Asymptotic properties of a nonparametric conditional density estimator in the local linear estimation for functional data via a functional single-index model

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ABSTRACT
This paper deals with the conditional density estimator of a real response variable given a functional random variable (i.e., takes values in an infinite-dimensional space). Specifically, we focus on the functional index model, and this approach represents a good compromise between nonparametric and parametric models. Then we give under general conditions and when the variables are independent, the quadratic error and asymptotic normality of estimator by local linear method, based on the single-index structure. Finally, we complete these theoretical advances by some simulation studies showing both the practical result of the local linear method and the good behaviour for finite sample sizes of the estimator and of the Monte Carlo methods to create functional pseudo-confidence area.

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1. Introduction
The nonparametric estimation of the conditional density function plays a crucial role in statistical analysis. This subject can be approached from multiple perspectives depending on the complexity of the problem. Many techniques were studied in the literature to treat these various situations but all treat only real or multidimensional explanatory random variables.

Focusing on functional data for the kernel-type, the first results on the nonparametric estimate of this model were got by Ferraty and Vieu (2006). They have studied the almost complete convergence the estimator of the conditional density and its derivates. Laksaci (2007) studied quadratic error of this estimator, and we return to Ferraty et al. (2010) which established the uniform almost complete convergence of this model always.

Now, we show a few results on the local linear smoothing for functional data, actually these results have been considered by many authors. Baillo and Grané (2009) first proposed a local linear smoothing of the regression estimator in a Hilbert space, and coming after them Barrientos-Marin et al. (2010) developed this method of local linear estimation of the regression in the semi-metric space for independent and identically distributed. Demongeot et al. (2013, 2014), has used this method to estimate conditional distribution and density function. In the case of spatial data (Laksaci et al., 2013) they established pointwise almost complete convergence rates.

Furthermore, the functional index model plays a major role in statistics. The interest of this approach comes from its use to reduce the dimension of the data by projection in fractal space. The literature on this topic is closely limited, the first work which was interested in the single-index model on the nonparametric estimation is Ferraty et al. (2003) which stated for i.i.d. variables and obtained the almost complete convergence under some conditions. Based on the cross-validation procedure, Ait Saidi et al. (2008) proposed an estimator of this parameter, where the functional single-index is unknown. Recently, Attouï et al. (2011) considered the nonparametric estimation of the conditional density in the single functional model. They established its pointwise and uniform almost complete convergence (a.co.) rates. In the same topic, Attouï and Ling (2016) proved the asymptotic results of a nonparametric conditional cumulative distribution estimator for time series data. More recently, Tabti and Ait Saidi (2018) obtained the almost complete convergence and the uniform almost complete convergence of a kernel estimator of the hazard function with quasi-association condition when the observations are linked with functional single-index structure.

In this paper, we focus on the local linear estimation with the single-index structure to compute under some...
conditions, the quadratic error of the conditional density function estimator. In practice, this study has great importance, because, it permits to construct a prediction method based on the maximum risk estimation with a single functional index.

In Section 2, we introduce the estimator of our model in the single functional index. In Section 3, we introduce assumptions and asymptotic properties are given. Simulations are given in Section 4. Finally, Section 5 is devoted to the proofs of the results.

2. The model
Let \{(X_i, Y_i), 1 \leq i \leq n\} be \(n\) random variables, independent and identically distributed as the random pair \((X, Y)\) with values in \(\mathcal{H} \times \mathbb{R}\), where \(\mathcal{H}\) is a separable real Hilbert space with the norm \(\| \cdot \|\) generated by an inner product \((., .)\). We consider the semi-metric \(d_0\) associated to the single-index \(\theta \in \mathcal{H}\) defined by \(d_0(x_1, x_2) := |\langle x_1 - x_2, \theta \rangle|, \forall x_1, x_2 \in \mathcal{H}.\) Assume that the explanation of \(Y\) given \(X\) is done through a fixed functional index \(\theta \in \mathcal{H}\). In the sense, there exists \(a\) \(\theta \in \mathcal{H}\) (unique up to a scale normalization factor) such that: \(E[Y|X] = E[Y|(\theta, X)]\). The conditional density of \(Y\) given \(X = x\) denoted by \(f_0(x, y)\) exists and is given by \(f_0(y|x) := \int f(y|x, \theta), \forall y \in \mathbb{R}\). In the following, we denote by \(f(\theta, ., x)\), the conditional density of \(Y\) given \((x, \theta)\) and we define the local linear estimator for single-index structure \(f(\theta, ., x)\) of \(f(\theta, ., x)\) by

\[
\hat{f}(\theta, y, x) = \frac{\sum_{1 \leq i \leq n} W_i(\theta, x) H(h_H^{-1}(y - Y_i))}{h_H \sum_{1 \leq i \leq n} W_i(\theta, x)} = \sum_{1 \leq i \leq n} \frac{\Omega_i K_j}{h_H \sum_{1 \leq i \leq n} \Omega_i K_j},
\]

with

\[
W_i(\theta, x) = \beta_0(X_i, x) \left( \beta_0(X_i, x) - \beta_0(X_j, x) \right)
\times K(h_H^{-1}d_0(x_i, X_i))K(h_H^{-1}d_0(x_j, X_j)),
\]

and \(\Omega_i K_j = \sum_{i=1}^{n} W_{ij}\) with \(\beta_0(X_i, x)\) is a known bi-functional operator from \(\mathcal{H}^2\) into \(\mathbb{R}\) where \(K\) and \(H\) are kernel functions and \(h_K := h_{n,K}\) (resp. \(h_H := h_{n,H}\)) is a sequence that decreases to zero as \(n\) goes to infinity.

