Twisted Conormal Bundles and Canonical Relations

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Abstract

I study a special type of canonical relations given by twisted conormal bundles, construct a “subcategory” of the symplectic “category” out of these canonical relations and quantize them into semi-classical Fourier integral operators. Furthermore, I give a description of the intrinsic line bundle of symbols of these operators and describe how the symbols compose when the operators compose.

1 Introduction

For a submanifold $Z$ of a smooth manifold $X$, let $f \in C^\infty(Z)$ be a smooth function on $Z$. Define the $f$-twisted conormal bundle of $Z$ to be the set

$$N^*_f Z = \{(x, \xi + df_x) : (x, \xi) \in N^*Z\}$$

where we also use $f$ to denote any extension of $f$ to $X$. It is easy to see that this is independent of the choice of the extension and is a Lagrangian submanifold of $T^*X$.

For a submanifold $Z \subset X \times Y$ and $f \in C^\infty(Z)$, we can act the map

$$\zeta_X : T^*X \to T^*X, (x, \xi) \mapsto (x, -\xi)$$

to

$$N^*_f Z \subset T^*(X \times Y) = T^*X \times T^*Y$$

and get the Lagrangian submanifold

$$\Gamma_{Z,f} = (\zeta_X \times id)N^*_f Z \subset T^*X^- \times T^*Y,$$

where $T^*X^-$ is the manifold $T^*X$ with its reversed symplectic form and $\zeta_X$ is the symplectomorphism from $T^*X$ to $T^*X^-$. Denote $\Gamma_{Z,f}$ by $\Gamma_Z$ in the non-twisted case $f = 0$. Note that as Lagrangian submanifolds of $T^*X \times T^*Y^-$, we can view $\Gamma_{Z,f}$ and $\Gamma_Z$ as canonical relations from $T^*X$ to $T^*Y$.

In section 2 we will show that, under certain conditions, the composition of canonical relations $\Gamma_Z$ (or $\Gamma_{Z,f}$) in the symplectic “category” $\mathcal{S}$ is again of this form. Specifically, we will prove:
Theorem 1 Suppose $X_1, X_2, X_3$ are smooth manifolds and $Z_1 \subset X_1 \times X_2, Z_2 \subset X_2 \times X_3$ are submanifolds. Suppose $\Gamma_{Z_1} = (\zeta_1 \times id)N^*Z_1 \in \text{Morph}(T^*X_1, T^*X_2)$, $\Gamma_{Z_2} = (\zeta_2 \times id)N^*Z_2 \in \text{Morph}(T^*X_2, T^*X_3)$ are cleanly composable canonical relations,

$$Z = \{(x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, \text{s.t.} (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2\}$$

is a submanifold of $X_1 \times X_3$, and the projection

$$\Gamma_{Z_2} \circ \Gamma_{Z_1} \to Z$$

is a submersion in the category of smooth manifolds. Then

$$\Gamma_{Z_2} \circ \Gamma_{Z_1} = \Gamma_Z$$

where $\Gamma_Z = (\zeta_1 \times id)N^*Z \in \text{Morph}(T^*X_1, T^*X_3)$.

Theorem 2 Suppose $X_1, X_2, X_3$ are smooth manifolds, $Z_1 \subset X_1 \times X_2, Z_2 \subset X_2 \times X_3$ are submanifolds and $f_1 \in C^\infty(Z_1), f_2 \in C^\infty(Z_2)$ are smooth functions. Suppose $\Gamma_{Z_1,f_1} = (\zeta_1 \times id)N^*_fZ_1 \in \text{Morph}(T^*X_1, T^*X_2)$, $\Gamma_{Z_2,f_2} = (\zeta_2 \times id)N^*_fZ_2 \in \text{Morph}(T^*X_2, T^*X_3)$ are cleanly composable canonical relations,

$$Z = \{(x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, \text{s.t.} (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2\}$$

is a simply connected submanifold of $X_1 \times X_3$, the projection

$$\Gamma_{Z_2,f_2} \circ \Gamma_{Z_1,f_1} \to Z$$

is a submersion in the category of smooth manifolds, and

$$d_2f_1(x_1, x_2) + d_2f_2(x_2, x_3) = 0 \in T^*X_2$$

for every $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$. Then there exists $f \in C^\infty(Z)$ such that

$$\Gamma_{Z_2,f_2} \circ \Gamma_{Z_1,f_1} = \Gamma_{Z,f}$$

where $\Gamma_{Z,f} = (\zeta_1 \times id)N^*_fZ \in \text{Morph}(T^*X_1, T^*X_3)$.

We then have a natural picture of a functor projecting the “upstairs category” to the “downstairs category”. Specifically, we can define $\mathcal{C}_0$ and $\mathcal{C}$ to be “categories” with cotangent bundles $T^*X$ as objects and morphisms

$$\text{Morph}_{\mathcal{C}_0}(T^*X, T^*Y) = \{\Gamma_Z : Z \subset X \times Y \text{ is a submanifold}\}$$

$$\text{Morph}_{\mathcal{C}}(T^*X, T^*Y) = \{\Gamma_{Z,f} : Z \subset X \times Y \text{ is a submanifold and } f \in C^\infty(Z)\}.$$ 

Two morphisms are defined to be composable when they satisfy the conditions in Theorem 1 or 2. Note that $\mathcal{C}_0$ and $\mathcal{C}$ are “subcategories” of the symplectic “category” $\mathcal{S}$. 

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Define the ‘downstairs category’ $\mathcal{D}$ to be the “category” with smooth manifolds as objects and morphisms

\[ \text{Morph}(X,Y) = \{ \text{submanifolds } Z \subset X \times Y \} \]

where $Z_1 \subset X_1 \times X_2$ and $Z_2 \subset X_2 \times X_3$ are composable if and only if

\[ Z = \{(x_1,x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, \text{s.t.} (x_1,x_2) \in Z_1, (x_2,x_3) \in Z_2 \} \]

is a submanifold of $X_1 \times X_3$, in which case we define their composition to be $Z$. We then have the natural “projection” functors

\[ C_0 \to \mathcal{D} \]

and

\[ C \to \mathcal{D} \]

sending the “upstairs objects” $T^*X$ to “downstairs objects” $X$ and “upstairs morphisms” $\Gamma_Z$ and $\Gamma_{Z,f}$ to “downstairs morphisms” $Z$.

