A DICHOTOMY FOR THE DIMENSION OF SRB MEASURE

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ABSTRACT. We study dynamical systems generated by skew products:

\[ T : [0, 1) \times \mathbb{R} \to [0, 1) \times \mathbb{R} \quad T(x, y) = (bx \mod 1, \gamma y + \phi(x)) \]

where integer \( b \geq 2 \), \( 0 < \gamma < 1 \) and \( \phi \) is a real analytic \( \mathbb{Z} \)-periodic function. We prove the following dichotomy for the SRB measure \( \omega \) for \( T \): Either the support of \( \omega \) is a graph of real analytic function, or the dimension of \( \omega \) is equal to \( \min\{2, 1 + \log b / \log \gamma\} \). Furthermore, given \( b \) and \( \phi \), the former alternative only happens for finitely many \( \gamma \) unless \( \phi \) is constant.

1. INTRODUCTION

In this paper, we consider dynamical systems generated by skew products:

\[ T : [0, 1) \times \mathbb{R} \to [0, 1) \times \mathbb{R} \quad T(x, y) = (bx \mod 1, \gamma y + \phi(x)) \]

where \( b \geq 2 \) is an integer, \( 0 < \gamma < 1 \) is a real number and \( \phi \) is a non-constant \( \mathbb{Z} \)-periodic Lipschitz function. There exists an ergodic probability measure \( \omega \) on \([0, 1) \times \mathbb{R}\) such that almost every point \( z \in [0, 1) \times \mathbb{R} \) is generic, that is,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(z)} = \omega \quad \text{weakly.} \]

So \( \omega \) is the unique SRB measure for \( T \). Let \( F_{b,\gamma}^\phi \) be the support of this measure, which is called the solenoidal attractor for \( T \) (See [11, Section 2]).

A probability measure \( \mu \) in a metric space \( X \) is called exact-dimensional if there exists a constant \( \beta \geq 0 \) such that for \( \mu - \text{a.e. } x \),

\[ \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \beta. \]

In this situation, we write \( \dim(\mu) = \beta \) and call it the dimension of \( \mu \). For any set \( K \subset \mathbb{R}^2 \), let \( \dim_H(K) \) be the Hausdorff dimension of \( K \). In the present paper, we study the dimension of the SRB measure \( \omega \) and the Hausdorff dimension of the attractor \( F_{b,\gamma}^\phi \). The main result of this paper is following.
Main Theorem. Let \( b \geq 2 \) be an integer, \( \gamma \in (0, 1) \) and let \( \phi \) be a \( \mathbb{Z} \)-periodic real analytic function. Then \( \omega \) is exact dimensional and exactly one of the following holds:

(i) \( F_{b, \gamma}^{\phi} \) is a graph of a real analytic function;
(ii) \( \dim_H(F_{b, \gamma}^{\phi}) = \dim(\omega) = \min\{2, 1 + \frac{\log b}{\log 1/\gamma}\} \).

Moreover, given \( b \) and non-constant \( \phi \), the first alternative only holds for finitely many \( \gamma \in (0, 1) \).

Historical remarks. In the work [14], Alexander and Yorke considered a class maps called generalized baker’s transformation:

\[
B : [-1, 1] \times [-1, 1] \ni (x, y) \mapsto B(x, y) = \begin{cases} 
(2x - 1, \gamma y + (1 - \gamma)) & x \geq 0 \\
(2x + 1, \gamma y - (1 - \gamma)) & x < 0.
\end{cases}
\]

Alexander and Yorke studied the case \( \frac{1}{2} < \gamma \leq 1 \) where \( T \) is locally area expanding. They showed that the map \( B \) admits an absolutely continuous ergodic measure (ACEM) if and only if the number \( \gamma \) satisfies a condition: absolute continuity of the corresponding infinitely convolved Bernoulli measure. Erdős [3] proved that \( B \) admits no ACEM if \( 1/\gamma \) is Pisot number. On the other hand, \( B \) admits an ACEM for Lebesgue almost every \( \gamma \in (1/2, 1] \) according to the results of Solomyak [10, 8]. Later Shmerkin [24] showed that the Hausdorff dimension of the exceptional set is zero. Recently Varjú [15] showed that \( B \) admits an ACEM for a class of algebraic parameters.

To study the class of dynamical systems that stably admits an ACEM with a negative Lyapunov exponents. Tsujii [11] introduced the class of dynamical systems generated by maps \( T \), which is a generalization of the generalized baker’s transformations \( B \) from the view of smoothness (See (1.1) for the definition of \( T \)).

In the case \( b\gamma < 1 \) where \( T \) contracts area, the SRB measure \( \omega \) is totally singular with respect to the Lebesgue measure, thus the natural questions in this situation are what are the dimension of \( \omega \) and the Hausdorff dimension of \( F_{b, \gamma}^{\phi} \). Our paper gives a complete answer when \( \phi \) is a real analytic \( \mathbb{Z} \)-periodic function. For the case \( b\gamma = 1 \) where \( T \) preserves area, our paper shows \( \dim(\omega) = 2 \) unless \( F_{b, \gamma}^{\phi} \) is a graph of a real analytic function. For the case \( b\gamma > 1 \), the natural questions are when \( \omega \) is absolutely continuous with respect to the Lebesgue measure and what are the geometric properties of \( F_{b, \gamma}^{\phi} \). In [11], Tsujii proved that the SRB measure is absolutely continuous respect to the Lebesgue measure for \( C^2 \) generic \( \phi \). Later Avila, Gouëzel and Tsujii [16] studied the smoothness of the SRB measure for \( C^r \) generic \( \phi \) ( for some integer \( r \geq 3 \)). For the geometric properties of \( F_{b, \gamma}^{\phi} \), Bamón et al. [17] proved the result: for any non-constant Lipschitz function \( \phi(x) \) and integer
$b \geq 2$, there exists $\gamma_3 \in (0, 1)$ such that the set $F_{b, \gamma}^\phi$ has non-empty interior for all $\gamma \in (\gamma_3, 1)$. Our results show weaker answers for these questions: we figure out when $\dim(\omega)$ and $\dim_H(F_{b, \gamma}^\phi)$ are equal to 2 for the situation that $\phi$ is a real analytic periodic function. Note that, if $\omega$ is absolutely continuous with respect to the Lebesgue measure, then $\dim(\omega) = 2$.

Another important reason to study the SRB measure $\omega$ is to study the Hausdorff dimension of the graph of Weierstrass-type functions
\[ W(x) = W_{\lambda, b}^\phi(x) = \sum_{n=0}^{\infty} \gamma^n \psi(b^n x), \quad x \in \mathbb{R} \]
where integer $b > 1$, $1/b < \lambda < 1$ and $\psi(x) : \mathbb{R} \to \mathbb{R}$ is a non-constant $\mathbb{Z}$-periodic Lipschitz function. The most famous example, with $\psi(x) = \cos(2\pi x)$, was introduced by Weierstrass as a continuous nowhere differentiable function, see [4]. Denote $\Gamma W_{\lambda, b}^\phi = \{ (x, W_{\lambda, b}^\phi(x)) \}$. In fact Ledrappier [6] show that $\dim_H(\Gamma W_{\lambda, b}^\phi)$ is equal to $2 + \frac{\log \lambda}{\log b}$ if $\dim(\omega) = 2$ where $\gamma = \frac{1}{by}$ and $\psi'(x) = \phi(x)$. Following this way, [1, 9] studied the values of $\dim_H(\Gamma W_{\lambda, b}^\phi)$ when $\psi(x) = \cos(2\pi x)$.

Let us also mention that Zhang [18] studied the smoothness of the SRB measures for the map $T : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1)$ when $\gamma = 1$. See [19] for more general fat Baker maps.

**Organization.** We shall introduce Theorem A and prove Main Theorem in Sect. 2. The rest of the paper is devoted to prove Theorem A by Hochman’s criterion on entropy increase (See Theorem 8.1). Thus we shall first recall Ledrappier-Young theorem and some basic properties of entropy of measures in Sect. 3, then we shall analyze the separation properties in Sect. 4, entropy porosity in Sect. 5 and transversality in Sect. 6. In Sect. 7 we will construct a nested sequence of partition of $\bigcup_{n=1}^{\infty} \{0, \ldots, b-1\}^n$. Finally we will assume the contrary and use Hochman’s criterion to obtain a contradiction in Sect. 8.

2. **Main findings and proof of Main Theorem**

In this section, we will first introduce Theorem A, then we will give some explanations for the idea of the proof of Theorem A. Finally we will use Theorem A to finish the proof of the Main Theorem.

2.1. **Theorem A.** Let $\mathbb{Z}_+$ denote the set of positive integers. Let $\mathbb{N}$ be the set of nonnegative integers. Let $\Lambda = \{0, 1, \ldots, b-1\}$, $\Lambda^n = \bigcup_{n=1}^{\infty} \Lambda^n$, $\Sigma = \Lambda^{\mathbb{Z}_+}$. For any word $j = j_1 j_2 \cdots j_p \in \Lambda^p$ of length $1 \leq p \leq \infty$ and $x \in [0, 1]$ define
\[ S(x, j) = S_{x, b}^\phi(x, j) = \sum_{n=1}^{p} \gamma^{n-1} \phi \left( \frac{x}{b^n} + \frac{j_1}{b^n} + \frac{j_2}{b^{n-1}} + \cdots + \frac{j_p}{b} \right), \]
and the map
\begin{equation}
G : [0, 1) \times \Sigma \to [0, 1) \times \mathbb{R}, \quad G(x, j) = (x, S(x, j)).
\end{equation}

let \( \nu \) denote even distributed probability measure on \( \Lambda \) and let \( \nu Z^+ \) be product measure on \( \Sigma \). Thus \( G(m \times \nu Z^+) \) is the SRB measure \( \omega \)

\[
\Omega_{b, \gamma} = \left\{(x, S(x, j)) : x \in [0, 1), j \in \Sigma \right\}
\]

where \( m \) is Lebesgue measure on \( [0, 1) \) (see [11, Section 2] for details).

In the work [11, 16, 1, 9], \( \omega \) has absolute continuity under suitable transversality conditions for all \( \gamma \in (1/b, 1) \). In the following, for \( \gamma \in (0, 1) \) we shall recall a gentle transversality the condition (H) and the degenerate situation the condition (H*) from the work [13].

**Definition 2.1.** Given an integer \( b \geq 2 \) and \( \gamma \in (0, 1) \), we say that a \( \mathbb{Z} \)-periodic \( C^1 \) function \( \phi(x) \) satisfies

- the condition (H) if

\[
S(x, j) - S(x, i) \neq 0, \quad \forall j \neq i \in \Sigma.
\]

- the condition (H*) if

\[
S(x, j) - S(x, i) \equiv 0, \quad \forall j, i \in \Sigma.
\]

The following is our main result in this work.

**Theorem A.** If a real analytic \( \mathbb{Z} \)-periodic function \( \phi(x) \) satisfies the condition (H) for an integer \( b \geq 2 \) and \( \gamma \in (0, 1) \), then

\[
\dim(\omega) = \min\{1 + \frac{\log b}{\log 1/\gamma}, 2\}.
\]

The idea of the proof of Theorem A is from the Hochman [5]’s breakthrough observation for entropy growth of measures under convolution. More specifically, a self-similar measure is the convolution of a measure with itself scaled down by some positive value, which allows Hochman to apply his criterion on entropy increase to get the dimension of self-similar measures in \( \mathbb{R} \) under the exponential separation condition. Later the methods are generalized to study the self-affine measures on the plane [2, 20]. Recently, following [2], Shen and the author [13] studied a class of measures induced by nonlinear IFS. Our paper will take the similar strategy in [2, 13].