3. Assumptions and main results
Throughout the paper, we will denote by \(C, C'\) and \(C_{0,x}\) some strictly positive generic constants and \(K(\theta, x) := K(h_H^{-1}d_0(X_i, X_j)), \forall \theta \in \mathcal{H}, i = 1, \ldots, n, H_i := H(h_H^{-1}(y - Y_i))\), \(\forall y \in \mathbb{R}, j = 1, \ldots, n\). \(\beta_0, \beta_1, \beta_2\) denote \(\beta_0(X_i, x), W_i(\theta, x) := W_{0,j} \) and we will use the notation \(B_0(x, h_K) := \{x_1 \in \mathcal{H} : 0 < |x - x_1| < h_K\}\), the ball centred at \(x\) with radius \(h_K\). Moreover, to find the results in our paper we denote \(\psi_l(., y) := \frac{\partial^2 f(\theta, x, y)}{\partial y^2}\), for any \(l \in \{0, 2\}\)

\[
\Phi_l(s) = E[\psi_l(X, Y) - \psi_l(x, y)]d_0(x, X) = s.
\]

In order to study our asymptotic results, we need the following assumptions:

(H1) \(\mathbb{P}(X \in B_0(x, h_K)) = \phi_{0,2}(h_K) > 0\), and assume that there exists a function \(\chi_{\theta,x}(\cdot)\) such that

\[
\lim_{h_k \to 0} \frac{\phi_{0,2}(sh_K, h_K)}{\phi_{0,2}(h_K)} = \chi_{\theta,x}(s), \forall s \in [-1, 1].
\]

(H2) For any \(l \in \{0, 2\}\), the quantities \(\Phi'_l(0)\) and \(\Phi''_l(0)\) exist, where \(\Phi'_l\) (resp. \(\Phi''_l\)) denotes the first (resp. the second) derivative of \(\Phi_l\).

(H3) The bi-functional \(\beta_0(., .)\) satisfies

\[
\forall x' \in \mathcal{F}, \quad C_1 d_0(x, x') \leq |\beta_0(x, x')| \leq C_2 d_0(x, x'), \quad \text{where} \quad C_1 > 0, C_2 > 0,
\]

\[
\sup_{u \in B(0, \mathcal{H})} |\beta_0(u, x) - d_0(x, u)| = o(a),
\]

\[
\int_{B(0, \mathcal{H})} \int_{B(0, \mathcal{H})} \beta_0^2(u, x) dP(u),
\]

where \(B_0(x, r) = \{x' \in \mathcal{H}/|d_0(x, x') \leq r\} \) and \(dP(x)\) is the cumulative distribution of \(X\).

(H4) The kernel \(K\) is a positive, differentiable function and its derivative \(K'\) exists and is such that there exist two constants \(C\) and \(C'\) with \(-\infty < C < K'\) \((t) < C' < 0\), for \(t \in [-1, 1]\) and \(K(1) > 0\).

(H5) The kernel \(H\) is a differentiable function and bounded, such that

\[
\int H(t) dt = 1, \quad \int t^2 H(t) dt < \infty \quad \text{and} \quad \int H^2(t) dt < \infty.
\]

(H6) The bandwidths \(h_K, h_H\) satisfy

\[
\begin{align*}
(\text{i}) & \quad \lim_{n \to \infty} h_K = 0, \quad \lim_{n \to \infty} h_H = 0 \quad \text{and} \quad \lim_{n \to \infty} nh_H \phi_{0,2}(h_K) = 0, \\
(\text{ii}) & \quad \lim_{n \to \infty} nh_H^5 \phi_{0,2}(h_K) = 0 \quad \text{and} \quad \lim_{n \to \infty} nh_H^5 \phi_{0,2}(h_K) = 0.
\end{align*}
\]

Comments on assumptions: Notice that, (H1) and (H2) are a simple adaptation of the conditions in Ferraty et al. (2007) on the regression operator, when we replace the semi-metric by some bi-functional \(d_0\). The second part of the condition (H3) is unrestricted and is verified, for instance, if \(d_0(\cdot, \cdot) = \beta_\theta(\cdot, \cdot)\); moreover

\[
\lim_{d_0(x, u) \to 0} \left| \frac{\beta_0(u, x)}{d_0(x, u)} - 1 \right| = 0.
\]

Assumptions (H4)–(H6) are classical in this context of quadratic errors and asymptotic normality in functional statistic.

3.1. Mean square convergence
In this part, we are going to show the asymptotic results of quadratic-mean convergence.
**Theorem 3.1:** Under assumptions (H1)–(H6), we obtain
\[
\mathbb{E} \left[ f(\theta, y, x) - f(\theta, y, x) \right]^2 = B_H(\theta, x, y) h_H^2 + B_K(\theta, x, y) h_K^2 + \frac{V_{HK}(\theta, x, y)}{n h_{\Phi_0}(h_K)} + o(h_H^2) + o(h_K^2) + o \left( \frac{1}{n h_{\Phi_0}(h_K)} \right),
\]
where
\[
B_H(\theta, x, y) = \frac{1}{2} \frac{\partial^2 f(\theta, y, x)}{\partial y^2} \int t^2 H(t) \, dt,
\]
and
\[
B_K(\theta, x, y) = \Phi_0(0) \frac{M_0}{M_1},
\]
and
\[
V_{HK}(\theta, x, y) = \frac{M_2 f(\theta, y, x)}{M_1} \left( \int H^2(t) \, dt \right),
\]
with
\[
M_0 = K(1) - \int_0^1 s K'(s) \chi_{\theta, x}(s) \, ds \quad \text{and}
\]
\[
M_j = K_j(1) - \int_0^1 (K_j)'(s) \chi_{\theta, x}(s) \, ds \quad \text{for } j = 1, 2.
\]
We set
\[
\hat{f}(\theta, y, x) = \frac{\hat{f}_N(\theta, y, x)}{\hat{f}_D(\theta, x)},
\]
where
\[
\hat{f}_N(\theta, y, x) = \frac{1}{n(n-1)h_H} \mathbb{E} \left[ W_{12}(\theta, x) \right] \times \sum_{1 \leq i \neq j \leq n} W_{ij}(\theta, x) H(h_H^{-1}(y - Y_j)),
\]
and
\[
\hat{f}_D(\theta, x) = \frac{1}{n(n-1)h_H} \mathbb{E} \left[ W_{12}(\theta, x) \right] \sum_{1 \leq i \neq j \leq n} W_{ij}(\theta, y, x).
\]

The following lemmas will be useful for proof of Theorem 3.1.

