Recursively Feasible Data-Driven Distributionally Robust Model Predictive Control with Additive Disturbances

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Abstract—In this paper we propose a data-driven distributionally robust Model Predictive Control framework for constrained stochastic systems with unbounded additive disturbances. Recursive feasibility is ensured by optimizing over a linearly interpolated initial state constraint in combination with a simplified affine disturbance feedback policy. We consider a moment-based ambiguity set with data-driven radius for the second moment of the disturbance, where we derive a minimum number of samples in order to ensure user-given confidence bounds on the chance constraints and closed-loop performance. The paper closes with a numerical example, highlighting the performance gain and chance constraint satisfaction based on different sample sizes.

I. INTRODUCTION

Model predictive control (MPC) is an optimization-based control method capable of dealing with general constraints and performance criteria [1]. In many MPC applications, decisions must be made under uncertainty, leading to robust [2] or stochastic MPC [3] approaches. While robust MPC requires an a-priori known bound on the disturbance to satisfy hard constraints, stochastic MPC relaxes this requirement by using knowledge of the underlying moment information of the stochastic disturbance to soften the constraints as chance constraints [12]. In practice, exact moment information is rarely available and must be estimated from data, which leads to distributionally robust optimization (DRO) methods [4].

The main idea of DRO is to define a so-called ambiguity set that contains plausible variations of the empirically estimated distribution (or moment information). In the DRO literature, a rough distinction is made between moment-based and discrepancy-based sets [4], both of which are considered in several works on distributionally robust MPC (DR-MPC). E.g., the Wasserstein discrepancy measure is used in [5]–[8], the total variation discrepancy measure in [9], whereas the authors of [10], [11] rely on moment-based ambiguity sets.

Contribution: In this paper we study a data-driven approach to DR-MPC for linear systems with additive unbounded disturbances under moment-based ambiguity sets. We use a simplified disturbance feedback (SADF) control parameterization [14] with the intention to reduce the number of decision variables of the MPC optimization problem. This further allows us to replace the expected value cost function and the chance constraints with their distributionally robust surrogates, where the constraints are cast as second order cone (SOC) constraints [13]. To ensure recursive feasibility we employ a similar technique as suggested by [20], where we optimize the initial state constrained on a line between a guaranteed feasible solution (the shifted optimal solution from the previous time step) and the state feedback initialization. In Proposition II.2 we derive a minimum number of disturbance samples, such that the ambiguity set contains the true moments with high probability. In comparison to [17] we thus require fewer samples by assuming a known first moment. This enables us to derive rigorous theoretical properties, such as an expected cost decrease, an asymptotic average performance bound and satisfaction of conditional chance constraint.

Coppens and Patrinos [11] recently proposed a comparable DR-MPC framework under the assumption of bounded disturbances, where the main difference lies in the handling of the distributionally robust chance constraints and in ensuring recursive feasibility. Van Parys et. al. [10] proposed a DR-MPC framework under the assumption of bounded disturbances, where the constraints are cast as second order cone (SOC) constraints [13]. To ensure recursive feasibility we employ a similar technique as suggested by [20], where we optimize the initial state constrained on a line between a guaranteed feasible solution (the shifted optimal solution from the previous time step) and the state feedback initialization. In Proposition II.2 we derive a minimum number of disturbance samples, such that the ambiguity set contains the true moments with high probability. In comparison to [17] we thus require fewer samples by assuming a known first moment. This enables us to derive rigorous theoretical properties, such as an expected cost decrease, an asymptotic average performance bound and satisfaction of conditional chance constraint.

The paper is organized as follows. In Section II we introduce the problem setting and our result on a data-driven ambiguity radius. Section III discusses the SADF control parameterization, the treatment of the distributionally robust chance constraints and cost function, the required terminal ingredients and initial conditions for the DR-MPC problem. Section IV is devoted to the theoretical analysis of the DR-MPC problem and Section V to a numerical example. The paper closes with some concluding remarks.

II. PRELIMINARIES

A. Notation

A probability space is defined by the triplet \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is the sample space, \(\mathcal{F}\) the \(\sigma\)-algebra on \(\Omega\) and \(\mathbb{P}\) the probability measure on \((\Omega, \mathcal{F})\). Given an event \(E_1\) we define the probability occurrence as \(\mathbb{P}(E_1)\) and the conditional probability given \(E_2\) as \(\mathbb{P}(E_1|E_2)\). For a random variable \(w\), we define the expected value as \(\mathbb{E}\{w\}\), whereas the conditional
expectation of $w$ conditional to a random variable $x$ is denoted as $\mathbb{E}\{w|x\}$. The weighted 2-norm w.r.t. a positive definite matrix $Q = Q^T$ is $\|x\|_Q^2 = x^T Q x$. Positive definite and semidefinite matrices are indicated as $A > 0$ and $A \succeq 0$, respectively. The ceiling function is denoted as $\lceil \cdot \rceil$. The pseudo inverse of a matrix $A$ is denoted as $A^\dagger$.

