STRESS-ENERGY-MOMENTUM TENSORS
IN CONSTRAINT FIELD THEORIES

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Abstract

One has not any conventional energy-momentum conservation law in Lagrangian field theory, but relations involving different stress-energy-momentum tensors associated with different connections. It is not obvious how to choose the true energy-momentum tensor. This problem is solved in the framework of the multimomentum Hamiltonian formalism which provides the adequate description of constraint field systems. The goal is that, for different solutions of the same constraint field model, one should choose different stress-energy-momentum tensors in general. Gauge theory illustrates this result. The stress-energy-momentum tensors of affine-metric gravity are examined.

1 Introduction

In analytical mechanics, there exists the conventional energy conservation law, otherwise in field theory. Here, we are concerned only with differential conservation laws.

Let $F$ be a manifold. In time-dependent mechanics on the phase space $\mathbb{R} \times T^*F$ coordinatized by $(t, y^i, \dot{y}_i)$ and on the configuration space $\mathbb{R} \times TF$ coordinatized by $(t, y^i, \dot{y}^i)$, the Lagrangian energy and the construction of the Hamiltonian formalism require the prior choice of a connection on the bundle $\mathbb{R} \times F \to \mathbb{R}$ [4]. Such a connection however is usually hidden by utilizing the natural trivial connection on this bundle. Given a Hamiltonian function $\mathcal{H}$ on the phase space $\mathbb{R} \times T^*F$, we therefore have the conventional energy conservation law

$$\frac{d\mathcal{H}}{dt} \approx \frac{\partial \mathcal{H}}{\partial t}$$

where by "$\approx$" is meant the weak identity modulo the Hamilton equations. Given a Lagrangian function $\mathcal{L}$ on the configuration space $\mathbb{R} \times TF$, there exists the fundamental identity

$$\frac{\partial \mathcal{L}}{\partial t} + \frac{d}{dt}(\dot{y}^i(t) \frac{\partial \mathcal{L}}{\partial \dot{y}^i} - \mathcal{L}) \approx 0$$

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modulo the motion equations. It is the energy conservation law in the following sense.

Let \( \hat{L} \) be the Legendre morphism
\[
\dot{y}_i \circ \hat{L} = \frac{\partial L}{\partial \dot{y}_i}
\]
and \( Q = \text{Im} \hat{L} \) the Lagrangian constraint space. Let \( \mathcal{H} \) be a Hamiltonian function associated with \( L \) and \( \hat{H} \) the momentum morphism
\[
\dot{y}^i \circ \hat{H} = \frac{\partial H}{\partial \dot{y}_i}.
\]
Every solution \( r \) of the Hamilton equations of \( \mathcal{H} \) which lives on \( Q \) yields the solution \( \hat{H} \circ r \) of the Euler-Lagrange equations of \( L \). Then, the identity (2) on \( \hat{H} \circ r \) recovers the energy tranformation law (1) on \( r \).

Note that there are different Hamiltonian functions associated with the same singular Lagrangian function as a rule. Given such a Hamiltonian function, the Lagrangian constraint space \( Q \) plays the role of the primary constraint space, and the Dirac-Bergmann procedure can be utilized in order to get the final constraint space where a solution of the Hamilton equations exists [2, 6].

In field theory, classical fields are described by sections of a fibered manifold \( Y \to X \). Their dynamics is phrased in terms of jet manifolds.

As a shorthand, one can say that the \( k \)-order jet manifold \( J^kY \) of a fibered manifold \( Y \to X \) comprises the equivalence classes \( j^k_x s \), \( x \in X \), of sections \( s \) of \( Y \) identified by the first \( (k + 1) \) terms of their Taylor series at a point \( x \).

We restrict ourselves to the first order Lagrangian formalism when the configuration space is \( J^1Y \). Given fibered coordinates \( (x^\mu, y^i) \) of \( Y \), the jet manifold \( J^1Y \) is endowed with the adapted coordinates \( (x^\mu, y^i, y^i_\mu) \). A first order Lagrangian density on \( J^1Y \) is defined to be the morphism
\[
L : J^1Y \to \wedge^n T^*X, \quad n = \dim X,
\]
\[
L = L(x^\mu, y^i, y^i_\mu) \omega, \quad \omega = dx^1 \wedge ... \wedge dx^n.
\]
The corresponding first order Euler-Lagrange equations for sections \( \bar{s} \) of \( J^1Y \to X \) read
\[
\partial_\lambda \bar{s}^i = \bar{s}_\lambda^i, \quad \partial_i L - (\partial_\lambda + \bar{s}_\lambda^j \partial_j + \partial_\lambda \bar{s}^j_\mu \partial_j^\mu) \partial_i^\lambda L = 0. \quad (3)
\]

We consider the Lie derivatives of Lagrangian densities in order to obtain differential conservation laws.

Let
\[
u = u^\mu(x) \partial_\mu + u^i(y) \partial_i
\]
be a projectable vector field on \( Y \to X \) and \( \pi \) its jet lift onto \( J^1Y \to X \). Given \( L \), let us computer the Lie derivative \( L_{\pi} L \). We get the identity

\[
\overline{s}^* L_{\pi} L \approx -\frac{d}{dx^\lambda} [\pi_i^\lambda (u^\mu \pi_i^\mu - u^i) - u^\lambda \mathcal{L}] \omega, \quad \pi_i^\mu = \partial_i^\mu \mathcal{L},
\]

modulo the Euler-Lagrange equations (4).

In particular, if \( u \) is a vertical vector field this identity comes to the current conservation law exemplified by the Noether identities in gauge theory [15].

Let now \( \tau = \tau^\lambda \partial_\lambda \) be a vector field on \( X \) and

\[
\tau_\Gamma = \tau^\mu (\partial_\mu + \Gamma_i^\mu \partial_i)
\]

its horizontal lift onto \( Y \) by a connection \( \Gamma \) on \( Y \to X \). In this case, the identity (4) takes the form

\[
\overline{s}^* L_{\tau_\Gamma} L \approx -\frac{d}{dx^\lambda} [\tau^\mu \tau^{\lambda}_{\mu} (\tau)] \omega
\]

where

\[
\tau^{\lambda}_{\mu} (\tau) = [\pi_i^\lambda (y_i^\mu - \Gamma_i^\mu) - \delta_i^\mu \mathcal{L}] \circ \pi_i^\mu
\]

is the stress-energy-momentum (SEM) tensor on a field \( \tau \) relative to the connection \( \Gamma \).

We here restrict ourselves to this particular case of SEM tensors [5, 7, 10].

For instance, let us choose the trivial local connection \( \Gamma_i^\mu = 0 \). In this case, the identity (6) recovers the well-known conservation law

\[
\frac{\partial \mathcal{L}}{\partial x^\lambda} + \frac{d}{dx^\lambda} \tau^{\lambda}_{\mu} (\tau) \approx 0
\]

of the canonical energy-momentum tensor

\[
\tau^{\lambda}_{\mu} (\tau) = \pi_i^\lambda \pi_i^\mu - \delta_i^\mu \mathcal{L}.
\]

Physicists lose sight of the fact that (7) fails to be a mathematical well-behaved object.

The crucial point lies in the fact that the Lie derivative

\[
L_{\tau_\Gamma} L = \{ \partial_\mu \tau^\mu \mathcal{L} + [\tau^\mu \partial_\mu + \tau^\mu \Gamma_i^\mu \partial_i + (\partial_\lambda (\tau^\mu \Gamma_i^\mu) + \tau^\mu y_i^j \partial_j \Gamma_i^\mu - y_i^j \partial_\lambda \tau^\mu \partial_i^j \mathcal{L}] \omega
\]

is almost never equal to zero. Therefore, it is not obvious how to choose the true energy-momentum tensor.

For instance, the canonical energy-momentum tensor (4) in gauge theory is symmetrized by hand in order to obtain the gauge invariant one. In gauge theory in the presence of a background world metric \( g \), the identity (3) is brought into the covariant conservation law

\[
\nabla_\lambda t^\lambda_{\mu} \approx 0
\]

(8)
for the metric energy-momentum tensor.

In Einstein’s General Relativity, the covariant conservation law (8) issues directly from the gravitation equations. It is concerned with the zero-spin matter in the presence of the gravitational field generated by this matter, though the matter is not required to satisfy the motion equations. The total energy-momentum conservation law for matter and gravity is introduced by hand. It is written

$$\frac{d}{dx^\mu} [(-g)^N (t^{\lambda \mu} + t_g^{\lambda \mu})] \approx 0$$  \hspace{1cm} (9)

where the energy-momentum pseudotensor $t_g^{\lambda \mu}$ of a metric gravitational field is defined to satisfy the relation

$$(-g)^N (t^{\lambda \mu} + t_g^{\lambda \mu}) \approx \frac{1}{2\kappa} \partial_\alpha \partial_\alpha [(-g)^N (g^{\lambda \mu} g^{\sigma \alpha} - g^{\sigma \mu} g^{\lambda \alpha})]$$

modulo the Einstein equations. However, the conservation law (9) appears satisfactory only in cases of asymptotic-flat gravitational fields and of a background metric.

Moreover, the covariant conservation law (8) fails to take place in the affine-metric gravitation theory and in the gauge gravitation theory, e.g., in the presence of fermion fields.

Thus, we have not any conventional energy-momentum conservation law in Lagrangian field theory. In particular, one may take different SEM tensors for different field models and, moreover, different SEM tensors for different solutions of the same field equations. Just the latter in fact is the above-mentioned symmetrization of the canonical energy-momentum tensor in gauge theory.

Gauge theory exemplifies constraint field theories. Contemporary field models are almost always the constraint ones. To describe them, let us turn to the Hamiltonian formalism.

