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Efficient multicut enumeration of $k$-out-of-$n$:F and consecutive $k$-out-of-$n$:F systems

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ABSTRACT

We study multiple simultaneous cut events for $k$-out-of-$n$:F and linear consecutive $k$-out-of-$n$:F systems in which each component has a constant failure probability. We list the multicuts of these systems and describe the structural differences between them. Our approach, based on combinatorial commutative algebra, allows complete enumeration of the sets of multicuts for both kinds of systems. We also analyze the computational issues of multicut enumeration and reliability computations.

A system $S$ is formed by several components each of which can perform at different levels with different probabilities. The performance of the system is the result of the combination of the performance states of its components. A system is called coherent if the degradation of the performance of any of its components cannot lead to an improvement of the system’s performance and the improvement of any component cannot lead to degradation of the system’s performance. Coherent systems include most industrial and biological systems, networks or workflow diagrams. The reliability of a coherent system $S$ is the probability that the system is performing at a certain working level. The unreliability of $S$ is defined as the probability that it is performing below the required level. Here we focus on system’s unreliability, but our methods can directly be applied to its reliability. The study of system (un)reliability is a central area in engineering. In the recent years new mathematical methods and approaches have been developed including combinatorial methods, the study of correlated probabilities or minimal path and cut enumeration, cf. the recent papers [Li et al. (2016); Gäsemeyer and Natvig (2017); Song et al. (2016); Wang et al. (2017); Forghani-Elahabad and Bonani (2017); Borganovo et al. (2016)].

Let $S$ be a coherent system with $n$ components, each of which has a constant probability of failure $p_i$, $i \in \{1, \ldots, n\}$. If all $p_i$’s are equal and independent then the system is said to have i.i.d. (independent and identically distributed) components. We will consider that the failure probabilities of the components are mutually independent but do not need to be identical. The method does however apply to more general situations in which independence is not assumed. For the system $S$ we define a set of minimal failure states, i.e. states under which the system is failed and such that the improvement of any of its components would lead the system to a working state. We call such minimal failure states minimal cuts, while any failure state is simply called a cut. The unreliability of $S$ is described as the probability that at least one minimal failure event occurred.

We are interested in the behaviour of the system $S$ under several simultaneous failure events. By studying the distribution of the variable $Y = \{\text{number of minimal failure events}\}$ we obtain further information about the structure and behaviour of $S$. This is relevant for the management of spare parts, supplies and maintenance of the system. The consideration of simultaneous failures is important for instance in high performance computational systems, see [Thanakornworaki et al. (2013)]. Analogous to cuts we define $i$-multicuts as those system states that correspond to $i$ simultaneous cuts. A minimal $i$-multicut is a state in which the system is suffering $i$ simultaneous cuts but the improvement of any component makes the number of simultaneous cuts strictly smaller than $i$. Consider for instance the series system $S$ with 3 components. A minimal cut of $S$ is for instance $\{1\}$ which means that the component 1 is failing. The cut given by $\{1, 2, 3\}$ is a non-minimal cut, a non-minimal 2-multicut and a minimal 3-multicut for $S$.

The probability that a system $S$ is suffering at least $i$ simul-
taneous failures is the union event of all minimal $i$-multicuts of $S$. A first step to compute this probability is to enumerate the combinations of $i$ simultaneous failures for each $i \geq 1$ and then compute the corresponding probabilities. This enumeration is hard in general for the number of minimal $i$-multicuts can grow exponentially.

$k$-out-of-$n$:F and consecutive $k$-out-of-$n$:F systems (Cons/$k$-out-of-$n$:F) are fundamental examples of coherent systems that generalize the basic series and parallel systems. These systems are ubiquitously studied and applied because of their redundancy, which is useful in a large variety of situations, see Zuo et al. (2000); Huang et al. (2003); Eryilmaz (2010); Gökdere et al. (2016). The tool we use to analyze multicuts and unreliability of those systems is the algebraic approach described by the authors in Sáenz-de-Cabezón and Wynn (2009, 2010, 2011, 2015). This approach consists in associating a monomial ideal $I_S$ to the system $S$ and analyze the features of $S$ by studying the properties of $I_S$. The first section of the paper describes the algebraic approach to system reliability and its application to the problem of simultaneous failure events. In §2 and §3 we focus on $k$-out-of-$n$:F systems and Cons/$k$-out-of-$n$:F systems, and give a description of their $i$-multicuts structures, demonstrating their significant differences. Finally in §4 we consider some of the computational aspects of the problem and show the results of computer experiments based on the implementation of the tools previously described.

