FINITE TYPE INVARIANTS OF CLASSICAL AND VIRTUAL KNOTS

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Abstract. We observe that any knot invariant extends to virtual knots. The isotopy classification problem for virtual knots is reduced to an algebraic problem formulated in terms of an algebra of arrow diagrams. We introduce a new notion of finite type invariant and show that the restriction of any such invariant of degree $n$ to classical knots is an invariant of degree $\leq n$ in the classical sense. A universal invariant of degree $\leq n$ is defined via a Gauss diagram formula. This machinery is used to obtain explicit formulas for invariants of low degrees. The same technique is also used to prove that any finite type invariant of classical knots is given by a Gauss diagram formula. We introduce the notion of $n$-equivalence of Gauss diagrams and announce virtual counter-parts of results concerning classical $n$-equivalence.

1. Virtualization

Recently L. Kauffman introduced a notion of a virtual knot, extending the knot theory in an unexpected direction. We show here that this extension motivates a new approach to finite type invariants. This approach leads to new results both for virtual and classical knots.

1.1. Diagrams and Gauss Diagrams. Knots (smooth simple closed curves in $\mathbb{R}^3$) are usually presented by knot diagrams which are generic immersions of the circle into the plane enhanced by information on overpasses and underpasses at double points. A generic immersion of a circle into the plane is characterized by its Gauss diagram, which consists of the circle together with the preimages of each double point of the immersion connected by a chord. To incorporate the information on overpasses and underpasses, the chords are oriented from the upper branch to the lower one. Furthermore, each chord is equipped with the sign of the corresponding double point (local writhe number). See Figure 1. The result is called a Gauss diagram of the knot.

![Figure 1. A diagram of the figure eight knot and its corresponding Gauss diagram.](image-url)
A Gauss diagram is usually considered up to orientation preserving homeomorphism of the underlying circle. The Gauss diagram defines (up to isotopy of $S^2$) a knot diagram on the sphere, i.e., a knot diagram embedded into $S^2$ via the embedding $\mathbb{R}^2 \to S^2$. Given a knot diagram on the sphere, a knot diagram on the plane can be recovered modulo a finite ambiguity (this involves the choice of which connected component of the diagram contains the point at infinity), but the underlying knot itself is recovered uniquely up to isotopy.

Thus Gauss diagrams can be considered as an alternative way to present knots. Of course, they cannot compete with knot diagrams in creating a visual impression of knots, but Gauss diagrams are simpler from the combinatorial point of view and provide numerous advantages when we want to calculate knot invariants.

Unfortunately, not every picture which looks like a Gauss diagram is indeed a Gauss diagram of some knot. Moreover, this is not easy to recognize. There is an obvious algorithm [2] for checking this, which is just the result of attempting to draw the corresponding knot diagram. However this requires a considerable amount of effort.

1.2. Virtual Knots. The starting point for the present work is the idea that for some purposes it is easier just to ignore the problem of whether a Gauss diagram represents a knot, rather than trying to solve it. This gives rise to a generalization of classical knot theory by replacing true knots with objects which generalize Gauss diagrams of knots, but which are not necessarily associated to a knot. Of course, these objects are to be considered up to an appropriate equivalence, which imitates knot isotopy.

Although we had been led to this generalization by the internal logic of our previous research on combinatorial formulae for Vassiliev knot invariants, as soon as we formulated it, we recognized that we had rediscovered the theory of virtual knots, which was announced last year by Louis Kauffman in several talks [4]. Our main contribution to this newborn theory is to turn it into a useful tool for studying classical knots.

A virtual knot diagram is a generic immersion of the circle into the plane, with double points divided into real crossing points and virtual crossing points, with the real crossing points enhanced by information on overpasses and underpasses (as for classical knot diagrams). At a virtual crossing the branches are not divided into an overpass and an underpass. The Gauss diagram of a virtual knot is constructed in the same way as for a classical knot, but all virtual crossings are disregarded, see Figure 2.

![Figure 2. A diagram of a virtual knot with three real crossings and one virtual crossing and its corresponding Gauss diagram.](image-url)
Virtuality of virtual knots is manifest through the fact that while the diagram of a real knot is a picture describing a curve in $\mathbb{R}^3$, a virtual knot diagram apparently does not pertain to any familiar 3-dimensional geometric object. However we would like to keep speaking about virtual knots in the same way we that speak about real knots: a virtual knot is that thing presented by a virtual knot diagram. To resolve this ambiguity, we introduce moves on virtual knot diagrams similar to the moves of real knot diagrams which happen during an isotopy of a knot, and we will use the term virtual knot to denote an equivalence class of virtual knot diagrams under these moves. Two knot diagrams represent the same virtual knot, if one can be obtained from the other by a sequence of these moves. This agrees with the tradition of classical knot theory, where the term knot is often taken to refer to the isotopy class of a knot.

1.3. Reidemeister Moves and Virtual Moves. As is well-known, when a knot changes by a generic isotopy, its diagram undergoes a sequence of Reidemeister moves of one of the three types shown in Figure 3.

![Figure 3. Reidemeister moves.](image)

A diagram of a virtual knot can undergo the same Reidemeister moves, as well as the moves shown in Figure 4. These additional moves are called virtual moves. The first three of them are versions of the Reidemeister moves, but with virtual crossings in place of crossings. The last one looks like the third Reidemeister move, but involves two virtual crossings and one usual crossing.

![Figure 4. Virtual moves.](image)

Similar moves, but with two real crossings and one virtual crossing (shown in Figure 5) are forbidden. If one allows these moves, this makes the theory trivial: any virtual knot diagram can be unknotted by a sequence of moves shown in Figures 3, 4 and 5, see Section 5.3.

As mentioned above, a virtual knot is a class of virtual knot diagrams consisting of diagrams which can be transformed into each other by sequences of Reidemeister and virtual moves. A sequence of this kind is called a virtual isotopy.
Virtual moves do not affect Gauss diagrams. On the other hand, virtual moves allow one to move the interior of any arc which does not pass through a real crossing quite arbitrarily. Therefore we obtain the

1.A. Theorem. A Gauss diagram defines a virtual knot diagram up to virtual moves.

This means that a virtual knot (modulo Reidemeister and virtual moves) is equivalent to the corresponding Gauss diagram considered up to moves which are the counter-parts of Reidemeister moves for Gauss diagrams, see Figure 5. Since in a Gauss diagram all the orientations and the cyclic ordering of the endpoints of arrows are essential, each type of Reidemeister moves splits. In Figure 6 all moves corresponding to the first and second Reidemeister moves are shown in the top and middle rows, respectively. There are eight moves corresponding to the third Reidemeister move, but we only show two of them in the bottom row. As Östlund [6] showed, the remaining six moves are unnecessary. That is, any sequence of moves of a Gauss diagram can be replaced by a sequence of moves appearing in Figure 6. Although in [6] this is proved for Gauss diagrams of knots, the same proof works for virtual knots.

