Forward sum rule for the $2\gamma$-exchange correction to the charge radius extraction from elastic electron scattering

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Two-photon exchange (TPE) contributions to elastic electron-proton scattering in the forward regime and in leading $\log \sim t \ln |t|$ approximation in the momentum transfer $|t|$ are considered. The imaginary part of TPE amplitude in these kinematics is related to the total photo absorption cross section. The real part of the TPE amplitude is obtained from the imaginary part by means of a fixed-$t$ dispersion relation. The dispersion integrals for the relevant elastic $ep$-scattering amplitude converge and do not need subtraction. This allows us to make clean a prediction for the real part of the TPE amplitude at forward angles with the leading term $\sim t \ln |t|$. Numerical estimates are comparable to the experimental precision in extracting the charge radius from experimental data.

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I. INTRODUCTION

Nucleon structure has been studied with elastic electron scattering since the 1950’s. By means of the Rosenbluth separation the measurement of the unpolarized cross section allows to extract the electromagnetic form factors of the nucleon. The interest in measuring the elastic cross section at low (negative) $t$ is, e.g., the extraction of the slope of the electric Sachs form factor $G_E$ that is related to the charge radius $R_E$ as

$$G_E(t \to 0) = 1 + \frac{1}{6} R_E^2 t + O(t^2)$$

(1)

Recent measurement at Mainz [1] lead to the most precise ($\lesssim 1\%$) determination of the proton charge radius with electron scattering experiments to date,

$$R_E^p = 0.879 \pm 0.008 \text{ fm},$$

(2)

where the uncertainty quoted above represents a combined statistical, systematical, model-dependent and group-dependent uncertainties defined in that Ref. Proton charge radius is extracted from hydrogen spectroscopy data with even higher precision [2]

$$R_E^p = 0.8775 \pm 0.0051 \text{ fm},$$

(3)

the two methods delivering results that are in a very nice agreement. The recent Lamb shift measurements in muonic hydrogen [3–4] lead to an extraction of the proton charge radius that is ten times more precise,

$$R_E^p = 0.84087 \pm 0.00039 \text{ fm},$$

(4)

and differs by seven standard deviations from the value obtained with electronic hydrogen and in scattering experiments. In the context of the “proton radius puzzle”, as this discrepancy was dubbed in the literature, nucleon structure-dependent corrections to the Lamb shift, most notably the two-photon exchange (TPE) correction, underwent a renewed scrutiny with two methods that provide a controlled estimate of the systematical uncertainty of such a calculation: the dispersion relations [5–7] and within effective theories [8–11], however the discrepancy is still there. For electron scattering, dispersion relations have the potential to provide model-independent calculations of the TPE effect [12–14], although these references only account for the ground state contribution to TPE. We refer the reader to a recent review of the TPE effects in electron scattering [15].

In this work, we revisit the two-photon exchange correction to the elastic electron scattering at low momentum transfer. The article is organized as follows. In Section II, kinematics and cross section for elastic electron-proton scattering are considered in the low-$t$ regime. Section III introduces the TPE effects and considers the modifications of the cross section in their presence. In Section IV, the imaginary part of the (nearly) forward TPE amplitude is obtained, and the sum rule for the term $t \ln |t|$ due to TPE is obtained in terms of an energy-weighted integral over the total photo absorption cross section. Numerical results are presented in Section V, and their consequences for the existing and upcoming experimental data are discussed. A short conclusion closes the article.

II. ELASTIC $ep$-SCATTERING AMPLITUDE

In this work, we consider elastic electron-proton scattering process $e(k) + p(p) \to e(k') + p(p')$ for which we define:

$$P = \frac{p + p'}{2},$$

$$K = \frac{k + k'}{2},$$

$$\Delta = k - k' = p' - p,$$

(5)

and choose the invariants $t = \Delta^2 = -Q^2 < 0$ and $
u = (P \cdot K)/M$ as the independent variables. $M$ denotes the nucleon mass. They are related to the Mandelstam variables $s = (p + k)^2$ and $u = (p - k')^2$ through
s – u = 4Mν and s + u + t = 2M2. For convenience, we also introduce the usual polarization parameter ε of the virtual photon, which can be related to the invariants ν and t (neglecting the electron mass m):

