Special solutions of nonlinear von Neumann equations

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Abstract
We consider solutions of the non-linear von Neumann equation involving Jacobi’s elliptic functions sn, cn, and dn, and 3 linearly independent operators. In two cases one can construct a state-dependent Hamiltonian which is such that the corresponding non-linear von Neumann equation is solved by the given density operator. We prove that in a certain context these two cases are the only possibilities to obtain special solutions of this kind. Well-known solutions of the reduced Maxwell-Bloch equations produce examples of each of the two cases. Also known solutions of the non-linear von Neumann equation in dimension 3 are reproduced by the present approach.

1 Introduction

Exact solutions of non-linear differential equations are often based on sets of special functions like tanh and sech or Jacobi’s double periodic functions sn, cn, and dn. Here, the algebraic relations between these functions are exploited to construct solutions of non-linear von Neumann equations. Given a Hamilton operator $H$, the von Neumann equation reads

$$i\dot{\rho} = \left[H, \rho\right].$$

(1)
The notations

\[ \dot{\rho} = \frac{d}{dt} \rho \quad \text{and} \quad [H, \rho] = H\rho - \rho H \]  

are used. A non-linear von Neumann equation is obtained when \( H \) is allowed to depend on \( \rho \)

\[ i\dot{\rho} = [H[\rho], \rho]. \]  

The notation \( H[\rho] \) is used instead of \( H(\rho) \) to stress that \( H[\rho] \) is not a function of \( \rho \) in the sense of spectral theory. Alternatively, a function \( f(x) \) is given and the equation is written as \( i\dot{\rho} = [H, f(\rho)]_\rho. \) The prototype of the latter, obtained with the choice \( f(x) = x^2 \), is the Euler top equation \[1\] and can be written as well in the form \[3\]

\[ i\dot{\rho} = [H, \rho^2] = \{H, \rho\}_\rho \] with \( \{H, \rho\} = H\rho + \rho H. \]  

Some explicit solutions \[1, 2\] of this equation depend on time \( t \) as sech(\( \omega t \)). They have been called \[2\] self-scattering solutions because they exhibit different behaviour in the \( t \to -\infty \) and \( t \to +\infty \) limits and make the transition from one behaviour to the other in a limited region of time. However, up to now, the reason why the sech function appears had not been investigated in a systematic manner.

It has been observed \[3\] that solutions of the von Neumann equation with time-dependent Hamiltonian are also solutions of a non-linear von Neumann equation. The main application of the present work supports this point of view. The reduced Maxwell-Bloch equations describe the time evolution of a spin variable in presence of an applied field. It was noted \[4\] that the well-known McCall-Hahn \[5, 6\] solution of these equations is a solution of a non-linear von Neumann equation as well. This link between time-dependent and state-dependent Hamiltonians is clarified in the present paper.

The non-linear von Neumann equation has mainly been investigated with Darboux transformations, trying to find non-trivial solutions (see e.g. \[1, 2, 4, 7, 8, 9, 10\]). Many of the solutions constructed in this way concern 3-by-3 matrices. The connection between these special solutions and Jacobi’s double periodic functions was noticed in the Appendix of \[2\]. Here, we reproduce these solutions as special cases of our approach.

In the next section, some properties of solutions of non-linear von Neumann equations are discussed. In Section 3 we review basic properties of
Jacobi’s double periodic functions and formulate methods to construct state-dependent Hamiltonians for which the corresponding non-linear von Neumann equations have special solutions. Two cases are formulated and for each case a theorem is formulated which shows that this case is the only possible solution of the non-linear von Neumann equation that has the given form. Section 4 treats different applications. The paper ends with a short summary in Section 5. The proofs of the two theorems are given in Appendix.

2 Non-linear von Neumann equation

2.1 Definition

We restrict ourselves to non-linear von Neumann equations of the form \( H[\rho] = \sum_{jk} \lambda_{jk} X_j^* \rho X_k \)\(^{(5)}\), with a Hamiltonian which depends linearly on the density operator \( \rho \). E.g., \( H[\rho] \) might be of the form

\[
H[\rho] = \sum_{jk} \lambda_{jk} X_j^* \rho X_k.
\]

The operators \( X_j \) and the matrix coefficients \( \lambda_{jk} \) do not depend on time.

2.2 Spectrum

A nice property of solutions of the non-linear von Neumann equation is conservation of spectrum. Indeed, let be given a solution \( \rho_t \) of (3), and let \( U_t \) be solution of

\[
\frac{d}{dt} U_t = i U_t H[\rho_t]
\]

with initial condition \( U_0 = \mathbb{I} \). Then one has

\[
\rho_t = U_t^* \rho_0 U_t.
\]

The adjoint operators \( U_t^* \) are isometric. To see this, note that

\[
\frac{d}{dt} U_t U_t^* = 0
\]

because \( H[\rho_t] \) is self-adjoint. Hence one has \( U_t U_t^* = \mathbb{I} \), i.e. \( U_t^* \) is isometric. Assume now \( \rho_0 \psi = \lambda \psi \). Then there follows \( \rho_t U_t^* \psi = \lambda U_t^* \psi \). Conversely, if \( \rho_t \psi = \lambda \psi \) then \( \rho_0 U_t \psi = \lambda U_t \psi \) follows. Hence, \( \rho_t \) and \( \rho_0 \) have the same spectrum.

