On Bond Portfolio Management

Vladislav Kargin

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Abstract

This paper describes a new method of bond portfolio optimization based on stochastic string models of correlation structure in bond returns. The paper shows how to approximate correlation function of bond returns, compute the optimal portfolio allocation using Wiener-Hopf factorization, and check whether a collection of bonds presents arbitrage opportunities.

Keywords: bond portfolio management, Toeplitz operators, Padé approximations, Wiener-Hopf factorization.
1 Introduction

A plot of monthly stock returns typically looks like the surface of the stormy sea: Although stocks move up and down together with the market as the sea level moves up and down reacting to the Moon’s gravitation, the relative movements of nearby points are independent. In contrast, returns on bonds plotted against maturity look like the sea on a breezy but calm day – the nearby points are close to each other and move in accord like a wave. This distinction is at the basis of the specificity of bond portfolio management: Bonds are more structured but also more difficult to diversify. The present paper contends that the best way to find the optimal bond portfolio is by approximating the correlation structure in bond returns with a rational function of the difference in maturities and reducing the optimization problem to the inversion of an operator in a Hilbert space.

The importance of bond portfolio management is difficult to overestimate. In 2000 the values of government and corporate debt outstanding were 7.7 and 5 trillions respectively, compared with 17 trillions of corporate equity outstanding. In addition, the relative importance of debt is rising: the equity value in 2000 decreased by 13 percent from its 1999 level, while the government and corporate debts were up 2 and 9 percent respectively.\(^1\) Despite all the effort to balance its budget, the federal government is still spending more than it receives in revenues and therefore the likelihood that the Treasury securities market will be shriveling is scant. So, given the importance of debt markets how should

\(^1\)Source: Securities Industry Association Factbook 2001, page 22.
investors optimize their bond portfolios?

In the early 1980s Heaney and Cheng applied to bonds the techniques for stock portfolio optimization. This approach, however, was soon abandoned because bonds move together to a much greater extent than stocks and modeling of these co-movements is harder.

Another idea appeared even earlier and got much greater application – the idea of immunization (Fisher and Weil (1971)). The immunization technique minimizes sensitivity of the portfolio with respect to small, parallel shifts in all interest rates. So, this approach directly takes into account the observation that interest rates are highly correlated. The modern development of this idea uses stochastic programming (Dembo (1993), Mulvey and Zenios (1994), Golub et al. (1995), Zenios et al. (1998), Consigli and Dempster (1998), Beltratti et al. (1999), Dupučová and Bertocchi (2001)), in which the investor formulates a set of scenarios for interest rate movements, prescribes them probabilities, and minimizes a certain loss function – for example, loss that can occur with 5% probability.

While practical, the immunization technique neglects certain aspects in the dynamics of interest rates. The probabilities of scenarios are usually extracted from a finite factor model, which does not capture all the information about the statistical correlations available from the data. In addition, the market models often used for generation of the scenario probabilities are internally inconsistent since they require continual re-calibration of parameters. What is needed is a better method for modeling interest rate correlations.

The present paper explores a synthesis of the two methods of portfolio op-
timization: The correlations are directly estimated using a plausible assumption on their structure similar to the intuitive assumption of the immunization technique. The estimated correlations are then used by the standard portfolio optimization model.

This new method has its provenance in the random field models of bond returns (Kennedy (1994), Kennedy (1997), Goldstein (2000), and Santa-Clara and Sornette (2001)), which provide a more flexible framework than finite-factor models for estimation correlations in bond returns. According to philosophy of these papers, correlations in bond returns should be approximated as a smooth function of the difference in maturities. Once the function is estimated, the problem of optimization can be solved by inverting a special operator in a Hilbert space spanned by bond returns, a task that was already extensively studied in the communications engineering literature.

After the solution of the optimization problem is found, it can be profitably applied to evaluate the opportunities that a given structure of interest rates presents. A classical arbitrage opportunity arises when the investor is willing to invest infinite amount in a security and his utility from the investment is not bounded. The paper explains how to check for existence of these opportunities. Since they rarely happen in the market, the paper also explains how to check for the existence of near-arbitrage opportunities, which are situations where the utility from the investment is not infinite but abnormally large. It will be shown that there is a norm in Hilbert space defined through parameters of the correlation structure such that near-arbitrage opportunities arise only if the
vector of expected returns has large size in terms of this norm.