$$\varepsilon = \frac{\nu^2 - M^2 \tau (1 + \tau) \nu^2 + M^2 \tau (1 + \tau)}{\nu^2 + M^2 \tau (1 + \tau)}, \quad (6)$$

with τ = −t/(4M2). Elastic scattering of a massless electron off a spin-1/2 target in the Born (one photon exchange, OPE) approximation is described by the familiar Dirac and Pauli form factors F1 and F2, respectively,

$$T_R = \frac{e^2}{-t} \bar{u}(k') \gamma_\mu u(k) \bar{u}(p') \left[ F_1 \gamma^\mu + F_2 \frac{i\sigma^{\mu\nu} \Delta_{\nu}}{2M} \right] u(p). \quad (7)$$

The unpolarized cross section is

$$\frac{d\sigma}{d\Omega_{Lab}} = \sigma_R \sigma_0 \frac{d\sigma_0}{d\Omega_{Lab}}, \quad (8)$$

with the usual Rutherford cross section

$$\frac{d\sigma_0}{d\Omega_{Lab}} = 4\alpha^2 \cos^2 \frac{\Theta}{2} \frac{E'^3}{E}, \quad (9)$$

Θ the electron Lab scattering angle and E(E') the incoming (outgoing) electron Lab energy. The reduced cross section σR is expressed in terms of electric and magnetic Sachs form factors G_E = F1 − τF2 and G_M = F1 + F2, respectively, as

$$\sigma_R = \frac{G_E^2 + \frac{z}{2} G_M^2}{1 + \tau}. \quad (10)$$

Before going on to discuss the two-photon exchange we wish to determine the level of accuracy that modern experiments set for this calculation. To this end, the reduced cross section taken in Born approximation can be expanded in a Taylor series in powers of negative t. Keeping the linear terms in this expansion, we write

$$\sigma_R^B = 1 + \frac{1}{3} R_E^2 t - \frac{\nu_p^2}{4M^2} + \frac{t}{4M^2} + O(t^2), \quad (11)$$

with μ_p = G_E^p(0) ≈ 2.793 the proton magnetic moment. Correspondingly, the 1% relative uncertainty in the charge radius is translated into the uncertainty in the reduced cross section

$$\delta \sigma_R^B = \frac{1}{3} R_E^2 \frac{25 R_E}{R_E} |t| \approx 0.120 \frac{|t|}{\text{GeV}^2}. \quad (12)$$

For the smallest values of |t| accessed in the A1 experiment, |t|_{min} = 4 × 10^{-3} GeV^2 the relative uncertainty of σ_R is of order 5 × 10^{-4}, similar to the natural size of the order α_{em} correction, ~ α_{em}/(4π). Most order α_{em} corrections can be calculated quite reliably, the exception being the two-photon exchange. The latter is included approximating the TPE graph by only the ground state contribution that is furthermore approximated according to Mo and Tsai [16] or Maximin and Tjon [17], as well as the so-called Feshbach correction [18], leading to a generic result

$$\delta \sigma_R^{B + R.C.} = (1 + \delta) \sigma_R^B \quad (13)$$

with the correction δ ~ α_{em}. We will discuss the two corrections in more detail in the following session. An inclusion of general nucleon structure in the TPE is complicated and is only possible in forward kinematics, as e.g. in the calculation of the polarizability correction to the Lamb shift. It is possible to show that such inelastic contributions should vanish for t = 0, but can lead to the behavior t ln |t|. This behavior was obtained in Ref. [21] that concentrated on high energy regime. Since terms ~ t ln |t| introduce a substantial non-linearity of the reduced cross section as function of t at low t, i.e. the opposite to the Born contribution in Eq. (11) that should become more linear at lower t, this correction needs to be assessed in the kinematics of the relevant experiments, at few hundreds MeV to few GeV beam energy and |t| ≲ 0.1 GeV^2. This is the task that this work is dedicated to. The same approach is expected to be of relevance for the measurement of the deuteron charge radius in elastic eD-scattering, and the respective estimates will be presented, as well.

