Two-point function reduction of four-point amputated functions and transformations in $F\bar{F}$ and $RA$ basis in a real-time finite temperature NJL model

Bang-Rong Zhou
Department of Physics, The Graduate School of The Chinese Academy of Sciences
Beijing 100039, China
and CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China

Abstract

Based on a general analysis of Green functions in the real-time thermal field theory, we have proven that the four-point amputated functions in a NJL model in the fermion bubble diagram approximation behave like usual two-point functions. We expound the thermal transformations of the matrix propagator for a scalar bound state in the $F\bar{F}$ basis and in the $RA$ basis. The resulting physical causal, advanced and retarded propagator are respectively identical to corresponding ones derived in the imaginary-time formalism and this shows once again complete equivalence of the two formalisms of thermal field theory on the discussed problem in the NJL model.

PACS: 11.10.Wx, 11.30.Qc, 14.80.Mz

Keywords: NJL model; real-time thermal field theory; four-point amputated functions; two-point function reduction; transformations in $F\bar{F}$ and $RA$ basis

1 Introduction

For discussion of the Nambu-Goldstone mechanism [1] of spontaneous symmetry breaking at finite temperature in a Nambu-Jona-Lasinio (NJL) model [2], one must calculate the propagators for scalar bound states based on four-point amputated Green functions [3,4]. In the fermion bubble diagram approximation, the calculation of a four-point amputated function in a NJL model at finite temperature can be effectively reduced to the one of a two-point function. This is obvious in the imaginary-time formalism [5] of thermal field theory [4]. However, in the real-time formalism [6,7], although the obtained four-point amputated functions show a structure of $2 \times 2$ matrix and one can simply diagonalize it by a $2 \times 2$ thermal transformation matrix, the derived thermal matrix is a mere effective one and has not yet been explained from fundamental principles of thermal field theory.

*This work is partially supported by the National Natural Foundation of China and by Grant No.LWTZ-1298 of the Chinese Academy of Sciences.
In this paper, we will examine theoretical origin of the above reduction. Starting from the general transformation of thermal matrix propagator in the real-time formalism, we will prove that the calculations of the four-point amputated functions in a NJL model in the fermion bubble diagram approximation can be indeed reduced to the ones of conventional two-point functions in the following meanings: 1) the thermal transformation of the four-point amputated functions is identical to the one of usual two-point functions and 2) the relations among the thermal components of the four-point amputated functions can be changed into the ones among the thermal components of corresponding two-point functions. Since the thermal transformations of the real-time Green functions can be generally made in different bases e.g. $F\bar{F}$ basis, $RA$ basis and Keldysh basis etc. [8,9], the reduction of the four-point amputated functions to two-point functions will make the above transformations become very simple and feasible.

The paper is arranged as follows. In Sect.2 we will prove the reduction stated above of the four-point amputated functions in a NJL model to usual two-point functions. In Sect.3 and Sect.4, as an example, we will discuss thermal transformation of the matrix propagator for a scalar bound state respectively in the $F\bar{F}$ basis and in the $RA$ basis and compare the derived physical propagators with the corresponding ones obtained in the imaginary-time formalism. Finally, in Sect.5 we come our conclusions.

## 2 Two-point function reduction of four-point amputated functions in NJL model

Consider a chiral $U_L(1) \times U_R(1)$ NJL model in the real-time formalism with the four-fermion interactions

$$\mathcal{L}_{4F}^R = \frac{G}{4} \sum_{a=1}^{2} \left\{ \left[ (\bar{\psi}\gamma^a \psi)^{(a)} \right]^2 - \left[ (\bar{\psi}\gamma_5 \psi)^{(a)} \right]^2 \right\} (-1)^{a+1},$$

where $\psi$ represents the fermion field with single flavor and $N$ colors and $G$ is the coupling constant; $a = 1$ denotes physical fields and $a = 2$ ghost fields. Denote the four external four-momenta corresponding to a four-point function respectively by incoming $p_1$ and $p_2$ and outgoing $p_3$ and $p_4$, then the four-point functions with four external propagators (legs) may have the following three forms:

$$G_4(p_1, p_2, -p_3, -p_4), \quad G_4(p_2, -p_3, p_1, -p_4) \quad \text{and} \quad G_4(p_2, -p_3, -p_4, p_1),$$

where the momentum with the minus sign ”-” means outgoing, otherwise incoming [9]. In the fermion bubble diagram approximation, the external legs corresponding to the left two momenta and the right two momenta in the arguments of $G_4$ will separately intersect at a common vertex.

Let us consider $G_4(p_1, p_2, -p_3, -p_4)$ first. It can be expressed by

$$G_4^{a_1a_2b_3b_4}(p_1, p_2, -p_3, -p_4) = [iS(p_1)]^{a_1a}[iS(p_2)]^{a_2a} \Gamma^{a_3a_4b_3b_4}(p_1, p_2, -p_3, -p_4) \times [iS^T(-p_3)]^{b_3b} [iS^T(-p_4)]^{b_4b}, \quad (1)$$

where $[iS(p)]^{a'a}$ is the elements of the thermal matrix propagator for free fermion, $[iS^T(-p)]^{b'b} = [iS(-p)]^{b'b}$, i.e. ”$T$” denotes transpose of the matrix, and $\Gamma^{a_3a_4b_3b_4}(p_1, p_2, -p_3, -p_4)$ represent the corresponding four-point amputated functions. Bearing in mind the special form of summing the indices in Eq. (1), we may compactly write Eq. (1) in the matrix form
where "\( \otimes \)" denotes external product of the matrices. Following the method in Ref. [9], we can make a general transformation of the matrix propagator \( iS(p) \) and obtain the transformed matrix propagator

\[
i\hat{S}(p) = U(p)iS(p)U^T(-p),
\]

