\( \eta \)-Ricci solitons on contact pseudo-metric manifolds

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Abstract

In this paper, we prove that a Sasakian pseudo-metric manifold which admits an \( \eta \)-Ricci soliton is an \( \eta \)-Einstein manifold, and if the potential vector field of the \( \eta \)-Ricci soliton is not a Killing vector field then the manifold is \( D \)-homothetically fixed, and the vector field leaves the structure tensor field invariant. Next, we prove that a \( K \)-contact pseudo-metric manifold with a gradient \( \eta \)-Ricci soliton metric is \( \eta \)-Einstein. Moreover, we study contact pseudo-metric manifolds admitting an \( \eta \)-Ricci soliton with a potential vector field point wise colinear with the Reeb vector field. Finally, we study gradient \( \eta \)-Ricci solitons on \((\kappa, \mu)\)-contact pseudo-metric manifolds.

1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric, which was introduced by Hamilton [17] as the fixed point of the Hamilton’s Ricci flow \( \frac{\partial}{\partial t} g = -2 \text{Ric} \). The Ricci flow is a nonlinear diffusion equation analogue of the heat equation for metrics. A Ricci soliton \((g, V, \lambda)\) on the pseudo-Riemannian manifold \((M, g)\) is defined by the following equation

\[
\mathcal{L}_V g + 2 \text{Ric} + 2\lambda g = 0,
\]

where \( \mathcal{L}_V \) is the Lie derivative along the potential vector field \( V \), and \( \lambda \) is a constant real number. The Ricci soliton is called shrinking, steady, expanding if \( \lambda < 0 \), \( \lambda = 0 \) and \( \lambda > 0 \), respectively. If \( V = Df \), where \( Df \) is the gradient of the smooth function \( f \), then the Ricci soliton is called a gradient Ricci soliton. Ricci solitons have been studied in many different contexts (see [2,3,10,11,14,16,22,23]). Also, they are interests of physicists because of their relations to string theory [1,20], and physicists refer to Ricci solitons as quasi-Einstein metrics [13].

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The $\eta$–Ricci soliton notion, as a generalization of a Ricci soliton, was introduced by Cho and Kimura [9]. An $\eta$–Ricci soliton on a manifold $M$ is a tuple $(g, V, \lambda, \mu)$, where $g$ is a pseudo-Riemannian metric, $V$ is the potential vector field, and $\lambda, \mu$ are constant real numbers satisfying

$$\mathcal{L}Vg + 2\text{Ric} + 2\lambda g + 2\mu \eta \otimes \eta = 0,$$

where $\eta$ is a 1–form on $M$. Moreover, if $V = Df$, the $\eta$–Ricci soliton is called a gradient $\eta$–Ricci soliton and Eq.1 becomes

$$\text{Hess} f + \text{Ric} + \lambda g + \mu \eta \otimes \eta = 0.$$  

The $\eta$–Ricci solitons have been studied in many different settings, Blaga studied $\eta$–Ricci solitons on para-Kenmotsu [4] and Lorentzian para-Sasakian manifolds [5]. Devaraja and Venkatesha studied $\eta$–Ricci solitons on para-Sasakian manifolds [23], etc.

Contact geometry is an odd-dimensional analogue of the symplectic geometry and has been studied in many different contexts (particularly) those related to physics. It has been used as a proper framework for classical thermodynamics [6,27], and as a geometrical approach to magnetic field [7]. Also, it was studied in relation with the Yang-Mills theory [19], quantum mechanics [18], gravitational waves [21], etc. Studying contact structures with pseudo-Riemannian metrics was started by Takahashi in [26], but he just studied the Sasakian case. Recently, Calvaruso and Perrone [8] have studied a contact pseudo-metric manifold in the general case. Ghafrarzadeh and second author studied nullity conditions on the contact pseudo-metric manifolds and have introduced the “$(\kappa, \mu)$–contact pseudo-metric manifold” notion [15]. The relevance for the general relativity of contact pseudo-metric manifolds was studied in [12]. All of these applications have motivated us to study $\eta$–Ricci solitons in the contact pseudo-Riemannian settings.

The present paper has been organized as follows. In Section 2, we recalled the contact pseudo-metric manifold notion and proved some lemmas that are used in the next sections. In Section 3, we studied $\eta$–Ricci solitons on Sasakian pseudo-metric manifolds and showed that a Sasakian pseudo-metric manifold, which admits an $\eta$–Ricci soliton, is an $\eta$–Einstein manifold and if the potential vector field of the $\eta$–Ricci soliton is not a Killing vector field, then the manifold is $\mathcal{D}$–homothetically fixed, and presented an example for it. Moreover, we showed a $K$–contact pseudo-metric manifold which admits a gradient $\eta$–Ricci soliton is an $\eta$–Einstein manifold. Also, we studied an $\eta$–Ricci soliton that has a potential vector field colinear to the Reeb vector field on a contact pseudo-metric manifold and showed that the manifold is $K$–contact. In the last section, we studied gradient $\eta$–Ricci solitons on a $(\kappa, \mu)$–contact pseudo-metric manifold and obtained some conditions on the curvature tensor of the manifold.
2 Preliminaries

In this section, we recall some definitions and results needed in the rest of the paper.

