**G₂ Holonomy Spaces from Invariant Three-Forms**

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**Abstract**

We construct several new G₂ holonomy metrics that play an important role in recent studies of geometrical transitions in compactifications of M-theory to four dimensions. In type IIA string theory these metrics correspond to D6 branes wrapped on the three-cycle of the deformed conifold and the resolved conifold with two-form RR flux on the blown-up two-sphere, which are related by a conifold transition. We also study a G₂ metric that is related in type IIA to the line bundle over S² × S² with RR two-form flux. Our approach exploits systematically the definition of torsion-free G₂ structures in terms of three-forms which are closed and co-closed. Besides being an elegant formalism this turns out to be a practical tool to construct G₂ holonomy metrics.

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1 Introduction

Compactifications of M-theory on seven-manifolds with $G_2$ holonomy have recently attracted increased attention. If the seven-manifold is smooth the low energy theory contains only four-dimensional $N = 1$ supergravity coupled to a number of $U(1)$ vector multiplets and neutral chiral multiplets [1]. Due to the lack of non-abelian gauge symmetries and chiral matter it might seem that such compactifications are physically uninteresting but dualities imply that the situation cannot be as dire. Indeed, it has recently be found that both non-abelian gauge symmetries and chiral fermions can be included if we allow the seven-manifold to be singular [2, 3, 4, 5, 6].

Non-abelian gauge symmetries can be explained most easily by fiberwise application of the duality between heterotic string on $T^3$ and M-theory on $K^3$. Enhanced gauge symmetry on the heterotic side translates to a singular limit of $K^3$ on the M-theory side where an $ADE$ singularity appears corresponding to $ADE$ gauge groups. Fibering this over a compact three-cycle produces examples of $G_2$ holonomy spaces. Duals of $N = 1$ supersymmetric gauge theories based on orbifolds of known $G_2$ metrics on the spin bundle of the three-sphere $\mathbb{R}^4/\Gamma_{ADE} \times S^3$ were studied in [2, 3]. Furthermore, it was suggested that after the singular $G_2$ manifold undergoes a flop transition it is replaced by the smooth orbifold $\mathbb{R}^4 \times S^3/\Gamma_{ADE}$ and describes the strong coupling regime of $N = 1$ SYM [2, 3]. In particular, the existence of a mass-gap, $\chi_{SB}$ and confining strings charged under the center of the gauge group can be identified in the supergravity dual [3, 4, 7]. Note that such singularities appear in codimension four and are not special to $G_2$ spaces, they also appear in Calabi-Yau three- and four-folds. These types of dualities and phase transitions have been further studied and generalized in [7] - [25].

More exotic are the codimension seven (pointlike) singularities that give rise to chiral matter [5]. On one hand they can be understood via duality with type IIA string theory where the singularity corresponds to the point where stacks of parallel D6 branes intersect [4]. Alternatively they can be described in heterotic string theory on Calabi-Yau three-folds in which the rank of the gauge bundle jumps over isolated points on the three-fold [6].

In this paper we provide a general formalism to construct $G_2$ holonomy metrics which are of interest to improve our understanding of the above mentioned dualities and geometric transitions. The approach we use differs from the conventional procedures which usually starts from an ansatz for the metric. Here we exploit the mathematical fact that a torsion-free $G_2$ structure is characterized by an invariant three-form which is closed and co-closed and the corresponding $G_2$ metric is a non-linear function of the three-form [26, 27, 28, 29]. (See also [30] for a related approach.) Hence, our starting point will be an ansatz for the three-form which incorporates all symmetries we wish to impose. This method has the advantage that it reduces the tangent space symmetry $GL(7)$ directly.
down to $G_2$, and not in a two step process first to $SO(7)$ for the metric ansatz and then down to $G_2 \subset SO(7)$ by imposing $G_2$. When using the latter approach solutions can be missed if the metric ansatz is incompatible with the $G_2$ structure one wants to impose. The formalism we present is not only of academic interest, since it turns out to be a practical tool to construct $G_2$ holonomy metrics. The ansatz for the three-form requires only a minimal number of unknown functions and it is straightforward, though in general very tedious, to determine the condition for $G_2$ holonomy. In general this condition boils down to a set of highly non-linear differential equations which can be solved exactly only in special cases and have to be studied numerically otherwise. New examples of metrics with $G_2$ holonomy have been constructed recently \cite{15,16,21,33,6}. Also the search for $Spin(7)$ metrics was revived recently \cite{31-35}.

The plan of the paper is as follows. In section 2 we show how the fact that $G_2$ holonomy metrics are characterized by a three-form that is closed and co-closed can be used to construct metrics with torsion-free $G_2$ structure. We illustrate this on a known example where the metric is asymptotic to a cone over $S^3 \times \tilde{S}^3$ by writing down the most general three-form ansatz compatible with $SU(2)^3$ symmetry. In this way we find the most general $G_2$ metric with this symmetry and by requiring in addition regularity we reproduce the three previously known asymptotically conical (AC) metrics. In section 3 we systematically search for new metrics by constructing the most general three-form ansatz that preserves an $SU(2)^2 U(1)$ symmetry. Using our method we find three metrics that correspond to Kaluza-Klein monopoles fibered over one of three possible three-spheres, one of which was constructed recently \cite{15}. In type IIA string theory these backgrounds correspond to a configuration of one (several) D6-brane(s) wrapped on the three-sphere inside a deformed conifold. Furthermore, we find a completely new class of metrics that have the interpretation of being the M-theory lift of the small resolution of the conifold with one (or more) unit(s) of RR two-form flux on the blown up two-sphere. This metric is interesting because it is related in weakly coupled string theory to the wrapped D6 brane solution via Vafa’s conifold transition \cite{36} and is the supergravity dual of $N = 1$ SYM at strong coupling \cite{2,3,4}. The M-theory lift at infinite string coupling, on the other hand, was described in \cite{3} using the $SU(2)^3$ symmetric $G_2$ spaces \cite{27,39}. Hence, the solutions we find describe the interpolation between these two pictures for arbitrary, finite string coupling. Furthermore, this new metric has the novel feature that it has an $U(1)$ isometry whose orbit is finite everywhere. It never blows up or shrinks to zero so the metric in M-theory and after reduction to type IIA the ten-dimensional solution is non-singular everywhere even when the number of units of RR flux is larger than one. Asymptotically, the metric is similar to the brane solution and corresponds to a $U(1)$ fibration over the conifold with a finite size fiber. A metric with similar features but a different blown up cycle and different asymptotics has been found recently \cite{21}. It is a new branch of solutions of the equations that were first found in \cite{15} in the search for new $G_2$ metrics. We present further numerical evidence for the existence
of this solution. Finally, in section 4 we conclude with a discussion of our results and remarks on generalizations of our formalism. In Appendix A we present the general \( SU(2)^2U(1) \) symmetric three-form ansatz and the corresponding metric and condition for \( G_2 \) holonomy. Appendix B summarizes similar information for a related ansatz with \( SU(2)^2 \) symmetry.

2 \( G_2 \) Holonomy Manifolds

\( G_2 \) holonomy metrics on a seven-manifold \( X \) are solutions of eleven-dimensional supergravity and can be used to describe four-dimensional vacua with \( N = 1, d = 4 \) supersymmetry of the type \( \mathbb{R}^{1,3} \times X \). If \( X \) has finite volume the four-dimensional Newton constant is finite, but since we will exclusively study non-compact \( X \) gravity lives in eleven dimensions. Another important ingredient of the metrics we study is that they have at least an \( U(1) \) isometry\(^1\) which allows to reduce the purely geometric background \( \mathbb{R}^{1,3} \times X \) in \( M \)-theory to a background in type IIA string theory using

\[
    ds_{11}^2 = e^{-2\phi/3} ds_{10}^2 + e^{4\phi/3} \left(dx_{11} + C \mu dx^\mu \right)^2
\]

where \( \phi \) and \( C \) are the type IIA dilaton and Ramond-Ramond (RR) one-form, respectively. In type IIA string theory these backgrounds involve intersecting D6 branes, D6 branes wrapped on supersymmetric cycles of Calabi-Yau three-folds, or RR two-form fluxes \( F = dC \) over non-trivial cycles of three-folds. For numerous examples of these types, see \([3, 4, 6, 15, 31, 21]\).

We want to find an effective method to construct new metrics with \( G_2 \) holonomy on seven-manifolds \( X \). As mentioned above we focus on non-compact examples, which are important to study \( M \)-theory on compact \( G_2 \) manifolds in the vicinity of singularities. Physical transitions can occur if the singularities are resolved in inequivalent ways. In many cases these compactifications also provide interesting supergravity duals of supersymmetric gauge theories \([2, 7, 4]\) similar to \([37, 38]\).

