THE ALGEBRAIC THEORY OF THE FUNDAMENTAL GERMS

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ABSTRACT. This paper introduces a notion of fundamental group appropriate for laminations.

INTRODUCTION

Let \( \mathcal{L} \) be a lamination: a space modeled on a “deck of cards” \( \mathbb{R}^n \times T \), where \( T \) is a topological space and overlap homeomorphisms take cards to cards continuously in the deck direction \( T \). One thinks of \( \mathcal{L} \) as a family of manifolds, the leaves, bound by a transversal topology prescribed locally by \( T \). Using this picture, many constructions familiar to the theory of manifolds can be extended to laminations via the ansatz:

Replace manifold object \( A \) by a family of manifold objects \( \{ A_L \} \) existing on the leaves of \( \mathcal{L} \) and respecting the transverse topology.

For example, one defines a smooth structure to be a family of smooth structures on the leaves in which the card gluing homeomorphisms occurring in a pair of overlapping decks vary transversally in the smooth topology. Continuing in this way, constructions over \( \mathbb{R} \), such as tensors, de Rham cohomology groups, etc. may be defined.

Identifying those constructions classically defined over \( \mathbb{Z} \) is not as straightforward, especially if one wishes to follow tradition and define them geometrically. To see why this is true, consider the case of an exceptionally well-behaved lamination: an inverse limit \( \hat{M} = \lim_{\leftarrow} M_\alpha \) of manifolds by covering maps. Such a system induces a direct limit of de Rham cohomology groups, and there is a canonical map from this limit into the tangential cohomology groups \( H^*(\hat{M}, \mathbb{R}) \) with dense image. In fact, here one may use the system to define – by completion of limits – tangential homology groups \( H_*(\hat{M}, \mathbb{R}) \) as well. If one endeavors to use this point of view to define the groups \( \pi_1, H_*(\cdot, \mathbb{Z}), H^*(\cdot, \mathbb{Z}) \), the result is failure since the systems they induce have trivial limits. The purpose of this paper is to introduce for certain classes of laminations \( \mathcal{L} \) a construction \([\pi_1](\mathcal{L}, x)\) called the fundamental germ, a generalization of \( \pi_1 \) which represents an attempt to address this omission in the theory of laminations.

The intuition which guides the construction is that of the lamination as irrational manifold. Recall that for a pointed manifold \( (M,x) \), the deck group of the universal cover \( (\hat{M},\hat{x}) \to (M,x) \) – which may be identified with \( \pi_1(M,x) \) – reveals through its action how to make identifications within \( (\hat{M},\hat{x}) \) so as to recover \( (M,x) \) by quotient. Let us imagine that we have disturbed the process of identifying \( \pi_1 \) orbits, so that instead, points in an orbit merely approximate one another through some auxiliary transversal space \( T \). The result is that \( (\hat{M},\hat{x}) \) does not produce a quotient manifold but rather coils upon itself, perhaps forming a leaf \( (L,x) \) of a lamination \( \mathcal{L} \). The germ of the transversal \( T \) about \( x \) may
be interpreted as the failed attempt of \((L,x)\) to form an identification topology at \(x\). The fundamental germ \([\pi]_1(L,x)\) is then a device which records algebraically the dynamics of \((L,x)\) as it approaches \(x\) through the topology of \(T\). See Figure 1.

\[
\tilde{M}
\]

**Figure 1.** The Lamination as Irrational Manifold

One might define an element of \([\pi]_1(L,x)\) as a tail equivalence class of a sequence of approaches \(\{x_\alpha\}\), where \(L \ni x_\alpha \rightarrow x\) through \(T\). In this paper, the laminations under
consideration (see §2) have the property that there is a group \( G \) acting on \( L \) in such a way that every approach is of the form \( \{g_\alpha x\} \), for \( g_\alpha \in G \). We then define \( \{\pi_1(\mathcal{L}, x)\} \) as the set of tail equivalence classes of sequences of the form \( \{g_\alpha h_{\alpha}^{-1}\} \), where \( g_\alpha x, h_{\alpha}x \to x \) in \( \mathcal{T} \). A groupoid structure on \( \{\pi_1(\mathcal{L}, x)\} \) is defined by component-wise multiplication of sequences, and \( \pi_1(L, x) \) is contained in \( \{\pi_1(\mathcal{L}, x)\} \) as a subgroup. In practice, \( \{\pi_1(\mathcal{L}, x)\} \) has no additional structure; but for many reasonably well-behaved laminations such as inverse limit solenoids, Sullivan solenoids and linear foliations of tori, it is a group. See §§3–7 for definitions and examples.

When \( \mathcal{L} = M \) is a manifold (a lamination with one leaf), \( \{\pi_1(M, x)\} \) is equal to \( \pi_1(M, x) \), the nonstandard version of \( \pi_1(M, x) \): the group of tail equivalence classes of all sequences in \( \pi_1(M, x) \). When \( \mathcal{L} \) is a lamination contained in a manifold \( M \), under certain circumstances, §7, there is a map \( \{\pi_1(\mathcal{L}, x)\} \rightarrow \pi_1(M, x) \) whose image consists of those classes of sequences in \( \pi_1(M, x) \) that correspond to the holonomy of \( \mathcal{L} \). Thus, in expanding \( \pi_1 \) to its nonstandard counterpart, it is possible to detect – algebraically – sublaminations invisible to \( \pi_1 \).

One can profitably think of \( \{\pi_1(\mathcal{L}, x)\} \) as made from sequences of “G-diophantine approximations”. In the case of an irrational foliation \( \mathcal{F} \) of the torus \( \mathbb{T}^2 \) by lines of slope \( r \in \mathbb{R} \setminus \mathbb{Q} \), §4.4, this is literally true: the elements of \( \{\pi_1(\mathcal{F}, x)\} \) are the equivalence classes of diophantine approximations of \( r \). More generally, in \( \{\pi_1\} \) one finds an algebraic-topological tool which enables systematic translation of the geometry of laminations into the algebra of (non-linear) diophantine approximation.

One can extend the definition of the fundamental germ to include accumulations of \( L \) on points of other leaves. Thus if \( \hat{x} \) is any point of \( \mathcal{L} \), we define \( \{\pi_1(\mathcal{L}, x, \hat{x})\} \) as the set of classes of sequences of the form \( \{g_\alpha \cdot h_{\alpha}^{-1}\} \) where \( g_\alpha x, h_{\alpha}x \to \hat{x} \). We suspect that, together with the topological invariants of the leaves, the fundamental germs \( \{\pi_1(\mathcal{L}, x, \hat{x})\} \) will play a central role in the topological classification of laminations.

By unwrapping the accumulations of \( L \) implied by the fundamental germ \( \{\pi_1(\mathcal{L}, x)\} \), one obtains the germ universal cover \( \{\mathcal{L}\} \), §9, which is a kind of nonstandard completion of \( \mathcal{L} \). If \( \{\pi_1(\mathcal{L}, x)\} \) is a group, then under certain circumstances one may associate lamination coverings \( \mathcal{L}_G \) to every conjugacy class of subgroup \( \mathcal{C} \subset \{\pi_1(\mathcal{L}, x)\} \), and when \( \mathcal{C} \) is a normal subgroup, the quotient \( \{\pi_1(\mathcal{L}, x)/\mathcal{C}\} \) can be identified with the automorphism group of \( \mathcal{L}_G \to \mathcal{L} \). These considerations give rise to the beginnings of a Galois theory of laminations, §10.

This first paper on the fundamental germ is foundational in nature. One should not expect to find in it hard theorems, but rather the description of a complex and mysterious object which reveals the explicit connection between the geometry of laminations and the algebra of diophantine approximation. Due to its somewhat elaborate construction, we shall confine ourselves here to the following themes:

- Basic definitions: §§1–3.
- Examples: §§4–7.
- Functoriality: §8.
- Covering space theory: §§9,10.

The focus will be on laminations which arise through group actions: suspensions, quasi-suspensions, double coset foliations and locally-free Lie group actions. The exposition will be characterized by a careful exploration of a number of concrete examples which serve not only to illustrate the definitions in action but also to indicate the richness of the algebra they produce. In a second installment [5], to appear elsewhere, the construction of \( \{\pi_1\} \) will be extended to any lamination whose leaves admit a smooth structure.
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1. Nonstandard Algebra

We review facts concerning nonstandard algebra, proofs of which may be found in the literature. References: [7], [11].

Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \mathfrak{U} \subset 2^\mathbb{N} \) an ultrafilter all of whose elements have infinite cardinality. Given \( \mathcal{S} = \{S_i\} \) a sequence of sets, write \( S_X = \prod_{i \in \mathbb{X}} S_i \). The ultraproduct is the direct limit

\[
[S_i] := \lim_{\rightarrow} S_X,
\]

where the system maps are the cartesian projections. If \( S_i = S \) for all \( i \), the ultraproduct is called the ultrapower of \( S \), denoted \( {}^*S \).

If \( \mathcal{S} \) consists of nested sets, denote by \( {}_\text{\bigcirc} \mathcal{S} \) the set of sequences which converge with respect to \( \mathcal{S} \). For each \( X \in \mathfrak{U} \), define a map \( P_X : {}_\text{\bigcirc} \mathcal{S} \to {}_\text{\bigcirc} \mathcal{S} \) by restriction of indices: \( P_X(\{x_\alpha\}) = \{x_\alpha\}_{\alpha \in X} \). The ultrascope is the direct limit

\[
\bigoplus S_i := \lim_{\rightarrow} {}_\text{\bigcirc} \mathcal{S}_X.
\]

There is a canonical inclusion \( [S_i] \to \bigoplus S_i \), and when \( S_i = S \) for all \( i \), the ultrascope coincides with the ultrapower. In general, we have \( \bigoplus S_i = \bigcap {}^*S_i \supseteq {}^* (\bigcap S_i) \), where the inclusion is an equality if and only if \( S_i \) is eventually equal to a fixed set.

If \( \mathcal{S} \) is a (nested) sequence of groups or rings, the induced component-wise operations on sequences descend to operations making the ultraproduct (the ultrascope) a group or ring. This is also true if \( \mathcal{S} \) is a (nested) sequence of fields: we remark here that the maximality property of ultrafilters is required to rule out zero divisors.

If one uses a different ultrafilter \( \mathfrak{U}' \) and if \( \mathcal{S} \) is a (nested) sequence of groups, rings or fields, then assuming the continuum hypothesis, it is classical [2] that the resulting ultraproduct is isomorphic to that formed from \( \mathfrak{U} \). The same can shown for the ultrascope, however we shall not pursue this point here.

The ultrapower \( {}^*\mathbb{R} \) is called nonstandard \( \mathbb{R} \). There is a canonical embedding \( \mathbb{R} \hookrightarrow {}^*\mathbb{R} \) given by the constant sequences, and we will not distinguish between \( \mathbb{R} \) and its image in \( {}^*\mathbb{R} \). For \( {}^*x, {}^*y \in {}^*\mathbb{R} \), we write \( {}^*x < {}^*y \) if there exists \( X \in \mathfrak{U} \) and representative sequences \( \{x_i\}, \{y_i\} \) such that \( x_i < y_i \) for all \( i \in X \). The non-negative nonstandard reals are defined \( {}^*\mathbb{R}_+ = \{ {}^*x \in {}^*\mathbb{R} : {}^*x \geq 0 \} \). The Euclidean norm \( |\cdot| \) on \( \mathbb{R} \) extends to a \( {}^*\mathbb{R}_+ \)-valued norm on \( {}^*\mathbb{R} \). An element \( {}^*x \) of \( {}^*\mathbb{R} \) is called infinite if for all \( r \in \mathbb{R} \), \( |{}^*x| > r \), otherwise \( {}^*x \) is called finite. \( {}^*\mathbb{R} \) is a totally-ordered, non-archimedean field.

Here are two topologies that we may give \( {}^*\mathbb{R} \):

- The enlargement topology \( {}^*\tau \), generated by sets of the form \( {}^*A \), where \( A \subset \mathbb{R} \) is open. \( {}^*\tau \) is 2\text{nd}-countable but not Hausdorff.
- The internal topology \( [\tau] \), generated by sets of the form \( [A_i] \) where \( A_i \subset \mathbb{R} \) is open for all \( i \). \( [\tau] \) is Hausdorff but not 2\text{nd}-countable.

We have \( {}^*\tau \subseteq [\tau] \), the inclusion being strict. It is not difficult to see that \( [\tau] \) is just the order topology.

**Proposition 1.** \( ({}^*\mathbb{R}, [\tau]) \) is a real, infinite dimensional topological vector space.
We note however that $^*\mathbb{R}$ is not a topological group with respect to $^*\tau$. Let $^*\mathbb{R}_{\text{fin}}$ be the set of finite elements of $^*\mathbb{R}$.

**Proposition 2.** $^*\mathbb{R}_{\text{fin}}$ is a topological subring of $^*\mathbb{R}$ with respect to both the $^*\tau$ and $[^\tau]$ topologies.

The set of infinitesimals is defined $^*\mathbb{R}_e = \{^*\varepsilon \mid |^*\varepsilon| < M \text{ for all } M \in \mathbb{R}_+\}$, a vector subspace of $^*\mathbb{R}$. If $^*x - ^*y \in ^*\mathbb{R}_e$, we write $^*x \simeq ^*y$ and say that $^*x$ is infinitesimal to $^*y$.

**Proposition 3.** $^*\mathbb{R}_{\text{fin}}$ is a local ring with maximal ideal $^*\mathbb{R}_e$ and

$$^*\mathbb{R}_{\text{fin}}/^*\mathbb{R}_e \cong \mathbb{R},$$

a homeomorphism with respect to the quotient $^*\tau$-topology.

We note that $^*\mathbb{R}_e$ is clopen in the $[^\tau]$-topology; the quotient $[^\tau]$-topology on $^*\mathbb{R}_{\text{fin}}/^*\mathbb{R}_e$ is therefore discrete. $^*\mathbb{R}_e$ is not an ideal in $^*\mathbb{R}$. The vector space

$$^*\mathbb{R} := ^*\mathbb{R}/^*\mathbb{R}_e,$$

equipped with the quotient $^*\tau$-topology, is called the extended reals. By Proposition 3, $^*\mathbb{R}$ contains a subfield isomorphic to $^*\mathbb{R}_{\text{fin}}$.

The results above show that neither topology $^*\tau$ or $[^\tau]$ can claim to be preferred. The lack of a canonical topology on $^*\mathbb{R}$ is a theme we will encounter again in §9, where we will see that $^*\mathbb{R}$ may be viewed as the universal cover of a host of 1-dimensional laminations, each one providing a different topology to $^*\mathbb{R}$ (and by pull-back to $^*\mathbb{R}$).