(3)

where \( U(p) \) is a 2 \( \times \) 2 matrix. A special form of \( U(p) \) is the thermal transformation matrix in usual real-time formalism which diagonalizes \( iS(p) \) and leads to the physical causal propagator for free fermion and its complex conjugate. From Eq. (3) it follows that

\[
i\hat{S}^T(-p) = U(p)iS^T(-p)U^T(-p).
\]

(4)

Now left multiply \( U(p_1) \otimes U(p_2) \) and right multiply \( U^T(-p_3) \otimes U^T(-p_4) \) in Eq. (2) then by means of Eqs. (3) and (4), we can obtain the transformed four-point function matrix with external legs

\[
\hat{G}_4(p_1, p_2, -p_3, -p_4) = i\hat{S}(p_1) \otimes i\hat{S}(p_2)\hat{\Gamma}_4(p_1, p_2, -p_3, -p_4)i\hat{S}^T(-p_3) \otimes i\hat{S}^T(-p_4),
\]

(5)

where

\[
\hat{\Gamma}_4(p_1, p_2, -p_3, -p_4) = V(p_1) \otimes V(p_2)\Gamma_4(p_1, p_2, -p_3, -p_4)V^T(-p_3) \otimes V^T(-p_4)
\]

is the transformed four-point amputated functions and the denotations

\[
V(p) = [U^T(-p)]^{-1}, \quad V^T(-p) = U^{-1}(p)
\]

has been used. When writing Eq. (5) through its components we must note that, as far as the discussed four-point functions in a NJL model are concerned, the transformed external legs \( i\hat{S}(p_1) \) and \( i\hat{S}(p_2) \) will intersect at a same vertex and \( i\hat{S}^T(-p_3) \) and \( i\hat{S}^T(-p_4) \) will also intersect at another same vertex, thus we can obtain

\[
\hat{G}_4^{a_1a_2b_1b_2}(p_1, p_2, -p_3, -p_4) = [i\hat{S}(p_1) \otimes i\hat{S}(p_2)]^{a_1a_2a_3a_4}\hat{\Gamma}_4^{a_3a_4}b_1b_2b_3b_4,
\]

where the transformed four-point amputated function will be

\[
\hat{\Gamma}_4^{a_3a_4}b_1b_2b_3b_4(p_1, p_2, -p_3, -p_4) = [V(p_1) \otimes V(p_2)]^{a_3a_4a_5a_6}\Gamma_4^{a_5a_6b_1b_2b_3b_4}(p_1, p_2, -p_3, -p_4)[V^T(-p_3) \otimes V^T(-p_4)]^{b_1b_2b_3b_4}.
\]

(6)

Since the four indices of all the matrices in Eq. (6) can always be divided into two pairs each with same entries, Eq. (6) will be reduced to the form of 2 \( \times \) 2 matrix. Noting that, owing to the four-momentum conservation, \( \Gamma_4(p_1, p_2, -p_3, -p_4) \) only depend on \( p = p_1 + p_2 = p_3 + p_4 \). Thus we can denote

\[
\hat{\Gamma}_4^{a_3a_4}b_1b_2(p_1, p_2, -p_3, -p_4) \equiv \hat{\Gamma}^{a_3a_4}b_1b_2(p), \quad [V(p_1) \otimes V(p_2)]^{a_3a_4a_5a_6} \equiv [O(p)]^{a_3a_5}, \quad [V^T(-p_3) \otimes V^T(-p_4)]^{b_1b_2b_3b_4} \equiv [O^T(-p)]^{b_1b_3}, \quad \Gamma_4^{a_5a_6b_1b_2b_3b_4}(p_1, p_2, -p_3, -p_4) \equiv \Gamma^{a_b}(p)
\]

(7)

and then obtain from Eq. (6)

\[
\hat{\Gamma}^{a_3a_4}b_1b_2(p) = [O(p)]^{a_3a_5}\Gamma^{a_b}(p)[O^T(-p)]^{b_1b_3}
\]

or in the form of 2 \( \times \) 2 matrix.
Eq. (8) indicates that the transformation of the four-point amputated function corresponding to \( G_4(p_1, p_2, -p_3, -p_4) \) is indeed effective to the one of a two-point function \( \Gamma(p) \). In this case, \( p^2 = (p_1 + p_2)^2 > 0 \), i.e. \( p \) is a time-like four-momentum.

We can make similar discussions for \( G_4(p_2, -p_3, -p_4, p_1) \) and \( G_4(p_2, -p_4, -p_3, p_1) \). For instance, the transformed function of \( G_4(p_2, -p_3, -p_4, p_1) \) will be expressed by

\[
\hat{G}_4(p_2, -p_3, -p_4, p_1) = i \hat{S}(p_2) \otimes i \hat{S}(-p_3) \hat{\Gamma}_4(p_2, -p_3, -p_4, p_1) i \hat{S}^T(-p_4) \otimes i \hat{S}^T(p_1),
\]

where the transformed four-point amputated function \( \hat{\Gamma}_4(p_2, -p_3, -p_4, p_1) \) have the following component form:

\[
\hat{\Gamma}_4^{a'a'bb'}(p_2, -p_3, -p_4, p_1) = \left[ V(p_2) \otimes V(-p_3) \right]^{a'a'a'b} \Gamma_4^{aabb}(p_2, -p_3, -p_4, p_1) \left[ V^T(-p_4) \otimes V^T(p_1) \right]^{bb'b'}. \tag{9}
\]