1. Algebraic reliability primer

Let $S$ be a coherent system with $n$ components. For ease of notation we assume that $S$ is a binary system, i.e. the system and each of its components $c_1, \ldots, c_n$ can be either failed (1) or working (0). However, this method works also for multistate components. Let $\mathcal{F}_S$ be the set of failure states (cuts) of $S$ and let $\mathcal{F}_S^c$ be the set of minimal cuts. Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{k}$. Each state of $S$ corresponds to a monomial in $R$: the monomial $x_{i_1} \cdots x_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$ corresponds to the state in which components $c_{i_1}, \ldots, c_{i_r}$ are failed and the rest are working. The coherence property of $S$ is equivalent to the fact that the elements of $\mathcal{F}_S$ correspond to the monomials in a monomial ideal, the failure ideal of $S$, which we denote by $I_S$. The unique minimal monomial generating set of $I_S$ is formed by the monomials corresponding to the elements of $\mathcal{F}_S$, see (Sáenz-de-Cabezón and Wynn 2009 §2). Obtaining the set of minimal cuts of $S$ amounts to compute the minimal generating set of $I_S$. In order to compute the unreliability of $S$ we can use the numerator of the Hilbert series of $I_S$ which gives a formula, in terms of $x_1, \ldots, x_n$ that enumerates all the monomials in $I_S$, i.e. the monomials corresponding to states in $\mathcal{F}_S$. Hence, computing the (numerator of the) Hilbert series of $I_S$ provides a way to compute the unreliability of $S$ by substituting $x_i$ by $p_i$, the failure probability of the component $i$ as explored in (Sáenz-de-Cabezón and Wynn 2009 §2). Furthermore, in order to have a formula that can be truncated to obtain bounds for the reliability in the same way that the inclusion-exclusion formula is truncated to obtain the so-called Bonferroni bounds, we need a special way to write the numerator of the Hilbert series of $I_S$. This convenient form is given by the alternating sum of the ranks in any free resolution of $I_S$. Every monomial ideal has a minimal free resolution, which provides the tightest bounds among the aforementioned ones. In general, the closer the resolution is to the minimal one, the tighter the bounds obtained, see e.g. section 3 in Sáenz-de-Cabezón and Wynn (2009).

In summary, the algebraic method for computing the unreliability of a coherent system $S$ works as follows:

- Associate to the system $S$ its failure ideal $I_S$ and obtain the minimal generating set of $I_S$ to get the set $\mathcal{F}_S$ of minimal cuts of $S$.

- Compute the Hilbert series of $I_S$ to have the unreliability of $S$.

- Compute any free resolution of $I_S$. The alternating sum of the ranks of this resolution gives a formula for the Hilbert series of $I_S$, i.e. the unreliability of $S$ which provides bounds by truncation at each summand. If the computed resolution is minimal, then these bounds are the tightest of this type.

Observe that this method works without alterations if the failure probabilities of the components of $S$ are mutually different. For more details and the proofs of the results enunciated here, see Sáenz-de-Cabezón and Wynn (2009). For more applications of this method in reliability analysis, see Sáenz-de-Cabezón and Wynn (2010, 2011, 2015).

When studying multiple simultaneous failures in a coherent system, the above methods can be extended to a filtration of ideals associated to the system under study as follows (cf. Mohammadi et al. (2017)): Let $S$ be a coherent system in which several minimal failures can occur at the same time. Let $Y$ be the number of such simultaneous failures. The event $\{Y \geq 1\}$ is the event that at least one elementary failure event occurs, which is the same as the event that the system fails. If $x^α$ and $x^β$ are the monomials corresponding to two elementary failure events, then $\text{lcm}(x^α, x^β) = x^{α∨β}$ corresponds to the intersection of the two events. The full event $\{Y \geq 2\}$ corresponds to the ideal generated by all such pairs. The argument extends to $\{Y \geq i\}$ and the study of the tail probabilities $\text{prob}(Y \geq i)$. For each $i$, the set of $i$-multicuts generates an ideal, which altogether form a filtration. Let $I \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be a monomial ideal and $\{m_1, \ldots, m_r\}$ its minimal monomial generating set. Let $I_i$ be the ideal generated by the least common multiples of all sets of $i$ distinct monomial generators of $I$, i.e. $I_i = \langle \text{lcm}(m_{i,j}, \sigma) : \sigma \subseteq \{1, \ldots, r\}, |\sigma| = i \rangle$. We call $I_i$ the $i$-fold lcm-ideal of $I$. The ideals $I_i$ form a descending filtration $I = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r$, which we call the lcm-filtration of $I$. If $I = I_S$ is the failure ideal of a system $S$ then the minimal generating set of $I_i$ is formed by the monomials corresponding to minimal $i$-multicuts of $S$. The survivor functions $F(i) = \text{prob}(Y \geq i)$ for a coherent system, are obtained from the multigraded Hilbert function of the $i$-fold lcm-ideal $I_i$. The above considerations can be summarized in the following proposition.
Proposition 1.1. Let $Y$ be the number of failure events of a system $S$. If $\{m_1, \ldots, m_l\}$ is the set of monomial minimal generators of the failure ideal $I_1 = I_1$ then $\text{prob}(Y \geq i) = E[I_1(\alpha)] = \mathcal{H}_l(x)$, where $1_{\mathcal{M}_i}$ is the indicator function of the exponents of monomials in the $i$-fold lcm-ideal $I_i$ and $\mathcal{H}_i(x)$ is the numerator of its multigraded Hilbert series.

