THE CALDERÓN PROBLEM IN THE $L^p$ FRAMEWORK ON RIEMANN SURFACES

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Abstract. The purpose of this article is to extend the uniqueness results for the two dimensional Calderón problem to unbounded potentials on general geometric settings. We prove that the Cauchy data sets for Schrödinger equations uniquely determines potentials in $L^p$ for $p > 4/3$. In doing so, we first recover singularities of the potential, from which point a $L^2$-based method of stationary phase can be applied. Both steps are done via constructions of complex geometric optic solutions and Carleman estimates.

Keywords: Calderón problems, Carleman estimates, Riemann surfaces

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1. Introduction

Let $(M_0, g)$ be a compact Riemannian manifold with smooth boundary $\partial M_0$ and dimension $n \geq 2$. Suppose that $V$ is a function in $L^p(M_0)$ for $p > 1$, and that 0 is not a Dirichlet eigenvalue of $\Delta_g + V$, then the famous Calderón problem for the Schrödinger equation

\begin{equation}
\begin{aligned}
(\Delta_g + V)u &= 0 & \text{in } M_0, \\
u u &= f & \text{on } \partial M_0
\end{aligned}
\end{equation}

(1.1)

asks whether or not the Dirichlet-Neumann map

\[ \Lambda : H^{1/2}(\partial M_0) \to H^{-1/2}(\partial M_0), \]

\[ f \mapsto \partial_{\nu} u_f|_{\partial M_0} \]

uniquely determines the potential $V$, where $\nu$ is the outward pointing unit normal vector field on $\partial M_0$ and $u_f$ solves (1.1) with Dirichlet condition $f$. If $M_0 = \Omega$ is a bounded domain in $\mathbb{R}^n$ with the Euclidean metric, $n \geq 3$ and $V \in C^\infty(\overline{\Omega})$, then the pioneering breakthrough accomplished in [14] by J. Sylvester and G. Uhlmann using the method of Complex Geometric Optic (CGO) solutions gave the positive answer. Since them, a considerable number of results towards this direction have appeared in the literature, with the method of CGO solutions becoming a standard tool in the subject. For dimension $n \geq 3$, D. Dos Santos Ferreira, C.E. Kenig, M. Salo and G. Uhlmann in [7] solved the Calderón problem with smooth potentials on certain admissible Riemannian manifolds which have at least one Euclidean direction. In the $L^p$ framework, S. Chanillo solved the Calderón problem for $V \in L^{n/2}$ on Euclidean bounded domains in [4] and Ferreira-Kenig-Salo on admissible manifolds in [6]. One could also consider the much related partial data problem, where one makes measurement on a specific open subset $\Gamma \subset \partial M_0$ instead of the entire boundary. All of the results mentioned above have their partial data counterparts, see [11, 5, 16].

It is known that the two dimensional case is formally determined and thus notably difficult, with the first uniqueness result by A. Nachman in [12] for a bounded domain $\Omega \subset \mathbb{C}$. For a potential $V \in W^{2,p}(\Omega)$, $p > 2$, the successful implementation of CGO solutions was due to Bukgheim in [1]. This was improved to $V \in L^p(\Omega)$ for $p > 2$ in [2] by E. Blåsten, O.Y. Imanuvilov and M. Yamamoto, and later on E. Blåsten, L. Tzou and J.-N. Wang in [3] did the case of $V \in L^p(\Omega)$ for $p > 4/3$. Not as much is known if $M_0$ is a compact Riemann surface with
smooth boundary. In this case, L. Tzou and C. Guillarou in [18] extended Bukgheim’s method to solve the Calderón problem for a potential \( V \in W^{2,p}(M_0) \) with \( p > 2 \) and partial data. However, their proof relied critically on the fact that \( V \) is continuous. For the partial data problem in Euclidean geometry, O.Y. Imanuvilov, G. Uhlmann and M. Yamamoto obtained this result for \( V \in W^{2,p}(\Omega), \ p \geq 2 \) in [3] [10].

To this day there has not been any work in establishing uniqueness for the Calderón problem on compact Riemann surfaces with smooth boundaries for unbounded potentials. Since the direct problem for the Schrödinger equation is well-posed for all \( V \in L^p(M_0), \ p > 1 \), it is reasonable to ask whether the inverse problem can be solved in this range as well. In this paper we take care of the cases for \( p > 4/3 \). It remains an interesting question to fill the gap for \( p \in [1, 4/3] \). Our main result, formulated in terms of the graph of the Dirichlet-Neumann map, is thus the following:

**Theorem 1.1.** Let \((M_0, g)\) be a compact Riemann surface with smooth boundary \( \partial M_0 \). Assume that \( V_1 \) and \( V_2 \) are two complex valued functions in \( L^p(M_0) \) for \( p > 4/3 \), such that their corresponding Cauchy data sets

\[
\mathcal{C}_j \overset{\text{def}}{=} \{(u|_{\partial M_0}, \partial_n u|_{\partial M_0}) / u \in H^1(M_0), (\Delta_g + V)u = 0\} \subset H^{1/2}(\partial M_0) \times H^{-1/2}(\partial M_0), \quad j = 1, 2
\]

satisfy \( \mathcal{C}_1 = \mathcal{C}_2 \), then \( V_1 = V_2 \).

To prove Theorem 1.1 we will extend the strategy of [3] which follows the philosophy of Bukgheim. There it was important to know a priori that the difference \( V_1 - V_2 \) already lives in \( L^2(M_0) \) in order to apply a generalised method of stationary phase. Hence we also prove the following

**Theorem 1.2.** Let \((M_0, g)\) be a compact Riemann surface with smooth boundary \( \partial M_0 \). Assume that \( V_1 \) and \( V_2 \) are two complex valued functions in \( L^p(M_0) \) for \( p > 4/3 \) such that \( \mathcal{C}_1 = \mathcal{C}_2 \), then \( V_1 - V_2 \) is in \( L^2(M_0) \).

Prior to the work of Nachman, Z. Sun and G. Uhlmann in their work [15] proved various versions of Theorem 1.2 on Euclidean geometry using the methods of higher dimensions. Later in [13], V.S. Serov and L. Päivärinta made improvements and established statements which are parallel to ours. In either cases, the methods of proof in these works relied on tools which had no obvious analogy for general geometries. Our proof of Theorem 1.2 demonstrates how the idea of Bukgheim can be adapted to obtain the same result on a Riemann surface with smooth boundary.

### 2. INHOMOGENEOUS CAUCHY-RIEMANN PROBLEMS

In this section we discuss solutions of the inhomogenous Cauchy-Riemann equation \( \bar{\partial}u = f \) which will be important in the process of constructing Green’s operators to various conjugated operators.

#### 2.1. Riemann Surfaces

We begin by establishing some notations on Riemann surfaces. If \( M_0 \) is a compact Riemann surface with smooth boundary, then we can identify it as the closure of a bounded subset contained in a larger compact Riemann surface \( \overline{M} \) with interior \( M \) and boundary \( \partial M \). The Hodge star operator * acts on the cotangent bundle \( T^*M \) with eigenvalues \( i, -i \) and their respective eigenspaces \( T^*_{1,0} M \) and \( T^*_{0,1} M \). In a holomorphic coordinate \( z = x + iy \) one has \( T^*_{1,0} M = \mathbb{C}dz \) and \( T^*_{0,1} M = \mathbb{C}d\bar{z} \) where \( dz = dx + idy \) and \( d\bar{z} = dx - idy \) and the complexified cotangent bundle admits the splitting \( \mathbb{C}T^*M = T^*_{1,0} M \oplus T^*_{0,1} M \). This splitting induces the natural projections \( \pi_{1,0} : \mathbb{C}T^*M \to T^*_{1,0} M \) and \( \pi : \mathbb{C}T^*M \to T^*_{0,1} M \). We then define the Cauchy-Riemann operators as \( \partial f = \pi_{1,0} df \) and \( \bar{\partial} f = \pi_{0,1} df \) for \( f \in \mathcal{C}^\infty(M) \). If we let
Λ^k M denote the real bundle of k-forms on M and Ξ M denote its complexified bundle, then \( \partial \) and \( \bar{\partial} \) also extend to \( \mathbb{C} \Lambda^1 M \to \mathbb{C} \Lambda^2 M \) by setting \( \partial(\sigma_{1,0} + \sigma_{0,1}) = d\sigma_{1,0} \) and \( \bar{\partial}(\sigma_{1,0} + \sigma_{0,1}) = d\sigma_{1,0} \) if \( \sigma_{1,0} \in T_{1,0}^* M \) and \( \sigma_{0,1} \in T_{0,1}^* M \). They satisfy \( d = \partial + \bar{\partial} \) and are expressed in local coordinates as \( \partial f = \partial_z f dz \) and \( \bar{\partial} f = \bar{\partial}_z f d\bar{z} \) for \( \partial_z = 2^{-1}(\partial_x - i\partial_y) \) and \( \bar{\partial}_z = 2^{-1}(\partial_x + i\partial_y) \) on functions, as well as \( \partial(udz + vdz) = \partial_z vdz + d\bar{z} \) and \( \bar{\partial}(udz + vdz) = \bar{\partial}_z ud\bar{z} \wedge dz \) for forms. The formal adjoints of \( \partial \) and \( \bar{\partial} \) are defined by \( \partial^* = -\ast \bar{\partial} \ast \) and \( \bar{\partial}^* = -\ast \partial \ast \). The Laplacian is defined by \( \Delta f = 2\bar{\partial}\partial \bar{\partial} f = 2\partial^* \partial^* f \). In local coordinates \( z \) we write \( dzd\bar{z} \) to denote the flat volume form \( 2^{-1}idz \wedge d\bar{z} \). If \( M \) has metric \( g \) then we let \( dv_g \) be the volume form of \( M \) with respect to \( g \).

2.2. Inverting the \( \bar{\partial} \) Operator. In this subsection we recall some facts regarding the inhomogeneous \( \bar{\partial} \) equations. For every \( k \in \mathbb{N} \) and \( p \in [1, \infty[ \), it was proved in Proposition 2.3 of [17] that there exists a bounded operator

\[
\bar{T} : W^{k,p}(M; T^*_1 M) \to W^{k+1,p}(M)
\]

such that \( \bar{T} \bar{T} = \text{Id} \). By adapting to the definition of \( \bar{\partial}^* \), we can also extend \( \bar{T} \) to obtain bounded right inverses of

\[
\bar{T}_* : W^{k+1,p}(M; T^*_1 M) \to W^{k,p}(M) \quad \text{and} \quad \bar{T} : W^{k+1,p}(M; T^*_1 M) \to W^{k,p}(M; \Lambda^2 M)
\]

by setting respectively

\[
\bar{T}_* : W^{k,p}(M) \to W^{k+1,p}(M; T^*_1 M) \quad \text{and} \quad \bar{T} : W^{k,p}(M; \Lambda^2 M) \to W^{k+1,p}(M; T^*_1 M)
\]

where \( G : W^{k,p}(M) \to W^{k+2,p}(M) \) by elliptic regularity is the Dirichlet Green’s operator for the Laplacian on \( M \). As it is the usual convention, we adopt the same notations \( \partial \) and \( \bar{T} \) for mappings between forms of various orders since they will be obvious from the contexts. The operators \( T \) and \( T^* \) are to be understood as the complex conjugates of \( \bar{T} \) and \( \bar{T}_* \) respectively.

Moreover, since \( 2\partial^* \partial G = \text{Id} \) and \( \partial^* : W^{k+1,p}(M; T^*_1 M) \to W^{k,p}(M) \) satisfies \( \partial^* = -\ast \bar{\partial} \ast \) for \( \bar{\partial} : W^{k+1,p}(M; T^*_1 M) \to W^{k,p}(M; \Lambda^2 M) \), we have that \( \partial^* T^* = \text{Id} \) and \( \bar{\partial} \bar{T} = \text{Id} \) in \( (2.3) \) as well. We remark that despite the notations, with our constructions the operators \( T \) and \( T^* \) in general are not adjoints of one another.

On a bounded domain \( \Omega \subseteq \mathbb{C} \), the operator \( \bar{T} \) in \( (2.1) \) has a well-known local analogy given by the operator \( \bar{R} \), which is bounded on \( W^{k,p}(\Omega) \to W^{k+1,p}(\Omega) \) and defined by

\[
\bar{R} f \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\Omega} f(\zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z},
\]

and in local coordinates \( (2.4) \) solves \( \partial_\zeta \bar{R} = \text{Id} \). It is obvious that \( \bar{R} \) can naturally be extended to \( W^{k,p}(\Omega; T^*_1 M) \to W^{k+1,p}(\Omega) \) via identification of \( W^{k,p}(\Omega) \) with \( W^{k,p}(\Omega; T^*_1 \Omega) \). As before, we let \( R \) denote the complex conjugate of \( \bar{R} \).

The following results were proved in Section 2 of [19].

**Lemma 2.1.** Let \( \chi \) and \( \chi' \) be smooth cut-off functions on \( M \). Assume that \( \chi \) is supported on a holomorphic chart \( \Omega \subseteq M \) and \( \chi' \) supported on a small open neighbourhood of \( \Omega \) and identically 1 on the support of \( \chi \), then there exists integral operators

\[
K : W^{k,p}(M; T^*_1 M) \to C^\infty(M) \quad \text{and} \quad L : W^{k,p}(M) \to C^\infty(M; T^*_1 M)
\]

respectively with smooth kernels on \( M^2 \) such that

\[
\bar{T} (\chi \sigma) = \chi' \bar{R} (\chi \sigma) + K (\chi \sigma) \quad \text{and} \quad \bar{T}_* (\chi f) = \chi' \bar{R} (|g|^{1/2} \chi f) d\bar{z} + L(|g|^{1/2} \chi f)
\]

for all \( \sigma \in W^{k,p}(M; T^*_1 M) \) and \( f \in W^{k,p}(M) \).
In particular, suppose that \( \{ \Omega_j \}_{j \geq 0} \) is a finite collection of holomorphic charts in \( M \). Let \( \{ \chi_j \}_{j \geq 0} \) be a partition of unity subordinate to \( \{ \Omega_j \}_{j \geq 0} \) such that for every \( j \geq 0 \), we can choose smooth cut-off function \( \chi_j' \) that is identically 1 on the support of \( \chi_j \). We have by (2.5) that

\[
(2.6) \quad \bar{T} \sigma = \sum_{j \geq 0} \chi_j' \bar{R}(\chi_j \sigma) + K_j(\chi_j \sigma) \quad \text{and} \quad \bar{T}^* f = \sum_{j \geq 0} \chi_j' \bar{R}(|g_j|^{1/2} \chi_j f) d\bar{z} + L_j(|g_j|^{1/2} \chi_j f)
\]

for all \( \sigma \in W^{k,p}(M; T_{0,1} M) \) and \( f \in W^{k,p}(M) \) whose supports are contained in \( \bigcup_{j \geq 0} \Omega_j \), where \( \{ K_j \}_{j \geq 0} \) and \( \{ L_j \}_{j \geq 0} \) are operators with smooth kernels.

### 2.3. Special Holomorphic Amplitudes

To solve the inverse problem, we will also be required to consider special holomorphic functions as well as 1-forms on \( M \) parametrised by an auxiliary variable.

Let us introduce some important constructions which will be used throughout this paper. Suppose that \( \tilde{p}_0 \) is an arbitrary point in \( M_0 \). By the result in [8], there exists a holomorphic function \( \Phi : M \to \mathbb{C} \) which can be extended to a larger open Riemann surface \( M' \) containing \( \overline{M} \) with non-vanishing derivative on \( M' \) and remains holomorphic there. Without loss of generality we may assume that \( \Phi(\tilde{p}_0) = 0 \). Thus by the inverse function theorem, for \( r > 0 \) we can choose small neighbourhoods \( \Omega_{0} \subset \tilde{\Omega}_{0} \subset M \) such that \( \Phi : \Omega_{0} \to D_{r} \) and \( \Phi : \tilde{\Omega}_{0} \to D_{2r} \) are biholomorphic maps, where \( D_{r} \subset \mathbb{C} \) is the disk with radius \( r \) centred at the origin.

