Double tracking control for the directed complex dynamic network via the state observer of outgoing links

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Abstract
From the perspective of large system, a directed complex dynamic network (DCDN) is regarded as being made up of the nodes subsystem (NS) and the links subsystem (LS), which are coupled to each other. Different from previous studies, which propose the dynamic model of LS with the matrix differential equations, this paper describes the dynamic behaviour of LS with the outgoing links vector at every node, by which the dynamic model of LS can be represented as the vector differential equation to form the outgoing links subsystem (OLS). Due to the fact that vectors have more flexible mathematical properties than matrices, this paper proposes the more convenient mathematic method to investigate the double tracking control problem of NS and OLS. Under the condition that the states of NS are available and the states of OLS are unavailable, the asymptotical state observer of OLS is designed, by which the tracking controllers of NS and OLS are synthesised to ensure achieving the double tracking goals. Finally, an example simulation for supporting the theoretical results is also provided.

1. Introduction
In the past twenty years, the complex dynamic network has been widely used in many real networks such as the power networks (Baggio et al., 2021; Zhu et al., 2020a, 2020b, 2019), the ego-networks (Colleran, 2020; X. M. Wang et al., 2020), and the boolean networks (Zhang et al., 2019), and so on. It is well known in graph theory that a complex dynamic network can be regarded as consisting of a group of dynamic nodes and the dynamic links that connect them. From the perspective of large system, all dynamic nodes can be regarded as the dynamic subsystem which is called the nodes subsystem (NS), and all dynamic links can be regarded also as the dynamic subsystem which is called the links subsystem (LS), and thus the complex dynamic network consists of NS and LS coupled with each other (Z. L. Gao et al., 2018; Liu, Wang, Gao, 2020; Y. H. Wang et al., 2020). By employing the above perspectives, the dynamic behaviours of the complex dynamic network can be reflected by the NS and the LS, respectively. For example, the synchronisation (Y. H. Wang et al., 2012; Zhang et al., 2019, 2015, 2020) and the stability (Zhao & Wang, 2020) are regarded as the dynamic behaviour of NS, and the asymptotic structural balance (Z. L. Gao & Wang, 2018; Liu, Wang & Li et al., 2020; Peng et al., 2020) is regarded as the dynamic behaviour of LS.

It is worth noting that the above dynamic behaviours of NS and LS can be interpreted as the tracking problem in the field of control theory. In other words, the synchronisation can be interpreted as the NS tracking a reference target, and the asymptotic structural balance can be interpreted as the LS tracking a reference relation matrix. Corresponding to the above tracking problem, the ‘mutual assistance’ plays an important role in the above literature, that is, through the mutual coupling relations between NS and LS, the tracking of NS is achieved with the help of LS, and the tracking of LS is achieved with the help of NS.

However, it is noticed that the tracking of NS and the tracking of LS are discussed separately in the above literature excepting (Z. L. Gao et al., 2018; Liu, Wang, 2020a, 2020b, 2019) and 2020).
In Z. L. Gao et al. (2018) and Liu, Wang, and Gao (2020), the tracking targets of NS and LS are required to satisfy certain relation equalities, which limits its widespread application. On the other hand, the weighted-values of links are defined as the state variables of LS in the above literature, and thus the dynamics of LS is represented as the Riccati matrix differential (or difference in discrete-time) equations, by which the difficult theoretical analysis results in the complex mathematic conditions for the asymptotic structural balance.

It is also noticed in the above literature that it is assumed that the states of NS are available and the states of LS are not fully available. The states of LS are more complex and difficult to be measured than the states of NS in the complex dynamic network so it is necessary to employ the measurable outputs of LS to achieve the estimation of uncertain state variables for LS. At present, some research results have been achieved on the state observer (Z. L. Gao et al., 2019; Z. L. Gao, Wang & Peng et al., 2020; Z. L. Gao, Wang & Xiong et al., 2020; Liu, Wang, Gao, 2020) design of LS. In Z. L. Gao et al. (2019), the state observer of LS is synthesised for the first time for the complex dynamic network. In Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020), the state observer employs the outputs of LS as its inputs to obtain the state observation information for LS, and then the LS tracks on a reference relation matrix by utilising the state observation information to synthesise the controller of LS, without considering the dynamic behaviour of NS. In view of this, the results in Liu, Wang, Gao (2020) extend the continuous state observer to the discrete state observer for LS. The LS tracks on a reference relation matrix via employing the controller which is designed by using the observation state information about LS. Nevertheless, the previous published studies have not dealt with the double tracking control problems of NS and LS in DCDN.

Inspired by the results of the above discussion, this paper mainly studies the dynamic evolution of the outgoing links at each node instead of the dynamic evolution of the overall links at all nodes. That is, the dynamics of the proposed outgoing links subsystem (OLS) in this paper is mainly represented by the outgoing links vectors at each node rather than the overall links matrix at all nodes, by which the outgoing links vectors represent the combination of weighted-values for a node pointing to other nodes (including self-linking). With this in mind, the main innovations and contributions in this paper compared with the existing research work are as follows.

- The dynamics of LS is represented by the vector differential equations accompanying the outgoing links, whose structure is more concise compared to the matrix differential equations as in Z. L. Gao et al. (2018), Y. H. Wang et al. (2020), Liu, Wang and Li et al. (2020), Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020). This representation can effectively avoid the theoretical analysis difficulties caused by the matrices representation (Barnett, 1973; Hao et al., 2019);
- From the angle of state space in control theory, the outgoing links vectors represent the set of the states of outgoing links at each node, which is more geometrically intuitive than the overall link matrix. On the other hand, the dynamic model of OLS is more likely to be extended to the network with nonlinear circumstances via using the vector algebra methods in this paper;
- By using the state observer of OLS, the double tracking controllers of NS and OLS are designed separately to achieve the double tracking goals for the DCDN. In other words, the state of NS tracks the given reference target while the state of OLS tracks the designed goal.