As a result, we also obtain identities and lower and upper bounds for multiple failure probabilities from free resolutions as in the single failure case.

2. Multiple simultaneous failures in $k$-out-of-$n$:F systems

A $k$-out-of-$n$:F system denoted by $S_{k,n}$ is a system with $n$ components that fails whenever $k$ of them fail. The failure ideal of $S_{k,n}$ is given by $I_{k,n} = \langle \prod_{i \in \sigma} x_i : \sigma \subseteq \{1, \ldots, n\}, |\sigma| = k \rangle$. Let $I_{k,n,i}$ be the $i$-fold lcm-ideal of $I_{k,n}$.

Theorem 2.1. For all $k < j \leq n$ and $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix}) < l \leq (\begin{smallmatrix} j \end{smallmatrix})$ we have that $I_{k,n,i} = I_{j,n} = \langle \prod_{i \in \sigma} x_i : \sigma \subseteq \{1, \ldots, n\}, |\sigma| = j \rangle$.

Proof. Let $k < j \leq n$. We want to see that $I_{j,n} = I_{k,n,i}$ for all $l$ such that $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix}) < l \leq (\begin{smallmatrix} j \end{smallmatrix})$. This amounts to show that for all such $l$ the following apply:

(i) Any subset of $\{1, \ldots, n\}$ of cardinality $j$ (i.e. every minimal generator of $I_{j,n}$) can be obtained as the union of $l$ subsets of $\{1, \ldots, n\}$ of cardinality $k$.

(ii) No such subset of cardinality $j$ can be obtained as the union of more than $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ subsets of cardinality $k$.

(iii) Taking any union of less than $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})+1$ subsets of cardinality $k$ produces some sets of cardinality less than $j$.

To see (i) observe that to obtain a set of cardinality $j$ we only need the union of $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ sets of cardinality $k$. Just consider $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ sets of $k$ disjoint elements, which is always possible, and add another one which contains the remaining $j-\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ elements. To see (ii) observe that any union of more than $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ sets of cardinality $k$ has at least cardinality $j+1$. This is because for every set of $j$ elements, we can choose $k$ of them in at most $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ ways. Hence if we have more than $\langle (\begin{smallmatrix} j \end{smallmatrix}) \rangle$ sets of cardinality $k$, their union must have more than $j$ elements. To see (iii) consider any set of $j-1$ elements. These can be chosen in sets of cardinality $k$ in $\langle (\begin{smallmatrix} j-1 \\ k \end{smallmatrix}) \rangle$ ways. The union of all these $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})$ sets has then $j-1$ elements.

Using (i), (ii) and (iii) we have that for all $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})+1 \leq l \leq (\begin{smallmatrix} j \end{smallmatrix})$ $I_{k,n,i}$ is generated by some subsets of $\{1, \ldots, n\}$ all of which have cardinality $j$. We still have to check that it is generated by all such sets. For this, take any subset $\sigma' \subseteq \{1, \ldots, n\}$ such that $|\sigma'| = j$. Let $i$ be an element of $\sigma$ and $\sigma'' = \sigma - i$. We can form $\sigma$ as the union of $l$ sets of cardinality $k$ for all $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})+1 \leq l \leq (\begin{smallmatrix} j \end{smallmatrix})$. Observe that $\sigma'$ can be obtained as the union of $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})$ sets of cardinality $k$. Now, we can add to this union any number of sets of cardinality $k$ that contribute only with the element $i$ to it. We can do this with any number of sets from 1 to $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix})$. Since $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix}) + (\begin{smallmatrix} j-1 \\ k-1 \end{smallmatrix}) = (\begin{smallmatrix} j \end{smallmatrix})$ the proof is complete.