By compactness we now find finitely many distinct points \( \{ \tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_N \} \subset M_0 \) which form the preimage of \( \Phi|_{\Omega_{0}} \) under \( 0 \), then there exists subsets \( \{ \tilde{\Omega}_{j'} \}_{1 \leq j' \leq N}, \{ \tilde{\Omega}'_{j'} \}_{1 \leq j' \leq N} \) in \( M \) such that \( \tilde{p}_j \in \tilde{\Omega}_{j'} \) and \( \tilde{\Omega}_{j'} \subset \tilde{\Omega}'_{j'} \). Moreover, we can choose them to be such that

\[
\bigcup_{1 \leq j' \leq N} \tilde{\Omega}_{j'} \cap M_0 = \Phi^{-1}_{|M_0} (D_{r}), \quad \Phi : \tilde{\Omega}_{j'} \to D_{r} \text{ is biholomorphic, and}
\]

\[
\bigcup_{1 \leq j' \leq N} \tilde{\Omega}'_{j'} \cap M_0 = \Phi^{-1}_{|M_0} (D_{2r}), \quad \Phi : \tilde{\Omega}'_{j'} \to D_{2r} \text{ is biholomorphic, } 0 \leq j' \leq N.
\]

Thus \( \{ \tilde{\Omega}_{j'} \}_{0 \leq j' \leq N}, \{ \tilde{\Omega}'_{j'} \}_{0 \leq j' \leq N} \) define holomorphic charts on \( M \). Let \( \{ \tilde{\chi}_{j'}, \tilde{\chi}'_{j'} \}_{0 \leq j' \leq N} \) be smooth cut-off functions such that \( \tilde{\Omega}_{j'} \subset \text{Supp} \tilde{\chi}_{j'}, \tilde{\Omega}'_{j'} \subset \text{Supp} \tilde{\chi}'_{j'}, \) and \( \tilde{\chi}_{j'}, \tilde{\chi}'_{j'} \) are identically 1 on these sets. Without loss of generality, we can choose \( \tilde{\chi}'_{j'} \) to be supported in a chart on which \( \Phi \) remains biholomorphic and \( \tilde{\chi}_{j'} \) and is identically 1 on the support of \( \tilde{\chi}'_{j'} \). In particular, for every \( 1 \leq j' \leq N \) we assume that \( \text{Supp} \tilde{\chi}'_{j'} \) contains no other point \( p \in M \) such that \( \Phi(p) \in D_{2r} \).

We now construct our special holomorphic amplitudes. For convenient we will often employ the notation \( \Omega_0 = \Omega_0 \).

**Lemma 2.2.** Let \( \tilde{p}_0 \) be an arbitrary point in \( M_0 \) for which \( \Phi(\tilde{p}_0) = 0 \). Suppose that \( \Omega_0 \subset M \) is a neighbourhood of \( \tilde{p}_0 \) such that \( \Phi : \Omega_0 \to D_{r} \) is biholomorphic for some \( r > 0 \). Let \( \{ \tilde{\chi}_{j'}, \tilde{\chi}'_{j'} \}_{0 \leq j' \leq N} \) be the holomorphic charts chosen above with coordinates \( \{ \tilde{\chi}_{j'} \}_{0 \leq j' \leq N} \), then the followings are true:

- For every \( p_0 \in \tilde{\Omega}_0 \) there exists a smooth function \( \tilde{a}(p; p_0) \) in \( p \in M \) such that the function defined by

\[
(2.7) \quad a(p; p_0) \overset{\text{def}}{=} \tilde{\chi}_0(p) \left( \Phi(p) - \Phi(p_0) \right) \tilde{a}(p; p_0)
\]

is holomorphic on \( M \times \tilde{\Omega}_0 \).
- For every $p_0 \in \tilde{\Omega}'_0$ there exists smooth $(1,0)$-forms $\tilde{b}_{j'}(p; p_0)$ in $p \in M$ such that the forms defined by

\begin{equation}
\tilde{b}_{j'}(p; p_0) \overset{\text{def}}{=} \tilde{\chi}'_{j'}(p)dz_{j'} + (\Phi(p) - \Phi(p_0))\tilde{b}_{j'}(p; p_0), \quad 0 \leq j' \leq N
\end{equation}

are holomorphic on $M$. Moreover, on every holomorphic chart $\Omega$ of $M$, the coefficients of $\tilde{b}_{j'}$ are holomorphic on $\Omega \times \tilde{\Omega}'_0$.

**Proof.** For every $p_0 \in \tilde{\Omega}'_0$ we let $N(p_0)$ be the finite set of points $p \in M$ such that $\Phi(p) = \Phi(p_0)$ by compactness, then on $M \setminus N(p_0)$ we can carry out the following calculations

\begin{equation}
\tilde{\partial}_p \left( \frac{\tilde{\chi}'_0(p)}{\Phi(p) - \Phi(p_0)} \right) = \tilde{\partial}_p \tilde{\chi}'_0(p) \quad \text{and} \quad \tilde{\partial}_p \left( \frac{\tilde{\chi}'_0(p)dz_{j'}}{\Phi(p) - \Phi(p_0)} \right) = \tilde{\partial}_p(\tilde{\chi}'_0(p)dz_{j'})/\Phi(p) - \Phi(p_0).
\end{equation}

Suppose that $\Phi(p) = \Phi(p_0)$, then by construction we must have either $p \in \tilde{\Omega}'_j$ for some $0 \leq j' \leq N$, or $p$ is contained in sets away from the supports of $\tilde{\chi}'_{j'}$ for all $j'$. Since $\partial_{\xi} \tilde{\chi}'_0$ vanishes identically on all such sets, the terms in (2.9) extend respectively to a smooth holomorphic 1-form and a 2-form on $M$ via extension by zero. Thanks to the existence in (2.1) and (2.3), we can choose $\tilde{a}$ and $\tilde{b}$ by

\begin{equation}
\tilde{a}(p; p_0) \overset{\text{def}}{=} \tilde{T}_p \tilde{\partial}_p' \left( \frac{\tilde{\chi}'_0(p')}{\Phi(p') - \Phi(p_0)} \right) \quad \text{and} \quad \tilde{b}_{j'}(p; p_0) \overset{\text{def}}{=} \tilde{T}_p \tilde{\partial}_p' \left( \frac{\tilde{\chi}'_{j'}(p')dz_{j'}}{\Phi(p') - \Phi(p_0)} \right).
\end{equation}

By the smoothness of (2.9) and the boundedness of $\tilde{T}$ and $T$ we deduce that both $\tilde{a}(\cdot; p_0)$ and $\tilde{b}(\cdot; p_0)$ are smooth for every $p_0 \in \tilde{\Omega}'_0$. Therefore, the prescriptions

\begin{equation}
\begin{aligned}
a(p; p_0) &= (\Phi(p) - \Phi(p_0)) \left( \frac{\tilde{\chi}'_0(p)}{\Phi(p) - \Phi(p_0)} - \tilde{a}(p; p_0) \right) = \tilde{\chi}'_0(p) - (\Phi(p) - \Phi(p_0))\tilde{a}(p; p_0), \\
b_{j'}(p; p_0) &= (\Phi(p) - \Phi(p_0)) \left( \frac{\tilde{\chi}'_{j'}(p)dz_{j'}}{\Phi(p) - \Phi(p_0)} - \tilde{b}_{j'}(p; p_0) \right) = \tilde{\chi}'_{j'}(p)dz_{j'} - (\Phi(p) - \Phi(p_0))\tilde{b}_{j'}(p; p_0)
\end{aligned}
\end{equation}

define respectively a holomorphic function and $(1,0)$-forms in $p \in M$ as required.

It remains to show that $\tilde{a}$ and the coefficients of $\tilde{b}_{j'}$ are holomorphic in $p_0 \in \Omega_0$, but it suffices to observe directly from (2.10) combined with Lemma 2.1 that for some locally supported smooth functions $\tilde{\chi}'_0$ and $\tilde{\chi}'_{j'}$, we can write $\tilde{a}$ and $\tilde{b}$ as

\begin{equation}
\begin{aligned}
\tilde{a}(p; p_0) &= (\tilde{\chi}'_0(p)\tilde{R}_{p'} + K_{p'}) \left( \frac{\tilde{\partial}_p'\tilde{\chi}_0(p')}{\Phi(p') - \Phi(p_0)} \right), \\
\tilde{b}_{j'}(p; p_0) &= -(\tilde{\chi}'_{j'}(p)\tilde{R}_{p'} + L_{p'}) \left( \frac{|g_{j'}|^{1/2} \tilde{\partial}_p'(\tilde{\chi}_{j'}(p')dz_{j'})}{\Phi(p') - \Phi(p_0)} \right), \quad 0 \leq j' \leq N
\end{aligned}
\end{equation}

where $K$ and $L$ are operators with smooth kernels. In particular, these are linear combinations of absolutely convergent integrals in the variable $p'$ since $\Phi(p') \neq \Phi(p_0)$ for any $p_0 \in \tilde{\Omega}'_0$ on the supports of their integrands, thus we can apply Lebesgue’s differentiation theorem to conclude that $\tilde{a}$ and $\tilde{b}$ depend holomorphically on $p_0 \in \tilde{\Omega}'_0$. The claim now follows directly from the structures of $a$ and $b_{j'}$. \hfill $\Box$

**Lemma 2.3.** Let $a$ and $\{b_{j'}\}_{0 \leq j' \leq N}$ be the holomorphic functions and $(1,0)$-forms constructed in Lemma 2.2. Suppose that $\{\Omega_j\}_{j \geq 0}$ is a holomorphic atlas of $M_0$ in $M$ such that $\Phi:\Omega_j \rightarrow\Phi(\Omega_j)$ is biholomorphic for each $j \geq 0$ and that $\Phi(\Omega_j)$ is sufficiently small. If $\{\chi_j\}_{j \geq 0}$ is a partition of unity subordinate to $\{\Omega_j\}_{j \geq 0}$ with coordinate $z_j$ and let $b_{j'}^{(j)}$ be the coefficients of...
by \( b_j \) in the coordinates \( z_j \), then there exists Taylor expansion
\[
a(p;p_0) = \sum_{\mu \geq 0} a_{\mu}(p) z_0^\mu \quad \text{and} \quad \\
\chi_j(p) b_j^{(j)}(p;p_0) = \sum_{\mu \geq 0} \chi_j(z) b_j^{(j)}(z) z_0^\mu
\]
for every \( z_0 \in D_r \) under the changes of variables \( \Phi_{|z_0}(p) = z_0 \) and \( \Phi_{|z_j}(p) = z_j \), where \( a_\mu \) is holomorphic in \( M \) and \( b_j^{(j)} \) is holomorphic in \( \Omega_j \) for every \( \mu \geq 0 \) and \( j,j' \geq 0 \). Moreover,
\[
\sup_{z_0 \in D_r} \sum_{\mu \geq 0} \|a_\mu\|_{L^\infty(M_0)} |z_0|^\mu \leq \sum_{\mu \geq 0} \|a_\mu\|_{L^\infty(M_0)} r^\mu < \infty, \quad \text{and} \\
\sup_{z_0 \in D_r} \sum_{\mu \geq 0} \|b_j^{(j)}\|_{L^\infty(\Omega_j)} |z_0|^\mu \leq \sum_{\mu \geq 0} \|b_j^{(j)}\|_{L^\infty(\Omega_j)} r^\mu < \infty
\]
for every \( j,j' \geq 0 \).

**Proof.** By Lemma 2.2, the local expression of \( a(p;\cdot) \) in the chart \( \tilde{\Omega}_j \) is holomorphic on \( D_{2r} \) so that its radius of convergence is greater than \( r \). Since \( a_\mu = \partial_{z_0} a(p;0)/\mu! \) we certainly have that \( a_\mu \) is holomorphic on \( M \). We can now cover \( M_0 \) by the finite collection of holomorphic charts \( \{\Omega_j, \chi_j\}_{j \geq 0} \) and assume that \( \Phi(\Omega_j) = \tilde{D}_{r_j} \) is a disk of radius \( r_j > 0 \) for every \( j \geq 0 \). In particular, we can assume that the local representation of \( a \) is holomorphic on \( \tilde{D}_{2r_j} \times D_{2r} \).

We obviously have
\[
\sup_{z_0 \in D_r} \sum_{\mu \geq 0} \|a_\mu\|_{L^\infty(M)} |z_0|^\mu \leq \sum_{j \geq 0} \sum_{\mu \geq 0} \|\chi_j a_\mu\|_{L^\infty(\Omega_j)} r^\mu.
\]
By standard power series theory, there exists constants \( C_j > 0 \) such that
\[
\sum_{\mu \geq 0} \|\chi_j a_\mu\|_{L^\infty(\Omega_j)} r^\mu \leq \sum_{\mu,\nu \geq 0} |a_{\mu,\nu}^j| r^\mu r_j^{\nu} \leq C_j < \infty
\]
where \( a_{\mu,\nu}^j \) are the Taylor coefficients of \( a_\mu \) in the chart \( \Omega_j \). Summing over \( j \) proves the first part of (2.14). The case of \( b_j \) proceeds similarly. \( \square \)

### 2.4. Inverting the Conjugated Operators.
Suppose that \( \Phi \) is the holomorphic function without critical point chosen in subsection 2.3 and let \( \varphi, \phi \) be respectively the real and imaginary parts of \( \Phi \). For every \( p_0 \in M \), we set \( \Psi(p;p_0) = (\Phi(p) - \Phi(p_0))^2 \) and let \( \psi \) be the real part of \( \Psi \). Assume also that \( \omega \) is a complex vector in \( \mathbb{C} \) such that \( |\omega|, \lambda > 0 \). In this subsection we establish the conjugated operators that will be important in constructing special solutions to the Schrödinger equations. We first consider the conjugated Laplacians
\[
P_\psi \defeq e^{-i\psi\lambda} \Delta e^{i\psi\lambda}, \quad P_\Phi \defeq e^{-i\Phi\omega} \Delta e^{i\Phi\omega}, \quad P_\psi \defeq e^{+\Phi\omega} \Delta e^{-\Phi\omega}, \quad P_\Phi \defeq e^{+\psi\lambda} \Delta e^{-\psi\lambda}.
\]
In the same way we may look at conjugated inverses to the Cauchy-Riemann operators
\[
T_\psi \defeq \frac{1}{2} e^{-2i\psi\lambda} Te^{2i\psi\lambda}, \quad T_\Phi \defeq \frac{1}{2} e^{-2i\Phi\omega} Te^{2i\Phi\omega}, \quad T_\psi \defeq \frac{1}{2} e^{+i\Phi\omega} Te^{-i\Phi\omega}, \quad T_\Phi \defeq \frac{1}{2} e^{+i\psi\lambda} Te^{-i\psi\lambda}.
\]
Let \( M'_0 \) be a bounded domain in \( M \) with smooth boundary such that \( M_0 \subset \overline{M'_0} \subset M \). We can choose \( \tilde{\rho} \in C^\infty(M'_0) \) such that \( \tilde{\rho} \) is identically 1 near \( M_0 \). In view of the factorisation \( \Delta_g = 2\partial^* \bar{\partial} = 2\bar{\partial}^* \partial \), it is clear that
\[
P_\psi T_\psi \tilde{\rho} T^* = P_\psi \tilde{T}_\psi \tilde{\rho} T^* = \text{Id} \quad \text{and} \quad P_\Phi T_\Phi \tilde{\rho} T^* = P_\Phi \tilde{T}_\Phi \tilde{\rho} T^* = \text{Id}
\]
on \( W^{k,p}(M_0) \).


