The Price of Diversity in Assignment Problems

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In this paper, we introduce and analyze an extension to the matching problem on a weighted bipartite graph (i.e. the assignment problem): Assignment with Type Constraints. Here, the two parts of the graph are each partitioned into subsets, called types and blocks respectively; we seek a matching with the largest sum of weights under the constraint that there is a pre-specified cap on the number of vertices matched in every type-block pair. Our primary motivation stems from the large-scale public housing program run by the state of Singapore, accounting for over 70% of its residential real estate. To promote ethnic diversity within its housing projects, Singapore imposes ethnicity quotas: the population is divided into ethnicity-based groups and each new housing development into blocks of flats such that each group must not own more than a certain percentage of flats in a block. However, other domains use similar hard capacity constraints to maintain diversity: these include matching prospective students to schools or medical residents to hospitals. Limiting agents’ choices for ensuring diversity in this manner naturally entails some welfare loss. One of our goals is to study the tradeoff between diversity and (utilitarian) social welfare in such settings. We first show that, while the classic assignment program is polynomial-time computable, adding diversity constraints makes the problem computationally intractable; however, we identify a $\frac{1}{2}$-approximation algorithm, as well as reasonable assumptions on the structure of utilities (or weights) which permit poly-time algorithms. Next, we provide two upper bounds on the price of diversity — a measure of the loss in welfare incurred by imposing diversity constraints — as functions of natural problem parameters. We conclude the paper with simulations based on publicly available data from two diversity-constrained allocation problems — Singapore Public Housing and Chicago School Choice — which shed light on how the constrained maximization as well as lottery-based variants perform in practice.

CCS Concepts: • Theory of computation → Problems, reductions and completeness; • Computing methodologies → Artificial intelligence;

Additional Key Words and Phrases: Assignment Problem; Diversity Constraints; Price of Diversity

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INTRODUCTION

Consider a mechanism that allocates a set of goods to agents; agents have utilities over items, and we are interested in finding a socially optimal allocation. This setting (known in the literature as the assignment problem) is often used to model real-world problems such as allocating public housing, assigning slots in schools, or courses to students. It is often the case in these contexts that one wishes to maintain a diverse allocation: it would be undesirable (from the mechanism designer’s perspective) to have certain apartment blocks that predominantly consist of a specific ethnic group, or to have a public school serving students from a specific district. In both cases, agents have different types, and goods are partitioned into blocks; our goal is to ensure that each block of goods is allocated to a diverse population of agent types. A diverse allocation of goods is desirable for many reasons (especially in the case of government funded public goods). First and foremost, it avoids the inadvertent creation of segregated communities; secondly, by ensuring equal access to a public resource, one avoids the risk of discriminatory funding: for example, systematically underfunding schools that serve certain segments of the population, or investing in parks and public facilities in neighborhoods comprising of certain ethnic groups.

In this work we study quota-based mechanisms for maintaining diversity; the initial motivation for this work stems from Singapore’s public housing system.

The state of Singapore operates a unique national public housing program, offering a variety of flats for sale at subsidized rates to Singapore citizens and permanent residents. The construction of public housing projects as well as the sale of the flats in these projects on a large-scale public market is centrally managed by a government body called the Housing and Development Board (HDB)\(^1\), a statutory board of the Ministry of National Development\(^2\). As per the latest reports available at the time of writing this paper, an estimated 82% of the resident population of Singapore live in HDB flats [Housing and Development Board, Singapore 2017] that constitute approximately 73% of all apartments in the country [Department of Statistics, Singapore 2017]. Since its inception in 1960, HDB has been providing a public good — affordable apartments in a small country with little real estate — but by 1989, the system began to exhibit an unforeseen side-effect: the emergence of de facto ethnic enclaves. Mr. S. Dhanabalan, then Minister for National Development, voiced the following concerns as he introduced the Ethnic Integration Policy (EIP) in parliament on 16 February 1989 [Parliament of Singapore. Parliament Debates: Official Report. 1989]:

> [P]roportionately more Chinese applied for flats in Ang Mo Kio/Hougang Zone and proportionately more Malays bought more than half (55\%) of the flats in the Bedok/Tampines Zone. In Bedok new town alone, if present trends continue, the proportion of Malays will reach 30\% by 1991, and will exceed 40\% in 10 years’ time. [...] There are clear signs that racial groupings are re-emerging. Although the problem has not reached crisis proportions, the experience in other multi-racial societies such as the United States shows that while racial groupings start slowly, once a critical point is passed, racial groupings accelerate suddenly.\(^3\)

\(^1\)http://www.hdb.gov.sg
\(^2\)https://www.mnd.gov.sg/
\(^3\)The authors note that this aligns with long-known models of segregation: in his seminal paper, Schelling [1971] shows how agents of two types, who are allowed to distribute themselves over an area based on their preferences for the composition of their immediate neighborhoods, lead to the emergence of segregated enclaves, even when each individual prefers a minority of neighbors of a different type to having all neighbors of the same type as herself.

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The EIP was officially implemented on 1 March 1989; it imposes quotas on the number of units occupied by each of the three ethnic groups: Chinese, Malay and Indian/Others. In 1989, when the percentages of the three ethnic groups (Chinese, Malay, and Indian/Others) in the population were 76.0%, 15.1%, and 8.9%, the corresponding block percentage caps were set at 87%, 25%, and 13% respectively; since 5 March 2010, the percentage for Indian/Others has been revised to 15% [Deng et al. 2013; Housing and Development Board, Singapore 2010].

Ethnic quotas add another layer of complexity to what should be, at its foundation, a straightforward allocation problem. HDB uses a lottery mechanism to allocate new developments: all applicants who apply for a particular development pick their flats in random order (see Section 1.3 for further details). Consider an applicant i of Chinese ethnicity applying for an estate with 100 flats per block, up to 87 of which may be assigned to ethnically Chinese applicants, and at most 25 of which may be assigned to ethnically Malay applicants. Assume that i is 90th in line to select an apartment; will she get a chance to pick a flat in a block she prefers? If at least 87 Chinese applicants were allowed to choose a flat before i and all of them picked flats in this block, the Chinese ethnic quota for the block will have been filled and applicant i will no longer be eligible for the block, even if it still has vacant flats. On the other hand, suppose that i is 105th in line to select an apartment; if 40 Malay applicants end up before i in the lottery, then 15 of them will be rejected, and i will have a spot.

As the example above shows, diversity constraints interact with the allocation mechanism in peculiar ways to affect the overall welfare of the allocation. This issue is not restricted to Singapore public housing; the following are a few examples where upper bounds similar to those in the above housing allocation problem are applied (see [Fragiadakis and Troyan 2017] and references therein for more detailed expositions). To circumvent a shortage of doctors in rural areas due to medical graduates’ preference for urban residency programs, the Japanese government places a “regional cap” on the total number of residents matched within each of its 47 prefectures [Kamada and Kojima 2015] – here, the population of residency applicants is not partitioned, but hospitals within a prefecture can be thought of as forming a block of items. Many school districts in the U.S.A. take active measures for the integration of students from families with differing socio-economic statuses (SES) [U.S. Department of Education, Office of Elementary and Secondary Education 2017], one of which is to allot a fraction of the vacant spots in schools via lotteries with percentage caps for all SES groups, as is done in the city of Chicago, Illinois (see Section 1.4 for further details). The United States Military Academy assigns newly graduated cadets to positions in the army branches, taking cadets’ preferences into account but under “artificial caps” on the number of assignments per branch [Fragiadakis and Troyan 2017].

The imposition of diversity constraints as above can naturally lead to a reduction in the total achievable utility/economic value by the assignment, but we must bear in mind that diversity is a social desideratum external to any such economic consideration. For purposes such as policy making and the proper functioning of diversity-inducing measures included in automated decision-making systems, it is imperative to deepen our understanding of the impact that these measures have on the underlying assignment mechanism. In this paper, we study this impact from both computational and economic angles.

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4In addition to these block capacities, the EIP also imposes neighborhood capacities, where each neighborhood comprises several blocks. Thus, an estate is partitioned into neighborhoods and a neighborhood into blocks with the neighborhood capacity being naturally smaller than the block capacity for each ethnic group: 84%, 22%, and 12% (increased from 10% in 2010) for Chinese, Malay, and Indian/Others respectively. Moreover, the EIP applies to both new and resale flats, e.g. a Chinese occupant of an HDB flat is free to resell it to another Chinese buyer but will not be able to resell to a Malay buyer if the Malay ethnic quota for that block is already filled. In this paper, we do not address these complications for the most part.

5While this example is, of course, highly stylized, the effects it describes are quite real: one often hears stories of young couples who arrive at the HDB office to select a flat, only to be notified that their ethnic quota had just been filled.

6These caps are called artificial since they are calculated by the Academy in such a way that any feasible assignment ex post satisfies the maximum and minimum quotas for each branch that are based on actual staffing needs.
1.1 Our Contributions

We study the interplay between diversity and utility in assignment problems; we set up a benchmark where a central planner (e.g., HDB) has access to the correct utilities (or, in general, weights) of all agents (e.g., applicant households) for all items (e.g., flats); agents are partitioned into types with respect to a single attribute (e.g., ethnic groups) and goods are also similarly divided into disjoint blocks (e.g., blocks of flats as defined by HDB); a limited number of goods in each block can be allocated to agents of each type. We call these upper bounds type-block capacities.

