A complete characterization of \((f_0, f_1)\)-pairs of 6-polytopes

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Introduction

Background

The \((f_0, f_1)\)-pairs for 6-polytopes

The \((f_0, f_1)\)-pairs for 7-polytopes

Conclusion
Motivations

- The $f$-vector of a $d$-polytope $P$ is the vector $(f_0, f_1, \cdots, f_{d-1})$ where faces of $P$ of dimension $0$, $1$, $2$, $d-2$ and $d-1$ are called vertices, edges, subfacets (or ridges), and facets of $P$, respectively.
- For example the $f$-vector of a tetrahedron $T$ (a 3-simplex) is $f(T) = (4, 6, 4)$ and the $f$-vector of the octahedron is $(6, 12, 8)$.
- For a simplicial complex $\Delta$ of dimension $d$, its $f$-vector is $(f_0(\Delta), \cdots, f_d(\Delta))$; $f_d(\Delta) = 1$. The $h$-vector is $(h_0(\Delta), \cdots, h_d(\Delta))$ where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1};$$

$\forall \ k = 0, \cdots, d+1.$
Motivations

- We set $f_{-1} = 1$. The $f$-vector and the $h$-vector uniquely determine each other through the linear relation

$$
\sum_{i=0}^{d} f_{i-1}(t - 1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}.
$$

- The $g$-theorem says that the $h$-vector increases until the middle ($g_i = h_i - h_{i-1} \geq 0$) and $h_i = h_{d-i}$. The $h$-vector of the tetrahedron is $(1, 1, 1, 1)$ and for the octahedron is $(1, 3, 3, 1)$ and is palindromic. $f_0 - 1, f_1 - (f_0 - 1), f_2 - f_1 + f_0 - 1 = 1$. Euler formula $\sum_{i=0}^{d} (-1)^i f_i = 1 - (-1)^d$. 

\[ 
\begin{array}{cccc}
1 \\
1 & 6 \\
1 & 5 & 12 \\
1 & 4 & 7 & 8 \\
h= & 1 & 3 & 3 & 1 \\
\end{array} \]
Motivations

- In 1980 Billera & Lee and Stanley have proved the characterization of the $f$-vectors of simplicial and of simple polytopes conjectured by McMullen in 1971 through the famous “$g$-theorem”.
- Grünbaum, Barnette and Barnette-Reay have characterized for any $0 \leq i < j \leq 3$ the following sets:
  \[ \left\{ (f_i, f_j) : P \text{ is a 4-polytope} \right\}. \]
- Steinitz found the characterization for $d = 3$, $\mathcal{E}^3 = \left\{ (f_0, f_1) : \frac{3}{2} f_0 \leq f_1 \leq 3f_0 - 6 \right\}$. 
Motivations

- For $d = 4, 5$ the results is given by the set $S = \{(f_0, f_1) : \frac{d}{2} f_0 \leq f_1 \leq \binom{f_0}{2}\}$ from which some exceptions have been removed.
- Grünbaum proved the case $d = 4$ by removing four exceptions: $(6, 12)$, $(7, 14)$, $(8, 17)$ and $(10, 20)$.
- The case $d = 5$ becomes more complicated and has been proved in two different ways by G. Pineda-Villavicencio, J. Ugon and D. Yost, and more recently by T. Kusunoki and S. Murai. For this case, exceptions are infinitely many:

$$E^5 = \left\{ (f_0, f_1) : \frac{5}{2} f_0 \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left( \left\{ \left( f_0, \left\lfloor \frac{5}{2} f_0 + 1 \right\rfloor \right) : f_0 \geq 7 \right\} \cup \left\{ (8, 20), (9, 25), (13, 35) \right\} \right) \right\},$$

(1)

where $\lfloor r \rfloor$ denotes the integer part of a rational number $r$. 


The excess degree or excess $\Sigma(P)$ of a $d$-polytope $P$ is defined as the sum of the excess degrees of its vertices and given by $\epsilon(P) = 2f_1 - df_0 = \sum_u (\deg(u) - d)$.

(Proposition 1 G. Pineda-Villavicencio, J. Ugon and D. Yost) Let $P$ be a $d$-polytope. Then the smallest values of $\Sigma(P)$ are 0 and $d - 2$.

We set for all $d$-dimensional polytopes $\phi(v, d) = \frac{1}{2}dv + \frac{1}{2}(v - d - 1)(2d - v)$.

(Proposition 2) Let $P$ be a $d$-polytope. If $f_0(P) \leq 2d$, then $f_1(P) \geq \phi(f_0(P), d)$. If $d \geq 4$, then $(f_0(P), f_1(P)) \neq (d + 4, \phi(d + 4, d) + 1)$.