In this case, \( \Gamma_4(p_2, -p_3, -p_4, p_1) \) and \( \hat{\Gamma}_4(p_2, -p_3, -p_4, p_1) \) are only the function of \( p = p_2 - p_3 = p_4 - p_1 \), thus Eq. (9) can be effectively written by

\[
\hat{\Gamma}^{a'bb'}(p) = [O(p)]^{a'a} \Gamma^{ab}(p) [O^T(-p)]^{bb'}, \tag{10}
\]

with

\[
O(p) \equiv V(p_2) \otimes V(-p_3), \quad O^T(p) \equiv V^T(-p_4) \otimes V^T(p_1), \tag{11}
\]

however, now \( p^2 < 0 \), i.e. \( p \) is a space-like four-momentum. For \( G_4(p_2, -p_4, -p_3, p_1) \), we have the similar conclusion. In fact, if exchanging \(-p_3 \) with \(-p_4 \) in \( \hat{G}_4(p_2, -p_3, -p_4, p_1) \) we will obtain the transformed \( \hat{G}_4(p_2, -p_4, -p_3, p_1) \), the corresponding four-point amputated function \( \hat{\Gamma}(p) \) will have similar effective transformation of \( 2 \times 2 \) matrix form to Eq. (10), where \( p = p_2 - p_4 = p_3 - p_1 \) with \( p^2 < 0 \), i.e. \( p \) is also a space-like momentum.

The above discussions show that no matter time-like or space-like the momentum \( p \) is, the thermal transformation of the four-point amputated functions in a NJL model in the fermion bubble diagram approximation can always be effectively reduced to the one of two-point functions.

We may further prove that the relations among the thermal components of the above four-point amputated function will also be changed into the ones among the thermal components of the corresponding two-point function. The four-point amputated functions \( \Gamma_{a_1a_2a_3a_4}(p_1, p_2, p_3, p_4) \) \( (a_1, a_2, a_3, a_4 = 1, 2) \) generally obey the following relations [9-11]:

\[
\sum_{a_i=1,2} \Gamma^{a_1...a_4}_{a_i}(p_1, \ldots, p_4) \prod_{a_i=2} e^{\sigma p_i^0} = 0, \tag{12}
\]

\[
\sum_{a_i=1,2} \eta_i e^{\sigma p_i^0 - x_i} = 0, \tag{13}
\]

where \( x = \beta (p_i^0 - \mu) \), \( \mu \) is the chemical potential and \( \eta = +1(-1) \) for boson (fermion). If one \( p_i \) in \( \Gamma^{a_1...a_4}_{a_i}(p_1, \ldots, p_4) \) is written \(-p_i\) (incoming is changed into outgoing), then the corresponding product factor in \( \prod_{a_i=2} \) in Eq. (13) should be replaced by \( \eta_i e^{x_i - \sigma p_i^0} \). \( \sigma \) is an arbitrary parameter, \( 0 \leq \sigma \leq \beta [6,7] \). In the following \( \sigma = \beta/2 \) will be specified.

We will consider in turn the above four-point amputated functions \( \Gamma_{aabb}(p_1, p_2, -p_3, -p_4) \), \( \Gamma_{aabb}(p_2, -p_3, -p_4, p_1) \) and \( \Gamma_{aabb}(p_2, -p_4, -p_3, p_1) \). For \( \Gamma_{aabb}(p_1, p_2, -p_3, -p_4) \), Eq. (12) becomes

\[
\Gamma_{1111}^{aabb}(p_1, p_2, -p_3, -p_4) + \Gamma_{1122}^{aabb}(p_1, p_2, -p_3, -p_4) e^{-\frac{\beta}{2}(p_3^0 + p_4^0)} + \Gamma_{2211}^{aabb}(p_1, p_2, -p_3, -p_4) e^{\frac{\beta}{2}(p_3^0 + p_4^0)}
\]
In view of $p = p_1 + p_2 = p_3 + p_4$ and that $\Gamma^{aabb}_{4}$ is only function of $p$, we can obtain from Eq. (14) that

$$\Gamma^{11}(p) + \Gamma^{12}(p)e^{-\beta p^0/2} + \Gamma^{21}(p)e^{\beta p^0/2} + \Gamma^{22}(p) = 0,$$

(15)

where the definition $\Gamma^{aabb}_{4}(p_1, p_2, -p_3, -p_4) \equiv \Gamma^{ab}(p)$ in Eq. (7) has been used. Next, Eq. (13) becomes

$$\Gamma^{1111}_{4}(p_1, p_2, -p_3, -p_4) + \Gamma^{1222}_{4}(p_1, p_2, -p_3, -p_4)$$

$$+ \Gamma^{2211}_{4}(p_1, p_2, -p_3, -p_4)e^{\frac{\beta}{2}(p_1^0 + p_2^0) - x_1 - x_2}$$

$$+ \Gamma^{2221}_{4}(p_1, p_2, -p_3, -p_4)e^{\frac{\beta}{2}(p_1^0 + p_2^0 - p_3^0 - p_4^0) - x_1 - x_2 + x_3 + x_4} = 0.$$  

(16)