Note that the solution of (6) can be written as

\[
U_t = \mathbb{I} + i \int_0^t dt_1 H[\rho_t] + i^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H[\rho_{t_2}] H[\rho_{t_1}] + \cdots.
\]

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2.3 Linear part

The Hamiltonian $H[\rho]$ may contain a contribution of the form $(\text{Tr} \rho)H_0$, which for properly normalized density operators leads to the standard linear von Neumann equations. For simplicity, this contribution is called here the linear part. Quite often it can be eliminated, leading to a problem which can be discussed more easily.

Let $\rho_t$ be a density operator satisfying the non-linear von Neumann equation (3). Let $H_0$ be a symmetric operator satisfying for all $t$

$$H[e^{itH_0} \rho_t e^{-itH_0}] = e^{itH_0} H[\rho_t] e^{-itH_0}. \quad (10)$$

Introduce a linear map $K[\sigma]$ by

$$K[\sigma] = H[\sigma] - (\text{Tr} \sigma) H_0. \quad (11)$$

Then $\sigma_t$, defined by

$$\sigma_t = e^{itH_0} \rho_t e^{-itH_0}, \quad (12)$$

satisfies the non-linear von Neumann equation (3) with $H[\rho]$ replaced by $K[\sigma]$

$$i\dot{\sigma}_t = [K[\sigma_t], \sigma_t]. \quad (13)$$

3 Special solutions

3.1 Jacobi’s elliptic functions

Solutions of non-linear equations are often based on Jacobi’s elliptic functions with elliptic modulus $k$. They satisfy the algebraic relations

$$1 = \text{sn}^2(x, k) + \text{cn}^2(x, k)$$

$$1 = \text{dn}^2(x, k) + k^2 \text{sn}^2(x, k)$$

$$\frac{d}{dx} \text{sn}(x, k) = \text{cn}(x, k) \text{dn}(x, k)$$

$$\frac{d}{dx} \text{cn}(x, k) = -\text{sn}(x, k) \text{dn}(x, k)$$

$$\frac{d}{dx} \text{dn}(x, k) = -k^2 \text{sn}(x, k) \text{cn}(x, k). \quad (14)$$

In the limit $k = 1$ the period of these elliptic functions diverges and $\text{sn}(x, k)$ converges to $\tanh(x)$, $\text{cn}(x, k)$ and $\text{dn}(x, k)$ both converge to $\text{sech}(x)$. The above relations then simplify to

$$1 = \text{sech}^2 u + \tanh^2 u$$
\[
\frac{d}{du} \text{sech} u = - \text{sech} u \tanh u,
\frac{d}{du} \tanh u = \text{sech}^2 u.
\] (15)

In the limit \( k = 0 \) the function \( \text{sn}(x, k) \) converges to \( \sin(x) \), \( \text{cn}(x, k) \) to \( \cos(x) \), and \( \text{dn}(x, k) \) converges to 1.

We next consider special solutions of the non-linear von Neumann equation involving three independent operators. Two different cases are considered.

### 3.2 Case 1

**Theorem 1** Let be given \( \rho(t) \) of the form

\[
\rho(t) = \theta + \text{cn}(\omega t, k)A + \text{sn}(\omega t, k)B + \text{dn}(\omega t, k)X.
\]

Assume \( \rho(t) \) satisfies the nonlinear von Neumann equation

\[
\dot{\rho} = i [\rho, H[\rho]]_-
\]

with \( H[\cdot] \) linear.

Assume that \( A, B, \) and \( X \) linearly independent operators.

Assume \( \theta \) commutes with \( A, B, \) and \( X \).

Assume \( \omega \neq 0 \) and \( 0 < k \leq 1 \).

Assume that the range of \( H[\cdot] \) is in the linear span of the operators \( A, B, \) and \( X \).

Then there exist constants \( \alpha \) and \( \beta \) such that

\[
\begin{align*}
i[B, X] & = \alpha A \quad \text{(16)} \\
i[A, B] & = k^2 \beta X \quad \text{(17)} \\
i[A, X] & = -\frac{\alpha \beta}{\alpha + \beta} B. \quad \text{(18)}
\end{align*}
\]

The Hamiltonian is of the form

\[
\begin{align*}
H[A] & = (\nu + \frac{\omega}{\beta}) A \quad \text{(19)} \\
H[B] & = \nu B \quad \text{(20)} \\
H[X] & = (\nu - \frac{\omega}{\beta}) X \quad \text{(21)} \\
H[\theta] & = 0. \quad \text{(22)}
\end{align*}
\]

Conversely, given operators \( A, B, X, \) and \( \theta \), satisfying the above assumptions, then the operator \( \rho(t) \) given by (16) solves the non-linear von Neumann equation.
One direction of the proof is given in Appendix A. The proof of the converse statement is a straightforward exercise.

### 3.3 Case 2

**Theorem 2** Let be given \( \rho(t) \) of the form

\[
\rho(t) = \theta + \text{cn}(\omega t, k)A + \text{sn}(\omega t, k)\text{dn}(\omega t, k)C + \text{cn}(\omega t, k)^2D,
\]

with \( \theta, A, C, \) and \( D \) linearly independent operators, and with \( \omega \neq 0 \) and \( 0 < k \leq 1 \).

Assume \( \rho(t) \) satisfies the nonlinear von Neumann equation

\[
\dot{\rho} = i[\rho, H[\rho]],
\]

with \( H[\cdot] \) linear. Assume that the range of \( H[\cdot] \) is in the linear span of the operators \( A, C, \) and \( D \).

Assume that an operator \( \theta_0 \) exists, which commutes with \( A, C, \) and \( D, \) such that \( \theta - \theta_0 \) belongs to the linear span of \( A, C, \) and \( D \).

