On minimal flows and definable amenability in some distal NIP theories

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Abstract

We study the definable topological dynamics \((G(M), S_G(M))\) of a definable group acting on its type space, where \(M\) is either an \(o\)-minimal structure or a \(p\)-adically closed field, and \(G\) a definably amenable group. We focus on the problem raised in [14] of whether weakly generic types coincide with almost periodic types, showing that the answer is positive when \(G\) has boundedly many global weakly generic types. We also give two “minimal counterexamples” where \(G\) has unboundedly many global weakly generic types, extending the main results in [22] to a more general context.

1 Introduction

In model theory, we study a group \(G\) definable in a structure \(M\) and the action of \(G\) on its type space \(S_G(M)\), which is the collection of all types over \(M\) containing the formula defining \(G\). The space of generic types, introduced by Poizat as a generalization of the notion of generic points in an algebraic group, plays a heart role when \(Th(M)\) is stable. But for the unstable cases, the generic types may not exist. So various of weakenings of the generic were introduced to the unstable environment to generalize the properties of stable groups to the unstable context. The notion of weakly generic types introduced by Newelski in [14], which exists in any context, is a suitable substitution for generic types. We say that a definable set \(X\) is weakly generic if there is a non-generic definable set \(Y\) such that \(X \cup Y\) is generic, where a definable set is generic if finitely many its translates cover the whole group. Newelski studied the action of a definable group on its type space in the topological dynamics point of view, and tried to link the invariants suggested by topological dynamics with model-theoretic invariants.

Almost periodic types are one of the “new” objects suggested by topological dynamics. We say that a type \(p \in S_G(M)\) is almost periodic if the closure of its \(G(M)\)-orbit is a minimal subflow of \(S_G(M)\). Newelski proved that the space of almost periodic types coincides with the closure of space of weakly generic types, and when the generic types exist, almost periodic coincides with the weakly generics (see Corollary 1.9 and Remark 1.10 in [14]). An example was given where the two classes differ and the problem was explicitly raised (Problem 5.4 of [14]) of finding an \(o\)-minimal or even just NIP example. Newelski’s question is restated in [4] (Question 3.35) in the special case of definably amenable groups in NIP theories.

When \(M\) is an \(o\)-minimal expansion of a real closed field and \(G\) is a definably amenable group definable over \(M\), Pillay and Yao proved in [22] that weakly generics coincide with almost periodics when the torsion free part of \(G\) has dimension one. They also construct a counter-example when \(G = S^1 \times (\mathbb{R}, +)^2\), to show that the set of weakly generic types properly
contains the set of almost periodic types. The existence of a non-stationary weakly generic type in \((\mathbb{R}^2, +)\) plays a crucial role in the construction of the counter-example, where a weakly generic type is by definition stationary if it has a unique global weakly generic extension (see [18], Definition 3.41).

The current paper continues this line of work, extending the main results of [22] to a rather broader context, where \(T\) is \(p\text{CF}\) of \(p\)-adically closed fields or a complete \(o\)-minimal expansion of the theory \(\text{RCF}\) of real closed fields. Let \(M\) be a model of \(T\) and \(M = \mathbb{Q}_p\) when \(T = p\text{CF}\). We consider the case that \(G\) is a \(M\)-definable group admitting a \(M\)-definable short exact sequence

\[ 1 \to H \to G \to C \to 1, \tag{1} \]

where \(C\) is definably compact (see [17] and [15] for definitions) and \(H\) is “totally non-compact”. We say that \(H\) is \underline{totally non-compact} if there is a definable normal sequence

\[ H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_i \triangleleft \ldots \triangleleft H_n = H \tag{2} \]

such that \(H_0\) is finite and each \(H_{i+1}/H_i\) is a one-dimensional non-definably compact group. Note that any group definable over \(M\) admitting a decomposition as in (1) is definably amenable (see Remark [24]).

If \(H\) is defined in an \(o\)-minimal expansion of \(\text{RCF}\), then \(H\) is torsion-free iff it is totally non-compact and each \(H_i\) as in (2) is definably connected (see [7]). Any group \(G\) definable in an \(o\)-minimal expansion of \(\text{RCF}\) is definably amenable iff it has a decomposition as in (1) (see [5]).

When \(H\) is definable over \(\mathbb{Q}_p\), \(H\) is eventually trigonalizable over \(\mathbb{Q}_p\) [21]. A recent result of [11] shows that any \(\mathbb{Q}_p\)-definable abelian group also has a decomposition as in (1).

We can also describe the “totally non-compact” and “definably compact” via the model-theoretic invariants of “dfg” and “fsg” respectively, in both \(o\)-minimal and \(p\text{CF}\) contexts. Recall that a group \(G\) has finitely satisfiable generics (fsg) or definable \(f\)-generics (dfg) if there is a global type \(p\) on \(G\) and a small model \(M\) such that every left translate of \(p\) is finitely satisfiable in \(M\) or definable over \(M\), respectively. Suppose that \(M\) is either \(\mathbb{Q}_p\) or an \(o\)-minimal expansion of a real closed field, and \(H\) is definable over \(M\), then \(H\) is totally non-compact iff \(H\) has dfg [5, 21], and \(C\) is definably compact iff \(C\) has fsg [9, 15]. So we assume in this paper that \(G\) admits a “dfg-fsg” short exact sequence

\[ 1 \to H \to G \to C \to 1, \tag{3} \]

namely, \(G\) is an extension of a fsg group \(C\) by a dfg group \(H\).

The structure of this paper is analogous to [22]. We study the topological dynamics in both global and local environments. What we call the global context is where \(M\) is a monster model of \(T\). What we call the local context is where \(M\) is any model of \(T\), and we pass to the Shelah expansion \(M_0 = M^{\text{ext}}\) of \(M\) by the externally definable sets and consider the action of \(G(M_0)\) on \(S_G(M_0)\).

We introduce a new terminology first. As it was mentioned above a weakly generic type over \(A\) is stationary if it has just one extension to a complete weakly generic type over the monster model \(M \supset A\). In this paper, we call a definable group \(X\) is \underline{stationary} if every weakly generic type over every small model \(M \prec M\) is stationary.

We give our positive results first, which generalize Theorem 1.1 and Theorem 1.2 of [22], in which \(T\) is an \(o\)-minimal expansion of \(\text{RCF}\) and \(H\) has dimension \(\leq 1\).
Theorem 1.1. Let $M$ be either $\mathbb{Q}_p$ or an o-minimal expansion of a real closed field, $G$ a group defined over $M$ admitting a “dfg-fsg” short exact sequence as in [3]. Then

- $H$ is stationary iff $G$ is stationary.
- (Global case) Suppose that either $G = H$ or $H$ is stationary, then every global weakly generic type is almost periodic.
- (Local case) Suppose that either $G = H$ or $H$ is stationary. Let $M_0 = M^{\text{ext}}$, then every weakly generic type in $S_G(M_0)$ is almost periodic.

Let $\mathbb{G}_m$ be the multiplicative group of $\mathbb{M} \models \text{RCF}$, then $\mathbb{G}_m$ and $\mathbb{G}_m^2$ are stationary dfg groups [18]. If $\mathbb{G}_m$ is the multiplicative group of $\mathbb{M} \models pCF$, then each $\mathbb{G}_m^n$ is a stationary dfg group for each $n \in \mathbb{N}^+$ [31]. Let $\mathbb{G}_a$ be the additive group of either $\mathbb{R}$ or $\mathbb{Q}_p$, then $\mathbb{G}_a$ is also a stationary dfg group. By comparison, both $\mathbb{G}_a^2$ and the borel subgroup $\mathbb{G}_m \times \mathbb{G}_a$ of $\text{SL}_2(\mathbb{M})$ are examples of non-stationary dfg groups. So it is reasonable to consider $\mathbb{G}_a^2$ and $\mathbb{G}_m \times \mathbb{G}_a$ as “minimal-non-stationary” dfg groups.

Note that if $M$ is either $(\mathbb{Q}_p, +, \times, 0, 1)$ or an o-minimal expansion of $(\mathbb{R}, +, \times, 0, 1)$, then $M^{\text{ext}} = M$ by [13] and [6]. In contrast to the above positive results, the next theorem gives “minimal counterexamples”, which generalizes Theorem 1.3 of [22].

Theorem 1.2. Let $M$ be $\mathbb{R}$ or $\mathbb{Q}_p$ in the language of rings. Let $H$ and $C$ be definable over $M$, where $H$ has dfg and $C$ has fsg.

1. If $H$ is bad (see Definition [4.7]) and $G = H \times C$, then working either in $S_G(M)$ or $S_G(\mathbb{M})$ for $\mathbb{M}$ a monster model, the set of weakly generic types properly contains the set of almost periodic types.

2. If $H$ is either $\mathbb{G}_a^2$ or a borel subgroup of $\text{SL}_2(\mathbb{M})$, then $H$ is bad.

The next conjecture is based on the above result:

Conjecture 1. Let $G$ be a definable, definably amenable group, defined over either an o-minimal structure or a p-adically closed field. If $G$ is not dfg, then the set of weakly generics coincides with the set of almost periodics iff $G$ is stationary.

1.1 Notation and conventions

$L$ will denote a language, $T$ a complete theory, $M,N...$ models of $T$. It will be convenient to assume that $T$ has models which are $\kappa$-saturated and of cardinality $\kappa$ for arbitrarily large $\kappa$. Such a model, $\mathbb{M}$ say, will have homogeneity properties in addition to saturation properties. We call $\mathbb{M}$ a monster model. Let us fix a monster model $\mathbb{M}$ in this paper. A subset $A$ of $\mathbb{M}$ is called small if $|A| < |\mathbb{M}|$. We sometimes pass to an $|\mathbb{M}|^+$-saturated extension $\mathbb{M}$ where types over $\mathbb{M}$ can be realized. We usually write tuples as $a,b,x,y...$ rather than $\bar{a},\bar{b},\bar{x},\bar{y}...$. A “type” is a complete type, and a “partial type” is a partial type. By a “global type” we mean a complete type over $\mathbb{M}$ (or $\mathbb{M}$). If $X$ is a definable set, defined over $M$ then we write $S_X(M)$ for the space of complete types concentrating on $X$. Let $\phi(x)$ be any $L_\mathbb{M}$-formula with $x = (x_1,...,x_n)$, and $A \subseteq \mathbb{M}$, then $\phi(A)$ is defined to be the set $\{a \in A^n | \mathbb{M} \models \phi(a)\}$. We sometimes use $X(x)$ to denote the formula which defines $X$, and identify $X$ with points in $\mathbb{M}$, namely $X = X(\mathbb{M})$. If $A \subseteq B$ and $p \in S(B)$, then by $p | A$ we mean the restriction of $p$ to $A$,
namely, the collection of all formulas in \( p \) with parameters from \( A \). Suppose that \( G \) is a group and \( g, h \in G \), then by \( g^h \) we mean the conjugate \( h^{-1}gh \).

