Majorana Fermions, Supersymmetry Breaking, and Born-Infeld Theory

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Abstract

This review is devoted to highlight some aspects of the relevance of Majorana fermions in rigid supersymmetry breaking in four spacetime dimensions. After introducing some basic facts on spinors, and on their symmetries and reality properties, we consider Goldstino actions describing partial breaking of rigid supersymmetry, then focussing on Born-Infeld non-linear theory, its duality symmetry, and its supersymmetric extensions, also including multi-field generalizations exhibiting doubly self-duality.

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Dedicated to the memory of Guido Altarelli
1 Introduction

The present review is aimed to contribute to the Proceedings of the 53rd Course of the International School of Subnuclear Physics (held in Erice in Summer 2015), which also celebrated the Triumph of Ettore Majorana, the great disappeared Sicilian Physicist whose ideas and invention have permeated both theoretical and experimental physics.

Section 2 is devoted to Majorana spinors, their definition in higher dimensions and their role in Supersymmetry. Example of such particles are the gravitino and the Goldstino in $D = 4$, since the related anticommuting supersymmetry parameters are Majorana spinors. The same holds true in $D = 3, 9, 11$ while in $D = 10$ (as in $D = 2$) the notion of Majorana-Weyl spinor occurs. As an application, the role of Majorana and Dirac neutrino masses in the see-saw mechanism of GUT’s is explained.

In Section 3 some basic features of spontaneously broken rigid supersymmetry and partial supersymmetry breaking are described. The Volkov-Akulov and the Born-Infeld Lagrangians naturally emerge in this framework, and they both contribute to the description of partially broken rigid $\mathcal{N} = 2$ supersymmetry.

The electric-magnetic self duality enjoyed by the Born-Infeld theory is then extended in Section 4, in order to describe more general theories, following the work of Gaillard and Zumino [1]. These
generalizations involve the Schrödinger formulation of Born-Infeld-type theories, forms of higher degree as well as geometrical mass terms in higher dimensions.

2 Majorana Fermions and Supersymmetry

Supersymmetry deals with basic spinorial entities called Majorana spinors. This notion originates from a fundamental observation of the Italian (Sicilian) physicist Ettore Majorana, who noticed that the four-dimensional Dirac equation admits “real” solutions if the $\gamma$-matrices are suitably chosen \cite{2}.

This is the so-called Majorana representation, in which the $\gamma$-matrices are real. Therefore the Dirac equation

$$ (\gamma^\mu \partial_\mu + m) \psi = 0 \quad (1) $$

admits manifestly real solutions.

In the Majorana representation, the $\gamma^i$ are symmetric, while $\gamma^0$ and $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ are antisymmetric, and they satisfy the Clifford algebra relations

$$ \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag} (-, +, +, +). \quad (2) $$

How about chiral spinors? They cannot be real, since they are eigenstates of $\gamma^5$, corresponding to the eigenvalues $\pm i \left( (\gamma^5)^2 = -1 \right)$:

$$ \gamma^5 \psi_\pm = \pm i \psi_\pm, \quad (3) $$

thus implying that\footnote{In the present Section, the upperscript “*” denotes complex conjugation.}

$$ \psi_\pm = \frac{1}{2} \left( 1 \mp i \gamma^5 \right) \psi; \quad (4) $$

$$ \psi_+ + \psi_- = \psi; \quad \psi^*_\pm = \psi_\mp. \quad (5) $$

Spinors which are eigenstates of $\gamma^5$ are called Weyl spinors. From the previous properties, it follows that Weyl spinors have two components instead of four, and they are complex in $D = 4$.

Symmetries of $\gamma$-matrices depend on the dimension $D = s + t$ of spacetime, and the reality properties of spinors on its signature $\rho := |s - t|$ (modulo 8 : Bott periodicity). There are eight cases:

- complex spinors : $\rho = 2, 6$;
- real spinors : $\rho = 0, 1, 7$;
- pseudoreal spinors : $\rho = 3, 4, 5$.

$$ (6) \quad (7) \quad (8) $$
where

\[ \psi \text{ real } \iff \psi^* = \psi; \quad (9) \]
\[ \psi \text{ pseudoreal } \iff (\psi^A)^* = \Omega^{AB} C_{\alpha\beta} \psi^B, \quad (10) \]

with \( \Omega \) and \( C \) respectively denoting the anti-involutive invariant structures of the automorphism group of the supersymmetry algebra and of the spinor space (\( \Omega^2 = -1, C^2 = -1 \)).

For ordinary \( D \)-dimensional Minkowski spacetime \((s,t) = (D-1,1)\), and thus \( \rho = D - 2 \). In this case, it holds that

\[
D \text{ even (mod 8) : } \begin{cases} 
\text{complex spinors : } D = 4, 8; \\
\text{real spinors : } D = 2, 10; \quad \text{applying to Weyl spinors } \psi_\pm. \\
\text{pseudoreal spinors : } D = 6,
\end{cases} \quad (11)
\]

\[
D \text{ odd (mod 8) : } \begin{cases} 
\text{real spinors : } D = 3, 9, 11; \\
\text{pseudoreal spinors : } D = 5, 7.
\end{cases} \quad (12)
\]

For instance, in \( D = 10 \) the Majorana representation is valid for both \( \psi_+ \) and \( \psi_- \), since \( (\gamma^{11})^2 = 1 \); thus, \( \psi_+ \) and \( \psi_- \) in \( D = 10 \)-dimensional ordinary Minkowski spacetime are named \textit{Majorana-Weyl spinors}.

Since we have been mentioning them a few lines above, it is here worth recalling also the anticommutator sectors of supersymmetry algebras (without central extensions) in \( D \)-dimensional Minkowski spacetime:

\[
\{ Q^A_\alpha, Q^B_\beta \} = \Omega^{AB} (\gamma^\mu)_{\alpha\beta} P_\mu \quad \text{in} \ D = 5, 6, 7; \quad (13)
\]
\[
\{ Q^A_\alpha, \bar{Q}^{\dot{A}B} \} = \delta^A_B (\gamma^\mu)_{\alpha\dot{a}} P_\mu \quad \text{in} \ D = 4, 8; \quad (14)
\]
\[
\{ Q^A_\alpha, Q^B_\beta \} = \delta^{AB} (\gamma^\mu)_{\alpha\beta} P_\mu \quad \text{in} \ D = 3, 9, 10, 11, \quad (15)
\]

where \( P_\mu \) is the momentum operator. The Latin capital indices run over the number \( N \) of independent spinor charges : \( A, B = 1, ..., N \); for \( D \) even, the automorphism group of the supersymmetry algebra (named \( R \)-symmetry group) reads

\[
SO(\mathcal{N}_+) \times SO(\mathcal{N}_-) \text{ in } D = 2 \text{ mod 8 (} \mathcal{N} = \mathcal{N}_+ + \mathcal{N}_-); \quad (16)
\]
\[
USp(2\mathcal{N}_+) \times USp(2\mathcal{N}_-) \text{ in } D = 6 \text{ mod 8 (} \mathcal{N} = 2\mathcal{N}_+ + 2\mathcal{N}_-); \quad (17)
\]
\[
U(\mathcal{N}) \text{ in } D = 4, 8 \text{ mod 8.} \quad (18)
\]

In \( D = 11 \), if the massless spectrum is bound to have 2 as the highest spin, then \( \mathcal{N}_{\text{max}} = 1 \), and the corresponding M-theory superalgebra with central extensions reads as follows:

\[
\{ Q_\alpha, Q_\beta \} = (\gamma^\mu)_{\alpha\beta} P_\mu + (\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (\gamma^{\mu\nu\rho\delta})_{\alpha\beta} Z_{\mu\nu\rho\delta}. \quad (19)
\]

[point particle][M2-brane][M5-brane]
Concerning the counting of (off-shell) independent components of spinors, it goes as follows:

\[ \text{D even : each } \psi_{\pm} \text{ has } 2^{D-2} \text{ components (which double for complex or pseudoreal ones)} \]  
(20)

\[ \text{D odd : each } \psi \text{ has } 2^{D-1} \text{ components (which double for pseudoreal ones).} \]  
(21)

Consequently, (pseudo)real spinor components of dimension 8, 16 and 32 are possible for \( D \leq 6 \), \( D \leq 10 \) and \( D \leq 11 \), respectively.