**Figure 5.** Forbidden moves.

**Figure 6.** Moves of Gauss diagrams corresponding to Reidemeister moves.
1.4. Kauffman’s Results on Extending Knot Invariants to Virtual Knots. Kauffman [4], [5] has proved that many knot invariants extend to invariants of virtual knots. In particular, the notions of knot group, quandle and rack, and the bracket polynomial, all extend in a straightforward manner. He also announced that with the use of a ”virtual framing” there are extensions of all quantum link invariants and the large collection of their corresponding Vassiliev invariants.

The extensions are done in a formal way, disregarding the original topological nature of these invariants. For example, the knot group, which is defined for classical knots as the fundamental group of the knot complement, is extended via a formal construction of a Wirtinger presentation. This construction can be written down in terms of a Gauss diagram as follows.

Let \( G \) be a Gauss diagram. If we cut the circle at each arrowhead (forgetting arrowtails), the circle of \( G \) is divided into a set of arcs. To each of these arcs there corresponds a generator of the group. Each arrow gives rise to a relation. Suppose the sign of an arrow is \( \varepsilon \), its tail lies on an arc labelled \( a \), its head is the final point of an arc labelled \( b \) and the initial point of an arc labelled \( c \). Then we assign to this arrow the relation \( c = a^{-\varepsilon}ba^{\varepsilon} \). The resulting group is called the group of the Gauss diagram. One can easily check that it is invariant under the Reidemeister moves shown in Figure 3. Moreover, the group system also extends. For the meridian, take the generator corresponding to any of the arcs. To write down the longitude, we go along the circle starting from this arc and write \( a^{\varepsilon} \), when passing the head of an arrow whose sign is \( \varepsilon \) and whose tail lies on the arc labelled \( a \).

The notion of quandle [3] is extended in the same way as the knot group: the generators remain the same, but each group relation \( c = a^{-\varepsilon}ba^{\varepsilon} \) is replaced with the corresponding quandle relation \( c = a \triangleright b \).

1.5. Knots Versus Virtual Knots. Any diagram of a classical knot can be considered to be a virtual knot diagram. A virtual isotopy can turn it into a diagram with virtual crossings, and then back again to a real knot diagram. Thus virtual isotopy is a new relation among classical knots, which apriori could differ from classical isotopy. However this is not the case.

1.B. Theorem (Virtual Isotopy Implies Isotopy). (See also Kauffman [4].) Virtually isotopic classical knots are isotopic.

Proof. The group system extends to virtual knots. Hence it is preserved under virtual isotopy, and virtually isotopic knots have isomorphic group systems. Now recall that the group system is a complete knot invariant: knots with isomorphic group systems are isotopic.

Any invariant of virtual knots is obviously an invariant of classical knots. On the other hand, by Theorem 1.B, any invariant of classical knots can be extended to an invariant of virtual knots. Nevertheless, for some invariants it is not easy to choose a natural extension. Even for the linking number, the extension to virtual 2-component links is not unique (see Section 1.7 below). A similar situation occurs for the degree 2 Vassiliev knot invariant considered in Section 3.2.

These examples are based on the same phenomenon. Unlike a classical knot, a virtual knot cannot be turned upside down. A rotation of a classical knot by the angle \( \pi \) around a horizontal line reverses all arrows of its Gauss diagram, while

\[ \text{Recall that the group system of a knot is the knot group together with the class of subgroups which are conjugate to the subgroup which is generated by a meridian and longitude.} \]
their signs do not change. An application of this operation to a Gauss diagram of a virtual knot gives rise to a virtual knot which may be non isotopic to the original one. The composition of this operation with an invariant of virtual knots may be another invariant.

A striking manifestation of this phenomenon comes from the knot group. Instead of the upper Wirtinger presentation of a knot group, which was generalized to virtual knots by Kauffman (see the preceding section), let us use the lower Wirtinger presentation, i.e. compose Kauffman’s construction with arrows reversal. We will call these groups the upper and the lower virtual knot groups, respectively. A virtual knot with different upper and lower groups is shown in Figure 7. The upper group of this knot is isomorphic to the group of the trefoil knot, while its lower group is $\mathbb{Z}$.

These examples may create an impression that virtual knot theory is more cumbersome than the classical knot theory. However, this is not the case. Due to its larger class of objects, the theory of virtual knots provides more flexibility. This leads to significant simplification, especially in the theory of finite type invariants.

1.6. Long Knots. By a (classical) long knot we mean a smooth embedding $\mathbb{R} \to \mathbb{R}^3$ which coincides with the standard embedding outside a compact set.

An isotopy of long knots is a smooth isotopy in the class of embeddings above. In the classical knot theory, long knots are introduced for purely technical reasons, since adding the point at infinity turns a long knot into a knot in the sphere $S^3$ and this construction establishes a one-to-one correspondence between the isotopy classes of long knots and the isotopy classes of knots.

Given a diagram of a long knot, the corresponding Gauss diagram is the line parameterizing the knot, together with arcs connecting the preimages of each crossing. As in the case of closed classical knots considered above, the arcs are oriented from the upper branch to the lower one and equipped with signs which are equal to the local writhe numbers of the corresponding crossing points. Each oriented signed arc is called an arrow.

A virtual long knot diagram is a generic immersions $\mathbb{R} \to \mathbb{R}^2$ with double points divided into real and virtual crossing points, where real crossing points are enhanced by information on overpasses and underpasses, as in a classical knot diagram. At a virtual crossing the branches are not divided into an overpass and an underpass. The Gauss diagram of a virtual long knot is constructed in the same way as for a classical long knot, but all the virtual crossings are disregarded. A virtual long knot is a class of diagrams which can be transformed into each other by sequences of Reidemeister and virtual moves (shown in Figures 3 and 4).

Surprisingly, virtual long knots differ from virtual knots. That is, there is no one-to-one correspondence between the virtual isotopy classes of virtual long knots and virtual knots. Addition of a point at infinity of the plane of the diagram turns
a diagram of a virtual long knot into a virtual knot diagram on $S^2$. Removal of a point from the complement of the diagram yields a virtual knot diagram on $\mathbb{R}^2$. One can easily prove that its virtual isotopy class does not depend on the choice of this point. Thus we have a natural map from the set of virtual long knots to virtual knots. This map is surjective but not injective. The simplest pair of virtual long knots which are not virtually isotopic, but give rise to isotopic virtual closed knots is shown in Figure 8. These virtual long knots are distinguished by the invariant $v_{2,2}$ defined in Section 3.2 below. The upper diagram can be transformed into the lower diagram by moving the underpassing arc of the leftmost crossing through the point at infinity. If these were classical knots, these transformation could be replaced by moving the same arc under the rest of the diagram by a sequence of Reidemeister moves. In our case this is impossible, since we cannot apply the move of Figure 5.

As a result, the theories for long and closed virtual knots are quite different, although their restrictions to usual knots coincide.

