Boundary conditions as constraints

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ABSTRACT: A new method to compute the symplectic structure of a quantum field theory with non trivial boundary conditions is proposed. Following the suggestion in [1, 2], we regard that the boundary conditions are second class constraints in the sense of the Dirac’s method. However, we show that this proposal is more useful if we consider an inverse of the Holographic map between a theory defined in the boundary to another with constraints but without boundary.
1. Introduction

In recent years, the study of the boundary conditions in Quantum Theory has produced several important results. For example, an interesting idea is to consider the black hole event horizon as a physical boundary [3], this induce an extra term in the action, having as consequence the existence of a central charge in the algebra of generators of gauge transformations [4]. Using this central charge it is possible to determine the asymptotic behavior of the density of states and in this way to get the entropy for a black hole in 2+1-dimensions [5].

In the context of string theory, the D-branes are natural boundaries for the open strings. These boundaries have very interesting effects in the theory among them is the non-commutativity of the D-brane coordinates [6, 7]. This non commutativity appears due to the presence of a constant B-field on the boundary that implies a change of the symplectic structure on the boundary. This change of the symplectic...
structure has been computed using different methods, the most simple is the direct solution of the Field equations subject to the Dirichlet boundary conditions on the 9-p transverse directions and Neumann boundary conditions on p+1 directions parallel to the Dp-brane. However, for some interesting systems is not possible to solve exactly this problem and the question appears if is there exist other method to compute the symplectic structure.

An alternative to solve this problem was to consider that the boundary conditions in a Quantum Field Theory can be interpreted as second class constraints in the sense of the Dirac’s Method \cite{1, 2}. This procedure has some interesting characteristics, the non commutativity appears naturally due to a modification of the symplectic structure given by the Dirac’s brackets. These brackets are constructed with the second class constraints that arise from the boundary conditions and the time evolution of these conditions. However, the procedure have some problems, see for example \cite{3}. First the Lagrange multipliers are fixed by hand and not following the standard Dirac’s method where the Lagrange multipliers are fixed in the case of second class constraints by the time evolution of the constraints. This has as consequence that appears an infinite number of constraints and then a minus infinite number of degrees of freedom in the boundary. To solve these problems we propose an alternative procedure that follows in many aspects the previous proposal. However we have a very different interpretation that allows solve most of the shortcomings. The key point of our procedure is to perform a mapping from the original problem with a given Hamiltonian and boundary conditions to another problems with the same Hamiltonian, but now with second class constraints equal to the boundary conditions and no boundary. Our conjecture is that the results that we get for the Dirac’s brackets projected to the interior of the new problem, is the symplectic structure of the original problem. We show that this conjecture is valid in all examples that we know and inclusive we check that in the example of PP-waves \cite{9} our result is fully consistent whereas the previous result it is not. In the literature exist several proposals \cite{10, 11, 12, 13, 14, 15, 16, 17, 18, 19} to solve the problems associated to the procedure of consider the boundary conditions as Dirac’s constraints. However, all the previous proposals are not useful in at least in one of the examples that we present, whereas our procedure is fully consistent and systematic in all these examples.

In section 2 we present an outline of our procedure, in the next section we analyze the case of the scalar field with Neumann and Dirichlet boundary conditions. In section 4 we present the case of the bosonic string with mixed boundary conditions and in section 5 we study the case of the bosonic string in a PP-wave background.
2. Boundary conditions as constraints

2.1 Boundary conditions and the action principle

Let $M$ be a $(d+1)$-dimensional manifold, with topology $\Sigma \times \mathbb{R}$, where $\Sigma$ is an oriented $d$-manifold with boundary $\partial \Sigma$. In $M$ we consider a Field Theory given by the action

$$S = \int_M dx^{d+1} L(\phi_a(x), \partial \phi_a(x)),$$

(2.1)

for the fields $\phi_a$. The conditions that the integral (2.1) be stationary implies

$$\delta S = \int_M dx^{d+1} \left[ \frac{\partial L}{\partial \phi_a(x)} - \frac{\partial L}{\partial (\partial_\alpha \phi_a(x))} \right] \delta \phi_a(x)$$

$$+ \int_M dx^{d+1} \partial_\alpha \left[ \frac{\partial L}{\partial (\partial_\alpha \phi_a(x))} \right] \delta \phi_a(x) = 0.$$

(2.2)

For a system without boundary the second term cancel automatically, but in our case we need to impose boundary conditions. There are three different ways to do that:

Dirichlet conditions:

$$\delta \phi_a(x) |_{x \in \partial \Sigma} = 0$$

(2.3)

Neumann conditions:

$$\left( \frac{\partial L}{\partial (\partial_\alpha \phi_a(x))} \right)(x) |_{x \in \partial \Sigma} = 0,$$

(2.4)

or one combination of both types for the components of the field $\phi_a$.