The rest of the paper is organized as follows. Section 2 explains assumptions and notation. Section 3 is about Padé approximations to the correlation function. Section 4 solves the portfolio optimization in terms of the Wiener-Hopf factorization. Section 5 is about arbitrage opportunities. Section 6 computes the correlations and the optimal portfolio for the Treasury interest rates in a certain period. And Section 7 concludes.

2 Assumptions and Notation

Recent research in the theory of interest rates treats the bond returns as a random field that has a two-dimensional correlation structure. As usual, bond returns follow a certain stochastic time process, but in addition the contemporaneous returns of the bonds with different maturities are also stochastically correlated. Most importantly, the contemporaneous correlations are functions of maturities. I will assume that the contemporaneous correlation of two bond returns is a smooth function of the difference of their maturities. Let me introduce some notation to formulate this assumption formally. Symbol $t$ will denote the time to maturity and symbol $s$ the calendar time. Let $R(t, s)$ be a one-period return on the bond with maturity $t$ at time $s$:

$$R(t, s) = \log \frac{P(t - \Delta t, s + \Delta t)}{P(t, s)},$$

(2.1)

where $P(t, s)$ is the price of the bond with maturity $t$ at time $s$, and $\Delta t$ is the minimal possible difference between maturities. For example, a month is a
realistic minimal difference between maturities of government bonds. Then the assumption claims that the correlation between returns \( R(t, s) \) and \( R(t+\tau, s+\varsigma) \) can be written as \( C(\tau, \varsigma) \), where \( \tau \) denotes difference between maturities and \( \varsigma \) is difference between calendar times.

The assumption may be motivated by analogy with assumption of stationarity in time series where autocovariance of the returns depends only on the difference between times of these returns. On a deeper level, the market perceives the bonds with close maturities as similar and the assumption says that the relevant measure of similarity is the difference between maturities. Another potentially useful measure of similarity for non-government bonds would be the difference in credit ratings. Since the question about appropriate measure of similarity is a question about market perceptions, it needs further empirical investigation. A priori, the assumption that is taken in this paper seems to be reasonable.

Let me in the following suppress the second argument in \( R(t, s) \) if the calendar time is unimportant, and in \( C(\tau, \varsigma) \) if the correlation of contemporaneous returns is considered, \( C(\tau) := C(\tau, 0) \). Also, let \( \Delta t = 1 \). This notation is less cumbersome and no confusion should arise.

It is useful to introduce artificial securities \( S_t \) that have unit variance and the following expected return

\[
E(t) = \frac{ER(t)}{\sqrt{V(t)}},
\]

(2.2)

where \( ER(t) \) is the expected return of bond with maturity \( t \), and \( V(t) \) is the
variance of this return. Let also

\[ Y(t) = X(t) \sqrt{\bar{V}(t)}, \]  

(2.3)

where \( X(t) \) is the holding in the bond with maturity \( t \). Then the bond portfolio that holds \( X(t) \) in the bond with maturity \( t \) has the same variance and expected return as the portfolio that holds \( Y(t) \) in the security \( S_t \). The usefulness of this reformulation is that securities \( S_t \) have unit variance of return, and the optimization depends solely on the correlation structure of the returns. Since it is the well-developed correlation structure that makes the bond portfolio management different from the stock portfolio management, the formulation it terms of normalized holdings \( Y(t) \) is useful as emphasizing the importance of the correlation structure. When the optimal portfolio of securities \( S_t \) is found, it is easy to translate it back into the optimal portfolio of bonds by inverting relationship (2.3).