A \((2n + 1)\)-dimensional manifold \(M\) is called an almost contact pseudo-metric manifold, if there exists an almost contact pseudo-metric structure \((\varphi, \xi, \eta, g)\) on \(M\), where \(\varphi, \xi, \eta, g\) are a \((1, 1)\)-tensor field, a vector field, a 1-form and a compatible pseudo-Riemannian metric, respectively, which satisfy the following equations

\[
\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad (3)
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad (4)
\]

where \(\epsilon = \pm 1\), and \(X, Y\) are arbitrary vector fields. Using the above equations, we have

\[
\varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad (5)
\]

\[
\eta(X) = \epsilon g(\xi, X), \quad g(\varphi X, Y) = -g(X, \varphi Y)
\]

and especially \(g(\xi, \xi) = \epsilon\). Notice that the signature of the metric \(g\) is \((2p + 1, 2n - 2p)\) if \(\xi\) is a spacelike vector field \((g(\xi, \xi) > 0)\), and is \((2p, 2n - 2p + 1)\) if \(\xi\) is a timelike vector field \((g(\xi, \xi) < 0)\).

The fundamental 2-form \(\Phi\) of an almost contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) is defined as \(\Phi(X, Y) = g(X, \varphi Y)\), where \(X, Y \in \Gamma(M)\). If

\[
g(X, \varphi Y) = (\text{d}\eta)(X, Y),
\]

then \(\eta\) is a contact form, \((\varphi, \xi, \eta, g)\) is a contact pseudo-metric structure and \(M\) is called a contact pseudo-metric manifold.

Throughout this paper, we use \(R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}\), where \(X, Y \in \Gamma(M)\), as the Riemannian curvature tensor definition. In a contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) the \((1, 1)\)-tensor field \(\ell\) and \(h\) are defined by

\[
\ell X = R(X, \xi)\xi, \quad hX = \frac{1}{2}(\ell \varphi)X.
\]

Also, notice the \(\ell\) and \(h\) are self-adjoint operators. In the contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\), we have the following equations [8][24]

\[
\text{trace}(h) = \text{trace}(h\varphi) = 0, \quad (6)
\]

\[
\eta \circ h = 0, \quad \ell \xi = 0, \quad (7)
\]

\[
h\varphi = -\varphi h, \quad h\xi = 0, \quad (8)
\]

\[
\nabla_\xi \varphi = 0, \quad (9)
\]

\[
\nabla_X \xi = -\epsilon \varphi X - \varphi hX, \quad (10)
\]

\[
\text{Ric}(\xi, \xi) = 2n - trh^2, \quad (11)
\]

where \(X\) is an arbitrary vector field.
A contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) is a \(K\)−contact pseudo-metric manifold if \(\xi\) is a Killing vector field or equivalently \(h = 0\). So, we have the following equations

\[
Q\xi = 2n\epsilon\xi,
\]
\[
\nabla_X\xi = -\epsilon\varphi X,
\]
where \(Q\) is the Ricci operator of the metric \(g\) and \(X \in \Gamma(M)\).

**Lemma 2.1.** Let \((M, \varphi, \xi, \eta, g)\) be a \((2n + 1)\)−dimensional \(K\)−contact pseudo-metric manifold, then

\[
(\nabla_X Q)\xi = -2n\epsilon\varphi X + \epsilon Q\varphi X,
\]
\[
(\nabla_\xi Q)X = \epsilon(Q\varphi - \varphi Q)X,
\]
where \(X\) is an arbitrary vector field.

**Proof.** First, differentiating 12 along an arbitrary vector field \(X\) and using 13, we obtain 14. Next Lie differentiating Ric, along \(\xi\), we find

\[
(\mathcal{L}_\xi \text{Ric})(X, Y) = g((\nabla_\xi Q)X + Q(\nabla_X\xi), Y) + g(QX, \nabla_Y\xi),
\]
where \(X, Y \in \Gamma(M)\). Because \(\xi\) is a Killing vector field, so \(\mathcal{L}_\xi \text{Ric} = 0\), using this and 13 in the above equation give 15, and it completes the proof.

An almost contact pseudo-metric structure \((\varphi, \xi, \eta, g)\) is called normal if

\[
[\varphi, \varphi] + 2d\eta \otimes \xi = 0.
\]
A normal contact pseudo-metric manifold is a Sasakian pseudo-metric manifold. A Sasakian pseudo-metric manifold is a \(K\)−contact pseudo-metric manifold, satisfying

\[
(\nabla_\varphi Y) = g(X, Y)\xi - \epsilon\eta(Y)X,
\]
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]
where \(X, Y \in \Gamma(M)\).

**Lemma 2.2.** Let \((M, \varphi, \xi, \eta, g)\) be a Sasakian pseudo-metric manifold then \(Q\varphi = \varphi Q\).

**Proof.** First, calculating the curvature tensor by 16 we have

\[
R(X, Y, \varphi Z, W) + R(X, Y, Z, \varphi W) =
\]
\[
\epsilon g(Z, \varphi Y)g(X, W) - \epsilon g(Z, \varphi X)g(Y, W)
\]
\[
- \epsilon g(X, Z)g(\varphi Y, W) + \epsilon g(Y, Z)g(\varphi X, W),
\]
where \(X, Y, Z, W \in \Gamma(M)\) and \(R(X, Y, Z, W) = g(R(X, Y, Z), W)\). Now, let \(X, Y, Z, W\) be orthogonal to \(\xi\), then using the above equation, we have

\[
R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W),
\]
and this gives

\[
\text{Ric}(X, \varphi Y) + \text{Ric}(\varphi X, Y) = 0,
\]
where \(X, Y\) are orthogonal vector fields. Using the last equation, we obtain \(Q\varphi = \varphi Q\), completing the proof.
A contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) is called an \(\eta\)--Einstein manifold if the Ricci curvature is of the form \(\text{Ric} = ag + b\eta \otimes \eta\), where \(a, b\) are smooth functions on the manifold \(M\). If the manifold \(M\) is a \(K\)--contact pseudo-metric manifold with dimension greater than three, then \(a, b\) are constants.