Our notation for model theory is standard, and we will assume familiarity with basic notions such as type spaces, heirs, coheirs, definable types etc. References are [23] as well as [25].

The paper is organized as follows: For the rest of the introduction we give precise definition and preliminaries relevant to our results.

In Section 2, we prove part (1) and (2) of Theorem [1.1] the positive results for global case.
In Section 3, we prove part (3) of Theorem [1.1] the positive result for Local case.
In Section 4, we show that when a dfg group \( H \) is bad (see Definition [4.1]), then, for both global and local cases, we can find a weakly generic type on \( H \times C \) which is not almost periodic whenever \( C \) is an infinite fsg group. We also show that \( \mathbb{G}_s^2 \) and any borel subgroup of \( SL_2(\mathbb{M}) \) are examples of bad dfg groups.

### 1.2 Definable Topological dynamics

Our reference for (abstract) topological dynamics are [1,8]. Let \( G \) be a topological (often discrete) group, by a \( G \)-flow we mean an action \( G \times X \to X \) of \( G \) on a compact Hausdorff topological space \( X \) by homeomorphisms, and denote it by \( (G, X) \). Often it is assumed that there is a dense orbit.

By a subflow of \( X \) we mean a closed \( G \)-invariant subset of \( X \). Minimal subflows of \( X \) always exist. A point \( x \in X \) is almost periodic if the closure \( \text{cl}(G \cdot x) \) of its \( G \)-orbit is a minimal subflow of \( X \). Equivalently, \( x \in X \) is almost periodic if \( x \) is in some minimal subflow of \( X \).

Given a \( G \)-flow \( (G, X) \), its enveloping semigroup \( E(X) \) is the closure in the space \( X^X \) (with the product topology) of the set of maps \( \pi_g : X \to X \), where \( \pi_g(x) = g \cdot x \), equipped with the composition (which is continuous on the left). So any \( e \in E(X) \) is a map from \( X \) to \( X \).

**Fact 1.3.** Let \( X \) be a \( G \)-flow. Then

- \( E(X) \) is also a \( G \)-flow and \( E(E(X)) \cong E(X) \) as \( G \)-flows.
- For any \( x \in X \), the closure of its \( G \)-orbit is exactly \( E(X)(x) \). Particularly, for any \( f \in E(X) \), \( E(X) \circ f \) is the closure of \( G \cdot f \).

In the model theoretic context, we consider a complete theory \( T \), model \( M \) of \( T \), group \( G \) definable over \( M \) and the action of \( G(M) \) on the type-space \( S_G(M) \) as \( gp = \text{tp}(ga/M) \) where \( a \) realizes \( p \). It is easy to see that \( S_G(M) \) is a \( G(M) \)-flow with a dense orbit \( \{\text{tp}(g/M) \mid g \in G(M)\} \).

Take a monster model \( \mathbb{M} \) and identify \( G \) with \( G(\mathbb{M}) \). We call a formula \( \varphi(x) \), with parameters in \( \mathbb{M} \), a \( G \)-formula if \( \varphi(\mathbb{M}) \) is a definable subset of \( G \). Suppose that \( \varphi(x) \) is a \( G \)-formula and \( g \in G \), then the left translate \( g \varphi(X) \) is defined to be \( \varphi(g^{-1}x) \). It is easy to check that \( (g\varphi)(\mathbb{M}) = gX \) if \( X = \varphi(\mathbb{M}) \). For \( p \in S_G(M) \), we have \( gp = \{g\varphi(x) \mid \varphi \in p\} \).

We recall some notions from [14].

**Definition 1.4.** 1. A definable subset \( X \subseteq G \) is generic if finitely many left translates of \( X \) cover \( G \). Namely, there are \( g_1, \ldots, g_n \in G \) such that \( \bigcup_{i=1}^{n} g_i X = G \).
2. A definable subset \( X \subseteq G \) is weakly generic if there is an non-generic definable subset \( Y \) such that \( X \cup Y \) is generic.

3. A formula \( \varphi(x) \) is generic if the definable set \( \varphi(M) \) is generic (similarly for weakly generic formulas).

4. A complete type \( p \in S_G(A) \) is generic if every formula in \( p \) is generic.

5. Likewise \( p \in S_G(A) \) is weakly generic if every formula in \( p \) is weakly generic.

**Fact 1.5.** ([11])

- Let \( AP(S_G(M)) \subseteq S_G(M) \) be the space of almost periodic types, and \( WG(S_G(M)) \subseteq S_G(M) \) the space of weakly generic types. Then \( WG(S_G(M)) = \text{cl}(AP(S_G(M))) \).

- If there is a generic type in \( S_G(M) \), then there is a unique minimal subflow of \( S_G(M) \) which moreover coincides with the set of generic types. So also generic types, almost periodic types, and weakly generic types coincide.

Let \( M^{\text{ext}} \) be a Shelah expansion of \( M \) in the language \( L^{\text{ext}} = \{ R_\varphi(x) \mid \varphi(x) \in L_M \} \) with \( R_\varphi(M) = \varphi(M) = \{ \varphi(a) \mid a \in M \} \), and \( T^{\text{ext}}_M \) the complete theory of \( M^{\text{ext}} \). We denote the collection of quantifier-free types over \( M^{\text{ext}} \) which concentrate on \( G \) by \( S_{G,\text{ext}}(M) \). The space \( S_{G,\text{ext}}(M) \) is naturally homeomorphic to the space \( S_{G,M}(M) \) of global complete types concentrating on \( G \) which are finitely satisfiable in \( M \), via the map

\[
p \in S_{G,M}(M) \mapsto \{ \psi(M) \mid \psi \in p \} \in S_{G,\text{ext}}(M).
\]

**Fact 1.6.** ([11]) The enveloping semigroup \( E(S_{G,\text{ext}}(M)) \) of \( S_{G,\text{ext}}(M) \) is isomorphic to \( (S_{G,M}(M), \ast) \) where \( \ast \) is defined as following: for any \( p, q \in S_{G,M}(M) \), \( p \ast q = \text{tp}(b \cdot c/M) \) with \( b \) realizes \( p \) and \( c \) realizes \( q \), and \( \text{tp}(b/M, c) \) is finitely satisfiable in \( M \).

**Remark 1.7.** It is easy to see from Fact 1.3 and Fact 1.6 that for any \( p \in S_{G,M}(M) \),

\[
\text{cl}(G(M) \cdot p) = S_{G,M}(M) \ast p.
\]

### 1.3 NIP, Definable amenability, and connected component

Recall that a theory \( T \) is said to be NIP if for any indiscernible sequence \( (b_i : i < \omega) \), formula \( \phi(x, y) \) and \( a \in M \), there is an eventual truth-value of \( \phi(a, b_i) \) as \( i \to \infty \). Now we assume that \( T \) has NIP throughout this paper.

Let \( \phi(x, y) \) be a formula. Recall that a formula \( \phi(x, b) \) divides over a set \( A \) if there is an infinite \( A \)-indiscernible sequence \( (b = b_0, b_1, b_2, \ldots) \) such that \( \{ \phi(x, b_i) \mid i < \omega \} \) is inconsistent. A type \( p \in S(B) \) divides over \( A \subseteq B \) if there is a formula \( \phi \in p \) divides over \( A \). By [10], a global type \( p \in S(M) \) does not divide over a small model \( M \) if and only if \( p \) is \( \text{Aut}(M)/M \)-invariant.

**Fact 1.8.** ([11]) Let \( M \) be a model of \( T \) and \( G \) a definable group defined over \( M \). Then \( T^{\text{ext}}_M \) has quantifier elimination and NIP. So \( S_{G,\text{ext}}(M) \) coincide with the space of \( S_G(M^{\text{ext}}) \) of complete types over \( M^{\text{ext}} \) concentrating on \( G \).

**Remark 1.9.** We see from Fact 1.6 and Fact 1.8 that the semigroup operation “\( \ast \)” on \( S_G(M^{\text{ext}}) \) is defined as following: for any \( p, q \in S_G(M^{\text{ext}}) \), \( p \ast q = \text{tp}(b \cdot c/M^{\text{ext}}) \) with \( b \) realizes \( p \) and \( c \) realizes the unique heir of \( q \) over \( M^{\text{ext}}, b \).
Let $G = G(M)$ be a definable group. Recall that a type-definable over $A$ subgroup $H$ is a type-definable over $A$ subset of $G$, which is also a subgroup of $G$. We say that $H$ has bounded index if $|G/H| < 2^{1+1/A}$. For groups definable in NIP structures, the smallest type-definable subgroup of bounded index exists [9], which is the intersection of all type-definable subgroups of bounded index, we write it as $G^{00}$, and call it the type-definable connected component.

Another model theoretic invariant is $G^{0}$, called the definable-connected component of $G$, which is the intersection of all definable subgroups of $G$ of finite index. Clearly, $G^{00} \leq G^{0}$.

Fact 1.10. [3] If $G$ is definable over $M$, then $G^{00}$ is the same whether computed in $T$ or $T^{\text{ext}}_{M}$.

Recall also that a Keisler measure over $M$ on $X$, with $X$ a definable subset of $M^{n}$, is a finitely additive measure on the Boolean algebra of $M$-definable subsets of $X$. When we take the monster model, i.e., $M = \mathbb{M}$, we call it a global Keisler measure. A definable group $G$ is said to be definably amenable if it admits a global (left) $G$-invariant probability Keisler measure. By [9] this is equivalent to the existence of a $G(M)$-invariant probability Keisler measure over $M$ on $G$, whenever $M$ is a model over which $G$ is defined.

