Global regularity for the 3D non-diffusive MHD-Boussinesq system with axisymmetric data

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Abstract

In this paper, we will show that the solution of the 3 dimensional non-diffusive MHD-Boussinesq system is globally regular if the initial data is axisymmetric and the swirl component of the velocity and the magnetic vorticity is trivial. Our method can also be applied to the magnetic Rayleigh-Bénard convection system.

Keywords: magnetohydrodynamics, Boussinesq, Rayleigh-Bénard convection, axisymmetric, global regularity.

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1 Introduction

In this paper, we consider the global regularity problem for the 3 dimensional (3D) magnetohydrodynamics(MHD) -Boussinesq system

$$\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = h \cdot \nabla h + \rho e_3, \\
\partial_t u + u \cdot \nabla h - h \cdot \nabla u - \eta \Delta h = 0, \\
\partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = 0, \\
\nabla \cdot u = \nabla \cdot h = 0.
\end{cases}$$

(1.1)

Here $u(t, x), h(x) \in \mathbb{R}^3, p(t, x) \in \mathbb{R}$ and $\rho(t, x) \in \mathbb{R}$ represent the velocity field, magnetic field, pressure and temperature fluctuation. The vector $e_3 = (0, 0, 1)$ is the unit vector in the vertical direction. $\nu \geq 0, \eta \geq 0$ and $\kappa \geq 0$ stands for the constant viscosity, magnetic diffusivity and thermal diffusivity, respectively. The MHD-Boussinesq system models the convection of an incompressible flow driven by the buoyant effect of a thermal field and the Lorenz force.

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generated by the magnetic field. This system is closely related to a type of the Rayleigh-Bénard convection, which occurs in a horizontal layer of conductive fluid heated from below, with a presence of a magnetic field. The only difference between the magnetic Rayleigh-Bénard convection system and the MHD-Boussinesq system is that (1.1) is replaced by the following equation

$$\partial_t \rho + u \cdot \nabla \rho - \kappa \Delta \rho = u^3.$$ 

Various physical theories and numerical experiments have been developed to study the magnetic Rayleigh-Bénard convection and related equations. See, for example, [31, 33] and references therein.

We say that the MHD-Boussinesq system is non-diffusive, which means $\nu > 0$ and $\eta = \kappa = 0$. Without loss of generality, we set $\nu = 1$ and system (1.1) becomes

$$\begin{align*}
&\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = h \cdot \nabla h + \rho e_3, \\
&\partial_t u + u \cdot \nabla h - h \cdot \nabla u = 0, \\
&\partial_t \rho + u \cdot \nabla \rho = 0, \\
&\nabla \cdot u = \nabla \cdot h = 0.
\end{align*}$$

The local well-posedness result of (1.2) can be founded in [27]. However, the global well-posedness is still wildly open even for the Navier-Stokes equations ($h = \rho \equiv 0$), let alone for system (1.2). In this paper, we will show that a family of axisymmetric solutions to (1.2) are globally as regular as the initial data.

In the following, we will carry out our proof in the cylindrical coordinates $(r, \theta, z)$. That is, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3.$$ 

And the axisymmetric solution of system (1.2) is given by

$$u = u^r(t, r, z)e_r + u^\theta(t, r, z)e_\theta + u^z(t, r, z)e_z,$$

$$h = h^r(t, r, z)e_r + h^\theta(t, r, z)e_\theta + h^z(t, r, z)e_z,$$

$$\rho = \rho(t, r, z),$$

where the basis vectors $e_r, e_\theta, e_z$ are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$

We will prove the global regularity of the following family of axisymmetric solutions

$$u = u^r(t, r, z)e_r + u^z(t, r, z)e_z, \quad h = h^\theta(t, r, z)e_\theta, \quad \rho = \rho(t, r, z).$$

(1.3)

Denote

$$\Phi_{k,c}(t) := c \exp(\cdots \exp(ct) \cdots).$$

More precisely, we have the following theorem.
Theorem 1.1. Let $u_0$, $h_0$ and $\rho_0$ are all axially symmetric data with $\nabla \cdot u_0 = \nabla \cdot h_0 = 0$. Besides, we assume that $u_0^\theta = h_0^\theta = h_0^z = 0$. If $(u_0, h_0, \rho_0) \in H^2(\mathbb{R}^3)$ and $J_0 := \frac{h_0^\theta}{\rho} \in L^\infty(\mathbb{R}^3)$, then there exists a unique global solution $(u, h, \rho)$ to the MHD-Boussinesq system (1.2) with data $(u_0, h_0, \rho_0)$, which satisfies

$$
\|(u, h, \rho)(t, \cdot)\|_{H^2}^2 + \int_0^t \|\nabla u(t, \cdot)\|_{H^2}^2 ds \leq \Phi_{3, c_0}(t),
$$

where $c_0$ is a positive constant depending only on $H^2$ norms of $u_0$, $h_0$, $\rho_0$ and $L^\infty$ norm of $J_0$.

Remark 1.1. It is not hard to extend the result of Theorem (1.1) to the case where $\nu > 0$, $\eta \geq 0$ and $\kappa \geq 0$ in (1.1) with the same initial data as in Theorem (1.1).

Remark 1.2. When $h_0^\theta \equiv 0$, the global well-posedness result for the axisymmetric Navier-Stokes-Boussinesq can be found in [2, 13]. While if $\rho \equiv 0$, see [25] for the global well-posedness result for the axisymmetric MHD system. Our main result can be viewed as an extension of those in the above papers.

Remark 1.3. Result in Theorem 1.1 can also be applied to the following non-diffusive magnetic Rayleigh-Bénard convection system

$$
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p - \Delta u &= h \cdot \nabla h + \rho e_3, \\
\partial_t h + u \cdot \nabla h - h \cdot \nabla u &= 0, \\
\partial_t \rho + u \cdot \nabla \rho &= u^3, \\
\nabla \cdot u &= \nabla \cdot h = 0.
\end{align*}
$$

The proof is essentially the same as that for (1.2) with little difference. We omit the details.

