The Correspondence between Discrete Surface and Difference Geometry of the KP-hierarchy

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Abstract

The correspondence between two geometrical descriptions of the KP-hierarchy, one by discrete surface and another by difference analogue of differential geometry, is given.

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In the traditional physics the fundamental laws of nature were described in terms of differential equations. Their corresponding discrete versions were considered being approximation appropriate to compute them numerically or to avoid divergence. If there exists an integrable discrete system, on the other hand, the situation becomes completely different. A single discrete integrable equation can be associated to infinitely many continuous ones. It will be more natural to consider the discrete integrable equation being more fundamental, rather than special deformations of continuous version.

Among other discrete integrable systems the Hirota bilinear difference equation, which we abbreviate as HBDE[1], or 2D Toda equation, plays an important role in theoretical physics. This single equation describes the totality of soliton equations in the KP hierarchy. The connections of this equation were shown to algebraic curves[2], to the string correlation function in particle physics[3], to the 2D Ising model[4], to the Yang-Baxter equation of statistical lattice models[5,6], to the Painlevé system[7], to the completely integrable cellular automatons[8].

Very recently HBDE has been given a geometrical interpretation by means of discrete geometry[9,11] of two dimensional surface. There the HBDE turns out to be a recurrence formula satisfied by projective invariants of the Laplace sequence of the discrete analogue of the conjugate nets. On the other hand we developed a formulation of difference analogue of differential geometry[12]. The purpose of our work was to describe the Moyal quantisation as a discretisation of phase space and to provide an overall view to integrable systems. To this end we defined difference analogue of vector fields, differential forms, Lie derivatives and so on. In ref.[13] Kemmoku showed that this formalism naturally induces the infinite symmetry which characterises the KP-system. These two works, one of geometry of discrete surface and the other of difference analogue of differential geometry, were done quite independently. In fact they have nothing in common apart from the fact that both describe discrete integrable systems.

Our question we discuss in this article is if they are related. The answer is yes. In fact they are the same.

Let us begin with writing HBDE:

\[ \alpha f_n(l + 1, m)f_n(l, m + 1) + \beta f_n(l, m)f_n(l + 1, m + 1) + \gamma f_{n+1}(l + 1, m)f_{n-1}(l, m + 1) = 0. \]  

where \( l, m, n \in \mathbb{Z} \) are discrete variables, \( \alpha, \beta, \gamma \in \mathbb{C} \) are parameters subject to the constraint \( \alpha + \beta + \gamma = 0 \).

The statement in ref.[1] is as follows. We consider in the 3D Euclidian space a quadrilateral \( (Y(l, m), Y(l + 1, m), Y(l, m + 1), Y(l + 1, m + 1)) \), called elementary quadrilateral. If they satisfy the discrete Laplace equation

\[ Y(l + 1, m + 1) - Y(l, m) = A(l, m)(Y(l + 1, m) - Y(l, m)) + B(l, m)(Y(l, m + 1) - Y(l, m)), \]

they form a planar quadrilateral. A collection of such planar quadrilaterals \( Y \) is called a discrete conjugate net. The Laplace transform of the net \( Y \equiv Y_0 \) defines the other two conjugate nets \( Y_1 \) and \( Y_{-1} \) by connecting the intersection points of the directions of the opposite edges of the four elementary quadrilaterals which intersect at \( Y_0(l, m) \). The same process can be repeated to the images of the map, hence
we obtain a sequence of conjugate nets, $Y_n$, $n = 0, \pm 1, \pm 2, \cdots$. At every step of the map there exists an invariant under the projective transformations, called a cross ratio $f_n(l, m)$, which is formed by the four points \{$Y_n(l, m - 1), Y_n(l, m), Y_{n+1}(l, m), Y_{n-1}(l - 1, m)$\} of the $n$th map. Since the cross ratio is determined by the coefficients $A_n$ and $B_n$ of the Laplace map of the form (2), the sequence of the maps define a recurrence formula among the cross ratios. This formula was shown in ref.[9] being equivalent to the HBDE after some normalisation. We notice that this sequence of the map is exactly the same one found in ref.[14] in connection with the linearisation of HBDE.

