A no-broadcasting theorem for quantum asymmetry and coherence and a trade-off relation for approximate broadcasting

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Symmetries of both closed and open-system dynamics imply many significant constraints. These generally have instantiations in both classical and quantum dynamics (Noether’s theorem, for instance, applies to both sorts of dynamics). We here provide an example of such a constraint which has no counterpart for a classical system, that is, a uniquely quantum consequence of symmetric dynamics. Specifically, we demonstrate the impossibility of broadcasting asymmetry (symmetry-breaking) relative to a continuous symmetry group, for bounded-size quantum systems. The no-go theorem states that if, during a symmetric dynamics, asymmetry is created at one subsystem, then the asymmetry of the other subsystem must be reduced. We also find a quantitative relation describing the tradeoff between the subsystems. These results cannot be understood in terms of additivity of asymmetry, because, as we show here, any faithful measure of asymmetry violates both sub-additivity and super-additivity. Rather, it is understood as a consequence of an information-disturbance principle, which is a quantum phenomenon. Our result also implies that if a bounded-size quantum reference frame for the symmetry group, or equivalently, a bounded-size reservoir of coherence (e.g., a clock with coherence between energy eigenstates in quantum thermodynamics) is used to implement any operation that is not symmetric, then the quantum state of the frame/reservoir is necessarily disturbed in an irreversible fashion, i.e., degraded.

Introduction– Finding the consequences of symmetries of a closed or open quantum dynamics is a problem that has a wide range of applications in physics, with Noether’s theorem being perhaps the most prominent example. It is notable that the consequences that physicists have focussed on, including the conservation of Noether charges and currents, have held in both quantum and classical contexts. A natural question, therefore, is whether there are consequences of symmetric dynamics that are unique to quantum theory.

Eugene P. Wigner pioneered the study of the consequences of symmetry in quantum theory and made various fundamental contributions to the topic. For instance, in 1952, he showed\textsuperscript{[1, 2]} that under the restriction of using only Hamiltonians which conserve an observable \( L \) that is additive across subsystems (e.g., the total angular momentum in a given direction), an exact measurement of another observable \( O \) becomes impossible unless \( O \) commutes with \( L \). This fundamental no-go result, known as the Wigner-Araki-Yanase (WAY) theorem\textsuperscript{[3, 4]}, can equivalently be phrased as a consequence of the restriction to Hamiltonians which are invariant under a continuous symmetry, namely the symmetry for which \( L \) is the generator.

In recent years, inspired by the success of entanglement theory\textsuperscript{[5]}, the problem of finding the consequences of symmetric dynamics has been further studied in the framework of quantum resource theories\textsuperscript{[6–9]}. In the resource theory of asymmetry\textsuperscript{[20]}, any state which breaks the symmetry under consideration, i.e., any state which has some asymmetry, is treated as a resource (similar to entangled states in entanglement theory). A particular case of interest, which is relevant in the context of the WAY theorem for instance, is when the symmetry under consideration is the continuous set of translations generated by a fixed observable \( H \), i.e., \( \{ e^{-iHt} : t \in \mathbb{R} \} \) (Note that \( H \) need not be the Hamiltonian, nor \( t \) the time parameter, although the notation is meant to bring to mind this example).

In this case a state contains asymmetry iff it contains coherence (off-diagonal terms) with respect to the eigenspaces of \( H \). It follows that the resource theory of asymmetry provides a natural framework to study this sort of coherence, which is known as unspeakable coherence\textsuperscript{[10, 11]}, and which is the notion that is relevant for quantum metrology\textsuperscript{[12]} and quantum thermodynamics\textsuperscript{[13–14]} (as argued in Ref.\textsuperscript{[10]}).

The resource-theoretic approach to the study of symmetric dynamics and asymmetry properties of quantum states has shed new light on earlier work. For instance, it was found that the skew information, a function introduced by Wigner and Yanase\textsuperscript{[15]} as a replacement for the von-Neumann entropy in the presence of symmetry, is, in fact, a measure of asymmetry\textsuperscript{[16–18]}. Also, it was found in Ref.\textsuperscript{[19]} that the WAY theorem can be understood as a corollary of a deep result in quantum information theory, known as the no-programming theorem\textsuperscript{[20, 22]}.

Another no-go theorem about continuous symmetries was uncovered in Ref.\textsuperscript{[23]}, the no-catalysis theorem\textsuperscript{[53]}. This result concerns state conversions using operations which are covariant (symmetric) with respect to a compact connected Lie group, and states that if the pure state conversion \( \psi \to \phi \) is not achievable, then the catalyzed version of this same conversion, \( \psi \otimes \eta \to \phi \otimes \eta \), is also not achievable for any choice of pure catalyst state, \( \eta \), in a finite-dimensional Hilbert space\textsuperscript{[23]} (See also\textsuperscript{[24]} for related observations).

Taking the perspective of resource theories has also made evident that existing results on symmetric dynamics, including the no-catalysis and WAY theorems, are not uniquely
quantum. This is because it has clarified that a key assumption in each of these no-go theorems is that the resource state being used is not perfectly asymmetric in the sense that the state and its translated versions (under the symmetry transformations) are not perfectly distinguishable. If one makes the analogous assumption classically—that the probability distribution over classical configurations constituting one’s resource is not perfectly asymmetric in the same sense—then one obtains similar no-go results.

In this Letter, we find an example of a consequence of symmetric dynamics that is uniquely quantum, namely, a no-broadcasting theorem for asymmetry. It asserts that if during a symmetric dynamics asymmetry is created at one subsystem, then the asymmetry of another subsystem should reduce. We also show that this result does not hold classically.

In fact, we prove a more general result, namely, that under symmetric dynamics, if one uses a bounded-size quantum system in an asymmetric state (a reference frame or coherence reservoir) as a resource to perform an asymmetric operation (i.e., a task which is impossible under symmetric dynamics), then one necessarily disturbs its state irreversibly, i.e., the frame/reservoir degrades. While it has been previously noted that quantum reference frames degrade when used to implement certain asymmetric operations [25][28], these works did not consider arbitrary target operations and also considered only the case where the frame starts in a pure state.

Our proofs rely on a new version of the information-disturbance principle [29], which itself relies on deep results in quantum information theory, such as the properties of Markov states [30], and information-preserving structures [31].

We also find a tradeoff relation for approximate broadcasting, namely, a lower bound on the amount of disturbance caused by the broadcasting of asymmetry/coherence in the case of pure states. This investigation also leads us to take note of a very general constraint on measures of asymmetry (See theorem 2).

