A FREE BOUNDARY PROBLEM FOR DEFAULTABLE CORPORATE BOND WITH CREDIT RATING MIGRATION RISK AND ITS ASYMPTOTIC BEHAVIOR

YUAN WU
School of Statistics and Mathematics
Shanghai Lixin University of Accounting and Finance
Shanghai 201209, China

JIN LIANG*
School of Mathematical Science
Tongji University
Shanghai 200092, China

BEI HU
Department of Applied Computational Mathematics and Statistics
University of Notre Dame, Notre Dame, IN 46556, USA

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Abstract. In this paper, valuation of a defaultable corporate bond with credit rating migration risk is considered under the structure framework by using a free boundary model. The existence, uniqueness and regularity of the solution are obtained. Furthermore, we analyze the solution's asymptotic behavior and prove that the solution is convergent to an closed form solution. In addition, numerical examples are also shown.

1. Introduction. In recent years, people pay more and more attentions to credit risks. Credit risks include not only default risks, but also credit rating migration ones. An upgrade or downgrade of credit rating affects the investors’ investment strategy and the corporate bond’s pricing. In fact, the outbreak and spread of the financial crisis make more and more people realize that credit rating migration plays an important role in risk managements.

There are two traditional models for studying default risks–structural models and reduced-form ones. In a reduced-form model, the default event is described by introducing an exogenous variant. In this approach, default time is modeled by a so called stochastic default intensity; see Jarrow and Turnbull ([10], 1995), Lando ([13], 1998), Duffie and Singleton ([3], 1999) and so forth. In a structural model, default is determined by the relationship between the firm’s value and some insolvency threshold. The default occurs once the firm’s value, which follows some stochastic
process, falls below the threshold. In Merton’s model ([21], 1974), a default event may only occur at the maturity. Later, Black and Cox ([1], 1976) extended Merton’s model to a first-passage-time model. That is, default may occur at any time before the debt’s maturity; see also Longstaff and Schwartz ([20], 1995), Leland and Toft ([16], 1996), Briys and de Varenne([2], 1997) and so forth.

In the credit rating migration studies, Markov chain is a main tool. The transferring intensity matrix is adapted, which usually comes from general statistical data; see Jarrow et al. ([11], 1997), Das and Tufano ([4], 1996), Lando ([13], 1998), Thomas et al. ([22], 2002) and so forth. Markov chain approach is one type of reduced-form method. However it does not take firm’s value into consideration when modeling credit rating migrations. In fact, firm’s value is also an important factor of the credit rating migrations. From this point of the view, Liang et al. ([18], [19]) started to use structural model to study credit rating migration risk based on the Merton’s model. As a first step, they set a predetermined migration threshold to divide firm’s value into high and low rating regions. The firm’s value follows different stochastic processes in different rating regions. In practice, the threshold is usually not predetermined. Then Hu et al. ([9], 2015) improved this model. The authors treated the migration boundary to be the proportion of the debt and the value of the firm. Thus the model becomes a free boundary problem. Liang et al. ([17], 2016) further incorporated a risk discount factor of the firm, and the authors showed that the problem admits an asymptotic traveling wave solution.

So far, valuations of the credit rating migration under the structure framework are based on Merton’s model, i.e., default may only happen at maturity. However, in more realistic situations, default may occur at any time up to maturity. In this paper, we relax the default restriction and set a predetermined default threshold as the first-passage-time model. Once the firm’s value falls below the threshold at any time, default happens. Based on the idea of the free boundary problem in [9], a new model with initial and boundary condition is established. The main theoretical difficulty for analysis of the model is that we must deal with estimates related to this barrier. In fact, the problem turns into an initial-boundary problem, in which barrier function method is used for estimates, where we must prove that the default boundary stays a positive distance away from the migration free boundary. By using PDE techniques, the existence, uniqueness and regularities of the solution are obtained. Furthermore, the uniqueness proof is quite non-standard, as the coefficient of the highest order derivative is discontinuous. Actually, the methods used in [9] and [17] are no longer applicable to this new model. This new proof can also be applied to the models in [9] and [17], without imposing any restrictive conditions on the second order derivatives there, which is a significant improvement and is of independent interests by itself. We have also obtained an asymptotic behavior of the solution when $t$ goes to infinity, where the solution converges to a steady-state solution, given in an explicit form.

The paper is organized as follows: in Section 2, a model is established; this model is reduced to a PDE free boundary with initial and boundary condition in Section 3; in Section 4, an approximated problem is analyzed and some preliminary lemmas are collected; through this approximated problem, the existence and uniqueness of the solution for the free boundary are obtained in Section 5. In Section 6, an asymptotic solution is obtained. Numerical simulation results are presented in Section 7 and conclusions are summarized in Section 8.
2. Model.

2.1. Assumption. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. We assume that the firm issues a corporate bond, which is a contingent claim of its value on the space \((\Omega, \mathcal{F}, P)\).

**Assumption 2.1** (the asset of a firm which is subject to credit rating migrations). Let \(S_t\) denote the firm’s value in the risk neutral world. It satisfies

\[
dS_t = \begin{cases} 
  rS_t dt + \sigma_H S_t dW_t, & \text{in high rating region,} \\
  rS_t dt + \sigma_L S_t dW_t, & \text{in low rating region,}
\end{cases}
\]

where \(r\) is the risk free interest rate, and \(\sigma_H < \sigma_L\) (2.1) represent volatilities of the firm under the high and low credit grades respectively. They are assumed to be positive constants. \(W_t\) is the Brownian motion which generates the filtration \(\mathcal{F}_t\).

It is reasonable to assume (2.1), namely, that the volatility in high rating region is lower than the one in the low rating region.

**Assumption 2.2** (the corporate bond). The firm issues only one corporate defaultable bond with face value \(F\). We focus on the effect of the firm’s value with credit rating migration to the bond, so the discount value of bond is considered. Denote \(\Phi_t\) the discount value of the bond at time \(t\).

