Hypermaps Over Non-Abelian Simple Groups and Strongly Symmetric Generating Sets

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Abstract
A generating pair $x, y$ for a group $G$ is said to be symmetric if there exists an automorphism $\varphi_{x,y}$ of $G$ inverting both $x$ and $y$, that is, $x^{\varphi_{x,y}} = x^{-1}$ and $y^{\varphi_{x,y}} = y^{-1}$. Similarly, a group $G$ is said to be strongly symmetric if $G$ can be generated with two elements and if all generating pairs of $G$ are symmetric.

In this paper we classify the finite strongly symmetric non-abelian simple groups. Combinatorially, these are the finite non-abelian simple groups $G$ such that every orientably regular hypermap with monodromy group $G$ is reflexible.

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1 Introduction
The aim of this note is to classify the finite strongly symmetric non-abelian simple groups.

Theorem 1. Let $S$ be a finite non-abelian simple group. Then $S$ is strongly symmetric if and only if $S \cong \text{PSL}(2, q)$ for some prime power $q$.

Interest on strongly symmetric groups stems from maps and hypermaps, which (roughly speaking) are embeddings of graphs on surfaces, see [6]. We give a brief account on this connection, for more details see [4, 10].

A map on a surface is a decomposition of a closed connected surface into vertices, edges and faces. The vertices and edges of this decomposition form the underlying graph of the map. An automorphism of a map is an automorphism of the underlying graph which can be extended to a homeomorphism of the whole surface. For the definition of hypermaps, which bring us closer to strongly symmetric groups, we need to take a more combinatorial point of view.

Each map on a orientable surface can be described by two permutations, usually denoted by $R$ and $L$, acting on the set of directed edges (that is, ordered pairs of adjacent edges).
vertices) of the underlying graph. The permutation $R$ permutes cyclically the directed edges starting from a given vertex and preserving a chosen orientation of the surface. The permutation $L$ interchanges the end vertices of a given directed edge. The monodromy group of the surface is the group generated by $R$ and $L$ and the map is said to be regular if the monodromy group acts regularly, that is, the identity is the only permutation fixing some element.

Observe that in a map, we have $L^2 = 1$. A hypermap is simply given by the combinatorial data $R$ and $L$, where $L$ is not necessarily an involution. Inspired by the topological and geometrical counterpart for maps, a hypermap is said to be reflexible if the assignment $R \mapsto R^{-1}$ and $L \mapsto L^{-1}$ extends to a group automorphism of $\langle R, L \rangle$; otherwise the hypermap is said to be chiral.

It was shown in [4, Lemma 7], that a finite group $G$ is strongly symmetric if and only if every orientably regular hypermap with monodromy group $G$ is reflexible. In particular, Theorem 1 classify the finite non-abelian simple groups $G$ with the property that every orientably regular hypermap with monodromy group $G$ is reflexible.

In our opinion, Theorem 1 suggests a natural problem, which in principle should give a measure of how chirality is abundant among regular hypermaps. Let $S$ be a non-abelian simple group and let $\delta(S)$ be the proportion of strongly symmetric generating sets of $S$, that is,

$$\delta(S) := \frac{|\{(x, y) \in S \times S \mid x, y \text{ symmetric generating set}\}|}{|\{(x, y) \in S \times S \mid S = \langle x, y \rangle\}|}.$$ 

The closer $\delta(S)$ is to 1, the more abundant reflexible hypermaps are among orientably regular hypermaps with monodromy group $S$. Indeed, Theorem 1 classifies the groups $S$ attaining 1. We do not have any “running conjecture”, but we wonder whether statistically it is frequent the case that $\delta(S) < 1/2$. Moreover, we wonder whether it is statistically significant the case that $\delta(S) \to 0$ as $|S| \to \infty$, as $S$ runs through a certain family of non-abelian simple groups.\footnote{During the refereeing process of this paper, Theorem 1 has proved to be useful in [7, page 2 and 3] for the proof of Cherlin’s conjecture on finite primitive binary permutation groups.}

2 Proof of Theorem 1

We start with a preliminary lemma.