When applied to field theory, the conventional Hamiltonian formalism takes the form of the instantaneous Hamiltonian formalism where canonical variables are field functions at a given instant of time. The corresponding phase space is infinite-dimensional, so that the Hamilton equations in the bracket form fail to be differential equations.

The true partners of the Lagrangian formalism in classical field theory are polysymplectic and multisymplectic Hamiltonian machineries where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the temporal one [4, 8, 11, 12]. We here follow the multimomentum Hamiltonian formulation of field theory when the phase space of fields is the Legendre bundle

$$\Pi = \wedge^n T^* X \otimes T X \otimes V^* Y$$  \hspace{1cm} (10)
over $Y$ which is coordinatized by $(x^\lambda, y^i, p^\lambda_i)$ \[12\, 13\, 14\]. Every Lagrangian density $L$ on $J^1Y$ implies the Legendre morphism

$$\hat{L} : J^1Y \to \Pi,$$

$$p^n_i \circ \hat{L} = \pi^n_i.$$

The Legendre bundle (10) carries the polysymplectic form

$$\Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial^\lambda.$$

(11)

We say that a connection $\gamma$ on the fibered Legendre manifold $\Pi \to X$ is a Hamiltonian connection if the form $\gamma \mid \Omega$ is closed. Then, a Hamiltonian form $H$ on $\Pi$ is defined to be an exterior form such that

$$dH = \gamma \mid \Omega$$

(12)

for some Hamiltonian connection $\gamma$. The key point lies in the fact that every Hamiltonian form admits splitting

$$H = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda \omega - \tilde{H} \Gamma \omega = p^\lambda_i dy^i \wedge \omega_\lambda - \mathcal{H}, \quad \omega_\lambda = \partial_\lambda \mid \omega,$$

(13)

where $\Gamma$ is a connection on $Y \to X$.

Given the splitting (13), the equality (12) comes to the Hamilton equations

$$\partial_\lambda r^i = \partial^\lambda_i \mathcal{H},$$

$$\partial_\lambda r^\lambda_i = -\partial_i \mathcal{H}$$

(14)

for sections $r$ of $\Pi \to X$.

The Hamilton equations (14) are the multimomentum generalization of the Hamilton equations in mechanics. The corresponding multimomentum generalization of the conventional energy conservation law (1) is the weak identity

$$\tau^\mu[(\partial_\mu + \Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu \partial^\lambda_j \tilde{H}_\Gamma - \frac{d}{dx^\lambda} T^\lambda_\mu(r)] \approx \tau^\mu r^i_\lambda R^\mu_\lambda,$$

(15)

$$T^\lambda_\mu(r) = [r^\lambda_i \partial^\mu_i \tilde{H}_\Gamma - \delta^\mu_i (r^\alpha_i \partial^\mu_\alpha \tilde{H}_\Gamma - \tilde{H}_\Gamma)]$$

(16)

where

$$R^i_\lambda_\mu = \partial_\lambda \Gamma^i_\mu - \partial_\mu \Gamma^i_\lambda + \Gamma^j_\lambda \partial_j \Gamma^i_\mu - \Gamma^j_\mu \partial_j \Gamma^i_\lambda$$

is the curvature of the connection $\Gamma$. One can think of the tensor (16) as being the Hamiltonian SEM tensor.

If a Lagrangian density is regular, the multimomentum Hamiltonian formalism is equivalent to the Lagrangian formalism, otherwise in case of degenerate Lagrangian densities. In field theory, if a Lagrangian density is not regular, the Euler-Lagrange equations...
become underdetermined and require supplementary gauge-type conditions. In gauge theory, they are the familiar gauge conditions. In general case, the gauge-type conditions however remain elusive. In the framework of the multimomentum Hamiltonian formalism, they appear automatically as a part of the Hamilton equations. The key point consists in the fact that, given a degenerate Lagrangian density, one must consider a family of different associated Hamiltonian forms in order to exhaust all solutions of the Euler-Lagrange equations.

Lagrangian densities of all realistic field models are quadratic or affine in the velocity coordinates $y^\mu_i$. Complete family of Hamiltonian forms associated with such a Lagrangian density always exists $[12, 13, 14]$. Moreover, these Hamiltonian forms differ from each other only in connections $\Gamma$ in the splitting (13). Different connections are responsible for different gauge-type conditions mentioned above. They are also the connections which one should utilize in construction of the Hamiltonian SEM tensors $[16]$.

Thus, the tools are now at hand in order to examine the energy-momentum conservation laws in constraint field models.

As a test case, we construct Hamiltonian SEM tensors in gauge theory and then in gravitation theory.

The identity (15) remains true in the first order Lagrangian theories of gravity. In this work, we examine the metric-affine gravity where independent dynamic variables are world metrics and general linear connections. The energy-momentum conservation law in the affine-metric gravitation theory is not widely discussed $[9, 16]$. We construct the Hamiltonian SEM tensor for gravity. In case of the affine Hilbert-Einstein Lagrangian density, it is equal to

$$T^\lambda_\mu = \frac{1}{2\kappa} \delta^\lambda_\mu R \sqrt{-g}$$

and the total conservation law (13) for matter and gravity is reduced to the conservation law for matter in the presence of a background world metric, otherwise in case of quadratic Lagrangian densities.

2 Technical preliminary

The first order jet manifold $J^1Y$ of $Y$ is both the fibered manifold $J^1Y \to X$ and the affine bundle $J^1Y \to Y$ modelled on the vector bundle $T^*X \otimes_Y VY$. Hereafter, $J^1Y$ is identified to its image under the contact map

$$\lambda : J^1Y \to T^*X \otimes_Y TY,$$

$$\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \quad (17)$$

Recall that every fibered morphism of $\Phi : Y \to Y'$ over a diffeomorphism of $X$ has
the jet prolongation to the fibered morphism

\[ J^1\Phi : J^1Y \to J^1Y', \]

\[ y^i_{\mu} \circ J^1\Phi = (\partial_{\lambda} \Phi^i + \partial_j \Phi^i y^j_{\lambda}) \frac{\partial x^\lambda}{\partial y^\mu}, \]

Every projectable vector field

\[ u = u^\lambda \partial_{\lambda} + u^i \partial_i \]
on \( Y \to X \) gives rise to the projectable vector field

\[ \tilde{u} = u^\lambda \partial_{\lambda} + u^i \partial_i + (\partial_{\lambda} u^i + y^j_{\lambda} \partial_j u^i - y^i_{\mu} \partial_{\lambda} u^\mu) \partial^\lambda_i, \quad (18) \]
on \( J^1Y \to X \).

The contact map \([17]\) implies the bundle monomorphism

\[ \hat{\lambda} : J^1Y \times TX \ni \partial_{\lambda} \mapsto \partial_{\lambda} \lambda \in J^1Y \times TY \]

and the canonical horizontal splitting of the pullback

\[ J^1Y \times TY = \hat{\lambda}(TX) \oplus VY, \]

\[ \hat{x}^\lambda \partial_{\lambda} + \hat{y}^i \partial_i = \hat{x}^\lambda (\partial_{\lambda} + y^i_{\lambda} \partial_i) + (\hat{y}^i - \hat{x}^\lambda y^i_{\lambda}) \partial_i. \]

Building on this splitting, one obtains the corresponding horizontal splittings

\[ u = u^\lambda \partial_{\lambda} + u^i \partial_i = u_H + u_V = u^\lambda (\partial_{\lambda} + y^i_{\lambda} \partial_i) + (u^i - u^\lambda y^i_{\lambda}) \partial_i. \quad (19) \]
on \( J^1Y \).

There is the 1:1 correspondence between the global sections

\[ \Gamma = dx^\lambda \otimes (\partial_{\lambda} + \Gamma^i_{\lambda} \partial_i) \]
of the affine jet bundle \( J^1Y \to Y \) and the connections on \( Y \to X \). These connections constitute the affine space modelled on the linear space of soldering forms \( Y \to T^*X \otimes VY \) on \( Y \).

The repeated jet manifold \( J^1J^1Y \), by definition, is the first order jet manifold of \( J^1Y \to X \). It is provided with the adapted coordinates \((x^\lambda, y^i, y^i_{\lambda}, y^i_{\mu})\). Its subbundle \( \hat{J}^2Y \) given by the coordinate relation \( y^i_{(\lambda)} = y^i_{\lambda} \) is called the sesquiholonomic jet manifold.

The second order jet manifold \( J^2Y \) of \( Y \) is the subbundle of \( \hat{J}^2Y \) where \( y^i_{\lambda \mu} = y^i_{\mu \lambda} \).
3 SEM tensors in the Lagrangian formalism

Given a Lagrangian density $L$, the jet manifold $J^1 Y$ carries the associated Poincaré-Cartan form
\[
\Xi_L = \pi^i dy^i \wedge \omega^\lambda - \pi^i y^i_\lambda \omega + \mathcal{L}\omega
\]  

and the Lagrangian polysymplectic form
\[
\Omega_L = (\partial_j \pi^i dy^j + \partial^\mu \pi^i dy^i_{\mu}) \wedge dy^i \wedge \omega \otimes \partial_\lambda.
\]

Using the pullback of these forms onto the repeated jet manifold $J^1 J^1 Y$, one can construct the exterior generating form
\[
\Lambda_L = d\Xi_L - \lambda|\Omega_L = [y^i_{\lambda} - y^i_\lambda) d\pi^i + (\partial_i - \delta_\lambda \partial^\lambda)i \mathcal{L}dy^i] \wedge \omega,
\]

\[
\lambda = dx^\lambda \otimes \tilde{\partial}_\lambda, \quad \tilde{\partial}_\lambda = \partial_\lambda + y^i_{\lambda} \partial_i + y^i_{\mu \lambda} \partial^\mu_i,
\]
on $J^1 J^1 Y$. Its restriction to the sesquiholonomic jet manifold $\hat{J}^2 Y$ defines the first order Euler-Lagrange operator
\[
\mathcal{E}'_L : \hat{J}^2 Y \rightarrow (T^* Y)^{n+1},
\]

\[
\mathcal{E}_L = \delta_i \mathcal{L}dy^i \wedge \omega = [\partial_i - (\partial_\lambda + y^i_{\lambda} \partial_i + y^i_{\mu \lambda} \partial^\mu_i)] \mathcal{L}dy^i \wedge \omega,
\]
corresponding to $L$. The restriction of the form \([21]\) to the second order jet manifold $J^2 Y$ of $Y$ recovers the familiar variational Euler-Lagrange operator
\[
\mathcal{E}_L : J^2 Y \rightarrow (T^* Y)^{n+1},
\]

It is given by the expression \([22]\), but with symmetric coordinates $y^i_{\mu \lambda} = y^i_{\lambda \mu}$.