**Assumption 2.3** (the default time). The firm can default before maturity time \(T\). The default time \(\tau_d\) is the first moment when the firm’s value falls below the threshold \(K\):

\[
\tau_d = \inf \{ t > 0 | S_0 > K, S_t \leq K \},
\]

where \(K < F \cdot D(t,T)\) and \(0 < D(t,T) < 1\) is the discount function. Once the firm defaults, the investor will get what is left. Therefore, \(\Phi_t(K,t) = K\) and on the maturity time \(T\), the investor can get \(\Phi_T = \min\{S_T, F\}\).

**Assumption 2.4** (the credit rating migration time). High and low rating regions are determined by the proportion of the debt and value. The credit rating migration time \(\tau_1\) and \(\tau_2\) are the first moment when the firm’s grade is upgraded and downgraded respectively as follows:

\[
\tau_1 = \inf \{ t > 0 | \Phi_0/S_0 > \gamma, \Phi_t/S_t \leq \gamma \}, \quad \tau_2 = \inf \{ t > 0 | \Phi_0/S_0 < \gamma, \Phi_t/S_t \geq \gamma \},
\]

where \(\Phi_t = \Phi_t(S_t, t)\) is a contingent claim with respect to \(S_t\) and

\[
0 < \gamma < 1 \quad (2.2)
\]

is a positive constant representing the threshold proportion of the debt and value of the firm’s rating.

2.2. Cash flow. Once the credit rating migrates before the maturity \(T\), a virtual substitute termination happens, i.e., the bond is virtually terminated and substituted by a new one with a new credit rating. There is a virtual cash flow of the
bond. Denoted by $\Phi_H(y,t)$ and $\Phi_L(y,t)$ the values of the bond in high and low grades respectively. Then, they are the conditional expectations as follows:

$$\Phi_H(y,t) = E_{y,t}[e^{-r(T-t)} \min(S_T, F) \cdot 1_{\min(\tau_1, \tau_d) \geq T}$$

$$+ e^{-r(\tau_1-t)} \Phi_L(S_{\tau_1}, \tau_1) \cdot 1_{t < \tau_1 < \min(T, \tau_d)}$$

$$+ e^{-r(\tau_d-t)} K \cdot 1_{t < \tau_d < \min(\tau_2, T)} | S_t = y > \frac{1}{\gamma} \Phi_H(y,t)],$$

(2.3)

$$\Phi_L(y,t) = E_{y,t}[e^{-r(T-t)} \min(S_T, F) \cdot 1_{\min(\tau_2, \tau_d) \geq T}$$

$$+ e^{-r(\tau_2-t)} \Phi_H(S_{\tau_2}, \tau_2) \cdot 1_{t < \tau_2 < \min(\tau_d, T)}$$

$$+ e^{-r(\tau_d-t)} K \cdot 1_{t < \tau_d < \min(\tau_2, T)} | S_t = y < \frac{1}{\gamma} \Phi_L(y,t)],$$

(2.4)

where $1_{\text{event}} = \begin{cases} 1, & \text{if "event" happens}, \\ 0, & \text{otherwise}. \end{cases}$

2.3. 

PDE problem. By Feynman-Kac formula, it is not difficult to derive that $\Phi_H$ and $\Phi_L$ are the functions of the firm value $S$ and time $t$. They satisfy the following partial differential equations in their regions:

$$\frac{\partial \Phi_H}{\partial t} + \frac{1}{2} \sigma_H^2 S^2 \frac{\partial^2 \Phi_H}{\partial S^2} + rS \frac{\partial \Phi_H}{\partial S} + r\Phi_H = 0, \quad S > \frac{1}{\gamma} \Phi_H, \quad t > 0,$$

(2.5)

$$\frac{\partial \Phi_L}{\partial t} + \frac{1}{2} \sigma_L^2 S^2 \frac{\partial^2 \Phi_L}{\partial S^2} + rS \frac{\partial \Phi_L}{\partial S} + r\Phi_L = 0, \quad K < S < \frac{1}{\gamma} \Phi_L, \quad t > 0,$$

(2.6)

with the terminal condition:

$$\Phi_H(S,T) = \Phi_L(S,T) = \min\{S, F\},$$

(2.7)

and the boundary condition:

$$\Phi_L(K,t) = K.$$

(2.8)

(2.3) and (2.4) imply that the value of the bond is continuous when it passes the rating threshold, i.e.

$$\Phi_H = \Phi_L \quad \text{on the rating migration boundary}.$$

(2.9)

Also, if we construct a risk free portfolio $\pi$ by longing a bond and shorting $\Delta$ amount asset value $S$, i.e., $\pi_t = \Phi_t - \Delta_t S_t$ and such that $d\pi_t = r\pi_t$, this portfolio is also continuous when it passes the rating migration boundary, i.e.,

$$\pi_H = \pi_L \quad \text{on the rating migration boundary},$$

(2.10)

or by (2.9),

$$\Delta_H = \Delta_L \quad \text{on the rating migration boundary}.$$

(2.11)

By Black-Scholes theory (e.g., [12]), it is equivalent to

$$\frac{\partial \Phi_H}{\partial S} = \frac{\partial \Phi_L}{\partial S} \quad \text{on the rating migration boundary}.$$

(2.12)
3. **Free boundary problem.** We introduce the standard change of variables \( x = \log \frac{S}{K} \), rename \( T - t \) as \( t \), and define

\[
\psi(x, t) = \begin{cases} 
\frac{e^{-x}}{K} \Phi_H(e^x, T - t) & \text{in high rating region}, \\
\frac{e^{-x}}{K} \Phi_L(e^x, T - t) & \text{in low rating region},
\end{cases}
\]

using also (2.9) and (2.12), we then derive the following equation from (2.5), (2.6):

\[
\frac{\partial \psi}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 \psi}{\partial x^2} - \left( r + \frac{1}{2} \sigma^2 \right) \frac{\partial \psi}{\partial x} = 0,
\]

\( 0 < x < \infty, \ t > 0, \) \hspace{1cm} (3.1)

where \( \sigma \) is a function of \( \psi \) and \( x \), i.e.,

\[
\sigma = \sigma(\psi, x) = \begin{cases} 
\sigma_H & \text{if } \psi < \gamma, \\
\sigma_L & \text{if } \psi \geq \gamma,
\end{cases}
\]

(3.2)

The constants \( \sigma_H, \sigma_L \) are defined in (2.1).