Then there exist constants \( \alpha \) and \( \delta \) such that

\[
\begin{align*}
    i[C,D] &= \alpha A \\
    i[A,C] &= \delta D \\
    i[A,D] &= -k^2\delta C.
\end{align*}
\]

The Hamiltonian is necessarily of the form

\[
\begin{align*}
    H[A] &= \left( \nu + \frac{2\omega}{\delta} \right) A \\
    H[C] &= \nu C \\
    H[D] &= \nu D \\
    H[\theta] &= \left( -\frac{\omega}{\alpha} + \frac{1 - 2k^2}{2k^2} \nu \alpha + \frac{1}{2} \delta \nu \right) D.
\end{align*}
\]

The operator \( \theta_0 \) is given by

\[
\theta_0 = \theta - \left( \frac{1 - 2k^2}{2k^2} - \frac{\delta}{2\alpha} \right) D.
\]

Conversely, given operators \( A, C, D, \theta_0 \) and \( \theta \), satisfying the above assumptions, then the operator \( \rho(t) \) given by \((23)\) solves the non-linear von Neumann equation.

The difficult direction of the proof is given in Appendix B. The proof of the converse statement is a straightforward exercise.
4 Examples

4.1 Reduced Maxwell-Bloch equations

Let $\sigma_1, \sigma_2, \sigma_3$ denote the Pauli matrices. Consider the density operator

$$\rho_t = \frac{1}{2} (1 + \sum_{\alpha=1}^{3} u_\alpha \sigma_\alpha) = \frac{1}{2} \begin{pmatrix} 1 + u_3 & u_1 - i u_2 \\ u_1 + i u_2 & 1 - u_3 \end{pmatrix}$$

with

$$u_1 = \frac{2 \tau \Delta}{1 + (\tau \Delta)^2} \text{sech} (t/\tau)$$

$$u_2 = \frac{\tau \Delta}{1 + (\tau \Delta)^2} \text{sech} (t/\tau) \tanh (t/\tau)$$

$$u_3 = -1 + \frac{2}{1 + (\tau \Delta)^2} \text{sech}^2 (t/\tau).$$

See (Eq. (4.21) of [6]). It solves the reduced Maxwell-Bloch equations

$$\dot{u}_1 = -\Delta u_2$$

$$\dot{u}_2 = \Delta u_1 + \kappa \mathcal{E} u_3$$

$$\dot{u}_3 = -\kappa \mathcal{E} u_2$$

with constants $\Delta, \kappa, \text{and } \tau$ and with

$$\mathcal{E}(t) = \frac{2}{\kappa \tau} \text{sech}(t/\tau).$$

This solution satisfies the requirement that $|u|$ is constant in time so that the eigenvalues of $\rho$, which equal $(1/2)(1 \pm |u|)$, are conserved. In fact, one has $|u| = 1$. Hence, $\rho$ is a one-dimensional projection operator.

This density operator is an example of Case 2, with $k = 1, \omega = 1/\tau$, and with operators

$$\theta_0 = \frac{1}{2} \mathbb{I}$$

$$A = \frac{\tau \Delta}{1 + (\tau \Delta)^2} \sigma_1$$

$$C = \frac{1}{1 + (\tau \Delta)^2} \sigma_2$$

$$D = \frac{1}{1 + (\tau \Delta)^2} \sigma_3.$$
They satisfy the commutation relations

\[ i[A, C]_+ = - \frac{2\tau \Delta}{1 + (\tau \Delta)^2} D \]  
(39)

\[ i[C, D]_+ = - \frac{2}{\tau \Delta 1 + (\tau \Delta)^2} A \]  
(40)

\[ i[D, A]_+ = - \frac{2\tau \Delta}{1 + (\tau \Delta)^2} C. \]  
(41)

From Theorem 2 follows that \( \rho_t \) satisfies the non-linear von Neumann equation with Hamiltonian given by

\[ H[\sigma_1] = - \frac{1}{\Delta} (\omega^2 + \Delta^2) \sigma_1 \]  
(42)

\[ H[\sigma_2] = 0 \]  
(43)

\[ H[\sigma_3] = 0 \]  
(44)

\[ H[\mathbb{I}] = - \frac{2\omega^3}{\omega^2 + \Delta^2} \sigma_3. \]  
(45)

4.2 Including phase modulation

A related problem is obtained by including phase modulation effects. The equations are

\[ \dot{u}_1 = \dot{\phi} u_2 \]

\[ \dot{u}_2 = -\dot{\phi} u_1 + \kappa \mathcal{E} u_3 \]

\[ \dot{u}_3 = -\kappa \mathcal{E} u_2 \]  
(46)

with

\[ \dot{\phi} = -\delta \tanh(t/\tau) \]

\[ \mathcal{E} = \frac{1}{\kappa \tau} \sqrt{1 + (\tau \delta)^2 \text{sech}(t/\tau)}. \]  
(47)

Solutions are (Eq. (4.49) of [3])

\[ u_1 = \tau \delta u_2 \]

\[ u_2 = - \frac{1}{\sqrt{1 + (\tau \delta)^2}} \text{sech}(t/\tau) \]

\[ u_3 = \tanh(t/\tau). \]  
(48)

Note that

\[ \dot{u}_1 = -\frac{1}{\tau} u_1 u_3 \]

8
\[
\begin{align*}
\dot{u}_2 &= -\frac{1}{\tau} u_2 u_3 \\
\dot{u}_3 &= \frac{1}{\tau} (1 - u_3^2) = \frac{1}{\tau} (u_1^2 + u_2^2).
\end{align*}
\] (49)

