Joint functional convergence of partial sums and maxima for linear processes*

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Abstract. For linear processes with independent identically distributed innovations that are regularly varying with tail index \( \alpha \in (0, 2) \), we study the functional convergence of the joint partial-sum and partial-maxima processes. We derive a functional limit theorem under certain assumptions on the coefficients of the linear processes, which enable the functional convergence in the space of \( \mathbb{R}^2 \)-valued càdlàg functions on \([0, 1]\) with the Skorokhod weak \( \mathcal{M}_2 \) topology. We also obtain a joint convergence in the \( \mathcal{M}_2 \) topology on the first coordinate and in the \( \mathcal{M}_1 \) topology on the second coordinate.

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1 Introduction

It is known that the joint partial-sum and partial-maxima processes constructed from i.i.d. regularly varying random variables with tail index \( \alpha \in (0, 2) \) satisfy the functional limit theorem with \( (V, W) \) as a limit, where \( V \) is a stable Lévy process, and \( W \) is an extremal process; see [8] and [13]. The convergence takes place in the space \( D([0, 1], \mathbb{R}^2) \) of \( \mathbb{R}^2 \)-valued càdlàg functions on \([0, 1]\) with the Skorokhod \( J_1 \) topology.

In this paper, we study the functional convergence of a special class of weakly dependent random variables, the linear processes or moving averages processes. Due to possible clustering of large values, the functional convergence fails to hold with respect to the \( J_1 \) topology, and hence we will have to use a somewhat weaker topology, namely the Skorokhod weak \( \mathcal{M}_2 \) topology. In the proofs of our results, we will use the methods and results of Basrak and Krizmanič [4], who obtained the functional convergence of partial-sum processes with respect to the Skorokhod (standard or strong) \( \mathcal{M}_2 \) topology.

We proceed by precisely stating the problem. Let \( (Z_i)_{i \in \mathbb{Z}} \) be an i.i.d. sequence of regularly varying random variables with index of regular variation \( \alpha \in (0, 2) \). In particular, this means that

\[
\mathbb{P}(|Z_i| > x) = x^{-\alpha}L(x), \quad x > 0,
\]

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where \( L \) is a slowly varying function at \( \infty \). Let \((a_n)\) be a sequence of positive real numbers such that

\[
nP\left( |Z_1| > a_n \right) \to 1 \quad \text{as } n \to \infty. \tag{1.1}\]

Then \( a_n \to \infty \). Regular variation of \( Z_i \) can be expressed in terms of vague convergence of measures on \( \mathbb{E} = \mathbb{R} \setminus \{0\} \):

\[
nP\left( a_n^{-1} Z_i \in \cdot \right) \overset{\text{v}}{\to} \mu \quad \text{as } n \to \infty \tag{1.2}\]

with the measure \( \mu \) on \( \mathbb{E} \) given by

\[
\mu(dx) = \left( p \mathbf{1}_{(0,\infty)}(x) + r \mathbf{1}_{(-\infty,0)}(x) \right) \alpha |x|^{-\alpha-1} \, dx, \tag{1.3}\]

where

\[
p = \lim_{x \to \infty} \frac{P(Z_i > x)}{P(|Z_i| > x)} \quad \text{and} \quad r = \lim_{x \to \infty} \frac{P(Z_i \leq -x)}{P(|Z_i| > x)}. \tag{1.4}\]

When \( \alpha \in (1, 2) \), we have that \( E(Z_1) < \infty \). We study the moving-average process of the form

\[
X_i = \sum_{j=-\infty}^{\infty} \varphi_j Z_{i-j}, \quad i \in \mathbb{Z},
\]

where the constants \( \varphi_j \) are such that the series is a.s. convergent. One sufficient condition, commonly used in the literature, for that is

\[
\sum_{j=-\infty}^{\infty} |\varphi_j|^\delta < \infty \quad \text{for some } 0 < \delta < \alpha, \quad \delta \leq 1 \tag{1.5}\]

(see [9, Thm. 2.1] or [14, Sect. 4.5]). As noted in [3], condition (1.5) excludes some important cases, for example, the case of strictly \( \alpha \)-stable random variables \((Z_i)\) with \( \sum_{j} |\varphi_j|^\alpha < \infty \) but \( \sum_{j} |\varphi_j|^\delta = \infty \) for every \( \delta < \alpha \). To resolve this issue, some new conditions weaker than (1.5) for \( \alpha \leq 1 \) were proposed in [3, Cors. 4.6 and 4.9]. In [1], it was observed that if, additionally,

\[
E(Z_1) = 0 \quad \text{if } \alpha \in (1, 2), \quad \text{or} \quad Z_1 \text{ is symmetric} \quad \text{if } \alpha = 1,
\]

then the series defining \( X_i \) is a.s. convergent if and only if

\[
\sum_{j=-\infty}^{\infty} |\varphi_j|^\alpha L(|\varphi_j|^{-1}) < \infty \tag{1.6}\]

(see also [3, Prop. 5.4]). Note that condition (1.5) implies \( \sum_{i=-\infty}^{\infty} |\varphi_i| < \infty \). The same holds if condition (1.6) is satisfied when \( \alpha \in (0, 1) \).

Our goal is to find sufficient conditions such that, with respect to some Skorokhod topology on \( D([0, 1], \mathbb{R}^2) \),

\[
\left( \sum_{i=1}^{n-1} \frac{X_i - b_n}{a_n}, \left\lfloor \frac{n-1}{a_n} \right\rfloor \right) \overset{d}{\to} (\beta V, W), \tag{1.7}\]
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in $D([0,1], \mathbb{R}^2)$, where $V$ is an $\alpha$-stable Lévy process, $W$ is an extremal process, $b_n$ are appropriate centering constants, $\beta = \sum_{j=-\infty}^{\infty} \varphi_j \neq 0$, and $D([0,1], \mathbb{R}^2)$ denotes the space of right-continuous $\mathbb{R}^2$-valued functions on $[0,1]$ with left limits.

Recall here some basic facts on Lévy processes and extremal processes. The distribution of a Lévy process $V$ is characterized by its characteristic triple (i.e., the characteristic triple of the infinitely divisible distribution of $V(1)$). The characteristic function of $V(1)$ and the characteristic triple $(a, \nu', b)$ are related in the following way:

$$E[e^{izV(1)}] = \exp\left(-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} (e^{ix} - 1 - i x 1_{[-1,1]}(x)) \nu'(dx)\right)$$

for $z \in \mathbb{R}$, where $a \geq 0$ and $b \in \mathbb{R}$ are constants, and $\nu'$ is a measure on $\mathbb{R}$ satisfying

$$\nu'\{0\} = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1) \nu'(dx) < \infty.$$ 

We refer to [16] for a textbook treatment of Lévy processes. The distribution of a nonnegative extremal process $W$ is characterized by its exponent measure $\nu''$ in the following way:

$$P(W(t) \leq x) = e^{-\nu''(x, \infty)}$$

for $t > 0$ and $x > 0$, where $\nu''$ is a measure on $(0, \infty)$ satisfying $\nu''(\delta, \infty) < \infty$ for any $\delta > 0$ (see [15, p. 161]).

If $X_i$ is a finite-order moving average with at least two nonzero coefficients, then the convergence in (1.7) cannot hold in the $J_1$ sense, since, as shown by Avram and Taqqu [2], the $J_1$ convergence fails to hold for the first components of the processes in (1.7), that is, for partial-sum processes. Astrauskas [1] and Davis and Resnick [10] showed that the normalized sums of $X_i$ under (1.5) converge in distribution to a stable random variable. Basrak and Krizmanić [4] replaced this convergence by weak convergence with respect to the Skorokhod $M_2$ topology, that is, for partial sums, they showed the convergence

$$\sum_{i=1}^{[n \cdot]} \frac{X_i - b_n}{a_n} \overset{d}{\to} \beta V$$

in the $M_2$ topology under the following assumption on the coefficients $\varphi_i$: $\varphi_j = 0$ for $j < 0$, $\varphi_0, \varphi_1, \ldots \in \mathbb{R}$, and for every $s = 0, 1, 2, \ldots$,

$$0 \leq \sum_{j=0}^{s} \varphi_j / \sum_{j=0}^{\infty} \varphi_j \leq 1.$$ 

The characteristic triple of the limiting process $V$ is of the form $(0, \mu, b)$ with $\mu$ as in (1.3) and

$$b = \begin{cases} 
0, & \alpha = 1, \\
(p-r) \frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2).
\end{cases}$$

As for the partial maxima, Resnick [14] showed that if $\varphi_+ p + \varphi_- r > 0$, then, as $n \to \infty$,

$$\sum_{i=1}^{[n \cdot]} \frac{X_i}{a_n} \overset{d}{\to} W$$

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in the \( J_1 \) topology, where
\[
\varphi_+ = \max\{\varphi_j \vee 0: j \in \mathbb{Z}\}, \quad \varphi_- = \max\{-\varphi_j \vee 0: j \in \mathbb{Z}\},
\]
and \( W \) is an extremal process with exponent measure
\[
\nu(dx) = (\varphi_+^a p + \varphi_-^a r) \alpha x^{-\alpha-1} 1_{(0,\infty)}(x) dx.
\]
(see [14, Prop. 4.28]).

In this paper, we show that, under assumptions (1.8) and \( \varphi_+ p + \varphi_- r > 0 \), relation (1.7) holds in the weak \( M_2 \) topology. To this end, in Section 2, we first recall the precise definition of the weak \( M_2 \) topology, then, in Section 3, we proceed by proving (1.7) for finite-order moving average processes, and, finally, we extend this to infinite-order moving-average processes. At the end in Remark 3 we discuss the joint convergence in (1.7) in the \( M_2 \) topology on the first coordinate and in the \( M_1 \) topology on the second coordinate.

## 2 Skorohod \( M_2 \) topologies

We start with a definition of the Skorohod weak \( M_2 \) topology in the general space \( D([0,1], \mathbb{R}^d) \) of \( \mathbb{R}^d \)-valued càdlàg functions on \([0,1]\).

