A THRESHOLD FOR CLUSTERS IN REAL-WORLD RANDOM NETWORKS

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Abstract. Recent empirical work [35] has suggested the existence of a size threshold for the existence of clusters within many real-world networks. We give the first proof that this clustering size threshold exists within a real-world random network model, and determine the asymptotic value at which it occurs.

More precisely, we choose the Community Guided Attachment (CGA) random network model of Leskovek, Kleinberg, and Faloutsos [34]. The model is non-uniform and contains self-similar communities, and has been shown to have many properties of real-world networks. To capture the notion of clustering, we follow Mishra et. al. [42], who defined a type of clustering for real-world networks: an \((\alpha, \beta)\)-cluster is a set that is both internally dense (to the extent given by the parameter \(\beta\)), and externally sparse (to the extent given by the parameter \(\alpha\)). With this definition of clustering, we show the existence of a size threshold of \((\ln n)^{1/2}\) for the existence of clusters in the CGA model. For all \(\epsilon > 0\), a.a.s. clusters larger than \((\ln n)^{1/2-\epsilon}\) exist, whereas a.a.s. clusters larger than \((\ln n)^{1/2+\epsilon}\) do not exist. Moreover, we show a size bound on the existence of small, constant-size clusters.
1. Introduction

Real-world networks are everywhere. Examples include the network formed by the connections between people in a city; the network of citations between academic papers; an electric power grid; and the network of physical interactions between proteins [44]. Despite their differing origins, empirical observation has shown these networks to share many properties. These include: the small-world effect (the average shortest path between two nodes in a real-world network is smaller than one might expect, and may shrink over time) [52, 30]; the scale-free property (the degree distribution of nodes in the network follows a power law) [10, 21]; and clustering (certain parts of the network are much more closely connected than their surrounding neighbourhood) [52, 11]. The present work concerns this last property of clustering. Clustering is greatly important in biology, sociology, and computer science (see e.g. [3, 2, 27, 43, 46, 45, 48, 23] lists hundreds of others), and plays an important role in understanding the structure of real-world networks [52, 4, 11, 23, 24]. Despite this, our work is the first we know of to study clustering analytically in any random model for real-world networks.

Clusters in real-world networks often overlap [42, 3]; that is, a single node may be a member of more than one cluster. One can imagine that a computer in a computer network may belong to multiple groups (corresponding to clusters); similarly, a person in a social network may have multiple groups of friends. However, most approaches to clustering partition the network without allowing for clusters to overlap. In 2007, Mishra, Schreiber, Stanton, and Tarjan [42] proposed the \((\alpha, \beta)\)-cluster as a new formulation of clustering. In this definition, a cluster is a set that is both externally sparse (each node outside the set is connected to only a few nodes outside the set, as determined by the parameter \(\alpha\)) and internally dense (each node in the set is connected to many others inside the set, as determined by the parameter \(\beta\)). This definition, motivated by real-world networks such as social networks, allows for overlapping clusters.

A measure related to clustering is that of conductance. The conductance of a set is the ratio between the number of “cut edges” (the edges between the set and its complement) and the number of internal edges in that set. A set with low conductance has many internal edges and few “cut” edges, and is therefore intuitively a good cluster. Recently, Leskovek, Lang, Dasgupta, and Mahoney [35] empirically examined how the conductance of the best conductance clusters changed as cluster size increased. They found similar behaviour in many existing real-world networks: below a certain size threshold, good clusters exist; moreover, increasing cluster size below the threshold improves the quality of the best cluster. Above the threshold, however, increasing the cluster size decreases the quality of the best cluster.

In other words, there appears to exist a size threshold for conductance clusters in real-world networks. In this work, we give the first proof of a clustering size threshold in any model. More precisely, we choose to work with \((\alpha, \beta)\)-clusters. Because \((\alpha, \beta)\)-clusters were created specifically for types of real-world networks, they are a natural formulation to choose. Instead of an empirical approach, our method will be to study clustering analytically within an existing model of a real-world random network. The benefits of this approach are two-fold: first, it allows us to consider the asymptotic behaviour of clusters as the network size grows; and second, it allows us to prove directly the existence of a threshold.

Many random graph models that contain properties of real-world networks have been proposed (e.g. [52, 12, 5, 30, 16, 32]). In 2005, Leskovek et. al. [34] observed that real-world networks obey the additional property of densification (the average degree of a node will grow over time), and proposed a model called Community Generated Attachment (CGA). Because this model is built from a self-similar structure of nested communities, it exhibits non-trivial clustering in a way that previous models do not. It was also the first model to exhibit densification in addition to being scale-free. Furthermore, its simple mathematical description makes it amenable to analysis; other models that exhibit densification have not tended to permit analysis [32]. For these reasons, we choose to work with the CGA model, which will be defined fully in Section 2.

The goal of the present work is to analyze conditions under which \((\alpha, \beta)\)-clusters occur in the CGA model for real-world networks. For every fixed \(0 < \alpha, \beta \leq 1\), we establish a cluster size threshold of...
(\ln n)^{1/2}$ (where $n$ is the size of the network): for all $\epsilon > 0$, a.a.s. there are clusters larger than $(\ln n)^{1/2-\epsilon}$, while a.a.s. there are no clusters larger than $(\ln n)^{1/2+\epsilon}$. Furthermore, we show a size bound on the existence of small, constant-size clusters.

Our work is the first instance we know of that studies the existence of any notion of clustering analytically in a random model for real-world networks.

2. Model and Definitions

2.1. CGA Random Graph Model. The community-guided attachment (CGA) random graph model was first proposed by Leskovec, Kleinberg, and Faloutsos [34], in response to the observation that real-world random graphs tend to have average node degrees that increase over time, a property known as densification. It was the first model proposed with this property. Several other models with densification exist: Leskovec et. al. also proposed the “forest fire” model [34]; the model in [36] can be shown to have densification and other real-world network properties. Lattanzi and Sivakumar’s recent affiliation network model [32] is an example of a densifying model that can admit analysis.

The CGA model is based on levels of nested, self-similar communities. This is natural, because real-world networks often exhibit some level of self-similarity. For example, a computer network may be decomposed into several sub-networks, based on geography or purpose. Likewise, each of these sub-networks may themselves be further decomposed into smaller groups. A pair of computers sharing membership in one of these small groups are much more likely to be connected than two computers chosen at random from within the large network. This self-similar hierarchical structure is also observable in other domains. For example, it has been argued to apply to social groupings [53], subject classification of patents [34], and topic classification of Web pages [39]. The CGA model itself has previously been used to describe peer-to-peer networks [18].

The nested form of CGA makes it a natural choice of model to study clustering. Random graph models using fixed power-law degree sequences (e.g. [40, 5]) are a.a.s. locally tree-like, and hence exhibit no clustering at all. Methods such as Watts and Strogatz’ small-world model [52] start with a regular local graph structure such as a cycle or grid, and then add or re-wire some number of edges at random to insert long-range edges. While these models exhibit more clustering than a uniform random graph, it is trivial clustering: clusters are determined by the original, deterministic local structure rather than the random edges. In CGA, the nested communities mean that small, more dense communities are more likely to contribute to clusters than larger, less dense ones. As we will show, this implies the existence of varied clusters of different sizes, formed by random edges.

We now define the model precisely.

Definition. Let $T$ be a complete tree of height $H$, with constant fan-out $b$ (that is, each non-leaf node has exactly $b$ children). Let $n = b^H$ be the number of leaves of $T$. We will construct a random un-directed graph $G = (V, E)$ whose nodes $V$ are the leaves of $T$. Given two nodes $u, v \in V$, we define the height $h(u, v)$ to be the height of the smallest subtree in $T$ that contains both $u$ and $v$. (In other words, $h(u, v)$ is one half of the distance between $u$ and $v$ in T.) For a parameter $c > 1$, the probability that our random graph $G$ will have an edge from $u$ to $v$ will be equal to $c^{-h(u,v)}$.

The edge probability function chosen here is the only natural choice. To see this, set $\Pr((u, v) \in E) = f(h(u, v))$ for some function $f$. For $G$ to have a power-law degree sequence, we require that $f(h)/f(h-1)$ is constant. Hence, we must have $f(h) = \gamma c^{-h}$, where $c > 1$ is the shrinking parameter, and $\gamma \leq 1$ indicates the initial density. For simplicity, we will take $\gamma = 1$, but the work that follows can be adapted for any value of $\gamma \leq 1$.

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1We say an event occurs asymptotically almost surely (a.a.s.) if the probability the event occurs approaches one as $n \to \infty$. 

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Intuitively, each internal node of $T$ defines a sub-community of $G$, given by the leaf nodes of the sub-tree of $T$ rooted at that internal node. The larger the sub-tree, the larger and less connected (on average) the sub-community.

We now define some terminology used within the CGA model:

**Definition.** Let $M \subseteq V(G)$, so that $M$ corresponds to a subset of the leaves of the tree $T$. Define the height of $M$ to be the height of the minimum complete subtree in $T$ containing all of $M$. If a set $M$ has height $h$, then we will call $M$ complete if it has $b^h$ nodes. For each $h' \geq h$, $M$ is a subset of exactly one complete set of height $h'$; we will denote this set by $S(M,h')$. $S(M,h)$ is the minimum complete set containing $M$, and will usually be denoted $S(M)$. If $u$ is a leaf node of $T$ disjoint from $S(M)$, notice that there exists a $j > h$ such that for all $v \in S(M)$, $h(u,v) = j$. Therefore, we will define $h(u,M) = j$ and refer to it as the height of $u$ from $M$.

Intuitively, the height of a pair of (leaf) nodes gives a prediction of the similarity between those two nodes. Notice that the height of a set is the maximum height over all pairs of vertices in the set.

The above definitions are for the undirected version of the CGA model. The directed CGA model is only a minor modification from this: instead of a single edge between two vertices $u$ and $v$, we will have two edges going in opposite directions, which occur independently with equal probability $c^{-h(u,v)}$. This version was the one originally proposed by Leskovec et. al. [34]. In this paper, we work with the undirected version, but all conclusions hold in both forms of the model with only trivial modifications.

2.2. $(\alpha, \beta)$-clusters. Intuitively, a cluster is a set of vertices that is more edge-dense than the graph average. However, the exact definition chosen often varies with application. For the real-world networks we consider, allowing overlapping clusters is natural [42] [3]. In other words, a single vertex should be allowed membership in more than one cluster. Furthermore, it may be possible that a vertex is not a member of any clusters at all. In 2007, Mishra et. al. [42] proposed a clustering definition for social networks (and applying to other real-world graph applications) that allows clusters to overlap. This definition is as follows:

**Definition.** For an undirected graph $G$, let $v \in V(G)$ and $M \subseteq V(G)$, and let $e(v,M)$ denote the number of edges between $v$ and $M$. For parameters $0 \leq \alpha, \beta \leq 1$, we say $M$ is internally dense if for all $v \in M$, $e(v,M) \geq \beta |M|$, and is externally sparse if for all $u \notin M$, $e(u,M) \leq \alpha |M|$. An $(\alpha, \beta)$-cluster is a set that is both internally dense and externally sparse.