Let us try to match this example to Case 1. Clearly needed is \(\omega = 1/\tau\), \(k^2 = 1\), \(\theta = (1/2)\mathbb{I}\), and \(B = (1/2)\sigma_3\). The choice of operators \(A\) and \(X\) is not so obvious. But the choice

\[
\begin{align*}
A &= -\frac{1}{2} \frac{\tau \delta}{\sqrt{1 + \tau^2 \delta^2}} \sigma_1 \\
X &= -\frac{1}{2} \frac{1}{\sqrt{1 + \tau^2 \delta^2}} \sigma_2
\end{align*}
\] (50) (51)

satisfies the requirements. From Theorem 1 then follows that \(\rho_t\) satisfies the non-linear von Neumann equation with Hamiltonian given by

\[
\begin{align*}
H[\sigma_1] &= \frac{\omega^2}{\delta} \sigma_1 \\
H[\sigma_2] &= -\delta \sigma_2 \\
H[\sigma_3] &= 0 \\
H[\mathbb{I}] &= 0.
\end{align*}
\] (53) (54) (55) (56)

4.3 Three-level system

Fix \(0 < k \leq 1\) and \(\alpha, \delta,\) and \(\phi\) real. Consider 3-by-3 matrices

\[
\begin{align*}
A &= k \frac{\delta}{\sqrt{2}} \begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & e^{i\phi} \\ e^{-i\phi} & e^{-i\phi} & 0 \end{pmatrix}, \\
C &= \sqrt{\frac{\alpha \delta}{2}} \begin{pmatrix} 0 & 0 & -ie^{i\phi} \\ 0 & 0 & ie^{i\phi} \\ ie^{-i\phi} & ie^{-i\phi} & 0 \end{pmatrix}, \\
D &= k \sqrt{\alpha \delta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\theta_0 &= \frac{1}{3} \mathbb{I}.
\end{align*}
\] (57) (58) (59) (60)

It is easy to verify that they satisfy the case 2 commutation relations \([23, 24, 25]\). Hence the density matrix \(\rho(t)\), given by \([23]\), satisfies the non-linear von...
Neumann equation (16) for any Hamiltonian of the form (29). The operator \( \theta_0 \) is given by (30).

This result can be transformed by adding a linear part (in the sense of Section 2.3). Let \( \sigma(t) = e^{-i\mu t P_3} \rho(t) e^{i\mu t P_3} \) with

\[
P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]  
(61)

Note that

\[
H[\sigma(t)] = e^{-i\mu t P_3} H[\rho(t)] e^{i\mu t P_3}
\]
(62)
is trivially satisfied because \( H[A] \) is proportional to \( A \), \( H[C] \) is proportional to \( C \), \( H[D] \) is proportional to \( D \), and \( \theta \) commutes with \( P_3 \). This implies that \( \sigma(t) \) satisfies the equation

\[
i\dot{\sigma} = [H[\sigma] + \mu P_3, \sigma]
\]
(63)
The latter equation can be written into the form

\[
i\dot{\sigma} = [H_0 + H_I, \sigma],
\]
(64)
where

\[
H_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},
\]  
(65)
and

\[
H_I = \epsilon \operatorname{cn}(\omega t, k) \begin{pmatrix} 0 & 0 & e^{i(\phi - \mu t)} \\ 0 & 0 & e^{i(\phi - \mu t)} \\ e^{-i(\phi - \mu t)} & e^{-i(\phi - \mu t)} & 0 \end{pmatrix},
\]  
(66)
and \( \lambda = -k \omega \sqrt{\frac{1}{\alpha}}, \epsilon = \frac{k \omega}{\sqrt{2}}, \nu = 0 \). It describes a three-level system interacting with an electromagnetic pulse \( E = E_0 e^{i(\phi - \mu t)} \operatorname{cn}(\omega t, k) \).
4.4 Known solutions in $d = 3$

Fix real constants $k$, $\omega$, $\phi$, $\lambda$ and $\mu$, satisfying $|\lambda| < \mu$ and $0 < k \leq 1$. Define operators $\theta$, $A$, $B$, and $X$, by

$$\theta = \frac{1}{3} \mathbb{I},$$
(67)

$$B = \frac{k \omega}{\sqrt{\mu^2 - \lambda^2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
(68)

$$A = \frac{k \omega}{\sqrt{2\mu(\mu + \lambda)}} \begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & 0 \\ e^{-i\phi} & 0 & 0 \end{pmatrix}$$
(69)

$$X = \frac{\omega}{\sqrt{2\mu(\mu - \lambda)}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\phi} \\ ie^{-i\phi} & 0 & 0 \end{pmatrix}.$$  
(70)

They satisfy the case-1 commutation relations with

$$\alpha = \frac{\omega}{\mu - \lambda}$$
(71)

$$\beta = \frac{\omega}{\mu + \lambda}.$$  
(72)

Hence, the results of the previous section imply that the density operator

$$\sigma_t = \theta + B \text{sn}(\omega t, k) + A \text{cn}(\omega t, k) + X \text{dn}(\omega t, k)$$
(73)

satisfies the non-linear von Neumann equation $i\dot{\sigma} = [H[\sigma], \sigma]$ for any Hamiltonian $H[\sigma]$ satisfying

$$H[\theta] = 0$$
(74)

$$H[A] = (\nu + \mu + \lambda)A$$
(75)

$$H[B] = \nu B$$
(76)

$$H[X] = (\nu - \mu + \lambda)X.$$  
(77)

A satisfactory choice is

$$H[\sigma] = \mu \begin{pmatrix} 0 & 0 & \sigma_{31} \\ 0 & 0 & -\sigma_{32} \\ \sigma_{13} & -\sigma_{23} & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & \sigma_{31} \\ 0 & 0 & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & 0 \end{pmatrix}.$$  
(78)

Note that

$$H[\sigma] = \{H_0, \sigma\} - \frac{2}{3} (\text{Tr} \sigma) H_0$$
(79)
with

\[ H_0 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \] (80)

In particular, \( \rho_t \) defined by

\[ \rho_t = e^{-\left(\frac{2}{3}\right)itH_0} \sigma_t e^{\left(\frac{2}{3}\right)itH_0} \] (81)

satisfies the non-linear von Neumann equation \( i\dot{\rho} = [H_0, \rho^2]_+ \). This equation with Hamiltonian (80) has been studied in [2]. Explicit solutions were obtained in the limit \( k = 1 \).