The difference analogue of differential geometry was discussed in ref.[12] as follows. Let $x$ and $a$ be vectors in $\mathbb{R}^n$ and define a difference vector field associated with a $C^\infty$ function $v(x, a)$ by

$$X^D := \int da \ v(x, a)\nabla a,$$

where the difference operator is given by

$$\nabla a := \frac{1}{\lambda} \sinh(\lambda \sum_j a_j \partial x_j).$$

We also define the operator $\Delta a$ dual of $\nabla a$ under the following inner product

$$\langle \Delta a | \nabla b \rangle = \delta(a - b).$$

Then a difference one form is defined by (see ref.[12] for more detail)

$$\Omega^D := \int da \ w(x, a)\Delta a,$$

which gives rise to an invariant upon taking the inner product with a vector field:

$$\langle \Omega^D | X^D \rangle = \int dx \int da \ w(x, a)\nabla a.$$

Extending the space to the $2n$ dimensional phase space, $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$, a difference analogue of Hamiltonian vector field is defined as

$$X^D_A := \int da_1 da_2 \ v_A(x, p; a_1, a_2)\nabla a_1, a_2,$$

where

$$v_A(x, p; a_1, a_2) = \left(\frac{\lambda}{2\pi}\right)^{2n} \int db_1 db_2 \ e^{-i(a_1 b_2 - a_2 b_1)} A(x + \lambda b_1, p + \lambda b_2).$$

The operation of the Hamiltonian vector field to a function on the phase space yields the Moyal bracket[15] :

$$X^D_A B(x, p) = \frac{1}{\lambda} \sin \left\{ \lambda \left( \partial x_A \partial p_B - \partial x_B \partial p_A \right) \right\} A(x_A, p_A)B(x_B, p_B) \bigg|_{x, p}$$

$$=: i\{A, B\}_M.$$

It is apparent from this expression that in the continuous limit of the phase space $\lambda \to 0$ we recover the classical Poisson bracket.
The above definition of the Hamiltonian vector fields enables us to extend the concept of the Lie derivative in the differential geometry to the discrete phase space. In fact we can show directly the following operator relation between two vector fields:

\[
[X_A^D, X_B^D] = X_{-s(A,B)}^D.
\]

This exhibits a large symmetry generated by the Hamiltonian vector fields, which should be common in the physical systems formulated in our prescription. This algebra itself was discussed\[^{17}\]\[^{18}\] in various contexts including some geometrical arguments.

As we consider a physical phase space the pairing of a vector field with a difference one form is expected to mean an expectation value of a physical quantity. It is accomplished by adopting the Wigner distribution function\[^{19}\] \(F(x,p)\) to specify the physical state and by defining the difference one form by

\[
\Omega^D_F := \int da_1 da_2 \int db_1 db_2 \ e^{i(a_1 b_2 - a_2 b_1)} F(x + \lambda b_1, p + \lambda b_2) \Delta a_1 a_2.
\]

The expectation value of a physical quantity, say \(A(x,p)\), is obtained by calculating the inner product 

\[
\langle \Omega^D_F | X_D^A \rangle.
\]

Two types of prescriptions of discrete analogue of differential geometry are explained in above briefly. We notice that in the geometrical approach a real discretisation of two dimensional surface is investigated while in the other approach a formal discretisation of phase space with arbitrary dimensionality is the subject of the study. The space which is discretised is different. Moreover the meaning of discretisation of the phase space do not correspond to real discretisation of \(x\) and \(p\) separately, since only their product must have a finite minimum \(\lambda\). In fact uncertainty relations can be proved for the product of their square mean values\[^{15}\]. There seems little hope to find direct correspondence between these two discretisations.

What we like to discuss in the following is that, contrary to this naive observation, they are simply different view of the same object. For this purpose let us recall the two different view of the KP-hierarchy. One is given by \(^\text{(1)}\) in which three discrete variables \(l, m, n\) are used. The other representation is possible by using the KP-flow parameters \(\{t_n | n \in N\}\) and splitting \(^\text{(1)}\) into many soliton equations through reduction of variables. The variables are related each other by the Miwa transformation\[^{16}\] :

\[
t_n = \frac{1}{n} \sum_j k_j z_j^n, \quad n \in N
\]

where \(z_j\)’s are arbitrary parameters and

\[
l = p + q + r - \frac{3}{2}, \quad m = -p - \frac{1}{2}, \quad n = p + q - 1
\]

with \((p, q, r)\) being arbitrary three out of \(k_j\)’s in \(^\text{(13)}\). We know that the geometry of the discrete surface was described by means of the shift operators \(e^{\partial l}, e^{\partial m}\), whereas the third shift operator \(e^{\partial n}\) generates the sequence of Laplace transforms. The question is if they correspond to some difference vector fields in the form of \(^\text{(8)}\) when they are translated into the language of the other variables \(t_n\)’s.