A nice result of [4] shows that:

Fact 1.11. Let $G$ be definable over $M$. Then $G$ is definably amenable iff there exists $p \in S_{G}(\mathbb{M})$ such that for every $g \in G = G(M)$, $gp$ does not divide over $M$.

Following the notation of [4] we call a type $p$ as in the right hand side a (global) strongly $f$-generic on $G$ over $M$. A global type is strongly $f$-generic if it is strongly $f$-generic over some small model.

Given a definable subset $X$ of $G$, we say that $X$ is $f$-generic if for some/any model $M$ over which $X$ is defined and any $g \in G$, $gX$ does not divide over $M$. Call a complete type $p$ (over some set of parameters) $f$-generic iff for every formula $\psi(x)$ in $p$, $\psi(\mathbb{M})$ is $f$-generic. In [4], the authors showed that in NIP theories:

Fact 1.12. Let $G$ be a definable group. Then the following are equivalent:

1. $G$ is definably amenable.
2. $G$ admits a global type $p \in S_{G}(\mathbb{M})$ with a bounded $G$-orbit.
3. $G$ admits a global strongly $f$-generic type.

Moreover,

Fact 1.13. For a definably amenable group $G$, we have that

1. Weakly generic definable subsets, formulas, and types coincide with $f$-generic definable subsets, formulas, and types, respectively.
2. $p \in S_{G}(\mathbb{M})$ is $f$-generic iff it has a bounded $G$-orbit.
3. $p \in S_{G}(\mathbb{M})$ is $f$-generic if and only if it is $G^{00}$-invariant. A type-definable subgroup $H$ of the stabilizer of a global $f$-generic type is exactly $G^{00}$.
4. A global type is strongly $f$-generic over $M$ iff it is weakly generic and $M$-invariant, or equivalently, does not divide over $M$.

Among the strongly $f$-generic types $p \in S_{G}(\mathbb{M})$, there are two extreme cases:
1. There is a small submodel $M$ such that every left $G$-translate of $p$ is finitely satisfiable in $M$, and we call such types the fsg (finitely satisfiable generic) types on $G$ over $M$;

2. There is a small submodel $M$ such that every left $G$-translate of $p$ is definable over $M$, and we call such types the dfg (definable f-generic) types on $G$ over $M$.

A definable group $G$ is called fsg or dfg if it has a fsg or dfg type, respectively. Clearly, both fsg and dfg groups are definably amenable. We now discuss these two cases. Let $\text{Stab}_l(p)$ denote the stabilizer of $p \in S_G(M)$ with respect to the left group action, and $\text{Stab}_r(p)$ the stabilizer of $p$ with respect to the right group action. By [9] we have:

**Fact 1.14.** Let $G$ be an $\emptyset$-definable fsg group witnessed by a global type $p \in S_G(M)$ and a small model $M$. Then

- $p$ is both left and right generic.
- Any left (right) translate of $p$ is finitely satisfiable in any small model.
- $G^{00} = \text{Stab}_l(p) = \text{Stab}_r(p)$.
- Let $\text{Gen}(G(M))$ be the space global generic types in $S_G(M)$, then $\text{Gen}(G(M))$ is the unique minimal subflow of $S_G(M)$.
- Every global left/right generic type is fsg.
- For any $N < M$, every generic type $q \in S_G(N)$ has a unique global generic extension.

**Remark 1.15.** Let $G$ be an $\emptyset$-definable fsg group and $p \in S_G(M)$ is $G^{00}$-invariant, then by Fact [L.13] $p$ is weakly generic, thus is $p$ generic by Fact [L.5].

**Fact 1.16.** [3] Assume that $T$ is NIP. Let $G$ be an $\emptyset$-definable fsg group and $M < M$. Then

- $G$ also has fsg when we compute it in $T_M^\text{ext}$.
- $q \mapsto \{\psi(M) | \psi \in q\}$ is a bijection between $\text{Gen}(G(M))$ and $\text{Gen}(G(M^\text{ext}))$.
- $\text{Gen}(G(M^\text{ext}))$ is a bi-ideal of (the semigroup) $S_G(M^\text{ext})$.

We now discuss the dfg groups.

**Fact 1.17.** [22] Assume that $T$ is NIP. Let $G$ be an $\emptyset$-definable group and $p \in S_G(M)$ an $f$-generic type. If $p$ is definable over $M$, then

- Every left translate of $p$ is definable over $M$;
- $G^{00} = G^0 = \text{Stab}_l(p)$.
- $G \cdot p$ is closed, and hence a minimal subflow of $S_G(M)$.

**Fact 1.18.** [3] Assume that $T$ is NIP. Let $G$ be a dfg group definable over $M$. Let $N^*$ be an $|M|^+$-saturated extension of $M^\text{ext}$. Then $p \in S_G(M^\text{ext})$ is almost periodic iff its unique heir $\overline{p}$ over $N^*$ is weakly generic. Moreover, any $G(N^*)$-translate of $\overline{p}$ is an heir of some $q \in S_G(M^\text{ext})$ over $N^*$. So $\overline{p}$ is a dfg type and thus $G$ is dfg with respect to $T_M^\text{ext}$. 

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**Fact 1.19.** [28] Assume that $T$ is NIP, and $G$ is a definably amenable group definable over $M$. Let $M \prec N$, $\pi : S_G(N) \to S_G(M)$ the canonical restriction map, and $\mathcal{M}$ a minimal $G(N)$-subflow of $S_G(N)$. Then $\pi(\mathcal{M})$ is a minimal $G(M)$-subflow of $S_G(M)$.

We conclude directly from Fact 1.18 and Fact 1.19 that

**Corollary 1.20.** Assume that $T$ is NIP. Let $G$ be a dfg group definable over $M$, $N^*$ an $|M|^+$-saturated extension of $M^{\text{ext}}$, and $\mathcal{J}$ a minimal $G(N^*)$-subflow of $S_G(N^*)$. Then $\pi(\mathcal{J})$ is a minimal $G(M)$-subflow of $S_G(M^{\text{ext}})$ and $\pi$ is a bijection from $\mathcal{J}$ to $\pi(\mathcal{J})$.

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## 2 Stationary dfg groups and the global case

Here we prove Theorem 1.1. We assume that $M$ is $\mathbb{Q}_p$ or an $o$-minimal expansion of a real closed field and $T = \text{Th}(M)$. Assume that $\mathbb{M}$ be a monster model of $T$ and $G$ is an $M$-definable group defined in $\mathbb{M}$. We also assume that $G$ admits a $M$-definable short exact sequence

$$1 \to H \to G \to C \to 1,$$

where $H$ has dfg and $C$ has fsg. As $T$ has definable Skolem functions (see [24] and Proposition 1.2 of Chapter 6 in [29]), let $f : C \to G$ be a definable section of $\pi$. We aim to show that if $H$ is stationary, then every weakly generic type on $G$ is almost periodic.

**Remark 2.1.** Note that in an arbitrary NIP theory, if $Y$ is a definable group and $X$ a definable normal subgroup of $Y$ such that both $X$ and $Y/X$ are definably amenable, then so is $Y$ (see Exercise 8.23 of [25]). So any group $G$ admitting a decomposition as in (4) is definably amenable.

The following Lemma is formally analogous to Lemma 2.2 in [22], replacing “$H$-invariant” with “$f$-generic”.

**Lemma 2.2.** Let $p = \text{tp}(a/\mathbb{M})$ be a weakly generic type on $G$ where $a \in G(\mathbb{M})$. Write $a$ uniquely as $h \cdot f(\pi(a))$ with $h \in H(\mathbb{M})$. Then $\text{tp}(\pi(a)/\mathbb{M})$ is a generic type on $C$ and $\text{tp}(h/\mathbb{M})$ is a weakly generic type on $H$.

**Proof.** Since $\text{Stab}(p) = G^{00}$, we see that $\text{Stab}(\pi(p)) = \pi(G^{00}) = C^{00}$ and so $\pi(p)$ is generic by Remark 1.15.

Let $\eta : G(\mathbb{M}) \to H(\mathbb{M})$ be the function given by $\eta(x) = x \cdot f(\pi(x))^{-1}$, then $h = \eta(a)$. For each $h_0 \in H$, we have

$$h_0 \cdot h = h_0 \cdot a \cdot f(\pi(a))^{-1} = (h_0 \cdot a) \cdot f(\pi(h_0 \cdot a))^{-1} = \eta(h_0 \cdot a).$$

Since $\text{tp}(a/\mathbb{M})$ is weakly generic, we see that

$$H \cdot \text{tp}(a/\mathbb{M}) = \{\text{tp}(h_0 \cdot a/\mathbb{M}) | h_0 \in H\}$$

is bounded. So the $H$-orbit

$$\{\text{tp}(h_0 \cdot h/\mathbb{M}) | h_0 \in H\} = \{\text{tp}(\eta(h_0 \cdot a)/\mathbb{M}) | h_0 \in H\}$$

of $\text{tp}(h/\mathbb{M})$ is bounded. So $\text{tp}(h/\mathbb{M})$ is a weakly generic as required.

\[\square\]
Lemma 2.3. Let $p \in S_H(M)$. If $p$ is a weakly generic type, then every global heir of $p$ is weakly generic.

Proof. If $H$ is defined in a model $M$ of an o-minimal expansion of RCF, then $p$ is weakly generic iff it is a global weakly generic extension $\bar{p} \in S_H(M)$. By Fact 1.13 and 1.17, $\bar{p}$ is $H^0$-invariant, where $H^0$ is an $M$-definable group of $H$ by DCC (see [20]), so $p$ is $H^0(M)$-invariant, and thus every global heir of $p$ is also $H^0(M)$-invariant, hence also weakly generic. If $H$ is defined in an $p$-adically closed field, then by [21] $H$ is eventually a trigonalizable algebraic groups. By Corollary 2.16 of [30], we see that every global heir of $p$ is weakly generic.

Recall from the Section of Introduction that a definable group $X$ is stationary if every weakly generic type over any model has a unique global weakly generic extension. By Fact 1.14, the fsg group $C$ is stationary.

In the rest of this section, we may assume that $G$ and its “dfg-fsg” short exact sequence are definable over $\emptyset$ by naming parameters.