### 4.5 Variations on a theme

Let us slightly modify the previous example. Fix real constants \( b \neq 0, \omega, \phi \), and \( 0 < k \leq 1 \). Define operators \( \theta, A, B, \) and \( X \), by

\[ \theta = \frac{1}{3}I, \] (82)

\[ A = \frac{k\omega}{b\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (83)

\[ B = \frac{k\omega}{b} \begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & 0 \\ e^{-i\phi} & 0 & 0 \end{pmatrix}, \] (84)

\[ X = \frac{\omega}{b\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\phi} \\ ie^{-i\phi} & 0 & 0 \end{pmatrix}. \] (85)

They satisfy the case-1 commutation relations with

\[ \alpha = \beta = -\frac{\omega}{b}. \] (86)

The density operator

\[ \sigma_t = \theta + B \text{sn}(\omega t, k) + A \text{cn}(\omega t, k) + X \text{dn}(\omega t, k) \] (87)

satisfies the non-linear von Neumann equation \( i\dot{\sigma} = [H[\sigma], \sigma]_+ \) for any Hamiltonian \( H[\sigma] \) satisfying

\[ H[\theta] = 0 \] (88)

\[ H[A] = (\nu - b)A \] (89)

\[ H[B] = \nu B \] (90)

\[ H[X] = (\nu + b)X. \] (91)
A satisfactory choice, corresponding with \( \nu = 4b \), is

\[
H[\sigma] = b \begin{pmatrix}
0 & 3\sigma_{21} & 4\sigma_{31} \\
3\sigma_{13} & 0 & 5\sigma_{32} \\
4\sigma_{13} & 5\sigma_{23} & 0
\end{pmatrix}.
\] (92)

With this choice is

\[
H[\sigma] = \{H_0, \sigma\} - \frac{2}{3} \langle \text{Tr } \sigma \rangle H_0,
\] (93)

with

\[
H_0 = b \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}.
\] (94)

In particular, \( \rho_t \) defined by

\[
\rho_t = e^{-\frac{2}{3}itH_0} \sigma_t e^{\frac{2}{3}itH_0}
\] (95)

satisfies the non-linear von Neumann equation \( i\dot{\rho} = [H_0, \rho^2]_- \). In the limit \( k = 1 \) this example has been discussed in [4].

5 Discussion

This paper studies the non-linear von Neumann equation under the restrictions that (1) the Hamiltonian \( H[\rho] \) depends linearly on the density operator \( \rho \); (2) the solution \( \rho_t \) involves Jacobi’s elliptic functions sn, cn and dn, or their limits tanh and sech; (3) the solution \( \rho_t \) involves 3 linearly independent operators, and possibly a fourth operator commuting with these three operators.

The paper does not focus on methods for solving non-linear von Neumann equations. It rather follows the opposite way. Starting from a special solution \( \rho_t \), it constructs the Hamiltonian \( H[\rho] \) for which \( \rho_t \) solves the corresponding non-linear von Neumann equation. Two cases have been treated. In both cases we have proved a theorem stating conditions under which the special solution is found. Two well-known solutions of the reduced Maxwell-Bloch equations are examples of these two cases. We have shown that the reduced Maxwell-Bloch equations can be generalized to three-level systems and that these have not only special solutions involving sech and tanh, but also periodic solutions involving Jacobi’s elliptic functions. Finally, some of the known 3-dimensional solutions of the non-linear von Neumann equation appear to be examples of these two cases as well.
The examples of the reduced Maxwell-Bloch equations show that non-linear von Neumann equations and their solutions appear in physics in a natural manner. The \( d = 3 \)-solutions of the equation \( i\dot{\rho} = [H_0, \rho^2] \), obtained here, are generalizations of those found in [4].

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A Proof of Theorem 1

A straightforward calculation leads to the set of equations

\[
0 = i[H[A], A]_+ + i[H[X], X]_-
\]
\[
0 = -i[H[A], A]_- + i[H[B], B]_- - k^2 i[H[X], X]_-
\]
\[
0 = i[H[\theta], A]_-
\]
\[
0 = i[H[\theta], B]_-
\]
\[
0 = i[H[\theta], X]_-
\]
\[
\omega A = i[H[B], X]_+ + i[H[X], B]_-
\]
\[
-\omega B = i[H[A], X]_- + i[H[X], A]_-
\]
\[
k^2 \omega X = i[H[A], B]_- + i[H[B], A]_-.
\]

Equations (98, 99, 100) imply that \( H[\theta] \) commutes with \( A, B, \) and \( X \).

