Universal Nonadiabatic Geometric Gates in Two-Qubit Decoherence-Free Subspaces

Guofu Xu1,2 & Guilu Long1,2

1State Key Laboratory of Low-dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, China, 2Tsinghua National Laboratory of Information Science and Technology, Beijing 100084, China.

Geometric quantum computation in decoherence-free subspaces is of great practical importance because it can protect quantum information from both control errors and collective dephasing. However, previous proposed schemes have either states leakage or four-body interactions problems. Here, we propose a feasible scheme without these two problems. Our scheme is realized in two-qubit decoherence-free subspaces. Since the Hamiltonian we use is generic, our scheme looks promising to be demonstrated experimentally in different systems, including superconducting charge qubits.

Decoherence has been regarded as one main practical obstacle in building a quantum computer. It makes the desired coherence of the system collapse and hence reduces the efficiency of quantum computation. So far, three main strategies, namely, quantum error corrections, decoherence-free subspaces (DFSs) and dynamical decoupling, have been proposed to combat decoherence. As one promising way to avoid quantum decoherence, DFSs can be constructed if the interaction between the system and its environment has some symmetry structures. The basic idea of DFSs is to encode information in a subspace of the system, over which the dynamics is unitary. DFSs have been experimentally realized in many physical systems.

Besides the requirement of coherence protection, one also needs to achieve highly accurate control to enact quantum computation. The requirement of highly accurate control has been regarded as another main practical obstacle in building a quantum computer. Fortunately, this requirement can be relaxed by using quantum computing paradigms with built-in fault tolerance. Geometric quantum computation (GQC) is one such quantum computing paradigm. It exploits different types of quantum holonomies and provides a resilient way of information processing through all-geometric control. Until now, GQC has attracted much attention and many efforts, both in theory and in practice.

To overcome both decoherence and control errors, the schemes of GQC in DFSs have been proposed. Although impressive progresses have been made in this field, two problems still persist. One problem is the states leakage problem. This problem happens in two different ways. In one way, the evolutions generated by the geometric gates can drive the logical states out of the DFSs and hence spoil the predicted protection against decoherence. In another way, when extra degrees of freedom, like center-of-mass vibrational mode of ions or cavity mode, are used to generate quantum holonomies, although the logical states are always protected by the DFSs, the extra degrees of freedom are affected by the environment. So, geometric gates generated in this way can only be partially protected by the DFSs. Another problem is the four-body interactions problem. There exist GQC-DFS schemes without the states leakage problem. However, four-body interactions are used to build the logical entangling gates in these schemes. Considering four-body interactions are very challenge in experiment, this kind of interactions should be avoided.

In this article, we propose a universal set of nonadiabatic geometric gates without the states leakage and four-body interactions problems. Specifically, we use the tunable XXZ Hamiltonian to realize a universal set of nonadiabatic geometric gates in two-qubit DFSs. This is the major merit of our scheme. We also investigate how to make our scheme resilient to arbitrary collective decoherence. Since the XXZ Hamiltonian is generic, our scheme looks promising to be demonstrated experimentally in different systems and we use superconducting charge qubits as an illustration.
Results

The XXZ model. Before proceeding further, we explain how nonadiabatic quantum holonomies arise. Consider a $M$-dimensional subspace $S(0)$ spanned by the orthonormal vectors $\{ |\phi_j(0)\rangle \}$. The evolution operator is a holonomic matrix acting on $S(0)$ if $|\phi_j(t)\rangle$ satisfy the following conditions: (i) $\sum_{j=1}^{M} \langle \phi_j(T_0)| \phi_j(T_0)\rangle = \sum_{j=1}^{M} |\phi_j(0)\rangle \langle \phi_j(0)|$; (ii) $\langle \phi_j(t)| \mathcal{H}(t)\langle \phi_j(t)| = 0$, $\mu, v = 1, \ldots, M$. In the above, $\mathcal{H}(t)$ is the Hamiltonian of the system, $T_0$ is the evolution period, and $|\phi_j(t)\rangle = T \exp\left(-i \int_0^t \mathcal{H}(t) dt\right) |\phi_j(0)\rangle$, where $T$ is the time ordering operator. While condition (i) means the subspace $S(0)$ completes a cyclic evolution, condition (ii) means this cyclic evolution is purely geometric.