Lemma 2.7. The following are equivalent:

1. $H$ is stationary.
2. $H$ has boundedly many global weakly generic types.
3. Every global weakly generic type is $\emptyset$-definable.
4. $H$ has boundedly many strongly $f$-generic types.

Proof. 
1 ⇒ 2: Let $N \prec M$ be any small submodel. Assume that $H$ is stationary, then every global weakly generic type $\bar{p}$ is the unique weakly generic extension of $\bar{p}|N$. As there are only boundedly many types over $N$, there are boundedly many global weakly generic types.
2 \Rightarrow 3: We first prove that every global weakly generic type is definable. Suppose that there are at most \( \lambda < |\mathcal{M}| \) weakly generic types over \( \mathcal{M} \). If \( p \in S_H(\mathcal{M}) \) is weakly generic but not definable, then \( p \) has unboundedly many heirs over an \( |\mathcal{M}|^{+} \)-saturated extension \( \mathcal{M} \) of \( \mathcal{M} \) (see Proposition 1.19 in [19]), and all of them are weakly generic by Lemma 2.9. Let \( \{ q_i \mid i < \lambda^+ \} \) be a set of distinct weakly generic types over \( \mathcal{M} \). Let \( \phi_{ij}(x, b_{ij}) \) be the formula such that \( \phi_{ij}(x, b_{ij}) \in p_i \) and \( \neg \phi_{ij}(x, b_{ij}) \in p_j \). Let \( N \) be a small submodel of cardinality \( \lambda^+ \) which contains the set \( \{ b_{ij} \mid i \neq j \in \lambda^+ \} \). We see that \( p_i|N \neq p_j|N \) for all \( i \neq j \in \lambda^+ \). So there are at least \( \lambda^+ \) many weakly generic types over \( N \). By the saturation of \( \mathcal{M} \), take some \( N' \prec \mathcal{M} \) such that \( \text{tp}(N) = \text{tp}(N') \). So there are at least \( \lambda^+ \) many weakly generic types over \( N' \). We conclude that there are at least \( \lambda^+ \) many weakly generic types over \( \mathcal{M} \), which is a contradiction.

Now take any global weakly generic type \( p \in S_H(\mathcal{M}) \) and any small submodel \( N_0 \). If \( p \) is not \( \emptyset \)-definable, then we see from the definability of \( p \) that \( \{ \sigma(p) \mid \sigma \in \text{Aut}(\mathcal{M}) \} \) is unbounded. But each \( \sigma(p) \) is weakly generic. So \( p \) is \( \emptyset \)-definable.

3 \Rightarrow 1: Let \( N \) be a small submodel and \( p \in S_H(\mathcal{M}) \) weakly generic. Suppose that \( q \) is a global weakly generic extension of \( p \). Then \( p \) and \( q \) are definable over \( N \). We see that \( q \) is the unique heir of \( p \). So the \( q \) is the unique global weakly generic extension of \( p \).

It is easy to see that 2 \Rightarrow 4. We now show that 4 \Rightarrow 1.

Suppose that \( H \) is non-stationary. Let \( N \) be any small submodel. Then there is a weakly generic type \( p \in S_H(N) \) such that \( p \) has unboundedly many global weakly generic extensions. For any cardinal \( \lambda < |\mathcal{M}| \), take a sufficiently large submodel \( N' \succ N \) such that \( |N'| = \lambda \) and \( p \) has \( \lambda \) many different weakly generic extensions \( \{ q_i \mid i < \lambda \} \) over \( N' \). By Lemma 2.6, each \( p_i \) has a strongly \( f \)-generic extension over \( \mathcal{M} \). Hence there are unboundedly many strongly \( f \)-generic extensions of \( p \).

Simon [26] has isolated a notion, distality, meant to express the property that a NIP theory \( T \) has “no stable part”, or is “purely unstable”. Examples include any \( \alpha \)-minimal theory and \( p\text{CF} [26, 27] \).

Let \( p(x), q(y) \) be global types such that \( p \) is definable over a small submodel and \( q(y) \) is finitely satisfiable in a small submodel, then \( p(x) \otimes q(y) = q(y) \otimes p(x) \) (see Lemma 2.23, [26]). By Lemma 2.16 in [26], we have the following fact:

**Fact 2.8.** Assume \( T \) is a distal NIP theory. Let \( \mathcal{M} \) be saturated, \( N \prec \mathcal{M} \) a small submodel, \( p(x) \in S(\mathcal{M}) \) definable over \( N \), and \( q(y) \in S(\mathcal{M}) \) finitely satisfiable in \( N \). Then \( p(x) \) and \( q(y) \) are orthogonal. Namely, \( p(x) \cup q(y) \) implies a complete global type. In fact, if \( a \models p \) and \( b \models q \), then \( \text{tp}(a/\mathcal{M}, b) \) is the unique heir of \( \text{tp}(a/N) \) and \( \text{tp}(b/\mathcal{M}, a) \) is finitely satisfiable in \( N \).

We now consider the case where \( H \) is stationary. The following Lemma says that if \( H \) is stationary, then we can exchange the positions of \( h \) and \( f(\pi(a)) \) in the “decomposition” of \( a = h \cdot f(\pi(a)) \) as in Lemma 2.2.

**Lemma 2.9.** Let \( p = \text{tp}(a/\mathcal{M}) \) be a weakly generic type on \( G \). Write \( a \) uniquely as \( f(\pi(a)) \cdot h' \) with \( h' \in H \). If \( H \) is stationary, then \( \text{tp}(\pi(a)/\mathcal{M}) \) is a generic type on \( C \) and \( \text{tp}(h'/\mathcal{M}) \) is a weakly generic type on \( H \).
Proof. We see from Lemma 2.2 that \( \text{tp}(a/\mathbb{M}) \) is of the form \( \text{tp}(h \cdot f(\pi(a))/\mathbb{M}) \) such that \( \text{tp}(h/\mathbb{M}) \) and \( \text{tp}(\pi(a)/\mathbb{M}) \) are global weakly generic types of \( H \) and \( C \) respectively. It suffices to show that \( \text{tp}(h^{f(\pi(a))}/\mathbb{M}) \) is a weakly generic type on \( H \).

We now show that the \( H \)-orbit of \( \text{tp}(h^{f(\pi(a))/\mathbb{M}}) \) is bounded. By Lemma 2.7 and Fact 2.8 \( \text{tp}(h/\mathbb{M}) \) and \( \text{tp}(\pi(a)/\mathbb{M}) \) are orthogonal, we conclude that \( \text{tp}(h/\mathbb{M}, \pi(a)) \) is the unique heir of \( \text{tp}(h/\mathbb{M}) \). So \( \text{tp}(h/\mathbb{M}, \pi(a)) \) is a weakly generic type, and thus its \( H(\text{dcl}(\mathbb{M}, \pi(a))) \)-orbit

\[
\{ \text{tp}(h_0 h/\mathbb{M}, \pi(a)) \mid h_0 \in H(\text{dcl}(\mathbb{M}, \pi(a))) \}
\]

is bounded. Since \( x \mapsto x^{f(\pi(a))} \) is a \( \pi(a) \)-definable automorphism of \( H \), we see that \( H(\text{dcl}(\mathbb{M}, \pi(a))) \)-orbit of \( \text{tp}(h^{f(\pi(a))/\mathbb{M}}) \) is bounded. So the \( H \)-orbit of \( \text{tp}(h^{f(\pi(a))/\mathbb{M}}) \) is bounded as required. \( \square \)

Remark 2.10. Let \( H \) be stationary. Then for any global weakly generic type \( p_H \) and \( q_C \) on \( H \) and \( C \) respectively, we can speak about the type \( f(q_C) \cdot p_H \) (resp. \( p_H \cdot f(q_C) \)) which is defined to be \( \text{tp}(f(c^*) \cdot h^*/\mathbb{M}) \) (resp. \( \text{tp}(h^* \cdot f(c^*)/\mathbb{M}) \)) where \( c^* \models q_C \) and \( h^* \) realizes \( p_H \). By Lemma 2.7 and Fact 2.8 \( p_H \) and \( q_C \) are orthogonal, so \( \text{tp}(h^*/\mathbb{M}, c^*) \) is the unique heir of \( p_H \) over \( \mathbb{M}, c^* \). Let \( \text{Gen}(C) \) be the space of generic types in \( S_C(\mathbb{M}) \), and \( H_{p_H} \) the \( H \)-orbit of \( p_H \), then \( f(\text{Gen}(C)) \cdot (H_{p_H}) \) will denote the set

\[
\{ f(q) \cdot p \mid q \in \text{Gen}(C) \text{ and } p \in H_{p_H} \}.
\]

Fact 2.11. \( [\text{Zil} \] \) Let \( X \) be a definable group and \( p \in S_X(\mathbb{M}) \). Then

\[
\text{cl}(X \cdot p) = \{ \text{tp}(b \cdot c/\mathbb{M}) \mid c \models p, \ b \in X, \ \text{and } \text{tp}(b/\mathbb{M}, c) \text{ is finitely satisfiable in } \mathbb{M} \}
\]

Lemma 2.12. Let \( r = f(q_C) \cdot p_H \) be a weakly generic type on \( G \), where \( q_C \in \text{Gen}(C) \) and \( p_H \in S_H(\mathbb{M}) \) a definable \( f \)-generic type on \( H \). Then

\[
\text{cl}(G \cdot r) = f(\text{Gen}(C)) \cdot H_{p_H}
\]

is a minimal \( G \)-flow, and in particular, \( r \) is almost periodic.

Proof. It is easy to see that \( \text{cl}(G \cdot r) = f(\text{Gen}(C)) \cdot H_{p_H} \) implies \( \text{cl}(G \cdot r) \) is a minimal subflow, since any \( r' \in \text{cl}(G \cdot r) \) is also of the form \( r' = f(q') \cdot p' \) with \( q' \in f(\text{Gen}(C)) \) and \( p' \in H_{p_H} = H_{p'} \) a definable \( f \)-generic type on \( H \).

We now show that \( \text{cl}(G \cdot r) = f(\text{Gen}(C)) \cdot H_{p_H} \). Firstly, \( \text{cl}(G \cdot r) \subseteq f(\text{Gen}(C)) \cdot H_{p_H} \) is contained in the proof of Lemma 2.4 in [\text{Zil}].