**Lemma 3.2:** Under the assumptions of Theorem 3.1, we obtain
\[
\mathbb{E} \left[ \hat{f}_N(\theta, y, x) - f(\theta, y, x) \right] = B_H(\theta, x, y) h_H^2 + B_K(\theta, x, y) h_K + o(h_H^2) + o(h_K).
\]

**Lemma 3.3:** Under the assumptions of Theorem 3.1, we obtain
\[
\text{Var} \left[ \hat{f}_N(\theta, y, x) \right] = \frac{V_{HK}(\theta, x, y)}{n h_{\Phi_0}(h_K)} + o \left( \frac{1}{n h_{\Phi_0}(h_K)} \right).
\]

**Lemma 3.4:** Under the assumptions of Theorem 3.1, we get
\[
\text{Cov} \left[ \hat{f}_N(\theta, y, x), \hat{f}_D(\theta, x) \right] = O \left( \frac{1}{n h_{\Phi_0}(h_K)} \right).
\]

**Lemma 3.5:** Under the assumptions of Theorem 3.1, we get
\[
\text{Var} \left[ \hat{f}_D(\theta, x) \right] = O \left( \frac{1}{n h_{\Phi_0}(h_K)} \right).
\]

**3.2. Asymptotic normality**

This section contains results on the asymptotic normality of \( \hat{f}(\theta, y, x) \). Before announcing our main results, we introduce the quantity \( N(a, b) \), which will appear in the bias and variance dominant terms:
\[
N(a, b) = K^a(1) - \int_{-1}^1 (u^b K^a(u))' \chi_\alpha(u) \, du
\]
for all \( a > 0 \) and \( b = 2, 4 \).

Then, we have the following theorem.

**Theorem 3.6:** Under assumptions (H1)–(H6), we obtain
\[
\sqrt{n h_{\Phi_0}(h_K)} \frac{\bar{f}(\theta, y, x) - f(\theta, y, x)}{\mathbb{E}_n(\theta, x, y)} \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta, x, y))
\]
where
\[
V_{HK}(\theta, x, y) = \frac{M_2 f(\theta, y, x)}{M_1} \left( \int H^2(t) \, dt \right)
\]
and
\[
\mathbb{E}_n(\theta, x, y) = \frac{\mathbb{E} \left[ \hat{f}_N(\theta, y, x)(y) \right]}{\mathbb{E} \left[ \hat{f}_D(\theta, x) \right]} - f(\theta, y, x)
\]
with \( \xrightarrow{D} \) denoting the convergence in distribution.

**Proof of Theorem 3.6:** Inspired by the decomposition given in Masry (2005), we set
\[
\frac{\bar{f}(\theta, y, x) - f(\theta, y, x) - \mathbb{E}_n(\theta, x, y)}{\hat{f}_D(\theta, x)} = \frac{\hat{f}_N(\theta, y, x) - f(\theta, y, x)\hat{f}_D(\theta, x) - \hat{f}_D(\theta, x)\mathbb{E}_n(\theta, x, y)}{\hat{f}_D(\theta, x)}.
\]
If we denote
\[
Q_n(\theta, x, y) = \hat{f}_N(\theta, y, x) - f(\theta, y, x)\hat{f}_D(\theta, x)
\]
and
\[
- \mathbb{E} \left[ \hat{f}_N(\theta, y, x) - f(\theta, y, x) \right] \hat{f}_D(\theta, x) = \hat{f}_N(\theta, y, x) - f(\theta, y, x)\hat{f}_D(\theta, x) - \mathbb{E}_n(\theta, x, y),
\]
since
\[
\hat{f}_N(\theta, y, x) - f(\theta, y, x)\hat{f}_D(\theta, x)
\]
As mentioned in Demongeot et al. (2013), Remark 3.9:

Finally we illustrate the Monte-Carlo methodology to test the efficiency of the asymptotic normality results parallel the practical experiment and build functional pseudo-confidence area.

For this purpose, we consider the following process explanatory functional variables for \( n = 350 \)

\[
X_j(t) = \sum_{j=1}^{3} V_{ij} \cos((3+j)t) + W_i(t-\pi)^2, \quad \forall t \in [0,100],
\]

where \( V_{ij} \) and \( W_i \) are \( n \) independent real random variables (r.r.v.) uniformly distributed over \([0.3;2]\) (resp. \([1;3]\)), and it is assumed that these curves are observed on a discretization grid of 100 points in the interval. These functional variables are represented in Figure 1.

For response variables \( Y_i \), we consider the following model for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, 100 \):

\[
Y = r(\theta_k, X) + \epsilon
\]

where \( r(U_i) = \int_{0}^{100} \frac{1}{1-U_i(v)^2} \, dv \) and \( \epsilon \) is a centred normal variable and assumed to be independent of \( (X_i)_i \). Then, we can get the corresponding conditional density, which is explicitly defined by

\[
f(\theta_k, y, x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-r(\theta_k, x))^2}.
\]

Our goal in this illustration is to show the usefulness of conditional density in a context of forecasting. Thus the use of optimal parameters of the conditional density is without theoretical validity.