Note that in order to guarantee

\[ Z = Z_2 \circ Z_1 = \{(x_1,x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, \text{s.t.} (x_1,x_2) \in Z_1, (x_2,x_3) \in Z_2 \} \]

is a submanifold, we can naturally define clean composition conditions of submanifolds $Z_1$ and $Z_2$ imitating the clean composition of canonical relations. Namely we call $Z_1 \subset X_1 \times X_2$ and $Z_2 \subset X_2 \times X_3$ cleanly composable if

\[ Z_2 \star Z_1 = \{(x_1,x_2,x_3) \in X_1 \times X_2 \times X_3 : (x_1,x_2) \in Z_1, (x_2,x_3) \in Z_2 \} \]

is a submanifold of $X_1 \times X_2 \times X_3$, and the projection

\[ \kappa : Z_2 \star Z_1 \to Z_2 \circ Z_1 \]

to be proper of constant rank, with each fiber simply connected.

**Remark 1** Now that since we have defined the clean composition conditions of submanifolds, we can rephrase Theorem 1 by saying that if the “downstairs objects” $Z_1$, $Z_2$ and “upstairs objects” $\Gamma_{Z_1}$ and $\Gamma_{Z_2}$ are both cleanly composable, and the projection

\[ \Gamma_{Z_2} \circ \Gamma_{Z_1} \to Z_2 \circ Z_1 \]

is a submersion, then the composition of the “upstairs objects” corresponds to the composition of “downstairs objects” by the simple formula

\[ \Gamma_{Z_2} \circ \Gamma_{Z_1} = \Gamma_{Z_2 \circ Z_1} \]

The second part of the paper will show how to quantize our type canonical relations via the techniques of Hörmander (see [1] and [2]). In section 3 we will give the following result useful in the quantization procedure.
Theorem 3 Suppose \( X \) is a smooth manifold of dimension \( n \), \( \Lambda \) is a Lagrangian submanifold of \( T^*X \) and \( p \in \Lambda \) such that \( \pi|_{\Lambda} \) locally at \( p \) is a submersion onto a \((n-k)\)-dimensional submanifold \( Z \subset X \) with every fiber \( \pi^{-1}(z), z \in Z \) connected. Then \( \Lambda \) can be described locally at \( p \) in Hörmander sense as a fiber bundle \( W = X \times \mathbb{R}^k \to X \) and a generating function \( \phi : W \to \mathbb{R} \).

Since twisted conormal bundles are exact Lagrangian submanifolds, we will quantize them by associating the space of oscillatory \( \frac{1}{2} - \)densities and semi-classical Fourier integral operators. Section 4 will compute these spaces explicitly. In section 5 we talk about symbols of such operators. Note that since the Maslov bundle is always trivial, Hörmander’s “intrinsic line bundle of symbols”

\[
L = \Lambda_{\text{Maslov}}(\Lambda) \otimes |T\Lambda|^{\frac{1}{2}}
\]
is generally isomorphic to the \( \frac{1}{2} - \)density bundle \( |T\Lambda|^{\frac{1}{2}} \). We will prove that there is a canonical isomorphism \( L \cong |T\Lambda|^{\frac{1}{2}} \) in the case when \( \Lambda \) is a twisted conormal bundle. Specifically, we will prove

**Theorem 4** Suppose \( Z \subset X \) is a submanifold and \( f \in C^\infty(Z) \). Let \( \Lambda = N_f^*Z \subset T^*X \) be the twisted conormal bundle. Then the intrinsic line bundle of symbols

\[
L \to \Lambda
\]
is canonically isomorphic to the \( \frac{1}{2} - \)density line bundle \( |T\Lambda|^{\frac{1}{2}} \) of \( \Lambda \).

We will further show that this isomorphism is functorial under the composition of canonical relations.

**Theorem 5** Under the assumptions of Theorem 2, we have a clean composition of (exact) canonical relations

\[
\Gamma_2 \circ \Gamma_1 = \Gamma
\]
where \( \Gamma_i = \Gamma_{Z_i, f_i} \in \text{Morph}(T^*X_i, T^*X_{i+1}) \) for \( i = 1, 2 \) and \( \Gamma = \Gamma_{Z, f} \in \text{Morph}(T^*X_1, T^*X_3) \). In this case the intrinsic line bundles of symbols

\[
L_1 \to \Gamma_1, L_2 \to \Gamma_2, L \to \Gamma
\]
can be identified with the \( \frac{1}{2} - \)density bundles \( |TT_1|^{\frac{1}{2}}, |TT_2|^{\frac{1}{2}} \) and \( |TT|^{\frac{1}{2}} \).

If \( F_1 \in \mathcal{F}^{m_1}(\Gamma_1) \) and \( F_2 \in \mathcal{F}^{m_2}(\Gamma_2) \) (so that \( F_2 \circ F_1 \in \mathcal{F}^{m_1 + m_2}(\Gamma) \)), then the symbol

\[
\sigma(F_2 \circ F_1) \in C^\infty(|TT|^{\frac{1}{2}})
\]
is the composition of \( \sigma(F_2) \in C^\infty(|TT_2|^{\frac{1}{2}}) \) with \( \sigma(F_1) \in C^\infty(|TT_1|^{\frac{1}{2}}) \) in the enhanced symplectic “category”.

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2 Twisted conormal bundles as canonical relations

In [2], the authors defined the symplectic "category", which has symplectic manifolds as objects and morphisms

\[ \text{Morph}(M_1, M_2) = \{ \text{Lagrangians submanifolds of } M_1^- \times M_2 \}. \]

The morphisms are also called canonical relations. For \( \Gamma_1 \in \text{Morph}(M_1, M_2) \) and \( \Gamma_2 \in \text{Morph}(M_2, M_3) \), we define their composition

\[ \Gamma_2 \circ \Gamma_1 = \{(x, z) \in M_1 \times M_3 : \exists y \in M_2, \text{s.t.} (x, y) \in \Gamma_1, (y, z) \in \Gamma_2 \}. \]

Note that this is not necessarily an Lagrangian submanifold of \( M_1^- \times M_3 \), so \( \Gamma_1 \) and \( \Gamma_2 \) may not compose, and the symplectic "category" is not strictly a category. Several conditions are imposed in [2] to make two morphisms compose. Define

\[ \Gamma_2 \ast \Gamma_1 = \{(m_1, m_2, m_3) \in M_1 \times M_2 \times M_3 : (m_1, m_2) \in \Gamma_1, (m_2, m_3) \in \Gamma_2 \}. \]

We call \( \Gamma_1, \Gamma_2 \) cleanly composable if \( \Gamma_2 \ast \Gamma_1 \) is a submanifold; the natural projection \( \kappa : \Gamma_2 \ast \Gamma_1 \to \Gamma_2 \circ \Gamma_1 \) is a smooth fiber bundle with compact fibers; and for every point \( (m_1, m_2, m_3) \in \Gamma_2 \ast \Gamma_1 \), the tangent space \( T_{(m_1, m_2, m_3)} \Gamma_2 \ast \Gamma_1 \) equals

\[ \{(v_1, v_2, v_3) \in T_{m_1} M_1 \times T_{m_2} M_2 \times T_{m_3} M_3 : (v_1, v_2) \in T_{(m_1, m_2)} \Gamma_1, (v_2, v_3) \in T_{(m_2, m_3)} \Gamma_2 \}. \]

In this case \( \Gamma_2 \circ \Gamma_1 \) is an (embedded) Lagrangian submanifold of \( M_1 \times M_3 \), and the canonical relations \( \Gamma_1, \Gamma_2 \) are defined to be composable with composition \( \Gamma_2 \circ \Gamma_1 \).