Indeed, for any \( x \in [0, 1] \) define the map
\begin{equation}
S_x : \Sigma \to \mathbb{R}, \quad S_x(j) = S(x, j)
\end{equation}

and measure \( m_x := S_x(\nu Z^+) \). Note that \( \omega = \int_{[0,1]} (\delta_x \times m_x) dx \). By Ledrappier-Young theory [7], \( m_x \) is exact dimensional and there exists a constant \( \alpha \in \mathbb{R} \).
[0, 1] such that \( \dim(m_x) = \alpha \) for Lebesgue almost every \( x \in [0, 1] \), thus we only need to show \( \alpha = \min\{1, \frac{\log b}{\log 1/\gamma}\} \), see Sect 3.1 For any \( n \in \mathbb{Z}_+ \), \( \mathbf{j} = j_1j_2 \ldots j_n \in \Lambda^n \) and \( x \in [0, 1] \), let

\[
\mathbf{j}(x) = \frac{x + j_1 + \ldots + j_nb^{n-1}}{b^n}
\]

and

\[
T^n(m_{\mathbf{j}(x)}) = f_{x, \mathbf{j}}(m_{\mathbf{j}(x)})
\]

where \( f_{x, \mathbf{j}}(y) = y^n + S(x, \mathbf{j}) \), \( \forall y \in \mathbb{R} \). By [11, (9)] the following holds

\[
m_x = \frac{1}{b^n} \sum_{\mathbf{j} \in \Lambda^n} T^n(m_{\mathbf{j}(x)})
\]

for \( n \geq 1 \) and \( x \in [0, 1] \). Since most of \( m_{\mathbf{j}(x)} \) are similar in the sense of entropy when \( n \) is large enough, see Sect 5. Thus \( m_x \) is approximate with

\[
\left( \frac{1}{b^n} \sum_{\mathbf{j} \in \Lambda^n} \delta_{S(x, \mathbf{j})} \right) \ast (\gamma^n m_x).
\]

Assuming the contrary, we shall apply the Hochman’s criterion on entropy growth [5] to obtain a contradiction.

In [5] they studied the linear systems and relative self-similar(affine) measures. But since \( T \) is a nonlinear map, we shall use some ideas in [13] to construct the convolution of measures by transversality. In [13] the authors considered the functions

\( S_j : [0, 1] \rightarrow \mathbb{R} \quad S_j(x) = S(x, \mathbf{j}) \)

for any \( \mathbf{j} \in \Sigma \). But in our paper we shall consider the functions \( S_x \) which is defined in symbolic space \( \Sigma \) for any \( x \in [0, 1] \) (See (2.3) for the definition of \( S_x \).) Thus we shall take a different way to analyze the separation, transversality and construct a different partition for symbolic space, which are important in this strategy to prove Theorem A.

2.2. Proof of Main Theorem. To prove our main result, we shall need the dichotomy between condition (H) and condition (H*). The following is an immediate consequence of [21] Theorem 2.1. For the case \( by > 1 \), see also [13] Theorem A.

**Theorem 2.1.** Fix \( b \geq 2 \) integer and \( \gamma \in (0, 1) \). Assume that \( \phi \) is analytic \( \mathbb{Z} \)-periodic function. Then exactly one of the following holds:

(i) \( \phi \) satisfies the condition (H*);

(ii) \( \phi \) satisfies the condition (H).
Proof. Let $\phi^* : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi^*(x) = \phi(x) - \int_{[0,1]} \phi(s) \, ds$ for $x \in \mathbb{R}$, then $\hat{f}(x) := \int_0^x \phi^*(s) \, ds, \forall x \in \mathbb{R}$ is an analytic $\mathbb{Z}$-periodic function. For any $w \in \Sigma$, we have
\[
\int_0^x S_{b,\gamma}^\phi(s, w) \, ds = \frac{1}{\gamma} \sum_{n=1}^\infty (by)^n \left( \hat{f} \circ \tau_{w, n}(x) - \hat{f} \circ \tau_{w, n}(0) \right)
\]
where $\tau_{w, n}(x) = \frac{x + w_1 + \cdots + w_n b^k 1_{n-1}}{b^n}$. Therefore by [21, Theorem 2.1], the following are equivalent:

(i) there exists $i \neq j \in \Sigma$ with $\int_0^x S_{b,\gamma}^\phi(s, i) \, ds \equiv \int_0^x S_{b,\gamma}^\phi(s, j) \, ds$;
(ii) $\int_0^x S_{b,\gamma}^\phi(s, i) \, ds \equiv \int_0^x S_{b,\gamma}^\phi(s, j) \, ds$ for any $j \in \Sigma$.

Proof of the Main Theorem. There exists $\gamma_1 \in (0, 1)$ such that $\omega$ is absolutely continuous with respect with Lebesgue measure for all $\gamma \in (\gamma_1, 1)$ (See the proof of Theorem 1.1 in [9] for this conclusion). Thus we only need to consider the case $\gamma \in (0, \gamma_1)$. The proof for the dimension of SRB measure $\omega$ is similar to the proof of [13, Main Theorem] by Theorem A and Theorem 2.1. For non-degenerate $\omega$, by the mass distribution principle, we have $dim_H(F_{b,\gamma}^\phi) \geq \min(2, 1 + \frac{\log b}{\log 1/\gamma})$ since $F_{b,\gamma}^\phi$ is the support of $\omega$. Also we have $dim_b(F_{b,\gamma}^\phi) \leq \min(2, 1 + \frac{\log b}{\log 1/\gamma})$ by the definition of $F_{b,\gamma}^\phi$. Thus our theorem holds.

3. Preliminaries

In this section we shall first recall some basic facts from Ledrappier-Young theory. Later we will recall the definition and basic properties of the entropy of measures, which is a basic tool in our paper.

3.1. Ledrappier’s Theorem. The following Theorem 3.1 can be proved by similar methods in [6] Proposition 2 (See [7, 22] for more general results on compact manifolds). For the completeness, we offer a proof in Appendix.

Theorem 3.1. If $\phi : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Z}$-periodic Lipschitz function, then

(i) $\omega$ is exact dimensional;
(ii) there is a constant $\alpha \in [0, 1]$ such that for Lebesgue a.e. $x \in [0, 1]$, $m_x$ is exact dimensional and $\dim (m_x) = \alpha$. 

(3) \( \dim(\omega) = 1 + \alpha. \)

So it suffices to show that \( \alpha = \min\{1, \frac{\log b}{\log 1/y}\} \) under the condition (H) for proving Theorem A.

3.2. **Entropy of measures.** For a probability space \((\Omega, \mathcal{B}, \mu)\), a countable partition \(\mathcal{Q}\) is a countable collection of pairwise disjoint measurable subsets of \(\Omega\) whose union is equal to \(\Omega\). Let \(\mathcal{Q}(x)\) be the member of \(\mathcal{Q}\) that contains \(x\). For \(\mu(\mathcal{Q}(x)) > 0\), we call the conditional measure
\[
\mu_{\mathcal{Q}(x)}(A) = \frac{\mu(A \cap \mathcal{Q}(x))}{\mu(\mathcal{Q}(x))}
\]
a \(\mathcal{Q}\)-component of \(\mu\). Define the entropy
\[
H(\mu, \mathcal{Q}) = \sum_{Q \in \mathcal{Q}} -\mu(Q) \log_b \mu(Q)
\]
where the common convention \(0 \log 0 = 0\) is adopted. For another countable partition \(\mathcal{P}\), define the condition entropy as
\[
H(\mu, \mathcal{Q} | \mathcal{P}) = \sum_{P \in \mathcal{P}, \mu(P) > 0} \mu(P) H(\mu_P, \mathcal{Q}).
\]
When \(\mathcal{Q}\) is a refinement of \(\mathcal{P}\), i.e., \(\mathcal{Q}(x) \subset \mathcal{P}(x)\) for every \(x \in \Omega\), we have
\[
H(\mu, \mathcal{Q} | \mathcal{P}) = H(\mu, \mathcal{Q}) - H(\mu, \mathcal{P}).
\]

If there exists a sequence of partitions \(\mathcal{Q}_i, i = 0, 1, 2, \cdots\), such that \(\mathcal{Q}_{i+1}\) is a refinement of \(\mathcal{Q}_i\), we shall denote \(\mu_{i, \mathcal{Q}} = \mu_{\mathcal{Q}_i(x)}\), and call it a \(i\)-th component measure of \(\mu\). For a finite set \(I\) of integers, if for every \(i \in I\), there is a random variable \(Y_i\) defined over \((\Omega, \mathcal{B}(\mathcal{Q}_i), \mu)\), where \(\mathcal{B}(\mathcal{Q}_i)\) is the sub-\(\sigma\)-algebra of \(\mathcal{B}\) which is generated by \(\mathcal{Q}_i\). Then we shall use the following notation
\[
\mathbb{P}_{i \in I}(K_i) = \mathbb{P}^\mu_{i \in I}(K_i) := \frac{1}{\#I} \sum_{i \in I} \mu(K_i),
\]
where \(K_i\) is an event for \(Y_i\). If \(Y_i\)’s are \(\mathbb{R}\)-valued random variable, we shall also use the notation
\[
\mathbb{B}_{i \in I}(Y_i) = \mathbb{B}^\mu_{i \in I}(Y_i) := \frac{1}{\#I} \sum_{i \in I} \mathbb{B}(Y_i).
\]

Therefore the following holds
\[
H(\mu, \mathcal{Q}_{m+n} | \mathcal{Q}_n) = \mathbb{E}(H(\mu_{x,n}, \mathcal{Q}_{m+n})) = \mathbb{E}_{x=n}(H(\mu_{x,i}, \mathcal{Q}_{i+m})).
\]

These notations were used extensively in [5] and [2].

In most of cases, we shall consider the situation that \(\Omega = \mathbb{R}\) and \(\mathcal{B}\) the Borel \(\sigma\)-algebra. Let \(\mathcal{L}_n\) be the partition of \(\mathbb{R}\) into \(b\)-adic intervals of level
$n$, i.e., the intervals $[j/b^n, (j+1)/b^n)$, $j \in \mathbb{Z}$. Let $\mathcal{P}(\mathbb{R})$ denote the collection of all Borel probability measures in $\mathbb{R}$. If an probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is exact dimensional, its dimension is closely related to the entropy. More specifically we have the following fact [12, Theorem 4.4]. See also [23, Theorem 1.3].

**Proposition 3.1.** If $\mu \in \mathcal{P}(\mathbb{R})$ is exact dimensional, then
\[
\dim(\mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu, \mathcal{L}_n).
\]

The following are some well-known facts about entropy and conditional entropy, which will be used a lot in our work. See [5, Section 3.1] for details.

**Lemma 3.1 (Concavity).** Consider a measurable space $(\mu, \mathcal{B})$ which is endowed with partitions $\mathcal{Q}$ and $\mathcal{P}$ such that $\mathcal{P}$ is a refinement of $\mathcal{Q}$. Let $\mu, \mu'$ be probability measures in $(\mu, \mathcal{B})$. For any $t \in (0, 1)$,
\[
t H(\mu, \mathcal{Q}) + (1 - t) H(\mu', \mathcal{Q}) \leq H(t\mu + (1 - t)\mu', \mathcal{Q}),
\]
\[
t H(\mu, \mathcal{P}|\mathcal{Q}) + (1 - t) H(\mu', \mathcal{P}|\mathcal{Q}) \leq H(t\mu + (1 - t)\mu', \mathcal{P}|\mathcal{Q}).
\]

**Lemma 3.2.** Let $\mu \in \mathcal{P}(\mathbb{R})$. There is a constant $C > 0$ such that for any affine map $f(x) = ax + c$, $a, c \in \mathbb{R}$, $a \neq 0$ and for any $n \in \mathbb{N}$ we have
\[
\left| H(f\mu, \mathcal{L}_n + \{\log a|a|\}) - H(\mu, \mathcal{L}_n) \right| \leq C.
\]

**Lemma 3.3.** Given a probability space $(\Omega, \mathcal{B}, \mu)$, if $f, g : \Omega \to \mathbb{R}$ are measurable and $\sup_x |f(x) - g(x)| \leq b^{-n}$ then
\[
\left| H(f\mu, \mathcal{L}_n) - H(g\mu, \mathcal{L}_n) \right| \leq C,
\]
where $C$ is an absolute constant.

### 4. Exponential separation

In this section, we deduce from the condition (H) the exponential separation properties, which is used in the section 8.1 to prove Theorem A. The standard method and Definition 4.1 are from [5]. Before the statement of the theorem, it is convenient to introduce the following notation.

**Notation.** For every integer $n \in \mathbb{N}$, let $\hat{n}$ be the unique integer such that
\[
\gamma^\hat{n} \leq b^{-n} < \gamma^{\hat{n} - 1}.
\]

In fact the following definition is a little different from [5], which is more convenient in this paper.
Definition 4.1. Let $E_1, E_2, \ldots$ be subsets of $\mathbb{R}$. For any $\varepsilon > 0$ and $Q \subset \mathbb{Z}_+$, we say that the sequence $(E_n)_{n \in \mathbb{Z}}$ is $(\varepsilon, Q)$-exponential separation if

$$|p - q| > \varepsilon^b \quad \forall p \neq q \in E_b$$

for each $n \in Q$.

