Weak essentially undecidable theories of concatenation

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Abstract
In the language \{0, 1, \circ, \leq\}, where 0 and 1 are constant symbols, \circ is a binary function symbol and \leq is a binary relation symbol, we formulate two theories, WD and D, that are mutually interpretable with the theory of arithmetic R and Robinson arithmetic Q, respectively. The intended model of WD and D is the free semigroup generated by \{0, 1\} under string concatenation extended with the prefix relation. The theories WD and D are purely universally axiomatised, in contrast to Q which has the \Pi_2-axiom \(\forall x \ (x = 0 \lor \exists y \ (x = Sy)\).

Keywords Theory of concatenation · First-order arithmetic · Interpretability

Mathematics Subject Classification 03B25 · 03B30 · 03F25 · 03F30 · 03F40

1 Introduction
This paper follows the line of work that focuses on determining whether there is a weakest theory that is essentially undecidable. We formulate two natural essentially undecidable theories in the language of concatenation that are purely universally axiomatised.

A countable first-order theory is called essentially undecidable if any consistent extension, in the same language, is undecidable (there is no algorithm for deciding whether an arbitrary sentence is a theorem). A countable first-order theory is called essentially incomplete if any recursively axiomatizable consistent extension is incomplete. It is known that a theory is essentially undecidable if and only if it is essentially incomplete. Indeed, if a theory is not essentially undecidable, then by definition it has a consistent decidable extension which can be extended to a decidable complete consistent theory (see Chapter 1 of Tarski et al. [18]). On the other hand, if a theory is not essentially incomplete, then by definition it has a complete consistent recursively
axiomatizable extension which clearly is decidable. Two theories that are known to be essentially undecidable are Robinson arithmetic $Q$ and the related theory $R$. Rosser’s generalization of Gödel’s first incompleteness theorem is usually taken as the statement that $Q$ is essentially undecidable.

The main objective of this paper is to show that two theories, $WD$ and $D$, are mutually interpretable with $R$ and $Q$, respectively. The axioms of $WD$ and $D$ are given below using juxtaposition instead of the binary function symbol of the formal language. Later we regard $D$ as the theory without the axiom $D_5$. We do this because the two versions of $D$ are mutually interpretable. The theories $WD$ and $D$ are theories in the language of concatenation extended with a binary relation symbol. That is, the language \{0, 1, $\circ$, $\leq$\} where 0 and 1 are constant symbols, $\circ$ is a binary function symbol and $\leq$ is a binary relation symbol. The intended structure $D$ is the free semigroup with two generators extended with the prefix relation which we denote $\leq_D$. As in number theory, each element $\alpha$ in the universe is associated with a canonical term $\overline{\alpha}$. The inclusion of a binary relation in the languages makes it easier to define $\Sigma_1$-formulas and to give a purely universal $\Sigma_1$-complete axiomatization of $D$. We observe that any theory that proves all true $\Sigma_1$-sentences is an extension of $WD$. Indeed, instances of $WD_1$ and $WD_2$ are true $\Sigma_1$-sentences, and each instance of $WD_3$ follows from the true $\Sigma_1$-sentences $\forall x \leq z [ \lor_{y \leq z} x = \overline{y} ]$ and $\land_{y \leq z} \overline{y} \leq \overline{z}$. The theory $WD$ is thus the weakest $\Sigma_1$-complete axiomatization of $D$ (modulo closure under logical implication) and the theory $D$ is a natural finitely axiomatizable extension of $WD$. A variant of $D$ where we have an identity element was introduced in Kristiansen and Murwanashyaka [9] as a $\Sigma_1$-complete axiomatization of the structure $D$ extended with the empty string. In [9, 10], we identify a number of decidable and undecidable fragments of $D$ and related structures.

The Axioms of $R$

| Axiom ID | Axiom |
|----------|--------|
| $R_1$   | $\overline{n + m} = \overline{n} + \overline{m}$ |
| $R_2$   | $\overline{n \times m} = \overline{n} \times \overline{m}$ |
| $R_3$   | $\overline{n} \neq \overline{m}$ if $n \neq m$ |
| $R_4$   | $\forall x \ [ x \leq \overline{n} \lor \overline{n} \leq x ]$ |
| $R_5$   | $\forall x \ [ x \leq \overline{n} \lor \overline{n} \leq x ]$ |

The Axioms of $Q$

| Axiom ID | Axiom |
|----------|--------|
| $Q_1$   | $\forall xy \ [ x \neq y \rightarrow Sx \neq Sy ]$ |
| $Q_2$   | $\forall x \ [ Sx \neq 0 ]$ |
| $Q_3$   | $\forall x \ [ x = 0 \lor \exists y \ [ x = S(y) ] ]$ |
| $Q_4$   | $\forall x \ [ x + 0 = x ]$ |
| $Q_5$   | $\forall xy \ [ x + Sy = S(x + y) ]$ |
| $Q_6$   | $\forall x \ [ x \times 0 = 0 ]$ |
| $Q_7$   | $\forall xy \ [ x \times Sy = (x \times y) + x ]$ |

The Axioms of $WD$

| Axiom ID | Axiom |
|----------|--------|
| $WD_1$  | $\overline{\alpha \beta} = \overline{\alpha} \overline{\beta}$ |
| $WD_2$  | $\overline{\alpha} \neq \overline{\beta}$ if $\alpha \neq \beta$ |
| $WD_3$  | $\forall x \ [ x \leq \overline{\alpha} \leftrightarrow \lor_{y \leq \overline{\alpha}} x = \overline{y} ]$ |

The Axioms of $D$

| Axiom ID | Axiom |
|----------|--------|
| $D_1$   | $\forall xyz \ [ (xyz)z = x(yz) ]$ |
| $D_2$   | $\forall xy \ [ x \neq y \rightarrow (x \neq y \land x \neq y) ]$ |
| $D_3$   | $\forall xy \ [ x \neq y \land x \neq y ]$ |
| $D_4$   | $\forall x \ [ x \leq 0 \leftrightarrow x = 0 ]$ |
| $D_5$   | $\forall x \ [ x \leq 1 \leftrightarrow x = 1 ]$ |
| $D_6$   | $\forall xy \ [ x \leq y \rightarrow (x = y \lor x \leq y) ]$ |
| $D_7$   | $\forall xy \ [ x \leq y \rightarrow (x = y \lor x \leq y) ]$ |
The theory of concatenation $TC$ was introduced by Grzegorczyk in [4] where he also showed that it is undecidable. The language of $TC$ consists only of the two constant symbols $0$ and $1$ and the binary function symbol $\circ$. The intended model of $TC$ is a free semigroup with at least two generators. That is, a structure of the form $(\Gamma^+, a_1, \ldots, a_n, \cdot )$ where $\Gamma = \{ a_1, \ldots, a_n \}$ is a finite alphabet with at least two symbols, $\Gamma^+$ is the set of all finite non-empty strings over $\Gamma$ and $\cdot$ is the binary operator that concatenates elements of $\Gamma^+$. Grzegorczyk’s motivation for introducing the theory $TC$ was that, as computation involves manipulation of text, the notion of computation can be formulated on the basis of discernibility of text without reference to natural numbers. Then, undecidability of first-order logic and essential undecidability can be explained using a theory of strings thereby avoiding complicated coding of syntax based on natural numbers. In [5], Grzegorczyk and Zdanowski showed that $TC$ is essentially undecidable. This was further improved in Ganea [3], Visser [19] and Švejdar [16] where it was shown that $TC$ is mutually interpretable with Robinson arithmetic. In Higuchi and Horihata [6], it was shown that $TC$ is minimal essentially undecidable. That is, removing any one of the axioms of $TC$ gives a theory that is not essentially undecidable. We will refer to $TC$ as $TC^{\varepsilon}$ and we will let $TC$ refer to the theory where we have an identity element. The two theories are known to be mutually interpretable (see Grzegorczyk and Zdanowski [5] and Visser [19]). In the article [6], it was also shown that a weak theory of concatenation $WTC^{\varepsilon}$ is minimal essentially undecidable and mutually interpretable with $\mathbb{R}$. That $WTC^{\varepsilon}$ is minimal essentially undecidable means that removal of any one of the axiom schemas of $WTC^{\varepsilon}$ gives a theory that is not essentially undecidable.

The Axioms of $TC^{\varepsilon}$

$$
\begin{align*}
TC_{1}^{\varepsilon} & \forall xyz \ [ x(yz) = (xy)z ] \\
TC_{2}^{\varepsilon} & \forall xyzw \ [ (xy = zw) \rightarrow ( ( x = z \land y = w) \lor \exists u \ [ ( z = xu \land uw = y ) \lor ( x = zu \land uy = w ) ] ) ] \\
TC_{3}^{\varepsilon} & \forall xy \ [ xy \neq 0 ] \\
TC_{4}^{\varepsilon} & \forall xy \ [ xy \neq 1 ] \\
TC_{5}^{\varepsilon} & 0 \neq 1
\end{align*}
$$

The Axioms of $WTC^{\varepsilon}$

$$
\begin{align*}
WTC_{1}^{\varepsilon} & \forall xyz \ [ x(yz) \subseteq_s \overline{\alpha} \lor (xy)z \subseteq_s \overline{\alpha} ) \rightarrow x(yz) = (xy)z ] \\
WTC_{2}^{\varepsilon} & \forall xyzw \ [ (xy = zw \land xy \subseteq_s \overline{\alpha} ) \rightarrow ( ( x = z \land y = w) \lor \exists u \ [ ( z = xu \land uw = y ) \lor ( x = zu \land uy = w ) ] ) ] \\
WTC_{3}^{\varepsilon} & \forall xy \ [ xy \neq 0 ] \\
WTC_{4}^{\varepsilon} & \forall xy \ [ xy \neq 1 ] \\
WTC_{5}^{\varepsilon} & 0 \neq 1
\end{align*}
$$

We use $x \subseteq_s y$ as shorthand for

$$
y = x \lor \exists uv \ [ y = ux \lor y = xv \lor y = (ux)v \lor y = u(xv) ] .
$$
Fig. 1 $S \rightarrow T$ means $S$ is interpretable in $T$ but $S$ does not interpret $T$. $S \leftrightarrow T$ means $S$ and $T$ are mutually interpretable.

The diagram in Fig. 1 summarizes the relationships between $R$, $Q$, $WD$, $D$, $WTC_{\neg \varepsilon}$ and $TC_{\neg \varepsilon}$.

Kristiansen and Murwanashyaka [11] have also introduced two essentially undecidable theories, $WT$ and $T$, that are purely universally axiomatised. While $WD$ and $D$ are theories of concatenation, $WT$ and $T$ are theories of terms. The intended model of $WT$ and $T$ is the extended term algebra given by the language $L_T = \{ \bot, \langle \cdot, \cdot \rangle, \sqsubseteq \}$ where $\bot$ is a constant symbol, $\langle \cdot, \cdot \rangle$ is a binary function symbol and $\sqsubseteq$ is a binary relation symbol. The universe is the set of all variable-free $L_T$-terms and $\langle \cdot, \cdot \rangle$ is interpreted as the function that maps a pair $(s, t)$ to the term $\langle s, t \rangle$. Every variable-free term is thus realized as itself. The relation symbol $\sqsubseteq$ is interpreted as the subterm relation. The theory $WT$ has a compact axiomatization that consists only of analogues of $WD_2$ and $WD_3$. An analogue of $WD_1$ is not necessary since it holds by pure logic. The theory $T$ has two axioms $T_1 \equiv \forall xy [ \langle x, y \rangle \neq \bot]$ and $T_2 \equiv \forall xyzw [ \langle x, y \rangle = \langle z, w \rangle \rightarrow (x = z \land y = w)]$

that describe $\langle \cdot, \cdot \rangle$; and two axioms $T_3 \equiv \forall x [ x \sqsubseteq \bot \leftrightarrow x = \bot]$ and $T_4 \equiv \forall xyz [ x \sqsubseteq \langle y, z \rangle \leftrightarrow (x = \langle y, z \rangle \lor x \sqsubseteq y \lor x \sqsubseteq z)]$

that describe $\sqsubseteq$. Kristiansen and Murwanashyaka give interpretations of $R$ and $Q$ in $WT$ and $T$, respectively. A result of Visser [20] ensures that $WT$ is interpretable in $R$ since it is locally finitely satisfiable. They conjecture that $T$ and $Q$ are mutually interpretable.

2 Preliminaries

In this section, we clarify a number of notions that we only glossed over in the previous section. We also introduce a number of intermediate theories that will be useful in showing that $WD$ and $R$ are mutually interpretable and that $D$ and $Q$ are mutually interpretable.

We consider the structures $\mathcal{D}_- = (\{0, 1\}^+, 0, \mathbf{1}, \cdot)$ and $\mathcal{D} = (\{0, 1\}^+, 0, \mathbf{1}, \cdot, \preceq^D)$

where $\{0, 1\}^+$ is the set of all finite non-empty strings over the alphabet $\{0, 1\}$, the binary operator $\cdot$ concatenates elements of $\{0, 1\}^+$ and $\preceq^D$ denotes the prefix relation, i.e., $x \preceq^D y$ if and only if $y = x$ or there exists $z \in \{0, 1\}^+$ such that $y = x \cdot z$.

The structure $\mathcal{D}_-$ is thus the free semigroup with two generators. We call elements of $\{0, 1\}^+$ bit strings. The structures $\mathcal{D}_-$ and $\mathcal{D}$ are first-order structures over the languages $\mathcal{L}_{\mathcal{BT}} = \{0, 1, \circ\}$ and $\mathcal{L}_{\mathcal{BT}} = \{0, 1, \circ, \preceq\}$, respectively.
In first-order theory, each natural number \( n \) is associated with a numeral \( \overline{n} \). Each non-empty bit string \( \alpha \in \{0, 1\}^+ \) is associated by recursion with a unique \( \mathcal{L}_{BT}^- \)-term \( \overline{\alpha} \), called a bilateral, as follows: \( \overline{0} \equiv 0, \overline{1} \equiv 1, \overline{\alpha\overline{0}} \equiv (\overline{\alpha} \circ 0) \) and \( \overline{\alpha\overline{1}} \equiv (\overline{\alpha} \circ 1) \). The biterals are important if we, for example, want to show that certain sets are definable since we then need to talk about elements of \( \{0, 1\}^+ \) in the formal theory.