D. W. Barnette proved that for all $d$-dimensional simplicial polytope the following inequality holds: $f_{d-1} \geq (d - 1)f_0 - (d + 1)(d - 2)$. 
(\(f_0, f_1\))-vectors pairs for 6-polytopes

- If \(P\) is a 6-polytope having a simple vertex \(v\) and \(Q\) the 6-polytope obtained from \(P\) by truncating the vertex \(v\) then

\[
f_0(Q) = f_0(P) + 5 \quad \text{and} \quad f_1(Q) = f_1(P) + 15.
\]

We can prove that if for a 6-polytope \(P\) we have \(f_1(P) \leq \frac{7}{2} f_0(P)\) then \(P\) has at least one simple vertex.

- (Theorem 1) The set of (\(f_0, f_1\))-vectors pairs for 6-polytopes is given by

\[
\mathcal{E}^6 = \left\{ (f_0, f_1) : 3f_0 \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left\{ (f_0, 3f_0 + 1) : f_0 \geq 7 \right\} \cup \left\{ (8, 24); (9, 27); (9, 29); (10, 30); (10, 32); (10, 34); (11, 33); (12, 38); (12, 39); (13, 39); (14, 42); (14, 44); (15, 47); (18, 54); (19, 57); (17, 53); (20, 62) \right\}.
\]
Proof

• If $P$ is a $d$-polytope with $d > 4$, then

$$f_1(P) \neq \left\lfloor \frac{d}{2} f_0(P) + 1 \right\rfloor.$$  \hspace{1cm} (3)

• Assume that $f_1(P) = \left\lfloor \frac{d}{2} f_0(P) + 1 \right\rfloor$. If $f_0(P)$ is even then $2f_1(P) - df_0(P) = 2$ and $0 < 2 < d - 2$ which is impossible since from Proposition 1, $\Sigma(P)$ can not take any value between 0 and $d - 2$. If $f_0(P)$ is odd then $2f_1(P) - df_0(P) = 1$ and $0 < 1 < d - 2$ which is also impossible.

• The following relations hold

$(8, 24); (9, 27); (9, 29); (10, 30); (10, 32); (10, 34); (11, 33) \notin E^6$ and $(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor) \notin E^6$ for $f_0 = 7, 8, 9.$
Proof

• The fact that \((10, 34) \notin \mathcal{E}^6\) is given by Proposition 2 (2) and all the remaining are given by Proposition 2 (1).

• (Lemma 1) The following result is obtained from pyramids over 5-polytopes

\[
\left\{ \left( f_0, f_1 \right) : \frac{7}{2} f_0 - \frac{7}{2} \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left\{ \left( f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor \right) : f_0 \geq 7 \right\} \cup \left\{ (9, 28), (10, 34), (14, 48) \right\} \subset \mathcal{E}^6. \tag{4}
\]

• There is no 6-polytope with 11 vertices and 36 edges and no 6-polytope with 12 vertices and 38 edges.
Proof

- The following exist $(13, 43), (14, 48) \in E^6$.
- There is no 6-polytope with 12 vertices and 39 edges.
- (Lemma 2) For an odd integer $f_0 \geq 12$ we have $(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor) \in E^6$. Furthermore if $(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor) \in E^6$, then $(f_0 + 7, \left\lfloor \frac{7}{2} (f_0 + 6) + 1 \right\rfloor) \in E^6$.
- Suppose that $f_0$ is odd. If $f_0 \geq 12$ then $f_0 - 4 \geq 8$ and from Lemma 1, $(f_0 - 4, \left\lfloor \frac{7}{2} (f_0 - 4) \right\rfloor) \in E^6$. Also $\left\lfloor \frac{7}{2} (f_0 - 4) \right\rfloor < \frac{7}{2} (f_0 - 4)$ as $f_0 - 4$ is odd then $(f_0 + 1, \left\lfloor \frac{7}{2} f_0 + 1 \right\rfloor) \in E^6$ by truncation of simple vertex.
• Let $P$ be a 6-polytope with $(f_0, f_1)$-pairs equal to $(f_0(P) + 1, \left\lceil \frac{7}{2} f_0(P) + 1 \right\rceil) \in E^6$ then after the truncation of a simple vertex of $P$ and a pyramid over a simplex facet of the resulting polytope we obtain a 6-polytope $Q$ with $f_0(Q) = f_0(P) + 7$ and $f_1(Q) = \left\lfloor \frac{7}{2} (f_0(P) + 6) + 1 \right\rfloor$.

