Nonequilibrium Zaklan model on Apollonian networks

F. W. S. Lima

*Dietrich Stauffer Computational Physics Lab, Departamento de Física, Universidade Federal do Piauí, 64049-550, Teresina - PI, Brazil.*

e-mail: fwslima@gmail.com

⋆ This paper is dedicated to Dietrich Stauffer

Abstract

The Zaklan model had been proposed and studied recently using the equilibrium Ising model on Square Lattices (SL) by Zaklan et al (2008), near the critical temperature of the Ising model presenting a well-defined phase transition; but on normal and modified Apollonian networks (ANs), Andrade et al. (2005, 2009) studied the equilibrium Ising model. They showed the equilibrium Ising model not to present on ANs a phase transition of the type for the 2D Ising model. Here, using agent-based Monte-Carlo simulations, we study the Zaklan model with the well-known majority-vote model (MVM) with noise and apply it to tax evasion on ANs, to show that differently from the Ising model the MVM on ANs presents a well defined phase transition. To control the tax evasion in the economics model proposed by Zaklan et al, MVM is applied in the neighborhood of the critical noise $q_c$ to the Zaklan model. Here we show that the Zaklan model is robust because this can be studied besides using equilibrium dynamics of Ising model also through the nonequilibrium MVM and on various topologies giving the same behavior regardless of dynamic or topology used here. Keywords: Opinion dynamics, Sociophysics, Majority vote, Nonequilibrium.
1 Introduction

The Ising model [1, 2] has become an excellent tool to study other models of social application. Therefore, following this line of reasoning the Zaklan model had been proposed and studied recently using the equilibrium Ising model on square Lattices by Zaklan et al. [3, 4, 12]. Lima [13], based on Grinstein et al. [5], made a proposal to extend the current model (Zaklan’s model) to nonequilibrium systems, using nonequilibrium Majority-Vote Model (MVM) [6] in order to make Zaklan’s model more realistic, because tax evasion is nonequilibrium.

Our simulation is based on the well-known Apollonian packing introducing Apollonian networks [7]. According to Andrade et al. [7] the ANs are simultaneously scale-free [8], small-world [9], Euclidean, space filling, and with matching graphs [10]. Therefore, the ANs have social connections that are often similar to scale-free or small-world networks [11] and have been studied e.g. for the Ising model and a magnetic model [15, 16]. The ANs here are defined and described in detail in the pioneering work by Andrade et al. [7].

As shown in [7, 15, 16, 17] on ANs, the Ising and Potts models do not present a phase transition. Therefore, the Ising model on this topology is not useful for the Zaklan model, because it does not have a phase transition on ANs. Therefore, our work is not only to show that the Zaklan model works another a topology as the ANs, but also to show that in some topologies the traditional equilibrium spin models, as the Ising model, are not appropriate to study socio-economic models as the model proposed by Zaklan et al [3, 4]. Therefore, different from Ising models, the nonequilibrium MVM model is presented here as an alternative model in the study of socio-economic Zaklan model on ANs. This similar behavior was also shown by Sumour et al [18] and Lima and Stauffer [19], where the Ising model do not present a phase transition on directed Babási-Albert networks and hypercubic lattices, respectively. Thus, we present an alternative proposed in the study of social and economics models via nonequilibrium MVM and also showed that in this case the criterion of Grinstein et al. [5] is not applicable, i.e, on ANs Ising and MVM do not belong to the same universality class.

Our paper is organised as follows. In section 2, we present the Zaklan model evolving with dynamics of MVM. In section 3 we make an analysis of tax evasion dynamics with the Zaklan model on ANs, using MVM for their temporal evolution under different enforcement regimes; we discuss the
results obtained. In section 4 we show that MVM also is capable to control the different levels of the tax evasion analysed in section 3, as it was made by Zaklan et al. [4] using Ising models. We use the enforcement mechanism cited above on ANs and discuss the resulting tax evasion dynamics. Finally in section 5 we present our conclusions about the study of the Zaklan model using MVM on ANs.

