Near-BPS-Saturated Rotating Electrically Charged Black Holes as String States

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Abstract

We construct generating solutions for general $D$-dimensional ($4 \leq D \leq 9$) rotating, electrically charged, black holes in the effective action of toroidally compactified heterotic (or Type IIA) string. The generating solution is parameterized by the ADM mass, two electric charges and $\left\lfloor \frac{D-1}{2} \right\rfloor$ angular momenta (as well as the asymptotic values of one toroidal modulus and the dilaton field). For $D \geq 6$, those are generating solutions for general black holes in toroidally compactified heterotic (or type IIA) string. Since in the BPS-limit (extreme limit) these solutions have singular horizons or naked singularities, we address the near extreme solutions with all the angular momenta small enough. In this limit, the thermodynamic entropy can be cast in a suggestive form, which has a qualitative interpretation as microscopic entropy of (near)-BPS-saturated charged string states of toroidally compactified heterotic string, whose target-space angular momenta are identified as $\left\lfloor \frac{D-1}{2} \right\rfloor U(1)$ left-moving world-sheet currents.

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I. INTRODUCTION

Recently, black holes in string theory have become a subject of intense research since it has now become possible to address their quantum properties, and in particular the microscopic origin of the entropy for certain classes of black holes. In particular, for certain five-dimensional [four-dimensional] BPS-saturated static and rotating black holes, which can be identified with particular D-brane configurations, their microscopic entropy can be calculated by applying the “D-brane technology” [12]. For a recent review of recent developments in the black hole physics in string theory see Ref. [13]. For some related approaches either from the point of view of M-theory and other pure D-brane configurations see [14–17].

An earlier complementary approach to calculate the microscopic entropy of four-dimensional BPS-saturated black holes was initiated in Ref. [21] and was further elaborated on in Refs. [18,19] and [20]. These approaches identify the microscopic black hole degrees of freedom as quantum hair [21] associated with particular small scale (marginal) perturbations of string theory, which does not change the large scale properties of the black hole solutions. On the one hand, the magnetic charges of such dyonic black hole solutions ensure that the classical solutions have regular horizons (and thus corrections are under control), while on the other hand they effectively renormalize the string tension of the underlying string theory. A large part of these ideas was build on an earlier proposal of Sen [24] (and further explored in Refs. [25–27]) to identify the area of BPS-saturated electrically charged spherically symmetric solutions, evaluated at the stretched horizon [28], as the microscopic entropy associated with the elementary BPS-saturated states of string theory. Since the stretched horizon is determined up to $\mathcal{O}(\alpha'^{1/2})$, the identification of the two quantities agrees up to $\mathcal{O}(1)$, only.

In this paper we would like to amplify the idea [24] to identify the near-BPS-saturated electrically charged black holes in string theory with elementary string excitations, by addressing such near-BPS-saturated rotating black holes. In particular, we would like to propose to trade the evaluation of the area of the BPS-saturated rotating solution at the stretched horizon for the evaluation of the area of the horizon of the near-BPS-saturated rotating black hole and equate the latter quantity with the microscopic entropy of the (near)-BPS-saturated charged string states with non-zero angular momenta. In this proposal, the role of the distance between the stretched and the (singular) horizon of the BPS-saturated states is traded for the non-extremality parameter, which parameterizes a deviation of the near-BPS-saturated solution from the BPS-saturated one.

With that in mind we construct the generating solution for the general rotating electrically charged solution of toroidally compactified heterotic (or Type IIA) string in $D$-dimensions ($4 \leq D \leq 9$). The generating solution is parameterized by the ADM mass $M_{BH}$ (or equivalently, by a non-extremality parameter $m$, the mass of the corresponding Kerr solution), two electric charges $Q_1^{(1),(2)}$ (associated with the Kaluza-Klein and the two-form $U(1)$ gauge fields of one compactified direction) and $\left[ \frac{D-1}{2} \right]$ angular momenta $J_i$, $i = 1, \ldots, \left[ \frac{D-1}{2} \right]$ (as well as the asymptotic values of one toroidal modulus $G_{11\infty}$ and the dilaton field $\varphi_{\infty}$). For $D \geq 6$, the general rotating black hole solutions can have only electric charges, and thus for $D \geq 6$ these solutions are the generating solutions for general black holes of toroidally compactified heterotic string theory.
compactified heterotic (or type IIA) string.

