A lattice Maxwell system with discrete space-time symmetry and local energy-momentum conservation

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A lattice Maxwell system is developed with gauge-symmetry, symplectic structure and discrete space-time symmetry. Noether’s theorem for Lie group symmetries is generalized to discrete symmetries for the lattice Maxwell system. As a result, the lattice Maxwell system is shown to admit a discrete local energy-momentum conservation law corresponding to the discrete space-time symmetry. These conservative properties make the discrete system an effective algorithm for numerically solving the governing differential equations on continuous space-time. Moreover, the lattice model, respecting all conservation laws and geometric structures, is as good as and probably more preferable than the continuous Maxwell model. Under the simulation hypothesis by Bostrom1 and in consistent with the discussion on lattice QCD by Beane et al.2, the two interpretations of physics laws on space-time lattice could be essentially the same.

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Maxwell’s equations are partial differential equations (PDEs) governing the dynamics of electromagnetic field on space-time. In this paper, we propose a lattice Maxwell system with gauge-symmetry, symplectic structure, discrete space-time symmetry and discrete local energy-momentum conservation. The electromagnetic field is only defined on a space-time lattice with a set of discrete rules governing its dynamics. A lattice Maxwell system can be interpreted in two ways. First, it can be viewed as a model for the electromagnetic field by itself. Second, it can be treated as a numerical algorithm of the continuous Maxwell’s equations. For both purposes, it is desirable for the lattice Maxwell system to conserve fundamental physics quantities locally, such as energy-momentum, and preserve important structures, such as space-time symmetry, gauge-symmetry and symplectic structure. Especially for the first purpose, the conservation and structure preserving properties are indispensable. One can argue that a lattice Maxwell system which respects all the important conservation laws and structures is a valid model for the electromagnetic field. It is as good as the continuous Maxwell’s equations, but with an added advantage that it can be easily calculated. For the second purpose, i.e., using the lattice Maxwell system as an algorithm, the conservative and structure-preserving properties of the algorithm, will render more reliable numerical solutions of the continuous Maxwell’s equations. This is especially true for simulations of long-term multi-scale dynamics. A well-known example in this respect is Yee’s algorithm\(^3\) and its generalizations using discrete exterior calculus (DEC). The outstanding performance of Yee’s algorithm is attributed to the fact that it preserves the differential form structure of the Maxwell system\(^4,5\). We emphasize that a local energy-momentum conservation law for numerical algorithms is a much more desirable property than global energy-momentum conservation law. This is because a local conservation law requires the numerical solutions satisfy the law at every space-time grid point, and the number of constraints is as many as the space-time grid points. For a global conservation law, there is only one conserved quantity, and this is not a strong constraint for the system which contains a large number of degrees of freedom defined on the space-time lattice.

Under the simulation hypothesis by Bostrom\(^1\) and in consistent with the discussion on lattice QCD by Beane et al.\(^2\), the two interpretations of physics laws on space-time lattice could be essentially the same.

How to design a lattice Maxwell system with these desirable conservative and structure-preserving properties? It is natural and probably necessary to adopt a field theoretical
approach. In this paper, the lattice Maxwell system with gauge symmetry, symplectic structure, discrete space-time symmetry and discrete local energy-momentum conservation is constructed by discretizing the Lagrangian of the electromagnetic field on a space-time lattice.

For continuous systems, Noether’s theorem connects Lie group symmetries and conservation laws. For example, energy-momentum conservation is the consequence of space-time symmetry. However, for discrete systems, such a connection has not been fully understood. Even though Noether’s theorem has been applied to discrete systems, the symmetries in these applications are continuous Lie groups, as required by the Noether’s theorem. In this paper, we demonstrate a generalized version of Noether’s theorem which establishes connections between discrete symmetries and discrete conservation laws for the lattice Maxwell system. As far as we know, such a generalized Noether’s theorem for discrete symmetries has not be discussed in the existing literature. We then show that the lattice Maxwell system admits a discrete space-time symmetry, and more importantly that the discrete space-time symmetry induces a discrete local energy-momentum conservation law.

We should mention that there are other methods to construct numerical schemes with conservation properties for partial differential equations, e.g., the finite volume method and the discontinuous Galerkin method. In particular, Poynting’s theorem for Yee’s algorithm has been discussed. However, these studies are only on the level of numerical algorithms, and are not related to the fundamental properties of discrete space-time symmetries.