**B. Problem setting**

In this paper, we consider a discrete linear time-invariant system

$$x(k + 1) = Ax(k) + Bu(k) + Ew(k)$$

with the state $x \in \mathbb{R}^{n_x}$, input $u \in \mathbb{R}^{n_u}$, disturbance $w \in \mathbb{R}^{n_w}$ and matrices $A$, $B$, and $E$ are of conformal dimensions. The stochastic disturbance $w(k)$ is assumed to be i.i.d. for all $k \in \mathbb{N}$ that follows an unknown distribution (push-forward measure) $\mu^*$. We assume that we have access to $i = 1, \ldots, N_s$ random samples of $w \sim \mu^*$, which we denote as $\hat{w}^i$. We further assume that the matrix pair $(A, B)$ is stabilizable, the matrix $E$ has full column rank and that perfect state measurement is available at each time instant $k$. To distinguish between closed-loop and open-loop we introduce the inverse of a matrix $A$ and $\mu^*$. The control problem of chance constraint satisfaction. Given an initial value $x(0)$, we impose chance constraints on the states and inputs

$$\mathbb{P}(h^T_{i,k} \bar{x}_k \leq 1 | x(k)) \geq p^r_t \quad t = 0, \ldots, N - 1 \quad (2a)$$

$$\mathbb{P}(l^T_{i,s} \bar{u}_k \leq 1 | x(k)) \geq p^s_t \quad t = 0, \ldots, N - 1 \quad (2b)$$

where $h^T_{i,k} \in \mathbb{R}^{(N+1)n_s}$ and $l^T_{i,s} \in \mathbb{R}^{n_u}$ denote the left-hand-side of the $i, s$, state and $s, i$, input half-space constraints and $p^r_t, p^s_t \in (0, 1)$ are the required levels of chance constraint satisfaction. Given an initial value $x(0)$ we opt to solve the following finite horizon stochastic optimal control problem

$$\min_{u_k} \mathbb{E}_{\mu^*} \left\{ \|x_{N|k}\|_P^2 + \sum_{t=0}^{N-1} \|x_{t|k}\|_Q^2 + \|u_{t|k}\|_T^2 | x(k) \right\}$$

subject to

$$x_{N|k} = \bar{A} x_{0|k} + \bar{B} \bar{u}_k + \bar{E} \bar{w}_k$$

$$\mathbb{P}(h^T_{i,k} \bar{x}_k \leq 1 | x(k)) \geq p^r_t \quad t = 0, \ldots, N - 1 \quad (3b)$$

$$\mathbb{P}(l^T_{i,s} \bar{u}_k \leq 1 | x(k)) \geq p^s_t \quad t = 0, \ldots, N - 1 \quad (3c)$$

where $Q > 0$, $R > 0$ and $P > 0$ are positive definite symmetric weighting matrices and $P$ additionally satisfies the Lyapunov inequality

$$(A + BK)^T P (A + BK) + Q + K^T R K \leq P$$

for some linear controller matrix $K \in \mathbb{R}^{n_u \times n_x}$.

Problem (3) represents an infinite dimensional optimization problem due to the control input $u$ and the additive disturbance $w$. This issue will be tackled in Section III-A by using a SADF parameterization. Furthermore, the cost function (3a) and chance constraints (3b)-(3c) are evaluated w.r.t. the true, but unknown distribution $\mu^*$. Thus, we formulate a distributionally robust optimization problem that uses a so-called ambiguity set $\mathcal{P}$, i.e. a set of probability distributions, where each $\mu \in \mathcal{P}$ lies within some distance to the sample covariance $\hat{\Sigma} = \frac{1}{N_s} \sum_{i=1}^{N_s}(\hat{w}^i(\hat{w}^i)^T)$ under the assumption that $\mathbb{E}_{\mu\{w\}} = 0$. In particular, the ambiguity set represents the uncertainty of the empirical estimator and is parameterized as in [15]

$$\mathcal{P} := \left\{ \mu \in \mathcal{M} \mid \mathbb{E}_{\mu\{w\}} = 0, \mathbb{E}_{\mu\{ww^T\}} \leq \kappa_{\beta} \hat{\Sigma} \right\}$$

where $\mathcal{M}$ denotes the set of all probability distributions defined on $(\mathbb{R}^{n_w}, B)$ with $B$ the associated Borel $\sigma$-algebra of $\mathbb{R}^{n_w}$. For some confidence level $\beta \in (0, 1)$ we define a constant $\kappa_{\beta} \geq 1$, such that $\mathbb{P}(\mu^* \in \mathcal{P}) \geq 1 - \beta$.

