A logarithm law for automorphism groups of trees

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Abstract

Let $\Gamma$ be a geometrically finite tree lattice. We prove a Khintchine-Sullivan type theorem for the Hausdorff measure of the set of points at infinity of the tree that are well approximated by the parabolic fixed points of $\Gamma$. Using Bruhat-Tits trees, an application is given for the Diophantine approximation of formal power series in the variable $X^{-1}$ over the finite field $\mathbb{F}_q$ by rational fractions in $X$ over $\mathbb{F}_q$, satisfying some congruence properties.

1 Introduction

Let $T$ be a locally finite tree, $\partial T$ be its space of ends, and $\text{Aut}(T)$ be its locally compact group of automorphisms. To simplify the statements in the introduction, we assume that $T$ has no degree 1 or 2 vertex and that $\text{Aut}(T) \setminus T$ is a finite graph. A lattice of $T$ is a discrete subgroup with a cofinite Haar measure in $\text{Aut}(T)$. It is uniform if $\Gamma \setminus T$ is a finite graph. In [2, chap. 4], H. Bass and A. Lubotzky gave numerous examples of non uniform lattices. In [17], the second author has introduced a special class of discrete subgroups of $\text{Aut}(T)$, where the interesting ones are non uniform lattices, called geometrically finite (see also Section 2). In particular, they contain all the algebraic examples. More precisely, let $\hat{K}$ be a non-archimedian local field, and $G$ be a connected semi-simple algebraic group over $\hat{K}$ with $\hat{K}$-rank 1. Let $T$ be the Bruhat-Tits tree of $(G, \hat{K})$ (see [6, 20]), endowed with its action of $G(\hat{K})$. By the work of J.-P. Serre, M. Raghunathan and A. Lubotzky [15], every lattice contained in the image of $G(\hat{K})$ in $\text{Aut}(T)$ is geometrically finite in the sense of [17].

Let $\Gamma$ be a fixed geometrically finite lattice in $\text{Aut}(T)$. Let $\Gamma_{\infty}$ be a parabolic subgroup of $\Gamma$, $\infty$ be its unique fixed point in $\partial T$, and $(H_r)_{r \in \Gamma_{\infty}}$ be its associated family of horoballs (see [17] and also Section 2 for definitions). For every $r$ which is different from $\infty$ in the orbit $\Gamma_{\infty}$, define $D(r) = d(H_{\infty}, H_r)$, where $d$ is the distance on $T$. Let $d_{\partial T}$ be a usual distance on $\partial T$, $\delta$ be its Hausdorff dimension and let $\mu$ be the associated Hausdorff measure in dimension $\delta$ (see Section 2).

Theorem 1.1 Let $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ be a map such that $\log \psi$ is Lipschitz. Let $E(\psi)$ be the set of points $\xi$ in $\partial T \setminus \{\infty\}$ such that there exist infinitely many $r$ in $\Gamma_{\infty}$ with

$$d_{\partial T}(\xi, r) \leq \psi(D(r)) e^{-D(r)}.$$ 

Then $\mu(E(\psi)) = 0$ (respectively $\mu(\{E(\psi)\} = 0$) if and only if the integral $\int_1^{+\infty} \psi(t)^\delta dt$ converges (respectively diverges).

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A more general statement is given by Theorem 2.2. The main consequence is the following corollary, called the logarithm law for $\Gamma$.

Define the height function $h: T \to [0, +\infty]$ for $\Gamma$ by $h(x) = 0$ if $x \in T - \bigcup_{r \in \Gamma} H_r$, and if $x \in H_r$, then $h(x) = d(x, \partial H_r)$.

**Corollary 1.2** For every vertex $x$ in $T$, let $c_\eta$ denote the geodesic ray starting from $x$ and ending at $\eta \in \partial T$. Then for $\mu$-almost every $\eta$ in $\partial T$,

$$\limsup_{t \to +\infty} \frac{h(c_\eta(t))}{\log t} = \frac{1}{\delta}.$$ 

Theorem 1.1 and this corollary are analogues, in the setting of trees and in particular for non archimedian rank one lattices, of the similar well-known results for

- $\text{PSL}_2(\mathbb{Z})$ acting on the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$, due to Khintchine [12],
- a lattice in the isometry group of a real hyperbolic space $\mathbb{H}^n_\mathbb{R}$, due to D. Sullivan [23],
- a lattice in the isometry group of a symmetric space of non compact type, due to D. Kleinbock and G. Margulis [13],
- a geometrically finite group of isometries acting on $\mathbb{H}^n_\mathbb{R}$ due to B. Stratmann and S.L. Velani [22], and
- more general geometrically finite groups of isometries of complete simply connected Riemannian manifolds with negative curvature, due to the authors [11].