We will often refer to a $(\alpha, \beta)$-cluster as just a cluster, with the parameters $\alpha$ and $\beta$ being implicit. Note that for a cluster to necessarily be connected, we would have to require $\beta \geq \frac{1}{2}$. For this reason, Mishra et. al. restrict $\beta$ to this range, but nothing in this work places any restriction on $\beta$. As well, it is natural to have $\alpha \leq \beta$, but we do not require this either.

Within the CGA model, clusters may be described with the same terminology as vertex sets:

**Definition.** A cluster of height $h$ is a complete cluster if it has $b^h$ vertices.

By using outgoing edges, the definition of $(\alpha, \beta)$-cluster carries over to directed networks:

**Definition.** For a directed graph $G$, let $v \in V(G)$ and $M \subseteq V(G)$, and let $e_d(v,M)$ denote the number of edges from $v$ to $M$. For parameters $0 \leq \alpha, \beta \leq 1$, we say $M$ is internally dense if for all $v \in M$, $e_d(v,M) \geq \beta |M|$, and is externally sparse if for all $u \notin M$, $e_d(u,M) \leq \alpha |M|$. A directed $(\alpha, \beta)$-cluster is a set that is both internally dense and externally sparse.

Note that the directed form of the clustering definition depends on only outgoing edges. This corresponds to situations where cluster membership depends only on outgoing intent (for example, online social networks where users may "subscribe" to other users). While we will work in the undirected CGA model using the undirected definition of clustering, our results hold as well for the directed CGA model using the directed definition of clustering.
3. Current Work

Our goal is to establish a size threshold of \((\ln n)^{\frac{1}{2}}\) for the existence of clusters in both the directed and undirected versions of the CGA model. We will work in the undirected version of the model, but all statements hold with trivial modifications in the directed model. Our main theorem is stated as follows:

**Theorem 1.** Let \(G\) be a graph chosen according to the undirected CGA model. Then for all \(0 < \alpha, \beta \leq 1\) and \(\epsilon > 0\):

- **a:** Let \(m^* = \frac{\ln b}{\alpha \ln c}\). There a.a.s. exists \((\alpha, \beta)\)-clusters of size larger than \((\ln n)^{\frac{1}{2} - \epsilon}\) in \(G\). Moreover, there exists a constant \(\gamma = \gamma(\alpha, b, c) > 0\) such that for each \(h\) satisfying \(\log_b m^* < h \leq \left(\frac{1}{2} - \epsilon\right) \frac{\ln n}{\ln b}\), there a.a.s. exists at least \((\ln n)^{\gamma b^h}\) complete \((\alpha, \beta)\)-clusters of size \(b^h\).
- **b:** There a.a.s. no \((\alpha, \beta)\)-clusters with more than \((\ln n)^{\frac{1}{2} + \epsilon}\) vertices.

We will try to give some intuition towards why such a threshold might occur. It turns out (see Lemma 3) that complete sets of increasing (i.e., non-constant) size tend to be externally sparse; we will attempt to intuitively justify why sets below the threshold are internally dense.

To give intuition, we therefore make the following simplifications. First, given a set \(M\) of fixed size \(b^h\), \(M\) is more likely to be internally dense if its height is small, because short-range edges are more likely to occur. In the most extreme case, \(M\) has height \(h\) and size \(b^h\), and forms a complete set. For now, we will consider only sets of this form. Second, instead of considering whether \(M\) is internally dense, we will examine the probability of the stronger event of \(M\) being a clique. Finally, let us pretend all vertices in \(M\) are at height \(h\) from each other, so that the probability of any edge inside \(M\) occurring is \(c^{-h}\). Note that this last simplification is not so extreme: for any vertex \(v \in M\), \(\frac{b-1}{b} |M|\) of the vertices in \(M\) (that is, a large fraction) are height exactly \(h\) from \(v\); of the remaining \(\frac{|M|}{b}\) vertices, most have height close to \(h\).

Since there are \(\binom{b^h}{2}\) potential edges in \(M\), the probability that \(M\) is a clique is therefore at least

\[
\left(c^{-h}\right)^{\binom{b^h}{2}} \geq c^{-hb^{2h}}
\]

There are \(\frac{n}{b^h}\) disjoint complete sets of height \(h\), so if we let \(X\) denote the number of complete cliques of height \(h\), we have that

\[
E[X] \geq \frac{n}{b^h} c^{-hb^{2h}} \approx \exp\left(\ln n - hb^{2h}\right)
\]

(3.1)

(Of course, this is not really correct due to the simplifications we have made, but turns out to be close enough to give the right asymptotic value.) For the expected number of cliques to be growing with \(n\), we require the positive term in the exponential in (3.1) to be growing faster than the negative term. Suppose the height is given \(h = a \ln \ln n\) for some constant \(a\), so that the size of \(M\) is \(b^h = (\ln n)^a\). Then (3.1) shows that the expected number of cliques will grow precisely when \(a < \frac{1}{2}\), the value of our cluster size threshold.

Of course, the preceding justification considers only complete sets, and ignores many details such as external sparseness. Many other types of sets may exist, and it is natural to suspect they could form clusters with sizes larger than \((\ln n)^{\frac{1}{2} + \epsilon}\). Much of the work in proving part b) of Theorem 1 is in showing that this is not the case.

Let us try to give intuition for why part b) of Theorem 1 should be true; that is, why clusters larger than \((\ln n)^{\frac{1}{2} + \epsilon}\) should not exist. As noted, the sets most likely to be internally dense are those with small heights, because the probability of edges is higher. In this simplified explanation, we will consider only sets of this form. Let \(S\) be a complete set of height \(h = (\frac{1}{2} + \epsilon) \ln \ln n\), so that \(S\) has \(b^h = (\ln n)^{\frac{1}{2} + \epsilon}\) vertices and \(\binom{b^h}{2} \approx (\ln n)^{1+2\epsilon}\) possible edges. Of these edges, only \(b^h \binom{b^h}{2} = o(\ln n)\) have height \(\frac{h}{2}\) or...
less. Ignoring these edges, which are negligible in number, the remaining edges occur with probability at most \( c^{-\frac{b}{2}} \). This turns out to be small enough to show that a.a.s. there are no complete sets like \( S \) with more than \( \ln n \) edges.

That is, each complete set \( S \) of height \( h = \left( \frac{1}{2} + \epsilon \right) \frac{\ln \ln n}{\ln n} \) does not contain many edges. Given a set \( M \), edges from \( M \cap S \) to \( M \setminus S \) occur with low probability, because the height between these two sets is high. If the intersection \( M \cap S \) is large enough, then \( M \) also cannot have many edges contained within \( S \), because \( S \) itself does not have many edges. On the other hand, if \( M \) does not overlap any complete set \( S \) significantly, then \( M \cap S \) is small enough to be ignored. In other words, if \( M \) is large enough, then \( M \) cannot be internally dense, and is therefore not a cluster.

Although this explanation gives the intuition behind the core idea of the proof of Theorem 1, the actual proof is much complicated, requiring multiple steps to complete.

Our result holds as well in the directed version of the CGA model using the directed definition of clusters. This version of the theorem is stated as follows:

**Theorem 2.** Let \( G \) be a graph chosen according to the directed CGA model. Then for all \( 0 < \alpha, \beta \leq 1 \) and \( \epsilon > 0 \):

- **a:** Let \( m^* = \frac{\ln b}{\ln m} \). There a.a.s. exists directed \((\alpha, \beta)\)-clusters of size larger than \((\ln n)^{\frac{1}{2} - \epsilon}\) in \( G \). Moreover, there exists a constant \( \gamma > 0 \) such that for each \( h \) satisfying \( \log_b m^* < h \leq \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln n} \), there a.a.s. exists at least \((\ln n)^{\gamma h}\) complete directed \((\alpha, \beta)\)-clusters of size \( b^h \).

- **b:** There are a.a.s. no \((\alpha, \beta)\)-clusters with more than \((\ln n)^{\frac{1}{2} + \epsilon}\) vertices.

**Remark.** In fact, the directed version is easier to work with than the undirected one, due to the added independence of having one potential edge in each direction between each node pair. A different argument than the one given here strengthens the number of clusters given in part a) of Theorem 2: we can show that there a.a.s. exists at least \( \frac{\ln b}{\ln n} \) complete directed \((\alpha, \beta)\)-clusters of size \( b^h \).

We give the proof of Theorem 1 but not Theorem 2. The proof of Theorem 2 is nearly identical to that of Theorem 1 and may be given with a few straightforward changes. The remainder of the paper will be organized as follows. Section 4 lists related work. Section 5 gives several results establishing the existence and non-existence of clusters of size less than \((\ln n)^{\frac{1}{2}}\), including a proof of part a) of Theorem 1. Section 6 gives a proof of part b) of Theorem 1.

### 4. Related Work

**4.1. Random Models for Real-World Networks.** Models for real-world networks have evolved in response to several important empirical observations. One of these is the small-world effect, the observation that network diameters are smaller than one might expect [52, 5, 15, 41]. Watts and Strogatz [52] and Kleinberg [30] proposed models that add random edges to a regular network to reduce the diameter. Another key property is the scale-free property: degree distributions tend to follow a power law. This was first observed in the Internet graph [21], and later in phone call graphs [11] and the web graph [9].

Preferential attachment models [10, 16, 15], in which a network grows by adding new edges with a preference for attachment to nodes with high degree, give one approach to explaining this. Another approach is edge copying [29, 31], in which newly added vertices copy the edges of existing vertices. Many other models exist (see e.g. [5, 51, 37, 17, 28, 19], among many others).

In 2005, Leskovek, Kleinberg, and Faloutsos [34] observed that average node degree increased polynomially as the network grows. Prior to this, models had assumed a constant (or possibly logarithmically increasing) average node degree. They proposed two models with densification: the Community Guided Attachment (CGA) model used in this work, and a “forest fire” model similar to edge copying. Another approach for densifying models is based on Kronecker graphs [36, 38]. Lattanzi and Sivakumar [32]...
4.2. Clustering. Detection of clusters is greatly important in sociology, biology, and computer science. Fortunato [23] gives a good review of some of the many hundreds of published works on the topic. Depending on the application, the definition of what constitutes a cluster can vary greatly. Empirical studies show that clustering is present in real-world networks [52, 41, 35], and that these clusters often overlap [47, 4]: that is, a single node in a real-world network may be part of multiple clusters at once.

Most popular approaches to clustering (e.g. [27, 50, 46, 25, 8]) do not allow overlapping clusters. The most popular approach to overlapping communities is the clique percolation method [3, 22, 33]. In this method, two \( k \)-cliques overlap if they share \( k - 1 \) vertices. A \( k \)-clique community (cluster) is the union of a \( k \)-clique with all other \( k \)-cliques that overlap it. One problem with this approach is that it is not clear initially which value of \( k \) should be chosen. Additionally, it presumes the existence of many \( k \)-cliques, which may not be the case. Mishra, Schreiber, Stanton, and Tarjan’s [42] \((\alpha, \beta)\)-clusters (the clustering definition used in this work) avoids these problems by instead parameterizing the fraction of edges that should be present inside the cluster (the parameter \( \beta \)). Additionally, they introduce the notion of external sparseness (the parameter \( \alpha \)). Other approaches to overlapping clusters exist (e.g. [13, 20, 54, 48]).