This construction thus adds to a completion of a program to obtain the general rotating black hole solutions of toroidally compactified heterotic (or Type IIA) string in $4 \leq D \leq 9$. The generating solution for the general rotating five-dimensional solutions (parameterized by the ADM mass, two angular momenta and three charges) were given in \cite{29,30}, while the four-dimensional rotating generating solution is missing one more charge parameter, i.e., the generating rotating solution obtained up to now is in terms the ADM mass, the angular momentum and four (two electric and two magnetic) charge parameters \cite{30}. Only. In the BPS-limit (extreme limit), these solutions have at most singular horizons, and thus, they are not black holes in the conventional sense. Consequently, for $D \geq 6$ there are no BPS-saturated solutions with regular horizons. Similar conclusions for spherically symmetric, static, BPS-saturated solutions were obtained in \cite{16,31} by considering a general sigma model with $O(D-1)$ spherical symmetry.

In the analysis of the classical solutions, we shall concentrate on near-BPS-saturated rotating solutions with the macroscopic values of the charges and angular momenta. For that purpose, $Q_1^{(1),(2)} \gg m = O(1)$ (measured in units of $\alpha'$). In addition, the inequalities $Q_1^{(1)}Q_1^{(2)} \gg J_1^{2^{[\frac{D-1}{2}]}} \geq \sqrt{Q_1^{(1)}Q_1^{(2)}}$ ensure that on the one hand the solution has regular horizons, and on the other hand that the contribution of the angular momenta to the entropy is still non-negligible macroscopically.

We then calculate the degeneracy of states in the (near)-BPS-saturated string states with the same target space values of the physical parameters. For that purpose the target-space angular momenta of the string states are identified with $\left[\frac{D-1}{2}\right] U(1)$ left-moving world-sheet currents. The logarithm of the degeneracy of such states is in qualitative agreement with the thermodynamic entropy of the near-BPS-saturated black holes, thus lending a support to the identification of the microscopic degrees of freedom of such black holes with the elementary string excitations.

The paper is organized in the following way. In Section II, the explicit form of the generating solution for the general $D$-dimensional rotating electrically charged black hole solution of toroidally compactified string is given. We address the singularity structure and write down the thermodynamic entropy for such near-BPS-saturated black holes. In Section III, we calculate the degeneracy of states of (near)-BPS-saturated string states with the same quantum numbers and obtain a (qualitative) agreement between the microscopic (statistical) and macroscopic (thermodynamic) entropies of these black holes.

**II. $D$-DIMENSIONAL GENERAL ROTATING CHARGED SOLUTION**

We shall present an explicit form of the (generating) solution for the general rotating electrically charged black hole of $D$-dimensional $(4 \leq D \leq 9)$ toroidally compactified heterotic (or type IIA) string. We choose to parameterize the generating solution in terms of massless fields of the heterotic string compactified on a $(10-D)$-torus (or Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of the Type IIA string compactified on a $(10-D)$-torus). A subset of $T$-duality symmetry transformations (which do not affect the $D$-dimensional space-time) allows one to obtain the most general rotating electrically charged solutions in this class.
Note also that the string-string duality relates these solutions to solutions of other string vacua.

**A. Effective Action of Heterotic String on Tori**

We briefly summarize the results of the effective action of toroidally compactified heterotic string in $D$-dimensions, following Refs. [32,33]. This subsection is essentially the same as the one of [33], however, we include it in this paper for the sake of completeness.

The compactification of the extra $(10 - D)$ spatial coordinates on a $(10 - D)$-torus can be achieved by choosing the following Abelian Kaluza-Klein Ansatz for the ten-dimensional metric

$$g_{MN} = \left( e^{a \phi} g_{\mu \nu} + G_{mn} A^{(1) m}_{\mu} A^{(1) n}_{\nu} \right),$$

where $A^{(1) m}_{\mu} (\mu = 0, 1, ..., D - 1; m = 1, ..., 10 - D)$ are $D$-dimensional Kaluza-Klein $U(1)$ gauge fields, $\phi \equiv \Phi - \frac{1}{2} \ln \det G_{mn}$ is the $D$-dimensional dilaton field, and $a \equiv \frac{2}{D - 2}$. Then, the effective action is specified by the following massless bosonic fields: the (Einstein-frame) graviton $g_{\mu \nu}$, the dilaton $e^{\phi}$, $(36 - 2D) U(1)$ gauge fields $A^{I}_{\mu} \equiv (A^{(1) m}_{\mu}, A^{(2) m}_{\mu}, A^{(3) I}_{\mu})$ defined as $A^{(2) m}_{\mu} \equiv B_{mn} A^{(1) n}_{\mu} + \frac{1}{2} i \hat{A}^{I}_{\mu} A^{(3) I}$, $A^{(3) I}_{\mu} \equiv \hat{A}^{I}_{\mu} - \hat{A}^{I}_{m} A^{(1) m}_{\mu}$, and the following symmetric $O(10 - D, 26 - D)$ matrix of the scalar fields (moduli):

$$M = \begin{pmatrix} G^{-1} & -G^{-1} C & -G^{-1} a^T \\ -C^T G^{-1} & G + C^T G^{-1} C + a^T a & C^T G^{-1} a^T + a^T \\ -a G^{-1} C + a & a G^{-1} C + a & I + a G^{-1} a^T \end{pmatrix},$$