Without loss of generality, we assume \( F = 1 \). Then we have \( K < 1 \) from Assumption 2.3. Equation (3.1) is supplemented with the initial and boundary conditions (derived from (2.7) and (2.8))

\[
\psi(x, 0) = \min\{1, \frac{e^{-x}}{K}\}, \quad 0 < x < \infty, \quad (3.3)
\]

\[
\psi(0, t) = 1, \quad t > 0. \quad (3.4)
\]

The domain will be divided into the high rating region \( \Omega_H \) where \( \psi < \gamma \) and a low rating region \( \Omega_L \) where \( \psi \geq \gamma \). We shall prove that these two domains will be separated by a free boundary \( x = s(t) \), and

\[
\Omega_H = \{ x > s(t) \}, \quad \Omega_L = \{ 0 < x < s(t) \}.
\]

In another word, \( s(t) \) is apriorily unknown since it should be solved by the equation

\[
\psi(s(t), t) = \gamma, \quad (3.5)
\]

where the solution \( \psi \) is also apriorily unknown.

Since we have assumed that equation (3.1) is valid across the free boundary \( x = s(t) \), equations (2.9), (2.12) are implied:

\[
\psi(s(t)^-, t) = \psi(s(t)^+, t) = \gamma, \quad (3.6)
\]

\[
\frac{\partial \psi}{\partial x}(s(t)^-, t) = \frac{\partial \psi}{\partial x}(s(t)^+, t). \quad (3.7)
\]

4. **Preliminaries.**

4.1. **Approximation.** Let \( H(\xi) \) be the Heaviside function, i.e., \( H(\xi) = 0 \) for \( \xi < 0 \) and \( H(\xi) = 1 \) for \( \xi > 0 \). Then we can rewrite (3.2) as

\[
\sigma = \sigma_H + (\sigma_L - \sigma_H)H(\psi - \gamma).
\]

We approximate \( H(\xi) \) by a \( C^\infty \) function \( H_\varepsilon \) such that

\[
H_\varepsilon(\xi) = 0 \quad \text{for } \xi < -\varepsilon, \quad H_\varepsilon = 1 \quad \text{for } \xi > 0, \quad H'_\varepsilon(\xi) \geq 0 \quad \text{for } -\infty < \xi < \infty.
\]

Consider the approximated problem

\[
\mathcal{L}_\varepsilon[\psi_\varepsilon] \equiv \frac{\partial \psi_\varepsilon}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 \psi_\varepsilon}{\partial x^2} - \left( r + \frac{1}{2} \sigma^2 \right) \frac{\partial \psi_\varepsilon}{\partial x} = 0, \quad 0 < x < \infty, \ t > 0, \quad (4.1)
\]
with the initial condition
\[ \psi_\varepsilon(x,0) = \min\{1, \frac{e^{-x}}{K}\}, \quad 0 < x < \infty, \quad (4.2) \]
and boundary condition
\[ \psi_\varepsilon(0,t) = 1, \quad t > 0, \quad (4.3) \]
where
\[ \sigma_\varepsilon = \sigma_H + (\sigma_L - \sigma_H)H'_\varepsilon(\psi_\varepsilon - \gamma). \quad (4.4) \]
The equation (4.1)–(4.3) admits a unique classical solution \( \psi = \psi_\varepsilon \). We now proceed to derive estimates for \( \psi_\varepsilon \).

4.2. Estimates for the approximating system.

**Lemma 4.1.** 
\[ 0 \leq \psi_\varepsilon \leq \min\{1, \frac{e^{-x}}{K}\}, \quad 0 < x < \infty, \quad 0 < t < T. \]

**Proof.** It is not difficult to verify that 0 is a lower solution while both 1 and \( \frac{e^{-x}}{K} \) are upper solutions. \( \square \)

**Lemma 4.2.** There exists a constant \( C > 0 \), independent of \( \varepsilon \) and \( T \), such that
\[ -C \leq \frac{\partial \psi_\varepsilon}{\partial x} \leq 0, \quad 0 < x < \infty, \quad 0 < t < T. \]

**Proof.** Differentiating equation (4.1) in \( x \), we obtain
\[ \mathcal{L}_1^\varepsilon \left[ \frac{\partial \psi_\varepsilon}{\partial x} \right] = \mathcal{L}_1^\varepsilon \left[ \frac{\partial \psi_\varepsilon}{\partial x} \right] - \left( \frac{\partial^2 \psi_\varepsilon}{\partial x^2} + \frac{\partial \psi_\varepsilon}{\partial x} \right) \cdot \sigma_\varepsilon \cdot (\sigma_L - \sigma_H) \cdot H'_\varepsilon(\psi_\varepsilon - \gamma) \cdot \frac{\partial \psi_\varepsilon}{\partial x} = 0. \quad (4.5) \]

It is known that \( \frac{\partial \psi_\varepsilon}{\partial x}(x,0) = 0 \) for \( 0 < x < \log \frac{1}{K} \) and \( \frac{\partial \psi_\varepsilon}{\partial x}(x,0) = -\frac{1}{K} e^{-x} \leq 0 \) for \( x > \log \frac{1}{K} \). And since
\[ \frac{\psi_\varepsilon(x,t) - \psi_\varepsilon(0,t)}{x} \leq 0, \]
then let \( x \to 0 \), we have \( \frac{\partial \psi_\varepsilon}{\partial x}(0,t) \leq 0 \). Thus it follows by maximum principle that \( \frac{\partial \psi_\varepsilon}{\partial x} \leq 0 \).