For $u = u_1 u_2 \ldots u_t \in \Lambda^\#$ and $j \in \Lambda^\# \cup \Sigma$, let $uj = u_1 u_2 \ldots u_{j_1} j_2 \ldots \in \Lambda^\# \cup \Sigma$ as usual. The main result of this section is the following theorem.

Theorem 4.1. If $\phi(x)$ is a real analytic $\mathbb{Z}$-periodic function which satisfies the condition (H) for some integer $b \geq 2$ and $\gamma \in (0, 1)$, then there exists $\ell_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that the following holds for Lebesgue-a.e. $x \in [0, 1]$.

For any integer $\ell \geq \ell_0$, there exists a set $Q_{x, \ell} \subset \mathbb{Z}_+$ such that

(i) $\#Q_{x, \ell} = \infty;$

(ii) for any $w \in \Lambda^\ell$, the sequence $(X_n^{w,x})_{n \in \mathbb{Z}}$ is $(\varepsilon_0, Q_{x, \ell})$-exponential separation

where $X_n^{w,x} = \{S(x, jw) : j \in \Lambda^{n-\ell}\}$ for $n > \ell$ and $X_n^{w,x} = \{0\}$ for $n \leq \ell$.

Before giving the proof, it is necessary to introduce the following formulas. For any $x \in [0, 1]$, $w = w_1 w_2 \ldots w_m \in \Lambda^\#$ and each $i, j \in \Lambda^\# \cup \Sigma$, we have

(4.2) $S(x, wi) = S(x, w) + \gamma^m S(w(x), i)$

by the definition of function $S(x, \cdot)$ where $w(x)$ is from (2.4). This implies, for any $k \in \mathbb{N},$

(4.3) $S^{(k)}(x, wi) - S^{(k)}(x, wj) = \left(\frac{\gamma}{b^k}\right)^m \left(S^{(k)}(w(x), i) - S^{(k)}(w(x), j)\right)$.

We also need to recall the following results in [5, Lemma 5.8] and [13, Lemma 5.2].

Lemma 4.1. Let $k \in \mathbb{N}$, and let $F$ a $k$-times continuously differentiable function on a compact interval $J \subset \mathbb{R}$. Let $M = \|F\|_{L^k}$, and let $0 < d < 1$ be such that for every $x \in J$ there is a $p \in \{0, 1, \ldots, k\}$ with $|F^{(p)}(x)| > d$. Then for every $0 < \rho < (d/2)^{1/k}$, the set $F^{-1}(-\rho, \rho) \subset F$ can be cover by $O_{k, M, \rho}(1/d^k)$ intervals of length $\leq 2(\rho/d)^{1/2k}$ each.

Lemma 4.2. If the condition (H) holds, then there exists a constant $\varepsilon_1 > 0$ and an integer $Q_1 \geq 0$ such that for every $i, j \in \Sigma$ such that $i_1 \neq j_1$ and any $x \in [0, 1]$, there exists $k \in \{0, 1, \ldots, Q_1\}$ such that

$$|S^{(k)}(x, i) - S^{(k)}(x, j)| \geq \varepsilon_1.$$
The proof of Theorem 4.1 Let $Q_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ be the values in Lemma 4.2. Let $\ell_0$ be a fixed integer such that
\[
\frac{(\gamma/b^n)^{\ell_0}}{1 - \gamma/b^n} \| \phi^{(q)} \|_\infty < \varepsilon_1/4
\]
for every $q \in \{0, 1, \ldots, Q_1\}$. Let $\ell, n$ be the integers such that $\ell \geq \ell_0$, $n > \ell$. Let $w \in \Lambda^n$ and $i \neq j \in \Lambda^{n-\ell}$, we shall consider the function $f_w^{(i)} : [0, 1] \to \mathbb{R}$
\[
f_w^{(i)}(x) = S(x, iw) - S(x, jw) \quad \forall x \in [0, 1].
\]
By (4.3) for each $q \in \{0, 1, \ldots, Q_1\}$ and $x \in [0, 1]$, we have
\[
(4.4) \quad (f_w^{(i)})^{(q)}(x) = \left(\gamma/b^n\right)^{\ell(q)}(S^{(q)}(u(x), i'w) - S^{(q)}(u(x), j'w))
\]
where $u = i \wedge j$ and $i' = \sigma^{|i|}i$, $j' = \sigma^{|i|}j$. Also by Lemma 4.2 there exists $k = k(x, w, i, j) \in \{0, 1, \ldots, Q_1\}$ such that
\[
|S^{(q)}(u(x), i'w_\infty) - S^{(q)}(u(x), j'w_\infty)| \geq \varepsilon_1
\]
where $0_\infty = 0 \ldots 0 \ldots \in \Sigma$. Combining this with (4.4) and the definition of $\ell_0$, then the following holds since $w \in \Lambda^\ell$ and $\ell \geq \ell_0$. For each $x \in [0, 1]$ there exists $k = k(x, w, i, j) \in \{0, 1, \ldots, Q_1\}$ such that
\[
|(f_w^{(i)})^{(k)}(x)| \geq b_n
\]
where $b_n = \frac{\varepsilon_1}{2} (\gamma/b^{Q_1})^n$.

Therefore we have the following property by Lemma 4.1. For every $0 < \varepsilon < (\gamma^2/b^{Q_1})^{2Q_1}$ let $n$ be an integer such that $n > \ell$ and $\varepsilon^n < (b_n/2)^{2Q_1}$, then the set
\[
E_{\varepsilon, n, \ell} = \bigcup_{w \in \Lambda^\ell} \bigcup_{i+j \in \Lambda^{n-\ell}} (f_w^{(i)})^{-1}(\varepsilon^n, \varepsilon^n)
\]
can be covered by $O(b^{2n-\ell}/(b_n)^{Q_1})$ intervals of length $\leq 2(\varepsilon^n/b_n)^{1/2Q_1}$. By the above for any integer $N$ such that $\hat{N} > \ell$, we have
\[
\text{dim}_0\left(\bigcap_{n \geq N} E_{\varepsilon, n, \ell}\right) \leq \lim_{n \to \infty} \log\left((b_n/\varepsilon)^{1/2Q_1}\right) \leq 2Q_1 \log(b^{2+Q_1}/\gamma^{Q_1}) / \log(\gamma/(b^{Q_1} \varepsilon)).
\]
Let $\varepsilon_0 = \varepsilon_0(\phi, \gamma, b, Q_1) > 0$ be small enough such that
\[
2Q_1 \log(b^{2+Q_1}/\gamma^{Q_1}) / \log(\gamma/(b^{Q_1} \varepsilon)) < 1,
\]
thus $m(\bigcap_{n \geq N} E_{\varepsilon, n, \ell}) = 0$ for all integer $N$ s.t. $\hat{N} > \ell$.

Denote $E_{\varepsilon_0} = \bigcup_{\ell \geq \ell_0} \bigcup_{\hat{N} > \ell} \bigcap_{n \geq N} E_{\varepsilon_0, n, \ell}$. Therefore, for any $x \in [0, 1] \setminus E_{\varepsilon_0}$ and $\ell \geq \ell_0$, there exists $Q_{x, \ell}$ such that (i) and (ii) holds. \qed
5. Entropy Porosity

In this section we shall analyze the entropy porosity of measures $m_x$ under the condition (H). This property will be used to obtain entropy growth under convolution by Hochman’s criterion (See Theorem 8.1) in Subsection 8.4.

**Definition 5.1 (Entropy porous).** A measure $\mu \in \mathcal{P}(\mathbb{R})$ is $(h, \delta, m)$-entropy porous from scale $n_1$ to $n_2$ if

$$\mathcal{P}^\mu_{n_1 \leq i \leq n_2}\left(\frac{1}{m}H(\mu_{x,i}, \mathcal{L}_{i+m}) < h + \delta\right) > 1 - \delta.$$ 

The main result of this section is the following Theorem 5.1.

**Theorem 5.1.** Fix an integer $b \geq 2$ and $\gamma \in (0, 1)$. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Z}$-periodic Lipschitz function such that condition (H) holds. Then for any $\varepsilon > 0$, $m \geq M_1(\varepsilon)$, $k \geq K_1(\varepsilon, m)$ and $n \geq N_1(\varepsilon, m, k)$, the following holds:

$$\nu^n (\{i \in \Lambda^n : m_{i(0)} \text{ is } (\alpha, \varepsilon, m) - \text{entropy porous from scale 1 to } k\}) > 1 - \varepsilon$$

where $\alpha$ is a constant in Theorem 3.1.

Recall that $i(0) = i_1 + i_2 b + \cdots + i_n b^{n-1}$ for $i \in \Lambda^n$. We shall follow the strategy in [2, Section 3] to prove Theorem 5.1 (See or [13, Section 4]). The only difficulty is to show $m_x$ is nonatomic.

### 5.1. Nonatomic measure $m_x$.

This subsection is devoted to prove the measure $m_x$ is nonatomic under the condition (H), which will be used to show the uniform continuity across scales of the measures $m_x$ (See the next subsection for the definition).

**Lemma 5.1.** If the condition (H) holds, the measure $m_x$ has no atom for any $x \in [0, 1]$.

**Proof.** By the definition of the measure $m_x$, there exists a constant $L = L(\phi, b, \gamma) > 0$ such that $m_x$ is support in $[-L, L]$ for all $x \in [0, 1]$. If the lemma fails, there exists $x_1 \in [0, 1]$ and $p \in \mathbb{R}$ such that

$$m_{x_1}([p]) = \max_{x \in [0, 1], z \in \mathbb{R}} m_x([z]) > 0$$

by the compactness of the probability measures $m_x$ in the weak star topology and the continuity of function $S(x, j)$. Also for any $n \in \mathbb{Z}_+$, by (2.6) we have

$$m_{x_1}([p]) = \frac{1}{b^n} \sum_{w \in \Lambda^n} T^n m_{w(x_1)}([p]).$$

Denote

$$p_w = \frac{p - S(x_1, w)}{\gamma^n},$$
thus $T^n m_{w(x)}(\{p\}) = m_{w(x)}(\{p_w\})$. Combining this with (5.1) and (5.2) we have

$$m_{w(x_0)}(\{p_w\}) = m_x(\{p\}) > 0 \quad \forall n \in \mathbb{Z}_+, \ w \in \Lambda^n.$$  

This implies

(5.3) \[ M_0 = \sup_{w \in \Lambda^n} |P_w| < \infty \]

since the supports of the family of probability measures $\{m_x\}_{x \in [0,1]}$ have uniform bound in $\mathbb{R}$.

For any $w \in \Lambda^n$ and $m \in \mathbb{Z}_+$, by the definition of $p_w$ we have

$$S(x_1, w_{0,m}) - S(x_1, w_{1,m}) = \gamma^{|w| + m}(p_{w_{1,m}} - p_{w_{0,m}})$$

where $1_m = 11 \ldots 1, 0_m = 00 \ldots 0 \in \Lambda^m$, which implies

$$S(w(x_1), 0_m) - S(w(x_1), 1_m) = \gamma^m (p_{w_{1,m}} - p_{w_{0,m}})$$

by (4.3). This and (5.3) implies that

$$|S(w(x_1), 0_m) - S(w(x_1), 1_m)| \leq 2\gamma^m M_0 \quad \forall m \in \mathbb{Z}_+.$$  

So when $m$ goes to infinity, we have

$$S(w(x_1), 0_\infty) - S(w(x_1), 1_\infty) = 0 \quad \forall w \in \Lambda^n.$$  

Therefore $S(x, 0_\infty) - S(x, 1_\infty) \equiv 0$ since the set $\{w(x_1)\}_{w \in \Lambda^n}$ is dense in $[0,1]$, which contradicts condition (H).

\[ \square \]

5.2. Uniform continuity across scales. Following [2], we call that a measure $\mu \in \mathcal{P}(\mathbb{R})$ is uniformly continuous across scales if for any $\varepsilon > 0$ there exists $\delta > 0$ satisfied that for each $x \in \mathbb{R}$ and $r \in (0,1)$, we have

(5.4) \[ \mu(B(x, \delta r)) \leq \varepsilon \mu(B(x, r)). \]

A family $M$ of measures in $\mathcal{P}(\mathbb{R})$ is called jointly uniformly continuous across scales if for each $\varepsilon > 0$ there exists $\delta > 0$ such that (5.4) holds for every $\mu \in M$, $x \in \mathbb{R}$ and any $r \in (0,1)$. The proof of the following Proposition [5.1] is similar to [13] proposition 4.1. For the readers’ convenience we will give the details.