Let us now elucidate our physical model. Suppose we can handle the following Hamiltonian

\[ H = \sum_k \tilde{J}_k \sigma_z + \sum_{k<l} (J_k^{xy} \sigma_x^k \sigma_x^l + J_k^{yz} \sigma_y^k \sigma_y^l), \]

where $\tilde{J}_k$ is the effective local field applied to the $k$th physical qubit, $J_k^{xy}$ and $J_k^{yz}$ are coupling parameters, $\sigma_x^k$ and $\sigma_y^k$ is the Pauli $\beta$ operator acting on the $k$th physical qubit and $\sigma_x^k \sigma_y^k - \sigma_y^k \sigma_x^k$. The Hamiltonian in Eq. (1) is the tunable XXZ Hamiltonian and can be realized in different systems. For example, the physical qubit can be a superconducting island which is coupled to a ring by two symmetric Josephson junctions and the states $|0\rangle$ and $|1\rangle$ are respectively the two charge states near the charging energy degeneracy point.

For the XXZ model, the major source of decoherence is dephasing. So, if two physical qubits are put close, the interaction Hamiltonian between these two physical qubits and its environment can be described by

\[ H_I = (\sigma^z \otimes I + I \otimes \sigma^z) \otimes B, \]

where $\sigma^z$ and $I$ are respectively the Pauli $Z$ and identity operators acting on the corresponding physical qubit and $B$ is an arbitrary environment operator. The symmetry of the interaction implies there exists a two-dimensional DFS $S = \text{Span}\{|0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle\}$, where $|0\rangle$ and $|1\rangle$ are the eigenstates of the Pauli $Z$ operator of the corresponding physical qubit. We encode a logical qubit in the subspace $S$ and respectively denote the logical states $|0\rangle_L$ and $|1\rangle_L$ as $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$.

For convenience, we respectively write $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |0\rangle$ as $|01\rangle$ and $|10\rangle$. In the whole article, we use the subscript $L$ to denote that the states and the operators are respectively logical states and logical operators. In the following paragraphs, we will show how to use the tunable XXZ Hamiltonian to realize a universal set of nonadiabatic geometric gates in two-qubit DFSs.

One-logical-qubit gates. As is well known, to achieve a universal set of quantum logical gates, one needs to realize two noncommutative one-logical-qubit gates (not two fixed one-logical-qubit gates but two classes of one-logical-qubit gates) and one nontrivial entangling logical gate. Firstly, we demonstrate how to realize two noncommutative one-logical-qubit gates. For convenience, the corresponding two physical qubits are respectively denoted as 1 and 2. We consider the following one-logical-qubit gate

\[ U_1(T_1, 0) = e^{-i \int_0^{T_1} H_1(t) dt} e^{-i \int_0^{T_0} H_0(t) dt}. \]

In the above equation, $T_1$ and $T_0$ are respectively an arbitrary intermediate time and the evolution period, the Hamiltonians $H_1(t)$ and $H_0(t)$ respectively read

\[ H_0(t) = J_0(t)(\cos \phi \sigma_z^{12} + \sin \phi \sigma_z^{12}), \]

\[ H_1(t) = J_1(t) \sigma_x^{12}, \]

where $\phi$ is an arbitrary parameter, $J_0(t)$ and $J_1(t)$ are controllable coupling parameters and satisfy the conditions

\[ \int_0^{T_1} J_0(t) dt = \int_0^{T_1} J_1(t) dt = \frac{\pi}{2}. \]

If one ignores the global phase, the logical gate $U(T_1, 0)$ can be written as

\[ U_1(T_1, 0) = e^{-i \theta Y_L}, \]

where $Y_L$ is the logical Pauli $Y$ operator and can be written as $Y_L = -i |0\rangle_L \langle 1|_L + i |1\rangle_L \langle 0|_L$.

One can verify that $U_1(T_1, 0)$ has both decoherence-free and geometric properties. $U_1(T_1, 0)$ has decoherence-free property because $S$ is an invariant subspace of the evolution operator $U_1(t, 0)$. The geometric property of $U_1(T_1, 0)$ can be verified by using the holonomic conditions (i) and (ii). Consider the eigenstates of the logical operator $Y_L$. According to Eq. (8), the two one-dimensional subspaces respectively complete cyclic evolutions and the evolution operators are phases. Then, condition (i) is satisfied. By calculating the matrix elements of $H_0(t)$ and $H_1(t)$ in the basis of the eigenstates of $Y_L$, one can verify condition (ii) is satisfied. Since both conditions (i) and (ii) are satisfied, the accumulated phases $\phi - \phi$ are geometric and hence $U_1(T_1, 0)$ is a geometric gate. The geometric property of $U_1(T_1, 0)$ can also be illustrated by Fig. 1 in which the Hamiltonians $H_0(t)$ and $H_1(t)$ drive the eigenstates of $Y_L$ from one starting pole to the opposite pole and then back to the initial pole. Since these trajectories are connected geodesics, the dynamical phases are zero and the logical gate $U_1(T_1, 0)$ is geometric.

