ON THE $p$-SCHATTEN ENERGY OF BIPARTITE GRAPHS

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Abstract. We give a Coulson integral formula and a Coulson–Jacobs formula for the $p$-Schatten energy of bipartite graphs. We use these formulas to compare the $p$-Schatten energy of different trees by using a quasiorder, establishing the maximality of paths and minimality of stars among all trees.

1. Introduction

For a graph $G = (V, E)$, we denote by $A_G$ the adjacency matrix of $G$. Let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, and $\phi(G, x)$ its characteristic polynomial. The $p$-Schatten energy of $G$ is defined by the formula

$$E_p(G) = \sum_{i=1}^{n} |\lambda_i|^p.$$

The purpose of this short note is to give an integral formula for the $p$-Schatten energy of a bipartite graph. In other words, we give a formula for the $p$-Schatten norm of its adjacency matrix.

THEOREM 1. Let $G$ be a bipartite graph on $n$ vertices. For $0 < p < 2$, the following integral formula holds:

$$E_p(G) = \frac{2 \sin \left( \frac{p\pi}{2} \right)}{\pi} \int_{0}^{\infty} z^{p-1} \left( n - iz \frac{\phi'(G, iz)}{\phi(G, iz)} \right) dz.$$

This formula generalizes the famous Coulson integral formula [1] for the energy (corresponding to $p = 1$) in the case of a bipartite graph, which has

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been broadly used in mathematical chemistry. One important feature of Coulson integral formula is that it has been proved to be very effective to compare the energies of different trees. Our formula allows to extend this feature for any \(0 < p < 2\). In particular we use such comparison to answer the following question of Nikiforov [9, Question 4.52.(b)].

**Theorem 2.** Let \(S_n\) be the star graph, \(P_n\) the path graph, and \(T_n\) a tree on \(n\) vertices. Then, for \(0 < p < 2\),

\[
\mathcal{E}_p(S_n) \leq \mathcal{E}_p(T_n) \leq \mathcal{E}_p(P_n).
\]

We note that this comparison theorem is in contrast with the result of Csikvári [3] proving that that for \(p \in 2\mathbb{N}\),

\[
\mathcal{E}_p(S_n) \geq \mathcal{E}_p(T_n) \geq \mathcal{E}_p(P_n).
\]

The general case \(p > 2\) is still open. See also Lovász and Pelikán [8] for the case \(p = \infty\).

While writing this paper we became aware of the papers [4,10,11] which provide formulas which are similar to (1). The method of proof here resembles the ones used in [4]. Since the formulas in [4] have no restriction on \(p\), it would be interesting to explore if their formulas allow to prove (3) for all \(p > 2\).

**2. A Coulson integral formula**

In this section, we will give a proof of Theorem 1. That is, we find a Coulson-like integral formula for the \(p\)-Schatten energy of a bipartite graph. We will consider the case where the number of vertices is even, the odd case follows from the even case. Indeed, on the one hand, the \(p\)-Schatten energy does not change when adding isolated vertices, since we only add 0’s to the spectrum. On the other hand, by noticing that, if \(G \cup \{v\}\) denotes the graph \(G\) together with an isolated vertex, then

\[
\frac{z(\phi'(G \cup \{v\}, z))}{\phi(G \cup \{v\}, z)} = \frac{z(z\phi(G, z))'}{z\phi(G, z)} = \frac{z\phi(G, z)' + \phi(G, z)}{\phi(G, z)} = \frac{z\phi'(G, z)}{\phi(G, z)} + 1.
\]

Thus, we may deduce that the right-hand side of (1) is also not affected by adding an isolated vertex.