The remaining part of the paper proceeds as follows. In Section 2, the dynamic models of NS and OLS are proposed respectively, and the directed characteristics of OLS are explained. The state observer for OLS is proposed in Section 3. In Section 4, by employing the state observation information of OLS, the tracking controllers of NS and OLS are synthesised separately, by which the complex dynamic network can achieve the double tracking goals based on Lyapunov stability theory and the vector algebra methods. Some numerical examples are given for supporting the theory results in Section 5. Finally, the conclusion is given in Section 6.

2. Model description

Consider a directed complex dynamic network with $N$ nodes, the weighted-value of the $i$th node pointing
to the \( j \)th node is represented as \( l_{ij} = l_{ij}(t) \in \mathbb{R}, \) where \( l_{ij}(t) \) is treated as the self-link, \( i, j = 1, 2, \ldots, N. \) The dynamic equation of the \( i \)th node is described as

\[
\dot{x}_i = A_j x_i + f_i(x_i) + c \sum_{k=1}^{N} l_{ik} \Lambda h_k(x) + u_i, \tag{1}
\]

where \( x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]^T \in \mathbb{R}^n \) is the state vector of the \( i \)th node, \( x = [x_1^T, x_2^T, \ldots, x_N^T]^T \in \mathbb{R}^{nN} \) is the overall state vector of NS, \( A_j \in \mathbb{R}^{n \times n} \) denotes the real constant matrix, \( f_i(x_i) = [f_1(x_i), f_2(x_i), \ldots, f_n(x_i)]^T \in \mathbb{R}^n \) stands the nonlinear continuous function, \( \Lambda = \text{diag}(\Lambda_1, \Lambda_2, \ldots, \Lambda_n) \in \mathbb{R}^{n \times n} \) is the diagonal matrix with the real constant \( \Lambda_l, l = 1, 2, \ldots, n, h_k(x) \in \mathbb{R}^n \) is the nonlinear smooth inner connection vector function, \( c \) is the common coupling connection strength, and \( u_i \in \mathbb{R}^n \) denotes the control input, \( i, k = 1, 2, \ldots, N. \)

Motivated by the outgoing and incoming degree of directed weighted networks (Pan et al., 2021), the following concepts are proposed.

**Definition 2.1:** \( L_i(t) = [l_{i1}(t), l_{i2}(t), \ldots, l_{IN}(t)]^T \in \mathbb{R}^n \) is called the outgoing links vector of the \( i \)th node. Likewise, the incoming links vector of the \( i \)th node is defined as \( \tilde{L}_i(t) = [\tilde{l}_{i1}(t), \tilde{l}_{i2}(t), \ldots, \tilde{l}_{NI}(t)]^T \in \mathbb{R}^n, i = 1, 2, \ldots, N. \)

Then, the dynamical equation (1) can be rewritten as follows

\[
\dot{x}_i = A_j x_i + f_i(x_i) + c \Lambda H(x) L_i(t) + u_i, \tag{2}
\]

where \( H(x) = [h_1(x), h_2(x), \ldots, h_N(x)] \in \mathbb{R}^{n \times N}, \) \( L_i(t) = [\tilde{l}_{i1}(t), \tilde{l}_{i2}(t), \ldots, \tilde{l}_{IN}(t)]^T \in \mathbb{R}^N, i = 1, 2, \ldots, N. \)

**Remark 2.1:** (i) In Definition 2.1, the outgoing links vector at a node is used to describe the links relationships of a node pointing to other nodes. This is inspired by the ones in P. T. Gao et al. (2021). (ii) As part of the coupling, the outgoing links vector \( L_i(t) \) is shown in Equation (2), which reflects the influence of the outgoing links of the \( i \)th node on its dynamics. (iii) Noticing the relation matrix \( Z = Z(t) = [l_{ij}]_{N \times N}, \) it can be easily observed that the outgoing links vector \( L_i(t) \) is the \( i \)th row of \( Z(t), \) and the incoming links vector \( \tilde{L}_i(t) \) is the \( i \)th column of \( Z(t), i = 1, 2, \ldots, N. \) (iv) Especially, if \( L_i(t) = \tilde{L}_i(t), \) the network is undirected.

The dynamical equation of the outgoing links sub-system (OLS) is regarded as follows

\[
\dot{L}_i = B_i L_i + \Phi_i(x) + U_i Y_i = C_i L_i, \tag{3}
\]

where \( B_i \in \mathbb{R}^{N \times N} \) is the constant matrix, \( C_i \in \mathbb{R}^{N_1 \times N} \) is the output matrix, \( \Phi_i(x) \in \mathbb{R}^N \) is the coupling vector, \( U_i \in \mathbb{R}^N \) stands the control input, \( Y_i \in \mathbb{R}^{N_1} \) denotes the output of OLS, \( i = 1, 2, \ldots, N. \)

**Remark 2.2:** (i) Equation (3) represents the outgoing links vector dynamics of the \( i \)th node, in which the vector \( \Phi_i(x) \) shows the coupling relationship between the outgoing links vector of the \( i \)th node and the state of all nodes. (ii) The vector representation (3) for LS is shown relatively rare in the existing literature on the observer design for LS, which helps to reduce the difficult theoretical analysis because of the vectors possessing more properties of flexible mathematical operations than matrices. (iii) Equation (3) (without controller \( U_i \)) can be rewritten as follows

\[
\begin{align*}
\dot{y}_i^s & = \sum_{k=1}^{N} c_{ik}^s L_k, \tag{4}
\end{align*}
\]

where \( B_i = (b_{jk})_{N \times N}, \) \( \Phi_i(x) = (\psi_i^j(x)) \in \mathbb{R}^N, \) \( Y_i = (y_i^s)_{N_1 \times 1}, C_i = (c_{ik})_{N_1 \times N}, i, j, k = 1, 2, \ldots, N, s = 1, 2, \ldots, N_1. \) The schematic diagram of Equation (4) may be explained as follows.