In the language \( \mathcal{L}_{BT}^- \), we can define the \( \Sigma_1 \)-formulas as follows: atomic formulas and their negations are \( \Sigma_1 \)-formulas. If \( \alpha \) and \( \beta \) are \( \Sigma_1 \)-formulas, then \( \alpha \land \beta, \alpha \lor \beta \) and \( \exists x \alpha \) are \( \Sigma_1 \)-formulas. Furthermore, if the variable \( x \) does not occur in the term \( t \), then \( \exists x \ [x \leq t \land \alpha] \) and \( \forall x \ [x \leq t \rightarrow \alpha] \) are also \( \Sigma_1 \)-formulas. We use \( \exists x \leq t \alpha \) and \( \forall x \leq t \alpha \) as abbreviations for \( \exists x \ [x \leq t \land \alpha] \) and \( \forall x \ [x \leq t \rightarrow \alpha] \), respectively. To define \( \Sigma_1 \)-formulas over the language \( \mathcal{L}_{BT}^- \), we first need to define a binary relation in order to have bounded quantifiers. Two natural choices are

\[
- x \leq_i y \equiv y = x \lor \exists z \ [y = xz] \\
- x \leq_s y \equiv y = x \lor \exists uv \ [y = ux \land y = xv \lor y = (ux)v \lor y = u(xv)].
\]

Over the structure \( \mathcal{D}^- \), the defined relation \( \leq_i \) is realized as the prefix relation while \( \leq_s \) is realized as the substring relation. Given a bit string \( \alpha \), the set of those bit strings that are substrings of \( \alpha \) is denoted \( \langle \ldots \alpha \rangle \) and consists of those bit strings \( \beta \) such that \( \beta = \alpha \) or there exist bit strings \( u \) and \( v \) such that \( \alpha = uv \) or \( \alpha = v \) or \( \alpha = u\beta \). We choose to work with \( \leq_s \) since the intended interpretation of \( \leq \) is the prefix relation. We observe that the number of substrings of a string \( \alpha \) is quadratic in the length of \( \alpha \), whereas the number of prefixes of \( \alpha \) is linear in the length of \( \alpha \). This means that the choice between \( \leq_i \) and \( \leq_s \) could make a difference in the context of very weak theories.

Having introduced \( \Sigma_1 \)-formulas, it is natural to try to find \( \Sigma_1 \)-complete axiomatizations of the structures \( \mathcal{D}^- \) and \( \mathcal{D} \). That is, to find theories that prove all true \( \Sigma_1 \)-sentences (sentences are formulas without free variables) and that are such that the non-logical axioms are true over the intended structure. A natural first step is to introduce the theories \( WBT \) and \( WD \) defined below. It is not difficult to see that these two theories are \( \Sigma_1 \)-complete. The theories \( WBT \) and \( WD \) are not finitely axiomatizable but they are the weakest possible \( \Sigma_1 \)-complete axiomatizations of \( \mathcal{D}^- \) and \( \mathcal{D} \), respectively, modulo closure under logical implication. Once we have \( WBT \) and \( WD \), the theories \( BTQ \) and \( \mathcal{D} + \forall x \ [x \leq 1 \leftrightarrow x = 1] \) defined below are natural finitely axiomatizable extensions \( (T \) is an extension of \( S \) if the language of \( S \) is a subset of the language of \( T \) and every theorem of \( S \) is a theorem of \( T \)). The reason for not having \( \forall x \ [x \leq 1 \leftrightarrow x = 1] \) as an axiom of \( \mathcal{D} \) is that it is not necessary for essential undecidability (we could, of course, very well have worked with the theory where we have \( \forall x \ [x \leq 1 \leftrightarrow x = 1] \) instead of \( \forall x \ [x \leq 0 \leftrightarrow x = 0] \)). Although the theories \( WBT, BTQ, WD \) and \( \mathcal{D} \) are \( \Sigma_1 \)-complete, it is not at all obvious that they are essentially undecidable. When proving that \( R \) is essentially undecidable, the axiom schema \( R_S \equiv \forall x \ [x \leq \overline{n} \lor \overline{n} \leq x] \) is essential. It is however not straightforward to define a binary relation that provably satisfies the analogue of \( R_S \). The method of relative interpretability then becomes important for establishing that these theories are

\[ Springer\]
Fig. 2 \( S \rightarrow T \) means \( S \) is interpretable in \( T \) but \( S \) does not interpret \( T \). \( S \leftrightarrow T \) means \( S \) and \( T \) are mutually interpretable. \( S \rightarrow axisshort/axisshort/arrowaxisright T \) means \( S \) is interpretable in \( T \) but it is unknown whether \( S \) interprets \( T \) essentially undecidable.

The Axioms of WBT

\[ \begin{align*}
\text{WBT}_1 & \; \; \alpha \beta = \alpha \beta \\
\text{WBT}_2 & \; \; \alpha \neq \beta \text{ if } \alpha \neq \beta \\
\text{WBT}_3 & \; \; \forall x \; [ x \subseteq \alpha \rightarrow \bigvee_{y \in (x: \alpha)} x = \bar{y} ]
\end{align*} \]

The Axioms of WD

\[ \begin{align*}
\text{WD}_1 & \; \; \alpha \beta = \alpha \beta \\
\text{WD}_2 & \; \; \alpha \neq \beta \text{ if } \alpha \neq \beta \\
\text{WD}_3 & \; \; \forall x \; [ x \preceq \alpha \leftrightarrow \bigvee_{y \preceq \alpha} x = \bar{y} ]
\end{align*} \]

The Axioms of BTQ

\[ \begin{align*}
\text{BTQ}_1 & \; \; \forall xyz \; [ (xy)z = x(yz) ] \\
\text{BTQ}_2 & \; \; \forall xy \; [ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1) ] \\
\text{BTQ}_3 & \; \; \forall xy \; [ x0 \neq y1 ] \\
\text{BTQ}_4 & \; \; \forall x \; [ x = 0 \lor x = 1 \lor \exists y \; [ x = y0 \lor x = y1 ] ]
\end{align*} \]

The Axioms of D

\[ \begin{align*}
\text{D}_1 & \; \; \forall xyz \; [ (xy)z = x(yz) ] \\
\text{D}_2 & \; \; \forall xy \; [ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1) ] \\
\text{D}_3 & \; \; \forall xy \; [ x0 \neq y1 ] \\
\text{D}_4 & \; \; \forall x \; [ x \leq 0 \leftrightarrow x = 0 ] \\
\text{D}_5 & \; \; \forall xy \; [ x \leq y0 \leftrightarrow (x = y0 \lor x \leq y) ] \\
\text{D}_6 & \; \; \forall xy \; [ x \leq y1 \leftrightarrow (x = y1 \lor x \leq y) ]
\end{align*} \]

The Axioms of BT

\[ \begin{align*}
\text{BTQ}_1, \; \text{BTQ}_2, \; \text{BTQ}_3, \; \text{WBT}_3 \; \text{D}_1, \; \text{D}_2, \; \text{D}_3, \; \text{WD}_3
\end{align*} \]

The overall relationship among the various theories introduced so far is summarized in Fig. 2. The constants 0 and 1 are atoms in BTQ. This means that BTQ \( \vdash \forall xy \; [ xy \neq 0 ] \land \forall xy \; [ xy \neq 1 ] \) (see Lemma 8 in Ganea [3] for a proof). This observation is used in the proof of Theorem 7 where we show that D is interpretable in BTQ.

We recall the method of relative interpretability introduced by Tarski [18] for showing that first-order theories are essentially undecidable. Let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be computable first-order languages. A relative translation \( \tau \) from \( \mathcal{L}_1 \) to \( \mathcal{L}_2 \) is a computable map given by:

1. An \( \mathcal{L}_2 \)-formula \( \delta(x) \) with exactly one free variable. The formula \( \delta(x) \) is called a domain.
2. For each \( n \)-ary relation symbol \( R \) of \( \mathcal{L}_1 \), an \( \mathcal{L}_2 \)-formula \( \psi_R(x_1, \ldots, x_n) \) with exactly \( n \) free variables. The equality symbol \( = \) is treated as a binary relation symbol.
3. For each \(n\)-ary function symbol \(f\) of \(\mathcal{L}_1\), an \(\mathcal{L}_2\)-formula \(\psi_f(x_1, \ldots, x_n, y)\) with exactly \(n + 1\) free variables.
4. For each constant symbol \(c\) of \(\mathcal{L}_1\), an \(\mathcal{L}_2\)-formula \(\psi_c(y)\) with exactly one free variable.

We extend \(\tau\) to a translation of atomic \(\mathcal{L}_1\)-formulas by mapping a \(\mathcal{L}_1\)-term \(t\) to a \(\mathcal{L}_2\)-formula \((t)^{\tau,w}\) with a free variable \(w\) that denotes the value of \(t\):

5. For each \(n\)-ary relation symbol \(R\) of \(\mathcal{L}_1\)

\[ (R(t_1, \ldots, t_n))^{\tau} \equiv \exists v_1 \ldots v_n \left[ \bigwedge_{i=1}^{n} \delta(v_i) \land \bigwedge_{j=1}^{n} (t_j)^{\tau,w_j} \land \psi_R(v_1, \ldots, v_n) \right] \]

where \(v_1, \ldots, v_n\) are distinct variable symbols that do not occur in \(t_1, \ldots, t_n\) and

(a) for each variable symbol \(x\) of \(\mathcal{L}_1\), \((x)^{\tau,w} \equiv w = x\)
(b) for each constant symbol \(c\) of \(\mathcal{L}_1\), \((c)^{\tau,w} \equiv \psi_c(w)\)
(c) for each \(n\)-ary function symbol \(f\) of \(\mathcal{L}_1\)

\[ (f(t_1, \ldots, t_n))^{\tau,w} \equiv \exists w_1 \ldots w_n \left[ \bigwedge_{i=1}^{n} \delta(w_i) \land \bigwedge_{j=1}^{n} (t_j)^{\tau,w_j} \land \psi_f(w_1, \ldots, w_n, w) \right] \]

where \(w_1, \ldots, w_n\) are distinct variable symbols that do not occur in \(\bigwedge_{j=1}^{n} (t_j)^{\tau,w}\).

We extend \(\tau\) to a translation of all \(\mathcal{L}_1\)-formulas as follows:

6. \((-\phi)^{\tau} \equiv \neg\phi^{\tau}\)
7. \((\phi \circ \psi)^{\tau} \equiv \phi^{\tau} \circ \psi^{\tau}\) for \(\circ \in \{\land, \lor, \to, \leftrightarrow\}\)
8. \((\exists x \, \phi)^{\tau} \equiv \exists x \left[ \delta(x) \land \phi^{\tau} \right]\)
9. \((\forall x \, \phi)^{\tau} \equiv \forall x \left[ \delta(x) \to \phi^{\tau} \right]\).

Let \(S\) be an \(\mathcal{L}_1\)-theory and let \(T\) be an \(\mathcal{L}_2\)-theory. We say that \(S\) is (relatively) interpretable in \(T\) if there exists a relative translation \(\tau\) such that

- \(T \vdash \exists x \, \delta(x)\)
- For each function symbol \(f\) of \(\mathcal{L}_1\)

\[ T \vdash \bigwedge_{i=1}^{n} \delta(x_i) \to \exists! y \left[ \delta(y) \land \psi_f(x_1, \ldots, x_n, y) \right]. \]

- For each constant symbol \(c\) of \(\mathcal{L}_1\)

\[ T \vdash \exists! y \left[ \delta(y) \land \psi_c(y) \right]. \]

- \(T\) proves \(\phi^{\tau}\) for each non-logical axiom \(\phi\) of \(S\). If equality is not translated as equality, then \(T\) must prove the translation of each equality axiom.
If $S$ is relatively interpretable in $T$ and $T$ is relatively interpretable in $S$, we say that $S$ and $T$ are *mutually interpretable*.

The following proposition summarizes important properties of relative interpretability (see Tarski et al. [18] for the details).

**Proposition 1** Let $S$, $T$ and $U$ be computably enumerable first-order theories.

1. If $S$ is interpretable in $T$ and $T$ is consistent, then $S$ is consistent.
2. If $S$ is interpretable in $T$ and $T$ is interpretable in $U$, then $S$ is interpretable in $U$.
3. If $S$ is interpretable in $T$ and $S$ is essentially undecidable, then $T$ is essentially undecidable.

### 3 Intermediate theories

We show in our Master’s thesis [12] that $\text{WBT}$ and $\text{WTC}^{-\varepsilon}$ are equivalent, i.e., they prove the same formulas. We do not include a proof of this result, but the interested reader may find the fairly straightforward proof in [12]. We also show in [12] that $\text{BTQ}$ and $\text{TC}^{-\varepsilon}$ are mutually interpretable. Although this result is not trivial, we omit a proof since there is another way of seeing that $\text{BTQ}$ is mutually interpretable with $Q$. The theory $\text{BTQ}$ is a fragment of the theory $\mathcal{F}$, introduced first by Alfred Tarski at the end of Chapter 3 of [18]. The theory $\text{BTQ}$ also resembles the theory $\mathcal{Q}_{\text{bin}}$ in Visser [19]. One can view $\mathcal{Q}_{\text{bin}}$ as the analogue of $\text{BTQ}$ where we have an identity element and associativity has been weakened to $\forall xyz [ (xy)0 = x(y0) \land (xy)1 = x(y1) ]$. The theory $\mathcal{F}$ differs from $\text{BTQ}$ in that the axiom $\text{BTQ}2$ is replaced with the axioms

$$\mathcal{F}_2 \equiv \forall xyz [ zx = zy \rightarrow x = y ] \quad \text{and} \quad \mathcal{F}_3 \equiv \forall xyz [ xz = yz \rightarrow x = y ].$$

In Ganea [3], it is shown that $\mathcal{F} - \mathcal{F}_2$ is mutually interpretable with $Q$. Clearly, $\text{BTQ}$ is interpretable in $\mathcal{F} - \mathcal{F}_2$. The other way, we can interpret $\mathcal{F} - \mathcal{F}_2$ in $\text{BTQ}$ by simply relativizing quantification to the domain

$$J = \{ z : \forall xy [ xz = yz \rightarrow x = y ] \}.$$

It follows from $\text{BTQ}2$ that $0, 1 \in J$. We now show that $J$ is closed under $\circ$. Suppose $z, w \in J$. We need to show that $zw \in J$. We have

$$x(zw) = y(zw) \Rightarrow (xz)w = (yz)w \quad \text{by BTQ1 }$$
$$\Rightarrow xz = yz \quad \text{since } w \in J$$
$$\Rightarrow x = y \quad \text{since } z \in J.$$
$$uy = vy \Rightarrow (uy)0 = (vy)0 \land (uy)1 = (vy)1$$
$$\Rightarrow u(y0) = v(y0) \land u(y1) = v(y1) \quad \text{(by BTQ}_1\text{)}$$
$$\Rightarrow ux = vx$$
$$\Rightarrow u = v$$

(hence $x \in J$ implies $y \in J$. Thus, BTQ holds on $J$. It is not difficult to verify that the other axioms also hold on $J$. We can thus state the following theorem which will be used implicitly to show that D and Q are mutually interpretable.

