On the existence of connecting orbits for critical values of the energy

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Abstract

We consider an open connected set Ω and a smooth potential $U$ which is positive in Ω and vanishes on $\partial \Omega$. We study the existence of orbits of the mechanical system

$$\ddot{u} = U_x(u),$$

that connect different components of $\partial \Omega$ and lie on the zero level of the energy. We allow that $\partial \Omega$ contains a finite number of critical points of $U$. The case of symmetric potential is also considered.

1 Introduction

Let $U : \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^2$. We assume that $\Omega \subset \mathbb{R}^n$ is a connected component of the set $\{x \in \mathbb{R}^n : U(x) > 0\}$ and that $\partial \Omega$ is compact and is the union of $N \geq 1$ distinct nonempty connected components $\Gamma_1, \ldots, \Gamma_N$. We consider the following situations

$\mathbf{H}$ $N \geq 2$ and, if $\Omega$ is unbounded, there is $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0.$$  \hspace{1cm} (1.1)

$\mathbf{H}_s$ $\Omega$ is bounded, the origin $0 \in \mathbb{R}^n$ belongs to $\Omega$ and $U$ is invariant under the antipodal map

$$U(-x) = U(x), \quad x \in \Omega.$$  \hspace{1cm} (1.2)

Condition (1.1) was first introduced in [7]. A sufficient condition for (1.1) is that $\lim \inf_{|x| \to \infty} U(x) > 0$.

We study non constant solutions $u : (T_-, T_+) \to \Omega$, of the equation

$$\ddot{u} = U_x(u), \quad U_x = \left(\frac{\partial U}{\partial x}\right)^T,$$  \hspace{1cm} (1.2)

that satisfy

$$\lim_{t \to T_{\pm}} d(u(t), \partial \Omega) = 0,$$  \hspace{1cm} (1.3)

with $d$ the Euclidean distance, and lie on the energy surface

$$\frac{1}{2} |\dot{u}|^2 - U(u) = 0.$$  \hspace{1cm} (1.4)

We allow that the boundary $\partial \Omega$ of $\Omega$ contains a finite set $P$ of critical points of $U$ and assume

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If $\Gamma \in \{\Gamma_1, \ldots, \Gamma_N\}$ has positive diameter and $p \in P \cap \Gamma$ then $p$ is a hyperbolic critical point of $U$. If $\Gamma$ has positive diameter, then hyperbolic critical points $p \in \Gamma$ correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (1.2). These points are organizing centers of complex dynamics, see [6].

Note that $H$, organizing centers of complex dynamics, see [6].

$\textbf{H}_3$ If $\Gamma \in \{\Gamma_1, \ldots, \Gamma_N\}$ has positive diameter and $p \in P \cap \Gamma$ then $p$ is a hyperbolic critical point of $U$. If $\Gamma$ has positive diameter, then hyperbolic critical points $p \in \Gamma$ correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (1.2). These points are organizing centers of complex dynamics, see [6].

A comment on $H$ and $H_3$ is in order. If $P$ is nonempty $u \equiv p$ for $p \in P$ is a constant solution of (1.2) that satisfies (1.3) and (1.4). To avoid trivial solutions of this kind we require $N \geq 2$ in $H$, and look for solutions that connect different components of $\partial \Omega$. In $H_3$ we do not exclude that $\partial \Omega$ is connected ($N = 1$) and avoid trivial solutions by restricting to a symmetric context and to solutions that pass through 0.

We prove the following results.

**Theorem 1.1.** Assume that $H$ and $H_3$ hold. Then for each $\Gamma_1 \in \{\Gamma_1, \ldots, \Gamma_N\}$ there exist $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\} \setminus \{\Gamma_1\}$ and a map $u^* : (T_-, T_+) \to \Omega$, with $-\infty \leq T_- < T_+ \leq +\infty$, that satisfies (1.2), (1.4) and

$$\lim_{t \to T_\pm} d(u^*(t), \Gamma_\pm) = 0. \tag{1.5}$$

Moreover, $T_+ > -\infty$ (resp. $T_+ < +\infty$) if and only if $\Gamma_-$ (resp. $\Gamma_+$) has positive diameter. If $T_- > -\infty$ it results

$$\lim_{t \to T_-} u^*(t) = x_-, \tag{1.6}$$
$$\lim_{t \to T_-} u^*(t) = 0,$$

for some $x_- \in \Gamma_- \setminus P$. An analogous statement holds if $T_+ < +\infty$.

**Theorem 1.2.** Assume that $H_3$ and $H_3$ hold. Then there exist $\Gamma_+ \in \{\Gamma_1, \ldots, \Gamma_N\}$ and a map $u^* : (0, T_+) \to \Omega$, with $0 < T_+ \leq +\infty$, that satisfies (1.2), (1.4) and

$$\lim_{t \to T_+} d(u^*(t), \Gamma_+) = 0. \tag{1.7}$$

Moreover, $T_+ < +\infty$ if and only if $\Gamma_+$ has positive diameter. If $T_+ < +\infty$ it results

$$\lim_{t \to T_+} u^*(t) = x_+,$$
$$\lim_{t \to T_+} u^*(t) = 0,$$

for some $x_+ \in \Gamma_+ \setminus P$.

We list a few straightforward consequences of Theorems [1.1] and [1.2]

**Corollary 1.3.** Theorem [1.1] implies that, if $\partial \Omega = P$, given $p_- \in P$ there is $p_+ \in P \setminus \{p_-\}$ and a heteroclinic connection between $p_-$ and $p_+$, that is a solution $u^* : \mathbb{R} \to \mathbb{R}^n$ of (1.2) and (1.4) that satisfies

$$\lim_{t \to \pm \infty} u^*(t) = p_{\pm}.$$

The problem of the existence of heteroclinic connections between two isolated zeros $p_{\pm}$ of a non-negative potential has been recently reconsidered by several authors. In [1] existence was established under a mild monotonicity condition on $U$ near $p_{\pm}$. This condition was removed in [8], see also [2].

The most general results, equivalent to the consequence of Theorem [1.1] discussed in Section 2.1, were recently obtained in [7] and in [11], see also [3]. All these papers establish existence by a variational approach. In [1], [8] and [2] by minimizing the action functional, and in [7] and [11] by minimizing the Jacobi functional.
Corollary 1.4. Theorem 1.1 implies that, if \( \Gamma_- = \{ p \} \) for some \( p \in P \) and the elements of \( \{ \Gamma_1, \ldots, \Gamma_N \} \) do not have all positive diameter, there exists a nontrivial orbit homoclinic to \( p \) that satisfies (1.2), (1.3).

*Proof.* Let \( v^* : \mathbb{R} \to \Omega \cup \{ x_+ \} \) be the extension defined by

\[
v^*(T_+ + t) = u^*(T_+ - t), \quad t \in (0, +\infty), \quad v^*(T_+) = x_+,
\]

of the solution \( u^* : (-\infty, T_+) \to \Omega \) given by Theorem 1.1. The map \( v^* \) so defined is a smooth non-constant solution of (1.2) that satisfies

\[
\lim_{t \to \pm \infty} v^*(t) = p.
\]

\[ \square \]

Corollary 1.5. Theorem 1.1 implies that, if all the sets \( \Gamma_1, \ldots, \Gamma_N \) have positive diameter, given \( \Gamma_- \in \{ \Gamma_1, \ldots, \Gamma_N \} \), there exist \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \) and a periodic solution \( v^* : \mathbb{R} \to \Omega \) of (1.2) and (1.4) that oscillates between \( \Gamma_- \) and \( \Gamma_+ \). This solution has period \( T = 2(T_+ - T_-) \).

*Proof.* The solution \( v^* \) is the \( T \)-periodic extension of the map \( w^* : [T_- - 2T_+ - T_-, 2T_+ - T_-] \to \Omega \) defined by \( w^*(t) = u^*(t) \) for \( t \in (T_- - T_+, T_-) \), where \( u^* \) is given by Theorem 1.1 and

\[
\begin{align*}
w^*(0) &= x_+ , \\
w^*(T_+) &= x_+ , \\
w^*(T_+ + t) &= u^*(T_+ - t), \quad t \in (0, T_+ - T_-) .
\end{align*}
\]

\[ \square \]

The problem of existence of heteroclinic, homoclinic and periodic solutions of (1.2), in a context similar to the one considered here, was already discussed in [2] where \( \partial \Omega \) is allowed to include continua of critical points. Our result concerning periodic solutions extends a corresponding result in [2] where existence was established under the assumption that \( P = \emptyset \).

The following result is a direct consequence of Theorem 1.2.