On the other hand, take an appropriate \( C > 0 \), such that
\[ \mathcal{L}_1^\varepsilon [e^{-Cx}] = \left[ -\frac{1}{2} \sigma^2 C^2 + \left( r + \frac{1}{2} \sigma^2 \right) C \right] \cdot e^{-Cx} < 0. \quad (4.6) \]
Clearly \( e^{-Cx} |_{x=0} = 1 \), \( e^{-Cx} \leq \min\{1, \frac{e^{-x}}{K}\} \) for \( C \) is big enough (recalling that \( K < 1 \)). Then \( \psi_\varepsilon(x,t) \geq e^{-Cx} \), and therefore
\[ \frac{\psi_\varepsilon(x,t) - \psi_\varepsilon(0,t)}{x} \geq \frac{e^{-Cx} - 1}{x}. \]
Letting \( x \to 0 \), we have \( \frac{\partial \psi_\varepsilon}{\partial x}(0,t) \geq -C \). Clearly,
\[ \mathcal{L}_1^\varepsilon [-C] = -C^2 \sigma_\varepsilon \cdot (\sigma_L - \sigma_H) \cdot H'_\varepsilon(\psi_\varepsilon - \gamma) \leq 0. \]
It follows by comparison principle that \( \frac{\partial \psi_\varepsilon}{\partial x} \geq -C. \) \( \square \)

**Lemma 4.3.** There exist constants \( c_1, C_2 \) and \( C_3 \), independent of \( \varepsilon \), such that
\[ -C_3 - \frac{C_2}{\sqrt{t}} \exp \left( -c_1 \frac{|x - \log(1/K)|^2}{t} \right) \leq \frac{\partial \psi_\varepsilon}{\partial t} \leq 0, \quad 0 < t < T. \]
Proof. Differentiating equation (4.1) in $t$, we obtain
\[
\mathcal{L}^\varepsilon \left[ \frac{\partial \psi_z}{\partial t} \right] = - \left( \frac{\partial^2 \psi_z}{\partial x^2} + \frac{\partial \psi_z}{\partial x} \right) \cdot \sigma_z \cdot (\sigma_L - \sigma_H) \cdot H'_\varepsilon(\psi_z - \gamma) \cdot \frac{\partial \psi_z}{\partial t} = 0. \tag{4.7}
\]

It is also clear that initially
\[
\frac{\partial \psi_z}{\partial t}(x, 0) = 0 \quad \text{for} \; x < \log \frac{1}{K}, \quad \frac{\partial \psi_z}{\partial t}(x, 0) = -\frac{r}{K} e^{-x} \quad \text{for} \; x \geq \log \frac{1}{K}.
\]

At $x = \log \frac{1}{K}$, $\frac{\partial^2 \psi_z}{\partial x^2}(x, 0)$ produces a dirac measure of density $-1$. Thus $\frac{\partial \psi_z}{\partial t}(x, 0) \leq 0$ in the distribution sense. Furthermore, since 2nd order compatibility condition is satisfied at $(0, 0), \frac{\partial \psi_z}{\partial t}$ is continuous at $(0, 0)$. By further approximating the initial data with smooth functions if necessary, and $\frac{\partial \psi_z}{\partial t}(0, t) = 0$, we conclude by maximum principle
\[
\frac{\partial \psi_z}{\partial t} < 0 \quad \text{for} \; 0 < x < \infty, \; t > 0.
\]

Next, since $\psi_z(\log \frac{1}{K}, 0) = 1 > \gamma$, and by Hölder continuity of the solution (e.g., [8]), there exists a $\rho > 0$, independent of $\varepsilon$, such that
\[
\psi_z(x, t) > (1 + \gamma)/2 \quad \text{for} \; |x - \log \frac{1}{K}| \leq \rho, \; 0 < t \leq \rho^2.
\]

Thus for small $\varepsilon < (1 - \gamma)/2$, $\sigma_z \equiv \sigma_L$ for $|x - \log \frac{1}{K}| \leq \rho, \; 0 < t \leq \rho^2$. It follows from the standard parabolic estimates (e.g., [6]) that
\[
\frac{\partial \psi_z}{\partial t} \geq -C_2 - \frac{C_2}{\varepsilon} \exp \left( -c_1 \frac{|x - \log(1/K)|^2}{t} \right) \quad \text{for} \; |x - \log \frac{1}{K}| < \rho, \; 0 < t \leq \rho^2/4. \tag{4.8}
\]

We now take $C_3 \geq rC$ such that on $\{ |x - \log \frac{1}{K}| = \frac{\rho}{2}, 0 < t < \frac{\rho^2}{4} \} \cup \{ |x - \log \frac{1}{K}| < \frac{\rho}{2}, t = \frac{\rho^2}{4} \}$,
\[
C_3 \geq C_2 + \frac{C_2}{\varepsilon} \exp \left( -c_1 \frac{|x - \log(1/K)|^2}{t} \right).
\]

Substituting (4.1) into (4.7) and using also the estimates $\frac{\partial \psi_z}{\partial t} \leq 0, \frac{\partial \psi_z}{\partial x} \geq -C$, we get if $\frac{\partial \psi_z}{\partial t} + rC < 0$,
\[
\mathcal{L}^\varepsilon \left[ \frac{\partial \psi_z}{\partial t} \right] = \frac{2}{\sigma_z} \left\{ \frac{\partial \psi_z}{\partial t} - r \frac{\partial \psi_z}{\partial x} \right\} \sigma_L - \sigma_H \cdot H'_\varepsilon(\psi_z - \gamma) \cdot \frac{\partial \psi_z}{\partial t} \geq \frac{2(\sigma_L - \sigma_H)}{\sigma_z} H'_\varepsilon(\psi_z - \gamma) \cdot \frac{\partial \psi_z}{\partial t} \left\{ \frac{\partial \psi_z}{\partial t} + rC \right\} \geq 0 = \mathcal{L}^\varepsilon[-C^3].
\]

We claim that $\{ \frac{\partial \psi_z}{\partial t} < -C^3 \} \{ |x - \log \frac{1}{K}| < \frac{\rho}{2}, 0 < t < \frac{\rho^2}{4} \}$ is an empty set. If not, in the region $\left\{ \frac{\partial \psi_z}{\partial t} < -C^3 \right\} \{ |x - \log \frac{1}{K}| < \frac{\rho}{2}, 0 < t < \frac{\rho^2}{4} \}$, we have $\mathcal{L}^\varepsilon \left[ \frac{\partial \psi_z}{\partial t} \right] > 0$.