Let $\vec{s}$ be a section of the fibered jet manifold $J^1 Y \rightarrow X$ such that its jet prolongation $J^1 s$ takes its values into $\text{Ker} \mathcal{E}'_L$ given by the coordinate relations
\[
\partial_i \mathcal{L} - (\partial_\lambda + y^i_{\lambda} \partial_j + y^i_{\mu \lambda} \partial^\mu_j) \partial^\lambda_i \mathcal{L} = 0.
\]

Then, $\vec{s}$ satisfies the first order Euler-Lagrange equations \([3]\). These equations are equivalent to the second order Euler-Lagrange equations
\[
\partial_i \mathcal{L} - (\partial_\lambda + \partial_\lambda s^i \partial_j + \partial_\lambda \partial_\mu s^i \partial^\mu_j) \partial^\lambda_i \mathcal{L} = 0.
\]

for sections $s$ of $Y \rightarrow X$ where $\vec{s} = J^1 s$.

We have the following differential conservation laws on solutions of the first order Euler-Lagrange equations.
Given a Lagrangian density \( L \) on \( J^1Y \), let us consider its pullback onto \( J^2Y \). Let \( u \) be a projectable vector field on \( Y \rightarrow X \) and \( \pi \) its jet lift \((18)\) onto \( J^1Y \rightarrow X \). Its pullback onto \( J^1J^1Y \) has the the canonical horizontal splitting \((13)\) given by the expression

\[
\pi = \pi_H + \pi_V = u^\lambda (\partial_\lambda + y(\lambda)\partial_i + y(\mu, \lambda)\partial_{\mu}) + [(u^i - y(\lambda)u^\lambda)\partial_i + (u^\mu - y(\mu, \lambda)u^\lambda)\partial_{\mu}].
\]

Let us compute the Lie derivative \( L_{\pi}L \). We have

\[
L_{\pi}L = \tilde{\partial}_\lambda [\pi^\lambda_i (u^i - u^\mu y(\mu)) + u^\lambda \mathcal{L}] + \partial_V \mathcal{L}.
\]

Being restricted to \( \ker E'_L \), the equality \((24)\) is written

\[
\partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i + y(\lambda, j)\partial_j u^i - y(\mu, \lambda)\partial_{\mu} u^\mu)\partial_{i}] \mathcal{L} \approx \tilde{\partial}_\lambda [\pi^\lambda_i (u^i - u^\mu y(\mu)) + u^\lambda \mathcal{L}].
\]

On solutions \( \pi \) of the first order Euler-Lagrange equations, the weak identity \((24)\) comes to the differential conservation law

\[
\pi^\lambda \mathcal{L} \approx d(\pi^\lambda \mathcal{L} \circ \pi).
\]

which takes the coordinate form \((1)\).

In particular, let \( \tau \) be the horizontal lift \((3)\) of a vector field \( \tau \) on \( Y \rightarrow X \) by a connection \( \Gamma \) on \( Y \). In this case, the identity \((25)\) is written

\[
\tau^\mu [\partial_\mu + \Gamma^i_\mu \partial_i + (\partial_\lambda \Gamma^i_\mu + y(\lambda, i)\partial_j \Gamma^i_\mu)\partial_{i}] \mathcal{L} + \tilde{\partial}_\lambda [\pi^\lambda_i (y(\mu) - \Gamma^i_\mu) - \delta^\lambda_\mu \mathcal{L}] \approx 0.
\]

On solutions \( \pi \) of the first order Euler-Lagrange equations, the identity \((26)\) comes to the differential conservation law \((3)\) where \( \tau^\lambda \mathcal{L}(\pi) \) are coefficients of the \( T^\star X \)-valued form

\[
T^\lambda \mathcal{L}(\pi) = -(\pi^\lambda \mathcal{L} \circ \pi) = [\pi^\lambda_i (\pi^\mu_i - \Gamma^i_\mu) - \delta^\lambda_\mu \mathcal{L}] dx^\mu \otimes \omega_\lambda
\]

on \( X \). This conservation law takes the coordinate form

\[
\tau^\mu [\partial_\mu + \Gamma^i_\mu \partial_i + (\partial_\lambda \Gamma^i_\mu + \pi^\lambda_i \partial_j \Gamma^i_\mu)\partial_{i}] \mathcal{L} + \frac{d}{dx^\lambda} [\pi^\lambda_i (\pi^\mu_i - \Gamma^i_\mu) - \delta^\lambda_\mu \mathcal{L}] \approx 0.
\]

### 4 Multimomentum Hamiltonian formalism

Let \( \Pi \) be the Legendre bundle \((10)\) coordinatized by \((x^\lambda, y^i, p^\lambda_i)\). By \( J^1\Pi \) is meant the first order jet manifold of \( \Pi \rightarrow X \). It is coordinatized by \((x^\lambda, y^i, p^\lambda_i, y^i(\mu), p^\lambda_i(\mu))\).

The Legendre manifold \( \Pi \) carries the generalized Liouville form

\[
\theta = -p^\lambda_i dy^i \wedge \omega \otimes \partial_\lambda.
\]
and the polysymplectic form $\Omega$. The Hamiltonian formalism in fibered manifolds is formulated intrinsically in terms of Hamiltonian connections which play the role similar to that of Hamiltonian vector fields in the symplectic geometry. A connection
\[
\gamma = dx^\mu \otimes (\partial_\mu + \gamma^i_\mu \partial_i + \gamma^\lambda_i \partial_\lambda)
\]
on the fibered Legendre manifold $\Pi \to X$, by definition above, is called the Hamiltonian connection if the exterior form $\gamma] \Omega$ is closed. An exterior $n$-form $H$ on $\Pi$ is called a Hamiltonian form if there exists a Hamiltonian connection satisfying the equation (12). Hamiltonian forms always exist as follows.

Every connection $\Gamma$ on $Y \to X$ gives rise to the connection
\[
\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda \partial_i + (-\partial_j \Gamma^i_\lambda p^\mu_i - K^\mu_\nu \lambda p^\nu_j + K^\alpha_\alpha p^\mu_j)\partial_\mu]
\]
on $\Pi \to X$ where
\[
K^* = dx^\lambda \otimes (\partial_\lambda + K^\mu_\nu \lambda \dot{x}_\mu \frac{\partial}{\partial \dot{x}_\nu})
\]
is a linear symmetric connection on $T^*X$. We have the equality
\[
\tilde{\Gamma]} \Omega = d(\Gamma] \theta).
\]
A glance at this equality shows that $\tilde{\Gamma}$ is a Hamiltonian connection and
\[
H_{\tilde{\Gamma}} = \Gamma] \theta = p^i_\lambda dy^i \wedge \omega^\lambda - p^\lambda_j \Gamma^i_\lambda \omega
\]
is a Hamiltonian form.

Lemma. Let $H$ be a Hamiltonian form. For any exterior horizontal density $\tilde{H} = \tilde{H}\omega$ on $\Pi \to X$, the form $H + \tilde{H}$ is a Hamiltonian form. Conversely, if $H$ and $H'$ are Hamiltonian forms, their difference $H - H'$ is an exterior horizontal density on $\Pi \to X$.

Thus, Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities on $\Pi \to X$. It follows that every Hamiltonian form on $\Pi$ can be given by the expression (13) where $\Gamma$ is some connection on $Y \to X$. Moreover, a Hamiltonian form has the canonical splitting (13) as follows.

Every Hamiltonian form $H$ implies the momentum morphism
\[
\hat{H} : \Pi \to J^1 Y,
\]
\[
y^i_\lambda \circ \hat{H} = \partial^i_\lambda \mathcal{H},
\]
and the associated connection $\Gamma_H = \hat{H} \circ \hat{0}$ on $Y$ where $\hat{0}$ is the global zero section of $\Pi \to Y$. As a consequence, we have the canonical splitting
\[
H = H_{\Gamma_H} - \hat{H}.
\]
The Hamilton operator $\mathcal{E}_H$ of a Hamiltonian form $H$ is defined to be the first order differential operator

$$\mathcal{E}_H : J^1\Pi \to T^*\Pi,$$

$$\mathcal{E}_H = dH - \hat{\Omega} = [(y^i_{(\lambda)} - \partial^i_{\lambda}\mathcal{H})dp^\lambda_i - (p_{i\lambda} + \partial_i\mathcal{H})dy^i] \wedge \omega$$

on $\Pi \to X$ where

$$\hat{\Omega} = dp^\lambda_i \wedge dy^i \wedge \omega + p^\lambda_i dy^i \wedge \omega - y^i_{(\lambda)} dp^\lambda_i \wedge \omega$$

is the pullback of the multisymplectic form $[\Pi]$ onto $J^1\Pi$.