Theorem 2.1 shows that the minimal multicuts of $k$-out-of-$n$:F systems are ordinary minimal cuts for other $j$-out-of-$n$:F systems. On the other hand, the simultaneous failure ideals follow a staircase pattern, in the sense that their reliability is the same for several numbers of simultaneous failures until a certain particular number occurs. Theorem 2.1 also enables us to study the reliability polynomial applying geometric techniques developed in Mohammadi (2016). The fact that the ideals in the lcm-filtration of $I_{k,n}$ correspond to the failure ideals of other $j$-out-of-$n$ systems allows us to obtain reliability identities and bounds by making use of the known formulae for the Betti numbers of these systems, cf. Sáenz-de-Cabañes and Wynn (2009).

Corollary 2.2. In the case of i.i.d. components, the graded Betti numbers of $I_{k,n,i}$ are given by

$$
\beta_{a,a+j}(I_{k,n,i}) = \beta_{a,a+j}(I_{j,n}) = \left( \begin{array}{c} n \\ j \end{array} \right) \frac{(a + j - 1)}{(j - 1)}
$$

for all $(\begin{smallmatrix} j-1 \\ k \end{smallmatrix}) < j \leq (\begin{smallmatrix} j \end{smallmatrix})$ and $0 \leq a \leq n - j$. For $b \neq a + j$ we have that $\beta_{a,b}(I_{k,n,i}) = 0$. For the multigraded Betti numbers, we have that

$$
\beta_{a,0}(I_{k,n,i}) = \beta_{a,0}(I_{j,n}) = \left( \begin{array}{c} a + j - 1 \\ j \end{array} \right)
$$

if $a$ is the product of $a + j$ variables and 0 otherwise.

Remark 2.3. From the graded Betti numbers we obtain the unreliability polynomial of the $j$-out-of-$n$ systems as

$$
f_{j,n}(x) = \sum_{a=0}^{n-j} (-1)^a \beta_{a,a+j}(x^{a+j}) = \sum_{a=0}^{n-j} (-1)^a \left( \begin{array}{c} n \\ j \end{array} \right) \left( \begin{array}{c} a + j - 1 \\ j - 1 \end{array} \right) \sum x^a
$$

By truncating this polynomial we obtain bounds for the reliability formula. The multigraded version is the following

$$
f_{j,n}(x) = \sum_{a=0}^{n-j} (-1)^a \left( \begin{array}{c} a + j - 1 \\ j - 1 \end{array} \right) \sum_{\mathcal{M}_i \subseteq (\begin{smallmatrix} n \\ j \end{smallmatrix})} x^a
$$

where $[n, j + a]$ denotes the set of vectors with 1 in the indices of the $(j + a)$-subsets of $\{1, \ldots, n\}$ and 0 in the other entries.

Example 2.4. Let us consider a $k$-out-of-$n$:F system for $k = 2$ and $n = 8$. The failure ideal $I_{2,8}$ of this system is generated by all the 28 possible pairs of 8 variables. The lcm-ideals that describe the multiple failures of this system are given in Table 1.

| $I_{2,8}$ | $I_{2,8,1}$ | $I_{2,8,2}$ | $I_{2,8,3}$ |
|----------|---------|---------|---------|
| $I_{2,8,4}$ | $I_{2,8,5}$ | $I_{2,8,6}$ |
| $I_{2,8,7}$ | $I_{2,8,8}$ |
| $I_{2,8,9}$ |
| $I_{2,8,10}$ |
| $I_{2,8,11}$ | $I_{2,8,12}$ |
| $I_{2,8,13}$ | $I_{2,8,14}$ |
| $I_{2,8,15}$ |
| $I_{2,8,16}$ | $I_{2,8,17}$ |
| $I_{2,8,18}$ | $I_{2,8,19}$ |
| $I_{2,8,20}$ | $I_{2,8,21}$ |
| $I_{2,8,22}$ | $I_{2,8,23}$ |
| $I_{2,8,24}$ |
| $I_{2,8,25}$ |
| $I_{2,8,26}$ |
| $I_{2,8,27}$ |
| $I_{2,8,28}$ |