Covariance condition— We begin with some formalism. Consider an arbitrary physical process with input systems $Q$ and $S$ and output systems $Q'$ and $S'$, and let $\Lambda_{QS\rightarrow Q'S'}$ (or simply $\Lambda$) be the corresponding Completely-Positive Trace-Preserving (CPTP) map from the density operators of $QS$ to the density operators of $Q'S'$. We are interested in the processes satisfying the covariance condition

$$\forall t \in \mathbb{R} : \Lambda \circ [U_Q(t) \otimes U_S(t)] = [U_{Q'}(t) \otimes U_{S'}(t)] \circ \Lambda .$$  \hfill (1)

Here, for each system $X \in \{Q, S, Q', S'\}$, we have defined $U_X(t) [\cdot] = e^{-iH_X t} [\cdot] e^{iH_X t}$, where $H_X$ is a (Hermitian) observable defined on system $X$. Note that for each system $X$, the map $\mathbb{R} \ni t \rightarrow U_X(t)$ can be interpreted as a representation of a group of translations. Eq. (1) means that the description of the process $\Lambda_{QS\rightarrow Q'S'}$ is independent of which reference frame for translations one uses.

A particular case of interest is when the operator $H_X$ is the Hamiltonian describing the closed-system dynamics of $X$, so that $U_X(t)$ represents evolution for time $t \in \mathbb{R}$. In this case, the tensor product form of $U_Q(t) \otimes U_S(t)$ (and $U_{Q'}(t) \otimes U_{S'}(t)$) reflects the fact that systems $Q$ and $S$ (and systems $Q'$ and $S'$) are not interacting with one another before (and after) the process $\Lambda$, and, therefore, can be treated as separate non-interacting subsystems. Then, the covariance condition in Eq. (1) means that the effect of the process $\Lambda$ on the inputs $Q$ and $S$, does not depend on the time at which the process acts on these systems. This property is satisfied, for instance, by any thermal machine that interacts a system with thermal baths and with work reservoirs (batteries).

Asymmetry as a resource— A simple consequence of a process satisfying the covariance condition in Eq. (1) is that it cannot generate asymmetry. Suppose the input state $\rho_{QS}$ is symmetric with respect to the symmetry represented by $t \rightarrow U_Q(t) \otimes U_S(t)$, i.e.,

$$\forall t \in \mathbb{R} : U_Q(t) \otimes U_S(t) [\rho_{QS}] = \rho_{QS} .$$  \hfill (2)

Note that this holds iff $\rho_{QS}$ is diagonal, or incoherent relative to the eigenspaces of $H_Q \otimes I_S + I_Q \otimes H_S$, where $I_S$ and $I_Q$ are the identity operators on $S$ and $Q$. Then, it can be easily seen that the covariance of process $\Lambda$ implies that incoherent states of the input systems are mapped to incoherent states of the output systems. In this sense, asymmetry, or coherence, is a resource which cannot be generated under covariant operations. Obviously, the physical interpretation of this resource depends on the nature of the symmetry. For instance, only for states that are asymmetric with respect to time-translations is a system useful as a clock and only for states that are asymmetric with respect to rotations is a system useful as a gyroscope.

Under the restriction to processes which satisfy the covariance condition in Eq. (1), having access to a resource of asymmetry allows one to perform operations that would otherwise be impossible. For any fixed state $\rho_Q$ of system $Q$, let $\mathcal{E}_{S\rightarrow S'}$ be the CPTP map from $S$ to $S'$ induced by the covariant operation $\Lambda_{QS\rightarrow Q'S'}$.

$$\mathcal{E}_{S\rightarrow S'}(\cdot) \equiv \text{Tr}_{Q'} [\Lambda_{QS\rightarrow Q'S'}(\rho_{QS} \otimes \cdot)] .$$  \hfill (3)

(Note that $Q$ and $S$ are assumed to be initially uncorrelated.) It can be easily seen that if $\rho_Q$ is a symmetric state, then the map $\mathcal{E}_{S\rightarrow S'}$ is covariant, i.e., satisfies $\forall t \in \mathbb{R} : \mathcal{E} \circ U_S(t) = U_{S'}(t) \circ \mathcal{E}$. On the other hand, using a state $\rho_Q$ which contains asymmetry, we can implement a non-covariant channel $\mathcal{E}_{S\rightarrow S'}$.

For instance, if the process $\Lambda$ satisfies time-translation symmetry, then using an input system $Q$, whose state $\rho_Q$ contains asymmetry with respect to time translations (or equivalently, contains coherence relative to the energy eigenspaces), one can nonetheless implement on $S$ operations which do not satisfy time-translation symmetry. Therefore, for an agent who seeks to implement an operation at a particular time relative to some time standard (i.e., reference clock) but who lacks access to it, such a system can constitute a token of the standard,
a quantum clock that is synchronized with the reference clock. **Irreversibility and Degradation**—Suppose that there is a covariant process under which \( \rho_Q \to \sigma_Q \). We say that the state conversion \( \rho_Q \to \sigma_Q \) is *reversible* in the resource theory if there exists a covariant process \( R_{Q' \to Q} \) which recovers \( \rho_Q \) from \( \sigma_Q \), i.e., \( R_{Q' \to Q}(\sigma_Q) = \rho_Q \); otherwise, we say that the state conversion is *irreversible* and that the asymmetry of \( \rho_Q \) is degraded under the state conversion.

**Degradation theorem**—The following theorem shows that using a system \( Q \) in an asymmetric state \( \rho_Q \) to implement a non-covariant operation on system \( S \) necessarily degrades the asymmetry of \( \rho_Q \).

**Theorem 1.** Let \( Q \) be a system with a finite-dimensional Hilbert space, prepared in state \( \rho_Q \). Suppose system \( S \), initially uncorrelated with \( Q \), interacts with system \( Q \) via a process \( \Lambda_{Q \to QS} \). Let \( \mathcal{E}_{S \to S'} \), defined in Eq. (3), be the effective map which determines how the reduced state of output \( S' \) depends on the state of \( S \) (for a fixed \( \rho_Q \)). If the process \( \Lambda_{Q \to QS} \) is covariant, but \( \mathcal{E}_{S \to S'} \) is not, then, for some states of \( S \) (including the completely mixed state) the conversion from \( \rho_Q \) to \( \sigma_Q \) is irreversible, i.e., state \( \rho_Q \) cannot be recovered from state \( \rho_{Q'} \), via a covariant process \( R_{Q' \to Q} \).

