Volume of metric balls in Liouville quantum gravity

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Abstract

We study the volume of metric balls in Liouville quantum gravity (LQG). For $\gamma \in (0, 2)$, it has been known since the early work of Kahane (1985) and Molchan (1996) that the LQG volume of Euclidean balls has finite moments exactly for $p \in (-\infty, 4/\gamma^2)$. Here, we prove that the LQG volume of LQG metric balls admits all finite moments. This answers a question of Gwynne and Miller and generalizes a result obtained by Le Gall for the Brownian map, namely, the $\gamma = \sqrt{8}/3$ case. We use this moment bound to show that on a compact set the volume of metric balls of size $r$ is given by $r^{d_\gamma + o(1)}$, where $d_\gamma$ is the dimension of the LQG metric space. Using similar techniques, we prove analogous results for the first exit time of Liouville Brownian motion from a metric ball. Our result implies that the metric measure space structure of $\gamma$-LQG determines its conformal structure.

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1 Introduction

Liouville quantum gravity (LQG) was introduced in the physics literature by Polyakov [25] as a canonical model of two-dimensional random geometry, and has also been shown to be the scaling limit of random planar maps in various topologies (see e.g. [17] [19] and references therein). Let \( h \) be an instance of the Gaussian free field (GFF) on the plane \( \mathbb{C} \), and fix \( \gamma \in (0, 2) \). Formally, the \( \gamma \)-LQG surface described by \((\mathbb{C}, h)\) is the Riemannian manifold with metric tensor given by \( e^{\gamma h}(dx^2 + dy^2) \). The conformal factor \( e^{\gamma h} \) only makes sense formally since the GFF \( h \) does not admit pointwise values. Nevertheless, one can make rigorous sense of the \( \gamma \)-LQG volume measure \( \mu_h \) through the following regularization and renormalization procedure by Duplantier and Sheffield [13]:

\[
\mu_h = \lim_{\varepsilon \to 0} \varepsilon^{-2/\gamma} e^{\gamma h_\varepsilon(z)} dz,
\]

where \( h_\varepsilon(z) \) is the average of \( h \) on the radius \( \varepsilon \) circle centered at \( z \). This falls into the framework of Gaussian multiplicative chaos, see [27, 46, 48, 5]. The circle average mollification can be replaced by other alternatives.

We now explain the recent construction of the LQG metric. For \( \varepsilon > 0 \), let

\[
D_h^\varepsilon(z,w) = \inf_{P: z \to w} \int_0^1 e^{(\gamma/d)\varepsilon^{\gamma}(P(t))} |P'(t)| \, dt,
\]

where \( h_\varepsilon^\gamma \) is a particular mollified version of \( h \) obtained by integrating against the heat kernel, \( d_\gamma \) is the dimension of \( \gamma \)-LQG [10, 25], and the infimum is taken over all piecewise continuously differentiable paths from \( z \) to \( w \). Ding, Dubédat, Dunlap and Falconet [7] proved that for all \( \gamma \in (0, 2) \) the laws of the suitably rescaled metrics \( D_h^\varepsilon \) are tight, so subsequential limits exist as \( \varepsilon \to 0 \) (see also the earlier tightness results [9, 12, 8]). Building on this and several other works [13, 20, 22], Gwynne and Miller [21] showed that all subsequential limits agree and satisfy a natural list of axioms uniquely characterizing the LQG metric. So it makes sense to speak of the LQG metric \( D_h \).

Now, we have the metric-measure space corresponding to \( \gamma \)-LQG. The main result of our paper is the following theorem concerning the volume of metric balls, which answers a question of [21] and generalizes estimates obtained by Le Gall [31] for the Brownian map.

**Theorem 1.1.** Fix \( \gamma \in (0, 2) \) and let \( h \) be a whole-plane GFF normalized to have average zero\(^1\) on the unit circle. Let \( B_s(z; D_h) \) be the \( D_h \)-ball of radius \( s \) centered at \( z \). Then

\[
\mathbb{E}[\mu_h(B_s(0; D_h))^p] < \infty \quad \text{for all } p \in \mathbb{R}.
\]

Moreover, for any compact set \( K \subset \mathbb{C} \) and \( \varepsilon > 0 \), we have almost surely that

\[
\sup_{z \in (0,1)} \sup_{s \in (0,1)} \frac{\mu_h(B_s(z; D_h))}{s^{d_\gamma+\varepsilon}} < \infty \quad \text{and} \quad \inf_{z \in (0,1)} \inf_{s \in (0,1)} \frac{\mu_h(B_s(z; D_h))}{s^{d_\gamma-\varepsilon}} > 0. \tag{1.1}
\]

Consequently, the Minkowski dimension of \( \gamma \)-LQG is \( d_\gamma \) almost surely.

This result is in stark contrast to the LQG volume of a deterministic bounded open set, which only has finite moments for \( p \in (-\infty, 4/\gamma^2) \). Roughly speaking, \( \mu_h(B_s(0; D_h)) \) has finite positive moments because the metric ball \( B_s(0; D_h) \) in some sense avoids regions where \( h \) (and thus \( \mu_h \)) is large.

Similar arguments allow us to prove an analogous result for the first exit time of the Liouville Brownian motion (LBM) from quantum balls. Classically, Brownian motion is well defined on smooth manifolds and on some random fractals. Formally, LBM is Brownian motion associated to the metric tensor \( e^{\gamma h}(dx^2 + dy^2) \), and can be rigorously constructed via regularization and renormalization [16, 3]. For a set \( X \subset \mathbb{C} \) and \( z \in \mathbb{C} \), denote by \( \tau_h(z; X) \) the first exit time of the Liouville Brownian motion started at \( z \) from the set \( X \). When \( X \) is a deterministic bounded open set, \( \tau_h(z; X) \) has finite moments for \( p \in (-\infty, 4/\gamma^2) \). Here, we study the case where \( X \) is given by a quantum ball.

**Theorem 1.2.** Fix \( \gamma \in (0, 2) \) and let \( h \) be a whole-plane GFF normalized to have average zero on the unit circle. Then

\[
\mathbb{E}[\tau_h(0; B_s(0; D_h))^p] < \infty \quad \text{for all } p \in \mathbb{R}.
\]

Moreover, for any compact set \( K \subset \mathbb{C} \) and \( \varepsilon > 0 \), we have at a rate uniform in \( z \in K \) that

\[
\lim_{s \to 0} \mathbb{P}[\tau_h(z; B_s(z; D_h)) \in (s^{d_\gamma+\varepsilon}, s^{d_\gamma-\varepsilon})] = 1.
\]

\(^1\)Throughout the paper we focus on this particular variant of GFF only for concreteness.
As an application of Theorem 1.1 we can extend results of [23] to the case of general $\gamma \in (0,2)$. The following theorem resolves another question of [21].

**Theorem 1.3.** Let $\gamma \in (0,2)$ and $h$ be a whole-plane GFF $h$ normalized to have average zero on the unit circle. Then the field $h$ up to rotation and scaling of the complex plane is almost surely determined by (i.e. measurable with respect to) the random pointed metric measure space $(\mathbb{C},0,D_h,\mu_h)$.

We emphasize that the input is $(\mathbb{C},0,D_h,\mu_h)$ as a pointed metric measure space, so in particular we forget the exact parametrization in the complex plane of $D_h$ and $\mu_h$. For the special case $\gamma = \sqrt{8/3}$, [23] proves an analogous theorem for the quantum disk [2]. Their results depend on the correspondence between the Brownian map and $\sqrt{8/3}$-LQG [37, 38, 10, 11], and rely on the estimates obtained by Le Gall [31] for the Brownian map. Theorem 1.1 provides the estimates needed to generalize the results of [23] to all $\gamma \in (0,2)$, yielding Theorem 1.3 and a statement of the convergence of the simple random walk on a Poisson-Voronoi tessellation of $\gamma$-LQG to Brownian motion (viewed as curves modulo time-parametrization) in the quenched sense; see Section 5.3.

**Paper outline.** In Section 2, we discuss preliminary material about LQG. We prove the finiteness of moments statement of Theorem 1.1 in Sections 3 and 4, which bound the positive and negative moments of the LQG metric $D$. Finally Section 5.3 discusses Theorem 1.3. In the appendix, we recollect some ingredients of the proof by Le Gall for the Brownian map case as a comparison.

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## 2 Background and preliminaries

### 2.1 Notation

For each $\gamma \in (0,2)$, we write $d_\gamma$ for the Hausdorff dimension of $\gamma$-LQG [23] (this was originally introduced in the literature as the “fractal dimension” of $\gamma$-LQG, a scaling exponent associated with models expected to converge to $\gamma$-LQG; see [10, 11, 18]). We also write set the $\gamma$-dependent constants

$$Q = \frac{\gamma}{2} + \frac{2}{\gamma} \quad \text{and} \quad \xi = \frac{\gamma}{d_\gamma}. \quad (2.2)$$

We write $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the floor and ceiling functions evaluated at $x$. We write $|E|$ for the cardinality of a finite set $E$. If $f$ is a function from a set $X$ to $\mathbb{R}^+$ for some $n \geq 1$, we denote the (generalized) $L^p$ norm of $f$ by $||f||_p \equiv \sup_{x \in X} |f(x)|$.

We will denote by $S(\mathbb{C})$ the space of Schwartz functions and by $L^2(\mathbb{C})$ the space of square integrable functions, on $\mathbb{C}$. For $f, g \in L^2(\mathbb{C})$, let $\langle f, g \rangle$ stands for the $L^2(\mathbb{C})$ inner product. Furthermore, $*$ denotes the convolution operator.

In our arguments, it is natural to consider both Euclidean balls and quantum balls. We use the notation $B_r(z)$ to denote the *Euclidean* ball of radius $r$ centered at $z$, and $B_r(z;D_h)$ to denote the *quantum* ball of radius $r$ centered at $z$ (i.e. the ball with respect to the metric $D_h$). We also distinguish the unit disk $\mathbb{D} \equiv B_1(0)$. We denote by $\overline{X}$ the closure of a set $X$. For any $r > 0$ and $z \in \mathbb{C}$, let $A_r(z)$ stand for the annulus $B_r(z) \setminus B_{r/2}(z)$. Furthermore, for $0 < s < r$, we set $A_r(z) := B_r(z) \setminus B_s(z)$.

We recall that a length metric is a metric such that the distance between two points is given by the infimum over the arc lengths of paths connecting the two points. The LQG metric $D_h$ is almost surely a length metric; we write $D_h^U$ for the internal metric on an open set $U \subset \mathbb{C}$, where the set of admissible paths in the variational problems are subset of $U$.

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2In the Brownian map case the first measurability result is due to [39]. However, the proof is non-constructive.
We write \( \int_C f \) for the average of \( f \) over the circle \( C \). For a GFF \( h \), we write \( h_*(z) \) for the average of \( h \) on the circle \( \partial B_r(z) \).

We write \( X \sim N(m, \sigma^2) \) to express that the random variable \( X \) is distributed according to a Gaussian probability measure with mean \( m \) and variance \( \sigma^2 \).

We say that an event \( E_\varepsilon \), depending on \( \varepsilon \), occurs with superpolynomially high probability if for every fixed \( p > 0 \), for all \( \varepsilon \) small enough, \( \mathbb{P}[E_\varepsilon] \geq 1 - \varepsilon^p \). We similarly define events which occur with superpolynomially high probability as a parameter tends to \( \infty \).

### 2.2 The whole-plane Gaussian free field

We give here a brief introduction to the whole-plane GFF. For more details see [42].

Let \( H_{s,0} \) be the Hilbert space closure of smooth compactly supported functions \( f \) on \( \mathbb{C} \), equipped with the Dirichlet inner product

\[
(f, g)_\mathbb{C} = (2\pi)^{-1} \int_C \nabla f(z) \cdot \nabla g(z) \, dz.
\]

Let \( \{f_n\} \) be any orthonormal basis of \( H_{s,0} \). The whole-plane GFF modulo additive constant \( h \) is a random equivalence class of distributions, a representative of which is given by \( \sum_n \alpha_n f_n \) where \( \{\alpha_n\} \) is a sequence of i.i.d. \( \mathcal{N}(0,1) \) random variables. The law of \( h \) does not depend on the choice of \( \{f_n\} \).

For any affine transformation of the complex plane \( \mathbb{A} \), it is easy to verify that \( (f \circ \mathbb{A}, g \circ \mathbb{A})_\mathbb{C} = (f, g)_\mathbb{C} \). Consequently, \( h \) has a law that is invariant under affine transformations: for each \( r, z \in \mathbb{C} \) we have \( h \overset{\text{d}}{=} h(r \cdot +z) \).

Write \( H_{s,0} \subset H_{s,0} \) for the subspace of functions \( f \) with \( \int_C f = 0 \). Although we cannot define \( \langle h, f \rangle \) for general \( f \in H_{s,0} \), the distributional pairing makes sense for \( f \in H_{s,0} \) (the choice of additive constant does not matter). Explicitly, for \( f \in H_{s,0} \) the pairing \( \langle h, f \rangle \) is a centered Gaussian with variance

\[
\text{Var}(\langle h, f \rangle) = \int_{\mathbb{C}^2} f(w) f(z) \log |w - z|^{-1} \, dw dz.
\]

It is easy to check that (2.3) in fact defines the whole-plane GFF modulo additive constant.

We will often fix the additive constant of \( h \), i.e. choose an equivalence class representative. This can be done by specifying the value of \( \langle h, f \rangle \) for some \( f \in H_{s,0} \) with \( \int_C f \neq 0 \), or the average of \( h \) on a circle.

Recalling that \( h_*(z) \) means the circle average of \( h \) on \( \partial B_1(z) \), we will typically work with a whole-plane GFF \( h \) normalized so \( h_1(0) = 0 \) (this is a distribution not modulo additive constant).

Let \( H_1 \subset H_{s,0} \) (resp. \( H_2 \subset H_{s,0} \)) be the Hilbert space completion of compactly supported functions which are constant (resp. have mean zero) on circles \( \partial B_r(0) \). It is easy to verify the orthogonal decomposition \( H_{s,0} = H_1 \oplus H_2 \). This allows us to write the whole-plane GFF \( h \) with \( h_1(0) = 0 \) as the sum of independent fields \( h^1 \) and \( h^2 \); these are respectively the projections of \( h \) to \( H_1 \) and \( H_2 \). Moreover, we can explicitly describe the law of \( h^1 \): Writing \( X_t = h_{e^{-t}}(0) \), the processes \( (X_t)_{t \geq 0} \) and \( (X_{1-t})_{t \geq 0} \) are independent Brownian motions started at zero. The strong Markov property tells us that for any stopping time \( T \) of \( (X_t)_{t \geq 0} \), the random process \( (X_{s+T-T})_{s \geq 0} \) is independent from \( X_T \). Also, by the scale invariance of the whole-plane GFF, the law of \( h^s \) is scale invariant. These observations (with the independence of \( h^1, h^2 \)) give us the following.

**Lemma 2.1.** Let \( h \) be a whole-plane GFF with \( h_1(0) = 0 \), and let \( T \geq 0 \) be a stopping time of the circle average process \( (h_{e^{-t}}(0))_{t \geq 0} \). Then we have, as fields on \( \mathbb{D} \),

\[
h(e^{-T})_{|\mathbb{D}} - h_{e^{-T}}(0) \overset{\text{d}}{=} h_{|\mathbb{D}}.
\]

Moreover, \( h(e^{-T})_{|\mathbb{D}} - h_{e^{-T}}(0) \) is independent of \( h_{e^{-T}}(0) \).

We note that there exist variants of the GFF on bounded domains \( D \subset \mathbb{C} \), such as the zero boundary GFF and the Neumann GFF; we do not go into further detail, but remark that their LQG measures (Section 2.3) are well defined.

Finally, we present a version of the Markov property for the whole-plane GFF, taken from [24] Lemma 2.2. It essentially follows from the orthogonal decomposition \( H_{s,0} = H_2 \oplus H_{\text{harmonic}} \), where \( H_2 \) (resp. \( H_{\text{harmonic}} \)) is the Hilbert space completion of functions which are compactly supported (resp. harmonic) in \( \mathbb{D} \).

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\(^3\)See [13] Section 3] for the construction and properties of the circle averages of \( h \).
Lemma 2.2 (Markov property of GFF). Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \). For each open set \( U \subset \mathbb{C} \) with harmonically non-trivial boundary and \( U \cap \partial \mathbb{D} = \emptyset \), we have the decomposition 

\[
\mu_h = \mu + \hat{h}
\]

where \( \mu \) is a random distribution which is harmonic on \( U \), and \( \hat{h} \) is independent from \( \mu \) and has the law of a zero-boundary GFF on \( U \) (in particular, \( h|_{\partial U} \equiv 0 \)).

2.3 Liouville quantum gravity and Gaussian multiplicative chaos

Fix \( \gamma \in (0, 2) \) and let \( h \) be a GFF plus a random continuous function on a domain \( D \subset \mathbb{C} \). We can define the \( \gamma \)-LQG volume measure or quantum volume measure via the almost sure limit in the vague topology 

\[
\mu_h(dz) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma/2} e^{\gamma h_{\varepsilon}(x)} \, dz
\]

where the limit \( \varepsilon \to 0 \) is taken along powers of two \([14]\). (The limit was shown to hold in probability without the dyadic constraint \([38, 5]\).) Two properties are clear from the form of the above limit. Firstly, \( \mu_h \) is locally defined by \( h \), i.e. for any open set \( U \), the volume \( \mu_h(A) \) is a.s. determined by \( h|_U \). Secondly, we have \( \mu_{h+\mathcal{C}}(\cdot) = e^{\gamma \mathcal{C}} \mu_h(\cdot) \) for any \( \mathcal{C} \in \mathbb{R} \), and slightly more generally, for any random continuous function \( f \) on a compact set \( D \), we have almost surely \( e^{\gamma \inf_D f} \mu_h(D) \leq \mu_{h+f}(D) \leq e^{\gamma \sup_D f} \mu_h(D) \).

Liouville quantum gravity is a special case of Gaussian multiplicative chaos (GMC) introduced by Kahane \([27]\), which considers more general log-correlated fields. More precisely, if \( D \subset \mathbb{C} \) is a bounded domain and \( g \) a continuous function on \( \overline{D} \times \overline{D} \) such that \( K(x, y) = \log |x - y|^{-1} + g(x, y) \) is a nonnegative definite kernel, then one can consider the log-correlated Gaussian field \( \phi \) with covariance kernel \( K \).

Consider then the approximating measures \( \mu_{h_\varepsilon}(dz) = e^{\gamma \varepsilon \phi_{\varepsilon}(z)} \frac{\text{Var} \phi_{\varepsilon}(z)}{\pi} \sigma(dz) \) where \( \sigma \) is a Radon measure on \( \overline{D} \) of dimension at least two and \( \phi_{\varepsilon}(x) \) denotes the circle average approximation of \( \phi \). Then, \( \mu_{h_\varepsilon} \) converges in probability towards a Borel measure \( \mu \) on \( D \) for the topology of weak convergence of measures on \( D \) and the limit is the same for different approximation schemes e.g. when replacing the circle average approximation by another mollification \([5, 48]\). The renormalizations for LQG and GMC are different. We will work with the LQG one when we use the GFF \( h \) and the GMC one when we consider another log-correlated Gaussian field \( \phi \).

We refer the reader to \([2, 4, 46]\) for excellent introductions to the domain.

2.4 LQG volume of Euclidean balls

Tails estimates for the LQG volume of Euclidean balls are quite well understood. It has been known since the work of Kahane \([27]\) and Molchan \([14]\) that it admits finite moments for \( p \in (-\infty, 4/\gamma^2) \). This result contrasts a very different behavior between the right tails and the left tails.

Negative moments. The finiteness of all negative moments goes back to Molchan \([14]\); moreover it is more generally true that for any base measure of the GMC, the total GMC mass has negative moments of all order \([15]\). Duplantier and Sheffield obtained the following more explicit tail behavior \([14, \text{Lemma } 4.5]\): writing \( \mu_h \) for the LQG measure corresponding to a zero boundary GFF \( h \) on \( \mathbb{D} \), they showed that if \( U \subset \subset \mathbb{D} \) is an open set, then there exists \( C, c > 0 \) such that for all \( s > 0 \),

\[
\mathbb{P} [\mu_h(U) \leq e^{-s}] \leq Ce^{-cs^2}. \tag{2.4}
\]

We note that this result is sharp in the sense that

\[
\mathbb{P} [\mu_h(U) \leq e^{-s}] \geq ce^{-Cs^2}.
\]

by a simple application of the Cameron-Martín formula. When \( h \) is replaced by \( h - \int_U h \, dz \), a sharper tail estimate is obtained in \([29]\).
Positive moments. Recently, Rhodes and Vargas [47] obtained a precise asymptotic result about the upper tails of GMC when $\gamma \in (0, 2)$. They obtained a power law and identified the constant. This result has been generalized to a more general family of Gaussian fields in [50], and extended to the critical case $\gamma = 2$ in [49].

As already mentioned, the LQG volume of Euclidean balls has finite $p$ moments for $p < 1/\gamma^2$. This can be easily seen for integers moments $k < 4/\gamma^2$, which we explain below for pedagogical purpose. Indeed, due to the logarithmic correlations of the field, the problem is essentially equivalent to the finiteness of

$$u_k := \int_{\mathbb{D}^k} \frac{dz_1 \ldots dz_k}{\prod_{i<j} |z_i - z_j|^\gamma}.$$

By introducing

$$u_k(r) := \int_{r\mathbb{D}^k} \frac{dz_1 \ldots dz_k}{\prod_{i<j} |z_i - z_j|^\gamma} \quad \text{and} \quad v_k(r) = \int_{r\mathbb{D}^k} \frac{1}{\prod_{i<j} |z_i - z_j|^\gamma} dz_1 \ldots dz_k,$$

we note that when $u_k < \infty$ then $u_k(r) = r^{2k-\gamma^2} k(k-1) u_k$. Furthermore, the $v_k$'s provide the following inductive inequality, obtained by splitting the points $\{z_1, \ldots, z_k\}$ into two well-separated clusters (see Lemma [6.1] in the Appendix for details):

$$v_k(r) \leq C \frac{k}{r^2} \sum_{i=1}^{k-1} r^{-\gamma^2(i-k)} u_k(4r) u_{k-i}(4r) \leq C \frac{k}{r^2} \sum_{i=1}^{k-1} u_i u_{k-i}.$$ 

Finally, we note that

$$k\gamma^2 - \frac{1}{2} \gamma^2 k^2 - 2 = k(2 + \frac{\gamma^2}{2}) - \frac{1}{2} \gamma^2 k^2 - 2 = 2(k-1) - \frac{\gamma^2}{2} k(k-1) > 0 \quad \text{if} \quad 1 < k < 4/\gamma^2,$$

and the conclusion follows from $u_k = \sum_{p \geq 1} v_k(2^{-p})$ and an induction on $k$.

Our later arguments in Section [3.1] follow a similar structure to the above, but also have to account for the random geometry of the quantum ball $B_1(0; D_h)$.

2.5 LQG metric

Recently, a metric for LQG was constructed and characterized in [7, 21], relying on [10, 18, 22, 20, 36]. It is also the limit of an approximation scheme similar to the one of the LQG measure. A Euclidean metric is the unique Euclidean metric $D_h$ determined by a field $h$ (a whole-plane GFF plus a possibly random bounded continuous function) such that the following holds.

I. Length space. $(\mathbb{C}, D_h)$ is almost surely a length space. That is, the $D_h$-distance between any two points in $\mathbb{C}$ is the infimum of the $D_h$-lengths of continuous paths between the two points.

II. Locality. Let $U \subset \mathbb{C}$ be a deterministic open set. Then the internal metric $D_h^U$ is almost surely determined by $h|U$.

