A PRIME DECOMPOSITION THEOREM FOR THE STRING LINK MONOID

RYAN BLAIR, JOHN BURKE, ROBIN KOYTCHEFF

ABSTRACT. In this paper we use 3-manifold techniques to illuminate the structure of the string link monoid. In particular, we give a prime decomposition theorem for string links on two components as well as give necessary conditions for string links to commute under the stacking operation.

1. INTRODUCTION

It is well known that isotopy classes of knots form a monoid via the operation of connected sum. This operation is well defined on both closed knots (embeddings of \(S^1\) into \(S^3\) or \(\mathbb{R}^3\)) and long knots (embeddings of \(\mathbb{R}\) into \(\mathbb{R}^3\) which agree with a fixed linear embedding outside of a compact set). However, if one tries to generalize this operation to links of more than one component, one finds an analogue of connect-sum not for closed links, but for string links. This operation is given by “stacking”, and the resulting monoid structure is the subject of this paper.

The prime decomposition theorem for knots proven by Schubert in the 1949 states that the monoid of isotopy classes of knots is the free commutative monoid on the isotopy classes of prime knots \([6]\). One main result of the current paper is an analogous theorem for 2-component string links. An analogue of Schubert’s theorem is well known for closed links of any number of components, but under the operation of connected summing along one component, which is quite different from the operation of stacking. Our main result is the following:

**Theorem** (Corollary 4.5). A 2-component string link \(L\) can be written as the product (under stacking) of prime factors \(L = L_1 \# ... \# L_n\), where this decomposition is unique up to permuting the order of central elements and multiplication by units.

In the 2-string link monoid, the center turns out to be generated by split string links and one-strand cables, while the units are braids.
Along the way, we consider the \(n\)-string link monoid for any \(n\), establishing the necessary conditions for commutativity and characterizing the units and the center of the monoid of \(n\)-component string links.

This work was motivated by work in preparation of the last two authors on operad actions on spaces of string links \([3]\). In turn, that work built upon papers of Budney \([1, 2]\), which roughly speaking generalize decomposition theorems about the monoid of isotopy classes of long knots (or \(\pi_0\) of the space of long knots) to the level of the whole space of long knots. In seeking generalizations from long knots to string links, it was both natural and necessary to understand the structure of the monoid of isotopy classes of string links. The results in the current paper regarding 2-string links allows the last two authors to prove a decomposition theorem for a large part of the whole space of 2-string links.

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2. Basic Notions

An \(n\)-string link is a properly embedded collection of \(n\) arcs \(T\) in \(M = D^2 \times I\) such that each arc has one endpoint in \(\partial_- M = D^2 \times \{0\}\) and one endpoint in \(\partial_+ M = D^2 \times \{1\}\) and each of \(\partial_- M \cap T\) and \(\partial_+ M \cap T\) are a standard collection of \(n\) points. An \(n\)-string link is pure if each component of \(T\) begins and ends at the same point in \(D^2\).

String links \(T_1\) and \(T_2\) are equivalent if there is an isotopy of \(M\) fixing \(\partial M\) that takes \(T_1\) to \(T_2\). A braid is a string link that is isotopic to an embedding whose restriction to each strand \(I \hookrightarrow D^2 \times I\) is monotone (increasing) in the second coordinate. Denote the stacking\(^1\) operation by \(T_1 \# T_2\) which is achieved by embedding \(T_1\) in \(D^2 \times [0, 1/2]\) and \(T_2\) in \(D^2 \times [1/2, 1]\) in the obvious ways. See Figure 1.

Definition 2.1. Given a compact 1-manifold \(T\) properly embedded in a compact 3-manifold \(M\), an embedded surface \(F\) in \(M\) is \(k\)-punctured if \(F\) meets \(T\) transversely in exactly \(k\) points. See \(F\) in Figure 7.

Definition 2.2. An \(n\)-string link \(L\) embedded in \(M = D^2 \times I\) is prime if \(L = J \# K\) implies \(J\) or \(K\) is a braid and if \(L\) is not a braid itself. Equivalently, \(L\) is prime if every \(n\)-punctured disk properly embedded in \(M\) with boundary isotopic to \(\partial(\partial_+ M)\) is boundary parallel, but \(\partial_+ M\) is not isotopic to \(\partial_- M\).

\(^{1}\)One might also call this operation connected sum, except that it is in general not commutative for string links with \(> 1\) component.
Definition 2.3. A decomposing disk for an \( n \)-string link \( T \) in \( M \) is an \( n \)-punctured disk which is properly embedded in \( M \), with its boundary isotopic to \( \partial(\partial_+ M) \) and which is isotopic to neither \( \partial_+ M \) nor \( \partial_- M \). See the disk \( F \) in Figure 1.

Thus, a string link is prime if and only if it has no decomposing disks. Note also that each strand of a string link passes through a decomposing disk exactly once.

**Notation:** A decomposing disk \( F \) for a string link \( T \) in \( M \) separates \( T \) into two string links \( K \) and \( L \). It also separates \( M \) into two pieces, each naturally decomposed as \( D^2 \times I \). We will let \( M_K^F \) and \( M_L^F \) denote the pieces containing \( K \) and \( L \) respectively.

Definition 2.4. A loop \( \gamma \) embedded in a punctured surface \( F \) is essential if it does not bound a 0-punctured disk or 1-punctured disk in \( F \).

Definition 2.5. Given a string link \( T \) in \( M \) and an embedded, possibly punctured surface \( F \) in \( M \), \( F \) is compressible if some essential loop in \( F \) bounds an embedded disk in \( M \) with interior disjoint from \( F \) and \( T \).
Such a disk is called a compressing disk. Otherwise, \( F \) is incompressible. A surface \( F \) is essential if \( F \) is incompressible and non-boundary parallel.

(Thus when we say [compressible, incompressible, essential], we mean what some authors might call “[compressible, incompressible, essential] in \( M \setminus T \).”)