(See Fig. 1)

\[
\begin{array}{c}
\rho_Q \\
\mathcal{E}_{S \to S'}(\rho_S) \\
\rho_S \\
\Lambda \\
\mathcal{R}_{Q' \to Q} \\
\rho_Q
\end{array}
\]

**FIG. 1:** If using a covariant operation \( \mathcal{R}_{Q' \to Q} \), state \( \rho_Q \) can be recovered from \( \sigma_Q \), then the effective operation \( \mathcal{E}_{S \to S'} \) is covariant, and therefore can be implemented without interacting with \( \rho_Q \).

Thus, if \( \rho_Q \) can be recovered from \( \sigma_Q \), then the effective channel \( \mathcal{E}_{S \to S'} \) is covariant, and therefore, can be implemented without interacting with the resource \( \rho_Q \). As we show later, this theorem follows from a new version of the information-disturbance principle.

It is worth noting that, unlike the no-catalysis theorem of \cite{23}, here we do not assume that systems \( Q \) and \( S' \) are uncorrelated after the recovery process \( \mathcal{R}_{Q' \to Q} \) is applied; rather, the result concerns the reduced state of \( Q \) itself. Such correlations become relevant, for instance, if we want to repeat this process to implement \( \mathcal{E}_{S \to S'} \) multiple times, i.e., to implement \( \mathcal{E}_{S \to S'}^n \), for arbitrary integer \( n \). As we see in the following, if one requires such a notion of repetitability, which amounts to assuming lack of correlations, then the proof of the no-go theorem becomes much simpler and can be achieved by using arguments similar to the no-catalysis theorem of \cite{23} or arguments of \cite{23}. However, interestingly, according to theorem 1 even if we relax this requirement and ignore correlations, the no-go theorem still holds, i.e., using state \( \rho_Q \) to implement a non-covariant process \( \mathcal{E}_{S \to S'} \), will necessarily imply that \( Q \) undergoes a state conversion \( \rho_Q \to \sigma_Q \) that is irreversible.

**No-broadcasting of asymmetry/coherence**—The special case of Theorem 1 that is the focus of this work concerns a map that incorporates both the process \( \Lambda_{Q \to QS} \) as well as any recovery operation \( \mathcal{R}_{Q' \to Q} \) on it, and which is specialized to the case where \( S \) is trivial. We can conceptualize such a map as a broadcast map from \( Q \) to the pair of systems \( Q \) and \( S' \). Unlike the usual discussions of broadcasting \cite{32}, where there is a set of possible states at the input and no restriction on the nature of the broadcast map, we are here interested in the case where there is a single state at the input, but the broadcast map is constrained to be covariant. We will say that asymmetry/coherence can be broadcast if there is a covariant broadcast map that takes any input state \( \rho_Q \) to a state \( \sigma_{QS'} \) with the property that (i) the input state \( \rho_Q \) is reproduced in the output \( Q \), i.e., \( \sigma_Q = \rho_Q \) where \( \sigma_Q = \text{Tr}_{S'}(\sigma_{QS'}) \), and (ii) the state of system \( S' \) has nontrivial asymmetry/coherence. Theorem 1 implies that for bounded-size \( Q \), such a map does not exist, that is,

\[
\rho_Q \to \sigma_{QS'} \implies \text{NOT}(\sigma_Q = \rho_Q \text{ AND } [\sigma_{S'},H_{S'}] \neq 0).
\]

(4)

We summarize this result as: asymmetry/coherence cannot be broadcast.

To see that this no-go result does not apply to classical asymmetry, it suffices to note that a map that clones any point distribution on a classical configuration space is covariant relative to any symmetry and consequently such a map achieves broadcasting of asymmetry when acted on any point distribution that breaks the symmetry of interest.

**Non-additivity of asymmetry**—At first glance, it may appear that the impossibility of broadcasting asymmetry should follow from an intuitive idea, namely, that asymmetry might be a kind of extensive quantity, so that to create asymmetry in the system \( S' \) one needs to reduce the asymmetry of \( Q \). This intuition can be formalized using the notion of measures of asymmetry (See e.g., \cite{17, 33, 34}): a function \( f \) from states to real numbers is called a measure of asymmetry if (i) it is non-increasing under covariant operations, i.e., \( \rho_A \to \sigma_B \) implies \( f(\rho_A) \geq f(\sigma_B) \), and (ii) it vanishes on symmetric states. A measure of asymmetry is called *faithful* if it vanishes only on symmetric states. The Wigner-Yanase Skew information, \( f(\rho_X) \equiv -\text{Tr}((\sqrt{\rho_X} H_X^2)/2) \), is an example of a faithful measure of asymmetry where \( H_X \) is the generator of the symmetry (e.g., \( H_X \) is the Hamiltonian if the symmetry is time translations).

A measure of asymmetry, \( f \), is called sub-additive if for any state \( \sigma_{AB} \) of a composite system \( AB \), \( f(\sigma_{AB}) \leq f(\sigma_A) + f(\sigma_B) \) where \( \sigma_A \) and \( \sigma_B \) are the reduced states of \( \sigma_{AB} \) on \( A \) and \( B \), respectively. It is called super-additive if \( f(\sigma_{AB}) \geq f(\sigma_A) + f(\sigma_B) \). A measure of asymmetry is called additive if it is both sub-additive and super-additive.

Suppose that there was even a single faithful super-additive measure of asymmetry, \( f \). In this case, \( \rho_Q \to \sigma_{QS'} \) would imply that \( f(\rho_Q) \geq f(\sigma_{QS'}) \geq f(\sigma_Q) + f(\sigma_{S'}) \). Since \( f \) is assumed to be faithful, if \( \sigma_{S'} \) is not symmetric, then \( f(\sigma_{S'}) > 0 \),
and we would be able to infer that $f(\rho_Q) > f(\sigma_Q)$ and consequently that $\rho_Q \rightarrow \sigma_Q$ is irreversible, which would prove the impossibility of broadcasting asymmetry.

However, interestingly, as we show in the SM,

**Theorem 2.** A faithful measure of asymmetry is neither super-additive, nor sub-additive.

It follows that the argument articulated above—wherein one seeks to justify no-broadcasting of asymmetry from super-additivity of asymmetry—is not sound. Indeed, the fact that our no-broadcasting result holds in spite of theorem [2] makes it more surprising. (Note that the failure of super-additivity for the skew information has been observed previously in [35,36].)