III. TWO-PHOTON EXCHANGE AMPLITUDE

In presence of the TPE effects, and in the approximation of small electron mass, the elastic ep-scattering amplitude is given by three scalar amplitudes $\tilde{F}_i(\nu, t)$,

$$T = \frac{e^2}{-t} \bar{u}(k') \gamma_\mu u(k) \times \bar{u}(p') \left[ \tilde{F}_1 \gamma^\mu \tilde{F}_2 \frac{i\sigma^{\mu\nu} \Delta_{\nu}}{2M} + \tilde{F}_3 \frac{K^\mu}{M^2} \right] u(p), \quad (14)$$

In the one-photon exchange (OPE) approximation, two of the amplitudes reduce to the Dirac and Pauli form factors,

$$\tilde{F}_1^{\text{Born}}(\nu, t) = F_1(t), \quad \tilde{F}_2^{\text{Born}}(\nu, t) = F_2(t), \quad \tilde{F}_3^{\text{Born}}(\nu, t) = 0. \quad (15)$$

It is useful to separate one- and two-photon exchange effects explicitly,

$$\tilde{F}_{1,2} = \tilde{F}_{1,2} + \delta \tilde{F}_{1,2}, \quad \tilde{G}_{E,M} = \tilde{G}_{E,M} + \delta \tilde{G}_{E,M}, \quad (16)$$

where the generalization of the Sachs form factors was introduced, $\tilde{G}_E = \tilde{F}_1 - \tau \tilde{F}_2$ and $\tilde{G}_M = \tilde{F}_1 + \tilde{F}_2$. The reduced cross section $\sigma_R$ in presence of TPE effects is
given by

\[
\sigma = G_E^2 + \frac{G_M^2}{1 + \tau} + 2 G_E^2 \text{Re}\left(\delta G_E + \frac{\nu}{M} \tilde{F}_3\right) + \frac{2}{\epsilon(1 + \tau)} G_M \text{Re}\left(\delta G_M + \frac{\nu}{M} \tilde{F}_3\right) + O(\alpha^2).
\]

(17)

It is straightforward to see that the TPE effect on the unpolarized cross section at low \( t \) depends on the same combination of the amplitudes as the elastic amplitude averaged over nucleon spins,

\[
\tilde{T} = -\frac{e^2}{-t} \bar{u}(k') \gamma_\mu u(k) \times \frac{\text{Tr}(\not \! p' + M) \left[ \tilde{F}_1 \gamma^\mu + \tilde{F}_2 \frac{i \omega_m}{2 \sqrt{s}} \not \! \omega + \tilde{F}_3 K^\mu_{2\tau} \right] (\not \! p + M)}{8M} = \frac{e^2}{-t} \bar{u}(k') \gamma_\mu u(k) \Phi(\nu, t),
\]

(18)

where upon keeping leading terms in \( t \) only a short hand was introduced,

\[
\Phi(\nu, t) = \delta G_E(\nu, t) + \frac{\nu}{M} \tilde{F}_3(\nu, t)
\]

(19)

### IV. NEAR-FORWARD ELASTIC \( e^-p \)-SCATTERING AMPLITUDE FROM A DISPERSION RELATION

![FIG. 1: Imaginary part of the 2γ-exchange diagram](image)

The imaginary part of the TPE diagram in Fig. 1 is given by the integral

\[
2\text{Im}T_{2\gamma} = e^4 \int \frac{d^3k_1}{(2\pi)^3} \frac{\epsilon_{\mu\nu}}{2E_1} \cdot \text{Im}W^{\mu\nu}(\not \! k_1 + m_e \not \! \gamma_\mu u(k) \approx \bar{u}(k') \gamma_\mu \not \! k_1 \gamma_\mu u(k),
\]

(20)

and the on-shell condition for the intermediate electron leads to \( E_1 = (\not \! k_1 + m_e)^2 = m_e^2 \). The hadronic tensor can be split into elastic and inelastic contributions, \( W^{\mu\nu} = W_{el}^{\mu\nu} + W_{inel}^{\mu\nu} \). This separation is possible because the former has a pole, \( \text{Im}W_{el} \sim \delta((p + q_1)^2 - M^2) \), whereas the latter has a unitarity cut starting at the pion production threshold \((p + q_1)^2 = (M + m_\pi)^2\).

