Fusion multiplicities as polytope volumes:
\( \mathcal{N} \)-point and higher-genus \( su(2) \) fusion

Jørgen Rasmussen\( ^1 \) and Mark A. Walton\( ^2 \)

Physics Department, University of Lethbridge, Lethbridge, Alberta, Canada T1K 3M4

Abstract
We present the first polytope volume formulas for the multiplicities of affine fusion, the fusion in Wess-Zumino-Witten conformal field theories, for example. Thus, we characterise fusion multiplicities as discretised volumes of certain convex polytopes, and write them explicitly as multiple sums measuring those volumes. We focus on \( su(2) \), but discuss higher-point (\( \mathcal{N} > 3 \)) and higher-genus fusion in a general way. The method follows that of our previous work on tensor product multiplicities, and so is based on the concepts of generalised Berenstein-Zelevinsky diagrams, and virtual couplings. As a by-product, we also determine necessary and sufficient conditions for non-vanishing higher-point fusion multiplicities. In the limit of large level, these inequalities reduce to very simple non-vanishing conditions for the corresponding tensor product multiplicities. Finally, we find the minimum level at which the higher-point fusion and tensor product multiplicities coincide.

\( ^1 \)rasmussj@cs.uleth.ca; supported in part by a PIMS Postdoctoral Fellowship and by NSERC
\( ^2 \)walton@uleth.ca; supported in part by NSERC
1 Introduction

In a recent paper [1] we have shown how a higher-point $su(r + 1)$ tensor product multiplicity may be expressed as a multiple sum measuring the discretised volume of a certain convex polytope. That work is an extension of our previous work [2] on ordinary three-point couplings where three highest weight modules are coupled to the singlet. The number of times the singlet occurs in the decomposition is the associated multiplicity. Both of these papers are based on generalisations of the famous Berenstein-Zelevinsky (BZ) triangles [3]. They also rely on the use of so-called virtual couplings, that relate different (true) couplings associated to the same tensor product.

Our long-term objective is to extend these results to affine $su(r + 1)$ fusions. Here we make a start by considering $su(2)$. It turns out that all our results on $N$-point tensor products [1] have analogous and level-dependent counterparts in $N$-point fusions. Firstly, a fusion multiplicity admits a polyhedral combinatorial expression, where it is characterised by the discretised volume of a convex polytope. Secondly, this volume may be measured explicitly expressing the fusion multiplicity as a multiple sum.

We also work out very simple, easily remembered conditions determining when an $N$-point $su(2)$ fusion exists, i.e., when the associated multiplicity is non-vanishing. For infinite level, these “mnemo-friendly” conditions reduce to even simpler ones, solving the analogous problem for tensor products.

The second part of the present work deals with the extension of the above results to higher-genus $su(2)$ fusions. The first result is a characterisation of a general genus-$h$ $N$-point fusion multiplicity as the discretised volume of a convex polytope. The volume is measured explicitly, whereby the fusion multiplicity is expressed as a multiple sum. In order to reduce the number of summations, we then modify our approach slightly. The main building blocks in these considerations are the genus-one two-point couplings. Combining these allows one to describe general higher-genus $N$-point fusion multiplicities using fewer parameters than inherent in our polytope description. In terms of this reduced set of parameters, we provide explicit multiple sum formulas for the generic genus zero-, one- and two-point fusion multiplicities.

Our expressions make manifest that the various fusion multiplicities are non-negative integers, and are non-decreasing functions of the affine level.

2 $su(2)$ $N$-point fusion multiplicities

Let $M_\lambda$ denote an integrable highest weight module of an untwisted affine Lie algebra. The affine highest weight is uniquely specified by the highest weight $\lambda$ of the simple horizontal subalgebra (the underlying Lie algebra), and the affine level $k$. Fusion of two such modules may be written as

$$M_\lambda \times M_\mu = \sum_\nu N^{(k)}_{\lambda,\mu} \nu M_\nu ,$$

(1)

where $N^{(k)}_{\lambda,\mu} \nu$ is the fusion multiplicity. Determining these multiplicities is equivalent to studying the more symmetric problem of determining the multiplicity of the singlet in the expansion of the triple fusion

$$M_\lambda \times M_\mu \times M_\nu \supset N^{(k)}_{\lambda,\mu,\nu} M_0 .$$

(2)
If $\nu^+$ denotes the highest weight conjugate to $\nu$, we have $N^{(k)}_{\lambda,\mu,\nu} = N^{(k)}_{\lambda,\mu,\nu^+}$.