**Lemma 2.6.** Any decomposing disk \( F \) for a string link \( T = K \# L \) in \( M \) is incompressible.

**Proof.** Suppose \( F \) is a decomposing disk. Then \( F \) separates \( M \) into \( M_+^F \) and \( M_-^F \). Suppose \( F \) is compressible. Let \( D \) be a compressing disk for \( F \). We can assume \( D \) is completely contained in \( M_+^F \) or \( M_-^F \). Without loss of generality, assume \( D \) is contained in \( M_+^F \). (If not, one can consider circles in \( D \cap F \) and, by a standard argument, replace \( D \) by an innermost disk bounded by these circles.) Since \( D \) is a compressing disk, then, by definition, \( \partial D \) is essential in \( F \). Since \( F \) is a punctured disk, this implies \( \partial D \) bounds a punctured disk \( D_1 \subset F \). Since the 3-ball is irreducible, \( D_1 \cup D \) bounds a 3-ball in \( M \). Hence, any arc of \( T \) in the 3-ball bounded by \( D_1 \cup D \) is an arc in \( T_1 \) which must have both endpoints in \( D_1 \subset \partial_+ M_+^K (= \partial_- M_-^L) \). This contradicts the fact that each strand of a string link passes through a decomposing disk only once. \( \square \)

**Proposition 2.7.** If \( K \) and \( L \) are \( n \)-string links such that \( K \# L \) is a braid, then both \( K \) and \( L \) are braids.

**Proof.** Let \( F \) be the \( n \)-punctured disk corresponding to \( \partial_+ M_+^F \) and \( \partial_- M_-^F \) in \( M \). By Lemma 2.6, \( F \) is incompressible. Since \( K \# L \) is a braid, \( M \) can be decomposed as an \( I \)-bundle over \( D^2 \) with \( K \# L \) the union of \( I \)-fibers. Since \( F \) is incompressible and separates \( \partial_- M \) and \( \partial_+ M \), then by standard results for \( I \)-bundles, \( F \) is transversely isotopic to both \( \partial_+ M \) and \( \partial_- M \). The isotopy taking \( F \) to \( \partial_+ M \) can be taken to be monotone with respect to the projection of \( M = D^2 \times I \) onto its \( I \) factor. Hence, \( M_+^F \) can be decomposed as an \( I \)-bundle over \( D^2 \) with \( K \) the union of \( I \)-fibers. Similarly, \( M_-^F \) can be decomposed as an \( I \)-bundle over \( D^2 \) with \( L \) the union of \( I \)-fibers. Thus, both \( K \) and \( L \) are braids. \( \square \)

**3. Necessary Conditions for Commutativity in the String Link Monoid**

Given a string link \( L \) we only consider isotopies of surfaces embedded in \( M \) that are everywhere transverse to \( L \).
Definition 3.1. An $n$-string link $L$ embedded in $M = D^2 \times I$ is split if there exists a compressing disk for $\partial M$ that meets $\partial(\partial_{\pm} M)$ minimally in exactly two points. Equivalently, $L$ can be decomposed as a $k$-strand string link and a $(n - k)$-strand string link placed side-by-side in $M$. Otherwise, $L$ is non-split.

Figure 2. Split Link

Definition 3.2. A splitting disk for an $n$-string link $T$ in $M = D^2 \times I$ is a properly embedded $k$-punctured disk $F$ in $M$ such that $\partial F$ is contained in $\partial_{\pm} M$ (resp. $\partial_{\pm} M$), each strand of $T$ meets the splitting disk at most once and the splitting disk is not transversely isotopic to a subdisk of $\partial_{\pm} M$ (resp. $\partial_{\pm} M$).

Figure 3. Splitting Disk

Note that for a prime $n$-string link with a splitting disk $F$, the union of the punctured annulus in $\partial_{\pm} M$ cobounded by $\partial \partial_{\pm} M$ and $\partial F$ together with the punctured disk $F$ is isotopic to $\partial_{\pm} M$. Note also that a split string link may have non-isotopic splitting disks, but that a non-split string link has a unique splitting disk up to isotopy.

Definition 3.3. If $T$ embedded in $M$ is a string link, then a properly embedded annulus $A$ in $M$ is a cabling annulus if one component of $\partial A$ is contained in $\partial_{-} M$, the other component is contained in $\partial_{+} M$ and,
if $N$ is the copy of $D^2 \times I$ that $A$ cobounds with two disks in $\partial M$, then $T \cap N \neq \emptyset$ and $T \cap N$ is a braid in $N$.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{cabling.png}
\caption{Cabling}
\end{figure}

Note that the boundary of a regular neighborhood of a single strand of a string link is a cabling annulus.

When stacking it is natural to weaken the notion of string-link equivalence to allow abient isotopies that move $D^2 \times \{0\}$ and $D^2 \times \{1\}$. To this end we introduce the following definition.

**Definition 3.4.** Two $n$-string links $T_1$ and $T_2$ are braid-equivalent if there exist braids $B_1$ and $B_2$ such that $T_1 = B_1 \# T_2 \# B_2$.

**Theorem 3.5.** Suppose $K$ and $L$ are prime $n$-string links. If $T = K \# L = L \# K$, then $K$ is braid equivalent to $L$ or, up to relabeling $K$ and $L$, $L$ is a string link with cabling annulus $A$ and $K$ has a splitting disk $F$ such that $\partial F$ is isotopic to a component of $\partial A$.

**Proof.** Suppose $K$ and $L$ are distinct prime $n$-string links such that $K \# L = L \# K$. Let $M$ be the copy of $D^2 \times I$ containing $K \# L$. Let $D$ be the decomposing disk for $K \# L$ and let $E$ be the decomposing disk for $L \# K$. By Lemma 2.6 and the definition of decomposing disk, both $D$ and $E$ are incompressible and non-boundary-parallel. Since $\partial D$ is isotopic to $\partial E$, we can isotope $E$ so that $\partial D \cap \partial E = \emptyset$ while simultaneously demanding that $D = D^2 \times \{\frac{1}{2}\} \subset M$. Subject to these constraints, isotope $E$ to minimize $|D \cap E|$.

If $|D \cap E| = 0$, then, since both $L$ and $K$ are prime, $D$ is isotopic to $E$. Hence, $K$ is braid-equivalent to $L$.

Suppose $|D \cap E| > 0$. Let $\{\alpha_i\}$ be the collection of all innermost curves of $D \cap E$ in $E$ and let $D_i, E_i$ be the possibly punctured disks that $\alpha_i$ bounds in $D, E$ respectively. Since $K$ and $L$ are string links, then $|D_i \cap T| = |E_i \cap T|$. So if one of $\{D_i, E_i\}$ has zero punctures, then
so does the other. In that case, \( D_i \cup E_i \) bounds a 3-ball, and \( D_i \) can be pushed across \( E_i \) to eliminate \( \alpha_i \), contradicting the minimality of \( |D \cap E| \). Thus every (innermost) component of \( D \cap E \) encloses at least one puncture.

Let \( D \) separate \( M \) into \( M^K_D \) and \( M^L_D \). Since \( \alpha_i \) is innermost, then \( D \) is disjoint from the interior of \( E_i \) for each \( i \) and \( E_i \) is properly embedded in one of \( M^K_D \) or \( M^L_D \) for each \( i \).