Next, we illustrate how to realize the second one-logical-qubit gate. One can see that the logical states $|0\rangle_L$ and $|1\rangle_L$ (the eigenstates

Figure 1 | Evolutions of $U_1(T_1, 0)$ and $U_2(T_2, 0)$ in logical Bloch sphere. For logical gate $U_1(T_1, 0)$, the $+1$ eigenstate of $Y_L$ represented by point $A$ completes a cyclic evolution on $ACBDA$ and the accumulated geometric phase is proportional to the solid angle $\alpha$; for logical gate $U_2(T_2, 0)$, the $+1$ eigenstate of $Z_L$ represented by point $C$ completes a cyclic evolution on $CBDAC$ and the accumulated geometric phase is also proportional to the solid angle $\alpha$. Similar evolutions exist for the $-1$ eigenstates of $Y_L$ and $Z_L$.
of $Z_L$ are respectively evolved to the logical states $\frac{1}{\sqrt{2}} \left( |0\rangle_L - i|1\rangle_L \right)$ and $\frac{1}{\sqrt{2}} \left( |0\rangle_L + i|1\rangle_L \right)$ (the eigenstates of $Y_L$) under the action of the operation $\exp \left( -\frac{\pi}{4} \sigma^y_{12} \right)$ while the inverse evolution can be realized by using the operation $\exp \left( i\frac{\pi}{4} \sigma^y_{12} \right)$. So, we realize the second one-logical-qubit gate by first using the operation $\exp \left( i\frac{\pi}{4} \sigma^y_{12} \right)$ to evolve the eigenstates of $Z_L$ to the eigenstates of $Y_L$, and then using the operation $U_1(T_1, 0)$ to evolve the eigenstates of $Y_L$ cyclically, and then using the operation $\exp \left( i\frac{\pi}{4} \sigma^y_{12} \right)$ to evolve the eigenstates of $Y_L$ back to the eigenstates of $Z_L$. Specifically, we realize the following one-logical-qubit gate

$$U_1(T_2, 0) = e^{-i \int_{t_2}^{t_1} H_1(t) dt} e^{-i \int_{t_1}^{t_2} H_1(t) dt} = e^{i \int_{t_1}^{t_2} H_1(t) dt},$$

(9)

where $t_1$, $t_2$ and $T_2$ are respectively arbitrary intermediate time parameters and the evolution period, the Hamiltonians $H_1(t)$ and $H_1(t)$ are described by Eq. (6). Here, the control coupling parameters $J_1(t)$ and $J_1(t)$ need to satisfy the conditions

$$\int_{t_1}^{t_2} J_1(t) dt = \int_{t_1}^{t_2} J_1(t) dt = \frac{\pi}{4}.$$

(10)

If one ignores the global phase, the second one-logical-qubit gate can be written as

$$U_1(T_1, 0) = e^{i B Z_L},$$

(11)

where $Z_L$ is the logical Pauli Z operator and can be written as $Z_L = |0\rangle_L \langle 0|_L - |1\rangle_L \langle 1|_L$.

The illustration of the decoherence-free and geometric properties of $U_1(T_2, 0)$ is similar to that of $U_1(T_1, 0)$. The logical gate $U_1(T_2, 0)$ is always protected by the subspace $S$ because $S$ is an invariant subspace of the evolution operator $U_1(T_2, 0)$. By using the holonomic conditions (i) and (ii), one can verify that the accumulated phases of the logical states $|0\rangle_L$ and $|1\rangle_L$ are geometric phases and hence the logical gate $U_1(t, 0)$ is geometric gate. The evolution of the logical gate $U_1(T_2, 0)$ can also be illustrated by Fig. 1 in which the logical eigenstates of $Z_L$ respectively act as the starting points of the evolution and the accumulate geometric phases are proportional to the solid angle $\phi$.