Now, we precise the different parameters of our estimators. Indeed, first of all, it is clear that the shape of the curves allows us to use

\[
d(x_1, x_2) = \sqrt{\int_0^1 (x_1(t) - x_2(t))^2 \, dt}; \quad \forall x_1, x_2 \in \mathcal{H}, \text{ where } \mathcal{H} \text{ is a semi-metric space.}
\]

We choose particularly the quadratic kernels defined by

\[
\frac{3}{2}(1-x^2)1_{[-1,1]} \quad \text{and} \quad K(1) > 0.
\]

In this illustration, we select the functional index \( \theta_k \) on the set of eigenvectors of the empirical covariance operator.

\[
\Gamma_n(X) = \frac{1}{200} \sum_{i=1}^{200} (X_i - \bar{X})^t ((X_i - \bar{X})).
\]

Indeed, we recall that the ideas of Ait Saidi et al. (2008) can be adapted to find a method of practical selection for \( \theta_k \). However, this adaptation in the case of the conditional density requires tools and additional preliminary results (See the discussion Attaoui et al. (2011) and Attaoui (2014)).
Figure 1. The curves $X_i, i = 1, \ldots, 200$.

For this purpose, we divide our observations on two packets learning sample $(X_i, Y_i)_{i=1,\ldots,200}$ and test sample $(X_i, Y_i)_{i=201,\ldots,250}$. For the choice of smoothing parameters $h_K$ and $h_H$, we will adopt the selection criterion used by Ferraty et al. (2006) in the case of the kernel method for which $h_K$ and $h_H$ are obtained by minimizing the next criterion

$$
\frac{1}{n} \sum_{i=1}^{n} W_1(X_i) \int \widehat{f}_{(h_K,h_H)}^{-1}(X_i,y) W_2(y) dy
- \frac{2}{n} \sum_{i=1}^{n} \widehat{f}_{(h_K,h_H)}^{-1}(X_i,Y_i) W_1(X_i) W_2(Y_i),
$$

(7)

where

$$
\widehat{f}_{(h_K,h_H)}^{-1}(X_k,y) = \frac{\sum_{k \neq i,j=1}^{n} W_j(X_k)H(h^{-1}_H(y - Y_j))}{h_H \sum_{i,j=1}^{n} W_j(X_i)}.
$$

A first way of assessing the quality of prediction is to compare predicted functional responses $\widehat{f}(\theta, y, x)$ for any $X$ in the testing sample versus the true of conditional density operator (i.e., $f(\theta, y, x)$) as in Figure 2.

For the next simulation algorithm, we used:

- Simulate a sample of size $n$.
- Calculate the smoothing parameters $h_K$ and $h_H$ that are varied over interval $[0,1]$ and which minimizes (7).
- Compute for $k = 1, 2, 3, 4$ the quantities

$$
(nh_H \phi_{\theta_k,x})^{1/2} \left( \frac{\widehat{f}(\theta_k, y, x) - f(\theta_k, y, x)}{\sqrt{ \sum_{i,j=1}^{n} W_j(X_i)W_j(X_j) }} \right).
$$

where $\widehat{f}(\theta_k, y, x)$ is the functional kernel estimator from the sample $(X_i, Y_i)_{i=1,\ldots,200}$, and $k = 1, 2, 3, 4$.

Figure 2. Predicted functional responses (solid lines); observed functional responses (dashed lines).
Compute a standard density estimator by local linear method.

Compute the estimated $\hat{f}(\theta_k, y, x)$ with the corresponding estimated $f(\theta_k, y, x)$.

The obtained results are shown in Figure 3. It can be seen that both densities are very well approximated and have good behaviours with respect to the standard normal distribution.

An application of results of Theorem 3.6 is to build the functional pseudo-confidence areas. To this aim, let us set for any component $k$, $(k = 1, 2, \ldots, K)$ and $\eta_k = \eta/K$ with $\eta \in [0, 1]$, confidence intervals $E_{\eta_k}$ such that

$$P \left( \bigcap_{k=1}^{K} r_k(U) \in E_{\eta_k} \right) \geq 1 - \eta$$

where $U = \langle X, \theta \rangle$ with $\theta_1, \ldots, \theta_K$ being a data-driven orthonormal basis, the $K$ eigenfunctions associated to the $K$ largest eigenvalues of $\Gamma$.

The results from the asymptotic normality of the conditional density are expressed in Corollary 3.10 and we can approximate $(1 - \eta)$ confidence interval of $f(\theta, y, x)$ by

$$\hat{f}(\theta, y, x) \pm t_{\eta/2} \times \left( \frac{\hat{V}_H(\theta, x, y)}{n h^2 \phi_{\theta, x}(h \hat{K})} \right)^{1/2},$$

where $t_{\eta/2}$ denotes the $\eta/2$ quantile of the standard normal $N(0, 1)$.

Figure 4 represents a functional pseudo-confidence zone for 9 different fixed curves with $\eta = 0.05$ and $K = 4$. We see that $r(\langle x, \theta_k \rangle)$ and its $K$-dimensional
projection onto $\hat{\theta}_1, \ldots, \hat{\theta}_K$ are very close. This conclusion shows the good performance of our asymptotic normality. Indeed, when one replaces the data-driven basis with the eigenfunctions of $\Gamma$, one gets very similar functional pseudo-confidence areas.

5. Conclusion

In this paper, we are mainly interested in the nonparametric estimation of the conditional density function by the local linear method for a variable explanatory functionally conditioned to an actual response variable via a functional single-index model. We show that the estimator provides good predictions under this model. One of the main contributions of this work is the choice of the semi-metric. Indeed, it is well known that, in nonparametric functional statistics, the semi-metric of the projection type is very important for increasing the concentration property. The functional index model is a special case of this family of semi-metrics because it is based on the projection on a functional direction which is important for the implementation of our method in practice. Therefore, we can draw zones of functional pseudo-confidence, which is a very interesting tool for assessing the quality of the prediction.