Now let $^*\mathcal{G}$ be any complete topological group. Some of the properties satisfied by $^*\mathbb{R}$ also hold for $^*^\mathcal{G}$. If $\tau$ denotes the topology of $^*\mathcal{G}$, then the topologies $^*\tau$ and $[^\tau]$ are defined exactly as above. $^*^\mathcal{G}$ is a topological group in the $[^\tau]$ topology, but not in the $^*\tau$ topology. Denote by $^*^\mathcal{G}_e$ the classes of sequences converging to the unit element 1. $^*^\mathcal{G}_e$ is a group since a product of sequences converging to 1 in a topological group is again a sequence converging to 1. Let $^*^\mathcal{G}_{\text{fin}}$ be the subset of $^*^\mathcal{G}$ all of whose elements are represented by sequences which converge to an element of $^*^\mathcal{G}$. We have the following analogue of Proposition 3.

**Proposition 4.** $^*^\mathcal{G}_e$ is a normal subgroup of $^*^\mathcal{G}_{\text{fin}}$ and

$$^*^\mathcal{G}_{\text{fin}}/^*^\mathcal{G}_e \cong ^*^\mathcal{G},$$

a homeomorphism with respect to the quotient $^*\tau$-topology.

The left coset space

$$^*^\mathcal{G} := ^*^\mathcal{G}/^*^\mathcal{G}_e,$$

with the quotient $^*\tau$-topology, is called the extended $^*^\mathcal{G}$. It contains $^*^\mathcal{G}$ as a topological subgroup. If $^*^\mathcal{G}$ is compact or abelian, then $^*^\mathcal{G}$ is a group, though in general it need not be. We will avail ourselves of its natural structure as a $^*^\mathcal{G}$-set with respect to the left multiplication action.

2. Laminations Associated to Group Actions

The laminations for which we shall define the fundamental germ arise from actions of groups: we review them here as a way of fixing notation. References: [1], [6], [9].

Let us begin by reviewing the definitions and terminology surrounding the concept of a lamination. A deck of cards is a product $\mathbb{R}^n \times T$, where $T$ is a topological space. A card is a subset of the form $C = O \times \{t\}$, where $O \subset \mathbb{R}^n$ is open and $t \in T$. A lamination of dimension $n$ is a space $\mathcal{L}$ equipped with a maximal atlas $\mathcal{A} = \{\phi_\alpha\}$ consisting of charts
with range in a fixed deck of cards \( \mathbb{R}^n \times T \), such that each transition homeomorphism \( \phi_{\alpha\beta} = \phi_{\beta} \circ \phi_{\alpha}^{-1} \) satisfies the following conditions:

1. For every card \( C \in \text{Dom}(\phi_{\alpha\beta}) \), \( \phi_{\alpha\beta}(C) \) is a card.
2. The family of homeomorphisms \( \{ \phi_{\alpha\beta}(\cdot, t) \} \) is continuous in \( t \).

If \( T \) is totally disconnected, we say that \( \mathcal{L} \) is a solenoid.

An open (closed) transversal in \( \mathcal{L} \) is a subset of the form \( \phi_{\alpha}^{-1}(\{x\} \times T') \) where \( T' \) is open (closed) in \( T \). Note that an open (closed) transversal need not be open (closed) in \( \mathcal{L} \) i.e. if \( \mathcal{L} \) is a manifold (viewed as a trivial lamination) then every point is an open transversal. An open (closed) flow box is a subset of the form \( \phi_{\alpha}^{-1}(O \times T') \), where \( O \) is open and \( T' \subset T \) is open (closed). A plaque in \( \mathcal{L} \) is a subset of the form \( \phi_{\alpha}^{-1}(C) \) for \( C \) a card in the deck \( \mathbb{R}^n \times T \). A leaf \( L \subset \mathcal{L} \) is a maximal continuation of overlapping plaques in \( \mathcal{L} \). Note that \( \mathcal{L} \) is the disjoint union of its leaves; we denote by \( L_x \) the leaf containing the point \( x \). A lamination is weakly minimal if it has a dense leaf; it is minimal if all of its leaves are dense. A transversal which meets every leaf is called complete. Unless we say otherwise, all transversals in this paper will be complete and open. Two laminations \( \mathcal{L} \) and \( \mathcal{L}' \) are said to be homeomorphic if there is a homeomorphism \( f : \mathcal{L} \to \mathcal{L}' \) mapping leaves homeomorphically onto leaves and transversals homeomorphically onto transversals.

2.1. Suspensions. Let \( B \) be a manifold in which \( \pi_1 B \) acts without fixed points. Let \( F \) be a topological space and \( \rho : \pi_1 B \to \text{Homeo}(F) \) a representation. The suspension of \( \rho \) is the space

\[
\mathcal{L}_\rho = \frac{\tilde{B} \times F}{\alpha \cdot (x, t) = (\alpha \cdot \tilde{x}, \rho_{\alpha}(t))}
\]

defined by quoting \( \tilde{B} \times F \) by the diagonal action of \( \pi_1 B \). \( \alpha \cdot (\tilde{x}, t) = (\alpha \cdot \tilde{x}, \rho_{\alpha}(t)) \). The suspension is a fiber bundle over \( B \) with model fiber \( F \). If \( F = \emptyset \) is a topological group and \( \phi : \pi_1 B \to \emptyset \) a homomorphism, then the representation \( \rho : \pi_1 B \to \text{Homeo}(\emptyset) \) defined \( \rho_{\emptyset}(g) = g \cdot \phi(g^{-1}) \) gives rise to what we call an \( \emptyset \)-suspension, denoted \( \mathcal{L}_{\emptyset} \), a principle \( \emptyset \)-bundle over \( B \).

The action of \( \pi_1 B \) used to define \( \mathcal{L}_\rho \) is properly discontinuous and leaf preserving, hence \( \mathcal{L}_\rho \) is a lamination modeled on the deck of cards \( \tilde{B} \times F \). If \( K = \text{ker}(\rho) \) and \( (L, x) \subset \mathcal{L}_\rho \) is a pointed leaf, we have \( K \leq \pi_1(L, x) \). \( \mathcal{L}_\rho \) is minimal (weakly-minimal) if and only if every (at least one) \( \rho(\pi_1 B) \) orbit is dense.

The restriction \( p|_L \) of the projection \( p : \mathcal{L}_\rho \to B \) to a leaf \( L \) is a covering map. Suppose that \( p_L \) is a Galois covering (we say that \( L \) is Galois). The deck group \( D_L \) of \( p|_L \) has the property that

\[
D_L \cdot x = L \cap F_x,
\]

where \( F_x \) is the fiber of \( p \) through \( x \). In particular, if we give \( (L \cap F_x) \subset F_x \) the subspace topology, we have an inclusion

\[
D_L \hookrightarrow \text{Homeo}(L \cap F_x).
\]

A manifold \( B \) is a suspension with \( F \) a point and \( \rho : \pi_1 B \to F \) trivial. The following subsections discuss examples which are more interesting.

2.1.1. Inverse Limit Solenoids. Let \( \mathcal{C} = \{ \rho_\alpha : M_\alpha \to M \} \) be an inverse system of pointed manifolds and Galois covering maps with initial object \( M \); denote by

\[
\tilde{M} = \tilde{M}_\mathcal{C} := \lim_{\leftarrow} M_\alpha
\]

the limit. By definition \( \tilde{M} \subset \prod M_\alpha \), so elements of \( \tilde{M} \) are denoted \( \tilde{x} = (x_\alpha) \), where \( x_\alpha \in M_\alpha \). The natural projection onto the base surface is denoted \( p : \tilde{M} \to M \). We may identify
the universal covers \( \tilde{M}_\alpha \) with \( \tilde{M} \) and choose the universal covering maps \( \tilde{M} \to M_\alpha \) to be compatible with the system \( \mathcal{C} \). By universality, there exists a canonical map \( i : \tilde{M} \to \tilde{M} \).

Let \( H_\alpha = (\rho_\alpha)_* (\pi_1 M_\alpha) < \pi_1 M \). Associated to \( \mathcal{C} \) is the inverse limit of deck groups

\[
\hat{\pi}_1 M := \varprojlim \pi_1 M / H_\alpha,
\]

a Cantor group since the \( \pi_1 M / H_\alpha \) are finite. By universality of inverse limits, the projections \( \pi_1 M / H_\alpha \to \hat{\pi}_1 M / \hat{\pi}_1 M \) yield a canonical homomorphism \( i : \hat{\pi}_1 M \to \hat{\pi}_1 M \) with dense image. The closures of the images \( i(H_\alpha) \) are clopen, and give a neighborhood basis about 1. Let \( \mathcal{L}_1 \) be the associated \( \hat{\pi}_1 M \)-suspension.

**Proposition 5.** \( \hat{M} \) is homeomorphic to \( \mathcal{L}_1 \). In particular, \( \hat{M} \) is a solenoid.

**Proof.** Let \( \Upsilon : \hat{M} \times \hat{\pi}_1 M \to \hat{M} \) be the map defined \( (\hat{x}, \hat{g}) \to \hat{g} \cdot i(\hat{x}) \). \( \Upsilon \) is invariant with respect to the diagonal action of \( \pi_1 M \), and descends to a homeomorphism \( \hat{M} \times \rho \hat{\pi}_1 M \to \hat{M} \).\( \square \)

2.1.2. **Linear Foliation of Torii.** Let \( V \) be a \( p \)-dimensional subspace of \( \mathbb{R}^{p+q} \). Denote by \( \tilde{\mathcal{F}}_V \) the foliation of \( \mathbb{R}^{p+q} \) by cosets \( v + V \). The image \( \mathcal{F}_V \) of \( \tilde{\mathcal{F}}_V \) in the torus \( \mathbb{T}^{p+q} = \mathbb{R}^{p+q} / \mathbb{Z}^{p+q} \) gives a foliation of the latter by Euclidean manifolds. \( V \) may be regarded as the graph of a \( q \times p \) matrix map

\[
\mathbb{R} : \mathbb{R}^p \to \mathbb{R}^q
\]

whose columns are independent. For \( y \in \mathbb{R}^q \), denote by \( \overline{y} \) its image in \( \mathbb{T}^q \). Let \( \varphi_\mathbb{R} : \mathbb{Z}^p \to \mathbb{T}^q \) be the homomorphism defined

\[
\varphi_\mathbb{R}(n) = \overline{\mathbb{R}n},
\]

and denote by \( \mathcal{L}_{\varphi_\mathbb{R}} \) the corresponding \( \mathbb{T}^q \)-suspension.

**Proposition 6.** \( \mathcal{F}_V \) is homeomorphic to \( \mathcal{L}_{\varphi_\mathbb{R}} \).

**Proof.** Let \( P_0 : \mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{T}^q \) be the map defined \( (x, y) \to (x, \overline{y} - \mathbb{R}x) \). Let \( P \) be the composition of \( P_0 \) with the projection \( \xi : \mathbb{R}^p \times \mathbb{T}^q \to \mathcal{L}_{\varphi_\mathbb{R}} \). Then \( P \) is a covering homomorphism with kernel \( \mathbb{Z}^{p+q} \), hence \( \mathcal{L}_{\varphi_\mathbb{R}} \approx \mathbb{T}^{p+q} \). Since \( V = (x, \mathbb{R}x) \), we have \( P(V) = \xi (\mathbb{R}^p \times \mathbb{R}) \); thus \( P(V) \) is a leaf of the suspension. It follows that \( P \) defines a map \( \tilde{\mathcal{F}}_V \to \mathcal{L}_{\varphi_\mathbb{R}} \) which descends to the desired homeomorphism.\( \square \)

Let \( r_i \) be the \( i \)th column vector of \( \mathbb{R} \). If \( r_i \in \mathbb{Q}^q \), for all \( i \), the leaves of \( \tilde{\mathcal{F}}_V \) are homeomorphic to \( \mathbb{T}^p \) and are not dense. If at least one of the \( r_i \) has an irrational coordinate, then the leaves of \( \tilde{\mathcal{F}}_V \) are non-compact and dense, homeomorphic to the quotient of \( \mathbb{R}^p \) by a discrete subgroup with as many generators as rational \( r_i \).

2.1.3. **Anosov Foliation.** Let \( \Sigma = \mathbb{H}^2 / \Gamma \) be a hyperbolic surface and let \( \rho : \Gamma \to \text{Homeo}(\mathbb{S}^1) \) be defined by extending the action of \( \Gamma \) on \( \mathbb{H}^2 \) to \( \partial \mathbb{H}^2 \approx \mathbb{S}^1 \). The suspension

\[
\mathcal{F}_\Gamma = \mathbb{H}^2 \times_\rho \mathbb{S}^1
\]

is called an **Anosov foliation**. Note that \( \mathcal{F}_\Gamma \) is not an \( \mathbb{S}^1 \)-suspension. It is classical that the underlying space of \( \mathcal{F}_\Gamma \) is homeomorphic to the unit tangent bundle \( T^1 \Sigma \).
2.2. **Quasisuspensions.** Let \( \mathcal{L}_p = B \times_p F \) be a suspension over a manifold \( B \). We say that \( \mathcal{L}_p \) is Galois if every leaf of \( \mathcal{L}_p \) is Galois. Throughout this section, \( \mathcal{L}_p \) will be a Galois suspension. We define an action of \( \pi_1 B \) on \( \mathcal{L}_p \) by

\[
\gamma \cdot x \mapsto \tilde{\gamma} \cdot x,
\]

where, for \( x \) contained in the leaf \( L \), \( \tilde{\gamma} \) is the image of \( \gamma \in \pi_1 B \) in \( \pi_1 B /\langle p_L \rangle, \langle \pi_1 L \rangle \cong D_L \) is the deck group of \( p|_L \).

Let \( \mathcal{X} \subset \mathcal{L}_p \) be any closed subset which is invariant with respect to the action of \( \pi_1 B \). Let \( \mathcal{L}_0 := \mathcal{L}_p \setminus \mathcal{X} \), which is a lamination mapping to \( B \). If \( \mathcal{L}_p \) is minimal, then \( \mathcal{X} \) is the preimage of a subset \( X \subset B \), hence \( \mathcal{L}_0 \) is a fiber bundle over \( B_0 = B \setminus X \). In general, we shall define the fibers of \( \mathcal{L}_0 \) over \( x \in B \) to be the preimages of the map \( \mathcal{L}_0 \to B \).

A lamination homeomorphism \( f : \mathcal{L}_0 \to \mathcal{L}_0 \) is weakly fiber-preserving if for every fiber \( F_x \) over \( B \),

\[
f(F_x) = \bigcup_{i=1}^n E_{x_i},
\]

where \( E_{x_i} \subset F_x \) denotes a subset of the fiber \( F_x \). The collection \( \text{Homeo}_{\omega \rightarrow \text{fib}}(\mathcal{L}_0) \) of weakly fiber-preserving homeomorphisms is clearly a group. Since the fibers are disjoint, each \( E_{x_i} \) occurring in (1) must be open in \( F_x \). In particular, if the fibers are connected, a weakly fiber-preserving homeomorphism is fiber-preserving. Thus, the concept of a weakly fiber-preserving homeomorphism differs from that of a fiber-preserving homeomorphism when the fibers are disconnected e.g. when \( \mathcal{L}_0 \) is a solenoid.

