THE STRUCTURE OF RADIAL SOLUTIONS
FOR A GENERAL MEMS MODEL

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Abstract. We investigate the structure of radial solutions corresponding to the equation
\[ \Delta u = \frac{1}{f(u)} \] in \( B_{r_0} \subset \mathbb{R}^N, \ N \geq 3, \ r_0 > 0, \]
where \( f \in C[0, \infty) \cap C^2(0, \infty), \ f(u) > 0 \) for \( u > 0, \ f(0) = 0 \) and \( f \) satisfies certain assumptions which include the standard case of pure power encountered in the study of Micro-Electromechanical Systems (MEMS). A particular attention is paid to degenerate solutions of the above equation, that is, solutions \( u^* \) which are positive in \( B_{r_0} \setminus \{0\} \) and vanish at the origin. We prove that a degenerate solution \( u^* \) exists, is unique and equals the limit of the regular solutions \( u(\cdot, \alpha) \) (with \( u(0, \alpha) = \alpha \)) in \( C^2_{\text{loc}}(0, r_0) \cap C^1_{\text{loc}}(0, r_0) \) as \( \alpha \to 0. \) The rate at which \( u^* \) approaches zero at the origin is also obtained. Further, we show that the number of intersection points between \( u^* \) and \( u(\cdot, \alpha) \) tends to infinity as \( \alpha \to 0. \) This leads to the complete bifurcation diagram of MEMS type problems.

1. Introduction and Main results

In this paper we are concerned with radial solutions of the semilinear elliptic equation
\[ \Delta u = \frac{1}{f(u)} \] in \( B_{r_0} \subset \mathbb{R}^N, \ N \geq 3, \ r_0 > 0, \] (P)
where, for any \( r > 0, \) we denote by \( B_r \) the open ball centred at the origin and having radius \( r > 0. \) Clearly, radial solutions \( u = u(r) \) of the above equation satisfy
\[ u'' + \frac{N-1}{r} u' = \frac{1}{f(u)} \] for all \( 0 < r < r_0. \) (1.1)
Our study is motivated by the problem
\[ \begin{cases} -\Delta U = \frac{\Lambda}{(1-U)^{\tau}} & \text{in } B_1, \\ U = 0 & \text{on } \partial B_1, \\ U > 0 & \text{in } B_1, \end{cases} \] (1.2)
which arises in the mathematical modelling of Micro-Electromechanical Systems (MEMS), a technology that designs various types of microscopic devices by combining the electronics with micro-size mechanical components. A valuable source on the MEMS modelling is the monograph [20]. For a mathematical account on this topic, the reader is referred to [4]. It
is easy to see that if $U$ is a radial solution of (1.2) in $B_1$, then $u(\sqrt{A}r) := 1 - U(r)$ satisfies (1.1) with $f(u) = u^2$ and $r_0 = 1$.

One key topic in the study of equation (P) is the investigation of rupture set, that is, the set $\{u = 0\}$. In the case of pure power nonlinearities $f(u) = u^p$, this has been carried out in [2, 10]. In the present work we shall be mainly interested in nonnegative radial solutions of (P) which are degenerate, that is, functions $r \mapsto -u(r)$ which satisfy:

- $u \in C^0[0, r_0) \cap C^2(0, r_0)$ for some $r_0 > 0$ and $u$ satisfies (1.1) in the classical sense;
- $\lim_{r \to 0} u(r) = 0$ and $u(r) > 0$ for $0 < r < r_0$.

Obviously, degenerate solutions are not $C^2$ at the origin. In fact, we shall see that degenerate solutions are not even $C^1$ at the origin and despite this lack of regularity we shall prove that degenerate solutions are unique.

In order to describe the assumptions on $f(u)$, let us first introduce $F[u] := \int_0^u f(s) \, ds$.

Throughout this paper we assume that $f$ satisfies the conditions (F1)-(F3) below:

(F1) $f \in C[0, \infty) \cap C^2(0, \infty)$, $f(0) = 0$ and there exists $\bar{u} > 0$ such that $f'(u) > 0$ and $f''(u) > 0$ for all $0 < u < \bar{u}$.

(F2) There exists $q := \lim_{u \to 0} \frac{f'(u)^2}{f(u)f''(u)}$.

(F3) If $q = 1$, then $\frac{f'(u)F[u]}{f(u)^2} \leq 1$ for small $u > 0$.

The growth rate of $f(u)$ at $u = 0$ can be defined by $p := \lim_{u \to 0} u f'(u)/f(u)$. Applying formally L'Hospital’s rule, we find

$$p = \lim_{u \to 0} \frac{u}{f(u)/f'(u)} = \lim_{u \to 0} \frac{1}{1 - f(u)f''(u)/f'(u)^2} = \frac{1}{1 - 1/q},$$

and hence $1/p + 1/q = 1$.

Thus, $q$ is the conjugate exponent of the growth rate of $f$ at $u = 0$. In Section 3 we show the following: $q \geq 1$ and

$$\text{the limit } \gamma := \lim_{u \to 0} \frac{f'(u)F[u]}{f(u)^2} \text{ exists and } \gamma = \frac{q}{2q - 1}.$$  

Since $q \geq 1$, we see that $1/2 < \gamma \leq 1$.

For $\alpha > 0$, let $u(r, \alpha)$ be a solution of the initial value problem

$$\begin{cases}
  u'' + \frac{N-1}{r} u' = \frac{\beta}{f(u)} & \text{for all } 0 < r < r_0, \\
  u(0) = \alpha, \\
  u'(0) = 0.
\end{cases}$$

For a continuous function $g(r)$ defined on the interval $I \subset \mathbb{R}$ we define the zero number of $g$ on $I$ by

$$\mathcal{Z}_I[g(\cdot)] = \#\{r \in I \mid g(r) = 0\}.$$
We are now in a position to state the main result of this paper.

**Theorem 1.1.** The following hold:

(i) Equation (1.1) has a unique degenerate solution \( u^*(r) \) and \( u^* \) satisfies

\[
u^*(r) = F^{-1} \left[ \frac{r^2}{2N - 4\gamma} (1 + o(r)) \right] \quad \text{as } r \to 0,
\]

where \( F^{-1} \) denotes the inverse function of \( F \) and \( o(r) \) is a continuous function such that \( o(r) \to 0 \) as \( r \to 0 \). Moreover, \( u^*(r) \) is not of class \( C^1 \) at \( r = 0 \) and

\[
u(r, \alpha) \to u^*(r) \quad \text{in } C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0) \quad \text{as } \alpha \to 0,
\]

where \( u(r, \alpha) \) is a regular solution of (1.4).

(ii) Let

\[
q_c := \frac{1}{2} + \frac{1}{N - 2 - 2\sqrt{N - 1}}.
\]

If \( q < q_c \), then, for each \( \rho \in (0, r_0) \) we have

\[
Z_{(0, \rho)} [u(\cdot, \alpha) - u^*(\cdot)] \to \infty \quad \text{as } \alpha \to 0.
\]

If \( N \geq 10 \), then \( q_c \leq 1 \) and the conclusion of Theorem 1.1 (ii) cannot occur. Theorem 1.1 (ii) states that the number of intersection points on \( (0, \rho) \) between \( u \) and \( u^* \) goes to infinity no matter how small we may choose \( \rho \in (0, r_0) \). Let us point out that the case \( q \geq q_c \) is not discussed here. As the results in the case \( f(u) = u^p \) suggests (see Proposition 5.3 in Section 5) the case \( q \geq q_c \) requires global assumptions on the nonlinearity \( f \) which are beyond the scope of the present work.

We define \( f_q(u) \) and \( F_q[u] \) as follows:

\[
f_q(u) := \begin{cases} 
  e^u, & \text{if } q = 1, \\
  u^p, & \text{if } q > 1,
\end{cases}
\]

\[
F_q[u] := \begin{cases} 
  e^u, & \text{if } q = 1, \\
  \frac{u^{p+1}}{p+1}, & \text{if } q > 1.
\end{cases}
\]

In the typical case \( f(u) = u^p \) the problem (1.4) becomes

\[
\begin{align*}
  v'' + \frac{N-1}{s} v' &= \frac{1}{f'_q(v)} & s > 0, \\
v(0) &= \beta, \\
v'(0) &= 0.
\end{align*}
\]

The degenerate solution of the equation in (1.7) can be written explicitly as

\[
v^*(s) = F_q^{-1} \left[ \frac{s^2}{2N - 4\gamma} \right] \quad \text{and} \quad \gamma := \frac{q}{2q - 1}.
\]

(Note that since \( 1/2 < \gamma \leq 1 \) we always have \( 2N - 4\gamma > 0 \)) Relevant to our approach is the number of intersection points between \( v(s, \beta_1) \) and \( v(s, \beta_2) \) respectively between \( v(s, \beta_1) \) and \( v^*(s) \) which was discussed in [8, 16]. We remind this result in Proposition 5.3 below.

As an application of Theorem 1.1 we consider the following bifurcation problem:

\[
\begin{cases}
  \Delta U + \frac{\Lambda}{f(1-u)} = 0 & \text{in } B_1, \\
  U = 0 & \text{on } \partial B_1, \\
  U > 0 & \text{in } B_1.
\end{cases}
\]


where $\Lambda > 0$. Here, in addition to (F1)–(F3) we assume the following (F4):

\[(F4) \quad f'(u) > 0 \quad \text{for} \quad 0 < u < \infty.\]

The nonlinear term $\Lambda f(1 - U)$ has a singularity at $U = 1$. If $0 \leq U < 1$ in $\bar{B}_1$, then the solution $U$ is classical. By the symmetry result of [6] we see that $U$ is radial and $\|U\|_{L^\infty(\bar{B}_1)} = U(0)$. It is known that the set of the classical solutions $C := \{(\Lambda, U)\}$ can be parametrized by $\tau := U(0)$, and that $C$ emanates from $(0, 0)$. By the results in [12], $C$ can be expressed as $C := \{(\Lambda(\tau), U(R, \tau))\}$.

The problem (1.9) can be reduced to the following:

\[
\begin{cases}
U'' + \frac{N-1}{R} U' + \frac{\Lambda}{f(1-U)} = 0, & 0 < R < 1, \\
U(1) = 0, & \\
U(R) > 0, & 0 \leq R < 1.
\end{cases}
\]

Using Theorem 1.1 we obtain:

**Corollary 1.2.** The following hold:

(i) The problem (1.10) has a unique degenerate solution $(\Lambda^*, U^*)$. Let $C := \{(\Lambda(\tau), U(R, \tau))| U(0) = \tau, 0 < \tau < 1\}$ be the bifurcation curve. Then, as $\tau \to 1$ one has

\[
\Lambda(\tau) \to \Lambda^* \quad \text{and} \quad U(R, \tau) \to U^*(R) \quad \text{in} \quad C[0, 1].
\]

(ii) If $q < q_c$, then the curve $C$ has infinitely many turning points around $\Lambda^*$, and hence (1.10) with $\Lambda = \Lambda^*$ has infinitely many classical solutions.