**Theorem 1** BTQ and Q are mutually interpretable.

The theories BT and C are examples of theories that lie strictly between R and Q w.r.t. relative interpretability. It follows from $\forall x y \ [ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1) ]$ and $\forall xy \ [ x0 \neq y1 ]$ that any model of BT or C is infinite. If BT (C) were interpretable in WBT (WD), it would be interpretable in a finite sub-theory. Since any finite sub-theory of WBT or WD clearly has a finite model, we cannot interpret BT (C) in WBT (WD). We now show that BTQ is not interpretable in BT. Similar reasoning shows that D is not interpretable in C. We let BT4 denote WBT3 and for $1 \leq i \leq 3$ we let BTi denote BTQi.

**Theorem 2** BTQ is not interpretable in BT.

**Proof** Suppose $\tau$ is an interpretation of BTQ in BT. Then, there is a finite subset $\Sigma$ of the axioms of BT such that

$$\Sigma \vdash \bigwedge_{j=1}^{4} (\text{BTQ}_j)^\tau.$$

To see that $\tau$ cannot exist, it suffices to show that the theory given by $\Sigma$ is interpretable in BT–BT4, which is not essentially undecidable by minimality of BTQ (see Lemma 11).

To see that $\Sigma$ is interpretable in BT–BT4, it suffices to show that for each natural number $n \geq 1$, the theory BT$_{\leq n}$ is interpretable in BT. The theory BT$_{\leq n}$ is like BT except that the axiom schema

$$\text{BT}_4 \equiv \forall x \ [ x \subseteq \bar{\alpha} \rightarrow \bigvee_{y \in (...\bar{\alpha})} x = \bar{y} ].$$

is limited to those $\alpha$ such that the length of $\alpha$, denoted $|\alpha|$, is bounded by $n$. To do this, we define by recursion a sequence of domains

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$$

such that we obtain an interpretation of BT$_{\leq n}$ in BT by simply relativizing quantification to $I_n$. So, we proceed to construct these domains. We will omit parentheses most of the time since we have the axiom

$$\text{BT}_1 \equiv \forall xyz \ [ (xy)z = x(yz) ].$$
Construction of $I_1$:

We let

$$A_1 = \{ y : \forall x [ xy0 \neq 0 \land xy1 \neq 1 ] \}.$$

Suppose $x00 = 0 \lor x01 = 1$. Then, $11x00 = 110 \lor 11x01 = 111$. By $BT_2$, we then have $11x0 = 11$, contradicting $BT_3$. Hence, $0 \in A_1$. By similar reasoning, we have $1 \in A_1$. We now show that $A_1$ is closed under $\circ$. Suppose $y, y' \in A_1$. By $BT_1$, we have

$$x(yy')0 = (xy)y'0 \quad \text{and} \quad x(yy')1 = (xy)y'1.$$

Hence, $x(yy')0 = 0 \lor x(yy')1 = 1$ contradicts $y' \in A_1$. Thus, $yy' \in A_1$. Hence, $A_1$ is closed under $\circ$.

We let

$$B_1 = \{ x \in A_1 : x0 \neq 0 \land x1 \neq 1 \}.$$

Suppose $00 = 0 \lor 01 = 1$. Then, $1100 = 110 \lor 1101 = 111$. By $BT_2$, we then have $110 = 11$, contradicting $BT_3$. Thus, $0 \in B_1$. By similar reasoning, we have $1 \in B_1$. We now show that $B_1$ is closed under $\circ$. Suppose $y, y' \in B_1$. We observe that

$$yy'0 = 0 \lor yy'1 = 1$$

contradicts $y' \in A_1$. Hence, $yy' \in B_1$. Thus, $B_1$ is closed under $\circ$.

We let

$$I_1 = \{ y \in B_1 : \forall x \in B_1 [ xy \neq 0 \land xy \neq 1 ] \}.$$

We clearly cannot have $x1 = 0 \lor x0 = 1$ since $1x1 = 10 \lor 1x0 = 11$ would follow, contradicting $BT_3$. It follows from the definition of $B_1$ that we have

$$\forall x \in B_1 [ x0 \neq 0 \land x1 \neq 1 ].$$

Hence, $0, 1 \in I_1$. We now show that $I_1$ is closed under $\circ$. Suppose $y, y' \in I_1$, $x \in B_1$ and $x(yy') = 0 \lor x(yy') = 1$. Since $y \in I_1 \subseteq B_1$ and $B_1$ is closed under $\circ$, we have $xy \in B_1$. By $BT_1$, we have have $(xy)y' = 0 \lor (xy)y' = 1$, contradicting $y' \in I_1$. Hence, $yy' \in I_1$. Thus, $I_1$ is closed under $\circ$. This means that $I_1$ satisfies the domain conditions.

We interpret $BT_{\leq 1}$ in $BT - BT_4$ by simply relativizing quantification to $I_1$. Since $BT_1$, $BT_2$ and $BT_3$ are universal sentences that are theorems of $BT - BT_4$, they hold in $I_1$. It remains to show that $BT - BT_4$ proves the translation of

1. $\forall x [ x \sqsubseteq 0 \rightarrow x = 0 ]$
2. $\forall x [ x \sqsubseteq 1 \rightarrow x = 1 ].$
We only show (1). The case (2) is handled similarly. So, suppose \( x \in I_1 \) and the translation of \( x \sqsubseteq s 0 \) holds. We then have one of the following cases for some \( z, w \in I_1 \):

(i) \( 0 = x \)
(ii) \( 0 = zx \)
(iii) \( 0 = xw \)
(iv) \( 0 = (zx)w \)
(v) \( 0 = z(xw) \).

We notice that (ii)-(v) contradict the definition of \( I_1 \). We thus see that the translation of (1) is a theorem of \( BT^{-BT_4} \). Similarly, the translation of (2) is a theorem of \( BT^{-BT_4} \).

**Construction of \( I_{n+1} \):**

Let \( I_n \) denote the set of all nonempty biterals of length at most \( n \). Let \( \sqsubseteq_s I_n \) denote the realization of \( \sqsubseteq_s \) in \( I_n \), i.e.

\[
x \sqsubseteq_s y \equiv y = x \lor \exists u v \in I_n \lt y = ux \lor y = xv \lor y = u xv \gt .
\]

We assume \( I_n \) has been constructed and satisfies

\[
\bigwedge_{\alpha \in I_n} \forall x \in I_n \lt x \sqsubseteq_s \overline{\alpha} \to \bigvee_{\gamma \in \langle ..\alpha \rangle} x = \overline{\gamma} \gt .
\]

We let

\[
I_{n+1} = \{ y \in I_n : \bigwedge_{\alpha \in I_n} \lt \forall x \in I_n \lt xy = \overline{\alpha}0 \to \bigvee_{\gamma, \delta \in \langle \ldots \alpha 0 \rangle} x = \overline{\gamma} \land y = \overline{\delta} \gt \gt \land \forall x \in I_n \lt xy = \overline{\alpha}1 \to \bigvee_{\gamma, \delta \in \langle \ldots \alpha 1 \rangle} x = \overline{\gamma} \land y = \overline{\delta} \gt \gt \gt \} .
\]

By \( BT_3 \), we cannot have \( x1 = \overline{\alpha}0 \lor x0 = \overline{\alpha}1 \). By \( BT_2 \), we have

\[
x0 = \overline{\alpha}0 \Rightarrow x = \overline{\alpha}
\]

and

\[
x1 = \overline{\alpha}1 \Rightarrow x = \overline{\alpha}.
\]

Hence, \( 0, 1 \in I_{n+1} \). We now show that \( I_{n+1} \) is closed under \( \circ \). Suppose \( y_0, y_1 \in I_{n+1} \). Let \( x \in I_n \). We observe that \( xy_0 \in I_n \) since \( x, y_0 \in I_n \) and \( I_n \) is closed under \( \circ \). By \( BT_1 \), we have \( x(y_0 y_1) = (xy_0) y_1 \). Then

\[
x(y_0 y_1) = \overline{\alpha}0 \Rightarrow \bigvee_{\gamma, \delta \in \langle \ldots \alpha 0 \rangle} (xy_0 = \overline{\gamma} \land y_1 = \overline{\delta}) \quad (y_1 \in I_{n+1} \text{ and } xy_0 \in I_n)
\]

\[
\Rightarrow \bigvee_{\zeta, \eta, \delta \in \langle \ldots \alpha 0 \rangle} (x = \overline{\zeta} \land y_0 = \overline{\eta} \land y_1 = \overline{\delta}) \quad (x, y_0 \in I_n \text{ and } \gamma \in I_n)
\]

and

\[
x(y_0 y_1) = \overline{\alpha}1 \Rightarrow \bigvee_{\gamma, \delta \in \langle \ldots \alpha 1 \rangle} (xy_0 = \overline{\gamma} \land y_1 = \overline{\delta}) \quad (y_1 \in I_{n+1} \text{ and } xy_0 \in I_n)
\]

\[
\Rightarrow \bigvee_{\zeta, \eta, \delta \in \langle \ldots \alpha 1 \rangle} (x = \overline{\zeta} \land y_0 = \overline{\eta} \land y_1 = \overline{\delta}) \quad (x, y_0 \in I_n \text{ and } \gamma \in I_n).
\]
It is not difficult to see that $BT - BT_4$ proves each instance of the axiom schemas

\begin{align*}
WBT_1 \quad &\overline{\alpha} \overline{\beta} = \overline{\alpha \beta} \\
WBT_2 \quad &\overline{\alpha} \neq \overline{\beta} \text{ if } \alpha \neq \beta.
\end{align*}

This implies

$$x(\langle \alpha_0, \alpha_1 \rangle) = \overline{\alpha} \Rightarrow \bigvee_{\beta, \zeta \in \langle \alpha \rangle} (x = \overline{\zeta} \wedge y_0 y_1 = \overline{\beta}).$$

and

$$x(\langle \alpha_0, \alpha_1 \rangle) = \overline{\alpha} \Rightarrow \bigvee_{\beta, \zeta \in \langle \alpha \rangle} (x = \overline{\zeta} \wedge y_0 y_1 = \overline{\beta}).$$

Hence, $\langle \alpha_0, \alpha_1 \rangle \in I_{n+1}$. Thus, $I_{n+1}$ is closed under $\circ$. This proves that $I_{n+1}$ satisfies the domain conditions. It is clear that we get an interpretation of $BT_{\leq n+1}$ in $BT - BT_4$ by simply relativizing quantification to $I_{n+1}$. \hfill \Box

By a similar argument, we see that $D$ is not interpretable in $C$.

**Theorem 3** \textit{D is not interpretable in C.}

The theories $C$ and $BT$ are strictly between $R$ and $Q$ w.r.t. interpretability. A natural question is whether they are mutually interpretable. It is not difficult to see that our interpretation of $WD$ in $WBT$ given at the end of Sect. 4 is also an interpretation of $C$ in $BT$. It is not clear however whether $BT$ is interpretable in $C$.

**Open Problem 1** \textit{Is BT interpretable in C?}

### 4 The theory WD

We now show that the theory $WD$ is mutually interpretable with $R$. We give an interpretation of $R$ in $WD$. To show that $WD$ is interpretable in $R$ we invoke the result of Visser [20]: a recursively enumerable theory is interpretable in $R$ if and only if it is locally finitely satisfiable (each finite sub-theory has a finite model). We also show how to interpret $WD$ in $WBT$, from which follows that $WD$ is interpretable in $R$ since $WBT$ and $WTC^{-\varepsilon}$ are equivalent, and $WTC^{-\varepsilon}$ and $R$ are mutually interpretable (see Fig. 2). The theory $WD$ is purely universally axiomatised, in contrast to $WTC^{-\varepsilon}$ which has $\Pi_2$-axioms.

**Definition 1** The first-order theory $WD$ contains the following non-logical axioms: for each $\alpha, \beta \in \{0, 1\}^+$

\begin{align*}
WD_1 \quad &\overline{\alpha} \overline{\beta} = \overline{\alpha \beta} \\
WD_2 \quad &\overline{\alpha} \neq \overline{\beta} \text{ if } \alpha \neq \beta \\
WD_3 \quad &\forall x \ [ x \leq \overline{\alpha} \iff \bigvee_{\gamma \leq \overline{\alpha}} x = \overline{\gamma} ].
\end{align*}
We start by showing that a theory of arithmetic we call $R^-$ is interpretable in $WD$. The theory $R^-$ has been shown to be mutually interpretable with $R$ (see Jones and Shepherdson [8]).