The new observation here is that \( \text{Gen}(C) \cdot H_{p_H} \subseteq \text{cl}(G \cdot r) \). Let \( q' \in \text{Gen}(C) \), \( p' \in H \cdot p_H \), and \( r' = f(q') \cdot p' \). Let \( \mathbb{M} \) be an \( |\mathbb{M}|^+ \)-saturated extension of \( \mathbb{M} \) and \( a^* \in G(\mathbb{M}) \) realize \( r \). By Fact 2.11 it suffices to show that there is \( q \in G(\mathbb{M}) \) such that \( r' = \text{tp}(q a^*/\mathbb{M}) \) where \( \text{tp}(q/\mathbb{M}, a^*) \) is finitely satisfiable in \( \mathbb{M} \).

Since \( \text{Gen}(C) \) is the unique minimal subflow of \( S_C(\mathbb{M}) \), we see that

\[
\text{cl}(C \cdot q') = \text{cl}(C \cdot q_C) = \text{Gen}(C).
\]

Take any \( b \in G(\mathbb{M}) \) such that \( q' = \text{tp}(\pi(b) \pi(a^*)/\mathbb{M}) \), where \( \text{tp}(b/\mathbb{M}, a^*) \) is finitely satisfiable in \( \mathbb{M} \).

Let \( b = f(\pi(b))h \) and \( a^* = f(\pi(a^*))h^* \), then

\[
ba^* = f(\pi(b))h f(\pi(a^*))h^* = f(\pi(b)) f(\pi(a^*)) h f(\pi(a^*)) h^*.
\]
Let \( h_0 \in H(\bar{M}) \) such that \( f(\pi(b))f(\pi(a^*)) = f(\pi(b)\pi(a^*))h_0 \). Then \( h_0 \in \text{dcl}(b, \pi(a^*)) \). Since both \( \text{tp}(b/\bar{M}, \pi(a^*), h^*) \) and \( \text{tp}(\pi(a^*)/\bar{M}, h^*) \) are finitely satisfiable in \( \bar{M} \), we see that \( \text{tp}(h^*/\bar{M}, \pi(a^*), b) \) is the unique heir of \( p_H \), which implies that
\[
\text{tp}(h_0h^{f(\pi(a^*))}h^*/\bar{M}) \in \text{cl}(H \cdot p_H) = H_p_H,
\]
By the orthogonality of \( \text{tp}(\pi(b)\pi(a^*)/\bar{M}) \in \text{Gen}(C) \) and \( \text{tp}(h_0h^{f(\pi(a^*))}h^*/\bar{M}) \in H_p_H \), we have that
\[
\text{tp}(ba^*/\bar{M}) = \text{tp}(f(\pi(b)\pi(a^*))h_0h^{f(\pi(a^*))}h^*/\bar{M}) = f(q') \cdot p,
\]
with \( p \in H_p_H \).

Take some small submodel \( N \) such that \( H(N) \) meets all coset of \( H^0 \). Then \( H(N)^g \) also meets all coset of \( H^0 \) for each \( g \in G(\bar{M}) \) as \( H^0 \) is normal in \( G \), Let \( h_s \models p, c_s \models q', \) and \( c_0 \in C \) such that \( \text{tp}(c_s/N) = \text{tp}(c_0/N) \). By orthogonality of \( \text{tp}(h_s/\bar{M}) \) and \( \text{tp}(c_s/\bar{M}) \), we have that \( \text{tp}(c_s, h_s/N) = \text{tp}(c_0, h_s/N) \). For each \( h \in H(N) \), we see that \( \text{tp}((h_1h_s)^{f(c_0)}/N) \) is weakly generic, and both \( \text{tp}((h_1h_s)^{f(c_s)}/\bar{M}) \) and \( \text{tp}((h_1h_s)^{f(c_0)}/\bar{M}) \) are weakly generic extensions of \( \text{tp}((h_1h_s)^{f(c_0)}/N) \). By the stationarity of \( H \), we conclude that
\[
\text{tp}((h_1h_s)^{f(c_s)}/\bar{M}) = \text{tp}((h_1h_s)^{f(c_0)}/\bar{M})\text{.}
\]
Since both \( H(N) \) and \( H(N)^{f(c_0)} \) meet every coset of \( H^0 \), we have
\[
p' \in H_p_H = H(N)p_H = H(N)^{f(c_0)}p_H.
\]
So there exist \( h_s, h_t \in H(N) \) such that
\[
\text{tp}(h_t^{f(c_0)}h_s/\bar{M}) = \text{tp}(h_t^{f(c_0)}h_s/\bar{M}) = p'.
\]
It follows that
\[
h_s\text{tp}(h_s^{f(c_s)^{-1}}/\bar{M}) = h_t\text{tp}(h_s^{f(c_0)^{-1}}/\bar{M}) = \text{tp}((h_s^{f(c_0)h_s})^{f(c_0)^{-1}}/\bar{M}) = \text{tp}((h_t^{h_s})^{f(c_s)^{-1}}/\bar{M}).
\]
Let \( g = h_3b \), then \( \text{tp}(g/\bar{M}, a^*) \) is finitely satisfiable in \( \bar{M} \). We have that
\[
\text{tp}(ga^*/\bar{M}) = h_s f(q') \cdot p = \text{tp}(h_s^g f(c_0)h_s/\bar{M})
\]
\[
= (h_s \text{tp}(h_s^{f(c_s)^{-1}}/\bar{M}) \cdot \text{tp}(f(c_s)/\bar{M}) \text{ (by orthogonality)}
\]
\[
= \text{tp}((h_t h_s)^{f(c_s)^{-1}}/\bar{M}) \cdot \text{tp}(f(c_s)/\bar{M}) \text{ (by orthogonality)}
\]
\[
= \text{tp}(f(c_s)/\bar{M}) \cdot \text{tp}(h_t h_s/\bar{M}) \text{ (by orthogonality)}
\]
\[
= f(q') \cdot p' = r'.
\]
This completes the proof. \( \Box \)

If \( H \) is stationary, then by Lemma 2.9 every weakly generic \( r \in S_G(\bar{M}) \) is of the form \( f(q_c) \cdot p_H \) with \( q_c \in \text{Gen}(C) \) and \( p_H \in S_H(\bar{M}) \) a definable \( f \)-generic type on \( H \). We conclude directly from Lemma 2.12 that:

**Theorem 2.13.** If \( H \) is stationary, then every global weakly generic type on \( G \) is almost periodic.
Lemma 2.14. Let \( \text{tp}(c^*/\mathcal{M}) \) be a generic type on \( C \) and \( \text{tp}(h^*/\mathcal{M}) \) is a strongly \( f \)-generic type on \( H \), over \( N \prec \mathcal{M} \), where \( \mathcal{M} \) is \( |\mathcal{M}|^+ \)-saturated. Then \( \text{tp}(f(c^*)h^*/\mathcal{M}) \) is a strongly \( f \)-generic type on \( G \) over \( N \).

Proof. For any \( c_0 \in C \) and \( h_0 \in H \), we have that

\[
\text{tp}(f(c_0^0)h_0) \cdot \text{tp}(f(c^*)h^*/\mathcal{M}) = \text{tp}(f(c_0)h_0f(c^*)h^*/\mathcal{M}).
\]

Let \( h_1 = f(c_0c^*)^{-1}f(c_0)f(c^*) \) and \( h_2 = h_1h_0f(c^*) \), then \( h_2 \in \text{dcl}(\mathcal{M}, c^*) \) and

\[
\text{tp}(f(c_0)f(c^*)h_0f(c^*)h^*/\mathcal{M}) = \text{tp}(f(c_0c^*)h_2h^*/\mathcal{M})
\]

Since \( \text{tp}(h^*/\mathcal{M}) \) is strongly \( f \)-generic over \( N \), we have that \( \text{tp}(h_2h^*/\mathcal{M}, c_0c^*) \) does not fork over \( N \). We also have that \( \text{tp}(c_0c^*/\mathcal{M}) \) is a fsg type on \( C \), and thus does not fork over \( N \). We conclude that \( \text{tp}(h_2h^*/\mathcal{M}, c_0c^*/\mathcal{M}) \in S_{H \times C}(\mathcal{M}) \) does not fork over \( N \), so \( \text{tp}(f(c_0c^*) \cdot h_2h^*/\mathcal{M}) \in S_G(\mathcal{M}) \) does not fork over \( N \) as required. \( \square \)

Proposition 2.15. \( H \) is stationary iff \( G \) is stationary.

Proof. Suppose that \( H \) is stationary. Let \( N \) be a small elementary submodel of \( \mathcal{M} \) and \( p(x) = \text{tp}(f(c)h/N) \in S_G(N) \) be any weakly generic type and \( \bar{p} = \text{tp}(f(c^*)h^*/\mathcal{M}) \) a global weakly generic extension of \( p \) where \( c^* \in C(\mathcal{M}) \) and \( h^* \in H(\mathcal{M}) \). Then by Lemma 2.9, we have that \( c^* \) realizes a global generic type on \( C \) and \( h^* \) realizes a global definable weakly generic type on \( H \). Since both \( C \) and \( H \) are stationary, we see that \( \text{tp}(c^*/\mathcal{M}) \) is the unique global generic extension of \( \text{tp}(c/N) \) and \( \text{tp}(h^*/\mathcal{M}) \) is the unique global weakly generic extension of \( \text{tp}(h/N) \). By the orthogonality of \( \text{tp}(c^*/\mathcal{M}) \) and \( \text{tp}(h^*/\mathcal{M}) \), \( \text{tp}(f(c^*)h^*/\mathcal{M}) \) is determined by \( \text{tp}(c^*/\mathcal{M}) \cup \text{tp}(h^*/\mathcal{M}) \). We conclude that \( \bar{p} = \text{tp}(f(c^*)h^*/\mathcal{M}) \) is the unique global weakly generic extension of \( p \). So \( G \) is stationary as required.