It is worth noting that some faithful measures of asymmetry, such as skew information, are additive on product states. Therefore, the argument articulated above does yield a proof of our no-broadcasting theorem, Eq. (4), for the special case where $\sigma_{QS'} = \sigma_Q \otimes \sigma_{S'}$. However, to prove the theorem in the general case we need more powerful tools from quantum information theory.

**Approximate broadcasting**—Next, we derive a quantitative version of our no-broadcasting theorem. Specifically, we assume that there is a covariant process which transforms $\rho_Q$ to $\sigma_{QS'}$, and we seek to find a quantitative limit on the degree of success in broadcasting in terms of the amount of asymmetry (unspeakable coherence) in the initial state $\rho_Q$. We quantify the success in broadcasting by a combination of (i) the degree of irreversibility of the state conversion $\rho_Q \rightarrow \sigma_Q$ (where $\sigma_Q \equiv \text{Tr}_R(\sigma_{QS'})$) and (ii) the amount of asymmetry (unspeakable coherence) left in state $\sigma_{S'}$ (where $\sigma_{S'} \equiv \text{Tr}_Q(\sigma_{QS'})$).

To quantify the degree of irreversibility in a state conversion $\rho_Q \rightarrow \sigma_Q$, we consider the minimum achievable infidelity in recovering the initial state $\rho_Q$ from the final state $\sigma_Q$.

$$\text{irrev}(\rho_Q, \sigma_Q) \equiv 1 - \max_R \text{Fid}^2(\rho_Q, R(\sigma_Q)),$$

where the maximization is over all covariant CPTP maps. Here, $\text{Fid}(\tau_1, \tau_2) \equiv \|\sqrt{\tau_1} \sqrt{\tau_2}\|_1$ is the fidelity of [37,39]. This definition implies that $\text{irrev}(\rho_Q, \sigma_Q)$ is between 0 and 1, and the state conversion $\rho_Q \rightarrow \sigma_Q$ is irreversible iff $\text{irrev}(\rho_Q, \sigma_Q) = 0$.

To quantify the asymmetry left in state $\sigma_{S'}$, we consider a measure of asymmetry defined in terms of the (Uhlmann) fidelity. For any $t \in \mathbb{R}$, define $f_t(\rho) \equiv 1 - \text{Fid}(\rho, e^{-iHt} \rho e^{iHt}) = 1 - \|\sqrt{\rho} e^{-iHt} \sqrt{\rho}\|_1$. As we show in the SM, $f_t$ is a measure of asymmetry for any $t \in \mathbb{R}$, and it takes values in $[0, 1]$. $f_t(\rho)$ quantifies how distinguishable $\rho$ is from $e^{-iHt} \rho e^{iHt}$.

The trade-off relation we prove, unlike our no-broadcasting theorem, is limited to the case where the initial state is pure, a fact which we denote by writing $\rho_Q = \psi_Q$. Specifically, if $\psi_Q \rightarrow \sigma_{QS'}$, then

$$\forall t \in \mathbb{R} : f_t(\sigma_{S'}) \leq 4 \sqrt{\text{irrev}(\psi_Q, \sigma_Q)} \frac{1 - f_t(\psi)}{1 - f_t(\psi)}.

This tradeoff relation states that for any $t \in \mathbb{R}$, the measure of asymmetry of $\sigma_{S'}$, relative to $f_t$, is upper bounded by a multiple of the degree of irreversibility of the state conversion $\psi_Q \rightarrow \sigma_Q$, as quantified by $\sqrt{\text{irrev}(\psi_Q, \sigma_Q)}$. Note that as $f_t(\sigma_{S'})$ increases, the derived lower bound on $\text{irrev}(\psi_Q, \sigma_Q)$ decreases.

The proof is given in the SM. There, we also demonstrate that in the special case where the state $\rho_Q$ is pure, this trade-off relation immediately implies our no-broadcasting theorem, Eq. (4).

Next, we present the proof of the no-broadcasting theorem in the general case, where $\rho_Q$ is possibly mixed, and then use this result to prove theorem [1].

**Information-disturbance principle**—Since the early days of quantum mechanics, various formulations of the information-disturbance principle have been proposed (See e.g. [40–46]). Roughly speaking, this principle states that any process which obtains information about an unknown state of a quantum system disturbs the state irreversibly. In Ref [29], we present a new formulation of this principle, which is summarized in the following lemma.

**Lemma 3.** (Information-disturbance principle) Consider a classical message $x$ drawn from a set $X$ (which may be discrete or continuous). Let $x \mapsto \rho_Q(x)$ be a quantum encoding of this message in system $Q$. Suppose that under a fixed CPTP channel (i.e., one that is independent of $x$), the state $\rho_Q(x)$ of system $Q$ is transformed to the state $\sigma_{QS'}$ of the composite system $QS'$. If the information encoded in $Q$ is preserved, in the sense that the initial state on $Q$, $\rho_Q(x)$, can be recovered from the final (marginal) state on $Q$, without knowing $x$, then there exists a complete set of orthogonal projectors $\{\Pi_\mu\}_\mu$ such that $\{\Pi_\mu\}_\mu$ commute with all states in $\{\rho_Q(x)\}_x$, and (ii) for any $x \in X$, the reduced state $\sigma_{S'}(x) = \text{Tr}_Q(\sigma_{QS'}(x))$ of $S'$ can be prepared by performing the non-disturbing projective measurement $\{\Pi_\mu\}_\mu$ on state $\rho_Q(x)$ of system $Q$, followed by a state preparation for system $S'$, which only depends on $\mu$.

Note that the map from each classical message $x$ to a corresponding probability distribution over the label $\mu$ is a classical encoding of the message. Therefore, the lemma implies that if all the information about a classical message $x$ that is initially encoded (in general, quantumly) in $Q$ is preserved in $Q$, then only the information about $x$ that is encoded classically in $Q$ can be transmitted to $S'$.

We now note that if there is a covariant operation that converts $\rho_Q$ to $\sigma_{QS'}$, then this map also achieves the conversion $U_Q(t) [\rho_Q] \rightarrow U_Q(t) \otimes U_{S'}(t) [\sigma_{QS'}]$ for all $t \in \mathbb{R}$, which one can conceptualize as the conversion of a quantum encoding of $t$ in $Q$ to a quantum encoding of $t$ in $QS'$. Therefore, if there is a map that satisfies our definition of a broadcast map for asymmetry, then it also preserves the information about $t$ encoded in $Q$ while also encoding some nontrivial information about $t$ in $S'$. 
Lemma 3 then implies that the only information about $t$ that is available in the preparation of $S^t$ is what can be obtained from the outcome of aprojective measurement on $Q$, $\{\Pi_\mu\}_\mu$, satisfying the constraint that for all $\mu$, $\Pi_\mu$ must commute with every element of the set $\{e^{-iH_0t}p\rho e^{iH_0t} : t \in \mathbb{R}\}$.