**Case A:** Assume \( |\{\alpha_i\}| \geq 2 \). Let \( E_j \) and \( E_k \) be distinct innermost disks in \( E \). Let \( D_j \) and \( D_k \) be the punctured disks in \( D \) that \( \alpha_j \) and \( \alpha_k \) bound respectively. Since \( |D \cap E| \) has been minimized, \( D_j \) is not isotopic to \( E_j \) and \( D_k \) is not isotopic to \( E_k \). Note that since \( E_j, E_k, D_j \) and \( D_k \) are all subdisks of \( E \) and \( D \) and both \( D \) and \( E \) meet every strand of \( T \) exactly once, then each of \( E_j, E_k, D_j \) and \( D_k \) must meet each strand of \( T \) exactly once.

**Note:** Diagrammatic representations of the following subcases are displayed in Figure 5. A rectangle is used to represent \( D^2 \times I \) while \( D \) is represented by a red line and the subdisks \( E_j \) and \( E_k \) of \( E \) are represented with green lines.

**Case A1:** Suppose both \( E_j \) and \( E_k \) are properly embedded in \( M^D_L \).

**Sub-case A1a:** Suppose \( D_j \) and \( D_k \) do not intersect in \( D \) then the \( n \)-punctured disk \( H \) that is the union of \( E_j \) and the annulus bounded by \( \alpha_j \) and \( \partial D \) in \( D \) is not isotopic to \( D \) since \( E_j \) is not isotopic to \( D_j \), and \( H \) is not isotopic to \( \partial D \) since \( E_k \) is not isotopic to \( D_k \). This is a contradiction to \( L \) being prime.

**Figure 5.**
**Sub-case A1b:** Suppose $D_j$ and $D_k$ do intersect in $D$. Up to relabeling, we can assume that $D_j \subset D_k$. Notice that each of $E_j$ and $D_j$ meet each strand of $T$ at most once. Thus every arc of $T$ contained in the 3-ball whose boundary is $D_j \cup E_j$ has one endpoint in $D_j$ and one endpoint in $E_j$. By Alexander’s Theorem, the punctured disks $E_j \cup E_k$ separate $M_L^D$ into three 3-ball components. Examine an arc $\beta \subset L$ in the component which is incident to both $E_j$ and $E_k$ such that $\beta$ has an end point on $E_j$. There are three possibilities for the other endpoint of $\beta$: it is contained in $E_j$, $E_k$, or $D$. If this other endpoint of $\beta$ is contained in either $E_j$ or $E_k$, then we immediately contradict that $\beta$ intersects $E$ only once.

If this other endpoint of $\beta$ is contained in $D$, then the strand of $L$ that contains $\beta$ has both endpoints on $D$, a contradiction to $L$ being a string link. Hence, in each case we derive a contradiction.

**Case A2:** Suppose both $E_j$ and $E_k$ are properly embedded in $M_K^D$.

The proof in this case is similar to the proof in Case A1.

**Case A3:** Suppose $E_j$ is properly embedded in $M_K^P$ and $E_k$ is properly embedded in $M_L^P$.

**Sub-case A3a:** Suppose $D_j$ and $D_k$ intersect in $D$, then we can assume $D_j \subset D_k$. Let $p$ be a point of intersection between $T$ and $D_j$. (We know such a point exists since no curve of $D \cap E$ bounds an unpunctured disk in $D$.) The point $p$ is the endpoint of an arc $\alpha$ in $K$ and an arc $\beta$ in $L$. Since the other endpoint of $\alpha$ is contained in $\partial_- M_K^P$, then $E_j$ separates the endpoints of $\alpha$ and $\alpha \cap E_j \neq \emptyset$. Since the other endpoint of $\beta$ is contained in $\partial_+ M_L^P$, then $E_k$ separates the endpoints of $\beta$ and $\beta \cap E_k \neq \emptyset$. Hence the arc $\alpha \cup \beta$ in $T$ meets $E$ twice, a contradiction.

**Sub-case A3b:** Suppose $D_j$ and $D_k$ do not intersect in $D$, then examine the $n$-punctured disk $D'$ that is the union of $E_j$ and the annulus in $D$ with boundary $\partial E_j \cup \partial D$. Since $K$ is prime and $E_j$ is not isotopic to $D_j$, then $D'$ must be isotopic to $\partial_- M_K^P$ via an isotopy $\iota$. Moreover, $\iota$ can be taken to be an embedding. This identifies the part of $M$ between $D'$ and $\partial_+ M_L^P$ with the product $D^2 \times I$ such that $K$ meets this product in $I$-fibers. Now for any curve $\gamma$ in $D$ which is disjoint from the interior of $D_j$ and which encloses a puncture in $D$, the image of the restriction of $\iota$ to $\gamma$ is a cabling annulus in $M_K^P$ with $\gamma$ a boundary component. Hence, $E_k$ is a splitting disk for $L$ such that $\alpha_k$ bounds a cabling annulus in $M_K^P$, which is what we wanted to show.

**Case A4:** Suppose $E_j$ is properly embedded in $M_L^P$ and $E_k$ is properly embedded in $M_K^P$. This case follows from the proof of Case A3.
Hence, it remains to consider the case that there is a unique innermost curve $\alpha_1$ of $D \cap E$ in $E$ bounding a punctured disk $E_1$. Let $D_1$ be the punctured disk $\alpha_1$ bounds in $D$. Since there is a unique innermost curve $\alpha_1$ of $D \cap E$ in $E$, every other curve in $D \cap E$ encloses $\alpha_1$, and hence there is also a unique curve $\beta$ which is outermost in $E$. Let $A_E$ be the possibly punctured annulus in $E$ with boundary $\beta \cup \partial E$. Let $A_D$ be the possibly punctured annulus in $D$ with boundary $\beta \cup \partial D$. $E_1$ is not isotopic to $D_1$ and $A_E$ is not isotopic to $A_D$, as otherwise we could decrease $|D \cap E|$. As noted previously, $|E_1 \cap T| = |D_1 \cap T|$ and, similarly, $|A_E \cap T| = |A_D \cap T|$. Without loss of generality, assume $A_E \subset M^D_K$. There are several cases to consider.

**Note:** The following subcases are diagrammatically represented in Figure 6. Again $D$ is represented by a red line while $D_1$ and $A_E$ are represented with green lines.

