Sparse Bayesian Imaging of Solar Flares
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Abstract. We consider imaging of solar flares from NASA RHESSI data as a parametric imaging problem, where flares are represented as a finite collection of geometric shapes. We set up a Bayesian model in which the number of objects forming the image is a priori unknown, as well as their shapes. We use a Sequential Monte Carlo algorithm to explore the corresponding posterior distribution. We apply the method to synthetic and experimental data, largely known in the RHESSI community. The method reconstructs improved images of solar flares, with the additional advantage of providing uncertainty quantification of the estimated parameters.

Key words. Sparse imaging, Bayesian inference, Sequential Monte Carlo, Astronomical imaging, Solar flares.

AMS subject classifications. 65R32, 62F15, 65C05, 85A04

1. Introduction. In the last couple of decades, sparsity has emerged as a key concept in a number of imaging applications including geophysics [23], magnetic resonance [29], photonoacoustics [22], image restoration problems [16], and also astronomical imaging [15]. The fundamental intuition underlying this revolution is that the information content of natural images is small compared to the number of pixels we use to represent them. As a consequence images can be compressed substantially by representing them on a proper basis, in which they turn out to have few non-zero coefficients, i.e. they are sparse; crucially, such compression typically entails little loss of information.

In many imaging problems, sparsity would be ideally obtained by introducing a regularization term with the $\ell^0$-norm of the image, represented on a suitable basis [8]. However, such regularization term is non-convex and leads to a problem of combinatorial complexity, for which two computational strategies have been proposed: greedy methods (also known as orthogonal matching pursuit) [35] and the relaxation into a convex term [36]. Since the relaxation to $\ell^p$-norm with $0 < p < 1$ leads again to a non-convex minimization problem, the most common approximation is the $\ell^1$-norm, which represents a good trade-off between sparsity–promotion and computational tractability. In the last decade this approach has been widely investigated, also thanks to the development of new efficient algorithms for convex optimization [38, 3].

In this study we present an alternative approach to sparse imaging, based on three ingredients: (i) the parametrization of the image in terms of a small set of objects, each one described by a small number of parameters; (ii) a Bayesian model in which sparsity is encouraged through the choice of a Poisson prior distribution on the number of objects with a small mean; (iii) a Monte Carlo algorithm which is able to sample the resulting complex posterior distribution.

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We apply the proposed approach to the problem of reconstructing images from data recorded by the NASA RHESSI satellite [26]. RHESSI [28], launched in 2002, records X-rays from the Sun, with the main objective of improving our understanding of the mechanisms underlying the generation of solar flares. Due to the specific hardware configuration, RHESSI imaging can be synthetically described as an inversion of the Fourier transform, with undersampled data. In the past fifteen years RHESSI data have been converted to images using several different methods: from old-fashioned, heuristic approaches such as CLEAN [25], MEM [11] and PIXON [31], to more mathematically sound regularization methods such as UV-smooth [30], Space-D [6], Expectation–Maximization [4], and Semi-blind deconvolution [5]. Only recently, sparse imaging has been proposed through compressed sensing approaches in a few studies [15, 19]. Specifically, the former (VIS_WV) proposes to use a finite isotropic wavelet transform combined with FISTA [3]; the second one (VIS_CS) uses a Coordinate Descent [21] algorithm and a dictionary of pre-defined Gaussian shapes.

These recent approaches are providing improved images to the solar physics community, particularly for low signal-to-noise ratios. However, one of the goals of RHESSI imaging is to make quantitative inference on the physical objects appearing in the images. For instance, one of the variables of interest is the total flux of X-rays coming from a flare; in order to estimate this quantity from an image, it is required to integrate over the area occupied by the flare; however, evaluating what part of the image actually contains the flare can be a difficult task, and may make results subjective. In addition, the two mentioned recent studies make use of optimization algorithms that do not allow directly for uncertainty quantification.

The novel approach proposed here has some similarities with the work of [19], inasmuch as both methods assume that the image can be described by a combination of very simple geometric shapes. However, in our method these shapes are parameterized and therefore more flexible than those in the finite database used in [19]; as a consequence, each object in our image has a direct interpretation in terms of a part of the solar flare, thus overcoming the integration problem mentioned above.

We set up a statistical model in the Bayesian framework, in which the number of objects in the image is itself a random variable. Technically, this is done by a so-called variable dimension model, i.e. a collection of spaces of differing dimensions. Since the posterior distribution is often non-concave, deterministic optimization algorithms would have difficulties in finding its maximum; here we resort to sampling, rather than optimization. While this approach has an inherently high computational cost, it provides additional benefits such as uncertainty quantification and possibly multi-mode detection.

We sample the posterior distribution using an adaptive Sequential Monte Carlo (SMC) sampler [12]. SMC samplers are efficient algorithms for sampling complex distributions of interest; the main idea behind SMC samplers is to construct an auxiliary sequence of distributions such that the first distribution is easy to sample, the last distribution is the target distribution, and the sequence is “smooth”. In the present case, we draw a random sample from the prior distribution and then let these sample points evolve according to properly selected Markov kernels, until they reach the posterior distribution. In less technical terms, SMC samplers start with a set of candidate solutions, and proceed in parallel with a stochastic local optimization – done through these Markov kernels. The key to the efficiency of the proposed algorithm is highly dependent on the construction of effective sets of transition kernels,
to explore the space. Here we adopt three kind of moves: individual–parameter–updating moves, such as modifying the abscissa of the location of an object; birth and death moves, where the number of objects is increased or decreased by addition or deletion; split and merge moves, where a single object is split into two, or two objects are merged into one.

