Gribov copies, avalanches and dynamic generation of a gluon mass

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Analytic calculations in the infrared regime of nonabelian gauge theories are hampered by the presence of Gribov copies which results in some ambiguity in the gauge-fixing procedure. This problem shares strong similarities with the issue of finding the true ground state among a large number of metastable states, a typical situation in the field of statistical physics of disordered systems. Building on this analogy, we propose a new gauge-fixing procedure which, we argue, makes more explicit the influence of the Gribov copies. A 1-loop calculation shows that the dynamics of these copies can lead to the spontaneous generation of a gauge-dependent gluon mass.

Introduction. Gauge invariance is a very important concept of modern theoretical physics which has been heavily used in the past to build and classify new theories. Nowadays, as is well known, all fundamental interactions (gravity, strong and electroweak interactions) involve such symmetries. However, as soon as analytic calculations are performed, one needs to break this invariance explicitly, through a procedure called gauge fixing. This is in particular true in the realm of quantum field theories, where gauge invariance leads to zero modes that make the propagator, one of the building blocks of field theory, ill-defined.

In its standard form, gauge fixing consists in grouping field configurations which are related one to another through a gauge transformation in equivalence classes, called gauge orbits. All these field configurations bear the same physical content and one can retain only one representative per gauge orbit. This choice is done through a gauge condition, such as the Lorenz condition \( \partial \mu A^\mu = 0 \) of electromagnetism (also called Landau gauge). In quantum field theory, gauge fixing is generally implemented by restricting the path integral to the subset of field configurations that fulfill the gauge condition. Once the gauge is fixed, the propagator is well defined and one can implement the standard techniques of field theory (perturbation theory, renormalization group, etc.) to access the physics of the system.

However, as first understood by Gribov [1] (see also [2]), this procedure fails for nonabelian gauge theories. Indeed, there typically exists several field configurations which are equivalent up to a gauge transformation and which satisfy the same gauge condition. These many intersections between a gauge orbit and the gauge condition, called Gribov copies, spoil the standard gauge-fixing procedure described above. In fact, Neuberger [3] showed that the textbook Faddeev-Popov gauge-fixing procedure, which overlooks the existence of Gribov copies is actually nonperturbatively ill-defined. In principle, this result invalidates the calculations based on the Faddeev-Popov procedure, at least in the long-distance regime. However, very few methods are available to overcome the Gribov ambiguity. (Even the Gribov-Zwanziger approach, which was built to overcome this problem [1-4], presents some residual ambiguities [5].) This probably explains why no consensus has been reached on the influence of the Gribov issue on actual calculations.

There is however a consensus in the community on one unambiguous way of treating the Gribov ambiguity in lattice simulations. This was heavily used in the past to obtain correlation functions in the Landau gauge, in the quenched approximation [11-14] as well as in full QCD [15,16]. We now briefly describe this method in the context of Euclidean Yang-Mills theory with a SU(2) group. Let us introduce the gauge field \( A^\mu_\mu = A^\mu_\mu \sigma^\alpha/2 \) where the \( \sigma \) are the Pauli matrices and a sum over repeated indices is implied. The gauge transformation acts on the gauge field as

\[
A^\mu_\mu = U(A^\mu_\mu + ig_0^{-1} \partial^\mu)U^{\dagger}
\]

where \( U(x) \) belongs to the SU(2) group and \( g_0 \) is the (bare) coupling constant of the theory. It is easy to show that an extremum \( U^* \) of

\[
f_A[U] = \int d^4x \text{ Tr } A^U \cdot A^U
\]

satisfies the Landau condition \( \partial^\mu A^\mu_{\mu} = 0 \). This observation can be used to implement in a numerically efficient way the Landau gauge on the lattice: Instead of solving the constrained problem \( \partial^\mu A^\mu_{\mu} = 0 \), it proves easier to search for a local minimum of \( f_A[U] \). It is found that there typically exist many local minima of the functional \( f_A \), which is the signature of the Gribov problem on the lattice.