On the other hand, we have the canonical formalism defined by the Hamiltonian,

$$H_c = \int_M dx^d H_c(\phi, \partial \phi, \Pi, \partial \Pi),$$

(2.5)

the symplectic structure,

$$\{\phi_a(x, t), \phi_b(x', t)\} = \{\Pi_a(x, t), \Pi_b(x', t)\} = 0$$

(2.6)

$$\{\phi^a(x, t), \Pi_b(x', t)\} = \delta^a_b \delta(x - x'),$$

(2.7)

and the boundary conditions,

$$F_a(\phi, \partial \phi, \Pi, \partial \Pi)|_{x \in \partial \Sigma} = 0.$$  

(2.8)

This set of boundary conditions is equivalent in the canonical formalism to the equations (2.3) or (2.4). In the definition of the canonical formalism we are assuming
that the boundary conditions are not in contradiction with the local symmetries of the system. So from the beginning we consider that our system does not have gauge degrees of freedom or that a gauge choice consistent with the boundary conditions has been done. Also, we are assuming that the symplectic structure (2.6)-(2.7) is defined until a zero measure set, so this structure can be changed in the boundary without affect the rest of the theory. So, if the boundary conditions (2.8) are not consistent with the symplectic structure, it is possible to modify the structure in the boundary in such way that be consistent with the boundary conditions. For example suppose that we have the boundary conditions,

$$F_a = [\Pi_a(x) - \phi_a(x)]|_{x \in \partial \Sigma} = 0,$$

these conditions are clearly inconsistent with (2.7), but we can modify the structure in the boundary, so we can introduce

$$\{\phi^a(x, t), \Pi_b(x', t)\}|_{x \in \partial \Sigma} = 0. \quad (2.9)$$

This symplectic structure is consistent with the boundary conditions. In principle, there are many ways to modify the symplectic structure in such way that these be consistent with the boundary conditions. However, no all of these modified structures are consistent with the exact solution of the equations of motion of the system. The main result of our paper is to propose a new procedure to obtain a symplectic structure that agrees with the symplectic structure obtained by the exact solution of the problem.

2.2 Outline of the method to compute the symplectic structure in the boundary

Following the reference [1, 2, 10], we assume that the boundary conditions are primary constraints in the sense of the Dirac’s method. However, we want to construct a procedure fully consistent with the Dirac’s method, avoiding problems with regularization. To achieve this we consider that the primary constraints are not only valid in the boundary \( \partial \Sigma \). Then, we suppose that these constraints are valid on all \( \Sigma \), i.e.,

$$F_a(\phi, \partial \phi, \Pi, \partial \Pi)|_{x \in \Sigma} \approx 0. \quad (2.10)$$

So, it make sense to write these constraints in smeared form

$$F[N] = \int_{\Sigma} dx d^dN^a(x) F_a(\phi, \partial \phi, \Pi, \partial \Pi) \approx 0, \quad (2.11)$$

for \( N^a(x) \) smeared functions with compact support. In consequence, we are mapping our original problem with boundary conditions (2.8) and canonical Hamiltonian (2.3)
to another problem where we have primary constraints (2.10) and a total Hamiltonian, given by

\[ H_T = H_c + \int \Sigma dx^d \lambda^a F_a. \]  

(2.12)

Using the total Hamiltonian (2.12) we evaluate the time evolution of the primary constraints,

\[ \dot{F}[N] = [F[N], H_T]. \]  

(2.13)

From this procedure we can obtain new constraints. To this secondary constraints, we applied the same procedure, we evolve the constraints and see if we get new constraints. We finish the procedure when we don’t get new constraints and determine the Lagrange’s multipliers. This must be so, because we are considering that in the original problem (2.1) does not have gauge degrees of freedom and furthermore that the boundary condition (or primary constraints), do not generate gauge freedom in the boundary. We conclude that the system only have second class constraints, then all Lagrange’s multipliers associated to the primary constraints must be determined. The set of second class constraints that we obtain are denote by,

\[ \chi_\alpha(\phi, \partial \phi, \pi, \partial \pi) |_{x \in \Sigma} \approx 0. \]  

(2.14)

With these constraints we obtain the invertible matrix

\[ C_{\alpha\beta}(x, x') = \{\chi_\alpha(x), \chi_\beta(x')\} \]  

(2.15)

and using (2.13) we construct the Dirac bracket,

\[ \{A(x), B(x')\}^* = \{A(x), B(x')\} - \{A(x), \chi_\alpha(y)\}C^{\alpha\beta}(y, z)\{\chi_\beta(z), B(x')\}. \]  

(2.16)