Before delving into the proof of Theorem 1 and 2, we first prove the following result given by Jingwen [3], which says that under certain regularity conditions, any Lagrangian submanifold of \( T^*X \) is locally a twisted conormal bundle.

**Lemma 1** Let \( X \) be a smooth manifold and \( T^*X \) be its cotangent bundle. Denote the canonical projection by \( \pi : T^*X \to X \). Suppose \( \Lambda \) is a Lagrangian submanifold of \( T^*X \) such that \( \pi|_{\Lambda} = Z \) is a simply connected submanifold of \( X \), and that the projection \( \pi|_{\Lambda} : \Lambda \to Z \) is a surjective submersion. Then there is a smooth function \( f \) on \( Z \) such that \( \Lambda \) is an open submanifold of \( N_f^*Z \).

**Proof:** Suppose \( \alpha \) is the canonical 1-form on \( T^*X \) and \( \iota : \Lambda \to T^*X \) is the inclusion map, then \( \iota^*\alpha \) is closed on \( \Lambda \). Furthermore, \( \iota^*\alpha \) is a horizontal form since, for any \( p = (x, \xi) \in \Lambda \) and vertical vector \( v \in T_p \Lambda \) (i.e. \( (d\pi)_p v = 0 \)), we have

\[ \iota^*\alpha_p(v) = \iota^*(d\pi)_p^*\xi(v) = \iota^*(\xi, (d\pi)_p v) = 0. \]

As a consequence, there exists a 1-form \( \beta \) on \( Z \) such that \( \pi^*\beta = \iota^*\alpha \). Since \( Z \) is simply connected, we can write \( \beta = d\phi_0 \) for some \( \phi_0 \in C^\infty(Z) \).

Denote an extension of \( \phi_0 \) to \( X \) by \( \phi \), and let

\[ \gamma_\phi : T^*X \to T^*X, (x, \xi) \mapsto (x, \xi + d\phi_x) \]
be the symplectomorphism with the property $\pi \circ \gamma_\phi = \pi$. By constructions we have

$$(\gamma_\phi)^{-1}^* \alpha = (\gamma_\phi)^{-1}^* d\pi^* \xi = d\pi^*(\xi - d\phi) = \alpha - \pi^* d\phi$$

and

$$\iota^* \alpha = \pi^* \beta = \pi^* d\phi_0 = \iota^* \pi^* d\phi.$$ 

Thus

$$\iota^* (\gamma_\phi)^{-1} \alpha = \iota^* (\alpha - \pi^* d\phi) = \iota^* \pi^* d\phi - \iota^* \pi^* d\phi = 0,$$

which means that the restriction of $\alpha$ to the image $\Lambda' = \gamma_\phi(\Lambda)$ is zero. Note that $\Lambda'$ is also a Lagrangian submanifold of $T^*X$ with $\pi(\Lambda') = Z$ and each fiber $\Lambda' \cap T^*_pX$ is a connected submanifold of $T^*_X$.

For any $p = (x, \xi) \in \Lambda'$, we have $\alpha_p = (d\pi)^*_p \xi$ by definition. Let $\iota_Z : Z \hookrightarrow X$ be the inclusion map, then we have $\iota_Z \circ \pi = \pi \circ \iota$. It follows that

$$0 = (\iota^* \alpha)_p = \iota^*(d\pi_p)^* \xi = (d\pi_p)^* (\iota_Z^* \xi)$$

Since $(d\pi_p)^* : T^*_xZ \to T^*_p\Lambda$ is injective, we get $\iota_Z^* \xi = 0$, i.e. $\xi \in T^*_xZ$. We have proved that $\Lambda' \subset T^*_xZ$. Thus $\Lambda = (\gamma_\phi)^{-1} \Lambda' \subset T^*_xZ$. \qed

**Remark 2** Note that in the step of finding 1-form $\beta$, we have implicitly used the $k = 1$ case of the result from [4]: For a surjective submersion $\pi : M \to N$ between smooth manifolds such that each fiber is connected, a tangent vector $v \in T_pM$ is called vertical if $d\pi_p(v) = 0$. Suppose $w \in \Omega^k(M)$, then there exists $\eta \in \Omega^k(N)$ such that $w = \pi^* \eta$ if and only if $i_v w_p = 0$ and $i_v dw_p = 0$ for all $p \in M$ and vertical vector $v \in T_pM$.

Now we begin the proof of Theorem 2. Calculate

$$N^*_xZ = \{(x, y, \xi, \eta) : (x, \xi) \in T^*_xX, (y, \eta) \in T^*_yY, (\xi, \eta) \text{ vanishes on } T_{(x,y)}Z\}$$

and

$$N^*_yZ = \{(x, y, \xi, \eta) : (x, \xi) \in T^*_xX, (y, \eta) \in T^*_yY, (\xi - d_X f, \eta - d_Y f) \text{ vanishes on } T_{(x,y)}Z\};$$

so

$$\Gamma_{Z,f} = \{(x, y, \xi, \eta) : (x, \xi) \in T^*_xX, (y, \eta) \in T^*_yY, (-\xi - d_X f, \eta - d_Y f) \text{ vanishes on } T_{(x,y)}Z\}.$$

In the context of Theorem 2, we have

$$\Gamma_2 \circ \Gamma_1 = \{(x_1, x_3, \xi_1, \xi_3) : \exists (x_2, \xi_2) \in T^*_xX_2, s.t. (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2, (-\xi_1 - d_1 f_1, \xi_2 - d_2 f_1) \text{ vanishes on } T_{(x_1,x_2)}Z_1, (-\xi_2 - d_2 f_2, \xi_3 - d_3 f_2) \text{ vanishes on } T_{(x_2,x_3)}Z_2\}.$$
and

\[(\zeta_1 \times id)(\Gamma_2 \circ \Gamma_1) = \{(x_1, x_3, \xi_1, \xi_3) : \exists (x_2, \xi_2) \in T^*X_2, s.t. (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2, \]

\[(\xi_1 - d_1f_1, \xi_2 - d_2f_2) \text{ vanishes on } T_{(x_1, x_2)}Z_1, (-\xi_2 - d_2f_2, \xi_3 - d_3f_2) \text{ vanishes on } T_{(x_2, x_3)}Z_2\}.