1.7. Links. For links, the basic notions of the virtual theory are introduced in a straightforward way. The only change is that the underlying circle of a Gauss diagram is replaced with several circles. Simple examples show that in many respects it is richer than the classical one and sometimes looks surprising. For instance, for 2-component links there are two independent versions of the linking number. The invariant $lk_{1/2}$ may be computed as a sum of signs of real crossings where the first component passes over the second one. Similarly, one can define $lk_{2/1}$ by exchanging the components in the definition of $lk_{1/2}$ above.

String links are related to links like long knots are to knots. A classical $n$-component string link is a smooth embedding of a disjoint union of $n$ copies of $\mathbb{R}$ into $\mathbb{R}^3$ which coincides with the standard embedding outside a compact set. Here, by the standard embedding, we mean the one given by the formula $t \mapsto (t, k, 0)$, with $t \in \mathbb{R}$ and $k = 1, \ldots, n$. All the basic notions of the virtual theory extend naturally to string links. A Gauss diagram in this case consists of $n$ parallel lines and signed arrows with end points on these lines.

2. Finite Type Invariants

2.1. Crossing Virtualization Versus Crossing Change. In the realm of virtual knots there is an elementary operation which does not exist for classical knots. A real crossing can be turned into a virtual one. In terms of Gauss diagrams it looks even simpler: we erase an arrow. This operation simplifies the knot in the sense that after applying it a sufficient number of times we eventually get to the unknot.
In the classical knot theory an operation with this property is widely used. This is the crossing change. However, it is more complicated in several ways. First, in order to turn a knot into the unknot, one must apply this move according to a certain pattern, say making the diagram descending or ascending, whereas virtualizing crossings leads to the unknot automatically. Second, the unknotting by crossing changes involves a choice: even if we have chosen to proceed towards a descending diagram, we still have to choose a point at which the descent begins. The result considered as a diagram depends on this choice. Unknotting by virtualization eliminates these technically unpleasant problems. Third, crossing changes do not diminish the number of crossings, while each virtualization diminishes the number of real crossings. Finally, virtualization is more elementary than crossing changing, since a crossing change can be presented as the composition of one crossing virtualization and the inverse of another.

A more general operation defined on Gauss diagrams of virtual knots is passage to subdiagrams. Here \( D' \) is a subdiagram of \( D \) if all the arrows of \( D' \) belong to \( D \). In this case we write \( D' \subset D \).

### 2.2. Classical Finite Type Invariants

The standard theory of finite type invariants is based on crossing change as the basic modification.

Recall that a function \( \nu \) defined on the set of knot isotopy types and taking values in an abelian group \( G \) is said to be a finite type invariant of degree \( \leq n \), if for any knot diagram \( D \) and \( n + 1 \) crossing points \( d_1, d_2, \ldots, d_{n+1} \) of \( D \)

\[
\sum_{\sigma} (-1)^{|\sigma|} \nu(D_\sigma) = 0.
\]

(1)

Here \( \sigma = \{\sigma_1, \ldots, \sigma_{n+1}\} \) runs over \( (n+1) \)-tuples of zeros and ones, \( |\sigma| \) is the number of ones in \( \sigma \), and \( D_\sigma \) is the diagram obtained from \( D \) by switching all crossings \( d_i \) with \( \sigma_i = 1 \).

This description can be simplified by extending a knot invariant to knots with double points (called also singular knots). A double point appears when at a crossing one moves the upper branch downwards through the lower branch. The knot with a double point is identified with the formal difference between the two knots obtained by resolving the double point in two ways. This can be formulated as the following formal relation:

\[
\begin{array}{c}
\includegraphics[width=1cm]{double_point}
\end{array}
\]

(2)

Double points are depicted with thick points, so as to distinguish them from virtual crossings.

Any knot invariant extends to formal linear combinations of knot diagrams by linearity. Under the identification in (2), the alternating sum in the left hand side of equality (1) becomes the value of \( \nu \) on a knot with \( n + 1 \) double points. Thus a knot invariant has degree at most \( n \) if its extension vanishes on every singular knot having at least \( n + 1 \) double points.

### 2.3. A New Notion of Finite Type Invariant

The counter-part in the virtual theory of the notion of finite type invariant can be described as follows. We introduce a new kind of crossing, which is called semi-virtual. At a semi-virtual crossing there are still over- and under-passes. In a diagram a semi-virtual crossing is shown
as a real one, but surrounded by a small circle. Semi-virtual crossings are related to the other types of crossings by the following formal relation:

\[(3)\] 

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
= \end{array}
\end{array}
\end{array}
\end{array}
\]

In a Gauss diagram a semi-virtual crossing is presented by a dashed arrow. The relation (3) becomes

\[(4)\] 

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
= \end{array}
\end{array}
\end{array}
\end{array}
\]

Let \(D\) be a virtual knot diagram and \(\{d_1, \ldots, d_n\}\) be an \(n\)-tuple of its real crossings points. For an \(n\)-tuple \(\sigma = \{\sigma_1, \ldots, \sigma_n\}\) of zeros and ones, define \(D_\sigma\) to be the diagram, obtained from \(D\), by switching all the crossings \(d_i\), with \(\sigma_i = 1\), to virtual crossings. Denote by \(|\sigma|\) the number of ones in \(\sigma\). The formal alternating sum

\[
\sum_\sigma (-1)^{|\sigma|} D_\sigma
\]

is called a diagram with \(n\) semi-virtual crossings. We depict the corresponding alternating sum of Gauss diagrams by the Gauss diagram of \(D\) with all the arrows associated to \(\{d_1, \ldots, d_n\}\) being dashed. This agrees with the convention (4) on semi-virtual crossings.

Denote by \(\mathcal{K}\) the set of virtual knots. Let \(\nu: \mathcal{K} \to G\) be an invariant of virtual knots with values in an abelian group \(G\). Extend it to \(\mathbb{Z}[\mathcal{K}]\) by linearity. We say that \(\nu\) is an invariant of finite type, if for some \(n \in \mathbb{N}\) it vanishes for any virtual knot \(K\) with more than \(n\) semi-virtual crossings. The minimal such \(n\) is called the degree of \(\nu\).

Note that (3) and (2) imply

\[(5)\] 

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
= \end{array}
\end{array}
\end{array}
\end{array}
\]

It follows that for any finite type invariant of the virtual theory, its restriction to classical knots is a finite type invariant (of at most the same degree) in the classical sense.

The definition of finite type invariants extends to virtual links in a natural way. A particularly simple example is given by the invariants \(\text{lk}_{1/2}\) and \(\text{lk}_{2/1}\) considered in Section 1.7. These invariants of 2-component virtual links have degree one.