Let \((M, \varphi, \xi, \eta, g)\) be a contact pseudo-metric manifold, for any constant real number \(t \neq 0\), is defined a contact pseudo-metric manifold \((\tilde{M}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\), where \(\tilde{\eta} = t\eta, \tilde{\xi} = \frac{1}{t}\xi\), \(\tilde{\varphi} = \varphi\) and \(\tilde{g} = tg + \epsilon(t - 1)\eta \otimes \eta\). This transition is called a \(D\)--homothetic deformation and it preserves some basic properties such as being \(K\)--contact and, in particular being Sasakian. A \(D\)--homothetic deformation of an \(\eta\)--Einstein \(K\)--contact pseudo-metric manifold with \(\text{Ric} = ag + b\eta \otimes \eta\) is an \(\eta\)--Ricci \(K\)--contact pseudo-metric manifold such that \(\tilde{\text{Ric}} = (a - 2\epsilon + 2\epsilon^2)\tilde{g} + (2n - \tilde{a})\tilde{\eta} \otimes \tilde{\eta}\), notice that when \(a = -2\epsilon\) then the Ricci tensor form is not changed. Thus, we have the following definition.

**Definition 1.** An \(\eta\)--Einstein \(K\)--contact pseudo-metric manifold with \(a = -2\epsilon\) is said to be \(D\)--homothetically fixed.

### 3 \(\eta\)--Ricci solitons on Sasakian pseudo-metric manifolds

In this section, we have studied \(\eta\)--Ricci solitons on Sasakian pseudo-metric manifolds.

**Theorem 3.1.** Let \((M, \varphi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional Sasakian pseudo-metric manifold. If \((g, V, \lambda, \mu)\) be an \(\eta\)--Ricci soliton on the manifold \(M\), then \(M\) is an \(\eta\)--Einstein manifold and

\[
\text{Ric} = (\frac{\mu \epsilon - \lambda}{2})g + \left(\frac{n}{2}(\epsilon + 1) + \frac{\lambda}{4}(\epsilon + 1) + \frac{(\epsilon - 3)}{4}\mu\right)\eta \otimes \eta, \tag{18}
\]

\[
r = \frac{1}{4}(\lambda - \mu + 8n^2 + (4\mu + 6)n) + \frac{1}{4}(-\lambda - \mu - 4\lambda n + 2n), \tag{19}
\]

where \(\text{Ric}\) and \(r\) are the Ricci tensor and the scalar curvature of the metric \(g\), respectively.

**Proof.** Using \([1]\) in the following formula \([25\text{ p. 23}]\)

\[
(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V, X]} g)(Y, Z) = -g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y),
\]

where \(X, Y\) and \(Z\) are arbitrary vector fields, we find

\[
g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z (\text{Ric} + \mu\eta \otimes \eta))(X, Y)
\]

\[
-(\nabla_X (\text{Ric} + \mu\eta \otimes \eta))(Y, Z)
\]

\[
-(\nabla_Y (\text{Ric} + \mu\eta \otimes \eta))(Z, X), \quad \forall X, Y, Z \in \Gamma(M).
\]

Using lemma \([2.1]\) and lemma \([2.2]\) we obtain \(\nabla_{\xi} Q = 0\). Substituting \(\xi\) for \(Y\) in \([20]\) using the foregoing equation and lemma \([2.1]\) give

\[
(\mathcal{L}_V \nabla)(X, \xi) = (4n + 2\mu)\varphi X - 2\epsilon Q\varphi X, \quad \forall X \in \Gamma(M). \tag{21}
\]
Differentiating \([21]\) along an arbitrary vector field \(Y\) and using \([10]\) yield
\[
(\nabla_Y \nabla)(X, \xi) - \epsilon(\nabla_Y \nabla)(X, \varphi Y) = 2\mu\eta(X, Y)\xi - (4n + 2\mu)\epsilon\eta(X)Y - 2\epsilon(\nabla_Y Q)(\varphi X) + 2\eta(X)QY, \quad \forall X, Y \in \Gamma(M).
\] (22)

Using \([22]\) in the following commutative formula \([28]\)
\[
(\nabla_V R)(X, Y)Z = (\nabla_X \nabla)(Y, Z) - (\nabla_Y \nabla)(X, Z),
\] (23)
where \(X, Y, Z\) are arbitrary vector fields, we find:
\[
(\nabla_V R)(X, Y)Z = 2\epsilon(\nabla_Y Q)(\varphi X) - 2\epsilon(\nabla_X Q)(\varphi Y) + 2\eta(Y)QX - 2\eta(X)QY, \quad \forall X, Y, Z \in \Gamma(M).
\] (24)

Substituting \(\xi\) for \(Y\) in \([24]\) and using \([21]\) we obtain
\[
(\nabla_V R)(X, \xi)\xi = 4QX - 4\epsilon(2n + \mu)X + 4\mu\epsilon(X)\xi, \quad \forall X \in \Gamma(M).
\] (25)