The outline of the paper is the following. In section 2 we provide the necessary background on RHESSI imaging. In section 3 we present the Bayesian model and the adaptive sequential Monte Carlo method (ASMC) used to approximate the posterior distribution. In section 4 we show the results obtained by our method on a set of synthetic and real data, and compare them to those obtained by commonly used techniques. Finally we draw our conclusions.

2. Parametric models for RHESSI imaging. The Reuven Ramaty High Energy Solar Spectroscopic Imager (RHESSI) is a hard X–ray imaging instrument which was launched by NASA on February 5, 2002. The goal of this NASA mission is to study solar flares and other energetic solar phenomena [28]. RHESSI observes X-ray emissions from the Sun through a set of nine coaligned pairs of rotating modulation collimators, and the transmitted radiation is recorded on a set of cooled high-purity germanium detectors [26]. The raw data provided by RHESSI are nine count profiles of the detected radiation as a function of time, modulated by the grid pairs. By combining rotation and modulation it is possible to estimate the visibilities, which are image Fourier components at specific spatial frequencies. Each detector samples Fourier components on a different circle in the Fourier space, over the course of the $\sim 4$ s spacecraft rotation period. More formally, if $F : \mathbb{R}^2 \to \mathbb{R}$ is the spatial flux distribution, the visibilities $V : \mathbb{R}^2 \to \mathbb{C}$ are defined as

$$V(u, v) = \int_{\mathbb{R}^2} F(x, y) e^{2\pi i (ux + vy)} dx dy ;$$

for more details we refer to [5] and [26].

![Figure 1: Left: sampling of the spatial frequency $(u, v)$ plane by seven detectors on RHESSI. Absolute value of amplitude (middle) and phase (right) of the visibilities for a particular event. For graphical reason, we do not show the detectors 1 and 2.](image)

In the left panel of Figure 1 we include the sampled $(u, v)$-plane, where each of the sampled circles corresponds to a single detector (here, only detectors from 3 to 9 are shown) and each dot of a circle represents a sampled $(u, v)$ frequency at which a visibility is measured. In the
middle and in the right panel of Figure 1 an example of the absolute value of the amplitude and the phase of some visibilities is shown.

Equation (1) defines a linear inverse problem, and indeed, in the last decade most of the methodological efforts have been devoted to devising novel linear inverse methods. However, early works such as [27] and [33] have shown that with a little simplification, which is acceptable given the limited spatial resolution provided by RHESSI, a solar flare can show up in an image under a very limited range of shapes: the lower parts of the flare, called footpoints, can be circular or elliptical depending on the viewing angle; the upper part of the flare may either appear as an elliptical source or as an arc, a sort of bent ellipsoid, often referred to as loop.

An image containing one of these geometrical shapes can be parameterized easily; for instance, a circular source $C$ centered at $(x_C, y_C)$, of radius $r_C$ and total flux $\phi_C$ can be properly represented by a 2-dimensional Gaussian distribution defined by

$$ F_C(x, y) = \phi_C \exp \left[ -\frac{(x-x_C)^2 + (y-y_C)^2}{2r_C^2} \right]. $$

An elliptical source $E$ oriented along the $x, y$ axes can be easily derived from (2), by adding the eccentricity parameter $\varepsilon_E$

$$ F_E(x, y) = \phi_E \exp \left[ -\frac{(x-x_E)^2}{2r_E^2} - \frac{(y-y_E)^2}{2r_E^2(\varepsilon_E + 1)^2} \right]. $$

One can obtain a generic elliptical shape by adding a rotation angle $\alpha$; and finally, a loop source is obtained by adding a loop angle $\beta$. In the following, we will be calling these distributions...
circle, ellipse and loop; we refer to Figure 2 for a pictorial representation of these geometric objects, and to [1] for the explicit definition of these type of sources.

The key point is that it is straightforward to compute the visibilities associated to these simple objects. For instance, by computing the 2D-Fourier transformation for the circular Gaussian source, we obtain the following

\[ V_C(u,v) = \phi_C \exp \left[ 2\pi i (ux_C + vy_C) - \frac{\pi^2 r_C^2}{4 \log 2} (u^2 + v^2) \right], \]

and similar results, although more complicated, hold for ellipses and loops.

Eventually, we can build an imaging model in which the final image is composed of an arbitrary but small number of elementary objects

\[ F(x,y) = \sum_{s=1}^{N_S} F_{T_s}(x_{T_s}, y_{T_s}, r_{T_s}, \phi_{T_s}, \ldots), \]

where \( N_S \) is the number of sources in the image, and the corresponding visibilities will be the sum of the visibilities of the single objects

\[ V(u,v) = \sum_{s=1}^{N_S} V(F_{T_s}(x_{T_s}, y_{T_s}, r_{T_s}, \phi_{T_s}, \ldots)). \]

It therefore makes sense to re-cast the imaging problem (1) as a parametric problem, in which one aims at reconstructing the parameters of one or more circles, ellipses, or loops. In [1] this approach was pursued using a non-linear optimization algorithm; however, the number and type of shapes has to be defined a priori within a very small set of pre-defined configurations (one circle, one ellipse, one loop, two circles).

We also notice that eq. (5) corresponds to a representation of the image on the basis of circles, ellipses and loops, in which the flux of each source can be seen as the coefficient of the corresponding basis element.

3. Bayesian sparse imaging with Sequential Monte Carlo. We cast the problem of reconstructing images of solar flares from the visibilities measured by RHESSI as a Bayesian inference problem. After formalization of the parameterized image space, we describe our choice in terms of prior distribution, embodying the sparsity penalty term, and the sequential Monte Carlo algorithm that we use to obtain samples of the posterior distribution of interest.