The dimension \( D \mod 8 \) determines the symmetry of the tensor products of spinor representations (denoted by \( \psi \)), which are defined by the morphism map

\[ A : \psi \times \psi \rightarrow \Lambda^k, \]  
(22)

where \( \Lambda^k \) denotes the space of \( k \)-forms. The resulting symmetry of \( A \), depending on \( D \) and \( k \), is presented in Table 1.

| \( D \mod 8 \) | \( k \) even | \( k \) odd |
|----------------|-----------|-----------|
| 0              | \( \psi_{\pm} \otimes \psi_{\pm} \rightarrow \Lambda^k \) | \( \psi_{\pm} \otimes \psi_{\mp} \rightarrow \Lambda^k \) |
| 1              | \( \psi \otimes \psi \rightarrow \Lambda^k \) | \( \psi \otimes \psi \rightarrow \Lambda^k \) |
| 2              | \( \psi_{\pm} \otimes \psi_{\mp} \rightarrow \Lambda^k \) | \( \psi_{\pm} \otimes \psi_{\pm} \rightarrow \Lambda^k \) |
| 3              | \( \psi \otimes \psi \rightarrow \Lambda^k \) | \( \psi \otimes \psi \rightarrow \Lambda^k \) |
| 4              | \( \psi_{\pm} \otimes \psi_{\pm} \rightarrow \Lambda^k \) | \( \psi_{\pm} \otimes \psi_{\mp} \rightarrow \Lambda^k \) |
| 5              | \( \psi \otimes \psi \rightarrow \Lambda^k \) | \( \psi \otimes \psi \rightarrow \Lambda^k \) |
| 6              | \( \psi_{\pm} \otimes \psi_{\mp} \rightarrow \Lambda^k \) | \( \psi_{\pm} \otimes \psi_{\pm} \rightarrow \Lambda^k \) |
| 7              | \( \psi \otimes \psi \rightarrow \Lambda^k \) | \( \psi \otimes \psi \rightarrow \Lambda^k \) |

Table 1: Properties of the morphism map \( A \) [3]

For a comprehensive treatment of spinors and Clifford algebras for arbitrary \((s,t)\)-signatures of spacetime, see e.g. [3].

It should be recalled that the massless Dirac equation (with \( m = 0 \)) has an extra symmetry, which in the Majorana representation takes the form

\[ \psi' = e^{\alpha \gamma^5} \psi; \]  
(23)

this is indeed a \( U(1) \) rotation, since \( \gamma^5 \) is anti-involutive : \( (\gamma^5)^2 = -1 \).

However, this symmetry is broken by the mass term for a Majorana fermion (called Majorana mass). Defining a complex Dirac spinor \( \psi_{D} \) as

\[ \psi_{D} : = \psi_1 + i \psi_2; \]  
(24)

\[ \psi_{D}^* = \psi_1 - i \psi_2, \]  
(25)
where \( \psi_1 \) and \( \psi_2 \) are Majorana spinors, one can them have a \( U(1) \) symmetry which rotates \( \psi_1 \) into \( \psi_2 \).

Then, one can define
\[
\psi_L := \frac{1}{2} (1 - i \gamma^5) \psi_D; \quad \psi_R := \psi_L^* = \frac{1}{2} (1 + i \gamma^5) \psi_D; \\
\chi_L := \frac{1}{2} (1 - i \gamma^5) \psi_D^*; \quad \chi_R := \chi_L^* = \frac{1}{2} (1 + i \gamma^5) \psi_D.
\]

Therefore, \( \psi_L \) and \( \chi_L \) (and \( \psi_R \) and \( \chi_R \)) have opposite \( U(1) \) phases, whereas \( \psi_L \) and \( \chi_R \) (and \( \psi_R \) and \( \chi_L \)) have identical \( U(1) \) phases. The \( U(1) \)-invariant Dirac equation in the chiral (Weyl) notation becomes
\[
\gamma^\mu \partial_\mu \psi_L + m \chi_R = 0 \quad (+h.c.); \\
\gamma^\mu \partial_\mu \chi_L + m \psi_R = 0 \quad (+h.c.).
\]

Thus, the Dirac mass term, in this notation, is \( m \psi_L \chi_L + h.c. \), which is of course \( U(1) \)-invariant.

In principle, if one gives up the \( U(1) \) symmetry, a Dirac fermion can have three types of mass terms, namely :
\[
m \psi_L \chi_L, \quad M \psi_L \psi_L, \quad N \chi_L \chi_L,
\]
where \( m \) is named Dirac mass, and \( M \) and \( N \) Majorana masses. As a result, the matrix
\[
\mathcal{M} := \begin{pmatrix} M & m \\ m & N \end{pmatrix}
\]
has two eigenvalues :
\[
m_{1,2} := \frac{1}{2} \left[ M + N \pm \sqrt{(M - N)^2 + 4m^2} \right].
\]

In particular, for \( N = 0 \) and \( M \gg m \), the eigenvalues become
\[
m_{1,2} \simeq \left( M, -\frac{m^2}{M} \right).
\]
This result is at the basis of the see-saw mechanism \([4, 5]\).

Concerning the superpartner of the graviton, this is a spin-\( \frac{3}{2} \) Rarita-Schwinger field, called gravitino. Under the Lorentz group \( SO(1, D - 1) \) it is a vector-spinor \( \psi_\mu \). If it is massless (namely, in the case of unbroken supersymmetry), it lies in an irreducible, \( \gamma \)-traceless representation of the little group \( SO(D - 2) \); so, assuming real (semi)spinors, the gravitino has a number of (on-shell) independent components given by
\[
D \text{ odd, } (\psi \text{ real}) : \quad \# (\psi_\mu) = (D - 2) 2^{D/2} - 2^{D-3} = (D - 3) 2^{D/2}; \\
D \text{ even, } (\psi_\pm \text{ real}) : \quad \# (\psi_\mu, \pm) = (D - 2) 2^{D/4} - 2^{D/2} = (D - 3) 2^{D/4},
\]
which for example yields
\[
D = 11 : \# (\psi_\mu) = 8 \cdot 2^4 = 128; \\
D = 10 : \# (\psi_\mu, \pm) = 7 \cdot 2^3 = 56.
\]
The simplest Lagrangian density of supergravity couples Einstein gravity with the Rarita-Schwinger density, yielding
\[ \mathcal{L}_{\text{sugra}} = R - \bar{\psi}_\mu \gamma^{\mu \nu \rho} D_\nu \psi_\rho. \] (37)

Assuming 2 as the highest spin in the massless spectrum (pertaining to the graviton), after [6] it is known that Poincaré supersymmetry coupled to gravity is possible for \( D \leq 11 \), while conformal supersymmetry is possible for \( D \leq 6 \); this result also stems from the classification of superalgebras done by Kac [7].

### 3 Spontaneous Symmetry Breaking of Rigid Supersymmetry: Goldstino Actions

As holding for any action invariant under continuous symmetries, in theories invariant under one or more supersymmetry transformations there are conserved \textit{Majorana vector–spinor Noether currents} [8] (in van der Waerden notation):
\[ \partial^\mu J^A_\mu \alpha (x) = 0 \quad (A = 1, ..., \mathcal{N}). \] (38)

The corresponding charges
\[ Q^A_\alpha = \int d^3x \ J^A_0 \alpha (x) \] (39)
generate the (\( \mathcal{N} \)-extended) supersymmetry algebra, whose non-trivial part in \( D = 4 \) (which we consider throughout this Section) is given by (14).

In \( \mathcal{N} = 1 \) spontaneous breaking, the order parameter \( \mu \) enters a term linear in the supercurrent [9, 10]
\[ J_{\mu \dot{\alpha}} = \mu (\gamma_\mu \mathcal{G})_{\dot{\alpha}} + ..., \] (40)
where \( \mathcal{G} \) is a Majorana field, the \textit{Goldstino}; note that \( \mu \) has dimension 2 in natural units.