Investigations into the size of clusters in real-world networks show that the tail of the cluster size distribution may follow a power law [37, 49, 45]. In other words, the relative sizes of the larger clusters in a network follow a certain distribution; unlike the current work, no observation is made about size relative to the overall size of the network. Recently, Leskovec, Lang, Dasgupta, and Mahoney [35] found empirical evidence for the existence of a size threshold for the “best” clusters in the network: beyond that size, the quality of clusters declines. This is also discussed in Section 4.

Despite the importance of clustering and the proliferation of random models for real-world networks, we are aware of no work that studies clustering analytically in any random model for real-world networks.

5. The Existence of Small Clusters

To prove the existence of clusters, we must establish that both external sparseness and internal denseness occur. We begin with external sparseness. Let \( h^* = (\frac{1}{2} - \epsilon) \frac{\ln \ln n}{\ln b} \). This value is important because the number of nodes in a complete set of height \( h^* \) is equal to \( \ln \ln n \), the bound in Theorem 1a). Let \( M \) be a set with height \( h \leq h^* \) and size \( m \). Recall that \( S(M) \) is the complete set of height \( h \) containing \( M \), and \( S(M, h^*) \) is the complete set of height \( h^* \) containing \( M \). For \( M \) to be externally sparse, we require three events to occur, defined as follows:

- \( E_1(M) \): The vertices of \( S(M) \setminus M \) must satisfy the external sparseness property with respect to \( M \). That is, \( \forall u \in S \setminus M, e(u, M) \leq \alpha m \).
- \( E_2(M) \): The vertices of \( S(M, h^*) \setminus S(M) \) must satisfy the external sparseness property with respect to \( M \). That is, \( \forall u \in S(M, h^*) \setminus S(M), e(u, M) \leq \alpha m \).
- \( E_3(M) \): The vertices of \( G \setminus S(M, h^*) \) must satisfy the external sparseness property with respect to \( M \). That is, \( \forall u \in G \setminus S(M, h^*), e(u, M) \leq \alpha m \).

When the set \( M \) is clear, we will sometimes denote these events as \( E_1, E_2, \) and \( E_3 \), without the parenthetical argument. Together, \( E_1 \cap E_2 \cap E_3 \) forms the event that \( M \) is externally sparse. This division will be used to show in steps when these different parts of the external sparseness property occur. In particular, this lessens the problem of dependence between sets. To see this, let \( M_1 \) and \( M_2 \) be sets such that \( S(M_1, h^*) \) and \( S(M_2, h^*) \) do not intersect. Because these sets do not intersect, the events \( E_1(M_1), E_1(M_2), E_2(M_1), \) and \( E_2(M_2) \) are all independent; the only events with dependence between each other are \( E_3(M_1) \) and \( E_3(M_2) \).

The following lemma establishes when \( E_2 \) can occur:

**Lemma 3.** Let \( m^* = \frac{\ln b}{\ln \ln c} \) and \( h^* = (\frac{1}{2} - \epsilon) \frac{\ln \ln n}{\ln b} \). Then:

- **a:** There are a.a.s. no externally sparse sets of size smaller than \( m^* \).
b: If \( M \) is a set of size \( m < m^* \) and height \( h \leq h^* \), then there exists a constant \( \alpha > 0 \) such that 
\[
\Pr \left( E_2(M) \right) > \alpha.
\]

The proof of Lemma 3 may be found in the appendix; a) follows from a first moment argument, and b) comes from the application of a concentration bound.

Because of the dependence problem, dealing with \( E_3 \) is more complicated. To give intuition, consider the following scenario. Suppose \( G \) is partitioned into complete sets of height \( h^* \), and from each of these sets we choose at most a single subset of size \( m \). Call this resulting collection of subsets \( \mathcal{M} \). Any single one of those subsets is likely to have \( E_3 \), because the probability of edges from that set to any of the other sets is small. In fact, by dealing with dependence appropriately, we can show that a.a.s. at least
\[
\min \left( \left| \mathcal{M} \right|, (\ln n)^{\frac{\alpha \ln c}{4 \ln b}(m-m^*)} \right)
\]
of the sets in \( \mathcal{M} \) have \( E_3 \) simultaneously. The proof of (5.1) may be found in the appendix.

We use (5.1) and Lemma 3 to prove the following corollary:

**Corollary 4.** Let \( m^* = \frac{\ln b}{\alpha \ln c} \), and let \( h = \Theta(1) \) be a fixed, constant value. Then:

a: For \( m < m^* \), there are a.a.s. no clusters of size \( m \).

b: For each \( m \) such that \( m^* < m \leq b^h \), there are a.a.s. at least \( (\ln n)^{\frac{\alpha \ln c}{4 \ln b}(m-m^*)} \) clusters of size \( m \) and height \( h \).

Note that Corollary 4 establishes a sharp size bound on the existence of small, constant-sized clusters. We give the proof of 4

**Proof.** Because a set must be externally sparse to be a cluster, part a) follows directly from Lemma 3. To show part b), let \( M \) be a complete set of height \( h \) and size \( m > m^* \), and let \( D \) denote the event that \( M \) is internally dense. Because there are only a constant number of vertices in \( M \), \( M \) has \( D \) with constant probability. Similarly, there are only a constant number of vertices in \( S(M) \setminus M \), so \( M \) has \( E_1 \) with constant probability. By Lemma 3 b), \( M \) has \( E_2 \) with at least constant probability. Hence, there is a constant \( q > 0 \) such that \( \Pr (D \cap E_1 \cap E_2) > q \).

Now, partition \( G \) into \( n/b^h \) sets of height \( h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln n}{\ln b} \), and from each set choose a complete set of height \( h \). Because each set lies in a different complete set of height \( h^* \), each set has \( D \cap E_1 \cap E_2 \) with probability at least \( a \) independently. Let those sets that have \( D \cap E_1 \cap E_2 \) form the collection of sets \( \mathcal{M} \). The expected size of \( \mathcal{M} \) is \( qn/b^h = \Omega (n^{3/4}) \), and a concentration bound (see appendix) shows that a.a.s. \( |\mathcal{M}| > n^{3/4} \). Applying (5.1), we see that at least \( (\ln n)^{\frac{\alpha \ln c}{4 \ln b}(m-m^*)} \) of the sets in \( \mathcal{M} \) are clusters.

We are now ready to show how (5.1) proves part a) of Theorem 1, which we now restate:

Let \( m^* = \frac{\ln b}{\alpha \ln c} \). There a.a.s. exists clusters of size larger than \( (\ln n)^{\frac{1}{2} - \epsilon} \) in \( G \). Moreover, there exists a constant \( \gamma = \gamma (\alpha, b, c) > 0 \) such that for each \( h \) satisfying \( \log_b m^* < h \leq \left( \frac{1}{2} - \epsilon \right) \frac{\ln n}{\ln b} \), there a.a.s. exists at least \((\ln n)^{\gamma b^h}\) complete clusters of size \( b^h \).

**Proof.** Note that the second part of the theorem statement implies the first, because a complete set of height \( h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln n}{\ln b} \) has \( b^h = (\ln n)^{\frac{1}{2} - \epsilon} \) vertices. Let \( M \) be a complete set of height \( h \). We will first examine the event \( D \) that \( M \) is internally dense.

In fact, let us consider the stronger event of \( M \) being a clique. Each potential edge in \( M \) occurs with probability at least \( c^{-h} \), so the probability of all \( (\frac{b^h}{2}) \) \( \leq b^{2h} \) edges occurring is at least \( c^{-hb^{2h}} \). Hence, 
\[
\Pr (D) \geq \exp \left( - \ln (c) hb^{2h} \right).
\]

Now, partition \( G \) into complete sets of height \( h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln n}{\ln b} \). From each such set, choose a single complete set of height \( h \), and let these \( n/b^h \) disjoint sets form the family \( \mathcal{A} \). Because these sets are complete, they automatically have \( E_1 \). Because they do not overlap and lie in different sets of height \( h^* \),
they will have $D$ and $E_2$ independently. Lemma 3.b) implies they have $E_2$ with probability at least $a$ for some positive constant $a$. Hence, for every $M \in A$, we have that

$$\Pr(D \cap E_1 \cap E_2) \geq \Pr(D) \Pr(E_1) \Pr(E_2) \geq a \exp\left(-\ln(e) hh^{2h}\right)$$

(5.2)

Now, let $X$ denote the number of sets in $A$ with $D \cap E_1 \cap E_2$. A fairly straightforward calculation (found in the appendix) shows that $E[X]$ is asymptotically larger than $\frac{n}{m}$, and moreover that $\Pr(X < \frac{n}{m}) \leq \exp\left(n^{-1/3}\right)$. In other words, a.a.s. there are at least $\frac{n}{m}$ sets in $A$ with $D \cap E_1 \cap E_2$. Let $M \subset A$ be this sub-family of sets. (5.1) implies that at least $(\ln n)^{\frac{\ln b}{\ln h} (b^h - m^*)}$ of the sets in $M$ also have $E_3$, and therefore are clusters. Recall that $m^* = \frac{\ln b}{\alpha n^c}$, and let $h_{\min}$ be the minimum integral value of $h$ such that $b^h > m^*$; setting

$$\gamma = \frac{\alpha \ln e \left(b^{h_{\min}} - m^*\right)}{4 \ln b \ b^{h_{\min}}}$$

guarantees that $(\ln n)^{\gamma b^h} \leq (\ln n)^{\frac{\ln b}{\ln h} (b^h - m^*)}$, and hence there are a.a.s. at least $(\ln n)^{\gamma b^h}$ clusters of height $h$ in $G$. □

6. THE NON-EXISTENCE OF LARGE CLUSTERS

The goal of this section is to prove part b) of Theorem 1 that, for all $\epsilon > 0$, there are no clusters with more than $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices.

The intuition of the proof may be outlined as follows. A set of a given size is more likely to form a cluster if the height of that set is small. More generally, a set is more likely to be a cluster if a large subset of the set has a small height, so that edges within that subset are more common. We will concentrate on ruling out these types of clusters.

We use the term thick to capture this notion of a set with small height containing many vertices. We will consider the following two types of thick sets:

**Definition.**

- A short $\epsilon$-thick set is a set with height at most $h_\epsilon = \left(\frac{1}{2} + \epsilon\right) \frac{\ln \ln n}{\ln b}$ and containing at least $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices.

- A tall $\epsilon$-thick set is a set with height at most $\frac{(\ln n)^{\frac{1}{2}}}{\ln b}$ and containing at least $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices.