In the context of compactifications of \( M \)-theory the condition of \( G_2 \) holonomy is simply the condition of \( d = 4, \mathcal{N} = 1 \) supersymmetry and requires the existence of precisely one covariantly constant spinor. For practical purposes this condition is not very helpful and we will use different, mathematically equivalent conditions. For this purpose let us first review some basic mathematical facts about \( G_2 \) structures and metrics of \( G_2 \) holonomy. In flat \( \mathbb{R}^7 \) with metric

\[
    ds^2 = dx_1^2 + \ldots + dx_7^2
\]

\(^1\)Note that compact examples in general lack any continuous symmetries.
the Lie group $G_2$, which is also the invariance group of the unit octonions, can be defined as the stabilizer of the three-form

$$\Phi_0 = dx_{123} + dx_{147} + dx_{165} + dx_{246} + dx_{257} + dx_{354} + dx_{367}$$  \hspace{1cm} (3)

and the four-form

$$\ast \Phi_0 = dx_{4567} + dx_{2356} + dx_{2374} + dx_{1357} + dx_{1346} + dx_{1276} + dx_{1245}$$  \hspace{1cm} (4)

under the natural action of $GL(7, \mathbb{R})$. In other words $\Phi_0$ and $\ast \Phi_0$ define a $G_2$ structure on $\mathbb{R}^7$. On a general curved manifold $X$ a $G_2$ structure is an identification of the tangent space $T_X$ with the unit quaternions. Equivalently, the geometry is determined by a stable three-form $\Phi$ for which at every point $p \in X$ there exists an isomorphism between $T_p X$ and $\mathbb{R}^7$ that identifies $\Phi$ with $\Phi_0$ in (3). Stability in the sense of Hitchin [30] means that $\Phi$ lies in a particular open orbit of the action of $GL(7, \mathbb{R})$. Most importantly for us, $\Phi$ determines a metric $g_{ij}$ on $X$ and hence a Hodge-star operator $\ast$. Furthermore, if $\Phi$ and $\ast \Phi$ are closed, then the metric $g_{ij}$ is Ricci flat and has holonomy $\text{Hol}(g) \subseteq G_2$. If in addition $\pi_1(X)$ is finite or $b_1(X) = 0$ then $\text{Hol}(g) = G_2$.

The following two equivalent definitions of $G_2$ holonomy are interesting for us:

- The $G_2$ structure $(\Phi, g)$ is torsion-free:

$$\nabla \Phi = 0 \, .$$  \hspace{1cm} (5)

- The three-form $\Phi$ is closed and co-closed:

$$d \Phi = 0 \quad \text{and} \quad \ast \Phi = 0 \, .$$  \hspace{1cm} (6)

We will use the latter definition, hence, the construction of $G_2$ holonomy metrics on seven-manifolds is equivalent to the construction of globally defined three-forms $\Phi$ that are closed and co-closed. Note that in the compact case this is equivalent to the condition that $\Phi$ be harmonic, however, in the non-compact case the two are not equivalent and we have to use the stronger condition (6). The $G_2$ holonomy metric can be expressed in terms of the three-form [26, 27]

$$g_{ij} = (\det s_{ij})^{-1/9} s_{ij} \, ,$$  \hspace{1cm} (7)

$$s_{ij} = -\frac{1}{144} \Phi_{m_1 m_2} \Phi_{m_3 m_4} \Phi_{m_5 m_6 m_7} \epsilon^{m_1 \ldots m_7}, \quad \epsilon^{1234567} = +1 \, .$$  \hspace{1cm} (7)

Although (3) and (4) appear linear in $\Phi$, in fact $\ast \Phi$ and $\nabla$ depend on the metric $g$, which depends on $\Phi$ through (7). Hence, the condition of $G_2$ holonomy is a highly nonlinear partial differential equation on the three-form $\Phi$. The general strategy pursued in this paper to construct $G_2$ holonomy metrics is based on (3) and (7). This method is quite
general and can naturally be generalized to other examples than those studied in this paper. Note that this approach was first used by [27] to construct the first complete non-singular examples of metrics with $G_2$ holonomy [27, 39]. See also [30] for a related approach that uses diffeomorphism-invariant functionals of forms to study geometrical structures in various dimensions.

We are interested in constructing new $G_2$ holonomy metrics on non-compact seven-manifolds. The simplest such metrics are cones over six-manifolds $Y$

$$ds_Y^2 = dr^2 + r^2 d\Omega_Y^2,$$

where $Y$ is a compact Einstein space with weak $SU(3)$ holonomy [40]. There are three simply connected examples known in the literature all of which are homogeneous. The examples are\n
$$\CP_3 = \frac{Sp(2)}{SU(2)U(1)}, \quad F_{1,2} = \frac{SU(3)}{U(1)U(1)}, \quad S^3 \times S^3 = \frac{SU(2)^3}{SU(2)}.$$  

(9)

To make things more interesting we want to study smooth deformations of these conical metrics, and will restrict our attention to cohomogeneity one. This means that the level surfaces is a six dimensional space $Y$ but the metrics on $Y$ is not Einstein anymore but depends on the radial coordinate $r$. Smooth deformations of the three examples which preserve the symmetries of the conical metrics and are asymptotically conical (AC) have already been constructed some time ago [27, 39]. They are the non-singular metrics on the $\R^3$ bundle over $S^4$, the $\R^3$ bundle over $\CP^2$, and the spin bundle over $S^3$ which is topologically $\R^4 \times S^3$.

The examples that we will study in this paper are further generalizations of the asymptotically conical metrics on $\R^4 \times S^3$ with $SU(2)^3$ symmetry. In particular we construct metrics that are not (AC) but only asymptotically locally conical (ALC). This means that for large $r$ one of the directions of the manifold does not blow up but stabilizes at a finite value. This direction is the orbit of an $U(1)$ isometry and this requires that the isometry group of the original (AC) metric is reduced. In compactifications of $M$-theory on such manifolds we use this particular $U(1)$ isometry to reduce to a vacuum solution of type IIA string theory. Because of (11) the size of this $U(1)$ corresponds to the dilaton which in contrast to the (AC) metrics is finite. Furthermore, because the $U(1)$ has to be non-trivially fibered we will get a non-trivial RR one-form gauge field which can be attributed to the presence of D6-branes or RR two-form flux. In order to have D6-branes the $U(1)$ isometry has to have fix points in co-dimension four. The fix point set has the nice interpretation as being the cycle on which the D6-brane wraps. If there are no fix points the background is a pure flux solution with flux over some non-trivial cycle.

In the search for new $G_2$ metrics we take the standpoint that the central object is the three-form $\Phi$ and the metric $g_{ij}$ is derived from it. In the first step we make an ansatz

5
for $\Phi$ which obeys all the required symmetries. Now the first equation in (6) can easily be solved by writing the three-form as the sum of a closed and an exact part

$$\Phi = \Phi_{cl} + \Phi_{ex} = \Phi_{cl} + d\Lambda$$

(10)

where $\Phi_{cl} \in H^3(X)$. It will turn out that choosing a specific representative of $H^3(X)$ determines which non-trivial cycle is blown up in $X$ to smooth out the singularity at the tip of the corresponding conical metric. The role of the exact piece is two-fold. It is needed to solve for co-closure (6) using (7). Furthermore, it ensures that the three-form $\Phi$ is stable in the sense of [30]. In simple terms this means that $\Phi$ has to be sufficiently generic to make the metric in (7) non-degenerate.

2 It is clear that the equations for co-closure impose highly non-linear conditions and in practise they can only be worked out using computer programs like Mathematica. In the rest of this section we will apply our method to the case of $SU(2)^3$ symmetric metrics on the spin bundle of $S^3$. This is meant as a warm-up for the new, $SU(2)^2U(1)$ symmetric examples of metrics that we will study in section 3.

2.1 Old $G_2$ Holonomy Metrics Revisited

The metrics we want to rederive here are smooth deformations of cones over $S^3 \times \tilde{S}^3$ and have an $SU(2)^3$ symmetry. It is most convenient to introduce two sets of $SU(2)$ left-invariant one-forms $\sigma_a$ and $\Sigma_a$ by

$$U^{-1}dU = T^a\sigma_a \equiv \sigma, \ V^{-1}dV = T^a\Sigma_a \equiv \Sigma$$

(11)

which makes an $SU(2)^2 = SU(2)_L \times \tilde{SU}(2)_L$ symmetry manifest. The $SU(2)$ valued matrices $U$ and $V$ parametrize the two three-spheres. The third $SU(2)$ symmetry acts by the diagonal subgroup of the right action

$$SU(2)^{diag}_R = \left( SU(2)_R \times \tilde{SU}(2)_R \right)_{diag}$$

(12)

The two three-spheres can be parametrized by two independent sets of Euler angles in terms of which the two sets of one-forms associated with the symmetries mentioned above become

$$\sigma_1 = \cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\phi, \quad \Sigma_1 = \cos \tilde{\psi} \, d\tilde{\theta} + \sin \tilde{\psi} \, \sin \tilde{\theta} \, d\tilde{\phi}$$

$$\sigma_2 = -\sin \psi \, d\theta + \cos \psi \, \sin \theta \, d\phi, \quad \Sigma_2 = -\sin \tilde{\psi} \, d\tilde{\theta} + \cos \tilde{\psi} \, \sin \tilde{\theta} \, d\tilde{\phi}$$

$$\sigma_3 = d\psi + \cos \theta \, d\phi, \quad \Sigma_3 = d\tilde{\psi} + \cos \tilde{\theta} \, d\tilde{\phi}$$

(13)

2 A small example should make clear what we mean by this. Assume we study $X = S^3 \times R^4$, then we could naively assume that the volume three-form on $S^3$ is a nice harmonic three-form. But it is not generic enough and leads to an $s_{ij}$ in (6) with less than maximal rank and hence leads to a degenerate metric.