On the parabolic boundary of this region we clearly have $\frac{\partial \psi_z}{\partial t} \geq -C^3$ in this region. This is a contradiction. □

As an immediate corollary, we have

**Corollary 4.4.** There exists a constant $C_3$, independent of $\varepsilon$ and valid for all $t^0 < t < \infty$, such that
\[
\frac{\partial \psi_z}{\partial t} \geq -C^3, \quad 0 < x < \infty, \quad t^0 < t < \infty.
\]
Proof. Since $\frac{\partial \psi_\varepsilon}{\partial t}(x,t_0^2) \geq -C_3$ for all $0 \leq x < \infty$, the proof in the above lemma can be carried out for all $t_0^2 < t < \infty$. \hfill \Box

**Lemma 4.5.** There exist constant $C_4$ and $C_5$, independent of $\varepsilon$ and valid for all $t_0^2 < t < \infty$,

$$-C_4 - \frac{C_5}{\sqrt{t}} \exp \left( -c_1 \frac{|x - \log(1/K)|^2}{t} \right) \leq \frac{\partial^2 \psi_\varepsilon}{\partial x^2} \leq C_5, \quad 0 < x < \infty, \quad t_0^2 < t < \infty.$$ 

Denote $s_\varepsilon(t)$ is the approximated free boundary, which is the solution of the equation

$$\psi_\varepsilon(s_\varepsilon(t), t) = \gamma. \quad (4.9)$$

Then we have the following estimates for the free boundary.

**Lemma 4.6.** The approximated free boundary defined in (4.9) satisfies

$$s_\varepsilon(t) \leq \log \frac{1}{\gamma K}. \quad (4.10)$$

*Proof.* From Lemma 4.1, we have

$$\psi_\varepsilon < \frac{1}{K} e^{-x},$$

so that,

$$\psi_\varepsilon < \gamma \quad \text{for} \quad x > \log \frac{1}{\gamma K}.$$ 

This means that the region $\{ x > \log \frac{1}{\gamma K} \}$ is in the high rating region and hence (4.10) holds. \hfill \Box

**Lemma 4.7.**

$$s_\varepsilon(t) \geq \frac{1}{1 + \frac{2r}{\sigma_H}} \log \frac{1 + \gamma}{2\gamma}. \quad (4.11)$$

*Proof.* Let $f(x) = \frac{1}{2} (1 + \gamma) e^{-(1 + \frac{2r}{\sigma_H}) x}$, then $f(0) = \frac{1}{2} (1 + \gamma) < 1 = \psi_\varepsilon(0, t)$. And $f(x) < f(0) < 1$, $f(x)/\frac{\sigma_H}{\sigma_H} = \frac{1}{2} K (1 + \gamma) e^{-\frac{2r}{\sigma_H} x} < 1$, then $f(x) < \min \{ 1, \frac{\sigma_H}{\sigma_H} \} = \psi_\varepsilon(x, 0)$. In addition

$$\mathcal{L}^\varepsilon [f(x)] = \frac{1}{2} (1 + \frac{2r}{\sigma_H})(1 + \gamma) e^{-(1 + \frac{2r}{\sigma_H}) x} [r - \frac{\sigma_H}{\sigma_H} r] \leq 0.$$ 

By Maximum Principle, we have $f(x) < \psi_\varepsilon(x, t)$, so that

$$\psi_\varepsilon > \frac{1}{2} (1 + \gamma) e^{-(1 + \frac{2r}{\sigma_H}) x} > \gamma \quad \text{for} \quad x < \frac{1}{1 + \frac{2r}{\sigma_H}} \log \frac{1 + \gamma}{2\gamma}.$$ 

This means that the region $\{ x < \frac{1}{1 + \frac{2r}{\sigma_H}} \log \frac{1 + \gamma}{2\gamma} \}$ is in the low rating region and hence (4.11) holds. \hfill \Box

**Lemma 4.8.** For any $T > 0$, there exists $C_T > 0$, independent of $\varepsilon$, such that

$$-C_T \leq s_\varepsilon'(t) \leq 0.$$
Proof. Clearly, $\frac{\partial \psi_s}{\partial x}(x,t) > 0$ for $t > 0$, and

$$s'_e(t) = -\frac{\partial \psi_s}{\partial x}(s_e(t),t).$$

By Lemmas 4.2 and 4.3,

$$s'_e(t) \leq 0.$$

Clearly $s(0) = \log(1/(\gamma K)) = \log(1/K) + \log(1/\gamma)$. By Lemma 4.3, there is a constant $\rho > 0$ independent of $\varepsilon$ such that $s_e(t) \geq \rho + \log(1/K)$ for $0 \leq t \leq \rho^2$. It follows from Lemma 4.3 that

$$- C^* \leq \frac{\partial \psi_s}{\partial t}(s_e(t),t), 0 \leq t \leq T$$

for some constant $C^*$ independent of $\varepsilon$. To finish the proof, it suffices to establish $-\frac{\partial \psi_s}{\partial x}(s_e(t),t) \geq c^*$ for some positive $c^*$ independent of $\varepsilon$.

As shown in Lemma 4.2, $L^\varepsilon_{\varepsilon} - \frac{\partial \psi_s}{\partial x} = 0$ and $-\frac{\partial \psi_s}{\partial x}(x,0) = 0$ for $0 < x < \log \frac{1}{K}$,

$$-\frac{\partial \psi_s}{\partial x}(x,0) = \frac{\partial \psi_s}{\partial x}(x,0) = \frac{1}{K}e^{-x}$$

for $x > \log \frac{1}{K}$. By Lemmas 4.3, 4.6 and 4.7, there exists $R > 0$, independent of $\varepsilon$, such that

$$\frac{2}{R} \leq s_e(t) \leq R - 1$$

for $0 < t \leq T$,

and

$$s_e(t) \geq \log \frac{1}{K} + \rho$$

for $0 \leq t \leq \rho^2$. Consider the region

$$\Omega_1 = \{x,0 \leq x < R, 0 < t < T\} \cup \{x,0 \leq x \leq R, \rho^2 \leq t \leq T\}.$$ 