III. Weyl scaling. Recall $\xi$ in (2.2). For each continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z, w) := \inf_{P: z \to w} \int_0^{\text{len}(P; D_h)} e^{\xi(f(P(t)))} dt, \quad \text{for all} \ z, w \in \mathbb{C},$$

where we take the infimum over all continuous paths from $z$ to $w$ parametrized by $D_h$-length. Then almost surely $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous $f : \mathbb{C} \to \mathbb{R}$.

IV. Coordinate change for translation and scaling. Recall $Q$ in (2.2). For fixed deterministic $z \in \mathbb{C}$ and $r > 0$ we have almost surely

$$D_h(ru + z, rv + z) = D_h(rz + z) = \|Q \cdot \log_r (u, v)\|_{\mathbb{C}} \quad \text{for all} \ u, v \in \mathbb{C}.$$
Basic estimates for distances. The main quantitative input we need when working with the LQG metric is the following estimate relating the $D_h$-distance between compact sets to circle averages of $h$.

**Proposition 2.3** (Concentration of side-to-side crossing distance [13 Proposition 3.1]). Let $U \subset \mathbb{C}$ be an open set (possibly $U = \mathbb{C}$) and let $K_1, K_2 \subset U$ be disjoint connected compact sets which are not singletons. Then for $r > 0$, it holds with superpolynomially high probability as $A \to \infty$ (at a rate uniform in $r$) that

$$A^{-1} r^{\xi_Q} e^{\xi h_r(0)} \leq D_h^U(r K_1, r K_2) \leq A r^{\xi_Q} e^{\xi h_r(0)}.$$  

This formulation is slightly different from that of [13 Proposition 3.1], but by [13 Remark 3.16] they are equivalent. Note that by taking $r = 1$, this includes the superpolynomial tails of side-to-side crossing distances.

Euclidean balls within LQG balls. The next lemma is an important input in the proof of the finiteness of the negative moments.

**Proposition 2.4** (LQG balls contain Euclidean balls of comparable diameter [23 Proposition 4.5]). Fix $\zeta \in (0, 1)$ and compact $K \subset \mathbb{C}$. Let $h$ be a whole-plane GFF normalized so $h_1(0) = 0$. With superpolynomially high probability as $\delta \to 0$, each $D_h$-metric ball $B \subset K$ with $\text{diam}(B) \leq \delta$ contains a Euclidean ball of radius at least $\delta^{1+\zeta}$.

**Proof.** [23 Proposition 4.5] gives this result with $K$ replaced by $\mathbb{D}$ and with the specific choice $\gamma = \sqrt{8/3}$. To get the result for $K$, we simply note that the whole-plane GFF (viewed modulo additive constant) is scale-invariant, and that the set of all $D_h$-metric balls (viewed as subsets of $\mathbb{C}$) does not depend on the choice of additive constant. To generalize to $\gamma \in (0, 2)$, we remark that the proof of [23 Proposition 4.5] uses only the following few inputs for the $\sqrt{8/3}$ LQG metric, which we ascertain hold for general $\gamma$:

- The scaling relation [23 Lemma 2.3]. In our setting, this is Axiom [11] (Weyl scaling), plus the following easy consequence of Weyl scaling: for $h$ a whole-plane GFF plus a bounded continuous function and $f : \mathbb{C} \to \mathbb{R}$ a (possibly random) bounded continuous function, then a.s.

  $$\exp \left( \xi \inf_{\zeta} f \right) D_h(z, w) \leq D_{h+f}(z, w) \leq \exp \left( \xi \sup_{\zeta} f \right) D_h(z, w)$$  

  for all $z, w \in \mathbb{C}$.

- With probability tending to 1 as $C \to \infty$, the $D_h$-distance from $S = [0, 1]^2$ to $\partial B_{1/2}(S)$ is at least $1/C$ (here, $B_{1/2}(S)$ is the Euclidean 1/2-neighborhood of $S$). This follows immediately from Proposition 2.3

- Fix $n \geq 1$. With probability tending to 1 as $C \to \infty$, each Euclidean ball of radius $e^{-Cn^{2/3}}$ which intersects $[0, 1]^2$ has $D_h$-diameter at most $e^{-n^{2/3}}$. This follows from the fact that $D_h$ is a.s. bi-H"older with respect to the Euclidean metric, and that $e^{-Cn^{2/3}} \to 0$ as $C \to \infty$.

We point out that this is possible to obtain a more quantitative version of this Proposition, with essentially the same arguments as in [23], which can then be used to obtain more precise lower tail estimates for the volume of LQG metric balls.

### 3 Positive moments

The main result of this section is the following.

**Proposition 3.1.** Let $h$ be a whole-plane GFF such that $h_1(0) = 0$. Then, $\mu_h(B_1(0); D_h)$ has finite $k$th moments for all $k \geq 1$. Furthermore, this result still holds if we add to the field $h$ an $\alpha$-log singularity at the origin for $\alpha < Q$, i.e., replace $h$ with $h + \alpha \log | \cdot |^{-1}$.

In the following paragraphs, we present heuristic arguments and an outline of the proof. Recall the definition of the annulus $A_k = B_1(0) \setminus B_{1/2}(0)$. The key difficulty to prove this result is in arguing that $\mathbb{E}[\mu_h(B_1(0); D_h) \cap A_k] < \infty$. Heuristically (since $h$ does not admit pointwise values), we want to prove

$$\mathbb{E} \left[ \int_{(A_k)^k} \prod_{i=1}^k e^{h(x_i)} 1_{D_h(0, x_i) < 1} dz_1 \ldots dz_k \right] < \infty.$$  


By a Gaussian tail estimate, introducing the term lower bound holds

\[ \int \exp \left( \frac{\gamma^2}{2} \sum_{i < j} \text{Cov}(h(z_i), h(z_j)) \right) P \left[ D_{h+\gamma \sum_i \text{Cov}(h(z_i), h(z_j))}(0, z_i) < 1 \right] dz_1 \ldots dz_k < \infty. \]  

(3.7)

A first heuristic. We present a heuristic explaining why \( E \left[ \mu_k(B_1(0; D_h) \cap A_k^k) \right] < \infty \). As remarked above and since \( h \) is log-correlated, this is bounded from above by

\[ \int \prod_{1 < j} \frac{P_{z_1, \ldots, z_k}}{|z_i - z_j|^2} dz_1 \ldots dz_k \quad \text{where} \quad P_{z_1, \ldots, z_k} = P \left[ D_{h+\gamma \sum_i \text{Cov}(h(z_i), h(z_j))}(z_i, \partial B_{1/2}(z_i)) < 1 \right] \text{for all } i. \]

The blows up of moments for Euclidean ball volume comes from the contribution of clusters at mutual distance \( r \). Indeed, for such clusters \( \{z_1, \ldots, z_k\} \), the singularities contributes as \( \prod_{1 < j} |z_i - z_j|^{-\gamma} \approx r^{-\gamma/2} \), on a macroscopic domain, we have \( r^{-2} \) possibilities for placing this cluster and the volume associated is \( r^{2k} \). The total contribution is then \( r^{-2+2k} \gamma^2 \) and the sum over dyadic \( r \) is finite if and only if \( k < 4/\gamma^2 \). Now, we explain how this is counterbalanced by the \( P_{z_1, \ldots, z_k} \) term when \( k \geq 4/\gamma^2 \). By the annulus crossing distance bound from Proposition 2.3 for any \( z \in K = \{z_1, \ldots, z_k\} \), the following lower bound holds

\[ 1 \geq D_{h+\gamma \sum_i \log |z_i|} (z, \partial B_{1/2}(z)) \geq r^{Q} c_k r^{-\gamma/2}. \]

By a Gaussian tail estimate, introducing the term \( c_k = k \gamma - Q \geq \frac{1}{\gamma} \gamma - Q = 2 \gamma / 2 > 0 \), we have

\[ P_{z_1, \ldots, z_k} \leq P [ h_r(z) \leq -c_k \log r^{-1} ] \approx r^{\frac{\gamma}{2} c_k}. \]

An elementary computation, namely \( -2 + 2k - \gamma/2 + \frac{1}{2} c_k^2 = \frac{1}{2} Q^2 - 2 \), gives then that for such a cluster, the scale \( r \) contribution is \( r^{\frac{1}{2} Q^2 - 2} \), which is summable for all \( k \) since \( Q = \frac{1}{2} + \frac{1}{2} > 2 \) for \( \gamma \in (0, 2) \) and this is essentially the reason of the finiteness of all moment.

Outline of the proof. To turn this argument into a proof requires us to take care of all configurations of clusters \( K = \{z_1, \ldots, z_k\} \). Similarly to the one presented in Section 2.4 our proof works by induction on \( k \). We will partition \( K = \{z_1, \ldots, z_k\} \) into two clusters \( I \) and \( J \) such that the pairwise distance of points between \( I \) and \( J \) is \( \geq r \), since both \( \prod_{1 < j} |z_i - z_j|^{-\gamma} \) and \( P_{z_1, \ldots, z_k} \) have a nice hierarchical clusters structure (see 3.13) for the exact splitting procedure partitioning \( K = I \cup J \) and the definition of \( r \). Indeed, for such a cluster, we can bound from above

\[ \prod_{1 < j} |z_i - z_j|^{-\gamma} \leq r^{-|I| |J| \gamma} \prod_l |z_a - z_b|^{-\gamma} \prod_l |z_a - z_b|^{-\gamma}. \]  

(3.8)

Now, we discuss \( P_{z_1, \ldots, z_k} \). The aforementioned annuli crossing distance bounds imply that for all \( z \in K \), \( x \in (0, 1/2) \),

\[ h_c(z) + \gamma \sum_{z_n \in K} \int_{\partial B_{\varepsilon}(z)} \log |\cdot - z_n|^{-1} + x \leq Q \log \varepsilon^{-1}, \]

(3.9)

for \( x = 0 \). From now, denote by \( \bar{P}_{z_1, \ldots, z_k}^x \) the circle average variant of \( P_{z_1, \ldots, z_k} \), where the conditions on distances defining \( P_{z_1, \ldots, z_k} \) are replaced by the conditions (3.9) with this extra parameter \( x \in \mathbb{R} \), which is necessary when deriving an inductive inequality. Note that when \( I \) and \( J \) are at distance at least \( r \) and the diameters of both \( I \) and \( J \) are smaller than \( O(r) \), for \( x \in (0, r) \), then

\[ \forall z, z_0 \in K, \int_{\partial B_{\varepsilon}(z)} \log |\cdot - z_0|^{-1} \approx \log r^{-1} \quad \text{and} \quad \forall z_i \in I, z_j \in J, \int_{\partial B_{\varepsilon}(z_i)} \log |\cdot - z_j|^{-1} \approx \log r^{-1}. \]

Therefore, we can rewrite the condition (3.9) for \( z \in I \) as follows

\[ (h_c(z) - h_r(z)) + \left( \gamma \sum_{z_{1, j} \in J} \int_{\partial B_{\varepsilon}(z)} \log |\cdot - z_{1, j}|^{-1} + |J| \gamma \log r^{-1} \right) - k \gamma \log r^{-1} 
\]

\[ + (x + h_r(z) + k \gamma \log r^{-1} - Q \log r^{-1}) \leq Q \log (\varepsilon/r)^{-1}. \]
Hence, after simplification, for \( z \in I \), we have
\[
(h_z(z) - h_{\ast}(z)) + \gamma \sum_{z_i \in J} \int_{\partial B_z(z)} \log |r - z_i/r|^{-1} + (x + h_{\ast}(z) + c_k \log r^{-1}) \leq Q \log (\varepsilon/r)^{-1},
\]
which is a variant of (3.9). Furthermore, note that the processes \((h_z(z) - h_{\ast}(z))_{z \in \{0, r\}} \) are approximately independent and \( h_{\ast}(z) \approx h_{\ast}(w) \) for all \( z, w \in K \), which we then denote by \( X_{\ast} \). By the properties of the circle average processes, we get
\[
\hat{P}_K^{X_{\ast}} \lesssim \mathbb{E} \left[ 1_{X_{\ast} + c_k \log r^{-1} \leq \hat{P}_I/r} \hat{P}_I^{X_{\ast} + c_k \log r^{-1}} \right], \tag{3.10}
\]
which is the hierarchical structure we were looking for. Altogether, (3.8) and (3.10) allow to inductively bound from above the term
\[
\int_{A_k} \prod_{i<j} |z_i - z_j|^\gamma \, dz_1 \ldots dz_k,
\]
by a quantitative estimate in term of \( x \). This provides not only \( \mathbb{E}[\mu_h(B_1(0; D_h) \cap A_1)^{k}] < \infty \) but also a quantitative estimate which allows to get \( \mathbb{E}[\mu_h(B_1(0; D_h) \cap A_1^{k}) < \varepsilon^{\alpha_k} \) for some \( \alpha_k > 0 \) and all \( s \in (0, 1) \), via a standard scaling/decoupling argument. An application of Hölder’s inequality shows \( \mathbb{E}[\mu_h(B_1(0; D_h) \cap \mathbb{D})^{k}] < \infty \) and similar techniques concludes that \( \mathbb{E}[\mu_h(B_1(0; D_h)^{k}) < \infty \), yielding the proof of Proposition 3.1.

In our implementation of these ideas, because we have to carry the Euclidean domains, we use *-scale invariant fields. The short-range correlation of the fine field gives independence between well-separated clusters, and invariance properties of the *-scale invariant field simplifies our multiscale analysis.

In Section 3.1, we prove a quantitative variant of (3.7) where the field \( h \) is replaced by a *-scale invariant field plus some constant, and the probability in the integrand is replaced by the probability of coarse-field distance approximations being less than 1. In Section 3.2, we use these estimates to first bound \( \mathbb{E}[\mu_h(B_1(0; D_h) \cap A_1^{k})] \), by using a truncated moment estimate, then extend our arguments to all annuli to deduce the finiteness of the \( k \)th moment \( M_k := \mathbb{E}[\mu_h(B_1(0; D_h)^{k})] \) for all \( k \geq 1 \). By keeping track of the \( k \) dependence, it turns out that it is possible to bound \( M_k \) by \( cK_{k}^{c_2} \) for some constants \( C, c \) depending only on \( \gamma \). To simplify the presentation of our arguments, we omit these precise estimates.

### 3.1 Inductive estimate for the *-scale invariant field

We derive a key estimate for the positive moments (Proposition 3.8), which is like a quantitative version of (3.7) where we add a constant to the field. We will use *-scale invariant fields, which satisfy properties convenient for multiscale analysis. Relevant references are [1, 2, 20].

**Proposition 3.2** (*-scale decomposition of \( h \)). The whole plane GFF \( h \) normalized so \( h_1(0) = 0 \) can be written as
\[
h = g + \phi = g + \phi_1 + \phi_2 + \ldots
\]
where the fields \( g, \phi_1, \phi_2, \ldots \) satisfy the following properties:

1. \( g \) and the \( \phi_n \)’s are continuous centered Gaussian fields.
2. The law of \( \phi_n \) is invariant under Euclidean isometries.
3. \( \phi_n \) has finite range dependence with range of dependence \( e^{-n} \), i.e. the restrictions of \( \phi_n \) to regions with pairwise distance at least \( e^{-n} \) are mutually independent.
4. \( (\phi_n(z))_{z \in \mathbb{R}^2} \) has the law of \( (\phi_1(ze^{-n-1}))_{z \in \mathbb{R}^2} \).
5. The \( \phi_n \)’s are mutually independent fields.
6. The covariance kernel of \( \phi \) is \( C_{0, \infty}(z, z') = -\log |z - z'| + q(z - z') \) for some smooth function \( q \).
7. We have \( \mathbb{E}[(\phi_n(z))^2] = 1 \) for all \( n, z \).

**Proof.** Lemma 6.3 gives the coupling \( h = g + \phi \) with \( g \) continuous. The fields \( \phi_n \) are defined in Appendix 6.2 and are shown to satisfy these properties there. \( \square \)
Define also the field $\phi_{a,b}$ from scales $a$ to $b$ via
\[
\phi_{a,b} := \begin{cases} 
\phi_{a+1} + \cdots + \phi_b & \text{if } a < b \\
0 & \text{if } a \geq b
\end{cases}
\] (3.11)
so that $\phi = \phi_{0,\infty}$ and set, for $z, z' \in \mathbb{C},$
\[
C_{a,b}(z, z') := \mathbb{E} \left( \phi_{a,b}(z) \phi_{a,b}(z') \right).
\] (3.12)
We will construct a hierarchical representation of a set of points $K = \{z_1, \ldots, z_k\} \subset \mathbb{C}$. Roughly speaking, starting with $K$, we will iteratively split each cluster into smaller clusters that are well separated. We formalize the splitting procedure below.

**Splitting procedure.** Define for any finite set $S$ of points in the plane (with $|S| \geq 2$) the separation distance $s(S)$ to be the largest $t \geq 0$ for which we can partition $S = I \cup J$ such that $d(I, J) \geq t$, i.e.
\[
s(S) := \max_{S=I\cup J, |I|, |J| \geq 1} d(I, J).
\] (3.13)
Define $I_S, J_S \subset S$ to be any partition of $S$ with $d(I, J) = s(S)$. Note that if diam $S$ denote the diameter of the set $S$, we have the following inequality
\[
\frac{\text{diam } S}{|S|} \leq s(S) \leq \text{diam } S.
\] (3.14)
For the edge case where $|S| = 1$ define $s(S) = 0$.

**Lemma 3.3.** For $|S| \geq 2$, we have $s(I_S), s(J_S) \leq s(S)$.

**Proof.** It suffices to prove the lemma for $S$ such that all pairwise distances in $S$ are distinct, then continuity yields the result for general $S$. Suppose for the sake of contradiction that $s(J) > s(S)$, then there is a partition $J = J_1 \cup J_2$ satisfying $d(J_1, J_2) > s(S)$. Since distances are pairwise distinct, we must have $d(I, J_1) = s(S)$ and $d(I, J_{3-i}) > s(S)$ for some $i$. Then $d(I \cup J_i, J_{3-i}) = \min(d(I, J_{3-i}), d(J_i, J_{3-i})) > s(S)$. This contradicts the definition of $s(S)$. \qed

**Hierarchical structure of $K = \{z_1, \ldots, z_k\}$ and definition of $T^K_{\gamma} (\{\phi\})$.** By iterating the splitting procedure above, we can decompose a set $K = \{z_1, \ldots, z_k\} \subset \mathbb{C}$ into a binary tree of clusters. This decomposition into hierarchical clusters is unique for Lebesgue typical points $\{z_1, \ldots, z_k\}$. Two vertices in this tree are separated by the separation distance of their first common ancestor. See Figure 1 for an illustration.

A labeled (binary) tree is a rooted binary tree with $k$ leaves. For each $K = \{z_1, \ldots, z_k\} \subset \mathbb{C}$, collection of fields $\{\phi\} = \{\phi_{a}\}_{a \geq 0}$, and nonnegative integer $a \leq \lfloor \log s(K)^{-1} \rfloor$ we will define a labeled binary tree denoted by $T^K_{\gamma} (\{\phi\})$. Each internal vertex of this tree is labeled with a quadruple $(S, m, \psi, \eta)$ with $S \subset K$ and $|S| \geq 2$, an integer $m$, and pairs $\psi, \eta \in \mathbb{R}$, whereas each leaf is labeled with just a singleton $(z) \subset K$. The truncated labels $(S, m)$ depend only on the recursive splitting procedure described above: $S$ is one of the clusters associated with this hierarchical cluster decomposition, and $m = \lfloor \log s(S)^{-1} \rfloor$.

For such a labeled tree $T$ we write $T + (\psi_0, \eta_0)$ to be the tree obtained by replacing each internal vertex label $(S, m, \psi, \eta)$ with $(S, m, \psi + \psi_0, \eta + \eta_0)$. We also write Left$(S)$ to denote the leftmost point of $S$, viz. arg min$_{z \in S} \Re(z)$, where $\Re(z)$ denotes the real part of the complex number $z$.

We explain how the remaining parts $(\psi, \eta)$ of the labels are obtained. For $(K, (\phi), a)$ as above, we proceed as follows to complete the definition of the labeled tree $T^K_{\gamma} (\{\phi\})$. For $|K| = 1$, we simply set $T^K_{\gamma} (\{\phi\})$ to be the tree with one vertex, labeled with the singleton $K$. For $K > 1$, setting $m := \lfloor \log s(K)^{-1} \rfloor \geq a$, the root vertex of $T^K_{\gamma} (\{\phi\})$ is labeled $(K, m, \phi_{a,m}(\text{Left}(K)), (m - a)k\gamma)$, and its two child subtrees are given by $T^{m}_{\gamma} (\{\phi\}) + \phi_{a,m}(\text{Left}(K)), (m - a)k\gamma)$ and $T_{\gamma}^{m-a} (\{\phi\}) + \phi_{a,m}(\text{Left}(K)), (m - a)k\gamma)$. Essentially, after making the split $K = I \cup J$, we add up the contribution of the coarse field $\phi_{a,m}$ and the contribution of the $\gamma$-log singularities to get the scale $m$ field approximation for the clusters $I$ and $J$.

We note that the tree structure of $T^K_{\gamma} (\{\phi\})$ is deterministic, and for each internal vertex with label $(S, m, \psi, \eta)$, only $\psi = \psi(\{\phi\})$ is random; the other components are deterministic. Roughly speaking, $S$ is a cluster in our hierarchical decomposition, $m$ is the scale of the cluster (i.e. $s(S) \approx e^{-m}$), $\psi$ (resp. $\eta$) approximates a radius $e^{-m}$ circle average of the field $\phi_{a,m}$ (resp. $\gamma \sum_{z \in K} \log |z - \cdot|^{-1} - \gamma k\gamma$) at the cluster.
Remark 3.4. For the labeled tree $\mathcal{T}_K^a(\{\phi\})$, at each internal vertex the field approximation $\psi$ can be explicitly described in terms of the fields $\{\phi\}$ as follows. Let $(S_i, m_i, \psi_i, \eta_i)$ for $i = 1, \ldots, n$ be a path from the root $(S_1, m_1, \psi_1, \eta_1)$ to $(S_n, m_n, \psi_n, \eta_n)$. Then, writing $m_0 = a$, we have

$$\psi_n = \sum_{i=1}^{n} \phi_{m_{i-1},m_i}(\text{Left}(S_i)).$$

(3.15)

The $\gamma$-singularity approximation $\eta$ can likewise be stated non-recursively, as

$$\eta_n = \gamma \sum_{i=1}^{n} (m_i - m_{i-1}) |S_i|.$$

(3.16)

Remark 3.5. The choice $\text{Left}(S_i)$ is arbitrary; any other deterministic choice of point in $S_i$ works. Replacing $\phi_{m_{i-1},m_i}(\text{Left}(S_i))$ with the average $|S_i|^{-1} \sum_{z \in S_i} \phi_{m_{i-1},m_i}(z)$ would also work without affecting our proofs much.

Definitions of key observables. In this paragraph, we provide analogous definitions of the quantities appearing in (3.7). The first one corresponds to a variant of $\mathbb{P}[\sum_{j} \text{Cov}(h(z_j),h(\cdot))(0,z_i) < 1$ for all $i]$, with an extra parameter $x$. For $x \in \mathbb{R}$, let $P_{K}^{a,x}$ be the probability that the tree with random labels $\mathcal{T}_K^a(\{\phi\})$ satisfies

$$\psi + \eta + x \leq Q(m - a) \quad \text{for each internal vertex labeled} \ (S, m, \psi, \eta).$$

(3.17)

Note that this probability is taken over the randomness of the fields $\{\phi\}$, and that this definition yields for $|K| = 1$ that $P_{K}^{a,x} = 1$. Let us comment a bit on this definition and its relation with the conditions $D_{h+\gamma} \sum_{j} \text{Cov}(h(z_j),h(\cdot))(0,z_i) < 1$. These distances being less than one implies upper bounds for annuli crossing distances for annuli separating the origin from the singularities. The $\psi$ term corresponds to field average over these annuli, $\eta$ is an approximation for the $\gamma$-singularities and the $Q$ term stands for the scaling of the metric. Altogether, roughly speaking, $P_{K}^{a,x}$ is the probability that for the field $\phi_{0,\infty} + \sum_{z \in K} \gamma \log |z - i|^{-1} + x$, for all clusters $S$ of $K$ the field-average approximation of annulus-crossing distances near $S$ is less than 1.