These restrictions result in several interesting outcomes. While the unconstrained optimal assignment problem is well-known to be poly-time solvable [Kuhn 1955], we show that imposing type-block constraints makes it computationally intractable (Section 3). However, we show that, in general, a polynomial-time $\frac{1}{2}$-approximation algorithm (Section 3.1) exists, and identify utility models for which one can find the optimal assignment with type-block constraints in polynomial time (Section 3.2). In Section 4, we study the potential utility loss from imposing type-block constraints, which we term the price of diversity as in Ahmed et al. [2017], and we show that it can be bounded by natural problem parameters. Finally, we analyze the empirical price of diversity as well as the welfare loss induced by the lottery mechanism on simulated instances generated from publicly available, real-world data pertaining to public housing in Singapore and school choice in Chicago, IL, USA (Section 5).

1.2 Related work

The problem we study is an extension to the bipartite matching problem [Lovász and Plummer 2009] where each edge joins an agent to an item and is weighted with the utility the agent will receive if she is allocated that item. There is a rich literature on weighted bipartite matching problems (also known as assignment problems [Munkres 1957]), and polynomial-time algorithms for the unconstrained version have long been known (e.g., [Kuhn 1955]). Several generalizations and/or constrained versions have been studied, e.g., recent work by Lian et al. [2018] who allow each agent (resp. item) to be matched to multiple items (resp. agents) but within upper and lower capacities. Some previously studied variants correspond to (polynomial-time) special cases of our problem. For example, the assignment problem with subset constraints studied by Bauer [2004] can be thought of as a special case of our problem, with a single block or a single type; if all agents of each type have identical utilities for all apartments in each block, and each type-block capacity is smaller than both the corresponding type and block sizes, then our problem reduces to a special case of the polynomial-time solvable capacitated b-matching on a bipartite graph [Ahn and Guha 2014].

In addition to our main motivating problem of HDB housing allocation and the other documented examples [Fragiadakis and Troyan 2017] noted in the introduction, type-block constraints can naturally arise in many other settings related to assignment/allocation problems with no monetary transfers [Hylland and Zeckhauser 1979; Zhou 1990]. For example, consider the course allocation problem analyzed by Budish and Cantillon [2012]; one might require that each course has students from different departments and impose maximal quotas to ensure this. Other examples include allocating subsidized on-campus housing to students [Abdulkadiroğlu and Sönmez 1998], appointing teachers at public schools in different regions as done by some non-profit organizations [Featherstone 2015], or assigning first year business school students to overseas programs [Featherstone 2015]. Our results apply to the seminal work on public school allocation [Abdulkadiroğlu et al. 2009; Abdulkadiroğlu and Sönmez 2003; Pathak and Sönmez 2013] and matching medical interns or residents to hospitals [Roth 1984] that does not concern itself with diversity/distributional constraints. This line of work mainly explores the interaction between individual selfish behavior and allocative efficiency (e.g., Pareto-optimality) of matching mechanisms, under either ordinal preferences or cardinal utilities, one-sided or two-sided.
(see, e.g. [Anshelevich et al. 2013; Bade 2016; Bhalgat et al. 2011; Bogomolnaia and Moulin 2001] and references therein); we, on the other hand, focus on the impact of type-block constraints on welfare loss, when agents’ utilities are known to a central planner.

Another relevant strand of literature is that on the fair allocation of indivisible goods (see, e.g., [Barman et al. 2017; Barman and Murthy 2017; Caragiannis et al. 2016; Kurokawa et al. 2016; Procaccia and Wang 2014] and references therein): fairness is usually quantified in terms of the utilities or preferences of agents for allocated items (e.g. proportionality, envy-freeness and the maximin share guarantee) but our contribution deals with a different notion of fairness: the proportionate representation of groups in the realized allocation, with no regard to agents’ utilities.

Some recent work has formally addressed diversity issues in computational social choice. Unlike our paper, Ahmed et al. [2017] treat “diversity as an objective, not a constraint” in a b-matching context (e.g. matching papers to reviewers with diverse interests): they minimize a supermodular objective function to encourage the matching of each item to agents of different types. Our diversity concept comes close to that of Bredereck et al. [2018] who also achieve diversity by imposing hard constraints on the maximization of a (submodular) objective that measures the quality of the solution; however, they work in a committee (subset) selection setting with variously structured agent labels while we solve a matching problem with both agents and items split into disjoint subsets. Lang and Skowron [2016] focus on multi-attribute proportional representation in committee selection where they essentially define diversity in terms of the divergence between the realized distribution of attribute values in the outcome and some target distribution, but admit no notion of solution quality in addition to diversity.

In a recent paper, Immorlica et al. [2017] study the efficiency of lottery mechanisms such as the ones used by HDB to allocate apartments; however, their work does not account for block ethnicity constraints; as we show both theoretically and empirically, these type-block constraints can have a significant effect on allocative efficiency.

1.3 The Singapore Public Housing Allocation System

A few facts about HDB public housing, a dominant force in Singapore, are in order. New HDB flats are purchased directly from the government, which offers them at a heavily subsidized rate. New apartments are typically released at quarterly sales launches; these normally consist of plans for several estates at various locations around Singapore, an estate consisting of four or five blocks (each apartment block has approximately 100 apartments) sharing some communal facilities (e.g. a playground, a food court, a few shops etc.). Estates take between 3 to 5 years to complete, during which HDB publicly advertises calls to ballot for an apartment in the new estate. A household (say, a newly married couple looking for a new house) would normally ballot for a few estates (balloting is cheap: only S$10 per application [Housing and Development Board, Singapore 2015]). HDB allocates apartments using a lottery: all applicants to a certain estate choose their flat in some random order; they are only allowed to select an apartment in a block such that their ethnic quota is not reached.

The lottery mechanism actually employed by HDB has further necessary complications: HDB has elaborate eligibility criteria as well as privilege and priority schemes that take into account sales launch types, flat types, and relevant attributes of the applicants, e.g. first-timers and low-income families usually have improved chances of being balloted for a flat; moreover, the same estate may have several balloting rounds in order to ensure that all apartments are allocated by the time of completion. However, the focus of this work is on the welfare effects of using ethnic quotas rather than the intricacies of the HDB lottery mechanism. Hence, we use a simplified version of the HDB lottery mechanism where

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7http://www.hdb.gov.sg/cs/infoweb/residential/buying-a-flat/new/hdb-flat
8http://www.hdb.gov.sg/cs/infoweb/residential/buying-a-flat/new/eligibility/priority-schemes
applicants are selected one by one uniformly at random from the remaining pool and assigned the available flat which they value the most, respecting ethnic quotas (see Section 5).

We must mention the existing literature on the documentation of Singapore’s residential desegregation policies [Chua 1991; Deng et al. 2013; Phang and Kim 2011] and the empirical evaluation of their impact on various socioeconomic factors [Sim et al. 2003; Wong 2014]; to the best of our knowledge, ours is the first formal approach towards this problem.

1.4 Public School Choice in Chicago, U.S.A.

Many school districts across the U.S.A. employ a variety of strategies for promoting student diversity [Kahlenberg 2016; U.S. Department of Education, Office of Elementary and Secondary Education 2017], e.g. controlled choice systems wherein parents are allowed to apply for options beyond their neighborhood schools, thereby counteracting underlying residential segregation. Following restrictions placed on the explicit use of race in defining diversity goals in school choice by the U.S. Supreme Court in 2007, it has been common to use some indicator of the socio-economic status (SES) of a family in integration efforts. The system in Chicago, IL, is a notable example.

Chicago Public Schools (CPS) is one of the largest school districts in the U.S.A.\(^9\), overseeing more than 600 schools of various types: neighborhood schools, selective schools, magnet schools, and charter schools\(^10\). The application and selection processes for these schools [Chicago Public Schools 2017] may involve a number of computerized lotteries with no diversity component, e.g. sibling lottery, proximity lottery, school staff preference lottery; however, a significant number of entry-level seats in magnet and selective enrollment schools are filled by lotteries based on a “tier system”. We briefly describe its operation as follows. A composite SES score is computed for each of the census tracts that Chicago is divided into, based on six factors (median family income, adult education level, home-ownership rate, single-parent family rate, rate of English-speaking, and neighborhood school performance), and each tract is placed in one of four tiers based on its score. The maximum and minimum scores defining a tier are set in such a way that (roughly) a quarter of school-aged children end up in each tier, with Tier 1 having the lowest scores. The tier of a child is determined by the address furnished by the parents. Of the seats in each school earmarked for a citywide SES lottery or general lottery, an equal number is allocated to each tier. There is an upper limit on the number of schools that a child can apply to, and each applicant is entered into a lottery for each school they apply to, for their own tier (thus, there is a lottery per school per tier); an applicant, who comes up in the lottery and accepts the offer from the school under consideration, is removed from all lotteries. If the size of the applicant pool from a tier to a school falls short of the number of its allocated seats for that tier at any stage, “the unfilled seats will be divided evenly and redistributed across the remaining tier(s) as the process continues ’ ”[Chicago Public Schools 2017].

For an empirical study of the impact of Chicago’s diversity-promoting measures on integration and student outcomes, the interested reader is referred to [Quick 2016] and citations therein.

2 PRELIMINARIES

We first describe a formal model for the allocation problem with diversity quotas. Throughout the paper, given \(s \in \mathbb{N}\), we denote the set \(\{1, 2, \ldots, s\}\) by \([s]\).

**Definition 2.1 (AssignTC).** An instance of the Assignment with Type Constraints (AssignTC) problem is given by:

(i) a set \(N\) of \(n\) agents partitioned into \(k\) types \(N_1, \ldots, N_k\),

\(^9\)http://www.cps.edu/About_CPS/At-a-glance/Pages/Stats_and_facts.aspx
\(^10\)http://cpstiers.opencityapps.org/about.html
(ii) a set $M$ of $m$ items/goods partitioned into $l$ blocks $M_1, \ldots, M_l$,

(iii) a utility $u(i, j) \in \mathbb{R}_+$ for each agent $i \in N$ and each item $j \in M$,

(iv) a capacity $\lambda_{pq} \in \mathbb{N}$ for all $(p, q) \in [k] \times [l]$, indicating the upper bound on the number of agents of type $N_p$ allowed in the block $M_q$.

