Hawking-Penrose Black Hole Model. Large Emission Regime.

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Abstract

The work devoted to an application of the large deviations theory to a Markov jump process describing a state dynamics of the Hawking-Penrose model of the black hole. To do this we introduce stochasticity in the black hole life. We study the large deviations asymptotics for a stochastic version of the model with special attention to the large emission regime. The stochasticity is defined by random dynamics of quanta out and in the black hole. It is assumed that there are $N$ quanta in total. The considered Markov process takes two dimension values: the number of the quanta $\xi(t)$ in the black hole at the time moment $t$ and the amount of the emission $\eta(t)$ until $t$. Applying the large deviations theory we consider a sequence of scaled versions of the processes $\xi_N(t)$, $\eta_N(t)$. The measures of these processes are concentrating on the space $\{(x(\cdot), y(\cdot))\}$ of pairs of the differentiable functions.

One of our goals is to find the most probable trajectory corresponding to a certain amount of the emission during the time interval. To find the trajectory we have to solve a Hamiltonian system of equations under proper boundary conditions. The Hamiltonian system is highly nonlinear. We present the solution of the system under the condition of the stationarity of the first component $x$.

1 Introduction

A state of any system is not always the result of a quiet and long evolution. Sometimes a very rare event drastically changes directions of the development. If randomness is presented in the system then the rare event can be studied by the large deviations theory. The large deviations theory is one of the well developed and often currently applied parts of the probability theory which gives means for asymptotical evaluations of the rare event probability.

The large deviations theory was started from the famous Cramér’s article [1]. It was the work which initiates elaboration of a new section of the probability theory. One of the next crucial contributions to the theory has been done by S. R. S. Varadhan in [2] where the notion of the large deviations principle was introduced. The theory is well developed at the present time. There exists a fairly large library of books devoted to the large deviations theory, ([3, 4, 5, 6, 7, 8, 9]). Some of the books contain the chapters about the applications of the large deviations.

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The goal of this article is an application of the large deviation theory to Markov processes with continuous time describing the evolution of the simplest model of the black hole (see [14], [15]). We do not claim that our result can help to understand some real physical problem. The models we investigate are more likely, imitating some physics. However, these studies may display various phenomena which are interesting from the large deviations point of view.

In essence, the concept of a black hole as a domain bounded by an event horizon was discovered by K. Schwarzschild in 1916 [10]. This concept had no connections with the statistical physics until the calculation of a black hole entropy and the discovery of a Hawking radiation, [11, 12, 13]. After this discovery, S. Hawking considered the model containing the black hole and the photon gas in the dynamical equilibrium [14]. In more detail, this dynamical model was developed by R. Penrose [15].

In this article we study the large deviations asymptotics for a stochastic version of the Hawking-Penrose model with special attention to the large emission regime. We start from the deterministic picture and then formulate the stochastic Markov model. Particularly we find the new relation between the size of a black hole and its emission power in a large emission regime. The stochastic model is, in fact, a Markov random process which describes dynamics of states of the black hole and emissions from the black hole.

When using the large deviations theory the rate function $I$ of the studied stochastic system is sought. We apply the developed in [4] approach to find the rate function of the studied system at appropriate scaling. Knowledge of the rate function allows finding the dynamic trajectory of the black hole state corresponding the given amount of the emission. The probability of the process is concentrated in any neighborhood of this trajectory when the scaling parameter is going to infinity. Finding this trajectory is reduced to solving a Hamiltonian system of equations. In the considered case the Hamiltonian system is highly nonlinear.

Presented work continues our works [22, 23, 24], where the similar problems concerning emission regime were studied.

2 Model

The goal of this article is to study stochastic version of the Hawking and Penrose black hole model introduced in [14] and [15].

The model in our considerations has two constituents: the black hole and a cloud of photons. A part of the photons are located in the hole, the rest photons are free and located in a box with reflected boundaries. There exists an exchange by the photons between the cloud and the hole: emission and absorption.

