ON COUPLED DIRAC SYSTEMS

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Abstract. In this paper, we show the existence of solutions for the coupled Dirac system
\[
\begin{align*}
Du &= \frac{\partial H}{\partial v}(x, u, v) \quad \text{on } M, \\
Dv &= \frac{\partial H}{\partial u}(x, u, v) \quad \text{on } M,
\end{align*}
\]
where $M$ is an $n$-dimensional compact Riemannian spin manifold, $D$ is the Dirac operator on $M$, and $H : \Sigma M \oplus \Sigma M \to \mathbb{R}$ is a real valued superquadratic function of class $C^1$ in the fiber direction with subcritical growth rates. Our proof relies on a generalized linking theorem applied to a strongly indefinite functional on a product space of suitable fractional Sobolev spaces. Furthermore, we consider the $\mathbb{Z}_2$-invariant $H$ that includes a nonlinearity of the form
\[
H(x, u, v) = f(x) \frac{|u|^{p+1}}{p+1} + g(x) \frac{|v|^{q+1}}{q+1},
\]
where $f(x)$ and $g(x)$ are strictly positive continuous functions on $M$ and $p, q > 1$ satisfy
\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}.
\]
In this case we obtain infinitely many solutions of the coupled Dirac system by using a generalized fountain theorem.

1. Introduction. Let $(M, g)$ be an $n$-dimensional compact oriented Riemannian manifold equipped with a spin structure $\rho : P_{\text{Spin}(M)} \to P_{\text{SO}(M)}$, and let $\Sigma M = \Sigma(M, g) = P_{\text{Spin}(M)} \times \sigma \Sigma_n$ denote the complex spinor bundle on $M$. The latter is a complex vector bundle of rank $2^{[n/2]}$ endowed with the spinorial Levi-Civita connection $\nabla$ and a pointwise Hermitian scalar product. In the following, let $\langle \cdot, \cdot \rangle$ always denote the real part of the Hermitian product on $\Sigma M$. It induces a natural

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inner product \((u,v)_{L^2} = \int_M \langle u(x), v(x) \rangle \, dx\) on the space \(C^\infty(M, \Sigma M)\) of all \(C^\infty\)-sections of the bundle \(\Sigma M\), where \(dx\) is the Riemannian measure of \(g\). Denote by \(L^2(M, \Sigma M)\) the space of \(L^2\)-sections of \(\Sigma M\). The Dirac operator is an elliptic differential operator of order one, \(D = D_g : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)\), locally given by \(D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi\) for \(\psi \in C^\infty(M, \Sigma M)\) and a local \(g\)-orthonormal frame \(\{e_i\}_{i=1}^n\) of the tangent bundle \(TM\). Consider Whitney direct sum \(\Sigma M \oplus \Sigma M\) of \(\Sigma M\) and itself, and write a point of it as \((x, \xi, \zeta)\), where \(x \in M\) and \(\xi, \zeta \in \Sigma_x M\). In this paper we study the following system of the coupled semilinear Dirac equations:

\[
\begin{cases}
Du = \frac{\partial H}{\partial v}(x, u, v) \quad &\text{on } M, \\
Dv = \frac{\partial H}{\partial u}(x, u, v) \quad &\text{on } M,
\end{cases}
\]

(1)

where \((u(x), v(x)) \in \Sigma_x M \forall x \in M\) and \(H : \Sigma M \oplus \Sigma M \to \mathbb{R}\) is \(C^1\) in the fiber direction. (1) is the Euler-Lagrange equation of the functional

\[
\mathcal{L}_H(u, v) = \int_M (\langle Du, v \rangle - H(x, u, v)) \, dx.
\]

(2)

The functional \(\mathcal{L}_H\) is strongly indefinite since the spectrum of the operator \(D\) is unbounded from below and above.

Nonlinear Dirac equations arise in many interesting problems in geometry and physics, such as the generalized Weierstrass representation of surfaces in three-manifolds [15] and the supersymmetric nonlinear sigma model in quantum field theory [10, 11]. In [2, 3] Ammann studied a class of Dirac equations of the form \(D\psi = \lambda|\psi|^{p-1}\psi\) for \(\lambda > 0\) and \(2 < p \leq 2n/(n-1)\). Raulot [24] extended Ammann’s results to the equations of the form \(D\psi = H(x)|\psi|^{p-1}\psi\), where \(H\) is a smooth positive function whose all partial derivatives at some maximum point of order less than or equal to \(n - 1\) vanish. Furthermore, Isobe [19] studied super-linear and sub-linear nonlinear Dirac equations on compact spin manifolds. Besides, Isobe [20] used the dual variational method to obtain the existence of nonlinear Dirac equations with critical nonlinearities. Recently, in [17] the authors extended Isobe’s results to a large class of nonlinear Dirac equations with positive \(C^1\)-potentials.

However, except for these works, the coupled Dirac systems like (1) are less studied in the literatures. In quantum physics, the problem (1) describes two coupled fermionic fields. Mathematically, it can be viewed as a spinorial analogue of other strongly indefinite variational problems such as infinite dynamical systems [5, 6] and elliptic systems [4, 12], and this is our main motivation for its study. A typical way to deal with such problems is the min-max method of Benci and Rabinowitz [9, 23], including the mountain pass theorem, linking theorem and so on. In this paper we use the techniques introduced by Felmer [13] to prove the existence of solutions of (1), and apply a generalized fountain theorem established by Batkam and Colin [8] to obtain infinitely many solutions of the coupled Dirac system provided the nonlinearity \(H\) is even.

In the following we assume that two real numbers \(p, q > 1\) satisfy

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}.
\]

(3)

It is not hard to verify that we can choose a real number \(s \in (0, 1)\) such that

\[
p < \frac{n+2s}{n-2s} \quad \text{and} \quad q < \frac{n+2-2s}{n+2s-2}.
\]

(4)
On nonlinearity $H$, we make the following hypotheses:

(H1) $H \in C^0(\Sigma M \oplus \Sigma M, \mathbb{R})$ is $C^1$ in the fiber direction.

(H2) For all $x \in M$ and $(0, 0) \neq (u, v) \in \Sigma_x M \oplus \Sigma_x M$

\[
\frac{1}{p+1} \langle H_u(x, u, v), u \rangle + \frac{1}{q+1} \langle H_v(x, u, v), v \rangle \geq H(x, u, v) > 0.
\]  

(H3) There exists a constant $c_1 > 0$ such that

\[
|H_u(x, u, v)| \leq c_1 \left(1 + |u|^p + |v|^{q(p+1)} \right),
\]

\[
|H_v(x, u, v)| \leq c_1 \left(1 + |u|^{q(p+1)} + |v|^q \right).
\]

(H4) $H(x, u, v) = o(|u|^2 + |v|^2)$ uniformly with respect to $x$ as $|u| + |v| \to 0$.

Consider the following typical examples satisfying the above (H1)-(H4),

\[
H(x, u, v) = f(x) \frac{|u|^{p+1}}{p+1} + g(x) \frac{|v|^{q+1}}{q+1},
\]

where $f(x)$ and $g(x)$ are strictly positive continuous functions on $M$. Then (1) reduces to the following form

\[
\begin{cases}
Du = g(x)|v|^{q-1}v & \text{on } M,

Dv = f(x)|u|^{p-1}u & \text{on } M.
\end{cases}
\]

Note that $\int_M H(x, u, v)dx$ is not well-defined on the Hilbert space $H^{\frac{1}{2}}(M, \Sigma M) \times H^{\frac{1}{2}}(M, \Sigma M)$ unless we make a stronger hypothesis on the exponents $p, q$ as in [12]. To overcome this difficulty, inspired by the ideas of Hulshof and Van der Vorst [18], we consider the following well-defined functional

\[
A_H(u, v) = \int_M \langle Du, v \rangle dx - \int_M H(x, u, v)dx
\]

on the fractional Sobolev space $H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M)$.