Without loss of generality, we assume that the inequality $\lambda_{pq} \leq |M_q|$ holds for all type-block pairs $(p, q) \in [k] \times [l]$, since it is not possible to assign more than $|M_q|$ agents of type $N_p$ to a block by definition. In general, agents types could be based on any criterion such as gender, profession, or geographical location. We consider the idealized scenario where we have a central planner who has access to the utilities of each agent for all items, and determines an assignment that maximizes social welfare under type-block constraints.

A few words about the type-block capacities are in order. Note that our analysis is agnostic to how these capacities are determined and just treats the vector $\{\lambda_{pq}\}_{p \in [k], q \in [l]}$ as a problem input. Moreover, neither do we assume inequalities of the form $\lambda_{pq} \leq |N_p|$ nor is there any positive lower bound on the number of assignments for any type-block pair: this is in keeping with the actual HDB housing problem where $\lambda_{pq}$’s are fixed by policy (as percentages of block size) even before observing the applicant pool so that capacities larger than the size of an ethnic group are possible. Adding lower bounds a priori may render the problem infeasible if there not enough applicants of a certain type.

An assignment of items to agents can be represented by a $(0, 1)$-matrix $X = (x_{ij})_{n \times m}$ where $x_{ij} = 1$ if and only if item $j$ is assigned to agent $i$; a feasible solution is an assignment in which each item is allocated to at most one agent, and each agent receives at most one item, respecting the type-block capacities defined in (iv). We define the objective value (or total utility) as the utilitarian social welfare, i.e. the sum of the utilities of all agents in an assignment $u(X) \triangleq \sum_{i \in N} \sum_{j \in M} x_{ij} u(i, j)$. Clearly, this optimization problem can be formulated as the following integer linear program:

$$\begin{align*}
\text{max} & \quad \sum_{i \in N} \sum_{j \in M} x_{ij} u(i, j) \\
\text{s.t.} & \quad \sum_{j \in M_q} x_{ij} \leq \lambda_{pq} \quad \forall p \in [k], \forall q \in [l] \\
& \quad x_{ij} \leq 1 \quad \forall i \in N \\
& \quad x_{ij} \leq 1 \quad \forall j \in M \\
& \quad x_{ij} \in \{0, 1\} \quad \forall i \in N, \forall j \in M
\end{align*}$$

where constraints (3-5) jointly ensure that $X$ is a matching of items to agents, and inequalities (2) embody our type-block constraints.

Finally, an instance of the decision version of AssignTC consists of parameters (i) to (iv) in Definition 2.1, as well as a positive value $U$: it is a ‘yes’-instance iff there exists a feasible assignment, satisfying constraints (2-5), whose objective value is at least $U$.

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11The Singapore EIP percentage caps consider various factors such as “[t]he racial composition of the population[,] . . . the rate at which new households are being formed in each one of the racial groups and the present composition of applications” [Parliament of Singapore. Parliament Debates: Official Report. 1989], but these aspects of the problem are beyond the scope of the present work.

12Fragiadakis and Troyan [2017] show that, for some assignment problems with actual floor and ceiling constraints for each type-block pair, where the agent population is known beforehand and there is a guarantee that no agent remains unassigned, it is possible to reformulate the problem constraints in terms of “artificial caps” (modifying block sizes as well as type-block ceilings) and no floors: our analysis applies to these problems in this modified form.
3 THE COMPLEXITY OF THE ASSIGNMENT PROBLEM WITH TYPE CONSTRAINTS

Our first main result is that the decision problem we introduce in Section 2 is NP-complete. We prove this by describing a polynomial-time reduction from the NP-complete Bounded Color Matching problem [Garey and Johnson 1979], defined as follows:

**Definition 3.1 (BCMatching).** An instance of the Bounded Color Matching (BCMatching) problem is given by (i) a bipartite graph $G = (A \cup B, E)$, where the set of edges $E$ is partitioned into $r$ subsets $E_1, \ldots, E_r$ representing the $r$ different edge colors, (ii) a capacity $w_t \in \mathbb{N}$ for each color $t \in [r]$, (iii) a profit $\pi_e \in \mathbb{Q}_+$ for each edge $e \in E$, and (iv) a positive integer $P$. It is a ‘yes’-instance if there exists a matching (i.e., a collection of pairwise non-adjacent edges) $E' \subseteq E$ such that the sum of the profits of all edges in the matching is at least $P$, and there are at most $w_t$ edges of color $t$ in it, i.e., $\sum_{e \in E'} \pi_e \geq P$ and $|E' \cap E_t| \leq w_t$ for all $t \in [r]$.

**Theorem 3.2.** The AssignTC problem is NP-complete.

**Proof.** That the problem is in NP is immediate: given an assignment, one can verify in poly-time that it satisfies the problem constraints and compute total social welfare. Given an instance $(G; \bar{w}; \bar{\pi}; P)$ of BCMatching, we construct an instance of the AssignTC problem as follows (see Example 3.3 for an illustration). Each edge $e \in E$ is an agent, whose type is its color. Items in our construction are partitioned into two blocks: $M_1$ and $M_2$. The items in block $M_1$ correspond to the vertices in $B$: there is one item $j_b$ for each node $b \in B$. For every $a \in A$, we add $\deg(a) - 1$ items $j_a^1, \ldots, j_a^{\deg(a)-1}$ to $M_2$, for a total of $|E| - |A|$ items. Thus, there is a total of $m = |B| + |E| - |A|$ items. Block $M_1$ accepts at most $\lambda_p$ agents of type $N_p$, whereas block $M_2$ has unlimited type-block capacity; in other words, $\lambda_{p1} = \lambda_p$ and $\lambda_{p2} = \min(|N_p|, |M_2|)$ for all $p \in [k]$. Given $e = (a, b)$, we define the utility function of agent $e$ as follows:

$$u(e, j) = \begin{cases} \pi_e & \text{if } j = j_b, \\ \Phi & \text{if } j = j_a^s \text{ for some } s \in [\deg(a) - 1], \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\Phi$ is an arbitrarily large constant, e.g., $\Phi = 1 + \sum_{e \in E} \pi_e$. Finally, let $U = P + \Phi(|E| - |A|)$; that is, our derived AssignTC instance is a “yes” instance iff there is some assignment of items to agents such that the social welfare exceeds $U$.

We begin by showing that if the original BCMatching instance is a ‘yes’ instance, then so is our constructed AssignTC instance. Let $E' \subseteq E$ be a valid matching whose value is at least $P$; let us construct an assignment $X$ of items to agents via $E'$ as follows. Observe some node $a \in A$: if $(a, b) \in E'$ then we assign the item $j_b \in M_1$ to the agent $(a, b)$; the remaining $\deg(a) - 1$ agents of the form $(a, b')$, with $b' \in B$, are arbitrarily assigned to the items $j_a^1, \ldots, j_a^{\deg(a)-1} \in M_2$. If $E'$ contains no edges adjacent to $a$, then we arbitrarily choose $\deg(a) - 1$ edges adjacent to $a$ and assign the corresponding agents to the items $j_a^1, \ldots, j_a^{\deg(a)-1}$. We now show that this indeed results in a valid assignment satisfying the type-block constraints.

First, by construction, every agent $(a, b)$ is assigned at most one item. Moreover, since $E'$ is a matching, every item $j_b \in M_1$ is assigned to at most one agent of the form $(a, b)$; hence, every item in $M_2$ is assigned to at most one agent.

Let $E'_p = E_p \cap E'$ be the edges of color $p$ in $E'$. Since the matching $E'$ satisfies the capacity constraints of the BCMatching instance, we have $|E'_p| \leq \lambda_p$ for all $p \in [k]$; in particular, the number of items in $M_1$ assigned to agents of type $p$ is no more than $\lambda_p = \lambda_{p1}$. Thus, the type-block constraints for $M_1$ are satisfied. On the other hand, the
type-block constraints for $M_2$ are trivially satisfied. We conclude that our constructed assignment is indeed valid, and satisfies the type-block constraints.

Finally, we want to show that total social welfare exceeds $U$ the prescribed bound. Let us fix a node $a \in A$. By our construction, if the edge $e = (a, b)$ is in the matching $E'$, then agent $e$ is assigned the item $j_b$ for a utility of $\pi_e$. Thus the total welfare of agents in $E'$ equals $\sum_{e \in E'} \pi_e$, which is at least $P$ by choice of $E'$. In addition, for every $a \in A$, there are exactly $\deg(a) - 1$ agents assigned to items in $M_2$ for a total utility of $\Phi(\deg(a) - 1)$. Summing over all $a \in A$, we have that the total utility derived by agents in $E \setminus E'$ is

$$\sum_{a \in A} \Phi(\deg(a) - 1) = \Phi \left( \sum_{a \in A} \deg(a) - \sum_{a \in A} 1 \right) = \Phi(|E| - |A|).$$

Putting it all together, we have that the total utility obtained by our assignment is at least $P + \Phi(|E| - |A|) = U$.