Note that the height $h_\epsilon$ is the height of a complete set with $(\ln n)^{\frac{1}{2} + \epsilon}$ nodes, as in Theorem 1.b).

Let $Q$ be a set of size $q \geq (\ln n)^{\frac{1}{2} + \epsilon}$. There are two cases: either $Q$ contains no tall $\epsilon$-thick sets, or it contains at least one such set as a subset. Existence of clusters in the former case is easy to rule out: given any vertex $v \in Q$, we must have at least $\beta q$ edges from $v$ to other vertices in $Q$ for $Q$ to be a cluster. Since there is no tall $\epsilon$-thick set, at most $(\ln n)^{\frac{1}{2} + \frac{1}{2}} = o(q)$ of these vertices are “close”—that is, within height $\frac{(\ln n)^{\frac{1}{2}}}{\ln b}$ of $v$. Even assuming all the edges from $v$ to the close vertices exist, we must have at least $\frac{\beta q}{2}$ edges from $v$ to the far vertices in $Q$, which occur with low probability.

The latter case is harder. Let us write $Q = T \cup R$, where $T$ is a tall $\epsilon$-thick set, and $R$ contains the other vertices. Given $v \in T$, we must also rule out the possibility of there being many edges between $v$ and $T$, which occur with much higher probability than edges between $v$ and $R$.

This is done by repeating the argument used above for tall $\epsilon$-thick sets on short $\epsilon$-thick sets. $T$ will either contain a short $\epsilon$-thick set as a subset, or it will not. Again, the latter case is more easy to deal with, because most of the required edges for $Q$ to form a cluster are of height $h_\epsilon = \left(\frac{1}{2} + \epsilon\right) \frac{\ln \ln n}{\ln b}$ or more, and so are unlikely to occur. In the former case, we partition $T = M \cup K$ so that $M$ is the subset of height $h_\epsilon$ with at least $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices, and $K$ are the remaining vertices. We now have $Q = M \cup K \cup R$; a simplified form of the argument for this case is as follows. We will show that there
is some \( v \in M \) such that \( e(v, M) \leq \frac{\beta}{2} m, e(v, K) \leq \frac{\beta}{2} k, \) and \( e(v, R) \leq \frac{\beta}{4} r, \) together implying that \( e(v, Q) \leq \frac{\beta}{4} q. \)

However, some care is needed to make the asymptotics of this argument work. There are more than \( \binom{n}{c} \) choices for the set \( Q; \) if we divide \( Q = M \cup K \cup R \) for each such set \( Q, \) there are far too many choices of the set \( M \) for a first moment bound to show directly that a.a.s. all sets \( M \) have the property we desire. For that reason, the argument will instead be given in the reverse order: first, we will show that a.a.s. all short \( \epsilon \)-thick sets have the property we desire. This will be used to show that tall \( \epsilon \)-thick sets also have a desired property. Finally, the result on tall \( \epsilon \)-thick sets will be used to prove the clustering threshold for all sets of size at least \( (\ln n)^{\frac{3}{4} + \epsilon} \) in general.

We point out the need to make similar arguments twice, once with a height of \( h_e = \left( \frac{1}{2} + \epsilon \right) \frac{\ln \ln n}{\ln b} \), and once with a height of \( \frac{\ln n}{\ln b} \). It is possible to show directly that a.a.s. all short \( \epsilon \)-thick sets are not clusters, but the asymptotics of this argument will not work with a set of larger size. By splitting sets of height \( \frac{\ln n}{\ln b} \) up into both sets containing short \( \epsilon \)-thick sets and those that do not, this in turn implies that a tall \( \epsilon \)-thick sets are also a.a.s. not clusters. Again, the asymptotics of this argument will not work for a sets of larger size. Finally, we consider all sets with at least \( (\ln n)^{\frac{3}{4} + \epsilon} \) vertices. By splitting the class of such sets up into both those that contain tall \( \epsilon \)-thick sets and those that do not, we may finally show no clusters with at least \( (\ln n)^{\frac{3}{4} + \epsilon} \) vertices exist.

We begin by precisely stating the property of short \( \epsilon \)-thick sets that we are interested in, and showing when it occurs:

**Lemma 5.** For all \( \epsilon \) such that \( \frac{\ln c}{\ln b} > \epsilon > 0 \), a.a.s. for each short \( \epsilon \)-thick set \( M \), there exists a set \( M_1 \subset M \) such that \( |M_1| \geq \frac{3}{4} |M| \) and \( \forall v \in M_1, e(v, M) \leq \frac{\beta}{4} |M| \).

The proof of Lemma 5 may be found in the appendix. Lemma 5 implies that there are a.a.s. no short \( \epsilon \)-thick clusters. The stronger notion used here (that many vertices in \( M \) have at most \( \frac{\beta}{4} |M| \) edges) is necessary for later steps of the proof.

The next lemma is the first step in showing tall \( \epsilon \)-thick sets (of height \( \frac{\ln n}{\ln b} \)) are not clusters:

**Lemma 6.** For all \( \epsilon > 0 \), a.a.s. for every tall \( \epsilon \)-thick set \( T = M \cup K \), where \( M \) is a short \( \epsilon \)-thick set and \( K \) is a set with \( |K| \geq \frac{\beta}{4} |M| \) such that \( K \) does not intersect \( S(M, h_e) \), the complete set of height \( h_e = \left( \frac{1}{2} + \epsilon \right) \frac{\ln \ln n}{\ln b} \), there exists a set \( M_2 \subset M \) such that \( |M_2| \geq \frac{3}{4} |M| \) and \( \forall v \in M, e(v, K) \leq \frac{\beta}{4} |K| \).

The proof of Lemma 6 may be found in the appendix. Intuitively, Lemma 5 and Lemma 6 go together as follows: let \( T = M \cup K \) be a tall \( \epsilon \)-thick set containing a short \( \epsilon \)-thick set \( M \) as well as some other vertices \( K \). Lemma 5 implies that there are not enough edges within \( M \) for \( T \) to be internally dense, and Lemma 6 implies similarly that the edges from \( M \) to \( K \) are not sufficient for \( T \) to be internally dense. Taken together, this shows that a.a.s. there are no tall \( \epsilon \)-thick clusters:

**Lemma 7.** For all \( \epsilon \) such that \( \frac{\ln c}{\ln b} > \epsilon > 0 \), a.a.s. for every tall \( \epsilon \)-thick set \( T \), there exists a set \( T' \subset T \) such that \( |T'| > (\ln n)^{\frac{3}{4} + \epsilon} \) and \( \forall v \in T', e(v, T) \leq \frac{\beta}{2} |T| \).

Because it is required for the proof of later steps, Lemma 7 shows a property stronger than that of not being a cluster. The proof of Lemma 7 is found in the appendix, but is sketched as follows:

**Proof Sketch.** First, suppose \( T \) contains a short \( \epsilon \)-thick set \( M \). Adding vertices in \( S(M, h_e) \setminus M \) to \( M \) will preserve the property that \( M \) is a short \( \epsilon \)-thick set, so we may assume that \( M \) is maximal. In other words, we may assume the set \( T \setminus M \) does not intersect the complete set of height \( h_e, S(M, h_e) \). Set \( K = T \setminus M \), and Let \( m = |M| \) and \( k = |K| \). By Lemma 5, there is a set \( M_1 \subset M \) such that \( |M_1| \geq \frac{3}{4} m \) and \( \forall v \in M_1, e(v, M) \leq \frac{\beta}{4} m \). If \( k < \frac{\beta}{2} m \), then even if every edge from \( v \) to \( K \) exists, we will still have \( e(v, M \cup K) \leq \frac{\beta}{4} m + k \leq \frac{\beta}{2} m \leq \frac{\beta}{2} k \) for every \( v \in M_1 \), and hence may choose \( T' = M_1 \). If \( k \geq \frac{\beta}{4} m \),
then by Lemma 6, there is some set $M_2 \subset M$ such that $|M_2| > \frac{3}{4}m$ and $\forall v \in M_2$, $e(v, K) \leq \frac{\beta}{2} k$. Taking $T' = M_1 \cap M_2$, it follows that $|T'| \geq \frac{m}{2}$ and $e(v, M \cup K) \leq \frac{\beta}{2} m + \frac{\beta}{2} k = \frac{\beta}{2} t$ for every $v \in T'$.

In the case that $T$ contains no short $\epsilon$-thick clusters, every $v \in T$ has at most $o(t)$ vertices that are close to $v$, and the rest are far away and therefore these edges occur with low probability. It is thus not hard to show with a first moment argument that $T$ has the desired property. \hfill \Box

We are now ready to prove part b) of Theorem 1, which we now restate:

For all $\epsilon > 0$, there are a.a.s. no clusters with more than $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices.

The proof will occur in two steps. First, we give the following lemma:

Lemma 8. For all $\epsilon$ such that $\frac{\ln c}{4 \ln b} > \epsilon > 0$, there are a.a.s. no clusters $Q = T \cup R$ of size at least $(\ln n)^{\frac{1}{2} + \epsilon}$, where $T$ is a tall $\epsilon$-thick set.

The proof of Lemma 8 achieved using the first moment method, is found in the appendix. This lemma rules out the most likely type of potential cluster, leaving only sets with at least $(\ln n)^{\frac{1}{2} + \epsilon}$ vertices that contain no tall $\epsilon$-thick subsets. We now prove part b) of Theorem 1 by considering this case.

Proof. Clearly if the theorem holds for all $\epsilon$ such that $\frac{\ln c}{4 \ln b} > \epsilon > 0$, then it will hold for all $\epsilon > 0$, so we may assume $\epsilon < \frac{\ln c}{4 \ln b}$. Let $Q$ be a potential cluster of size $q \geq (\ln n)^{\frac{1}{2} + \epsilon}$. By Lemma 8, we may assume $Q$ contains no tall $\epsilon$-thick sets; that is, $Q$ contains no sets of height at most $h = \frac{(\ln n)^{\frac{1}{2}}}{\ln b}$ with at least $z = (\ln n)^{\frac{1}{2} + \epsilon}$ vertices. Hence, if we subdivide the vertices of $G$ into $\frac{q}{h}$ sets of height $h$, $Q$ must have fewer than $z$ vertices inside each set. This implies that for all $v \in Q$, at least $q - z > (\ln n)^{\frac{1}{2} + \epsilon}$ vertices in $Q$ are at height $h$ or more from $v$; for $Q$ to be a cluster, at least $\beta q - z > \frac{\beta}{2} q$ edges from $v$ to these distant vertices must exist. Let $X_v$ denote the number of edges from $v$ to $Q$ that have height more than $h$, and let $X_Q = \frac{1}{2} \sum_{v \in Q} X_v$ be the total number of such edges in $Q$. If $X_Q \leq \frac{\beta}{4} q^2$, then it follows that at least one vertex $v$ has $X_v \leq \frac{\beta}{2} q$, implying that $Q$ is not a cluster.