The lattice Maxwell system also admits a gauge symmetry and conserves a finite dimensional symplectic structure. As in the continuous case, the gauge symmetry on the lattice generates a discrete charge conservation law. The conservation of the symplectic structure inherited from the variational structure guarantees the conservation of phase-space volume and bounds long-term errors in energy. Since these topics have been well studied, they will not be the main focus of the present paper.

We start our discussion from the action of the electromagnetic field on the continuous space-time,

$$\mathcal{A}[\mathbf{A}] = \int \! dt \! dx \mathcal{L}(\mathbf{A}, \phi) ,$$

$$\mathcal{L}(\mathbf{A}, \phi) = \frac{1}{2} \left( (\nabla \phi - \dot{\mathbf{A}})^2 - (\nabla \times \mathbf{A})^2 \right) ,$$

where $\mathcal{L}$ is the Lagrangian density. For simplicity, permeability $\mu_0$ and permittivity $\varepsilon_0$ are
set to 1. The Euler-Lagrange equations for $A, \phi$ are

$$\begin{align*}
\frac{\partial}{\partial t} \left( \dot{A} + \nabla \phi \right) - \nabla \times \nabla \times A &= 0, \\
\nabla \cdot \left( \dot{A} + \nabla \phi \right) &= 0.
\end{align*}$$

(3) (4)

We select a “cubic” lattice in space-time. The discrete Lagrangian density is chosen to be

$$\mathcal{L}_{J,l} = \frac{1}{2} (\text{Dt} (A_{J,l-1}) - \nabla_d \phi_{J,l})^2 - \frac{1}{2} (\text{curl}_d A_{J,l})^2,$$

(5) $\text{Dt} (f_l) \equiv \frac{f_{l+1} - f_l}{\Delta t},$

(6)

where $J$ is the spatial grid indices $[i, j, k]$, and $l$ is the temporal index, and $\nabla_d$ and $\text{curl}_d$ are discrete gradient and discrete curl operators, respectively. They are defined by Eqs. (A1) and (A2) in the Appendix. The discrete action is the sum of the discrete Lagrangian density,

$$\mathcal{A}_d[A, \phi] = \sum_{J,l} \mathcal{L}_{J,l} \Delta t \Delta x^3.$$  

(7)

The Lagrangian density $\mathcal{L}_{J,l}$ admits a gauge symmetry, meaning that the following transformation

$$\begin{align*}
A_{J,l} &\rightarrow A_{J,l} + \nabla_d \psi_{J,l}, \\
\phi_{J,l} &\rightarrow \phi_{J,l} - \text{Dt}^* (\psi_{J,l}),
\end{align*}$$

(8) (9)

will leave $\mathcal{L}_{J,l}$ invariant. Here, $\text{Dt}^*$ is the dual operator of $\text{Dt}$,

$$\text{Dt}^* (f_l) \equiv \frac{f_l - f_{l-1}}{\Delta t}.$$ 

(10)

We note that this symmetry is defined on the space-time lattice. However, the symmetry group itself is continuous and forms a Lie group. Due to the gauge symmetry, we can choose a discrete gauge to simplify the calculation. In the present study, the temporal gauge is adopted, i.e., $\phi_{J,l} = 0$. In this gauge, the discrete Euler-Lagrange equation for $A_{J,l}$ is

$$\text{Dt}^* (\text{Dt} (A_{J,l})) + \text{curl}_d^T \text{curl}_d A_{J,l} = 0.$$ 

(10)

Now we introduce the concept of discrete symmetry. A discrete transformation of the lattice field $A_{J,l}$,

$$A_{J,l} \rightarrow A'_{J,l},$$

(11)
is a discrete symmetry, if the resulting variation of the Lagrangian density is at most a discrete 4-divergence of a discrete 4-vector field \( (\mathcal{L}_{t,J,l}, \mathcal{L}_{x,J,l}) \), i.e.,

\[
\Delta L_{J,l} = L_{J,l}(A_{J,l}) - L_{J,l}(A'_{J,l}) = \text{Dt}^* (\mathcal{L}_{t,J,l}) + \text{div}_d^* \mathcal{L}_{x,J,l} .
\]  

(12)

