STABILITY OF THE COMPOSITE WAVE FOR THE INFLOW PROBLEM ON THE MICROPOLAR FLUID MODEL

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Abstract. In this paper, we study the asymptotic behavior of solutions to the initial boundary value problem for the micropolar fluid model in half line \( \mathbb{R}_+ := (0, \infty) \). Inspired by the relationship between the micropolar fluid model and Navier-Stokes system, we can prove that the composite wave consisting of the subsonic BL-solution, the contact wave, and the rarefaction wave for the inflow problem on micropolar fluid model is time-asymptotically stable. Meanwhile, we obtain the global existence of solutions based on the basic energy method.

1. Introduction. The 1-D compressible viscous micropolar fluid model in half line \( \mathbb{R}_+ := (0, +\infty) \) reads in Eulerian coordinates:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \quad x > 0, \quad t > 0, \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) &= \mu \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0, \\
\frac{\partial (\rho \omega)}{\partial t} + \frac{\partial}{\partial x}(\rho u \omega) + A \omega &= A \frac{\partial^2 \omega}{\partial x^2}, \quad x > 0, \quad t > 0, \\
\frac{\partial}{\partial t} \left[ \rho (e + \frac{u^2}{2}) \right] + \frac{\partial}{\partial x} \left[ \rho (e + \frac{u^2}{2}) + pu \right] &= \mu \frac{\partial^2 (u \partial_x u)}{\partial x^2} + \kappa \frac{\partial^2 \theta}{\partial x^2} + (\partial_x \omega)^2 + \omega^2, \quad x > 0, \quad t > 0,
\end{align*}
\]

where \( \rho, u, \omega \) and \( \theta \) denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid, and \( A, \mu, \kappa \) are positive constants. Assuming that the fluid is perfect and polytropic, we have the state equations for pressure \( p \) and internal energy \( e \):

\[
p = R \rho \theta, \quad e = \frac{R}{\gamma - 1} \theta,
\]

where \( R \) and \( \gamma > 1 \) are positive constants.

Initial data for system (1) are given by

\[
(\rho, u, \omega, \theta)(x, 0) = (\rho_0, u_0, \omega_0, \theta_0)(x), \quad \inf_{x \in \mathbb{R}_+} \rho_0(x) > 0, \quad \inf_{x \in \mathbb{R}_+} \theta_0(x) > 0.
\]

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We assume that the initial data at the far field \( x = +\infty \) are constants, namely
\[
\lim_{x \to +\infty} (\rho_0, u_0, \omega_0, \theta_0)(x) = (\rho_+, u_+, \omega_+, \theta_+)
\]
and the boundary data for \( \rho, u, \omega \) and \( \theta \) at \( x = 0 \) are given by
\[
(\rho, u, \omega, \theta)(0, t) = (\rho_-, u_-, \omega_-, \theta_-), \quad \forall \ t \geq 0,
\]
where \( \rho_-, u_-, \omega_- \) and \( \theta_- \) are constants and the following compatibility conditions hold
\[
\rho_0(0) = \rho_-, \quad u_0(0) = u_-, \quad \omega_0(0) = \omega_-, \quad \theta_0(0) = \theta_-.
\]
Note that \( \rho_\pm, u_\pm, \omega_\pm \) and \( \theta_\pm \) could be distinct.

The boundary condition \( u_- > 0 \) means that gas blows into the region from the boundary \( x = 0 \) with the velocity \( u_- \). Thus this problem is called an inflow problem (see [18]). The inflow boundary condition \( u_- > 0 \) implies that the characteristic speeds of the hyperbolic equation (1) for the density \( \rho \) is positive around the boundary. Hence not only \( u, \omega \) and \( \theta \) but also \( \rho \) has to be imposed boundary conditions for the well-posedness of inflow problem (1). In addition, the boundary condition \( u_- < 0 \) means that fluid blows out from the boundary \( x = 0 \) with the velocity \( u_- \). Thus this problem is called an outflow problem (see [18]). The outflow boundary condition \( u_- < 0 \) implies that the characteristic speeds of the hyperbolic equation (1) for the density \( \rho \) is negative around the boundary so that boundary conditions on \( u, \omega \) and \( \theta \) to parabolic equations (1)\(_2\), (1)\(_3\) and (1)\(_4\) are necessary and sufficient for the well-posedness of outflow problem.

The study of micropolar fluid model started from 1966 ([6]). The micropolar fluid model has many potential applications not only in physics but also in mathematics. In physics, the micropolar fluid may represent fluid consisting of rigid, randomly oriented (or spherical particles) suspended in a viscous medium, where the deformation of fluid particles is ignored, such as liquid crystals, polymeric fluids, ferro liquids, animal blood, among others. For more background, we refer to [17] and references therein. Because of their microstructures, viscous micropolar fluids are non-Newtonian with non-symmetric tensor and cannot be described only by using Navier-Stokes equations. Now more and more mathematicians are devoted to the research of micropolar fluid. Here we only list some results related to our paper. In [1, 4, 17], the authors proved the existence of weak solutions and strong solutions for the compressible micropolar fluid model. Nowakowski in [28] considered the strong solutions of incompressible micropolar fluid when \( \text{div} u = 0 \). When the initial data with vacuum was allowed, the blowup criterion of solutions to the three dimensional compressible micropolar fluid model was obtained in [2, 3]. The large time behavior and stability of solutions for the compressible micropolar fluid model were obtained in [22, 23, 25, 29]. The regularity of solutions to the initial boundary value problem for compressible micropolar fluid model was proved in [21, 24]. Recently, the authors in [15] obtained the optimal time decay rate of the three-dimensional compressible micropolar fluid model.

In order to study the large time behavior of solutions to the initial boundary value problem (1), (3), (4), (5), (6), we assume that the microrotation velocity asymptotically converges to zero. Then the micropolar fluid model (1) can be
reduced to the following Navier-Stokes system in the form

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p) &= \mu \partial_x^2 u, \\
\partial_t [\rho (e + \frac{u^2}{2})] + \partial_x [\rho u (e + \frac{u^2}{2}) + pu] &= \mu \partial_x (u \partial_x u) + \kappa \partial_x^2 \theta.
\end{align*}
\]

(7)

Moreover, when the dissipation effects are neglected for the large time behavior, Navier-Stokes system (7) can be reduced to the following Euler system in the form

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2 + p) &= 0, \\
\partial_t [\rho (e + \frac{u^2}{2})] + \partial_x [\rho u (e + \frac{u^2}{2}) + pu] &= 0.
\end{align*}
\]

(8)

The Euler system (8) is a typical example of the hyperbolic conservation laws. The Riemann solutions for the Euler system (8) are shock waves, rarefaction waves, contact discontinuity waves and the linear combinations of these basic waves. It is well known that the large time behavior of solutions to the Cauchy problem of Navier-Stokes system are basically described by the viscous versions of three basic waves. Here we only mention several results related to Navier-Stokes system: the asymptotic stability of shock wave [7]; the asymptotic stability of rarefaction wave [16]; the asymptotic stability of contact discontinuity wave [11, 12]; the asymptotic stability of combination of viscous contact discontinuity wave with rarefaction waves [9]. Later, the initial-boundary value problem (IBVP) of Navier-Stokes system has attracted many mathematicians’s attention because it has more physical meanings and of course produces some new mathematical difficulties due to the boundary effect. Not only basic waves but also a new wave, which is called the boundary layer solution (BL-solution for brevity), may appear in the IBVP of Navier-Stokes system. So far, a great number of mathematical results about the BL-solution for the Navier-Stokes system have been studied by many authors. For details, please refer to [8, 13, 19, 20, 26, 27, 30] and references therein.

Recently, for one dimensional compressible micropolar fluid model (1), Liu-Yin [14] have obtained the stability of contact discontinuity wave for the Cauchy problem and Cui-Yin also [5] have obtained the stability of BL-solution to the outflow problems. However, to our knowledge, there are few results on the stability of composite wave for one dimensional compressible micropolar fluid model (1). Hence, in this paper, we pay our attention on the stability of composite wave consisting of the subsonic BL-solution, the contact discontinuity wave, and the rarefaction wave for the inflow problem on the micropolar fluid model (1) to the Riemann problem on Euler system (8). Compared with the previous results about the micropolar fluid model for the global existence of solutions near constant states, this paper is concerned with nonlinear stability of solutions near non-constant states, and thereby yields many nonlinear hard terms. In one word, the main difficulties in our proofs are how to deal with the boundary terms, the interactions of three different waves and the external term, i.e. microrotation velocity \(\omega\). Moreover, the case for \(\omega_+ \neq \omega_-\) which leads to more complex structure is left to study in the future.

For the purpose of obtaining the large time behavior of solutions to the initial boundary value problem (1), (3), (4), (5) and (6), it is more convenient to use the
Lagrangian coordinates. That is, we consider the following coordinate transformation:

\[ x \Rightarrow \int_{(0,0)}^{(x,t)} \rho(y,\tau)dy - \rho u(y,\tau)d\tau, \quad t \Rightarrow t. \]

We still denote the Lagrangian coordinates by \((x,t)\) for simplicity of notation. Thus the system (1) can be transformed into the following moving boundary problem for the micropolar fluid model in the Lagrangian coordinates

\[
\begin{align*}
\partial_t v - \partial_x u &= 0, \quad x > \sigma -, t > 0, \\
\partial_t u + \partial_x p &= \mu \partial_x \left( \frac{\partial_x u}{v} \right), \quad x > \sigma -, t > 0, \\
\partial_t \omega + A v \omega &= A \partial_x \left( \frac{\partial_x \omega}{v} \right), \quad x > \sigma -, t > 0, \\
\partial_t (e + \frac{u^2}{2}) + \partial_x (p u) &= \mu \partial_x \left( \frac{u \partial_x u}{v} + \kappa \partial_x \left( \frac{\partial_x \theta}{v} \right) \right) + \frac{(\partial_x \omega)^2}{v} + v \omega^2, \quad x > \sigma -, t > 0, \\
(v, u, \omega, \theta)(x = \sigma -, t) &= (v_-, u_-, 0, \theta_-), \quad u_- > 0, \\
(v, u, \omega, \theta)(x, 0) &= (v_0, u_0, \omega_0, \theta_0)(x) \rightarrow (v_+, u_+, 0, \theta_+), \quad \text{as } x \rightarrow +\infty,
\end{align*}
\]

where \(v(x,t) = \frac{1}{\rho(x,t)}\) represents the specific volume of the fluid, and the boundary moves with the constant speed \(\sigma_- = \frac{u_-}{\rho_-} < 0\). For the perfect gas, we have

\[ p = \frac{R \theta}{v}, \quad (10) \]

In order to fix the moving boundary \(x = \sigma -, t\), we introduce a new variable \(\xi = x - \sigma - t\). Then we have the half-space problem

\[
\begin{align*}
\partial_t v - \sigma_- \partial_\xi v - \partial_\xi u &= 0, \quad \xi > 0, t > 0, \\
\partial_t u - \sigma_- \partial_\xi u + \partial_\xi p &= \mu \partial_\xi \left( \frac{\partial_\xi u}{v} \right), \quad \xi > 0, t > 0, \\
\partial_t \omega - \sigma_- \partial_\xi \omega + A v \omega &= A \partial_\xi \left( \frac{\partial_\xi \omega}{v} \right), \quad \xi > 0, t > 0, \\
\partial_t (e + \frac{u^2}{2}) - \sigma_- \partial_\xi (e + \frac{u^2}{2}) + \partial_\xi (p u) &= \mu \partial_\xi \left( \frac{u \partial_\xi u}{v} + \kappa \partial_\xi \left( \frac{\partial_\xi \theta}{v} \right) \right) + \frac{(\partial_\xi \omega)^2}{v} + v \omega^2, \quad \xi > 0, t > 0, \\
(v, u, \omega, \theta)(\xi = 0, t) &= (v_-, u_-, 0, \theta_-), \quad u_- > 0, \\
(v, u, \omega, \theta)(\xi, 0) &= (v_0, u_0, \omega_0, \theta_0)(\xi) \rightarrow (v_+, u_+, 0, \theta_+), \quad \text{as } \xi \rightarrow +\infty.
\end{align*}
\]

Next, in thermodynamics, we know that by any given two of the five thermodynamical variables, \(v, p, e, \theta\) and entropy \(s\), the remaining three variables can be expressed. Without loss of generality, we define the entropy \(s\) as follows

\[ s = R \ln v + \frac{R}{\gamma - 1} \ln \theta + 1, \quad (12) \]

which obeys the second law of thermodynamics

\[ \theta ds = de + pdv. \]
Then due to (12), the entropy of the initial data is expressed as follows
\[ s(v_0(x), \theta_0(x)) = R \ln v_0(x) + \frac{R}{\gamma - 1} \ln \theta_0(x) + 1. \] 
Thus \( s_+ = \lim_{x \to +\infty} s(v_0(x), \theta_0(x)) \) satisfies
\[ s_+ = s(v_+, \theta_+) = R \ln v_+ + \frac{R}{\gamma - 1} \ln \theta_+ + 1. \] 

The rest of the paper is arranged as follows. In Section 2, we state some important results in [30] which will be used in this paper. In Section 3, we re-formulate the original system (1), then introduce our main theorem concerning the global existence and asymptotic stability of solutions. The proof of Theorem 3.1 is concluded in Section 4. In Appendix, we present the details which are left in the proofs of the previous sections for the completeness of the paper.