**Proposition 5.1.** If the condition (H) holds, the family of measures $\{m_x\}_{x \in [0,1]}$ is jointly uniformly continuous across scales.

**Proof.** By Lemma 5.1, for any $\varepsilon > 0$ there exists $\delta' = \delta'(\varepsilon) > 0$ such that for each $x \in [0,1]$ and $y \in \mathbb{R}$, we have

(5.5) \[ m_x(B(y, \delta')) < \varepsilon \]
since the family of probability measures $m_x$ is compact in the weak star topology. Let $\delta > 0$ be some constant such that the following holds. For any $r \in (0, 1)$, there exists $n_r \in \mathbb{N}$ satisfying that

\[(5.6) \quad r\delta + 2\gamma^n ||\phi||_{\infty}/(1 - \gamma) < r\]

and

\[(5.7) \quad \delta r/\gamma^n < \delta'.\]

Now we shall consider $w \in \Lambda^n$ satisfying that

\[T^n m_{w(x)}(B(y, \delta r)) > 0.\]

By the definition of $T^n m_{w(x)}$ there exists $j \in \Sigma$ such that $|S(x, w_j) - y| \leq \delta r$,

which implies

\[|S(x, w_i) - y| \leq \delta r + 2\gamma^n ||\phi||_{\infty}/(1 - \gamma) \quad \forall i \in \Sigma.\]

Combining this with (5.6) we have

\[(5.8) \quad T^n m_{w(x)}(B(y, r)) = 1.\]

Also (5.5) and (5.7) implies

\[(5.9) \quad T^n m_{w(x)}(B(y, \delta r)) = m_{w(x)}\left(B(\frac{y - S(x, w)}{\gamma^n}, \delta r/\gamma^n)\right) < \varepsilon.\]

Finally by (5.8) and (5.9), we have

\[
m_x(B(y, \delta r)) = \frac{1}{b^n} \sum_{w \in \Lambda^n} T^n m_{w(x)}(B(y, \delta r))
\leq \frac{\varepsilon}{b^n} \#\left\{w \in \Lambda^n : T^n m_{w(x)}(B(y, \delta r)) > 0\right\}
\leq \frac{\varepsilon}{b^n} \sum_{w \in \Lambda^n} T^n m_{w(x)}(B(y, r))
= \varepsilon m_x(B(y, r)).
\]

\[\square\]

5.3. Entropy porosity of $m_x$. In this subsection we shall complete the proof of Theorem [5.1]

**Lemma 5.2.** For any $\varepsilon > 0, m \geq M_2(\varepsilon), n \geq N_2(\varepsilon, m)$,

\[
\inf_{x \in [0,1]} y^n \left\{ i \in \Lambda^n : \alpha - \varepsilon < \frac{1}{m}H(m_{i(x)}, \mathcal{L}_m) < \alpha + \varepsilon \right\} > 1 - \varepsilon.
\]

**Proof.** We may consider $h_m(x) = \frac{1}{m}H(m_x, \mathcal{L}_m)$ and the proof is the same as [13, Lemma 4.2].

\[\square\]
Lemma 5.3. Under the condition (H), for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( m \geq M_3(\varepsilon) \) and \( k \geq K_3(\varepsilon, m) \) and if \( \left| \frac{1}{k}H(m_x, \mathcal{L}_k) - \alpha \right| < \frac{\delta}{2} \), then \( m_x \) is \((\alpha, \varepsilon, m)\)-entropy porous from scale 1 to \( k \).

Proof. Combining Proposition 5.1 and Lemma 5.2 with (2.6), the proof is similar to [13, Lemma 4.5]. \( \square \)

Proof of Theorem 5.1. Given \( \varepsilon > 0 \), let \( \delta, M_3(\varepsilon) \) and \( K_3(\varepsilon, m) \) be given by Lemma 5.3. For this \( \delta > 0 \), by Lemma 5.2, when \( k \geq M_2(\delta/2) \) and \( n \geq N_2(\delta/2, k) \),

\[
\nu^n\left( \left\{ i \in A^n : \left| \frac{1}{k}H(m_{i(0)}, \mathcal{L}_k) - \alpha \right| < \frac{\delta}{2} \right\} \right) > 1 - \delta.
\]

Therefore, when \( m \geq M_3(\varepsilon) \), \( k \geq \max(K_3(\varepsilon, m), K_2(\delta/2)) \) and \( n \geq N_2(\delta/2, k) \), the Theorem holds. \( \square \)

6. Transversality

In this section, under the condition (H) we deduce some quantified estimates for transversality, which will be used to construct a sequence of partitions \( \mathcal{L}_n^A \) in Sect. 7. These partitions allow us to construct the convolution structures and control the entropy of measures to prove Theorem A.

For any \( i, j \in A^\# \cup \Sigma \) and every integer \( 1 \leq k \leq |j| \), let \( j_k = j_1j_2\ldots j_k \) as usual. We denote \( i < j \) if \( i = j_i \) holds. When \( |j| < \infty \), let \( I_j \) be an interval in \([0, 1]\) such that

\[
I_j = \left[ \frac{i_1 + i_2 b + \cdots + i_n b^{n-1}}{b^n}, 1 + \frac{i_1 + i_2 b + \cdots + i_n b^{n-1}}{b^n} \right).
\]

The main result of this section is the following Theorem 6.1.

Theorem 6.1. Fix an integer \( b \geq 2 \) and \( \gamma \in (0, 1) \). Assume \( \phi(x) \) is a real analytic \( \mathbb{Z} \)-periodic function such that \( \alpha < \min\{1, \frac{\log b}{\log 1/\gamma}\} \) and the condition (H) holds. For any \( t_0 > 0 \), there exists an integer \( t > t_0 \), real number \( \Delta_1 > 0 \) and \( h, h' \in A^t \) with the following property.

For every \( z \in I_a \) and \( i, j \in A^t \), if \( h < i, h' < j \), then

(A.1) \( |S'(z, i)|, |S'(z, j)| > \Delta_1 \);

(A.2) \( |S'(z, i) - S'(z, j)| > \Delta_1 \).

In fact, (A.2) implies that \( h \neq h' \) in Theorem 6.1.

Lemma 6.1. Under the assumption of Theorem 6.1 there exists \( x_2 \in [0, 1] \) and \( u \in \Sigma \) such that

\[ S'(x_2, u) \neq 0. \]
Proof. We only need to prove the following claim.

Claim 1. There exists $i, j \in \Sigma$ and $x_2 \in [0, 1]$ such that $i_1 \neq j_1$ and

$$S'(x_2, i) - S'(x_2, j) \neq 0.$$ 

Let us consider the function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{k=1}^{\infty} \gamma^{k-1} \phi\left(\frac{x}{b^k}\right) \forall x \in \mathbb{R}.$$ 

Thus $f(x) = S(x, 0_\infty)$ and $f(x+1) = S(x, 10_\infty)$ for each $x \in [0, 1]$ where $0_\infty = 000 \ldots \in \Sigma$ and $1 = 1 \in \Lambda$.

If the claim fails, then we have $f'(x) - f'(x+1) = 0$ for each $x \in \mathbb{R}$ since $f'(x) - f'(x+1)$ is a real analytic function in $\mathbb{R}$. Combining this with $\sup_{x \in \mathbb{R}} |f(x)| < \infty$ we have $f(x) - f(x+1) = 0$ for all $x \in \mathbb{R}$, which contracts the condition (H). Thus the claim holds. \hfill \Box

Lemma 6.2. Under the assumption of Theorem 6.1 for any $v, w \in \Lambda^\#$ there exists $i \neq j \in \Sigma$ and $x_{v, w} \in I_v$ such that

$$S'(x_{v, w}, wi) - S'(x_{v, w}, wj) \neq 0.$$ 

Proof. If the lemma fails, then there exists $v, w \in \Lambda^\#$ such that the following holds. For any $i, j \in \Sigma$ s.t. $i_1 \neq j_1$, we have

$$S'(x, wi) - S'(x, wj) = 0 \quad \forall x \in I_v.$$ 

Thus, by (4.3) we have

$$S'(x, i) - S'(x, j) = 0 \quad \forall x \in I_v.$$ 

Since $S'(x, i) - S'(x, j)$ is a real analytic function, we have

$$S'(x, i) - S'(x, j) = 0 \quad \forall x \in [0, 1].$$ 

This contradicts the claim and the Lemma holds. \hfill \Box

The proof of Theorem 6.1. By Lemma 6.1 and $\| \phi' \|_\infty < \infty$, there exists $\Delta > 0$ and $y, w \in \Lambda^\#$ such that

(6.1) $|S'(x, wt)| > \Delta$

for any $x \in I_y$ and every $t \in \Lambda^\# \cup \Sigma$. Also there exists $\Delta' > 0$ and $y', i \neq j \in \Sigma$ such that the following holds by Lemma 6.2 and $\| \phi' \| < \infty$

1. $I_{y'} \subset I_y$;
2. for any $x \in I_{y'}$ we have $|S'(x, wi) - S'(x, wj)| \geq \Delta'$.

Choose an integer $t \geq t_0 + |w| + |y'|$ such that

(6.2) $|S'(x, wi_{t-|w|}) - S'(x, wj_{t-|w|})| \geq \Delta'/2 \quad \forall x \in I_y$. 

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and
\[
\frac{2(\gamma/b)^\gamma \| \phi' \|_\infty}{1 - \gamma/b} < \Delta'/4.
\]
Let \( y'' \in \Lambda' \) s.t. \( I_{y''} \subset I_y \) and \( \Delta_1 = \min \{ \Delta, \Delta'/4 \} \). Thus \( h = w_{i-r-|w|}, h' = w_{j-r-|w|} \) and \( a = y'' \) are what we need by (6.1), (6.2) and (6.3). \( \square \)

7. The partitions of the space \( \Lambda^\# \)

In this section, we construct a sequence of partitions \( \mathcal{L}_{n^\#}^{\Lambda^\#} \) of the space \( \Lambda^\# \). The method is similar to [13, Section 6]. Also we give some useful properties about these partitions with Theorem 6.1.

In the rest of paper, we fix an integer \( b \geq 2 \) and \( \gamma \in (0, 1) \). Also suppose that \( \phi(x) \) is a real analytic \( \mathbb{Z}\)-periodic function such that the condition (H) holds. Combining Theorem 3.1 with Theorem 4.1 and Theorem 6.1, there exists an integer \( t > 0 \), some constants \( \Delta_1, C > 0 \), a point \( x_0 \in [0, 1) \), set \( M \subset \mathbb{Z}_+ \) and \( a, h, h' \in \Lambda' \) with the following properties.

(B.1) For each \( w \in \Lambda' \), the sequence \( (X_{w, x_0}^n)_{n \in \mathbb{Z}_+} \) is \( (\gamma^{C/2}, M) \)-exponential separation where \( X_{w, x_0}^n \) is from Theorem 4.1.

(B.2) \( \dim(\mathcal{M}_n) = \alpha' \);

(B.3) For every \( z \in I_a \) and \( i, j \in \Lambda^\# \), if \( h < i, h' < j \) then (A.1), (A.2) hold;

(B.4) \( \#M = \infty \).

In the rest of paper, we shall fix such elements \( \{ t, x_0, C, \Delta_1, M, a, h, h' \} \).

Let \( \pi : \Lambda^\# \to \mathbb{N} \times \mathbb{R}^3 \) be the map such that
\[
\pi(w) = (|w|, S(w(x_0), h), S(w(x_0), h'), S(x_0, w)).
\]

**Definition 7.1.** For any integer \( n \geq 1 \), let \( \mathcal{L}_{n^\#}^{\Lambda^\#} \) be the union of all the non-empty subsets of \( \Lambda^\# \) of the following form
\[
\pi^{-1} \left( \{ m \} \times I_1 \times I_2 \times J \right),
\]
where \( m \in \mathbb{N}, I_1, I_2 \in \mathcal{L}_n, J \in \mathcal{L}_{n+[n \log_3 1/\gamma]} \). The partition \( \mathcal{L}_{0^\#}^{\Lambda^\#} \) consists of non-empty subsets of \( \Lambda^\# \) of the following form
\[
\pi^{-1} \left( \{ m \} \times \mathbb{R} \times \mathbb{R} \times J \right),
\]
where \( m \in \mathbb{N}, J \in \mathcal{L}_{[n \log_3 1/\gamma]} \).