Table 1. lcm- ideals (left) and multicut ideals (right) of the failure ideals of $S_{2,8}$

Figure 1 shows the unreliability of i.i.d. j-out-of-8:F systems for $j$ from 2 to 8, each of which corresponds to the probability of multiple simultaneous failures of the original system. For example, the unreliability of 5-out-of-8 system corresponds to
For any $i$-th responding to $j$-th, when the failure probability is lower, the probability of multiple failures drops rapidly as the number of simultaneous failures grows. On the other hand, when the failure probability is higher, the probability of multiple failures appears for a lower number of simultaneous failures. Figures 2 shows the probability that multiple failures appear in $S_{2,8}$ for three different component failure probabilities $p = 0.2, 0.5$ and $0.8$. This graph shows the staircase behaviour of $k$-out-of-$n$:F systems subject to several simultaneous failures. For high failure probability, e.g. $p = 0.8$, the probability of multiple failures drops slowly at the beginning but decreases more rapidly as the number of simultaneous failures grows. On the other hand, when the failure probability is lower, the probability of simultaneous failures drops drastically with fewer simultaneous failure events.

3. Cons/$k$-out-of-$n$:F systems

A Cons/$k$-out-of-$n$:F system $T_{k,n}$ has $n$ components and fails whenever $k$ consecutive components fail. The ideal of $T_{k,n}$ is given by $J_{k,n} = \langle x_1 \cdots x_{k}, x_2 \cdots x_{k+1}, \ldots, x_{n-k+1} \cdots x_n \rangle$. Let $J_{k,n,i}$ be the $i$-fold lcm-ideal of $J_{k,n}$, generated by the monomials corresponding to $i$ simultaneous minimal failures of $T_{k,n}$.

Let us denote by $S$ the set of subsets of $\{1, \ldots, n - k + 1\}$. For any $\sigma$ in $S$ let $\min(\sigma)$ and $\max(\sigma)$ denote the minimum and maximum element of $\sigma$ respectively. Let $S_i$ denote the elements of $S$ of cardinality $i$. We say that $\sigma \in S$ has a gap of size $s$ if there is a subset of $s$ consecutive elements of $\{\min(\sigma), \ldots, \max(\sigma)\}$ that are not in $\sigma$. For instance, $\sigma = \{2, 3, 6, 8\}$ has a gap of size 2 (because 4 and 5 are not in $\sigma$) and a gap of size 1 because 7 is not in $\sigma$. Let $S_i$ be the set of those $\sigma \in S$ such that the smallest gap in $\sigma$ has size $a$. Let $S_i^a$ be the set of all elements in $S_i$ of cardinality $i$.

**Theorem 3.1.** The ideal $J_{k,n,i}$ is minimally generated by the monomials $m_{\sigma}$ with $\sigma \in S_i^0 \cup S_i^1 \cup S_{i+1}^1 \cup \cdots$. In other words, the minimal generators of $J_{k,n,i}$ correspond to the lcm’s of sets of monomials of cardinality $i$ with no gaps of sizes 1, 2, \ldots, $k-1$.

**Proof.** Let $m_{\sigma} = \text{lcm}(m_j)_{j \in \sigma}$. We know that the set $\{m_{\sigma}, \sigma \in S, |\sigma| = i\}$ is a generating set of $J_{k,n,i}$. Let $G_{k,n,i}$ be the minimal set of generators of $J_{k,n,i}$. To prove the theorem we must show that

(i) $S_i^0 \subseteq G_{k,n,i}$.

(ii) If $\sigma \in S_i^0$ with $1 \leq a \leq k - 1$ then there is another element $\tau$ in $S_i^0$ with $b = 0$ or $b > a$ such that $m_{\tau}$ divides $m_{\sigma}$ (and hence $m_{\sigma} \notin G_{k,n,i}$ i.e. it is not minimal).

(iii) If $\sigma \in S_i^a$ with $a \geq k$ then $m_{\sigma} \in G_{k,n,i}$.

To see (i) observe that if $\sigma \in S_i^0$ then $m_{\sigma}$ is the lcm of $i$ consecutive minimal generators of $J_{k,n}$, say $m_{\sigma_1}, \ldots, m_{\sigma_{k-1}}$. Hence $m_{\sigma} = x_{i_1} \cdots x_{i_{k-1}}$ which is a product of $i + k - 1$ variables. Using $i$ different subsets of $\{1, \ldots, n - k + 1\}$ we obtain at least $i + k - 1$ variables. Note that we can only achieve this minimum if the $i$ elements of $\sigma$ are consecutive, i.e. if we are in $S_i^0$. Hence, if there is another $\sigma' \in S_i^0$ that divides $\sigma$ then it must be $\sigma'$ itself as any other element of $S_i^0$ produces a different set of variables and would not divide $m_{\sigma}$.