The associated and level-independent tensor product multiplicity is denoted $T_{\lambda,\mu,\nu}$. It is related to the fusion multiplicity as

$$T_{\lambda,\mu,\nu} = \lim_{k \to \infty} N^{(k)}_{\lambda,\mu,\nu}. \quad (3)$$

All of this extends readily to $N$-point couplings:

$$M_{\lambda^{(1)}} \times \ldots \times M_{\lambda^{(N)}} \supset N^{(k)}_{\lambda^{(1)},\ldots,\lambda^{(N)},M_0}, \quad (4)$$

which are the subject of the present work. In particular, we have the relation

$$T_{\lambda^{(1)},\ldots,\lambda^{(N)}} = \lim_{k \to \infty} N^{(k)}_{\lambda^{(1)},\ldots,\lambda^{(N)}}. \quad (5)$$

In the following we will focus on $su(2)$.

For $su(2)$ the three-point fusion multiplicity is

$$N^{(k)}_{\lambda,\mu,\nu} = \begin{cases} 1 & \text{if } 0 \leq S - \lambda_1, S - \mu_1, S - \nu_1, k - S, \quad S \equiv \frac{1}{2}(\lambda_1 + \mu_1 + \nu_1) \in \mathbb{Z} \geq 0 \text{ otherwise} \end{cases} \quad (6)$$

$\lambda_1$ denotes the finite or first Dynkin label of the weight $\lambda$. The level-independent information is encoded in the trivial BZ triangle

$$b \quad \quad c \quad \quad a \quad \quad (7)$$

where

$$a = \frac{1}{2}(-\lambda_1 + \mu_1 + \nu_1) \in \mathbb{Z} \geq, \quad b = \frac{1}{2}(\lambda_1 - \mu_1 + \nu_1) \in \mathbb{Z} \geq, \quad c = \frac{1}{2}(\lambda_1 + \mu_1 - \nu_1) \in \mathbb{Z} \geq, \quad (8)$$

and hence

$$\lambda_1 = b + c, \quad \mu_1 = c + a, \quad \nu_1 = a + b. \quad (9)$$

The level dependence is contained in the affine condition

$$k \geq a + b + c. \quad (10)$$

In Ref. [1] we outlined a general method for computing higher-point tensor product multiplicities. It is based on gluing BZ triangles together using “gluing roots” (we refer to Ref. [1] for details). An illustration is provided by the following $N$-point diagram (in this example $N$ is assumed odd):

$$\lambda^{(N-2)} \quad \lambda^{(N)} \quad \ldots \quad \lambda^{(N-3)} \quad \lambda^{(N-1)} \quad \lambda^{(2)} \quad \lambda^{(1)} \quad (11)$$
The role of the gluing is to take care of the summation over internal weights in a tractable way. The dual picture of ordinary (Feynman tree-) graphs is shown in thinner lines. Along a gluing, the opposite weights must be identified (for higher rank $su(r + 1)$ one must identify a weight with the conjugate weight to the opposite one, cf. (8)). The weights are simply given by sums of two entries (9). Our starting point (1) was to relax the constraint that the entries (8) should be non-negative integers. A diagram of that kind is called a generalised diagram. Any such generalised diagram, respecting the gluing constraints and the outer weight constraints (11), will suffice as an initial diagram. All other diagrams (associated to the same outer weights) may then be obtained by adding integer linear combinations of so-called virtual diagrams: adding a basis virtual diagram changes the weight of a given internal weight by two, leaving all other internal weights and all outer weights unchanged. Thus, the basis virtual diagram associated to a particular gluing is of the form:

$$G = \begin{pmatrix} 1 & -1 \\ \vdots & \ddots & \ddots & \ddots \\ -1 & 1 & & \end{pmatrix}$$

(12)

Enumerating the gluing roots (12) in (11) from right to left, the associated integer coefficients in the linear combinations are $-g_1, \ldots, -g_{N-3}$. Now, re-imposing the condition that all entries must be non-negative integers, results in a set of inequalities in the entries defining a convex polytope in the euclidean space $\mathbb{R}^{N-3}$:

$$0 \leq g_1, \lambda_1^{(2)} - g_1, \lambda_1^{(1)} - g_1,$$
$$0 \leq g_2 - g_1, \lambda_1^{(3)} - g_2 + g_1, \lambda_1^{(1)} + \lambda_1^{(2)} - g_2 - g_1,$$
$$\vdots$$
$$0 \leq g_{N-3} - g_{N-4}, \lambda_1^{(N-2)} - g_{N-3} + g_{N-4}, \lambda_1^{(1)} + \ldots + \lambda_1^{(N-3)} - g_{N-3} - g_{N-4},$$
$$0 \leq S - \lambda_1^{(N-1)} - g_{N-3}, S - \lambda_1^{(N)} - g_{N-3}, -S + \lambda_1^{(N-1)} + \lambda_1^{(N)} + g_{N-3}.$$  

(13)

By construction, its discretised volume is the tensor product multiplicity $T_{\lambda^{(1)}, \ldots, \lambda^{(N)}}$. In (13) we have introduced the quantity

$$S \equiv \frac{1}{2} \sum_{l=1}^{N} \lambda_1^{(l)} \in \mathbb{Z}_\geq.$$  

(14)

That $S$ is an integer is a consistency condition, i.e., for $S$ a half-integer the multiplicity vanishes.