**C. Data-driven ambiguity set**

In the following we derive an explicit value for the constant $\kappa_{\beta}$ under the assumption of sub-Gaussianity of the random variables $w(k)$, which was similarly done by [17]. This extends the results from Delage and Ye [15], who provide an explicit value $\kappa_{\beta}$ for bounded random variables.

**Definition II.1.** A random variable $\xi$ is sub-Gaussian with variance proxy $\sigma^2$ if $\mathbb{E}\{\xi\} = 0$ and its moment generating function satisfies

$$\mathbb{E}\{e^{\lambda\xi}\} \leq e^{\frac{1}{2}\lambda^2\sigma^2} \quad \forall \lambda \in \mathbb{R}.$$ 

We denote this by $\xi \sim \text{subG}(\sigma^2)$.

**Proposition II.2.** Let $w \in \mathbb{R}^{n_w}$ be a zero-mean sub-Gaussian random variable with $\mathbb{E}\{ww^T\} = \Sigma$. Let \{\hat{w}^i\}_{i=1}^{N_s}$ be $N_s$ i.i.d. samples obtained from the true distribution of $w$ and define $\hat{\Sigma} = \frac{1}{N_s} \sum_{i=1}^{N_s}(\hat{w}^i)(\hat{w}^i)^T$ as the empirical covariance matrix. Let $\epsilon \in (0, 0.5)$, $\beta \in (0, 1)$, $c_1(\sigma, \epsilon) = \sigma^2/(1 - 2\epsilon)$, $c_2(\beta, \epsilon, n_w) = n_w \log(1+2/\epsilon)+\log(2/\beta)$, then for all $N_s \in \mathbb{N}$ satisfying

$$N_s \geq \max \left[ 2c_1 c_2 \left( 8c_1 + 4\sqrt{4c_1^2 + c_1 + 1} \right) \right],$$

the covariance bound $\Sigma \preceq \frac{1}{1-\epsilon(N_s, \beta)^2} \hat{\Sigma}$ holds with a probability of at least $1 - \beta$, where

$$\gamma(N_s, \beta) := c_1(\sigma, \epsilon) \left( \frac{32 c_2(\beta, \epsilon, n_w)}{N_s} + \frac{2c_2(\beta, \epsilon, n_w)}{N_s} \right).$$
Proof. The proof follows from [17, Thm. 8]. Define a random variable \( \xi = \Sigma^{-1/2} w \sim \text{subG}(\sigma^2) \) such that
\[
E\{\xi\} = \Sigma^{-1/2} E\{w\} = 0 \\
E\{\xi\xi^T\} = \Sigma^{-1/2} E\{ww^T\} \Sigma^{-1/2} = I
\]
and let \( \tilde{I} = N^{-1} \sum_{i=1}^{N} \tilde{\xi}_i (\tilde{\xi}_i)^T \) be the empirical covariance matrix of \( \xi \). Consider now the empirical covariance matrix \( \hat{\Sigma} = N^{-1} \sum_{i=1}^{N} \tilde{w}_i (\tilde{w}_i)^T \) of the actual random variable \( w \), which, after substitution of \( \tilde{w}_i \) equals \( \Sigma^{1/2} \xi \) equals
\[
\hat{\Sigma} = \Sigma^{1/2} \left( N^{-1} \sum_{i=1}^{N} \tilde{\xi}_i (\tilde{\xi}_i)^T \right) \Sigma^{1/2} = \Sigma^{1/2} \tilde{I} \Sigma^{1/2}.
\] (7)

From [16, Lem. A.1] we have with probability of at least \( 1 - \beta \) that \( \| \tilde{I} - I \|_2 \leq \gamma(N_s, \beta/2) \), which is equivalent to
\[
(1 - \gamma(N_s, \beta/2)) I \leq \tilde{I} \leq (1 + \gamma(N_s, \beta/2)) I.
\] (8)