Now, to see (ii) let us assume without loss of generality that $\sigma$ is formed by two blocks $\sigma_1$ and $\sigma_2$ of consecutive elements, separated by a gap of size $a$ with $1 \leq a \leq k - 1$. For ease of notation we can consider $\sigma_1 = \{1, \ldots, i_1\}$ and $\sigma_2 = \{i_1 + a, \ldots, i_2\}$. Then $m_{\sigma} = x_{i_1} \cdots x_{i_{k-1}} \cdot (x_{i_1+a} \cdots x_{i_{k-1}+a})$. But now, since $a < k$ we have that $\sigma' = \{1, \ldots, i_1 + 1\} \cup \{i_1 + a + 1, \ldots, i_2\}$ is in $S_i^0$ with $b = 0$ (in fact it is either in $S_0^0$ or in $S_0^a$) and $m_{\sigma'}$ divides $m_{\sigma}$. In case $b \neq 0$, we repeat this procedure until we arrive at $b = 0$, where $m_{\{1, \ldots, i_2\}}$ divides $m_{\sigma}$ and hence $m_{\sigma}$ is not a minimal generator of $J_{k,n,i}$. Observe that the set $\{1, \ldots, i\}$ is in $S_0$. We could have also proceeded in a symmetric way, i.e. using $\sigma' = \{1, \ldots, i_1 - 1\} \cup \{i_1 + a - 1, \ldots, i_2\}$ in $S_i^0$ and we would obtain another element in $S_i^0$ that divides $\sigma$.

Finally, to see (iii) let us assume again, without loss of generality, that $\sigma_1 = \{1, \ldots, i_1\}$ and $\sigma_2 = \{i_1 + a, \ldots, i_2\}$ with $a \geq k$. Now, if $\tau \in S_i^0$ is different from $\sigma$ then there is at least one index $\tau_j$ such that $\tau_j \neq \sigma_j$. If $\tau_j > i_2$ then there exists at least a variable bigger than $x_{i_1+k-1}$ in $\tau$ which is not in $\sigma$ and hence $m_{\sigma}$ does not divide $m_{\tau}$. The same happens if $i_1 < j < i_1 + a$: since $a \geq k$ every $m_j$ with $i_1 < j < i_1 + a$ has a variable that does not belong to $\{x_{i_1}, \ldots, x_{i_1+k-1}\} \cup \{x_{i_1+a}, \ldots, x_{i_1+k-1}\}$. To see this observe that $i_1 + a > i + k - 1$ and every $m_j$ with $i_1 < j < i_1 + a$ has at least one variable in the nonempty set $\{x_{i_1}, \ldots, x_{i_1+k-1}\}$. Hence $\tau$ does not divide $\sigma$ and so $\sigma$ is in $G_{k,n,i}$. \[\square\]

**Remark 3.2.** Each $\tau \subseteq \{1, \ldots, n\}$ is given by blocks formed by subsequent indices (i.e. with no gaps among them), for instance, $123679$ is formed by a block of three indices $(123)$. a
block of two indices (67) and a block of one index (9). From these blocks we can read the degree of the corresponding generator: if \( \tau \) is formed by \( l \) blocks of sizes \( s_1, s_2, \ldots, s_l \) then \( m_{\tau} \) has degree \((s_1 + 1) + (s_2 + 1) + \cdots + (s_l + 1)\).

**Example 3.3.** \( J_{2:9:4} \) is minimally generated by 26 monomials that correspond to taking lcm’s of the following sets of generators of \( J_{2:9} \):

| Block sizes | sets | deg. of gens. |
|-------------|------|---------------|
| 4           | 1234,2345,3456,4567,5678 | 5 |
| 3,1         | 1236,1237,1238,2347,2348,3458 | 6 |
| 2,2         | 1256,1267,1278,2367,2378,3478 | 6 |
| 2,1,1       | 1258,1458,1478 | 7 |

Observe that 1236 in the table means lcm of generators 1, 2, 3 and 6 (i.e. \( x_1 x_2 x_3 x_4 x_5 x_6 x_7 \)), so 1236 corresponds to \( x_1 x_2 x_3 x_4 x_6 x_7 \), which means that the failure of the components 1, 2, 3, 4, 6 and 7 is a minimal 4-multicut for \( T_{2:9} \).