Using \([1]\) we have
\[
(\nabla_V g)(X, \xi) + (4n + 2\lambda\epsilon + 2\mu)\eta(X) = 0, \quad \forall X \in \Gamma(M),
\]
and this equation yields
\[
\epsilon(\nabla_V \eta)(X) - g(X, \nabla_V \xi) + 2(2n + \lambda\epsilon + \mu)\eta(X) = 0,
\] (26)
\[
\eta(\nabla_V \xi) = (2n\epsilon + \mu\epsilon + \lambda),
\] (27)
where \(X\) is an arbitrary vector field. Next Lie-differentiating the formula \(R(X, \xi)\xi = X - \eta(X)\xi\) along the vector field \(V\) and using \([20]\) \([21]\) and \([17]\) we have
\[
QX = \frac{(\epsilon - 1)}{4}((\nabla_V \eta)X)\xi + (n\epsilon + \frac{\mu\epsilon - \lambda}{2})X + (n + \frac{\lambda\epsilon + (1 - 2\epsilon)\mu}{2})\eta(X)\xi,
\] (28)
where \(X \in \Gamma(M)\). Now using the foregoing equation and symmetry of the Ricci tensor, we deduce
\[
\frac{\epsilon(\epsilon - 1)}{4}(\nabla_V \eta)X\eta(Y) = \frac{\epsilon(\epsilon - 1)}{4}(\nabla_V \eta)Y\eta(X), \quad \forall X, Y \in \Gamma(X).
\]

Using the above equation and \([27]\) we find \([18]\) and in turn it yields \([19]\) completing the proof.

Theorem 3.1 imposes strong condition on the potential vector field of an \(\eta\)-Ricci soliton on a Sasakian pseudo-metric manifold. We need the following lemma to further study.
Lemma 3.2. Let \((M, \varphi, \xi, \eta, g)\) be a \((2n+1)\)-dimensional contact pseudo-metric manifold. If \(\text{Ric} = ag + b\eta \otimes \eta\), where \(a, b \in \mathbb{R}\), then
\[
\text{Ric}^i_j \text{Ric}_{ij} + \lambda r + \mu (a + b) = 0,
\]
where \(r\) is the scalar curvature of the metric \(g\).

Proof. Using \([1]\) in the following formula \([28]\)
\[
\mathcal{L}_V \Gamma^h_{ij} = \frac{1}{2} g^{ht} (\nabla_j (\mathcal{L}_V g_{it}) + \nabla_i (\mathcal{L}_V g_{jt}) - \nabla_t (\mathcal{L}_V g_{ij})),
\]
where \(\Gamma^h_{ij}\) are the Christoffel symbols of the metric \(g\), we deduce
\[
\mathcal{L}_V \Gamma^h_{ij} = \nabla_h (\text{Ric}_{ij} + \mu \eta \eta_j) - \nabla_i (\text{Ric}^h_j + \mu \eta^h \eta_j) - \nabla_j (\text{Ric}^h_i + \mu \eta^h \eta_i).
\]
Next using the above equation in the following equation \([28]\)
\[
\mathcal{L}_V R^h_{kji} = \nabla_k (\mathcal{L}_V \Gamma^h_{ij}) - \nabla_j (\mathcal{L}_V \Gamma^h_{ki}).
\]
We obtain
\[
\mathcal{L}_V R^h_{kji} = \nabla_k \nabla^h (\text{Ric}_{ij} + \mu \eta \eta_j) - \nabla_i \nabla_j (\text{Ric}^h_i + \mu \eta^h \eta_j) - \nabla_j \nabla^h (\text{Ric}^h_i + \mu \eta^h \eta_i) + \nabla_i \nabla_j (\text{Ric}^h_i + \mu \eta^h \eta_j) + \nabla_j \nabla^h (\text{Ric}^h_i + \mu \eta^h \eta_i).
\]
The foregoing equation and the lemma’s assumption yield
\[
\mathcal{L}_V \text{Ric}_{ij} = \nabla_h \nabla^h (\text{Ric}_{ij} + \mu \eta \eta_j) - \nabla_h \nabla_i (\text{Ric}^h_i + \mu \eta^h \eta_i) - \nabla_h \nabla_j (\text{Ric}^h_i + \mu \eta^h \eta_j).
\]
Eq.\([1]\) gives \(\mathcal{L}_V g^{ij} = 2 \text{Ric}^{ij} + 2 g^{ij} + 2 \mu \eta^i \eta^j\), using this, \([1]\) and the above equation we obtain \([29]\) and it completes the proof. \(\square\)

Theorem 3.3. Let \((M, \varphi, \xi, \eta, g)\) be a Sasakian pseudo-metric manifold and let \((g, V, \lambda, \mu)\) be an \(\eta\)--Ricci soliton on \(M\).

(a) If \(\xi\) is a timelike vector field then, \(V\) is a Killing vector field.

(b) If \(\xi\) is a spacelike vector field and \(V\) is not a Killing vector field, then \(M\) is \(\mathcal{D}\)--homothetically fixed and \(\mathcal{L}_V \varphi = 0\).

Proof. In the case of (a), using \([12]\) we find \(\lambda - \mu = 2n\), this, \([18]\) and \([1]\) yield \(V\) is Killing.