3. Carleman Estimates and Complex Geometric Optic Solutions

A standard procedure in solving the Calderón problem is based on the Alessandrini's integral identity. For two potentials $V_1$ and $V_2$ in $L^p(M_0)$ and $p > 1$, it is well known that if $C_1 = C_2$, then

\begin{equation}
\int_{M_0} u V_1 v dV_g = 0, \quad V \overset{\text{def}}{=} V_1 - V_2
\end{equation}

for all $H^1(M_0)$ solutions $u \in \text{Ker} (\Delta_g + V_1)$ and $v \in \text{Ker} (\Delta_g + V_2)$. Thus the orthogonality (3.1) implies that the identification of $V_1 = V_2$ hinges on the extent to which we can find such solutions. In this section we construct various special CGO solutions to the Schrödinger equation, which will enable us to solve the Calderón problem from (3.1).

Let $a$ be the holomorphic function constructed in subsection 2.3 The CGO solutions we look for will be of the forms

\[ u = e^{i\Psi \lambda}(a + r), \quad v = e^{i\Psi \lambda}(\bar{a} + s), \quad \tilde{u} = e^{-\frac{i}{2} \Phi \omega}(a + \tilde{r}), \quad \tilde{v} = e^{-\frac{i}{2} \Phi \omega}(\bar{a} + \tilde{s}) \]

for sufficiently large $\lambda$ and $\omega$. Since $e^{i\Psi \lambda} a$ and $e^{-\frac{i}{2} \Phi \omega} a$ are holomorphic, $e^{i\Psi \lambda} \bar{a}$ and $e^{-\frac{i}{2} \Phi \omega} \bar{a}$ are antiholomorphic, they must also be harmonics. Thus the conditions we impose on the remainders should be

\begin{equation}
P_{\Phi} r = -V_1(a + r), \quad P_{\Phi} s = -V_2(a + s), \quad P_{\Phi} \tilde{r} = -V_2(a + \tilde{r}), \quad P_{\Phi} \tilde{s} = -V_2(a + \tilde{s}).
\end{equation}

Corresponding to any $\tilde{p}_0 \in M_0$ we recall the construction of holomorphic $(1, 0)$-forms $b_j^+$ for $1 \leq j \leq N$ in Lemma 2.2. Let $\{Q_j^+\}_{0 \leq j' \leq N}$ and $\{Q_j^-\}_{0 \leq j' \leq N}$ be sequences of $C^\infty_c(M)$ functions which will be chosen depending on $\epsilon > 0$ later. We set

\[ u_1 \overset{\text{def}}{=} T_{\Phi} \tilde{p} \left( T^* V_1 a - \sum_{1 \leq j' \leq N} Q_j^+ \rho(p_0) b_j^+ \right), \quad \tilde{u}_1 \overset{\text{def}}{=} T_{\Phi} \tilde{p} T^* V_1 a, \]

\[ v_1 \overset{\text{def}}{=} T_{\Phi} \tilde{p} \left( T^* V_2 \bar{a} - \sum_{1 \leq j' \leq N} Q_j^- \rho(p_0) b_j^- \right), \quad \tilde{v}_1 \overset{\text{def}}{=} T_{\Phi} \tilde{p} T^* V_2 \bar{a}. \]

One then easily observes from (3.2) and a direct computation that if

\[ u_j \overset{\text{def}}{=} T_{\Phi} \tilde{p} \left( T^* V_1 u_{j-1} \right), \quad \tilde{u}_j \overset{\text{def}}{=} T_{\Phi} \tilde{p} T^* (V_1 \tilde{u}_{j-1}), \]

\[ v_j \overset{\text{def}}{=} T_{\Phi} \tilde{p} \left( T^* V_2 v_{j-1} \right), \quad \tilde{v}_j \overset{\text{def}}{=} T_{\Phi} \tilde{p} T^* (V_2 \tilde{v}_{j-1}) \]

for all $j \geq 0$, then the functions defined by

\begin{equation}
r \overset{\text{def}}{=} \sum_{j \geq 1} (-1)^j u_j, \quad \tilde{r} \overset{\text{def}}{=} \sum_{j \geq 1} (-1)^j \tilde{u}_j, \quad s \overset{\text{def}}{=} \sum_{j \geq 1} (-1)^j v_j, \quad \tilde{s} \overset{\text{def}}{=} \sum_{j \geq 1} (-1)^j \tilde{v}_j
\end{equation}

will formally satisfy conditions (3.2). We complement these definitions by setting $u_0 = \tilde{u}_0 = a$ and $v_0 = \tilde{v}_0 = \bar{a}$. We elaborate on these findings in the followings.

**Proposition 3.1.** Let $V_1, V_2$ be in $L^p(M_0)$ for $p > 4/3$, then there exists $\lambda_0 > 0$ and $\omega_0 \in \mathbb{C}$ such that for all $\lambda > \lambda_0$ and $|\omega| > |\omega_0|$, we can find functions $u, v, \tilde{u}, \tilde{v}$ in $L^\infty(M_0) \cap H^1(M_0)$ of the forms

\begin{equation}
u = \sum_{j \geq 0} (-1)^j e^{i\Psi \lambda} u_j, \quad \tilde{u} = \sum_{j \geq 0} (-1)^j e^{-\frac{i}{2} \Phi \omega} \tilde{u}_j,
\end{equation}

\begin{equation}
v = \sum_{j \geq 0} (-1)^j e^{i\Psi \lambda} u_j, \quad \tilde{v} = \sum_{j \geq 0} (-1)^j e^{-\frac{i}{2} \Phi \omega} \tilde{v}_j,
\end{equation}

with $u, \tilde{u}$ belonging to $\text{Ker} (\Delta_g + V_1)$ and $v, \tilde{v}$ belonging to $\text{Ker} (\Delta_g + V_2)$.
Proof. We have shown that the functions defined by (3.4) satisfy formally their corresponding Schrödinger equations. Thus it suffices to show that the series converge in appropriate spaces. We do so by first showing that they satisfy sufficiently nice asymptotic behaviour. These will be done via the following estimates.

Recall the surface $M_0'$ defined in Subsection 2.4.

**Proposition 3.2.** Assume that $(p, q, r)$ is in $]4/3, 2[ \times ]4, \infty[ \times ]2, \infty[ \times ]2 + 1/q \geq 1/p > 1/2$, then there exists a constant $C > 0$ independent of $p_0 \in \Omega_0$ such that

\[
(3.5) \quad \|T_\psi \sigma\|_{L^q(M_0')} \leq \frac{C \|\sigma\|_{W^{1,p}(M; T_{0,1}^* M)}}{\lambda^{1-\left(\frac{1}{p} - \frac{1}{q}\right)}} , \quad \|T_\psi \sigma\|_{L^\infty(M_0')} \leq \frac{C \|\sigma\|_{W^{1,r}(M; T_{0,1}^* M)}}{\lambda^{\frac{r}{2}}}
\]

for all $\sigma \in W_0^{1,p}(M_0'; T_{0,1}^* M_0')$. Alternatively, if $(p, q)$ is in $]4/3, 2[ \times ]2, 4[ \times ]1/p + 1/q = 1$, then we have

\[
(3.6) \quad \|T_\psi \sigma\|_{L^q(M_0')} \leq \frac{C \|\sigma\|_{W^{1,p}(M; T_{0,1}^* M)}}{\omega} , \quad \|T_\psi \sigma\|_{L^\infty(M_0')} \leq \frac{C \|\sigma\|_{W^{1,r}(M; T_{0,1}^* M)}}{\omega}
\]

for all $\sigma \in W_0^{1,p}(M_0'; T_{0,1}^* M_0')$.

By equation (2.7), it is sufficient to prove (3.5) and (3.6) locally for $R$ and globally for an operator $\mathcal{K}$ with smooth kernel. Moreover, by density we may assume that $\sigma \in C_0^\infty(M_0'; T_{0,1}^* M_0')$, and so by partition of unity and the fact that $\Phi$ has non-vanishing derivative on $M$, it is enough to assume that the support of $\sigma$ is compactly contained in a holomorphic chart $\Omega \subset M$ on which $\Phi_{\Omega}(p) = \lambda$.

Let us first consider the local case, the proof of which we partially recall from [3].

**Lemma 3.1.** Assume that $(p, q, r)$ is in $]4/3, 2[ \times ]4, \infty[ \times ]2, \infty[ \times ]2 + 1/q \geq 1/p > 1/2$, then there exists a constant $C > 0$ independent of $p_0 \in \Omega_0$ such that

\[
(3.7) \quad \|\text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda f\|_{L^p(\Omega)} \leq \frac{C \|f\|_{W^{1,p}(\Omega)}}{\lambda^{1-\left(\frac{1}{p} - \frac{1}{q}\right)}} , \quad \|\text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda f\|_{L^\infty(\Omega)} \leq \frac{C \|f\|_{W^{1,r}(\Omega)}}{\lambda^{\frac{r}{2}}}
\]

for all $f \in C_0^\infty(\Omega)$. Alternatively, if $(p, q)$ is in $]4/3, 2[ \times ]2, 4[ \times ]1/p + 1/q = 1$, then we have

\[
(3.8) \quad \|\text{Re}^{-iz}\omega f\|_{L^q(\Omega)} \leq \frac{C \|f\|_{W^{1,p}(\Omega)}}{\omega} , \quad \|\text{Re}^{-iz}\omega f\|_{L^\infty(\Omega)} \leq \frac{C \|f\|_{W^{1,r}(\Omega)}}{\omega}
\]

for all $f \in C_0^\infty(\Omega)$.

**Proof.** Let $\chi$ in $C_0^\infty(\mathbb{C})$ be identically 1 for $|z| \geq 2$ and vanishes for all $|z| \leq 1$. Setting $\chi_\lambda = \chi(\lambda^{1/2}(z - z_0))$, we can write

\[
(3.9) \quad \text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda f = \text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda \chi f + \text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda (1 - \chi_\lambda) f.
\]

Assume first $1/2 + 1/q \geq 1/p > 1/2$ and set $(p', q', r_1, r_2)$ to be such that

\[
(10) \quad \frac{1}{p'} = \frac{1}{p} - \frac{1}{2}, \quad \frac{1}{q'} = \frac{1}{q} - \frac{1}{2}, \quad \frac{1}{r_1} = \frac{1}{q} - \frac{1}{p'}, \quad \frac{1}{r_2} = \frac{1}{q} - \frac{1}{p'} = \frac{1}{q'} - \frac{1}{p}.
\]

The final term in (3.9) can easily be estimated from Sobolev and Hölder's inequalities

\[
(11) \quad \|\text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda (1 - \chi_\lambda) f\|_{L^{q'}} \lesssim \|\text{Re}^{2i}\text{Re}(z-z_0)^2 \lambda (1 - \chi_\lambda) f\|_{W^{1,q'}} \leq \|f\|_{W^{1,p(x)}} \frac{1}{\lambda^{\frac{1}{2}}} = \frac{C \|f\|_{W^{1,p}}}{\lambda^{\frac{1}{2}}},
\]
and likewise we can get that
\[
\|\tilde{R}e^{2i\text{Re}(z-z_0)^2}\lambda(1-\chi)f\|_{L^\infty} \lesssim \|\tilde{R}e^{2i\text{Re}(z-z_0)^2}\lambda(1-\chi)f\|_{W^{1,r}}
\]
\[
\lesssim \|1-\chi\|_{L^r} \leq \|1-\lambda\|_{L^r} \|f\|_{L^\infty} \lesssim \frac{\|f\|_{W^{1,r}}}{\lambda^r}.
\]
For the first term in the expansion (3.9), we integrate by parts to see that
\[
\|\tilde{R}e^{2i\text{Re}(z-z_0)^2}\lambda\chi f\|_{L^q} \lesssim \frac{1}{\lambda} \left( \|\frac{\lambda f}{z-z_0}\|_{L^q} + \|f\partial_z f\|_{L^{r'}} + \|\lambda \partial_z f\|_{L^{r'}} \right).
\]
It follows again from boundedness of \(\tilde{R}\) that
\[
\|\tilde{R}e^{2i\text{Re}(z-z_0)^2}\lambda\chi f\|_{L^q} \lesssim \frac{1}{\lambda} \left( \|\frac{\lambda f}{z-z_0}\|_{L^q} + \|f\partial_z f\|_{L^{r'}} + \|\lambda \partial_z f\|_{L^{r'}} \right).
\]
Since \(\lambda\) is supported away from the set on which \(z = z_0\), the first term on the right hand side in (3.14) can be estimated by
\[
\|\frac{\lambda f}{z-z_0}\|_{L^q} \lesssim \|\frac{\lambda f}{z-z_0}\|_{L^{r'}} \lesssim \frac{\|f\|_{W^{1,p}}}{\lambda^{r-2}} = \frac{\|f\|_{W^{1,p}}}{\lambda^r - \frac{2}{p}}.
\]
and for the last term, we have
\[
\|\lambda \partial_z f\|_{L^{r'}} \lesssim \|\frac{\lambda f}{z-z_0}\|_{L^{r'}} \|\partial_z f\|_{L^p} \lesssim \frac{\|f\|_{W^{1,p}}}{\lambda^{r-2}} = \frac{\|f\|_{W^{1,p}}}{\lambda^r - \frac{2}{p}}.
\]
Putting everything back into (3.14) and combined with (3.11), we arrive at the first estimate in (3.7). Applying the same strategy with respect to the supremum norm, we also have from (3.13) that
\[
\|\tilde{R}e^{2i\text{Re}(z-z_0)^2}\lambda\chi f\|_{L^\infty} \leq \frac{1}{\lambda} \left( \|\frac{\lambda f}{z-z_0}\|_{L^\infty} + \|f\partial_z f\|_{L^{r'}} + \|\lambda \partial_z f\|_{L^{r'}} \right)
\]
\[
\leq \frac{1}{\lambda} \left( \|\frac{\lambda f}{z-z_0}\|_{L^\infty} + \|f\partial_z f\|_{L^{r'}} + \|\lambda \partial_z f\|_{L^{r'}} \right) \lesssim \frac{\|f\|_{W^{1,r}}}{\lambda^r}.
\]
Combining the above with inequality (3.12) yields the second estimate in (3.7).

To obtain (3.8), it is enough to note that in this case, the identity
\[
\tilde{R}e^{-i\omega}f = \frac{2}{i\omega} \left( e^{-i\omega}f + \tilde{R}(e^{-i\omega}\partial_z f) \right).
\]
holds conveniently without the need for localisation. Since \(p' \geq q\) for all \(p \geq 4/3\) and \(1/p + 1/q = 1\), one can estimate directly using Sobolev’s inequality to get that
\[
\|f\|_{L^q} \lesssim \|f\|_{L^{p'}} \lesssim \|f\|_{W^{1,p}} \quad \text{and} \quad \|\tilde{R}e^{-i\omega}\partial_z f\|_{L^q} \lesssim \|\partial_z f\|_{L^p} \leq \|f\|_{W^{1,p}}.
\]
Combining the above with (3.16) proves the first estimate in (3.8) and the second one follows from the same argument by applying the embedding \(W^{1,r} \hookrightarrow L^\infty\). Notice also that after changing variables and taking modulus, the bounds we obtain are independent of \(z_0\). Thus we have arrived at the required claims.

