Strain Tensors and Matching Property on Surfaces with the Gauss curvature changing sign

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Abstract  We prove the regularity of solutions to the strain tensor equation on a region $S$ with the Gauss curvature changing sign. Furthermore, we obtain the density property that smooth infinitesimal isometries are dense in the $W^{2,2}(S, \mathbb{R}^3)$ infinitesimal isometries. Finally, the matching property is established. Those results are important tools in obtaining recovery sequences ($\Gamma$-lim sup inequality) for dimensionally-reduced shell theories in elasticity.

Keywords  shell, nonlinear elasticity, Riemannian geometry, tensor analysis

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1 Introduction and Main Results

Let $M \subset \mathbb{R}^3$ be a surface with a normal $\vec{n}$ and let the middle surface of a shell be an open set $S \subset M$. Let $T_k^kS$ denote all the $k$-order tensor fields on $S$ for an integer $k \geq 0$. Let $T^2_{\text{sym}}S$ be all the 2-order symmetrical tensor fields on $S$. For $y \in W^{1,2}(S, \mathbb{R}^3)$, we decompose it into $y = W + w\vec{n}$, where $w = \langle y, \vec{n} \rangle$ and $W \in TS$. For $U \in T^2_{\text{sym}}S$ given, linear strain tensor of a displacement $y \in W^{1,2}(S, \mathbb{R}^3)$ of the middle surface $S$ takes the form

$$\text{sym } DW + w\Pi = U \quad \text{for} \quad x \in S,$$

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where $D$ is the connection of the induced metric in $M$, $2 \text{sym}DW = DW + D^TW$, and $\Pi$ is the second fundamental form of $M$. Equation (1.1) plays a fundamental role in the theory of thin shells, see [4, 6, 5, 7, 13, 15] and many others. When $U = 0$, a solution $y$ to (1.1) is referred to as an \textit{infinitesimal isometry}.

The type of equation (1.1) depends on the sign of the curvature on the region $S$: It is elliptic if $S$ has positive curvature; it is parabolic if the curvature is zero but $\Pi \neq 0$ on $S$; it is hyperbolic if $S$ has negative curvature. When the curvature of the region $S$ changes its sign, problem (1.1) is of changing type.

Here we establish the regularity of solutions to (1.1) when $S$ is a region with its curvature changing sign, which will be specified below. Then it is proved that smooth infinitesimal isometries are dense in the $W^{2,2}(S, \mathbb{R}^3)$ infinitesimal isometries on the region $S$. Finally, the matching property is derived that smooth enough infinitesimal isometries can be matched with higher order infinitesimal isometries. Those results are important tools in obtaining recovery sequences ($\Gamma$-lim sup inequality) for dimensionally-reduced shell theories in elasticity, when the elastic energy density scales like $h^\beta$, $\beta \in (2, 4)$, that is, intermediate regime between "pure bending” ($\beta = 2$) and the von-Kármán regime ($\beta = 4$). Such results have been obtained for elliptic surfaces [7], developable surfaces [4], and hyperbolic surfaces [13, 15]. Moreover, the case of degenerated hyperbolic surfaces has been studied in [1]. A survey on this topic is presented in [5].

Mixed type equations also arise naturally in many other areas. A detailed account of the historical background and known results on mixed type equations and transonic flows is given in [8]. A detailed review on mixed type equations and Riemannian-Lorentzian metrics is presented in [11]. The most intensively studied equation of mixed type is the Tricomi equation [10].

In this paper we study the mixed type equation (1.1) which is very different from all the above cases. Equation (1.1) is equivalent to a mixed type scalar equation of the form

$$
\langle D^2w, Q^\ast \Pi \rangle + \frac{1}{\kappa} \langle Dw, X_0 \rangle + \kappa (\text{tr}_g \Pi) w = \kappa f + \frac{1}{\kappa} \langle X_0, F \rangle + \langle DF, Q^\ast \Pi \rangle \quad \text{for } x \in S, (1.2)
$$

$\kappa \neq 0$, where $w \in L^2(S)$ is the unknown, $f \in L^2(S)$ and $F \in L^2(S, TS)$ are given, and $\kappa$ is the Gauss curvature. The type of (1.2) is subject to the sign of $\kappa$. When $\kappa = 0$, there are two terms degenerating in (1.2): The coefficient of $\langle Dw - F, X_0 \rangle$ becomes from $+\infty$ to $-\infty$ as a point crosses a zero curvature curve from the positive curvature to the negative. The coefficient of a second derivative of $w$ along the direction of the zero principal curvature changes from positive to negative. Those situations challenge the analysis of (1.2).

We observe that problem (1.2) is equivalent to a vector field system as

$$
\begin{cases}
Dv = \nabla \bar{n}V + F \quad \text{for } x \in S, \\
\text{div}_g V = -v \text{tr}_g \nabla \bar{n} + f \quad \text{for } x \in S,
\end{cases} (1.3)
$$
where $\nabla\vec{n} : S_x \rightarrow S_x$ is the second fundamental form operator, $(V,v)$ is the unknown where $V$ is a vector field and $v$ is function, and $(F,f)$ is as in (1.2). Problem (1.3) changes type when the curvature changes sign. In order to have solutions to problem (1.1), one normally starts to solve problem (1.2) in the previous works as in [1, 4, 7, 13]. However, it makes the solvability of (1.2) extremely difficult that the coefficient of $\langle Dw - F, X_0 \rangle$ varies between $+\infty$ and $-\infty$ when the curvature changes sign. Instead of analysing (1.2), here we begin from (1.3) and then obtain solutions to (1.1). The hard task is to achieve regularity of solutions to (1.3) by some priori estimates near the zero curvature curve. Thanks to the Bochner technique and the tensor analysis, the regularity analysis has been complete in Section 3.

We state our main results as follows. Let $S \subset M$ be given by

$$S = \{ \alpha(t,s) \mid (t,s) \in [0,a) \times (-b_0, b_1) \}, \quad a > 0, \ b > 0, \ b_0 > 0, \ (1.4)$$

where $\alpha : [0,a) \times [0,b] \rightarrow M$ is an imbedding map which is a family of regular curves with two parameters $t, s$ such that

$$\Pi(\alpha_t(t,s),\alpha_t(t,s)) > 0, \quad \text{for all} \quad (t,s) \in [0,a) \times [-b_0,0], \quad (1.5)$$

where $\alpha(t,s)$ is a closed curve with period $a$ for each $s \in [-b_0,b_1]$. Set

$$S = S^+ \cup \Gamma_0 \cup S^-,$$

where

$$S^+ = \{ \alpha(t,s) \mid (t,s) \in [0,a) \times (0,b) \}, \quad \Gamma_0 = \{ \alpha(t,0) \mid t \in [0,a] \}, \quad S^- = \{ \alpha(t,s) \mid (t,s) \in [0,a) \times (-b_0,0) \}.$$

**Curvature assumptions** Let $\kappa$ be the Gaussian curvature function on $M$. We assume that $S$ satisfies the following curvature conditions:

$$\kappa(x) > 0 \quad \text{for} \quad x \in S^+ \cup \Gamma_{b_1}; \quad (1.6)$$

$$\kappa = 0, \quad D\kappa(x) \neq 0 \quad \text{for} \quad x \in \Gamma_0; \quad (1.7)$$

$$\kappa(x) < 0 \quad \text{for} \quad x \in S^- \cup \Gamma_{-b_0}, \quad (1.8)$$

where

$$\Gamma_{b_1} = \{ \alpha(t,b_1) \mid t \in [0,a) \}, \quad \Gamma_{-b_0} = \{ \alpha(t,-b_0) \mid t \in [0,a) \}.$$

Our main results are the following.

**Theorem 1.1.** Let $S$ be of class $C^{m+5}$ for some integer $m \geq 1$. For $U \in W^{m+1,2}(S,T_{\text{sym}}^2 S)$, there exists a solution $y = W + w\vec{n} \in W^{m,2}(S,\mathbb{R}^3)$ to equation (1.1) satisfying

$$\|W\|_{W^{m+1,2}(S,T_{\text{sym}}^2 S)}^2 + \|w\|_{W^{m,2}(S)}^2 \leq C\|U\|_{W^{m+1,2}(S,T_{\text{sym}}^2 S)}^2.$$
Remark 1.1. The curvature of $S$ affects the regularity of (1.1) as
\[
\begin{cases}
\|y\|^2_{W^{m,2}(S,IR^3)} \leq C\|U\|^2_{W^{m,2}(S,T^2_{sym}S)} & \text{if } \kappa > 0 \text{ on } \Sigma; \\
\|y\|^2_{W^{m,2}(S,IR^3)} \leq C\|U\|^2_{W^{m+1,2}(S,T^2_{sym}S)} & \text{if } \kappa < 0 \text{ on } \Sigma.
\end{cases}
\]

By the imbedding theorem [3, P. 158], the following corollary is immediate.

Corollary 1.1. Let $S$ be of $S \in C^{m+5}$ for some integer $m \geq 0$. Then problem (1.1) admits a solution $y = W + w\vec{n} \in C^m_B(S, IR^3)$ satisfying
\[
\|W\|_{C^{m+1}_B(S,T^3S)} + \|w\|_{C^m_B(S)} \leq C\|U\|_{C^{m+3}_B(S,T^2_{sym}S)},
\]
where
\[
C^m_B(S, IR^3) = \{ y \in C^m(S, IR^3) | D^\alpha y \in L^\infty(S, IR^3) \text{ for } |\alpha| \leq m \}.
\]

For $y \in W^{1,2}(S, IR^3)$, we denote the left hand side of equation (1.1) by $\text{sym} \nabla y$. Let
\[
\mathcal{V}(S, IR^3) = \{ y \in W^{2,2}(S, IR^3) | \text{sym} \nabla y = 0 \}.
\]

Theorem 1.2. Let $S$ be of class $C^{m+5,1}$. Then, for every $y \in \mathcal{V}(S, IR^3)$ there exists a sequence $\{ y_k \} \subset \mathcal{V}(S, IR^3) \cap C^m_B(S, IR^3)$ such that
\[
\lim_{k \to \infty} \|y - y_k\|_{W^{2,2}(S, IR^3)} = 0.
\]

A one parameter family $\{ y_\varepsilon \}_{\varepsilon > 0} \subset C^1_B(\Sigma, IR^3)$ is said to be a (generalized) $m$th order infinitesimal isometry if the change of metric induced by $y_\varepsilon$ is of order $\varepsilon^{m+1}$, that is,
\[
\|\nabla^T \varepsilon \nabla y_\varepsilon - g\|_{L^\infty(S,T^2)} = \mathcal{O}(\varepsilon^{m+1}) \quad \text{as} \quad \varepsilon \to 0,
\]
where $g$ is the induced metric of $M$ from $IR^3$, see [4]. A given $m$th order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order $m_1 > m$, a property to which we refer to by matching property of infinitesimal isometries, [4, 7].

Theorem 1.3. Let $S$ be of class $C^{4m,1}$. Given $y \in \mathcal{V}(S, IR^3) \cap C^{4m-2}_B(S, IR^3)$, there exists a family $\{ z_\varepsilon \}_{\varepsilon > 0} \subset C^2_B(S, IR^3)$, equi-bounded in $C^2_B(S, IR^3)$, such that for all small $\varepsilon > 0$ the family:
\[
y_\varepsilon = \text{id} + \varepsilon y + \varepsilon^2 z_\varepsilon
\]
is a $m$th order infinitesimal isometry of class $C^2_B(S, IR^3)$.

Application to elasticity of thin shells Let $\vec{n}$ be the normal field of surface $M$. Consider a family $\{ S_h \}_{h > 0}$ of thin shells of thickness $h$ around $S$,
\[
S_h = \{ x + t\vec{n}(x) \mid x \in S, \ |t| < h/2 \}, \quad 0 < h < h_0,
\]
where $h_0$ is small enough so that the projection map $\pi : S_h \to S$, $\pi(x + t\vec{n}) = x$ is well defined. For a $W^{1,2}$ deformation $u_h : S_h \to \mathbb{R}^3$, we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

$$E_h(u_h) = \frac{1}{h} \int_{S_h} W(\nabla u_h) dz.$$  

The stored-energy density function $W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is $C^2$ in an open neighborhood of $SO(3)$, and it is assumed to satisfy the conditions of normalization, frame indifference and quadratic growth: For all $F \in \mathbb{R}^3 \times \mathbb{R}^3$, $R \in SO(3)$,

$$W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq C \text{dist}^2(F, SO(3)),$$

with a uniform constant $C > 0$. The potential $W$ induces the quadratic forms ([2])

$$Q_3(F) = D^2W(Id)(F,F), \quad Q_2(x,F_{\tan}) = \min\{ Q_3(\hat{F}) \mid \hat{F} = F_{\tan} \}.$$  

Let $A : S \to \mathbb{R}^{3 \times 3}$ be a matrix field. We define $A \in T^2S$ by

$$A(\alpha, \beta) = \langle A(x)\alpha, \beta \rangle \quad \text{for} \quad \alpha, \beta \in T_xS, \quad x \in S.$$  

For given $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$, there exists a unique $A \in W^{1,2}(S,T^2)$ such that

$$\nabla_\alpha V = A(x)\alpha \quad \text{for} \quad \alpha \in T_xS, \quad A(x) = -A^T(x), \quad x \in \Omega. \quad (1.9)$$  

We shall consider a sequence $e_h > 0$ such that:

$$0 < \lim_{h \to 0} e_h/h^3 < \infty \quad \text{for some} \quad 2 < \beta \leq 4. \quad (1.10)$$  

Let

$$\beta_m = 2 + 2/m.$$  

Recall the following results.

**Theorem 1.4.** [6] Let $S$ be a surface embedded in $\mathbb{R}^3$, which is compact, connected, oriented, of class $C^{1,1}$, and whose boundary $\partial S$ is the union of finitely many Lipschitz curves. Let $u_h \in W^{1,2}(S_h, \mathbb{R}^3)$ be a sequence of deformations whose scaled energies $E_h(u_h)/e_h$ are uniformly bounded. Then there exist a sequence $Q_h \in SO(3)$ and $c_h \in \mathbb{R}^3$ such that for the normalized rescaled deformations

$$y_h(z) = Q_h u_h(x + \frac{h}{h_0} t\vec{n}(x)) - c_h, \quad z = x + t\vec{n}(x) \in S_{h_0},$$

the following holds.

(i) $y_h$ converge to $\pi$ in $W^{1,2}(S_{h_0}, \mathbb{R}^3)$. 

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(ii) The scaled average displacements

\[ V_h(x) = \frac{h}{h_0 \sqrt{e_h}} \int_{-h_0/2}^{h_0/2} [y_h(x + t\vec{n}) - x] dt \]

converge to some \( V \in \mathcal{V}(S, \mathbb{R}^3) \).

(iii) \( \liminf_{h \to 0} \frac{E_h(u_h)}{e_h} \geq I(V) \), where

\[ I(V) = \frac{1}{24} \int_S Q_2 \left( x, (\nabla(A\vec{n}) - A\nabla\vec{n})_{\tan} \right) dg, \tag{1.11} \]

where \( A \) is given in (1.9).

The above result proves the lower bound for the \( \Gamma \)-convergence. With Theorems 1.2 and 1.3 a recover sequence can be constructed in the \( \Gamma \)-limit for thin shells as in [4, 7, 13] such that the following theorem holds true. The details of the proof are omitted.

**Theorem 1.5.** Let \( S \subset M \) be of class \( C^{5,1} \) given in (1.4) with (1.5) – (1.8). Suppose that

\[ e_h = o(h^{5-n}). \]

Then for every \( V \in \mathcal{V}(S, \mathbb{R}^3) \) there exists a sequence of deformations \( \{ u_h \} \subset W^{1,2}(S, \mathbb{R}^3) \) such that (i) and (ii) of Theorem 1.4 hold. Moreover,

\[ \lim_{h \to 0} \frac{1}{e_h} E_h(u_h) = I(V), \tag{1.12} \]

where \( I(V) \) is given in (1.11).

2 \( L^2 \) Solutions of the tensor equation of mixed type

Let \( \nabla \) and \( D \) denote the connection of \( \mathbb{R}^3 \) in the Euclidean metric and the one of \( M \) in the reduced metric, respectively. We have to treat the relationship between \( \nabla \) and \( D \) carefully.

Let \( m \geq 1 \) be an integer. Let \( T \in T^m M \) be an \( m \)th order tensor field on \( M \). We define a \( m - 1 \)th order tensor field by

\[ i_\alpha T(Y_1, \cdots, Y_{m-1}) = T(Y, Y_1, \cdots, Y_{m-1}) \quad \text{for} \quad Y_1, \cdots, Y_{m-1} \in TM, \]

which is called an inner product of \( T \) with \( Y \). For any \( T \in T^2 S \) and \( \alpha \in T_x M \),

\[ \mathrm{tr}_g i_\alpha DT \]

is a linear functional on \( T_x M \), where \( \mathrm{tr}_g i_\alpha DT \) is the trace of the 2-order tensor field \( i_\alpha DT \) in the induced metric \( g \). Thus there is a vector, denoted by \( \mathrm{div}_g T \), such that

\[ \langle \mathrm{div}_g T, \alpha \rangle = \mathrm{tr}_g i_\alpha DT \quad \text{for} \quad \alpha \in T_x M, \ x \in M. \]
Clearly, the above formula defines a vector field $\div gT \in TM$.

We need a linear operator $Q$ ([13], [15]) as follows. For each point $p \in M$, the Riesz representation theorem implies that there exists an isomorphism $Q : T_p M \to T_p M$ such that

$$\langle \alpha, Q\beta \rangle = \det (\alpha, \beta, \vec{n}(p)) \quad \text{for} \quad \alpha, \beta \in T_p M. \quad (2.1)$$

Let $e_1, e_2$ be an orthonormal basis of $T_p M$ with positive orientation, that is,

$$\det (e_1, e_2, \vec{n}(p)) = 1.$$

Then $Q$ can be expressed explicitly by

$$Q\alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in T_p M. \quad (2.2)$$

Clearly, $Q$ satisfies

$$Q^T = -Q, \quad Q^2 = -\text{Id}.$$

Operator $Q$ plays an important role in our analysis.

Here we sum up some formulas about operator $Q$ and their proofs have been given in [1].

**Lemma 2.1.** The following identities hold.

$$QD_X Y = D_X (QY) \quad \text{for} \quad X, Y \in TM. \quad (2.3)$$

$$\langle X, Y \rangle Z = \langle Z, Y \rangle X + \langle Z, QX \rangle QY \quad \text{for} \quad X, Y, Z \in TM. \quad (2.4)$$

Let $P \in T^2 M$. Let $X$ and $Y$ be vector fields and $f$ be a function. Then

$$\div g (P X) = \langle P, DX \rangle + \langle \div g P, X \rangle, \quad (2.5)$$

$$\div g (f P) = f \div g P + P^T Df.$$

Let $P \in T^2 M$ and let $p \in M$ be given. Then

$$\langle Q v, w \rangle P - \langle P v, w \rangle Q = Q v \otimes P w - P v \otimes Q w, \quad (2.6)$$

$$\langle Q v, w \rangle Q P + \langle P v, w \rangle \text{id} = Q v \otimes Q P w + P v \otimes w \quad \text{for} \quad v, w \in T_p M. \quad (2.7)$$

For any $X, Y \in TS$,

$$\nabla \vec{n} [X, Y] = D_X \nabla \vec{n} Y - D_Y \nabla \vec{n} X, \quad (2.8)$$

$$\div g [X, Y] = X \div g Y - Y \div g X. \quad (2.9)$$

**Lemma 2.2.** For any $Z, X, V, W \in TM$,

$$\langle X, Z \rangle (\langle W, QX \rangle Q \nabla \vec{n} V + \langle W, \nabla \vec{n} X \rangle V, Z)$$

$$= \Pi(X, X) \langle V, Z \rangle \langle W, Z \rangle - \Pi(Q Z, QZ) \langle V, QX \rangle \langle W, QX \rangle \quad \text{for} \quad p \in M. \quad (2.10)$$
Proof. Using (2.4) where \( Y = Z \) and \( Z = V \), we have

\[
\langle X, Z \rangle V = \langle V, Z \rangle X + \langle V, QX \rangle QZ \quad \text{for} \quad p \in M.
\]

Similarly,

\[
\langle X, Z \rangle W = \langle W, Z \rangle X + \langle W, QX \rangle QZ \quad \text{for} \quad p \in M.
\]

Thus we obtain

\[
\langle X, Z \rangle (\langle W, QX \rangle Q\nabla \bar{n}V + \langle W, \nabla \bar{n}X \rangle V)
\]

\[
= \langle W, QX \rangle Q\nabla \bar{n}(\langle X, Z \rangle V) + \langle \langle X, Z \rangle W, \nabla \bar{n}X \rangle V
\]

\[
= \langle W, QX \rangle Q\nabla \bar{n}(\langle V, Z \rangle X + \langle V, QX \rangle QZ) + \langle \langle W, Z \rangle X + \langle W, QX \rangle QZ, \nabla \bar{n}X \rangle V
\]

\[
= \langle V, Z \rangle \langle W, QX \rangle Q\nabla \bar{n}X + \langle V, QX \rangle \langle W, QX \rangle Q\nabla \bar{n}QZ + \langle W, Z \rangle \Pi(X, X) V
\]

\[
\quad + \langle W, QX \rangle \langle QZ, \nabla \bar{n}X \rangle V \quad \text{for} \quad p \in M.
\]

It follows by \( Q^T = -Q \) and \( \langle QZ, Z \rangle = 0 \) that

\[
\langle X, Z \rangle (\langle W, QX \rangle Q\nabla \bar{n}V + \langle W, \nabla \bar{n}X \rangle V), Z)
\]

\[
= \langle V, Z \rangle \langle W, QX \rangle Q\nabla \bar{n}X, Z) + \langle V, QX \rangle \langle W, QX \rangle \langle Q\nabla \bar{n}QZ, Z) + \langle W, Z \rangle \Pi(X, X) \rangle V, Z)
\]

\[
\quad - \langle W, QX \rangle \langle Z, Q\nabla \bar{n}X \rangle \langle V, Z)\rangle
\]

\[
= \Pi(X, X) \langle V, Z) \langle W, Z) - \Pi(QZ, QZ) \langle V, QX \rangle \langle W, QX) \quad \text{for} \quad p \in M.
\]

\[
\square
\]

For given \( \varepsilon > 0 \) small, let

\[
\Omega_{-\varepsilon} = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (-\varepsilon, b_1) \},
\]

\[
\Gamma_{-\varepsilon} = \{ \alpha(t, -\varepsilon) \mid t \in [0, a) \}, \quad \Gamma_{b_1} = \{ \alpha(t, b_1) \mid t \in [0, a) \}.
\]

Let \( X \in TS \) be a given vector field and \( \nu \) be the outside normal of \( \Omega_{-\varepsilon} \). We consider solvability of system

\[
\begin{align*}
\text{div}_g(Q\nabla \bar{n}V &= f_1 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div}_gV &= f_2 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\langle W, QX \rangle |_{\Gamma_{-\varepsilon}} &= q, \quad \langle W, \nu \rangle |_{\Gamma_{b_1}} &= p,
\end{align*}
\]

(2.11)

where \( V \in T\Omega_{-\varepsilon} \) is the unknown, and \( f_i \in L^2(\Omega_{-\varepsilon}), q \in L^2(\Gamma_{-\varepsilon}), \) and \( p \in L^2(\Gamma_{b_1}) \) are given.