Next, we assume that our constructed $\text{AssignTC}$ instance is a 'yes' instance, and show that the original $\text{BCMatching}$ instance must also be a 'yes' instance. Let $X$ be a constrained assignment whose social welfare is at least $U = P + \Phi(|E| - |A|)$. Let $E'$ be the set of edges corresponding to agents $(a, b)$ assigned to items in $M_1$; we show that $E'$ is a valid matching whose value is at least $P$. First, for any $b \in B$, $X$ must assign the item $j_b$ to at most one agent $e \in E'$. Next, since $\Phi$ is greater than the total utility obtainable from assigning all items in $M_1$, it must be the case that $X$ assigns all items $j_1^b, \ldots, j_{\deg(a) - 1}^b$ to $\deg(a) - 1$ agents of the form $(a, b)$, with $b \in B$, for every node $a \in A$; thus, there can be one edge in $E'$ that is incident on $a$ for every $a \in A$. Next, since $X$ satisfies the type-block constraints, we know that for every $p \in [k]$, there are at most $\lambda_{p1} = w_p$ agents from $E_p$ that are assigned items in $M_1$; thus, $E'$ satisfies the capacity constraints. Finally, the utility extracted from the agents assigned to items in $M_2$ is exactly $\Phi(|E| - |A|)$; the total utility of the matching $X$ is at least $U = P + \Phi(|E| - |A|)$, thus $E'$ has a total profit of at least $P$ in the original $\text{BCMatching}$ instance, and we are done.

**Example 3.3.** In Figure 1, the graph $G = (A \cup B, E_1 \cup E_2)$, with $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$, $E_1 = \{(a_1, b_1), (a_2, b_2)\}$ and $E_2 = \{(a_1, b_2), (a_2, b_1), (a_2, b_3)\}$, is an instance of the $\text{BCMatching}$ problem; edge labels are profits. The associated instance of the $\text{AssignTC}$ problem is defined by $N = N_1 \cup N_2$ and $M = M_1 \cup M_2$, where $N_1 = \{(a_1, b_1), (a_2, b_2)\}$, $N_2 = \{(a_1, b_2), (a_2, b_1), (a_2, b_3)\}$, $M_1 = \{j_{b_1}, j_{b_2}, j_{b_3}\}$ and $M_2 = \{j_{a_1}^1, j_{a_2}^1, j_{a_2}^2\}$; the utility of an agent for an item is equal to 0 if there is no edge between them, to $\Phi$ if the edge is dashed, and to the edge label otherwise.

### 3.1 A Polynomial-Time Constant Factor Approximation Algorithm

Having established that the $\text{AssignTC}$ problem is computationally intractable in general, we next present an efficient constant-factor approximation algorithm: we construct an approximation-preserving reduction [Orponen and Mannila 1987] — in fact, an $S$-reduction [Crescenzi 1997] — from this problem to the $\text{BCMatching}$ problem (Definition 3.1), for which a polynomial-time approximation algorithm is known.

**Theorem 3.4.** There exists a poly-time $\frac{1}{2}$-approximation algorithm for the $\text{AssignTC}$ problem.

**Proof.** Given an instance of the $\text{AssignTC}$ problem, we define a complete bipartite graph whose nodes correspond to the sets of agents $N$ and items $M$, and give the edge joining agent-node $i$ to item-node $j$ a profit equal to the utility $u(i, j)$ for all $i \in N, j \in M$. We also give all edges joining agents of one type to items in one block the same color, so that there are $kl$ colors indexed lexicographically by pairs $(p, q) \in [k] \times [l]$; let the capacity for color $(p, q)$ be $\lambda_{pq}$. This produces, in $O(mn)$ time, an instance of $\text{BCMatching}$; the size of this instance is obviously polynomial in that of the original, and, by construction, there is a one-to-one correspondence between the sets of feasible solutions of the original
and reduced instances with each corresponding pair having the same objective value (sum of edge-profits/utilities), so that the optimal values of the instances are also equal. We can now apply the polynomial-time \( \frac{1}{2} \)-approximation algorithm introduced by Stamoulis [2014] for \textsc{BCMatching} on general weighted graphs. □

Theorem 3.4 offers a \( \frac{1}{2} \)-approximation to the \textsc{AssignTC} problem; whether a better poly-time approximation algorithm exists is left for future work.

3.2 Uniformity Breeds Simplicity: Polynomial-Time Special Cases

Our results thus far make no assumptions on agent utilities; as we now show, the \textsc{AssignTC} problem admits a poly-time algorithm under some assumptions on the utility model.

\textbf{Definition 3.5 (Type-uniformity and Block-uniformity).} A utility model \( u \) is called \textit{type-uniform} if all agents of the same type have the same utility for each item, i.e. for all \( p \in [k] \) and for all \( j \in M \), there exists \( U_{pj} \in \mathbb{R}_+ \) such that \( u(i,j) = U_{pj} \) for all \( i \in N_p \). A utility model \( u \) is called \textit{block-uniform} if all items in the same block offer the same utility to every agent; that is, for all \( q \in [l] \) and for all \( i \in N \), there exists \( U_{iq} \in \mathbb{R}_+ \) such that \( u(i,j) = U_{iq} \) for all \( j \in M_q \).

In the context of the HDB allocation problem, type uniformity implies that Singaporeans of the same ethnicity share the same preferences over apartments (perhaps due to cultural or socioeconomic factors). Cases that deal with uniform goods satisfy the block-uniformity assumption: e.g. students applying for spots in public schools or job applicants applying for multiple (identical) positions; in the HDB domain, block-uniformity captures purely location-based preferences, i.e. a tenant does not care which apartment she gets as long as it is in a specific block close to her workplace, family, or favorite public space.

\textbf{Theorem 3.6.} The \textsc{AssignTC} problem can be solved in \( \text{poly}(n,m) \) time under either a type-uniform or a block-uniform utility model.

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We prove the result for a type-uniform utility model; the result for block-uniform utilities can be similarly derived. We propose a polynomial time algorithm based on the Minimum-Cost Flow problem which is known to be solvable in polynomial time. Recall that a flow network is a directed graph $G = (V, E)$ with a source node $s \in V$ and a sink node $t \in V$, where each arc $(a, b) \in E$ has a cost $\gamma(a, b) \in \mathbb{R}$ and a capacity $\psi(a, b) > 0$ representing the maximum amount that can flow on the arc; for convenience, we set $\gamma(a, b) = 0$ and $\psi(a, b) = 0$ for all $a, b \in V$ such that $(a, b) \notin E$. Let us denote by $\Gamma$ and $\Psi$ the matrices of costs and capacities respectively defined by $\Gamma = (\gamma(a, b))_{V \times V}$ and $\Psi = (\psi(a, b))_{V \times V}$. A flow in the network is a function $f : V \times V \to \mathbb{R}_+$ satisfying:

(i) $f(a, b) \leq \psi(a, b)$ for all $a, b \in V$ (capacity constraints),
(ii) $f(a, b) = -f(b, a)$ for all $a, b \in V$ (skew symmetry), and
(iii) $\sum_{b \in V} f(a, b) = 0$ for all $a \in V \setminus \{s, t\}$ (flow conservation).

The value $v(f)$ of a flow $f$ is defined by $v(f) = \sum_{a \in V} f(s, a) = \sum_{a \in V} f(a, t)$ and its cost is given by $\gamma(f) = \sum_{(a, b) \in E} f(a, b)\gamma(a, b)$. The optimization problem can be formulated as follows. Given a value $F$, find a flow $f$ that minimizes the cost $\gamma(f)$ subject to $v(f) = F$. This optimization problem that takes as input the graph $G = (V, E)$, the matrices $\Gamma$ and $\Psi$, and the value $F$, will be denoted by MinCostFlow hereafter; given an instance $(G, \Gamma; \Psi; F)$ of the MinCostFlow problem, we let $\gamma(G, \Gamma, \Psi, F)$ be the cost of the optimal flow for that instance.

Given an instance $\mathcal{I}$ of AssignTC, we construct a flow network $G_f(V, E)$ and matrices $\Gamma_f$ and $\Psi_f$ as follows (see Figure 2 for an illustration). The node set $V$ is partitioned into layers: $V = \{s\} \cup A \cup B \cup C \cup \{t\}$. $A$ is the agent type layer: there is one node $a_p \in A$ for all agent types $N_p, p \in [k]$. $B$ is the type-block layer: it has a node $b_{pq} \in B$ for every type-block pair $(p, q) \in [k] \times [l]$. Finally, $C$ is the item layer: there is one node $c_j \in C$ for all items $j \in M$. The arcs in $E$ are as follows: for every $a_p \in A$, there is an arc from $s$ to $a_p$ whose capacity $\psi(s, a_p) = |N_p|$. Fixing $p \in [k]$, there is an arc from $a_p \in A$ to every $b_{pq} \in B$, where the capacity of $(a_p, b_{pq})$ is the quota for type $N_p$ in block $M_q$ (i.e., $\psi(a_p, b_{pq}) = \lambda_{pq}$). Finally, given $q \in [l]$, there is an arc from $b_{pq}$ to $c_j$ iff $j \in M_q$; in that case, we have $\psi(b_{pq}, c_j) = 1$. The costs associated with arcs from $B$ to $C$ (i.e. arcs of the form $(b_{pq}, c_j)$ where $j \in M_q$) are $-U_{pq}$; recall that $U_{pq}$ is the utility that every agent of type $N_p$ assigns to item $j$. All other arc costs are set to 0. We begin by proving a few technical lemmas on the above network.

Given a positive integer $F$, there exists an optimal flow that is integer-valued since $(G_f, \Gamma_f; \Psi_f; F)$ is integer-valued as well. Let $f^*$ be an integer-valued optimal flow, taken over all possible values of $F$; that is:

$$f^* \in \arg\min_{F \in [n]} v(G_f, \Gamma_f, \Psi_f, F)$$

(6)

Finding the flow $f^*$ involves solving $n$ instances of MinCostFlow by definition; thus, one can find $f^*$ in polynomial time. Given $f^*$ as defined in (6), let $X^* = (x_{ij}^*)_{n \times m}$ be defined as follows: for every item $j \in M_q$, if $f^*(b_{pq}, c_j) = 1$ for some $p \in [k]$, then we choose an arbitrary unassigned agent $i \in N_p$ and set $x_{ij}^* = 1$.