Here, \( \mathcal{L}_{t,J,l} \) and \( \mathcal{L}_{x,J,l} \) are the temporal and spatial components of the 4-vector field, which only depends on the values of discrete vector potential \( A \) near the grid point \( J,l \), e.g., \( A_{J,l}, A_{J,l+1}, A_{i,j-1,k,l}, \ldots \). The operator \( \text{div}_d^* \) is defined in Eq. (A6). This definition of discrete symmetry is similar to that of continuous (Lie group) symmetry in Noether’s theorem. We emphasize again that the gauge symmetry of Eqs. (8) and (9) is a continuous Lie group symmetry, instead of a discrete symmetry. We now show that for the lattice Maxwell system proposed here, a discrete symmetry will induce a discrete local conservation law. This result can be viewed as a discrete generalization of Noether’s theorem. However, the fundamental difference is that the symmetry group in the current context is a discrete group, instead of a Lie group for the standard Noether’s theorem.

To prove this fact, we first calculate the change of the Lagrangian density due to the transformation (11),

\[
\Delta L_{J,l} = \frac{1}{2} \text{Dt} \left( (A_{J,l-1})^2 - \frac{1}{2} (\text{curl}_d A_{J,l})^2 \right) - \frac{1}{2} \text{Dt} \left( (A'_{J,l-1})^2 + \frac{1}{2} (\text{curl}_d A'_{J,l})^2 \right) \\
= \text{Dt} \left( A^\dagger_{J,l-1} \right) \cdot \text{Dt} \left( \left( A_{J,l-1} - A'_{J,l-1} \right) \right) - \text{curl}_d A^\dagger_{J,l} \cdot \text{curl}_d \left( A_{J,l} - A'_{J,l} \right) ,
\]  

(13)

where

\[
A^\dagger_{J,l} \equiv \frac{1}{2} \left( A_{J,l} + A'_{J,l} \right) .
\]  

(15)

Note that \( \Delta L_{J,l} \) cannot be made arbitrarily small as in the case of Lie group symmetries in the standard Noether’s theorem. Similar to Eq. (10), the Euler-Lagrange equation for \( A'_{J,l} \) derived from \( \mathcal{L}_{J,l}(A'_{J,l}) \) is

\[
\text{Dt}^* \left( \text{Dt} \left( A'_{J,l} \right) \right) + \text{curl}_d^T \text{curl}_d A'_{J,l} = 0 .
\]  

(16)

The combination of Eq. (14) + (Eq. (10) + Eq. (16)) \( \cdot \frac{1}{2} \left( A_{J,l} - A'_{J,l} \right) \) gives

\[
\Delta L_{J,l} = \text{Dt} \left( A^\dagger_{J,l-1} \right) \cdot \text{Dt} \left( \left( A_{J,l-1} - A'_{J,l-1} \right) \right) - \text{curl}_d A^\dagger_{J,l} \cdot \text{curl}_d \left( A_{J,l} - A'_{J,l} \right) \\
+ \text{Dt}^* \left( \text{Dt} \left( A^\dagger_{J,l} \right) \right) \cdot \left( A_{J,l} - A'_{J,l} \right) + \left( \text{curl}_d^T \text{curl}_d A^\dagger_{J,l} \right) \cdot \left( A_{J,l} - A'_{J,l} \right) \\
= \text{Dt}^* \left( \text{Dt} \left( A^\dagger_{J,l} \right) \cdot \left( A_{J,l} - A'_{J,l} \right) \right) - \text{div}_d^* \left( \left( A_{J,l} - A'_{J,l} \right) \ast \text{curl}_d A^\dagger_{J,l} \right) .
\]  

(17)

(18)
In deriving Eq. (18), use has been made of the operator identity Eq. (A18) in the Appendix. Now, if the discrete transformation \( A_{J,l} \rightarrow A'_{J,l} \) is a discrete symmetry, then by definition \( \Delta L_{J,l} \) is the discrete 4-divergence of a discrete 4-vector \( (L_{t,J,l}, L_{x,J,l}) \),

\[
\Delta L_{J,l} = D^* t (L_{t,J,l}) + \text{div}_d^* L_{x,J,l} .
\] (19)

Then, Eqs. (18) and (19) can be combined to give a discrete local conservation law,

\[
D^* t \left( D^* t (A^\dagger_{J,l}) \cdot (A_{J,l} - A'_{J,l}) - L_{t,J,l} \right) - \text{div}_d^* \left( (A_{J,l} - A'_{J,l}) \times \text{curl}_d A^\dagger_{J,l} + L_{x,J,l} \right) = 0 .
\] (20)

This completes the proof of the generalized Noether’s theorem for discrete symmetries of the lattice Maxwell system.