Note that \( \pi(x_0) \) is defined by (2.4).

**Lemma 7.1.** There exists \( R > 0 \) such that the following holds.

Let \( n, i \) be positive integers such that \( n > t \). For each \( u, v \in \Lambda^{\hat{n}+1} \), if \( ua \) and \( va \) belong to the same element of \( \mathcal{L}_{i}^{\Lambda^\#} \), then for any \( q \in \Lambda^\# \) and \( j, i \in \Sigma \), we have
\[
|S(x_0, uaqj) - S(x_0, vaqi)| \leq Rb^{-(n+i)}.
\]
Proof. We only need to consider the case \( u \neq v \). By the definition of the partition \( \mathcal{L}_n \), we have
\[
|S(u \alpha(x_0), h) - S(v \alpha(x_0), h)| \leq 1/b^j.
\]
Combining this with (A.1) we have
\[
|u \alpha(x_0) - v \alpha(x_0)| = O(1/b^j)
\]
since \( u \alpha(x_0), v \alpha(x_0) \in I_a \) (\( I_a \) is defined in Sect. 6). Thus
\[
(7.2) \quad |S(u \alpha(x_0), q) - S(v \alpha(x_0), q)| = O(1/b^j).
\]
Also we have
\[
(7.3) \quad |S(x_0, u \alpha) - S(x_0, v \alpha)| = O(1/b^{n+i})
\]
by the definition of the partition \( \mathcal{L}_n^\# \). Combining (7.3), (7.2) with
\[
S(x_0, u \alpha j) = S(x_0, u \alpha) + \gamma_i S(u \alpha(x_0), q) + \gamma_i^{j+1} S(u \alpha q(x_0), j)
\]
and
\[
S(x_0, v \alpha i) = S(x_0, v \alpha) + \gamma_i S(v \alpha(x_0), q) + \gamma_i^{j+1} S(v \alpha q(x_0), i),
\]
we have (7.1) holds for some constant \( R > 0 \) large enough. \( \square \)

It is necessary to introduce the following notation. For any probability measure \( \xi \in \mathcal{P}(\Lambda^\#) \) and \( u \in \Lambda^\# \), let probability measure \( A_u(\xi) \in \mathcal{P}(\mathbb{R}) \) be such that
\[
(7.4) \quad A_u(\xi) = \sum_{w \in \text{supp}(\xi)} \xi(w) \delta_{S(x_0, w u)}.
\]

Lemma 7.2. There exists a constant \( L_1 > 0 \) such that the following holds for positive integers \( n, k, i \) and \( q \in \Lambda^\# \). If \( \xi \) is a probability measure supported in an element of \( \mathcal{L}_n^\# \) such that \( |w| = n \) and \( I_w \subset I_a \) for each element \( w \) in the support of \( \xi \), then
\[
H(\xi, \mathcal{L}_n^\#) \leq H(A_{h'q}(\xi), \mathcal{L}_{i+k+n}) + H(A_{hq}(\xi), \mathcal{L}_{i+k+n}) + L_1.
\]

Proof. Define \( F : \text{supp}(\xi) \to \mathbb{R}^2 \), by
\[
w \to (S(x_0, whq), S(x_0, wh'q)).
\]

Claim 2. There exists a constant \( L_1 > 0 \) such that
\[
H(\xi, \mathcal{L}_n^\#) \leq H(F \xi, \mathcal{L}_{i+k+n}^{\mathcal{L}_n^\#}) + L_1.
\]

To prove this claim, take \( I \in \mathcal{L}_{i+k+n}^{\mathcal{L}_n^\#} \). It suffices to show that the cardinality of the set \( \{ J \in \mathcal{L}_n^\# | J \cap F^{-1}(I) \neq \emptyset \text{ and } J \cap \text{supp}(\xi) \neq \emptyset \} \) is uniformly bounded. For any \( w^{(m)} \in \text{supp}(\xi) \), with \( F(w^{(m)}) \in I \), \( m = 1, 2 \), we have
\[
|S(x_0, w^{(1)} hq) - S(x_0, w^{(1)} h'q)| - |S(x_0, w^{(2)} hq) - S(x_0, w^{(2)} h'q)| = O(b^{-(i+k+n)})
\]
by the definition of the function $F$, which implies
\[
|S(\mathbf{w}^{(1)}(x_0), \mathbf{h}q) - S(\mathbf{w}^{(1)}(x_0), \mathbf{h}'q)| - |S(\mathbf{w}^{(2)}(x_0), \mathbf{h}q) - S(\mathbf{w}^{(2)}(x_0), \mathbf{h}'q)| = O(b^{-(k+i)})
\]
by (4.3). Therefore
\[
|\mathbf{w}^{(1)}(x_0) - \mathbf{w}^{(2)}(x_0)| = O(b^{-(k+i)})
\]
by (A.2) and $\mathbf{w}^{(1)}(x_0), \mathbf{w}^{(2)}(x_0) \in I_a$ (A.2 is in Sect. 6). So we have
\[
|S(\mathbf{w}^{(1)}(x_0), \mathbf{h}) - S(\mathbf{w}^{(2)}(x_0), \mathbf{h})| = O(b^{-(k+i)})
\]
(7.5)
and
\[
|S(\mathbf{w}^{(1)}(x_0), \mathbf{h}') - S(\mathbf{w}^{(2)}(x_0), \mathbf{h}')| = O(b^{-(k+i)})
\]
(7.6)
Combining this with Claim 2 this lemma holds. □
8. Proof of Theorem A

In this section, we will apply Hochman’s criterion on entropy increasing to complete the proof of Theorem A. For a discrete measure \( \xi \in \mathcal{P}(\Lambda^\#) \) and \( \mathbf{q} \in \Lambda^\# \), let \( B_\mathbf{q}(\xi) \) denote the Borel probability measures in \( \mathbb{R} \) such that for any Borel subset \( A \) of \( \mathbb{R} \),

\[
B_\mathbf{q}(\xi)(A) = \xi \times \nu^z((\{ (\mathbf{w}, \mathbf{j}) \in \Lambda^\# \times \Sigma : S(x_0, \mathbf{w} \mathbf{q} \mathbf{j}) \in A \}).
\]

The key point is to introduce the discrete measure in \( \Lambda^\# \)

\[
(8.1) \quad \theta_n^\mathbf{u} := \frac{1}{b^{\hat{n} - \hat{i}}} \sum_{\mathbf{w} \in \Lambda^{\hat{n} - \hat{i}}} \delta_{\mathbf{w} \mathbf{u}}
\]

for any \( \mathbf{u} \in \Lambda^\hat{i} \) and integer \( n \) such that \( \hat{n} > \hat{i} \). Let \( n, i \) be integers satisfying that \( \hat{n}, \hat{i} > t \). \( \theta_n^\mathbf{u} \) implies that, for any Borel subset \( A \) of \( \mathbb{R} \),

\[
m_{x_0}(A) = \frac{1}{b^{2t}} \sum_{(\mathbf{u}, \mathbf{v}) \in \Lambda^\#	imes\Lambda^\#} \frac{1}{b^{\hat{n} - \hat{i}}} \sum_{\mathbf{q} \in \Lambda^{\hat{n} - \hat{i}}} \nu^z((\mathbf{j} \in \Sigma : S(x_0, \mathbf{w} \mathbf{u} \mathbf{v} \mathbf{q} \mathbf{j}) \in A)).
\]

Therefore we have

\[
m_{x_0}(A) = \frac{1}{b^{2t}} \sum_{(\mathbf{u}, \mathbf{v}) \in \Lambda^\#	imes\Lambda^\#} \frac{1}{b^{\hat{n} - \hat{i}}} \sum_{\mathbf{q} \in \Lambda^{\hat{n} - \hat{i}}} \theta_n^\mathbf{u} \times \nu^z((\{ (\mathbf{i}, \mathbf{j}) \in \Lambda^\# \times \Sigma : S(x_0, \mathbf{i} \mathbf{v} \mathbf{q} \mathbf{j}) \in A \}))
\]

by the definition of \( \theta_n^\mathbf{u} \). Thus we have

\[
(8.3) \quad m_{x_0}(A) = \frac{1}{b^{2t}} \sum_{(\mathbf{u}, \mathbf{v}) \in \Lambda^\#	imes\Lambda^\#} \frac{1}{b^{\hat{n} - \hat{i}}} \sum_{\mathbf{q} \in \Lambda^{\hat{n} - \hat{i}}} B_{\mathbf{v} \mathbf{q}}(\theta_n^\mathbf{u})
\]

by the definition of \( B_{\mathbf{v} \mathbf{q}}(\theta_n^\mathbf{u}) \).

8.1. The entropy of \( \theta_n^\mathbf{u} \). We start with analyzing the entropy of \( \theta_n^\mathbf{u} \) with respect to the partitions \( \mathcal{L}_n^\mathbf{A} \).

\textbf{Lemma 8.1.}

\[
\lim_{n \to \infty} \frac{1}{n} H(\theta_n^\mathbf{u}, \mathcal{L}_n^\mathbf{A}) = \lim_{n \to \infty} \frac{1}{n} H(m_{x_0}, \mathcal{L}_n) = \alpha.
\]

\textbf{Proof.} For \( \hat{n} > t \), define \( \pi_n : \Sigma \to \Lambda^\hat{n} \) by \( \pi(\mathbf{j}) = j_{i+1} \ldots j_{\hat{n} - t} \mathbf{a} \) and \( f_n : \Lambda^\hat{n} \to \mathbb{R} \) by \( f_n(\mathbf{w}) = S(x_0, \mathbf{w}) \) where \( j_{\hat{n} - t} = j_1 j_2 \ldots j_{\hat{n} - t} \). Recall the map \( S_{x_0} : \Sigma \to \mathbb{R} ; \mathbf{j} \mapsto S(x_0, \mathbf{j}) \), so we have \( f_n \pi_n - S_{x_0} = O(b^{-n}) \). Thus

\[
H(m_{x_0}, \mathcal{L}_n) = H(S_{x_0} \nu^z, \mathcal{L}_n) = H(f_n \pi_n \nu^z, \mathcal{L}_n) + O(1) = H(f_n \theta_n^\mathbf{a}, \mathcal{L}_n) + O(1)
\]

since \( \pi_n \nu^z = \theta_n^\mathbf{a} \). By \( \lim_{n \to \infty} \frac{1}{n} H(m_{x_0}, \mathcal{L}_n) = \alpha \), we have

\[
\lim_{n \to \infty} \frac{1}{n} H(f_n \theta_n^\mathbf{a}, \mathcal{L}_n) = \alpha.
\]

Since \( H(f_n \theta_n^\mathbf{a}, \mathcal{L}_n) = H(\theta_n^\mathbf{a}, \mathcal{L}_0^\mathbf{A}) \), the lemma holds. \( \square \)
Lemma 8.2.

\[
\lim_{n \to \infty, n \in M} \frac{1}{n} H(\theta_n^a, \mathcal{L}_{C_n}^{A_t}) = \frac{\log b}{\log(1/\gamma)}
\]

where \( M \) is from Sect. [7]

Proof. For \( n \in M \) large enough, we have

\[
|S(x_0,j a) - S(x_0,i a)| > b^{-cn} \quad \forall j \neq i \in \Lambda^{a-t}
\]

by (B.1). Thus

\[
H(\theta_n^a, \mathcal{L}_{C_n}^{A_t}) = \hat{n} - t
\]

by the definition of \( \mathcal{L}_{C_n}^{A_t} \). Since \( \lim_{n \to \infty} \hat{n}/n = \log b/\gamma \), the lemma holds. \( \square \)

8.2. Decomposition of entropy. In the following Lemma 8.3, we decompose the entropy of \( \theta_n^a \) and \( m_{x_0} \) into small scales.