For any connection $\gamma$ on $\Pi \to X$, we have

$$\mathcal{E}_H \circ \gamma = dH - \gamma \lceil \Omega.$$  

It follows that $\gamma$ is a Hamiltonian connection for a Hamiltonian form $H$ iff it takes its values into $\text{Ker} \mathcal{E}_H$ given by the coordinate relations

$$y^i_{(\lambda)} = \partial^i_{\lambda}\mathcal{H}, \quad p^\lambda_i = -\partial_i\mathcal{H}. \quad (29)$$

Let a Hamiltonian connection $\gamma$ has an integral section $r$ of $\Pi \to X$, that is, $\gamma \circ r = J^1r$. Then, the algebraic equations (29) are brought into the first order differential Hamilton equations (14).

Now we consider relations between Lagrangian and Hamiltonian formalisms. A Hamiltonian form $H$ is defined to be associated with a Lagrangian density $L$ if it satisfies the relations

$$\tilde{L} \circ \tilde{H}|_Q = \text{Id}_Q, \quad Q = \tilde{L}(J^1Y), \quad \tilde{H} = H \hat{\tilde{H}} + L \circ \hat{\tilde{H}}$$

which take the coordinate form

$$\partial^i_\mu L(x^\lambda, y^j, \partial^j_\lambda\mathcal{H}) = p_i^\mu,$$

$$L(x^\lambda, y^j, \partial^j_\lambda\mathcal{H}) = p_i^\mu \partial_\mu \mathcal{H} - \mathcal{H}.$$  

Note that there are different Hamiltonian forms associated with the same singular Lagrangian density as a rule.

Bearing in mind physical application, we restrict our consideration to so-called semiregular Lagrangian densities $L$ when the preimage $\tilde{L}^{-1}(q)$ of each point of $q \in Q$ is the connected submanifold of $J^1Y$. In this case, all Hamiltonian forms associated with a
semiregular Lagrangian density $L$ coincide on the Lagrangian constraint space $Q$, and the Poincaré-Cartan form $\Xi_L$ is the pullback

$$
\Xi_L = H \circ \hat{L},
$$

$$
\pi^\lambda y^\lambda_i - \mathcal{L} = \mathcal{H}(x^\mu, y^i, \pi^\lambda_i),
$$
of any associated multimomentum Hamiltonian form $H$ by the Legendre morphism $\hat{L}$ [17]. Also the generating form (21) is the pullback of

$$
\Lambda_L = \mathcal{E}_H \circ J^1\hat{L}
$$
of the Hamilton operator (28) of any Hamiltonian form $H$ associated with a semiregular Lagrangian density $L$. As a consequence, we obtain the following correspondence between solutions of the Euler-Lagrange equations and the Hamilton equations [13, 17].

Let a section $r$ of $\Pi \to X$ be a solution of the Hamilton equations (14) for a Hamiltonian form $H$ associated with a semiregular Lagrangian density $L$. If $r$ lives on the Lagrangian constraint space $Q$, the section $\overline{s} = \hat{H} \circ r$ of $J^1Y \to X$ satisfies the first order Euler-Lagrange equations (3). Conversely, given a semiregular Lagrangian density $L$, let $\bar{s}$ be a solution of the first order Euler-Lagrange equations (3). Let $H$ be a Hamiltonian form associated with $L$ so that

$$
\hat{H} \circ \hat{L} \circ \bar{s} = \bar{s}.
$$

Then, the section $r = \hat{L} \circ \bar{s}$ of $\Pi \to X$ is a solution of the Hamilton equations (14) for $H$. For sections $\bar{s}$ and $r$, we have the relations

$$
\bar{s} = J^1s, \quad s = \pi_{i\Pi} \circ r
$$
where $s$ is a solution of the second order Euler-Lagrange equations (23).

We shall say that a family of Hamiltonian forms $H$ associated with a semiregular Lagrangian density $L$ is complete if, for each solution $\bar{s}$ of the first order Euler-Lagrange equations (3), there exists a solution $r$ of the Hamilton equations (14) for some Hamiltonian form $H$ from this family so that

$$
r = \hat{L} \circ \bar{s}, \quad \bar{s} = \hat{H} \circ r, \quad \bar{s} = J^1(\pi_{i\Pi} \circ r).
$$

Such a complete family exists iff, for each solution $\bar{s}$ of the Euler-Lagrange equations for $L$, there exists a Hamiltonian form $H$ from this family so that the condition (31) holds.

We do not discuss here existence of solutions of Euler-Lagrange and Hamilton equations [3]. Note that, in contrast with mechanics, there are different Hamiltonian connections associated with the same multimomentum Hamiltonian form in general. Moreover, in field theory when the primary constraint space is the Lagrangian constraint space $Q$, there is a family of Hamiltonian forms associated with the same Lagrangian density as a rule. In practice, one can choose either the Hamilton equations or solutions of the Hamilton equations such that these solutions live on the constraint space.
5 Hamiltonian SEM tensors

Let $H$ be a Hamiltonian form on the Legendre bundle $\Pi$ over a fibered manifold $Y \to X$. We have the following differential conservation law on solutions of the Hamilton equations.

Let $r$ be a section of the fibered Legendre manifold $\Pi \to X$. Given a connection $\Gamma$ on $Y \to X$, we consider the $T^*X$-valued $(n-1)$-form

$$T_\Gamma(r) = -(\Gamma [H]) \circ r,$$

$$T_\Gamma(r) = [r_i^\lambda (\partial_\mu r^i - \Gamma^i_\mu) - \delta^\lambda_\mu (r_i^\alpha (\partial_\alpha r^i - \Gamma^i_\alpha) - \tilde{H}_\Gamma)] dx^\mu \otimes \omega_\lambda,$$  \hspace{1cm} (33)

on $X$ where $\tilde{H}_\Gamma$ is the Hamiltonian density in the splitting (13) of $H$ with respect to the connection $\Gamma$.

Let $\tau = \tau^\lambda \partial_\lambda$ be a vector field on $X$. Given a connection $\Gamma$ on $Y \to X$, it gives rise to the projectable vector field

$$\tilde{\tau}_\Gamma = \tau^\lambda \partial_\lambda + \tau^\lambda \Gamma^j_\lambda \partial_i + (-\tau^\mu p_j^\lambda \partial_i \Gamma^j_\mu - p_i^\lambda \partial_\mu \tau^\mu + p_i^\mu \partial_\mu \tau^\lambda) \partial_\lambda$$

on the Legendre bundle $\Pi$. Let us calculate the Lie derivative $L_{\tilde{\tau}_\Gamma} \tilde{H}_\Gamma$ on a section $r$. We have

$$(L_{\tilde{\tau}_\Gamma} \tilde{H}_\Gamma) \circ r = \{\partial_\mu \tau^\lambda \tilde{H}_\Gamma + [\tau^\lambda \partial_\lambda + \tau^\lambda \Gamma^j_\lambda \partial_i + (-\tau^\mu p_j^\lambda \partial_i \Gamma^j_\mu - p_i^\lambda \partial_\mu \tau^\mu + p_i^\mu \partial_\mu \tau^\lambda) \partial_\lambda] \tilde{H}_\Gamma\} \omega$

$$= \tau^\mu r_i^\lambda R^i_{\lambda \mu} \omega + d(\tau^\mu T^\lambda_\mu (r) \omega_\lambda) - (\tilde{\tau}_\Gamma V) \mathcal{E}_H \circ r$$ \hspace{1cm} (34)

where $\tilde{\tau}_\Gamma V$ is the vertical part of the canonical horizontal splitting (19) of the vector field $\tilde{\tau}_\Gamma$ on $\Pi$ over $J^1 \Pi$. If $r$ is a solution of the Hamilton equations, the equality (34) comes to the conservation law (19). The form (33) modulo the Hamilton equations reads

$$T_\Gamma(r) \approx [r_i^\lambda (\partial_\mu H - \Gamma^i_\mu) - \delta^\lambda_\mu (r_i^\alpha \partial_\alpha H - H)] dx^\mu \otimes \omega_\lambda.$$ \hspace{1cm} (35)

For instance, if $X = \mathbb{R}$ and $\Gamma$ is the trivial connection, we have

$$T_\Gamma(r) = \mathcal{H} dt$$

where $\mathcal{H}$ is a Hamiltonian function. Then, the identity (15) comes to the conventional energy conservation law (1) in mechanics.

Unless $n = 1$, the identity (15) can not be regarded directly as the energy-momentum conservation law. To clarify its physical meaning, we turn to the Lagrangian formalism.

**Lemma.** Let a Hamiltonian form $H$ be associated with a semiregular Lagrangian density $L$. Let $r$ be a solution of the Hamilton equations of $H$ which lives on the Lagrangian constraint space $Q$ and $\pi$ the associated solution of the first order Euler-Lagrange equations of $L$ so that they satisfy the conditions (22). Then, we have

$$T_\Gamma(r) = \mathcal{T}_\Gamma(\tilde{H} \circ r),$$

$$T_\Gamma(L \circ \pi) = \mathcal{T}_\Gamma(\pi)$$

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where $T_{\Gamma}$ is the SEM tensor (27).

It follows that, on the Lagrangian constraint space $Q$, the form (33) can be treated the Hamiltonian SEM tensor relative to the connection $\Gamma$ on $Y \to X$.

At the same time, the examples below show that, in several field models, the equality (15) is brought into the covariant conservation law (8) for the metric energy-momentum tensor.

