From Uniform Boundedness to the Boundary Between Convergence and Divergence

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Three of the fundamental ideas Stefan Banach introduced in functional analysis together lead to his discovery of three fundamental results [6, 7, section VI.84]. The ideas were abstract points (functions as points, leading to operators and function spaces), abstract sizes (norms of functions, leading to distances between functions), and abstract limits (limits of sequences of functions, leading to completeness of function spaces). The results were the uniform boundedness principle, the open mapping theorem, and the closed graph theorem, which are all interrelated, in the sense that in complete normed vector spaces (known as Banach spaces), Baire’s category theorem leads to several equivalences between qualitative properties (e.g., finiteness, surjectivity, regularity) and quantitative properties (e.g., estimates) of continuous (or equivalently bounded) linear operators [11, section 1.7]. One of these equivalences is captured by the uniform boundedness principle, also known as the Banach-Steinhaus theorem. In this article, we introduce a dual of the uniform boundedness principle which does not require completeness and gives an indirect means for testing the boundedness of a set. The dual principle, although known to analysts and despite its applications in establishing results such as the Hellinger-Toeplitz theorem, is often missing from elementary treatments of functional analysis. In Example 1, we indicate a connection between the dual principle and a question in the spirit of du Bois-Reymond regarding the boundary between convergence and divergence of sequences. This example is intended to illustrate why the statement of the principle is natural and clarify what the principle claims and what it does not.

Unbounded sets in normed spaces

We begin with a proposition of linear algebraic flavor about the relation between unbounded subsets of a normed space and the linear functionals on that space. In what follows, we shall assume that all vector spaces are over the field $\mathbb{R}$, and all linear maps between them are real-linear, although our results carry over easily to the field of complex numbers.

**Proposition 1.** Let $S$ be an unbounded subset of a normed vector space $X$. Then there exists a linear functional $\phi: X \to \mathbb{R}$ whose restriction to $S$ has an unbounded image in $\mathbb{R}$.

We include a simple proof here for the sake of completeness.

**Proof.** First assume $X$ is finite dimensional and let $n = \dim X$. Then since all norms on any finite-dimensional normed space are equivalent, we will assume that the norm of $X$ is induced by an inner product. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ with respect...
to this inner product for $X$, and for $i = 1, \ldots, n$ let $\text{Proj}_{e_i} : X \to \mathbb{R}$ denote the scalar projection onto the $i$th coordinate:

$$\text{Proj}_{e_i}(c_1e_1 + \cdots + c_ne_n) = c_i \quad \text{for any } c_1, \ldots, c_n \in \mathbb{R}.$$ 

We claim that for some value of $i$, the restriction of $\text{Proj}_{e_i}$ to $S$ has an unbounded image. Indeed, if we had

$$\sup_{x \in \text{Proj}_{e_i} S} |x| \leq M_i \quad \text{with } M_i \geq 0 \text{ for all } i = 1, \ldots, n,$$

then it would follow that $\sup_{s \in S} \|s\| \leq \sqrt{\sum M_i^2} < \infty$, contrary to the unboundedness of $S$.

The same argument proves the proposition that if $S$ (or an unbounded subset of $S$) is contained in a finite-dimensional subspace of $X$, or equivalently, if $\dim \text{Span } S < \infty$.

So we will assume that $S$ is not contained in any finite-dimensional subspace of $X$ and then proceed to find an infinite linearly independent subset $\{b_1, b_2, \ldots\}$ of $S$ and extend $\{b_1, b_2, \ldots\}$ to a possibly uncountable Hamel basis $B$ of $X$. Now define a function $\phi$ on $B$ by

$$\phi(b) = \begin{cases} 
  k & \text{if } b = b_k, \\
  0 & \text{otherwise,}
\end{cases}$$

for $b \in B$, and extend $\phi$ linearly to a functional, also denoted by $\phi$, on the entire space $X$. By construction, the restriction of $\phi$ to $\{b_1, b_2, \ldots\} \subset S$ has an unbounded image. This completes the proof.