Notice that \(\Lambda = (\zeta_1 \times id)(\Gamma_2 \circ \Gamma_1)\) is a Lagrangian submanifold of \(T(X_1 \times X_3)\) whose projection on \(X_1 \times X_3\) inside

\[\{(x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, s.t. (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2\} = Z.\]

Conversely, under the condition \(d_2f_1 + d_2f_2 = 0\), we have at least one point

\[(x_1, x_3, d_1f_1, d_3f_2) \in \Lambda \cap T_{(x_1, x_3)}(X_1 \times X_3)\]
in the fiber of \(\pi|_\Lambda\) on each \((x_1, x_3) \in Z\). So \(\pi|_\Lambda\) is a surjective submersion onto \(Z\).

We further notice that all its fibers are connected since they are unions of some affine subspaces of \(T_{(x_1, x_3)}(X_1 \times X_3)\) containing the same point \((d_1f_1, d_3f_2)\). Since \(Z\) is simply connected, from Lemma 1 we know that there exists \(f \in C^\infty(Z)\) such that \(\Lambda\) is an open submanifold of \(N_f^*Z\). Furthermore, since \(N_f^*Z\) has all fibers affine subspaces and \(\Lambda\) has all fibers unions of affine subspaces containing the same point, we have

\[\Lambda = N_f^*Z \subset T^*X_1 \times T^*X_3\]

thus

\[\Gamma_2 \circ \Gamma_1 = (\zeta_1 \times id)N_f^*Z \subset T^*X_1^- \times T^*X_3.\]

Theorem 2 has been proved. In the case when \(Z_1\) and \(Z_2\) are not twisted we have each fiber

\[\Lambda \cap T_{(x_1, x_3)}^*(X_1 \times X_3) = \{(\xi_1, \xi_3) : \exists (x_2, \xi_2) \in T^*X_2, s.t. (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2, (\xi_1, \xi_2) \text{ vanishes on } T_{(x_1, x_2)}Z_1, (-\xi_2, \xi_3) \text{ vanishes on } T_{(x_2, x_3)}Z_2\}

a union of linear subspaces of \(T_{(x_1, x_3)}^*(X_1 \times X_3)\). So every fiber of \(\Lambda = N_f^*Z\) contains the origin and thus \(N_f^*Z = N^*Z\), i.e. the composition is not twisted. The simply connected condition of \(Z\) can be removed in this case because \(Z\) as a submanifold is always locally simply connected. From our proof,

\[\Gamma_{Z_2} \circ \Gamma_{Z_1} = \Gamma_Z\]
is true locally near any point of \(Z\), which means that it is globally true. Thus Theorem 1 is also proved.

**Example 2.1** For smooth manifolds \(X_1, X_2, X_3\), pick a point \(* \in X_2\) and let \(Z_1 = X_1 \times \{*\}\) and \(Z_2 = \{*\} \times X_3\) in Theorem 2. Suppose \(f_1 \in C^\infty(X_1)\) and \(f_2 \in C^\infty(X_3)\) viewed as functions on \(Z_1\) and \(Z_2\), then we can easily see

\[\Gamma_{Z_2,f_2} \circ \Gamma_{Z_1,f_1} = \Gamma_Z,f\]

for \(Z = X_1 \times X_3\) and \(f = f_1 + f_2 \in C^\infty(Z)\).
Example 2.2 Consider a trivial type of twisted conormal bundles
\[ \Lambda_f = \{ (x, df(x)) : x \in X \} \subset T^*X \]
for \( f \in C^\infty(X) \). For smooth manifolds \( X_1, X_2, X_3 \) and \( f_1 \in C^\infty(X_1 \times X_2) \), \( f_2 \in C^\infty(X_2 \times X_3) \), we have canonical relations \( \Gamma_{f_1} = (\zeta_1 \times id)\Lambda_f \) and \( \Gamma_{f_2} = (\zeta_2 \times id)\Lambda_f \). Suppose
\[ d_2 f_1(x_1, x_2) + d_2 f_2(x_2, x_3) = 0 \in T^*X_2 \]
for every \( (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3 \). Note that this condition means \( f_1(x_1, x_2) + f_2(x_2, x_3) \) is constant in the variable \( x_2 \) and we can write it \( f(x_1, x_3) \). In this case we have
\[ \Gamma_{f_2} \circ \Gamma_{f_1} = \Gamma_f \]
where \( \Gamma_f = (\zeta_1 \times id)\Lambda_f \).

Example 2.3 Consider twisted conormal bundles of graphs
\[ \Gamma_{g, f} := (\zeta_X \times id)N^*_f(\text{graph}(g)) \subset T^*X^{-} \times T^*Y \]
as canonical relations, where \( g : X \to Y \) is a smooth map and \( f \in C^\infty(X) \) deemed as a function on graph \( g \). We can get a smaller subcategory (without quotation marks) of the symplectic “category” from this. Specifically, suppose \( X_1, X_2, X_3 \) are smooth manifolds, \( g_1 : X_1 \to X_2 \), \( g_2 : X_2 \to X_3 \) are maps and \( f_1 \in C^\infty(X_1) \), \( f_2 \in C^\infty(X_2) \). Then \( \Gamma_{g_2, f_2} \) and \( \Gamma_{g_1, f_1} \) are always composable, and we have
\[ \Gamma_{g_2, f_2} \circ \Gamma_{g_1, f_1} = \Gamma_{g, f} \]
where \( g = g_2 \circ g_1 : X_1 \to X_3 \) and \( f = f_1 + f_2 \circ g_2 \in C^\infty(X_1) \).

Remark 3 In the composition
\[ \Gamma_{Z_2, f_2} \circ \Gamma_{Z_1, f_1} = \Gamma_{Z, f} \]
from Theorem 2, the submanifold \( Z \subset X_1 \times X_3 \) is given explicitly by
\[ Z = \{ (x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2, \text{s.t.} (x_1, x_2) \in Z_1, (x_2, x_3) \in Z_2 \} , \]
but the function \( f \in C^\infty(Z) \) can be somewhat mysterious. From Example 2.3, we know that \( f \) cannot be expressed explicitly by a calculation of \( f_1 \) and \( f_2 \), since it also depends on \( g_1 \) and \( g_2 \), which are datas representing the “shapes of submanifolds” \( Z_1 \) and \( Z_2 \).