3 The $T^a$ are $SU(2)$ generators with $T^aT^b = \delta_{ab} + i\epsilon_{abc}T^c$. The triplet of one-forms $\sigma_a$ can be extracted from the $SU(2)$ valued $\sigma$ using $\sigma^a = -(i/2)\text{Tr}T^a\sigma$. This relation will be used at various occasions throughout the paper.
which, furthermore, satisfy the $SU(2)$ algebras
\[ d\sigma_1 = -\sigma_2 \wedge \sigma_3 + \text{cyclic perms.}, \quad d\Sigma_1 = -\Sigma_2 \wedge \Sigma_3 + \text{cyclic perms.} \tag{14} \]

The next step is to construct an ansatz for the 3-form $\Phi$ which is invariant under the symmetries
\[ \Phi = \Phi_{\text{cl}} + d\Lambda \tag{15} \]

It contains a closed piece $\Phi_{\text{cl}} \in H^3(X)$ and an exact piece, which guarantees closure of the 3-form and the exact piece will be fixed by imposing co-closure. In the case at hand $X = \mathbb{R} \times S^3 \times \tilde{S}^3$ and $H^3(X) = \mathbb{Z} \oplus \mathbb{Z}$. The closed piece of the 3-form is given by a linear combination of the two volume forms of the 3-spheres
\[ \Phi_{\text{cl}} = r_0^3 (p\sigma_1 \wedge \sigma_2 \wedge \sigma_3 + q\Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3) \tag{16} \]
where $(p, q)$ are integers and label different elements of $H^3(X)$. In general the metric on $X$ is a smooth deformation of the cone metric over $S^3 \times \tilde{S}^3$. To be more specific it is the spin bundle over $S^3$ which has topology $S^3 \times \mathbb{R}^4$. The zero section of the total bundle space is a three-sphere with radius proportional to $r_0$. Hence, in the limit $r_0 \to 0$ we obtain the singular cone metric and the three-form $\Phi$ becomes exact.

The exact piece of the three-form consistent with the symmetries is
\[ d\Lambda = d(a(r) \sigma_a \wedge \Sigma_a) \tag{17} \]

For this highly symmetric case imposing co-closure of the three-form
\[ d*_{\Phi} \Phi = 0 \tag{18} \]
does not impose any additional constraints on $a(r)$ which is consistent with the fact that we have not fixed reparametrization invariance yet. Using (17) we can present the most general metric which depends on the choice of a representative of the third cohomology class labelled by $(p, q)$, the size of the blown up cycle $r_0$ and an arbitrary function $a(r)$
\[ ds^2 = \left[ a(a - pr_0^3) (\sigma_a)^2 + a(a + qr_0^3) (\Sigma_a)^2 \right. \]
\[ \quad - (pqr_0^6 + a^2) \sigma_a \Sigma_a + (a')^2 dr^2 \bigg] / \Omega \tag{19} \]
\[ \Omega = 2^{-\frac{2}{3}} (3a^4 - 4(p - q)r_0^3a^3 - 6pqr_0^6a^2 - p^2q^2r_0^{12})^{\frac{1}{3}}. \tag{20} \]

We can choose the arbitrary function to be linear $a(r) \equiv r$ using the reparametrization invariance of $r$
\[ ds^2 = \left[ r(r - pr_0^3) (\sigma_a)^2 + r(r + qr_0^3) (\Sigma_a)^2 \right. \]
\[ \quad - (r^2 + pqr_0^6) \sigma_a \Sigma_a + dr^2 \bigg] / \Omega \tag{21} \]
\[ \Omega = 2^{-\frac{2}{3}} (3r^4 - 4(p - q)r_0^3r^3 - 6pqr_0^6r^2 - p^2q^2r_0^{12})^{\frac{1}{3}}. \tag{22} \]
It is easy to see that the metric (21) is in general singular in the interior and we want to get constraints on the parameters \((p, q)\) by requiring the metric to be regular. A singularity appears when \(\Omega\) vanishes because of the fractional power of the warp factor \(\Omega^{-1}\). The only situation when we can get smooth solutions occur when \(\Omega\) vanishes linearly. For this to happen the forth order polynomial

\[
P = 3r^4 - 4(p - q)r^3r_0^3 - 6pqr^6r^2 - p^2q^2r_0^{12}
\]

must have a third order zero. Inspection of the discriminant

\[
\Delta \sim p^4q^4(p + q)^4
\]

reveals that there are only three possible solutions namely \((p, q) = (-1, 0), (1, -1)\) or \((0, 1)\) up to overall signs. Indeed only metrics corresponding to these three combinations lead to smooth solutions and all other combinations have singularities. (See [16] for a different derivation of this result.)

For large \(r\) the metrics are conical which can be made manifest if we choose a different coordinate in which (for large \(r\)) \(a(r) \propto r^3\). Adding a suitable constant we can bring the \((-1, 0)\) solution to the form as it is presented in [39]. We find

\[
a(r) = \frac{4}{3}(r^3 - r_0^3).
\]

The \((-1, 0)\) solution corresponds to the well known metric of \(G_2\) holonomy on the spin bundle of \(S^3\) [27, 39]

\[
ds^2 = 12dr^2/\left(1 - \frac{r_0^3}{r^3}\right) + r^2(\sigma_a)^2 + \frac{r^2}{3}\left(1 - \frac{r_0^3}{r^3}\right)(2\Sigma_a - \sigma_a)^2.
\]

The \((0, 1)\) solution is obtained by a simple exchange of the left invariant 1-forms \(\sigma_a \leftrightarrow \Sigma_a\) in the metric (26). Finally we come to the third metric, which was also found in [15, 16, 4], and which corresponds to the \((1, -1)\) solution. In this case the \(Z_2\) flip \(\sigma_a \leftrightarrow \Sigma_a\), which exchanges the \((-1, 0)\) with the \((0, 1)\) solution, becomes a symmetry. It leaves the metric invariant but transforms the three-form \(\Phi \rightarrow -\Phi\). The corresponding metric is given by

\[
ds^2 = 12dr^2/\left(1 - \frac{r_0^3}{r^3}\right) + r^2(\sigma_a - \Sigma_a)^2 + \frac{r^2}{3}\left(1 - \frac{r_0^3}{r^3}\right)(\sigma_a + \Sigma_a)^2.
\]

and \(a(r) = (4r^3 - r_0^3)/3\).

As shown in [16] the existence of three solutions can be nicely explained by the triality symmetry of the conical metric \(r_0 = 0\). This symmetry is broken down to a residual \(Z_2\) symmetry by blowing up one of three possible three-spheres. The broken part of the triality group permutes the three solutions, and the action of the unbroken \(Z_2\) symmetry on the one-forms is \(\sigma \rightarrow -\sigma\), \(\Sigma \rightarrow \Sigma - \sigma\) for the \((-1, 0)\) solution, \(\Sigma \rightarrow -\Sigma\), \(\sigma \rightarrow \sigma - \Sigma\) for the \((0, 1)\) solution, and \(\sigma \leftrightarrow \Sigma\) for the \((1, -1)\) solution.
3 New $G_2$ Holonomy Metrics

The goal of this section is to construct generalizations of the (AC) metrics (26) and (27) using the method outlined in the previous section. In particular we are interested in metrics which are less symmetric and are not asymptotically conical. We search for metrics that can be reduced to type IIA solutions with finite string coupling. This requires the existence of a $U(1)$ isometry whose orbits have finite radius for large radius $r$. As explained in the previous section such backgrounds correspond to the $M$-theory lift of type IIA compactifications on Calabi-Yau manifolds involving wrapped branes or RR two-form flux. Note that after taking into account the backreaction of the branes or flux the six-manifold is not Calabi-Yau anymore.

In particular, we will study in turn three cases that correspond to the $M$-theory lift of

- D6 branes wrapped on the deformed conifold
- The resolved conifold with RR two-form flux on the blown up $S^2$
- The line bundle over $S^2 \times S^2$ with RR two-form flux on the blown up $S^2 \times S^2$

Topologically the first two examples are $S^3 \times R^4$, whereas the third example [21] corresponds to the $R^2$ bundle over $T^{11}$.

