CONSTRUCTION OF LOW-ENERGY EFFECTIVE ACTION IN \( \mathcal{N}=4 \) SUPER YANG-MILLS THEORIES

I.L. Buchbinder
Department of Theoretical Physics Tomsk State Pedagogical University
Tomsk, 634041, Russia

Abstract
We review a recent progress in constructing low-energy effective action in \( \mathcal{N}=4 \) super Yang-Mills theories. Using harmonic superspace approach we consider \( \mathcal{N}=4 \) SYM in terms of unconstrained \( \mathcal{N}=2 \) superfield and apply \( \mathcal{N}=2 \) background field method to finding effective action for \( \mathcal{N}=4 \) SU(N) SYM broken down to U(1)\(^{N-1} \). General structure of leading low-energy corrections to effective action is discussed.

1 Introduction
Low-energy structure of quantum supersymmetric field theories is described by the effective lagrangians of two types: chiral and general or holomorphic and non-holomorphic. Non-holomorphic or general contributions to effective action are given by integrals over full superspace while holomorphic or chiral contributions are given by integrals over chiral subspace of superspace. As a result the effective action in low-energy limit is defined by the chiral superfield \( \mathcal{F} \) which is called holomorphic or chiral effective potential and real superfield \( \mathcal{H} \) which is called non-holomorphic or general effective potential.

Possibility of holomorphic corrections to effective action was firstly demonstrated in [1] ( see also [2]) for \( \mathcal{N}=1 \) SUSY and in [3] for \( \mathcal{N}=2 \) SUSY. The modern interest to structure of low-energy effective action in extended supersymmetric theories was inspired by the seminal papers [4] where exact instanton contribution to holomorphic effective potential has been found for \( \mathcal{N}=2 \) SU(2) super Yang-Mills theory. These results have later been extended for various gauge groups and for coupling to matter (see e.g. [5]). One can show that in generic \( \mathcal{N}=2 \) SUSY models namely the holomorphic effective potential is leading low-energy contribution. Non-holomorphic potential is next to leading correction. A detailed investigation of structure of low-energy effective action for various \( \mathcal{N}=2 \) SUSY theories has been undertaken in [6-9].

A further study of quantum aspects of supersymmetric field models leads to problem of effective action in \( \mathcal{N}=4 \) SUSY theories. These theories being maximally extended global supersymmetric models possess the remarkable properties on quantum level: (i) \( \mathcal{N}=4 \) super Yang-Mills model is finite quantum field theory, (ii) \( \mathcal{N}=4 \) super Yang-Mills model is superconformal invariant.
theory and hence, its effective action can not depend on any scale. These
properties allow to analyze a general form of low-energy effective action and
see that it changes drastically in compare with generic $\mathcal{N}=2$ super Yang-Mills
theories.

Analysis of structure of low-energy effective action in $\mathcal{N}=4$ SU(2) SYM
model spontaneously broken down to U(1) has been fulfilled in recent paper
by Dine and Seiberg [10]. They have investigated a part of effective action de-
pending on $\mathcal{N}=2$ superfield strengths $W, \bar{W}$ and shown (i) Holomorphic quan-
tum corrections are trivial in $\mathcal{N}=4$ SYM. Therefore, namely non-holomorphic
effective potential is leading low-energy contribution to effective action, (ii)
Non-holomorphic effective potential $\mathcal{H}(W, \bar{W})$ can be found on the base of the
properties of quantum $\mathcal{N}=4$ SYM theory up to a coefficient. All perturbative
or non-perturbative corrections do not influence on functional form of $\mathcal{H}(W, \bar{W})$
and concern only of this coefficient.

The approaches to direct calculation of non-holomorphic effective potential
including the above coefficient have been developed in [11-13], extensions for
gauge group SU(N) spontaneously broken to maximal torus have been given in
[15-17] (see also [14] where some bosonic contributions to low-energy effective
action have been found).