In Figure 1, the node \( k \) represents any node except nodes \( i \) and \( j. \) Figure 1 means that the time derivative of \( l_{ij} \) can be influenced directly by the linear combination operation of \( l_{ik} \) for all \( 1 \leq k \leq N \) with the coupling.
3. Design of asymptotical state observer for OLS

**Definition 3.1:** Consider the DCDN with Equations (2) and (3). If the state of NS in (2) can be available and there exists the dynamical system \( \hat{L}_i = F_i(\hat{L}_i, Y_i, x) \) such that \( E_i = L_i - \hat{L}_i \xrightarrow{t \to +\infty} 0 \) holds, then the dynamical system \( \hat{L}_i = F_i(\hat{L}_i, Y_i, x) \) is called the \( i \)th asymptotical state observer of the \( i \)th OLS, \( i = 1, 2, \ldots, N \).

According to Definition 3.1, our hypothesis is that the \( i \)th asymptotical state observer of the \( i \)th OLS is as follows

\[
\hat{L}_i = B_i\hat{L}_i + \Phi_i(x) - K_i(Y_i - C_i\hat{L}_i) + U_i, \tag{5}
\]

where \( K_i \in \mathbb{R}^{N \times N_i} \) is the gain matrix to be determined, \( i = 1, 2, \ldots, N \).

**Assumption 3.1.** Consider the OLS (3). The double matrices \( (B_i, C_i) \) is completely observable.

If Assumption 3.1 holds, there exists a matrix \( K_i \) such that \( B_i + K_iC_i \) is a Hurwitz stable matrix, and thus there exists the unique symmetric positive definition matrix solution \( M_i \in \mathbb{R}^{N \times N} \) satisfying the following Lyapunov equation for the given positive definition matrix \( Q_i \in \mathbb{R}^{N \times N} \), \( i = 1, 2, \ldots, N \)

\[
(B_i + K_iC_i)^T M_i + M_i(B_i + K_iC_i) = -Q_i. \tag{6}
\]

**Remark 3.1:** In Lyapunov equation (6), two matrices \( K_iC_i \) and \( M_i \) can be obtained by solving the linear matrix inequality (LMI) \( B_i^T M_i + M_iB_i + W_1 + W_1^T < 0 \), where \( M_i > 0, K_iC_i = M_i^{-1}W_1 \). The details for the solving steps can be referred to in Tuan et al. (2001) and Vanantwerp and Braatz (2000).

**Lemma 3.1.** If Assumption 3.1 holds, then Equation (5) is the asymptotical state observer for OLS (3).

**Proof 1.** Let the observation error \( E_i = L_i - \hat{L}_i \), then the following equation can be obtained, \( i = 1, 2, \ldots, N \).

\[
\dot{E}_i = \hat{L}_i - \hat{\hat{L}}_i = B_iL_i + \Phi_i(x) + U_i - [B_i\hat{L}_i + \Phi_i(x) - K_i(Y_i - C_i\hat{L}_i) + U_i] \\
= B_i(L_i - \hat{L}_i) + K_i(C_iL_i - C_i\hat{L}_i) \\
= (B_i + K_iC_i)E_i. \tag{7}
\]

Noticing that \( K_i \) can be chosen such that \( B_i + K_iC_i \) is a Hurwitz stable matrix based on Assumption 3.1, it is seen that \( E_i \) is bounded and \( E_i = L_i - \hat{L}_i \xrightarrow{t \to +\infty} 0 \). Lemma 3.1 is proved.

By using Lemma 3.1, the control input \( U_i \) in OLS (3) can be synthesised by employing the state in Equations (5) and (2), which helps to synthesise the double tracking control scheme for NS and OLS.

4. Double tracking controllers design based on observer

Let \( x_i^* = x_i^*(t) \in \mathbb{R}^N \) and \( L_i^* = L_i^*(t) \in \mathbb{R}^N \) be the state reference tracking goals of NS and OLS, respectively, and suppose that they are the differentiable bounded; In addition, let \( e_i = x_i - x_i^* \) be the tracking error of NS; \( i = 1, 2, \ldots, N \).

**Definition 4.1:** Consider the DCDN with Equations (1) and (3). If \( E_i = L_i - L_i^* \xrightarrow{t \to +\infty} 0 \) and \( e_i = x_i - x_i^* \xrightarrow{t \to +\infty} 0 \) hold for all \( i = 1, 2, \ldots, N \), the DCDN is known to achieve the double tracking goals.

**Control objective:** Suppose that the states \( x_i \) of NS are available and the states \( L_i \) of OLS are unavailable for \( i = 1, 2, \ldots, N \). By employing the observation states of observer (5), synthesise the controller \( u_i \) for NS (2) and the controller \( \hat{U}_i \) for OLS (3), respectively, such that the DCDN can achieve the double tracking goals.