We notice that the Dirac bracket (2.16) is valuated on the surface \( \Sigma \), and it is not smeared. So, to obtain a Dirac bracket that is valid only in the interior of sigma \( \Sigma - \partial \Sigma \), we need to eliminate from the computation of the bracket the part that is different of zero only in the boundary \( \partial \Sigma \). Finally, we affirm that the correct symplectic structure in the boundary is obtained from the projection of this bracket to the boundary. Our procedure is in some sense equivalent to a Wick rotation, where one maps the original problem to the Euclidean space to make sense to the integrals, here we map the original problem defined with a boundary to another problem without boundary but with primary constraints that reflects the presence of the boundary. We do that in order to make sense of the Poisson brackets and in this way avoiding the problems with the regularization and obtaining a Dirac’s method fully consistent. Finally, when we obtain the symplectic structure in the new problem it is possible to obtain the symplectic structure of the original problem by projecting the structure resulting from the Dirac brackets to the boundary. We see, that our procedure is in some sense equivalent to an inverse of the Holographic map, since we obtain information about the boundary using computations in the bulk.
We don’t have a rigorous proof that the outlined procedure is always correct. However, we show in the following section that for several examples that our results coincide with the exact results obtained using the exact solution of the problem. In the example of the bosonic string in a background of PP-waves we disagree with one part of the result recently published. Nevertheless, we show that our symplectic structure is fully in the boundary whereas the result of [9] is inconsistent in this region.

3. The scalar field with Dirichlet and Neumann conditions

We introduce our procedure in a very simple example, the scalar field. This example is quite useful because we have an exact solution for boundary conditions of the type of Dirichlet and Neumann, and we show that our method reproduce correctly the symplectic structure in both cases. We start from the action

\[ S = \frac{1}{2} \int_0^\pi dt \int_{t_i}^{t_f} dx [ (\partial_t \phi(x,t))^2 - (\partial_x \phi(x,t))^2 ] . \] (3.1)

From this action we get the equations of motion

\[ (\partial_t^2 - \partial_x^2) \phi(x,t) = 0 , \] (3.2)

and we have two possible choices of boundary conditions

\[ \phi_i(x,t)|_{x=\pi,0} = 0 \quad \text{Dirichlet}, \] (3.3)
\[ (\partial_x \phi_i)(x,t)|_{x=\pi,0} = 0 \quad \text{Neumann}. \] (3.4)

and the canonical Hamiltonian is given by

\[ H_c = \frac{1}{2} \int_0^\pi dx [ (\Pi(x,t))^2 + (\partial_x \phi(x,t))^2 ] . \] (3.5)

3.1 Scalar Field with Dirichlet boundary conditions

For the Dirichlet boundary condition we have the primary constraint

\[ \phi(x,t) \approx 0 . \] (3.6)

That we can rewrite in densityzied form as,

\[ \Phi_D^{(1)}(N) = \int_0^\pi dx N(x) \phi(x,t) \approx 0, \] (3.7)

and the total Hamiltonian is,

\[ H_T = H_c + \int_0^\pi dx \lambda(x) \phi(x,t) . \] (3.8)
From, the time evolution of the primary constraint (3.6), we get
\[ \dot{\Phi}_D^{(1)}(N) = \{\Phi^{(1)}(N), H_T\} = \int_0^\pi dx N(x)\pi(x) \approx 0, \] (3.9)
which implies that we have the secondary constraint,
\[ \Phi_D^{(2)}(M) = \int_0^\pi dx M(x)\Pi(x,t) \approx 0. \] (3.10)
The time evolution of the constraint (3.10) produce no new constraints and we get the Lagrange multiplier,
\[ \lambda = \partial^2_x \phi(x,t). \] (3.11)
Computing the Poisson bracket of the constraints (3.6) and (3.10),
\[ \{\Phi_D^{(1)}(N), \Phi_D^{(2)}(M)\} = \int_0^\pi dx M(x)N(x) \neq 0, \] (3.12)
we see that the constraints are second class, we have only two constraints, and if we make a count of degrees of freedom in the boundary we obtain, zero degrees of freedom, that is the correct result. Furthermore, the Dirac brackets of the variables are,
\[ \{\phi(x,t), \phi(x',t)\}^* = \{\Pi(x,t), \Pi(x',t)\}^* = 0, \] (3.13)
\[ \{\phi(x,t), \Pi(x',t)\}^* = 0. \] (3.14)
Then the projection of this brackets to the boundary, results in the symplectic structure
\[ \{\phi(x,t), \phi(x',t)\}|_{x=0}\pi = \{\Pi(x,t), \Pi(x',t)\}|_{x=0}\pi = 0, \] (3.15)
\[ \{\phi(x,t), \Pi(x',t)\}|_{x=0}\pi = \delta(x-x')|_{x=0}\pi = 0. \] (3.16)
This structure agree with the results of the exact solution, see appendix.

