data [9, 19, 20] but the position and the depth of the minima are rather far from those of the minimum in real $^4$He films [4] which requires a description with Dirichlet BC because of the vanishing of the order parameter at the boundaries [21].
In this Letter we develop an analytic theory of the Casimir force that is in substantially improved agreement with the observed depths and positions of the minima of systems with free or Dirichlet BC in the \( n = 1 \) and \( n = 2 \) universality classes. We also predict a minimum of the Casimir force scaling function for the \( (n = 3) \) Heisenberg universality class whose position is close to that found recently in the large \( n \) limit \([16]\). Our theory with Dirichlet BC should also be an appropriate basis for a quantitative description of Casimir forces in superconducting films \([6]\) provided that the theory includes the effects of lattice anisotropy \([22]\).

We start from the \( O(n) \) symmetric \( \varphi^4 \) Hamiltonian

\[
H = \int_V d^d x \left[ \frac{r_0}{2} \varphi^2 + \frac{1}{2} (\nabla \varphi)^2 + u_0 (\varphi^2)^2 \right]
\]  

(1)

where \( \varphi(x) \) is an \( n \)-component field in a finite \( d \) dimensional \( L_{\parallel}^{d-1} \times L \) slab geometry with a finite aspect ratio \( \rho = L/L_{\parallel} \) and a finite volume \( V = L_{\parallel}^{d-1}L \). We consider periodic BC in the \( d-1 \) "horizontal" directions but Dirichlet BC in the \( d \)th "vertical" direction. Accordingly \( \varphi(x) \equiv \varphi(y,z) \) is represented as \( \varphi(x) = \sqrt{2} \sum_{n,m} \hat{\varphi}_{n,m} e^{ip_y y} \sin(qz) \) where the sum \( \sum_{n,m} \) runs over \((d-1)\)-dimensional \( p \) vectors with components \( p_\alpha = 2\pi n_\alpha/L_{\parallel} \), \( \alpha = 1, 2, ..., d-1 \), with integers \( n_\alpha = 0, \pm 1, \pm 2, ... \), and over wave numbers \( q = \pi m/L \) with integers \( m = 1, 2, ... \) (up to some cutoff \( \Lambda \)). Our system differs fundamentally from the periodic slabs studied previously \([17]\) in that now there exist surface contributions to the free energy and that the lowest mode \( \psi(z) = \Phi \sqrt{2} \sin(\pi z/L) \) with \( \Phi \equiv \hat{\varphi}_{0,1} \) is inhomogeneous.

Our applications will be focussed on the universality classes of Ising-like and XY-like systems. The latter case includes the superfluid transition of \(^4\)He whose complex order parameter is equivalent to a two-component field \( \varphi(x) \). Eventually we shall take the idealized film limit \( \rho \to 0 \), i.e., \( L_{\parallel} \to \infty \) at fixed finite \( L \) in which case the film systems undergo a phase transition for \( n = 1, d > 2 \) and for \( n = 2, d \geq 3 \) at a finite temperature \( 0 < T_{c,\text{film}}(L) < T_c \) below the bulk critical temperature \( T_c \). For \( n = 2, d = 3 \), this is a Kosterlitz-
Thouless transition. No finite temperature $T_{c,film}(L)$ exists for $n > 2$.

The fundamental quantity from which the Casimir force per unit area $F_{Cas} = -\partial[F_{ex}]/\partial L$ can be derived is the excess free energy density (divided by $k_B T$) $f_{ex} = f - f_b$ where

$$f(T, L, L_{\parallel}) = -V^{-1} \ln \int \mathcal{D}\Phi \exp(-H)$$

and $f_b \equiv \lim_{V \to \infty} f$ are the free energy densities of the finite system and the bulk system, respectively. It is expected that, for isotropic systems near bulk criticality and for large $L$ and $L_{\parallel}$, $F_{Cas}$ can be written in a finite-size scaling form [23]