The parabolic boundary of this region $\Omega_1$ consists of 5 line segments. On the initial line segment $\{(x,0), 0 \leq x < R\}$, $-\frac{\partial \psi_s}{\partial x}(x,0) = \frac{1}{K}e^{-x}$. The remaining 4 parabolic boundaries $\{(x,t), 0 \leq t \leq T\} \cup \{(x,\rho^2 + \log \frac{1}{K}, t), 0 \leq t \leq \rho^2\} \cup \{(x,\rho^2), 0 \leq t \leq T\}$ are completely and uniformly within the high or low rating region (independent of $\varepsilon$). Thus by compactness and strong maximum principle, on these 4 boundaries, $-\frac{\partial \psi_s}{\partial x} \geq \tilde{c} > 0$ for some $\tilde{c}$ independent of $\varepsilon$. It follows that

$$-\frac{\partial \psi_s}{\partial x} \geq \min(1, \tilde{c}) \equiv c^*$$

on $\Omega_1$, and this establishes the lemma. \hfill \square

5. Existence & uniqueness.

5.1. Existence. Lemmas 4.1-4.5 and Corollary 4.4 provide estimates of approximated solution $\psi_s$. By taking a limit as $\varepsilon \to 0$ (along a subsequence if necessary), we derive the existence of problem (3.1)-(3.7).

Lemmas 4.6-4.8 show that there is a uniform estimate in space $C^1([0,T])$ for the approximated free boundary $s_e(t)$. Therefore, the limit of $s_e(t)$ as $\varepsilon \to 0$ exists, which is denoted by $s(t)$. This $s(t)$ is the free boundary of our original problem.

**Theorem 5.1.** The free boundary problem (3.1)-(3.7) admits a solution $(\psi, s)$ with $\psi$ in $W^{2,1}_\infty((0,\infty) \times [0,T] \setminus Q_{t_0}) \cap W^{1,0}(\infty, \infty) \times [0,T])$ and $s \in W^{1,\infty}[0,T]$, for any $t_0 > 0$, where $Q_{t_0} = (-t_0 + \log(1/K), t_0 + \log(1/K)) \times (0, t_0)$, and $s \in W^{1,\infty}[0,T]$. 

By the classical parabolic theory, it is also clear that the solution is in $C^\infty(\Omega_L) \cap C^\infty(\Omega_H)$, where $\Omega_L = \{(x,t); 0 < x < s(t), 0 < t \leq T\}$ and $\Omega_H = \{(x,t); s(t) < x < \infty, 0 < t \leq T\}$.

5.2. **Uniqueness.** Next, we prove that the solution $(\psi, s)$ we obtained in the last section is unique.

**Theorem 5.2.** The solution $(\psi, s)$ with $\psi$ in $W^{2,1}_\infty((-\infty, \infty) \times [0,T] \setminus \bar{Q}_{t_0}) \cap W^{1,0}_\infty((-\infty, \infty) \times [0,T]), s \in C[0,T]$ is unique.

**Proof.** Suppose $(\psi_1(x,t), s_1(t)), i = 1, 2$ are two solutions, then $\psi_1(s_1(t), t) = \psi_2(s_2(t), t) = \gamma$, and

$$\psi_1(s_1(t), t) - \psi_2(s_1(t), t) = \psi_2(s_2(t), t) - \psi_2(s_1(t), t).$$

Besides, at $t = 0$, $s_1(0) = s_2(0) = \log \frac{1}{\gamma K}$. By Lemma 4.8, there exists a constant $c^* > 0$, such that $(\psi_i)_x < -c^*, i = 1, 2$ on the region $\Omega_1$, which is defined by (4.13) in Lemma 4.8. Then by the implicit function theorem, there exists $\rho > 0$, such that when $0 < t < \rho$,

$$|s_2(t) - s_1(t)| \leq C \max |\psi_1(x,t) - \psi_2(x,t)|. \quad (5.2)$$

Let $w(x,t) = \psi_1(x,t) - \psi_2(x,t)$ and denote by $\sigma_1$ and $\sigma_2$ the corresponding coefficients, then $w$ satisfies

$$\frac{1}{\sigma_1} w_t(x,t) - \frac{1}{2} w_{xx}(x,t) - \left(\frac{r}{\sigma_1} + \frac{1}{2}\right) w_x(x,t) = \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) [(\psi_2(x,t))_t - r(\psi_2(x,t))_x]. \quad (5.3)$$

$\psi(x,t)$ and its derivatives decay exponentially fast to 0 as $x \to +\infty$. Multiplying the equation by $w$ on both sides and integrating $x$ from 0 to $+\infty$, we obtain

$$\int_0^{+\infty} \left| \frac{1}{\sigma_1} w_t(x,t) w(x,t) - \frac{1}{2} w_{xx}(x,t) w(x,t) - \left(\frac{r}{\sigma_1} + \frac{1}{2}\right) w_x(x,t) w(x,t) \right| dx$$

$$= \int_0^{+\infty} \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) [(\psi_2(x,t))_t - r(\psi_2(x,t))_x] w(x,t) dx. \quad (5.4)$$

Since $\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \equiv 0$ for $x \notin [s_1(t) \land s_2(t), s_1(t) \lor s_2(t)]$, and by assumption, $(\psi_2(x,t))_t$ and $(\psi_2(x,t))_x$ are uniformly bounded outside $Q_{t_0}$, we conclude that they are bounded for $x \notin [s_1(t) \land s_2(t), s_1(t) \lor s_2(t)]$. It follows that

$$\int_0^{+\infty} \left| \frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right| [(\psi_2(x,t))_t - r(\psi_2(x,t))_x] w(x,t) dx$$

$$\leq C \max_{0 < x < +\infty} |w(x,t)| \int_{s_1(t) \lor s_2(t)}^{s_1(t) \land s_2(t)} \left| \frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right| dx$$