Let $p_1, p_4$ correspond to fermions, $p_2, p_3$ to antifermions, then we will have $\mu_3 = -\mu = -\mu_4$, $\mu_1 = \mu = -\mu_2$ and

$$x_3 + x_4 - \frac{\beta}{2}(p_3^0 + p_4^0) = \frac{\beta}{2}p^0,$$

$$\frac{\beta}{2}(p_1^0 + p_2^0) - x_1 - x_2 = -\frac{\beta}{2}p^0,$$

hence Eq. (16) is reduced to

$$\Gamma^{11}(p) + \Gamma^{12}(p)e^{\beta p^0/2} + \Gamma^{21}(p)e^{-\beta p^0/2} + \Gamma^{22}(p) = 0.$$  

(17)

This proves that the relations (12) and (13) for the four-point amputated functions $\Gamma^{aabb}_{4}(p_1, p_2, -p_3, -p_4)$ are changed into the relations (15) and (17) obeyed by the two-point functions $\Gamma^{ab}(p)$ with $p^2 > 0$.

By similar way we can prove that the relations (12) and (13) for the four-point amputated functions $\Gamma^{aabb}_{4}(p_2, -p_3, -p_4, p_1)$ and $\Gamma^{aabb}_{4}(p_2, -p_4, -p_3, p_1)$ will also be reduced to the same relations (15) and (17) obeyed by the two-point functions $\Gamma^{ab}(p)$, but in the two cases we will have $p^2 < 0$. Therefore, the relations among the four-point amputated functions $\Gamma^{aabb}_{4}$ in a NJL model may always be expressed by the relations (15) and (17) among the two-point functions $\Gamma^{ab}(p)$, no matter whether the squared transfer momentum $p^2 > 0$ or $p^2 < 0$. It is indicated that if $\Gamma^{ab}(p)$ is a symmetric matrix, i.e. $\Gamma^{12}(p) = \Gamma^{21}(p)$, then the two equations (15) and (17) will combine into the single one

$$\Gamma^{11}(p) + \Gamma^{22}(p) + 2 \cosh(\beta p^0/2)\Gamma^{12}(p) = 0.$$  

(18)

Since both the thermal transformation equation of the four-point amputated function matrix $\Gamma^{aabb}_{4}$ ($a, b = 1, 2$) in a NJL model and the relations obeyed by its elements can be reduced to the ones of the two-point functions $\Gamma^{ab}(p)$, this makes it possible to discuss the thermal transformations of these four-point amputated functions by the same way to deal with two-point functions.

3 Diagonalization in the $F \bar{F}$ basis of the matrix propagator for scalar bound state

The propagator for the scalar bound state $\langle \bar{\psi} \psi \rangle = \phi_S$ is a typical example of four-point amputated function in the NJL model. Its effective $2 \times 2$ matrix form can be expressed as:
by [12]

\[
\Gamma_S(p) = \begin{pmatrix} \Gamma_{S}^{11}(p) & \Gamma_{S}^{12}(p) \\ \Gamma_{S}^{21}(p) & \Gamma_{S}^{22}(p) \end{pmatrix} = \frac{1}{[K(p) + H(p) - iS(p)]^2 - R^2(p)} 
\times \begin{pmatrix} S(p) - i[K^*(p) + H(p)] & -(p^2 - 4m^2)R(p) \\ -(p^2 - 4m^2)R(p) & S(p) + i[K(p) + H(p)] \end{pmatrix} 
\]

(19)

where \(K(p)\) is the zero-temperature loop integral and can be divided into real and imaginary part, i.e. \(K(p) = \text{Re}K(p) + i\text{Im}K(p), H(p), S(p)\) and \(R(p)\) are all real and even functions of \(p\). The explicit expressions of these functions will not given here because they are not important for discussions of thermal transformation except the relation [12] \[S'(p) = S(p) - \text{Im}K(p) = R(p) \cosh(\beta p^0/2). \] (20)

It is seen from Eq. (19) that the matrix elements \(\Gamma^{ab}(p)\) obey the relations

\[
\Gamma_{S}^{22}(p) = [\Gamma_{S}^{11}(p)]^*, \quad \Gamma_{S}^{21}(p) = \Gamma_{S}^{12}(p) = \Gamma_{S}^{12}(p)^*, \quad \text{which include five relations among the eight real components of } \Gamma^{ab}(p), \text{ thus only three of } \Gamma^{ab}(p) \text{ are independent.}
\]

Eq. (21) can be combined into

\[
\Gamma_{S}^{11}(p) + \Gamma_{S}^{22}(p) + A \Gamma_{S}^{12}(p) = 0. \quad (22)
\]

By means of Eqs. (19) and (20) we will obtain

\[
A = \frac{-2 \text{Re}\Gamma_{S}^{11}(p)}{\Gamma_{S}^{22}(p)} = 2 \left[ \frac{S'(p) - \varepsilon \text{Re}K(p) + H(p)}{R(p)} \right] = 2 \cosh(\beta p^0/2) - 2\varepsilon \frac{\text{Re}K(p) + H(p)}{(p^2 - 4m^2)R(p)}. \]

(23)

In Eq. (23) the term containing \(\varepsilon = 0^+\) only affects the position of the pole \(p^2 = 4m^2\); if it is ignored, then Eq. (22) will become

\[
\Gamma_{S}^{11}(p) + \Gamma_{S}^{22}(p) + 2 \cosh(\beta p^0/2) \Gamma_{S}^{12}(p) = 0. \quad (24)
\]

This implies that \(\Gamma^{ab}(p)\) \((a, b = 1, 2)\) in Eq. (19) does satisfy the relation (18).