The extension to fusion is provided by supplementing the set of inequalities (13) with the associated affine conditions (cf. (10)), one for each triangle, i.e., one for each line in (13). This results in the following definition of a convex polytope in the euclidean space $\mathbb{R}^{N-3}$ (the affine conditions are written on separate lines):

$$0 \leq g_1, \lambda_1^{(2)} - g_1, \lambda_1^{(1)} - g_1,$$

(15)

We are using a slightly different notation for these variables than that employed in (8).
Here we shall present necessary and sufficient conditions determining when an fusion multiplicity is non-vanishing.

### 2.1 Conditions for non-vanishing fusion and tensor product multiplicities

We have made a straightforward choice:

\[ k - \lambda_1^{(1)} - \lambda_1^{(2)} + g_1, \]
\[ 0 \leq g_2 - g_1, \lambda_1^{(3)} - g_2 + g_1, \lambda_1^{(1)} + \lambda_1^{(2)} - g_2 - g_1, \]
\[ k - \lambda_1^{(1)} - \cdots - \lambda_1^{(3)} + g_2 + g_1, \]

\[ \vdots \]
\[ 0 \leq g_{N-3} - g_{N-4}, \lambda_1^{(N-2)} - g_{N-3} + g_{N-4}, \lambda_1^{(1)} + \cdots + \lambda_1^{(N-3)} - g_{N-3} - g_{N-4}, \]
\[ k - \lambda_1^{(1)} - \cdots - \lambda_1^{(N-2)} + g_{N-3} + g_{N-4}, \]
\[ 0 \leq S - \lambda_1^{(N-1)} - g_{N-3}, S - \lambda_1^{(N)} - g_{N-3}, -S + \lambda_1^{(N-1)} + \lambda_1^{(N)} + g_{N-3}, \]
\[ k - S + g_{N-3}. \]

By construction, its discretised volume is the associated \( N \)-point fusion multiplicity \( N_{\lambda_1^{(1)}, \ldots, \lambda_1^{(N)}}^{(k)} \).

This characterisation of the fusion multiplicity is a new result.

It is stressed that (15) (and also (13)) is non-unique as it reflects our choice of initial diagram when deriving (13), cf. [2, 1]. Any choice will define a convex polytope of the same shape and hence discretised volume, however. Changing the initial triangle merely corresponds to shifting the origin, or translating the entire polytope.

We have seen that the fusion polytope (15) corresponds to “slicing out” a convex polytope embedded in the tensor product polytope (13). Thus, our approach offers a geometrical illustration of the statement that fusion is a truncated tensor product.

The discretised volume of the convex polytope (15) may be measured explicitly. In order to avoid discussing intersections of faces we have to choose an “appropriate order” of summation (see Ref. [3, 4]). However, such an order is easily found. In the following multiple sum formula we have made a straightforward choice:

\[
N_{\lambda_1^{(1)}, \ldots, \lambda_1^{(N)}}^{(k)} = \min\{S - \lambda_1^{(N-1)}, S - \lambda_1^{(N)}\} - k + S \]
\[ = \sum_{g_{N-3}=\max\{S - \lambda_1^{(N-1)} - \lambda_1^{(N)}, -k+S\}} \min\{g_{N-3}, \lambda_1^{(1)} + \cdots + \lambda_1^{(N-3)} - g_{N-3}\} \times \sum_{g_{N-4}=\max\{-\lambda_1^{(N-2)} + g_{N-3}, -k + \lambda_1^{(1)} + \cdots + \lambda_1^{(N-2)} - g_{N-3}\}} \cdots \]
\[ \times \min\{g_3, \lambda_1^{(1)} + \cdots + \lambda_1^{(3)} - g_3\} \times \sum_{g_2=\max\{-\lambda_1^{(4)} + g_3, -k + \lambda_1^{(1)} + \cdots + \lambda_1^{(4)} - g_3\}} \min\{\lambda_1^{(1)}, \lambda_1^{(2)}, g_2, \lambda_1^{(1)} + \lambda_1^{(2)} - g_2\} \times \sum_{g_1=\max\{0, -k + \lambda_1^{(1)} + \lambda_1^{(2)}, -\lambda_1^{(3)} + g_2, -k + \lambda_1^{(1)} + \cdots + \lambda_1^{(3)} - g_2\}} 1. \] (16)

### 2.1 Conditions for non-vanishing fusion and tensor product multiplicities

Here we shall present necessary and sufficient conditions determining when an \( N \)-point fusion multiplicity is non-vanishing, \( N \geq 2 \). A similar result for the associated tensor product multi-
plicity is easily read off. Both sets of conditions are given as inequalities in the (finite) Dynkin labels. The conditions for fusion depend on the level \( k \).