2 $\mathcal{N}=4$ super Yang-Mills theory in harmonic superspace

As well known, the most powerful and adequate approach to investigate the
quantum aspects of supersymmetric field theories is formulation of these theo-
ries in terms of unconstrained superfields carrying out a representation of the
supersymmetry. Unfortunately such a manifestly $\mathcal{N}=4$ supersymmetric formulation for $\mathcal{N}=4$ Yang-Mills theory is still unknown. A purpose of this paper is
study a structure of low-energy effective action for $\mathcal{N}=4$ SYM as a functional of
$\mathcal{N}=2$ superfield strengths. In this case it is sufficient to realize the $\mathcal{N}=4$ SYM
theory as a theory of $\mathcal{N}=2$ unconstrained superfields. It is naturally achieved
within harmonic superspace. The $\mathcal{N}=2$ harmonic superspace [19] is the only
manifestly $\mathcal{N}=2$ supersymmetric formalism allowing to describe general $\mathcal{N}=2$
supersymmetric field theories in terms of unconstrained $\mathcal{N}=2$ superfields. This
approach has been successfully applied to problem of effective action in various
$\mathcal{N}=2$ models in recent works [7, 9, 12, 13, 16, 18, 20].

From point of view of $\mathcal{N}=2$ SUSY, the $\mathcal{N}=4$ Yang-Mills theory describes
interaction of $\mathcal{N}=2$ vector multiplet with hypermultiplet in adjoint representation. Within harmonic superspace approach, the vector multiplet is realized by unconstrained analytic gauge superfield $V^{++}$. As to hypermultiplet, it can be described either by a real unconstrained superfield $\omega$ ($\omega$-hypermultiplet)
or by a complex unconstrained analytic superfield \( q^+ \) and its conjugate \(( q^-)\). In the \( \omega \)-hypermultiplet realization, the classical action of \( N=4 \) SYM model has the form

\[
S[V^+, \omega] = \frac{1}{2g^2} \text{tr} \int d^4x d^4\theta W^2 - \frac{1}{2g^2} \text{tr} \int d\zeta^{(-4)} \nabla^+ \omega \nabla^+ \omega
\]  

(1)

The first terms here is pure \( N=2 \) SYM action and the second term is action \( \omega \)-hypermultiplet. In \( q \)-hypermultiplet realization, the action of the \( N=4 \) SYM model looks like this

\[
S[V^+, q^+, q^-] = \frac{1}{2g^2} \text{tr} \int d^4x d^4\theta W^2 - \frac{1}{2g^2} \text{tr} \int d\zeta^{(-4)} q^+ q^- \nabla^+ q^+ q^- 
\]  

(2)

where

\[
q_i^+ = (q^+, \tilde{q}^+), \quad q^+ = \varepsilon^{ij} q_j^+ = (\tilde{q}^+, -q^+)
\]  

(3)

All other denotions are given in [19]. Both models (1,2) are equivalent and manifestly \( N=2 \) supersymmetric by construction. However, as has been shown in [19], both these models possess hidden \( N=2 \) supersymmetry and as a result they actually are \( N=4 \) supersymmetric.

3 General form of non-holomorphic effective potential

We study the effective action \( \Gamma \) for \( N=4 \) SYM with gauge group SU(2) spontaneously broken down to U(1). This effective action is considered as a functional of \( N=2 \) superfield strengths \( W \) and \( \bar{W} \). Then holomorphic effective potential \( \mathcal{F} \) depends on chiral superfield \( W \) and it is integrated over chiral subspace of \( N=2 \) superspace with the measure \( d^4x \, d^4\theta \). Non-holomorphic effective potential \( \mathcal{H} \) depends on both \( W \) and \( \bar{W} \). It is integrated over full \( N=2 \) superspace with the measure \( d^4x \, d^8\theta \). Taking into account the mass dimensions of \( W \), \( \mathcal{F}(W) \), \( \mathcal{H}(W, \bar{W}) \) and the superspace measures \( d^4x \, d^4\theta \) and \( d^4x \, d^8\theta \) ones write

\[
\mathcal{F}(W) = W^2 f\left( \frac{W}{\Lambda} \right), \quad \mathcal{H}(W, \bar{W}) = \mathcal{H}\left( \frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda} \right)
\]  