A particular case of the theorem was announced by J. Athreya [1], when $T$ is the Bruhat-Tits tree of $(\text{SL}_2, \mathbb{F}_q((X^{-1})))$ and $\Gamma = \text{SL}_2(\mathbb{F}_q[X])$.

Denote by $|\cdot|_\infty$ the usual absolute value on the field $\tilde{K} = \mathbb{F}_q((X^{-1}))$ of formal Laurent series in $X^{-1}$ over $\mathbb{F}_q$. Endowed $\tilde{K}$ with its Haar measure. Let $Q_0$ be a fixed non zero element in the ring $A = \mathbb{F}_q[X]$ of polynomials over $\mathbb{F}_q$. By using [16] [17], we obtain the following corollary, concerning the Diophantine approximation of elements of $\tilde{K}$ by rational fractions.

**Corollary 1.3** Let $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a map such that $u \mapsto \log \varphi(e^u)$ is Lipschitz. If the integral $\int_1^{+\infty} \varphi(t)/t \, dt$ diverges (resp. converges), then for almost every (resp. almost no) $f$ in $\tilde{K}$, there exist infinitely many couples $(P, Q)$ in $A \times (A - \{0\})$, where $P$ is coprime with $Q$ and $Q$ is divisible by $Q_0$, such that $|f - P/Q|_\infty \leq \varphi(|Q|_\infty)/|Q_0|_\infty^2$.

A more general statement is given by Corollary 2.3. The special case of Corollary 1.3 when $Q_0$ is the constant polynomial 1 was announced by J. Athreya [1].

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2 Notations, statements and applications

We refer to \[8\] for basic definitions about CAT(-1) geodesic metric spaces (among which are trees), their horospheres, their boundaries and their discrete groups of isometries.

Let \( T \) be a locally finite tree, \( \partial T \) be its space of ends and \( T \cup \partial T \) be its compactification by its ends. Such a tree is called uniform if there exists a discrete subgroup \( \Gamma \) in Aut(\( T \)) such that the quotient graph \( \Gamma \setminus T \) is finite. The tree \( T \) is endowed with the maximal geodesic distance \( d \) making each edge isometric to the interval \([0,1]\). For every \( x,y \) in \( T \cup \partial T \), we denote by \([x,y]\) the geodesic segment, ray or line between them, with the usual convention concerning the endpoints. Fix a point \( x_0 \) in \( T \). Let \( d_{x_0} \) be the visual distance on \( \partial T \) seen from \( x_0 \). It is defined by \( d_{x_0}(\eta,\eta') = e^{-d(x_0,p)} \) where \([x_0,\eta]\cap[x_0,\eta'] = [x_0,p] \) if \( \eta \neq \eta' \). The Busemann function \( \beta_\xi : X \times X \to \mathbb{R} \) of a point \( \xi \) in \( \partial T \) is defined by \( \beta_\xi(x,y) = d(x,u) - d(y,u) \) for every \( u \) in \( T \) close enough to \( \xi \). An horoball centered at \( \xi \) is the preimage by \( y \mapsto \beta_\xi(x_0,y) \) of \([t,\infty[\) for some \( t \in \mathbb{R} \).

Let \( \Gamma \) be a discrete subgroup of Aut(\( T \)). We assume that \( \Gamma \) is non-elementary, i.e. that \( \Gamma \) preserves no point nor pair of points in \( T \cup \partial T \). Let \( \Lambda \Gamma \) be its limit set, i.e. the smallest non empty closed \( \Gamma \)-invariant subset in \( \partial T \). Let \( CA\Lambda \) be the minimal non empty subtree of \( T \) which is invariant by \( \Gamma \).

The critical exponent of \( \Gamma \) is the unique number \( \delta \) in \([0,\infty[\) such that the Poincaré series \( \sum_{\gamma \in \Gamma} e^{-s d(x_0,\gamma x_0)} \) of \( \Gamma \) converges for \( s > \delta \) and diverges for \( s < \delta \). The group \( \Gamma \) is called of divergent type if its Poincaré series diverges at \( s = \delta \).

