A SUM RULE APPROACH TO THE VIOLATION OF DASHEN’S THEOREM

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Abstract

A classic sum rule by Das et al. is extended to seven of the low-energy constants $K_i$, introduced by Urech, which parameterize electromagnetic corrections at chiral order $O(e^2 p^2)$. Using the spurion formalism, a simple convolution representation is shown to hold and the structure in terms of the chiral renormalization scale, the QCD renormalization scale and the QED gauge parameter is displayed. The role of the resonances is studied as providing rational interpolants to relevant QCD n-point functions in the euclidean domain. A variety of asymptotic constraints must be implemented which have phenomenological consequences. A current assumption concerning the dominance of the lowest-lying resonances is shown clearly to fail in some cases.

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1. Introduction:

Thirty years ago, Das et al.\cite{DGMLY} (DGMLY) derived a remarkable relation between the mass difference of the charged and neutral pions and the masses of the lightest vector and axial vector resonances,

\[ M_{\pi^+}^2 - M_{\pi^0}^2 = \frac{3e^2}{16\pi^2} \frac{M_A^2 M_V^2}{M_A^2 - M_V^2} \log \frac{M_A^2}{M_V^2}. \]  

(1)

This relation follows from a sum rule which is exact in the chiral limit (i.e. \( m_u = m_d = m_s = 0 \)) under the only extra assumption that the lowest-lying vector and axial vector meson resonances make the essential contribution to the integral. The physical \( \pi^+ - \pi^0 \) mass difference is nearly purely electromagnetic in origin and happens to be rather accurately described by eq. (1). The analogous mass difference of kaons, \( M_{K^+}^2 - M_{K^0}^2 \), has an electromagnetic contribution and a purely QCD contribution proportional to \( m_u - m_d \) which are approximately of the same magnitude. Knowledge of the electromagnetic contribution allows one to access the value of the quark mass difference \( m_u - m_d \) (divided by, say, \( m_u + m_d \)) using chiral perturbation theory\cite{2}.

It has long been believed that estimating the EM contribution to \( M_{K^+}^2 - M_{K^0}^2 \) in the chiral limit was sufficient. In this limit, it is given by Dashen’s theorem\cite{3} (DT) to be equal to \( M_{\pi^+}^2 - M_{\pi^0}^2 \).

This approximation is now known to fail for the purpose of extracting \( m_u - m_d \). In particular, the value of the \( \eta \) decay rate \( \Gamma(\eta \to 3\pi) \) that one would predict (at one loop and including estimates of higher loop corrections \cite{4}\cite{5}) would be too small by as much as a factor of two compared to experiment.

Thus, it is necessary to estimate the EM contributions to \( M_{K^+}^2 - M_{K^0}^2 \) beyond the chiral limit (the difference from \( M_{\pi^+}^2 - M_{\pi^0}^2 \) is customarily, but somewhat inappropriately, referred to as the violation of Dashen’s theorem). There have been many attempts over the years in this direction\cite{6}\cite{7}\cite{8}\cite{9}\cite{10}\cite{11}\cite{12} (a representative, but non-exhaustive list) and, very recently, a lattice calculation has appeared\cite{13}. Several rather different approaches to this problem have been followed. The work of refs.\cite{9} and \cite{11} is based on the assumption that most of the information on DT violation is contained in the parameters of the lowest lying resonances, in a way similar to eq. (1). Unfortunately, the results of these two papers (which should be identical) are not in very good agreement with each other\footnote{A diagram was incorrectly evaluated in ref.\cite{9}. An update of this calculation has appeared very recently\cite{38}, which obtains a very similar result numerically.}. One of the purposes of the present paper is to investigate in some detail the validity of this assumption.

As shown by Urech\cite{14}, at low energies, electromagnetic effects can be parametrized in a chiral lagrangian framework\cite{15}\cite{16}\cite{2}. In this formalism, the leading electromagnetic contributions are computed from the effective lagrangian at tree level, which contains a single low-energy constant (LEC): the result of Das et al. provides a sum rule for this LEC. At next-to-leading order, the electromagnetic corrections are obtained by computing the photon loop (as well as the pion loops) from the leading order lagrangian and adding the contributions at tree level from the next order terms in the lagrangian, which involves (essentially\cite{14}) 13 new low-energy constants \( K_1, K_2, \ldots, K_{13} \). In this paper, we will propose a sum rule evaluation of the seven parameters \( K_7, \ldots, K_{13} \). The \( K^+ - K^0 \) mass difference actually involves two more constants, \( K_5 \) and \( K_6 \), which we will not attempt to evaluate. Little was known, up to now, on the individual\footnote{There are further constants which either play no role at low energy or correspond to \( e^4 \) contributions.}
values of these LEC’s (estimates for some combinations were given recently\textsuperscript{12}) except for their order of magnitude which, for consistency of the chiral counting for the electric charge, should be the same as that of the ordinary $O(p^4)$ constants $L_i$. Individual knowledge of these constants is useful for the purpose of evaluating radiative corrections. Indeed, as precision increases in the predictions of chiral perturbation theory, the computation of radiative corrections is becoming a topic of increased interest. This is already true for the $\pi\pi$ scattering amplitude which is now known at the level of two loops\textsuperscript{17} \textsuperscript{18}, and which will be subjected to a precision test in the planned experiment DIRAC\textsuperscript{19}.

In general, we will show that the $K_i$’s can be expressed as a convolution of a QCD correlation function with the electromagnetic propagator, plus a contribution from the QED counterterms which remove the divergence of the integral. The LEC’s $K_1,...,K_6$ are related to QCD 4-point functions, while $K_7,...,K_{13}$ are related to QCD two- and three-point functions. In that case, we will show that the contribution of the resonances can be discussed independently of any specific lagrangian model for resonances. An important question concerns the validity of the approximation of retaining only the lightest multiplet of resonances in each channel. One may view resonance saturation as a method of constructing rational interpolants to the QCD n-point functions in the euclidian region. These interpolants must satisfy certain asymptotic constraints, in order for the QED divergencies to cancel out, and one also expects them to be reasonably precise at low momenta, i.e. to some extent in the resonance region as well. It is not clear that low order interpolants are capable of satisfying all these constraints.

The plan of the paper is as follows. In the next section, we discuss the form of the QED counterterms when the spurion formalism is applied to the electric charge. The use of this formalism is a key technical ingredient in the derivation of the sum rules. In sec.3, we discuss a set of sum rules for the parameters $K_{11}, K_{12}$ and $K_{13}$ which involve QCD correlators in the chiral limit. We will show that a single multiplet of vector and of axial-vector resonances is enough to obey adequately all the asymptotic constraints, and we obtain expressions which are rather neat generalizations of eq.(1). Next, in sec.4, we discuss $K_7, K_8, K_9$ and $K_{10}$. The first two are suppressed by the Zweig rule and the latter two can be expressed in terms of flavour symmetry breaking differences of vector and axial-vector spectral functions. The phenomenology in terms of resonances is then discussed in sec.5 and the application to the violation of Dashen’s theorem in sec.6.

2. The spurion formalism and the QED counterterms

The spurion formalism applied to the electric charge matrix\textsuperscript{20} $q = \text{diag}(2/3, -1/3, -1/3)$ consists in calling $q$ by two different names $q_L$ and $q_R$ such that the QED interaction lagrangian for the quarks can be split into two pieces:

$$
\mathcal{L}_{QED} = i\bar{\psi}(\partial + i q A)\psi \equiv i\bar{\psi}_L(\partial + iq_L A)\psi_L + i\bar{\psi}_R(\partial + iq_R A)\psi_R
$$

(with $\psi^t = (u, d, s)$ ). One may thus render the QED lagrangian invariant under the chiral group by assuming appropriate chiral transformation properties for the two spurions $q_L$ and $q_R$. This is the same method as has been applied to the mass matrix\textsuperscript{16}. In order to take quantum electrodynamical effects into account, the chiral lagrangian must include the photon as a dynamical field, in addition to the octet of pseudo-Goldstone boson fields. As usual, the chiral lagrangian will consist of the most general set of local interaction terms which are invariant under the chiral group and which are classified according to increasing chiral order.
In the EM sector, the correct counting rules, introduced by Urech\cite{14}, state that the photon field $A_\mu$ is of order $O(p^0)$ and the charge spurions $q_L$ and $q_R$ are of order $O(p)$. At order two, the chiral lagrangian consists, firstly, of the usual terms,

$$\mathcal{L}_a^{(2)} = \frac{F_0^2}{4} < D_\mu U D^\mu U^\dagger + U X^\dagger + \chi U^\dagger >,$$

where, however, the covariant derivatives contain $q_L$ and $q_R$ in addition to the usual vector and axial-vector external sources

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu + q_R A_\mu)U + iU(v_\mu - a_\mu + q_L A_\mu).$$

In addition to that, one must include the purely photonic lagrangian,

$$\mathcal{L}_b^{(2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

and, finally, there is a single invariant term with two spurions that one can form\cite{20},

$$\mathcal{L}_c^{(2)} = C < q_R U q_L U^\dagger > .$$

Equations (3), (5), (6) define the most general lagrangian of chiral order two which satisfies EM gauge invariance in the limit of constant spurions. In order to quantize the electromagnetic field, one must include a gauge fixing term and it is also useful to introduce a small photon mass $M_\gamma$ in order to regulate infrared divergencies at intermediate steps of the calculations, this is performed by adding

$$\mathcal{L}_d^{(2)} = \frac{1}{2} M_\gamma^2 A_\mu A^\mu - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 .$$

The corresponding photon propagator is

$$- iD_{\mu\nu}(p) = \frac{-i}{p^2 - M_\gamma^2} \left[ g_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2 - \xi M_\gamma^2} \right].$$

At chiral order two, Green’s functions and observables are computed by using $\mathcal{L}^{(2)}$ at tree level. At the next chiral order, $O(p^4)$, one must firstly compute the one-loop graphs with vertices generated from $\mathcal{L}^{(2)}$. The loops will involve both pion and photon internal lines in the present case. Secondly, one must add the tree contributions from $\mathcal{L}^{(4)}$. The lagrangian $\mathcal{L}^{(4)}$ will consist, firstly, of the usual terms of Gasser and Leutwyler\cite{4}. In addition, one can construct a set of terms with two EM spurions\cite{3} which were classified by Urech. For convenience we will separate the terms which involve two derivatives of the pion fields,

$$\mathcal{L}^{(4)}_{1-6} = \frac{1}{2} K_1 F_0^2 < D_\mu U D^\mu U^\dagger > < q_L q_L + q_R q_R >$$

$$+ K_2 F_0^2 < D_\mu U D^\mu U^\dagger > < q_L U^\dagger q_R U >$$

$$+ K_3 F_0^2 (< D_\mu U q_L U^\dagger >^2 + < D_\mu U^\dagger q_R U >^2)$$

$$+ K_4 F_0^2 < D_\mu U q_L U^\dagger > < D_\mu U^\dagger q_R U >$$

$$+ K_5 F_0^2 < q_L U^\dagger D^\mu U q_L U + D_\mu U D^\mu U^\dagger q_R U >$$

$$+ K_6 F_0^2 < D_\mu U^\dagger D^\mu U q_L U^\dagger q_R U + q_L D_\mu U D^\mu U^\dagger > .$$

Symmetry considerations alone do not rule out terms with a single spurion and no $A_\mu$, for example a term like $< q_L D_\mu U U^\dagger + q_R D_\mu U U^\dagger >$. The technique to be used repeatedly below immediately shows that the coefficients of such terms vanish because the vacuum expectation value of the photon field does.
from the remaining ones

\[ L_{T-13}^{(4)} = \frac{1}{2} K_7 F_0^2 <\chi^\dagger U + U^\dagger \chi > q_L q_L + q_R q_R > \]  

\[ + K_8 F_0^2 <\chi^\dagger U + U^\dagger \chi > q_L U^\dagger q_R U > \]  

\[ + K_9 F_0^2 < (\chi^\dagger U + U^\dagger \chi) q_L q_L + (\chi U^\dagger + U^\dagger \chi) q_R q_R > \]  

\[ + K_{10} F_0^2 < (\chi^\dagger U + U^\dagger \chi) q_L U^\dagger q_R U + (\chi U^\dagger + U^\dagger \chi) q_R U q_L U^\dagger > \]  

\[ + K_{11} F_0^2 < (\chi^\dagger U - U^\dagger \chi) q_L U^\dagger q_R U + (\chi U^\dagger - U^\dagger \chi) q_R U q_L U^\dagger > \]  

\[ + K_{12} F_0^2 < U D^\mu U^\dagger [D_\mu q_R, q_R] + U^\dagger D^\mu U [D_\mu q_L, q_L] > \]  

\[ + K_{13} F_0^2 < D_\mu q_R U D^\mu q_L U^\dagger >, \]  

where

\[ D_\mu q_L = \partial_\mu q_L - i[v_\mu - a_\mu, q_L], \quad D_\mu q_R = \partial_\mu q_R - i[v_\mu + a_\mu, q_R]. \]  

Further terms which contain four EM spurions as well as terms which vanish when \( A_\mu \) is set to zero will not be considered in the following.