Let \( Y \in TS \) be a given vector field. In order to solve problem (2.11), we introduce an operator \( \mathcal{L}_Y : L^2(S, T) \to L^2(S, T) \) by

\[
\mathcal{L}_Y V = e^{-\gamma \kappa}[(\text{div}_gQ\nabla \bar{n}V + \langle V, Y - \eta QX \rangle)QX + (\text{div}_gV - \eta \langle V, \nabla \bar{n}X \rangle)\nabla \bar{n}X],
\]

(2.12)

\[
D(\mathcal{L}_Y) = \{ V \in L^2(S, T) \mid \text{div}_gV \in L^2(S), \text{div}_gQ\nabla \bar{n}V \in L^2(S) \},
\]

(2.13)
where $\gamma \in [0, \infty)$ and $\eta \in C^m(\Omega_\varepsilon)$.

For $V, W \in W^{1,2}(S, T)$, we compute

$$
\langle W, L_Y V \rangle = e^{-\gamma^*}[\langle \text{div}_g \nabla \bar{n} V + \langle V, Y - \eta Q X \rangle \rangle \langle W, Q X \rangle
+ \langle \text{div}_g V - \eta \langle V, \nabla \bar{n} X \rangle \rangle \langle W, \nabla \bar{n} X \rangle]
= \text{div}_g(e^{-\gamma^*}\langle W, Q X \rangle \nabla \bar{n} V) - \langle Q\nabla \bar{n} V, D(e^{-\gamma^*}\langle W, Q X \rangle) \rangle + \text{div}_g(e^{-\gamma^*}\langle W, \nabla \bar{n} X \rangle V)
- \langle V, D(e^{-\gamma^*}\langle W, \nabla \bar{n} X \rangle) \rangle - e^{-\gamma^*}(\langle V, Y - \eta Q X \rangle \langle W, Q X \rangle - \eta \langle V, \nabla \bar{n} X \rangle \langle W, \nabla \bar{n} X \rangle)
= \text{div}_g[e^{-\gamma^*}(\langle W, Q X \rangle \nabla \bar{n} V + \langle W, \nabla \bar{n} X \rangle V)]
- \gamma e^{-\gamma^*}(\langle V, (W, Q X) \nabla \bar{n} Q D \rangle - \langle W, \nabla \bar{n} X \rangle D \rangle)
+ e^{-\gamma^*}(\langle V, \nabla \bar{n} Q D \rangle \langle W, Q X \rangle - D \langle W, \nabla \bar{n} X \rangle)
+ e^{-\gamma^*}(\langle V, (W, Q X) (Y - \eta Q X) - \eta \langle W, \nabla \bar{n} X \rangle \nabla \bar{n} X \rangle)
= \text{div}_g[e^{-\gamma^*}(\langle W, Q X \rangle \nabla \bar{n} V + \langle W, \nabla \bar{n} X \rangle V)] + \langle V, L_Y^* W \rangle,
$$

where

$$
L_Y^* W = e^{-\gamma^*}[\nabla \bar{n} Q D \langle W, Q X \rangle - D \langle W, \nabla \bar{n} X \rangle - \gamma (\langle W, Q X \rangle \nabla \bar{n} Q D \rangle - \langle W, \nabla \bar{n} X \rangle D \rangle)
+ \langle W, Q X \rangle (Y - \eta Q X) - \eta \langle W, \nabla \bar{n} X \rangle \nabla \bar{n} X \rangle].
$$

**Lemma 2.3.** Let $X, Y \in TS$ be given. Suppose that $\Omega \subset S$ be a subregion of $S$ such that

$$
\langle X, \nu \rangle \neq 0 \text{ for } x \in \partial \Omega,
$$

where $\nu$ is the outside normal of $\Omega$. Then

$$
(W, L_Y V)_{L^2(\Omega, T)} = (V, L_Y^* W)_{L^2(\Omega, T)}
+ \int_{\partial \Omega} \langle L_1 \langle V, \nu \rangle \langle W, \nu \rangle - L_2 \langle V, Q X \rangle \langle W, Q X \rangle \rangle d\Gamma,
$$

for $V, W \in T \Omega$, where

$$
L_1 = \frac{e^{-\gamma^*}}{\langle X, \nu \rangle} \Pi(X, X), \quad L_2 = \frac{e^{-\gamma^*}}{\langle X, \nu \rangle} \Pi(Q \nu, \nu) \text{ for } x \in \partial \Omega.
$$

**Proof.** We integrate (2.14) over $\Omega$ and use (2.10), where $Z = \nu$, to have (2.16). \(\square\)

**Lemma 2.4.** Let $X, Y \in TS$ be given. Let $L_Y$ and $L_Y^*$ be given in (2.12) and (2.15),
respectively. Then
\[-e^{-\gamma}\langle W, \mathcal{L}_Y W + \mathcal{L}_X W \rangle\]
\[= \gamma (\langle W, QX \rangle \langle W, \nabla \bar{n} Q D\kappa \rangle - \langle W, \nabla \bar{n} X \rangle \langle W, D\kappa \rangle) + \langle W, D_W (\nabla \bar{n} X) \rangle\]
\[+ 2\eta (\langle W, QX \rangle^2 + \langle W, \nabla \bar{n} X \rangle^2) - \langle W, 2Y + \text{div}_g \nabla \bar{n} \rangle \langle W, QX \rangle\]
\[+ \langle W, D_Q \nabla \bar{n} W (QX) \rangle \text{ for } x \in \Omega_{-\varepsilon}. \tag{2.18}\]

**Proof.** From (2.12) and (2.5), we have
\[e^{-\gamma}\langle W, \mathcal{L}_Y W \rangle = (\text{div}_g Q \nabla \bar{n} W + \langle W, Y - \eta QX \rangle) \langle W, QX \rangle\]
\[+ (\text{div}_g W - \eta \langle W, \nabla \bar{n} X \rangle) \langle W, \nabla \bar{n} X \rangle\]
\[= \langle DW, (W, QX) Q \nabla \bar{n} + \langle W, \nabla \bar{n} X \rangle \text{id} \rangle - \eta (\langle W, QX \rangle^2 + \langle W, \nabla \bar{n} X \rangle^2)\]
\[+ \langle W, Y + \text{div}_g Q \nabla \bar{n} \rangle \langle W, QX \rangle \text{ for } p \in \Omega_{-\varepsilon}, \tag{2.19}\]

where formula \(\text{div}_g W = \langle DW, \text{id} \rangle\) has been used. In addition, it follows from (2.15) that
\[e^{-\gamma}\langle W, \mathcal{L}_X W \rangle = \langle W, \nabla \bar{n} Q D \langle W, QX \rangle - D \langle W, \nabla \bar{n} X \rangle \rangle\]
\[- \gamma (\langle W, QX \rangle \langle W, \nabla \bar{n} Q D\kappa \rangle - \langle W, \nabla \bar{n} X \rangle \langle W, D\kappa \rangle)\]
\[+ \langle W, QX \rangle \langle W, Y - \eta QX \rangle - \eta \langle W, \nabla \bar{n} X \rangle^2\]
\[= -\langle DW, QX \otimes Q \nabla \bar{n} W + \nabla \bar{n} X \otimes W \rangle - \langle W, D_Q \nabla \bar{n} W (QX) + D_W (\nabla \bar{n} X) \rangle\]
\[- \gamma (\langle W, QX \rangle \langle W, \nabla \bar{n} Q D\kappa \rangle + \langle W, \nabla \bar{n} X \rangle \langle W, D\kappa \rangle)\]
\[+ \langle W, QX \rangle \langle W, Y \rangle - \eta (\langle W, QX \rangle^2 + \langle W, \nabla \bar{n} X \rangle^2) \text{ for } p \in \Omega_{-\varepsilon}. \tag{2.20}\]

Moreover, formula (2.7) yields
\[
\langle W, QX \rangle Q \nabla \bar{n} + \langle W, \nabla \bar{n} X \rangle \text{id} = QX \otimes Q \nabla \bar{n} W + \nabla \bar{n} X \otimes W.
\]

Thus (2.18) follows from (2.19) and (2.20).

For given \(\varepsilon > 0\), small, set
\[S_{\varepsilon} = \{ \alpha(t, s) \in (t, s) \in [0, a] \times (-\varepsilon, \varepsilon) \}.
\]

Let \(x \in \Gamma_0\). Since \(\kappa(x) = 0\) and \(D\kappa(x) \neq 0\), from [14, Lemma 2.6], there exist vector fields \(X_1, X_2\) in a neighborhood of \(x\) satisfying \(\nabla \bar{n} X_i = \lambda_i X_i\), where \(\lambda_i\) are the principal curvatures. Clearly we may extend the vector fields \(X_i\) to the region \(\Omega_{\varepsilon}\) when \(\varepsilon > 0\) is given small. We assume that \(X_i\) are vector fields such that
\[\nabla \bar{n} X_i = \lambda_i X_i, \quad |X_i| = 1, \quad \langle X_1, X_2 \rangle = 0 \text{ for } x \in \Omega_{\varepsilon}, \tag{2.21}\]
where
\[\lambda_1 > 0 \text{ for } x \in S_{\varepsilon}, \quad \lambda_2 = 0 \text{ for } x \in \Gamma_0,
\]
\[\lambda_2 > 0 \text{ for } x \in S^+ \cap \Omega_{\varepsilon}, \quad \lambda_2 < 0 \text{ for } x \in S^- \cap \Omega_{\varepsilon}.
\]
Lemma 2.5. For given $\varepsilon > 0$ small

$$X_2(\lambda_2) \neq 0 \text{ for } x \in S_\varepsilon.$$ 

Proof. It will suffice to prove

$$X_2(\lambda_2) \neq 0 \text{ for } x \in \Gamma_0.$$ 

First, we claim that $\langle Q\alpha_t, X_2 \rangle(x) \neq 0$ for $x \in \Gamma_0$. If not, then $\langle Q\alpha_t, X_2 \rangle(x) = 0$ implies that $X_2 = \eta\alpha_t$ with $\eta \neq 0$, and thus

$$\eta^2 \Pi(\alpha_t, \alpha_t) = \langle \nabla X_2 \vec{n}, X_2 \rangle = \lambda_2(x) = 0,$$ 

which contradicts assumption $\Pi(\alpha_t, \alpha_t) \neq 0$.

In addition, assumption (1.7) implies

$$\langle D\kappa, \alpha_t \rangle = 0 \text{ for } x \in \Gamma_0, \quad D\kappa = \iota Q\alpha_t \text{ for } x \in \Gamma_0,$$ 

for some $\iota \neq 0$. Thus we obtain

$$X_2(\lambda_2) = \frac{1}{\lambda_1} X_2(\kappa) = \frac{\iota}{\lambda_1} \langle X_2, Q\alpha_t \rangle \neq 0 \text{ for } x \in \Gamma_0.$$ 

(2.23)

From (1.5) and (1.6), we have

$$\Pi(\alpha_t, \alpha_t) > 0 \text{ for } x \in \overline{S}.$$ 

(2.24)

We assume that for each $s \in [-b_0, b_1]$ the closed curve $\alpha(\cdot, s)$ goes in anticlockwise with $t \in [0, a)$ increasing. We further assume that

$$X_2(\lambda_2) > 0 \text{ for } \overline{S}_\varepsilon,$$ 

(2.25)

for given $\varepsilon > 0$ small. For otherwise, we replace $X_2$ with $-X_2$. Furthermore, we assume that $X_1, X_2$ has positive orientation. For otherwise, we replace $X_1$ with $-X_1$. Thus

$$QX_2 = X_1, \quad QX_1 = -X_2 \text{ for } x \in \overline{S}_\varepsilon.$$ 

(2.26)

Lemma 2.6. We have

$$\begin{cases} 
\langle X_2, Q\alpha_t \rangle < 0 & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) > 0, \\
\langle X_2, Q\alpha_t \rangle > 0 & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) < 0 
\end{cases} \text{ for } x \in \Gamma_0.$$ 

(2.27)
Proof. Let \( \det(\alpha_t, \alpha_s, \vec{n}) < 0 \). The case of \( \det(\alpha_t, \alpha_s, \vec{n}) > 0 \) can be treated similarly.

Since
\[
\frac{Q\alpha_t}{|\alpha_t|}, \quad \frac{\alpha_t}{|\alpha_t|}
\]
forms an orthonormal basis with positive orientation, the curvature assumptions (1.6)-(1.8) yield
\[
\langle Q\alpha_t, D\kappa \rangle > 0 \text{ for } x \in \Gamma_0.
\]
It follows from (2.22) that \( \iota > 0 \) for \( x \in \Gamma_0 \). Thus in the case of \( \det(\alpha_t, \alpha_s, \vec{n}) < 0 \) (2.27) follows from (2.25) and (2.23).

We set
\[
X_{\sigma_0} = \begin{cases} 
\sigma_0 \alpha_t - Q\alpha_t & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) > 0, \\
\sigma_0 \alpha_t + Q\alpha_t & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) < 0,
\end{cases}
\]
where \( \sigma_0 \) is a constant satisfying
\[
\sigma_0 \geq 1 + \sup_{x \in \Gamma_0} \frac{|\Pi(\alpha_t, Q\alpha_t)|}{\Pi(\alpha_t, \alpha_t)}.
\]

Lemma 2.7. We have
\[
\Pi(X_{\sigma_0}, X_{\sigma_0}) > 0 \text{ for } x \in S^+, \\
\langle X_{\sigma_0}, \nu \rangle < 0 \text{ for } x \in \Gamma_0, \quad \langle X_{\sigma_0}, \nu \rangle > 0 \text{ for } x \in \Gamma_{b_1},
\]
where \( \nu \) is the outside normal of region \( S^+ \).

Proof. It is easy to check that
\[
\begin{cases} 
\nu = \frac{Q\alpha_t}{|\alpha_t|} & \text{for } x \in \Gamma_0, \quad \text{if } \det(\alpha_t, \alpha_s, \vec{n}) > 0 \\
\nu = -\frac{Q\alpha_t}{|\alpha_t|} & \text{for } x \in \Gamma_{b_1},
\end{cases}
\]
and
\[
\begin{cases} 
\nu = -\frac{Q\alpha_t}{|\alpha_t|} & \text{for } x \in \Gamma_0, \quad \text{if } \det(\alpha_t, \alpha_s, \vec{n}) < 0 \\
\nu = \frac{Q\alpha_t}{|\alpha_t|} & \text{for } x \in \Gamma_{b_1},
\end{cases}
\]
Thus (2.29) follows. Moreover, we have
\[
\Pi(X_{\sigma_0}, X_{\sigma_0}) \geq \Pi(\alpha_t, \alpha_t)(\sigma_0 \pm \frac{\Pi(\alpha_t, Q\alpha_t)}{\Pi(\alpha_t, \alpha_t)})^2 > 0 \text{ for } x \in \Gamma_0.
\]

By Lemma 2.6 we fix \( \varepsilon > 0 \) small such that
\[
\begin{cases} 
\langle X_2, Q\alpha_t \rangle < 0 & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) > 0, \\
\langle X_2, Q\alpha_t \rangle > 0 & \text{if } \det(\alpha_t, \alpha_s, \vec{n}) < 0
\end{cases} \quad \text{for } x \in \Gamma_{-\varepsilon}.
\]

(2.30)
Moreover, since
\[ \langle \alpha_t, X_1 \rangle \neq 0 \quad \text{for} \quad x \in \Gamma_0, \]
we assume that the above \( \varepsilon \) is such that
\[ \langle \alpha_t, X_1 \rangle \neq 0 \quad \text{for} \quad x \in S_{2\varepsilon}. \]

Then
\[
\begin{cases}
\nu = \frac{Q\alpha_t}{|\alpha_t|} & \text{for} \quad x \in \Gamma_{-\varepsilon}, \quad \text{if} \quad \det(\alpha_t, \alpha_s, \vec{n}) > 0, \\
\nu = -\frac{Q\alpha_t}{|\alpha_t|} & \text{for} \quad x \in \Gamma_{-\varepsilon}, \quad \text{if} \quad \det(\alpha_t, \alpha_s, \vec{n}) > 0.
\end{cases}
\] (2.31)

Thus, by (2.30),
\[ \langle X_2, \nu \rangle < 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}. \] (2.32)

We take \( \zeta \in C_0^\infty(-\infty, \infty) \) such that
\[ 0 \leq \zeta \leq 1; \quad \zeta = 1 \quad \text{for} \quad s \in (-\varepsilon, \varepsilon); \quad \zeta = 0 \quad \text{for} \quad s \geq 2\varepsilon. \]

Then we take
\[ \eta = s\eta_0(s) \] (2.33)
in (2.12) where \( \eta_0 \in C_0^\infty(-\infty, \infty) \) such that
\[ 0 \leq \eta_0 \leq 1; \quad \eta_0 = 0 \quad \text{for} \quad s \leq \varepsilon/2; \quad \eta_0 = 1 \quad \text{for} \quad s \geq \varepsilon. \]

In addition, we take \( X \) in (2.12) by
\[ X = \zeta \left[ \varepsilon^{1/3} \langle \text{sign}(\alpha_t, X_1) \rangle X_1 + X_2 \right] + (1 - \zeta) X_{\sigma_0} \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \] (2.34)
where \( \langle \text{sign}(\alpha_t, X_1) \rangle \) is the sign function for \( x \in S_{2\varepsilon}, \) \( X_{\sigma_0} \) is given in (2.28), and \( \sigma_0 \) is given such that
\[ \sigma_0 \geq 1 + \sup_{x \in \Gamma_0} \frac{\|\Pi(\alpha_t, Q\alpha_t)\|}{\Pi(\alpha_t, \alpha_t)} + \sup_{x \in \overline{S}_{2\varepsilon}} \frac{|\langle Q\alpha_t, X_1 \rangle|}{|\langle \alpha_t, X_1 \rangle|}. \] (2.35)

Then
\[ \Pi(X, X) > 0, \quad \langle X, \nu \rangle < 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}, \] (2.36)
when \( \varepsilon > 0 \) is small enough.

**Lemma 2.8.** Let \( \eta \) and \( X \) be given in (2.33) and (2.34), respectively. Let \( Y \in C^0(\overline{\Omega}_{-\varepsilon}, T). \) Then there are constants \( \sigma > 0, \gamma > 0, \) and \( s > 0 \) such that
\[ -\langle W, \mathcal{L}_Y W + \mathcal{L}_Y^* W \rangle \geq \sigma|W|^2 \quad \text{for} \quad W \in T\Omega_{-\varepsilon}, \quad x \in \Omega_{-\varepsilon}, \] (2.37)
for given \( \varepsilon > 0 \) small.
Proof. Step 1. First, we prove

\[ |X| > 0 \quad \text{for} \quad x \in \overline{S^+ \setminus S_\varepsilon}. \]  

(2.38)

If not, there were a point \( x \in \overline{S^+ \setminus S_\varepsilon} \) such that

\[ X = 0. \]

If \( \zeta = 0 \), then

\[ |X| = |X_0| \geq \sigma_0|\alpha_t| > 0, \]

which is not possible. Thus \( \zeta \neq 0 \). By (2.35),

\[ 0 = \langle X, X_1 \rangle = (1 - \zeta)(\sigma_0(\alpha_t, X_1) + \langle Q\alpha_t, X_1 \rangle) + \varepsilon \text{sign} \langle \alpha_t, X_1 \rangle \]

implies that

\[ \zeta = \frac{\sigma_0(\alpha_t, X_1) + \langle Q\alpha_t, X_1 \rangle}{\sigma_0(\alpha_t, X_1) + \langle Q\alpha_t, X_1 \rangle - \varepsilon \text{sign} \langle \alpha_t, X_1 \rangle} > 1, \]

which contradicts \( 0 \leq \zeta \leq 1 \) when given \( \varepsilon > 0 \) is small.

Since \( \kappa > 0 \) for \( x \in \overline{S^+ \setminus S_\varepsilon} \), \( \nabla \bar{n}X \) forms an vector field basis on \( \overline{S^+ \setminus S_\varepsilon} \). Then there exists \( \sigma > 0 \) such that

\[ \langle W, QX \rangle^2 + \langle W, \nabla \bar{n}X \rangle^2 \geq \sigma|W|^2 \quad \text{for} \quad x \in \overline{S^+ \setminus S_\varepsilon}. \]

(2.39)

Step 2. Consider the region \( S_\varepsilon \). Noting that \( \zeta = 1 \) for \( x \in S_\varepsilon \), we have

\[ X = \varepsilon^{1/3}(\text{sign} \langle \alpha_t, X_1 \rangle)X_1 + X_2 \quad \text{for} \quad x \in \overline{S_\varepsilon}. \]

Denote \( \langle W, X_i \rangle = W_i \) for \( i = 1, 2 \). We compute

\[
\gamma ((W,QX_2)\langle W, \nabla \bar{n}QD\kappa \rangle - \langle W, \nabla \bar{n}QX_2 \rangle \langle W, D\kappa \rangle) + \langle W, D_W(\nabla \bar{n}X_2) \rangle - \langle W, 2Y + \text{div}_g Q \nabla \bar{n} \rangle \langle W, QX_2 \rangle + \langle W, D_{\nabla \bar{n}W}(QX_2) \rangle \\
= \gamma ((W,X_1)\langle W, \nabla \bar{n}QD\kappa \rangle - \lambda_2 \langle W, X_2 \rangle \langle W, D\kappa \rangle) + \langle W, D_W(\lambda_2 X_2) \rangle - \langle W, 2Y + \text{div}_g Q \nabla \bar{n} \rangle \langle W, X_1 \rangle + \langle W, D_{\nabla \bar{n}W}(X_1) \rangle \\
\geq \gamma [\lambda_1 X_2(\kappa) - \lambda_2 \| X_1(\kappa) \| W_1^2 + \| X_2(\lambda_2) - \lambda_2 \| (\| X_2, D_{X_1}(X_1) \| + \gamma X_2(\kappa)) W_2^2 - C(W_1^2 + \| W_1 \| W_2) \quad \text{for} \quad x \in \overline{S_\varepsilon}. \]

(2.40)

Similarly, we have

\[
\gamma ((W,QX_1)\langle W, \nabla \bar{n}QD\kappa \rangle - \langle W, \nabla \bar{n}X_1 \rangle \langle W, D\kappa \rangle) + \langle W, D_W(\nabla \bar{n}X_1) \rangle - \langle W, 2Y + \text{div}_g Q \nabla \bar{n} \rangle \langle W, QX_1 \rangle + \langle W, D_{\nabla \bar{n}W}(QX_1) \rangle \\
\geq - \gamma \lambda_1 \| X_1(\kappa) \| W_1^2 - C(W_1^2 + \| W_1 \| W_2) - \gamma \| \lambda_2 \| X_1(\kappa) \| + \lambda_2 X_2(\kappa) \| \rangle W_2^2. \]