If \( \Gamma \) is of divergent type with a finite non zero critical exponent, then there exists (see \[8\]) a family \((\mu_x)_{x \in T}\) of finite measures on \( \partial T \), with support \( \Lambda \Gamma \), called the Patterson-Sullivan density, which is unique up to a positive scalar factor, such that

\[ \forall \gamma \in \Gamma, \quad \gamma_* \mu_x = \mu_{\gamma x} , \]

\[ \forall x,y \in T, \forall \xi \in \partial T, \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta \beta_\xi(x,y)}. \]

An element \( \omega \) in \( \Lambda \Gamma \) is a conical limit point if there exist a sequence \((\gamma_n)_{n \in \mathbb{N}}\) in \( \Gamma \), a geodesic ray \( c \) in \( T \) ending at \( \omega \) and \( A \geq 0 \) such that \( d(\gamma_n x_0,c) \leq A \) for every \( n \) in \( \mathbb{N} \). An element \( \omega \) in \( \Lambda \Gamma \) is called a bounded parabolic point if its stabilizer \( \Gamma_\omega \) in \( \Gamma \) acts properly discontinuously with compact quotient on \( \partial T \setminus \{ \omega \} \). A parabolic subgroup \( \Gamma_\infty \) of \( \Gamma \) is a maximal infinite locally finite subgroup (see \[2\] for other characterizations and the following properties). In particular, there is one and only one point in \( T \cup \partial T \) fixed by \( \Gamma_\infty \) and it belongs to \( \partial T \).

A discrete subgroup \( \Gamma \) of Aut(\( T \)) is geometrically finite if it is non elementary and if every point in \( \Lambda \Gamma \) is either a conical limit point or a bounded parabolic point. A characterization in terms of the quotient graph of groups \( \Gamma \setminus T \) (see \[2\] for a definition when \( \Gamma \) acts without inversion) and the basic properties of geometrically finite subgroups \( \Gamma \) of Aut(\( T \)) are given in \[17\]. In particular, the point fixed by a parabolic subgroup of \( \Gamma \) is then a bounded parabolic point.

From now on, we fix a geometrically finite subgroup \( \Gamma \) in Aut(\( T \)) and a parabolic subgroup \( \Gamma_\infty \) of \( \Gamma \). Let \( \infty \) be the unique fixed point of \( \Gamma_\infty \) in \( \partial T \) and let \( \Gamma_\infty \) be its orbit under \( \Gamma \). In \[17\], it is proved that there exists a unique \( \Gamma \)-invariant family of horoballs \((H_r)_{r \in \Gamma_\infty}\), with pairwise disjoint interiors, with \( H_r \) centered at \( r \) and maximal with respect to inclusion.
Let \( d_\infty \) be the Hamenstädt distance on \( \partial T - \{ \infty \} \) associated to \( H_\infty \) (see [10, Appendix], [17]). It is defined by \( d_\infty(\eta,\eta') = e^{-d_\pm(p,q)} \) where \( \{ p \} = \partial H_\infty \cap \{ \infty, \eta \} \backslash \{ \infty, q \} \) and \( d_\pm(p,q) \) is the signed distance between \( p \) and \( q \) on \( \infty, \eta \backslash \{ \infty, \eta' \} \). The Hausdorff dimension and the Hausdorff measure class of \( \Lambda \Gamma - \{ \infty \} \) with respect to the distance \( d_\infty \) coincides with the Hausdorff dimension and the Hausdorff measure class of \( \Lambda \Gamma - \{ \infty \} \) with respect to the Hamenstädt distance \( d_\infty \) respectively, as locally on \( \partial T - \{ \infty \} \), the distances \( d_\infty \) and \( d_\infty \) differ by a positive scalar factor.

If \( \rho \) is the geodesic ray in \( T \) starting from \( x_0 \) and converging to \( \infty \), then the measures \( e^{\delta t}\mu_\rho(t) \) on \( \partial T - \{ \infty \} \) converge, as \( t \to +\infty \), to a measure \( \mu_\infty \) on \( \partial T - \{ \infty \} \), called the Patterson-Sullivan measure of \( \Gamma \) on \( \partial T - \{ \infty \} \), supported on \( \Lambda \Gamma - \{ \infty \} \). The measure \( \mu_\infty \) has the same measure class as \( \mu_{x_0} \) on \( \partial T - \{ \infty \} \) (see [11]).

The following theorem summarizes the known results on \( \Gamma \) that we will use throughout this paper.

**Theorem 2.1** Let \( \Gamma \) be a geometrically finite subgroup of \( \text{Aut}(T) \) such that \( C\Lambda \Gamma \) is a uniform tree without degree 2 vertices. Let \( \Gamma_\infty \) be a parabolic subgroup of \( \Gamma \). Then the following assertions hold.

1. **(1)** The group \( \Gamma \) has a finite non zero critical exponent, and is of divergent type. Its action on \( \Lambda \Gamma \) is ergodic with respect to the measure \( \mu_{x_0} \). This measure has no atom.

2. **(2)** There exists a constant \( c_1 > 0 \) such that for every \( n \) in \( \mathbb{N} \),
   \[
   \frac{1}{c_1} e^{n\delta} \leq \text{Card}\{ \gamma \in \Gamma : d(x_0, \gamma x_0) \leq n \} \leq c_1 e^{n\delta}.
   \]

3. **(3)** The critical exponent \( \delta \) equals the Hausdorff dimension of \( \Lambda \Gamma \) with respect to the distance \( d_{x_0} \).