The most striking result obtained by this method concerns the gluon-gluon correlation function which is found to saturate at low momentum. This is surprising because the gluons are massless in Yang-Mills theory, which would \textit{a priori} lead to a divergence of the correlation function at low momentum. Again, no consensus has been reached concerning the origin of this “massive” behavior. Calculations based on Dyson-Schwinger equations [6,8] and functional renormalization group [9] now indicate that the mass is generated through nonperturbative
effects while the refined Gribov-Zwanziger approach relates this phenomenon to the appearence of a condensate\cite{10}. In any case, this massive behavior is a welcome feature because it tends to regularize the infrared behavior of the theory. In particular, it was found that adding a mass term to the gluon on phenomenological ground leads to infrared-safe renormalization-group trajectories \cite{18}. This opens the way to perturbative calculations of the low-momentum sector of QCD correlation functions, in good agreement with those measured in lattice calculations \cite{17,19}.

We can gain new intuition on the Gribov issue by considering a priori unrelated models of statistical physics in the presence of quenched disordered, such as random manifolds \cite{20,21} or random field \cite{22} problems. These models share the property that their long-distance physics are governed by low-temperature properties. Indeed, thermal fluctuations are found to be negligible as compared to the sample-to-sample fluctuations of the quenched disorder. This shows up in the renormalization-group analysis because the fixed points that govern the critical properties are found at vanishing running temperature. As a consequence, extracting the critical properties of these systems requires to characterize their ground state. This proves difficult because there exists typically a very large number of local minima, with almost degenerate energies, very much like the extrema of \( f_A[U] \).

It is now well established that the existence of the many local extrema is a fundamental ingredient to understand the physics of these disordered systems. Let us briefly discuss this point in the case of the Random-Field Ising Model (RFIM) (see \cite{23,24} for more details). In order to characterize the properties of the ground state, a field-theoretical description was proposed by Parisi and Sourlas \cite{25} that closely resembles the Faddeev-Popov construction. It naturally leads to the famous property of dimensional reduction \cite{26} (that is, the critical exponents of the RFIM in \( d \) space dimensions are equal to those of the pure Ising model in \( d - 2 \) dimensions) to all orders in perturbation theory, a property which is however known to be wrong in three dimensions \cite{27}. Why does the Parisi-Sourlas construction fail to reproduce the breaking of dimensional reduction? Because, as the Faddeev-Popov construction, it does not properly take into account the many metastable states of the system \cite{28}, and in particular the following physical effect. Take one sample of the system, and submit it to a small external magnetic field. Since the hamiltonian presents many local minima, the ground state is very sensitive to the external field and there occur discontinuities in the magnetization when the external field is changed. These collective spin flips, called avalanches \cite{30}, can involve a large number of spins. It was shown in \cite{31} that the dimensional reduction property is violated if the strength of these avalanches is large enough. (Here, the strength is related to the Hausdorff dimension of spanning avalanches at criticality.) This implies that, if the strength of avalanches is small enough, although not justified because of the many metastable states, the Parisi-Sourlas construction leads to the correct prediction of dimensional reduction.

The analogy between disordered systems and the gauge-fixing procedure relies on the fact that, in both cases, one has to face a large number of extrema. From our knowledge on disordered system, it may or may not be true that the Faddeev-Popov construction is a good starting point for describing the long-distance physics of QCD. This strongly depends on the properties of the Gribov copies, in particular on the “size” of the avalanches that take place when the system jumps from one Gribov copy to another. The aim of this letter is to study this issue.

Gauge fixing. To avoid the Gribov-Singer no-go theorem, we follow the idea of \cite{32} of modifying the way the gauge-fixing is implemented. Instead of trying to pick-up just one representant per gauge orbit, we sum instead over all Gribov copies and weight their contributions. More precisely, we fix the gauge by looking for extrema of

\[
\exp\left[\int_{x} \left( A_{\mu} \partial_{\mu} A_{\nu} - \partial_{\mu} A_{\nu} + g_0 [A_{\mu}, A_{\nu}] \right) \right]
\]

where \( \eta \) is an external field, similar to the one introduced to generate the linear gauge. For an operator \( \mathcal{O}[A] \), the gauge-fixing procedure (that we note with brackets) reads:

\[
\langle \mathcal{O}[A] \rangle = \frac{\sum_{i} s(i) \mathcal{O}[A_{U_i}] e^{\rho_0 f_{A,\eta}[U_i]}}{\sum_{i} s(i) e^{\rho_0 f_{A,\eta}[U_i]}}
\]

where the sums run over all the Gribov copies, \( \rho_0 \) is a gauge parameter (of mass dimension 2), \( s(i) = 1 \) if \( f_{A,\eta} \) has an even number of unstable directions around \( U_i \), and \( s(i) = -1 \) otherwise.