3.1. The image space. In our model an image is a collection of pre-defined shapes, or objects. The number of objects is not known a priori, nor are the object types; therefore our aim is to make inference on the following unknown:

\[ F = (N_S, T_{1:N_S}, \theta_{1:N_S}) \]

where \( N_S = 0, \ldots, N_{\text{max}} \) represents the number of sources, \( T_{1:N_S} = (T_1, \ldots, T_{N_S}) \) the source types and \( \theta_{1:N_S} = (\theta_1, \ldots, \theta_{N_S}) \) the source parameters for each source.
We call $\mathcal{T}$ the set of source types $\mathcal{T} = \{C, E, L\}$; as explained in the previous section, different source types are characterized by different numbers of parameters: circles are defined by four parameters, ellipses by six parameters and loops by seven parameters. As a consequence, and for convenience, we define the single–source parameter space as the union of these fixed–dimensional spaces

\[(7) \quad \Theta = \mathbb{R}^4 \cup \mathbb{R}^5 \cup \mathbb{R}^7 \]

The most complex source, the loop, is defined by seven parameters, i.e. $\theta_s = (x_s, y_s, r_s, \phi_s, \alpha_s, \varepsilon_s, \beta_s)$, where $(x_s, y_s) \in \mathbb{R}^2$ indicates the position of the source, $r_s \in \mathbb{R}_{>0}$ the full width at half maximum (FWHM), $\phi_s \in \mathbb{R}_{>0}$ the flux, $\alpha_s \in [0, 360^\circ]$ the rotation angle, $\varepsilon_s \in [0, 1]$ the eccentricity and $\beta_s \in [-180^\circ, 180^\circ]$ the curvature. The circle only has the first four parameters, the ellipse the first six.

Since the number of sources is not known \textit{a priori}, we need to consider a variable-dimension model, i.e., the state space $\mathcal{F}$ of the sources is defined as follows:

\[(8) \quad \mathcal{F} := \bigcup_{s=1}^{N_S} \{s\} \times \mathcal{T}^s \times \Theta^s, \]

where $\Theta^s$ is the Cartesian product of $\Theta$, $s$ times (and analogously for $\mathcal{T}^s$).

3.2. The statistical model. In a Bayesian setting, we aim at characterizing the posterior distribution

\[(9) \quad \pi(f|\nu) \propto \pi(f)\pi(\nu|f), \]

where: $\nu$ and $f$ indicate the measured visibilities and the image parameters, respectively; $\pi(f)$ is the prior distribution, which represents the information that is known before any measurements; and $\pi(\nu|f)$ is the likelihood function.

3.2.1. Prior. Our choice of the prior probability density $\pi(f)$ reflects basic expectations that are based on available knowledge about the solar flare geometries. In our prior distribution we consider the number of sources, the source types and the parameter values of each source and we assume that they are all independent. Thus, the prior distribution $\pi$ can be written as follows

\[(10) \quad \pi(f) = \pi(N_S, T_1:N_S, \theta_1:N_S) = \rho_1(N_S) \prod_{s=1}^{N_S} \rho_2(T_s) \rho_3(\theta_s), \]

where $\rho_j$, with $j \in \{1, 2, 3\}$, indicates the distribution of the number of sources, the shape of the sources, and the parameters, respectively.
We assume that the number of sources in an image is Poisson distributed with mean $\lambda > 0$,
\begin{equation}
\rho_1(N_S) = \frac{e^{-\lambda} \lambda^{N_S}}{N_S!}.
\end{equation}
Past observations suggest that the maximum number of sources can be limited to five, see [17].
In the simulations below we will be using $\lambda \in [1, 4]$, and in section 4 we study the influence of the parameter $\lambda$ in the reconstruction.

We adopt the following distribution for the type of the sources
\begin{equation}
\rho_2(T_s) = \begin{cases} 
  p_C & \text{if } T_s \text{ is a circle} \\
  p_E & \text{if } T_s \text{ is an ellipse} \\
  p_L & \text{if } T_s \text{ is a loop,}
\end{cases}
\end{equation}
where $p_C, p_E, p_L \in [0, 1]$. In general the morphology of solar flare hard X-ray sources is dependent on the photon energy observed: images at lower energies $\lesssim 15$ keV generally are dominated by extended loop-like (or elliptical) structures, while images at higher photon energies are more likely to include compact sources (possibly exclusively so), see [37] and [7]. Thus, in our simulations, for low energy levels we set $(p_C, p_E, p_L) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, while for high energy levels we use $(p_C, p_E, p_L) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. However, we observed that moderate changes to these values do not have an impact on the result.

Given the $s$-th source, the associated source parameters have uniform distributions
\begin{align}
(x_s, y_s) &\sim \mathcal{U}([x_m - FOV/2, x_m + FOV/2], [y_m - FOV/2, y_m + FOV/2]) \\
\phi_s &\sim \mathcal{U}(0, V_{\text{max}}) \\
\epsilon_s &\sim \mathcal{U}(0, 1) \\
\alpha_s &\sim \mathcal{U}(0, 360^\circ) \\
\beta_s &\sim \mathcal{U}([-180^\circ, 180^\circ]),
\end{align}
where $FOV$ is the field of view, $(x_m, y_m)$ is the center of the image, and $V_{\text{max}} = \max(\text{Re}(V))$ represents the maximum of the real component of the visibilities.

**3.2.2. Likelihood.** We make the standard assumption that the visibilities are affected by additive Gaussian noise; although this may not be an accurate model, it is widely used in literature for lack of a better one. Therefore, the likelihood is given by
\begin{equation}
\pi(f|\nu) \propto \exp\left(-\sum_j \frac{(\nu - V(f))^2}{2\sigma_j^2}\right)
\end{equation}
where $V(f)$ indicates the forward operator, which maps the image parameters $f$ onto its exact visibilities, and $\sigma_j$ the noise standard deviation. Since uncertainty on the measured visibilities comes mainly from the statistical error on the X–ray counts, plus a known systematic error, we can assume knowledge of the standard deviations $\{\sigma_j\}_j$ [1]. In practice, these values are regularly provided together with the visibilities themselves.
3.2.3. Parallel with regularization theory. We now provide an informal parallel between the Bayesian model just described and the more well-known regularization approach to imaging.