**Notation:** Throughout the paper, we denote generally large and small positive constants independent of \( t \) by \( C \) and \( c \), respectively. And the character “\( C \)” and “\( c \)” may take different values in different places. \( L^p = L^p(\mathbb{R}_+) \) \( (1 \leq p \leq \infty) \) denotes the usual Lebesgue space on \([0, \infty)\) with its norm \( \| \cdot \|_{L^p} \), and when \( p = 2 \), we write \( \| \cdot \|_{L^2(\mathbb{R}_+)} = \| \cdot \| \). \( H^s = H^s(\mathbb{R}_+) \) is used to represent the usual \( s \)-th order Sobolev space with its norm \( \| f \|_{H^s(\mathbb{R}_+)} = \left( \sum_{i=0}^{s} \| \partial^i f \|^2 \right)^{\frac{1}{2}} \).

2. **Some preliminaries of the Navier-Stokes system.** Since we expect the solutions of the micropolar fluid model behave as that of Navier-Stokes system, we assume \( \omega(x, t) = 0 \) for the large time behavior. Therefore, when \( t \to \infty \), the micropolar fluid model (9) and (11) respectively become the following Navier-Stokes system
\[
\begin{aligned}
\partial_t v - \partial_x u &= 0, \quad x > \sigma - t, \quad t > 0, \\
\partial_t u + \partial_x p &= \mu \partial_x \left( \frac{\partial_x u}{v} \right), \quad x > \sigma - t, \quad t > 0, \\
\partial_t (e + \frac{u^2}{2}) + \partial_x (pu) &= \mu \partial_x \left( \frac{u \partial_x u}{v} \right) + \kappa \partial_x \left( \frac{\partial_x \theta}{v} \right), \quad x > \sigma - t, \quad t > 0, \\
(v, u, \theta)(x = \sigma - t, t) &= (v_-, u_-, \theta_-), \quad u_- > 0, \\
(v, u, \theta)(x, 0) &= (v_0, u_0, \theta_0)(x) \to (v_+, u_+, \theta_+), \quad \text{as } x \to +\infty,
\end{aligned}
\] 
and
\[
\begin{aligned}
\partial_t v - \sigma \partial_x v - \partial_x u &= 0, \quad \xi > 0, \quad t > 0, \\
\partial_t u - \sigma \partial_x u + \partial_x p &= \mu \partial_x \left( \frac{\partial_x u}{v} \right), \quad \xi > 0, \quad t > 0, \\
\partial_t (e + \frac{u^2}{2}) - \sigma \partial_x (e + \frac{u^2}{2}) + \partial_x (pu) &= \mu \partial_x \left( \frac{u \partial_x u}{v} \right) + \kappa \partial_x \left( \frac{\partial_x \theta}{v} \right), \quad \xi > 0, \quad t > 0, \\
(v, u, \theta)(\xi = 0, t) &= (v_-, u_-, \theta_-), \quad u_- > 0, \\
(v, u, \theta)(\xi, 0) &= (v_0, u_0, \theta_0)(\xi) \to (v_+, u_+, \theta_+), \quad \text{as } \xi \to +\infty.
\end{aligned}
\]
Notice that Navier-Stokes system (15) and (16) have been studied by Qin and Wang in [30]. They obtained the existence (or nonexistence) of the BL-solution for the inflow problem when the right end state \((v_+, u_+, \theta_+)\) respectively belongs to the subsonic region \( \Omega_{\text{sub}} \), transonic region \( \Gamma_{\text{trans}} \), and supersonic region \( \Omega_{\text{super}} \).
where \( M \) and \( x \) (curve, contact discontinuity wave curve and 3-Rarefaction wave curve) in terms of

\[ \Omega_{\text{sub}}, \Omega_{\text{trans}} \text{ and } \Omega_{\text{super}} \text{ are defined in Section 2.1}. \]

Moreover, they proved the asymptotic stability of the single contact discontinuity wave and the composite wave consisting of the subsonic BL-solution, the contact discontinuity wave, and the rarefaction wave. Hence, in order to prove the asymptotical stability of composite wave consisting of the subsonic BL-solution, the contact discontinuity wave, and the rarefaction wave for the inflow problem on the micropolar fluid model (11), we first review some known results about Navier-Stokes system in [30] which will be used repeatedly in this paper.

For any given right end state \((v_+, u_+, \theta_+)\), we can define wave curves (BL-solution curve, contact discontinuity wave curve and 3-Rarefaction wave curve) in terms of \((v, u, \theta)\) with \(v > 0\) and \(\theta > 0\) in the phase space as follows

\[ BL(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \bigg| \frac{u}{v} = -\sigma = \frac{u_+}{v_+}, \ (u, \theta) \in \mathcal{M}(u_+, \theta_+) \right\}, \]

\[ CD(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \bigg| p = p_+, \ u = u_+, \ v \neq v_+ \right\}, \]

and

\[ R_3(v_+, u_+, \theta_+) \equiv \left\{ (v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \bigg| s(v, \theta) = s_+, \right. \]

\[ \left. u = u_+ - \int_{v_+}^{v} \lambda_3(z, s_+)dz, \ v > v_+, \ u < u_+ \right\}, \]

where \(\mathcal{M}\) is a center-stable manifold defined in Section 2.1 (BL-solution), \(p_+ = \frac{n_0}{v_+}\), and \(\lambda_3 = \lambda_3(v, s)\) is the third characteristic speed given in (17).

Before we present our main results, we first define \(\Omega_{\text{sub}}^+ := \{(u, \theta) | 0 < u < \sqrt{R\gamma\theta_+}\}.\) In this paper, we expect to prove that if the left end state \((v_-, u_-, \theta_-) \in BL - CD - R_3(v_+, u_+, \theta_+)\), then there exist a unique state \((v_*, u_*, \theta_*) \in \Omega_{\text{sub}}^+\) and a unique state \((v_*, u_*, \theta_*)\) such that \((v_-, u_-, \theta_-) \in BL(v_*, u_*, \theta_*)\), \((v_+, u_+, \theta_+) \in CD(v_*, u_*, \theta_*)\) and \((v_*, u_*, \theta_*) \in R_3(v_+, u_+, \theta_+)\). Moreover, the superposition of the BL-solution, the viscous contact discontinuity wave and the 3-rarefaction wave for the inflow problem to the micropolar fluid model (11) is asymptotically stable, provided that \(|(u_--u_*, \theta_- - \theta_*)|\) and \(|v_* - v_*|\) are suitably small and the conditions in Theorem 3.1 hold. It is remarked that the BL-solution and the viscous contact discontinuity wave must be weak, but the rarefaction wave is not necessarily weak.

2.1. BL-solution. The characteristic speeds of the hyperbolic part of (15) are

\[ \lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \ \lambda_2 = 0, \ \lambda_3 = \sqrt{\frac{\gamma p}{v}}. \] (17)

The sound speed \(C(v, \theta)\) and the Mach number \(M(v, u, \theta)\) are defined by

\[ C(v, \theta) = v \sqrt{\frac{\gamma p}{v}} = \sqrt{R\gamma\theta}, \]

and

\[ M(v, u, \theta) = \frac{|u|}{\sqrt{R\gamma\theta}}. \]

Let \(C_+ = C(v_+, \theta_+) = \sqrt{R\gamma\theta_+}\) and \(M_+ = \frac{|u_+|}{C_+}\) be the sound speed and the Mach number at the far field \(x = +\infty\), respectively. The phase plane \(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+\) of
(v, u, θ) can be divided into three subsets:

\[ \Omega_{\text{sub}} := \{(v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+; \ M(v, u, \theta) < 1\} , \]

\[ \Gamma_{\text{trans}} := \{(v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+; \ M(v, u, \theta) = 1\} , \]

and

\[ \Omega_{\text{super}} := \{(v, u, \theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+; \ M(v, u, \theta) > 1\} , \]

where \( \Omega_{\text{sub}}, \Gamma_{\text{trans}} \) and \( \Omega_{\text{super}} \) are called the subsonic, transonic and supersonic regions, respectively. If we add the alternative condition \( u > 0 \) or \( u < 0 \), then we have six connected subsets \( \Omega_{\text{sub}}^\pm, \Gamma_{\text{trans}}^\pm \) and \( \Omega_{\text{super}}^\pm \).

When \( (v_-, u_-, \theta_-) \in \Omega_{\text{sub}}^+ = \{(u, \theta)|0 < u < \sqrt{k\gamma}\theta_+\} \), we have \( \lambda_1(v_-, u_-, \theta_-) < \sigma_- < 0 \). Hence the existence of the traveling wave solution

\[
\begin{cases}
(V^B, U^B, \Theta^B)(\xi) \quad \xi = x - \sigma_- t, \\
(V^B, U^B, \Theta^B)(0) = (v_-, u_-, \theta_-), \quad (V^B, U^B, \Theta^B)(+\infty) = (v_+, u_+, \theta_+),
\end{cases}
\]

(18)

to (15) or the stationary solution (BL-solution) to (16) is expected. From (18), BL-solution \( (V^B, U^B, \Theta^B)(\xi) \) satisfies the following ODE system

\[
\begin{cases}
- \sigma_- \partial_\xi V^B - \partial_\xi U^B = 0, \quad \xi > 0, \\
- \sigma_- \partial_\xi U^B + \partial_\xi P^B = \mu \partial_\xi \left( \frac{\partial_\xi U^B}{V^B} \right), \quad \xi > 0, \\
- \sigma_- \partial_\xi \left( \frac{R}{\gamma - 1} \Theta^B + \frac{(U^B)^2}{2} \right) + \partial_\xi (P^B U^B) = \mu \partial_\xi \left( \frac{U^B \partial_\xi U^B}{V^B} \right) + \kappa \partial_\xi \left( \frac{\partial_\xi \Theta^B}{V^B} \right), \quad \xi > 0, \\
(V^B, U^B, \Theta^B)(0) = (v_-, u_-, \theta_-), \quad (V^B, U^B, \Theta^B)(+\infty) = (v_+, u_+, \theta_+),
\end{cases}
\]

(19)

where \( P^B = p(V^B, \Theta^B) = \frac{R\Theta^B}{V^B} \). Integrating the system (19) over \( (\xi, +\infty) \), and then taking \( \xi = 0 \) in the resulting equality, it is easy to get

\[
\sigma_- = - \frac{u_-}{v_-} = - \frac{U^B}{V^B} = - \frac{u_+}{v_+}.
\]

(20)

Then the existence and uniqueness for the ODE system (19) are given as follows. For later use, we only list some useful properties of solutions of (19).

**Proposition 1** (See [30]). Assume that \( v_\pm > 0, u_- > 0, \theta_\pm > 0 \) and define \( \delta^B = |(u_+ - u_-, \theta_+ - \theta_-)| \). If \( u_+ \leq 0 \), then there is no solution to (19). If \( u_+ > 0 \), then there exists a suitable small constant \( \delta > 0 \) such that if \( 0 < \delta^B \leq \delta \), then note the following cases.