**Definition 2** The first-order theory $R^-$ contains the following non-logical axioms: for each $n, m \in \mathbb{N}$

\[
R^\sim_1 \quad \bar{n} \times \bar{m} = \bar{nm} \\
R^\sim_2 \quad \bar{n} \neq \bar{m} \quad \text{if } n \neq m \\
R^\sim_3 \quad \forall x \left[ x \leq \bar{n} \leftrightarrow \bigvee_{k \leq n} x = k \right].
\]

The most difficult step in interpreting $R^-$ in $WD$ is translating $\times$. We start by proving two lemmas which show how we intend to translate $\times$. In the structure $\mathcal{D}$, we can associate each natural number $k$ with the bit string $1^k0$. For $n \neq 0$ and $m > 1$, we can then translate

$$\bar{n} \times \bar{m} = \bar{nm}$$

by saying that the sequence

$$\left(1^n, 1^{n+n}, 1^{n+n+n}, \ldots, 1^{nm}\right)$$

exists. The formulas given in Lemmas 1 and 2 try to capture this way of viewing multiplication. But we have to be careful since we are reasoning in a weak theory. For example, we lack full associativity. For readability, we dispense with parentheses whenever possible and expect terms to be read from right to left. That is, $xyz$ should be regarded as shorthand for $(xy)z$. Although we do not have full associativity, by $WD_1$, we have $WD \vdash (\bar{\alpha} \bar{\beta})\bar{\gamma} = \bar{\alpha}(\bar{\beta}\bar{\gamma})$ for all $\alpha, \beta, \gamma \in \{0, 1\}^+$. Then, by $WD_3$, we have

$$WD \vdash \forall x \leq \bar{\alpha} \forall y \leq \bar{\beta} \forall z \leq \bar{\gamma} \left[ (xy)z = x(yz) \right].$$

The bounded quantifiers in Lemmas 1 and 2 are there to make full use of the axiom scheme $WD_3$.

**Lemma 1** Let $n \neq 0$ and $m \geq 2$. Then, $WD$ proves

1. $1^{nm} \leq 1^{nm}$ and $1^n, 1^n$ and $1^{nm}$ are the unique elements such that

   a. $1^n \leq 1^n0 \land 1^n0 = 1^n0$
   b. $1^m \leq 1^m0 \land 1^m0 = 1^m0$
   c. $1^{nm} \leq 1^{nm}0 \land 1^{nm}0 = 1^{nm}0$.

Furthermore, let

$$\omega \equiv 0001^n01001^{n+n}011001^{n+n+n}011100 \ldots 001^{nm}01^{m}00.$$ 

Then, $\omega$ provably (in $WD$) satisfies:
\[
\exists u \leq w \left[ u001^{nm}01^{m}00 \leq w \right]
\]

(3) it holds that \((\forall w' \leq w)(\forall y' \leq \overline{1}^{m})(\forall z' \leq \overline{1}^{nm}) \Psi_0(w', y', z')\) where

\[
\Psi_0(w', y', z') \equiv (y' \neq 1 \land w'00z'0y'y'00 \leq w) \rightarrow (\exists w'' \leq w)(\exists y'' \leq y') (\exists z'' \leq \overline{1}^{nm}) [z' = z''\overline{1}^{n} \land y' = y''1 \land w''00z''0y''00 \leq w \land \Phi_0(y'', z'') ]
\]

and

\[
\Phi_0(y'', z'') \equiv y'' = 1 \rightarrow z'' = \overline{1}^{n}.
\]

**Proof** Since we have the axiom schema WD\(_1\), we can skip parentheses in \(w\). Clause (1) holds due to the axiom schemas WD\(_3\) and WD\(_2\). We verify clause (a). By the right-left implication of WD\(_3\), we have \(\overline{1}^{n} \leq \overline{1}^{n}0\). By how biterals are defined, we have \(\overline{1}^{n}0 = \overline{1}^{n}0\). For uniqueness, suppose

\[
x \leq \overline{1}^{n}0 \land x0 = \overline{1}^{n}0.
\]

By WD\(_3\), we have

\[
x \leq \overline{1}^{n}0 \rightarrow \bigvee_{y \leq \overline{1}^{n}0} x = \overline{y}.
\]

Then, by WD\(_2\), in order for \(x0 = \overline{1}^{n}0\) to hold we must have \(x = \overline{1}^{n}\). By similar reasoning, one verifies clauses (b) and (c).

Clause (2) holds due to the axiom schema WD\(_1\) and the right-left implication of WD\(_3\). Let

\[
u = 000\overline{1}^{n}0100\overline{1}^{n+n}01100\overline{1}^{n+n+n}011100\ldots 001^{n(m-1)}01^{m-1}.
\]

By the right-left implication of WD\(_3\), we have \(u \leq w\). By WD\(_1\), we have

\[
u001^{nm}01^{m}00 = w.
\]

Then, by the right-left implication of WD\(_3\), we have

\[
u001^{nm}01^{m}00 \leq w.
\]

To see that clause (3) holds, suppose

\[
w' \leq w \land z' \leq \overline{1}^{nm} \land y' \leq \overline{1}^{m} \land y' \neq 1 \land w'00z'0y'y'00 \leq w \tag{*}
\]

holds. By the axiom schemas WD\(_3\) and WD\(_2\), the third and fourth conjunct of (*) imply \(y' = \overline{1}^{k+1}\) where \(2 \leq k + 1 \leq m\). Given \(w' \leq w \land z' \leq \overline{1}^{nm}\), the axiom schema WD\(_3\) gives us a set \(\Gamma\) of pairs \((a, b)\) where \(a \leq w\) and \(b \leq \overline{1}^{nm}\) such that \((w', z') \in \Gamma\). For
each pair \((a, b) \in \Gamma\), we use WD\(_1\) to compute \(a00b0y'00\) and then use the left-right implication of WD\(_3\) and WD\(_2\) to determine whether \(a00b0y'00 \leq w\). We are then led to the conclusion

\[
\begin{align*}
- w' &= 00111001^{n+n+1}011001^{n+n+1}011100 \ldots 001^{n+k}01^k \\
- z' &= 1^{n(k+1)}.
\end{align*}
\]

By WD\(_1\) and the right-left implication of WD\(_3\), we have

(I) \(w' = 00111001^{n+n+1}011001^{n+n+1}011100 \ldots 001^{n+k}01^k \leq w\)

(II) \(z' = 1^{n(k+1)} = 1^{nk}/1^n\)

(III) \(y' = 1^{k+1} = 1^k 1\).

We can then let

\[
\begin{align*}
- w'' &= 00111001^{n+n+1}011001^{n+n+1}011100 \ldots 001^{n(k-1)}01^{k-1} \\
- z'' &= 1^{nk} \land y'' = 1^k.
\end{align*}
\]

and \(z'' = 1^{nk} \land y'' = 1^k\). If \(k = 1\), we let \(w'' = 0\). If \(1^k = 1\), then by WD\(_2\) we must have \(k = 1\). We thus see that clause (3) holds.

\(\square\)

Lemma 2

Let

\[
\phi_M(X, Y, Z) \equiv \big( (X = 0 \lor Y = 0) \land Z = 0 \big) \lor \big( Y = 10 \land Z = X \big) \lor (X \neq 0 \land Y \neq 0 \land Y \neq 10 \land \exists X \leq X \exists Y \leq Y \exists Z \leq Z \exists w \\
X = x0 \land Y = y0 \land Z = z0 \land z \leq z \\
\land \exists u \leq w \{ u00z0y00 \leq w \} \\
\land (\forall w' \leq w)(\forall y' \leq y)(\forall z' \leq z) (\Psi(w', y', z', x, w) \big)\)
\]

where

\[
\Psi(w', y', z', x, w) \equiv (y' \neq 1 \land w'00z'0y'00 \leq w) \\
\rightarrow (\exists w'' \leq w)(\exists y'' \leq y')(\exists z'' \leq z) \\
[ z' = z''x \land y' = y''1 \land w''00z''0y''00 \leq w \land (\Phi(y'', z'', x) ]
\]

and

\[
\Phi(y'', z'', x) \equiv y'' = 1 \rightarrow z'' = x.
\]

Then, for each natural number \(n\) and \(m\), we have

\[
WD \vdash \forall Z [ \phi_M(1^{n0}, 1^{m0}, Z) \leftrightarrow Z = 1^{nm0} ].
\]

Proof We start by showing that

\[
WD \vdash \forall Z [ Z = 1^{nm0} \rightarrow \phi_M(1^{n0}, 1^{m0}, Z) ] . \quad (**)
\]

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– If \( n = 0 \) or \( m = 0 \), then the first disjunct in the definition of \( \phi_M(X, Y, Z) \) holds.
– If \( m = 1 \), then the second disjunct in the definition of \( \phi_M(X, Y, Z) \) holds.
– If \( n \neq 0 \) and \( m \geq 2 \), then by WD\(_1\) and WD\(_2\) we have

\[
\overline{1^n0} = \overline{1^n0} \neq 0 \land \overline{1^m0} = \overline{1^m0} \neq 0 \land \overline{1^m0} \neq 10. 
\]

Then, by Lemma 1, the third disjunct in the definition of \( \phi_M(X, Y, Z) \) holds since we can choose

\[
w = 000\overline{1^n0}100\overline{1^n0}1100\ldots 00\overline{1^m0}1000. 
\]

Thus, (***) holds.

We now prove that

\[
\text{WD} \vdash \forall Z \left[ \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \rightarrow Z = \overline{1^m0} \right]. \quad (***)
\]

So, suppose \( \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \) holds. If \( n = 0 \) or \( m \leq 1 \), then it follows from the definition of \( \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \) that \( Z = \overline{1^m0} \). We therefore assume that \( n \neq 0 \) and \( m > 1 \). Then, there exist \( x \leq \overline{1^n0}, y_1 \leq \overline{1^m0} \) and \( z_1 \leq Z \) such that

\[
\overline{1^n0} = x0 \land \overline{1^m0} = y_10 \land Z = z_10 \land z_1 \leq z_1. 
\]

By WD\(_3\) and WD\(_2\), we have

\[
x = \overline{1^n} \land y_1 = \overline{1^m}. 
\]

Furthermore, by the definition of \( \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \), there exist \( w \) and \( w_1 \leq w \) such that

\[
w_10z_10y_100 \leq w. 
\]

By WD\(_1\), it suffices to show that \( z_1 = \overline{1^m} \) to conclude that \( Z = \overline{1^m0} \). By the axiom schema WD\(_2\), we have \( \overline{1^m} \neq 1 \) since \( m > 1 \). By the axiom schema WD\(_3\), we have \( \overline{1^m} \leq \overline{1^m} \). Then, \( \Psi(w_1, y_1, z_1, x, w) \) in the definition of \( \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \) implies that there exist \( w_2 \leq w, y_2 \leq y_1 \) and \( z_2 \leq z_1 \) such that

\[
z_1 = z_2\overline{1^n} \land y_1 = y_21 \land w_200z_20y_200 \leq w. 
\]

By the axiom schemas WD\(_3\) and WD\(_2\), we must have

\[
y_2 = \overline{1^m-1}. 
\]

If \( m - 1 = 1 \), then \( \Phi(y_2, z_2, x) \) in the definition of \( \phi_M(\overline{1^n0}, \overline{1^m0}, Z) \) implies \( z_2 = \overline{1^n} \).

By WD\(_1\), this implies

\[
z_1 = z_2\overline{1^n} = \overline{1^n} \overline{1^n} = \overline{1^n+n} = \overline{1^m}.
\]

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since \( m = 2 \). If \( m - 1 \neq 1 \), we use \( \Psi(w_2, y_2, z_2, x, w) \) to conclude that there exist
\[
w_3 \leq w, \ y_3 \leq y_2 \text{ and } z_3 \leq z_1
\]
such that
\[
z_2 = z_3 \overline{1^n} \land y_2 = y_3 1 \land w_3 0 0 z_3 0 y_3 0 0 \leq w.
\]
By the axiom schemas WD\(_3\) and WD\(_2\), we must have
\[
y_3 = \overline{1^{m-2}}.
\]
If \( m - 2 = 1 \), then by \( \Phi(y_3, z_3, x) \) we have \( z_3 = \overline{1^n} \). By WD\(_1\), this implies
\[
z_1 = z_2 \overline{1^n} = (z_3 \overline{1^n}) \overline{1^n} = \overline{1^n} \overline{1^n} = \overline{1^{n+n}} = \overline{1^{nm}}
\]
since \( m = 3 \). If \( m - 2 \neq 1 \), we use \( \Psi(w_3, y_3, z_3, x, w) \) to repeat this procedure. We notice that after a finite number of steps, we have that there exist \( w_m \leq w, y_m \leq y_{m-1} \) and \( z_m \leq z_1 \) such that
\[
z_{m-1} = z_m \overline{1^n} \land y_{m-1} = \overline{1^{n-(m-2)}} \land y_{m-1} = y_m 1 \land w_m 0 0 z_m 0 y_m 0 0 \leq w.
\]
By the axiom schemas WD\(_3\) and WD\(_2\), we must have
\[
y_m = 1.
\]
The condition \( \Phi(y_m, z_m, x) \) in \( \Psi(w_{m-1}, y_{m-1}, z_{m-1}, x, w) \) then implies
\[
z_m = \overline{1^n}.
\]
By backtracking, we observe that
\[
z_1 = \overline{1^n} \overline{1^n} \ldots \overline{1^n}
\]
where \( \overline{1^n} \) occurs \( m \) times. It then follows from the axiom schema WD\(_1\) that \( z_1 = \overline{1^{nm}} \).
Thus, by WD\(_1\), we have \( Z = \overline{1^{nm}} 0 \). Hence, (***) holds. \( \square \)

**Theorem 4** \( R^- \) is interpretable in WD.

**Proof** We choose the domain \( \delta(x) \equiv x = x \). We translate, 0, S, \( \times \) and \( \leq \) as follows:
\[
\begin{align*}
\phi_0(x) & \equiv x = 0 \\
\phi_S(x, y) & \equiv y = 1 x \\
\phi_\times(x, y, z) & \equiv (\exists! w [ \phi_M(x, y, w) ] \land \phi_M(x, y, z)) \lor \\
& \quad (\neg \exists! w [ \phi_M(x, y, w) ] \land z = 0)
\end{align*}
\]
\[ \phi_\leq(x, y) \equiv x = 0 \lor \exists z \exists w \leq y \left[ x = z \cdot 0 \land y = w \cdot 0 \land z \leq w \right]. \]

The formula \( \phi_M \) is defined in Lemma 2. By the axiom schema \( \text{WD}_1 \), our translation of \( S \) implies that we translate each numeral \( \overline{n} \) as the biteral \( \overline{1^n0} \). By the axiom schema \( \text{WD}_2 \), the translation of each instance of the axiom schema \( R^{-2} \) is a theorem of \( \text{WD} \).