This exchange we describe by a Markov process with discrete phase space.

2.1 Deterministic picture

Let $V$ be a volume with mirror boundaries containing the radiation with the total energy $E$. Some amount of the energy $e$ is absorbed by the black hole. The black hole emits a radiation by the Hawking process. It means that the amount of the energy in the black hole depends on time.

Remark 2.1. In this subsection we assume that the values of $E$, $e$ and $m$ take real values. Further, when the random version is studied, the values of $E$, $e$ will be discrete.
The Schwarzschild radius of the black hole equals
\[ R = \frac{2Gm}{c^2} = \frac{2G}{c^4}e, \]
where \( m = \frac{e}{c^2} \) and \( G \) is the gravitational constant. The radius \( R \) depends on the energy \( e \) of the black hole. We denote the coefficient connecting \( R \) and \( e \) by \( a \),
\[ R = ae, \quad (2.1) \]
where
\[ a = \frac{2G}{c^4}. \quad (2.2) \]

The energy \( e \) satisfies the balance equation,
\[ \frac{de}{dt} = W_{\text{abs}} - W_{\text{em}}. \quad (2.3) \]

In this equation the absorbed by the black hole power is
\[ W_{\text{abs}} = \frac{1}{4}c A \frac{E - e}{V}, \quad (2.4) \]
where
\[ A = 4\pi R^2 \quad (2.5) \]
is the horizon area, and the factor \( 1/4 \) appears by geometrical reasons (see [19]).

**Remark 2.2.** The factor \( 1/4 \) in (2.4) reflect a fact that the absorbed power falls into black hole at some angle \( \theta \) to the surface. The absorbed power is proportional to \( \cos(\theta) \). An average value of \( \cos(\theta) \) on the hemisphere \( 0 \leq \theta < \pi/2 \) equals to \( 1/2 \). Additional factor \( 1/2 \) appears because we have to consider only rays directed towards the surface ([16], Vol.1, Ch. 45).

Here we ignore the gravitational light deflection. An elementary discussion of the gravitational light deflection as a consequence of the equivalence principle is contained in [17], vol.1, Ch. 14.

The considering of the light deflection gives \((27\pi/4)R^2\) instead of \(\pi R^2\) (see [18], Ch. 12, Section 102: gravitational collapse of spherical body, pp. 338).

Using (2.1) and (2.5) we obtain
\[ W_{\text{abs}} = \frac{\pi c}{V} a^2 e^2 (E - e). \quad (2.6) \]

The black hole emission \( W_{\text{em}} \) was calculated in [20] (eq. (146))
\[ W_{\text{em}} = \sigma AT^4, \]
where
\[ T = \frac{\hbar c}{4\pi R} \]
is the Hawking temperature and \( \sigma \) is the emission constant (see [20], eq.(146)). Note that the emission constant \( \sigma \) does not coincide with the classical Stefan-Boltzmann constant ([19]).
Using expressions of $A$ and $R$ via $e$ we obtain

$$W_{em} = \sigma \frac{(\hbar c)^4}{(4\pi^3 a^2 e^2}.$$  

Let

$$b = \frac{\hbar c}{4\pi a},$$  

then

$$T = \frac{b}{e}.$$  

We obtain (see (2.3))

$$\frac{de}{dt} = a_1 a^2 e^2 \frac{E - e}{V} - a_2 a^2 b^4 e^2,$$  

(2.8)

where

$$a_1 = \pi c, \quad a_2 = 4\pi \sigma.$$  

(2.9)

This equation can have a stationary solution if the equation

$$a_1 e^4 (E - e) = a_2 b^4 V$$  

(2.10)

has a solution. The condition for it is

$$\frac{4^4}{5^5} E^5 \geq \frac{a_2 b^4}{a_1} V.$$  

If this inequality is strict, then the equation (2.10) has two solutions. One of them corresponds to the stable and another to the unstable black hole ([15]).