where $G \equiv [\hat{G}_{mn}]$, $C \equiv [\frac{1}{2} \hat{A}^{(I)}_{m} \hat{A}^{(I)}_{n} + B_{mn}]$ and $a \equiv [\hat{A}^{I}_{m}]$ are defined in terms of the internal parts of ten-dimensional fields. Then the $D$-dimensional effective action takes the form:

$$L = \frac{1}{16 \pi G_D} \sqrt{-g} \mathcal{R}_g - \frac{1}{D - 2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{8} g^{\mu \nu} \text{Tr}(\partial_\mu ML \partial_\nu ML) - \frac{1}{12} e^{-2a \phi} g^{\mu \nu} g^{\rho \sigma} H_{\mu \nu \rho} H_{\mu \nu \rho} - \frac{1}{4} e^{-a \phi} g^{\mu \nu} g^{\rho \sigma} \mathcal{F}_{\mu \nu} \mathcal{F}^{ij}_{\rho \sigma},$$

where $g \equiv \det g_{\mu \nu}$, $\mathcal{R}_g$ is the Ricci scalar of $g_{\mu \nu}$, and $\mathcal{F}_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the $U(1)^{36-2D}$ gauge field strengths.

We take the units such that $c = \frac{\hbar}{2 \pi} = 1$. The $D$-dimensional gravitational constant $G_D$ is related to the 10-dimensional gravitational constant $G_{10}$ as: $G_{10} = \text{(volume of $T^{10-D}$)} \times G_D$. The canonical choice of the asymptotic values of scalar fields, i.e., $G_{ij = \delta_{ij}}$ (and $\varphi = 0$), corresponds to the compactification on $(10 - D)$-self-dual circles with radius $R = \sqrt{\alpha'}$. In this case, $G_D = G_{10}/(2\pi R)^{10-D} = G_{10}/(2\pi \sqrt{\alpha'})^{10-D}$.

The $D$-dimensional effective action (3) is invariant under the $O(10 - D, 26 - D)$ transformations ($T$-duality) [32,33]:

$$M \rightarrow \Omega M \Omega^T, \quad A_{\mu}^i \rightarrow \Omega_{ij} A_{\mu}^j, \quad g_{\mu \nu} \rightarrow g_{\mu \nu}, \quad \varphi \rightarrow \varphi, \quad B_{\mu \nu} \rightarrow B_{\mu \nu},$$

where $\Omega$ is an $O(10 - D, 26 - D)$ invariant matrix, i.e., with the following property:
\[
\Omega^T L \Omega = L, \quad L = \begin{pmatrix}
0 & I_{10-D} & 0 \\
I_{10-D} & 0 & 0 \\
0 & 0 & I_{26-D}
\end{pmatrix},
\] (5)

where \( I_n \) denotes the \( n \times n \) identity matrix.

At the quantum level, the parameters of \( T \)-duality transformation become integer-valued, corresponding to the exact symmetry of the perturbative string theory.

### B. Explicit Form of the Generating Solution

In order to obtain the explicit form of the general rotating electrically charged solution, we employ the solution generating technique, by performing symmetry transformations on a known (neutral) solution. Within toroidally compactified heterotic string, an approach to obtain the charged solutions from the neutral one was spelled out in Ref. [14]. This method was used to obtain, e.g., general rotating electrically charged solutions in four-dimensions [34], higher dimensional general electrically charged static solutions [25] and rotating solutions with one rotational parameter in \( D \)-dimensions [35], which constitute subsets of solutions obtained in this section.

In particular, we perform two \( SO(1,1) \subset O(11-D,27-D) \) boosts [29] on the \( D \)-dimensional Kerr solution, specified by the mass \( m \) and \( \{D-1\} \) angular momenta \( l_1, \ldots, \{\frac{D-1}{2}\} \). Here \( O(11-D,27-D) \) is a symmetry of the effective \( (D-1) \)-dimensional action for stationary solutions of \( D \)-dimensional heterotic string compactified on a circle (with the self dual radius \( R = \sqrt{\alpha'} \)). The two \( SO(1,1) \) boosts with boost parameters \( \delta_1, \delta_2 \) generate the electric charges \( Q^{(1),(2)} \) of the Kaluza-Klein \( U(1) \) gauge field (associated with the string momentum modes on a circle) \( A^{(1)}_{\mu} \), and the two-form \( U(1) \) gauge field (associated with the string winding modes on a circle) \( A^{(2)}_{\mu_1} \), respectively. The solution obtained in that manner is specified by the ADM mass, two \( U(1) \) charge parameters, and \( \{\frac{D-1}{2}\} \) angular momenta \( J_1, \ldots, \{\frac{D-1}{2}\} \), while the asymptotic values of the scalar fields assume canonical values. Note, however, that one can subsequently rescale the asymptotic values of the dilaton field and the toroidal moduli.