All these terms are counted as \( O(p^4) \) according to the standard chiral counting rules[16]. According to the so-called generalized CHPT (see e.g. ref.[21]) quark masses are counted as \( O(p) \) rather than \( O(p^2) \) such that the terms corresponding to \( K_7 \) up to \( K_{11} \) would be counted as of chiral order three. In this scheme, the terms corresponding to \( K_1 \) to \( K_6 \) as well as \( K_{12}, K_{13} \) would appear at \( O(p^3) \) as before but, to the same order, a number of additional terms with two occurrences of the scalar field \( \chi \) must also be included. This modification of the chiral counting corresponds to a weak form of spontaneous chiral symmetry breaking in which \( F_\pi \neq 0 \) while the quark condensate \( <\bar{q} q> \) could be small or even vanishing. Whether this scheme is of physical relevance or not can be decided experimentally by studying the low-energy \( \pi \pi \) amplitude, which is very sensitive to the size of the quark condensate[21][17]. In this paper, the discussion will be restricted to the case of the standard chiral counting, which is the most predictive framework.

The effective lagrangian generates a low-energy representation for the generating functional of QCD Green’s functions \( W \). In addition to the sources \( s(x), p(x), v_\mu(x), a_\mu(x) \) that one customarily introduces, \( W \) now depends on the two new sources \( q_L \) and \( q_R \),

\[ W \equiv W(v_\mu, a_\mu, s, p, q_L, q_R). \]  

Accordingly, one can introduce Q-currents,

\[ Q_L^a = \frac{\delta W}{i \delta q_L^a}, \quad Q_R^a = \frac{\delta W}{i \delta q_R^a}. \]  

and one may identify the low-energy constants by taking functional derivatives of \( W \) with respect to \( q_L \) and \( q_R \). It is useful, in connection with the LEC’s \( K_{12} \) and \( K_{13} \), to further extend this formalism by letting \( q_L \) and \( q_R \) become space-time dependent. This generalization allows one to identify \( K_{13} \), for instance, from the two-point function \( \delta^2 W / \delta q_L(x) \delta q_R(0) \) instead of a four-point function if the spurions are restricted to be constants. There is an apparent drawback to this extension, which is that we are allowing the photon to couple to a nonconserved current. This is an unusual situation and such a theory is not guaranteed to be renormalizable. It is not clear whether it is possible to define the generating functional in full generality in the presence
of $x$ dependent spurions. For our restricted purposes (i.e. the derivation of sum rules which are UV finite for $K_{12}$ and $K_{13}$) we simply need the expansion of $W$ to quadratic order in $q_L$, $q_R$ and we can switch off the external electromagnetic field. In this case, the UV divergencies are removed by the following finite set of counterterms

$$\mathcal{L}_q = \frac{i}{2} Z_2 \bar{\psi}_L q_L \slashed{D} q_L \psi_L + (L \leftrightarrow R) - Z_s \bar{\psi}_R q_R (s + ip) q_L \psi_L - Z_s \bar{\psi}_L q_L (s - ip) q_R \psi_R \quad (14)$$

The existence of such a set of local counterterms guarantees the validity of the idea of using space-time dependent spurions at this order. For our purposes, (14) will be used at tree level. We may then use the equations of motion generated by (2), thereby obtaining the counterterms in the form

$$\mathcal{L}_q = \frac{i}{2} Z_2 \bar{\psi}_L [q_L, D^\mu q_L] \gamma_\mu \psi_L + (L \leftrightarrow R)$$

$$+ \bar{\psi}_R \left[ \frac{1}{2} Z_2 \left( q_R^2 (s + ip) + (s + ip) q_L \right) - Z_s q_R (s + ip) q_L \right] \psi_L$$

$$+ \bar{\psi}_L \left[ \frac{1}{2} Z_2 \left( q_R^2 (s - ip) + (s - ip) q_L \right) - Z_s q_L (s - ip) q_R \right] \psi_R .$$

Setting $q_L = q_R = q$, one recovers the usual QED counterterms. For a free fermion, the renormalization constants $Z_2$ and $Z_s$ are determined from the requirement that the fermion self energy $\Sigma(\not{p})$ is normalized in the way appropriate for a free particle of mass $m$. For instance, in dimensional regularization, $Z_2$ and $Z_s$ read

$$Z_2(m) = \frac{\mu_0^{-2\epsilon}}{16\pi^2} \left\{ -\xi \left[ \Gamma(\epsilon) + 1 - \log \frac{m^2}{4\pi\mu_0^2} \right] + (\xi - 3) \log \frac{M^2}{m^2} + \xi \log \xi - 3 \right\} \quad (16)$$

$$Z_s(m) = \frac{\mu_0^{-2\epsilon}}{16\pi^2} \left\{ -(\xi + 3) \left[ \Gamma(\epsilon) + 1 - \log \frac{m^2}{4\pi\mu_0^2} \right] + (\xi - 3) \log \frac{M^2}{m^2} + \xi \log \xi + 2 \right\} ,$$

Quarks, however, are not free particles but on the contrary, are confined, so it seems unwise to use the above values of $Z_2$ and $Z_s$ in the present context. In fact, the QED counterterms play a role in a kinematical region where QCD becomes perturbative. The most reasonable thing to do, then, seems to resort to $\overline{MS}$ renormalization, as one customarily does in perturbative QCD calculations. Such a mass independent renormalization scheme is in fact implicitly assumed in the definition of the “current” quark masses, which appear in the effective lagrangian (see e.g. the review by Manohar in the PDG[22]). As a consequence, the corresponding renormalization scale $\mu_0$ will appear in the expression of some of the low-energy constants. This was pointed out in ref.[12]. According to this prescription, $Z_2$ and $Z_s$ have the following expression,

$$Z_2 = \frac{\mu_0^{-2\epsilon}}{16\pi^2} \left[ -\xi \left[ \Gamma(\epsilon) + \log(4\pi) \right] \right] \quad (17)$$

$$Z_s = \frac{\mu_0^{-2\epsilon}}{16\pi^2} \left[ -\xi - 3 \left[ \Gamma(\epsilon) + \log(4\pi) \right] \right] .$$

which will be adopted in the following. Further divergences involving the strong coupling constant $\alpha_s$ will be assumed to be removed by the $\overline{MS}$ prescription as well.
3. QCD in the chiral limit: sum rules for $C$, $K_{11}$, $K_{12}$ and $K_{13}$

3.1 The constants $C$ and $K_{13}$: spectral representation

Let us consider the correlation function $\langle Q_L Q_R \rangle$ constructed from the currents associated with the spurion fields $q_L$ and $q_R$ introduced in sec.2,

$$
\begin{align*}
\Pi^{Q}_{LR}(p^2) &= 4i \int d^4xe^{ipx} \langle 0|TQ^3_L(x)Q^3_R(0)|0 \rangle \\
&= i \int d^4xe^{ipx} \langle 0|TQ^3_L(x)Q^3_R(0) - Q^3_A(x)Q^3_A(0)|0 \rangle 
\end{align*}
$$

We have introduced vector and axial vector Q-currents:

$$
Q_V = Q_L + Q_R \quad Q_A = -Q_L + Q_R .
$$

On the one hand, we can use the chiral lagrangian, and compute the chiral expansion of $\Pi^{Q}_{LR}$ up to chiral order four. It receives contributions at tree level involving the constants $C$ and $K_{13}$ and a photon loop contribution. There is no pion loop contribution. Explicitly, a simple calculation gives, in an arbitrary gauge\footnote{We follow the usual chiral $\overline{MS}$ convention of relating bare and renormalized constants by: $K_i = \mu^{-2}[K_i^\prime(\mu) - \Sigma_i/32\pi^2(\Gamma(\epsilon) + 1 + \log(4\pi))]$}

$$
\Pi^{Q}_{LR}(p^2) = 2C + p^2F_0^2 \left\{ 2K_{13}(\mu) + \frac{1}{16\pi^2} 3 \left[ (\xi - 1) \left( \log \frac{M_0^2}{\mu^2} + \frac{1}{6} \right) + \xi \log \xi \right] + O(p^2) \right\} .
$$

On the other hand, it is clear from (2) that each Q-current is nothing but a QCD current multiplied by the quantum photon field, so that the above correlator can be written as a convolution of a QCD correlation function with the photon propagator:

$$
\Pi^{Q}_{LR}(p^2) = \int \frac{d^4k}{(2\pi)^4} \left( \Pi^{\alpha\beta}_{V^3}(k-p) - \Pi^{\alpha\beta}_{A^3}(k-p) \right) (-iD_{\alpha\beta}(k)) .
$$

In this case, the QED counterterms (see [3]) make no contribution, implying that the integral must be finite. This can be verified by using the operator product expansion for the difference $\Pi^{\alpha\beta}_{V^3} - \Pi^{\alpha\beta}_{A^3}$ which, indeed, starts with dimension six operators in the chiral limit. The vector and axial-vector currents are defined, as usual, as

$$
V^i_\mu(x) = \bar{\psi}(x)\gamma_\mu \frac{\lambda^i}{2} \psi(x), \quad A^i_\mu(x) = \bar{\psi}(x)\gamma_\mu \gamma^5 \frac{\lambda^i}{2} \psi(x) .
$$

The $\langle VV \rangle$ and $\langle AA \rangle$ correlation functions which appear in (21) satisfy once-subtracted dispersive representations (which we display in unsubtracted form for simplicity and for the reason that we will only consider combinations which need no subtractions in the sequel). Assuming the T-product to be covariantly defined,

$$
\Pi^{\mu\nu}_{V^3}(q) = i \int d^4xe^{iqx} < 0|TV^{3\mu}(x)V^{3\nu}(0)|0 > = (q^\mu q^\nu - g^{\mu\nu} q^2) \frac{1}{\pi} \int_0^\infty ds \frac{\rho_{V^3}(s)}{s - q^2} 
$$

and

$$
\Pi^{\mu\nu}_{A^3}(q) = i \int d^4xe^{iqx} < 0|TA^{3\mu}(x)A^{3\nu}(0)|0 > = (q^\mu q^\nu - g^{\mu\nu} q^2) \frac{1}{\pi} \int_0^\infty ds \frac{\rho_{A^3}(s)}{s - q^2} + g^{\mu\nu} \frac{1}{\pi} \int_0^\infty ds \frac{\sigma_{A^3}(s)}{s - q^2} .
$$
In the chiral limit, the spectral function \( \sigma_{A^3}(s) \) vanishes identically. It is convenient to separate explicitly the pion contribution in the axial spectral function,

\[
\rho_{A^3}(s) = \bar{\rho}_{A^3}(s) + F_0^2 \delta(s) .
\]

It is now a simple matter to insert the spectral representations (23)-(24) into the expression (21) for \( \Pi_{LR}^Q \) and compute the photon loop integral. Comparing with the chiral expansion form of \( \Pi_{LR}^Q \) (20), one readily identifies the constants \( C \) and \( K_{13} \) for which one obtains the following representations:

\[
C = \frac{-1}{16\pi^2} \frac{3}{2\pi} \int_0^\infty dx \log \frac{x}{\mu^2} (\rho_{V^3}(x) - \bar{\rho}_{A^3}(x)) ,
\]

which is a re-derivation of the DGMLY sum rule[1] (using the relation \( M_{V+}^2 - M_{V-}^2 = 2e^2C/F_0^2 \) which is exact in the chiral limit and to order \( O(e^2) \)), and its immediate generalization,

\[
K_{13}^r(\mu) = \frac{1}{16\pi^2} \frac{3}{4} \left\{ 1 + (1 - \xi) \left[ \frac{1}{12} + \frac{1}{2\pi F_0^2} \int_0^\infty dx \log \frac{x}{\mu^2} (\rho_{V^3}(x) - \bar{\rho}_{A^3}(x)) \right] \right\} .
\]

The expected order of magnitude for the constants \( K_{13}^r(\mu = M_V) \approx 1/16\pi^2 \) is indeed confirmed by this explicit calculation, but one must be careful since these constants are gauge dependent in general. The dependence of \( C \) and \( K_{13} \) upon the chiral renormalization scale \( \mu \) can be deduced from eq.(20) since the left-hand side is independent of \( \mu \): \( C \) is a constant and \( K_{13} \) satisfies

\[
\mu \frac{dK_{13}^r(\mu)}{d\mu} = \frac{1}{16\pi^2} \frac{3}{4} (\xi - 1) .
\]