• For any integer $f_0$ satisfying $f_0 \geq 12$, $(f_0 + 1, \left\lceil \frac{7}{2} f_0 + 1 \right\rceil) \in E^6$.

• Assume that $f_0 \geq 12$. From Lemma 2 it is enough to check the result for $f_0 = 12, 13, 14, 15, 16, 17$. The cases $f_0 = 12, 13, 15, 17$ come from Lemma 2. We now consider $f_0 = 14, 16$ which are $(15, 50)$ and $(17, 57)$. 
• Consider the 6-polytope $P$ with $f_0(P) = 10$ and $f_1(P) = 35$ obtained from a pyramid over a 5-polytope $Q$.

• If we assume that $P$ has no simple vertex then each of its vertices has degree 7 since $\sum_{v \in P} \deg(v) = 70$ and this is impossible since taking a pyramid over $Q$ implies that $P$ has a vertex of degree 9.

• Then $P$ has a simple vertex which truncation gives a polytope $P'$ with $f_0(P') = 15$ and $f_1(P) = 50$. Hence $(15, 50) \in \mathcal{E}^6$.

Let $R$ be a 6-polytope with $f_0(R) = 12$ and $f_1(P) = 42$ obtained from a pyramid over a 5-polytope. The same procedure as above gives $(17, 57) \in \mathcal{E}^6$.

• The following polytopes pairs do not exist:

  $$(13, 39); (14, 42); (14, 44); (15, 47); (18, 54), (19, 57) \notin \mathcal{E}^6.$$
• Consider the case \((19, 57)\) which is a simple polytope if it exists. Let \(P\) be such polytope. The dual \(P^*\) of \(P\) is a simplicial polytope with \(f\)-vector sequence \((f_0, f_1, f_2, f_3, f_4, f_5)\) where \(f_4 = 57\) and \(f_5 = 19\).

• For all \(d\)-dimensional simplicial polytope the following inequality holds: 
  \[ f_{d-1} \geq (d - 1)f_0 - (d + 1)(d - 2). \] Then \(f_5 \geq 5f_0 - 28\) implies that \(f_0 = 8\) or \(f_0 = 9\).

• The \(g\)-theorem for simplicial polytopes says that the sequence of integers 
  \((h_0, \cdots, h_7)\) is the \(h\)-vector of \(P^*\). We also have \(h_i = h_{7-i} \forall i = 0, \cdots, 7\) and now compute the numbers \(h_i's\) and obtain:

\[
\begin{align*}
h_1 &= -7 + f_0, \\
h_2 &= 21 - 6f_0 + f_1, \\
h_3 &= -35 + 15f_0 - 5f_1 + f_2, \\
h_4 &= 35 - 20f_0 + 10f_1 - 4f_2 + f_3, \\
h_5 &= -21 + 15f_0 - 10f_1 + 6f_2 - 3f_3 + f_4, \\
h_6 &= 7 - 6f_0 + 5f_1 - 4f_2 + 3f_3 - 2f_4 + f_5. \\
\end{align*}
\]
From $h_1 = h_6$ and $h_2 = h_5$ we get $f_2 = \frac{1}{2}(28 - 14f_0 + 6f_1 + f_4 - f_5)$ and the system of equations $h_3 = h_4; h_2 = h_5$ also gives $f_2 = \frac{1}{9}(168 - 84f_0 + 34f_1 + f_4)$.

Equaling these two expressions of $f_2$ we get $f_1 = \frac{1}{14}(-84 + 42f_0 + 7f_4 - 9f_5)$ which is not an integer for $f_0 = 8, 9$. In conclusion $(19, 57) \notin \mathcal{E}^6$.

The following pairs are possible:

$$(15, 45); (15, 49); (16, 48); (17, 54); (19, 59); (23, 69); (24, 72); (27, 83); (35, 107) \in \mathcal{E}^6.$$

We set

$$X' = \left\{(8, 24); (9, 27); (9, 29); (10, 30); (10, 32); (10, 34); (11, 33); (11, 36); (12, 38); (12, 39); (13, 39); (14, 42); (14, 44); (15, 47); (18, 54); (17, 53); (19, 57); (20, 62)\right\}$$
• For \( f_0 \geq 7 \); if \((f_0, f_1) \notin X'\) and \( f_1 \in \{3f_0 \cup ]3f_0 + 1, \frac{7}{2}f_0 - \frac{7}{2}\] \) then \((f_0, f_1) \in \mathcal{E}^6\).

• The cases \((17, 53); (20, 62)\) are unfeasible and \((22, 68); (25, 77); (30, 92) \in \mathcal{E}^6\) holds.