**Example 3.4.** Let us consider a Cons/2-out-of-9:F system. Its failure ideal is \( J_{2:9} = (x_1 x_2 x_3 x_4 x_5 x_6 x_7) \). Figure 3 shows on one hand the unreliability polynomials of Cons/i-out-of-9:F systems for \( i \) between 2 and 9, and on the other hand the polynomials corresponding to \( i \) multiple simultaneous failures of a Cons/2-out-of-9:F system for \( i \) from 1 to 8. Observe that the multiple failures are, unlike in the ordinary \( k \)-out-of-\( n \):F case, different from other Cons/\( k \)-out-of-\( n \):F systems; the graphs reflect this different behaviour.

If we consider the probability of \( i \) simultaneous failure events for a Cons/\( k \)-out-of-\( n \):F system with i.i.d components we observe that this system does not show the staircase behaviour of ordinary \( k \)-out-of-\( n \):F systems. This is seen in Figure 4, that shows how the probability that multiple failures appear in the i.i.d. Cons/2-out-of-9:F system for three different component probabilities \( p = 0.2, 0.5 \) and 0.8.

4. **Computational experiments**

4.1. **Number of generators**

One of the main drawbacks of the use of enumerative methods for the reliability analysis of systems with multiple simultaneous failures is that the number of possible combinations of such simultaneous failures grows exponentially. For a system with \( r \) minimal failure events (cuts) the number of \( i \)-multicuts can be \( \binom{r}{i} \). However, in most systems, the actual number is much lower, there is redundancy in those \( \binom{r}{i} \) combinations of cuts. We then face two problems: the big number of possible cut combinations and how to discard the redundant ones.

In the case of \( k \)-out-of-\( n \):F and Cons/\( k \)-out-of-\( n \):F systems, our methods efficiently enumerate the minimal \( i \)-multicuts of the system. Since \( k \)-out-of-\( n \):F systems with \( i \) simultaneous failures correspond to another \( j \)-out-of-\( n \):F system, we can count the generators of the corresponding ideal using Theorem 2.1. The corresponding number of generators is much smaller than the possible number \( \binom{n - 1}{i - 1} \) as indicated in Table 2 for the 2-out-of-8 system (showing the systems up to 10 simultaneous failures).

The list of generators of the iterated Cons/\( k \)-out-of-\( n \):F ideals are given by Theorem 3.1. Counting them for given \( k, n \) and \( i \) amounts to count all the subsets of \( \{1, \ldots, n - k + 1\} \) of cardinality \( i \) having no gaps of sizes between 1 and \( k - 1 \).

- Since the sets we are considering are sorted, the initial element can only be 1 up to \( n - k - i + 1 \). For each such initial element \( m \), we can consider the sum of all the gaps in the set to be at most \( n - k - i + 1 \).

- Each of such total gaps can be obtained using at most \( i - 1 \) summands (the gaps in a set of cardinality \( i \) can at most be \( i - 1 \) each of which has to be at least \( k \). For each number \( a \leq i - 1 \) of summands this is equivalent to count all the compositions of the integer \( k \) in at most \( i \) parts of smallest size \( k \). This number is known to be \( \binom{m - (k - 1)a - 1}{a - 1} \), cf. Andrews (1984), Knopfmacher and Munagi (2013).

- Each of these compositions can be used in \( \binom{i - 1}{a - 1} \) ways.

From these considerations we obtain the following formula for the number of generators of the \( i \)-th lcm-ideal of a Cons/\( k \)-out-of-\( n \):F system. This formula has a summand corresponding to the sets in \( S_0^k \) and another one to those in \( S_m^k \) for \( m > 0 \).

**Proposition 4.1.** The number of generators of the \( i \)-fold lcm-ideal of a Cons/\( k \)-out-of-\( n \):F system is

\[
N_{k,n,i} = \sum_{m=1}^{n-k-i+1} (n-k-i+2-m) \binom{i-1}{a-1} \left( \binom{m-(k-1)a-1}{a-1} \right) + (n-k-i+2).
\]
Table 2. Number of minimal generators for the ideals of 2-out-of-8 system with \( i \) simultaneous failures, \( i = 1 \ldots 10 \).