After fixing the gauge, we average the gluon field \( A \) with the Yang-Mills action,

\[
S_{YM} = \int \frac{1}{2} \left( \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + g_0 [A_{\mu}, A_{\nu}] \right)^2,
\]

and the \( \eta \) field with a gaussian distribution \( \mathcal{P}[\eta] \propto \exp[-g_0^2/\xi_0 \int A \left( \partial^2 \eta \right)] \). A similar procedure was used in \cite{33}. The difference is that we now average simultaneously over \( \eta \) and \( A \). In the language of statistical mechanics, the \( \eta \) fields is a quenched disorder while it was annealed in \cite{33}.

To rewrite the corresponding action in a manageable form, a few technical steps must be performed (see \cite{32} for more details). First, we rewrite the numerator and denominator of Eq. (4) in terms of a field theory with a matrix field \( U \), a Lagrange multiplier and a pair of ghost-antighost fields. All these fields nicely rearrange in a SU(2) superfield \( \mathcal{V}(x, \theta, \bar{\theta}) \) where \( \theta \) and \( \bar{\theta} \) are (anticommuting) Grassmann variables. Second, we use the
replica trick to take care of the denominator of Eq. [4]. This amounts to introducing \( n - 1 \) replica of the superfield: \( \{ \mathcal{V}_i, i \in \{ 2, \ldots, n \} \} \) and taking the limit \( n \to 0 \) at the end of the calculation. Third, we factorize the gauge group by integrating over one (out of \( n \)) matrix fields \( \mathcal{U} \). We end up with a local action that can be decomposed according to the number of replica sums, \( S = S_{YM} + S_0 + S_1 + S_2 \), with

\[
S_0 = \int x \left( \frac{1}{2} (D_a \overline{c} \partial_\mu c^a + \partial_\mu c^a D_a) + h^a \partial_\mu A_\mu^a \right) + \frac{\xi_0}{2} (h^a)^2 - \frac{g_0^2 \xi_0}{4} (\epsilon^{abc} c^b c^c)^2 + \frac{\rho_0}{2} (\mathcal{A}_\mu^a)^2 + \rho_0 \xi_0 c^a c^a
\]

\[
S_1 = \sum_{i=0}^{n} \int \mathcal{U} \frac{1}{2} \left( \mathcal{A}_\mu^a - \frac{i}{g_0} \mathcal{V}_i^\dagger \partial_\mu \mathcal{V}_i \right)^2 + \xi_0 (n_i^a c^a - \frac{4}{g_0} n_i^0)\rho_0 + \frac{2}{g_0} n_i^a h^a
\]

\[
S_2 = -\frac{2\xi_0}{g_0} \sum_{i,j=0}^{n} \int \mathcal{U} \left( n_i^a n_j^a + n_i^a n_j^a \right)
\]

where we have introduced the representation of SU(2) matrices in terms of 4-components unit-vectors \( \mathcal{V}_i = n_i^a \mathbb{1} + i \sigma^a c^a_i \). \( S_0 \) is the Curci-Ferrari action \[35\]. Each replicated field \( \mathcal{V}_i \) (or \( n_i \)) in \( S_1 \) and \( S_2 \) comes with its own set of Grassmann variables which are integrated over with a nontrivial measure: \( \int \mathcal{U} = \int d\mathcal{U} (\rho_0 \mathcal{U} - 1) \) \[34\]. The two-replica action Eq. (7) is of particular importance in what follows because this is where the dynamics of the aforementioned avalanches is incoded. This term originates from the quenched average over the \( \eta \) field and was not present in the similar implementation of \[33\].

**1-loop calculation.** We now want to perform a perturbative analysis of this action. However, it proves important to first determine the general structure of the divergent part of the average action \( \Gamma \). This can be achieved by using the Ward (or Slavnov-Taylor) identities associated with the symmetries of the bare action. Apart from the obvious invariance under translation, rotations and rotations in color space, the action is invariant under a BRST transformation:

\[
s A_\mu^a = \partial_\mu c^a + g_0 \epsilon^{abc} A_\mu^b c^c \quad s c^a = -\frac{g_0}{2} \epsilon^{abc} c^b c^c
\]

\[
s c^a = h^a - \frac{g_0}{2} \epsilon^{abc} h^b c^c \quad s \mathcal{V}_i = -i \frac{g_0}{2} \mathcal{V}_i e^{a} \sigma^a
\]

\[
s h^a = \rho_0 c^a + \frac{g_0}{2} \epsilon^{abc} \left( h^b c^c + \frac{g_0}{4} \epsilon^{dec} c^d c^e \right)
\]

and a similar anti-BRST transformation. Other constraints can be obtained by considering a transformation acting simultaneously on all superfields:

\[
\delta^a \mathcal{V}_i = i \sigma^a \mathcal{V}_i
\]