**Remark 4.1:** The control objective of this paper is largely inspired by Pagilla et al. (2006), Chu et al. (2018) and Chu et al. (2021). The web processing line may be treated to be composed of the roller (nodes) and the coupled web (the outgoing links), in which the roller velocity \( s_i \) and web tension \( T_i \) are regarded as the state variables of NS and OLS, respectively. Let \( \tilde{T}_i = T_i - T_{ri} \) and \( \tilde{s}_i = s_i - s_{ri} \) where \( T_{ri} \) and \( s_{ri} \) are the reference tension and velocity, respectively. It can be observed that the control objective for the web processing line is consistent with the one in this paper.
For the NS (2), there exists a matrix \( \tilde{K}_i \in R^{n \times n} \) such that \( A_i + \tilde{K}_i \) is a Hurwitz matrix, that is, the following Lyapunov equation has the unique symmetric positive definite matrix solution \( P_i \in R^{n \times n} \) for the given positive definite matrix \( \tilde{Q}_i \in R^{n \times n} \), \( i = 1, 2, \ldots, N \).

\[
(A_i + \tilde{K}_i)^T P_i + P_i (A_i + \tilde{K}_i) = -\tilde{Q}_i. \tag{8}
\]

**Remark 4.2:** In Lyapunov equation (8), the matrices \( \tilde{K}_i \) and \( P_i \) can be obtained by solving LMI \( A_i^T P_i + P_i A_i + W_2 + W_2^T < 0 \), where \( P_i > 0, \tilde{K}_i = P_i^{-1} W_2 \).

**Assumption 4.1.** Consider the OLS (3). The coupling function \( f_i(x_i) \) is bounded, that is, \( \|f_i(x_i)\| \leq \eta_i(x_i) \), where \( \eta_i(x_i) \geq 0 \) is a known function, and the inner connection function \( H(x) \) is known, which meets

\[
\|H(x)\| < \min_{1 \leq i \leq N} \left\{ \frac{\lambda_{\min}(\tilde{Q}_i)}{c_i}, \frac{\lambda_{\min}(Q_i)}{\sigma} \right\}
\]

and \( 0 < \sigma < \lambda_{\min}(\tilde{Q}_i) \),

where \( \sigma \) is an adjustable positive parameter, \( \| \cdot \| \) stands for the Euclidean norm of the matrix or the vector ‘\( \cdot \)’, \( \lambda_{\min}(\cdot) \) represents the minimum eigenvalue of matrix ‘\( \cdot \)’, \( i = 1, 2, \ldots, N \).

**Assumption 4.2.** Consider the OLS (3). The coupling function \( \Phi_i(x) \) is bounded, that is, \( \|\Phi_i(x)\| \leq g_i(x) \), where \( g_i(x) \geq 0 \) is a known function, \( i = 1, 2, \ldots, N \).

Based on the above control objective, the following double tracking controllers \( u_i \) and \( U_i \) are proposed for NS (2) and OLS (3), respectively.

\[
u_i = \tilde{K}_i x_i + \tilde{x}_i - (A_i + \tilde{K}_i) x_i^* - c \Delta H(x) \hat{L}_i + v_i, \tag{9}
\]

\[
\nu_i = -\eta_i(x_i) \text{sign}(P_i e_i), \tag{10}
\]

\[
U_i = K_i Y_i - (B_i + K_i C_i) L_i - \hat{L}_i^* + \tilde{V}_i, \tag{11}
\]

\[
\tilde{V}_i = -g_i(x) \text{sign}(M_i \dot{E}_i), \tag{12}
\]

where the vector sign function denotes

\[
\text{sign}(\xi^T) = \frac{\xi^T}{\|\xi\|} \xi \neq O \xi \neq O,
\]

it is easily verified that the vector sign function is continuous about \( \xi \). \( \tilde{E}_i = \hat{L}_i - L_i^* \) denotes the estimation error between the observer state \( \hat{L}_i \) and the reference signal \( L_i^* \). \( O \) denotes the zero vector of suitable dimensions.

Consider the DCDN with NS (2) and OLS (3), by using controllers (9) and (11), two error dynamical equations are obtained as follows

\[
\begin{align*}
\dot{e}_i &= \dot{x}_i - \dot{x}_i^* \\
&= A_i x_i + f_i(x_i) + c \Delta H(x) L_i + u_i - \dot{x}_i^* \\
&= (A_i + \tilde{K}_i)(x_i - x_i^*) + c \Delta H(x)(L_i - \hat{L}_i) + f_i(x_i) + v_i, \tag{13} \\
\dot{E}_i &= \dot{L}_i - \dot{L}_i^* \\
&= B_i \dot{L}_i + \Phi_i(x) - K_i (Y_i - C_i \hat{L}_i) + U_i - \dot{L}_i^* \\
&= B_i \dot{L}_i + K_i C_i \hat{L}_i - K_i Y_i + \Phi_i(x) + U_i - \dot{L}_i^* \\
&= (B_i + K_i C_i) \tilde{E}_i + \Phi_i(x) + \tilde{V}_i. \tag{14}
\end{align*}
\]

Considering a positive definite function \( V_1 = V_1(t) = \sum_{i=1}^{N} e_i^T P_i e_i + \sum_{i=1}^{N} E_i^T M_i E_i + \sum_{i=1}^{N} E_i^T M_i E_i \), where the positive definite matrices \( P_i \) and \( M_i \) are determined by Equations (8) and (6), respectively. The orbit derivative of \( V_1 = V_1(t) \) along three error systems (7), (13) and (14) is obtained as follows.