After the change of variables according to \(^\text{(13)}\) we have

\[
e^{\partial z_j} = \exp \left[ \sum_{n=1}^\infty \frac{1}{n^2} \frac{a_n}{n} \frac{\partial}{\partial t_n} \right].
\]
The right hand side can be interpreted as a simultaneous shifts to all directions. The amount of the shift along the $t_n$ direction is $\frac{1}{n}z_j^n$. The antisymmetric combination of $e^{\partial_{k_j}}$ forms a vector field with a constant weight function. Since $l, m, n$ are linear combinations of $k_j$'s, the shifts to these directions are produced by the shifts to the $t_n$ directions by $\frac{1}{n}(z_p^n + z_q^n + z_r^n)$, $-\frac{1}{n}z_p^n$, $\frac{1}{n}(z_p^n + z_q^n)$ respectively.

From this consideration it is convenient to introduce a shift vector field:

$$X^S_A := \int da \ A(t, a) e^{\lambda \sum_n a_n \partial_{t_n}}. \quad (16)$$

We can convince ourselves that the shift vector fields form an algebra among themselves:

$$[X_A^S, X_B^S] = X^S_{\{A,B\}}, \quad (17)$$

where

$$\{A, B\} := \int db \ \left( \exp \left[ \lambda \left( \frac{a}{2} - b \right) \partial_{t_A} \right] - \exp \left[ \lambda \left( \frac{a}{2} + b \right) \partial_{t_B} \right] \right) \times A(t_A, a + b) B(t_B, a - b) \bigg|_{t_A = t_B = t}. \quad (18)$$

We also introduce a gauge covariant shift operator, which is defined by

$$X^S_{k_j} [U] := U(k) e^{\partial_{k_j}} U^{-1}(k)$$

where

$$u(t) = U(k) U^{-1}(k'), \quad k' := \{k_1, k_2, \ldots, k_j, k_j + 1, k_{j+1}, \ldots\}. \quad (19)$$

The correspondence of $X^S_{k_j} [U]$ to $X^S_A$ of (16) is achieved by choosing

$$A(t, a) = u(t) \prod_n \delta \left( a_n - \frac{1}{n} z_j^n \right). \quad (20)$$

These new operators enable us to translate various terms in the KP-hierarchy in the language of $t_n$'s into the language of $k_j$'s. We will give a list of such translation in the following.

1. Vertex operator I:

First we consider the case

$$U(k) = \prod_{l \neq j} \left( 1 - q \frac{z_l}{z_j} \right)^{k_l k_j} \quad (22)$$

and let

$$t_n = \frac{1}{n} \sum_j k_j q^n z_j^n, \quad |q| < 1, \quad n = 1, 2, \ldots. \quad (23)$$

Then we find

$$X^S_{k_j} [U] = \prod_{l \neq j} \left( 1 - q \frac{z_l}{z_j} \right)^{-k_l} e^{k_j}$$

$$= (1 - q)^{k_j} \exp \left[ \sum_{n=1}^\infty t_n z_j^{-n} \right] \exp \left[ \sum_{n=1}^\infty \frac{1}{n} q^n z_j^n \partial_{t_n} \right]. \quad (24)$$

This last expression, after divided by the factor $(1 - q)^{k_j}$ and taking the $q \to 1$ limit, is the one called the vertex operator in the soliton theory.
2. Vertex operator II:

We can generalise the vertex operator by making double the space of \( t_n \)’s. For this we define

\[
U(k) := \prod_{l \neq j} \left(1 - \frac{\bar{z}_l}{z_j}\right)^{k_l k_j} \left(1 - \frac{\bar{z}_j}{z_l}\right)^{-k_l k_j}
\]  

(25)

and

\[
t_n := \frac{1}{n} \sum_j k_j z_j^n, \quad n = 1, 2, \cdots, \quad t_0 := \sum_j k_j \ln z_j
\]

\[
\bar{t}_n := \frac{1}{n} \sum_j k_j \bar{z}_j^{-n}, \quad n = 1, 2, \cdots, \quad \bar{t}_0 := -\sum_j k_j \ln \bar{z}_j,
\]