The correct scale dependence emerges from the spectral representations (24) and (27) provided the spectral functions satisfy the two Weinberg sum rules[23] (a modern discussion can be found in ref.[24])

\[
\frac{1}{\pi} \int_0^\infty dx (\rho_{V^3}(x) - \bar{\rho}_{A^3}(x)) = F_0^2, \quad \int_0^\infty dx x (\rho_{V^3}(x) - \bar{\rho}_{A^3}(x)) = 0 .
\]

### 3.2 The LEC’s \( K_{11} \) and \( K_{12} \) and the correlator \( \langle VAP \rangle \)

We will follow the same approach as in sec. 3.1. In order to isolate contributions from the LEC’s \( K_{11} \) and \( K_{12} \) we may consider a correlation function of two Q-currents with the pseudoscalar current,

\[
\Pi_{WP}^Q(p,q) = \int d^4x d^4y e^{ipx+ipy} \langle 0|\mathcal{T}Q^1_V(x)Q^2_A(y)\mathcal{P}_3^A(0)|0 \rangle ,
\]

where the pseudoscalar current is defined as

\[
P^a(x) = i\bar{\psi}(x)\gamma^5\lambda^a \frac{\lambda^a}{2} \psi(x) .
\]

It will also be useful to introduce the related pion to vacuum matrix element,

\[
\Pi_{WP}^Q(p,l) = \int d^4x e^{ipx} \langle 0|\mathcal{T}Q^1_V(x)Q^2_A(y)\mathcal{P}_3^A(l)|0 \rangle .
\]
We first construct the chiral expansions of these objects, up to one loop, using the chiral lagrangian. Setting \( q = 0 \) in (30), we obtain
\[
\Pi_Q^p(0, q) = \frac{-2B_0C}{q^2} - 2B_0F_0^2(2K_{11} - K_{12}) + O(p^2) .
\] (33)
The contributions from the photon loop and from the pion loop both vanish, implying that the combination \( 2K_{11} - K_{12} \) is finite in any gauge. Setting \( p = 0 \) now in (30), we obtain the following chiral expansion, which involves the combination \( 2K_{11} + K_{12} \),
\[
\Pi_Q^W(0, q) = \frac{-2B_0C}{q^2} - B_0F_0^2 \left\{ 2 \left( 2K_{11}^r(\mu) + K_{12}^r(\mu) \right) + \frac{1}{16\pi^2} \left[ (\xi - \frac{3}{2}) \log \frac{M^2}{\mu^2} + \xi \log \xi - \frac{1}{4} \right] + O(q^2) \right\} .
\] (34)
We will also use the expansion of \( \Pi_Q^W \):
\[
\Pi_Q^W(p, l) = i\frac{2C}{F_0} + iF_0, p \cdot l \left\{ -4K_{12}^r(\mu) - 2K_{31}^r(\mu) \right\} + \frac{1}{16\pi^2} \left[ (-7\xi + 9) \log \frac{M^2}{\mu^2} - 7\xi \log \xi - \frac{1}{2} \xi + \frac{3}{2} \right] + O(p^2, p \cdot l) + O(p^4) .
\] (35)
We may now express \( \Pi_Q^W \) and \( \Pi_Q^Q \) by functionally differentiating the QCD generating functional with respect to the sources \( q_L(x), q_R(x) \) and \( p(x) \), and one obtains,
\[
\Pi_Q^W(p, q) = \int \frac{d^4k}{(2\pi)^4} W_{\mu\nu}(p - k, q + k) (-iD^{\mu\nu}(k)) + \frac{1}{2}B_0F_0^2 \left[ Z_2 + Z_s - 2Z_2 \frac{p(p + q)}{(p + q)^2} \right] ,
\] (36)
where \( W^{\mu\nu}(p, q) \) is the correlator of one vector, one axial-vector and one pseudoscalar current \( < V A P > \) in the chiral limit,
\[
W_{\mu\nu}(p, q) = \int d^4x d^4y e^{ipx+iqy} < 0|TV^\dagger_\mu(x)A^2_\nu(y)P^3(0)|0 > .
\] (37)
In eq.(36) the convolution integral simply arises from replacing the currents \( Q_V \) and \( Q_A \) from their definitions. The additional contribution gets generated upon functionally differentiating the QED counterterms. This contribution implies that the convolution integral is UV divergent in QCD, in such a way that the sum of the two terms in eq.(36) is finite.

Equating expression (36) with the chiral expansions (33) and (34) one obtains an expression for the LEC’s \( K_{11} \) and \( K_{12} \) as a convolution of the QCD 3-point function \( < V A P > \) and the electromagnetic propagator. An additional expression, which we have used to cross-check the results, can be found which involves the pion to vacuum matrix element \( < 0|VA|\pi > \),
\[
\hat{W}_{\mu\nu}(p, l) = \int d^4x e^{ipx} < 0|TV^\dagger_\mu(x)A^2_\nu(0)|\pi^3(l) > = \frac{-i}{B_0F_0} \lim_{l^2\to 0} l^2 \hat{W}_{\mu\nu}(p, q) .
\] (38)
The Q-current matrix element introduced in eq.(32) can be expressed in terms of \( < 0|VA|\pi > \) as
\[
\Pi_Q^W(p, l) = \int \frac{d^4k}{(2\pi)^4} \hat{W}_{\mu\nu}(p - k, l)(-iD^{\mu\nu}(k)) + iF_0Z_2 p \cdot l ,
\] (39)
and using the chiral expansion (35) one obtains a convolution representation for the combination $2K_{12} + K_{13}$.

To summarize, at this point we have shown that the constants $K_{11}$, $K_{12}$ and $K_{13}$ can be expressed as convolution representations in terms of QCD correlators in the chiral limit. Using the same ideas it is not difficult to see that the constants $K_1$ to $K_6$ must obey similar representations, involving the QCD correlators $<VVAA>$ and $<AAAA>$ in the chiral limit as well. The difficulty in this case is the implementation of resonance saturation. We will now show that a rather neat implementation can be performed for the correlators $VV - AA$ (which is very simple) and $<VAP>$ (which is somewhat less trivial) and defer a brief discussion of $K_1$ to $K_6$ to sec. 3.6.

### 3.3 Resonance saturation of $<VV - AA>$

The spectral densities $\rho_V$ and $\rho_A$ which appear in the expressions (26) and (27) for $C$ and $K_{13}$ can be related to experimentally measurable quantities like $e^+e^- \to \text{hadrons}$ cross sections and $\tau$ decay rates. Analyses of the DGMLY and Weinberg sum rules in terms of available experimental data have been performed \cite{25} \cite{26}. A physically appealing and time honoured approximation is that of “resonance saturation” of the spectral integrals. There are, in fact, two aspects in this approximation. One is to represent the spectral functions as a sum over delta functions, and the second is to retain just a few terms in the sum. The first aspect corresponds to a leading large $N_c$ limit and was actually shown to be rather precise in practice \cite{26}. However, even in the large $N_c$ limit it is not clear how many resonances should be included. A current assumption is that minimal saturation, i.e. in the present case including only the contribution of the $\rho(770)$ and the $a_1(1260)$ resonances, does provide a reasonable approximation. In this case, the spectral functions are given by

$$\rho_{\nu^3}(x) = F_{\nu}^2 \delta(x - M_{\nu}^2), \quad \tilde{\rho}_{\nu^3}(x) = F_{\nu}^2 \delta(x - M_{\nu}^2) \quad (40)$$

One may at least verify that this lowest order approximation is not grossly inconsistent. Indeed, if one uses experimental values for $F_{\nu}$ and $M_{\nu}$ then the Weinberg sum rules are satisfied with values of $F_A$ and $M_A$ which are within 30% of those of experiment. Furthermore, the corresponding result for the constant $C$ seems to be rather accurate as one can judge from the resulting value of $M_{\pi^+}^2 - M_{\pi^0}^2$. In the dispersive representation, one may attribute the success of the minimal resonance approximation to the fact that the various integrands are oscillatory. In that situation, numerical analysis teaches us that an optimal sequence of approximations is obtained by cutting the integrand after an even number of oscillations. The precision could be much better than one could expect from the asymptotic behaviour of the absolute value of the integrand. This also suggests that in order to improve on the first order approximation one should include an additional pair of resonances $\rho'$ and $a'_1$, which means integrating up to a rather high energy.

An alternative point of view on resonance saturation emerges from considering the correlation function $<VV - AA>$ itself. Resonance saturation amounts to constructing rational parametrizations for the correlation function. In the minimal version, for instance, one would have

$$\Pi_{\nu^3}^{\mu\nu}(p) - \Pi_{\nu^3}^{\mu\nu}(p) = (p^\mu p^\nu - g^{\mu\nu} p^2) \left\{ \frac{F_6^2}{p^2} + \frac{F_A^2}{p^2 - M_A^2} - \frac{F_{\nu}^2}{p^2 - M_{\nu}^2} + \text{polynomial} \right\} \quad (41).$$
This is the most general form which satisfies the chiral Ward identities (which dictate the form of the tensor structure) and which has a specified number of resonance poles. When computing the four-dimensional convolution integrals for $C$ or $K_{13}$, one may perform a Wick rotation such that the correlator $<VV - AA>$ is needed only for euclidian values of $p^2$. In this region, the correlator is a smooth function so it makes perfectly good sense to employ a rational approximation. The parameters of the rational function $(11) (M_V, M_A, F_V,...)$ may be constrained both from the asymptotic region $p^2 \to -\infty$, using information from the operator product expansion, and by extrapolating to positive values of $p^2$ using physical information on the resonances. In the asymptotic region, the behaviour must ensure that the convolution integral defining $\Pi_{LR}^{Q}$ eq.(21) is finite. This implies that the polynomial in eq.(11) must vanish and one must impose the two conditions that the coefficients of $1/p^2$ and $1/p^4$ both vanish. These conditions are exactly equivalent to the two Weinberg sum rules. We recover here the necessity of imposing asymptotic matching conditions when discussing resonances in the effective lagrangian context, a fact which has been appreciated relatively recently [29] (see also [30]). The usefulness of the effective lagrangian for the resonances is to guarantee that the n-point functions that one might compute automatically obey all the chiral Ward identities. When discussing two- or three-point functions it is sometimes more expedient to write directly the most general form compatible with a given set of resonance poles and impose that the relevant Ward identities be obeyed by hand. This is what we have done for $<VV - AA>$ and we will proceed in this way also for $<VAP>$ below.

Returning to the evaluation of the low-energy constants, after imposing the two asymptotic conditions on the parametrization $(11)$, performing the Wick rotation and integrating over angles one obtains $C$ in the form of an integral

$$C = \frac{3F_0^2M_A^2M_V^2}{32\pi^2} \int_0^\infty \frac{dx}{(x + M_V^2)(x + M_A^2)}.$$  

The interest of this expression is the observation that a substantial part of the integral (approximately 40%) comes from the high energy region $x > 1 \text{ GeV}^2$. This means that the first nonvanishing term in the asymptotic expansion of $<VV - AA>$, call it $c/p^6$, plays an important role. In the minimal saturation approximation $c$ is predicted to be $c = F_0^2M_A^2M_V^2$. In QCD, $c$ is not exactly a constant because of the anomalous dimension of the dimension six operator involved. Ignoring this fact, and using the vacuum saturation approximation$^6$ to estimate the value of the dimension six condensate$[28]$, one indeed finds a value consistent with that above, within a factor of two. This suggests an algorithm for systematically improving the calculation. Adding more resonance poles to the rational parametrization $(11)$, one can constrain the extra parameters so as to improve the interpolation function at both ends. On the one hand one can ask that more terms of the asymptotic expansion in $1/p^2$ be reproduced (in practice a better determination of the dimension six and dimension eight condensates than available at present is needed) and on the other hand, in the Minkowski region, one can improve the agreement with the experimental resonance parameters such as $M_A$ and $F_A$.

3.4 Resonance saturation of $<VAP>$

Let us now investigate the approximation of resonance saturation for the three-point function $W_{\mu\nu}(p,q)$ (37). As explained above, this approximation consists in constructing a rational

$^6$ This approximation is exact in the limit $N_c \to \infty$. In practice, however, it seems to be valid within a factor 2-3 [27].
interpolation function, the parameters of which will be constrained from both the asymptotic euclidian region and from the physical resonance region. In the asymptotic region one should impose the constraints which are necessary to insure finiteness of the result for the low-energy constants $K_{11}$ and $K_{12}$. As before, the minimal version of resonance saturation has the polynomial form with one resonance content of the form

$$p^\mu W_{\mu\nu}(p, q) = (-B_0 F_0^2) \left( \frac{q_\nu}{q^2} \left( \frac{(p + q)_{\nu}}{(p + q)^2} \right) \right),$$

$$q^\nu W_{\mu\nu}(p, q) = (-B_0 F_0^2) \frac{(p + q)_\mu}{(p + q)^2},$$

which imply that $W_{\mu\nu}$ must have the following tensor structure,

$$W_{\mu\nu}(p, q) = -B_0 F_0^2 \left( \frac{(p_\mu + 2q_\mu)q_\nu}{q^2 l^2} - \frac{g_{\mu\nu}}{l^2} + F(p^2, q^2, l^2) P_{\mu\nu} + G(p^2, q^2, l^2) Q_{\mu\nu} \right)$$

with $l^2 = (p + q)^2$ and where $P_{\mu\nu}$ and $Q_{\mu\nu}$ are the two independent tensors which vanish under contraction with $p_\mu$ as well as with $q_\nu$.

$$F$$ and $G$ are analytic functions of the three variables $p^2$, $q^2$ and $l^2 = (p + q)^2$ with appropriate poles and cuts. In the simple approximate representation used here, $F$ and $G$ are meromorphic functions.