### 3.2 Scalar Field with Neumann boundary conditions

For the Neumann boundary condition (3.4), we have the primary constraint
\[ \partial_x \phi(x,t) \approx 0, \] (3.17)
that we can rewrite in the form
\[ \Phi_N^{(1)}(N) = \int_0^\pi dx N(x)\partial_x \phi(x,t) = 0. \] (3.18)
Now the total Hamiltonian is given by
\[ H_T = H_c + \int_0^\pi dx \lambda(x)\partial \phi(x,t). \] (3.19)
From the evolution of the primary constraint $\Phi^{(1)}_{N}(N)$ we obtain the secondary constraint,

$$\Phi^{(2)}_{N}(M) = \int_{0}^{\pi} dx M(x) \partial_x \Pi(x, t) \approx 0. \quad (3.20)$$

From the stabilization of this constraint results that we don’t have more constraints and the Lagrange multiplier is,

$$\lambda(x) = 0.$$

Computing the Poisson bracket between the constraints

$$\{\Phi^{(1)}_{N}(N), \Phi^{(2)}_{N}(M)\} = -\int_{0}^{\pi} dx M(x) \partial^2_x N(x) \neq 0,$$

we see that are second class constraints, then we construct the matrix (2.15) that in this case have the form,

$$C_{\alpha\beta}(x, x') = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x \partial_x', \delta(x - x') \quad \text{and} \quad C_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F(x, x'), \quad (3.21)$$

where $F(x, x')$ is a function with compact support that satisfies

$$\partial^2_x F(x, x') = -\delta(x - x'). \quad (3.22)$$

The solution to this equation is

$$F(x, x') = \sum_{n \geq 1} \frac{1}{n^2 \pi} \sin(nx) \sin(nx'). \quad (3.23)$$

With the inverse matrix (3.21), we get the Dirac brackets,

$$\{\phi(x, t), \phi(x', t)\}^* = \{\Pi(x, t), \Pi(x', t)\}^* = 0, \quad (3.24)$$

$$\{\phi(x, t), \Pi(x', t)\}^* = \Delta(x - x') \quad (3.25)$$

with

$$\Delta(x - x') = \delta(x - x') - \partial_x \partial_x' F(x, x'). \quad (3.26)$$

To the function $\Delta(x - x')$ we call the Dirac’s delta transverse, because this have the property

$$\partial_x \Delta(x - x') = \partial_x' \Delta(x - x') = 0.$$

This implies

$$\{\partial_x \phi(x, t), \Pi(x', t)\}^* = \{\phi(x, t), \partial_x' \Pi(x', t)\}^* = \{\partial_x \phi(x, t), \partial_x' \Pi(x', t)\}^* = 0.$$

So, the symplectic structure that we get for the scalar field with Neumann boundary conditions is given by,

$$\{\phi(x, t), \phi(x', t)\}_{|x=0, \pi} = \{\Pi(x, t), \Pi(x', t)\}_{|x=0, \pi} = 0, \quad (3.27)$$

$$\{\partial_x \phi(x, t), \Pi(x', t)\}_{|x=0, \pi} = \partial_x \delta(x - x')_{|x=0, \pi} = 0. \quad (3.28)$$

Comparing (3.27-3.28) with the result obtained in the appendix from the exact solution (A.7- A.8). We see that both agree completely.
4. Bosonic string with mixed boundary conditions

Now we consider the case of the bosonic string in a constant magnetic field. The action for this string in the conformal gauge is,

\[ S = \frac{1}{2} \int_{-\pi}^{\pi} dx dt \left[ (\partial_t \phi_i(x,t))^2 - (\partial_x \phi_i(x,t))^2 + F_{ij} \partial_t \phi_i(x,t) \partial_x \phi_j(x,t) \right], \] (4.1)

with \( F_{ij} \) an antisymmetric constant matrix, and \( i = 1, \ldots, N \). The equations of motion for the system are,

\[ (\partial_t^2 - \partial_x^2)\phi_i(x,t) = 0. \] (4.2)

From the action (4.1) the canonical Hamiltonian is given by,

\[ H_c = \frac{1}{2} \int_{-\pi}^{\pi} dx \left[ \Pi_i(x,t) - T_{ij} \partial_x \phi_i(x,t) \right]^2 + (\partial_x \phi_i(x,t))^2 \] (4.3)

and we choose Neumann boundary conditions,

\[ (\partial_x \phi_i + T_{ij} \partial_t \phi_j)(x,t)|_{x=\pm \pi} = 0 \quad \text{with} \quad T_{ij} = \frac{F_{ij}}{2}. \] (4.4)