### A. Elastic contribution

The imaginary part of the elastic part is due to the on-shell nucleon in the intermediate state,

\[
\text{Im}W_{el}^{\mu\nu} = 2\pi \delta((P + K - k_1)^2 - M^2) \times \bar{u}(p') \Gamma^{\nu}(q_2)(P + K - k_1 + M)\Gamma^\mu(q_1)u(p),
\]

(22)

with the on-shell nucleon electromagnetic vertex \( \Gamma^{\mu}(q) = F_1(q^2)\gamma_\mu + F_2(q^2)i\sigma^{\mu\nu}q_\nu / 2M \). It contains the infrared (IR) divergent part that is logarithmic in the fictitious photon mass, \( \sim \ln \lambda^2 \), the coefficient in front of it is model-independent, and has been calculated in Refs. [15, 17] using the soft photon approximation in the loop. The former reference used the approximation \( q_1 \approx 0, q_2 \approx \Delta \) and vice versa both in the numerator and the denominator of the integral, the result simply factorizing the one-photon exchange (Born) amplitude as

\[
\text{Im}\Phi(\text{soft}, a) = \alpha \frac{(E_{cm})^2}{2\pi} G_E(t) \int d\Omega_1 \left[ \frac{1}{Q_1^2} + \frac{1}{Q_2^2} \right] = \alpha \ln \left( \frac{4(E_{cm})^2}{\lambda^2} \right) G_E(t),
\]

(23)

with c.m. energy of both the external and the intermediate electron \( E_{cm} \approx s - M^2 \), neglecting the electron mass.

On the other hand, the latter Ref. applied the soft photon approximation in the numerator only leading to

\[
\text{Im}\Phi(\text{soft}, b) = -\alpha t \frac{(E_{cm})^2}{2\pi} G_E(t) \int d\Omega_1 Q_1^2 Q_2^2 = \alpha \ln \left( \frac{-t}{\lambda^2} \right) G_E(t).
\]

(24)

The limit of small electron mass was taken in the above results. The real part can be obtained from a dispersion relation at fixed \( t \),

\[
\Re\Phi(\nu, t) = \frac{2\nu}{\pi} \int_{\nu_0^d}^{\infty} \frac{d\nu'}{\nu'^2 - \nu^2} \text{Im}\Phi(\nu', t),
\]

(25)

with \( \nu_0^d = t/(4M) \leq 0 \) the threshold for the \( s \)-channel unitarity cut. The evaluation of the dispersion integral with the imaginary part of, e.g., Eq. [24] yields the real part

\[
\Re\Phi(\text{soft}, b) = \frac{\alpha}{\pi} \ln \left( \frac{-t}{\lambda^2} \right) G_E(t) \ln \left( \frac{4M\nu + t}{4M\nu - t} \right) \approx \frac{\alpha}{\pi} \frac{t}{2ME} \ln \left( \frac{-t}{\lambda^2} \right),
\]

(26)

where the LAB energy \( E = \frac{s - M^2}{2M} \), the \( t = 0 \) limit of \( \nu = \frac{s - M^2 + t^2/2}{2M} \) was introduced. We observe here that while the imaginary part of \( \Phi(\text{soft}, b) \) behaves as \( \ln(-t/\lambda^2) \), its real part is suppressed by an extra power of \( t \). This is
a consequence of the principal value integral vanishing identically at \( t = 0 \),
\[
P \int_{t/(4M)}^{\infty} \frac{2\nu d\nu'}{\nu'^2 - \nu^2} = \ln \left( \frac{4M\nu + t}{4M\nu - t} \right) = O(t). \quad (27)
\]