The next step of the argument is where the restriction of scope to continuous symmetries occurs. We appeal to the following lemma, proved in the SM.

**Lemma 4.** Consider a given state $\rho$, observable $H$, and set of projectors $\{\Pi_\mu\}_\mu$. If it is the case that $[e^{-iH_0t}p\rho e^{iH_0t}, \Pi_\mu] = 0$ for all $t \in \mathbb{R}$ and for all $\mu$, and if it is the case that $e^{-iH_0t}p\rho e^{iH_0t}$ is a differentiable function of $t$ (which holds if $H$ is a bounded operator), then the probability $\text{Tr}(\Pi_\mu e^{-iH_0t}p\rho e^{iH_0t})$ is independent of $t$.

This lemma implies that the outcome variable $\mu$ cannot encode any information about the parameter $t$. Consequently, neither can $S^t$, i.e., $e^{-iH_0t}\sigma S^t e^{iH_0t}$ is independent of $t$, hence $[\sigma S^t, H S^t] = 0$. In conclusion, we have shown that if $\sigma Q = \rho Q$ then $[\sigma S^t, H S^t] = 0$, which implies Eq. (4), the impossibility of broadcasting asymmetry.

In the SM, we show how no-broadcasting of asymmetry can be leveraged into a proof of the more general result described in theorem 1.

**Conclusion**—In this work we have used a new formulation of the information-disturbance principle to prove a strong constraint on the manipulation of asymmetry (equivalently, unspeakable coherence) and we have discussed some of its applications in the context of quantum clocks and quantum thermodynamics. In addition to our no-go theorem, we have also found a tradeoff relation which quantifies the amount of irreversibility in a covariant state conversion that broadcasts asymmetry/coherence. It is worth noting that any continuous symmetry (i.e., one associated with a Lie group) includes translational symmetries (because every Lie group has one or more Abelian continuous subgroups). Consequently, the constraints, the hypothesis has been discussed here are generic to continuous symmetries. The results are also specific to continuous symmetries in that they generally do not hold for discrete symmetries. This parallels the situation with the celebrated WAY theorem [14], and the no-catalysis theorem of [23].

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[52] Originally called the resource theory of “frameness” in Ref. [43], but renamed in Ref. [49].
[53] The question of whether there is nontrivial catalysis in the resource theory of asymmetry was inspired by the fact that there is nontrivial catalysis in entanglement theory [50][51].
Proof of theorem 2

To prove that a faithful measure of asymmetry is not sub-additive, we note that for any symmetry, there exists states $\sigma_{AB}$ of the composite system $AB$, such that $\sigma_{AB}$ is asymmetric, while the reduced states of $A$ and $B$ can both be symmetric. For instance, $\sigma_{AB}$ can be a maximally entangled state which is asymmetric, while $\sigma_A$ and $\sigma_B$ are completely mixed states, and therefore symmetric. The assumption of faithfulness of the measure of asymmetry $f$ implies that $f(\sigma_{AB}) > 0$, while $f(\sigma_A) = f(\sigma_B) = 0$. It follows that $f(\sigma_{AB}) > f(\sigma_A) + f(\sigma_B)$, implying that $f$ is not sub-additive.

Note that the presence of entanglement is not necessary to observe a violation of sub-additivity. For instance, for any finite group $G$, we can define the state

$$\sigma_{AB} = \frac{1}{|G|} \sum_{g \in G} |g\rangle \langle g| \otimes U_B(g)\rho_B U_B^\dagger(g) ,$$  

(7)

where $G \ni g \to U_B(g)$ is the representation of symmetry on system $B$, and $\{|g\rangle : g \in G\}$ is a set of orthonormal states on $A$. We assume the representation of symmetry on $A$ is trivial. Then, it can be easily seen that if state $\rho_B$ is asymmetric, then the state $\sigma_{AB}$ will also be asymmetric. However, the reduced state on $B$, which is given by $\sigma_B = \frac{1}{|G|} \sum_{g \in G} U_B(g)\rho_B U_B^\dagger(g)$, is not asymmetric. The reduced state of $A$ also cannot be asymmetric, because the representation of symmetry on $A$ is trivial. It follows that for any measure of asymmetry $f$, $f(\sigma_A) = f(\sigma_B) = 0$. However, since $\sigma_{AB}$ is asymmetric and $f$ is faithful, $f(\sigma_{AB}) > 0$. This implies that $f(\sigma_{AB}) > f(\sigma_A) + f(\sigma_B)$, i.e., $f$ is not sub-additive. This argument can be easily generalized to the case of continuous groups as well.

To prove that a faithful measure of asymmetry is not super-additive, we use the fact that starting with any asymmetric state, a covariant map can distribute its asymmetry to an arbitrary number $n$ of systems, such that the reduced state of each system is a fixed (i.e., independent of $n$) asymmetric state. Specifically, one can consider the universal cloner map $[47]$, which approximately clones its input state in a $d$-dimensional Hilbert space $A$ to arbitrary many output systems $A_1 \cdots A_n$ with Hilbert spaces identical with $A$. The universal cloner $\mathcal{E}_{A\to A_1 \cdots A_n}$ of $[47]$ is covariant with respect to $SU(d)$ symmetry, i.e., for any unitary $U$, it satisfies

$$\mathcal{E}_{A\to A_1 \cdots A_n}(U(\cdot)U^\dagger) = U^{\otimes n}\mathcal{E}_{A\to A_1 \cdots A_n}(\cdot)U^\dagger^{\otimes n} .$$

(8)

Therefore, it also satisfies the covariance condition for any symmetry group.