Corollary 1.6. Theorem 1.2 implies that, if all the sets \( \Gamma_1, \ldots, \Gamma_N \) have positive diameter, there exists \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \) and a periodic solution \( v^* : \mathbb{R} \to \Omega \) of (1.2) and (1.4) that satisfies

\[
v^*(-t) = -v^*(t), \quad t \in \mathbb{R} .
\]

This solution has period \( T = 4T_+ \), with \( T_+ \).

*Proof.* The solution \( v^* \) is the \( T \)-periodic extension of the map \( w^* : [-2T_+, 2T_+] \to \Omega \) defined by \( w^*(t) = u^*(t) \) for \( t \in (0, T_+) \), where \( u^* \) is given by Theorem 1.2 and by

\[
\begin{align*}
w^*(t) &= -w^*(-t), \quad t \in (-T_+, 0) , \\
w^*(0) &= 0, \quad w^*(\pm T_+) = \pm x_+ , \\
w^*(T_+ + t) &= u^*(T_+ - t), \quad t \in (0, T_+), \\
w^*(-T_+ + t) &= w^*(-T_+ - t), \quad t \in [-T_+, 0) .
\end{align*}
\]

In particular the solution oscillates between \( x_+ \) and \( -x_+ \) and this is true also when \( \partial \Omega \) is connected (\( N = 1 \)).

\[ \square \]
2 Proof of Theorems 1.1 and 1.2

We recall a classical result.

**Lemma 2.1.** Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded and non-negative potential, $I = (a, b)$ a bounded interval. Define the Jacobi functional

$$J_G(q, I) = \sqrt{2} \int_I \sqrt{G(q(t))} |\dot{q}(t)| dt$$

and the action functional

$$A_G(q, I) = \int_I \left( \frac{1}{2} |\dot{q}(t)|^2 + G(q(t)) \right) dt.$$ 

Then

(i) $J_G(q, I) \leq A_G(q, I), \quad q \in W^{1,2}(I; \mathbb{R}^n)$

with equality sign if and only if

$$\frac{1}{2} |\dot{q}(t)|^2 - G(q(t)) = 0, \quad t \in I.$$

(ii) $\min_{q \in Q} J_G(q, I) = \min_{q \in Q} A_G(q, I),$

where

$$Q = \{ q \in W^{1,2}(I; \mathbb{R}^n) : q(a) = q(a), q(b) = q(b) \}.$$ 

When $G = U$ we shall simply write $J, A$ for $J_U, A_U$.

We now start the proof of Theorem 1.1. Choose $\Gamma_1 \in \{ \Gamma_1, \ldots, \Gamma_N \}$ and set

$$d = \min \{|x - y| : x \in \Gamma_1, y \in \partial \Omega \setminus \Gamma_1 \}.$$ 

For small $\delta \in (0, d)$ let $O_\delta = \{ x \in \Omega : d(x, \Gamma_1) < \delta \}$ and let $U_0 = \frac{1}{2} \min_{x \in \partial O_\delta = \partial \Omega} U(x)$. We note that $U_0 > 0$ and define the admissible set

$$U = \{ u \in W^{1,2}((T_w^u, T_w^{-u}); \mathbb{R}^n) : -\infty < T_w^u < T_w^{-u} < +\infty, \quad u((T_w^u, T_w^{-u})) \subset \Omega, \ u(u(0)) = U_0, \ u(T_w^u) \in \Gamma_1, \ u(T_w^{-u}) \in \partial \Omega \setminus \Gamma_1 \}. \quad (2.1)$$

We determine the map $u^*$ in Theorem 1.1 as the limit of a minimizing sequence $\{ u_j \} \subset U$ of the action functional

$$A(u, (T_w^u, T_w^{-u})) = \int_{T_w^u}^{T_w^{-u}} \left( \frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \right) dt,$$

Note that in the definition of $U$ the times $T_w^u$ and $T_w^{-u}$ are not fixed but, in general, change with $u$. Note also that the condition $U(u(0)) = U_0$ in (2.1) is a normalization which can always be imposed by a translation of time and has the scope of eliminating the loss of compactness due to translation invariance. Let $\bar{x}_- \in \Gamma_1$ and $\bar{x}_+ \in \partial \Omega \setminus \Gamma_1$ be such that $|\bar{x}_+ - \bar{x}_-| = d$ and set

$$\dot{u}(t) = (1 - (t + \tau))\bar{x}_- + (t + \tau)\bar{x}_+, \quad t \in [\tau, 1 - \tau],$$

where $\tau \in (0, 1)$ is chosen so that $U(\dot{u}(0)) = U_0$. Then $\dot{u} \in U$, $T_w^u = -\tau$, $T_w^{-u} = 1 - \tau$ and

$$A(\dot{u}, (-\tau, 1 - \tau)) = a < +\infty.$$
Next we show that there are constants \( M > 0 \) and \( T_0 > 0 \) such that each \( u \in \mathcal{U} \) with
\[
\mathcal{A}(u, (T_-^u, T_+^u)) \leq a,
\]
satisfies
\[
\|u\|_{L^\infty((T_-^u, T_+^u); \mathbb{R}^n)} \leq M, \\
T_-^u \leq -T_0 < T_0 \leq T_+^u.
\]
(2.3)
The \( L^\infty \) bound on \( u \) follows from \( \mathbf{H} \) and from Lemma 2.1 in fact, if \( \Omega \) is unbounded, \( |u(t)| = M \) for some \( t \in (T_-^u, T_+^u) \) implies
\[
a \geq \mathcal{A}(u, (T_-^u, t)) \geq \int_{T_-^u}^t \sqrt{2U(u(t))|\dot{u}(t)|} dt \geq \sqrt{2} \int_{T_-^u}^T \sigma(s)ds.
\]
The existence of \( T_0 \) follows from
\[
\frac{d_1^2}{|T_+^u|} \leq \int_{T_+^u}^0 |\dot{u}(t)|^2 dt \leq 2a, \quad \frac{d_2^2}{|T_-^u|} \leq \int_{T_-^u}^0 |\dot{u}(t)|^2 dt \leq 2a,
\]
where \( d_1 = d(\partial \Omega, \{ x : U(x) > U_0 \}) \).
Let \( \{ u_j \} \subset \mathcal{U} \) be a minimizing sequence
\[
\lim_{j \to +\infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) = \inf_{u \in \mathcal{U}} \mathcal{A}(u, (T_-^u, T_+^u)) := a_0 \leq a.
\]
(2.4)
We can assume that each \( u_j \) satisfies (2.2) and (2.3). By considering a subsequence, that we still denote by \( \{ u_j \} \), we can also assume that there exist \( T_-^\infty, T_+^\infty \) with \( -\infty \leq T_-^\infty \leq -T_0 < T_0 \leq T_+^\infty \leq +\infty \) and a continuous map \( u^* : (T_-^\infty, T_+^\infty) \to \mathbb{R}^n \) such that
\[
\lim_{j \to +\infty} T_-^{u_j} = T_-^\infty, \\
\lim_{j \to +\infty} T_+^{u_j} = T_+^\infty, \\
\lim_{j \to +\infty} u_j(t) = u^*(t), \ t \in (T_-^\infty, T_+^\infty),
\]
(2.5)
and in the last limit the convergence is uniform on bounded intervals. This follows from (2.3) which implies that the sequence \( \{ u_j \} \) is equi-bounded and from (2.2) which implies
\[
|u_j(t_1) - u_j(t_2)| \leq \int_{t_1}^{t_2} |\dot{u}_j(t)| dt \leq \sqrt{a}|t_1 - t_2|^{\frac{1}{2}},
\]
(2.6)
so that the sequence is also equi-continuous.
By passing to a further subsequence we can also assume that \( u_j \to u^* \) in \( W^{1,2}((T_1, T_2); \mathbb{R}^n) \) for each \( T_1, T_2 \) with \( T_-^\infty < T_1 < T_2 < T_+^\infty \). This follows from (2.2) which implies
\[
\frac{1}{2} \int_{T_-^{u_j}}^{T_+^{u_j}} |\dot{u}_j|^2 dt \leq \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) \leq a,
\]
and from the fact that each map \( u_j \) satisfies (2.3) and therefore is bounded in \( L^2((T_-^{u_j}, T_+^{u_j}); \mathbb{R}^n) \).
We also have
\[
\mathcal{A}(u^*, (T_-^\infty, T_+^\infty)) \leq a_0.
\]
(2.7)
Indeed, from the lower semicontinuity of the norm, for each \( T_1, T_2 \) with \( T_-^\infty < T_1 < T_2 < T_+^\infty \) we have
\[
\int_{T_1}^{T_2} |\dot{u}_j|^2 dt \leq \liminf_{j \to +\infty} \int_{T_1}^{T_2} |\dot{u}_j|^2 dt.
\]
This and the fact that \( u_j \) converges to \( u^* \) uniformly in \([T_1, T_2]\) imply

\[
\mathcal{A}(u^*, (T_1, T_2)) \leq \liminf_{j \to +\infty} \mathcal{A}(u_j, (T_1, T_2)) \leq \liminf_{j \to +\infty} \mathcal{A}(u_j, (T_j^{u^*}, T_j)) = a_0.
\]

Since this is valid for each \( T_\infty < T_1 < T_2 < T_\infty \) the claim (2.7) follows.