3 Hörmander descriptions at regular points

Suppose \( \pi : W \to X \) is a smooth fiber bundle and \( \phi \) is a smooth function on \( W \). View the Lagrangian submanifold
\[ \Lambda_\phi = \{ (z, d\phi(z)) : z \in W \} \]
of $T^*W$ as a canonical relation in $\text{Morph}(pt, T^*W)$ and the graph

$$\Gamma_\pi = \{(z, \xi, x, \eta) \in T^*W \times T^*X : x = \pi(z), \xi = (d\pi_z)^*\eta\}$$

of $\pi : W \to X$ as a canonical relation in $\text{Morph}(T^*W, T^*X)$. According to [2], the assumption that $\Gamma_\pi$ and $\Lambda_\phi$ are transversally composable is equivalent to

$$C_\phi := \{z \in W : (d_{\text{vert}}\phi)_z = 0\}$$

being a submanifold of $W$ and

$$C_\phi \to T^*X, z \mapsto (\pi(z), \eta)$$

being a Lagrangian embedding, where $\eta$ is the unique vector in $T_{\pi(z)}^*X$ such that $d\phi_z = (d\pi_z)^*\eta$. The image of this embedding is the Lagrangian submanifold $\Lambda = \Gamma_\pi(\Lambda_\phi) = \Gamma_\pi \circ \Lambda_\phi$ of $T^*X$.

When this happens we say $\phi$ is a (transverse) generating function of $\Lambda$ with respect to the fiber bundle $(W, \pi)$, and the data $(W, \pi, \phi)$ is a Hörmander description of the Lagrangian submanifold $\Lambda \subset T^*X$. There is a deep result called Hörmander’s Theorem, proved in [2], saying that every Lagrangian submanifold of $T^*X$ has a Hörmander description locally. However, the construction in this theorem can be somewhat abstract, so it would be nice to actually find the Hörmander description assuming that $\Lambda \subset T^*X$ is locally regular.

Suppose $\Lambda \subset T^*X$ is regular at $p \in \Lambda$ in the sense of Theorem 3. Then there is a neighborhood $U$ of $p$ such that $\pi|_{U \cap \Lambda}$ is a submersion onto a $(n - k)$-dimensional submanifold $Z \cap \pi(U) \subset X$ with connected fibers, where $\pi : T^*X \to X$ is the canonical projection. By restricting to a smaller neighborhood one can assume further that $Z \cap \pi(U)$ is simply connected and

$$Z = \{x \in X : u_1(x) = u_2(x) = \cdots = u_k(x) = 0\}$$

where $u_1, \ldots, u_k \in C^\infty(X)$ are functionally independent near $p$. From Lemma 1 we know that $U \cap \Lambda = U \cap N_f^*Z$ for some $f \in C^\infty(Z)$. We show that the fiber bundle $W = X \times \mathbb{R}^k \to X$ and the generating function

$$\phi = \sum_{i=1}^k s_i u_i + f \in C^\infty(W)$$

is a Hörmander description for $N_f^*Z$, thus for $\Lambda$ locally at $p$, where the $s_i$ for $i = 1, 2, \ldots, k$ are coordinates for $\mathbb{R}^k$. In fact in this case

$$C_\phi := \{z \in X \times \mathbb{R}^k : (d_{\text{vert}}\phi)_z = 0\}$$

$$= \left\{z \in X \times \mathbb{R}^k : \frac{\partial \phi}{\partial s_i} = 0, i = 1, 2, \ldots, k\right\}$$

$$= \left\{z \in X \times \mathbb{R}^k : u_i = 0, i = 1, 2, \ldots, k\right\}$$

$$= Z \times \mathbb{R}^k.$$
The transversality assumption amounts to saying that the equations \( \frac{\partial \phi}{\partial s_i} = u_i = 0, i = 1, 2, \ldots k \) are functionally independent, and the Lagrangian submanifold of \( T^*X \) determined by \( W = X \times \mathbb{R}^k \to X \) with \( \phi : W \to \mathbb{R} \) is the image of the embedding

\[
C_\phi \to T^*X, (x, s) \mapsto (x, \sum_{i=1}^k s_i d_X u_i + d_X f)
\]

Since the differentials \( d_X u_i \) span the conormal bundle of \( Z \) in \( X \), the image of this embedding is \( N^*_f Z \subset T^*X \). So we have proved Theorem 3.

4 The quantization process

In semi-classical analysis we quantize canonical relations by associating semi-classical Fourier integral operators to them. In this section we will get simple formulas of these operators when the canonical relation takes the form of a twisted conormal bundle in a phase space.

Recall that as in [2], for every exact Lagrangian submanifold \((\Lambda, \psi)\) of \( T^*X \) and \( r \in \mathbb{Z} \) we can associate them to a space \( I^r(X, \Lambda) \) of rapidly oscillating \( \frac{1}{2} \)-densities on \( X \). In the special case when \( \Lambda \subset T^*X \) admits a global Hörmander description \((W, \pi, \phi)\), we fix the arbitrary constant in \( \phi \) so that \( \psi(x, \xi) = \phi(z) \) if \( d\phi_z = \pi^* \xi \) where \( \pi(z) = x \) and pick an enhancement \( \sigma \) of \( \pi \). We define the space of compactly supported \( \frac{1}{2} \)-densities on \( X \) to be

\[
I^r_\sigma(X, \Lambda) = \left\{ \mu = \hbar^{r - \frac{1}{2}} \pi(z)(ae^{i \frac{1}{\hbar} \tau})|a = a(z, \hbar) \in C_\infty^0(W \times \mathbb{R}) \right\}
\]

where \( \tau \) is a nowhere vanishing \( \frac{1}{2} \)-density on \( W \). We can check that the space \( I^r_\sigma(X, \Lambda) \) is independent of the Hörmander description \((W, \pi, \phi)\), the enhancement \( \sigma \) of \( \pi \) and the \( \frac{1}{2} \)-density \( \tau \). In the general case when \((\Lambda, \psi)\) is an exact Lagrangian submanifold of \( T^*X \), we can find a locally finite cover of \( \Lambda \) by open sets \( \Lambda_i \) such that each \( \Lambda_i \) is defined by a generating function \( \phi_i \) relative to a fiber bundle \( \pi_i : W_i \to U_i \) where the \( U_i \) are open subsets of \( X \). Define \( I^r(X, \Lambda) \) to consist of those \( \frac{1}{2} \)-densities \( \mu \) on \( X \) such that \( \rho \mu \in I^r_\sigma(X, \Lambda) \) for every \( \rho \in C_\infty^0(X) \). It is easy to see that these definitions are independent of the choice of open cover and the local Hörmander descriptions.