The result of Eq. (26) was used in the analysis of the low-t data from Mainz [11] (without the low-t approximation), and we use the IR part of the TPE amplitude in this form to define the IR finite part of the elastic box as
\[
\Phi_{hard}^{el} \equiv \Phi^{el} - \Phi^{(soft, b)},
\]
that should be added to the full set of radiative corrections included in the experimental analysis. A straightforward calculation using the hadronic tensor of Eq. (22), the leptonic tensor of Eq. (21), and the definition of the amplitude \( \Phi \) in Eq. (18), we obtain
\[
\text{Im}\Phi_{hard}^{el}(\nu, t) = -\frac{\alpha t}{2\pi} \frac{(E_{cm})^2}{Q_1^2 Q_2^2} \int d\Omega_1 \frac{t + Q_1^2 + Q_2^2 + \frac{Q_1^2 F_{21} F_{22} + Q_2^2 F_{11} F_{22} + t F_2}{4M^2}}{t + Q_1^2 + Q_2^2 + \frac{(F_{11} + F_{21}) (F_{12} + F_{22}) - F_1 F_{12} \left( 1 + \frac{4s M^2}{(s - M^2)^2 + st} \right)}} \quad (29)
\]
where terms that cannot lead to \( t \ln t \)-behavior were dropped. For compactness, the shorthand was introduced \( F_{ij} = F_i(q_j^2) \) and \( F_i = F_i(t) \). The above integral is IR finite as it depends on one master integral (at the needed precision) over the solid angle of the intermediate electron,
\[
\int d\Omega_1 \frac{t + Q_1^2 + Q_2^2}{Q_1^2 Q_2^2} = \frac{2\pi}{(E_{cm})^2} \ln \frac{(E_{cm})^2}{-t}, \quad (30)
\]
and after expanding the form factors under the integral as \( F_i(q_j^2) = F_i(0) + q_j^2 F_i'(0) + \ldots \) where necessary, we obtain for the imaginary part
\[
\text{Im}\Phi_{hard}^{el}(\nu, t) = \frac{\alpha t}{2} \ln \frac{(E_{cm})^2}{t} \left[ \frac{1}{3 + \frac{1}{4(E_{cm})^2 + t}} \right], \quad (31)
\]
where we used the relation \( F_i'(0) = \frac{1}{3} \pi^2 E_i^2 - \frac{1}{347\pi^2} F_2(0) \). The real part obtains through a principal value integral and leads to
\[
\text{Re}\Phi_{hard}^{el}(\nu, t) \approx \frac{\alpha t}{2} \left[ \frac{\sqrt{-t/2}}{\nu + \sqrt{-t/2}} + \frac{t}{3} \pi^2 E_i^2 \right]. \quad (32)
\]
The first term in the square bracket is the well-known Feshbach correction [18], the second term was recently found in Ref. [19]. We see this calculation as a valuable check for the method of calculation (the Feshbach term was also found in a similar manner in Ref. [12] [15]). On the other hand, it is obvious that due to approximations made the term proportional to \( t_E^2 \) is not model-independent: other terms that do not have the log behavior were omitted, but they would lead to terms contributing at the order of \( \alpha t \). Therefore, while we support the statement of Ref. [19] that Feshbach correction alone is not enough to warrant the precision of the charge radius extraction in Bernauer et al, inclusion of the correction \( \sim \alpha t_E^2 \) is not enough, either. An important lesson that we learned is that no terms \( \sim t \ln t \) arise from the elastic box, only \( \sqrt{-t} \) and \( \sim const \cdot t \). In the following section we show that the term \( \sim t \ln t \) does arise from the inelastic states in the box, and that the coefficient in front of that term can be calculated model-independently.

