Thermofield Dynamics of Time-Dependent Boson and Fermion Systems

Sang Pyo Kim
Department of Physics, Kunsan National University, Kunsan 573-701, Korea and
Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea

F. C. Khanna
Theoretical Physics Institute, Department of Physics,
University of Alberta, Edmonton, Alberta, Canada T6G 2J1 and
TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada, V6T 2A3

(Dated: April 1, 2022)

We formulate the thermofield dynamics for time-dependent systems by combining the Liouville-von Neumann equation, its invariant operators, and the basic notions of thermofield dynamics. The new formulation is applied to time-dependent bosons and fermions by using the time-dependent annihilation and creation operators that satisfy the Liouville-von Neumann equation. It is shown that the thermal state is the time- and temperature-dependent vacuum state and a general formula is derived to calculate the thermal expectation value of operators.

PACS numbers: PACS numbers: 03.65.Ca, 05.30.Jp, 05.30.Fk, 11.10.Wx

I. INTRODUCTION

The thermofield dynamics (TFD) introduced by Takahashi and Umezawa three decades ago is a canonical formalism for finite temperature theory to describe quantum systems in thermal equilibrium. The great merit of TFD is that it preserves many useful properties of the zero-temperature field theory. The central concept in TFD is the thermal state, a pure state, in some extended Hilbert space, which corresponds to the thermal equilibrium, a mixed state, in the original Hilbert space. The three essential ingredients of TFD are (i) the tilde conjugation rule, (ii) the Heisenberg equation and (iii) the thermal state conditions. In a simple physical terminology, the TFD of a quantum system doubles the degrees of freedom by introducing a fictitious Hamiltonian without any interaction with the system and uses an extended Hilbert space of the direct product of the Hilbert spaces of the system plus the fictitious system. In the oscillator representation there is a temperature-dependent Bogoliubov transformation between the annihilation and creation operators of the total system and those of temperature-dependent ones. Then the thermal state is a two-mode squeezed vacuum state (temperature-dependent vacuum state) of the extended Hilbert space, which in turn is annihilated by the temperature-dependent annihilation operators. There have been numerous diverse applications of TFD to systems in condensed matter, nuclear physics, particle physics, quantum optics and cosmology in thermal equilibrium (for review and references, see [3] and [4]).

On the other hand, there are also many systems exhibiting nonequilibrium characteristics. An open system interacting with a reservoir or a time-dependent system is such a nonequilibrium system. As the time-translational invariance of such a time-dependent system is broken, there is an obstacle in applying the Matsubara’s imaginary-time method. Neither does the conventional wisdom work well using the basis of the time-dependent energy eigenstates in evaluating thermal quantities since the density operator is not given by $e^{-\beta H(t)}$. However, the closed-time path method by Schwinger and Keldysh is well-known and is widely used for such nonequilibrium systems. Another canonical theory based on the (functional) Schrödinger equation has been employed to study nonequilibrium evolution of time-dependent systems. Keeping the three ingredients of TFD, requiring the Hamiltonian to be tildian, and using the Heisenberg picture, Umezawa et al attempted to extend the TFD to such nonequilibrium systems.

In this paper we show that when (ii) the Heisenberg equation, one of the ingredients of TFD, is replaced by the Liouville-von Neumann (LvN) equation, the TFD has a direct generalization to time-dependent boson and fermion systems, in particular, time-dependent oscillators whose mass and frequency may change explicitly in time. The invariant operators satisfying the LvN equation not only lead to the Hilbert (Fock) space of exact quantum states but also provide the correct density operator. Thus we are able to extend the TFD to time-dependent bosons (fermions) first by using the time-dependent annihilation and creation operators for the bosons (fermions)
and the fictitious bosons (fermions), all linear invariant operators\textsuperscript{10,11,12,13}, and then by finding the time- and temperature-dependent annihilation and creation operators through temperature-dependent Bogoliubov transformation of TFD. Then the thermal state is a two-mode squeezed state of the time-dependent vacuum state for the bosons (fermions) plus the fictitious bosons (fermions). We find a general formula for evaluating the thermal expectation values of operators and finally discuss the distribution which evolves from an initial boson distribution.

The organization of this paper is as follows: In Sec. II, we briefly review the TFD for boson and fermion systems. In Sec. III, we introduce the time-dependent annihilation and creation operators for a time-dependent boson. In Sec. IV, we extend the TFD to time-dependent boson and discuss the physical implication of the TFD for a boson which evolves from an initial thermal state to a final one through a time-dependent interaction. In Sec. V, TFD is extended to time-dependent fermions. Additional comments and conclusion are given in Sec. VI.