Let $\sigma_{A_1 \cdots A_n} = \mathcal{E}_{A\to A_1 \cdots A_n}(\rho_A)$ be the joint state of $A_1 \cdots A_n$ for input $\rho_A$ and

$$\sigma_{A_i} = Tr_{A\setminus A_i}(\sigma_{A_1 \cdots A_n})$$

(9)

be the reduced state of system $A_i$, where the partial trace is over all subsystems except $A_i$. Then, the covariance of the universal cloner implies that for any input state $\rho_A$, the reduced state of each system $A_i$ is in the form

$$\sigma_{A_i} = c_n \rho_A + (1 - c_n) \frac{I}{d} ,$$

(10)

where $0 < c_n < 1$ determines the fidelity of cloning and is bounded away from 0, even in the limit $n \to \infty$ [47]. Therefore, if state $\rho_A$ breaks a given symmetry, then state $\sigma_{A_i}$ will also break that symmetry.

Suppose $f$ is a measure of asymmetry. Then, the covariance of the universal cloner $\mathcal{E}_{A\to A_1 \cdots A_n}$ implies

$$f(\sigma_{A_1 \cdots A_n}) = f(\mathcal{E}_{A\to A_1 \cdots A_n}(\rho_A)) \leq f(\rho_A) .$$

(11)

If $f$ is super-additive then

$$f(\sigma_{A_1 \cdots A_n}) \geq \sum_{i=1}^n f(\sigma_{A_i}) = n \times f(c_n \rho_A + (1 - c_n) \frac{I}{d}) .$$

(12)

Combining these two inequalities, we conclude that

$$f(\rho_A) \geq n \times f(c_n \rho_A + (1 - c_n) \frac{I}{d}) .$$

(13)

In the limit that $n$ goes to infinity $c_n$ converges to a fixed non-zero constant $c_\infty$, and therefore state $c_\infty \rho_A + (1 - c_\infty) \frac{I}{d}$ still breaks the symmetry. Therefore, if $f$ is faithful then $f(c_\infty \rho_A + (1 - c_\infty) \frac{I}{d})$ is strictly larger than zero. But this implies that, in
the limit that \( n \) goes to infinity, the right-hand side of Eq. (13) diverges, while the left-hand side remains finite. This leads to a contradiction and proves that a faithful measure of asymmetry cannot be super-additive.

It is worth noting that, although in this proof we used the universal cloner map \[47\], the proof does not rely on the specific properties of this map. For instance, rather than the universal cloner, we could have used a covariant measure-and-prepare map, which first performs a covariant measurement on input \( A \), and then prepares states of \( A_1 \cdots A_n \) according to the outcome of the measurement. In this case, one can also show that \( \sigma_{A_i} \) can be an asymmetric state, independent of the choice of \( n \), and the above argument can be applied to prove the result.

**Proof that \( f_t \) is a measure of translation asymmetry for any \( t \in \mathbb{R} \)**

For any \( t \in \mathbb{R} \), we have defined

\[
 f_t(\rho) \equiv 1 - \text{Fid}(\rho, e^{-iHt}\rho e^{iHt}), \tag{14}
\]

where \( \text{Fid}(\rho, e^{-iHt}\rho e^{iHt}) = \| \sqrt{\rho} \sqrt{e^{-iHt}\rho e^{iHt}} \|_1 \) is the (Uhlmann) fidelity between states \( \rho \) and \( e^{-iHt}\rho e^{iHt} \).

To see that \( f_t(\rho) \) takes values in the range \([0, 1]\), it suffices to note that the fidelity between any two quantum states is a value in the range \([0, 1]\).

To prove that \( f_t \) is a measure of translational asymmetry for any \( t \in \mathbb{R} \), one must show (i) that it is zero for incoherent states, and (ii) that it is non-increasing under translationally covariant operations.

To see that \( f_t \) is zero for incoherent states, it suffices to note that for incoherent \( \rho, [\rho, H] = 0 \), which immediately implies \( f_t(\rho) = 0 \) by Eq. (14).

To see that \( f_t \) is non-increasing under translationally covariant operations, it suffices to note that for any translationally covariant operation \( \mathcal{E} \),

\[
 f_t(\mathcal{E}(\rho)) = 1 - \text{Fid}(\mathcal{E}(\rho), e^{-iHt}\mathcal{E}(\rho)e^{iHt}) \\
 = 1 - \text{Fid}(\mathcal{E}(\rho), \mathcal{E}(e^{-iHt}\rho e^{iHt})) \\
 \leq 1 - \text{Fid}(\rho, e^{-iHt}\rho e^{iHt}) \tag{17}
\]

where in the second line we have used the covariance of \( \mathcal{E} \), in the third line we have used the monotonicity of the fidelity, and in the fourth line we have used the definition (14).

**Proof of the tradeoff relation in Eq. (6)**

Recall that we have defined a degree of irreversibility of a state conversion \( \rho_Q \rightarrow \sigma_Q \) by

\[
 \text{irrev}(\rho_Q, \sigma_Q) \equiv 1 - \max_{\mathcal{R}_Q} \text{Fid}^2(\rho_Q, \mathcal{R}_Q(\sigma_Q)), \tag{19}
\]

where \( \mathcal{R}_Q \) is a recovery operation. Relative to this definition, and the definition of the measure of asymmetry \( f_t \) in Eq. (14), the tradeoff relation we seek to prove here is as follows: if the state conversion \( \psi_Q \rightarrow \sigma_Q S' \) (where \( \psi_Q \) is pure) is achievable by a translationally covariant operation, then

\[
 \forall t \in \mathbb{R} : \quad f_t(\sigma_{S'}) \leq 4\sqrt{\frac{\text{irrev}(\psi_Q, \sigma_Q)}{1 - f_t(\psi_Q)}}. \tag{20}
\]

If it can be shown that for all translationally covariant candidates for the recovery operation, \( \mathcal{R}_Q \), it holds that

\[
 \forall t \in \mathbb{R} : \quad (1 - f_t(\psi_Q)) f_t(\sigma_{S'}) \leq 4\sqrt{1 - \text{Fid}^2(\psi_Q, \mathcal{R}_Q(\sigma_Q))}, \tag{21}
\]

then this equation also holds for the particular recovery operation that achieves the maximum value of \( \text{Fid}^2(\psi_Q, \mathcal{R}_Q(\sigma_Q)) \), and then the tradeoff relation follows directly from Eq. (19) whenever \( f_t(\psi_Q) < 1 \). It suffices, therefore, to establish Eq. (21) for all translationally covariant operations \( \mathcal{R}_Q \).
Consider the post-recovery state $\omega_{QS'}$ associated to the recovery operation $\mathcal{R}_Q$.

$$\omega_{QS'} \equiv (\mathcal{R}_Q \otimes \mathcal{I}_S')(\sigma_{QS'}), \tag{22}$$

where $\mathcal{I}_S'$ is the identity operation on $S'$. We denote the marginals on $S'$ and $Q$ of the post-recovery state by $\omega_{S'} \equiv \text{Tr}_Q(\omega_{QS'})$ and $\omega_Q \equiv \text{Tr}_{S'}(\omega_{QS'})$ respectively. Using $\omega_Q$, we can rewrite Eq. (22) as

$$\forall t \in \mathbb{R}: \quad (1 - f_t(\psi)) f_t(\sigma_{S'}) \leq 4\sqrt{1 - \text{Fid}^2(\psi_Q, \omega_Q)} . \tag{23}$$

This is what will be proven below.