### 2.2 Stochastic picture

In this section we study a discrete version of the system outlined above. Moreover, we impose stochasticity on the system.

As in the previous section, $E$ is the total energy in the volume $V$, and $e$ is the part of $E$ which is assumed to be contained in the black hole. The discreteness assumes that the total energy $E$ is split in quanta. Let $N$ be the number of quanta. Then the energy $\varepsilon$ of each quanta is

$$\varepsilon = \frac{E}{N}.$$  

From now on $e$ is also a discrete variable. Later on, $E$ is fixed while $N$ is growing.

The volume $V$ split into two parts: the black hole interior and its exterior. Arbitrary positive part $k = 1, 2, ..., N$ of the quanta can be absorbed by the black hole, and be contained in it. The energy of the black hole is $e = k\varepsilon$ if $k$ quanta are in the hole.
2.2.1 Markov Process

The dynamics consists of the emission and the absorption of the quanta from and to the black hole. This dynamics is a random process and is determined by a Markov process $\xi(t)$. Values of the Markov process $\xi(t), t \in [0, T]$ are from the state space $\mathbb{N} = \{1, \ldots, N\}$ and mean the number of the quanta absorbed by the black hole.

The transition intensities of $\xi(t)$ are as the following:

If $\xi(t) = k > 1$ then the rate of the transition $k \to k - 1$ (the emission rate) equals to $W_{\text{em}} = a_2 a^2 b^4 \frac{E^3}{k} N^2 k^2$.

If $\xi(t) = k < N$ then the rate of the transition $k \to k + 1$ (the absorption rate) equals to $W_{\text{abs}} = \frac{a_1 a_2 b^4}{V} \frac{E^3}{k^2} (1 - \frac{k}{N})$.

Thus, the generator of the jump Markov process $\xi(t)$ is

$$L f(k) = \frac{a_1 a_2 E^2}{V} N^2 \frac{k^2}{N^2} (1 - \frac{k}{N}) [f(k + 1) - f(k)] + \frac{a_2 a^2 b^4}{E^3} N^2 \frac{E^3}{k^2} (1 - \delta(k - 1))[f(k - 1) - f(k)].$$

(2.11)

Here $\delta(k) = 1$ for $k = 0$, and $\delta(k) = 0$ otherwise.

**Remark 2.3.** We introduce the term $1 - \delta(k - 1)$ which do not allow the black hole to completely evaporate.

We further use the following notations

$$\mu = a_2 a^2 b^4 \frac{1}{E},$$

$$\lambda = \frac{a_1 a_2 E^2}{V}.$$

(2.12)

2.2.2 Markov process with emission

Next we consider the joint process $\psi = (\xi, \eta)$ where the second component $\eta$ describes emissions from the hole that happen over the time interval $[0, T]$. The process $\eta(t)$ takes its values in $\mathbb{Z}_+$ and it is monotone increasing process. The value $\eta(t)$ counts the number of the radiations from the black hole on the time interval $[0, t], t \leq T$. Therefore the generator of the joint process is

$$L f(k, m) = \lambda N^2 \frac{k^2}{N^2} (1 - \frac{k}{N}) [f(k + 1, m) - f(k, m)] + \mu N^2 \frac{E^3}{k^2} (1 - \delta(k - 1))[f(k - 1, m + 1) - f(k, m)],$$

(2.13)

where $k \in \mathbb{N}, m \in \mathbb{Z}_+$.

Considering the large deviations of the black hole emissions on the interval $[0, T]$ we scale $(\xi(t), \eta(t))$:

$$\gamma_N(t) = \frac{\xi(t)}{N}, \quad \nu_N(t) = \frac{\eta(t)}{N}.$$

(2.14)

In this scaling we study the large emission when $N \to \infty$.