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Appendix

Proof of Theorem 3.1: We know the theorem is a consequence of separately computing two quantities (bias and variance) of $\hat{f}(\theta, y, x)$, and we have

$$
\mathbb{E} \left[ \hat{f}(\theta, y, x) - f(\theta, y, x) \right]^2 = \mathbb{E} \left[ \hat{f}(\theta, y, x) - f(\theta, y, x) \right]^2 + \text{Var} \left[ \hat{f}(\theta, y, x) \right].
$$
By classical calculations, we obtain
\[
\hat{f}(\theta, y, x) - f(\theta, y, x) \\
= \left( \hat{f}_N(\theta, y, x) - f(\theta, y, x) \right) \\
- \int t^2 H(t) dt \\
\hat{f}_N(\theta, y, x) - \mathbb{E}[\hat{f}_N(\theta, y, x)] \\
- \mathbb{E}[\hat{f}_N(\theta, y, x)] \hat{f}_N(\theta, x) \\
+ \hat{f}_D(\theta, x) - 1)^2 \hat{f}(\theta, y, x)
\]
which implies that
\[
\mathbb{E}[\hat{f}(\theta, y, x)] - f(\theta, y, x) \\
= \left( \mathbb{E}[\hat{f}_N(\theta, y, x)] - f(\theta, y, x) \right) \\
- \text{Cov} \left( \hat{f}_N(\theta, y, x), \hat{f}_D(\theta, x) \right) \\
+ \left( \mathbb{E}[\hat{f}_N(\theta, y, x)] \right)^2 \text{Var} \left( \hat{f}_D(\theta, x) \right)
\]
Under the assumption (H5), we can bound \( \hat{f}(\theta, y, x) \) by a constant \( C > 0 \) where \( \hat{f}(\theta, y, x) \leq C/h_{\mathcal{H}} \). Hence
\[
\mathbb{E}[\hat{f}(\theta, y, x)] - f(\theta, y, x) \\
= \left( \mathbb{E}[\hat{f}_N(\theta, y, x)] - f(\theta, y, x) \right) \\
- \text{Cov} \left( \hat{f}_N(\theta, y, x), \hat{f}_D(\theta, x) \right) \\
+ \left( \mathbb{E}[\hat{f}_N(\theta, y, x)] \right)^2 \text{Var} \left( \hat{f}_D(\theta, x) \right)
\]
Now, by similar techniques as those of Sarda and Vieu (2000) and by Bozig and Lecoutre (1987), the variance term is
\[
\text{Var} \left[ \hat{f}(\theta, y, x) \right] = \frac{1}{n_{h_{\mathcal{H}}} \phi_{0,x}(h_{\mathcal{H}})}.
\]