For a graph \(G\), if \(A_G\) is its adjacency matrix, then \(\phi(G, z) = \det(zI - A_G)\) which in terms of the eigenvalues of \(A_G\) is given by \(\phi(G, z) = \prod_{j=1}^{n}(z - \lambda_j)^{\alpha_j}\). We will use the following identity which is easy to prove

\[
\sum_{j=1}^{n} \frac{1}{z - \lambda_j} = \frac{d}{dz} \left[ \log(\phi(z)) \right] = \frac{\phi'(G, z)}{\phi(G, z)}.
\]

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Now, we may proceed to prove Theorem 1. We start from the elementary identity
\[
\frac{a}{it + a} - \frac{a}{it - a} = \frac{a(it - a) - a(it + a)}{(it - a)(it + a)} = \frac{-2a^2}{(it - a)(it + a)} = \frac{2a^2}{t^2 + a^2}.
\]

If \( G \) is a bipartite graph with \( 2n \) vertices, we can write its spectrum as \( \text{Spec}(G) = \{±\lambda_j\}_{j=1}^n \), for some \( \lambda_i \geq 0 \). From the above identity, we have
\[
\sum_{j=1}^n \frac{2\lambda_j^2}{z^2 + \lambda_j^2} = -\sum_{j=1}^n \lambda_j \left( \frac{1}{iz - \lambda_j} - \frac{1}{iz + \lambda_j} \right)
= -\sum_{\lambda \in \text{Spec}(G)} \frac{\lambda}{iz - \lambda} = 2n - iz \frac{\phi'(G, iz)}{\phi(G, iz)}, \tag{4}
\]
where \( \phi = \phi_G \).

On other hand, let us define
\[
f(\alpha) = \int_0^\infty \frac{t^\alpha}{t^2 + 1} \, dt
\]
for \( \alpha \in (-1, 1) \). Using a simple change of variables, we have
\[
\int_0^\infty \frac{t^\alpha}{t^2 + a^2} \, dt = \frac{\alpha^\alpha}{a^2((t/a)^2 + 1)} \, dt = \alpha^{-1} \int_0^\infty \frac{s^\alpha}{s^2 + 1} \, ds = a^{\alpha-1} f(\alpha). \tag{5}
\]

With the property of \( f \) given in equation (5), we can integrate (4) and obtain a formula involving the \((\alpha + 1)\)-Schatten energy
\[
\int_0^\infty z^\alpha \left( 2n - iz \frac{\phi'(G, iz)}{\phi(G, iz)} \right) \, dz = \sum_{j=1}^n 2\lambda_j^2 \int_0^\infty \frac{z^\alpha}{z^2 + \lambda_j^2} \, dz
= \sum_{j=1}^n 2\lambda_j^2 \lambda_j^{\alpha-1} f(\alpha) = 2 \sum_{j=1}^n \lambda_j^{\alpha+1} f(\alpha) = f(\alpha) \mathcal{E}_{\alpha+1}(G).
\]

Replacing \( p = \alpha + 1 \), we can write
\[
\mathcal{E}_p(G) = \frac{1}{f(p-1)} \int_0^\infty z^{p-1} \left( 2n - iz \frac{\phi'(G, iz)}{\phi(G, iz)} \right) \, dz, \tag{6}
\]
and we arrive at the stated formula by realizing that
\[
f(\alpha) = \int_0^\infty \frac{t^\alpha}{t^2 + 1} \, dt = \frac{\pi}{2 \cos \left( \frac{\alpha \pi}{2} \right)},
\]

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which finishes the proof by the simple identity relation \( \cos t = \sin(t + \frac{\pi}{2}) \).

**Remark 1.** Let us mention that equation (1) holds for any graph polynomial of the form \( P(z) = \beta z^c \prod(z^2 - \alpha_i^2) \). To be precise

\[
\sum_{i=1}^{n} |\alpha_i|^p = \frac{2 \sin\left(\frac{p\pi}{2}\right)}{\pi} \int_{0}^{\infty} z^{p-1} \left(n - iz\frac{P'(iz)}{P(iz)}\right) dz.
\]

A particular case which may be of interest is the matching polynomial [5], defined as

\[
M_G(x) := \sum_{k \geq 0} (-1)^k m_k x^{n-2k},
\]

where \( m_k \) denotes the number of matchings of the graph \( G \).