Since we are only interested in an upper bound for the covariance matrix \( \Sigma \), e.g. as required by (5), we find from the left inequality in (8) to
\[
I \leq \frac{1}{1 - \gamma(N_s, \beta/2)} \tilde{I} \Sigma \leq \frac{1}{1 - \gamma(N_s, \beta/2)} \hat{\Sigma},
\]
where we used the fact that condition (6) implies \( \gamma(N_s, \beta/2) < 1 \). Finally, condition (6) follows from assuming that \( 1 - \gamma(N_s, \beta/2) > 0 \), which is a quadratic inequality in the sample size \( N_s \). \( \Box \)

For a fixed \( \beta \in (0, 1) \), the mapping from \( \epsilon \mapsto N_s \) in condition (6) is convex on the interval \( \epsilon \in (0, 0.5) \). Thus, to obtain the smallest number \( N_s \) satisfying (6), we solve a nonlinear (convex) optimization problem
\[
e^* = \arg\min_{\epsilon \in (0,0.5)} 2c_1 c_2 \left( 8c_1 + 4\sqrt{4c_1^2 + c_1} + 1 \right).
\] (9)

Finally, by setting \( \kappa_\beta = 1/(1 - \gamma(N_s, \beta/2)) \) we obtain from optimization problem (9) and Proposition II.2 an explicit number of samples \( N_s \) to ensure that the ambiguity set (5) contains the true distribution \( \mu^* \) with \( 1 - \beta \) confidence.

Remark II.3. The result of Proposition II.2 is a special case of [17, Thm. 8] with known first moment information. As a consequence we require fewer samples compared to the mean and variance ambiguity set proposed by [17] to achieve the \( 1 - \beta \) confidence of the ambiguity set (5).

III. DISTRIBUTIONALLY ROBUST MPC

A. Control parameterization

We resort to a SADF parameterization [14] of the form
\[
u_{i|k} = \tilde{v}_{i|k} + \sum_{l=0}^{i-1} M_{i-l|k} w_{l|k},
\] (10)
where \( w_{l|k} \in \mathbb{R}^{n_u} \) is the predicted control input and \( M_{i-l|k} \in \mathbb{R}^{n_u \times n_w} \) are feedback matrices, both of which are decision variables in the resulting MPC optimization problem. Thus, \( \nu_{i|k} \) depends affinely on the past \( i \) disturbance variables \( w_{0|k}, \ldots, w_{i-1|k} \). To streamline the presentation we consider the matrix \( M_k = \sum_{i=0}^{\infty} M_{i|k} w_{i|k} \) and the vector \( \nu_k \in \mathbb{R}^{n_u} \)
\[
M_k := \begin{bmatrix}
0 & 0 & \cdots & 0 \\
M_{1|k} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
M_{N-1|k} & \cdots & M_{1|k} & 0
\end{bmatrix}, \quad \nu_k := \begin{bmatrix}
v_{0|k} \\
v_{1|k} \\
\vdots \\
v_{N-1|k}
\end{bmatrix}
\]
such that \( \tilde{u}_k = \nu_k + \hat{M}_k \tilde{w}_k \). The vector \( \nu_k \) can be interpreted as the predicted nominal control input that corresponds to the predicted nominal state trajectory \( \tilde{x}_k = A x_{0|k} + B \tilde{v}_k \).

B. Distributionally robust chance constraints

In the following we replace the individual chance constraints (5b)-(5c) with distributionally robust chance constraints of the form
\[
\inf_{\mu \in \mathbb{P}} \mathbb{P}(h_{t,r}^T \nu_k \leq 1 | x(k)) = \inf_{\mu \in \mathbb{P}} \mathbb{P} \left( \begin{bmatrix} \tilde{w}_k^T \ 1 \end{bmatrix} \begin{bmatrix} h_{t,r}^T (B \tilde{M}_k + \hat{E}) \end{bmatrix} \right) \leq 1 \geq p_n^t,
\] (11a)
\[
\inf_{\mu \in \mathbb{P}} \mathbb{P}(l_{t,s}^T \nu_k \leq 1 | x(k)) \geq p_n^s,
\] (11b)
where we substituted the expressions for \( \tilde{x}_k, \tilde{u}_k \) and \( \tilde{z}_k \). By definition of the ambiguity set (5) it holds that \( E\mu \{w\} = 0 \) and \( sup_{\mu \in \mathbb{P}} E\mu \{w w^T\} = \kappa_2 \hat{\Sigma} \). Thus, \( E\mu \{d\} = [0 \ 1]^T \) and \( sup_{\mu \in \mathbb{P}} E\mu \{(d - E\mu \{d\})(d - E\mu \{d\})^T\} = [\kappa_2 \hat{S}_N \ 0] \) with \( \hat{S}_N = I_N \otimes \hat{\Sigma} \). It remains to apply [13, Thm. 3.1] to express (11a)-(11b) as SOC constraints
\[
h_{t,r}^T \tilde{z}_k \leq 1 - \sqrt{\kappa_2} \sqrt{\frac{p_n^t}{1 - p_n^t}} \| h_{t,r}^T (B \tilde{M}_k + \hat{E}) \hat{S}_N^{1/2} \|_2
\] (12)
\[
l_{t,s}^T \tilde{v}_k \leq 1 - \sqrt{\kappa_2} \sqrt{\frac{p_n^s}{1 - p_n^s}} \| l_{t,s}^T \tilde{M}_k \hat{S}_N^{1/2} \|_2.
\] (13)