We state our main result as

**Theorem 1.1.** Assume that $n \geq 2$ and $0 \notin \text{Spec}(D)$. Problem (1) has at least a nontrivial solution $(u, v) \in W^{1,(q+1)/q}(M, \Sigma M) \times W^{1,(p+1)/p}(M, \Sigma M)$ if $H : \Sigma M \oplus \Sigma M \to \mathbb{R}$ satisfies (H1)-(H4).

For even nonlinearities, we have the following multiplicity result:

**Theorem 1.2.** Assume that $n \geq 2$ and $0 \notin \text{Spec}(D)$. If $H : \Sigma M \oplus \Sigma M \to \mathbb{R}$ satisfies (H1)-(H4). Furthermore, assume $H$ is even in the fiber direction, i.e., $H(x, -u, -v) = H(x, u, v)$ for each $x \in M$ and $(u, v) \in \Sigma_x M \oplus \Sigma_x M$. Then there exists a sequence of solutions $\{z_k\}_{k=1}^\infty \subset W^{1,(q+1)/q}(M, \Sigma M) \times W^{1,(p+1)/p}(M, \Sigma M)$ to (1) with $A_H(z_k) \to \infty$ as $k \to \infty$.

A direct application of Theorem 1.2 is the following corollary:

**Corollary 1.** Let $H$ be as in (8). Assume that $n \geq 2$ and $0 \notin \text{Spec}(D)$. Then (9) has infinitely many solutions $z_k = (u_k, v_k) \in W^{1,(q+1)/q}(M, \Sigma M) \times W^{1,(p+1)/p}(M, \Sigma M)$ ($k = 1, 2, \ldots$) with $A_H(z_k) \to \infty$ as $k \to \infty$.

**Organization of the paper.** In Section 2, we define a functional on a suitable product space of fractional Sobolev spaces and give some regularity results on the functional $A_H$ and solutions of (1) obtained by its critical points. The aim of Section 3 is to prove the Palais-Smale condition. In Section 4, we give some
The induced norm $\| \cdot \|$ and its spectrum consists of an unbounded sequence of real numbers (cf. [14, 22]). The well known Schrödinger-Lichnerowicz formula implies that all eigenvalues of $D$ are nonzero if $M$ has positive scalar curvature. Hereafter, we assume:

$$0 \notin \text{spec}(D) \quad \text{and} \quad \int_M dx = 1 \quad \text{i.e., the volume of } (M, g) \text{ equals to 1.}$$

(The second assumption is only for simplicity, it is actually unnecessary for our result!)

Let $(\psi_k)_{k=1}^\infty$ be a complete $L^2$- orthonormal basis of eigenspinors corresponding to the eigenvalues $(\lambda_k)_{k=1}^\infty$ counted with multiplicity such that $|\lambda_k| \rightarrow \infty$ as $k \rightarrow \infty$. For each $s \geq 0$, let $H^s(M, \Sigma M)$ be the Sobolev space of fractional order $s$, its dual space is denoted by $H^{-s}(M, \Sigma M)$. We have a linear operator $|D|^s : H^s(M, \Sigma M) \subset L^2(M, \Sigma M) \rightarrow L^2(M, \Sigma M)$ defined by

$$|D|^s u = \sum_{k=1}^\infty a_k |\lambda_k|^s \psi_k, \quad (11)$$

where $u = \sum_{k=1}^\infty a_k \psi_k \in H^s(M, \Sigma M)$. Since $0 \notin \text{spec}(D)$ the inverse $|D|^{-s} \in \mathcal{L}(L^2(M, \Sigma M))$ is compact and self-adjoint. $|D|^s$ can be used to define a new inner product on $H^s(M, \Sigma M)$,

$$(u, v)_{s,2} := (|D|^s u, |D|^s v)_2. \quad (12)$$

The induced norm $\| \cdot \|_s = \sqrt{(\cdot, \cdot)_{s,2}}$ is equivalent to the usual one on $H^s(M, \Sigma M)$ (cf. [1, 2]). For $r \in \mathbb{R}$ consider the Hilbert space

$$\tilde{\omega}^{2r} = \left\{ a = (a_1, a_2, \cdots) \mid \sum_{k=1}^\infty a_k^2 \lambda_k^{2r} < \infty \right\}$$

with inner product

$$\langle \langle a, b \rangle \rangle_{2r} = \sum_{k=1}^\infty \lambda_k^{2r} a_k b_k.$$ 

Then $H^s(M, \Sigma M)$ can be identified with the Hilbert space $\tilde{\omega}^{2s}$. Hence

$$H^{-s}(M, \Sigma M) = (H^s(M, \Sigma M))'$$

can be identified with $\tilde{\omega}^{-2s}$, where the pairing between $\tilde{\omega}^{-2s}$ and $\tilde{\omega}^{2s}$ is given by

$$\langle \langle a, b \rangle \rangle = \sum_{k=1}^\infty a_k b_k. \quad (13)$$

It follows that $|D|^{-2s}$ gives a Hilbert space isomorphism from $H^{-s}(M, \Sigma M)$ to $H^s(M, \Sigma M)$ with respect to the equivalent new inner products as in (12). Moreover we have a continuous inclusion $L^2(M, \Sigma M) \rightarrow H^{-s}(M, \Sigma M)$ and

$$(|D|^{-2s} u, v)_{s,2} := (u, v)_2 \quad \forall u, v \in L^2(M, \Sigma M). \quad (14)$$

Consider the Hilbert space

$$E_s := H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M)$$

(15)
Proposition 1. Assume that $u$ are constants independent of $H$ for some constant $C > H$ we see that the nonlinearity $H$ that is of class $C$ for all $E$. Then

$$D_s := \begin{pmatrix} |D|^{-2s} & 0 \\ 0 & |D|^{-2(1-s)} \end{pmatrix} : E_s^* \to E_s$$

is a Hilbert space isomorphism by the arguments above (14) and

$$(D_s z_1, z_2)_{E_s} = (z_1, z_2)_{L^2}$$

for all $z_1, z_2 \in L^2(M, \Sigma M) \times L^2(M, \Sigma M)$.

Since $M$ is compact, by the assumption (H2) we have constants $C_1, C_2 > 0$ such that

$$|H(x, u, v)| \geq C_1(|u|^{p+1} + |v|^{q+1}) - C_2 \forall (x, u, v),$$

and by the assumption (H3) we can use Young’s inequality to derive

$$|H(x, u, v)| \leq C(1 + |u|^{p+1} + |v|^{q+1}) \forall (x, u, v)$$

for some constant $C > 0$. (Later on, we also use $C$ to denote various positive constants independent of $u$ and $v$ without special statements). From (18) and (19) we see that the nonlinearity $H$ is superquadric.

**Proposition 1.** Assume that $H \in C^0(\Sigma M \oplus \Sigma M)$ satisfies (H1) and (H3). Then the functional $\mathcal{H} : E_s \to \mathbb{R}$ defined by

$$\mathcal{H}(u, v) = \int_M H(x, u(x), v(x))dx,$$ (20)

is of class $C^1$, its derivation at $(u, v) \in E_s$ is given by

$$D\mathcal{H}(u, v)(\xi, \zeta) = \int_M \left( (H_u(x, u, v), \xi) + (H_v(x, u, v), \zeta) \right)dx \ \forall (\xi, \zeta) \in E_s,$$ (21)

and $D\mathcal{H} : E_s \to E_s^*$ is a compact map, where $E_s^*$ consists of all bounded linear functionals on $E_s$.