In view of Eq. (24), one may always find a transformation matrix \(O(p)\) which will make some elements of the transformed matrix \(\hat{\Gamma}_S(p)\) are identical to zeroes [9]. The diagonalization in the \(\bar{F}\) basis means that we can seek a \(2 \times 2\) matrix \(O(p)\) which, through the equation like Eq. (8), make \(\hat{\Gamma}_S(p)\) become a diagonal matrix, i.e.

\[
O(p)\Gamma_S(p)O^T(-p) = \hat{\Gamma}_S(p) = \begin{pmatrix} \Gamma_{\phi S}(p) & 0 \\ 0 & \Gamma_{\phi S}^*(p) \end{pmatrix}. \quad (25)
\]

Assume that

\[
O(p) = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix}, \quad (26)
\]

then, noting that \(\Gamma_{S}^{21}(p) = \Gamma_{S}^{12}(p)\) in Eq. (21), we will obtain respectively from \([O(p)\Gamma_S(p)O^T(-p)]^{21} = 0\) and \([O(p)\Gamma_S(p)O^T(-p)]^{21} = 0\) that

\[
\begin{align*}
(a(p) & b(p) \\
c(p) & d(p) \end{align*} = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix},
\]

(27)
and
\[ c(p)a(-p) \left\{ \Gamma^1_S(p) + \frac{b(-p)}{a(-p)} + \frac{d(p)}{c(p)} \right\} \Gamma^2_S(p) + \frac{d(p)}{c(p)} \frac{b(-p)}{a(-p)} \Gamma^2_S(p) = 0. \] (28)

Comparing Eq. (27) with Eq. (24), we will have
\[ \frac{d(-p)}{c(-p)} = \alpha(p), \quad \frac{b(p)}{a(p)} = \gamma(p) = 2 \cosh(\beta p^0/2) - \alpha(p), \quad \frac{b(-p)}{a(-p)} \cdot \frac{d(-p)}{c(-p)} = \gamma(p) \cdot \alpha(p) = 1, \]
which result in that
\[ \alpha(p) = e^{\beta|p^0|/2}, \quad \gamma(p) = e^{-\beta|p^0|/2}. \] (29)

Comparing Eq. (28) with Eq. (24), we will have
\[ \frac{d(p)}{c(p)} = \alpha(p), \quad \frac{b(-p)}{a(-p)} = \gamma(p) = 2 \cosh(\beta p^0/2) - \alpha(p), \quad \frac{b(-p)}{a(-p)} \cdot \frac{d(p)}{c(p)} = \gamma(p) \cdot \alpha(p) = 1, \]
which lead to Eq. (29) once again. The above results can be summarized as
\[ \frac{d(p)}{c(p)} = \frac{d(-p)}{c(-p)} = \alpha(p), \quad \frac{b(p)}{a(p)} = \frac{b(-p)}{a(-p)} = \gamma(p), \] (30)
which are consistent with the fact that \( \alpha(p) \) and \( \gamma(p) \) are both even functions of \( p \). Furthermore, owing to \( \alpha(p) \cdot \gamma(p) = 1 \), we will have
\[ b(p)d(p) = a(p)c(p). \] (31)

On the other hand, from Eq. (25) we must have
\[ [O(p) \Gamma_S(p)O^T(-p)]^{11} = [O(p) \Gamma_S(p)O^T(-p)]^{22*} \] (32)
where
\[ [O(p) \Gamma_S(p)O^T(-p)]^{11} = a(p)a(-p) \left[ \Gamma^1_S(p) + \frac{b(-p)}{a(-p)} \Gamma^2_S(p) + \frac{b(p)}{a(p)} \Gamma^2_S(p) \right. \]
\[ + \frac{b(p)}{a(p)} \cdot \frac{b(-p)}{a(-p)} \Gamma^2_S(p) \\right] \]
\[ = a(p)a(-p) \left[ \Gamma^1_S(p) + \gamma(p) \Gamma^2_S(p) + \gamma(p) \Gamma^2_S(p) + \gamma^2(p) \Gamma^2_S(p) \right. \]. (33)

It is incidentally indicated that when comparing Eq. (28) with Eq. (24), it is also possible to take \( b(-p)/a(-p) = \alpha(p) \) and \( d(p)/c(p) = \gamma(p) \), however, this will lead to
\[ [O(p) \Gamma_S(p)O^T(-p)]^{11} = a(p)a(-p) \left[ \Gamma^1_S(p) + 2 \cosh(\beta p^0/2) \Gamma^2_S(p) + \Gamma^2_S(p) \right] = 0 \] due to Eq. (24), obviously inconsistent with the requirement of Eq. (25). Next,
\[ [O(p) \Gamma_S(p)O^T(-p)]^{22} = d(p)d(-p) \left[ \frac{c(p)}{d(p)} \cdot \frac{c(-p)}{d(-p)} \Gamma^1_S(p) + \frac{c(p)}{d(p)} \Gamma^2_S(p) \right. \]
\[ + \frac{c(-p)}{d(-p)} \Gamma^2_S(p) \\right] \]
\[ = d(p)d(-p) \left[ \frac{1}{O^2(p)} \Gamma^1_S(p) + \frac{1}{O^2(p)} \Gamma^2_S(p) + \frac{1}{O^2(p)} \Gamma^2_S(p) + \Gamma^2_S(p) \right. \].
Assume that \( a(p), b(p), c(p) \) and \( d(p) \) are all real functions of \( p \), then by means of \( \Gamma_{22}^S(p) = \Gamma_{11}^S(p) \) in Eq. (21), Eq. (32) will lead to

\[
 a(p)a(-p) = d(p)d(-p). \tag{34}
\]