The following observable stands for a variant of the integral in (3.7). Writing $K = \{z_1, \ldots, z_h\}$ and $dz_K = dz_1 \ldots dz_h$, we define

$$u_{k}^{x}(x) := \int_{B(0,a)^h} \frac{P_{K}^{a,x}}{\prod_{i<j} |z_i - z_j|^2} 1_{s(K) \leq x} dz_K.$$

(3.18)

In Proposition 3.8 we show that $u_{k}^{x}(x) < \infty$, and bound it in terms of $x$. Note that the statement $u_{k}^{x}(x) < \infty$ is comparable to (3.7) by the fact that $\exp(\gamma \text{Cov}(h(z_i),h(z_j))) \asymp |z_i - z_j|^{-\gamma^2}$.

The next lemma establishes basic properties of $P_{K}^{a,x}$. To state it, we first define

$$c_{k} := k\gamma - Q.$$

(3.19)

Lemma 3.6. The $P_{K}^{a,x}$’s satisfy the following properties:
1. Monotonicity: $P_{K}^{0,x}$ is decreasing in $x$.

2. Markov decomposition: for the partition $I_{k} \cup J_{k} = K$ with separation distance satisfying $e^{-m} \leq s(K) < e^{-m+1}$ (i.e. for $r = e^{-m}$, we have $r \leq s(K) < cr$) we have

$$P_{K}^{0,x} = \mathbb{E} \left[ 1_{X_{r} + x + c_{k} \log r^{-1} \leq 0} P_{I_{k}}^{\log r^{-1} X_{r} + x + c_{k} \log r^{-1}, J_{k}} \right],$$

where $X_{r} = \phi_{0,m} (\text{Left}(K))$ is a centered Gaussian with variance $\log r^{-1}$.

3. Scaling: $P_{r_{1} \cdots r_{k}}^{0,x} = P_{r_{1} \cdots r_{k}}^{0,x}$ for any $r = e^{-m}$ with $m \in \mathbb{Z}$.

4. Invariance by translation: $P_{r_{1} \cdots r_{k} + w \cdots w}^{0,x} = P_{r_{1} \cdots r_{k}}^{0,x}$.

The first condition corresponds to a shift of the field. The second condition is an identity with three terms in the right-hand side: the term $X_{r}$ represents the coarse field, the indicator says that the “coarse field approximation of quantum distances” at Euclidean scale $r$ are less than 1, and the product of the two other terms represent a Markovian decomposition conditional on the coarse field. Properties 3 and 4 are clear from the translation invariance and scaling properties of $\phi_{n}$.

**Proof.** The monotonicity Property 1 is clear from the definition.

Property 2 follows from the inductive definition of $P_{K}^{0,x}$, by looking at the first split $K = I \cup J$. Indeed, we can take $X_{r} = \phi_{0,m} (\text{Left}(K))$. The event $\{X_{r} + x + c_{k} \log r^{-1} \leq 0\}$ corresponds to inequality (3.17) for the root vertex $(K, m, \phi_{0,m}(\text{Left}(K)), mk\gamma)$.

Then, if the set $K$ is decomposed as $K = I \cup J$, note that the trees $T_{r}^{m}(\{\phi\})$ and $T_{r}^{m}(\{\phi\})$ are independent. Indeed, $d(I, J) \geq e^{-m}$, so the restrictions of the field $\phi_{m}$ (and each finer field) to $I$ and $J$ are independent. Therefore, since $(\phi_{0,m}(\text{Left}(K)), T_{r}^{m}(\{\phi\}), T_{r}^{m}(\{\phi\}))$ are independent, conditionally on $\phi_{0,m}(\text{Left}(K))$, the trees $T_{r}^{m}(\{\phi\}) + (\phi_{0,m}(\text{Left}(K)), mk\gamma)$ and $T_{r}^{m}(\{\phi\}) + (\phi_{0,m}(\text{Left}(K)), mk\gamma)$ are independent. Thus, all conditions in the definition of $P_{K}^{0,x}$ associated to the child subtrees are conditionally independent. To conclude, we just have to explain that this is indeed the term $f_{r}^{m, X_{r} + x + c_{k} m}$ which appears. For a non-root vertex $(S, b, \psi, \eta)$ of $T_{K}^{m}$ belonging to the genealogy of $I$, the condition (3.17) can be rewritten,

$$\psi + \eta + x = (X_{r} + \psi') + (mk\gamma + \eta') + x \leq Qb = Q(b - m) + Qm,$$

hence $\psi' + \eta' + (X_{r} + x + c_{k} m) \leq Q(a - m)$, which is exactly the condition we were looking for at the vertex $(S, b, \psi', \eta')$ in the tree $T_{r}^{m}(\{\phi\})$.

The scaling Property 3 follows from the scaling property of the $\phi_{m}$ and the observation that $s(rK) = rs(K)$ (and hence $\lceil \log s(rK)^{-1} \rceil = \log r^{-1} + \lceil \log s(K)^{-1} \rceil$).

The invariance by translation Property 4 follows from the translation invariance of the fields $\phi_{m}$. \[\square\]

Using these properties, we derive the following inductive inequality.

**Lemma 3.7.** For each $n, k > 0$, there exists a constant $C_{n,k}$ such that the following inductive inequality holds, for all $x \in \mathbb{R}$, where $X_{r} \sim \mathcal{N}(0, \log r^{-1})$.

$$u_{k}^{n}(x) \leq C_{n,k} \sum_{r = e^{-m}, m \geq 0} \sum_{r = e^{-m}, m \geq 0} r^{k+1} \mathbb{E} \left[ 1_{X_{r} + x + c_{k} \log r^{-1} \leq 0} u_{k}^{6}(X_{r} + x + c_{k} \log r^{-1}) u_{k}^{6}(X_{r} + x + c_{k} \log r^{-1}) \right].$$

We now turn to the proof of the inductive relation. The argument is close to that of Lemma 6.1, the difference being that we have to take care of the decoupling of $P_{K}^{0,x}$. \[\square\]

**Proof.** We first introduce some notation. In what follows we will be integrating over $k$-tuples of points $z_{1}, \ldots, z_{k}$; write $K$ for this collection of points and $dz_{K} = dz_{1} \cdots dz_{k}$. Write $f(K) := \prod |z - z'|^{-\gamma z^{2}/2}$ where the product is taken over all pairs $z, z' \in K$ with $z \neq z'$.

We first split the integral as

$$u_{k}^{n}(x) = \sum_{r \in (0, 1]} v_{k}^{n}(x, r)$$

where for $r \in (0, 1]$, $v_{k}^{n}(x, r)$ is defined by

$$v_{k}^{n}(x, r) := \int_{B(0, r)^{k}} P_{K}^{0,x} f(K) 1_{r \leq s(K) \leq e^{-m}} dz_{K}. \quad (3.20)$$
Notice that $s(K) \leq er$ implies $\text{diam } K \leq ekr$, so any $K$ contributing to the integral in (3.20) is contained in a ball of radius $6kr$ centered in $r\mathbb{Z}^2 \cap B(0,n)$. Taking a sum over the $O(n^2r^{-2})$ such balls and by translation invariance, we get the bound

$$v_n^k(x,r) \leq O(n^2r^{-2}) \int_{B(0,6kr)^c} P_{K}^{0,z} f(K)1_{r \leq s(K) \leq er} dz_K.$$ 

Write $K = I_K \cup J_K$ for the partition described before Lemma 3.3. For $z \in I_K$ and $z' \in J_K$ we have $|z - z'|^{-2} \leq s(K)^{-2} \leq r^{-2}$, and $s(I_K), s(J_K) \leq s(K) \leq er$ by Lemma 3.3 so

$$v_n^k(x,r) \leq O(n^2r^{-2}) \int_{B(0,6kr)^c} r^{-2}|I_K||J_K| P_{K}^{0,z} f(I_K)1_{s(I_K) \leq er} f(J_K)1_{s(J_K) \leq er} dz_K.$$ 

The Markov property decomposition Lemma 3.6 allows us to split $P_{K}^{0,z}$ into an expectation over a product of terms, yielding an upper bound of $v_n^k(x,r)$ as an integral of terms which 'split' into $z_{I_K}$ and $z_{J_K}$ parts. This expression is in terms of the partition $I_K \cup J_K = K$; we can upper bound it by summing over all $I, J \subset K$. To be precise, for each $i = 1, \ldots, k-1$ we sum over all pairs $I, J \subset K$ with $|I| = i$ and $|J| = k - i$. Absorbing the combinatorial terms like $\binom{k}{i}$ and the prefactor $n^2$ into the constant $C_{n,k}$, we get

$$v_n^k(x,r) \leq C_{n,k}r^{-2} \sum_{i=1}^{k-1} r^{-\gamma i(k-1)} \mathbb{E}_{x, \gamma} \left[ 1_{X_r + x + c_k \log r^{-1} < \gamma i(k-1)} \right] \int_{B(0,6kr)^c} \frac{P_{\log r^{-1}, X_r + x + c_k \log r^{-1}}^{0,z, s(z_1, \ldots, z_i) \leq \epsilon r} dz_1 \cdots dz_i }{\prod_{a < b} |z_a - z_b|^\gamma}.$$ 

We analyze the first integral. Changing the domain of integration from $B_{6kr}(0)^i$ to $B_{6k}(0)^i$, we get

$$\int_{B_{6kr}(0)^i} \frac{P_{\log r^{-1}, X_r + x + c_k \log r^{-1}}^{0,z, s(z_1, \ldots, z_i) \leq \epsilon r} dz_1 \cdots dz_i }{\prod_{a < b} |z_a - z_b|^\gamma} = \int_{B_{6k}(0)^i} \frac{P_{\log r^{-1}, X_r + x + c_k \log r^{-1}}^{0,z, s(z_1, \ldots, z_i) \leq \epsilon r} dz_1 \cdots dz_i }{\prod_{a < b} |z_a - z_b|^\gamma},$$

and then applying the scaling property of $P$, the integral on the right hand side is equal to

$$\int_{B_{6k}(0)^i} \frac{P_{\gamma r \log r^{-1}, \epsilon r, s(z_1, \ldots, z_i) \leq \epsilon r} dz_1 \cdots dz_i }{\prod_{a < b} |z_a - z_b|^\gamma} = u^k_4(X_r + x + c_k \log r^{-1}).$$

By gathering the previous bounds and identities, and noting that the power of $r$ is

$$r^{-2 - \gamma i(k-1) + 2k - \gamma^2 i(k-1) - \gamma^2(k-1)} = r^{\gamma k - \frac{1}{2} \gamma^2 k^2 - 2},$$

we have

$$v_n^k(x,r) \leq C_{n,k}r^{\gamma k Q - \frac{1}{2} \gamma^2 k^2 - 2} \sum_{i=1}^{k-1} \mathbb{E}_{x, \gamma} \left[ 1_{X_r + x + c_k \log r^{-1} < \gamma i(k-1)} u^k_4(X_r + x + c_k \log r^{-1}) \right].$$

This completes the proof of the inductive inequality.

We use the inductive relation and the base case, we derive the following proposition, which provides a bound on the quantity (3.18) introduced at the beginning of the section.

**Proposition 3.8.** Recall that $c_k = k \gamma - Q$. For $x \in \mathbb{R}$ we have

$$u_n^k(x) \leq C_{n,k} e^{-\gamma k x} \quad \text{when } k \geq 4/\gamma^2,$$

and

$$u_n^k(x) \leq C_{n,k} \quad \text{when } k < 4/\gamma^2,$$

where $C_{n,k}$ is a constant depending only on $n, k$. 

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\[ u_k(x) \leq \int_{B_n(0)} \prod_{i<j} |z_i - z_j|^{-\gamma^2} \, dz_1 \ldots dz_k, \]

and the right-hand side is finite by the discussion in Section 2.4.

Now consider \( k \geq 4/\gamma^2 \). We proceed inductively, assuming that Proposition 3.8 has been shown for all \( k' < k \). Lemma 3.7 gives us the bound

\[ u_k^n(x) \leq C_n,k \sum_{i=1}^{k-1} \sum_{r=\epsilon^m, m \geq 0} r^{k|Q| - \frac{1}{2} \gamma^2 k^2 - 2} \mathbb{E} \left[ 1_{X_r + x + c_k \log r^{-1} - 1 \leq 0} u_k^{6k}(X_r + x + c_k \log r^{-1}) \right], \]

where \( X_r \sim \mathcal{N}(0, \log r^{-1}) \). We bound each term \( u_k^{6k} \) using the inductive hypothesis. We need to split into cases based on which bound of Proposition 3.8 is applicable (i.e. based on the sizes of \( i, k - i \)), but the different cases are almost identical, so we present the first case in detail and simply record the computation for the remaining cases.

**Case 1:** \( i, k - i \geq 4/\gamma^2 \). By the inductive hypothesis we can bound the \( i \)th term of (3.21) by a constant times

\[ \sum_{r=\epsilon^m, m \geq 0} r^{k|Q| - \frac{1}{2} \gamma^2 k^2 - 2 + c_k - Q} e^{-(c_k - Q)x} \mathbb{E} \left[ e^{-(c_k - Q)x} 1_{X_r + x + c_k \log r^{-1} - 1 \leq 0} \right], \]

where we have used the identity \( c_k - c_{k-1} = c_k - Q \). For each \( r \) we can write the expectation in the equation (3.22) by a Cameron-Martin shift as

\[ \mathbb{E}[e^{-(c_k - Q)x} \mathbb{P}[X_r + x + c_k \log r^{-1} - (c_k - Q)] \mathbb{V}(X_r) \leq 0] = r^{-(c_k - Q)^2} \mathbb{P}[X_r \leq -(Q \log r^{-1} + x)]. \]

We claim that

\[ \mathbb{P}[X_r \leq -(Q \log r^{-1} + x)] \leq r^{{\frac{1}{4}}Q^2} e^{-Qx}. \]

Indeed, in the case where \( Q \log r^{-1} + x \geq 0 \), we have by a standard Gaussian tail bound that

\[ \mathbb{P}[X_r \leq -(Q \log r^{-1} + x)] \leq e^{-(Q \log r^{-1} + x)^2} = r^{{\frac{1}{2}}Q^2} e^{-Qx} \leq r^{{\frac{1}{4}}Q^2} e^{-Qx}, \]

and in the cases where \( Q \log r^{-1} + x < 0 \) we have

\[ \mathbb{P}[X_r \leq -(Q \log r^{-1} + x)] \leq 1 \leq e^{-Q(Q \log r^{-1} + x)} = r^{{\frac{1}{2}}Q^2} e^{-Qx} \leq r^{{\frac{1}{4}}Q^2} e^{-Qx}. \]

Finally, we combine (3.22), (3.23) and (3.24) to upper bound the \( i \)th term of (3.21). This upper bound is a sum over \( r \) of terms of the form \( r^{k|Q| - \frac{1}{2} \gamma^2 k^2 - 2 + c_k - Q} e^{-c_k x} \), where the power is

\[ k|Q| - \frac{1}{2} \gamma^2 k^2 - 2 + c_k - Q = \frac{1}{2} (c_k - Q)^2 + \frac{1}{2} Q^2 = \frac{1}{2} Q^2 - 2 > 0. \]

So we can bound the \( i \)th term of (3.21) by a constant times

\[ \sum_{r=\epsilon^m, m \geq 0} r^{{\frac{1}{4}}Q^2} e^{-c_k x} = O(e^{-c_k x}). \]
Case 2: \( i \geq 4/\gamma^2 \) and \( k - i < 4/\gamma^2 \). By the inductive hypothesis we can bound the \( i \)th term of (3.21) by a constant times
\[
\sum_{r=\epsilon-m} r^{k \gamma Q - \frac{1}{2} \gamma^2 k^2 - 2} e^{r^{-1} \left( X_r + x + c_k \log r^{-1} \right)} 1_{X_r + x + c_k \log r^{-1} \leq 0}
\]
\[
= \sum_{r=\epsilon-m} r^{k \gamma Q - \frac{1}{2} \gamma^2 k^2 - 2 + c_k} e^{-c_k x} e^{-c_k x} \mathbb{P} \left[ X_r \leq -(c_k - c_i) \log r^{-1} + x \right]
\]
\[
\leq \sum_{r=\epsilon-m} r^{k \gamma Q - \frac{1}{2} \gamma^2 k^2 - 2 + c_k} e^{-c_k x} \frac{1}{2} e^{-c_k x} e^{-c_k x}
\]
\[
= \sum_{r=\epsilon-m} r^{1/2} Q^2 - 2 e^{-c_k x} = O(e^{-c_k x}).
\]

Note that by symmetry Case 2 also settles the case where \( i < 4/\gamma^2 \) and \( k - i \geq 4/\gamma^2 \).

Case 3: \( i, k - i < 4/\gamma^2 \). By the inductive hypothesis we can bound the \( i \)th term of (3.21) by a constant times
\[
\sum_{r=\epsilon-m} r^{k \gamma Q - \frac{1}{2} \gamma^2 k^2 - 2} \mathbb{P} \left[ X_r \leq -(c_k \log r^{-1} + x) \right] \leq \sum_{r=\epsilon-m} r^{k \gamma Q - \frac{1}{2} \gamma^2 k^2 - 2 + \frac{1}{2} \gamma^2} e^{-c_k x}
\]
\[
= \sum_{r=\epsilon-m} r^{1/2} Q^2 - 2 e^{-c_k x} = O(e^{-c_k x}).
\]

This completes the proof. \( \square \)

The proof of Proposition 3.8 depends on the exponent \( \frac{1}{2} Q^2 - 2 = \frac{1}{2} \left( \frac{2}{\gamma} - \frac{2}{\gamma^2} \right)^2 \) being positive. If we make a slight perturbation to our definitions, so long as the resulting exponent is still positive, we get a variant of Proposition 3.8. In particular, for \( \delta > 0 \), we define \( P_k^{\mu, \xi, \delta} \) similarly to \( P_k^{\mu, \xi} \) by replacing the inequality (3.17) with \( \psi + \eta + x \leq (Q + \delta)(m - a) \), and define \( u_k^{\mu, \xi, \delta} \) analogously to (3.18) with \( P_k^{\mu, \xi, \delta} \). We record the following result as a corollary since the proof follows the same steps as in the proof of Proposition 3.8.

**Corollary 3.9.** For \( k \geq 1, \delta \in (0, 1/2) \) and \( n \geq 1 \), there exist constants \( C_{n,k,\delta} \) and \( c_{k,\delta} \) such that,
\[
u_k^{n,\delta}(x) \leq C_{n,k,\delta} e^{-c_{k,\delta} x} \]

for all \( x \in \mathbb{R} \) when \( k \geq 4/\gamma^2 \),

and
\[
u_k^{n,\delta}(x) \leq C_{n,k,\delta} \]

for all \( x \in \mathbb{R} \) when \( k < 4/\gamma^2 \).

Furthermore, \( \lim_{\delta \to 0} c_{k,\delta} = k \gamma - Q \) for fixed \( k \).

**Remark 3.10.** Alternatively, one could modify the definition of \( u_k^{n}(x) \) in (3.18) to have a different denominator \( |z_i - z_j|^{\gamma^2 + \delta} \).

### 3.2 Moment bounds for the whole-plane GFF

In this section, we use our previous estimate to obtain the moment bounds for a whole-plane GFF \( h \) such normalized such that \( h_1(0) = 0 \) and therefore prove Proposition 3.1. For the reader’s convenience, we have broken the proof in several steps. Additionally, in this section we write \( C \) or \( C_{k,\delta} \) to represent large constants depending only on \( k \) and \( \delta \), and may not necessarily represent the same constant in different contexts or equations.
Proxy estimate for whole-plane GFF

Recall the notation $A_{s,r} := B_s(0) \setminus B_r(0)$ for $0 < s < r$. We introduce the following proxy

$$P^c_{s,r} := \{ z \in \mathbb{C} : D_h(z, \partial B_{r/s}(z)) \leq d \}.$$  \hfill (3.25)

The set $P^c_{s,r}$ contains points whose “local distances” are small. We work with $P^c_{s,r}$ because the event $z \in P^c_{s,r}$ depends only on the field $h|_{\partial B_{r/s}(z)}$, and is thus more tractable than the event $z \in B_s(0) \cap B_r(0)$ (which depends on the field in a more “global” way). Moreover we have $B_s(0) \cap A_r \subset P^c_{s,r} \cap A_r$, so to bound from above $\mu_h(B_s(0) \cap B_r(0))$ it suffices to bound from above the volume of the proxy set.

**Proposition 3.11.** Let $h$ be a whole-plane GFF such that $h_1(0) = 0$. For $k \geq 4/\gamma^2$, $\delta \in (0, 1/2)$, there exists a constant $C_{k, \delta}$ such that for all $x \in \mathbb{R}$,

$$E \left[ \mu_h \left( B_{10}(0) \cap P^1_{h, e^{-c_x}} k \right) \right] \leq C_{k, \delta} e^{-c_k \delta^2},$$

where we recall that $c_k = k \gamma - Q$ and $c_k, \delta \rightarrow c_k$ as $\delta \rightarrow 0$.

In fact, for $x > 0$ it is possible, by using tail estimates for side-to-side distances, to show that the decay is Gaussian in $x$. We do not need this result so we omit it.

**Proof.** In order to keep the key ideas of the proof transparent, we postpone the proofs of some intermediate elementary lemmas to the end of this section. Consider the collection of balls

$$\mathcal{B} = \left\{ B_{e^{-\ell}}(z) : \ell \in \mathbb{N}_0, z \in e^{-\ell - 2\gamma^2}B_{e^{-\ell}}(z) \cap B_{10}(0) \neq \emptyset \right\}.$$  \hfill (3.26)

We will work with three events in the proof: $E_{\delta, M}$ is a global regularity event, $F_{K, \delta, M}$ is a field average approximation of the event $\{ K \subset P^1_{h, e^{-c_x}} \}$, and $F_{K, \delta, M}$ is a variant of $F_{K, \delta, M}$ where $\gamma$-log singularities are added to the field at the points $z \in K$ (this is related to $P^\delta_{K, x}$. The integer $k$ is fixed throughout the proof, so the events are allowed to depend on $k$ and we omit it in the notation.

**Step 1: truncating over a global regularity event** $E$. The event $E_{\delta, M}$ is given by the following criteria:

1. For all $\ell \geq 0$, the annulus crossing distance of $B\setminus 0.99 B$ is at least $M^{-\ell}e^{-\ell^2/4 + \delta}e^{-\ell Q}e^\ell \int_{0B} h$ for all $B \in \mathcal{B}$ with radius $e^{-\ell}$.
2. For all integers $\ell > \ell' \geq 0$, for all $B \in \mathcal{B}$ of radius $e^{-\ell'}$, we have $e^{-\ell'} \sup_{g \in B} |\nabla \phi_{r, \ell'}| \leq e^{-\ell} + \log M$.
3. For all $\ell \geq 0$ and all $B \in \mathcal{B}$ of radius $e^{-\ell - 2}$$e^\ell \int_{0B} \phi_{r, \ell} \leq e^{-\ell} + \log M$.
4. $\| h - h \|_{10} = \| g \|_{10} \leq \log M$.

As we see later in Lemma 3.13, for fixed $\delta$ the event $E_{\delta, M}$ occurs with superpolynomially high probability in $M$ as $M \rightarrow \infty$. Therefore, when looking at moments of $\mu_h(0; D_h) \cap D)$, one can restrict to moments truncated on $E_{\delta, M}$.