In the case of (b), using lemma \([5, 2]\) we obtain \((-\lambda + \mu + 2n)(\lambda + \mu + 2n) = 0\). According to the theorem’s assumption \(V\) is not a Killing vector field, so \(\lambda - \mu = 2n + 4\), using this in \([18]\) we deduce
\[
\text{Ric} = -2g + 2(n + 1)\eta \otimes \eta,
\]
so \(M\) is \(\mathcal{D}\)--homothetically fixed. Using the foregoing equation and \([20]\) we obtain
\[
(\mathcal{L}_V \nabla)(Y, Z) = 2(2n + 2 + \mu)(\eta(Z)\varphi Y + \eta(Y)\varphi Z), \quad \forall Y, Z \in \Gamma(M).
\]
Differentiating the above equation along an arbitrary vector field \( X \), using \( 23 \) and contracting at \( X \), we have
\[
(\mathcal{L}_V \text{Ric})(Y, Z) = 2(2n + 2 + \mu)(2g(Y, Z) - (4n + 2)\eta(Y)\eta(Z)),
\]
where \( Y, Z \) are arbitrary vector fields.

Next using \( 30 \) in \( 1 \), we find
\[
(\mathcal{L}_V g)(Y, Z) = -(2n + \lambda + \mu)(g + \eta \otimes \eta)(Y, Z), \quad \forall Y, Z \in \Gamma(M).
\]

Lie-differentiating \( 30 \) along the vector field \( V \) gives us
\[
(\mathcal{L}_V \eta)(Y) = 2(2 + 2n + \lambda + \mu)g(Y, \phi Y) - (2n + 1)\eta(Y)\eta, \quad \forall Z, Y \in \Gamma(M).
\]

Substituting \( \xi \) for \( Y \) in \( 33 \) and \( 31 \) using \( 27 \) we obtain
\[
(\mathcal{L}_V \eta)(X, Y) = -2(2 + 2n + \mu)\eta(Y), \quad \forall Y \in \Gamma(M).
\]

Operating the above equation by \( d \) and noticing the fact that \( d \) commutes with Lie-derivative we deduce
\[
(\mathcal{L}_V d\eta)(X, Y) = -2(2n + \lambda + \mu)g(X, \phi Y), \quad \forall X, Y \in \Gamma(M).
\]

Example 1. Consider \( \mathbb{R}^3 \) with the standard coordinate system \((x, y, z)\). Let
\[
\xi = 2\frac{\partial}{\partial z}, \quad \eta = \frac{1}{2}(-ydx + dz), \quad \phi(x) = -\frac{\partial}{\partial y}, \quad \phi(y) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \quad \text{and} \quad \phi(z) = 0.
\]
If \( g = \epsilon\eta \otimes \eta + \frac{1}{2}(dx^2 + dy^2) \), then \((M, \phi, \xi, \eta, g)\) is a Sasakian pseudo-metric manifold. By direct calculation, we have \( \text{Ric} = -2\epsilon g + 4\eta \otimes \eta \). Now, let \( V \) be a vector field defined by
\[
V = ((2 - 6\epsilon + (\epsilon - 1)\lambda)\lambda + (1 - 2\epsilon)\mu)x\frac{\partial}{\partial x} + (2\epsilon - \lambda)y\frac{\partial}{\partial y} - (2 + \epsilon\lambda + \mu)z\frac{\partial}{\partial z}.
\]
If \( \xi \) be a spacelike vector field and \( \lambda - \mu = 6 \) then \((g, V, \lambda, \mu)\) is an \( \eta \)-Ricci soliton on \( M \), \( \mathcal{L}_V \phi = 0 \) and \( V \) is not a Killing vector field. But if \( \xi \) is a timelike vector field then \((g, V, \lambda, \mu)\) is an \( \eta \)-Ricci soliton on \( M \) iff \( V \) is a Killing vector field, and this condition is satisfied if \( \lambda = -2 \) and \( \mu = -4 \).

Proposition 3.4. Let \((M, \phi, \xi, \eta, g)\) be a \( K \)-contact pseudo-metric manifold. If \((g, V, \lambda, \mu)\) is a gradient \( \eta \)-Ricci soliton on \( M \) then \( M \) is an \( \eta \)-Einstein manifold and \( \text{Ric} = -\lambda g - \mu \eta \otimes \eta \), where \( -\epsilon\lambda - \mu = 2n \).

Proof. First \( 2 \) gives
\[
\nabla_X Df + QX + \lambda X + \epsilon \mu \eta(X)\xi = 0, \quad \forall X \in \Gamma(M).
\]
Calculating $R(X,Y)Df$ by the above equation, we deduce

$$R(X,Y)Df = \epsilon\mu(\nabla Y \eta)X\xi + \epsilon\mu\eta(X)\nabla Y \xi + (\nabla Y Q)X - \epsilon\mu(\nabla X \eta)Y\xi - \epsilon\mu\eta(Y)\nabla X \xi - (\nabla X Q)Y, \quad \forall X,Y \in \Gamma(M).$$

(36)

Substituting $\xi$ for $Y$ in the last equation and using lemma [2.1, theorem 2.1] we find

$$R(X,\xi)Df = (\mu + 2n)\varphi X - \epsilon\varphi QX, \quad X \in \Gamma(M).$$

Scalar product of the above equation with $\xi$ gives $df = (\xi(f))\eta$, operating $d$ on this equation, we obtain $d\eta \wedge (\xi(f)) + \eta \wedge d(\xi(f)) = 0$, taking exterior product of the last equation with $\eta$ and using $\eta \wedge d\eta \neq 0$, we have $\xi(f) = 0$, so $f$ is a constant function. Next using this consequence in (35) we find $\text{Ric} = -\lambda g - \mu \eta \otimes \eta$ and this gives $-\epsilon \lambda - \mu = 2n$, completing the proof.