**Lemma 3.7.** $X^*$ is a feasible solution of the AssignTC instance $\mathcal{I}$.

**Proof.** First, we assign at most one item to every agent by construction; next, let us show that each item $j \in M_q$ is assigned to at most one agent. Since $f^*$ is a flow, we have $\sum_{p=1}^{k} f^*(b_{pq}, c_j) = f^*(c_j, t)$ due to flow conservation; note that the capacity of the arc $(c_j, t)$ is 1, thus at most one arc $(b_{pq}, c_j)$ has $f^*(b_{pq}, c_j) = 1$. Finally, since item $j$ is assigned to an agent in $N_p$ iff $f^*(b_{pq}, c_j) = 1$, we conclude that item $j$ is assigned to at most one of the agents in $N$. 

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Next, let us prove that assignment $X^*$ satisfies the type-block constraints; in other words, we need to show that:

$$\sum_{i \in N_p} \sum_{j \in M_q} x_{ij}^* \leq \lambda_{pq}, \forall p \in [k], \forall q \in [l]$$  \hspace{1cm} (7)

Since $f^*$ is a flow, we have $f^*(a_p, b_{pq}) = \sum_{j \in M_q} f^*(b_{pq}, c_j)$ for every type-block pair $(p, q) \in [k] \times [l]$ due to flow conservation; moreover, we have $f^*(a_p, b_{pq}) \leq \psi(b_{pq}, c_j) = \lambda_{pq}$ by construction. As a consequence, we necessarily have $\sum_{j \in M_q} f^*(b_{pq}, c_j) \leq \lambda_{pq}$ for all $p \in [k]$. Since an item $j \in M_q$ is matched with some agent $i \in N_p$ if and only if we have $f^*(b_{pq}, c_j) = 1$, we conclude that (7) indeed holds. \hfill \square

Now, let us establish a relation between the cost of $f^*$ and the utility of the feasible assignment $X^*$.

**Lemma 3.8.** The cost of the flow $f^*$ satisfies $\gamma(f^*) = -u(X^*)$.

**Proof.** By construction, the cost of $f^*$ can only be induced by arcs from nodes in $B$ to nodes in $C$, where the cost of all arcs of the form $(b_{pq}, c_j)$, with $j \in M_q$, is equal to $-U_{pj}$ (the negative of the uniform utility derived from item $j$ by members of $N_p$). In other words, the cost of $f^*$ can be written as follows:

$$\gamma(f^*) = - \sum_{p=1}^{k} \sum_{q=1}^{l} \sum_{j \in M_q} f^*(b_{pq}, c_j) U_{pj}$$

As previously argued, we have that $f^*(b_{pq}, c_j) \in \{0, 1\}$ for all arcs $(b_{pq}, c_j)$; moreover, $f^*(b_{pq}, c_j) = 1$ iff item $j$ is assigned to some agent in $N_p$. Therefore, we obtain:

$$\gamma(f^*) = - \sum_{p=1}^{k} \sum_{i \in N_p} \sum_{j \in M_q} x_{ij}^* U_{pj} = - \sum_{i \in N} \sum_{j \in M} x_{ij}^* u(i, j) = -u(X^*)$$

where the second equality holds since all agents in $N_p$ have the same utility by assumption. \hfill \square

Finally, we show that for every feasible solution to the ASSIGNTC instance $I$, there exists a flow with a matching cost.

**Lemma 3.9.** Let $X$ be a feasible assignment for the ASSIGNTC instance $I$; there exists some feasible flow $f$ such that $\gamma(f) = -u(X)$. Moreover, we have $v(f) = |\{i \in N : \sum_{j \in M} x_{ij} = 1\}|$.

**Proof.** Given a feasible assignment $X = (x_{ij})_{n \times m}$, we define $f : V \times V \to \mathbb{R}_+$ as follows:

\[
\begin{align*}
  f(s, a_p) &= \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} \quad \forall a_p \in A \\
  f(a_p, b_{pq}) &= \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} \quad \forall (a_p, b_{pq}) \in E \\
  f(b_{pq}, c_j) &= \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} \quad \forall (b_{pq}, c_j) \in E \\
  f(c_j, t) &= \sum_{i \in N} \sum_{j \in M_q} x_{ij} \quad \forall c_j \in C \\
  f(a, b) &= -f(b, a) \quad \forall (a, b) \in E \\
  f(a, b) &= 0 \quad \forall (a, b) \not\in E
\end{align*}
\]

The function $f$ is indeed a flow: $f$ trivially satisfies the skew symmetry condition by construction; next, we show that $f$ satisfies flow conservation. For all $a_p \in A$, the incoming flow to node $a_p$ from node $s$ is $f(s, a_p) = \sum_{i \in N_p} \sum_{j \in M_q} x_{ij}$, and the outgoing flow to every $b_{pq}$ is $\sum_{q=1}^{l} f(a_p, b_{pq}) = \sum_{i \in N_p} \sum_{j \in M_q} x_{ij}$. Since $M$ is partitioned into $M_1, \ldots, M_l$, hence flow is conserved. For a node $b_{pq} \in B$, the incoming flow equals $f(a_p, b_{pq}) = \sum_{i \in N_p} \sum_{j \in M_q} x_{ij}$ and an amount of
we have \( f(b_{pq}, c_j) = \sum_{i \in N_p} x_{ij} \) flows to every node \( c_j \) such that \( j \in M_q \), thus flow is conserved. For a node \( c_j \in C \) such that \( j \in M_q \), its incoming flow equals \( f(b_{pq}, c_j) = \sum_{i \in N_p} x_{ij} \) from every \( b_{pq} \), for a total flow of \( \sum_{p=1}^{K} \sum_{i \in N_p} x_{ij} \), which equals its outgoing flow to \( t \). To conclude, \( f \) satisfies flow conservation.

Now let us prove that \( f \) satisfies the capacity constraints (i.e. \( f(a, b) \leq \psi(a, b) \) for all arcs \( (a, b) \in E \)). For all \( (s, a_p) \in E \), we have \( f(s, a_p) = \sum_{i \in N_p} \sum_{j \in M} x_{ij} \leq |N_p| = \psi(s, a_p) \) since every agent \( i \in N_p \) is matched with at most one item. For all \( (a_p, b_{pq}) \in E \), we have \( f(a_p, b_{pq}) = \sum_{i \in N_p} \sum_{j \in M_i} x_{ij} \leq \lambda_{pq} = \psi(a_p, b_{pq}) \) since \( X \) satisfies the type-block constraints. For all arcs \( (b_{pq}, c_j) \in E \), we have \( f(b_{pq}, c_j) = \sum_{i \in N_p} x_{ij} \leq 1 = \psi(b_{pq}, c_j) \) since item \( j \) is matched with at most one of the agents in \( N_p \). For all \( (c_j, t) \in E \), we have \( f(c_j, t) = \sum_{i \in N} x_{ij} \leq 1 = \psi(c_j, t) \) since item \( j \) is matched with at most one of the agents in \( N \). Hence, \( f \) satisfies the capacity constraints and is a valid flow. Note that we have:

\[
v(f) = \sum_{a \in V} f(s, a) = \sum_{p=1}^{k} f(s, a_p) = \sum_{p=1}^{k} \sum_{i \in N_p} \sum_{j \in M} x_{ij} = \sum_{i \in N} \sum_{j \in M} x_{ij}
\]

Then, since \( X \) is a feasible assignment of the ASSIGNTC instance \( I \), we conclude that we have \( v(f) = |\{i \in N : \sum_{j \in M} x_{ij} = 1\}| \). We just need to prove that we have \( \gamma(f) = -u(X) \), and we are done. By definition of the flow network, only arcs of the form \((b_{pq}, c_j)\) contribute to the cost \( \gamma(f) \) and we have \( \gamma(b_{pq}, c_j) = -U_{pj} \); therefore, \( \gamma(f) = -\sum_{i \in N_p} x_{ij} \) (by definition of \( f \)) and \( u(i, j) = U_{pj} \) for all agents \( i \in N_p \) (by hypothesis), we finally obtain \( \gamma(f) = -\sum_{j \in M} \sum_{p=1}^{k} \sum_{i \in N_p} x_{ij} u(i, j) = -\sum_{j \in M} \sum_{i \in N} x_{ij} u(i, j) = -u(X) \). \( \square \)

We are now ready to prove Theorem 3.6.

PROOF OF THEOREM 3.6. We begin by observing the flow \( f^* \) as defined in (6), and the assignment \( X^* \) derived from it. First, according to Lemma 3.7, \( X^* \) is a feasible assignment of the ASSIGNTC instance \( I \). Moreover, we have \( u(X^*) = -\gamma(f^*) \) according to Lemma 3.8. Finally, for any feasible assignment \( X \) of the ASSIGNTC instance \( I \), there exists a flow \( f \) such that \( \gamma(f) = -u(X) \); furthermore, since \( v(f) = |\{i \in N : \sum_{j \in M} x_{ij} = 1\}| \in [n] \), flow \( f \) is a feasible
solution of the MinCostFlow instance \((G_f; \Gamma_f; \Psi_f; F)\) for some \(F \in [n]\). Therefore, we have:
\[
\gamma(f) \leq \gamma(G_f, \Gamma_f, \Psi_f, \nu,f) \leq \gamma(f^*) = u(X^*)
\]
Thus, \(X^*\) is an optimal solution of the AssignTC instance \(I\); since \(X^*\) can be computed in poly-time (Proposition 6), we are done.