From Eq. (30), it is reasonable to assume that \( a(p), b(p), c(p) \) and \( d(p) \) are all even functions of \( p \), hence Eq. (34) is reduced to

\[
 a^2(p) = d^2(p).
\]

Taking \( a(p) = d(p) \) and then from Eq. (31), we will have \( b(p) = c(p) \). Consequently, the transformation matrix \( O(p) \) in Eq. (26) can be written by

\[
 O(p) = \begin{pmatrix} a(p) & b(p) \\ b(p) & a(p) \end{pmatrix}. \tag{35}
\]

If further assuming that the determinant of the matrix \( O(p) \) is unity, i.e. \( \det O(p) = a^2(p) - b^2(p) = 1 \), then we can obtain from Eqs. (29) and (30) that

\[
 a^2(p) = 1/[1 - \gamma^2(p)] = n(p^0) + 1, \quad b^2(p) = n(p^0) = 1/(e^{\beta p^0} - 1). \tag{36}
\]

Then \( O(p) \) will take the following form:

\[
 O(p) = \begin{pmatrix} \cosh \Theta & \sinh \Theta \\ \sinh \Theta & \cosh \Theta \end{pmatrix}, \quad \sinh \Theta = [n(p^0)]^{1/2} = b, \quad \cosh \Theta = [n(p^0) + 1]^{1/2} = a. \tag{37}
\]

Therefore, the resulting transformation matrix \( O(p) \) in the \( F\bar{F} \) basis is just the thermal transformation matrix \( M \) of the two-point function matrix for a real scalar field in the usual real-time formalism [6]. The form of \( M \) was given in Ref. [12], but the details of its derivation were omitted there. From Eq. (33), the physical causal propagator for the scalar bound state \( \phi_S \) will be

\[
 \Gamma_{\phi_S}^S(p) = [O(p)\Gamma_S(p)O^T(-p)]^{11} = a^2(p)\Gamma_{11}^{S'}(p) + 2a(p)b(p)\Gamma_{12}^{S'}(p) + b^2(p)\Gamma_{22}^{S'}(p) = [a^2(p) + b^2(p)]\text{Re}\Gamma_{11}^S(p) + i\text{Im}\Gamma_{11}^S(p) + 2a(p)b(p)\Gamma_{12}^S(p) = \frac{\cosh(\beta |p^0|/2)}{\sinh(\beta |p^0|/2)}\text{Re}\Gamma_{11}^S(p) + i\frac{1}{\sinh(\beta |p^0|/2)}\Gamma_{12}^S(p), \tag{38}
\]

where Eq. (37) has been used. By means of Eqs. (19) and (20), we can obtain from Eq. (38)

\[
 \Gamma_{\phi_S}^S(p) = \frac{-i}{[\text{Re}K(p) + H(p)]^2 + R^2(p)\sinh^2(\beta |p^0|/2)} \cdot \frac{1}{(p^2 - 4m^2)^2 + \varepsilon^2} \times \left( p^2 - 4m^2 - i\varepsilon \frac{\cosh(\beta |p^0|/2)}{\sinh(\beta |p^0|/2)} \right) [\text{Re}K(p) + H(p) + iR(p)\sinh(\beta |p^0|/2)]
\]

\[
 = -i/[\text{Re}K(p) + H(p) - iR(p)\sinh(\beta |p^0|/2)](p^2 - 4m^2 + i\varepsilon) \tag{39}
\]

which is precisely the physical causal propagator for the scalar bound state \( \phi_S \) given in Ref. [12] and can be obtained from the Matsubara frequency \( \Omega_m \)'s analytic continuations \(-i\Omega_m \rightarrow p^0 + i\Omega_m \) in the imaginary-time formalism.
4 Transformation in the RA basis of the matrix propagator for scalar bound state

For fixing the positions of the propagators' poles more rigorously, we will use the relation (22) instead Eq. (24). Owing to this relation among the matrix elements $\Gamma_S^{ab}(p)$ ($a, b = 1, 2$), we may also seek a transformation matrix $\hat{O}(p)$ which makes all the diagonal elements of the transformed matrix $\hat{\Gamma}_S(p)$ be equal to zeroes. This is so called the transformation in the RA basis [9]. Its explicit form is

$$\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p) = \hat{\Gamma}_S(p) = \begin{pmatrix} 0 & \Gamma_{2S}^{\phi S}(p) \\ \Gamma_{2S}^{\phi S}(p) & \Gamma_S^{aS}(p) \end{pmatrix}. \quad (40)$$

Assume that

$$\hat{O}(p) = \begin{pmatrix} \tilde{a}(p) & \tilde{b}(p) \\ \tilde{c}(p) & \tilde{d}(p) \end{pmatrix}, \quad (41)$$

then, owing to $\Gamma_S^{21}(p) = \Gamma_S^{12}(p)$, Eq. (40) indicates that

$$[\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p)]^{11} = \tilde{a}(p)\bar{a}(-p) \left\{ \Gamma_S^{11}(p) + \left[ \frac{\bar{b}(p)}{\bar{a}(-p)} + \frac{\bar{b}(p)}{\bar{a}(p)} \right] \Gamma_S^{12}(p) + \frac{\bar{b}(p)}{\bar{a}(p)} \cdot \frac{\bar{b}(p)}{\bar{a}(-p)} \Gamma_S^{22}(p) \right\}$$