Suppose \( X_1, X_2 \) are manifolds and \((\Gamma, \Psi)\) is an exact canonical relation from \( T^*X_1 \) to \( T^*X_2 \), then by the following definition we can associate \( \Gamma \) to a space of semi-classical Fourier integral operators \( \mathcal{F}_{\Gamma}^m(X_1) \). Let \( X = X_1 \times X_2 \) and

\[
\zeta_1 : T^*X_1 \to T^*X_1, \zeta_1(x_1, \xi_1) = (x_1, -\xi_1),
\]
then $\Lambda = (\zeta_1 \times id)(\Gamma)$ and $\psi = \Psi \circ (\zeta_1 \times id)$ give an exact Lagrangian submanifold $(\Lambda, \psi)$ of

$$T^*X = T^*X_1 \times T^*X_2.$$  

Associated with $(\Lambda, \psi)$ we have the space of compactly supported oscillatory $\frac{1}{2}$-densities $I^r_0(X, \Lambda)$. We can write a typical element $\mu \in I^r_0(X, \Lambda)$ as

$$\mu = u(x_1, x_2, h)dx_1^{\frac{1}{2}}dx_2^{\frac{1}{2}},$$

where $dx_i^{\frac{1}{2}}$ is a nowhere vanishing $\frac{1}{2}$-density on $X_i$ and $u$ is a smooth function of compact support in all three “variables”. Define the Fourier integral operator

$$F_{\mu, \hbar} : L^2(X_1) \to L^2(X_2), gdx_1^{\frac{1}{2}} \mapsto \left( \int_{X_1} g(x_1)u(x_1, x_2, h)dx_1 \right) dx_2^{\frac{1}{2}}$$

and denote the space of such operators by $F^m_r(\Gamma)$ where $m = r + \frac{n_2}{2}$ and $n_2 = \dim X_2$.

For the rest of this section we focus on the case when $\Lambda = (\zeta_1 \times id)(\Gamma)$ is a twisted conormal bundle. Suppose $X$ is a smooth manifold with a nowhere vanishing $\frac{1}{2}$-density $dx_i^{\frac{1}{2}}$ and $u_1, \ldots, u_k$ are functionally independent functions on $X$. Then $Z = \{ x \in X : u_1(x) = u_2(x) = \ldots = u_k(x) = 0 \}$ is a submanifold of $X$. From section 3, $\Lambda = N^*_fZ$ has a Hörmander description of a fiber bundle $W = X \times \mathbb{R}^k \to X$ and a generating function

$$\phi = \sum_{i=1}^k s_i u_i + f \in C^\infty(W).$$

In this case the space of compactly supported oscillating $\frac{1}{2}$-densities is

$$I^r_0(X, \Lambda) = \left\{ \mu = \hbar^{r - \frac{1}{2}} \left( \int_{\mathbb{R}^k} a(x, s, h)e^{i \sum_{i=1}^k s_i u_i} ds \right)dx_1^{\frac{1}{2}} : a(x, s, h) \in C^\infty_0(X \times \mathbb{R}^k \times \mathbb{R}) \right\}$$

$$= \left\{ \mu = \hbar^{r - \frac{1}{2}}e^{i \sum_{i=1}^k s_i u_i} \left( \int_{\mathbb{R}^k} a(x, s, h) e^{\frac{1}{\hbar} \sum_{i=1}^k s_i u_i} ds \right)dx_1^{\frac{1}{2}} : a(x, s, h) \in C^\infty_0(X \times \mathbb{R}^k \times \mathbb{R}) \right\}$$

for every $r \in \mathbb{Z}$. Note that in the special case when $a(x, s, h) = a(x, h)$ and $u_i(x) = x_i$, this integration is just the Fourier transform of $a(x, h)$ in the variables $x_1, \ldots, x_k$.

Suppose $X = X_1 \times X_2$ and $dx_i^{\frac{1}{2}} = dx_i^1dx_i^2$, where $dx_i^1$ and $dx_i^2$ are nowhere vanishing $\frac{1}{2}$-densities on $X_1$ and $X_2$. Then the space $F^m_r(\Gamma)$ of compactly supported semi-classical Fourier integral operators associated to the canonical relation

$$\Gamma = \Gamma_{Z, f} = (\zeta_1 \times id)N^*_fZ \in Morph(T^*X_1, T^*X_2)$$

has elements

$$F_{\mu, \hbar} : L^2(X_1) \to L^2(X_2), gdx_1^{\frac{1}{2}} \mapsto \left( \int_{X_1} g(x_1)u(x_1, x_2, h)dx_1 \right) dx_2^{\frac{1}{2}}$$

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for every
\[
\mu = u(x_1, x_2, h)dx_1^\frac{1}{2}dx_2^\frac{3}{2} \in I_0^0(X, \Lambda),
\]
which in this case is
\[
F_{\mu, h}(gdx_1^\frac{1}{2}) = \hbar^{-\frac{h}{2}}(\int_{X_1} g(x_1)e^{i\sum_{k=0}^{m_1} \frac{n_1}{k} s_i u_i} a(x_1, x_2, s, h)e^{i\sum_{k=0}^{m_2} \frac{n_2}{k} s_i u_i} ds)dx_2^\frac{3}{2}
\]
for every \(a(x_1, x_2, s, h) \in C_0^\infty(X_1 \times X_2 \times \mathbb{R}^{k_2} \times \mathbb{R})\), where \(m = r + \frac{n_2}{2}, n_2 = \dim X_2\).

Note that in the special case when \(a(x_1, x_2, s, h) = a(x_1, x_2, h)\) does not depend on the \(s\)-variables and \(u_i\) are coordinate functions, the inner integraion is just the Fourier transform of some \(x_i\)-variables.