The first step to the formulation of the investor optimization problem is to introduce generating functions for correlations, holdings and expected returns on securities \( S_t \):

\[
\hat{C}(z) = \sum_{\tau=0}^{\infty} C(\tau) z^\tau, \\
\hat{Y}(z) = \sum_{t=0}^{\infty} Y(t) z^t, \\
\hat{E}(z) = \sum_{t=0}^{\infty} E(t) z^t.
\]  

(2.4)

The second step is to introduce some machinery of Hilbert spaces. The space is needed because the number of bonds with different maturities is potentially
infinite. Let $\mathcal{H}$ be the linear space of the formal series in variable $z$ with the
bounded sum of squared coefficients:

$$a(z) = \sum_{-\infty}^{\infty} a_k z^k \text{ such that } \sum_{-\infty}^{\infty} |a_k|^2 < \infty.$$  \hfill (2.5)

Scalar product $\langle a | b \rangle =: \sum a_i b_i$ turns $\mathcal{H}$ into a Hilbert space. This scalar
product can also be written in an integral form often useful in computations:

$$\langle a | b \rangle = \frac{1}{2\pi i} \int_{|z|=1} a(z) \overline{b(z)} \frac{d(z)}{z}.$$  \hfill (2.6)

Let $\mathcal{H}_+$ be a subspace of series with non-negative coefficients, $\mathcal{H}_-$ its orthog-
onal complement, and $P_+$ the orthogonal projector on $\mathcal{H}_+$. Clearly, $\hat{C}(z), \hat{Y}(z),$ and $\hat{E}(z)$ are elements of $\mathcal{H}_+$. In addition, the variance of a bond portfolio can
be seen as a norm on $\mathcal{H}_+$. To write down this norm explicitly and relate it to the
function $\hat{C}(z)$, we need to introduce multiplication operators: To any function

$$F(z) = \sum_{-\infty}^{\infty} f_i z^i \text{ such that } \sup |f_i| < \infty,$$

corresponds an operator of multiplication by this function,

$$a(z) \rightarrow F(z) a(z).$$

To make notation for multiplication operators distinct from notation for corre-
sponding functions, the operators will have a multiplication sign in the subscript:

$F_x$. So, function $F$ maps complex numbers to complex numbers, and operator
$F_x$ maps the Hilbert space $\mathcal{H}$ to itself.

Using this notation it is easy to write the variance of the portfolio $Y(t)$ :

$$\text{Var}(Y) = \langle \hat{Y} | P_+ A_x \hat{Y} \rangle,$$  \hfill (2.7)
Assume that the investor optimizes the short-run performance of his portfolio. The myopia assumption allows to neglect correlation of bond returns across time and reduce the problem to the static case. The dynamic problem with non-myopic investor is considerably more difficult and deserves further investigation.

Formally, the investor aims to maximize a certain linear combination of the expected return and the variance in the return. We can write it as follows:

$$ U(Y) = \langle \hat{E}|\hat{Y} > - \gamma \langle \hat{Y}|P_+A_x\hat{Y} >, $$

(2.9)

where $\gamma$ is a coefficient of risk aversion. This is the traditional Markowitz portfolio optimization problem, written in the language of Hilbert spaces. Note that it is precisely because of the assumption that the correlations depend only on the difference in bond maturities that the variance of the portfolio can be written as a scalar product of the portfolio vector and its operator image.

The solution to this problem is:

$$ \hat{Y} = \frac{1}{2\gamma} [P_+A_x]^{-1} \hat{E}, $$

$$ U = \frac{1}{4\gamma} \langle \hat{E}|[P_+A_x]^{-1}\hat{E} >. $$

(2.10)

It is important to note that similar solutions arise also for optimization problems different from the problem considered here. For example, solutions have the similar form for the problem of minimization of portfolio variance with certain constraints on the portfolio composition. The problem in this paper was chosen
as the most familiar representative of this class of problems but the method of solution is useful for all of them.

In all these problems the solution requires estimation of correlation function \( C(\tau) \) and inversion of the operator \( P_+ A_\times \). We will address the problem of inversion later and first explain how to estimate the correlation function.

3 Padé Approximations

Estimation of the correlation function is not a trivial task because the data about returns of the bond with particular maturity are scarce. Therefore, the data must be fitted by a smooth function. One way to approach the fitting problem is to use Padé approximations – ratios of polynomials constrained to have specific initial terms in their Taylor expansions. Using the appropriate degrees of polynomials in numerator and denominator, it is also possible to match any conjectured asymptotic behavior of the function.