If $N \geq 10$, then $q_c \leq 1$ and the conclusion of Corollary 1.2(ii) cannot occur.

Based on techniques developed in [1], the authors in [9] obtain the existence of infinitely many turning points for $-\Delta U = \Lambda |x|^\alpha/(1 - U)^p$. However, the locations of turning points were not obtained and in particular, the infinite multiplicity of classical solutions at $\Lambda = \Lambda^*$ was not proved (see also [7]). In [13] a similar conclusion to that in Corollary 1.2 above is obtained except the uniqueness of the degenerate solution for the equation $-\Delta_m U = \Lambda |x|^\alpha/(1 - U)^p$. Here, $\Delta_m$ denotes the $m$-Laplace operator. We also refer the interested reader to [14] for the detailed asymptotic behavior of the bifurcation curve near $\tau = 1$ and numerical experiments. Quasilinear equations including the $k$-Hessian operator were considered in [3].

Turning back to our study of (1.1), let us briefly describe the main tools of our approach. The degenerate solution $u^*$ of (1.1) is constructed as the limit of the regular solutions $u(r, \alpha)$ of (1.4) for $\alpha \to 0$. The uniqueness of the degenerate solution requires a detailed ODE analysis of (1.1) near the origin, in which a major role is played by the change of variables

$$F_q[\zeta(t)] = \frac{F[u(r)]}{F_q[u^{\ast}(r)]} \quad \text{and} \quad t := -\log r.$$

This is a generalization of the Emden transformation that corresponds to the case $f(u) = u^p$. The construction of the degenerate solution does not rely on the contraction mapping theorem as mentioned above but rather on a limiting process. Thus, the asymptotic expansion of the degenerate solution near $r = 0$ is difficult to achieve in this method. However, the
detailed ODE analysis we develop in Section 4 provides us with the leading term of the degenerate solution expansion around the origin and it leads to its uniqueness.

Our method is applicable to the study of the singular solution \( u^*(r) \) of supercritical problems \(-\Delta u = f(u)\), where \( u^*(r) \to \infty \) as \( r \to 0 \). See [5, 17, 18, 19] for the counterpart of Theorem 1.1 in such a setting.

Finally, in the study of the number of intersection points between the degenerate and regular solutions, the following transformation plays a key role:

\[
F_q[\tilde{u}(s, \beta)] = \lambda^{-2} F[u(r, \alpha)] \quad \text{where} \quad s := \frac{r}{\lambda} \quad \text{and} \quad F_q[\beta] = \lambda^{-2} F[\alpha].
\]

Although the original equation (1.1) does not have a scaling invariance, through a limiting process given by (1.12) we are led to (1.7) whose structure of solution set is known. In this way, the analysis of (1.4) can be reduced to that of (1.7).

The remaining of the paper is organised as follows. In order to show the full strength of Theorem 1.1 we provide some relevant examples of nonlinearities \( f(u) \) and determine the explicit behavior around the origin of the degenerate solution \( u^* \). This will be done in Section 2. In Section 3 we prove the existence of the degenerate solution to (1.1) while in Section 4 we prove the uniqueness of the degenerate solution and the convergence property (1.5). In Section 5 we study the intersection properties using a blow-up argument and prove Theorem 1.1 (ii). In Section 6 we prove Corollary 1.2.

2. Examples

For two functions \( f(u), g(u) \) defined in a positive neighborhood of the origin, by \( f(u) \asymp g(u) \) we understand that \( f(u)/g(u) \to 1 \) as \( u \to 0^+ \).

2.1. Example 1. Let

\[
f(u) := \begin{cases} u^p |\log u - C|^d & \text{if } u > 0, \\ 0 & \text{if } u = 0, \end{cases}
\]

where \( p > 1, d \geq 0 \) and \( C \in \mathbb{R} \). We have \( q = p/(p-1) > 1 \) and by L’Hospital’s rule we find

\[ F(u) \asymp \frac{u^{p+1}}{p+1} |\log u|^d. \]

Using the above asymptotic behavior of \( F(u) \) we find

\[
\lim_{u \to 0} \frac{F^{-1}(u)}{u^{p+1}} = \lim_{v = F^{-1}(u) \to 0} \frac{v}{F(v)^{-1/p+1}} = (p + 1)^{d+1}. \]

Hence,

\[ F^{-1}(u) \asymp \left[ \frac{(p + 1)^{d+1} u}{|\log u|^d} \right]^{1/p+1}. \]

Using Theorem 1.1 we find:
Proposition 2.1. Let $f$ be defined by (2.1), $p > 1$, $d \geq 0$ and $C \in \mathbb{R}$. Then, there exists a unique degenerate solution $u^*$ of
\[ u'' + \frac{N-1}{r} u' = \frac{1}{u^p |\log u - C|^d} \quad \text{for all } 0 < r < r_0. \]
Furthermore,
\[ u^*(r) = \left[ \frac{1}{N - 2p/(p+1)} \left( \frac{p+1}{2} \right)^{d+1} \frac{r^2}{|\log r|^d} \right]^{1/(p+1)} (1 + o(1)) \quad \text{as } r \to 0. \]

If $u(\cdot, \alpha)$ denotes the regular solution of (1.4) with $f(u)$ defined in (2.1), then

(i) $u(\cdot, \alpha) \to u^*$ in $C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0]$ as $\alpha \to 0$;
(ii) If $q_c$ is the exponent defined in (1.6), $p > q_c/(q_c - 1)$ and $2 < N < 10$, then for any $\rho \in (0, r_0)$ one has
\[ \mathcal{Z}_{(0,\rho)}[u(\cdot, \alpha) - u^*(\cdot)] \to \infty \quad \text{as } \alpha \to 0. \]

2.2. Example 2. Let
\[ f(u) := \begin{cases} e^{-\frac{1}{u^p}} & \text{if } u > 0, \\ 0 & \text{if } u = 0, \end{cases} \]
where $p > 0$. We have $q = 1$ and $f'(u)^2/f(u)f''(u) \geq 1$ for small $u > 0$. By Lemma 3.1 (iii) the condition (F3) holds. By L’Hospital’s rule we obtain
\[ F(u) \simeq \frac{1}{p} u^{p+1} e^{-\frac{1}{u^p}}. \]

From here we deduce
\[ \lim_{u \to 0} \frac{F^{-1}(u)}{(\log \frac{1}{u})^{-1/p}} = \lim_{v = F^{-1}(u) \to 0} \frac{v}{(\log \frac{1}{F(v)})^{-1/p}} = 1. \]
Thus, $F^{-1}(u) \simeq |\log u|^{-1/p}$. Using Theorem 1.1 we find:

Proposition 2.2. Let $f$ be defined by (2.2) where $p > 0$. Then, there exists a unique degenerate solution $u^*$ of
\[ u'' + \frac{N-1}{r} u' = e^{\frac{1}{u^p}} \quad \text{for all } 0 < r < r_0. \]
Furthermore,
\[ u^*(r) = |2 \log r|^{-\frac{1}{p}} (1 + o(1)) \quad \text{as } r \to 0. \]

If $u(\cdot, \alpha)$ denotes the regular solution of (1.4) with $f(u)$ defined in (2.2), then

(i) $u(\cdot, \alpha) \to u^*$ in $C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0]$ as $\alpha \to 0$;
(ii) For $2 < N < 10$ and any $\rho \in (0, r_0)$ we have
\[ \mathcal{Z}_{(0,\rho)}[u(\cdot, \alpha) - u^*(\cdot)] \to \infty \quad \text{as } \alpha \to 0. \]
2.3. Example 3. Let
\begin{equation}
(2.3) \quad f(u) := \begin{cases} 
\exp\{-e^{1/u}\} & \text{if } u > 0, \\
0 & \text{if } u = 0,
\end{cases}
\end{equation}
where \(\exp\{v\} = e^v\). An easy calculation yields \(q = 1\) and \(f'(u)^2/f(u)f''(u) \geq 1\) for small \(u > 0\). By Lemma 3.1 the condition (F3) holds. By L'Hospital's rule we find
\[ F(u) \sim u^2 e^{-1/u} f(u). \]
Further, we compute
\[ \lim_{u \to 0} \frac{F^{-1}(u)}{\log(\log u)} = \lim_{v=F^{-1}(u)\to 0} \frac{v}{\log(\log F(v))} = 1. \]
Hence, \(F^{-1}(u) \sim \frac{1}{\log(\log u)}\). Using Theorem 1.1 we find:

**Proposition 2.3.** Let \(f\) be defined by \((2.3)\). Then, there exists a unique degenerate solution \(u^*\) of
\[ u'' + \frac{N-1}{r} u' = \exp\{e^{1/u}\} \quad \text{for all } 0 < r < r_0. \]
Furthermore,
\[ u^*(r) = \frac{1 + o(1)}{\log(\log r)} \quad \text{as } r \to 0. \]
If \(u(\cdot, \alpha)\) denotes the regular solution of \((1.4)\) with \(f(u)\) defined in \((2.3)\), then
(i) \(u(\cdot, \alpha) \to u^* \) in \(C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0)\) as \(\alpha \to 0\);
(ii) For \(2 < N < 10\) and any \(\rho \in (0, r_0)\) we have
\[ \mathcal{Z}_{(0, \rho)}[u(\cdot, \alpha) - u^*(\cdot)] \to \infty \quad \text{as } \alpha \to 0. \]

3. Existence

**Lemma 3.1.** The following hold:
(i) Let \(q\) be defined in (F2). Then, \(q \geq 1\).
(ii) The limit \(\gamma\) given by \((1.3)\) exists and \(\gamma = q/(2q - 1)\). In particular, \(1/2 < \gamma \leq 1\).
(iii) If \(f'(u)^2/f(u)f''(u) \geq 1\), then (F3) holds, that is, \(f'(u)F[u]/f(u)^2 \leq 1\).