Since there are less than $\binom{\ln n}{2}$ edges in $Q$ with height more than $h$, each occurring with probability at most $e^{-h}$, $X_Q$ is stochastically dominated by the random variable $\text{Bin} \left( \binom{\ln n}{2}, e^{-h} \right)$, which has expected value $\binom{\ln n}{2} e^{-h} = o \left( q^2 \right)$. A concentration bound (see the appendix) therefore shows that

$$
\Pr \left( X_Q \geq \frac{\beta}{4} q^2 \right) \leq \exp \left( -\frac{\beta \ln c}{8} q^2 h \right).
$$

Now let $X$ be the number of sets $Q$ of size at least $(\ln n)^{\frac{1}{2} + \epsilon}$ such that $X_Q \geq \frac{\beta}{4} |Q|^2$. As noted above, it suffices to show that a.a.s. $X = 0$. Since there are less than $\binom{n}{q}$ choices of size $q$ for the set $Q$, we have

$$
E[X] \leq \sum_{q \geq (\ln n)^{\frac{1}{2} + \epsilon}} \binom{n}{q} \Pr \left( X_Q \geq \frac{\beta}{4} q^2 \right) \leq \sum_{q \geq (\ln n)^{\frac{1}{2} + \epsilon}} \exp \left( q \ln n - \frac{\beta \ln c}{8} q^2 h \right)
$$

Since $h = \frac{(\ln n)^{\frac{1}{2}}}{\ln b}$ and $q \geq (\ln n)^{\frac{1}{2} + \epsilon}$, it follows that $q \ln n = o \left( q^2 h \right)$ and the term in the exponential is negative. Hence, it is maximized when $q$ is minimized; that is, when $q = (\ln n)^{\frac{1}{2} + \epsilon}$. The above becomes

$$
E[X] \leq \sum_{q \geq (\ln n)^{\frac{1}{2} + \epsilon}} \exp \left( (\ln n)^{\frac{1}{2} + \epsilon} - \frac{\beta \ln c}{8 \ln b} (\ln n)^{\frac{3}{2} + 2\epsilon} \right)
$$

$$
\leq n \exp \left( -\frac{\beta \ln c}{10 \ln b} (\ln n)^{\frac{3}{2} + 2\epsilon} \right)
$$

which tends to zero as $n \to \infty$. By the first moment method, this implies that a.a.s. $X = 0$. \hfill \Box
7. Conclusions

This work deals with the question: what form do clusters take in real-world networks? In this case, we considered the existence of \((\alpha, \beta)\)-clusters in the CGA real-world random network model. We showed the existence of a size threshold of \((\ln n)^{\frac{1}{2}}\) for the existence of such clusters. As noted in Section I, the CGA model captures many of the properties observed in real-world networks. \((\alpha, \beta)\)-clusters capture a particular notion of clustering in real-world networks, in which clusters are denser than their surrounding neighbourhood, and in which clusters may overlap. Therefore, the choice of model and clustering definition seem valid for approaching the motivating question.

Thus, it is interesting to ask the extent to which the clustering threshold of \((\ln n)^{\frac{1}{2}}\) extends beyond the CGA model to random models for real-world networks in general. Can a threshold for \((\alpha, \beta)\)-clusters be observed in real-world data? Can other models for real-world networks be shown to have thresholds for cluster size? Do such size thresholds exist for other notions of clustering?

One goal of this work was to achieve our result analytically. Many real-world random network models are prohibitively hard to analyze, so simulation is often needed to establish the existence of desirable properties. We have avoided this approach and concentrated on analytic results.

Turning now to our result, an open question is whether or not a sharp threshold for \((\alpha, \beta)\)-clusters exists: we have not addressed the existence of \((\alpha, \beta)\)-clusters in the range \((\ln n)^{\frac{1}{2}+f(n)}\), where \(f(n) = o(1)\). Part a) of Theorem 1 shows the existence of complete clusters at each height less than \((\frac{1}{2} - \epsilon) \ln \ln n\), but a more complete treatment might consider the existence of other (i.e. non-complete) clusters of this size.

8. Appendix

This section gives the full proof of the results in this work. We begin by introducing some probabilistic tools that will be required.

8.1. Probabilistic Tools. We aim to characterize the asymptotic behaviour of \(G\) as the number of vertices \(n\) increases. Since \(G\) is only defined when \(n\) is a power of \(b\), it is more correct to let \(H\), the height of \(G\), increase, and take \(n = b^H\). However, little clarity is lost when taking asymptotics in relation to \(n\).

The binomial random variable given by the number of successes over \(r\) independent trials, each succeeding with probability \(p\), is denoted \(\text{Bin}(r, p)\).

The main probabilistic idea used is that of the first moment method: Let \(X\) be a non-negative random variable that takes integral values, with expected value \(E[X]\). If \(E[X] = o(1)\), then by Markov’s Inequality, \(\Pr(X \geq 1) = o(1)\). In other words, a.a.s. \(X = 0\). This technique will be used repeatedly to establish that events a.a.s. do not occur.

The following lemma gives a bound on the upper tail of a Binomial random variable:

**Lemma 9.** Let \(X = \text{Bin}(n, p)\) be a binomial random variable. Let \(t > 1\) and \(1 \leq s = \lfloor tpn \rfloor \leq n - 1\). Then

\[
\Pr(X \geq tpn) < \frac{t}{t-1} \binom{n}{s} p^s (1-p)^{n-s}
\]

A proof may be found in [14]. We will use a simplified form of this. Since \(\binom{n}{s} \leq \left(\frac{en}{s}\right)^s\) (which follows from Stirling’s approximation), Lemma 9 implies that if \(s \geq 2tn\), then
The final tool we will make use of is a Chernoff-type bound from Janson [26], which gives concentration bounds on a sum of independent Bernoulli random variables.

Lemma 10. Let \( X = X_1 + \ldots + X_k \), where the \( X_i \) are independent Bernoulli random variables with \( \Pr (X_i = 1) = p_i \). Let \( \mu = E[X] = \sum p_i \). Then for \( t \geq 0 \) we have:

\[
\Pr (X \geq \mu + t) \leq \exp \left( -\frac{t^2}{2(\mu + t/3)} \right)
\]

and

\[
\Pr (X \leq \mu - t) \leq \exp \left( -\frac{t^2}{2\mu} \right).
\]

A proof may be found in [26].

8.2. Proofs.

Lemma 3. Let \( m^* = \frac{\ln b}{\alpha \ln c} \) and \( h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln b} \). Then:

a: There are a.a.s. no externally sparse sets of size smaller than \( m^* \).

b: If \( M \) is a set of size \( m > m^* \) and height \( h \leq h^* \), then there exists a constant \( a > 0 \) such that

\[
\Pr (\mathcal{E}_2 (M)) > a.
\]

Proof. To prove a), first suppose \( M \) is a set of size \( m < m^* \). The event \( \mathcal{E}_2 (M) \) that \( M \) is externally sparse holds if there does not exist any vertex \( v \in V(G) \setminus M \) such that \( e(v, M) > \alpha m \). Since \( m = O(1) \), we have \( |V(G) \setminus M| \geq \frac{n}{2} \) for large enough \( n \). The probability of an edge from \( v \) to a vertex in \( M \) is at least \( p = e^{-\log_b n} = n^{-\log_b c} \). It follows that \( e(v, M) \) stochastically dominates the random variable \( \text{Bin}(m, p) \), and hence

\[
\Pr (e(v, M) > \alpha m) > p^{\alpha m}.
\]

\( \mathcal{E}(M) \) will hold only if all of the at least \( \frac{n}{2} \) vertices in \( V(G) \setminus M \) have \( \alpha m \) or fewer links, so we have

\[
\Pr (\mathcal{E}(M)) \leq \left( 1 - p^{\alpha m} \right)^{\frac{n}{2}}
\]

\[
\leq \exp \left( -\frac{1}{2} np^{\alpha m} \right)
\]

\[
= \exp \left( -\frac{1}{2} n^{1-\alpha \log_b c} \right).
\]

Because \( m < m^* \), \( n^{1-\alpha \log_b c} \) goes to zero as \( n \) increases, so this probability is exponentially small.

Now, we wish to show that a.a.s. for every set \( M \) of size less than \( m^* \), \( \mathcal{E}(M) \) does not hold. Let \( X \) be the number of clusters of size smaller than \( m^* \); since there are at most \( \binom{n}{m} \) sets of size \( m \), it follows from (8.4) that

\[
E[X] \leq \sum_{m < m^*} \binom{n}{m} \exp \left( -\frac{1}{2} n^{1-\alpha \log_b c} \right).
\]
For each $m < m^*$, $1 - \alpha m \log_b c > 0$, so the term
\[
\binom{n}{m} \exp \left( -\frac{1}{2} n^{1-\alpha m \log_b c} \right) \leq \exp \left( m \ln n - \frac{1}{2} n^{1-\alpha m \log_b c} \right) = o(1)
\]
Hence, each of the $O(1)$ terms inside $(8.5)$ is $o(1)$. It follows that $E[X] = o(1)$, so by the first moment method, a.a.s. $X = 0$.

This proves a).

To prove b), suppose $m > m^*$, and recall $h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln n}$. Let $S = S(M)$ be the minimum complete set containing $M$. To show that $E_2$ holds for $M$, we need to show that $\forall u \in S(M, h^*) \setminus S(M)$, $e(u, M) \leq \alpha m$. We wish to choose a new complete subset $S'$ that also contains $M$, and whose height $h'$ is large enough that the vertices in $S(M, h^*) \setminus S'$ are likely to have at most $\alpha m$ edges to vertices in $M$. More explicitly, for our choice of $h'$ and $S'$, let $A$ be the event that $\forall u \in S \setminus S(M)$, $e(u, M) \leq \alpha m$, and $B$ be the event that $\forall u \in S(M, h^*) \setminus S$, $e(u, M) \leq \alpha m$. Then the events $A$ and $B$ are disjoint and independent, and $E_2(M) = A \cap B$.

In particular, we will choose $h'$ to be larger than some constant $\delta = \delta(\alpha, b, c)$. If $m$ (and hence $h$) is increasing with $n$ then surely we have $h > \delta$. In this case, taking $S' = S(M)$ and $h' = h$, we have that $A$ trivially occurs. If $m$ and $h$ are constant, then $h'$ will also be a constant that is possibly larger than $h$. Since $S'$ is of constant size, there are a constant number of vertices in $S' \setminus S(M)$. Furthermore, each of these vertices will have at most $\alpha m$ neighbours in $M$ with some constant probability, because $M$ is of constant size. Hence, $A$ will occur with at least constant probability, say $\Pr(A) \geq \gamma$.