We now move on to the smoothing terms in (2.6). We do not provide much details because the argument will be analogous to the proof of Lemma 3.1. Since we only consider \(\sigma \in C_c^\infty(M_0', T_{0,1}^1 M_0')\) which are supported on a local chart \(\Omega\), by identifying \(\sigma\) with its coefficients
in this chart, it is sufficient to consider an operator $K : W^{k,p} (\Omega) \to C^\infty (M)$ with smooth kernel, thus we prove

**Lemma 3.2.** Assume that $(p, q, r)$ is in $[4/3, 2[ \times ]4, \infty[ \times ]2, \infty[$ and $1/2 + 1/q \geq 1/p > 1/2$, then there exists a constant $C > 0$ independent of $p_0 \in \Omega_0$ such that

$$
(3.17) \quad \| K e^{2i \text{Re}(z-z_0)^2} f \|_{L^q (M_0)} \leq \frac{C \| f \|_{W^{1,p} (\Omega)}}{\lambda^1 (\frac{1}{p} - \frac{1}{q})}, \quad \| K e^{2i \text{Re}(z-z_0)^2} f \|_{L^\infty (M_0)} \leq \frac{C \| f \|_{W^{1,r} (\Omega)}}{\lambda^1 (\frac{1}{r} - \frac{1}{q})}
$$

for all $f \in C^\infty_c (\Omega)$. Alternatively, if $(p, q, r)$ is in $[4/3, 2[ \times ]2, 4[ \times ]1/p + 1/q = 1$, then we have

$$
(3.18) \quad \| K e^{-i \omega} f \|_{L^q (M_0)} \leq \frac{C \| f \|_{W^{1,p} (\Omega)}}{\| \omega \|}, \quad \| K e^{-i \omega} f \|_{L^\infty (M_0)} \leq \frac{C \| f \|_{W^{1,r} (\Omega)}}{\| \omega \|}
$$

for all $f \in C^\infty_c (\Omega)$.

**Proof.** Let $\chi_\lambda$ be the compactly supported function defined in the proof of Lemma 3.1. Since $K$ has smooth kernel, by Minkowski’s inequality it is obvious that $K$ satisfies the same boundedness properties as $\bar{K}$. For $1/2 + 1/q \geq 1/p > 1/2$ we may split as before to get

$$
(3.19) \quad K e^{2i \text{Re}(z-z_0)^2} f = K e^{2i \text{Re}(z-z_0)^2} \chi_\lambda f + K e^{2i \text{Re}(z-z_0)^2} (1 - \chi_\lambda) f.
$$

The local term in (3.19) clearly satisfies

$$
\| K e^{2i \text{Re}(z-z_0)^2} (1 - \chi_\lambda) f \|_{L^q} \lesssim \frac{\| f \|_{W^{1,p}}}{\lambda^1 (\frac{1}{p} - \frac{1}{q})}.
$$

Integrating by parts, we also see that there exists an operator $K'$ with smooth kernel so that

$$
K e^{2i \text{Re}(z-z_0)^2} \chi_\lambda f = \frac{i}{2 \lambda} K' e^{2i \text{Re}(z-z_0)^2} \chi_\lambda f z - z_0 + \frac{i}{2 \lambda} K e^{2i \text{Re}(z-z_0)^2} \partial_z (\chi_\lambda f z - z_0).
$$

This is analogous to (3.13), so the proof now proceeds exactly as in Lemma 3.1 and we have

$$
\| K e^{2i \text{Re}(z-z_0)^2} \chi_\lambda f \|_{L^q} \lesssim \frac{\| f \|_{W^{1,p}}}{\lambda^1 (\frac{1}{p} - \frac{1}{q})}
$$

as well. The other claims follow similarly. $\square$

**Proof of Proposition 3.2.** By compactness and the fact that $\Phi$ has non-vanishing derivative on $M$, we can find a finite collection of holomorphic charts $\{ \Omega_j \}_{j \geq 0}$ in $M$ which covers $M'$, such that $\Phi|_{\Omega_j} (p) = z_j$ defines holomorphic coordinates on $\Omega_j$ for each $j \geq 0$. Let $\{ \lambda_j \}_{j \geq 0}$ be a partition of unity subordinate to $\{ \Omega_j \}_{j \geq 0}$ and choose $\{ \lambda_j' \}_{j \geq 0}$ so that for each $j \geq 0$, $\lambda_j'$ is supported in a neighbourhood of $\Omega_j$ on which $\Phi$ remains biholomorphic, and $\lambda_j'$ is identically 1 on the support of $\lambda_j$, then by (2.6) we have that

$$
e^{-2i \psi} T e^{2i \psi} \sigma = \sum_{j \geq 0} e^{-2i \psi} \lambda_j' e^{2i \text{Re}(z-z_0)^2} \lambda_j \sigma + e^{-2i \psi} K_j e^{2i \text{Re}(z-z_0)^2} \lambda_j \sigma$$

and

$$
e^{\Phi} T e^{-i \Phi} \omega = \sum_{j \geq 0} e^{i \omega} \lambda_j' e^{-i \omega} \lambda_j \sigma + e^{i \omega} K_j e^{-i \omega} \lambda_j \sigma$$

for all $\sigma \in C^\infty (M'_0; T_{0,1} M'_0)$. We now apply Lemma 3.1 and Lemma 3.2 to get the required claims by density. $\square$

We also deduce from Proposition 3.2 the following Carleman estimates.

**Corollary 3.1.** Assume that $(p, q, r)$ is in $[4/3, 2[ \times ]4, \infty[ \times ]2, \infty[$ and $1/2 + 1/q \geq 1/p > 1/2$, then there exists a constant $C > 0$ independent of $p_0 \in \Omega_0$ such that

$$
(3.20) \quad \| \bar{T} \psi \bar{T}^* f \|_{L^q (M_0)} \leq \frac{C \| f \|_{L^p (M_0)}}{\lambda^1 (\frac{1}{p} - \frac{1}{q})}, \quad \| \bar{T} \psi \bar{T}^* f \|_{L^\infty (M_0)} \leq \frac{C \| f \|_{L^p (M_0)}}{\lambda^0 (\frac{1}{p} - \frac{1}{q})}
$$


for all $f \in L^p(M_0)$. Alternatively, if $(p, q)$ is in $[4/3, 2[ \times ]2, 4[ $ and $1/p + 1/q = 1$, then we have

$$
(3.21) \quad \|T_\theta \tilde{p} T^* f\|_{L^q(M_0)} \leq \frac{C\|f\|_{L^p(M_0)}}{|\omega|^0}, \quad \|T_\theta \tilde{p} T^* f\|_{L^\infty(M_0)} \leq \frac{C\|f\|_{L^p(M_0)}}{|\omega|^0+}
$$

for all $f \in L^p(M_0)$.

Here we adopt the notation $\alpha+$ to denote $\alpha + \epsilon$ for some $\epsilon > 0$ whenever $\alpha$ is real.

**Proof.** Suppose first that $f \in C^\infty_c(M_0)$, then $\tilde{p} T^* f \in C^\infty_c(M_0; T_0^1 M_0)$, thus the proofs for the $L^p \to L^q$ estimates in this case are obvious from Proposition 3.2. On the other hand, we set $(p_0, p_1)$ to be such that $1 < p_0 < 4/3 < p < 2 < p_1$ and let $p'_0 = [2, 4[$ be defined by $1/p'_0 = 1/p_0 - 1/2$, then by Sobolev inequalities and the boundedness of $T$ and $T^*$, we have

$$
(3.22) \quad \|\tilde{T}_\theta \tilde{p} T^* f\|_{L^\infty(M_0)} \leq \|\tilde{T}_\theta \tilde{p} T^* f\|_{W^{1,p'_0}(M)}
$$

$$
\leq \|\tilde{p} T^* f\|_{L^{p'_0}(M'_0; T^1_0 M'_0)} \leq \|T^* f\|_{W^{1,p_0}(M; T^1_0 M)} \leq \|f\|_{L^{p_0}(M_0)}.
$$

On the other hand, a direct application of (3.5) yields

$$
(3.23) \quad \|\tilde{T}_\theta \tilde{p} T^* f\|_{L^\infty(M_0)} \leq \frac{\|f\|_{L^{p_0}(M_0)}}{\lambda^{\frac{1}{4}}},
$$

Interpolating between (3.22) and (3.23) implies the second estimate in (3.20). We can get the other one using the same strategy. The claim now follows from density. \hfill \square

It remains to show that the sums in (3.3) indeed converge. By iterating the $L^p \to L^\infty$ estimates in Corollary 3.1, we see that there exists a constant $C > 0$ such that

$$
\|u_j\|_{L^\infty(M_0)} \leq \left( \frac{C\|V_1\|_{L^p}}{|\lambda|^{0+}} \right)^{-j} \|u_1\|_{L^\infty(M_0)} \text{ for all } j \geq 2, \text{ and}
$$

$$
\|\tilde{u}_j\|_{L^\infty(M_0)} \leq \left( \frac{C\|V_1\|_{L^p}}{|\lambda|^{0+}} \right)^{-j} \|a\|_{L^\infty(M_0 \times \Omega)} \text{ for all } j \geq 0.
$$

Inserting the above inequalities into (3.4), we get

$$
(3.24) \quad \|\sum_{j \geq 0} (-1)^j u_j\|_{L^\infty(M_0)} \leq \sum_{0 \leq j < 2} \|u_j\|_{L^\infty(M_0)} + \sum_{j \geq 2} \left( \frac{C\|V_1\|_{L^p}}{|\lambda|^{0+}} \right)^{-j} \|u_1\|_{L^\infty(M_0)}, \quad \text{and}
$$

$$
\|\sum_{j \geq 0} (-1)^j \tilde{u}_j\|_{L^\infty(M_0)} \leq \sum_{j \geq 0} \left( \frac{C\|V_1\|_{L^p}}{|\lambda|^{0+}} \right)^{-j} \|a\|_{L^\infty(M_0 \times \Omega)}.
$$

We can further bound the $L^\infty$ norm of $u_1$ by Sobolev embedding so that

$$
\|u_1\|_{L^\infty} \lesssim \|V_1\|_{L^p} \|a\|_{L^\infty(M_0 \times \Omega)} + \max_{1 \leq j' \leq N, p_0 \in \Omega} \sup_{1 \leq j' \leq N, p_0 \in \Omega} |Q^+_{j', \epsilon}(p_0)| \|b_j\|_{L^\infty(M_0; T^1_0 M_0)} < \infty
$$

which is finite and so is $\|u_0\|_{L^\infty(M_0 \times \Omega)}$. Thus we can find $\lambda_0 > 0$ and $\omega_0 \in \mathbb{C}$ such that for all $\lambda > \lambda_0$ and $|\omega| > |\omega_0|$, the right hand sides of (3.24) converge by geometric series. Hence $u$ and $\tilde{u}$ are in $L^\infty(M_0)$ for all sufficiently large $\lambda$ and $|\omega|$. Since $T_\theta \tilde{p} T^*$ and $T_\theta \tilde{p} T^*$ are bounded $L^p(M_0) \to W^{2,p}(M_0)$, we can now write

$$
u = e^{i\Psi \lambda} a + T_\theta \tilde{p} T^* e^{i\Psi \lambda} V_1 \tilde{r} \quad \text{and} \quad \tilde{\nu} = e^{-\frac{i}{2} \Phi \omega} a + T_\theta \tilde{p} T^* e^{-\frac{i}{2} \Phi \omega} V_1 \tilde{r}.
$$

The embedding $W^{2,p}(M_0) \to H^1(M_0)$ then yields that $\nu$ and $\tilde{\nu}$ are $H^1(M_0)$ functions. Similar calculations work for $\nu$ and $\tilde{\nu}$. This concludes the proof of Proposition 3.1. \hfill \square
4. Improving Regularity

In this section we prove Theorem 1.2. Observe by compactness it suffices to show that \( V \in L^2_{\text{loc}}(M_0) \). Indeed, we will show that

**Proposition 4.1.** Any point in \( M_0 \) admits an open neighbourhood \( \Omega_0 \subset M \) such that \( V \in L^2(\Omega_0) \).

**Proof.** Implementing the \( L^\infty(M_0) \cap H^1(M_0) \) solutions \( \tilde{u} \) and \( \tilde{v} \) from Proposition 3.1 into identity (3.1), we have

\[
0 = \int_{M_0} \tilde{u} \tilde{v} \, dv_g = \int_{M_0} e^{-i\Phi \cdot \omega} |a|^2 V \, dv_g + \sum_{k+k' \geq 1} \int_{M_0} e^{-i\Phi \cdot \omega} \tilde{u}_k V \tilde{v}_{k'} \, dv_g
\]

where we switched the sum and integral by boundedness. Let \( \Phi : M \to \mathbb{C} \) be the holomorphic function without critical point chosen in Section 2 and \( \tilde{p}_0 \) be an arbitrary point in \( M_0 \). Assume without loss of generality that \( \Phi(\tilde{p}_0) = 0 \). Let \( \{\Omega_{j'}\}_{1 \leq j' \leq N} \) be the holomorphic charts constructed at the beginning of Section 2.3, and \( \{\Omega_j\}_{j \geq 0} \) be an open covering of \( M_0 \) in \( M \) such that \( \Phi : \Omega_j \to \Phi(\Omega_j) \) is biholomorphic for each \( j \geq 0 \). Fix a partition of unity \( \{\chi_{j'}\}_{j \geq 0} \) subordinate to \( \{\Omega_{j'}\}_{j \geq 0} \) and choose \( \{\chi'_j\}_{j \geq 0} \) so that \( \chi'_j \) is supported on a holomorphic chart with coordinate map \( \Phi \) and is identically 1 on the support of \( \chi_j \). We can modify the definitions by setting \( \Omega_0 = \Omega_0 \).

Choose \( \omega_0 \in \mathbb{C} \) so that \( |\omega_0| \) is sufficiently large. Let \( \rho \) be a smooth function which vanishes on the set \( |\omega| \leq |\omega_0| \). For \( \epsilon > 0 \) we can multiply both sides of equation (4.1) by the weight \( 1_{\Omega_0}(p_0) \rho(\omega)e^{-\epsilon |\omega|^2} e^{i\Phi(p_0) \cdot \omega} \) and integrate to get

\[
1_{\Omega_0}(p_0) \int_{|\omega| > |\omega_0|} \rho(\omega)e^{-\epsilon |\omega|^2} e^{i\Phi(p_0) \cdot \omega} \int_{M_0} e^{-i\Phi \cdot \omega} |a|^2 V \, dv_g
\]

\[
= -1_{\Omega_0}(p_0) \int_{|\omega| > |\omega_0|} \rho(\omega)e^{-\epsilon |\omega|^2} e^{i\Phi(p_0) \cdot \omega} \sum_{k+k' \geq 1} \int_{M_0} e^{i\Phi \cdot \omega} \tilde{u}_k V \tilde{v}_{k'} \, dv_g.
\]

We now want to take a limit as \( \epsilon \to 0 \) in order to apply a Fourier inversion argument. By our construction of \( a \), we show that the left hand side of (4.2) converges in \( L^1(M) \) to \( 1_{\Omega_0} V |g|^{1/2} \) where \( |g| \) is the volume component of \( g \) in \( \Omega_0 \), while the right hand side converges to a \( L^2(\Omega_0) \) function. The extra complexity is that the amplitude \( a \) depends nonlinearly on both \( p \) and \( p_0 \) while \( V \) might not even be continuous. We resolve this problem by relying on the Taylor expansion introduced in Lemma (2.3) which reduces the problem to the usual Fourier inversion theorem. Nevertheless, a more careful computation is required.