A fusion multiplicity is non-vanishing if and only if the associated convex polytope has a non-vanishing discretised volume. In particular, the multiplicity is one when the polytope is a point. An analysis of the polytope \((\mathcal{P})\), or equivalently of the multiple sum formula \((\sigma)\), leads to the following necessary and sufficient conditions for the fusion multiplicity to be non-vanishing:

\[
0 \leq \lambda^{(l)}_1, \quad S - \lambda^{(l)}_1, \quad k - \lambda^{(l)}_1, \quad l = 1, \ldots, N, \\
0 \leq dk - S + \lambda^{(l)}_1 + \ldots + \lambda^{(l_{N-2d-1})}_1, \quad l_m < l_n \text{ for } m < n; \quad 1 \leq d \leq \left\lfloor \frac{N-1}{2} \right\rfloor. \tag{17}
\]

\([x]\) denotes the integer value of \( x \), i.e., the greatest integer less than or equal to \( x \). Note that for \( d = 0 \) the associated inequalities reduce to \( 0 \leq S - \lambda^{(l)}_1 \). These latter inequalities have been written separately for clarity. The upper bound on \( d \) is included to avoid redundancies.

The conditions \((17)\) may be proved by induction. In the set of inequalities \((\mathcal{I})\) (or equivalently in the multiple sum formula \((\sigma)\)), one eliminates one after the other the variables \( g_1, \ldots, g_{N-3} \). The inequalities involving \( g_1 \) and \( g_{N-3} \) are different in form from those for the remaining \( N - 5 \) variables \((\mathcal{F})\). Thus, the induction concerns the elimination of the middle \( N - 5 \) variables, \( g_2, \ldots, g_{N-4} \). First we eliminate \( g_1 \), then \( g_2 \) etc. After having eliminated the first \( n - 1 \) variables, \( 2 \leq n - 1 \leq N - 4 \), we have obtained the following set of inequalities

\[
0 \leq \lambda^{(l)}_1, \quad k - \lambda^{(l)}_1 \quad \text{for} \quad l \leq n, \\
\max\{-\lambda^{(n+1)}_1 + g_n, \quad -k + \lambda^{(1)}_1 + \ldots + \lambda^{(n+1)}_1 - g_n, \quad 0, \quad \frac{1}{2}(-k + \lambda^{(1)}_1 + \ldots + \lambda^{(n)}_1), \quad -dk + \lambda^{(l_1)}_1 + \ldots + \lambda^{(l_{2d})}_1\} \quad \text{for} \quad l_m < l_{m'} \leq n \text{ for } m < m'; \quad 2d \leq n \\
\leq \min\{g_n, \quad \lambda^{(1)}_1 + \ldots + \lambda^{(n)}_1 - g_n, \quad \frac{1}{2}(\lambda^{(1)}_1 + \ldots + \lambda^{(n)}_1), \quad \lambda^{(1)}_1 + \ldots + \lambda^{(n)}_1 - \lambda^{(l)}_1, \quad Dk + \lambda^{(l_1)}_1 + \ldots + \lambda^{(l_{n-2d-1})}_1\} \quad \text{for} \quad l \leq n; \quad l_m < l_{m'} \leq n \text{ for } m < m'; \quad 2D \leq n - 1, \tag{18}
\]

in addition to the original inequalities \((\mathcal{I})\) involving only \( g_n, \ldots, g_{N-3} \). It is when proving \((\mathcal{I})\) that we use induction in \( n \), and we conclude that it is true for \( 2 \leq n - 1 \leq N - 4 \). Eliminating the final variable \( g_{N-3} \) results in the asserted conditions \((\mathcal{I})\), which we believe are new.

For high level \( k \) a fusion reduces to a tensor product \((\mathcal{F})\). Necessary and sufficient conditions for a non-vanishing tensor product multiplicity are therefore easily read off \((\mathcal{I})\):

\[
0 \leq \lambda^{(l)}_1, \quad S - \lambda^{(l)}_1, \quad l = 1, \ldots, N. \tag{19}
\]

As discussed in Ref. \([\mathcal{I}]\), this result is easily verified for \( N \leq 4 \). For general \( N \) it is believed to be a new result.

We note that \((\mathcal{I})\) and \((\mathcal{F})\) are also valid for \( N = 2 \), despite the fact that, a priori, the inequalities were derived for \( N \geq 3 \) only.

### 2.2 Conditions on the level

The lower bound on \( k \) is immediately read off \((\mathcal{I})\). In ordinary three-point fusion the analogous bound is sometimes referred to as the minimum threshold level, and is denoted \( \mu_{\text{min}} \). It specifies
the minimum value of $k$ for which $N^{(k)}_{\lambda^{(1)},...\lambda^{(N)}}$ is non-vanishing:

$$N^{(k<\text{min})}_{\lambda^{(1)},...\lambda^{(N)}} = 0, \quad N^{(k\geq\text{min})}_{\lambda^{(1)},...\lambda^{(N)}} > 0.$$ \hspace{1cm} (20)

It does not make sense to assign a minimum threshold level to a fusion for which the associated tensor product multiplicity $T_{\lambda^{(1)},...\lambda^{(N)}}$ vanishes.