In the Lagrangian formalism, the metric energy-momentum tensor is defined to be

$$\sqrt{-g} t_{\alpha\beta} = 2 \frac{\partial L}{\partial g^{\alpha\beta}}.$$  

In case of a background world metric $g$, this object is well-behaved. In the framework of the multimomentum Hamiltonian formalism, one can introduce the similar tensor

$$\sqrt{-g} H^{\alpha\beta} = 2 \frac{\partial H}{\partial g^{\alpha\beta}}. \quad (36)$$

If a Hamiltonian form $H$ is associated with a semiregular Lagrangian density $L$, there are the equalities

$$t_{H}^{\alpha\beta}(q) = -g^{\alpha\mu} g^{\beta\nu} t_{\mu\nu}(x^\lambda, y^i, \partial_i \mathcal{H}(q)), \quad q \in Q,$$

$$t_{H}^{\alpha\beta}(x^\lambda, y^i, \pi^\lambda_i(z)) = -g^{\alpha\mu} g^{\beta\nu} t_{\mu\nu}(z), \quad \tilde{H} \circ \tilde{L}(z) = z.$$  

In view of these equalities, we can think of the tensor (36) restricted to the Lagrangian constraint space $Q$ as being the Hamiltonian metric energy-momentum tensor. On $Q$, the tensor (36) does not depend upon choice of a Hamiltonian form $H$ associated with $L$. Therefore, we shall denote it by the common symbol $t$. Set

$$t^\lambda_{\alpha} = g_{\alpha\nu} t^\lambda_{\nu}.$$  

In the presence of a background world metric $g$, the identity (15) takes the form

$$t^\lambda_{\alpha}\{^\alpha_{\lambda\mu}\} \sqrt{-g} + (\Gamma^i_{\mu} \partial_i - \partial_i \Gamma^j_{\mu} \partial^i_{\lambda} \tilde{\mathcal{H}}_{\Gamma}) \approx \frac{d}{dx^\lambda} T_{\Gamma}^\lambda_{\mu} + r^\lambda_i R^i_{\lambda\mu} \quad (37)$$

where by $\{^\alpha_{\lambda\mu}\}$ are meant the Cristoffel symbols of the world metric $g$.

### 6  SEM tensors in gauge theory

As a test case, let us consider the gauge theory of principal connections treated the gauge potentials.
In the rest of the article, the manifold $X$ is assumed to be oriented and provided with a nondegenerate fibre metric $g_{\mu\nu}$ in the tangent bundle of $X$. We denote $g = \det(g_{\mu\nu})$.

Let $P \to X$ be a principal bundle with a structure Lie group $G$ which acts freely and transitively on $P$ on the right:

$$r_g : p \mapsto pg, \quad p \in P, \quad g \in G.$$  

A principal connection $A$ on $P \to X$ is defined to be a $G$-equivariant connection on $P$ such that

$$J^1r_g \circ A = A \circ r_g$$

for each canonical morphism (3). There is the 1:1 correspondence between the principal connections on a principal bundle $P \to X$ and the global sections of the quotient bundle

$$C := J^1P/G \to X. \quad (38)$$

The bundle (38) is the affine bundle modelled on the vector bundle

$$\overline{C} = T^*X \otimes (VP/G).$$

Given a bundle atlas $\Psi^P$ of $P$, the bundle $C$ is provided with the fibered coordinates $(x^\mu, k^m_\mu)$ so that

$$(k^m_\mu \circ A)(x) = A^m_\mu(x)$$

are coefficients of the local connection 1-form of a principal connection $A$ with respect to the atlas $\Psi^P$. The first order jet manifold $J^1C$ of the bundle $C$ is coordinatized by $(x^\mu, k^m_\mu, k^m_{\mu\lambda})$.

There exists the canonical splitting

$$J^1C = C_+ \oplus C_- = (J^2P/G) \oplus (\wedge^2T^*X \otimes V^G P), \quad (39)$$

$$k^m_{\mu\lambda} = \frac{1}{2}(k^m_{\mu\lambda} + k^m_{\lambda\mu} + c^m_{n\lambda}k^n_\lambda k^l_\mu) + \frac{1}{2}(k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{n\lambda}k^n_\lambda k^l_\mu),$$

over $C$. The corresponding surjections are written

$$S : J^1C \to C_+, \quad S^m_{\lambda\mu} = k^m_{\mu\lambda} + k^m_{\lambda\mu} + c^m_{n\lambda}k^n_\lambda k^l_\mu,$$

$$F : J^1C \to C_-, \quad F^m_{\lambda\mu} = k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{n\lambda}k^n_\lambda k^l_\mu.$$

The Legendre bundle over the bundle $C$ is

$$\Pi = \wedge^nT^*X \otimes TX \otimes [C \times \overline{C}]^*.$$

It is coordinatized by $(x^\mu, k^m_\mu, p^m_{\mu\lambda})$.  

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On the configuration space (39), the conventional Yang-Mills Lagrangian density $L_{YM}$ of gauge potentials in the presence of a background world metric is given by the expression

$$L_{YM} = \frac{1}{4\varepsilon^2} a^G_{mn} g^{\lambda\mu} g^{\beta\nu} F_{\lambda\beta}^m F_{\mu\nu}^n \sqrt{|g|} \omega$$

where $a^G$ is a nondegenerate $G$-invariant metric in the Lie algebra of $G$. The Legendre morphism associated with the Lagrangian density (40) takes the form

$$p_m^{(\mu\lambda)} \circ \hat{L}_{YM} = 0,$$  

$$p_m^{[\mu\lambda]} \circ \hat{L}_{YM} = \varepsilon^{-2} a^G_{mn} g^{\lambda\alpha} g^{\mu\beta} F_{\alpha\beta}^n \sqrt{|g|}.$$  

The equation (41a) defines the constraint space of gauge theory.

Following the general procedure [12, 13, 14], let us consider connections on the bundle $C$ which take their values into Ker $\hat{L}_{YM}$:

$$\Gamma : C \rightarrow C_+, \quad \Gamma^m_{\mu\lambda} - \Gamma^m_{\lambda\mu} - c^m_{nl} k^n_{\lambda\mu} = 0.$$  

Moreover, we can restrict ourselves to connections of the following type. Every principal connection $B$ on $P$ gives rise to the connection $\Gamma_B$ (42) on $C$ such that

$$\Gamma_B \circ B = \mathcal{S} \circ J^1 B,$$

$$\Gamma^m_{B\mu\lambda} = \frac{1}{2} [c^m_{nl} k^n_{\lambda\mu} + \partial_\mu B^m_{\lambda} + \partial_\lambda B^m_{\mu} - c^m_{nl} (k^n_{\mu B}^l_{\lambda} + k^n_{\lambda B}^l_{\mu})] - \{^\beta_{\mu\lambda}\} (B^m_{\beta} - k^m_{\beta}).$$

For all these connections, the Hamiltonian forms

$$H_B = p_m^{\mu\lambda} d k^m_{\mu\lambda} \wedge \omega - p_m^{\mu\lambda} \Gamma^m_{B\mu\lambda} \omega - \widetilde{H}_{YM} \omega,$$

$$\widetilde{H}_{YM} = \varepsilon^2 a^G_{mn} g^{\lambda\mu} g^{\beta\nu} p_m^{[\mu\lambda]} p_n^{[\nu\beta]} |g|^{-1/2},$$

are associated with the Lagrangian density $L_{YM}$ and constitute a complete family. The corresponding Hamilton equations for sections $r$ of $\Pi \rightarrow X$ read

$$\partial_\lambda k^m_{\mu\nu} = -c^m_{lm} k^l_{\mu\nu} + c^m_{nl} B^l_{\mu\nu} (k^n_{\lambda B}^l_{\lambda}) - \{^\mu_{\lambda\nu}\} p_m^{(\lambda\nu)},$$

$$\partial_\nu k^m_{\mu} + \partial_\mu k^m_{\lambda} = 2 \Gamma^m_{B\mu\lambda}.$$  

plus the equation (41a). The equations (41a) and (44) restricted to the constraint space (41a) are the familiar Yang-Mills equations. Different Hamiltonian forms (43) lead to the different equations (45). The equation (45) is independent of canonical momenta and plays the role of the gauge-type condition. Its solution is $k(x) = B$.

Turn now to the energy-momentum conservation law.
Let $A$ be a solution of the Yang-Mills equations. There exists the Hamiltonian form $H_{B=A}$ (13) such that $r_A = \hat{L}_Y M \circ A$ is a solution of the corresponding Hamilton equations (44), (13) and (111) on the constraint space (11a).

Let us examine the conservation law (37) where we take $\tilde{H}_{Y M} = \tilde{H}_{Y M}$ and $\Gamma = \Gamma_{B=A}$ from the splitting (13).