If the fluid is not affected by the temperature, then our system (1.1) is reduced to the classical MHD system. There already have been many studies and fruitful results related to the well-posedness of MHD system. Sermange-Temam [24] established the local existence and uniqueness of the solution and particularly the 2D local strong solution was proved to be global. Cao et al. in [10, 9] proved the global regularity of the MHD system for a variety of combinations of partial dissipation and diffusion in 2D space. Lin-Xu-Zhang [28] proved the global well-posedness of classical solutions for 2D non-resistive MHD under the assumption that the initial data is a small perturbation of a nonzero constant magnetic field. See also [23] for similar results. For the 3D case, readers can see [29, 35] for related results. Cai-Lei [7] and He-Xu-Yu [19] proved the global well-posedness of small initial data for the idea (non-viscous and non-resistive) MHD system. Lei [25] proved the global regularity of classical solutions to the idea (non-viscous and non-resistive) MHD system with a family of axisymmetric large data. We also emphasized some partial regularity results and blow up criteria in [17, 18, 8, 30] and references therein.

On the other hand, if the fluid is not affected by the Lorentz force, then our system (1.2) is the classical Boussinesq system without diffusion. Many works and efforts have been made to study the well-posedness of the Cauchy problem for the Boussinesq system. In 2D case, Chae [12] and Hou-Li [15] independently proved the global regularity of solutions to the 2D Boussinesq
system. And also Chae [12] consider the case of zero viscosity and non-zero diffusion. See [1, 16] for related results in critical space. For 3D case, Abidi et al. [2] and Hmidi-Rousset [13, 14] proved the global well-posedness of the Cauchy problem for the 3D axisymmetric Boussinesq system without swirl. Readers can see [21, 11] and references therein for more regularity results on Boussinesq system.

For the full MHD-Boussinesq system, recently, there are also some works concentrated on the global well-posedness of weak and strong solutions. See [3, 4] and references therein for 2D cases. In the 3D case, Larios-Pei [27] proved the the local well-posedness results in Sobolev space. Liu-Bian-Pu [20] proved the global well-posedness of strong solutions with nonlinear damping term in the momentum equations. Regarding the MHD-Bénard system, some progress has also been made in 2D and 3D case. See, e.g., [37, 5, 36, 38] and references therein.

Define

\[ J := \frac{h^\theta}{r}, \quad \Omega := \frac{w^\theta}{r}, \quad w^\theta = \partial_z u^r - \partial_r u^z. \]

The proof of Theorem 1.1 strongly depends on the special structure of the MHD-Boussinesq system in axisymmetric case with zero swirl component of the velocity and the magnetic vorticity. We will show that \( J, \rho \) satisfy the same transport equations and \( \Omega \) satisfies a linear diffusive equation with inhomogeneous term involving only in \( J \) and \( \rho \). See (2.3). Then the \( L^\infty_t L^2_x \) norm of \( \Omega \) will be obtained. This is a key step for us to bootstrap the regularity of \( u, h \) and \( \rho \).

Our paper is organized as follows. In Section 2, we reformulate our system in cylindrical coordinates and proved a a priori \( L^\infty_t L^2_x \) estimate for \( \Omega \). In Section 3, we give the \( H^1 \) a priori estimate of the solution. In Section 4, we give the \( H^2 \) a priori estimate of the solution and prove Theorem 1.1. Throughout the paper, we use \( C \) or \( c \) to denote a generic constant which may be different from line to line. We also apply \( A \lesssim B \) to denote \( A \leq CB \).

2 Reformulation of the system and \( L^\infty_t L^2_x \) estimate of \( \Omega \)

The axisymmetric MHD-Boussinesq system (1.2) in cylindrical coordinates read

\[
\begin{align*}
\partial_t u^r + (u^r \partial_r + u^z \partial_z) u^r - \frac{(u^\theta)^2}{r} + \partial_r P &= (h^r \partial_r + h^z \partial_z) h^r - \frac{(h^\theta)^2}{r} + (\Delta - \frac{1}{r^2}) u^r, \\
\partial_t u^\theta + (u^r \partial_r + u^z \partial_z) u^\theta + \frac{u^\theta u^r}{r} &= (h^r \partial_r + h^z \partial_z) h^\theta + \frac{h^r h^\theta}{r} + (\Delta - \frac{1}{r^2}) u^\theta, \\
\partial_t u^z + (u^r \partial_r + u^z \partial_z) u^z + \partial_z P &= (h^r \partial_r + h^z \partial_z) h^z + \Delta u^z + \rho, \\
\partial_t h^r + (u^r \partial_r + u^z \partial_z) h^r - (h^r \partial_r + h^z \partial_z) u^r &= 0, \\
\partial_t h^\theta + (u^r \partial_r + u^z \partial_z) h^\theta - (h^r \partial_r + h^z \partial_z) u^\theta &= 0, \\
\partial_t h^z + (u^r \partial_r + u^z \partial_z) h^z - (h^r \partial_r + h^z \partial_z) u^z &= 0, \\
\partial_r \rho + (u^r \partial_r + u^z \partial_z) \rho &= 0, \\
\nabla \cdot u &= \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \quad \nabla \cdot h = \partial_r h^r + \frac{h^r}{r} + \partial_z h^z = 0,
\end{align*}
\] (2.1)
where the pressure is $P = p + \frac{1}{2} |h|^2$ and $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the usual Laplacian operator. By the uniqueness of local solutions, it is easy to see that if the initial data satisfy $u_0^\theta = h_0^\theta = h_0^z = 0$, then the solution of (2.1) will be the form of (1.3). In this situation, (2.1) can be simplified as

$$\frac{\partial_t}{r} u^\theta + (u^r \partial_r + u^z \partial_z) u^\theta + \partial_r p = -\frac{(h^\theta)^2}{r} + (\Delta - \frac{1}{r^2}) u^\theta,$$

$$\frac{\partial_t}{r} u^z + (u^r \partial_r + u^z \partial_z) u^z + \partial_z p = \Delta u^z + \rho,$$

$$\partial_t h^\theta + (u^r \partial_r + u^z \partial_z) h^\theta - \frac{u^r}{r} h^\theta = 0,$$

$$\partial_t \rho + (u^r \partial_r + u^z \partial_z) \rho = 0,$$

$$\frac{1}{r} \partial_r (ru^r) + \partial_z u^z = \frac{1}{r} \partial_r (rh^\theta) + \partial_z h^\theta = 0.$$ (2.2)

Denote $J := \frac{h^\theta}{r}$ and $\Omega := \frac{u^\theta}{r}$. From (2.2), we can get

$$\begin{cases}
\frac{\partial_t}{r} \Omega + u \cdot \nabla \Omega = (\Delta + \frac{2}{r} \partial_r) \Omega - \partial_z J^2 - \frac{\partial_r \rho}{r}, \\
\frac{\partial_t}{r} J + u \cdot \nabla J = 0, \\
\frac{\partial_t}{r} \rho + u \cdot \nabla \rho = 0.
\end{cases}$$ (2.3)