**Definition 1.** Let \( \mathcal{L}_0 \) be as above and suppose \( H < \text{Homeo}_{\omega \rightarrow \text{fib}}(\mathcal{L}_0) \) is a subgroup acting properly discontinuously on \( \mathcal{L}_0 \). The quotient

\[
\mathcal{L} = H \setminus \mathcal{L}_0
\]

is a lamination called a **quasisuspension** (over \( B \)).

We consider now two examples.

2.2.1. **The Sullivan Solenoid.** The following important example comes from holomorphic dynamics. Let \( U, \mathcal{V} \subset \mathbb{C} \) be regions conformal to the unit disc, with \( \mathcal{V} \subset \mathcal{V} \). Recall that a **polynomial-like map** is a proper conformal map \( f : U \to \mathcal{V} \). The conjugacy class of \( f \) is uniquely determined by a pair \((p, \partial f)\), where \( p \) is a complex polynomial of degree \( d \) and \( \partial f : \mathbb{S}^1 \to \mathbb{S}^1 \) is a smooth, expanding map of degree \( d \). The space

\[
\hat{\mathbb{S}} = \lim_{\leftarrow} \left( \mathbb{S}^1 \xleftarrow{\partial f} \mathbb{S}^1 \xleftarrow{\partial f} \mathbb{S}^1 \right)
\]

is an inverse limit solenoid which may be identified with the \( \hat{\mathbb{Z}}_d \)-suspension \( \mathcal{L}_i = \mathbb{R} \times_\rho \hat{\mathbb{Z}}_d \), where \( \hat{\mathbb{Z}}_d \) is the group of \( d \)-adic integers and \( \iota : \mathbb{Z} \to \hat{\mathbb{Z}}_d \) is the canonical inclusion. Every leaf of \( \hat{\mathbb{S}} \) is homeomorphic to \( \mathbb{R} \). \( \partial f \) defines a self map of the inverse system in (2), inducing a homeomorphism \( \hat{\partial f} : \hat{\mathbb{S}} \to \hat{\mathbb{S}} \).

Consider the suspension

\[
\hat{\mathbb{S}} = \mathbb{H}^2 \times_\rho \hat{\mathbb{Z}}_d
\]

obtained by extending to \( \mathbb{H}^2 \times \hat{\mathbb{Z}}_d \) the identification used to define \( \mathcal{L}_i \) e.g. \((z, \hat{n}) \sim (\gamma^n \cdot z, \rho_m(\hat{n}))\) for \( m \in \mathbb{Z} \), where

\[
\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
is the affine extension of the map $x \mapsto x + 1$ to $\mathbb{H}^2$. The base of the suspension $\hat{D}$ is the punctured hyperbolic disc $\mathbb{D}^2 = \gamma \setminus \mathbb{H}^2$, and its ideal boundary may be identified with $\mathbb{S}$.

The map $\partial \hat{f}$ extends to a weakly fiber-preserving homeomorphism $\hat{f} : \hat{D} \to \hat{D}$ which acts properly discontinuously on $\hat{D}$. The quotient

$$\hat{D}_f := \langle \hat{f} \rangle \backslash \hat{D}$$

is a quasisuspension called the *Sullivan solenoid* [12].

### 2.2.2. The Reeb Foliation

Let $\mathbb{R}_+ = [0, \infty)$, consider the trivial suspension $\mathbb{C} \times \mathbb{R}_+$ over $\mathbb{C}$, and denote $(\mathbb{C} \times \mathbb{R}_+)^* = \mathbb{C} \times \mathbb{R}_+ \setminus \{(0,0)\}$. Fix $(\mu, \lambda) \in (\mathbb{C} \times \mathbb{R}_+)^*$ with $|\mu|, \lambda > 1$, $\mu \neq \lambda$. Then multiplication by $(\mu, \lambda)$ in $(\mathbb{C} \times \mathbb{R}_+)^*$ is a fiber-preserving lamination homeomorphism giving rise to an action by $\mathbb{Z}$. The resulting quasisuspension

$$\mathcal{F}_{\text{Reeb}} = \mathbb{Z} \backslash (\mathbb{C} \times \mathbb{R}_+)^*$$

has underlying space a solid torus, and is called the *Reeb foliation*.

Let $P : (\mathbb{C} \times \mathbb{R}_+)^* \to \mathcal{F}_{\text{Reeb}}$ denote the projection map. The leaves of $\mathcal{F}_{\text{Reeb}}$ are of the form:

1. $L_\epsilon = P(\mathbb{C} \times \{t\}) \cong \mathbb{C}$, for $t > 0$.
2. $L_0 = P(\mathbb{C}^* \times \{0\}) \cong \mathbb{C}^*/\langle \mu^m \rangle$. For every leaf of $\mathcal{F}_{\text{Reeb}}$ intersects $T_0$.

The fiber traversals of $\mathcal{F}_{\text{Reeb}}$ are of the form:

1. $T_\epsilon = P(\{z\} \times \mathbb{R}_+) \approx \mathbb{R}_+, \epsilon > 0$. Every leaf of $\mathcal{F}_{\text{Reeb}}$ intersects $T_\epsilon$.
2. $T_0 = P(\{0\} \times (0, \infty)) \approx S^1$. Every leaf except $L_0$ intersects $T_0$.

There is an action of $\mathbb{Z}$ on $\mathcal{F}_{\text{Reeb}}$ induced by the map $(z, t) \mapsto (\mu^n z, t)$. For $x \in \mathcal{F}_{\text{Reeb}}$, we write this action $x \mapsto n \cdot x$. For every $t$ we have $n \cdot L_t = L_t$ and for all $z$, $n \cdot T_\epsilon = T_\epsilon$. Note that this action is the identity on $L_0$.

### 2.3. Double Coset Foliations

Let $\mathfrak{G}$ be a Lie group, $\mathfrak{H}$ a closed Lie subgroup, $\Gamma \subset \mathfrak{G}$ a discrete subgroup. The foliation of $\mathfrak{G}$ by right cosets $\mathfrak{H} g$ descends to a foliation $\mathcal{F}_{\mathfrak{G}, \Gamma}$ of $\mathfrak{G}/\Gamma$, called a *double coset foliation*.

Let $\Gamma$ be a co-finite volume Fuchsian group. Denote by $\Sigma = \mathbb{H}^2/\Gamma$, and by $T^1 \Sigma$ the unit tangent bundle of $\Sigma$. Recall that every $v \in T^1 \mathbb{H}^2$ determines three oriented, parametrized curves: a geodesic $\gamma$ and two horocycles $h_+, h_-$ tangent to, respectively, $\gamma(+\infty)$ and $\gamma(-\infty)$. By parallel translating $v$ along these curves, we obtain three flows on $T^1 \mathbb{H}^2$. The three flows are $\Gamma$-invariant, and define flows on $T^1 \Sigma$. The corresponding foliations are denoted $\text{Geod}_\Gamma$, $\text{Hor}^\pm_\Gamma$, and $\text{Hor}^+_\Gamma$.

Now let $\mathfrak{G} = \text{SL}(2, \mathbb{R})$ and take $\mathfrak{H}$ to be one of the 1-parameter subgroups $H^+ = \{A_r^+\}$, $H^- = \{A_r^-\}$ and $G = \{B_r\}$, where

$$A_r^+ = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \quad A_r^- = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad \text{and} \quad B_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$$

for $r \in \mathbb{R}$. Then it is classical that the foliations $\mathcal{F}_{\text{Geod}, \Gamma}$ and $\mathcal{F}_{\text{Hor}^\pm, \Gamma}$ are homeomorphic to $\text{Geod}^\pm_\Gamma$ and $\text{Hor}^+_\Gamma$, respectively. Note also that the Anosov foliation $\mathcal{F}_{\Gamma}$ is homeomorphic to the sum $\text{Geod}^+ \oplus \text{Hor}^+_{\Gamma}$.

### 2.4. Locally-Free Lie Group Actions

Let $\mathfrak{B}$ be a Lie group of dimension $k$, $M^n$ an $n$-manifold, $n > k$, $X$ a subspace of $M^n$. A continuous representation $\Theta : \mathfrak{B} \to \text{Homeo}(X)$ is called *locally free* if for all $x \in X$, the isotropy subgroup $I_x < \mathfrak{B}$ is discrete. If for any pair $x, y \in X$, their $\mathfrak{B}$-orbits are either disjoint or coincide, then $X$ has the structure of a lamination $\mathcal{L}_\mathfrak{B}$ whose leaves are the $\mathfrak{B}$-orbits.
For example, let $M^n$ be a Riemannian manifold. Fix a tangent vector $v \in T_x M$. Let $l \subset M^n$ be the complete geodesic determined by $v$, $X$ its closure (itself a union of geodesics). Then there is a locally free action of $R$ given by geodesic flow along $X$, and $X$ is a lamination when $l$ is simple. When $M^n = \Sigma$ is a hyperbolic surface and $l$ is simple, we obtain a geodesic lamination in $\Sigma$ in the sense of [3], a solenoid since its transversals are totally-disconnected.

3. The Fundamental Germ

Let $\mathcal{L}$ be any of the laminations considered in the previous section and let $L \subset \mathcal{L}$ be a fixed leaf. If $\mathcal{L} = H \setminus \mathcal{L}_0$ is a quasisuspension let $L_0 \subset \mathcal{L}_0$ be a leaf lying over $L$. The diophantine group $G_L$ of $\mathcal{L}$ with respect to $L$ is

- $\pi_1 B$ if $\mathcal{L}$ is a suspension.
- The group generated by $\pi_1 B$, $H_L = \{ h \in H \mid h(L_0) = L_0 \}$ and $\pi_1 L$ (viewed as groups acting on $\bar{L}$) if $\mathcal{L}$ is a quasisuspension.
- The group $\bar{B}$ if $\mathcal{L}$ is a double coset.
- The group $\bar{B}$ if $\mathcal{L}$ is a locally free group action.

Note that in every case, $\pi_1 L < G_L$.

Let $\hat{x} \in \mathcal{L}$ and $T$ a transversal containing $\hat{x}$. Denote by $\bar{T}_L \subset \bar{L}$ the set of points lying over $T \cap L$. Then $T$ is said to be a diophantine transversal if for every leaf $L$ and $\hat{x} \in \bar{T}_L$, any $\hat{y} \in \bar{T}_L$ may be written in the form $\hat{y} = g \cdot \hat{x}$ for some $g \in G_L$. For $\hat{x} \in \bar{T}_L$ fixed, we call $\{g_\alpha\} \subset G_L$ a $G_L$-diophantine approximation of $\hat{x}$ along $T$ based at $\hat{x}$ if $\{g_\alpha \cdot \hat{x}\}$ projects in $L$ to a sequence converging to $\hat{x}$ in $T$. The image of all such $G_L$-diophantine approximations in $^* G_L$ is denoted

$$^* D(\hat{x}, \hat{x}, T),$$

and when $\hat{x} = x$ we write $^* D(\hat{x}, T)$. If there are no $G_L$-diophantine approximations of $\hat{x}$ along $T$ based at $\hat{x}$, we define $^* D(\hat{x}, \hat{x}, T) = 0$. Note that if $\hat{x}' = \hat{\gamma} \cdot \hat{x}$ for $\hat{\gamma} \in \pi_1 L < G_L$ then

$$^* D(\hat{x}', \hat{x}, T) \cdot \hat{\gamma} = ^* D(\hat{x}, \hat{x}, T).$$

Let $^* D(\hat{x}, \hat{x}, T)^{-1}$ consist of the set of inverses $^* g^{-1}$ of classes belonging to $^* D(\hat{x}, \hat{x}, T)$.

**Definition 2.** Let $\mathcal{L}$, $L$, $x$, $\hat{x}$ and $T$ be as above. The fundamental germ of $\mathcal{L}$ based at $\hat{x}$ along $x$ and $T$ is

$$[\pi]_1(\mathcal{L}, x, \hat{x}, T) = ^* D(\hat{x}, \hat{x}, T) \cdot [\pi]_1(\mathcal{L}, x, T)^{-1}$$

where $\hat{x}$ is any point in $\bar{L}$ lying over $x$.

By [3], $[\pi]_1(\mathcal{L}, x, \hat{x}, T)$ does not depend on the choice of $\hat{x}$ over $x$. When $x = \hat{x} \in L$, we write $[\pi]_1(\mathcal{L}, x, x, T)$. Observe in this case that $[\pi]_1(\mathcal{L}, x, T)$ contains a subgroup isomorphic to $^* \pi_1(L, x)$.

We now describe a groupoid structure on $[\pi]_1(\mathcal{L}, x, \hat{x}, T)$. To do this, we define a unit space on which it acts: let $^* D(\hat{x}, \hat{x}, T)$ be the image of $^* D(\hat{x}, \hat{x}, T)$ in $^* G_L$, for any $\hat{x}$ over $x$. We say that $^* u \in [\pi]_1(\mathcal{L}, x, \hat{x}, T)$ is defined on $^* g \in ^* D(\hat{x}, \hat{x}, T)$ if $^* u \cdot ^* g \in ^* D(\hat{x}, \hat{x}, T)$. Here we are using the left action of $^* G_L$ on $^* G_L$. Having defined the domain and range of elements of $[\pi]_1(\mathcal{L}, x, \hat{x}, T)$, it is easy to see that $[\pi]_1(\mathcal{L}, x, \hat{x}, T)$ is a groupoid, as every element has an inverse by construction. This groupoid structure does not depend on the choice of $\hat{x}$ over $x$. 

4. The Fundamental Germ of a Suspension

In the case of a suspension $\mathcal{L}_\rho = \tilde{B} \times_\rho F$, any fiber over the base $B$ is a diophantine transversal. Conversely, any diophantine transversal is an open subset of a fiber transversal. It follows that any two diophantine transversals $T, T'$ through a given point $\hat{x}$ define the same set of $G_1$-diophantine approximations. Thus

Proposition 7. If $T$ and $T'$ are diophantine transversals through $\hat{x}$ then

$$[\pi]_1(\mathcal{L}_\rho, x, \hat{x}, T) = [\pi]_1(\mathcal{L}_\rho, x, \hat{x}, T').$$

Accordingly for suspensions we drop mention of the transversal and write $[\pi]_1(\mathcal{L}, x, \hat{x}).$

We note that since the diophantine group $G_1 = \pi_1B$ is discrete, $^*G_1 = ^*G_L$ and the unit space for the groupoid structure is just $^*D(\hat{x}, \hat{x}).$

4.1. Manifolds. A manifold is a lamination with just one leaf, which can be viewed as the suspension of the trivial representation of its fundamental group. Since a fiber transversal is just a point, we have immediately

Proposition 8. If $M$ is a manifold then

$$[\pi]_1(M, x) = ^*D(\hat{x}) = ^*\pi_1(M, x).$$

4.2. $\mathcal{G}$-Suspensions. Let $\varphi : \pi_1B \to \mathcal{G}$ be a homomorphism, $\mathcal{L}_\varphi$ the corresponding $\mathcal{G}$-suspension. Let $\{U_i\}$ be a neighborhood basis about $1$ in $\mathcal{G}$ and define a collection of nested sets $\{G_i\}$ by $G_i = \{\gamma \in \pi_1B \mid h(\gamma) \in U_i\}$. Note that the ultrascope $\bigcirc G_i$ is a subgroup of $^*\pi_1B$. In fact, if $^*\varphi : ^*\pi_1B \to ^*\mathcal{G}$ is the nonstandard version of $^*\varphi$, then

$$\bigcirc G_i = ^*\varphi^{-1}(^*\mathcal{G}_\epsilon).$$

Theorem 1. If $\varphi$ has dense image, then for any pair $x, \hat{x}$ belonging to a diophantine transversal, $[\pi]_1(\mathcal{L}_\varphi, x, \hat{x})$ is a group isomorphic to $\bigcirc G_i$.