The regularization approach consists in minimizing a functional with the following form

$$
\|\nu - V(f)\|_2^2 + \lambda R(f)
$$

where the first term represents the data fitting term (in this case with the $\ell^2$–norm), $\lambda$ is the regularization parameter and $R(f)$ is the regularization term, which is responsible for promoting sparsity when sparse solutions are sought.

It is well known that this functional can be interpreted, up to a normalizing constant, as the negative logarithm of a posterior distribution, where the likelihood is of the form (14); we now discuss what sort of regularization term is implied by our choice of the prior distribution.

We first observe that all parameters, except for the number of sources, are assigned a uniform prior distribution, which corresponds to a specific choice of the set of basis elements but with no explicit regularization term. As far as the number of sources is concerned, the Poisson distribution with mean $\lambda$ corresponds to the following regularization term

$$
R(f) = -\log \left( \frac{e^{-\lambda} \lambda^{N_S}}{N_S!} \right) = \lambda + \log(N_S!) - N_S \log(\lambda) \approx \log(N_S!) - N_S \log(\lambda) .
$$

(15)

which, for small values of $\lambda$ (less than 4, the maximum value used below) and reasonable values of $N_S$, grows almost linearly with $N_S > \lambda$ (see Figure 3). Therefore, our regularization term is counting the number of sources, i.e. the number of non–zero components in our basis expansion, exceeding the expected value $\lambda$; in this sense, for $\lambda < 1$ it corresponds to an $\ell^0$–regularization term. We also remark, however, that the proposed approach is not an optimization approach, but rather a Monte Carlo sampling approach, that aims at characterizing the whole posterior distribution rather than just finding its mode; in this way, our method provides more rich information about the unknown that is not typically provided in regularization.

3.3. Sequential Monte Carlo samplers.

3.3.1. Monte Carlo sampling. In order to approximate the posterior distribution we employ a sequential Monte Carlo method [12]. The general goal of Monte Carlo sampling is to obtain a set of $N_P$ sample points $f^{(p)}$, also called particles, with corresponding weights $w^{(p)}$, such that the following approximation holds for a general enough class of functions $h(\cdot)$:

$$
\sum_{p=1}^{N_P} w^{(p)} h(f^{(p)}) \approx \int h(f) \pi(f) df .
$$

(16)

the relevance of (16) is manifest, since all the moments of the distribution can be obtained by simple choices of $h(f)$, such as $h(f) = f$ for the mean; in addition, (16) can be interpreted as
providing an approximation of the distribution \( \pi(f) \):

\[
\sum_{p=1}^{N_P} w^{(p)} \delta(f, f^{(p)}) \approx \pi(f).
\]

Ideally, one would like to have uniformly weighted samples, as small–weight samples provide little contribution to the sum; therefore a standard measure of the quality of the sample set \( W = \{w^{(p)}\}_{p=1,...,N_P} \) is given by the so–called Effective Sample Size, defined as

\[
\text{ESS}(W) := \left( \sum_{p=1}^{N_P} \left( w^{(p)} \right)^2 \right)^{-1};
\]

the ESS ranges between 1, for bad choices of the sample points, and \( N_P \), the optimal value attained when all samples carry the same probability mass.

### 3.3.2. Sequential Monte Carlo samplers.

Sequential Monte Carlo samplers produce such weighted sample set through a smooth iterative procedure, and are particularly useful when the target distribution is complex and/or highly concentrated in few areas of the state–space. The derivation of SMC samplers is relatively complex and will not be reported here; instead, we provide here the minimal amount of details that should allow reproduction of the results.

The key point is to consider a sequence of distributions \( \{\pi_i\}_{i=1}^{N_I} \), with \( N_I \) the number of iterations, such that \( \pi_1(f) = \pi(f) \) is the prior distribution and \( \pi_{N_I}(f) = \pi(f|\nu) \) is the posterior distribution; a natural choice is given by

\[
\pi_i(f) \propto \pi(\nu|f)^{\gamma_i} \pi(f),
\]

where \( \gamma_i \in [0, 1] \) for \( i = 1, \ldots, N_I \), \( \gamma_{i+1} > \gamma_i \), \( \gamma_1 = 0 \) and \( \gamma_{N_I} = 1 \). Our SMC sampler starts by drawing an i.i.d. sample set \( \{f_i^{(p)}\}_{p=1}^{N_P} \) from \( \pi_1(f) \), and then produces, at iteration \( i \), an approximation of \( \pi_i(f) \) with the sample points \( f_i^{(p)} \) and the corresponding weights \( w_i^{(p)} \) through the following procedure:

- the sample \( f_i^{(p)} \) is drawn from the \( \pi_i \)–invariant Markov kernel \( K_i(f|f_{i-1}^{(p)}) \); technically, these Markov kernels are built as a composition of vanilla Metropolis–Hastings [10] and
reversible jump [24] moves; in practice, the particle at the next iteration is obtained
by perturbing the current particle in such a way that the new state is neither too
different nor too similar to the current one; the choice of these kernels is crucial in
determining the effectiveness of the procedure, and will be discussed more in detail in
the next subsections;
ii the weights are given by
\[ w_i^{(p)} \propto w_{i-1}^{(p)} \frac{\pi_i(f_i^{(p)})}{\pi_{i-1}(f_{i-1}^{(p)})}, \]
i.e. the right side is computed and then weights get normalized; in order to obtain
good performances, it is important that the sequence is smooth, i.e. that \( \pi_i \simeq \pi_{i-1} \).
We control such smoothness by increasing adaptively the exponent: we first attempt
\( \gamma_{i+1} = \gamma_i + \delta_{\text{max}} \) and compute the ratio
\[ \frac{\text{ESS}(W_{i+1})}{\text{ESS}(W_i)}; \]
if the value of this ratio is in the interval \([0.90, 0.99]\) we confirm the choice \( \gamma_{i+1} \),
otherwise we proceed with \( \gamma_{i+1} = \gamma_i + \delta \) by bisection on \( \delta \) until the ESS ratio falls
in the prescribed interval. In the simulations below we use \( \delta_{\text{max}} = 0.1 \). Notice that
the particle weights do not depend on the current particle, but only on the previous
particle; this makes this adaptive calculation very cheap.
iii finally, we notice that the Effective Sample Size naturally diminishes as the iterations
proceed: when its value gets below \( \frac{N_P}{2} \), we perform a resampling step [13], where
the particles with high weights are copied multiple times and the ones with low weights
are discarded. At the end of the resampling, all the new particles are assigned equal
weights.

Importantly, since in our model the likelihood is Gaussian, every distribution of the se-
quence (18) can be interpreted as the posterior distribution corresponding to a different (de-
creasing) scaling factor of the noise (co)variance. Although in our application the noise on the
visibilities is assumed to be known, in the simulations below we exploit this fact to investigate
what happens when such estimate is not correct.

3.3.3. Transition kernels. As discussed in the previous subsection, the generation of new
samples from the existing ones is the key point of the algorithm, as it allows exploration of
the parameter space. Due to the relative complexity of the state space, which is a variable–
dimension model with an unknown number of objects of unknown type, we had to construct
a relatively complex transition kernel. Specifically, the transition kernel has the following
abstract form:
\[ K_i(f_i|f_{i-1}) = K_i^{\text{bd}}(f_i|f_{i-1})K_i^{\text{cm}}(f''|f''')K_i^{\text{cm}}(f''|f')K_i^{\text{bd}}(f'|f_{i-1}); \]
Namely, the whole transition kernel is given by the composition of several \( \pi_i \)–invariant kernels
implementing different types of move, specifically: \( K_i^{\text{bd}}(\cdot, \cdot) \) implements a birth/death move,
$K^c_i(\cdot, \cdot)$ implements a change move, $K^{sm}_i(\cdot, \cdot)$ implements a split/merge move and $K^u_i(\cdot, \cdot)$ implements an update move.

In practice, all these moves are attempted in sequence, starting from the current state $f^{(p)}_{i-1}$ and ending up in the next state $f^{(p)}_i$, through several intermediate states $f', \ldots, f''$. In order for each move to be a draw from a $\pi_i$–invariant kernel, we use the classical Metropolis–Hastings accept/reject, where proposal distributions and acceptance probabilities are defined according to the theory of reversible jumps [24].

The general mechanism for moving from $f'$ to $f''$ is as follows. Given the particle $f'$, a new particle $f^*$ is proposed from a proposal distribution $q(\cdot | f')$; notice that this can also be viewed as the random variable $F^*$ being a deterministic function of the current state and of the random quantity $u$: $F^* = g(f', u)$. The proposed particle is then either accepted, with probability

$$\alpha(f^*, f') = \min \left\{ 1, \frac{\pi_i(f^*) q(f' | f^*)}{\pi_i(f') q(f^* | f')} \frac{\partial g}{\partial (f', u)} \right\},$$

or rejected: in case of acceptance, we set $f'' = f^*$; otherwise, we set $f'' = f'$. We now explain more in detail how these kernels work, by providing the explicit form of the birth/death case, which is the most complex one.

**Birth and death moves.** Birth and death moves consist in adding/deleting one source to/from the image. We realize this by using the following proposal

$$q(f'' | f') = \frac{1}{3} \rho_2(T_{N'}+1) \mathcal{U}(\theta_{N'+1}) \prod_{s=1}^{N'} \delta(\theta_{s}, \theta_{nk}) +$$

$$\frac{1}{3} \frac{1}{N'} \sum_{t=1}^{N'} \prod_{s=1}^{N'-1} \delta(\theta_{a(t,s)}, \theta_{nk}) +$$

$$\frac{1}{3} \prod_{k} \delta(\theta_{nk}, \theta_{nk})$$

In the first row a birth move is proposed with probability $1/3$, and the new $(N' + 1)$–th source type and the new source parameters are drawn from their (uniform) prior distributions, collectively denoted here as $\mathcal{U}(\theta_{N'+1})$; the other sources remain identical, as represented here by the delta functions. In the second row a death move is attempted with probability $1/3$, and one of the existing sources is removed at random; this entails a re–organization of the indices from the disappeared source, with $a(t, s)$ being the ancestor of source $s$ when the $t$–th source has disappeared, $a(t, s) = s$ for $s < t$, $a(t, s) = s - 1$ for $s > t$. In the last row, the state remains the same.

**The change move.** Change moves are done systematically at each iteration, and consist in proposing the transformation of a circle into an ellipse (and vice versa) and an ellipse into a loop (and vice versa). We do not consider the transition from a circle to a loop (and vice versa) because we retain only “smooth” transformations. From the mathematical perspective,
these moves resemble birth/death moves, but they foresee the appearance/disappearance of just one or two parameters, rather than of a whole source; like in the birth/death move, the new parameters are drawn from the prior.

**The merge and split moves.** The merge and split moves are attempted at each iteration if the current state has at least one or two ellipses, respectively. The merge move consists of combining two close ellipses to create a new one. The proposal here is deterministic: given two ellipses, the proposed source is defined by the combination of their parameters, i.e. the position and the rotation angle are the mean of the corresponding parameters of the two ellipses, while the flux, FHWM and the eccentricity are the sum. The split move generates two ellipses from a single one. This move is semi-deterministic: the centers of the new ellipses are drawn along the major axis of the old ellipse, at a random distance; the flux is split equally, and the major axis is a random quantity around half the major axis of the old ellipse.