**Lemma 1.** Let $n$ be an integer with $n \geq 3$, let $q$ be a prime power with $(n, q) \neq (3, 4)$, let $g \in \text{GL}(n, q)$ be a Singer cycle of order $q^n - 1$, let $x := g^{\gcd(n, q-1)}$ and let $a \in \text{GL}(n, q)$ such that $x^a = z z^\varepsilon$, for some $z \in \mathbb{Z}((\text{GL}(n, q))$ and $\varepsilon \in \{-1, 1\}$. Then $z = 1$, $\varepsilon = 1$ and $a \in \langle g \rangle$.

**Proof.** Let $e_1, \ldots, e_n$ be the canonical basis of the $n$-dimensional vector space $\mathbb{F}_q^n$ of row vectors over the finite field of cardinality $q$. Set $v := e_1$ and let $P_1$ be the stabilizer in $\text{GL}(n, q)$ of the vector $v$. As $\langle g \rangle$ acts transitively on the set of non-zero vectors of $\mathbb{F}_q^n$ and as $P_1$ is the stabilizer of the non-zero vector $v$, we deduce from the Frattini argument...
that $\text{GL}(n, q) = \langle g \rangle P_1$. In particular, as $a \in \Gamma \text{L}(n, q)$, we have $a = g^i bc$, where $i \in \mathbb{Z}$, $b \in P_1$ and $c$ lies in the Galois group $\text{Gal}(\mathbb{F}_q)$ of the field $\mathbb{F}_q$. Set $a' := bc$. Observe that $x^a = x'^a$ because $g^i$ centralizes $x \in \langle g \rangle$. Moreover, $a \in \langle g \rangle$ if and only if $a' \in \langle g \rangle$. Therefore, replacing $a$ with $a'$ if necessary, in the rest of the argument we may suppose that $a = a' = bc$.

As $Z(\text{GL}(n, q))$ consists of scalar matrices, we may identify the matrix $z$ with an element in the field $\mathbb{F}_q$. We show that, for every $\ell \in \mathbb{N}$, we have $(v x^\ell)^a = z^\ell v x^\ell$. When $\ell = 0$, $v^a = v^b c = v$, because $b$ and $c$ fix the vector $v = \epsilon_1$. When $\ell > 0$, we have

$$(v x^\ell)^a = v^a(x^\ell)^a = v(x^\ell)^a = v(vx^\ell)^a = v(zx^\ell)^a = v(z^\ell x^\ell) = z^\ell v x^\ell.$$

Observe that $v, vx, \ldots, vx^{n-1}$ is a basis of $\mathbb{F}_q$ and hence there exists $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}_q$ with

$$vx^n = a_0 v + a_1 vx + \cdots + a_{n-1} vx^{n-1}. \quad (2.1)$$

Now, by applying $a$ on both sides of this equality and using the previous paragraph, we obtain

$$z^n v x^{\varepsilon n} = a_0^\varepsilon v + a_1^\varepsilon z v x^\varepsilon + \cdots + a_{n-1}^\varepsilon z^{n-1} v x^{(n-1)\varepsilon}. \quad (2.2)$$

We let $f(T) := T^n - a_{n-1}T^{n-1} - a_{n-2}T^{n-2} - \cdots - a_1 T - a_0 \in \mathbb{F}_q[T]$ be the characteristic polynomial of the matrix $x$. Observe that $f(T)$ is irreducible in $\mathbb{F}_q[T]$ because $x = g^{\gcd(n, q-1)}$ acts irreducibly on $\mathbb{F}_q^n$. Let $\lambda \in \mathbb{F}_q^*$ be a root of $f(T)$ and observe that $\lambda$ generates the field extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. Observe that $\lambda$ has order $(q^n - 1)/\gcd(n, q-1)$ in the multiplicative group $\mathbb{F}_q^*$, because so does $x$. Now, let $f^c(T) = T^n - a_{n-1}^c T^{n-1} - a_{n-2}^c T^{n-2} - \cdots - a_1^c T - a_0^c \in \mathbb{F}_q[T]$ be the image of the polynomial $f(T) \in \mathbb{F}_q[T]$ under the Galois automorphism $c \in \text{Gal} \left( \mathbb{F}_q \right)$. Clearly, the roots of $f(T)$ are

$$\lambda, \lambda^q, \ldots, \lambda^{q^{n-1}}$$

and the roots of $f^c(T)$ are

$$\lambda^c, \lambda^{cq}, \ldots, \lambda^{cq^{n-1}}.$$

Moreover, let $\kappa \in \mathbb{N}$ with $q = p^\kappa$, for some prime number $p$, and let $j \in \{0, \ldots, \kappa - 1\}$ with $\omega^c = \omega^j p^\kappa$, $\forall \omega \in \mathbb{F}_q$.