Thus, it will suffice to show the event $B$ occurs with at least constant probability. Given a vertex $u$ of height $j > h'$ from $M$, there is a uniform probability $c^{-j}$ of an edge between $u$ and a particular vertex in $M$. Let $X_u = \text{Bin}(m, c^{-j})$ be the number of such links. Since, $j > h' > \delta$, by taking $\delta = \delta(\alpha, b, c)$ to be large enough, we can require that $c^{-j} \leq \frac{\alpha}{2}$. Hence, $\alpha m > 2mc^{-j}$ and it follows from $(8.1)$ with $s = \alpha m$ that

\[
\Pr(X_u \geq \alpha m) < 2 \left( \frac{e}{\alpha c^{j}} \right)^{\alpha m}.
\]

Now let $R_j$ be the event that there exists a $u \in G$ of height $j > h'$ from $M$ such that $X_u \geq \alpha m$. There are fewer than $b^j$ such vertices, so by the union bound,

\[
\Pr(R_j) \leq b^j 2 \left( \frac{e}{\alpha c^{j}} \right)^{\alpha m}.
\]

$B$ occurs only if each of the disjoint, independent events $R_j$, $h' < j \leq h^*$, does not occur. Therefore

\[
\Pr(B) \geq 1 - \sum_{j > h'} \Pr(R_j) \geq 1 - 2e^{\alpha m} \left( \frac{e^{\alpha m}}{e^{\alpha m} - b} \right) \left( \frac{b}{e^{\alpha m}} \right)^{h'}
\]

Now, it suffices to show that

\[
2e^{\alpha m} \left( \frac{e^{\alpha m}}{e^{\alpha m} - b} \right) \left( \frac{b}{e^{\alpha m}} \right)^{h'} < \frac{1}{2}
\]

which is true when

\[
h' > \frac{4\alpha m (1 + \ln e) - \ln (e^{\alpha m} - b)}{\alpha m \ln c - \ln b}
\]
There are (\(m > m\) is already exposed.) Then a.a.s. at least \(4m(1 + \ln c) - \ln(e^{am} - b) = \frac{1 + \ln c}{\ln c}\) so that the right side of (8.7) does not depend on \(m\).

Hence, (8.6) gives that \(\Pr(B) \geq \frac{1}{2}\), which suffices to prove b). 

The next lemma proves (5.1):

**Lemma.** Let \(m^* = \frac{\ln h}{\alpha \ln c}\) and \(h^* = (\frac{1}{2} - \varepsilon) \frac{\ln \ln n}{\ln b}\). Let \(\mathcal{M}\) be a family of sets such that each set \(M \in \mathcal{M}\) is of size \(m > m^*\) and height \(h \leq h^*\). Furthermore, suppose for every \(M_1, M_2 \in \mathcal{M}\), the complete sets of height \(h^*, S(M_1, h^*)\) and \(S(M_2, h^*)\), do not intersect, and no edges between \(S(M_1, h^*)\) and \(S(M_2, h^*)\) have yet been exposed. (For each \(M \in \mathcal{M}\), we allow any number of internal edges in \(S(M, h^*)\) to have been already exposed.) Then a.a.s. at least

\[
\min \left(|\mathcal{M}|, (\ln n)^{\frac{\alpha \ln n}{\ln b} (m - m^*)}\right)
\]

sets in \(\mathcal{M}\) have \(E_3\).

**Proof.** Consider a set \(M \in \mathcal{M}\) so that \(m = |M|\). Given a vertex \(u \notin M\) of height \(j > h^*\) from \(M\), there is a probability \(c^{-j}\) of an edge between \(u\) and a particular vertex in \(M\). Let \(X_u = \text{Bin}(m, c^{-j})\) be the total number of such edges. Since \(c^{-j} = o(1), \alpha m > 2mc^{-j}\) and it follows from (8.1) with \(s = \alpha m\) that

\[
\Pr(X_u \geq \alpha m) < 2 \left(\frac{e}{\alpha c^j}\right)^{\alpha m}.
\]

Now let \(A_j\) be the event that there exists a \(u \in G\) of height \(j > h^*\) from \(M\) such that \(X_u \geq \alpha m\). There are \((b - 1)b^{j-1} \leq b^j\) such vertices, so by the union bound,

\[
\Pr(A_j) \leq b^j 2 \left(\frac{e}{\alpha c^j}\right)^{\alpha m}.
\]

\(E_3(M)\) occurs if and only if each of the disjoint, independent events \(A_j, j > h^*\), does not occur. Therefore, since \(m > m^*\) implies that \(\frac{b}{e^{am}} < 1\), we have by the union bound that

\[
\Pr(E_3(M)) \geq 1 - \sum_{j > h^*} \Pr(A_j)
\]

\[
= 1 - 2 e^{am} \sum_{j > h^*} \left(\frac{b}{e^{am}}\right)^j
\]

\[
\geq 1 - \frac{2e^{am} (\frac{b}{e^{am}})^{h^*}}{1 - (\frac{b}{e^{am}})}
\]

(8.8)

Now, considering the term \(e^{am} (\frac{b}{e^{am}})^{h^*} = b^{h^*} (\frac{e}{c^h})^{am}\) in the numerator of (8.8), we have:

\[
b^{h^*} (\frac{e}{c^h})^{am} = \exp(\alpha m + h^* (\ln b - \alpha m \ln c))
\]

\[
= \exp\left(\alpha m - \frac{1}{2} - \varepsilon\right) \frac{\ln c}{\ln b} (m - m^*) \ln \ln n
\]

\[
\leq \exp\left(-\frac{\alpha \ln c}{2.5 \ln b} (m - m^*) \ln \ln n\right).
\]
Since $1 - \left( \frac{b}{c^m} \right)$ is a constant, we have from (8.3) that

$$\Pr \left( \mathbf{E}_3 \ (M) \right) \geq 1 - \exp \left( -\frac{\alpha \ln c}{3 \ln b} (m - m^*) \ln \ln n \right). \tag{8.9}$$

Because removing edges only increases the probability that $\mathbf{E}_3$ occurs, it is a monotone property. More precisely, let $G_1$ and $G_2$ be graphs on the same vertex set $V$ and let $E(G_1) \subset E(G_2)$. For any set $M \subset V$, if $\mathbf{E}_3 \ (M)$ holds in $G_2$ then it also holds in $G_1$. Hence, for two sets $M_1$ and $M_2$, Proposition 6.3.1 in [7] implies that $\mathbf{E}_3 \ (M_1)$ and $\mathbf{E}_3 \ (M_2)$ are positively correlated; that is:

$$\Pr \left( \mathbf{E}_3 \ (M_1) \cap \mathbf{E}_3 \ (M_2) \right) \geq \Pr \left( \mathbf{E}_3 \ (M_1) \right) \Pr \left( \mathbf{E}_3 \ (M_2) \right). \tag{8.10}$$

Now, let $M_1, M_2, \ldots$ be any ordering of the sets in $\mathcal{M}$. We will expose the edges necessary for $\mathbf{E}_3 \ (M_i)$ using this order. (8.10) implies that for any $i$,

$$\Pr \left( \mathbf{E}_3 \ (M_i) \big| \cap_{j<i} \mathbf{E}_3 \ (M_j) \right) \geq \Pr \left( \mathbf{E}_3 \ (M_i) \right).$$

The union bound and (8.9) thus implies that for some value $k$,

$$\Pr \left( \cap_{i<k} \mathbf{E}_3 \ (M_i) \right) \geq 1 - \sum_{i<k} \Pr \left( -\mathbf{E}_3 \ (M_i) \right) \geq 1 - k \exp \left( -\frac{\alpha \ln c}{3 \ln b} (m - m^*) \ln \ln n \right)$$

Hence, if

$$\ln k < \frac{\alpha \ln c}{3 \ln b} (m - m^*) \ln \ln n$$

then it follows that a.a.s. every set $M_i$ for $1 \leq i \leq k$ has the property $\mathbf{E}_3 \ (M_i)$. Taking $k = \min \left( |\mathcal{M}|, (\ln n)^{\frac{\alpha \ln c}{3 \ln b} (m - m^*)} \right)$ proves the lemma. \qed

In the proof of [4], the expected number of sets with $D \cap E_1 \cap E_2$ is $\frac{qn}{b^{h^*}} = \Omega \left( n^{3/4} \right)$. Setting $t = n^{2/3}$ in Lemma [10] gives that a.a.s. there is a subset $\mathcal{M}' \subset \mathcal{M}$ of size at least $\Omega \left( n^{3/4} - n^{2/3} = \Omega \left( n^{3/4} \right) \right)$ such that each set in $\mathcal{M}'$ has $D \cap E_1 \cap E_2$.

Next, we give the proof of Theorem 1a) in more detail, with the omitted calculations inserted:

**Theorem 1a.** Let $m^* = \frac{\ln b}{\alpha \ln c}$. There a.a.s. exists clusters of size larger than $(\ln n)^{\frac{1}{2} - \epsilon}$ in $G$. Moreover, there exists a constant $\gamma = \gamma (\alpha, b, c) > 0$ such that for each $h$ satisfying $\log_b m^* < h \leq \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln b}$, there a.a.s. exists at least $(\ln n)^{b^h}$ complete clusters of size $b^h$.

**Proof.** Note that the second part of the theorem statement implies the first, because a complete set of height $h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln b}$ has $b^{h^*} = (\ln n)^{\frac{1}{2} - \epsilon}$ vertices. Let $M$ be a complete set of height $h$. We will first examine the event $D$ that $M$ is internally dense.

In fact, let us consider the stronger event of $M$ being a clique. Each potential edge in $M$ occurs with probability at least $c^{-h}$. so the probability of all $(b^h)$ edges occurring is at least

$$\left( c^{-h} \right)^{(b^h)} \geq c^{-hb^{2h}}$$

Hence, $\Pr (D) \geq \exp (-\ln (c) hb^{2h})$.

Now, partition $G$ into complete sets of height $h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln b}$. From each such set, choose a single complete set of height $h$, and let these $n/b^{h^*}$ disjoint sets form the family $\mathcal{M}$. Because these sets are complete, they automatically have $E_1$. Because they do not overlap and lie in different sets of height $h^*$,
they will have $D$ and $E_2$ independently. Lemma 3b) implies they have $E_2$ with probability at least $a$ for some positive constant $a$. Hence, for every $M \in \mathcal{M}$, we have that

$$\Pr (D \cap E_1 \cap E_2) \geq \Pr (D) \Pr (E_1) \Pr (E_2)$$

(8.11)

Now, the number of sets in $\mathcal{M}$ with $D \cap E_1 \cap E_2$ stochastically dominates the random variable $X = \text{Bin} \left( \frac{n}{b^M}, a \exp \left( - \ln (c) hb^{2h} \right) \right)$. Since $h \leq h^*$, we have that

$$E[X] \geq \frac{n}{b^M} a \exp \left( - \ln (c) hb^{2h} \right) = n \exp \left( \ln a - \ln (b) h^* - \ln (c) hb^{2h} \right) \geq n \exp \left( -h^* b^{2h^*} \right)$$

Since $h^* = \left( \frac{1}{2} - \epsilon \right) \frac{\ln \ln n}{\ln b}$, this becomes

$$E[X] \geq \frac{n}{\left( \frac{1}{2} - \epsilon \right)^2 \left( \ln \ln n \right)^2} \frac{n}{\ln (n) \left( \ln n \right)^{1-2\epsilon}}$$

Notice that $E[X]$ is asymptotically larger than $\frac{n}{\ln n}$. Setting $t = n^{2/3}$ so that $E[X] - t \geq \frac{n}{\ln n}$, Lemma 10 gives that

$$\Pr \left( X < \frac{n}{\ln n} \right) \leq \exp \left( \frac{n^{4/3}}{2E[X]} \right) \leq \exp \left( n^{-1/3} \right)$$

In other words, a.a.s. there are at least $\frac{n}{\ln n}$ sets in $\mathcal{M}$ with $D \cap E_1 \cap E_2$. Let $\mathcal{M}' \subset \mathcal{M}$ be this sub-family of sets. Because each set in $\mathcal{M}'$ lies in a different complete set of height $h^*$, and we have only exposed edges inside these complete sets of height $h^*$, the conditions of Lemma 7 apply to $\mathcal{M}'$. Hence at least $(\ln n)^{\gamma b^h} \alpha c (\beta - m^*)$ sets in $\mathcal{M}'$ also have $E_3$, and therefore are clusters.