**The update move.** The last move is the classical parameter update move of the Metropolis - Hastings algorithm, and is systematically applied at each iteration. Each parameter is updated independently, i.e. a separate move is attempted for each parameter. We highlight that even if the prior distribution of the rotation angle and the loop angle are uniform on a bounded set, during the update move we let these parameters evolve freely without any constraints, in order to exploit their periodic nature.

Finally, let us observe that ellipses with small eccentricity are not practically different from circles, and loops with small loop angle are not practically different from ellipses. Therefore, at the end of each iteration the source type $T_s$ of each source is updated based on the parameter values, according to the following scheme:

$$T_s = \begin{cases} 
\text{circle:} & \text{if } \varepsilon_s \leq 0.1 \\
\text{ellipse:} & \text{if } 0.1 \leq \varepsilon_s < 0.4, |\beta_s| \leq 45^\circ \\
& \text{and } \varepsilon_s \geq 0.4 \text{ and } |\beta_s| \leq 10^\circ \\
\text{loop:} & \text{otherwise.}
\end{cases}$$

**3.4. Point estimates.** At every iteration we can compute the point estimates for each parameter. The number of sources at the $i$–th iteration ($\hat{N}_S$) is given by the mode of the marginal distribution of the number of sources in the sample, i.e.

$$\hat{\pi}_i(N_S = k|\nu) = \sum_{p=1}^{N_P} w_i^{(p)} \delta\left(k, (N_S^{(p)})_i\right),$$

where $\delta$ indicates the Kronecker delta function and $(N_S^{(p)})_i$ is the number of sources in the $p$-th particle at the iteration $i$.

In the same way, the estimated locations of the sources are given by the local modes of the intensity measure for the source locations, conditioned on the estimated number of sources,
Algorithm 1 Adaptive sequential Monte Carlo algorithm

Initialization of the sample:

\begin{equation}
\text{for } p = 1, \ldots, N_P \text{ do}
\begin{align*}
\text{draw } f_1^{(p)} \text{ from } \pi \\
\text{set } w_1^{(p)} = \frac{1}{N_P}
\end{align*}
\text{end for}
\end{equation}

Set $i = 1$ and $\gamma_1 = 0$

while $\gamma_i \leq 1$ do

Increase $i$

Possible resampling step:

if $\text{ESS}(W_i) \leq N_S/2$ then

Apply the systematic resampling

end if

MCMC sampling:

\begin{equation}
\text{for } p = 1, \ldots, N_P \text{ do}
\begin{align*}
\text{propose birth/death/split/merge/change moves, then accept/reject} \\
\text{for each parameter, update the value, then accept/reject} \\
\text{compute the weights } w_i^{(p)} \propto w_{i-1}^{(p)} \pi \left( \frac{f_i^{(p)}}{f_{i-1}^{(p)}} \right),
\end{align*}
\text{end for}
\end{equation}

Normalize weights and compute the effective sample size:

\begin{equation}
\text{compute the } \text{ESS}(W_{i+1})
\end{equation}

Adaptive determination of the next exponent:

\begin{equation}
\text{while } \frac{\text{ESS}(W_{i+1})}{\text{ESS}(W_i)} \notin [0.9, 0.99] \text{ do}
\begin{align*}
\text{modify } \delta \text{ and set } \gamma_{i+1} = \gamma_i + \delta \\
\text{recompute weights } w_i^{(p)} \text{ and } \text{ESS}(W_{i+1})
\end{align*}
\text{end while}

Type re-assignment

end while

\begin{equation}
\text{return } \{ f_i^{(p)}, w_i^{(p)} \}_{p=1,\ldots,N_P}
\end{equation}

\begin{equation}
i.e.
\hat{\pi}(c|\nu, (\hat{N}_S)_i) = \sum_{p=1}^{N_P} w_i^{(p)} \delta \left( \left( N_S^{(p)} \right)_i, (\hat{N}_S)_i \right) \left( \sum_{s=1}^{N_S^{(p)}} \delta(c, c_{is}^{(p)}) \right),
\end{equation}

where $c = (x, y)$ and $c_{is}^{(p)} = (x_{i,s}^{(p)}, y_{i,s}^{(p)})$ indicate the center of the $s$-th source in the $p$-th particle at the iteration $i$.

The source types are determined as the modes of the type distributions, conditioned on the estimated source locations and the number of sources. The flux, FWHM, and the eccentricity are estimated by the mean values of the conditional distribution conditioned on the sources location, the type and number of sources. To exploit the periodicity, the rotation and loop angles are estimated by taking the mode of the conditional distribution on the sources location,
4. Numerical results. We now demonstrate the effectiveness of the proposed technique in reconstructing images of solar flares. We present first the results obtained by applying the Bayesian method to simulated data which have been already used for testing and validating imaging methods [30, 2, 18, 19]. In order to prove that the method is feasible also on real data, we then proceed with the analysis of four experimental data sets.

In all the numerical tests below, unless otherwise specified, we set \((p_C, p_E, p_L) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\) and \(\lambda = 1\). As we will see later (Figure 7 and Figure 10), in most of the cases, especially when the noise level is low, the value of \(\lambda\) does not influence the final result. We systematically compare the results obtained by our Bayesian approach with those obtained by two alternative methods available in SSW (Solar SoftWare, [20]): CLEAN and VIS_WV. We used CLEAN algorithm with the default parameter values\(^1\), because the number of parameters is high and appropriate tuning requires expert knowledge; for VIS_WV we tune the number of scales and the regularization parameter such that the reconstruction is as close as possible to the expected image.