Proof of Lemma 3.2: We have
\[
\mathbb{E}[\hat{f}_N(\theta, y, x)] \\
= \frac{1}{n(n-1)h_{\mathcal{H}}} \mathbb{E}[W_{12}(\theta, x)] \\
\times \sum_{1 \leq j \leq n} W_j(\theta, x) H(h_{\mathcal{H}}^{-1}(y - Y_j)) \\
= \frac{1}{h_{\mathcal{H}}} \mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1].
\]
By using a Taylor’s expansion and under assumption (H5), we have
\[
\mathbb{E}[H_2|X_1] = f(\theta, y, x) \\
+ \frac{h_{\mathcal{H}}^2}{2} \left( \int t^2 H(t) dt \right) \frac{\partial^2 f(\theta, y, x)}{\partial y^2} + o(h_{\mathcal{H}}^2).
\]
Now, we can rewrite the above equation as
\[
\mathbb{E}[H_2|X_1] = \psi_0(X_2, y) \\
+ \frac{h_{\mathcal{H}}^2}{2} \left( \int t^2 H(t) dt \right) \psi_2(X_2, y) + o(h_{\mathcal{H}}^2).
\]
Thus, we obtain
\[
\mathbb{E}[\hat{f}_N(\theta, y, x)] = \frac{1}{\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1]} \\
= \frac{1}{\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1]} \mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1] \\
= \frac{1}{\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1]} \mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1] \\
\]
According to Ferraty et al. (2007), for \( I \in [0, 2] \), we show that
\[
\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1] = \frac{1}{\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1]} \mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1] \\
= \frac{1}{\mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1]} \mathbb{E}[W_{12}] \mathbb{E}[H_2|X_1] \\
\]
Therefore, it remains to determine the quantities \( \mathbb{E}[d_0(X_2, x) W_{12}] \) and \( \mathbb{E}[W_{12}] \). According to the definition of \( W_{12} \), the two quantities \( \mathbb{E}[d_0(X_2, x) W_{12}] \) and \( \mathbb{E}[W_{12}] \) are based on the evaluation asymptotic of \( K_2^\beta X_2 | X_1 \). To do that, we treat firstly, the case \( b = 1 \). For this case, we use the assumptions (H3) and (H4) to get
\[
h_{\mathcal{H}} \mathbb{E}[K_2^\beta | X_1] = o \left( \int_{B_{\mathcal{H}}(\mathcal{H})} h_{\mathcal{H}} \phi_{0,x}(h_{\mathcal{H}}) \right).
\]
So, we obtain that,
\[
\mathbb{E}[K_2^\beta | X_1] = o(h_{\mathcal{H}} \phi_{0,x}(h_{\mathcal{H}))}.
\]
Moreover, for all \( b > 1 \), and after simplifications of the expressions, it is permitted to write that
\[
\mathbb{E}[K_2^\beta | X_1] = \mathbb{E}[K_2^\beta | X_1] + o(h_{\mathcal{H}} \phi_{0,x}(h_{\mathcal{H}})).
\]
Concerning the first term, we write
\[
\mathbb{E}[K_2^\beta | X_1] = \int \psi_0(X_2, y) d\mathbb{E}[H_2|X_1] \\
= \int \psi_0(X_2, y) d\mathbb{E}[H_2|X_1] \\
= \int_0^1 \left( K(1) - \int_0^1 \left( \psi_0(X_2, y) \right) \phi_{0,x}(h_{\mathcal{H}}) ds \right) d\mathbb{E}[H_2|X_1] \\
= \phi_{0,x}(h_{\mathcal{H}}) \left( K(1) - \int_0^1 \left( \psi_0(X_2, y) \right) \phi_{0,x}(h_{\mathcal{H}}) ds \right) \\
\]
Finally, under assumptions (H1), we get
\[
\mathbb{E}[K_2^\beta | X_1] = h_{\mathcal{H}} \phi_{0,x}(h_{\mathcal{H}}) \left( K(1) - \int_0^1 \left( \psi_0(X_2, y) \right) \phi_{0,x}(h_{\mathcal{H}}) ds \right) \\
\]
By direct calculations, we get
\[
E[W_{0,12}^{2}] = O(h_{K}^{2}f_{\theta}^{2}(h_{K})).
\]
Thus, by the change of variables \( t = h_{H}^{-1}(y - z) \), we get
\[
E[K_{1}^{2}H_{1}^{2}] = h_{H}E \left[ K_{1}^{2} \int H^{2}(t)f(\theta, y - h_{H}t, X_{1})dt \right].
\]
By using Taylor’s expansion of order 1 of \( f(\theta, \cdot, y) \) we get
\[
f(\theta, y - h_{H}t, X_{1}) = f(\theta, y, X_{1}) + O(h_{H}) = f(\theta, y, X_{1}) + o(1).
\]
Then
\[
E[K_{1}^{2}H_{1}^{2}] = h_{H} \left( \int H^{2}(t)dt \right) E[K_{1}^{2}f(\theta, y, X_{1})] + o(h_{H}E[K_{1}^{2}]).
\]
Also, by the same steps in the proof of Lemma 3.2, we obtain
\[
E[K_{1}^{2}f(\theta, y, X_{1})] = f(\theta, y)E[K_{1}^{2}] + o(E[K_{1}^{2}])
\]
which give that
\[
E[K_{1}^{2}H_{1}^{2}] = h_{H}f(\theta, y)E[K_{1}^{2}] \int H^{2}(t)dt + o(h_{H}E[K_{1}^{2}]).
\]
Finally, we obtain from (A2), (A4) and (A5), that
\[
\text{Var}(\hat{f}_{N}(\theta, y, x)) = \frac{f(\theta, y, x)}{nh_{H}f_{\theta_{0},x}(h_{K})} \left( \int H^{2}(t)dt \right)
\]
By direct calculations, we get
\[
\left\{ \begin{array}{l}
E[W_{0,12}^{2}H_{1}^{2}] = O(h_{K}^{2}H_{1}^{2}f_{\theta}^{2}(h_{K})).
E[W_{0,12}H_{0,12}^{2}H_{1}^{2}]
\end{array} \right\}
\]
By the same arguments used in the proof of Lemma 3.2, we obtain
\[
\text{Var}(\hat{f}_{N}(\theta, y, x)) = \frac{E[K_{1}^{2}H_{1}^{2}]}{nh_{H}E[K_{1}^{2}]} + o \left( \frac{1}{nh_{H}f_{\theta_{0},x}(h_{K})} \right).
\]
Observe that
\[
E[K_{1}^{2}H_{1}^{2}] = E \left[ K_{1}^{2} \int H^{2}(t)dt \right].
\]
By direct calculations, we get
\[
\begin{align*}
\mathbb{E}[W_{0,12}H_2] &= \mathbb{E}[W_{0,12}W_{0,21}H_2] = O(h^2_k h_{0,2}(h_k)), \\
\mathbb{E}[W_{0,12}W_{0,13}H_2] &= \mathbb{E}[W_{0,12}W_{0,31}H_2] \\
&= O(h^2_k h_{0,2}(h_k)), \\
\mathbb{E}[W_{0,12}W_{0,23}H_2] &= \mathbb{E}[W_{0,12}W_{0,32}H_2] \\
&= O(h^2_k h_{0,2}(h_k)).
\end{align*}
\]
Since \(\mathbb{E}[W_{0,12}] = O(h^2_k \phi_{0,2}(h_k))\), we obtain
\[
\text{Cov}(\tilde{f}_S(\theta, y, x), \tilde{f}_D(\theta, x)) = O\left(\frac{1}{n \phi_{0,2}(h_k)}\right).
\]

**Proof of Lemma 3.5:** We have that
\[
\text{Var}(\tilde{f}_D(\theta, x)) = \frac{1}{(n(n-1))} \mathbb{E}[W_{0,12}]^2 \text{Var}\left(\sum_{1 \leq i \neq j \leq n} W_{0,i}\right).
\]
Similarly to the proof of Lemma 3.3, we get
\[
\text{Var}(\tilde{f}_D(\theta, x)) = \mathbb{E}[K_1^2] n \mathbb{E}[K_1^1]^2 + O\left(\frac{1}{n \phi_{0,2}(h_k)}\right).
\]
We have as \(n \rightarrow \infty, \frac{1}{\phi_{0,2}(h_k)} \mathbb{E}[K_1^1] \rightarrow \Omega_j, j = 1, 2\) (see Ferraty et al., 2007). Then, we can write finally
\[
\text{Var}(\tilde{f}_D(\theta, x)) = \frac{M_2 \phi_{0,2}(h_k)}{n M_1 \phi_{0,2}(h_k)^2} + O\left(\frac{1}{n \phi_{0,2}(h_k)}\right).
\]