**Case B1:** $E_1$ is contained in $M^D_K$, and $\alpha_1$ can be isotoped to be disjoint from $A_D$. The $n$-punctured disk $H$ in $M^D_K$ that is the union of $E_1$ and the annulus bounded by $\partial E_1$ and $\partial D$ in $D$ is not isotopic to $D$ since $E_1$ is not isotopic to $D_1$ and $H$ is not isotopic to $\partial M^D_K$ since $A_D$ is not isotopic to $A_E$. This is a contradiction to $K$ being prime.

**Case B2:** $E_1$ is contained in $M^D_K$, and $\alpha_1$ cannot be isotoped to be disjoint from $A_D$. The annulus $A_E$ separates $M^D_K$ into two pieces. (This can be seen by doubling the 3-ball $M^D_K$ and the annulus along their boundaries and applying the Generalized Jordan Curve Theorem [5].)
Since $E$ is embedded, $E_1$ is contained in one of these pieces, necessarily the one between $A_E$ and $A_D$. This implies that $D_1 \subset A_D$. Then any arc of $K$ that intersects $E_1$ is forced to also intersect $A_E$, a contradiction to the fact that every arc of $T$ meets $E$ exactly once.

**Case B3:** $E_1$ is contained in $M_D$, and $\alpha_1$ cannot be isotoped to be disjoint from $A_D$. Here $A_D$ and $D_1$ are forced to meet $T$ in a common puncture $p$. The arc of $K$ with endpoint $p$ must meet $A_E$ since $A_E$ separates the endpoints of this arc. The arc of $L$ with endpoint $p$ must meet $E_1$ since $E_1$ separates the endpoints of this arc. This is a contradiction to the fact that $E$ meets every arc of $T$ exactly once.

**Case B4:** $E_1$ is contained in $M_D$, and $\alpha_1$ can be isotoped to be disjoint from $A_D$. Note that $E_1$ is a splitting disk for $L$. Since $A_E$ is not isotopic to $A_D$ and $K$ is prime, the $n$-punctured disk $A_E \cup (D - A_D)$ is isotopic to $\partial_- M_K$. Thus, $\alpha_1$ is the boundary of a cabling annulus for $K$, which is what we wanted to show.

The following essentially provides the converse to the above theorem.

**Proposition 3.6.** Suppose $T = K \# L$, where $K$ and $L$ are prime, $n$-string links. Let $D$ be the decomposing disk that separates $T$ into $K$ and $L$. If $L$ is a string link with cabling annulus $A$ and $K$ has a splitting disk $F$ such that, after forming $K \# L$, $\partial F$ is isotopic to a component of $\partial A$ in $\partial_+ M_K^D$, then $K \# L$ is braid equivalent to $L \# K$.

**Proof.** The portion of $L \# K$ in the 3-ball bounded by $F$ and a subdisk of $\partial_+ M_K^D$ in $M_K^D$ can be isotoped through the complementary component of $A$ in $M_L^D$ homeomorphic to $D^2 \times I$ so that it lies inside an $\epsilon$ neighborhood of $\partial_+ M_K^D$. Since $K$ is prime, then, after this isotopy, the portion of $K \# L$ in $M_K^D$ is a braid and the portion of $K \# L$ in an $\epsilon$ neighborhood of $\partial_- M_L^D$ is braid equivalent to $K$. After the isotopy, the portion of $K \# L$ in $M_L^D$ that meets the $D^2 \times I$ complementary component of $A$ in $M_L^D$ remains a braid while the portion of $K \# L$ in $M_L^D$, but outside of this $D^2 \times I$ is fixed by this isotopy. Hence, $K \# L$ is braid-equivalent to $L \# K$. □

**Corollary 3.7.** Let $K$, $L$, $P$, $Q$ be prime, $n$-string links such that $K \# L = P \# Q$, then $K$ is braid-equivalent to $P$ and $L$ is braid equivalent to $Q$ or $K$ is braid-equivalent to $Q$ and $L$ is braid equivalent to $P$.

**Proof.** This follows an argument very similar to the proof of Theorem 3.5 and the proof of Proposition 3.6. Let $D$ be the decomposing disk for $K \# L$ and $E$ the decomposing disk for $P \# Q$. If $D$ can be isotoped
to be disjoint from $E$, then $K$ is braid-equivalent to $P$ and $L$ is braid equivalent to $Q$. If $D$ can not be isotoped to be disjoint from $E$, then we arrive at a contradiction exactly as in the proof of Theorem 3.5, except for case B4 and sub-case A3b, which require only slight modification, as follows.

Recall the terminology from the proof of Theorem 3.5. In case B4, $E_1$ is contained in $M_D^L$, and $\alpha_1$ can be isotoped to be disjoint from $A_D$. We can then conclude that $\alpha_1$ is the boundary of a cabling annulus for $K$. After applying the proof of Proposition 3.6, we see that this implies that $K$ is braid-equivalent to $Q$ and $L$ is braid equivalent to $P$.

In the second part of case A3, $E_j$ is properly embedded in $M^D_K$, $E_k$ is properly embedded in $M^L_D$, and $D_j$ and $D_k$ do not intersect in $D$. We can conclude that $E_k$ is a splitting disk for $L$ such that $\partial E_k$ bounds a cabling annulus in $M^D_K$. After applying the proof of Proposition 3.6 with the roles of $K$ and $L$ switched, we see that this implies that $K$ is braid-equivalent to $Q$ and $L$ is braid equivalent to $P$.

□

Definition 3.8. An $n$-string link with a non-boundary parallel cabling annulus such that one of the complementary components of the annulus contains all strands of the string link is known as a one-strand cable.

Corollary 3.9. If $K \subset M \cong D^2 \times I$ is a prime element in the center of the monoid of $n$-string links, then $K$ is a one-strand cable or the exterior (i.e., complement of an open neighborhood) of $K$ in $M$ contains an essential 2-punctured sphere.

Proof. Sketch: It is easy to show that there exists an $n$-string link $K$ for any $n > 1$ such that the only cabling annulus for $K$ is the boundary of a regular neighborhood of a single strand of $K$ and $K$ admits no splitting disks. An example of such a string link when $n = 2$ is given in the upper right hand side of Figure 1. Hence, by the Proposition 3.5 the only $n$-string links that can commute with $K$ are one-strand cables or string links with a 1-punctured splitting disk with boundary bounding a 1-punctured subdisk of $\partial M$. For this second class of string links the union of the 1-punctured splitting disk and the 1-punctured subdisk of $\partial M$ is an essential 2-punctured sphere. □

Corollary 3.10. The prime elements of the center of the monoid of pure 2-string links consist of

(1) split links
(2) one-strand cables
(3) braids.
To prove this corollary, we first show that the monoid of all 2-string links splits as braids and the remaining quotient by braids. Let $\mathcal{L}_2$ denote the monoid of (isotopy classes of) 2-string links.