We now distinguish the cases, depending on whether $\varepsilon = 1$ or $\varepsilon = -1$. Assume $\varepsilon = 1$.

Using (2.1) and (2.2) and using the fact that $f(\lambda) = 0$, we get $f^c(z \lambda) = 0$. Therefore, we deduce $z \lambda$ is a root of $f^c(T)$ and hence, there exists $i \in \{0, \ldots, n-1\}$, with $\lambda^{q^i} = z \lambda$. This gives $\lambda^{q^{i+1}} = z \in \mathbb{F}_q^*$ and hence $\lambda^{(q^{i+1})(q-1)} = 1$. Since $\lambda$ has order $(q^n - 1)/\gcd(n, q-1)$, this implies

$$\frac{q^n - 1}{\gcd(n, q-1)} \text{ divides } (p^i q^i - 1)(q-1). \quad (2.3)$$

If $p^i q^i - 1 = 0$, then $j = 0$ and $i = 0$. This implies $c = 1$ and $z = 1$ and the lemma follows in this case. Suppose $p^i q^i - 1 \neq 0$. Assume further that $(\kappa n, p) \neq (6, 2)$. Then Zsigmondy’s theorem guarantees the existence of a primitive prime divisor $r$ of $p^{kn} - 1$. Clearly $r$ does not divide $p^i q^i - 1 = p^{i+\kappa i} - 1$ and hence we contradict (2.3). Finally,
Proof of Theorem 1. Macbeath has proved in [11] that, for every prime power \( q \), \( \text{PSL}(2, q) \) is strongly symmetric; see also [4, Proposition 8]. In particular, for the rest of the proof, we let \( S \) be a finite strongly symmetric non-abelian simple group and our task is to show that \( S \cong \text{PSL}(2, q) \), for some prime power \( q \).

Observe that, if \( S = \langle s_1, s_2 \rangle \) and \( \alpha \in \text{Aut}(S) \) inverts both \( s_1 \) and \( s_2 \), then

\[
\alpha^2 \in C_{\text{Aut}(S)}(s_1) \cap C_{\text{Aut}(S)}(s_2) = C_{\text{Aut}(S)}(\langle s_1, s_2 \rangle) = C_{\text{Aut}(S)}(S) = 1.
\]

If \( \alpha \) is the identity automorphism, then \( s_1, s_2 \) are involutions and hence \( S = \langle s_1, s_2 \rangle \) is a dihedral group, contradicting the fact that \( S \) is a non-abelian simple group. Therefore \( \alpha \) has order 2, that is, \( \alpha \) is an involution of \( \text{Aut}(S) \).

In [10, Theorem 1.1], Leemans and Liebeck have proved that, if \( T \) is a finite non-abelian simple group that is not isomorphic to \( \text{Alt}(7) \), to \( \text{PSL}(2, q) \), to \( \text{PSL}(3, q) \) or to \( \text{PSU}(3, q) \), then there exist \( x, s \in S \) such that the following hold:

(i) \( T = \langle x, s \rangle \);

(ii) \( s \) is an involution;

(iii) there is no involution \( \alpha \in \text{Aut}(T) \) such that \( x^\alpha = x^{-1}, s^\alpha = s \).

In particular, if \( S \) is not isomorphic to \( \text{Alt}(7) \), to \( \text{PSL}(2, q) \), to \( \text{PSL}(3, q) \) or to \( \text{PSU}(3, q) \), then \( S \) is not strongly symmetric. In the rest of this proof, we deal with each of these cases separately.