A subset of \( T \)-duality transformations, i.e., \([SO(10-D) \times SO(26-D)]/[SO(9-D) \times SO(25-D)] \subset O(10-D,26-D) \) transformations, which do not affect the \( D \)-dimensional space-time (and the canonical asymptotic values of the scalar fields), provides \( (9-D) + (25-D) \) additional electric charge parameters, which allow for a general rotating electrically charged solution specified by the ADM mass, \( \{\frac{D-1}{2}\} \) angular momenta and \( 36-2D \) electric charges, thus consistent with the no-hair theorem [1]. Thus, we shall present the generating solution for the most general \( D \)-dimensional rotating electrically charged black holes.

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\(^1\) A similar analysis is possible in the case of toroidally compactified Type IIA string.
1. General Rotating Neutral Solution — Kerr Solution

The starting point is the Kerr solutions in $D$-dimensions. Following Ref. [30], we parameterize it as:

$$
 ds^2 = -\frac{(\Delta - 2N)}{\Delta} dt^2 + \frac{\Delta}{\prod_{i=1}^{[D/2]} (r^2 + l_i^2) - 2N} dr^2 + (r^2 + l_i^2 \cos^2 \theta + K_i \sin^2 \theta) d\theta^2 \\
+ (r^2 + l_i^2 \cos^2 \psi_i + K_{i+1} \sin^2 \psi_i) \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} d\psi_i^2 \\
- 2(\Delta - K_{i+1}) \cos \theta \sin \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i d\theta d\psi_i \\
- 2 \sum_{i<j} (\Delta - K_j) \cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1} \cos \psi_i \sin \psi_i \cdots \cos^2 \psi_{j-1} \cos \psi_j \sin \psi_j d\psi_i d\psi_j \\
+ \frac{\mu_i^2 ((r^2 + l_i^2) \Delta + 2l_i^2 N) d\phi_i^2}{\Delta} - \frac{4l_i \mu_i^2 N}{\Delta} d\phi_i + \sum_{i<j} \frac{4l_i l_j \mu_i^2 \mu_j^2 N}{\Delta} d\phi_i d\phi_j,
$$

(6)

where for

- Even dimensions:
  $$
  \Delta \equiv \alpha^2 \prod_{i=1}^{[D/2]} (r^2 + l_i^2) + r^2 \sum_{i=1}^{[D/2]} \mu_i^2 (r^2 + l_i^2)(r^2 + l_{i-1}^2)(r^2 + l_{i+1}^2) \cdots (r^2 + l_{D-2}^2), \\
  K_i \equiv l_{i+1}^2 \sin^2 \psi_i + \cdots + l_{D-3}^2 \cos^2 \psi_i \cdots \cos^2 \psi_{D-6} \sin^2 \psi_{D-4}, \quad N = mr,
  $$

and

$$
\mu_1 \equiv \sin \theta, \quad \mu_2 \equiv \cos \theta \sin \psi_1, \quad \cdots, \quad \mu_{D-3} \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-6} \sin \psi_{D-4}, \\
\alpha \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-4}.
$$

(7)

- Odd dimensions:
  $$
  \Delta \equiv r^2 \sum_{i=1}^{[D/2]} l_i^2 (r^2 + l_i^2)(r^2 + l_{i-1}^2)(r^2 + l_{i+1}^2) \cdots (r^2 + l_{D-2}^2), \quad N = mr^2, \\
  K_i \equiv l_{i+1}^2 \sin^2 \psi_i + \cdots + l_{D-3}^2 \cos^2 \psi_i \cdots \cos^2 \psi_{D-7} \sin^2 \psi_{D-5} + l_{D-1}^2 \cos^2 \psi_i \cdots \cos^2 \psi_{D-5},
  $$

(9)

and

$$
\mu_1 \equiv \sin \theta, \quad \mu_2 \equiv \cos \theta \sin \psi_1, \quad \cdots, \quad \mu_{D-3} \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-7} \sin \psi_{D-5}, \\
\mu_{D-1} \equiv \cos \theta \cos \psi_1 \cdots \cos \psi_{D-5}.
$$

(10)

Here, the repeated indices are summed over. ($i, j$ in $\psi$ and $\phi$ run from 1 to $[D-3]/2$ and from 1 to $[D-1]/2$, respectively.)

The Kerr solution is thus parameterized by the ADM mass $m$ and $[D-1]/2$ angular momenta $l_1, \ldots, [D-1]/2$. 