**Proposition 3.11.** There is a splitting $\mathcal{L}_2 \cong \mathcal{L}_2^0 \oplus \mathbb{Z}$, where the $\mathbb{Z}$ factor corresponds to the braids in $\mathcal{L}_2$.

**Proof.** On the submonoid $\mathcal{L}_2^{\text{pure}} \subset \mathcal{L}_2$ of pure string links, we have a map $2\ell : \mathcal{L}_2^{\text{pure}} \to \mathbb{Z}$, given by twice the linking number. We can extend this to all string links as follows: Let $\tau$ be the string link obtained by composing a standard unlink $I \sqcup I \hookrightarrow D^2 \times I$ with the map $D^2 \times I \to D^2 \times I$ given by $(z, t) \mapsto (e^{\pi i t}z, t)$. (Thus $\tau^2$ has linking number $+1$.)

For a non-pure 2-string link $L$, define $2\ell$ by twice the linking number of $L \# \tau$ minus one. This gives a monoid homomorphism $2\ell : \mathcal{L}_2 \to \mathbb{Z}$. On the other hand, we also have a monoid homomorphism $b : \mathbb{Z} \to \mathcal{L}_2$ given by $b(n) = \tau^n$, whose image is clearly the braids in $\mathcal{L}_2$. The composite $2\ell \circ b : \mathbb{Z} \to \mathbb{Z}$ is the identity, so $b$ gives a right-splitting of the short exact sequence

$$0 \to \ker(2\ell) \to \mathcal{L}_2 \to \mathbb{Z} \to 0.$$ 

Thus $\mathcal{L}_2 \cong \ker(2\ell) \oplus \mathbb{Z}$, where the $\mathbb{Z}$ factor is the braids in $\mathcal{L}_2$. So if we let $\mathcal{L}_2^0$ denote the quotient of $\mathcal{L}_2$ by the braids, then $\mathcal{L}_2 \cong \mathcal{L}_2^0 \oplus \mathbb{Z}$. (We chose this notation because $\mathcal{L}_2^0$ is also isomorphic to $\ker(2\ell)$, which is the 2-string links with linking number zero.)

**Figure 7.** Commuting Split links

**Proof of Corollary 3.10.** From two previous statements, we understand the center of $\mathcal{L}_2^0$. In fact, by Corollary 3.9, the center is contained in the set of one-strand cables and split string links. Conversely, Proposition
applied to the case of 2-string links implies that one-strand cables and split string links are central, as illustrated in Figure 7 and Figure 8. Thus, the center of $\mathcal{L}_0$ is precisely one-strand cables and split links. Since $\mathcal{L}_2 \cong \mathcal{L}_0^2 \times \mathbb{Z}$ where $\mathbb{Z}$ is the braids, the center of $\mathcal{L}_2$ is precisely 1-strand cables, split links, and braids.

4. PRIME DECOMPOSITION FOR 2-STRING LINKS

We now turn our attention to pure 2-string links, in which case we can prove the following prime decomposition theorem.

**Theorem 4.1.** A pure 2-string link $L$ has a prime decomposition $L = L_1 \# \ldots \# L_n$, where each $L_i$ is a prime pure string link. Such a decomposition is unique up to reordering the factors in the center and up to multiplication by units (pure braids).

**Proof.** The following theorem of Freedman and Freedman implies the existence statement [4]. In fact, we get the existence statement for $n$-string links for any $n$, i.e., any $n$-string link has a finite decomposition into prime factors. We just apply the theorem below with $M$ the exterior (i.e., complement of an open neighborhood) of the $n$-string link, $F_1, \ldots, F_k$ is any collection of decomposing disks, and $b = n$.

**Theorem 4.2.** [4] Let $M$ be a compact 3-manifold with boundary and $b$ an integer greater than zero. There is a constant $c(M, b)$ so that if $F_1, \ldots, F_k$, $k > c$, is a collection of incompressible surfaces such that all the Betti numbers $b_i F_i < b$, $1 \leq i \leq k$, and no $F_i$, $1 \leq i \leq k$, is a boundary parallel annulus or a boundary parallel disk, then at least two members $F_i$ and $F_j$ are parallel.
Before proving the uniqueness statement, we prove a lemma.

**Lemma 4.3** (Splitting Disk Lemma). Suppose an $n$-string link $L$ in $M$ has a nontrivial splitting disk with punctures corresponding to strands $1, \ldots, k$. Then any prime decomposition of $L$ must have a prime factor which is a split string link such that the sublink of this prime factor in one of the complementary components of the splitting disk is an $n$-string braid and the splitting disk only meets strands with labels $1, \ldots, k$.

**Proof.** Suppose $L$ has a nontrivial splitting disk $D$ with punctures corresponding to the first $k$ strands and whose boundary (without loss of generality) is contained in $\partial M$. By possibly replacing $D$ by a “smaller” disk (lying between $D$ and $\partial M$), we may assume that the $k$-strand link $L'$ enclosed by $D$ is prime. Suppose we have a prime decomposition of $L$, and let $E_1, \ldots, E_n$ be the decomposing disks for this decomposition. Isotop the $E_i$ so as to minimize intersection with $D$. Let $\alpha$ be a circle of intersection between the union of the $E_i$ and $D$ which is innermost in $D$. The loop $\alpha$ bounds a punctured disk $D_\alpha$ in $D$ and a punctured disk $E_\alpha$ in some $E_j$ such that $D_\alpha$ is disjoint from the union of the $E_i$ in its interior. As the intersection between the $E_i$ and $D$ is was taken to be minimal, $D_\alpha$ is not isotopic to $E_\alpha$. Since $D$ is incident to only strands labeled $1, \ldots, k$, so is $D_\alpha$. Let $M_j$ be the complementary component of the union of the $E_i$ in $M$ that contains $D_\alpha$ and let $K_j$ be the portion of $L$ in $M_j$. $D_\alpha$ is a splitting disk for $K_j$ in $M_j$. Since $K_j$ is prime, the restriction of $K_j$ to the one of the complementary components of $D_\alpha$ in $M_j$ is an $n$-string braid. \hfill \Box

Returning to the proof of the uniqueness statement of the Theorem, suppose we have a prime decomposition of a 2-string link $L = L_1 \# \ldots \# L_n$. Let $T_1$ be the product of all the $L_i$ which are split links with the first component unknotted. Similarly, let $T_2$ be the product of all the $L_i$ which are split links with the second component unknotted. Let $T_3$ be the product of all the $L_i$ which are one-strand cables. Let $T_4$ be the product of the remaining $L_i$ in the given prime decomposition of $L$. Since all the $L_i$ in $T_1, T_2, T_3$ are central by Corollary 3.10, we can write $L = T_1 \# T_2 \# T_3 \# T_4$.