(2.41)
It follows from (2.18), (2.40), and (2.41) that
\[ -\langle W, \mathcal{L}_Y W + \mathcal{L}_Y^* W \rangle \geq \gamma [\lambda_1 X_2(\kappa) - |\lambda_2||X_1(\kappa)| - \varepsilon \lambda_1(|X_1(\kappa)| + X_2(\kappa))] W_1^2 + \{|X_2(\lambda_2) - |\lambda_2||X_1(\kappa)| + \gamma X_2(\kappa)|
- \varepsilon[\gamma (|\lambda_2||X_1(\kappa)| + \lambda_1 X_2(\kappa)) + |\langle X_2, D_X(\lambda_1 X_1) \rangle| + |\langle \text{div}_g Q \nabla \bar{n}, X_2 \rangle|| W_2^2
- C(W_1^2 + |W_1||W_2|)
\geq \{ \gamma [\lambda_1 X_2(\kappa) - |\lambda_2||X_1(\kappa)| - \varepsilon \lambda_1(|X_1(\kappa)| + X_2(\kappa))] - C\} W_1^2
+ \{|X_2(\lambda_2) - |\lambda_2||X_1(\kappa)| + \gamma X_2(\kappa)|/2
- \varepsilon[\gamma (|\lambda_2||X_1(\kappa)| + \lambda_1 X_2(\kappa)) + |\langle X_2, D_X(\lambda_1 X_1) \rangle| + |\langle \text{div}_g Q \nabla \bar{n}, X_2 \rangle|| W_2^2, \quad (2.42) \]
for \( x \in \overline{S}_\varepsilon \), where \( C > 0 \) is independent of \( W \in TS_\varepsilon \). Noting that \( X_2(\kappa) = \lambda_1 X_2(\lambda_2) > 0 \) for \( x \in \Gamma_0 \) and
\[ |\lambda_2| = \mathcal{O}(\varepsilon) \quad \text{for} \quad x \in \overline{S}_\varepsilon, \]
we assume that the \( \varepsilon > 0 \) has been given so small such that
\[ \lambda_1 X_2(\kappa) - |\lambda_2||X_1(\kappa)| - \varepsilon \lambda_1(|X_1(\kappa)| + X_2(\kappa)) > 0 \quad \text{for} \quad x \in \overline{S}_\varepsilon. \]
Then we fix \( \gamma > 0 \) such that
\[ \gamma [\lambda_1 X_2(\kappa) - |\lambda_2||X_1(\kappa)| - \varepsilon \lambda_1(|X_1(\kappa)| + X_2(\kappa)) - C > 0 \quad \text{for} \quad x \in \overline{S}_\varepsilon. \]
Finally, we move the \( \varepsilon > 0 \) to zero again such that
\[ \{|X_2(\lambda_2) - |\lambda_2||X_1(\kappa)| + \gamma X_2(\kappa)|/2
- \varepsilon[\gamma (|\lambda_2||X_1(\kappa)| + \lambda_1 X_2(\kappa)) + |\langle X_2, D_X(\lambda_1 X_1) \rangle| + |\langle \text{div}_g Q \nabla \bar{n}, X_2 \rangle|| W_2^2 > 0 \]
for \( x \in \overline{S}_\varepsilon \). Thus for the above given \( \gamma > 0 \) and \( \varepsilon > 0 \), from (2.42), we have obtained \( \sigma > 0 \) such that
\[ -e^{-\gamma \kappa}\langle W, \mathcal{L}_Y W + \mathcal{L}_Y^* W \rangle \geq \sigma |W|^2 \quad \text{for} \quad W \in T\overline{S}_\varepsilon, \quad x \in \overline{S}_\varepsilon. \quad (2.43) \]

**Step 3.** Consider region \( S^+ \setminus S_\varepsilon \). From (2.18) and (2.39), we have
\[ -e^{-\gamma \kappa}\langle W, \mathcal{L}_Y W + \mathcal{L}_Y^* W \rangle \geq s\eta_0((W, QX)^2 + \langle W, \nabla \bar{n}X \rangle^2) - C|W|^2 \geq (s - C)|W|^2 \quad \text{for} \quad s > 0, \quad W \in TS^+ \setminus S_\varepsilon, \quad (2.44) \]
for \( x \in \overline{S^+ \setminus S_\varepsilon} \). Thus (2.37) follows from (2.43) and (2.44) after we take \( s = C + \sigma \) in (2.44). \( \square \)

Next, we consider solvability of problem (2.11) on region \( \Omega_{-\varepsilon} \). Let \( F \in L^2(\Omega_{-\varepsilon}, T) \), \( p \in L^2(\Gamma_{b_1}) \), and \( q \in L^2(\Gamma_{-\varepsilon}) \) be given. Clearly, system (2.11) is equivalent to problem
\[ \begin{cases}
\mathcal{L}_Y V = F & \text{for} \quad x \in \Omega_{-\varepsilon},
\langle V, QX \rangle = q & \text{for} \quad x \in \Gamma_{-\varepsilon},
\langle V, \nu \rangle = p & \text{for} \quad x \in \Gamma_{b_1},
\end{cases} \quad (2.45) \]
where $L_Y$ is defined in (2.12) and vector field $X$ is given in (2.34).

Let $m \geq 0$ be an integer. Set

$$C^m_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T) = \{ W \in C^m(\Omega_{-\varepsilon}, T) \mid \langle W, QX \rangle_{\Gamma_{-\varepsilon}} = \langle W, \nu \rangle_{\Gamma_1} = 0 \},$$

$$C^m_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T) = \{ W \in C^m(\Omega_{-\varepsilon}, T) \mid \langle W, QX \rangle_{\Gamma_{b_1}} = \langle W, \nu \rangle_{\Gamma_{-\varepsilon}} = 0 \}.$$

We denote the completions of $C^m(\Omega_{-\varepsilon}, T)$ by $L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T)$ and $L^2_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)$ in the norms of

$$\| W \|_{L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T)}^2 = \| W \|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \| \langle W, QX \rangle \|_{L^2(\Gamma_{-\varepsilon}, T)}^2 + \| \langle W, \nu \rangle \|_{L^2(\Gamma_{b_1}, T)}^2$$

and

$$\| W \|_{L^2_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)}^2 = \| W \|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \| \langle W, QX \rangle \|_{L^2(\Gamma_{b_1}, T)}^2 + \| \langle W, \nu \rangle \|_{L^2(\Gamma_{-\varepsilon}, T)}^2,$$

respectively.

**Theorem 2.1.** Let vector field $X$ be given in (2.34) and $Y \in C^0(\overline{\Omega_{-\varepsilon}, T})$. Let $F \in L^2(\Omega_{-\varepsilon}, T)$, $p \in L^2(\Gamma_{b_1})$, and $q \in L^2(\Gamma_{-\varepsilon})$ be given. Then problem (2.45) admits a unique solution $V \in D(L_Y) \cap L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T)$, which satisfies

$$(V, L_Y^* W)_{L^2(\Omega_{-\varepsilon}, T)} = (W, F)_{L^2(\Omega_{-\varepsilon}, T)} + \int_{\Gamma_{-\varepsilon}} L_2 q \langle W, QX \rangle d\Gamma - \int_{\Gamma_{b_1}} L_1 p \langle W, \nu \rangle d\Gamma \quad (2.46)$$

for $W \in C^1_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)$, where $L_1$ and $L_2$ are given in (2.17). Furthermore, the following estimate holds true.

$$\| V \|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \| V \|_{L^2(\Gamma_{-\varepsilon}, T)}^2 + \| V \|_{L^2(\Gamma_{b_1}, T)}^2 \leq C (\| F \|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \| q \|_{L^2(\Gamma_{-\varepsilon}, T)}^2 + \| p \|_{L^2(\Gamma_{b_1}, T)}^2). \quad (2.47)$$

**Proof.** **Step 1.** Denote the completion of $C^1_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)$ in the norm of $W^{1,2}(\Omega_{-\varepsilon}, T)$ by $W^{1,2}_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)$.

We define a bilinear functional,

$$\alpha : L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T) \times W^{1,2}_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T) \to \mathbb{R},$$

by

$$\alpha(V, W) = -2(V, L_Y^* W)_{L^2(\Omega_{-\varepsilon}, T)}.$$

Clearly, we have

$$| \alpha(V, W) | \leq C \| V \|_{L^2(\Omega_{-\varepsilon}, T)} \| W \|_{W^{1,2}(\Omega_{-\varepsilon}, T)}$$

for $(V, W) \in L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T) \times W^{1,2}_{b_1, -\varepsilon}(\Omega_{-\varepsilon}, T)$. On the other hand, it follows from (2.29) and (2.32) that

$$\langle X, \nu \rangle < 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}, \quad \langle X, \nu \rangle > 0 \quad \text{for} \quad x \in \Gamma_{b_1}. \quad (2.48)$$
Thus, from (2.16) and (2.37), we have
\[
\alpha(W,W) = -(W, \mathcal{L}_Y W + \mathcal{L}_Y^* W)_{L^2(\Omega_{-\varepsilon},T)} + \int_{\partial \Omega_{-\varepsilon}} (\mathcal{L}_1(W,\nu)^2 - \mathcal{L}_2(W,QX)^2)d\Gamma
\]
\[
\geq \sigma \|W\|^2_{L^2(\Omega_{-\varepsilon},T)} - \int_{\Gamma_{-\varepsilon}} \mathcal{L}_2(W,QX)^2d\Gamma + \int_{\Gamma_{b_1}} \mathcal{L}_1(W,\nu)^2d\Gamma
\]
\[
\geq \sigma \|W\|^2_{L^2_{b_1,-\varepsilon}(\Omega_{-\varepsilon},T)} \quad \text{for} \quad W \in C^1_{b_1,-\varepsilon}(\Omega_{-\varepsilon},T). \tag{2.49}
\]

Next, we define a functional on \( L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon},T) \) by
\[
\mathcal{F}(W) = -2(W,F)_{L^2(\Omega_{-\varepsilon},T)} - 2\int_{\Gamma_{-\varepsilon}} \mathcal{L}_2(W,QX)d\Gamma + 2\int_{\Gamma_{b_1}} \mathcal{L}_1 p(W,\nu)d\Gamma.
\]
It is clearly, \( \mathcal{F} \) is bounded on \( L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon},T) \). By Theorem A given in Appendix there exists a \( V \in L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon},T) \) such that
\[
-2(V, \mathcal{L}_Y^* W)_{L^2(\Omega_{-\varepsilon},T)} = -2(W,F)_{L^2(\Omega_{-\varepsilon},T)} - 2\int_{\Gamma_{-\varepsilon}} \mathcal{L}_2(W,QX)d\Gamma
\]
\[
+ 2\int_{\Gamma_{b_1}} \mathcal{L}_1 p(W,\nu)d\Gamma \tag{2.50}
\]
for all \( W \in C^1_{b_1,-\varepsilon}(\Omega_{-\varepsilon},T) \), which yield, by (2.16) again,
\[
-2(W, \mathcal{L}_Y V)_{L^2(\Omega_{-\varepsilon},T)} = -2(W,F)_{L^2(\Omega_{-\varepsilon},T)}
\]
\[
+ 2\int_{\Gamma_{-\varepsilon}} \mathcal{L}_2(V,QX - q)(W,QX)d\Gamma + 2\int_{\Gamma_{b_1}} \mathcal{L}_1 (p - \langle V,\nu \rangle)(W,\nu)d\Gamma
\]
for all \( W \in C^1_{b_1,-\varepsilon}(\Omega_{-\varepsilon},T) \). Thus \( V \in L^2_{-\varepsilon,b_1}(\Omega_{-\varepsilon},T) \) solves problem (2.45) and identity (2.46) holds.

Noting that \( \zeta = 1 \) for \( x \in \overline{S}_\varepsilon \), from (2.34), we have
\[
|\nabla \bar{n}X|^2 = \varepsilon^2 \lambda_1^2 + \lambda_2^2 > 0 \quad \text{for} \quad x \in \overline{S}_\varepsilon.
\]
Then, by (2.39), \( QX, \nabla \bar{n}X \) forms a vector field basis on \( \overline{\Omega}_\varepsilon \). Thus \( V \in D(\mathcal{L}_Y) \).

**Step 2.** From (2.49) and (2.50), we obtain (2.47). Thus the uniqueness follows. \( \square \)

We consider a duality problem of system (2.45)
\[
\begin{cases}
\mathcal{L}_Y^* V = F \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\langle V,\nu \rangle = p \quad \text{for} \quad x \in \Gamma_{-\varepsilon}, \quad \langle V,QX \rangle = q \quad \text{for} \quad x \in \Gamma_{b_1}.
\end{cases} \tag{2.51}
\]
By a duality of operators \( \mathcal{L}_Y \) and \( \mathcal{L}_Y^* \) a similar argument as for Theorem 2.1 yields the following.
Theorem 2.2. Let vector field $X$ be given in (2.34) and $Y \in C^0(\Omega_{-\varepsilon}, T)$. Let $F \in L^2(\Omega_{-\varepsilon}, T)$, $p \in L^2(\Gamma_{-\varepsilon})$, and $q \in L^2(\Gamma_{b_1})$ be given. Then problem (2.51) admits a unique solution $V \in D(\mathcal{L}_Y^*) \cap L^2_{0_{b_1,-\varepsilon}}(\Omega_{-\varepsilon}, T)$, which satisfies

$$
(V, \mathcal{L}_Y W)_{L^2(\Omega_{-\varepsilon}, T)} = (W, F)_{L^2(\Omega_{-\varepsilon}, T)} + \int_{\Gamma_{-\varepsilon}} \mathcal{L}_1 p(W, \nu) d\Gamma \\
- \int_{\Gamma_{b_1}} \mathcal{L}_2 q(W, QX) d\Gamma
$$

(2.52)

for $W \in C^1_{-\varepsilon,b_1}(\Omega_{-\varepsilon}, T)$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are given in (2.17). Furthermore, the following estimate holds true.

$$
\|V\|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \|V\|_{L^2(\Gamma_{-\varepsilon}, T)}^2 + \|V\|_{L^2(\Gamma_{b_1}, T)}^2 \\
\leq C(\|F\|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \|p\|_{L^2(\Gamma_{-\varepsilon})}^2 + \|q\|_{L^2(\Gamma_{b_1})}^2).
$$

(2.53)

3. $W^{m,2}$ Solutions of the tensor equation of mixed type

Consider problem

$$
\begin{cases}
\mathcal{L}_0 V = F & \text{for } x \in \Omega_{-\varepsilon}, \\
\langle V, QX \rangle = p & \text{for } x \in \Gamma_{-\varepsilon}, \quad \langle V, \nu \rangle = q & \text{for } x \in \Gamma_{b_1},
\end{cases}
$$

(3.1)

where $X$ is given in (2.34) and $\mathcal{L}_0$ is given in (2.12) with $Y = 0$.

In this section, we shall establish the following.

Theorem 3.1. Let $S$ be of $C^{m+3}$. Let $F \in W^{m,2}(\Omega_{-\varepsilon}, T)$, $p \in W^{m,2}(\Gamma_{-\varepsilon})$, and $q \in W^{m,2}(\Gamma_{b_1})$ be given. Then problem (3.1) admits a unique solution $V \in W^{m,2}(\Omega_{-\varepsilon}, T)$ satisfying

$$
\|V\|_{W^{m,2}(\Omega_{-\varepsilon}, T)}^2 \leq C(\|F\|_{W^{m,2}(\Omega_{-\varepsilon}, T)}^2 + \|p\|_{W^{m,2}(\Gamma_{-\varepsilon})}^2 + \|q\|_{W^{m,2}(\Gamma_{b_1})}^2).
$$

(3.2)

By a similar argument and the duality, we have

Theorem 3.2. Let $S$ be of $C^{m+3}$. Let $F \in W^{m,2}(\Omega_{-\varepsilon}, T)$, $p \in W^{m,2}(\Gamma_{b_1})$, and $q \in W^{m,2}(\Gamma_{-\varepsilon})$ be given. Then problem

$$
\begin{cases}
\mathcal{L}_0^* V = F & \text{for } x \in \Omega_{-\varepsilon}, \\
\langle V, \nu \rangle = p & \text{for } x \in \Gamma_{-\varepsilon}, \quad \langle V, QX \rangle = q & \text{for } x \in \Gamma_{b_1},
\end{cases}
$$

(3.3)

admits a unique solution $V \in W^{m,2}(\Omega_{-\varepsilon}, T)$ satisfying (3.2).

Theorem 3.3. Let $S$ be of $C^3$. Let $p \in W^{1,2}(\Gamma_{-\varepsilon})$ and $q \in W^{1,2}(\Gamma_{b_1})$ be given. Suppose that $F \in L^2(\Omega_{-\varepsilon}, T)$ be given such that

$$
F|_{\Sigma_{-\varepsilon}/2} \in W^{1,2}(\Sigma_{-\varepsilon}/\varepsilon^2, T),
$$

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where
\[ \Sigma^{s_2}_{s_1} = \{ \alpha(t, s) \, | \, (t, s) \in [0, a) \times (s_1, s_2) \} \quad \text{for} \quad -b_0 \leq s_1 < s_2 \leq b_1. \]

Then problem (3.1) admits a unique solution \( V \in W^{1,2}(\Omega_{-\epsilon}, T) \) satisfying
\[
\|V\|^2_{W^{1,2}(\Omega_{-\epsilon}, T)} \leq C(\|F\|^2_{L^2(\Omega_{-\epsilon}, T)} + \|F\|^2_{W^{1,2}(\Sigma_{-\epsilon}^{\epsilon/2}, T)}) + \|p\|^2_{W^{1,2}(\Gamma_{-\epsilon})} + \|q\|^2_{W^{1,2}(\Gamma_{b_1})}). \tag{3.4}
\]

If \( S \) is of \( C^{m+3} \), then the above \( V \) satisfies
\[
\|V\|^2_{W^{m,2}(\Omega_{-\epsilon}, T)} \leq C(\|F\|^2_{W^{m-1,2}(\Omega_{-\epsilon}, T)} + \|F\|^2_{W^{m,2}(\Sigma_{-\epsilon}^{\epsilon/2}, T)}) + \|p\|^2_{W^{m,2}(\Gamma_{-\epsilon})} + \|q\|^2_{W^{m,2}(\Gamma_{b_1})}). \tag{3.5}
\]

By the duality and a similar argument in the proof of Theorem 3.3, we have the following. The proof is omitted.

**Theorem 3.4.** Let \( S \) be of \( C^3 \). Let \( p \in W^{1,2}(\Gamma_{-\epsilon}) \) and \( q \in W^{1,2}(\Gamma_{b_1}) \) be given. Suppose that \( F \in L^2(\Omega_{-\epsilon}, T) \) be given such that
\[
F|_{\Sigma_{-\epsilon}^{\epsilon/2}} \in W^{1,2}(\Sigma_{-\epsilon}^{\epsilon/2}, T).
\]

Then problem (3.3) admits a unique solution \( V \in W^{1,2}(\Omega_{-\epsilon}, T) \) satisfying (3.4). If \( S \) is of \( C^{m+3} \), then the above \( V \) satisfies (3.5).

The proofs of Theorems 3.1 and 3.3 will be given in the end of this section.

We assume that \( V \in C^1(\Omega_{-\epsilon}, T) \) satisfies equation
\[
L_0 V = F \quad \text{for} \quad x \in \Omega_{-\epsilon}. \tag{3.6}
\]

For convenience, we denote
\[
\begin{align*}
\div_g Q \nabla \vec{n} V - \eta \langle V, Q X \rangle &= f_1 \quad \text{for} \quad x \in \Omega_{-\epsilon}, \\
\div_g V - \eta \langle V, \nabla \vec{n} X \rangle &= f_2 \quad \text{for} \quad x \in \Omega_{-\epsilon}.
\end{align*} \tag{3.7}
\]

Then
\[
F = e^{-\gamma \kappa}(f_1 Q X + f_2 \nabla \vec{n} X).
\]

Let \( \Phi \in TS \) be given such that
\[
\langle \nabla \vec{n} \Phi, \Phi \rangle \neq 0 \quad \text{for} \quad x \in \overline{S}.
\]

We define an operator \( \mathcal{R}_\Phi : L^2(S, T) \to L^2(S, T) \) by
\[
\mathcal{R}_\Phi V = [\Phi, Q \nabla \vec{n} V] - Q \nabla \vec{n} [\Phi, V] \quad \text{for} \quad V \in TS. \tag{3.8}
\]
Lemma 3.1. Set
\[ h_1 = \frac{\langle R_\Phi \Phi, Q\Phi \rangle}{\langle \nabla \bar{n} \Phi, \Phi \rangle}, \quad h_2 = \frac{\langle R_\Phi Q\Phi, Q\Phi \rangle - h_1 \langle \nabla \bar{n} Q\Phi, Q\Phi \rangle}{|\Phi|^2}, \] (3.9)
\[ Z = \frac{1}{|\Phi|^2} (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id})^T \Phi. \] (3.10)

Then
\[ R_\Phi = h_1 Q \nabla \bar{n} + h_2 \text{id} + Z \otimes \Phi. \] (3.11)

Proof. We have
\[ \langle (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id}) \Phi, Q\Phi \rangle = \langle R_\Phi \Phi, Q\Phi \rangle - h_1 \langle Q \nabla \bar{n} \Phi, Q\Phi \rangle - h_2 \langle \Phi, Q\Phi \rangle = \langle R_\Phi \Phi, Q\Phi \rangle - h_1 \langle \nabla \bar{n} \Phi, \Phi \rangle = 0, \] (3.12)
and
\[ \langle (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id}) Q\Phi, Q\Phi \rangle = \langle R_\Phi Q\Phi, Q\Phi \rangle - h_1 \langle Q \nabla \bar{n} Q\Phi, Q\Phi \rangle - h_2 \langle Q\Phi, Q\Phi \rangle - \langle (R_\Phi Q\Phi, Q\Phi) - h_1 \langle \nabla \bar{n} Q\Phi, \Phi \rangle \rangle = 0. \] (3.13)

Since \( \frac{Q\Phi}{|\Phi|^2}, \Phi \) forms an orthonormal basis, it follows from (3.12) and (3.13) that
\[ \langle (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id}) W, Q\Phi \rangle = 0 \quad \text{for} \quad W \in TS. \]

Thus
\[ (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id}) W = \frac{1}{|\Phi|^2} \langle (R_\Phi - h_1 Q \nabla \bar{n} - h_2 \text{id}) W, \Phi \rangle \Phi = \langle W, Z \rangle \Phi = (Z \otimes \Phi) W \]
for all \( W \in TS \), that is, (3.10).

Lemma 3.2. For given \( V \in C^1(S,T) \), we have
\[ \mathcal{L}_Z[\Phi, V] = F_\Phi, \] (3.14)
where \( Z \) is given in (3.10) and
\[ F_\Phi = e^{-\gamma \kappa} (f_1 \Phi Q X + f_2 \Phi \nabla \bar{n} X), \] (3.15)
\[ f_1 \Phi = \Phi f_1 - h_1 f_1 - h_2 f_2 + \langle V, H_1 \rangle, \quad f_2 \Phi = \Phi f_2 + \langle V, H_2 \rangle, \]
\[ H_1 = \nabla \bar{n} Q D \text{ div}_g \Phi + [\Phi(\eta) - h_1 \eta] Q X - h_2 \eta \nabla \bar{n} X + \eta D_\Phi (Q X) + h_1 \text{ div}_g Q \nabla \bar{n} - \text{ div}_g R_\Phi - D_\Phi (Z, \cdot), \]
\[ H_2 = \Phi(\eta) \nabla \bar{n} X - D \text{ div}_g \Phi + \eta D_\Phi (\nabla \bar{n} X, \cdot). \]
Proof. From (3.8), (2.9), (2.5), (3.11), and (3.7), we have

\[ \text{div}_g Q \nabla \bar{n} [\Phi, V] = \text{div}_g [\Phi, Q \nabla \bar{n} V] - \text{div}_g R \Phi V \]

\[ = \Phi(\text{div}_g Q \nabla \bar{n} V) - Q \nabla \bar{n} V (\text{div}_g \Phi) - \langle D V, R \Phi \rangle - \langle V, \text{div}_g R \Phi \rangle \]

\[ = \Phi(\text{div}_g Q \nabla \bar{n} V) - Q \nabla \bar{n} V, D \text{div}_g \Phi - \langle D V, h_1 Q \nabla \bar{n} + h_2 \text{id} + Z \otimes \Phi \rangle - \langle V, \text{div}_g R \Phi \rangle \]

\[ = \Phi(\text{div}_g Q \nabla \bar{n} V) + \langle V, \nabla \bar{n} Q D \text{div}_g \Phi \rangle - h_1(\text{div}_g Q \nabla \bar{n} V - \langle V, \text{div}_g Q \nabla \bar{n} \rangle) \]

\[ - h_2 \text{div}_g V - \langle D \Phi V, Z \rangle - \langle V, \text{div}_g R \Phi \rangle \]

\[ = \Phi f_1 - h_1 f_1 - h_2 f_2 + \langle [\Phi, V], \eta Q X - Z \rangle - \langle D V, \Phi, Z \rangle \]

\[ + \langle V, \nabla \bar{n} Q D \text{div}_g \Phi + [\Phi(\eta) - h_1 \eta] Q X - h_2 \eta \nabla \bar{n} X \]

\[ + \eta D \Phi(Q X) + h_1 \text{div}_g Q \nabla \bar{n} - \text{div}_g R \Phi \].