6
2. Generating Rotating Charged Solution

The two electric charges \(Q_1^{(1)}\) and \(Q_1^{(2)}\) necessary in parameterizing the generating solution can be induced by imposing two \(SO(1, 1) \subset O(11 - \Delta, 27 - \Delta)\) transformations \(\Omega_1\) and \(\Omega_2\) on the Kerr solution \([\mathbb{I}]\). The boost transformations \(\Omega_{1,2}\) have the following forms:

\[
\Omega_1 \equiv \begin{pmatrix}
\cosh \delta_1 & \cdot & \cdot & \cdot & - \sinh \delta_1 \\
\cdot & I_{9-D} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cosh \delta_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & I_{25-D} & \cdot \\
- \sinh \delta_1 & \cdot & \cdot & \cdot & \cosh \delta_1
\end{pmatrix},
\]

\[
\Omega_2 \equiv \begin{pmatrix}
\cosh \delta_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & I_{9-D} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cosh \delta_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & I_{25-D} & \cdot \\
- \sinh \delta_2 & \cdot & \cdot & \cdot & \cosh \delta_2
\end{pmatrix},
\]

where the dot denotes the corresponding zero entry.

The final expressions for the non-trivial \(D\)-dimensional fields take the following forms:

\[
A_i^{(1)} = \frac{N \sinh \delta_1 \cosh \delta_1}{2N \sinh^2 \delta_1 + \Delta}, \quad A_i^{(2)} = \frac{N \sinh \delta_2 \cosh \delta_2}{2N \sinh^2 \delta_2 + \Delta},
\]

\[
A_{\phi_i}^{(1)} = \frac{N l_i \mu_i^2 \sinh \delta_1 \cosh \delta_2}{2N \sinh^2 \delta_1 + \Delta}, \quad A_{\phi_i}^{(2)} = \frac{N l_i \mu_i^2 \sinh \delta_2 \cosh \delta_1}{2N \sinh^2 \delta_2 + \Delta},
\]

\[
e^{2\varphi} = \frac{\Delta^2}{W}, \quad G_{11} = \frac{2N \sinh^2 \delta_1 + \Delta}{2N \sinh^2 \delta_2 + \Delta},
\]

\[
B_{\phi_i \phi_j} = -\frac{2N l_i l_j \mu_i^2 \mu_j^2 \sinh \delta_1 \sinh \delta_2 \cosh \delta_1 \cosh \delta_2 [N(\sinh^2 \delta_1 + \sinh^2 \delta_2) + \Delta]}{W} \times [2N^2 \sinh^2 \delta_1 \sinh^2 \delta_2 + N(\sinh^2 \delta_1 + \sinh^2 \delta_2 - 1) + \Delta^2] / [(\Delta - 2N) W^2],
\]

\[
ds^2 = \Delta \frac{\Delta^2}{W} \left[ -\frac{\Delta - 2N}{W} dt^2 + \frac{dr^2}{\Pi_{i=1}^{(D-1)} (r^2 + l_i^2) - 2N} + \frac{r^2 + l_i^2 \cos^2 \theta + K_1 \sin^2 \theta}{\Delta} d\theta^2 + \frac{\cos^2 \theta \cos^2 \psi_1 \cdots \cos^2 \psi_{i-1}}{(r^2 + l_i^2) \cos \psi_i \sin \psi_i \cdots \cos \psi_j \cos \psi_j \sin \psi_j \sin \psi_j \sin \psi_j} d\psi_i^2 \\
- \frac{2 \sum_{i<j} l_j^2 - K_j}{\Delta} \cos \theta \cos \psi_1 \cdots \cos \psi_{i-1} \cos \psi_i \sin \psi_i \cdots \cos \psi_j \cos \psi_j \sin \psi_j \sin \psi_j \sin \psi_j \sin \psi_j \sin \psi_j d\theta^2 d\psi_i \\
- \frac{2 l_i l_{i+1} - K_{i+1}}{\Delta} \cos \theta \sin \theta \cos \psi_1 \cdots \cos \psi_{i-1} \cos \psi_i \sin \psi_i \sin \psi_i \sin \psi_i \sin \psi_i \sin \psi_i \sin \psi_i d\theta^2 d\psi_i \\
+ \frac{\mu_i^2}{\Delta W} (r^2 + l_i^2) \Delta^2 + 2N l_i \mu_i^2 + 4N^2 \sinh^2 \delta_1 \sinh^2 \delta_2 \{r^2 + l_i^2 (1 - \mu_i^2)\} \\
+ 2N(\sinh^2 \delta_1 + \sinh^2 \delta_2) (r^2 + l_i^2) d\phi_i^2 - \frac{4N l_i \mu_i^2 \cosh \delta_1 \cosh \delta_2}{W} dtd\phi_i \\
+ \sum_{i<j} 4N l_i l_j \mu_i^2 \mu_j^2 (\Delta - 2N \sinh^2 \delta_1 \sinh^2 \delta_2) d\phi_i d\phi_j \right],
\]
where

$$W \equiv (2N \sinh^2 \delta_1 + \Delta)(2N \sinh^2 \delta_2 + \Delta)$$  \hspace{1cm} (13)

and the other quantities are defined for even and odd dimensions in (7), (8) and (10), (11), respectively.