On the solution $r_A$, the curvature of the connection $\Gamma_A$ is reduced to

$$R^m_{\lambda \alpha \mu} = \frac{1}{2} (\partial_\lambda F^m_{\alpha \mu} - c^m_{qn} k^q F^m_{\alpha \mu} - \{^\beta \alpha \lambda \} F^m_{\beta \mu} - \{^\beta \mu \lambda \} F^m_{\alpha \beta} ) =$$

$$\frac{1}{2} [ (\partial_\alpha F^m_{\lambda \mu} - c^m_{qn} k^q F^m_{\lambda \mu} - \{^\beta \lambda \alpha \} F^m_{\beta \mu} - (\partial_\mu F^m_{\lambda \alpha} - c^m_{qn} k^q F^m_{\lambda \alpha} - \{^\beta \lambda \mu \} F^m_{\alpha \beta} )]$$

where $F = F \circ A$ is the strength of $A$. Set

$$S^\lambda \mu = p_m^{[\alpha \lambda]} \partial^m_{\alpha \mu} \tilde{H}_{Y M} = \frac{\varepsilon^2}{2 \sqrt{|g|}} a^m_{\alpha \alpha} g_{\mu \nu} g_{\alpha \beta} p_m^{[\alpha \lambda]} p^\nu_{[\beta \nu]}.$$

We have

$$S^\lambda \mu = \frac{1}{2} p^{[\alpha \lambda]} F^m_{\alpha \mu}, \quad \tilde{H}_{Y M} = \frac{1}{2} S^\alpha \alpha.$$

In virtue of Eqs. (113), (111) and (114), we obtain the relations

$$\partial_n \Gamma_A^{\lambda \alpha \mu} p_m^{\alpha \lambda} \partial^m_{\beta \lambda} \tilde{H}_{Y M} = \{^\beta \alpha \mu \} S^\alpha \beta,$$

$$r_{A[m}^{\lambda \alpha \mu} R^m_{\alpha \lambda \mu} = \partial_\lambda S^\lambda \mu (r_A) - \{^\beta \mu \lambda \} S^\lambda \beta (r_A)$$

and find that

$$t^\lambda \mu \sqrt{|g|} = 2 S^\lambda \mu - \frac{1}{2} \delta^\lambda \mu S^\alpha \alpha,$$

$$T_{\Gamma_A}^\lambda \mu (r_A) = S^\lambda \mu (r_A) - \frac{1}{2} \delta^\lambda \mu S^\alpha \alpha (r_A),$$

$$T^\lambda \mu (r_A) \sqrt{|g|} = T_{\Gamma_A}^\lambda \mu (r_A) + S^\lambda \mu (r_A).$$

Hence, the identity (37) in gauge theory is brought into the covariant energy-momentum conservation law

$$\nabla_\lambda t^\lambda \mu (r_A) \approx 0.$$
7 SEM tensors of matter fields

In gauge theory, matter fields possessing only internal symmetries are described by sections of a vector bundle

\[ Y = (P \times V)/G \]

associated with a principal bundle \( P \). It is provided with a \( G \)-invariant fibre metric \( a^Y \).

Because of the canonical vertical splitting \( VY = Y \times Y \), the metric \( a^Y \) is a fibre metric in the vertical tangent bundle \( VY \to X \). Every principal connection \( A \) on a principal bundle \( P \) yields the associated connection

\[ \Gamma = dx^\lambda \otimes \left[ \partial_\lambda + A^m_\mu(x)I_m^i jy^j \partial_i \right] \]

where \( A^m_\mu(x) \) are coefficients of the local connection 1-form and \( I_m \) are generators of the structure group \( G \) on the standard fibre \( V \) of the bundle \( Y \).

On the configuration space \( J^1Y \), the regular Lagrangian density of matter fields in the presence of a background connection \( \Gamma \) on \( Y \) reads

\[ L(m) = \frac{1}{2} a^Y_{ij}[g^{\mu\nu}(y^i_\mu - \Gamma^i_\mu)(y^j_\nu - \Gamma^j_\nu) - m^2 y^i y^j] \sqrt{|g|} \omega. \]

The Legendre bundle of the vector bundle \( Y \) is

\[ \Pi = \bigwedge^n T^*X \otimes TY \otimes Y^*. \]

The unique Hamiltonian form on \( \Pi \) associated with the Lagrangian density \( L(m) \) is written

\[ H(m) = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma^i_\lambda \omega - \frac{1}{2} (a^Y_{ij} g_{\mu\nu} p_i^\mu p_j^\nu |g|^{-1} + m^2 a^Y_{ij} y^i y^j) \sqrt{|g|} \omega \]

where \( a_Y \) is the fibre metric in \( V^*Y \) dual to \( a^Y \). There is the 1:1 correspondence between the solutions of the first order Euler-Lagrange equations of the regular Lagrangian density \( L(m) \) and the solutions of the Hamilton equations of the Hamiltonian form \( H(m) \).

To examine the conservation law \( (37) \), let us take the same Hamiltonian SEM tensor relative to the connection \( \Gamma \) for all solutions \( r \) of the Hamilton equations. The following equality motivates the option above. We have

\[ T^\lambda_{\Gamma, \mu}(r) = t^\lambda_{\mu}(r) \sqrt{|g|} = [a^Y_{ij} g_{\mu\nu} r^\lambda_i jp^\nu_j |g|^{-1} - \frac{1}{2} \delta^\lambda_\mu (a^Y_{ij} g_{\mu\nu} r^\lambda_i j r^\nu_j |g|^{-1} + m^2 a^Y_{ij} r^i j r^j)] \sqrt{|g|}. \]

The gauge invariance condition

\[ I_m^i j r^\lambda_i \partial_\lambda \tilde{H} = 0 \]

also takes place. Then, it easily observed that the identity \( (37) \) reduces to the familiar covariant energy-momentum conservation law

\[ \sqrt{|g|} \nabla_\lambda t^\lambda_{\mu}(r) \approx -r^\lambda_i F^m_{\lambda, \mu} I_m^i j y^j. \]
8 SEM tensors in affine-metric gravitation theory

After testing above, we apply the Hamiltonian SEM tensor machinery to gravitation theory.

In this Section, $X$ is a 4-dimensional world manifold which obeys the well-known topological conditions in order that a gravitational field exists on $X^4$.

The contemporary concept of gravitation interaction is based on the gauge gravitation theory with two types of gravitational fields. These are tetrad gravitational fields and Lorentz gauge potentials. In absence of fermion matter, one can choose by gravitational variables a pseudo-Riemannian metric $g$ on a world manifold $X^4$ and a general linear connections $K$ on the tangent bundle of $X^4$. We call them a world metric and a world connection respectively. Here we are not concerned with the matter interacting with a general linear connection, for it is non-Lagrangian and hypothetical as a rule.

Let $LX \to X^4$ be the principal bundle of linear frames in the tangent spaces to $X^4$. Its structure group is $GL^+(4, \mathbb{R})$. The world connections are associated with the principal connections on the principal bundle $LX \to X^4$. Hence, there is the 1:1 correspondence between the world connections and the global sections of the quotient bundle

$$C = J^1 LX/GL^+(4, \mathbb{R}).$$ (49)

We therefore can apply the standard procedure of the Hamiltonian gauge theory in order to describe the configuration and phase spaces of world connections [12, 13].

There is the 1:1 correspondence between the world metrics $g$ on $X^4$ and the global sections of the bundle $\Sigma$ of pseudo-Riemannian bilinear forms in tangent spaces to $X^4$. This bundle is associated with the $GL_4$-principal bundle $LX$. The 2-fold covering of the bundle $\Sigma$ is the quotient bundle $LX/SO(3, 1)$.

The total configuration space of the affine-metric gravitational variables is the product

$$J^1 C \times J^1 \Sigma.$$ (50)

coordinatized by $(x^\mu, g^{\alpha\beta}, k^\alpha_{\beta\mu}, g^{\alpha\beta}_\lambda, k^\alpha_{\beta\mu\lambda})$.

Also the total phase space $\Pi$ of the affine-metric gravity is the product of the Legendre bundles over the above-mentioned bundles $C$ and $\Sigma$. It is provided with the corresponding canonical coordinates $(x^\mu, g^{\alpha\beta}, k^\alpha_{\beta\mu}, p_{\alpha\beta}^\lambda, p_{\alpha\beta\mu\lambda})$.

On the configuration space (50), the Hilbert-Einstein Lagrangian density of General Relativity reads

$$L_{HE} = -\frac{1}{2\kappa} g^{\beta\lambda} F^\alpha_{\beta\alpha\lambda} \sqrt{-g_\omega},$$ (51)

$$F^\alpha_{\beta\nu\lambda} = k^\alpha_{\beta\nu\lambda} - k^\alpha_{\beta\nu\lambda} + k^\alpha_{\nu\epsilon\lambda} k^\epsilon_{\beta\lambda} - k^\alpha_{\nu\epsilon\lambda} k^\epsilon_{\beta\nu}. $$
The corresponding Legendre morphism is given by the expressions
\[ p_{\alpha\beta}^\lambda \circ \hat{L}_{\text{HE}} = 0, \]
\[ p_{\alpha}^{\beta\nu\lambda} \circ \hat{L}_{\text{HE}} = \pi_{\alpha}^{\beta\nu\lambda} = \frac{1}{2\kappa} (\delta_{\alpha}^{\nu} g_{\beta\lambda} - \delta_{\alpha}^{\lambda} g_{\beta\nu}) \sqrt{-g}. \]  
(52)

They define the constraint space of General Relativity in the affine-metric variables.

Let us consider the following connections on the bundle \( C \times \Sigma \) in order to construct a complete family of Hamiltonian forms associated with the Lagrangian density (51).

Let \( K \) be a world connection and
\[ \Gamma_{\alpha\beta}^{\lambda} = \frac{1}{2} [k_{\alpha}^{\epsilon\nu} k_{\beta\lambda}^{\epsilon
u} - k_{\alpha}^{\epsilon\lambda} k_{\beta\nu}^{\epsilon\nu} + \partial_{\lambda} K_{\alpha\beta}^{\epsilon\nu} + \partial_{\nu} K_{\alpha\beta}^{\epsilon\lambda}] \]
the corresponding connection on the bundle \( C \) (49). Let \( K' \) be another symmetric world connection. Building on these connections, we set up the connection
\[ \Gamma_{\alpha\beta}^{\lambda} = - K'_{\alpha\beta}^{\epsilon\lambda} g_{\epsilon\beta} + K'_{\beta\epsilon}^{\lambda} g_{\epsilon\alpha}, \]
\[ \Gamma_{\alpha\beta}^{\nu\lambda} = \Gamma_{K_{\alpha\beta}^{\nu\lambda}} - \frac{1}{2} R_{\alpha\beta}^{\nu\lambda}, \]  
(53)
on the bundle \( C \times \Sigma \) where \( R_{\alpha\beta}^{\nu\lambda} \) is the curvature of \( K \).