Conversely, suppose that \( H \) is not stationary. It suffices to show that \( G \) has unboundedly many global weakly generic types. By Lemma 2.7, we see that \( H \) has unboundedly many strongly \( f \)-generic types. Let \( \text{tp}(c^*/\mathcal{M}) \) be a generic type on \( C \) with \( c^* \in C(\mathcal{M}) \). Let \( \{\text{tp}(h_i/\mathcal{M})\mid i < \lambda\} \) be a set of distinct strongly \( f \)-generic types on \( H \), with each \( \text{tp}(h_i/\mathcal{M}) \) strongly \( f \)-generic over \( N_i \prec \mathcal{M} \). By Lemma 3.11 in [4], each \( \text{tp}(h_i/\mathcal{M}) \) extends to a type \( \text{tp}(h_i^*/\mathcal{M}) \), which is also strongly \( f \)-generic over \( N_i \). Then each \( \text{tp}(f(c^*)h_i^*/\mathcal{M}) \) is a strongly \( f \)-generic type on \( G \) by Lemma 2.14.

Let \( \eta(x) = f(\pi(x))^{-1}x \). Then \( \eta \) is an is a \( \emptyset \)-definable function. Since each \( h_i^* \) is \( \eta((f(c^*))h_i^*) \), we see that \( \text{tp}(f(c^*)h_i^*/\mathcal{M}) \neq \text{tp}(f(c^*)h_j^*/\mathcal{M}) \) when \( i \neq j \). So there are at least \( \lambda \) many strongly \( f \)-generic types of \( G \) for each \( \lambda < |\mathcal{M}| \). We conclude that there are unboundedly many global weakly generic types on \( G \), and this completes the proof. \( \square \)

Theorem 2.16. Assume the assumptions of Theorem 1.1 hold. If \( G \) is stationary, then the space of global almost periodics coincides with the space of global weakly generics.

Proof. It is immediately from Theorem 2.13 and Proposition 2.15. \( \square \)

3 Local case

We assume here that \( M \) is an arbitrary model of \( T \), where \( T \) is \( p \text{CF} \) or an \( o \)-minimal expansion of \( \text{RCF} \), in the language \( L \).
We denote $M^{\text{ext}}$ by $M_0$, and $\text{Th}(M^{\text{ext}})$ by $T^{\text{ext}}$. Let $M_0$ be a monster model of $T^{\text{ext}}$, and $M$ the restriction of $M_0$ to $L$. Note that $M$ is also a monster of $T$ since $M$ is also a saturated model of cardinality arbitrary large. We assume again that $G$ is a definable group over $M$, and admits a $M$-definable short exact sequence

$$1 \to H \to G \to \pi C \to 1,$$

with $C$ a fsg group, $H$ a dfg group, and $f : C \to G$ a definable section of $\pi$. So we can write any $g \in G$ uniquely as $f(\pi(g))h$ or $h'f(\pi(g))$. Note that by Fact 1.16 and Fact 1.18, $H$ and $C$ also have dfg and fsg, respectively, when we compute them in $T^{\text{ext}}$.

The following Facts appear in [3].

**Fact 3.1.** $p \in S_H(M_0)$ is almost periodic iff its unique global heir is a weakly generic type.

**Fact 3.2.** Suppose $p(x) \in S_H(M)$ is definable. Then $p(x)$ implies a unique complete type $p^*(x) \in S_H(M_0)$. Moreover, if $\bar{p}$ is the unique heir of $p$ over $M_0$, then $\bar{p}$ implies a unique complete type over $M_0$, which is precisely the the unique heir of $p^*$.

Given $p \in S(M_0)$ we define $p_L = \{ \phi(x,b) \in p | \phi \in L, b \in M \}$, which is the restriction of $p$ to the language $L$.

**Lemma 3.3.** Suppose that $H$ is stationary with respect to $T$. Let $\text{WG}(S_H(M_0))$ and $\text{WG}(S_H(M))$ be the space of weakly generic types of $S_H(M_0)$ and $S_H(M)$ respectively. Then $p \mapsto p_L$ is a bijection from $\text{WG}(S_H(M_0))$ to $\text{WG}(S_H(M))$. Particularly, $H$ is stationary with respect to $T^{\text{ext}}$.

**Proof.** If $p \in S_H(M_0)$ is weakly generic, then $H(M_0)p$ is bounded, so $H(M)p_L$ is also bounded, we see that $p_L \in S_H(M)$ is also weakly generic type on $H$, hence is definable over $M$. By Fact 3.2 $p|M_0$ is determined by $p_L|M$, where $p|M_0$ and $p_L|M$ are the restrictions of $p$ and $p_L$ to $M_0$ and $M$ respectively. Using Fact 3.2 again, we see that $p_L$ determines a complete type over $M_0$, which is the unique heir of $p|M_0$. It follows that $p$ is the unique heir of $p|M_0$. As every weakly generic type $p$ on $H$ over $M_0$ is the unique heir of $p|M_0$, we conclude that $H$ is stationary with respect to $T^{\text{ext}}$.

**Remark 3.4.** We see from the previous Lemma that $H$ is stationary with respect to $T$ iff it is stationary with respect to $T^{\text{ext}}$.

**Lemma 3.5.** Suppose that $H$ is stationary (in the sense of $T$ or $T^{\text{ext}}$), $p(x) \in S_H(M_0)$ is a weakly generic type, $q(y) \in S_C(M_0)$ is a finitely satisfiable generic type. Then $p(x)$ and $q(y)$ are orthogonal.

**Proof.** Let $p(x) = \text{tp}(h^*/M_0)$ and $q(x) = \text{tp}(c^*/M_0)$. It suffices to show that $\text{tp}(h^*/M_0, c^*)$ is the unique heir of $\text{tp}(h^*/M_0)$. By Fact 2.8 we see that $p_L$ and $q_L$ are orthogonal. So $\text{tp}(h^*/M, c^*)$ is the unique heir of $\text{tp}(h^*/M)$. By Fact 3.2 we see that $\text{tp}(h^*/M_0, c^*)$ is the unique heir of $\text{tp}(h^*/M_0)$ as required.

**Corollary 3.6.** Suppose that $H$ is stationary (in the sense of $T$ or $T^{\text{ext}}$). Then $G$ is stationary with respect to $T^{\text{ext}}$ and every global weakly generic type in $S_G(M_0)$ is almost periodic.
Proof. By Fact 1.14, Fact 1.16, Lemma 3.3, and Lemma 3.5, we see that every weakly generic type $r \in S_G(M_0)$ is of the form $f(q_C) \cdot p_H$, where $q_C \in S_C(M_0)$ is generic, $p_H \in S_H(M_0)$ is definable $f$-generic over $M_0$, and $q_C$, $p_H$ are orthogonal. A similar argument as in the proof of Lemma 2.12 shows that

$$\text{cl}(G(M_0) \cdot r) = G(M_0) \cdot r = f(\text{Gen}(C)) \cdot (H(M_0) \cdot p_H).$$

So $r$ is almost periodic as required. Similarly, the proof of Proposition 2.15 shows that $G$ is stationary with respect to $T_M^{\text{ext}}$. \hfill \square

Theorem 3.7. Suppose that $H$ is stationary (in the sense of $T$ or $T_M^{\text{ext}}$). Then every weakly generic type in $S_G(M_0)$ is almost periodic (in the sense of $T_M^{\text{ext}}$).

Proof. By Corollary 4.7 in [28], the restriction of a global almost periodic type to any submodel is also almost periodic. We conclude that every type in $S_G(M_0)$ is almost periodic by Corollary 3.6.

Also, with Proposition 2.15, it proves the local case of Theorem 1.11.

For the rest of this section, $H$ need not to be stationary. Let $\mathcal{I}$ be the space of generic types in $S_C(M_0)$, then $\mathcal{I}$ is the unique minimal subflow of $S_C(M_0)$, which is also a bi-ideal of the semigroup $(S_C(M_0), \ast)$. We use $f(\mathcal{I})$ to denote the set \{ $f(q) | q \in \mathcal{I}$ \}. Let $\mathcal{J}$ be the union of all minimal subflow of $S_H(M_0)$. For any $p \in \mathcal{J}$, $\mathcal{J}(p)$ denotes the minimal subflow generated by $p$. We are going to describe the space of almost periodic types in $S_G(M_0)$ via $\mathcal{I}$ and $\mathcal{J}$.

Lemma 3.8. $\mathcal{J}$ is a bi-ideal of $S_H(M_0)$.

Proof. Clearly, $\mathcal{J}$ is a left ideal. We now show that $\mathcal{J}$ is also a right ideal. Let $p_0 \in \mathcal{J}$ and $p \in S_H(M_0)$. Then it suffices to show that $p_0 \ast p$ is almost periodic. It is easy to see that

$$\text{cl}(H(M_0) \cdot p_0 \ast p) = S_H(M_0) \ast p_0 \ast p = \mathcal{J}(p_0) \ast p.$$

For any $p_1 \in \mathcal{J}(p_0)$, we have that

$$\text{cl}(H(M_0) \cdot p_1 \ast p) = S_H(M_0) \ast p_1 \ast p = \mathcal{J}(p_1) \ast p = \mathcal{J}(p_0) \ast p.$$

We conclude that $\text{cl}(H(M_0) \cdot p_0 \ast p)$ is a minimal $H(M_0)$-flow, and hence $p_0 \ast p$ is almost periodic as required. \hfill \square

Lemma 3.9. For any $p \in \mathcal{J}$, $f(\mathcal{I}) \ast \mathcal{J}(p)$ is a minimal subflow of $S_G(M_0)$.