Given another prime decomposition $L = L'_1 \# \ldots \# L'_m$, we similarly can write $L = T'_1 \# \ldots \# T'_4$, where each $T'_i$ is a product of $L'_i$’s of the same type as those in $T_i$. The first step is to show that $T_i = T'_i$ modulo braid equivalence for each $i$. The prime decomposition theorem for knots will then imply equality of the factors in $T_i$ and $T'_i$ modulo braid equivalence for $i = 1, 2, 3$. It will then remain to prove that the factors
in $T_4$ and $T_4'$ agree up to braid equivalence. Since 2-string pure braids are the units, the theorem will follow.

**Step 1:** We will first show $T_1$ and $T_1'$ agree up to stacking with a pure braid. Let $D_1$ be the 2-punctured decomposing disk properly embedded in $M$ and separating $T_1$ from $T_2\#\ldots\#T_4$ in $L$. Similarly, let $E_1$ be such a 2-punctured decomposing disk separating $T_1'$ from $T_2'\#\ldots\#T_4'$ in $L$. Since $T_1$ is split, $D_1$ is isotopic to the union of a once-punctured annulus in $\partial_- M$ with a once-punctured, properly embedded disk $D$ whose boundary is contained in $\partial_- M$. Similarly, $E_1$, is isotopic to the union of a once-punctured annulus in $\partial_- M$ with a 1-punctured disk $E$ whose boundary is contained in $\partial_- M$.

By an isotopy, we can take $\partial D$ and $\partial E$ to be disjoint in $\partial_- M$. Perform a further proper isotopy of $D$ and $E$ so as to minimize the number of components of $D \cap E$. Suppose that this number is nonzero. Let $\alpha$ be a component of $D \cap E$ which is innermost in $E$. Let $D_\alpha$ be the possibly punctured disk in $D$ bounded by $\alpha$, and let $E_\alpha$ be the possibly punctured disk in $E$ bounded by $\alpha$. Now $D$ and $E$ each have exactly one puncture. Thus, $D_\alpha$ and $E_\alpha$ must both have either zero punctures or one puncture. In the case of zero, $D_\alpha \cup E_\alpha$ bounds a 3-ball, and we can get rid of $\alpha$ by pushing $D_\alpha$ across $E_\alpha$, thus contradicting the minimality of $|D \cap E|$. Hence $D_\alpha$ and $E_\alpha$ each have one puncture.

Since $D$ and $E$ themselves have just one puncture, the argument just made shows that $\alpha$ is the unique curve in $D \cap E$ which is innermost in $E$. Let $d_\beta$ be the punctured disk in $D$ bounded by $\beta$, and let $e_\beta$ be the punctured disk in $E$ bounded by $\beta$. Now $D$ and $E$ each have exactly one puncture. Thus, $D_\beta$ and $E_\beta$ must both have either zero punctures or one puncture. In the case of zero, $D_\beta \cup E_\beta$ bounds a 3-ball, and we can get rid of $\beta$ by pushing $D_\beta$ across $E_\beta$, thus contradicting the minimality of $|D \cap E|$. Hence $D_\beta$ and $E_\beta$ each have one puncture.

Since $D$ and $E$ themselves have just one puncture, the argument just made shows that $\alpha$ is the unique curve in $D \cap E$ which is innermost in $E$. Reversing the roles of $D$ and $E$ shows that among the curves in $D \cap E$, there is a unique curve $\beta$ that is innermost in $D$. Let $D_\beta$ denote the punctured disk that $\beta$ bounds in $D$. We can so far see that the circles in $D \cap E$ are all concentric in both $D$ and $E$.

In other words, the components of $D \cap E$ separate each of $D$ and $E$ into $|D \cap E| + 1$ regions which look like regions of a dartboard as shown in Figure 9. We now label each region of $D$ by a "1" and "2" according as the region is in $M_{T_1}^{E_1}$ or $M_{T_2\#T_3\#T_4}^{E_1}$. Similarly, we label each region of $E$ by a "1" or "2" according as the region is in $M_{T_1}^{D_1}$ or $M_{T_2\#T_3\#T_4}^{D_1}$. The labels of the outermost regions of $D$ and $E$ must be different. In each of $D$ and $E$, the labels of regions must alternate. Since $D$ and $E$ have the same number of regions, the labels of $D_\beta$ and $E_\alpha$ must be different. Hence one of these disks must have the label "2". Without loss of generality, suppose it is $E_\alpha$ that has the label "2". Then $E_\alpha$ is a once-punctured splitting disk in $M_{T_2\#T_3\#T_4}^{D_1}$. (If it is not a splitting disk, we contradict the minimality of $|D \cap E|$.) By the Splitting Disk Lemma, one of the prime factors of $T_2\#T_3\#T_4$ must be a split link with
the first strand knotted. This contradicts the definition of the $T_i$. We conclude that $D$ and $E$ can be isotoped to be disjoint.

**Figure 9.**

If the disjoint punctured disks $D$ and $E$ are not isotopic to each other, then there must be a knot between them, which again contradicts either the definition of the $T_i$ or the definition of the $T'_i$.

Thus, $D$ and $E$, and hence $D_1$ and $E_1$, are isotopic. By definition of $D_1$ and $E_1$, this shows that both $T_1$ and $T'_1$ agree up to stacking with a pure braid and that $T_2#T_3#T_4$ and $T'_2#T'_3#T'_4$ agree up to stacking with a pure braid. This completes Step 1.

**Step 2:** We now want to show that $T_2 = T'_2$ modulo braid equivalence. We know that $T_2#T_3#T_4$ and $T'_2#T'_3#T'_4$ agree up to a pure braid. We apply to this string link $(T_2#T_3#T_4)$ the same argument as in Step 1, but with the roles of the first and second strands reversed. This shows that $T_2 = T'_2$ modulo braid equivalence and also that $T_3#T_4 = T'_3#T'_4$ modulo braid equivalence. This completes Step 2.

**Step 3:** We now know that $T_3#T_4 = T'_3#T'_4$ modulo braid equivalence so it suffices to consider the case where $L = T_3#T_4 = T'_3#T'_4$ and where $T_3#T_4$ has a prime decomposition where none of the factors are split links.