II. TFD FOR BOSON AND FERMION SYSTEMS IN EQUILIBRIUM

We briefly review the TFD for static (time-independent) bosons and fermions in a way that can be readily applied to time-dependent ones in the following sections. The static boson (fermion) has the Hamiltonian of the form

\[ H = \hbar \omega a^\dagger a, \]

where the standard annihilation and creation operators satisfy the commutator (anticommutator)

\[ [a, a^\dagger]_\pm = 1. \]

Quantum statistics of the boson (fermion) in thermal equilibrium is described by the density operator

\[ \rho = \frac{1}{Z} e^{-\beta H}, \quad Z = \text{Tr}[e^{-\beta H}], \]

where \( \beta \) is the inverse temperature. For instance, the thermal expectation value of an operator \( A \) is given by

\[ \langle A \rangle_T = \text{Tr}[\rho A] = \frac{1}{Z} \sum_n e^{-\beta \omega n} \langle n | A | n \rangle, \]

with \( |n\rangle \) being number states. The thermal equilibrium is a mixed state with the probability \( p_n = e^{-\beta \omega n} \) for each projector \( |n\rangle \langle n| \).

The idea of TFD is to double the system by adding a fictitious system and extend the thermal equilibrium in the system’s Hilbert space to a thermal state, a pure state, in the extended Hilbert space of the total system. For that purpose, let us introduce a fictitious boson (fermion) using the tilde conjugation rule, with the Hamiltonian,

\[ \tilde{H} = \hbar \tilde{\omega} \tilde{a}^\dagger \tilde{a}, \]

with the number state \( |\tilde{n}\rangle \) and the commutator (anticommutator)

\[ [\tilde{a}, \tilde{a}^\dagger]_\pm = 1. \]

As \( \{a, a^\dagger\} \) and \( \{\tilde{a}, \tilde{a}^\dagger\} \) describe two independent systems, \( H \) and \( \tilde{H} \), respectively, they commute (anticommutate) with each other

\[ [a, \tilde{a}]_\pm = [a^\dagger, \tilde{a}]_\pm = [a^\dagger, \tilde{a}^\dagger]_\pm = 0. \]

The total Hamiltonian

\[ \hat{H} := H - \tilde{H} = \hbar \omega (a^\dagger a - \tilde{a}^\dagger \tilde{a}) \]

now carries an extended Hilbert space \( \hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}} \). Each state of the extended Hilbert space consists of a product of number states

\[ |n, \tilde{n}\rangle = |n\rangle \otimes |\tilde{n}\rangle. \]
In TFD the thermal equilibrium corresponds to the thermal state in the extended Hilbert space, described by a pure state

$$|0(\beta)\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta \hbar \omega n / 2} \frac{1}{n!} a^\dagger_n a_n |0, 0\rangle = (1 \mp e^{-\beta \hbar \omega} \frac{1}{2} e^{-(\beta \hbar \omega / 2) a^\dagger a}) |0, 0\rangle,$$

where the upper (lower) sign is for bosons (fermions) and $|0\rangle = |0, 0\rangle$ in the second line. In fact, the thermal state is a two-mode squeezed vacuum state obtained by applying a Bogoliubov transformation to the vacuum state

$$|0(\beta)\rangle = e^{-iG} |0\rangle,$$

where

$$G = -i\theta(\beta) (\tilde{a}a - a^\dagger \tilde{a}^\dagger).$$

(12)

Here $\beta(\theta)$ is a temperature-dependent parameter determined by

$$\cosh \theta(\beta) = (1 - e^{-\beta \hbar \omega})^{-1/2},$$
$$\sinh \theta(\beta) = e^{-\beta \hbar \omega / 2} (1 - e^{-\beta \hbar \omega})^{-1/2},$$

(13)

for bosons and

$$\cos \theta(\beta) = (1 + e^{-\beta \hbar \omega})^{-1/2},$$
$$\sin \theta(\beta) = e^{-\beta \hbar \omega / 2} (1 + e^{-\beta \hbar \omega})^{-1/2},$$

(14)

for fermions.

The two-mode squeeze operator introduces the temperature-dependent annihilation and creation operators through the Bogoliubov transformations

$$a(\beta) = e^{-iG} a e^{iG} = \cosh \theta(\beta) a - \sinh \theta(\beta) \tilde{a}^\dagger,$$
$$\tilde{a}(\beta) = e^{-iG} \tilde{a} e^{iG} = \cosh \theta(\beta) \tilde{a} - \sinh \theta(\beta) a^\dagger,$$

(15)

for bosons and

$$a(\beta) = e^{-iG} a e^{iG} = \cos \theta(\beta) a - \sin \theta(\beta) \tilde{a}^\dagger,$$
$$\tilde{a}(\beta) = e^{-iG} \tilde{a} e^{iG} = \cos \theta(\beta) \tilde{a} + \sin \theta(\beta) a^\dagger,$$

(16)

for fermions. Similar relations follow for $a^\dagger(\beta)$ and $\tilde{a}^\dagger(\beta)$ for bosons and fermions. The thermal state in Eq. (11) is nothing but a temperature-dependent vacuum state

$$a(\beta) |0(\beta)\rangle = \tilde{a}(\beta) |0(\beta)\rangle = 0.$$

(17)

Now the expectation value of an operator $A$ with respect to thermal equilibrium is equivalent to that with respect to the thermal state

$$\langle A \rangle_T = \langle 0(\beta)|A|0(\beta)\rangle.$$

(18)

The expectation value of the number operator leads to the boson distribution

$$\langle 0(\beta)|a^\dagger a|0(\beta)\rangle = \sinh^2 \theta(\beta) = \frac{1}{e^{\beta \hbar \omega} - 1},$$