Assume \( S = \text{Alt}(7) \); in particular, \( \text{Aut}(S) = \text{Sym}(7) \). Let \( s_1 := (1, 2, 3, 4, 5, 6, 7) \) and \( s_2 := (1, 2, 3, 4, 6, 7, 5) \) and, for \( i \in \{1, 2\} \), let \( \Delta_i := \{ \alpha \in \text{Sym}(7) \mid s_i^\alpha = s_i^{-1} \} \). It can be easily checked that \( S = \langle s_1, s_2 \rangle \) and

\[
\Delta_1 = \{(2, 7)(3, 6)(2, 4), (1, 7)(2, 6)(3, 5), (1, 6)(2, 5)(3, 4), (1, 5)(2, 4)(6, 7), (1, 4)(2, 3)(5, 7), (1, 3)(4, 7)(5, 6), (1, 2)(3, 7)(4, 6)\},
\]

\[
\Delta_2 = \{(2, 5)(3, 7)(4, 6), (1, 5)(2, 7)(3, 6), (1, 7)(2, 6)(3, 4), (1, 6)(2, 4)(5, 7), (1, 4)(2, 3)(5, 6), (1, 3)(4, 5)(6, 7), (1, 2)(3, 5)(4, 7)\}.
\]
Since $\Delta_1 \cap \Delta_2 = \emptyset$, the generating pair $s_1, s_2$ of $\text{Alt}(7)$ witnesses that $\text{Alt}(7)$ is not strongly symmetric.

Assume $S = \text{PSL}(3, q)$. Since $\text{PSL}(3, 2) = \text{PSL}(2, 7)$, we may assume $q > 2$. Moreover, we have verified with a computer that $\text{PSL}(3, 4)$ is not strongly symmetric.

Let $A := \text{Aut}(S)$, let $d := \gcd(3, q - 1)$ and let $\iota$ be the graph automorphism of $\text{PSL}(3, q)$ defined via the inverse-transpose mapping $x \mapsto x^T = (x^{-1})^T$, for every $x \in \text{PSL}(3, q)$, where $x^T$ denotes the transpose of the element $x$ of $\text{PSL}(3, q)$. Since $x \in \text{PSL}(3, q)$ is not a single matrix, but a coset of the center $\mathbb{Z}(\text{SL}(3, q))$ in $\text{SL}(3, q)$, there is a slight abuse of notation when we talk about the transpose of the coset $x$. However, since $\mathbb{Z}(\text{SL}(3, q))$ consists of diagonal matrices, this should cause no confusion.

Next, let $\Omega_1$ be the set of cyclic subgroups of $S$ generated by a Singer cycle of order $(q^2 + q + 1)/d$ and, for any $K \in \Omega_1$, let

$$\Delta_K := \{ \alpha \in A \mid \alpha^2 = 1, k^\alpha = k^{-1} \forall k \in K \}. $$

Observe that the set $\Omega_1$ consists of a single $S$-conjugacy class.

Let $K \in \Omega_1$, let $k \in K$ be a generator of $K$ and let $\alpha, \beta \in \Delta_K$. Then $k^\alpha = k^{-1} = k^\beta$ and hence $\beta^{-1} \alpha \in C_{\text{Aut}(S)}(k)$. This shows that $\Delta_K \subseteq C_{\text{Aut}(S)}(k)\alpha$ and that $\Delta_K$ consists of the involutions in $C_{\text{Aut}(S)}(k)\alpha$. From [2, Theorem 8], we see that there exists a symmetric matrix $g \in \text{GL}(3, q)$ having order $q^2 - 1$. Let $\bar{g}$ be the projection of $g$ in $\text{PGL}(3, q)$. The element $h := \bar{g}^d$ generates a subgroup $H \in \Omega_1$. Since $g$ is symmetric, $g = g^T$ and hence $h^\iota = h^{-1}$, that is, $\iota \in \Delta_H$ and $\Delta_H$ consists of the involutions contained in $C_{\text{Aut}(S)}(h)\iota$. From Lemma 1, we deduce that, if $a \in \text{PGL}(3, q)$ and $h^a = h^\varepsilon$ with $\varepsilon \in \{ 1, -1 \}$, then $\varepsilon = 1$ and $a \in \langle \bar{g} \rangle$. As $\bar{g}^{-1} = \bar{g}^T$, we deduce $C_{\text{Aut}(S)}(h) = \langle \bar{g} \rangle$ and that $\langle \bar{g}, \iota \rangle$ is a dihedral group of order $2(q^2 + q + 1)$. Thus

$$|\Delta_H| = q^2 + q + 1.$$  \hspace{1cm} (2.5)