**Remark 1.** If the real numbers $p, q$ satisfy

$$1 < p, q < \min \left\{ \frac{n + 2s}{n - 2s}, \frac{n + 2(1-s)}{n - 2(1-s)} \right\}$$

for some $s \in (0, 1)$, which implies (3), then the above space $E_s$ can be replaced by $E_{s'}$.

For the sake of completeness we shall give the proof of Proposition 1 in Appendix A.

**Definition 2.1.** We say that $z = (u, v) \in E_s$ is a weak solution of (1) if it satisfies

$$\int_M \langle D\zeta, u \rangle dx + \int_M \langle D\xi, v \rangle dx - \int_M (H_u(x, u, v), \xi) dx - \int_M (H_v(x, u, v), \zeta) dx = 0$$

(22)

for all $\xi, \zeta \in C^\infty(M, \Sigma M)$.

We prove a regularity result for weak solutions $(u, v) \in E_s$ to (1).
Theorem 2.2. If \((u, v) \in E_s\) is a weak solution of (1) then \(u \in W^{1,(q+1)/q}(M, \Sigma M)\), \(v \in W^{1,(p+1)/p}(M, \Sigma M)\) and it holds that
\[
\begin{align*}
Du &= \frac{\partial H}{\partial v}(x, u, v), \\
Dv &= \frac{\partial H}{\partial u}(x, u, v)
\end{align*}
\]
a.e. in \(M\).

Proof. Taking \(\zeta = 0\) in (22) leads to
\[
\int_M \langle D\zeta, v \rangle dx = \int_M \langle H_u(x, u, v), \zeta \rangle dx
\]
for all \(\xi \in C^\infty(M, \Sigma M)\). We deduce from the Sobolev embedding theorem, (6) and the H"older’s inequality that
\[
H_u(x, u, v) \in L^{p+1}_p(M, \Sigma M) \quad \forall (u, v) \in E_s.
\]
Since \(0 \notin \text{spec}(D)\) by our assumption,
\[
D : W^{1,\frac{p+1}{p}}(M, \Sigma M) \to L^{\frac{p+1}{p}}(M, \Sigma M)
\]
is invertible. Therefore, by (25), there exists a unique solution \(\psi \in W^{1,\frac{p+1}{p}}(M, \Sigma M)\) to the equation
\[
D\psi = H_u(x, u, v)
\]
a.e. on \(M\). Thus we have
\[
\int_M \langle D\psi, \xi \rangle dx = \int_M \langle H_u(x, u, v), \xi \rangle dx
\]
for all \(\xi \in C^\infty(M, \Sigma M)\). Combining (24) with (27) we arrive at
\[
\int_M \langle D(v - \psi), \xi \rangle dx = 0 \quad \forall \xi \in C^\infty(M, \Sigma M)
\]
from which it follows that \(v = \psi\). So we obtain \(v \in W^{1,(p+1)/p}(M, \Sigma M)\) and \(v\) satisfies the second equation of (23). Similarly, we prove that \(u \in W^{1,(q+1)/q}(M, \Sigma M)\) and satisfies the first equation of (23). \(\square\)

It follows from Proposition 1 that the functional \(A_H\) in (10) is of class \(C^1\) on Hilbert space \(E_s\) with inner product
\[
\langle (u_1, v_1), (u_2, v_2) \rangle_{E_s} = (u_1, v_1)_s + (u_2, v_2)_{1-s, 2}
\]
for \((u_i, v_i) \in E_s, i = 1, 2\). Moreover, \((u, v) \in E_s\) is a critical point of \(A_H\) if and only if
\[
\begin{align*}
Du &= H_v(x, u, v) \quad \text{on } M, \\
Dv &= H_u(x, u, v) \quad \text{on } M.
\end{align*}
\]
Since the operator \(D\) is self-adjoint, the functional \(A_H\) can be written as
\[
A_H(z) = \frac{1}{2} \int_M \langle Lz(x), z(x) \rangle dx - \int_M H(x, z(x))dx,
\]
where
\[
L = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix},
\]
Note that \( \int_M (L z(x), z(x)) \, dx = (L z, z)_2 = (D_s L z, z)_{E_s} \) by (17). Since \( D_s L : E_s \to E_s \) is a self-adjoint isometry operator and
\[
(D_s L) \circ (D_s L) = \begin{pmatrix} |D|^{-2s} D |D|^{-2(1-s)} D & 0 \\ 0 & |D|^{-2(1-s)} D |D|^{-2s} D \end{pmatrix} = \text{Id}_{E_s},
\]
we can split \( E_s \) into
\[
E_s = E_+ \oplus E_- = \{ z = z^+ + z^-, \ z^\pm \in E_\pm \}
\]
with \( (D_s L)|_{E_\pm} = \pm \text{Id}_{E_\pm} \) and
\[
E_\pm = \{ (u, \pm |D|^{-2(1-s)} D u) | u \in H^s(M, \Sigma M) \}.
\]
The orthogonal projections \( \Pi_\pm : E_s \to E_\pm \) are given by
\[
\Pi_\pm(u, v) = \frac{1}{2} (u \pm |D|^{-2s} D v, v \pm |D|^{-2(1-s)} D u).
\]
Denote by
\[
Q(z) = (D_s L z, z)_{E_s}.
\]
Then we can write \( \mathcal{A}_H \) as
\[
\mathcal{A}_H(z) = \frac{1}{2} Q(z) - \mathcal{H}(z).
\]
For each \( z = z^+ + z^- \) we have
\[
Q(z) = \| z^+ \|^2 - \| z^- \|^2.
\]

3. Palais-Smale condition. In this section we prove the Palais-Smale condition for \( \mathcal{A}_H \).

**Definition 3.1.** Let \( \Phi \) be a \( C^1 \)-functional on a Banach space \( E \), and let \( c \in \mathbb{R} \). A sequence \( \{ x_n \}_{n=1}^\infty \subset E \) is called a Palais-Smale sequence for \( \Phi \) if \( \Phi(x_n) \to c \) and \( \| D\Phi(x_n) \|_{E^*} \to 0 \). If every Palais-Smale sequence \( \{ x_n \}_{n=1}^\infty \subset E \) for \( \Phi \) has a convergent subsequence then \( \Phi \) is said to satisfy the Palais-Smale condition.

**Proposition 2.** Assume that \( H \) satisfies (H1) – (H3). Then the Palais-Smale condition is satisfied for \( \mathcal{A}_H \) on \( E_s \).