### B. Inelastic contribution

We study first the behavior of two representative integrals over the solid angle in the inelastic case,
\[
I_1 = \int d\Omega_1 \frac{1}{Q_1^2 Q_2^2} = \frac{2\pi}{(E_{cm})^2} \ln \frac{(E_{cm})^2}{-t} \left( \frac{t}{(E_{cm} - E_1^2)^2 m_e^2} \right), \quad (33)
\]
where the c.m. energy of the intermediate electron is distinct from the external electron energy, \( E_{cm}^1 = \frac{s - W^2}{2\sqrt{s}} \), and the invariant mass squared of the intermediate hadronic system, \( W^2 = (p + q_1)^2 \) lies above the pion production threshold, \( W^2 \geq W_p^2 = (M + m_\pi)^2 \). Due to this threshold, the soft IR divergence is not possible, however the collinear divergence (emission of a real photon collinear to the electron line) would be possible if the electron were massless. Keeping the finite mass of the electron makes the individual integrals finite, but a potential chiral divergence is introduced. It cannot appear in the final result, and we expect this log dependence on the electron mass to vanish. Again, we will be looking for the leading \( t \)-behavior that is expected to be \( \sim t \ln t \), and that behavior can only come from the integral \( I_1 \). We will keep the integral \( I_2 \) contribution that cancels the \( \ln m_e^2 \) dependence, and neglect the terms \( \sim t \), consistently with the rest of this work.

The spin-averaged part of the hadronic tensor with real photons in general (non-forward) kinematics is expressed in terms of two scalar amplitudes \( f_{12}(P \cdot t, t) \) [20]
The tensor contraction leads to
\[
\ell_{\mu\nu} \cdot \text{Im} W^{\mu\nu} = f_1(Q_1^2 + Q_2^2) \tilde{u}(k') \bar{\ell} (u(k) + f_2 \left\{ (P \cdot q_1)^2 - (P \cdot q_2)^2 \right\} \tilde{u}(k') \bar{\ell} (u(k) + [2(P \cdot q_1)(P \cdot q_2) - (P \cdot q_1)(Q_1^2 + Q_2^2)] \tilde{u}(k') \bar{\ell} (u(k)) \right) \tag{35}
\]

With the help of the relation
\[
\tilde{u}(k') \bar{\ell} (u(k)) = \frac{Q_1^2 + Q_2^2 + t(P \cdot k_1)}{4(P \cdot K)} \tilde{u}(k') \bar{\ell} (u(k)), \tag{36}
\]
and consistently neglecting terms \( tQ_1^2, Q_2^2 \), the result for the imaginary part of the TPE amplitude becomes
\[
\text{Im} T_{2\gamma} = e^4 \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{E_1} \frac{1}{Q_1^2 Q_2^2} \tilde{u}(k') \bar{\ell} (u(k)) \tag{37}
\]
\[
\times \frac{(PK^2) + (P_k)^2}{2(P \cdot K)} \left[ Q_1^2 + Q_2^2 + t(P \cdot k_1) \right] \text{Im} f_2,
\]
while the amplitude \( f_1 \) does not contribute at the leading log accuracy. According to the power counting used throughout this calculation, \( \text{Im} f_2(P, t, Q_1^2, Q_2^2) \) should be taken at \( t = Q_1^2 = Q_2^2 = 0 \). In this kinematics, the optical theorem relates this imaginary part to the total real photo absorption cross section \( \sigma_T \) as
\[
\text{Im} f_2(P, q_1, 0, 0, 0) = -\frac{2}{(P q_1) e^2} \sigma_T. \tag{38}
\]

Using the definition of Eq. [18] and identifying the solid angle integrals in Eq. [37] with the previously introduced scalar integrals \( I_{1,2} \) in Eq. [39], we can finally express
\[
\text{Re} \Phi(E, t) = -\frac{t}{4\pi^3} \int_{E \to E_n} \frac{d\omega}{\omega} \sigma_T(\omega) \ln \frac{\omega^2}{2E^2} \left[ \left( 1 + \frac{\omega^2}{2E^2} \right) \ln \frac{E + \omega}{E - \omega} + \frac{\omega^2}{E^2} \ln \left| 1 - \frac{E^2}{\omega^2} \right| - \frac{\omega^2}{E} \right]. \tag{42}
\]

This is the master formula that is a more general result than that quoted in Ref. [21] where the integral over \( \omega \) was evaluated in the high energy approximation for the cross section. In the next section we present the numerical evaluation of this integral.