**Case I. Supersonic case:** \( M_+ > 1 \). Then there is no solution to (19).

**Case II. Transonic case:** \( M_+ = 1 \). Then there exists a unique trajectory \( \Gamma \) tangent to the line

\[
\mu v_+ (U^B - u_+) - \kappa (\gamma - 1)(\Theta^B - \theta_+) = 0
\]

at the point \((u_+, \Theta^B)\). For each \((u_-, \Theta^B) \in \Gamma\), there exists a unique solution \((U^B, \Theta^B)\) satisfying

\[
U^B > 0, \quad \Theta^B > 0,
\]

and

\[
\left| \frac{d^n}{d\xi^n} (U^B - u_+, \Theta^B - \theta_+) \right| \leq C \frac{(\delta^B)^{n+1}}{(1 + \delta^B \xi)^{n+1}}, \quad n = 0, 1, 2, \ldots
\]

(21)
Case III. Subsonic case: \( M_+ < 1 \). Then there exists a center-stable manifold \( \mathcal{M} \) tangent to the line
\[
(1 + a_2 c_2 u_+)(U^B - u_+) - a_2 (\Theta^B - \theta_+) = 0
\]
on the opposite directions at the point \((u_+, \theta_+)\), where \( a_2 \) and \( c_2 \) are some positive constants, see [30] for their definitions. Only when \((u_-, \theta_-) \in \mathcal{M} \), does there exist a unique solution \((U^B, \Theta^B) \subset \mathcal{M} \) satisfying
\[
\left| \frac{d^n}{d\xi^n}(U^B - u_+, \Theta^B - \theta_+) \right| \leq C\delta^B e^{-c_0 \xi}, \quad n = 0, 1, 2, \ldots
\]

2.2. Viscous contact wave. If \((v_-, u_-, \theta_-) \in CD(v_+, u_+, \theta_+), \) i.e.,
\[
u_— = u_+, \quad p_— = p_+
\]
then the following Riemann problem of the Euler system
\[
\begin{aligned}
\partial_t v - \partial_x u &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
\partial_t u + \partial_x p &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
\partial_t (e + \frac{u^2}{2}) + \partial_x (pu) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
(v, u, \theta)(x, 0) &= \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\
(v_+, u_+, \theta_+), & x > 0, \end{cases}
\end{aligned}
\]

admits a single contact discontinuity solution
\[
(v, u, \theta)(x, t) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \quad t > 0, \\
(v_+, u_+, \theta_+), & x > 0, \quad t > 0. \end{cases}
\]

From [11], we know that the viscous version of the above contact discontinuity, called viscous contact discontinuity wave \((V^{CD}, U^{CD}, \Theta^{CD})(x, t)\), could be defined by
\[
\begin{aligned}
\Theta^{CD}(x, t) &= \Theta^{Sim}(\frac{x}{\sqrt{1 + t}}), \\
V^{CD}(x, t) &= \frac{R \Theta^{CD}(x, t)}{p_+}, \\
U^{CD}(x, t) &= u_+ + \frac{\kappa(g - 1) \partial_x \Theta^{CD}(x, t)}{\Theta^{CD}(x, t)},
\end{aligned}
\]

where \( \Theta^{Sim}(\eta) (\eta = \frac{x}{\sqrt{1 + t}}) \) is the unique self-similar solution of the nonlinear diffusion equation
\[
\partial_t \Theta = \frac{\kappa(g - 1)p_+}{R^2 g} \partial_x \left( \frac{\partial_x \Theta}{\Theta} \right), \quad \Theta(\pm \infty, t) = \theta_\pm.
\]

Thus the viscous discontinuity contact wave defined in (27) satisfies the following property
\[
(1 + t)\frac{3}{2} |\partial^2 \Theta^{CD}| + (1 + t) |\partial^2 \Theta^{CD}| + \frac{\kappa(g - 1)}{R^2 g} |\partial_x \Theta^{CD}| + |\Theta^{CD} - \theta_\pm| = O(1) \delta^{CD} e^{-c_0 \xi}, \text{ as } x \to \pm \infty,
\]

where \( \delta^{CD} = |\theta_+ - \theta_-| \) is the amplitude of the viscous contact discontinuity wave and \( c_0 \) is some positive constant. Note that \( \xi = x - \sigma t. \) Then the viscous contact
discontinuity wave \( (V^{CD}, U^{CD}, \Theta^{CD}) (\xi, t) \) satisfies
\[
\begin{align*}
\partial_t V^{CD} - \sigma_{-} \partial_x V^{CD} - \partial_x U^{CD} &= 0, \\
\partial_t U^{CD} - \sigma_{-} \partial_x U^{CD} + \partial_x P^{CD} &= \mu \partial_x \left( \frac{\partial_x U^{CD}}{V^{CD}} \right) + \bar{Q}_1, \\
\frac{R}{\gamma - 1} (\partial_x \Theta^{CD} - \sigma_{-} \partial_x \Theta^{CD}) + P^{CD} \partial_x U^{CD} &= \mu \left( \frac{\partial_x U^{CD}}{V^{CD}} \right)^2 + \kappa \partial_x \left( \frac{\partial_x \Theta^{CD}}{V^{CD}} \right) + \bar{Q}_2,
\end{align*}
\]
where \( P^{CD} := p(V^{CD}, \Theta^{CD}) = \frac{R \theta^{CD}}{V^{CD}} \) and the error terms \( \bar{Q}_1, \bar{Q}_2 \) are given by
\[
\begin{align*}
\bar{Q}_1 &= \partial_t U^{CD} - \sigma_{-} \partial_x U^{CD} - \mu \partial_x \left( \frac{\partial_x U^{CD}}{V^{CD}} \right) \\
&= O(1) \left( |\partial_x \Theta^{CD}| + |\partial_x^2 \Theta^{CD}| \right) + O(1) \delta^{CD} (1 + t)^{-2} e^{-\gamma_{0}(t+\sigma_{-}t)^{2} \gamma_{1}}, \quad \text{as} \quad |\xi + \sigma_{-}t| \to +\infty,
\end{align*}
\]
and
\[
\begin{align*}
\bar{Q}_2 &= - \mu \left( \frac{\partial_x U^{CD}}{V^{CD}} \right)^2 \\
&= O(1) \left( |\partial_x \Theta^{CD}|^4 \right) + O(1) \delta^{CD} (1 + t)^{-2} e^{-\gamma_{0}(t+\sigma_{-}t)^{2} \gamma_{1}}, \quad \text{as} \quad |\xi + \sigma_{-}t| \to +\infty.
\end{align*}
\]

2.3. Rarefaction wave. If \( (v_{-}, u_{-}, \theta_{-}) \in R_{3}(v_{+}, u_{+}, \theta_{+}) \), then there exists a 3-rarefaction wave \( (v^{r}, u^{r}, \theta^{r}) (\frac{\xi}{t}) \) which is the global (in time) weak solution of the following Riemann problem
\[
\begin{align*}
\partial_t v^{r} - \partial_x u^{r} &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
\partial_t u^{r} + \partial_x p(v^{r}, \theta^{r}) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
\frac{R}{\gamma - 1} \partial_x \theta^{r} + p(v^{r}, \theta^{r}) \partial_x u^{r} &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
(v^{r}, u^{r}, \theta^{r})(x, 0) &= \begin{cases}
(v_{-}, u_{-}, \theta_{-}), & x < 0, \\
(v_{+}, u_{+}, \theta_{+}), & x > 0.
\end{cases}
\end{align*}
\]

For the sake of constructing the smooth approximated rarefaction wave, we consider the Riemann problem on the Burgers equation
\[
\begin{align*}
\partial_t \bar{w} + \bar{w} \partial_x \bar{w} &= 0, \\
\bar{w}(x, 0) &= \bar{w}_0(x) = \begin{cases}
w_{-}, & x < 0, \\
w_{+}, & x > 0
\end{cases}
\end{align*}
\]
for \( w_{-} < w_{+} \). It is well-known that the Riemann problem (33) admits a continuous weak solution \( w^{r}(\frac{\xi}{t}) \) connecting \( w_{-} \) and \( w_{+} \), taking the form of
\[
\begin{align*}
w^{r}(\frac{\xi}{t}) &= \begin{cases}
w_{-}, & x \leq w_{-}t, \\
x, & w_{-}t < x < w_{+}t, \\
w_{+}, & w_{+}t \leq x.
\end{cases}
\end{align*}
\]
Moreover, \( w'(\xi) \) is approximated by a smooth function \( w(x,t) \) satisfying
\[
\begin{align*}
\partial_t w + w\partial_x w &= 0, \\
w(x,0) &= w_0(x) = \begin{cases} w_-, & x < 0, \\
w_- + C_q\delta^r \int_0^x y^q e^{-y} dy, & x > 0,
\end{cases}
\end{align*}
\]
where \( \delta^r := w_+ - w_- \), \( q \geq 16 \) is a constant, \( C_q \) is a constant such that \( C_q \int_0^\infty y^q e^{-y} dy = 1 \), and \( \epsilon \leq 1 \) is a small positive constant. Then the solution \( w(x,t) \) of the Burgers equation (35) have the following properties.

**Lemma 2.1** (**[10]**). Let \( 0 < w_- < w_+ \), then Burgers equation (35) has a unique smooth solution \( w(x,t) \) which satisfies the following properties.

(i) \( w_- \leq w(x,t) \leq w_+ \), \( \partial_x w \geq 0 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \).

(ii) For any \( p \ (1 \leq p \leq \infty) \), there exists a positive constant \( C_{p,q} \) such that for \( t \geq 0 \)
\[
\|\partial_t w(t)\|_{L^p} \leq C_{p,q} \min\{\delta^r \epsilon^{1-\frac{1}{p}}, (\delta^r)^{\frac{1}{p}}(1+\frac{1}{p})\},
\]
\[
\|\partial_x^2 w(t)\|_{L^p} \leq C_{p,q} \min\{\delta^r \epsilon^{2-\frac{1}{p}}, [(\delta^r)^{\frac{1}{p}}+(\delta^r)^{\frac{1}{p}}(1+\frac{1}{p})]\},
\]
(iii) When \( x < w_- t \), \( w(x,t) \equiv w_- \).

(iv) \( \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |w(x,t) - w'(\frac{x}{t})| = 0 \).

Thus we construct the smooth approximated rarefaction wave \( (V^R, U^R, \Theta^R)(x,t) \) by
\[
\begin{align*}
S^R(x,t) &= s(V^R(x,t), \Theta^R(x,t)) = s_+,
\lambda_3(V^R(x,t), s_+) &= w(x,1+t),
U^R(x,t) &= u_+ - \int_{v_+}^{V^R(x,t)} \lambda_3(z,s_+) dz.
\end{align*}
\]

Note that \( \xi = x - \sigma_- t \). Then the smooth 3-rarefaction wave \( (V^R, U^R, \Theta^R)(\xi,t) \) defined above satisfies
\[
\begin{align*}
\partial_t V^R - \sigma_- \partial_\xi V^R - \partial_\xi U^R &= 0, \quad \xi > 0, \ t > 0, \\
\partial_t U^R - \sigma_- \partial_\xi U^R + \partial_\xi P^R &= 0, \quad \xi > 0, \ t > 0, \\
R \frac{R - 1}{\gamma - 1} (\partial_\xi \Theta^R - \sigma_- \partial_\xi \Theta^R) + P^R \partial_\xi U^R &= 0, \quad \xi > 0, \ t > 0,
\end{align*}
\]
where \( P^R := p(V^R, \Theta^R) = \frac{R \Theta^R}{\Theta^R} \).

**Lemma 2.2**. Let \( \delta^R = |(v_+, u_+, \theta_+) - (v_-, u_-, \theta_-)| \), then the smooth approximate rarefaction wave \( (V^R, U^R, \Theta^R)(\xi,t) \) satisfies the following properties:

(i) \( \partial_\xi U^R \geq 0 \) for \( \xi \in \mathbb{R}_+ \) and \( t \geq 0 \).

(ii) For any \( 1 \leq p \leq +\infty \), there exists a constant \( C_{p,q} \) such that for \( t \geq 0 \)
\[
\begin{align*}
\|\partial_\xi (V^R, U^R, \Theta^R)\|_{L^p(\mathbb{R}_+)} &\leq C_{p,q} \min\{\delta^R \epsilon^{1-\frac{1}{p}}, (\delta^R)^{\frac{1}{p}}(1+\frac{1}{p})\}, \\
\|\partial_\xi^2 (V^R, U^R, \Theta^R)\|_{L^p(\mathbb{R}_+)} &\leq C_{p,q} \min\{\delta^R \epsilon^{2-\frac{1}{p}}, [(\delta^R)^{\frac{1}{p}}+(\delta^R)^{\frac{1}{p}}(1+\frac{1}{p})]\},
\end{align*}
\]
(iii) If \( \xi + \sigma_- t \leq \lambda_3(v_-, u_-, \theta_-)(1+t) \), then \( (V^R, U^R, \Theta^R)(\xi,t) \equiv (v_-, u_-, \theta_-) \).
3. Reformulation of the problem and main results. Define the composite wave \((V, U, \Theta)(\xi, t)\) by

\[
\begin{pmatrix}
V \\
U \\
\Theta
\end{pmatrix}(\xi, t) = \begin{pmatrix}
V^B + V^{CD} + V^R \\
U^B + U^{CD} + U^R \\
\Theta^B + \Theta^{CD} + \Theta^R
\end{pmatrix}(\xi, t) - \begin{pmatrix}
v_+ + v^* \\
u_+ + u^* \\
\theta_+ + \theta^*
\end{pmatrix},
\]

where \((V^B, U^B, \Theta^B)(\xi, t)\) is the subsonic BL-solution (Case III) defined in Proposition 1 with the right state \((v_+, u_+, \theta_+)\) replaced by \((v_*, u_*, \theta_*)\), \((V^{CD}, U^{CD}, \Theta^{CD})(\xi, t)\) is the viscous contact discontinuity wave defined in (26) with the end states \((v_-, u_-, \theta_-)\), \((v_+, u_+, \theta_+)\) replaced by \((v_*, u_*, \theta_*)\) and \((v^*, u^*, \theta^*)\), respectively, and \((V^R, U^R, \Theta^R)(\xi, t)\) is the smooth 3-rarefaction wave defined in (36) with the left state \((v_-, u_-, \theta_-)\) replaced by \((v^*, u^*, \theta^*)\).