• Let \( \mathcal{E}^d_{>3d-10} \) be the set of \( d \)-polytopes whose excess degree is larger than \( 3d - 10 \). For \( d = 4 \), the set \( \mathcal{E}^4_{>2} \) of 4-polytopes whose excess degree is larger than 2 is given by:

\[
\mathcal{E}^4_{>2} = \left\{ (f_0, f_1) : 1 + 2f_0 < f_1 \leq \binom{f_0}{2} \right\}.
\]
The case \( d = 7 \)

- In the same way for \( d = 5, 6 \) we obtain:

\[
E_{>5}^5 = \left\{ (f_0, f_1) : \frac{5}{2} + \frac{5}{2} f_0 < f_1 \leq \binom{f_0}{2} \right\},
\]

and

\[
E_{>8}^6 = \left\{ (f_0, f_1) : 4 + 3f_0 < f_1 \leq \binom{f_0}{2} \right\}.
\]

- (Theorem 2) Let \( E^7 \) be the set of \((f_0, f_1)\)-pairs of 7-polytopes. For \( v = (p, q) \) such that \( p \geq 8 \) and \( \frac{7}{2} p \leq q \leq \binom{p}{2} \), if \( v \notin E^7 \) then \( \epsilon_7(v) \leq 4 \times 7 - 10 = 11 \). In other words the set of \((f_0, f_1)\)-vector pairs for 7-polytopes with excess strictly larger than 11 is given by

\[
E_{>11}^7 = \left\{ (f_0, f_1) : \frac{7}{2} f_0 + \frac{11}{2} < f_1 \leq \binom{f_0}{2} \right\}.
\]

With \( \epsilon_d(v) = 2q - dp \).
The case \( d = 7 \)

- From the previous section we had

\[
\mathcal{E}^6 = \left\{ (f_0, f_1) : 3f_0 \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left( \left\{ \left( f_0, 3f_0 + 1 \right) : f_0 \geq 7 \right\} \cup \left\{ (8, 24); (9, 27); (9, 29); (10, 30); (10, 32); (10, 34); (11, 33); (12, 38); (12, 39); (13, 39); (14, 42); (14, 44); (15, 47); (18, 54); (19, 57); (17, 53); (20, 62) \right\} \right) \]

- A pyramid over the 6-polytopes gives:

\[
\left\{ (f_0, f_1) : 4f_0 - 4 \leq f_1 \leq \binom{f_0}{2} \right\} \setminus \left( \left\{ (f_0 + 1, 4f_0 + 1) : f_0 \geq 7 \right\} \cup \left\{ (9, 32); (10, 36); (10, 38); (11, 40); (11, 42); (11, 44); (12, 44); (13, 50); (13, 51); (14, 52); (15, 56); (15, 58); (16, 62); (18, 70); (19, 72); (20, 76); (21, 82) \right\} \right) \subset \mathcal{E}^7.
\]
Proof

- A direct computation shows that $\epsilon_7((f_0 + 1, 4f_0 + 1)) > 11$ if and only if $f_0 > 17$. Assume that $f_0 > 17$ and let us prove that $(f_0 - 6, 3f_0 - 14) \in E_6$.

- We have $\epsilon_6((f_0 - 6, 3f_0 - 14)) = 8$ and if $(f_0 - 6, 3f_0 - 14) \notin E_6$ then $(f_0 - 6, 3f_0 - 14) = (10, 34)$, because $(10, 34)$ is the only vector not in $E_6$ with excess equal to 8.

- Therefore we get $f_0 = 16$ which is a contradiction. In conclusion for $f_0 > 17$ there is a 6-polytope $P$ with $(f_0, f_1)$-pair $(f_0 - 6, 3f_0 - 14)$; and a pyramid over $P$ give a 7-polytope $Q$ having $(f_0, f_1)$-vector which is equal to $(f_0 - 5, 4f_0 - 20)$.

- As $4(f_0 - 5) < (4f_0 - 20) + 1$ the polytope $Q$ has a simple vertex whose truncation gives a 7-polytope having $(f_0, f_1)$-pair equals $(f_0 + 1, 4f_0 + 1)$. 

\(19/24\)
Proof

- We can conclude that all the 7-polytopes with excess greater than 11 and with $(f_0, f_1)$-pairs in $\left\{ (f_0 + 1, 4f_0 + 1) : f_0 \geq 7 \right\}$ exist.