Thus

\[ f_1 \Phi = \text{div}_g Q \nabla \bar{n} [\Phi, V] + \langle [\Phi, V], Z - \eta Q X \rangle = \Phi f_1 - h_1 f_1 - h_2 f_2 + \langle V, H_1 \rangle. \]

Similarly, we compute

\[ \text{div}_g[\Phi, V] = \Phi \text{div}_g V - V \text{div}_g \Phi = \Phi f_2 + \eta(\langle [\Phi, V], \nabla \bar{n} X \rangle + \langle V, H_2 \rangle). \]

Consider an operator on \( L^2(\Omega, T) \):

\[ L_0 V = e^{-\gamma} \left( \text{div}_g Q \nabla \bar{n} V - \langle V, \eta Q X \rangle \right) Q X + \left( \text{div}_g V - \eta(\langle V, \nabla \bar{n} X \rangle) \right) \nabla \bar{n} X, \] (3.16)

\[ D(L_0) = \{ V \in L^2(\Omega, T) \mid \text{div}_g V \in L^2(\Omega), \text{div}_g Q \nabla \bar{n} V \in L^2(\Omega), \]

\[ \langle V, Q X \rangle |_{\Gamma_{\eta \xi}} = 0, \text{div}_g V |_{\Gamma_{\eta \xi}} = 0 \}. \]

By Theorem 2.1, \( L_0^{-1} : L^2(\Omega, T) \rightarrow L^2(\Omega, T) \) is bounded. Similarly, Theorem 2.2 implies that \( L_0^* : L^2(\Omega, T) \rightarrow L^2(\Omega, T) \) is bounded, where the domain of operator \( L_0^* \) is

\[ D(L_0^*) = \{ V \in L^2(\Omega, T) \mid \langle V, Q X \rangle |_{\Gamma_{\eta \xi}} = 0 \}, \]

where \( L_0^* \) is given in (2.15) with \( Y = 0 \).

Let \( X \) be given in (2.34). Then \( Q X, \nabla \bar{n} X \) forms a vector field basis on \( \Omega_{\eta \xi} \). Denote its conjugate basis by \( F_1, F_2 \), where

\[ F_1 = \frac{Q \nabla \bar{n} X}{(\nabla \bar{n} X, X)}, \quad F_2 = \frac{X}{(\nabla \bar{n} X, X)} \] for \( x \in \Omega_{\eta \xi}. \) (3.17)

We define operator \( K : L^2(\Omega, T) \rightarrow L^2(\Omega, T) \) by

\[ KF = D \Phi F + \mathcal{M} F + \mathcal{N} L_0^{-1}(F), \] (3.18)
where
\[
\mathcal{M} = [D_{\Phi}F_1 + (\gamma \langle \Phi, D\kappa \rangle - h_1)F_1 - h_2F_2] \otimes QX + (D_{\Phi}F_2 + \gamma \langle \Phi, D\kappa \rangle F_2) \otimes \nabla \vec{n}X,
\]
\[
\mathcal{N} = e^{-\gamma \kappa}(H_1 \otimes QX + H_2 \otimes \nabla \vec{n}X).
\]

**Lemma 3.3.** Let \( K \) be given in (3.18). Then
\[
(KF,G)_{L^2(\Omega_{-\varepsilon},T)} = (F,K^*G)_{L^2(\Omega_{-\varepsilon},T)} + \int_{\partial \Omega_{-\varepsilon}} \langle F,G \rangle_{\Phi} \nu d\Gamma,
\]
where
\[
K^*G = -D_{\Phi}G - (\text{div}_g \Phi)G + M^T G + L_0^{-1}(N^T G).
\]

**Proof.** We have
\[
\langle KF,G \rangle = \langle D_{\Phi}F + \mathcal{M}F + \mathcal{N}L_0^{-1}(F), G \rangle
\]
\[
= \text{div}_g((F,G)\Phi) + \langle F, - (\text{div}_g \Phi)G - D_{\Phi}G \rangle + \langle F, M^T G \rangle + \langle L_0^{-1}F, N^T G \rangle.
\]
We integrate the above identity over \( \Omega_{-\varepsilon} \) to have (3.19).

**Lemma 3.4.** We have
\[
\mathcal{L}_Z[\Phi, V] = K\mathcal{L}_0 V \quad \text{for} \quad V \in W^{1,2}(\Omega_{-\varepsilon}),
\]
where \( Z \) is given in (3.10).

**Proof.** For given \( V \in W^{1,2}(\Omega_{-\varepsilon}) \), set
\[
\text{div}_g Q\nabla \vec{n}V - \langle V, \eta QX \rangle = f_1, \quad \text{div}_g V - \eta \langle V, \nabla \vec{n}X \rangle = f_2.
\]
By Lemma 3.2
\[
\mathcal{L}_Z[\Phi, V] = F_{\Phi},
\]
where \( F_{\Phi} \) is given in Lemma 3.2.

Next, we shall prove
\[
K\mathcal{L}_0 V = F_{\Phi}.
\]
Noting that
\[
\mathcal{L}_0 V = e^{-\gamma \kappa}(f_1 QX + f_2 \nabla \vec{n}X),
\]
we have
\[
D_{\Phi}\mathcal{L}_0 V = -\gamma \langle \Phi, D\kappa \rangle\mathcal{L}_0 V + e^{-\gamma \kappa}(\Phi f_1 QX + \Phi f_2 \nabla \vec{n}X) + e^{-\gamma \kappa}[f_1 D_{\Phi}(QX) + f_2 D_{\Phi}(\nabla \vec{n}X)].
\]
Since \( F_1, F_2 \) is the conjugate basis of \( QX, \nabla \vec{n}X \), it follows that
\[
\mathcal{M}\mathcal{L}_0 V = \langle D_{\Phi}F_1 + (\gamma \langle \Phi, D\kappa \rangle - h_1)F_1 - h_2F_2, \mathcal{L}_0 V \rangle QX + \langle D_{\Phi}F_2 + \gamma \langle \Phi, D\kappa \rangle F_2, \mathcal{L}_0 V \rangle \nabla \vec{n}
\]
\[
= e^{-\gamma \kappa}[-f_1 D_{\Phi}(QX) - f_2 D_{\Phi}(\nabla \vec{n}X)] + \gamma \langle \Phi, D\kappa \rangle\mathcal{L}_0 V - e^{-\gamma \kappa}(h_1 f_1 + h_2 f_2)QX.
\]
Thus, by (3.15), we obtain
\[
KL_0V = D\Phi L_0V + ML_0V + N(L_0^{-1}(L_0V)) = e^{-\gamma \kappa}\{(\Phi f_1 - (h_1 f_1 + h_2 f_2) + (H_1, V)]QX + (\Phi f_2 + (H_2, V))\nabla nX \} = F_\Phi.
\]

Now we construct a special \( \Phi \) as follows. Define a function \( \psi \) on \( S \) by
\[
\psi(\alpha(t, s)) = s \quad \text{for} \quad x = \alpha(t, s) \in S.
\]
Then
\[
\langle D\psi, \alpha_t \rangle = 0, \quad \langle D\psi, \alpha_s \rangle = 1 \quad \text{for} \quad x \in S.
\]
Let
\[
\Phi_0 = QD\psi \quad \text{for} \quad x \in S.
\]
Then
\[
\langle \Phi_0, \nu \rangle = \langle QD\psi, \frac{Q\alpha_t}{|\alpha_t|} \rangle = 0 \quad \text{for} \quad x \in \partial\Omega_{-\varepsilon}.
\]
Set
\[
\Phi = e^\varphi \Phi_0 \quad \text{for} \quad x \in \Omega_{\varepsilon},
\]
where \( \varphi \) is a function given in Lemma 3.5 below. Since
\[
D\psi = \frac{\langle D\psi, Q\alpha_t \rangle}{|\alpha_t|^2}Q\alpha_t \quad \text{for} \quad x \in \Omega_{-\varepsilon}
\]
with \( \langle D\psi, Q\alpha_t \rangle \neq 0 \), from (2.24), we have
\[
\langle \nabla n\Phi, \Phi \rangle = e^{2\varphi}\langle \nabla n\Phi_0, \Phi_0 \rangle = e^{2\varphi}\frac{(D\psi, Q\alpha_t)^2}{|\alpha_t|^4}\Pi(\alpha_t, \alpha_t) > 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}.
\]

**Lemma 3.5.** There exists a function \( \varphi \) on \( \Omega_{-\varepsilon} \) such that
\[
\langle D\Phi D\psi + MD\psi, QD\psi \rangle = 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon} \setminus S^+\,,
\]
\[
\langle [\Phi, X], QX \rangle = 0 \quad \text{for} \quad x \in \Gamma_{b_1},
\]
where \( X \) is given in (2.34). Furthermore, for \( W \in C^1(\Omega_{-\varepsilon}, T) \), we have
\[
\langle [\Phi, W], D\psi \rangle = \Phi \langle W, D\psi \rangle \quad \text{for} \quad x \in \Omega_{-\varepsilon} \setminus S^+\,,
\]
\[
\langle [\Phi, W], QX \rangle = \Phi \langle W, QX \rangle + \langle W, QX \rangle \langle [\Phi, \frac{1}{|X|^2}QX], QX \rangle \quad \text{for} \quad x \in \Gamma_{b_1}.
\]
Proof. Noting that $\Phi = e^{\varphi}\Phi_0 = e^{\varphi}QD\psi$, we have

$$
\langle D_\Phi D\psi + MD\psi, QD\psi \rangle = e^{\varphi}\langle D_{\Phi_0}D\psi, \Phi_0 \rangle + \langle D_\Phi F_2 + \gamma \langle \Phi, D\kappa \rangle F_2, D\psi \rangle \langle QD\psi, \nabla \bar{n}X \rangle
$$

$$
+ \langle D_\Phi F_1 + (\gamma \langle \Phi, D\kappa \rangle - h_1)F_1 - h_2 F_2, D\psi \rangle \langle QD\psi, QX \rangle
$$

$$
e^{\varphi}h - (h_1\langle F_1, D\psi \rangle + h_2\langle F_2, D\psi \rangle)\langle D\psi, QX \rangle,
$$

(3.27)

where

$$
h = \langle D_{\Phi_0}D\psi, \Phi_0 \rangle + \langle D_{\Phi_0}F_2 + \gamma \langle \Phi_0, D\kappa \rangle F_2, D\psi \rangle \langle QD\psi, \nabla \bar{n}X \rangle
$$

$$
+ \langle D_{\Phi_0}F_1 + \gamma \langle \Phi_0, D\kappa \rangle F_1, D\psi \rangle \langle D\psi, QX \rangle.
$$

On the other hand, from (3.8), we have

$$
\langle R_\Phi \Phi, Q\Phi \rangle = \langle D_\Phi (Q\nabla \bar{n}\Phi) - D_{Q\nabla \bar{n}\Phi} \Phi, Q\Phi \rangle
$$

$$
e^{3\varphi}[\langle R_{\Phi_0}\Phi_0, Q\Phi_0 \rangle + \Phi_0(\varphi)\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle].
$$

Similarly, we obtain

$$
\langle R_\Phi Q\Phi, Q\Phi \rangle = e^{3\varphi}[\langle R_{\Phi_0}Q\Phi_0, Q\Phi_0 \rangle + (Q\Phi_0)(\varphi)\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle].
$$

Thus

$$
h_1 = e^{\varphi}\left[\frac{\langle R_{\Phi_0}\Phi_0, Q\Phi_0 \rangle}{\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle} + \Phi_0(\varphi)\right],
$$

$$
h_2 = \frac{e^{\varphi}}{|\Phi_0|^2}\left\{\langle R_{\Phi_0}Q\Phi_0, Q\Phi_0 \rangle - \frac{\langle R_{\Phi_0}\Phi_0, Q\Phi_0 \rangle}{\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle}\langle \nabla \bar{n}Q\Phi_0, \Phi_0 \rangle
$$

$$
- (Q\Phi_0)(\varphi)\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle + \Phi_0(\varphi)\langle \nabla \bar{n}Q\Phi_0, \Phi_0 \rangle\right\}.
$$

We then obtain

$$
h_1\langle F_1, D\psi \rangle + h_2\langle F_2, D\psi \rangle = e^{\varphi}\left\{\hat{h} + \Phi_0(\varphi)\langle (F_1, D\psi) + \frac{\langle \nabla \bar{n}Q\Phi_0, \Phi_0 \rangle}{|\Phi_0|^2}\langle F_2, D\psi \rangle
$$

$$
- (Q\Phi_0)(\varphi)\frac{\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle}{|\Phi_0|^2}\langle F_2, D\psi \rangle\right\},
$$

where

$$
\hat{h} = \frac{\langle R_{\Phi_0}\Phi_0, Q\Phi_0 \rangle}{\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle}\langle F_1, D\psi \rangle + \frac{\langle F_2, D\psi \rangle}{|\Phi_0|^2}\left\{\langle R_{\Phi_0}Q\Phi_0, Q\Phi_0 \rangle - \frac{\langle R_{\Phi_0}\Phi_0, Q\Phi_0 \rangle}{\langle \nabla \bar{n}\Phi_0, \Phi_0 \rangle}\langle \nabla \bar{n}Q\Phi_0, \Phi_0 \rangle\right\}.
$$

It follows from (3.27) that

$$
\langle D_\Phi D\psi + MD\psi, QD\psi \rangle = e^{\varphi}[\hat{h} + H(\varphi)],
$$

(3.28)

where

$$
H = \langle (F_1, D\psi) - \frac{\langle \nabla \bar{n}D\psi, QD\psi \rangle}{|D\psi|^2}\langle F_2, D\psi \rangle \rangle QD\psi + \frac{\langle \nabla \bar{n}QD\psi, QD\psi \rangle}{|D\psi|^2} \langle F_2, D\psi \rangle D\psi.
$$
By (2.31) and (3.22),
\[ \langle QD\psi, \nu \rangle = 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}. \]

From (3.22), (3.17), and (2.48), we have
\[ \langle H, \nu \rangle = \langle D\psi, Q\alpha_t \rangle^2 \left| \frac{\alpha_t}{\Pi(X, X)} \right| \Pi(\alpha_t, \alpha_t) \langle X, \nu \rangle < 0 \quad \text{for} \quad x \in \Gamma_{-\varepsilon}. \]

Thus, when given \( \varepsilon > 0 \) is small, the integral curves of the vector field \( H \) initiating from \( \Gamma_{-\varepsilon} \) direct to the interior of \( \Omega_{-\varepsilon} \) can across \( \Gamma_0 \). We solve equation (3.28) along those integral curves to have \( \psi \) to satisfy (3.23).

By a similar argument, we obtain a function \( \psi \) in a neighborhood \( \Gamma_{b_1} \) such that (3.24) holds.

Next, we prove (3.25) and (3.26). By the symmetry of \( D^2\psi \), we have
\[ \langle [\Phi, W], D\psi \rangle = \Phi \langle W, D\psi \rangle - W \langle \Phi, D\psi \rangle - \langle W, D\Phi D\psi \rangle = \Phi \langle W, D\psi \rangle. \]

Moreover, since \( QX \mid X \mid, \frac{X}{X} \) forms an orthonormal frame, it follows from (3.24) that
\[ \langle [\Phi, W], QX \rangle = \langle [\Phi, W] \frac{X}{X^2}, QX \rangle + \langle W, QX \rangle \langle \Phi, \frac{X}{X^2}, QX \rangle QX \]
\[ = \langle W, QX \rangle \left( \Phi \frac{X}{X^2} + \frac{1}{X^2} QX \right) \]
\[ = \Phi \langle W, QX \rangle + \langle W, QX \rangle \langle \Phi, \frac{1}{X^2} QX \rangle QX \quad \text{for} \quad x \in \Gamma_{b_1}. \]

**Lemma 3.6.** Let \( k > 0 \) be given large and \( W \in C^1(\Omega_{-\varepsilon}, T) \). Then
\[ \langle K^*[\Phi, W] + kW, \nu \rangle |_{\Gamma_{-\varepsilon}} = 0 \quad \Rightarrow \quad \langle W, \nu \rangle |_{\Gamma_{-\varepsilon}} = \langle [\Phi, W], \nu \rangle |_{\Gamma_{-\varepsilon}} = 0, \]
\[ \langle K^*[\Phi, W] + kW, QX \rangle |_{\Gamma_{b_1}} = 0 \quad \Rightarrow \quad \langle W, QX \rangle |_{\Gamma_{b_1}} = \langle [\Phi, W], QX \rangle |_{\Gamma_{b_1}} = 0, \]
where operator \( K^* \) is given in (3.20).

**Proof.** Since \( \frac{QD\psi}{|D\psi|}, \frac{D\psi}{|D\psi|} \) forms an orthonormal frame, from (3.23), we have
\[ D\Phi D\psi + MD\psi = \frac{D\Phi D\psi + MD\psi + D\psi}{|D\psi|^2} D\psi. \]

Noting that \( D\psi |_{\Gamma_{-\varepsilon}} = \pm \frac{D\psi, Q\alpha_t}{|\alpha_t|} \nu \), we have
\[ \langle L_0^{-1}(\mathcal{N}^T[\Phi, W], D\psi) |_{\Gamma_{-\varepsilon}} = 0. \]
It follows from (3.25) that
\[
\langle \mathcal{K}^*[\Phi, W], D\psi \rangle = \langle -D\Phi[\Phi, W] - (\text{div} \Phi)[\Phi, W] + M^T[\Phi, W], D\psi \rangle \\
= -\Phi^2(W, D\psi) - (\text{div} \Phi)(W, D\psi) + \langle [\Phi, W], D\Phi D\psi + M D\psi \rangle \\
= -\Phi^2(W, D\psi) - [\text{div} \Phi - \frac{\langle D\Phi D\psi + M D\psi, D\psi \rangle}{|D\psi|^2}]\langle W, D\psi \rangle.
\]
Thus we obtain
\[
\langle \mathcal{K}^*[\Phi, W] + kW, D\psi \rangle \langle W, D\psi \rangle \geq -\Phi \langle (W, D\psi) \Phi(W, D\psi) \rangle + (\Phi(W, D\psi))^2 + (k-C)\langle W, D\psi \rangle^2.
\]
We integrate the above inequality over $\Gamma_{-\varepsilon}$ to have
\[
\|\langle \mathcal{K}^*[\Phi, W] + kW, D\psi \rangle \|_{L^2(\Gamma_{-\varepsilon})}^2 \geq \sigma(\|\Phi(W, D\psi)\|_{L^2(\Gamma_{-\varepsilon})}) + \|\langle W, D\psi \rangle\|_{L^2(\Gamma_{-\varepsilon})}^2
\]
for given $k > 0$ large, that is what we want.

A similar computation yields the second conclusion.

We introduce a Hilbert space by
\[
\mathcal{V}_\Phi = \{ W \in L^2(\Omega, T) \mid \langle W, \nu \rangle|_{\Gamma_{b_1}} \in W^{1,2}(\Gamma_{b_1}), (W, QX)|_{\Gamma_{-\varepsilon}} \in W^{1,2}(\Gamma_{-\varepsilon}) \},
\]
\[
\|W\|_{\mathcal{V}_\Phi} = \|\Phi(W)\|_{L^2(\Omega, T)} + \|W\|_{L^2(\Gamma_{b_1})} + \|\langle W, QX \rangle\|_{W^{1,2}(\Gamma_{-\varepsilon})} + \|\langle W, \nu \rangle\|_{W^{1,2}(\Gamma_{b_1})}.
\]

**Lemma 3.7.** Let $W_0 \in C^1(\Omega_{-\varepsilon}, T)$ be given with $\langle W_0, \nu \rangle|_{\Gamma_{-\varepsilon}} = 0$ and $(W_0, QX)|_{\Gamma_{b_1}} = 0$. Then there exists a unique $W \in \mathcal{V}_\Phi$ such that
\[
\mathcal{K}^*[\Phi, W] + kW = W_0 \text{ for } x \in \Omega_{-\varepsilon}
\]
for given $k > 0$ large.

**Proof.** Define a bilinear form $\alpha : \mathcal{V}_\Phi \times W^{1,2}_0(\Omega, T) \to \mathbb{R}$ by
\[
\alpha(W, V) = \langle \mathcal{K}^*[\Phi, W] + kW, V \rangle_{L^2(\Omega_{-\varepsilon})}.
\]
Since $\langle \Phi, \nu \rangle|_{\partial\Omega} = 0$, from (3.19), we have
\[
\alpha(W, V) = (\mathcal{K}^*[\Phi, W], KV)_{L^2(\Omega_{-\varepsilon})} + k\langle W, V \rangle_{L^2(\Omega_{-\varepsilon})}
\]
which yield
\[
|\alpha(W, V)| \leq C\|W\|_{\mathcal{V}_\Phi}\|V\|_{W^{1,2}(\Omega, T)} \text{ for } W \in \mathcal{V}_\Phi, W \in W^{1,2}_0(\Omega, T).
\]
By (3.18), we have
\[
\alpha(W, W) = (\mathcal{K}^*[\Phi, W], \mathcal{K}W)_{L^2(\Omega_{-\varepsilon})} + k\|W\|_{L^2(\Omega_{-\varepsilon})}^2 \\
= (\mathcal{K}^*[\Phi, W], D\Phi W + MW + NL^{-1}_0(W))_{L^2(\Omega_{-\varepsilon})} + k\|W\|_{L^2(\Omega_{-\varepsilon})}^2 \\
= (\mathcal{K}^*[\Phi, W], \Phi W + D\Phi + MW + NL^{-1}_0(W))_{L^2(\Omega_{-\varepsilon})} + k\|W\|_{L^2(\Omega_{-\varepsilon})}^2 \\
\geq \|\mathcal{K}^*[\Phi, W]\|_{L^2(\Omega_{-\varepsilon})}^2 + (k-C)\|W\|_{L^2(\Omega_{-\varepsilon})}^2
\]
(3.29)
\[
\geq \|W\|_{\mathcal{V}_\Phi}^2 \text{ for } W \in W^{1,2}_0(\Omega_{-\varepsilon}, T).
\]
Let \( \mathcal{F}(V) = (W_0, V) \) for \( V \in \mathcal{V}_\Phi \). Then \( \mathcal{F} \) is bounded on \( \mathcal{V}_\Phi \). By Theorem A given in Appendix there exists \( W \in \mathcal{V}_\Phi \) such that

\[
\mathcal{F}(V) = \alpha(W, V) \quad \text{for} \quad V \in W^{1,2}_0(\Omega_-, T),
\]

which implies

\[
\mathcal{K}^*[\Phi, W] + kW = W_0 \quad \text{for} \quad x \in \Omega_-.
\]

Moreover, a similar computation as in (3.29) gives

\[
(\mathcal{K}^*[\Phi, W] + kW, W)_{L^2(\Omega_-)} \geq \|\Phi, W\|_{L^2(\Omega_-)}^2 + (k - C)\|W\|_{L^2(\Omega_-)}^2.
\]

Thus the uniqueness follows.