$$F_{Cas}(t, L, L_{\parallel}) = L^{-d} X(\tilde{x}, \rho)$$

with the scaling variable $\tilde{x} = t(L/\xi_{0+})^{1/\nu}$, $t = (T - T_c)/T_c$ where $\xi_{0+}$ is the amplitude of the bulk correlation length above bulk $T_c$. So far no satisfactory analytic calculation of the function $X(\tilde{x}, 0)$ for the film systems with Dirichlet boundary conditions has been performed that describes $X$ in the low-temperature region $\tilde{x} < 0$ where $X$ exhibits minima for both $n = 1$ and $n = 2$ that are much more pronounced than those in systems with periodic BC [9]. It is the goal of this Letter to derive a single scaling function $X(\tilde{x}, 0)$ for general $n$ that predicts the depths of the minima below bulk $T_c$ without any adjustment of parameters as well as the entire critical behavior above the minima up to the region far above $T_c$. No attempt will be made to describe the (weak) singularities [9] at $T_{c,film}(L)$ below the minima.

We first present our approach for $n = 1$. Our strategy is to set up a lowest-mode separation approach at finite $\rho > 0$ and then consider $0 < \rho \ll 1$ including the limit $\rho \to 0$ at the end of the calculations. We decompose $\varphi(x) = \psi(z) + \hat{\varphi}(x)$ with the higher-mode fluctuations $\hat{\varphi}(x) = \sum_{n,m} \sqrt{2} \hat{\varphi}_{n,m} e^{ipy} \sin(qz)$ where the sum $\sum_{n,m}$ does not include the lowest mode $(0, 1)$. Accordingly we decompose $H = H_0 + H^{(2)} + H^{(3)} + H^{(4)}$,

$$H_0(\Phi^2) = V \left[ \frac{1}{2}(r_0 + \pi^2/L^2)\Phi^2 + \frac{3}{2}u_0\Phi^4 \right],$$

(4)
\[ H^{(2)}(\Phi, \hat{\varphi}) = \sum_{n,m} \left\{ \frac{1}{2} \left[ \hat{r}(\Phi^2) + p^2 + q^2 \right] \hat{\varphi}_{n,m} \hat{\varphi}_{-n,m} \ight. \\
+ b(\Phi^2) \left[ \hat{\varphi}_{n,m} \hat{\varphi}_{-n,m} \delta_{m,1} - \hat{\varphi}_{n,m} \hat{\varphi}_{-n,m+2} \ight. \\
- \left\{ \hat{\varphi}_{n,m} \hat{\varphi}_{-n,m} \delta_{m,1} - \hat{\varphi}_{n,m} \hat{\varphi}_{-n,m+2} \right\} - w(\Phi) \hat{\varphi}_{0,3}, \quad (5) \]

with \( b(\Phi^2) = 3u_0\Phi^2 \), \( w(\Phi) = 2u_0\Phi^3 \), and the "longitudinal" parameter \( \hat{r}(\Phi^2) = r_0 + 12u_0\Phi^2 \). For the explicit form of \( H^{(3)} \sim O(u_0\Phi^3) \) and \( H^{(4)} \sim O(u_0\Phi^4) \) we refer to [24]. After integration over \( \hat{\varphi} \), we obtain the unrenormalized free energy density

\[ f = f_0 = \frac{1}{V} \ln \left\{ \int_{-\infty}^{\infty} d\Phi \exp \left[ -H_0(\Phi^2) - \Gamma(\Phi^2) \right] \right\}, \quad (6) \]

\[ \Gamma(\Phi^2) = \sum_{n,m} \left\{ \frac{1}{2} \ln a_{n,m}(\hat{r}, b) - \frac{2b^2}{a_{n,m}(\hat{r}, b) a_{n,m+2}(\hat{r}, b)} \right\} \]

\[ -6u_0\Phi w \left\{ \sum_n \frac{1}{a_{n,1}(\hat{r}, b)} - \sum_n \frac{1}{a_{n,2}(\hat{r}, b)} \right\} \quad (7) \]