(19)

and the fermion distribution

$$\langle 0(\beta)|a^\dagger a|0(\beta)\rangle = \sin^2 \theta(\beta) = \frac{1}{e^{\beta \hbar \omega} + 1}.$$

(20)
III. TIME-DEPENDENT BOSON SYSTEM

We now consider a time-dependent boson system and find the ingredients necessary for TFD extension. The most general time-dependent quadratic Hamiltonian for bosons takes the form

\[ H(t) = \hbar \left[ \omega_0(t) a \dagger a + \frac{1}{2} \omega_+(t) a^2 \dagger + \frac{1}{2} \omega_+(t) a^2 \right], \]  

where \( a \) and \( a \dagger \) are the Schrödinger (time-independent) annihilation and creation operators, and \( \omega_0 \) is real and \( \omega_+ \) real or complex. In particular, the oscillator with time-dependent mass and frequency,

\[ H(t) = \frac{p^2}{2m(t)} + \frac{m(t)}{2} \omega^2(t) q^2, \]  

belongs to the Hamiltonian, Eq. (21). Note that \( p \) and \( q \) in Eq. (22) are also the Schrödinger operators, but the Hamiltonian depends explicitly on time through \( m \) and \( \omega \). An instantaneous energy eigenstate \( |e_n, t\rangle \) of the Hamiltonian, Eq. (21), is definitely not an exact quantum state of the Schrödinger equation. Moreover, the density operator is not given by \( e^{-\beta H(t)} \). So the set \( \{ |e_n, t\rangle \} \) is not a good basis to study the nonequilibrium evolution of the thermal system.

We explain a physical motivation for replacing the Heisenberg equation in the ingredient (ii) by the LvN equation. In the Heisenberg picture, it is the Heisenberg equation that determines the operators evolving in time. In fact, in terms of the evolution operator \( U \) of the Schrödinger equation, each Schrödinger operator \( A \) leads to the Heisenberg operator

\[ A_H = U^\dagger A U. \]  

Therefore, to know thermal properties of the system, either the Heisenberg operator \( A_H \) or the density operator \( \rho(t) = U \rho U^\dagger \) should be given in advance.

On the other hand, there exist the invariant operators that satisfy the LvN equation. A complete set of invariant operators provide another picture for a time-dependent system and such a set of invariant operators are known for time-dependent oscillators in terms of classical solutions. In this sense the LvN equation may replace the Heisenberg equation in TFD. Further, the linearity of the LvN equation allows the construction of the density operator from an invariant operator. In fact, we may find the time-dependent annihilation operator, an invariant operator, of the form

\[ a(t) = f^(-)(t) a + f^+(t) a \dagger \]  

and its Hermitian conjugate \( a \dagger(t) \), and impose the LvN equations

\[ i \hbar \frac{\partial a(t)}{\partial t} + [a(t), H(t)]_+ = 0, \]
\[ i \hbar \frac{\partial a \dagger(t)}{\partial t} + [a \dagger(t), H(t)]_+ = 0. \]  

The time-dependent creation operator \( a \dagger(t) \) is another invariant operator. The pair \( \{ a(t), a \dagger(t) \} \) form a complete set. Therefore any invariant operator can be constructed out of them.

The LvN equation leads to the vector equation

\[ i \frac{\partial V}{\partial t} + M_V V = 0, \]  

where

\[ V(t) = \begin{pmatrix} f^(-) \\ f^+ \end{pmatrix}, \]  

and

\[ M_V(t) = \omega_0 \sigma_3 - \frac{1}{2} (\omega_+^* - \omega_+) \sigma_1 - i \frac{1}{2} (\omega_+^* + \omega_+) \sigma_2. \]
Here σ’s are the Pauli spin matrices. The solution to Eq. (26) provides the time-dependent annihilation and creation operators. Further, the equal-time commutator

$$[a(t), a^\dagger(t) ] = 1$$

can hold by choosing the initial data $V^\dagger \sigma_3 V = 1$ since Eq. (26) leads to the relation

$$\frac{\partial}{\partial t} (V^\dagger \sigma_3 V) = 0.$$  (30)

For the time-dependent oscillator, Eq. (22), the following form of the time-dependent annihilation operator is known

$$a(t) = \frac{i}{\sqrt{\hbar}} [v^*(t) p - m(t) \dot{v}^* q].$$  (31)

The time-dependent creation operator $a^\dagger(t)$ is the Hermitian conjugate of $a(t)$. In fact, these operators are invariant operators satisfying the LvN equation (25) for the Hamiltonian, Eq. (22), when $v$ is a complex solution to the classical equation of motion

$$\ddot{v}(t) + \frac{\dot{m}(t)}{m(t)} v(t) + \omega^2(t) v(t) = 0,$$  (32)

and satisfies the Wronskian condition

$$m(\dot{v}^* v - \dot{v} v^*) = i.$$  (33)

The number states of the number operator $N(t) = a^\dagger(t)a(t)$, another invariant operator, defined as,

$$N(t)|n, t\rangle = n|n, t\rangle,$$  (34)

are exact quantum states of the Schrödinger equation and constitute the Hilbert space [9, 10, 11, 12]. The state $|0, t\rangle$ is the time-dependent vacuum that is annihilated by $a(t)$, and the number states are obtained by applying the creation operators on it

$$|n, t\rangle = \frac{a^\dagger n(t)}{\sqrt{n!}} |0, t\rangle.$$  (35)