The supersymmetry current algebra implies that
\[ \int d^3y \ \{ J^A_{0 \dot{\alpha}} (y), J^A_{\mu \alpha} (x) \} = (\sigma^{\nu})_{\alpha \dot{\alpha}} T_{\nu \mu} (x) + \text{derivatives}, \] (41)
so that
\[ \langle 0 | \{ \bar{Q}_\alpha , J_{\mu \alpha} (x) \} | 0 \rangle = (\sigma^{\nu})_{\alpha \dot{\alpha}} \langle 0 | T_{\nu \mu} | 0 \rangle = (\sigma_\mu)_{\alpha \dot{\alpha}} \mu^2. \] (42)

By generalizing this for several supercharges, one can conclude that (in absence of central extensions, as well as in the rigid case - no gravity coupled -, as we assume throughout this Section), either all supersymmetries are unbroken or they are all broken at the same scale \( f \):
\[ J_{\mu \dot{\alpha} A} = \mu (\gamma_\mu \mathcal{G}_A)_{\dot{\alpha}} + ... . \] (43)
One can evade this result by modifying the current algebra via terms not proportional to $\delta^B_A$, namely by adding a contribution proportional to a constant matrix $C^B_A$ in the adjoint of $SU(N)$ (i.e., traceless and Hermitian):

$$\langle 0 | \{ \bar{Q}_{\dot{a}}, J_{\mu a}(x) \} | 0 \rangle = (\sigma^\nu)_{\alpha \dot{\alpha}} \langle 0 | T_{\nu \mu} | 0 \rangle + (\sigma_\mu)_{\alpha \dot{\alpha}} C^B_A.$$ (44)

As it holds in general for Goldstone particles, the Goldstino $G$ is a massless fermion, with low-energy self–interactions described by a Lagrangian invariant under a non–linear realization of supersymmetry, and depending on both $N$ and $C^B_A$, because it can be shown that

$$\delta_A \chi^i \delta^B \chi_i = V \delta^B_A + C^B_A \left( \delta \chi^i = \delta_A \chi^i \epsilon^A \right).$$ (45)

If $k$ supersymmetries are unbroken, the Hermitian matrix $V 1 + C$ has rank $N - k$ in the vacuum, and the $N - 1$ possible scales of partial supersymmetry breaking are the classified by the $N - 1$ Casimirs of $SU(N)$. Amusingly, the resulting characteristic equation can be solved in algebraically closed form up to the quartic order, corresponding in this setting to $N = 4$, the maximal value of $N$ for rigid supersymmetry.

For $N = 2$, there are various examples realizing a spontaneous supersymmetry breaking [11, 12, 13, 14, 15], and the previous analysis in terms of $SU(2)$ invariants was performed in [16]. The rigid limit of $N = 2$ supergravity theory reproducing partial supersymmetry breaking was considered in [17]. The unique Casimir of $SU(2)$ is the squared norm of a 3–vector $\xi^x$ constructed via electric and magnetic Fayet–Iliopoulos (FI) terms

$$\xi^x : = (Q_y \wedge Q_z) \epsilon^{xyz};$$ (46)

$$Q_y \wedge Q_z : = m^\Lambda_y e_{\Lambda z} - m^\Lambda_z e_{\Lambda y},$$ (47)

where $Q_x = (m^\Lambda_x, e_{x \Lambda})$, and $\Lambda = 1, \ldots, n$, with $n$ denoting the number of $N = 2$ vector multiplets. The wedge product $\wedge$ is with respect to the symplectic structure inherited from the rigid special Kähler geometry endowing the vector multiplets’ scalar manifold, and the matrix $C^B_A$ acquires the explicit form

$$\delta_A \chi^i \delta^B \chi_i = V \delta^B_A + (\sigma_x)^B_A \xi^x = \begin{pmatrix} V - \xi^3 & \xi^1 + i \xi^2 \\ \xi^1 - i \xi^2 & V + \xi^3 \end{pmatrix},$$ (48)

with eigenvalues

$$\lambda_\pm = V \mp \sqrt{\xi^x \xi^x}.$$ (49)

Thus, when $V = \sqrt{\xi^x \xi^x}$, $\lambda_+ = 0$ and $N = 2$ rigid supersymmetry is broken down to $N = 1$, with order parameter

$$\mu = (\xi^x \xi^x)^{1/4}.$$ (50)
The last term in the r.h.s. of Eq. (44) corresponds to a “vector” central charge \( Z^A_{\mu|B} \) \[18, 19\]; the only field not being inert is the gauge field of \( \mathcal{N} = 2 \) vector multiplets.

For a general \( \mathcal{N} \)-extended rigid supersymmetry broken down to \( \mathcal{N} = 0 \) (corresponding, from the previous reasoning, to \( C_B^A = 0 \)), the low–energy effective Lagrangian for \( \mathcal{N} \) Goldstino fields is an obvious extension of the Volkov–Akulov (VA) action \[20\] derived in the \( \mathcal{N} = 1 \) case, and in general reads

\[
\mathcal{L}_{VA}(\mu, \mathcal{G}^A_\alpha) = \mu^2 \left[ 1 - \sqrt{-\det(\eta_{\mu\nu} + \frac{i}{\mu^2} (\overline{\mathcal{G}}_A \gamma^\mu \partial_\nu \mathcal{G}^A - \text{h.c.}))} \right].
\]

(51)

It should be stressed that this Lagrangian contains a finite number of terms, because of the nilpotency of the Goldstino fields \( \mathcal{G}^A \). Furthermore, \( \mathcal{L}_{VA} \) has a \( U(\mathcal{N}) \) \( \mathcal{R} \)-symmetry, being invariant under the following non–linear supersymmetry transformations:

\[
\delta \mathcal{G}^A_\alpha(x) = \mu \epsilon^A_\alpha + \frac{i}{\mu} \left( \overline{\epsilon}_B \gamma^\mu \epsilon^B - \epsilon_B \gamma^\mu \mathcal{G}^B \right) \partial_\mu \mathcal{G}^A_\alpha.
\]

(52)

On the other hand, the Goldstino action for \( \mathcal{N} = 2 \) rigid supersymmetry partially broken down to \( \mathcal{N} = 1 \) is described by the supersymmetric Born–Infeld (SBI) theory: this is given by a non-linear Lagrangian \( \mathcal{L}_{SBI}(\mathcal{G}_\alpha, F_{\mu\nu}) \) for an \( \mathcal{N} = 1 \) vector multiplet whose chiral superfield strength \( W_\alpha = \overline{D}^2 D_\alpha V \) contains the goldstino \( \mathcal{G} \) at \( \theta = 0 \) and the self–dual Maxwell field strength at the next order in \( \theta \), with the following two properties:

\[
\mathcal{L}_{SBI}(\mathcal{G}_\alpha, F_{\mu\nu} = 0) \rightarrow \mathcal{L}_{VA};
\]

(53)

\[
\mathcal{L}_{SBI}(\mathcal{G}_\alpha = 0, F_{\mu\nu}) = \mu^2 \left[ 1 - \sqrt{-\det(\eta_{\mu\nu} + \frac{1}{\mu} F_{\mu\nu})} \right] = \mathcal{L}_{BI},
\]

(54)

where \( \mathcal{L}_{BI} \) is given by \[94\] further below.

\( \mathcal{L}_{SBI} \) exhibits two types of super–invariances: a manifest \( \mathcal{N} = 1 \) supersymmetry, linearly realized in \( \mathcal{N} = 1 \) superspace, and a second supersymmetry which is non–manifest and non–linearly realized. In terms of microscopic parameters, the scale of the broken supersymmetry is the \( SU(2)- \) and symplectic–invariant quantity \( 50 \); see Sec. \[4.4\] further below.