apart from contributions of \( O(b^3, w^2, u_0) \), with \( a_{n,m}(\hat{r}, b) = \hat{r} + 2b\delta_{m,1} + 4\pi^2n^2/L^2 + \pi^2m^2/L^2 \). The sum \( \sum_n' \) does not include \( n = 0 \). The constant \( f_0 \) is independent of \( r_0 \) and \( u_0 \). The main contribution of the integration over \( \Phi \) comes from the region around \( \Phi^2 \approx M_0^2 \) where \( M_0^2 = \int_{-\infty}^{\infty} d\Phi \Phi^2 \exp \left[ -H_0(\Phi^2) \right] \) is the lowest-mode average. This provides the justification for approximating \( \Gamma(\Phi^2) \) by \( \Gamma(M_0^2) \). In the film limit \( \rho \to 0 \), \( M_0^2 \) vanishes for \( r_0 \geq -\pi^2/L^2 \). For finite \( V \), \( M_0^2 \) and \( \hat{r}(M_0^2) \) are positive for arbitrary \( r_0 \), and \( f \) has finite bulk limits above and below \( T_c \). The latter differs from that of the theory with periodic BC \([12, 17, 22]\) because of the factor 3/2 in \( H_0 \) and the term \( \propto b(M_0^2)^2 \) in \( \Gamma(M_0^2) \).

Since the smallest value of \( q \) is finite, namely \( \pi/L \), \( f \) is an analytic function of \( r_0 \), for both \( \rho > 0 \) and \( \rho = 0 \), at the bulk transition temperature \( r_0 = 0 \). (The shifted temperature variable \( r_0 - r_{0c} = a_0t \) with \( r_{0c} \sim O(u_0) \) will be incorporated subsequently at the renormalized stage of the theory.) On the level of
the bare theory, the analyticity for \( \rho = 0 \) extends down to the film transition temperature at \( r_0 = -\pi^2/L^2 \). The focus of our theory is the (renormalized counterpart of the) range \( r_0 > -\pi^2/L^2 \) above the film transition. We shall see that this range fully includes the minimum of the Casimir force scaling function and that this function is analytic at its minimum, in contrast to the MF results [8, 10, 15].

We have performed an exact analytic calculation of \( \Gamma(M_0^2) \) not only for \( \tilde{r}(M_0^2) \equiv \hat{r} > 0 \) but also for its full range of existence \( \hat{r} > -\pi^2/L^2 \) for finite \( L \gg \Lambda^{-1}, L_\parallel \gg \Lambda^{-1} \) and arbitrary \( \rho > 0 \) above and below \( T_c \) in \( 2 < d < 4 \) dimensions including the limits \( L \to \infty, L_\parallel \to \infty \). We apply this calculation to the excess free energy in the region above \( T_{c,\text{film}}(L) \) for small \( 0 < \rho \ll 1 \) where \( M_0^2 \) is small. The result above (+) and below (−) bulk \( T_c \) reads \( f^{\text{ex}}(r_0, L, L_\parallel) = f_s(\hat{r}, L, L_\parallel) - f_{b,s}^{\pm}(r_0) \) with the bare singular parts

\[
f_s(\hat{r}, L, L_\parallel) = -\frac{A_d}{d\varepsilon/2} \left\{ \frac{\hat{r}^2}{\varepsilon} + \frac{\pi^2}{2L^2} \left[ \hat{r} - \frac{(d-2)\pi^2}{4L^2} \right] \right\} + \frac{A_{d-1}}{2(d-1)(5-d)} \left[ \hat{r} - \frac{(d-3)\pi^2}{2L^2} \right] + L^{-d}\mathcal{P}(\tilde{r}L^2, \rho), \tag{8}
\]

\[
f_{b,s}^{+}(r_0) = -\frac{A_d}{d\varepsilon} r_0^{d/2}, \tag{9a}
\]

\[
f_{b,s}^{-}(r_0) = -\frac{r_0^2}{24u_0} - \frac{A_d}{d\varepsilon} (-r_0)^{d/2} \left[ 3 - \frac{\varepsilon(d+2)}{4} \right] \tag{9b}
\]