2 Zaklan model and evolution dynamics

2.1 Zaklan model

The Zaklan model [3] consists of agents located on a regular or irregular structure. Each agent is represented by an individual spin $S_i = \pm 1$, who can either be an honest tax payer $+1$ or a cheater $-1$. Initially everybody is assumed honest. In each iteration individuals can rethink their behaviour and have the opportunity to become the opposite type of agent they were in the previous period. Each agent’s environment may prefer tax evasion or reject it. The agent depends on two factors: First, the agent’s environment exerts influence on the agent in the next period. Second, people’s decisions are partly autonomous, independent of their environment. This autonomous part is responsible for the emergence of tax evasion, because some initially honest tax payers decide to evade taxes and then exert influence on others to do so as well. Tax evaders have the greatest influence to turn honest citizens into tax evaders if they constitute a majority in the respective neighbourhood. On the other hand, if most people in the vicinity are honest, the respective individual is likely to become a tax payer if (s)he was a tax evader before.

The model also presents an enforcement mechanism that consists of two components: a probability of an efficient audit $p$. If tax evasion is detected, the individual must remain honest for a number $k$ of periods to be specified. One iteration is one sweep through the entire lattice. The temporal evolution this model can be performed by using an equilibrium or, in the present work, by nonequilibrium dynamics.

2.2 Zaklan model via nonequilibrium MVM model

The Apollonian networks contain $N = 3 + (3^{n-1} - 1)/2$ nodes (sites, agents, spins) where $n$ is the generation number [15]. Our MVM dynamics contains
a noise parameter $q$. For the case of four neighbors and $q = 0$, if three or four neighbors disagree with the central site, the center flips; if one or none disagrees, the center does not flip; if two agree and two disagree, it flips with probability $1/2$. If $q > 0$ the center may disobey this majority rule. More precisely, in each time period the system evolves by a single spin-flip dynamics with a probability $w$ given by

$$ w(\sigma_i) = \frac{1}{2} \left[ 1 - (1 - 2q)\sigma_i S\left( \sum_{\delta=1}^{k_i} \sigma_{i,\delta} \right) \right]. $$

(1)

where $S(x)$ is the sign $\pm 1$ of $x$ if $x \neq 0$, $S(x) = 0$ if $x = 0$, and the summation runs over all $k_i$ nearest-neighbour sites $\sigma_{i,\delta}$ of $\sigma_i$. In this model an agent assumes the value $\pm 1$ depending on the opinion of the majority of its neighbors. The noise parameter $q$ plays the role of the temperature in equilibrium systems and measures the probability of aligning $\sigma_i$ antiparallel to the majority of its neighbors $\sigma_{i,\delta}$.

3 Controlling the tax evasion dynamics

![Graph](image)

Figure 1: Reciprocal logarithm of the relaxation times on ANs for versus $q$. 

4
In order to test if there is a phase transition in MVM models on ANs, we measured the relaxation time $\tau$ as a function of the noise parameter $q$, independent of our tax question. We start the system with all spins up and a number $N$ of spins equal to 7, 174, 456 ($n = 16$). We determine the time $\tau$ after which the magnetisation $\sum_i \sigma_i$ has flipped its sign for the first time, and then take the median value of nine samples. As one can see in Fig. 1, the relaxation time goes to infinity at some positive $q$ value near 0.18, indicating a second order phase transition. On contrast, the Ising model on ANs [15, 16] and directed BA networks has no phase transition and agrees with the modified Arrhenius law for relaxation time [18]. In order to improve earlier affirmation the magnetisation and susceptibility for $N = 3, 283; 9, 844; 29, 527; 88, 576,$ and 265, 723 sites and with $n = 8, 9, 10, 11,$ and 12 generation are plotted in Fig. 2.

![Figure 2: Plot of the magnetisation (a) and susceptibility (b) for different generations of the Apollonian Network. (circle) $G_8$, (square) $G_9$, (star) $G_{10}$, (triangle up) $G_{11}$, (triangle down) $G_{12}$. The number after $G$ gives the generation number $n$.](image)

In order to calculate the rate of tax evaders, we use

$$\text{tax evasion} = \frac{[N - N_{\text{honest}}]}{N},$$

(2)

where $N$ is the total number and $N_{\text{honest}}$ the honest number of agents. The tax evasion is calculated at every time step $t$ of system evolution; one time step is one sweep through the entire network.

Here, we follow the same steps we did in a previous work [13]. Therefore, we first will present the baseline case $k = 0$ and $p = 0.0\%$, i.e., no use of
enforcement, at \( q = 0.80 q_c \) and with \( N = 367 \) \((n = 7)\) sites for ANs. All simulation are performed over 25,000 time steps, as shown in Fig. 3. For very low noise the part of autonomous decisions almost completely disappears. The individuals then base their decision solely on what most of their neighbours do. A rising noise has the opposite effect. Individuals then decide more autonomously. Therefore, Figure 3 was expanded to four examples, in order to show how much the results differ if one changes the random numbers. Error bars cannot describe this randomness properly. (For the later figures the error bars are visible from the fluctuations in time which show a band of fractions.) Although everybody is honest initially, it is impossible to predict roughly which level of tax compliance will be reached at some time step in the future.