C. Distributionally robust cost function

Similar to the previous section we robustify the cost function (5a) against distributional ambiguity. To this end, we reformulate (5a) by leveraging the matrices introduced in Section III-B and define the distributionally robust cost function as
\[
J_k(k) = sup_{\mu \in \mathbb{P}} \mathbb{E}\mu \left\{ \|x_N|k\|^2_p + \sum_{t=0}^{N-1} \|x_{t|k}\|^2_Q + \|u_{t|k}\|^2_R \right\}(x(k))
\]
\[
= sup_{\mu \in \mathbb{P}} \mathbb{E}\mu \{d^T (\tilde{H}_{k}^T \tilde{Q} \tilde{H}_k + F_{k}^T \tilde{R} \tilde{F}_k) \ d | x(k))
\]
\[
= tr \left( \sum_{\mu \in \mathbb{P}} \mathbb{E}\mu \{d^T | x(k)) \} \tilde{H}_{k}^T \tilde{Q} \tilde{H}_k + F_{k}^T \tilde{R} \tilde{F}_k \right),
\] (14)
where \( \tilde{H}_k = [B \tilde{M}_k + \hat{E}, \tilde{z}_k], \tilde{F}_k = [\tilde{M}_k, \tilde{u}_k], \tilde{Q} = \text{diag}(I_N \otimes Q, P), \tilde{R} = I_N \otimes R \). The third equality applies the trace trick.
for quadratic forms, which can be further simplified with
to choose a scalar \( \alpha \) terminal cost function
the terminal controller, whereas the second and third conditions
interpolates linearly between Mode 1 and Mode 2 resulting
in the constraint
\[
\begin{align*}
z_{0|k} &= (1 - \lambda_k)x(k) + \lambda_k z_{1|k-1},
\end{align*}
\]
where \( \lambda_k \in [0, 1] \). The advantage is that only one optimization
problem needs to be solved, where \( \lambda_k = 1 \) reflects the guar-
anteed feasible solution (Mode 2) and \( \lambda_k = 0 \) the feedback
strategy (Mode 1).