The joint process $\psi_N(t) = (\gamma_N(t), \nu_N(t))$ takes its value in $D_N = \left(\frac{1}{N} \mathbb{N} \times \frac{1}{N} \mathbb{Z}_+\right)$. Since $D_N \subset D = [0, 1] \times \mathbb{R}_+$ for every $N$ we will assume that the processes $\psi_N$ takes their values in $D$. 
The process $\psi_N$ is a jump process with two types of jumps: $\left(\frac{1}{N}, 0\right)$ and $\left(-\frac{1}{N}, \frac{1}{N}\right)$. Let $(x_N, y_N) \in D_N$. Then the infinitesimal operator of $\psi_N$ is

$$L_{\psi_N} f(x_N, y_N) = \lambda x_N^2 \left(1 - x_N\right) N \left[f(x_N + \frac{1}{N}, y_N) - f(x_N, y_N)\right] + \frac{1}{x_N^2} N \left(1 - \delta(x_N - \frac{1}{N})\right) \left[f(x_N - \frac{1}{N}, y_N + \frac{1}{N}) - f(x_N, y_N)\right].$$

(2.15)

Let $(x, y) \in D$ and a sequence $(x_N, y_N) \in D_N$ be such that $(x_N, y_N) \to (x, y)$. Assuming differentiability of $f$ we obtain a limit

$$L_{\infty} f(x, y) = \lim L_{\psi_N} f(x_N, y_N) = \lambda x^2 \left(1 - x\right) \frac{\partial f}{\partial x} + \mu \frac{1}{x^2} \left(\frac{\partial f}{\partial y} - \frac{df}{dx}\right).$$

Let

$$S = \{(x(t), y(t)) : [0, T] \to D\}$$

be Skorohod space, it means that the paths $x(\cdot)$ and $y(\cdot)$ are continuous from the right and have limits from the left. This space is equipped with the Skorohod topology ([21]). Let also $C_1 \subset S$ be a subset of pairs of absolutely continuous functions $(x(\cdot), y(\cdot))$ such that $x(t) \in [0, 1]$ and $y(t) \in \mathbb{R}_+$ is non-decreasing and $y(0) = 0$.

The process $\psi_N$ induces a measure $\bar{\mathbb{P}}_N$ on $S$.

The operator $L_{\infty} f(x, y)$ can be considered as an infinitesimal generator of a deterministic dynamics. This dynamics is described by the following ordinary differential equations

$$\frac{dx}{dt} = \lambda x^2(1 - x) - \mu \frac{1}{x^2} \theta(x),$$

(2.17)

$$\frac{dy}{dt} = \mu \frac{1}{x^2} \theta(x)$$

(2.18)

for $(x, y) \in C_1$, where $\theta(\cdot)$ is the Heaviside step function whose value is zero for negative arguments and one for positive arguments, and we defined $\theta(0) = 0$. The equation (2.17) coincides with the equation (2.8). The equation (2.18) counts an amount of emitted energy.

For large finite $N$ the paths of a random process having the generator $L_{\psi_N}$ (see (2.15)) fluctuate around of the solutions of (2.17) and (2.18).

The probability of these fluctuations are governed by the rate function $I(x(\cdot), y(\cdot))$, which we define in the next section about the large deviation theory. Here we outline the role of the rate function $I$. The probability that the process $\psi_N$ is close to a path $(x(t), y(t))$ has an asymptotics

$$\Pr(\psi_N(t) \approx (x(t), y(t)), t \in [0, T]) \approx \exp[-NI(x(\cdot), y(\cdot))]$$

as $N \to \infty$. The sign $\approx$ means that the process $\psi_N$ is located in a neighbourhood of the path $(x(t), y(t))$, and the neighbourhood is shrinking to this path with growing $N$. 

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3 Large Deviations

We use the large deviation theory to find the probability of the large emission on \([0, T]\). The large deviation theory is especially useful when looking at the asymptotic probability of rare events.