First we have the following Proposition

**Proposition 2.1.** Let $(u, h, \rho)$ be a smooth solution of (2.2), then we have

1. for $p \in [1, \infty]$ and $t \in \mathbb{R}_+$, we have

$$\| (J(t), \rho(t)) \|_{L^p} \leq \| (J_0, \rho_0) \|_{L^p},$$ (2.4)

2. for $u_0, h_0, \rho_0 \in L^2$ and $t \in \mathbb{R}_+$, we have

$$\| (u(t), h(t)) \|_{L^2} + \int_0^t \| \nabla u(s) \| ds \leq C_0 (1 + t)^2,$$ (2.5)

where $C_0$ depends only on $\| (u_0, h_0) \|_{L^2}$ and $\| \rho_0 \|_{L^2}$.

**Proof of Proposition 2.1.**

The estimate in (2.4) is classical for the transport equation with finite $p$. While if $p = \infty$, it is just the maximum principle. For the estimate in (2.5), we proceed the standard $L^2$ inner product estimate of system (1.2). Then we have

$$\frac{1}{2} \frac{d}{dt} \| (u(t), h(t)) \|_{L^2}^2 + \| \nabla u(t) \|_{L^2}^2 \leq \| u(t) \|_{L^2} \| \rho(t) \|_{L^2}.$$ (2.6)

This indicates that

$$\frac{d}{dt} \| (u(t), h(t)) \|_{L^2} \leq 2 \| \rho(t) \|_{L^2}.$$
Integration on time indicates that
\[
\| (u(t), h(t)) \|_{L^2} \leq \| (u_0, h_0) \|_{L^2} + 2 \int_0^t \| \rho(\tau) \|_{L^2} d\tau
\]
\[
\leq \| (u_0, h_0) \|_{L^2} + 2 \| \rho_0 \|_{L^2} t.
\]
Inserting this into (2.6) and integration on time, we have
\[
\frac{1}{2} \| (u(t), h(t)) \|_{L^2}^2 + \int_0^t \| \nabla u(s) \|_{L^2}^2 ds
\]
\[
\leq \frac{1}{2} \| (u_0, h_0) \|_{L^2}^2 + \{ \| (u_0, h_0) \|_{L^2} + 2 \| \rho_0 \|_{L^2} t \} \| \rho_0 \|_{L^2} t.
\]
This gives (2.5). \qed

Based on Proposition 2.1, we have the following Proposition which gives the a priori $L_t^\infty L_x^2$ estimate of $\Omega$.

**Proposition 2.2.** Suppose $(u, h, \rho)$ be the smooth solution of (1.2) with initial data $(u_0, h_0, \rho_0)$ satisfying assumptions in Theorem 1.1, then we have, for $t \in \mathbb{R}^+$,
\[
\| \Omega(t) \|_{L^2} \leq \Phi_{1,c_0}(t),
\] (2.7)
where $c_0$ is a positive constant depending only on $H^2$ norms of $u_0, h_0, \rho_0$ and $L^\infty$ norm of $J_0$.

Before proving Proposition 2.2, we collect some useful estimates and identities.

**Lemma 2.1** (Proposition 3.1, 3.2 and Lemma 3.3 of [13]). Denote $\mathcal{L} = (\Delta + \frac{2}{r} \partial_r)^{-1} \partial_r$ and $\tilde{\mathcal{L}} = (\Delta + \frac{2}{r} \partial_r)^{-1} \frac{2}{r} \partial_r$. Suppose $\rho \in H^2(\mathbb{R}^3)$ be axisymmetric, then for every $p \in [2, +\infty)$, there exists an absolute constant $C_p > 0$ such that
\[
\| \mathcal{L} \rho \|_{L^p} \leq C_p \| \rho \|_{L^p}, \quad \| \tilde{\mathcal{L}} \rho \|_{L^p} \leq C_p \| \rho \|_{L^p}.
\] (2.8)
Moreover, for any smooth axisymmetric function $f$, we have the identity
\[
\mathcal{L} \partial_r f = \frac{\tilde{f}}{r} - \mathcal{L} \left( \frac{\tilde{f}}{r} \right) - \partial_z \tilde{\mathcal{L}} f.
\] (2.9)

**Lemma 2.2.** For $1 < p < +\infty$, there exists an absolute constant $C_p > 0$ such that
\[
\| \nabla \frac{u^r}{r} \|_{L^p} \leq C_p \| \Omega \|_{L^p}.
\] (2.10)

The proof of this lemma can be founded in many literatures, such as [25] (A.5 on page 3213), [6] (Lemma 2.3) or [32] (Proposition 2.5).

**Proof of Proposition 2.2**

Applying $\mathcal{L}$ to (2.3)_3, we get
\[
\partial_t \mathcal{L} \rho + u \cdot \nabla \mathcal{L} \rho = -[\mathcal{L}, u \cdot \nabla] \rho.
\] (2.11)
where $[A, B] = AB - BA$ is the commutator.

Denote $\Gamma := \Omega - L\rho$. Subtract (2.11) from (2.3), we have

$$\partial_t \Gamma + u \cdot \nabla \Gamma - (\Delta + \frac{2}{r} \partial_r) \Gamma = [L, u \cdot \nabla] \rho - \partial_z J^2. \quad (2.12)$$

Taking $L^2$ inner product of (2.12), using integration by parts and divergence-free condition of $u$, we get

$$\frac{1}{2} \frac{d}{dt} \|\Gamma(t)\|_{L^2}^2 + \|\nabla \Gamma(t)\|_{L^2}^2 \leq \int_{\mathbb{R}^3} L(u \cdot \nabla \rho) \Gamma dx - \int_{\mathbb{R}^3} u \cdot \nabla (L\rho) \Gamma dx - \int_{\mathbb{R}^3} \partial_z J^2 \Gamma dx$$

Next we will estimate $I_i (i = 1, 2, 3)$ term by term. For $I_1$, first we make some computation on $L(u \cdot \nabla \rho)$.

$$L(u \cdot \nabla \rho) = L(\nabla \cdot (u\rho)) = L \left( \partial_r (u^r \rho) + \frac{1}{r} (u^r \rho) + \partial_z (u^z \rho) \right).$$

From (2.9), we have

$$L(u \cdot \nabla \rho) = L \partial_r (u^r \rho) + L \left( \frac{u^r \rho}{r} \right) + L \partial_z (u^z \rho) = \frac{u^r \rho}{r} - \partial_z L(u^r \rho) + \partial_z L(u^z \rho),$$

where we have used the fact that $\partial_z$ is commutated with $L$.