Formally, let \( P_M(z) \) and \( Q_N(z) \) be a couple of polynomials with the ratio that have the Tailor expansion \( f(z) \):

\[
\frac{P_M(z)}{Q_N(z)} = \sum_{i=0}^{\infty} f(i)z^i.
\] (3.1)

This couple of polynomials is a Padé approximation to the correlation function \( \hat{C}(z) \) if the first \( N + M + 1 \) coefficients of the Tailor expansion coincide with the corresponding coefficients of \( \hat{C}(z) \):

\[
f(i) = C(i) \text{ for } i = 0, 1, ..., M + N.
\] (3.2)
Padé approximations can be easily found by solving a system of linear equations (see Baker and Graves-Morris (1996) for more information about Padé approximations).

A generalization of Padé approximations may be helpful when the data is noisy. The generalization requires that the coefficients of Taylor expansions be only approximately equal:

\[ f(i) = \hat{C}(i) + \varepsilon_i \text{ for } i = 0, 1, ..., M + N + K. \]  

(3.3)

By definition, the \([M, N, K]\)-order generalized Padé approximation minimizes the sum of squared errors in (3.3). The benefit of this generalization is that it allows using correlations of bonds with larger differences in maturities for more precise estimation of the coefficients of the approximation.

### 4 Wiener-Hopf factorization

In the engineering literature the operator \(P_+A_\times\) is called rational filter, and its inversion is a well-known problem. It can be solve by several efficient methods (Kailath et al. (2000)), from which one of the most elegant is given by the Wiener-Hopf factorization. Let \(\ln A(z)\) be decomposed as follows:

\[ \ln A(z) = A_+(z) + A_-(z), \text{ where } a_+(z) \in \mathcal{H}_+ \text{ and } a_-(z) \in \mathcal{H}_-. \]  

(4.1)

Then the Wiener-Hopf factorization theorem claims that

\[ [P_+A_\times]^{-1} = [\exp(-A_+)]_\times P_+ [\exp(-A_-)]_\times. \]  

(4.2)
See [Lax (2002)] for the proof and Appendix A for computational details. The benefit of this theorem is that it allows explicitly inverting the infinite matrix corresponding to operator $P_+ A_\times$, which greatly reduces demand for necessary computational resources.

The technique of the Wiener-Hopf factorization allows writing analytic expressions for the optimal bond portfolio allocation and corresponding utility:

**Theorem 4.1** The optimal allocation is

$$\hat{Y} = \frac{1}{2\gamma} \left[ \exp(-A_+) \right]_\times P_+ \left[ \exp(-A_-) \right]_\times \hat{E}$$

The corresponding utility function is

$$U = \frac{1}{4\gamma} < \hat{E} | \left[ \exp(-A_+) \right]_\times P_+ \left[ \exp(-A_-) \right]_\times \hat{E} > .$$

**Proof:** This theorem is a direct consequence of the Wiener-Hopf factorization theorem and expressions for optimal portfolio and utility (2.10).

**Example 4.1** AR(1) correlations and expectations

Let correlations between bond returns be as they are in AR(1) time series model:

$$\hat{C}(z) = 1 + \sum_{i=1}^{\infty} \alpha^i z^i = \frac{1}{1 - \alpha z},$$

(4.3)

Then

$$A(z) = \hat{C}(z^{-1}) + \hat{C}(z) - 1 = \frac{1 - \alpha^2}{(1 - \alpha z)(1 - \alpha z^{-1})},$$

(4.4)

$$A_+ = \ln \frac{1 - \alpha^2}{1 - \alpha z}, \text{ and } A_- = \ln \frac{1}{1 - \alpha z^{-1}}.$$

Therefore,

$$[P_+ A_\times]^{-1} = \frac{1}{1 - \alpha^2 (1 - \alpha z)} P_+ (1 - \alpha z^{-1})_\times.$$  

(4.5)
Assume also for the purposes of this example that the normalized expectations for bonds with longer maturities are smaller – perhaps because of large variance of the returns on longer maturity bonds. More precisely, let the normalized expectations decline exponentially:

\[ \hat{E}(z) = E_0(1 + \sum_{i=1}^{\infty} \beta^i z^i) = \frac{E_0}{1 - \beta z}, \]  

(4.6)

where \( \beta < 1 \) is the rate of decline. Then, according to Theorem 4.1,

\[ \hat{Y} = \frac{E_0}{2\gamma} \frac{1 - \alpha \beta}{1 - \alpha^2} \frac{1 - \alpha z}{1 - \beta z}, \]  