Proof. Let $^*g \in ^*D(\hat{x}, \hat{x})$. Then any other element $^*g' \in ^*D(\hat{x}, \hat{x})$ may be written in the form $^*g \cdot ^*h$ where $^*h \in \bigcirc G_i$. It follows immediately that

$$[\pi]_1(\mathcal{L}_\varphi, x, \hat{x}) = ^*g \cdot (\bigcirc G_i) \cdot ^*g^{-1} \cong \bigcirc G_i.$$

Because the unit space $^*D(\hat{x}, \hat{x})$ is invariant under left-multiplication by its elements, it follows that $[\pi]_1(\mathcal{L}_\varphi, x, \hat{x})$ acts on it as a group, its groupoid law coinciding with multiplication in $\bigcirc G_i$. $\square$

For $\mathcal{G}$-suspensions with $\varphi$ having dense image, we can thus reduce our notation to $[\pi]_1(\mathcal{L}_\varphi)$.

Let $^*\varphi : ^*\pi_1B \to ^*\mathcal{G}$ be the induced map of nonstandard groups, and denote by $^*\pi_1B_{\text{fin}}$ the subgroup $^*\varphi^{-1}(^*\mathcal{G}_{\text{fin}})$. The following theorem can be used to display many familiar topological groups as algebraic quotients of nonstandard versions of discrete groups.

Theorem 2. If $\varphi$ has dense image, then $[\pi]_1(\mathcal{L}_\varphi)$ is a normal subgroup of $^*\pi_1B_{\text{fin}}$ with

$$^*\pi_1B_{\text{fin}}/\pi_1(\mathcal{L}_\varphi) \cong \mathcal{G}.$$

Proof. Since $\varphi$ has dense image, the composition of homomorphisms $^*\pi_1B_{\text{fin}} \to ^*\mathcal{G}_{\text{fin}} \to \mathcal{G}$ – where the first arrow is $^*\varphi$ – is surjective with kernel $^*\varphi^{-1}(^*\mathcal{G}_\epsilon) = [\pi]_1(\mathcal{L}_\varphi).$ $\square$
4.3. Inverse Limit Solenoids. Let $\hat{M}$ be an inverse limit solenoid over the base $M$, and let $\{H_i\}$ be a sequence of subgroups of $\pi_1 M$ cofinal in the collection of subgroups in the defining inverse system. By the discussion in §2.1.1, the collection of closures $\{\hat{H}_i\} \subset \hat{H}_i M$ defines a neighborhood basis about 1. Since $\hat{M}$ is a $\pi_1 M$-suspension in which $\varphi$ is dense, it follows from Theorem 1 that $[\pi_1]_1(\hat{M}, x, \hat{x})$ is a group isomorphic to $\bigodot H_i$.

For example, consider a solenoid $\hat{S}$ over $S^1$. Here, each $H_i$ is an ideal in $Z$, hence $[\pi_1]_1(\hat{S})$ is an ideal in the ring $*Z = nonstandard Z$. When $H_i = (d_i)$ for $d_i \in Z$ fixed, we denote the resulting germ $*Z_{d_i}(d)$ and when $H_i = \{i\}$ we write $*Z_{\hat{d}}$. Being uncountable, these ideals are not principal, so $*Z$, unlike $Z$, in not a PID. By Theorem 2 we have $*Z/\text{nonstandard } Z$ and $*Z/\text{nonstandard } Z_{\hat{d}}(d) \cong 2.$

4.4. Linear Foliation of Tori and Classical Diophantine Approximation. Let $\mathcal{F}_V$ be the linear foliation of $\mathbb{T}^p$ associated to the subspace $V \subset \mathbb{R}^{p+d}$. As in §2.1.2, we regard $V$ as the graph of the $q \times p$ matrix $R$. Let $\varphi_R : Z^p \to \mathbb{T}^q$ be the homomorphism used to define $\mathcal{F}_V$. Let $\{U_i\}$ be a neighborhood basis in $\mathbb{T}^q$ about $0$. We define a nested set $\{G_i\} \subset \mathbb{Z}^p$ by $n \in G_i$ if and only if $\varphi_R(n) \in U_i$. Denote

$$*Z^p_R := \bigodot G_i = *\varphi_R^{-1}(\mathbb{T}^q),$$

a subgroup of $*Z^p$. If $p = q = 1$ and $R = r \in \mathbb{R}$, we write instead $*Z_r$.

Theorem 3. If $R \notin M_{q,p}(\mathbb{Q})$, then $[\pi_1]_1(\mathcal{F}_V, x, \hat{x}) = *Z^p_R$. Otherwise,

$$[\pi_1]_1(\mathcal{F}_V, x, \hat{x}) = \begin{cases} 0 & \text{if } x \neq \hat{x} \\ *Z^p_R & \text{otherwise} \end{cases}$$

Proof. If $R \notin M_{q,p}(\mathbb{Q})$, then $\varphi_R$ has dense image and the result follows by Theorem 1. If not, then all of the leaves are torii so $*D(\hat{x}, \hat{x}) = 0$ unless $x = \hat{x}$, in which case, if $L$ is the leaf containing $x$, $*D(\hat{x}, \hat{x}) = *\pi_1 L = *Z^p_R$. 

If $R$ is a vector with at least one irrational entry, then Theorems 2 and 3 give:

Corollary 1. Every finite dimensional torus $\mathbb{T}^q$ is algebraically isomorphic to a quotient of the nonstandard integers $*Z$.

Theorem 4. $*Z^p_R$ is an ideal in $*Z^p$ if and only if $R \in M_{q,p}(\mathbb{Q})$.

Proof. Suppose that $R \in M_{q,p}(\mathbb{Q})$ and let $a_k$ be the l.c.d. of the entries of $R_k$ = the $k$th column of $R$. Write

$$a = (a_1) \oplus \cdots \oplus (a_p)$$

where $(a_k)$ is the ideal generated by $a_k$. Note that $*a \subset *Z^p_R$. On the other hand, rationality of the entries of the $R_k$ implies that a sequence $\{n_{a_k}\} \subset Z^p$ defines an element of $*Z^p_R$ if and only if there exists $X \in \mathbb{Q}$ such that $\varphi_R(n_{a_k}) = \hat{0}$ for all $\alpha \in X$. This is equivalent to $n_{a_k} = \hat{a}$ for all $\alpha \in X$. Thus $*Z^p_R = *a$ which is an ideal in $*Z^p$.

Suppose now that $R = R_k \notin \mathbb{Q}^q$ for some $k$, $1 \leq k \leq p$. Let $\{n_{a_k}\}$ represent an element $*n \subset *Z^p_R$, and denote by $\{m_{a_k}\}$ the sequence of $k$-th coordinates of the $n_{a_k}$. Note that $m_{a_k} = \hat{0}$ for all $\alpha$ since $R$ is not rational. In fact, for any $\delta > 0$ we may find a sequence of integers $\{m_{a_k}\}$ such that $m_{a_k} = \hat{0}$ and $m_{a_k} = m_{a_k}$ is not within $\delta$ of $0$. Let $m_{a_k} \in Z^p$ be the vector whose $k$th coordinate is $m_{a_k}$ and whose other coordinates are 0. Then the sequence $\{m_{a_k} \cdot n_{a_k}\}$ does not converge with respect to $\{G_i\}$ i.e. $\text{for all } \epsilon > 0$ there exists $\delta > 0$, so $*Z^p_R$ is not an ideal.

Theorem 3 draws another sharp distinction between $Z$ and $*Z$: every subgroup of the former is an ideal, while this is false for the latter.
We spend the rest of this section studying $^{*}\mathbb{Z}_R^p$, in and of itself a complicated and intriguing object. Let us begin with the following alternate description of $^{*}\mathbb{Z}_R^p$:

\[(4)\quad ^{*}\mathbb{Z}_R^p = \left\{ *n \in ^{*}\mathbb{Z}^p \mid \exists *n^\perp \in ^{*}\mathbb{Z}^q \text{ such that } R(*n) - *n^\perp \in ^{*}\mathbb{R}_d^p \right\}.
\]

Given $*n \in ^{*}\mathbb{Z}_R^p$, the corresponding element $*n^\perp \in ^{*}\mathbb{Z}^q$ is called the dual of $*n$; it is uniquely determined. From (4), it is clear that the set \[(^{*}\mathbb{Z}_R^p)^\perp := \left\{ *n^\perp \mid *n^\perp \text{ is the dual of } *n \in ^{*}\mathbb{Z}_R^p \right\}
\]
is a subgroup of $^{*}\mathbb{Z}^q$, called the dual of $^{*}\mathbb{Z}_R^p$. Note that when $R \in M_{q,p}(\mathbb{R} \setminus \mathbb{Q})$ has a left-inverse $S$, we have $(^{*}\mathbb{Z}_R^p)^\perp \cong ^{*}\mathbb{Z}_S^q$.

Similarly, the set \[^{*}\mathbb{R}_e^q := \left\{ ^{*}\epsilon \in ^{*}\mathbb{R}_e^q \mid \exists *n \in ^{*}\mathbb{Z}_R^p \text{ such that } R(*n) - *n^\perp = ^{*}\epsilon \right\}
\]
is a subgroup of $^{*}\mathbb{R}_e^q$, called the group of rates of $R$.

The following proposition is an immediate consequence of (4).

**Proposition 9.** The maps $*n \mapsto *n^\perp$ and $*n \mapsto ^{*}\epsilon$ define isomorphisms \[^{*}\mathbb{Z}_R^p \cong (^{*}\mathbb{Z}_R^p)^\perp \quad \text{and} \quad ^{*}\mathbb{Z}_R^p \cong ^{*}\mathbb{R}_e^q.
\]

**Note 1** (A. Verjovsky). Using formulation (4) of $^{*}\mathbb{Z}_R^p$, it follows that every triple \[(^{*}\mathbb{n}, *n^\perp, ^{*}\epsilon)\]
represents a diophantine approximation of $R$. Thus we may regard $^{*}\mathbb{Z}_R^p$ as the group of diophantine approximations of $R$.

For example, when $p = q = 1$ and $r \in \mathbb{R} \setminus \mathbb{Q}$, $*n$ and $*n^\perp$ are equivalence classes of sequences $\{x_\alpha\}$ and $\{y_\alpha\} \subset \mathbb{Z}$, and $^{*}\epsilon$ an equivalence class of sequence $\{\epsilon_\alpha\} \subset \mathbb{R}, \epsilon_\alpha \to 0$, such that

\[
\left| r - \frac{y_\alpha}{x_\alpha} \right| = \left| \frac{\epsilon_\alpha}{x_\alpha} \right| \to 0.
\]

Conversely, every diophantine approximation of $r$ defines uniquely a triple $(^*n, *n^\perp, ^{*}\epsilon)$.

Recall that two irrational numbers $r, s \in \mathbb{R} \setminus \mathbb{Q}$ are equivalent if there exists

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]
such that $s = A(r) = (ar + b)/(cr + d)$.

**Proposition 10.** If $r$ and $s$ are equivalent irrational numbers, then $^{*}\mathbb{Z}_r \cong ^{*}\mathbb{Z}_s$.

*Proof.* Given $*n \in ^{*}\mathbb{Z}_r$, observe that $(cr + d)^*n \simeq c^n + d^*n \in ^{*}\mathbb{Z}$. Write $*m = c^n + d^*n$. Then $*m \in ^{*}\mathbb{Z}_s$, since

\[
s^*m \simeq (ar + b)^*n \simeq a^*n^\perp + b^*n \in ^{*}\mathbb{Z}.
\]

The association $*n \mapsto *m$ defines an injective homomorphism $\psi : ^{*}\mathbb{Z}_r \to ^{*}\mathbb{Z}_s$, with inverse defined $\psi^{-1}(^*m) \simeq (-cs + a)^*m$. \[\square\]

**Note 2.** Two irrational numbers $r, s$ are called virtually equivalent if there exists $A \in \text{SL}(2, \mathbb{Q})$ such that $A(r) = s$. In this case, there exists a pair of monomorphisms \[\psi_1 : ^{*}\mathbb{Z}_r \to ^{*}\mathbb{Z}_s \quad \text{and} \quad \psi_2 : ^{*}\mathbb{Z}_s \to ^{*}\mathbb{Z}_r,
\]
defined as in Proposition (10). In other words, $^{*}\mathbb{Z}_r$ and $^{*}\mathbb{Z}_s$ are virtually isomorphic. These maps are mutually inverse to each other if and only if $A \in \text{SL}(2, \mathbb{Z})$. 

We are led to make the following conjecture.

**Conjecture 1.** If \( *\mathbb{Z}_r \cong *\mathbb{Z}_s \) for irrational numbers \( r, s \), then \( r \) and \( s \) are equivalent.

A verified Conjecture 1 would augur a group theoretic approach to diophantine approximation.

### 4.5. Anosov Foliations and Hyperbolic Diophantine Approximation

Let \( \Gamma \) be a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \) with no elliptics, \( \Sigma = \Gamma \backslash \mathbb{H}^2 \) the corresponding Riemann surface. Let \( \rho : \Gamma \to \text{Homeo}(\mathbb{S}^1) \) be the representation of \( \Gamma \) on \( \mathbb{S}^1 = \partial \mathbb{H}^2 \) and denote as in §2.1.3 the associated Anosov foliation by \( \mathcal{F}_\Gamma \). Fix \( t, \xi \in \mathbb{S}^1 \), consider a neighborhood basis \( \{ U_i(\xi) \} \) about \( \xi \), and define the nested set \( \{ G_i(t; \xi) \} \subset \Gamma \) by

\[
G_i(t; \xi) = \{ A \in \Gamma \mid \rho_A(t) \in U_i(\xi) \}.
\]

**Proposition 11.** Let \( \hat{x} \in \mathcal{F}_\Gamma \) be contained in a leaf covered by \( \mathbb{H} \times \{ \xi \} \) and let \( x \) be contained in a leaf covered by \( \mathbb{H} \times \{ \xi \} \). Then

\[
[[\pi]]_1(\mathcal{F}, x, \hat{x}) = \bigcirc (G_i(t; \xi) \cdot G_i(t; \xi)^{-1}).
\]

**Proof.** Immediate from the definition of \( [[\pi]]_1 \).

Classically \[10\], given \( \xi \in \mathbb{S}^1 \) in the limit set of \( \Gamma \) and \( t \in \mathbb{S}^1 \), a \( \Gamma \)-hyperbolic diophantine approximation of \( \xi \) based at \( t \) is a sequence \( \{ A_\alpha \} \subset \Gamma \) such that \( |\xi - A_\alpha(t)| \to 0 \), where \( | \cdot | \) is the norm induced by the inclusion \( \mathbb{S}^1 \subset \mathbb{R}^2 \). It follows from our definitions that \( *D(\hat{x}, \hat{x}) \) consists precisely of equivalence classes of \( \Gamma \)-hyperbolic diophantine approximations.