### 3. Applications

In this section, we will extend the technique of quasi-order to compare the \( p \)-Schatten energy of bipartite graphs. In order to do this, recall that, if \( G \) is a bipartite graph with \( n \) vertices, then its characteristic polynomial has the following form

\[
\sum_{k \geq 0} (-1)^k b_{2k} x^{n-2k},
\]

where \( b_{2k} \geq 0 \) for all \( k \). The quasi-order \( \preceq \) is defined for bipartite graphs as follows: \( G_1 \preceq G_2 \) if \( b_{2k}(G_1) \leq b_{2k}(G_2) \) for all \( k \), see [6,12] or [7, Section 4.3].

We note that, for \( z \in \mathbb{R} \), we have

\[
\phi(G, iz) = \sum_{k \geq 0} (-1)^k b_{2k}(iz)^{n-2k} = i^n \sum_{k \geq 0} b_{2k} z^{n-2k}
\]

and thus \( i^{-n} \phi(G, iz) > 0 \) and \( i^{-n} \phi(G_2, iz) \leq i^{-n} \phi(G_1, iz) \) if \( G_2 \preceq G_1 \), for \( z \in [0, \infty) \).

Our aim is to use the integral formula of Theorem 1, to compare \( p \)-Schatten energy of two graphs, \( G_1 \) and \( G_2 \). For this, we will need to modify (1) as done in the classical paper of Coulson and Jacobs [2].

**Theorem 3** (Coulson–Jacobs formula for \( p \)-energy). Let \( G_1 \) and \( G_2 \) be bipartite graphs on \( n \) vertices, and \( 0 < p < 2 \). Then

\[
\mathcal{E}_p(G_1) - \mathcal{E}_p(G_2) = \frac{2p \sin\left(\frac{p\pi}{2}\right)}{\pi} \int_{0}^{\infty} z^{p-1} \log\left(\frac{\phi(G_1, iz)}{\phi(G_2, iz)}\right) dz.
\]
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PROOF. From Theorem 1 we see that

$$\mathcal{E}_p(G_1) - \mathcal{E}_p(G_2) = \frac{2\sin\left(\frac{p\pi}{2}\right)}{\pi} \int_0^\infty iz^p \left( \frac{\phi'(G_2, iz)}{\phi(G_2, iz)} - \frac{\phi'(G_1, iz)}{\phi(G_1, iz)} \right) dz$$

$$= \frac{2\sin\left(\frac{p\pi}{2}\right)}{\pi} \int_0^\infty z^p \frac{d}{dz} \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) dz.$$

Now, integration by parts with the functions $u = z^p$ and $v = \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right)$ yields the identity

$$\int_0^\infty z^p \frac{d}{dz} \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) dz = \int_0^\infty p z^{p-1} \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) dz + \left[ z^p \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) \right]_0^\infty.$$

We claim that the last summand is 0 since

$$\lim_{z \to 0} z^p \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) = 0 \quad \text{and} \quad \lim_{z \to \infty} z^p \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) = 0.$$

The first limit is clear by writing

$$z^p \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) = z^p \left( \log(i^n \phi(G_2, iz)) - \log(i^{-n} \phi(G_1, iz)) \right),$$

and observing that for any $p > 0$ and any non trivial polynomial with positive coefficients $P(z)$, we have the convergence $z^p \log(P(z)) \to 0$ as $z \to 0$ (which may be proved by an application of L’Hôpital’s rule).