4 THE PRICE OF DIVERSITY

We now turn to the allocative efficiency of the constrained assignment. As before, an instance of the AssignTC problem is given by a set of \(n\) agents \(N\) partitioned into types \(N_1, \ldots, N_k\), a set of \(m\) items \(M\) partitioned into \(M_1, \ldots, M_l\), a list of capacity values \((\lambda_{pq})_{k \times l}\), and agent utilities for items given by \(u = (u(i,j))_{n \times m}\). We denote the set of all assignments \(X\) of items to agents satisfying only the matching constraints (3-5) of Section 2 by \(X\), and that of all assignments additionally satisfying the type-block constraints (2) by \(X_C\); the corresponding optimal social welfares for any given utility matrix \((u(i,j))_{n \times m}\) are:
\[
OPT(u) \triangleq \max_{X \in X} u(X); \quad OPT_C(u) \triangleq \max_{X \in X_C} u(X).
\]
Clearly, \(OPT_C(u) \leq OPT(u)\) since \(X_C \subseteq X\); we define the following natural measure of this welfare loss that lies in \([1, \infty]\):

**Definition 4.1.** For any instance of the AssignTC problem, we define the Price of Diversity as follows, along the lines of Ahmed et al. [2017] and Bredereck et al. [2018]:
\[
PoD(u) \triangleq \frac{OPT(u)}{OPT_C(u)}.
\]
The main result of this section is to establish an upper bound on \(PoD(u)\) that is independent of the utility model. Denote the ratio of a type-block capacity to the size of the corresponding block by:
\[
\alpha_{pq} \triangleq \frac{\lambda_{pq}}{|M_q|}.
\]

**Theorem 4.2.** For any instance of AssignTC, we have:
\[
PoD(u) \leq \frac{1}{\min_{(p,q) \in [k] \times [l]} \alpha_{pq}}
\]
and the above upper bound is tight.

In general, the bound in Theorem 4.2 grows linearly in \(m\) (e.g. if the capacities \(\lambda_{pq}\) are fixed constants). However, type-block capacities are determined by a central planner in our model; a natural way of setting them is to fix the proportional capacities or quotas \(\alpha_{pq}\) in advance, and then compute \(\lambda_{pq} = \alpha_{pq} \times |M_q|\) when block sizes become available: by committing to a fixed minimum type-block quota \(\alpha^*\) (i.e. \(\alpha_{pq} \geq \alpha^*\) for all \((p,q) \in [k] \times [l]\)), the planner can ensure a \(PoD(u)\) of at most \(1/\alpha^*\), regardless of the problem size and utility function. Higher values of \(\alpha^*\) reduce the upper bound on \(PoD(u)\) but also increase the capacity of a block for every ethnicity, potentially affecting the diversity objective adversely: it thus functions as a tunable tradeoff parameter between ethnic integration and worst-case welfare loss. In fact, in the Singapore allocation problem, the Ethnic Integration Policy fixes a universal percentage cap for each of the three ethnicities in all blocks; these percentages are set slightly higher than the actual respective population proportions: the current block quotas \(\alpha_{pq}\) are 0.87 for Chinese, 0.25 for Malays and 0.15 for Indian/Others [Deng et al. Manuscript submitted to ACM]
This bound makes no assumptions on agent utilities; in other words, it holds under any utility model.\footnote{In practice, the effective value of each fractional capacity $\lambda_{pq}/|M_q|$ might be smaller than the corresponding pre-specified fraction $\alpha_{pq}$. Since each $\lambda_{pq}$ must be an integer, we need to set $\lambda_{pq} = \lfloor \alpha_{pq} \times |M_q| \rfloor$, which depends on the $|M_q|$-values and may be higher than $\lfloor \alpha_{pq} \times |M_q| \rfloor$. For example, if we have a uniform block size of 10, the actual numerical capacities for Chinese, Malay, and Indian/Others based on EIP quotas become 8, 2, and 1 respectively, so that the effective PoD bound is 10. However, the effective bound is still independent of utility values as well as the population size of agents of any type; moreover, larger block sizes reduce the discrepancy between the effective bound and the theoretical bound $1/\min_{p,q} \alpha_{pq}$.}

The price of diversity in assignment problems holds under any utility model.\footnote{In practice, the effective value of each fractional capacity $\lambda_{pq}/|M_q|$ might be smaller than the corresponding pre-specified fraction $\alpha_{pq}$. Since each $\lambda_{pq}$ must be an integer, we need to set $\lambda_{pq} = \lfloor \alpha_{pq} \times |M_q| \rfloor$, which depends on the $|M_q|$-values and may be higher than $\lfloor \alpha_{pq} \times |M_q| \rfloor$. For example, if we have a uniform block size of 10, the actual numerical capacities for Chinese, Malay, and Indian/Others based on EIP quotas become 8, 2, and 1 respectively, so that the effective PoD bound is 10. However, the effective bound is still independent of utility values as well as the population size of agents of any type; moreover, larger block sizes reduce the discrepancy between the effective bound and the theoretical bound $1/\min_{p,q} \alpha_{pq}$.}

The proof relies on the following lemma. Given an assignment $X \in X_C$, let $u_p(X)$ denote the total utility of agents in $N_p$ under $X$:

$$u_p(X) \triangleq \sum_{i \in N_p} \sum_{j \in M} x_{ij} u(i,j) = \sum_{q \in [1]} \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j). \quad (8)$$

**Lemma 4.3.** For any instance of AssignTC and any optimal unconstrained assignment $X^* \in X$, we have:

$$\text{PoD}(u) \leq \frac{\sum_{p \in [k]} u_p(X^*) \min_{q \in [l]} \alpha_{pq}}{\sum_{p \in [k]} u_p(X^*) \min_{q \in [l]} \alpha_{pq}}.$$ \n
**Proof.** Based on the optimal assignment $X^*$, we can construct an assignment $X \in X_C$ satisfying the type-block constraints, by carefully ‘revoking’ the smallest-utility items in $M_q$ from agents in $N_p$ for every $(p,q)$-pair that violates the corresponding type-block constraint. In other words, let $n_{pq}$ denote the number of items in $M_q$ assigned to agents in $N_p$ under $X^*$. If $n_{pq} \leq \lambda_{pq}$, we leave that type-block pair untouched, so that $\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j) = \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j)$ and $\lambda_{pq}$ is given by $\lambda_{pq} = \min_{q \in [l]} (\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j))$. If $n_{pq} > \lambda_{pq}$, we order these $n_{pq}$ agents according to their utilities for the items they are assigned and retain only the top $\lambda_{pq}$ agents in that order (breaking ties lexicographically), setting $x_{ij} = 0$ for the remaining agents. This change increases the average utility of assignments for this type-block pair:

$$\frac{\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j)}{\lambda_{pq}} \geq \frac{\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j)}{n_{pq}}.$$ \n
(To see why this is true, consider a sequence $z_1 \geq z_2 \geq \ldots$, where $z_i \geq 0$ and $z_i \geq z_{i+1}$ for all $i \geq 1$. Clearly, $(v - \mu) \sum_{i=1}^{\mu} z_i \geq (v - \mu) z_\mu \geq \mu \sum_{i=1}^{\mu} z_i$ since $z_i \geq z_\mu$ for all $i \geq 1$. Rearranging and simplifying, we get $\frac{1}{\mu} \sum_{i=1}^{\mu} z_i \geq \frac{1}{\mu} \sum_{i=1}^{\mu} z_i$.)

Further, since $n_{pq} \leq |M_q|$, the above inequality implies that

$$\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j) \geq \lambda_{pq} \frac{|M_q|}{|M_q|} \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j) \geq \alpha_{pq} \sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j), \quad \text{since } \alpha_{pq} = \frac{\lambda_{pq}}{|M_q|}.$$
Thus, for every $p \in [k]$ and every $q \in [l]$, we have
\[
\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j) \geq \min_{q \in [l]} \alpha_{pq} \sum_{i \in N_p} \sum_{j \in M_q} x_{ij}^* u(i,j),
\]
since $\min_{q \in [l]} \alpha_{pq} \leq \alpha_{pq} \leq 1$.

Summing over blocks, we obtain from Definition (8):
\[
\sum_{i \in N_p} \sum_{j \in M_q} x_{ij} u(i,j) \geq \min_{q \in [l]} \alpha_{pq}.\]

By definition, $u(X^*) = OPT(u)$. Moreover, since $X \in X_C$, we have $u(X) \leq OPT_C(u)$. Hence, by Definition 4.1,
\[
\text{PoD}(u) \leq \frac{u(X^*)}{u(X)} \leq \frac{u(X^*)}{\sum_{p \in [k]} u_p(X^*) \min_{q \in [l]} \alpha_{pq}}.
\]

We can now complete the proof of the theorem.

**Proof of Theorem 4.2.** Since we have $\min_{p,q \in [k] \times [l]} \alpha_{pq} \leq \min_{q \in [l]} \alpha_{pq}$ for all $p' \in [k]$, Lemma 4.3 implies that:
\[
\text{PoD}(u) \leq \frac{u(X^*)}{\sum_{p \in [k]} u_p(X^*) \min_{q \in [l]} \alpha_{pq}} = \frac{1}{\min_{(p,q) \in [k] \times [l]} \alpha_{pq}}.
\]

Depending on the utility matrix $u$, this upper bound can be tight whenever $|N_{p_0}| \geq |M_{q_0}|$ for some type-block pair $(p_0, q_0)$ in the set $\text{argmin}_{(p,q) \in [k] \times [l]} \alpha_{pq}$. We identify an agent utility matrix for which the bound holds with equality:
\[
u(i,j) = \begin{cases} 1 & \text{if } i \in N_{p_0} \text{ and } j \in M_{q_0}, \\ 0 & \text{otherwise}. \end{cases}\]

The optimal assignment without type-block constraints fully allocates the items in block $M_{q_0}$ to agents in $N_{p_0}$, for a total utility of $|M_{q_0}|$; furthermore, we know that any optimal constrained assignment allocates exactly $\lambda_{p_0,q_0}$ items in $M_{q_0}$ to agents in $N_{p_0}$ for a total utility of $\lambda_{p_0,q_0}$. Since $\lambda_{p_0,q_0} = \alpha_{p_0,q_0} \times |M_{q_0}|$, we have:
\[
\text{PoD}(u) = \frac{|M_{q_0}|}{\alpha_{p_0,q_0} \times |M_{q_0}|} = \frac{1}{\alpha_{p_0,q_0}} = \frac{1}{\min_{(p,q) \in [k] \times [l]} \alpha_{pq}}.
\]

\[\square\]