### 5. THE FUNDAMENTAL GERm OF A QUASISUSPENSION

Let \( \mathcal{L}_0 \) be a Galois suspension, \( \mathcal{X} \subset \mathcal{L}_0 \) a \( \pi_1B \) invariant closed set, \( \mathcal{Z}_0 = \mathcal{L}_0 \setminus \mathcal{X} \). Let \( H < \text{Homeo}_{\omega - \text{rb}}(\mathcal{L}_0) \) be a subgroup acting properly discontinuously and let \( \mathcal{L} = H \setminus \mathcal{Z}_0 \) be the resulting quasisuspension. See §2.2. We have the following analogue of Proposition 7.

**Proposition 12.** If \( T \) and \( T' \) are diophantine transversals containing \( x \) and \( \hat{x} \) then

\[
[[\pi]]_1(\mathcal{L}, x, \hat{x}, T) = [[[\pi]]_1(\mathcal{L}, x, \hat{x}, T').
\]

**Proof.** First suppose that the leaf \( L \) containing \( x \) has the same topology as any leaf \( L_0 \) lying above it in \( \mathcal{Z}_0 \); in other words, \( H_L = 1 \). Then the diophantine group \( G_L \) is generated only by elements of \( \pi_1B \) and \( \pi_1L_0 \). We may assume that the transversal \( T \) lifts to an \( H \) orbit of disjoint \( \pi_1B \) transversals \( H \cdot T_0 \) in \( \mathcal{L}_0 \), wherein it follows that

\[
[[\pi]]_1(\mathcal{L}, x, \hat{x}, T) = [[[\pi]]_1(\mathcal{L}, x_0, \hat{x}_0, T_0)
\]

where \( (x_0, \hat{x}_0, T_0) \) is a triple that covers \( (x, \hat{x}, T) \). On the other hand, since the \( \pi_1B \)-invariant set \( \mathcal{X} \) which we removed from \( \mathcal{L}_0 \) to get \( \mathcal{Z}_0 \) is closed, we may assume that \( T_0 \) is a diophantine transversal for \( \mathcal{L}_0 \). It follows then that

\[
[[\pi]]_1(\mathcal{L}_0, x_0, \hat{x}_0, T_0) = *\pi_1L_0 \cdot [[[\pi]]_1(\mathcal{L}, x, \hat{x}, T_0).
\]

The same is true for \( T' \) so by Proposition 7 the result follows.

Now suppose that \( H_L \neq 1 \). Then there are \( \pi_1B \) transversals \( T_0, T'_0 \subset \mathcal{Z}_0 \) covering \( T, T' \) such that every \( G_L \)-diophantine approximation of \( \hat{x} \) along \( T \) resp. \( T' \) is of the form

\[
*\gamma \cdot *h \cdot *g, \quad \text{resp.} \quad *\gamma \cdot *h \cdot *g',
\]

where \( *g \in *D(\hat{x}, \hat{x}_0, T_0) \), \( *g' \in *D(\hat{x}, \hat{x}_0, T'_0) \) (here \( \hat{x}_0 \in T_0 \cap T'_0 \) covers \( \hat{x} \)), \( *h \in *H_L \) and \( *\gamma \in *\pi_1L_0 \). By the previous paragraph, we have \( *D(\hat{x}, \hat{x}_0, T_0) = *D(\hat{x}, \hat{x}_0, T'_0) \) and the result follows.
Accordingly, we drop mention of $T$ and write $\langle \pi \rangle_1(\mathcal{D}, x, \tilde{x})$.

**Note 3.** The proof of Proposition [12] shows that $^*\pi_1(L)$ is a subgroup of $\langle \pi \rangle_1(\mathcal{D}, x, \tilde{x})$. In addition, there is a monomorphism $\langle \pi \rangle_1(\mathcal{D}, x, \tilde{x}) \hookrightarrow \langle \pi \rangle_1(\mathcal{D}, x, \tilde{x})$, an isomorphism if $H_L = \{1\}$.

5.1. **Sullivan Solenoids and the Baumslag-Solitar Groups.** Consider the Baumslag-Solitar group

$$G_{BS} = G_{BS}(d) = \langle f, x : fx^{-1}d = x^d \rangle.$$ 

Define a nested set about 1 by

$$G_i = \{ f^mx^d | m, r \in \mathbb{Z} \},$$

and denote

$$\langle G_{BS} \rangle := \bigcirc(G_i \cdot G_i^{-1}).$$

**Theorem 5.** $\langle G_{BS} \rangle$ is a group.

**Proof.** Observe by induction that in $G_{BS}$,

$$x^{-d_2}f = fx^{-d_1}$$

for all $\alpha > 0$. To see that $\langle G_{BS} \rangle$ is a group, it suffices to check that $G_i \cdot G_i^{-1}$ is a group for all $i$. Write a generic element $g \in G_i \cdot G_i^{-1}$ in the form $g = f^l x^d f^m$ for $l, m, r \in \mathbb{Z}$. Then an element $gh^{-1}, g, h \in G_i \cdot G_i^{-1}$ may be written (using (7))

$$gh^{-1} = f^l x^d f^m x^d f^n = \begin{cases} f^{l_x (r+xd^m)d} f^{m+n} & \text{if } m > 0 \\
^{f^l x^d (x^d + s) d} f^n & \text{if } m \leq 0, \end{cases}$$

where $l, m, n, r, s \in \mathbb{Z}$. It follows that $gh^{-1} \in G_i \cdot G_i^{-1}$. □

**Note 4.** The ultrascope $\bigcirc G_i$ is not even a groupoid as elements do not have inverses. Indeed, consider the sequence $\{g_\alpha\} = \{ f^{-m_\alpha} x^{d_\alpha} \}$, where $m_\alpha > \alpha > 0, \alpha = 1, 2, \ldots$. Note that $\{g_\alpha\}$ defines an element of $\bigcirc G_i$. Using (7), we may write the inverse sequence

$$\{g_\alpha^{-1}\} = \{ x^{-d_\alpha} f^{m_\alpha} \} = \{ f^{\alpha} x^{-1} f^{m_\alpha - \alpha} \}.$$ 

Since $m_\alpha > \alpha$, we cannot use the defining relation of $G_{BS}$ to move the remaining $f^{m_\alpha - \alpha}$ to the left of the $x$-term. It follows that $\{g_\alpha^{-1}\}$ does not define an element of $\bigcirc G_i$, so the latter does not have the structure of a groupoid.

**Theorem 6.** For all $x, \tilde{x} \in D_f$ with $x \in L$,

$$\pi \langle D_f, x, \tilde{x} \rangle \cong \begin{cases} \langle \pi \rangle \text{ if } L \text{ is an annulus} \\
^*\mathbb{Z}_d \text{ if } L \text{ is a disk} \end{cases}$$

In either event, $\langle \pi \rangle \langle D_f, x, \tilde{x} \rangle$ is a group.

**Proof.** First suppose $L$ is an annulus. The action of $\pi_1 D^* \cong \mathbb{Z}$ on $\tilde{D}$ is generated by $(z, \tilde{n}) \mapsto (z, \tilde{n} + 1)$, where $(z, \tilde{n}) \in \mathbb{H} \times \tilde{D}$. Then if $\gamma$ is the generator of $\pi_1 D^*$, we have $\hat{f} \gamma \hat{f}^{-1} = \gamma^d$. It follows that the diophantine group is isomorphic to $G_{BS}$. The set of diophantine approximations $^*D(\tilde{x}, \tilde{y})$ is equal to $\bigcirc G_i$, where $G_i$ is the nested set $\{ \tilde{\gamma} \}$. The result now follows by definition of $\pi \langle D \rangle$. If $L$ is a disk, then $\pi \langle D_f, x, \tilde{x} \rangle = \pi \langle D, x_0, \tilde{x}_0 \rangle$ where $(x_0, \tilde{x}_0)$ covers $(x, \tilde{x})$. By the results of §4.3 we have $\pi \langle D_f, x_0, \tilde{x}_0 \rangle = ^*\mathbb{Z}_d(d)$. □
The example of \( \hat{\mathcal{D}}_f \) illustrates the advantage of the “nonabelian Grothendieck group” type construction used in Definition \( \text{II.4} \) by Note \( \text{II.4} \) the naive choice \( [\pi]_1 = \pi \) would not even have produced a groupoid.

5.2. **Reeb Foliations.** Let \( \mathcal{F}_{\text{Reeb}} \) be a Reeb foliation. The diophantine group here is \( \mathbb{Z} \). Recall that \( L_0 \) is the torus leaf.

**Theorem 7.** For any pair \( x, \hat{x} \in \mathcal{F}_{\text{Reeb}} \) contained in a diophantine transversal with \( x \in L \)

\[
[\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) \cong \begin{cases} *\mathbb{Z}^2 & \text{if } x = \hat{x} \in L_0 = L \\ *\mathbb{Z} & \text{if } \hat{x} \in L_0 \neq L \\ 0 & \text{otherwise} \end{cases}
\]

In every case, \( [\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) \) is a group.

**Proof.** Suppose first that \( x = \hat{x} \in L_0 \). Then \( [\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) = *\pi_1L_0 = *\mathbb{Z}^2 \). If \( x \neq \hat{x} \) and \( L = L_0 \), there is no diophantine transversal containing the two points hence the fundamental germ is undefined. Now if \( x \in L, \hat{x} \in L_0 \neq L \) are contained in a diophantine transversal, then a sequence \( \{n_\alpha\} \) is a diophantine approximation if and only if it is infinite. Thus \( *D(\hat{x}, \hat{x}) = *\mathbb{Z}_\infty := *\mathbb{Z} \setminus *\mathbb{Z}_{\text{fin}}, \) the infinite nonstandard integers. Then

\[
[\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) = *\mathbb{Z}_\infty = *\mathbb{Z}.
\]

If \( \hat{x} \in L' \neq L_0 \), there are no accumulations of \( L \) on \( L' \) so the fundamental germ is 0. \( \square \)

Intuitively, when \( \hat{x} \in L_0 \n L, [\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) \) records the approximation by the dense leaf of the circumferential cycle \( c \subset L_0 \) through \( \hat{x} \). On the other hand, \( [\pi]_1(\mathcal{F}_{\text{Reeb}}, x, \hat{x}) \) does not predict the meridian cycle \( \hat{c}' \subset L_0 \). Instead, \( \hat{c}' \) is approximated by a sequence of inessential loops in \( L \) that move off to infinity, and such sequences are not the stuff of \( [\pi]_1 \).

6. **The Fundamental Germ of A Double Coset Foliation**

Let \( \mathcal{G} \) be a Lie group, \( \hat{\mathcal{H}} \subset \mathcal{G} \) a closed subgroup, \( \Gamma < \mathcal{G} \) a discrete subgroup and \( \mathcal{F}_{\mathcal{H}, \Gamma} \) the associated double coset foliation. The situation is considerably more subtle due to the fact that the diophantine group is no longer discrete. Thus two choices of diophantine transversal \( T_1, T_2 \) through \( x, \hat{x} \) yield distinct sets of diophantine approximations, in contrast with the case of a (quasi)suspension. Note on the other hand that every transversal is diophantine, since the universal covers of the leaves are homogeneous with respect to the left action of the diophantine group \( \hat{\mathcal{H}} \). In fact, if \( x_1 \) and \( x_2 \) are contained in the same leaf, then \( \hat{a} \cdot \hat{x}_1 = \hat{x}_2 \) for some \( \hat{a} \in \hat{\mathcal{H}} \). This yields a bijection of diophantine sets

\[
*\mathcal{D}(\hat{x}_1, \hat{x}, T_1) \longrightarrow *\mathcal{D}(\hat{x}_2, \hat{x}, T_2)
\]

defined \( *g_1 \mapsto *g_2 \) if \( *g_1 = *g_2 \cdot \hat{a} \) in \( \hat{\mathcal{H}} \). That is, the bijection is given by the equality \( *\mathcal{D}(\hat{x}_1, \hat{x}, T_1) = *\mathcal{D}(\hat{x}_2, \hat{x}, T_2) \cdot \hat{a} \). However, it is not clear that the following prescription for a map of fundamental germs:

\[
*u_1 \mapsto *u_2 \text{ iff } *u_1 = *g_1 \cdot h_1^{-1} \text{, } *u_2 = *g_2 \cdot h_2^{-1}, \text{ and } *g_1 = *g_2 \cdot \hat{a}, \text{ } *h_1 = *h_2 \cdot \hat{a}
\]

is well-defined since there might be, say, another representation \( *u_1 = *g_1 \cdot (h_1')^{-1} \) which leads to a different assignment. Even if \( (8) \) were well-defined, there is no reason to expect that it should respect the groupoid structure. When \( \hat{\mathcal{H}} \) is a group, one can say more:
Lemma 1. If \( \ast \tilde{\mathcal{H}} \) is a group then \( \ast u \circ \ast v = \ast w \) in \( [\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, x, \hat{x}, T) \) implies \( \ast u \cdot \ast v = \ast w \) in \( \ast \tilde{\mathcal{H}} \).

Proof. This follows immediately since the groupoid structure of the fundamental germ is defined in terms of left multiplication on the unit space \( \ast \mathcal{D}(\hat{x}, T) \).

Proposition 13. If \( \ast \tilde{\mathcal{H}} \) is a group and \( T_1 \) and \( T_2 \) are diophantine transversals through \( x_1, \hat{x} \) and \( x_2, \hat{x} \), respectively, where \( x_1, x_2 \) belong to the same leaf \( L \), then

\[
[\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, x_1, \hat{x}, T_1) \cong [\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, x_2, \hat{x}, T_2).
\]

Proof. It is clear now that the bijection \( \tilde{\mathcal{F}}_\Gamma \) is well-defined: in fact, since \( \ast \tilde{\mathcal{H}} \) is a group, we have \( \ast u_1 = \ast u_2 \). From this it follows that \( \text{Dom}(\ast u_1) = \text{Dom}(\ast u_2) \cdot \tilde{a} \), and that the bijection \( \tilde{\mathcal{F}}_\Gamma \) defines a groupoid isomorphism.

We shall assume from this moment on that \( \ast \tilde{\mathcal{H}} \) is a group. We will then not mention the base point \( x \) and the transversal \( T \) and write \( [\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, L, \hat{x}) \) where \( L \) is the leaf along which diophantine approximations are taking place. If \( \hat{x} \in L \) we write simply \( [\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, L) \).