Lemma 8.3. For any \( \varepsilon > 0 \), there exists \( C_0(\varepsilon) > 0 \) such that the following holds. For any positive integers \( k, n \) satisfying that \( n > C_0(\varepsilon)k \), we have

\[
\frac{1}{C_n} H(\theta_n^a, \mathcal{L}_{C_n}^{A_t}) \leq \mathbb{E}_{0 \leq i < C_n} \left[ \frac{1}{k} H((\theta_n^a)_w,i, \mathcal{L}_{i+k}^{A_t}) \right] + \varepsilon,
\]

\[
\frac{1}{C_n} H(m_{x_0}, \mathcal{L}_{(C+1)n} \| \mathcal{L}_n) \geq \frac{1}{b^{2t}} \sum_{(u,v) \in A \times A} \frac{1}{C_n} \sum_{0 \leq i < C_n, t \in \mathbb{Z}} \frac{1}{b^{j-t}} \sum_{q \in A^{-t}} \left[ \frac{1}{k} H(B_{vq}(\theta_n^a), \mathcal{L}_{i+k+n} \| \mathcal{L}_{i+n}) \right] - \varepsilon.
\]

Proof. The proof is similar to [13, Lemma 7.3], thus we only give sketchy proof for (8.5). By the same method as [13, Lemma 7.3], when \( n/k \) is large enough, we have

\[
\frac{1}{C_n} H(m_{x_0}, \mathcal{L}_{(C+1)n} \| \mathcal{L}_n) \geq \frac{1}{C_n} \sum_{0 \leq i < C_n} \left[ \frac{1}{k} H(m_{x_0}, \mathcal{L}_{i+k+n} \| \mathcal{L}_{i+n}) \right] - \varepsilon.
\]

Combining this with (8.3), thus (8.5) holds by concavity of conditional entropy.

\( \square \)

Corollary 8.1. If \( \alpha < \frac{\log b}{\log(1/\gamma)} \), then there exists \( \Delta_2 = \Delta_2(\alpha) > 0 \), \( p_0 = p_0(\alpha, \Delta_2) \in (0, 1) \) and \( C_1 = C_1(\alpha, \Delta_2) \), \( N_0 = N_0(\alpha, \Delta_2) > 0 \) such that the following holds. For any \( k \in \mathbb{Z}_+ \), \( n \in M \) satisfied that \( n/k > C_1, n \geq N_0 \), we have

\[
\mathbb{P}_{0 \leq i < C_n} \left[ \frac{1}{k} H((\theta_n^a)_w,i, \mathcal{L}_{i+k}^{A_t}) > \Delta_2 \right] > p_0.
\]
Proof. Let \( \Delta_3 = \frac{1}{4} (\log b - \alpha) > 0 \). Combining (8.4) with Lemma 8.1 and Lemma 8.2 we have

\[
\frac{1}{k} H((\theta_n^a)_{w,i}, \mathcal{L}_{i+k}^A) \geq \Delta_3
\]

when \( n > C_0(\Delta_3)k \) and \( n \in M \) is large enough. Also there is constant \( L_2 = L_2(\phi, \gamma) > 0 \) such that each element of \( \mathcal{L}_{i+k}^A \) contains at most \( b^{L_2k} \) elements of \( \mathcal{L}_{i+k}^A \) (See [13, Lemma 6.1] for the idea of the proof). Therefore for any \( w \in \text{supp}(\theta_n^a) \), we have

\[
\frac{1}{k} H((\theta_n^a)_{w,i}, \mathcal{L}_{i+k}^A) \leq L_2 \quad \forall \ 0 \leq i < Cn.
\]

Combining this with (8.6) our claim holds. \( \square \)

8.3. The lower bound estimate of the condition entropy of \( B_q(\theta_n^a) \). The following Lemma 8.4 is a trivial lower bound estimate of condition entropy by concavity of conditional entropy. Recall \( q(0) = \sum_{i \in \alpha} q(w_{i+1}) b^{-i} \) for each \( i \in \mathbb{Z} \) and \( q \in \Lambda^i \). The following \( B_q(\theta_n^a) \) is defined by (8.1).

Lemma 8.4. For any \( \varepsilon > 0 \), if \( k, n, i \) are positive integers with \( \hat{n} > t, i > C_2k \) and \( k \geq K_4(\varepsilon) \), then for any \( u \in \Lambda^i \) and \( q \in \Lambda^j \), we have

\[
\frac{1}{k} H(B_q(\theta_n^a), \mathcal{L}_{i+k+n} \mathcal{L}_{i+n}) \geq \frac{1}{k} H(m_q(0), \mathcal{L}_k) - \varepsilon
\]

where \( C_2 = 2 \log \gamma^{-1} / \log b \).

Proof. By the concavity of conditional entropy and the definition of \( \theta_n^a \),

\[
\frac{1}{k} H(B_q(\theta_n^a), \mathcal{L}_{i+k+n} \mathcal{L}_{i+n}) \geq \frac{1}{b^{j-i}} \sum_{w \in \Lambda^{j-i}} \frac{1}{k} H(B_q(\delta_{wu}), \mathcal{L}_{i+k+n} \mathcal{L}_{i+n}).
\]

Since the measure \( B_q(\delta_{wu}) \) is support in an interval of length \( O(b^{-j-i+n}) \) by (4.2) and the definition of \( B_q(\delta_{wu}) \), we have

\[
H(B_q(\delta_{wu}), \mathcal{L}_{i+n}) = O(1).
\]

Also for any \( j \in \mathbb{S} \), we have

\[
S(x_0, \text{wuq}j) - (S(x_0, \text{wuq}) + \gamma^{\hat{i}+\hat{j}} S(q(0), j)) = \gamma^{\hat{i}+\hat{j}} (S(\text{wuq}(x_0) j) - S(q(0), j)).
\]

Combining this with

\[
|\text{wuq}(x_0) - q(0)| = O(b^{\hat{i}})
\]

and \( \hat{i} \geq \frac{\log b}{\log 1/\gamma} > \frac{\log b}{\log 1/\gamma} C_2 k > k \) we have

\[
|S(x_0, \text{wuq}j) - f \circ S(q(0), j)| = O(b^{-n+i+k})
\]
where \( f(s) = S(x_0, w u q) + \gamma h^i s, \forall s \in \mathbb{R} \). Thus the following holds

\[
H(B_q(\delta_w), \mathcal{L}_{i+n+k}) = H(f m_{q(0)}, \mathcal{L}_{i+n+k}) + O(1) = H(m_{q(0)}, \mathcal{L}_k) + O(1)
\]

Combining this with (8.8) we have

\[
H(B_q(\delta _w), \mathcal{L}_{i+k+n}\mathcal{L}_{i+n}) = H(m_{q(0)}, \mathcal{L}_k) + O(1).
\]

Let \( k \) be large enough, thus the lemma holds by (8.7). \( \square \)

We shall use the Hochman’s criterion on entropy increase (See Theorem 8.1) to give the nontrivial lower bound estimate of the entropy of \( B_q(\theta_n^a) \) when \( h < q \) or \( h' < q \) in Sect. 8.4. Thus we need the following result. For any measure \( \mathbb{P} \in \mathcal{P}(\mathbb{R}) \) and \( r > 0 \), let

\[
r^\mathbb{P} := h_r^\mathbb{P}
\]

where \( h_r(s) = rs, \forall s \in \mathbb{R} \).

**Lemma 8.5.** For any \( \varepsilon > 0 \), there exists \( K_\varepsilon(\varepsilon) > 0 \) such that when \( h > t, i > C_2 k \) and \( k \geq K_\varepsilon(\varepsilon) \), the following holds. For any \( q \in \Lambda \) if \( h < q \) or \( h' < q \), then

\[
\frac{1}{k} H(B_q(\theta_n^a), \mathcal{L}_{i+k+n}\mathcal{L}_{i+n}) \geq \mathbb{E}^\mathbb{P}_i \left[ \frac{1}{k} H\left( \left. \gamma^{-i}\hat{\gamma}^n A_q\left((\theta_n^a)_w,i\right) \right| m_{q(0)}, \mathcal{L}_k \right) \right] - \varepsilon
\]

where \( C_2 \) is from Lemma 8.4.

Recall that \( A_q((\theta_n^a)_w,i) \) is defined by (7.4) and \( B_q(\theta_n^a) \) is defined by (8.1).

**Proof.** By concavity of conditional entropy we have

\[
(8.9) \quad \frac{1}{k} H(B_q(\theta_n^a), \mathcal{L}_{i+k+n}\mathcal{L}_{i+n}) \geq \mathbb{E}^\mathbb{P}_i \left[ \frac{1}{k} H\left( B_q((\theta_n^a)_w,i), \mathcal{L}_{i+k+n}\mathcal{L}_{i+n} \right) \right].
\]

For each \( w \in \text{supp}(\theta_n^a) \), let \( \Psi = (\theta_n^a)_w,i \).

**Claim 3.**

\[
(8.10) \quad H(B_q(\Psi), \mathcal{L}_{i+n}) = O(1).
\]

For each \( w', \ w'' \in \text{supp}(\Psi) \) and \( j', j'' \in \Sigma \), we have

\[
|S(x, w' q j') - S(x, w'' q j'')| \leq R b^{-i(n+i)}
\]

by Lemma 7.1 which implies \( B_q(\Psi) \) is supported in an interval of length \( O(b^{-i(n+i)}) \). Thus the claim holds.

**Claim 4.**

\[
H(B_q(\Psi), \mathcal{L}_{i+n+k}) = H(A_q(\Psi) * [\gamma h^i m_{q(0)}], \mathcal{L}_{i+n+k}) + O(1).
\]
To prove the claim let us consider the map
\[ F : \text{supp}(\Psi) \times \Sigma \to \mathbb{R} \quad (w', j) \to S(x_0, w'q_j) \]
and the map
\[ Q : \text{supp}(\Psi) \times \Sigma \to \mathbb{R} \quad (w', j) \to S(x_0, w'q) + \gamma^{\hat{j}+\hat{i}}S(q(0), j). \]
Since \( |q(0) - w'q(x_0)| = O(b^{-\hat{i}}) \) and \( \hat{i} > k \), we have
\[ |F(w', j) - Q(w', j)| = O(b^{-(n+i+k)}), \]
which implies that
\[ H(F(\Psi \times \nu_{Z}^{-}), \mathcal{L}_{i+n+k}) = H(Q(\Psi \times \nu_{Z}^{-}), \mathcal{L}_{i+n+k}) + O(1). \]
Combining this with
\[ H(F(\Psi \times \nu_{Z}^{-}), \mathcal{L}_{i+n+k}) = H(B_q(\Psi), \mathcal{L}_{i+n+k}) \]
and
\[ H(Q(\Psi \times \nu_{Z}^{-}), \mathcal{L}_{i+n+k}) = H(A_q(\Psi) * m_q(0), \mathcal{L}_{i+n+k}), \]
the claim holds.

Finally Claim 3 and Claim 4 imply that
\[ H(B_q(\Psi), \mathcal{L}_{i+n+k}|_{\mathcal{L}_{i+n}}) = H([\gamma^{\hat{i}}A_q(\Psi)] * m_q(0), \mathcal{L}_{k}) + O(1). \]
Combining this with (8.9) the lemma holds when \( k \) is large enough. \( \square \)

8.4. The proof of Theorem A. The following Theorem 8.1 is a version of Hochman’s entropy increasing criterion, see [5, Theorem 2.8] and [2, Theorem 4.1].

**Theorem 8.1** (Hochman). *For any \( \epsilon > 0 \) and \( m \in \mathbb{Z}_+ \) there exists \( \delta = \delta(\epsilon, m) > 0 \) such that for \( k > K(\epsilon, \delta, m) \), \( n \in \mathbb{N} \), and \( \tau, \theta \in \mathcal{P}(\mathbb{R}) \), if*

1. \( \text{diam}(\text{supp}(\tau)), \text{diam}(\text{supp}(\theta)) \leq b^{-n} \),
2. \( \tau \) is \((1 - \epsilon, \frac{\epsilon}{2}, m)\)-entropy porous from scales \( n \) to \( n + k \),
3. \( \frac{1}{k}H(\theta, \mathcal{L}_{n+k}) > \epsilon \),

*then*
\[ \frac{1}{k}H(\theta * \tau, \mathcal{L}_{n+k}) \geq \frac{1}{k}H(\tau, \mathcal{L}_{n+k}) + \delta. \]

We have the following Lemma.