3.1 Wrapped D6 branes

A first example of that type was found recently [13] and we will review it here although we will derive it in a different fashion. It corresponds to the uplift of a background of D6 branes wrapped on the three-sphere inside the deformed conifold $T^*S^3$ at finite string coupling. As before we choose to work with the basis of one-forms $\sigma_a$, $\Sigma_a$ defined in (11). Then the continuous symmetry of the deformed conifold which is not altered by the presence of the D6 branes is

$$SU(2)_L \times \tilde{SU(2)}_L.$$ (28)

In addition there is an extra $U(1)$ symmetry corresponding to the M-theory circle which in our basis is implemented as

$$U(1)_{R}^{\text{diag}} = \left(U(1)_R \times \tilde{U(1)}_R\right)_{\text{diag}},$$ (29)

which corresponds to the Killing vector $\partial_\psi + \partial_{\tilde{\psi}}$. The $U(1)_{R}^{\text{diag}}$ symmetry is a left-over of the diagonal right $SU(2)$ symmetry (12). It acts as an $SO(2)$ rotation simultaneously on $\sigma_a$, $\Sigma_a$, $a = 1, 2$ but leaves $\sigma_3$, $\Sigma_3$ unchanged.
At this stage we can present the most general ansatz for the 3-form $\Phi$ invariant under the $SU(2)_L \times SU(2)_L \times U(1)_R^{\text{diag}}$ symmetry. It has the form

$$
\Phi = r_0^3 (p \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + q \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3)
+ \mathcal{d}(a(r)(\sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2) + b(r)\sigma_3 \wedge \Sigma_3)
.$$  

(30)

A few comments are in order. To write down the three-form we have made a particular choice for $\sigma_i$, $\Sigma_i$, $i = 1, 2$ since we could replace them by $\sigma'_i = M^i_j \sigma_j$ and $\Sigma'_i = N^i_j \Sigma_j$ without changing the equation for $a(r)$ and $b(r)$, where $M^i_j$ and $N^i_j$ are independent, constant $SO(2)$ matrices. This allows us to get rid of terms of the form $d(f(r)(\sigma_1 \wedge \Sigma_2 - \sigma_2 \wedge \Sigma_1))$. Furthermore, we have excluded terms of the form $d(g(r)\sigma_1 \wedge \sigma_2 + h(r)\Sigma_1 \wedge \Sigma_2)$ although they are allowed by the symmetries. The reason is that they lead to terms in the metric that mix radial and angular directions. With this in mind (30) is the most general three-form ansatz that obeys the symmetries (28) and (29).

Finally, we have to impose the discrete $\mathbb{Z}_2$ symmetry of the conifold in which our basis exchanges the two sets of left-invariant one-forms $\sigma \leftrightarrow \Sigma$. This symmetry leaves the metric invariant but flips the sign of the three-form $\Phi \rightarrow -\Phi$, hence we have to take

$$
(p, q) = (1, -1).
$$  

(31)

The general expression for the metric following from (19), (30) and (31) is

$$
\begin{align*}
\mathcal{d}^2 &= \left[\frac{1}{4} (b - r_0^3)(2a + b + r_0^3)a' [ (\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2 ] \\
+ \frac{1}{4} (b - r_0^3)(2a - b - r_0^3)a' [ (\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2 ] \\
+ \frac{1}{4} (4a^2 - (b + r_0^3)^2)b'(\sigma_3 - \Sigma_3)^2 \\
+ \frac{1}{4} (b - r_0^3)^2b'(\sigma_3 + \Sigma_3)^2 + (a')^2b'dr^2 \right] / \Omega \\
\Omega &= \frac{1}{2^{2/3}} (b - r_0^3)^{2/3} (4a^2 - (b + r_0^3)^2)^{1/3} (a')^{2/3} (b')^{1/3}.
\end{align*}
$$  

(32)

Imposing co-closure (13) leads to a non-linear second order differential equation for $a(r)$ and $b(r)$

$$
4a'b'(a(r_0^3 - b)a' + (2a^2 - b(r_0^3 + b))b') \\
+ (r_0^3 - b)(r_0^3 - 2a + b)(r_0^3 + 2a + b)(a'b'' - a''b') = 0,
$$  

(33)

where we used $' \equiv \frac{d}{dr}$. The differential equation (33) is second order which seems to contradict the fact that closure and co-closure impose first order equations on the three-form. However, in writing the ansatz (30) we already imposed closure thus replacing...
certain functions by first derivatives of others. We also obtained only one equation in two functions, since reparametrization invariance allows to choose one of them freely. However, this choice is not completely arbitrary and the function has to be chosen such that the metric (32) does not become degenerate.

Despite the formidable form of the equation, it is possible to find solutions in special cases. In general we expect a two parameter family of non-singular solutions once one of the functions is fixed by reparametrizations. In type IIA these two parameters correspond to the asymptotic value of the dilaton and the size of the blown up three-sphere in the deformed conifold.

The simplest solution was already discussed in the previous section which corresponds to the $SU(2)^3$ symmetric $(1,-1)$ solution (27) with

\[ a = b = (4r^3 - r_0^3)/3 . \] (34)

Things become more interesting when the two functions are not equal. A particular one parameter solution can be found if we assume $a$ and $b$ to be polynomials of finite degree in the radial coordinate $r$. Inserting this ansatz into (33) we find

\[ a = \frac{1}{18}(r^3 - 3r_0^2 r) , \quad b = \frac{1}{9}(2r_0 r^2 - 9r_0^3) . \] (35)

This gives the solution found in [15]

\[ ds^2 = A^2 \left[ (\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2 \right] + B^2 \left[ (\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2 \right] + D^2(\sigma_3 - \Sigma_3)^2 + 4r_0^2 C^2 (\sigma_3 + \Sigma_3)^2 + dr^2/C^2 \] (36)

with

\[ A(r)^2 = \frac{1}{12} (r - r_0)(r + 3r_0) , \]
\[ B(r)^2 = \frac{1}{12} (r + r_0)(r - 3r_0) , \]
\[ C(r)^2 = \frac{(r - 3r_0)(r + 3r_0)}{(r - r_0)(r + r_0)} , \]
\[ D(r)^2 = \frac{r^2}{9} . \] (37)

The metric is complete and well-defined in the range $r \in [3r_0, \infty)$. This metric can be thought of as a Taub-NUT space fibered over $S^3$ and the base of this fibration can be found at $r = 3r_0$. It is parametrized by $(U,V) = (g,g^{-1})$ with $g \in SU(2)$ and the solution is a deformation of the $(1,-1)$ solution (27) of the previous section.

---

The mismatch of certain numerical coefficients is due to different conventions used in this paper.

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of this metric is that it does not behave conically in all directions but, as we can see from
the behaviour of \( C(r) \) for large \( r \), there is one direction that stabilizes at large radii. The
corresponding \( U(1) \) isometry is generated by the Killing vector \( \partial_\psi + \partial_{\tilde{\psi}} \).

The triality symmetry \([4]\) of the conical metric predicts that there exist similar metrics
which are deformations of the \((-1,0)\) and \((0,1)\) solutions. The metrics we will present
in the following are new, however, the differential equations and the coefficient functions
turn out to be the same, so we do not have to repeat them.

It is interesting to note that we cannot obtain these metrics by using the ansatz \([30]\)
with the same one-forms \([13]\) and replacing \([31]\) by \((-1,0)\) or \((0,1)\). However, such
ansätze will play a vital role in the next subsection when we construct M-theory uplifts
of the type IIA background of the resolved conifold with RR two-form flux.

The new solutions can be expressed in terms of the known one \([36]\) by replacing the
two sets of left-invariant one-forms \([13]\) in an appropriate fashion. Before presenting
the result let us explain the reason behind these replacements. The level surfaces of our
metrics are homogeneous spaces of the form

\[
\frac{SU(2) \times SU(2) \times SU(2)}{SU(2)_D} \sim S^3 \times \bar{S}^3.
\]

By virtue of the coset structure there is an \( SU(2)^3 \) symmetry which acts by left multi-
lication on each of the three \( SU(2) \) factors. Furthermore, there is a triality symmetry
which acts by permuting the three \( SU(2) \) factors \([4]\). If we want to deform the conical
metric we can blow up one of three three-spheres corresponding to one of three \( SU(2) \)
factors. In homology these three three-spheres are not independent and they obey a linear
relation \( D_1 + D_2 + D_3 = 0 \), where the \( D_i \), \( i = 1, 2, 3 \) denote the homology classes of the
three-spheres. The coset \([38]\) can be represented in homogeneous coordinates by three
\( SU(2) \) elements \( a, b, c \) modulo the identification \( (a, b, c) \sim (a\lambda, b\lambda, c\lambda) \) where \( \lambda \in SU(2) \).

Now by choosing the particular patch \( c = \lambda^{-1} \) we can match this to the parametriza-
tion we chose in the definition of our one-forms \([11]\) in terms of \( U, V \)

\[
(ac^{-1}, bc^{-1}, 1) = (U,V,1).
\]

In this parametrization the \((-1,0)\) sphere is parametrized by \((U,V) = (g^{-1},1)\), the \((0,1)\)
sphere by \((U,V) = (1,g)\), and the \((1,-1)\) sphere by \((U,V) = (g,g^{-1})\) with \( g \in SU(2) \).

Our solution \([36]\) corresponds to \((1,-1)\), how do we get the other ones?