### 2.4. The Algebra of Arrow Diagrams

An arrow diagram (on a circle) is an abstract diagram, which consists of an oriented circle with pairs of distinct points connected by dashed arrows. Each arrow is equipped with a sign. The algebra of arrow diagrams \(\mathcal{A}\) is the free abelian group generated by all arrow diagrams. We call \(\mathcal{A}\) an algebra, because there is indeed a natural multiplication in \(\mathcal{A}\) making it into an associative algebra.\(^2\) The algebra \(\mathcal{A} \otimes \mathbb{Q}\) is isomorphic to the one introduced in \(\[\mathcal{A}\]\). However in this paper we will not make use of the multiplicative structure in \(\mathcal{A}\).

\(^2\) The product of arrow diagrams \(A_1, A_2\) is the sum (with appropriate multiplicities, cf. \(\[\mathcal{A}\]\)) of all diagrams each of which is the union of subdiagrams isomorphic to \(A_1\) and \(A_2\).
Denote the set of all Gauss diagrams by $D$. Starting from any Gauss diagram we get an arrow diagram just by making all its arrows dashed. The extension of this map to $\mathbb{Z}[D]$ defines a natural isomorphism $i : \mathbb{Z}[D] \to A$.

There is another important map $I : D \to A$, assigning to a Gauss diagram $D$ the sum of all its subdiagrams and then making each of them dashed:

$$I(D) = \sum_{D' \subset D} i(D').$$

Thus the map $I$ can be described by the following symbolic formula:

$$I : \left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right) \mapsto \left( \begin{array}{c}
-\varepsilon \\
\varepsilon
\end{array} \right) + \left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right).$$

(6)

The reason for using the same dashed arrows both for semi-virtual crossings in $\mathbb{Z}[D]$ and for arrows in $A$ becomes clear if one compares formulas (6) and (4).

Extend $I$ to $\mathbb{Z}[D]$ by linearity.

2.A. Proposition. $I : \mathbb{Z}[D] \to A$ is an isomorphism. The inverse map $I^{-1} : A \to \mathbb{Z}[D]$ is defined on the generators of $A$ by

$$I^{-1}(A) = \sum_{A' \subset A} (-1)^{|A-\begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array}|} i^{-1}(A'),$$

where $|A - A'|$ is the number of arrows of $A$ which do not belong to $A'$.

A Gauss diagram is called semi-virtual if each of its arrows is dashed.

2.B. Corollary. Semi-virtual diagrams form a basis of $\mathbb{Z}[D]$.

2.C. Remark. We can now explain an additional reason for the dual use of dashed arrows: $I$ maps each semi-virtual Gauss diagram to the arrow diagram with the same arrows. This observation extends to diagrams containing both solid and dashed arrows. Consider such a diagram $D$ as an element of $\mathbb{Z}[D]$. Then each diagram appearing in $I(D) \in A$ contains all the dashed arrows of $D$.

Thus we see that $I$ can be interpreted as a presentation of a Gauss diagram by a linear combination of semi-virtual diagrams.

2.5. The Polyak Algebra. The Polyak algebra is the quotient of $A$ by the following relations:

$$\left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right) = 0$$

(7)

$$\left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right) + \left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right) + \left( \begin{array}{c}
\varepsilon \\
-\varepsilon
\end{array} \right) = 0$$

(8)
Here we follow the common convention that the unshown parts of all diagrams involved in each of the relations coincide. The embeddings of the shown parts into the whole diagrams should preserve the orientations in (9).

The same relations define analogous algebras for long knots, links and string links.

The quotient of $A$ by the relations (7) - (9) is an algebra, since the relations generate an ideal of $A$, but we shall not go into further detail on this point. Denote this algebra by $P$. This algebra is closely related to the algebra $A$ introduced in [8].

The isomorphism $I$ induces an isomorphism $I : \mathbb{Z}[K] \rightarrow P$ of quotient algebras. Indeed, the equivalence relation induced in $D$ by the Reidemeister moves shown in Figure 3 can be rewritten in terms of diagrams with semi-virtual crossings as follows

(10) \[ \epsilon = 0 \]

(11) \[ \epsilon + \epsilon + \epsilon = 0 \]

(12) \[ \epsilon + \epsilon + \epsilon + \epsilon = \]

Note that the map $I$ turns (10) – (12) into (7) – (9). Thus for any Gauss diagram $D$ of a virtual knot $K$, $I(D)$ defines a $P$-valued invariant of $K$. Moreover, since the Gauss diagram determines $K$, the invariant $I(D)$ distinguishes virtual knots. Thus we obtain the following theorem.

2.D. **Theorem.** Let $D$ be any diagram of a virtual knot $K$. The formula $K \mapsto I(D) \in P$ defines a complete invariant of virtual knots.

2.6. **The Truncated Algebras $P_n$ and the Universal Finite Type Invariant.**
Define the truncated algebra $P_n$ by putting $A = 0$ for any diagram $A \in P$ with more than $n$ arrows. Denote by $I_n : K \rightarrow P_n$ the composition of $I : K \rightarrow P$ with the projection $P \rightarrow P_n$. 
Let $P$ be an abelian group and $p : K \to P$ be a $P$-valued invariant of virtual knots. We call $p$ a universal invariant of degree $n$, if for every abelian group $G$ and every invariant $\nu : K \to G$ of degree at most $n$ factors through $p$, i.e. there exists a map $\pi : P \to G$, such that $\nu = \pi \circ p$.

2.E. Theorem. The map $I_n : K \to P_n$ defines a universal invariant of degree $n$.

Proof. Since, by Remark 2.C, $I$ preserves all dashed arrows, $I_n$ maps a knot with more than $n$ semi-virtual crossings to zero. Therefore $I_n$ is an invariant of degree at most $n$.

Let $\nu : K \to G$ be an invariant of degree at most $n$. We have to prove that $\nu \circ I^{-1} : P \to G$ factors through $P_n$. Observe that, by Remark 2.C, for any arrow diagram $A$ its image $I^{-1}(A) \in \mathbb{Z}[D]$ can be identified with the same diagram $A$, but considered as a semi-virtual Gauss diagram. Since the degree of $\nu$ is at most $n$, $\nu$ vanishes on each diagram with more than $n$ semi-virtual crossings. Therefore $\nu \circ I^{-1}$ vanishes on each arrow diagram with more than $n$ arrows and $\nu \circ I^{-1}$ factors through $P_n$. □

2.F. Corollary. The space of $\mathbb{Q}$-valued invariants of degree at most $n$ is finite-dimensional, of dimension equal to $\text{rk}(P_n)$. It can be identified with the dual space $P_n^*$ of $\mathbb{Q}$-valued linear functions on $P_n$.