Next we talk about compositions of these operators. Suppose
\[
(X_1, dx_1^\frac{1}{2}), (X_2, dx_2^\frac{3}{2}), (X_3, dx_3^\frac{3}{2})
\]
are smooth manifolds with nowhere vanishing \(\frac{1}{2}\)-densities on them,
\[
Z_1 = \{u_1 = \cdots = u_{k_1} = 0\} \subset X_1 \times X_2 \text{ and } Z_2 = \{v_1 = \cdots v_{k_2} = 0\} \subset X_2 \times X_3
\]
are submanifolds and \(f_1 \in C^\infty(Z_1)\) and \(f_2 \in C^\infty(Z_2)\). Denote the canonical relations by
\[
\Gamma_1 = \Gamma_{Z_1, f_1} \in \text{Morph}(T^*X_1, T^*X_2) \text{ and } \Gamma_2 = \Gamma_{Z_2, f_2} \in \text{Morph}(T^*X_2, T^*X_3).
\]

Suppose
\[
\Gamma_2 \circ \Gamma_1 = \Gamma \in \text{Morph}(T^*X_1, T^*X_3)
\]
is a clean composition. According to a theorem in [2], the composition of Fourier integral operators in \(\mathcal{F}^{m_2}_0(\Gamma_2)\) with operators in \(\mathcal{F}^{m_1}_0(\Gamma_1)\) are the operators in \(\mathcal{F}^{m_1 + m_2 - \frac{n}{2}}_0(\Gamma)\), where \(\epsilon\) is the fiber dimension of projection \(\kappa : \Gamma_1 \times \Gamma_2 \to \Gamma_1 \circ \Gamma_2\). Since the Hörmander descriptions of \(N^*_{f_1}Z_1\) and \(N^*_{f_2}Z_2\) can induce a Hörmander description of \((\zeta_1 \times \text{id})\Gamma_1\), it is thus easily computable that the compositions of \(\mathcal{F}^{m_1}_0(\Gamma_2)\) with \(\mathcal{F}^{m_1}_0(\Gamma_1)\) are of the form
\[
F_{\mu, h} : L^2(X_1) \to L^2(X_3),
\]
\[
F_{\mu, h}(gdx_1^\frac{1}{2}) = \hbar^{-\frac{h}{2}}(\int_{X_1 \times X_2 \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}} g(x_1) a(x_1, x_2, x_3, s, t, h)e^{i\sum_{k=0}^{m_1} \frac{n_1}{k} s_i u_i} a(x_1, x_2, x_3, s, t, h)e^{i\sum_{k=0}^{m_2} \frac{n_2}{k} s_i u_i} ds dt dx_2^\frac{3}{2}
\]
for some \(a(x_1, x_2, x_3, s, t, h) \in C^\infty(X_1 \times X_2 \times X_3 \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R})\), where
\[
m_1 + m_2 - \frac{\epsilon}{2} = r + \frac{n_3}{2} \text{ and } k = n_2 + k_1 + k_2.
\]
5 The intrinsic line bundle of symbols

Recall that for any Lagrangian submanifold \( \Lambda \subset T^*X \), we can associate the Maslov line bundle to it. Specifically, suppose \( \Lambda \) has a Hörmander description \((W, \pi, \phi)\). Let \( z \) be any point in the critical set \( C_{\phi} \subset W \) and \( x = \pi(z) \), then \( z \) must be the critical point of \( \phi|_{\pi^{-1}(x)} \). Denote the signature of \( \phi \) at \( z \in \pi^{-1}(x) \) to be \( \text{sgn}^z \), which gives an integer valued function \( \text{sgn}^z \) on \( C_{\phi} \). By pulling back the function \( \text{sgn}^z \) via the diffeomorphism \( \lambda_{\phi} : C_{\phi} \to \Lambda \)

we get a \( \mathbb{Z} \)-valued function \( \text{sgn}_\phi \) on \( \Lambda \). Let

\[
s_{\phi} : \Lambda \to \mathbb{C}^*, s_{\phi} = e^{\frac{i\pi}{4} \text{sgn}_{\phi}}
\]

and define the Maslov bundle \( \mathbb{L}_{\text{Maslov}} \to \Lambda \) to be the trivial flat bundle having \( s_{\phi} \) as a flat section.

Suppose we have two Hörmander descriptions \((W_i, \pi_i, \phi_i)\) of \( \Lambda \), \( i = 1, 2 \). By the Hörmander-Morse Lemma in [2], each two Hörmander descriptions of \( \Lambda \) are related by the three types of moves. The functions \( s_{\phi_1} \) and \( s_{\phi_2} \) are equal in the type 1) and 2) move, and in the type 3) move the functions \( \text{sgn}_{\phi_1}^z \) and \( \text{sgn}_{\phi_2}^z \) are related by

\[
\text{sgn}_{\phi_1}^z = \text{sgn}_{\phi_2}^z + \text{signature of } \Lambda,
\]

thus \( s_{\phi_1} = s_{\phi_2} e^{\frac{i\pi}{4} \text{signature of } \Lambda} \), i.e. \( s_{\phi_1} \) and \( s_{\phi_2} \) differ by multiple of a constant of norm 1. Thus our definition of Maslov bundle is intrinsic.

For a general Lagrangian submanifold \( \Lambda \subset T^*X \) we can cover \( \Lambda \) by open sets \( U_i \) admitting generating functions \( \phi_i \). By the previous arguments we get functions \( s_{\phi_i} : U_i \to \mathbb{C} \) such that each pair \( s_{\phi_i} \) and \( s_{\phi_j} \) differ by multiple of a constant of norm 1 on \( U_i \cap U_j \). Thus we can patch these local definitions together to get the globally defined Maslov bundle

\[
\mathbb{L}_{\text{Maslov}} \to \Lambda.
\]

According to Hörmander [1], we construct the intrinsic line bundle of symbols

\[
\mathbb{L} = \mathbb{L}_{\text{Maslov}} \otimes |TA|^{\frac{1}{2}}
\]

for any exact Lagrangian submanifold \( \Lambda = (\Lambda, \psi) \subset T^*X \). Suppose \( \Lambda \) has a Hörmander description \((W, \pi, \phi)\) and we pick a nowhere vanishing section of the vertical subbundle \( |V|^{\frac{1}{2}} \) of the fiber bundle \( \pi : W \to X \), which can be deemed as an enhancement of \( \pi \). This enhancement does to things; it both gives us a nowhere vanishing \( \frac{1}{2} \)-density \( \rho_\pi \) on the canonical relation \( \Gamma_\pi \) and a fiber integration

\[
\pi_\pi : C^\infty_0(|TW|^{\frac{1}{2}}) \to C^\infty_0(|TX|^{\frac{1}{2}}).
\]

By the definition in section 3,

\[
\Lambda = \Gamma_\pi \circ \Lambda_{\phi}
\]