The ADM mass, the angular momenta and electric $U(1)$ charges carried by the generating solutions are given by

$$M_{BH} = \frac{\Omega_{D-2} m}{8\pi G_D} [(D - 3)(\cosh^2 \delta_1 + \cosh^2 \delta_2) - (D - 4)],$$

$$J_i = \frac{\Omega_{D-2}}{4\pi G_D} m \gamma_i \cosh \delta_1 \sinh \delta_1,$$

$$Q^{(1)}_1 = \frac{\Omega_{D-2}}{8\pi G_D} (D - 3)m \sinh \delta_1,$$

$$Q^{(2)}_1 = \frac{\Omega_{D-2}}{8\pi G_D} (D - 3)m \sinh \delta_2,$$  \hspace{1cm} (14)

where $\Omega_{D-2} \equiv \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}$ is the area of a unit $(D - 2)$-sphere and $G_D$ is the $D$-dimensional gravitational constant. Recall, that for the canonical choice of the asymptotic value of the internal metric (toroidal moduli) $G_{ij} = \delta_{ij}$, i.e., the compactification is on $(10 - D)$ self-dual circles with radius $R = \sqrt{\alpha'}$, $G_D = G_{10}/(2\pi \sqrt{\alpha'})^{10-D}$. Also, the Kaluza-Klein (momentum mode) $Q_1^{(1)}$ and the two-form field (winding mode) $Q_1^{(2)}$ charges are quantized in units $p/\sqrt{\alpha'}$ and $q/\sqrt{\alpha'}$, respectively. Here $(p, q) \in \mathbb{Z}$.

The surface area of the generating solution is given by

$$A_D = 2mr_H \Omega_{D-2} \cosh \delta_1 \cosh \delta_2,$$  \hspace{1cm} (15)

where $r_H$ is the outer horizon determined by

$$\left[ \prod_{i=1}^{(\frac{D}{2} - 1)} (r^2 + l^2_i) - 2N \right]_{r=r_H} = 0.$$  \hspace{1cm} (16)

Again $N$ is defined in (7) and (10) for even and odd dimensions $D$, respectively.

The above generating solution has a ring-like singularity at $(r, \theta) = (0, \frac{\pi}{2})$, and therefore the space-time structure for the case $m > 0$ is that of the Kerr solution.

With all the angular momenta turned on, the explicit form of the thermodynamic entropy, i.e., $S_{thermo} \equiv A_D/(4G_D)$, (as well as temperature) can be expressed in a compact form only for four- and five-dimensional black holes [30] and it becomes progressively complicated as the dimensionality increases. In the case of only one rotational parameter non-zero, such explicit expressions for the entropy and the temperature was obtained in Ref. [9].

\textsuperscript{2}We use the conventions of Ref. [36], however, we keep in mind that the matter Lagrangian in [3] contains $1/(16\pi G_D)$ prefactor.
C. Thermodynamic Entropy of Near-BPS-Saturated Solutions

The BPS saturated solutions are those whose ADM masses satisfy BPS mass formula, and are of special interest. Such a limit is achieved by letting \( m \to 0 \) and \( \delta_i \to \infty \) while keeping \( me^{2\delta_i} (i = 1, 2) \) as finite constants. However, in general with all the angular momenta \( J_{1,\ldots,[D-1]} \) non-zero the BPS-saturated solution has a naked singularity, except in the case when at most only one angular momentum is turned on (for \( D \geq 6 \)). The latter property was observed in Ref. \[35\]. However, even in this special case, the solution has a singular horizon, \( i.e. \), a null singularity.

Therefore, the study of the thermodynamic properties of the BPS-saturated rotating black holes is faced with difficulties, since the thermodynamic quantities, such as the entropy and the temperature are defined at the “regular” horizons of black holes. Alternatively, one may want to evaluate such quantities at the “stretched” horizons as proposed by Sen \[24\] for the static electrically charged BPS-saturated black holes. In this case the stretched horizon can be chosen to be by \( O(\sqrt{\alpha'}) \) distance away from the singular horizon and the thermodynamic entropy is in qualitative agreement (up to a factor of \( O(1) \)) \[24,25\] with the statistical entropy, \( i.e. \), the logarithm of the degeneracy of BPS-saturated elementary string states with the same quantum numbers. On the other hand, the area of the rotating BPS-saturated black holes, evaluated at the stretched horizon (whose value is chosen to be independent of the angular momenta and electric charges), turns out to be independent of the angular momenta. This result is therefore not in accordance with the expectations that it should depend on the angular momenta, in order to be at least in qualitative agreement with the logarithm of the degeneracy of the corresponding rotating BPS-saturated string states (see the subsequent Section). It may turn out that in order to obtain such an agreement, the stretched horizon of rotating BPS-saturated black holes should be chosen to depend on the physical parameters \( 3 \).