**Proof.** Let \( \{ z_k \}_{k=1}^\infty \subset E_s \) be a sequence satisfying
\[
\mathcal{A}_H(z_k) \to c \in \mathbb{R} \quad \text{and} \quad \| D\mathcal{A}_H(z_k) \|_{E^*_s} \to 0 \quad \text{as} \quad k \to \infty,
\]
We prove first that \( \{ z_k \}_{k=1}^\infty \) is bounded. For each \( z_k = (u_k, v_k) \) we take
\[
\xi_k = \left( \frac{q+1}{p+q+2} u_k, \frac{p+1}{p+q+2} v_k \right).
\]
It follows from (36) that there is a sequence \( \{ \varepsilon_k \} \) converging to 0 such that
\[
\begin{align*}
c + \varepsilon_k \| \xi_k \| & \geq \mathcal{A}_H(z_k) - D\mathcal{A}_H(z_k) \xi_k \\
& = \frac{(p+1)(q+1)}{p+q+2} \int_M \frac{1}{p+1} \langle H_u(x, u_k, v_k), u_k \rangle \\
& \quad + \frac{1}{q+1} \langle H_v(x, u_k, v_k), v_k \rangle - H(x, u_k, v_k) \, dx \\
& \quad + \left( \frac{(p+1)(q+1)}{p+q+2} - 1 \right) \int_M H(x, u_k, v_k) \, dx
\end{align*}
\]
By (H2) and (18) we find a constant $c_1$ so that
\[
\int_M |u_k|^{p+1} + |v_k|^{q+1} \, dx \leq c_1 (1 + \|u_k\|_s + \|v_k\|_{1-s}). \tag{38}
\]
Writing $z_k^\pm = (u_k^\pm, v_k^\pm)$, we deduce from (36) and (H3) that
\[
\|z_k^\pm\|^2 - \varepsilon_k\|z_k^\pm\| \leq \left| (D_s L z_k, z_k^\pm)_{E_s} - D_A H(z_k) z_k^\pm \right|
= \left| \int_M H_u(x, z_k) u_k^\pm \, dx + \int_M H_v(x, z_k) v_k^\pm \, dx \right|
\leq C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|u_k^\pm\|_r + C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|v_k^\pm\|_r
\leq C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|u_k^\pm\|_s + C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|v_k^\pm\|_{1-s}
\leq C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|z_k^\pm\|
+ C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|z_k^\pm\|_{1-s}.
\]
Thus
\[
\|z_k^\pm\| \leq C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|z_k^\pm\|_s + C \left( 1 + \|u_k\|_p + \|v_k\|_{p+1} \right) \|z_k^\pm\|_{1-s}. \tag{39}
\]
Combining (38) with (40) for $z_k = z_k^+ + z_k^-$, we obtain
\[
\|z_k\| \leq C \left( 1 + \|z_k\|_p + \|z_k\|_p \right), \tag{41}
\]
which implies that $\{z_k\}_{k=1}^\infty$ is bounded. Passing to a subsequence, one may assume that $z_k$ converges weakly in $E_s$ to $z = (u, v)$. Since the operator $D_s L$ is isometric and $DH$ is compact (see proposition (1)), we conclude that
\[
z_k = (D_s L)^{-1} (DA_H(z_k) + DH(z_k)) \tag{42}
\]
converges in $E_s$ and $\|z_k - z\| \to 0$ as $k \to \infty$. This completes the proof. \hfill \square

4. **Linking geometry.** In this section, we give some geometric conditions for two linking properties of $A_H$. A suitable framework for the infinite dimensional linking geometry was first given by [9]. Before giving the geometric conditions for the first linking property, we define linking subsets as follows:

For $R_1, R_2, \rho > 0$ with $0 < \rho < R_2$, we define
\[
S_\rho = \{ z \in E_+ | \|z\| = \rho \} \tag{43}
\]
and
\[
Q_{R_1, R_2} = \{ z \in E_- | \|z\| \leq R_1 \} \oplus \{ r e^+ | 0 \leq r \leq R_2 \} \tag{44}
\]
where \( e^+ = (\xi^+, \eta^+) \in E_+ \) with \( \xi^+ \) some eigenspinor of \( D \) corresponding to the first positive eigenvalue \( \lambda_1 \). We assume \( \|e^+\| = 1 \). Denote by \( \partial Q_{R_1, R_2} \) the boundary of \( Q_{R_1, R_2} \) relative to the subspace
\[
V = \{ z + re^+ | z \in E_-, \ r \in \mathbb{R} \}.
\]

**Lemma 4.1.** There exist \( \rho > 0 \) and \( \delta > 0 \) such that
\[
\mathcal{A}_H(z) \geq \delta \quad \forall z \in S_{\rho}.
\]

**Proof.** Assumptions (H1), (H3) and (H4) imply that for any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that
\[
|H(x, u, v)| \leq \varepsilon (|u|^p + |v|^q) + C_\varepsilon (|u|^{p+1} + |v|^{q+1})
\]
for all \( x \in M \) and \( u, v \in \Sigma_x M \). By (45) and the Sobolev embeddings we obtain
\[
\mathcal{A}_H(z^+) \geq \left( \frac{1}{2} - c_1 \varepsilon \right) \|z^+\|^2 - c_2 C_\varepsilon (\|z^+\|^{p+1} + \|z^+\|^{q+1}) \quad \forall z^+ \in E_+
\]
for some constants \( c_1 > 0 \) and \( c_2 > 0 \). Thus, choosing \( \varepsilon = \frac{1}{4c_1} \), we can take \( \rho > 0 \) and \( \delta > 0 \) small enough such that \( \mathcal{A}_H(z) \geq \delta \) on \( S_{\rho} \).

**Lemma 4.2.** There are two constants \( R_1 > 0 \) and \( R_2 > 0 \) with \( 0 < \rho < R_2 \) such that
\[
\mathcal{A}_H(z) \leq 0 \quad \forall z \in \partial Q_{R_1, R_2}.
\]

**Proof.** For \( z = z^- + re^+ \in V \), we have
\[
\mathcal{A}_H(z^- + re^+) = \frac{1}{2} r^2 - \frac{1}{2} \|z^-\|^2 - \mathcal{H}(z^- + re^+).
\]
We set \( z^- = (u^-, v^-) \). By definition of \( E_\pm \) we have
\[
\eta^+ = |D|^{-2(1-s)} D\xi^+ \quad \text{and} \quad v^- = -|D|^{-2(1-s)} D u^-.
\]
Using (18), for \( z^- + re^+ = (u^- + r\xi^+, v^- + r\eta^-) \), we get
\[
\mathcal{H}(z^- + re^+) \geq C_1 \int_M |u^- + r\xi^+|^p + C_1 \int_M |v^- + r\eta^+|^q - C_2.
\]
We write \( u^- = t\xi^+ + \hat{u} \), where \( \hat{u} \) is orthogonal to \( \xi^+ \) in \( L^2(M, \Sigma M) \). By the Hölder’s inequality,
\[
(r + t) \int_M |\xi^+|^2 dx = \int_M \langle u^- + r\xi^+, \xi^+ \rangle dx
\leq \|u^- + r\xi^+\|_{L^{p+1}} \|\xi^+\|_{L^{\frac{p+1}{p}}},
\]
which implies
\[
(r + t) \leq C_3 \|u^- + r\xi^+\|_{L^{p+1}}
\]
for some constant \( C_3 \) depending on \( \xi^+ \). Similarly, since \( \eta^+ = \lambda_1^{\frac{q-s}{2}} \xi^+ \), where \( \lambda_1 \) is the first positive eigenvalue of \( D \) in \( H^1(M, \Sigma M) \), we have
\[
(r - t) \lambda_1^{\frac{q-s}{2}} \int_M |\xi^+|^2 dx = \int_M \langle v^- + r\eta^+, \xi^+ \rangle dx
\leq \|v^- + r\eta^+\|_{L^{q+1}} \|\xi^+\|_{L^{\frac{q+1}{q}}},
\]
or
\[
(r - t) \leq C_4 \|v^- + r\eta^+\|_{L^{q+1}}
\]
for some constant $C_4$ depending on $\xi^+$ and $\lambda_1$. If $t \geq 0$ we deduce from (47), (49) and (51) that
\[ \mathcal{A}_H(z^- + r e^+) \leq \frac{1}{2} r^2 - C_5 r^{q+1} + C_2. \] (54)
If $t \leq 0$ it follows from (47), (49) and (53) that
\[ \mathcal{A}_H(z^- + r e^+) \leq \frac{1}{2} r^2 - C_5 r^{q+1} + C_2. \] (55)
Taking $r = R_2 > p$ large enough, we deduce from (54) and (55) that $\mathcal{A}_H(z^- + r e^+) < 0$. Since $H \geq 0$, for $r \in [0, R_2]$ it holds that
\[ \mathcal{A}_H(z^- + r e^+) \leq \frac{1}{2} r^2 - \frac{1}{2} \|z^-\|^2. \] (56)
Taking $\|z^-\| = R_1$ large enough we get $\mathcal{A}_H(z^- + r e^+) < 0$. For the remaining part of the boundary, we show that when $r = 0$ we have $\mathcal{A}_H(z^- + r e^+) \leq 0$, since $Q(z) \leq 0$ in $E_-$ and $\mathcal{H}(z) \geq 0$ for each $z \in E_s$. □