\[
\begin{align*}
\dot{V}_1 &= \sum_{i=1}^{N} e_i^T P_i e_i + \sum_{i=1}^{N} e_i^T P_i \dot{e}_i + \sum_{i=1}^{N} \dot{E}_i^T M_i E_i + \sum_{i=1}^{N} \dot{E}_i^T M_i E_i - \sum_{i=1}^{N} \dot{E}_i^T M_i E_i \\
&= \sum_{i=1}^{N} \left[ (A_i + \tilde{K}_i) e_i + c \Delta H(x) E_i + f_i(x_i) + v_i \right]^T P_i e_i \\
&+ \sum_{i=1}^{N} e_i^T P_i \left[ (A_i + \tilde{K}_i) e_i + c \Delta H(x) E_i \right. \\
&+ f_i(x_i) + v_i \\
&+ \sum_{i=1}^{N} \left[ (B_i + K_i C_i) \dot{E}_i + \Phi_i(x) + \dot{V}_i \right]^T M_i \dot{E}_i \\
&+ \sum_{i=1}^{N} \dot{E}_i^T M_i \left[ (B_i + K_i C_i) \dot{E}_i + \Phi_i(x) + \dot{V}_i \right] \\
&+ \sum_{i=1}^{N} \left[ (B_i + K_i C_i) \tilde{E}_i \right]^T M_i E_i
\end{align*}
\]
Inequality (15) yields

\[
+ \sum_{i=1}^{N} E_i^T M_i (B_i + K_i C_i) E_i
\]

Then, substituting Equations (10) and (12) into Equation (18) is a negative definite function about \( \dot{V}_1 \).

According to Young's inequality, Inequality (16) can be rewritten as

\[
\dot{V}_1 \leq - \sum_{i=1}^{N} \lambda_{\min} (\tilde{Q}_i) \| e_i \|^2 - \sum_{i=1}^{N} \lambda_{\min} (Q_i) \| E_i \|^2
\]

Let \( d_1 = \lambda_{\min} (\tilde{Q}_i) - c \| [\Lambda H(x)]^T P_i \sigma \| ) \) and \( d_2 = \lambda_{\min} (Q_i) - c \| (\Lambda H(x))^T P_i \| \) be rewritten as

\[
\dot{V}_1 \leq - \sum_{i=1}^{N} \| E_i \| \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} \| e_i \| \\ \| E_i \| \\ \| \tilde{E}_i \| \end{pmatrix}.
\]
Consider the DCDN with Equations (2), (3) and the state observer (5) for the OLS (3). If Assumptions 3.1–4.2 are satisfied, the DCDN can achieve the double tracking goals.

Remark 4.3: For using Theorem 4.1 to realise the double tracking control of the DCDN, the steps are presented as follows

(i) Determining the reference tracking signals $x_i^*(t)$ and $L_{i}^{*}$.

(ii) Utilising the outputs of OLS to design its observer (5), and the determined matrices $K_i$, $M_i$, $P_i$ and $\bar{K}_i$ can be obtained by solving the LMI in Remark 3.1 and the Lyapunov equation (8) in Remark 4.2, respectively.

(iii) By using the observed state information of OLS from Step (ii), the controller (9)–(10) for NS and the controller (11)–(12) for OLS are designed separately.

(iv) By employing the controllers (9)–(12) of NS and OLS from Step (iii), the DCDN can achieve the double tracking goals.

Remark 4.4: (i) If the state reference tracking signal $x_i^*$ is the same one for all nodes, the results in Theorem 4.1 can be regarded as the synchronisation in Zhang et al. (2019), Zhang et al. (2015), Y. H. Wang et al. (2012) and Zhang et al. (2020). Therefore, the results in Theorem 4.1 are more general due to the different reference tracking signals. (ii) Compared to the existing literature about the tracking problem in control theory, it is observed in this paper that if $L^*_i \neq 0$, the eventual network structure exhibits that all nodes are not isolated when the double tracking goals have happened ($t \to +\infty$). Especially, this implies that all nodes are not isolated when the synchronisation has happened ($t \to +\infty$), which is different from the results (all nodes are isolated when the synchronisation has happened ($t \to +\infty$) in the existing literature (Cai et al., 2014; Chen et al., 2018; Duan et al., 2008; Feng et al., 2018; Huang et al., 2020; Liu et al., 2009; J. L. Wang et al., 2017; Xing et al., 2019; Yu et al., 2009; Zhou et al., 2006).

5. Simulation example

In this paper, we consider the $N$ continuous chaotic Chua’s circuits in Thamilmaran et al. (2004) as the isolate nodes, which are coupled in the form of (2), the dynamics of which is as follows

$$\dot{x}_i = A_i x_i + B \alpha_1 f_i(x_i) + c \Lambda H(x) L_i(t) + u_i,$$

where $x_i = [x_{i1}, x_{i2}, x_{i3}, x_{i4}]^T$ is the state vector of the $i$th Chua’s circuit, $f_i(x_i) = bx_{i1} + 0.5(a-b) (|x_{i1} + 1| - |x_{i1} - 1|)$ denotes the nonlinear vector function,

$$A_i = \begin{bmatrix} 0 & 0 & \alpha_1 & 0 \\ 0 & \alpha_2 & -1 & -1 \\ -\beta_1 & \beta_1 & -\beta_1 & 0 \\ 0 & \beta_2 & 0 & 0 \end{bmatrix}$$

is a constant matrix, $\alpha_1 = 2.1428, \alpha_2 = -0.2025, \beta_1 = 0.09798, \beta_2 = 0.00382, a = -0.04725$ and $b = 3.15$, $\Lambda = \text{diag}(g_1, g_2, g_3, g_4)$ stands the diagonal matrix, where $g_1, g_2, g_3$ and $g_4$ are randomly generated in the interval $[-4, 4]$, the control input $u_i$ is given as (9)–(10); Furthermore, $H(x) = \frac{1-|x|}{1+|x|}$, $X$ denotes the inner connection matrix function, where $X = [x_1, x_2, \ldots, x_N] \in R^{1 \times N}$. Let the common coupling strength $c$ be randomly generated in the interval $[0, 1]$.