It is here worth pointing out that the Goldstino action for a partial breaking \( \mathcal{N} \rightarrow \mathcal{N} - k \) of rigid supersymmetry is only known for \( \mathcal{N} = 2 \), corresponding to the non–linear limit of a quadratic action of \( \mathcal{N} = 2 \) vector multiplets endowed with FI terms. The simplest case is given by a single \( \mathcal{N} = 2 \) vector multiplet, which under the spontaneous breaking \( \mathcal{N} = 2 \rightarrow 1 \) reduces to an \( \mathcal{N} = 1 \) vector multiplet \( W_\alpha \) and to an \( \mathcal{N} = 1 \) chiral multiplet \( X \). The presence of a non-vanishing matrix \( C_B^A \) allows the chiral multiplet \( X \) to acquire a non-zero mass \( m_X \) :

\[
\begin{pmatrix}
1, 2 \\
\frac{1}{2}
\end{pmatrix}_{\mathcal{N}=2 \text{ vector mult.}} \xrightarrow{\mathcal{N}=2 \rightarrow 1} \begin{pmatrix}
1, 1 \\
\frac{1}{2}
\end{pmatrix}_{\mathcal{N}=1 \text{ vector mult.}, m_V=0} + \begin{pmatrix}
1, 2 \\
\frac{1}{2}, 2 (0)
\end{pmatrix}_{\mathcal{N}=1 \text{ chiral mult.}, m_X \neq 0}.
\]

(55)
For \( m_X \) large enough, \( X \) can be integrated out, giving rise to a non-linear theory for the fields \( G_\alpha \) and \( F_{\mu\nu} \) of the \( \mathcal{N} = 1 \) vector multiplet, described by the aforementioned \( \mathcal{L}_{SB}\)I \( (G_\alpha, F_{\mu\nu}) \) \cite{21, 23}.

### 4 Electric-Magnetic Duality, Born–Infeld and Generalizations

*Electric-magnetic (e.m.) duality* is one of the most fascinating symmetries of non-linear (and thus interacting) theories of gauge fields.

#### 4.1 Maxwell Theory

*Free Maxwell theory is the prototype of e.m. duality invariant theory.* In the vacuum, the Eqs. of motion read \((F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu)\)

\[
\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \star F^{\mu\nu} = 0
\]  

(56)

where

\[
\star F_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},
\]

(57)

and they are invariant under \( SO(2) \) rotations:

\[
\left( \begin{array}{c} F \\ \star F \end{array} \right) ' = \left( \begin{array}{cc} \cos \alpha & - \sin \alpha \\ \sin \alpha & \cos \alpha \end{array} \right) \left( \begin{array}{c} F \\ \star F \end{array} \right)
\]

(58)

This latter transformation, in terms of electric and magnetic fields \( F_{\mu\nu} = \left( \mathbf{E}, \mathbf{B} \right) \) reads

\[
\left( \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right) ' = \left( \begin{array}{cc} \cos \alpha & - \sin \alpha \\ \sin \alpha & \cos \alpha \end{array} \right) \left( \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right)
\]

(59)

Also in this notation it is immediate to see that Maxwell Equations in absence of sources \((F_{oi} = E_i, F_{ij} = \epsilon_{ijk}B_k)\)

\[
\nabla \cdot \mathbf{E} = 0, \quad \partial_t \mathbf{E} = \nabla \times \mathbf{B},
\]

(60)

\[
\nabla \cdot \mathbf{B} = 0, \quad \partial_t \mathbf{B} = - \nabla \times \mathbf{E}
\]

(61)

are invariant.

Note that the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \left( |\mathbf{E}|^2 + |\mathbf{B}|^2 \right)
\]

(62)

is e.m. duality invariant, while the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \left( |\vec{E}|^2 - |\vec{B}|^2 \right)
\]

is not. It should also be pointed out that the e.m. duality symmetry is not an internal symmetry, since it rotates tensors with pseudo-tensors (as such, it can be regarded as a sort of “bosonic chiral transformation”).

Electric-magnetic dualities are transformations among two-form field strengths and their duals. These transformations extend to \((p + 2)\)-form field strengths and their duals, which, in \(D\) spacetime dimensions, are \((D - p - 2)\)-forms (for \(D = 4\) and \(p = 0\), one obtains that \(p + 2 = D - p - 2\)). In order to generalize the e.m. duality to \(p\)-forms, one demands that \(D/2 = p + 2\), so that one has \((p + 1)\)-form gauge fields coupled to sources, which are \(p\)-extended objects: \(p = 0\) (pointlike), \(p = 1\) (string), \(p = 2\) (membrane), \(p = 3\) (three-brane), etc.:

\[
F_{\mu_1...\mu_{p+2}} = \partial_{[\mu_1} A_{\mu_2...\mu_{p+2}]}.
\]  

When \(D/2 = p + 2\) we have that \(D - p - 2 = p + 2\), and \(p\)-dimensional extended objects can source both electric and magnetic fields. We thus have dyons, extended objects that carry both an electric and a magnetic charge. In all other cases, electric and magnetic objects extends in different space dimensions, \((p, D - p - 4)\) respectively, so dyons cannot exist.

The Dirac-Schwinger-Zwanziger quantization condition reads as follows:

- for \(D/2 \neq p + 2\):
  \[
  em' = 2\pi k, \quad k \in \mathbb{Z},
  \]

- for \(D/2 = p + 2\)
  \[
  em' + e'm = 2k\pi, \quad k \in \mathbb{Z} \quad (p \text{ odd});
  \]
  \[
  em' - e'm = 2k\pi, \quad k \in \mathbb{Z} \quad (p \text{ even}).
  \]

Note that the latter condition is not only invariant under \(SO(2)\), but also under \(Sp(2, \mathbb{R})\), isomorphic to \(SL(2, \mathbb{R})\).

### 4.2 Non-Linear Electric–Magnetic Duality

The very idea of duality is to generalize the duality of free Maxwell theory to the case of interactions. Interactions introduce non-linearities, which can be due either to the Maxwell field itself, or to other matter fields (as in electrodynamics) which interact with the e.m. field.
Non-linear (pure) electromagnetism can be seen as an effective theory of Maxwell theory in a medium that has a nonlinear response to the electric and magnetic fields. These nonlinearities are captured by the relations expressing the electric displacement $\vec{D}$ and the magnetic field $\vec{H}$ in terms of the magnetic induction $\vec{B}$ and the electric field $\vec{E}$. The field equations (in absence of sources) are thus divided in a set of linear ones that are independent from the medium

\[
\vec{\nabla} \cdot \vec{D} = 0, \quad \partial_t \vec{D} = \vec{\nabla} \times \vec{H}, \tag{68}
\]

\[
\vec{\nabla} \cdot \vec{B} = 0, \quad \partial_t \vec{B} = -\vec{\nabla} \times \vec{E}, \tag{69}
\]

and a set describing the nonlinearities, i.e. the properties of the medium, hence these equations are called constitutive relations,

\[
\vec{D} = \vec{D}(\vec{E}, \vec{B}) = \vec{E} + ..., \tag{70}
\]

\[
\vec{H} = \vec{H}(\vec{E}, \vec{B}) = \vec{B} + ..., \tag{71}
\]

Maxwell theory is recovered when

\[
\vec{D} = \vec{E}, \quad \vec{H} = \vec{B}. \tag{72}
\]

The dots ... in Eq. (71) stand for some higher powers of the electric field $\vec{E}$ and of the magnetic induction $\vec{B}$, so that the nonlinear theories we consider are deformations of linear electromagnetism because (70) and (71) reduce to (72) for weak fields.

In a theory based on a Lagrangian density $\mathcal{L} = \mathcal{L}(\vec{E}, \vec{B})$, one has the Eqs.

\[
\vec{D} = 2 \frac{\delta \mathcal{L}(\vec{E}, \vec{B})}{\delta \vec{E}}, \quad \vec{H} = -2 \frac{\delta \mathcal{L}(\vec{E}, \vec{B})}{\delta \vec{B}}. \tag{73}
\]

Note that Eqs. (68) are invariant under the linear transformations

\[
\begin{pmatrix}
\vec{B} \\
-\vec{D}
\end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix}
\vec{B} \\
-\vec{D}
\end{pmatrix}, \tag{74}
\]

\[
\begin{pmatrix}
\vec{E} \\
\vec{H}
\end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix}
\vec{E} \\
\vec{H}
\end{pmatrix}. \tag{75}
\]

Hence the full set of equations of motion are invariant under the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ duality rotations if the constitutive relations are compatible with these rotations, i.e. if the transformations $\vec{D}(\vec{E}, \vec{B}) \rightarrow \vec{D}(\vec{E'}, \vec{B'})$ and $\vec{H}(\vec{E}, \vec{B}) \rightarrow \vec{H}(\vec{E'}, \vec{B'})$ are the same as the rotations (74), (75).
It is convenient to rewrite the nonlinear equations and the duality rotations using a relativistic formalism. We set as usual \( F_{\mu\nu} = (\vec{E}, \vec{B}) \) and \( G_{\mu\nu} = (\vec{H}, -\vec{B}) \), so that in the vacuum:

\[
\vec{D} = \vec{E}, \quad \vec{H} = \vec{B} \iff G_{\mu\nu} = \ast F_{\mu\nu}.
\] (76)