From (19), (29), (37) and (38), and by a careful calculation, we have

\[
\begin{aligned}
\partial_t V - \sigma_\partial \partial_\xi V - \partial_\xi U &= 0, \quad \xi > 0, \ t > 0, \\
\partial_t U - \sigma_\partial \partial_\xi U + \partial_\xi P &= \mu \partial_\xi \left(\frac{(\partial_\xi U)^2}{V}\right) + Q_1, \quad \xi > 0, \ t > 0, \\
\frac{R}{\gamma - 1}(\partial_\xi \Theta - \sigma_\partial \partial_\xi \Theta) + P \partial_\xi U &= \mu \left(\frac{(\partial_\xi U)^2}{V}\right) + \kappa \partial_\xi \left(\frac{(\partial_\xi \Theta)^2}{V}\right) + Q_2, \quad \xi > 0, \ t > 0, \\
(V, U, \Theta)(\xi = 0, t) &= (v_+ + V^{CD} - v_+, u_+ + U^{CD} - u_+, \theta_+ + \Theta^{CD} - \theta_+)(0, t), \\
(V, U, \Theta)(\xi = +\infty, t) &= (v_+, u_+, \theta_+),
\end{aligned}
\]

where \(P := p(V, \Theta) = \frac{R \Theta}{\gamma - 1}\) and the error terms \(Q_1, Q_2\) are given by

\[
Q_1 = \partial_\xi \left[ P - P^B - P^{CD} - P^R \right] - \mu \partial_\xi \left(\frac{(\partial_\xi U)^2}{V}\right) - \partial_\xi \left(\frac{(\partial_\xi U^{CD})^2}{V^{CD}}\right) + \tilde{Q}_1,
\]

\[
Q_2 = \left[ P \partial_\xi U - P^B \partial_\xi U^B - P^{CD} \partial_\xi U^{CD} - P^R \partial_\xi U^R \right] + \tilde{Q}_2
\]

\[
\begin{aligned}
&= \mu \left(\frac{(\partial_\xi U)^2}{V} - \frac{(\partial_\xi U^B)^2}{V^B} - \frac{(\partial_\xi U^{CD})^2}{V^{CD}}\right) - \kappa \left(\frac{(\partial_\xi \Theta)^2}{V} - \frac{(\partial_\xi \Theta^B)^2}{V^B} - \frac{(\partial_\xi \Theta^{CD})^2}{V^{CD}}\right),
\end{aligned}
\]

where \(\tilde{Q}_1, \tilde{Q}_2\) are the error terms defined in (30) and (31) to the viscous contact discontinuity wave.

By a direct calculation and using Proposition 1, Lemma 2.2 and (28), we have the following estimates about error terms \(Q_1\) and \(Q_2\):

\[
Q_1 = O(1) \left[ \left| (\partial_\xi U^B, \partial_\xi V^B, \partial_\xi \Theta^B, \partial_\xi^2 U^B) \right| \times \left| (V - V^B, \Theta - \Theta^B, \partial_\xi V^{CD}, \partial_\xi U^{CD}, \partial_\xi V^R, \partial_\xi U^R) \right| \right. \\
+ \left| (\partial_\xi U^{CD}, \partial_\xi V^{CD}, \partial_\xi \Theta^{CD}, \partial_\xi^2 U^{CD}) \times \left| (V - V^{CD}, \Theta - \Theta^{CD}, \partial_\xi V^R, \partial_\xi U^R) \right| \right. \\
+ \left. \left| (\partial_\xi^2 U^R, \partial_\xi \Theta^R) \times \left| (V - V^R, \Theta - \Theta^R) \right| + \left| (\partial_\xi^2 U^R, \partial_\xi U^R \partial_\xi V^R) \right| \right] + \tilde{Q}_1
\]
\[= O(1)(\delta^B + \delta^{CD})e^{-c(\xi + t)} + O(1) \left| \left( \partial^2_{\xi} U_R, \partial_{\xi} U_R \partial_{\xi} V_R \right) \right| + \bar{Q}_1, \quad (40)\]

\[Q_2 = O(1) \left[ \left| \left( \partial_{\xi} U_B, \partial_{\xi} V_B, \partial_{\xi} \Theta_B, \partial^2_{\xi} \Theta_B \right) \right| \times \left| (V - V^B, \Theta - \Theta^B, \partial_{\xi} V^{CD}, \partial_{\xi} \Theta^{CD}, \partial^2_{\xi} \Theta^{CD}) \right| \right. \]
\[\left. + \left| (V - V^{CD}, \Theta - \Theta^{CD}, \partial_{\xi} V^R, \partial_{\xi} \Theta^R) \right| \times \left| (V - V^R, \Theta - \Theta^R) \right| + \left| \left( \partial^2_{\xi} \Theta^R, \partial_{\xi} \Theta^R \partial_{\xi} V^R, | \partial_{\xi} U_R|^2 \right) \right| \right] + \bar{Q}_2, \quad (41)\]

where \( c \) is a positive constant independent of \( \xi \) and \( t \).

We first define the perturbation

\[ [\varphi, \psi, \omega, \zeta](\xi, t) = [v - V, u - U, \omega - 0, \theta - \Theta](\xi, t). \]

Then from (11) and (39), it is easy to obtain that \([\varphi, \psi, \omega, \zeta](\xi, t)\) satisfies

\[\begin{aligned}
\partial_t \varphi - \sigma_{-} \partial_\xi \varphi - \partial_{\xi} \psi &= 0, \quad \xi > 0, \quad t > 0, \\
\partial_t \psi - \sigma_{-} \partial_\xi \psi + \partial_\xi (p - P) &= \mu \partial_{\xi} \left( \frac{\partial_\xi u}{v} - \frac{\partial_{\xi} U}{V} \right) - Q_1, \quad \xi > 0, \quad t > 0, \\
\partial_\xi \omega - \sigma_{-} \partial_{\xi} \omega + A \nu \omega &= A \partial_\xi \left( \frac{\partial_\xi \omega}{v} \right), \quad \xi > 0, \quad t > 0, \\
\frac{R}{\gamma - 1} (\partial_\xi \zeta - \sigma_{-} \partial_{\xi} \zeta) + (p \partial_{\xi} u - p \partial_{\xi} U) &= \kappa \partial_{\xi} \left( \frac{\partial_\xi \theta}{v} - \frac{\partial_{\xi} \Theta}{V} \right) \\
&\quad + \mu \left( \frac{(\partial_\xi u)^2}{v} - \frac{(\partial_{\xi} U)^2}{V} \right) + \left( \frac{(\partial_{\xi} \omega)^2}{v} + \nu \omega^2 \right) - Q_2, \quad \xi > 0, \quad t > 0, \\
[\varphi, \psi, \omega, \zeta](\xi, 0) &= [\varphi_0, \psi_0, \omega_0, \zeta_0](\xi) = [v_0(\xi) - V(\xi, 0), u_0(\xi) - U(\xi, 0), \\
&\quad \omega_0(\xi) - 0, \theta_0(\xi) - \Theta(\xi, 0)] \rightarrow (0, 0, 0, 0), \quad \text{as } \xi \rightarrow +\infty, \\
[\varphi, \psi, \omega, \zeta](0, t) &= (v_-, u_- - U, 0, \theta_+ - \Theta)(0, t). \end{aligned}\]

Concerning the global existence and time-asymptotic properties of solutions to the above reformulated half-space problem (42), one has the following theorem.

**Theorem 3.1.** For any given \([v_\pm, u_\pm, \omega_\pm, \theta_\pm]\) with \(v_\pm > 0, u_+ > 0 \text{ and } \theta_+ > 0\), we suppose that \(u_+ > 0, \omega_+ = 0 \text{ and } \theta_+ > 0\). Let \([V, U, \Theta](\xi, t)\) be the composite wave consisting of the subsonic BL-solution, the viscous contact discontinuity wave, and the rarefaction wave defined in (38) with the BL-solution amplitude \(\delta^B\) and the contact discontinuity wave amplitude \(\delta^{CD}\). There exist positive constants \(\delta_0 > 0\) and \(C_0 > 0\), such that if

\[\begin{aligned}
[\varphi_0, \psi_0, \omega_0, \zeta_0](\xi) &\in H^1(\mathbb{R}_+) , \\
\end{aligned}\]

and

\[\|[\varphi_0, \psi_0, \omega_0, \zeta_0](\xi)\|_{H^1(\mathbb{R}_+)}^2 + \delta^B + \delta^{CD} + \epsilon^{\frac{1}{2}} \leq \delta_0, \quad (43)\]

then the micropolar fluid model to the inflow problem (9) or the half-space problem (11) admits a unique global solution \([v, u, \omega, \theta](\xi, t)\) satisfying

\[\begin{aligned}
[v - V, u - U, \omega, \theta - \Theta] &\in C(0, +\infty; H^1(\mathbb{R}_+)), \\
\end{aligned}\]
and
\[
\sup_{t \geq 0} \|v - V, u - U, \omega, \theta - \Theta\|_{H^1(0, +\infty)} \leq C_0 \delta_0^{1/2}.
\] (44)
Moreover, it holds that
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} \|v - V, u - U, \omega, \theta - \Theta\| = 0.
\] (45)

Remark 1. \(\epsilon\) is a small positive constant defined in (35).

4. Global existence and large time behavior. In this section, compared with previous results about the global existence of solutions near constant states for the micropolar fluid model, we need to prove global existence and large time behavior of smooth solutions near non-constant states. The key to the proof of global existence part of Theorem 3.1 is to derive the uniform \textit{a priori} estimates of solutions to the half-space problem (42). Our \textit{a priori} assumption is defined as follows:
\[
\sup_{0 \leq \tau \leq t} \|\varphi, \psi, \omega, \zeta(\tau)\|_{H^{1}(\mathbb{R}_+)} \leq \epsilon_1,
\] (46)
where \(\epsilon_1\) is a small positive constant.

**Proposition 2** (A priori estimates). Assume all the conditions listed in Theorem 3.1 hold. Let \([\varphi, \psi, \omega, \zeta](\xi, t)\) be a solution to the half-space problem (42) on \(0 \leq t \leq T\) for some positive constant \(T\). There are constants \(\delta_0 > 0\) and \(C > 0\), such that if \([\varphi, \psi, \omega, \zeta] \in C(0, T; H^1(\mathbb{R}_+))\) and
\[
\|\varphi_0, \psi_0, \omega_0, \zeta_0\|_{H^1(\mathbb{R}_+)} \leq \delta_0,
\]
then for any \(t \in [0, T]\), the solution \([\varphi, \psi, \omega, \zeta](\xi, t)\) satisfies
\[
\sup_{0 \leq \tau \leq t} \|\varphi, \psi, \omega, \zeta(\tau)\|_{H^{1}(\mathbb{R}_+)} \leq C (\epsilon_1^{1/4} + \delta^{1/2} + \delta^{2CD}) + \epsilon_1^{1/2} \leq \delta_0
\] (47)
and
\[
\sup_{0 \leq \tau \leq t} \|\varphi, \psi, \omega, \zeta(\tau)\|_{H^{1}(\mathbb{R}_+)} \leq C (\epsilon_1^{1/4} + \delta^{1/2} + \delta^{2CD}) + \epsilon_1^{1/2}.
\] (48)

From \textit{a priori} assumption (46), it is easy to get
\[
\|\varphi, \psi, \omega, \zeta\|_{L^\infty} \leq \sqrt{2}\epsilon_1,
\] (49)
where the following Sobolev inequality
\[
\|h(\xi)\|_{L^\infty} \leq \sqrt{2}\|h\|^{1/2}_1\|h\|^{1/2}_1, \quad \text{for} \, \, h(\xi) \in H^{1}(\mathbb{R}_+)
\] (50)
is used.

**Lemma 4.1** (Boundary estimates). There exists a positive constant \(C\) such that for any \(t > 0\),
\[
\int_0^t \|\varphi, \psi, \zeta\|^2 + |\partial_\tau(\varphi, \psi, \zeta)|^2(0, \tau)d\tau \leq C(\delta^{2CD}),
\] (51)
\[
\int_0^t \left[\mu \left(\frac{\partial u}{v} - \frac{\partial U}{V}\right) \right](0, \tau)d\tau \leq \nu \int_0^t (\|\partial_\xi \varphi\|^2 + \|\partial_\xi^2 \varphi\|^2)d\tau + C_\nu (\delta^{2CD})^2,
\] (52)
\[
\int_0^t \left[\kappa \left(\frac{\partial \varphi}{v} - \frac{\partial U}{V}\right) \right](0, \tau)d\tau \leq \nu \int_0^t (\|\partial_\xi \varphi\|^2 + \|\partial_\xi^2 \varphi\|^2)d\tau + C_\nu (\delta^{2CD})^2,
\] (53)
and
\[
\int_0^t \left( |\partial_\xi \psi|^2 + (\partial_\xi \varphi)^2 \right) \, d\tau \leq \nu \int_0^t |\partial_\xi^2 \psi|^2 \, d\tau + C_\nu \int_0^t |\partial_\xi \psi|^2 \, d\tau + C(\delta^{CD})^2
\]
where \( \nu \) is a positive small constant to be determined later, and \( C_\nu \) is a positive constant depending on \( \nu \).