Consider now the asymptotic euclidian region. The OPE provides an expansion valid in the regime where all three momenta squared $p^2$, $q^2$ and $l^2$ are large and negative. In other terms, scaling $p \to \lambda p$ and $q \to \lambda q$ one obtains an expansion in inverse powers of the scale parameter $\lambda$. The leading part in this expansion is controlled by the dimension three $\bar{q}q$ condensate and scales as $1/\lambda^2$ (this scaling behaviour is exact as anomalous dimensions cancel out). We have also worked out the subleading part in this expansion, scaling as $1/\lambda^4$, which is proportional to the so-called mixed condensate,

$$\lim_{\lambda \to \infty} W_{\mu\nu}(\lambda p, \lambda q) = \frac{\langle \bar{u}u \rangle}{\lambda^2} \left\{ \frac{(p_\mu + 2q_\mu)q_\nu}{q^2 l^2} - \frac{g_{\mu\nu}}{l^2} + \frac{p^2 - q^2 - l^2}{2p^2 q^2 l^2} P_{\mu\nu} - \frac{1}{p^2 q^2 l^2} Q_{\mu\nu} \right\}$$

$$- \frac{\langle \bar{u}\sigma_{\mu\nu}G^{\mu\nu}u \rangle}{6\lambda^4} \left\{ \frac{l^2}{p^4 q^4} + \frac{p^2 + q^2 - p^2 + q^2}{p^4 l^4 - q^4 l^4} \right\} P_{\mu\nu} - 2 \left( \frac{1}{p^4 l^4} - \frac{1}{q^4 l^4} \right) Q_{\mu\nu} + O \left( \frac{1}{\lambda^6} \right).$$

There are also $O(\alpha_s)$ corrections which we have not evaluated. With the minimal resonance content, it proves perfectly possible to match the leading part of this asymptotic expansion. This may be viewed as a surprise as this matching is much more constraining than that for the two-point function $<VV - AA>$. We have now three independent variables instead of just one and two independent amplitudes $F$ and $G$. In fact, this asymptotic constraint entirely determines the rational approximant up to two constants, $a$ and $b$

$$F(p^2, q^2, l^2) = \frac{(p^2 - q^2 - l^2 + 2a)}{2(p^2 - M_V^2)(q^2 - M_A^2)l^2}, \quad G(p^2, q^2, l^2) = \frac{-q^2 + b}{(p^2 - M_V^2)(q^2 - M_A^2)q^2 l^2}.$$

Note that the effective lagrangian model used in refs. is inconsistent with this asymptotic constraint. Indeed, using this model one finds

$$F(p^2, q^2, l^2) = \frac{1}{F_0^2} \left( \frac{F_0^2 - 2F_V G_V}{(p^2 - M_V^2)l^2} - \frac{F_0^2}{(q^2 - M_A^2)l^2} \right), \quad G(p^2, q^2, l^2) = \frac{1}{F_0^2} \frac{-2F_V G_V}{(p^2 - M_V^2)q^2 l^2}. \quad (48)$$
Comparing with the QCD prediction in the asymptotic region, eq. (46) one observes that the scaling behaviour is correct but one does not match the individual terms exactly. Worse than that, this model for $<VAP>$ does not satisfy the weaker constraint to reproduce the correct QED divergence in the calculation of $2K_{11} \pm K_{12}$ (in other terms, after taking due account of the counterterms, the result for $K_{11}$ and $K_{12}$ is infinite in this model). The reason for these problems is that no $p_\rho a_1$ coupling has been introduced. Allowing for such couplings one can recover the result of eq. (47). If one now attempts to match not only the leading asymptotic terms but also the subleading ones in the expansion (46) one finds that this is no longer possible unless the resonance content is enlarged. Obviously, for instance, in order to produce terms proportional to $1/l^4$ we need not only the $\pi$ but also the $\pi'$ resonance, coupling to the pseudoscalar current.

Returning to the minimal model, it turns out that there are further chiral symmetry constraints as well as asymptotic constraints. These constraints concern the pion to vacuum matrix elements that one can deduce as residues of the pion poles in $<VAP>$. The minimal model will prove capable of obeying all these constraints and the two constants $a$ and $b$ will be determined. Consider chiral constraints first: the pion to vacuum matrix element $<0|VA|\pi>$ must satisfy a soft pion theorem \[ \hat{W}^{\mu\nu}(p,0) = \frac{i}{F_0} (\Pi^{a\nu}_{\mu\nu}(p) - \Pi^{\mu\nu}_{A^3}(p)) \, , \] (49) Using the rational parametrization for $<VAP>$, eq. (17), and the corresponding one for $<VV - AA>$, eq. (11), one finds that the soft pion theorem is obeyed provided the first Weinberg sum rule holds and the parameters $a$ and $b$ satisfy

$$ a - b = \frac{1}{F_0^2} (M_V^2 F_A^2 - M_A^2 F_V^2) = -(M_V^2 + M_A^2) \, . $$

(50)

where the second equality follows from imposing the second Weinberg sum rule. A second soft pion theorem associated with the pion to vacuum matrix element $<0|VP|\pi>$ is satisfied without bringing new constraints.

We are thus left with a single arbitrary constant, $b$. The physical meaning of this constant is made clear by identifying the vector form-factor of the pion. For that purpose, let us consider the residue of the pion pole $(p - l)^2 = 0$ in $\hat{W}_{\mu\nu}$,

$$ \lim_{(p - l)^2 \to 0} (p - l)^2 \hat{W}_{\mu\nu}(p,l) = iF_0 (p_\mu - 2l_\mu)(p_\nu - l_\nu) F_V(p^2) \, . $$

(51)

Here, the function $F_V(p^2)$ is the vector form factor of the pion defined in a standard way

$$ <\pi^a|V^b_\mu|\pi^c> = if^{abc}(p^a_\mu + p^c_\mu) F_V(p^2) \, . $$

(52)

Using our model for $\hat{W}$, we obtain the following expression for $F_V(p^2)$

$$ F_V(p^2) = 1 - \frac{bp^2}{2M_A^2(p^2 - M_V^2)} \, . $$

(53)

It is thus tempting to determine $b$ in order to reproduce the standard VMD form of the vector form factor of the pion. Amusingly, this property need not be imposed by hand here, but can
be deduced from the operator product expansion\(^7\). Consider, indeed, the pion matrix element of the product of the vector and pseudoscalar currents,

\[
\hat{W}_\mu(p,l) = \int d^4 x \, e^{ipx} \langle 0 | TV_\mu^I(x) P^3(0) | \pi(l) > .
\]  

(54)

The OPE implies that for large euclidian values of \(p^2\), \(\hat{W}_\mu(p,l)\) behaves as

\[
\hat{W}_\mu(p,l) = B_0 F_0 \left\{ \frac{p_\mu}{p^2} + O\left( \frac{1}{p^3} \right) \right\} .
\]  

(55)

This is to be compared with the result obtained from the rational parametrization of \(< VAP >\):

\[
\hat{W}_\mu(p,l) = B_0 F_0 \left\{ \frac{p_\mu}{p^2} \left[ 1 + \frac{p \cdot l}{p^2} \left( -2 + \frac{b}{M^2_A} \right) + \frac{l_\mu}{p^2} \left[ 2 - \frac{b}{M^2_A} \right] + O\left( \frac{1}{p^3} \right) \right] \right\} .
\]  

(56)

Clearly, this will match with the OPE result provided one takes

\[
b = 2M^2_A ,
\]  

(57)

which is the same value that also insures that the pion form factor satisfies VMD exactly. Furthermore, one can work out the asymptotic expansion of the pion to vacuum matrix element \(\hat{W}_{\mu\nu}(p,l)\) when \(p^2 \to -\infty\),

\[
\hat{W}_{\mu\nu}(p,l) = iF_0 \left\{ \frac{-p_\mu l_\nu}{p^2} + \frac{g_{\mu\nu}}{p^2} \left( 1 + \frac{p \cdot l}{p^2} \right) + \frac{p^2 l_\mu l_\nu - p \cdot l p_\mu l_\nu}{p^4} + O\left( \frac{1}{p^3} \right) \right\} .
\]  

(58)

It can be checked that this result is exactly reproduced in our model provided eq.(50) holds. Inserting this asymptotic expansion into eq.(39) one verifies explicitly that the UV divergence cancels out with the counterterm contribution.

At this point, we have verified that the rational parametrization of \(< VAP >\) in terms of two resonance poles (and the pion pole) is capable of matching the leading terms in the asymptotic expansion of \(< VAP >\), as well as the leading asymptotic terms of the two related pion to vacuum matrix elements, while obeying all the chiral symmetry constraints. Using this parametrization (see \(44, 47, 50, 57\)) it is not difficult to compute the momentum integrals in \(36, 39\) and match the result with that of the chiral expansion for small values of external momenta. One finds that the QED infinities do cancel out exactly, and one obtains the following finite results for the low energy constants \(K_{11}\) and \(K_{12}\):

\[
K^r_{11}(\mu) = \frac{1}{16\pi^2} \frac{1}{8} \left\{ \frac{(\xi - \frac{3}{2}) \log \frac{\mu^2}{M^2_V} - (\xi + 3) \log \frac{\mu^2}{M^2_V} - \xi - \frac{3}{4}}{16} \right\} + \frac{3\log(1 + \frac{z}{2})}{16(z - \frac{1}{2})} + \frac{3\log(1 + \frac{z}{2})}{16(z - \frac{1}{2})} \right\} .
\]  

(59)

where

\[
z = \frac{M^2_A}{M^2_V} .
\]  

(60)

\(^7\) This actually holds only for the minimal rational approximation.
and
\[ K_{12}(\mu) = \frac{1}{16\pi^2} \{ (\xi - \frac{3}{2}) \log \frac{\mu^2}{M_V^2} - \xi \log \frac{\mu_0^2}{M_V^2} - \xi + \frac{7}{4} \frac{\log z}{(z-1)^2} - \frac{3z}{z-1} \}. \] (61)

The dependence upon the chiral renormalization scale \( \mu \) agrees with that derived in ref.[14] (see also [34]) in the gauge \( \xi = 1 \). As was anticipated, one observes that \( K_{11} \) and \( K_{12} \) also depend on the QCD renormalization scale \( \mu_0 \). The result on this dependence agrees with ref.[12] concerning \( K_{11} \) but not \( K_{12} \). We stress that once the assumption of saturation from a single multiplet of vector and of axial-vector resonances is made, the result for \( K_{11} \) and \( K_{12} \) is uniquely determined, in much the same way as it was for the \( O(p^2) \) constant \( C \), in terms of just \( M_A \) and \( M_V \). We also note that once the expression for \( K_{12} \) is obtained, the three-point function \( <VA\pi> \) may be used for getting \( K_{13} \) via (35). We checked that the same result for \( K_{13} \) which was found before from a two-point function (i.e. eq.(27), plus resonance saturation) was indeed recovered.