The generation rules of matrices $B_i, K_i, C_i, \bar{K}_i, P_i, M_i$, the double tracking goals $x_i^*, L_i^{*}$ and the initial state values $x_i^*(0), L_i^*(0), \bar{L}_i(0)$ are as follows.

(i) Let $B_i = m_1 \text{randn}(N, N)$, where $m_1$ is an adjustable parameter, and $m_1 = 8$.

(ii) Solving for the positive definite matrices $M_i$ and $K_iC_i$ such that LMI $B_i^2 M_i + M_i B_i + W_i + W_i^T < 0$ and $M_i > 0$ holds, it can be obtained the observed gain matrix $K_i C_i = M_i^{-1} W_i$. Similarly, solving for the positive definite matrix $P_i$ and $\bar{K}_i$ such that LMI $A_i^2 P_i + P_i A_i + W_2 + W_2^T < 0$ and $P_i > 0$ holds, it can be obtained $\bar{K}_i = P_i^{-1} W_2$.

(iii) Let the double tracking goals $x_i^* = [\theta \sin(t), \theta \cos(t), \theta \tan(t), \theta \tanh(t)]^T$ and $L_i^* = \text{randn}(N, 1)$, $\sin(t)$, where $\theta$ is an adjustable parameter, and its value is randomly generated in the interval $[-8, 8]$, $i = 1, 2, \ldots, N$. 

(iv) The initial state values of NS and OLS are $x_i(0) = \text{randn}(n, 1)$ and $L_i(0) = \text{randn}(N, 1)$, respectively. Moreover, the initial state values of observer are $\hat{L}_i(0) = \text{randn}(N, 1)$.

The dynamics of $L_i(t)$ is chosen as (3). The coupling term $\Phi_i(x)$ meets Assumption 4.2, the control input $U_i$ are given as (11)–(12). In the process of simulation, other parameters are selected as follows. $N = 30$, $\sigma = \text{rand}(1)$ is an adjustable parameter, $\bar{Q}_i = I_i$, where $I_i$ is the four-dimensional identity matrix. In order to show the advantage of this paper, the simulation results in this paper are compared with the results in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020), respectively.

By using the norm $\|\vartheta\| = \sqrt{\sum^m_k |\vartheta_k|^2}$ for the real vector $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_m)^T \in \mathbb{R}^m$, the compared simulation results are shown in Figures 2–5.

From Figures 2–5, we are enabled to make the following conclusions.

(i) From Figures 2 and 4, it can be seen that the tracking controllers in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) are designed by utilising the observer’s state of LS to control NS to track the reference signal, but tracking error curves show not

![Figure 2](image1)

**Figure 2.** (a) The tracking error curves of NS with the control scheme in Z. L. Gao et al. (2019); (b) The tracking error curves of NS with the control scheme in Z. L. Gao, Wang and Peng et al. (2020); (c) The tracking error curves of NS with the control scheme in Z. L. Gao, Wang and Xiong et al. (2020); (d) The tracking error curves of NS with the control scheme in this paper.

![Figure 3](image2)

**Figure 3.** (a) The tracking error curves of OLS with the control scheme in Z. L. Gao et al. (2019); (b) The tracking error curves of OLS with the control scheme in Z. L. Gao, Wang and Peng et al. (2020); (c) The tracking error curves of OLS with the control scheme in Z. L. Gao, Wang and Xiong et al. (2020); (d) The tracking error curves of OLS with the control scheme in this paper.
Figure 4. (a) The curves of tracking error norm $\|e_i\|$ of NS with the control schemes in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) and this paper respectively; (b) The curves of tracking error norm $\|E_i\|$ of OLS with the control scheme in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) and this paper respectively.

Figure 5. The orbit curves of $L_i^*$'s in this paper.

only the larger chattering problem but also the slower convergence speeds than the ones in this paper. Similarly, it can be observed from Figures 3 and 4 that the tracking controllers are synthesised by employing the observer's state of LS to control LS to track the differentiable bounded reference goal, but the tracking response speeds of OLS in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) are also slower than one in this paper.

(ii) By utilising the norms $\|e_i\|$ and $\|E_i\|$ in Figure 4, it is shown intuitively that this paper has more advantage over Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) in the convergence speeds of tracking for NS and LS.

(iii) Figure 5 shows that the given tracking signal $L_i^*$ does not converge to zero, which means that the eventual network structure ($t \to +\infty$) is shown as all nodes are not isolated when synchronisation happens, in other words, the connections are still connected in an eventual complex network. This is different from the results in Z. L. Gao et al. (2019), Z. L. Gao, Wang and Peng et al. (2020) and Z. L. Gao, Wang and Xiong et al. (2020) where the connections are disconnected in an eventual complex network.
6. Conclusion

The double tracking control of the DCDN has been achieved via synthesising the double tracking controllers of NS and OLS based on the state observer for OLS. Compared to the existing literature about the tracking problems of complex dynamic networks in control theory, the main advantage in this paper is that the outgoing links vector at every node is utilised to describe the dynamics of LS, by which the mathematical method of this paper is shown the more convenient than the ones in the existing literature. The results in this paper show that the double tracking problems of NS and OLS can be regarded as a generalisation of synchronisation in the existing literature. Moreover, it is observed in this paper that the eventual network structure exhibits the new topological feature, that is, all nodes are not isolated when the double tracking control happens, which is regarded as the new one of the eventual topological structures of networks with synchronisation. The results in simulation examples also support the above observation. How to achieve the tracking control of NS and OLS for DCDN under links topology constrained bit rates (Li et al., 2022) or attacks (Tajudeen et al., 2022) is one of the problems worthy of research in the future work.