for \( r_0 > 0 \) and \( r_0 < 0 \), respectively, apart from terms of \( O(M_0^2, \rho^{d-1}) \), with \( \tilde{r} = \hat{r} + \pi^2/L^2 \) and the geometric factor \( A_d = \Gamma(3d/2)[2^{d-2}\pi^{d/2}(d-2)]^{-1} \).
The function $P$ is given by

$$P(\bar{r}L^2, \rho) = \frac{1}{2^{d+1} \pi} \int_0^\infty dz \left\{ \frac{\pi}{z} \right\}^{1/2} \left( 1 + z + \frac{z^2}{2} \right)$$

$$- \left[ 2\rho K(4\rho^2 z) \right]^{d-1} e^z [K(z) - 1] \left( \frac{\pi}{z} \right)^{(1-d)/2}$$

$$- (1 + z) \left\{ \left( \frac{\pi}{z} \right)^{(d+1)/2} \exp \left[ -\bar{r}L^2 z/\pi^2 \right] \right\}$$  \hspace{1cm} (10)$$

with $K(z) = \sum_{m=-\infty}^{\infty} \exp(-zm^2)$ and $2\rho K(4\rho^2 z) \rightarrow (\pi/z)^{1/2}$ for $\rho \rightarrow 0$. For finite $\bar{r}L^2 > 0$, the integral $P(\bar{r}L^2, \rho)$ exists in $1 < d < 5$ dimensions for $\rho \geq 0$. Note that $f_s$ is divergent at $d = 3$ because of the $d = 3$ pole contained in the first surface contribution $\propto A_{d-1}$ in eq. (8). The origin of this pole term is well understood as an artifact of low-order perturbation theory due to the vanishing of the critical exponent of the Gaussian surface energy density in three dimensions [25]. In the present context, the $d = 3$ pole is not problematic because it is canceled in the quantity $-f^ex - L \partial f^ex / \partial L = F_{Cas}$. Our function $f_s$, (8), is an analytic function at $r_0 = 0$ for both $\rho > 0$ and $\rho = 0$ at finite $L$, in agreement with general analyticity requirements [26]. In fact, $f_s(r_0, L, L_{\parallel})$ constitutes the analytic continuation of an earlier result (as given by the singular part of eqs. (66),(67), and (69) of [25] that is valid only for $r_0 \geq 0$) to the region $r_0 > -\pi^2/L^2$ [27].

The bare expressions (8) - (10) do, of course, not yet correctly describe the finite-size scaling behavior in terms of the scaling variable $\tilde{x}$ with the correct critical exponent $\nu$. This will be achieved by appropriate renormalizations that we perform within the minimal subtraction scheme at fixed dimension $2 < d < 4$ [22, 28].

It is straightforward to extend our calculation to $n > 1$ as far as the disordered phase above $T_{c,\text{film}}$ is concerned. Then $n - 1$ transverse contributions exist which depend on the "transverse" parameter $\bar{r}_T(M_0^2) = r_0 + 4u_0M_0^2$ rather than then "longitudinal" parameter $\bar{r}(M_0^2)$ defined above. Especially in the
limit \( \rho \to 0 \), each of the \( n \) components of \( \varphi(x) \) contributes equally to the free energy which amounts to multiplying both \( f_s \) and the bulk part \( f_{b,s}^+ \) by \( n \).

As far as the transverse finite-size contributions are concerned, our approach is not applicable to \( T < T_{c,\text{film}}(L) \) where \( \tilde{r}_T(M_0^2) \) would become negative. This can be traced back to the factor \( 3/2 \) in the \( \Phi^4 \) term of \( H_0 \). As far as the transverse bulk contribution is concerned we argue, however, that no bare transverse bulk contributions below \( T_c \) exists at \( O(u_0^{-1}) \) and \( O(1) \) as is known from bulk perturbation theory \([29]\). This remains true also for the renormalized bulk theory in terms of the renormalized coupling \( u \). Thus, on our level of the theory which neglects terms of \( O(u) \) and for the application restricted to \( T_{c,\text{film}}(L) < T < T_c \), we shall approximate the bulk part below \( T_c \) for general \( n \geq 1 \) by the longitudinal bulk contribution below \( T_c \) as given in eq. \((12b)\) below (where the \( n \) dependence enters only through the fixed point value \( u^* \) and the flow parameter \( l_- \)).