Here we would like to propose to consider instead the near-BPS-saturated black holes, \( i.e. \), those with non-extremality parameter \( m \) small compared with the electric charges, and which have regular horizons. In this case, the role of the stretched horizon of the BPS-saturated states is traded for the non-extremality parameter \( m \) of the near-BPS-saturated states.

The near-BPS-saturated black holes have regular horizons at \( r_H \) (defined by (16)), provided that the angular momentum parameters \( l_{1,\ldots,[D-1]} \) have the magnitude which is smaller than that of \( m \). More precisely, one has to take \( m \) much smaller compared to \( e^{\delta_i} \), such that (when measured in units of \( \alpha' \)) \( Q_1^{(1),(2)} \gg m = O(1) \). In addition, the angular momenta are kept small compared to charges, so that (when charges are measured in units of \( \alpha' \)) \( Q_1^{(1)}Q_2^{(2)} \gg J_{1,\ldots,[D-1]}^2 \gg \sqrt{Q_1^{(1)}Q_2^{(2)}} \). The first inequality ensures the regular horizon, while the second inequality ensures that the contribution from the angular momenta to the entropy is still non-negligible macroscopically.

In such a near-BPS-saturated limit, the thermodynamic entropy \( S_{themro} = \frac{A_D}{4G_D} \) can be

\[ \text{We thank A. Sen for a discussion on this point.} \]
cast in a suggestive form:

\[
S_{\text{thermo}} = 2\pi \left[ \frac{4}{(D-3)^2} Q_1^{(1)} Q_1^{(2)} (2m)^{\mu-1} - \frac{2}{(D-3)} \sum_{i=1}^{[\frac{D-1}{2}]} J_i^2 \right]^{\frac{1}{2}}. \tag{17}
\]

This thermodynamic entropy coincides with the entropy of the four- and five-dimensional rotating black holes \[30\] in the near-extreme limit and with two electric charges, only.

III. NEAR-BPS-SATURATED ROTATING ELECTRICALLY CHARGED ELEMENTARY STRING STATES

In this section we evaluate the degeneracy of states of the (near)-BPS-saturated string states with the same quantum numbers as the near-BPS-saturated rotating black holes. For the sake of simplicity, we shall address string excitations of toroidally compactified heterotic string \[1\]. The aim is to show that the logarithm of the degeneracy of such string states has the same (qualitative) structure as the thermodynamic entropy \[17\] of the near-BPS-saturated black hole, thus providing an evidence that these string states should serve as microscopic degrees of freedom, specifying the microscopic (statistical) origin of the black hole thermodynamic entropy.

The string states specified in terms of the the Kaluza-Klein (momentum modes) and two-form field (winding modes) charges \(Q_1^{(1)},Q_1^{(2)}\) associated with the NS-NS sector of the heterotic string compactification on a self-dual circle with radius \(R = \sqrt{\alpha'}\) have the mass:

\[
M_{\text{string}}^2 = (Q_1^{(1)} + Q_1^{(2)})^2 + \frac{4}{\alpha'} (N_R - \frac{1}{2}) = (Q_1^{(1)} - Q_1^{(2)})^2 + \frac{4}{\alpha'} (N_L - 1), \tag{18}
\]

where \(N_{L,R}\) are the left- and [right-] moving oscillator numbers. We would like to determine the degeneracy of states with the mass \(M_{\text{string}} = M_{\text{BH}} \[14\]\), and the physical charges \(Q_1^{(1)},Q_1^{(2)} \gg \frac{1}{\sqrt{\alpha'}}\) (quantized in units of \(1/\sqrt{\alpha'}\)) and \([\frac{D-1}{2}]\) angular momenta \(J_1,...,[\frac{D-1}{2}]\) \(\gg 1\) being in the classical regime.

In the region of \(Q_1^{(1)},Q_2 \gg \frac{1}{\sqrt{\alpha'}}\), the number of the left-moving oscillator modes \(N_L\) is large and is related to the number of right-moving oscillator modes \(N_R\) through \[18\] as:

\[
N_L = \alpha' Q_1 Q_2 + N_R + \frac{1}{2}, \tag{19}
\]

For the BPS-saturated states \(N_R = \frac{1}{2}\), while for the near-BPS-saturated state \(N_L \gg N_R > \frac{1}{2}\) and the relationship \[13\] becomes of the form:

\[
N_L = \alpha' Q_1 Q_2 + O(N_R) \gg N_R, \tag{20}
\]

\[4\]A related analysis should be possible in the case of elementary string excitations of Type IIA string.
and is thus, in the leading order, the same as that for the BPS-saturated states.