Using the complete $L^2$- orthonormal basis of eigenspinors $(\psi_k)_{k=1}^\infty$ corresponding to the eigenvalues $(\lambda_k)_{k=1}^\infty$, we see from (34) that an orthonormal basis consisting of eigenspinors of $E_{\pm}$ is given by
\[ \left\{ \frac{1}{\sqrt{2}}(|\lambda_k|^{-s} \psi_k, \pm |\lambda_k|^{s-2} \lambda_k \psi_k) \right\}_{k=1}^\infty. \]
Then
\[ E_+ = \bigoplus_{j=1}^\infty \mathbb{R} e_j \quad \text{with} \quad e_j = \frac{1}{\sqrt{2}}(|\lambda_j|^{-s} \psi_j, |\lambda_j|^{s-2} \lambda_j \psi_j). \]

For giving the geometric conditions of the second linking property we use the following notation:
\[ Y_k : = E_- \ominus \left( \bigoplus_{j=1}^k \mathbb{R} e_j \right), \quad Z_k : = \bigoplus_{j=k}^\infty \mathbb{R} e_j, \] (57)
\[ B_k : = \{ z \in Y_k \|z\| \leq \rho_k \}, \quad N_k : = \{ z \in Z_k \|z\| = \rho_k \} \] (58)
\[ \partial B_k : = \{ z \in Y_k \|z\| = \rho_k \} \quad \text{where} \quad 0 < \rho_k < \rho_k, \quad k = 3, 4, \ldots. \] (59)

**Lemma 4.3.** There exists $\rho_k > r_k > 0$ such that
(A1): $a_k : = \inf_{z \in N_k} \mathcal{A}_H(z) \to \infty$, $k \to \infty$,
(A2): $b_k : = \sup_{z \in \partial B_k} \mathcal{A}_H(z) \leq 0$ and $d_k : = \sup_{z \in B_k} \mathcal{A}_H(z) < \infty$.

**Proof.** Set
\[ t_1 : = \max\{p + 1, q + 1\}, \quad t_2 : = \min\{p + 1, q + 1\}, \] (60)
\[ \alpha_k : = \sup_{(u,v) \in Z_k} \|u\|_{L^{p+1}} + \|v\|_{L^{q+1}}. \] (61)

Then $t_1 > 2$ and $t_2 > 2$. Let $z = (u, v) \in Z_k$. Using (45), by the Sobolev embeddings, we obtain
\[ \mathcal{A}_H(z) = \frac{1}{2} \|z\|^2 - \int_M H(x, z(x)) dx \]
\[ \geq \frac{1}{2} \|z\|^2 - \varepsilon(\|u\|_2^2 + \|v\|_2^2) - C_\varepsilon(\|u\|_{L^{p+1}} + \|v\|_{L^{q+1}}) \]
\[ \frac{1}{2} - c_1 \|z\|^2 - c_2 C_2 (\alpha_k^{p+1} \|z\|^{p+1} + \alpha_k^{q+1} \|z\|^{q+1}) \] (62)

where \( c_1 \) and \( c_2 \) are two constants. Choosing \( \varepsilon = \frac{1}{4c_1} \), for \( 0 < \alpha_k \leq 1 \) and \( \|z\| \geq 1 \), we deduce from (62) that

\[ A_H(z) \geq \frac{1}{4} \|z\|^2 - 2c_2 C_2 \alpha_k^{q+1} \|z\|^{q+1}. \] (63)

Hence for \( \|z\| = r_k = (4t_1 c_2 C_2 \alpha_k^{q+1} r_k^{q+1}) \frac{1}{r_k^{q+1}} \), it holds that

\[ A_H(z) \geq \left( \frac{1}{4} - \frac{1}{2t_1} \right) (4t_1 c_2 C_2 \alpha_k^{q+1} r_k^{q+1}) \frac{1}{r_k^{q+1}}. \] (64)

Similar to the proof of Lemma 3.8 in [25] we see that \( \alpha_k \to 0 \) as \( k \to \infty \). Indeed, \( 0 < \alpha_{k+1} < \alpha_k \) implies that \( \alpha_k \to \alpha \geq 0 \) as \( k \to \infty \). For each each \( k \geq 1 \) there exists \( z_k = (u_k, v_k) \in Z_k \) such that \( \|z_k\| = 1 \) and \( \|u_k\|_{L^{p+1}} + \|v_k\|_{L^{q+1}} \geq \frac{1}{2} \alpha_k \). The definition of \( Z_k \) implies that \( z_k \) weakly converges to 0 in \( E_k \). Then by the Sobolev imbedding theorem we obtain \( z_k \to 0 \) in \( L^{p+1}(M, \Sigma M) \times L^{q+1}(M, \Sigma M) \). Therefore, by (64), relation \( (A_1) \) is proved.

Let \( z = y + w \in Y_k \) with \( y \in E_- \) and \( w \in \bigoplus_{j=1}^k \mathbb{R} e_j \). Assumptions \( (H2) \) and \( (H4) \) imply that for every \( \delta > 0 \) there exists a constant \( C_\delta > 0 \) such that

\[ H(x, u, v) \geq \delta (|u|^2 + |v|^2) - C_\delta \] (65)

and then

\[-\int_M H(x, u, v) dx \leq -\delta (|u|_{L^2}^2 + |v|_{L^2}^2) + C_\delta. \] (66)

Since \( E_- \) is orthogonal to \( E_- \) in \( L^2(M, \Sigma M \oplus \Sigma M) \), and in a finite-dimensional vector space all norms are equivalent, there exists a constant \( c_3 \) such that

\[ |u|_{L^2}^2 + |v|_{L^2}^2 \geq \|w\|_{L_x^2 \times L^2}^2 \geq c_3 \|w\|^2 \] (67)

Combining (66) with (67) yields

\[ A_H(z) \leq -\frac{1}{2} \|y\|^2 + \frac{1}{2} \|w\|^2 - \delta c_3 \|w\|^2 + C_\delta. \] (68)

Taking \( \delta > \frac{1}{2c_3} \), we obtain \( A_H(z) \to -\infty \) as \( \|z\| \to \infty \), and consequently relation \( (A_2) \) is satisfied.