Then, using integration by parts, we get

$$I_1 = \int_{\mathbb{R}^3} \frac{u^r}{r} \rho \Gamma dx + \int_{\mathbb{R}^3} \tilde{L}(u^r \rho) \partial_z \Gamma dx - \int_{\mathbb{R}^3} L(u^z \rho) \partial_z \Gamma dx$$

Using H"older inequality, Sobolev embedding and (2.10), we have

$$|I_1^1| \leq \frac{\|u^r\|_{L^6}}{r} \|\rho\|_{L^3} \|\Gamma\|_{L^2} \leq \frac{\|\nabla u^r\|_{L^2} \|\rho\|_{L^3} \|\Gamma\|_{L^2}}{r}$$

Using (2.8), (2.4) and Sobolev embedding, we have

$$|I_1^1| \leq C \left( \|\Gamma\|_{L^2} + \|\rho\|_{L^2} \right) \|\rho\|_{L^3} \|\Gamma\|_{L^2} \leq C \|\rho_0\|_{H^2} \|\Gamma\|_{L^2} + C \|\rho_0\|_{H^2} \|\Gamma\|_{L^2} \leq C \left( \|\rho_0\|_{H^2} + 1 \right) \|\Gamma\|_{L^2}^2 + C \|\rho_0\|_{H^2}^4.$$
From (2.8), Proposition 2.1 and using Hölder inequality, Young inequality, we have

\[ |I_1^2| + |I_3^1| \leq \left( \| \mathcal{L}(u^r \rho) \|_{L^2} + \| \mathcal{L}(u^r \rho) \|_{L^2} \right) \| \partial_z \Gamma \|_{L^2} \]
\[ \leq C \| u^r \rho \|_{L^2} \| \partial_z \Gamma \|_{L^2} \]
\[ \leq C \| \rho_0 \|_{L^\infty} \| u \|_{L^2} \| \partial_z \Gamma \|_{L^2} \]
\[ \leq C \| \rho_0 \|_{L^2}^2 \| u \|_{L^2}^2 + \frac{1}{4} \| \partial_z \Gamma \|_{L^2}^2 \]
\[ \leq C_0(1 + t)^2 + \frac{1}{4} \| \partial_z \Gamma \|_{L^2}^2. \]

Also, the same techniques as above imply

\[ |I_2^2| + |I_3^0| \leq \left( \| \mathcal{L}(\mathcal{L}(\rho)) u \|_{L^2} + \| J_2 \|_{L^2} \right) \| \nabla \Gamma \|_{L^2} \]
\[ \leq \left( \| \mathcal{L}(\rho) \|_{L^3} \| u \|_{L^6} + \| J_0 \|_{L^\infty} \| J \|_{L^2} \right) \| \nabla \Gamma \|_{L^2} \]
\[ \leq \left( \| \rho \|_{L^2} \| \nabla u \|_{L^2} + \| J_0 \|_{L^\infty} \| \nabla u \|_{L^2} \right) \| \nabla \Gamma \|_{L^2} \]
\[ \leq \left( \| \rho_0 \|_{L^2}^2 + \| J_0 \|_{L^\infty}^2 \right) \| \nabla u \|_{L^2}^2 + \frac{1}{4} \| \nabla \Gamma \|_{L^2}^2 \]
\[ \leq C \left( \| \rho_0 \|_{H^2}^2 + \| J_0 \|_{L^\infty}^2 \right) \| \nabla u \|_{L^2}^2 + \frac{1}{4} \| \nabla \Gamma \|_{L^2}^2. \]

The above estimates indicate that

\[ \frac{d}{dt} \| \Gamma(t) \|_{L^2}^2 + \| \nabla \Gamma(t) \|_{L^2}^2 \leq C \left( \| \rho_0 \|_{H^2}^2 + \| J_0 \|_{L^\infty}^2 \right) \| \nabla u \|_{L^2}^2 + C_0(1 + t)^2 + C \| \rho_0 \|_{H^2}^4. \]

Gronwall inequality indicates that

\[ \| \Gamma(t) \|_{L^2}^2 + \int_0^t \| \nabla \Gamma(s) \|_{L^2}^2 ds \leq \Phi_{1,c_0}(t). \]

Then we have

\[ \| \Omega(t) \|_{L^2} \leq \| \Gamma \|_{L^2} + \| \mathcal{L} \rho \|_{L^2} \]
\[ \leq \| \Gamma \|_{L^2} + C \| \rho \|_{L^2} \]
\[ \leq \| \Gamma \|_{L^2} + \| \rho_0 \|_{L^2} \leq \Phi_{1,c_0}(t). \]

This proves Proposition 2.2 and (2.7) is valid. \( \square \)
\section{H^1 estimate of the solution}

In this section, we give a prior \(H^1\) estimate for the solution of system 2.2. We have the following Proposition.

\textbf{Proposition 3.1.} Suppose \((u, h, \rho)\) be the smooth solution of (1.2) with initial data \((u_0, h_0, \rho_0)\) satisfying assumptions in Theorem 1.1, then we have, for \(t \in \mathbb{R}_+\),

\[\|(\nabla u(t), \nabla h(t), \nabla \rho(t))\|_{L^2}^2 + \int_0^t \|\nabla^2 u(s)\|_{L^2}^2 ds \leq \Phi_{2,c_0}(t).\]  

(3.1)

where \(c_0\) is a positive constant depending only on \(H^2\) norms of \(u_0, h_0, \rho_0\) and \(L^\infty\) norm of \(J_0\).