It follows that the density operator may be given by

$$\rho(t) = \frac{1}{Z} e^{-\beta \omega a^\dagger(t)a(t)},$$  (36)

where $\beta$ and $\omega$ are constants. In the case of the oscillator, Eq. (22), the position and momentum operators have the oscillator representation

$$q = \sqrt{\hbar}[v(t)a(t) + v^*(t)a^\dagger(t)],$$

$$p = \sqrt{\hbar m(t)}[\dot{v}(t)a(t) + \dot{v}^*(t)a^\dagger(t)].$$  (37)

The merit of using the LvN equation is that we can keep all the steps of the time-independent boson case in finding the Hilbert space, the density operator and so on.

### IV. TFD FOR TIME-DEPENDENT BOSON SYSTEM

To extend TFD to the time-dependent boson system, Eq. (21), we use the tilde conjugation rule $(cA) = c^* \tilde{A}$ and introduce a fictitious boson system with the Hamiltonian

$$\tilde{H}(t) = h[\tilde{\omega}_0(t)\tilde{a}^\dagger \tilde{a} + \frac{1}{2}\tilde{\omega}_+^*(t)\tilde{a}^\dagger \tilde{a} + \frac{1}{2}\omega_+^2].$$  (38)
As for the time-dependent boson case, we introduce the time-dependent annihilation operator for the fictitious boson

\[ \hat{a}(t) = f^{(-)}(t)\hat{\alpha} + f^{(+)}(t)\hat{\alpha}^\dagger, \] (39)

where \( f^{(\pm)} \) satisfy Eq. (26). Then \( \hat{\alpha}(t) \) and its Hermitian conjugate \( \hat{\alpha}^\dagger(t) \) satisfy the LvN equations

\[
\begin{align*}
\hat{\rho}(t) &= \frac{1}{Z} e^{-\beta \hbar \omega a^\dagger(t)a(t)}, \\
\hat{\bar{\rho}}(t) &= \frac{1}{Z} e^{\beta \hbar \omega \tilde{a}^\dagger(t)\tilde{a}(t)},
\end{align*}
\] (47)

which obviously satisfy the LvN equations. Here \( \beta \) and \( \omega \) are constants that may be fixed by the initial temperature and frequency. The density operator in the extended Hilbert space is given by

\[
\hat{\rho}(t) = \hat{\rho}(t) \otimes \hat{\bar{\rho}}(t) = \frac{1}{Z^2} e^{-\beta \hbar \omega (a^\dagger(t)a(t) - \tilde{a}^\dagger(t)\tilde{a}(t))}.
\] (49)

The thermal expectation value of the operator \( A \) of the system now takes the form

\[
\langle A \rangle = \text{Tr}[\rho(t)A] = \langle 0(\beta), t | A | 0(\beta), t \rangle,
\] (50)

where the thermal vacuum state is given by

\[
|0(\beta), t\rangle = \frac{1}{Z^{1/2}} \sum_n e^{-\beta \hbar \omega n/2} \frac{1}{n!} a^{\dagger n}(t)\tilde{a}^\dagger(\tilde{\rho})|0, \tilde{\rho}, t\rangle
\] (51)
with \(|0, t\rangle = |0, 0, t\rangle\). The thermal state is an exact eigenstate of the Schrödinger equation for the total system, Eq. (53). The thermal state is also written as a time-dependent two-mode squeezed state of the vacuum state

\[ |0(\beta), t\rangle = e^{-iG(t)}|0, t\rangle, \]

(52)

where

\[ G(t) = -i\theta(\beta)[\hat{a}(t)a(t) - a^\dagger(t)\hat{a}(t)]. \]

(53)

Here \(\theta(\beta)\) is the same parameter fixed by Eq. (53).

As the density operator in Eq. (53) or (55) involves a constant \(\beta\), we may find the time- and temperature-dependent annihilation and creation operators through the Bogoliubov transformation

\[
a(\beta, t) = \cosh \theta(\beta)a(t) - \sinh \theta(\beta)\hat{a}(t),
\]

(54)

\[
a(\beta, t) = \cosh \theta(\beta)a(t) + \sinh \theta(\beta)\hat{a}(t),
\]

and their inverse transformation

\[
\hat{a}(\beta, t) = \cosh \theta(\beta)\hat{a}(t) - \sinh \theta(\beta)a^\dagger(t),
\]

(55)

\[
\hat{a}(\beta, t) = \cosh \theta(\beta)\hat{a}(t) + \sinh \theta(\beta)a^\dagger(t).
\]

We get similar equations for \(a^\dagger(\beta, t), \hat{a}^\dagger(\beta, t), a^\dagger(t)\) and \(\hat{a}^\dagger(t)\) by using the Hermitian conjugate of these equations. As \(\theta(\beta)\) is a constant, \(a(\beta, t), \hat{a}(\beta, t)\) are invariant operators. Then the thermal state is the time- and temperature-dependent vacuum

\[
a(\beta, t)|0(\beta), t\rangle = \hat{a}(\beta, t)|0(\beta), t\rangle = 0.
\]