Let $\Omega_2$ be the set of the conjugates of $\iota$ in $A$. Given $y \in \Omega_2$, we want to determine the number $\delta_y$ of subgroups $K \in \Omega_1$ with the property that $y \in \Delta_K$. Consider the bipartite graph having vertex set $\Omega_1 \cup \Omega_2$ and having edge set consisting of the pairs $\{ K, y \}$ with $K \in \Omega_1, y \in \Omega_2$ and $y \in \Delta_K$. Since $\Omega_1$ and $\Omega_2$ both consist of a single $A$-conjugacy class, the group $A$ acts as a group of automorphisms on our bipartite graph with orbits $\Omega_1$ and $\Omega_2$. Thus, the number of edges of the bipartite graph is $|\Omega_1||\Delta_H| = |\Omega_2|\delta_y$. Therefore, for every $y \in \Omega_1$, we have

$$\delta_y = \frac{|\Omega_1||\Delta_H|}{|\Omega_2|}. \hspace{1cm} (2.6)$$

Let $\omega_H$ be the number of $K \in \Omega_1$ with $\Delta_H \cap \Delta_K \neq \emptyset$. Clearly

$$\omega_H \leq \delta_y|\Delta_H|. \hspace{1cm} (2.7)$$

We claim that there exists $K \in \Omega_1$ with $\Delta_H \cap \Delta_K = \emptyset$. From (2.5), (2.6) and (2.7), it suffices to show that

$$|\Omega_1| \geq \delta_y|\Delta_H| = \frac{|\Omega_1||\Delta_H|^2}{|\Omega_2|} \geq \omega_H.$$
or, equivalently, that

\[ |A : C_A(\iota)| = |\Omega_2| > |\Delta_H|^2 = (q^2 + q + 1)^2. \]

Let \( G = \text{InnDiag}(S) = \text{PGL}(3, q) \). From \cite[Chapter 4]{[5]} or \cite[Proposition 3.2.11]{[3]}, we have \( C_G(\iota) = \text{Sp}(2, q) \) when \( q \) is even and \( C_G(\iota) = \text{PGO}(2, q) \) when \( q \) is odd. Thus, in both cases, we have

\[ |A : C_A(\iota)| \geq |G : C_G(\iota)| = \frac{(q^3 - 1)(q^2 - 1)q^3}{(q^2 - 1)q} = (q^3 - 1)q^2. \]

As \( q > 2 \), it follows \( |A : C_A(\iota)| > (q^2 + q + 1)^2 \) and our claim is now proved.

Now choose \( K = (y) \in \Omega_1 \) such that \( \Delta_H \cap \Delta_K = \emptyset \). We use the list of the maximal subgroups of \( S = \text{PSL}(3, q) \), see \cite[Table 8.3]{[3]}. When \( q \neq 4 \), \( N_S(H) \) is a maximal subgroup of \( S \) isomorphic to \( H : 3 \) and hence \( \langle h, y \rangle = S \). In particular, \( h, y \) is a generating pair of \( S \) witnessing that \( S \) is not strongly symmetric. When \( q = 4 \), we have used the computer algebra system \textit{magma} \cite[1]{[1]} to show that \( \text{PSL}(3, 4) \) is not strongly symmetric.

Let \( S = \text{PSU}(3, q) \) and \( A = \text{Aut}(S) \). Since \( \text{PSU}(3, 2) \) is solvable, \( q > 2 \). Let \( A := \text{Aut}(S) \), let \( d := \gcd(3, q + 1) \) and let \( \Omega_1 \) be the set of cyclic subgroups of \( S \) generated by a Singer cycle of order \( (q^3 - q + 1)/d \) and, for any \( K \in \Omega_1 \), let

\[ \Delta_K := \{ \alpha \in A \mid \alpha^2 = 1, k^\alpha = k^{-1} \forall k \in K \}. \]

Observe that the set \( \Omega_1 \) consists of a single \( S \)-conjugacy class.