By using Property 4 of $E_{\delta, M}$ and the definition of $\mu_\phi$ as a Gaussian multiplicative chaos (see Section 2.3), we get

$$E \left[ \mathbb{I}_{E_{\delta, M}} \mu_h \left( B_{10}(0) \cap P^1_{h, e^{-c_x}} k \right) \right] \leq C_k M^{\gamma k} E \left[ \mathbb{I}_{E_{\delta, M}} \mu_\phi \left( B_{10}(0) \cap P^1_{h, e^{-c_x}} k \right) \right] = C_k M^{\gamma k} E \left[ \int_{B_{10}(0)^{\times k}} \mathbb{I}_{E_{\delta, M}} \mathbb{I}_{\{ z_i \in P^1_{h, e^{-c_x}} \}} \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right] \leq C_k M^{\gamma k} E \left[ \int_{B_{10}(0)^{\times k}} \mathbb{I}_{F_{K, \delta, M}} \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right],$$

where the event $F_{K, \delta, M}$ is defined in the following lemma. In the first inequality above, the constant $C_k$ appears from the difference of definition between Gaussian multiplicative chaos measures and the Liouville quantum gravity measure; the former one is defined by renormalizing by a pointwise expectation whereas the latter one by $e^{\frac{\gamma^2}{2}}$. 

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Lemma 3.12. For $k \geq 2$, there exists a constant $C$ so that for any $k$-tuple of points $K = \{z_1, \ldots, z_k\} \subset \mathbb{D}$ we have the inclusion of events

$$E_{\delta,M} \cap \{z_i \in P_h^{1,\epsilon-x}\text{ for all } i = 1, \ldots, k\} \subset F_{K,\delta,M}$$

where $F_{K,\delta,M}$ is the event that for all vertices $(S, m, \psi, \eta)$ of $T_h^0(\{\phi\})$ we have

$$\psi + x < Qm + Cm^{1+\delta} + C \log M. \quad (3.27)$$

Essentially, Lemma 3.12 holds because $K \subset P_h^{1,\epsilon-x}$ implies that distances near each cluster are small. Then for each cluster, Property 1 of $E_{\delta,M}$ lets us convert bounds on distances to bounds on circle averages of $h$, Property 2 lets us replace the coarse field circle average with the coarse field evaluated at any nearby point, and Properties 3 and 4 allow us to neglect the fine field and the random continuous function $h - \phi$; this gives (3.27).

Step 2: shifting LQG mass as $\gamma$-singularities. We then use the following lemma to replace the terms $\mu_\phi(dz_i)'s$ by $dz_i$ and $\gamma$-singularities.

Lemma 3.13. If $f$ is a bounded nonnegative measurable function, and $C_{a,b}$ are the covariances of $\phi_{a,b}$ (defined as in [3.12]), we have

$$E \left[ \int_{B_{10}(0)^k} f(\phi, z_1, \ldots, z_k, \phi_1, \ldots, \phi_k, \ldots) \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right] \leq \int \int_{B_{10}(0)^k} \mathbb{P}[E_{\delta,M}] \exp(\frac{\gamma^2}{2} \sum_{i \neq j} C_{0,\infty}(z_i, z_j)) dz_1 \ldots dz_k,$$

where $E_{\delta,M}$ is the event that in the labeled tree $T_h^0(\{\phi\})$, for any path from the root $(S_1, m_1, \psi_1, \eta_1)$ to $(S_n, m_n, \psi_n, \eta_n)$, we have

$$\psi_n + \gamma \sum_{i=1}^n \sum_{z \in K} C_{m_{i-1},m_i}(z, \text{Left}(S_i)) + x \leq Qm_n + Cm_n^{1+\delta} + C \log M. \quad (3.28)$$

Note that by Lemma 3.14 below, (3.28) implies that for each vertex $(S_n, m_n, \psi_n, \eta_n)$ we have

$$\psi_n + \eta_n + x \leq (Q + \delta)m_n + C \log M + 2C. \quad (3.29)$$

(The term $2C$ comes from Lemma 3.14 and the bound $Cm_n^{1+\delta} \leq \delta m_n + C$, using that $\delta \in (0,1/2)$.) Now, the probability that (3.29) occurs for each vertex is precisely $P_{K,\delta,M}^{0,x-C \log M - 2C\delta}$, defined in just before the Corollary 3.9, so we conclude that $\mathbb{P}[E_{\delta,M}] \leq P_{K,\delta,M}^{0,x-C \log M - 2C\delta}$.

Lemma 3.14. For $k \geq 2$, there exists $C_k$ such that for $K \subset B_{10}(0)^k$, for any path from the root $(S_1, m_1, \psi_1, \eta_1)$ to $(S_n, m_n, \psi_n, \eta_n)$ in the labeled tree $T_h^0(\{\phi\})$ we have, writing $m_0 = 0$,

$$|\eta_n - \gamma \sum_{i=1}^n \sum_{z \in K} C_{m_{i-1},m_i}(z, \text{Left}(S_i))| = |\gamma \sum_{i=1}^n (m_i - m_{i-1}) |S_i| - \gamma \sum_{i=1}^n \sum_{z \in K} C_{m_{i-1},m_i}(z, \text{Left}(S_i))| < C.$$

By Proposition 3.2 for $K \subset B_{10}(0)$ we have $\exp(\frac{\gamma^2}{2} \sum_{i \neq j} C_{0,\infty}(z_i, z_j)) \leq C \prod_{i<j} |z_i - z_j|^{-\gamma^2}$, and the above bounds yields

$$E \left[ \mathbb{P}_{E_{\delta,M}} \left( B_{10}(0) \cap P_h^{1,\epsilon-x} \right)^k \right] \leq C_k M^{\gamma k} \int_{B_{10}(0)^k} \prod_{i<j} \frac{P_{0,x-C \log M - 2C\delta}^{m_i-m_j}}{|z_i - z_j|^2} dz_1 \ldots dz_k.$$

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Finally, by Corollary 3.9 we conclude that for all $x \in \mathbb{R}$ we have

$$
\mathbb{E} \left[ \mathbb{I}_{E_{\delta,M}} \mu_h \left( B_{10}(0) \cap P_h^{1, e^{-c_0 x}} \right)^k \right] \leq C_{k,\delta} M^C e^{-c_k \delta x}.
$$

(3.30)

**Step 3: concluding the proof.** By Markov’s inequality, we get,

$$
\mathbb{P}[\mu_h(B_{10}(0) \cap P_h^{1, e^{-c_0 x}}) \geq t] \leq \mathbb{P}[E_{\delta,M}^c] + \mathbb{P}[E_{\delta,M}, \mu_h(B_{10}(0) \cap P_h^{1, e^{-c_0 x}}) \geq t] \\
\leq \mathbb{P}[E_{\delta,M}^c] + t^{-k} \mathbb{E}[1_{E_{\delta,M}} \mu_h(B_{10}(0) \cap P_h^{1, e^{-c_0 x}})^k].
$$

The second term is bounded by (3.30). To control the first term, we use the following lemma.

**Lemma 3.15.** For fixed $\delta \in (0,1/2)$, the regularity event $E_{\delta,M}$ occurs with superpolynomially high probability as $M \to \infty$.

Combining these bounds we get, for all $\delta, k, p$, a constant $C_{k,p}$ such that for all $x \in \mathbb{R}$ and for all $M, t > 0$,

$$
\mathbb{P}[\mu_h(B_{10}(0) \cap P_h^{1, e^{-c_0 x}}) \geq t] \leq C_{k,\delta,p} \left( M^{-p} + t^{-k} M^C e^{-c_k \delta x} \right).
$$

By taking $M = t^{k/(p+C)} e^{c_k x/(p+C)}$ and choosing $p$ large, we get $\mathbb{E}[\mu_h(B_{10}(0) \cap P_h^{1, e^{-c_0 x}})^{k-\delta}] \leq C e^{-(c_k \delta - \delta)x}$. Then, by (3.30) and the Cauchy-Schwartz inequality, we get

$$
\mathbb{E} \left[ \mu_h \left( B_{10}(0) \cap P_h^{1, e^{-c_0 x}} \right)^k \right] \leq C_{k,\delta,M} e^{-c_k \delta x} + \mathbb{E}[E_{\delta,M}^c]^{1/2} \mathbb{E} \left[ \mu_h \left( B_{10}(0) \cap P_h^{1, e^{-c_0 x}} \right)^{2k} \right]^{1/2},
$$

and we conclude by taking $M = e^{[x]}$ for some small $x > 0$ (indeed, for this choice of $M$ we have $\mathbb{P}[E_{\delta,M}] \leq e^{-a|x|}$ for any $a > 0$, and our earlier bound says that the 2kth moment is at most exponential in $x$).

**Annuli contributions and $\alpha$-singularities.**

Here, we use the proxy estimate to study moments of metric balls when the field has singularities. The link is made with the following deterministic remark. Recall that $A_{r/2} := B_{r/2}(0) \setminus B_{r/4}(0)$. If $z \in B_h(0; D_h) \cap A_{r/2}$ then $D_h(0, \partial B_{r/4}(0)) \leq 1$ and $z \in P_h^{r-1-D_h(0, \partial B_{r/4}(0))}$ (recall (3.25) for the definition of $P_h^{r-1}$).

**Lemma 3.16 (Small annuli).** Let $h$ be a whole-plane GFF such that $h_1(0) = 0$. Then for $\alpha < Q$,

$$
\mathbb{E} \left[ \mu_{h+\alpha \log |z|-1} \left( B_{10}(0; D_h+\alpha \log |z|-|z|) \cap \mathbb{D} \right)^k \right] < \infty.
$$

**Proof.** Note that $B_h(0; D_h) \cap A_{r/2} \subset P_h^{r-1} \cap A_{r/2}$ and that the latter one is measurable with respect to the field $h|_{B_{r/2}(0)}$. We use a decoupling/scaling argument as follows. We write,

$$
\mu_h(B_1(0; D_h) \cap A_{r/2}) \leq 1_{D_h(0, \partial B_{r/4}(0)) \leq \mu_h(P_h^{r-1} \cap A_{r/2})} \leq e^{c_{h,r}(0) \mu_h - h_r(0)} \left( A_{r/2} \cap P_h^{r, e^{-c_{h,r}(0)}} \right).
$$

and set $\tilde{h} := h(r) - h_r(0)$. By Lemma 2.1 we have the equality in law $\tilde{h} \stackrel{(d)}{=} h_1 \| h \|_2$, and also $\tilde{h} \| h \|_2$ is independent of $h_r(0)$. Using the scaling of the metric and of the measure, we get

$$
\mathbb{E} \left[ \mu_h(B_1(0; D_h) \cap A_{r/2})^k \right] \leq e^{c_{h,r}(0) \mu_h - h_r(0)} \left( A_{r/2} \cap P_h^{r, e^{-c_{h,r}(0)}} \right)^k \leq e^{c_{h,r}(0) \mu_h - h_r(0)} \left( A_{r/2} \cap P_h^{r, e^{-c_{h,r}(0)}} \right)^k.
$$

In what follows, we will forget about the term $D_h(0, \partial B_{r/4}(0))$. Indeed, one can split the expectation with $1_{D_h(0, \partial B_{r/4}) \leq \epsilon}$ and $1_{D_h(0, \partial B_{r/4}) \geq \epsilon}$. Note first that for $p > 1$, by Proposition 3.11 and a moment computation for the exponential of a Gaussian variable with variance constant times $\log r^{-1}$,

$$
\mathbb{E} \left[ e^{c_{h,r}(0) \mu_h} \left( A_{r/2} \cap P_h^{r, e^{-c_{h,r}(0)}} \right)^k \right] \leq C r^{\text{power}},
$$

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for some power whose value does not matter. Indeed, because of the superpolynomial decay of the event \( \{ D_k(0, \partial B_1) \leq r^2 \} \), the quantity
\[
E \left[ 1_{D_k(0, \partial B_1) \leq r^2} e^{\gamma_k h_{19}(0)} \mu_{\tilde{h}} \left( A_{1/2} \cap P^1_{\tilde{h}} e^{-\gamma_k h_{19}(0)} \right)^k \right]
\]
\[
\leq P[D_k(0, \partial B_1) \leq r^2] 1/4 \leq r^{k1/4} E \left[ e^{\gamma_k h_{19}(0)} \mu_{\tilde{h}} \left( A_{1/2} \cap P^1_{\tilde{h}} e^{-\gamma_k h_{19}(0)} \right)^k \right]^{1/p}
\]
decays superpolynomially fast in \( r \), by using Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \). From now, we truncate the event \( \{ D_k(0, \partial B_1) \geq r^2 \} \) and we bound from above
\[
r^{k1/4} E \left[ 1_{\mu_{\tilde{h} + \beta h_{19}(0)} \sum_{\xi \leq 1} e^{\gamma_k h_{19}(0)} \mu_{\tilde{h}} \left( A_{1/2} \cap P^1_{\tilde{h}} e^{-\gamma_k h_{19}(0)} \right)^k \right].
\]
By Proposition 3.11, since \( A_{1/2} \subset B_{10}(0) \) and \( h_{19}(0) \) is independent of \( \tilde{h} \), by writing \( c_{k, \beta} = k\gamma - Q + \alpha \beta \) for some small \( \alpha, \beta, k \), we get
\[
r^{k1/4} E \left[ 1_{\mu_{\tilde{h} + \beta h_{19}(0)} \sum_{\xi \leq 1} e^{\gamma_k h_{19}(0)} \mu_{\tilde{h}} \left( A_{1/2} \cap P^1_{\tilde{h}} e^{-\gamma_k h_{19}(0)} \right)^k \right] \]
\[
\leq C_k r^{k1/4} \left( e^{(Q - \alpha) h_{19}(0)} \right) \leq C_k r^{Q - \alpha} E \left[ e^{(Q - \alpha) h_{19}(0)} \right].
\]
Furthermore, by the Cameron–Martin formula for a Gaussian variable,
\[
E \left[ 1_{\mu_{\tilde{h} + \beta h_{19}(0)} \sum_{\xi \leq 1} e^{\gamma_k h_{19}(0)} \mu_{\tilde{h}} \left( A_{1/2} \cap P^1_{\tilde{h}} e^{-\gamma_k h_{19}(0)} \right)^k \right] \leq E \left[ e^{(Q - \alpha) h_{19}(0)} \right],
\]
so, altogether, we get the bound
\[
E \left[ \mu_h (B_1(0; D_h) \cap A_{r/2})^k \right] \leq C_{k\gamma} r^{Q^2 + \beta \delta},
\]
for some arbitrarily small \( \beta \). Furthermore, note (details are left to the reader) that when one replaces \( h \) by \( h + \alpha \log |\cdot|^{-1} \) for \( \alpha < Q \) (this additional term is easily bounded from above and from below on \( A_r \)), we get
\[
E \left[ \mu_h + \alpha \log |\cdot|^{-1} \left( B_1(0; D_h + \alpha \log |\cdot|^{-1}) \cap A_{r/2} \right)^k \right] \leq C_{k\gamma} r^{Q^2 + \beta \delta}. \quad (3.31)
\]
We can conclude as follows. Set \( V_{\gamma, \alpha} := \mu_h + \alpha \log |\cdot|^{-1} \left( B_1(0; D_h + \alpha \log |\cdot|^{-1}) \cap A_r \right) \). By monotone convergence,
\[
E \left[ \mu_h + \alpha \log |\cdot|^{-1} \left( B_1(0; D_h + \alpha \log |\cdot|^{-1}) \cap D \right)^k \right] = \lim_{n \to \infty} E \left[ \sum_{i=0}^n V_{\gamma, \alpha}^k \right].
\]
We introduce some deterministic \( \Lambda > 1 \) to be chosen. By Hölder’s inequality we get
\[
\left( \sum_{i=0}^n V_{\gamma, \alpha}^k \right)^k \leq \left( \sum_{i=0}^n \Lambda^i (V_{\gamma, \alpha}^k)^i \right) \left( \sum_{i=0}^n \Lambda^{-i} \right)^{k-1}.
\]
Taking expectations, and using the bound \( (3.31) \), we get, uniformly in \( n \),
\[
E \left[ \sum_{i=0}^n V_{\gamma, \alpha}^k \right] \leq \left( \frac{1}{1 - \Lambda^k} \right)^{k-1} \sum_{i=0}^{\infty} \Lambda^{k2^{-i}} \left( Q - \alpha \right)^2 + \beta \delta.
\]
Taking \( \Lambda \) close enough to one such that \( \Lambda^{k2^{-i}} (Q - \alpha)^2 + \beta \delta < 1 \), this series is absolutely convergent, as desired. \( \square \)

**Lemma 3.17** (Large annuli). Let \( h \) be a whole-plane GFF such that \( h_1(0) = 0 \). Then, for \( \alpha < Q \),
\[
E \left[ \mu_h + \alpha \log |\cdot|^{-1} \left( B_1(0; D_h + \alpha \log |\cdot|^{-1}) \cap C \right)^k \right] < \infty.
\]
Proof. The proof follows the same approach as before by using the proxy estimate and a decomposition over annuli with a scaling argument. We point out here only the main differences with the previous proof.

Write $D_h(0, \partial B_{R/4}(0)) =: R^Q \mathcal{E}^{h, R/4(0)} X_R$. By using the remark at the beginning of this subsection,

$$
E[\mu_h(B_1(0; D_h) \cap A_R)] \leq E[1_{D_h(0, \partial B_{R/4}(0))} \leq 1 \mu_h(P_{R}^{h, \alpha}) \cap A_R]^k
$$

Again because of the superpolynomial decay of $P(X_R < R^{-\delta})$ and the room at the level of exponent (and a Hölder inequality) we will continue the computation without $X_R$ as follows. By using that $h - h_{R/4(0)}|_{A_{R/2 R}}$ is independent of $h_{R/4}(0)$ and that the proxy $P_{R,x}^{h, x} \cap A_r$ is measurable with respect to $h|_{A_{R/2, R}}$, we get by scaling,

$$
E(1_{R^Q \mathcal{E}^{h, R/4(0)}} \leq 1 \mu_{h-R/4(0)}(P_{h-R/4(0)}^{h, \alpha} \cap A_R)] = R^{k \gamma Q} E(1_{R^Q \mathcal{E}^{h, R/4(0)}} \leq 1 \mu_{h}(P_{h}^{h, \alpha}) \cap A_{1})^k
$$

At this stage we use the estimate from Proposition 3.11. Since we will have some room at the level of exponent, we don’t carry on the $\delta$ in the computation. Therefore, we compute

$$
R^{k \gamma Q} E(1_{h_{R/4}(0)} \leq -Q \log R e^{k \gamma h_{R/4(0)}(h - c_R h_{R/4(0)} + Q \log R)}) = R^{k \gamma Q} e^{-c_R Q \log R} E(1_{h_{R/4}(0)} \leq -Q \log R e^{k \gamma h_{R/4(0)}})
$$

and by using the Cameron-Martin formula we get

$$
R^{k \gamma Q} e^{-c_R Q \log R} E(1_{h_{R/4}(0)} \leq -Q \log R e^{k \gamma h_{R/4}(0)}) \approx R^{k \gamma Q} E(1_{h_{R/4}(0)} \leq -Q \log R e^{k \gamma h_{R/4}(0)} - \frac{1}{2} Q \log R)
$$

$$
\approx R^{k \gamma Q} E(1_{h_{R/4}(0)} \leq -2Q \log R) \approx R^{k \gamma Q} R^Q
$$

The rest of the proof follows the same lines.

\[ \square \]

Proof of Proposition 3.11. Let $h$ be a whole-plane GFF such that $h_1(0) = 0$ and fix $\alpha < Q$. The proof follows easily by writing

$$
\mu_{h+\alpha \log |\cdot|^{-1}}(B_1(0; D_{h+\alpha \log |\cdot|^{-1}})) = \mu_{h+\alpha \log |\cdot|^{-1}}(B_1(0; D_{h+\alpha \log |\cdot|^{-1}}) \cap C) + \mu_{h+\alpha \log |\cdot|^{-1}}(B_1(0; D_{h+\alpha \log |\cdot|^{-1}}) \cap C \setminus D)
$$

and using the inequality $(x + y)^k \leq 2^{k-1}(x^k + y^k)$ together with Lemma 3.16 and Lemma 3.17.

\[ \square \]

Lemma 3.18 (Upper bound for small metric balls). For $\varepsilon \in (0, 1)$, $k \geq 1$, there exists a constant $C_{k, \varepsilon}$ such that for all $s \in (0, 1)$,

$$
E[\mu_h(B_s(0; D_h))] \leq C_{k, \varepsilon} s^{k d_y - \varepsilon}
$$

Proof. The proof is very similar to the one of Lemma 3.16, therefore we omit the details and just provide the differences. By replacing 1 by $s$ in the proof, we get $E[\mu_h(B_h^0 \cap A_r)] \leq C_s^{k d_y - \varepsilon} Q^2$ where $c_y = \frac{d_y}{2} Q$. By using Hölder’s inequality, we get $E[\mu_h(B_h^0 \cap A_r)^k] \leq C_s^{k d_y - \varepsilon} Q^2$. We then take $p$ such that $c_y/p < \varepsilon$ and the rest of the proof follows the same line as those of Lemma 3.16.

\[ \square \]

Proofs of the intermediate lemmas for Proposition 3.11

We recall here the definition of the event $E_{\delta, M}$ (recall the definition of $\mathcal{B}$ in (3.26)). It is given by the following criteria:

1. For all $\ell \geq 0$, the annulus crossing distance of $B \setminus 0.99 B$ is at least $M^{-\varepsilon} \varepsilon^{1+\delta} e^{-\varepsilon Q\ell} e^{\varepsilon h \mathcal{B}}$ for all $B \in \mathcal{B}$ with radius $e^{-\varepsilon}$.
2. for all integers $\ell > \ell' > 0$, for all $B \in \mathcal{B}$ of radius $e^{-\ell}$, we have $e^{-\ell} \sup_{\|B \|B} |\nabla \phi_{\ell, t}| \leq \ell^{1+\delta} + \log M$.
3. for all $\ell \geq 0$ and for all $B \in \mathcal{B}$ of radius $e^{-\ell'}$, $\int_B \phi_{\ell, t} \leq \ell^{1+\delta} + \log M$.

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4. and \( \| \phi - h \|_D = \| g \|_D \leq \log M \).

**Proof of Lemma 3.12.** We prove here that for any \( k \)-tuple of points \( K = \{ z_1, \ldots, z_k \} \subset D \) we have

\[
E_{k,M} \cap \{ z_i \in P \_1 \_e^{-\xi} \} \quad \text{for all } i = 1, \ldots, k
\]

\[
\subset \{ \psi + x \leq Qm + 8k^2 (m^{\frac{1}{2} + \delta} + \log M) \} \quad \text{for each vertex } (S, m, \psi, \eta) \text{ of } T_K^0 \{ \{ \phi \} \}.
\]

Fix \( K \) and consider any vertex \( (S, m, \psi, \eta) \) of \( T_K^0 \{ \{ \phi \} \} \). Recall first that by (3.15),

\[
\psi = \psi_n = \sum_{i=1}^{n} \phi_{m_{i-1}, m_i}(\text{Left}(S_i)),
\]

where we write \( (S_i, m_1, \psi_i, \eta_i) \) for the path from the root \( (S_1, m_1, \psi_1, \eta_1) \) to \( (S_n, m_n, \psi_n, \eta_n) = (S, m, \psi, \eta) \).

The proof is to compare a circle average around \( z \in S \) (which can be bounded since \( z \in P \_1 \_e^{-\xi} \)) with the right-hand side above. Pick any point \( z \in S \). Since \( z \in P \_1 \_e^{-\xi} \),

\[
D_h(z, \partial B_{z-m-1}(z)) \leq D_h(z, \partial B_{1/4}(z)) \leq e^{-\xi},
\]

and we can find a ball \( B \in B \), centered at a point in \( e^{-m-2}Z^2 \) with radius \( e^{-m-2} \) whose boundary separates \( z \) from \( \partial B_{z-m-1}(z) \). Hence its crossing distance is at most \( e^{-\xi} \). By Property 1 we have,

\[
M^{-\xi} e^{-\xi}(m+2\xi) e^{-\xi}(m+2\xi) e^{-\xi} f_{AB} h \leq e^{-\xi} e^{-\xi},
\]

or equivalently

\[
\int_{\partial B} h + x \leq Q(m + 2) + (m + 2)^{\frac{1}{2} + \delta} + \log M.
\]

Now we lower bound \( \int_{\partial B} h \) in term of (3.32) by using properties 2, 3 and 4 of \( E_{k,M} \).