According to (17) we have

$$t^{\text{min}} = \max\{\lambda^{(l)}_1, \frac{1}{d}(\lambda^{(l_{N-2})}_1 + ... + \lambda^{(l_{N})}_1 - S)\},$$ \hspace{1cm} (21)

with the parameters specified as in (17).

The maximum threshold level, denoted $t^{\text{max}}$, is defined as the minimum level $k$ for which the fusion multiplicity equals the tensor product multiplicity:

$$N^{(k<\text{max})}_{\lambda^{(1)},...\lambda^{(N)}} < T_{\lambda^{(1)},...\lambda^{(N)}}, \quad N^{(k\geq\text{max})}_{\lambda^{(1)},...\lambda^{(N)}} = T_{\lambda^{(1)},...\lambda^{(N)}}.$$ \hspace{1cm} (22)

Again, it is not natural to assign a maximum threshold level to a fusion if $T_{\lambda^{(1)},...\lambda^{(N)}}$ vanishes. Though in this case, one could define it as $t^{\text{max}} = 0$, since by assumption $k \in \mathbb{Z}_{\geq}$, and (22) would still be respected.

To compute $t^{\text{max}}$ in our case, we first observe that all affine conditions in (15) are redundant when $k \geq S$. As an illustration, we have (assuming $3 \leq m \leq N - 3$)

$$0 \leq (g_1) + (g_2 - g_1) + ... + (g_{m-2} - g_{m-3}) + (\lambda^{(m+1)}_1 - g_m + g_{m-1}) + ... + (\lambda^{(N-2)}_1 - g_{N-3} + g_{N-4}) + (-S + \lambda^{(N-1)}_1 + \lambda^{(N)}_1 + g_{N-3})$$

$$\quad = \lambda^{(m+1)}_1 + ... + \lambda^{(N)}_1 - S + g_{m-1} + g_{m-2}$$

$$\leq k - \lambda^{(1)}_1 - ... - \lambda^{(m)}_1 + g_{m-1} + g_{m-2}.$$ \hspace{1cm} (23)

This means that $t^{\text{max}} \leq S$. In order to show that

$$t^{\text{max}} = S,$$ \hspace{1cm} (24)

we first assume that there exists integer $n$, $2 \leq n \leq N - 2$, $(n = 1$ is trivial) such that

$$\lambda^{(1)}_1 + ... + \lambda^{(n-1)}_1 - S < 0 \leq \lambda^{(1)}_1 + ... + \lambda^{(n)}_1 - S.$$ \hspace{1cm} (25)

We then consider the point defined by

$$g_l = ... = g_{n-2} = 0,$$

$$g_{n-1} = \lambda^{(1)}_1 + ... + \lambda^{(n)}_1 - S = S - \lambda^{(n+1)}_1 - ... - \lambda^{(N)}_1,$$

$$\vdots$$

$$g_{N-3} = \lambda^{(1)}_1 + ... + \lambda^{(N-2)}_1 - S = S - \lambda^{(N-1)}_1 - \lambda^{(N)}_1.$$ \hspace{1cm} (26)

It is straightforward to show that it is in the fusion polytope (13) when $k \geq S$, and that it is not when $k < S$.

Finally, if there does not exist an $n$, $2 \leq n \leq N - 2$, satisfying (25), we must have $S \leq \lambda^{(N-1)}_1 + \lambda^{(N)}_1$. In that case we consider the point $g_l = 0$, $l = 1, ..., N - 3$. For this point to be in the polytope, the condition on $k$ (13) is $S \leq k$, and we conclude that the maximum threshold level is given by (24).
3 Higher-genus $su(2)$ fusion multiplicities

Here we will discuss the extension of our results above on genus-zero fusion to generic genus-$h$ fusion. $N^{(k,h)}_{\lambda^{(1)},\ldots,\lambda^{(N)}}$ denotes the genus-$h$ $\mathcal{N}$-point fusion multiplicity.

Just as in the case of vanishing genus, we may choose the channel freely. A simple extension of (11) is the following genus-$h$ $\mathcal{N}$-point diagram (in this example $\mathcal{N}$ is assumed even, while $h$ is arbitrary):

Again, the dual trivalent fusion graph is represented by thinner lines and loops. $h$ is the number of such loops or handles. The role of the two zeros in (27) will be discussed below.

Independent of the choice of channel, the number of internal weights or gluings is $\mathcal{N}+3(h-1)$, while the number of vertices or triangles is $\mathcal{N}+2(h-1)$.

The basis diagram associated to the “self-coupling” or tadpole diagram

is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We call (28) a loop-gluing diagram. It is stressed that it differs from the gluing root (12) since it adds only one to the internal weight and not two. This discrepancy follows from the fact that the Dynkin labels satisfy $\lambda_1 + \mu_1 + \nu_1 \in 2\mathbb{Z}_{\geq 0}$, so if two weights are changed simultaneously and equally, we can only require an even change of the sum of them.