In the present study, we will consider the following spatial and the temporal translation symmetries,

\[
A_{J,l} \rightarrow A_{J-x,l} ,
\] (21)

\[
A_{J,l} \rightarrow A_{J-y,l} ,
\] (22)

\[
A_{J,l} \rightarrow A_{J-z,l} ,
\] (23)

\[
A_{J,l} \rightarrow A_{J,l+1} ,
\] (24)

where \( J - x, J - y, J - z \) denote \([i - 1, j, k], [i, j - 1, k], [i, j, k - 1], \) respectively. First, let’s look at the symmetry in the Eq. (21), which is the discrete translation symmetry in the \( e_x \)-direction. In this case, the change of Lagrangian density is

\[
\Delta L_{J,l} = \frac{1}{2} \left( D^* t (A_{J,l+1})^2 - D^* t (A_{J-x,l} - 1)^2 - \text{curl}_d A_{J,l}^2 + (\text{curl}_d A_{J-x,l})^2 \right) ,
\] (25)

which is the discrete 4-divergence of the following discrete 4-vector

\[
L_{t,J,l} = 0 ,
\] (26)

\[
L_{x,J,l} = \left[ \frac{1}{2} D^* t (A_{J,l+1})^2 - \frac{1}{2} (\text{curl}_d A_{J,l})^2 , 0, 0 \right] .
\] (27)

This verifies that Eq. (21) is indeed a discrete symmetry. Therefore, according to Eq. (20), we obtain the following discrete conservation law,

\[
D^* t \left( D^* t \left( A^{x/2}_{J,l} \right) \cdot (A_{J,l} - A_{J-x,l}) \right) - \text{div}_d^* \left( (A_{J,l} - A_{J-x,l}) \times \text{curl}_d A^{x/2}_{J,l} + \left[ \frac{1}{2} D^* t (A_{J,l+1})^2 - \frac{1}{2} (\text{curl}_d A_{J,l})^2 , 0, 0 \right] \right) = 0 .
\] (28)
Here,
\[
A_{J,l}^{x/2} \equiv \frac{A_{J,l} + A_{J-x,l}}{2}.
\] (29)

Equation (28) is the discrete local momentum conservation law in the \(e_x\)-direction. It can be transformed into a familiar form expressed in terms of \(E\) and \(B\). Taking a discrete divergence \(\text{div}_{\text{d}}^*\) of Eq. (10) and using the discrete operator identity Eq. (A16), we can see that
\[
\text{Dt}^* (\text{div}_{\text{d}}^* (\text{Dt} (A_{J,l}))) = 0 .
\] (30)

If initially \(\text{div}_{\text{d}}^* (\text{Dt} (A_{J,0}))\) is zero, which will be automatically satisfied when there is no charge in the space, then it remains zero all time. Therefore,
\[
\text{div}_{\text{d}}^* (\text{Dt} (A_{J,l})) = 0 ,
\] (31)

The electromagnetic fields on the lattice are defined as
\[
E_{J,l} = -\text{Dt} (A_{J,l}) ,
\] (32)
\[
B_{J,l} = (\text{curl}_{\text{d}} A)_{J,l} .
\] (33)

In terms of \(E_{J,l}\) and \(B_{J,l}\), the discrete Maxwell equations are
\[
\text{Dt}^* (E_{J,l}) = (\text{curl}_{\text{d}}^T B)_{J,l} ,
\] (34)
\[
\text{Dt} (B_{J,l}) = -(\text{curl}_{\text{d}} E)_{J,l} .
\] (35)

With the help of Eq. (31) and discrete operator identities (A19) and (A20), Eq. (28) can be transformed into
\[
\text{Dt}^* (E_{J,l}^{x/2} \times B_{J,l})_x + \text{div}_{\text{d}}^* (\text{Dt}^* (E_{J,l}^{x/2}) \otimes A_{x,J,l} - (E_{J,l-1}^{x/2} \otimes E_{x,J,l-1})) - \text{div}_{\text{d}}^* (-(A_{J,l} - A_{J-x,l}) \times \text{curl}_{\text{d}} A_{J,l}^{x/2} - \frac{1}{2} E_{J,l-1}^2 - \frac{1}{2} (\text{curl}_{\text{d}} A_{J,l})^2 , 0, 0)_x = 0 .
\] (36)