(4)

where \( \Lambda \) is some scale and \( f\left( \frac{W}{\Lambda} \right) \) and \( \mathcal{H}\left( \frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda} \right) \) are the dimensionless functions of their arguments. The effective action is scale independent, therefore

\[
\Lambda \frac{d}{d\Lambda} \int d^4x \, d^4\theta W^2 f\left( \frac{W}{\Lambda} \right) = 0, \quad \Lambda \frac{d}{d\Lambda} \int d^4x \, d^8\theta \mathcal{H}\left( \frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda} \right) = 0
\]  

(5)
First of eqs (5) leads to $f\left(\frac{W}{\Lambda}\right) = \text{const}$. Second of eqs (5) reads

$$\Lambda \frac{d}{d\Lambda} \mathcal{H} = g\left(\frac{W}{\Lambda}\right) + \bar{g}\left(\frac{\bar{W}}{\Lambda}\right)$$

(6)

Here $g$ is arbitrary chiral function of chiral superfield $\frac{W}{\Lambda}$ and $\bar{g}$ is conjugate function. Since $f\left(\frac{W}{\Lambda}\right) = \text{const}$ the holomorphic effective potential $\mathcal{F}(W)$ is proportional to classical lagrangian $W^2$. General solution to eq (6) is written as follows

$$\mathcal{H}\left(\frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda}\right) = c \log \frac{W^2}{\Lambda^2} \log \frac{\bar{W}^2}{\Lambda^2}$$

(7)

with arbitrary coefficient $c$. As a result, holomorphic effective potential is trivial in $\mathcal{N}=4$ SYM theory. Therefore, namely non-holomorphic effective potential is leading low-energy quantum contribution to effective action. Moreover, the non-holomorphic effective potential is found exactly up to coefficient and given by eq (7) [10]. Any perturbative or non-perturbative quantum corrections are included into a single constant $c$. However, this result immediately face the problems: 1) is there exist a calculational procedure allowing to derive $\mathcal{H}(W/\Lambda, \bar{W}/\Lambda)$ in form (7) within a model? 2) what is value of $c$? If $c = 0$, the non-holomorphic effective potential vanishes and low-energy effective action in $\mathcal{N}=4$ SYM is defined by the terms in effective action depending on the covariant derivatives of $W$ and $\bar{W}$, 3) what is a structure of non-holomorphic effective potential for the other than SU(2) gauge groups?

The answers all these questions have been given in [11-17]. Further we are going to discuss a general manifestly $\mathcal{N}=2$ supersymmetric and gauge invariant procedure of deriving the non-holomorphic effective potential in one-loop approximation [13,16]. This procedure is based on the following points: 1) formulation of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=2$ unconstrained superfields in harmonic superspace [19], 2) $\mathcal{N}=2$ background field method [9] providing manifest gauge invariance on all steps of calculations, 3) identical transformation of path integral for effective action over $\mathcal{N}=2$ superfields to path integral over some $\mathcal{N}=1$ superfields. This point is nothing more then replacement of variables in path integral, 4) superfield proper-time technique (see first of refs [2]) which is manifestly covariant method for evaluating effective action in superfield theories.