**Proof of Lemma 3.8:** We have
\[
\sqrt{n h_{11} \phi_{0,2}(h_k)} Q_n(\theta, x, y) = \sqrt{n h_{11} \phi_{0,2}(h_k)} n \mathbb{E}[\Omega_1 K_1] \sum_{j=1}^{n} \Omega_j K_j(H_j - f(\theta, y, x))
\]
\[
- \mathbb{E}\left(\sum_{j=1}^{n} \Omega_j K_j(H_j - f(\theta, y, x))\right).
\]
Then, combined with (4) implies that
\[
\sqrt{n h_{11} \phi_{0,2}(h_k)} Q_n(\theta, x, y) = \frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{j=1}^{n} \beta_j^2 K_j \sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1^2 K_1)
\]
\[
\times \mathbb{E}(\Omega_1 K_1)
\]
\[
- \frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{j=1}^{n} \beta_j K_j \sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1 K_1)
\]
\[
\times \mathbb{E}(\Omega_1 K_1)
\]
\[
\times \sum_{j=1}^{n} \beta_j K_j(H_j - f(\theta, y, x))
\]
\[
- \mathbb{E}\left(\frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{j=1}^{n} \beta_j^2 K_j \sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1^2 K_1)
\]
\[
\times \mathbb{E}(\Omega_1 K_1)
\]
\[
\times \sum_{j=1}^{n} K_j(H_j - f(\theta, y, x))\right)
\]
\[
+ \mathbb{E}\left(\frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{j=1}^{n} \beta_j K_j \sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1 K_1)
\]
\[
\times \mathbb{E}(\Omega_1 K_1)
\]
\[
\times \sum_{j=1}^{n} K_j(H_j - f(\theta, y, x))\right).
\]

Denote
\[
S_1 = \frac{1}{n \mathbb{E}(\beta_1^2 K_1)} \sum_{i=1}^{n} \beta_i^2 K_i,
\]
\[
S_2 = \frac{\sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)} \sum_{j=1}^{n} K_j(H_j - f(\theta, y, x)),
\]
\[
S_3 = \frac{1}{n \mathbb{E}(\beta_1 K_1)} \sum_{i=1}^{n} \beta_i K_i \quad \text{and}
\]
\[
S_4 = \frac{\sqrt{n h_{11} \phi_{0,2}(h_k)} \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \sum_{j=1}^{n} K_j(H_j - f(\theta, y, x)).
\]
It remains to show that,
\[
\sqrt{n h_{11} \phi_{0,2}(h_k)} Q_n(\theta, x, y)
\]
\[
= S_1 S_2 - S_3 S_4 - \mathbb{E}(S_1 S_2 - S_3 S_4)
\]
\[
= (S_1 S_2 - \mathbb{E}(S_1 S_2)) - (S_3 S_4 - \mathbb{E}(S_3 S_4)).
\]
Hence by Slutsky’s theorem, to show (A3), it suffices to prove the following two claims:
\[
S_1 S_2 - \mathbb{E}(S_1 S_2) \xrightarrow{D} \mathcal{N}(0, V_{11}(\theta, x, y)), 
\]
\[
S_3 S_4 - \mathbb{E}(S_3 S_4) \xrightarrow{D} 0.
\]

**Proof of (A6)** We can write that
\[
S_1 S_2 - \mathbb{E}(S_1 S_2) = S_2 - \mathbb{E}(S_2) + (S_1 - 1)S_2 - \mathbb{E}((S_1 - 1)S_2).
\]
By Slutsky’s theorem, we get the following intermediate results:
\[
(S_1 - 1)S_2 - \mathbb{E}((S_1 - 1)S_2) \xrightarrow{D} 0
\]
\[
S_2 - \mathbb{E}(S_2) \xrightarrow{D} \mathcal{N}(0, V_{11}(\theta, x, y)).
\]
Concerning the proof of (A8), by applying the Bienaymé–Tchebychev’s inequality, we obtain for all \(\epsilon > 0\)
\[
\mathbb{P}(|(S_1 - 1)S_2 - \mathbb{E}((S_1 - 1)S_2)| > \epsilon) \leq \frac{\mathbb{E}((S_1 - 1)S_2 - \mathbb{E}((S_1 - 1)S_2))}{\epsilon}.
\]
Then, the Cauchy–Schwarz inequality implies that
\[
\mathbb{E}((S_1 - 1)S_2 - \mathbb{E}((S_1 - 1)S_2)) \leq 2\mathbb{E}((S_1 - 1)S_2) \leq 2\sqrt{\mathbb{E}((S_1 - 1)^2)\mathbb{E}((S_2)^2)}.
\]
On one side, by using (A1) and (A2), we obtain
\[
\mathbb{E}((S_1 - 1)^2) = \text{Var}(S_1) = \frac{1}{n \mathbb{E}(\beta_1^2 K_1)} n \mathbb{E}(\beta_1^2 K_1)
\]
\[
\leq \frac{1}{n O(h^2_k \phi_{0,2}(h_k))} \mathbb{E}(\beta_1^2 K_1)^2
\]
\[
= O\left(\frac{1}{n h_{11} \phi_{0,2}(h_k)}\right).
\]
And on the other side, we obtain
\[
\mathbb{E}((S_2)^2) = \frac{n h_{11} \phi_{0,2}(h_k) \mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)}.
\]
Concerning the second term on the right-hand side of (A10),

\[
\mathbb{E}(K(H_1 - f(\theta, y, x)))^2 = (\mathbb{E}(K(H_1 - f(\theta, y, x)))^2, \\
\mathbb{E}(H_1 | X_1) - f(\theta, y, x)) \to 0 \quad \text{as } n \to \infty. \quad \text{(A11)}
\]

Now let us return to the first term of the right hand of (A10). We have

\[
\begin{align*}
&\frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} (\mathbb{E}(K_1^2 (H_1 - f(\theta, y, x)))^2 \\
&= \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} (\mathbb{E}(E((H_1 - f(\theta, y, x))^2 | X_1))^2 \\
&= \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \mathbb{E}(\text{var}(H_1 | X_1))^2 \\
&+ \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \times (\mathbb{E}(E((H_1 | X_1) - f(\theta, y, x))^2))^2.
\end{align*}
\]

By using (A9), we have as \(n \to \infty\)

\[
\frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} (\mathbb{E}(E((H_1 | X_1) - f(\theta, y, x))^2))^2 \to 0.
\]