The restriction of the functional $\phi$ to $S$ in the above proof had an unbounded image, as we required. However, the functional itself might also be “unbounded” or discontinuous, an unwelcome phenomenon in analysis. Therefore, we can ask whether unboundedness of $S$ can be captured by a continuous linear functional. Although it may not be clear a priori, this question is closely related to the famous uniform boundedness principle in analysis. We explore this relation in the next two sections. As we shall see, a central role in this regard is played by the notion of the operator norm, denoted by $\|T\|_{op}$, of a linear map of normed spaces $T : X \to Y$. We say that $T$ is bounded when the operator norm defined by

$$\|T\|_{op} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

is finite. In words, $T$ is said to be bounded (as a function) if the image of the unit ball under $T$ is bounded (as a set). A simple observation that shows the importance of this definition in analysis is that boundedness and continuity are equivalent properties for linear maps of normed spaces.

**Uniform boundedness principle**

Before introducing its dual, let us first give a “quantitative” version of the uniform boundedness principle itself.

**Theorem 1** (Uniform boundedness principle). Let $X$ be a Banach space and let $Y$ be a normed space. Consider a family $F$ of bounded linear operators $T : X \to Y$. If $F$ is pointwise bounded, then it is uniformly bounded.
If $F$ is not uniformly bounded, then there exists a point $x \in X$ and a sequence $(T_n)$ of operators in $F$ satisfying

$$\|T_{n+1}\|_{op} > \|T_n\|_{op}, \quad \|T_{n+1}x\|_Y > \|T_nx\|_Y$$

for all $n$, and $\|T_nx\|_Y \to \infty$.

**Proof.** Suppose $F$ is not uniformly bounded. Then we can find a sequence $(T_n)$ of nonzero operators in $F$ such that

$$\|T_{n+1}\|_{op} \geq 4^{2n+1} \|T_n\|_{op}.$$ 

Choose unit vectors $x_n$ such that

$$\|T_nx_n\|_Y \geq \frac{1}{2} \|T_n\|_{op}.$$ 

For any $a, b \in X$, by the triangle inequality, at least one of the two inequalities

$$\|a + b\|_X \geq \|b\|_X \quad \text{and} \quad \|a - b\|_X \geq \|b\|_X$$

must hold. Thus, we may define a vector $x$ by

$$x = \sum_{k=1}^{\infty} \sigma(k)4^{-k}x_k,$$

where $\sigma(k)$ takes its values from $\{\pm 1\}$ and is defined recursively so that

$$\left\| \sum_{k=1}^{n} \sigma(k)4^{-k}T_nx_k \right\|_Y \geq \|4^{-k}T_nx_n\|_Y \geq \frac{1}{2} 4^{-n} \|T_n\|_{op}.$$ 

Note that the series defining $x$ is absolutely convergent and hence convergent by completeness of $X$. The triangle inequality then implies that

$$\|T_nx\|_Y \geq \left( \sum_{k=1}^{n} \sigma(k)4^{-k}T_nx_k \right) - \left( \sum_{k=n+1}^{\infty} \sigma(k)4^{-k}T_nx_k \right) \geq \frac{1}{2} 4^{-n} \|T_n\|_{op} - \frac{1}{3} 4^{-n} \|T_n\|_{op} = \frac{1}{6} 4^{-n} \|T_n\|_{op}.$$ (1)

Since $\|x\|_X \leq \frac{1}{3}$, we have $\|T_nx\|_Y \leq \frac{1}{3} \|T_n\|_{op}$ and hence equation (1) yields

$$\|T_{n+1}x\|_Y \geq \frac{1}{6} 4^{-n-1} \|T_{n+1}\|_{op} \geq \frac{1}{6} 4^{-n-1} 4^{2n+1} \|T_n\|_{op} = \frac{1}{6} 4^n \|T_n\|_{op} > \frac{1}{3} \|T_n\|_{op} \geq \|T_nx\|_Y,$$

as desired.