**Proof.** The proof of (51)-(53) could be found in [30](Lemma 3.1). Here we only prove (54). It is easy to obtain that
\[
\int_0^t |\partial_\xi \psi|^2 \, d\tau \leq \nu \int_0^t |\partial_\xi^2 \psi|^2 \, d\tau + C_\nu \int_0^t |\partial_\xi \psi|^2 \, d\tau
\]
where we have used Sobolev’s inequality (50) in the second inequality.

From (42), we have
\[
\partial_\xi \varphi(0, \tau) = \frac{\partial_\tau \varphi(0, \tau) - \partial_\xi \psi(0, \tau)}{\sigma_-}
\]
Then using the Cauchy–Schwarz’s inequality, (55), (42) and (28), we have
\[
\int_0^t |\partial_\xi \psi|^2 \, d\tau \leq C \int_0^t |\partial_\xi^2 \psi|^2 \, d\tau + C_\nu \int_0^t |\partial_\xi \psi|^2 \, d\tau + C \int_0^t |\partial_\tau V|^2 \, d\tau
\]
where we have used
\[
\partial_\tau V(0, \tau) = \partial_\tau V^{CD}(0, \tau) = O(1)\delta^{CD}(1 + \tau)^{-1}e^{-\frac{\theta}{\left(\frac{1}{\sigma_+} + \frac{1}{\sigma_-}\right)}}
\]
and
\[
\partial_\tau V^B(0, \tau) = \partial_\tau V^R(0, \tau) = 0.
\]

**Lemma 4.2.** Assume the conditions of Proposition 2 hold, then we have the following energy estimate for any \( t \in [0, T] \),
\[
||[\psi, \varphi, \zeta, \omega]||^2 + \int_0^t \left( ||\partial_\xi [\psi, \zeta, \omega]||^2 + ||\omega||^2 \right) \, d\tau + \int_0^t \sqrt{\partial_\xi U^R[\varphi, \zeta]} ||^2 \, d\tau
\]
\[
\leq C ||[\psi_0, \varphi_0, \zeta_0, \omega_0]||^2 + C(\epsilon^\gamma + \delta^B + \delta^{CD}) + C(\epsilon^\gamma + \delta^B) \int_0^t ||\partial_\xi \varphi||^2 \, d\tau
\]
\[
+ C\delta^{CD} \int_0^t \int_\mathbb{R}_+ (1 + \tau)^{-1}e^{-\frac{\theta}{\left(\frac{1}{\sigma_+} + \frac{1}{\sigma_-}\right)}} (\varphi^2 + \zeta^2) \, d\xi \, d\tau.
\]

**Proof.** Multiplying (42), (42), (42) and (42) by \(- R \Theta \left( \frac{1}{\gamma} - \frac{1}{\gamma} \right)\), \(\psi\), \(\omega\) and \(\zeta\), respectively, and taking the summation of the resulting equations, we obtain
\[
\partial_\xi \left( \frac{1}{2} \psi^2 + R \Theta \Phi \left( \frac{v}{V} \right) \right) + \frac{R \Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} + \partial_\xi H_1 + \mu \frac{\Theta(\partial_\xi \psi)^2}{V^2} + \kappa \frac{\Theta}{v^2 \Theta^2} (\partial_\xi \zeta)^2
\]
\[
+ Av^2 + \frac{A}{v} (\partial_\xi \omega)^2 + P \partial_\xi U^R \left[ \Phi \left( \frac{\Theta V}{v \Theta} \right) + \gamma \Phi \left( \frac{v}{V} \right) \right]
\]
\[ Q_3 = Q_1 \psi - \frac{\zeta}{\theta} Q_2 + \frac{\zeta}{\theta} \left[ \frac{(\partial_\xi \omega)^2}{v} + v \omega^2 \right], \]  

(58)

where

\[ \Phi(s) = s - 1 - \ln s, \]

\[ H_1 = - \sigma - \left( \frac{1}{2} \psi^2 + R \Theta \Phi \left( \frac{V}{V} \right) + \frac{R \Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} \right) + (p - P) \psi \]

\[ - \mu \left( \frac{\partial_\xi u}{v} - \frac{\partial_\xi U}{V} \right) \psi - \kappa \frac{\zeta}{\theta} \left( \frac{\partial_\xi \theta}{V} - \frac{\partial_\xi \Theta}{V} \right) - A \omega \partial_\xi \omega \cdot v, \]

and

\[ Q_3 = - P \left( \partial_\xi U^B + \partial_\xi U^{CD} \right) \left[ \Phi \left( \frac{\theta V}{v \Theta} \right) + \gamma \Phi \left( \frac{v}{V} \right) \right] \]

\[ + \left[ \frac{1}{2} (\partial_\xi U)^2 \frac{1}{V} + \kappa \partial_\xi \left( \frac{\partial_\xi \Theta}{V} \right) + Q_2 \right] \left[ (\gamma - 1) \Phi \left( \frac{V}{V} \right) - \Phi \left( \frac{\Theta}{\Theta} \right) \right] + \kappa \frac{\Theta \psi \partial_\xi \epsilon}{\theta^2 V} \partial_\xi \Theta - \kappa \frac{\zeta \varphi}{\theta^2 V} (\partial_\xi \Theta)^2 + \mu \frac{\partial_\xi U}{v \Theta} \psi - \mu (\partial_\xi U)^2 \frac{\varphi}{v \Theta} + 2 \mu \frac{\partial_\xi U}{v \Theta} \zeta \partial_\xi \psi. \]

Then integrating the resulting identity (58) over \( \mathbb{R} \times [0, t] \), we thus arrive at

\[ \int_{\mathbb{R}^+} \frac{\psi^2}{2} + R \Theta \Phi \left( \frac{V}{V} \right) + \frac{R \Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} \right) d\xi + \mu \int_0^t \int_{\mathbb{R}^+} \frac{\Theta (\partial_\xi \psi)^2}{v \Theta} d\xi d\tau \]

\[ + \kappa \int_0^t \int_{\mathbb{R}^+} \frac{\Theta (\partial_\xi \epsilon)^2}{v \theta^2} d\xi d\tau + \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} A v \omega^2 d\xi d\tau + \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} A (v \partial_\xi \omega)^2 d\xi d\tau \]

\[ + \int_0^t \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} P \partial_\xi U^R \left[ \Phi \left( \frac{\theta V}{v \Theta} \right) + \gamma \Phi \left( \frac{v}{V} \right) \right] d\xi d\tau \]

\[ = \int_{\mathbb{R}^+} \left( \frac{\psi^2}{2} + R \Theta \Phi \left( \frac{V}{V} \right) + \frac{R \Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} \right) (\xi, 0) d\xi + \int_0^t \int_{\mathbb{R}^+} H_1 (0, \tau) d\tau \]

\[ + \int_0^t \int_{\mathbb{R}^+} Q_3 d\xi d\tau - \int_0^t \int_{\mathbb{R}^+} Q_1 \psi d\xi d\tau - \int_0^t \int_{\mathbb{R}^+} \frac{\zeta}{\theta} Q_2 d\xi d\tau \]

\[ + \int_0^t \int_{\mathbb{R}^+} \frac{\zeta}{\theta} \left[ \frac{(\partial_\xi \omega)^2}{v} + v \omega^2 \right] d\xi d\tau. \]

From the definition of \( \Phi(\cdot) \) and the smallness of perturbation solutions \([\varphi, \psi, \omega, \zeta]\), we have

\[ \frac{\psi^2}{2} + R \Theta \Phi \left( \frac{V}{V} \right) + \frac{R \Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) + \frac{\omega^2}{2} = O(1)(\varphi^2 + \psi^2 + \omega^2 + \zeta^2), \]

(60)

and

\[ \Phi \left( \frac{\theta V}{v \Theta} \right) + \gamma \Phi \left( \frac{v}{V} \right) = O(1)(\varphi^2 + \zeta^2). \]

(61)

Since \( \partial_\xi U^R \geq 0 \), we know

\[ \int_0^t \int_{\mathbb{R}^+} P \partial_\xi U^R \left[ \Phi \left( \frac{\theta V}{v \Theta} \right) + \gamma \Phi \left( \frac{v}{V} \right) \right] d\xi d\tau \geq c \int_0^t \int_{\mathbb{R}^+} \partial_\xi U^R \varphi^2 d\xi d\tau, \]

(62)

where (61) has been used.

By applying the \textit{a priori} assumption (46), (42), (40), (41), (30), (31), Cauchy-Schwarz’s inequality with \( 0 < \nu < 1 \), (60), (61), (49), Sobolev’s inequality (50) and
Lemma 4.1[(51)-(53)], we obtain the estimates for the right hand side of (59) as follows:

\[\int_{\mathbb{R}^+} \left( \frac{\psi^2}{2} + R\Theta \Phi \left( \frac{v}{\sqrt{\nu}} \right) + \frac{R}{\gamma - 1} \phi \left( \frac{\theta}{\Theta} \right) + \omega^2 \right) \xi, 0) d\xi \leq c ||[\varphi, \psi, \omega, \zeta]||^2, \tag{63}\]

\[\int_0^t H_1(0, \tau) d\tau \leq \nu \int_0^t (||\partial_\xi [\psi, \zeta]||^2 + ||\partial^2_\xi [\psi, \zeta]||^2) d\tau + C\nu (\delta^{CD})^2, \tag{64}\]

\[\int_0^t \int_{\mathbb{R}^+} \xi \left[ (\partial_\xi^2 \omega)^2 \nu + v(v^2 + \nu^2) \right] d\xi d\tau \leq C \int_0^t ||\xi||_{\infty} (||\partial_\xi \omega||^2 + ||\omega||^2) d\tau \tag{65}\]

and

\[\int_0^t \int_{\mathbb{R}^+} Q_3 d\xi d\tau \leq \nu \int_0^t ||\partial_\xi [\psi, \zeta]||^2 d\tau + (C\nu + C) \int_0^t \int_{\mathbb{R}^+} (|\partial_\xi^2 \Theta^B, \partial_\xi \Theta^B, \partial_\xi V^B, \partial_\xi U^B|)(\varphi^2 + \zeta^2) d\xi d\tau \tag{66}\]

\[+ (C\nu + C) \int_0^t \int_{\mathbb{R}^+} |(\partial_\xi^2 \Theta^R, (\partial_\xi \Theta^R)^2, (\partial_\xi V^R)^2, (\partial_\xi U^R)^2)| (\varphi^2 + \zeta^2) d\xi d\tau \]

\[+ (C\nu + C) \int_0^t \int_{\mathbb{R}^+} |(\partial_\xi^2 \Theta^{CD}, (\partial_\xi \Theta^{CD})^2)| (\varphi^2 + \zeta^2) d\xi d\tau \]

\[+ C \int_0^t \int_{\mathbb{R}^+} |Q_2|(\varphi^2 + \zeta^2) d\xi d\tau. \]

For $I_1$, we have

\[I_1 \leq C\delta^B \int_0^t \int_{\mathbb{R}^+} e^{-\xi} ((\varphi, \zeta)^2(0, \tau) + \xi ||\partial_\xi [\varphi, \zeta]||^2) d\xi d\tau \]

\[\leq C\delta^B \int_0^t \int_{\mathbb{R}^+} (||\varphi, \zeta||^2(0, \tau)) d\tau + C\delta^B \int_0^t \int_{\mathbb{R}^+} ||\partial_\xi [\varphi, \zeta]||^2 d\xi d\tau \tag{67}\]

\[\leq C\delta^B (\delta^{CD})^2 + C\delta^B \int_0^t \int_{\mathbb{R}^+} ||\partial_\xi [\varphi, \zeta]||^2 d\xi d\tau, \]

where we have used (22), (51) and the fact that

\[|f(\xi)| = |f(0) + \int_0^\xi \partial_\xi f dy| \leq |f(0)| + \sqrt{\xi} ||\partial_\xi f||. \tag{68}\]

From Lemma 2.2, we have

\[I_2 \leq \int_0^t \int_{\mathbb{R}^+} (||\partial_\xi [V^R, U^R, \Theta^R]||^2 + ||\partial_\xi^2 \Theta^R||_{L^1} ||[\varphi, \zeta]||^2_{L^\infty} ) d\xi d\tau \]

\[\leq C\epsilon^\frac{1}{2} \int_0^t (1 + \tau)^{-\frac{1}{2}} ||[\varphi, \zeta]|| ||\partial_\xi [\varphi, \zeta]|| d\tau \leq C\epsilon^\frac{1}{2} + C\epsilon^\frac{1}{2} \int_0^t ||\partial_\xi [\varphi, \zeta]||^2 d\tau, \tag{69}\]
where we have used
\[ \| \partial_\xi [V^R, U^R, \Theta^R] \|^2 \leq C \epsilon^\frac{1}{5} (1 + t)^{-\frac{7}{5}}, \]
and
\[ \| \partial_\xi^2 \Theta^R \|_{L^1} \leq C \epsilon^\frac{1}{5} (1 + t)^{-\frac{7}{5}}. \]

From the properties of the viscous contact wave, we can get
\[ I_3 \leq C \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\varphi + \sigma + \tau)^2}{1 + \tau}} (\varphi^2 + \zeta^2) d\xi d\tau. \]

Similar to the estimates of \( I_2 \), we have
\[ I_4 \leq C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) + C \epsilon^\frac{1}{5} \int_0^t \| \partial_\xi [\varphi, \zeta] \|^2 d\tau, \]
where we have used (41) and (31).