\subsection{L^\infty L^2 estimate of \(w = \nabla \times u\)}

In cylindrical coordinates, the vorticity of the swirl-free axisymmetric velocity \(u\) is given by \(w = \nabla \times u = w^\theta e_\theta\) and \(w^\theta\) satisfies

\[\partial_t w^\theta + u \cdot \nabla w^\theta - (\Delta - \frac{1}{r^2})w^\theta - \frac{u^r}{r}w^\theta = -\partial_z (\frac{h^\theta}{r})^2 - \partial_r \rho.\]

Performing the standard \(L^2\) inner product, we have

\[\frac{1}{2} \frac{d}{dt}\|w^\theta\|_{L^2}^2 + \|\nabla w^\theta\|_{L^2}^2 + \frac{1}{r}\|w^\theta\|_{L^2}^2 \leq \int_{\mathbb{R}^3} \frac{u^r}{r}(w^\theta)^2 dx - \int_{\mathbb{R}^3} \partial_z (\frac{h^\theta}{r})^2 w^\theta dx - \int_{\mathbb{R}^3} \partial_r \rho w^\theta dx\]

\[= I_1 + I_2 + I_3.\]

We estimate \(I_i\) \((i = 1, 2, 3)\) separately. the Hölder inequality and Gagliardo-Nirenberg interpolation inequality imply that

\[I_1 \leq \|u^r\|_{L^3}\|\frac{w^\theta}{r}\|_{L^2}\|w^\theta\|_{L^6}\]

\[\leq \|u^r\|_{L^3}\|\Omega\|_{L^2}\|\nabla w^\theta\|_{L^2}\]

\[\leq C\|w^\theta\|_{L^3}^2\|\Omega\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2\]

\[\leq C\|u^r\|_{L^2}\|\nabla u^r\|_{L^2}\|\Omega\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2,\]

and

\[I_2 = \int_{\mathbb{R}^3} \frac{(h^\theta)^2}{r} \partial_z w^\theta dx\]

\[\leq \|J\|_{L^\infty}\|h^\theta\|_{L^2}\|\nabla w^\theta\|_{L^2}\]

\[\leq C\|J\|_{L^\infty}^2\|h^\theta\|_{L^2}^2 + \frac{1}{4}\|\nabla w^\theta\|_{L^2}^2.\]
Also

\[ I_3 = -2\pi \int_{\mathbb{R}} \int_0^\infty \partial_r \rho w^\theta r dr dz \]
\[ = 2\pi \int_{\mathbb{R}} \int_0^\infty \rho \partial_r (w^\theta r) dr dz \]
\[ = 2\pi \int_{\mathbb{R}} \int_0^\infty \rho \partial_r w^\theta r dr dz + 2\pi \int_{\mathbb{R}^3} \rho w^\theta dr dz \]
\[ \leq \|\rho\|_{L^2} \|\nabla w^\theta\|_{L^2} + \|\rho\|_{L^2} \left| \frac{w^\theta}{r} \right|_{L^2} \]
\[ \leq C \|\rho\|_{L^2}^2 \|\nabla w^\theta\|_{L^2}^2 + \frac{1}{4} \left( \|\nabla w^\theta\|_{L^2}^2 + \left| \frac{w^\theta}{r} \right|_{L^2}^2 \right). \]

The above estimates and Proposition 2.1, Proposition 2.2 indicate that

\[ \frac{d}{dt} \|w^\theta\|_{L^2}^2 + \|\nabla w^\theta\|_{L^2}^2 + \left| \frac{w^\theta}{r} \right|_{L^2}^2 \]
\[ \leq C \|u^r\|_{L^2} \|\nabla u^r\|_{L^2} \left| \frac{\Omega}{r} \right|_{L^2}^2 + C \|J\|_{L^\infty}^2 \|h\|_{L^2}^2 + C \|\rho\|_{L^2}^2 \]
\[ \leq C_0 (1 + t) \Phi_{1,c_0}(t) \|\nabla u^r\|_{L^2} + C_0 \|J_0\|_{L^\infty}^2 (1 + t)^2 + C \|\rho_0\|_{L^2}^2. \]

Integration on time implies that

\[ \|w^\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla w^\theta(s)\|_{L^2}^2 ds + \int_0^t \left| \frac{w^\theta}{r}(s) \right|_{L^2}^2 ds \leq \Phi_{1,c_0}(t). \] (3.2)

Using the identity \( \nabla \times \nabla \times u = -\Delta u + \nabla \nabla \cdot u \) and \( u \) is divergence-free condition of \( u \), we have

\[ \nabla u = \nabla (-\Delta)^{-1} \nabla \times w = \nabla (-\Delta)^{-1} \nabla \times (w^\theta e_\theta). \] (3.3)

Calderón-Zygmund theorem implies that for any \( 1 < p < +\infty \), we have

\[ \|\nabla u(t)\|_{L^p} \leq C_p \|w^\theta\|_{L^p}. \] (3.4)

from (3.2) and (3.4), we see that

\[ \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla^2 u(s)\|_{L^2}^2 ds \leq \Phi_{1,c_0}(t). \] (3.5)

In order to bootstrap our energy estimates, we need the \( L_t^1 L_x^\infty \) estimate of \( u \). Before getting the that, we first performing the \( L_t^\infty L_x^4 \) estimates of \( h^\theta \) and \( w^\theta \).
3.2 $L_t^\infty L^4$ estimate of $h^\theta$ and $w^\theta$

Performing $L^4$ inner product of $h^\theta$ and using Hölder inequality, Gagliardo-Nirenberg interpolation inequality, we see that

$$\frac{d}{dt} \|h^\theta(t)\|_{L^4}^4 \leq 4 \int_{\mathbb{R}^3} \frac{u^r}{r} (h^\theta)^4 \, dx$$

$$\leq 4 \|J\|_{L^\infty} \int_{\mathbb{R}^3} |u^r|(h^\theta)^3 \, dx$$

$$\leq 4 \|J_0\|_{L^\infty} \|u^r\|_{L^4} \|h^\theta\|_{L^4}^3$$

$$\leq C \|J_0\|_{L^\infty} \|\nabla u^r\|_{L^2}^{3/4} \|u^r\|_{L^2}^{1/4} \|h^\theta\|_{L^4}^3.$$ 