Introduce the notations

\[
H[A] = a_A A + b_A B + x_A X
\]
\[
H[B] = a_B A + b_B B + x_B X
\]
\[
H[X] = a_X A + b_X B + x_X X
\]
\[
H[\theta] = a_0 A + b_0 B + x_0 X.
\]
Then the remaining equations become

\[ 0 = -b_A K_{AB} + (a_X - x_A) K_{AX} + b_X K_{BX} \]  
(108)

\[ 0 = (a_B + b_A) K_{AB} + (x_A - k^2 a_X) K_{AX} - (x_B + k^2 b_X) K_{BX} \]  
(109)

\[ 0 = -b_0 K_{AB} - x_0 K_{BX} \]  
(110)

\[ 0 = a_0 K_{AB} - x_0 K_{BX} \]  
(111)

\[ 0 = a_0 K_{AX} + b_0 K_{BX} \]  
(112)

\[ \omega A = a_X K_{AB} + a_B K_{AX} + (b_B - x_X) K_{BX} \]  
(113)

\[ -\omega B = -b_X K_{AB} + (a_A - x_X) K_{AX} + b_A K_{BX} \]  
(114)

\[ k^2 \omega X = (a_A - b_B) K_{AB} - x_B K_{AX} - x_A K_{BX}. \]  
(115)

Because \( A \), \( B \), and \( X \) are linearly independent and \( \omega \) and \( k \) do not vanish the last three equations imply that \( K_{AB} \), \( K_{AX} \), and \( K_{BX} \) are linearly independent. Then the first five equations imply \( a_B = a_X = b_A = b_X = x_A = x_B = a_0 = b_0 = c_0 = 0 \). The three last equations then read

\[ \omega A = (b_B - x_X) K_{BX} \]  
(116)

\[ -\omega B = (a_A - x_X) K_{AX} \]  
(117)

\[ k^2 \omega X = (a_A - b_B) K_{AB}. \]  
(118)

This implies (11, 17, 18) with

\[ \alpha = \frac{\omega}{b_B - x_X} \]  
(119)

\[ \beta = \frac{\omega}{aA - b_B}. \]  
(120)

Let \( \nu = b_B \). Then one finds

\[ a_A = \nu + \frac{\omega}{\beta} \]  
(121)

\[ x_X = \nu - \frac{\omega}{\alpha}. \]  
(122)

The equations for \( H[\cdot] \) then follow.
B Proof of Theorem 2

B.1 Equations

A straightforward calculation leads to the set of equations

\[ 0 = i[D, H[D]] - k^2 i[C, H[C]] \] (123)
\[ 0 = i[\theta, H[\theta]] + (1 - k^2) i[C, H[C]] \] (124)
\[ 0 = i[\theta, H[D]] + i[D, H[\theta]] + i[A, H[A]] + (2k^2 - 1) i[C, H[C]] \] (125)
\[ 0 = i[C, H[D]] + i[D, H[C]] \] (126)
\[ -\omega A = i[\theta, H[C]] + i[C, H[\theta]] \] (127)
\[ 2k^2 \omega C = i[A, H[D]] + i[D, H[A]] \] (128)
\[ \omega C = i[\theta, H[A]] + i[A, H[\theta]] + i[A, H[D]] + i[D, H[A]] \] (129)
\[ -2 \omega D = i[A, H[C]] + i[C, H[A]] \] (130)

Introduce the notations \( K_{AB} = i[A, B] \), and similar notation for other commutators. Because the range of \( H[\cdot] \) is in the span of \( A, C, \) and \( D \), one can write

\[ H[A] = a_A A + c_A C + d_A D \] (131)
\[ H[C] = a_C A + c_C C + d_C D \] (132)
\[ H[D] = a_D A + c_D C + d_D D \] (133)
\[ H[\theta] = a_0 A + c_0 C + d_0 D. \] (134)

By assumption there exist numbers \( t_A, t_C, \) and \( t_D \), and an operator \( \theta_0 \) commuting with \( A, C, \) and \( D \), such that

\[ \theta = \theta_0 + t_A A + t_C C + t_D D. \] (135)

This implies

\[ K_{A\theta} = t_C K_{AC} + t_D K_{AD} \] (136)
\[ K_{C\theta} = -t_A K_{AC} + t_D K_{CD} \] (137)
\[ K_{D\theta} = -t_A K_{AD} - t_C K_{CD}. \] (138)

Then the equations (127, 128, 130) become

\[ -\omega A = (c_C t_A - a_C t_C - a_0) K_{AC} + (d_C t_A - a_C t_D) K_{AD} \]
\[ + (d_C t_C - c_C t_D + d_0) K_{CD} \] (139)
\[ 2k^2 \omega C = c_D K_{AC} + (d_D - a_A) K_{AD} - c_A K_{CD} \] (140)
\[ -2 \omega D = (c_C - a_A) K_{AC} + d_C K_{AD} + d_A K_{CD}. \] (141)
Because $A$, $C$ and $D$, are linearly independent and $k$ and $\omega$ do not vanish one concludes that the operators $K_{AC}$, $K_{AD}$, and $K_{CD}$, are linearly independent.