It remains to find a constant $\gamma = \gamma (\alpha, b, c) > 0$ such that there are a.a.s. at least $(\ln n)^{\gamma b^h}$ clusters of size $b^h$. Recall that $m^* = \frac{\ln b}{\alpha \ln c}$, and let $h_{\text{min}}$ be the minimum integral value of $h$ such that $b^h > m^*$; setting

$$\gamma = \frac{\alpha \ln c (\beta h_{\text{min}} - m^*)}{4 \ln b} \frac{b^{h_{\text{min}}}}{b^h}$$

guarantees that $(\ln n)^{\gamma b^h} \leq (\ln n)^{\frac{\ln c}{4 \ln b} (\beta h_{\text{min}} - m^*)}$, and hence there are a.a.s. at least $(\ln n)^{\gamma b^h}$ clusters of height $h$ in $G$. \hfill \square

**Lemma 5.** For all $\epsilon$ such that $\frac{\ln c}{4 \ln b} > \epsilon > 0$, a.a.s. for each short $\epsilon$-thick set $M$, there exists a set $M_1 \subset M$ such that $|M_1| = \frac{3}{4} |M|$ and $\forall v \in M_1$, $e(v, M) \leq \frac{\beta}{4} |M|$.

**Proof.** Let $m = |M|$, and let $S = S (M, h_\epsilon)$ be the complete set of height $h_\epsilon = (\frac{1}{2} + \epsilon) \frac{\ln n}{\ln b}$ containing $M$. If $S$ contains less than $\frac{\beta}{4} m^2$ internal edges, then it follows that there can be no set $M' \subset M$ of size $|M'| \geq \frac{m}{4}$ such that $\forall v \in M'$, $e(v, M) \geq \frac{\beta}{4} m$. Hence, it suffices to show that a.a.s. all complete sets of height $h_\epsilon$ have less than $\frac{\beta}{4} m^2$ internal edges.

Consider a complete set of height $j$. It contains $b$ sets of height $j - 1$, each of which contains $b^{j-1}$ vertices. Hence, there are $\left( \frac{1}{2} \right) b^j$ potential edges of height $j$ in a single complete set of height $j$. Let
the actual number of such edges be given by \( X_j \); since each occurs with probability \( c^{-j} \), we have that 

\[
E[X_j] = \left( \frac{b}{2} \right) b^{2(j-1)} c^{-j},
\]

Now, let \( X_S \) be the number of edges in \( S \). We wish to show that \( E[X_S] = O(\ln n) \). Since there are \( b^{h_c-j} \) complete sets of height \( j \) in \( S \), we have that 

\[
E[X_S] = \sum_{j=1}^{h_c} b^{h_c-j} E[X_j] \]

(8.12)

The sum in this expression is bounded as follows:

\[
\sum_{j=1}^{h_c} \left( \frac{b}{c} \right)^j \leq \begin{cases} 
\frac{c}{c-b} & \text{if } b < c \\
\frac{1}{h_c} & \text{if } b = c \\
b - b \left( \frac{b}{c} \right)^{h_c+1} & \text{if } b > c
\end{cases}
\]

In the first two cases, this sum is small enough that (8.12) combined with the fact that \( h_c = (\ln n)^{\frac{1}{2} + \epsilon} \) easily gives that \( E[X_S] = O(\ln n) \). In the case that \( b > c \), we may rewrite (8.12) as follows:

\[
E[X_S] \leq \frac{b(b-1)}{2c(b-c)} b^{h_c} \left( \frac{b}{c} \right)^{h_c} = O \left( b^{h_c} \left( \frac{b}{c} \right)^{h_c} \right)
\]

(8.13)

Now, since \( h_c = \left( \frac{1}{2} + \epsilon \right) \frac{\ln \ln n}{\ln b} \), we have \( b^{h_c} = (\ln n)^{\frac{1}{2} + \epsilon} \) and \( \left( \frac{b}{c} \right)^{h_c} = (\ln n)^{(\frac{1}{2} + \epsilon)(1 - \frac{\ln c}{\ln b})} \). Since \( \epsilon < \frac{\ln c}{4 \ln b} \), it follows that \( (1 + 2c)(1 - \frac{\ln c}{\ln b}) < 1 \), so

\[
b^{h_c} \left( \frac{b}{c} \right)^{h_c} = (\ln n)^{(1+2c)(1 - \frac{\ln c}{\ln b})} \leq \ln n.
\]

Hence, from (8.13) we have \( E[X_S] = O(\ln n) \) holds as well when \( b > c \), and therefore holds in all cases.

We now use this bound on the expected value of \( X_S \) to bound the probability that \( X_S \geq \frac{\beta}{32} m^2 = \frac{\beta}{32} (\ln n)^{1 + \frac{\epsilon}{4}} \). Set \( t = (\ln n)^{(1 + \frac{\epsilon}{4})} \), so that \( E[X_S] + t < 2(\ln n)^{1 + \frac{\epsilon}{4}} \leq \frac{\beta}{32} m^2 \), since \( m \geq (\ln n)^{\frac{1}{2} + \frac{\epsilon}{4}} \). Applying Lemma 10 with this \( t \), we have

\[
\Pr \left( X_S \geq \frac{\beta}{32} m^2 \right) \leq \exp \left( -\frac{1}{4} (\ln n)^{1 + \frac{\epsilon}{4}} \right).
\]

There are \( \frac{h_c}{\ln c} \leq n \) complete sets of height \( h_c \). Letting \( X \) denote the number of such sets with at least \( \frac{\beta}{32} m^2 \) internal edges, it follows that

\[
E[X] = n \Pr \left( X_S \geq \frac{\beta}{32} m^2 \right) \leq \exp \left( \ln n - \frac{1}{4} (\ln n)^{1 + \frac{\epsilon}{4}} \right).
\]

This goes to zero as \( n \to \infty \), showing that a.a.s. \( X = 0 \). \( \square \)
Lemma 6. For all \( \epsilon > 0 \), a.a.s. for every tall \( \epsilon \)-thick set \( T = M \cup K \), where \( M \) is a short \( \epsilon \)-thick and \( K \) is a set with \( |K| > \frac{\beta}{2} |M| \) such that \( K \) does not intersect \( S(M, h_\epsilon) \), the complete set of height \( h_\epsilon = \left( \frac{1}{2} + \epsilon \right) \frac{\ln n}{\ln b} \), there exists a set \( M_2 \subset M \) such that \(|M_2| > \frac{\beta}{4} |M| \) and \( \forall v \in M \), \( e(v, K) \leq \frac{\beta k}{2} \) for large enough \( n \). Setting \( t = \frac{\beta k}{4} \), by Lemma 10 we have that

\[
\Pr \left( e(v, K) > \frac{3}{8} t \right) \leq \exp \left( - \frac{3}{8} t \right).
\]

(8.14)

Now, let \( M_2 = \left\{ v \in M : e(v, K) \leq \frac{\beta k}{2} \right\} \), and let \( X_{M,K} = |M - M_2| \). We wish to show that a.a.s. for all valid choices of \( M \) and \( K \), \( X_{M,K} \leq \frac{m}{4} \). Let \( p_{M,K} \) denote the probability that \( X_{M,K} > \frac{m}{4} \). From (8.14), it follows that \( X_{M,K} \) is stochastically dominated by the random variable \( \text{Bin} \left( m, \exp \left( - \frac{\beta k}{16} \right) \right) \). Therefore, by (8.2), setting \( s = \frac{\beta m}{4} \), we have that

\[
p_{M,K} \leq 2 \exp \left( \frac{m}{4} \left( \ln m + 1 - \ln \left( \frac{m}{4} \right) - \frac{\beta k}{16} \right) \right)
\]

\[
\leq \exp \left( - \frac{\beta}{100} mk \right).
\]

Now, let \( S \) be a complete set of height \( \frac{(\ln n)^{1/2}}{\ln b} \); we will count the expected number \( X_S \) of tall \( \epsilon \)-thick sets \( T = M \cup K \) inside \( S \) such that \( X_{M,K} > \frac{m}{4} \). We have that \( E[X_S] \leq \sum_{M,K} p_{M,K} \), where this sum ranges over all valid choices of \( T = M \cup K \) inside \( S \). Fixing the set \( M \) and a size \( k \), we have that there are at most

\[
\binom{|S|}{k} = \binom{(\ln n)^{1/2}}{k} \leq \exp \left( k \left( \ln n \right)^{1/2} \right)
\]

sets \( K \) of size \( k \); hence,

\[
E[X_S] \leq \sum_{M} \sum_{k} \exp \left( k \left( \ln n \right)^{1/2} - \frac{\beta}{100} m \right).
\]

Since \( m \geq (\ln n)^{1/2} \), the term in the exponential is negative, and hence is maximized when \( k \) is minimized; that is, when \( k = \frac{\beta}{4} m \). Furthermore, the height of \( K \) is at most \( \frac{(\ln n)^{1/2}}{\ln b} \), so \( k \leq \exp \left( (\ln n)^{1/2} \right) \). Hence,

\[
E[X_S] \leq \sum_{M} \sum_{k} \exp \left( \frac{\beta m}{4} \left( \ln n \right)^{1/2} - \frac{\beta}{100} m \right)
\]

\[
\sum_{M} \exp \left( (\ln n)^{1/2} \right) \exp \left( \frac{\beta m}{4} \left( \ln n \right)^{1/2} - \frac{\beta}{100} m \right)
\]

\[
\sum_{M} \exp \left( - \frac{\beta^2}{500} m^2 \right)
\]

since \( (\ln n)^{1/2} = o(m) \).
Now, within $S$ there are $b \binom{\ln n}{1/2}^{1/2} \leq \exp \left( (\ln n)^{1/2} \right)$ complete sets of height $h_e$, and each has at most ${\ln n \choose 1/2}^{1/2} \leq \exp (m \ln n)$ subsets $M$ of size $m$, so there are at most $\exp \left( (\ln n)^{1/2} + m \ln \ln n \right) \leq \exp (2m \ln \ln n)$ choices total for the set $M$ within $S$. Hence,

$$E[X_S] \leq \sum_m \exp (2m \ln n) \exp \left( -\frac{\beta^2 m^2}{500} \right) \leq \sum_m \exp \left( -\frac{\beta^2 m^2}{1000} \right),$$

since $m \ln \ln n = o \left( m^2 \right)$. Again, the term in the sum is maximized when $m$ is minimized; since $(\ln n)^{1/2 + \epsilon} \leq m \leq (\ln n)^{1/2 + \epsilon}$, it follows

$$E[X_S] \leq (\ln n)^{1/2 + \epsilon} \exp \left( -\frac{\beta^2}{2000} (\ln n)^{1/2 + \epsilon} \right) \leq \exp \left( -\frac{\beta^2}{2000} (\ln n)^{1/2 + \epsilon} \right).$$