In deriving Eq. (36), use is also made of
\[
\text{Dt}^* (\text{div}_{\text{d}}^* (E_{J,l}^{x/2} \otimes A_{x,J,l})) = \text{div}_{\text{d}}^* (\text{Dt}^* (E_{J,l}^{x/2}) \otimes A_{x,J,l} - (E_{J,l-1}^{x/2} \otimes E_{x,J,l-1})) .
\] (37)

The second term in Eq. (36) can be simplified by identities (A21) and (A16),
\[
\text{div}_{\text{d}}^* (\text{Dt}^* (E_{J,l}^{x/2} \otimes A_{x,J,l}) = \text{div}_{\text{d}}^* (-\nabla_d A_{x,J,l} \times E_{J,l}^{x/2} ) .
\] (38)
Finally, using identity \((A22)\), the discrete local momentum conservation law in the \(e_x\)-direction in terms of \(E_{J,l}\) and \(B_{J,l}\) is

\[
Dt^* \left( E_{J,l}^{x/2} \times *B_{J,l} \right)_x + \text{div}^*_d \left[ \frac{1}{2} E_{J,l-1} \cdot E_{J,l-1}, 0, 0 \right] - \text{div}^*_d \left( E_{J,l}^{x/2} \otimes E_x \right)_{J,l-1} + \text{div}^*_d \left[ \frac{B_{J-x,l} \cdot B_{J,l}}{2}, 0, 0 \right] - \text{div}^*_d \left( B \otimes *B_{x/2}^{x/2} \right)_{J,l} = 0 .
\]

(39)

It is straightforward to verify that it recovers the familiar momentum conservation law in the continuous space-time when the grid-size goes to zero. The discrete local momentum conservation law in the \(e_y\) or \(e_z\) direction can be obtained in a similar way.

Next, we look at the discrete local energy conservation due to the discrete temporal symmetry specified by Eq. \((24)\). In this case, the change of the Lagrangian density in terms of \(E_{J,l}\) and \(B_{J,l}\) is

\[
\Delta L_{J,l} = \frac{1}{2} \left( E_{J,l-1}^{2} - E_{J,l-2}^{2} - B_{J,l}^{2} + B_{J,l-1}^{2} \right) ,
\]

(40)

and it is the discrete 4-divergence of the discrete 4-vector field

\[
L_{t,J,l} = \frac{1}{2} \left( E_{J,l-1}^{2} - B_{J,l}^{2} \right) ,
\]

(41)

\[
L_{x,J,l} = [0, 0, 0] .
\]

(42)

This verifies that \(A_{J,l} \rightarrow A_{J,l+1}\) is a discrete symmetry. According to Eq. \((20)\), the local discrete conservation law in terms of \(E_{J,l}\) and \(B_{J,l}\) for this discrete symmetry is

\[
Dt^* \left( \frac{1}{2} E_{J,l} \cdot E_{J,l-1} + \frac{1}{2} B_{J,l}^{2} \right) + \text{div}^*_d \left( \frac{B_{J,l} + B_{J,l-1}}{2} \right) = 0 .
\]

(43)

Of course, this is the discrete local energy conservation law. It recovers the well-known continuous local energy conservation law, a.k.a. Ponyting’s theorem, when the grid-size approaches zero.

In conclusion, we reported three important advances in the study of lattice model and structure-preserving geometric algorithm for the Maxwell system. (i) A lattice Maxwell system is developed with gauge symmetry, symplectic structure, and discrete space-time symmetry. (ii) Noether’s theorem is generalized to the case of discrete symmetry for the lattice Maxwell system, which establishes the correspondence between discrete symmetries and discrete local conservation laws. (iii) Applying the discrete Noether’s theorem, the lattice Maxwell system is shown to admit a discrete local energy-momentum conservation law.
Appendix A: Discrete difference operators and identities

In this appendix, we list the definitions of discrete operators and identities that are needed in the present study.