4. **(4)** The critical exponent of \( \Gamma_\infty \) is equal to \( \delta/2 \).

5. **(5)** There exists a constant \( c_2 > 0 \) such that for every \( n \) in \( \mathbb{N} \),
   \[
   \frac{1}{c_2} e^{n\delta/2} \leq \text{Card}\{ \gamma \in \Gamma_\infty : d(x_0, \gamma x_0) \leq n \} \leq c_2 e^{n\delta/2}.
   \]

**Proof.** Let \( T' = C\Lambda \Gamma \). Then \( T' \), endowed with the restricted action of \( \Gamma \), satisfies the hypotheses of the theorem if and only if \( T \), endowed with the action of \( \Gamma \), does. If the assertions (1)-(5) are true for \( T' \), then they are also true for \( T \). Hence we may assume that \( T = C\Lambda \Gamma \). Up to minor changes in the following geometric series argument when replacing \( T \) by its barycentric subdivision, we may assume that \( \Gamma \) acts without inversion.

Recall that the quotient graph of groups \( \Gamma \backslash T \) is the union of a finite graph of finite groups and of finitely many cuspidal rays of groups (see [2], [17] for the definitions and [17] for the proof). The covolume of \( \Gamma \) in \( \text{Aut}(T) \) is (see [2]) a multiple of the sum of the inverses of the cardinals of the vertex groups in \( \Gamma \backslash T \). By an easy geometric series argument, the fact that \( T \) has no vertex of degree 2 (here and below, the assumption that there are no degree 2 vertices inside an horoball of an equivariant family of pairwise disjoint horoballs centered at all parabolic fixed points is sufficient) implies in particular that \( \Gamma \) is a lattice in \( T \). (See for instance [7], or simply note that if \( \Gamma_n \) is the stabilizer
of the $n$-th vertex of a cuspidal ray, then $[\Gamma_{n+1} : \Gamma_n] \geq 2$ as $T$ has no vertex of degree 2, hence $\sum_n 1/|\Gamma_n| \leq \sum_n 1/2^n$ is finite.)

(1) The finiteness (and non vanishing) of the critical exponent follows for instance from [7, Prop. 1.7]. By [7, Coro. 6.5] for instance, the group $\Gamma$ is of divergent type. By [18, Theorem 1.7] and [18, Corollary 1.8] for instance, the measure $\mu_{x_0}$ is ergodic and has no atom.

(4) See for instance [5, Prop. 3.1].

(2) As every vertex of $T$ has degree at least 3, and as there exists at least one parabolic subgroup, the subgroup of $\mathbb{Z}$ generated by the hyperbolic translation lengths of the elements of $\Gamma$ equals to $\mathbb{Z}$ (see [5]). By (1), (4) and [18, Théorème 1.11], the Bowen-Margulis measure of $\Gamma$ (see for instance [18] for a definition of this measure) is finite. Hence by corollaire 2 following théorème 4.1.1 in [18], the result holds.

(3) This follows for instance from [7, Coro. 6.5].

(5) This follows for instance from [5, Prop. 3.1].

Before providing a more general statement than Theorem 1.1, we will introduce the class of functions we will work with.

A map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called slowly varying (see [23]) if it is measurable and if there exist constants $B > 0$ and $A \geq 1$ such that for every $x, y \in \mathbb{R}_+$, if $|x - y| \leq B$, then $\psi(y) \leq A \psi(x)$. Recall (see for instance [11, Sec. 5]) that this implies that $\psi$ is locally bounded, hence it is locally integrable; also, if $\log \psi$ is Lipschitz, then $\psi$ is slowly varying.

Theorem 2.2 Let $\Gamma$ be a geometrically finite subgroup of $\text{Aut}(T)$ such that $C \Lambda \Gamma$ is a uniform tree without degree 2 vertices. Let $\Gamma_\infty$ be a parabolic subgroup whose fixed point will be denoted by $\infty$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a slowly varying map. Let $E(\psi)$ be the set of points $\xi$ in $\partial T - \{\infty\}$ such that there exist infinitely many $r$ in $\Gamma_\infty$ with $d_\infty(\xi, r) \leq \psi(D(r))e^{-D(r)}$. Then $\mu_\infty(E(\psi)) = 0$ (respectively $\mu_\infty(cE(\psi)) = 0$) if and only if the integral $\int_1^{+\infty} \psi(t)^d dt$ converges (respectively diverges).