The dynamical equations then read \( dF = 0, dG = 0 \) i.e.,

\[
\partial_\mu \ast F^{\mu\nu} = 0, \quad \partial_\mu \ast G^{\mu\nu} = 0,
\] (77)

while the constitutive relations become

\[
G_{\mu\nu} = G_{\mu\nu}(F, \ast F),
\] (78)

\[
G_{\mu\nu} = \ast F_{\mu\nu} + \ldots,
\] (79)

and, if we have a Lagrangian,

\[
\ast G^{\mu\nu} = 2 \frac{\delta L(F)}{\delta F_{\mu\nu}}.
\] (80)

In order to have compatibility with the equations of motion, this demands the following integrability conditions:

\[
\frac{\delta \ast G^{\mu\nu}}{\delta F^{\rho\sigma}} = \frac{\delta \ast G_{\rho\sigma}}{\delta F_{\mu\nu}}.
\] (81)

The duality rotations correspond to transform

\[
\begin{pmatrix} F \\ G \end{pmatrix} \rightarrow \begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix},
\] (82)

and are a symmetry of the theory if the transformation \( G(F) \rightarrow G(F') = G(AF + BG) \) is the same as the rotation \( G(F) \rightarrow CF + DG(F) \). This highly non-linear constraint restricts the possible forms of \( G \) (the constitutive relations), and also the linear transformations \([82]\).

Gaillard and Zumino \([1]\) have proven that the most general e.m. duality rotation for a nonlinear theory, also depending on extra matter fields (such as fermions, scalars, and possibly in a curved gravitational background) is \( Sp(2n, \mathbb{R}) \), where \( n \) is the number of vector field strengths \( F^\Lambda \) \((\Lambda = 1, \ldots, n)\). This means that the matrix

\[
\mathcal{S} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\] (83)

instead of being a \( GL(2n, \mathbb{R}) \) matrix is a symplectic matrix, indeed it satisfies

\[
\mathcal{S}^T \Omega \mathcal{S} = \Omega, \quad \Omega := \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},
\] (84)
which equivalently reads

\[ A^T D - C^T B = I, \quad A^T C = C^T A, \quad B^T D = D^T B. \]

In absence of scalars, or if scalars are inert under duality rotations, the e.m. duality group is at most \( U(n) \) (in particular, for \( n = 1 \), one retrieves Maxwell e.m. duality group: \( U(1) \sim SO(2) \)). The constraints on the \( n \) field strength \( F^A \) and \( G^A \) in this case read

\[
F^A \star F^\Sigma + G^A \star G^\Sigma = 0; \tag{85}
\]
\[
F^A \star G^\Sigma - F^\Sigma \star G^A = 0. \tag{86}
\]

For \( n = 1 \) (namely, \( U(1) \)), one gets

\[
F \star F + G \star G = 0. \tag{87}
\]

For general e.m. theories where also matter fields are present the constitutive relations will also depend on these fields, that we generically denote \( \zeta^i \), (and on their derivatives that we omit writing) so that we have \( G_{\mu\nu} = G_{\mu\nu}(F, \zeta^i) \). Duality rotations now will include also a (nonlinear) transformation on the matter fields \( \zeta^i \).

As shown by Gaillard and Zumino [1] (and reviewed in [24]) in this general case, under an infinitesimal duality rotation parametrized by

\[
s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp}(2n, \mathbb{R}), \text{ i.e. } a = -d^T, b = b^T, c = c^T, \tag{88}
\]

the Lagrangian varies by

\[
\delta L = \frac{1}{4} (Fc \star F + Gb \star G),
\]

where \( \delta L \equiv L(F + \delta F, \zeta^i + \delta \zeta^i) - L(F, \zeta^i) \) is the total infinitesimal variation of the Lagrangian under the duality rotations of the field strengths and of the matter fields. If we expand \( \delta F \) in terms of \( F \) and \( G \) using the infinitesimal version of the duality rotation (82) then we obtain

\[
\delta \zeta L := L(F, \zeta^i + \delta \zeta^i) - L(F, \chi^i) = \frac{1}{4} (Fc \star F - Gb \star G) + \frac{1}{2} Fd \star G. \tag{89}
\]

If there are no matter fields the left hand side is zero, the duality rotation group reduces then to \( U(n) \), and in this case the infinitesimal parameters satisfy \( c = -b \) and \( d = -d^T \). Then Eq. (89) reproduces just the previous Eqs. (85) and (86) for \( U(n) \).

Note that for e.m. duality rotations the total variation \( \delta L \) never vanishes. If \( c \neq 0, b = 0 \) the infinitesimal variation \( \delta L \) is a total derivative because of the equations of motion \( dF = 0 \). If
\( c = 0, b \neq 0 \) it is a total derivative because of the equations of motion \( dG = 0 \). If both \( b \neq 0 \) and \( c \neq 0 \) we have to use both equations of motion \( dF = 0 \) and \( dG = 0 \). We therefore see that the Lagrangians of theories with duality rotation symmetry change by a term that is a total derivative only on shell of the equations of motion \( dF = 0 \) and \( dG = 0 \).

We also have that under a finite duality rotation the action changes by a term that is a total derivative only on shell of both \( dF = 0 \) and \( dG = 0 \) equations of motions. Indeed under a finite \( U(1) \) rotation of angle \( \alpha \) we have \[25\]

\[
\mathcal{L}(F') - \mathcal{L}(F) = \frac{1}{8} \left( \sin(2\alpha) \left( F \star F - G \star G \right) - 4 \sin^2(\alpha) F \star G \right). \tag{90}
\]

We notice that in the special case of Maxwell theory \( G = \ast F \) and therefore the infinitesimal variation \( \mathcal{L}(F') - \mathcal{L}(F) \) is a total derivative just on shell of the relation \( G = \ast F \) and of the equation \( dF = 0 \) (i.e. without using the equation of motion \( dG = 0 \)). In other words, if we solve the equation \( dF = 0 \) by introducing a gauge potential so that \( F = dA \) and hence see the Lagrangian as dependent on \( A \), then for infinitesimal angle \( \alpha \) the variation \( \mathcal{L}(F') - \mathcal{L}(F) \) is a total derivative off shell of the equations of motion for the gauge potential \( A \). This off shell infinitesimal symmetry of Maxwell theory (that acts non-locally on the gauge potential \( A \)) was studied in \[26\]; however it does not correspond to a finite off shell symmetry. As we see from the second term in \( \mathcal{L}(F') - \mathcal{L}(F) \) in order for \( F \ast G \) to be a total derivative we have to use also the equations of motion \( dG = 0 \), that in Maxwell theory simply read \( d \ast F = 0 \).

Two important results are obtained from the previous formulæ for general nonlinear theories.

1. Any Lagrangian density can be written as \( \frac{1}{4} F \ast G \) plus a duality invariant Lagrangian \( \mathcal{L}_{inv} \),

\[
\mathcal{L} = \frac{1}{4} F \ast G + \mathcal{L}_{inv} , \quad \delta \mathcal{L}_{inv} = 0. \tag{91}
\]

The term \( \mathcal{L}_{inv} \) provides information about the constitutive relations (i.e., the medium), in particular \( \mathcal{L}_{inv} = 0 \) vanishes in linear Maxwell theory without coupling to matter. On the other hand, for (quadratic) theories coupled to matter \( \mathcal{L}_{inv} \) provides information about some matter couplings (such as Pauli-like terms). In supergravity it is in fact possible to construct pairs of antisymmetric Lorentz tensors \( (H_{\mu \nu} (\zeta), I_{\mu \nu} (\zeta)) \) (typically bilinear in fermions) which transforms as the couple \( (F, G) \). In this case, one has

\[
\mathcal{L}_{inv} = \frac{1}{4} (FI - GH) + \mathcal{L}_{inv} (\zeta). \tag{92}
\]
2. The energy-momentum tensor

\[ \Theta^{\mu}_{\lambda} = -\partial_{\lambda} \zeta^{i} \frac{\partial L}{\partial (\partial_{\mu} \zeta^{i})} + \delta^{\mu}_{\lambda} L + \ast G^{\lambda \mu \nu} F_{\lambda \nu} \]  

is duality invariant if duality is a symmetry of the theory. In the absence of matter fields the trace of the energy momentum tensor is proportional to \( L_{\text{inv}} : \frac{1}{4} \Theta^{\mu}_{\mu} = L_{\text{inv}} \).