The weak \( M_2 \) topology on \( D([0,1], \mathbb{R}^d) \) is defined using completed graphs. For \( x \in D([0,1], \mathbb{R}^d) \), the completed (thick) graph of \( x \) is the set
\[
G_x = \\{ (t,z) \in [0,1] \times \mathbb{R}^d: z \in \left[ [x(t-),x(t)] \right] \},
\]
where \( x(t-) \) is the left limit of \( x \) at \( t \), and \([a, b]\) is the product segment, that is, \([a, b] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \) for \( a = (a_1, a_2, \ldots, a_d) \) and \( b = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d \). We define an order on the graph \( G_x \) by saying that \((t_1, z_1) \leq (t_2, z_2)\) if either (i) \( t_1 < t_2 \) or (ii) \( t_1 = t_2 \) and \( |x_j(t_1-) - z_{1j}| \leq |x_j(t_2-) - z_{2j}| \) for all \( j = 1,2,\ldots, d \). The relation \( \leq \) induces only a partial order on the graph \( G_x \). A weak \( M_2 \) parametric representation of the graph \( G_x \) is a continuous function \((r,u)\) mapping \([0,1]\) into \( G_x \) such that \( r \) is nondecreasing, \( r(0) = 0 \), \( r(1) = 1 \), and \( u(1) = x(1) \) (\( r \) is the time component, and \( u \) is the spatial component). Let \( \Pi_w(x) \) denote the set of weak \( M_2 \) parametric representations of the graph \( G_x \). For \( x_1, x_2 \in D([0,1], \mathbb{R}^d) \), define
\[
d_w(x_1, x_2) = \inf \left\{ \|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi_w(x_i), \ i = 1,2 \right\},
\]
where \( \|x\|_{[0,1]} = \sup \{ \|x(t)\| : t \in [0,1] \} \). We say that \( x_n \to x \) in \( D([0,1], \mathbb{R}^d) \) for a sequence \( (x_n) \) in the weak Skorokhod \( M_2 \) (or shortly \( WM_2 \)) topology if \( d_w(x_n, x) \to 0 \) as \( n \to \infty \).

If we replace the graph \( G_x \) with the completed (thin) graph
\[
\Gamma_x = \{ (t,z) \in [0,1] \times \mathbb{R}^d: z = \lambda x(t-) + (1 - \lambda)x(t) \text{ for some } \lambda \in [0,1] \}
\]
and a weak \( M_2 \) parametric representation with a strong \( M_2 \) parametric representation (i.e., a continuous function \((r,u)\) mapping \([0,1]\) onto \( \Gamma_x \) such that \( r \) is nondecreasing), then we obtain the standard (or strong) \( M_2 \) topology. This topology is stronger than the weak \( M_2 \) topology, but they coincide for \( d = 1 \). Both topologies are weaker than the more frequently used Skorokhod \( J_1 \) and \( M_1 \) topologies. The \( M_2 \) topology on \( D([0,1], \mathbb{R}) \) can be generated using the Hausdorff metric on the spaces of graphs. For \( x_1, x_2 \in D([0,1], \mathbb{R}) \), define
\[
d_{M_2}(x_1, x_2) = \left( \sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a,b) \right) \vee \left( \sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a,b) \right),
\]
where \( d \) is the metric on \( \mathbb{R}^2 \) defined by \( d(a,b) = |a_1 - b_1| \vee |a_2 - b_2| \) for \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \in \mathbb{R}^2 \).
The weak $M_2$ topology on $D([0,1], \mathbb{R}^2)$ coincides with the (product) topology induced by the metric

\[ d_p(x_1, x_2) = \max_{j=1,2} d_{M_2}(x_{1j}, x_{2j}) \]  

for $x_i = (x_{i1}, x_{i2}) \in D([0,1], \mathbb{R}^2)$, $i = 1, 2$. For a detailed discussion of the strong and weak $M_2$ topologies, we refer to [17, Sects. 12.10–12.11].

In the next section, we will use the following lemma.

**Lemma 1.** Let $(A_n, B_n)$, $n = 0, 1, 2, \ldots$, be stochastic processes in $D([0,1], \mathbb{R}^2)$ such that, as $n \to \infty$,

\[ (A_n, B_n) \overset{d}{\to} (A_0, B_0) \]  

in $D([0,1], \mathbb{R}^2)$ with weak $M_2$ topology. Suppose $x_n$, $n = 0, 1, 2, \ldots$, are elements of $D([0,1], \mathbb{R})$ with continuous $x_0$ such that, as $n \to \infty$,

\[ x_n(t) \to x_0(t) \]  

uniformly in $t$. Then

\[ (A_n + x_n, B_n) \overset{d}{\to} (A_0 + x_0, B_0) \]  

in $D([0,1], \mathbb{R}^2)$ with weak $M_2$ topology.

**Proof.** Let $C_n := (A_n, B_n)$. For $n = 0, 1, 2, \ldots$, define the functions $y_n : [0,1] \to \mathbb{R}^2$ by $y_n(t) = (x_n(t), 0)$. Then clearly $y_n \in D([0,1], \mathbb{R}^2)$. Since $x_0$ is continuous, by Corollary 12.11.5 in [17] and the definition of the metric $d_p$ in (2.1) it follows that the function $h : D([0,1], \mathbb{R}^2) \to D([0,1], \mathbb{R}^2)$ defined by $h(x) = x + y_0$ is continuous with respect to the weak $M_2$ topology. Therefore by the continuous mapping theorem from (2.2) we obtain, as $n \to \infty$, $h(C_n) \overset{d}{\to} h(C_0)$, that is,

\[ C_n + y_0 \overset{d}{\to} C_0 + y_0 \]  

in $D([0,1], \mathbb{R}^2)$ with weak $M_2$ topology.

If we show that

\[ \lim_{n \to \infty} \mathbf{P}[d_p(C_n + y_n, C_n + y_0) > \delta] = 0 \]

for any $\delta > 0$, then from (2.3) by Slutsky’s theorem (see [15, Thm. 3.4]) we have $C_n + y_n \overset{d}{\to} C_0 + y_0$ in $D([0,1], \mathbb{R}^2)$ with weak $M_2$ topology. Recalling the definition of the metric $d_p$ and the fact that the Skorokhod $M_2$ metric on $D([0,1], \mathbb{R})$ is bounded above by the uniform metric on $D([0,1], \mathbb{R})$, we have

\[ \mathbf{P}[d_p(C_n + y_n, C_n + y_0) > \delta] \]

\[ = \mathbf{P}[d_{M_2}(x_n, x_0) > \delta] \leq \mathbf{P}\left( \sup_{t \in [0,1]} |x_n(t) - x_0(t)| > \delta \right). \]

Since $x_n(t) \to x_0(t)$ uniformly in $t$, we immediately obtain $\mathbf{P}[d_p(C_n + y_n, C_n + y_0) > \delta] \to 0$ as $n \to \infty$, and hence $C_n + y_n \overset{d}{\to} C_0 + y_0$ as $n \to \infty$, that is,

\[ (A_n + x_n, B_n) \overset{d}{\to} (A_0 + x_0, B_0) \]

in $D([0,1], \mathbb{R}^2)$ with weak $M_2$ topology. \( \square \)
3 Functional limit theorem

Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \(\alpha \in (0, 2)\). When \(\alpha = 1\), assume further that \(Z_1\) is symmetric. Let \(\{\varphi_i, i = 0, 1, 2, \ldots\}\) be a sequence of real numbers satisfying

\[
0 \leq \sum_{i=0}^{s} \varphi_i / \sum_{i=0}^{\infty} \varphi_i \leq 1 \quad \text{for every } s = 0, 1, 2 \ldots
\]

(3.1)

and such that the series defining the moving-average process

\[
X_i = \sum_{j=0}^{\infty} \varphi_j Z_{i-j}, \quad i \in \mathbb{Z},
\]

is a.s. convergent. We assume also that \(\sum_{i=0}^{\infty} |\varphi_i| < \infty\). Hence \(\beta = \sum_{i=0}^{\infty} \varphi_i\) is finite. Without loss of generality, we may assume that \(\beta > 0\) (the case \(\beta < 0\) is treated analogously and is therefore omitted). Let \(\varphi_+ = \max\{\varphi_j \vee 0: j \geq 0\}, \quad \varphi_- = \max\{-\varphi_j \vee 0: j \geq 0\}\).

Define further the corresponding partial-sum and maxima processes

\[
V_n(t) = \frac{1}{a_n} \left( \sum_{i=1}^{\lfloor nt \rfloor} X_i - \lfloor nt \rfloor b_n \right), \quad W_n(t) = \frac{1}{a_n} \sqrt{\sum_{i=1}^{\lfloor nt \rfloor} X_i}, \quad t \in [0, 1],
\]

(3.2)

where the normalizing sequence \((a_n)\) satisfies (1.2), and

\[
b_n = \begin{cases} 0, & \alpha \in (0, 1], \\ \beta \mathbb{E}(Z_1), & \alpha \in (1, 2). \end{cases}
\]

**Theorem 1.** Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \(\alpha \in (0, 2)\). When \(\alpha = 1\), suppose further that \(Z_1\) is symmetric. Let \(\{\varphi_i, i = 0, 1, 2, \ldots\}\) be a sequence of real numbers satisfying (3.1), \(\sum_{j=0}^{\infty} |\varphi_j| < \infty\), and \(\varphi_+ p + \varphi_- r > 0\) with \(p\) and \(r\) as in (1.4). Then, as \(n \to \infty\),

\[
L_n := (V_n, W_n) \xrightarrow{d} (\beta V, W)
\]

in \(D([0, 1], \mathbb{R}^2)\) with weak \(M^2\) topology, where \(V\) is an \(\alpha\)-stable Lévy process with characteristic triple \((0, \mu, b)\) with \(\mu\) as in (1.3) and

\[
b = \begin{cases} 0, & \alpha = 1, \\ (p - r) \frac{\alpha}{1-\alpha}, & \alpha \in (0, 1) \cup (1, 2), \end{cases}
\]

and \(W\) is an extremal process with exponent measure

\[
\nu(dx) = (\varphi_+ p + \varphi_- r) \alpha x^{-\alpha-1} \mathbf{1}_{(0, \infty)}(x) \, dx.
\]

In the proof of the theorem, we will need the following lemma.
Lemma 2. Let

\[ V_n^Z(t) := \sum_{i=1}^{\left\lfloor nt \right\rfloor} \beta_i - b_n, \quad W_n^Z(t) := \frac{\left\lfloor nt \right\rfloor}{a_n} \left( \varphi + \mathbb{1}_{\{Z_i > 0\}} + \varphi - \mathbb{1}_{\{Z_i < 0\}} \right), \quad t \in [0, 1]. \]

Then, as \( n \to \infty \),

\[ I_n^Z := (V_n^Z, W_n^Z) \overset{d}{\to} (\beta V, W) \quad (3.3) \]

in \( D([0, 1], \mathbb{R}^2) \) with weak \( M_2 \) topology, where \( V \) is an \( \alpha \)-stable Lévy process with characteristic triple \((0, \mu, b)\), and \( W \) is an extremal process with exponent measure \( \nu(dx) = (\varphi_\alpha^r + \varphi_\alpha^{-r})\alpha x^{-\alpha-1} \mathbb{1}_{(0, \infty)}(x) \) \( dx \).