For MVM it is known that for \( q > q_c \), half of the people are honest and the other half cheat, while for \( q < q_c \) either one opinion or the other opinion dominates. Because of this behavior we set at fixed ”Social Temperature” \((q)\) to some values slightly below \( q_c \), where the case that agents distribute in equal proportions onto the two alternatives is excluded. Then having set the noise parameter \( q \) close to \( q_c \approx 0.18 \) on the ANs, we vary the degrees of punishment \((k = 1, 10 \text{ and } 50)\) and audit probability rate \((p = 0.5\%, 10\% \text{ and } 90\%)\). Therefore, if tax evasion is detected by the enforcement mechanism \( p \), the period of punishment \( k \) is triggered in order to control the tax evasion level. The punished individuals remain honest for a certain number \( k \) of periods, as explained before in section 2.

Figure 4 illustrates different simulation settings for ANs, for each considered combination of degree of punishment \((k = 1, 10 \text{ and } 50)\) and audit probability \((p = 0.5\%, 10\% \text{ and } 90\%)\), where the tax evasion is plotted over 25,000 time steps. Both a rise in audit probability \((\text{greater } p)\) and a higher penalty \((\text{greater } k)\) work to flatten the time series of tax evasion and to shift the band of possible non-compliance values towards more compliance. However, the simulations show that even extreme enforcement measures \((p = 90\% \text{ and } k = 50)\) cannot fully solve the problem of tax evasion.

In Fig. 5 we plot tax evasion for ANs, but now with \( N = 3,283 \), again for different enforcement \( k \) and audit probability \( p \). Now the fluctuations are much smaller since the network is nearly nine times larger. For case (a) we plot the baseline case \( k = 0 \) and \( p = 0 \), i.e., no use of enforcement for ANs and parameters as in Fig. 3. The probable error for part (c) fluctuates near 0.0031 and is much smaller than the symbols (circle). Case (b) with
Figure 3: Baseline case: $k = p = 0$ and with three different seed (a), (b), (c) and the average over twenty different seeds (d). We use $q = 0.80q_c$ on ANs and perform all simulations over 25,000 time steps, also in the later figures.

$k = 1$, $p = 0.5\%$ shows already a strong reduction of tax evasion. In case (c) we show the tax evasion level decreases, on ANs, for a more realistic set of possible values degrees of punishment $k = 10$ and audit probability $p = 4.5\%$ [14, 3]. In case (d) we also show the tax evasion level decreases much more for an extreme set of punishment $k = 50$ and audit probability $p = 90\%$ [3]. Therefore, our model also works for large networks.
Figure 4: Tax evasion for different enforcement regimes ANs and for degrees of punishment $k = 0, 1, 10$ and 50 and audit probability $p = 0.5\%, 10\%$ and 90\%.

To understand statistical errors, in Fig. 5 we plot tax evasion for ANs with $N = 3,283$ now for the case $k = 10$ and $p = 4.5\%$. We found from 20 samples in part (c) that the tax evasion remains at around 20\%, but with fluctuations in time larger than from sample to sample: The probable errors are much smaller than the fluctuations seen in part (c).
Figure 5: Tax evasion for different enforcement regimes ANs and for degrees of punishment $k = 0, 1, 10$ and $50$ and audit probability $p = 0.0\%, 0.5\%, 4.5\%$, and $90\%$ for $N = 3,283$ sites (nodes) of ANs and use $50,000$ time steps. Here, for $k = 0$ and $p = 4.5\%$ (c), we present the average over twenty different seeds.

4 Conclusion

Less developed countries may have more tax evasion because of less trust in government [14]. To study this problem Zaklan et al. [3, 4] proposed a model, called here the Zaklan model, using Monte Carlo simulations and a equilibrium dynamics (Ising model) on square lattices. Their results are in good agreement with analytical and experimental results obtained by [14]. In this work we show that the Zaklan model of tax evasion is very robust because we use a nonequilibrium dynamics (MVM) to simulate the Zaklan model, with results similar to equilibrium dynamics (Ising model) [3, 4], and also on various topologies [13]. Also here we found the plausible result that tax evasion is diminished by higher audit probability $p$ and stronger
punishment $k$.

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