Figure 4: Ground truth images: loop (S1L), ellipse with circle (S1E1C), two circles (S2C), four circles (S4C).

4.1. Simulated data. In this section we use a well–known simulated data set to test the proposed Bayesian method. The data set comprises several sets of synthetic visibilities, generated from four underlying source configurations, that have been constructed in order to mimick four real solar flares happened between 2002 and 2005. The four configurations are characterized by different degrees of complexity: one single loop (S1L), one ellipse with a small circular source (S1E1C), two circles (S2C) and four circles (S4C); see Figure 4 for the true configurations. For each flaring configuration, two different synthetic calibrated event lists mimicking a RHESSI acquisition were at our disposal, corresponding to two different levels of statistics (an average of 1000 counts per detector for the low level – high noise, 100000 for the high level – low noise).

4.2. Reconstructions. In Figure 5 and Figure 6, the reconstructions given by CLEAN, VIS_WV and the Bayesian approach are shown in the case of low and high noise, respectively.

\(^1\)Some of the default parameters of RHESSI imaging algorithms can be found at [https://hesperia.gsfc.nasa.gov/ssw/hessi/doc/params/hsi_params_image.htm](https://hesperia.gsfc.nasa.gov/ssw/hessi/doc/params/hsi_params_image.htm)
Figure 5: Low noise case. Reconstructions obtained by CLEAN (first row), VIS_WV (second row) and ASMC (third row). True sources in Figure 4.

We can clearly see that, even though CLEAN has been extensively used, it does not perform very well: the images look too smooth and different sources are recovered as only one (see for instance the S4C). The wavelet-based reconstruction method works better: most of the reconstructed images are close to the original test images, but some spurious sources appear, probably due to the noise in the data (see for instance the top right corner).

In the low noise case, the reconstructions given by the proposed Bayesian approach (third row in Figure 5) are very similar to the original ones (Figure 4), and the method seems to overcome some of the issues of the VIS_WV. In fact, for the configuration S1E1C (second column in Figure 4), it is possible to recognize both the ellipse and the circle, while in VIS_WV only one source seems to be recovered. Furthermore, the shape of the loop-source (first column in Figure 4) is more accurate than with the other methods.

When the noise level is high, all the reconstructions get worse; the Bayesian reconstructions, in particular, tend to lose some degree of complexity, either missing entire sources or estimating them as simpler geometrical objects (e.g. a circle rather than a loop in S1L). Some of these drawbacks can be overcome by tuning the parameters, as discussed below; we notice however that this behaviour can be simply interpreted as a consequence of the lower information content of the data, due to the presence of noise.
CLEAN

VIS_WV

ASMC

Figure 6: High noise case. Reconstructions obtained by CLEAN (first row), VIS_WV (second row) and ASMC (third row). True sources in Figure 4.

4.3. Influence of the parameters. The results of the Bayesian method presented here can be tuned by changing the parameters in the prior distribution and in the likelihood function, in the same way as the results of a regularization algorithm can be tuned by changing the regularization parameter(s). In this Section, we study the impact of (i) the mean number of sources $\lambda$ in the Poisson prior distribution, (ii) the prior probabilities for the source types and (iii) the noise variance. For (i) and (ii) we run the ASMC method with different values of the parameters; for (iii), on the other hand, we exploit the fact that at each iteration the ASMC algorithm approximates a different distribution, that can be interpreted as the posterior distribution under a different noise standard deviation (see the paragraph at the end of subsection 3.3.2).

To study the influence of the parameter $\lambda$, in Figure 7 we show the posterior probability distribution of the number of sources for the simulated loop S1L and the simulated four sources S4C with different noise levels, and four different values of $\lambda$. The first manifest result is that, expectedly, the posterior distribution depends weakly on the value of $\lambda$ for low noise levels, i.e. when the data are highly informative; on the other hand, the posterior probabilities are more affected by the prior when the noise level is high. For example, for the S4C the posterior mode indicates three sources for the high noise case; for increasing values of $\lambda$, however, the posterior probability for the two–source model goes from 20% to zero, while the posterior probability for the four–source model increases from zero to about 20%.
Figure 7: Posterior probabilities for the number of sources for the single loop (left column) and the four circles (right column), with different values of the prior parameter $\lambda \in [1, 4]$.

Figure 8: Posterior probabilities for the source types for the single loop configuration with high noise level, using different a priori probability sets: $(p_C, p_E, p_L) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ (left); $(p_C, p_E, p_L) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{4}\right)$ (middle); $(p_C, p_E, p_L) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ (right).

To study the influence of the prior on the source type, we considered the S1L case and run the algorithm three times; the prior values were set to favour, each time, a different source type: the preferred source type was given a prior probability of $\frac{1}{2}$, while the other two were given a prior probability of $\frac{1}{4}$. In the low noise case, the results do not depend on the choice of the prior, as the posterior probability assigns more than 99% to the loop type. The high noise case is more delicate: in Figure 8 we show the influence of the prior on the source types for the S1L with high noise level. From the histograms, we see that the posterior probabilities are now affected by the priors. This suggests that noise perturbs the data enough to reduce the information on the source shape to such an extent that the prior has a non-negligible impact. This is somewhat confirmed also by visual inspection of the reconstructions obtained by the other methods (see Figure 6 and figures in [19]).
Figure 9: Distribution of the particles (first row) and their respective X–ray reconstructions (second row) at different iteration. The black stars represent the position of the true sources.