4.1. Analysis of Principle Terms. We first consider the left hand side of (4.2). By the change of variables \( \Phi_{|\tilde{p}_0}(p) = z_0 \), we can integrate over the \( \omega \) variable to see that

\[
1_{\Omega_0}(p_0) \int_{|\omega| > |\omega_0|} \rho(\omega)e^{-\epsilon |\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_{M_0} e^{-i\Phi \cdot \omega} |a|^2 V \, dv_g \right) d\omega d\bar{\omega}
\]

\[
= 1_{D_\epsilon}(z_0) \int_{|\omega| > |\omega_0|} e^{-\epsilon |\omega|^2} e^{-iz_0 \cdot \omega} \left( \int_{M_0} e^{-i\Phi \cdot \omega} |a|^2 V \, dv_g \right) d\omega d\bar{\omega}
\]

\[
- 1_{D_\epsilon}(z_0) \int_{|\omega| < |\omega_0|} (1-\rho)(\omega)e^{-\epsilon |\omega|^2} e^{-iz_0 \cdot \omega} \left( \int_{M_0} e^{-i\Phi \cdot \omega} |a|^2 V \, dv_g \right) d\omega d\bar{\omega}.
\]

We first look at the second line of (4.3), from which we shall obtain local information about \( V \). This is summarised in the following technical lemma. Notice that by extending \( V_1, V_2 \) to \( M \) via zero, it is enough to show
Lemma 4.1. We have that

\[ \lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_{\mathbb{C}} e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_M e^{-i\Phi \cdot \omega}|a|^2 V \, dv_g \right) \, d\omega d\bar{\omega} = 1_{\Omega_0} V |g|^{1/2}. \]

in \( L^1(M) \).

Proof. Set

\[ \tilde{M} \overset{\text{def}}{=} M \setminus \bigcup_{1 \leq j' \leq N} \tilde{\Omega}_{j'} \]

and introduce the splitting

\[ 1_{\Omega_0}(p_0) \int_{\mathbb{C}} e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_M e^{-i\Phi \cdot \omega}|a|^2 V \, dv_g \right) \, d\omega d\bar{\omega} \]

\[ = 1_{\Omega_0}(p_0) \sum_{1 \leq j' \leq N} \int_{\tilde{\Omega}_{j'}} e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_{\tilde{\Omega}_{j'}} e^{-i\Phi \cdot \omega}|a|^2 V \, dv_g \right) \, d\omega d\bar{\omega} \]

\[ + 1_{\Omega_0}(p_0) \sum_{j \geq 0} \int_{\mathbb{C}} e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_{\tilde{M}} \chi_j e^{-i\Phi \cdot \omega}|a|^2 V \, dv_g \right) \, d\omega d\bar{\omega}. \]

On each \( \tilde{\Omega}_j \) we make the change of variable so that \( \Phi|_{\tilde{\Omega}_{j'}}(p) = z_{j'} \) and write

\[ 1_{\Omega_0}(p_0) \int_{\mathbb{C}} e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \left( \int_{\tilde{\Omega}_{j'}} e^{-i\Phi \cdot \omega}|a|^2 V \, dv_g \right) \, d\omega d\bar{\omega} \]

\[ = 1_{D_r}(z_0) \int_{\mathbb{C}} e^{-|\omega|^2} e^{iz_0 \cdot \omega} \left( \int_{D_r} e^{-iz_{j'} \cdot \omega} a(\Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_{j'}); \Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_0))^2 \right. \]

\[ \times V(\Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_{j'})) |g_{j'}|^{1/2} dz_{j'} d\bar{z}_{j'} \] \( d\omega d\bar{\omega}. \)

We recall the Taylor expansion from Lemma 2.3

\[ a(p; \Phi^{-1}_{|\tilde{\Omega}_{0}}(z_0)) = \sum_{\mu \geq 0} a_\mu(p) z_0^\mu, \quad z_0 \in D_r \]

with holomorphic coefficients on \( M \). One thus observe from (2.14) and the dominated convergence theorem that (4.6) can be written as

\[ \sum_{\mu, \mu' \geq 0} 1_{D_r}(z_0) z_0^\mu \bar{z}_0^{\mu'} \int_{\mathbb{C}} e^{-|\omega|^2} e^{iz_0 \cdot \omega} \]

\[ \times \left( \int_{D_r} e^{-iz_{j'} \cdot \omega} a_\mu(\Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_{j'})) a_{\mu'}(\Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_{j'})) V(\Phi^{-1}_{|\tilde{\Omega}_{j'}}(z_{j'})) |g_{j'}|^{1/2} dz_{j'} d\bar{z}_{j'} \right) \, d\omega d\bar{\omega}, \]
and so it can be easily seen from (2.14) combined with (4.8) that

\[ \left\| e^{-|\omega|^2} e^{iz_0 \cdot \omega} \left( \int_{D_r} e^{-iz_j' \cdot \omega} |a(\Phi^{-1}_{|z_j'}(z_j'); \Phi^{-1}_{|z_0}(z_0))|^2 V(\Phi^{-1}_{|z_1}(z_1')) |g_{j'}|^{1/2} dz_{j'} d\bar{z}_{j'} \right) d\omega d\bar{\omega} \right\|_{L^1(D_r)} \]

\[ \leq \sum_{\mu, \mu' \geq 0} r^{|\mu|+|\mu'|} \|o_{L^1, \phi}^{(\mu, \mu')}(1)\|_{L^1(D_r)} \]

\[ \leq \left( \sum_{\mu, \mu' \geq 0} r^{|\mu|+|\mu'|} \|a_\mu\|_{L^\infty(M_0)} \|a_{\mu'}\|_{L^\infty(M_0)} \right) \left( e^{-1} \left\| e^{-|\omega|^2/\delta} \left( L^1 + 1 \right) \right\| L^1(M_0) \right) \]

can be bounded independently of \( \epsilon \) in view of the Gaussian approximation of the identity, and

\[ o_{L^1, \phi}^{(\mu, \mu')}(1) = \mathcal{F}^{-1} e^{-|\omega|^2} \mathcal{F}[1_{D_r} a_\mu(\Phi^{-1}_{|z_j'}(z_j')) \bar{\alpha}_{\mu'}(\Phi^{-1}_{|z_0}(z_0)) V(\Phi^{-1}_{|z_1}(z_1')) |g_{j'}|^{1/2}] \]

\[ - 1_{D_r}(z_0) [a_\mu(\Phi^{-1}_{|z_j'}(z_j')) \bar{\alpha}_{\mu'}(\Phi^{-1}_{|z_0}(z_0)) V(\Phi^{-1}_{|z_1}(z_1')) |g_{j'}|^{1/2}]_{z_{j'} = z_0} \to 0 \text{ as } \epsilon \to 0 \]

in \( L^1 \) by the Fourier inversion theorem. Thus the limit as \( \epsilon \to 0 \) can be switched with the infinite sum in the second line of (4.9), from which we may conclude that

\[ \lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_C e^{-|\omega|^2} e^{i\omega \cdot \Phi(p_0)} \left( \int_{\Omega_j} e^{i\Phi \cdot \omega} |a|^{2} V dV_g \right) d\omega d\bar{\omega} \]

\[ = 1_{D_r}(z_0) [a(\Phi^{-1}_{|z_j'}(z_j'); \Phi^{-1}_{|z_0}(z_0))] V(\Phi^{-1}_{|z_1}(z_1')) |g_{j'}|^{1/2}]_{z_{j'} = z_0} \]

in \( L^1(M_0) \). On the other hand, since

\[ \Phi(\Phi^{-1}_{|z_j'}(z_0)) = \Phi(\Phi^{-1}_{|z_0}(z_0)) \]

for all \( 1 \leq j' \leq N \), by the construction of \( a \) in (2.7), we obviously have the property

\[ a(\Phi^{-1}_{|z_j'}(z_0); \Phi^{-1}_{|z_0}(z_0)) = \begin{cases} 1, & \text{if } j' = 0, \\ 0, & \text{if } j' \neq 0, \end{cases} \]

thus we have from (4.10) that

\[ \lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \sum_{1 \leq j' \leq N} \int_C e^{-|\omega|^2} e^{i\omega \cdot \Phi(p_0)} \left( \int_{\Omega_j} e^{i\Phi \cdot \omega} |a|^{2} V dV_g \right) d\omega d\bar{\omega} = 1_{\Omega_0} V |g|^{1/2} \]

in \( L^1(M_0) \). To complete the proof, we note that by construction, \( \tilde{M} \) contains no point \( p \in M_0 \) for which \( \Phi(p) \in D_r \), so that \( \Phi(M \cap M_0) \) is disjoint from \( D_r \). Thus with the same calculations as above we can conclude that

\[ \lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_C e^{-|\omega|^2} e^{i\omega \cdot \Phi(p_0)} \left( \int_{\tilde{M}} \chi_j e^{-i\Phi \cdot \omega} |a|^{2} V dV_g \right) d\omega d\bar{\omega} = \sum_{\mu, \mu' \geq 0} 1_{D_r} 1_{\Phi(M \cap M_0 \cap \Omega_j)} o_{\mu, \mu'}(1) \]

\[ \times [\chi_j(\Phi^{-1}_{|z_j}(z_j')) a_\mu(\Phi^{-1}_{|z_j}(z_j')) \bar{\alpha}_{\mu'}(\Phi^{-1}_{|z_j}(z_j')) V(\Phi^{-1}_{|z_j}(z_j')) |g_j|^{1/2}]_{z_{j'} = z_0} = 0 \]

for each \( j \geq 0 \). Applying the above equation and (4.12) we arrive from (4.5) at (4.4). \( \square \)
On the other hand, to take care of the last line in (4.3), we note that since \((1 - \rho)\) is compactly supported, one can take the limit in \(\epsilon\) directly inside the integral to arrive at

\[
\lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_{|\omega| < |\omega_0|} (1 - \rho)(\omega)e^{-|\omega|^2}e^{-i\Phi(p_0)\omega} \left( \int_{M_0} e^{-i\Phi(\omega)}|a|^2 V d\nu_g \right) d\omega d\bar{\omega} = 1_{D_\rho}(z_0) \int_{|\omega| < |\omega_0|} (1 - \rho)(\omega)e^{-i\omega_0 \cdot \omega} \left( \int_{M_0} \chi_j e^{-i\Phi(\omega)}|a|^2 V d\nu_g \right) d\omega d\bar{\omega}.
\]

(4.13)

One is therefore able to differentiate under the integrals to conclude that the later integral depends smoothly on \(z_0 \in D_\rho\). By compactness it follows that (4.13) belongs to \(L^\infty(\Omega_0)\). This concludes our analysis of the principle terms.

4.2. Analysis of Remainder Terms. To take care of the right hand side of (4.2), we refine the structure of the solutions constructed in Section 3. Let \(\{a_\mu\}_{\mu \geq 0}\) be the holomorphic coefficients of \(a\). We set

\[
\tilde{u}_0^{(\mu)} \overset{\text{def}}{=} a_\mu, \quad \tilde{v}_0^{(\mu)} \overset{\text{def}}{=} \bar{a}_\mu \quad \text{and} \quad \tilde{u}_j^{(\mu)} = T_\Phi \tilde{v}^*(V_1 \tilde{v}_{j-1}^{(\mu)}), \quad \tilde{v}_j^{(\mu)} = \tilde{T}_\Phi \tilde{v}^*(V_2 \tilde{v}_{j-1}^{(\mu)}), \quad j \geq 1
\]

and introduce the notations

\[
\tilde{I}_{k,k'}^{(\mu,\mu')} \overset{\text{def}}{=} \int_{M_0} e^{-i\Phi(\omega)} V \tilde{u}_{k,\nu} \tilde{v}_{k'} d\nu_g \quad \text{and} \quad \tilde{I}_{k,k'}^{(\mu')} \overset{\text{def}}{=} \int_{M_0} e^{-i\Phi(\omega)} V \tilde{u}_{k,\nu} \tilde{v}_{k'}^{(\mu')} d\nu_g.
\]

Since \(T_\Phi \tilde{T}^*: L^p(M_0) \to L^\infty(M_0)\) is bounded. Using (2.14) it is clear that we have

\[
\tilde{u}_j = \sum_{\mu \geq 0} \tilde{u}_j^{(\mu)} \tilde{z}_0^{\mu} \quad \text{and} \quad \tilde{v}_j = \sum_{\mu \geq 0} \tilde{v}_j^{(\mu)} \tilde{z}_0^{\mu}
\]

with convergence in \(L^\infty(M_0)\) for all \(p_0 \in \Omega_0\) and every \(j \geq 0\). Our procedure for the cases \((k,k') = (1,0)\) or \((k,k') = (0,1)\) require additional arguments.

Lemma 4.2. The limits

\[
\lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_C e^{-|\omega|^2} e^{i\Phi(p_0)\omega} \tilde{I}_{1,0} d\omega d\bar{\omega} \quad \text{and} \quad \lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_C e^{-|\omega|^2} e^{i\Phi(p_0)\omega} \tilde{I}_{0,1} d\omega d\bar{\omega}
\]

exist in \(L^2(M)\).

Proof. As before we extend \(V_1, V_2\) to \(M\) by zeros. Decomposing via the partition of unity \(\{\Omega_j, \chi_j, \chi_j'\}_{j \geq 0}\), since \(\tilde{\rho}\) is compactly supported in \(M_0'\), we have

\[
\tilde{I}_{1,0} = \sum_{j \geq 0} \int_M \tilde{a}_V T(\chi_j e^{-i\Phi(\omega)} \tilde{v}^* V_1 a) d\nu_g
\]

\[
= \sum_{j \geq 0} \int_M \tilde{a}_V R(\chi_j e^{-i\Phi(\omega)} \tilde{v}^* V_1 a) d\nu_g + \sum_{j \geq 0} \int_M \tilde{a}_V K_j(\chi_j e^{-i\Phi(\omega)} \tilde{v}^* V_1 a) d\nu_g.
\]

Making the change of variables \(\Phi_{|\Omega_0}(\rho) = z_j\) and apply again the Taylor expansion of \(a\), we can obtain from Fubini’s theorem that

\[
1_{\Omega_0}(p_0) \int_M \chi_j \tilde{a}_V R(\chi_j e^{-i\Phi(\omega)} \tilde{v}^* V_1 a) d\nu_g + 1_{\Omega_0}(p_0) \int_M \tilde{a}_V K_j(\chi_j e^{-i\Phi(\omega)} \tilde{v}^* V_1 a) d\nu_g
\]

\[
= -1_{D_\rho}(z_0) \sum_{\mu, \mu' \geq 0} \int_{\Omega_j} \chi_j e^{-i z_j \cdot \omega} R(\chi_j V \tilde{a}_\mu |g_j|^{1/2}) \tilde{v}^* (V_1 a_{\mu'}) d\omega_0 dz_j d\bar{\omega}_0
\]

\[
+ 1_{D_\rho}(z_0) \sum_{\mu, \mu' \geq 0} \int_{\Omega_j} \chi_j e^{-i z_j \cdot \omega} K_j(\chi_j V \tilde{a}_\mu) \tilde{v}^* (V_1 a_{\mu'}) d\nu_g,
\]

(4.15)
for each $j \geq 0$, where we identify the 1-forms $\chi_j T^*(Va_\mu)$ with their coefficients in the coordinates $z_j$ and $K_j^*$ has smooth kernel. Multiplying on both sides of (4.15) by $e^{-|\omega|^2} e^{i\Phi(p_0)\cdot \omega}$ and integrate over $\omega \in \mathbb{C}$, we have by the dominated convergence theorem that the second line of (4.15) becomes

$$-1_{D_\epsilon}(z_0) \sum_{\mu, \mu' \geq 0} z_0^\mu z_0^{\mu'} \int_C e^{-|\omega|^2} e^{iz_0 \cdot \omega} \left( \int_{\Omega_j} \chi_j e^{-iz_j \cdot \omega} R(\chi_j^\epsilon V \bar{a}_\mu | g_j |^{1/2}) \tilde{\rho} T^*(V_1 a_{\mu'}) d\bar{z}_j dz_j \right) d\omega d\bar{\omega}.$$ 