Then, the primary constraints that we have are,

\[ \Theta^{(1)}_i(x,t) = M_{ij} \partial_x \phi_j(x,t) + T_{ij} \Pi_j(x,t) \approx 0, \quad \text{with} \quad M_{ij} = (I - T^2)_{ji}. \] (4.5)

In the smeared version we have

\[ \Theta^{(1)}_i[N] = \int_{-\pi}^{\pi} dx N^i(x) \Theta^{(1)}_i(x,t). \] (4.6)

The Poisson brackets between these constraints are,

\[ \{\Theta^{(1)}[N], \Theta^{(1)}[M]\} = 0, \] (4.7)

where we use that the smeared vectors \( N^i(x) \) and \( M^i(x) \) have compact support. From the evolution of these constraints with the total Hamiltonian, we obtain the secondary constraints,

\[ \Theta^{(2)}[M] = \int_{-\pi}^{\pi} dx M^i(x) \partial_x \Pi_i(x,t) \approx 0. \] (4.8)

The stabilization of these constraints, don’t imply new constraints. So, the complete set of constraints is,

\[ \chi_\alpha(x,t) = (M_{ij} \partial_x \phi_j(x,t) + T_{ij} \Pi_j(x,t), \partial_x \Pi_k(x,t)) \quad \text{with} \quad \alpha = 1, \ldots, 2N \] (4.9)

The algebra algebra of these constraints is

\[ \{\chi_\alpha(x,t), \chi_\beta(y,t)\} = C_{\alpha\beta}(x,y), \] (4.10)
where the invertible matrix $C_{\alpha\beta}(x, y)$ is given by,

$$C_{\alpha\beta}(x, y) = \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix} \partial_x \partial_y \delta(x - y).$$

(4.11)

For the inverse matrix $C^{\alpha\beta}(x, y)$ we obtain,

$$C^{\alpha\beta}(x, y) = \begin{pmatrix} 0 & -M^{-1} \\ M^{-1} & 0 \end{pmatrix} F(x, y),$$

(4.12)

with $F(x, y)$ defined in (3.23). Using (4.12) we obtain the Dirac brackets,

$$\{\phi_i(x, t), \phi_j(y, t)\}^* = -(TM^{-1})_{ij} \partial_y F(x, y) + \partial_x F(x, y),$$

(4.13)

$$\{\Pi_i(x, t), \Pi_j(y, t)\}^* = 0,$$

(4.14)

$$\{\phi_i(x, t), \Pi_j(y, t)\}^* = \delta_{ij} \Delta(x - y).$$

(4.15)

Notice that the Dirac bracket (4.13) vanishes in the boundary, but we have still an extra step in our procedure that implies to remove from the brackets the contribution that is different of zero only in the boundary. In order to obtain a Dirac bracket that is valid only in the bulk we take into account (3.23) and comparing with $A(x, y)$ defined in (A.15), we have

$$\partial_y F(x, y) + \partial_x F(x, y) = A(x, y) - \frac{(x + y)}{2\pi}.$$  

(4.16)

To compare with the exact solution of the original problem we take the limit of our Dirac brackets to boundary, in this case $x \to x' \to \pm \pi$. So, we have

$$\{\phi_i(x = \pm \pi, t), \phi_j(x' = \pm \pi, t)\}^* = \pm(TM^{-1})_{ij}.$$  

(4.17)

In consequence using our procedure for the boundary we have the results

$$\{\phi_i(x, t), \phi_j(x', t)\}_{x' = x = \pm \pi} = \pm(TM^{-1})_{ij},$$

(4.18)

$$\{\Pi_i(x, t), \Pi_j(x', t)\}_{x' = x = \pm \pi} = 0,$$  

(4.19)

$$\{\partial_x \phi_i(x, t), \Pi_j(x', t)\}_{x' = x = \pm \pi} = \delta_{ij} \partial_x \delta(x - x'),$$

(4.20)

with the first derivatives of the deltas null in the boundary. This result for the symplectic structure in the boundary agrees with [1] and the exact solution (A.13)-(A.14) projected in the boundary.