The following are definitions of discrete operators.

\[
(\nabla_d \phi)_{i,j,k} = \left[ \phi_{i+1,j,k} - \phi_{i,j,k}, \phi_{i,j+1,k} - \phi_{i,j,k}, \phi_{i,j,k+1} - \phi_{i,j,k} \right], \quad (A1)
\]

\[
(\text{curl}_d \mathbf{A})_{i,j,k} = \begin{bmatrix}
(A_{z,i,j+1,k} - A_{z,i,j,k}) - (A_{y,i,j,k+1} - A_{y,i,j,k}) \\
(A_{x,i,j,k+1} - A_{x,i,j,k}) - (A_{z,i+1,j,k} - A_{z,i,j,k}) \\
(A_{y,i,j+1,k} - A_{y,i,j,k}) - (A_{x,i,j+1,k} - A_{x,i,j,k})
\end{bmatrix}^T, \quad (A2)
\]

\[
(\text{div}_d \mathbf{B})_{i,j,k} = (B_{x,i+1,j,k} - B_{x,i,j,k}) + (B_{y,i,j+1,k} - B_{y,i,j,k})
+ (B_{z,i,j,k+1} - B_{z,i,j,k}), \quad (A3)
\]

\[
(\nabla_d^* \phi)_{i,j,k} = \left[ \phi_{i,j,k} - \phi_{i-1,j,k}, \phi_{i,j,k} - \phi_{i,j,k-1}, \phi_{i,j,k} - \phi_{i,j,k+1} \right], \quad (A4)
\]

\[
(\text{curl}_d^T \mathbf{A})_{i,j,k} = \begin{bmatrix}
(A_{z,i,j,k} - A_{z,i,j-1,k}) - (A_{y,i,j,k} - A_{y,i,j,k-1}) \\
(A_{x,i,j,k} - A_{x,i,j,k-1}) - (A_{z,i,j,k} - A_{z,i,j-1,k}) \\
(A_{y,i,j,k} - A_{y,i,j-1,k}) - (A_{x,i,j,k} - A_{x,i,j-1,k})
\end{bmatrix}^T, \quad (A5)
\]

\[
(\text{div}_d^* \mathbf{B})_{i,j,k} = (B_{x,i,j,k} - B_{x,i-1,j,k}) + (B_{y,i,j,k} - B_{y,i,j-1,k})
+ (B_{z,i,j,k} - B_{z,i,j,k-1}), \quad (A6)
\]

\[
(M \times \mathbf{B})_{i,j,k} = \begin{bmatrix}
M_{y,i+1,j,k}B_{z,i,j,k} - M_{z,i+1,j,k}B_{y,i,j,k} \\
M_{x,i,j+1,k}B_{z,i,j,k} - M_{z,i,j+1,k}B_{x,i,j,k} \\
M_{x,i,j,k}B_{y,i,j+1,k} - M_{y,i,j,k}B_{x,i,j,k}
\end{bmatrix}^T, \quad (A7)
\]

\[
(M \times 2 \mathbf{B})_{i,j,k} = \begin{bmatrix}
M_{y,i,j,k}B_{z,i,j,k} - M_{z,i,j,k}B_{y,i,j,k}
M_{x,i-1,j+1,k}B_{x,i,j,k} - M_{x,i-1,j+1,k}B_{x,i,j,k}
M_{x,i-1,j,k+1}B_{y,i,j,k} - M_{y,i-1,j,k+1}B_{x,i,j,k}
\end{bmatrix}^T, \quad (A9)
\]

\[
(M \times \mathbf{B})_{i,j,k} = \begin{bmatrix}
M_{y,i,j,k}B_{z,i,j,k} - M_{z,i,j,k}B_{y,i,j,k} \\
M_{x,i,j+1,k}B_{x,i,j,k} - M_{x,i,j+1,k}B_{x,i,j,k} \\
M_{x,i,j,k}B_{y,i,j+1,k} - M_{y,i,j,k}B_{x,i,j,k}
\end{bmatrix}^T, \quad (A8)
\]
\[(M \otimes Q_x)_{i,j,k} = [M_{x,i,j,k} Q_{x,i,j,k}, M_{y,i,j,k} Q_{x,i-1,j,k}, M_{z,i,j,k} Q_{x,i-1,j,k+1}], \quad (A10)\]
\[M_J \otimes_2 Q_{x,J} = [M_{x,J} Q_{x,i-1,j,k+1}, M_{y,J} Q_{x,i,j,k+1}, M_{z,J} Q_{x,i,j,k+1}], \quad (A11)\]
\[(M \otimes^* Q_x)_{i,j,k} = [M_{x,i,j,k} Q_{x,i,j,k}, M_{y,i-1,j,k+1} Q_{x,i,j,k}, M_{z,i-1,j,k+1} Q_{x,i,j,k}], \quad (A12)\]
\[(\ast Q)_{i,j,k} = [Q_{x,i-1,j,k}, Q_{y,i,j-1,k}, Q_{z,i,j,k-1}], \quad (A13)\]