- Let us focus on the set

  \[ L = \left\{ (9, 32); (10, 36); (10, 38); (11, 40); (11, 42); (11, 44); (12, 44); (13, 50); (13, 51); (14, 52); (15, 56); (15, 58); (16, 62); (18, 70); (19, 72); (20, 76); (21, 82) \right\}. \]

- The only vectors $v = (p, q) \in L$ with $\epsilon_7(v) > 11$ are

  \[ v = (p, q) = (16, 62); (18, 70); (20, 76); (21, 82); (23, 90); (26, 102); (31, 123). \]
Proof

- For \( v = (p, q) = (16, 62); (20, 76); (21, 82); (26, 102); (31, 123) \), we compute \( v' = (p - 8, q - p - 20) = (8, 26); (12, 36); (13, 41); (18, 56); (23, 72) \) \( \in \mathcal{E}^6 \).

- Then there exist 6-polytopes \( P_{v'} \) whose \((f_0, f_1)\)-pairs are equal to \( v' \). A pyramid over them give 7-polytopes having \((f_0, f_1)\)-pairs equal to \((p - 7, q - 28) = (9, 34); (13, 48); (14, 54); (19, 74); (24, 95) \).

- In each case we observe that \( q - 28 < 4(p - 7) \) which means that each of them has a simple vertex whose truncation give 7-polytopes with \((f_0, f_1)\)-pairs equal to \((p - 1, q - 7) \). As truncations of simple vertices generate simplex facets then pyramids on these give the result.
Proof

- Consider \( v = (18, 70) \). There is a 6-polytope \( R \) with \((f_0, f_1) = (10, 35)\). A pyramid over \( R \) gives a 7-polytope \( R' \) having \((f_0, f_1)\)-vector equal to \((11, 42)\). As \( 42 < 4 \times 11 \) then \( R' \) has a simple vertex whose truncation gives a 7-polytope \( R'' \) with \((f_0(R''), f_1(R'')) = (17, 63)\).

- The truncation of a simple vertex in \( R'' \) with generate a simplex facet \( F \) and a pyramid other \( F \) gives a 7-polytope with \((f_0, f_1)\)-vector equal to \((18, 70)\). The same method works for \((23, 90)\).

- We now turn to the pair \( v = (f_0, f_1) \) with \( f_0 \geq 8 \) and \( f_1 \in ]\frac{7}{2} f_0, 4 f_0 + 1[ \). The condition \( \epsilon_7(v) > 11 \) implies that \( f_1 \geq \frac{11}{2} + \frac{7}{2} f_0 \) and then we need to discuss two cases: \( \frac{11}{2} + \frac{7}{2} f_0 > 4 f_0 - 4 \) and \( \frac{11}{2} + \frac{7}{2} f_0 < 4 f_0 - 4 \). 

Proof

- If $\frac{11}{2} + \frac{7}{2} f_0 > 4f_0 - 4$ then there is nothing else to prove as we end up in the pyramid case. Suppose that $\frac{11}{2} + \frac{7}{2} f_0 < 4f_0 - 4$ i.e. $f_0 > 19$ and set for $k$, $X_k^7 = \{(k, f_1); \frac{11}{2} + \frac{7}{2} k < f_1 < 4k - 4\}$.

- We can prove by truncation that if $X_k^7 \subset \mathcal{E}_{>11}$, then $X_{k+6}^7 \subset \mathcal{E}_{>11}$. To prove that each vector $(f_0, f_1)$ satisfying this condition defines a 7-polytope it is sufficient to show that $X_k^7 \subset \mathcal{E}_{>11}$ for $k = 8, \cdots, 13$. Which have already been solved.

- Finally we conclude that all the pairs $(p, q)$ with $p \geq 8$, $\epsilon_7(v) > 11$ and $\frac{7}{2} p \leq q \leq \binom{p}{2}$, characterize 7-polytopes. In other words the set of $(f_0, f_1)$-vectors pair for 7-polytopes with excess strictly larger than 11 is given by

$$\mathcal{E}_{>11}^7 = \left\{(f_0, f_1): \frac{7}{2} f_0 + \frac{11}{2} < f_1 \leq \binom{f_0}{2}\right\}.$$
Conjecture

Let $d \geq 4$ be an integer and $\mathcal{E}^d$ be the set of $(f_0, f_1)$-pairs of $d$-polytopes. For $v = (p, q)$ such that $p \geq d + 1$ and $\frac{d}{2}p \leq q \leq \binom{p}{2}$, if $v \not\in \mathcal{E}^d$ then $2q - dp \leq 4d - 10$. In other words the set of $(f_0, f_1)$-pairs for $d$-polytopes; $d \geq 4$ with excess strictly larger than $3d - 10$ is given by

$$
\mathcal{E}^d_{>3d-10} = \left\{(f_0, f_1) : \frac{d}{2} f_0 + \frac{3d - 10}{2} < f_1 \leq \binom{f_0}{2}\right\}.
$$