(4.7)

and

\[ U = \frac{E_0^2}{4\gamma} \frac{1 - \alpha \beta}{1 - \alpha^2} \frac{1}{1 - \beta z} \int_{|z|=1} \frac{1}{1 - \beta z^{-1}} \frac{1 - \alpha z}{1 - \beta z} dz = \frac{E_0^2}{4\gamma} \frac{(1 - \alpha \beta)^2}{(1 - \alpha^2)(1 - \beta^2)}. \]  

(4.8)

Note the wonderful symmetry of the expression relative to parameters that govern expectations and correlations of bond returns. The symmetry illustrates the idea that the investor will take into account both the expectations and correlations of future returns.

Examination of the expression for the optimal portfolio allocation [4.7] reveals that the investor will sell short all bonds except the bond with the shortest maturity provided that \( \beta < \alpha \). In this case the bonds with larger maturities are valid only as hedging instruments. On the contrary, he will buy the bonds with all the maturities if \( \beta > \alpha \).
5 Arbitrage opportunities

Classically, the absence of arbitrage opportunities means that there is no investment with zero risk and positive return. In terms of the Hilbert space formalism, zero-risk investments lie in the kernel of the operator $P_+A_x$, and investments with zero expected return are orthogonal to the vector of normalized expectations $\hat{E}$. Therefore, the no-arbitrage condition on the structure of interest rates is

$$ \ker P_+A_x \perp \hat{E}. $$ (5.1)

The classical arbitrage, however, can exist only if there is a perfect correlation between a bond and a linear combination of other bonds, which is unlikely to happen in reality. A formal tool to check for existence of perfect correlations is given by a theorem of Hartman-Wintner-Widom [Hartman and Wintner (1954) and Widom (1960)], according to which the operator $P_+A_x$ is invertible if and only if $0 \notin [\inf_{|z|=1} A(z), \sup_{|z|=1} A(z)]$. In the likely event that the operator $P_+A_x$ is invertible, the classical arbitrage opportunities are absent.

However, it seems natural to rule out also the near-arbitrage opportunities, which are interest rates structure that permit to get utility greater than a certain “normality” threshold. The near-arbitrage structures would allow either getting a moderate excess return with very small risk, or a large excess return with moderate risk. The formal criterion for deciding whether an interest rate structure presents a near-arbitrage opportunity is presented in the following Theorem:
Theorem 5.1  The market of bonds do not present near arbitrage opportunities if and only if

\[ < \hat{E}[\exp(-A_+)] \times P_+ [\exp(-A_-)] \times \hat{E} > \leq C, \]

where \( C \) is a constant that depends on risk aversion of a typical investor.

Proof: This theorem directly follows from the definition of near-arbitrage opportunity, and the expression for utility of optimal portfolio.

6  Application

We use Treasury interest rates data by J. Huston McCulloch that represent the 67 months from 8/1985 to 2/1991. These data give the zero-coupon yield curve implicit in coupon bond prices. The yields have been defined for each month using interpolation by cubic splines. From these data the returns on holding a particular bond for one month have been computed.

The correlations have been estimated according to the formula

\[ C(\tau) = \frac{1}{N(s)N(t)} \sum_{s,t} (E_{s,t} - \bar{E}_{s,t})(E_{s,t+\tau} - \bar{E}_{s,t+\tau}), \] (6.1)

where \( E_{s,t} \) are returns normalized by their standard deviation, and \( N(s) \) and \( N(t) \) are number of dates and maturities available for estimation.

Figure 1 shows correlations predicted by a classical Padé approximation and the actual estimates of the correlations. Figure 2 shows correlations from a generalized Padé approximation. From the comparison of these figures, it is clear that the classical approximation is good for small differences in maturities.
but severely underestimate the correlation between bonds with larger difference in maturities. The generalized Padé approximation is more balanced in the sense that it approximates equally well the correlations for all differences in maturities. On the other hand, the generalized approximation underestimate the correlations between bonds with small difference in maturities.