The following are discrete vector analysis identities used in the paper. For any discrete vector fields \(M_J\) and \(Q_J\),

\[\text{curl}_d \nabla_d M_{x,J} = 0, \quad (A14)\]
\[\text{div}_d \text{curl}_d M_J = 0, \quad (A15)\]
\[\text{curl}_d T \nabla_d^* M_{x,J} = 0, \quad (A16)\]
\[\text{div}_d^* \text{curl}_d^T M_J = 0, \quad (A17)\]

\[M_J \cdot \text{curl}_d Q_J - \left(\text{curl}_d^T M\right) \cdot Q_J - \text{div}_d^* (M_J \ast \times Q_J) = 0, \quad (A18)\]
\[(M \times \ast \text{curl}_d Q)_J - (\nabla_d^* Q)_J \cdot M_J + M_J \cdot (\nabla_d^* Q)_J = 0, \quad (A19)\]
\[(\text{div}_d^* (M \otimes Q_x))_J - (M_J \cdot (\nabla_d^* Q))_J + (\text{div}_d^* M)_J (\ast Q)_J = 0, \quad (A20)\]
\[\text{curl}_d^T M_J \otimes Q_{x,J} - \text{curl}_d^T (M_J \otimes_2 Q_{x,J}) + \nabla_d Q_{x,J} \times_2 M_J = 0, \quad (A21)\]
\[(\text{curl}_d Q_J) \otimes^* M_{x,J} - [(\text{curl}_d Q_J) \cdot M_J, 0, 0] + (Q_J - Q_{J-x}) \ast \times M_J - (\nabla_d Q_{x,J}) \times_2 M_J = 0. \quad (A22)\]

We note that in the continuous limit Eqs. (A14) and (A16) both recover the familiar identity \(\nabla \times \nabla f = 0\) for a scalar field \(f\), and Eqs. (A15) and (A17) both recover the identity \(\nabla \cdot \nabla \times M = 0\) for a vector field \(M\). The structures embedded in these discrete vector analysis identities are rich. A general discussion on this topic is beyond the scope of the paper. However, we give a hint on these structures by demonstrating that Eqs. (A14)-(A17) can be derived from two more general identities. Define discrete operators \(D^1_a, D^2_a\) and \(D^3_a\) as follows,
\[ D^1_a \phi(x, y, z) = [\phi(x + a_x, y, z) - \phi(x, y, z), \phi(x, y + a_y, z) - \phi(x, y, z), \phi(x + a_z, y, z) - \phi(x, y, z)] , \]

\[ D^2_a A(x, y, z) = \begin{bmatrix}
(A_x(x + a_x, y, z) - A_x(x, y, z)) - (A_y(x + a_y, z) - A_y(x, y, z)), \\
(A_y(x, y + a_y, z) - A_y(x, y, z)) - (A_z(x + a_z, y, z) - A_z(x, y, z)), \\
(A_z(x + a_z, y, z) - A_z(x, y, z)) - (A_y(x, y + a_y, z) - A_y(x, y, z))
\end{bmatrix} , \]

\[ D^3_a B(x, y, z) = (B_z(x + a_z, y, z) - B_z(x, y, z)) + (B_z(x, y + a_y, z) - B_z(x, y, z)) + \\
(B_z(x, y, z + a_z) - B_z(x, y, z)) , \]

where \( \phi, A \) and \( B \) are any scalar and vector fields on \( \mathbb{R}^3 \), and \( a = [a_x, a_y, a_z] \) is an arbitrary vector in \( \mathbb{R}^3 \). It can be proven that \( D^1_a, D^2_a \) and \( D^3_a \) satisfy the following identities,

\[ D^2_a D^1_a \phi(x, y, z) = 0, \]  \hspace{1cm} (A26)

\[ D^3_a D^2_a A(x, y, z) = 0. \]  \hspace{1cm} (A27)

If we let \( a = [1, 1, 1] \) and \( a = [-1, -1, -1] \), Eqs. (A14)-(A17) are immediately recovered.

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