4 Non-holomorphic effective potential for SU(N)-gauge group

We study effective action for the classically equivalent theories (1, 2) within $\mathcal{N}=2$ background field method [9]. We assume also that the gauge group of
these theories is SU(N). In accordance with background field method \[9\], the one-loop effective action in both realizations of $\mathcal{N}=4$ SYM is given by

$$\Gamma(1)[V++] = \frac{i}{2} \text{Tr}_{(2,2)} \log \hat{\Box} - \frac{i}{2} \text{Tr}_{(4,0)} \log \hat{\Box}$$  \hspace{1cm} (8)$$

where $\hat{\Box}$ is the analytic d’Alambertian introduced in \[9\].

$$\hat{\Box} = \mathcal{D}^m \mathcal{D}_m + \frac{i}{2} (\mathcal{D}^{+\alpha} W) \mathcal{D}_\alpha + \frac{i}{2} (\mathcal{D}^{+\dot{\alpha}} \bar{W}) \mathcal{D}^{\dot{\alpha}} -$$

$$- \frac{i}{4} (\mathcal{D}^{+\alpha} \mathcal{D}_\alpha W) \mathcal{D}^- - \frac{i}{8} [\mathcal{D}^{+\alpha}, \mathcal{D}^-] W + \frac{i}{2} \{W, W\}$$  \hspace{1cm} (9)$$

The formal definitions of the $\text{Tr} (2,2) \log \hat{\Box}$ and $\text{Tr} (4,0) \log \hat{\Box}$ are given in \[12\]. Our purpose is finding of non-holomorphic effective potential $\mathcal{H}(W, \bar{W})$ where the constant superfields $W$ and $\bar{W}$ belong to Cartan subalgebra of the gauge group SU(N). Therefore, for calculation of $\mathcal{H}(W, \bar{W})$ it is sufficient to consider on-shell background, $\mathcal{D}^\alpha (\mathcal{D}_\alpha W) = 0$. In this case the one-loop effective action (8) can be written in the form \[16\]

$$\Gamma(1) = \sum_{k<l} \Gamma_{kl}, \quad \Gamma_{kl} = i \text{Tr} \log \Delta_{kl}$$  \hspace{1cm} (10)$$

$$\Delta_{kl} = \mathcal{D}^m \mathcal{D}_m - (W^{k\alpha} - W^{l\alpha}) \mathcal{D}_\alpha + (\bar{W}^{\dot{k}\dot{\alpha}} - \bar{W}^{\dot{l}\dot{\alpha}}) \mathcal{D}^{\dot{\alpha}} + |\Phi^k - \Phi^l|^2$$  \hspace{1cm} (11)$$

and $\mathcal{D}_m, \mathcal{D}_\alpha, \mathcal{D}^{\dot{\alpha}}$ are the $\mathcal{N}=1$ supercovariant derivatives. Here

$$\Phi = \text{diag}(\Phi^1, \Phi^2, \ldots, \Phi^N), \quad \sum_{k=1}^{N} \Phi^k = 0.$$  \hspace{1cm} (12)$$

$$W_\alpha = \text{diag}(W^1_\alpha, \ldots, W^N_\alpha), \quad \sum_{k=1}^{N} W^k_\alpha = 0.$$  \hspace{1cm} (13)$$

$\Phi$ and $W_\alpha$ are the $\mathcal{N}=1$ projections of $W$. The operator (11) has been introduced in \[16\]. Evaluation of the $\text{Tr} \log \Delta_{kl}$ leads to

$$\Gamma_{kl} = \frac{1}{(4\pi)^2} \int d^8 z \frac{W^{\alpha kl} W^{\dot{\alpha} kl} W^{\dot{\beta} kl} (\Phi^{kl})^2}{(\Phi^{kl})^2}$$  \hspace{1cm} (14)$$

where

$$\Phi^{kl} = \Phi^k - \Phi^l, \quad W^{kl} = W^k - W^l.$$  \hspace{1cm} (15)$$

Eqs (10, 13, 14) define the non-holomorphic effective potential of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=1$ projections of $\mathcal{N}=2$ superfield strengths. The last step
is restoration of $\mathcal{N} = 2$ form of effective action (13). For this purpose we write

$$
\int d^{12}z \mathcal{H}(W, \bar{W}) = \int d^8z W^\alpha \bar{W}_\alpha \bar{W}_\beta \partial^4 \mathcal{H}(\bar{\Phi}, \Phi) + \text{derivatives}
$$

(15)