3. Gauss Diagram Formulas for Finite Type Invariants

3.1. Gauss Diagram Formulas. Since the algebra $\mathcal{A}$ has a distinguished basis, consisting of arrow diagrams, there is a natural orthonormal scalar product $(\cdot, \cdot)$ on $\mathcal{A}$. Namely, on the generators of $\mathcal{A}$ we put $(D_1, D_2)$ to be 1, if $D_1 = D_2$, and 0 otherwise and then extend $(\cdot, \cdot)$ bilinearly. This allows us to define the pairing $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{D} \to \mathbb{Z}$ in the following way. For any $D \in \mathcal{D}$ and $A \in \mathcal{A}$ put

\begin{equation}
\langle A, D \rangle = \langle A, I(D) \rangle = \langle A, \sum_{D' \subset D} i(D') \rangle.
\end{equation}

(13)

Informally speaking, we count subdiagrams of $D$ with weights, where the weight of a diagram $D'$ is the coefficient of $i(D')$ in $A$.

In the case of Gauss diagrams corresponding to usual knots, this pairing (in a slightly different form) was introduced in [7] as a tool for writing down Gauss diagram formulas for knot invariants. We will use it below for the same purpose in the framework of virtual knots.

Using equation (13) and Theorem 2.D it is easy to see that any $\mathbb{Z}$-valued invariant of finite type of virtual knots can be obtained by a Gauss diagram formula $\langle A, \cdot \rangle : K \to \mathbb{Z}$ for some $A \in \mathcal{A}$. The maximal number of arrows of the diagrams in the linear combination giving $A$ is an upper bound for the degree. However, in general the expression $\langle A, D \rangle$ depends on the choice of the Gauss diagram $D$ of a virtual knot. In our earlier work [8] we did not present any systematic method for producing arrow polynomials $A$ which give invariants, and we posed the following question: “Which arrow polynomials define knot invariants...?" We can now answer this question in the framework of the virtual theory: $A$ defines an invariant of degree at most $n$ if and only if all diagrams in $A$ have at most $n$ arrows and $A$ satisfies the equations $\langle A, R \rangle = 0$, where $R$ runs over the left hand sides of the relations $R = 0$ defining $P_n$ in $\mathcal{A}$, see Section 2.E.
The general method for producing all invariants of degree $n$ requires a computation of the algebra $\mathcal{P}_n$. Some simple observations allow one to reduce this computation. First, by a repeated use of (11), one can eliminate arrows with the negative sign. Second, since all the diagrams with more than $n$ arrows vanish, for diagrams with $n$ arrows equations (11) and (12) become simpler and contain only 2 and 6 terms respectively, all of them with exactly $n$ arrows. The simplified version of (11) implies the following rule for elimination of negative arrows in diagrams with exactly $n$ arrows: if two such diagrams differ only by the signs of $k$ arrows, they differ in $\mathcal{P}_n$ by multiplication by $(-1)^k$. This allows one to drop the signs of arrows in diagrams with $n$ arrows, using the convention that a diagram with $k$ negative signs of arrows is counted with coefficient $(-1)^k$. The simplified version of (12) involves only diagrams with exactly $n$ arrows and looks as follows:

\begin{equation}
\begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array}
+ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array}
+ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array}
= \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram4.png}
\end{array}
\end{array}
\end{equation}

This 6T-equation (introduced earlier in [8]) is an oriented version of the well-known 4T-relation for chord diagrams. The 4T-relation can be recovered from (14) by repeated use of the following formula which follows from (15):

\begin{equation}
\left\{ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram5.png}
\end{array} \right\} = \left\{ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram6.png}
\end{array} \right\} + \left\{ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram7.png}
\end{array} \right\}.
\end{equation}

3.2. Invariants of Small Degree. Computation of $\mathcal{P}_n$ for small $n$ can be done by hand and leads to some interesting results. Similarly to the case of classical knots, there are no invariants of degree one. More surprisingly, the algebra $\mathcal{P}_2$ is also trivial, so there are no invariants of degree two! However, for long knots the corresponding algebra is 2-dimensional, so there are two independent invariants $v_{2,1}$ and $v_{2,2}$ of degree 2. These invariants are given by

\begin{equation}
v_{2,1}(\cdot) = \left\langle \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diag21.png}
\end{array} , \cdot \right\rangle; \quad v_{2,2}(\cdot) = \left\langle \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diag22.png}
\end{array} , \cdot \right\rangle
\end{equation}

This illustrates a curious feature of the theory of virtual knots which was discussed in Section 1.6. For classical knots, there is one-to-one correspondence between the isotopy classes of knots and long knots, hence any invariant of long knots is an invariant of closed knots. We now see that for virtual knots this is no longer true.

Another interesting feature of this theory is that many invariants, which coincide for usual knots (due to the existence of certain symmetries), are different on the larger class of virtual knots. The invariants $v_{2,1}$ and $v_{2,2}$ provide a good illustration.

In degree three there is only one invariant, given by

\begin{equation}
\left\langle \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diag31.png} + \includegraphics[width=0.2\textwidth]{diag32.png} + \includegraphics[width=0.2\textwidth]{diag33.png} - \includegraphics[width=0.2\textwidth]{diag34.png} - \includegraphics[width=0.2\textwidth]{diag35.png} - \includegraphics[width=0.2\textwidth]{diag36.png} - \includegraphics[width=0.2\textwidth]{diag37.png} - \includegraphics[width=0.2\textwidth]{diag38.png} + \includegraphics[width=0.2\textwidth]{diag39.png} + \includegraphics[width=0.2\textwidth]{diag310.png} , \cdot \right\rangle.
\end{equation}

It vanishes on real knots. Similarly to degree two, for long virtual knots there are several invariants of degree three, which give the same degree three invariant of real
knots. Here is an example of such an invariant:

\[ \langle A, \cdot \rangle, \]

3.3. The Case of Classical Knots. Our work in this direction started with a search for combinatorial formulas for finite type invariants of classical knots. The first results were summarized in the paper [7] of the second and third authors. There a class of combinatorial formulas similar to (13) was introduced, and numerous special formulas of this sort were found. In [7] we posed the following question: “Can any Vassiliev invariant be calculated as a function of arrow polynomials evaluated on the knot diagram?” In the terminology used above, an arrow polynomial evaluated on the knot diagram is an expression of the type given in (13).

This question has been answered in the affirmative by the first author. The result is formulated as follows.

3.A. Theorem (Goussarov). Let \( G \) be an abelian group and let \( \nu \) be a \( G \)-valued invariant of degree \( n \) of long (real) knots. Then there exists a function \( \pi : A \rightarrow G \) such that \( \nu = \pi \circ I \) and such that \( \pi \) vanishes on any arrow diagram with more than \( n \) arrows.

3.B. Corollary. Any integer-valued finite type invariant of degree \( n \) of long knots can be presented as \( \langle A, \cdot \rangle \), where \( A \) is a linear combination of arrow diagrams on a line with at most \( n \) arrows.

The next section is devoted to the proof of this theorem.

To a large extent, the present paper was motivated by an analysis of the proof of 3.A, which originally was rather cumbersome. The main difficulties in this proof were caused by the necessity of requiring all the numerous Gauss diagrams involved to be realizable. The desire to get rid of this restriction motivated our interest in virtual knots. Indeed, for virtual knots the problem stated in [7] is solved by Theorem 2.E above. The universal invariant of Theorem 2.E is essentially \( \sum_{A \in P_n} \langle A, D \rangle A \), so any \( G \)-valued invariant can be presented by a Gauss diagram formula.