**Proof.** (i) We prove the lemma by contradiction. We may assume that
\[ \frac{f'(u)^2}{f(u)f''(u)} \leq q_0 < 1 \quad \text{for } 0 < u \leq u_0 < \bar{u}. \]
Then \(f'(u)/(q_0 f(u)) \leq f''(u)/f'(u)\). Integrating both sides of this inequality over \([u, u_0]\) we have \(f'(u)/f(u)^{1/q_0} \leq f'(u_0)/f(u_0)^{1/q_0}\). Then
\[ \frac{q_0}{1 - q_0} \left( f(u)^{1 - \frac{1}{q_0}} - f(u_0)^{1 - \frac{1}{q_0}} \right) \leq \frac{f'(u_0)}{f(u_0)^{1/q_0}} (u_0 - u) \quad \text{for } 0 < u \leq u_0. \]
Letting \(u \to 0\), we obtain a contradiction, since the left hand side diverges as \(u \to 0\). Thus if the limit \(q\) exists, then \(q \geq 1\).
(ii) We show that \( \lim_{u \to 0} f(u)^2/f'(u) = 0 \). Using (F1) we find that \( f' \) is increasing on \( (0, \hat{u}) \) so \( f(u) = \int_0^u f' (s) ds \leq uf'(u) \) for all \( 0 < u < \hat{u} \). Then,
\[
0 \leq \lim_{u \to 0} \frac{f(u)^2}{f'(u)} \leq \lim_{u \to 0} \frac{uf(u)f'(u)}{f'(u)} = 0.
\]

Applying the L'Hospital rule, we have
\[
\gamma = \lim_{u \to 0} \frac{F[u]}{f(u)^2/f'(u)} = \lim_{u \to 0} \frac{f(u)}{uf(u)f''(u)/f'(u)^2} = \frac{1}{2 - 1/q}.
\]

Hence, \( \gamma = q/(2q - 1) \). Since \( q \geq 1 \), we see that \( 1/2 < \gamma \leq 1 \).

(iii) By (F3) we have \( f(u) \leq 2f(u) - f(u)^2 f''(u)/f'(u)^2 \). Integrating it over \( [0, u] \), by (ii) we have
\[
F[u] \leq \left[ \frac{f(u)^2}{f'(u)} \right]_0^u = \frac{f(u)^2}{f'(u)} \text{ for small } u > 0.
\]

The conclusion follows.

\( \square \)

**Lemma 3.2.** The following hold:

(i) The regular solution \( u(r, \alpha) \) satisfies
\[
r^{N-1} u'(r, \alpha) = \int_0^r \frac{s^{N-1}}{f(u(s, \alpha))} ds \quad \text{and} \quad u'(r, \alpha) \geq 0 \text{ for small } r > 0.
\]

(ii) The regular solution \( u(r, \alpha) \) is nondecreasing in \( r \), and
\[
F[u(r, \alpha)] - F[\alpha] \geq \frac{r^2}{2N} \quad \text{and} \quad u(r, \alpha) \geq F^{-1} \left[ \frac{r^2}{2N} + F[\alpha] \right].
\]

**Proof.** (i) This follows after multiplication with \( r^{N-1} \) in the main equation of (1.3) and then an integration over \( [0, r] \).

(ii) We see that \( u(r, \alpha) \) is nondecreasing, because of (i). Since \( r \mapsto f(u(r)) \) is nondecreasing, for small \( r > 0 \) one has
\[
r^{N-1} u'(r, \alpha) = \int_0^r \frac{s^{N-1}}{f(u(s, \alpha))} ds \geq \frac{1}{f(u(r, \alpha))} \int_0^r s^{N-1} ds = \frac{r^N}{Nf(u(r, \alpha))}.
\]

Integrating the inequality \( f(u)u' \geq r/N \), we obtain the conclusion. \( \square \)

**Lemma 3.3.** The following hold:

(i) \( u(\cdot, \alpha) \) satisfies
\[
0 \leq u'(r, \alpha) \leq \frac{2N}{N-2} \frac{u(r, \alpha)}{r} \quad \text{for small } r > 0.
\]

(ii) Let \( \eta := 2 \{ 1 + 2N/(N - 2)^2 \} \) and for any small \( \delta > 0 \) define
\[
\delta_0 := \frac{\delta}{\eta} \quad \text{and} \quad r_\delta := (N\delta f(\delta_0))^{1/2}.
\]

If \( 0 < \alpha < \delta_0 \), then \( u(r, \alpha) \) satisfies
\[
u(r, \alpha) < \delta \quad \text{for} \quad 0 < r < r_\delta.
\]
Proof. (i) By (F3) and Lemma 3.2 (i) we have
\[
\frac{d}{dr} \left( \frac{F[u(r, \alpha)]}{f(u(r, \alpha))} \right) = \left(1 - \frac{F[u(r, \alpha)]f'(u(r, \alpha))}{f(u(r, \alpha))^2} \right) u'(r, \alpha) \geq 0 \quad \text{for small } r > 0.
\]
Thus, \( r \mapsto F[u(r, \alpha)]/f(u(r, \alpha)) \) is nondecreasing for small \( r > 0 \). Using this fact and Lemma 3.2 (ii) one has
\[
(3.3) \quad r^{N-1}u'(r, \alpha) = \int_0^r \frac{F[u(s, \alpha)]}{f(u(s, \alpha))} s^{N-1} ds \leq \frac{F[u(r, \alpha)]}{f(u(r, \alpha))} \int_0^r s^{N-1} ds \leq \frac{F[u(r, \alpha)]}{f(u(r, \alpha))} \int_0^r 2Ns^{N-1} ds = \frac{2N}{N-2} \frac{F[u(r, \alpha)]}{f(u(r, \alpha))} r^{N-2}.
\]
Since \( f'(u) \geq 0 \), we have
\[
(3.4) \quad F[u] = \int_0^u f(s) ds \leq uf(u).
\]
By (3.3) and (3.4) we derive (3.1).

(ii) Assume by contradiction that there exist \( \alpha \in (0, \delta_0) \) and \( r_* \in (0, \delta) \) such that
\[
(3.5) \quad u(r, \alpha) < \delta \quad \text{for } 0 \leq r \leq r_* \quad \text{and} \quad u(r_*, \alpha) = \delta.
\]
Since \( \alpha < \delta_0 < \delta \), there exists \( r_1 \in (0, r_*) \) such that
\[
\begin{align*}
\alpha < \delta_0 & < \delta \quad \text{there exists } r_1 \in (0, r_*) \quad \text{such that} \\
u(r_1, \alpha) &= \delta_0 \quad \text{and} \quad u(r, \alpha) \geq \delta_0 \quad \text{for } r_1 \leq r \leq r_*.
\end{align*}
\]

Let \( v \) be the solution of the initial value problem
\[
\begin{align*}
(3.6) \quad (r^{N-1}v')' &= \frac{r^{N-1}}{f(\delta_0)} \quad r_1 < r < r_* \\
v(r_1) &= u(r_1, \alpha) = \delta_0, \\
v'(r_1) &= u'(r_1, \alpha).
\end{align*}
\]

We claim that
\[
(3.7) \quad u(r, \alpha) \leq v(r) \quad \text{for } r_1 \leq r \leq r_*.
\]
Indeed, let \( w(r) := v(r) - u(r, \alpha) \). Then \( w \) satisfies
\[
(r^{N-1}w')' = r^{N-1} \left( \frac{1}{f(\delta_0)} - \frac{1}{f(u(r, \alpha))} \right) \quad \text{for } r_1 < r < r_*
\]
and \( w(r_1) = 0 \). Since \( 1/f(\delta_0) - 1/f(u(r, \alpha)) \geq 0 \), we see that \( w(r) \geq 0 \) for \( r_1 \leq r \leq r_* \), which implies (3.7).

Integrating the equation in (3.6) over \([r_1, r]\), we have
\[
r^{N-1}v'(r) - r_1^{N-1}v'(r_1) = \int_{r_1}^r \frac{s^{N-1}}{f(\delta_0)} ds \leq \frac{r^N}{Nf(\delta_0)} \quad \text{for all } r_1 < r < r_*.
\]
Hence,
\[
(3.8) \quad v'(r) < \frac{r_1^{N-1}v'(r_1)}{r^{N-1}} + \frac{r}{Nf(\delta_0)} \quad \text{for all } r_1 < r < r_*.
\]
Integrating (3.8) over $[r_1, r_*]$, we have
\[ v(r_*) - v(r_1) < -\frac{r_1^{N-1}v'(r_1)}{N-2}(r_*^{-N+2} - r_1^{-N+2}) + \frac{r_*^2 - r_1^2}{2Nf(\delta_0)}. \]

Since $v'(r_1) \geq 0$, $f(\delta_0) > 0$ and $r_* \leq r_\delta$, from the above estimate and (3.2) we find
\[ v(r_*) - \delta_0 < \frac{r_1v'(r_1)}{N-2} + \frac{r_1^2}{2Nf(\delta_0)} \left( \frac{r_1v'(r_1)}{N-2} + \frac{\delta}{2} \right). \]

Using $v'(r_1) = u'(r_1, \alpha)$ and (3.1) we have
\[ \delta = u(r_*, \alpha) \leq v(r_*) < \delta_0 + \frac{r_1u'(r_1)}{N-2} + \frac{\delta}{2} \leq \delta_0 \left( 1 + \frac{2N}{(N-2)^2} \right) + \frac{\delta}{2} = \delta, \]

which is a contradiction. Thus, (3.5) does not occur and the statement (ii) holds. \(\square\)

**Proof of Theorem 1.1 (i) Existence part.** Let \(\{\alpha_n\}\) be a positive decreasing sequence such that \(\alpha_n \to 0\) as \(n \to \infty\). By Lemma 3.3 (ii), there exists \(r_0 > 0\) such that \(\{u(r, \alpha_n)\}\) is uniformly bounded on any compact subset \(I \subset (0, r_0)\). By Lemma 3.3 (i) we see that \(\{u(r, \alpha_n)\}\) is also uniformly bounded on \(I\). Now, Lemma 3.2 (ii) implies that \(\{u(r, \alpha_n)\}\) is uniformly bounded away from zero on \(I\), and hence \(\{1/f(u(r, \alpha_n))\}\) is uniformly bounded on \(I\). Since \(f \in C^1\), it follows that \(\{u'(r, \alpha_n)\}\) and \(\{u''(r, \alpha_n)\}\) are uniformly bounded on \(I\).

By the Ascoli-Arzelà theorem and with a diagonal argument, there exists \(u^* \in C^2(0, r_0) \cap C^0[0, r_0]\) and a subsequence, which is again denoted by \(\{u(r, \alpha_n)\}\), such that
\[ u(r, \alpha_n) \to u^*(r) \quad \text{in} \quad C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0] \quad \text{as} \quad n \to \infty. \]

It is clear that \(u^*\) solves (1.1) for \(0 < r < r_0\). By Lemma 3.2 (ii) we see that \(u^*(r) > 0\) for \(0 < r < r_0\). By Lemma 3.3 (ii) we have \(\lim_{r \to 0} u^*(r) = 0\). Therefore, \(u^*\) is a degenerate solution of (1.1).