4.3 Born-Infeld Theory

The Born-Infeld (BI) model [27]

\[ L_{\text{BI}} = \mu^{2} \left( 1 - \sqrt{1 - \mu^{2} F^{2} - \frac{1}{16 \mu^{4}} (F \ast F)^{2}} \right) \]  

\[ = \mu^{2} \left( 1 - \sqrt{1 + \frac{1}{2 \mu^{2}} F^{2} - \frac{1}{16 \mu^{4}} (F \ast F)^{2}} \right) \]  

\[ = \mu^{2} \left( 1 - \sqrt{1 + \frac{1}{\mu^{2}} \left( |\vec{B}|^{2} - |\vec{E}|^{2} \right) - \frac{1}{\mu^{4}} (\vec{E} \cdot \vec{B})^{2}} \right) \]

is the simplest non-linear e.m. duality invariant theory. It was introduced to remove the divergence of the electrons self-energy in classical electrodynamics. This is obtained by having an upper bound on the electric field; indeed, for \( \vec{B} = 0 \) \( L_{\text{BI}} \) yields the bound

\[ |\vec{E}| \leq \mu . \]  

In fact, \( L_{\text{BI}} \) has unique properties against instabilities created by the medium, and not many generalizations are known in the multi-field case (i.e., in presence of more than one Maxwell field). Furthermore, by expanding \( L_{\text{BI}} \) in power series of the coupling constant \( \frac{1}{\mu} \) one obtains

\[ \ast G_{\mu \nu} (F, \ast F; \mu) = -F_{\mu \nu} + ... \]  

and hence for \( \mu \to \infty \), i.e., for weak fields, one recovers Maxwell theory.

The equations of motion of BI theory are a particular realization of the constitutive relations

\[ \ast G_{\mu \nu} = \ast G_{\mu \nu} (F, \ast F; \mu) , \]  

(here \( \mu \) is a dimensionful coupling constant typically present in nonlinear theories, which was previously omitted for sake of brevity). Moreover, they satisfy the e.m. duality conditions (87). By recalling (94) and the definition (80), the explicit dependence of \( G \) on \( F \) can be computed to read

\[ G_{\mu \nu} (F, \ast F; \mu) = \frac{\ast F_{\mu \nu} + \frac{1}{4 \mu^{2}} (F \ast F) F_{\mu \nu}}{\sqrt{1 + \frac{1}{2 \mu^{2}} F^{2} - \frac{1}{16 \mu^{4}} (F \ast F)^{2}}} \]  

(100)
The Hamiltonian depends on the variables \((\vec{D}, \vec{B})\), and is obtained by a Legendre transform with respect to \(\vec{E}\), namely by defining
\[
\vec{D} = \frac{\delta \mathcal{L}(E, B)}{\delta \vec{E}} , \quad \mathcal{H} = \vec{D} \cdot \vec{E} - \mathcal{L} .
\] (101)

It is worth stressing that the Hamiltonian is perfectly regular for all values of these fields variables:
\[
\mathcal{H}_{BI} = \mu^2 \left( \sqrt{1 + \frac{1}{\mu^2} \left( |\vec{D}|^2 + |\vec{B}|^2 \right) + \frac{1}{\mu^4} |\vec{D} \wedge \vec{B}|^2} - 1 \right) .
\] (102)

In particular in the limits of weakly and of strongly valued fields:
\[
\lim_{\mu \to \infty} \mathcal{H}_{BI} = \frac{1}{2} \left( |\vec{D}|^2 + |\vec{B}|^2 \right) ;
\] (103)
\[
\lim_{\mu \to 0} \mathcal{H}_{BI} = |\vec{D} \wedge \vec{B}| .
\] (104)

As yielded by the property (54), the BI Lagrangian \(\mathcal{L}_{BI}\) is the bosonic part of the \(\mathcal{N} = 1\) non-linear Goldstino Lagrangian for the aforementioned partial breaking \(\mathcal{N} = 2 \to 1\) of rigid supersymmetry; in this context, the Maxwell field is the \(\mathcal{N} = 1\) superpartner of the spin-\(\frac{1}{2}\) Goldstino field [28, 29, 11]. For various recent developments on BI theory, supersymmetry, and e.m. duality, see e.g. [30, 31, 32, 33, 34].

### 4.4 Supersymmetry, Special Geometry, and Multi–Field Extension

The supersymmetric BI Lagrangian follows from an algebraic constraint in Superspace among various multiplets [35, 36].

This constraint enforces a non–linear relation between the two \(\mathcal{N} = 1\) supermultiplets \(X\) and \(W_\alpha\) that build the \(\mathcal{N} = 2\) vector multiplet [11, 37, 38]:
\[
W^2 + X \left( m_1 - \vec{D}^2 \vec{X} \right) = 0 ,
\] (105)
where \(m_1\) corresponds to a particular choice of a magnetic FI term. The non–linear relation (105) determines \(X\) as a non–linear function of \(D^2 W^2\) and \(D^2 \overline{W}^2\) and implies the nilpotency constraints
\[
X^2 = 0 , \quad X W_\alpha = 0 .
\] (106)

The supersymmetric BI Lagrangian then acquires the form
\[
\mathcal{L}_{SB} = Im \int d^2 \theta \left( e_1 + ie_2 \right) X \left( m_1, D^2 W^2, \overline{D}^2 \overline{W}^2 \right) ,
\] (107)
where \( e_1 \) and \( e_2 \) are electric FI terms whose labels correspond to the first two directions in an \( SU(2) \) triplet. For canonically normalized vectors, one can then recognize that the scale \( \mu \) and the theta–angle \( \vartheta \) can be expressed in terms of \( m_1, e_1 \) and \( e_2 \) as

\[
\mu = \sqrt{m_1 e_2}, \quad \vartheta = \frac{e_1}{e_2}.
\]  

(108)

In the bosonic limit, the quadratic constraint becomes

\[
F_+^2 + F (m_1 - F) = 0,
\]  

(109)

where \( F_+ \) is the self-dual curvature of the Maxwell field:

\[
F_{+\mu\nu} := \frac{1}{2} (F_{\mu\nu} + i * F_{\mu\nu})
\]  

(110)

and \( F \) is an auxiliary complex scalar field such that

\[
\mathcal{L}_{BI} = e_1 \text{Im}(F) + e_2 \text{Re}(F) = -\vartheta F_{\mu\nu} * F^{\mu\nu} + \frac{\mu^2}{2} \left[ 1 - \sqrt{1 + \frac{4}{\mu^2} F_{\mu\nu} F^{\mu\nu} - \frac{4}{\mu^4} (F_{\mu\nu} * F^{\mu\nu})^2} \right].
\]  

(111)

In the multi-field case, the generalization of \( \mathcal{L}_{SBI} \) rests on the constraints \( (A = 1, ..., n) \) \cite{21, 22}

\[
d_{ABC} \left[ W^B, W^C + Y^B (m^C - \bar{D}^2 \bar{Y}^C) \right] = 0,
\]  

(112)

and the complete Lagrangian reads

\[
\mathcal{L} = -\text{Im} \int d^2 \theta \left[ U_{AB} W^A W^B + \mathcal{W}(X) + \frac{1}{2} \bar{D}^2 \left( X^A \bar{U}_A - \bar{X}^A U_A \right) \right].
\]  

(113)

The \( \theta^2 \) component of eq. (112),

\[
d_{ABC} \left[ F_+^B, F_+^C + \mathcal{F}^B (m^C - \mathcal{F}^C) \right] = 0,
\]  

(114)

where \( F_+^A \) are self–dual field strength combinations and \( \mathcal{F}^A \) is the auxiliary field, is the multi–field generalization of the BI constraint (109) that is induced by the Special Geometry. Moreover, it holds that

\[
\Re F_+^A F_+^B = F^A F^B, \quad \Im F_+^A F_+^B = F^A * F^B.
\]  