In the above, we have realized two two noncommutative one-logical-qubit gates $U_1(T_1, 0) = e^{-i Y Z_L}$ and $U_1(T_2, 0) = e^{-i Y Z_L}$. It is well known that arbitrary rotations around two orthogonal axes are sufficient to realize any one-logical-qubit rotation. Since the rotation axes $Y$ and $Z$ are orthogonal and the parameter $\phi$ can be chosen arbitrarily, our scheme is sufficient to realize any one-logical-qubit rotation.

### Two-logical-qubit gate.

We now demonstrate how to realize a nontrivial entangling logical gate. For convenience, the corresponding four physical qubits are respectively denoted as 1, 2, 3, and 4. The DFS of this four-qubit system reads

$$S = \text{Span} \{ |00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L \}.$$  

(12)

The nontrivial entangling logical gate we realize reads

$$U(T, 0) = e^{-i \int_T H(t) dt} e^{i \int_{-T} H(t) dt}.$$  

(13)

In the above, $T$ and $T$ respectively being an arbitrary intermediate time and the evolution period, the Hamiltonians $H(t)$ and $H(t)$ respectively read

$$H(t) = f(t) \left( \cos \theta \sigma^x_{12} + \sin \theta \sigma^x_{12} \right),$$

$$H'(t) = f'(t) \sigma^y_{12},$$

(14)

where $\theta$ is an arbitrary parameter, $f(t)$ and $f'(t)$ are controllable coupling parameters and satisfy the conditions

$$\int_0^T f(t) dt = \int_0^T f'(t) dt = \frac{\pi}{2}.$$  

(15)

If one ignores the global phase, the logical gate generated by the Hamiltonians $H(t)$ and $H'(t)$ can be written as

$$U(T, 0) = e^{-i Y_{12}}.$$  

(16)

One can see that $U(T, 0)$ is a nontrivial entangling logical gate if $\sin \theta$ and $\cos \theta$ are nonzero.

The decoherence-free and geometric properties of $U(T, 0)$ can be illustrated as follows. The logical gate $U(T, 0)$ is always protected by the subspace $S$ because $S$ is an invariant subspace of the evolution operators $U(t, 0)$. In other words, the logical gate $U(T, 0)$ has decoherence-free property. To ensure $U(T, 0)$ is a geometric gate, one needs to use the holonomic conditions (i) and (ii). Letting $U_1 = [U_1 - i (X_2 + Y_2 - Z_2)]/2$, $|0\rangle_L = U_1|0\rangle_L$ and $|1\rangle_L = U_1|1\rangle_L$. In the basis $\{|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L \}$, the entangling logical gate $U(T, 0)$ has the following form

$$U(T, 0) = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$  

(17)

where $0$ is a two-dimensional zero matrix, $U_1$ and $U_2$ respectively read

$$U_1 = \begin{pmatrix} -\cos \theta & i \sin \theta \\ -i \sin \theta & -\cos \theta \end{pmatrix},$$

$$U_2 = \begin{pmatrix} -\cos \theta & -i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}.$$  

(18)

We are splitting $S$ into the following two orthogonal subspaces

$$S_1 = \text{Span} \{ |00\rangle_L, |01\rangle_L \}, S_2 = \text{Span} \{ |10\rangle_L, |11\rangle_L \}.$$  

(19)

According to Eq. (17), the subspaces $S_1$ and $S_2$ undergo cyclic evolutions and the evolution operators are $U_1$ and $U_2$, respectively. Thus, the matrices $U_1$ and $U_2$ satisfy condition (i). The second condition of $U_1$ can be reduced to $0 = |\langle k' \rangle_{H(t)}|0\rangle_L = 0$ and $|\langle k' H(t) |1\rangle_L = 0$ because $H(t)$ and $H^t(t)$ respectively commute with their evolution operators and $\exp \left( -i \int_0^T H(t) dt \right) S_1 = S_2$, where $k, k' \in \{ 0, 1 \}$. So, the matrix $U_1$ satisfies condition (ii). Similarly, one can verify that the matrix $U_2$ also satisfies condition (ii). Since the matrices $U_1$ and $U_2$ satisfy both conditions (i) and (ii), they are holonomic matrices. Observing that $U(T, 0)$ is direct sum of $U_1$ and $U_2$, the logical gate $U(T, 0)$ has the required geometric property. The geometric property of $U(T, 0)$ may also be illustrated by Fig. 1 if one replaces the semi-great circles by two-dimensional holonomies.