We have shown that for any sequence \(\{\alpha_n\} \subset (0, \infty), \alpha_n \to 0, \{u(r, \alpha_n)\}\) converges in \(C^2_{\text{loc}}(0, r_0) \cap C_{\text{loc}}[0, r_0]\) as \(n \to \infty\) to a degenerate solution of (1.1). The convergence (1.5) follows once we prove the uniqueness of a degenerate solution. This will be done in the next section. \(\square\)

### 4. Uniqueness and Convergence

In this section we prove the uniqueness and the behaviour at the origin of a degenerate solution to (1.1). Since \(q \geq 1\) and \(N \geq 3\) we see that \(N + 2 - 4\gamma > 0\) and \(2N - 4\gamma > 0\), which will be used. We start with several preliminary results on degenerate solutions.

**Lemma 4.1.** Let \(u(r)\) be a degenerate solution. Then the following hold:

(i) \(u'(r) > 0\) for \(r > 0\) and \(u\) satisfies
\[ r^{N-1}u'(r) = \int_0^r s^{N-1}ds/f(u(s)). \]

(ii) The degenerate solution \(u\) is not of class \(C^1\) at the origin.
Proof. Since
\[(r^{N-1}u')' = \frac{r^{N-1}}{f(u)} \geq 0,\]
r\(^{N-1}u'\) is nondecreasing. The limit \(l := \lim_{r \to 0} r^{N-1}u'(r)\) exists and \(l \in [-\infty, \infty)\). If \(l < 0\), then \(u'(r) < 0\) in a neighborhood of \(r = 0\). This implies that \(u(r) < u(0) = 0\), which is a contradiction. If \(l > 0\), then there are \(\varepsilon > 0\) and \(\rho > 0\) such that \(r^{N-1}u'(r) \geq \varepsilon\) for \(0 < r < \rho\).

Integrating it over \([r, \rho]\), we have \(u(\rho) - u(r) \geq \frac{\varepsilon}{N-2} (r^{2-N} - \rho^{2-N})\). Then,
\[u(\rho) > \frac{\varepsilon}{N-2} (r^{2-N} - \rho^{2-N}) \quad \text{for} \quad 0 < r < \rho.\]

Letting \(r \to 0\), we have \(u(\rho) = \infty\), which is a contradiction. Therefore, \(l = 0\). Now (4.1) follows by integrating (4.2) over \([0, r]\).

(ii) Assume by contradiction that there exists \(\ell = \lim_{r \to 0} u'(r) \in \mathbb{R}\). By (i) one has \(\ell \geq 0\). Since \(\lim_{r \to 0} \frac{F(r)F'[r]}{f(r)^2} = \gamma \in (1/2, 1]\), we may find \(\sigma \in (1/2, \gamma]\) and \(\rho \in (0, r_0]\) such that \(f\) is increasing on \((0, \rho)\) and
\[\frac{f'(r)F'[r]}{f(r)^2} \geq \sigma \quad \text{for all} \quad 0 < r \leq \rho.\]

We rewrite the above inequality in the form \(f'(r)/f(r) \geq \sigma f(r)/F[r]\) and integrate it over \([r, \rho]\). We obtain
\[(4.3) \quad f(r)F[r]^{-\sigma} \leq C \quad \text{for all} \quad 0 < r \leq \rho,\]
where \(C = f(\rho)F[\rho]^{-\sigma} > 0\). On the other hand, since \(f\) is increasing on \((0, \rho)\) we have \(F[r] \leq r f(r)\) and from (4.3) we deduce
\[r^{-\sigma} f(r)^{1-\sigma} \leq C \quad \text{for all} \quad 0 < r \leq \rho.\]

This yields \(f(r) \leq C r^\sigma/(1-\sigma)\) for all \(0 < r \leq \rho\), where \(C > 0\). In particular, since \(\sigma > 1/2\), one has \(\lim_{r \to 0} \frac{r}{f(r)} = \infty\). Now, by (i) and L'Hospital's rule we derive
\[(4.4) \quad \ell = \lim_{r \to 0} u'(r) = \lim_{r \to 0} \frac{\int_0^r \frac{s^{N-1}}{f(u(s))} ds}{r^{N-1}} = \frac{1}{N-1} \lim_{r \to 0} \frac{u(r)}{f(u(r))} \frac{r}{u(r)}.\]

Since
\[\lim_{r \to 0} \frac{r}{u(r)} = \begin{cases} 1/\ell \quad & \text{if} \quad \ell > 0, \\ \infty \quad & \text{if} \quad \ell = 0 \end{cases} \quad \text{and} \quad \lim_{r \to 0} \frac{u(r)}{f(u(r))} = \infty,\]
the equality (4.4) cannot hold for finite \(\ell \geq 0\). Hence, \(u\) is not of class \(C^1\) at the origin. \(\Box\)

Lemma 4.2. Let \(u\) be a degenerate solution. Then, for small \(r > 0\) we have
\[(4.5) \quad F[u(r)] \geq \frac{r^2}{2N}, \quad \text{and hence} \quad u(r) \geq F^{-1} \left[ \frac{r^2}{2N} \right].\]

Proof. Since by Lemma 4.1 (i) \(u' > 0\), we have
\[\frac{d}{dr} \left( \frac{1}{f(u(r))} \right) = - \frac{f'(u(r))}{f(u(r))^2} u'(r) \leq 0 \quad \text{for} \quad r > 0 \text{ small.}\]
Thus, $1/f(u(r))$ is nonincreasing. Then, by (4.1) we have
\[ r^{N-1}u'(r) = \int_0^r s^{N-1}ds \geq \frac{1}{f(u(r))} \int_0^r s^{N-1}ds = \frac{r^N}{Nf(u(r))}. \]
Integrating $f(u') \geq r/N$ over $[0, r]$, we have (4.5).

**Lemma 4.3.** Let $u$ be a degenerate solution. Then,
\[ \limsup_{r \to 0} \frac{r^2}{F[u(r)]} > 0. \]

**Proof.** We show that
\[ \frac{F[u(r)]}{f(u(r))} \] is nondecreasing for small $r > 0$.
By (F3), (4.3) and Lemma 4.1 (i) we have
\[ \frac{d}{dr} \left( \frac{F[u(r)]}{f(u(r))} \right) = \left( 1 - \frac{f'(u(r))F[u(r)]}{f(u(r))^2} \right) u'(r) \geq 0. \]
We prove (4.6) by contradiction. We assume that
\[ \lim_{r \to 0} \frac{r^2}{F[u(r)]} = 0. \]
For each small $\varepsilon > 0$, there is $\rho > 0$ such that $r^2/F[u(r)] < (N - 2)\varepsilon$ for all $0 < r \leq \rho$. By (4.1) and (4.7) we have
\[ r^{N-1}u'(r) = \int_0^r s^{N-3}F[u(s)]ds < (N - 2)\varepsilon \int_0^r \frac{F[u(s)]}{f(u(s))} s^{N-3}ds \leq (N - 2)\varepsilon \frac{F[u(r)]}{f(u(r))} \int_0^r s^{N-3}ds = \varepsilon \frac{F[u(r)]}{f(u(r))} r^{N-2}. \]
Integrating $f(u')/u \leq \varepsilon/r$ over $[r, \rho]$, we have $\log F[u(\rho)]/F[u(r)] \leq \varepsilon \log(\rho/\rho)$. Hence, $r^\varepsilon \leq CF[u]$ for small $r > 0$. We have
\[ r^{N-1}u'(r) = \int_0^r s^{N-1}F[u(s)]ds \leq C \int_0^r \frac{F[u(s)]}{f(u(s))} s^{N-1-\varepsilon}ds \leq C \frac{F[u(r)]}{f(u(r))} \int_0^r s^{N-1-\varepsilon}ds = \frac{C}{N-\varepsilon} F[u(r)] r^{N-\varepsilon}. \]
Integrating $f(u')/u \leq Cr^{1-\varepsilon}/(N - \varepsilon)$ over $[0, \rho]$, we find
\[ \log \frac{F[u(\rho)]}{F[u(r)]} \leq \frac{C}{(N-\varepsilon)(2-\varepsilon)} (\rho^{2-\varepsilon} - r^{2-\varepsilon}). \]
Taking the limit $r \to 0$, we have $\infty \leq C\rho^{2-\varepsilon}/(N - \varepsilon)(2 - \varepsilon)$ which is a contradiction. The assumption (4.8) does not hold, and thus (4.6) follows. □
Let $u$ be a degenerate solution and let $\gamma$ be defined by \[113\]. We use the following two transformations:

\begin{equation}
(4.9) \quad e^z(t) = \frac{2N - 4\gamma}{r^2}F[u(r)] \quad \text{and} \quad t := -\log r,
\end{equation}

\begin{equation}
(4.10) \quad F_q[\zeta(t)] = \frac{2N - 4\gamma}{r^2}F[u(r)] \quad \text{and} \quad t := -\log r.
\end{equation}

Note that $\zeta(t)$ and $z(t)$ are related by

\begin{equation}
(4.11) \quad e^z(t) = F_q[\zeta(t)],
\end{equation}

and $\frac{\partial}{\partial t} = -e^t \frac{\partial}{\partial t}$. From now on, by $'$ we denote the derivative with respect to $r$ variable while by subscript notation we denote the derivative with respect to $t$ variable.