**Remark.** Over the years, there have been numerous proofs of the uniform boundedness principle. These proofs may be categorized into those which use Baire’s category theorem (“non-elementary” proofs) and those which do not (“elementary” proofs). Out of the “elementary” proofs, the “simple” ones are of special interest; they usually
make use of a “gliding hump argument,” such as the ones given by [3, p. 51] or [10]. Carothers [3] reports that the original proof of the principle by Steinhaus and his protege Banach must have been an elementary proof of this kind, but apparently it was lost during the war. The nonelementary proof that survived was suggested as an alternative proof by Saks, who refereed their paper! Some other elementary proofs (such as the one given by Riesz and Sz.-Nagy [9, p. 63]) make use of a “nested ball” argument, similar to the argument used in the proof of Baire’s category theorem. The advantage of the proof given above is its “constructive” nature (as opposed to most other proofs that are proofs by contradiction) which allows us to give a quantitative version of the uniform boundedness principle that we shall use in proving Theorem 2.

Norms in the codomain and its dual Since the dual involves the codomain \( Y \), let us say a few words about \( Y \) in the uniform boundedness principle, which is merely a normed space. One of the easy consequences of the Hahn-Banach theorem is the duality between the definitions of the norm in \( Y \) and in its dual \( Y^* \) consisting of bounded linear maps \( y^*: Y \to \mathbb{R} \). More precisely, for any \( y^* \in Y^* \),

\[
\|y^*\|_{op} = \sup_{\|y\| \leq 1} |y^*(y)|,
\]

and for any \( y \in Y \),

\[
\|y\| = \sup_{\|y^*\|_{op} \leq 1} |y^*(y)|,
\]

where in the second equality the supremum is attained.

Remark. The most basic examples of normed spaces are, of course, the scalar fields \( \mathbb{R} \) and \( \mathbb{C} \). It is perhaps interesting to note that the uniform boundedness principle for linear functionals (i.e., in the special case that the codomain is a scalar field) implies the same theorem for all linear operators using the above remark about the computation of norms. To see this, let \( F \) be a family of bounded operators \( T: X \to Y \) between normed spaces with \( X \) complete. Suppose \( F \) is pointwise bounded so that \( \|Tx\| \leq M_x \) for some \( M_x \geq 0 \) depending on each \( x \) in the unit ball of \( X \). Then for each \( y^* \) in the unit ball of \( Y^* \), the functional \( y^* \circ T \) is bounded and \( \|(y^* \circ T)(x)\| \leq M_x \). Thus by the uniform boundedness principle for functionals applied to the family \( \{y^* \circ T \mid T \in F, y^* \in Y^* \text{ with } \|y^*\|_{op} \leq 1\} \), we conclude that \( \|(y^* \circ T)(x)\| \leq M \) for some \( M \geq 0 \) independent of \( x \). Taking the supremum over \( y^* \), we obtain \( \|Tx\| \leq M \), as claimed.

A dual for the uniform boundedness principle

The appearance of the Hahn-Banach theorem, which is applicable to general (i.e., not necessarily complete) normed spaces, in the last section is not completely accidental. It suggests the idea that the uniform boundedness principle might have some applications in the context of general normed spaces as well. Our Hahn-Banach argument proves, in particular, the following theorem, which is where Hahn (1879–1934), Banach (1892–1945), and Steinhaus (1887–1972) meet, posthumously!

**Theorem 2** (Dual for the uniform boundedness principle). Let \( S \) be a subset of a normed space \( X \). If \( \phi(S) \) is bounded for each \( \phi \in X^* \), then \( S \) is bounded.

If \( S \) is unbounded, then there exists \( \phi \in X^* \) and a sequence \( (s_n) \) in \( S \) satisfying \( \|s_{n+1}\|_X > \|s_n\|_X, |\phi(s_{n+1})| > |\phi(s_n)| \) for all \( n \), and \( |\phi(s_n)| \to \infty \).
Theorem 2 can be thought of as a dual for the uniform boundedness principle since the boundedness of $\phi(S)$ can be rephrased as the finiteness of $\sup_{s \in S} |\phi(s)|$. Note that this theorem gives an affirmative answer to the question that we raised after Proposition 1.