This transformation is not a symmetry, but the variation of the action can be expressed in terms of the fields and their BRST variations. We can thus deduce Slavnov-Taylor identities. Finally, we can find strong constraints by performing a shift of the \( h \) field in the partition function, which implies a local Ward identity.

By using all these constraints, a straightforward but lengthy calculation shows that the microscopic action is not renormalizable. It can however be made renormalizable (up to some reparametrization of the field \( \mathcal{V} \)) by considering more general forms for the 1 and 2 replica parts:

\[
S_1 = v_0 \sum_{i=2}^{n} \int x \mathcal{U} \left( \mathcal{A}_\mu - \frac{i}{g_0} \mathcal{V}_i^\dagger \partial_\mu \mathcal{V}_i \right)^2 - \frac{4\xi_0 v_0}{g_0} W_0(n_i^0) - \xi_0 W_0'(n_i^0) (\epsilon^{abc} n_i^a n_i^b)
\]

\[
S_2 = -\frac{2\xi_0}{g_0} \sum_{i,j=2}^{n} \int x \mathcal{U} R_0(n_i^0 n_j^0 + n_i^0 n_j^0)
\]

where \( W_0 \) and \( R_0 \) are arbitrary functions. We recover the original action for

\[
v_0 = 1, \quad R_0(x) = W_0(x) = x.
\]

We have checked that the action obtained by choosing \( v_0 = 1 \) and \( R_0 = W_0 \) a generic function can be interpreted as a gauge-fixed action for a constraint more general than that of \[33\].

The one-loop calculation of the divergent part of the effective action is now rather standard except maybe for the fact that we need to renormalize whole functions (\( R \) and \( W \)). This is a usual feature of disordered systems and the most adapted method consists in performing the calculations in a nontrivial background for the fields \( n_i \). This is clearly efficient because we can extract the renormalization of the whole \( R \) and \( W \) functions in a single calculation. It also gives nontrivial tests of the loop calculations because a same renormalization factor can be extracted by different means. We have checked that all these definitions lead to the same renormalization factors in our calculation.

The one-loop calculation leads, in the Minimal Scheme, to the following \( \beta \) functions:

\[
\beta_u = -\frac{22}{3} u^2 \quad \beta_\xi = \frac{\xi u}{6} (26 - 3\xi)
\]

\[
\beta_\rho = \rho \left( -35 + 3\xi \right) \quad \beta_v = v \frac{\xi}{2} \left( R'(1) - 1 \right)
\]

\[
\beta_R(x) = \frac{\xi}{8} \left( 4 R(x) - 8 R(x) R'(1) + (1 + 2 x^2) R'(1)^2 - 2 [R'(x) - x R'(x)]^2 - [R'(1) - x R'(x) + (1 - x^2) R'(x)]^2 \right)
\]
where the interaction is expressed in terms of \( u = g^2 / (8\pi^2) = \alpha / (2\pi) \). We retrieve the Curci-Ferrari \( \beta \) functions \([35]\) for the coupling constant \( u \), gauge parameter \( \xi \) and gluon mass \( \rho \). Finally, the flow of \( R \) closely resembles that of the two-replica function \( R \) in the random field \( O(4) \) model \([36]\), except for the multiplicative factor and the first term in braces. This is not a surprise because, as already mentioned, the SU(2) matrices can be parametrized in terms of a 4-component unit vector, which yields a nonlinear representation of the \( O(4) \) group.

Integrated flow of the RG flow Let us now study the renormalization-group flow and its consequences for the theory under study. We initialize the flow at some ultraviolet scale \( \Lambda_{\text{UV}} \) which would correspond to the inverse lattice spacing of a lattice simulation with some initial value of the coupling constant \( g_{\text{UV}} \), gauge parameter \( \xi_{\text{UV}} \) and weight \( \rho_{\text{UV}} \). The flows of \( u \), \( \xi \) and \( \rho \) decouple from the rest and we can integrate analytically these expressions. As usual, \( u \) presents a Landau pole at a scale \( \Lambda_{\text{QCD}} = \Lambda_{\text{UV}} e^{-3/(2u_{\text{UV}})} \).