**Lemma 2.2.** Define \( T_\infty^* \leq T_- \leq -T_0 < T_0 \leq T_+ \leq T_\infty^* \) by setting

\[
T_- = \inf\{ t \in (T_\infty^*, 0) : u^*([t, 0]) \subset \Omega \}
\]

\[
T_+ = \sup\{ t \in (0, T_\infty^*) : u^*([0, t]) \subset \Omega \}.
\]

Then

(i) \( \mathcal{A}(u^*, (T_-, T_+)) = a_0 \).  

(ii) \( T_+ < +\infty \) implies \( \lim_{t \to T_+} u^*(t) = x_+ \) for some \( x_+ \in \Gamma_+ \) and \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \).

(iii) \( T_+ = +\infty \) implies

\[
\lim_{t \to +\infty} d(u^*(t), \Gamma_+) = 0,
\]

for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \setminus \{ \Gamma_- \} \).

Corresponding statements apply to \( T_- \).

**Proof.** We first prove (ii), (iii). If \( T_+ < +\infty \) the existence of \( \lim_{t \to T_+} u^*(t) \) follows from (2.6) which implies that \( u^* \) is a \( C^{0, \frac{1}{2}} \) map. The limit \( x_+ \) belongs to \( \partial \Omega \) and therefore to \( \Gamma_+ \) for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \). Indeed, \( x_+ \not\in \partial \Omega \) would imply the existence of \( \tau > 0 \) such that, for \( j \) large enough,

\[
d(u_j([T_+, T_+ + \tau]), \partial \Omega) \geq \frac{1}{2}d(x_+, \partial \Omega),
\]

in contradiction with the definition of \( T_+ \). If \( T_+ = +\infty \) and (iii) does not hold there is \( \delta > 0 \) and a diverging sequence \( \{t_j\} \) such that

\[
d(u^*(t_j), \partial \Omega) \geq \delta.
\]

Set \( U_m = \min_{d(x, \partial \Omega) = \delta} U(x) > 0 \). From the uniform continuity of \( U \) in \( \{|x| \leq M\} \) \( (M \text{ as in (2.3)}) \) it follows that there is \( l > 0 \) such that

\[
|U(x_1) - U(x_2)| \leq \frac{1}{2}U_m, \quad \text{for } |x_1 - x_2| \leq l, \ x_1, x_2 \in \{|x| \leq M\}.
\]

This and \( u^* \in C^{0, \frac{1}{2}} \) imply

\[
U(u^*(t)) \geq \frac{1}{2}U_m, \quad t \in I_j = \left(t_j - \frac{l^2}{a}, t_j + \frac{l^2}{a}\right),
\]

and, by passing to a subsequence, we can assume that the intervals \( I_j \) are disjoint. Therefore for each \( T > 0 \) we have

\[
\sum_{t_j \leq T} \frac{l^2 U_m}{a} \leq \int_0^T U(u^*(t))dt \leq a_0,
\]

which is impossible for \( T \) large. This establishes (2.9) for some \( \Gamma_+ \in \{ \Gamma_1, \ldots, \Gamma_N \} \). It remains to show that \( \Gamma_+ \neq \Gamma_- \). This is a consequence of the minimizing character of \( \{u_j\} \). Indeed, \( \Gamma_+ = \Gamma_- \) would imply the existence of a constant \( c > 0 \) such that \( \lim_{j \to +\infty} \mathcal{A}(u_j, (T_j^{u^*}, T_j)) \geq a_0 + c. \)
Now we prove (i). $T_+ - T_- < +\infty$, implies that $u^*$ is an element of $\mathcal{U}$ with $T^u_+ = T_+$. It follows that $\mathcal{A}(u^*, (T_-, T_+)) \geq a_0$, which together with (2.7) imply (2.8). Assume now $T_+ - T_- = +\infty$. If $T_+ = +\infty$, (2.9) implies that, given a small number $\epsilon > 0$, there are $t_0$ and $\bar{x}_0$ in $\partial \Omega$ such that $|u^*(t_0) - \bar{x}_0| = \epsilon$ and the segment joining $u^*(t_0)$ to $\bar{x}_0$ belongs to $\Omega$. Set
\[v(t) = (1 - (t - t_0))u^*(t_0) + (t - t_0)\bar{x}_0, \quad t \in (t_0, t_0 + 1].\]
From the uniform continuity of $U$ there is $\eta_0 > 0$, $\lim_{t \to 0} \eta_0 = 0$, such that $U(u(t)) \leq \eta_0$, for $t \in \left[t_0, t_0 + 1\right]$. Therefore we have
\[\mathcal{A}(v, (t_0, t_0 + 1)) \leq \frac{1}{2} \epsilon^2 + \eta_0.
\]
If $T_- > -\infty$ the map $u_0 = 1_{[T_-, t_0]}u^* + 1_{(t_0, t_0 + 1]}v$ belongs to $\mathcal{U}$ and it results
\[a_0 \leq \mathcal{A}(u_0, (T_-, t_0 + 1)) = \mathcal{A}(u^*, (T_-, t_0)) + \mathcal{A}(v, (t_0, t_0 + 1)) \leq \mathcal{A}(u^*, (T_-, T_+)) + \frac{1}{2} \epsilon^2 + \eta_0.
\]
Since this is valid for all small $\epsilon > 0$ we get
\[a_0 \leq \mathcal{A}(u^*, (T_-, T_+)),
\]
that together with (2.7) establishes (2.8) if $T_- > -\infty$ and $T_+ = +\infty$. The discussion of the other cases where $T_+ - T_- = +\infty$ is similar.

We observe that there are cases with $T_+ < T_+^{\infty}$ and/or $T_- > T_-^{\infty}$, see Remark 2.

**Lemma 2.3.** The map $u^*$ satisfies (1.2) and (1.4) in $(T_-^-, T_+)$.

**Proof.** 1. We first show that for each $T_1, T_2$ with $T_- < T_1 < T_2 < T_+$ we have
\[\mathcal{A}(u^*, (T_1, T_2)) = \inf_{v \in \mathcal{V}} \mathcal{A}(v, (T_1, T_2)),
\]
(2.10)
where
\[\mathcal{V} = \{v \in W^{1,2}((T_1, T_2); \mathbb{R}^n) : v(T_1) = u^*(T_1), i = 1, 2; v([T_1, T_2]) \subset \Omega\}.
\]
Suppose instead that there are $\eta > 0$ and $v \in \mathcal{V}$ such that
\[\mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta.
\]
Set $w_j : (T_-^{(u_j)}, T_+^{(u_j)}) \to \Omega$ defined by
\[w_j(t) = \begin{cases} u_j(t), & t \in (T_-^{(u_j)}, T_1) \cup [T_2, T_+^{(u_j)}), \\ v(t) + \frac{T_2 - t}{T_2 - T_1} \delta_j + \frac{t - T_1}{T_2 - T_1} \delta_{2j}, & t \in (T_1, T_2), \end{cases}
\]
where $\delta_{ij} = u_j(T_i) - u^*(T_i)$, $i = 1, 2$, with $u_j$ as in (2.4). Define $v_j : [T_-^{(v_j)}, T_+^{(v_j)}] \to \mathbb{R}^n$ by
\[v_j(t) = w_j(t - \tau_j),
\]
where $\tau_j$ is such that $U(v_j(0)) = U_0$, as in (2.1). Note that
\[\mathcal{A}(v_j, (T_-^{(v_j)}, T_+^{(v_j)})) = \mathcal{A}(w_j, (T_-^{(u_j)}, T_+^{(u_j)})).
\]
(2.11)
From (2.5) we have $\lim_{j \to +\infty} \delta_{ij} = 0$, $i = 1, 2$, so that
\[\lim_{j \to +\infty} \mathcal{A}(w_j, (T_1, T_2)) = \mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta \leq \liminf_{j \to +\infty} \mathcal{A}(u_j, (T_1, T_2)) - \eta.
\]
Therefore we have
\[
\liminf_{j \to +\infty} A(w_j, (T_+^u, T_+^u)) = \lim_{j \to +\infty} A(w_j, (T_1, T_2)) + \liminf_{j \to +\infty} A(u_j, (T_+^u, T_1)) + \liminf_{j \to +\infty} A(u_j, (T_2, T_+^u)) \\
\leq \liminf_{j \to +\infty} A(u_j, (T_1, T_2)) - \eta + \liminf_{j \to +\infty} A(u_j, (T_+^u, T_1)) + \liminf_{j \to +\infty} A(u_j, (T_2, T_+^u)) \leq a_0 - \eta,
\]
that, given (2.11), is in contradition with the minimizning character of the sequence \( \{u_j\} \).