5. Proof of Theorem 1.1. In this section we use a variant version of Benci-Rabinowitz’s generalized saddle point theorem in [9] to prove the existence of critical points of \( A_H \). Assume that \( E \) is a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \) and it has a splitting \( E = X \oplus Y \), where both of the subspaces \( X \) and \( Y \) can be infinite dimensional. We denote by \( \Pi_X \) the projection of \( E \) onto \( X \). Let \( \Phi \in C^1(E, \mathbb{R}) \) be a functional having the form

\[ \Phi(z) = \frac{1}{2} \langle Az, z \rangle + F(z) \]

satisfying

\[(T1): A : E \to E \text{ is a bounded, linear, selfadjoint operator}, \]
\[(T2): F' \text{ is compact}, \]
\[(T3): \text{For each real number } \theta \geq 0, \text{ the linear operator } \]
\[ B = \Pi_X \exp(\theta A) : X \to X \]

is invertible.
Let \( e^+ \in Y, \|e^+\| = 1 \). For \( \rho, R_1, R_2 > 0 \) with \( R_2 > \rho \) we set
\[
S = \{z|\|z\| = \rho, z \in Y\}
\]
and
\[
Q = \{z \in X \oplus \text{span}\{e^+\}|z = z^- + re^+, \|z^-\| \leq R_1, 0 \leq r \leq R_2\}.
\]
Denote by \( \partial Q \) the boundary of \( Q \) relative to the subspace
\[
\{z + re^+|z \in X, r \in \mathbb{R}\}.
\]
We present the following critical point theorem which is essentially own to Felmer [13].

**Theorem 5.1.** (Generalized linking theorem) Let \( \Phi \in C^1(E, \mathbb{R}) \) be a functional satisfying the Palais-Smale condition. Assume that the above (T1)-(T3) hold and there exists a constant \( \delta > 0 \) such that
(i): \( \Phi(z) \geq \delta \quad \forall z \in S \),
(ii): \( \Phi(z) \leq 0 \quad \forall z \in \partial Q \).
Then \( \Phi \) has a critical point with critical value \( c_\rho \geq \delta \).

For a proof of this theorem please refer to Theorem 3.1 of [13]. We consider a class of deformations
\[
\Gamma = \{\gamma \in C^0([0, 1] \times E, E)|\gamma \text{ satisfies } \Gamma_1, \Gamma_2 \text{ and } \Gamma_3\},
\]
where
\[
\Gamma_1: \gamma \text{ is given by } \gamma(t, z) = \exp(\theta(t, z)A)z + K(t, z). \text{ Here } \theta(t, z) : [0, 1] \times E \to \mathbb{R}^+ \text{ is continuous and } K(t, z) : [0, 1] \times E \to E \text{ is compact.}
\]
\[
\Gamma_2: \gamma(t, z) = z \quad \forall z \in \partial Q, \forall t \in [0, 1].
\]
\[
\Gamma_3: \gamma(0, t) = z \quad \forall z \in Q.
\]
Then the minimax value
\[
c_\rho = \inf_{\gamma \in \Gamma} \sup_{z \in Q} \Phi(\gamma(1, z))
\]
is a critical value which is a variational characterization of critical points given in Theorem 5.1. In order to use Theorem 5.1 to find critical points of our functional \( A_H \), taking \( E = E_s, A = D_sL, F = H, X = E_- \) and \( Y = E_+ \), by Lemma 4.1 and Lemma 4.2 we only need to verify hypothesis (T3).

**Lemma 5.2.** The operator \( B = \Pi_- \exp(\theta A) : E_- \to E_- \) is invertible.

**Proof.** For \( (u, v) \in E_s \),
\[
D_sL(u, v) = (|D|^{-2s}Du, |D|^{-2(1-s)}Du).
\]
From (32) we see \( (D_sL)^2 = \text{Id}_{E_s} \). Then by writing explicitly the exponential operator \( \exp(\theta D_sL) \) as a series and reordering the terms we obtain
\[
\exp(\theta D_sL)(u, v) = \cos(\theta)(u, v) + \sinh(\theta)D_sL(u, v).
\]
Given \( z \in E_- \) by (34) we have \( z = (u, -|D|^{-2(1-s)}Du) \) with \( u \in H^s(M, \Sigma M) \). Let
\[
(x, y) = \exp(\theta D_sL)z.
\]
We deduce from (70) that
\[
\begin{align*}
x &= (\cosh(\theta) - \sinh(\theta))u, \\
y &= (\sinh(\theta) - \cosh(\theta))|D|^{-2(1-s)}Du.
\end{align*}
\]
Set \((\xi, \zeta) = \Pi_-(x, y)\). Combining (35) with (71) yields

\[
\begin{align*}
\xi &= (\cosh(\theta) - \sinh(\theta))u, \\
\zeta &= -(\cosh(\theta) - \sinh(\theta))|D|^{2(1-s)}Du.
\end{align*}
\]

(72)

So \(Bz = e^{-\theta}z\) for each \(z \in E_-\) and thus \(B = \Pi_- \exp(\theta A) : E_- \to E_-\) is invertible.

**Proof of Theorem 1.1.** Let \(R_1, R_2, p, \delta > 0\) be as in Lemma 4.1 and Lemma 4.2. We apply Theorem 5.1 to the functional \(A_H\). The Palais-Smale condition is proved in Proposition 2. Conditions (T1) and (T2) are satisfied by the considerations made in Section 2, while Lemma 5.2 gives (T3). The geometric conditions leading to (i) and (ii) are proved in Lemma 4.1 and Lemma 4.2 respectively. Therefore, there exists \(z \in E_s\) such that \(A_H(z) = 0\) with \(A_H(z) \geq c_p > 0\), and hence \(z\) is a weak solution of (1). By Theorem 2.2, \(z = (u, v)\) is a solution of (1) such that \(u \in W^{1,(q+1)/q}(M, \Sigma M)\) and \(v \in W^{1,(p+1)/p}(M, \Sigma M)\). Hypothesis (H4) implies that \((0, 0)\) is a solution of (1) satisfying \(A_H((0, 0)) = 0\). So \(z \neq (0, 0)\) and this concludes the proof.

6. **Proof of Theorem 1.2.** In this section we use a generalized fountain theorem to prove Theorem 1.2. Let \(Y\) be a separable closed subspace of a Hilbert space \(X\) with inner product \(\langle \cdot, \cdot \rangle\) and norm \(\|\cdot\|\), and \(Z = Y^\perp = \bigoplus_{j=1}^{\infty} \mathbb{R}e_j\), where \(\{e_j\}_{j=1}^{\infty}\) is a total orthonormal sequence in \(Z\). For \(k \geq 3\) and \(\rho_k > r_k > 0\) set

\[
Y_k : = Y \oplus \left( \bigoplus_{j=1}^{k} \mathbb{R}e_j \right), \quad Z_k : = \bigoplus_{j=k}^{\infty} \mathbb{R}e_j,
\]

\[
B_k : = \{z \in Y_k \|z\| \leq \rho_k\}.
\]

On \(X\) we will use the \(\tau\)-topology introduced by Kryszewski and Szulkin [21], i.e.,

the topology is generated by the norm

\[
\|\|z\|\| = \max_{k=1}^{\infty} \left( \frac{1}{2k+1} |\langle P_- z, \theta_k \rangle|, \|P_+ z\| \right)
\]

where \(P_- : X \to Y\) and \(P_+ : X \to Z\) are the orthogonal projections, and \(\{\theta_k\}_{k=1}^{\infty}\) is an orthonormal basis of \(Y\). Let \(\Phi \in C^1(X)\), recall that \(\Phi\) is \(\tau\)-upper semicontinuous if \(z_n \tau \to z\) implies

\[
\Phi(z) \geq \lim_{n \to \infty} \Phi(z_n),
\]

while \(\nabla \Phi\) is weakly sequentially continuous if \(z_n \rightharpoonup z\) implies \(\nabla \Phi(z_n) \rightharpoonup \nabla \Phi(z)\).