Let \( \phi \) be the automorphism of \( S \) induced by the Frobenius automorphism \( x \mapsto x^q \) of the underlying finite field \( \mathbb{F}_{q^2} \) of order \( q^2 \). We recall now some main facts about Singer cycles. Let \( \mathbb{F}_{q^6} \) be the field with \( q^6 \) elements and let \( a \in \mathbb{F}_{q^6} \) with \( a \neq 0 \). Consider the multiplication \( \pi_a : \mathbb{F}_{q^6} \rightarrow \mathbb{F}_{q^6} \) defined by \( \pi_a(x) = ax \), for all \( x \in \mathbb{F}_{q^6} \). For every divisor \( d \) of 6, the set \( V = \mathbb{F}_{q^6} \) can be interpreted as a vector space of dimension \( 6/d \) over the field \( \mathbb{F}_{q^6} \) and the map \( \pi_a \) is a \( \mathbb{F}_{q^6} \)-linear transformation of \( V \). Thus, once a base is fixed, \( \pi_a \) induces a matrix belonging to \( \text{GL}(6/d, q^d) \). Now, let \( a \) be a generator of the multiplicative field of \( \mathbb{F}_{q^6} \). Then, by \cite[Theorem 5.2]{[9]}, the multiplication \( \pi_{q^3-1} \) seen as a \( \mathbb{F}_{q^2} \)-linear transformation of \( \mathbb{F}_{q^6} \) induces a Singer cycle \( g \) for \( \text{GU}(3, q) \) having order \( q^3 + 1 \). Moreover,

\[ g^\phi = \pi_{q^3-1} = \pi_{a(q^4-1)q^3} = \pi_{aq^6-q^3} = \pi_{a^{-1}q^3} = \pi_{a^{-1}(q^3-1)} = g^{-1}. \]

Let \( \tilde{g} \) be the projection of \( g \) in \( \text{PGU}(3, q) \) and let \( h := \tilde{g}^d \). Thus \( H := \langle h \rangle \in \Omega_1 \) and \( \phi \in \Delta_H \). Since \( C_{\text{PGU}(3, q)}(\langle \tilde{g} \rangle) = \langle \tilde{g} \rangle \) and since no field automorphism centralizes \( H \), we deduce that \( \Delta_H \) is the set of the \( 2(q^2 - q + 1) \) involutions in the dihedral group \( \langle \tilde{g}, \phi \rangle \) of order \( 2(q^2 - q + 1) \) (we are omitting some details here, but these are similar to the arguments in the case of \( \text{PSL}(3, q) \)). In particular, \( |\Delta_H| = q^2 - q + 1 \).

Let \( \Omega_2 \) be the set of the conjugates of \( \phi \) in \( A \). Given \( y \in \Omega_2 \), we want to determine the number \( \delta_y \) of subgroups \( K \in \Omega_1 \) with \( y \in \Delta_K \). Arguing as in the previous paragraph, we deduce that

\[ \delta_y = \frac{|\Omega_1||\Delta_H|}{|\Omega_2|}. \]
Let $\omega_H$ be the number of $K \in \Omega_1$ with $\Delta_H \cap \Delta_K \neq \emptyset$. Clearly

$$\omega_H \leq \delta_y|\Delta_H|.$$ 

We claim that there exists $K \in \Omega_1$ with $\Delta_H \cap \Delta_K = \emptyset$. It suffices to show that

$$|\Omega_1| > \delta_y|\Delta(H)| = \frac{|\Omega_1||\Delta_H|^2}{|\Omega_2|} \geq \omega_H$$

or, equivalently, that

$$|A : C_A(\phi)| = |\Omega_2| > |\Delta_H|^2 = (q^2 - q + 1)^2.$$

Let $G = \text{InnDiag}(S) = \text{PGU}(3,q)$. From [8, Chapter 4] or [5, Proposition 3.3.15], we have $C_G(\phi) = \text{Sp}(2,q)$ when $q$ is even and $C_G(\phi) = \text{PGO}(2,q)$ when $q$ is odd. In both cases, it follows

$$|A : C_A(\phi)| \geq |G : C_G(\phi)| = \frac{(q^3 + 1)(q^2 - 1)q^3}{(q^2 - 1)q} = (q^3 + 1)q^2.$$

It follows $|A : C_A(\phi)| > (q^2 - q + 1)^2$ and our claim is now proved.

Now choose $K = \langle y \rangle \in \Omega_1$ such that $\Delta_H \cap \Delta_K = \emptyset$. We use the list of the maximal subgroups of $S = \text{PSU}(3,q)$, see [3, Table 8.3]. When $q \notin \{3,5\}$, $N_S(H)$ is a maximal subgroup of $S$ isomorphic to $H : 3$ and hence $\langle h, y \rangle = S$. In particular, $h, y$ is a generating pair of $S$ witnessing that $S$ is not strongly symmetric. When $q \in \{3,4\}$, we have used the computer algebra system magma [1] to show that $\text{PSU}(3,3)$ and $\text{PSU}(3,5)$ are not strongly symmetric.

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