$$= 0 \quad (42)$$

and

$$[\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p)]^{22} = \tilde{c}(p)\bar{c}(-p) \left\{ \Gamma_S^{11}(p) + \left[ \frac{\bar{d}(p)}{\bar{c}(-p)} + \frac{\bar{d}(p)}{\bar{c}(p)} \right] \Gamma_S^{12}(p) + \frac{\bar{d}(p)}{\bar{c}(p)} \cdot \frac{\bar{d}(p)}{\bar{c}(-p)} \Gamma_S^{22}(p) \right\}$$

$$= 0. \quad (43)$$

For the use of Eq. (22), we may assume that

$$\frac{\bar{b}(\pm p)}{\bar{a}(\pm p)} = e^{\mp \eta(p^0)\Theta}, \quad \frac{\bar{d}(\pm p)}{\bar{c}(\pm p)} = e^{\mp \eta(p^0)\Theta}, \quad \eta(p^0) = \frac{p^0}{|p^0|}. \quad (44)$$

where

$$2 \cosh \Theta = A. \quad (45)$$

It is indicated that $\bar{d}(\pm p)/\bar{c}(\pm p) = e^{\mp \eta(p^0)\Theta}$, even though allowed by Eq. (43), should be removed because it will lead to $[\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p)]^{12} = [\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p)]^{21} = 0$ due to Eq. (22) and these do not satisfy the requirement of Eq. (40). Substituting Eq. (44) into Eq. (42), we can obtain the relations

$$\Gamma_S^{22}(p) + e^{-\eta(p^0)\Theta} \Gamma_S^{12}(p) = - \left[ \Gamma_S^{11}(p) + e^{\eta(p^0)\Theta} \Gamma_S^{12}(p) \right] \quad (46)$$

or

$$\Gamma_S^{22}(p) + e^{\eta(p^0)\Theta} \Gamma_S^{12}(p) = - \left[ \Gamma_S^{11}(p) + e^{-\eta(p^0)\Theta} \Gamma_S^{12}(p) \right] \quad (47)$$

By means of Eqs. (40),(41) and (44), we have

$$\Gamma_S^{22}(p) = [\hat{O}(p)\Gamma_S(p)\hat{O}^T(-p)]^{12}$$

$$= \tilde{a}(p)\bar{c}(-p) \left\{ \Gamma_S^{11}(p) + \frac{\bar{d}(p)}{\bar{c}(-p)} \Gamma_S^{12}(p) + \frac{\bar{b}(p)}{\bar{a}(p)} \Gamma_S^{21}(p) + \frac{\bar{b}(p)}{\bar{a}(p)} \cdot \frac{\bar{d}(p)}{\bar{c}(-p)} \Gamma_S^{22}(p) \right\}$$

$$= \tilde{a}(p)\bar{c}(-p) \left[ \Gamma_S^{11}(p) + 2e^{\eta(p^0)\Theta} \Gamma_S^{12}(p) + e^{2\eta(p^0)\Theta} \Gamma_S^{22}(p) \right].$$
where Eq. (46) has been used and
\[
\Gamma^\phi_S(p) = [\bar{O}(p) \Gamma_s(p) \bar{O}^T(-p)]^{21}
\]
\[
= \bar{c}(p)\bar{a}(-p) \left[ \Gamma^{S1}(p) + \frac{\tilde{b}(-p)}{\bar{a}(-p)} \Gamma^{S2}(p) + \frac{\tilde{d}(p)}{\tilde{c}(p)} \Gamma^{S21}(p) + \frac{\tilde{d}(p)}{\tilde{c}(p)} \frac{\tilde{b}(-p)}{\bar{a}(-p)} \Gamma^{S22}(p) \right]
\]
\[
= \bar{c}(p)\bar{a}(-p) \left[ \Gamma^{S1}(p) + 2e^{-\eta(p^0)}\bar{\Theta}\Gamma^{S2}(p) + e^{-2\eta(p^0)}\bar{\Theta}\Gamma^{S22}(p) \right]
\]
\[
= -\bar{c}(p)\bar{a}(-p) \left[ 1 - e^{-2\eta(p^0)}\bar{\Theta} \right] \left[ \Gamma^{S2}(p) + e^{\eta(p^0)}\bar{\Theta}\Gamma^{S12}(p) \right],
\] (49)

where Eq. (47) has been used. Now define
\[
\Gamma_a^\phi_S(p) \equiv \Gamma^{S1}(p) + e^{\eta(p^0)}\bar{\Theta}\Gamma^{S12}(p)
\] (50)

and
\[
\Gamma_r^\phi_S(p) \equiv -[\Gamma^{S2}(p) + e^{\eta(p^0)}\bar{\Theta}\Gamma^{S12}(p)]
\] (51)

which obey the relation
\[
[\Gamma_a^\phi_S(p)]^* = -\Gamma_r^\phi_S(p),
\] (52)

then from Eqs. (48) and (49) we are led to that
\[
\bar{a}(p)\bar{c}(-p) \left[ 1 - e^{-2\eta(p^0)}\bar{\Theta} \right] = 1,
\]
\[
\bar{c}(p)\bar{a}(-p) \left[ 1 - e^{-2\eta(p^0)}\bar{\Theta} \right] = 1.
\] (53)