Proof. It suffices to show that $f(\mathcal{I}) \ast \mathcal{J}(p) \subseteq S_G(M_0) \ast f(r) \ast s$ for any $r \in \mathcal{I}$ and $s \in \mathcal{J}(p)$. Let $u \in \mathcal{I}$ and $v \in \mathcal{J}(p)$, then there are $c \in C$ and $h \in H$ such that $u = \text{tp}(c/M_0) \ast r$ and $v = \text{tp}(h/M_0) \ast s$. We assume that $b$ realizes the heir of $r$ over $dcl(M_0, c)$, $\text{tp}(h/M_0, c, b)$ is the heir of $\text{tp}(h/M_0)$, and $h^*$ realizes the global heir of $s$. Let $h' \in H$ such that $f(c)f(b)h' = f(cb)$. Then

$$f(u) \ast v = \text{tp}(f(cb)hh^*/M_0) = \text{tp}(f(c)f(b)h'hh^*/M_0)$$

$$= \text{tp}(f(c)f(b)/M_0) \ast \text{tp}(h'hh^*/M_0) \quad \text{(by Fact 1.18)}$$

$$= \text{tp}(f(c)/M_0) \ast \text{tp}(f(b)/M_0) \ast \text{tp}(h'hh^*/M_0)$$

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Let \( h_0 \in (hh')^{-1} H^0 \) such that \( \text{tp}(h_0/M_0, b, h^*) \) is finitely satisfiable in \( M_0 \). Let \( \bar{c} \) realize a coheir of \( \text{tp}(c/M_0) \) over \( M_0, h_0, b, h^* \). Then

\[
\text{tp}(f(\bar{c})/M_0) \ast \text{tp}(h_0/M_0) \ast \text{tp}(f(b)/M_0) \ast \text{tp}(h^*/M_0) \\
= \text{tp}(f(\bar{c})h_0f(b)h^*/M_0) \\
= \text{tp}(f(\bar{c})f(b)h_0f(b)h^*/M_0) \\
= \text{tp}(f(\bar{c})f(b)/M_0) \ast \text{tp}(h_0f(b)h^*/M_0) \\
= \text{tp}(f(\bar{c})/M_0) \ast \text{tp}(f(b)/M_0) \ast \text{tp}(hh'h^*/M_0) \\
= \text{tp}(f(c)/M_0) \ast \text{tp}(f(b)/M_0) \ast \text{tp}(hh'h^*/M_0)
\]

So \( f(u) \ast v = \text{tp}(f(c)h_0/M_0) \ast f(r) \ast s \) as required. \( \square \)

**Lemma 3.10.** Let \( r \in S_G(M_0) \). Then \( r \) is almost periodic iff \( r = f(q) \ast p \ast r \) for some generic type \( q \in \mathcal{I} \) and almost periodic type \( p \in \mathcal{J} \).

**Proof.** For any \( r \in S_G(M_0) \) and \( p \in \mathcal{J} \), the previous lemma shows that \( \text{cl}(G(M_0) \cdot f(\mathcal{I}) \ast p \ast r) = f(\mathcal{I}) \ast \mathcal{J}(p) \ast r \). It is also easy to see that \( \text{cl}(G(M_0) \cdot f(q') \ast p' \ast r) = f(\mathcal{I}) \ast \mathcal{J}(p') \ast r = f(\mathcal{I}) \ast \mathcal{J}(p) \ast r \) for any \( q' \in \mathcal{I} \) and \( p' \in \mathcal{J}(p) \). So we conclude that \( f(\mathcal{I}) \ast \mathcal{J}(p) \ast r \) is a minimal subflow. As \( \text{cl}(G(M_0) \cdot r) \supset f(\mathcal{I}) \ast \mathcal{J}(p) \ast r \), it follows that \( r \) is almost periodic iff \( r \in f(\mathcal{I}) \ast \mathcal{J}(p) \ast r \), which completes the proof. \( \square \)

We now consider the case where \( G \) is a product of \( C \) and \( H \). We identify \( C \) with \( C \times \{1_H\} \) and \( H \) with \( \{1_C\} \times H \), subgroups of \( G = C \times H \)

**Lemma 3.11.** Suppose that \( G = C \times H \). Then \( r \in S_G(M_0) \) is almost periodic iff \( r \in \mathcal{I} \ast \mathcal{J} \).

**Proof.** Let \( q \in S_C(M_0) \) and \( p \in S_H(M_0) \). We see from Lemma 3.10 that \( q \ast p \) is almost periodic if \( q \in \mathcal{I} \) and \( p \in \mathcal{J} \).

Conversely, suppose that \( r \in S_G(M_0) \) is almost periodic, then by Lemma 3.10 we have \( r = q \ast p \ast r \) for some \( q \in \mathcal{I} \) and \( p \in \mathcal{J} \). Let \( N \succ M_0 \) be any \( |M_0|^+ \)-saturated extension. Take \( c \in C(N) \) and \( h \in H(N) \) such that \( ch \) realizes \( r \). Let \( h^* \in H(N) \) realize the coheir of \( p \) over \( M_0, c, h \) and \( c^* \in C \) realize the coheir of \( q \) over \( N \). Then

\[
q \ast p \ast r = \text{tp}(c^*h^*ch/M_0) = \text{tp}(c^*ch^*h/M_0),
\]

which is precisely \( \text{tp}(c^*c/M_0) \ast \text{tp}(h^*h/M_0) \) since \( \text{tp}(c^*c/N) \) is a generic type over \( N \). It is easy to see from Lemma 3.8 that \( \text{tp}(h^*h/M_0) = \text{tp}(h^*/M_0) \ast \text{tp}(h/M_0) \) is almost periodic in \( H \). Hence \( r \in \mathcal{I} \ast \mathcal{J} \) as required. \( \square \)

4 Bad dfg groups and examples

In this section, we assume that \( M \) is the field \( \mathbb{R} \) of real numbers, or \( \mathbb{Q}_p \) of \( p \)-adic numbers, and \( \mathbb{M} \succ M \) a monster model. As every type over \( M \) is definable, we have that \( M_{\text{ext}} = M \).

**Definition 4.1.** Let \( H \) be a dfg group definable over \( M \). We say that \( H \) is bad if there are \( \text{tp}(a/\mathbb{M}) \) strongly \( f \)-generic over \( M \) and an \( M \)-definable function \( \theta \) such that \( \text{tp}(\theta(a)/\mathbb{M}) \) is a non-realized type finitely satisfiable in \( M \).
Remark 4.2. The badness of \( H \) is witnessed by any monster model \( \mathbb{M} \supseteq \mathbb{M} \). Let \( \text{tp}(a/\mathbb{M}) \) be as in Definition 4.1 then \( \text{tp}(a/\mathbb{M}) \) is \( M \)-invariant by Fact 1.13 (4). By Lemma 3.11 of \( [1] \), \( \text{tp}(a/\mathbb{M}) \) has an extension \( \text{tp}(a^*/\mathbb{M}) \) which is also strongly \( f \)-generic over \( M \), so is \( M \)-invariant too. We conclude that \( \text{tp}(\theta(a^*)/\mathbb{M}) \) is an \( M \)-invariant extension of \( \text{tp}(\theta(a)/\mathbb{M}) \). By the saturation of \( \mathbb{M} \), \( \text{tp}(\theta(a^*)/\mathbb{M}) \) is also finitely satisfiable in \( M \).

We aim to show that every bad dfg group \( H \) yields a counterexample \( G = H \times C \), where there is weakly generic type on \( G \) which is not almost periodic whenever \( C \) is an infinite fsg group. We will also give two “minimal” examples of non-stationary dfg groups, and show that they are bad.

We denote the additive group and the multiplicative group of \( \mathbb{M} \) by \( \mathbb{G}_a \) and \( \mathbb{G}_m \), respectively. We use \( |a| \) to denote the norm (or absolute value) of \( a \in \mathbb{M} \). Note that relation \( |x| \leq |y| \) is definable in both \( \text{Th}(\mathbb{R}) \) and \( \text{Th}(\mathbb{Q}_p) \) (see [12]) in the language of rings. For any \( a \in \mathbb{M} \), we say that \( a \) is bounded over \( M \) if \( |a| < |b| \) for some \( b \in M \). If \( a \in \mathbb{M} \) is bounded over \( M \), then there is \( st(a) \in M \) which is infinitesimally close to \( a \) over \( M \), namely, \( 0 \leq |a - st(a)| < |b| \) for all \( b \in M \setminus \{0\} \). We call \( st(a) \) the standard part of \( a \). Clearly, \( a \in \mathbb{M} \) is unbounded over \( M \) iff \( a^{-1} \) is bounded and \( st(a^{-1}) = 0 \).

The following is a folklore. Nevertheless we give a proof here for convenience.

Fact 4.3. Let \( e \in \mathbb{M} \setminus M \) and \( p \in S_1(M) \) be a non-algebraic type. Suppose that \( p_1 \) is the unique heir \( p \) over \( M, e \), then \( p_1 \) is not a finitely satisfiable in \( M \).

Proof. Suppose that \( p(x) = \text{tp}(a/M) \). Then \( a \) is either unbounded over \( M \) or infinitesimally close to \( st(a) \) over \( M \). Suppose for example that \( a \) is unbounded over \( M \). Let \( a^* \models p_1(x) \), then \( a^* \) is also unbounded over \( dcl(M,e) \) since \( p_1 \) is the heir of \( p \). Since \( e \notin M \), we see that \( e \) or \( d \neq (e - st(e))^{-1} \) is unbounded over \( M \). Now \( |a^*| > \max\{|e|,|d|\} \) but the formula \( |x| > \max\{|e|,|d|\} \) is not satisfiable in \( M \). So \( \text{tp}(a^*/M,e) \) is not finitely satisfiable in \( M \). \( \square \)

Proposition 4.4. Let \( H \) be a bad dfg group, \( C \) a definably compact group over \( M \) with \( \text{dim}(C) \geq 1 \), and \( G = C \times H \). Then \( S_G(M) \) has a weakly generic type which is not almost periodic.

Proof. Let \( \mathbb{M} \supset \mathbb{M} \) be \( [\mathbb{M}]^+ \)-saturated. Let \( c^* \in C(\mathbb{M}) \) realize a generic type on \( C \) over \( \mathbb{M} \). Let \( \text{tp}(h^*/\mathbb{M}) \in S_H(\mathbb{M}) \) be a strongly \( f \)-generic type on \( H \) over \( M \) such that \( \theta(h^*) \) is finitely satisfiable in \( M \), where \( \theta \) is an \( M \)-definable function. We see from Lemma 2.14 that \( \text{tp}((c^*,h^*)/\mathbb{M}) \) is strongly \( f \)-generic over \( M \). By Fact 1.13 \( \text{tp}((c^*,h^*)/\mathbb{M}) \) is weakly generic.