Combining (5) and (6), we obtain as \(n \to \infty\)

\[
\mathbb{E}(\text{var}(H_1 | X_1))^2 \to \mathbb{E}(K_1^2 f(\theta, y, x) \left( \int H^2(t) \, dt \right)) = M_2 f(\theta, y, x) \left( \int H^2(t) \, dt \right) \phi_0(x H).
\]

Therefore, by using (5) and (6), Equation (A10) becomes

\[
\mathbb{E}(\text{var}(P_n(x, y))) = \frac{n^2 \phi_0(x H) (N(1, 2) \mathbb{E}(\beta_1 K_1))^2}{((n - 1) N(1, 2) M_2 \mathbb{E}(\phi_0(x H)))^2} \times \left( \int H^2(t) \, dt \right) \mathbb{E}(\text{var}(\mu_n(x, y)))
\]

where

\[
\mathbb{E}(\text{var}(P_n(x, y))) = \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \mathbb{E}(\text{var}(\mu_n(x, y)))
\]

By the fact that \(\mu_n(x, y)\) are i.i.d., it follows that

\[
\text{var}(P_n(x, y)) = \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \mathbb{E}(\text{var}(\mu_n(x, y)))
\]

Thus

\[
\text{var}(P_n(x, y)) = \frac{n^2 \phi_0(x H) \mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)} \mathbb{E}(K_1^2 (H_1 - f(\theta, y, x))^2) \\
- \mathbb{E}(K_1 (H_1 - f(\theta, y, x))^2).
\]

Concerning the second term on the right-hand side of (A10), we have

\[
\mathbb{E}(K(H_1 - f(\theta, y, x)))^2 = (\mathbb{E}(K(H_1 - f(\theta, y, x)))^2, \\
(\mathbb{E}(K_1 (H_1 - f(\theta, y, x)))^2, \\
\mathbb{E}(\text{var}(H_1 | X_1) - f(\theta, y, x)))^2.
\]

Proofs of (A7) To use the same arguments as those invoked to prove (A6), let us write

\[
S_3 S_4 = (S_3 S_4) \\
S_4 = S_4 + (S_3 - 1) S_3 - E(S_3 - 1) S_4.
\]

By applying Bienaymé–Tchebychev’s inequality, we obtain for all \(\varepsilon > 0\)

\[
\mathbb{P}((S_3 S_4 - E(S_3 S_4)) > \varepsilon) \leq \frac{(S_3 S_4 - E(S_3 S_4))}{\varepsilon}.
\]

And the Cauchy–Schwarz inequality implies that

\[
E((S_3 - 1) S_4 - E((S_3 - 1) S_4))
\]
Taking into account the assumptions H(5) and H(6), we get
\[
\mathbb{E}((S_3 - 1)^2) = \text{Var}(S_3) = \frac{n}{n^2 \text{Var}(\beta_1 K_1)} \text{var}(\beta_1 K_1)
\]
\[
\leq \frac{1}{n O(\text{Var}(\beta_1 K_1))} \mathbb{E}(\beta_1^4 K_1^2)
\]
\[
= O\left(\frac{1}{n h_{\text{II}} \phi_{\omega}(h_K)}\right).
\]

On the other hand,
\[
\mathbb{E}((S_4)^2) = \frac{n h_{\text{II}} \phi_{\omega}(h_K) \mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)}
\times \mathbb{E}\left(\sum_{j=1}^{n} \beta_j K_j (H_j - f(\theta, y, x))\right)^2
\]
\[
= \frac{n h_{\text{II}} \phi_{\omega}(h_K) O(\text{Var}(\beta_1 K_1))}{(n - 1)^2 O(\text{Var}(\beta_1 K_1))}
\times (n \mathbb{E}(\beta_1 K_1 (H_1 - f(\theta, y, x))))^2
+ n(n - 1) \mathbb{E}^2 (\beta_1 K_1 (H_1 - f(\theta, y, x)))
= o(1) + o(n h_{\text{II}} \phi_{\omega}(h_K)).
\]

It remains to show
\[
\mathbb{E}((S_3 - 1)S_4 - \mathbb{E}((S_3 - 1)S_4))
\leq 2 \sqrt{\mathbb{E}((S_3 - 1)^2) \mathbb{E}((S_4)^2)} = o(1)
\]
which implies that
\[
\mathbb{E}((S_3 - 1)S_4 - \mathbb{E}((S_3 - 1)S_4)) = o_p(1).
\]
Therefore,
\[
\mathbb{P}(|S_3 S_4 - \mathbb{E}(S_3 S_4)| > \epsilon)\]
\[
\leq \frac{\mathbb{E}((S_3 S_4 - \mathbb{E}(S_3 S_4)))}{\epsilon} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

So, to prove (A6), it suffices to show \(S_4 - \mathbb{E}(S_4) = o(1)\), while
\[
\mathbb{E}(S_4 - \mathbb{E}(S_4))^2 = \text{Var}(S_4)
\]
\[
= \frac{n^2 \phi_{\omega}(h_K) \mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)}
\times \text{Var}(\beta_1 K_1 (H_1 - f(\theta, y, x))).
\]

We arrive finally at
\[
\text{Var}(\beta_1 K_1 (H_1 - f(\theta, y, x)))
\]
\[
= f(\theta, y, x)\left(\int H^2(t)dt\right) \mathbb{E}(\beta_1^2 K_1^2).
\]

This last result together with (5) and (6) leads directly to
\[
\mathbb{E}(S_4 - \mathbb{E}(S_4))^2
\]
\[
= \frac{n^2 \phi_{\omega}(h_K) \mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} f(\theta, y, x)\left(\int H^2(t)dt\right) \mathbb{E}(\beta_1^2 K_1^2)
\]
\[
= f(\theta, y, x)\left(\int H^2(t)dt\right) o(1),
\]
which allows finishes the proof. \(\square\)