**Lemma 8.6.** *If \( \alpha < \{1, \frac{\log b}{\log 1/g}\} \), then there exists a constant \( A_4 = A_4(\alpha) > 0 \) such that the following holds.*
For any \( \varepsilon > 0 \) and positive integers \( n, i, k \) when \( \hat{n} > t, k \geq K_6 = K_6(\varepsilon, \Delta_4, \alpha) \) and \( i \geq I_1 = I_1(\varepsilon, \Delta_4, \alpha, k) \), we have

\[
\frac{1}{b^{i-t}} \sum_{q \in \Lambda^{i-t}} \left[ \frac{1}{k} \mathsf{H}(\mathbb{B}_n^*(\theta^a_n), \mathcal{L}_{i+k+n} \mid \mathcal{L}_{i+n}) + \frac{1}{k} \mathsf{H}(\mathbb{B}_n^q(\theta^a_n), \mathcal{L}_{i+k+n} \mid \mathcal{L}_{i+n}) \right]
\]

(8.11)

\[
\geq \Delta_4 P_{i,k} + \frac{1}{b^{i-t}} \sum_{q \in \Lambda^{i-t}} \left[ \frac{1}{k} \mathsf{H}(m_{hq(0)}, \mathcal{L}_k) + \frac{1}{k} \mathsf{H}(m_{hq(0)}, \mathcal{L}_k) \right] - \varepsilon
\]

where \( \Delta_2 > 0 \) is from Corollary [8.1] and

\[
P_{i,k} = \mathbb{P}_{i} \left[ \frac{1}{k} \mathsf{H}(\theta^a_n(\omega, i), \mathcal{L}_{i+k}) > \Delta_2 \right].
\]

Proof. By Lemma [8.5] if \( k, n, i \) are positive integers with \( \hat{n} > t, i > C_2 k \) and \( k \geq K_5(\varepsilon/4) \), for any \( q \in \Lambda^{i-t} \), we have

\[
\frac{1}{k} \mathsf{H}(\mathbb{B}_n^*(\theta^a_n), \mathcal{L}_{i+k+n} \mid \mathcal{L}_{i+n}) + \frac{1}{k} \mathsf{H}(\mathbb{B}_n^q(\theta^a_n), \mathcal{L}_{i+k+n} \mid \mathcal{L}_{i+n}) \geq \mathbb{E}_{i} \left[ \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq}(\theta^a_n)) * m_{hq(0)}, \mathcal{L}_k) + \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq(0)}, \mathcal{L}_k) \right] - \varepsilon/2.
\]

For each \( w \in \text{supp}(\theta^a_n) \), let \( \Psi = (\theta^a_n)_{w,i} \). For the case \( \frac{1}{k} \mathsf{H}(\Psi, \mathcal{L}_{i+k}^\Lambda) > \Delta_2 \), by Lemma [7.2] we have

\[
\frac{1}{k} \mathsf{H}(\Psi, \mathcal{L}_{i+k}^\Lambda) \geq \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq}(\Psi), \mathcal{L}_k) + \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq}(\Psi), \mathcal{L}_k) + O\left(\frac{1}{k}\right),
\]

thus there exist \( h^* \in \{h, h'\} \) such that

\[
(8.13) \quad \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq}(\Psi), \mathcal{L}_k) \geq \Delta_2/4
\]

when \( k \geq K_7(\Delta_2) > 0 \).

Let \( \varepsilon_4 = \min\{\frac{1-\alpha}{3}, d_2, \frac{1}{16b}\} \). By Theorem [5.1] let \( m = M_1(\varepsilon_4) \). For any \( k \geq K_1(\varepsilon_4, m) \) and \( i \geq t + N_1(\varepsilon_4, m, k) \), thus we have

\[
(8.14) \quad \gamma^{i-t} \left\{ q \in \Lambda^{i-t} : m_{hq(0)}, m_{hq(0)}(q) \text{ are } (1 - \varepsilon_4, \varepsilon_4/2, m) \text{ - entropy porous from scale } 1 \text{ to } k \right\} > \frac{1}{2}.
\]

Also Lemma [7.1] implies that \( A_{hq}(\Psi) \) is supported in an interval of length \( O(b^{-(i+n)}) \). Combining this with Theorem [8.1] we have following property.

There exists \( \Delta_5 = \delta(\varepsilon_4, m) > 0 \) such that for any \( k \geq K(\varepsilon_4, \Delta_5, m) \), if

1. \( m_{hq(0)} \) is \( (1 - \varepsilon_4, \frac{\varepsilon_4}{2}, m) \)-entropy porous from scales 1 to k;
2. \( \frac{1}{k} \mathsf{H}(\gamma^{-(i+k)} A_{hq}(\Psi), \mathcal{L}_k) \geq \Delta_2/4,
\]
then

\[ \frac{1}{k} H(\{y^{-(i+b)} A_{h'2}(P)\} * m_{h'2}(0), \mathcal{L}_k) \geq \frac{1}{k} H(m_{h'2}(0), \mathcal{L}_k) + A_5. \]

Let \( A_4 = A_5/2 \), \( K_6 = \max\{K_3(\varepsilon), K_5(A_4), K_1(\varepsilon_4, m), K(\varepsilon_4, A_4, m)\} \) and \( I_1 = kC_2(t+N(\varepsilon_4, m, k)) \). Thus Combining (8.15) with (8.12), (8.13) and (8.14), our claim holds.

**The proof of Theorem A.** Assume that \( \alpha < \min\{1, \frac{\log b}{\log 1/\gamma}\} \). Let \( p_0 \) be the value in Corollary 8.1. \( A_4 \) is from Lemma 8.6. Denote \( \varepsilon = \frac{A_4p_0}{10b^2}. \) By (8.5), Lemma 8.4 and Lemma 8.6, if \( k, n \) are positive integers with \( n > \max\{C_0(\varepsilon), C_2k\}, k \geq \max\{K_4(\varepsilon), K_6(\varepsilon, A_4, \alpha)\} \) and \( n \geq I_1(\varepsilon, A_2, \alpha, k) \), then we have

\[ \frac{1}{Cn} H(m_{x_0}, \mathcal{L}_{(C+1)n}|\mathcal{L}_n) \geq \frac{1}{Cn} \sum_{i=l_i}^{Cn-1} b^i \sum_{q \in \Lambda^i} \frac{1}{k} H(m_{q(0)}, \mathcal{L}_k) - 4\varepsilon + \frac{A_4}{b^2} \mathbb{P}_{k,Cn} \]

where

\[ \mathbb{P}_{k,Cn} = \mathbb{P}_{I_1 \leq i < Cn} \left\{ \frac{1}{k} H((\theta_n^{x_0})_{w,i}, \mathcal{L}_{(C+1)n_i}) > A_2 \right\}. \]

Also by Lemma 8.2, when \( k, n \) are large enough we have

\[ \frac{1}{Cn} \sum_{i=l_i}^{Cn-1} b^i \sum_{q \in \Lambda^i} \frac{1}{k} H(m_{q(0)}, \mathcal{L}_k) \geq \alpha - \varepsilon. \]

By Corollary 8.1 for \( n \in M \) large enough, we have

\[ \mathbb{P}_{I_1 \leq i < Cn} \left\{ \frac{1}{k} H((\theta_n^{x_0})_{w,i}, \mathcal{L}_{(C+1)n_i}) > A_2 \right\} \geq \frac{P_0}{2}. \]

Combining this with (8.17) and (8.16) we have

\[ \frac{1}{Cn} H(m_{x_0}, \mathcal{L}_{(C+1)n}|\mathcal{L}_n) \geq \alpha + \varepsilon \]

for \( n \in M \) large enough. However, as \( n \to \infty \), the left hand side of (8.18) converges to \( \alpha \), a contradiction!

9. **Appendix: Proof of Theorem 3.1**

Let us consider the bijection

\[ G : \Sigma \times \mathbb{R} \times [0, 1) \to \Sigma \times \mathbb{R} \times [0, 1) \quad (j, y, x) \to (\sigma_j, y - \frac{y}{\gamma} + \frac{x - y}{b}, \frac{x + y}{b}) \]

where \( \sigma \) is the shift transformation on \( \Sigma \), and the projection map

\[ \Pi : \Sigma \times [0, 1) \to \Sigma \times \mathbb{R} \times [0, 1) \quad (j, x) \to (j, S(x, j), x). \]
Let $\mu = \Pi(v_{\mathbb{Z}} \times m)$ and it is easy to see that the measure $\mu$ is invariant and ergodic with respect to $G$.

For $n \in \mathbb{Z}$ define $\mathcal{P}^n = \{ [w] \times \mathbb{R} \times [0, 1) : w \in \Lambda^n \}$. Let $\mathcal{P}^n(j, y, x)$ be the only element of $\mathcal{P}^n$ that contains $(j, y, x)$ and $\mathcal{P}^0 := \Sigma \times \mathbb{R} \times [0, 1)$. For any $x \in [0, 1)$, let

$$Y_s : \Sigma \to \Sigma \times \mathbb{R} \times [0, 1) \quad j \to (j, S(x, j), x)$$

and

$$\mu_s := Y_s(v_{\mathbb{Z}}).$$

We shall recall a basic property of measures $\mu_s$ in this system.

**Lemma 9.1.** For every $(j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1)$, $n \in \mathbb{N}$ and $r > 0$, we have

$$\mu_s(B^T_{j, y, x}(r) \cap \mathcal{P}^{n+1}(j, y, x)) = \frac{1}{b^n} \mu_s(B^T_{j, y, x}(r) \cap \mathcal{P}^n(G(j, y, x)))$$

where $B^T_{j, y, x}(r) = \{(j', y', x') \in \Sigma \times \mathbb{R} \times [0, 1) : |y - y'| \leq r\}$.

**Proof.** We only need to show the case $n = 0$ and others are similar. Let us consider the sets

$$E = \{j' \in \Sigma : |S(x, j') - y| \leq r, j_1 = j_1\}$$

and

$$F = \{i \in \Sigma : |S(x + j_1, i) - y - \frac{\phi(d_1 + i)}{b} + \frac{y}{\gamma}| \leq \frac{r}{\gamma}\}.$$

By the definition of the function $S$, we have $\sigma E = F$. Thus

$$v_{\mathbb{Z}}(E) = \frac{1}{b} v_{\mathbb{Z}}(F)$$

by the property of Bernoulli measure $v_{\mathbb{Z}}$, which implies (9.1) holds.

Let us consider the measurable partition of $\Sigma \times \mathbb{R} \times [0, 1)$ defined by

$$\zeta := \left\{ \Sigma \times \{y\} \times \{x\} : y \in \mathbb{R}, x \in [0, 1) \right\}.$$

Then by the result of Rokhlin [25], there exists a canonical system of conditional measure. For $\mu$ almost every $(j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1)$ there exists conditional measure $\mu_{(j, y, x)} \in \mathcal{P}(\Sigma \times \mathbb{R} \times [0, 1))$ such that

(C.1) $\text{supp} \mu_{(j, y, x)} \subset \zeta(j, y, x)$

(C.2) $\mu_{(j, y, x)} = \mu_{(j', y, x)}$, for all $(j', y, x) \in \zeta(j, y, x)$.

(C.3) for every $A \in \mathcal{B}(\Sigma \times \mathbb{R} \times [0, 1))$,

$$\mu(A) = \int_{\Sigma \times \mathbb{R} \times [0, 1)} \mu_{(j, y, x)}(A) d\mu(j, y, x)$$

(9.2)
where $0 = 00 \ldots \in \Sigma$.

9.1. The exact properties of $m_x$. In this subsection we shall show that $m_x$ is exact dimensional for almost everywhere $x \in [0, 1)$, and give a formula for $\text{dim}(m_x)$.

The idea of the proof in this part is from [6]. Define on $\Sigma \times \mathbb{R} \times [0, 1)$ the mapping $g_k$ and $g$ by

$$g_k(j, y, x) = -\log \frac{\mu_x \left( B^j_{(j, y, x)}(y^k) \cap P(j, y, x) \right)}{\mu_x \left( B^j_{(j, y, x)}(y^k) \right)}$$

and

$$g(j, y, x) = -\log \left( \mu_{(j, y, x)} \left( P(j, y, x) \right) \right)$$

for $k \in \mathbb{Z}_+$.