- By Property 4 we have

\[
\int_{\partial B} h \geq \sum_{i=1}^{n} \int_{\partial B} \phi_{m_{i-1}, m_i} + \int_{\partial B} \phi_{m, \infty} - \log M.
\]

- For each \( i \), notice that \( z \in S_i \), and so \( d(z, \text{Left}(S_i)) \leq eke^{-m_i} \) by (3.14). Consequently, by Property 2 we have for each \( i = 1, \ldots, n \)

\[
\int_{\partial B} \phi_{m_{i-1}, m_i} \geq \phi_{m_{i-1}, m_i}(\text{Left}(S_i)) - 4km^{\frac{1}{2} + \delta} - 4k \log M.
\]

- By Property 4 we have

\[
\int_{\partial B} \phi_{m, \infty} \geq -m^{\frac{1}{2} + \delta} - \log M.
\]

Combining these yields (see Remark 3.4)

\[
\int_{\partial B} h \geq \sum_{i=1}^{n} \phi_{m_{i-1}, m_i}(\text{Left}(S_i)) - 6k^2 m^{\frac{1}{2} + \delta} - 6k^2 \log M = \psi - 6k^2 m^{\frac{1}{2} + \delta} - 6k^2 \log M.
\]

Together with (3.33), this gives \( \psi + x \leq Qm + 8k^2 (m^{\frac{1}{2} + \delta} + \log M) \) and concludes the proof. \( \square \)

**Proof of Lemma 3.13** This is an application of the Cameron-Martin theorem. We outline here the main idea, assuming for notational simplicity that the function \( f \) depends only on \( \phi, z_1, \ldots, z_k \). The argument works the same way for \( f \) depending also on \( (\phi_n)_{n \geq 0} \).
Assume first that \( f \) is continuous. Fix \( k \geq 2 \), \( \delta > 0 \) and set \( C_\delta := \{(z_1, \ldots, z_k) \in B_{10}(0)^k : \min_{i,j} |z_i - z_j| \geq \delta \} \). Then, by using Fatou’s lemma and the Cameron-Martin formula, we have
\[
E \left[ \int_{B_{10}(0)^k \cap C_\delta} f(\phi, z_1, \ldots, z_k) \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right] \\
\leq \liminf_{\varepsilon \to 0} E \left[ \int_{B_{10}(0)^k \cap C_\delta} e^{\gamma \phi, \varepsilon}(z_1) \ldots e^{\gamma \phi, \varepsilon}(z_k) \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right] \\
= \liminf_{\varepsilon \to 0} \int_{B_{10}(0)^k \cap C_\delta} e^{\gamma \phi, \varepsilon}(\sum_{i \neq j} Cov(\phi(z_i), \phi(z_j))) \mathbb{E} \left[ f(\phi, z_1, \ldots, z_k) \right] dz_1 \ldots dz_k \\
= \int_{B_{10}(0)^k \cap C_\delta} e^{\gamma \phi, \varepsilon}(\sum_{i \neq j} Cov(\phi(z_i), \phi(z_j))) \mathbb{E} \left[ f(\phi + \gamma \sum_{i \leq k} Cov(\phi(\cdot), \phi(z_i)), z_1, \ldots, z_k) \right] dz_1 \ldots dz_k.
\]
where we used the dominated convergence theorem in the last equality (indeed, \( \sum_{i \neq j} Cov(\phi(z_i), \phi(z_j)) \) is uniformly bounded for \((z_1, \ldots, z_n) \in C_\delta \)). The Cameron-Martin formula is used by writing
\[
\gamma \sum_{i \leq k} \phi(\varepsilon, z_i) = \langle \phi, \gamma \sum_{i \leq k} \rho_{\varepsilon, z_i} \rangle
\]
where \( \rho_{\varepsilon, z_i} \) denote the uniform probability measure on the circle \( \partial B_{\varepsilon}(z_i) \). Note that the above inequality was only shown for continuous \( f \), but we can approximate general bounded nonnegative measurable \( f \) by a sequence of continuous \( f_n \) which converge pointwise to \( f \), and apply the dominated convergence theorem. Thus the above inequality holds for general \( f \).

Finally, letting \( \delta \) going to zero and using the monotone convergence theorem, we get
\[
E \left[ \int_{B_{10}(0)^k} f(\phi, z_1, \ldots, z_k) \mu_\phi(dz_1) \ldots \mu_\phi(dz_k) \right] \\
\leq \int_{B_{10}(0)^k} e^{\gamma \phi, \varepsilon}(\sum_{i \neq j} Cov(\phi(z_i), \phi(z_j))) \mathbb{E} \left[ f(\phi + \gamma \sum_{i \leq k} Cov(\phi(\cdot), \phi(z_i)), z_1, \ldots, z_k) \right] dz_1 \ldots dz_k.
\]
This concludes the proof. \( \square \)

**Proof of Lemma 3.14** It suffices to show that for some constant \( C \), for each \( z \in K \) and each \( i = 1, \ldots, n \), writing \( w = \text{Left}(S_i) \) we have
\[
|C_{m_i-1, m_i}(z, w) - (m_i - m_{i-1})1_{z \in S_i}| < C.
\]
If \( z \notin S_i \), then by definition \( d(z, w) \geq d(z, S_i) \geq e^{-m_{i-1}} \). This is larger than the range of dependence of \( \phi_{m_{i-1}, m_i} \), so \( C_{m_i-1, m_i}(z, w) = 0 \) as desired.

Now suppose \( z \in S_i \). By (3.14), we know that \( S_i \) is contained in a ball of radius \( 6ke^{-m_i} \); by translation invariance we may assume this ball is centered at the origin. On \( B_{6k}(0) \times B_{6k}(0) \), the correlation of \( \phi_{0, \infty} \) is \( C_{0, \infty}(\cdot, \cdot) = |\cdot - \cdot|^{-1} + q(\cdot - \cdot) \) for some bounded continuous \( q \). Thus, by scale invariance, we can write
\[
C_{m_i-1, m_i}(z, w) = C_{0, m_i-1, m_i-1}(e^{m_{i-1}z}, e^{m_{i-1}w}) = \log |e^{m_{i-1}(z - w)}|^{-1} - C_{m_i-1, m_i-1, \infty}(e^{m_{i-1}z}, e^{m_{i-1}w}) + O(1).
\]
But again by scale invariance we have
\[
C_{m_i-1, m_i-1, \infty}(e^{m_{i-1}z}, e^{m_{i-1}w}) = C_{0, \infty}(e^{m_{i-1}z}, e^{m_{i-1}w}) = \log |e^{m_{i-1}(z - w)}|^{-1} + O(1).
\]
Comparing these two equations we conclude that \( C_{m_i-1, m_i}(z, w) = m_i - m_{i-1} + O(1) \), as needed. \( \square \)

Finally we check the bound on the regularity event \( E \).
Proof of Lemma 3.15. We prove here the estimate of the occurrence of the event $E_{\delta,M}$.

For all integers $\ell > \ell'$, for all $B \in \mathcal{B}$ of radius $e^{-\ell-2}$, the probability that $e^{-\ell} \sup_{\theta \in \partial B} |\nabla \phi_{\ell,\ell'}| > e^{\ell \frac{1}{2} + \delta} + \log M$ is $\leq C e^{-c(\log M)^2} e^{-\ell \alpha + \beta}$ by Lemma 6.2. Therefore, the probability that Condition 2 does not hold is $\leq C e^{-c(\log M)^2} \sum_{\ell \geq 0} e^{2\ell} e^{-\ell \alpha + \beta}$.

For Condition 3, for a $B \in \mathcal{B}$ of size $e^{-\ell-2}$, by scaling $\int_{\partial B} \phi_{\ell,\ell'}$ is distributed as $\int_{\partial B_0} \phi_{0,\ell'}$ where $B_0$ is of size $e^{-2}$ and this is a centered Gaussian variable with bounded variance. Therefore, the probability it is at least $e^{\ell \frac{1}{2} + \delta} + \log M$ is less than $C e^{-c(\ell \frac{1}{2} + \delta) + \log M} e^{-\ell \alpha + \beta}$. For each $\ell$, there are $O(e^{\ell \delta})$ balls of size $e^{-\ell-2}$ in $\mathcal{B}$, hence the probability that Condition 3 does not hold is less than $C e^{-c(\log M)^2} \sum_{\ell \geq 0} e^{2\ell} e^{-\ell \alpha + \beta}$.

For Condition 4, since $\phi - h$ is continuous by Proposition 3.2 and applying Fernique’s theorem, the probability that $\|\phi - h\|_B \leq \log M$ occurs is $\geq 1 - C e^{-c(\log M)^2}$. For Condition 1, we use Proposition 2.3 and again a union bound.

4 Negative moments

In this section, we prove the following lower bound on the LQG volume of the unit quantum ball.

Proposition 4.1 (Negative moments of LQG ball volume). Let $h$ be a whole-plane GFF normalized so $h_1(0) = 0$. Then

$$\mathbb{E} \left[ \mu_h(B_1(0; D_h))^{-p} \right] < \infty \text{ for all } p \geq 0.$$

This result also holds if we instead consider the LQG measure and metric associated with the field $\tilde{h} = h - \alpha \log |\cdot|$ for $\alpha < Q$.

In Section 4.1, we prove the finiteness of negative moments of $\mu_h(B_1(0; D_h^0))$, the unit ball with respect to the $\mathbb{D}$-internal metric $D_h^0$. This immediately implies Proposition 4.1 since $B_1(0; D_h^0) \subset B_1(0; D_h)$. In Section 4.2, we bootstrap our results to obtain lower bounds on $\mu_h(B_s(0; D_h))$ for $s \in (0, 1)$; these lower bounds will be useful in our applications in Section 5.

4.1 Lower tail of the unit quantum ball volume

The goal of this section is the following result.

Proposition 4.2 (Superpolynomial decay of internal metric ball volume lower tail). Let $h$ be a whole-plane GFF normalized so $h_1(0) = 0$. Let $D_h^0 : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ be the internal metric in $\mathbb{D}$ induced by $D_h$, and $B_1(0; D_h^0) \subset \mathbb{D}$ the $D_h^0$-metric ball. Then for any $p > 0$, for all sufficiently large $C > 0$ we have

$$\mathbb{P} \left[ \mu_h(B_1(0; D_h^0)) \geq C^{-1} \right] \geq 1 - C^{-p}.$$

This result also holds if we instead consider the LQG measure and metric associated with the field $\tilde{h} = h - \alpha \log |\cdot|$ for $\alpha < Q$.

Let $N > 1$ be a parameter which we keep fixed as $C \to \infty$ (taking $N$ large yields $p$ large in Proposition 4.2) and define

$$k_0 = \left\lceil \frac{1}{N} \log C \right\rceil, \quad k_1 = \left\lfloor N \log C \right\rfloor.$$

Let $P$ be the $D_h$-geodesic from 0 to $\partial B_{e^{-k_0}}(0)$. See Figure 2 (left) for the setup.

Proof sketch of Proposition 4.2. The proof follows several steps. Each step below holds with high probability.

- We find an annulus $B_{e^{-k_1+1}}(0) \setminus B_{e^{-k}}(0)$ with $k > k_0$ not too large, such that the annulus-crossing length of $P$ is not too small. This is possible because the $D_h$-length of $P$ between $\partial B_{e^{-k_1}}(0)$ and $\partial B_{e^{-k_0}}(0)$ is at least $C^{-\beta}$ for some fixed $\beta > 0$. We conclude that the circle average $\bar{h}_{e^{-k}}(0)$ is not small ($\bar{h}_{e^{-k}} \gtrsim -\log C$).
• We find a $D_h$-metric ball which is “tangent” to $\partial B_{e^{-k}}(0)$ and $\partial B_{e^{-k-1}}(0)$. Then, by Proposition 2.4 this metric ball (and hence $B_1(0; D^C_h)$) contains a Euclidean ball $B$ with Euclidean radius not too small (say $e^{-(1+\zeta)k}$ for small $\zeta > 0$). Since $h_{e^{-k}}(0)$ is not small, neither is the average of $\bar{h}$ on $\partial B$ (i.e. $\int_{\partial B} \bar{h} \geq -\log C$).

• Finally, we have a good lower bound on $\mu_\infty(B)$ in terms of the average of $\bar{h}$ on $\partial B$, so we find that $B$ has not-too-small LQG volume. Since $B$ lies in $B_1(0; D^C_\infty)$, we obtain a lower bound $\mu_\infty(B_1(0; D^C_\infty)) \gtrsim C^{-\text{power}}$. This last exponent does not depend on $N$, so we may take $N \to \infty$ to conclude the proof of Proposition 4.2.

We now turn to the details of the proof. Let $L_k$ be the $D_h$-length of the subpath of $P$ from 0 until the first time one hits $\partial B_{e^{-k}}(0)$. We emphasize that $L_k$ is not the $D_h$ distance from 0 to $\partial B_{e^{-k}}(0)$.

**Lemma 4.3** (Length bounds along $P$). There exist positive constants $c = c(\gamma, \alpha)$ and $\beta = \beta(\gamma, \alpha)$ independent of $N$ such that for sufficiently large $C$, with probability $1 - O(C^{-cN})$ the following all hold:

\begin{align}
L_{k_0} &> C^{-\beta}, \quad (4.34) \\
L_{k_1} &< C^{-\beta - 1}, \quad (4.35) \\
L_{k_1} - L_k < C \exp (-k(\xi(Q - \alpha) + \xi h_{e^{-k}}(0))) & \text{ for all } k \in [k_0 + 1, k_1]. \quad (4.36)
\end{align}

**Proof.** We focus first on (4.34). Using Proposition 2.3 to bound the crossing distance of $B_{e^{-k_0}}(0) \setminus B_{e^{-k_0-1}}(0)$, we see that with superpolynomially high probability as $C \to \infty$ we have

$$L_{k_0} \geq C^{-\beta} \left( e^{-\mu_{k_0}} \right)^{(Q - \alpha)} \exp(\xi h_{e^{-k_0}}(0)).$$

Note that since $\text{Var}(h_{e^{-k_0}}(0)) = k_0 \leq N^{-1} \log C$, we have

$$\mathbb{P}(\xi h_{e^{-k_0}}(0) < -\log C) \leq \exp \left( -\frac{(\log C)^2}{2\xi^2 N^{-1} \log C} \right) = C^{-cN}$$

for $c = 1/(2\xi^2)$. Notice that when we have both (4.37) and $\{\xi h_{e^{-k_0}} \geq -\log C\}$, then

$$L_{k_0} \geq C^{1 \cdot C^{-\xi(Q - \alpha)/N}} \cdot C^{-\beta} \geq C^{-\beta}$$

for the choice $\beta = 2 + \xi(Q - \alpha)$. Thus (4.34) holds with probability $1 - O(C^{-cN})$.

To prove the upper bound (4.36), we glue paths to bound $L_{k_1} - L_k$. By Proposition 2.3 and a union bound, with superpolynomially high probability as $C \to \infty$ the following event $E_C$ holds:

• For each $k \in [k_0 + 1, k_1]$, there exists a path from $\partial B_{e^{-k-1}}(0)$ to $\partial B_{e^{-k}}(0)$ and paths in the annuli $B_{e^{-k}} \setminus B_{e^{-k-1}}$ and $B_{e^{-k-1}} \setminus B_{e^{-k-2}}$ which separate the circular boundaries of the annuli, such that each path has $D_h$-length at most $\frac{1}{4} \epsilon h_{e^{-k_0}}(0)$. Since the segment on $P$ measured by $L_{k_1} - L_k$ is the restriction of a geodesic which crosses a larger annulus, by triangular equality (4.36) holds on $E_C$.

Finally, we check that for our choice of $\beta$, the inequality (4.35) holds with probability $1 - O(C^{-cN})$ (possibly by choosing a smaller value of $c > 0$). By the triangle inequality, $L_{k_1}$ is bounded from above by the sum of the $D_h$-distance from the origin to $\partial B_{e^{-k_1+1}}(0)$ plus the $D_h$-length of any circuit in the annulus $B_{e^{-k_1+1}} \setminus B_{e^{-k_1}}(0)$. Hence, using the circuit bound on $E_C$, we have

$$L_{k_1} \leq D_h(0, \partial B_{e^{-k_1+1}}(0)) + C e^{-k_1(\xi(Q - \alpha))} \exp(h_{e^{-k_1}}(0)).$$

By scaling of the metric, $D_h(0, \partial B_{e^{-k_1+1}}(0))$ is bounded from above by $e^{\xi h_{e^{-k_1+1}}(0)} e^{(\xi h_{e^{-k_1+1}}(0)) Y}$ where $Y$ is distributed as $D_h(0, \partial B_1(0))$. Now, since $k_1 = \lceil N \log C \rceil$ and $h_{e^{-k_1}}(0)$ has variance $N \log C$, by a Gaussian tail estimate we get

$$\mathbb{P} \left[ h_{e^{-k_1}}(0) > \frac{1}{4} k_1(Q - \alpha) \right] \leq C^{-cN}.$$

Furthermore, since $Y$ has some finite small moments for $\alpha < Q$ (by Theorem 1.10), the Markov’s inequality provides

$$\mathbb{P} \left[ Y e^{\frac{1}{4} k_1(\xi(Q - \alpha))} > 1 \right] \leq C^{-cN}.$$

Altogether, we obtain (4.35) with probability $1 - O(C^{-cN})$. \qed
As an immediate consequence of the above lemma, we can find a scale \( k \in (k_0, k_1) \) such that \( B_1(0; D^2_h) \) intersects \( \partial B_{e^{-k}}(0) \), and the field average at scale \( k \) is large. We introduce here a small parameter \( \zeta > 0 \) which does not depend on \( C \), whose value we fix at the end.

**Lemma 4.4** (Existence of large field average near \( B_1(0; D^2_h) \)). Consider \( c \) and \( \beta \) as in Lemma 4.3. With probability \( 1 - O(C^{-cN}) \), there exists \( k \in [k_0, k_1] \) such that \( D^i_h(0, \partial B_{e^{-k}}(0)) < 1 \) and

\[
-k(Q - \alpha) + h_{e^{-k}}(0) \geq -e^{-1}(\beta + 2) \log C; \tag{4.38}
\]

moreover, there exists a Euclidean ball \( B_r(z) \) with \( r = e^{-k(1 + \zeta)} \) and \( z \in \mathbb{R}^2 \) such that \( B_r(z) \subset B_{e^{-k}}(0) \cap B_e^{-k-1}(0) \) and \( B_r(z) \subset B_1(0; D^2_h) \).

**Proof.** To prove (4.38), we first claim that when the event of Lemma 4.3 holds, there exists \( k \in [k_0, k_1] \) such that \( L_k < 1 \) and \( L_{k-1} - L_k \geq C^{-\beta - 1} \). Let \( k_* \) be the smallest \( k \in (k_0, k_1) \) such that \( L_{k_*} < C^{-\beta} \), then

\[
\sum_{k = k_*}^{k_1} L_{k-1} - L_k = L_{k_*-1} - L_{k_*} \geq C^{-\beta} - C^{-\beta - 1}.
\]

Since the LHS is a sum over at most \( N \log C \) terms, we indeed find some index \( k \in [k_*, k_1] \) such that

\[
L_{k-1} - L_k \geq C^{-\beta} - C^{-\beta - 1} > C^{-\beta - 1}.
\]

For this choice of \( k \), we have \( D^i_h(0, \partial B_{e^{-k}}(0)) \leq L_k \leq L_{k_*} < C^{-\beta} < 1 \), and by (4.36) we have (4.38) also.

![Figure 2: Left: Setup of Lemma 4.3. Given \( C \) that we eventually sent to \( \infty \), we take the circles with radii \( e^{-k_0} \approx C^{-1/N} \) and \( e^{-k_1} = C^{-N} \), and draw all circles with radii \( e^{-k} \) with \( k_0 \leq k \leq k_1 \). In Lemma 4.4 we follow the geodesic \( P \) from the outer circle to the inner until we find an annulus on which the geodesic segment is long. Right: Illustration of the second assertion of Lemma 4.4. We find a \( D_h \)-quantum ball \( U \subset B_1(0; D^2_h) \) such that \( U \) is “tangent” to \( \partial B_{e^{-k}} \) and \( \partial B_{e^{-k-1}} \), then apply Proposition 2.3 to find a Euclidean ball \( B_r(z) \subset U \).](image)

Now we turn to the second assertion of the lemma; see Figure 2(right). Let \( P' \) be a \( D_h \)-geodesic from 0 to \( \partial B_{e^{-k}}(0) \). By the continuity of \( D^i_h \), we can find a point \( p \in P' \) in the annulus \( B_{e^{-k}}(0) \) such that \( D_{h+1}(p, \partial B_{e^{-k}}(0)) = D_{h+1}(p, \partial B_{e^{-k-1}}(0)) \); let \( U \) be the \( D_{h+1}(p, \partial B_{e^{-k}}(0)) \)-ball with this radius centered at \( p \).

We claim that \( U \subset B_1(0; D^2_h) \). We assume that \( \alpha \geq 0 \) (the other case is similar). Since \( (k + 1)\alpha \geq \alpha \log | \cdot |^{-1} \geq k \alpha \) on \( B_{e^{-k}}(0) \), we have for all \( w \in U \) that

\[
D^i_h(p, w) \leq e^{\alpha} D^i_{h+1}(p, w) \leq e^{\alpha} D^i_{h+1}(p, \partial B_{e^{-k}}(0)) \leq e^{\alpha} D^i_h(p, \partial B_{e^{-k}}(0)),
\]

and consequently

\[
D^i_h(0, w) \leq D^i_h(0, p) + D^i_h(p, w) \leq D^i_h(0, p) + e^{\alpha} D^i_h(p, \partial B_{e^{-k}}(0)) \leq e^{\alpha} D^i_h(0, \partial B_{e^{-k}}(0));
\]
this last inequality follows from the fact that \( p \) lies on \( P' \) so \( D_h^p(0,p)+D_h^p(p,\partial B_{e^{-k}}(0)) = D_h^p(0,\partial B_{e^{-k}}(0)) \).

Since \( D_h^p(0,\partial B_{e^{-k}}(0)) \leq L_k, \gamma < C^{-\beta} \), we conclude that \( D_h^p(0,w) < e^{c\alpha} C^{-\beta} \leq 1 \), and hence \( U \subset B_1(0; D_h^p) \).

Since \( U \) is a \( D_{h^k(\alpha + 1)} \) metric ball, it is also a \( D_h \) metric ball. Furthermore, since \( \text{diam}(U) \in (\frac{1}{2} e^{-k}, 2 e^{-k}) \), Proposition 2.4 gives us a Euclidean ball of radius \( e^{-k(1+\zeta/2)} \) in \( U \), and hence a Euclidean ball \( B_r(z) \subset U \) with \( z \in r \mathbb{Z}^2 \). Since \( U \) lies in \( B_{e^{-k}}(0) \setminus \overline{B}_{e^{-k-1}}(0) \) and in \( B_1(0; D_h^p) \), so does \( B_r(z) \), so we have shown Lemma 4.3.

Finally, we need a regularity event to say that the \( \mu_{\zeta} \)‐volumes of Euclidean balls are close to their field average approximations, and that the field does not fluctuate too much on each scale. The bounds in the following lemma are standard in the literature. We introduce a large parameter \( q > 0 \) that does not depend on \( C \), and fix its value at the end.