The starting inequality for our proof is obtained by an application of the following lemma:

**Lemma 5**. For any pairs of states $\tau_1$ and $\tau_2$ and unitary $U$, it holds that

$$\left| \text{Fid}(U \tau_1 U^\dagger, \tau_1) - \text{Fid}(U \tau_2 U^\dagger, \tau_2) \right| \leq 4\sqrt{1 - \text{Fid}(\tau_1, \tau_2)} . \tag{24a}$$

Applying this lemma with $\tau_1$ as the post-recovery state $\omega_{QS'}$, $\tau_2$ as the initial state $|\psi\rangle \langle \psi| \otimes \sigma_{S'}$ and $U$ as the time-translation $e^{-iH_{QS'}t}$, and recalling the definition of $f_t$ from Eq. (14), we obtain

$$|f_t(\psi \otimes \sigma_{S'}) - f_t(\omega_{QS'})| \leq 4\sqrt{1 - \text{Fid}(\psi_Q \otimes \sigma_{S'}, \omega_{QS'})} . \tag{25}$$

It remains only to show that Eq. (25) implies Eq. (23).

We begin with the left-hand side of Eq. (25). We note that by the definition of $f_t$ in Eq. (14), and the assumption that $Q$ and $S'$ are noninteracting ($H_{QS'} = H_Q + H_{S'}$), we have

$$f_t(\psi) = 1 - \text{Fid}(\psi(t) U_Q(t) |\psi\rangle \langle \psi| U_Q^\dagger(t) \otimes U_{S'}(t) \sigma_{S'} U_{S'}^\dagger(t)) \tag{26}$$

$$= 1 - \text{Fid}(\psi(t) U_Q(t) |\psi\rangle \langle \psi|) \text{Fid}(\sigma_{S'} U_{S'}(t) \sigma_{S'} U_{S'}^\dagger(t)) \tag{27}$$

$$= 1 - [1 - f_t(\psi_Q)] [1 - f_t(\sigma_{S'})] \tag{28} .$$

It follows that the left-hand side of Eq. (25) can be rewritten as

$$|f_t(\psi_Q \otimes \sigma_{S'}) - f_t(\omega_{QS'})| = \left| 1 - [1 - f_t(\psi_Q)] [1 - f_t(\sigma_{S'})] - f_t(\omega_{QS'}) \right| \tag{29}$$

Next, we note that

$$f_t(\psi_Q) \geq f_t(\omega_{QS'}) \tag{30}$$

based on the monotonicity of $f_t$ under translationally covariant operations together with the fact that $\omega_{QS'}$ is obtained from $\psi_Q$ via a translationally covariant operation.

Given that the value of $f_t$ is in the range $[0, 1]$, we obtain the following lower bound for the left-hand side of Eq. (25),

$$|f_t(\psi_Q \otimes \sigma_{S'}) - f_t(\omega_{QS'})| \geq \left| 1 - [1 - f_t(\psi_Q)] [1 - f_t(\sigma_{S'})] - f_t(\psi_Q) \right| \tag{31}$$

and then by simple algebra,

$$|f_t(\psi_Q \otimes \sigma_{S'}) - f_t(\omega_{QS'})| \geq \left| [1 - f_t(\psi_Q)] f_t(\sigma_{S'}) \right| \tag{32} .$$

Next, we turn to the right-hand side of Eq. (25).

It remains only to show that

$$\text{Fid}(\psi) \geq \text{Fid}(\psi_Q \otimes \sigma_{S'}, \omega_{QS'}) \tag{33}$$

because this, together with Eq. (32), establishes what we need to show, namely, that Eq. (25) implies Eq. (23).

The fact that fidelity is a concave function of each of its arguments implies that if Eq. (35) holds when $\omega_{QS'}$ is pure, then it holds when $\omega_{QS'}$ is mixed as well. Consequently, it suffices to prove Eq. (35) in the case of $\omega_{QS'}$ being pure.
Letting
\[ \omega_{QS'} = |\Omega\rangle\langle \Omega|_{QS'}. \] (34)
and recalling that if one of the arguments of the fidelity is a pure state \( \Psi = |\Psi\rangle\langle \Psi| \), then \( \text{Fid}(\tau, \Psi) = \langle \Psi|\tau|\Psi\rangle \), it follows that we must show that
\[ \langle \Omega| (|\psi\rangle\langle \psi|_{Q} \otimes \sigma_{S'}) |\Omega\rangle_{QS'} \geq \langle \psi| \omega_{Q}|\psi\rangle^2. \] (35)

Consider the decomposition of \( |\Omega\rangle_{QS'} \) into its component within the subspace associated to the projector \( |\psi\rangle\langle \psi|_{Q} \otimes \mathbb{1}_{S'} \) and its component orthogonal to this subspace. This can be written as
\[ |\Omega\rangle_{QS'} = \sqrt{p}|\psi\rangle \otimes |\eta\rangle_{S'} + \sqrt{1-p}|\gamma\rangle_{QS'}, \] (36)
where
\[ p \equiv \langle \Omega| (|\psi\rangle\langle \psi|_{Q} \otimes \mathbb{1}_{S'}) |\Omega\rangle_{QS'}, \] (37)
\[ = \langle \psi| \omega_{Q}|\psi\rangle. \] (38)
and where \( |\gamma\rangle_{QS'} \) is the component of \( |\Omega\rangle_{QS'} \) which is orthogonal to \( |\psi\rangle_{Q} \otimes |\eta\rangle_{S'} \).