The quantity of primary interest is the Casimir force scaling function \( X(\tilde{x}) = \lim_{\rho \to 0} X(\tilde{x}, \rho) \) in the film limit. From \((8) - (10)\) we derive its analytic form for general \( n \) above (+) and below (−) \( T_c \) and above \( T_{c,\text{film}}(L) \)

\[
X(\tilde{x}) = -A_d l_\pm^4 \left[ \frac{1}{4} - \frac{l_\pm^2}{d l_\pm^{d-\varepsilon}} \right] + \tilde{F}_b^\pm(\tilde{x}) \\
+ \frac{A_d n \pi^2 l_\pm^4}{d} \hat{\rho}_\pm^{d-\varepsilon} - \frac{A_d n \pi^2}{2d} \hat{\rho}_\pm^{d-\varepsilon}
\]

\[
+ \frac{\pi^4}{4 l_\pm^2} (d - 4)(d + 2) \right] - n \frac{\pi (9 - d)/2}{2d} \Gamma \left( \frac{5 - d}{2} \right) \hat{\rho}_\pm^{d-5}
\]

\[
+ \frac{n}{2^{d+1} \pi} \int_0^\infty dz \left[ d - 1 \pm 2 \frac{l_\pm^2}{\pi^2 z} \right] \left( \frac{\pi}{z} \right)^{(d+1)/2} \exp \left[ -l_\pm^2 z / \pi^2 \right],
\]

\[
\right]
\]

\[
(11)
\]
\[
\tilde{F}_b^+ (\tilde{x}) = -Ad^d n/(4d), \\
\tilde{F}_b^- (\tilde{x}) = -Ad^d [1/(24u^*) + 1/(4d) - 1/4],
\]

(12a)

(12b)

where \( l_\pm = (\pm \tilde{x}Q^*)^\nu \) and \( \hat{l}_\pm = (\pm l_\pm^2 + \pi^2)^{1/2} \). The quantity \( Q^* = Q(1, u^*, d) \) is the fixed point value of the \( n \) dependent amplitude function \( Q(1, u, d) \) of the second-moment bulk correlation length above \( T_c \) [28].

Equations (11) and (12) are the central result of this Letter. They contain no adjustable parameters. They are valid in \( 2 < d < 4 \) dimensions (with a finite limit for \( d \to 4 \)) including \( d = 3 \) in the range \( \tilde{x} > \tilde{x}_{c, \text{film}} \) which is the renormalized counterpart of the bare range \( r_0 > -\pi^2/L^2 \) mentioned above. The film transition occurs at \( l_- = \pi \), i.e.,

\[
\tilde{x}_{c, \text{film}} = -\pi^{1/\nu}/Q^*.
\]

(13)

Our function \( X(\tilde{x}) \) is an analytic function in the entire region \( \tilde{x}_{c, \text{film}} < \tilde{x} < 0 \) and \( 0 < \tilde{x} < \infty \). By definition, \( X(\tilde{x}) \) has a weak singularity at \( \tilde{x} = 0 \) due to the subtraction of the singular bulk part of \( f^{ex} \).

Our result eqs. (11), (12) is compared with experimental and MC data in figs. 1 (a) and (b). In three dimensions we employ the following numerical values [28, 30, 31, 32]

\[
\begin{align*}
u &= 0.6335, 0.671 \\
\end{align*}
\]

for \( n = 1, 2 \), respectively. We obtain \( \tilde{x}_{c, \text{film}} = -6.44, -5.86 \) for \( n = 1, 2 \), respectively. This is not far from the observed transitions at \( \tilde{x}_{c, \text{film}} = -7.6 \) for both the Ising \( (n = 1) \) and the XY \( (n = 2) \) universality classes [9]. The positions of the minima predicted by our theory are \( \tilde{x}_{\text{min}} = -5.53, -4.73 \) for \( n = 1, 2 \), respectively. This is in excellent agreement with the position \( \tilde{x}_{\text{MC}}^{\text{MC}} = -5.7 \) observed by MC simulations for \( n = 1 \) [9] and in reasonable agreement with \( \tilde{x}_{\text{min}}^{\text{exp}} = -5.7 \) measured by experiments for \( n = 2 \) [4], as shown in figs. 1 (a) and (b). The position predicted by MF theory [8, 10]