Unfortunately, for classical knots the new technique does not give a universal invariant. However it gives powerful and simple machinery to generate Gauss diagram formulas for any invariant which can be extended to a finite-type invariant of virtual knots. Hoping for the best, we conjecture

3.C. Conjecture. Every finite-type invariant of classical knots can be extended to a finite-type invariant of long virtual knots.

This may require the consideration of virtual framing. The extension given by Kauffman [4] of numerous invariants to virtual knots strongly supports this conjecture.

The main open problem concerning finite type invariants of classical knots is whether such invariants distinguish non-isotopic knots. The positive solution of this problem would follow from the positive solution of the corresponding problem
for virtual knots. By Theorems \(2.D\) and \(2.E\), the latter can be reformulated in purely algebraic terms as the question whether the natural map
\[
P \rightarrow \lim_{\leftarrow} P_n
\]
is injective.

4. Proof of Goussarov’s Theorem

4.1. Scheme of the Proof. The standard method for calculation of a Vassiliev invariant \(\nu\) goes as follows (see [11]). One picks a set of singular knots which span (using relation (2)) the free abelian group generated by all nonsingular knots. The invariant \(\nu\) is determined by its actuality table, i.e. its values on this set. Given a knot diagram, one unknots it, making it descending by a sequence of crossing changes. Under each crossing change the invariant jumps. The jump is equal to the value of \(\nu\) on the knot with a double point by (2). Then each of these singular knots is deformed to a knot with a single double point from the actuality table by an isotopy and a sequence of crossing changes. The jumps of \(\nu\) correspond to knots with two double points. They are again deformed to the knots from the actuality table. The process eventually stops when the number of double points exceeds the degree of \(\nu\) (by definition of the degree).

In the proof of \(3.A\), both the actuality table and the procedure of expansion described above are made canonical. This is done by generalizing the notion of a descending diagram to singular knots.

More importantly, this is done in terms of Gauss diagrams, so that the notion of descending diagram and the procedure of expansion extend to virtual knots. For a real descending knot the isotopy class and hence the value of \(\nu\) is determined by the part of the Gauss diagram encoding the double points. For a virtual descending diagram the isotopy type is not determined by this part of the Gauss diagram. Nevertheless we extend \(\nu\) to virtual descending diagrams literally in the same way. We do not know whether the result is an invariant of virtual knots. However, for our purposes, this formal extension turns out to be sufficient.

Next we use the isomorphism \(I^{-1} : A \rightarrow \mathbb{Z}[D]\) (see Proposition \(2.A\)) to define \(\pi : A \rightarrow G\) as \(\nu \circ R\). Some special properties of the extended map \(\nu : \mathbb{Z}[D] \rightarrow G\) are then used to prove that \(\pi\) vanishes on diagrams with more than \(n\) chords.

4.2. Descending Singular Diagrams. On a diagram of a long virtual singular knot each double point is naturally equipped with a sign. Indeed, the branches at a double point are ordered and the sign is the intersection number of the branches (taken in this order). On a Gauss diagram of a long singular knot, each double point is shown by a dashed chord equipped with the above sign. A diagram \(D'\) is called a subdiagram of a diagram \(D\) if \(D'\) consists of all the chords and some arrows of \(D\).

Recall that a diagram of a real long knot is descending if going along the knot in the positive direction we pass each crossing first going over and then under. In terms of Gauss diagrams it means that all the arrows are directed to the right.

We now extend this notion to virtual long knots with double points. We still require that all the arrows are directed to the right. There is also an additional

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\(^\text{4}\) A similar notion called almost monotone diagram was considered by Bar-Natan [1], but the procedure of expansion in [1] involves some choices.

\(^\text{4}\) A positive answer to this question would imply Conjecture \(3.C\).
condition: there is no chord whose left endpoint has an endpoint of an arrow as immediate left neighbor. In other words, the situations shown in Figure 9 are forbidden.

![Figure 9.](image)

A real long knot with a Gauss diagram of this type can be presented by a diagram such that

- all the double points are in the left half-plane,
- all the crossings are in the right half-plane,
- the intersection of the diagram with the left half-plane is an embedded tree,
- the intersection with the right half-plane is an ordered collection of arcs; each of them is descending and lies below all the previous ones.

See Figure 10.

![Figure 10. A descending long real knot and its Gauss diagram.](image)

4.A. Lemma. Let $D_1$ and $D_2$ be Gauss diagrams of real descending long knots and let $\nu$ be an invariant of long knots. If the chord parts of $D_1$ and $D_2$ coincide then $\nu(D_1) = \nu(D_2)$.

Proof. One can see that the isotopy class of a real descending long knot is determined by the chord part of its Gauss diagram. Indeed, the chord part determines the tree in the left half-plane (recall that the chords have signs, which define the embedding locally). The rule for connecting the endpoints of the tree by arcs in the right half-plane is determined by the mutual position of the chords in the Gauss diagram. Since the diagram is descending, the connection by the arcs is unique up to isotopy.
4.3. **Reduction to Descending Diagrams.** There is an algorithm for expressing the Gauss diagram of a long knot with double points as a linear combination of descending diagrams. This algorithm consists of steps of two types. At each step, one inspects the Gauss diagram from the left to the right looking for the first fragment where the diagram fails to be descending. Such a fragment may either be a bad arrow or a bad chord. An arrow is bad if it is directed to the left. A chord is bad if an immediate left neighbor of its left endpoint is an endpoint of an arrow, as in Figure 9.

In the case of a bad arrow the step of the algorithm is the replacement of the diagram with the sum of two diagrams according to the formula:

\[
\begin{align*}
&\varepsilon \\
&\varepsilon
\end{align*}
\]

In terms of Gauss diagrams this replacement is as follows:

\[
\begin{align*}
&\varepsilon \\
&\varepsilon
\end{align*}
\]

In the case of a bad chord the step of the algorithm is the pulling of the crossing over or under the appropriate branch by isotopy:

\[
\begin{align*}
&\varepsilon \\
&\varepsilon
\end{align*}
\]

In terms of Gauss diagrams, this corresponds to one of the transformations shown in Figure 11. The different cases in Figure 11 correspond to different orientations and possible orderings of the three arcs.

Since we deal with an invariant of degree \(n\), the diagrams with more than \(n\) chords are disregarded. Thus, when one applies a step of the algorithm to a bad arrow in a diagram with \(n\) chords, the summand with \(n + 1\) chords disappears. Denote by \(D_n\) the free abelian group generated by Gauss diagrams of virtual long singular knots with at most \(n\) chords (note that \(\mathbb{Z}[D] = D_0 \subset D_n\)). We will think of a step of our algorithm as of an operator acting on \(D_n\). Denote this operator by \(P\).