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Disclosure statement

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Data availability statement

The data used to support the findings of this study are available from the first author upon request.

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References

Baggio, G., Bassett, D. S., & Pasqualetti, F. (2021). Data-driven control of complex networks. Nature Communications, 12(1), 1–13. https://doi.org/10.1038/s41467-021-21554-0
Barnett, S. (1973). Matrix differential equations and Kronecker products. SIAM Journal on Applied Mathematics, 24(1), 1–5. https://doi.org/10.1137/0124001
Cai, C. X., Wang, Z. D., Xu, J., Liu, X. H., & Alsaadi, F. E. (2014). An integrated approach to global synchronization and state estimation for nonlinear singularly perturbed complex networks. IEEE Transactions on Cybernetics, 45(8), 1597–1609. https://doi.org/10.1109/tcyb.2014.2356560
Chen, Y. G., Wang, Z. D., Shen, B., & Dong, H. L. (2018). Exponential synchronization for delayed dynamical networks via intermittent control: Dealing with actuator saturations. IEEE Transactions on Neural Networks and Learning Systems, 30(4), 1000–1012. https://doi.org/10.1109/tnnls.2018.2854841
Chu, X. Y., Nian, X. H., Sun, M. P., Wang, H. B., & Xiong, H. Y. (2018). Robust observer design for multi-motor web-winding system. Journal of the Franklin Institute, 355(12), 5217–5239. https://doi.org/10.1016/j.jfranklin.2018.05.002
Chu, X. Y., Nian, X. H., Xiong, H. Y., & Wang, H. B. (2021). Robust fault estimation and fault tolerant control for three-motor web-winding systems. International Journal of Control, 94(11), 3009–3021. https://doi.org/10.1080/00207199.2020.1749887
Colleran, H. (2020). Market integration reduces kin density in women’s ego-networks in rural Poland. Nature Communications, 11(1), 1–9. https://doi.org/10.1038/s41467-019-14158-2
Duan, Z. S., Chen, G. R., & Huang, L. (2008). Synchronization of weighted networks and complex synchronized regions. Physics Letters A, 372(21), 3741–3751. https://doi.org/10.1016/j.physleta.2008.02.056
Feng, Y. T., Duan, Z. S., Lv, Y. Z., & Ren, W. (2018). Some necessary and sufficient conditions for synchronization of second-order interconnected networks. IEEE Transactions on Cybernetics, 49(12), 4379–4387. https://doi.org/10.1109/tcyb.2018.2864625
Gao, P. T., Wang, Y. H., Liu, L. Z., Zhang, L. L., & Tang, X. (2021). Asymptotical state synchronization for the controlled directed complex dynamic network via links dynamics. Neurocomputing, 448, 60–66. https://doi.org/10.1016/j.neucom.2021.03.095
Gao, Z. L., & Wang, Y. H. (2018). The structural balance analysis of complex dynamical networks based on nodes’ dynamical couplings. PLoS One, 13(1), e0191941. https://doi.org/10.1371/journal.pone.0191941
Gao, Z. L., Wang, Y. H., Peng, Y., Liu, L. Z., & Chen, H. G. (2020). Adaptive control of the structural balance for a class of complex dynamical networks. Journal of Systems Science and Complexity, 33(3), 725–742. https://doi.org/10.1007/s11424-020-8093-4
Gao, Z. L., Wang, Y. H., Xiong, J., Pan, Y., & Huang, Y. Y. (2020). Structural balance control of complex dynamical networks based on state observer for dynamic connection relationships. Complexity, 2020, 5075487. https://doi.org/10.1155/2020/5075487
Gao, Z. L., Wang, Y. H., Xiong, J., Zhang, L. L., & Wang, W. L. (2019). Robust state observer design for dynamic connection relationships in complex dynamical networks. International Journal of Control, Automation and Systems, 17(2), 336–344. https://doi.org/10.1007/s12555-018-0315-3
Gao, Z. L., Wang, Y. H., & Zhang, L. L. (2018). Adaptive control of structural balance for complex dynamical networks based on dynamic coupling of nodes. International Journal of Modern Physics B, 32(04), 1850042. https://doi.org/10.1142/s021797921850042X
Hao, Y. Q., Wang, Q. Y., Duan, Z. S., & Chen, G. R. (2019). Controllability of Kronecker product networks. Automatica, 110, 108597. https://doi.org/10.1016/j.automatica.2019.108597
Huang, Y. Y., Huang, L. W., Wang, Y. H., Peng, Y. X., & Yu, F. (2020). Shape synchronization in driver-response of 4-D chaotic system and its application in image encryption. IEEE Access, 8, 135308–135319. https://doi.org/10.1109/access.2020.3011524
Li, J. Y., Wang, Z. D., Lu, R. Q., & Xu, Y. (2022). Distributed filtering under constrained bit rate over wireless sensor networks: Dealing with bit rate allocation protocol. IEEE Transactions on Automatic Control. https://doi.org/10.1109/tac.2022.3159486
Liu, C., Duan, Z. S., Chen, G. R., & Huang, L. (2009). L2 norm performance index of synchronization and LQR control synthesis of complex networks. Automatica, 45(8), 1879–1885. https://doi.org/10.1016/j.automatica.2009.04.004
Liu, L. Z., Wang, Y. H., & Gao, Z. L. (2020). Tracking control for the dynamic links of discrete-time complex dynamical network via state observer. Applied Mathematics and Computation, 369, 124857. https://doi.org/10.1016/j.amc.2019.124857
Liu, L. Z., Wang, Y. H., Li, X. X., & Gao, Z. L. (2020). Structural balance for discrete-time complex dynamical network associated with the controlled nodes. Modern Physics Letters B, 34(10), 2050098. https://doi.org/10.1142/s0217984920500980
Pagilla, P. R., Siraskar, N. B., & Dwivedula, R. V. (2006). Decentralized control of web processing lines. IEEE Transactions on Control Systems Technology, 15(1), 106–117. https://doi.org/10.1109/tcst.2006.883345
Pan, L. L., Shao, H. B., Xi, Y. G., & Li, D. W. (2021). Bipartite consensus problem on matrix-valued weighted directed networks. *Science China Information Sciences, 64*(4), 1–3. https://doi.org/10.1007/s11432-018-9710-8