$$\leq C \max_{0 < x < +\infty} |w(x,t)||s_1(t) - s_2(t)|$$

$$\leq C \max_{0 < x < +\infty} |w(x,t)|^2 \quad (\text{by equation (5.2)})$$

$$\leq \frac{C}{\varepsilon} \int_0^{+\infty} w(x,t)^2 dx + \varepsilon \int_0^{+\infty} w_2^2(x,t) dx. \quad (5.5)$$
The last inequality of (5.5) is derived as follows. Since \( w(x,t) \) decays exponentially to 0 as \( x \to +\infty \), for any \( t > 0 \), there exists \( x_0 < \infty \), such that \( \max_{0 < x < +\infty} w^2(x,t) = w^2(x_0,t) \). Take \( \overline{w} = \frac{1}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} w(x,t)dx = w(x^*,t) \), for some \( x^* \in (x_0, x_0 + \varepsilon) \). Then
\[
\max_{0 < x < +\infty} w^2(x,t) = w^2(x_0,t) = |w(x_0,t) - \overline{w} + \overline{w}|^2 
\leq 2|w(x_0,t) - \overline{w}|^2 + 2|\overline{w}|^2 
= 2 \left( \int_{x_0}^{x_0+\varepsilon} w_x(x,t)dx \right)^2 + 2 \frac{1}{\varepsilon^2} \left( \int_{x_0}^{x_0+\varepsilon} w(x,t)dx \right)^2 
\leq 2\varepsilon \int_{x_0}^{x_0+\varepsilon} w_x^2(x,t)dx + 2 \frac{\varepsilon}{\varepsilon} \int_{x_0}^{x_0+\varepsilon} w^2(x,t)dx 
\text{(by Hölder's inequality)}
\leq 2\varepsilon \int_0^{+\infty} w_x^2(x,t)dx + \frac{2}{\varepsilon} \int_0^{+\infty} w^2(x,t)dx.
\]

We now proceed to estimate the left side of the equation (5.4).
\[
\int_0^{+\infty} \frac{1}{\sigma_L^2} w_t(x,t)w(x,t)dx 
= \int_0^{s_1(t)} \frac{1}{\sigma_L^2} w_t(x,t)w(x,t)dx + \int_{s_1(t)}^{+\infty} \frac{1}{\sigma_L^2} w_t(x,t)w(x,t)dx 
= \frac{d}{dt} \left[ \int_0^{s_1(t)} \frac{1}{\sigma_L^2} w^2(x,t)dx \right] + \int_{s_1(t)}^{+\infty} \frac{1}{\sigma_L^2} w^2(x,t)dx 
- \left( \frac{1}{\sigma_L^2} - \frac{1}{\sigma_H^2} \right) s_1'(t) w^2(s_1(t),t),
\]
where \( s_1'(t) = -\frac{(w_1)_t}{(w_1)(s_1(t),t)} \) is bounded for \( 0 < t < t_0 \).
\[
\int_0^{+\infty} -\frac{1}{\sigma_1} w_{xx}(x,t)w(x,t)dx = \frac{1}{2} \int_0^{+\infty} w_x^2(x,t)dx 
\int_0^{+\infty} \left( \frac{1}{\sigma_1} + \frac{1}{2} \right) w_x(x,t)w(x,t)dx \leq \varepsilon \int_0^{+\infty} w_x^2(x,t)dx + \frac{C}{\varepsilon} \int_0^{+\infty} w^2(x,t)dx.
\]

Combining the above inequalities, taking also into account (5.4) and (5.5), we derive
\[
\frac{d}{dt} \left[ \int_0^{s_1(t)} \frac{1}{\sigma_L^2} w^2(x,t)dx \right] + \int_{s_1(t)}^{+\infty} \frac{1}{\sigma_H^2} w^2(x,t)dx 
\leq \int_0^{+\infty} \left( \frac{1}{\sigma_1} w_t(x,t)w(x,t) - \frac{1}{2} w_{xx}(x,t)w(x,t) - \left( \frac{1}{\sigma_1^2} + \frac{1}{2} \right) w_x(x,t)w(x,t) \right)dx 
\leq \varepsilon \int_0^{+\infty} w_x^2(x,t)dx + \frac{C}{\varepsilon} \int_0^{+\infty} w^2(x,t)dx.
\]
It follows that, for a small $\varepsilon$,
\[
c_0 \int_{0}^{+\infty} \frac{w^2(x,t)}{2} dx \leq \int_{0}^{s_1(t)} \frac{1}{\sigma_L^2} w^2(x,t) dx + \int_{s_1(t)}^{+\infty} \frac{1}{\sigma_H^2} w(x,t)^2 dx
\]
\[
\leq \int_{0}^{t_0} \left( \frac{C}{\varepsilon} \int_{0}^{+\infty} w^2(x,s) dx \right) ds.
\]

By applying Gronwall's inequality, we conclude $w \equiv 0$. This establishes the uniqueness for $0 \leq t \leq \rho$. A close examination of the proof indicates that the uniqueness result be extended to any time interval where $\psi_x$ is strictly negative, and this is already established in (4.14) the existence part of the proof.

**Remark 5.1.** Unlike [9], [17], no additional assumptions on the sign of second order derivative are needed. The proof provided here represents a significant progress of the uniqueness theorem. Since this method can also be applied in [9] and [17], the uniqueness theorem there are valid without the assumption on the sign of second derivative.