5. Klein-Gordon equation with mixed boundary conditions

In this section we analyze the case of the open string ending on a D-brane in the pp-wave background. This example has been recently studied in [9]. Where the
bosonic action for the open string in the light cone gauge is

\[ S = \frac{1}{2} \int_{-\pi}^{\pi} dt \int_{t_1}^{t_2} dx dt [ (\partial_x \phi_i(x,t))^2 - (\partial_t \phi_i(x,t))^2 - m^2 (\phi_i(x,t))^2 + F_{ij} \partial_t \phi_i(x,t) \partial_x \phi_j(x,t) ]. \]  

(5.1)

From this action the canonical Hamiltonian is

\[ H_c = \frac{1}{2} \int_{-\pi}^{\pi} dx [ (\Pi_i(x,t) - T_{ij} \partial_x \phi_j(x,t))^2 + (\partial_x \phi_i(x,t))^2 + m^2 (\phi_i(x,t))^2 ] , \]  

(5.2)

with \( T_{ij} \) given in (4.4). The primary constraints are the Neuman boundary conditions,

\[ \Theta^{(1)}_i(x,t) = M_{ij} \partial_x \phi_j(x,t) + T_{ij} \Pi_j(x,t) \approx 0 , \]  

(5.3)

with \( M_{ij} \) given in (4.5). The variation in time of these constraints produce the secondary constraints

\[ \Theta^{(2)}[\mathbf{M}] = \int_{-\pi}^{\pi} dx M(x) [ \partial_x \Pi_i(x,t) - m^2 T_{ib} \phi_b(x,t) ] \approx 0 . \]  

(5.4)

These constraints in the same way that \( \Theta^{(1)}[\mathbf{N}] \) also satisfy

\[ \{ \Theta^{(2)}[\mathbf{N}], \Theta^{(2)}[\mathbf{M}] \} = 0 . \]

However, the matrix \( C_{\alpha\beta}(x,y) \) given by the Poisson bracket of all constraints is invertible and have the form

\[ C_{\alpha\beta}(x,y) = \begin{pmatrix} 0 & (M \partial_y \partial_x - m^2 T^2)_{ij} \\ -(M \partial_y \partial_x - m^2 T^2)_{ij} & 0 \end{pmatrix} . \]  

(5.5)

Then the inverse matrix \( C^{\alpha\beta}(x,y) \) is given

\[ C^{\alpha\beta}(x,y) = \begin{pmatrix} 0 & -R_{ij}(x,y) \\ R_{ij}(x,y) & 0 \end{pmatrix} , \]  

(5.6)

with \( R_{ij}(x,x') \) a matrix with compact support, that satisfies

\[ (M \partial_x^2 + m^2 T^2)_{ib} R_{bj}(x,x') = -\delta_{ij} \delta(x-x') . \]  

(5.7)

The solution to this boundary problem is

\[ R(x,x') = \sum_{n \geq 1} \frac{\sin(nx) \sin(nx')}{\pi [m^2 - m^2 T^2]} . \]  

(5.8)

Then the Dirac brackets are,

\[ \{ \phi_i(x,t), \phi_j(x',t) \}^* = -T_{ia}(x) R_{aj}(x',x') + \partial_x R_{aj}(x,x') \neq 0 , \]

\[ \{ \Pi_i(x,t), \Pi_j(x',t) \}^* = -m^2 (MT)_{ia} [ \partial_x R_{aj}(x',x') + \partial_x R_{aj}(x,x') ] , \]

\[ \{ \phi_i(x,t), \Pi_j(x',t) \}^* = \delta_{ij} (x-x') + m^2 T_{ib}^2 R_{bj}(x,x') - \partial_x \partial_x' R_{ib}(x,x') M_{bj} . \]  

(5.9)
Notice that, the massive term is included in all brackets. To compare with the exact solution, we see that in the term,
\[
\partial_x R(x, x') + \partial_{x'} R_{ij}(x, x') = \sum_{n \geq 1} \frac{n \sin n(x + x')}{\pi [M^2 - m^2 T^2]}
\]
\[
= M^{-1} \left[ \sum_{n \geq 1} \frac{1}{n \pi} \sin n(x + x') + \sum_{n \geq 1} \frac{m^2 M^{-1} T^2 \sin n(x + x')}{n \pi [n^2 - m^2 M^{-1} T^2]} \right].
\]
(5.10)

That, using \(A(x, x')\) defined in (A.13)
\[
\partial_x R(x, x') + \partial_{x'} R_{ij}(x, x') = M^{-1} \left[ \frac{(x + x')}{2\pi} + A(x, x') + \sum_{n \geq 1} \frac{m^2 M^{-1} T^2 \sin n(x + x')}{n \pi [n^2 - m^2 M^{-1} T^2]} \right].
\]
Let us remember that before to take the projection to the boundary, we eliminate all terms that take values only in the boundary. Then to compare (5.10) with the exact solution we don’t take into account the term \(A(x, x')\). Then taking the limit \(x \to x' \to \pm \pi\) we get
\[
[\partial_x R(x, x') + \partial_{x'} R_{ij}(x, x')]_{\pm \pi} = \pm M^{-1}.
\]
(5.11)
In this way we arrive to the Dirac brackets projected in the boundary
\[
\{\phi_i(x = \pm \pi, t), \phi_j(x' = \pm \pi, t)\} = \pm (TM^{-1})_{ij},
\]
\[
\{\Pi_i(x = \pm \pi, t), \Pi_j(x' = \pm \pi, t)\} = \mp m^2 T_{ij},
\]
\[
\{\partial_x \phi_i(x = \pm \pi, t), \Pi_j(x' = \pm \pi, t)\} = \mp m^2 (TM^{-1})_{ij}.
\]
(5.12)