**Lemma 4.4.** The following hold:

(i) The function $z(t)$ satisfies

\begin{equation}
(4.12) \quad z_{tt} - (N + 2 - 4\gamma)z_t + (2N - 4\gamma)(1 - e^{-z}) + (1 - \gamma)z_t^2 + \left(\gamma - \frac{f'(u)F[u]}{f(u)^2}\right)(z_t - 2)^2 = 0.
\end{equation}

(ii) The function $\zeta(t)$ satisfies

\begin{equation}
(4.13) \quad \zeta_{tt} - (N + 2 - 4\gamma)\zeta_t + (2N - 4\gamma) \frac{F_q[\zeta] - 1}{f_q(\zeta)} + \left(\gamma - \frac{f'(u)F[u]}{f(u)^2}\right)\left(\zeta_t - 2\frac{F_q[\zeta]}{f_q(\zeta)}\right)^2 \frac{f_q(\zeta)}{F_q[\zeta]} = 0.
\end{equation}

**Proof.** (i) Differentiating

\begin{equation}
(4.14) \quad F[u] = \frac{e^{-2t+z}}{2N - 4\gamma}
\end{equation}

with respect to $r$, we have

\begin{equation}
(4.15) \quad f(u)u' = \frac{-e^{-t+z}}{2N - 4\gamma}(z_t - 2).
\end{equation}

Differentiating (4.15) with respect to $r$, we obtain

\begin{equation}
(4.16) \quad f(u)u'' + f'(u)u'^2 = \frac{e^z}{2N - 4\gamma}\{z_{tt} + (z_t - 2)(z_t - 1)\}.
\end{equation}

By (4.14) and (4.15) we have

\begin{equation}
(4.17) \quad f'(u)u'^2 = \frac{f'(u)(z_t - 2)^2}{f(u)^2(2N - 4\gamma)}e^{-2t+2z} = \frac{1}{2N - 4\gamma} \frac{f'(u)F[u]}{f(u)^2}(z_t - 2)^2 e^z.
\end{equation}

By (4.16) and (4.17) we deduce

\begin{equation}
(4.18) \quad f(u)u'' = \frac{e^z}{2N - 4\gamma}\left\{z_{tt} - 3z_t + 2 + \frac{f'(u)F[u]}{f(u)^2}(z_t - 2)^2\right\}.
\end{equation}

By (111), (4.15) and (4.18) we have

\begin{equation}
0 = f(u)u'' + \frac{N - 1}{r}f(u)u' - 1
= \frac{e^z}{2N - 4\gamma}\left\{z_{tt} - (N + 2 - 4\gamma)z_t + (2N - 4\gamma)(1 - e^{-z}) + (1 - \gamma)z_t^2 + \left(\gamma - \frac{f'(u)F[u]}{f(u)^2}\right)(z_t - 2)^2\right\}.
\end{equation}
Lemma 4.6. Assume that (ii) \( z = \log F_q[\zeta] \), we have \( z_t = \frac{f_q}{F_q} \) and

\[
z_{tt} = \frac{f_q}{F_q} \zeta_{tt} + \frac{f_q^2}{F_q^2} \left( \frac{f_q F_q}{f_q^2} - 1 \right) \zeta_t^2 = \frac{f_q}{F_q} \zeta_{tt} + \frac{f_q^2}{F_q^2} (\gamma - 1) \zeta_t^2.
\]

Substituting \( z, z_t \) into (4.12), we obtain (4.13).

\[\Box\]

Lemma 4.5. \( z_t \) satisfies

\[-\frac{4}{N - 2} \leq z_t(t) \leq 2 \quad \text{for large } t.\]

In particular, \( z_t \) is bounded for large \( t \).

Proof. Since \( z_t(t) = 2 - \rho f(u(r))u'(r)/F[u(r)] \) and \( u'(r) > 0 \) by Lemma 4.1 (i), we have

\[z_t(t) \leq 2 \quad \text{for large } t.
\]

We combine now Lemma 4.1 (i) with Lemma 4.2 and (4.7) to deduce

\[
\frac{f(u(r))u'(r)r}{F[u(r)]} = \frac{f(u(r))}{F[u(r)]} \frac{1}{r^{N-2}} \int_0^r s^2 \frac{F[u(s)]}{f(u(s))} s^{N-3} ds
\leq \frac{f(u(r))}{F[u(r)]} \frac{2N}{r^{N-2}} \int_0^r s^{N-3} ds
\leq \frac{2N}{N - 2}.
\]

Thus,

\[z_t(t) = 2 - \frac{f(u(r))u'(r)r}{F[u(r)]} \geq 2 - \frac{2N}{N - 2} = -\frac{4}{N - 2},
\]

which completes our proof.

We divide our argument into two cases.
(i) \( z_t(t) \geq 0 \) or \( z_t(t) \leq 0 \) for large \( t \) (nonoscillatory at \( t = \infty \)).
(ii) \( z_t(t) \) changes sign infinitely many times as \( t \to \infty \) (oscillatory at \( t = \infty \)).

First, we consider the case (i).

Lemma 4.6. Assume that \( z_t(t) \geq 0 \) or \( z_t(t) \leq 0 \) for large \( t \). Then, \( z(t) \to 0 \) as \( t \to \infty \).

Proof. We consider the case where \( z_t(t) \leq 0 \) for large \( t \). Due to Lemma 4.2, \( z(t) \) is bounded from below. Hence, \( z(t) \) converges as \( t \to \infty \). We consider the case where \( z_t(t) \geq 0 \) for large \( t \). Suppose that \( \lim_{t \to \infty} z(t) = \infty \). Then,

\[
\limsup_{r \to \infty} \left( \frac{r^2}{F[u(r)]} \right) = \limsup_{t \to \infty} \left( \frac{2N - 4\gamma}{e^{z(t)}} \right) = 0,
\]

which contradicts Lemma 4.3. Thus, \( z(t) \) is bounded from above. Since \( z_t(t) \geq 0 \) for large \( t \), \( z(t) \) converges as \( t \to \infty \). We see that in both cases \( z(t) \to c \) as \( t \to \infty \), for some \( c \in \mathbb{R} \).

We claim that \( c = 0 \). Suppose the contrary, i.e.,

\[
(4.19) \quad c \neq 0.
\]

We show that

\[
(4.20) \quad \lim_{t \to \infty} z_t(t) = 0.
\]
Since $z(t)$ is bounded, it is clear that
\begin{equation}
\liminf_{t \to \infty} |z_t(t)| = 0.
\end{equation}
In order to prove (4.20) it is enough to show that
\begin{equation}
\limsup_{t \to \infty} |z_t(t)| = 0.
\end{equation}
Suppose that (4.22) does not hold, so
\begin{equation}
\limsup_{t \to \infty} |z_t(t)| > 0.
\end{equation}
By (4.21) and (4.23) we see that there is \( \{t_n\}_{n=1}^{\infty} \) such that \( z_t(t_n) \to 0 \) \( (n \to \infty) \) and \( z_{tt}(t_n) = 0 \). Then, by (4.12) we obtain
\begin{align*}
0 &= z_{tt}(t_n) - (N+2-4\gamma)z_t(t_n) + (2N-4\gamma)(1-e^{-z(t_n)}) \\
&\quad + (1-\gamma)z_t(t_n)^2 + \left( \frac{F[u(r)]f'(u(r))}{f(u(r))^2} \right) (z_t(t_n) - 2)^2 \\
&\quad \to (2N-4\gamma)(1-e^{-c}) \neq 0 \text{ as } n \to \infty.
\end{align*}
This is a contradiction which shows that (4.22) holds and thus (4.20) follows.

Passing to the limit with \( t \to \infty \) in (4.12) we find
\begin{align*}
z_{tt}(t) &= (N+2-4\gamma)z_t(t) - (2N-4\gamma)(1-e^{-z(t)}) \\
&\quad - (1-\gamma)z_t(t)^2 - \left( \frac{F[u(r)]f'(u(r))}{f(u(r))^2} \right) (z_t(t) - 2)^2 \\
&\quad \to (2N-4\gamma)(1-e^{-c}) \neq 0 \text{ as } t \to \infty.
\end{align*}
Therefore, \( |z_t(t)| \) diverges, which contradicts (4.20). The assumption (4.19) does not hold, and hence \( \lim_{t \to \infty} z(t) = 0 \). The proof is complete.

**Lemma 4.7.** Assume that \( z_t(t) \geq 0 \) or \( z_t(t) \leq 0 \) for large \( t \). Then, \( z_t(t) \to 0 \) as \( t \to \infty \).

**Proof.** Let \( \zeta(t) \) be defined by (4.10). Since \( F_{q}[\zeta(t)] = e^{\zeta(t)} \to 1 \) \( (t \to \infty) \), we can easily see that
\begin{equation}
f_{q}(\zeta(t)) \to f_{q}(F_{q}^{-1}[1]) = \begin{cases} 
1 & \text{if } \gamma = 1, \\
\frac{(p+1)^{\frac{1}{p}}}{p^{\frac{1}{p}}} & \text{if } \gamma < 1.
\end{cases}
\end{equation}
In particular, \( f_{q}(\zeta(t)) \) is uniformly bounded away from 0 for large \( t \). By (4.11) we have \( \zeta_t = F_{q}[\zeta]z_t/f_{q}(\zeta) \). Then
\begin{equation}
\left( \zeta_t - 2 \frac{F_{q}[\zeta]}{f_{q}(\zeta)} \right)^2 \frac{f_{q}(\zeta)}{F_{q}[\zeta]} = (z_t - 2)\left( \frac{F_{q}[\zeta]}{f_{q}(\zeta)} \right)^2 \frac{e^{\zeta}}{F_{q}[\zeta]}.
\end{equation}
Since \( z_t \) is bounded (thanks to Lemma 4.5), \( e^{\zeta} \) is bounded (by Lemma 4.6) and \( f_{q}(\zeta) > C_3 > 0 \), we see that \( (z_t - 2)^2 e^{\zeta}/f_{q}(\zeta) \) is bounded for large \( t \). By (4.24) we see that \( \left( \zeta_t - 2 \frac{F_{q}[\zeta]}{f_{q}(\zeta)} \right)^2 \frac{f_{q}(\zeta)}{F_{q}[\zeta]} \) is bounded. Hence
\begin{equation}
\left( \gamma - \frac{f'(u)F[u]}{f(u)^2} \right) \left( \zeta_t - 2 \frac{F_{q}[\zeta]}{f_{q}(\zeta)} \right)^2 \frac{f_{q}(\zeta)}{F_{q}[\zeta]} \to 0 \text{ as } t \to \infty.
\end{equation}
It is clear from Lemma 4.6 that \( \liminf_{t \to \infty} |\zeta_t(t)| = 0 \), otherwise we would have \( \lim_{t \to \infty} |\zeta(t)| = \infty \), which contradicts (4.11) and Lemma 4.6.