The fact that \( u^* \) satisfies (1.2) follows from (2.10) and regularity theory, see [5]. To show that \( u^* \) satisfies (1.4) we distinguish the case \( T_+ - T_- = +\infty \) from the case \( T_+ - T_- = +\infty \).

2. \( T_+ - T_- = +\infty \). Given \( t_0, t_1 \) with \( T_- < t_0 < t_1 < T_+ \), let \( \phi : [t_0, t_1 + \tau] \to [t_0, t_1] \) be linear, with \( |\tau| \) small, and let \( \psi : [t_0, t_1] \to [t_0, t_1 + \tau] \) be the inverse of \( \phi \). Define \( u_\tau : [T_-, T_+ + \tau] \to \mathbb{R}^n \) by setting
\[
|\phi(t)|^2 + U(u^*(t)) dt - \int_{t_0}^{t_1} \frac{1}{2} |\dot{u}^*(t)|^2 + U(u^*(t)) dt
\]
\[
= \int_{t_0}^{t_1} \left( \frac{1 - \dot{\psi}(t)}{2\psi(t)} |\dot{u}^*(t)|^2 + (\dot{\psi}(t) - 1)U(u^*(t)) \right) dt
\]
\[
= \int_{t_0}^{t_1} \left( \frac{1}{2(1 + \frac{\tau}{t_1 - t_0})} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0}U(u^*(t)) \right) dt
\]
\[
= - \frac{\tau}{t_1 - t_0} \int_{t_0}^{t_1} \left( \frac{1}{2(1 + \frac{\tau}{t_1 - t_0})} - U(u^*(t)) \right) dt.
\]
This and (2.13) imply
\[
\int_{t_0}^{t_1} \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) dt = 0.
\]
Since this holds for all \( t_0, t_1 \), with \( T_- < t_0 < t_1 < T_+ \), then (1.4) follows.

3. \( T_+ - T_- = +\infty \). We only consider the case \( T_+ = +\infty \). The discussion of the other cases is similar.
Let \( T \in (T_-, +\infty) \), let \( T_- < t_0 < t_1 < T \) and let \( \phi : [t_0, T] \to [t_0, T] \) be linear in the intervals \( [t_0, t_1 + \tau], [t_1 + \tau, T] \), with \( |\tau| \) small, and such that \( \phi([t_0, t_1 + \tau]) = [t_0, t_1] \). Define \( u_\tau : (T_- + \infty) \to \mathbb{R}^n \) by setting
\[
u_\tau(t) = \begin{cases} u^*(t), & t \in (T_-, t_0] \cup [T, +\infty) \\ u^*(\phi(t)), & t \in [t_0, T]. \end{cases}
\]
We have
\[
\mathcal{A}(u_\tau, (T_-, T)) - \mathcal{A}(u^*, (T_-, T))
\[
= \int_{t_0}^{t_1} \left( \frac{\tau}{2(1 + \frac{\tau}{t_1 - t_0})} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0}U(u^*(t)) \right) dt + \int_{t_1}^{T} \left( \frac{\tau}{2(1 + \frac{\tau}{t_1 - t_0})} |\dot{u}^*(t)|^2 - \frac{\tau}{T - t_1}U(u^*(t)) \right) dt.
\]
Since \( u^* \) restricted to the interval \([t_0, T]\) is a minimizer of \(2.10\), by differentiating with respect to \(\tau\) and setting \(\tau = 0\) we obtain

\[
-\frac{1}{t_1} \int_{t_0}^{t_1} \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \, dt + \frac{1}{T - t_1} \int_{t_1}^{T} \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \, dt = 0.
\]

From \(2.7\) it follows that the second term in this expression converges to zero when \(T \to +\infty\). Therefore, after taking the limit for \(T \to +\infty\), we get back to \(2.14\) and, as before, we conclude that \(1.4\) holds.

**Lemma 2.4.** Assume that \(\lim_{t \to T_+} u^*(t) = p \in P\). Then

\[ T_+ = +\infty. \]

**Proof.** Since \(U\) is of class \(C^2\) and \(p\) is a critical point of \(U\) there are constants \(c > 0\) and \(\rho > 0\) such that

\[ U(x) \leq c|x - p|^2, \quad x \in B_\rho(p) \cap \Omega. \]

Fix \(t_0\) so that \(u^*(t) \in B_\rho(p) \cap \Omega\) for \(t \geq t_0\). Then \(T_+ = +\infty\) follows from \(1.4\) and

\[
\frac{d}{dt} |u^* - p| \geq -|\dot{u}^*| = -\sqrt{2U(u^*)} \geq -\sqrt{2c|u^* - p|}, \quad t \geq t_0.
\]

We now show that if \(\Gamma_+\) has positive diameter then \(T_+ < +\infty\). To prove this we first show that \(T_+ = +\infty\) implies \(u^*(t) \to p \in P\) as \(t \to +\infty\), then we conclude that this is in contrast with \(2.8\).

**Lemma 2.5.** If \(T_+ = +\infty\), then there is \(p \in P\) such that

\[
\lim_{t \to +\infty} u^*(t) = p. \tag{2.15}
\]

An analogous statement applies to \(T_-\).

**Proof.** If \(\Gamma_+ = \{p\}\) for some \(p \in P\), then \(2.15\) follows by \(2.9\). Therefore we assume that \(\Gamma_+\) has positive diameter. The idea of the proof is to show that if \(u^*(t)\) gets too close to \(\partial \Gamma_+ \setminus P\) it is forced to end up on \(\Gamma_+ \setminus P\) in a finite time in contradiction with \(T_+ = +\infty\).

If \(2.15\) does not hold there is \(q > 0\) and a sequence \(\{\tau_j\}\), with \(\lim_{j \to \infty} \tau_j = +\infty\), such that \(d(u^*(\tau_j), P) \geq q\), for all \(j \in \mathbb{N}\). Since, by \(2.3\) \(u^*\) is bounded, using also \(2.9\), we can assume that

\[
\lim_{j \to +\infty} u^*(\tau_j) = \bar{x}, \quad \text{for some } \bar{x} \in \Gamma_+ \setminus \cup_{p \in P} B_\rho(p). \tag{2.16}
\]

The smoothness of \(U\) implies that there are positive constants \(\bar{r}, r, c\) and \(C\) such that

(i) the orthogonal projection on \(\pi : B_{\bar{r}}(\bar{x}) \to \partial \Omega\) is well defined and \(\pi(B_{\bar{r}}(\bar{x})) \subset \partial \Omega \setminus P\);

(ii) we have

\[ B_{\bar{r}}(x_0) \subset B_{\bar{r}}(\bar{x}), \quad \text{for all } x_0 \in \partial \Omega \cap B_{\bar{r}}(\bar{x}); \]

(iii) if \((\xi, s) \in \mathbb{R}^{n-1} \times \mathbb{R}\) are local coordinates with respect to a basis \(\{e_1, \ldots, e_n\}\), \(e_j = e_j(x_0)\), with \(e_n(x_0)\) the unit interior normal to \(\partial \Omega\) at \(x_0 \in \partial \Omega \cap B_{\bar{r}}(\bar{x})\) it results

\[ \frac{1}{2}cs \leq U(x(x_0, (\xi, s))) \leq 2cs, \quad |\xi|^2 + s^2 \leq r^2, \quad s \geq h(x_0, \xi), \tag{2.17} \]

where

\[ x = x(x_0, (\xi, s)) = x_0 + \sum_{j=1}^n \xi_j e_j(x_0) + se_n(x_0), \]

and \(h : \partial \Omega \cap B_{\bar{r}}(\bar{x}) \times \{ |\xi| \leq r \} \to \mathbb{R}, \quad |h(x_0, \xi)| \leq C|\xi|^2\), for \( |\xi| \leq r\), is a local representation of \(\partial \Omega\) in a neighborhood of \(x_0\), that is \(U(x(x_0, (\xi, h(x_0, \xi)))) = 0\) for \( |\xi| \leq r\).
Fix a value $j_0$ of $j$ and set $t_0 = \tau_{j_0}$. If $j_0$ is sufficiently large, setting $t_0 = \tau_{j_0}$ we have that $x_0 = \pi(u^*(t_0))$ is well defined. Moreover $x_0 \in \partial \Omega \cap B \bar{r}_{k\delta}(\bar{x})$ and
\[ u^*(t_0) = x_0 + \delta e_n(x_0), \quad \delta = |u^*(t_0) - x_0|. \]

For $k = \frac{8}{3} \sqrt{2}$ let $Q_0$ be the set
\[ Q_0 = \{ x(x_0, (\xi, s)) : |\xi|^2 + (s - \delta)^2 < k^2 \delta^2, \ s > \delta/2 \}. \]

Since $\delta \to 0$ as $j_0 \to +\infty$ we can assume that $\delta > 0$ is so small ($\delta < \min \{ \frac{1}{2c\sqrt{2}}, \frac{r}{12} \}$ suffices) that $\overline{Q}_0 \subset \Omega \cap B_r(x_0)$.