(26)

where we added the 0-th components \( t_0 \) and \( \bar{t}_0 \) as well. In terms of \( t_n \)’s

\[
X^S_{k_j}[U] = \exp \left[-\sum_{n=0}^{\infty} (t_n \bar{z}_j^{-n} - \bar{t}_n z_j^n)\right]
\]

\[
\times \exp \left[\ln z_j \partial_{t_0} - \ln \bar{z}_j \partial_{\bar{t}_0} + \sum_{n=1}^{\infty} \frac{1}{n} (z_j^n \partial_{t_n} + \bar{z}_j^{-n} \partial_{\bar{t}_n})\right]
\]

\[= e^{iX_+(z_j, \bar{z}_j)}e^{iX_-(z_j, \bar{z}_j)}.\]

(27)

The last expression is given to remind the relation of the operator to the string coordinates \( X_\pm \).

Note that this has a symplectic structure such that \((t_n, \bar{t}_n)\) form a symplectic pair of the infinite dimensional phase space as it is implied in (3). Therefore it satisfies the Moyal algebra (11), rather than (17).

3. Bäcklund transformation:

The Bäcklund transformation is generated by the operators of the form:

\[
B(z_j, \bar{z}_j) = X^S_{k_j}[U]X^S_{-k_j}[U].
\]

(28)

They form an algebra which follows to (11).

4. Linearisation of HBDE:

HBDE can be derived as an integrability condition of the following linear problem [20]:

\[
X^S_l[U]|\Phi\rangle = c_+ X^S_{l+n}[V]|\Phi\rangle
\]

\[
X^S_m[V]|\Phi\rangle = c_- X^S_{m-n}[U]|\Phi\rangle
\]

(29)

It was shown in ref. [20] that under the gauge condition \( V(l, m, n) = U(l, m, n - 1) \), the gauge potential \( U(l, m, n) \) satisfies HBDE (3) with \( \gamma = -c_+ c_- \alpha \). The gauge potential \( U \) in the covariant shift operator has an obvious geometrical meaning as a connection. HBDE is nothing but a geometrical constraint imposed on the connection. We have also shown that this set of linear equations reduce to the Lax type of equation for the Toda lattice in an appropriate continuous limit [21]. It is apparent that writing the equations by using \( t_n \)’s we will get infinitely many inverse problems depending on the way of reduction of the variables.
5. Baker-Akhiezer function:

The Baker-Akhiezer function of the KP-hierarchy was given explicitly in terms of the solution $\tau$ of HBDE. It can be interpreted within our formulation as a wave function in the form:

$$|\Psi(z_j)\rangle = X^{S}_{k_j}[\tau^{-1}U_0]|0\rangle$$

(30)

where $U_0$ is the one of (22).

We have shown various objects in the KP theory expressed differently by using two different coordinates, one of discrete surface $k_j$'s and the other the soliton coordinates $t_n$'s. In all examples they are translated from one to the other by the operator $X^{S}_{k_j}[U]$. This operator itself possesses two faces, one is a gauge covariant shift operator in the former coordinates. At the same time it represents a difference analogue of vector field in the latter. If we let $U$ be an arbitrary function of $t_n$'s or $k_j$'s, it generates a large symmetry algebra as given by (11). From the viewpoint of geometry this algebra should describe some topological nature of the discrete surface as $k_j$'s are restricted to $p, q, r$. An extension to higher dimensional hyper surface must be straightforward. We know that these two sets of coordinates, $t_n$'s and $k_j$'s, are linearly dependent each other by the Miwa transformation (13). Nevertheless it is still not known what is the meaning of $k_j$'s in the space of soliton coordinates and what is the role played by $t_n$'s on the discrete surface.

Besides two approaches discussed in this paper there exist some other discrete approaches to the KP hierarchy or its generalisations (22) (23). There have been also many works which explore geometrical nature of Moyal quantisation (24) (25) (26) (27). An interpretation of the analytical approach to rather general integrable systems (28) by means of geometry of higher dimensional surfaces was pointed out recently in ref[10]. It is desirable to clarify correlation among these different approaches and get insight underlying all discrete integrable systems.

In the forthcoming paper we will discuss the symmetry generated by the covariant shift operators and also their connection to the Yang-Baxter relation from the view we discussed recently(28).

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