We claim that
\[
\lim_{t \to \infty} \zeta_t(t) = 0.
\]
Suppose to the contrary that \( \limsup_{t \to \infty} |\zeta_t(t)| > 0 \). Then, there is a sequence \( \{t_n\}_{n=1}^{\infty} \) such that
\[
|\zeta_t(t_n)| > \delta, \quad \zeta_t(t_n) = 0 \quad \text{and} \quad t_n \to \infty.
\]
By (4.13) and (4.25) we have
\[
\begin{align*}
(N + 2 - 4\gamma)\zeta_t(t_n) &= \zeta_{tt}(t_n) + (2N - 4\gamma)\frac{F_q[\zeta(t_n)]}{f_q(\zeta(t_n))} - 1 \\
&\quad + \left( \gamma - \frac{f^2(u)}{f(u)^2} \right) \left( \zeta_t(t_n) - 2 \frac{F_q[\zeta(t_n)]}{f_q(\zeta(t_n))} \right)^2 \frac{f_q(\zeta(t_n))}{F_q[\zeta(t_n)]}
\end{align*}
\]
\[
\to 0 \quad \text{as} \quad n \to \infty.
\]
Since \( N + 2 - 4\gamma > 0 \), it follows that \( \zeta_t(t_n) \to 0 \) as \( n \to \infty \), which contradicts the condition \( |\zeta_t(t_n)| > \delta \).

Thus, \( \limsup_{t \to \infty} |\zeta_t(t)| = 0 \), and hence (4.26) holds. Since \( e^{z(t)}z_t(t) = f_q(\zeta(t))\zeta_t(t) \), by Lemma 4.8 and (4.26) we see that \( z_t(t) \to 0 \) as \( t \to \infty \). □

Next, we consider the case (ii).

**Lemma 4.8.** Assume that \( z_t(t) \) changes sign infinitely many times as \( t \to \infty \). Then, there is a compact set \( K \) in the \( zz_t \)-plane and \( t_0 \in \mathbb{R} \) such that \( \{(z(t), z_t(t)) \mid t > t_0\} \subset K \).

**Proof.** The boundedness of \( z_t(t) \) follows from Lemma 4.5. It follows from (4.9) and Lemma 4.2 that
\[
\begin{align*}
\log \left( \frac{2N - 4\gamma}{2N} \right) \\
\leq \log \left( \frac{2N - 4\gamma}{2N} \right).
\end{align*}
\]
It is enough to show that
\[
\limsup_{t \to \infty} z(t) < \infty.
\]
Suppose to the contrary that \( \limsup_{t \to \infty} z(t) = \infty \). Due to Lemma 4.3 we have
\[
\liminf_{t \to \infty} \frac{e^{z(t)}}{2N - 4\gamma} = \liminf_{t \to 0} \frac{F[u]}{r^2} = \frac{1}{\limsup_{t \to \infty} F[u]} < \infty,
\]
and hence \( \liminf_{t \to \infty} z(t) < \infty \). Therefore, one can find two sequences \( \{t_n\}_{n=1}^{\infty} \) and \( \{\tilde{t}_n\}_{n=1}^{\infty} \) such that:
- \( t_n \) is a local maximum point for \( z_t \), \( z_t(t_n) = 0 \) and \( z(t_n) \to \infty \) as \( n \to \infty \);
- \( \tilde{t}_n > t_n \), \( z_t(\tilde{t}_n) = 0 \) and \( z_t < 0 \) on \( (t_n, \tilde{t}_n) \).

Then, \( z(\tilde{t}_n) < z(t_n) \). Let
\[
J(z, z_t) := \frac{1}{2} z_t^2 + (2N - 4)z + (2N - 4\gamma)e^{-z}.
\]
Then from (4.12), (13) and (F3), for large \( n \geq 1 \) we have
\[
\frac{d}{dt} J(z(t), z_t(t)) = \left\{ (N + 2 - 4\gamma) + 4 \left( \gamma - \frac{f'(u)F[u]}{f(u)^2} \right) \right\} z_t^2 \\
+ \left( 1 - \frac{f'(u)F[u]}{f(u)^2} \right) (-z_t^3) + 4 \left( 1 - \frac{f'(u)F[u]}{f(u)^2} \right) (-z_t)
\geq 0 \quad \text{for} \quad t_n < t < \tilde{t}_n.
\]
Hence, \( J(z(t), z_t(t)) \) is nondecreasing. Since \( J(z(\tilde{t}_n), z_t(\tilde{t}_n)) \geq J(z(t_n), z_t(t_n)) \), for \( n \geq 1 \) large we have
\[
(4.29) \quad (N - 2)z(\tilde{t}_n) + (N - 2\gamma)e^{-z(\tilde{t}_n)} \geq (N - 2)z(t_n) + (N - 2\gamma)e^{-z(t_n)} \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.
\]
Since \( z(\tilde{t}_n) < z(t_n) \) and
\[
\left[ \log \frac{N - 2\gamma}{N - 2}, \infty \right) \ni s \mapsto (N - 2)s + (N - 2\gamma)e^{-s} \quad \text{is increasing,}
\]
we see that \( z(\tilde{t}_n) < \log \frac{N - 2\gamma}{N - 2} \). In particular, \( \{z(\tilde{t}_n)\} \) is bounded from above. It follows from (4.29) that \( z(\tilde{t}_n) \rightarrow -\infty \) as \( n \rightarrow \infty \) which contradicts (4.27). Thus, (4.28) follows and this completes our proof.

We study the limit equation of (4.12). More precisely we consider the initial value problem
\[
\begin{cases}
\tilde{z}_t = \tilde{w}, \\
\tilde{w}_t = (N + 2 - 4\gamma)\tilde{w} - (1 - \gamma)\tilde{w}^2 - (2N - 4\gamma)(1 - e^{-\tilde{z}}), \\
(\tilde{z}(0), \tilde{w}(0)) = (z_0, w_0).
\end{cases}
\]

**Lemma 4.9.** Let \((\tilde{z}(t), \tilde{w}(t))\) be the solution of (4.30), and let \( K \) be a compact set in the \( zw\)-plane. If \((z_0, w_0) \neq (0, 0)\), then there exists \( T > 0 \) such that \((\tilde{z}(T), \tilde{w}(T)) \notin K\).

**Proof.** We can easily see that (4.30) has the unique equilibrium \((0, 0)\) and that the two eigenvalues of the linearization problem are
\[
\lambda_{\pm} := \frac{1}{2} \left\{ N + 2 - 4\gamma \pm \sqrt{(N + 2 - 4\gamma)^2 - 4(2N - 4\gamma)} \right\}.
\]
Since \( N + 2 - 4\gamma > 0 \) and \( 2N - 4\gamma > 0 \), we see that \( \text{Re}(\lambda_{\pm}) > 0 \). Therefore, \((0, 0)\) is an unstable node or a spiral-out. The orbit \( \{(\tilde{z}(t), \tilde{w}(t))\} \) does not converge to \((0, 0)\).

When \( q = 1, \tilde{z} \) satisfies
\[
\frac{d}{dt} \left\{ \frac{\tilde{z}_t^2}{2} + (2N - 4)(\tilde{z} + e^{-\tilde{z}}) \right\} = (N - 2)\tilde{z}_t^2 \geq 0.
\]
This inequality indicates that (4.30) has no nontrivial periodic orbit. When \( q > 1 \), let \( \tilde{\zeta} \) be defined by \( e^{\tilde{\zeta}} = F_q[\tilde{\zeta}] \). Then, \( \tilde{\zeta} \) satisfies
\[
\tilde{\zeta}_{tt} - (N + 2 - 4\gamma)\tilde{\zeta}_t + (2N - 4\gamma) \frac{F_q[\tilde{\zeta}] - 1}{f_q(\tilde{\zeta})} = 0.
\]
Since $F_q[\tilde{z}] = e^{\tilde{z}} > 0$, the denominator does not vanish. Then,
\[
\frac{d}{dt} \left\{ \frac{\tilde{z}^2}{2} + (2N - 4\gamma) \left( \frac{\tilde{z}^2}{2(p+1)} + \frac{\tilde{z}^{1-p}}{p-1} \right) \right\} = (N + 2 - 4\gamma)\tilde{z}_t^2 \geq 0,
\]
where $p := q/(q-1)$. Therefore, (4.30) has no nontrivial periodic orbit. We see that (4.30) has no limit cycle in both cases $q = 1$ and $q > 1$.

If $(\tilde{z}(t), \tilde{w}(t)) \in K$ for all $t \geq 0$, then from the Poincaré-Bendixon theorem it follows that $(\tilde{z}(t), \tilde{w}(t))$ converges to either an equilibrium or approaches a limit cycle, which is impossible. Thus, there exists $T > 0$ such that $(\tilde{z}(T), \tilde{w}(T)) \notin K$. □

Lemma 4.10. Let $z(t)$ be defined by (4.9) and assume that $z_t(t)$ changes sign infinitely many times as $t \to \infty$. Then, as $t \to \infty$,
\[
z(t) \to 0 \quad \text{and} \quad z_t(t) \to 0.
\]

Proof. Suppose by contradiction that $(z(t), z_t(t)) \not\to (0,0)$ as $t \to \infty$. By Lemma 4.8 one can find a compact set $K$ and $t_0 > 0$ such that $\{(z(t), z_t(t)) | t > t_0\} \subset K$. There exist $\{t_n\}_{n=1}^\infty$ and $(z_0, w_0) \neq (0,0)$ such that $t_n \to \infty$ and $(z(t_n), z_t(t_n)) \to (0,0)$ as $n \to \infty$. Let $(\tilde{z}(t), \tilde{w}(t))$ be the solution of (4.30) with the initial data $(z_0, w_0)$. If $t$ is large, then $|\gamma - f'(u)F[u]/f(u)^2| < 2$ is arbitrarily small, since $z_t$ is bounded for large $t$. Hence, for $n$ large, $(z(t_n + t), z_t(t_n + t))$ is close to the solution $(\tilde{z}(t), \tilde{w}(t))$ in a finite interval $0 \leq t \leq T$, where $T$ is given in Lemma 4.9. It follows from Lemma 4.9 that $(\tilde{z}(T), \tilde{w}(T)) \notin K$, and hence $(z(t_n + T), z_t(t_n + T)) \notin K$ for large $n$. This clearly contradicts Lemma 4.8 and thus, the conclusion of the Lemma 4.10 follows. □

By Lemmas 4.6, 4.7 and 4.10 we obtain the following:

Corollary 4.11. Let $u$ be a degenerate solution and let $z(t)$ be defined by (4.9). Then, as $t \to \infty$,
\[
z(t) \to 0 \quad \text{and} \quad z_t(t) \to 0.
\]

In particular,
\[
u(r) = F^{-1} \left[ \frac{r^2}{2N - 4\gamma} (1 + o(1)) \right] \quad \text{as} \quad r \to 0.
\]

One important tool in the proof of the uniqueness is the following result from [15].

Proposition 4.12. (see [15] Lemma 4.2) Suppose that $A(t)$ and $B(t)$ are continuous functions satisfying $\lim_{t \to \infty} A(t) = A > 0$ and $\lim_{t \to \infty} B(t) = B > 0$. Let $z(t)$ be a solution of
\[
z_{tt} - A(t)z_t + B(t)z = 0 \quad \text{for large} \; t.
\]

If $z(t)$ is bounded as $t \to \infty$, then $z(t) \equiv 0$.