Now, we can compare with the results of [3], where the use the exact solution. They get for \(i = 1, 2\)
\[
\{\phi_i(x, t), \phi_j(x', t)\}_{|x'=\pm\pi} \sim \pm (TM^{-1})_{ij},
\]
(5.13)
\[
\{\Pi_i(x, t), \Pi_j(x', t)\}_{|x'=\pm\pi} \sim \pm m^2 T_{ij},
\]
(5.14)
\[
\{\phi_i(x, t), \Pi_j(x', t)\} = \delta_{ij} \delta(x - x')
\]
(5.15)
with \(\partial_x \delta(x - x')_{|x'=\pm\pi} = 0\). Here we see that our results agree in (5.13) and (5.14) but not for (5.15). However is easy to see that the result (5.15) is inconsistent with the boundary condition. Taking the partial derivative with respect to \(x\) in (5.15), we get
\[
\{\partial_x \phi_i(x, t), \Pi_j(x', t)\}_{|x'=\pm\pi} = \delta_{ij} \partial_x \delta(x - x')_{|x'=\pm\pi} = 0.
\]
On the other hand using the boundary condition (5.3) in (5.14) we get
\[
\{\partial_x \phi^i(x, t), \Pi^j(x', t)\}_{|x'=\pm\pi} \sim \pm m^2 (M^{-1} T^2)^{ij}.
\]
Then, the results (5.14) and (5.15) are inconsistent, whereas (5.12) is fully consistent with the boundary conditions.
6. Conclusions

In the present paper was developed a procedure to compute the symplectic structure for a field theory with boundary. We consider that the boundary conditions can be interpreted as Dirac’s constraints and we construct a procedure that following the ordinary steps of the Dirac’s method, produce the symplectic structure on the boundary, for several examples. In our procedure, we have the problem that does not exist a definition for a smeared Dirac bracket so to compute this bracket we consider the prescription that the Dirac bracket in the interior of \( \Sigma \) is given by the ordinary Dirac bracket valued on \( \Sigma \) minus the contributions that are no null only in the boundary \( \partial \Sigma \). Using this prescription we are capable to compute the symplectic structure in several examples always obtaining fully consistent results. In a future we will try to extend our procedure to the case of General Relativity and compare our results with the recently published papers \cite{20, 21, 22}.

Acknowledgments

The authors acknowledge partial support from DGAPA-UNAM grant IN117000.

A. Appendix

In this appendix, we show the results obtained by using the exact solutions.

A.1 Scalar field

The general solution of the equation of motion for the scalar field (3.2) with Dirichlet boundary conditions (3.3) is

\[
\phi(x, t) = \sum_{n \geq 1} q_n(t) \sqrt{\frac{2}{\pi}} \sin(nx),
\]

where \( q_n(t) \) satisfies

\[
\ddot{q}_n(t) = -n^2 q_n(t).
\]

The inverse relation to (A.1) is given by

\[
q_n(t) = \frac{\sqrt{2}}{\pi} \int_0^l dx' \phi(x', t) \sin(nx).
\]

Using (A.2) in (A.1) we found

\[
\phi(x, t) = \int_0^\pi dx' \phi(x', t) \frac{2}{\pi} \sum_{n \geq 1} \sin(nx') \sin(nx) = \int_0^l dx' \phi(x', t) \delta(x - x'),
\]
that means that the Dirac delta is,

$$\delta(x - x') = \frac{2}{\pi} \sum_{n \geq 1} \sin(nx') \sin(nx).$$

On the other hand, the canonical momentum is,

$$\Pi(x, t) = \frac{\partial L}{\partial \dot{\phi}(x, t)} = \dot{\phi}(x, t) = \sum_{n \geq 1} \dot{q}_n(t) \sqrt{\frac{2}{\pi}} \sin(nx), \quad (A.3)$$

form this follows the notation $\dot{q}_n(t) = p_n(t)$, and then we have

$$\Pi(x, t) = \sqrt{\frac{2}{\pi}} \sum_{n \geq 1} p_n(t) \sin(nx) \quad (A.4)$$

with inverse given by

$$p_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi dx \Pi(x, t) \sin(nx). \quad (A.5)$$

Now, considering that the Poisson brackets between the normal modes are,

$$\{q_n(t), q_m(t)\} = \{p_n(t), p_m(t)\} = 0 \quad \text{and} \quad \{q_n(t), p_m(t)\} = \delta_{nm}. \quad (A.6)$$

In this way, the Poisson brackets that follows from the exact solution are,

$$\{\phi(x, t), \Pi(x', t)\} = \frac{2}{\pi} \sum_{n \geq 1} \sin(nx') \sin(nx) = \delta(x - x') \quad (A.6)$$

$$\{\phi(x, t), \phi(x', t)\} = \{\Pi(x, t), \Pi(x', t)\} = 0. \quad (A.6)$$

The projection of this result to the boundary, gives exactly the same result, that we get using our procedure, see (3.15) and (3.16).