For all connections (53), the Hamiltonian forms
\[ H_{\text{HE}} = (p_{\alpha\beta}^{\nu\lambda} dg^{\alpha\beta} + p_{\alpha}^{\beta\nu\lambda} dk^{\alpha\beta}) \land \omega_{\lambda} - \mathcal{H}_{\text{HE}} \omega, \]
\[ \mathcal{H}_{\text{HE}} = -p_{\alpha\beta}^{\nu\lambda}(K_{\alpha\beta}^{\nu\lambda} g_{\epsilon\beta} + K_{\beta\epsilon}^{\nu\lambda} g_{\epsilon\alpha}) + p_{\alpha}^{\beta\nu\lambda} \Gamma_{K_{\alpha\beta}^{\nu\lambda}} \pi_{\alpha}^{\beta\nu\lambda} - \frac{1}{2} R_{\alpha\beta}^{\nu\lambda} (p_{\alpha}^{\beta\nu\lambda} - \pi_{\alpha}^{\beta\nu\lambda}) \]
\[ \hat{\mathcal{H}}_{\text{HE}} = \frac{1}{2\kappa} R \sqrt{-g}, \]  
(54)
are associated with the Lagrangian density \( L_{\text{HE}} \) and constitute a complete family.

Given the Hamiltonian form \( H_{\text{HE}} \) (54) plus a Hamiltonian form \( H_{M} \) for matter, the corresponding Hamilton equations read
\[ \partial_{\lambda} g^{\alpha\beta} + K^{\alpha\beta}_{\lambda} g_{\epsilon\beta} + K^{\beta\nu}_{\epsilon\lambda} g^{\alpha\epsilon} = 0, \]  
(56a)
\[ \partial_{\lambda} k_{\alpha\beta}^{\nu\lambda} = \Gamma_{K_{\alpha\beta}^{\nu\lambda}} - \frac{1}{2} R_{\alpha\beta}^{\nu\lambda}, \]  
(56b)
\[ \partial_{\lambda} p_{\alpha\beta}^{\nu\lambda} = p_{\epsilon\beta}^{\nu\gamma} K_{\epsilon\nu}^{\alpha\gamma} + p_{\epsilon\alpha}^{\nu\gamma} K_{\epsilon\nu}^{\beta\gamma} - \frac{1}{2\kappa} (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) \sqrt{-g} - \frac{\partial H_{M}}{\partial g_{\alpha\beta}}, \]  
(56c)
\[ \partial_{\lambda} p_{\alpha}^{\beta\nu\lambda} = -p_{\alpha}^{\epsilon\nu\gamma} k_{\epsilon\nu}^{\beta\gamma} + p_{\epsilon}^{\beta\nu\alpha} k_{\epsilon\alpha}^{\beta\gamma} - p_{\alpha}^{\beta\nu\alpha} K_{\epsilon\alpha}^{\nu\gamma}, \]  
(56d)
plus the motion equations of matter. The Hamilton equations (56a) and (56b) are independent of canonical momenta and so, reduce to the gauge-type conditions. The equation (56b) breaks into the following two parts

\[ F_{\alpha \beta \lambda \nu} = R_{\alpha \beta \nu \lambda}, \]

\[ \partial_\nu (K^{\alpha \beta \lambda} - k^{\alpha \beta \lambda}) + \partial_\lambda (K^{\alpha \beta \nu} - k^{\alpha \beta \nu}) - 2K^\varepsilon_{\nu \lambda} (K^{\alpha \beta \varepsilon} - k^{\alpha \beta \varepsilon}) + K^\varepsilon_{\beta \lambda} k^{\alpha \varepsilon \nu} + K^\varepsilon_{\beta \nu} k^{\alpha \varepsilon \lambda} - K^\varepsilon_{\varepsilon \lambda} k^{\alpha \beta \nu} - K^\alpha_{\varepsilon \lambda} k^{\varepsilon \beta \nu} = 0 \]  

(58)

where \( F \) is the curvature of the connection \( k(x) \). It is readily observed that the gauge-type conditions (56a) and (56b) are satisfied by

\[ k(x) = K, \quad K'^{\alpha \beta \lambda} = \{^{\alpha \beta \lambda}\}. \]

(59)

When restricted to the constraint space (52), the Hamilton equations (56c) and (56d) come to

\[ \frac{1}{\kappa} (R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R) \sqrt{-g} = -\frac{\partial H_M}{\partial g^{\alpha \beta}}. \]

(60)

\[ D_\alpha (\sqrt{-g} g^{\nu \beta}) - \delta^\nu_\alpha D_\lambda (\sqrt{-g} g^{\lambda \beta}) + \sqrt{-g} g^{\nu \beta} (k_{\lambda \alpha} - k^\lambda_{\alpha \lambda}) + g^{\lambda \beta} (k^{\nu \lambda} - k^{\nu \lambda \beta}) + \delta^\nu_\alpha g^{\lambda \beta} (k^\mu_{\mu \lambda} - k^{\mu \mu \lambda}) = 0 \]

(61)

where

\[ D_\lambda g^{\alpha \beta} = \partial_\lambda g^{\alpha \beta} + k^\alpha_{\mu \lambda} g^{\mu \beta} + k^\beta_{\mu \lambda} g^{\alpha \mu}. \]

Substituting the equation (57) into the equation (60), we obtain the Einstein equations

\[ \frac{1}{\kappa} (F_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} F) = -t_{\alpha \beta} \]

(62)

where \( t_{\alpha \beta} \) is the metric energy-momentum tensor of matter. The equations (61) and (62) are the familiar equations of affine-metric gravity. In particular, the former is the equation for torsion and nonmetricity terms of the general linear connection \( k(x) \). In the absence of matter sources of a general linear connection, it admits the well-known solution

\[ k^{\alpha \beta} = \{^{\alpha \beta}\} - \frac{1}{2} \delta^\alpha_\nu V_\beta, \]

\[ D_\alpha g^{\beta \gamma} = V_\alpha g^{\beta \gamma}, \]

where \( V_\alpha \) is an arbitrary covector field corresponding to the well-known projective freedom.

Turn now to the identity (15).

Let \( s = (k(x), g(x)) \) be a solution of the Euler-Lagrange equations of the first order Hilbert-Einstein Lagrangian density (54) and \( r \) the corresponding solution of the Hamilton equations of the Hamiltonian form (54), where \( K \) and \( K' \) are given by the expressions (57).
For this solution \( r \), let us take the SEM tensor \( T_s \) relative to the connection \( (53) \) where \( K \) and \( K' \) are given by the expressions \( (59) \). It reads

\[
T_s^{\lambda \mu} = \delta_\mu^\lambda \tilde{\mathcal{H}}_{HE} = \frac{1}{2\kappa} \delta_\mu^\lambda R \sqrt{-g}
\]

and the identity \( (13) \) takes the form

\[
(\partial_\mu + \Gamma_\alpha^\beta_\mu \partial_\alpha \beta + \Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^\lambda_j \partial_\lambda)(\tilde{\mathcal{H}}_{HE} + \tilde{\mathcal{H}}_M) \approx \frac{d}{dx^{\lambda}}(T_s^{\lambda \mu} + T_M^{\lambda \mu}) + p^{\beta \nu \lambda} R^\alpha_\beta \nu \lambda \mu + p^\lambda_\mu R^i_\lambda \mu
\]

where \( T_M \) is the SEM tensor for matter.

One can verify that the SEM tensor \( T_s \) meets the condition

\[
(\partial_\mu + \Gamma_\alpha^\beta_\mu \partial_\alpha \beta) \tilde{\mathcal{H}}_{HE} = \frac{d}{dx^{\lambda}} T_s^{\lambda \mu}
\]

and, on solutions \( (54) \), the curvature of the connection \( (53) \) vanishes. Hence, the identity \( (63) \) is reduced to the conservation law \( (37) \) of matter in the presence of a background metric. The gravitation SEM tensor is eliminated from the conservation law because the Hamiltonian form \( \mathcal{H}_{HE} \) is affine in all canonical momenta. Note that only gauge-type conditions \( (56a), (56b) \) and the motion equations of matter have been used.

At the same time, since the canonical momenta \( p_{\alpha \beta}^\lambda \) of the world metric are equal to zero, the Hamilton equation \( (56c) \) on the Lagrangian constraint space comes to

\[
\partial_\alpha \beta (\tilde{\mathcal{H}}_{HE} + \tilde{\mathcal{H}}_M) = 0.
\]

Hence, the equality \( (53) \) takes the form

\[
\pi_\alpha^{\beta \nu \lambda} \partial_\mu R^\alpha_\beta \nu \lambda + (\partial_\mu + \Gamma^i_\mu \partial_i - \partial_i \Gamma^j_\mu p^\lambda_j \partial_\lambda) \tilde{\mathcal{H}}_M \approx \frac{d}{dx^{\lambda}}(T_s^{\lambda \mu} + T_M^{\lambda \mu}) + p^\lambda_\mu R^i_\lambda \mu.
\]

It is the form of the energy-momentum conservation law which we observe also in case of quadratic Lagrangian densities of affine-metric gravity. Substituting the equality \( (64) \) into \( (38) \), we obtain the above mentioned result.