We describe the large deviation approach in terms of the studied here system. The large emission from the black hole on interval \([0, T]\) is described by the event

\[
\mathcal{E}_N = \{(y_N(\cdot), v_N(\cdot)) : v_N(T) \geq BT\},
\]

where \(B > 0\). The first component \(\gamma_N(t)\) (see (2.14)) is irrelevant in this event, same the values of \(v_N(t)\) for \(t < T\) except \(t = T\), where \(v_N(T) \geq BT\).

Let

\[
\mathbf{E} = \{(x(\cdot), y(\cdot)) : y(T) \geq BT, y(0) = 0\}.
\]

Further we consider an event \(\mathcal{G}_N\) which is a part of \(\mathcal{E}_N\). In the definition of \(\mathcal{G}_N\), restrictions on the first component \(\gamma_N(t)\) are introduced:

\[
\mathcal{G}_N = \{(y_N(\cdot), v_N(\cdot)) : \gamma_N(0) = \gamma_N(T), v_N(T) \geq BT\}.
\]

Then

\[
\mathbf{G} = \{(x(\cdot), y(\cdot)) : x(0) = x(T), y(T) \geq BT, y(0) = 0\} \subseteq \mathbf{E}.
\]

The asymptotic of the probabilities of \(\Pr(\mathcal{E}_N)\) and \(\Pr(\mathcal{G}_N)\) as \(N \to \infty\) is the subject of the large deviation theory.

The large deviation theory states the existence of the function

\[
I(x, y) : \mathbb{Z} \to \mathbb{R}_+,
\]

such that \(I(x, y) = \infty\) if \((x, y) \notin C_1\). In the large deviation theory, the function \(I\) is called the *rate function* which was mentioned in the previous section. The rate function has a set of the properties that can be found in the literature (for example, see [4]).

Applying the large deviations theory ([5]) we can find the logarithmic asymptotics of \(\Pr(\mathcal{E}_N)\) and \(\Pr(\mathcal{G}_N)\) that is

\[
\lim_{N \to \infty} \frac{1}{N} \ln \Pr(\mathcal{E}_N) = \inf_{(x, y) \in \mathbf{E}} I(x, y),
\]

\[
\lim_{N \to \infty} \frac{1}{N} \ln \Pr(\mathcal{G}_N) = \inf_{(x, y) \in \mathbf{G}} I(x, y),
\]

because the boundaries of the sets \(\mathcal{E}\) and \(\mathcal{G}\) have the probabilities equal to 0.

Looking for the rate function \(I(x, y)\) in our case we follow the way by Feng and Kurtz ([4]). The rate function, according to this way, is constructed by a Hamiltonian \(H\). In first step, the non-linear Hamiltonian has to be found, for \((x, y) \in D\):

\[
(\mathcal{H}_N f)(x, y) := \frac{1}{N} \exp(-N f(x, y)) \times L_{\psi_N} \exp(N f)(x, y)
\]

\[
= \lambda x^2 (1 - x) \left[ \exp \left\{ N \left( f \left( x + \frac{1}{N}, y \right) - f(x, y) \right) \right\} - 1 \right] + \mu \frac{1}{x^2} (1 - \delta(x - \frac{1}{N}) \left[ \exp \left\{ N \left( f \left( x - \frac{1}{N}, y + \frac{1}{N} \right) - f(x, y) \right) \right\} - 1 \right],
\]

\[
\text{lim}_{N \to \infty} \frac{1}{N} \ln \Pr(\mathcal{E}_N) = \inf_{(x, y) \in \mathbf{E}} I(x, y),
\]

\[
\text{lim}_{N \to \infty} \frac{1}{N} \ln \Pr(\mathcal{G}_N) = \inf_{(x, y) \in \mathbf{G}} I(x, y),
\]
of Hamiltonian (3.5), the extremals of (3.7) and (3.8) should satisfy a Hamiltonian system

\[
\lim_{N \to \infty} (H_N f)(x, y) = \lambda x^2(1 - x) \left[ \exp\left( \frac{\partial}{\partial x} f(x, y) \right) - 1 \right] + \mu \frac{1}{x^2} \left[ \exp\left( -\frac{\partial}{\partial x} f(x, y) + \frac{\partial}{\partial y} f(x, y) \right) - 1 \right].
\] (3.4)