**Claim 1.** $u^*(t)$ leaves $\overline{Q}_0$ through the disc $D_0 = \partial Q_0 \setminus \partial B_{k\delta}(u^*(t_0))$.

From (2.4) we have $a_0 \leq \mathcal{A}(v, (T_-, T^+_x))$ for each $W^{1,2}$ map $v : (T_-, T^+_x) \to \mathbb{R}^n$ that coincides with $u^*$ for $t \leq t_0$, and satisfies $v((t_0, T^+_x)) \subset \Omega$, $v(T^+_x) \in \partial \Omega$ and (1.4). Therefore if we set
\[ w(s) = x_0 + se_n(x_0), \]
$s \in [0, \delta]$, we have
\[ a_0 \leq \mathcal{A}(w^*, (T_-, t_0)) + \mathcal{J}(w, (0, \delta)). \quad (2.18) \]

On the other hand, if $u^*(t'_0) \in \partial Q_0(x_0) \cap \partial B_{k\delta}(u^*(t_0))$, where
\[ t'_0 = \sup \{ t > t_0 : u^*(t_0, t) \subset \overline{Q}_0 \setminus \partial B_{k\delta}(u^*(t_0)) \}, \]
from (2.7) it follows
\[ \mathcal{A}(u^*, (T_-, t_0)) + \mathcal{J}(u^*, (t_0, t'_0)) \leq a_0. \quad (2.19) \]

Using (2.17) we obtain
\[ \mathcal{J}(w, (0, \delta)) \leq \frac{4}{3} \zeta \delta^2, \quad (2.20) \]
and, since
\[ \zeta \delta \leq U(x(x_0, (\xi, s))), \ (\xi, s) \in \overline{Q}_0(x_0), \]

**Figure 1:** The coordinates $(\xi, s)$ and the domain $Q_0$ in Lemma 2.5.
we also have, with \( k \) defined above,
\[
\frac{8}{3} c^3 \delta^2 = \frac{k}{\sqrt{2}} c^2 \delta^2 \leq \frac{c^2 \delta^2}{\sqrt{2}} \int_{t_0}^{t_1} |\dot{u}^*(t)| dt \leq \sqrt{2} \int_{t_0}^{t_1} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt.
\] (2.21)

From (2.20) and (2.21) it follows
\[
J(u^*(0, \delta)) \leq \frac{1}{2} J(u^*, (t_0, t_0')),
\]
and therefore (2.18) and (2.19) imply the absurd inequality \( a_0 < a_0 \). This contradiction proves the claim.

From Claim 1 it follows that there is \( t_1 \in (t_0, +\infty) \) with the following properties:
\[
\begin{align*}
&u^*([t_0, t_1)) \subset Q_0(x_0), \\
u(t_1) \in D_0.
\end{align*}
\]
Set \( x_{0,1} = \pi(u^*(t_1)) \) and \( \delta_1 = |u^*(t_1) - x_{0,1}| \). Since \( h(x_0, 0) = h_x(x_0, 0) = 0 \) and the radius \( \rho_\delta = (k^2 - \frac{1}{4}) \delta \) of \( D_0 \) is proportional to \( \delta \), we can assume that \( \delta \) is so small that the ratio \( \frac{2t_1}{c^2} \) and \( \frac{|x_0 - x_{0,1}|}{|u^*(t_1) - x(x_0, 0)|} \) are near 1 so that we have
\[
\delta_1 \leq \rho \delta, \text{ for some } \rho < 1, \\
|x_{0,1} - x_0| \leq k \delta.
\]
We also have
\[
t_1 - t_0 \leq k' \delta^{1/2}, \quad k' = \frac{8k}{c^2}.
\]
This follows from
\[
(t_1 - t_0) \frac{c}{4} \delta \leq A(u^*, (t_0, t_1)) = J(u^*, (t_0, t_1)) = \sqrt{2} \int_{t_0}^{t_1} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt \leq 2 \sqrt{c^2 \delta} |u^*(t_1) - u^*(t_0)| \leq 2c^{1/2} k \delta^{1/2}.
\]
where we used (2.17) to estimate \( J \) on the segment joining \( u^*(t_0) \) with \( u^*(t_1) \).
We have \( u^*(t_1) = x_{0,1} + \delta_1 \epsilon_n(x_{0,1}) \) and we can apply Claim 1 to deduce that there exists \( t_2 > t_1 \) such that
\[
\begin{align*}
&u^*([t_1, t_2)) \subset Q_1(x_{0,1}), \\
u^*(t_2) \in D_1,
\end{align*}
\]
where \( Q_1 \) and \( D_1 \) are defined as \( Q_0 \) and \( D_0 \) with \( \delta_1 \) and \( x(x_{0,1}, (\xi, s)) \) instead of \( \delta \) and \( x(x_0, (\xi, s)) \). Therefore an induction argument yields sequences \( \{t_j\}, \{x_{0,j}\}, \{\delta_j\} \) and \( \{Q_j(x_{0,j})\} \) such that
\[
\begin{align*}
&u^*([t_j, t_{j+1})) \subset Q_j(x_{0,j}), \quad x_{0,j} = \pi(u^*(t_j)), \\
&\delta_{j+1} \leq \rho \delta_j \leq \rho^{j+1} \delta, \\
&|x_{0,j+1} - x_{0,j}| \leq k \delta_j \leq k \rho^j \delta, \\
&(t_{j+1} - t_j) \leq k' \delta_{j+1}^{1/2} \leq k' \rho^{j/2} \delta^{1/2}, \\
&u^*(t_j) = x_{0,j} + \delta_j \epsilon_n(x_{0,j}) \in D_j.
\end{align*}
\] (2.22)

We can also assume that \( Q_j(x_{0,j}) \subset \Omega \cap B_r(x_0) \), for all \( j \in \mathbb{N} \). This follows from \( |u^*(t_{j+1}) - u^*(t_j)| \leq k \delta_j \leq k \rho^j \delta \).
From (2.22) we obtain that there exists $T$ with $t_0 < T \leq \frac{k\delta^2}{1-\rho^2}$ such that
\[ u^*(T) = \lim_{t \to T} u^*(t) = \lim_{j \to +\infty} x_{0,j} \in \partial \Omega \setminus P, \]
\[ |u^*(T) - x_0| \leq \frac{k\delta}{1-\rho}. \]
This contradicts the existence of the sequence $\{\tau_j\}$, with $\lim_{j \to +\infty} \tau_j = +\infty$, appearing in (2.16) and establishes (2.15). The proof of the lemma is complete.

We continue by showing (2.15) contradicts (2.8).

**Lemma 2.6.** Assume that $\Gamma_+$ has positive diameter. Then
\[ T_+ < +\infty. \]
An analogous statement applies to $\Gamma_-$ and $T_-$. 