The proof of this theorem will be given in Section 3. Note that for some of the Bruhat-Tits trees (the ones that are biregular of type $(2, q)$, see [6]), we have to remove the vertices of degree 2 in order to verify the hypothesis of the above theorem. In the new tree, the critical exponent is multiplied by 2, the complexities $D(r)$’s are divided by 2 and the distance $d_\infty$ is replaced by its square root. Hence the conclusion of Theorem 2.2 is also true for these Bruhat-Tits trees.

Let us show now that Theorem 1.1 and Corollary 1.2 in the introduction follows from Theorem 2.2

Proof of Theorem 1.1 Since $\Gamma$ in the statement of Theorem 1.1 is assumed in addition to be a lattice, we have $\Lambda \Gamma = \partial T$. Hence, since $T$ has no terminal vertex, $C \Lambda \Gamma = T$. A tree whose quotient by its automorphism group is a finite graph, and whose automorphism group contains a (non-uniform) lattice is unimodular, hence uniform (see [2, para. 1.2, 0.5 (6)]). As we noticed above, a map whose logarithm is Lipschitz is slowly varying. Hence, if the hypotheses of Theorem 1.1 are satisfied, then so are the hypotheses of Theorem 2.2. The critical exponent of $\Gamma$ coincides with the Hausdorff dimension of $\Lambda \Gamma$ with respect to the $d_{x_0}$ metric. The distances $d_\infty$ and $d_{x_0}$ are locally equivalent, and the measure classes of the measure $\mu_\infty$ and of the Hausdorff measure of $d_{x_0}$ coincide. Hence Theorem 1.1 does follow from Theorem 2.2. \hfill $\square$
Proof of Corollary 1.2. The proof follows from Theorem 1.1 along the same line as in the other cases we recalled in the introduction (see for instance [11, Sect. 6]). We use the maps $\psi(t) = t^{-\kappa}$, and the fact that for a geodesic line starting from $\infty$, ending at $\xi$, and entering in $H_r$ for some $r \in \Gamma \cap \{\infty\}$ with highest penetration point $\xi_r$, we have $-\log d_\infty(\xi, r) = D(r) + d(\xi_r, \partial H_r)$. \hfill \Box

We now give some applications of our results. We refer for instance to [14, 19] for nice surveys of the Diophantine approximation properties of elements in $\tilde{K} = \mathbb{F}_q((X^{-1}))$ by elements in $K = \mathbb{F}_q(X)$. Recall the definition of the absolute value of $f \in \tilde{K} - \{0\}$. Let $f = \sum_{i=n}^\infty a_i X^{-i}$ where $a_n \neq 0$. Then we define $\nu(f) = n \in \mathbb{Z}$ and $|f|_\infty = q^{-\nu(f)}$.

Endow the locally compact additive group $\tilde{K}$ with its (unique up to a constant factor) Haar measure. Let $A = \mathbb{F}_q[X]$.

**Corollary 2.3** Let $\Gamma$ be a finite index subgroup of $\text{SL}_2(A)$. Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a map with $u \mapsto \varphi(e^u)$ slowly varying. If the integral $\int_1^{+\infty} \varphi(t)/t \ dt$ diverges (resp. converges), then for almost every (resp. almost no) $f$ in $\tilde{K}$, there exist infinitely many couples $(P, Q)$ in $A \times (A - \{0\})$, where $P$ is coprime with $Q$ and $P/Q \in \Gamma \infty$, such that $|f - P/Q|_\infty \leq \varphi(|Q|_\infty)/|Q|^2_\infty$.

**Proof.** In [19], a geometric interpretation of the above Diophantine approximation is given in terms of the Bruhat-Tits tree $\mathcal{T}_q$ of $(\text{SL}_2, \tilde{K})$ (see also [17, 11]). Identify as usual $\partial \mathcal{T}_q$ and $\tilde{K} \cup \{\infty\}$, and let $x_0$ be the standard base point in $\mathcal{T}_q$ (see [20]). Note that the Hausdorff dimension of $d_{x_0}$ is $\log q$, as $\mathcal{T}_q$ is a regular tree of degree $q + 1$ (see [20]). Recall that the kernel of the action of $\text{SL}_2(\tilde{K})$ on $\mathcal{T}_q$ is finite, hence denoting by the same letter a lattice in $\text{SL}_2(\tilde{K})$ and its image in $\text{Aut}(\mathcal{T}_q)$ causes no problem.

Now $\Gamma$ is a lattice of $\mathcal{T}_q$, hence $C\Gamma = \mathcal{T}_q$ is a uniform tree without degree 2 vertices. Let $\Gamma \infty$ be the stabilizer in $\Gamma$ of the point $\infty$ in $\partial \mathcal{T}_q = \tilde{K} \cup \{\infty\}$, which is a parabolic subgroup of $\Gamma$. Since $\Gamma$ is contained in $\text{SL}_2(A)$, the subset $\Gamma \infty - \{\infty\}$ is contained in $K$. By [16] [17], the following assertions hold.