V. RESULTS

The integral of Eq. [42] can be evaluated numerically using the phenomenological fit [22] of the world data on real photo absorption on the proton target [23]. The evaluation was done with the Mathematica numerical integration routine. Results are presented in Figs. [3] at three values of the electron beam energy relevant for the Mainz A1 experiment with the proton [11] and the deuteron [24] target, the latter being currently under analysis, for the quantity
\[
\delta \sigma^{TPE}_R / |t| = \frac{2}{|t|} \text{Re} \Phi(E, t), \tag{43}
\]
that features the logarithmic behavior at low \(|t|\). It is compared to the experimental sensitivity of the Mainz experiments that is obtained as

\[
\delta\sigma_R^{E,\text{exp.}}/|t| = -\frac{1}{3} R_E^2 \left( 1 \pm \frac{\delta R_E}{R_E} \right).
\] (44)

For the proton, the experimental result of \(R_E = 0.879(8)\) fm translates into

\[
\delta\sigma_R^{E,\text{exp.}}/|t| = -6.61(12) \text{GeV}^{-2}.
\] (45)

This experimental sensitivity is compared in Fig. 2 to the numerical evaluation of Eq. (42) in the kinematics of the A1 Mainz experiment. The energy dependence (difference between the solid, dashed and dash-dotted lines) reflects the energy dependence of the photo absorption cross section around the \(\Delta(1232)\) region. For the deuteron the projected precision of 0.25% \[24\] together with the most recent global extraction of the deuteron radius from scattering and spectroscopy data \(R_E^{d} = 2.1424(21)\) fm \[2\] leads to

\[
\delta\sigma_R^{E,\text{exp.}}/|t| = -39.278(196) \text{GeV}^{-2},
\] (46)

the uncertainty corresponding to 0.25% projected precision of the scattering experiment. A comparison of this sensitivity to the TPE correction in the kinematics of A1 Mainz experiment is displayed in Fig. 3. The result for higher energy relevant for the proposed JLab experiment \[25\] at higher energies is shown in Fig. 4. As already mentioned, the TPE results in the leading log approximation are model-independent modulo a \(\text{const.}x|t|\) offset that translates into a constant in Figs. 2, 3, 4 therefore what really matters is not the absolute value of the TPE correction but rather the difference between the lowest and highest values of \(t\) (i.e., non-linearity). This non-linearity is close to the experimental precision for the proton and the deuteron at moderate energies, as in the A1 Mainz kinematics, but is seen to be roughly a triple of the experimental sensitivity for GeV-ish energy and \(|t|\) between \(10^{-4}\) and \(5 \times 10^{-2}\) GeV² as in the proposed measurement at JLab. This suggests that the leading log TPE correction has to be included in the experimental analyses that aim at extracting the charge radius from electron scattering with the accuracy below 1%. It is seen that the inclusion of the TPE correction leads to a stronger \(t\)-dependence at low momentum transfer. Upon subtracting the positive-definite \(|t|\ln(4E^2/|t|)\) correction from the experimental data, the extracted value of the charge radius will necessarily go down. For Mainz kinematics, this effect will not affect the extracted value of

![FIG. 2: Results for the dispersive TPE effect on the reduced cross section \(\delta\sigma_R(E,t)/|t|\) for the proton, as function of \(|t|\) in GeV² for three LAB beam energies: 180 MeV (solid black curve), 315 MeV (long-dashed red curve), and 450 MeV (dot-dashed blue curve). For comparison, the experimental sensitivity is shown as a thin dotted horizontal line.](image1)

![FIG. 3: Results for the dispersive TPE effect on the reduced cross section \(\delta\sigma_R(E,t)/|t|\) for the deuteron. Notation as in Fig. 2](image2)

![FIG. 4: Results for the dispersive TPE effect on the reduced cross section \(\delta\sigma_R(E,t)/|t|\) for the proton, as function of \(|t|\) in GeV² for two values of the LAB beam energies: 1.1 GeV (solid black curve) and 2.2 GeV (long-dashed red curve) corresponding to the kinematics of the proposed JLab experiment \[25\]. For comparison, the experimental sensitivity is shown as a thin dotted horizontal line.](image3)
the charge radius beyond a per cent level at the most. For higher energies realized in the proposed JLab experiment, the TPE effect is as important as three per cent.