The answer is

\[
\sigma \equiv T^a \sigma_a = U^{-1}dU \rightarrow V(\sigma - \Sigma)V^{-1} = (UV^{-1})^{-1}d(UV^{-1})
\]

\[
\Sigma \equiv T^a \Sigma_a = V^{-1}dV \rightarrow -V \Sigma V^{-1} = Vd(V^{-1})
\]
which can be understand by going to a different patch \((U, V, 1) \rightarrow (UV^{-1}, 1, V^{-1})\). For the sums and differences that appear in the metric this means

\[
\begin{align*}
\sigma - \Sigma & \rightarrow V\sigma V^{-1} \\
\sigma + \Sigma & \rightarrow V(\sigma - 2\Sigma)V^{-1}
\end{align*}
\]

and, therefore, in this case we find at \(r = 3r_0\) a finite size three-sphere parametrized by \(U\) i.e. the \((-1, 0)\) three-sphere. Compared to the \(SU(2)^3\) symmetric case \((\text{26})\) the one-forms are rotated non-trivially. But in the \(SU(2)^3\) symmetric case these rotation would never show up in the expressions for the metric and \(\Phi\).

Finally, the \((0, 1)\) case can be obtained in the same fashion by going to the patch \((U, V, 1) \rightarrow (1, VU^{-1}, U^{-1})\). This translates into replacing

\[
\begin{align*}
\sigma - \Sigma & \rightarrow U\Sigma U^{-1} \\
\sigma + \Sigma & \rightarrow U(\Sigma - 2\sigma)U^{-1}
\end{align*}
\]

which is just a \(U \leftrightarrow V\) flip of the solution obtained from \((\text{36})\) using \((\text{41})\).

Therefore, we are able to identify three solutions that are fibrations of a Kaluza-Klein monopole over a three-sphere which agrees with expectations from triality. The metrics are smooth and complete. They can be used to describe four-dimensional vacua with \(\mathcal{N} = 1\) supersymmetry of the type \(R^4 \times X\) where \(X\) is any of the three metrics. If we reduce this solution along the particular \(U(1)\) isometry to type IIA we get a background that describes a D6 brane wrapped on the three-sphere inside the deformed conifold. In this case the dilaton interpolates between zero and a finite value set by \(r_0\) as \(r\) varies from \(r = 3r_0\) to infinity. The string frame metric has small curvature over most of the manifold but blows up over the three-sphere at \(r = 3r_0\) because the dilaton \(e^\phi = r_0^{3/4}C^{3/2}\) vanishes there \([15]\).

It is straightforward to generalize these metrics so that they describe a stack of \(N\) wrapped D6 branes. For this we have to mod out the the \(G_2\) holonomy metric by a \(Z_N\) action as in \([15]\). In this case also the M-theory solution is singular, there is a \(R^4/Z_N\) orbifold singularity at \(r = 3r_0\), which gives rise to \(SU(N)\) enhanced gauge symmetry. At low energies we find \(\mathcal{N} = 1\) SYM with \(SU(N)\) gauge group in four dimensions coupled to eleven-dimensional supergravity. For more details on the metric \((\text{36})\) and its reduction to type IIA we refer the reader to \([17]\). (The \((-1, 0)\) and \((0, 1)\) metrics have the same properties.)

### 3.2 Resolved conifold with RR two-form flux

We continue our search for new \(G_2\) metrics and study the three-form ansatz \((\text{30})\) with

\[
(p, q) = (-1, 0) .
\]

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This will lead us to the construction of a novel class of metrics that correspond to the M-theory uplift of type IIA backgrounds of the resolved conifold metric with RR two-form flux turned on over the blown up $S^2$. From (30) and (43) we obtain the metric

$$ds^2 = A^2 \left[ (\sigma_1)^2 + (\sigma_2)^2 \right] + B^2 (\sigma_3)^2 + C^2 \left[ (\Sigma_1 - f\sigma_1)^2 + (\Sigma_2 - f\sigma_2)^2 \right] + D^2 (\Sigma_3 - g\sigma_3)^2 + E^2 dr^2$$

(44)

with

$$A^2 = \frac{a'}{4a} (4a^2(b + r_0^3) - b^3)/\Omega, \quad B^2 = \frac{bb'}{4a^2} (4a^2(b + r_0^3) - b^3)/\Omega,$$

$$C^2 = aba'/\Omega, \quad D^2 = a^2 b'/\Omega, \quad E^2 = (a')^2 b'/\Omega,$$

$$f = \frac{b}{2a}, \quad g = 1 - 2f^2,$$

$$\Omega = \frac{1}{2^{2/3}} b^{1/3} (4a^2(b + r_0^3) - b^3)^{1/3} (a')^{2/3} (b')^{1/3},$$

(45)

and a second order differential equation for $a(r)$ and $b(r)$

$$4a'b' \left[ ab(b + r_0^3)a' + (b^3 - a^2(r_0^3 + 2b))b' \right] + b(b^3 - 4a^2(r_0^3 + b))(a'b'' - a''b') = 0.$$  

(46)

A particular solution of this rather complicated equation (46) was already presented in section 2. It corresponds to the $(1,0)$ solution (26) with

$$a = b = \frac{4}{3} (r^3 - r_0^3).$$

(47)

A second analytic solution can be obtained by assuming that $a$ and $b$ are polynomials of finite degree in $r$. One solution that can be found easily is

$$a = r^3, \quad b = -\frac{2r_0^3}{r^{1/3}} t^2,$$

(48)

however, the corresponding metric turns out to be singular at $r = 0$. Therefore, we will not discuss it further and move on to study numerical solutions.

In order to study (46) numerically we solve it perturbatively in the interior which we choose to be at $r = 0$ and integrate numerically to large radii to obtain the asymptotic behaviour. We will compare these numerically obtained asymptotics with a perturbative solution at $r = \infty$. For convenience we choose $a = r^3$ and with this input we study (46) perturbatively around $r = 0$. At this point it is important to understand which boundary conditions we have to impose and this requires knowledge of the geometry in the interior. First of all we do not want the finite size circle to shrink to zero at $r = 0$ because we seek a solution that corresponds in type IIA to a background with no D6
branes and only flux. This means that the circle we use to reduce to type IIA must not lie in the fiber direction but has to mix with the base direction of the spin bundle over $S^3$. The circle is again generated by the Killing vector $\partial_\psi + \partial_{\bar{\psi}}$. Hence the size of the circle is given by

$$B^2 + (1 - g)^2 D^2$$

(49)

which means that $B^2$ has to remain finite, since $D^2$ being part of the fiber directions vanishes in the interior. The coefficient function $B^2$ sets the size of the $U(1)$ Hopf-fiber of the base $S^3$ of the fibration. As we want to keep the expression in (49) finite for all radii, $B^2$ has to stabilize at $r \to \infty$ whereas $D^2(1 - g)^2 \propto 1/r^2$ so that $D^2(1 - g)^2 \propto 1/r^2$. The coefficients $A^2$ and $B^2$ in general do not agree in the interior although they do so in the $SU(2)^3$ symmetric solution. This means that the $U(1)$ fiber of the base three-sphere is in general squashed.

The boundary condition at $r = 0$ that guarantees that $A^2$ and $B^2$ are finite and that the fiber part of the metric approaches the flat metric on $\mathbb{R}^4$ is

$$b \sim (1 - y) \rho^3 + \mathcal{O}(\rho^6) .$$

(50)

The parameter $y$ measures the deviation from the $SU(2)^3$ symmetric solution. As numerical studies with the boundary condition (50) show $y$ is related to the asymptotic value of the dilaton. Note that in general we have two integration constants, one of which is fixed by the boundary condition (50) and one, $y$, is left as a free parameter. In addition we have the parameter $r_0$ from the three-form ansatz which sets the scale for the blown up two-sphere in the resolved conifold. So we expect a two-parameter family of smooth solutions.

Next we wish to solve (46) perturbatively with the boundary condition (50) and use these solutions as starting values for a numerical integration from $r \gtrsim 0$. In the interior it is useful to use the coordinate $\rho \equiv r^3$. The perturbative solution around $\rho = 0$ is

$$b = (1 - y) \rho - \frac{1}{3r_0^3}(2y - 5y^2 + 4y^3 - y^4) \rho^2 + \ldots .$$

(51)

The corresponding metric to lowest order in $\rho$ is

$$ds^2 \sim \frac{1}{(1 - y)^{2/3}} \left\{ r_0^2 \left[ (\sigma_1)^2 + (\sigma_2)^2 \right] + r_0^2 (1 - y)^2 (\sigma_3)^2 \right\}$$

$$+ \frac{(1 - y)^{1/3}}{r_0} \rho \left\{ \left( \Sigma_1 - \frac{1 - y}{2} \sigma_1 \right)^2 + \left( \Sigma_2 - \frac{1 - y}{2} \sigma_2 \right)^2 \right.$$}

$$+ \left. \left( \Sigma_3 - \frac{1 + 2y - y^2}{2} \sigma_3 \right)^2 + \frac{d\rho^2}{\rho^2} \right\} ,$$

(52)

which corresponds to a squashed three-sphere with an $\mathbb{R}^4$ fibered over it. Numerical integration of (46) shows that this perturbative solution can be smoothly extended to
where we find $b \sim c(y)r^2$. The positive constant $c(y)$ depends on the coefficient $y$ in the boundary condition (50). The large $r$ behaviour of $b$ is consistent with the existence of a one parameter family of perturbative solutions at $r \to \infty$

$$b = c(y)r^2 - \frac{7c(y)^3 + 8r_0^3}{16} + \frac{81c(y)^6 + 160c(y)^3r_0^3 + 64r_0^6}{256c(y)r^2}$$

$$- \frac{9c(y)(63c(y)^6 + 128c(y)^3r_0^3 + 64r_0^6)}{1024r^4} \log(r) + O(r^4). \quad (53)$$

Note that the asymptotic solution contains logarithmic terms which are necessary to obtain solutions with positive $c(y)$. If we do not allow for log terms we can only get solutions with negative $c(y)$ e.g. the singular solution (48).