We now give a “diophantine” description of \( \ast \mathcal{D}(\hat{x}, T) \), similar in spirit to that of \( \ast \mathbb{Z}_R \) appearing in \( \mathcal{F} \). Denote by \( p : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}} \) the universal cover of \( \tilde{\mathcal{H}} \). Suppose that \( L \) is covered by a coset \( \tilde{\mathcal{H}}g \) and \( \tilde{g} \in \mathcal{G} \) is an element covering \( \hat{x} \). A subset \( \mathcal{F} \subset \mathcal{G} \) is called a local section at \( \tilde{g} \) for the quotient map \( \mathcal{G} \to \mathcal{F} \) if \( \mathcal{F} \) maps homeomorphically onto an open subset containing \( \tilde{\mathcal{H}}g \). We may assume without loss of generality that the transversal \( T \) through \( \hat{x} \) lifts to a local section \( \mathcal{F} \) through \( \tilde{g} \). As our interest is in sequences which converge to \( \tilde{g} \) in \( \mathcal{F} \), we may assume also that \( \mathcal{F} = \tilde{g} \cdot \mathcal{F} \) for some local section \( \mathcal{F} \) about \( 1 \). Let \( \ast \mathcal{F} \subset \ast \mathcal{G} \) denote the set of infinitesimals which are represented by sequences in \( \mathcal{F} \).

Now let \( \ast \tilde{h} \) be a diophantine approximation of \( \tilde{x} \) based at \( \hat{x} \) along \( T \), which is characterized by the property that \{ \( p(\ast \tilde{h}) \cdot g \) \} lies in \( \tilde{g} \cdot \ast \mathcal{F} \cdot \ast \Gamma \). This gives the following diophantine description of \( \ast \mathcal{D}(\hat{x}, T) \):

\[
\ast \mathcal{D}(\hat{x}, T) = \left\{ \ast \tilde{h} \in \ast \tilde{\mathcal{H}} \mid \exists \ast \gamma \in \ast \Gamma, \ast \in \ast \mathcal{F} \text{ such that } \tilde{g}^{-1} \cdot p(\ast \tilde{h}) \cdot g \cdot \ast \gamma = \ast \in \ast \mathcal{F} \right\}.
\]

The element \( \ast \tilde{h} \) associated to \( \ast \tilde{h} \) in \( \mathcal{F} \) is called the dual of \( \ast \tilde{h} \). When \( \tilde{g} = g \), we let \( \ast \tilde{\mathcal{H}} := \ast \mathcal{D}(\hat{x}, T) \) denote the set of diophantine approximations and let \( \ast \tilde{\mathcal{H}}_g \) denote the set of duals. Thus if \( \mathcal{G}_g \) denotes the conjugation map \( a \mapsto g^{-1}ag \),

\[
\ast \tilde{\mathcal{H}}_g = \left\{ \ast \tilde{h} \in \ast \tilde{\mathcal{H}} \mid \exists \ast \gamma \in \ast \Gamma, \ast \in \ast \mathcal{F} \text{ such that } \mathcal{G}_g(p(\ast \tilde{h})) \cdot \ast \gamma = \ast \in \ast \mathcal{F} \right\}.
\]

In general, whether \( g = \tilde{g} \) or not, it follows that

\[
[\pi]_1(\mathcal{F}_{\mathcal{H}, \Gamma}, L, \hat{x}) \subset \left\{ \ast \tilde{u} \in \ast \tilde{\mathcal{H}} \mid \exists \ast \gamma, \ast \in \ast \Gamma, \ast \in \ast \mathcal{F} \text{ s.t. } \mathcal{G}_g(p(\ast \tilde{u})) \cdot \ast \eta = \ast \omega \right\},
\]

where \( \ast \mathcal{F} = \ast \mathcal{F} \cdot \ast \mathcal{F}^{-1} \). The inclusion is in general strict as the following example shows:

Example 1. Consider the double coset foliation \( \text{Geod}_\Gamma \), which possesses a noncompact leaf \( L \) and a pair of cycles \( c_-, c_+ \) such that \( L \) coils about \( c_- \) (about \( c_+ \)) as one goes to negative (positive) infinity in \( L \), and has no other accumulations. If \( \hat{x} \) belongs to either \( c_- \) or \( c_+ \), we have

\[
[\pi]_1(\text{Geod}_\Gamma, L, \hat{x}) \cong \ast \mathbb{Z},
\]

but if \( \hat{x} \in L \), we have \( [\pi]_1(\text{Geod}_\Gamma, L, \hat{x}) = 0 \).
Note 5. Since $p^{-1}(e) \cong \pi_1 \mathcal{F}$, we have $^{*} \pi_1 \mathcal{F} < \llbracket \pi \rrbracket_1 (\mathcal{F}, \Gamma, L, \hat{x})$.

One can understand the description of $^{*} \mathcal{F}_g$ appearing in (10) as a nonlinear version of (4). In fact, if $\mathcal{O}$ is a linear group of $p \times p$ matrices and $g \in \mathcal{O}$, then one can think of $^{*} \mathcal{O}_g$ as defined in (4) as the set of linear diophantine approximations of $g$ (approximations of $g$ by pairs of vectors with respect to linear algebra), whereas $^{*} \mathcal{F}_g$ can be thought of as a set of nonlinear diophantine approximations of $g$ (approximations of $g$ by pairs of matrices with respect to matrix algebra).

We now consider the horocyclic and geodesic flows on the unit tangent bundle of a riemannian surface, which are, as is widely appreciated, deep mathematical objects. It should come as no surprise that this deepness is reflected in their fundamental germs, which present the most complex and intractable diophantine algebra we have encountered thus far. In the remainder of this section, we will attempt to give the reader a feel for the complexity of these fundamental germs by walking through a sample calculation.

We restrict to the case $\mathcal{O} = SL(2, \mathbb{R})$ and $\Gamma = SL(2, \mathbb{Z})$. See §2.3 for the relevant notation. Consider first the case of the (positive) horocyclic flow $\text{Hor} = \text{Hor}_{SL(2, \mathbb{R})}$, that is, $\mathcal{F} = H = H^+$. If $D$ is the subgroup of matrices of the form

$$
\begin{pmatrix}
    e^{t/2} & 0 \\
    \frac{t}{e^{-t/2}} & e^{-t/2}
\end{pmatrix}
$$

$s, t \in \mathbb{R}$, then $D$ defines a local section about 1 so we take $\mathcal{F} = D$. Finally, since $H \cong (\mathbb{R}, +)$, we shall simplify notation by identifying $r$ with the matrix $A_r$ and write $^{*} \mathcal{R}_g = H_g$ for the set of diophantine approximations.

Let us consider the relatively simple choice

$$
g = \begin{pmatrix}
\sqrt{2} & 1 \\
1 & \sqrt{2}
\end{pmatrix}.
$$

The right coset of $g$ is

$$
H_g = \left\{ \begin{pmatrix} r + \sqrt{2} & 2r + 1 \\
1 & \sqrt{2}
\end{pmatrix} \mid r \in \mathbb{R} \right\}.
$$

Since $Hg$ does not define a cycle in $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ it must be dense by a theorem of Hedlund [8], so we can expect from $g$ a nontrivial set of diophantine approximations. The conjugate of $H$ by $g$ is

$$
\sigma_g(H) = \left\{ \begin{pmatrix} 1 + \sqrt{2}r & 2r \\
-r & 1 - \sqrt{2}r
\end{pmatrix} \mid r \in \mathbb{R} \right\}.
$$

In order to characterize the elements of $^{*} \mathcal{R}_g$, we shall need the following generalization of $^{*} \mathcal{R}_r$. Let $\mathcal{O}$ be the ring of integers of a number field. For $^{*} r \in ^{*} \mathcal{R}$, define

$$
^{*} \mathcal{O}_r = \{ ^{*} n \in ^{*} \mathcal{O} \mid \exists ^{*} n^\perp \in ^{*} \mathcal{O} \text{ such that } ^{*} r \cdot ^{*} n - ^{*} n^\perp \in ^{*} \mathcal{O} \}.
$$

Clearly $^{*} \mathcal{O}_r$ is a subgroup of $^{*} \mathcal{O}$.

Theorem 8. Let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\sqrt{2})$. Then $^{*} r \in ^{*} \mathcal{R}_g$ if and only if there exists $^{*} \gamma = \begin{pmatrix} ^{*} a & ^{*} b \\
^{*} c & ^{*} d \end{pmatrix} \in SL(2, ^{*} \mathbb{Z})$ for which

- $\sqrt{2} a + 2^{*} c, \sqrt{2} b + 2^{*} d \in ^{*} \mathcal{O}_r$ and $(\sqrt{2} a + 2^{*} c)^\perp = 1 - ^{*} a,
  (\sqrt{2} b + 2^{*} d)^\perp = -^{*} b$. 


Theorem 9. Let \( c \in \mathbb{Z}, d = 1 - a \), \( a \in \mathbb{R} \) and \( c = 1 - b \).

\[
\ast c \ast d \in \mathbb{Z} \quad \text{and} \quad \ast c^\perp = 1 - \ast a, \ast d^\perp = 1 - \ast b.
\]

\[
\ast b = - (\sqrt{2} b + 2 \ast d) \ast r \quad \text{and} \quad \ast a = (a + (\sqrt{2} a + 2 \ast c) \ast r) (\ast d - (b + \sqrt{2} \ast d) \ast r) = 1.
\]

**Proof.** From (10), \( \ast r \in \mathbb{R}_\Gamma \) if and only if there exists \( * \gamma \in * \Gamma \) and \( * \epsilon, * \delta \in * \mathbb{R}_\epsilon \) with

\[
\begin{pmatrix}
* a(1 + \sqrt{2} r) + 2 * c r & * b(1 + \sqrt{2} r) + 2 * d r \\
- * a r + * c(1 - \sqrt{2} r) & - * b r + * d(1 - \sqrt{2} r)
\end{pmatrix}
= \begin{pmatrix}
1 + * \epsilon & 0 \\
* \delta & (1 + * \epsilon)^{-1}
\end{pmatrix}.
\]

The first and third items follow immediately. The second item follows upon noting that we may eliminate \( * r \) by multiplying the second row equations by \( \sqrt{2} \) and adding them to the first row equations.

\[ \square \]

Theorem 9 illustrates why it is so difficult to say anything about the algebraic structure of \( * \mathbb{R}_\epsilon \) or \( \| \pi \|_1(\text{Hor}, L) \). In order to determine whether the sum \( * r + * s \) defines an element of \( * \mathbb{R}_\epsilon \), we must find a way to "compose" the corresponding duals \( * r^\perp, * s^\perp \in \mathbb{R}_\frac{1}{*} \) to obtain one for their sum, and it is not even clear what this operation on matrices should be. One could reverse the logic and ask if the product \( * r^\perp \ast s^\perp \) defines an element of \( \mathbb{R}_\frac{1}{*} \), however this seems just as hopeless since the diophantine conditions spelled out in the statement of Theorem 8 are not stable with respect to matrix multiplication.

As for the geodesic flow, we leave it to the reader to formulate the appropriate analogue of Theorem 8, e.g. using the local section \( T \) for which

\[
* T \epsilon = \left\{ \left( \begin{array}{cc}
1 & * \delta \\
* \delta & 1 + * \delta^* \delta
\end{array} \right) \mid * \delta, * \delta^* \epsilon \in * \mathbb{R}_\epsilon \right\}.
\]

The result would be a set of diophantine conditions at least as daunting as that obtained for the horocyclic flow.

7. The Fundamental Germ of a Locally Free Lie Group Action

The discussion here is very similar to that for a double coset, so we will be brief. Let \( \mathcal{B} \) be a Lie group of dimension \( k, M^n \) an \( n \)-manifold, \( n > k, X \subset M^n \). Let \( \theta : \mathcal{B} \rightarrow \text{Homeo}(X) \) be a locally-free representation whose orbits either coincide or are disjoint and let \( \mathcal{L}_\mathcal{B} \) be the associated lamination on \( X \). Any diophantine transversal through \( x, x^* \) may be obtained as the intersection of \( \mathcal{L}_\mathcal{B} \) with a submanifold \( T \) of \( M^n \) of dimension \( n - k \) such that \( x, x^* \in T \) and \( T \cap (\theta(\mathcal{B}) \cdot x) \) is discrete in \( \theta(\mathcal{B}) \cdot x \). As in the case of a double coset foliation, when \( * \mathcal{B} \) is group,

(1) Groupoid multiplication in the fundamental germ corresponds to multiplication in \( * \mathcal{B} \).

(2) If \( T_1, T_2 \) are transversals through \( x_1, x_2 \) and \( x_2, x_2^* \) where \( x_1, x_2 \) belong to the same leaf \( L \) then

\[
[\pi]_1(\mathcal{L}_\mathcal{B}, x_1, x, T_1) \cong [\pi]_1(\mathcal{L}_\mathcal{B}, x_2, x, T_2).
\]

Accordingly we shorten to \( [\pi]_1(\mathcal{L}_\mathcal{B}, L, x) \).

**Theorem 9.** Let \( \Sigma = \Gamma \bigsqcup \mathbb{H}^2 \) be a compact hyperbolic surface, \( \mathcal{L} \subset \Sigma \) a geodesic lamination, \( \hat{x} \in \mathcal{L} \) and \( l \subset \mathcal{L} \) a leaf. Then

\[
[\pi]_1(l, \hat{x}) = [\pi]_1(\text{Geom}, L, \hat{v})
\]

where \( L \) is a leaf covering \( l \) and \( \hat{v} \) is a tangent vector to \( l \) at \( \hat{x} \).
8. Functoriality

We begin by recalling the notion of morphism in the category of laminations. A lamination map \( F : \mathcal{L} \to \mathcal{L}' \) is a map satisfying the following conditions:

1. For every leaf \( L \subset \mathcal{L} \), there exists a leaf \( L' \subset \mathcal{L}' \) with \( F(L) \subset L' \).
2. For all \( x \in \mathcal{L} \), there exist open transversals \( T \ni x, T' \ni F(x) \), such that \( F(T) \subset T' \).

The projection \( P : \mathcal{L} \to B \) of a suspension onto its base is a lamination map. On the other hand, let \( \mathcal{F} \) be a foliation, \( M \) the underlying manifold. Then the canonical inclusion \( \iota : \mathcal{F} \to M \) is a map which maps leaves into the unique leaf \( M \), yet is not a lamination map since no open transversal of \( \mathcal{F} \) is mapped into a point, an open transversal of \( M \).

Let

\[
F : (\mathcal{L}, x, \hat{x}) \to (\mathcal{L}', x', \hat{x}')
\]

be a lamination map. We say that \( F \) is diophantine if there exist diophantine transversals \( T \ni x, \hat{x} \) and \( T' \ni x', \hat{x}' \) such that \( F(T) \subset T' \). Note that this condition is always satisfied if either \( \mathcal{L} \) or \( \mathcal{L}' \) are laminations defined by double cosets or locally free Lie group actions.