Proof. Fix \( 0 < u < \infty \) and define the sum-maximum functional

\[ \Phi^{(u)} : \mathbb{M}_p([0, 1] \times \mathbb{E}) \to D([0, 1], \mathbb{R}^2) \]

by

\[ \Phi^{(u)} \left( \sum_i \delta_{(t_i, x_i)} \right)(t) = \left( \beta \sum_{t_i \leq t} x_i \mathbb{1}_{\{u < |x_i| < \infty\}}, \left\lfloor x_i \right\rfloor \left( \varphi + \mathbb{1}_{\{x_i > 0\}} + \varphi - \mathbb{1}_{\{x_i < 0\}} \right) \right) \]

for \( t \in [0, 1] \) (here we setsup \( \emptyset = 0 \) for convenience), where the space \( \mathbb{M}_p([0, 1] \times \mathbb{E}) \) of Radon point measures on \([0, 1] \times \mathbb{E}\) is equipped with vague topology. Let \( \mathbb{E}_u = \mathbb{E} \setminus [-u, u] \) and \( A = A_1 \cap A_2 \), where

\[ A_1 = \{ \eta \in \mathbb{M}_p([0, 1] \times \mathbb{E}) : \eta(\{(0, 1) \times \mathbb{E}\} = 0 = \eta([0, 1] \times \{\pm \infty, \pm u\}) \}, \]

\[ A_2 = \{ \eta \in \mathbb{M}_p([0, 1] \times \mathbb{E}) : \eta(\{t\} \times \mathbb{E}_u) \leq 1 \text{ for all } t \in [0, 1] \}. \]

The elements of \( A_2 \) have no two atoms in \([0, 1] \times \mathbb{E}_u\) with the same time coordinate.

The functional \( \Phi^{(u)} \) is continuous on the set \( A \) when \( D([0, 1], \mathbb{R}^2) \) is endowed with weak \( M_2 \) topology. Indeed, take an arbitrary \( \eta \in A \) and suppose that \( \eta_n \overset{\mathbb{M}_p}{\to} \eta \) in \( \mathbb{M}_p([0, 1] \times \mathbb{E}) \). We need to show that \( \Phi^{(u)}(\eta_n) \to \Phi^{(u)}(\eta) \) in \( D([0, 1], \mathbb{R}^2) \) according to the \( WM_2 \) topology. By Theorem 12.5.2 in [17] it suffices to prove that, as \( n \to \infty \),

\[ d_p(\Phi^{(u)}(\eta_n), \Phi^{(u)}(\eta)) = \max_{k=1,2} d_{M_2}(\Phi^{(u)}_k(\eta_n), \Phi^{(u)}_k(\eta)) \to 0. \]

Following, with small modifications, the arguments in the proof of Lemma 3.2 in [5], we obtain \( d_{M_2}(\Phi^{(u)}_1(\eta_n), \Phi^{(u)}_1(\eta)) \to 0 \) as \( n \to \infty \). Let

\[ T = \{ t \in [0, 1] : \eta(\{t\} \times \mathbb{E}) = 0 \}. \]

Since \( \eta \) is a Radon point measure, the set \( T \) is dense in \([0, 1]\). Fix \( t \in T \) and take \( \epsilon > 0 \) such that \( \eta([0, t] \times \{\pm \epsilon\}) = 0 \). Later, as \( \epsilon \to 0 \), we assume the convergence to 0 through a sequence of values \( \epsilon_j \) such that \( \eta((0, t] \times \{\pm \epsilon_j\}) = 0 \) for all \( j \in \mathbb{N} \) (this can be arranged since \( \eta \) is a Radon point measure). Since the set \([0, t] \times \mathbb{E}_\epsilon\) is relatively compact in \([0, 1] \times \mathbb{E}\), there exists a nonnegative integer \( k = k(\eta) \) such that

\[ \eta([0, t] \times \mathbb{E}_\epsilon) = k < \infty. \]

By assumption, \( \eta \) does not have any atoms on the border of the set \([0, t] \times \mathbb{E}_\epsilon\). Therefore, by Lemma 7.1 in [15] there exists a positive integer \( n_0 \) such that

\[ \eta_n([0, t] \times \mathbb{E}_\epsilon) = k \quad \text{for all } n \geq n_0. \]

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Let \((t_i, x_i)\) for \(i = 1, \ldots, k\) be the atoms of \(\eta\) in \([0, t] \times \mathbb{E}\). By the same lemma the \(k\) atoms \((t^{(n)}_i, x^{(n)}_i)\) of \(\eta_n\) in \([0, t] \times \mathbb{E}\) (for \(n \geq n_0\)) can be labeled so that, for every \(i \in \{1, \ldots, k\}\), we have

\[
(t^{(n)}_i, x^{(n)}_i) \to (t_i, x_i) \quad \text{as} \ n \to \infty.
\]

In particular, for any \(\delta > 0\), we can find a positive integer \(n_\delta \geq n_0\) such that, for all \(n \geq n_\delta\),

\[
|t^{(n)}_i - t_i| < \delta \quad \text{and} \quad |x^{(n)}_i - x_i| < \delta \quad \text{for} \ i = 1, \ldots, k.
\]

If \(k = 0\), then (for large \(n\)) the atoms of \(\eta\) and \(\eta_n\) in \([0, t] \times \mathbb{E}\) are all situated in \([0, t] \times (-\epsilon, \epsilon)\). Hence \(\Phi^{(u)}(\eta)(t) \in [0, \epsilon)\) and \(\Phi^{(u)}(\eta_n)(t) \in [0, \epsilon)\), which implies

\[
|\Phi^{(u)}(\eta_n)(t) - \Phi^{(u)}(\eta)(t)| < \epsilon.
\]  

(3.4)

If \(k \geq 1\), then take \(\delta = \epsilon\). Note that \(|x^{(n)}_i - x_i| < \delta\) implies \(x^{(n)}_i > 0\) iff \(x_i > 0\). Hence we have

\[
|\Phi^{(u)}(\eta_n)(t) - \Phi^{(u)}(\eta)(t)|
\]

\[
= \left| \bigvee_{i=1}^{k} |x^{(n)}_i| \left( \varphi_+ 1_{\{x^{(n)}_i > 0\}} + \varphi_- 1_{\{x^{(n)}_i < 0\}} \right) \right| - \left| \bigvee_{i=1}^{k} |x_i| \left( \varphi_+ 1_{\{x_i > 0\}} + \varphi_- 1_{\{x_i < 0\}} \right) \right|
\]

\[
\leq \bigvee_{i=1}^{k} \left| (|x^{(n)}_i| - |x_i|) \left( \varphi_+ 1_{\{x_i > 0\}} + \varphi_- 1_{\{x_i < 0\}} \right) \right| \leq (\varphi_+ \vee \varphi_-) \bigvee_{i=1}^{k} |x^{(n)}_i - x_i|,
\]

(3.5)

where the first inequality follows from the inequality

\[
\bigvee_{i=1}^{k} a_i - \bigvee_{i=1}^{k} b_i \leq \bigvee_{i=1}^{k} |a_i - b_i|
\]

for arbitrary real numbers \(a_1, \ldots, a_k, b_1, \ldots, b_k\). Therefore form (3.4) and (3.5) we obtain

\[
\lim_{n \to \infty} |\Phi^{(u)}(\eta_n)(t) - \Phi^{(u)}(\eta)(t)| < (\varphi_+ \vee \varphi_- \vee 1) \epsilon,
\]

and letting \(\epsilon \to 0\), it follows that \(\Phi^{(u)}(\eta_n)(t) \to \Phi^{(u)}(\eta)(t)\) as \(n \to \infty\). Note that \(\Phi^{(u)}(\eta)\) and \(\Phi^{(u)}(\eta_n)\) are nondecreasing functions. Since by Corollary 12.5.1 in [17] the \(M_1\) convergence for monotone functions is equivalent to the pointwise convergence in a dense subset of points plus convergence at the endpoints, and the \(M_1\) convergence implies the \(M_2\) convergence, we conclude that \(d_{M_2}(\Phi^{(u)}(\eta_n), \Phi^{(u)}(\eta)) \to 0\) as \(n \to \infty\). Hence \(\Phi^{(u)}\) is continuous at \(\eta\).