**Lemma 4.5** (Regularity of field averages and ball volumes). Fix \( \zeta \in (0,1) \) and \( q > 0 \). Then for sufficiently large \( C \), with probability \( 1 - C^{-\zeta (\frac{2}{\sqrt{q}} - 2N^-1)} \) the following is true. For each \( k \in [k_0,k_1] \), writing \( r = e^{-k(1+\zeta)} \), for all \( z \in r \mathbb{Z}^2 \) such that \( B_r(z) \subset B_{e^{-k}}(0) \), we have

\[
|h_r(z) - h_{e^{-k}}(0)| < kq\zeta
\]

and

\[
\mu_{\zeta}(B_r(z)) \geq C^{-\gamma q} \exp(\gamma h_r(z)).
\]

**Proof.** By standard GFF estimates, we have \( \text{Cov}(h_r(z), h_{e^{-k}}(0)) = k + O(1) \), \( \text{Var}(h_r(z)) = -\log r + O(1) = k(1+\zeta) + O(1) \) and \( \text{Var}(h_{e^{-k}}(0)) = k + O(1) \). Consequently,

\[
\text{Var}(h_r(z) - h_{e^{-k}}(0)) = \zeta k + O(1),
\]

and hence by the Gaussian tail bound,

\[
\mathbb{P}(|h_r(z) - h_{e^{-k}}(0)| < kq\zeta) \geq 1 - O(e^{-\frac{q^2 k}{2}}).
\]

Taking a union bound over all \( O(e^{2k\zeta}) \) points in \( r \mathbb{Z}^2 \cap B_{e^{-k}}(0) \), then summing over all \( k \in [k_0,k_1] \), we see that the probability (4.39) holds for all \( k \) and all suitable \( z \) at least

\[
1 - O\left( \sum_{k=k_0}^{k_1} e^{2k\zeta} e^{-q^2 k/2} \right) \geq 1 - O\left( N \log C \cdot e^{2k_1\zeta} e^{-q^2 k_0/2} \right) \geq 1 - C^{-\zeta (\frac{2}{\sqrt{q}} - 2N^-1)}.
\]

Now, we establish that for each fixed choice of \( k,z \), the inequality (4.40) holds with superpolynomially high probability as \( C \to \infty \) (then we are done by a union bound over a collection of polynomially many \( k,z \)); since \( -\alpha \log |z| - \alpha k \) is bounded on the annulus, it suffices to show (4.40) with \( h \) replaced by \( h + ak \) (or equivalently by \( h \), since both sides of the equation (4.40) scale the same way under adding a constant to the field). By the Markov property of the GFF (Lemma 2.2) we can decompose \( h = h + \tilde{h} \), where \( h \) is a distribution which is harmonic in \( B_{2r}(z) \), and \( \tilde{h} \) is a zero boundary GFF in the domain \( B_{2r}(z) \); moreover \( h \) and \( \tilde{h} \) are independent. We can then write

\[
\mu_{\zeta}(B_r(z)) \geq e^{\gamma \inf_{B_{2r}(z)} h} \mu_{\zeta}(B_r(z))
\]

\[
= (2r)^\gamma e^{-\gamma h_r(z)} e^{-\gamma \tilde{h}_r(z)} e^{\gamma \inf_{B_{2r}(z)} h - \gamma h(z)} \mu_{\zeta}(B_{\frac{r}{2}}(0)),
\]

where \( g := \tilde{h}(2r \cdot +z) \) has the law of a zero boundary GFF on \( \mathbb{D} \). (This follows from an affine change of coordinates mapping \( B_{2r}(z) \to \mathbb{D} \); then by the coordinate change formula \( \mu_{\zeta}(B_r(z)) = (2r)^\gamma \mu_{\zeta}(B_{\frac{r}{2}}(0)) \).

Since \( \tilde{h}_r(z) \) is a mean zero Gaussian with fixed variance, and by the quantum volume lower bound (4.4), we have \( e^{-\gamma \tilde{h}_r(z)} \geq C^{-1/3} \) and \( \mu_{\zeta}(B_{\frac{r}{2}}(0)) \geq C^{-1/3} \) with superpolynomially high probability in \( C \). Combining these bounds with the above estimate, with superpolynomially high probability in \( C \) we have

\[
\mu_{\zeta}(B_r(z)) \geq (2r)^\gamma C^{-2/3} e^{\gamma \inf_{B_{2r}(z)} h - \gamma h(z)}.
\]
Hence we are done once we check that with superpolynomially high probability in $C$,  
\[ e^{-\gamma \inf_{B_r(z)} h - \gamma h(z)} \geq C^{-1/3}. \]  
(4.41)

Since $h = h + \widehat{h}$ and $h, \widehat{h}$ are independent, for $x, x' \in B_r(z)$ we have  
\[ \text{Var} \left( h(x) - h(x') \right) \leq \text{Var} \left( h_r(x) - h_r(x') \right) = O(1). \]

Moreover, by the scale and translation invariance of the GFF modulo additive constant and the fact that $h$ is continuous in $B_{2r}(z)$, we know that $h(z) - \inf_{B_r(z)} h > -\infty$ and has a law independent of $r, z$, so by the Borell-TIS inequality we see that for some absolute constants $m, c$, we have  
\[ \mathbb{P} \left( h(z) - \inf_{B_r(z)} h > u + m \right) \leq e^{-cu^2} \quad \text{for all } u > 0. \]

This immediately implies (4.41). Thus, for each fixed choice of $k, z$, the inequality (4.40) holds with superpolynomially high probability as $C \to \infty$. Taking a union bound, we obtain (4.40).

Proof of Proposition 4.2. Let $c, \beta$ be as in Lemma 4.3. We will work with parameters $N, \zeta, q$, and choose their values at the end. Assume that the events of Lemmas 4.4 and 4.5 hold; this occurs with probability at least $1 - C^{-cN} - C^{-C(N^{2/3} - 2N^{1/2} - 1)}$. Let $k, r, z$ be as in Lemma 4.4.

We now lower bound the quantum volume of $B_r(z)$. By (4.38) and (4.39), we see that  
\[ r^\gamma \exp \left( \gamma \bar{h}_r(z) \right) \geq \exp \left( -\gamma k Q(1 + \zeta) + \gamma h_r(z) + \gamma \alpha k \right) \]
\[ \geq \exp \left( -\gamma \zeta k(Q + q) - \gamma k(Q - \alpha) + \gamma h_{c^{-1}}(0) \right) \]
\[ \geq \exp \left( -\gamma \zeta k(Q + q) \right) C^{-2(\beta + 2)} \]
\[ \geq C^{-\gamma N(Q + q) C^{-2(\beta + 2)}}. \]

The last inequality follows from $k \leq k_1 = \lfloor N \log C \rfloor$. Choose $q = N^3$ and $\zeta = N^{-4}$. Then by the above inequality, (4.40), and $B_r(z) \subset B_1(0; \mathcal{D}_{h_k}^\beta)$, we see that for a constant $\beta' = \beta'(\gamma) > 0$ we have  
\[ \mu_{h_k}(B_1(0; \mathcal{D}_{h_k}^\beta)) \geq \mu_{h}(B_r(z)) \geq C^{-\beta'}. \]

Since this occurs with probability $1 - C^{-cN} - C^{-C(N^{2/3} - 2N^{1/2} - 1)} = 1 - O(C^{-cN})$, and $N$ can be made arbitrarily large, we have proved Proposition 4.2.

4.2 Lower tail of small quantum balls

Using Proposition 4.2 and the scaling properties of the LQG metric and measure, we can easily prove a similar result for quantum balls centered at the origin of all radii $s \in (0, 1)$. We emphasize that in the following proposition, we are considering the $D_h$-metric balls, rather than $D_{h_k}^\beta$-metric balls.

Lemma 4.6. Let $h$ be a whole-plane GFF normalized so $h_1(0) = 0$. For any $p > 0$, there exists $C_p$ such that for all $C > C_p$ and $s \in (0, 1)$, we have  
\[ \mathbb{P} \left( \mu_h(B_s(0; D_h)) \geq C^{-1} s^{d_s} \right) \geq 1 - C^{-p}. \]

Proof. The process $t \mapsto h_{c^{-1}}(0)$ for $t \geq 0$ evolves as standard Brownian motion started at 0. Fix $s \in (0, 1)$ and let $T > 0$ be the first time $t > 0$ that $-Qt + h_{c^{-1}}(0) = \xi^{-1} \log s$. Notice that  
\[ h(e^{-T}) + Q \log e^{-T} = \left( h(e^{-T}) - h_{c^{-1}}(0) \right) - QT + h_{c^{-1}}(0) = \left( h(e^{-T}) - h_{c^{-1}}(0) \right) + \xi^{-1} \log s. \]

By Lemma 2.1, conditioned on $T$, we have $(h(e^{-T}) + Q \log e^{-T})_{\hat{h}} \overset{d}{=} \left( \hat{h} + \xi^{-1} \log s \right)_{\hat{h}}$, where $\hat{h}$ is a whole-plane GFF normalized to have mean zero on $\partial \mathbb{D}$. Couple these fields to agree. By the Weyl scaling relations and the change of coordinates formula for quantum volume and distances, and the locality property of the internal metric (Axiom 1), we have the internal metric relation  
\[ D_{h_k}^\beta (e^{-T} z, e^{-T} w) = D_{h_{k+1}}^\beta (z, w) = s D_{h_k}^\beta (z, w) \]

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and the volume measure relation
\[ \mu_h(e^{-z}) = \mu_{\hat{h} + \xi}^{-1}(\cdot) = s^d \mu_{\hat{h}}(\cdot). \]
Thus we can relate the quantum volume of the internal metric balls \( B_e(0; D_h^{e^{-z}}) \) to \( B_1(0; D_h^0) \):
\[ \mu_h\left( B_e(0; D_h^{e^{-z}}) \right) = s^d \mu_{\hat{h}}(B_1(0; D_h^0)), \]
and consequently we have
\[ \left\{ \mu_h(B_e(0; D_h^{e^{-z}})) \geq C^{-1} s^{d_1} \right\} = \left\{ \mu_h(B_1(0; D_h^0)) \geq C^{-1} \right\}. \]
Since \( \mu_h(B_e(0; D_h^0)) \geq \mu_h(B_e(0; D_h^{e^{-z}})) \), our claim follows from Proposition 4.2.

## 5 Applications and other results

### 5.1 Uniform volume estimates and Minkowski dimension

In this section, we prove the remaining assertions of Theorem 1.1. Namely, the Minkowski dimension of a bounded open set \( S \) is almost surely equal to \( d_r \) and for any compact set \( K \subset \mathbb{C} \) and \( \varepsilon > 0 \), we have, almost surely
\[ \sup_{s \in (0,1)} \sup_{z \in K} \frac{\mu_h(B_s(z; D_h))}{s^{d_r - \varepsilon}} < \infty \quad \text{and} \quad \inf_{s \in (0,1)} \inf_{z \in K} \frac{\mu_h(B_s(z; D_h))}{s^{d_r + \varepsilon}} > 0. \]

Since the whole-plane GFF modulo additive constants has a translation invariant law, we can deduce a similar result as the one in the previous section for quantum balls centered at \( z \neq 0 \).

**Proposition 5.1** (Uniform lower tail for \( \mu_h(B_s(z; D_h)) \)). Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \), and \( K \subset \mathbb{C} \) be any compact set. For any \( p > 0 \), there exists \( C_{p,K} > 0 \) such that
\[ \sup_{s \in (0,1), z \in K} \mathbb{P} \left[ \mu_h(B_s(z; D_h)) \geq C^{-1} s^{d_r} \right] \geq 1 - C^{-p} \quad \text{for each } C > C_{p,K}. \]

**Proof.** Fix \( z \in K \). We can write \( h = \hat{h} + X \) where \( \hat{h} \) is a whole-plane GFF normalized so \( \hat{h}_1(z) = 0 \), and \( X = h_1(z) \) is a random real number. On the event \( \{ |X| \leq \gamma^{-1} \log C \} \) we have \( C^{-1} \leq e^{-\gamma X} \leq C \), so
\[ \{ \mu_h(B_s(z; D_h)) < C^{-3} s^{d_r} \} \subseteq \{ e^{-\gamma X} \mu_h(B_{\gamma^{-1} s} X; D_h) < C^{-3} s^{d_r} \} \subseteq \{ C^{-1} \hat{h}_1(B_{C^{-1} s} X; D_h) < C^{-3} s^{d_r} \} \cup \{ |X| > \gamma^{-1} \log C \} \]
\[ = \{ \mu_h(B_{C^{-1} s} X; D_h) < C^{-1} (C^{-1} s)^{d_r} \} \cup \{ |X| > \gamma^{-1} \log C \}. \]

In the last line, the first event is superpolynomially rare in \( C \) by Lemma 4.6 (with \( p = 1 \) instead of \( p = 3 \)), and the second because \( X \) is a centered Gaussian. Note that \( \text{Var} X = \text{Var} h_1(z) \) is uniformly bounded for all \( z \in K \), so the decay of the second event is uniform for \( z \in K \). This completes the proof.

Similarly, we can bootstrap Lemma 3.18 to a statement uniform for \( D_h \)-balls centered in a compact set.

**Proposition 5.2** (Uniform upper tail for \( \mu_h(B_s(z; D_h)) \)). Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \). For any compact set \( K \subset \mathbb{C} \), \( p > 0 \), \( \varepsilon \in (0,1) \), there exists a constant \( C_{p,\varepsilon,K} > 0 \) such that
\[ \sup_{s \in (0,1), z \in K} \mathbb{P} \left[ \mu_h(B_s(z; D_h)) \leq Cs^{d_r - \varepsilon} \right] \geq 1 - C^{-p} \quad \text{for each } C > C_{p,\varepsilon,K}. \]

**Proof.** We note that Lemma 3.18 implies an upper bound version of Lemma 4.6 (with an exponent of \( d_r - \varepsilon \) instead of \( d_r \)), and we deduce Proposition 5.2 in the same way that we obtain Proposition 5.1 from Lemma 4.6.

Before moving to the proof of the almost sure uniform estimate, we first prove volume bounds on a countable collection of quantum balls.
Lemma 5.3. For any $\varepsilon > 0$ and bounded open set $2\mathcal{D}$, the following is true almost surely. For all sufficiently large $m$, for all $z \in 2^{-m}\mathbb{Z}^2 \cap 2\mathcal{D}$, and for all dyadic $s = 2^{-k} \in (0, 1]$ we have
\[ s^{d_+ - \varepsilon} 2^{-cm} > \mu_h(B_s(z; D_h)) > s^{d_+ + \varepsilon} 2^{-cm}. \]

Proof. The proof is a straightforward application of Propositions 5.2 and 5.1 and the Borel-Cantelli lemma. We prove the lower bound; the upper bound follows the same argument.

Pick any large $p > 0$, and let $C_{p,2D}$ be the constant from Proposition 5.1. Consider any $m$ such that $2^{-cm} > C_{p,2D}$, then for any $z \in 2\mathcal{D}$ we have
\[ \mathbb{P} \left[ \mu_h(B_s(z; D_h)) > s^{d_+ + \varepsilon} 2^{-cm} \text{ for all dyadic } s \in (0, 1] \right] > 1 - 2^{-cm} \sum_{\text{dyadic } s} s^{\varepsilon p}. \]

Taking a union bound over all the $O(2^{2m})$ points in $2^{-m}\mathbb{Z}^2 \cap 2\mathcal{D}$ yields
\[ \mathbb{P} \left[ \mu_h(B_s(z; D_h)) > s^{d_+ + \varepsilon} 2^{-cm} \text{ for all dyadic } s \in (0, 1] \text{ and } z \in 2^{-m}\mathbb{Z}^2 \cap 2\mathcal{D} \right] > 1 - O(2^{(cp-2)m}) \sum_{\text{dyadic } s} s^{\varepsilon p}. \]

For $p$ large enough we have $\varepsilon p - 2 > 0$, so by the Borel-Cantelli lemma, a.s. at most finitely many of the above events fail, i.e. the lower bound of Lemma 5.3 holds. The upper bound follows the same argument.

With this lemma and the bi-Hölder continuity of $D_h$ with respect to Euclidean distance, we can prove the second part of Theorem 1.1.

Proof of Theorem 1.1 part 2. We first prove that a.s. for some random $r \in (0, 1)$, we have
\[ \inf_{s \in (0, r]} \inf_{\mathcal{D} \in B} \frac{\mu_h(B_s(z; D_h))}{s^{d+\varepsilon}} > 0. \quad (5.42) \]

We use the bi-Hölder continuity of $D_h$ with respect to Euclidean distance (see e.g. [13, Theorem 1.7]) and the Borel-Cantelli lemma to obtain the following. There exist deterministic constants $\chi, \chi' > 0$ and random constant $c, C$ such that, almost surely,
\[ c|u - v|^\chi \leq D_h(u, v) \leq C|u - v|^\chi \quad \text{for all } u, v \in 2\mathcal{D}. \]

Moreover, Proposition 2.4 and Borell-Cantelli yield that a.s. every quantum ball $B$ contained in $2\mathcal{D}$ and having sufficiently small Euclidean diameter contains a Euclidean ball of radius at least $\text{diam}(B)^2$.

Consequently, for all sufficiently small $s$ and any $z \in \mathcal{D}$, we have
\[ \frac{s}{2} \leq C \text{diam}(B_s(z; D_h))^\chi, \]
and since any two points in $B_s(z; D_h)$ have $D_h$-distance at most $s$, the bi-Hölder lower bound gives
\[ c \text{diam}(B_s(z; D_h))^\chi' \leq s. \]

Since the ball $B_s(z; D_h)$ has a small diameter, it a.s. contains a Euclidean ball of radius at least $\text{diam}(B_s(z; D_h))^2 \geq (s/2C)^{2/\chi}$ hence contains a $w \in 2^{-m}\mathbb{Z}^2$ with $m = \left\lceil -\frac{\chi}{2} \log_2(s/2C) \right\rceil < -\frac{\chi}{2} \log_2(s/2C)$.

Thus, for a random constant $c'$, for sufficiently small $s$, applying Lemma 5.3 to $m$ as above and dyadic $s \in (\frac{s}{4}, \frac{s}{2}]$, we have
\[ \mu_h(B_s(z; D_h)) \geq \mu_h(B_{s_1}(z; D_h)) \geq \left( \frac{s}{4} \right)^{d_+ + \varepsilon} \cdot 2^{-cm} \geq \left( \frac{s}{2C} \right)^{d_+ + \varepsilon} = c' s^{d_+ + \varepsilon + \frac{2\chi}{\chi'}}. \]

Since $w \in B_s(z; D_h)$, by the triangle inequality we have $B_s(z; D_h) \subset B_s(z; D_h)$, so
\[ \mu_h(B_s(z; D_h)) > c' s^{d_+ + \varepsilon + \frac{2\chi}{\chi}}. \]

Almost surely, this holds for all sufficiently small $s > 0$ and all $z \in \mathcal{D}$. Choosing $\varepsilon > 0$ so that $\varepsilon + \frac{2\chi}{\chi} < \zeta$, we obtain (5.42).
The supremum analog of (5.42) follows almost exactly the same proof, except that instead of finding a “dyadic” quantum ball inside each radius \( s \) quantum ball, we find a dyadic quantum ball \( \tilde{B} \) (with dyadic radius \( s_1 \in [2^s, 4^s) \)) around each quantum ball \( B \), then apply Lemma 5.3 to upper bound \( \mu_h(\tilde{B}) \) (and hence \( \mu_h(B) \)).

Now, we extend (5.42) to a supremum/infimum over all \( s \in (0, 1] \). For any \( s \in (r, 1] \) and \( z \in \mathbb{D} \), we have
\[
\frac{\mu_h(B_s(z; D_h))}{\gamma^{d_s + \epsilon}} \geq \mu_h(B_s(z; D_h)) \geq r^{d_s + \epsilon} \frac{\mu_h(B_s(z; D_h))}{r^{d_s + \epsilon}},
\]
and noting that a.s. for sufficiently large \( R \) we have \( D_h(\mathbb{D}, \partial B_R(0)) > 1 \),
\[
\frac{\mu_h(B_s(z; D_h))}{\gamma^{d_s - \epsilon}} \leq r^{-d_s + \epsilon} \mu_h(B_R(0)) < \infty.
\]
This concludes the proof of the uniform volume estimates.

Finally, we prove the statement from Theorem 1.1 about the Minkowski dimension of a set.

**Proof of Theorem 1.1, part 3.** Consider any bounded measurable set \( S \) containing an open set and fix \( \delta \in (0, 1) \). Let \( N^S_\varepsilon \) be the minimal number of LQG metric balls with radius \( \varepsilon \) needed to cover the set \( S \) and denote by \( C_\varepsilon \) the set of centers associated to such a covering. Then, since
\[
\mu_h(S) \leq \sum_{z \in \mathbb{D}_\varepsilon} \mu_h(B_s(z; D_h)) \leq N^S_\varepsilon \max_{z \in \mathbb{D}_\varepsilon} \mu_h(B_s(z; D_h)),
\]
the uniform volume estimate and the fact that \( \mu_h(S) > 0 \) a.s. imply that for every \( \delta > 0 \), we have the a.s. lower bound \( \liminf_{\varepsilon \to 0} \frac{\log N^S_\varepsilon}{\log \varepsilon} \geq d_s - \delta \). Now, denote by \( M^S_\varepsilon \) the maximal number of pairwise disjoint LQG metric balls with radius \( \varepsilon \) whose union is included in \( S \). Denote by \( D_\varepsilon \) the set of centers associated to such a collection of metric balls. Note that \( M^S_\varepsilon \geq N^S_\varepsilon \). Therefore,
\[
\mu_h(S) \geq \sum_{z \in D_\varepsilon} \mu_h(B_s(z; D_h)) \geq M^S_\varepsilon \min_{z \in \mathbb{D}_\varepsilon} \mu_h(B_s(z; D_h)) \geq N^S_\varepsilon \min_{z \in \mathbb{D}_\varepsilon} \mu_h(B_s(z; D_h))
\]
from which we get the a.s. upper bound \( \limsup_{\varepsilon \to 0} \frac{\log N^S_\varepsilon}{\log \varepsilon} \leq d_s + \delta \) by the uniform volume estimate and the fact that \( \mu_h(S) < \infty \) almost surely. Letting \( \delta \to 0 \) completes the proof. 

\[\square\]

### 5.2 Estimates for Liouville Brownian motion metric ball exit times

Liouville Brownian motion is, roughly speaking, Brownian motion associated to the LQG metric tensor \( \gamma e^{2h}(dx^2 + dy^2) \), and was rigorously constructed independently in the works \[15\] and \[3\]. These papers consider fields different from our field \( h \) (a whole-plane GFF normalized so \( h_1(0) = 0 \)), but their results are applicable in our setting\footnote{\[3\] considers a GFF in a bounded planar domain, and \[16\] discusses a whole-plane massive free field but explains how to adapt their framework to the setting of a GFF in a bounded planar domain. Since \( h \) is locally absolutely continuous with respect to a zero boundary GFF in a bounded planar domain (modulo additive constant), the results of \[16\] hold in our setting.}

Liouville Brownian motion was defined in \[16\] by applying an \( h \)-dependent time-change to standard planar Brownian motion. Letting \( B_t \) be standard planar Brownian motion from the origin sampled independently from \( h \), we can define Liouville Brownian motion as \( X_t = B_{F^{-1}(t)} \) for \( t \geq 0 \), where \( F \) is a random time-change defined \( h \)-almost surely. The function \( F(t) \) should be understood as the quantum time elapsed at Euclidean time \( t \), and has the following explicit description. Defining the approximation
\[
F^\varepsilon(t) = \int_0^t \varepsilon^{2/2} e^{\gamma h_\varepsilon(B_s)} ds,
\]
and writing \( T_R \) for the Euclidean time that \( B_t \) exits the ball \( B_R(0) \), the sequence \( F^\varepsilon|_{[0, T_R]} \) converges almost surely as \( \varepsilon \to 0 \) to \( F|_{[0, T_R]} \) in the uniform metric \[3\, \text{Theorem 1.2} \].