Peng, Y., Wang, Y. H., & Liu, L. Z. (2020). Asymptotically tracking control for discrete time-varying link system to structural balance via determined external stimulations. *International Journal of Modern Physics B, 34*(17), 2050144. https://doi.org/10.1142/S0217979220501441

Tajudeen, M. M., Ali, M. S., Subkrajang, S. A., Jirawattanapanit, K., & Rajchakit, G. (2022). Adaptive event-triggered control for complex dynamical network with random coupling delay under stochastic deception attacks. *Complexity, 2022*, 8761612. https://doi.org/10.1155/2022/8761612

Thamilmaran, K., Lakshmanan, M., & Venkatesan, A. (2004). Hyperchaos in a modified canonical Chua’s circuit. *International Journal of Bifurcation and Chaos, 14*(01), 221–243. https://doi.org/10.1142/S0218127404009119

Wang, J. L., Wei, P. C., Wu, H. N., Huang, T. W., & Xu, M. (2017). Pinning synchronization of complex dynamical networks with multiveights. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 49*(7), 1357–1370. https://doi.org/10.1109/TSMC.2017.2754466

Wang, X. M., Ran, Y. J., & Jia, T. (2020). Measuring similarity in co-occurrence data using ego-networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science, 30*(1), 013101. https://doi.org/10.1063/1.5129036

Wang, Y. H., Fan, Y. Q., Wang, Q. Y., & Zhang, Y. (2012). Stabilization and synchronization of complex dynamical networks with different dynamics of nodes via decentralized controllers. *IEEE Transactions on Circuits and Systems I: Regular Papers, 59*(8), 1786–1795. https://doi.org/10.1109/TCSI.2011.2180439

Wang, Y. H., W. L. Wang, & Zhang, L. L. (2020). State synchronization of controlled nodes via the dynamics of links for complex dynamical networks. *Neurocomputing, 384*, 225–230. https://doi.org/10.1016/j.neucom.2019.12.055

Xing, W., Shi, P., Agarwal, R. K., & Zhao, Y. X. (2019). A survey on global pinning synchronization of complex networks. *Journal of the Franklin Institute, 356*(6), 3590–3611. https://doi.org/10.1016/j.jfranklin.2019.02.021

Yu, W. W., Chen, G. R., & Lü, J. H. (2009). On pinning synchronization of complex dynamical networks. *Automatica, 45*(2), 429–435. https://doi.org/10.1016/j.automatica.2008.07.016

Zhang, L. L., Y. H. Wang, & Wang, Q. Y. (2015). Synchronization for time-varying complex dynamical networks with different-dimensional nodes and non-dissipative coupling. *Communications in Nonlinear Science and Numerical Simulation, 24*(1–3), 64–74. https://doi.org/10.1016/j.cnsns.2014.12.012

Zhang, L. L., Wang, Y. H., Wang, Q. Y., Lei, Y. F., & Wang, F. (2019). Matrix projective cluster synchronization for arbitrarily coupled networks with different dimensional nodes via nonlinear control. *International Journal of Robust and Nonlinear Control, 29*(11), 3650–3665. https://doi.org/10.1002/rnc.4574

Zhang, L. L., Wang, Y. H., Wang, Q. Y., Qiao, S. H., & Wang, F. (2020). Generalized projective synchronization for networks with one crucial node and different dimensional nodes via a single controller. *Asian Journal of Control, 22*(4), 1471–1483. https://doi.org/10.1002/asjc.2053

Zhao, P., & Wang, Y. H. (2020). Asymptotical stability for complex dynamical networks via link dynamics. *Mathematical Methods in the Applied Sciences, 43*(15), 8706–8713. https://doi.org/10.1002/mma.6533

Zhou, J., Lu, J. A., & Lü, J. H. (2006). Adaptive synchronization of an uncertain complex dynamical network. *IEEE Transactions on Automatic Control, 51*(4), 652–656. https://doi.org/10.1109/TAC.2006.872760

Zhu, S. B., Zhou, J., Chen, G. R., & Lu, J. A. (2019). A new method for topology identification of complex dynamical networks. *IEEE Transactions on Cybernetics, 51*(4), 2224–2231. https://doi.org/10.1109/TCYB.2019.2894838

Zhu, S. B., Zhou, J., Yu, X. H., & Lu, J. A. (2020a). Bounded synchronization of heterogeneous complex dynamical networks: A unified approach. *IEEE Transactions on Automatic Control, 66*(4), 1756–1762. https://doi.org/10.1109/TAC.2020.2995822

Zhu, S. B., Zhou, J., Yu, X. H., & Lu, J. A. (2020b). Synchronization of complex networks with nondifferentiable time-varying delay. *IEEE Transactions on Cybernetics, 52*(5), 3342–3348. https://doi.org/10.1109/TCYB.2020.3022976