Since the random variables \(Z_i\) are i.i.d. and regularly varying, Corollary 6.1 in [15] yields

\[
N_n := \sum_{i=1}^{n} \delta_{(i/n, Z_i/a_n)} \xrightarrow{d} N := \sum_{i} \delta_{(t_i, j_i)} \quad \text{as} \ n \to \infty
\]

(3.6)

in \(\mathcal{M}_p([0, 1] \times \mathbb{E})\), where the limiting point process \(N\) is a Poisson process with intensity measure \(\text{Leb} \times \mu\). Since \(\mathbf{P}(N \in A) = 1\) (see [15], p. 221) and the functional \(\Phi^{(u)}\) is continuous on the set \(A\), from (3.6) by the
continuous mapping theorem we obtain \( \Phi^{(u)}(N_n) \xrightarrow{d} \Phi^{(u)}(N) \) as \( n \to \infty \), that is,

\[
L^{(u)}_n := \left( \beta \sum_{i=1}^{[n\cdot]} \frac{Z_i}{a_n} 1_{|Z_i|/a_n > u} + \beta \sum_{i=1}^{[n\cdot]} \left( \varphi_{+1} 1_{\{Z_i > 0\}} + \varphi_{-1} 1_{\{Z_i < 0\}} \right) \right)
\]

\[
\xrightarrow{d} L^{(u)}_0 := \left( \beta \sum_{t_i \leq t} j_i 1_{\{|j_i| > u\}} + \beta \sum_{t_i \leq t} \left( \varphi_{+1} 1_{\{j_i > 0\}} + \varphi_{-1} 1_{\{j_i < 0\}} \right) \right)
\]

(3.7)

in \( D([0, 1], \mathbb{R}^2) \) with weak \( M_2 \) topology. From (1.2) we have, as \( n \to \infty \),

\[
\frac{|nt|}{n} \mathbb{E}\left( \frac{Z_i}{a_n} 1_{u < |Z_i|/a_n \leq 1} \right) = \frac{|nt|}{n} \int_{u < |x| \leq 1} x_n \mathbb{P}\left( \frac{Z_i}{a_n} : \frac{Z_i}{a_n} \in dx \right) \to t \int_{u < |x| \leq 1} x \mu(dx)
\]

(3.8)

uniformly for \( t \in [0, 1] \). From (3.7) and (3.8), applying Lemma 1, we obtain, as \( n \to \infty \),

\[
L_n^{(u)} \xrightarrow{d} L_0^{(u)} - x^{(u)}
\]

(3.9)

in \( D([0, 1], \mathbb{R}^2) \) with weak \( M_2 \) topology, where

\[
\tilde{L}_n^{(u)}(t) = \left( \beta \sum_{i=1}^{[nt]} \frac{Z_i}{a_n} 1_{|Z_i|/a_n > u} - \beta \frac{|nt|}{n} \mathbb{E}\left( \frac{Z_i}{a_n} 1_{u < |Z_i|/a_n \leq 1} \right) \right)
\]

for \( t \in [0, 1] \), and

\[
x^{(u)}(t) = (ta_u, 0), \quad a_u = \beta \int_{u < |x| \leq 1} x \mu(dx).
\]

By the Itô representation of a Lévy process (see [15, Sect. 5.5.3] or [16, Thm. 19.2]) there exists a Lévy process \( V_0 \) with characteristic triple \((0, \mu, 0)\) such that, as \( u \to 0 \),

\[
\sup_{t \in [0, 1]} \left| L_n^{(u)}(t) - ta_u - \beta V_0(t) \right| \xrightarrow{a.s.} 0.
\]

Since uniform convergence implies (weak) \( M_2 \) convergence, it immediately follows that

\[
d_p\left( L_0^{(u)} - x^{(u)}, L \right) \to 0
\]

almost surely as \( u \to 0 \), where

\[
L(t) := \left( \beta V_0(t), \sqrt{\sum_{t_i \leq t} |j_i|\left( \varphi_{+1} 1_{\{j_i > 0\}} + \varphi_{-1} 1_{\{j_i < 0\}} \right) } \right), \quad t \in [0, 1].
\]

From this, since almost sure convergence implies convergence in distribution, we obtain, as \( u \to 0 \),

\[
L_n^{(u)} - x^{(u)} \xrightarrow{d} L
\]

(3.10)
in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology. Since $\sum_t \delta_{(t, j_t)}$ is a Poisson process with intensity measure $Leb \times \mu$, the process

$$W(t) := \bigvee_{t_i \leq t} |j_i| \left( \varphi_+ 1_{\{j_i > 0\}} + \varphi_- 1_{\{j_i < 0\}} \right), \quad t \in [0, 1],$$

is an extremal process with exponent measure $\nu(dx) = (\varphi^\alpha p + \varphi^\alpha r) x^{-\alpha - 1} 1_{(0, \infty)}(x) dx$ (see [14, Sect. 4.5] and [15, p. 161]).

Let

$$\tilde{L}_n^Z(t) := \left( \sum_{i=1}^{[nt]} \beta Z_i/a_n - \beta [nt] \mathbb{E} \left( \frac{Z_i}{a_n} \mathbb{1}_{\{Z_i/a_n \leq u\}} \right) \right) \bigvee_{i=1}^{[nt]} \left( \frac{|Z_i|}{a_n} (\varphi_+ 1_{\{Z_i > 0\}} + \varphi_- 1_{\{Z_i < 0\}}) \right)$$

for $t \in [0, 1]$. If we show that

$$\lim_{u \to 0} \limsup_{n \to \infty} P \left[ d_p(\tilde{L}_n^Z, \tilde{L}_n^u) > \delta \right] = 0$$

for any $\delta > 0$, then from (3.9), (3.10), and a generalization of Slutsky’s theorem (see [15, Thm. 3.5]) we will have $\tilde{L}_n^Z \overset{d}{\to} L$ as $n \to \infty$ in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology. Recalling the definitions and the fact that the metric $d_{M_2}$ is bounded above by the uniform metric, we have

$$P \left[ d_p(\tilde{L}_n^Z, \tilde{L}_n^u) > \delta \right] \leq P \left( \sup_{t \in [0, 1]} \left( \sum_{i=1}^{[nt]} \beta Z_i/a_n - \beta [nt] \mathbb{E} \left( \frac{Z_i}{a_n} \mathbb{1}_{\{Z_i/a_n \leq u\}} \right) \right) > \delta \right)$$

$$= P \left( \max_{k=1, \ldots, n} \left( \sum_{i=1}^{k} \frac{Z_i}{a_n} \mathbb{1}_{\{Z_i/a_n \leq u\}} - k \mathbb{E} \left( \frac{Z_1}{a_n} \mathbb{1}_{\{Z_1/a_n \leq u\}} \right) \right) > \delta \beta^{-1} \right).$$

In the i.i.d. case, we have

$$\lim_{u \to \infty} \limsup_{n \to \infty} P \left( \max_{k=1, \ldots, n} \left( \sum_{i=1}^{k} \frac{Z_i}{a_n} \mathbb{1}_{\{Z_i/a_n \leq u\}} - k \mathbb{E} \left( \frac{Z_1}{a_n} \mathbb{1}_{\{Z_1/a_n \leq u\}} \right) \right) > \delta \beta^{-1} \right) = 0$$

(see the proof of Proposition 3.4 in [13]), and therefore, as $n \to \infty$,

$$\tilde{L}_n^Z \overset{d}{\to} L$$

(3.11)

in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology.

Note that when $\alpha = 1$, we have $\tilde{L}_n^Z = L_n^Z$ (since $Z_1$ is symmetric), and the statement of the lemma holds. Therefore, first, assume that $\alpha \in (0, 1)$. By Karamata’s theorem, as $n \to \infty$,

$$[nt] \mathbb{E} \left( \frac{Z_1}{a_n} \mathbb{1}_{\{Z_1/a_n \leq 1\}} \right) \to t(p - r) \frac{\alpha}{1 - \alpha}$$

for every $t \in [0, 1]$. From this and (3.11), applying Lemma 1, we obtain, as $n \to \infty$,

$$\tilde{L}_n^Z + \left( \beta \left[ nt \right] \mathbb{E} \left( \frac{Z_1}{a_n} \mathbb{1}_{\{Z_1/a_n \leq 1\}} \right), 0 \right) \overset{d}{\to} L + \left( \beta (p - r) \frac{\alpha}{1 - \alpha}, 0 \right),$$
that is,

\[ L_n^Z \xrightarrow{d} \left( \beta V_0 + \beta(p - r) \frac{\alpha}{1 - \alpha}, W \right) \]

(3.12)
in \( D([0,1], \mathbb{R}^2) \) with weak \( M_2 \) topology. Put

\[ V(t) := V_0(t) + t(p - r) \frac{\alpha}{1 - \alpha}, \quad t \in [0,1], \]

and note that (3.3) holds in this case, since the characteristic triple of the Lévy process \( V \) is \((0, \mu, (p - r)\alpha/ (1 - \alpha))\).

Finally, assume that \( \alpha \in (1, 2) \). By Karamata’s theorem, as \( n \to \infty \),

\[ [nt] E \left( \frac{Z_1}{a_n} \mathbf{1}_{\{|Z_1/a_n| > 1\}} \right) \to t(p - r) \frac{\alpha}{\alpha - 1} \]

for every \( t \in [0,1] \). Therefore a new application of Lemma 1 to (3.11) yields, as \( n \to \infty \),

\[ \tilde{L}_n^Z - \left( \beta \mathbb{E} \left( \frac{Z_1}{a_n} \mathbf{1}_{\{|Z_1/a_n| > 1\}} \right), 0 \right) \xrightarrow{d} L - \left( \beta(p - r) \frac{\alpha}{\alpha - 1}, 0 \right), \]

that is, \( L_n^Z \xrightarrow{d} (\beta V, W) \) in \( D([0,1], \mathbb{R}^2) \) with weak \( M_2 \) topology, and this concludes the proof. \( \square \)

**Remark 1.** From the proof of Lemma 2 it follows that the components of the limiting process \((\beta V, W)\) can be expressed as functionals of the limiting point process \( N = \sum_i \delta_{(t_i, j_i)} \) from relation (3.6), that is,

\[ V = \lim_{u \to 0} \left( \sum_{t_i \leq u} j_i \mathbf{1}_{\{|j_i| > u\}} - \int_{u < |x| \leq 1} x \mu(dx) \right) + (p - r) \frac{\alpha}{1 - \alpha} \mathbf{1}_{\{|t_i| \neq 0\}}, \]

where the limit holds almost surely uniformly on \([0,1] \), and

\[ W = \bigvee_{t_i \leq u} |j_i| (\varphi_+ \mathbf{1}_{\{j_i > 0\}} + \varphi_- \mathbf{1}_{\{j_i < 0\}}). \]

The process \( N \) is a Poisson process with intensity measure \( Leb \times \mu \), and by using standard Poisson point process transformations (see [15, Props. 5.2, 5.3]) it can also be represented as

\[ N = \sum_i \delta_{(t_i, j_i, Q_i)}, \]

where

(i) \( \sum_{i=1}^{\infty} \delta_{(t_i, j_i)} \) is a Poisson point process on \([0,1] \times (0, \infty) \) with intensity measure \( Leb \times d(-x^{-\alpha}) \);

(ii) \( (Q_i)_{i \in \mathbb{N}} \) is a sequence of i.i.d. random variables, independent of \( \sum_{i=1}^{\infty} \delta_{(t_i, j_i)} \), such that \( P(Q_1 = 1) = p \) and \( P(Q_1 = -1) = r \).