Proof. Since $\phi(S)$ is bounded, for $\phi \in X^*$,

$$\sup_{s \in S} |s^{**}(\phi)| = \sup_{s \in S} |\phi(s)| < \infty.$$  

This shows that the hypotheses of the uniform boundedness principle are satisfied for $F = \{s^{**} : X^* \to \mathbb{R} \mid s \in S\}$, thanks to the fact that the dual of any normed space is complete. Therefore, by the uniform boundedness principle and the fact that the map $x \mapsto x^{**}$ is an isometry from $X$ into the Banach space $X^{**}$, we have that

$$\sup_{s \in S} \|s\|_X = \sup_{s \in S} \|s^{**}\|_{X^{**}} < \infty,$$

as desired. The last assertion follows from the second part of Theorem 1. □

Let us finish with a question about a possible strengthening of Theorem 2 that we shall pick up in the next section.

Question 1. Suppose $S = \{s_1, s_2, \ldots\}$ is a subset of a normed space $X$ such that

$$\|s_{n+1}\|_X > \|s_n\|_X$$

for each $n$, and $\|s_n\|_X \to \infty$. Can we necessarily find a functional $\phi \in X^*$ satisfying $|\phi(s_{n+1})| > |\phi(s_n)|$ for each $n$, and $|\phi(s_n)| \to \infty$?

## Boundary between convergence and divergence

We begin with a question concerning the boundary between convergence and divergence of series that first appeared in the work of Abel [1], Dini [4], and du Bois-Reymond [5].

**Question 2.** Suppose $\sum_{n=1}^{\infty} x_n$ is a convergent series with positive terms. Does there exist a sequence $(y_n)$ such that $y_n \to 0$ and $\sum_{n=1}^{\infty} x_n y_n < \infty$? Similarly, suppose $\sum_{n=1}^{\infty} x_n$ is a divergent series with positive terms. Does there exist a sequence $(y_n)$ such that $y_n \to 0$ and $\sum_{n=1}^{\infty} x_n y_n = \infty$?

The answer to both of these, as is well-known, is affirmative [2,8]. That is to say, there is neither a fastest convergent series nor a slowest divergent series. One can, of course, make analogous claims about sequences and, for instance, easily show that there is no slowest divergent sequence. Generalizing this, we pose a more restrictive question about the boundary between convergence and divergence of sequences.

**Question 3.** Suppose $(x_n)$ is a sequence of numbers diverging to infinity. Does there exist a sequence $(y_n)$ such that $\sum_{n=1}^{\infty} y_n < \infty$ and $x_n y_n \to \infty$? What if we require $(y_n) \in \ell^p$?

Let us provide a quick comparison of the claims made in Questions 2 and 3.

|     | Given       | Wanted Convergence | Wanted Divergence |
|-----|-------------|--------------------|-------------------|
| Q2  | $\sum_{n=1}^{\infty} x_n = \infty$ | $y_n \to 0$         | $\sum_{n=1}^{\infty} x_n y_n = \infty$ |
| Q3  | $x_n \to \infty$ | $\sum_{n=1}^{\infty} y_n < \infty$ | $x_n y_n \to \infty$ |
Example 1. Let $x_n = \sqrt{n}$ for $n = 1, 2, \ldots$. Now we ask whether there exists a sequence $(y_n) \in \ell^2$ such that $x_n y_n \to \infty$ as $n \to \infty$. What makes the sequence $(x_n)$ worth studying in this context is the fact that $(1/\sqrt{n^{1+\varepsilon}}) \not\in \ell^2$ for all $\varepsilon > 0$, and $(1/\sqrt{n^{1-\varepsilon}}) \not\in \ell^2$ for all $\varepsilon \geq 0$. Before answering this question, we indicate its connection with Theorem 2, the dual for the uniform boundedness principle.