**Proof.** From Lemma 2.5 if $T_+ = +\infty$ there exists $p \in P$ such that $\lim_{t \to +\infty} u^*(t) = p$. We use a local argument to show that this is impossible if $\Gamma_+$ has positive diameter. By a suitable change of variable we can assume that $p = 0$ and that, in a neighborhood of $0 \in \mathbb{R}^n$, $U$ reads
\[ U(u) = V(u) + W(u), \]
where $V$ is the quadratic part of $U$:
\[ V(u) = \frac{1}{2} \left( -\sum_{i=1}^{m} \lambda_i^2 u_i^2 + \sum_{i=m+1}^{n} \lambda_i^2 u_i^2 \right), \quad \lambda_i > 0 \tag{2.23} \]
and $W$ satisfies,
\[ |W(u)| \leq C|u|^3, \quad |W_x(u)| \leq C|u|^2, \quad |W_{xx}(u)| \leq C|u|. \tag{2.24} \]
Consider the Hamiltonian system with
\[ H(p, q) = \frac{1}{2} |p|^2 - U(q), \quad p \in \mathbb{R}^n, \quad q \in \Omega \subset \mathbb{R}^n. \]
For this system the origin of $\mathbb{R}^{2n}$ is an equilibrium point that corresponds to the critical point $p = 0$ of $U$. Set $D = \text{diag}(-\lambda_1^2, \ldots, -\lambda_m^2, \lambda_{m+1}^2, \ldots, \lambda_n^2)$. The eigenvalues of the symplectic matrix
\[ \begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix} \]
are
\[ -\lambda_i, \quad i = m + 1, \ldots, n \]
\[ \lambda_i, \quad i = m + 1, \ldots, n \]
\[ \pm i\lambda_i, \quad i = 1, \ldots, m. \]
Let $\{e_1, 0\}, \ldots, \{e_n, 0\}, \{0, e_1\}, \ldots, \{0, e_n\}$ be the basis of $\mathbb{R}^{2n}$ defined by $e_j = (\delta_{j1}, \ldots, \delta_{jn})$, where $\delta_{ji}$ is Kronecker’s delta. The stable $S^s$, unstable $S^u$ and center $S^c$ subspaces invariant under the flow of the linearized Hamiltonian system at $0 \in \mathbb{R}^{2n}$ are
\[ S^s = \text{span}\{(-\lambda_j e_j, e_j)\}_{j=m+1}^{n}, \]
\[ S^u = \text{span}\{(\lambda_j e_j, e_j)\}_{j=m+1}^{n}, \]
\[ S^c = \text{span}\{(e_j, 0), (0, e_j)\}_{j=1}^{m}. \]
From (2.15) and (1.4) we have
\[ \lim_{t \to +\infty} (\dot{u}^*(t), u^*(t)) = 0 \in \mathbb{R}^{2n}. \]

Let \( W^s \) and \( W^u \) be the local stable and unstable manifold and let \( W^c \) be a local center manifold at \( 0 \in \mathbb{R}^{2n} \). From the center manifold theorem \([4], [10]\), there is a constant \( \lambda_0 > 0 \) such that, for each solution \( (p(t), q(t)) \) that remains in a neighborhood of \( 0 \in \mathbb{R}^{2n} \) for positive time, there is a solution \( (p'(t), q'(t)) \in W^c \) that satisfies
\[ |(p(t), q(t)) - (p'(t), q'(t))| = O(e^{-\lambda_0 t}). \] (2.25)

Since \( W^c \) is tangent to \( S^c \) at \( 0 \in \mathbb{R}^{2n} \), the projection \( W^c_0 \) on the configuration space is tangent to \( S^c_0 = \text{span}\{e_j\}_{j=1}^m \), which is the projection of \( S^c \) on the configuration space. Therefore, if \( (p', q') \neq 0 \), given \( \gamma > 0 \), by (2.25) there is \( t_\gamma \) such that \( d(q(t), S^c_0) \leq \gamma |q(t)| \), for \( t \geq t_\gamma \). For \( \gamma \) small, this implies that \( q(t) \notin \Omega \) for \( t \geq t_\gamma \). It follows that \( (p', q') \equiv 0 \) and from (2.25) \( (p(t), q(t)) \) converges to zero exponentially. This is possible only if \( (p(t), q(t)) \in W^s \) and, in turn, only if \( q(t) \in W^s_0 \), the projection of \( W^s \) on the configuration space. This argument leads to the conclusion that the trajectory of \( u^* \) in a neighborhood of \( 0 \) is of the form
\[ u^*(t(s)) = u^*(s) = s\eta + z(s), \] (2.26)
where
\[ \eta = \sum_{i=m+1}^n \eta_ie_i \]
is a unit vector \( s \in [0, s_0] \) for some \( s_0 > 0 \), and \( z(s) \) satisfies
\[ z(s) \cdot \eta = 0, \quad |z(s)| \leq c|s|^2, \quad |z'(s)| \leq c|s| \] (2.27)
for a positive constant \( c \).

We are now in the position of constructing our local perturbation of \( u \). We first discuss the case \( U = V, z(s) = 0 \). We set
\[ \bar{u}(s) = s\eta \]
and, in some interval \([1, s_1]\), construct a competing map \( \bar{v} : [1, s_1] \to \mathbb{R}^n \),
\[ \bar{v} = \bar{u} + ge_1, \quad g : [1, s_1] \to \mathbb{R}, \]
with the following properties:
\[ V(\bar{v}(1)) = 0, \]
\[ \bar{v}(s_1) = \bar{u}(s_1), \]
\[ J_V(\bar{v}, [1, s_1]) < J_V(\bar{u}, [0, s_1]). \] (2.28)

The basic observation is that, if we move from \( \bar{u} \) in the direction of one of the eigenvectors \( e_1, \ldots, e_m \) corresponding to negative eigenvalues of the Hessian of \( V \), the potential \( V \) decreases and therefore, for each \( s_0 \in (1, s_1) \) we can define the function \( g \) in the interval \([1, s_0]\) so that
\[ J_V(\bar{u} + ge_1, (1, s_0)) = J_V(\bar{u}, (1, s_0)). \] (2.29)

Indeed it suffices to impose that \( g : (1, s_0] \to \mathbb{R} \) satisfies the condition
\[ \sqrt{V(\bar{u}(s))} = \sqrt{1 + g^2(s)} \sqrt{V(\bar{u}(s) + g(s)e_1)}, \quad s \in (1, s_0]. \]

\(^1\) Actually \( \eta \) coincides with one of the eigenvectors of \( U''(0) \).
According with this condition we take $g$ as the solution of the problem

$$
\begin{align*}
g'(s) &= -\frac{\lambda_1 g}{\sqrt{s^2 \lambda_0^2 - \lambda_1^2 g^2}} = -\frac{\lambda_2 g}{\sqrt{1 - \frac{\lambda_2^2 g^2}{s^2 \lambda_0^2}}}, \\
g(1) &= \frac{\lambda_2}{\lambda_0}.
\end{align*}
$$

(2.30)

where we have used (2.23) and set

$$
\lambda_\eta = \sqrt{\sum_{i=m+1}^{n} \lambda_i^2 \eta_i^2}.
$$

Note that the initial condition in (2.30) implies $V(\bar{v}(1)) = 0$. The solution $g$ of (2.30) is well defined in spite of the fact that the right hand side tends to $-\infty$ as $s \to 1$. Since $g$ defined by (2.30) is positive for $s \in [1, +\infty)$, to satisfy the condition $\bar{v}(s_1) = \bar{u}(s_1)$, we give a suitable definition of $g$ in the interval $[s_0, s_1]$ in order that $g(s_1) = 0$. Choose a number $\alpha \in (0, 1)$ and extend $g$ with continuity to the interval $[s_0, s_1]$ by imposing that

$$
\sqrt{V(\bar{u}(s))} = \alpha \sqrt{1 + g'^2(s)} \sqrt{V(\bar{u}(s)) + g(s)e_1}, \quad s \in (s_0, s_1].
$$

(2.31)

Therefore, in the interval $(s_0, s_1]$, we define $g$ by

$$
g'(s) = -\frac{1}{\alpha} \sqrt{1 - \frac{\lambda_1^2 g^2}{s^2 \lambda_0^2}} \leq -\frac{\sqrt{1 - \alpha^2}}{\alpha}.
$$

(2.32)

Since (2.31) implies

$$
\mathcal{J}_V(\bar{v}, [s_0, s_1]) = \frac{1}{\alpha} \mathcal{J}_V(\bar{u}, [s_0, s_1]),
$$

from (2.29) we see that $\bar{v}$ satisfies also the requirement (2.28) above if we can choose $\alpha \in (0, 1)$ and $1 < s_0 < s_1$ in such a way that

$$
\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V(\bar{u}, (s_0, s_1)).
$$

Since (2.32) implies $s_1 < s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}}$ a sufficient condition for this is

$$
\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V(\bar{u}, (s_0, s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}})).
$$
or equivalently

$$1 > \frac{1 - \alpha}{\alpha} \left( \left( s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}} \right)^2 - s_0^2 \right) = 2s_0g(s_0)\sqrt{\frac{1 - \alpha}{1 + \alpha}} + \frac{\alpha g^2(s_0)}{1 + \alpha}. \quad (2.33)$$

By a proper choice of $s_0$ and $\alpha$ the right hand side of (2.33) can be made as small as we like. For instance we can fix $s_0$ so that $g(s_0) \leq \frac{1}{4}$ and then choose $\alpha$ in such a way that $\frac{1}{2}s_0^2\sqrt{\frac{1 - \alpha}{1 + \alpha}} \leq \frac{1}{4}$ and conclude that (2.28) holds.