Integration on time implies that

$$\|h^\theta(t)\|_{L^4} \leq \Phi_{1,c_0}(t). \quad (3.6)$$

Next performing the standard $L^4$ inner product of the $w^\theta$ equation, we have

$$\frac{1}{4} \frac{d}{dt} \|w^\theta\|_{L^4}^4 + \frac{3}{4} \|\nabla |w^\theta|^2\|_{L^2}^2 + \||w^\theta|^2\|_{L^2}^2$$

$$\leq \int_{\mathbb{R}^3} \frac{u^r}{r} (w^\theta)^4 \, dx - \int_{\mathbb{R}^3} \partial_z \frac{(h^\theta)^2}{r} (w^\theta)^3 \, dx - \int_{\mathbb{R}^3} \partial_r \rho (w^\theta)^3 \, dx$$

$$:= I_1 + I_2 + I_3.$$

By the Hölder inequality, Gagliardo-Nirenberg interpolation inequality and Young inequality, we have

$$I_1 \leq \|u^r\|_{L^4} \|w^\theta\|_{L^2} \|(w^\theta)^3\|_{L^4}$$

$$\leq C \|u^r\|_{L^2}^{1/4} \|\nabla u^r\|_{L^2}^{3/4} \|\Omega\|_{L^2} \|(w^\theta)^2\|_{L^6}^{3/2}$$

$$\leq C \|u^r\|_{L^2}^{1/4} \|\nabla u^r\|_{L^2}^{3/4} \|\Omega\|_{L^2} \|\nabla (w^\theta)^2\|_{L^2}^{3/2}$$

$$\leq C \|u^r\|_{L^2} \|\nabla u^r\|_{L^2}^{3/4} \|\Omega\|_{L^2}^{1/4} + \frac{1}{8} \|\nabla (w^\theta)^2\|_{L^2}^2.$$ 

Also, Hölder inequality and Young inequality imply

$$I_2 = \int_{\mathbb{R}^3} \frac{(h^\theta)^2}{r} \partial_z (w^\theta)^3 \, dx$$

$$= 3 \int_{\mathbb{R}^3} \frac{(h^\theta)^2}{r} (w^\theta)^2 \partial_z w^\theta \, dx$$

$$\leq C \|J\|_{L^\infty} \|h^\theta\|_{L^4} \|w^\theta\|_{L^2} \|\partial_z w^\theta\|_{L^2} \|w^\theta\|_{L^4}$$

$$\leq C \|J_0\|_{L^\infty} \|h^\theta\|_{L^4} + \frac{1}{8} \|\partial_z (w^\theta)^2\|_{L^2}^2 + \|w^\theta\|_{L^4}^4,$$
and the same, we have

\[ I_3 = -2\pi \int_0^\infty \int_{-\infty}^\infty \partial_r \rho (w^\theta)^3 r dr dz \]

\[ I_3 = 2\pi \int_0^\infty \int_{-\infty}^\infty \rho \partial_r ((w^\theta)^3 r) dr dz \]

\[ I_3 = 6\pi \int_0^\infty \int_{-\infty}^\infty \rho (w^\theta)^2 \partial_r w^\theta r dr dz + 2\pi \int_{\mathbb{R}^3} \rho (w^\theta)^3 dr dz \]

\[ \leq C \| \rho \|_{L^\infty} \| \nabla (w^\theta)^2 \|_{L^2} \| w^\theta \|_{L^2} + \| \rho \|_{L^\infty} \left( \frac{(w^\theta)^2}{r} \right) \| \rho \|_{L^2} \| w^\theta \|_{L^2} \]

\[ \leq C \| \rho \|_{L^\infty} \| w^\theta \|^2_{L^2} + \frac{1}{4} \| \nabla (w^\theta)^2 \|^2_{L^2} + \frac{1}{4} \left( \frac{(w^\theta)^2}{r} \right) \| \rho \|_{L^2} \| w^\theta \|^2_{L^2} \].

Using (3.5), (3.6) and Proposition 2.1, the above inequalities implies

\[ \frac{d}{dt} \| w^\theta \|^4_{L^4} + \| \nabla |w^\theta|^2 \|^2_{L^2} + \| \frac{|w^\theta|^2}{r} \|^2_{L^2} \]

\[ \leq C \| w^\theta \|^4_{L^4} + C \| u^\theta \|_{L^2} \| \nabla |u^\theta|^3 \|^4_{L^2} + C \| J_0 \|^4_{L^\infty} \| h^\theta \|^4_{L^4} + C \| \rho \|^2_{L^\infty} \| w^\theta \|^2_{L^2} \]

\[ \leq C \| w^\theta \|^4_{L^4} + \Phi_{1,c_0}(t) \].

Gronwall inequality implies that

\[ \| w^\theta (t) \|^4_{L^4} + \int_0^t \| \nabla |w^\theta (s)|^2 \|^2_{L^2} ds + \int_0^t \left( \frac{(w^\theta)^2}{r} (s) \right) \|^2_{L^2} ds \leq \Phi_{1,c_0}(t). \]

The above inequality implies that

\[ \| \nabla u(t) \|_{L^4} \leq \Phi_{1,c_0}(t). \]  

(3.7)

Next we give a crucial estimate for bootstrapping the regularity of the solution.

### 3.3 \( L^1_t L^\infty \) estimate of \( \nabla u \)

Applying \( \nabla \times \) to (1.2)\(_1\), we have

\[ \partial_t w - \Delta w = -\nabla \times [u \cdot \nabla u - h \cdot \nabla h - \rho e_3]. \]  

(3.8)

For a \( H^1 \) vector function \( f \), we have

\[ (\nabla \times f) \times f = f \cdot \nabla f - \frac{1}{2} \nabla \| f \| \].

Then we have

\[ \nabla \times (f \cdot \nabla f) = \nabla \times [(\nabla \times f) \times f]. \]