\[
\tilde{x}_{\text{min}}^{\text{MF}} = -\pi^2 = -9.87 \]

differs considerably from the observed position. Also the shape of \( X(\tilde{x}) \) and the depth of the minimum \( X_{\text{min}} = -1.53 \) predicted by
our theory for $n = 1$ are in excellent agreement with the MC data (fig. 1 (a)) while semiquantitative agreement with the experimentally measured depth for $n = 2$ (fig. 1 (b)) is found. The RG improved MF theory [8] (dashed line in fig. 1 (b)) has a minimum $X_{\text{min}}^{\text{MF}} = -6.92$ that is far from the experimental value and is outside the range of the vertical scale shown in fig. 1 (b).

As a shortcoming of our theory, eq. (11) does not capture the weak singularity at the film transition [9] for $n = 1, 2$ but yields a divergence of $X$ at $\tilde{x}_{c,\text{film}}$, eq. (13), for $n \geq 1$. Nevertheless we expect that our function $X(\tilde{x})$ provides a reasonable description at a semiquantitative level for general $n > 2$ in the region $\tilde{x}_{\text{min}} \lesssim \tilde{x} \leq \infty$. An application of our result (11), (12) to $n = 3$ (with parameters $u^* = 0.0327, Q^* = 0.937, \nu = 0.7112$ taken from [30, 31, 33]) yields a pronounced minimum $X_{\text{min}} = -2.07$ at $\tilde{x}_{\text{min}} = -4.20$ as shown in fig. 1 (c). We note that the latter value is close to $\tilde{x}^{(\infty)}_{\text{min}} = -4.56$ of the pronounced minimum found recently in the large - $n$ limit [16]. It would be interesting to test our $n = 3$ prediction by MC simulations for Heisenberg models with free BC.

Our analytic theory provides the opportunity of studying separately the contributions arising from bulk and finite-size parts. For $n = 1, 2, 3$, the occurrence of the pronounced minimum can be understood as the result of a competition between a decreasing bulk contribution and an increasing $L$-dependent fluctuation contribution to the Casimir force as the temperature is lowered below bulk $T_c$. An analysis of eqs. (11), (12) for larger $n > 3$ can answer the question whether this feature persists up to $n = \infty$ [16]. The fluctuation contribution is missing in MF theory which explains why no minimum exists in MF theory (dashed line in fig. 1 (b)) above the MF film transition temperature $\tilde{x}_{c,\text{film}}^{\text{MF}} = -\pi^2 = -9.87$.

To summarize, we have shown that the pronounced minima of the Casimir force scaling function of $O(n)$ symmetric film systems observed in experiments [4] and MC simulations [7, 9, 13] can be described analytically within
a finite-size RG approach on the basis of the $\phi^4$ model with Dirichlet BC. By appropriate modification of the mode functions, our finite-size RG approach may be applicable to the low-temperature phase of superfluid films and superconducting films where Goldstone modes play an important role [5, 17].

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Figure 1: (Color online) Scaling function $X(\tilde{x})$ of the Casimir force as a function of $\tilde{x} = t(L/\xi_{0+})^{1/\nu}$ in three dimensions for $n = 1, 2, 3$. Thick solid lines: RG theory from eqs. (11), (12) [4] MC data (i) and (ii) in (a) from ref. [9] for the Ising model with $L = 20$. Thin line in (b): $^4$He data from ref. [4] with $\xi_{0+} = 1.43 \times 10^{-8}$ cm. Dashed line in (b): RG improved MF theory from refs. [8, 10] with a minimum $X_{\text{MF}}^{\text{min}} = -6.92$ at $\tilde{x}_{\text{MF}}^{\text{min}} = -9.87$. 