We now show that the translation of each instance of the axiom schema \( R^{-1} \) is a theorem of \( \text{WD} \). By Lemma 2, for each natural number \( n \) and \( m \), we have

\[
\text{WD} \vdash \exists! w \left[ \phi_M(\overline{1^n0}, \overline{1^m0}, w) \right] \land \phi_M(\overline{1^n0}, \overline{1^m0}, \overline{1^{nm}0}).
\]

Hence

\[
\text{WD} \vdash \phi_x(\overline{1^n0}, \overline{1^m0}, \overline{1^{nm}0}).
\]

Thus, the translation of each instance of the axiom schema \( R^{-1} \) is a theorem of \( \text{WD} \). We observe that \( \text{WD} \) can have models where not all elements in the universe are realization of terms of the form \( 1^k0 \). Hence, the condition

\[
\neg \exists! w \left[ \phi_M(x, y, w) \right] \land z = 0
\]

in the definition of \( \phi_x(x, y, z) \) ensures that \( \times \) is translated as a total function.

We now show that the translation of each instance of the axiom schema \( R^{-3} \) is a theorem of \( \text{WD} \). We have

\[
\phi_\leq(x, \overline{1^n0}) \iff x = 0 \lor \exists z \exists w \leq \overline{1^n0} \left[ x = z \cdot 0 \land \overline{1^n0} = w \cdot 0 \land z \leq w \right]\]
\[
\iff x = 0 \lor \exists z \left[ x = z \cdot 0 \land \overline{1^n0} = \overline{1^n} \right] \quad \text{(by \( \text{WD}_3, \text{WD}_2 \))}
\]
\[
\iff x = 0 \lor \bigvee_{y \leq \overline{1^n0}} x = \overline{y0} \quad \text{(by \( \text{WD}_2 \))}
\]
\[
\iff x = 0 \lor \bigvee_{y \leq \overline{1^n0}} x = \overline{y0}.
\]

The last equivalence is due to how biterals are defined. We thus see that the translation of each instance of the axiom schema \( R^{-3} \) is a theorem of \( \text{WD} \).

\[ \Box \]

**Theorem 5** \( \text{WD} \) and \( R \) are mutually interpretable.

**Proof** We have shown that \( R^{-} \) is interpretable in \( \text{WD} \). Since \( R^{-} \) and \( R \) are mutually interpretable, \( R \) is interpretable in \( \text{WD} \). To see that \( \text{WD} \) is interpretable in \( R \), we first observe that \( \text{WD} \) is locally finitely satisfiable, i.e., any finite subset of the axioms has a finite model. In [20], Albert Visser shows that a recursively enumerable theory is interpretable in \( R \) if and only if it is locally finitely satisfiable. Hence, \( \text{WD} \) is interpretable in \( R \). Thus, \( \text{WD} \) and \( R \) are mutually interpretable. \[ \Box \]

We could have shown that \( \text{WD} \) is interpretable in \( R \) by showing that it is interpretable in \( \text{WTC}^{-\varepsilon} \), which we know is mutually interpretable with \( R \). Since we in our Master’s thesis [12] show that \( \text{WBT} \) and \( \text{WTC}^{-\varepsilon} \) are equivalent, it suffices to show how to interpret \( \text{WD} \) in \( \text{WBT} \). We choose the domain \( \delta(x) \equiv x = x \). We translate \( 0, 1, \circ \) and \( \leq \) as follows:

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\[ \begin{align*}
\phi_0(x) &\equiv x = 0 \\
\phi_1(x) &\equiv x = 1 \\
\phi_0(x, y, z) &\equiv z = xy \\
\phi_<= (x, y) &\equiv y = x \lor \exists z \ [ y = xz ] .
\end{align*} \]

Clearly, the translation of each instance of the axiom schemas WD_1 and WD_2 is a theorem of WBT. We now show that the translation of each instance of the axiom schema WD_3 is a theorem of WBT. We have

\[ \phi_<= (x, \alpha) \iff \alpha = x \lor \exists z \ [ \alpha \subseteq 5 \alpha \land z \subseteq 5 \alpha \land \alpha = xz ] \] (by def. of \( \subseteq 5 \))

\[ \iff \alpha = x \lor \bigvee_{\beta, \gamma \in \ldots \alpha} (x = \beta \land z = \gamma \land \beta \gamma = \alpha) \] (WBT_3)

\[ \iff \bigvee_{\beta \leq^c 5 \alpha} x = \beta \] (WBT_1, WBT_2).

We thus see that the translation of each instance of the axiom schema WD_3 is a theorem of WBT. Hence, WD is interpretable in WBT.

5 The theory D

We now show that the theory D is mutually interpretable with Q. In contrast to Q and TC^{-\varepsilon}, which have \( \Pi_2 \)-axioms, the theory D is purely universally axiomatised. When interpreting one theory into another, handling existential quantifiers can become cumbersome. This is clearly illustrated in the proof of Theorem 6. Therefore, having a theory with purely universal axiomatization and that is mutually interpretable with Q could be advantageous in some circumstances.

**Definition 3** The first-order theory D is defined by the following non-logical axioms:

\[ \begin{align*}
D_1 &\forall xyz \ [ (xy)z = x(yz) ] \\
D_2 &\forall xy \ [ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1 ) ] \\
D_3 &\forall xy \ [ x0 \neq y1 ] \\
D_4 &\forall x \ [ x \leq 0 \iff x = 0 ] \\
D_5 &\forall xy \ [ x \leq y0 \iff (x = y0 \lor x \leq y ) ] \\
D_6 &\forall xy \ [ x \leq y1 \iff (x = y1 \lor x \leq y ) ] .
\end{align*} \]

For D to be an extension of WD, we need the axiom \( \forall x \ [ x \leq 1 \iff x = 1 ] \). The theory D extended with this axiom is what we call D at page3. The next lemma shows why we have decided to not include this axiom. We could very well have replaced D_4 by \( \forall x \ [ x \leq 1 \iff x = 1 ] \). The proof of the next lemma also illustrates some of the advantages of not having to worry about existential quantifiers when defining a domain.

**Lemma 3** \( D + \forall x \ [ x \leq 1 \iff x = 1 ] \) is interpretable in D.
**Proof** We translate 0, 1, $\circ$ and $\leq$ as

\[
\begin{align*}
\phi_0(x) & \equiv x = 0100 \\
\phi_1(x) & \equiv x = 0110 \\
\phi_{\leq}(x, y) & \equiv x \leq y \\
\phi_{\circ}(x, y, z) & \equiv xy = z .
\end{align*}
\]

We choose the domain

\[
I = \{x : x = 0100 \lor x = 0110 \lor \exists y \ [ x = y0100 \lor x = y0110 ] \} .
\]

Clearly, $I$ contains 0100, 0110 and is closed under $\circ$. We use $D_1$ when showing that $I$ is closed under $\circ$. We now proceed to show that the translation of each non-logical axiom of $D + \forall x \ [ x \leq 1 \iff x = 1 ]$ is a theorem of $D$.

By $D_1$ and the definition of $\phi_{\circ}(x, y, z)$, the translation of $D_1$ is a theorem of $D$. The translation of $D_2$ is a theorem of $D$ since by $D_2$ we have

\[
x0100 = y0100 \lor x0110 = y0110 \Rightarrow x = y .
\]

The translation of $D_3$ is also a theorem of $D$ since by $D_2$ we have

\[
x0100 = y0110 \Rightarrow x010 = y011
\]

and $x010 = y011$ contradicts $D_3$.

We now show that the translation of $D_4$ is a theorem of $D$. We have

\[
\begin{align*}
x \leq 0100 & \iff x = 0100 \lor x = 010 \lor x \leq 01 \quad \text{(by $D_5$)} \\
& \iff x = 0100 \lor x = 010 \lor x = 01 \lor x \leq 0 \quad \text{(by $D_6$)} \\
& \iff x = 0100 \lor x = 010 \lor x = 01 \lor x = 0 \quad \text{(by $D_4$)} .
\end{align*}
\]

We need to show that we cannot have

\[
x = 010 \lor x = 01 \lor x = 0 .
\]

Since $x \in I$, by the definition $I$, we have that $x = z0100 \lor x = z0110$ where $z$ could possibly be empty ($x = zu$ with $z$ empty means $x = u$ since we do not have an empty string in $D$). We have

\[
\begin{align*}
z0100 & = 010 \lor z0100 = 01 \lor z0100 = 0 \\
\downarrow
\end{align*}
\]

\[
\begin{align*}
z0100 & = 010 \lor z0100 = 01 \lor 11z0100 = 110 \\
\downarrow \text{ (by $D_2$) }
\end{align*}
\]

\[
z010 = 01 \lor z0100 = 01 \lor 11z0100 = 11
\]

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contradicting $D_3$. Hence, $x \preceq 0100$ if and only if $x = 0100$. Thus, the translation of $D_4$ is a theorem of $D$. By similar reasoning, the translation of $\forall x [ x \leq 1 \iff x = 1 ]$ is a theorem of $D$.

We now show that the translation of $D_5$ is a theorem of $D$. By $D_5$, we have

$$x \preceq y0100 \iff x = y0100 \lor x = y010 \lor x \preceq y01 .$$

By $D_6$, we have

$$x \preceq y0100 \iff x = y0100 \lor x = y010 \lor x = y0 \lor x \preceq y0 .$$

By $D_5$, we have

$$x \preceq y0100 \iff x = y0100 \lor x = y010 \lor x = y01 \lor x = y0 \lor x \preceq y .$$

We need to show that we cannot have

$$x = y010 \lor x = y01 \lor x = y0 .$$

Since $x, y \in I$, by the definition $I$, we have

$$x = z0100 \lor x = z0110 \text{ and } y = w0100 \lor y = w0110$$

where $z$ and $w$ could possibly be empty. Reasoning as in the preceding paragraph shows that $x = y010 \lor x = y01 \lor x = y0$ leads to a contradiction. Hence,

$$x \preceq y0100 \iff x = y0100 \lor x \preceq y .$$

Thus, the translation of $D_5$ is a theorem of $D$. By similar reasoning, the translation of $D_6$ is a theorem of $D$.

Since $D$ proves the translation of each axiom of $D + \forall x [ x \leq 1 \iff x = 1 ]$, it follows that $D + \forall x [ x \leq 1 \iff x = 1 ]$ is interpretable in $D$. $\square$

We now proceed to show that $Q$ and $D$ are mutually interpretable. We do this indirectly by showing that $D$ is mutually interpretable with the theory $BTQ$ which we have seen is mutually interpretable with $Q$.

**Theorem 6** $BTQ$ is interpretable in $D$. 
Proof By Lemma 3, it suffices to show that BTQ is interpretable in $D + D'_4$ where

$$D'_4 \equiv \forall x \ [ x \leq 1 \leftrightarrow x = 1 ] .$$

We also observe that $D$ proves the axioms BTQ$_1$, BTQ$_2$ and BTQ$_3$ (the axioms are identical with $D_1, D_2, D_3$). So, to translate BTQ in $D + D'_4$, we simply define a domain $K$ such that the axiom

$$\text{BTQ}_4 \equiv \forall x \ [ x = 0 \lor x = 1 \lor \exists y \ [ x = y0 \lor x = y1 ] ]$$

holds restricted to $K$. Before defining $K$, we define auxiliary classes $A \supseteq B \supseteq C \supseteq I \supseteq J \supseteq K$. We need to ensure that there is $y \in K$ such that $x = y0$ or $x = y1$ if $x \in K$ and $x \neq 0, 1$. We do this by relying on $\leq$. The idea is to first let $I$ be such that if $x \in I$ and $x \neq 0, 1$; then we can find $y \leq x$ such that $x = y0$ or $x = y1$. It will not necessarily be the case that $y \in I$. What we then do is to restrict $I$ so that we have a subclass $K$ that is downward closed under $\leq$, that is, $x \in K$ and $y \leq x$ implies $y \in K$. Since $K$ is a subclass of $I$, this immediately ensures that $\text{BTQ}_4$ holds in $K$. We realize that in order for $I$ to be closed under $\circ$, it is useful if for all $x_0, x_1 \in I$, we have that $x_0 \leq x_0 x_1$ and that $y \leq x_1$ implies $x_0 y \leq x_0 x_1$. We therefore let $I$ be a subclass of a class $C$ with this property.

We let

$$A = \{ y : \forall x \ [ xy \leq xy ] \} .$$

By $D_5$ and $D_6$, we have $D \vdash \forall x \ [ x0 \leq x0 \land x1 \leq x1 ]$. Hence, $0, 1 \in A$. We now show that $A$ is closed under $\circ$. Suppose $y_0, y_1 \in A$. Since $y_1 \in A$, we have

$$\forall x \ [ (xy_0)y_1 \leq (xy_0)y_1 ] .$$

By $D_1$, we then have $\forall x \ [ x(y_0y_1) \leq x(y_0y_1) ]$. Hence, $y_0y_1 \in A$. Thus, $A$ is closed under $\circ$.

We let

$$B = \{ x \in A : x \leq x \} .$$

By $D_4$ and $D'_4$, we have $D \vdash 0 \leq 0 \land 1 \leq 1$. Hence, $0, 1 \in B$. We now show that $B$ is closed under $\circ$. Suppose $x_0, x_1 \in B$. Since $x_1 \in B \subseteq A$, we have $x_0 x_1 \leq x_0 x_1$. Hence, $x_0 x_1 \in B$. Thus, $B$ is closed under $\circ$.