**Lemma 9.2.** $\sup_{x \in [0, 1)} \int_{\Sigma \times \mathbb{R} \times \{x\}} \sup_{k \geq 1} g_k(j, y, x) \, d\mu_x(j, y, x) < \infty$

**Proof.** Let $g_{\text{sup}}(j, y, x) = \sup_{m \geq 1} g_m(j, y, x)$. It is easy to see that $g_{\text{sup}}(j, y, x)$ is a constant when $j_1$, $y$ and $x$ are fixed, which is reasonable to consider the set

$$E_{R, k}^x = \left\{ (j, y, x) \in \Sigma \times \mathbb{R} \times \{x\} : g_{\text{sup}}(j, y, x) > R, j_1 = k \right\}$$

and

$$\overline{E_{R, k}^x} = \left\{ y \in \mathbb{R} : \exists j \text{ s.t. } g_{\text{sup}}(j, y, x) > R \text{ and } j_1 = k \right\}$$

for any $R > 0$ and $k \in \Lambda$.

So for any $y \in \overline{E_{R, k}^x}$, there exists $t_y \in \mathbb{Z}_+$ and $j_y \in \Sigma$ such that $g_{t_y}(j_y, y, x) > R$, which implies that

$$\mu_x \left( [k] \times B(y, \gamma^k) \times \{x\} \right) \leq e^{-R} \mu_x \left( \Sigma \times B(y, \gamma^k) \times \{x\} \right).$$

So for any $y \in \overline{E_{R, k}^x}$, we could find a ball $B(y, \lambda^k) \in \mathbb{R}$ and let $\mathcal{A}_{R, k}^x$ be the union of all the above balls when $y$ take all possible elements in $\overline{E_{R, k}^x}$. By Besicovitch covering theorem there exists a constant integer $\hat{K} = \hat{K}(\mathbb{R})$ and

(9.3) $\mu(A) = \int_{(0, 1)} \int_{\Sigma \times \mathbb{R}} \mu_{(j, y, x)}(A) \, d\mu_x(j, y, x) \, dx$

(9.4) $\mu(A) = \int_{(0, 1)} \int_{\mathbb{R}} \mu_{\theta_{x, y, x}}(A) \, dm_x(y) \, dx$

where $\theta_0 = 00 \ldots \in \Sigma$. 

(9.5) $\sup_{x \in [0, 1)} \int_{\Sigma \times \mathbb{R} \times \{x\}} \sup_{k \geq 1} g_k(j, y, x) \, d\mu_x(j, y, x) < \infty$
$A_j \subset A_{R,k}$ for $j = 1, 2, \ldots, \hat{K}$ such that all elements in $A_j$ are pair disjoint and $\bigcup_{j=1}^{\hat{K}} A_j$ is a cover of $E_{R,k}^x$. Therefore

$$\mu_x(E_{R,k}^x) \leq \sum_{j=1}^{\hat{K}} \sum_{B \in A_j} \mu_x([k] \times B \times \{x\}) \leq \sum_{j=1}^{\hat{K}} e^{-R} \mu_x(\Sigma \times B \times \{x\}) \leq \hat{K} e^{-R}$$

with the disjoint property of the elements of $A_j$ and (9.6). Let

$$E_R^x = \left\{ (j, y, x) \in \Sigma \times \mathbb{R} \times S^1 : g_{sup}(j, y, x) > R \right\},$$

thus we have

$$\mu_x(E_R^x) \leq b \hat{K} e^{-R}.$$  

By the basic formula in probability theorem, thus

$$\int_{\Sigma \times \mathbb{R} \times \{x\}} g_{sup}(j, y, x) \, d\mu_x(j, y, x) = \int_0^\infty \mu_x(E_R^x) \, dR \leq b \hat{K} \int_0^\infty e^{-R} \, dR < \infty.$$  

□

By the standard method we have the following fact.

**Lemma 9.3.** $h := \int_{\Sigma \times \mathbb{R} \times [0,1)} g(j, y, x) \, d\mu(j, y, x) < \infty$

**Proof.** By (9.4) and (C.2), we have

$$h = \int_{[0,1)} \int_{\mathbb{R}} \int_{\Sigma} g(j, y, x) \, d\mu_{(0_\infty, y, x)}(j, y, x) \, dm_x(y) \, dx$$

where $0_\infty = 00 \cdots 0 \cdots \in \Sigma$. Thus we have

(9.7)  

$$h = \int_{[0,1)} \int_{\mathbb{R}} \sum_{k \in A} F\left(\mu_{(0_\infty, y, x)}([k] \times \{y\} \times \{x\})\right) \, dm_x(y) \, dx$$

where

$$F(s) = s \log \frac{1}{s} \quad \forall s \geq 0.$$  

Since $\sum_{k \in A} \mu_{(0_\infty, y, x)}([k] \times \{y\} \times \{x\}) = 1$, we have

$$\sum_{k \in A} F\left(\mu_{(0_\infty, y, x)}([k] \times \{y\} \times \{x\})\right) \leq \log b.$$  

Combining this with (9.7) we have

$$h \leq \log b.$$  

□

**Lemma 9.4.** For $m$ a.e. $x \in [0, 1)$, the following holds

1) $m_x$ is exact dimensional;

2) $\dim(m_x) = \frac{\log b}{\log y} \left(\frac{h}{\log b} - 1\right)$.
Proof. We first show the claim:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_{n-k} \circ G^k(j, y, x) = h \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1).
\]

By Birkhoff ergodic theorem \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g \circ G^k(j, y, x) = h \quad \mu \text{a.e.} \quad \mu(a, t, z, i) \in \Sigma \times \mathbb{R} \times [0, 1), \) since \( g \in L^1(\mu) \) by Lemma 9.3.

Thus we only need to prove that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |g_{n-k} - g| \circ G^k(j, y, x) = 0 \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1).
\]

By lemma 9.2, we can define

\[
\Delta_N(j, y, x) = \sup_{k \geq N} |g_k - g|(j, y, x)
\]

for \( \forall N \in \mathbb{Z}_+ \).

For fixed \( N \), with lemma 9.2 and Birkhoff ergodic theorem, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |g_{n-k} - g| \circ G^k(j, y, x) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta_N \circ G^k(j, y, x)
\]

for \( \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1). \) Also by Birkhoff ergodic theorem we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Delta_N \circ G^k(j, y, x) = \mathbb{B} \Delta_N \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1).
\]

Then for every \( N \in \mathbb{Z}_+ \),

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |g_{n-k} - g| \circ G^k(j, y, x) \leq \mathbb{B} \Delta_N \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1).
\]

Finally combining \( \Delta_N(j, y, x) \leq (\sup_{k \geq 1} g_k + g)(j, y, x) \) and

\[
\lim_{k \to \infty} g_k(j, y, x) = g(j, y, x) \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1)
\]

by measure differential theorem and (9.3), which implies

\[
\lim_{n \to \infty} E \Delta_n = 0
\]

by Lebesgue dominated convergent theorem. So we have shown

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |g_{n-k} - g| \circ G^k(j, y, x) = 0 \quad \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1).
\]

Now we shall finish the proof. Without loss of generality we assume \( \mu_x[B_{(j, y, x)}(1)] = 1 \) for \( \mu \text{a.e.} \quad (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1). \) Thus we have
(9.9) \[ \mu_x\left(B^T_{(j,y,x)}(\gamma^n)\right) = \prod_{k=0}^{n-1} \frac{\mu_{j,x}^k(B^T_{G^k(j,y,x)}(\gamma^{n-k}))}{\mu_{j,x}^k(B^T_{G^k(j,y,x)}(\gamma^{n-k-1}))}. \]

for \( \mu \text{ a.e. } (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1) \). (9.9) implies

\[ \mu_x\left(B^T_{(j,y,x)}(\gamma^n)\right) = \prod_{k=0}^{n-1} \frac{\mu_{j,x}^k(B^T_{G^k(j,y,x)}(\gamma^{n-k}))}{b \cdot \mu_{j,x}^k(B^T_{G^k(j,y,x)}(\gamma^{n-k-1})) \cap \mathscr{P}(G^k(j, y, x))}. \]

by lemma 9.1. Thus

\[ b \cdot \mu_x\left(B^T_{(j,y,x)}(\gamma^n)\right) = n \log b + \sum_{k=0}^{n-1} g_{n-k} \circ G^k(j, y, x). \]

Combining this with (9.8) we have

\[ \lim_{n \to \infty} \frac{\log \mu_x(B_{(y,r)}(\gamma^n))}{\log \gamma^n} = \log b \left( \frac{h}{\log h} - 1 \right) \mu \text{ a.e. } (j, y, x) \in \Sigma \times \mathbb{R} \times [0, 1). \]

Since \( m_x(B(y, \gamma^n)) = \mu_x\left(B^T_{(j,y,x)}(\gamma^n)\right) \) for any \( j \in \Sigma \), the lemma holds. \( \Box \)

9.2. The proof of Theorem 3.1. In this subsection, we will use Lemma 9.4 to prove Theorem 3.1 by a standard method. For convenience let

\[ \alpha = \frac{\log b}{\log y} \left( \frac{h}{\log b} - 1 \right) \]

and

\[ M^\phi = M^\phi_{y, b} = \sup_{j \in \Sigma \cup \Lambda^y, x \in [0, 1]} |S'(x, j)|. \]

Proof of Theorem 3.1. For any \( \epsilon, \delta > 0 \), by Lemma 9.4, Egoroff theorem and differentiation theorems for measures, there exists set \( E \subset \Sigma \times \mathbb{R} \times [0, 1) \) and \( r_0 \in (0, 1) \) such that the following holds. For all \( r \in (0, r_0) \) and \( z = (x, y) \in E \), we have

(1) \( \omega(E) > 1 - \delta; \)

(2) \( \omega(E \cap B(z, r)) \geq \frac{1}{2} \omega(B(z, r)); \)

(3) \( r^{\alpha+\epsilon} \leq m_x(B(y, r)) \leq r^{\alpha-\epsilon}. \)
We first give the upper bound estimate. By (C.2) and Rokhlin decomposition of $\omega$ we have

\begin{equation}
\omega\left(B(z, r)\right) \leq 2\omega\left(E \cap B(z, r)\right) \leq \int_{B(x,r)} m_s\left(E_s \cap B(y, r)\right) ds
\end{equation}

where $E_s = \{y' \in [0, 1) : (s, y') \in E\}$.

If $m_s\left(E_s \cap B(y, r)\right) > 0$, there exists $y_s \in \mathbb{R}$ such that $(s, y_s) \in E$. Thus

$$m_s\left(E_s \cap B(y, r)\right) \leq m_s\left(B(y_s, 2r)\right) \leq (2r)^{\alpha - \epsilon}$$

by (C.3) for $r < \frac{\omega}{2}$. Combining this with (9.10) we have

$$\omega\left(B(z, r)\right) \leq 2r(2r)^{\alpha - \epsilon}.$$

Then we have

\begin{equation}
\liminf_{r \to 0^+} \frac{\log \omega\left(B(z, r)\right)}{\log r} \geq 1 + \alpha - \epsilon
\end{equation}

for every $z \in E$.

Finally we give the lower bound estimate. By (C.2) and Rokhlin decomposition of $\omega$ we have

\begin{equation}
\omega\left(B\left(z, \sqrt{2}(M^{\phi} + 1)r\right)\right) \geq \int_{B(x,r)} m_s\left(B(y, (M^{\phi} + 1)r)\right) ds.
\end{equation}

For any $x' \in B(x, r)$ and $j \in \Sigma$ such that $|S(x, j) - y| \leq r$, we have

$$|S(x', j) - y| \leq M^\phi r + r.$$

Thus

$$m_s\left(B\left(y, (M^{\phi} + 1)r\right)\right) \geq m_s\left(B(y, r)\right) \geq r^{\alpha + \epsilon}$$

by (3). Combining this with (9.12) we have

$$\omega\left(B\left(z, \sqrt{2}(M^{\phi} + 1)r\right)\right) \geq 2r(r)^{\alpha + \epsilon}.$$

Thus

\begin{equation}
\limsup_{r \to 0^+} \frac{\log \omega\left(B(z, r)\right)}{\log r} \leq 1 + \alpha + \epsilon
\end{equation}

For all $z \in E$. Since $\epsilon, \delta$ can be arbitrarily close to 0, then (9.11) and (9.13) imply

$$\lim_{r \to 0^+} \frac{\log \omega\left(B(z, r)\right)}{\log r} = 1 + \alpha$$
For $\mu \text{a.e. } z \in [0, 1) \times \mathbb{R}$.

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