The TPE effect for the deuteron is somewhat larger than for the proton in comparison with the respective experimental sensitivity. This can be understood recalling that the total photo absorption cross section for the deuteron is roughly twice that for the proton in the hadronic range. On the other hand, the quantity \(\frac{r^2_{E} \delta r_{E}}{r_{P}}\) is only about 1.5 times larger for the deuteron, hence a larger relative effect for the latter target. Nuclear effects were neglected for this estimate. Moreover, the deuteron quasi elastic break-up was not included: the derivation is based on treating \(t\) as small compared to all other scales, an approximation that would not be obeyed if \(t\) be compared to the characteristic scale \(\sim MB_d \sim 2 \times 10^{-3} GeV^2\), with the deuteron binding energy \(B_d \approx 2.224 MeV\). An exact calculation would have to be carried out to account for the nuclear part of the photo excitation of the deuteron.

In summary, we considered elastic electron-proton (deuteron) scattering at low momentum transfer \(t\) and in presence of the two-photon exchange (TPE). We calculated the TPE effect on the unpolarized cross section in the hadronic range. On the other hand, the quantity \(r^2_{E} \delta r_{E}/r_{P}\) is only about 1.5 times larger for the deuteron, hence a larger relative effect for the latter target. Nuclear effects were neglected for this estimate. Moreover, the deuteron quasi elastic break-up was not included: the derivation is based on treating \(t\) as small compared to all other scales, an approximation that would not be obeyed if \(t\) be compared to the characteristic scale \(\sim MB_d \sim 2 \times 10^{-3} GeV^2\), with the deuteron binding energy \(B_d \approx 2.224 MeV\). An exact calculation would have to be carried out to account for the nuclear part of the photo excitation of the deuteron.

In summary, we considered elastic electron-proton (deuteron) scattering at low momentum transfer \(t\) and in presence of the two-photon exchange (TPE). We calculated the TPE effect on the unpolarized cross section in the low-\(t\) asymptotics. For the TPE effect with just the nucleon degrees of freedom inside the loop (elastic contribution), the leading behavior \(\sim \sqrt{-t}\) is given by the model-independent Feshbach correction. For the TPE effect with inelastic states, the leading low-\(t\) behavior is \(t \ln t\), and the coefficient in front of this term can be calculated model-independently, as well. This coefficient is given by a weighted integral over the total photo absorption cross section. This integral was evaluated numerically using the recent parametrization of world total photo absorption data on the proton and the deuteron, and the result was confronted to the experimental accuracy in extracting the charge radius from electron scattering data at low \(t\) in the kinematics of recent and upcoming experiments. We found that while at the beam energy of a few hundreds MeV, as in A1 Mainz the non-linearity introduced by the \(t \ln t\) TPE correction is comparable to the experimental precision. At higher energy of 1-2 GeV corresponding to the experiment planned at Jefferson Lab, this effect becomes about three times larger than the experimental precision for extracting the proton charge radius, and thus it is a must to include this effect in the experimental analysis to warrant the extraction of the charge radius from unpolarized cross section data at the level of one per cent or below.

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The reader may note that the amplitude $\Phi$ is an odd function of energy, quite opposite to the Lamb shift calculation where the TPE amplitude is an even function of energy. These two facts are not in contradiction since there are indeed two different amplitudes for a scattering of an electron off a spinless target: $\Phi \bar{u} P u$ and $\phi m_e \bar{u} u$. The latter will also contribute to the scattering process considered here for finite $m_e$, but is additionally suppressed by the small $m_e/M \sim 10^{-3}$ ratio and is neglected. In the kinematics relevant for atomic calculations $E \approx 0$, this latter amplitude $\phi$ is the one of interest, while the former $\Phi$ vanishes when evaluated at $E = 0$. 