We let

$$C = \{ z \in B : \forall xy \ [ x \leq yz \leftrightarrow ( x \leq y \lor \exists u \leq z [ x = yu ] ) ] \} .$$

By $D_5$ and $D_4$, we have

$$x \leq y0 \ \Leftrightarrow \ x = y0 \lor x \leq y \ \Leftrightarrow \ x \leq y \lor \exists u \leq 0 [ x = yu ] .$$
Similarly, by $D_6$ and $D'_4$, we have

\[ x \leq y1 \iff x = y1 \lor x \leq y \]
\[ \iff x \leq y \lor \exists u \leq 1 \left[ x = yu \right]. \]

Hence, $0, 1 \in C$. Next we show that $C$ is closed under $\circ$. Suppose $z_0, z_1 \in C$. Then

\[ x \leq y(z_0z_1) \iff x \leq (yz_0z_1) \tag{D_1} \]
\[ \iff x \leq yz_0 \lor \exists u_1 \leq z_1 \left[ x = (yz_0)u_1 \right] \tag{z_1 \in C} \]
\[ \iff x \leq y \lor \exists u_0 \leq z_0 \left[ x = yu_0 \right] \lor \exists u_1 \leq z_1 \left[ x = y(z_0u_1) \right] \tag{z_0 \in C} \]
\[ \iff x \leq y \lor \exists u \leq z_0z_1 \left[ x = yu \right] \tag{z_1 \in C}. \]

We justify the last equivalence as follows:

($\Rightarrow$) Suppose $u_0 \leq z_0 \land x = yu_0$. Since $z_1 \in C$ and $u_0 \leq z_0$, the right-left implication in the definition of $C$ tells us that $u_0 \leq z_0z_1$. We can thus let $u = u_0$. Suppose now $u_1 \leq z_1 \land x = y(z_0u_1)$. Since $u_1 \leq z_1$ and $z_1 \in C$, the right-left implication in the definition of $C$ tells us that $z_0u_1 \leq z_0z_1$. We can thus let $u = z_0u_1$.

($\Leftarrow$) Suppose $u \leq z_0z_1 \land x = yu$. Since $z_1 \in C$, by the left-right implication in the definition of $C$, we have that $u \leq z_0$ or there exists $u' \leq z_1$ such that $u = z_0u'$. If $u \leq z_0$, then $u_0 \leq z_0 \land x = yu_0$ by setting $u_0 = u$. If $u' \leq z_1 \land u = z_0u'$, then $u_1 \leq z_1 \land x = y(z_0u_1)$ by setting $u_1 = u'$.

Hence, $z_0z_1 \in C$. Thus, $C$ is closed under $\circ$.

We let

\[ I = \left\{ x \in C : x = 0 \lor x = 1 \lor \exists y \leq x \left[ x = y0 \lor x = y1 \right] \right\}. \]

By definition of $I$, we have $0, 1 \in I$. We now show that $I$ is closed under $\circ$. Suppose $x_0, x_1 \in I$. Since $x_1 \in I$, we have one of the following cases:

1. $x_1 = 0 \lor x_1 = 1$
2. $x_1 = y10 \lor x_1 = y11$ for some $y1 \leq x_1$.

We first consider (1). Since $x_0 \in C \subseteq B$, we have $x_0 \leq x_0$. Since $x_1 \in C$, this implies $x_0 \leq x_0x_1$. Hence, $\exists y \leq x_0x_1 \left[ x_0x_1 = y0 \lor x_0x_1 = y1 \right]$. We now consider (2). We have $\exists y \leq x \left[ x0x_1 = (x0y1)0 \lor x0x_1 = (x0y1)1 \right]$. Since $x_1 \in C$, this implies $\exists y \leq x0x_1 \left[ x0x_1 = y0 \lor x0x_1 = y1 \right]$.

We thus see that both cases imply $x0x_1 \in I$. Hence, $I$ is closed under $\circ$.

We let

\[ J = \left\{ v \in I : \forall x \leq v \left[ x \in I \right] \right\}. \]
By $D_4$ and $D'_4$, we have $D \vdash \forall x \leq 0 \ [ \ x = 0 \ ] \land \forall x \leq 1 \ [ \ x = 1 \ ]$. Hence, $0, 1 \in J$. We now show that $J$ is closed under $\circ$. Suppose, $v_0, v_1 \in J$ and $x \leq v_0v_1$. Since $v_1 \in C$, we have one of the following:

(a) $x \leq v_0$
(b) $x = v_0y$ for some $y \leq v_1$.

In case of (a), we have $x \in I$ since $v_0 \in J$. In case of (b), we have $y \in I$ since $v_1 \in J$. We also have $v_0 \in I$ since $v_0 \in J \subseteq I$. Since $I$ is closed under $\circ$, we then have $x = v_0y \in I$. We thus see that both cases imply $v_0v_1 \in J$. Hence, $J$ is closed under $\circ$.

We let

$$K = \{ v \in J : \forall w \leq v \forall x y [ \ x \leq y \land y \leq w \rightarrow x \leq w ] \}.$$ 

By $D_4$, we have

$$x \leq y \land y \leq w \land w \leq 0 \Rightarrow x = y = w = 0 \leq 0.$$ 

Hence, $0 \in K$. By $D'_4$, we have

$$x \leq y \land y \leq w \land w \leq 1 \Rightarrow x = y = w = 1 \leq 1.$$ 

Hence, $1 \in K$.

We now show that $K$ is closed under $\circ$. Suppose $v_0, v_1 \in K$ and

$$w \leq v_0v_1 \land x \leq y \land y \leq w.$$ 

We need to show that $x \leq w$. From $w \leq v_0v_1$ and $v_1 \in K \subseteq C$, we have one of the following:

(I) $w \leq v_0$
(II) $w = v_0u$ for some $u \leq v_1$.

In case of (I), since $v_0 \in K$, we have

$$w \leq v_0 \land x \leq y \land y \leq w \Rightarrow x \leq w.$$ 

In case of (II), we observe that $y \leq w = v_0u$ and $u \leq v_1$. Furthermore, since $v_1 \in J$, we have that $u \leq v_1$ implies $u \in I \subseteq C$. Then, $y \leq w = v_0u$ and $u \in C$ implies that we have one of the following:

(IIa) $y \leq v_0$
(IIb) $y = v_0u'$ for some $u' \leq u$.

In case of (IIa), since $v_0 \in K \subseteq B$, we have

$$v_0 \leq v_0 \land x \leq y \land y \leq v_0 \Rightarrow x \leq v_0.$$
since $v_0 \in K$. Then, since $u \in C$, we have

$$x \leq v_0 \Rightarrow x \leq v_0 u = w.$$  

In case of (IIb), we observe that $v_1 \leq v_1$ since $v_1 \in K \subseteq B$. Then, $v_1 \in K$ implies

$$v_1 \leq v_1 \land u' \leq u \land u \leq v_1 \Rightarrow u' \leq v_1.$$  

Since $v_1 \in K \subseteq J$, we have that $u' \leq v_1$ implies $u' \in I \subseteq C$. Then, $x \leq y = v_0 u'$ and $u' \in C$ implies that we have one of the following:

(IIbi) $x \leq v_0$

(IIbii) $x = v_0 u''$ for some $u'' \leq u'$.

In case of (IIbi), since $u \leq v_1$ and $v_1 \in K \subseteq J$ implies $u \in I \subseteq C$, we have

$$x \leq v_0 \Rightarrow x \leq v_0 u = w.$$  

In case of (IIbii), we first observe that

$$u \leq v_1 \land u'' \leq u' \land u' \leq u \Rightarrow u'' \leq u$$  

since $v_1 \in K$. Then, since $u \in C$, we have

$$x = v_0 u'' \land u'' \leq u \Rightarrow x \leq v_0 u = w.$$  

We thus see that

$$x \leq y \land y \leq w \land w \leq v_0 v_1 \Rightarrow x \leq w.$$  

Hence, $v_0 v_1 \in K$. Thus, $K$ is closed under $\circ$.

We now show that the class $K$ has the following important property:

$$\forall v \in K \forall w \left[ w \leq v \rightarrow w \in K \right]. \quad (*)$$  

Indeed, suppose $v \in K$ and $w \leq v$. We need to show that $w \in K$. By definition of $K$, we need to prove

$$z \leq w \land x \leq y \land y \leq z \Rightarrow x \leq z.$$  

Since $v \in K \subseteq B$, we have $v \leq v$. Then,

$$v \in K \land v \leq v \land z \leq w \land w \leq v \Rightarrow z \leq v.$$  

This in turn implies

$$v \in K \land z \leq v \land x \leq y \land y \leq z \Rightarrow x \leq z.$$  

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Hence, \( w \in K \). Thus, (*) holds.

We are now ready to give an interpretation of BTQ in \( D \). We choose the domain \( K \).

We translate 0, 1 and \( \circ \) as

\[
\begin{align*}
\phi_0(x) & \equiv x = 0 \\
\phi_1(x) & \equiv x = 1 \\
\phi_\circ(x, y, z) & \equiv z = xy .
\end{align*}
\]

It is clear that the translations of BTQ\(_1\), BTQ\(_2\) and BTQ\(_3\) are theorems of \( D \). We now show that the translation of BTQ\(_4\) is a theorem of \( D \). Let \( x \in K \). Since \( K \subseteq I \), we have

\[
x = 0 \lor x = 1 \lor \exists y \leq x \ [ x = y0 \lor x = y1 ] .
\]

If the third disjunct is the case, then we have \( y \in K \) by (*). Thus, the translation of BTQ\(_4\) is a theorem of \( D \). Hence, BTQ is interpretable in \( D \).

\[\square\]

**Theorem 7** \( D \) is interpretable in BTQ.

**Proof** We choose the domain \( J(x) \equiv x = x \). We translate 0, 1, \( \circ \) and and \( \leq \) as follows:

\[
\begin{align*}
\phi_0(x) & \equiv x = 0 \\
\phi_1(x) & \equiv x = 1 \\
\phi_\circ(x, y, z) & \equiv z = xy \\
\phi_\leq(x, y) & \equiv y = x \lor \exists z \ [ y = xz ] .
\end{align*}
\]

It is clear that the translation of \( D_1 \), \( D_2 \) and \( D_3 \) are theorems of BTQ. We now show that the translation of

\[
D_4 \equiv \forall x \ [ x \leq 0 \leftrightarrow x = 0 ]
\]

is a theorem of BTQ. So, suppose \( 0 = x \lor \exists z \ [ 0 = xz ] \). It is not difficult to see that we cannot have \( \exists z \ [ 0 = xz ] \). Hence, the translation of \( D_4 \) is a theorem of BTQ.

We now show that the translation of

\[
D_5 \equiv \forall xy \ [ x \leq y0 \leftrightarrow ( x = y0 \lor x \leq y ) ]
\]

is a theorem of BTQ. We need to show that

\[
y0 = x \lor \exists z \ [ y0 = xz ] \iff y0 = x \lor y = x \lor \exists u \ [ y = xu ] .
\]

It suffices to show that

\[
\exists z \ [ y0 = xz ] \iff y = x \lor \exists u \ [ y = xu ] . \quad (\ast)
\]
The right-left implication of (*) is trivial. The left-right implication holds since

\[
y_0 = xz \Rightarrow (y_0 = x_0 \land z = 0) \lor \\
\exists z' [y_0 = x(z')0 \land z = z'0] \quad \text{(by BTQ}_4, \text{BTQ}_3)
\]

\[
\Rightarrow (y_0 = x_0 \land z = 0) \lor \\
\exists z' [y_0 = (xz')0 \land z = z'0] \quad \text{(by BTQ}_1)
\]

\[
\Rightarrow (y = x \land z = 0) \lor \\
\exists z' [y = xz' \land z = z'0] \quad \text{(by BTQ}_2)
\]

\[
\Rightarrow y = x \lor \exists u [y = xu].
\]

Thus, the translation of \(D_5\) is a theorem of BTQ.

By similar reasoning, the translation of \(D_6\) is a theorem of BTQ.

Thus, \(D\) is interpretable in BTQ. \qed

6 Minimality results

This section is devoted to show that the axiomatizations of WD, WBT and BTQ are minimal essentially undecidable, which is to say that removing any one of the axioms (axiom schemas) gives a theory that is not essentially undecidable. We are not able to show that our axiomatization of \(D\) is minimal essentially undecidable, but we reduce the problem to showing that \(D\)–\(D_5\) and \(D\)–\(D_6\) are not essentially undecidable. However, as \(D\) has a finite axiomatization, we can make it minimal essentially undecidable by replacing some of the axioms with their conjunction.

We now proceed to show that WBT—WBT\(_1\), WD—WD\(_1\) and D—D\(_1\) are not essentially undecidable by interpreting them in S2S. S2S is a monadic second order theory whose language is \(\{e, 0, 1, S_0, S_1\}\), where \(e\), 0 and 1 are constant symbols and \(S_0\) and \(S_1\) are unary function symbols. The axioms of S2S are the true sentences in the standard second-order structure where the universe is \(\{0, 1\}^*\). The symbol \(e\) is interpreted as the empty string, 0 is interpreted as 0 and 1 is interpreted as 1. The function symbol \(S_0\) is interpreted as the function that takes a bit string and concatenates it with the bit 0, and the function symbol \(S_1\) is interpreted as the function that takes a bit string and concatenates it with the bit 1. We have quantifiers that range over \(\{0, 1\}^*\), and we have quantifiers that range over subsets of \(\{0, 1\}^*\). It was proved in Rabin [13] that S2S is decidable. Our interpretation of WBT—WBT\(_1\) in S2S does not use the monadic second order part, and this makes the induced algorithm more efficient. It is known that extending S2S with the prefix relation does not change the expressive power of S2S (see Börger et al. [1] p. 317). We also show this when we interpret WD—WD\(_1\) and D—D\(_1\) in S2S.

We recall that biterals are associated to the left. So, \((0 \circ 0) \circ 0\) is a biteral while \((0 \circ (0 \circ 0))\) is not. Although we have so far not needed to take this into account, it now becomes important.