(56)

The thermal state \(|0(\beta), t\rangle\), as an eigenstate of the invariant operators \(a(\beta, t)\) and \(\hat{a}(\beta, t)\), is an exact eigenstate of the total system. At each moment, the boson still keeps the same boson distribution since the expectation value of the time-dependent number operator yields

\[
\langle 0(\beta), t|a^\dagger(t)a(t)|0(\beta), t\rangle = \sinh^2 \theta(\beta) = \frac{1}{e^{\beta \hbar c} - 1}.
\]

(57)

Using TFD for time-dependent bosons, we are able to find the thermal expectation values of operators. In general, through the Bogoliubov transformations from \(\{a(t), a^\dagger(t)\}\) to \(\{a(\beta, t), a^\dagger(\beta, t)\}\), we find the formula

\[
\langle F(a(t), a^\dagger(t))\rangle_T = \langle 0(\beta), t|F(\cosh \theta(\beta)a(\beta, t) + \sinh \theta(\beta)\hat{a}(\beta, t),
\]

\[
cosh \theta(\beta)a^\dagger(\beta, t) + \sinh \theta(\beta)\hat{a}(\beta, t))|0(\beta), t\rangle.
\]

(58)

This provides the basic rule for calculating matrix element of any operator in TFD. For instance, in the case of the oscillator, Eq. (22), using the position representation in Eq. (37), we obtain

\[
q = \sqrt{n} \cosh \theta(\beta)[v(t)a(\beta, t) + v^*(t)a^\dagger(\beta, t)]
\]

\[
+ \sqrt{n} \sinh \theta(\beta)[v^*(t)\hat{a}(\beta, t) + v(t)\hat{a}^\dagger(\beta, t)],
\]

(59)

we obtain

\[
\langle 0(\beta), t|q^{2n}|0(\beta), t\rangle = \hbar^n \sum_{k=0}^{n} \begin{pmatrix} 2n \\ 2k \end{pmatrix} \langle 0, t| \cosh^{2k} \theta(\beta)[v(t)a(\beta, t) + v^*(t)a^\dagger(\beta, t)]^{2k}
\]

\[
\times \sinh^{2n-2k} \theta(\beta)[v^*(t)\hat{a}(\beta, t) + v(t)\hat{a}^\dagger(\beta, t)]^{2n-2k}|0, t\rangle.
\]

(60)

After normal ordering, we finally obtain the result

\[
\langle q^{2n}\rangle_T = \langle 0(\beta), t|q^{2n}|0(\beta), t\rangle
\]

\[
= \frac{(2n)!}{2^n n!} [\hbar v^*(t)v(t)]^n (1 + 2 \sinh^2 \theta(\beta))^n.
\]

(61)

We discuss the physical implication of TFD for a time-dependent boson, when it evolves with a time-dependent interaction from initial \(\omega_i\)'s at \(t_i\) to final ones, \(\omega_f\)'s at \(t_f\). That is, all \(\omega\)'s change from \(\omega_i\)'s to \(\omega_f\)'s. The solution to
and the invariant creation operators $a_i = \mu a_f + \nu a_f^\dagger$, $a_i^\dagger = \mu^* a^\dagger_f + \nu^* a_f$, where \{a_i, a_i^\dagger\} for $\omega_i$'s and \{a_f, a_f^\dagger\} for $\omega_f$'s. Here $\mu$ and $\nu$, which should be determined by the solution $f(\pm)$ to Eq. 20, carry all the information about the history of interaction and may take the form

$$\mu = \mu(t_i, t_f; \omega_i, \omega_f), \quad \nu = \nu(t_i, t_f; \omega_i, \omega_f),$$

and satisfy

$$\mu^* \mu - \nu^* \nu = 1.$$

If the boson is initially in thermal equilibrium with the inverse temperature $\beta$ and has the boson distribution $\bar{n}_i = 1/(e^{\beta \hbar \omega_i} - 1)$, then, according to Secs. III and IV, the boson in the final state has a different distribution

$$(0, \beta), t_f | a_i^\dagger a_i | 0(\beta), t_f) = \nu^* \nu + \frac{1 + 2 \nu^* \nu}{e^{\beta \hbar \omega} - 1}.$$ (65)

The first term is originated from the particle production from vacuum fluctuations \[1\], $\langle 0, t_f | a_i^\dagger a_i | 0, t_f \rangle = \nu^* \nu$, and the second term is a purely thermal result, having an overall amplification factor $(1 + 2 \nu^* \nu)$ to the boson distribution. Thus the evolution of the time-dependent system leads to a distribution quite different from the boson distribution function.