**Proposition 3.1.** Let \( F \in L^2(\Omega_-, T) \) be given with \( \Phi, F \in L^2(\Omega_-, T) \). Let \( p \in W^{1,2}(\Gamma_-) \) and \( q \in W^{1,2}(\Gamma_b) \) be given. Then problem (3.1) admits a unique solution \( V \in \mathcal{V}_\Phi \) satisfying

\[
\|V\|_{\Phi} \leq C(\|\Phi, F\|_{L^2(\Omega_-)} + \|F\|_{L^2(\Omega_-)} + \|p\|_{W^{1,2}(\Gamma_-)} + \|q\|_{W^{1,2}(\Gamma_b)}).
\]

**Proof.** Let \( k > 0 \) be given large. Let

\[
\mathcal{D}_0 = \{ W \in C^1(\Omega_-, T) \mid \langle W, \nu \rangle|_{\Gamma_-} = \langle W, QX \rangle|_{\Gamma_b} = 0 \},
\]

\[
\mathcal{W} = \{ W \in \mathcal{V}_\Phi \mid \text{there is } W_0 \in \mathcal{D}_0 \text{ such that } \mathcal{K}^*[\Phi, W] + kW = W_0 \}.
\]

By Lemma 3.7, we introduce an inner product on \( \mathcal{W} \) by

\[
(V, W)_{\mathcal{W}} = (V_0, W_0)_{W^{1,2}(\Omega_-, T)},
\]

where \( \mathcal{K}^*[\Phi, V] + kV = V_0 \) and \( \mathcal{K}^*[\Phi, W] + kW = W_0 \). Then \( \mathcal{W} \subset \mathcal{V}_\Phi \) is an inner product space. It follows from Lemma 3.6 that

\[
\langle W, QX \rangle|_{\Gamma_b} = \langle [\Phi, W], QX \rangle|_{\Gamma_b} = \langle W, \nu \rangle|_{\Gamma_-} = \langle [\Phi, W], \nu \rangle|_{\Gamma_-} = 0
\]

for \( W \in \mathcal{W} \).

**Step 1.** We define a bilinear form \( \alpha : \mathcal{V}_\Phi \times \mathcal{W} \to \mathbb{R} \) by

\[
\alpha(V, W) = -(V, \mathcal{L}_0^*(\mathcal{K}^*[\Phi, W] + kW))_{L^2(\Omega_-)}.
\]

Clearly, we have

\[
|\alpha(V, W)| \leq C\|V\|_{\Phi}\|W\|_{\mathcal{W}} \quad \text{for} \quad V \in \mathcal{V}_\Phi, \ W \in \mathcal{W}.
\]
Next, we prove the coerciveness. From (2.16), (3.32), (3.19), (3.21), (2.16) (again), we compute

$$-2(W, L_0^*(K^*[\Phi, W] + kW))_{L^2(\Omega_{-\varepsilon})} = -2(L_0 W, K^*[\Phi, W] + kW)_{L^2(\Omega_{-\varepsilon})} + 2 \int_{\Gamma_{b_1}} L_1(W, \nu) \langle K^*[\Phi, W] + kW, \nu \rangle d\Gamma$$

$$-2 \int_{\Gamma_{-\varepsilon}} L_2(W, QX) \langle K^*[\Phi, W] + kW, QX \rangle d\Gamma = -2(KL_0[\Phi, W], [\Phi, W])_{L^2(\Omega_{-\varepsilon})} - 2k(L_0 W, W)_{L^2(\Omega_{-\varepsilon}, T)} + 2 \int_{\Gamma_{b_1}} L_1(W, \nu) \langle K^*[\Phi, W] + kW, \nu \rangle d\Gamma$$

$$-2 \int_{\Gamma_{-\varepsilon}} L_2(W, QX) \langle K^*[\Phi, W] + kW, QX \rangle d\Gamma = -(([\Phi, W], L_2[\Phi, W] + L_2^*[\Phi, W])_{L^2(\Omega_{-\varepsilon})} - k(W, L_0 W + L_0^* W)_{L^2(\Omega_{-\varepsilon}, T)} + \Gamma_{b_1}(W) + \Gamma_{-\varepsilon}(W),$$

where

$$\Gamma_{b_1}(W) = \int_{\Gamma_{b_1}} |L_1|[2\langle W, \nu \rangle \langle K^*[\Phi, W] + kW, \nu \rangle - \langle [\Phi, W], \nu \rangle^2 - k(W, \nu)^2] d\Gamma,$$

$$\Gamma_{-\varepsilon}(W) = \int_{\Gamma_{-\varepsilon}} |L_2|[2\langle W, QX \rangle \langle K^*[\Phi, W] + kW, QX \rangle - \langle [\Phi, W], QX \rangle^2 - k\langle W, QX \rangle^2] d\Gamma.$$

We estimate $\Gamma_{-\varepsilon}(W)$ and $\Gamma_{b_1}(W)$ below. By (2.48), $QX, \nu$ forms a vector field basis along $\Gamma_{-\varepsilon}$. It follows from (3.32) that

$$W = O(\langle W, QX \rangle), \quad [\Phi, W] = O(\langle [\Phi, W], QX \rangle) \quad \text{for} \quad x \in \Gamma_{-\varepsilon}.$$

Set $\tau = \Phi/|\Phi|$. Then

$$2|L_2|\langle W, QX \rangle (-D\Phi[\Phi, W], QX)$$

$$= -2\tau(\langle [\Phi, W], |L_2|\Phi(W, QX)QX \rangle + 2|\langle [\Phi, W], D\tau(|L_2|\Phi(W, QX)QX) \rangle$$

$$= -2\tau(\langle [\Phi, W], |L_2|\Phi(W, QX)QX \rangle + 2|L_2|\Phi(W, QX)\langle [\Phi, W], QX \rangle)$$

$$+ 2\langle W, QX \rangle \langle [\Phi, W], D\tau(|L_2|\Phi QX) \rangle$$

$$= \tau(\tilde{h}) + 2|L_2|\langle [\Phi, W], QX \rangle^2 + O(\langle W, QX \rangle \langle [\Phi, W], QX \rangle) \quad \text{for} \quad x \in \Gamma_{-\varepsilon},$$

where

$$\tilde{h} = -2|\langle [\Phi, W], |L_2||\Phi(W, QX)QX \rangle.$$
from (3.20), we obtain

\[
|L_2^2[2\langle W, QX \rangle \langle K^*[\Phi, W] + kW, QX \rangle - \langle \Phi, W \rangle, QX \rangle^2 - k\langle W, QX \rangle^2]
= 2|L_2[(W, QX)\langle - D \Phi, W \rangle - (\text{div}_y \Phi)\langle \Phi, W \rangle + M^T \Phi, W] + L_0^{-1}(N^T \Phi, W), QX)
\]

\[
+k|L_2[(W, QX)^2 - |L_2|(|\Phi, W|, QX)^2
\]

\[
= \tau(\tilde{h}) + |L_2|(|\Phi, W|, QX)^2 + k|L_2|(|W, QX)^2 + \mathcal{O}(\langle W, QX \rangle \langle \Phi, W \rangle, QX))
\]

\[
\geq \tau(\tilde{h}) + \frac{|L_2|}{2}(|\Phi, W|, QX)^2 + (k - C)\langle W, QX \rangle^2
\]

Thus

\[
\Gamma_{-\varepsilon}(W) \geq \sigma \|\langle W, QX \rangle\|^2_{W_{1,2}(\Gamma_{-\varepsilon})} \quad \text{for } W \in \mathcal{W}
\]

for given \(k > 0\) large. By (3.22) a similar argument yields

\[
\Gamma_{b_1}(W) \geq \sigma \|\langle W, \nu \rangle\|^2_{W_{1,2}(\Gamma_{b_1})} \quad \text{for } W \in \mathcal{W}
\]

for given \(k > 0\) large.

Using (2.37) in (3.33), we obtain

\[
\alpha(W, W) \geq \sigma \|W\|^2_{V_\Phi} \quad \text{for } W \in \mathcal{W}.
\]

**Step 2.** Let \(F \in L^2(\Omega_{-\varepsilon}, T)\) be given with \([\Phi, F] \in L^2(\Omega_{-\varepsilon}, T)\). Set

\[
\mathcal{F}(W) = -(F, K^*[\Phi, W] + kW)_{L^2(\Omega_{-\varepsilon}, T)} + \int_{\Gamma_{-\varepsilon}} |L_2|p(K^*[\Phi, W] + kW, QX)d\Gamma
\]

\[
+ \int_{\Gamma_{b_1}} |L_1|q(K^*[\Phi, W] + kW, \nu)d\Gamma \quad \text{for } W \in \mathcal{V}_\Phi.
\]

Since \((F, K^*[\Phi, W]_{L^2(\Omega_{-\varepsilon}, T)} = (K^*, [\Phi, W])_{L^2(\Omega_{-\varepsilon}, T)}\), it is easy to check that \(\mathcal{F}\) is bounded on \(\mathcal{V}_\Phi\). By Theorem A given in Appendix there exists a \(V \in \mathcal{V}_\Phi\) such that

\[
\mathcal{F}(W) = \alpha(V, W) \quad \text{for } W \in \mathcal{W},
\]

\[
\|V\|_\Phi \leq C\|\mathcal{F}\|_{\mathcal{V}_\Phi}.
\]

**Step 3.** We shall show that \(V \in \mathcal{V}_\Phi\) given in (3.34) is the solution to problem (3.1). Let \(W_0 \in C^1(\Omega_{-\varepsilon}, T)\) be given with \(\langle W_0, \nu \rangle_{\Gamma_{-\varepsilon}} = \langle W_0, QX \rangle_{\Gamma_{b_1}} = 0\). By Lemma 3.7, there is \(W \in \mathcal{V}_\Phi\) such that \(K^*[\Phi, W] + kW = W_0\). Then (3.34) becomes

\[
(V, L_0^2W_0)_{L^2(\Omega_{-\varepsilon})} = (F, W_0)_{L^2(\Omega_{-\varepsilon}, T)} - \int_{\Gamma_{-\varepsilon}} |L_2|p(W_0, QX)d\Gamma
\]

\[
- \int_{\Gamma_{b_1}} |L_1|q(W_0, \nu)d\Gamma \quad \text{for } W_0 \in \mathcal{D}_0,
\]

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where $D_0$ is given in (3.31). Using (2.16) in (3.36), one has

$$\langle W_0, L_0V \rangle_{L^2(\Omega, T)} = \langle F, W_0 \rangle_{L^2(\Omega, T)} + \int_{\Gamma_{b1}} |L_1|(q - \langle V, \nu \rangle)\langle W_0, \nu \rangle d\Gamma + \int_{\Gamma_{-\epsilon}} |L_2|(p - \langle V, QX \rangle)\langle W_0, QX \rangle d\Gamma$$

(3.37)

for all $W_0 \in D_0$. Thus $V \in V_s$ solves problem (3.1). Finally, (3.30) follows from (3.35). □

**Lemma 3.8.** Let $Y \in TM$ be with $|Y| = 1$. Then

$$i_Y DZ = D(Z, Y) - \langle Z, QY \rangle QY$$

for $Z \in TM$. (3.38)

In addition,

$$-id \otimes Q\nabla\bar{n}Z + Q\nabla\bar{n} \otimes Z = QZ \otimes Q\nabla\bar{n}Q - \nabla\bar{n}Z \otimes Q$$

for $Z \in TM$. (3.39)

**Proof.** $|Y| = 1$ implies that $QY, Y$ is an orthonormal frame. Then

$$i_Z DY = \langle D_Z Y, QY \rangle QY = \langle i_Y DZ, Z \rangle QY$$

for $Z \in TM$.

Since

$$\langle D_Y QY, QY \rangle = \langle D_{QY} Y, QY \rangle = 0,$$

we have

$$D\langle Z, Y \rangle = QY \langle Z, Y \rangle QY + Y \langle Z, Y \rangle Y$$

$$= (\langle i_Y DZ, QY \rangle + \langle D_Y QY, QY \rangle \langle Z, QY \rangle) QY + (\langle i_Y DZ, Y \rangle + \langle D_Y QY, QY \rangle \langle Z, QY \rangle) Y$$

$$= i_Z DY + \langle Z, QY \rangle (\langle D_{QY} Y - D_Y QY, QY \rangle QY + \langle D_Y QY - D_Y QY, Y \rangle Y)$$

$$= i_Z DY + \langle Z, QY \rangle [QY, Y],$$

that is (3.38).

Let $E_1, E_2$ be an orthonormal frame. Letting $v = E_i, w = QE_j$, and $P = \nabla\bar{n}$ in (2.7) yields

$$\langle E_i, E_j \rangle Q\nabla\bar{n} + \langle \nabla\bar{n}E_i, QE_j \rangle \text{id} = QE_i \otimes Q\nabla\bar{n}QE_j + \nabla\bar{n}E_i \otimes QE_j.$$

Thus

$$\langle E_i, E_j \rangle Q\nabla\bar{n}Z + \langle \nabla\bar{n}E_i, QE_j \rangle Z = \langle Z, QE_i \rangle Q\nabla\bar{n}QE_j + \langle Z, \nabla\bar{n}E_i \rangle QE_j.$$

Then

$$\langle E_i, E_j \rangle \langle Q\nabla\bar{n}Z, E_k \rangle - \langle Q\nabla\bar{n}E_i, E_j \rangle \langle Z, E_k \rangle = -\langle QZ, E_i \rangle \langle Q\nabla\bar{n}QE_j, E_k \rangle + \langle \nabla\bar{n}Z, E_i \rangle \langle QE_j, E_k \rangle$$

for all $1 \leq i, j, k \leq 2$, that is (3.39). □
Lemma 3.9. Let $\Omega \subset \Omega_{-\varepsilon}$ be a subregion and let $\theta \in C^1(\Omega)$. Let $E \in T\Omega$ be given with $|E| = 1$. Let $\sigma > 0$ be given small. Then

$$
\int_{\Omega} \theta^2 |Q^*\Pi(i_E DV, i_E DV) - \sigma |DV|^2|dg \\
\leq C \int_{\Omega} \theta^2 [\langle \text{div}_g Q \nabla \vec{n}V \rangle^2 + \langle \text{div}_g V \rangle^2]dg + C\|V\|_{L^2(\Omega,T)}^2
$$

$$
- \int_{\partial\Omega} \theta^2 \langle \nabla \vec{n}Q, E \rangle [\frac{1}{2} |V|^2 (\nu \wedge |Q, E|, \mathcal{E}) + \langle V, Q, E \rangle (\nu \wedge d(V, E), \mathcal{E})]d\Gamma
$$

(3.40)

for all $V \in W^{1,2}(\Omega,T)$, where $\nu$ is the outside normal of $\Omega$ and $\mathcal{E}$ is the volume element of $M$.

Proof. Let $E_1, E_2$ be an orthonormal frame. For given $W \in TM$, using (3.39), we have

$$
\langle \langle DV, Q\nabla \vec{n}\rangle Q, E - \langle DV, \text{id}\rangle Q\nabla \vec{n}Q, W \rangle = \langle DV, Q\nabla \vec{n}\rangle (W, Q, E) - \langle DV, \text{id}\rangle (Q\nabla \vec{n}Q, W) \\
= \langle DV \otimes W, Q\nabla \vec{n} \otimes Q \rangle - \langle DV \otimes W, \text{id} \otimes Q\nabla \vec{n}Q \rangle \\
= \langle DV \otimes W, -E \otimes Q\nabla \vec{n}Q - \nabla \vec{n}QE \otimes Q \rangle \\
= -\sum DV(E_i, E_j)(W, E_k)(\langle E, E_i \rangle (Q\nabla \vec{n}Q, E_j, E_k) + \langle \nabla \vec{n}QE, E_i \rangle (QE, E_k)) \\
= \langle \nabla \vec{n}Q, i_E DV, QW \rangle + \langle i_{\nabla \vec{n}Q}DV, QW \rangle.
$$

Letting $W = i_E DV$ in the above identity gives

$$
\langle \langle DV, Q\nabla \vec{n}\rangle Q, E - \langle DV, \text{id}\rangle Q\nabla \vec{n}Q, i_E DV \rangle \\
= Q^*\Pi(i_E DV, i_E DV) + \langle i_{\nabla \vec{n}Q}DV, Qi_E DV \rangle.
$$

(3.41)

Since $QE, E$ forms an orthonormal frame, we obtain

$$
\langle i_{\nabla \vec{n}Q}DV, Qi_E DV \rangle = \langle \nabla \vec{n}Q, E \rangle \langle i_{Q, E}DV, Qi_E DV \rangle \\
= \langle \nabla \vec{n}Q, E \rangle [DV(Q, E)E + DV(Q, Q)QE, DV(E, E)QE - DV(Q, E)E] \\
= \langle \nabla \vec{n}Q, Q \rangle [DV(Q, E)DV(E, E) - DV(Q, E)DV(E, Q)] \\
= \langle \nabla \vec{n}Q, Q \rangle \langle i_{Q, E}DV \wedge i_E DV, \mathcal{E} \rangle.
$$

(3.42)

On the other hand, it follows from (3.38) that

$$
i_E DV \wedge i_{Q, E}DV = (D(V, E) - \langle V, Q \rangle [QE, E]) \wedge (D(V, Q) + \langle V, E \rangle [QE, E]) \\
= D(V, E) \wedge D(V, Q) + (\langle V, E \rangle D(V, E) + \langle V, Q \rangle D(V, Q)) \wedge [QE, E] \\
= d(V, E) \wedge d(V, Q) + \frac{1}{2} d(|V|^2) \wedge [QE, E].
$$

(3.43)
Moreover, noting that $d^2 = 0$, we compute

$$
\begin{align*}
d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle \langle V, Q E \rangle d(V, E)) + \langle V, Q E \rangle d(V, E) & \wedge d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle) \\
& = \frac{1}{2} d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q|^2) - \frac{1}{2} |V|^2 d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q, E]) \\
& = \frac{1}{2} d(\langle \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q E, E]) + \frac{1}{2} d(|V|^2) \wedge \langle \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q E, E]) \\
& = \frac{1}{2} d(|V|^2) \wedge \langle \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q E, E]).
\end{align*}
$$

(3.44)

and

$$
\begin{align*}
\frac{1}{2} d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |V| \langle Q, E, E]) & \wedge d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q, E, E]) \\
& = \frac{1}{2} d(\langle \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q, E, E]) \wedge d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q, E, E]) \\
& \Rightarrow \frac{1}{2} d(|V|^2) \wedge \langle \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |Q E, E]).
\end{align*}
$$

(3.45)

Since

$$
\text{div}_g Q \nabla \tilde{n} V = \langle DV, Q \nabla \tilde{n} \rangle, \quad \text{div}_g V = \langle DV, \text{id} \rangle,
$$

inserting (3.45), (3.44), (3.43), and (3.42) into (3.41), we obtain

\begin{align*}
\theta^2 & \langle (\text{div}_g Q PV) Q E - (\text{div}_g V)QP Q E, i_E DV \rangle \\
& = \theta^2 \langle (DV, Q \nabla \tilde{n}) Q E - (DV, \text{id}) Q P Q E, i_E DV \rangle \\
& = \theta^2 Q \Pi(i_E DV, i_E DV) + \theta^2 \langle i \nabla \tilde{n} Q E, Q W \rangle \\
& = \theta^2 Q \Pi(i_E DV, i_E DV) + \theta^2 \langle i \nabla \tilde{n} Q E, Q E \rangle \langle i_Q E V \wedge i_E DV, E \rangle \\
& = \theta^2 Q \Pi(i_E DV, i_E DV) + \theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle \langle d(V, E) \wedge d(V, Q E) + \frac{1}{2} d(|V|^2) \wedge [Q E, E], E \rangle \\
& = \theta^2 Q \Pi(i_E DV, i_E DV) + \langle d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle Q E \langle d(V, E) \rangle, E \rangle \\
& \quad + \langle V, Q E \rangle \langle d(V, E) \wedge d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle, E \rangle + \frac{1}{2} \langle d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |V|^2 Q E, E \rangle, E \rangle \\
& \quad - \frac{1}{2} |V|^2 \langle d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle Q E, E \rangle, E \rangle \\
& \geq \theta^2 Q \Pi(i_E DV, i_E DV) + \langle d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle Q E \langle d(V, E), E \rangle, E \rangle \\
& \quad + \frac{1}{2} \langle d(\theta^2 \langle \nabla \tilde{n} Q E, Q E \rangle |V|^2 Q E, E \rangle, E \rangle - C(|V||\theta DV| + |V|^2).}
\end{align*}
We integrate the above inequality over $\Omega$ to have
\[
\int_{\Omega} \theta^2 \langle (\text{div}_g QP \nu \nu Q)Q - (\text{div}_g V)QPQ, \tau \rangle d\theta + \int_{\Omega} \left( \frac{d}{d\theta} \langle \nabla \theta QP \nu \nu Q, \theta Q \rangle \right) d\theta = \int_{\Omega} \left( \frac{d}{d\theta} \langle \nabla \theta QP \nu \nu Q, \theta Q \rangle \right) d\theta
\]
where the following formula has been used
\[
(dY, T)_{L^2(\Omega, d\theta)} = (Y, \delta T)_{L^2(\Omega, d\theta)} + \int_{\Gamma} \langle \nu \wedge Y, T \rangle d\Gamma \quad \text{for} \quad Y \in TM, \quad T \in \Lambda^2 M,
\]
see [12, Theorem 1.29]. Thus (3.40) follows. \qed

**Proof of Theorem 3.1.** We shall prove the case of $m = 1$. The cases of $m \geq 2$ can be treated by repeating a similar argument.