Comparison of eqs (14) and (15) leads to

$$
\Gamma^{(1)} = \int d^4xd^8\theta \mathcal{H}(\bar{W}, W)
$$

$$
\mathcal{H}(W, \bar{W}) = \frac{1}{(8\pi)^2} \sum_{k<l} \log \left( \frac{(\bar{W}^k - \bar{W}^l)^2}{\Lambda^2} \right) \log \left( \frac{(W^k - W^l)^2}{\Lambda^2} \right)
$$

(16)

Eq (16) is our final result. In partial case of SU(2) group spontaneously broken down to U(1) eq (16) coincides with eq (7) where $c = 1/(8\pi^2)$. Another approach to deriving the effective potential (16) for SU(2)-group was developed in recent paper [13].

5 Discussion

Eq (16) defines the non-holomorphic effective potential depending on $\mathcal{N} = 2$ superfield strengths for $\mathcal{N} = 4$ SU(N) super Yang-Mills theories. As a result we answered all the questions formulated in section 3. First, we have presented the calculational procedure allowing to find non-holomorphic effective potential. Second, we calculated the coefficient $c$ in eq (7) for SU(2) group. It is equal to $1/(8\pi^2)$. Third, a structure of non-holomorphic effective potential for the gauge group SU(N) has been established.

It is interesting to point out that the scale $\Lambda$ is absent when the non-holomorphic effective potential (16) is written in terms of $\mathcal{N} = 1$ projections of $W$ and $\bar{W}$ (see eqs (15, 16)). Therefore, the $\Lambda$ will be also absent if we write the non-holomorphic effective potential through the component fields. We need in $\Lambda$ only to present the final result in manifestly $\mathcal{N} = 2$ supersymmetric form. $\mathcal{N} = 1$ form of non-holomorphic effective potential (10) allows very easy to get leading bosonic component contribution. Schematically it has the form $F^4/|\phi|^4$, where $F_{mn}$ is abelian strength constructed from vector component and $\phi$ is a scalar component of $\mathcal{N} = 2$ vector multiplet. It means that non-zero expectation value of scalar field $\phi$ plays a role of effective infrared regulator in $\mathcal{N} = 4$ SYM theories.
Generalization of low-energy effective action containing all powers of constant \( F_{mn} \) has recently been constructed in [18]. The direct proof of absence of three- and four-loop corrections to \( \mathcal{H} \) was given in [20].

It was recently noticed [17, 21] that R-symmetry and scale independence do not prohibit the terms of the form \( G(W^{ij}/W^{kl}, \bar{W}^{ij}/\bar{W}^{kl}) \) in non-holomorphic effective potential for SU(N)-models with \( N > 2 \) where \( G \) is a real function and \( W^{ij} = W^{i} - W^{j} \). However, the calculations [20] did not confirm an emergence of such terms at one-, two-, three- and four loops. One can expect that \( N=4 \) supersymmetry imposes more rigid restrictions on a structure of effective potential then only R-symmetry and scale independence taking place for any \( N=2 \) superconformal theory.

Acknowledgments. I am very grateful to E.I. Buchbinder, E.I. Ivanov, S.M. Kuzenko, B.A. Ovrut, A.Yu Petrov, A.A. Tseytlin for collaboration. The work was supported in part by the RFBR grant 99-02-16617, RFBR-DFG grant 99-02-04022, INTAS grant 991-590 and 001-254.