By the definition of \(P\), for any descending Gauss diagram \(D\) we have \(P(D) = D\).

4.B. **Lemma.** For any diagram \(D \in D_n\) there exists \(m\) such that \(P^m(D)\) is a sum of descending diagrams.

**Proof.** Let \(l(D)\) be the number of chords of \(D\) which have one of the endpoints to the left of the first bad fragment. As is easy to see, \(l(D') \geq l(D)\) for each diagram in the expansion of \(P(D)\). However the number of such chords in a non-descending diagram is at most \(n\). Therefore it suffices to prove that the diagram cannot change infinitely many times in subsequent iterations of \(P\) without changing \(l\).

Consider the number of arrowheads on the ray to the left of the left endpoint of the \((l(D) + 1)\)-th chord. For any diagram involved in the expansion of \(P(D)\) this number is not greater than that for \(D\). If it is the same for one of these diagrams,
then it has less arrowtails on the same ray. This can happen only finitely many times.

4.4. The Extension of $\nu$ and the Construction of $\pi$. Denote by $\mathcal{D}_n^{re}$ the subgroup of $\mathcal{D}_n$ generated by Gauss diagrams of real long singular knots. Any finite type invariant of classical knots of degree at most $n$ extends to $\mathcal{D}_n^{re}$ by linearity.

4.C. Obvious Lemma. The operator $P : \mathcal{D}_n \to \mathcal{D}_n$ preserves $\mathcal{D}_n^{re}$. The restriction of $P$ to $\mathcal{D}_n^{re}$ preserves any invariant of degree at most $n$.

We now extend an invariant $\nu$ of degree at most $n$ to virtual descending diagrams. For any such diagram $D$ there exists a descending diagram $D^{re}$ of a real knot with the same double points, i.e., the same chord part of the Gauss diagram. $D^{re}$ can be obtained by turning all the virtual crossings of $D$ into appropriate real ones. Put $\nu(D) = \nu(D^{re})$. By Lemma 4.A, $\nu(D^{re})$ does not depend on the choice of $D^{re}$.

Next we extend $\nu$ to all virtual diagrams. By Lemma 4.B, for any diagram $D \in \mathcal{D}_n$ there exists $m$ such that $P^m(D)$ is a sum of descending diagrams and hence $\nu(P^m(D))$ is already defined. Put $\nu(D) = \nu(P^m(D))$. Lemma 4.C implies that on $D^{re}$ this agrees with the initial definition of $\nu$. Since $P^{m+1}(D) = P^m(D)$ we get:

4.D. Obvious Lemma. The operator $P : \mathcal{D}_n \to \mathcal{D}_n$ preserves $\nu$, i.e. $\nu \circ P = \nu$. 

We are now in a position to construct the map $\pi : A \rightarrow G$ of Theorem 3.4. Define $\pi : A \rightarrow G$ as the composition

$$\mathcal{A} \xrightarrow{I^{-1}} \mathbb{Z}[\mathcal{D}] \subset \mathcal{D}_n \xrightarrow{\nu} G,$$

where $I^{-1}$ is the isomorphism of Proposition 2.4 and $\nu$ is the extension of the original finite type invariant to $\mathcal{D}_n$. Then for any diagram $D$ of a long knot

$$\nu(D) = \pi(I(D)) = \sum_{D' \subset D} \pi(i(D')).$$

In order to prove Theorem 3.4, we must show that $\pi(A) = 0$ for any arrow diagram $A$ with more than $n$ arrows. The rest of this section is devoted to the proof of this fact.

4.5. The Analogues of $\mathcal{D}_n$ and $P$ for Arrow Diagrams. The algebra $\mathcal{A}$ of arrow diagrams on the line is generated by diagrams consisting of the line and dashed arrows (oriented signed arcs). Consider now diagrams which in addition to arrows also contain dashed chords, i.e. unoriented signed arcs. Denote by $\mathcal{A}_n$ the free abelian group generated by such diagrams with at most $n$ chords.

The maps $i, I : \mathbb{Z}[\mathcal{D}] \rightarrow \mathcal{A}$ defined in Section 2.4 on Gauss diagrams without chords extend to isomorphisms $i, I : \mathcal{D}_n \rightarrow \mathcal{A}_n$. The chord parts of the diagrams remain intact under both $i$ and $I$, while the arrows are dealt with as in Section 2.4.

We now define an operator $Q : \mathcal{A}_n \rightarrow \mathcal{A}_n$, which is an analogue of $P$. A diagram $A \in \mathcal{A}_n$ is called descending, if $i^{-1}(A)$ is descending. Put $Q(A) = A$ if $A$ is descending. Otherwise, find the leftmost bad fragment of $A$ (the notion of a bad fragment is borrowed from $\mathcal{D}_n$ via $i$). If it is a bad arrow, we define $Q(A) = iP_i^{-1}(A)$. If it is a bad chord, put $Q(A) = \sum A'$ where the sum runs over all the subdiagrams of $iP_i^{-1}(A)$, each of which contains all the arrows not shown in Figure 11, all the chords and at least one more arrow. In other words, we sum up all seven subdiagrams of $iP_i^{-1}(A)$ which contain all the arrows and chords also belonging to $A$ plus at least one more arrow.

4.E. Remark. Observe that in both cases, we sum up all subdiagrams of diagrams in $iP_i^{-1}(A)$ which are not subdiagrams of $A$, but contain all arrows of $A$ except for the arrow involved into the bad fragment. The arrows of $A$ which are not in the leftmost bad fragment play a passive role in the construction of $Q$: if $A'$ is a subdiagram of $A$ obtained by removing arrows which are not in the leftmost bad fragment, then $Q(A')$ is obtained from $Q(A)$ by removing the same arrows from each of the summands.

4.F. Obvious Lemma. For any diagram $A \in \mathcal{A}_n$, the total number of arrows and chords in each diagram appearing in $Q(A)$ is at least the total number of arrows and chords in $A$. \hfill \Box

4.G. Lemma. For any diagram $A \in \mathcal{A}_n$, there exists $m$ such that $Q^m(A)$ is a sum of descending diagrams.

The proof of this Lemma is completely analogous to the proof of Lemma 4.B. \hfill \Box

4.H. Lemma. For any non-descending diagram $D \in \mathcal{D}_n$, there is a splitting $I(D) = U + V$ with $U, V \in \mathcal{A}_n$ such that

$$I(P(D)) = Q(U) + V$$

(17)
and such that $U = i(D) + U'$, where $U'$ is a sum of diagrams each of which has fewer arrows than $D$.