Next, we consider the flow of \( v \). It involves \( R'(1) \) whose flow can be easily deduced from Eq. \([15]\):

\[
\beta_{R(1)} = (u\xi/8)R'(1)[1 - R'(1)]
\]

(16)

(Note that the flow of \( R'(1) \) does not involve higher derivatives of \( R \)! We would naively deduce that the initial value \( R'(1) = v = 1 \) [see Eq. \([14]\)] is a fixed point. However, we should keep in mind that we need to renormalize the whole function \( R \). In fact, experience from disordered systems shows that \( R''(1) \) is of interest \([36, 38]\). Using again Eq. \([15]\) and \( R'(1) = 1 \) we find:

\[
\beta_{R''(1)} = -(u\xi/4)[11R''(1)^2 + 4R''(1) + 1]
\]

(17)

Note that \( \beta_{R''(1)} < 0 \) for all values of \( R''(1) \) and that it increases quadratically for large \( R''(1) \). Consequently, the flow of \( R''(1) \) may eventually end up in a singularity at some finite length scale, called Larkin scale in the context of disordered systems \([37]\). We find that if \( 26/3 > \xi_{\text{UV}} > \xi_c = 26/3(1 - \arctan(2/\sqrt{7}\pi) / \pi) \approx 4.35 \), the Larkin scale is larger than the Landau pole scale. In this regime, we find that

\[
\Lambda_{\text{Larkin}} = \Lambda_{\text{QCD}} \exp\left[ -\frac{3}{22u_{\text{UV}}} \left( 1 - \frac{\xi_c}{\xi_{\text{UV}}} \right)^{22/13} \right].
\]

(18)

What have we gained? It seems, at this point that the perturbative solution disappears either at the Landau pole (for \( \xi_{\text{UV}} < \xi_c \)) or at an even more ultraviolet RG scale. However, the appearance of a singularity at the Larkin scale is not as disastrous as the Landau pole. Indeed, we can treat Eq. \([15]\) as a partial differential equation and go through the Larkin scale. A careful analysis shows that, as in the random field case \([38]\), beyond this scale, \( R'(x) \) behaves as \( C\text{te} - \alpha\sqrt{1-x} \). This spoils the Taylor expansion which leads to \([16, 17]\) and the function \( R'(1) \) starts departing from 1. As a consequence, \( v \) itself starts to flow away from 1. We illustrate these behaviors in Fig. 1.

We are now in position to discuss the generation of the gluon mass which, looking at the renormalizable action, receives contributions from \( S_0 \) and \( \tilde{S}_1 \). In total, the “gluon square mass” reads \( \rho + (n - 1)/nu \to \rho(1 - v) \) in the limit \( n \to 0 \). This implies that, as long as the coefficient \( v \) is strictly equal to 1 (i.e. in the range \( \Lambda_{\text{Larkin}} < \mu < \Lambda_{\text{UV}} \)), the gluon mass vanishes, while it departs from zero at lower momenta. Since \( v > 1 \) in this regime, the gluon square mass is negative, which implies that the gluon field acquires a nonzer expectation value.

Conclusion. We have proposed a gauge-fixing procedure where the Gribov ambiguity is under control. It makes possible to study the influence of avalanches which correspond to collective changes when the system jumps from one Gribov copy to another. A 1-loop renormalization-group study shows that these avalanches can lead to a singularity in the 2-replica potential, which, in turn generates a gauge-dependent gluon mass. This scenario only work for large enough gauge parameter \( \xi \), which exclude the widely studied case of the Landau gauge. Further investigations are needed to study whether this gluon mass is sufficient to avoid the Landau pole, as was already shown in the Curci-Ferrari model \([17, 18]\). Another important study consists in understanding the consequences of a nonvanishing expectation value for the gauge field.

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![FIG. 1: Numerical integration of the flow equations with initial condition \( g = 0.1, \xi = 7, \rho = 0.5 \) and \( R'(x) = 1 \). \( R'(1) \) (dash-dotted line) diverges at \( t = t_{\text{Larkin}} \). For \( t < t_{\text{Larkin}} \), \( R'(1) \) (full curve) departs from 1 and the gluon square mass (dashed curved, in arbitrary units) departs from zero.](image-url)