For the second limit, note that since $G_1$ and $G_2$ are both bipartite graphs of size $n$, then

$$\phi(G_1, z) = z^n + m_1 z^{n-2} + o(z^{n-2}) \quad \text{and} \quad \phi(G_2, z) = z^n + m_2 z^{n-2} + o(z^{n-2}).$$

This implies that

$$\phi(G_2, iz)/\phi(G_1, iz) = 1 + (m_1 - m_2) z^{-2} + o(z^{-2})$$

as $z \to \infty$. Now, using the approximation $\log(1 + x) = x$ for $x \approx 0$, we see that

$$z^p \log \left( \frac{\phi(G_2, iz)}{\phi(G_1, iz)} \right) = (m_1 - m_2) z^{p-2} + o(z^{p-2}),$$

which converges to 0, as $z \to \infty$, for any $p < 2.$

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Coming back to our original integral, we have obtained that

$$
\mathcal{E}_p(G_1) - \mathcal{E}_p(G_2) = -\frac{2 \sin \left(\frac{p\pi}{2}\right)}{\pi} \int_0^\infty p z^{p-1} \log \left(\frac{\phi(G_2, iz)}{\phi(G_1, iz)}\right) dz,
$$

as desired. \(\square\)

**Corollary 1.** Let \(G_1\) and \(G_2\) be two bipartite graphs. If \(G_1 \preceq G_2\) then \(\mathcal{E}_p(G_1) \leq \mathcal{E}_p(G_2)\).

**Proof.** Notice that if \(G_1 \preceq G_2\) then \(i^{-n}\phi(G_2, iz) \geq i^{-n}\phi(G_1, iz)\) for all \(z \in \mathbb{R}\) and then \(\log \left(\frac{\phi(G_2, iz)}{\phi(G_1, iz)}\right) \geq 1\), from where we see that

$$
\mathcal{E}_p(G_2) - \mathcal{E}_p(G_1) = \frac{2p \sin \left(\frac{p\pi}{2}\right)}{\pi} \int_0^\infty z^{p-1} \log \left(\frac{\phi(G_2, iz)}{\phi(G_1, iz)}\right) dz \geq 0, \tag{10}
$$

since the above integral must be positive. \(\square\)

Now Theorem 2 is a direct consequence of the following known relation in the quasi-order for trees (see for example [6] or [7, Theorem 4.6]).

**Theorem 4** [6]. Let \(T_n\) be a tree on \(n\) vertices, then

$$
S_n \preceq T_n \preceq P_n. \tag{11}
$$

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**References**

[1] C. A. Coulson, On the calculation of the energy in unsaturated hydrocarbon molecules, *Math. Proc. Cambridge Philos. Soc.*, 36 (1940), 201–203.

[2] C. A. Coulson and J. Jacobs, 591. conjugation across a single bond, *J. Chem. Soc. (Resumed)* (1949), 2805–2812.

[3] P. Csikvári, On a poset of trees, *Combinatorica*, 30 (2010), 125–137.

[4] Z. Du, Coulson-type integral formulas for the sum of powers of absolute values of roots of polynomials, *J. Math. Anal. Appl.*, 494 (2021), Paper No. 124650, 31 pp.

[5] E. J. Farrell, An introduction to matching polynomials, *J. Combin. Theory, Ser. B*, 27 (1979), 75–86.

[6] I. Gutman and F. J. Zhang, On a quasi-ordering of bipartite graphs, *Publ. Inst. Math. (Beograd) (N.S.)*, 40 (1986), 11–15.

[7] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer (New York, 2012).

[8] L. Lovász and J. Pelikán, On the eigenvalues of trees, *Period. Math. Hungar.*, 3 (1973), 175–182.
[9] V. Nikiforov, Beyond graph energy: norms of graphs and matrices, *Linear Algebra Appl.*, **506** (2016), 82–138.

[10] L. Qiao, S. Zhang and J. Li, Coulson-type integral formulas for the general Laplacian energy-like invariant of graphs. II, *J. Math. Anal. Appl.*, **449** (2017), 1725–1740.

[11] L. Qiao, S. Zhang, B. Ning and J. Li, Coulson-type integral formulas for the general Laplacian-energy-like invariant of graphs. I, *J. Math. Anal. Appl.*, **435** (2016), 1249–1261.

[12] F. J. Zhang, Two theorems of comparison of bipartite graphs by their energy, *Kexue Tongbao (English Ed.)*, **28** (1983), 726–730.

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