Denote by \( L \) and \( L' \) the leaves containing \( x, x' \) and let \( \hat{F} : \hat{L} \to \hat{L}' \) be the lift of the restriction \( F|_{L} \). Let \( \hat{T} \subset \hat{L}, \hat{T}' \subset \hat{L}' \) be the pre-images of \( T \cap L, T' \cap L' \). Then for \( F \) diophantine there is a well-defined map

\[
*DF : *D(\hat{x}, \hat{x}, T) \to *D(\hat{x}', \hat{x}', T')
\]

of diophantine approximations. If the assignment

\[
*u = *g \cdot *h^{-1} \mapsto *DF(*g) \cdot (*DF(*h))^{-1}
\]

leads to a well-defined map

\[
\[ F \] : [\pi]_1(\mathcal{L}, x, \hat{x}) \to [\pi]_1(\mathcal{L}', x', \hat{x})
\]

we say that \( F \) is germ.

**Proposition 14.** Let \( \mathcal{L} = \hat{B} \times_{\rho} F \) be a suspension with \( x, \hat{x} \) lying over \( x_0 \in B \). Then the projection \( \xi : (\mathcal{L}, x, \hat{x}) \to (B, x_0) \) is germ, and the induced map \( \[ \xi \] \) is a groupoid monomorphism.

**Proof.** It is clear from the definitions that \( *D \xi \) is the inclusion

\[
*D(\hat{x}, \hat{x}) \subset *\pi_1(B, x).
\]

In particular, it follows that \( \[ \xi \] \) is well-defined. Since the product in \( [\pi]_1(\mathcal{L}, \hat{x}, L) \) is induced by multiplication in \( *\pi_1(B, x) \), \( \[ \xi \] \) is a groupoid homomorphism as well. □

Unfortunately, we cannot assert in general that the map \( [F] \) induced by a germ laminations map \( F \) defines a groupoid homomorphism. We now introduce a class of lamination maps which is sufficiently well-behaved so as to allow us to say more.

Let \( \mathcal{F} \) be a foliation, \( M \) the underlying space of \( \mathcal{F} \), and \( \iota : \mathcal{F} \to M \) the inclusion. Although \( \iota \) is not a lamination map, we may nevertheless define a map of diophantine approximations as follows. An element \( *g \in *D(\hat{x}, \hat{x}, T) \), represented say by \( \{g_{\alpha}\} \), may be regarded as made up from an equivalence class of sequence \( \{y_{\alpha}\} \) where the \( y_{\alpha} \) are homotopy classes of curves lying within \( L \) whose endpoints converge to \( \hat{x} \). One may assume that there is an open disc \( O \subset M \) about \( \hat{x} \) such that the endpoints of these sequences lie entirely in \( O \). By connecting their endpoints to \( \hat{x} \) by paths contained in \( O \), we obtain a sequence
of homotopy classes of curves \( \{ \eta_{n} \} \subset \Pi_{1}(M, x, \tilde{x}) \) = the set of homotopy classes of paths from \( x \) and \( \tilde{x} \), hence a map

\[ *D_{t} : *D(\tilde{x}, x, T) \rightarrow *\Pi_{1}(M, x, \tilde{x}), \quad *g \mapsto \eta_{g} \]

which depends neither on \( O \) nor on the choice of connecting paths. More generally, given \( \mathcal{L} \) a lamination and \( i : \mathcal{L} \rightarrow X \) a map into a path-connected space, we may define a map

\[ *D_{t} : *D(\tilde{x}, x, T) \rightarrow *\Pi_{1}(X, i(x), i(\tilde{x})) \].

We say that the map \( i \) is germ if \( *D_{t} \) induces a well-defined map

\[ [i] : [\pi]_{1}(\mathcal{L}, x, \tilde{x}, T) \rightarrow *\pi_{1}(X, x), \quad *u = *g * h^{-1} \mapsto *(D_{t}(g) \cdot (D_{t}(h))^{-1} \].

**Definition 3.** Let \( \mathcal{L} \) be a lamination arising from a group action, \( X \) a path connected space. A map \( i : (\mathcal{L}, x, \tilde{x}) \rightarrow (X, i(x), i(\tilde{x})) \) is called a fidelity if it is germ and \([i] \) is a groupoid monomorphism. We say that \( \mathcal{L} \) is faithful if it has a fidelity.

For example, by Proposition 14 any suspension is faithful, however if the underlying space of a suspension \( \mathcal{L} \) is a manifold \( M \), we shall see that it is much more useful to be able to assert that the inclusion \( \mathcal{L} \hookrightarrow M \) is a fidelity.

For the remainder of the section, the base points \( x \) and \( \tilde{x} \) will be supressed in order to simplify notation.

**Proposition 15.** Let \( \mathcal{F}_{V} \) be the foliation of \( \mathbb{T}^{p+q} \) induced by the \( p \)-plane \( V \subset \mathbb{R}^{p+q} \). Then the inclusion \( i : \mathcal{F}_{V} \rightarrow \mathcal{T}^{p+q} \) is a fidelity.

**Proof.** Recall that for some \( q \times p \) matrix \( R \), \([\pi]_{1}(\mathcal{F}_{V}) = *\mathbb{Z}^{p}_{R} \). Then for \( *n \in *\mathbb{Z}^{p}_{R} \), the map \([i] \) is

\[ [i]([n] = ([n], *n^{-1}) \in *\mathbb{Z}^{p+q} = *\pi_{1}\mathbb{T}^{p+q} \]

where \( *n^{-1} \) is the dual to \( *n \). \([i] \) is then clearly an injective homomorphism. \( \square \)

The problem of the existence of fidelities for laminations arising from group actions is interesting but seems difficult.

**Conjecture 2.** Every lamination arising form a group action is faithful.

**Definition 4.** A germ lamination map \( F : \mathcal{L} \rightarrow \mathcal{L}' \) is trained if \( \mathcal{L} \) and \( \mathcal{L}' \) are faithful, and there exist fidelities \( i : \mathcal{L} \rightarrow X \), \( i' : \mathcal{L}' \rightarrow X' \) and a map \( f : X \rightarrow X' \) such that

\[ *f \circ [i] = [i'] \circ [F]. \]

The triple \((i, i', f)\) is called a training for \( F \).

**Theorem 10.** Let \( F : \mathcal{L} \rightarrow \mathcal{L}' \) be a trained lamination map. Then the induced map \([F] \) is a groupoid homomorphism.

**Proof.** Let \((i, i', f)\) be a training for \( F \). Then for all \(*u, *v \in [\pi]_{1}(\mathcal{L}) \) such that \(*u \cdot *v \) is defined we have

\[ [i'] \circ [F](*u \cdot *v) = [i']([F] * u \cdot [F] * v) \]

Since \([i'] \) is injective, \([F](*u \cdot *v) = [F] * u \cdot [F] * v \). \( \square \)

**Corollary 2.** Let \( F : (\mathcal{F}, x) \rightarrow (\mathcal{F}', x') \) be a map of foliations. Suppose that the inclusions \( \iota : \mathcal{F} \rightarrow M \), \( \iota' : \mathcal{F}' \rightarrow M' \) are fidelities. Then \([F] \) is a groupoid homomorphism.

**Proof.** Take \( f : M \rightarrow M' \) to be \( F \), viewed as a map on underlying manifolds. Then \((i, i', f)\) is a training. \( \square \)
Corollary 3. Any map $F : \mathcal{F}_V \to \mathcal{F}_V$ of linear foliations of torii induces a homomorphism $[F]$ of fundamental germs.

9. THE GERMAN UNIVERAL COVER

We assume throughout this section that

1. $\mathcal{L}$ is a weakly-minimal lamination arising from a group action.
2. $x = \hat{x} \in L$ a fixed dense leaf.

We abbreviate the associated fundamental germ to $[\pi]_1(\mathcal{L})$. An ultrafilter $\mathcal{U}$ is fixed throughout.

Let $p : \hat{L} \to L$ be the universal cover. A sequence $\{\hat{x}_n\} \subset \hat{L}$ is called $\mathcal{L}$-convergent if it projects to a sequence in $L$ converging to some $\hat{x} \in \mathcal{L}$. Two $\mathcal{L}$-convergent sequences $\{\hat{x}_n\}$ and $\{\hat{x}_m\} \subset \hat{L}$ are called $\mathcal{L}$-asymptotic if their projections converge to the same point $\hat{x}$ and if for every flowbox $O$ in $\mathcal{L}$ about $\hat{x}$, there exists $X \in \mathcal{U}$ such that $\hat{x}_n$ and $\hat{x}_m$ lie in a common lift of a plaque of $O$, for all $\alpha \in X$. The asymptotic class corresponding to $\{\hat{x}_n\}$ is denoted $^*\hat{x}$; we refer to $\hat{x}$ as the limit of $^*\hat{x}$ and write $\lim^*\hat{x} = \hat{x}$. The set of $^*\hat{x}$ with limit $\hat{x}$ is denoted $\lim^*_x$.

Definition 5. The germ universal cover of $\mathcal{L}$ with respect to $L$ is

$$[\hat{\mathcal{L}}] = \left\{ \text{classes } ^*\hat{x} \text{ of } \mathcal{L}\text{-convergent sequences in } \hat{L} \right\}.$$

Note that for any $\hat{x} \in \hat{\mathcal{L}}$, every $G_\mathcal{L}$-diophantine approximation $^*g$ of $\hat{x}$ determines an element of $[\hat{\mathcal{L}}]$, and the sets $\lim^*_x$ and $^*\mathcal{D}(\hat{x}, \hat{x}, T)$ are in bijective correspondence, for any diophantine transversal $T$ through $\hat{x}$.

Proposition 16. Let $\mathcal{L}$ be compact and suppose that $L = \mathcal{G}$ is a topological group for which $^*\hat{b}, ^*\hat{c} \in ^*\mathcal{G}$ are $\mathcal{L}$-asymptotic if and only if $^*\hat{b} \cdot ^*\hat{c}^{-1} \in ^*\mathcal{G}_F$. Then $[\hat{\mathcal{L}}] = ^*\mathcal{G}$.

Proof. Suppose that there is some $^*\hat{b} \in ^*\mathcal{G}$ represented by a sequence $\{\hat{b}_n\}$ which is not $\mathcal{L}$-convergent. Thus if $\{b_n\}$ is the projection of this sequence to $\mathcal{G} \subset \mathcal{L}$, then for all $\hat{x} \in \hat{\mathcal{L}}$, $\hat{x}$ has a neighborhood $U_\epsilon \subset \hat{\mathcal{L}}$ for which there is no $X \in \mathcal{U}$ with $\{b_n\}|_X \subset U_\epsilon$. The $U_\epsilon$ cover $\mathcal{L}$ so that there is a subcover $U_{\epsilon_1}, \ldots, U_{\epsilon_n}$; this implies that there exists a partition $X_1 \sqcup \cdots \sqcup X_n$ of $\mathcal{G}$ with $\{b_n\}|_{X_i} \subset U_{\epsilon_i}$. Since $\mathcal{U}$ is an ultrafilter, one of the $X_i$ belongs to $\mathcal{U}$, contradiction. Thus every element $^*\hat{b} \in ^*\mathcal{G}$ defines an element of $[\hat{\mathcal{L}}]$. Since the relation of being $\mathcal{L}$-asymptotic coincides with differing by an infinitesimal, we are done. \qed

For example, if $\mathcal{F}_V$ is a linear $n$-foliation of a torus, $[\mathcal{F}_V] = ^*\mathbb{R}^n$.

Denote by

$$^*p : [\hat{\mathcal{L}}] \to \mathcal{L},$$

the natural projection defined $^*\hat{x} \mapsto \lim^*\hat{x}$. The leaf $L_{^*\hat{x}}$ through $^*\hat{x}$ is defined to be the set of $^*\hat{y}$ such that

1. If $^*\hat{x} = \lim^*\hat{x}$ and $^*\hat{y} = \lim^*\hat{y}$ then $L_{^*\hat{x}} = L_{^*\hat{y}}$.
2. There are representative sequences $\{\hat{x}_n\}$, $\{\hat{y}_n\}$, and paths $\hat{\eta}_n$ connecting $\hat{x}_n$ to $\hat{y}_n$ so that $p(\hat{\eta}_n)$ converges to a path connecting $\hat{x}$ to $\hat{y}$.

Theorem 11. $[\hat{\mathcal{L}}]$ may be given the structure of a lamination whose leaves are nowhere dense and for which $^*p$ is an open lamination map.
Proof. Denote by \([T] \subset \tilde{\mathcal{L}}\) the pre-image of a transversal \(T \subset \mathcal{L}\) and well-order each \(\text{Lim}_x\) for \(x \in T\). Note that the cardinalities of the \(\text{Lim}_x\) are the same: that of the continuum, since \(L\) is dense and \(L \cap T\) is countable. We define a decomposition

\[
[T] = \bigsqcup T_a
\]

where \(T_a\) is the section over \(T\) defined by \(\hat{x} \mapsto \alpha\)th element of \(\text{Lim}_x\). By definition of the leaves of \([\tilde{\mathcal{L}}]\), given \(\hat{x}, \hat{y} \in T\),

\[
\left(\bigcup_{\hat{x} \in \text{Lim}_x} L_{\hat{x}}\right) \cap \left(\bigcup_{\hat{y} \in \text{Lim}_y} L_{\hat{y}}\right) \neq \emptyset
\]

if and only if \(L_{\hat{x}} = L_{\hat{y}}\). In the latter event the two unions of leaves appearing in (13) are equal, so in particular, given \(\hat{x} \in \text{Lim}_x\), there is a unique \(\hat{y} \in \text{Lim}_y\) for which \(L_{\hat{x}} = L_{\hat{y}}\). Since \(T \cap L_{\hat{x}}\) is countable, we may thus choose the ordering of each \(\text{Lim}_x\), \(\hat{y} \in T \cap L_{\hat{x}}\), so that all of the \(\alpha\)th elements lie on distinct leaves. In this way we may assume that the associated section \(T_a\) intersects any leaf of \([\tilde{\mathcal{L}}]\) no more than once. We topologize each section \(T_a\) through its identification with \(T\), and give \([\tilde{\mathcal{L}}]\) the associated product laminar structure. By construction of this topology, \(*p\) becomes an open laminar map.

The topology constructed in Theorem \(\ref{thm:germUniversalCover}\) is called a germ universal cover topology: it is not unique and depends on the choice of decomposition \(\ref{eq:decomposition}\). From now on, we assume that \([\tilde{\mathcal{L}}]\) has been equipped with such a topology.

There is a canonical simply connected leaf corresponding to the inclusion \(\tilde{L} \hookrightarrow [\tilde{\mathcal{L}}]\), however the other leaves need not be simply connected. For example, if \(L\) is one of the simply connected leaves of the Sullivan solenoid \(\mathbb{D}_f\), then leaves of the associated germ universal cover that correspond to accumulations of \(L\) on an annular leaf will not be simply connected. Thus \([\tilde{\mathcal{L}}]\) can be thought of as the ordinary universal cover \(\tilde{L}\) surrounded by a nonstandard cloud of leaves corresponding to the laminar accumulations of \(L\), since these leaves are nowhere dense, one might say that on passing to \([\tilde{\mathcal{L}}]\) all of the diophantine approximations within \(\mathcal{L}\) have been “unwrapped”.