Proof. Let $U$ be the sum of all the subdiagrams of $i(D)$ which include the first bad fragment of $i(D)$. These subdiagrams contain the same bad fragment as the whole diagram $i(D)$. As follows from Remark 4.I, $Q(U)$ is the sum of all subdiagrams of diagrams in $iP(D)$ which are not subdiagrams of $i(D)$. Then $V$ is the sum of the subdiagrams of $i(D)$ which do not contain the arrow from the bad fragment (in the case of a bad chord, this is the arrow shown on the left hand side of Figure 11) and these subdiagrams of $i(D)$ remain unchanged, when one applies $P$ to $D$. Thus $I(P(D)) = Q(U) + V$. 

4.I. Lemma. The operator $Q : A_n \to A_n$ preserves $\pi$, i.e. $\pi \circ Q = \pi$.

Proof. Let $A \in A_n$ be a diagram and $D = i^{-1}(A)$. Let us prove that $\pi(Q(A)) = \pi(A)$ by induction on the number of arrows in $A$. If this number equals 0, then $A$ is descending and $Q(A) = A$ by definition of $Q$. Suppose inductively that the statement is correct for any diagram whose number of arrows is less than the number of arrows in $A$ and let us prove the statement for $A$. Apply $\pi$ to (17):

$$\pi \circ Q(U) + \pi(V) = \pi \circ I \circ P(D) = \nu \circ P(D).$$

By Lemma 4.I and the definition of $\pi$

$$\nu \circ P(D) = \nu(D) = \pi \circ I(D) = \pi(U) + \pi(V).$$

Thus $\pi \circ Q(U) = \pi(U)$. By the induction assumption, $\pi \circ Q(U') = \pi(U')$, where $U' = U - A$ (as in Lemma 4.H), and we obtain the desired equality $\pi(Q(A)) = \pi(A)$. This completes the induction step.

4.J. Lemma. Let $A \in A_n$ be a descending diagram such that the total number of arrows and chords in $A$ is greater than $n$. Then $\pi(A) = 0$.

Proof. Let $D = i^{-1}(A)$. By the definition of $\pi$ and Proposition 2.A,

$$\pi(A) = \nu \circ R(A) = \sum_{D' \subset D} (-1)^{|D-D'|} \nu(D').$$

Since any subdiagram $D'$ of $D$ is descending and has the same chord part, $\nu(D') = \nu(D)$ by the construction of $\nu$. Therefore

$$\pi(A) = \left( \sum_{D' \subset D} (-1)^{|D-D'|} \right) \nu(D).$$

As one can easily check by induction on the number of arrows in $A$, the sum in parentheses is equal to 1 if $A$ has no arrows and is 0 otherwise. Since all the diagrams in $A_n$ have at most $n$ chords and the total number of arrows and chords in $A$ is greater than $n$, it has at least one arrow. Hence $\pi(A) = 0$.

4.K. Lemma. Let $A \in A_n$ be a diagram such that the total number of arrows and chords in $A$ is greater than $n$. Then $\pi(A) = 0$.

Proof. Let $m$ be the number which exists for $A$ by Lemma 4.G. By Lemma 4.I, $\pi(A) = \pi(Q^m(A))$. By Lemma 4.F, the expansion of $Q^m(A)$ contains only descending diagrams with the total number of chords and arrows greater than $n$. Then by Lemma 4.J, $\pi(A) = 0$. 


This concludes the proof of Theorem 3.A.

5. $n$-Equivalence

5.1. $n$-Trivial Gauss Diagrams. Let $D$ be a Gauss diagram and let its arrows be colored with $n$ colors. Consider all subdiagrams which can be obtained from $D$ by removing all arrows colored with one or several colors. If all arrows of each of these diagrams can be removed by the second Reidemeister moves, then the coloring is said to be destroying.

A Gauss diagram, based on a union of several disjoint segments, is called $n$-trivial if it admits a destroying coloring with $n + 1$ colors.

The property of $n$-triviality does not change if one reverses the orientation of some of the segments. It also does not change if one simultaneously reverses the orientations or the signs of all arrows connecting two segments.

5.2. $n$-Variations. On a Gauss diagram $D$, choose several segments which do not contain an endpoint of any arrow. Adjoin to $D$ arrows of an $(n - 1)$-trivial Gauss diagram based on the chosen segments. This transformation of $D$ is called an $n$-variation.

It is easy to see that an addition of any number of arrows is a 1-variation. On a virtual knot diagram an addition of an arrow can be realized as follows:

In Figure 12, we show the simplest 2-variations. To get the corresponding destroying coloring, one colors all arrows connecting the first two strings with one color and the other arrows with the other color. It is easy to see that these 2-variations do not change $\text{lk}_{i/j}$.

On a virtual knot diagram these 2-variations can be realized as

Observe that these modifications coincide, up to isotopy, with the forbidden moves of Figure 5.

Some obvious properties of $n$-variations are:

1. An $n$-variation is a $k$-variation for any $k < n$.
2. Composition of several $n$-variations is an $n$-variation.
A less obvious property is: the result of an isotopy followed by an \( n \)-variation can be presented as the result of other \( n \)-variations followed by an isotopy.

The following proposition is a key property of \( n \)-variations.

**5.A. Proposition.** After any \( n \)-variation, one can apply another \( n \)-variation such that the final result is the initial diagram, up to a sequence of second Reidemeister moves.

### 5.3. \( n \)-Equivalence

Two Gauss diagrams are said to be \( n \)-equivalent if they can be transformed to each other by a sequence of isotopies and \((n+1)\)-variations. For example, the Gauss diagrams shown in Figure 12 are 1-equivalent. Moreover, one can prove that any two 1-equivalent Gauss diagrams can be transformed to each other by a sequence of isotopies and the 2-variations of Figure 12. A 1-equivalence class of string links is completely determined by the invariants \( \text{lk}_{ij} \). Therefore any two closed virtual knots are 1-equivalent and can be transformed to each other by a sequence of the Reidemeister moves and the forbidden moves of Figure 5.

The transition from the use of Gauss diagrams to that of \( n \)-equivalence classes yields better results when the set of Gauss diagrams is equipped with a natural multiplication. For example, in the case of virtual string links (and, in particular, long knots) \( n \)-equivalence classes form a group.

In the cases of virtual knots and (closed) links the set of \( n \)-equivalence classes has a more complicated algebraic structure. As in the case of classical links, the following trick works. Consider a virtual string link with \( 2n \) strings. It can be turned into a closed one by adding \( n \) arcs from above and below (this generalizes the plat presentation of a link with braids replaced by string links). This gives rise to a map from the group of \( n \)-equivalence classes of virtual string links to the set of \( n \)-equivalence classes of closed virtual links. This map is a double coset factorization. From the left we quotient out by string links which become \( n \)-equivalent to the trivial one by adding only arcs from below, and from the right, similarly with arcs from above. Both sets of string links give rise to subgroups which are not normal in general.

The value of a finite type invariant of degree \( \leq n \) depends only on the \( n \)-equivalence class. Usually, in a non-group situation, invariants of degree \( \leq n \) do not separate all the \( n \)-equivalence classes. For instance, virtual closed knots do not admit an invariant of degree 2.

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