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where $\Lambda_\phi$ is the Lagrangian submanifold \{$(p, d\phi_p) : p \in Z$\} of $T^*Z$. Pick a $\frac{1}{2}$-density $\tau$ on $W$ and write a typical element in $I^r_0(X, \Lambda)$ as

$$\mu = h^{-\frac{1}{2}}\pi_\ast \nu, \ \nu = ae^{i\phi}$$

where $a = a(z, h) \in C^\infty_0(W \times \mathbb{R})$. Denote by $\mathcal{P}$ the projection of $\Lambda_\phi$ to $W$. We define the “symbol” of $\nu$ to be $\frac{1}{2}$-density $\sigma(\nu) = \mathcal{P}^\ast (a(z, 0) \tau)$ on $\Lambda$ and the symbol of $\mu$ to be

$$\sigma(\mu) = s_\phi \otimes (\rho_\pi \circ \sigma(\nu)) \in C^\infty(\mathbb{L})$$

where $s_\phi$ is the section of $L_{\text{Maslov}}$ associated with $\phi$ and $\rho_\pi \circ \sigma(\nu) \in C^\infty(|T\Lambda|^\frac{1}{2})$ is a composition in the enhanced symplectic “category”. Call

$$\sigma : I^r_0(X, \Lambda) \to C^\infty(\mathbb{L})$$

the symbol map.

For a general exact Lagrangian submanifold $\Lambda \subset T^*X$ one can add together the symbol map on local Hörmander pieces to get the global symbol map

$$\sigma : I^r(X, \Lambda) \to C^\infty(\mathbb{L})$$

which is proved to be intrinsically defined and surjective with kernel $I^{k+1}(X, \Lambda)$. Hence

$$I^k(X, \Lambda)/I^{k+1}(X, \Lambda) \cong C^\infty(\mathbb{L}).$$

For an exact canonical relation $(\Gamma, \Psi) \in \text{Morph}(T^*X_1, T^*X_2)$ we have the corresponding exact Lagrangian submanifold $(\Lambda, \psi) \subset T^*X_1 \times T^*X_2$ where

$$\Lambda = (\zeta_1 \times id)(\Gamma), \ \psi = \Psi \circ (\zeta_1 \times id).$$

Define the intrinsic line bundle of symbols $\mathbb{L}_\Gamma \to \Gamma$ to be the pullback of the line bundle $L_\Lambda \to \Lambda$ via the diffeomorphism $\zeta_1 \times id$,

$$\mathbb{L}_\Gamma = (\zeta_1 \times id)^\ast \mathbb{L}_\Lambda = L_{\text{Maslov}}(\Gamma) \otimes |T\Gamma|^\frac{1}{2}.$$

For a semi-classical Fourier integral operator $F \in \mathcal{F}^m_0(\Gamma, \Psi)$ given by a oscillatory $\frac{1}{2}$-density $\mu \in I^r_0(X, \Lambda)$, we define its symbol to be

$$\sigma(F) = (\zeta_1 \times id)^\ast \sigma(\mu) \in C^\infty(\mathbb{L}_\Gamma).$$

Suppose $\Gamma_2$ and $\Gamma_1$ are cleanly composable exact canonical relations and $\Gamma = \Gamma_2 \circ \Gamma_1$. Note that the Maslov bundle is functorial under the composition:

$$\kappa^\ast L_{\text{Maslov}}(\Gamma_2 \circ \Gamma_1) \cong \text{pr}_1^\ast L_{\text{Maslov}}(\Gamma_1) \otimes \text{pr}_2^\ast L_{\text{Maslov}}(\Gamma_2)$$
where $\kappa : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$, $\text{pr}_1 : \Gamma_2 \star \Gamma_1 \to \Gamma_1$ and $\text{pr}_2 : \Gamma_2 \star \Gamma_1 \to \Gamma_2$ are canonical projections. This means that given a section of $L_{\text{Maslov}}(\Gamma_2)$ and a section of $L_{\text{Maslov}}(\Gamma_1)$, one can compose them to get a section of $L_{\text{Maslov}}(\Gamma)$. Also we can compose sections of $|T\Gamma_2|^{\frac{1}{2}}$ with sections of $|T\Gamma_1|^{\frac{1}{2}}$ as morphisms in the enhanced symplectic “category” to get a section of $|T(\Gamma_2 \circ \Gamma_1)|^{\frac{1}{2}}$. Thus by tensoring these two compositions we can compose sections of $\mathbb{L}_{\Gamma_2}$ with sections of $\mathbb{L}_{\Gamma_1}$. By a deeper examination of all the composition laws we have

$$\sigma(F_2 \circ F_1) = \sigma(F_2)\sigma(F_1)$$

which means that the composition of symbols defined above is consistent with the composition of Fourier integral operators.

Next we look at the special case when the Lagrangian submanifolds are twisted conormal bundles. Suppose $Z \subset X$ is a submanifold and $f \in C^\infty(Z)$, then the twisted conormal bundle $\Lambda = N^*_fZ$ is an exact Lagrangian submanifold of $T^*X$. As in section 3, on any open set $U \subset X$ where $Z \cap U$ is the common zero set of functionally independent functions $u_1, \ldots, u_k \in C^\infty(U)$, we have a Hörmander description $(W, \pi, \phi)$ of $\Lambda$ on $U$, where $\pi : W \to U$ is the projection from $W = U \times \mathbb{R}^k$ to $U$ and $\phi = \sum_{i=1}^k s_i u_i + f \in C^\infty(W)$. Note that on any fiber $\pi^{-1}(x)$ of $x \in U$, the function $\phi$ is just a linear functional in the variables $s_i$, which has Hessian zero, so the functions $\text{sgn} s_i \equiv 0$ and $s_i \equiv 1$ on $U$. Patching the local data together we get a canonical (global) isomorphism between $L_{\text{Maslov}}(\Lambda)$ and the trivial line bundle $\mathbb{C}$ on $\Lambda$. Theorem 4 follows. Theorem 5 is true because for a twisted conormal bundle $\Lambda = N^*_fZ$, the symbol map

$$\sigma : \Gamma^r(X, \Lambda) \to C^\infty(L_{\text{Maslov}} \otimes |T\Lambda|^{\frac{1}{2}})$$

always has first argument the 1-section, which means that the composition of the Maslov factor is always trivial in the composition of symbols for canonical relations of the form $\Gamma = (\zeta_1 \times \text{id})N^*_fZ$. Thus via the isomorphism given in Theorem 4, the composition of symbols of such operators only work as composition of $\frac{1}{2}$-densities in the enhanced symplectic “category”.

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