Using the following notations

\[
\kappa_1 = \frac{\partial}{\partial x} f(x, y), \quad \kappa_2 = \frac{\partial}{\partial y} f(x, y).
\]

we obtain from (3.4) the Hamiltonian \( H \) of the system

\[
H(x, y, \kappa_1, \kappa_2) = \lambda x^2(1 - x) [e^{\kappa_1} - 1] + \mu \frac{1}{x^2} [e^{-\kappa_1 + \kappa_2} - 1].
\] (3.5)

To define the rate function for the considered system we introduce paths \((\kappa_1, \kappa_2)\) on \([0, T]\); \((\kappa_1(t), \kappa_2(t)) \in \mathbb{R}^2\). Then the rate function is, (see (3.5)),

\[
I(x, y) = \int_0^T \mathcal{L}(x, y) dt
= \int_0^T \sup_{\kappa_1(t), \kappa_2(t)} \{ \kappa_1(t)\dot{x}(t) + \kappa_2(t)\dot{y}(t) - \lambda x^2(t)(1 - x(t))[e^{\kappa_1(t)} - 1] - \mu \frac{1}{x(t)} [e^{-\kappa_1(t) + \kappa_2(t)} - 1] \} dt,
\] (3.6)

where

\[
\mathcal{L}(x(t), y(t)) = \sup_{\kappa_1(t), \kappa_2(t)} \{ \kappa_1(t)\dot{x}(t) + \kappa_2(t)\dot{y}(t) - H(x(t), y(t), \kappa_1(t), \kappa_2(t)) \}
\]

is Legendre transform of Hamiltonian \( H \) ((3.5)). Recall that \((x(t), y(t)) \in \mathbb{Z}\).

### 3.1 Result

Our goal is to study the large emission that is the large \( y(T) \). To this end we have to find

\[
\inf_{(x, y) \in E} I(x, y),
\] (3.7)

(see (3.2)). Here there are no any constraints on the particle number in the black hole. The infimum

\[
\inf_{(x, y) \in G} I(x, y),
\] (3.8)

also gives the asymptotical behavior of the large emission probability with restrictions on the value of the number of quanta in the black hole. Namely, in this case the quanta number satisfies periodic boundary conditions on the \([0, T]\).

Since the rate function is the non-linear integral functional which integrand is the Legendre transform of Hamiltonian (3.5), the extremals of (3.7) and (3.8) should satisfy a Hamiltonian system

\[
\begin{align*}
\dot{x} &= \lambda x^2(1 - x) \exp[\kappa_1] - \mu \frac{1}{x^2} \exp[-\kappa_1 + \kappa_2], \\
\dot{y} &= \mu \frac{1}{x^2} \exp[-\kappa_1 + \kappa_2], \\
\dot{\kappa}_1 &= -\lambda(2x - 3x^2)[\exp[\kappa_1] - 1] + \mu \frac{2}{x^3} [\exp[-\kappa_1 + \kappa_2] - 1], \\
\dot{\kappa}_2 &= 0,
\end{align*}
\] (3.9)

with suitable boundary conditions.
Theorem 3.1 (hypothetical). Consider the following boundary conditions
\[
\begin{align*}
x(0) &= x(T), \\
y(0) &= 0, \quad y(T) = BT.
\end{align*}
\]
At this boundary conditions the solution of the minimization problem is the following: there exists a constant \( x_B \in [0, 1] \) such that the infimum in (3.8) is achieved at
\[
x(t) \equiv x_B \in [0, 1],
\]
and
\[
y(t) = Bt, \quad t \in [0, T].
\]