Let $D$ be the 2-punctured disk decomposing $L$ into $T_3$ and $T_4$, and let $E$ be the 2-punctured decomposing $L$ into $T'_3$ and $T'_4$. We may assume $\partial D \cap \partial E = \emptyset$. Subject to this constraint, isotope $D$ and $E$ so as to minimize the number of circles in $D \cap E$. Suppose that this number is nonzero. Let $\alpha$ be one of these circles which is innermost in $E$. Let $E_\alpha$ be the punctured disk in $E$ bounded by $\alpha$, and let $D_\alpha$ be the punctured disk in $D$ bounded by $\alpha$. Since each strand of $L$ intersects each of $D$ and $E$ once, $D_\alpha$ and $E_\alpha$ have the same number of punctures. If the number of punctures is zero, then $D_\alpha$ can be pushed across $E_\alpha$ to eliminate $\alpha$, contradicting minimality of $|D \cap E|$. If the number of punctures is one, then $T_3$ or $T'_3$ has a nontrivial splitting disk, which by the same
argument given at the end of Step 1, contradicts the assumption that 
$L$ has no split factors. So the number of punctures in $D_\alpha$ (and hence 
also in $E_\alpha$) is exactly 2. This implies that $\alpha$ is the only circle in $D \cap E$ 
which is innermost in $E$. By a similar argument there is a circle $\beta$ in $D \cap E$ which is the unique innermost circle in $D$. If $D_\beta$ and $E_\beta$ are the 
punctured disks bounded by $\beta$ in $D$ and $E$, this similar argument also 
shows that each of these disks has exactly 2 punctures.

So the circles in $D \cap E$ are concentric in both $D$ and $E$, that is, 
$D \cap E$ separates each of $D$ and $E$ into an inner punctured disk and $|D \cap E|$ concentric annuli, and the 2 punctures in each of $D$ and $E$ are 
contained in the innermost punctured disks $D_\beta$ and $E_\alpha$.

Let $A$ be the outermost annulus in $E$. By the above conclusion, $A$ 
has no punctures. Without loss of generality, $A$ is properly embedded 
in $M_{T_4}^L$. (Otherwise, reverse the roles of $D,T_1,...,T_4$ and $E,T_1',...,T_4'$ so 
that $A$ is an embedded subsurface of $D$ in $M_{T_4}^E$.) If $A$ is boundary par-
allel in $M_{T_4}^D$, then there is an isotopy of $E$ eliminating $\partial A$ from $D \cap E$, 
contradicting the minimality of $|D \cap E|$. Furthermore, $A$ is incompressible since it $E(\supset A)$ is incompressible. Hence $A$ is an essential annulus 
in $M_{T_4}^D$.

Let $F$ be the (disjoint) union of the decomposing punctured disks 
$F_1,...,F_k$ for the prime decomposition of $T_4$.

Claim 1: After an isotopy, we can assume $A \cap F$ is a collection of circles 
none of which bounds an unpunctured disk in either surface. 
Proof: suppose such a circle $\gamma$ bounds an unpunctured disk in 
one of $\{E,F_i\}$. If the disk that $\gamma$ bounds in the other of $\{E,F_i\}$ 
has at least one puncture, then it must have an even number of 
punctures, contradicting that each strand of $L$ intersects each 
of $E$ and $F_i$ only once. So $\gamma$ bounds unpunctured disks in both $E$ and $F_i$. But then we can eliminate $\gamma$ by isotopy.

Claim 2: No circle $\gamma$ of $A \cap F$ bounds a 1-punctured disk in a compo-
nent of $F$. Proof: If some $\gamma$ does bound a 1-punctured disk, 
then, by taking $\gamma$ to be innermost, the union of the this disk 
together with the disk which $\gamma$ bounds in $E$ is a punctured immers-
ioned sphere (with transverse intersections), which meets $L$ 
either once (if $\gamma$ is inessential in $A$) or 3 times (if $\gamma$ is essential 
in $A$); this contradicts the fact that a properly embedded 1-
manifold in a 3-ball must intersect an immersed closed surface 
(with transverse self-intersections) in an even number of points. 
(The latter fact can be shown by resolving self-intersections and 
the Generalized Jordan Curve Theorem [5].)
Now $\mathcal{F}$ separates $M_{D_4}^D$ into components $M_1, \ldots, M_k$ corresponding to the prime factors $L_1, \ldots, L_k$ in our given decomposition of $T_4$, where each $M_i \cong D^2 \times I$. Since $A$ has no punctures, Claim 1 above implies that no circle in $A \cap \mathcal{F}$ bounds a disk in $A$. Thus all the circles of $A \cap \mathcal{F}$ are essential in $A$, and each $M_i$ meets $A$ in a (possibly empty) collection of annuli. Since $A$ is essential and non-boundary-parallel in $M_{D_4}^D$, $A$ meets some $M_i$ in an essential, non-boundary-parallel annulus $A^*$, i.e., an annulus $A^*$ which is knotted. If $A^*$ is not a cabling annulus, then $L_i$ must not be prime, a contradiction. So $A^*$ is a cabling annulus.

By Claims 1 and 2, every circle of $A \cap \mathcal{F}$ bounds a 2-punctured disk in a component of $\mathcal{F}$. Thus $A^*$ encloses all of $L_i$, and $L_i$ is a one-strand cable. This contradicts the definition of $T_4$, since none of the prime factors in its given decomposition were one-strand cables. So we may now assume that $D$ and $E$ are disjoint.

Suppose $D$ and $E$ are not isotopic. Since $D \cap E = \emptyset$, either $D$ is contained in $M_{E_4}^E$ or $E$ is contained in $M_{D_4}^D$; without loss of generality, suppose $E$ is contained in $M_{D_4}^D$. As before, let $\mathcal{F} = \bigcup F_i$ be the union of the decomposing disks for the prime factors $L_1, \ldots, L_k$ of $T_4$ in our given decomposition, and let $M_1, \ldots, M_k$ be the components that the $F_i$ separate $M_{D_4}^D$ into. If $E$ can be isotoped to be disjoint from $F$, then $E$ must be isotopic to some $F_j$. Hence, the cabling annulus for the portion of $L$ in $M_{E_4}^E$ meets each of $M_1, \ldots, M_j$ demonstrating that each $L_1, \ldots, L_j$ is a one-strand cable and contradicting the definition of $T_4$. So $E$ must intersect $\mathcal{F}$. By the same argument as at the beginning of this Step, after minimizing $|E \cap \mathcal{F}|$ there is a unique circle among those in $E \cap \mathcal{F}$ which is innermost in $E$. Let $F_i$ be the other surface which this circle is contained in. We can now apply the same argument as above with $F_i$ playing the role of $D$, as follows:

We deduce that the circles in $E \cap F_i$ separate $E$ into a 2-punctured disk and concentric (unpunctured) annuli. We consider the outermost annulus in $E$. As in the argument given above, this annulus implies one of the $L_i$ is a one-strand cable, a contradiction to the definition of $T_4$. Thus, $F_i$ and $E$ must be isotopic, which completes Step 3.