A similar situation arises when considering the genus-one two-point coupling

$$\lambda \quad \mu$$

A simple analysis shows that there are two basis loop-gluing associated to this coupling, and that they may be represented by the diagrams

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathcal{L}' = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$
It is now easy to write down the inequalities defining the convex polytope. Our choice of initial diagram is indicated in (27) by the two zeros: all entries of the higher-genus part to the right of them are zero, while the $\mathcal{N}$-point part follows the pattern of the initial diagram associated to (11) and (13) - see [1] for details. Enumerating the (loop-)gluings from right to left (and $\mathcal{L}$ before $\mathcal{L}'$), the integer coefficients in the linear combinations are $g_1, \ldots, g_h, -g_{h+1}, \ldots, -g_{N+h-2}$ (the sign convention is merely for convenience), and $l_1, l'_1, \ldots, l_{h-1}, l'_{h-1}$, while $l$ is associated to the tadpole at the extreme right. Listing the inequalities associated to the triangles from right to left, we have the following convex polytope (assuming $h \geq 1$):

$$0 \leq l - g_1, g_1, g_1,$$
$$k - g_1 - l,$$
$$0 \leq l_1 - g_1, g_1 + l'_1, g_1 - l'_1,$$
$$k - g_1 - l_1,$$
$$0 \leq l_1 - g_2, g_2 + l'_1, g_2 - l'_1,$$
$$k - g_2 - l_1,$$

$$\vdots$$

$$0 \leq l_{h-1} - g_{h-1}, g_{h-1} + l'_{h-1}, g_{h-1} - l'_{h-1},$$
$$k - g_{h-1} - l_{h-1},$$
$$0 \leq l_{h-1} - g_h, g_h + l'_{h-1}, g_h - l'_{h-1},$$
$$k - g_h - l_{h-1},$$
$$0 \leq g_{h+1} + g_h, -g_{h+1} + g_h, \lambda_1^{(1)} - g_{h+1} - g_h,$$
$$k - \lambda_1^{(1)} + g_{h+1} - g_h,$$
$$0 \leq g_{h+2} - g_{h+1}, \lambda_1^{(1)} - g_{h+2} - g_{h+1}, \lambda_1^{(2)} - g_{h+2} + g_{h+1},$$
$$k - \lambda_1^{(1)} - \lambda_1^{(2)} + g_{h+2} + g_{h+1},$$

$$\vdots$$

$$0 \leq g_{N+h-2} - g_{N+h-3}, \lambda_1^{(N-1)} + \ldots + \lambda_1^{(N-3)} - g_{N+h-2} - g_{N+h-3},$$
$$\lambda_1^{(N-2)} - g_{N+h-2} + g_{N+h-3},$$
$$k - \lambda_1^{(1)} - \ldots - \lambda_1^{(N-2)} + g_{N+h-2} + g_{N+h-3},$$
$$0 \leq S - \lambda_1^{(N-1)} - g_{N+h-2}, S - \lambda_1^{(N)} - g_{N+h-2}, -S + \lambda_1^{(N-1)} + \lambda_1^{(N)} + g_{N+h-2},$$
$$k - S + g_{N+h-2}. \tag{32}$$

By construction, its discretised volume is the fusion multiplicity $N_{\lambda_1^{(1)}, \ldots, \lambda_1^{(N)}}^{(k,h)}$, which then provides a new way of characterising fusion multiplicities. The volume may be measured explicitly expressing $N_{\lambda_1^{(1)}, \ldots, \lambda_1^{(N)}}^{(k,h)}$ as a multiple sum:

$$N_{\lambda_1^{(1)}, \ldots, \lambda_1^{(N)}}^{(k,h)} = \sum_{g_{N+h-2}} \ldots \sum_{g_h} \left( \sum_{l_{h-1}} \sum_{l_{h-1}} \sum_{g_{h-1}} \right) \ldots \left( \sum_{l'_1} \sum_{l_1} \sum_{g_1} \right) \sum_{l} 1. \tag{33}$$
The summation variables are bounded according to
\[ g_1 \leq l \leq k - g_1, \]
\[ |l'_1| \leq g_1 \leq \min\{l_1, k - l_1\}, \]
\[ g_2 \leq l_1 \leq k - g_2, \]
\[ -g_2 \leq l'_1 \leq g_2, \]
\[ |l'_{h-1}| \leq g_{h-1} \leq \min\{l_{h-1}, k - l_{h-1}\}, \]
\[ g_h \leq l_{h-1} \leq k - g_h, \]
\[ -g_h \leq l'_{h-1} \leq g_h, \]
\[ |g_{h+1}| \leq g_h \leq \min\{\lambda_1^{(2)} - g_{h+1}, k - \lambda_1^{(2)} + g_{h+1}\}, \]
\[ \max\{-\lambda_1^{(2)} + g_{h+2}, -k + \lambda_1^{(1)} + \lambda_1^{(2)} - g_{h+2}\} \leq g_{h+1} \leq \min\{g_{h+2}, \lambda_1^{(1)} - g_{h+2}\}, \]
\[ \max\{-\lambda_1^{(N-2)} + g_{N+h-2}, -k + \lambda_1^{(1)} + \ldots + \lambda_1^{(N-2)} - g_{N+h-2}\} \leq g_{N+h-3} \leq \min\{g_{N+h-2}, \lambda_1^{(1)} + \ldots + \lambda_1^{(N-3)} - g_{N+h-2}\}, \]
\[ \max\{S - \lambda_1^{(N-1)} - \lambda_1^{(N)}, -k + S\} \leq g_{N+h-2} \leq \min\{S - \lambda_1^{(N-1)}, S - \lambda_1^{(N)}\}. \] (34)