F. Optimization problem
At each time instant \( k \geq 0 \) we solve the following MPC
optimization problem
\[
\begin{align}
& \min_{\hat{v}_k, M_k, \lambda_k, z_{0|k}} \quad \text{tr} \left( \Sigma_k^T \hat{H}_k \hat{H}_k + F_k^T \hat{F}_k \right) \\
& \text{s.t.} \quad \hat{z}_k = \tilde{A} z_{0|k} + B \hat{u}_k \quad (12), (13), (15), \lambda_k \in [0, 1] \\
& \quad z_{N|k} \in \mathbb{Z}_f. 
\end{align}
\]
The solution to problem (16) is the optimal SADF pair
\( (\hat{v}_k^*, M_k^*) \) and the nominal states \( \hat{z}_k^* \). To obtain the control
input at time \( k \), we recall [14, Thm. 1], which establishes an
equivalence between the SADF parameterization (10) and the
state feedback parameterization. By linear superposition, we
can thus establish also an equivalence to the error-feedback
(EF) parameterization
\[
\begin{align}
& u_{i|k} = g_{i|k} + \sum_{i=0}^{k} K_{t-i|k}(x_{i|k} - z_{i|k}) \\
& (17a)
\end{align}
\]
In other words, the state and input trajectories \( (\hat{z}_k, \hat{u}_k) \) result
from SADF with \( (\hat{v}_k^*, M_k^*) \) are equivalent to the ones
obtained from EF with \( (\tilde{v}_k, \tilde{M}_k^*) \), where
\[
\begin{align}
& \tilde{K}_k := \begin{bmatrix} K_{0|k} & K_{1|k} & \cdots & K_{0|k} \\
                          & K_{1|k} & \cdots & K_{0|k} \\
                          & \vdots & \ddots & \vdots \\
                          & K_{N-1|k} & \cdots & K_{0|k} \\
\end{bmatrix}, \quad \tilde{g}_k := \begin{bmatrix} g_{0|k} \\
                                               \vdots \\
                                               0 \\
\end{bmatrix}. 
\end{align}
\]
Similar to [14], the optimal pair \( (\tilde{v}_k, \tilde{M}_k) \) is obtained by
\[
\begin{align}
& \tilde{K}_k = (I + M_k^* E^\dagger B)^{-1} M_k^* E^\dagger \\
& \tilde{g}_k = (I + M_k^* E^\dagger B)^{-1} (\hat{v}_k - M_k^* E^\dagger A z_{0|k}). 
\end{align}
\]
while the input to system (1) is defined with the EF parameter-
ization at time \( t = 0 \), resulting in
\[
\begin{align}
& u(k) = u_{0|k} = g_{0|k} + K_{0|k}^*(x(k) - z_{0|k}^*). 
\end{align}
\]
Remark III.3. Note that the chance constraints (11a) - (11b)
depend on the information available at time \( k \). In view of
the initial condition (15), this implies that whenever the
MPC problem (16) is feasible with \( \lambda_k = 0 \), the probability
operator in (11a) - (11b) is conditioned on time \( k \), resulting
in closed-loop constraint satisfaction, while for \( \lambda_k \in (0, 1] \)
the constraints are verified in prediction, i.e. conditioned on
the last time instant \( k - \tau \) when problem (16) was feasible
with \( \lambda_k - \tau = 0 \). Note that the same conditioning appears in
the expectation operator of the cost function (14).

By leaving \( \lambda_k \) un-penalized in the objective function (16a),
we mimic a so-called hybrid scheme [3] with the intention to
minimize the open-loop cost despite feasibility of \( x(k) \). This
can lead to an increase in constraint violations in presence of
unmodeled disturbances, as we will demonstrate in Section IV.

However, adding an additional penalty term \( c \lambda_k^2 \) with \( c > 0 \)
to the objective function \((16a)\) causes the MPC controller to favor feedback initialization \(z_{0|k} = x(k)\) with the intention of introducing as much feedback as possible into the constraints, i.e. conditioning the probability operator in (11a) - (11b) on time \(k\) as often as possible. A possible drawback is the degradation of transient closed-loop performance, since the initial state cannot be freely chosen and the optimization problem therefore has fewer degrees of freedom, see Section V.2 for a numerical comparison.

IV. THEORETICAL PROPERTIES

A. Recursive feasibility

**Proposition IV.1.** Let Assumption III.1 hold. If at time \(k = 0\) the MPC optimization problem (16) admits a feasible solution with \(\lambda_k = 0\), then it is recursively feasible for all \(k \geq 0\).

**Proof.** Suppose that at time \(k\) problem (16) is feasible with \(\lambda_k = 0\), \((\bar{u}_k, M_k^x, \bar{z}_k^*)\) and equivalently with the EF parameterized input \(\tilde{u}_k = \bar{g}_k + K_k^x(\bar{z}_k + \bar{z}_k^*)\) with \((\bar{g}_k, K_k^x)\) due to [14, Thm. 1]. Now we construct the usual shifted candidate sequence \(u_t(k+1) = u_t+1(k)\) for \(t = 0, \ldots, N-2\) and append the terminal controller \(\bar{u}_{N-1|k+1} = K_{N-1|k+1} \bar{z}_{N-1|k+1}\). The shifted mean states and controller gains satisfy \((\bar{z}_{t+1|k}, \bar{K}_{t+1|k}) = (\bar{z}_{t|k+1}, \bar{K}_{t|k+1})\) for \(t = 0, \ldots, N-1\) appended with \((\bar{z}_{|k}, \bar{K}_{|k}) = ((A+BK)\bar{z}_{N|k}, K)\). Recursive feasibility is then a consequence of Assumption III.1. By stacking the shifted candidate sequences into the corresponding matrix and vector form, we obtain the triplet \((\tilde{g}_{k+1}, \tilde{K}_{k+1}, \tilde{z}_{k+1})\). A feasible input pair \((\bar{v}_{k+1}, \bar{M}_{k+1})\) for problem (16) is then simply found by [14, eq. (24)], i.e.