By the inequalities of Hölder and Sobolev, we can estimate

$$\|R(\chi_j^\epsilon V \bar{a}_\mu | g_j |^{1/2})\chi_j \tilde{\rho} T^*(V_1 a_{\mu'})\|_{L^2(\Omega_j)}$$

$$\lesssim \|R(\chi_j^\epsilon V \bar{a}_\mu | g_j |^{1/2})\|_{W^{1,4/3}(\Omega_j)} \|\chi_j \tilde{\rho} T^*(V_1 a_{\mu'})\|_{W^{1,4/3}(\Omega_j)}$$

$$\lesssim \|\tilde{a}_\mu\|_{L^\infty(M_0)} \|a_{\mu'}\|_{L^\infty(M_0)} \|V\|_{L^{4/3}(M_0)} \|V_1\|_{L^{4/3}(M_0)}.$$ 

Thus we may argue as in (4.13) via the Plancherel theorem and Fourier inversion to see that there exists a constant $C > 0$ independent of $\epsilon > 0$ and functions of order $o_{L^2, \epsilon}(1)$ such that

$$\left\| \int_C e^{-|\omega|^2} e^{iz_0 \cdot \omega} \left( \int_M \chi_j^\epsilon \tilde{a} V R(\chi_j^\epsilon e^{-i\Phi \cdot \omega} \tilde{\rho} T^* V_1 a) d\nu_\nu \right) d\omega d\bar{\omega} \right\|_{L^2(\mathbb{C})}$$

$$+ \sum_{\mu, \mu' \geq 0} z_0^\mu z_0^{\mu'} \left| \int_M \chi_j^\epsilon \tilde{a} V R(\chi_j^\epsilon e^{-i\Phi \cdot \omega} \tilde{\rho} T^* (V_1 a_{\mu'})) |_{z_j = z_0} \right\|_{L^2(D_\epsilon)} \lesssim \sum_{\mu, \mu' \geq 0} r^{\mu + \mu'} |o_{L^2, \epsilon}(1)|_{L^2(\mathbb{C})}$$

$$\lesssim \sum_{\mu, \mu' \geq 0} r^{\mu + \mu'} \sup_{\omega \in \mathbb{C}} |e^{-|\omega|^2} - 1| \|F(\chi_j^\epsilon R(\chi_j^\epsilon \tilde{a} V g_j |^{1/2}) \tilde{\rho} T^* (V_1 a_{\mu'}))\|_{L^2(\mathbb{C})} \leq C < \infty,$$

where $o_{L^2, \epsilon}(1) \to 0$ as $\epsilon \to 0$ in $L^2$. Summing over all $j \geq 0$ and taking the limit as $\epsilon \to 0$ in the above, we have form (4.15) that

$$\lim_{\epsilon \to 0} \sum_{j \geq 0} \int_C e^{-|\omega|^2} e^{iz_0 \cdot \omega} \left( \int_M \chi_j^\epsilon \tilde{a} V R(\chi_j^\epsilon e^{-i\Phi \cdot \omega} \tilde{\rho} T^* V_1 a) d\nu_\nu \right) d\omega d\bar{\omega}$$

$$= -1_{D_\epsilon}(z_0) \sum_{j \geq 0} \sum_{\mu, \mu' \geq 0} z_0^\mu z_0^{\mu'} \left| \int_M \chi_j^\epsilon \tilde{a} V R(\chi_j^\epsilon e^{-i\Phi \cdot \omega} \tilde{\rho} T^* (V_1 a_{\mu'})) |_{z_j = z_0} \right|$$

in $L^2(M)$. The right hand side of (4.18) lives in $L^2(D_\epsilon)$ by estimate (4.17) and (2.14). Thus we have arrived at the required limit. The last line in (4.15) involving smoothing operators can be taken cared of in the same way, and the obvious modification to the argument works for the term which contains $\tilde{I}_{0,1}$. 

Finally we can again write

$$1_{\Omega_0}(p_0) \int_{|\omega| \geq |\omega_0|} \rho(\omega) e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \tilde{I}_{1,0} d\omega d\bar{\omega}$$

$$= 1_{\Omega_0}(p_0) \int_C e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \tilde{I}_{1,0} d\omega d\bar{\omega} - 1_{\Omega_0}(p_0) \int_{|\omega| < |\omega_0|} (1 - \rho)(\omega) e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \tilde{I}_{1,0} d\omega d\bar{\omega}.$$ 

Notice from (2.6) that $\tilde{I}_{1,0}$ depends smoothly on $p_0 \in \Omega_0$. Thus we can apply Lemma 4.2 to the first term above and argue as in (4.13) for the second shows that (4.19) converges to a limit in $L^2(M)$. A similar calculation works for the case of $\tilde{I}_{0,1}$.

It remains to show that the limit

$$\lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \sum_{k + k' \geq 2} \int_{|\omega| \geq |\omega_0|} \rho(\omega) e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \tilde{I}_{k, k'} d\omega d\bar{\omega}$$

$$= 1_{\Omega_0}(p_0) \sum_{k + k' \geq 2} \int_{|\omega| \geq |\omega_0|} (1 - \rho)(\omega) e^{-|\omega|^2} e^{i\Phi(p_0) \cdot \omega} \tilde{I}_{k, k'} d\omega d\bar{\omega}.$$
exists in \( L^2(M_0) \). For this we will make use of the Carleman estimates proved in Section 3. For the cases where \( k = 0 \) or \( k' = 0 \) we formulate the following result.

**Lemma 4.3.** There exists constants \( \tilde{C}, C > 0 \) independent of \( \lambda \) and \( p_0 \) such that

\[
|\tilde{I}_{k,0} | \leq \tilde{C} \left( \frac{C}{|\omega|^0} \right) |a_{\mu'}|_{L^\infty(M_0)} \|ar{u}_{\mu'}\|_{L^\infty(M_0)}
\]

for all \( k, k' \geq 2 \).

*Proof.* It is sufficiently to note from Corollary 3.1 and Hölder’s inequalities that

\[
\int_{M_0} |V| |\bar{u}_{k}^{(\mu)}| |a_{\mu'}| dv_g \lesssim \|ar{u}_{\mu'}\|_{L^p} \|V\|_{L^p} \|T_\Phi \bar{\rho} T^*(V_1 T_\Phi \bar{\rho} T^*(\bar{u}_{k-2}^{(\mu)} V_1))\|_{L^q}
\]

\[
\lesssim \|ar{u}_{\mu'}\|_{L^\infty} \|V\|_{L^p} \|V_1\|_{L^p} |T_\Phi \bar{\rho} T^*(\bar{u}_{k-2}^{(\mu)} V_1)|_{L^\infty} \lesssim \|ar{u}_{\mu'}\|_{L^\infty} \|V\|_{L^p} \|V_1\|_{L^p} \|ar{u}_{k-2}^{(\mu)}\|_{L^\infty}
\]

where \( q \) is the Hölder conjugate of \( p \). Now by iterating the \( L^p \to L^\infty \) estimate obtained in (3.20), we easily see that there exists a constant \( C > 0 \) such that

\[
\|\bar{u}_{k}^{(\mu)}\|_{L^\infty} \leq \left( \frac{C}{|\omega|^0} \right)^k |a_{\mu}|_{L^\infty}
\]

for all \( k \geq 0 \). Putting the two estimates together we arrive at the required claim. A similar calculation works for the case of \( \tilde{I}_{0,k'} \). \( \square \)

The remaining terms in (1.20) can be estimated in similar manners.

**Lemma 4.4.** There exists constant \( \tilde{C}, C > 0 \) independent of \( \lambda \) and \( p_0 \) such that

\[
|\tilde{I}_{k,k'} | \leq \tilde{C} \left( \frac{C}{|\omega|^0} \right)^{k+k'-2} |a_{\mu}|_{L^\infty(M_0)} \|ar{u}_{\mu'}\|_{L^\infty(M_0)}
\]

for all \( k + k' \geq 2 \) with \( k, k' \neq 0 \).

*Proof.* Applying again Corollary 4.3 we see that if \( 1/p + 1/q = 1 \), then

\[
\int_{M_0} |V| |\bar{u}_{k}^{(\mu)}| |\bar{v}_{k'}^{(\mu')}| dv_g \leq \|V\|_{L^p} \|T_\Phi \bar{\rho} T^*(\bar{u}_{k-1}^{(\mu)} V_1)\|_{L^q} \|T_\Phi \bar{\rho} T^*(\bar{v}_{k'-1}^{(\mu')} V_2)\|_{L^\infty}
\]

\[
\lesssim \|V\|_{L^p} \|ar{u}_{k-1}^{(\mu)}\|_{L^\infty} \|ar{v}_{k'-1}^{(\mu')}\|_{L^\infty} \lesssim \|ar{u}_{k-1}^{(\mu)}\|_{L^\infty} \|ar{v}_{k'-1}^{(\mu')}\|_{L^\infty}
\]

Now (4.22) yields the existence of a constant \( C > 0 \) such that

\[
\|ar{u}_{k-1}^{(\mu)}\|_{L^\infty} \|ar{v}_{k'-1}^{(\mu')}\|_{L^\infty} \lesssim \left( \frac{C}{|\omega|^0} \right)^{k+k'-2} |a_{\mu}|_{L^\infty} \|ar{u}_{\mu'}\|_{L^\infty}
\]

which implies the claim. \( \square \)

Combining the results of Lemma 4.3 and Lemma 4.4 we see that there exists constants \( \tilde{C}, C > 0 \) such that whenever \( k + k' \geq 2 \), we have

\[
|\tilde{I}_{k,k'} | \leq \tilde{C} \left( \frac{C}{|\omega|^0} \right)^{k+k'-2} |a_{\mu}|_{L^\infty(M_0)} \|ar{u}_{\mu'}\|_{L^\infty(M_0)}
\]
Moreover, from (1.24) we deduce that \( \rho \tilde{I}_{k,k'}^{\mu,\mu'} \in L^2 \) in the \( \omega \) variable, so that by the Plancherel theorem we know \( \mathcal{F}^{-1} \rho \tilde{I}_{k,k'}^{\mu,\mu'} \) exists in \( L^2 \). Since we have

\[
1_{\Omega_0}(p_0) \int_{|\omega| \geq |\omega_0|} e^{-|\omega|^2} e^{i\Phi(p_0) - \omega} \rho(\omega) \tilde{I}_{k,k'} d\omega d\bar{\omega} = \sum_{\mu,\mu' \geq 0} 1_{D_{\epsilon}}(z_0) z_0^\mu \bar{z}_0^\mu' \int_{|\omega| \geq |\omega_0|} e^{-|\omega|^2} e^{i\omega_0 - \omega} \rho(\omega) \tilde{I}_{k,k'}^{\mu,\mu'} d\omega d\bar{\omega},
\]

it follows from (2.14) that there exists \( C' > 0 \) such that

\[
\| \int_{|\omega| \geq |\omega_0|} e^{-|\omega|^2} e^{i\omega_0 - \omega} \rho(\omega) \sum_{k+k' \geq 2} \tilde{I}_{k,k'} d\omega d\bar{\omega} - \sum_{k+k' \geq 2} \sum_{\mu,\mu' \geq 0} z_0^\mu \bar{z}_0^\mu' \mathcal{F}^{-1} \rho \tilde{I}_{k,k'}^{\mu,\mu'} \|_{L^2(D_{\epsilon})} \leq \sum_{k+k' \geq 2} \sum_{\mu,\mu' \geq 0} r^{\mu+\mu'} \| \mathcal{F}^{-1} (e^{-|\omega|^2} - 1) \rho \tilde{I}_{k,k'}^{\mu,\mu'} \|_{L^2(D_{\epsilon})} \]

\[
\leq \sup_{\omega \in \mathbb{C}} \frac{|e^{-|\omega|^2} - 1|}{|\omega|^1} \sum_{k+k' \geq 2} \left( \frac{C'}{|\omega_0|} \right)^{k+k'-2} \sum_{\mu,\mu' \geq 0} r^{\mu+\mu'} \|a_\mu\|_{L^\infty(M_0)} \|a_{\mu'}\|_{L^\infty(M_0)} < \infty
\]

for all \( |\omega| > |\omega_0| \). In particular, the last line above is uniformly bounded in \( \epsilon > 0 \), so Fourier inversion yields

\[
\lim_{\epsilon \to 0} 1_{\Omega_0}(p_0) \int_{|\omega| \geq |\omega_0|} e^{-|\omega|^2} e^{i\Phi(p_0) - \omega} \rho(\omega) \sum_{k+k' \geq 2} \tilde{I}_{k,k'} d\omega d\bar{\omega} = \sum_{k+k' \geq 2} \sum_{\mu,\mu' \geq 0} 1_{D_{\epsilon}}(z_0) z_0^\mu \bar{z}_0^\mu' \mathcal{F}^{-1} \rho \tilde{I}_{k,k'}^{\mu,\mu'}
\]

in \( L^2(M) \) as expected. From (1.24) we also have

\[
\| \sum_{k+k' \geq 2} \sum_{\mu,\mu' \geq 0} |z_0|^\mu \mathcal{F}^{-1} \rho \tilde{I}_{k,k'}^{\mu,\mu'} \|_{L^2(D_{\epsilon})} \leq \sum_{k+k' \geq 2} \sum_{\mu,\mu' \geq 0} r^{\mu+\mu'} \| \rho \tilde{I}_{k,k'}^{\mu,\mu'} \|_{L^2} \leq \left( \sum_{k+k' \geq 2} 2^{k-k'+2} \sum_{\mu,\mu' \geq 0} \mathcal{C} r^{\mu+\mu'} \|a_\mu\|_{L^\infty(M_0)} \|a_{\mu'}\|_{L^\infty(M_0)} \left( \int_{|\omega| > |\omega_0|} \left| \rho(\omega) \right|^2 \frac{d\omega d\bar{\omega}}{|\omega|^2} \right)^{1/2} \right)^{1/2}
\]

which is finite for sufficiently large \( |\omega_0| \), hence we have arrived at the required convergence in (4.20). Putting everything together and noting that \( L^2(M) \hookrightarrow L^1(M) \), we can take the \( L^1(M) \) limit on both sides of (1.1) to deduce that

\[
1_{\Omega_0} V |g|^{1/2} \in L^2(M).
\]

Since \( |g|^{1/2} \) is non-vanishing, this in particular implies that \( V \) is \( L^2 \) on \( \Omega_0 \) and so we have arrived at the required claim. \( \square \)

5. Identification of the Potential

In this final section we prove Theorem 1.1. The procedure will be similar to what was done in Section 4, with the key difference being that we now have \( V \in L^2(M_0) \). As in section 4 it suffices to show

**Proposition 5.1.** Any point in \( M_0 \) admits an open neighbourhood \( \Omega_0 \subset M \) such that \( V = 0 \) almost everywhere on \( \Omega_0 \).

**Proof.** For an arbitrary point \( \bar{p}_0 \) in \( M_0 \) we adopt the same convention introduced at the beginning of Section 4. We recall that this means we let \( \{ \Omega_j^i, \chi_j^i \}_{1 \leq j' \leq N} \) be constructed as in Subsection 2.3. The collection \( \{ \Omega_j \}_{j \geq 0} \) defines an open covering of \( M_0' \) in \( M \) so that for each \( j \geq 0 \), the map \( \Phi : \Omega_j \to \Phi(\Omega_j) \) is biholomorphic. We let \( \{ \chi_j \}_{j \geq 0} \) be a partition of unity subordinate to \( \{ \Omega_j \}_{j \geq 0} \), and choose \( \{ \chi_j' \}_{j \geq 0} \) so for that each \( j \geq 0 \), \( \chi_j' \) supported on a
Let \( \Omega_j \) be a holomorphic coordinate neighbourhood of \( \Omega \) with coordinate map \( \Phi \) and is identically 1 on the support of \( \chi_\gamma \). Lastly, \( M \) is the complement of the union of \( \{ \Omega_j \}_{1 \leq j \leq N} \) in \( M \).