**Step 4:** We now know that $T_i = T_i'$ modulo braid equivalence for all $i = 1, \ldots, 4$. We also know that the prime factors in $T_i$ and $T_i'$ agree modulo braid equivalence for $i = 1, 2, 3$ by the prime decomposition theorem for knots. Thus, it remains to show that the prime factors in $T_4$ and $T_4'$ agree modulo braid equivalence.

Suppose we have two decompositions of this link, $K_1 \# \ldots \# K_m = T_4 = L_1 \# \ldots \# L_n$. Let $D_1, \ldots, D_{m-1}$ be the decomposing 2-punctured
 disks for the $K_i$ and let $E_1, \ldots, E_{n-1}$ be the decomposing 2-punctured disks for the $L_i$. Here we mean for the punctured disks to be in order, i.e., $D_1$ is the closest $D_i$ to $\partial_- M$ and $E_1$ is the closest $E_i$ to $\partial_- M$.

We will show that $D_i$ is isotopic to $E_i$ for every $i$. This will give us uniqueness of the decomposition. We start with $i = 1$. We may assume that the circles $\partial D_1$ and $\partial E_1$ are disjoint.

Suppose that we have minimized the number of circles in $D_1 \cap E_1$ (subject to the constraint $\partial D_1 \cap \partial E_1 = \emptyset$), and suppose first that this number is nonzero. Let $\alpha$ be a component of $D_1 \cap E_1$ that is innermost in $E_1$. Let $E_\alpha$ be the possibly punctured disk in $E_1$ bounded by $\alpha$. If $E_\alpha$ has no punctures, then by an argument that we have previously made several times, we contradict the minimality of $|D_1 \cap E_1|$.

If $E_\alpha$ has just one puncture, then by the minimality of $|D_1 \cap E_1|$ and the same arguments given at the end of Step 1, we find a split prime factor, contradicting the definition of $T_4$. Hence by the same arguments as in the previous Steps, all circles of $D_1 \cap E_1$ are concentric in $E_1$, and the innermost circle $\alpha$ bounds a 2-punctured disk in $E_1$.

Let $\beta$ be a component of $D_1 \cap E_1$ which is outermost in $E_1$. Let $A$ be the annulus bounded by $\beta$ and $\partial E_1$. By the above paragraph, $\beta$ encloses both punctures of $E_1$, and $A$ has no punctures. If $A$ is boundary parallel in $M^D_{K_1}$ or $M^D_{K_1 \# \cdots \# K_m}$, then we can eliminate $\beta$ (by moving $\partial E_1$ past $\partial D_1$), contradicting our minimality assumption. Thus $A$ is knotted, i.e., $A$ is an essential annulus, contained in either $M^D_{K_1}$ or $M^D_{K_1 \# \cdots \# K_m}$. In the first case, the primeness of $K_1$ and the fact that $\beta$ encloses both punctures of $E$ imply (as in Step 3) that $K_1$ is a 1-strand cable, contradicting the definition of $T_4$. In the second case, we can argue (again, as in Step 3) that some other $K_i$ is a 1-strand cable, a contradiction.

So we may now assume that $D_1$ and $E_1$ can be isotoped to be disjoint. Then one of the punctured disks lies between $\partial_- M$ and the other disk. Without loss of generality, suppose $E_1$ lies between $\partial_- M$ and $D_1$. Then since $E_1$ is not isotopic to $\partial_- M$ and since $K_1$ is prime, we conclude that $E_1$ must be isotopic to $D_1$. This implies $K_1 = L_1$ and $K_2 \# \cdots \# K_m = L_2 \# \cdots \# L_n$.

We may now repeat this argument for the string link $K_2 \# \cdots \# K_m = L_2 \# \cdots \# L_n$ to conclude $K_2 = L_2$ and $K_3 \# \cdots \# K_m = L_3 \# \cdots \# L_n$, and so on. If $m < n$, then $E_{m-1}$ are decomposing disks between $D_{m-1}$ and $\partial_+ M$ but isotopic to neither $D_{m-1}$ nor $\partial_+ M$. This contradicts the primeness of $K_m$. If $n < m$, we reach a contradiction by a similar argument, by the primeness of $L_n$. Thus $m = n$, and $D_i$ is isotopic to
Finally we point out an equivalent restatement of this theorem in terms of the language developed in the proof of Corollary 3.10. In this alternative formulation, we essentially “remove” all the pure braids from a 2-string link.

**Theorem 4.4** (Reformulation of Theorem 4.1). Any element of \( L_0^2 \) can be written as a product of primes which is unique up to only reordering the split link and one-strand cable factors.

**Corollary 4.5.** A 2-component string link \( L \) has a prime decomposition \( L = L_1 \# \ldots \# L_n \), where each \( L_i \) is a prime string link. Such a decomposition is unique up to reordering the factors in the center and up to multiplication by units (pure braids).

**Proof.** If \( L \) is pure, the corollary follows from Theorem 4.1. Assume that \( L \) is not pure. The existence of a prime decomposition again follows from Theorem 4.2. Given two prime decompositions for \( L \), each is defined (up to braid equivalence of the factors) by a complete set of decomposing disks \( E \) and \( D \) respectively. If \( \tau \) is the generator of the 2-strand braids, then \( L \# \tau \) is a pure braid. However, \( (D^2 \times I, L) \) is homeomorphic to \( (D^2 \times I, L \# \tau) \) in a natural way that takes decomposing disks to decomposing disks. So the image of \( E \) and \( D \) under this homeomorphism are complete sets of decomposing disks for \( L \# \tau \) that decompose \( L \# \tau \) into the exact same factors (up to braid equivalence) as in the original decompositions of \( L \). By Theorem 4.1, such decompositions for \( L \# \tau \) are unique up to reordering the factors in the center and up to multiplication by units. Since the prime factors in each of the decompositions of \( L \) are taken homeomorphically and in order to the prime factors of two decompositions of \( L \# \tau \), then the two decompositions of \( L \) are related via reordering the factors in the center and up to multiplication by units.

\[ \square \]

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