As a test case of quadratic Lagrangian densities of affine-metric gravity, let us examine the sum

\[
L = \left( -\frac{1}{2\kappa} g^{\beta \lambda} \mathcal{F}_\beta^{\alpha \lambda} + \frac{1}{4\varepsilon} g_{\alpha \gamma} g^{\beta \sigma} g^{\mu \nu} g^{\lambda \epsilon} \mathcal{F}_\beta^{\alpha \lambda} \mathcal{F}_\gamma^{\sigma \mu \epsilon} \right) \sqrt{-g} \omega
\]

of the Hilbert-Einstein Lagrangian density and the Yang-Mills one. The corresponding Legendre morphism reads

\[
p_{\alpha \beta}^\lambda \circ \hat{L} = 0, \quad p_\alpha^{\beta (\nu \lambda)} \circ \hat{L} = 0, \quad p_\alpha^{\beta [\nu \lambda]} \circ \hat{L} = \pi_\alpha^{\beta \nu \lambda} + \frac{1}{\varepsilon} g_{\alpha \gamma} g^{\beta \sigma} g^{\nu \mu} g^{\lambda \epsilon} \mathcal{F}_\gamma^{\epsilon \sigma \mu \epsilon} \sqrt{-g}.
\]
The relations (67a) and (67b) define the Lagrangian constraint space. Let us consider connections
\[\Gamma^{\alpha\beta}_\lambda = -K'^\alpha_{\varepsilon\lambda}g^{\varepsilon\beta} - K'^{\varepsilon\beta}_{\varepsilon\lambda}g^{\alpha\varepsilon},\]
\[\Gamma^{\alpha}_{\beta\nu\lambda} = \Gamma^{\alpha}_{K^{\beta}_{\nu\lambda}},\] (68)
on the bundle \(C \times \Sigma\) where the notations of the expression (53) are utilized. The corresponding Hamiltonian forms
\[H = (p^{\alpha\beta}_\lambda dg^{\alpha\beta}_\lambda + p^{\alpha}_{\beta\nu\lambda}dk^{\alpha}_{\beta\nu\lambda}) \wedge \omega_\lambda - H_\omega,\]
\[\tilde{H} = \frac{1}{4}g^{\gamma\sigma}g_{\gamma\sigma}g_{\nu\lambda}(p^{\alpha}_{\gamma\sigma}[\mu\varepsilon] - \pi_\alpha^{\gamma\sigma}[\mu\varepsilon] - \pi_\gamma^{\beta\varepsilon})(70)\]
are associated with the Lagrangian density (66) and constitute a complete family.

Given the Hamiltonian form (69) plus the Hamiltonian form \(H_M\) for matter, the corresponding Hamilton equations read
\[\partial_\lambda g^{\alpha\beta}_\lambda + K'^\alpha_{\varepsilon\lambda}g^{\varepsilon\beta} + K'^{\varepsilon\beta}_{\varepsilon\lambda}g^{\alpha\varepsilon} = 0,\] (70a)
\[\partial_\lambda k^{\alpha}_{\beta\nu} = \Gamma^{\alpha}_{K^{\beta}_{\nu\lambda}} + \varepsilon g^{\gamma\nu}g_{\beta\sigma}g_{\nu\lambda}(p^{\sigma}_{\gamma\sigma}[\mu\varepsilon] - \pi_\sigma^{\gamma\sigma}[\mu\varepsilon]),\] (70b)
\[\partial_\lambda p^{\alpha\beta}_\lambda = -\frac{\partial H}{\partial g^{\alpha\beta}} - \frac{\partial H_M}{\partial g^{\alpha\beta}},\] (70c)
\[\partial_\lambda p^{\beta\varepsilon}_{\alpha\lambda} = -p^{\varepsilon}_{\alpha\varepsilon}[\nu\lambda]k^{\beta}_{\varepsilon\gamma} + p^{\varepsilon\beta[\nu\varepsilon]}k^{\alpha\gamma} - p^{\varepsilon\beta[\nu\varepsilon]}k^{\alpha\gamma} - p^{\varepsilon}_{\alpha\varepsilon}[\nu\lambda]k^{\beta}_{\varepsilon\gamma} - p^{\varepsilon\beta[\nu\varepsilon]}k^{\alpha\gamma},\] (70d)
plus the motion equations for matter. The equation (70a) breaks into the equation (67a) and the gauge-type condition (58). The gauge-type conditions (70a) and (58) have the solution (59). Substituting the equation (67a) into the equation (70a) on the constraint space (67a), we get the quadratic Einstein equations. Substitution of the equations (67b) and (67c) into the equation (70d) results into the Yang-Mills generalization
\[\partial_\lambda p^{\beta\varepsilon}_{\alpha\lambda} + p^{\varepsilon}_{\alpha\varepsilon}[\nu\lambda]k^{\beta}_{\varepsilon\gamma} - p^{\varepsilon\beta[\nu\varepsilon]}k^{\alpha\gamma} = 0\]
of the equation (61).

Turn now to the energy-momentum conservation law. Let us consider the splitting of the Hamiltonian form (69) with respect to the connection (53) and the Hamiltonian density
\[\tilde{H}_\Gamma = \tilde{H} + \frac{1}{2}p^{\alpha}_{\beta\nu\lambda}R^\alpha_{\beta\nu\lambda}.\]

Let \(s = (k(x), g(x))\) be a solution of the Euler-Lagrange equations of the Lagrangian density (66) and \(r\) the corresponding solution of the Hamilton equations of the Hamiltonian form (69) where \(K\) and \(K'\) are given by the expressions (59). For this solution \(r\), let
us take the SEM tensor $T_s$ (16) relative to the connection (53) where $K$ and $K'$ are given by the expressions (59). It reads

$$T_s^{\lambda \mu} = \frac{1}{2} P_\alpha \beta_\nu \lambda R_\alpha^{\beta \nu \lambda} + \frac{\varepsilon}{2} g^{\alpha \gamma} g_{\lambda \mu} g_{\rho \delta} P_\sigma \beta_\nu \lambda (P_\gamma^{\sigma \delta \epsilon} - \pi_\gamma^{\sigma \delta \epsilon})$$

and is equal to

$$\frac{1}{2} R^{\beta \nu \lambda} R_\alpha^{\beta \nu \lambda} + \pi_\alpha^{\beta \nu \lambda} R_\alpha^{\beta \nu \lambda} - \delta^{\lambda \mu} \left( \frac{1}{4 \varepsilon} R^{\beta \nu \lambda} R_\alpha^{\beta \nu \lambda} + \frac{1}{2 \kappa} R \right).$$

The identity (15) takes the form

$$(\partial_\mu + \Gamma_i^j \partial_i - \partial_i \Gamma_j^i p_j^{\lambda \rho \delta} - p_\alpha^{\beta \nu \lambda} \frac{\partial}{\partial k^{\sigma \gamma \delta}} \Gamma_{K \beta \nu \lambda} \frac{\partial}{\partial p^{\sigma \gamma \delta \lambda}} \tilde{H}_\Gamma + \tilde{H}_M) \approx \frac{d}{dx^\lambda} (T_s^{\lambda \mu} + T_M^{\lambda \mu}) + p_\alpha^{\beta \nu \lambda} R_\alpha^{\beta \nu \lambda} + p_i^\lambda R_{i \mu}$$

and is simplified to

$$(\partial_\mu + \Gamma_i^j \partial_i - \partial_i \Gamma_j^i p_j^{\lambda \rho \delta} - p_\alpha^{\beta \nu \lambda} \frac{\partial}{\partial k^{\sigma \gamma \delta}} \Gamma_{K \beta \nu \lambda} \frac{\partial}{\partial p^{\sigma \gamma \delta \lambda}} \tilde{H}_\Gamma \approx \frac{d}{dx^\lambda} (T_s^{\lambda \mu} + T_M^{\lambda \mu}) + p_i^\lambda R_{i \mu}$$

where

$$p_\alpha^{\beta \nu \lambda} \frac{\partial}{\partial k^{\sigma \gamma \delta}} \Gamma_{K \beta \nu \lambda} \frac{\partial}{\partial p^{\sigma \gamma \delta \lambda}} \tilde{H}_\Gamma = \frac{1}{\kappa} k^{\gamma \rho \sigma} (g^{\beta \nu} R_\alpha^{\gamma \rho \sigma} - g^{\alpha \nu} R_\beta^{\gamma \rho \sigma}) \sqrt{-g} - k^{\gamma \rho \sigma} p_\alpha^{\nu \beta \lambda} R_\alpha^{\nu \gamma \lambda}.$$

Note that if $kx$ is a Lorentz connection, the term (72) comes to

$$-k^{\gamma \rho \sigma} p_\alpha^{\nu \beta \lambda} R_\alpha^{\nu \gamma \lambda}.$$

Let us choose the local geodetic coordinate system at a point $x \in X$. Relative to this coordinate system, the equality (71) at $x$ comes to the conservation law

$$(\partial_\mu + \Gamma_i^j \partial_i - \partial_i \Gamma_j^i p_j^{\lambda \rho \delta} - p_\alpha^{\beta \nu \lambda} \frac{\partial}{\partial k^{\sigma \gamma \delta}} \Gamma_{K \beta \nu \lambda} \frac{\partial}{\partial p^{\sigma \gamma \delta \lambda}} \tilde{H}_\Gamma \approx \frac{d}{dx^\lambda} (T_s^{\lambda \mu} + T_M^{\lambda \mu}) + p_i^\lambda R_{i \mu}$$

For instance, in gauge theory, we have

$$\frac{d}{dx^\lambda} (T_{\Gamma \lambda \mu} + t_M^{\lambda \mu}) = 0$$

where $t_M$ is the metric energy-momentum tensor of matter.

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