Theorem 4.13. The equation (1.1) has at most one degenerate solution.

Proof. Let $u_j(r)$, $j = 1, 2$, be two degenerate solutions of (1.1). Let $z_j(t)$, $j = 1, 2$, be defined by the transformation (1.9), that is,
\[
e^{z_j(t)} = \frac{F[u_j(r)]}{r^2} \quad \text{and} \quad t = -\log r.
\]
By (4.12) we see that \( z_j \) satisfies
\[
z_{jt} - az_t + b(1 - e^{-z_j}) + (1 - \gamma)(z_{2t} + z_{1t})z + \left( \frac{\gamma - \frac{f'(u_j)F[u_j]}{f(u_j)^2} - \frac{f'(u_1)F[u_1]}{f(u_1)^2}}{f(u_2)^2} \right) (z_{jt} - 2)^2 = 0,
\]
where \( a := N + 2 - 4\gamma > 0 \) and \( b := 2N - 4\gamma > 0 \). Here and in the subsequent arguments \( z_{jt} \) and \( z_{jt}, j = 1, 2 \), stand for \( \frac{d^2z_j}{dt^2} \) and \( \frac{dz_j}{dt} \) respectively. Let \( z(t) := z_2(t) - z_1(t) \). By Corollary 4.11 we see that
\[
(4.31) \quad z(t) \text{ is bounded as } t \to \infty.
\]

Then, \( z \) satisfies
\[
z_{tt} - az_t + b \left( \frac{e^{-z_2} - e^{-z_1}}{z_2 - z_1} \right) z + (1 - \gamma)(z_{2t} + z_{1t})z + \left( \frac{\gamma - \frac{f'(u_2)F[u_2]}{f(u_2)^2} - \frac{f'(u_1)F[u_1]}{f(u_1)^2}}{f(u_2)^2} \right) (z_{tt} - 2)^2 = 0,
\]
where we define \( (e^{-z_2} - e^{-z_1})/(z_2 - z_1) = -e^{-z_2} \) if \( z_1 = z_2 \). We consider the last term of the above equation. Since
\[
\frac{d}{dw} \left( \frac{w f'(F^{-1}[w])}{f(F^{-1}[w])^2} \right) = \left\{ 1 + \left( \frac{f(F^{-1}[w])f''(F^{-1}[w])}{f'(F^{-1}[w])^2} - 2 \right) \frac{w f'(F^{-1}[w])}{f(F^{-1}[w])^2} \right\} \frac{w f'(F^{-1}[w])}{f(F^{-1}[w])^2} \frac{1}{w} (w_2 - w_1),
\]

it follows from the mean value theorem that there exists \( \bar{w} \) between \( w_2 \) and \( w_1 \) such that
\[
\frac{w_2 f'(F^{-1}[w_2])}{f(F^{-1}[w_2])^2} - \frac{w_1 f'(F^{-1}[w_1])}{f(F^{-1}[w_1])^2} = \left\{ 1 + \left( \frac{f(U)f''(U)}{f'(U)^2} - 2 \right) \frac{f'(U)F[U]}{f(U)^2} \right\} \frac{f'(U)F[U]}{f(U)^2} \frac{1}{U} (U_2 - U_1).
\]

Let \( w_j = F[u_j], j = 1, 2 \), in the above equality. Since \( F \) is continuous and increasing there exists \( \bar{U} \) between \( U_2 \) and \( U_1 \) so that \( \bar{w} = F(\bar{U}) \) and
\[
\frac{f'(U_2)F[U_2]}{f(U_2)^2} - \frac{f'(U_1)F[U_1]}{f(U_1)^2} = \left\{ 1 + \left( \frac{f(U)f''(U)}{f'(U)^2} - 2 \right) \frac{f'(U)F[U]}{f(U)^2} \right\} \frac{f'(U)F[U]}{f(U)^2} \frac{(2N - 4\gamma)e^{-2t}}{F[U]} e^{z_2} - e^{z_1},
\]
where we define \( (e^{z_2} - e^{z_1})/(z_2 - z_1) = e^{z_2} \) if \( z_1 = z_2 \). Therefore, \( z \) satisfies
\[
z_{tt} - A(t)z_t + B(t)z = 0,
\]
where
\[
A(t) = a - \left( \gamma - \frac{f'(u_2)F[u_2]}{f(u_2)^2} \right) (z_{2t} + z_{1t} - 4),
\]
\[
B(t) = -b \left( \frac{e^{-2z_2} - e^{-2z_1}}{z_2 - z_1} \right) + (1 - \gamma)(z_{2t} + z_{1t}) + \left\{ 1 + \left( \frac{f(U)f''(U)}{f'(U)^2} - 2 \right) \frac{f'(U)F[U]}{f(U)^2} \right\} \frac{f'(U)F[U]}{f(U)^2} \frac{(2N - 4\gamma)e^{-2t}}{F[U]} e^{z_2} - e^{z_1}.
\]
Since $f'(u_2)F[u_2]/f(u_2)^2 \to \gamma$, by Corollary 4.11 we see that
\begin{equation}
A(t) \to a \text{ as } t \to \infty.
\end{equation}

Since $\bar{u}$ is between $u_2$ and $u_1$, we see that $\bar{u}(r) \to 0$ as $r \to 0$, and hence
\begin{equation}
1 + \left( \frac{f(\bar{u})f''(\bar{u})}{f'(\bar{u})^2} - 2 \right) \frac{f'(\bar{u})F[\bar{u}]}{f(\bar{u})^2} \to 1 + \left( \frac{1}{q} - 2 \right) \gamma = 0 \text{ as } r \to 0.
\end{equation}

By Corollary 4.11 we see that $F[u_j(r)] = e^{-2t(1 + o(1))/(2N - 4\gamma)}$ as $t \to \infty$. Since $F$ is monotone, we see that $F[\bar{u}] = e^{-2t(1 + o(1))/(2N - 4\gamma)}$ as $t \to \infty$. Then, we see
\begin{equation}
B(t) \to b \text{ as } t \to \infty.
\end{equation}

Because of (4.31), (4.32) and (4.33), by Proposition 4.12 we see that $z(t) \equiv 0$, and hence the conclusion holds. \hfill \Box

**Proof of Theorem 1.1 (i) Uniqueness and convergence (1.5).** The existence of the degenerate solution is already proved in Section 3. The uniqueness of a degenerate solution of (1.1) follows from Theorem 4.13. The convergence (1.5) follows from the uniqueness of the degenerate solution and (3.9). By Lemma 4.1 (ii) the degenerate solution is not of class $C^1$ at $r = 0$. The proof of Theorem 1.1 (i) is now complete. \hfill \Box

5. **Proof of Theorem 1.1 (ii)**

Let $u(r, \alpha)$ be a solution of (1.4). We define
\begin{equation}
\tilde{u}(s) := F_q^{-1} \left[ \lambda^{-2} F[u(r, \alpha)] \right], \quad s := \frac{r}{\lambda} \quad \text{and} \quad \lambda := \sqrt{\frac{F[\alpha]}{F_q[1]}}.
\end{equation}

Then $\tilde{u}(s)$ satisfies
\begin{equation}
\begin{cases}
\tilde{u}''(s) + \frac{N-1}{s} \tilde{u}'(s) - \frac{1}{F_q(\tilde{u}(s))} \frac{f_q(\tilde{u}(s))}{f_q[1]} \left( \gamma - \frac{f'([u(r)])F[u(r)]}{f(u(r))^2} \right) \tilde{u}'(s)^2 = 0, & s > 0, \\
\tilde{u}(0) = 1, \\
\tilde{u}'(0) = 0.
\end{cases}
\end{equation}

**Lemma 5.1.** Let $v(s, 1)$ be a solution of (1.7) with $\beta = 1$. Then,
\[ \tilde{u}(s) \to v(s, 1) \text{ in } C_{\text{loc}}[0, \infty) \text{ as } \alpha \to 0. \]

**Proof.** Let $s_0 > 0$ be fixed. We show that
\begin{equation}
u(\lambda s, \alpha) \to 0 \text{ uniformly in } s \in [0, s_0] \text{ as } \alpha \to 0.
\end{equation}

Let $\delta > 0$ be small. By Lemma 3.3 (ii) there exists $r_\delta > 0$ such that
\begin{equation}
\text{if } 0 < \alpha < \delta/\eta, \text{ then } 0 \leq u(r, \alpha) < \delta \text{ for all } 0 < r < r_\delta,
\end{equation}
where $\eta$ is defined in Lemma 3.3 (ii). If $\alpha > 0$ is small, then $s_0 < r_\delta/\lambda$, because $\lim_{\alpha \to 0} \lambda = \lim_{\alpha \to 0} \sqrt{F[\alpha]/F_q[1]} = 0$. By (5.4) we see that if $\alpha > 0$ is small, then $0 \leq u(\lambda s, \alpha) < \delta$ for $0 \leq s \leq s_0$. Since $\delta > 0$ can be chosen arbitrarily small, we see that (5.3) follows.

By (5.3) we have
\[ \frac{f'(u(\lambda s, \alpha))F[u(\lambda s, \alpha)]}{f(u(\lambda s, \alpha))^2} \to \gamma \text{ uniformly in } s \in [0, s_0] \text{ as } \alpha \to 0. \]
Clearly $F_q[\tilde{u}] \geq F_q[1] > 0$ and the denominator $F_q[\tilde{u}]$ in (5.2) is uniformly bounded away from 0. Because of the continuity of $\tilde{u}(s) \in C[0, s_0]$ with respect to the nonlinearity in (5.2), we see that $\tilde{u}(s) \rightarrow v(s, 1)$ in $C[0, s_0]$ as $\alpha \rightarrow 0$. Since $s_0 > 0$ can be chosen arbitrarily large, the conclusion follows. \hfill \Box

Lemma 5.2. Let $u^*(r)$ be the degenerate solution given by Theorem 1.1, and let $q$ and $\lambda$ be defined by (5.1). Let $\tilde{u}^*(s) := F_q^{-1}\left[\lambda^{-2}F[u^*(r)]\right]$. Then

$$
\tilde{u}^*(s) \rightarrow v^*(s) \text{ in } C_{loc}(0, \infty) \text{ as } \alpha \rightarrow 0,
$$

where $v^*(s) = F_q^{-1}[s^2/(2N - 4\gamma)]$ which is defined by (1.3).