Definition 3.2. For a constant \( B > 0 \), the path \((x_B(t), y_B(t))\) is called a stationary emission regime if

1. there is a constant \( x_B \) such that
   \[
x_B(t) \equiv x_B,
   \]
   \( t \in [0, T] \).
2. \[
y_B(t) = Bt,
   \]
   \( t \in [0, T] \).
3. the path \((x_B(t), y_B(t))\) are extremal of \( I \) with the boundary conditions \( x_B(0) = x_B(T) = x_B \) and \( y_B(0) = 0, \ y_B(T) = BT \).

Theorem 3.3. For any \( B > 0 \), there exists a constant \( x_B \) such that the path \( x(t) \equiv x_B, y(t) = Bt \) is the stationary emission regime. We have \( x_B \to 0 \) as \( B \to \infty \) with the asymptotics
\[
x_B \sim \frac{\sqrt[3]{2a^2a^2b^4}}{E} \frac{1}{B}.
\]

Proof. We see from the definition of the stationary emission regime that
\[
\begin{align*}
0 &= \lambda x_B^2(1 - x_B) \exp[\kappa_1] - \mu x_B^3 \exp[-\kappa_1 + \kappa_2], \\
B &= \mu x_B^3 \exp[-\kappa_1 + \kappa_2], \\
\dot{\kappa}_1 &= -\lambda(2x_B - 3x_B^2)[\exp[\kappa_1] - 1] + \mu x_B^2 [\exp[-\kappa_1 + \kappa_2] - 1], \\
\dot{\kappa}_2 &= 0.
\end{align*}
\]
From the fourth and second equations of (3.10) it follows that \( \kappa_2 \) and \( \kappa_1 \) do not depend on time. Besides, the following equality
\[
\lambda x_B^2 (1 - x_B) e^{\kappa_1} = \mu \frac{1}{x_B} e^{-\kappa_1} e^{\kappa_2} = B,
\] (3.11)
where \( \kappa_1 \equiv \kappa_1 \) and \( \kappa_2 \equiv \kappa_2 \), follows from the first and second equations of (3.10). We obtain from these equations
\[
x_B \left[ \lambda x_B (1 - x_B) e^{\kappa_1} - \mu \frac{1}{x_B} e^{-\kappa_1} e^{\kappa_2} \right] = 0 \tag{3.12}
\]
Next we prove the equality
\[
\lambda x_B^2 e^{\kappa_1} - 2 \mu \frac{1}{x_B^3} + \lambda (2 x_B - 3 x_B^2) = 0. \tag{3.13}
\]
To this end we use the third equation of (3.10) which we rewrite in the following way
\[
-2 \lambda x_B (1 - x_B) e^{\kappa_1} + \mu \frac{1}{x_B} e^{-\kappa_1} e^{\kappa_2} + \lambda x_B^2 e^{\kappa_1} - 2 \mu \frac{1}{x_B} + \lambda (2 x_B - 3 x_B^2) = 0.
\]
Using now (3.12), we obtain (3.13). Substitute in (3.13) the value of \( e^{\kappa_1} = \frac{B}{\lambda x_B (1 - x_B)} \) from (3.11) to obtain
\[
\frac{B}{1 - x_B} = 2 \mu \frac{1}{x_B^3} + \lambda (2 x_B - 3 x_B^2) = 0. \tag{3.14}
\]
It is the equation to find \( x_B \) via \( B \).
Assuming now that \( B \to \infty \) we obtain from (3.14)
\[
x_B \sim \left( \frac{2 \mu}{B} \right)^{\frac{1}{3}}
\]
since \( \lambda \) is a constant and \( x_B \in [0, 1] \).

\[ \square \]

**Remark 3.4.** The asymptotics of \( x_B \) is determined only by the \( \mu \) which depends only on the emission constant \( \sigma \) and the coefficient in the Hawking’s formula for the temperature.

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