As usual in finance, evaluation of expected returns is more tricky than estimation of covariances. In particular, it depends on what theory the researcher holds about formation of interest rates. The results would be different for segmentation, liquidity preference and expectation theories. One possibility, assumed here for the purposes of illustration, is that expectations of the future interest rate curve coincide with the current interest rate curve. This allows to estimate expected return as follows:

\[
ER(t, s) = \log \frac{P(t - 1, s)}{P(t, s)},
\]

where \( P(t, s) \) is the price of the bond with maturity \( t \) at time \( s \). It should be emphasized that this is only a possible choice among many others. It is appropriate for illustrative purposes because of its simplicity.

Figure 3 compares results of investment in optimal portfolio calculated using a generalized Padé approximation with a benchmark. The benchmark is results of the portfolio calculated under assumption that the bond returns are uncorrelated. In the calculation of the portfolios, instead of introducing a specific risk aversion parameter, the sum of investments is constrained to be 1. Intuitively, this assumption means that even if the investors buys more than his capital the excess is offset by a short sale of a similar security. So, the return on such a
portfolio is a return on the unleveraged capital.

From Figure 3 it is clear that the optimal portfolio performs much better than the benchmark portfolio. It has much lower variance and its monthly returns are always positive in a striking difference with the returns of benchmark portfolio. These results suggest that modelling the correlation structure pays off.

7 Conclusion

An investor entering the business of bond portfolio management faces the business that was regulated as early as in the time of Hammurabi when the law required putting to death as a thief any man who received a deposit from a minor or a slave without power of attorney, but that is still shaken by demises of huge hedge funds, the business that attracts more bright mathematicians and physicists than all mathematics and physics departments in the country, that is operated by traders who play toy machine guns during their lunch, and that demands deeper economic insight than stock portfolio management will ever require, – but when he faces this fascinating business, the investor may perhaps be comforted by the thought that the business is the most scientific and precise in the whole area of financial speculation.

The present article is a contribution that shows how engineering techniques can be applied to calculating optimal bond portfolios, and illustrates the obtained formulas by developing a test for existence of arbitrage opportunities.
The method used in the article is a blend of the traditional portfolio optimization with the techniques for estimation of correlations between bonds of the similar maturities. While the present paper uses a particular form of approximation, the method has more general applicability and can be used with other estimation techniques.

A Computation of Wiener-Hopf Factorization

Let $A(z)$ be a ratio of polynomials:

$$A(z) = a_0 \frac{\prod_{i=1}^{M}(z - \theta_i^+)}{\prod_{j=1}^{N}(z - \eta_j^-)}.$$  

Let $\theta_i^+$ and $\eta_i^+$ be zeros and poles outside the unit circle, and $\theta_i^-$ and $\eta_i^-$ be zeros and poles inside the unit circle. Then

$$A_+(z) = \ln \frac{a_0 \prod(z - \theta_i^+)}{\prod(z - \eta_i^-)},$$

$$A_-(z) = \ln \frac{\prod(z - \theta_i^-)}{\prod(z - \eta_i^+)}.$$

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Figure 1: Actual and Fitted Correlation Functions for Classical Padé Approximation

The solid line marked by circles is the estimate of the actual correlation function for bonds of different maturities. The dashed line marked by pluses is the fitted correlations from the classical Padé $[1/2]$ approximation. The vertical axis is correlations. The horizontal axis is the differences between maturity times measured in years.
The solid line marked by circles is the estimate of the actual correlation function for bonds of different maturities. The dashed line marked by pluses is the fitted correlations from the generalized Padé \([0/5/28]\) approximation. The vertical axis is correlations. The horizontal axis is the differences between maturity times measured in years.
Figure 3: Returns for Optimal and Benchmark Portfolios

The solid line shows the returns of the optimal portfolio computed using [0/5/28] generalized Padé approximation. The dashed line is the returns of the benchmark portfolio computed under assumption of zero correlations between bond returns. The horizontal axes shows time in months; the vertical axes shows annualized monthly returns.
The graph shows the function $C(\tau)$ as a function of $\tau$.

- The blue line represents $C(\tau)$.
- The green dashed line represents another function, possibly $C(\tau) - 0.1$.

The y-axis represents the value of $C(\tau)$, ranging from 0.7 to 1.0.

The x-axis represents $\tau$, ranging from 0 to 30.