### 4.1 The Impact of Disparity among Types

Theorem 4.2 offers a worst-case tight bound on the price of diversity, making no assumptions on agent utilities. However, its proof suggests that this upper bound is attained when social welfare is solely extracted from a single agent type and a single block. Intuitively, we can obtain a better bound on the price of diversity if a less ‘disparate’ optimal assignment exists. To formalize this notion, we introduce a new parameter:

**Definition 4.4.** For an optimal unconstrained assignment $X^* \in X$, denote by $\beta_p(X^*)$ the ratio of the average utility of agents in $N_p$ to the average utility of all agents under $X^*$. The *inter-type disparity parameter* $\beta(X^*)$ is defined as:
\[
\beta(X^*) = \min_{p \in [k]} \beta_p(X^*) = \min_{p \in [k]} \frac{u_p(X^*)/|N_p|}{u(X^*)/n}.
\]

Notice that $\beta(X^*) \in (0, 1]$ can be computed in polynomial time and is fully independent of the type-block capacities. The closer $\beta(X^*)$ is to 1, the lower the disparity between average agents of different types under $X^*$. 

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Theorem 4.5. For any AssignTC instance and any unconstrained optimal assignment \(X^* \in X\), we have:

\[
\text{PoD}(u) \leq \frac{1}{\beta(X^*)} \sum_{p \in [k]} v_p \min_{q \in [l]} \alpha_{pq},
\]

where \(v_p = \frac{|N_p|}{n}\) is the proportion of type \(p\) in the agent population, for every \(p \in [k]\).

Proof. By definition of \(\beta(X^*)\), for every \(p \in [k]\), we have:

\[
u_p(X^*) \geq \beta(X^*) \frac{|N_p|}{n} u(X^*) = \beta(X^*) v_p u(X^*),\]

Substituting this in Lemma 4.3, we obtain the desired bound. □

Let us now apply the result to the Singapore public housing domain; we use the ethnic proportions reported in the 2010 census report [Department of Statistics, Singapore 2010] to obtain \(|N_1|/n = 0.741\) (Chinese), \(|N_2|/n = 0.134\) (Malay), and \(|N_3|/n = 0.125\) (Indian/Others). Using the same block quotas \(\alpha_{pq}\) as before and assuming \(\beta(X^*) = 1\), we have:

\[
\text{PoD}(u) \leq \frac{1}{0.87 \times 0.741 + 0.25 \times 0.134 + 0.15 \times 0.125} \approx 1.43.
\]

Combining Theorems 4.2 and 4.5, we obtain the following upper bound on the price of diversity of any instance of AssignTC:

\[
\text{PoD}(u) \leq \min \left( \frac{1}{\min_{p,q \in [k] \times [l]} \alpha_{pq}} \cdot \frac{1}{\beta(X^*)} \cdot \sum_{p \in [k]} v_p \min_{q \in [l]} \alpha_{pq} \right). \tag{9}
\]

Thus, if we plot the \(\text{PoD}(u)\) against the disparity parameter \(\beta(X^*)\), the point corresponding to any AssignTC instance with above block quotas and ethnic proportions (from data on Singapore) must lie in the shaded region of Figure 3.

![Fig. 3. PoD vs disparity parameter for the HDB problem for ethnic proportions \(|N_1|/n = 0.741\) (Chinese), \(|N_2|/n = 0.134\) (Malay), and \(|N_3|/n = 0.125\) (Indian/Others), and corresponding quotas \(\alpha_{1q} = 0.87, \alpha_{2q} = 0.25\) and \(\alpha_{3q} = 0.15\) for every block \(M_q\).](image)

5 EXPERIMENTAL ANALYSIS

In this section, we simulate instances of the AssignTC problem using recent, publicly available Singaporean demographic and housing allocation statistics and Chicago public school admission data. We compare the welfare of three assignment mechanisms: the optimal unconstrained mechanism, the optimal constrained mechanism, and the lottery-based mechanism (see Section 5.1 below). Both the unconstrained and constrained social welfare maximizations are solved using the Gurobi Optimizer. We refer the reader to https://git.io/fNhlm for full implementation details.
5.1 The Lottery Mechanism

Sections 2 and 3 study an optimal mechanism for assigning goods to agents under diversity constraints. To the best of our knowledge, this mechanism is not used for allocating goods in practice; rather, lotteries are used to allocate items in both real-world instances that inspire this work. The mechanisms randomly order agents, and let each agent pick their favorite item in turn, while respecting predetermined quotas. In this section, we formulate a simple one-shot lottery-based mechanism that captures the aspect of the problems described in Sections 1.3 and 1.4 that we are most interested in: the impact of type-block constraints (as defined by inequalities (2)) on assignment by lottery. Algorithm 1 is not the actual mechanism used in Singapore public housing or Chicago school choice (see the discussion in the respective sections). It is easy to see that the algorithm takes \( \text{poly}(mn) \) time to run.

**Algorithm 1:** Lottery Mechanism for Assignment with Type-Block Constraints

| Input: | Agents \( N \) grouped into types \( N_1, N_2, \ldots, N_k \); items \( M \) split into blocks \( M_1, M_2, \ldots, M_l \); type-block capacities \( \lambda_{pq} \forall (p, q) \in [k] \times [l] \); utility matrix \( (u(\text{i}, j))_{n \times m} \). |
|--------|----------------------------------------------------------------------------------|
| Initialize: | Allocation matrix \( X = (x_{ij})_{n \times m} \leftarrow (0)_{n \times m} \); remaining agents \( N_{\text{rem}} \leftarrow N \); unassigned items \( M_{\text{rem}} \leftarrow M \). |
| for \( t \in \{1, 2, \ldots, n\} \) do |
| Draw agent uniformly at random: \( \text{i}_t \sim \mathcal{U}(N_{\text{rem}}) \). |
| Find type of \( \text{i}_t : p_t \leftarrow p \in [k] \) s.t. \( \text{i}_t \in N_p \). |
| Find blocks that have not hit capacity for type \( p_t \): \( Q_t \leftarrow \{ q \in [l] : \sum_{j \in N_{\text{pq}}} \sum_{i \in M_q} x_{ij} < \lambda_{pq} \} \). |
| Find available items: \( M_t \leftarrow M_{\text{rem}} \cap (\bigcup_{q \in Q_t} M_q) \). |
| if \( M_t \neq \emptyset \) then |
| Assign to \( \text{i}_t \) available item for which she has highest utility, breaking ties lexicographically: |
| \( \text{j}_t \leftarrow \arg \max_{j \in M_t} u(\text{i}_t, j) \). |
| \( x_{\text{i}_t \text{j}_t} \leftarrow 1 \). |
| \( M_{\text{rem}} \leftarrow M_{\text{rem}} \setminus \{j_t\} \). |
| end |
| \( N_{\text{rem}} \leftarrow N_{\text{rem}} \setminus \{\text{i}_t\} \). |
| end |
| return \( X \). |

5.2 The Singapore Public Housing Allocation Problem

**Data Collection.** In order to create realistic instances of the \textsc{AssignTC} problem within the Singaporean context, we collected data on the location and number of flats of recent HDB housing development projects advertised over the second and third quarters of 2017\(^4\). Each of these developments corresponds to a block in our setup, for a total of \( m = 1350 \) flats partitioned into \( l = 9 \) blocks (a detailed map is given in Figure 4). Moreover, each flat in any of these blocks belongs to one of several pre-specified categories, viz. 2-room flexi, 3-room, 4-room, and 5-room; our data set includes lower and upper bounds, \( LB(t, q) \) and \( UB(t, q) \) respectively, on the monthly cost (loan) for a flat of category \( t \) in block \( M_q \) for every \( t \) and \( q \). Then, we consider a pool of \( n \in \{1350, 3000\} \) applicants whose ethnic composition follows the 2010 Singapore census report [Department of Statistics, Singapore 2010]: we have \( |N_1| = 1000 \approx 74.1\% \) Chinese), \(|N_2| = 180 \approx 13.4\% \) Malay and \(|N_3| = 170 \approx 12.5\% \) Indian/Others) for \( n = 1350 \) and we have \(|N_1| = 2223, |N_2| = 402 \) and \(|N_3| = 375 \) for \( n = 3000 \). From the 2010 Singapore census report, we also collected the average salary \( S(p) \) of each ethnicity group \( p \in \{k\} \), given in Singapore dollars: \( S(1) = 7,326, S(2) = 4,575 \) and \( S(3) = 7,664 \). Finally,\(^4\)

\(^4\)http://www.hdb.gov.sg/cs/infoweb/residential/buying-a-flat/new/bto-shf

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we use a uniform block capacity using the latest HDB block quotas [Deng et al. 2013]: for every block $M_q$, we have $\alpha_{1q} = 0.87, \alpha_{2q} = 0.25$ and $\alpha_{3q} = 0.15$.

Utility Models. All parameters used to generate AsstToTC instances in our simulations are based on real data, except for agent utilities over apartments. Conducting large-scale surveys that elicit user preferences over apartments is beyond the scope of this work; thus, we base our agent utility models on simulated utilities. We examine three utility models, each characterized by a variance parameter $\sigma^2$ whose value does not come from the data: distance-based ($\text{Dist}(\sigma^2)$), type-based ($\text{Type}(\sigma^2)$) and price-based ($\text{Price}(\sigma^2)$).