Consider next (123) [126]. They can be written as

\[ 0 = -(c_D + k^2d_C)K_{CD} - a_D K_{AD} + k^2a_C K_{AC} \]  
\[ 0 = (d_D - c_C)K_{CD} - a_C K_{AD} - a_D K_{AC}. \]  

Because of the linear independence of $K_{AC}$, $K_{AD}$, and $K_{CD}$, they imply $a_C = a_D = 0$ and $c_D = -k^2d_C$ and $d_D = c_C$. The equations (123) and (125), the latter simplified with (128), become

\[ -\omega A = (c_C t_A - a_0)K_{AC} + d_C t_A K_{AD} \]
\[ + (d_C t_C - c_C t_D + d_0)K_{CD} \]
\[ 2k^2\omega C = -k^2d_C K_{AC} + (c_C - a_A)K_{AD} - a_C K_{CD} \]
\[ -2\omega D = (c_C - a_A)K_{AC} + d_C K_{AD} + d_A K_{CD}. \]

Finally, (124), (125), and (129), the latter simplified with (128), become

\[ 0 = (c_0 t_A - a_0 t_C)K_{AC} + (d_0 t_A - a_0 t_D)K_{AD} \]
\[ + (d_0 t_C - c_0 t_D + (1 - k^2)d_C)K_{CD} \]
\[ 0 = (c_A - k^2d_C t_A)K_{AC} + (c_C t_A + d_A - a_0)K_{AD} \]
\[ + (k^2d_C t_D + c_C t_D + (2k^2 - 1)d_C - a_0)K_{CD} \]
\[ 0 = -(c_A - k^2d_C t_A)(1 - 2k^2)d_C - 2k^2(c_0 - a_A t_C + c_A t_A)K_{AC} \]
\[ + ((1 - 2k^2)(c_C - a_A) - 2k^2(d_0 - a_A t_D + d_A t_A))K_{AD} \]
\[ + (2k^2(c_A - 2k^2d_C t_D - d_A t_C))K_{CD}. \]

Because of the independence of $K_{AC}$, $K_{AD}$, and $K_{CD}$, nine equations follow

\[ 0 = c_0 t_A - a_0 t_C \]
\[ 0 = d_0 t_A - a_0 t_D \]
\[ 0 = d_0 t_C - c_0 t_D + (1 - k^2)d_C \]
\[ 0 = c_A - k^2d_C t_A \]
\[ 0 = c_C t_A + d_A - a_0 \]
\[ 0 = k^2d_C t_D + c_C t_D + (2k^2 - 1)d_C - c_0 \]
\[ 0 = -k^2(1 - 2k^2)d_C - 2k^2(c_0 - a_A t_C + c_A t_A) \]
\[ 0 = (1 - 2k^2)(c_C - a_A) - 2k^2(d_0 - a_A t_D + d_A t_A) \]
\[ 0 = -(1 - 2k^2)(A + 2k^2(c_A t_D - d_A t_C). \]

These equations fix $a_0$, $c_0$, $d_0$, $c_A$, and $d_C$ in terms of the remaining parameters. The latter are constraint by another 4 equations. Their analysis is done in the next subsection.
B.2 Analysis

Lemma 1 \( c_A = 0 \).

**Proof** Multiply (152) to obtain
\[
(1 - k^2)a_0 d_C = a_0 c_0 t_D - a_0 d_0 t_c. \tag{159}
\]
Then (150, 151) imply that the r.h.s. vanishes. One concludes that \( a_0 d_C = 0 \).

This leaves two cases

1) \( d_C = 0 \); the lemma follows from (153).

2) \( d_C \neq 0 \) and \( a_0 = 0 \); then (152) implies that \( c_0 \) and \( d_0 \) cannot both vanish.

But this implies \( t_A = 0 \) via (150, 151). Again \( c_A = 0 \) follows from (153).

\[\square\]

Lemma 2 Either one of the following cases holds

- a) \( d_A = 0 \) and \( a_0 d_C = d_C t_A = 0 \);
- b) \( d_A \neq 0 \) and \( t_C = d_C = c_0 = 0 \).

**Proof** From (158), using \( c_A = 0 \), follows \( d_A t_C = 0 \). Hence either \( d_A = 0 \) or \( d_A \neq 0 \) and \( t_C = 0 \).

Use that \( a_0 d_C = d_C t_A = 0 \) from the proof of the previous lemma. Then (154), multiplied with \( d_C \), implies that \( d_A d_C = 0 \). Hence the assumption that \( d_A \neq 0 \) implies \( d_C = 0 \). But \( d_A \neq 0 \) implies also \( t_C = 0 \). Finally, (155) implies \( c_0 = 0 \).

Next assume \( d_A = 0 \). The results \( a_0 d_C = d_C t_A = 0 \) are taken from the proof of the previous lemma.

\[\square\]

The first possibility of the previous lemma is the only one that can occur.

This is proved in following lemma.

Lemma 3 \( d_A = 0 \).

**Proof** The proof goes ex absurdo. Assume that the second case of the previous lemma holds, i.e. \( d_A \neq 0 \) and \( c_A = t_C = d_C = c_0 = 0 \). Then the operator expressions (144, 145, 146) become

\[
\begin{align*}
- \omega A &= (c_C t_A - a_0) K_{AC} + (-c_C t_D + d_0) K_{CD} \tag{160} \\
2k^2 \omega C &= (c_C - a_A) K_{AD} \tag{161} \\
-2 \omega D &= (c_C - a_A) K_{AC} + d_A K_{CD}. \tag{162}
\end{align*}
\]
The solution of this set of equations is

\[
K_{AC} = \frac{1}{N} [-d_A \omega A + 2(d_0 - c_C t_D) \omega D]
\]  \hspace{1cm} (163)

\[
K_{AD} = \frac{2k^2}{c_C - a_A} \omega C
\]  \hspace{1cm} (164)

\[
K_{CD} = \frac{1}{N} [(c_C - a_A) \omega A - 2(c_C t_A - a_0) \omega D],
\]  \hspace{1cm} (165)

with

\[
N = d_A (c_C t_A - a_0) - (c_C - a_A)(d_0 - c_C t_D).
\]  \hspace{1cm} (166)