### 3.5 Phenomenological implications

Our construction of the Green’s function \( <VA\pi> \) has phenomenological implications which are interesting to consider in order to assess the reliability of the result for \( K_{11} \) and \( K_{12} \). Let us firstly consider the predictions in the region of low momenta and then, further away, in the resonance region. For small momenta, comparing with the chiral expansion of \( <VA\pi> \) (which can be found in ref.[16]), one finds that the model reproduces the vector meson dominance formulas for the constants \( l_5(M_V) \) and \( l_6(M_V) \) which are known to be reasonably accurate [16]. Obviously, however, the model does not generate the logarithmic singularity caused by the pion loop. It is perfectly feasible to improve on this by taking a more sophisticated form for the vector meson propagator. However, this will complicate the computation of the photon loop integral while bringing corrections which are subleading in the large \( N_c \) counting, and should thus be rather small. Concerning the resonance sector, for the \( \rho \) meson, the model is found to embody a reasonable value for the \( \pi\pi \) decay width. For the \( a_1 \) resonance, the model predicts a value for the \( \rho\pi \) decay width of the order of 200 MeV, which is within 50% of the experimental value. A surprising result emerges for the radiative width, \( a_1 \rightarrow \gamma\pi \). Extracting the decay amplitude from \( <VA\pi> \) using the reduction formula, one obtains the following expression

\[ T(a_1(q) \rightarrow \gamma(p)\pi) = (\epsilon.p\epsilon'.q - p.q\epsilon.\epsilon') F_0 \left( \frac{2M_V^2 - M_A^2}{M_A F_A} \right) \left( \frac{2M_V^2 - M_A^2}{2M_V^2} \right). \] (62)

Because of the factor \( 2M_V^2 - M_A^2 \), this amplitude is strongly suppressed and one obtains a value of a few tens of KeV for the width, which is one order of magnitude smaller than the experimental value obtained in ref.[12]. Clearly, since the \( a_1 \) pole is rather far from the integration region relevant for the evaluation of the LEC’s \( K_{11} \) and \( K_{12} \) it is plausible that these constants could be reasonably well evaluated using the rational approximant for \( <VA\pi> \) and at the same time, the properties of the \( a_1 \) need not be too precisely reproduced by the approximant. Still, a mismatch by one order of magnitude would be disturbing. A closer look at the literature, however, reveals that a small value for the radiative width is not ruled out. Indeed, a recent photoproduction experiment has found no trace of \( a_1 \) production [33] (while observing very clean evidence for \( a_2(1320) \) production). In this experiment, the photon is on shell, which is not necessarily exactly satisfied in the experiment of ref. [22] which is a Primakov-type experiment. Our amplitude is very strongly energy dependent and it is nearly vanishing only for exactly
massless photons. At any rate, if the $a_1$ radiative width turns out to be unsuppressed, this would be clear evidence for the necessity of including further resonances in the parametrization of $<VAP>$ and would invalidate the above estimate of $K_{11}$ and $K_{12}$. On the contrary, confirmation of the suppression would be a nice experimental support for these estimates.

3.6 Some remarks on $K_1$ to $K_6$

Let us discuss here some aspects of the convolution representations for the constants $K_1$ to $K_6$, with some emphasis on the combination $K_5 + K_6$ which is of interest for DT violation. In order to illustrate the procedure, consider the correlator

$$
\Pi^{abcd}(p) = \int d^4x \, d^4y \, d^4z \, e^{ip\cdot(x+y)} <0|T \left(A^a_\alpha(x)A^b_\beta(y) - V^a_\alpha(x)V^b_\beta(y)\right)Q_V(z)Q^d_\nu(0)|0> \tag{63}
$$

$$
= g_{\alpha\beta} \Pi_0^{abcd}(p^2) + p_\alpha p_\beta \Pi_1^{abcd}(p^2).
$$

We will concentrate in the following on the first tensor component, $\Pi_0^{abcd}$. Using the chiral lagrangian, we can compute its chiral expansion, which involves a photon loop as well as a pion loop. Choosing flavour indices in order to isolate $K_5 + K_6$, one obtains,

$$
\Pi_0^{1425}(p^2) = iF_0^2 \left\{ K^r_5(\mu) + K^r_6(\mu) - \frac{3}{2}K^r_{13}(\mu) + 3K^r_{12}(\mu) \right\}
$$

$$
+ \frac{1}{16\pi^2} \frac{3}{4} \left( \log \frac{M^2}{\mu^2} + \frac{\xi - 1}{4} \left[ \log \frac{M^2}{\mu^2} + \xi \log \xi - 1 \right] + C \frac{\log \left(-\frac{p^2}{\mu^2} + 1\right)}{d^2} \right) + O(p^2).
$$

This expression allows one to deduce the scale dependence of $K_5 + K_6$, utilizing the previous results for $K_{12}$ (eq. (27)) and $K_{13}$ (eq. (27)),

$$
\mu \frac{d}{d\mu}(K^r_5(\mu) + K^r_6(\mu)) = \frac{1}{16\pi^2} \left( \frac{9}{4} - \frac{3C}{2F_0^2} \right), \tag{65}
$$

which reproduces the result of ref. [4] with the new feature that it is found not to depend on the gauge parameter $\xi$ (we note in passing that the correlator $<V_\alpha V_\beta Q_V Q_V>$ is finite at one loop, which immediately implies that $d/d\mu(K^r_{13}(\mu) + 2K^r_{14}(\mu)) = 0$). In a similar way, it is easy to pick up two sets of flavour indices such that the chiral expansion of the correlation function will involve the combinations $K_1 + K_2$,

$$
\Pi_0^{388}(p^2) = 4iF_0^2 \left\{ K^r_1(\mu) + K^r_2(\mu) + \frac{1}{3}(K^r_5(\mu) + K^r_6(\mu)) \right\}
$$

$$
+ \frac{3C}{4F_0^4} \left[ -\log \left(-\frac{p^2}{\mu^2} + 1\right) \right]
$$

and $-2K_3 + K_4$,

$$
\Pi_0^{383}(p^2) = 2iF_0^2 \left\{ -2K^r_3(\mu) + K^r_4(\mu) + \frac{2}{3}(K^r_5(\mu) + K^r_6(\mu)) \right\}
$$

$$
+ \frac{3C}{2F_0^4} \left[ -\log \left(-\frac{p^2}{\mu^2} + 1\right) \right]. \tag{67}
$$

The next step is to express the correlator above (63) by taking functional derivatives of the QED and QCD actions. One observes, firstly, that the counterterms make a contribution which has the following form,

$$
\Pi^{abcd}_{\alpha\beta}(p)|_{counter} = iZ_2(f^{acl} f^{dbl} + f^{adl} f^{cbl}) \left( \Pi^{\alpha\beta}_{A^2}(p/2) - \Pi^{\alpha\beta}_{V^2}(p/2) \right). \tag{68}
$$
From this, one deduces the finite convolution representation

\[ \Pi^{1425}_{\alpha\beta}(p) = -i \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) \left[ \Pi^{\alpha\beta\mu\nu}_{A^a A^b V^c V^d}(p, k) - \Pi^{\alpha\beta\mu\nu}_{V^c V^d V^e V^f}(p, k) \right] \]  

(69)

in terms of the QCD 4-point functions,

\[ \Pi^{\alpha\beta\mu\nu}_{A^a A^b V^c V^d}(p, k) = \int d^4x d^4yd^4z e^{ipA^a - ikz} < 0|TA^b_{\alpha}(x)A^b_{\beta}(y))V^c_{\mu}(z)V^d_{\nu}(0)|0 > \]  

(70)

and the similar correlator with \( A^a, A^b \) replaced by \( V^a, V^b \).

The analogous representations for \( \Pi^{3^{388}}_{\alpha\beta} \) and \( \Pi^{3^{388}}_{\alpha\beta} \) have no contribution from the counterterms implying that the convolution integrals must be finite. One must extract the tensor component \( g^{\alpha\beta} \) from the four dimensional integral and one obtains \( K_5 + K_6 \) using eq.(64).

The part proportional to \( \xi - 1 \) in the convolution integral above (69) can be worked out in a model independent way. Indeed, the contraction of \( k^\mu k^\nu \) with the 4-point function can be expressed in terms of the 2-point function \( <VVAA> \) using the chiral Ward identities. In this manner, the dependence upon \( \xi \) of \( \Pi^{1425}_{0}(0) \) can be exhibited as,

\[ \Pi^{1425}_{0}(0) = \Pi^{1425}_{0}(0)|_{\xi=0} + \frac{iF_0^2}{16\pi^2} \frac{3}{16} \xi \left\{ \log \xi + \log \frac{M^2}{\mu^2_0} - 4 + \frac{3}{\pi F_0^2} \int_0^{\infty} dx \log \frac{x}{\mu^2_0} [\rho^\nu(x) - \tilde{\rho}_A(x)] \right\} \]  

(71)

in which the expression of the counterterm constant \( Z_2 \) (see (17)) has been used. This formula, together with the expressions of \( K_{12} \) and \( K_{13} \) (21), (27) allows one to show that the combination \( K_5 + K_6 \) is independent of the gauge parameter \( \xi \). Furthermore, the dependence on the QCD renormalization scale \( \mu_0 \) is found to be entirely absorbed by that of the parameter \( K_{12} \) such that \( K_5 + K_6 \) is independent of \( \mu_0 \). In a similar way, one can also show that the two combinations \( K_1 + K_2 \) and \( -2K_3 + K_4 \) are gauge independent and \( \mu_0 \) independent. Correlation representations for all of \( K_1 \) to \( K_6 \) can be obtained by further considering the correlators \( <A^a_{\alpha} A^b_{\beta} Q^c_L Q^d_R> \) from which one can isolate \( K_2, K_4 \) and \( K_6 \). The derivation exactly parallels the discussion of the correlator (13) and will not be detailed. Let us simply mention that there will be no contribution from the counterterms, implying that the convolution integrals must be finite. Again, the dependence on the gauge parameter \( \xi \) can be worked out explicitly and it turns out that the six parameters \( K_1 \) to \( K_6 \) are independent of the value of \( \xi \).

In order now to obtain phenomenological estimates, for instance for \( K_5 + K_6 \), we must construct a resonance saturated model for the QCD 4-point function \( <AAVVVV> \). The asymptotic constraint to be satisfied in this case is that the convolution integral must be finite in the particular gauge \( \xi = 0 \) (Landau gauge). The model must also obey a set of chiral Ward identities. The simplest way to implement these in the present situation is to first write down a chirally invariant effective lagrangian for the resonances. For this purpose, one may use, for instance, the convenient method advocated in ref.[20]. However, it will be necessary to investigate more couplings than were considered in this reference (where only those couplings actually contributing to the LEC’s \( L_i \) were discussed). We already pointed out that in order to fulfill the asymptotic constraints associated with \( <VAP> \), vertices associated with \( a_1 \rho \pi \) couplings were needed. Here, we can also expect contributions associated with \( \omega \rho \pi \) coupling.
(its contribution to \(\Delta M_K^2\) was considered long ago \([3]\)) but possibly also from \(b_1\omega\pi\) coupling. Also four-particle couplings and exchanges of scalar or tensor mesons could be considered. It is clear that a rather extensive study is necessary in order to carefully investigate all these possibilities. This study will not be undertaken here. We have simply performed the exercise to reevaluate the LEC \(K_{13}\), which can be isolated from the 4-point function \(< V^a_\alpha V^b_\beta Q^c_L Q^d_R \rangle\), using the resonance effective lagrangian of Ecker et al.\([20]\). The number of graphs to consider is strongly reduced by working in the Landau gauge (the gauge dependence can be worked out explicitly as above). The calculation is much more tedious than the one based on a two-point function in sec.3.1, but one indeed recovers exactly the same result.

4. Flavour symmetry breaking and the constants \(K_7, \ldots, K_{10}\)

4.1 The constants \(K_7\) and \(K_8\)

We can isolate the contributions of the two constants \(K_7\) and \(K_8\) by taking the derivative with respect to \(m_s\) of the Q-correlators \(< Q V^3 \rangle\) and \(< Q A^3 \rangle\) already considered. After a simple calculation, one obtains from the chiral lagrangian,

\[
\frac{d}{dm_s} \Pi^{Q V^3}_{V^3}(0) = 2B_0 F_0^2 \left\{ 2(K_7^r(\mu) + K_8^r(\mu)) - \frac{1}{16\pi^2} \frac{C}{F_0} (1 + \log \frac{\mu^2}{M^2 K}) \right\} + O(m_q) \tag{72}
\]

and

\[
\frac{d}{dm_s} \Pi^{Q A^3}_{A^3}(0) = 2B_0 F_0^2 \left\{ 2(K_7^r(\mu) - K_8^r(\mu)) + \frac{1}{16\pi^2} \frac{C}{F_0} (1 + \log \frac{\mu^2}{M^2 K}) \right\} + O(m_q) . \tag{73}
\]

These expressions display the scale dependence of \(K_7^r\) and \(K_8^r\). In particular \(K_7^r\) is finite and scale independent. One may then use the convolution representation for the Q-correlators \(\Pi^{Q V^3}_{V^3}\) and \(\Pi^{Q A^3}_{A^3}\) in terms of QCD correlators but this does not lead to an expression which can easily be evaluated. However, it is not difficult to show that the derivatives with respect to the strange quark mass of the isospin one vector and axial-vector spectral functions are suppressed in the large \(N_c\) counting. This is because the graphs which contribute involve one extra (strange) quark loop with respect to the leading order graphs. The fact that \(K_7^r\) and \(K_8^r\) are subleading in \(N_c\) could, of course, have been suspected from the double trace structure of the corresponding lagrangian terms (see ref.\([12]\) for more details on this aspect). One may assume, then, that \(K_7^r\) and \(K_8^r\) are suppressed for scales \(\mu \simeq M_V\) and infer the following estimate,

\[
K_7^r \simeq 0, \quad K_8^r(\mu) \simeq \frac{1}{16\pi^2} \frac{C}{2F_0^2} \frac{M^2 V}{\mu^2} \log \frac{M^2 V}{\mu^2} . \tag{74}
\]