(115)

d_{ABC} is a rank-3 completely symmetric tensor, encoding the coefficients of the cubic term of the holomorphic prepotential of the aforementioned rigid special Kähler geometry characterizing the \( \mathcal{N} = 2 \) vector multiplets’ scalar manifold, whose classification is based on the singularity structure of cubic varieties \cite{22, 23}.
The real parts of Eqs. (114) are $n$ quadratic equations that are generally coupled. Letting

$$H^A = \frac{m^A}{2} - \Re A^A, \quad R^{AB} = A^A A^B + \frac{m^A m^B}{4} - \Im A^B \Im A^C,$$

they take the form

$$d_{ABC} \left( H^B H^C - R^{BC} \right) = 0.$$  \hspace{0.5cm} (117)

On the other hand, the imaginary parts of Eqs. (114) are $n$ linear equations for $\Im A^A$:

$$d_{ABC} \left( A^B \ast A^C + \Im A^B m^C \right) = 0.$$  \hspace{0.5cm} (118)

The bosonic part of the multi-field Lagrangian (113) can be expressed in terms of the $A^A$ and of the real magnetic changes $m^A$. It can also be written in terms of additional complex charges $e_A = e_{1A} + i e_{2A}$, as

$$L_{\text{Bose}} = e_{2A} \left( \frac{m^A}{2} - H^A \right) + C_{AB} \left( H^A H^B - R^{AB} \right) + e_{1A} \Im A^A.$$  \hspace{0.5cm} (119)

The matrix $C_{AB}$ is needed whenever the matrix $d_{AB} := d_{ABC} q^C$ is not positive definite. Moreover, by a change of symplectic basis one could also eliminate the real parts $e_{1A}$ of the electric charges, which multiply combinations of the fields that are total derivatives. These Lagrangians combine, in general, a quadratic Maxwell–like term with additional higher–order contributions. For $n = 1$, or whenever the matrix $d_{AB}$ is positive definite, one is not compelled to introduce the $C_{AB}$ and the Lagrangian takes the simpler form (111). In all cases, however, the difficult step in the construction of the Lagrangians is the solution of the quadratic constraints, and in particular of the non–linear ones given in (117).

### 4.5 Schrödinger Formulation

The general study and classification of constitutive relations admitting e.m. duality symmetry can be attacked by adopting complex and chiral variables, indeed these variables simply transform with a phase under duality symmetry rotations. We therefore define

$$T := F - i G \iff \overline{T} := F + i G,$$

so that the e.m. duality condition (87) equivalently reads

$$\ast T \overline{T} = 0.$$  \hspace{0.5cm} (121)

\footnote{Here and below, the bar denotes complex conjugation.}
The corresponding chiral combinations read

\[ T^\pm := \frac{1}{2} \left( F^\pm - iG^\pm \right) \Leftrightarrow \overline{T}^\pm := \frac{1}{2} \left( F^\pm + iG^\pm \right), \tag{122} \]

where \( F^\pm = \frac{1}{2} (F \pm i \star F) \), \( G^\pm = \frac{1}{2} (G \pm i \star G) \). The equations of motion of linear electromagnetism in these variables read

\[ T^+ = 0 \Leftrightarrow (F - iG) + i(\star F - i \star G) = 0 \Leftrightarrow G = \star F. \tag{123} \]

The constitutive relations in these new variables, studied in [39], and independently and more generally in [40, 41], read as follows:

\[ T^+ = \frac{\partial I \left( T^-, \overline{T}^- \right)}{\partial T^-}, \quad \overline{T}^+ = \frac{\partial I \left( T^-, \overline{T}^- \right)}{\partial T^-}. \tag{124} \]

These are six relations if the arbitrary function \( I \) (the “Lagrangian” of these new variables) is real. Moreover, duality invariance is obtained if \( I \) is invariant under duality rotations. This is the case if

\[ I \left( T^-, \overline{T}^-, g \right) = I(u), \]

where

\[ u = 2g^2 |(T^-)|^2 = g(|T^2| + |T^*T|), \quad i.e., \quad u^2 = 4g^4 (T^-)^2 (\overline{T}^-)^2. \tag{125} \]

We note that the function \( I \) is \( U(1) \)-duality-invariant because the Lorentz scalar \((T^-)^2 (\overline{T}^-)^2\) is duality invariant, indeed under a \( U(1) \) duality rotation of angle \( \alpha \), the fields \( T \) and \( \overline{T} \) transform with opposite phases \( e^{-i\alpha} \) and \( e^{i\alpha} \).

By use of a Legendre transform [39, 42, 25] between the functions \( \mathcal{L}(F) \) and \( I(T^-, \overline{T}^-) \), the constitutive relations (124) in the \( T \)-variables can be shown to be equivalent to the original constitutive relations (80).

To summarize, the freedom in writing a self-dual and non-linear theory of electromagnetism can be traced back to a real function \( I \) of a real (Lorentz-invariant and \( U(1) \)-duality invariant) variable \((T^-)^2 (\overline{T}^-)^2\), or its square root \( u \). Arbitrary functions \( I(u) \) are in one-to-one correspondence with Lagrangians \( \mathcal{L}(F) \) such that the equations (124) are equivalent to the original constitutive relations and self duality condition (80) and (87).

Furthermore, it was shown by Schrödinger [43] (see also [44]) that the constitutive relations for the BI theory are

\[ \star T_{\mu\nu} = -\frac{T^2}{\star TT} T_{\mu\nu} - \frac{g^2}{8} \star TT \overline{T}_{\mu\nu}. \tag{126} \]
In the general case the following constitutive relations à la Schrödinger define a nonlinear and self-dual theory \[42\]

\[
\star T_{\mu\nu} = -\frac{T^2}{\star T} T_{\mu\nu} - \frac{g^2 f(u)}{8} \star T \star T_{\mu\nu},
\]

(127)

where we recall that \(u = 2|\langle T\rangle^2| = |T^2| + |T \star T|\), and where we require \(\lim_{u \to 0} f(u) = 0\), so to recover Maxwell theory for weak fields. For example the duality conditions (121) follow contracting (127) with \(\star T_{\mu\nu}\). Contracting (127) with \(T_{\mu\nu}\) and with \(T_{\mu\nu}\) it can also be shown that

\[
(1 - t^2)^2 f(u) = 32t;
\]

(128)

where

\[
t = \frac{T\star T}{|T^2| + |T \star T|}.
\]

(129)

In the first Ref. of [41] the function \(t(u)\) corresponding to the BI Lagrangian density was found through an iterative procedure order by order in \(u\); the first coefficients of its power series expansion were observed to match those of a particular generalized hypergeometric function, yielding to the conclusion that

\[
t(u) = \frac{u}{32} \frac{3F_2}{3F_2} \left(\frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \frac{5}{3}, \frac{5}{3}, -\frac{u^2}{3^3 \cdot 4^2}\right)
\]

(130)

\[
= \frac{u}{16} \sum_{k=0}^{\infty} (-1)^k \frac{(4k + 1)!}{(3k + 2)!k!} \left(\frac{u^2}{4^5}\right)^k.
\]

(131)

The relation between the constitutive relations à la Schrödinger and those in terms of the \(I(u)\) function follows from the relations between the \(T\) variables and the chiral \(T^\pm\) variables. This allows to determine \(f(u)\) from \(I(u)\) and vice versa. Explicitly, given \(f(u)\), one obtains \(t = t(u)\) from (131); the function \(I(u)\) is then given by

\[
I(u) = \frac{1}{2} \int t(u) du.
\]

(132)

4.6 Generalizations

Other non-linear (square-root) BI-like type Lagrangian densities in \(D\) dimensions have been suggested in [45, 46], starting from alternative \(D = 4\) Goldstino Lagrangians studied in [12, 37, 47].