### Discussion

We have succeeded in constructing two noncommutative one-logical-qubit gates and one nontrivial entangling logical gate by using the tunable XXZ Hamiltonian. Since two-qubit DFSs (or the direct product of two-qubit DFSs) are invariant subspaces of the proposed geometric gates, these logical gates are always protected by the two-qubits DFSs. The one-logical-qubit gates $U_1(T_1, 0)$ and $U_2(T_2, 0)$ are
Specifically, one can use the simultaneous Pauli consideration how to reduce collective dissipation. To do this, one can obtain collective dephasing by using DFSs, one only need to protect one logical qubit against dissipation. It should be noted that the following equations for different logical qubits do not need to be simultaneous. Since the simultaneous Pauli Z pulse commutes with the Hamiltonians of our scheme and preserves the two-qubit DFSs, the advantages of quantum holonomies, DFSs and dynamical decoupling can be usefully combined. In other words, our scheme can be resilient to both control errors and arbitrary collective decoherence.

The XXZ Hamiltonian can be demonstrated in different systems and we here briefly illustrate how to use the superconducting charge qubits (SCQs) to realize our scheme. For each SCQ, a superconducting island (Cooper-pair box) is coupled to a ring by two symmetric Josephson junctions characterized by coupling energy $E_J$ and capacitance $C_F$. We operate the system in the charging regime, then the extra Cooper-pairs $n$ in the box is a good quantum number. Near the charging energy degeneracy point, only two charge states ($n = 0, 1$) play a dominant role and we use this two charge states as the physical qubit states.  

For the kth SCQ, a control gate voltage $V_{gk}$ is applied to the box through a capacitance $C_k$ and an external magnetic flux $\Phi_k$ is used to modulate the Josephson coupling Hamiltonian. The Hamiltonian of the kth SCQ reads

$$H_k = \epsilon_k \sigma_k^x - E_{Jk} \sigma_k^z,$$  

where the charging energy is $\epsilon_k = E_g (1 - 2n_{gk})/2$ and the effective Josephson coupling energy is $E_{Jk} = E_g \cos(\pi \phi_k/\phi_0)$, with the charging energy scale being $E_g = 2e^2/\Ck$ the corresponding gate charge being $n_{gk} = C_k V_{gk}/2e$ and the fluxon being $\Phi_k = h/2e$.

We realize a logical qubit by connecting two SCQs $k$ and $k'$ with a superconducting quantum interference device (SQUID), see Fig. 2(a). The SQUID is pierced by a magnetic flux $\Phi_{k'}$ which can be used to modulate the Josephson coupling. Choosing small junction capacitances of the SQUID, the electrostatic energy between boxes $k$ and $k'$ is much smaller than the corresponding Josephson energy, and hence the effect of electrostatic coupling energy can be ignored. The interaction Hamiltonian can be written as

$$H_{k\k'} = -E_{Jk} \sigma_k^x \sigma_{k'}^x,$$  

where the tunable Josephson coupling is $E_{Jk} = E_{gk}' \cos (\Phi_{k'} / \phi_0) / 2$.

To implement nonlocal operations, different logical qubits are coupled by a variable electrostatic transformer $C_m$, see Fig. 2(b). $C_m$ is one part of $C_{ms}$, $E_m$ and $E_c$ are the charging and Josephson coupling energies, and $V_c$ is the voltage. If we respectively denote the two logical qubits as $a_d$ and $b_d$ the corresponding four physical qubits as $a_1, a_2, b_1, b_2$, the interaction Hamiltonian reads

$$H_{ab} = \frac{\Delta_m}{2} \cos(2\pi \phi_0) \sigma_{a_1}^x \sigma_{a_2}^x \sigma_{b_1}^x \sigma_{b_2}^x,$$
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**Acknowledgments**
This work was supported by the National Natural Science Foundation of China under Grant Nos. 11175094 and 91221205, the National Basic Research Program of China under Grant Nos. 2009CB929402 and 2011CB921602, and the China Postdoctoral Science Foundation under Grant No. 2013M530595. G.L.L. also thanks the support of Center of Atomic and Molecular Nanoscience of Tsinghua University.

**Author contributions**
G.-F.X. and G.-L.L. contributed to the paper equally.

**Additional information**
Competing financial interests: The authors declare no competing financial interests.

*How to cite this article:* Xu, G. & Long, G. Universal Nonadiabatic Geometric Gates in Two-Qubit Decoherence-Free Subspaces. *Sci. Rep.* 4, 6814; DOI:10.1038/srep06814 (2014).

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