4.2 Constraints on symmetry breaking spectral function differences

In order to discuss the constants \(K_9\) and \(K_{10}\), we consider the correlator

\[
\Delta \Pi^{Q V^3}_{V^3}(p^2) = \frac{-3i}{2(\bar{m} - m_s)} < \bar{u}u > \int dx e^{ipx} < 0 | T(Q^3_V(x)^\dagger Q^3_V(0) - Q^3_V(0)^\dagger Q^3_V(x)) | 0 >_{lin} . \tag{75}
\]

and a similar correlator \(\Delta \Pi^{Q A^3}_{A^3}\) with \(V\) replaced everywhere by \(A\) in the above formula. The normalization factor, in front of the integral is introduced for convenience. The subscript \(lin\) means that we must keep only terms linear (or logarithmic) in the quark masses and drop the
quadratic or higher order terms (the reason for this will be given below). Using the definition of the Q-currents in terms of the QCD vector or axial currents and the spectral representation of the correlator $\Pi_{V^3}(q)$ (see (23)) and the analogous definition for $\Pi_{V^8}(q)$, the following spectral function difference will appear,

$$\Delta \rho_{V}(x) = \frac{-3}{2(\hat{m} - m_s)} \langle \bar{u}u \rangle \left( \rho_{V^3}(x) - \rho_{V^8}(x) \right)_{\text{lin}} .$$  

(76)

Again, we can also define $\Delta \rho_{A}(x)$ by replacing $\rho_{V^a}$ by $\rho_{A^a}$ everywhere in (76). In the latter case, as we are away from the chiral limit, there is a second spectral function $\sigma_A$ (see (24)) to be considered and we define

$$\Delta \sigma_{A}(x) = \frac{-3}{2(\hat{m} - m_s)} \langle \bar{u}u \rangle \left( \sigma_{A^3}(x) - \sigma_{A^8}(x) \right)_{\text{lin}} .$$  

(77)

With these definitions $\Delta \rho_{V}$ and $\Delta \rho_{A}$ must satisfy sum rules analogous to the two Weinberg sum rules. Indeed, at large $-q^2$, the leading term in the asymptotic behaviour of the QCD correlation function difference $<V^3 V^3 - V^8 V^8>$ is given by [28]

$$\lim_{p^2 \to -\infty} p^4 \left( \Pi_{V^3}(p^2) - \Pi_{V^8}(p^2) \right)_{\text{lin}} = \frac{2}{3} (\hat{m} - m_s) \langle \bar{u}u \rangle \left( p^\mu p^\nu - g^{\mu\nu} p^2 \right) .$$  

(78)

This implies the two sum rules,

$$\int_0^\infty dx \, \Delta \rho_{V}(x) = 0, \quad I_V \equiv \frac{1}{\pi} \int_0^\infty dx \, x \, \Delta \rho_{V}(x) = 1 .$$  

(79)

For these sum rules to hold, it is essential to drop the quadratic mass terms in the definition of $\Delta \rho_{V}(x)$. Otherwise, the first sum rule would still be valid but the second one would diverge. The first of the above sum rules was established long ago [31] (and is customarily referred to as the DMO sum rule) while, curiously, we could not find a trace of the second one in the literature. Analogous sum rules also hold for the axial current spectral functions

$$\int_0^\infty dx \, \Delta \rho_{A}(x) = 0, \quad I_A \equiv \frac{1}{\pi} \int_0^\infty dx \, x \, \Delta \rho_{A}(x) = 1 ,$$  

(80)

together with

$$\frac{1}{\pi} \int_0^\infty dx \, \Delta \sigma_{A}(x) = 2 .$$  

(81)

The latter sum rule is in fact saturated by the pseudo-Goldstone boson contributions to $\Delta \sigma_{A}$,

$$\Delta \sigma_{A}(x) = 2 \frac{M_\pi^2 \delta(x - M_\pi^2) - M_\eta^2 \delta(x - M_\eta^2)}{M_\pi^2 - M_\eta^2} + \Delta \bar{\sigma}_{A}(x) .$$  

(82)

Because the divergence of the axial current is linear in the quark masses, the piece $\Delta \bar{\sigma}_{A}(x)$ is of higher order in the quark expansion than the first term in (82) and must be dropped.

Instead of the neutral vector currents $V^3$ and $V^8$ (or the corresponding axial vector currents) one could equally well employ charged currents $V_{ud}$ and $V_{us}$

$$V_{\mu}^{ud} = \bar{u} \gamma_{\mu} d, \quad V_{\mu}^{us} = \bar{u} \gamma_{\mu} s ,$$  

(83)

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and the corresponding definition for the axial-vector case. The charged vector currents are not conserved but this effect is proportional to the quark masses and can be ignored here.

### 4.3 Sum rules for $K_9$ and $K_{10}$

Let us now return to the Q-current correlator $\Delta \Pi^Q_V(p^2)$ defined in (77). As before, we first compute this correlator from the chiral lagrangian, up to chiral order four. Setting $p^2 = 0$, one obtains:

$$\Delta \Pi^Q_V(0) = 8 \left[ K^r_9(\mu) + K^r_10(\mu) - \frac{1}{16\pi^2} \frac{3}{4} Z^0(\mu) \right] + O(m_q) \ ,$$

where

$$Z^0(\mu) = \frac{C}{F_4^2} \frac{M_K^2 \log(M_K^2/\mu^2) - M_{\pi}^2 \log(M_{\pi}^2/\mu^2)}{M_K^2 - M_{\pi}^2}$$

is the pion (and kaon) tadpole contribution. In terms of QCD currents, we obtain the convolution representation

$$\Delta \Pi^Q_V(0) = \frac{-3}{2(\bar{m} - m_s) < \bar{u}u>} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\mu^2} \left( \Pi^\mu_{V3}(k) - \Pi^\mu_{V8}(k) \right) (-iD^\mu(k)) + Z_s - Z_2 \ .$$

Next, inserting the spectral function representation for the correlator $< V^3V^3 - V^8V^8 >$ yields

$$\Delta \Pi^Q_V(0) = \frac{1}{16\pi^2} \frac{3}{\pi} \int_0^\infty dx x \Delta \rho_V(x) \left[ \Gamma(\epsilon) + \log 4\pi + \frac{1}{3} - \log x \right] + Z_s - Z_2 \ .$$

Some remarks are in order here concerning the ultraviolet divergence. From the large momentum expansion of the correlator $< V^3V^3 - V^8V^8 >$, one observes that there are two terms which will lead to a divergence in the photon loop integral eq.(87),

$$\Pi^\mu_{V3}(p) - \Pi^\mu_{V8}(p) = \left( p^\mu p^\nu - g^\mu\nu p^2 \right) \left\{ \frac{2}{3p^4} (\bar{m} - m_s) < \bar{u}u> \left[ 1 + \frac{\alpha_s(p^2)}{3\pi} + \ldots \right] + O(1/p^4) \right\} \ .$$

(retaining only terms linear in the quark masses). The first term in the square bracket in eq.(89) generates a $1/\epsilon$ pole in dimensional regularization. This pole appears explicitly in the spectral representation (88). Using the expression of the QED counterterms $Z_s$ and $Z_2$ (17) one easily verifies that the $1/\epsilon$ infinities cancel exactly. This cancellation requires that the second sum rule (23) be satisfied. It is important to keep this point in mind in phenomenological applications based on resonance saturation. In particular, it is not consistent to ignore flavour symmetry breaking in the resonance sector as has been done in ref.[11]. If one does not impose that this sum rule be obeyed, the result will be infinite. Now the second term in the square bracket (89), proportional to $\alpha_s$, also generates a divergence in the photon loop integral, of the form $\log \epsilon$ in dimensional regularization (such a smooth singularity is the result of renormalization-group improvement). This divergence should be removed by minimal subtraction. This means that the integral $\int_0^\infty dx x \Delta \rho_V(x)$ appearing in eq.(88) is, in fact, divergent (the same holds true for the axial spectral function $\Delta \rho_A$). Since the integral $\int_0^\infty dx x \Delta \rho_V(x)$ is convergent, this suggests
that the spectral function difference behaves as $\Delta \rho_V \sim 1/x^2 \log^2 x$ asymptotically. This remark can be used to minimally subtract the $\log \epsilon$ singularity in the spectral representation, which should be done in practice if one were to use realistic spectral functions. The simple models which will be considered below do not lead to such singularities, so we will ignore this subtlety in the following.

In a similar way, one obtains for the corresponding axial current correlator, firstly from the chiral lagrangian

$$\Delta \Pi_A^Q(0) = 8 \left[ K_9^r(\mu) - K_{10}^r(\mu) + \frac{\xi}{16\pi^2} \frac{M_{\eta}^2 \log(M_{\eta}^2/\mu^2) - M_{\pi}^2 \log(M_{\pi}^2/\mu^2) - \frac{3}{4} \log(M_{\eta}^2 - M_{\pi}^2)}{4(M_{\eta}^2 - M_{\pi}^2)} + \frac{1}{16\pi^2} Z^0(\mu) \right] + O(m_{\eta}) \tag{90}$$

and secondly from the QCD action

$$\Delta \Pi_A^Q(0) = \frac{-3}{2(m - m_s)} \int \frac{d^4k}{(2\pi)^4} \left( \Pi_{A^3}^\mu(\eta) - \Pi_{A^3}^{\mu\nu}(\eta) \right) (-iD^\mu(\eta)) - Z_s - Z_2 \tag{91}$$

Again here, the sum rules (81) and (82) for $\Delta \rho_A$ and $\Delta \sigma_A$ ensure the correct cancellation of infinities. One notices that the pion contribution in the photon loop in eq.(90) appears also in eq.(71) in spectral representation so that this contribution cancels out in the expression for $K_9 - K_{10}$. One also expects that the chiral logarithms in the function $Z^0(\mu)$ in eqs.(83) and (90) should cancel out with similar terms present in the spectral functions but we will not attempt to elucidate exactly how this happens and we will keep this contribution (which is subleading in the large $N_c$ counting) as it is for the moment.

It is now easy to deduce the sum rules for $K_9$ and $K_{10}$. Defining the two integrals

$$Z_V = \int_0^\infty dx \log \frac{x}{M_V^2} \Delta \rho_V(x), \quad Z_A = \int_0^\infty dx \log \frac{x}{M_V^2} \Delta \rho_A(x) \tag{92}$$

where $\Delta \rho_A$ is the part of the spectral function with no pion contribution,

$$\Delta \rho_A(x) = \frac{2F_{\pi \eta}^2(x-M_{\pi}^2) - F_{\eta}^2(x-M_{\eta}^2)}{(M_{\pi}^2-M_{\eta}^2)F_{\pi}^2} + \Delta \rho_A(x) \tag{93}$$

(and the worrisome asymptotic tail related to $\alpha_s$ is assumed to be removed). We find the following result for $K_9$.

$$K_9^r(\mu) = \frac{1}{16\pi^2} \left\{ \frac{\xi \log M_{\eta}^2}{M_{\eta}^2} - \xi - \frac{3}{2}(Z_V + Z_A) \right\} \tag{94}$$

and for $K_{10}$,

$$K_{10}^r(\mu) = \frac{1}{16\pi^2} \left\{ - \xi \log M_{\eta}^2 - \frac{3}{2}(Z_A - Z_V) + 1 + 6Z^0(\mu) \right\} \tag{95}$$

The same expression for the constants $K_9$ and $K_{10}$ holds in which the spectral function differences related to the neutral currents $\Delta \rho_V$ and $\Delta \rho_A$ are replaced by the charged ones $\Delta \rho^+_V$.
and $\Delta \rho^+_{\pi}$ (see eq. (84)). The only modification to eqs. (94) and (95) is that the tadpole function $Z^0(\mu)$ must be replaced by $Z^+(\mu)$, with

$$Z^+(\mu) = \frac{C}{6 F_\pi^2 (M_K^2 - M_\pi^2)} \left\{ 3 M_\eta^2 \log \frac{M_\eta^2}{\mu^2} + 2 M_K^2 \log \frac{M_K^2}{\mu^2} - 5 M_\pi^2 \log \frac{M_\pi^2}{\mu^2} \right\}. \quad (96)$$