One may ask, what will happen if the potential vector field of an $\eta$--Ricci soliton on a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is $\xi$, we have answered this question in the following theorem.

**Theorem 3.5.** Let $(M, \varphi, \xi, \eta, g)$ be a contact pseudo-metric manifold, and let $(g, \varphi, \lambda, \mu)$ be an $\eta$--Ricci soliton on the manifold $M$. If $V$ is colinear with $\xi$ and $Q\varphi = \varphi Q$, then $M$ is an $\eta$--Einstein $K$--contact pseudo-metric manifold and $\text{Ric} = -\lambda g - \mu \eta \otimes \eta$, where $-\epsilon \lambda - \mu = 2n$.

**Proof.** Let $V = f\xi$, where $f$ is a non-zero smooth function on the manifold $M$. Using this in (11) we have

$$\epsilon X(f)\eta(Y) + \epsilon Y(f)\eta(X) - 2fg(\varphi hX, Y) + 2\text{Ric}(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,$$

(37)

for $X, Y \in \Gamma(M)$. Substituting $\xi$ for $Y$ in (37) we deduce

$$\epsilon Df + 2Q\xi + (\xi(f) + 2\lambda + 2\epsilon\mu)\xi = 0.$$

(38)

By assumption $Q\varphi = \varphi Q$, so $\varphi Q\xi = 0$, using this and (11) we have $Q\xi = \epsilon(2n - tr h^2)\xi$. Substituting this consequence in (38) we find

$$\epsilon Df + (2\epsilon(2n - tr h^2) + \xi(f) + 2\lambda + 2\epsilon\mu)\xi = 0.$$

(39)

Next, substituting $\xi$ for $X, Y$ in (37) we obtain

$$2n - tr h^2 = -\epsilon(\xi(f)) - \lambda \epsilon - \mu.$$

The above equation and (39) give $Df = \epsilon(\xi(f))\xi$, differentiating this equation along an arbitrary vector field $X$ and using (10) we find

$$g(\nabla X(Df), Y) = X(\xi(f))\eta(Y) - \epsilon\xi(f)\{g(\epsilon\varphi X, Y) + g(\varphi hX, Y)\}, \quad \forall X, Y \in \Gamma(M).$$

Using the above equation, (5) and the known formula $g(\nabla X(Df), Y) = g(\nabla Y(Df), X)$, where $X, Y \in \Gamma(M)$, we deduce

$$X(\xi(f))\eta(Y) - Y(\xi(f))\eta(X) = -2\xi(f)d\eta(X, Y), \quad \forall X, Y \in \Gamma(M).$$

9
Considering $X,Y$ as arbitrary orthogonal vector fields to $\xi$ in the above equation and noticing that $d\eta \neq 0$, we deduce $X(f) = 0$, so $f$ is a constant function on the manifold $M$. Using this consequence in (37) gives

$$-f\varphi h X + QX + \lambda X + \epsilon \mu \eta(X)\xi = 0, \quad \forall X \in \Gamma(M). \quad (40)$$

Substituting $\varphi X$ for $X$ in the above equation, we find

$$-f\varphi h \varphi X + QX + \lambda X = 0, \quad \forall X \in \Gamma(M). \quad (41)$$

Operating $\varphi$ on (40) and using $\varphi h = -h\varphi$, we have

$$f\varphi h \varphi X + QX + \lambda X = 0, \quad \forall X \in \Gamma(M). \quad (42)$$

Using the above equation, (41) and $Q\xi = (-\lambda - \mu\epsilon)\xi$, we obtain:

$$\text{Ric} = -\mu \eta \otimes \eta - \lambda g.$$  

Using the above equation in (1) gives $\mathcal{L}_\xi g = 0$, so $M$ is a $K$–contact pseudo-metric manifold and $-\epsilon \lambda - \mu = 2n$, completing the proof.

4 $\eta$–Ricci solitons on $(\kappa, \mu)$–contact pseudo-metric manifolds

Studying nullity conditions on manifolds is one of the interesting topics in differential geometry, specially in the context of contact pseudo-metric manifolds. In [15], Ghaffarzadeh and second author introduced the notion of a $(\kappa, \mu)$–contact pseudo-metric manifold. According to them a contact pseudo-metric manifold $(M, \varphi, \xi, \eta)$ is called a $(\kappa, \mu)$–contact pseudo-metric manifold if it satisfies

$$R(X, Y)\xi = \epsilon \kappa (\eta(Y)X - \eta(X)Y) + \epsilon \mu (\eta(Y)hX - \eta(X)hY), \quad (43)$$

where $R$ is the Riemannian curvature tensor of $M$, $\kappa, \mu$ are constant real numbers, and $X, Y$ are arbitrary vector fields. For a $(\kappa, \mu)$–contact pseudo-metric manifold we have the following formulas [15]

$$h^2 = (\epsilon \kappa - 1)\varphi^2, \quad (44)$$

$$Q\xi = 2n\kappa \xi, \quad (45)$$

$$(\nabla_\xi h) = -\epsilon \mu \varphi h, \quad (46)$$

Furthermore if $\epsilon \kappa < 1$ then we have [15]

$$QX = \epsilon [2(n - 1) - n\mu]X + (2(n - 1) + \mu)hX + [2(1 - n)\epsilon + 2n\kappa + n\mu][\eta(X)\xi, \quad (47)$$

$$r = 2n(\kappa - 2\epsilon) + 2n^2\epsilon (2 - \mu), \quad (48)$$

where $X$ and $r$ are, an arbitrary vector field and the scalar curvature of the manifold, respectively.
Lemma 4.1. Let \((M, \varphi, \xi, \eta, g)\) be a \((\kappa, \mu)\)-contact pseudo-metric manifold, and let \(\epsilon \kappa < 1\). If \((g, V, \lambda, \tau)\) is a gradient \(\eta\)-Ricci soliton on the manifold \(M\) then
\[
\epsilon \kappa (-2 + \mu) = n \mu + \mu + \tau. \tag{49}
\]