Inserting this into (3.8), we have

\[ \partial_t w - \Delta w = -\nabla \times [(\nabla \times u) \times u - (\nabla \times h) \times h - \rho e_3]. \]
Then we can write it as
\[
w = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} (\nabla \times [(\nabla \times u) \times u - (\nabla \times h) \times h - \rho e_3]) \, ds
\]
\[
= e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times u) \times u] \, ds
\]
\[
+ \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times h) \times h] \, ds + \int_0^t e^{(t-s)\Delta} \nabla \times [\rho e_3] \, ds.
\]
By a direct computation, if \( h = h^\theta e_\theta \), we can get
\[
\nabla \times [(\nabla \times h) \times h] = -2\frac{\partial h^\theta}{r} \partial_z h^\theta e_\theta = -\partial_z (Jh^\theta e_\theta).
\]
Then we have
\[
w = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} (\nabla \times [(\nabla \times u) \times u - (\nabla \times h) \times h - \rho e_3]) \, ds
\]
\[
= e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \nabla \times [(\nabla \times u) \times u] \, ds
\]
\[
- \int_0^t e^{(t-s)\Delta} \partial_z (Jh^\theta e_\theta) \, ds + \int_0^t e^{(t-s)\Delta} \nabla \times [\rho e_3] \, ds.
\]
Then by using (3.7), the \( L_t^q L_x^q \) estimates for the parabolic equation of singular integral and potentials (see, for example, [26, 34]) give that
\[
\| \nabla u \|_{L^1([0, t], L^4(\mathbb{R}^3))} \leq \| w_0 \|_{L^4(\mathbb{R}^3)} + C \| (\nabla \times u) \times u \|_{L^1([0, t], L^4(\mathbb{R}^3))} + \| Jh^\theta \|_{L^1([0, t], L^4(\mathbb{R}^3))} + \| \rho \|_{L^1([0, t], L^4(\mathbb{R}^3))}
\]
\[
\leq \| w_0 \|_{L^4(\mathbb{R}^3)} + C \| \nabla u \|_{L^\infty(\mathbb{R}^3)} \| \nabla \times u \|_{L^1([0, t], L^4(\mathbb{R}^3))} + \| Jh^\theta \|_{L^1([0, t], L^4(\mathbb{R}^3))} + \| \rho \|_{L^1([0, t], L^4(\mathbb{R}^3))}
\]
\[
\leq \| w_0 \|_{L^4(\mathbb{R}^3)} + C \| \nabla u \|_{L^\infty(\mathbb{R}^3)} \| \nabla u \|_{L^1([0, t], L^4(\mathbb{R}^3))}^{1/7} \| \nabla u \|_{L^1([0, t], L^4(\mathbb{R}^3))}^{6/7} + \| \rho \|_{L^1([0, t], L^4(\mathbb{R}^3))}
\]
\[
\leq \partial \Phi_{1, c_0}(t).
\]
This, combining with (3.3), implies
\[
\| \nabla^2 u \|_{L^1([0, t], L^4(\mathbb{R}^3))} \leq C \| \nabla u \|_{L^1([0, t], L^4(\mathbb{R}^3))} \leq \partial \Phi_{1, c_0}(t).
\]
Then by using Holder inequality and Gagliardo-Nirenberg interpolation inequality, we have
\[
\| \nabla u \|_{L^1([0, t], L^\infty(\mathbb{R}^3))} \leq \int_0^t \| \nabla u(s) \|_{L^4}^{1/4} \| \nabla^2 u(s) \|_{L^4}^{3/4} \, ds
\]
\[
\leq \| \nabla u(s) \|_{L^4}^{1/4} \left( \int_0^t \| \nabla^2 u(s) \|_{L^4} \, ds \right)^{3/4} \left( \int_0^t ds \right)^{1/4}
\]
\[
\leq \partial \Phi_{1, c_0}(t).
\]
(3.9)
Next we will use \( L_t^1 L_x^\infty \) estimate of \( \nabla u \) to bootstrap the regularity of the solution.
3.4 Estimate of $\nabla \rho$ and $\nabla h$

Applying $\nabla$ to the third equation of (1.2), we have
\[
\partial_t \nabla \rho + u \cdot \nabla \nabla \rho = -\nabla u \cdot \nabla \rho.
\]
We can have for $1 \leq p \leq \infty$,
\[
\|\nabla \rho(t)\|_{L^p} \leq \|\nabla \rho_0\|_{L^p} + C \int_0^t \|\nabla u\|_{L^\infty}\|\nabla \rho(s)\|_{L^p} ds.
\]
Using the estimate (3.9), Gronwall inequality indicates that
\[
\|\nabla \rho(t)\|_{L^p} \leq \Phi_{2,c_0}(t).
\tag{3.10}
\]
For the estimate of $\nabla h$, first we write the second equation of (1.2) as
\[
\partial_t h + u \cdot \nabla h = \frac{u^r}{r} h.
\]
Applying $\nabla$ to the above equality, we have
\[
\partial_t \nabla h + u \cdot \nabla \nabla h = -\nabla u \cdot \nabla h + \frac{u^r}{r} \nabla h + \nabla u^r \mathbf{e}_\theta + \left(\frac{1}{r^2}\right) u^r h.
\]
Noting
\[
\left(\frac{1}{r^2}\right) u^r h = -\frac{1}{r^2} e_r u^r h = -\frac{u^r}{r} \mathbf{e}_r \otimes e_\theta.
\]
We can have for $1 \leq p \leq \infty$
\[
\|\nabla h(t)\|_{L^p} \leq \|\nabla h_0\|_{L^p} + C \int_0^t \|\nabla \rho(s)\|_{L^p} ds
+ C \int_0^t \|\nabla u(s)\|_{L^\infty}\|\nabla h(s)\|_{L^p} ds.
\]
Also using the estimate (3.9), Gronwall inequality indicates that
\[
\|\nabla h(t)\|_{L^p} \leq \Phi_{2,c_0}(t).
\tag{3.11}
\]
Combining the estimates in (3.5), (3.10) and (3.11), we finish the proof of Proposition 3.1 and (3.1) is valid.

\[ \square \]

4 $H^2$ estimates of the solution and proof of Theorem 1.1

In this section, we give a prior $H^2$ estimate for the solution of system 2.2. We have the following Proposition.

**Proposition 4.1.** Suppose $(u, h, \rho)$ be the smooth solution of (1.2) with initial data $(u_0, h_0, \rho_0)$ satisfying assumptions in Theorem 1.1, then we have, for $t \in \mathbb{R}_+$,
\[
\|\nabla^2 u(t), \nabla^2 h(t), \nabla^2 \rho(t)\|_{L^2}^2 + \int_0^t \|\nabla^3 u(s)\|_{L^2}^2 ds \leq \Phi_{3,c_0}(t).
\tag{4.1}
\]
where $c_0$ is a positive constant depending only on $H^2$ norms of $u_0, h_0, \rho_0$ and $L^\infty$ norm of $J_0$. 
4.1 Estimate of $\nabla^2 u$, $\nabla^2 h$

Applying $\nabla^2$ to (1.2), we have

\[
\left\{ \begin{array}{ll}
\partial_t \nabla^2 u + u \cdot \nabla \nabla^2 u + \nabla \nabla^2 u - \Delta \nabla^2 u - h \cdot \nabla \nabla^2 h = -[\nabla^2, u \cdot \nabla]u + \nabla^2 (\rho e_3), \\
\partial_t \nabla^2 h + u \cdot \nabla \nabla^2 h - h \cdot \nabla \nabla^2 u = -[\nabla^2, u \cdot \nabla]h + [\nabla^2, h \cdot \nabla]u. 
\end{array} \right. 
\]

(4.2)

Next we will use the following commutator estimate due to Kato-Ponce [22],

\[
\| \Lambda^m (fg) - f \Lambda^m g \|_{L^p} \leq C \left( \| \nabla f \|_{L^{p_1}} \| \Lambda^{m-1} g \|_{L^{p'_1}} + \| \Lambda^m f \|_{L^{p_2}} \| g \|_{L^{p'_2}} \right) 
\]

(4.3)

with $m \in \mathbb{N}$, $\Lambda = (-\Delta)^{1/2}$ and $1/p = 1/p_1 + 1/p'_1 = 1/p_2 + 1/p'_2$.