V. TFD FOR TIME-DEPENDENT FERMION SYSTEM

The time-dependent fermion system, quadratic in the annihilation and creation operators, has the Hamiltonian

$$H(t) = \hbar [\omega_0(t)(a^\dagger a - b^\dagger b) + \omega_+(t)a^\dagger b^\dagger - \omega_-(t)ab + \omega_-(t)ab^\dagger - \omega_+(t)a^\dagger b],$$

where $a, a^\dagger$ for the particles and $b, b^\dagger$ for the antiparticles are all the Schrödinger (time-independent) operators, and $\omega_0$ is real and $\omega_\pm$ are real or complex. They satisfy the anticommutators

$$[a, a^\dagger]_+ = [b, b^\dagger]_+ = 1, \quad [a, b]_+ = [a, b^\dagger]_+ = [a^\dagger, b^\dagger]_+ = 0.$$ (67)

The Hamiltonian, Eq. (66), is a Hermitian operator and its time-dependence comes only from the parameters $\omega_i$'s.

As for the boson case, we may find a pair of time-dependent invariant annihilation operators \[13\]

$$a(t) = f_a^{(-)}(t)a + f_a^{(+)}(t)a^\dagger + g_a^{(-)}(t)b + g_a^{(+)}(t)b^\dagger,$$

$$b(t) = f_b^{(-)}(t)a + f_b^{(+)}(t)a^\dagger + g_b^{(-)}(t)b + g_b^{(+)}(t)b^\dagger,$$ (68)

and the invariant creation operators $a^\dagger(t)$ and $b^\dagger(t)$ are the Hermitian conjugates of $a(t)$ and $b(t)$, respectively. The LvN equations for the operators in Eq. (68) with the Hamiltonian, Eq. (66), lead to the following vector equations

$$i \frac{\partial W}{\partial t} + \omega_0 \sigma_1 W + M_w Z = 0,$$

$$i \frac{\partial Z}{\partial t} + \omega_0 \sigma_1 Z + M_z W = 0,$$ (69)

where

$$W(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} f^{(-)} + f^{(+)} \\ f^{(-)} - f^{(+)} \end{pmatrix},$$

$$Z(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} g^{(-)} + g^{(+)} \\ g^{(-)} - g^{(+)} \end{pmatrix}.$$ (70)
and

\[ M_W(t) = \frac{1}{2}(\omega^+ - \omega^-)I - \frac{i}{2}(\omega^+_+ + \omega^-_+)\sigma_1 - \frac{1}{2}(\omega^+_+ + \omega^-_+)\sigma_2 - \frac{1}{2}(\omega^+_+ - \omega^-_+)\sigma_3, \]

\[ M_Z(t) = \frac{1}{2}(\omega^+ - \omega^-)I - \frac{i}{2}(\omega^+_+ + \omega^-_+)\sigma_1 + \frac{i}{2}(\omega^+_+ + \omega^-_+)\sigma_2 + \frac{1}{2}(\omega^+_+ - \omega^-_+)\sigma_3, \]

(71)

where \( I \) is the identity matrix. The equal-time anticommutators

\[ [a(t), a^\dagger(t)]_+ = [b(t), b^\dagger(t)]_+ = 1, \quad [a(t), b(t)]_+ = [a^\dagger(t), b^\dagger(t)]_+ = 0, \]

(72)

are guaranteed by the relations

\[ \frac{\partial}{\partial t}(W^\dagger W + Z^\dagger Z) = 0, \]

\[ \frac{\partial}{\partial t}(W^T_a \sigma_3 W_b + Z^T_a \sigma_3 Z_b) = 0, \]

\[ \frac{\partial}{\partial t}(W^\dagger_a W_b + Z^\dagger_a Z_b) = 0. \]

(73)

Each set of \( \{W_a, Z_a\} \) and \( \{W_b, Z_b\} \) gives rise to \( a(t) \) and \( b(t) \), and their Hermitian conjugates. The number operators

\[ N_a(t) = a^\dagger(t)a(t) \quad \text{and} \quad N_b(t) = b^\dagger(t)b(t), \]

which are also invariant operators, span the state vector space of the fermions

\[ |0, t\rangle, \quad a^\dagger(t)|0, t\rangle, \quad b^\dagger(t)|0, t\rangle, \quad a^\dagger(t)b^\dagger(t)|0, t\rangle. \]

(74)

Here the vacuum state, \( |0, t\rangle = |0, \hat{0}, t\rangle \), implies no particles and antiparticles. The density operator may take the form

\[ \rho(t) = \frac{1}{Z}e^{-\beta \omega (a^\dagger(t)a(t)-b^\dagger(t)b(t))}, \]

(75)

where \( \beta \) and \( \omega \) are again constants.

To construct the TFD for fermions, we introduce the fictitious fermion Hamiltonian

\[ \hat{H}(t) = \hbar[\omega_0(t)(\tilde{a}^\dagger \tilde{a} - \tilde{b}^\dagger \tilde{b}) + \omega^+_+(t)\tilde{a}^\dagger \tilde{b} - \omega_+(t)\tilde{a}^\dagger \tilde{b} + \omega^+_-(t)\tilde{a}^\dagger \tilde{b} - \omega_-(t)\tilde{a}^\dagger \tilde{b}]. \]

(76)