Thus substituting (67)-(71) into (66), we have
\[
\int_0^t \int_{\mathbb{R}^+} Q_3 d\xi d\tau \leq [\nu + (C + C)(\epsilon^\frac{1}{5} + \delta_B)] \int_0^t \| \partial_\xi [\varphi, \psi, \zeta] \|^2 d\tau + C \epsilon^\frac{1}{5} \int_0^t \| \partial_\xi \varphi \|^2 d\tau,
\]
\[
+ (C' + C) \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\varphi + \sigma + \tau)^2}{1 + \tau}} (\varphi^2 + \zeta^2) d\xi d\tau.
\]

Now we estimate the last two terms as follows:
\[
\int_0^t \int_{\mathbb{R}^+} Q_1 d\xi d\tau \leq C \int_0^t \| \psi \|_{L^\infty} \| Q_1 \|_{L^1} d\tau
\]
\[
\leq C \int_0^t \| \psi \|_{L^\infty} \| \partial_\xi \psi \|_{L^1} \left[ (\delta_B + \delta^{CD}) e^{-c_0 \varphi^2} + (1 + \tau)^{-\frac{7}{5}} \right] d\tau
\]
\[
\leq C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) + C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) \int_0^t \| \partial_\xi \psi \|^2 d\tau,
\]
and
\[
\int_0^t \int_{\mathbb{R}^+} Q_2 d\xi d\tau \leq C \int_0^t \| \zeta \|_{L^\infty} \| Q_2 \|_{L^1} d\tau
\]
\[
\leq C \int_0^t \| \zeta \|_{L^\infty} \| \partial_\xi \zeta \|_{L^1} \left[ (\delta_B + \delta^{CD}) e^{-c_0 \varphi^2} + (\delta^{CD})^2 (1 + \tau)^{-\frac{7}{5}} + \epsilon^\frac{1}{5} (1 + \tau)^{-\frac{7}{5}} \right] d\tau
\]
\[
\leq C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) + C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) \int_0^t \| \partial_\xi \zeta \|^2 d\tau.
\]

Substituting the above estimates into (59), and letting \( \nu, \epsilon, \delta_B, \delta^{CD} \) and \( \epsilon_1 \) be suitably small, we obtain (57) and thus complete the proof of Lemma 4.2.

**Lemma 4.3.** Assume the conditions in Proposition 2 hold, then we have the following energy estimate for \( t \in [0, T] \),
\[
\| \partial_\xi \varphi \|^2 + \int_0^t \| \partial_\xi \varphi \|^2 d\tau
\]
\[
\leq C \| [\psi_0, \zeta_0, \omega_0] \|^2 + C \| \varphi_0 \|^2_{H^1} + C (\epsilon^\frac{1}{5} + \delta_B + \delta^{CD}) + \nu \int_0^t \| \partial_\xi^2 \psi \|^2 d\tau
\]
\[
+ C \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\varphi + \sigma + \tau)^2}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau.
\]
Proof. We first differentiate (42)1 with respect to \( \xi \) and then obtain
\[
\partial_\xi \partial_\xi \varphi - \sigma_i \partial_\xi^2 \varphi - \partial_\xi^2 \psi = 0.
\] (76)

Then multiplying (42)2 and (76) by \(-\nu \partial_\xi \varphi \) and \(\mu \partial_\xi \varphi \), respectively, and integrating the resulting equalities over \( \mathbb{R}_+ \times [0, t] \), one has
\[
- \int_0^t \int_{\mathbb{R}_+} \partial_\xi \psi v \partial_\xi \varphi d\xi d\tau + \sigma_- \int_0^t \int_{\mathbb{R}_+} \partial_\xi \psi v \partial_\xi \varphi d\xi d\tau
- \int_0^t \int_{\mathbb{R}_+} \partial_\xi (\nu - P) \psi v \partial_\xi \varphi d\xi d\tau + \mu \int_0^t \int_{\mathbb{R}_+} \partial_\xi^2 \psi v \partial_\xi \varphi d\xi d\tau
- \mu \int_0^t \int_{\mathbb{R}_+} \partial_\xi (\nu - 1) \partial_\xi uv \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi^2 U (\frac{1}{v} - \frac{1}{v}) \psi v \partial_\xi \varphi d\xi d\tau
+ \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi U \partial_\xi (V - 1) \psi v \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} Q_1 \psi v \partial_\xi \varphi d\xi d\tau,
\] (77)
and
\[
\mu \int_0^t \int_{\mathbb{R}_+} (\partial_\xi \partial_\xi \varphi - \sigma_i \partial_\xi^2 \varphi - \partial_\xi^2 \psi) \partial_\xi \varphi d\xi d\tau = 0.
\] (78)

The summation of (77) and (78) further implies
\[
- \int_{\mathbb{R}_+} \psi v \partial_\xi \varphi d\xi + \mu \int_{\mathbb{R}_+} (\partial_\xi \varphi)^2 d\xi + \int_0^t \int_{\mathbb{R}_+} P(\partial_\xi \varphi)^2 d\xi d\tau
= - \int_{\mathbb{R}_+} \psi_0(\xi) v_0(\xi) \partial_\xi \varphi_0(\xi) d\xi + \mu \int_{\mathbb{R}_+} (\partial_\xi \varphi_0(\xi))^2 d\xi + \frac{\mu |\sigma_-|}{2} \int_0^t (\partial_\xi \varphi)^2 (0, \tau) d\tau
- \int_0^t \int_{\mathbb{R}_+} \psi \partial_\xi v \partial_\xi \varphi d\xi d\tau - \int_0^t \int_{\mathbb{R}_+} \psi v \partial_\xi \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} R \partial_\xi \left[ \frac{\Theta}{\nu} \right] v \partial_\xi \varphi d\xi d\tau
- \mu \int_0^t \int_{\mathbb{R}_+} \partial_\xi (\nu - 1) \partial_\xi u v \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi U \partial_\xi (V - 1) \psi v \partial_\xi \varphi d\xi d\tau
+ \int_0^t \int_{\mathbb{R}_+} \mu \partial_\xi^2 U (\frac{1}{v} - \frac{1}{v}) \psi v \partial_\xi \varphi d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} Q_1 \psi v \partial_\xi \varphi d\xi d\tau.
\] (79)

By applying the a priori assumption (46), Cauchy-Schwarz’s inequality with \( 0 < \nu < 1 \), Sobolev’s inequality (50) and Lemma 4.1[51,54], we turn to estimate \( I_i (5 \leq i \leq 14) \) as follows
\[
|I_5| \leq \nu \int_0^t \| \partial_\xi^2 \psi \|^2 d\tau + |C_\nu \int_0^t \| \partial_\xi \psi \|^2 d\tau + C(\delta^{CD})^2,\]
\[
|I_6| \leq C \int_0^t \int_{\mathbb{R}^+} |\psi \partial_\xi \psi \partial_\xi \varphi| \, d\xi \, dt + C \int_0^t \int_{\mathbb{R}^+} |\psi \partial_\xi U \partial_\xi \varphi| \, d\xi \, dt + C \int_0^t \int_{\mathbb{R}^+} |\psi (\partial_\xi \varphi)^2| \, d\xi \, dt + C \int_0^t \int_{\mathbb{R}^+} |\psi \partial_\xi V \partial_\xi \varphi| \, d\xi \, dt \\
\leq C \nu (\delta^{CD} + \delta^B + \epsilon^2) + C (\epsilon_1 + \nu + \delta^B) \int_0^t \|\partial_\xi [\psi, \varphi]\|^2 \, dt \\
+ C \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0 (\xi + \tau - \tau)^2}{1 + \tau}} \psi^2 \, d\xi \, dt,
\]

\[
|I_7| \leq \int_0^t \int_{\mathbb{R}^+} \frac{c_0 (\xi + \tau - \tau)^2}{1 + \tau} \psi^2 \, d\xi \, dt,
\]

\[
|I_8| + |I_9| + |I_{10}|
\leq (\nu + C \epsilon_1) \int_0^t \|\partial_\xi \varphi\|^2 \, dt + C \nu \int_0^t \|\partial_\xi [\zeta, \psi]\|^2 \, dt + C_\nu \int_0^t \|\partial_\xi [\varphi, \zeta]\|^2 \, dt + C_\nu \int_0^t \|\partial_\xi \varphi\|^2 \, dt + C \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0 (\xi + \tau - \tau)^2}{1 + \tau}} (\varphi^2 + \zeta^2) \, d\xi \, dt,
\]

\[
|I_{11} + I_{12} + I_{13}|
\leq C \int_0^t \int_{\mathbb{R}^+} (\varphi^2 + \zeta^2) (\partial_\xi \Theta)^2 \, d\xi \, dt
\]

\[
\leq C (\epsilon^2 + \delta^B) + (\nu + C \epsilon_1 + C \epsilon^2 + C \delta^B) \int_0^t \|\partial_\xi [\varphi, \zeta]\|^2 \, dt
\]

\[
+ C \nu \int_0^t \|\partial_\xi [\varphi, \zeta]\|^2 \, dt + C \delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0 (\xi + \tau - \tau)^2}{1 + \tau}} (\varphi^2 + \zeta^2) \, d\xi \, dt,
\]

and

\[
|I_{14}| \leq \nu \int_0^t \|\partial_\xi \varphi\|^2 \, dt + C \nu \int_0^t \|Q_1\|^2 \, dt \leq \nu \int_0^t \|\partial_\xi \varphi\|^2 \, dt + C_\nu (\epsilon^2 + \delta^B + \delta^{CD}).
\]

Substituting the above estimates for \( I_i \) (5 \( \leq i \leq 14 \)) and (57) into (79), letting \( \nu, \epsilon, \delta^B, \delta^{CD} \) and \( \epsilon_1 \) be suitably small, and using Cauchy-Schwarz’s inequality, we obtain (75). Thus we complete the proof of Lemma 4.3. \( \square \)
Lemma 4.4. Assume the conditions in Proposition 2 hold, then we have the following energy estimate for \( t \in [0, T] \),
\[
\|\partial_t [\psi, \omega, \zeta]\|^2 + \int_0^t \|\partial_x^2 [\psi, \omega, \zeta]\|^2 d\tau \\
\leq C\|\varphi_0, \psi_0, \zeta_0, \omega_0\|^2_{H^1(\mathbb{R}_+)} + C(\epsilon \delta + \delta^B + \delta^{CD}) \\
+ C\delta^{CD} \int_0^t \int_{\mathbb{R}_+} (1 + \tau)^{-1} e^{-\frac{c_0(\epsilon + \delta + \tau)^2}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau.
\]

Proof. Multiplying (42) by \(-\partial_x^2 \psi\), and integrating the resulting equality over \( \mathbb{R}_+ \times [0, t] \), one has
\[
\frac{1}{2} \int_{\mathbb{R}_+} (\partial_x \psi)^2 d\xi + \mu \int_0^t \int_{\mathbb{R}_+} \frac{(\partial_x^2 \psi)^2}{\nu} d\xi d\tau \\
= \frac{1}{2} \int_{\mathbb{R}_+} (\partial_x \psi_0)^2 d\xi - \int_0^t (\partial_x \psi \partial_x \psi)(0, \tau) d\tau + \frac{\sigma}{2} \int_0^t (\partial_x \psi)^2(0, \tau) d\tau \\
+ \int_0^t \int_{\mathbb{R}_+} \partial_x (\nu - P) \partial_x^2 \psi d\xi d\tau + \mu \int_0^t \int_{\mathbb{R}_+} \partial_x \psi \partial_x \psi d\xi d\tau \\
+ \mu \int_0^t \int_{\mathbb{R}_+} \frac{\partial_x U}{\nu} \partial_x^2 \psi d\xi d\tau - \mu \int_0^t \int_{\mathbb{R}_+} \partial_x \left( \frac{\partial_x U}{\nu} - \frac{\partial_x U}{V} \right) \partial_x^2 \psi d\xi d\tau \\
+ \int_0^t \int_{\mathbb{R}_+} Q_1 \partial_x^2 \psi d\xi d\tau.
\]