- There exists a constant $c > 0$ such that the Hamenstädt distance on $\partial \mathcal{T}_q - \{\infty\} = \tilde{K}$ satisfies $d_\infty(f, f') = c|f - f'|_\infty^{\log q}$, for every $f, f'$ in $\tilde{K}$ (see [16 Coro. 5.2]).

- There exists a constant $c' > 0$ such that for every $P/Q$ in $\Gamma \infty - \{\infty\}$, where $P$ and $Q$ are coprime polynomials such that $\nu(P) \geq \nu(Q)$, we have $D(P/Q) = -2\nu(Q) + c' = (2 \log |Q|_\infty)/\log q + c'$ (see [16 Coro. 6.1]).

The constants are due to the fact that for $\gamma \in \Gamma$, the horoball centered at a point $\gamma \infty$ in the family associated to $\Gamma$ contains, but is not necessarily equal to, the horoball centered at a point $\gamma \infty$ in the family associated to $\text{SL}_2(A)$. Hence, for every $c' > 0$, we have $d_\infty(f, P/Q) \leq c'e^{-D(P/Q)}$ if and only if $|f - P/Q|_\infty \leq (e^{-c'}e^{-c'})^{\log q}/|Q|^2_\infty$. Define $\psi(t) = ce^c \varphi(e^{(t-c')/2})^{\log q}$, for every $t > 0$. So that $\psi$ is slowly varying by the assumption on $\varphi$. And $d_\infty(f, P/Q) \leq \psi(D(P/Q))e^{-D(P/Q)}$ if and only if $|f - P/Q|_\infty \leq \varphi(|Q|_\infty)/|Q|^2_\infty$.

By an easy change of variables, the integral $\int_1^{+\infty} \varphi(t)/t \ dt$ diverges if and only if $\int_1^{+\infty} \psi(t)\log q \ dt$ diverges. By invariance, the Haar measure of $\tilde{K}$ is equal to the Hausdorff measure of $d_\infty$, up to a constant positive factor. In particular, their measure classes are the same. The result follows. \hfill \Box
Proof of Corollary 1.3. Apply Corollary 2.3 to the congruence subgroup $\Gamma$ of $\text{SL}_2(A)$, consisting of the elements \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] such that $c$ is divisible by $Q_0$. The subgroup $\Gamma$ has finite index in $\text{SL}_2(A)$, as it is the preimage of the upper triangular subgroup by the reduction modulo $Q_0$ map:

\[
\text{Ker}(\text{SL}_2(F_q[X]) \to \text{SL}_2(F_q[X]/Q_0F_q[X])),
\]

with the convention that $\text{SL}_2(R)$ is the trivial group if $R$ is the trivial one-point ring. Note that $\Gamma_\infty - \{\infty\}$ is the set of fractions $P/Q$ where $P$ is coprime with $Q$ and $Q$ is divisible by $Q_0$.

Similar results to Corollary 1.3 can be obtained for instance by varying the congruence subgroup of $\text{SL}_2(F_q[X])$.

3 Proof of the main result

Throughout this section, we will assume that the hypotheses of Theorem 2.2 are satisfied. As $E(\psi)$ is contained in $\Lambda \Gamma$, and as $\mu_\infty$ has support in $\Lambda \Gamma$, we may assume, up to replacing $T$ by $C \Lambda \Gamma$, that $T = C \Lambda \Gamma$.

We first claim that for every constant $\eta > 0$, in order to prove Theorem 2.2, it is sufficient to prove the assertion if $\psi(t) \leq \eta$ for every $t$ in $\mathbb{R}_+$. Indeed, since $\infty$ is a bounded parabolic point and $\Gamma$ acts discretely on $T$, there exist, modulo the action of the parabolic subgroup $\Gamma_\infty$, only finitely many elements $r$ in $\Gamma_\infty - \{\infty\}$ with $D(r)$ less than some constant. The above reduction is then justified by the same arguments as in the proof of the similar reduction in [11, Lem. 5.2]. In particular, from now on, we will assume that $\psi \leq 1$.

Let $\Delta$ be a compact subset of $\partial T - \{\infty\}$ whose images under $\Gamma_\infty$ cover $\partial T - \{\infty\}$, and whose interior contains a representative of every element of $\Gamma_\infty - \{\infty\}$ modulo $\Gamma_\infty$. The existence of such a subset $\Delta$ follows from the fact that $\infty$ is a bounded parabolic point. The following proposition gives counting estimates on the number of points in $(\Gamma_\infty) \cap \Delta$.