(8.15)

Now, there are fewer than $n$ choices for the complete set $S$. Let $X = \sum_S X_S$ denote the total number of tall $\epsilon$-thick sets $T = M \cup K$ such that $X_{M,K} > \frac{m}{t}$. Then $E[X] \leq nE[X_S]$. Since $\ln n = o \left( (\ln n)^{1/2 + \epsilon} \right)$, (8.15) implies that $E[X] = o(1)$, and hence a.a.s. $X = 0$. \hfill \Box

**Lemma 7.** For all $\epsilon$ such that $\frac{\ln n}{\ln n} \epsilon > 0$, a.a.s. for every tall $\epsilon$-thick set $T$, there exists a set $T' \subset T$ such that $|T'| > (\ln n)^{1/2 + \epsilon}$ and $\forall v \in T'$, $e(v, T) \leq \frac{\beta}{\beta} |T|$.

**Proof.** We consider two cases: either $T$ contains a short $\epsilon$-thick set as a subset, or it does not. Let $t = |T|$.

First, suppose $T$ contains a short $\epsilon$-thick set $M$. Adding vertices in $S(M, h_e) \setminus M$ to $M$ will preserve the property that $M$ is a short $\epsilon$-thick set, so we may assume that $M$ is maximal. In other words, we may assume the set $T \setminus M$ does not intersect the complete set of height $h_e$, $S(M, h_e)$. Set $K = T \setminus M$, and let $m = |M|$ and $k = |K|$. By Lemma 5 there is a set $M_1 \subset M$ such that $|M_1| \geq \frac{3}{2}m$ and $\forall v \in M_1$, $e(v, M) \leq \frac{\beta}{\beta} m$. If $k < \frac{3m}{4}$, then even if every edge from $v$ to $K$ exists, we will still have $e(v, M \cup K) \leq \frac{\beta}{4} m + k \leq \frac{\beta}{2} (m + k)$ for every $v \in M_1$, and hence may choose $T' = M_1$. If $k \geq \frac{3m}{4}$, then by Lemma 6, there is some set $M_2 \subset M$ such that $|M_2| \geq \frac{3}{2}m$ and $\forall v \in M_2$, $e(v, K) \leq \frac{\beta}{4} k$. Taking $T' = M_1 \cap M_2$, it follows that $|T'| \geq \frac{m}{4}$ and $e(v, M \cup K) \leq \frac{\beta}{4} m + \frac{\beta}{4} k = \frac{\beta}{2} t$ for every $v \in T'$.

Otherwise, suppose $T$ contains no short $\epsilon$-thick sets. That is, each set of height $h_e = (1/2 + \epsilon) \frac{\ln n}{\ln n}$ can contain at most $\binom{\ln n}{1/2}^{1/2} \leq \frac{\beta}{4} t$ vertices. For any $v \in T$, there are at least $\frac{3\beta}{4} t$ vertices in $T$ with height more than $h_e$ from $v$. Let $X_v$ denote the number of edges from $v$ to $T$ of height more than $h_e$. If $X_v \leq \frac{\beta}{4} t$, then certainly $e(v, T) \leq \frac{\beta}{4} t$. Now, set $X_T = \frac{1}{2} \sum_{v \in T} X_v$ to be the total number of edges of height more than $h_e$ in $T$. If $X_T < \frac{\beta}{10} t^2$, then by a counting argument there must be a set $T' \subset T$ of size at least $\frac{1}{2} t$ such that $\forall v \in T'$, $X_v \leq \frac{\beta}{4} t$. Hence, it suffices to show that a.a.s. for all $T$, $X_T < \frac{\beta}{10} t^2$.

Since there are less than $\left( \frac{1}{2} \right)$ edges of height greater than $h_e$ in $T$, each occurring with probability at most $e^{-h_e}$, $X_T$ is stochastically dominated by the random variable Bin \left( \left( \frac{1}{2} \right), e^{-h_e} \right)$. Since \left( \frac{1}{2} \right) e^{-h_e} =
We may assume \( o(t^2) \), we may apply (8.2) with \( s = \frac{\beta}{16} t^2 \), giving

\[
\Pr \left( X_T \geq \frac{\beta}{16} t^2 \right) \leq 2 \exp \left( \frac{\beta}{16} t^2 \left( \ln \left( \frac{t}{2} \right) + 1 - \ln \left( \frac{\beta}{16} t^2 \right) - h_c \ln c \right) \right)
\]

\[
\leq \exp \left( - \frac{\beta \ln c}{32} t^2 \right).
\]

Finally, let \( X \) denote the number of sets \( T \) such that \( X_T \geq \frac{\beta}{16} t^2 \). Then since there are at most 

\[
\left( \exp \left( \frac{\ln n}{t} \right) \right) \leq \exp \left( t (\ln n)^{1/2} \right)
\]

sets \( T \) of size \( t \) contained within a single complete set of height \( \frac{\ln n}{\ln b} \), and there are less than \( n \) such complete sets, we have that

\[
E[X] \leq n \sum_{t=(\ln n)^{1/2}}^{n} \exp \left( (t (\ln n)^{1/2} - \frac{\beta \ln c}{32} (\ln n)^{3/2} h_c) \right)
\]

and adding from the vertices in \( T \) to \( T \) will preserve the property that \( T \) is a tall \( \epsilon \)-thick set. Hence, we may assume \( T \) is maximal; that is, that the vertices in \( R \) are at height at least \( \frac{\ln n}{\ln b} \) from the vertices in \( T \). By Lemma 7, there exists a set \( T' \subset T \), \( |T'| \geq (\ln n)^{2+\epsilon} \), such that \( e(v, T) \leq \frac{\beta}{16} t^2 \) for every \( v \in T' \). For \( T' \cup R \) to be a cluster, we must have that \( e(v, T' \cup R) \geq \beta (t + r) \). Hence, for \( v \in T' \), it follows that \( e(v, R) \geq \frac{\beta}{2} t + \beta r \). This implies first that \( r \geq \frac{\beta}{2} t \); and second, that we must have \( e(v, R) \geq \beta r \) for every \( v \in T' \). The remainder of the proof will show that a.a.s. no sets \( T' \cup R \) satisfy this condition.

Since \( v \in T' \) is height at least \( \frac{\ln n}{\ln b} \) from each vertex in \( R \), \( e(v, R) \) is stochastically dominated by the random variable \( \text{Bin} \left( r, \exp \left( - \frac{\ln c}{\ln b} (\ln n)^{1/2} \right) \right) \). By (8.2) with \( s = \beta r \), we have

\[
\Pr \left( e(v, R) \geq \beta r \right) \leq 2 \exp \left( \beta r \left( 1 - \ln \beta - \frac{\ln c}{\ln b} (\ln n)^{1/2} \right) \right)
\]

\[
\leq \exp \left( \frac{\beta \ln c \cdot \beta r}{2 \ln b} (\ln n)^{1/2} \right).
\]

Since the edges of \( T' \subset T \) are independent of the edges between \( T \) and \( R \), this is true independently for all vertices \( v \in T' \). Since \( |T'| \geq (\ln n)^{2+\epsilon} \), the probability \( p_T \) that \( e(v, R) \geq \beta r \) holds for all \( v \in T' \) is therefore bounded.
Now, let $X$ denote the total number of clusters of the form $T \cup R$. There are at most $(n \binom{r}{t}) \leq \exp (r \ln n)$ sets of size $r$, so

$$E[X] \leq \sum_{T} \sum_{r=\frac{\beta t}{2}}^{n} \binom{n}{r} p_T \leq \sum_{T} \sum_{r} \exp \left( r \left( \ln n - \frac{\beta \ln c}{2 \ln b} \left( \ln n \right)^{1+\frac{\epsilon}{4}} \right) \right).$$

The term in the exponential is negative, and therefore maximized when $r = \frac{\beta t}{2}$. For a fixed size $t$, there are less than $n \left( e^{(\ln n)^{\frac{1}{2}}} \right) \leq n \exp \left( t \left( \ln n \right)^{\frac{1}{2}} \right)$ choices for the tall $\epsilon$-thick set $T$, and hence

$$E[X] \leq \sum_{T} \exp \left( \frac{\beta t}{2} \left( \ln n - \frac{\beta \ln c}{8 \ln b} \left( \ln n \right)^{1+\frac{\epsilon}{4}} \right) \right) \leq \sum_{t=(\ln n)^{\frac{1}{2}}}^{n} \exp \left( t \left( \ln n \right)^{\frac{1}{2}} - \frac{\beta^2 \ln c}{8 \ln b} \left( \ln n \right)^{1+\frac{\epsilon}{4}} \right).$$

Since $t \left( \ln n \right)^{\frac{1}{2}} = o \left( t \left( \ln n \right)^{1+\frac{\epsilon}{4}} \right)$, this exponential term is decreasing and hence maximized for the minimum value of $t$; that is, for $t = (\ln n)^{\frac{1}{2}}$. Thus

$$E[X] \leq \sum_{t=(\ln n)^{\frac{1}{2}}}^{n} \exp \left( \frac{\beta^2 \ln c}{10 \ln b} \left( \ln n \right)^{\frac{3}{2}+\frac{\epsilon}{4}} \right) \leq n \exp \left( -\frac{\beta^2 \ln c}{10 \ln b} \left( \ln n \right)^{\frac{3}{2}+\frac{\epsilon}{4}} \right).$$

Since $\ln n = o \left( \left( \ln n \right)^{\frac{3}{2}+\frac{\epsilon}{4}} \right)$ we have that $E[X] = o \left( 1 \right)$, implying that a.a.s. $X = 0$. □

Finally, we fill a detail from the proof of Theorem 1b). To get (6.1), we use (8.2) with $s = \frac{2}{\epsilon} q^2$, giving:

$$\Pr \left( X_Q \geq \frac{\beta}{4} q^2 \right) \leq 2 \exp \left( \frac{\beta}{4} q^2 \left( \ln \left( \frac{q}{2} \right) + 1 - \ln \left( \frac{\beta}{4} q^2 \right) - h \ln c \right) \right) \leq \exp \left( -\frac{\beta \ln c}{8} q^2 h \right).$$
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