The first class of Lagrangians is given by \(D\)-dimensional generalizations containing pairs field strengths forms of degrees \(p + 1\) and \(D - p - 1\), corresponding to gauge fields coupling to \((p - 1)\)- and \((D - p - 3)\)- branes, respectively. These Lagrangians generalize the \(D = 4, p = 2\) case, corresponding to a non-linear Lagrangian for a tensor multiplet regarded as an \(\mathcal{N} = 2 \to 1\)
Goldstino multiplet in rigid supersymmetry \[47, 46\]. They read

\[
\mathcal{L} (X,Y^2; \mu) = \mu^2 \left(1 - \sqrt{1 + \frac{1}{\mu^2} X - \frac{1}{\mu^4} Y^2}\right); \quad (133)
\]

\[
X : = - \star (H_{p+1} \wedge \star H_{p+1} + V_{D-p-1} \wedge \star V_{D-p-1}) ; \quad (134)
\]

\[
Y : = \star (H_{p+1} \wedge V_{D-p-1}) ; \quad (135)
\]

and they have the property of being \textit{doubly self-dual} under

\[
V'_{D-p-1} = \star H_{p+1}; \quad H'_{p+1} = \star V_{D-p-1}. \quad (136)
\]

Moreover, after a single duality the Lagrangian ends up containing two forms of the same degree, and it can be recast in the following manifestly \(U(1)\)–\textit{e.m.}–invariant form \[46\] :

\[
\mathcal{L}' (W, \bar{W}; \mu) = \mu^2 \left[1 - \sqrt{1 + \frac{(W_{D-p-1} \bar{W}_{D-p-1})^2 - W_{D-p-1}^2 \bar{W}_{D-p-1}^2}{\mu^2}}\right]. \quad (137)
\]

Actions with the full \(U(n)\) (where \(n\) is the number of Maxwell vector fields) were proposed in \[48\], but they are currently not available in closed form, even for \(n = 2\).

On the other hand, we can present three relatively simple extensions of the BI Lagrangian to a pair of curvatures \(F^1\) and \(F^2\) whose field equations possess \(U(1)\), \(U(1) \times U(1)\) and \(SU(2)\) duality symmetries. All these models solve subsets of the non-linear Gaillard-Zumino constraints (85) and (86) in the case \(\Lambda, \Sigma = 1, 2, \text{ i.e.} \) of :

\[
F^1 \star F^2 + G^1 \star G^2 = 0; \quad (138)
\]

\[
F^1 \star F^1 + G^1 \star G^1 = 0; \quad (139)
\]

\[
F^2 \star F^2 + G^2 \star G^2 = 0; \quad (140)
\]

\[
F^1 \star G^2 - F^2 \star G^1 = 0. \quad (141)
\]

The Lagrangian with \(U(1)\) invariance reads

\[
\mathcal{L} = \mu^2 \left[1 - \sqrt{1 + \frac{(F^1)^2 + (F^2)^2}{\mu^2}} - \frac{\star [F^1 \wedge F^2]^2}{\mu^4}\right], \quad (142)
\]

and was obtained as an application of double duality between two forms of the same degree. This \(U(1)\) corresponds to the constraint (85) for \(\Lambda = 1\) and \(\Sigma = 2\), namely (138).

The other two examples are simply different complexifications of the two–field system, whose manifest \(U(1)\) invariance corresponds to the single constraint of Eq. (86). In particular, letting

\[
F = F^1 + i F^2, \quad (143)
\]
the case with $U(1) \times U(1)$ invariance, where the second $U(1)$ constraint corresponds to the equation

$$F^1 \ast F^1 + F^2 \ast F^2 + G^1 \ast G^1 + G^2 \ast G^2 = 0,$$

reads

$$\mathcal{L} = \mu^2 \left[ 1 - \sqrt{1 + \frac{F \bar{F}}{\mu^2} - \left( \ast [F \wedge \bar{F}] \right) \left( \ast [\bar{F} \wedge F] \right)} \right].$$

(145)

This is actually a particular case of the $U(n, n)$ dualities of [48] for $n = 1$, where in the absence of scalars $U(1, 1)$ reduces to its maximal compact subgroup $U(1) \times U(1)$. Finally, the case with $SU(2)$ invariance reads

$$\mathcal{L} = \mu^2 \left[ 1 - \sqrt{1 + \frac{F \bar{F}}{\mu^2} - \left( \ast [F \wedge \bar{F}] \right) \left( \ast [\bar{F} \wedge F] \right)} \right].$$

(146)

where the other two constraints correspond to the $SU(2)$ generators

$$F^1 \ast F^1 - F^2 \ast F^2 + G^1 \ast G^1 - G^2 \ast G^2 = 0,$$

(147)

$$F^1 \ast F^2 + G^1 \ast G^2 = 0.$$

(148)

From these examples it is manifest that the standard BI system admits several types of inequivalent complexifications that differ in their quartic couplings.

A fourth option, with maximal $U(2)$ duality, is naturally captured by the construction of [48], but its Lagrangian, as have been mentioned before, is not known in closed form.

All the above non-linear actions can be massive by introducing Green-Schwarz terms [45], i.e. couplings to another gauge field of the form

$$m H_{p+1} \wedge A_{D-p-1}.$$  

(149)

A Stückelberg mechanism is then generated, and the non-linear mass terms turn out to take the same functional form as the original non-linear curvature terms; in other words, the $(D - p - 1)$-form gauge field has been eaten to give mass to the original $p$-form gauge field (whose field strength is $H_{p+1}$). The simplest example of this class of Lagrangians is the $D = 4$ BI action used to make an antisymmetric field $B_{\mu \nu}$ massive: starting point is thus the master action

$$\mathcal{L} = -\frac{k^2}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} - \frac{m}{4} \epsilon_{\mu \nu \rho \sigma} B_{\mu \nu} F_{\rho \sigma}$$

$$+ \frac{\mu^2}{8 g^2} \left[ 1 - \sqrt{1 + \frac{4}{\mu^2} F_{\mu \nu} F^{\mu \nu} - \frac{4}{\mu^4} (F_{\mu \nu} \ast F^{\mu \nu})^2} \right],$$

(150)

22
where we have introduced a dimensionless parameter $g$, the counterpart of the parameter $k$ that accompanies the two–form kinetic term. The parameter $\mu$ is the BI scale factor, with mass–squared dimension, which sizes the non–linear corrections.

As above, a massive variant of the BI action principle would be obtained by eliminating $H$ after moving to a first–order form where it is unconstrained. However, as we have seen, the additional field is just a standard Stückelberg mode, so that for brevity we can just display the gauge–fixed Proca–like Lagrangian for the massive BI vector,

$$\mathcal{L} = -\frac{m^2}{2k^2} A_\mu A^\mu + \mathcal{L}_{BI}(g, \mu, F_{\mu\nu}), \quad (151)$$

where

$$\mathcal{L}_{BI}(g, \mu, F) = \frac{\mu^2}{8g^2} \left[ 1 - \sqrt{1 + \frac{4}{\mu^2} F_{\mu\nu} F^{\mu\nu} - \frac{4}{\mu^4} (F_{\mu\nu} \star F^{\mu\nu})^2} \right]. \quad (152)$$

In the massless case, the self–duality of the BI Lagrangian would translate into the condition that

$$\mathcal{L}_{BI}\left(g, \mu, F_{\mu\nu}(A)\right) = \mathcal{L}_{BI}\left(g' = \frac{1}{g}, \mu' = \frac{\mu}{g^2}, F_{\mu\nu}(C)\right), \quad (153)$$

where $C$ is the dual gauge field. On the other hand, in the presence of the Green–Schwarz term ($m \neq 0$), one can eliminate the vector altogether and work, in the dual formulation, solely in terms of the two–form $B_{\mu\nu}$. The self–duality of the massless BI theory then implies that the dual action involves a BI mass term and reads

$$\mathcal{L} = -\frac{k^2}{12} H_{\mu\nu\rho}(B) H^{\mu\nu\rho}(B) + \frac{\mu^2}{8g^2} \left[ 1 - \sqrt{1 + \frac{4m^2 g^4}{\mu^2} B_{\mu\nu} B^{\mu\nu} - \frac{4 m^4 g^8}{\mu^4} (B_{\mu\nu} \star B^{\mu\nu})^2} \right]. \quad (154)$$

The massless limit can be recovered reintroducing the gauge invariant combination $mB + dC$ before letting $m \to 0$. In this fashion, the limiting Lagrangian describes a massless two–form, dual to a scalar, and a dual massless vector $C$.

This procedure could be relevant when the system is coupled to $\mathcal{N} = 2$ supergravity, in which case one of the two gravitino would belong to a massive spin-$\frac{3}{2}$ multiplet.

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