4.4 Difficulties of minimal resonance saturation

As one can see from eqs. (94), (95) and from (108), (109) below, in order to make a definite prediction concerning the violation of Dashen’s theorem one must be able to evaluate, in a reliable way, the the integrals $Z_A$ and $Z_V$ defined in (92). A priori, we will envisage to estimate these integrals in the approximation of resonance saturation, retaining the contributions of the lowest-lying vector meson and axial-vector meson octets. One must first verify that the spectral functions $\Delta \rho_V(x)$ and $\Delta \rho_A(x)$ approximated in this way obey the sum rules (79) and (80) with acceptable values of the resonance parameters. The first sum rule for $\Delta \rho_V$ and the equivalent sum rule for $\Delta \rho^+_V$ imply, respectively,

$$F_{\rho^0} = (3 F_\omega^2 + 3 F_\phi^2)^{\frac{1}{2}}, \quad F_{\rho^+} = F_{K^{*+}} \quad (97)$$

where the couplings of the neutral vector mesons are defined with respect to the electromagnetic current, i.e., $< 0| j_{\mu}^e V_\mu^0 > = M_{V^0} F_{V^0} e_\mu$. Assuming exact isospin conservation allows one to extract the matrix elements of the currents $V_{\mu\pi}^0$ and $V_{\mu\eta}^0$. Using the $e^+ e^-$ branching ratios, one obtains

$$F_{\rho^0} = 152.9 \pm 4, \quad F_\omega = 45.9 \pm 1.0, \quad F_\phi = 79.1 \pm 1.5 \quad (\text{MeV}). \quad (98)$$

If one now extracts the value of $F_{\rho^0}$ using the first relation (97), one obtains $F_{\rho^0} \approx 158$. This is in rather reasonable agreement with the experimental value. We can also extract the experimental values of the charged decay constants from $\tau$ decay data. Using the most recent compilation $^{22}$ one gets

$$F_{\rho^+} = 146 \pm 2, \quad F_{K^{*+}} = 151.5 \pm 4.7. \quad (99)$$

The values of $F_{\rho^+}$ and $F_{K^{*+}}$ turn out indeed to be nearly equal as is demanded by the DMO sum rule for $\Delta \rho^+_V$ saturated by a single resonance multiplet. This is not a completely trivial result if one thinks that typical coupling constants such as $F_\pi$ and $F_K$ differ by 20%. In conclusion, we seem to find that minimal resonance saturation is a good approximation as far as the first sum rule for $\Delta \rho_V$ is concerned. Let us now examine the analogous sum rule for the axial currents. Experimentally, the only accessible data concern the charged currents. Using the axial version of the DMO sum rule for $\Delta \rho^+_A$, together with resonance saturation, implies

$$F_{K_{1}(1270)}^2 + F_{K_{1}(1400)}^2 - F_{a_1}^2 = F_{\pi}^2 - F_{K}^2. \quad (100)$$

In this relation $F_{\pi}$ and $F_K$ are, of course, rather accurately known. Concerning $F_{a_1}$, an estimate can be made under the hypothesis that the decay amplitude $\tau \rightarrow \pi^- \pi^0 \pi^0$ (the most recent PDG $^{22}$ value for the branching fraction is $9.27 \pm 0.14\%$) is dominated by the $a_1$ resonance and proceeds via $a_1 \rightarrow \rho \tau$ (in other terms, we take $\Gamma(\tau \rightarrow a_1 \nu_\tau) = 2 \Gamma(\tau \rightarrow \pi^- \pi^0 \pi^0 \nu_\tau)$), which gives $F_{a_1} = 165 \pm 13 \text{ MeV}$ with $M_{a_1} = 1230 \pm 40 \text{ MeV}$. The first published data for the tau decay rate into $K_{1}(1270)$ and $K_{1}(1400)$ by the TPC/2$\gamma$ collaboration $^{35}$ have rather large error bars. Much more precise results obtained by the ALEPH collaboration were presented very recently $^{36}$. Transcribed into coupling constants these results read

$$F_{K_{1}(1270)}^2 = 123^2 \pm 63^2, \quad F_{K_{2}(1400)}^2 = 97^2 \pm 68^2 \quad (\text{MeV}^2). \quad (101)$$
The errors are certainly larger than in the vector case, still it appears plausible that the axial DMO sum rule is reasonably well saturated from the lowest lying resonance contributions.

Let us now turn to the second sum rule that $\Delta \rho_V$ and $\Delta \rho_A$ must satisfy (79), (80). Consider the vector currents first. Computing the integral $I_V$ using minimal resonance saturation and enforcing the first sum rule gives

$$I_V = \frac{F_V^2 (M^2_\rho - M^2_\omega)}{F^2_\pi (M^2_\pi - M^2_K)} .$$

(102)

The second sum rule states that $I_V = 1$. Unfortunately, this relation completely fails to be satisfied as, using experimental values, one finds $I_V \approx 5.4$. Neither experimental uncertainties nor the use of the narrow width approximation can explain such a large discrepancy. One source of uncertainty stems from the requirement of expanding the numerator to linear order in the quark masses. We have assumed that the term linear in the quark masses dominates the chiral expansion of the vector meson masses, as is suggested by the success of the Gell-Mann-Okubo mass formula for the vector nonet. The only possible explanation, then, is that the integral $I_V$ picks up significant contributions from the energy region above 1 GeV, implying that the simplest resonance approximation method appears to fail in this case. As has been discussed in sec.3 this approximation consists in performing a rational interpolation of the correlator $< V^3 V^3 - V^8 V^8 >$ with two poles. In the present case, the interpolation is not capable of correctly matching both the region of large $p^2$ and the region of small $p^2$. In order to improve it, we must increase the number of poles. If one nevertheless insists on employing a single resonance multiplet, then one must use an unphysical value for the product $F^2_V (M^2_\rho - M^2_\omega)$ such that $I_V$ in (102) satisfies $I_V = 1$ (recall that if the sum rule were not satisfied the result for $K_9 + K_{10}$ would be infinite). Doing this we find the following value for the integral $Z_V$ (104) which occurs in the expressions of $K_9$ and $K_{10}$:

$$Z^\text{min}_V = I_V = 1 .$$

(103)

Equality of $Z^\text{min}_V$ and $I_V$ results from using the narrow width approximation, together with the first sum rule, and expanding linearly in the quark masses. This result, however, is not stable against inclusion of higher mass resonances. Indeed, let us include one additional multiplet of vector resonances. One can fit the value of $F^2_V (M^2_\rho' - M^2_\omega')$ as well as the splitting $F^2_\rho' - F^2_\phi'$ in order that the first two sum rules be obeyed while keeping physical values for $F_V$ and $M^2_\rho - M^2_\omega$. This certainly improves the validity of the interpolation function for small values of the $p^2$. Whether the values of the new couplings $F_\rho'$, $F_\phi'$ are at all realistic cannot be decided since not much is experimentally known about these quantities. This improved model yields

the following evaluation of $Z_V$:

\footnote{This is possible provided $F^2_\rho' - 3F^2_\omega - 3F^2_\phi$ is negative which, fortunately, happens to be the case.}
where the sum rule $I$ taking $M$ and $I$ one finds instead $F$ rules (i.e. $F$ difference between $I$ obtains, approximation of minimal resonance saturation. Assuming the first sum rule to be satisfied, $Z$ of both $K$ Dashen’s theorem violation one really needs to be able to estimate spectral integrals are too slowly converging. However, if one is only interested in the amount of must satisfy two sum rules obtained by combining (79) and (80)

$$F \left( \frac{M^2}{M^2} - \log \frac{M^2}{M^2} - 1 \right) + \log \frac{M^2}{M^2} \left( 1 - \frac{(M^2 - M^2)F^2}{(M^2 - M^2)F^2} \right) \simeq -3.9,$$

taking $M_{V'} = 1.6$ GeV, which differs considerably from the estimate based on one resonance multiplet.

Turning to the axial-vector sector now, let us evaluate the integral $I_A$ in (80) under the approximation of minimal resonance saturation. Assuming the first sum rule to be satisfied, one obtains,

$$I_A = \frac{1}{(M^2 - M_K^2)F^2} \left[ F^2(M^2_{a1(1260)} - M^2_{f1(1510)}) + 2M_A^2(F^2_K - F^2) \right], \quad (105)$$

and $I_A$ should be equal to one according to the sum rule (80). Using experimental numbers, one finds instead $I_A \simeq 6 \pm 2$. If one uses for $M_A$ and $F_A$ the values from the Weinberg sum rules (i.e. $F_A = 122$ MeV, $M_A = 966$ MeV with our choice of $F_V$ and $M_V$ and ignoring the difference between $F_0$ and $F_\pi$) rather than the experimental ones, then one obtains $I_A \simeq 4$. The approximation of minimal saturation is again found to be in conflict with the second sum rule, if one takes physically reasonable values for the resonance parameters.

At this point, the conclusion would be that it is not possible to obtain a reliable estimate of both $Z_V$ and $Z_A$ (and consequently of the two LEC’s $K_9$ and $K_{10}$) at present, because the spectral integrals are too slowly converging. However, if one is only interested in the amount of Dashen’s theorem violation one really needs to be able to estimate $K_{10}$, because $K_9$ happens to be multiplied by $M^2_\pi$ and will contribute very little. Now the convolution representation for $K_{10}$ involves the double difference $\Pi_{V^3} - \Pi_{A^3} - \Pi_{V^8} + \Pi_{A^8}$. The corresponding spectral functions must satisfy two sum rules obtained by combining (79) and (80)

$$ \int_0^\infty dx (\Delta \rho_V(x) - \Delta \rho_A(x)) = 0, \quad \int_0^\infty dx x (\Delta \rho_V(x) - \Delta \rho_A(x)) = 0. \quad (106)$$

The second sum rule implies that convergence of the spectral representation should be faster than it is for $\Pi_{V^3} - \Pi_{V^8}$ and $\Pi_{A^3} - \Pi_{A^8}$ individually. In fact, one can observe that the second sum rule in (106), which can be stated as $I_A = I_V$, is approximately satisfied in the minimal resonance approximation as one can check from the values of $I_A$ and $I_V$ obtained above. Experience with the Weinberg sum rules and the corresponding DGMLY sum rule for $C$ suggests that minimal resonance saturation is likely to be a reasonable approximation as far as $K_{10}$ is concerned. In this approximation, the value of the difference $Z_A - Z_V$ to be used in eq.(105) for $K_{10}$ is given by

$$Z_A - Z_V = \frac{1}{F^2(M^2 - M_K^2)} \left[ 2M_A^2(F^2 - F^2) + F^2(F^2 - M^2) \log \frac{M^2}{M^2} \right] - 2 \left( 1 + \log \frac{M^2}{M^2} \right), \quad (107)$$

where the sum rule $I_A - I_V = 0$ has been used to simplify the expression.
5. Consequences for Dashen’s theorem violation

Up to corrections which are quadratic in the light quark masses or quartic in the electric charge, the contribution to the $K^+ - K^0$ mass difference proportional to $e^2$ can be written as a sum of six terms [14], as follows:

$$\Delta M_K^2 \equiv M_{K^+}^2 - M_{K^0}^2|_{em} = t_0 + t_1 + t_2 + t_3 + t_4 + t_5$$ \hspace{1cm} (108)
with

$$t_0 = e^2 \frac{2C}{F_0^2}$$

$$t_1 = \frac{-e^2}{16\pi^2} \left[3M_K^2 \log \frac{M_K^2}{\mu^2} - 4M_K^2 + \frac{2C}{F_0^4} \left(M_\pi^2 \log \frac{M_\pi^2}{\mu^2} + 2M_K^2 \log \frac{M_K^2}{\mu^2}\right)\right]$$

$$t_2 = \frac{-e^2}{F_0^4} \left[(M_\pi^2 + 2M_K^2)L_4 + M_K^2L_5^r\right]$$

$$t_3 = e^2 M_\pi^2 \left[4K_r^8 + \frac{4}{3}(K_{10}^r + K_{11}^r)\right]$$

$$t_4 = e^2 M_K^2 \left[8K_r^8 + 8(K_{10}^r + K_{11}^r)\right]$$

$$t_5 = -e^2 M_K^2 \frac{4}{3}(K_5^r + K_6^r)$$.