**Theorem 6.1.** (Generalized fountain theorem) Let \(\Phi \in C^1(X)\) be an even functional, i.e., \(\Phi(z) = \Phi(-z)\) for all \(z \in X\). Assume \(\Phi\) is \(\tau\)-upper semicontinuous and \(\nabla \Phi\) is weakly sequentially continuous. If for every \(k \in \mathbb{N}\), there exists \(\rho_k > r_k > 0\) such that

\[
\begin{align*}
(A_1): \quad & a_k := \inf_{z \in Z_k} \Phi(z) \to \infty, \quad k \to \infty, \\
(A_2): \quad & b_k := \sup_{z \in Y_k} \Phi(z) \leq 0 \text{ and } d_k := \sup_{z \in Y_k} \Phi(z) < \infty, \\
(A_3): \quad & \Phi \text{ satisfies the Palais-Smale condition,}
\end{align*}
\]

then \(\Phi\) has an unbounded sequence of critical values.
In [8] Batkam and Colin generalized the well-known fountain theorem obtained first by Bartsch [7], and showed Theorem 6.1 in a more general case that \( \Phi \) is invariant under an admissible action of a finite group \( G \). Since the antipodal action \( Z_2 \) is a particular case, we will not repeat the proof here. The critical values given in the above theorem have the following variational characterization:
\[
c_k = \inf_{\gamma \in \Gamma_k} \sup_{z \in B_k} \Phi(\gamma(z)), \quad k = 3, 4, \ldots
\]
which satisfies \( c_k \geq a_k \), where
\[
\{ \gamma : B_k \to X | \gamma \text{ is } \tau \text{-continuous and satisfies } \gamma|_{\partial B_k} = \text{id} \text{ and } \gamma(-z) = -\gamma(z) \forall z \in B_k \text{. Every } z \in \text{int}(B_k) \text{ has a } \tau \text{-neighborhood } N_z \text{ in } Y_k \text{ such that } (\text{id} - \gamma)(N_z \cap \text{int}(B_k)) \text{ is contained in a finite-dimensional subspace of } X. \text{ Moreover, } \Phi(\gamma(z)) \leq \Phi(z) \forall z \in B_k \).
\]

Now we use Theorem 6.1 to obtain infinitely many critical points of the even functional \( A_H \) in Theorem 1.2. Take \( X = E_s, Y_k \) and \( Z_k \) as in (57) and \( \Phi = A_H \).

Since Proposition 1 implies \( \nabla A_H \) is weakly sequentially continuous, by Proposition 2 and Lemma 4.3, it remains to verify that \( A_H \) is \( \tau \)-upper semicontinuous.

**Lemma 6.2.** Assume that \( H \) is nonnegative. Then the functional \( A_H \) is \( \tau \)-upper semicontinuous.

**Proof.** Assume that \( z_n \xrightarrow{\tau} z \) and \( A_H(z_n) \geq c \). Since \( \Pi_+ z_n \to \Pi_+ z \) and \( H \) is nonnegative, \( \|\Pi_- z_n\| \to \|\Pi_- z\| \) is bounded and hence \( \Pi_- z_n \to \Pi_- z \). Then by the Sobolev embedding theorem \( z_n \to z \) in \( L^{p+1}(M, \Sigma M) \times L^{q+1}(M, \Sigma M) \) and, passing if necessary to a subsequence, \( z_n \to z \) a.e. on \( M \). Using the Fatou lemma, we obtain
\[
-A_H(z) = \frac{1}{2}\|\Pi_- z\|^2 - \frac{1}{2}\|\Pi_+ z\|^2 + \int_M H(x, z)dx \\
\leq \lim_{n \to \infty} \left[ \frac{1}{2}\|\Pi_- z_n\|^2 - \frac{1}{2}\|\Pi_+ z_n\|^2 + \int_M H(x, z_n)dx \right] \\
= \lim_{n \to \infty} -A_H(z_n) \leq -c,
\]
which imply \( A_H(z) \geq \liminf_{n \to \infty} A_H(z_n) \geq c \). This completes the proof. \( \square \)

**Proof of Theorem 1.2.** Let \( Y_k \) and \( Z_k \) be two separable closed subspaces of \( E_s \) as in (57). Let \( \rho_k > r_k > 0 \) be as in Lemma 4.3. We apply Theorem 6.1 to the even functional \( A_H \). Proposition 1 implies \( \nabla A_H \) is weakly sequentially continuous and by Lemma 6.2 \( A_H \) is \( \tau \)-upper semicontinuous. Lemma 4.3 gives the geometric conditions \( (A_1) \) and \( (A_2) \) of linking geometry, while the Palais-Smale condition \( (A_3) \) is proved in Proposition 2. Thus there exists a sequence of critical points \( \{z_k\}_{k=1}^{\infty} \subset E_s \text{ of } A_H \text{ such that } A_H(z_k) = c_k \geq a_k \to \infty \text{ as } k \to \infty. \text{ Then, by Theorem 2.2, } \{z_k = (u_k, v_k)\}_{k=1}^{\infty} \text{ is a sequence of solutions of } (1) \text{ satisfying } u_k \in W^{1,(q+1)/q}(M, \Sigma M) \text{ and } v_k \in W^{1,(p+1)/p}(M, \Sigma M). \)

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**Appendix A. Proof of Proposition 1.** Step 1. \( \mathcal{H} \) is Gâteaux differentiable. Given \( z = (u, v) \in E_s, h = (\xi, \zeta) \in E_s \text{ and } t \in (-1, 1) \setminus \{0\} \) we have, by the mean value theorem,
\[
H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x)) = H(x, u(x)+, v(x)+) - H(x, u(x), v(x))
\]
for some $\theta_j = \theta(t, x, z(x), h(x)) \in (0, 1)$, $j = 1, 2$. It follows from the condition (H3) that

$$
\left| \frac{H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x))}{t} \right| 
\leq c_1 \left( 1 + |u(x) + \theta_1\xi(x)|^p + |v(x) + \theta_2\zeta(x)|^{\frac{p(q+1)}{p+1}} \right) |\xi(x)|
+ c_1 \left( 1 + |u(x)|^{\frac{p(q+1)}{p+1}} + |v(x) + \theta_2\zeta(x)|^q \right) |\zeta(x)|
\leq c_1 \left( 1 + 2^p|u(x)|^p + 2^p|\xi(x)|^p + 2^{\frac{p(q+1)}{p+1}}|v(x)|^{\frac{p(q+1)}{p+1}} + 2^{\frac{p(q+1)}{p+1}}|\zeta(x)|^{\frac{p(q+1)}{p+1}} \right) |\xi(x)|
+ c_1 \left( 1 + |u(x)|^{\frac{p(q+1)}{p+1}} + 2^q|v(x)| + 2^q|\zeta(x)|^q \right) |\zeta(x)|.
$$

Hereafter $c_1, c_2, \ldots$, denote constants only depending on $H$ and $M$. By the Hölder’s inequality and the Sobolev embeddings theorem it is easily checked that the last two lines of the above inequalities are integrable. Using the Lebesgue Dominated Convergence Theorem we deduce

$$
\lim_{t \to 0} \frac{\mathcal{H}(x, u + t\xi, v + t\zeta) - \mathcal{H}(x, u, v)}{t} = \int_M \lim_{t \to 0} \frac{H(x, u(x) + t\xi(x), v(x) + t\zeta(x)) - H(x, u(x), v(x))}{t} \, dx
= \int_M \left( \langle H_u(x, u(x), v(x)), \xi(x) \rangle + \langle H_v(x, u(x), v(x)), \zeta(x) \rangle \right) \, dx
$$

because $H$ is $C^1$ in the fiber direction. Notice that (H3) implies

$$
\int_M |\langle H_u(x, u, v), \xi \rangle| \, dx \leq c_1 \int_M \left( 1 + |u|^p + |v|^{\frac{p(q+1)}{p+1}} \right) |\xi| \, dx,
\int_M |\langle H_v(x, u, v), \zeta \rangle| \, dx \leq c_1 \int_M \left( 1 + |u|^{\frac{p(q+1)}{p+1}} + |v|^q \right) |\zeta| \, dx.
$$