Let $F = e^{-\gamma} \left( f_1 Q \nu X + f_2 \nabla \theta Q \nu X \right)$ be given with $f_1$, $f_2 \in W^{1,2}(\Omega, \nu)$ and let $V \in L^2(\Omega, T)$ be the solution to problem (3.1). By Proposition 3.1,
\[
D_{\tau} V \in L^2(\Omega, \nu X, T), \quad \tau = \frac{\alpha_{\nu}}{\left| \alpha_{\nu} \right|}
\]
since $\Phi = -e^{\nu} \frac{D_{\nu} \Omega \nu X}{\left| \alpha_{\nu} \right|}$. Noting that $\text{div}_g QP \nu \nu Q = 0$ and $Q \nu$, $\tau$ forms an orthonormal frame on $\nu \nu X$ with positive orientation, from (3.7) and (2.5), we have
\[
\begin{cases}
\langle D_{\nu} V, Q \nu \rangle = \eta \langle V, \nabla \theta Q \nu X \rangle + f_2 - \langle D_{\nu} V, \tau \rangle \quad \text{for} \quad x \in \Omega, \\
\langle D_{\nu} V, \tau \rangle \Pi(\tau, \tau) = (\langle D_{\nu} V, Q \nu \rangle + \langle D_{\nu} V, \tau \rangle) \Pi(\tau, \tau) + \langle D_{\nu} V, Q \nu \rangle \Pi(\tau, \tau) (3.46) \\
+ f_1 + \eta \langle V, Q \nu X \rangle \quad \text{for} \quad x \in \Omega,
\end{cases}
\]
which yield, by (2.24),
\[
\|D_{\nu} V\|_{L^2(\Omega, \nu X, T)}^2 \leq C\left( \|D_{\nu} V\|_{L^2(\Omega, \nu X, T)}^2 + \|V\|_{L^2(\Omega, \nu X, T)}^2 + \|F\|_{L^2(\Omega, \nu X, T)}^2 \right).
\]
Thus $V \in W^{1,2}(\Omega, T)$. In the case of $m = 1$ (3.2) follows from (3.30) and (2.47). \qed

**Proof of Theorem 3.3.** We shall prove the case of $m = 1$. The cases of $m \geq 2$ can be treated by repeating a similar argument. By Theorem 3.1, we may assume that $F \in C^1(\Omega, T)$. We take two cut-off functions $\theta_1, \theta_2 \in C^1(-\varepsilon, b_1)$ as follows:
\[
\theta_1(s) = 1 \quad \text{for} \quad s \in (-\varepsilon/3, b_1), \quad \theta_1(s) = 0 \quad \text{for} \quad s \in (-\varepsilon, \varepsilon/4);
\]
\[ \theta_2(s) = 1 \quad \text{for} \quad s \in (-\varepsilon, \varepsilon/3), \quad \theta_2(s) = 0 \quad \text{for} \quad s \in (\varepsilon/2, b_1). \]

Let \( \sigma_0 > 0 \) be given such that
\[ Q^* \Pi \geq 3\sigma_0 \text{id} \quad \text{for} \quad x \in S^+ \setminus S_{\varepsilon/3}. \]

We take \( \sigma = \sigma_0, \theta = \theta_1, E = \alpha_{\ell}/|\alpha_{\ell}|, \) and \( E = Q\alpha_{\ell}/|\alpha_{\ell}| \) in (3.40), respectively. Then we obtain
\[
\int_{\Omega_{-\varepsilon}} \theta^2[Q^* \Pi(i_E DV, i_E DV) + Q^* \Pi(i_{QE} DV, i_{QE} DV) - 2\sigma_0 |DV|^2] dg \\
\leq C(\|F\|^2_{L^2(\Omega_{-\varepsilon}, T)} + \|V\|^2_{L^2(\Omega_{-\varepsilon}, T)} + \|q\|^2_{W^{1,2}(\Gamma_{b_1})}).
\]

Thus
\[
\|V\|^2_{W^{1,2}(\Sigma_{1/3, T})} \leq C(\|F\|^2_{L^2(\Omega_{-\varepsilon}, T)} + \|V\|^2_{L^2(\Omega_{-\varepsilon}, T)} + \|q\|^2_{W^{1,2}(\Gamma_{b_1})}). \tag{3.47}
\]

Set
\[ V_2 = \theta_2 V. \]

Then \( V_2 \) satisfies problem
\[
\begin{cases}
L_0 V_2 = \theta_2 F + e^{-\gamma \varepsilon}(\langle D\theta_2, Q\nabla n\bar{V}\rangle QX + \langle D\theta_2, \nabla nX\rangle \nabla nX) \quad \text{for} \quad x \in \Sigma_{1/2}, \\
(V_2, QX)|_{r_{-\varepsilon}} = p, \quad \langle V_2, \nu \rangle |_{r_{1/2}} = 0.
\end{cases} \tag{3.48}
\]

Noting that \( \text{supp} \, D\theta_2 \subset (\varepsilon/3, \varepsilon/2) \), applying Theorem 3.1 to problem (3.48) yields, by (3.47),
\[
\|\theta_2 V\|^2_{W^{1,2}(\Sigma_{1/2, T})} \leq C(\|F\|^2_{W^{1,2}(\Sigma_{1/2, T})} + \|V\|^2_{W^{1,2}(\Sigma_{1/2, T})} + \|p\|^2_{W^{1,2}(\Gamma_{-\varepsilon})}) \\
\leq C(\|F\|^2_{L^2(\Omega_{-\varepsilon}, T)} + \|F\|^2_{W^{1,2}(\Sigma_{1/2, T})} + \|p\|^2_{W^{1,2}(\Gamma_{-\varepsilon})}). \tag{3.49}
\]

Thus (3.4) follows from (3.47) and (3.49). \qed

4 \( W^{m,2} \) Solutions of a coupling tensor system of mixed type

For given \( (F, f) \in L^2(\Omega_{-\varepsilon}, T) \) arbitrarily, we directly seek some appropriate conditions on the boundary data \((p, q) \in L^2(\Gamma_{-\varepsilon}) \times L^2(\Gamma_{b_1})\) such that problem
\[
\begin{cases}
Dv = \nabla n\bar{V} + F \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div}_g V = \rho v + f \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
(V, QX)|_{r_{-\varepsilon}} = p, \quad \langle V, \nu \rangle |_{r_{b_1}} = q, \quad \int_{\Omega_{-\varepsilon}} vdg = 0,
\end{cases}
\]

where \( \rho = -\text{tr}_g \Pi \), admits a solution \((V, v)\) on \( \Omega_{-\varepsilon} \).

For convenience, we denote
\[
CV = \eta e^{-\gamma \kappa}(\langle V, QX\rangle QX + \langle V, \nabla nX\rangle \nabla nX) \quad \text{for} \quad V \in L^2(\Omega_{-\varepsilon}, T), \tag{5.1}
\]

where the function \( \eta \) and the vector field \( X \) are given in (2.33) and (2.34), respectively.

We consider a uniqueness problem.
Proposition 4.1. Let $S$ be of $C^5$. Problem

\[
\begin{cases}
\mathcal{L}_0^* W + CW = 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
W|_{\Gamma_{b_1}} = 0
\end{cases}
\]  \tag{5.2}

admits a unique zero solution, where $\mathcal{L}_0^*$ is given in (2.15).

Proof. Let

\[ w_1 = e^{-\gamma \kappa} \langle W, QX \rangle, \quad w_2 = e^{-\gamma \kappa} \langle W, \nabla \bar{n}X \rangle. \]

From (2.15), we have

\[ \mathcal{L}_0^* W = \nabla \bar{n} Q Dw_1 - Dw_2 - CW, \]

that is,

\[ \nabla \bar{n} Q Dw_1 - Dw_2 = 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}. \]  \tag{5.3}

Thus we obtain

\[ \text{div}_g \nabla \bar{n} Q Dw_1 = 0, \quad \text{div}_g (\nabla \bar{n})^{-1} Dw_2 = 0 \quad \text{for} \quad x \in S_. \]

In addition $W|_{\Gamma_{b_1}} = 0$ implies

\[ w_1|_{\Gamma_{b_1}} = w_2|_{\Gamma_{b_1}} = 0. \]

Then

\[ Dw_i = \langle Dw_i, \nu \rangle \nu \quad \text{for} \quad x \in \Gamma_{b_1}. \]

It follows from (5.3) that

\[ \langle Dw_1, \nu \rangle \nabla \bar{n} Q \nu = \langle Dw_2, \nu \rangle \nu \quad \text{for} \quad x \in \Gamma_{b_1}, \]

this is,

\[ \langle Dw_1, \nu \rangle = \langle Dw_2, \nu \rangle = 0 \quad \text{for} \quad x \in \Gamma_{b_1}, \]

since $\Pi(Q\nu, Q\nu) > 0$ for $x \in \Gamma_{b_1}$. Noting that $- \text{div}_g \nabla \bar{n} Q Dw_1$ and $\text{div}_g (\nabla \bar{n})^{-1} Dw_2$ are elliptic in $S_+$, the uniqueness theorem in [9] implies that

\[ w_1 = w_2 = 0 \quad \text{for} \quad x \in S_. \]  \tag{5.4}

Next, we need to prove that when given $\varepsilon > 0$ is small,

\[ w_1 = w_2 = 0 \quad \text{for} \quad x \in \Sigma^0_{-\varepsilon}, \]  \tag{5.5}

where

\[ \Sigma^0_{-\varepsilon} = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (-\varepsilon, 0) \}. \]

Let $X_1$ and $X_2$ be given in (2.21). Set

\[ H = \frac{1}{\kappa^2} X_2 \quad \text{for} \quad x \in \Sigma^0_{-\varepsilon}. \]

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From (5.3), we have
\[
\langle QD_w, D(Hw) \rangle = \langle Q\nabla \bar{u}QDw, D(Hw) \rangle = \langle D_Q\nabla \bar{u}QDw, H, Dw \rangle + \langle DHw, Q\nabla \bar{u}QDw \rangle
\]
= \langle D_Q\nabla \bar{u}QDw, H, Dw \rangle + \text{div}_g (\langle Dw, Q\nabla \bar{u}QDw \rangle H) - \langle Dw, DH(Q\nabla \bar{u}QDw) \rangle - \langle Dw, Q\nabla \bar{u}QDw \rangle \text{div}_g H
= 2\langle D_Q\nabla \bar{u}QDw, H, Dw \rangle + \langle QDw, DH(\nabla \bar{u})QDw \rangle)
- \langle \langle Q\nabla \bar{u}QDw, DH Dw \rangle + \langle D_Q\nabla \bar{u}QDw, H, Dw \rangle \rangle
+ \langle QDw, \nabla \bar{u}QDw \rangle \text{div}_g H - \text{div}_g (\langle QDw, \nabla \bar{u}QDw \rangle H),
\]
that is,
\[
2\langle QDw, D(Hw) \rangle = \Upsilon(w) - \text{div}_g (\langle QDw, \nabla \bar{u}QDw \rangle H) \quad \text{for} \quad x \in \Sigma^0_{-\epsilon}, \quad (5.6)
\]
where
\[
\Upsilon(w) = 2\langle D_Q\nabla \bar{u}QDw, H, Dw \rangle + \langle QDw, DH(\nabla \bar{u})QDw \rangle + \langle QDw, \nabla \bar{u}QDw \rangle \text{div}_g H.
\]
Moreover, using \( H = \kappa^{-2} X_2 \), we compute
\[
\kappa^3 \langle D_Q\nabla \bar{u}QDw, H, Dw \rangle = -2\langle Q\nabla \bar{u}QDw, D\kappa X_2(w) \rangle + \kappa \langle D_Q\nabla \bar{u}QDw, X_2, Dw \rangle
\]
= \[2\kappa X_2(w)\kappa X_2(w) + \kappa X_2(w)\kappa X_2(w)\]
- \[2\kappa X_2(w)\kappa X_2(w) + \kappa X_2(w)\kappa X_2(w)\]
= \[2\kappa X_2(w)\kappa X_2(w) + \kappa X_2(w)\kappa X_2(w)\]
where the formula \( \langle DX_2, X_2, X_2 \rangle = 0 \) is used,
\[
\kappa^3 \langle QDw, DH(\nabla \bar{u})QDw \rangle = \kappa \langle QDw, D_X(\nabla \bar{u})QDw \rangle = \kappa X_2(\kappa X_2(w)) + \mathcal{O}(\kappa)[X_2(w)]^2 + \mathcal{O}(\kappa)[X_2(w)]^2 + \mathcal{O}(\kappa)[X_2(w)]^2,
\]
where the formula \( DX_2(\nabla \bar{u})(X_2, X_2) = X_2(\kappa X_2) \) is used, and
\[
\kappa^3 \langle QDw, \nabla \bar{u}QDw \rangle \quad \text{div}_g H = \kappa \{2\lambda X_2(w)\kappa X_2(w) + \lambda X_2(w)\kappa X_2(w)\} \text{div}_g X_2
\]
- \[2\lambda X_2(w)\kappa X_2(w) + \lambda X_2(w)\kappa X_2(w)\]
= \[2\lambda X_2(w)\kappa X_2(w) + \lambda X_2(w)\kappa X_2(w)\]
respectively. Since \( \kappa X_2(\kappa X_2) = \kappa X_2(\kappa X_2) + \mathcal{O}(\kappa) \), we obtain
\[
\kappa^3 \Upsilon(w) = \kappa \{2\lambda X_2(w)\kappa X_2(w) + \lambda X_2(w)\kappa X_2(w)\} \quad \text{div}_g X_2
\]
- \[2\lambda X_2(w)\kappa X_2(w) + \lambda X_2(w)\kappa X_2(w)\]
\[\mathcal{O}(\kappa)[X_2(w)]^2 + \mathcal{O}(\kappa)[X_2(w)]^2 + \mathcal{O}(\kappa)[X_2(w)]^2,
\]
which yields
\[
- \Upsilon(w) \geq \frac{c}{\kappa^2}|Dw|^2 \quad \text{for} \quad x \in \Sigma^0_{-\epsilon}, \quad (5.7)
\]
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when $c > 0$ and $\epsilon > 0$ are small enough, since $X_2(\kappa) > 0$ by (2.25).

Since $S$ is of $C^5$, by Theorem 3.4 $W \in W^{5,2}(\Omega_\epsilon)$ and thus

$$w_1, \ w_2 \in C^3(\Omega_{-\epsilon}, T).$$

From (5.4) we have

$$\lim_{s \to 0^-} \frac{Dw_1}{\kappa^2}(\alpha(t, s)) = \lim_{s \to 0^-} \frac{Dw_2}{\kappa^2} = 0 \text{ for } t \in [0, a).$$

Thus

$$\lim_{s \to 0^-} [2H(w_1)Q Dw_2 + \langle Q Dw_1, \nabla \bar{n} Q Dw_1 \rangle H](\alpha(t, s)) = 0 \text{ for } t \in [0, a). \quad (5.8)$$

Moreover, since $\text{div}_g Q Dw_2 = 0$, we have

$$\langle Q Dw_2, D(Hw_1) \rangle = \text{div}_g(Hw_1 Q Dw_2). \quad (5.9)$$

For given $x \in \Gamma_{-\epsilon}$ and by (5.3), we obtain

$$\kappa^2[2H(w_1)(Q Dw_2, \nu) + \langle Q Dw_1, \nabla \bar{n} Q Dw_1 \rangle \langle H, \nu \rangle]$$

$$= \langle X_2, \nu \rangle \{ \lambda_2[X_1(w_1)]^2 - \lambda_1[X_2(w_1)]^2 \} - 2\lambda_2(X_1, \nu)X_1(w_1)X_2(w_1),$$

that implies, by (2.32),

$$\kappa^2[2H(w_1)(Q Dw_2, \nu) + \langle Q Dw_1, \nabla \bar{n} Q Dw_1 \rangle \langle H, \nu \rangle] \geq c_1|Dw_1|^2 \text{ for } x \in \Gamma_{-\epsilon}, \quad (5.10)$$

when given $\epsilon > 0$ is small enough.

For given $-\epsilon < s < 0$, we integrate $\Upsilon(w_1)$ over $\Sigma_{s-\epsilon}$ to have, by (5.7), (5.6) and (5.9),

$$c \int_{\Sigma_{s-\epsilon}} \frac{|Dw_1|^2}{\kappa^2} dg \leq - \int_{\Gamma_{-\epsilon} \cup \Gamma_s} [2H(w_1)(Q Dw_2, \nu) + \langle Q Dw_1, \nabla \bar{n} Q Dw_1 \rangle \langle H, \nu \rangle]d\Gamma,$$

from which we obtain, (5.8) and (5.10),

$$c \int_{\Sigma_{-\epsilon}} \frac{|Dw_1|^2}{\kappa^2} dg \leq - \int_{\Gamma_{-\epsilon}} [2H(w_1)(Q Dw_2, \nu) + \langle Q Dw_1, \nabla \bar{n} Q Dw_1 \rangle \langle H, \nu \rangle]d\Gamma \leq 0,$$

that is, (5.5) is true. \qed

Let

$$\mathcal{W}_0 = \{ W \in L^2(\Omega_{-\epsilon}, T) \mid L_0^0 W + CW = 0, \langle W, QX \rangle_{\Gamma_{b_1}} = \langle W, \nu \rangle_{\Gamma_{-\epsilon}} = 0 \}. \quad (5.11)$$

Set

$$\Gamma(\mathcal{W}_0) = \{ L_1 \langle W, \nu \rangle_{\Gamma_{b_1}} \mid W \in \mathcal{W} \},$$

where $L_1$ is given in (2.17). Consider the direct sum composition

$$L^2(\Gamma_{b_1}) = \Gamma(\mathcal{W}_0) \oplus \Gamma^\perp(\mathcal{W}_0) \text{ in the norm of } L^2(\Gamma_{b_1}).$$

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Lemma 4.1. Problem

\[
\begin{cases}
L_0 V + CV = 0, \\
\langle V, \nu \rangle|_{\Gamma_1} = q, \\
\langle V, QX \rangle|_{\Gamma_{-\varepsilon}} = 0
\end{cases}
\]  

(5.12)

admits a solution \(V \in L^2(\Omega_{-\varepsilon}, T)\) if and only if

\[q \in \Gamma^\perp(W_0)\].

Proof. Let \(V\) satisfy (5.12) and \(W \in W_0\). From (2.16), we have

\[-(W, CV)_{L^2(\Omega, T)} = -(V, CW)_{L^2(\Omega, T)} + \int_{\Gamma_1 \cup \Gamma_{-\varepsilon}} (L_1(V, \nu)(W, \nu) - L_2(V, QX)(W, QX)) d\Gamma,\]

which yield \(q \in \Gamma^\perp(W_0)\).

Conversely, for given \(q \in \Gamma^\perp(W_0)\), by Theorem 2.1 there exists a unique solution \(V_0 \in L^2(\Omega_{-\varepsilon}, T)\) to problem

\[
\begin{cases}
L_0 V_0 = 0, \\
\langle V_0, \nu \rangle|_{\Gamma_1} = q, \\
\langle V_0, QX \rangle|_{\Gamma_{-\varepsilon}} = 0
\end{cases}
\]

For given \(Z \in L^2(\Omega_{-\varepsilon}, T)\), by Theorem 2.1 problem

\[
\begin{cases}
L_0 V = -CZ, \\
\langle V, \nu \rangle|_{\Gamma_1} = \langle V_0, QX \rangle|_{\Gamma_{-\varepsilon}} = 0
\end{cases}
\]

admits a unique solution \(V \in L^2(\Omega_{-\varepsilon}, T)\). By Theorem 3.3

\[V \in W^{1,2}(\Omega_{-\varepsilon}, T)\].

We define a linear operator \(B : L^2(\Omega_{-\varepsilon}, T) \to L^2(\Omega_{-\varepsilon}, T)\) by

\[BZ = V.\]

Then \(B\) is compact on \(L^2(\Omega_{-\varepsilon}, T)\). By Fredholm’s theorem, problem

\[(\text{id} - B)V = V_0\]  

(5.13)

admits a solution in \(L^2(\Omega_{-\varepsilon}, T)\) if and only if

\[\langle V_0, U \rangle_{L^2(\Omega_{-\varepsilon}, T)} = 0 \quad \text{for all} \quad U \in \mathcal{Z}(\text{id} - B^*).\]  

(5.14)

In addition, it is easy to check that

\[\mathcal{Z}(\text{id} - B^*) = \{ U \in L^2(\Omega_{-\varepsilon}, T) | U + C(L_0^*)^{-1}U = 0 \}.\]
For $U \in \mathcal{Z}\left(\text{id} - \mathcal{B}^*\right)$, let $W = (\mathcal{L}_0^*)^{-1}U$. Then

$$\mathcal{L}_0^*W + CW = 0,$$

that is, $W \in \Gamma(W_0)$. Thus (5.14) is equivalent to

$$0 = (V_0, \mathcal{L}_0^*W)_{L^2(\Omega_{-\epsilon}, T)} = (\mathcal{L}_0V_0, W)_{L^2(\Omega_{-\epsilon}, T)} - \int_{\Gamma_{b_1}} \mathcal{L}_1q(W, \nu) d\Gamma$$

that is, $q \in \Gamma^\perp(W_0)$.

Then there exists a solution $V \in L^2(\Omega_{-\epsilon}, T)$ to problem (5.13) that is also a solution to problem (5.12).

Set

$$\mathcal{V}_0 = \{ V \in L^2(\Omega_{-\epsilon}, T) \mid \langle V, \nu \rangle|_{\Gamma_{b_1}} = \langle V, QX \rangle|_{\Gamma_{-\epsilon}} = 0 \}. $$

By a similar argument and the duality, we have the following.

**Lemma 4.2.** Problem

$$\begin{cases}
\mathcal{L}_0^*W + CW = 0, \\
\langle W, \nu \rangle|_{\Gamma_{-\epsilon}} = 0, \quad \langle W, QX \rangle|_{\Gamma_{b_2}} = p
\end{cases}$$

admits a solution $W \in L^2(\Omega_{-\epsilon}, T)$ if and only if

$$\int_{\Gamma_{-\epsilon}} \mathcal{L}_1p(V, \nu) d\Gamma = 0 \quad \text{for} \quad V \in \mathcal{V}_0.$$  

**Proposition 4.2.** Let $S$ be of $C^{m+5}$. There exists a $H_0 \in W^{m+5,2}(\Omega_{-\epsilon}, T)$ satisfying

$$\begin{cases}
\text{div}_gQ\nabla nH_0 = \text{div}_gH_0 = 0 \quad \text{for} \quad x \in \Omega_{-\epsilon}, \\
\langle H_0, QX \rangle|_{\Gamma_{-\epsilon}} = 0, \quad \int_{\Gamma_{-\epsilon}} \langle \nabla nH_0, Q\nu \rangle d\Gamma = 1.
\end{cases}$$

**Proof.** Step 1. We consider a similar problem on a bigger region as

$$\tilde{\Omega}_{-\epsilon} = \{ \alpha(t, s) \mid (t, s) \in [0, a) \times (-\epsilon, b_2) \},$$

where $b_2 > b_1$ such that

$$\kappa(\alpha(t, s)) > 0 \quad \text{for} \quad (t, s) \in [0, a) \times [b_1, b_2].$$

We claim that there is a solution $H_0 \in L^2(\Omega_{-\epsilon}, T)$ to problem

$$\begin{cases}
\mathcal{L}_0H_0 + CH_0 = 0 \quad \text{for} \quad x \in \tilde{\Omega}_{-\epsilon}, \\
\langle H_0, \nu \rangle|_{\Gamma_{b_2}} = q, \quad \langle H_0, QX \rangle|_{\Gamma_{-\epsilon}} = 0
\end{cases}$$

(5.17)
such that
\[ \int_{\Gamma - \varepsilon} \langle \nabla \tilde{n}H_0, Q\nu \rangle d\Gamma = 1. \]

By contradiction. We suppose that for all \( q \in \Gamma^\perp(W_0) \) solutions \( V \) to problem (5.17) satisfy
\[ \int_{\Gamma - \varepsilon} \langle \nabla \tilde{n}V, Q\nu \rangle d\Gamma = 0. \]
(5.18)

Set
\[ p_0 = e^{\gamma \kappa} \frac{\Pi(X, Q\nu)}{\Pi(X, X)} \text{ for } x \in \Gamma - \varepsilon. \]

From (2.36), \( p_0 \neq 0 \) for \( x \in \Gamma - \varepsilon \). By (2.4)
\[ \langle X, \nu \rangle V = \langle V, QX \rangle Q\nu + \langle V, \nu \rangle X \text{ for } x \in \Gamma - \varepsilon, \quad V \in V_0. \]

Then \( V \in V_0 \) and (5.18) imply that
\[ \int_{\Gamma - \varepsilon} \mathcal{L}_1 p_0 \langle V, \nu \rangle d\Gamma = \int_{\Gamma - \varepsilon} \frac{\Pi(X, Q\nu)}{\Pi(X, X)} \langle V, \nu \rangle d\Gamma = \int_{\Gamma - \varepsilon} \langle \nabla \tilde{n}V, Q\nu \rangle d\Gamma = 0 \]
for all \( V \in V_0 \), where \( \mathcal{L}_1 \) is given in (2.17). By Lemma 4.2 problem (5.15) admits a solution \( W_0 \in L^2(\Omega - \varepsilon, T) \) with \( p = p_0 \), where \( \Omega - \varepsilon \) and \( b_1 \) are replaced with \( \tilde{\Omega} - \varepsilon \) and \( b_2 \), respectively.