Proposition 3.1 There exists an integer $N \geq 1$ and a constant $c_3 > 0$ such that for every $n$ in $\mathbb{N}$, if $\mathcal{N}(n)$ denotes the number of elements $r$ in $(\Gamma_\infty) \cap \Delta$ such that $n \leq D(r) < n + N$, then

\[
\frac{1}{c_3} e^{\delta n} \leq \mathcal{N}(n) \leq c_3 e^{\delta n}.
\]

Proof. Let $\mathcal{N}'(n) = \text{Card}\{[\gamma] \in \Gamma_\infty \setminus (\Gamma - \Gamma_\infty) / \Gamma_\infty : d(H_\infty, \gamma H_\infty) \leq n\}$. We first claim that there exists a constant $c'_1 > 0$ such that $e^{\delta n}/c'_1 \leq \mathcal{N}'(n) \leq c'_1 e^{\delta n}$. The proof of this claim is the same (with simplifications due to the tree structure) as the proof of [11 Theo. 3.4]. The crucial ingredients we used in that proof were that $\infty$ is a bounded parabolic point, that the orbit $\Gamma x_0$ satisfies some growth property (which is given here by Theorem 2.1 (2)) and that the critical exponent of $\Gamma_\infty$ is strictly less than $\delta$ (this is given here by Theorem 2.1 (4)).

By the properties of $\Delta$, there exists a constant $c > 0$ such that

\[
\mathcal{N}'(n) \leq \text{Card}\{r \in (\Gamma_\infty) \cap \Delta : D(r) \leq n\} \leq c \mathcal{N}'(n).
\]
Hence there exists \( c''_1 > 0 \) such that \( e^{\delta n}/c''_1 \leq \text{Card}\{r \in (\Gamma \infty) \cap \Delta : D(r) \leq n\} \leq c''_1 e^{\delta n} \).

Finally, Proposition 3.1 follows from this, and from [11, Lem. 3.3], as in the proof of [11, Theo. 3.4].

If \( H \) is an horoball and \( t \geq 0 \), define \( H(t) \) to be the horoball contained in \( H \), whose boundary is at distance \( t \) from the boundary of \( H \). To simplify the notation, for every \( r \in \Gamma \infty - \{\infty\} \), let \( H_{r,\psi} = H_r(-\log \psi \circ D(r)) \).

For every subset \( E \) of \( T \), let \( O_{\infty} E \) be the set of endpoints of the geodesic lines starting from \( \infty \) and passing through \( E \). Let \( A_n \) be the subset of \( \partial T - \{\infty\} \), which is the union of the \( O_{\infty} H_{r,\psi} \)'s where \( r \) ranges over the elements in \((\Gamma \infty) \cap \Delta\) with \( Nn \leq D(r) < (n+1)N \) and \( N \) is as in Proposition 3.1.

The proof of Theorem 2.2 is based on the following three propositions. The first one is an adaptation for trees of the fluctuating density property due to Sullivan [23] in the case of finite volume, constant curvature manifolds, to Stratmann-Velani [22] in the geometrically finite, constant curvature manifold case, and to [11] for more general geometrically finite manifolds with variable negative curvature. The proof of the following proposition greatly simplifies in the case of homogeneous trees, but the general case requires some care.

**Proposition 3.2** There exists a constant \( c > 0 \) such that for every \( r \in \Gamma \infty - \{\infty\} \) and every \( t \geq 0 \), one has

\[
\frac{1}{c} e^{-\delta(D(r)+t)} \leq \mu_{\infty}(O_{\infty} H_r(t)) \leq c e^{-\delta(D(r)+t)} .
\]

**Proof.** It is the same as the proof of [11, Theo. 4.1], except for the second step of the proof of [11, Prop. 4.3]. To check the hypotheses of [11, Theo. 4.1], we use the following claims: the point \( \infty \) is a bounded parabolic point, \( \Gamma \) is of divergent type (see Theorem 2.1 (1)) and for every \( n \in \mathbb{N} \),

\[
\frac{1}{c_2} e^{n\delta/2} \leq \text{Card}\{\gamma \in \Gamma \infty : d(x_0,\gamma x_0) \leq n\} \leq c_2 e^{n\delta/2},
\]

(see Theorem 2.1 (5)), so that the constant \( \delta_0 \) that appears in the statement of [11, Theo. 4.1] is now \( \delta/2 \). Furthermore, Proposition 4.2 in [11], which is needed in the proof of Theorem 4.1 in [11], is a bit simpler in the setting of trees.

Step 2 in the proof of [11 Prop. 4.3] (which is also needed for the proof of [11 Theo. 4.1]) needs to be adjusted. The arguments that used the hypothesis of pinched curvature are no longer valid. But, with the notation of Step 2 in the proof of [11 Prop. 4.3], the convergence of \((X_i, *_i, d_i, G_i)_{i \in \mathbb{N}} \) follows from the fact that these spaces are uniformly proper (see [9]), as \( T \) is a uniform tree. The limit is again a locally finite tree, and the rest of the proof is unchanged.