It follows from Hölder’s inequalities, (4) and Sobolev embeddings that

$$
\int_M |\langle H_u(x, u, v), \xi \rangle| \, dx \leq C \left( 1 + \|u\|_{L^p}^p + \|v\|_{L^{\frac{p(q+1)}{p+1}}}^{\frac{p(q+1)}{p+1}} \right) \|\xi\|_s,
\int_M |\langle H_v(x, u, v), \zeta \rangle| \, dx \leq C \left( 1 + \|u\|_{L^{\frac{p(q+1)}{p+1}}}^q + \|v\|_{L^{\frac{p(q+1)}{p+1}}}^q \right) \|\zeta\|_{1-s}.
$$

Hence the Gâteaux derivative $D\mathcal{H}(z) = D\mathcal{H}(u, v)$ exists and

$$
D\mathcal{H}(z) h = D\mathcal{H}(u, v)(\xi, \zeta)
= \int_M \left( \langle H_u(x, u(x), v(x)), \xi(x) \rangle + \langle H_v(x, u(x), v(x)), \zeta(x) \rangle \right) \, dx
= \int_M H_z(x, z(x)) h(x) \, dx.
$$

**Step 2.** $D\mathcal{H} : E_s \to E'_s$ is continuous and thus $\mathcal{H}$ has continuous (Fréchet) derivative $\mathcal{H}' = D\mathcal{H}$. Let

$$
\hat{r}_1 = \frac{2n}{n - 2s}, \quad \hat{r}_2 = \frac{2n}{n - 2(1 - s)}.
$$
Obviously, the above two inequalities yield

\[ |H_u(x, u, v)| \leq c_3 (1 + |u|^{r_1-1} + |v|^{2(r_1-1)/r_1}), \quad (75) \]
\[ |H_v(x, u, v)| \leq c_3 (1 + |u|^{r_2-1}/r_2 + |v|^{r_2-1}). \quad (76) \]

By the definition and the Hölder’s inequality we get

\[ \sup_{|u| \leq M/\sqrt{c}} \sup_{|v| \leq M/\sqrt{c}} \left| \left| \sum_{i=1}^{n} \frac{h_i}{r_i} \right| \right| \leq c_4 (1 + |u|^{r_1} + |v|^{r_2}). \quad (77) \]
\[ \sup_{|u| \leq M/\sqrt{c}} \sup_{|v| \leq M/\sqrt{c}} \left| \left| \frac{1}{r_1^{n-1}} \right| \right| \leq c_4 (1 + |u|^{r_1} + |v|^{r_2}). \quad (78) \]

By the Sobolev embedding we can find two constants \( C_i, i = 1, 2 \), such that

\[ \|u\|_{L^{2n/(n-2)}} \leq C_1 \|u\|_s \quad \forall u \in H^s(M, \Sigma M), \]
\[ \|v\|_{L^{2n/(n-2)(1-s)}} \leq C_2 \|v\|_{1-s} \quad \forall v \in H^{1-s}(M, \Sigma M). \]

In particular, there is a constant \( a_{r_1, r_2} > 0 \) such that for any \( z = (u, v) \in E_s = H^s(M, \Sigma M) \times H^{1-s}(M, \Sigma M) \) we have

\[ \|u\|_{L^{r_1}} + \|v\|_{L^{r_2}} \leq a_{r_1, r_2} \|z\|. \quad (79) \]

Given \( z = (u, v), h = (h_1, h_2) \in E_s \), combining (75) and (77) gives

\[ \int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} \, dx \leq c_5 \int_M (1 + |u|^{r_1} + |h_1|^{r_1} + |v|^{r_2} + |h_2|^{r_2}) \, dx \]
\[ \leq c_5 (1 + \|z\|^{r_1} + \|z\|^{r_2} + \|h\|^{r_1} + \|h\|^{r_2}). \quad (80) \]

Similarly, from (76) and (77) we arrive at

\[ \int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} \, dx \leq c_6 (1 + \|z\|^{r_1} + \|z\|^{r_2} + \|h\|^{r_1} + \|h\|^{r_2}). \quad (81) \]

By the definition and the Hölder’s inequality we get

\[ \|D\mathcal{H}(z + h) - D\mathcal{H}(z)\|_{E_s} = \sup_{\|g\| \leq 1} |D\mathcal{H}(z + h) - D\mathcal{H}(z), g| \]

\[ = \sup_{\|g\| \leq 1} \left[ \int_M (|H_u(x, z(x) + h(x)) - H_u(x, z(x))| |g_1(x)| + |H_v(x, z(x) + h(x)) - H_v(x, z(x))| |g_2(x)|) \, dx \right] \]
\[ \leq \sup_{\|g\| \leq 1} \left[ \|g_1\|_{L^{r_1}} \times \left( \int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} \, dx \right)^{r_1-1/r_1} \right] \]
\[ + \|g_2\|_{L^{r_2}} \times \left( \int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_2/(r_2-1)} \, dx \right)^{r_2-1/r_2} \]
\[ \leq a_{r_1, r_2} \left[ \left( \int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} \, dx \right)^{r_1-1/r_1} \right] \]
\[ + \left( \int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} \, dx \right)^{r_2-1/r_2}. \quad (82) \]
where we have used (77) in the last inequality. We deduce from (77) and (80) that the Nemytski map
\[ N_{H_u} : L^{r_1}(M, \Sigma M) \times L^{r_2}(M, \Sigma M) \to L^{r_1/(r_1-1)}(M, \Sigma M), \quad h \mapsto H_u(\cdot, h(\cdot)) \]
is continuous. Similarly, (78) and (81) imply the Nemytski map
\[ N_{H_v} : L^{r_1}(M, \Sigma M) \times L^{r_2}(M, \Sigma M) \to L^{r_2/(r_2-1)}(M, \Sigma M), \quad h \mapsto H_v(\cdot, h(\cdot)) \]
is also continuous. Then by the Sobolev embedding we have
\[
\int_M |H_u(x, z(x) + h(x)) - H_u(x, z(x))|^{r_1/(r_1-1)} dx \to 0, \quad (83)
\]
\[
\int_M |H_v(x, z(x) + h(x)) - H_v(x, z(x))|^{r_2/(r_2-1)} dx \to 0 \quad (84)
\]
as \( \|h\| \to 0 \). Combining (82), (83) and (84) yields
\[
\|DH(z + h) - DH(z)\|_{L(E_r, E_{r'})} \to 0 \quad \text{as} \quad \|h\| \to 0.
\]
**Step 3.** \( \mathcal{H}' \) is a compact map. Suppose that \( \{z_k\}_{k=1}^\infty \subset E_s \) is bounded. Passing to a subsequence, one may assume that \( z_k \) converges weakly in \( E_s \) to \( z = (u, v) \). By the Rellich embedding theorem and the proof of the continuousness of \( \mathcal{H}' = DH \) we obtain that \( \mathcal{H}'(z_k) \to \mathcal{H}'(z) \) as \( k \to \infty \). The desired result is proved. \( \square \)

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