The asymptotic form of the metric for $r \to \infty$ is

$$ds^2 = \frac{r^2}{6} [(\sigma_1)^2 + (\sigma_2)^2] + \frac{c^2}{9}(\sigma_3)^2$$

$$+ \frac{r^2}{6} \left[ \left( \Sigma_1 - \frac{c}{2r}\sigma_1 \right)^2 + \left( \Sigma_2 - \frac{c}{2r}\sigma_2 \right)^2 \right]$$

$$+ \frac{r^2}{9} \left( \Sigma_3 - \frac{c}{2r^2}\sigma_3 \right)^2 + dr^2. \quad (54)$$

This is the standard conifold metric with a finite size circle fibered over it. The allowed range of the parameter $y$ is $0 \leq y \leq 1$. If $y = 0$ we see from the first line of (52) that the three-sphere is not squashed and the full solution corresponds to the asymptotically conical solution with $SU(2)^3$ symmetry (29). For increasing $y$ the three-sphere in the interior is more and more squashed and the value of $c(y)$, which is related to the asymptotic value of the dilaton, decreases. Finally at $y = 1$ the circle shrinks to zero size and the remaining six-dimensional metric becomes the standard metric on the resolved conifold.

In order to find the metric in the limit $y \to 1$ we have to perform the following rescaling

$$ds^2 \to ds^2/(1-y)^{2/3}, \ r \to r(1+y), \quad (55)$$

as can be seen from (32).

**Reduction to type IIA string theory**

Here we want to provide some details on the reduction of the new $G_2$ holonomy metric (14) to type IIA string theory using (11) since this is slightly more involved in this case than in (32). The relevant $U(1)$ isometry is generated by the Killing vector $\partial_{\psi} + \partial_{\tilde{\psi}}$. Having this in mind we rewrite the metric (14) which makes this manifest

$$ds_{11}^2 = dx_4^2 + A^2 [(\sigma_1)^2 + (\sigma_2)^2] + C^2 [(\Sigma_1 - f\sigma_1)^2 + (\Sigma_2 - f\sigma_2)^2]$$

$$+ \frac{B^2D^2}{B^2 + (1-g)^2D^2} (\sigma_3 - \Sigma_3)^2 + E^2 dr^2$$

$$+ \frac{1}{4} [B^2 + (1-g)^2D^2] \left[ \sigma_3 + \Sigma_3 + \frac{B^2 - D^2(1-g^2)}{B^2 + (1-g)^2D^2} (\sigma_3 - \Sigma_3) \right]^2. \quad (56)$$
where $dx_4^2$ denotes flat $d = 4$ Minkowski space. Note that in this metric nothing depends of $\psi + \tilde{\psi}$. Now Kaluza-Klein reduction simply amounts to dropping the last line in (50) which has been written as a complete square for that purpose. In particular we can now read off the dilaton

$$e^\phi = 2^{-3/2} \left[ B^2 + (1 - g)^2 D_2^2 \right]^{3/4},$$

(57)

and the RR one-form gauge field

$$A_\mu dx^\mu = \frac{B^2 - D_2^2(1 - g)\sigma_3 - \Sigma_3}{B^2 + (1 - g)^2 D_2^2} \times \cos \theta d\phi + \cos \tilde{\theta} d\tilde{\phi}. $$

(58)

The ten-dimensional metric in string frame is given by

$$ds_{IIA}^2 = \frac{1}{2} \left\{ dx_4^2 + A^2 \left[ (\sigma_1)^2 + (\sigma_2)^2 \right] + C^2 \left[ (\Sigma_1 - f \sigma_1)^2 + (\Sigma_2 - f \sigma_2)^2 \right] 
+ \frac{B_2^2 D_2^2}{B^2 + (1 - g)^2 D_2^2} \left( \sigma_3 - \Sigma_3 \right)^2 + E^2 dr^2 \right\} \times \left[ B^2 + (1 - g)^2 D_2^2 \right]^{1/2}$$

(59)

From the asymptotic solution (54) we can read off the asymptotic value of the dilaton and measure the RR two-form flux at infinity

$$e^\phi_\infty = \left( \frac{c(y)}{6} \right)^{3/2}, \quad F_\infty = dA_\infty = \sin \theta d\phi \wedge d\theta \ldots$$

(60)

where the dots denote terms that do not contribute to the integral of the two-form over the blown-up two-sphere which is parametrized by $\phi, \theta$. Hence, we get precisely one unit of RR two-form flux over the two-sphere. Since the warp factor in (59) is finite everywhere the type IIA metric is completely non-singular. It is straightforward to generalize this metric to describe in ten dimensions a metric with $N$ units of RR flux. We simply have to mod the original eleven-dimensional metric by a $\mathbb{Z}_N$ action that acts in the direction of the $U(1)$ isometry as

$$\mathbb{Z}_N : (\psi, \tilde{\psi}) \rightarrow (\psi + 4\pi \frac{N}{N}, \tilde{\psi} + 4\pi \frac{N}{N}).$$

(61)

In contrast to the metrics of section 3.1 the $U(1)$ action has no fixed points. Therefore, the $\mathbb{Z}_N$ orbifold of the eleven-dimensional metric and the corresponding type IIA solution are perfectly smooth. They provide (finite string coupling version) of smooth supergravity duals of four-dimensional $\mathcal{N} = 1$ SYM at strong coupling [2, 3, 7, 4].

**More $G_2$ metrics**

By similar transformations as in section 3.1 we can construct five more $G_2$ holonomy metrics from (54). First of all the $\mathbb{Z}_2$ flip $\sigma \leftrightarrow \Sigma$, which is a symmetry of the $(1, -1)$ solution, generates another solution that corresponds to $(p, q) = (0, 1)$. In type IIA string theory this new metric is related to (54) by a flop transition [4].

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Furthermore, we can construct from the \((-1, 0)\) and \((0, 1)\) solution four more solutions using the replacements (41) and (42)

\[
\begin{align*}
\sigma & \rightarrow V(\sigma - \Sigma)V^{-1}, \\
\Sigma & \rightarrow -V\Sigma V^{-1},
\end{align*}
\]

and

\[
\begin{align*}
\sigma & \rightarrow -U\sigma U^{-1}, \\
\Sigma & \rightarrow U(\Sigma - \sigma)U^{-1}.
\end{align*}
\]

In total we obtain three metrics (from section 3.1) that correspond to wrapped D6 branes and six metrics that correspond to the resolved conifold with RR two-form flux. Under triality they naturally can be arranged into three groups with three metrics each. Each group is distinguished by a unique \(U(1)\) isometry with finite orbit and the three different metrics in each group correspond to blow-ups of one of the three possible spheres. In each group there is one deformed conifold and two resolved conifolds which are related by the familiar flop and conifold transitions in string theory.

### 3.3 Flux on the line bundle over \(S^2 \times S^2\)

In [21] evidence was found that for metrics of the type (32) there exists a new branch of solutions for the \(G_2\) holonomy conditions that has a quite different geometry than the metrics we discussed in section 3.1. On that branch the behaviour in the interior of the space is such that only one circle direction shrinks to zero size. The geometry of the space is that of the \(R^2\) bundle over \(T^{11}\). In order to make the metric nonsingular in the interior the periodicity of \(\psi + \tilde{\psi}\) has to be changed from its original value of \(4\pi\) to \(2\pi\), so that asymptotically the metric becomes a \(U(1)\) fibration over the line bundle over \(S^2 \times S^2\) [41] which is asymptotic to a cone over \(T^{11}/\mathbb{Z}_2\). The particular feature of this solution is that the size of the \(U(1)\) fiber is always finite similar to the \(U(1)\) isometry of the metrics we found in section 3.2. Note that the fibration is non-trivial and this metric is not a simple product manifold of a circle times a six-dimensional manifold. This would contradict the fact that this space carries a metric with \(G_2\) holonomy. Since the \(U(1)\) action has no fixed points the corresponding type IIA background does not involve D6-branes. It corresponds to a compactification on the line bundle over \(S^2 \times S^2\) [11] with RR two-form flux over the blown-up \(S^2 \times S^2\) cycle.