- In the distance-based utility model, each agent $i \in N$ has a preferred geographic location $\widehat{a}_i \in \mathbb{R}^2$ (chosen uniformly at random within the physical landmass of Singapore) that she would like to live as close as possible to (say, the location of her parents’ apartment, workplace or preferred school). For every block $M_q$, the utility agent $i$ derives from apartment $j \in M_q$ is generated according to the normal distribution $\mathcal{N}(1/d(\widehat{a}_i, \text{loc}(M_q)), \sigma^2)$; here, $\text{loc}(M_q)$ is the geographical location of block $M_q$, and $d(\cdot)$ is the Euclidean distance between $\widehat{a}_i$ and $\text{loc}(M_q)$.

- In the type-based utility model, we assume that all agents of the same type (i.e. ethnic group) have the same preferred location (i.e. $\forall p \in [k], \forall i, i' \in N_p, \widehat{a}_i = \widehat{a}_{i'}$).

- In the price-based utility model, each agent $i \in N_p$ has a salary $s_i$ that is generated according to the normal distribution $\mathcal{N}(S(p), \sigma^2)$. Each flat $j \in M_q$ of category $t$ has a monthly cost $p_j$ that is chosen uniformly in $[\text{LB}(t, q), \text{UB}(t, q)]$. The utility that agent $i$ derives from flat $j$ is then defined by

\[
 u(i, j) = \frac{1}{(p_j - \frac{\sigma^2}{4})^2},
\]

assuming that agent $i$ is willing to pay one-third of her monthly salary on mortgage installments\(^{15}\); the rationale for the utility formula is that a much higher cost relative to the budget makes the flat unaffordable while a much lower cost indicates unsatisfactory quality, making the agent unhappy in both scenarios.

Evaluation. First, we want to compare the distance-based utility model ($\text{Dist}(\sigma^2)$) and the type-based model ($\text{Type}(\sigma^2)$) in order to estimate the welfare loss due to imposing ethnicity constraints. To do so, we vary both $\sigma^2$ in $\{1, 5, 10\}$ and $n \in \{1350, 3000\}$; the results reported in Figure 5 and Figure 6 are on average performance over 100 randomly generated instances. For each treatment, we report $\text{PoD}(u)$ the price of diversity (green), the theoretical upper bound on $\text{PoD}(u)$ as

\(^{15}\)The choice of the one-third fraction is inspired by the “3-3-5 rule” for deciding whether one can afford a flat given one’s income (https://www. areyouready.sg/YourInfoHub/Pages/News-How-to-use-the-3-3-5-rule-to-consider-if-you-can-afford-your-new-home.aspx), endorsed by the Central Provident Fund Board of Singapore (https://www.cpf.gov.sg/members)
Thus, these numerical tests show that the lottery mechanism may perform rather poorly as compared to the optimal constrained allocation mechanism, even in allocation problems with a very low price of diversity.

The welfare loss induced by the lottery mechanism exceeds the theoretical upper bound on \( \text{PoD} \) when the ethnicity group have identical preferences when \( \sigma = 0 \); for instance, it only extracts 65% of the optimal unconstrained welfare under the \( \text{Dist}(1) \) with \( n = 3000 \). However, we observe that the welfare loss induced by the lottery mechanism is negatively impacted by the number of agents (see Figure 6); for instance, it only extracts 65% of the optimal unconstrained welfare under the \( \text{Dist}(1) \) with \( n = 3000 \).

Now, we focus on the price-based utility model (\( \text{Price}(\sigma^2) \)) and we vary \( \sigma^2 \) in \( \{0, 10, 50\} \); the results obtained by averaging over 100 runs are given in Figure 7. While the price of diversity is almost equal to one in all instances, the welfare loss observed with the lottery mechanism drastically increases with \( \sigma^2 \) (recall that agents from the same ethnicity group have identical preferences when \( \sigma^2 = 0 \)); for instance, it extracts 98% of the optimal unconstrained welfare under \( \text{Price}(0) \) while it only extracts 35% of this value under \( \text{Price}(50) \). Moreover, for \( \text{Price}(50) \), we observe that the welfare loss induced by the lottery mechanism exceeds the theoretical upper bound on \( \text{PoD}(u) \) in some instances. Thus, these numerical tests show that the lottery mechanism may perform rather poorly as compared to the optimal constrained allocation mechanism, even in allocation problems with a very low price of diversity.
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Fig. 7. Averaged utility losses obtained for $\text{Price}(\sigma^2)$ with $n = 1350$ in our simulated instances of the Singapore public housing allocation problem.

5.3 Chicago Public School Admissions

Data Collection. To create realistic instances, we collected public data from the Chicago public schools webpage\[^{16}\]. More precisely, we have first collected data on the location of the magnet schools in Chicago; we focus on these schools because they use a lottery mechanism to select students. We collect the total number of students enrolled in these schools in 2018; this number was then divided by 9 to obtain an approximation of the number of students that can be accepted in the first grade (there are 9 grades in total). This leads us to instances with $l = 37$ blocks (the schools) and $m = 2261$ items in total (the available spots). In this school admission problem, the students are partitioned into $k = 4$ types, viz. Tiers 1, 2, 3 and 4, depending on their residence (see Figure 8). In our experiments, we consider a pool of $n = 2261, 5000$ students whose type composition follows the actual proportion: we have $|N_1| = 613 (\approx 27.1\% \text{ Tier 1}), |N_2| = 622 (\approx 27.5\% \text{ Tier 2}), |N_3| = 533 (\approx 23.6\% \text{ Tier 3})$ and $|N_4| = 493 (\approx 21.8\% \text{ Tier 4})$ for $n = 2261$ and we have $|N_1| = 1355, |N_2| = 1375, |N_3| = 1180$ and $|N_4| = 1090$ for $n = 5000$. Finally, we use the same type-block capacities $\lambda_{pq, p, q} \in [k] \times [l]$ as in the Chicago public school admission problem, which are simply defined by $\lambda_{pq} = 0.25|M_q|$.

Utility model. Since we do not have access to students’ utilities over schools, we base our utility model on simulated utilities. Similarly to our experiments on Singapore public housing allocation problem, we consider the distance based utility model $\text{Dist}(\sigma^2)$ with the following important changes:

- The preferred location of a student is here chosen uniformly at random within the physical landmass of her Tier (see Figure 8); the position of every polygon in Figure 8 is approximated by taking the average of the coordinates of its extreme points.
- Students have a utility of 0 for any school ranked 21st or more in her preference ordering; this is because students may apply to at most 20 schools.

Evaluation. In our experiments, we vary both $\sigma^2$ in $\{0, 10, 50\}$ and $n$ in $\{2261, 5000\}$, and for each setting, we compute the following quantities: $\text{PoD}(u)$ the price of diversity (green), the theoretical upper bound on $\text{PoD}(u)$ as per Theorem 4.5 (blue), and the relative loss of the lottery mechanism averaged over 100 agent permutations (red). Results obtained

\[^{16}\]http://cps.edu/Pages/home.aspx
by averaging over 100 runs are given in Figure 9. First, we observe that both the price of diversity and the loss of the lottery mechanism decrease as $\sigma^2$ increases; this is unsurprising since utilities are less type-dependent as $\sigma^2$ grows. However, we observe that the loss of the lottery mechanism is quite high in all instances. Moreover, just as in the Singapore public housing allocation problem, we observe that the loss of the lottery mechanism is negatively impacted by the number of students; therefore, we can conclude that this mechanism seems to be better suited to problems with an equal number of agents and items.

6 CONCLUSIONS AND FUTURE WORK

Our work constitutes a first step towards a better understanding of the effect of diversity constraints on social welfare. We offer computational insights, providing a general hardness result, sufficient conditions for tractability, and a $\frac{1}{2}$-approximation algorithm for the AssignTC problem. Our S-reduction essentially shows that AssignTC is a special (still NP-complete) case of the BCMATCHING problem and also implies some easy generalizations of our results (although the question of a better approximation remains open). For example, there is a PTAS for BCMATCHING if one allows $(1 + \epsilon)$-violations of the color constraints [Grandoni et al. 2009]; this immediately implies a PTAS for AssignTC where one allows $(1 + \epsilon)$-violations of the type-block constraints.

We derive two upper bounds on the price of diversity defined as the ratio of the optimal welfare achievable with and without type-block constraints: the first is in terms of block quotas only, independent of the utility model, hence under the planner’s control; the second is parametrized in terms of inter-type disparity, which shows that when the disparity is low, the welfare loss is much closer to its ideal value of 1 than the first bound would suggest.

We analyze our model’s behavior in simulation: the fundamental experimental framework is based on Singapore census and HDB sales data. Simulating agent utilities is still a major challenge: ideally, one would elicit applicants’
utilities directly via large-scale national surveys. Our simulations tested two ‘extreme’ cases: one where there is no correlation between ethnicity and utility, and one where utility was artificially correlated to ethnicity. The truth is likely somewhere in between. Ethnic groups in Singapore most likely do have some correlation between their utility models; this can be due to socioeconomic factors (there is some correlation between ethnicity and socioeconomic status), the location of cultural or religious centers, or other unknown factors. Developing a more refined utility model is an interesting direction for future work.

Obviously, the HDB lottery mechanism cannot offer better welfare guarantees than the allocation based on constrained optimization under known utilities; however, in our experiments, it performs surprisingly well for several utility models. Offering theoretical guarantees on the performance of the lottery mechanism with diversity constraints (and in some sense, complementing the analysis by Immorlica et al. [2017]) would provide better insights on our experimental results.

Our results describe an inevitable tradeoff between diversity and social welfare; however, we emphasize that this does not constitute a moral judgment on the authors’ part. Economic considerations are certainly important, but they are by no means an exclusive nor a first order consideration. That said, understanding the impact of diversity constraints on social welfare is key if one is to justify their implementation.

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