\[
\bar{M}_{k+1} = \hat{K}_{k+1}(I - \tilde{B}\hat{K}_{k+1})^{-1}\hat{E}
\]

\[
\bar{v}_{k+1} = \hat{K}_{k+1}(I - \tilde{B}\hat{K}_{k+1})^{-1}(A\hat{z}_{k+1} + \tilde{B}\hat{g}_{k+1}) + \tilde{g}_{k+1}
\]

with \(\lambda_{k+1} = 1\). This concludes the proof. \(\blacksquare\)

B. Convergence

The following theorem establishes a quadratic stability result of the closed-loop system (1) under control law (18). By adding an additional penalty term to the cost function \((16a)\) (cf. Remark III.3), the performance bound (19) changes to \(\kappa_P \text{tr}(PE\Sigma^E) + c\).

**Theorem IV.2.** Let Assumption I hold and choose \(\beta \in (0, 1), \epsilon \in (0, 0.5)\) and \(N_\epsilon\), such that [6] holds true and let \(\Sigma\) be the corresponding empirical covariance matrix. Suppose that at time \(k = 0\) there exists a feasible solution to problem (16). Then, for all \(k \geq 0\) the optimal cost \(J_k(k+1)\) satisfies

\[
J_k(k+1) - J_k(k) \leq -\mathbb{E}(\|x(k)\|^2_Q + ||u(k)||^2_R|x(k)) + \kappa_P \text{tr}(PE\Sigma^E).
\]

Furthermore, with a probability of at least \(1 - \beta\) the closed-loop system achieves the following asymptotic average bound

\[
P\left(\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}(\|x(k)\|^2_Q + ||u(k)||^2_R|x(0)) \right) \leq \kappa_P \text{tr}(PE\Sigma^E). \geq 1 - \beta. \tag{19}
\]

**Proof.** We establish an expected cost decrease condition in case of \(\lambda_{k+1} = 1\). First, consider the cost function (14), which, due its quadratic form and in view of Proposition V.1, can equivalently be written as \(J_k(k) = J^m(\bar{z}_k, \bar{g}_k) + J^v(\beta_k, \hat{K}_k)\), where the mean and variance part satisfy

\[
J^m(\bar{z}_k, \bar{g}_k) = \|\bar{z}_{N|k}\|^2_P + \sum_{t=0}^{N-1} \|z_{t|k}\|^2_Q + \|\bar{g}_{t|k}\|^2_R
\]

\[
J^v(\beta_k, \hat{K}_k) = \text{tr}(PS_{\Sigma^E|N}|k) + \sum_{t=0}^{N-1} \text{tr}(Q \Sigma^E_{t|k} + R \Sigma^w_{t|k})
\]

with \(\Sigma^E_{t+1|k} = (A + BK_{t+1})\Sigma^E_{t|k} + \kappa_P \text{tr}(PE\Sigma^E)\) and \(\Sigma^w_{t|k} = \sum_{t=0}^{N-1} K_t - \hat{K}_t \Sigma^E_{t|k} K_t^\top + \kappa_P \text{tr}(PE\Sigma^E)\).

Using the feasible solution from Proposition V.1 we can argue by optimality that

\[
J_k(k) \leq J^m(\bar{z}_{k+1}, \bar{g}_{k+1}) + J^v(\beta_k, \hat{K}_{k+1}) = J^m(\bar{z}_k, \bar{g}_k) - \|z_{0|k}\|^2_Q + \|\bar{g}_{0|k}\|^2_R
\]

\[
+ \|\bar{z}_{N|k}\|^2_P + \|K_{N|k}\|^2 - \|\bar{z}_{N|k}\|^2_P + \|\bar{A}\bar{z}_{N|k}\|^2_P + J^v(\beta_k, \hat{K}_k)
\]

\[
= \kappa_P \text{tr}(PE\Sigma^E) - \|\bar{A}\bar{z}_{N|k}\|^2_P + \|\bar{A}\bar{z}_{N|k}\|^2_P + J^v(\beta_k, \hat{K}_k)
\]

\[
\leq J^m(\bar{z}_k, \bar{g}_k) + J^v(\beta_k, \hat{K}_k) - \|z_{0|k}\|^2_Q + \|\bar{g}_{0|k}\|^2_R
\]

\[
- \kappa_P \text{tr}(PE\Sigma^E) + \kappa_P \text{tr}(PE\Sigma^E).
\]

where \(A = A + BK\).