Proof. Let $\kappa := (2N - 4\gamma)^{-1}$. There exists a continuous function $\theta(r)$ such that $u^*(r) = F^{-1}[\kappa r^2(1 + \theta(r))]$ and $\theta(r) \rightarrow 0$ as $r \rightarrow 0$. We have

$$
\tilde{u}^*(s) = F_q^{-1}\left[\lambda^{-2}F[F^{-1}[\kappa r^2(1 + \theta(s))]\right] = F_q^{-1}[\kappa s^2(1 + \theta(s))].
$$

Let $0 < s_0 < s_1$ be fixed. Since the same transformation is applied to both $\tilde{Z}$ and $\tilde{u}$, we see that $\tilde{u}(s) \rightarrow v(s, 1)$ in $C[0, s_0]$ as $\alpha \rightarrow 0$. Since $s_0 > 0$ can be chosen arbitrarily large, the conclusion follows. \hfill \Box

In the proof of Theorem 1.1 (ii) we use the following:

**Proposition 5.3.** Let $q_c$ be defined by (1.4). Assume that $0 < \beta_1 < \beta_2$ (resp. $\beta_1 < \beta_2$) if $q > 1$ (resp. if $q = 1$). Let $v(s, \beta_j)$, $j = 1, 2$, be a solution of (1.7) with $\beta = \beta_j$, and let $v^*(s)$ be the degenerate solution given by (1.6). Then the following hold:

(i) If $q \geq q_c$, then $Z_{(0, \infty)}[v(\cdot, \beta_1) - v(\cdot, \beta_2)] = 0$ and $Z_{(0, \infty)}[v(\cdot, \beta_1) - v^*(\cdot)] = 0$.

(ii) If $q < q_c$, then $Z_{(0, \infty)}[v(\cdot, \beta_1) - v(\cdot, \beta_2)] = \infty$ and $Z_{(0, \infty)}[v(\cdot, \beta_1) - v^*(\cdot)] = \infty$.

When $q = 1$, then $w(r) = -v(r)$ satisfies $\Delta w + e^w = 0$. Proposition 5.3 follows from [21, Theorem 1.1]; see also [11]. Note that if $N \geq 10$, then $q_c \leq 1$ and Proposition 5.3 (ii) does not occur. When $q > 1$, see [31, 16] for details of Proposition 5.3.

**Proof of Theorem 1.1 (ii).** Let

$$
\bar{u}(s) := F_q^{-1}[\lambda^{-2}F[u(r, \alpha)]], \quad \tilde{u}^*(s) := F_q^{-1}[\lambda^{-2}F[u^*(r)]], \quad s := \frac{r}{\lambda} \text{ and } \lambda := \sqrt{\frac{F[\alpha]}{F_q[1]}}.
$$

By Lemmas 5.1 and 5.2 we see that

(5.5) \hspace{1cm} \bar{u}(s) \rightarrow v(s, 1) \text{ in } C_{loc}(0, \infty) \text{ as } \alpha \rightarrow 0,

(5.6) \hspace{1cm} \tilde{u}^*(s) \rightarrow v^*(s) \text{ in } C_{loc}(0, \infty) \text{ as } \alpha \rightarrow 0.

By Proposition 5.3 we have

(5.7) \hspace{1cm} Z_{(0, \infty)}[v(\cdot, 1) - v^*(\cdot)] = \infty.

Let $r_0 > 0$ be fixed. Since the same transformation is applied to both $\bar{u}(s)$ and $\tilde{u}^*(s)$, we have $Z_{(0, r_0)}[u(\cdot, \alpha) - u^*(\cdot)] = Z_{(0, r_0/\lambda)}[\bar{u}(\cdot) - \tilde{u}^*(\cdot)]$. For each $M > 0$, there are $s_M > 0$ and $\alpha_M > 0$ such that $Z_{(0, s_M)}[\bar{u}(\cdot) - \tilde{u}^*(\cdot)] \geq M$ for $0 < \alpha < \alpha_M$, because of (5.5), (5.6) and (5.7). If $\alpha > 0$ is small, then $(0, s_M) \subset (0, r_0/\lambda)$, and hence

$$
Z_{(0, r_0)}[u(\cdot, \alpha) - u^*(\cdot)] = Z_{(0, r_0/\lambda)}[\bar{u}(\cdot) - \tilde{u}^*(\cdot)] \geq Z_{(0, s_M)}[\bar{u}(\cdot) - \tilde{u}^*(\cdot)] \geq M.
$$

Since $M$ can be arbitrarily large, we see that $Z_{(0, r_0)}[u(\cdot, \alpha) - u^*(\cdot)] \rightarrow \infty$ as $\alpha \rightarrow 0$. \hfill \Box
6. Bifurcation diagram

Let $(\Lambda, U)$ be a solution of (1.10). Set $u(r) := U(R)$ and $r := \sqrt{\Lambda} R$. Then $u$ satisfies
\[
\begin{aligned}
  & \frac{u''}{r} + \frac{N-1}{r} u' - \frac{1}{f(u)} = 0, \quad 0 < r < \sqrt{\Lambda}, \\
  & u(\sqrt{\Lambda}) = 1, \\
  & 0 \leq u(r) < 1, \quad 0 \leq r < \sqrt{\Lambda}.
\end{aligned}
\]

Proof of Corollary 1.2 (i). By Theorem 1.1 (i) we have that (1.1) has a unique degenerate solution $u^*(r)$. Since $u''(r) > 0$ by Lemma 4.1 (i) and $u^*(r) \geq F^{-1}[r^2/2N]$ by Lemma 4.2 and (F4), there exists a unique $r_0^* > 0$ such that $u^*(r_0^*) = 1$. Then $(\Lambda^*, U^*(r)) := ((r_0^*)^2, 1 - u^*(r_0^* R))$ is a degenerate solution of (1.10).

Let us prove the uniqueness. Suppose that there are two degenerate solutions $(\Lambda_j, U_j^*(r))$, $j = 1, 2$, of (1.10). Then the two functions $u_j(r) := 1 - U_j^*(r/\sqrt{\Lambda_j})$, $j = 1, 2$, are degenerate solutions of (1.1). Because of the uniqueness of the degenerate solution $u^*$ of (1.1), we see that $u^*(r) = 1 - U_j^*(r/\sqrt{\Lambda_j})$. Since $U_j^*(R)$ satisfies the Dirichlet boundary condition, we see that $0 = U_j^*(1) = 1 - u^*(\sqrt{\Lambda_j})$. By the uniqueness of $r_0^*$ we have that $\sqrt{\Lambda_1} = \sqrt{\Lambda_2} = r_0^*$. Hence, $\Lambda_1 = \Lambda_2$. Since $u^*(r) = 1 - U_j^*(r/\sqrt{\Lambda_j})$, we see that $U_j^*(R) = U_2^*(R)$ which shows the uniqueness of a degenerate solution to $(\Lambda^*, U^*(R))$.

We next establish (1.11). Let $\delta > 0$ be fixed. Since $u^''(r) > 0$ and $u^''(r) \geq F^{-1}[r^2/2N]$, there exists $r_3^* > 0$ such that $u^*(r_3^*) = 1 + \delta$. Let $u(r, \alpha)$ be the solution of (1.4) and let $r_0(\tau) > 0$ be the first positive zero of the function $1 - u(\cdot, 1 - \tau)$. Then
\[
(\Lambda(\tau), U(R, \tau)) := (r_0(\tau)^2, 1 - u(r_0(\tau) R, 1 - \tau))
\]
is a solution of (1.10). If $\tau$ is close to 0, then the solution $u(\cdot, 1 - \tau)$ exists in $[0, r_3^*]$ and
\[
(6.1) \quad u(r, 1 - \tau) \to u^*(r) \text{ in } C[0, r_3^*] \text{ and } \tau \to 1.
\]
Since $1 - u^*(r_3^*) < 0$, we see that $r_0(\tau) < r_3^*$ provided that $\tau$ is close to 1. By (6.1) we have $r_0(\tau) \to r_3^*$ as $\tau \to 1$. Since $\Lambda^* = (r_0^*)^2$ and $\Lambda(\tau) = r_0(\tau)^2$, we see that $\Lambda(\tau) \to \Lambda^*$ as $\tau \to 1$. Since $\Lambda(\tau) \to \Lambda^*$, we have
\[
(6.2) \quad |u^*(\sqrt{\Lambda^*} R) - u^*(\sqrt{\Lambda(\tau)} R)| \to 0 \text{ in } C[0, 1] \text{ as } \tau \to 1.
\]
By Theorem 1.1 (i) we have
\[
(6.3) \quad |u(\sqrt{\Lambda(\tau)} R, 1 - \tau) - u^*(\sqrt{\Lambda(\tau)} R)| \to 0 \text{ in } C[0, 1] \text{ as } \tau \to 1.
\]
By (6.2) and (6.3), we have
\[
|U(R, \tau) - U^*(R)| \leq |u^*(\sqrt{\Lambda^*} R) - u^*(\sqrt{\Lambda(\tau)} R)| + |u(\sqrt{\Lambda(\tau)} R, 1 - \tau)| \to 0 \text{ in } C[0, 1] \text{ as } \tau \to 1.
\]
This finishes our proof. \qed
Proof of Corollary 1.2 (ii). Let $\hat{u}(r, \alpha) := U(R, \alpha)$ and $r := \sqrt{\Lambda}$, and let $r_0(\alpha)$ be the first positive zero of $\hat{u}(\cdot, \alpha)$. Then, $\hat{u}$ satisfies

$$
\begin{aligned}
\hat{u}'' + \frac{N-1}{r} \hat{u}' &= \frac{1}{f(1-\hat{u})} \quad 0 < r < r_0(\alpha), \\
\hat{u}(0, \alpha) &= 1 - \alpha, \\
\hat{u}(r_0(\alpha), \alpha) &= 0.
\end{aligned}
$$

(6.4)

Let $\Lambda(\alpha) := r_0(\alpha)^2$. By Theorem 1.1 the equation in (6.4) has a unique degenerate solution $\hat{u}^*(r)$ such that $\hat{u}^*(0) = 1$. Let $r_0^*$ be the first positive zero of $\hat{u}^*(\cdot)$, and let $\Lambda^* := (r_0^*)^2$. Let $r_1 > 0$ be small. By Theorem 1.1 we see that $Z(0, r_1)[\hat{u}(\cdot, \alpha) - \hat{u}^*(\cdot)] \to \infty$ as $\alpha \to 0$. This also implies the oscillation of $\Lambda(\alpha)$ around $\Lambda^*$. The rest of the proof follows the same line as in [17, Lemma 8.1].

□

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