Now we claim that
\[ \int_{\Gamma - \varepsilon} \mathcal{L}_1 \langle W_0, \nu \rangle q d\Gamma = 0 \text{ for all } q \in \Gamma^\perp(W_0). \]
(5.19)

By Lemma 4.1, for each \( q \in \Gamma^\perp(W_0) \), problem (5.17) admits a solution \( V \). Then (5.19) follows from the formula (2.16).

The formula (5.19) implies that
\[ \mathcal{L}_1 \langle W_0, \nu \rangle \big|_{\Gamma_b_2} \in \Gamma(W_0), \]
that is, there exists \( W_1 \in \mathcal{W}_0 \) such that
\[ \langle W_1, \nu \rangle = \langle W_0, \nu \rangle \text{ for } x \in \Gamma_b_2. \]

Since \( \langle W_1, QX \rangle = \langle W_0, QX \rangle = 0 \) for \( x \in \Gamma_b_2 \), we have
\[ W_0 = W_1 \text{ for } x \in \Gamma_b_2. \]

By Proposition 4.1, \( W_1 = W_0 \) for all \( x \in \tilde{\Omega} - \varepsilon \). This is a contradiction since \( \langle W_1, \nu \rangle \big|_{\Gamma - \varepsilon} = \langle W_0, \nu \rangle \big|_{\Gamma - \varepsilon} = p_0 \neq 0 \). The proof is complete.
**Step 2.** Next, we show that $H_0 \in W^{1,2}(\Omega_{-\varepsilon}, T)$. Let $\varphi \in C^{m+5}(\hat{\Omega}_{-\varepsilon})$ be a cutoff function satisfying

$$ \varphi(x) = 1 \quad \text{for} \quad x \in \Omega_{-\varepsilon}; \quad \varphi(x) = 0 \quad \text{for} \quad x \in \Gamma_{b_2}. $$

Set

$$ H_1 = \varphi H_0. $$

Then $H_1 \in L^2(\hat{\Omega}_{-\varepsilon}, T)$ solves problem

$$ \begin{cases} L_0 H_1 + CH_1 = \langle Q \nabla \bar{n} H_0, D \varphi \rangle Q X + \langle H_0, D \varphi \rangle \nabla \bar{n} X & \text{for} \quad x \in \hat{\Omega}_{-\varepsilon}, \\ \langle H_1, \nu \rangle |_{\Gamma_{b_2}} = \langle H_1, Q X \rangle |_{\Gamma_{-\varepsilon}} = 0. \end{cases} $$

It follows from Theorem 3.3 that $H_1 \in W^{m+5,2}(\hat{\Omega}_{-\varepsilon}, T)$. The proof is complete. \(\square\)

Let

$$ \hat{W}^{1,2}(\Omega_{-\varepsilon}) = \{ w \in W^{1,2}(\Omega_{-\varepsilon}) | \int_{\Omega_{-\varepsilon}} w d\bar{g} = 0 \} $$

with the norm

$$ \| w \|_{\hat{W}^{1,2}(\Omega_{-\varepsilon})} = \| Dw \|_{L^2(\Omega_{-\varepsilon}, T)}. $$

For given $V \in L^2(\Omega_{-\varepsilon}, T)$, we solve problem

$$ \begin{cases} \Delta_g g = \eta \langle V, Q X \rangle & \text{for} \quad x \in \Omega_{-\varepsilon}, \\ g |_{\Gamma_{-\varepsilon}} = g |_{\Gamma_{b_1}} = 0, \end{cases} \quad (5.20) $$

and define an operator $g$. : $L^2(\Omega_{-\varepsilon}, T) \to W^{2,2}(\Omega_{-\varepsilon})$ by

$$ g V = g. $$

For given $(V, v) \in L^2(\Omega_{-\varepsilon}, T) \times \hat{W}^{1,2}(\Omega_{-\varepsilon})$, we solve problem

$$ \begin{cases} L_0 W + CV = e^{-\gamma} \rho v \nabla \bar{n} X, \\ \langle W, \nu \rangle |_{\Gamma_{b_1}} = \langle W, Q X \rangle |_{\Gamma_{-\varepsilon}} = 0, \end{cases} \quad (5.21) $$

by Theorem 2.1, to have $W \in L^2(\Omega_{-\varepsilon}, T)$, where $\rho = - \mathrm{tr}_g \Pi$. Moreover, by Theorem 3.3

$$ W \in W^{1,2}(\Omega_{-\varepsilon}, T). $$

Then we have

$$ \mathrm{div}_g Q \nabla \bar{n} W = \eta \langle W - V, Q X \rangle = \Delta_g g W - V. $$

Thus we obtain

$$ \mathrm{div}_g Q [\nabla \bar{n} W + Q D g W - V + \phi \nabla \bar{n} H_0] = 0 \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \quad (5.22) $$

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where $H_0 \in W^{1,2}(\Omega_{-\varepsilon}, T)$ is given in Proposition 4.2 and
\[
\phi = -\int_{\Gamma_{-\varepsilon}} \langle \nabla \bar{n} W + Q D g W_{-V}, Q \nu \rangle d\Gamma.
\]
Since (5.22) means that $d[\nabla \bar{n} W + Q D g W_{-V} + \phi \nabla \bar{n} H_0] = 0$ where $d$ is the exterior derivative, by the de Rham cohomology group theorem there exists a unique $w \in \hat{W}^{1,2}(\Omega_{-\varepsilon})$ such that
\[
Dw = \nabla \bar{n} W + Q D g W_{-V} + \phi \nabla \bar{n} H_0.
\] (5.23)
Clearly, $w \in W^{2,2}(\Omega_{-\varepsilon})$. We define a linear operator $A : L^2(\Omega_{-\varepsilon}, T) \times \hat{W}^{1,2}(\Omega_{-\varepsilon}) \to L^2(\Omega_{-\varepsilon}, T) \times \hat{W}^{1,2}(\Omega_{-\varepsilon})$ by
\[
A(V, \nu) = (W, w).
\] (5.24)
Then $A$ is compact on $L^2(\Omega_{-\varepsilon}, T) \times \hat{W}^{1,2}(\Omega_{-\varepsilon})$.

We consider the structure of
\[
Z(\text{id} - A^*) = \{ (U, u) \in L^2(\Omega_{-\varepsilon}, T) \times \hat{W}^{1,2}(\Omega_{-\varepsilon}) \mid (U, u) = A^*(U, u) \}.
\]
Clearly, dim $Z(\text{id} - A^*) < \infty$. Suppose that $W_0$ satisfies problem
\[
\begin{aligned}
\mathcal{L}_0^* W_0 &= \eta_0 Q X, \\
(W_0, \nu)|_{\Gamma_{-\varepsilon}} &= \frac{e^{\gamma \kappa} \Pi(X, Q \nu)}{\Pi(X, X)}, \\
(W_0, Q X)|_{\Gamma_{b_1}} &= 0,
\end{aligned}
\]
where
\[
\begin{aligned}
\Delta_g \eta_0 &= 0, \\
\eta_0|_{\Gamma_{b_1}} &= 0, \\
\eta_0|_{\Gamma_{-\varepsilon}} &= 1.
\end{aligned}
\] (5.25)

**Lemma 4.3.** Let $H_0$ be given in Proposition 4.2. Then $(U, u) \in Z(\text{id} - A^*)$ if and only if $(U, u)$ satisfies
\[
\begin{aligned}
U &= -C(\mathcal{L}_0^*)^{-1}(U + \nabla \bar{n} Du) + c(u)(CW_0 + \eta_0 Q X), \\
-\Delta_g u &= e^{-\gamma \kappa} \rho(\mathcal{L}_0^*)^{-1}(U + \nabla \bar{n} Du, \nabla \bar{n} X) - e^{-\gamma \kappa} c(u) \rho(W_0, \nabla \bar{n} X), \\
(Du, \nu)|_{\partial \Omega_{-\varepsilon}} &= 0, \\
\int_{\Omega_{-\varepsilon}} udg &= 0,
\end{aligned}
\] (5.26)
where $c(u) = (\nabla \bar{n} H_0, Du)_{L^2(\Omega_{-\varepsilon}, T)}$.

**Proof.** Let $W$ solve problem (5.21). It follows from the formula $\langle X, \nu \rangle W = \langle W, Q X \rangle Q \nu + \langle W, \nu \rangle X$ that
\[
\langle \nabla \bar{n} W, Q \nu \rangle = \frac{\Pi(X, Q \nu)}{\langle X, \nu \rangle} \langle W, \nu \rangle \quad \text{for} \quad x \in \Gamma_{-\varepsilon}.
\]
By (2.16) we obtain
\[
\int_{\Gamma_{-\varepsilon}} \langle \nabla \bar{n} W, Q \nu \rangle d\Gamma = \int_{\Gamma_{-\varepsilon}} \mathcal{L}_1 (W, \nu) \langle W_0, \nu \rangle d\Gamma
\]
\[
= (W_0, \mathcal{L}_0 W)_{L^2(\Omega_{-\varepsilon}, T)} - (W, \mathcal{L}_0^* W_0)_{L^2(\Omega_{-\varepsilon}, T)}.
\] (5.28)
In addition, from (5.26) we have
\[
\int_{\Gamma_{-\varepsilon}} \langle Dg_{\mathcal{W} - V}, \nu \rangle d\Gamma = \int_{\Omega_{-\varepsilon}} \text{div} g_0 Dg_{\mathcal{W} - V} dg = \int_{\Omega_{-\varepsilon}} g_0 \eta (W - V, QX) dg. \tag{5.29}
\]

It follows from (5.21)-(5.29) that
\[
\phi = \int_{\Omega_{-\varepsilon}} \left[ \eta g_0 (V, QX) + (V, CW_0) - e^{-\gamma \kappa} \rho v (W_0, \nabla nX) \right] dg
= (V, CW_0 + \eta g_0 QX)_{L^2(\Omega_{-\varepsilon}, T)} + (v, -e^{-\gamma \kappa} \rho (W_0, \nabla nX))_{L^2(\Omega_{-\varepsilon})}. \tag{5.30}
\]

For given \((V, v)\), noting that
\[
\int_{\Omega_{-\varepsilon}} \langle QDg_{\mathcal{W} - V}, Du \rangle dg = -\int_{\Omega_{-\varepsilon}} \text{div}_g (g_{\mathcal{W} - V} QD) dg = 0,
\]
using (5.23), (5.21), and (5.30), we have
\[
(V, U)_{L^2(\Omega_{-\varepsilon}, T)} + (v, -\Delta g u)_{L^2(\Omega_{-\varepsilon})} + \int_{\partial \Omega_{-\varepsilon}} v \langle Du, \nu \rangle d\Gamma = \left( (V, v), (U, u) \right)_{L^2(\Omega_{-\varepsilon}, T) \times \tilde{W}^{1,2}(\Omega_{-\varepsilon})}
= \left( (\mathcal{L}_0)^{-1} (-Cv + e^{-\gamma \kappa} \rho v \nabla nX), U + \nabla nDu \right)_{L^2(\Omega_{-\varepsilon}, T)} + \left( \phi \nabla nH_0, Du \right)_{L^2(\Omega_{-\varepsilon}, T)}
= \left( V, -C(\mathcal{L}_0)^{-1} (U + \nabla nDu) + c(u)(CW_0 + \eta g_0 QX) \right)_{L^2(\Omega_{-\varepsilon}, T)}
+ \left( v, -e^{-\gamma \kappa} \rho (\mathcal{L}_0)^{-1} (U + \nabla nDu), \nabla nX - e^{-\gamma \kappa} c(u) \rho (W_0, \nabla nX) \right)_{L^2(\Omega_{-\varepsilon})},
\]
which yield (5.27).

By Theorem 3.3 and a similar argument as in the proof of Proposition 4.1, we have the following. The details are omitted.

**Proposition 4.3.** Let \( S \) be of \( C^5 \). Let \( \mathcal{U} \) consist of all \((Z, z) \in L^2(\Omega_{-\varepsilon}, T) \times \tilde{W}^{1,2}(\Omega_{-\varepsilon})\) satisfying
\[
\begin{cases}
\mathcal{L}_0^* Z + CZ = \nabla nDz, \\
-\Delta g z = e^{-\gamma \kappa} \rho (Z, \nabla nX), \\
\langle Z, \nu \rangle_{\Gamma_{-\varepsilon}} = \langle Z, QX \rangle_{\Gamma_{b_1}} = 0, \\
\langle Dz, \nu \rangle_{\Gamma_{-\varepsilon} \cup \Gamma_{b_1}} = 0, \quad \int_{\Omega_{-\varepsilon}} z dg = 0.
\end{cases} \tag{5.31}
\]

Then \( \dim \mathcal{U} < \infty \) and
\[
\langle Z, \nu \rangle_{\Gamma_{b_1}} \neq 0.
\]
Set
\[ W^0(\Omega_{-\varepsilon}, T) = \{ F \in L^2(\Omega_{-\varepsilon}, T) \mid \text{div} g Q F \in L^2(\Omega_{-\varepsilon}) \} , \]
with the norm
\[ \| F \|_{W^0(\Omega_{-\varepsilon}, T)}^2 = \| F \|_{L^2(\Omega_{-\varepsilon}, T)}^2 + \| \text{div} g Q F \|_{L^2(\Omega_{-\varepsilon})}^2 , \]
and
\[ H^0 = W^0(\Omega_{-\varepsilon}, T) \times L^2(\Omega_{-\varepsilon}) \times L^2(\Gamma_{-\varepsilon}) \times L^2(\Gamma_{b_1}) . \]

By a similar argument as for (5.23), we have the following.

**Lemma 4.4.** Let \( H_0 \) be given in Proposition 4.2. For given \((F, f, p, q) \in H_0\), problem
\[
\begin{aligned}
Dv &= \nabla n(V + \beta(F, f, p, q) H_0) + QD\varrho V + F, \\
\text{div} g(V + \beta(F, f, p, q) H_0) &= \eta(V, \nabla n X) + f, \\
\int_{\Omega_{-\varepsilon}} v dg &= 0,
\end{aligned}
\]
\[ (5.32) \]
admits a unique solution \((V, v) \in L^2(\Omega_{-\varepsilon}, T) \times \dot{W}^{1,2}(\Omega_{-\varepsilon})\), where \(\varrho V\) is given by (5.20) and
\[ \beta(F, f, p, q) = -\int_{\Gamma_{-\varepsilon}} (\nabla n V + QD\varrho V + F, QV)d\Gamma. \]

Consider problem
\[
\begin{aligned}
Dv &= \nabla n V + F \quad \text{for } x \in \Omega_{-\varepsilon}, \\
\text{div} g V &= \rho v + f \quad \text{for } x \in \Omega_{-\varepsilon}, \\
(V, QX)|_{\Gamma_{-\varepsilon}} &= p, \quad (V, \nu)|_{\Gamma_{b_1}} = q, \quad \int_{\Omega_{-\varepsilon}} v dg = 0,
\end{aligned}
\]
\[ (5.33) \]
where \(\rho = -\text{tr} g \Pi\). Let \(\mathcal{N}_0\) consist of solutions to
\[
\begin{aligned}
Dv &= \nabla n V \quad \text{for } x \in \Omega_{-\varepsilon}, \\
\text{div} g V &= \rho V \quad \text{for } x \in \Omega_{-\varepsilon}, \\
(V, QX)|_{\Gamma_{-\varepsilon}} &= (V, \nu)|_{\Gamma_{b_1}} = \int_{\Omega_{-\varepsilon}} v dg = 0.
\end{aligned}
\]

By Theorems 3.3 and 3.1
\[ \mathcal{N}_0 \subset W^{m+4,2}(\Omega_{-\varepsilon}, T) \times W^{m+5,2}(\Omega_{-\varepsilon}) \]
if \(S\) is of \(C^{m+5}\). Set
\[
W^m(\Omega_{-\varepsilon}, T) = \{ F \in W^{m,2}(\Omega_{-\varepsilon}, T) \mid \text{div} g Q F \in W^{m,2}(\Omega_{-\varepsilon}) \} ,
\]
\[ (5.34) \]
\[ \| F \|_{W^m(\Omega_{-\varepsilon}, T)}^2 = \| F \|_{W^{m,2}(\Omega_{-\varepsilon}, T)}^2 + \| \text{div} g Q F \|_{W^{m,2}(\Omega_{-\varepsilon})}^2 . \]
Theorem 4.1. Let $S$ be of $C^{m+5}$. Let $(F, f, p) \in W^0(\Omega_{-\varepsilon}, T) \times L^2(\Omega_{-\varepsilon}) \times L^2(\Gamma_{-\varepsilon})$ be given. Then there exists

$$q \in W^{m+9/2,2}(\Gamma_{b_1})$$

such that problem (5.33) admits a unique solution $\mathcal{N}_{0}^\varepsilon \cap [L^2(\Omega_{-\varepsilon}, T) \times W^{1,2}(\Omega_{-\varepsilon})]$. Moreover, if $(F, f, p) \in W^m(\Omega_{-\varepsilon}, T) \times W^{m,2}(\Omega_{-\varepsilon}) \times W^{m,2}(\Gamma_{-\varepsilon})$, then

$$\left(V, v, V|_{\Gamma_{-\varepsilon}}\right) \in W^{m,2}(\Omega_{-\varepsilon}, T) \times W^{m+1,2}(\Omega_{-\varepsilon}) \times W^{m,2}(\Gamma_{-\varepsilon}, T).$$

(5.35)

Proof. Step 1. Let $(F, f, p, q) \in \mathcal{H}_0$ be given. We solve problem (5.32) to have the solution $(V_0, v_0) \in L^2(\Omega_{-\varepsilon}, T) \times \dot{W}^{1,2}(\Omega_{-\varepsilon})$. Then

$$\mathcal{L}_0 V_0 = G(F, f), \quad \langle V_0, QX\rangle|_{\Gamma_{-\varepsilon}} = p, \quad \langle V_0, \nu\rangle|_{\Gamma_{b_1}} = q,$$

where $G(F, f) = e^{-\gamma \kappa} (\langle -\text{div}_g QF \rangle QX + f \nabla \eta X)$.

We compute $\beta(F, f, p, q)$ as follows. Let $W_0$ be given in (5.25). Noting that $\langle X, \nu\rangle V_0 = \langle V_0, QX\rangle Q\nu + \langle V_0, \nu\rangle X$, we have, by (5.25) and (2.16),

$$\int_{\Gamma_{-\varepsilon}} \langle \nabla \eta V_0, Q\nu \rangle d\Gamma = \int_{\Gamma_{-\varepsilon}} \frac{1}{2} \langle [p\Pi(Q
u, Q\nu) + \Pi(Q
u, Q\nu) \langle V_0, \nu \rangle] d\Gamma$$

$$= \int_{\Gamma_{-\varepsilon}} (e^\gamma \kappa L_2 p + L_1 (V_0, \nu) \langle W_0, \nu \rangle) d\Gamma$$

$$= (p, L_2 (e^\gamma \kappa + \langle W_0, QX \rangle))_{L^2(\Gamma_{-\varepsilon})} - (q, L_1 (W_0, \nu))_{L^2(\Gamma_{b_1})}$$

$$+ \langle W_0, \mathcal{L}_0 V_0 \rangle_{L^2(\Omega_{-\varepsilon}, T)} - \langle V_0, \mathcal{L}_0^2 W_0 \rangle_{L^2(\Omega_{-\varepsilon}, T)}.$$

$$\int_{\Gamma_{-\varepsilon}} \langle D\varrho_0, \nu \rangle d\Gamma = \int_{\Omega_{-\varepsilon}} \text{div} \varrho_0 D\varrho_0 d\eta = \int_{\Omega_{-\varepsilon}} \varrho_0 \eta \langle V_0, QX \rangle d\eta.$$

Noting that

$$\int_{\Gamma_{-\varepsilon}} \langle F, Q\nu \rangle d\Gamma = - \int_{\Omega_{-\varepsilon}} \text{div} \varrho_0 QF d\eta,$$

we obtain

$$- \beta(F, f, p, q) = \left( p, L_2 (e^\gamma \kappa + \langle W_0, QX \rangle) \right)_{L^2(\Gamma_{-\varepsilon})} - \left( q, L_1 (W_0, \nu) \right)_{L^2(\Gamma_{b_1})}$$

$$+ \langle W_0, G(F, f) \rangle_{L^2(\Omega_{-\varepsilon}, T)} - \langle \text{div} g QF, \varrho_0 \rangle_{L^2(\Omega_{-\varepsilon})}$$

$$+ \langle F, QD\varrho_0 \rangle_{L^2(\Omega_{-\varepsilon}, T)}.$$

(5.36)

Step 2. Consider problem

$$(\text{id} - \mathcal{A})(V, v) = (V_0, v_0) \quad \text{in} \quad L^2(\Omega_{-\varepsilon}, T) \times \dot{W}^{1,2}(\Omega_{-\varepsilon}),$$

(5.37)

where $\mathcal{A}$ is defined by (5.24). By the Fredholm theorem problem (5.37) admits a solution in $L^2(\Omega_{-\varepsilon}, T) \times \dot{W}^{1,2}(\Omega_{-\varepsilon})$ if and only if

$$\left( (V_0, v_0), (U, u) \right)_{L^2(\Omega_{-\varepsilon}, T) \times W^{1,2}(\Omega_{-\varepsilon})} = 0 \quad \text{for} \quad (U, u) \in Z(\text{id} - \mathcal{A}^*).$$

(5.38)
Set
\[ Z = (L_0^*)^{-1}(U + \nabla \overline{n} Du), \quad z = u \quad \text{for} \quad (U, u) \in \mathcal{Z}(\text{id} - \mathcal{A}^*). \quad (5.39) \]

Then \((Z - c(z)W_0, z)\) satisfies (5.31) and \((Z - c(z)W_0, z)\) ∈ \(\mathcal{U}\). Using (5.32), (5.36) and (2.16), we have
\[
\left( (V_0, v_0), (U, u) \right)_{L^2(\Omega_{\varepsilon}, T)^{\times \mathcal{W}^{1, 2}(\Omega_{\varepsilon})}} = \left( V_0, U \right)_{L^2(\Omega_{\varepsilon}, T)} + (Dv_0, Du)_{L^2(\Omega_{\varepsilon}, T)}
\]
\[
= \left( V_0, U + \nabla \overline{n} Du \right)_{L^2(\Omega_{\varepsilon}, T)} + \beta(F, f, p, q)c(u) + (F, Du)_{L^2(\Omega_{\varepsilon}, T)}
\]
\[
= \left( V_0, L_0^* Z \right)_{L^2(\Omega_{\varepsilon}, T)} + \beta(F, f, p, q)c(z) + (F, Dz)_{L^2(\Omega_{\varepsilon}, T)}
\]
\[
= \left( G(F, f), Z - c(z)W_0 \right)_{L^2(\Omega_{\varepsilon}, T)} + \left( p, \mathcal{L}_2(e^\gamma c(z) + (Z - c(z)W_0, QX)) \right)_{L^2(\Gamma_{\varepsilon})}
\]
\[
- \left( q, \mathcal{L}_1(Z - c(z)W_0, \nu) \right)_{L^2(\Gamma_{b_1})} + \left( \text{div}_g QF, c(z)\nu \right)_{L^2(\Omega_{\varepsilon})} + \left( F, Dz - c(z)QD\nu \right)_{L^2(\Omega_{\varepsilon}, T)}
\]
\[
= I(U, u) - \left( q, \mathcal{L}_1(Z - c(z)W_0, \nu) \right)_{L^2(\Gamma_{b_1})},
\]
where \(I(U, u) = \left( G(F, f), Z - c(z)W_0 \right)_{L^2(\Omega_{\varepsilon}, T)} + \left( p, \mathcal{L}_2(e^\gamma c(z) + (Z - c(z)W_0, QX)) \right)_{L^2(\Gamma_{\varepsilon})}
\]
\[
+ \left( \text{div}_g QF, c(z)\nu \right)_{L^2(\Omega_{\varepsilon})} + \left( F, Dz - c(z)QD\nu \right)_{L^2(\Omega_{\varepsilon}, T)}.\]

Set
\[
\mathcal{X} = \{ (Z - c(z)W_0, \nu) |_{\Gamma_{b_1}} | Z = (L_0^*)^{-1}(U + \nabla \overline{n} Du), z = u, (U, u) \in \mathcal{Z}(\text{id} - \mathcal{A}^*) \}.
\]

By Proposition 4.3
\[
\dim \mathcal{Z}(\text{id} - \mathcal{A}^*) = \dim \mathcal{X}.
\]

Let \(\{ (Z_i - c(z_i)W_0, \nu) \}_{i=1}^k\) be a basis of \(\mathcal{X}\) such that
\[
\left( (Z_i - c(z_i)W_0, \nu), \mathcal{L}_1(Z_j - c(z_j)W_0, \nu) \right)_{\Gamma_{b_1}} = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq k.
\]

For given \((F, f, p) \in \mathcal{W}^0(\Omega_{\varepsilon}, T) \times L^2(\Omega_{\varepsilon}) \times L^2(\Gamma_{\varepsilon})\), let
\[
q = \sum_{i=1}^k c_i (Z_i - c(z_i)W_0, \nu) |_{\Gamma_{b_1}},
\]
where
\[
c_i = \left( (V_0, v_0), (U_i, u_i) \right)_{L^2(\Omega_{\varepsilon}, T)^{\times \mathcal{W}^{1, 2}(\Omega_{\varepsilon})}} - I(U_i, u_i),
\]
and \((U_i, z_i) \in \mathcal{Z}(\text{id} - \mathcal{A}^*)\) such that
\[
Z_i = (L_0^*)^{-1}(U_i + \nabla \overline{n} Du_i), \quad z_i = u_i \quad \text{for} \quad 1 \leq i \leq k.
\]
Then (5.38) hold true. Thus problem (5.37) admits a unique solution \((V,v) \in L^2(\Omega_-, T) \times \dot{W}^{1,2}(\Omega_-)\), which implies that \((V + \beta_0 H_0, v) \in L^2(\Omega_-, T) \times \dot{W}^{1,2}(\Omega_-)\) is a solution to problem (5.33) with 

\[ q = q_0 + \beta_0 (H_0, \nu) \in W^{m+9/2,2}(\Gamma_b), \]

where

\[
\beta_0 = -\int_{\Gamma_-} \langle \nabla \bar{n}V + F, Q\nu \rangle d\Gamma.
\]

Finally, (5.35) follows from Theorems 3.3 and 3.1.