**Remark 2.** Lemma 2 shows that the process \( L_n^Z \) converges to \((\beta V, W)\) in the space \( D([0,1], \mathbb{R}^2) \) endowed with weak \( M_2 \) topology. If we show that \( L_n^Z \) is close to \( L_n \) in the weak \( M_2 \) sense, then by the so-called converging together result (Slutsky’s theorem) this will follow that \( L_n \) converges to the same limiting process. This is carried out in detail in the proof of Theorem 1.

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Heuristically, for a finite-order moving average \( X_t = \sum_{j=0}^{q} \varphi_j Z_{t-j} \), most of the sequence \( Z_{t,n} := Z_t / \sigma_n \) is negligible, except for “big values” \( Z_{t,n}, Z_{t+1,n}, \ldots, Z_{t+n,n}, \ldots \), which are spread far apart. Note that a “big value” \( Z_{t,n} \) produces \( q + 1 \) consecutive “big values” in the sequence \( X_{t,n} = \sum_{j=0}^{q} \varphi_j Z_{t-j,n} \): 

\[
X_{t,n} \approx \varphi_0 Z_{t,n}, \quad X_{t+1,n} \approx \varphi_1 Z_{t,n}, \quad \ldots, \quad X_{t+q,n} \approx \varphi_q Z_{t,n}.
\]  

(3.13)

These values cover an interval on the \( x \) axis of length \( q/n \), and their sum is well approximated by 

\[
\sum_{j=0}^{q} \varphi_j Z_{t,n} = \beta Z_{t,n} \quad \text{as} \quad n \to \infty,
\]

showing that \( V_n^Z \) is a suitable approximation of \( V_n \).

As for the maxima process, a “big value” \( \varphi_j Z_{t,n} \) has an effect on \( W_n \) only if it is positive, that is, \( \varphi_j \) and \( Z_{t,n} \) are of the same sign. Hence the maximum of the values \( X_{t+n} \) in (3.13) is well approximated by

\[
\lim_{n \to \infty} \left( \frac{1}{Z_{t,n}} \right) \sum_{j=0}^{q} \varphi_j Z_{t,n} = \beta \frac{Z_{t,n}}{n} = \beta Z_{t,n} \quad \text{as} \quad n \to \infty,
\]

showing that \( V_n^Z \) is an appropriate approximation of \( V_n \).

**Proof of Theorem 1.** We first prove the theorem for finite-order moving-average processes and then for infinite-order moving averages. Fix \( q \in \mathbb{N} \) and let \( X_i = \sum_{j=0}^{q} \varphi_j Z_{i-j}, \quad i \in \mathbb{Z} \). In this case, condition (3.1) reduces to

\[
0 \leq \sum_{i=0}^{s} \varphi_i / \sum_{i=0}^{q} \varphi_i \leq 1 \quad \text{for every} \quad s = 0, 1, \ldots, q.
\]  

(3.14)

If we show that, for every \( \delta > 0 \),

\[
\lim_{n \to \infty} \mathbb{P} \left( d_p(L_n^Z, L_n) > \delta \right) = 0,
\]

then from Lemma 2 by an application of Slutsky’s theorem we will obtain \( L_n \xrightarrow{d} (\beta V, W) \) as \( n \to \infty \) in \( D([0,1], \mathbb{R}^2) \) with weak \( M_2 \) topology. From the definition of the metric \( d_p \) in (2.1) it suffices to show that

\[
\lim_{n \to \infty} \mathbb{P} \left( d_{M_2}(V_n^Z, V_n) > \delta \right) = 0
\]  

(3.15)

and

\[
\lim_{n \to \infty} \mathbb{P} \left( d_{M_2}(W_n^Z, W_n) > \delta \right) = 0.
\]  

(3.16)

Relation (3.15) is established in the proof of Theorem 2.1 in [4]. It remains to show (3.16).

Fix \( \delta > 0 \) and let \( n \in \mathbb{N} \) be large enough, \( n > \max\{2q, 2q/\delta\} \). Then by the definition of the metric \( d_{M_2} \) we have

\[
d_{M_2}(W_n^Z, W_n) = \left( \sup_{v \in F_n^Z} \inf_{z \in F_n^Z} d(v, z) \right) \vee \left( \sup_{v \in F_n^Z} \inf_{z \in F_n^Z} d(v, z) \right) =: Y_n \vee T_n.
\]

Hence

\[
\mathbb{P} \left( d_{M_2}(W_n^Z, W_n) > \delta \right) \leq \mathbb{P}(Y_n > \delta) + \mathbb{P}(T_n > \delta).
\]  

(3.17)
Now, we estimate the first term on the right-hand side of (3.17). Let

$$D_n = \{ \exists v \in \Gamma_{W_n^Z} \text{ such that } d(v, z) > \delta \text{ for every } z \in \Gamma_{W_n} \}.$$ 

Then by the definition of $Y_n$

$$\{Y_n > \delta\} \subseteq D_n. \quad (3.18)$$

On the event $D_n$, we have $d(v, \Gamma_{W_n}) > \delta$. Let $v = (t_v, x_v)$. Then

$$\left| W_n^Z \left( \frac{i^*}{n} \right) - W_n \left( \frac{i^*}{n} \right) \right| > \delta, \quad (3.19)$$

where $i^* = \lfloor nt_v \rfloor$ or $i^* = \lfloor nt_v \rfloor - 1$. Indeed, $t_v \in \{i/n, (i+1)/n\}$ for some $i \in \{1, \ldots, n-1\}$ (or $t_v = 1$). If $x_v = W_n^Z(i/n)$ (i.e., $v$ lies on a horizontal part of the completed graph), then clearly

$$\left| W_n^Z \left( \frac{i}{n} \right) - W_n \left( \frac{i}{n} \right) \right| \geq d(v, \Gamma_{W_n}) > \delta,$$

and we put $i^* = i$. On the other hand, if $x_v \in \{W_n^Z((i-1)/n), W_n^Z(i/n)\}$ (i.e., $v$ lies on a vertical part of the completed graph), then we can similarly show that

$$\left| W_n^Z \left( \frac{i-1}{n} \right) - W_n \left( \frac{i-1}{n} \right) \right| \geq \delta \quad \text{if } W_n \left( \frac{i^*}{n} \right) > x_v$$

and

$$\left| W_n^Z \left( \frac{i}{n} \right) - W_n \left( \frac{i}{n} \right) \right| \geq \delta \quad \text{if } W_n \left( \frac{i^*}{n} \right) < x_v.$$

In the first case, put $i^* = i - 1$ and, in the second, $i^* = i$. Note that $i = \lfloor nt_v \rfloor$, and therefore (3.19) holds. Moreover, since $|i^*/n - (i^* + l)/n| \leq q/n \leq \delta$ for every $l = 1, \ldots, q$ (such that $i^* + l \leq n$), from the definition of the set $D_n$ we can similarly conclude that

$$\left| W_n^Z \left( \frac{i^*}{n} \right) - W_n \left( \frac{i^* + l}{n} \right) \right| > \delta. \quad (3.20)$$

Put $\gamma = \varphi_+ \lor \varphi_-$. We claim that

$$D_n \subseteq H_{n,1} \cup H_{n,2} \cup H_{n,3} \cup H_{n,4}, \quad (3.21)$$

where

$$H_{n,1} = \{ \exists l \in \{-q, \ldots, 0\} \text{ such that } \left| \frac{Z_l}{a_n} \right| > \frac{\delta}{4(q+1)\gamma} \},$$

$$H_{n,2} = \{ \exists l \in \{1, \ldots, q\} \cup \{n - q + 1, \ldots, n\} \text{ such that } \left| \frac{Z_l}{a_n} \right| > \frac{\delta}{4(q+1)\gamma} \},$$

$$H_{n,3} = \left\{ \exists k \in \{1, \ldots, n\} \text{ and } \exists l \in \{k - q, \ldots, k + q\} \setminus \{k\} \text{ such that } \left| \frac{Z_k}{a_n} \right| > \frac{\delta}{4(q+1)\gamma} \text{ and } \left| \frac{Z_l}{a_n} \right| > \frac{\delta}{4(q+1)\gamma} \right\},$$

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\[ H_{n,4} = \left\{ \exists k \in \{1, \ldots, n\}, \exists j \in \{1, \ldots, n\} \setminus \{k, \ldots, k+q\}, \exists l_1 \in \{0, \ldots, q\} \right. \]
\[
\text{and } \exists l \in \{0, \ldots, q\} \setminus \{l_1\} \text{ such that } \left| \frac{Z_j-l_1}{a_n} \right| > \frac{\delta}{4(q+1)\gamma} \text{ and } \left| \frac{Z_j-l}{a_n} \right| > \frac{\delta}{4(q+1)\gamma}. \]