To achieve the asymptotic cost bound we follow standard arguments in stochastic MPC [12], i.e.

\[
0 \leq \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E}(\|x(k)\|^2_Q + ||u(k)||^2_R|x(k))
\]

\[
+ \kappa_P \text{tr}(PE\Sigma^E),
\]

whereas the probability bound (19) follows by definition of the ambiguity set (5), i.e. \(\mathbb{P}(\Sigma \leq \beta_k \Sigma) \geq 1 - \beta. \square\)

V. NUMERICAL EXAMPLE

In this section, we carry out a numerical example. We consider the following system

\[
x(k+1) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} w(k),
\]

where \(w(k) \sim N(0, \Sigma)\) with \(\Sigma = 0.01^2 I\). For the ambiguity radius we select \(\beta = 0.05\), \(\epsilon = 0.0428\). Since \(u(w)\) is a zero-mean Gaussian it follows that \(\xi = \Sigma^{1/2} w\) is sub-Gaussian with variance \(\sigma^2 = 1\). According to Proposition II.2 we require \(N_\epsilon \geq 516\) samples to give the guarantee that \(\mathbb{P}(\mu^* \in \mathcal{P}) \geq 1 - \beta\). For the MPC cost function we select the weighting matrices \(Q = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}\), \(R = 1\) and \(P = \begin{bmatrix} 5.0000 & 5.0000 \\ 5.0000 & 14.2204 \end{bmatrix}\). We impose a single constrant \(\mathbb{P}(x_2(k) \leq 1) \geq P_x\) and use an ellipsoidal terminal set \(\mathcal{Z}_T = \{z | z^\top P z \leq \alpha\}\), where \(\alpha = 0.5293\) is obtained from \(N_\epsilon = 517\) samples. We keep \(\alpha\) constant for each experiment and select a prediction horizon of \(N = 10\). Note that the choice of \(\alpha\) is quite conservative, i.e. for \(N_\epsilon = 10^3\) the resulting terminal set is already 20.4
times larger, whereas under exact moment information we could enlarge the terminal set about 21.9 times.

1) Performance and constraint satisfaction: Starting at an initial condition $x(0) = [0, 0]^T$ we performed $10^5$ Monte-Carlo runs for different sample sizes $N_s$ computed over $10^3$ Monte-Carlo simulations (black) and optimal cost under exact moment information (red). (Right) Closed-loop trajectories for different $N_s$ with $p_x = 0.9$. The black dotted line denotes the constraint $x_2 \leq 1$.

![Figure 1](image_url)

**TABLE I**: Effect of sample size $N_s$ on the worst-case empirical probability of satisfying the constraint $P(x_2 \leq 1) \geq p_x$. $p_x$ | $N_s = 520$ | $N_s = 800$ | $N_s = 10^3$ | $N_s = 10^6$  
--- | --- | --- | --- | ---  
0.7 | 100% | 99.25% | 86.95% | 85.99%  
0.8 | 100% | 99.95% | 93.81% | 93.29%  
0.9 | 100% | 100% | 99.17% | 98.83%

2) Unmodeled disturbances: In the following we investigate the benefits of adding a penalty term for $\lambda_k$ to the objective function (16a). To this end we keep the same simulation setup as before and introduce an unmodeled larger disturbance at time step $k = 5$ with $w(5) \sim N(0, 6\Sigma)$. We add $c\lambda_k^2$ to the objective function (16a) to force the MPC optimization problem to prefer the feedback initialization over open-loop cost reduction (Remark III.3) and opt to satisfy the chance constraint with 70% probability.

**TABLE II**: Comparison of different controller configurations.

| Method | $c = 0$ | $c = 10^3$ | $c = 10^6$ | $c = 10^0$ | $\mathbb{E}\{l(x, u)\}$ | $\mathbb{P}(x_2(5) \leq 1)$  
--- | --- | --- | --- | --- | --- | ---  
783.20 | 784.47 | 784.50 | 784.74 | 68.76% | 73.37% | 73.91% | 75.08%

Table II reveals that penalization of $\lambda_k$ increases the constraint satisfaction rate by sacrificing transient closed-loop performance compared to the unpenalized case $c = 0$. Additionally, for $c > 0$ the chance constraint is empirically verified, whereas $c = 0$ violates the prescribed level of 70%.

**VI. CONCLUSION**

We have presented a DR-MPC framework for linear systems with additive disturbances under moment-based ambiguity sets, providing guarantees on closed-loop performance and recursive feasibility. The chance constraints are replaced with distributionally robust chance constraints in form of SOC constraints, whereas the cost function is minimized subject to the worst-case distribution across the ambiguity set. We used a simple numerical example to highlight the properties of the resulting controller.

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