Let \(W_0\) solve problem (5.25). We define

\[
\mathcal{N} = \{ (L_2(e^{\gamma x} c(z) + (Z - c(z)W_0, QX)), L_1(Z - c(z)W_0, \nu)) | (Z, z) is given in (5.39) \}.
\]

By Theorem Theorems 3.3 and 3.1

\[
\mathcal{N} \subset W^{m+9/2,2}(\Gamma_-) \times W^{m+9/2,2}(\Gamma_b)
\]

if \(S\) is of \(C^{m+5}\).

Consider problem

\[
\begin{aligned}
Dv & = \nabla \bar{n}V \quad \text{for} \quad x \in \Omega_-,
\text{div} gV & = \rho v \quad \text{for} \quad x \in \Omega_-,
\langle V, QX \rangle |_{\Gamma_-} & = p, \quad \langle V, \nu \rangle |_{\Gamma_b} = q.
\end{aligned}
\]

Let \(\mathcal{N}_1\) consist of solutions to

\[
\begin{aligned}
Dv & = \nabla \bar{n}V \quad \text{for} \quad x \in \Omega_-,
\text{div} gV & = \rho v \quad \text{for} \quad x \in \Omega_-,
\langle V, QX \rangle |_{\Gamma_-} & = \langle V, \nu \rangle |_{\Gamma_b} = 0.
\end{aligned}
\]

Then \(\mathcal{N}_1 \subset W^{m+4,2}(\Omega_-) \times W^{m+5,2}(\Omega_-)\) if \(S\) is of \(C^{m+5}\).

**Theorem 4.2.** Let \(S\) be of \(C^{m+5}\). Problem (5.42) admits a unique solution

\[ (V,v) \perp \mathcal{N}_1 \quad \text{in} \quad L^2(\Omega_-, T) \times W^{1,2}(\Omega_-) \]

if and only if

\[ (p,q) \perp \mathcal{N} \quad \text{in} \quad L^2(\Gamma_-) \times L^2(\Gamma_b). \]

Moreover, if \((p,q) \in W^{j,2}(\Gamma_-) \times W^{j,2}(\Gamma_b)\) with (5.44), then

\[ (V,v) \in W^{j,2}(\Omega_-, T) \times W^{j+1,2}(\Omega_-) \quad \text{for} \quad 0 \leq j \leq m + 4. \]
Proof. Let \((V,v)\) be a solution to problem (5.42). Then \((V,v - \theta)\) solves problem (5.33) with
\[
(F,f,p,q) = (0,\theta\rho,p,q),
\]
where \(\theta = (v,1)_{L^2(\Omega_{-\varepsilon})}/m(\Omega_{-\varepsilon}).\) Let \(P\) be the orthogonal projection operator from \(L^2(\Gamma_{-\varepsilon}) \times L^2(\Gamma_{b_1})\) on to \(\mathbb{N}.\) By (5.40) problem (5.33) admits a solution if and only
\[
\left(p, E_2(e^{-\gamma_\kappa}c(z) + \langle Z - c(z)W_0, QX \rangle) \right)_{L^2(\Gamma_{-\varepsilon})} + \left(q, E_1\langle Z - c(z)W_0, \nu \rangle \right)_{L^2(\Gamma_{b_1})} = -\theta \int_{\Omega_{-\varepsilon}} \rho e^{-\gamma_\kappa}Z - c(z)W_0, \nabla nX \rangle dg = \theta \int_{\Omega_{-\varepsilon}} \Delta gZdg = 0,
\]
where \(Z - c(z)W_0\) satisfies (5.31). Thus the desired results follow. \(\square\)

5 Proofs of the Main Results

Proof of Theorem 1.1. Let \(U \in T^2_{\text{sym}} S.\)

Consider problem
\[
\begin{align*}
Dv &= \nabla nV + F & \text{for} & \ x \in S, \\
\text{div}_g V + v \text{tr}_g \Pi &= f & \text{for} & \ x \in S,
\end{align*}
\]
where \((V,v) \in L^2(S,T) \times W^{1,2}(S)\) is the unknown and
\[
F = Q[D(\text{tr}_g U) - \text{div}_g U], \quad f = -\text{tr}_g U(Q\nabla nX, \cdot) & \text{ for } \ x \in S.
\]

For \(y \in W^{1,2}(S, IR^3)\), let
\[
2v = \nabla y(e_2,e_1) - \nabla y(e_1,e_2) & \text{ for } \ x \in S, \\
V = (\nabla n)^{-1}(Dv - F) & \text{ for } \ x \in S,
\]
where \(e_1, e_2\) is an orthonormal basis of \(T_x S\) with positive orientation.

By [13, Section 2], there is a \(y \in W^{1,2}(S, IR^3)\) to solve problem (1.1) if and only if \((V,v),\) being given in (6.4) and (6.3), respectively, solves problem (6.1). In that case, we have
\[
\begin{align*}
\nabla e_1 y &= U(e_1,e_1)e_1 + [v + U(e_1,e_2)]e_2 - \langle QV,e_1 \rangle \nabla n, \\
\nabla e_2 y &= [-v + U(e_1,e_2)]e_1 + U(e_2,e_2)e_2 - \langle QV,e_2 \rangle \nabla n, \quad \text{for } \ x \in S.
\end{align*}
\]
Moreover, from [13, Theorem 2.1], \((V,v)\) is a solution to problem (6.1) if and only if \(v\) solves problem
\[
\langle D^2 v, Q^* \Pi \rangle + \frac{1}{\kappa}X_0 v + \nu \text{tr}_g \Pi = \kappa f + \frac{1}{\kappa} \langle X_0, F \rangle + \langle DF, Q^* \Pi \rangle \quad \text{for } \ x \in S, \quad \kappa \neq 0
\]
where \( f \) and \( F \) are given in (6.2).

Let \( U \in W^{m+1,2}(S,T^2_{sym}) \) be given. We will find a solution

\[
(V,v) \in W^{m-1,2}(S,T) \times W^{m,2}(S)
\]

to problem (6.1) as follows.

Consider the region \( \Omega_{-\varepsilon} \) in Theorem 4.1. By (6.2)

\[
(F,f,0) \in W^{m-1}(\Omega_{-\varepsilon},T) \times W^{m-1,2}(\Omega_{-\varepsilon}) \times W^{m-1,2}(\Gamma_{-\varepsilon})
\]

By Theorem 4.1 there exists \( q \in W^{m+9/2,2}(\Gamma_{b_1}) \) such that problem

\[
\begin{align*}
Dv &= \nabla \vec{n}V + F \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\text{div}_g V &= \rho v + f \quad \text{for} \quad x \in \Omega_{-\varepsilon}, \\
\langle V,QX \rangle |_{\Gamma_{-\varepsilon}} &= p, \\
\langle V,\nu \rangle |_{\Gamma_{b_1}} &= q, \\
\int_{\Omega_{-\varepsilon}} vdg &= 0,
\end{align*}
\]

(6.7)

where \( \rho = -\text{tr}_g \Pi \), admits a solution \((U,u)\) satisfying

\[
(U,u,U|_{\Gamma_{-\varepsilon}}) \in W^{m-1,2}(\Omega_{-\varepsilon},T) \times W^{m,2}(\Omega_{-\varepsilon}) \times W^{m-1,2}(\Gamma_{-\varepsilon},T).
\]

The above solution can be extended to the region \( S_1 = S/\Omega_{-\varepsilon} \) below. Since \( S_1 \) is a noncharacteristic region, the boundary operators \( T_1 \) and \( T_2 \) as in [13] can defined. Let \( x \in \Gamma_{-\varepsilon} \) be given. \( \mu \in T_x S_1 \) with \(|\mu| = 1\) is said to be the noncharacteristic normal outside \( S_1 \) if there is a curve \( \zeta : (0,\iota) \to S \) such that

\[
\zeta(0) = x, \quad \zeta'(0) = -\mu, \quad \Pi(\mu,Y) = 0 \quad \text{for} \quad Y \in T_x S_1.
\]

Let \( \mu \) be the the noncharacteristic normal field along \( \Gamma_{-\varepsilon} \). We define boundary operators

\[
T_i : T_x M \to T_x M \quad \text{by}
\]

\[
T_i X = \frac{1}{2} \left[ Y + (-1)^i \chi(\mu,Y)\omega(Y)Q\nabla \vec{n}Y \right] \quad \text{for} \quad Y \in T_x M, \quad i = 1, 2,
\]

(6.8)

where

\[
\chi(\mu,Y) = \text{sign det} \left( \mu,Y,\vec{n} \right), \quad \omega(Y) = \frac{1}{\sqrt{-\kappa}} \text{sign} \Pi(Y,Y),
\]

(6.9)

and sign is the sign function. By a similar argument for [13, Theorem 4.2] problem

\[
\begin{align*}
\langle D^2 v, Q^* \Pi \rangle + \frac{\kappa}{\sqrt{\kappa}} X_0 v + v \kappa \text{tr}_g \Pi = \kappa f + \frac{1}{\kappa} \langle X_0, F \rangle + \langle DF, Q^* \Pi \rangle \quad \text{for} \quad x \in S_1, \\
w = u, \quad \frac{1}{\sqrt{2}} \langle Dw, (T_1 - T_2)\alpha_i \rangle = \frac{1}{\sqrt{2}} \langle Du, (T_1 - T_2)\alpha_i \rangle \quad \text{for} \quad x \in \Gamma_{-\varepsilon}
\end{align*}
\]

admits a unique solution \( w \in W^{m+1,2}(S_1) \), where \( u \) is the component of the solution \((U,u)\) to problem (6.7). Let

\[
W = (\nabla \vec{n})^{-1}(Dw - F) \quad \text{for} \quad x \in S_1.
\]
We define 

\[(V, v) = \begin{cases} (U, u) & \text{for } x \in \Omega - \varepsilon, \\ (W, w) & \text{for } x \in S_1. \end{cases} \]

Then \((V, v) \in W^{m-1,2}(S, T) \times W^{m,2}(S)\) is a solution to (6.1). Then by (6.5) \(y \in W^{m,2}(S, \mathbb{R}^3)\). Thus by \([12, \text{Lemma 4.3}]\)

\[\text{sym } DW = U - \langle y, \vec{n} \rangle \Pi \in W^{m,2}(S, T).\]

The proof is complete. \(\square\)

**Proof of Theorem 1.2** Let \(y \in W^{2,2}(S, \mathbb{R}^3)\) be an infinitesimal isometry. Let

\[2v = \nabla y(e_2, e_1) - \nabla y(e_1, e_2), \quad V = (\nabla \vec{n})^{-1} Dv \quad \text{for } x \in S,
\]

where \(e_1, e_2\) is an orthonormal basis of \(T_x S\) with positive orientation. Then \((V, v)\) solves problem

\[
\begin{cases}
Dv = \nabla \vec{n} V & \text{for } x \in S, \\
\text{div}_g V + v \text{tr}_g \Pi = 0 & \text{for } x \in S.
\end{cases}
\]

(6.10)

By (6.5) with \(U = 0, \ y \in W^{2,2}(S, \mathbb{R}^3)\) implies \((V, v) \in W^{1,2}(S, T) \times W^{2,2}(S)\).

Consider the region \(\Omega_{-\varepsilon}\) as in Theorem 4.2. By \([3, \text{Theorem 9.19}]\) \(v|_{\Gamma_{b_1}} \in C^{m+4,\alpha}(\Gamma_{b_1})\) due to that \(v\) solves an elliptic equation on \(S_+\). In addition, since \(v\) solves a hyperbolic equation on \(S_-\), \(v|_{\Gamma_{-\varepsilon}} \in W^{2,2}(\Gamma_{-\varepsilon})\) by \([13]\). Thus

\[V|_{\Gamma_{b_1}} \in C^{m+1}(\Gamma_{b_1}), \quad V|_{\Gamma_{-\varepsilon}} \in W^{1,2}(\Gamma_{-\varepsilon}).\]

Set

\[p = \langle V, QX \rangle|_{\Gamma_{-\varepsilon}}, \quad q = \langle V, v \rangle|_{\Gamma_{b_1}}.\]

We decompose as the direct sum

\[(V, v) = (V_1, v_1) + (V_0, v_0) \quad \text{in } L^2(\Omega_{-\varepsilon}, T) \times W^{1,2}(\Omega_{-\varepsilon}),\]

where \((V_0, v_0) \in \mathcal{N}_1\) and \(\mathcal{N}_1\) is given by (5.43). Then

\[(V_0, v_0) \in W^{m+4,2}(\Omega_{-\varepsilon}, T) \times W^{m+5,2}(\Omega_{-\varepsilon}),\]

and \((V_1, v_1) \in L^2(\Omega_{-\varepsilon}, T) \times W^{1,2}(\Omega_{-\varepsilon})\) solves problem (5.42). By Theorem 4.2

\[\langle p, q \rangle \perp \mathcal{N} \quad \text{in } L^2(\Gamma_{-\varepsilon}) \times L^2(\Gamma_{b_1}),\]

and

\[\|(V_1, v_1)\|_{W^{1,2}(\Omega_{-\varepsilon}, T) \times W^{2,2}(\Omega_{-\varepsilon})} \leq C\|\langle p, q \rangle\|^2_{W^{1,2}(\Gamma_{-\varepsilon}) \times W^{1,2}(\Gamma_{b_1})},\]

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where $\mathcal{N}$ is given in (5.41).

Next, for given $\iota > 0$ arbitrarily, we take $(p_\iota, q_\iota) \in W^{m+1,2}((\Gamma_1 - \varepsilon) \times W^{m+1,2}(\Gamma_{b_1})$ such that

$$
\|(p, q) - (p_\iota, q_\iota)\|_{W^{1,2}((\Gamma_1 - \varepsilon) \times W^{1,2}(\Gamma_{b_1}))} \leq \iota.
$$

Denote by $(\hat{p}_\iota, \hat{q}_\iota)$ by the orthogonal projection of $(p_\iota, q_\iota)$ from $L^2((\Gamma_1 - \varepsilon) \times L^2(\Gamma_{b_1})$ to $\mathcal{N}^\perp$ in $L^2((\Gamma_1 - \varepsilon) \times L^2(\Gamma_{b_1})$. Since $\dim \mathcal{N} < \infty$, we have

$$
\|(p, q) - (\hat{p}_\iota, \hat{q}_\iota)\|_{W^{1,2}((\Gamma_1 - \varepsilon) \times W^{1,2}(\Gamma_{b_1}))} \leq C\iota.
$$

By Theorem 4.2 problem (5.42) admits a unique solution $(V_\iota, v_\iota) \in \mathcal{N}^\perp_1$ in $L^2((\Omega_1 - \varepsilon) \times W^{1,2}(\Omega_{b_1})$, and

$$
\|(V_\iota, v_\iota)\|_{W^{m+1,2}(\Omega_1 - \varepsilon, T) \times W^{m+2,2}(\Omega_{b_1})} \leq C\|(\hat{p}_\iota, \hat{q}_\iota)\|_{W^{m+1,2}(\Gamma_1 - \varepsilon) \times W^{m+2,2}(\Gamma_{b_1})}.
$$

Thus

$$(V_\iota, v_\iota) \in C^{m-1}_B(\Omega_1 - \varepsilon),$$

and

$$
\|(V_\iota, v_\iota) - (V_\iota, v_\iota)\|_{W^{1,2}(\Gamma_1 - \varepsilon) \times W^{1,2}(\Gamma_{b_1})} \leq C\iota.
$$

Now we extend the domains of $(V_\iota, v_\iota)$, $(V_0, v_0)$, and $(V_\iota, v_\iota)$ from $\Omega_1 - \varepsilon$ to $S$, respectively, as in the proof of Theorem 1.1, with the same notations such that

$$(V, v) = (V_\iota, v_\iota) + (V_0, v_0) \quad \text{in} \quad W^{1,2}(S, T) \times W^{2,2}(S),$$

$$(V_\iota, v_\iota) \in C^{m-1}_B(S),$$

$(V_1, v_1)$, $(V_0, v_0)$, and $(V_\iota, v_\iota)$ satisfy problem (6.10) on the region $S$, respectively, and

$$
\|(V, v) - (V_\iota + V_0, v_\iota + v_0)\|_{W^{1,2}(S, T) \times W^{2,2}(S)} \leq C\iota.
$$

The proof is complete by the formula (6.5). \qed

**Proof of Theorem 1.3** As in [4] we conduct in $2 \leq i \leq m$. Let

$$
y_{\varepsilon} = \sum_{j=0}^{i-1} \varepsilon^j z_j
$$

be an $(i - 1)$th order isometry of class $C^{2+4(m-i+1)}_B(S, \mathbb{R}^3)$, where $z_0 = \text{id}$ and $z_2 = y$ for some $i \geq 2$. Then

$$
\sum_{j=0}^{k} \nabla^T z_j \nabla z_{k-j} = 0 \quad \text{for} \quad 1 \leq k \leq i - 1.
$$

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Next, we shall find out \( z_i \in C_B^{2+4(m-i)}(S, \mathbb{R}^3) \) such that
\[
\phi_\varepsilon = y_\varepsilon + \varepsilon^i z_i
\]
is an \( i \) th order isometry. By Corollary 1.1 there exists a solution \( z_i \in C_B^{2+4(m-i)}(S, \mathbb{R}^3) \) to problem
\[
sym \nabla z_i = -\frac{1}{2} \text{sym} \sum_{j=1}^{i-1} \nabla^T z_j \nabla z_{i-j}
\]
which satisfies
\[
\|z_i\|_{C_B^{2+4(m-i)}(S, \mathbb{R}^3)} \leq C \| \sum_{j=1}^{i-1} \text{sym} \nabla^T z_j \nabla z_{i-j} \|_{C_B^{2+4(m-i)+3}(S, \mathbb{R}^3)}
\]
\[
\leq C \| \sum_{j=1}^{i-1} z_j \|_{C_B^{2+4(m-i+1)}(S, \mathbb{R}^3)} \| z_{i-j} \|_{C_B^{2+4(m-i+1)}(S, \mathbb{R}^3)}
\]
The conduction completes. \( \square \)

Appendix: A theorem of Lax-Milgram’s type

The following theorem is an improved version of the Lax-Milgram theorem.

**Theorem A** Let \( (V, \langle \cdot , \cdot \rangle_V) \) be a Hilbert space. Suppose that \( W \subset V \) is a linear subspace of \( V \) such that \( (W, \langle \cdot , \cdot \rangle_W) \) is an inner product space, which may not be complete. Let \( \beta : V \times W \to \mathbb{R} \) be a bilinear functional satisfying
\[
|\beta(v, w)| \leq C \|v\|_V \|w\|_W \quad \text{for} \quad v \in V, \ w \in W, \quad (6.11)
\]
\[
\beta(w, w) \geq \sigma \|w\|_V^2 \quad \text{for} \quad w \in W. \quad (6.12)
\]

Then there is a closed linear subspace \( V' \subset V \) such that for given bounded linear functional \( F \) on \( V \), there exists a unique \( v \in V' \) satisfying
\[
F(w) = \beta(v, w) \quad \text{for} \quad w \in W,
\]
\[
\|v\| \leq \frac{1}{\sigma} \|F\|_{V'}. \quad (6.13)
\]

**Proof.** For given \( w \in W \), from (6.11) \( \beta(\cdot, w) \) is a bounded linear functional on \( V \). The Riesz representation theorem implies that there exists a bounded linear operator \( A : W \to V \) such that
\[
\beta(v, w) = \langle v, Aw \rangle_V \quad \text{for} \quad v \in V, \ w \in W.
\]
Then \( \|A\| \leq C \). Thus
\[
\langle w, Aw \rangle_V = \beta(w, w) \geq \sigma \|w\|_V^2 \quad \text{for} \quad w \in \mathcal{W},
\]
which yield
\[
\|Aw\|_V \geq \sigma \|w\|_V \quad \text{for} \quad w \in \mathcal{W}.
\]

Let \( \mathcal{V}' \) be the closure of \( R(A) \) in \( V \). Then for given \( y \in \mathcal{V}' \) there is a sequence \( \{w_k\}_{k=1}^\infty \subset \mathcal{W} \) such that
\[
\|y - Aw_k\|_V \to 0 \quad \text{as} \quad k \to \infty.
\]
Define
\[
\varphi(y) = \lim_{k \to \infty} \mathcal{F}(w_k).
\]
By (6.14) \( \varphi \) is a bounded linear functional on \( \mathcal{V}' \). By the Riesz representation theorem there exists a unique \( v \in \mathcal{V}' \) such that
\[
\varphi(y) = \langle v, y \rangle_V \quad \text{for} \quad y \in \mathcal{V}'.
\]
In particular, letting \( y = Aw \) for \( w \in \mathcal{W} \) yields
\[
\mathcal{F}(w) = \varphi(y) = \langle v, Aw \rangle_V = \beta(v, w) \quad \text{for} \quad w \in \mathcal{W}.
\]
Finally, (6.13) follows from (6.14).

Compliance with Ethical Standards
Conflict of Interest: The author declares that there is no conflict of interest.
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