We now posit \([\tilde{\mathcal{L}}]\) as the unit space of an enhanced groupoid structure for \([\pi]_1(\mathcal{L})\). Let \(*u \in [\pi]_1(\mathcal{L})\) and \(*\hat{x} \in [\tilde{\mathcal{L}}]\). We say that \(*u\) acts on \(*\hat{x}\) if there exist representative sequences such that \((u_\alpha \cdot \hat{x}_\alpha)\) defines an \(\mathcal{L}\)-convergent sequence \(*u \cdot *\hat{x}\) with

\[
\lim(*u \cdot *\hat{x}) = \lim(*\hat{x}).
\]

Defining the domain \(\text{Dom}(*u)\) and range \(\text{Ran}(*u)\) of \(*u\) through this notion of action, we see that \([\tilde{\mathcal{L}}]\) yields a new groupoid structure on \([\pi]_1(\mathcal{L})\), called the geometric groupoid structure. It is clear that both \(\text{Dom}(*u)\) and \(\text{Ran}(*u)\) are sublaminations of \([\tilde{\mathcal{L}}]\), since \(*\hat{x} \in \text{Dom}(*u)\) implies that \(L_{\hat{x}} \subset \text{Dom}(*u)\). Thus we may view \([\pi]_1(\mathcal{L})\) as a groupoid of partially defined bijections of \([\tilde{\mathcal{L}}]\). Note that the unit space for the old groupoid structure, \(\star \mathcal{D}(\hat{x}, T)\), maps into the new unit space \([\tilde{\mathcal{L}}]\) via its bijection with \(\text{Lim}_x\). There is a canonical inclusion of the old groupoid structure into the geometric groupoid structure, given by extension of domain and range, however in general this map need not be a groupoid homomorphism.

**Assumption.** For the remainder of the paper, we will assume that \([\pi]_1(\mathcal{L})\) is endowed with the geometric groupoid structure.
Definition 6. We say that $[\pi_1](\mathcal{L})$ is tame if whenever $\lim^\ast \tilde{x} = \lim^\ast \tilde{y}$, there exists $^u \in [\pi_1](\mathcal{L})$ such that $^u \cdot \tilde{x} = \tilde{y}$.

Proposition 17. If $[\pi_1](\mathcal{L})$ is tame, then the quotient

$$[\pi_1](\mathcal{L})/\mathcal{L}'$$

is homeomorphic to $\mathcal{L}$.

Proof. The equivalence relation enacted by the action of $[\pi_1](\mathcal{L})$ identifies precisely those points of $\mathcal{L}'$ which map to the same point $\tilde{x} \in \mathcal{L}$ by $^p$. Since $^p$ is open, it follows that quotient topology is that of $\mathcal{L}$.

Theorem 12. If $[\pi_1](\mathcal{L})$ is tame and a group, then there is a germ universal cover topology on $\mathcal{L}'$ for which $[\pi_1](\mathcal{L})$ acts as a group of homeomorphisms.

Proof. Let $[T]$ be the preimage of a transversal $T$ of $\mathcal{L}$. As $[\pi_1](\mathcal{L})$ is a group, $\text{Dom}^\ast u = [\mathcal{L}']$ for every element $^u \in [\pi_1](\mathcal{L})$, and moreover $^u([T]) = [T]$. Let $i : T \to [T]$ be a section so that for all $^x \in [\mathcal{L}]$, $i(T) \cap L^\ast_x$ contains at most one point. Since $[\pi_1](\mathcal{L})$ acts without fixed points and is tame, we have a decomposition as disjoint union

$$[T] = \bigsqcup_{^u \in [\pi_1](\mathcal{L})} ^u(i(T)).$$

Now construct as in Theorem 11 a lamination structure on $\mathcal{L}'$ based on this decomposition. It follows then that each $^u \in [\pi_1](\mathcal{L})$ acts homeomorphically on $\mathcal{L}'$.

Proposition 18. Let $F : (\mathcal{L}, L) \to (\mathcal{L}', L')$ be a lamination map, where $L$ and $L'$ are dense leaves. Then $F$ induces a map

$$[\tilde{F}] : [\mathcal{L}'] \to [\mathcal{L}'],$$

continuous with respect to appropriate choices of germ universal cover topologies.

Proof. Denote by $p' : L' \to \mathcal{L}'$ the universal cover. The map $[\tilde{F}]$ is defined by representing $^\ast \tilde{x}$ by a sequence $\{\tilde{x}_\alpha\}$ and taking $[\tilde{F}]([^\ast \tilde{x}])$ to be the asymptotic class of $\{F(\tilde{x}_\alpha)\}$. Now let $[\mathcal{L}']$ be any germ universal cover topology on $[\mathcal{L}']$, say constructed from a transversal $T'$. Since $F$ is a lamination map, there exists a transversal $T$ with $F(T) \subset T'$. We may thus find a decomposition $[T] = \sqcup T_\alpha$ compatible with that of $[T']$ i.e. so that $[\tilde{F}](T_\alpha) \subset T'_\alpha$ for all $\alpha$. Let $[\tilde{F}]$ to be the associated germ universal cover topology. Then $[\tilde{F}]$ is continuous with respect to $[\tilde{F}]$ and $[\tilde{T}]$.

We now return to the question of functoriality, which we must address in view of our adoption of a new groupoid structure. If we reconsider the notions of fidelities and trainings with regard to the geometric groupoid structure, then the analogue of Theorem 11–as well as its corollaries–remain true with identical proofs. For the remainder of the paper, the concepts of fidelity and training will be understood in the context of the geometric groupoid structure.

The classical universal cover enjoys the property that the lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of a map $f : X \to Y$ is $\pi_1 X$-equivariant. We now describe conditions under which the same can be said for a lamination map. A germ lamination map $F : \mathcal{L} \to \mathcal{L}'$ is said to be geometric if for all $^u \in [\pi_1](\mathcal{L}), [\tilde{F}](\text{Dom}^\ast u) \subset \text{Dom}([\tilde{F}]^\ast u), [\tilde{F}] : [\pi_1](\mathcal{L}) \to [\pi_1](\mathcal{L}')$ is a homomorphism and

$$[\tilde{F}]^\ast (^u \cdot \tilde{x}) = [\tilde{F}]^\ast u \cdot [\tilde{F}]^\ast (\tilde{x}).$$
Examples of geometric maps are the projection $\mathcal{L}_P \to B$ of a suspension onto its base and any map of manifolds $f: M \to M'$.

We say that a lamination $\mathcal{L}$ is geometrically faithful if it has a geometric fidelity: a fidelity $\iota: \mathcal{L} \to X$ which is geometric and for which $[\bar{\iota}]: [\bar{\mathcal{L}}] \to [\bar{X}]$ is injective. In addition $F: \mathcal{L} \to \mathcal{L}'$ is said to be geometrically trained if it possesses a training $(\iota, \iota', f)$ where $\iota, \iota'$ are geometric fidelities. For example, the fidelity $\iota: \mathcal{T}_V \to \mathbb{T}^{p+q}$ of a linear foliation of a torus is geometric, as well as the projection of a suspension onto a compact base.

**Theorem 13.** Let $F: \mathcal{L} \to \mathcal{L}'$ be geometrically trained. Then $F$ is geometric.

**Proof.** Let $(\iota, \iota', f)$ be a geometric training. Then we have

$$[\bar{\iota}] \circ [\bar{F}] ([u] \cdot [\bar{x}]) = [\bar{\iota}] ([F]([u] \cdot [\bar{F}]) ([\bar{x}]))$$

which implies the result as $[\bar{\iota}]$ is injective. □

**Corollary 4.** Suppose $F: \mathcal{F} \to \mathcal{F}'$ is a lamination map of foliations such that the inclusions into the underlying manifolds are geometric fidelities. Then $F$ is geometric. In particular any lamination map of linear foliations of torii is geometric.

10. COVERING SPACE THEORY

A surjective lamination map $P: \mathcal{L} \to \mathcal{L}'$ is called a lamination covering if $P|_{\mathcal{L}}$ is a covering map for every leaf $L \subset \mathcal{L}$. A lamination map which is a covering map in the classical sense is a lamination covering but not all lamination coverings occur this way e.g. the projection $\xi: \mathcal{L} \to B$ of a suspension onto its base. We say that $P$ is cover trained if it has a training $(\iota, \iota', \rho)$ in which $\rho: X \to X'$ is a covering map.

**Theorem 14.** Let $P: \mathcal{L} \to \mathcal{L}'$ be a germ lamination covering that is cover trained. Then

1. The induced map of fundamental germs

$$[P]: [\pi]_1(\mathcal{L}) \to [\pi]_1(\mathcal{L}')$$

is a groupoid monomorphism.

2. The induced map of germ universal covers

$$[\tilde{P}]: [\tilde{\mathcal{L}}] \to [\tilde{\mathcal{L}}']$$

is an open, injective map with respect to appropriate choices of germ universal cover topologies.

**Proof.** The first statement follows from the definition of training and the fact that $^*p$ is injective on $^*\pi_1$. Let $L, L'$ be dense leaves in $\mathcal{L}, \mathcal{L}'$ containing $x, x'$. Then the lift of the restriction $P|_{L}, P|_{L}: \tilde{L} \to \tilde{L}'$, is a homeomorphism. It follows that the induced map $[\tilde{P}]$ is injective. $[\tilde{P}]$ is automatically open with respect to the germ universal cover topologies constructed as in Proposition 13. □

**Note 6.** Here is an example when the map $[\tilde{P}]$ is not surjective. Take $\mathcal{L} = \mathbb{R}, \mathcal{L}' = S^1$ and $P: \mathbb{R} \to S^1$ the universal cover. Then $[\tilde{\mathcal{L}}] \approx \mathbb{R}$ but $[\tilde{\mathcal{L}}'] \not\approx {^*\mathbb{R}}$.

Thus when $P$ is a cover trained, the image

$$C = [P]( [\pi]_1(\mathcal{L}) )$$

is a subgroupoid of $[\pi]_1(\mathcal{L}')$. We shall now construct lamination coverings from subgroups, restricting attention to the case where $[\pi]_1(\mathcal{L})$ is tame and a group. Assume that
has been given a germ universal cover topology \([\tilde{\pi}]\) of the type guaranteed by Theorem 12. Consider a subgroup \(G < [\pi](L)\) and denote by \(L_G\) the quotient \(\mathcal{C}\setminus\{\tilde{\mathcal{L}}\}\). Note that \(L_G\) decomposes into a disjoint union of leaves. Let \(L_C\) be any leaf over the dense leaf \(L \subset \mathcal{L}\). Consider the set \(X\) of topologies \([\pi_c]\) on \([\tilde{\mathcal{L}}]\) that satisfy the following conditions.

1. The induced topology \(\pi_c\) on \(L_C\) defines a (possibly non Hausdorff) lamination structure for which \(L_C\) is dense and \(L_C \to \mathcal{L}\) is a lamination map.
2. Let \(T_C\) be any transversal of \(L_G\), and denote by \([\tilde{T}_C]\) its preimage with the induced topology. If \(G \cdot \hat{x}\) is contained in \([T_C]\), then \(G \cdot \hat{x}\) is not open in the topology of \([T_C]\).
3. The identity map \((\tilde{\pi}, [\pi_c]) \to ([\tilde{\mathcal{L}}], [\pi])\) is open and \([\pi](\mathcal{L})\) acts by homeomorphisms on \(([\tilde{\mathcal{L}}], [\pi])\).

\(X\) is not empty, as it contains \([\pi]\). If we order the elements of \(X\) with respect to inclusion, then \(X\) is closed under chains and so contains a maximal element which we also denote \([\pi_c]\), called a covering topology. Denote by \(\pi_c\) the quotient topology induced by \([\pi_c]\) on \(L_C\).

**Theorem 15.** \(L_C\) is Hausdorff with respect to \(\pi_c\) and the map \(L_C \to \mathcal{L}\) is a lamination covering.

**Proof.** Since \([\pi_c]\) is maximal, for any \(\hat{x}\),

\[
G \cdot \hat{x} = \bigcap_{G \cdot \hat{y} \subseteq [\pi_c]} \mathcal{U}.
\]

It follows that \(L_C\) is Hausdorff. By construction, \(L_C \to \mathcal{L}\) is surjective and a covering when restricted to any leaf. \(\square\)

Two lamination coverings \(P_i : \mathcal{L}_i \to \mathcal{L}, i = 1, 2\), are isomorphic if there exists a geometric homeomorphism \(F : \mathcal{L}_1 \to \mathcal{L}_2\) such that \(P_1 = P_2 \circ F\). The group of automorphisms of a lamination cover \(P\) is denoted Aut\((P)\).

**Proposition 19.** Let \(\hat{u} \in [\pi](\mathcal{L}, x)\) and \(G' = \hat{u} \cdot G \cdot \hat{u}^{-1}\). Then there exist covering topologies \([\pi_c]\) and \([\pi_{c'}]\) so that \(L_C \to \mathcal{L}\) and \(L'_C \to \mathcal{L}\) are isomorphic.

**Proof.** Choose \([\pi_c]\) a covering topology for \(L_C\) and let \([\pi_{c'}]\) be the image of \([\pi_c]\) by \(\hat{u}\). Then \([\pi_{c'}]\) is a covering topology for \(L'_C\). With respect to these choices, the bijection \(\hat{x} \mapsto \hat{u} \cdot \hat{x}\) defines a homeomorphism

\[
([\tilde{\mathcal{L}}], [\pi_c]) \to ([\tilde{\mathcal{L}}], [\pi_{c'}])
\]

which descends to a geometric homeomorphism of covers. \(\square\)

Now suppose \(G < [\pi](\mathcal{L}, x)\) is a normal subgroup, \([\pi_c]\) the covering topology and \(P_G : L_C \to G\) the associated covering.

**Theorem 16.** Aut\((P_G)\) is isomorphic to the quotient \([\pi](\mathcal{L}, x)/C\). The quotient of \(L_C\) by \([\pi](\mathcal{L}, x)/G\) is \(\mathcal{L}\).

**Proof.** Every element of \(L_C\) is a class \(G \cdot \hat{x}\), for \(\hat{x} \in \tilde{\mathcal{L}}\). The action of \([\pi](\mathcal{L}, x)/G\) on such classes is well-defined and yields a subgroup of Aut\((P_G)\). On the other hand, the set \(\text{Lim}_G\) is a \([\pi](\mathcal{L}, x)/C\)-set on which any geometric automorphism acts automorphically. However the automorphism group of \(\text{Lim}_G\) is \([\pi](\mathcal{L}, x)/C\), so it follows that Aut\((P_G)\) \(\subseteq\) \([\pi](\mathcal{L}, x)/C\). It is clear that the quotient of \(L_C\) by \([\pi](\mathcal{L}, x)/C\) is \(\mathcal{L}\). \(\square\)
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