By implementing solutions \( u \) and \( v \) from Proposition 3.1 into (3.1), we have

\[
0 = \int_{M_0} u V v d\gamma = \int_{M_0} e^{2i\nu\lambda} |a|^{2} V \gamma d\gamma + \sum_{k+k' \geq 1} \int_{M_0} e^{2i\nu\lambda} u_k v_{k'} d\gamma.
\]

Multiplying both sides of (5.1) by \( 2 \) and rearranging gives

\[
\frac{2\lambda \Omega_0(p_0)}{\pi} \int_{M_0} e^{2i\nu\lambda} |a|^{2} V \gamma d\gamma = -\frac{2\lambda \Omega_0(p_0)}{\pi} \sum_{k+k' \geq 1} \int_{M_0} e^{2i\nu\lambda} u_k v_{k'} d\gamma.
\]

In order to identify \( V \) from this expression, we will exploit the following \( L^2 \)-based method of stationary phase, the proof of which we recall from Lemma 3.3 in [2].

**Lemma 5.1.** Let \( V \) be in \( L^2(\mathbb{C}) \) and \( (V_\delta)_{\delta > 0} \) be a smooth approximation of \( V \) in \( L^2(\mathbb{C}) \), then for any \( s \in [0,1] \) we have

\[
\|V - \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V \|_{L^2} \lesssim \|V - V_\delta\|_{L^2} + \|V_\delta\|_{H^s_{\lambda^{\nu/2}}}.
\]

In particular, we have

\[
\lim_{\lambda \to \infty} \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V = V
\]

in \( L^2(\mathbb{C}) \).

**Proof.** By a standard calculation of the complex exponential and convolution theorem, we have

\[
\mathcal{F} \left( \frac{2\lambda}{\pi} e^{\pm 2i\nu\lambda \gamma} \ast V \right) = \mathcal{F} e^{\pm 2i\nu\lambda \gamma} \mathcal{F} V = e^{\pm \frac{2\nu e^{2i\lambda}}{\lambda}} \mathcal{F} V.
\]

Using the Fourier-Plancherel Theorem, we have

\[
\|V - \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V \|_{L^2} = \|\left(1 - e^{\pm \frac{2\nu e^{2i\lambda}}{\lambda}}\right) \mathcal{F} V\|_{L^2}.
\]

Now for all \( s \in [0,1] \) we may estimate that \( |1 - e^{\pm 2i\nu\lambda \gamma}| \approx 2^{\nu/2} |\gamma|^s \). Indeed, if \( |\gamma| \geq 1 \), then it is easy to see that \( |1 - e^{\pm 2i\nu\lambda \gamma}| \leq 2 \lesssim 2^{\nu/2} |\gamma|^s \) and so the result is obvious. On the other hand, if \( |\gamma| \leq 1 \), then a direct computation yields

\[
|1 - e^{\pm 2i\nu\lambda \gamma}|^2 = 4 \sin(|\gamma|^2 - |\gamma|^2)^2,
\]

therefore we have that

\[
|1 - e^{\pm 2i\nu\lambda \gamma}|^2 \lesssim |\gamma|^2 - |\gamma|^2 \leq |\gamma|^2 + |\gamma|^2 \leq 2^s |\gamma|^2
\]

By combining the above inequalities, we can extend (5.5) to

\[
\|V - \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V \|_{L^2} \lesssim \|\left(1 + |\gamma|^2\right) \mathcal{F} V\|_{L^2} = \|V\|_{H^s_{\lambda^{\nu/2}}}.
\]

Now let \( \{V_\delta\}_{\delta > 0} \) be a sequence of smooth functions in \( H^s \) such that \( \|V - V_\delta\|_{H^s} < \delta \). By inequality (5.6) we now have

\[
\|V - V_\delta\|_{L^2} \lesssim \|V - V_\delta\|_{L^2} \lesssim \|V - V_\delta\|_{L^2} < \delta.
\]

Triangle inequality now gives

\[
\|V - \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V\|_{L^2} \lesssim \|V - V\|_{L^2} + \|V_\delta - \frac{2\lambda}{\pi} e^{2i\nu\lambda \gamma} \ast V_\delta\|_{L^2} < \delta + \|V_\delta\|_{H^s_{\lambda^{\nu/2}}}.
\]

This proves (5.3). Letting \( \lambda \to \infty \) followed by \( \delta \to 0 \) and noticing the left hand side of (5.7) is independent of \( \delta \) concludes the proof of (5.1).
Our strategy now follows in a similar way as Section 4. Using Lemma 5.4, the left hand side of (5.2) will converge to $1_{\Omega_0} V |g|^{1/2}$ in $L^2(M)$ as $\lambda \to \infty$ while the right hand side vanishes in the limit. The same problem regarding the dependency of $a$ on both $p$ and $p_0$ remains, and we get over this issue with again the Taylor expansion of $a$ derived from Lemma 2.3.

5.1. Analysis of Principle Terms. In this subsection we study the integral

(5.8) \[ \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_M e^{2i\psi \lambda} |a|^2 V \, dv_g \]

from which we will recover the information of $V$ on $\Omega_0$.

Extending $V_1, V_2$ to $M$ by zero, we prove the following result analogous to Lemma 4.1.

**Lemma 5.2.** We have that

\[ \lim_{\lambda \to 0} \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_M e^{2i\psi \lambda} |a|^2 V \, dv_g = 1_{\Omega_0} V |g|^{1/2} \]

in $L^2(M)$.

**Proof.** We may write (5.8) as

(5.9) \[ \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_M e^{2i\psi \lambda} |a|^2 V \, dv_g \]

\[ = \sum_{j' \geq 0} \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_{\Omega_j} e^{2i\psi \lambda} |a|^2 V \, dv_g + \sum_{j \geq 0} \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_M \chi_j e^{2i\psi \lambda} |a|^2 V \, dv_g. \]

By making the change of variables $\Phi|_{\Omega_{j'}} (p) = z_{j'}$, we can apply Lemma 2.3 and the dominated convergence theorem to see that for every $1 \leq j' \leq N$, we have

(5.10) \[ \frac{2\lambda}{\pi} \Omega_0 (p_0) \int_{\Omega_{j'}} e^{2i\psi \lambda} |a|^2 V \, dv_g \]

\[ = \frac{2\lambda}{\pi} D_r (z_0) \int_{D_r} e^{2i\psi \lambda} |a|^2 V \, dv_g + \sum_{\mu, \mu' \geq 0} \frac{2\lambda}{\pi} D_r (z_0) \int_{D_r} e^{2i\psi \lambda} |a|^2 V \, dv_g \]

\[ \times a_{\mu} (\Phi|_{\Omega_{j'}} (z_j)) a_{\mu'} (\Phi|_{\Omega_{j'}} (z_j)) V (\Phi|_{\Omega_{j'}} (z_j)) |g_{j'}|^{1/2} d z_{j'} d \bar{z}_{j'}. \]

We want to take a $L^2(M)$ limit as $\lambda \to \infty$ in the last sum of (5.10). For this we note that since $V$ is in $L^2(M_0)$, we have from inequality (3.24), (5.3) and (5.4) the estimate

\[ \left\| \frac{2\lambda}{\pi} \int_{D_r} e^{2i\psi \lambda} |a|^2 V \, dv_g \right\| L^2(D_r) \]

\[ \leq \sum_{\mu, \mu' \geq 0} r^{\mu + \mu'} \left\| o_{L^2, \lambda} (1) \right\| L_2 \sum_{\mu, \mu' \geq 0} r^{\mu + \mu'} \left\| a_{\mu} \right\| L^\infty(M_0) \left\| a_{\mu'} \right\| L^\infty(M_0) \left\| V \right\| L^2 \lesssim \left\| V \right\| L^2(M_0) < \infty, \]
where \( o_{L^2,\lambda}(\mu,\mu') \to 0 \) as \( \lambda \to \infty \) in \( L^2 \) depending on \( \mu, \mu' \geq 0 \). It follows from (5.10) that the above calculation implies

\[
\lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} \int_{\Omega_j'} e^{2i\psi \lambda} |a|^2 V \, dv_g = 1_{D_r}(z_0) [a(\Phi^{-1}_{|\tilde{\Omega}_j'} (z_j'); \Phi^{-1}_{|\tilde{\Omega}_0} (z_0))]^2 V(\Phi^{-1}_{|\tilde{\Omega}_j'} (z_j')) |g_{j'}|^{1/2} |z_{j'} = z_0
\]

in \( L^2(M) \) for every \( 1 \leq j' \leq N \). Thanks to (5.11), summing over all such \( j' \) in (5.11) yields

\[
\lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} \sum_{j' \geq 0} \int_{\Omega_j'} e^{2i\psi \lambda} |a|^2 V \, dv_g = 1_{\Omega_0} V |g|^{1/2}
\]

which is the required asymptotic. Now since \( \bar{M} \) contains no point \( p \in M_0 \) for which we have \( \Phi(p) \in D_r \), we must have \( \Phi(M \cap M_0) \) and \( D_r \) are disjoint. Thus it is easy to see from the same arguments as above that

\[
\lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} \int_{\tilde{M}} \chi_j e^{2i\psi \lambda} |a|^2 V \, dv_g = \sum_{\mu, \mu' \geq 0} 1_{D_r} \mathbf{1}_{\Phi(M \cap M_0 \cap \Omega_r)}^{\mu, \mu'} \times [\chi_j (\Phi^{-1}_{|\tilde{\Omega}_j} (z_j)) a_{\mu}(\Phi^{-1}_{|\tilde{\Omega}_j} (z_j)) \bar{a}_{\mu'}(\Phi^{-1}_{|\tilde{\Omega}_j} (z_j)) V(\Phi^{-1}_{|\tilde{\Omega}_j} (z_j)) |g_j|^{1/2} |z_j = z_0 = 0
\]

for each \( j \geq 1 \). Combining (5.9) with (5.12) and (5.13) concludes the proof of the claim. \( \square \)

### 5.2. Analysis of Remainder Terms

In this subsection we show that

\[
\lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} \sum_{k+k' \geq 1} I_{k,k'} = o_{L^2}(1)
\]

where \( o_{L^2}(1) \to 0 \) as \( \epsilon \to 0 \) in \( L^2(M) \). As in Section 4, additional arguments are required for the lower order terms. Nevertheless, for the cases \( k + k' = 1 \) we need to argue more carefully since we now require these terms to be of order \( o_{L^2}(1) \). For this we will make use of the construction of \( \{ b_{j'} \}_{1 \leq j' \leq N} \) introduced in Lemma 2.2.

**Lemma 5.3.** We can choose sequences \( \{ Q_{j', \epsilon}^+ \}_{1 \leq j' \leq N} \) and \( \{ Q_{j', \epsilon}^- \}_{1 \leq j' \leq N} \) so that

\[
\lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} I_{1,0} = \lim_{\lambda \to \infty} \frac{2\lambda \mathbf{1}_{\Omega_0}(p_0)}{\pi} I_{0,1} = o_{L^2}(1)
\]

where \( o_{L^2}(1) \to 0 \) as \( \epsilon \to 0 \) in \( L^2(M) \).

**Proof.** Extending \( V_1, V_2 \) to \( M \), since \( \tilde{\rho} \) is compactly supported in \( M_0' \), we can write \( I_{1,0} \) as

\[
I_{1,0} = \sum_{1 \leq j' \leq N} \int_M V \tilde{a} T_1 \mathbf{1}_{\tilde{\Omega}_j'} e^{2i\psi \lambda} \tilde{\rho} (T^* V_1 a - \sum_{0 \leq k \leq N} Q_{j', \epsilon}^+(p_0) b_k) \, dv_g
\]

\[
+ \sum_{j \geq 0} \int_M V \tilde{a} T_1 \tilde{\rho} (T^* V_1 a - \sum_{0 \leq k \leq N} Q_{j', \epsilon}^+(p_0) b_k) \, dv_g,
\]

(5.14)
Since $\tilde{\chi}_j$ is identically 1 on $\tilde{\Omega}_{j'}$, for each $0 \leq j' \leq N$ we have that

\begin{equation}
(5.15)
\int_M V a T_1 \tilde{\Omega}_{j'} e^{2i\psi \lambda} \tilde{\rho} \left( T^* V_1 a - \sum_{0 \leq k \leq N} Q^+_{k,\epsilon}(p_0) b_k \right) dv_g
= \int_M V a \tilde{\chi}'_j R(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} T^* V_1 a) dv_g - \sum_{0 \leq k \leq N} Q^+_{k,\epsilon}(p_0) \int_M V a \tilde{\chi}'_j R(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} b_k) dv_g
+ \int_M V a K_{j'}(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} T^* V_1 a) dv_g - \sum_{0 \leq k \leq N} Q^+_{k,\epsilon}(p_0) \int_M V a K_{j'}(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} b_k) dv_g.
\end{equation}

We analyse the two differences in (5.15). By making the change of variables $\Phi_{\tilde{\Omega}_{j'}}(p) = z_{j'}$ and Fubini’s theorem, for each $j'$ we have

\[1_{\tilde{\Omega}_{0}}(p_0) \int_M V a \tilde{\chi}'_j R(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} T^* V_1 a) dv_g\]

\[= -1_{D_{\epsilon}}(z_0) \int_{D_{\epsilon}} R(V a \tilde{\chi}'_j |g_{j'}|^{1/2}) e^{2i\text{Re}(z_{j'} - z_0)^2 \lambda} \tilde{\chi}'_j \tilde{\rho} T^* (V_1 a) dz_{j'} d\tilde{z}_{j'}\]

\[= - \sum_{\mu, \mu' \geq 0} 1_{D_{\epsilon}}(z_0) \tilde{z}_0^\mu \tilde{z}_0^{\mu'} \int_{D_{\epsilon}} R(V a \mu \tilde{\chi}'_j |g_{j'}|^{1/2}) e^{2i\text{Re}(z_{j'} - z_0)^2 \lambda} \tilde{\chi}'_j \tilde{\rho} T^* (V_1 a_{\mu'}) dz_{j'} d\tilde{z}_{j'},\]

where the final expansion in the above can be justified with Lemma 2.3 and Sobolev embeddings. We also made the identification $\tilde{\chi}'_j T^* (V_1 a) = \tilde{\chi}'_j T^* (V_1 a) dz_{j'}$ on local charts. Recall from (4.17) that

\[\|R(V a \mu \tilde{\chi}'_j |g_{j'}|^{1/2}) \tilde{\chi}'_j \tilde{\rho} T^* (V_1 a_{\mu'}) \|_{L^2(D_{\epsilon})} \lesssim \|a_{\mu} \|_{L^\infty(M_0)} \|a_{\mu'} \|_{L^\infty(M_0)} \|V\|_{L^4(3(M_0))} \|V_1\|_{L^4(3(M_0))} \]

can be bounded independently of $\mu, \mu' \geq 0$. Thus we can apply Lemma 5.2 to find functions of order $o_{L^2, \lambda}((\mu, \mu'))$ so that

\begin{equation}
(5.16)
\left\| \frac{2\lambda}{\pi} \int_M V a \tilde{\chi}'_j R(1_{D_{\epsilon}} e^{2i\text{Re}(z_{j'} - z_0)^2 \lambda} \tilde{\chi}'_j \tilde{\rho} T^* V_1 a) dv_g
+ \sum_{\mu, \mu' \geq 0} \tilde{z}_0^\mu \tilde{z}_0^{\mu'} \left[ R(V a \mu \tilde{\chi}'_j |g_{j'}|^{1/2}) \tilde{\chi}'_j \tilde{\rho} T^* (V_1 a_{\mu'}) \right]_{|z_{j'} = z_0} \right\|_{L^2(D_{\epsilon})}
\lesssim \left( \sum_{\mu, \mu' \geq 0} \tilde{z}_0^\mu \tilde{z}_0^{\mu'} \|a_{\mu} \|_{L^\infty(M_0)} \|a_{\mu'} \|_{L^\infty(M_0)} \right) \|V\|_{L^4(3(M_0))} \|V_1\|_{L^4(3(M_0))} < \infty
\end{equation}

In particular, (5.16) implies that

\begin{equation}
\lim_{\lambda \to \infty} \frac{2\lambda}{\pi} \int_M V a \tilde{\chi}'_j R(1_{\tilde{\Omega}_{j'}} e^{2i\psi \lambda} \tilde{\chi}'_j \tilde{\rho} T^* V_1 a) dv_g
\end{equation}

\[= - \sum_{1 \leq j' \leq N} \sum_{\mu, \mu' \geq 0} 1_{D_{\epsilon}}(z_0) \tilde{z}_0^\mu \tilde{z}_0^{\mu'} \left[ R(V a \mu \tilde{\chi}'_j |g_{j'}|^{1/2}) \tilde{\chi}'_j \tilde{\rho} T^* (V_1 a_{\mu'}) \right]_{|z_{j'} = z_0}\]

in $L^2(M)$. On the other hand, on the support of $\tilde{\chi}'_j$ we may exploit the Taylor expansion of $b$ introduced in Lemma 2.3 so that for every $p_0 \in \tilde{\Omega}_{0}$, we have

\[\tilde{\chi}'_j(p)b_k(p; p_0) = \sum_{\mu \geq 0} \tilde{\chi}'_j(z_{j'}) b_{k,\mu}^{(j')} (z_{j'}) \tilde{z}_0^\mu dz_{j'}\] for all $1 \leq j', k \leq N$. 