To prove (3.21), it suffices to show that
\[ D_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c \subseteq H_{n,4}. \]
Thus assume that the event \( D_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c \) occurs. Then necessarily \( W_n^Z(i^*/n) > \delta/[4(q+1)] \).
Indeed, if \( W_n^Z(i^*/n) \leq \delta/[4(q+1)] \), that is,
\[
\sum_{j=1}^{i^*} \left| \frac{Z_j}{a_n} \right| (\varphi_{+1}Z_{j>0} + \varphi_{-1}Z_{j<0}) = W_n^Z\left(\frac{i^*}{n}\right) \leq \frac{\delta}{4(q+1)},
\]
then for every \( s \in \{q+1, \ldots, i^*\} \), we have
\[
\frac{X_s}{a_n} \leq \sum_{j=0}^{q} \left| \varphi_j \right| \frac{|Z_{s-j}|}{a_n} \leq \sum_{j=0}^{q} \left| \varphi_j \right| \frac{|Z_{s-j}|}{a_n} (\varphi_{+1}Z_{s-j>0} + \varphi_{-1}Z_{s-j<0}) \leq \frac{\delta}{4(q+1)} (q+1) = \frac{\delta}{4}. \quad (3.22)
\]
Since the event \( H_{n,1}^c \cap H_{n,2}^c \) occurs, for every \( s \in \{1, \ldots, q\} \), we also have
\[
\left| \frac{X_s}{a_n} \right| \leq \sum_{j=0}^{q} \left| \varphi_j \right| \frac{|Z_{s-j}|}{a_n} \leq \frac{\delta}{4(q+1)\gamma} \sum_{j=0}^{q} \left| \varphi_j \right| \leq \frac{\delta}{4(q+1)\gamma} (q+1) \gamma = \frac{\delta}{4}. \quad (3.23)
\]
yielding
\[
-\frac{\delta}{4} \leq \frac{X_1}{a_n} \leq W_n\left(\frac{i^*}{n}\right) = \bigvee_{s=1}^{i^*} \frac{X_s}{a_n} \leq \frac{\delta}{4}. \quad (3.24)
\]
Hence
\[
W_n^Z\left(\frac{i^*}{n}\right) - W_n\left(\frac{i^*}{n}\right) \leq \left| W_n^Z\left(\frac{i^*}{n}\right) \right| + W_n\left(\frac{i^*}{n}\right) \leq \frac{\delta}{4(q+1)} + \frac{\delta}{4} \leq \frac{\delta}{2},
\]
which is in contradiction with (3.19).
Therefore \( W_n^Z(i^*/n) > \delta/[4(q+1)] \). This implies the existence of \( k \in \{1, \ldots, i^*\} \) such that
\[
W_n^Z\left(\frac{i^*}{n}\right) = \left| \frac{Z_k}{a_n} \right| (\varphi_{+1}Z_{k>0} + \varphi_{-1}Z_{k<0}) > \frac{\delta}{4(q+1)} \quad (3.25)
\]
Therefore
\[
\left| \frac{Z_k}{a_n} \right| \geq \left| \frac{Z_k}{a_n} \right| (\varphi_{+1}Z_{k>0} + \varphi_{-1}Z_{k<0}) > \frac{\delta}{4(q+1)\gamma},
\]
and since $H_{n,2}^c$ occurs, it follows that $q + 1 \leq k \leq n - q$. Since $H_{n,3}^c$ occurs, it holds that
\[
\frac{|Z_l|}{a_n} \leq \delta \frac{1}{4(q + 1)\gamma} \quad \text{for all } l \in \{k - q, \ldots, k + q\} \setminus \{k\}.
\]

(3.26)

Now we claim that $W_n(i^*/n) = X_j/a_n$ for some $j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\}$. If this is not the case, then $W_n(i^*/n) = X_j/a_n$ for some $j \in \{k, \ldots, k + q\}$ (with $j \leq i^*$). Here we distinguish two cases.

(i) $k + q \leq i^*$. On the event \{$Z_k > 0$\}, we have
\[
|Z_k| (\varphi_+1_{\{Z_k > 0\}} + \varphi_-1_{\{Z_k < 0\}}) = \varphi_+Z_k = \varphi_{j_0}Z_k
\]
for some $j_0 \in \{0, \ldots, q\}$ (with $\varphi_{j_0} > 0$). Since $k + j_0 \leq i^*$, we have
\[
\frac{X_j}{a_n} = W_n \left( \frac{i^*}{n} \right) \geq \frac{X_{k+j_0}}{a_n}.
\]

(3.27)

Taking into account the assumptions that hold in this case, we can write
\[
\frac{X_j}{a_n} = \frac{\varphi_{j-k}Z_k}{a_n} + \sum_{s=0}^{q} \frac{\varphi_sZ_{j-s}}{a_n} =: \frac{\varphi_{j-k}Z_k}{a_n} + F_1
\]
and
\[
\frac{X_{k+j_0}}{a_n} = \frac{\varphi_{j_0}Z_k}{a_n} + \sum_{s=0}^{q} \frac{\varphi_sZ_{k+j_0-s}}{a_n} =: \frac{\varphi_{j_0}Z_k}{a_n} + F_2.
\]

From relation (3.26) (similarly as in (3.23)) we obtain
\[
|F_1| \leq \frac{\delta}{4(q + 1)\gamma} q < \frac{\delta}{4},
\]
and similarly $|F_2| < \delta/4$. Since $\varphi_{j_0} - \varphi_{j-k} = \varphi_+ - \varphi_{j-k} \geq 0$, from (3.27) it follows that
\[
0 \leq \frac{\varphi_{j_0}Z_k - \varphi_{j-k}Z_k}{a_n} \leq F_1 - F_2 \leq |F_1| + |F_2| < \frac{\delta}{2}.
\]

By (3.19) we have
\[
\left| \frac{\varphi_{j_0}Z_k}{a_n} - \frac{X_j}{a_n} \right| = \left| W_n \left( \frac{i^*}{n} \right) - W_n \left( \frac{i^*}{n} \right) \right| > \delta,
\]
and hence
\[
\delta < \left| \frac{\varphi_{j_0}Z_k}{a_n} - \frac{\varphi_{j-k}Z_k}{a_n} - F_1 \right| \leq \left| \frac{\varphi_{j_0}Z_k}{a_n} - \frac{\varphi_{j-k}Z_k}{a_n} \right| + |F_1| < \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4},
\]
which is not possible. On the event \{$Z_k < 0$\}, we have that $|Z_k| (\varphi_+1_{\{Z_k > 0\}} + \varphi_-1_{\{Z_k < 0\}}) = \varphi_-|Z_k| = \varphi_{i_0}Z_k$ for some $i_0 \in \{0, \ldots, q\}$ (with $\varphi_{i_0} \leq 0$). Repeating the arguments as before, we similarly arrive at a contradiction. Therefore this case cannot happen.
(ii) \( k + q > i^* \). Note that, in this case, \( k \leq j \leq i^* < k + q \). Let \( s_0 \in \{1, \ldots, q\} \) be such that \( i^* + s_0 = k + q \). Let

\[
W_n\left( \frac{i^* + s_0}{n} \right) = \frac{X_p}{a_n}
\]

for some \( p \leq k + q \). Since \( W_n(i^*/n) \leq W_n((i^* + s_0)/n) \), we have that \( j \leq p \). Then

\[
\frac{X_p}{a_n} = W_n\left( \frac{k + q}{n} \right) \geq \frac{X_{k+j_0}}{a_n} \vee \frac{X_{k+i_0}}{a_n}
\]

for \( j_0 \) and \( i_0 \) as in (i). By (3.20) we have

\[
\left| \frac{|Z_k|}{a_n} (\varphi - 1_{\{Z_k > 0\}} + \varphi - 1_{\{Z_k < 0\}}) - \frac{X_p}{a_n} \right| = \left| W_n\left( \frac{i^*}{n} \right) - W_n\left( \frac{i^* + s_0}{n} \right) \right| > \delta,
\]

and repeating the arguments as in (i) (with \( p \) instead of \( j \) and \( i^* + s_0 \) instead of \( i^* \)), we conclude that this case also cannot happen.

Hence, indeed, \( W_n(i^*/n) = X_j/a_n \) for some \( j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\} \). Now we have three cases: (A) all random variables \( Z_{j-Q}, \ldots, Z_j \) are “small”; (B) exactly one is “large,” and (C) at least two of them are “large” (\( Z \) is “small” if \( |Z_j|/a_n \leq \delta/[4(q + 1)\gamma] \); otherwise, it is “large”). We will show that the first two cases are not possible.

**Case A.** \( |Z_{j-l}|/a_n \leq \delta/[4(q + 1)\gamma] \) for every \( l = 0, \ldots, q \). This yields (as in (3.23))

\[
\left| W_n\left( \frac{i^*}{n} \right) \right| = \frac{|X_j|}{a_n} \leq \frac{\delta}{4}.
\]

Let \( j_0 \) and \( i_0 \) be as in (i) (we take \( j_0 \) on the set \( \{Z_k > 0\} \) and \( i_0 \) on the set \( \{Z_k < 0\} \)). If \( k + q \leq i^* \), then

\[
\frac{X_j}{a_n} > \frac{X_{k+j_0}}{a_n} = \frac{\varphi_{j_0}Z_k}{a_n} + F_2,
\]

where \( F_2 \) is as in (i) with \( |F_2| < \delta/4 \). Therefore

\[
\frac{\varphi_{j_0}Z_k}{a_n} \leq \frac{X_j}{a_n} - F_2 \leq \frac{|X_j|}{a_n} + |F_2| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}
\]

and

\[
\left| W_n\left( \frac{i^*}{n} \right) - W_n\left( \frac{i^*}{n} \right) \right| = \left| \frac{\varphi_{j_0}Z_k}{a_n} - \frac{X_j}{a_n} \right| \leq \frac{\varphi_{j_0}Z_k}{a_n} + \frac{|X_j|}{a_n} < \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4},
\]

which is in contradiction with (3.19). The same conclusion follows if \( j_0 \) is replaced by \( i_0 \). On the other hand, if \( k + q > i^* \), then let \( s_0 \) be as in (ii). Then, when \( W_n((i^* + s_0)/n) = X_j/a_n \), we similarly obtain a contradiction with (3.20). Alternatively, when \( W_n((i^* + s_0)/n) = X_p/a_n \) for some \( p \in \{i^*, \ldots, i^* + s_0\} \), in the same manner as in (ii), we get a contradiction. Thus this case cannot happen.