**Lemma 4** WBT—WBT\(_1\) and WD—WD\(_1\) are not essentially undecidable.
Proof We interpret the two theories in $S_2S$ as follows: We choose the domain $\delta(x) \equiv x = x$. We map 0 and 1 to

$$
\phi_0(x) \equiv x = 0 \\
\phi_1(x) \equiv x = 1.
$$

So, 0 and 1 are realized as 0 and 1, respectively. We map $\circ$ to

$$
\phi_\circ(x, y, z) \equiv \left( (x = e \lor (y \neq 0 \land y \neq 1)) \land z = e \right) \lor \\
\left[ x \neq e \land \left( (y = 0 \land z = S_0x) \lor \\
(y = 1 \land z = S_1x) \right) \right].
$$

This means that $\circ$ is realized as the function

$$(x, y) \mapsto \begin{cases} 
\varepsilon & \text{if } x = \varepsilon \text{ or } y \notin \{0, 1\} \\
x0 & \text{if } x \neq \varepsilon \text{ and } y = 0 \\
x1 & \text{if } x \neq \varepsilon \text{ and } y = 1.
\end{cases}$$

Recalling that biterals are associated to the left, it is clear that the translation of each instance of $WBT_2$ and $WD_2$ is a theorem of $S_2S$. We now show that the translation of each instance of

$$
WBT_3 \equiv \forall x \ [x \subseteq S \alpha \rightarrow \bigvee_{\gamma \in \langle \ldots \alpha \rangle} x = \gamma]
$$

is a theorem of $S_2S$. We recall that

$$
x \subseteq S \ y \equiv y = x \lor \exists u v \ [y = ux \lor y = xv \lor y = (ux)v \lor y = u(xv)].
$$

Let $\alpha \neq \varepsilon$. Suppose $\beta$ is a substring of $\alpha$ w.r.t. $\phi_\circ(x, y, z)$. We show that $\beta \neq \varepsilon$ and that $\beta$ is a substring of $\alpha$ in the actual sense. We have one of the following cases:

- $\alpha = \beta$
- There exists $u$ such that $\phi_\circ(u, \beta, \alpha)$. Then, $\alpha = u\beta$, $\beta \in \{0, 1\}$ and $u \neq \varepsilon$.
- There exists $v$ such that $\phi_\circ(\beta, v, \alpha)$. Then, $\alpha = \beta v$, $\beta \neq \varepsilon$ and $v \in \{0, 1\}$.
- There exist $u, v$ and $w$ such that $\phi_\circ(u, \beta, w) \land \phi_\circ(w, v, \alpha)$. Then, $\alpha = u\beta v$, $u \neq \varepsilon$ and $\beta, v \in \{0, 1\}$.
- There exist $u, v$ and $w$ such that $\phi_\circ(\beta, v, w) \land \phi_\circ(u, w, \alpha)$. This is not possible since $\phi_\circ(u, w, \alpha)$ implies $w \in \{0, 1\}$ while we cannot have $v$ such that $\phi_\circ(\beta, v, 0)$ or $\phi_\circ(\beta, v, 1)$.

We thus see that $\beta$ is a substring of $\alpha$ in the actual sense.

We map $\preceq$ to

$$
\phi_\preceq(x, y) \equiv \exists Y \left[ e \notin Y \land x \in Y \land y \in Y \land S_0y \notin Y \land S_1y \notin Y \land \
\right].
$$
\[ \forall z \left( \left( z \neq e \land \left( S_0 z \in Y \lor S_1 z \in Y \right) \right) \to z \in Y \right) \land \\
\forall w \left( S_0 w \in Y \to S_1 w \notin Y \right) \]

This formula forces \( \leq \) to be realized as the prefix relation on \( \{0, 1\}^+ \). Indeed, suppose \( \alpha, \beta \in \{0, 1\}^+ \) and \( \alpha \) is related to \( \beta \) as defined by \( \phi_{\leq}(x, y) \). The first line in \( \phi_{\leq}(x, y) \) tells us that there exists \( Y \subseteq \{0, 1\}^+ \) such that

\[ \alpha, \beta \in Y \quad \text{and} \quad \beta 0, \beta 1 \notin Y. \]

The second line in \( \phi_{\leq}(x, y) \) tells us that if \( \gamma \in Y \), then all the non-empty prefixes of \( \gamma \) are also in \( Y \). Let

\[ \{0, 1\} \ni \alpha_0 < \alpha_1 < \ldots < \alpha_k = \alpha \quad \text{and} \quad \{0, 1\} \ni \beta_0 < \beta_1 < \ldots < \beta_m = \beta \]

denote all the prefixes of \( \alpha \) and \( \beta \). The notation \( \gamma < \delta \) then means that \( \gamma \) is the longest proper prefix of \( \delta \). We show by induction that for each \( 0 \leq j \leq k \), there exists \( 0 \leq i \leq m \) such that \( \alpha_j = \beta_i \), which implies that \( \alpha \) is a prefix of \( \beta \). The third line in \( \phi_{\leq}(x, y) \) tells us that exactly one of 0 and 1 belongs to \( Y \) since we get for free that

\[ \forall w \left[ S_1 w \in Y \to S_0 w \notin Y \right] \].

Hence, \( \alpha_0 = \beta_0 \). Suppose now that for \( 0 \leq j < k \) there exists \( 0 \leq i \leq m \) such that \( \alpha_j = \beta_i \). We observe that we must have that \( i < m \) since \( \alpha_j = \beta_i = \beta \) and \( \alpha_{j+1} \in Y \) would contradict \( \beta 0, \beta 1 \notin Y \). It remains to show that \( \alpha_{j+1} = \beta_{i+1} \). The third line in \( \phi_{\leq}(x, y) \) tells us that exactly one of \( \alpha_j 0 \) and \( \alpha_j 1 \) belongs to \( Y \). Hence, \( \alpha_{j+1} = \beta_{i+1} \). Thus, by induction, each prefix of \( \alpha \) is a prefix of \( \beta \).

We thus see that this translation has the desired properties. This translation shows us how to define decidable models of the theories in question. \( \square \)

**Lemma 5**  \( D - D_1 \) is not essentially undecidable.

**Proof** We modify the translation in Lemma 4 as follows: We map \( \circ \) to

\[ \phi_{\circ}(x, y, z) \equiv \left( \left( y \neq 0 \lor y \neq 1 \right) \land z = x \right) \lor \left( y = 0 \land z = S_0 x \right) \lor \left( y = 1 \land z = S_1 x \right). \]

This means that \( \circ \) is realized as the function

\[ (x, y) \mapsto \begin{cases} 
  x & \text{if } y \notin \{0, 1\} \\
  x0 & \text{if } y = 0 \\
  x1 & \text{if } y = 1.
\end{cases} \]

We need this modification to ensure that the axiom \( D_3 \equiv \forall x y \left[ x0 \neq y1 \right] \) holds. \( \square \)

We observe that the simple translation of \( \circ \) we give in the preceding lemma does not work in the case of Lemma 4 since the axiom scheme \( \text{WT} \text{3} \) then fails. Indeed, for any non-empty bit string \( \alpha \) and any bit string \( \beta \neq 0, 1 \), we have \( \phi_{\circ}(\vec{\alpha}, \vec{\beta}, \vec{\alpha}) \). This means that all substring different from 0 and 1 are substrings of \( \alpha \) w.r.t. \( \phi_{\circ}(x, y, z) \).
Lemma 6  \textit{BTQ}−\textit{BTQ}_1 is not essentially undecidable.

\textbf{Proof} Let $\Sigma$ denote the set of all sentences in first order logic with equality and the language \{0, 1, +, $\times$, $\leq$\} true in the structure $(\mathbb{R}, 0, 1, +, \times, \leq)$. It was shown by Tarski [17] that $\Sigma$ is decidable. We show that \textit{BTQ}−\textit{BTQ}_1 is interpretable in $\Sigma$.

We now define a relative interpretation of $\textit{BTQ}−\textit{BTQ}_1$ in $\Sigma$. We relativize quantification to $\lambda(x) \equiv x = x$ and use the following translation of symbols

$$
0 \mapsto \psi_0(x) \equiv x = 0 \\
1 \mapsto \psi_1(x) \equiv x = 1 \\
\circ \mapsto \psi_c(x, y, z)
$$

where $\psi_c(x, y, z)$ is a formula which defines in $(\mathbb{R}, 0, 1, +, \times, \leq)$ the function $h = h_3 \circ h_2 \circ h_1 : \mathbb{R}^2 \to \mathbb{R}$ where

$$
h_1(r, u) = \begin{cases} 
-2 + \frac{2r}{1+\sqrt{1+4r^2}} & \text{if } u = 0 \\
2r & \text{if } u = 1 \\
0 & \text{otherwise}
\end{cases}
$$

$$
h_2(r) = \begin{cases} 
\frac{1+r}{2} & \text{if } r \in (-3, -1) \cup (-1, 1) \\
0 & \text{otherwise}
\end{cases}
$$

$$
h_3(r) = \begin{cases} 
\frac{1}{r} - 1 & \text{if } r \in (0, 1) \\
\frac{1}{r} + 1 & \text{if } r \in (-1, 0) \\
0 & \text{otherwise}
\end{cases}
$$

We first observe that the map $\mathbb{R} \to (-1, 1)$ defined by

$$
r \mapsto \frac{2r}{1 + \sqrt{1 + 4r^2}}
$$

is a bijection. The sections $h(-, 0)$ and $h(-, 1)$ are injective and have images $(-\infty, 0)$ and $(0, \infty)$. This is because we have the following sequences of bijections:

$$
\begin{align*}
\mathbb{R} & \xrightarrow{h_1(-, 0)} (-3, -1) \xrightarrow{h_2} (-1, 0) \xrightarrow{h_3} (-\infty, 0) \\
\mathbb{R} & \xrightarrow{h_1(-, 1)} (-1, 1) \xrightarrow{h_2} (0, 1) \xrightarrow{h_3} (0, \infty).
\end{align*}
$$

The fact that $h(-, 0)$ and $h(-, 1)$ are injective and have disjoint images implies that the translation of

$$
\textit{BTQ}_2 \equiv \forall x y [ x \neq y \to (x 0 \neq y 0 \land x 1 \neq y 1 ) ]
$$
and

$$
\text{BTQ}_3 \equiv \forall x y [ x0 \neq y1 ]
$$

are theorems of $\Sigma$. The translation of

$$
\text{BTQ}_4 \equiv \forall x [ x = 0 \lor x = 1 \lor \exists y [ x = y0 \lor x = y1 ] ]
$$

is also a theorem of $\Sigma$ since we have defined $\psi_0(x) \equiv x = 0$ and since the union of the images of $h(-, 0)$ and $h(-, 1)$ is

$$
(-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}.
$$

The translation shows how to define a decidable model of $\text{BTQ} - \text{BTQ}_1$. \hfill \Box

One of the referees has observed that we get a simpler proof of the preceding lemma by considering the translation given in Lemma 4 and restricting the domain to the set of all non-empty strings. We can thus for example translate $\circ$ as follows

$$(x, y) \mapsto \begin{cases} x0 & \text{if } y = 0 \\ x1 & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7 $\text{WBT} - \text{WBT}_2$ and $\text{WD} - \text{WD}_2$ are not essentially undecidable.

**Proof** We obtain a one element model $\mathfrak{A}$ of $\text{WBT} - \text{WBT}_2$ and $\text{WD} - \text{WD}_2$ as follows:

- the universe is $\{0\}$
- $0^\mathfrak{A} = 0$, $1^\mathfrak{A} = 0$
- $0 \circ^\mathfrak{A} 0 = 0$
- $\leq^\mathfrak{A} = \{(0, 0)\}$.

Since $\circ^\mathfrak{A}$ is associative, $\text{WBT}_1$ and $\text{WD}_1$ hold in $\mathfrak{A}$. We observe that $\mathfrak{A}$ satisfies

$$
\forall x [ x \subseteq_s \overline{\alpha} \rightarrow x = \overline{\alpha} ] \quad \text{and} \quad \forall x [ x \leq \overline{\alpha} \leftrightarrow \bigvee_{\beta \leq D_{\alpha}} x = \overline{\beta} ]
$$

since there is only one element in the universe. Thus, $\text{WBT}_3$ and $\text{WD}_3$ hold in $\mathfrak{A}$. \hfill \Box

Lemma 8 $\text{BTQ} - \text{BTQ}_2$ is not essentially undecidable.

**Proof** We obtain a two element model $\mathfrak{A}$ of $\text{BTQ} - \text{BTQ}_2$ as follows:

- the universe is $\{0, 1\}$
- $0^\mathfrak{A} = 0$, $1^\mathfrak{A} = 1$
- $\forall x, y \in \{0, 1\}( x \circ^\mathfrak{A} y = y )$. 

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The operator $\circ^{\mathbb{N}}$ is clearly associative. The axiom $\forall x y \ [ x 0 \neq y 1 ]$ holds since

$$x \circ^{\mathbb{N}} 0 = 0 \neq 1 = y \circ^{\mathbb{N}} 1.$$  

The axiom $\forall x \ [ x = 0 \lor x = 1 \lor \exists y \ [ x = y 0 \lor x = y 1 ] ]$ holds since the universe is $\{0^{\mathbb{N}}, 1^{\mathbb{N}}\}$.

The following proof of the next Lemma was suggested by one of the referees. We show that $D - D_2$ is not essentially undecidable by interpreting it in Presburger Arithmetic. Presburger Arithmetic refers to all sentences in first-order logic with equality and the language $\{0, 1, +, <\}$ true in the structure $(\mathbb{N}, 0, 1, +, <)$. It is shown that Presburger Arithmetic is decidable in Chapter 3.4 of Smoryński [15]. The idea is to consider the free monoid generated by 0 and 1 modulo the equations $00 \equiv 0$, $01 + 10 \equiv 0$, $10 1 + 10 \equiv 0$. The non-empty string, we have $n > 0$ when the string is of the form $(01)^n$. The concatenation operator on this set is described as follows

$$x * y = \begin{cases} xy & \text{if } x = ua, \ y = bv, \ a \neq b \text{ and } a, b \in \{0, 1\} \\ uav & \text{if } x = ua, \ y = av \text{ and } a \in \{0, 1\} \end{cases}.$$  

Observe that $u$ and $v$ denote the empty string in the cases $x, y \in \{0, 1\}$. It is clear that axiom $D_3 \equiv \forall x y \ [ x 0 \neq x 1 ]$ holds in this model. It also follows easily from the definition of $\ast$ that $(x * y) * z = x * (y * z)$ when $y$ has length at least two. To see that $\ast$ is indeed associative, we observe that given distinct $a, b \in \{0, 1\}$, we have

$$(x * a) * z = \begin{cases} xaz & \text{if } x = ub, \ z = bv \\ xav & \text{if } x = ub, \ z = av \\ uaz & \text{if } x = ua, \ z = bv \\ uav & \text{if } x = ua, \ z = av \end{cases} = x * (a * z).$$  

Thus, axiom $D_1 \equiv \forall x y z \ [ (xy)z = x(yz) ]$ holds.