Then there are the time-dependent annihilation operators, invariant operators,

\[ \tilde{a}(t) = f^{(-)*}_a(t)\tilde{a} + f^{(+)a}_a(t)\tilde{a}^\dagger + g^{(-)*}_a(t)\tilde{b} + g^{(+)a}_a(t)\tilde{b}^\dagger, \]

\[ \tilde{b}(t) = f^{(-)*}_b(t)\tilde{a} + f^{(+)b}_b(t)\tilde{a}^\dagger + g^{(-)*}_b(t)\tilde{b} + g^{(+)b}_b(t)\tilde{b}^\dagger. \]

(77)

These operators satisfy the Lévy–Nirenberg equations when \( f^{\prime} \)’s and \( g^{\prime} \)’s satisfy the vector equations. We are then equipped with the time-dependent annihilation and creation operators for the total system

\[ \hat{H}(t) = H(t) - \hat{H}(t). \]

(78)

The time-dependent vacuum state of the total system is annihilated by all the annihilation operators of the fermions and the fictitious fermions:

\[ a(t)|0, \hat{0}, t\rangle = b(t)|0, \hat{0}, t\rangle = \tilde{a}(t)|0, \hat{0}, t\rangle = \tilde{b}(t)|0, \hat{0}, t\rangle = 0. \]

(79)

Here the vacuum state, \( |0, \hat{0}, t\rangle = |0, \hat{0}, \hat{0}, t\rangle \), implies no particles or antiparticles and their counterparts. However, we will continue to use the abbreviation \( |0, \hat{0}, t\rangle \) for this state.

The thermal state of TFD is the two-mode squeezed state of the time-dependent vacuum state

\[ |0(\beta), t\rangle = e^{-iG_F(t)}|0, t\rangle, \]

(80)

where \( |0, t\rangle = |0, \hat{0}, t\rangle \), and

\[ G_F(t) = -i\theta(\beta) (\tilde{a}\tilde{a} - a^\dagger a^\dagger + \tilde{b}\tilde{b} - b^\dagger b^\dagger). \]

(81)
Here $\theta(\beta)$ is the same parameter fixed by Eq. 13. The two-mode squeeze operator introduces the time- and temperature-dependent annihilation and creation operators for the particles through the Bogoliubov transformation

$$a(\beta, t) = \cos \theta(\beta)a(t) - \sin \theta(\beta)\tilde{a}(t),$$
$$\tilde{a}(\beta, t) = \cos \theta(\beta)\tilde{a}(t) + \sin \theta(\beta)a^\dagger(t),$$

and for the antiparticles

$$b(\beta, t) = \cos \theta(\beta)b(t) - \sin \theta(\beta)\tilde{b}(t),$$
$$\tilde{b}(\beta, t) = \cos \theta(\beta)\tilde{b}(t) + \sin \theta(\beta)b^\dagger(t).$$

We get similar equations for $a^\dagger(\beta, t), \tilde{a}^\dagger(\beta, t), b(\beta, t)$ and $\tilde{b}^\dagger(\beta, t)$ by using Hermitian conjugate on these equations. Their inverse transformations are

$$a(t) = \cos \theta(\beta)a(\beta, t) + \sin \theta(\beta)\tilde{a}(\beta, t),$$
$$\tilde{a}(t) = \cos \theta(\beta)\tilde{a}(\beta, t) - \sin \theta(\beta)a^\dagger(\beta, t),$$

and

$$b(t) = \cos \theta(\beta)b(\beta, t) + \sin \theta(\beta)\tilde{b}(\beta, t),$$
$$\tilde{b}(t) = \cos \theta(\beta)\tilde{b}(\beta, t) - \sin \theta(\beta)b^\dagger(\beta, t).$$

Further, the thermal state is annihilated by

$$a(\beta, t)|0(\beta), t\rangle = \tilde{a}(\beta, t)|0(\beta), t\rangle = b(\beta, t)|0(\beta), t\rangle = \tilde{b}(\beta, t)|0(\beta), t\rangle.$$

Now the expectation value of the particle number operator

$$\langle 0(\beta), t|a^\dagger(t)a(t)|0(\beta), t\rangle = \sin^2 \theta = \frac{1}{e^{\beta\hbar \omega} + 1}.$$  

The thermal expectation value of operators, for instance, of the particles is given by the formula

$$\langle F(a(t), a^\dagger(t)) \rangle_T = \langle 0(\beta), t|F(\cos \theta(\beta)a(\beta, t) + \sin \theta(\beta)\tilde{a}(\beta, t),$$
$$\cos \theta(\beta)a^\dagger(\beta, t) + \sin \theta(\beta)\tilde{a}(\beta, t))|0(\beta), t\rangle.$$  

VI. CONCLUSION

To summarize, in this paper we have used the time-dependent annihilation and creation operators of the LVN equation to complete TFD for time-dependent boson and fermion systems. The first ingredient, (i) the tilde conjugation rule, has been accomplished by introducing the fictitious boson or fermion operators and appropriately constructing the extended Hilbert space. The second ingredient, (ii) the Heisenberg equation, is replaced by the LVN equation. The annihilation and creation operators from the LVN equation have led to exactly the same procedures as for TFD of time-independent boson and fermion systems. The third ingredient, (iii) the thermal state condition, is guaranteed by the time- and temperature-dependent Bogoliubov transformations, Eqs. 54, 82 and 83. The thermal state for time-dependent bosons is the time- and temperature-dependent vacuum as in Eq. 10, which can be written as

$$a(t)|0(\beta), t\rangle = \tanh \theta(\beta)\tilde{a}(t)|0(\beta), t\rangle, \quad \tilde{a}(t)|0(\beta), t\rangle = \tanh \theta(\beta)a^\dagger(t)|0(\beta), t\rangle.$$  