One notices first, using expressions (94), (95) for $K_9$ and $K_{10}$ and (59) for $K_{11}$ and the discussion in sec. 3.6 on $K_5 + K_6$ that the dependence on the QED gauge parameter $\xi$ drops out, as it should. In fact, the sum $t_0 + t_1 + t_2 + t_3 + t_4$ and $t_5$ are separately gauge independent. The dependence upon the QCD renormalization scale $\mu_0$ drops out in the term $t_4$ but not in the term $t_3$. This indicates that the electromagnetic contribution to the $K^+ - K^0$ mass difference is not, strictly speaking, an observable quantity: only the full mass difference is independent of $\mu_0$. Fortunately, this term $t_3$ makes a negligibly small contribution to the mass difference, being proportional to $M_\pi^2$. In order to estimate numerically the value of $\Delta M_K^2$ one needs, essentially, an estimate of the three constants $K_{10}$, $K_{11}$ and $K_5 + K_6$ ignoring the terms which are proportional to $M_\pi^2$ or suppressed by the Zweig rule. The estimate of $K_{10}$ based on resonance saturation is contained in the expressions (94) and (107) and a similar estimate for $K_{11}$ is given in (59). We were not able to evaluate $K_5 + K_6$ but one notices that this combination appears in the expression for $\Delta M_K^2$ with a much smaller numerical coefficient than $K_{10}$ or $K_{11}$. Assuming that $K_5$ and $K_6$ have the order of magnitude typical of $O(p^4)$ LEC’s the lack of precise evaluation generates a rather small uncertainty. Based on these estimates one obtains the following decomposition of $\Delta M_K^2$:

$$M_{K^+}^2 - M_{K^0}^2|_{em} = (M_{\pi^+}^2 - M_{\pi^0}^2) \left\{1.02 + 1.13 - 0.20 + [-2, 2]10^{-2} + 0.80 + [-0.3, 0.3]\right\} \hspace{1cm} (110)$$

where the six contributions to the six contributions $t_0,...,t_5$ displayed in eq. (109). Individual contributions are scale dependent: they are shown at the CHPT scale $\mu = M_V$ and the QCD scale $\mu_0 = 1$ GeV. The values of $M_A$ and $F_A$ wherever they appear are taken from the Weinberg sum rules. The contribution from $t_3$ (fourth entry in eq. (110)) has been estimated to lie in a range, assuming that $Z_V$ could take values between $\pm 4$ (see sec.4.4). The last entry corresponds to the contribution of $K_5$ and $K_6$ assuming that each of these LEC’s lies in the range $\pm 1/16\pi^2$.
It is interesting to verify the stability of the result if one drops systematically all terms which are known to be subleading in the large $N_c$ counting. This is easily accomplished in eq.(108) knowing that $F^2_0$, $L^2_5$ and $C$ are $O(N_c)$, $L^4_4$ is $O(1)$, $K^5_5$, $K^6_6$, $K^9_9$, $K^6_{10}$, $K^7_{11}$ are also $O(1)$ while $K^5_8$ is $O(1/N_c)$. The variation of $L^2_5(\mu)$ with respect to the scale $\mu$ is subleading in $N_c$ but, in practice, it is not negligible since $L^2_5$ varies by a factor of two as $\mu$ is varied between $M_\eta$ and $M_\rho$. This raises the question of which scale should one choose for performing a leading large $N_c$ calculation. We bypassed this difficulty by expressing directly $F^{\pi}$ or to the calculation performed on the lattice [13] in the quenched approximation. We note that neither [9], [38] or [11] have included enough terms (such as $\pi\rho a_1$ coupling, see sec 3.4) in their resonance lagrangian such as to satisfy the asymptotic conditions needed to evaluate separately $K_{10}$, $K_{11}$ and $K_5 + K_6$. A less constraining set of two sum rules were implemented in ref.[38] which are derived by requiring UV finiteness of $\Delta M^2_K$ (i.e. no contribution from QED counterterms), which holds in the limit $m_u = m_d = 0$. This is obviously a minimal requirement for any realistic model but it is not satisfied by the estimate of ref.[11]. In ref.[38] these two constraints are enforced by adjusting the two values of the axial-vector coupling constant $F_{K_1}$ (assuming $F_{K_2} = F_{K_1}$) and $F_{K'_1}$, the coupling constant of an excited axial resonance which they find necessary to include. These two sum rules are reminiscent of

$$L_5 = \frac{F^2_K - F^2_{\pi}}{8(M^2_K - M^2_{\pi})} \simeq 2.310^{-3} \quad (111)$$

Using this result, and dropping all subleading terms in $N_c$ we obtain the $K^+ - K^0$ mass difference in the following form,

$$M^2_{K^+} - M^2_{K^0}|_{em} = (M^2_{\pi^+} - M^2_{\pi^0}) \left\{ 1.02 + 0.75 - 0.51 + [-1,2]10^{-2} + 1.20 + [-0.3,0.3] \right\}. \quad (112)$$

Interestingly, though individual terms are substantially modified when removing these contributions which are subleading in $N_c$, the overall result is reasonably stable. The contribution of $K_5 + K_6$ still remains to be explored in detail, before a quantitative estimate of $\Delta M^2_K$ can be provided by this approach. Still, the results above are strongly suggestive of a rather large amount of violation of Dashen’s theorem, similar in magnitude to the results of [9], [38], [10] or to the calculation performed on the lattice [13] in the quenched approximation.

The cause for the different result obtained in ref.[11] (negligible DT violation) can be better understood with the help of a recent preprint by the same authors[39] in which they have computed, from exactly the same resonance lagrangian, the contributions to each of the constants $K_i$. Firstly, their neglect of flavour symmetry breaking in the resonance sector amounts to setting $K^7_{10}(M_T) = 0$. Doing this bring the contribution of $t_4$ to $\Delta M^2_K$ down from approximately one to 0.3, using our value for $K_{11}$. However, their estimate of $K_{11}$ is much smaller than ours such that $t_4$ is practically zero in their evaluation. As mentioned in ref.[39], a remainder contribution is needed in order to restore the correct chiral scale and QCD scale dependences. Comparison with our results shows that this remainder is not negligible, and may even be dominant as in the case of $K_{11}$.

We note that neither [9], [38] or [11] have included enough terms (such as $\pi\rho a_1$ coupling, see sec 3.4) in their resonance lagrangian such as to satisfy the asymptotic conditions needed to evaluate separately $K_{10}$, $K_{11}$ and $K_5 + K_6$. A less constraining set of two sum rules were implemented in ref.[38] which are derived by requiring UV finiteness of $\Delta M^2_K$ (i.e. no contribution from QED counterterms), which holds in the limit $m_u = m_d = 0$. This is obviously a minimal requirement for any realistic model but it is not satisfied by the estimate of ref.[11]. In ref.[38] these two constraints are enforced by adjusting the two values of the axial-vector coupling constant $F_{K_1}$ (assuming $F_{K_2} = F_{K_1}$) and $F_{K'_1}$, the coupling constant of an excited axial resonance which they find necessary to include. These two sum rules are reminiscent of

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9The correspondance of their results with our formulas is as follows: one must drop the counterterm contributions and define the convolution integrals using the chiral $\overline{MS}$ prescription. The QCD correlators must, of course, be computed using their resonance lagrangian. In this way one immediately recovers that $K^7_{10} = 0$ and we also checked their results for $K_{11}$ and $K_{12}$. We could not, however, reproduce their result for $K_{13}$. 

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those obtained above, eq. (106), in connection with the parameter $K_{10}$, the difference is that they pick up additional contributions associated with the combination $8K_{11} - 4/3(K_5 + K_6)$ which occurs in the expression of $\Delta M^2_R$. We recall that the two sum rules (106) did not seem to require significant contributions from excited resonances. A better experimental determination of the coupling constants $F_{K_1}$ and $F_{K_2}$ would help to clarify these issues. We note also that the calculation of ref. [38] includes contributions which are quadratic in $m_s$, it would be of interest to know the relative importance of these contributions. Our result for $K_{10}$ is in reasonable agreement with that of ref. [12]: dropping the tadpole contribution, setting $\mu = m_\rho$, $\mu_0 = 0.7$ GeV and $\xi = 1$, we obtain $K_{10}(\mu) = 5.2 \times 10^{-3}$, to be compared with the value $K_{10}'(\mu) = (4 \pm 1.5) \times 10^{-3}$ quoted in this paper. Finally, it would be desirable to estimate the error on the evaluations of $C$, $K_{10}$, $K_{11}$ and $K_5 + K_6$. In the case of $C$ and $K_{10}$ one can use experimental data from $\tau$ decay directly into the sum rules and this should allow for an estimate of the error as well. For the others, one must rely on resonance saturation and the reliability of the estimates can be assessed, in principle, by performing evaluations using higher order rational approximants (i.e. including additional resonance multiplets) and checking the stability of the result.

6. Conclusions

We have discussed an approach to the question of DT violation (and more generally, to the evaluation of the low-energy constants $K_i$ which parametrize all electromagnetic effects at order $O(\epsilon^2 p^2)$) which is a direct generalization of the classic sum rule of Das et al. [1]. Instead of the technique of current algebra, as used in [1], we have made use of the chiral lagrangian and the charge spurions. This technique is rather powerful, as one can judge from the simplicity of the rederivation of the DGMLY sum rule in sec. 3. As an immediate consequence of this method, one finds that all the $K_i$’s can be expressed as a convolution of a QCD n-point function (with n=2,3 and 4) with the photon propagator. We have displayed explicitly the contributions of the QED counterterms to these constants which allows one, if one wishes, to perform the calculation using an arbitrary regularization and renormalization scheme. In the $\overline{MS}$ scheme, the dependence of the parameters $K_1$ to $K_{13}$ on the gauge parameter $\xi$ and on the QCD renormalization scale $\mu_0$ was entirely determined.

A current prejudice, based in part on the phenomenology of the usual $O(p^4)$ constants $L_i$ (see [24]) is that the values of the constants $K_i$ (and, as a consequence, the amount of DT violation) should be essentially controlled by resonance physics at a scale of 1 GeV. The great simplicity and the accuracy of the DGMLY expression [1] certainly make it worthwhile to investigate this prejudice in more detail. This assumption concerning the role of the lowest lying resonances amounts to approximating in a minimal way the QCD correlation functions (which occur in the convolution expression of the constants $K_i$) with rational functions having the corresponding resonance poles. The parameters of these resonances are then subject to stringent constraints which can be expressed either in terms of sum rules that the exact QCD correlation function satisfies or in terms of matching conditions at asymptotic momenta. A well known example is the correlation function $<VV - AA>$ in the chiral limit, which leads to the two Weinberg sum rules: these can be saturated to a reasonable level of accuracy by the $\rho$ and the $a_1$ resonances. This model correlator leads to the DGMLY formula and, as we have shown, to a similar expression for the constant $K_{13}$. We have displayed a generalization of this construction to the QCD three-point function $<VAP>$ in the chiral limit. Again, we have shown that a model containing as poles only those of the $\rho$ and $a_1$ resonances (together with the pion pole) can exactly match all the relevant asymptotic constraints. The properties of
the $\rho$ and the $a_1$ resonances are then predicted to be in reasonable agreement with experiment provided the $a_1$ radiative width is indeed strongly suppressed as has been suggested by a recent experiment[33]. These asymptotic constraints cannot be avoided as they ensure that the QED ultraviolet divergencies are exactly cancelled by the QED counterterms. We have shown the necessity of not ignoring the $\pi\rho a_1$ couplings (as was done in earlier work) if the contribution of the $a_1$ resonance is to be correctly evaluated. Under the assumption of dominance of the lowest resonance multiplet in the vector and the axial-vector sectors, the constants $K_{11}$ and $K_{12}$ are uniquely determined in terms of $M_\rho$ and $M_{a_1}$. The constant $K_{12}$ does not participate in DT violation but it arises in other interesting isospin violating phenomena. For instance, as shown in ref.[34], it is the only constant $K_i$ which appears in the $Kl_3$ form factor $f_{K^+\pi^0}$ and it also appears in the decay constants $F_\pi$ and $F_K$ (note that these constants are no longer physically meaningful quantities in the presence of electromagnetism: they are gauge dependent and infrared divergent). These estimates can in principle be improved by using rational approximants of higher order, i.e. including more resonances.

A second class of LEC’s that we have considered are $K_9$ and $K_{10}$. We displayed sum rule expressions in terms of the spectral function differences $\rho_{V^3} - \rho_{V^8}$ and $\rho_{A^3} - \rho_{A^8}$ or, alternatively, $\rho_{V^{ud}} - \rho_{V^{us}}$ and $\rho_{A^{ud}} - \rho_{A^{us}}$ which are quantities measurable from $\tau$ decay data. In principle, then, $K_9$ and $K_{10}$ could be evaluated without any recourse to resonance saturation models using data from $e^+e^-$ scattering as well as $\tau$ decay. A number of classic chiral sum rules have been already explored in this manner [20]. At present, however, there are rather large uncertainties as far as $\rho_{V^8}$ (or $\rho_{A^8}$) is concerned as is discussed in ref.[37], but the data is constantly improving. In this sector, the constant $K_{10}$ is expressed in terms of the double difference $\rho_{V^{ud}} - \rho_{V^{us}} - (\rho_{A^{ud}} - \rho_{A^{us}})$ for which simple resonance approximation can be argued to be a good approximation and yields a simple expression. Convergence is, on the contrary, slower in the case of $K_9$ and one would need several multiplets of resonances in order to adequately estimate $K_9$. Fortunately, as far as DT violation is concerned, $K_9$ happens to be multiplied by a very small coefficient. Finally, all the LEC’s which are important for DT violation were estimated except for the combination $K_5 + K_6$. The difficulty of estimating $K_5 + K_6$ comes from the complexity of modelling the QCD four-point functions $<AAVV>$ and $<VVVV>$ in terms of resonances. One can think of a number of contributions, for instance tensor meson exchange, which were never accounted for in DT violation. We have contented ourselves with a simple order of magnitude estimate which, after all, could be more reliable than an incomplete evaluation.

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