The Jacobi identity then implies

\[
0 = i[A, K_{CD}] - i[C, K_{AD}] + i[D, K_{AC}]
\]

\[
= \frac{1}{N} [-2(c_C t_A - a_0) + d_A] \omega K_{AD}.
\]  \hspace{1cm} (167)

Hence there follows \(d_A = 2(c_C t_A - a_0)\). Comparison with (154) then implies \(d_A = 0\), which contradicts the assumption that \(d_A \neq 0\).

\[\square\]

**Lemma 4** \(d_C = 0\).

**Proof** From Lemma 2 follows \(d_C t_A = 0\). This implies \(d_C = 0\) or \(t_A = 0\). Let us assume \(t_A = 0\). Then, using the results of the previous lemmas, (150 — 158) reduce to

\[
0 = d_0 t_C - c_0 t_D + (1 - k^2) d_C
\]  \hspace{1cm} (168)

\[
0 = a_0
\]  \hspace{1cm} (169)

\[
0 = k^2 d_C t_D + c_C t_C + (2k^2 - 1)d_C - c_0
\]  \hspace{1cm} (170)

\[
0 = (1 - 2k^2) d_C + 2(c_0 - a_A t_C)
\]  \hspace{1cm} (171)

\[
0 = (1 - 2k^2)(c_C - a_A) - 2k^2 (d_0 - a_A t_D).
\]  \hspace{1cm} (172)

Use the last two equations to eliminate \(c_0\) and \(d_0\) from the first and the third equation. This gives

\[
0 = (1 - 2k^2) [k^2 d_C t_D + (c_C - a_A) t_C] + 2k^2 (1 - k^2) d_C
\]  \hspace{1cm} (173)

\[
0 = k^2 d_C t_D + (c_C - a_A) t_C + \frac{1}{2} (2k^2 - 1) d_C.
\]  \hspace{1cm} (174)

Combining these two equations gives \(d_C = 0\).  \[\square\]
Lemma 5 \( c_0 = t_C = 0 \).

**Proof** From \((145)\), using \( c_A = d_C = 0 \), follows \( c_C \neq a_A \). But from \((155), (156)\) follows

\[
0 = (c_C - a_A)t_C. \tag{175}
\]

Hence, \( t_C = 0 \) follows. From \((155)\) or \((156)\) then follows \( c_0 = 0 \).

\( \square \)

Lemma 6 \( a_0 = t_A = 0 \).

**Proof** From \((151), (154)\) follows

\[
0 = (d_0 - cc t_D)t_A. \tag{176}
\]

Note that \( d_0 \neq cc t_D \) follows from \((144)\). One concludes therefore that \( t_A = 0 \). Finally, \( d_A = 0 \) and \((154)\) imply \( cc t_A - a_0 = 0 \). Hence \( t_A = 0 \) implies \( a_0 = 0 \).

\( \square \)

**B.3 Completing the proof**

Thus far, the desired form of the operators in \((23 — 30)\) has been proved. The actual relations between the coefficients follow by inserting these expressions into the equations \((123—130)\). In particular, the operator expressions \((144), (145), (146)\) become

\[
\begin{align*}
2k^2\omega C &= (c_C - a_A) K_{AD} \tag{177} \\
-2\omega D &= (c_C - a_A) K_{AC} \tag{178} \\
\omega A &= (c_C t_D - d_0) K_{CD}. \tag{179}
\end{align*}
\]

The solution of this set of equations is of the form \((23), (24), (25)\) with

\[
\begin{align*}
\delta &= -\frac{2\omega}{c_C - a_A} \tag{180} \\
\alpha &= \frac{\omega}{c_C t_D - d_0}. \tag{181}
\end{align*}
\]
References

[1] S.B. Leble, M.Czachor, *Darboux-integrable nonlinear Liouville-von Neumann equation*, Phys. Rev. E **58**, 7091-7100 (1998).

[2] M. Czachor, M. Kuna, S.B. Leble, J. Naudts, *Nonlinear von Neumann-type equations*, in: New Trends in quantum mechanics, H.-D. Doebner, S.T. Ali, M. Keyl, R.F. Werner (eds.) (World Scientific, Singapore, 2000), 209-226.

[3] M. Czachor, H.-D. Doebner, M. Syty, K. Wasylka, *von Neumann equations with time-dependent Hamiltonians and supersymmetric quantum mechanics*, Phys. Rev. E **61**, 3325-3329 (2000).

[4] M. Kuna, *Construction of exact solutions of Bloch-Maxwell equation based on Darboux transformation*, arXiv:quant-ph/0408048.

[5] S.L. McCall, E.L. Hahn, *Self-induced transparency by pulsed coherent light*, Phys. Rev. Lett. **18**, 908-912 (1967).

[6] L. Allen and J.H. Eberly, *Optical resonance and two-level atoms* (Dover publications, 1987)

[7] M. Kuna, M. Czachor, S. Leble, *Nonlinear von Neumann-type equations: Darboux invariance and spectra*, Phys. Lett. A **255**, 42-48 (1999).

[8] M. Syty, K. Wasylka, M. Czachor, *The beauty of Harzians*, in: Quantum Theory and Symmetries, eds. H.-D. Doebner et al., (World Scientific, Singapore, 2000), pp. 171-175.

[9] N.V. Ustinov, M. Czachor, M. Kuna, S. Leble, *Darboux integration of* $i\dot{\rho} = [H, f(\rho)]$, Phys. Lett. A **279**, 333-340 (2001).

[10] J. Ciesliński, M. Czachor, N.V. Ustinov, *Darboux covariant equations of von Neumann type and their generalizations*, J. Math. Phys. **44**, 1763-1780 (2003).