The thermal state condition for time-dependent fermions is given by

$$a(t)|0(\beta), t\rangle = \tan \theta(\beta)\tilde{a}(t)|0(\beta), t\rangle, \quad \tilde{a}(t)|0(\beta), t\rangle = -\tan \theta(\beta)a^\dagger(t)|0(\beta), t\rangle,$$
$$b(t)|0(\beta), t\rangle = \tan \theta(\beta)\tilde{b}(t)|0(\beta), t\rangle, \quad \tilde{b}(t)|0(\beta), t\rangle = -\tan \theta(\beta)b^\dagger(t)|0(\beta), t\rangle.$$  

Note that the thermal state conditions, Eqs. 89 and 90, still have the same form for the time-independent boson and fermion systems. We may thus conclude that the LVN equation provides a direct generalization of TFD to nonequilibrium systems such as time-dependent boson and fermion systems. Finally it should be stressed that the replacement of the Heisenberg equation by the LVN equation leads us from the Heisenberg picture to the Schrödinger picture. Such a procedure yields interesting and useful results for quantum mechanical time-dependent systems. Extension to quantum field theory may be possible and this would bring the open systems to a treatment similar to the case of systems in equilibrium. That would provide a true extension of TFD, a la Takahashi and Umezawa that has proved so very successful for treating equilibrium problems, to systems out of equilibrium. Hopefully this will provide a useful perspective on such a class of problems. This topic is under active consideration.
Acknowledgments

We would like to thank M. Revzen for useful discussions. S.P.K. also would like to express his appreciation for the warm hospitality of the Theoretical Physics Institute, University of Alberta. The work of S.P.K. was supported by the Korea Research Foundation under Grant No. KRF-2002-041-C00053.

[1] Y. Takahashi and H. Umezawa, Collect. Phenem. 2, 55 (1975) [reprinted in Int. J. Mod. Physics B 10, 1755 (1996)].
[2] H. Umezawa, H. Matsumoto, and M. Tachiki, Thermofield Dynamics and Condensed States (North-Holland, Amsterdam, 1982); H. Umezawa, Advanced Field Theory: Micro, Macro and Thermal Physics (AIP, New York, 1993).
[3] H. Umezawa, Prog. Theor. Phys. Suppl. 80, 26 (1984).
[4] P.A. Henning, Phys. Rep. 253, 235 (1995).
[5] T. Matsubara, Prog. Theor. Phys. 14, 351 (1955).
[6] J. Schwinger, J. Math. Phys. 2, 407 (1961); L.V. Keldysh, JETP 20, 1018 (1965).
[7] O. Eboli, R. Jackiw, and S.-Y. Pi, Phys. Rev. D 37, 3557 (1988); R. Floreanini and R. Jackiw, ibid. 37, 2206 (1988).
[8] T. Arimitsu and H. Umezawa, Prog. Theor. Phys. 74, 429 (1985); ibid. 77, 32 (1987); ibid. 77, 53 (1987); T. Arimitsu, M. Guida, and H. Umezawa, Physica A 148, 1 (1988); Y. Yamanaka, H. Umezawa, K. Nakamura, and T. Arimitsu, Int. Mod. Phys. A 9, 1153 (1994); P.A. Henning and H. Umezawa, Nucl. Phys. B 417, 463 (1994).
[9] H.R. Lewis, Jr., Phys. Rev. Lett. 27, 510 (1967); J. Math. Phys. 9, 1976 (1968); H.R. Lewis, Jr. and W.B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).
[10] K.H. Cho, J.-Y. Ji, S.P. Kim, C.H. Lee, and J.Y. Ryu, Phys. Rev. D 56, 4916 (1997); S.P. Kim and C.H. Lee, ibid. 62, 125020 (2000); S.P. Kim, J. Korean Phys. Soc. 41, 643 (2002).
[11] I.A. Malkin, V.I. Man'ko, and D.A. Trifonov, Phys. Rev. D 2, 1371 (1970); J. Math. Phys. 14, 576 (1973); V.V. Dodonov and V.I. Man'ko, Phys. Rev. A 20, 550 (1979).
[12] S.P. Kim, Class. Quantum Grav. 13, 1377 (1996); S.P. Kim, J.-Y. Ji, H.-S. Shin, and K.-S. Soh, Phys. Rev. D 56, 3756 (1997); J.K. Kim and S.P. Kim, J. Phys. A 32, 2711 (1999); S.P. Kim and D.N. Page, Phys. Rev. A 64, 012104 (2001); S.P. Kim, J. Korean Phys. Soc. 43, 11 (2003).
[13] S.P. Kim, A.E. Santana, and F.C. Khanna, Phys. Lett. A 272, 46 (2000).
[14] L. Parker, Phys. Rev. Lett. 21, 562 (1968); Phys. Rev. 183, 1057 (1969).