The leading order contribution to the logarithm of degeneracy $d_{N_L}$ of the (near)-BPS-saturated states at the mass level $M_{\text{string}}$ can then be written in the form:

$$\log d_{N_L} \sim 2\pi \sqrt{\frac{1}{6} c_{\text{eff}} N_L} = 4\pi \sqrt{N_L},$$

where $c_{\text{eff}} = 26 - 2 = 24$. However, this degeneracy of states includes all the ((near)-BPS-saturated) states, i.e., those with zero and non-zero spins, at a particular mass level $M_{\text{string}}$.

In order to derive the degeneracy of states at the mass level $M_{\text{string}}$ and with the particular values of spins $J_1, \ldots, [\frac{D-1}{2}]$, we employ the conformal field theory technique for describing such string states. Note that the target space quantum numbers of such states are described by $[\frac{D-1}{2}]$ angular momenta $J_i$, which correspond to the eigenvalues of the $[\frac{D-1}{2}] U(1)$ generators of the Cartan sub-algebra of the $O(D - 1)$ rotational symmetry.

In the (near)-BPS-saturated limit, these target space quantum numbers have a map onto the left-moving world-sheet current algebra, specified by $U(1)^{[\frac{D-1}{2}]}$ left-moving world-sheet currents $j_i = i \partial_s H^i$ ($i = 1, \ldots, [\frac{D-1}{2}]$). Consequently, the string states are eigenstates of the world-sheet currents whose conformal fields can be written in the form:

$$\Phi_{J_1, \ldots, [\frac{D-1}{2}]} = \prod_{i=1}^{[\frac{D-1}{2}]} e^{j_i H^i} \Phi_0,$$

with the conformal field $\Phi_0$ specifying the current-algebra independent part, thus also specifying states without target space spins. The (left-moving) conformal dimensions of the two types of conformal fields $\Phi_{J_1, \ldots, [\frac{D-1}{2}]}$ and $\Phi_0$ are related in the following way:

$$\bar{h}_{\Phi_{J_1, \ldots, [\frac{D-1}{2}]}} = \frac{1}{2} \sum_{i=1}^{[\frac{D-1}{2}]} J_i^2 + \bar{h}_{\Phi_0},$$

which in turn implies that the degeneracy of states with non-zero spins is reduced (in comparison with the degeneracy of states with all spins included) precisely by the amount:

$$N_L \rightarrow \tilde{N}_L = N_L - 1 \left\lfloor \frac{D-1}{2} \right\rfloor \sum_{i=1}^{[\frac{D-1}{2}]} J_i^2.$$

5In principle, one should be able to determine the degeneracy of states by explicitly evaluating the partition function with spins included, as initiated in Ref. [37].

6Since the right-moving oscillator modes is $N_R = O(1)$ in the (near)-BPS-saturated limits, the string states have negligible (if any) right-moving spins.

7Note that similar argument (within the $D$-brane world-sheet current algebra) was used in [3] to determine the microscopic entropy of rotating five-dimensional BPS saturated black holes.
which in turn specifies the degeneracy $d_{\tilde{N}_L}$ of the (near)-BPS-saturated states at specific mass level $M_{\text{string}}$ (18) and non-zero spins as:

$$S_{\text{stat}} \equiv \log d_{\tilde{N}_L} \sim 4\pi \sqrt{\tilde{N}_L} = 4\pi \left( N_L - \frac{1}{2} \sum_{i=1}^{\lfloor \frac{D_{\text{max}}}{2} \rfloor} J^2_i \right)^{\frac{1}{2}} . \quad (25)$$

In order to ensure the statistical nature of the entropy we have to maintain $N_L \gg \sum_{i=1}^{\lfloor \frac{D_{\text{max}}}{2} \rfloor} J^2_i$, while still allowing for the statistically significant contribution from spins, i.e.,

$$\sqrt{N_L \sum_{i=1}^{\lfloor \frac{D_{\text{max}}}{2} \rfloor} J^2_i} \gg 1 .$$

When $N_L$ is expressed in terms of charges (20), the statistical entropy assumes the form:

$$S_{\text{stat}} = 2\pi \left( 4\alpha'Q_1Q_2 - 2 \sum_{i=1}^{\lfloor \frac{D_{\text{max}}}{2} \rfloor} J^2_i \right)^{\frac{1}{2}} . \quad (26)$$

There is qualitative agreement between the thermodynamic entropy (17) and the statistical entropy (26). Note however, that since the thermodynamic entropy explicitly depends on the non-extremality parameter $m$, which can be determined only up to $O(1)$ (in units of $\alpha'$), the agreement between the two entropies can only be qualitative.

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