**Case B.** There exists \( l_1 \in \{0, \ldots, q\} \) such that \( |Z_{j-l_1}|/a_n > \delta/[4(q + 1)\gamma] \) and \( |Z_{j-l_1}|/a_n \leq \delta/[4(q + 1)\gamma] \) for every \( l \in \{0, \ldots, q\} \setminus \{l_1\} \). First, assume that \( k + q \leq i^* \). Here we analyze only what happens on the event \( \{Z_k > 0\} \) (the event \( \{Z_k < 0\} \) can be treated analogously and is therefore omitted). Then

\[
\frac{X_j}{a_n} > \frac{X_{k+j_0}}{a_n} = \frac{\varphi_{j_0}Z_k}{a_n} + F_2,
\]

(3.28)
where \( j_0 \) and \( F_2 \) are as in (i) with \( |F_2| < \delta/4 \). Write
\[
\frac{X_j}{a_n} = \frac{\varphi_l, Z_{j-l}}{a_n} + \sum_{s=0}^{q} \frac{\varphi_s Z_{j-s}}{a_n} =: \frac{\varphi_l, Z_{j-l}}{a_n} + F_3.
\]

Similarly as before, we obtain \( |F_3| < \delta/4 \). Since
\[
W_n^Z\left(\frac{i^*}{n}\right) \geq \frac{|Z_{j-l}|}{a_n} (\varphi_+ 1_{\{Z_{j-l} > 0\}} + \varphi_- 1_{\{Z_{j-l} < 0\}}) \geq \frac{\varphi_l, Z_{j-l}}{a_n},
\]
we have
\[
\frac{\varphi_{j_n} Z_k}{a_n} = \frac{|Z_k|}{a_n} (\varphi_+ 1_{\{Z_k > 0\}} + \varphi_- 1_{\{Z_k < 0\}}) = W_n^Z\left(\frac{i^*}{n}\right) \geq \frac{\varphi_l, Z_{j-l}}{a_n},
\]
which yields
\[
\frac{\varphi_{j_n} Z_k}{a_n} - \frac{X_j}{a_n} \geq \frac{\varphi_l, Z_{j-l}}{a_n} - \frac{X_j}{a_n} = -F_3.
\]
Relations (3.28) and (3.29) yield
\[
-\left( |F_2| + |F_3| \right) \leq -F_3 \leq \frac{\varphi_{j_n} Z_k}{a_n} - \frac{X_j}{a_n} \leq -F_2 \leq |F_2| + |F_3|,
\]
that is,
\[
\left| W_n^Z\left(\frac{i^*}{n}\right) - W_n\left(\frac{i^*}{n}\right) \right| \leq \left| \frac{\varphi_{j_n} Z_k}{a_n} - \frac{X_j}{a_n} \right| \leq |F_2| + |F_3| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2},
\]
which is in contradiction with (3.19). Alternatively, assume that \( k + q > i^* \) and let \( s_0 \) be as in (ii). If \( W_n((i^* + s_0)/n) = X_j/a_n \), then we similarly obtain a contradiction with (3.20), and if \( W_n((i^* + s_0)/n) = X_p/a_n \) for some \( p \in \{i^*, \ldots, i^* + s_0\} \), with the same reasoning as in (ii) we arrive at a contradiction. Hence this case also cannot happen.

**Case C.** There exist \( l_1 \in \{0, \ldots, q\} \) and \( l \in \{0, \ldots, q\} \backslash \{l_1\} \) such that \( |Z_{j-l}|/a_n > \delta/[4(q+1)\gamma] \) and \( |Z_{j-l}|/a_n > \delta/[4(q+1)\gamma] \). In this case, the event \( H_{n,4} \) occurs.

Therefore only Case C is possible, and this yields \( D_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c \subseteq H_{n,4} \). Hence (3.21) holds. By stationarity we have
\[
P(H_{n,1}) \leq (q + 1)P\left(\frac{|Z_1|}{a_n} > \frac{\delta}{4(q + 1)\gamma}\right),
\]
and hence by the regular variation property we observe
\[
\lim_{n \to \infty} P(H_{n,1}) = 0. \tag{3.30}
\]
Similarly,
\[
P(H_{n,2}) \leq 2qP\left(\frac{|Z_1|}{a_n} > \frac{\delta}{4(q + 1)\gamma}\right)
\]
and
\[
\lim_{n \to \infty} P(H_{n,2}) = 0. \tag{3.31}
\]
Since $Z_k$ and $Z_l$ that appear in the formulation of $H_{n,3}$ are independent, it follows that

$$P(H_{n,3}) \leq \frac{2q}{n} \left[ nP \left( \frac{|Z_l|}{a_n} > \frac{\delta}{4(q+1)\gamma} \right) \right]^2,$$

and hence

$$\lim_{n \to \infty} P(H_{n,3}) = 0. \quad (3.32)$$

From the definition of the set $H_{n,4}$ it follows that $k$, $j-l_1$, $j-l$ are all different, which implies that the random variables $Z_k$, $Z_{j-l_1}$, and $Z_{j-l}$ are independent. Using this and stationarity, we obtain

$$P(H_{n,4}) \leq \frac{q(q+1)}{n} \left[ nP \left( \frac{|Z_l|}{a_n} > \frac{\delta}{4(q+1)\gamma} \right) \right]^3,$$

and hence we conclude

$$\lim_{n \to \infty} P(H_{n,4}) = 0. \quad (3.33)$$

Now from (3.21) and (3.30)–(3.33) we obtain $\lim_{n \to \infty} P(D_n) = 0$, and hence (3.18) yields

$$\lim_{n \to \infty} P(Y_n > \delta) = 0. \quad (3.34)$$

It remains to estimate the second term on the right-hand side of (3.17). Let

$$E_n = \{ \exists v \in \Gamma_{W_n} \text{ such that } d(v, z) > \delta \text{ for every } z \in \Gamma_{W_n^Z} \}.$$

Then by the definition of $T_n$

$$\{ T_n > \delta \} \subseteq E_n. \quad (3.35)$$

On the event $E_n$, we have $d(v, \Gamma_{W_n^Z}) > \delta$. Interchanging the roles of the processes $W_n$ and $W_n^Z$, in the same way as before, for the event $D_n$, we can show that

$$\left| W_n^Z \left( \frac{i^* - l}{n} \right) - W_n \left( \frac{i^*}{n} \right) \right| > \delta \quad (3.36)$$

for all $l = 0, \ldots, q$ (such that $i^* - l \geq 0$), where $i^* = \lfloor nt_v \rfloor$ or $i^* = \lfloor nt_v \rfloor - 1$, and $v = (t_v, x_v)$.

Now we want to show that $E_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c \subseteq H_{n,4}$ and hence assume that the event $E_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c$ occurs. Since (3.36) (for $l = 0$) is in fact (3.19), repeating the arguments used for $D_n$, we conclude that (3.25) holds. Here we also claim that $W_n(i^*/n) = X_j/a_n$ for some $j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k+q\}$. Hence assume that this is not the case, that is, $W_n(i^*/n) = X_j/a_n$ for some $j \in \{k, \ldots, k+q\}$ (with $j \leq i^*$). We can repeat the arguments from (i) to conclude that $k + q \leq i^*$ is not possible. It remains to see what happens when $k + q > i^*$. Let

$$W_n \left( \frac{i^* - q}{n} \right) = \frac{|Z_s|}{a_n} (\varphi + 1_{\{Z_s > 0\}} + \varphi - 1_{\{Z_s < 0\}})$$

for some $s \in \{1, \ldots, i^* - q\}$. Note that $i^* - q \geq 1$ since $q + 1 \leq k \leq i^*$. We distinguish two cases.

(a) $W_n^Z(i^*/n) > W_n(i^*/n)$. In this case, the definition of $i^*$ implies that $W_n(i^*/n) \leq x_v \leq W_n^Z(i^*/n)$. Since $|t_v - (i^* - q)/n| < (q+1)/n \leq \delta$, from $d(v, \Gamma_{W_n^Z}) > \delta$ we conclude that

$$\tilde{d} \left( \left[ W_n^Z \left( \frac{i^* - q}{n} \right), W_n \left( \frac{i^*}{n} \right) \right] \right) > \delta.$$
where \( d \) is the Euclidean metric on \( \mathbb{R} \). This yields

\[
W_n Z \left( \frac{i^* - q}{n} \right) > W_n \left( \frac{i^*}{n} \right),
\]

and from (3.36) we obtain

\[
W_n Z \left( \frac{i^* - q}{n} \right) > W_n \left( \frac{i^*}{n} \right) + \delta. \tag{3.37}
\]

From this, taking into account relation (3.24), we obtain

\[
\frac{|Z_s|}{a_n} \geq \frac{1}{\gamma} W_n \left( \frac{i^*}{n} \right) > \frac{1}{\gamma} \left( -\frac{\delta}{4} + \delta \right) = \frac{3\delta}{4\gamma} > \frac{\delta}{4(q + 1)\gamma},
\]

and since \( H_{n,3}^c \) occurs, it follows that

\[
\frac{|Z_l|}{a_n} \leq \frac{\delta}{4(q + 1)\gamma} \quad \text{for every } l \in \{s - q, \ldots, s + q\} \setminus \{s\}. \tag{3.38}
\]

Let \( p_0 \in \{0, \ldots, q\} \) be such that \( \varphi_{p_0} Z_s = |Z_s| (\varphi_+ 1_{\{Z_s > 0\}} + \varphi_- 1_{\{Z_s < 0\}}) \). Since \( s + p_0 \leq i^* \), we have that

\[
\frac{X_j}{a_n} = W_n \left( \frac{i^*}{n} \right) \geq \frac{X_{s + p_0}}{a_n} = \frac{\varphi_{p_0} Z_s}{a_n} + F_4, \tag{3.39}
\]

where

\[
F_4 = \sum_{m=0}^{q} \frac{\varphi_m Z_{s + p_0 - m}}{a_n}.
\]

From (3.37) and (3.39) we obtain

\[
\frac{\varphi_{p_0} Z_s}{a_n} > \frac{X_j}{a_n} + \delta \geq \frac{\varphi_{p_0} Z_s}{a_n} + F_4 + \delta,
\]

that is, \( \delta < -F_4 \). However, this is not possible since by (3.38) \( |F_4| \leq \delta/4 \), and we conclude that this case cannot happen.

(b) \( W_n Z (i^* / n) \leq W_n (i^* / n) \). Then from (3.36) we get

\[
W_n \left( \frac{i^* + s_0}{n} \right) \geq W_n \left( \frac{i^*}{n} \right) \geq W_n Z \left( \frac{i^*}{n} \right) + \delta, \tag{3.40}
\]

where \( s_0 \in \{1, \ldots, q\} \) is such that \( i^* + s_0 = k + q \). Hence

\[
\left| W_n Z \left( \frac{i^*}{n} \right) - W_n \left( \frac{i^* + s_0}{n} \right) \right| > \delta,
\]

and repeating the arguments from (ii), we conclude that this case also cannot happen.