For a set \( X \subset \mathbb{C} \) and \( z \in \mathbb{C} \), denote by \( \tau_h(z; X) \) the first exit time of the Liouville Brownian motion started at \( z \) from the set \( X \). We discuss now the results of \[16\] on the moments of \( \tau_h(z; B_1(z)) \) and of \( F(t) \), i.e. the moments of the elapsed quantum time at some Euclidean time. These results are analogous to the moments of the LQG volume of a Euclidean ball (Section 2.4).
Proposition 5.4 (Moments of quantum time [16, Theorem 2.10, Corollary 2.12, Corollary 2.13]). For all \( q \in (-\infty, 1/\gamma^2) \), \( t > 0 \), the following holds,

\[
E[\tau_h(0; B_t(0))^q] + E[F(t)^q] < \infty.
\]

Heuristically, the nonexistence of large moments is due to the Brownian motion hitting regions of small Euclidean size but large quantum size. On the other hand, the random set \( B_t(0; D_h) \) in some sense avoids such regions.

In this section we prove the finiteness of all moments of the LBM first exit time of \( B_t(0; D_h) \), which we abbreviate as \( \tau \), and discuss the moments of \( \tau_h(0; B_s(0; D_h)) \) for small \( s \in (0, 1) \).

Upper bound for LBM exit time of quantum balls

Theorem 5.5 (Positive moments for quantum exit time of quantum ball). Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \), and consider Liouville Brownian motion associated to \( h \). Let \( \tau \) be the first exit time of the Liouville Brownian motion started at the origin from the ball \( B_1(0; D_h) \), i.e.

\[
\tau = \inf \{ t \geq 0 : X_t \notin B_1(0; D_h) \}.
\]

Then

\[
E[\tau^k] < \infty \quad \text{for all} \quad k \geq 0.
\]

Proof sketch: In computing \( E[\tau^k] \), by first averaging out the randomness of \( (B_t)_{t \geq 0} \), we obtain an expectation in \( h \) of an integral over \( k \)-tuples of points in \( B_t(0; D_h) \); this is similar to the integral in Step 1 of the proof of Proposition 3.11, but with additional log-singularities between these points. Because the arguments of Proposition 3.11 had some room in the exponents, the log-singularities pose no issue for us, and we can carry out the same arguments from Section 3. We will be succinct when adapting these arguments.

Let \( \tau_n \) be the quantum time LBM spends in the annulus \( A_{2^n} := B_{2^n}(0) \setminus B_{2^{n-1}}(0) \) before exiting \( B_1(0; D_h) \). As in [16, (B.2)], we have the following representation of \( E[\tau_n^k] \) for \( k \) a positive integer, which follows from taking an expectation over the standard Brownian motion \( (B_t)_{t \geq 0} \) used to define \( (X_t)_{t \geq 0} \) (see (5.43)),

\[
E[\tau_n^k] = \mathbb{E}\left[ \int_{(A_{2^n})^k} f(z_1, \ldots, z_k, h) \mathbb{I}\{z_1, \ldots, z_k \in B_1(0; D_h)\} \mu_h(dz_1) \cdots \mu_h(dz_k) \right],
\]

and where, writing \( t_0 = 0 \) and \( z_0 = 0 \) for notational convenience, \( f \) is given by

\[
f(z_1, \ldots, z_k, h) := \int_{0 \leq t_1 \leq \cdots \leq t_k < \infty} \frac{k!}{(2\pi)^{k/2}} \prod_{i=1}^k (t_i - t_{i-1}) \exp \left( -\sum_{i=1}^k \frac{|z_i - z_{i-1}|^2}{2(t_i - t_{i-1})} \right) \times \mathbb{P}[B_{[0,t_k]} \text{ stays in } B_1(0; D_h) | h, B_{t_i} = z_i \text{ for } i = 1, \ldots, k] \, dt_1 \cdots dt_k.
\]

The function \( f(z_1, \ldots, z_k) \) is an integral of the Brownian motion transition density at times \( t_1, \ldots, t_k \) times the conditional probability that the Brownian motion does not escape \( B_1(0; D_h) \). We will need the following bound on \( f \), whose proof is postponed to the end of the section.

Lemma 5.6. There exists a constant \( C > 0 \) such that for all sufficiently large \( R > 0 \), on the event \( \{B_1(0; D_h) \subset B_R(0)\} \) we have

\[
f(z_1, \ldots, z_k, h) \leq C (\log R)^k g(z_1, \ldots, z_k) \quad \text{for all } z_1, \ldots, z_k \in \mathbb{R}^d,
\]

where, recalling \( z_0 = 0 \),

\[
g(z_1, \ldots, z_k) = \prod_{i=1}^k \max \left( -\log |z_i - z_{i-1}|, 1 \right).
\]

Proof of Theorem 5.5. Our strategy is to fix some large \( R > 0 \) then truncate on the event \( E_R' := \{B_1(0; D_h) \subset B_R(0)\} \). Subsequently, we show an analog of Proposition 3.11 and use it to bound \( E[\tau_n^k 1_{E_R'}] \) for all \( n \). Combining these, we obtain a bound on \( E[\tau^k 1_{E_R'}] \). Finally, we verify that \( P[E_R'] \) decays sufficiently quickly in \( R \), and we are done.
Step 1: Proving an analog of Proposition 3.11. The argument there bounded
\[ E \left[ \int (A_1)^k g(z_1, \ldots, z_k) \mathbb{1}_{\{z_1, \ldots, z_k \in A_1 \cap P_h^{1, e^{-\epsilon x}}\}} \mu_h(dz_1) \ldots \mu_h(dz_k) \right] \]
by using a Cameron-Martin shift (placing \( \gamma \)-log singularities at each \( z_i \) and replacing \( \prod \mu_h(dz_i) \) by \( \prod_{i<j} |z_i - z_j|^{-\gamma} \prod dz_i \)), then using Proposition 3.8 to bound the integral. Recalling Remark 3.10 Proposition 3.11 can be proved even if the exponent \( \gamma \) is made slightly larger. Any such exponent increase will upper bound the log-singularities of \( g \), hence we have the following analog of Proposition 3.11
\[ E \left[ \int (A_1)^k g(z_1, \ldots, z_k) \mathbb{1}_{\{z_1, \ldots, z_k \in A_1 \cap P_h^{1, e^{-\epsilon x}}\}} \mu_h(dz_1) \ldots \mu_h(dz_k) \right] \lesssim e^{-\epsilon_k x^2}. \]

Step 2: Bounding \( E[\tau_n^k] \) for each \( n \). We start with \( n = 0 \). Using Lemma 5.6 and 5.44 (and noting that \( B_1(0; D_h) \cap A_1 \subset A_1 \cap P_h^{1,1} \)), we obtain
\[ E[\tau_0^k] \lesssim (\log R)^k E \left[ \int (A_1)^k g(z_1, \ldots, z_k) \mathbb{1}_{\{z_1, \ldots, z_k \in B_1(0; D_h)\}} \mu_h(dz_1) \ldots \mu_h(dz_k) \right] \lesssim (\log R)^k, \]
where the last inequality follows from Step 1. Likewise, building off of Step 1, similar arguments as in Lemmas 3.16 and 3.17 yield
\[ E \left[ \tau_n^k \right] \lesssim \begin{cases} (\log R)^k 2^{-\frac{2^n |\alpha| \alpha}{n}} & \text{if } n < 0, \\ (\log R)^k 2^{-\frac{2^\alpha}{\alpha n}} & \text{if } n > 0. \end{cases} \]
for some arbitrarily small \( \alpha > 0. \)

Step 3: Bounding the upper tail of \( \tau \). By Hölder’s inequality (see end of proof of Lemma 3.16), the above bounds on \( E \left[ \tau_n^k \right] \) yield
\[ E \left[ \tau_n^k \right] \lesssim (\log R)^k. \]
By Lemma 5.7 (see end of section) we also have for some fixed \( a > 0 \) that
\[ \mathbb{P}(\{E_h^k\}^c) \leq R^{-a} \]
Combining these assertions, we have
\[ \mathbb{P}(\tau > t) \lesssim \mathbb{P}(\{E_h^k\}^c) + E \left[ \tau_n^k \right] t^{-k} \lesssim R^{-a} + (\log R)^k t^{-k}. \]
Taking \( R \) equal to some large power of \( t \), we conclude that for all \( p < k \) we have \( E[\tau^p] < \infty. \) Taking \( k \to \infty \), we obtain Theorem 5.5. \( \square \)

Proof of Lemma 5.6. We instead prove the stronger statement
\[ f(z_1, \ldots, z_k, h) \leq C \prod_{i=1}^k (\log R - \log |z_i - z_{i-1}|) \quad \text{for all } z_1, \ldots, z_k \in A_1. \]
We split the integral \( 5.45 \) into two parts (integrating over \( t_k < R^2 \) and \( t_k \geq R^2 \) respectively), and bound each part separately.

There exists \( p > 0 \) such that the following is true: Let \( t \geq 1/k \) and consider a Brownian bridge of duration \( t \) with endpoints \( B_0, B_t \) specified in \( \mathbb{D} \). Then this Brownian bridge stays in \( \mathbb{D} \) with probability at most \( e^{-pt} \). If \( t_k \geq R^2 \), then there exists some \( i \in \{1, \ldots, k\} \) such that \( t_i - t_{i-1} \geq t_k/k \geq R^2/k \) and
so \( B_{[t_{i-1}, t_i]} \) conditioned on \( B_{t_{i-1}} = z_{i-1} \) and \( B_{t_i} = z_i \) stays in \( R^D \) with probability at most \( e^{-\rho t_k/kR^2} \).

This allows us to upper bound the integral \( 5.45 \) on the restricted domain with \( t_k \geq R^2 \):

\[
\int_{0\leq t_1 \leq \cdots \leq t_k < \infty} \frac{k!}{(2\pi)^{k/2}} \prod_{i=1}^k \int_0^\infty \frac{1}{t} \exp \left( -\frac{|z_i - z_{i-1}|^2}{2t} - \frac{p}{kR^2} (t_i - t_{i-1}) \right) dt \, \ldots \, dt_k
\]

by using the bound \( \int_0^\infty e^{-t/\delta} e^{-1/t} dt \leq \int_0^1 e^{-1/t} dt \leq C + \log x \) for \( x \geq 1 \) and a change of variable.

Now we upper bound the integral \( 5.45 \) on the restricted domain \( 0 \leq t_1 \leq \cdots \leq t_k < R^2 \):

\[
\int_{0\leq t_1 \leq \cdots \leq t_k < R^2} \frac{k!}{(2\pi)^{k/2}} \prod_{i=1}^k \int_0^{R^2} \frac{1}{t} \exp \left( -\frac{|z_i - z_{i-1}|^2}{2t} \right) dt = O \left( \prod_{i=1}^k \left( \log R - \log |z_i - z_{i-1}| \right) \right).
\]

where the final inequality follows from \( \int_0^{R^2} e^{-a/2t} dt = \int_0^1 e^{-a/2u} du + \int_1^{R^2} e^{-a/2u} du \leq C + \log R^2 a^{-2} \).

Combining these two upper bounds, we are done.

\[ \square \]

**Lemma 5.7** (Polynomial tail for Euclidean diameter of \( B_1(0; D_h) \)). Let \( h \) be a whole-plane GFF with \( h_1(0) = 0 \). Then for all \( a \in (0, Q^2/2) \), for all sufficiently large \( R \) we have

\[
P \left[ B_1(0; D_h) \subset B_R(0) \right] \geq 1 - R^{-a}.
\]

**Proof.** Fix \( \varepsilon > 0 \) small. By Proposition 2.3 we have with superpolynomially high probability as \( R \to \infty \) that

\[
D_h(0, \partial B_R(0)) \geq D_h(\partial B_{R/\sqrt{2}}(0), \partial B_R(0)) \geq R^{(Q-\varepsilon)} e^{\varepsilon h_R(0)}.
\]

By a standard Gaussian tail bound we also have

\[
P [h_R(0) > -(Q-\varepsilon) \log R] \leq \exp \left( -\frac{(Q-\varepsilon)^2 \log R}{2} \right) = R^{-(Q-\varepsilon)^2/2}.
\]

Combining these two bounds, we see that with probability \( 1 - O(R^{-(Q-\varepsilon)^2/2}) \) we have \( D_h(0, \partial B_R(0)) \geq 1, \) as desired.

\[ \square \]

**Lower bound for LBM exit time of quantum balls**

**Theorem 5.8.** Recall that \( \tau \) is the first exit time of the Liouville Brownian motion \( (X_t)_{t \geq 0} \) from the LQG metric ball \( B_1(0; D_h) \). For all \( k \geq 1 \), we have

\[
E[\tau^{-k}] < \infty.
\]

Consider standard Brownian motion \( (B_t)_{t \geq 0} \) started at the origin, and recall that Liouville Brownian motion is given by a random time-change: \( X_t = B_{t_1} \), where the quantum clock \( F \) is formally given by \( F(t) = \int_0^t e^{\gamma h(B_s)} \, ds \) (see [5.43]). Consider an annulus \( A_{r/e,r}(z) \) with \( 0 \not\in A_{r/e,r}(z) \). Define \( \tau_r(z) \) to be the quantum passage time of the annulus. That is, for the case where the annulus encircles the origin, writing \( t_1 \) for the first time \( B_t \) hits \( \partial B_r(z) \), and \( t_0 \) for the last time before \( t_1 \) that \( B_t \) hits \( \partial B_{r/e}(z) \), we set \( \tau_r(z) = F(t_1) - F(t_0) \), and define it analogously in the case that the annulus does not encircle the origin.

We need the following input, which can be seen as a variant of [16] Proposition 2.12 combined with the scaling relation [16] Equation (2.25) and which can be obtained by using the same techniques.

**Proposition 5.9.** For any compact set \( K \subset C \), there exists a random variable \( X \geq 0 \) having all negative moments such that the following is true. For fixed \( r \in (0, 1) \) and \( z \in K \) such that \( 0 \not\in A_{r/e,r}(z) \), the quantum passage time \( \tau_r(z) \) is stochastically dominated by \( r^{\gamma Q}\gamma e^{\gamma h_r(z)} X \).
As an immediate consequence of the \( r = 1 \) case of this proposition, we have the following:

**Corollary 5.10.** The event \( \{ X_r \not\in \mathbb{D} \text{ and } \tau < C^{-1} \} \) is superpolynomially unlikely as \( C \to \infty \).

Similarly to Section 4.1, we set \( k_1 = \lfloor N \log C \rfloor \).

**Lemma 5.11.** There exist \( \gamma \)-dependent constants \( \chi_1, \chi > 0 \) so that the following holds. Consider the event \( E_C \) that each ball \( B_{\gamma - k_1}(z) \) included in \( 2\mathbb{D} \) has quantum diameter at most \( 2e^{-\chi k_1} \). Then, \( E_C \) occurs with probability at least \( 1 - 2e^{-\chi N} \).

**Proof.** This is an application of the Hölder estimate [13, Proposition 3.18] which implies that there exist positive constants \( \chi_1, \chi \) such that, as \( \varepsilon \to 0 \), with probability at least \( 1 - 2e^{-\chi N} \),

\[
D_h(u,v) \leq |u - v|^{\chi_1}, \quad \forall u,v \in 2\mathbb{D} \text{ with } |u - v| \leq \varepsilon.
\]

Therefore, taking \( \varepsilon = e^{-k_1} \), for \( z \) such that \( B_{\gamma - k_1}(z) \subset 2\mathbb{D} \), for all \( w \in B_{\gamma - k_1}(z) \), \( D_h(z,w) \leq e^{-k_1} \) and the quantum diameter of that ball is bounded from above by twice this upper bound.

We consider the grid \( \mathbb{Z} \) divided into \( 2e^{-k_1} \mathbb{Z}^2 \).

**Lemma 5.12.** Consider the event \( F_C \) that for every point \( z \in \mathbb{Z} \cap 2\mathbb{D} \), for all \( k \in [0, k_1] \), the following conditions hold. There is a circuit of \( D_h \)-length at most \( e^{-k_1} Q e^{\chi_h - k(\gamma Q)} C \) in the annulus \( \Lambda_{\gamma-k_1 - k} \), the crossing length \( D_h(\partial B_{\gamma - k_1}(z), \partial B_{\gamma - k+1}(z)) \) is at most \( e^{-k_1} Q e^{\chi_h - k(\gamma Q)} C \), \( \tau_{\gamma - k}(z) \geq e^{-k_1} Q e^{\chi_h - k(\gamma Q)} C \) and, finally, \( |h_{\gamma - k_1}(z) - h_{\gamma - k+1}(z)| \leq \xi^{-1} \log C \). Then, \( F_C \) occurs with superpolynomially high probability as \( C \to \infty \).

**Proof.** This follows from Proposition 5.9 and Proposition 2.8 together with a union bound.

**Proof of Theorem 5.8.** We will show that \( P[\tau > C^{-1}] \) occurs with superpolynomially high probability. By Corollary 5.10 and Lemmas 5.11 and 5.12, we see that the probability of \( \{ \tau < C^{-1} \text{ and } X_r \not\in \mathbb{D} \} \cup E_C \cup F_C \) is at most \( C^{-N} \) for some fixed \( N \).

Now restrict to the event \( \{ X_r \in \mathbb{D} \} \cap E_C \cap F_C \); we show that for some constant \( \alpha \) not depending on \( C,N \) we have \( \tau > C^{-\alpha} \) for sufficiently large \( C \), then we are done since \( N \) is arbitrary. On this event the distances \( D_h(0, \partial B_{\gamma - k_1}(0)) \) and \( D_h(X_r, \partial B_{\gamma - k_1}(X_r)) \) are small, so we have \( D_h(\partial B_{\gamma - k_1}(0), \partial B_{\gamma - k_1}(X_r)) \geq \frac{2}{3} \). Let \( w \in \mathbb{Z} \) be the closest point to \( X_r \), denoted by \( w \), and grow the annuli centered at 0 and \( w \) until they first hit; let \( k_* \in [0, k_1] \) satisfy \( 2e^{-k_*} \leq |w| < 2e^{-k_* + 1} \). By Lemma 5.12 we get

\[
\tau \geq \sum_{k \in [k_*, k_1]} \tau_{\gamma - k}(w) \geq C^{-1} \sum_{k \in [k_*, k_1]} e^{-k_1 Q e^{\chi_h - k(\gamma Q) w}}
\]

and, by taking an additional annulus crossing and circuit, using the circle average regularity between two annuli,

\[
\frac{1}{2} \leq D_h(\partial B_{\gamma - k_1}(0), \partial B_{\gamma - k_1}(X_r)) \leq 10C^2 \sum_{k \in [k_*, k_1]} e^{-k_1 Q e^{\chi_h - k(\gamma Q) w}}.
\]

Therefore, by raising the inequality above to the power \( d_\tau \) and using Jensen’s inequality for the right-hand side, as well as the lower bound for \( \tau \), we get

\[
\frac{1}{2d_\tau} \leq (10C^2)^d_\tau k_1^{-d_\tau - 1} \sum_{k \in [k_*, k_1]} e^{-k_1 Q e^{\chi_h - k(\gamma Q) w}} \leq (10C^2)^d_\tau k_1^{-d_\tau - 1} C \tau.
\]

hence \( \tau \geq C^{-\alpha} \) for some fixed power \( \alpha \) and \( C \) large enough. Since \( N \) is arbitrary (\( \alpha \) does not depend on \( N \), we conclude the proof of Theorem 5.8.
Scaling relations for small balls. Finally we explain the behavior of small ball exit times. Recall that \( \tau_h(z; B_s(z; D_h)) \) is the first time that Liouville Brownian motion started at \( z \) exits the ball \( B_s(z; D_h) \).

**Theorem 5.13.** Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \), and let \( K \subset \mathbb{C} \) be any compact set. For any \( \epsilon \in (0, 1) \), there exists a constant \( C_{p,\epsilon,K} \) so that for \( C > C_{p,\epsilon,K} \), for all \( s \in (0, 1) \) and \( z \in K \) we have

\[
P[\tau_h(z; B_s(z; D)) \leq C_s^{d_s-\epsilon}] \geq 1 - C^p, \quad (5.47)
\]

and

\[
P[\tau_h(z; B_s(z; D)) \geq C^{-1}s^{d_s}] \geq 1 - C^p. \quad (5.48)
\]

**Proof.** We first discuss the proofs of (5.47) and (5.48) for the specific case \( z = 0 \). For the upper bound, recall that we proved the finiteness of positive moments of \( \tau \) by slightly modifying the arguments of Section 3. An extension of these arguments (see Lemma 3.18) yields (5.47) for \( z = 0 \). For the lower bound, though the proof of Theorem 5.12 is quite different from that of the finiteness of negative moments of \( \mu_h(B_1(0; D_0^h)) \), the exact same rescaling argument in Section 4.2 is applicable here, yielding (5.48) for \( z = 0 \). Finally, the arguments of Section 5.1 allow us to extend the \( z = 0 \) case to obtain (5.47) and (5.48) for all \( z \in K \).

\( \square \)

5.3 Recovering the conformal structure from the metric measure space structure of \( \gamma \)-LQG

The Brownian map is constructed as a random metric measure space (see \[32, 33\]) and has been proved mostly carries over directly to the general setting the conformal structure of a Brownian map from its metric measure space structure, and their proof \[37, 38, 40, 41\], but this construction was non-explicit. The work of \[23\] gives an explicit way to recover \( \gamma \) structure on \( \{ \lambda \} \) \( \cdot \) cells \( \lambda \) \( B \) (i.e. the union of \( \lambda \) \( B \) \( \cdot \) with that of a \( 0 \)-quantum cone. By comparing

\( \cdot \) of Le Gall \[31\]. The missing ingredient for general \( \gamma \) was exactly the uniform volume estimates \[1, 1\] (cf. \[23\] Lemma 4.9).

As an immediate consequence of (1.1) and the arguments of \[23\] (see discussion before \[23\] Remark 1.3), we obtain the following generalization of \[23\] Theorem 1.1 to all \( \gamma \in (0, 2) \). Let \( h \) be a whole-plane GFF normalized so \( h_1(0) = 0 \), and write \( B_s^h(0; D_h) \) for the filled \( D_h \) ball centered at 0 with radius \( R \) (i.e. the union of \( B_R(0; D_h) \) and all \( h \) finite complementary regions). Let \( P^\lambda \) be a sample from the intensity \( \lambda \) Poisson point process associated to \( \mu_h \). We can obtain a \( D_h \)-Voronoi tessellation of \( \mathbb{C} \) into cells \( \{ H^\lambda_w \}_{w \in P^\lambda} \) by defining \( H^\lambda_w = \{ w \in \mathbb{C} : D_h(w, \bar{z}) \leq \text{dist}(w, \bar{z}) \forall \bar{z} \in P^\lambda \} \). We define a graph structure on \( P^\lambda \) by saying that \( w, \bar{z} \in P^\lambda \) are adjacent if their Voronoi cells \( H^\lambda_w, H^\lambda_{\bar{z}} \) intersect along their boundaries. Let \( Y^\lambda \) be a simple random walk on \( P^\lambda \) started from the point whose Voronoi cell contains 0, extend \( Y^\lambda \) from the integers to \( [0, \infty) \) by interpolating along \( D_h \)-geodesics, and finally stop \( Y^\lambda \) when it hits \( \partial B^\lambda_0(0; D_h) \).

**Theorem 5.14** (Generalization of \[23\] Theorem 1.1). As \( \lambda \to \infty \), the conditional law of \( Y^\lambda \) given \( (C, \gamma, D_h, \mu_h) \) converges in probability as \( \lambda \to 0 \) to standard Brownian motion in \( C \) started at 0 and stopped when it hits \( \partial B^\lambda_0(0; D_h) \) (viewed as curves modulo time parametrization).