**Proposition 3.3** The sum \( \sum_{n=0}^{\infty} \mu_{\infty}(A_n) \) diverges if and only if the integral \( \int_{1}^{\infty} \psi^\delta \) diverges.

**Proof.** The proof of this proposition is completely similar to the proof of [11 Prop. 5.3]. Lemma 5.4 used in that proof is first proved for the case of trees. Proposition 3.2 above replaces [11 Theo. 4.1].
Proposition 3.4 There exists a constant $c > 0$ such that if $n, m$ are distinct integers, then
\[ \mu_\infty(A_n \cap A_m) \leq c \mu_\infty(A_n) \mu_\infty(A_m). \]

Proof. The proof of this proposition is completely similar to the proof of \cite{11} Prop. 5.5. Lemma 5.6 used in that proof is even first proved for the case of trees. Proposition 3.2 above replaces \cite{11} Theo. 4.1.

We now end the proof of Theorem 2.2. Recall that in any tree $T'$, given an horoball $H$ with point at infinity $\xi$, and distinct points $\eta, \eta' \in \partial T'$ different from $\xi$, if $x$ is the intersection with $\partial H$ of the geodesic line $[\eta, \xi]$, then $\eta'$ belongs to $O_\eta H$ is and only if the geodesic line $[\eta, \eta']$ goes through $x$.

In particular, as the distance between $H_\infty$ and $H_{r, \psi}$ is equal to $D(r) - \log \psi \circ D(r)$, a point $\xi$ in $\partial T - \{\infty\}$ belongs to $O_\infty H_{r, \psi}$ if and only if $d_\infty(\xi, r) \leq \psi \circ D(r) e^{-D(r)}$.

Define $A_\infty = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k$, which is the set of points in $\partial T - \{\infty\}$ belonging to infinitely many $A_n$'s. With the notation in the statement of Theorem 2.2 as the images of $\Delta$ by $\Gamma_\infty$ cover $\partial T - \{\infty\}$, up to enlarging $\Delta$, we then have $E(\psi) = \Gamma_\infty A_\infty$. As $\Gamma_\infty$ is countable, we have $\mu_\infty(E(\psi)) > 0$ if and only if $\mu_\infty(A_\infty) > 0$.

The following result is well known (see for instance \cite{21}).

Proposition 3.5 Let $(Y, \nu)$ be a measurable space with a finite measure. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of measurable subsets of $Y$ such that there exists a constant $c > 0$ with $\nu(B_n \cap B_m) \leq c \nu(B_n) \nu(B_m)$ for every distinct integers $n, m$. Let $B_\infty = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} B_k$.

Then $\nu(B_\infty) > 0$ if and only if $\sum_{n=0}^{\infty} \nu(B_n)$ diverges.

Let $Y$ be a big enough compact subset of $\partial T - \{\infty\}$ which contains $\bigcup_{n \in \mathbb{N}} A_n$. Let $\nu$ be the restriction of $\mu_\infty$ to $Y$, and let $B_n = A_n$. We now apply Proposition 3.4. Proposition 3.3 and the result above, to obtain that $\mu_\infty(E(\psi)) > 0$ if and only if $\int_0^{\infty} \psi(t) \, dt$ diverges. This gives the first conclusion of Theorem 2.2.

Conversely, assume that $\int_0^{\infty} \psi(t) \, dt$ diverges. Let us prove that $\mu_\infty(E'(\psi)) = 0$.

Let $g : [0, +\infty[ \rightarrow [0, +\infty[ \,$ be a map decreasing to 0 such that $\int_0^{\infty} (g(t)) \, dt$ diverges. Let $E'(\psi)$ be the set of $\xi$ in $\partial T - \{\infty\}$ such that there exist $c > 0$ and infinitely many $r$ in $\Gamma_\infty - \{\infty\}$ with $d_\infty(\xi, r) \leq c \psi(D(r)) e^{-D(r)}$. Since $E(g\psi) \subset E'(g\psi)$, the first conclusion of Theorem 2.2 implies that $\mu_\infty(E'(g\psi)) > 0$. It is clear that $E'(g\psi)$ is a measurable subset of $\partial T$ which is invariant under $\Gamma$.

By Theorem 2.1 (1), the action of $\Gamma$ on $\partial T$ for the measure $\mu_{x_0}$ is ergodic and without atom. By ergodicity, $\mu_\infty(E'(g\psi)) = 0$. But $E'(g\psi) \subset E(\psi)$ since $g$ is decreasing to 0. Hence $\mu_\infty(E(\psi)) = 0$. This ends the proof of Theorem 2.2.

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