Thus we have proved that \( W_n (i^* / n) = X_j / a_n \) for some \( j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\} \). Similarly as before, we can prove now that Cases A and B cannot happen (when \( k + q > i^* \), we also use the arguments

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from (a) and (b)), which means that only Case C is possible. In that case the event $H_{n,4}$ occurs, and thus we have proved that $E_n \cap (H_{n,1} \cup H_{n,2} \cup H_{n,3})^c \subseteq H_{n,4}$. Hence

$$E_n \subseteq H_{n,1} \cup H_{n,2} \cup H_{n,3} \cup H_{n,4},$$

and from (3.30)–(3.33) we obtain $\lim_{n \to \infty} P(E_n) = 0$. Therefore (3.35) yields

$$\lim_{n \to \infty} P(T_n > \delta) = 0.$$  \hfill (3.41)

From (3.17), (3.34), and (3.41) we obtain (3.16) and conclude that $L_n \overset{d}{\to} (\beta V, W)$ in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology.

Therefore we proved the theorem for finite-order moving-average processes. Using this, we will now obtain the functional convergence of $L_n$ for infinite-order moving averages. Let $X_i = \sum_{j=0}^{\infty} \varphi_j Z_{i-j} \quad i \in \mathbb{Z}$, and put

$$\lambda = \begin{cases} \varphi_+ \wedge \varphi_- & \text{if } \varphi_+ > 0 \text{ and } \varphi_- > 0, \\ \varphi_+ & \text{if } \varphi_- = 0, \\ \varphi_- & \text{if } \varphi_+ = 0. \end{cases}$$

Since $\sum_{i=0}^{\infty} |\varphi_i| < \infty$, for large $q \in \mathbb{N}$, we have $\sum_{i=q}^{\infty} |\varphi_i| < \lambda$. Fix such $q$ and define

$$X_i^q = \sum_{j=0}^{q-1} \varphi_j Z_{i-j} + \varphi_q' Z_{i-q}, \quad i \in \mathbb{Z},$$

where $\varphi_q' = \sum_{i=q}^{\infty} \varphi_i$, and

$$V_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i^q - b_n}{a_n}, \quad W_{n,q}(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i^q}{a_n}, \quad t \in [0, 1],$$

where the sequence $(a_n)$ satisfies (1.2), and

$$b_n = \begin{cases} 0, & \text{if } \alpha \in (0, 1], \\ \beta \mathbb{E}(Z_1) & \text{if } \alpha \in (1, 2). \end{cases}$$

The coefficients $\varphi_0, \ldots, \varphi_{q-1}, \varphi_q'$ satisfy relation (3.14), and from the definition of $\lambda$ it follows that

$$\max \{ \varphi_j \vee 0 : j = 0, \ldots, q-1 \} \vee (\varphi_q' \vee 0) = \varphi_+$$

and

$$\max \{ -\varphi_j \vee 0 : j = 0, \ldots, q-1 \} \vee (-\varphi_q' \vee 0) = \varphi_-.$$

Therefore for the finite-order moving average process $(X_i^q)_i$, we have

$$L_{n,q} := (V_{n,q}, W_{n,q}) \overset{d}{\to} (\beta V, W) \quad \text{as } n \to \infty$$

in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology. If we show that, for every $\epsilon > 0$,

$$\lim_{q \to \infty} \limsup_{n \to \infty} P \left[ d_p(L_{n,q}, L_n) > \epsilon \right] = 0,$$

(3.42)
then by a generalization of Slutsky’s theorem (see [15, Thm. 3.5]) it will follow that $L_n \overset{d}{\to} (\beta V, W)$ as $n \to \infty$ in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology. By the definition of the metric $d_p$ in (2.1) and the fact that the metric $d_{M_2}$ on $D([0, 1], \mathbb{R})$ is bounded above by the uniform metric on $D([0, 1], \mathbb{R})$ it suffices to show that

$$\lim_{q \to \infty} \lim_{n \to \infty} \mathbf{P}\left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) = 0$$

and

$$\lim_{q \to \infty} \lim_{n \to \infty} \mathbf{P}\left( \sup_{0 \leq t \leq 1} |W_{n,q}(t) - W_n(t)| > \epsilon \right) = 0.$$ 

Recalling the definitions, we have

$$\mathbf{P}\left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) \leq \mathbf{P}\left( \sum_{i=1}^{n} \frac{|X_i^q - X_i^1|}{a_n} > \epsilon \right)$$

and

$$\mathbf{P}\left( \sup_{0 \leq t \leq 1} |W_{n,q}(t) - W_n(t)| > \epsilon \right) \leq \mathbf{P}\left( \sqrt[n]{\sum_{i=1}^{n} \frac{X_i^q - X_i^1}{a_n}} > \epsilon \right) \leq \mathbf{P}\left( \sum_{i=1}^{n} \frac{X_i^q - X_i^1}{a_n} > \epsilon \right).$$

In the proof of Theorem 3.1 in [4], it has been shown that

$$\lim_{q \to \infty} \lim_{n \to \infty} \mathbf{P}\left( \sum_{i=1}^{n} \frac{|X_i^q - X_i^1|}{a_n} > \epsilon \right) = 0.$$ 

Hence (3.42) holds, which means that $L_n \overset{d}{\to} (\beta V, W)$ as $n \to \infty$ in $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology. This concludes the proof. $\Box$

**Remark 3.** Theorem 1 gives the functional convergence of the joint stochastic process $L_n$ in the space $D([0, 1], \mathbb{R}^2)$ with weak $M_2$ topology induced by the metric $d_p$ given in (2.1). Since for the second coordinate of $L_n$, that is, the partial maxima process, the functional convergence in fact holds in the stronger $M_1$ topology (see, e.g., [6] and [11]), we can raise the question whether it is possible to obtain a sort of joint convergence of $L_n$ in the $M_2$ topology on the first coordinate and in the $M_1$ topology on the second coordinate. Precisely, does the functional convergence hold in the topology induced by the metric

$$\tilde{d}_p(x, y) = \max\{d_{M_2}(x_1, y_1), d_{M_1}(x_2, y_2)\}$$

for $x = (x_1, x_2), y = (y_1, y_2) \in D([0, 1], \mathbb{R}^2)$? Here $d_{M_1}$ denotes the $M_1$ metric defined by

$$d_{M_1}(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \lor \|u_1 - u_2\|_{[0,1]}: (r_i, u_i) \in \Pi(x_i), i = 1, 2\}$$

for $x_1, x_2 \in D([0, 1], \mathbb{R})$, where $\Pi(x)$ is the set of $M_1$ parametric representations of the completed graph $\Gamma_x$, that is, continuous nondecreasing functions $(r, u)$ mapping $[0, 1]$ onto $\Gamma_x$.

If the space $D([0, 1], \mathbb{R})$ with $M_2$ topology is a Polish space (which to our best knowledge is still an open question; see [7, Rem. 4.1]), we could proceed similarly as in [12], and the answer to the above question would be affirmative.

We follow another approach. Repeating the arguments from the proof of Lemma 2, but with $d_{M_1}$ for the second components of the corresponding processes instead of $d_{M_2}$, we derive immediately that

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The following metric is a complete metric topologically equivalent to $d_p$. To obtain $L_n \overset{d}{\to} (\beta V, W)$ in the same topology, as in the proof of Theorem 1, it remains to show that
\[
\lim_{n \to \infty} P[d_{M_1}(W_n^Z, W_n) > \delta] = 0
\]
for all $\delta > 0$ (compare this relation to (3.16)). We will not pursue it here, since it would presumably require a lot of technical details connected to parametric representation machinery, but instead we will use relation (3.16) and the fact that the second coordinate of $L_n$ refers to nondecreasing functions. By Remark 12.8.1 in [17] the following metric is a complete metric topologically equivalent to $d_{M_1}$:
\[
d^*_p(x_1, x_2) = d_{M_2}(x_1, x_2) + \lambda(\hat{\omega}(x_1, \cdot), \hat{\omega}(x_2, \cdot)),
\]
where $\lambda$ is the Lévy metric on the space of distributions,
\[
\lambda(F_1, F_2) = \inf \{ \epsilon > 0 : F_2(x - \epsilon) - \epsilon \leq F_1(x) \leq F_2(x + \epsilon) + \epsilon \text{ for all } x \},
\]
and
\[
\hat{\omega}(x, z) = \begin{cases} \omega(x, e^z), & z < 0, \\ \omega(x, 1), & z \geq 0, \end{cases}
\]
with
\[
\omega(x, \delta) = \sup_{0 \leq t \leq 1} \sup_{0 \vee (t - \delta) \leq t_1, t_2 < t_3 \leq (t + \delta) \wedge 1} \{ \| x(t_2) - [x(t_1), x(t_3)] \| \}
\]
for $x \in D([0, 1], \mathbb{R})$ and $\delta > 0$. Here $\| z - A \|$ denotes the distance between a point $z$ and a subset $A \subseteq \mathbb{R}$.

Since $W_n$ and $W_n^Z$ are nondecreasing, for $t_1 < t_2 < t_3$, we have that $\| W_n(t_2) - [W_n(t_1), W_n(t_3)] \| = 0$, which yields $\omega(W_n, \delta) = 0$ for all $\delta > 0$, and similarly $\omega(W_n^Z, \delta) = 0$. Hence $\lambda(W_n^Z, W_n) = 0$ and $d^*_p(W_n^Z, W_n) = d_{M_2}(W_n^Z, W_n)$. Now from (3.16) we obtain
\[
\lim_{n \to \infty} P[d^*_p(W_n^Z, W_n) > \delta] = 0
\]
and conclude that $L_n$ converges in distribution to $(\beta V, W)$ in the topology induced by the metric
\[
d^*_p(x, y) = \max \{ d_{M_2}(x_1, y_1), d^*_p(x_2, y_2) \}
\]
for $x = (x_1, x_2), y = (y_1, y_2) \in D([0, 1], \mathbb{R}^2)$, that is, in the $M_2$ topology on the first coordinate of $L_n$ and in the $M_1$ topology on the second coordinate.

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