We remark that in fact this convergence holds uniformly for the random walk and Brownian motion started in a compact set, and moreover holds for a range of quantum surfaces such as quantum spheres, quantum cones, quantum wedges, and quantum disks; see \[23\] Theorem 3.3. Consequently, the Tutte embedding of the Poisson-Voronoi tessellation of the quantum disk converges to the quantum disk as \( \lambda \to \infty \) (see the proof of \[23\] Theorem 1.2).

**Proof.** Since we have the estimates (1.1), the general \( \gamma \in (0, 2) \) version of \[23\] Theorem 3.3 holds. In particular, Theorem 5.14 holds if we replace the field \( h \) with that of a 0-quantum cone. By comparing \( h \) to the field of a 0-quantum cone and using local absolute continuity arguments, we obtain Theorem 5.14.

\( \square \)

\( \ast \)The metric on curves modulo time parametrization is given as follows. For curves \( \eta_j : [0, T_j] \to \mathbb{C} \) \( (j = 1, 2) \), we set

\[
d(\eta_1, \eta_2) = \inf_{\phi} \sup_{t \in [0, T_1]} |\eta_1(t) - \eta_2(\phi(t))|
\]

where the infimum is over increasing homeomorphisms \( \phi : [0, T_1] \to [0, T_2] \).
Notice that the construction of $Y^\lambda$ involves only the pointed metric measure space structure of $(\mathbb{C}, 0, D_h, \mu_h)$, so Theorem 5.14 roughly tells us that we can recover the conformal structure of $(\mathbb{C}, 0, D_h, \mu_h)$ from its metric measure space structure. The following variant of [23, Theorem 1.2] makes this observation explicit, resolving a question of [21].

**Theorem 5.15** (Pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ determines conformal structure). Let $h$ be a whole-plane GFF normalized so $h_1(0) = 0$. Almost surely, given the pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$, we can recover its conformal embedding into $\mathbb{C}$ and hence recover $h$ (both modulo rotation and scaling).

**Proof.** To simplify notation, assume that we are given the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$, then we show we can recover exactly the embedding of $\mu_h$ in $\mathbb{C}$ (otherwise, one can arbitrarily pick any other point from the pointed metric measure space and use that in place of 1, and only recover the encoded measure modulo rotation and scaling). Since $\mu_h$ (with its embedding in $\mathbb{C}$) determines $h$ [61] and hence $D_h$, it suffices to recover $\mu_h$.

Consider $R$ large so $1 \in B^*_R(0; D_h)$. In the same way that [23, Theorem 1.1] is used to prove [23, Theorem 1.2], using Theorem 5.14 we can obtain an embedding of the two-pointed metric measure space $(B^*_R(0; D_h), 0, 1, D_h, \mu_h)$ into the unit disk $\mathbb{D}$ with the correct conformal structure and sending 0 to 0. Rotate and rescale this embedding (and forget the metric) to obtain an equivalent two-pointed space $(cR\mathbb{D}, 0, 1, \mu^R)$ with the LQG measure and conformal structure. That is, there exists a conformal map $\varphi^R : B^*_R(0; D_h) \to cR\mathbb{D}$ such that $\varphi^R(0) = 0, \varphi^R(1) = 1$, and the pushforward $(\varphi^R)^* \mu_h$ equals $\mu^R$. We emphasize that since we are only given $(\mathbb{C}, 0, 1, D_h, \mu_h)$ as a two-pointed metric measure space, we know neither the embedding $B^*_R(0; D_h) \subset \mathbb{C}$ nor the conformal map $\varphi^R$, but we do know $cR$ and $\mu^R$.

Now, by a simple estimate on the distortion of conformal maps [43, Lemma 2.4], we see that for any compact $K \subset \mathbb{C}$ we have $\lim_{R \to \infty} \sup_{z \in K} |(\varphi^R(z) - z| = 0$ and $\lim_{R \to \infty} \sup_{z \in K} |(\varphi^R)^{-1}(z) - z| = 0$. Thus, for any fixed rectangle $A$, the measure of the symmetric difference $\mu_h(A \Delta \varphi^R(\mu^R)^{-1}(A))$ converges to zero as $R \to \infty$; this implies $\lim_{R \to \infty} |\mu^R(A) - \mu_h(A)| = 0$. Since $\mu^R$ is a function of the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$, we conclude that $\mu_h(A)$ is also. Therefore the two-pointed metric measure space $(\mathbb{C}, 0, 1, D_h, \mu_h)$ determines $\mu_h$ and hence $h$. □

6 Appendix

6.1 Proof of the inductive relation for small moments

**Lemma 6.1.** Recall $v_k(r)$ and $u_k(r)$ from (2.5). The following relation holds.

$$v_k(r) \leq Cr^{-2} \sum_{i=1}^{k-1} \binom{k}{i} (4k)^{2i} u_i(4r) u_{k-i}(4r). \tag{6.49}$$

**Proof.** Set $f_k(z_1, \ldots, z_k) := \prod_{i<j} |z_i - z_j|^{-2}$. Note that when $\max_k |z_i - z_j| \leq r$, the $k$ points are included in $B(z_1, r)$ which itself is included in a ball of radius $4r$ centered at point of $r\mathbb{Z}^2 \cap \mathbb{D}$. Since $f_k$ is a function of the pairwise distance, which is translation invariant, we get

$$v_k(r) = \int_{\mathbb{D}} 1_{r/2 \leq \max_{i<j} |z_i - z_j| \leq r} d\gamma \cdots d\gamma \leq Cr^{-2} \int_{4r\mathbb{D}} 1_{r/2 \leq \max_{i<j} |z_i - z_j|} f_k(z_1, \ldots, z_k) d\gamma \cdots d\gamma$$

Then, take two points at distance $r/2$ in $4r\mathbb{D}$, say $z$ and $w$ among $\{z_1, \ldots, z_k\}$. We cut $k + 1$ orthogonal sections of same width to the segment $[z, w]$. At least one should be empty and this separates two clusters of points, $I = \{z_{p_1}, \ldots, z_{p_l}\}$ and $J = \{z_{q_1}, \ldots, z_{q_{k-l}}\}$ for some $1 \leq i \leq k - 1$. All points between the two clusters $I$ and $J$ are separated by $|z - w|/(k + 1) \geq r/4k$. We decouple $f_k(z_1, \ldots, z_k)$ for two clusters $I$.

---

7 Roughly speaking, one obtains a conformal embedding of the metric measure space $(B^*_R(0; D_h), 0, 1, D_h, \mu_h)$ by taking a $\lambda$-intensity Poisson-Voronoi tessellation and harmonically embedding it in the disk. Taking $\lambda \to \infty$, the counting measure on the vertices of the embedded graph normalized by $\lambda^{-1}$ converges weakly in probability to the desired conformally embedded measure. See [23, Section 3.3] for details.

8 [43, Lemma 2.4] is stated for domains in the cylinder $\mathbb{R} \times [0, 2\pi]$; we need to map to \mathbb{C} via $z \to e^{-z}$. This distortion estimate is an easy consequence of the area theorem.
and \( J \) of size \( i \) and \( k - i \) by \( f_k(z_1, \ldots, z_k) \leq (4k)^{\gamma_2(i)}\gamma_2(e^{-k})f_k(I)f_k-z(I) \). In particular, splitting over the possibilities we get

\[
v_k(r) \leq C r^{2 \sum_{i=1}^{k} \gamma_2(i)} \int_{4r^2k} f_k(I)f_k-z(I)dz_1 \ldots dz_k,
\]

where for each \( i, I \) ranges over all subsets of \( \{z_1, \ldots, z_k\} \) with \( i \) elements. This gives

\[
v_k(r) \leq C r^{2 \sum_{i=1}^{k} \gamma_2(i)} u_k(4r)v_k(4r).
\]

\[\square\]

### 6.2 Whole-plane GFF and \( \ast \)-scale invariant field

In this section we recall some properties of \( \ast \)-scale invariant fields and explain that the whole-plane GFF modulo constants can be seen as a \( \ast \)-scale invariant field.

**\( \ast \)-scale invariant field \( \phi \).** We introduce here the field \( \phi = \sum_{k \geq 1} \phi_k \) we work with in Section 3.1. The notation and definition are close to the one in [12, Section 2.1] and we refer the reader to this Section for more details.

Consider \( k \), a smooth, radially symmetric and nonnegative bump function supported in \( B_{1/2}(0) \), such that \( k \) is normalized in \( L^2(\mathbb{C}) \). We set \( c = k * k \) which has therefore compact support included in \( B_1(0) \) and satisfies \( c(0) = 1 \). We consider a space-time white noise \( \xi(dx, dt) \) on \( \mathbb{C} \times [0, \infty) \) and define the random Schwartz distribution

\[
\phi(x) := \int_0^1 \int_C k \left( \frac{x-y}{t} \right) t^{-3/2} \xi(dy, dt).
\]

The covariance kernel of \( \phi \) is given by \( \mathbb{E}(\phi(x)\phi(x')) = \int_0^1 c(x-x') dt \). We decompose \( \phi = \sum_{k \geq 1} \phi_k \) where \( \phi_k(x) := \int_{e^{-k}(1)} e^{-k} k \left( \frac{x-y}{t} \right) t^{-3/2} \xi(dy, dt) \) and whose covariance kernel is given by \( C_k(x, x') := \int_{e^{-k}(1)} e^{-k} c(x-x') dt \). Note that \( C_k(x, x') = C_1(e^{-k-1}x, e^{-k-1}x') \) and that if \( |x-x'| \geq e^{-1} \), \( C_k(x, x') = 0 \) hence \( \phi_k \) has finite range dependence with range of dependence \( e^{-k} \). Note also that the pointwise variance of \( \phi_{0,n} := \sum_{1 \leq k \leq n} \phi_k \) is equal to \( n \).

**Lemma 6.2.** There exists \( C, c > 0 \) such that for all \( k \geq 0, x > 0, \mathbb{P}(c^{-k} \| \nabla \phi_{0,k} \|_{L^2(S)} \geq x) \leq Ce^{-cx^2} \), where \( S \) denotes the unit square \([0, 1] \times [0, 1] \).

**Proof.** This is essentially the argument as in the proof of Lemma 10.1 in [12] which we recall. By Fernique’s theorem, \( \mathbb{P}(\| \nabla \phi \|_{L^2(S)} \geq x) \leq Ce^{-cx^2} \). Therefore, by scaling, \( \mathbb{P}(c^{-k} \| \nabla \phi \|_{L^2(S)} \geq x) \leq Ce^{-cx^2} \) for \( \ell \geq 1 \). By setting \( X_{\ell} := c^{-\ell} \| \nabla \phi \|_{L^2(S)} \), by the triangle inequality and since \( e^{-k} S \subseteq e^{-\ell} S \) for \( \ell \leq k \), \( e^{-k} \| \nabla \phi_{0,k} \|_{L^2(S)} \leq \sum_{\ell \leq k} c^{-k-\ell} X_{\ell} \). By inspecting the Laplace functional, and using that the \( X_{\ell} \)'s are independent and identically distributed, we conclude the proof of the Lemma.

**Whole-plane GFF.** We explain here why \( \int_0^\infty k \left( \frac{x-y}{t^2} \right) t^{-3/2} \xi(dy, dt) \) is a whole-plane GFF modulo constants. Set \( \phi_\epsilon(x) = \int_e^\infty \int_C k \left( \frac{x-y}{t^2} \right) t^{-3/2} \xi(dy, dt) \) and take \( f \in S(\mathbb{C}) \) such that \( \int f dx = 0 \). We look for the convergence of

\[
\mathbb{E}(\langle \phi_\epsilon, f \rangle^2) = \int_{\mathbb{C} \times \mathbb{C}} f(x)\overline{f(y)} \xi(dy, dt) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{C}_\epsilon(\xi)\overline{\hat{f}(\xi)}^2 d\xi
\]

where our convention for the Fourier transform is \( \hat{g}(\xi) := \int g(x) e^{-i\xi \cdot x} \).

The kernel of \( \phi_\epsilon \) is given by \( C_\epsilon(x) = \int_0^\infty e^{-\epsilon} c(\frac{x}{t}) dt = \int_0^\infty e^{-\epsilon} c_k(x) \frac{dt}{t} \) with \( c_k(x) = c(x/t) \) thus its Fourier transform satisfies \( \hat{C}_\epsilon(\xi) = \int_0^\infty \hat{c}_k(\xi) \frac{dt}{t} = \int_0^\infty \hat{c}(t\xi) dt \) and since \( c = k * k \), \( \hat{c} = \hat{k}^2 \), then...
\[ \dot{C}_ε(ξ) = \int_{ε}^{1} t \hat{k}(tξ)^2 dt = \|ξ\|^{-2} \int_{\|ξ\|}^{1} \hat{u}(u) du. \] By monotone convergence, we get,

\[ \mathbb{E} \left( (\phi_ε, f)^2 \right) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \|ξ\|^{-2} \int_{\|ξ\|}^{1} \hat{u}(u) du |\hat{f}(ξ)|^2 dξ \rightarrow \left( \int_{0}^{\infty} \hat{u}(u) du \right) \times \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \|ξ\|^{-2} |\hat{f}(ξ)|^2 dξ. \]

Since \( \hat{k} \) is radially symmetric and \( \hat{k} \) is normalized in \( L^2 \), by Parseval \( \int_{0}^{\infty} \hat{u}(u) du = 2\pi \). Furthermore, by setting \( g(x) = \int_{\mathbb{C}} \log |x - y| f(y) dy \) we get \( \Delta g = 2\pi f \) and in Fourier modes, \( \|ξ\|^2 \hat{g}(ξ) = 2\pi |\hat{f}(ξ)| \)

\[ \int_{\mathbb{C}} f(x)(-\log |x-y|) f(y) dx dy = -\int_{\mathbb{C}} f(x) g(x) dx = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(ξ) \hat{g}(ξ) dξ = \frac{1}{2\pi} \int_{\mathbb{R}^2} \|ξ\|^{-2} |\hat{f}(ξ)|^2 dξ. \]

Note that this term is finite because under the assumption \( \int_{\mathbb{C}} f dx = 0 \), we have \( \hat{f}(0) = 0 \) so the above singularity at the origin is compensated by the first term in the development of \( \hat{f} \). Altogether, we get

\[ \mathbb{E} \left( (\phi_ε, f)^2 \right) \rightarrow \int_{\mathbb{C}} f(x)(-\log |x-y|) f(y) dx dy \]

Hence the convergence of the characteristic functionals:

\[ \mathbb{E}(e^{i(\phi_ε, f)}) = e^{-\frac{1}{2} \mathbb{E}(\phi_ε, f)^2} \rightarrow e^{-\frac{1}{2} \mathbb{E}(h, f)^2}. \]

The following lemma will be useful when working with the whole plane GFF not modulo additive constant.

**Lemma 6.3.** There exists a coupling of the whole-plane GFF \( h \) normalized such that \( h_1(0) = 0 \) and the \( * \)-scale invariant field \( \phi \) such that the difference \( h - \phi \) is a continuous field.

**Proof.** Recall the notation \( \phi_{k,t} = \int_{0}^{-k} k(\frac{t-s}{2\pi})t^{-3/2} \xi(dy, dt) \). We know \( \phi_{-∞,∞} \) is a whole-plane GFF modulo constant. The fine field \( \phi = \phi_{0,∞} \) is a well-defined Schwartz distribution. Also, the gradient field \( \nabla \phi_{-∞,0} \) is a well-defined continuous Gaussian vector (this can be checked by inspecting the covariance kernel).

Therefore, \( \phi_L := \phi_{-∞,0} - \int_{B_1(0)} \phi_{-∞,0} \) is a well-defined continuous Gaussian field, independent of \( \phi \). By setting \( g := \phi_L - \int_{B_1(0)} \phi \), we get that \( h := \phi + g \) is a whole-plane GFF normalized such that \( h_1(0) = 0 \).

### 6.3 Volume of small balls in the Brownian map

We do not use any material in this section in our proofs, but include it to facilitate an easier comparison between our argument in Section 6 and the analogous result for the Brownian map case. Le Gall obtained the following uniform estimate on the volume of small balls in the Brownian map. For \( \beta \in (0, 1) \), there exists a random \( K_β > 0 \) such that for every \( r > 0 \), the volume of any ball of radius \( r \) in the Brownian map is bounded from above by \( K_β r^{4-\beta} \). Our proof of the finiteness of LQG ball volume positive moments (Section 4) shares some similarities with his only at a very high level; no explicit formulas are available in our framework, and the techniques are very different. We discuss some of the arguments used in the Brownian map setting and we refer the reader to [29] [34] [39] [51] for details. This estimate was used in the proofs of the uniqueness of the Brownian map [33] [32].

**Tree of Brownian paths.** A binary marked tree is a pair \( (τ, (h_v)_{v ∈ τ}) \) where \( τ \) is a binary plane tree and where for \( v ∈ τ \), \( h_v \) is the length of the branch associated to \( v \). We denote by \( \Lambda_k(dθ) \) the uniform measure on the set of binary marked trees with \( k \) leaves (uniform measure over binary plane trees and Lebesgue measures for the length of the branches). \( I(θ) \) and \( L(θ) \) will denote respectively the internal nodes and leaves of \( θ \). One can define a Brownian motion on such a tree: the process is a standard Brownian motion over a branch, and after an intersection, the two processes evolve independently conditioning on the value at the node. We will denote by \( P_θ^x \) this process, started from the root of the tree with initial value \( x \). Similarly, instead of using a Brownian motion, one can consider a 9-dimensional Bessel process and we will denote it by \( Q_θ^x \).

Similarly, for trees given by a contour function \( (h(s))_{s ≤ σ} \) with lifetime \( σ \), one can associate the so-called Brownian snake given by the process \( (W_s)_{s ≤ σ} \) of Brownian type path (for each \( s \), \( W_s \) is a Brownian type path with lifetime \( h(s) \), its last value is denoted by \( \hat{W}_s \) and corresponds to the Brownian label above.
the point of the tree corresponding to $s$). We can add another level of randomness by taking $h$ given by a Brownian type excursion: $\mathbb{N}_0$ is the measure associated to the unconditioned lifetime Itô excursion, $\mathbb{N}_0$ is also associated to the unconditioned lifetime Itô excursion but the Brownian labels are conditioned to stay positive.

**Explicit formulas.** The following explicit formula (see [29], Proposition IV.2), relates the objects of the previous paragraph. For $p \geq 1$, $x \in \mathbb{R}$ and $F$ a symmetric nonnegative measurable function on $W^p$, where $W$ denotes the space of finite continuous paths,

$$\mathbb{N}_x \left[ \int_{(0, x)^p} F(W_{s_1}, \ldots, W_{s_p}) ds_1 \ldots ds_p \right] = 2^{p-1}! \int A_p(d\theta) P_x^{\theta} \left[ F((W^{(a)})_{a \in L(\theta)}) \right].$$

(6.50)

Here, $w$ is the tree-indexed Brownian motion with law $P_x^\theta$ and $w^{(a)}$ the restriction of $w$ to the path joining $a$ to the root, and $\mathbb{N}_x$ is the measure $\mathbb{N}_0$ where each Brownian snake has its labels incremented by $x$. This formula involves combining the branching structure of certain discrete trees with spatial displacements. It relies on nice Markovian properties, in particular on specific properties of the Itô measure. The proof of the uniform volume bound for metric ball is based on an explicit formula obtained in [34] for the finite-dimensional marginal distributions of the Brownian tree under $\mathbb{N}_0$, 

$$\mathbb{N}_0 \left[ \int_{(0, a)^p} F(W_{s_1}, \ldots, W_{s_p}) ds_1 \ldots ds_p \right] = 2^{p-1}! \int A_p(d\theta) Q_0^\theta \left[ F((\overline{w}^{(a)})_{a \in L(\theta)}) \prod_{b \in I(\theta)} \overline{V}_b \prod_{c \in L(\theta)} \overline{V}_c^{-4} \right].$$

(6.51)

Here, we write $\overline{w}$ and $\overline{w}^{(a)}$ for the nine-dimensional Bessel process counterparts of $w$ and $w^{(a)}$, and $\overline{V}_v$ for the value of the Itô process at the vertex $v$. Because of the conditioning of $\mathbb{N}_0$, the spatial displacements are given by nine-dimensional Bessel processes rather than linear Brownian motions. To derive such a formula, in [34] the authors generalize (6.50) to functionals including the range of labels and lifetime $\sigma$ and then use results on absolute continuity relations between Bessel processes, which are consequences of the Girsanov theorem (note that integrals over time of Brownian motions are integral over branches of tree motions).

**Positive moment estimates.** In the proof of the upper bound on small ball volumes of the Brownian map in [31], a key estimate is to show that, for $k \geq 1$, $c_k < \infty$ where

$$c_k := \mathbb{N}_0 \left[ \left( \int_0^\sigma 1_{\{\overline{W}_s \leq 1\}} ds \right)^k \right] = 2^{k-1}! k! \int A_k(d\theta) Q_0^\theta \left( \prod_{a \in I(\theta)} \overline{V}_a^4 \prod_{b \in L(\theta)} \overline{V}_b^{-4} 1_{\{\overline{V}_b \leq 1\}} \right) =: 2^{k-1}! \hat{a}_k.$$ 

(6.52)

Note that the second inequality is obtained by using (6.51) with $F(W_{s_1}, \ldots, W_{s_k}) = 1_{\{\overline{W}_{s_1} \leq 1\}} \ldots 1_{\{\overline{W}_{s_k} \leq 1\}}$. The proof works by induction, introducing an additional parameter to take care of the value of the label at the splitting node in the branching structure, by setting

$$\hat{d}_k(r) := \int A_k(d\theta) Q_0^\theta \left( \prod_{a \in I(\theta)} \overline{V}_a^4 \prod_{b \in L(\theta)} \overline{V}_b^{-4} 1_{\{\overline{V}_b \leq 1\}} \right) \Lambda_k(d\theta).$$

In this framework, the base case and inductive relation are quite straightforward because of the exact underlying branching structure. Let $R$ denote a 9-dimensional Bessel process that starts from $r$. The base case corresponds to

$$\hat{d}_1(r) = \mathbb{E} \left[ \int_0^\infty R_t^{-4} 1_{(R_t \leq 1)} dt \right] = c \int \mathbb{E} \left[ |r - z|^{-7} |z|^{-4} 1_{\{|z| \leq 1\}} \right] dz$$

(6.53)

and the inductive relation states

$$\hat{d}_k(r) = \mathbb{E} \left[ \int_0^\infty R_t^4 \left( \sum_{j=1}^{k-1} \hat{d}_j(R_t) \hat{d}_{k-j}(R_t) \right) \right].$$

(6.54)

Now, one can easily derive the bounds $\hat{d}_1(r) \leq M r^{-2} \wedge r^{-7}$ and for $j \geq 2$ $\hat{d}_j(r) \leq M_j 1 \wedge r^{-7}$. We underline that the exact branching structure of the framework is expressed through the equality (6.54).
Comparison. Let us compare our proof of the finiteness of positive moments with the one in the Brownian map setting. In our setup, no nice branching structure for distances is known. Furthermore, by working with a given embedding or a restriction to a specific domain, we have to carry in the analysis information about the Euclidean domain, including an additional layer of difficulty.

In the case of the Brownian map, when one considers the “volume” associated with the explicit formulas (6.50) and (6.51), one ends up with branching Bessel processes on uniform trees. In our framework, analogous observables of “distances” are not well understood so far. Instead, circle averages processes are tractable. They evolve as correlated Brownian motions. These are a good proxy for the metric because of the superconcentration of side-to-side crossing distances. Furthermore, when one weights the distribution with singularities (after a Cameron-Martin argument), these Brownian motions are shifted by drifts. (Note that the passage from (6.50) to (6.51) uses Girsanov.)

Similarities can be seen as the level of induction where the value of the Bessel process at the first node can be compared with the value of the circle average of the field in at the first branching as well in our hierarchical decomposition. Therefore, Lemma 3.7 is similar to (6.54) and Proposition 3.8 to (6.52).

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