Proof. Differentiating 45 along an arbitrary vector field \(X\) and using 10, we deduce
\[
(\nabla_X Q)\xi = Q(\epsilon \varphi + \varphi h)X - 2n \kappa (\epsilon \varphi + \varphi h)X, \quad \forall X \in \Gamma(M). \tag{50}
\]

Taking scalar product of 38 and \(\xi\), and using 50, we have
\[
g(R(X, Y)Df, \xi) = \epsilon g((Q \varphi + \varphi Q)Y, X) + g((Q \varphi h + h \varphi Q)Y, X)
+ (-4n \kappa \epsilon - 2 \tau)g(\varphi Y, X), \quad \forall X, Y \in \Gamma(M). \tag{51}
\]

Substituting \(\varphi X\) for \(X\) and \(\varphi Y\) for \(Y\) in 43 gives \(R(\varphi X, \varphi Y)\xi = 0\), using this, 3 and the above equation, we obtain
\[
\epsilon (\varphi Q + Q \varphi)X - (\varphi Q h + h \varphi Q)X + (-4n \kappa \epsilon - 2 \tau)\varphi X = 0, \tag{52}
\]
where \(X\) is an arbitrary vector field. Now, substituting \(\varphi X\) for \(X\) in 47 we have
\[
Q \varphi X = \epsilon [2 (n - 1) - n \mu] \varphi X + (2(n - 1) + \mu)h \varphi X, \quad \forall X \in \Gamma(M). \tag{53}
\]

Next, operating \(\varphi\) on 47 we obtain
\[
\varphi Q X = \epsilon [2 (n - 1) - n \mu] \varphi X + (2(n - 1) + \mu) \varphi h X, \quad \forall X \in \Gamma(M). \tag{54}
\]

Substituting \(hX\) for \(X\) in 44 using 44 and 3 give
\[
\varphi Q h X = \epsilon [2 (n - 1) - n \mu] \varphi h X - (\epsilon \kappa - 1)(2(n - 1) + \mu) \varphi X, \quad X \in \Gamma(M). \tag{55}
\]

Operating \(h\) on 53 using 44 and 3 we have
\[
h Q \varphi X = \epsilon [2 (n - 1) - n \mu] \varphi h X - (\epsilon \kappa - 1)(2(n - 1) + \mu) \varphi X, \quad X \in \Gamma(M). \tag{56}
\]

Using the last four equations in 52 and \(h \varphi = - \varphi h\) give 49 completing the proof. \(\square\)

Theorem 4.2. Let \((M, \varphi, \xi, \eta, g)\) be a \((\kappa, \mu)\)-contact pseudo-metric manifold and let \(\epsilon \kappa < 1\). If \((g, V, \lambda, \tau)\) is a gradient \(\eta\)-Ricci soliton on \(M\), then \(\mu = 0, \tau = -2 \epsilon \kappa\), or \(\text{Ric} = - \lambda g - \tau \eta \otimes \eta\) and \(\mu = 2 - 2n, \tau = 2n(-\frac{1}{n} + n - \epsilon \kappa)\).

Proof. First, substituting \(\xi\) for \(X\) in 51 using 43 and 45 we have
\[
\kappa (\xi(f))\xi = \epsilon \kappa Df - \epsilon \mu h Df = 0. \tag{51}
\]

Differentiating the above equation along vector field \(\xi\) and using 46 we have
\[
\kappa \xi(\xi(f))\xi + \epsilon \kappa (2n \kappa + \lambda + \tau \epsilon) \xi + (\epsilon \mu)^2 \varphi h Df = 0. \tag{51}
\]
Now, operating $\phi$ on the last equation we find $\mu^2 hDf = 0$, taking $h$ from this and using (14) we obtain

$$\mu^2(\epsilon \kappa - 1)(-Df + \eta(Df)\xi) = 0.$$  

Examining the above equation we have either, i) $\mu = 0$ or ii) $\mu \neq 0$.

In the case i), using (49) we obtain $\tau = -2\epsilon \kappa$. In the case ii), we have $Df = \eta(Df)\xi$; differentiating this along arbitrary vector field $X$ and using (10), we have

$$g(\nabla_X Df, Y) = X(\xi(f))\eta(Y) - \xi(f)g(\phi X, Y) - \epsilon \xi(f)f(\phi hX, Y),$$

where $X, Y$ are arbitrary vector fields. Using the above equation and $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, we find

$$X(\xi(f))\eta(Y) - Y(\xi(f))\eta(X) + 2\xi(f)d\eta(X, Y) = 0, \quad \forall X, Y \in \Gamma(M).$$

Substituting $\phi X$ for $X$ and $\phi Y$ for $Y$, and noticing the fact that $d\eta \neq 0$, we have $\xi(f) = 0$. So $f$ is a constant function and $\text{Ric} = -\lambda \eta - \tau \eta \otimes \eta$. Using this gives $r = (2n + 1)(-\lambda) - \epsilon \tau$, comparing the last consequence and (48) we have

$$n\mu = -2 + 2n - 2n\epsilon \kappa - \tau.$$  

Now using the above equation and (49) we obtain, $\mu = 2 - 2n$ and $\tau = 2n(-\frac{1}{n} + n - \epsilon \kappa)$, completing the proof.  