6. **Asymptotic behavior of solution.**

6.1. **Asymptotic solution.** Suppose $u$ satisfies the steady-state problem
\[
-\frac{1}{2} \sigma^2 u'' - (r + \frac{1}{2} \sigma^2) u' = 0, \quad 0 < x < \infty,
\]
\[
u(0) = 1,
\]
\[
u(+\infty) = 0,
\]
\[
u(\eta^*) = \gamma, \quad \nu'(\eta^*) = \nu'(\eta^-).
\]

Then, by solving the above ODE, we get an unique explicit steady-state solution
\[
u(x) = \begin{cases}
A_1 \exp \left[ -(\frac{2r}{\sigma_L^2} + 1)x \right] & \text{if } x > \eta^*,
A_2 + (1 - A_2) \exp \left[ -(\frac{2r}{\sigma_L^2} + 1)x \right] & \text{if } x \leq \eta^*,
\end{cases}
\]
where
\[
A_1 = \gamma \left( 1 + \frac{1 - \gamma}{\gamma} \cdot \frac{2r}{\sigma_L^2} + 1 \right)^{\frac{2r}{\sigma_L} + 1},
\]
\[
A_2 = \frac{2r \gamma \cdot \left( \frac{1}{\sigma_L^2} - \frac{1}{\sigma_H^2} \right)}{\frac{2r}{\sigma_L^2} + 1}, \quad 0 < A_2 < \gamma,
\]
\[
\eta^* = \frac{1}{\frac{2r}{\sigma_L^2} + 1} \cdot \log \left( 1 + \frac{1 - \gamma}{\gamma} \cdot \frac{2r}{\sigma_L^2} + 1 \right) > 0.
\]
6.2. Convergence. Equations (6.5) – (6.8) define an ODE solution \( u(x) \). In this section we prove \( \psi(x,t) \) converges to \( u(x) \) as \( t \to +\infty \). It is established that \( 0 \leq \psi_\varepsilon(x,t) \leq \min\{1, \frac{\varepsilon}{\sqrt{\pi}} \} \) in Lemma 4.1, which implies that \( \psi_\varepsilon(x,t) \to 0 \) as \( x \to +\infty \), uniformly in \( \varepsilon \).

Since \( \psi(x,t) \) and \( s(t) \) are bounded and decreasing with \( t \), there exists a function \( \tilde{u}(x) \) such that \( \lim_{t \to +\infty} \psi(x,t) = \tilde{u}(x) \), and a constant \( \bar{n} \) such that \( \lim_{t \to +\infty} s(t) = \bar{n} \).

Because this convergence is not on a sub-sequential limit, Lemmas 4.1, 4.2, and 4.5 and the compactness imply that

\[
\psi_x(x,t) \to \tilde{u}'(x) \quad \text{uniformly on } [0, L] \text{ for any } L > 0,
\]

\[
\psi_{xx} \to \tilde{u}'' \quad \text{weak-star,}
\]

and \( \tilde{u} \in W^{2,\infty}(0, \infty) \). The above convergence implies that \( \tilde{u} \) satisfies (6.2), (6.3) and (6.4). It remains to prove \( \tilde{u}(x) \) satisfies the equation (6.1).

Take a test function \( f \in C^\infty_c(0, \infty) \), this is a function of \( x \) only and is independent of \( t \), then

\[
\int_0^\infty \psi_t f dx = \int_0^\infty \frac{1}{2} \sigma^2(\psi) \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \right] f dx + \int_0^\infty r \frac{\partial \psi}{\partial x} f dx. \tag{6.9}
\]

Clearly the second term on the right-hand side converges to the corresponding integral of \( \tilde{u} \) as \( t \to +\infty \). For the first term on the right-hand side of (6.9),

\[
\int_0^\infty \frac{1}{2} \sigma^2(\psi) \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \right] f dx = \int_0^\infty \frac{1}{2} \sigma^2(\tilde{u}) \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \right] f dx
\]

\[
+ \int_0^\infty \left( \frac{1}{2} \sigma^2(\psi) - \frac{1}{2} \sigma^2(\tilde{u}) \right) \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x} \right] f dx.
\]

By weak-star convergence, the first term on the right-hand side of the above equation converges to the corresponding integral of \( \tilde{u} \). The second term on the right-hand side of the above equation is bounded by \( C \int_0^\infty |\sigma^2(\psi) - \sigma^2(\tilde{u})| |f| dx \), which converges to 0 by the Dominated Convergence Theorem.

Since \( \lim_{t \to +\infty} \psi(x,t) = \tilde{u}(x) \), we deduce

\[
\lim_{T \to +\infty} \int_{T-1}^T \int_0^\infty \psi_t f dx = \lim_{T \to +\infty} \left\{ \int_0^\infty \psi(x,T) f(x) dx - \int_0^\infty \psi(x,T-1) f(x) dx \right\} = 0.
\]

Thus integrate (6.9) over \( [T - 1, T] \) and let \( T \to \infty \), we obtain

\[
\int_0^\infty \frac{1}{2} \sigma^2(\tilde{u}) \left[ \tilde{u}''(x) + \tilde{u}'(x) \right] f(x) dx + \int_0^\infty r \tilde{u}'(x) f(x) dx = 0.
\]

It follows that \( \tilde{u} \) satisfies

\[
- \frac{1}{2} \sigma^2(\tilde{u}) \tilde{u}''(x) - \left( r + \frac{1}{2} \sigma^2(\tilde{u}) \right) \tilde{u}'(x) = 0. \tag{6.10}
\]

Since the problem (6.1)-(6.4) admits a unique solution, we have \( u \equiv \tilde{u} \). \( \square \)

7. Numerical results. We use explicit difference scheme to present the numerical results of \( \psi(x,t) \) and \( s(t) \), which are shown in Figures 1 and 2. And the parameters are chosen as follows

\[
r = 0.05, \sigma_L = 0.4, \sigma_H = 0.2, F = 1, \gamma = 0.8, K = 0.03, T = 5.
\]
From the graphs, the value of the bond price function $\psi(x, t)$ and free boundary $s(t)$ behave as we expected.

Next, the relationship between value function $\psi(x, T)$ with different maturity $T$ and asymptotic solution is shown in Figure 3. The parameters are chosen the same with those in Figure 1.
Fig 3: asymptotic behavior

From Figure 3, it is seen that the value function approaches the asymptotic solution as $T$ goes larger.

8. Conclusion. From the view of the proportion of the debt to value of a firm, considering default before maturity, a free boundary model for pricing corporate bonds with credit rating migration is established. By using PDE techniques, the existence, uniqueness and regularities of the solution are obtained. In addition, the proof of uniqueness represents a significant progress over [9] and [17] by loosening the restriction there. Furthermore, the asymptotic behavior of the solution is analyzed and the convergence is proved. Finally, numerical simulations are also presented.

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E-mail address: uuyuan0720@hotmail.com
E-mail address: liang_jin@tongji.edu.cn
E-mail address: blihu@nd.edu