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By \[2,14\] such an expansion satisfies the same convergence property of \(a\), thus we have

\[
\begin{align*}
1_{\Omega_0}(p_0)Q_{k,\epsilon}^+(p_0) & \int_M V\bar{\alpha}\tilde{\chi}_j^v R(1_{\Omega_j^v}e^{2i\psi\lambda}\tilde{\chi}_j^v\tilde{p}b_k)\, dv_g \\
= & -\sum_{\mu,\mu'\geq 0} 1_{D_r}(z_0)Q_{k,\epsilon}^+(z_0)\bar{z}_{\mu',0}^\mu \int_{D_r} R(V\bar{\alpha}\mu\tilde{\chi}_j^v|g_j'|^{1/2})e^{2i(z_j'-z_0)^2\lambda\tilde{\chi}_j^v\tilde{p}(\mu\tilde{b}_k)}\, dz_j'dz_j'
\end{align*}
\]

for each \(1 \leq j', k \leq N\). Now arguing in exactly the same way as in \[5,17\], it is easy to see that \[5,18\]

\[
\lim_{\lambda \to \infty} \frac{2\lambda 1_{\Omega_0}(p_0)Q_{k,\epsilon}^+(p_0)}{\pi} \int_M V\bar{\alpha}\tilde{\chi}_j^v R(1_{\Omega_j^v}e^{2i\psi\lambda}\tilde{\chi}_j^v\tilde{p}b_k)\, dv_g = -\sum_{\mu\geq 0} 1_{D_r}(z_0)Q_{k,\epsilon}^+(z_0)\bar{z}_{\mu,0}^\mu [R(V\bar{\alpha}\mu\tilde{\chi}_j^v|g_j'|^{1/2})]_{z_j'=z_0}(\tilde{\chi}_j^v\tilde{p})(\tilde{\Phi}^{-1}_{|\Omega_j^v}(z_0))\tilde{b}_k^{j'}(\tilde{\Phi}^{-1}_{|\Omega_j^v}(z_0); \tilde{\Phi}^{-1}_{|\Omega_0}(z_0))
\]

in \(L^2(M)\). By formula \[2,8\] we obviously have

\[
b_{j'}^{j'}(\tilde{\Phi}^{-1}_{|\Omega_j^v}(z_0); \tilde{\Phi}^{-1}_{|\Omega_j^v}(z_0)) = \begin{cases} 
1 & \text{if } j' = k, \\
0 & \text{if } j' \neq k
\end{cases}
\]

for all \(z_0 \in D_r\) and \(1 \leq j', k \leq N\). Summing over all such \(j', k\) we see from \[5,19\] that \[5,20\]

\[
\lim_{\lambda \to \infty} \sum_{1 \leq j', k \leq N} \frac{2\lambda 1_{\Omega_0}(p_0)Q_{k,\epsilon}^+(p_0)}{\pi} \int_M V\bar{\alpha}\tilde{\chi}_j^v R(1_{\Omega_j^v}e^{2i\psi\lambda}\tilde{\chi}_j^v\tilde{p}b_k)\, dv_g
\]

To take care of the smoothing terms in \[5,15\], we apply the exact same procedure to get that

\[
\lim_{\lambda \to \infty} \frac{2\lambda 1_{\Omega_0}(p_0)}{\pi} \sum_{1 \leq j', k \leq N} \int_M V\bar{\alpha}K_j'(1_{\Omega_j^v}e^{2i\psi\lambda}\tilde{\chi}_j^v\tilde{p}T^*V_1\alpha)\, dv_g
\]

\[
= \sum_{1 \leq j', k \leq N} \sum_{\mu,\mu'\geq 0} 1_{D_r}(z_0)\bar{z}_{\mu',0}^\mu [K_j'(V\bar{\alpha}\mu)(\tilde{\chi}_j^v\tilde{p}T^*(V_1a_{\mu'}))|g_j'|^{1/2}]_{z_j'=z_0},
\]

\[5,21\]

in \(L^2(M)\) where \(K_j'\) has smooth kernel. By Sobolov’s inequality, we have

\[
\sum_{\mu'\geq 0} \|z_{\mu}^\mu[T^*(V_1a_{\mu'})]\|_{L^4} \leq \sum_{\mu'\geq 0} r_{\mu'}\|T^*(V_1a_{\mu'})\|_{W^{1,4/3}} \leq \sum_{\mu'\geq 0} \|V_1\|_{L_{4/3}} r_{\mu'}\|a_{\mu'}\|_{L^\infty} < \infty.
\]

Since \(\{Q_{j',\epsilon}\}_{1 \leq j' \leq N} \subset C_c^\infty(M)\) were fixed arbitrarily, putting \(5,17\), \(5,20\) and \(5,21\) together, we see from \[5,15\] that if we choose them to be smooth approximations such that

\[
\lim_{\epsilon \to 0} 1_{D_r}(z_0)Q_{j',\epsilon}^+(z_0) = \sum_{\mu'\geq 0} 1_{D_r}(z_0)\bar{z}_{\mu',0}^{\mu} [T^*(V_1a_{\mu'})]_{z_j'=z_0}, \quad 1 \leq j' \leq N
\]
in $L^2(M)$, then we have
\[
\lim_{\lambda \to \infty} \sum_{1 \leq j' \leq N} \int_M V \tilde{a} T \chi_j e^{2i\nu \lambda} \tilde{\rho} T^*(V_1 a) \, dv_g = o_{L^2,L}(1)
\]
where $\lim_{\epsilon \to 0} o_{L^2,L}(1) = 0$ in $L^2(M)$. By construction, $\tilde{M}$ contains no point $p \in M_0$ for which we have $\Phi(p) \in D_\rho$. By taking $M'_0$ small enough, we may assume without loss of generality that $\text{Supp} \tilde{\rho}$ is disjoint from those neighbourhood $\Omega \subset M \setminus M_0$ such that $\Phi(\Omega) \cap D_\rho \neq \emptyset$. Thus $\Phi(\tilde{M} \cap \text{Supp} \tilde{\rho})$ and $D_\rho$ are disjoint, and the same procedure yields
\[
\lim_{\lambda \to \infty} \int_M V \tilde{a} T \chi_j e^{2i\nu \lambda} \tilde{\rho} (T^* V_1 a - \sum_{1 \leq k \leq N} Q_{\lambda k}^+(p_0) b_k) \, dv_g = 0, \quad j \geq 0.
\]
Indeed, it is sufficient to split $T \chi_j$ into linear combinations of $\chi_j R \chi_j'$ and $K_j$ and apply Lemma 5.2 and Fubini’s theorem as before, at which point the expression $(5.22)$ appears in the resulting limits, and we conclude from the remarks above that $(5.22)$ vanishes. The claim for the case of $I_{1,0}$ now follows from expression $(5.14)$. The obvious modifications hold for the case of $I_{0,1}$.

Finally we show that
\[
\lim_{\lambda \to \infty} \frac{2\lambda_1 \Omega_0(p_0)}{\pi} \sum_{k+k' \geq 2} I_{k,k'} = 0
\]
in $L^2(M)$. For this we rely only on the Carleman estimates derived in Section 3.

**Lemma 5.4.** There exists constants $C, \tilde{C} > 0$ independent of $\lambda$ and $p_0$ such that
\[
|I_{k,k'}| \leq \frac{\tilde{C}}{\lambda^{1+/(4-\frac{1}{p})}} \left( \frac{C \max\{\|V_1\|_{L^p(M_0)}, \|V_2\|_{L^p(M_0)}\}}{\lambda^{\theta+}} \right)^{k+k'-4}
\]
for all $k + k' \geq 4$. If $2 \leq k + k' < 4$, then we have
\[
|I_{k,k'}| \leq \frac{\tilde{C}}{\lambda^{1+}}.
\]

**Proof.** Let $p'$ be defined by $1/p' = 1/p - 1/2$. Set $1/q = 1/p + 1/p' = 2/p - 1/2$. Since $p \in [4/3, 2]$ [we can choose $r \in [2, 4]$ by $1/r = 1/p - 1/4$ so that $1/2 + 1/r \geq 1/q > 1/2$. Hence if $k' = 0$, then by Corollary 3.11 we have
\[
|I_{k,0}| \leq \int_{M_0} |V| |u_k| |a| \, dv_g \leq \|aV\|_{L^2} \|T \tilde{\rho} T^*(V_1 u_{k-1})\|_{L^{r'}} \leq \|a\|_{L^1} \|V\|_{L^2} \|V_1 u_{k-1}\|_{L^2} \leq \|a\|_{L^1} \|V\|_{L^2} \|V_1\|_{L^p} \|u_{k-1}\|_{L^{p'}}.
\]
If $k = 2$, then we can directly estimate from Sobolev embedding and (3.3) that
\[
\|u_{k-1}\|_{L^{p'}} \leq \|T \tilde{\rho} T^*(V_1 a)\|_{L^{p'}} + \sum_{1 \leq j' \leq N} \|Q_{j',\epsilon}^+\|_{L^\infty} \|T \tilde{\rho} b_{j'}\|_{L^{p'}} \leq \|a\|_{L^1} \|V_1\|_{L^p} \frac{\max_{1 \leq j' \leq N} \{\|Q_{j',\epsilon}^+\|_{L^\infty} \|b_{j'}\|_{L^{p'}}\}}{\lambda^{1-(\frac{1}{2}+\frac{1}{2})}}.
\]
Alternatively if \( k \geq 3 \), then we iterate the \( L^p \to L^\infty \) estimate obtained in (3.20) to get that there exists \( C > 0 \) such that

\[
\|u_{k-1}\|_{L^{p'}} \leq \frac{\|V_1\|_{L^p} \|u_{k-2}\|_{L^\infty}}{\lambda^{1-\left(\frac{2}{p'}\right)}} \leq \frac{\|a\|_{L^\infty} \|V_1\|_{L^p} \left( \frac{C \|V_1\|_{L^p}}{\lambda^{0+}} \right)^{k-2} + \max_{1 \leq j' \leq N} \{ \|Q_{j',\epsilon}^+\|_{L^\infty} \|b_{j'}\|_{L^\infty} \} \left( \frac{C \|V_1\|_{L^p}}{\lambda^{0+}} \right)^{k-3}}{\lambda^{1-\left(\frac{2}{p'}\right)}}.
\]  

(5.27)

A similar calculation works for the case \( k = 0 \).

In the cases where \( k, k' \neq 0 \), we can apply Hölder’s inequality to see that

\[
\int_{M_0} |V| u_k v_{k'} \, dv_g \leq \|V\|_{L^2} \|T_\Psi \tilde{\partial}_T^*(V_1 u_{k-1})\|_{L^4} \|\bar{T}_\Psi \tilde{T}^*(V_2 v_{k'-1})\|_{L^4}
\]

\[
\lesssim \frac{\|V\|_{L^2} \|V_1\|_{L^p} \|V_2\|_{L^p} \|u_{k-1}\|_{L^\infty} \|v_{k'-1}\|_{L^\infty}}{\lambda^{1+}}.
\]

As above we can now enumerate the \( L^p \to L^\infty \) estimates to get that

\[
\|u_{k-1}\|_{L^\infty} \lesssim \|a\|_{L^\infty} \|V_1\|_{L^p} \left( \frac{C \|V_1\|_{L^p}}{\lambda^{0+}} \right)^{k-1} + \max_{1 \leq j' \leq N} \{ \|Q_{j',\epsilon}^+\|_{L^\infty} \|b_{j'}\|_{L^\infty} \} \left( \frac{C \|V_1\|_{L^p}}{\lambda^{0+}} \right)^{k-2},
\]

\[
\|v_{k'-1}\|_{L^\infty} \lesssim \|a\|_{L^\infty} \|V_2\|_{L^p} \left( \frac{C \|V_2\|_{L^p}}{\lambda^{0+}} \right)^{k'-1} + \max_{1 \leq j' \leq N} \{ \|Q_{j',\epsilon}^-\|_{L^\infty} \|b_{j'}\|_{L^\infty} \} \left( \frac{C \|V_2\|_{L^p}}{\lambda^{0+}} \right)^{k'-2}
\]

for all \( k, k' \geq 2 \). In particular, putting together estimates (5.27) and (5.28) implies (5.24) for sufficiently large \( \lambda \). On the other hand, if \( k = k' = 1 \), then

\[
\int_{M_0} |V| u_1 v_1 \, dv_g \leq \frac{\|V\|_{L^2}^2 \left( \|V_1\|_{L^p} \|a\|_{L^\infty} + \max_{1 \leq j' \leq N} \|Q_{j',\epsilon}^+\|_{L^\infty} \|b_{j'}\|_{L^\infty} \right)}{\lambda^{1+}} \times \left( \|V_2\|_{L^p} \|a\|_{L^\infty} + \max_{1 \leq j' \leq N} \|Q_{j',\epsilon}^-\|_{L^\infty} \|b_{j'}\|_{L^\infty} \right).
\]

(5.29)

Combining (5.20), (5.29) and the other estimates now implies (5.25). □

By applying the estimates in Lemma 5.1 we easily see that

\[
\left| \frac{2 \lambda \Omega_0(p_0)}{\pi} \right| \sum_{k+k' \geq 2} I_{k,k'} \leq \frac{\tilde{C}}{\lambda^{0+}} + \frac{\tilde{C}}{\lambda^{0+}} \sum_{k+k' \geq 4} \left( \frac{C \max \{ \|V_1\|_{L^p(M_0)}, \|V_2\|_{L^p(M_0)} \}}{\lambda^{0+}} \right)^{k+k'-4}.
\]

(5.30)

The last term in (5.30) converges for sufficiently large \( \lambda > \lambda_0 \) and is in particular of order \( o(1) \) as \( \lambda \to \infty \). Since \( \tilde{C} \) and \( \tilde{C} \) are independent of \( p_0 \), we can take this limit in (5.2) and deduce from the sequence of lemmas proved above that

\[
\Omega_0 |V|^1_{L^2} = o_{L^2}(1)
\]

for all \( \epsilon > 0 \). Letting \( \epsilon \to 0 \) now yields \( \Omega_0 |V|^1_{L^2} = 0 \). Since \( |g| \) is non-vanishing we must therefore have \( V = 0 \) almost everywhere on \( \Omega_0 \). This concludes the proof of the claim. □

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