The six independent relations contained in Eqs. (44) and (53) allow us to determine the eight unknown \( \bar{a}(\pm p) \), \( \bar{b}(\pm p) \), \( \bar{c}(\pm p) \) and \( \bar{d}(\pm p) \) up to their being expressed by \( \bar{a}(p) \) and \( \bar{a}(-p) \). The results are as follows.
\[
\bar{b}(\pm p) = e^{\pm\eta(p^0)}\bar{\Theta}\bar{a}(\pm p), \quad \bar{c}(\pm p) = e^{\pm\eta(p^0)}\bar{\Theta}\bar{d}(\pm p),
\]
\[
\bar{d}(\pm p) = \pm\frac{1}{2\eta(p^0)}\frac{1}{\sinh \Theta \bar{a}(\mp p)}.
\]

As a result, the transformation matrix \( \bar{O}(p) \) in the RA basis may be written by
\[
\bar{O}(p) = \frac{1}{\bar{a}(-p)} \left( e^{\eta(p^0)\bar{\Theta}} \bar{a}(p)\bar{a}(-p) \quad e^{\eta(p^0)\bar{\Theta}} \bar{a}(p)\bar{a}(-p) \right)
\] (54)

which will give the transformed result in Eq. (40). From Eqs. (45) and (23), we have
\[
e^{\eta(p^0)\bar{\Theta}} = \cosh(\beta p^0/2) - \varepsilon \frac{\text{Re}K(p) + H(p)}{(p^2 - 4m^2)R(p)} + \eta(p^0) \sinh(\beta |p^0|/2),
\] (55)

thus \( \Gamma_a^\phi_S(p) \) and \( \Gamma_r^\phi_S(p) \) have the following expressions:
\[
\Gamma_a^\phi_S(p) \equiv \Gamma^{S1}(p) + e^{\eta(p^0)\bar{\Theta}}\Gamma^{S12}(p)
\]
\[
= \frac{1}{[\text{Re}K(p) + H(p)]^2 + R^2(p) \sinh^2(\beta |p^0|/2) \cdot (p^2 - 4m^2)^2 + \varepsilon^2}
\times \left( \{S'(p) - i[\text{Re}K(p) + H(p)]\}(p^2 - 4m^2 - i\varepsilon)
\right).
\]
\[
\begin{align*}
\Gamma_{\phi}^S(p) &= -i/[\text{Re}K(p) + H(p) - iR(p)\sinh(\beta p^0/2)][p^2 - 4m^2 - i\varepsilon(p^0)], \quad (57)
\end{align*}
\]

Therefore, through the transformation (40) in the RA basis, we have just obtained the advanced and retarded propagator \(\Gamma_a^S(p)\) and \(\Gamma_r^S(p)\) for the scalar bound state \(\phi_s\). It is indicated that the rigorous definition (55) of \(e^{i\phi(p)\bar{\Theta}}\) ensures the correct positions of the poles of \(\Gamma_a^S(p)\) and \(\Gamma_r^S(p)\). These results coincide with the ones obtained in the imaginary-time formalism respectively by the Matsubara frequency \(\Omega_m\)'s analytic continuations \(-i\Omega_m \rightarrow p^0 - i\varepsilon\) and \(-i\Omega_m \rightarrow p^0 + i\varepsilon\) [12].

5 Conclusions

Based a general analysis of four-point Green functions in the real-time thermal field theory, we have proven that the calculation of the four-point amputated functions in a NJL model in the fermion bubble diagram approximation is equivalent to the one of two-point functions in the following meanings that the thermal transformation of a four-point amputated function matrix can be reduced to the one of corresponding two-point function matrix and the same reduction is true to the relations satisfied by the elements of the two matrices. We further discuss in detail the thermal transformations of the matrix propagator for a scalar bound state in the \(\bar{F}F\) basis and in the RA basis and find out corresponding transformation matrices. In the former case, we obtain physical causal propagator and in the latter case, physical advanced and retarded propagator. All the results coincide with the ones derived in the imaginary-time formalism by analytic continuations of the discrete Matsubara frequency. This shows once again complete equivalence of the two formalisms of thermal field theory on the problem of the propagators for scalar bound states in the NJL model.

References

[1] Y. Nambu, Phys. Rev. Lett. 4 (1960) 380; J. Goldstone, Nuovo Cimento 19 (1961) 154; J. Goldstone, A. Salam, and S. Weinberg, Phy. Rev. 127 (1962) 965; S. Bludman and A. Klein, ibid. 131 (1962) 2363.

[2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.

[3] B.R. Zhou, Phys. Rev. D59 (1999) 065007; Commun. Theor. Phys. 33 (2000) 113.

[4] B. R. Zhou, Phys. Rev. D62 (2000) 105004.

[5] J.I. Kapusta, Finite-temperature field theory, Cambridge University Press, Cambridge, England, (1989).
[6] N.P. Landsman and Ch.G. van Weert, Phys. Rep. 145 (1987) 141 and the references therein.

[7] H. Umezawa, H. Matsumoto, and M. Tachiki, Thermofield dynamics and condensed matter states, North-Holland, Amsterdam, (1982);

[8] K. C. Chou, Z. B. Su, B. L. Hao and L. Yu, Phys. Rep. 118 (1985) 1.

[9] M. A. van Eijck, R. Kobes and Ch. G. van Weert, Phys. Rev. D50 (1994) 4097.

[10] G. 't Hooft and M. Veltman, in Particle interactions at very high energies, edited by D. Speiser, F. Halzen and J. Weyers, Plenum, New York, (1974), Part B, p.177.

[11] R. L. Kobes and G.W. Semenoff, Nucl. Phys. B260 (1985) 714; ibid. B272 (1986) 329.

[12] B.R. Zhou, Commun. Theor. Phys. 37 (2002) 303.