Here we would like to present some more numerical evidence that these solutions exist by analyzing eqn. (33) perturbatively in the interior. Once, we have identified the local solution we integrate (33) numerically using the perturbative solution to obtain
initial values. This provides us with an asymptotic solution that we can compare with perturbative solutions around \( r = \infty \). In the following we will parametrize the metric as

\[
d s^2 = A^2 \left[ (\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2 \right] + B^2 \left[ (\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2 \right] + D^2(\sigma_3 - \Sigma_3)^2 + C^2(\sigma_3 + \Sigma_3)^2 + E^2 dr^2 ,
\]

(64)

where the relation of the coefficient functions \( A, B, C, D, E \) to \( a \) and \( b \) can be read off from (32).

At this point it is convenient to perform a coordinate transformation on the angular coordinates as in [15] which allows us to rewrite the metric as

\[
d s^2 = A^2 \left[ (g_1)^2 + (g_2)^2 \right] + B^2 \left[ (g_3)^2 + (g_4)^2 \right] + C^2(g_6)^2 + D^2(g_5)^2 + E^2 \rho^2,
\]

(65)

with

\[
\begin{align*}
g_1 &= -\sin \theta_1 d\phi_1 - \cos \psi_1 \sin \theta_2 d\phi_2 + \sin \psi_1 d\theta_2 \\
g_2 &= d\theta_1 - \sin \psi_1 \sin \theta_2 d\phi_2 - \cos \psi_1 d\theta_2 \\
g_3 &= -\sin \theta_1 d\phi_1 + \cos \psi_1 \sin \theta_2 d\phi_2 - \sin \psi_1 d\theta_2 \\
g_4 &= d\theta_1 + \sin \psi_1 \sin \theta_2 d\phi_2 + \cos \psi_1 d\theta_2 \\
g_5 &= d\psi_1 + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2 \\
g_6 &= d\psi_2 + \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2
\end{align*}
\]

(66)

where the \( \psi_1 \) and \( \psi_2 \) coordinates are \( 4\pi \) periodic for the metrics studied in section 3.1 and [15]. We will see in a moment that this has to be modified.

As a first step we set \( a = r^3 \) and solve (33) perturbatively for \( b \) around \( r = 0 \). For \( r \to 0 \) we impose the boundary conditions

\[
b = -r_0^3 + \text{const.} \times r^6
\]

(67)

which ensures that only \( D^2 \) and \( E^2 \) vanish. Since \( A^2, B^2 \) and \( C^2 \) remain finite we can see that the first line of (64) corresponds to a (in general non-Einstein) metric on \( T^{11} \).

The power series solution we find around \( r = 0 \) is

\[
b \sim -r_0^3 - yr^6 + \frac{y^2(8 - 5r_0^3y)}{16r_0^3}r^{12} - \frac{y^3(80 - 88r_0^3y + 35r_0^6y^2)}{192r_0^6}r^{18} + \ldots .
\]

(68)

It is convenient to write the metric near \( r = 0 \) in a new coordinate \( \rho \equiv r^3 \). To lowest order in \( \rho \) we find

\[
d s^2 \sim \frac{1}{y^{1/3}} \left\{ \frac{r_0}{2} \left[ (g_1)^2 + (g_2)^2 + (g_3)^2 + (g_4)^2 \right] + r_0^4 y (g_6)^2 \\
+ \frac{y}{r_0^4} \left[ d\rho^2 + \rho^2 (g_5)^2 \right] \right\}
\]

(69)
The first line of this metric corresponds to five dimensional manifold which is a $S^1$ bundle over $S^2 \times S^2$. This space is also called $T^{11}$ and carries a non-Einstein metric. The second line of the metric becomes the flat metric on $\mathbb{R}^2$ if we change the periodicity of $\psi_1$ from $4\pi$ to $2\pi$. This is the reason behind the change of periodicity and gives rise to the change in the asymptotic behaviour [21].

Since we were not able to find analytic solutions for the boundary conditions [21] we have to work numerically. For this we use the perturbative solution [38] as initial conditions for a numerical integration starting from $r = \delta \gtrsim 0$.

Numerical investigations reveal that only solutions in a particular parameter regime $0 \leq y \leq y_{\text{max}}$ are non-singular and extend to $r = \infty$. For large $r$ we find numerically $b \sim -c(y)r^2$ where $c(y)$ is a positive constant that increases with increasing $y$. This matches nicely onto perturbative solutions for large $r$

$$b = -c(y)r^2 + \frac{7c(y)^3 + 16r_6^3}{16} - \frac{c(y)^2(91c(y)^3 + 320r_6^3)}{256r^2}$$

$$+ \frac{c(y)(567c(y)^6 + 2304c(y)^3r_6^3 + 1024r_6^6)}{1024r^4}\log(r) + O(r^4), \quad (70)$$

where $c(y)$ is a $y$ dependent positive constant. Note also the presence of logarithmic terms. We denoted only the first term with a log but at higher order in the expansion terms with higher powers of logs appear. Asymptotically, we get the following metric

$$ds^2 \sim 36^{2/3}\left[\frac{r^2}{12}(g_1^2 + g_2^2 + g_3^2 + g_4^2) + \frac{r^2}{9}(g_5^2 + \frac{c(y)^2}{36}(g_6^2 + dr^2)\right] \quad (71)$$

which corresponds to the line bundle over $S^2 \times S^2$ [1] with a finite size circle parametrized by $g_6$ fibered over it non-trivially. The metric on the line bundle over $S^2 \times S^2$ is asymptotic to a $\mathbb{Z}_2$ orbifold of the conifold. The interesting feature of this metric is the finite size circle that does not shrink to zero anywhere like the metrics found in section 3.2. The corresponding $U(1)$ action that acts by shifts of $\psi_2$ has no fixed points and therefore the type IIA background we obtain by Kaluza-Klein reduction on the circle has no D6-branes that source the flux. It is a solution with pure RR-twoform flux over the base of the line bundle over $S^2 \times S^2$ which is given by two two-spheres parametrized by $\theta_i, \phi_i; \ i = 1, 2$. From the explicit form of $g_6$ [39] we see that the charges over the two-spheres are $+1$ and $-1$ respectively. Note that the type IIA background is completely non-singular, even if we increase the number of units of flux $(1, -1) \rightarrow (N, -N)$ by changing the periodicity of $\psi_2$ to

$$\psi_2 \sim \psi_2 + 4\pi/N. \quad (72)$$

The function $c(y)$ is related to the asymptotic value of the dilaton in the type IIA solution and grows monotonically with $y$. So in the range $0 \leq y \leq y_{\text{max}}$ the metric is asymptotically locally conical (ALC), however for the borderline case $y = y_{\text{max}}$ the metric becomes asymptotically conical. The numerical value for the limiting value is $y_{\text{max}} \sim 0.42$
and agrees with the result found in [21] for which we have to identify \( y_{\text{max}} \) with \( q_0 \) of [21] as 
\[
y_{\text{max}} = \left( q_0 \right)^2 / 2.
\]

## 4 Discussion

In this paper we made progress in the construction of new \( G_2 \) holonomy manifolds using

directly the definition of torsion-free \( G_2 \) structures in terms of closed and co-closed three-forms. This approach avoids shortcomings of other approaches that start from an ansatz for the metric since it reduces the symmetry group of the tangent space directly to \( G_2 \). The examples we study in detail are \( G_2 \) structures on the space \( X = S^3 \times \mathbb{R}^4 \), but it should be stressed that this method can be generalized to other cases. We generalize the previously known metrics that are conical or asymptotically conical (AC) [27, 39]. By constructing a general ansatz for the three-form we derive the general conditions for \( G_2 \) holonomy with \( SU(2) \times SU(2) \times U(1) \) symmetry. The general form of the three-form, metric and the condition for \( G_2 \) holonomy in form of a second order differential equation are summarized in appendix A. We study three classes of solutions in some detail. The one example of the first class was found recently [15] using a less general method and can be thought of as a fibration of the KK monopole over \( S^3 \). In fact there exist two more examples which correspond to the blow up of one of three possible three-spheres in \( X \). They all have a \( U(1) \) isometry whose orbit stabilizes asymptotically and vanishes in the interior over a three-sphere. Hence the M theory background \( \mathbb{R}^4 \times X \) can be reduced to type IIA where it corresponds to the full metric of one or a group of \( N \) D6 branes wrapped on the blown up three-sphere inside the deformed conifold. The second class describes a completely new class of solutions which, similarly to the first class, has a \( U(1) \) isometry whose orbit stabilizes at large radii. But in the interior the behaviour is quite different because the orbit does not go to zero but is of finite size everywhere. In type IIA these solutions correspond to backgrounds without D6 branes but with RR two-form flux over the blown up two-sphere inside the resolved conifold \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1 \). The third class of solutions is a new branch of solutions of the \( G_2 \) conditions of the first class and were found in [21]. Locally the equations are the same but the manifold is different globally. In this case one finds a metric on the \( \mathbb{R}^2 \) bundle over \( T^{11} \) which asymptotically looks like a \( U(1) \) bundle over a \( \mathbb{Z}_2 \) orbifold of the deformed conifold. The interesting feature of this metric is the existence of a \( U(1) \) isometry with everywhere finite size orbit [21], like in the second class of solutions we found. Hence, also this corresponds to a type IIA vacuum solution with pure RR two-form flux. In this case the flux is over the four-cycle \( \mathbb{P}^1 \times \mathbb{P}^1 \) inside the line-bundle over \( \mathbb{P}^1 \times \mathbb{P}^1 \) [41].