Eq. (36) implies that
\[ \langle \Omega| (|\psi\rangle\langle \psi|_{Q} \otimes \omega_{S'}) |\Omega\rangle_{QS'} = p\langle \eta| \omega_{S'}|\eta\rangle_{S'}. \] (40)

However, noting that
\[ \langle \Omega| (\mathbb{1}_{Q} \otimes |\eta\rangle_{S'} \langle \eta|_{S'} ) |\Omega\rangle_{QS'} \geq \langle \Omega| (|\psi\rangle\langle \psi|_{Q} \otimes \mathbb{1}_{S'}) |\Omega\rangle_{QS'}, \] (41)
and that
\[ \langle \Omega| (\mathbb{1}_{Q} \otimes |\eta\rangle \langle \eta|_{S} ) |\Omega\rangle_{QS'} = \langle \eta| \omega_{S'}|\eta\rangle_{S'}. \] (42)
it follows that
\[ \langle \eta| \omega_{S'}|\eta\rangle_{S'} \geq p. \] (43)

Substituting this into Eq. (40) and recalling the definition of \( p \) from Eq. (37), we arrive at Eq. (35). This concludes the proof.

**Recovering no-broadcasting of asymmetry from the tradeoff relation for the case of pure states**

Here we prove that the tradeoff relation of Eq. (6) implies our no-broadcasting theorem, Eq. (4), in the special case where the state \( \rho_{Q} \) is pure.

We take \( \rho_{Q} = \psi_{Q} \) to denote the purity assumption. Eq. (6) implies that whenever \( f_{t}(\psi_{Q}) < 1 \), if the state conversion \( \psi_{Q} \rightarrow \sigma_{Q} \) is reversible, so that \( \text{irrev}(\psi_{Q}, \sigma_{Q}) = 0 \), then \( f_{t}(\sigma_{S'}) = 0 \). However, because \( f_{t} \) is not a faithful measure of asymmetry, this is not sufficient to infer that \( \sigma_{S'} \) is symmetric. However, this conclusion does follow if \( f_{t}(\sigma_{S'}) = 0 \) for a finite neighbourhood around \( t = 0 \), and the latter is the case whenever the expectation value of the energy on the initial state, \( \langle \psi_{Q}|H_{Q}|\psi_{Q}\rangle \), is finite (i.e., \( \psi_{Q} \) is of bounded-size). This last inference follows from the fact that for the neighbourhood of \( t = 0 \) defined by \( |t| \leq |\langle \psi_{Q}|H_{Q}|\psi_{Q}\rangle| \), \( \psi_{Q} \) cannot be perfectly distinguishable from \( e^{-iH_{Q}t}\psi_{Q}e^{iH_{Q}t} \), so that for this finite neighborhood, \( f_{t}(\psi_{Q}) < 1 \), and therefore, by the tradeoff relation, \( f_{t}(\sigma_{S'}) \equiv 0 \) as well.
Proof of lemma 4

Proof. Let $\rho(x) = e^{ixL}e^{-ixL}$. By assumption, the map $x \to \rho(x)$ is differentiable, i.e. $\frac{d\rho(x)}{dx}$ exists. Furthermore

$$\frac{d\rho(x)}{dx} = i[L, \rho(x)].$$

Furthermore, by assumption

$$\forall j \in \{1, \cdots N\} : [\Pi_j, \rho(x)] = 0.$$  

Differentiating this, we find

$$\forall j \in \{1, \cdots N\} : \frac{d}{dx} [\Pi_j, \frac{d\rho(x)}{dx}] = 0.$$  

Combining these equations, we have

$$\frac{d\rho(x)}{dx} = \sum_j \Pi_j \frac{d\rho(x)}{dx} \Pi_j$$

where in the first line we have used Eq. (46), in the second line we have used Eq. (44), in the third line we have used Eq. (45), and in the fourth line we have used the definition

$$\tilde{L} = \sum_j \Pi_j L \Pi_j.$$  

Integrating $\frac{d\rho(x)}{dx} = i[\tilde{L}, \rho(x)]$, we find

$$\rho(x) = e^{ix\tilde{L}}e^{-ix\tilde{L}},$$

This in turn implies that

$$\text{Tr}(\Pi_j \rho(x)) = \text{Tr}(\Pi_j e^{ix\tilde{L}}e^{-ix\tilde{L}}) = \text{Tr}(\Pi_j \rho),$$

where in the second equality we have used the fact that $\tilde{L}$ commutes with $\Pi_j$, which follows immediately from the definition in Eq. (51).

Proof of theorem 1

The no-broadcasting of asymmetry is a special case of theorem 1 where system $S$ is trivial and system $Q'$ is isomorphic to $Q$. Here, we use this special case, which was proven in the main text, together with the result of Ref. [48] to prove theorem 1 in the general case.

Suppose we apply the covariant process $\Lambda_{Q' \rightarrow Q'}$ to system $Q$ initially in state $\rho_Q$ and system $S$ which is initially in a maximally-entangled state with a reference system $S$, denoted $\Psi_{SS'}$. Let $\sigma_{Q'S'S} = \Lambda \otimes \mathcal{I}_S(\rho_Q \otimes \Psi_{SS'}$) be the joint state of $Q'$, $S'$ and $S$ at the end of the process, where $\mathcal{I}_S$ is the identity map on system $S$.

From [48], we know that the representation of symmetry on $S$ can be chosen such that $\Psi_{SS'}$ is a symmetric state. This implies that for the fixed state $\Psi_{SS'}$, the quantum operation $\tilde{\Lambda}_{Q' \rightarrow Q'}(\cdot) = \Lambda \otimes \mathcal{I}_S(\cdot \otimes \Psi_{SS'})$ from $Q$ to $Q'S'S$ is covariant. In other words, the transformation $\rho_Q \rightarrow \sigma_{Q'S'S}$ can be implemented by a covariant process. Hence, we can apply our no-broadcasting
of asymmetry theorem to conclude that if $\rho_Q$ can be recovered from $\sigma_{Q'} = \text{Tr}_{S' \overline{S}}(\sigma_{Q'S' \overline{S}'})$, then the reduced state of $S' \overline{S}$, i.e., $\sigma_{S' \overline{S}} = \text{Tr}_{Q'}(\sigma_{Q'S' \overline{S}'})$, should be a symmetric state.

Finally, we note that $\sigma_{S' \overline{S}} = \mathcal{E}_{S \rightarrow S'} \otimes \mathcal{I}_{\overline{S}}(\Psi_{S' \overline{S}})$. Using the result of [48], we know that this state is symmetric iff $\mathcal{E}_{S \rightarrow S'}$ is covariant. We conclude that if $\rho_Q$ can be recovered from the state of $Q'$, then $\mathcal{E}_{S \rightarrow S'}$ is a covariant channel, which completes the proof of theorem 1.