For the Neumann boundary conditions, using the exact solution we get

$$\{\phi(x, t), \Pi(x', t)\} = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n \geq 1} \cos(nx') \cos(nx) = \delta(x - x') \quad (A.7)$$

$$\{\partial_x \phi(x, t), \Pi(x', t)\}|_{x=\pi, 0} = 0 = \partial_x \delta(x - x')|_{x=\pi, 0}. \quad (A.8)$$

If we compare this result with our method of computation (3.27) and (3.28) we see that both result agree. So, we see that in the case of the scalar field the projection of the Dirac brackets obtained from our procedure agrees completely with the result obtained using the exact solution.
A.2 Mixed Conditions

The general solution for the equations of motion in this case is of the form,

$$\phi_i(x, t) = M_i + B_i t + T_{ij} B_j x + \sum_{n \geq 1} \frac{q_{ni}(t)}{\sqrt{n\pi}} \cos(nx) - \frac{T_{ij} p_{nj}(t)}{n\sqrt{n\pi}} \sin(nx).$$

With $\ddot{q}_i(t) = -q_i(t)$ and $\dot{q}_i = p_i(t)$. Furthermore, from the definition of the canonical momenta we have

$$\Pi_i(x, t) = M_{ij}[B_j + \sum_{n \geq 1} \frac{p_{nj}(t)}{\sqrt{n\pi}} \cos(nx)].$$

Then, follows that the Fourier coefficients are,

$$q_{in}(t) = \frac{1}{\sqrt{n\pi}} \int_{-\pi}^{\pi} dx \phi_i(x, t) \cos(nx), \quad (A.9)$$

$$p_{in}(t) = \frac{1}{\sqrt{n\pi}} \int_{-\pi}^{\pi} dx M_{ij}^{-1} \Pi_j(x, t) \cos(nx), \quad (A.10)$$

$$B_i = \frac{2}{\pi} \int_{-\pi}^{\pi} dx M_{ij}^{-1} \Pi_j(x, t), \quad (A.11)$$

$$A_i = \frac{2}{\pi} \int_{-\pi}^{\pi} dx [\phi_i(x, t) - M_{ij}^{-1} \Pi_j(x, t)]. \quad (A.12)$$

For the Poisson brackets we get,

$$\{q_{in}(t), p_{jm}(t)\} = M_{ij}^{-1} \delta_{nm}, \quad \{A_i, B_j\} = \frac{M_{ij}^{-1}}{2\pi},$$

and the brackets vanish for the other cases. Using, the exact solution we obtain

$$\{\phi_i(x, t), \Pi_j(x', t)\} = \delta_{ij} \left[ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n \geq 1} \cos(nx') \cos(nx) \right] = \delta_{ij} \delta(x - x'), \quad (A.13)$$

and

$$\{\phi_i(x, t), \phi_j(x', t)\} = (M^{-1}T)_{ij} \left[ \frac{(x + x')}{2\pi} + \sum_{n \geq 1} \frac{\sin n(x + x')}{n\pi} \right], \quad (A.14)$$

In particular on the boundary

$$\{\phi_i(\pm \pi, t), \phi_j(\pm \pi, t)\} = \pm (M^{-1}T)_{ij},$$

and we see that the fields do not commute on the boundary, whereas the canonical momenta are commutative.
On the other hand, from the consistency with the Poisson brackets we see that

\[ A(x, x') = \left[ \frac{(x + x')}{2\pi} + \sum_{n \geq 1} \frac{\sin n(x + x')}{n\pi} \right], \]  

(A.15)

must be vanish in all interval \((-\pi, \pi)\), except a zero measure set, in the boundary satisfies

\[ A(\pm\pi, \pm\pi) = \pm1. \]

In fact we see that \( A \) is a distribution given by

\[ \partial_x A(x, x') = \partial_{x'} A(x, x') = \delta_a(x - x') - \delta_b(x - x'). \]

With

\[ \delta_a(x - x') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n \geq 1} \cos(nx') \cos(nx), \]

and

\[ \delta_b(x - x') = \frac{1}{\pi} \sum_{n \geq 1} \sin(nx') \sin(nx). \]

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