We now turn to compute \( I_i \) (15 \leq i \leq 21) term by term. For brevity, we directly give the following computations:
\[
|I_{15}| + |I_{16}| \leq \nu \int_0^t \|\partial_x^2 \psi\|^2 d\tau + C_\nu \int_0^t \|\partial_x \psi\|^2 d\tau + C(\delta^{CD})^2,
\]
\[
|I_{17}| \leq C \int_0^t \int_{\mathbb{R}_+} |\partial_x [\zeta, \varphi]| \partial_x^2 \psi | d\xi d\tau + C \int_0^t \int_{\mathbb{R}_+} ||[\zeta, \varphi]| \partial_x [\varphi, V]| \partial_x^2 \psi | d\xi d\tau \\
\leq C(\epsilon \delta + \delta^B) + (C_\nu + \nu) \int_0^t \|\partial_x [\varphi, \partial_x \psi]\|^2 d\tau \\
+ [C_\nu + C(\epsilon \delta + \delta^B)] \int_0^t \|\partial_x [\zeta, \varphi]\|^2 d\tau \\
+ C\delta^{CD} \int_0^t \int_{\mathbb{R}_+} (1 + \tau)^{-1} e^{-\frac{c_0(\epsilon + \delta + \tau)^2}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau,
\]
\[
|I_{18}| + |I_{19}| \leq C(\epsilon + \delta^B + \delta^{CD} + \epsilon_1) \int_0^t \|\partial_x [\psi, \partial_x \psi]\|^2 d\tau,
\]
\[ |I_{20}| \leq C \int_0^t \int_{\mathbb{R}_+} |\partial^2_{\xi} U \varphi \partial^2_{\xi} \psi| \, d\xi \, d\tau + C \int_0^t \int_{\mathbb{R}_+} |\partial_\xi U \partial_\xi \varphi \partial^2_{\xi} \psi| \, d\xi \, d\tau \]

\[ \leq C(\epsilon \hat{\gamma} + \delta^B + \delta^{CD}) + C(\epsilon + \delta^B + \delta^{CD}) \int_0^t \|\partial_\xi [\varphi, \partial_\xi \psi]\|^2 \, d\tau, \]

and
\[ |I_{21}| \leq \nu \int_0^t \|\partial^2_{\xi} \psi\|^2 \, d\tau + C\nu \int_0^t \|Q_1\|^2 \, d\tau \leq \nu \int_0^t \|\partial^2_{\xi} \psi\|^2 \, d\tau + C\nu(\epsilon \hat{\gamma} + \delta^B + \delta^{CD}). \]

Plug the above estimations for \( I_i \) \((15 \leq i \leq 21)\) into (81), and recall (75) and (57), then choose \( \epsilon > 0, \delta^B > 0, \delta^{CD} > 0 \) and \( \nu > 0 \) suitably small, to derive
\[ \|\partial_\xi \psi\|^2 + \int_0^t \|\partial^2_{\xi} \psi\|^2 \, d\tau \leq C\|\xi, \omega_0\|^2 + C\|\varphi_0, \psi_0\|^2_{H^1(\mathbb{R}_+)} + C(\epsilon \hat{\gamma} + \delta^B + \delta^{CD}) \]
\[ + C\delta^{CD} \int_0^t \int_{\mathbb{R}_+} (1 + \tau)^{-1} e^{\frac{\|\omega_0(\xi, \tau)\|^2}{\gamma + \tau}} (\varphi^2 + \psi^2 + \xi^2) \, d\xi \, d\tau. \]  

Multiplying (42) by \(-\partial^2_{\xi} \omega\), and integrating the resulting equality over \( \mathbb{R}_+ \times [0, t] \), we obtain
\[
\frac{1}{2} \int_{\mathbb{R}_+} (\partial_\xi \omega)^2 \, d\xi + A \int_0^t \int_{\mathbb{R}_+} \frac{(\partial^2_{\xi} \omega)^2}{v} \, d\xi \, d\tau + \frac{\|\sigma_\gamma\|}{2} \int_0^t (\partial_\xi \omega)^2(0, \tau) \, d\tau
\]
\[
= \frac{1}{2} \int_{\mathbb{R}_+} (\partial_\xi \omega_0)^2 \, d\xi + A \int_0^t \int_{\mathbb{R}_+} \frac{\partial_\xi \omega \partial_\xi \varphi}{v^2} \partial^2_{\xi} \omega d\xi \, d\tau + A \int_0^t \int_{\mathbb{R}_+} \nu \omega \partial^2_{\xi} \omega d\xi \, d\tau,
\]
where we have used \( \sigma_\gamma < 0 \) to deal with the boundary term.

To obtain the estimates for \( I_i \) \((22 \leq i \leq 24)\), we use Cauchy-Schwarz’s inequality with \( 0 < \nu < 1 \), Sobolev’s inequality (50) and the a priori assumption (46) to obtain
\[
|I_{22}| + |I_{23}| + |I_{24}|
\]
\[
\leq C \int_0^t \|\partial_\xi \omega\| \|\partial_\xi \varphi\| \|\partial^2_{\xi} \omega\| \, d\tau + C(\epsilon + \delta^{CD} + \delta^B + \nu) \int_0^t \|\partial_\xi [\omega, \partial_\xi \omega]\|^2 \, d\tau
\]
\[
+ C\nu \int_0^t \|\omega\|^2 \, d\tau
\]
\[
\leq C \int_0^t (\|\partial_\xi \omega\| + \|\partial^2_{\xi} \omega\| \|\partial_\xi \varphi\| \|\partial^2_{\xi} \omega\| \, d\tau + C(\epsilon + \delta^{CD} + \delta^B + \nu) \int_0^t \|\partial_\xi [\omega, \partial_\xi \omega]\|^2 \, d\tau
\]
\[
+ C\nu \int_0^t \|\omega\|^2 \, d\tau
\]
\[
\leq C(\epsilon_1 + \nu + \epsilon + \delta^{CD} + \delta^B) \int_0^t \|\partial_\xi [\omega, \partial_\xi \omega]\|^2 \, d\tau + C\nu \int_0^t \|\omega\|^2 \, d\tau.
\]

Plug the above estimations into (83), and recall (57), (75), (82), then choose \( \epsilon > 0, \epsilon_1 > 0, \delta^B > 0, \delta^{CD} > 0 \) and \( \nu > 0 \) suitably small, to derive
\[
\|\partial_\xi \omega\|^2 + \int_0^t \|\partial^2_{\xi} \omega\|^2 \, d\tau \leq C\|\xi_0\|^2 + C\|\varphi_0, \psi_0, \omega_0\|^2_{H^1(\mathbb{R}_+)} + C(\epsilon \hat{\gamma} + \delta^B + \delta^{CD})
\]
Sobolev’s inequality (50). This ends the proof of Theorem 3.1. Consequently, (89) together with (48) gives (88). Then (45) follows from (88) and Lemma 4.2-4.4 with Lemma 5.2 in the appendix, and if the wave strength $\delta$ is to prove the large time behavior (45). For this, we first justify the following estimate (48).

\[
\|\partial_\xi \zeta\|^2 + \int_0^t \|\partial_\xi^2 \zeta\|^2 d\tau \leq C\|\varphi_0, \psi_0, \omega_0, \zeta_0\|_{H^1(\mathbb{R}^+)}^2 + C(e^{\|\epsilon\|_B} + \delta^B + \delta^{CD})
\]

\[
+ C\delta^{CD} \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{\alpha(\epsilon + \tau - \tau)}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau.
\]

Summing up (85), (84) and (82), we get the desired estimate (80). Thus we complete the proof of Lemma 4.4.

Proof of Proposition 2. Now, we are ready to prove Proposition 2. Combining Lemma 4.2-4.4 with Lemma 5.2 in the appendix, and if the wave strength $\delta$ and the constants $\epsilon$, $\epsilon_1$ are small enough, then for all $t \in [0, T]$, we have

\[
\|[\varphi, \psi, \omega, \zeta](t)\|_{H^1(\mathbb{R}^+)}^2 + \int_0^t \left(\|\partial_\xi \varphi\|^2 + \|\partial_\xi [\psi, \zeta]\|^2_{H^1(\mathbb{R}^+)} + \|\omega\|^2_{H^2(\mathbb{R}^+)}\right) d\tau
\]

\[
\leq C\|[\varphi_0, \psi_0, \omega_0, \zeta_0]\|_{H^1(\mathbb{R}^+)}^2 + C(e^{\|\epsilon\|_B} + \delta^B + \delta^{CD}),
\]

which gives desired estimate (48).

Proof of Theorem 3.1. We are now in a position to complete the proof of Theorem 3.1. In view of the energy estimates obtained in Proposition 2, one sees that

\[
\sup_{0 \leq \tau \leq t} \|[\varphi, \psi, \omega, \zeta](\tau)\|_{H^1(\mathbb{R}^+)}^2 \leq C\|[\varphi_0, \psi_0, \omega_0, \zeta_0]\|_{H^1(\mathbb{R}^+)}^2 + C(e^{\|\epsilon\|_B} + \delta^B + \delta^{CD}).
\]

Notice that $\epsilon$, $\delta^B$ and $\delta^{CD}$ are parameters independent of $\epsilon_1$. By letting $\epsilon$, $\delta^B$ and $\delta^{CD}$ be small enough, then the global existence of solution to the half-space problem (42) follows from the standard continuation argument based on local existence and the a priori estimate (48). Moreover, (87) and (43) imply (44). Next, our intention is to prove the large time behavior (45). For this, we first justify the following limits:

\[
\lim_{t \to +\infty} \|[\varphi, \psi, \omega, \zeta](t)\|_{L^2}^2 = 0.
\]

To prove (88), we get from (42), (48), (28), Lemma 2.2 and (22) that

\[
\int_0^{+\infty} \left| \frac{d}{dt} \|\partial_\xi [\varphi, \psi, \omega, \zeta]\|^2 \right| dt = 2 \int_0^{+\infty} \|\partial_\xi [\varphi, \psi, \omega, \zeta] - \partial_\xi \partial_\xi [\psi, \zeta, \omega]\| dt
\]

\[
\leq C + C \int_0^{+\infty} \|\partial_\xi [\varphi, \psi, \omega, \zeta, \partial_\xi [\psi, \zeta, \omega]]\|^2 dt < +\infty.
\]

Consequently, (89) together with (48) gives (88). Then (45) follows from (88) and Sobolev’s inequality (50). This ends the proof of Theorem 3.1.

5. Appendix. In this appendix, we will give some basic results used in the paper. Lemmas 5.1-5.2 are borrowed from [9] and [30], and we omit some details here.

Lemma 5.1. Suppose that $h(\xi, t)$ satisfies

$h \in L^\infty(0, T; L^2(\mathbb{R}^+)), \quad \partial_\xi h \in L^2(0, T; L^2(\mathbb{R}^+)), \quad \partial_\xi h - \sigma \partial_\xi h \in L^2(0, T; H^{-1}(\mathbb{R}^+)),$
then the following estimate holds for \( t \in [0, T] \),
\[
\int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} h^2 d\xi d\tau \leq C_\alpha \left[ \|h(\xi, 0)\|^2 + \int_0^t h^2(0, \tau) d\tau + \int_0^t \|\partial_\xi h\|^2 d\tau \right.
+ \left. \int_0^t \langle \partial_\tau h - \sigma_\tau \partial_\tau h, hg^2 \rangle_{H^{-1} \times H^1} d\tau \right],
\]
where
\[
g(\xi, t) = -(1 + t)^{-\frac{1}{2}} \int_{\xi + \tau - t}^{+\infty} e^{-\frac{\alpha_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi,
\]
\( \sigma_\tau = -\frac{\nu}{1 + \tau} \), and \( \alpha > 0 \) is a constant to be determined later.

We now give the following estimate concerning the delicate term \( \int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau \) by using Lemma 5.1.

**Lemma 5.2.** Under the conditions of Proposition 2, then there exists a constant \( C > 0 \) such that
\[
\int_0^t \int_{\mathbb{R}^+} (1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} (\varphi^2 + \psi^2 + \zeta^2) d\xi d\tau \leq C + C \int_0^t (\|\partial_\xi [\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2) d\tau + C \int_0^t \|\partial_\xi^2 [\psi, \zeta]\|^2 d\tau
\]
provided that the wave strength \( \delta_B \), \( \delta^{CD} \) and the constant \( \epsilon \) are small enough.

**Proof.** For any \( \nu > 0 \), the proof of inequality (91) consists of the following two parts:
\[
\int_0^t \int_{\mathbb{R}^+} [(R\zeta - P\varphi)^2 + \psi^2](1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi d\tau
\leq C + C(\epsilon + \delta_B + \delta^{CD}) \int_0^t \int_{\mathbb{R}^+} (\varphi^2 + \zeta^2)(1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi d\tau
+ C \int_0^t (\|\partial_\xi [\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2) d\tau + \nu \int_0^t \|\partial_\xi^2 [\psi, \zeta]\|^2 d\tau,
\]
and
\[
\int_0^t \int_{\mathbb{R}^+} (R\zeta + (\gamma - 1)P\varphi)^2 (1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi d\tau
\leq C + C(\epsilon + \delta_B + \delta^{CD}) \int_0^t \int_{\mathbb{R}^+} (\varphi^2 + \zeta^2)(1 + \tau)^{-1} e^{-\frac{c_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi d\tau
+ C_\nu \int_0^t (\|\partial_\xi [\varphi, \psi, \zeta, \omega]\|^2 + \|\omega\|^2) d\tau + \nu \int_0^t \|\partial_\xi^2 \zeta\|^2 d\tau,
\]
In fact, multiplying inequality (92) by \( \gamma - 1 \), adding the resulting inequality to (93) and taking \( \delta^{CD}, \delta_B \) and \( \epsilon \) suitably small, then implies (91) easily.