Next we use the function $g$ to define a comparison map $v$ that coincides with $u^*$ outside an $\epsilon$-neighborhood of 0 and show that the assumption that the trajectory of $u^*$ ends up in some $p \in P$ must be rejected. For small $\epsilon > 0$ we define

$$v(\epsilon s) = \epsilon \eta + z(\epsilon s) + \epsilon g(s - \sigma)e_1, \quad s \in [1 + \sigma, s_1 + \sigma], \quad (2.34)$$

where $\sigma = \sigma(\epsilon)$ is determined by the condition

$$U(\epsilon(1 + \sigma))) = 0,$$

which, using (2.23), (2.24), (2.27) and $g(1) = \frac{\lambda}{\lambda_1}$, after dividing by $\epsilon^2$, becomes

$$\frac{1}{2}\alpha^2((1 + \sigma)^2 - 1) = \epsilon f(\sigma, \epsilon), \quad (2.35)$$

where $f(\sigma, \epsilon)$ is a smooth bounded function defined in a neighborhood of $(0,0)$. For small $\epsilon > 0$, there is a unique solution $\sigma(\epsilon) = O(\epsilon)$ of (2.35). Note also that (2.34) implies that

$$v(\epsilon(1 + \sigma))) = u^*(\epsilon(1 + \sigma)).$$

We now conclude by showing that, for $\epsilon > 0$ small, it results

$$\mathcal{J}_U(u^*(\epsilon), (0, s_1 + \sigma)) > \mathcal{J}_U(v(\epsilon), (1 + \sigma, s_1 + \sigma)). \quad (2.36)$$

From (2.26) and (2.34) we have

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| = 1, \quad \lim_{\epsilon \to 0^+} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| = \sqrt{1 + g^2(s)}, \quad (2.37)$$

and, using also (2.24) and $\sigma = O(\epsilon)$,

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} U(u^*(\epsilon s)) = V(\bar{u}(s)), \quad s \in (0, s_1),$$

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} U(v(\epsilon s)) = V(\bar{v}(s)), \quad s \in (1, s_1) \quad (2.38)$$

uniformly in compact intervals.

The limits (2.37) and (2.38) imply

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} \mathcal{J}_U(u^*(\epsilon), (0, s_1 + \sigma)) = \lim_{\epsilon \to 0^+} \sqrt{2} \int_0^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(u^*(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| ds,$$

$$= \sqrt{2} \int_0^{s_1} \sqrt{V(\bar{u}(s))} ds = \mathcal{J}_V(\bar{u}, (0, s_1))$$

$$\lim_{\epsilon \to 0^+} \epsilon^{-2} \mathcal{J}_U(v(\epsilon), (1 + \sigma, s_1 + \sigma)) = \lim_{\epsilon \to 0^+} \sqrt{2} \int_{1 + \sigma}^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(v(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| ds,$$

$$= \sqrt{2} \int_1^{s_1} \sqrt{V(\bar{v}(s))} \sqrt{1 + g^2(s)} ds = \mathcal{J}_V(\bar{v}, (1, s_1)).$$

This and (iii) above imply that, indeed, the inequality (2.36) holds for small $\epsilon > 0$. The proof is complete.
We can now complete the proof of Theorem 1.1. We show that the map \( u^* : (T_-, T_+) \to \mathbb{R}^n \) possesses all the required properties. The fact that \( u^* \) satisfies (1.2) and (1.4) follows from Lemma 2.3. Lemma 2.2 implies (1.5) and, if \( T_- > -\infty \), also (1.6). The fact that \( x_- \in \Gamma_- \setminus P \) is a consequence of Lemma 2.4 and implies that \( \Gamma_- \) has positive diameter. Vice versa, if \( \Gamma_- \) has positive diameter, Lemmas 2.5 and 2.6 imply that \( T_- > -\infty \) and that (1.6) holds for some \( x_- \in \Gamma_- \setminus P \). The proof of Theorem 1.1 is complete.

Remark. From Theorem 1.1 it follows that if \( N \) is even then there are at least \( N/2 \) distinct orbits connecting different elements of \( \{ \Gamma_1, \ldots, \Gamma_N \} \). If \( N \) is odd there are at least \( (N + 1)/2 \). Simple examples show that, given distinct \( \Gamma_i, \Gamma_j \in \{ \Gamma_1, \ldots, \Gamma_N \} \), an orbit connecting them does not always exist. Let

\[
\mathcal{U}_{ij} = \{ u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : u((T_-^u, T_+^u)), u(T_-^u), u(T_+^u) \in \Gamma_i, u(T_-^u), u(T_+^u) \in \Gamma_j \}
\]

with \( i \neq j \) and

\[
d_{ij} = \inf_{u \in \mathcal{U}_{ij}} \mathcal{A}(u, (T_-^u, T_+^u)).
\]

An orbit connecting \( \Gamma_i \) and \( \Gamma_j \) exists if

\[
d_{ij} < d_{ik} + d_{kj}, \quad \forall k \neq i, j.
\]

The proof of Theorem 1.2 uses, with obvious modifications, the same arguments as in the proof of Theorem 1.1 to characterize \( u^* \) as the limit of a minimizing sequence \( \{ u_j \} \) of the action functional

\[
\mathcal{A}(u, (0, T_-^u)) = \int_0^{T_-^u} \left( \frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \right) dt.
\]

in the set

\[
\mathcal{U} = \{ u \in W^{1,2}((0, T_-^u); \mathbb{R}^n) : 0 < T_-^u < +\infty, \ u(0) = 0, \ u([0, T_-^u]) \subset \Omega, \ u(T_-^u) \in \partial \Omega \}. \quad (2.40)
\]

Remark. In the symmetric case of Theorem 1.2 it is easy to construct an example with \( T_+ < T_+^\infty \). For \( U(x) = 1 - |x|^2 \), \( x \in \mathbb{R}^2 \), the solution \( u : [0, \pi/2] \to \mathbb{R}^2 \) of (1.2) determined by (1.4) and \( u([0, \pi/2]) = \{(s, 0) : s \in [0, 1] \} \) is a minimizer of \( \mathcal{A} \) in \( \mathcal{U} \). For \( \epsilon \) small, let \( t_\epsilon = \arcsin(1 - \epsilon) \) and define \( u_\epsilon : [0, T_-^u) \to \mathbb{R}^2 \) as the map determined by (1.4), \( u_\epsilon([0, t_\epsilon]) = \{(s, 0) : s \in [0, 1 - \epsilon] \} \) and \( u_\epsilon((t_\epsilon, T_-^u)) = \{(1 - \epsilon, s) : s \in (0, \sqrt{2\epsilon - \epsilon^2}] \}. \) In this case \( T_+ = \pi/2 \) and \( T_+^\infty = 3\pi/4 \).  

2.1 On the existence of heteroclinic connections

Corollary 1.3 states the existence of heteroclinic connections under the assumptions of Theorem 1.1 and, in particular, that \( U \in C^2 \). Actually, by examining the proof of Theorem 1.1 we can establish an existence result under weaker hypotheses. In the special case \( \partial \Omega = P, \# P \geq 2 \), given \( p_- \in P \), the set \( \mathcal{U} \) defined in (2.31) takes the form

\[
\mathcal{U} = \{ u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty, \ u((T_-^u, T_+^u)) \subset \Omega, \ u(0) = U_0, \ u(T_-^u) = p_-, \ u(T_+^u) \in P \setminus \{ p_- \} \}.
\]

In this section we slightly enlarge the set \( \mathcal{U} \) by allowing \( T_+^u = \pm \infty \) and consider the admissible set

\[
\mathcal{U} = \{ u \in W^{1,2}_{loc}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty \leq T_-^u < T_+^u \leq +\infty, \ u((T_-^u, T_+^u)) \subset \Omega, \ u(0) = U_0, \ \lim_{t \to T_-^u} u(t) = p_-, \ \lim_{t \to T_+^u} u(t) \in P \setminus \{ p_- \} \}.
\]
Proposition 2.7. Assume that $U$ is a non-negative continuous function, which vanishes in a finite set $P$, $\#P \geq 2$, and satisfies
\[ \sqrt{U(x)} \geq \sigma(|x|), \, x \in \Omega, \, |x| \geq r_0 \]
for some $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{\infty} \sigma(r)dr = +\infty$.

Given $p_-, p_+ \in P \setminus \{p_-, p_+\}$ and a Lipschitz-continuous map $u^* : (T_-, T_+) \to \Omega$ that satisfies (1.4) almost everywhere on $(T_-, T_+)$,
\[ \lim_{t \to T_{\pm}} u^*(t) = p_{\pm}, \]
and minimizes the action functional $A$ on $\tilde{U}$.