Let $(x_n)$ be a sequence of numbers such that

\[
\left| x_n \right| > \left| x_n \right|
\]

and let $\{e_1, e_2, \ldots\}$ be the standard orthonormal basis for $\ell^2$. Define a set $S$ by

\[
S = \{x_1e_1, x_2e_2, \ldots\}.
\]

Then $S$ is an unbounded subset of $\ell^2$ and hence, by Theorem 2, we can find $\phi \in (\ell^2)^*$ and a sequence $(s_n)$ in $S$ satisfying

\[
\|s_{n+1}\|_2 > \|s_n\|_2, \quad |\phi(s_{n+1})| > |\phi(s_n)|
\]

for all $n$, and

\[
|\phi(s_n)| \to \infty.
\]

But since $|x_{n+1}| > |x_n|$, we must have $s_k = x_{n_k}e_{n_k}$ for a subsequence $(x_{n_k})$ of $(x_n)$. Thus,

\[
|\phi(x_{n_k+1}e_{n_k+1})| > |\phi(x_{n_k}e_{n_k})|
\]

for all $n$, and

\[
|\phi(x_{n_k}e_{n_k})| \to \infty.
\]

To reveal the connection between the dual for the uniform boundedness principle and Question 3, we use the Riesz representation theorem to establish the existence of a sequence $y = (y_n) \in \ell^2$ such that $\phi(x) = \langle x, y \rangle$ for all $x \in \ell^2$. Thus, for each $k$ we find

\[
\phi(x_{n_k}e_{n_k}) = \langle x_{n_k}e_{n_k}, y \rangle = x_{n_k}\langle e_{n_k}, y \rangle = x_{n_k}y_{n_k}.
\]

In conclusion, Theorem 2 implies the existence of a square-summable sequence $(y_k) \in \ell^2$ such that

\[
|x_{n_k+1}y_{k+1}| > |x_{n_k}y_k| \quad \text{and} \quad |x_{n_k}y_k| \to \infty
\]
as $k \to \infty$ for a subsequence $(x_{n_k})$ of $(x_n)$. But this statement is rather obvious! For instance, in the case of $(x_n) = (\sqrt{n})$ the subsequence $(x_{n_k})$ defined by $x_{n_k} = \sqrt{k^4} = k^2$ and the sequence $(y_k) = (\frac{1}{k})$ has the desired properties.

This cannot be done, however, if we do not allow the passage to subsequences as in Questions 1 and 3. To see this, we return to $(x_n) = (\sqrt{n})$ and let $(y_n)$ be any sequence such that $x_ny_n \to \infty$. Then

\[
x_n^2y_n^2 = ny_n^2 \to \infty.
\]

But since the harmonic series $\sum_{n=1}^\infty n^{-1}$ is divergent, and we are assuming that

\[
y_n^2/n^{-1} \to \infty,
\]

the limit comparison theorem for series implies that $\sum_{n=1}^\infty y_n^2$ must also be divergent, i.e., $(y_n) \not\in \ell^2$.

Questions of this nature arise in Fourier analysis and provide a means for measuring regularity of functions. For instance, for a periodic function $f \in L^2(\mathbb{T})$, we have
\( \hat{f} \in \ell^2(\mathbb{Z}) \), and if \( f \in C^k(\mathbb{T}) \), \( k \geq 0 \), then \( \hat{f} \in o(n^{-k}) \), where \( \hat{f}(n) \) is the \( n \)th Fourier coefficient of \( f \) given by

\[
\hat{f}(n) = \int_{0}^{1} f(x) e^{-2\pi i nx} \, dx
\]

for each \( n \in \mathbb{Z} \). See the Riesz-Fischer theorem and the Riemann-Lebesgue lemma \[11\]. Thus, “the smoother the function, the faster the decrease of its Fourier coefficients.” Indeed, a standard application of integration by parts shows that if \( f \in C^k(\mathbb{T}) \), then \( (n^k \hat{f}(n)) \in \ell^2(\mathbb{Z}) \). It would be nice if the converse were true, but it is false. It turns out that a weaker form of the converse is true, but we shall not state it here. Instead, we invite the interested reader to explore how this set of ideas leads to the definition of a Sobolev space.

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