Performing the $L^2$ energy estimate of (4.2), we have

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla^2 u(t) \|_{L^2}^2 + \| \nabla^2 h \|_{L^2}^2 \right) + \| \nabla^3 u(t) \|_{L^2}^2 \\
= - \int_{\mathbb{R}^3} [\nabla^2, u \cdot \nabla]u \nabla^2 u dx - \int_{\mathbb{R}^3} [\nabla^2, u \cdot \nabla]h \nabla^2 h dx \\
+ \int_{\mathbb{R}^3} [\nabla^2, h \cdot \nabla]u \nabla^2 h dx + \int_{\mathbb{R}^3} \nabla^2 (\rho e_3) \nabla^2 u dx \\
:= I_1 + I_2 + I_3 + I_4.
\]

We estimate $I_i$ ($i = 1, 2, 3, 4$) term by term. Using (4.3), we have

\[
I_1 \leq \| [\nabla^2, u \cdot \nabla]u \|_{L^2(\mathbb{R}^3)} \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} \\
\leq \| \nabla u \|_{L^\infty} \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} \\
\leq \| \nabla u \|_{L^\infty} \| \nabla^2 u \|_{L^2(\mathbb{R}^3)}^2,
\]

and the commutator estimate (4.3), Gagliardo-Nirenberg interpolation inequality and Young inequality imply

\[
I_2 \leq \| [\nabla^2, u \cdot \nabla]h \|_{L^2(\mathbb{R}^3)} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \\
\leq \left( \| \nabla u \|_{L^\infty} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} + \| \nabla^2 u \|_{L^3(\mathbb{R}^3)} \| \nabla h \|_{L^6(\mathbb{R}^3)} \right) \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \\
\leq \| \nabla u \|_{L^\infty} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} + \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} \| \nabla^3 u \|_{L^2(\mathbb{R}^3)} + \| \nabla h \|_{L^6(\mathbb{R}^3)} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \\
\leq \| \nabla u \|_{L^\infty} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} + C \| \nabla h \|_{L^6(\mathbb{R}^3)} \left( \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} + \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \right)^2 + \frac{1}{4} \| \nabla^3 u \|_{L^2(\mathbb{R}^3)}^2 \\
\leq \| \nabla u \|_{L^\infty} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} + \Psi_{2,0}(t) \left( \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} + \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \right)^2 + \frac{1}{4} \| \nabla^3 u \|_{L^2(\mathbb{R}^3)}^2.
\]

The same, we can get

\[
I_3 \leq \| \nabla u \|_{L^\infty} \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} + \Phi_{2,0}(t) \left( \| \nabla^2 u \|_{L^2(\mathbb{R}^3)} + \| \nabla^2 h \|_{L^2(\mathbb{R}^3)} \right)^2 + \frac{1}{4} \| \nabla^3 u \|_{L^2(\mathbb{R}^3)}^2,
\]
\[ |I_4| \leq \left| \int_{\mathbb{R}^3} \nabla (\rho e_3) \nabla^3 u dx \right| \leq \|\nabla \rho\|_{L^2} \|\nabla^3 u\|_{L^2} \leq \frac{1}{4} \|\nabla^3 u\|_{L^2}^2 + C\|\nabla \rho\|_{L^2}^2. \]

The above estimates indicate that
\[ \frac{1}{2} \frac{d}{dt} \left( \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 h\|_{L^2}^2 \right) + \|\nabla^3 u(t)\|_{L^2}^2 \leq \left( \|\nabla u\|_{L^\infty} + \Phi_{2,c_0} \right) \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) + \Phi_{2,c_0}. \]

Gronwall inequality indicates that
\[ \left( \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 h\|_{L^2}^2 \right) + \int_0^t \|\nabla^3 u(s)\|_{L^2}^2 ds \leq \Phi_{3,c_0}(t). \] (4.4)

Next we give the estimate of \( \nabla^2 \rho \). Applying \( \nabla^2 \) to the third equation of (1.2), we have
\[ \partial_t \nabla^2 \rho + u \cdot \nabla \nabla^2 \rho = -[\nabla^2, u \cdot \nabla] \rho. \]

Standard \( L^2 \) energy estimate implies that
\[ \|\nabla^2 \rho(t)\|_{L^2} \leq \|\nabla^2 \rho_0\|_{L^2} + C \int_0^t \|\nabla^2, u \cdot \nabla \| \rho\|_{L^2} ds \]
\[ \leq \|\nabla^2 \rho_0\|_{L^2} + C \int_0^t \left( \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} + \|\nabla^2 u\|_{L^6} \|\nabla \rho\|_{L^6} \right) ds \]
\[ \leq \|\nabla^2 \rho_0\|_{L^2} + C \int_0^t \left( \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} + \|\nabla^2 u\|_{L^2}^{1/2} \|\nabla^3 u\|_{L^2}^{1/2} \|\nabla \rho\|_{L^6} \right) ds \]
\[ \leq \|\nabla^2 \rho_0\|_{L^2} + C \int_0^t \|\nabla u\|_{L^\infty} \|\nabla^2 \rho\|_{L^2} ds + \Phi_{3,c_0}(t). \]

Gronwall inequality indicates that
\[ \|\nabla^2 \rho(t)\|_{L^p} \leq \Phi_{3,c_0}(t). \] (4.5)

The combination of (4.4) and (4.5) proves Proposition 4.1 and (4.1) is valid.

**Proof of Theorem 1.1** Combining Proposition 2.1, Proposition 3.1 and Proposition 4.1, we can get the a priori estimate (1.4). Then the local existence and uniqueness theorem in [27] and the a priori estimate (1.4) together prove Theorem 1.1.

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