Suppose for a contradiction that \( \text{tp}((c^*,h^*)/\mathbb{M}) \) is almost periodic. Then by Lemma 3.11 \( \text{tp}(c^*/M,h^*) \) is finitely satisfiable in \( M \). We conclude that both \( \text{tp}(c^*/M,\theta(h^*)) \) and \( \text{tp}(\theta(h^*)/M,c^*) \) are finitely satisfiable in \( M \), which contradicts to Fact 4.3. \( \square \)

We say that \( \text{tp}(a/M) \in S_1(M) \) is a type of infinite (resp. type of infinitesimal) if and \( |a| > |b| \) for all \( b \in M \) (resp. \( 0 < |a| < |b| \) for all nonzero \( b \in M \)).

We now consider a definably compact subgroup \( D \) of \( M \), where

\[
D = \text{SO}_2(M) = \{(x,y) \mid x^2 + y^2 = 1\}
\]

if \( M = \mathbb{R} \), and

\[
D = \mathbb{Z}_p = \{x \mid |x| \leq 1\}
\]

if \( M = \mathbb{Q}_p \). We tend to identify a point of \( \text{SO}_2 \) with its \( x \)-coordinate, working above the \( x \)-axis.

The following Facts can be found in [16] and [32]:

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Fact 4.5.  

- $D$ has fsg.

- Let $p \in S_1(M)$ be infinitesimally close to $0 \in M$. Then its global coheir is a generic type on $D$, and its global heir is a definable $f$-generic type on $G_m$.

Fact 4.6. If $a \in \bar{M}$ is unbounded over $M$ ($|a| > |b|$ for all $b \in M$). Then $tp(a/\bar{M})$ is a definable $f$-generic type over $\emptyset$, on both $G_m$ and $G_a$.

Fact 4.7. $G_m$ and $G_a$ are stationary dfg groups.

Fact 4.8. If $D = SO_2$, then

$$D^{00} = \{ x \in M \mid x \text{ is infinitesimal close to } 0 \text{ over } M \}.$$ 

If $D = \mathbb{Z}_p$, then

$$D^{00} = \{ x \in M \mid x \text{ is infinitesimal close to } 1 \text{ over } M \}.$$ 

A result of [11] shows that

Fact 4.9. (Assuming NIP) Suppose that $Y$ is a definable group and $X$ is a definable normal subgroup of $Y$. If both $X$ and $Y/X$ have dfg, then $Y$ has dfg.

Lemma 4.10. $G_a \times G_a$ is a bad dfg group.

Proof. Clearly, $G_a \times G_a$ has dfg by Fact 4.9. Let $p \in S_1(M)$ be a type of infinite, $a^*$ realize the global coheir of $p$, $b^*$ realize the global heir of $p$, and $h^* = a*b^*$. Then it suffices to show that $tp(b^*, h^*/M) \in S_{G_a \times G_a}(M)$ is a strongly $f$-generic type over $M$.

Since both $tp(a^*/M)$ and $tp(b^*/M)$ are $M$-invariant, so is $tp(a^*, b^*/M)$ by the orthogonality of $tp(a^*/M)$ and $tp(b^*/M)$. Now $(b^*, h^*)$ and $(a^*, b^*)$ are interdefinable, we conclude that $tp(b^*, h^*/M)$ is also $M$-invariant.

We now prove that $tp(b^*, h^*/M)$ is weakly generic. Note that $G_a^2 = (G_a^2)^{00}$, so it suffices to show that

$$tp(a^*, b^*/M) = tp\left(\frac{b + h}{b}, b + b^*/M\right)$$

for any $b, h \in M$. By the orthogonality, we only need to prove that $tp(a^*/M) = tp\left(\frac{b + h}{b}, b^*/M\right)$. Clearly, both $tp(b^*/M)$ and $tp(h^*/M)$ are unbounded over $M$, and hence definable $f$-generic types on $G_a$.

It is easy to see that $tp((b/h^*)/M)$ is infinitesimally close to $0$ over $M$, thus is a definable $f$-generic type on $G_m$ We now consider $tp(a^{*}^{-1}/M)$ as global generic type on $D$. The it is easy to see from the orthogonality that $tp(a^{*}^{-1}/M, (b/h^*))$ is a generic extension of $tp(a^{*}^{-1}/M)$, thus is finitely satisfiable in $M$, and $tp((b/h^*)/M, a^*)$ is a definable $f$-generic type on $G_m$ (definable over $M$).

Keep in mind that $tp(a^{*}^{-1}/M)$ is a global generic type on $D$.

- If $D$ is $SO_2$, then $a^{*}^{-1} + (b/h^*) = a^{*}^{-1}(1 + (a*b/h^*))$. It is easy to see from the orthogonality that $tp(a^{*}b/h^*)/M, a^*)$ is a definable $f$-generic type on $G_m$ over $M$, which is also infinitesimally close to $0$ over $dcl(M, a^*)$. We conclude that $1 + (a*b/h^*) \in D^{00}$ and thus

$$tp(a^{*}^{-1}/M) = tp(a^{*}^{-1}(1 + \frac{a^*b}{h^*})/M) = tp(a^{*}^{-1} + \frac{b}{h^*}/M).$$
as \( \text{tp}(a^* - 1/\mathbb{M}, (a*b/h^*)) \) is generic.

- If \( D = \mathbb{Z}_p \), then \( \text{tp}(a^* - 1/\mathbb{M}) = \text{tp}(a^* - 1 + (b/h^*)/\mathbb{M}) \) since \( b/h^* \in D^{00} \) and \( \text{tp}(a^* - 1/\mathbb{M}, (b/h^*)) \) is generic.

  So we conclude that

  \[
  \text{tp}(a^* - 1/\mathbb{M}) = \text{tp}(a^* - 1 + \frac{b}{h^*}/\mathbb{M}) = \text{tp}(\frac{b + b^*}{h^*}/\mathbb{M}).
  \]

  On the other side,

  \[
  \frac{h + h^*}{b + b^*} = \frac{h^*}{b + b^*} + \frac{h}{b + b^*} = \frac{h^*}{b + b^*}(1 + \frac{b + b^*}{h^*} \frac{h}{b + b^*}).
  \]

  Using orthogonality again, we see that \( \text{tp}(1 + \frac{h^*}{b + b^*} \frac{h}{b + b^*}/\mathbb{M}) \) is infinitesimally close to 1 over \( \mathbb{M} \). So \( \text{tp}((1 + \frac{h^*}{b + b^*} \frac{h}{b + b^*})^{-1}/\mathbb{M}) \) is also infinitesimally close to 1 over \( \mathbb{M} \). We conclude that

  \[
  \text{tp}(\left(\frac{h^*}{b + b^*}(1 + \frac{b + b^*}{h^*} \frac{h}{b + b^*})\right)^{-1}/\mathbb{M}) = \text{tp}(\frac{b + b^*}{h^*}(1 + \frac{b + b^*}{h^*} \frac{h}{b + b^*})^{-1}/\mathbb{M}) = \text{tp}(a^* - 1/\mathbb{M}),
  \]

  which implies that

  \[
  \text{tp}(a^*/\mathbb{M}) = \text{tp}(\frac{h^*}{b + b^*}(1 + \frac{b + b^*}{h^*} \frac{h}{b + b^*})/\mathbb{M}) = \text{tp}(\frac{h + h^*}{b + b^*}/\mathbb{M}).
  \]

  We have that \( \text{tp}(b^*, h^*/\mathbb{M}) \) is weakly generic and \( M \)-invariant, so it is strongly \( f \)-generic over \( M \) as required.

  Let \( H \) be the borel subgroup of \( \text{SL}_2(\mathbb{M}) \) consisting of the matrices of the form \( \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \).

  We identify \( \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \) with a pair \( (t, u) \). Then the group operation is given by

  \[(t, u) \cdot (t', u') = (tt', tu' + t'^{-1}u).\]

**Lemma 4.11.** Let \( H \) be the borel subgroup of \( \text{SL}_2(\mathbb{M}) \) consisting of the matrices of the form \( \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \). Then \( H \) is a bad dfg group.

**Proof.** Since \( H \) is an extension of \( \mathbb{G}_a \) by \( \mathbb{G}_m \), it has dfg by Fact 4.9. Let \( p \in S_1(M) \) be a type of infinite, \( a^* \) realize the global coher of \( p \), and \( t^* \) realize the global heir of \( p \). Let \( u^* = a^*t^* \). Then it suffices to show that \( \text{tp}(t^*, u^*/\mathbb{M}) \in S_H(\mathbb{M}) \) is a strongly \( f \)-generic type over \( M \). As we have showed in Lemma 4.10 we only need to prove that

  \[
  \text{tp}(a^*/\mathbb{M}) = \text{tp}(\frac{tt^* + u(t^* - 1)}{tt^*}/\mathbb{M})
  \]

  for any \( u \in \mathbb{G}_a \) and \( t \in \mathbb{G}_m^0 \).

  Since \( \text{tp}(t^2/\mathbb{M}) \) is a global definable \( f \)-generic type on \( \mathbb{G}_m \), we see that \( \text{tp}((t/u)t^{*2}/\mathbb{M}) \) is also a global definable \( f \)-generic type on \( \mathbb{G}_m \). Now

  \[
  \frac{tt^* + u(t^* - 1)}{tt^*} = a^* + \frac{u(t^* - 1)}{tt^*} = a^*(1 + a^*-1 u(t^* - 1)).
  \]
By orthogonality, letting $\epsilon = (a^*-1 u(t^*-1))/(tt^*)$, we have that $tp(1 + \epsilon/M)$ is infinitesimally close to 1 over $M$, so is $\emptyset$-definable. Clearly,

$$tp(a^*/M) = tp(a^*(1 + \epsilon)/M) \iff tp(a^*-1/M) = tp(a^*-1(1 + \epsilon)^{-1}/M).$$

Since $tp(a^*-1/M)$ is a generic type on $D$ and $tp((1 + \epsilon)^{-1}/M) \vdash D^{00}$, we see from the orthogonality that $tp(a^*-1/M) = tp(a^*-1(1 + \epsilon)^{-1}/M)$. So

$$tp(a^*/M) = tp(a^* + \frac{u(t^*-1)}{tt^*}/M) = tp(\frac{tu^* + u(t^*-1)}{tt^*}/M),$$

which completes the proof. \(\square\)

Finally, we conjecture that

**Conjecture 2.** Assume that $H$ is a dfg group definable in an o-minimal structure or a $p$-adically closed field, then $H$ is bad iff $H$ is non-stationary.

**References**

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