| \( i \) | \( \binom{8}{i} \) | \( \binom{i}{2} \) |
|---|---|---|
| 1 | 28 | 26 |
| 2 | 378 | 56 |
| 3 | 3276 | 56 |
| 4 | 20475 | 70 |
| 5 | 98280 | 70 |
| 6 | 376740 | 70 |
| 7 | 1184040 | 56 |
| 8 | 3108105 | 56 |
| 9 | 6906900 | 56 |
| 10 | 13123110 | 56 |

5. Conclusions

The main computational tasks that one has to face when computing the reliability of systems subject to multiple simultaneous failures are two: the first is to compute the minimal failure events, which grow exponentially with respect to the number of minimal failures of the initial system. In algebraic terms, this is equivalent to the enumeration of the minimal generating set of the corresponding monomial ideals. The second task is to obtain reliability formulae and bounds.

In systems with low redundancy, such as consecutive 2-out-of-\( n \) for large \( n \), the first problem is computationally expensive. In fact it can be even more expensive than the actual reliability formulae computations. This can be seen by comparing the Total row of columns Binomial and \( \text{MVT} \) in Table 4. The
importance of results such as Theorem 3.1 and Theorem 2.1 is clearer at the sight of the computation times.

With respect to reliability computations, a general approach such as the algebraic one can be useful when closed form formulae are not available, in particular when the components have different failure probabilities, which is the main assumption in many approaches, cf. [Daus and Beiu (2015)]. Since this method is general, improvements in its implementations, like the MVT algorithm in CoCoALib may result in a useful tool for reliability analysis.

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Table 3. Number of minimal generators for Cons/5-out-of-20:F ideals with i simultaneous failures, i = 1 . . . , 10.

| i | # of gens. | Binomials | Formula | Hilbert | Resolution | MVT |
|---|---|---|---|---|---|---|
| 1 | 19 | 0.001 | 0.001 | 0.209 | 2.139 | 0.024 |
| 2 | 154 | 0.008 | 0.024 | 0.096 | 27.840 | 0.660 |
| 3 | 712 | 0.063 | 0.103 | 0.168 | 81.742 | 1.432 |
| 4 | 2138 | 0.324 | 0.651 | 0.333 | 144.514 | 2.676 |
| 5 | 4537 | 1.753 | 1.163 | 0.524 | 201.975 | 6.452 |
| 6 | 7248 | 4.531 | 2.242 | 0.753 | 195.633 | 14.424 |
| 7 | 9143 | 10.012 | 5.086 | 0.746 | 148.765 | 25.952 |
| 8 | 9434 | 18.112 | 5.208 | 0.725 | 103.777 | 24.664 |
| 9 | 8169 | 32.283 | 7.065 | 0.556 | 60.028 | 15.932 |
| 10 | 6046 | 34.076 | 4.138 | 0.336 | 23.037 | 7.036 |
| 11 | 3874 | 33.413 | 2.566 | 0.246 | 10.849 | 2.652 |
| 12 | 2164 | 25.540 | 2.025 | 0.075 | 4.573 | 0.780 |
| 13 | 1067 | 15.375 | 1.501 | 0.039 | 0.484 | 0.176 |
| 14 | 448 | 6.946 | 0.263 | 0.016 | 0.097 | 0.032 |
| 15 | 180 | 2.086 | 0.113 | 0.006 | 0.014 | 0.012 |
| 16 | 49 | 0.449 | 0.032 | 0.002 | 0.002 | 0 |
| 17 | 19 | 0.082 | 0.011 | 0.001 | 0.001 | 0 |
| 18 | 2 | 0.001 | 0.001 | 0 | 0 | 0 |
| 19 | 1 | 0.001 | 0.001 | 0 | 0 | 0 |
| Total: | – | 183.065 | 32.194 | 4.831 | 1005.168 | 102.908 |

Table 4. Time for computing the generators, Hilbert series and resolutions of multiple failure ideals of the Cons/2-out-of-20 system.

| i | Time (s) |
|---|---|
| 1 | 16 |
| 2 | 120 |
| 3 | 560 |
| 4 | 1820 |
| 5 | 4368 |
| 6 | 8008 |
| 7 | 11440 |
| 8 | 12870 |
| 9 | 11440 |
| 10 | 8008 |

| # of gens. | Time (s) |
|---|---|
| 16 | 7.036 |
| 70 | 4.138 |
| 124 | 0.032 |
| 151 | 0.001 |
| 148 | 0.014 |
| 125 | 0.001 |
| 100 | 0.014 |
| 79 | 0.001 |
| 56 | 0.001 |
| 34 | 0.001 |