**Corollary 1.** Let $(M, \phi, \xi, \eta, g)$ be a $(\kappa, \mu)$–contact pseudo-metric manifold and let $\epsilon \kappa < 1$. If $(g, V, \lambda, 0)$ is a gradient $\eta$–Ricci soliton (in fact a gradient Ricci soliton) on $M$, then $R(X, Y)\xi = 0$, where $X, Y$ are arbitrary vector fields.

**References**

[1] M. M. Akbar and E. Woolgar, *Ricci solitons and Einstein-scalar field theory*, Classical Quantum Gravity, 26 (2009), pp. 055015, 14.

[2] C. S. Bagewadi and G. Ingalahalli, *Ricci solitons in Lorentzian $\alpha$-Sasakian manifolds*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 28 (2012), pp. 59–68.

[3] C. L. Bejan and M. Crasmareanu, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Global Anal. Geom., 46 (2014), pp. 117–127.

[4] A. M. Blaga, *Eta-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl., 20 (2015), pp. 1–13.

[5] A. M. Blaga, *$\eta$-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, 30 (2016), pp. 489–496.
[6] A. Bravetti, C. S. Lopez-Monsalvo, and F. Nettel, Contact symmetries and Hamiltonian thermodynamics, Ann. Physics, 361 (2015), pp. 377–400.

[7] J. L. Cabrerozo, M. Fernández, and J. S. Gómez, The contact magnetic flow in 3D Sasakian manifolds, J. Phys. A, 42 (2009), pp. 195201, 10.

[8] G. Calvaruso and D. Perrone, Contact pseudo-metric manifolds, Differential Geom. Appl., 28 (2010), pp. 615–634.

[9] J. T. Cho and M. Kimura, Ricci solitons and real hypersurfaces in a complex space form, Tohoku Mathematical Journal, Second Series, 61 (2009), pp. 205–212.

[10] O. Chodosh and F. T.-H. Fong, Rotational symmetry of conical Kähler-Ricci solitons, Math. Ann., 364 (2016), pp. 777–792.

[11] C. Călin and M. Crasmareanu, From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds, Bull. Malays. Math. Sci. Soc. (2), 33 (2010), pp. 361–368.

[12] K. Duggal, Space time manifolds and contact structures, International Journal of Mathematics and Mathematical Sciences, 13 (1990), pp. 545–553.

[13] D. Friedan, Nonlinear models in 2 + ε dimensions, Phys. Rev. Lett., 45 (1980), pp. 1057–1060.

[14] A. Futaki, H. Ono, and G. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, J. Differential Geom., 83 (2009), pp. 585–635.

[15] N. Ghaffarzadeh and M. Faghiouri, On contact pseudo-metric manifolds satisfying a nullity condition, J. Math. Anal. Appl., 497 (2021), pp. 124849, 16.

[16] A. Ghosh and R. Sharma, Sasakian metric as a Ricci soliton and related results, Journal of Geometry and Physics, 75 (2014), pp. 1–6.

[17] R. S. Hamilton, The Ricci flow on surfaces, in Mathematics and general relativity (Santa Cruz, CA, 1986), vol. 71 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.

[18] G. Herczeg and A. Waldron, Contact geometry and quantum mechanics, Physics Letters B, 781 (2018), pp. 312–315.

[19] J. Källén and M. Zabzine, Twisted supersymmetric 5d yang-mills theory and contact geometry, Journal of High Energy Physics, 2012 (2012), p. 125.
[20] A. L. Kholodenko, *Towards physically motivated proofs of the Poincaré and geometrization conjectures*, J. Geom. Phys., 58 (2008), pp. 259–290.

[21] R. Low, *Stable singularities of wave-fronts in general relativity*, J. Math. Phys., 39 (1998), pp. 3332–3335.

[22] H. G. Nagaraja and C. R. Premalatha, *Ricci solitons in Kenmotsu manifolds*, J. Math. Anal., 3 (2012), pp. 18–24.

[23] D. M. Naik and V. Venkatesha, *η-Ricci solitons and almost η-Ricci solitons on para-Sasakian manifolds*, International Journal of Geometric Methods in Modern Physics, 16 (2019), p. 1950134.

[24] D. Perrone, *Curvature of K-contact semi-Riemannian manifolds*, Canad. Math. Bull., 57 (2014), pp. 401–412.

[25] R. Sharma, *Certain results on k-contact and (κ, μ)-contact manifolds*, Journal of Geometry, 89 (2008), pp. 138–147.

[26] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tohoku Math. J. (2), 21 (1969), pp. 271–290.

[27] A. Van der Schaft and B. Maschke, *Geometry of thermodynamic processes*, Entropy, 20 (2018), p. 925.

[28] K. Yano, *Integral formulas in Riemannian geometry*, Pure and Applied Mathematics, No. 1, Marcel Dekker, Inc., New York, 1970.