What remains as a challenge is to find analytic solutions for the last two classes of metrics which reduce to the pure flux solutions in type IIA. So far we were only able to find numerical evidence for the existence of these metric. Combined with the knowledge
of asymptotic solutions and solutions in the interior to high orders this evidence is very strong but it would be nice to get better analytic control. In this respect we hope that our approach will turn out to be useful, too, since it reduces the problem to a minimum of unknown functions and a single second-order differential equation. Another application would be to study metrics on other manifolds known in the literature. For example one could try to generalize the metrics on the $\mathbf{R}^3$ bundles over $\mathbf{S}^4$ and $\mathbf{P}^2$ found by [27, 39]. These would correspond to the M-theory lift of the full solutions of localized intersecting D6 branes. Furthermore, in a recent paper [3] new singular conical $G_2$ metrics were conjectured to exist on spaces which are cones over weighted projective spaces $\mathbf{WP}[n,n,m,m]$. Naturally it would be interesting to find the explicit $G_2$ structure on those spaces.

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While this paper was being written, preprint [42] appeared where the metric (44) in section 3.2 is found independently.

\footnote{Note that in this context the solutions in [27, 39] correspond to the near horizon limit.}
A  General $SU(2) \times SU(2) \times U(1)$ symmetric ansatz

In this appendix we present the general form of the three-form ansatz and the corresponding metric and second order equation for $G_2$ holonomy used in section 3. The ansatz that we consider has $SU(2)_L \times \widetilde{SU(2)}_L \times U(1)^{\text{diag}}_R$ symmetry

$$
\Phi = r_0^3 (p \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + q \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3) + d (a(r)(\sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2) + b(r)\sigma_3 \wedge \Sigma_3), \quad (73)
$$

where $U(1)^{\text{diag}}_R \sim SO(2)$ acts diagonally from the right on the left-invariant one-forms $\sigma_{1,2}$ and $\Sigma_{1,2}$. The metric resulting from (7) and (73) is

$$
\text{ds}^2 = \left[ a(b - pr_0^3)a' \left( (\sigma_1)^2 + (\sigma_2)^2 \right) \\
- (b^2 + pq r_0^6)a' \left( \sigma_1 \Sigma_1 + \sigma_2 \Sigma_2 \right) \\
+ a(b + qr_0^3)a' \left( (\Sigma_1)^2 + (\Sigma_2)^2 \right) \\
+ (a^2 - pr_0^3 b)b' \left( \sigma_3 \right)^2 \\
+ (b^2 - 2a^2 - pqr_0^6)b' \left( \sigma_3 \Sigma_3 \right) \\
+ (a^2 + qr_0^3 b)b' \left( \Sigma_3 \right)^2 + (a')^2 b'dr^2 \right] / \Omega, \quad (74)
$$

with

$$
\Omega = \frac{(a')^{2/3}(b')^{1/3}}{2^{2/3}} \left[ 4a^2(b - pr_0^3)\left( b + qr_0^3 \right) - (pq r_0^6 + b^2)^2 \right]^{1/3}. \quad (75)
$$

The functions $a$ and $b$ obey a second order differential equation

$$
4a'b' \left[ aa'(b - pr_0^3)(b + qr_0^3) + \\
b'\left( a^2((p - q)r_0^3 - 2b) + pq r_0^6 b + b^3 \right) \right] + \\
(a'b'' - a''b') \left[ 4a^2(pr_0^3 - b)(qr_0^3 + b) + (pq r_0^6 + b^2)^2 \right] = 0. \quad (76)
$$

B  A $SU(2) \times SU(2)$ symmetric ansatz

In this appendix we present a three-form ansatz with $SU(2)_L \times \widetilde{SU(2)}_L$ symmetry and the corresponding metric and second order equation for $G_2$ holonomy. Note that contrary to (73) this is not the most general three-form ansatz consistent with the symmetries. It is
merely a natural generalization of the ansatz (73) by introducing one additional function which breaks the $U(1)^{\text{diag}}_R$ symmetry. The three-form ansatz is

$$
\Phi = r_0^3 (p \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + q \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3) \\
+ d (a(r) \sigma_1 \wedge \Sigma_1 + b(r) \sigma_2 \wedge \Sigma_2 + c(r) \sigma_3 \wedge \Sigma_3). 
$$

The metric resulting from (7) and (77) is

$$
\begin{align*}
\text{ds}^2 &= \left[(bc - pr_0^3 a) a' \left(\sigma_1\right)^2 + (bc + qr_0^3 a) a' \left(\Sigma_1\right)^2 \right. \\
&\left. + (a^2 - b^2 - c^2 - pr_0^6) a' \left(\sigma_1 \Sigma_1\right) \right. \\
&\left. + (ac - pr_0^3 b) b' \left(\sigma_2\right)^2 + (ac + qr_0^3 b) b' \left(\Sigma_2\right)^2 \right. \\
&\left. + (b^2 - a^2 - c^2 - pr_0^6) b' \left(\sigma_2 \Sigma_2\right) \right. \\
&\left. + (ab - pr_0^3 c) c' \left(\sigma_3\right)^2 + (ab + qr_0^3 c) c' \left(\Sigma_3\right)^2 \right. \\
&\left. + (c^2 - a^2 - b^2 - pr_0^6) c' \left(\sigma_3 \Sigma_3\right) + a' b' c' d r^2 \right]/\Omega, 
\end{align*}
$$

with

$$
\Omega = -\frac{(a' b' c')^{1/3}}{2^{2/3}} \left[ a^4 + b^4 + c^4 - 2(a^2 b^2 + a^2 c^2 + b^2 c^2) \\
+ 4(p - q)r_0^3 abc + 2pqr_0^6 (a^2 + b^2 + c^2) + p^2 q^2 r_0^{12} \right]^{1/3}.
$$

The functions $a$, $b$ and $c$ obey a set of two second order differential equations

$$
\begin{align*}
& a' (b' (4a' (-a^3 a' + b(-(p - q)r_0^3 c a' + 2(pqr_0^6 + \\
& b^2 - c^2) b') - c(pqr_0^6 - b^2 + c^2) c') + \\
& a^2 (-2bb' + cc') + a((-pqr_0^6 + b^2 + c^2) a' + \\
& (p - q)r_0^3 (2cb' - bc')))) - \\
& (a^4 + b^4 + 4(p - q)r_0^3 abc + 2b^2(pqr_0^6 - c^2) + \\
& 2a^2(pqr_0^6 - b^2 - c^2) + (pqr_0^6 + c^2)^2 a'') + \\
& 2(a^4 + b^4 + 4(p - q)r_0^3 abc + 2b^2(pqr_0^6 - c^2) + \\
& 2a^2(pqr_0^6 - b^2 - c^2) + (pqr_0^6 + c^2)^2 a' b'') - \\
& (a^4 + b^4 + 4(p - q)r_0^3 abc + 2b^2(pqr_0^6 - c^2) + \\
& 2a^2(pqr_0^6 - b^2 - c^2) + (pqr_0^6 + c^2)^2 a' c'') \\
& = 0,
\end{align*}
$$

(80)
and

\[
\begin{align*}
  b'(c'(2b'(2a'd' + b(2(p - q)r_0^3cd' - (pq r_0^6 + b^2 - c^2)b') - c(pqr_0^6 - b^2 + c^2)c' + \\
  a^2(bb' + cc') + a(2(pqr_0^6 - b^2 - c^2)a' - (p - q)r_0^3(cb' + bc')))) + \\
  (a^4 + b^4 + 4(p - q)r_0^3abc + 2b^2(pqr_0^6 - c^2) + 2a^2(pqr_0^6 - b^2 - c^2) + (pq r_0^6 + c^2)^2)a'' - \\
  (a^4 + b^4 + 4(p - q)r_0^3abc + 2b^2(pqr_0^6 - c^2) + 2a^2(pqr_0^6 - b^2 - c^2) + (pq r_0^6 + c^2)^2)a'b'') - \\
  (a^4 + b^4 + 4(p - q)r_0^3abc + 2b^2(pqr_0^6 - c^2) + 2a^2(pqr_0^6 - b^2 - c^2) + (pq r_0^6 + c^2)^2)a'b'c'') \\
  = 0 .
\end{align*}
\]

(81)

Again one of the functions \(a\), \(b\), \(c\) or a combination of them can be eliminated using reparametrization invariance in \(r\).
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