This constitutes the first explicit result for the general genus-\( h \) \( N \)-point fusion multiplicities. In the following we will discuss a few examples, where the convex polytope characterisation is sacrificed in order to reduce the number of summation variables.

### 3.1 Two-point couplings

Let us first consider the genus-one two-point coupling (30). According to the general discussion above, one may express the associated fusion multiplicity in terms of two parameters. A further analysis shows that
\[ N_{\lambda, \mu}^{(k, 1)} = \begin{cases} \min\{\lambda_0, \mu_0\} + 1, & |\lambda_1 - \mu_1| \in 2\mathbb{Z}_+ \\ 0, & |\lambda_1 - \mu_1| + 1 \in 2\mathbb{Z}_+ \end{cases}, \] (35)

where the zero'th Dynkin label of the affine weight \( \lambda \) is \( \lambda_0 = k - \lambda_1 \). Now, it is straightforward to construct higher-genus two-point diagrams by gluing together diagrams like (30)

\[ \lambda \quad \begin{array}{c} \vdots \end{array} \quad \mu \] (36)

When computing the associated fusion multiplicities one uses the result (35), paying attention to the finite Dynkin labels being odd or even. For example, when \( \lambda_1 \) and \( \mu_1 \) are both even, the
sum formula reads

\[ N_{\lambda,\mu}^{(k,h)} = \sum_{m_1,\ldots,m_{h-1}=0}^{[k/2]} (k - \max\{\lambda_1,2m_1\} + 1)(\min\{\lambda_1,2m_1\} + 1) \times (k - \max\{2m_1,2m_2\} + 1)(\min\{2m_1,2m_2\} + 1) \]

\[ \vdots \]

\[ \times (k - \max\{2m_{h-2},2m_{h-1}\} + 1)(\min\{2m_{h-2},2m_{h-1}\} + 1) \times (k - \max\{2m_{h-1},\mu_1\} + 1)(\min\{2m_{h-1},\mu_1\} + 1) . \quad (37) \]

It is easily adjusted to cover the situation when both labels are odd (see also (36)). If one label is odd and the other is even, the associated fusion multiplicity vanishes. Note that the number of summation variables is \( h - 1 \), while the number of summations in our previous treatment (33) was \( 3h - 1 \). Thus, from that point of view (37) is a considerable simplification.

The summations in (37) are, in principle, straightforward to evaluate using the formula

\[ \sum_{m=1}^{M} (m)_s = \frac{1}{s+1} (M)_{s+1} , \quad (38) \]

where

\[ (a)_n \equiv a(a+1)\ldots(a + n - 1) . \quad (39) \]

(38) is easily proven by induction.

### 3.2 One-point couplings

A one-point coupling simply corresponds to putting one of the weights of a two-point coupling equal to zero. It may be illustrated by the diagram

![Diagram](image)

and the associated fusion multiplicity is

\[ N_{\lambda}^{(k,h)} = \sum_{m_1,\ldots,m_{h-1}=0}^{[k/2]} (k - 2m_1 + 1) \times (k - \max\{2m_1,2m_2\} + 1)(\min\{2m_1,2m_2\} + 1) \]

\[ \vdots \]

\[ \times (k - \max\{2m_{h-2},2m_{h-1}\} + 1)(\min\{2m_{h-2},2m_{h-1}\} + 1) \times (k - \max\{2m_{h-1},\lambda_1\} + 1)(\min\{2m_{h-1},\lambda_1\} + 1) , \quad \lambda_1 \in 2\mathbb{Z}_\ge . (41) \]
It is noted that the Dynkin label \( \lambda_1 \) must be even. For \( h = 1 \) (41) reduces to

\[
N_{\lambda}^{(k,1)} = \begin{cases} 
  k - \lambda_1 + 1 , & \lambda_1 \in 2\mathbb{Z} \\
  0 , & \lambda_1 + 1 \in 2\mathbb{Z} 
\end{cases}
\] (42)