We now show the impact of changing the noise variances in (14) by a common factor, which is an other way of tuning regularization by “trusting” the data more or less. As previously stated, we do this simply by showing the posterior estimates computed at different, increasing iteration numbers of one single run of the ASMC sampler: this corresponds to decreasing values of the noise standard deviations, i.e. to trusting the data more. In Figure 9 we show, in the top row, the spatial distribution of particles at different iterations, specifically: when the first, the second, the third and the fourth source are first estimated; in the row below we show the corresponding estimated configuration. At first (left column), a large source is estimated around the true location of the central source, and its location is still uncertain (particles spread relatively high). Then particles begin to concentrate around the two slightly weaker sources (left and bottom right); however, the posterior mode for the number of sources is two, and therefore the final estimate contains only the central and the bottom right source. Subsequently, the third source is reconstructed, and finally also the weaker top right source is recovered, while the uncertainties on the location of the other sources get smaller and smaller.

We finally provide an insight about how different values of λ impact the iterative procedure: in Figure 10 we show the posterior probability of different number of sources, as a function of the iteration number for λ = 1 (left) and λ = 4 (right). At the beginning, the number of sources is distributed according to the prior, and then it smoothly shifts towards the final posterior distribution: when λ = 1, this translates in monotonically increasing complexity; when λ = 4, in the first iterations the complexity actually decreases, because most sampled high–dimensional solutions will have a poor fit with the data; around iteration 100, however, the two cases become very similar, as expected because of the low noise level/high information content in the data.
4.4. Application to real data. In order to show that the proposed method can work in practice, we finally show an application to experimental data. We select four real flare events: the August 23, 2005 (14:27:00 - 14:31:00 UT, 14 - 16 keV), the December 02, 2003 (22:54:00 - 22:58:00 UT, 18-20 KeV), the February 20, 2002 (11:06:02 - 11:06:34 UT, 22-26 KeV) and the July 23, 2002 events (00:29:10 - 00:30:19 UT, 36-41 KeV). Over the years, these events have been widely studied, and used as test bed for image reconstruction methods (see e.g. [15, 17, 32, 34] and references therein). In addition, these four real events have been used to inspire the generation of the synthetic data analyzed in the previous subsection. Therefore, this analysis represents the perfect complement to the previous subsections, as we study similar source configurations immersed in true solar background noise.

In Figure 11, we show the images of the four flaring events obtained by employing CLEAN, VIS_WV and the Bayesian method proposed here. For all the events, we observe that CLEAN tends to produce overly smooth images, with little separation between different sources, even when these sources are relatively far from each other (third column). In addition, it can be difficult to tell the difference between weaker sources and artifacts, such as in the case of the bottom right source in the fourth column. The reconstructions obtained with VIS_WV are better, inasmuch as they provide more detailed images and allow to better isolate different sources; however, they suffer from the presence of a relatively high number of small spots which should not be present (similar behaviour as in the synthetic case). The Bayesian approach confirms its capability of recovering tidy images even from experimental data. Despite the simplicity of the underlying geometrical model, the reconstructed images do not appear substantially simpler than those obtained with the other methods, while being less affected by spurious sources. Locations and intensities of the estimated sources are in accordance with those obtained with the other methods; the size of the sources estimated by our approach is, at times, smaller than that estimated by the other methods; this fact needs to be investigated further.

5. Conclusions. We described a Bayesian approach to RHESSI imaging that is based on the assumption that the image can be represented by a small set of parametric objects, the
Figure 11: From left to right: comparison of the images obtained for the 23 August 2005, the 02 December 2003, the 20 February 2002, and the 23 July 2002 events, by using CLEAN (first row), VIS_WV (second row) and ASMC (third row).

number of objects being a priori unknown. This setting can be considered as an alternative to classical regularization methods with sparse penalty; here, the parameters in the prior distribution and in the likelihood function (noise standard deviation) play the role of regularization parameters, and the prior on the number of sources is the sparsity-promoting element. We set up a Monte Carlo algorithm that approximates the posterior distribution of our Bayesian model, by constructing an artificial sequence of distributions that starts from the prior distribution and evolves smoothly towards the desired posterior. Interestingly, each distribution of the sequence can be interpreted as a different posterior distribution corresponding to a different, decreasing value of the noise standard deviation; as a consequence, the algorithm explores models of increasing complexity as iterations proceed.

We applied the proposed method to a set of synthetic data, already used to assess the reliability of RHESSI imaging algorithms, and to experimental data. In both cases, the Bayesian approach proved to perform similarly to or better than well known and state-of-the-art methods. Importantly, the Bayesian solutions appeared to have a relatively weak dependence on the parameters in the prior distribution, which implies that little or no tuning is needed to run the algorithm, while the other RHESSI reconstruction methods require an in-depth knowledge of the algorithms to fine-tune the parameters, which can be impractical for certain users and applications. In addition, the proposed algorithm provides posterior probabilities for the estimated parameters, thus allowing to quantify the uncertainty of the
estimated values; such additional information can be important in astrophysics studies (see [34]).

The main drawback of the method is perhaps the relatively high computational cost which is inherent in the sampling procedure. However, taking into account that almost no parameter tuning is needed, the proposed method can be considered relatively competitive in terms of efficiency when compared to the other well known methods. In addition, the algorithm can be quite strongly parallelized in order to reduce the computing time, if necessary.

The positive results of this work encourage further investigation and methodological development. For instance, one could exploit the linear dependence of the data on the source fluxes, and marginalize the posterior distribution using a technique known as Rao–Blackwellization [9], that entails sampling less parameters and therefore having smaller Monte Carlo variance. On a different level, RHESSI images are always obtained for a specific energy interval; nearby energy intervals typically produce similar, but different images; such “dynamic” characteristic of RHESSI data could be exploited by setting up a Monte Carlo filtering procedure [14], in which the posterior distribution at a given energy is used as a prior distribution for the next energy interval.

Finally we remark that, even though in this work we considered only application to RHESSI imaging, the proposed method can be generalized/adapted to any sparse imaging problem, provided that the image can be represented by a relatively small number of parametric objects.

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