References

1. P. West, Phys. Lett. B 258, 357 (1991); I. Jack, D.R.T. Jones, P. West, Phys. Lett. B 258, 382 (1991); M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 359, 571 (1991); I.L. Buchbinder, S.M. Kuzenko, A.Yu. Petrov, Phys. Lett. B 321, 372 (1994).
2. I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity (IOP Publishing, 1995; Revised Edition, 1998); I.L. Buchbinder, A.Yu. Petrov, Phys. Lett. B 416, 209 (1999); N.G. Pletnev, A.T. Banin, Phys. Rev. D 60, 105017 (1999); I.L. Buchbinder, M. Cvetic, A.Yu. Petrov, Mod. Phys. Lett. A 15, 783 (2000); Nucl. Phys. B 571, 358 (2000)
3. P. De Vecchia, R. Musto, F. Nicodemi, R. Pettorino, Nucl. Phys. B 252, 635 (1984); N. Seiberg, Phys. Lett. B 206, 75 (1988).
4. N. Seiberg, E. Witten, Nucl. Phys. B 426, 19 (1994); B 430, 485 (1994).
5. d’Hoker, D.H. Phong, Lectures on Supersymmetric Yang-Mills Theory and Integrable Systems, [hep-th/991227].
6. B. de Wit, M.T. Grisaru, M. Roček, Phys. Lett. B 374, 297 (1996); A. Pickering, P. West, Phys. Lett. B 383, 54 (1996); M.T. Grisaru, M. Roček, R. von Unge, Phys. Lett. B 383, 415 (1996); U. Lindström, F. Gonzalez-Rey, M. Roček, R. von Unge, Phys. Lett. B 388, 581 (1996); A. De Giovanni, M.T. Grisaru, M. Roček, R. von Unge, D.Zanon, Phys. Lett. B 409, 251 (1997); A.T. Banin, I.L. Buchbinder, N.G. Pletnev, Nucl. Phys. B 598, 371 2001; S.M. Kuzenko, I.N. McArthur,
Hypermultiplet effective action: N=2 approach, hep-th/0105121.

7. I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 412, 309 (1997); E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, Mod. Phys. Lett. A 13, 1071 (1998); S.V. Ketov, Phys. Rev. D 57, 1277 (1998); S. Eremin, E. Ivanov, Mod. Phys. Lett. A 15, 1859 (2000); I.L. Buchbinder, I.B. Samsonov, Mod. Phys. Lett. A 14, 2537 (1999); E.A. Ivanov, S.V. Ketov, B.M. Zupnik, Nucl. Phys. B 509, 53 (1997).

8. M. Chaichian, W.F. Chen, C. Montonen, Nucl. Phys. B 537, 161 (1999); D.G.C. McKeon, I. Sachs, I.A. Shovkovy, Phys. Rev. D 59, 105010 (1999); J. Wirstam, Phys. Rev. D 60, 065014 (1999).

9. I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 417, 61 (1998); I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, Phys. Lett. B 433, 335 (1998).

10. M. Dine, N. Seiberg, Phys. Lett. B 409, 239 (1997).

11. V. Periwal, R. von Unge, Phys. Lett. B 430, 71 (1998); F. Gonzalez-Rey, M. Roček, Phys. Lett. B 434, 303 (1989).

12. I.L. Buchbinder, S.M. Kuzenko, Mod. Phys. Lett. A 13, 1629 (1998).

13. S.M. Kuzenko, I.N. McArthur, Phys. Lett. B 506, 140 (2001).

14. I. Chepelev, A.A. Tseytlin, Nucl. Phys. B 511, 629 (1998).

15. F. Gonzalez-Rey, B. Kulik, I.Y. Park, M. Roček, Nucl. Phys. B 544, 218 (1999).

16. E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko, Phys. Lett. B 446, 216 (1999).

17. D.A. Lowe, R. von Unge, JHEP, 9811, 014, (1998).

18. I.L. Buchbinder, S.M. Kuzenko, A.A. Tseytlin, Phys. Rev. D 62, 045001 (2000).

19. A. Galperin, E. Ivanov, S. Kalitzyn, V. Ogievetsky, E. Sokachev, Class. Quant. Grav. 1, 469 (1984); A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokachev, Class. Quant. Grav. 2, 601 (1985); 2, 617 (1985); B. Zupnik, Phys. Lett. B 183, 175 (1987).

20. I.L. Buchbinder, A.Yu. Petrov, Phys. Lett. B 482, 429 (2000).

21. M. Dine, J. Gray, Phys. Lett. B 481, 427 (2000).