We firstly prove (92). Define
\[
\eta(\xi, t) = -(1 + t)^{-\frac{1}{2}} \int_{\xi + \sigma - t}^{+\infty} e^{-\frac{\alpha_0(\xi + \sigma - r)^2}{1 + \tau}} d\xi.
\]
One rewrites (42) as follows:
\[
(\partial_t \psi - \sigma \partial_x \psi) + \partial_x \left( \frac{R\zeta - P\varphi}{v} \right) = \mu \partial_x \left( \frac{\partial_x u}{v} - \frac{\partial_x U}{V} \right) - Q_1. \tag{94}
\]

Multiplying (94) by \((R\zeta - P\varphi) v\eta\), integrating the resulting equation over \(\mathbb{R}_+ \times [0, t]\) leads to
\[
\frac{1}{2} \int_0^t \int_{\mathbb{R}_+} (R\zeta - P\varphi)^2 \partial_x v \sigma d\xi d\tau \\
= - \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} [(R\zeta - P\varphi)^2 \eta](0, \tau) d\tau + \mu \int_0^t \left[ \frac{\partial_x u}{v} - \frac{\partial_x U}{V} \right] (R\zeta - P\varphi) v\eta \bigg| (0, \tau) d\tau \\
+ \sigma \int_0^t \left[ \psi (R\zeta - P\varphi) v\eta \right](0, \tau) d\tau + \int_{\mathbb{R}_+} \psi (R\zeta - P\varphi) v\eta d\xi \\
- \int_{\mathbb{R}_+} \psi_0 (R\zeta_0 - P(\xi, 0) v_0) v\eta(\xi, 0) d\xi \\
- \int_0^t \int_{\mathbb{R}_+} [\partial_t (R\zeta - P\varphi) - \sigma \partial_x (R\zeta - P\varphi)] \psi v\eta d\xi d\tau \\
\kappa_1 \\
- \int_0^t \int_{\mathbb{R}_+} \psi (R\zeta - P\varphi) (\partial_t v - \sigma \partial_x v) d\xi d\tau \\
- \int_0^t \int_{\mathbb{R}_+} \psi (R\zeta - P\varphi) v(\partial_t \eta - \sigma \partial_x \eta) d\xi d\tau \\
- \int_0^t \int_{\mathbb{R}_+} \frac{\partial_x (V + \varphi)}{v} (R\zeta - P\varphi)^2 \eta d\xi d\tau \\
+ \mu \int_0^t \int_{\mathbb{R}_+} \left( \frac{\partial_x u}{v} - \frac{\partial_x U}{V} \right) \partial_x [(R\zeta - P\varphi) v\eta] d\xi d\tau \\
+ \int_0^t \int_{\mathbb{R}_+} Q_1 (R\zeta - P\varphi) v\eta d\xi d\tau. \tag{95}
\]

For the delicate term \(\kappa_1\), it can be rewritten as
\[
\kappa_1 = - \int_0^t \int_{\mathbb{R}_+} [(R\partial_t \zeta - R\sigma \partial_x \zeta) - (\partial_t P - \sigma \partial_x P) \varphi - P(\partial_t \varphi - \sigma \partial_x \varphi)] \psi v\eta d\xi d\tau \\
= - \frac{1}{2} \int_0^t \left( \gamma P v\eta \psi^2 \right) (0, \tau) d\tau - \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \gamma P v\eta \psi^2 d\xi d\tau \\
- \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \gamma \partial_x (P v) \eta \psi^2 d\xi d\tau + \int_0^t \int_{\mathbb{R}_+} (\partial_t P - \sigma \partial_x P) \varphi \psi v\eta d\xi d\tau \\
- (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \left[ -(\rho - P) \partial_x u + \kappa \partial_x \left( \frac{\partial_x u}{v} - \frac{\partial_x U}{V} \right) \right] d\xi d\tau \\
+ \mu \left( \frac{(\partial_x u)^2}{v} - \frac{(\partial_x U)^2}{V} \right) + (\partial_x \omega)^2 + v\omega^2 - Q_2 \psi v\eta d\xi d\tau \\
= - \frac{1}{2} \int_0^t \left( \gamma P v\eta \psi^2 \right) (0, \tau) d\tau + \kappa (\gamma - 1) \int_0^t \left[ \left( \frac{\partial_x u}{v} - \frac{\partial_x U}{V} \right) \psi \eta \right] (0, \tau) d\tau
\[-\frac{1}{2} \int_0^t \int_{R^+} \gamma P \psi \delta \eta \eta^2 d\xi d\tau - \frac{1}{2} \int_0^t \int_{R^+} \gamma \partial \xi (P \psi) \eta \psi^2 d\xi d\tau \]

\[+ \int_0^t \int_{R^+} (\partial \xi P - \sigma \partial \xi P) \varphi \psi \eta \eta d\xi d\tau - (\gamma - 1) \int_0^t \int_{R^+} \left[ - (p - P) \partial \xi u \right. \]

\[+ \mu \left( \frac{(\partial \xi u)^2}{V} - \frac{(\partial \xi U)^2}{V} \right) + \frac{(\partial \xi \omega)^2}{v} + v \omega^2 - Q_2 \right] \psi \eta \eta d\xi d\tau \]

\[+ \kappa(\gamma - 1) \int_0^t \int_{R^+} \left( \frac{\partial \xi \Theta}{v} - \frac{\partial \xi \Theta}{V} \right) \partial \xi (\psi \eta \eta) d\xi d\tau, \tag{96} \]

where we have used (42) \(_1\) and (42) \(_4\) in the second equality.

Since

\[\partial \xi \eta(\xi, t) = (1 + t)^{-1} e^{- \frac{c_\alpha(\xi - \gamma \tau)}{1 + \tau}}, \]

combining (95) and (96), we have

\[\frac{1}{2} \int_0^t \int_{R^+} [(R \xi - P \varphi)^2 + \gamma P \psi^2](1 + \tau)^{-1} e^{- \frac{c_\alpha(\xi + \varphi \tau)}{1 + \tau}} d\xi d\tau = \int_0^t H_2(0, \tau) d\tau + Q_3, \tag{97} \]

where

\[H_2(0, \tau) = - \frac{1}{2} [(R \xi - P \varphi)^2 \eta](0, \tau) + \mu \left[ \left( \frac{\partial \xi u}{v} - \frac{\partial \xi U}{V} \right) (R \xi - P \varphi) \eta \right](0, \tau) \]

\[+ \sigma \left[ \psi (R \xi - P \varphi) \psi \eta \right](0, \tau) - \frac{1}{2} \left( \gamma P \psi \eta \psi^2 \right)(0, \tau) \]

\[+ \kappa(\gamma - 1) \left[ \left( \frac{\partial \xi \Theta}{v} - \frac{\partial \xi \Theta}{V} \right) \psi \eta \right](0, \tau), \tag{98} \]

and

\[Q_3 = \int_{R^+} \psi (R \xi - P \varphi) \psi \eta d\xi - \int_{R^+} \psi_0 (R \xi_0 - P(\xi_0) \varphi_0) \psi_0 \eta(\xi, 0) d\xi \]

\[- \int_0^t \int_{R^+} \psi (R \xi - P \varphi) \partial \xi \eta \eta d\xi d\tau - \int_0^t \int_{R^+} \psi (R \xi - P \varphi) \psi (\partial \xi \eta - \sigma \partial \xi \eta) d\xi d\tau \]

\[- \int_0^t \int_{R^+} \partial \xi U \frac{(R \xi - P \varphi)^2}{v} \eta \eta d\xi d\tau \]

\[+ \mu \left[ \frac{1}{2} \int_0^t \int_{R^+} \left( \frac{\partial \xi u}{v} - \frac{\partial \xi U}{V} \right) \partial \xi \left( \psi \eta \right) d\xi d\tau \right] \]

\[+ \mu \left[ \frac{1}{2} \int_0^t \int_{R^+} (\partial \xi P - \sigma \partial \xi P) \varphi \psi \eta \eta d\xi d\tau \right] \]

\[- (\gamma - 1) \int_0^t \int_{R^+} \left[ - (p - P) \partial \xi u + \mu \left( \frac{(\partial \xi u)^2}{v} - \frac{(\partial \xi U)^2}{V} \right) \right. \]

\[+ \frac{(\partial \xi \omega)^2}{v} + v \omega^2 - Q_2 \right] \psi \eta \eta d\xi d\tau \]

\[+ \kappa(\gamma - 1) \int_0^t \int_{R^+} \left( \frac{\partial \xi \Theta}{v} - \frac{\partial \xi \Theta}{V} \right) \partial \xi (\psi \eta \eta) d\xi d\tau. \tag{99} \]
From Lemma 4.1 (boundary estimates), we have
\[
\int_0^t H_2(0, \tau) d\tau \leq \nu \int_0^t (\|\partial_t [\psi, \zeta]\|^2 + |\partial^2_x [\psi, \zeta]|^2) d\tau + C_v (\delta^{CD})^2.
\]
(100)

Note that \(\|\eta(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-\frac{1}{2}}\) and applying (22), (28), Cauchy-Schwarz’s inequality, Sobolev’s inequality (50) and Young’s inequality, we can successfully estimate \(Q_3\). Then combining this with (100) and (96), we obtain (92).

Next we prove inequality (93) by using Lemma 5.1. Let \(h = R\zeta + (\gamma - 1)P\varphi\), then from (42)_1 and (42)_4, we get
\[
\int_0^t \langle \partial_t h - \sigma \cdot \partial_t h, h g^2 \rangle_{H^{-1} \times H^1} d\tau = - (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} (p - P) \partial_t \psi h g^2 d\xi d\tau
\]
\[
+ (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} [\langle \partial_t P - \sigma \cdot \partial_t P \rangle \varphi - (p - P) \partial_t U] h g^2 d\xi d\tau
\]
\[
- (\gamma - 1) \int_0^t \left\{ \kappa_1 \left[ \partial_t \frac{\partial_t \theta}{\nu} - \frac{\partial_t \Theta}{V} \right] h g^2 \right\} (0, \tau) d\tau
\]
\[
- (\gamma - 1) \int_0^t \left\{ \kappa_2 \left[ \partial_t \frac{\partial_t \theta}{\nu} - \frac{\partial_t \Theta}{V} \right] \partial_t (h g^2) \right\} d\xi d\tau
\]
\[
+ (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \mu \left( \frac{(\partial_t \zeta)^2}{\nu} - (\partial_t U)^2 \right) h g^2 d\xi d\tau
\]
\[
+ (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \left( \frac{(\partial_t \omega)^2}{\nu} + v \omega^2 \right) h g^2 d\xi d\tau - (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} Q_2 h g^2 d\xi d\tau.
\]

Noticing that \(\|g(\cdot, t)\|_{L^\infty} \leq C_\alpha\), we can directly estimate \(K_i (3 \leq i \leq 8)\). In order to estimate \(K_2\), by the mass equation (42)_1 and \(p - P = \frac{R\zeta - P\varphi}{v}\), one knows
\[
K_2 = - (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \frac{h - \gamma P\varphi}{v} h g^2 (\partial_t \varphi - \sigma \cdot \partial_t \varphi) d\xi d\tau
\]
\[
= - (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \left\{ \frac{h^2 g^2}{v} (\partial_t \varphi - \sigma \cdot \partial_t \varphi) - \frac{\gamma P h g^2}{2v} (\partial_t (\varphi^2) - \sigma \cdot \partial_t (\varphi^2)) \right\} d\xi d\tau
\]
\[
= - (\gamma - 1) \int_{\mathbb{R}_+} \frac{2h^2 g^2 \varphi - \gamma P h g^2 \varphi^2}{2v} d\xi
\]
\[
+ (\gamma - 1) \int_{\mathbb{R}_+} \left[ \frac{2h^2 g^2 \varphi - \gamma P h g^2 \varphi^2}{2v} \right] (\xi, 0) d\xi
\]
\[
- (\gamma - 1) \sigma - \int_0^t \left[ \frac{2h^2 g^2 \varphi - \gamma P h g^2 \varphi^2}{2v} \right] (0, \tau) d\tau.
\]
\[
- (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \frac{\gamma g^2 v^2 h}{2v} (\partial_t P - \sigma - \partial_x P) d\xi d\tau \\
- (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \frac{\gamma Ph v^2 - 2h^2 \varphi g (\partial_t g - \sigma - \partial_x g)}{v} d\xi d\tau \\
+ (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \frac{\gamma Ph^2 - 2h^2 \varphi g v (\partial_t v - \sigma - \partial_x v)}{2v} d\xi d\tau \\
- (\gamma - 1) \int_0^t \int_{\mathbb{R}_+} \frac{\gamma Pg^2 v^2 - 4g^2 h \varphi}{2v} (\partial_t h - \sigma - \partial_x h) d\xi d\tau.
\]  

Now each term in (101) can be estimated directly, and the detailed proof can be seen in [9]. Note that here we need to compute the boundary terms. Hence after taking \( \alpha = \frac{\epsilon_0}{2} \), estimate (93) thus easily follows from Lemma 5.1 and Lemma 4.1(boundary estimates).

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