### 3.3 Zero-point couplings

As for any other \( \mathcal{N} \), there are many possible choices of channels when discussing zero-point couplings. An immediate application of our discussion on two-point couplings (36) corresponds to the diagram

![Diagram](image)

This is obtained by putting both weights in (36) equal to zero, and the associated fusion multiplicity may be expressed as

\[
N^{(k,h)} = \sum_{m_1, \ldots, m_{h-1} = 0}^{[k/2]} (k - 2m_1 + 1) \times (k - \max\{2m_1, 2m_2\} + 1)(\min\{2m_1, 2m_2\} + 1) \times \cdots \times (k - \max\{2m_{h-2}, 2m_{h-1}\} + 1)(\min\{2m_{h-2}, 2m_{h-1}\} + 1) \times (k - 2m_{h-1} + 1) .
\] (44)

Another “natural” channel is governed by the diagram

![Diagram](image)

Following our general prescription above for computing the associated fusion multiplicity, results in the expression

\[
N^{(k,h)} = \sum_{m_1, \ldots, m_{g-1} = 0}^{[k/2]} (k - \max\{2m_{h-1}, 2m_1\} + 1)(\min\{2m_{h-1}, 2m_1\} + 1) \times (k - \max\{2m_1, 2m_2\} + 1)(\min\{2m_1, 2m_2\} + 1) \times \cdots \times (k - \max\{2m_{h-2}, 2m_{h-1}\} + 1)(\min\{2m_{h-2}, 2m_{h-1}\} + 1)
\]
\[ \sum_{m_1,\ldots,m_{h-1}=0}^{[(k-1)/2]} (k - \max\{2m_{h-1}, 2m_1\}) (\min\{2m_{h-1}, 2m_1\} + 2) \]

\[ \times (k - \max\{2m_1, 2m_2\}) (\min\{2m_1, 2m_2\} + 2) \]

\[ \vdots \]

\[ \times (k - \max\{2m_{h-2}, 2m_{h-1}\}) (\min\{2m_{h-2}, 2m_{h-1}\} + 2) \]  (46)

which differs considerably in form from (44). Nevertheless, by construction, the two multiple sums must be identical. We will not attempt to prove that explicitly. This identity provides a simple example of the result of identifying the fusion multiplicities computed using different channels.

Examples of non-trivial zero-point fusion multiplicities are

\[ N^{(k,1)} = k + 1, \quad (47) \]

and

\[ N^{(k,2)} = \frac{(k + 1)^3}{6} = \frac{1}{6} (k + 1)(k + 2)(k + 3). \quad (48) \]

4 Comments

We conclude by adding a few comments, primarily on the existing literature. In Ref. [4] Dowker discusses results on fusion multiplicities based on the Verlinde formula [5]. The results are expressed in terms of twisted cosec sums and Bernoulli polynomials, and pertain essentially to two-point couplings (and therefore also to one- and zero-point couplings). Particular emphasis is put on the classical limit where the level \( k \) tends to infinity, and previous results on that limit are recovered ([6, 7] for zero-point couplings and [7] for one-point couplings). In the language employed in [4], fusion multiplicities correspond to dimensions of certain vector bundles over the moduli space of an \( N \)-punctured Riemann surface of genus \( h \).

The results of [4] are essentially obtained by trigonometric manipulations of the Verlinde formula. They do not, therefore, display any transparent relationship with our convex polytope approach. Nevertheless, a comparison of results leads to interesting identities between different types of multiple sums, and some similarities of the final expressions are apparent. One could try to prove their equivalence by brute force. That is beyond the scope of the present work, though.

The results of [4] do not offer an immediate resolution to the question of when a fusion multiplicity is non-vanishing. By construction, a characterisation in terms of a convex polytope, on the other hand, is “almost” designed to address such problems. Furthermore, our approach seems amenable to the treatment of higher rank \( su(r + 1) \) fusions, whereas an application of the Verlinde formula appears technically very complicated. We are currently considering such an extension of our approach based on previous results on the role of BZ triangles in affine \( su(3) \) and \( su(4) \) fusions [8, 9]. A different approach to fusion based on the depth rule and the correspondence to three-point functions in Wess-Zumino-Witten conformal field theory may be found in our recent work [10, 11].

In Ref. [12], Kirillov provides a combinatorial formula for the \( N \)-point \( su(2) \) fusion multiplicities. It is a fermionic-type formula, a sum of products of binomial coefficients, derived by
applying the Bethe ansatz to certain solvable lattice models. (For a nice, brief review of formulas of fermionic and bosonic type, see the introduction to [13].) No formulas for higher-genus multiplicities are given, however.

Kirillov’s fermionic formula has also been generalised somewhat. See Theorem 6.2 of [14] for a $q$-deformed $su(r + 1)$ generalisation, and the extensive bibliography of [15]. Although interesting for other reasons, these formulas are only valid for certain representations at the $\mathcal{N}$-points, and they are also restricted to $h = 0$. Such restrictions do not appear to be necessary in our method.

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