We interpret $\leq$ as the prefix relation, i.e., $x \leq y$ if and only if $x = y$ or there exists $z$ such that $y = x * z$. Axiom $D_4 \equiv \forall x \ [ x \leq 0 \leftrightarrow x = 0 ]$ holds since $x * z = 0$ implies $x = z = 0$. Axiom $D_5 \equiv \forall x y \ [ x \leq y 0 \leftrightarrow (x = y 0 \lor x \leq y ) ]$ holds by the following reasoning:

- Suppose $x \leq y * 0$ and $y * 0 = x * z$ for some $z$. If $y$ has 0 as a suffix, then $y = x * z$, which implies $x \leq y$. Assume now $y$ has 1 as a suffix. We have that $y = 0$ or $z = u 10$ where $u$ is possibly empty. We first assume $z = 0$. If $x$ has 0 as a suffix, then $y * 0 = x$. If $y$ and $x$ both have 1 as suffixes, then $y = x$, which in turn implies $x \leq y$. Suppose now $z = u 10$. Then $y = x * u 1$, which in turn implies $x \leq y$.
- Suppose $x = y * 0 \lor y = x \lor \exists z \ [ y = x * z ]$. Then, by associativity of $\ast$, we have $x = y * 0 \lor y * 0 = x * 0 \lor \exists z \ [ y * 0 = x * (z * 0) ]$. Hence $x \leq y * 0$.  

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By similar reasoning, axiom $D_6 \equiv \forall xy \ [ x \leq y1 \leftrightarrow (x = y1 \lor x \leq y) ]$ holds.

Now, to interpret $D - D_2$ in Presburger Arithmetic, we associate strings with natural numbers as follows

\[(01)^n \mapsto 4n, \quad 1(01)^n \mapsto 4n + 1, \quad (01)^n0 \mapsto 4n + 2, \quad 1(01)^n0 \mapsto 4n + 3.\]

It is then not difficult to see that $\star$ is definable in Presburger Arithmetic. This in turn implies that the prefix relation is definable. We thus have the following result.

**Lemma 9** $D - D_2$ is not essentially undecidable.

**Lemma 10** $BTQ - BTQ_3$ and $D - D_3$ are not essentially undecidable.

**Proof** Since Presburger Arithmetic is decidable, we obtain a decidable model of $BTQ - BTQ_3$ and $D - D_3$ as follows:

- the universe is the set of natural numbers $\mathbb{N}$
- $0^{\mathfrak{A}} = 0, \ 1^{\mathfrak{A}} = 0$
- $\forall x, y \in \mathbb{N} \ (x \circ^{\mathfrak{A}} y = x + y)$
- $x \leq^{\mathfrak{A}} y \iff x = y$.

It is not difficult to see that axioms of $BTQ - BTQ_3$ hold. Axiom

\[D_4 \equiv \forall x \ [ x \leq 0 \leftrightarrow x = 0 ]\]

holds since $x \leq^{\mathfrak{A}} 0^{\mathfrak{A}}$ if and only if $x = 0^{\mathfrak{A}}$. The axioms

\[D_5 \equiv \forall xy \ [ x \leq y0 \leftrightarrow (x = y0 \lor x \leq y) ] \quad \text{and} \quad D_6 \equiv \forall xy \ [ x \leq y1 \leftrightarrow (x = y1 \lor x \leq y) ]\]

hold since $x \leq^{\mathfrak{A}} x$ and $y \circ^{\mathfrak{A}} 1^{\mathfrak{A}} = y = y \circ^{\mathfrak{A}} 0^{\mathfrak{A}}$.

**Lemma 11** $WD - WD_3$, $WBT - WBT_3$, $BTQ - BTQ_4$ and $D - D_4$ are not essentially undecidable.

**Proof** We start by showing that $BTQ - BTQ_4$ has a decidable model. We consider the set $M_2(\mathbb{R}_{\geq 0})$ of $2 \times 2$ matrices with coefficients in $\mathbb{R}_{\geq 0}$ and which are such that the first entry is strictly positive. So, the elements of $M_2(\mathbb{R}_{\geq 0})$ are of the form

\[
\begin{pmatrix}
  x \in \mathbb{R}_{\geq 0} & y \in \mathbb{R}_{\geq 0} \\
  z \in \mathbb{R}_{\geq 0} & w \in \mathbb{R}_{\geq 0}
\end{pmatrix}
\]

We consider the binary operation of matrix multiplication, denoted $\times$. It is not difficult to verify that $M_2(\mathbb{R}_{\geq 0})$ is closed under matrix multiplication. We can consider this structure as a first-order structure for the language $\{c_0, c_1, \times\}$ where $c_0$ and $c_1$ are constant symbols for the matrices

\[
\begin{pmatrix}
  1 & 0 \\
  1 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  1 & 1 \\
  0 & 1
\end{pmatrix}
\]
respectively. Let $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ denote the set of sentences true in this structure. We show that $\text{BTQ} - \text{BTQ}_4$ is interpretable in $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ and that $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ is decidable. Since relative interpretability preserves the property of being essentially undecidable, this means that $\text{BTQ} - \text{BTQ}_4$ is not essentially undecidable.

We interpret $\text{BTQ} - \text{BTQ}_4$ in $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ as follows:

- We choose the domain $\delta(x) \equiv x = x$.
- We map $0$ to $\phi_0(x) = x = c_0$.
- We map $1$ to $\phi_1(x) \equiv x = c_1$.
- We map $\circ$ to $\phi_0(x, y, z) \equiv x \times y = z$.

Since matrix multiplication is associative, the translation of

$$\text{BTQ}_1 \equiv \forall xyz \ [ (xy)z = x(yz) ]$$

is a theorem of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$. To see that the translation of

$$\text{BTQ}_2 \equiv \forall xy \ [ x \neq y \rightarrow (x0 \neq y0 \land x1 \neq y1) ]$$

is a theorem of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$, we observe that

$$
\begin{align*}
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + b & b \\ c + d & d \end{pmatrix} \\
&\Rightarrow x = a \land y = b \land z = c \land w = d
\end{align*}
$$

and

$$
\begin{align*}
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + b & b \\ c + d & d \end{pmatrix} \\
&\Rightarrow x = a \land y = b \land z = c \land w = d .
\end{align*}
$$

To see that translation of

$$\text{BTQ}_3 \equiv \forall xy \ [ x0 \neq y1 ]$$

is a theorem of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$, we observe that

$$
\begin{align*}
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + b & b \\ c + d & d \end{pmatrix} \\
&\Rightarrow x + y = a \land y = a + b \\
&\Rightarrow x + a + b = a \\
&\Rightarrow x + b = 0 \\
&\Rightarrow x = 0 .
\end{align*}
$$
But we cannot have \( x = 0 \) by how the set \( M_2(\mathbb{R}_{\geq 0}) \) is defined. We then see that the translation of \( BTQ_3 \) is a theorem of \( \text{Th}(M_2(\mathbb{R}_{\geq 0})) \).

To see that \( \text{Th}(M_2(\mathbb{R}_{\geq 0})) \) is decidable, we use Tarski’s result that the set \( \text{Th}(\mathbb{R}) \) of sentences which are true in the ordered field of real numbers \( (\mathbb{R}, 0, 1, +, \times, \leq) \) is decidable (see Tarski [17]). To see that this is the case, we associate each sentence \( \phi \) in the language of \( \text{Th}(M_2(\mathbb{R}_{\geq 0})) \) with a sentence \( \phi^* \) in the language of \( \text{Th}(\mathbb{R}) \) such that

\[
\phi \in \text{Th}(M_2(\mathbb{R}_{\geq 0})) \iff \phi^* \in \text{Th}(\mathbb{R}).
\]

We start by associating \( c_0 \) and \( c_1 \) with

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

respectively, and by associating each variable \( x \) with

\[
\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}
\]

We recall that matrix multiplication is defined by

\[
\begin{pmatrix} x & y \\ z & w \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xa + yc & xb + yd \\ za + wc & zb + wd \end{pmatrix}.
\]

By following this definition, we associate each term \( s \) in the language of \( \text{Th}(M_2(\mathbb{R}_{\geq 0})) \) with a matrix

\[
\begin{pmatrix} p_{s,1} & p_{s,2} \\ p_{s,3} & p_{s,4} \end{pmatrix}
\]

where each \( p_{s,i} \) is a term in the language of \( \text{Th}(\mathbb{R}) \). So, for example, the term \( x \times y \) is associated with the matrix

\[
\begin{pmatrix} x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \\ x_3y_1 + x_4y_3 & x_3y_2 + x_4y_4 \end{pmatrix}
\]

Next we define \( \phi^* \) by recursion on the structure of \( \phi \) as follows:

1. If \( \phi \equiv s = t \), then

\[
\phi^* \equiv \bigwedge_{x \in S \cup T, 1 \leq i \leq 4} x_i \geq 0 \wedge \bigwedge_{y \in S \cup T} y_1 \neq 0 \wedge \bigwedge_{i=1}^{4} p_{s,i} = p_{t,i}
\]

where \( S \) and \( T \) denotes the set of variables occurring in \( s \) and \( t \), respectively.
2. If \( \phi \equiv \neg \alpha \), then \( \phi^* \equiv \neg \alpha^* \)
If $\phi \equiv \alpha \lor \beta$, then $\phi^* \equiv \alpha^* \lor \beta^*$

If $\phi \equiv \alpha \land \beta$, then $\phi^* \equiv \alpha^* \land \beta^*$

If $\phi \equiv \exists x \alpha$, then $\phi^* \equiv \exists x_1 x_2 x_3 x_4 \alpha^*$

If $\phi \equiv \forall x \alpha$, then $\phi^* \equiv \forall x_1 x_2 x_3 x_4 \alpha^*$.

The first big conjunct in clause (1) reflects the fact that matrices in $M_2(\mathbb{R}_{\geq 0})$ have entries in $\mathbb{R}_{\geq 0}$. The second big conjunct in (1) reflects the fact that matrices in $M_2(\mathbb{R}_{\geq 0})$ are such that the first entry is in $\mathbb{R} > 0$. The last big conjunct in (1) states that the matrix associated with $s$ equals the matrix associated with $t$. It is obvious that $\phi$ is a theorem of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ if and only if $\phi^*$ is a theorem $\text{Th}(\mathbb{R})$. Thus, $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ is decidable.

It is not difficult to see that the model of $\text{BTQ} - \text{BTQ}_4$ we have defined is also a model of $\text{WD} - \text{WD}_3$ and $\text{WBT} - \text{WBT}_3$. We extend it to a decidable model of $\text{D} - \text{D}_4$ by interpreting $\leq$ as $M_2(\mathbb{R}_{\geq 0}) \times M_2(\mathbb{R}_{\geq 0})$.

Our proof of decidability of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ is actually a 4-dimensional interpretation of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ in $\text{Th}(\mathbb{R})$. This means that each object in the language of $\text{Th}(M_2(\mathbb{R}_{\geq 0}))$ is associated with a quadruple of objects in the language of $\text{Th}(\mathbb{R})$. For more on this more general notion of interpretability, see Visser [20].

We can now state the main theorem of this section.

**Theorem 8** $\text{WD}$, $\text{WBT}$ and $\text{BTQ}$ are minimal essentially undecidable.

The only thing that is lacking to show that $D$ is minimal essentially undecidable is to show that $D - D_5$ and $D - D_6$ are not essentially undecidable. A negative solution of the following problem would thus show that $D$ is minimal essentially undecidable.

**Open Problem 2** Show that neither $D - D_5$ nor $D - D_6$ is essentially undecidable.

We have not put much focus on the theories BT and C. The proofs of Lemma 5 and Lemma 11 show that $\text{C} - \text{C}_1$, $\text{C} - \text{C}_4$ and $\text{BT} - \text{BT}_4$ are not essentially undecidable. Beyond that, the minimality of BT and C is an open problem.

**Open Problem 3** Are BT and C minimal essentially undecidable?

In this section, we investigated whether our axiom sets are minimal w.r.t set inclusion and the property of being essentially undecidable. A different notion of minimality that we have been implicitly investigating is minimality w.r.t. interpretability. We have seen that WD is interpretable in all the essentially undecidable theories we have studied. It is however not minimal in the interpretability pre-order. In [7], Jeřábek shows that there is an essentially undecidable theory that is interpretable in $\mathbb{R}$ but that does not interpret $\mathbb{R}$. The theory Jeřábek gives is such that all partially recursive functions are representable. In Section 3 of [2], Yong Cheng uses results in [7] to give many examples of essentially undecidable theories that are interpretable in $\mathbb{R}$ but that do not interpret $\mathbb{R}$. One of the referees observed that the existence of theories strictly below $\mathbb{R}$ w.r.t. interpretability also follows from the work of Shoenfield [14].

Although there are many essentially undecidable theories below $\mathbb{R}$ w.r.t interpretability, to the best of our knowledge, it is not known whether there exists a minimal computably enumerable essentially undecidable theory, w.r.t. interpretability. For a more detailed discussion of this problem, we refer the reader to Yong Cheng [2].
follows from the idea behind the proof of Theorem 4.7 of [2] that the interpretability degrees of computably enumerable essentially undecidable theories have infima. Hence, if there exists a minimal essentially undecidable theory, then that theory is also the minimum essentially undecidable theory.

7 Summary

We have formulated essentially undecidable theories \( WD \subset C \subset D \) and \( WBT \subset BT \subset BTQ \) (see Fig. 3). The theories \( WD, WBT \) and \( R \) are mutually interpretable while the theories \( D, BTQ \) and \( Q \) are mutually interpretable. The theories \( WD, WBT \) and \( BTQ \) have minimal essentially undecidable axiomatizations, but it not clear whether the same is true of \( D \). Both \( WD \) and \( D \) are purely universally axiomatised.

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