Proof. We begin by showing that
\[ a_0 = \inf_{u \in \tilde{U}} A = \inf_{u \in \tilde{U}} \tilde{A} = \tilde{a}_0. \quad (2.41) \]
Since $\mathcal{U} \subset \tilde{U}$ we have $a_0 \geq \tilde{a}_0$. On the other hand arguing as in the proof of Lemma 2.2 if $T_+ - T_- = +\infty$, given a small number $\epsilon > 0$, we can construct a map $u_\epsilon \in \mathcal{U}$ that satisfies
\[ a_0 \leq A(u_\epsilon, (T_{u^-}^\epsilon, T_{u^+}^\epsilon)) \leq A(u, (T_u^-, T_u^+)) + \eta_\epsilon \]
where $\eta_\epsilon \to 0$ as $\epsilon \to 0$. This implies $a_0 \leq \tilde{a}_0$ and establishes (2.41). It follows that we can proceed as in the proof of Theorem 1.1 and define $u^* \in \tilde{U}$ as the limit of a minimizing sequence $\{u_j\} \subset \tilde{U}$. The arguments in the proof of Lemma 2.2 show that (2.8) holds. It remain to show that $u^*$ is Lipschitz-continuous. Looking at the proof of Lemma 2.3 we see that the continuity of $U$ is sufficient for establishing that (1.4) holds almost everywhere on $(T_-, T_+)$, and the Lipschitz character of $u^*$ follows. The proof is complete. \hfill \Box

Remark. Without further information on the behavior of $U$ in a neighborhood of $p_{\pm}$ nothing can be said on $T_{\pm}$ being finite or infinite and it is easy to construct examples to show that all possible combinations are possible. As shown in Lemma 2.7 a sufficient condition for $T_{\pm} = \pm \infty$ is that, in a neighborhood of $p = p_{\pm}$, $U(x)$ is bounded by a function of the form $c|x-p|^2$, $c > 0$. $U$ of class $C^2$ is a sufficient condition in order that $u^*$ is of class $C^2$ and satisfies (1.2).

3 Examples

In this section we show a few simple applications of Theorems 1.1 and 1.2. Our first application describes a class of potentials with the property that, in spite of the existence of possibly infinitely many critical values, (1.2) has a nontrivial periodic orbit on any energy level.

Proposition 3.1. Assume that $U : \mathbb{R}^n \to \mathbb{R}$ satisfies
\[ U(-x) = U(x), \, x \in \mathbb{R}^n, \]
\[ U(0) = 0, \, U(x) < 0 \text{ for } x \neq 0, \]
\[ \lim_{|x| \to \infty} U(x) = -\infty. \]
Assume moreover that each non zero critical point of $U$ is hyperbolic with Morse index $i_m \geq 1$. Then there is a nontrivial periodic orbit of (1.2) on the energy level $\frac{1}{2} |\dot{u}|^2 - U(u) = \alpha$ for each $\alpha > 0$.

Proof. For each $\alpha > 0$ we set $\tilde{U} = U(x) + \alpha$ and let $\Omega \subset \{\tilde{U} > 0\}$ be the connected component that contains the origin. $\Omega$ is open, nonempty and bounded and, from the assumptions on the properties of the critical points of $U$, it follows that $\partial \Omega$ is connected and contains at most a finite number of critical points. Therefore we are under the assumptions of Corollary 1.6 for the case $N = 1$ and the existence of the periodic orbit follows. \hfill \Box
An example of potential $U : \mathbb{R}^2 \to \mathbb{R}$ that satisfies the assumptions in Proposition 3.1 is, in polar coordinates $r, \theta$,

$$U(r, \theta) = -r^2 + \frac{1}{2} \tanh^4(r) \cos^2(r^{-1}) \cos^2(2\theta),$$

where $k > 0$ is a sufficiently large number.

Next we give another application of Corollary 1.6. For the potential $U : \mathbb{R}^2 \to \mathbb{R}$, with

$$U(x) = \frac{1}{2}(1 - x_1^2)^2 + \frac{1}{2}(1 - 4x_2^2)^2,$$  \hskip 2cm \hskip 2cm (3.1)

the energy level $\alpha = -\frac{1}{2}$ is critical and corresponds to four hyperbolic critical points $p_1 = (1, 0)$, $-p_1$, $p_2 = (0, \frac{1}{2})$ and $-p_2$. The connected component $\Omega \subset \{U > 0\}$, $(\bar{U} = U(x) - \frac{1}{2})$ that contains the origin is bounded by a simple curve $\Gamma$ that contains $\pm p_1$ and $\pm p_2$. In spite of the presence of these critical points, from Theorem 1.2 it follows that there is a minimizer $u \in U$, with $\bar{U}$ as in (2.40) and $u(T^u) \in \Gamma \setminus \{\pm p_1, \pm p_2\}$, and Corollary 1.6 implies the existence of a periodic solution $v^*$.

Note that there are also two heteroclinic orbits, solutions of (1.2) and (1.4):

$$u_1(t) = (\tanh(t), 0), \quad u_2(t) = (0, \frac{1}{2} \tanh(2t)).$$

These orbits connect $p_j$ to $-p_j$, for $j = 1, 2$. By Theorem 1.2 both $u_1$ and $u_2$ have action greater than $v^* |_{(-\tau_u, \tau_u)}$.

Our last example shows that Theorems 1.1 and 1.2 can be used to derive information on the rich dynamics that (1.2) can exhibit when $U$ undergoes a small perturbation. We consider a family of potentials $U : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}$. We assume that $U(x, 0) = x_1^4 + x_2^2$ which from various points of view is a structurally unstable potential and, for $\lambda > 0$ small, we consider the perturbed potential

$$U(x, \lambda) = 2\lambda^4 x_1^2 + x_2^2 - 2\lambda^2 x_1 x_2 - 3\lambda^2 x_1^4 + x_6^6.$$  \hskip 2cm \hskip 2cm (3.2)

This potential satisfies $U(-x, \lambda) = U(x, \lambda)$ and, for $\lambda > 0$, has the five critical points $p_0, \pm p_1$ and $\pm p_2$ defined by

$$p_0 = (0, 0), \quad p_1 = (\lambda(1 - (\frac{2}{3})^\frac{1}{2}), \lambda^3(1 - (\frac{2}{3})^\frac{1}{2})^\frac{1}{2}), \quad p_2 = (\lambda(1 + (\frac{2}{3})^\frac{1}{2}), \lambda^3(1 + (\frac{2}{3})^\frac{1}{2})^\frac{1}{2}),$$

which are all hyperbolic.

We have $U(p_2, \lambda) < 0 = U(p_0, \lambda) < U(p_1, \lambda)$ and $p_0$ is a local minimum, $p_1$ a saddle and $p_2$ a global minimum. Let $\alpha$ be the energy level. For $-\alpha < U(p_2, \lambda)$ or $-\alpha \geq U(p_1, \lambda)$ no information can be
derived from Theorems 1.1 and 1.2 therefore we assume \(-\alpha \in [U(p_2,\lambda), U(p_1,\lambda))\). For \(-\alpha = U(p_2,\lambda)\) Corollary 1.3 or Corollary 1.6 yields the existence of a heteroclinic connection \(u_2\) between \(-p_2\) and \(p_2\). For \(-\alpha \in (U(p_2,\lambda),0)\) Corollary 1.6 implies the existence of a periodic orbit \(u_\alpha\). This periodic orbit converges uniformly in compact intervals to \(u_2\) and the period \(T_\alpha \to +\infty\) as \(-\alpha \to U(p_2,\lambda)^+\).

For \(\alpha = 0\) Corollary 1.4 implies the existence of two orbits \(u_0\) and \(-u_0\) homoclinic to \(p_0 = 0\). We can assume that \(u_0\) satisfies the condition \(u_0(-t) = u_0(t)\) and that \(u_0(0) = 0\). Then we have that \(u_\alpha(\cdot \pm \frac{T_\alpha}{2})\) converges uniformly in compact intervals to \(\mp u_0\) and \(T_\alpha \to +\infty\) as \(-\alpha \to 0^\pm\). For \(-\alpha \in (0,U(p_1,\lambda))\), \(\partial \Omega\) is the union of three simple curves all of positive diameter: \(\Gamma_0\) that includes the origin and \(\pm \Gamma_2\) which includes \(\pm p_2\) and Corollary 1.5 together with the fact that \(U(\cdot, \lambda)\) is symmetric imply the existence of two periodic solutions \(\tilde{u}_\alpha\) and \(-\tilde{u}_\alpha\) with \(\tilde{u}_\alpha\) that oscillates between \(\Gamma_0\) and \(\Gamma_2\) in each time interval equal to \(\frac{T_\alpha}{2}\). Assuming that \(\tilde{u}_\alpha(0) \in \Gamma_2\) we have that, as \(-\alpha \to 0^+, \tilde{u}_\alpha \to u_0\) uniformly in compacts and \(T_\alpha \to +\infty\). Finally we observe that, in the limit \(-\alpha \to U(p_1,\lambda)^-, \tilde{u}_\alpha\) converges uniformly in \(\mathbb{R}\) to the constant solution \(u \equiv p_1\).
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