MATHEMATICAL INTERPRETATION OF PLATO’S THIRD MAN ARGUMENT BASED ON THE NOTION OF CONVERGENCE

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ABSTRACT. The main aim of this article is to defend the thesis that Plato apprehended the structure of incommensurable magnitudes in a way that these magnitudes correspond in a unique and well defined manner to the modern concept of the Dedekind cut. Thus, the notion of convergence is consistent with Plato’s apprehension of mathematical concepts, and in particular these of density of magnitudes and the complete continuum in the sense that they include incommensurable cuts. For this purpose I discuss and interpret, in a new perspective, the mathematical framework and the logic of the Third Man Argument (TMA) that appears in Plato’s Parmenides as well as mathematical concepts from other Platonic dialogues. I claim that in this perspective the apparent infinite sequence of $F$-Forms, that it is generated by repetitive applications of the TMA, converges (in a mathematical sense) to a unique $F$-Form for the particular predicate. I also claim and prove that within this framework the logic of the TMA is consistent with that of the Third Bed Argument (TBA) as presented in Plato’s Republic. This supports Plato’s intention for assuming a unique Form per Predicate; that is, the Uniqueness thesis.

1. INTRODUCTION

In this article I aim to provide an adequate mathematical interpretation of the classical Third Man Argument (TMA) which is strongly related with the mathematical approach of Plato’s theory of Forms.

In this section I introduce the problem, and in the second section I present the framework within which my arguments are based on, and the Main Claim is founded. In the third section I analyze, defend and eventually provide an adequate proof of the Main Claim. In the fourth section I present in detail the mathematical concepts and the topological framework within which my arguments and the main thesis are comprehended and analyzed. In the last section, summarizing the work done in the previous sections, I briefly present the main conclusions of my work.

2010 Mathematics Subject Classification. 62A01, 97E20, 03A05, 97E30.

Key words and phrases. Platonic Philosophy, Convergence, Complete Continuum, Infinity.
The literature dealing with TMA, already large in 1954, has become enormous since then and almost all of the authors have followed Gregory Vlastos, where in his famous paper of 1954 [31] pointed out that the argument is formally a non sequitur.

The classical TMA appears in the Plato’s dialogue Parmenides 132a1 – b1, and essentially its logic is present in other arguments in Platonic dialogues, such as the TMA version in Parm.132d1 – 133a6, where it is applied on the ‘Form-idea’ of ‘Resemblance’. I aim to adequately explain the apparent plurality of Forms (for a certain predicate) that appears in the TMA and to show that it is compatible with Plato’s thesis for the existence of a single form per predicate; that is the uniqueness thesis. For supporting this, I take into account the Platonic apprehension of fundamental mathematical concepts, the Mathematics developed in Plato’s Academy as well as Plato’s dialectic.

The Theory of Forms is also a theory of judgment. Judging involves consulting Forms: To judge that an object $x$, either a sensible particular or a Form, is $F$, is to consult the form of $F$-ness and to perceive $x$ as being sufficiently like $F$-ness to qualify for the predicate $F$. Alan Code in [5] suggests that the TMA raises an objection to this theory of judgment. I quote:

‘The TMA is designed to reduce to absurdity the claim that it is the consultation of Forms which enables us to make judgments. It does this by showing that if that were the case, we would have to perform an infinite number of such consultations to make just one judgment.’

Since the theory of Forms tries to explain predication, the TMA is also a challenge to it as a theory of predication. It is evident though that ‘participating in a Form’ is supposed to explain

1Text: ‘...This, I suppose, is what leads you to believe that each form is one. Whenever many things seem to you to be large, some one form probably seems to you to be the same when you look at them all. So you think that largeness is one. . . . But what about largeness itself and the other large things? If you look at them all in your mind in the same way, wont some one largeness appear once again, by virtue of which they all appear large? . . . So another form of largeness will have made an appearance, besides largeness itself and its participants. And there will be yet another over all these, by virtue of which they will all be large. So each of your forms will no longer be one, but an infinite multitude...’. (The translation is taken from Cohen and Keyt [8]).

2Plato’s intention in defending the uniqueness of a Form per predicate was clearly introduced in the Third Bed Argument, TBA, Republic 597c – d, and is also present in other Platonic texts. The related phrase ‘ἐν ἕκαστον εἶδος’ in Parm. 132a1 and its relation to the uniqueness thesis is analyzed by Cohen [6], pp.433 – 466. The uniqueness thesis should be expressed as:

‘There is exactly one Form corresponding to every predicate that has a Form’.

For a different version of the uniqueness thesis and an extensive discussion the reader should consult G. Fine [10].
predication. And the upshot of the TMA, as was presented by many authors, is that there is something defective about this explanation: Since trying to explain predication in terms of the notion of participating in a paradigmatic Form leads to an infinite regress, and hence is no explanation at all.

In developing and defending my arguments, I shall be consistent with the interpretation of the presence of a property in a thing, as well as the recurrence of a single property in different things. According to Scaltsas [26], the things are $F$ by participating in a Form $F$-ness is the answer to two different questions that Plato implies in *Phaedo* 100c9 – d8, *Parm.* 128e6 – 129a4, 130e5 – 131a2, as well as earlier in *Meno* 72c. The first is ‘Why is a thing $F$?’ That is, the first question concerns the predication of $F$-ness. In *Phaedo* 100c9 – d8 the forms are introduced as the causes of things being $F$. The second question is ‘Why are different things similar?’ This question that appears clearly in *Parm.* 128e6 – 129a4, 130e5 – 131a2 concerns recurrence of a single property in different things and considers the quality identity with respect to $F$-ness.

In this article I defend the thesis that the apparent infinite sequence of Forms, $\{F_i\}_{i=0}^{\infty}$, that appears by repetitive applications of the TMA, for either explaining predication or making a judgment that $x$ is $F$, it is increasing (ascending) and has a (mathematical) limit; that is, an attainable least upper bound. In other words, this infinite sequence $\{F_i\}_{i=0}^{\infty}$ converges to a unique Form $F$ for a particular predicate. Henceforth, $\lim_{i \to +\infty} F_i = F$, where the convergence is understood as a mathematical one. This terminating-limit $F$-Form should be also apprehended as compatible to the ‘anupotheton arxēn’ -‘ἀνυπόθετον ἀρχὴν’ - in Republic’s language, but applied here for each particular predicate $F$. Moreover, this $F$-Form should be also considered as analogous to the ‘final rung of Diotima’s ladder’ as presented in *Symposium* 210e ff. (In

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3In *Phaedo* the Form is also referred as the *F itself* (74a11 – 12), which is the ἄνωθεν $\alpha\iota\gamma\theta\sigma\upsilon$ $F$ (74b3 – 4), the cause ($\alpha\iota\nu\alpha\iota\alpha$) that makes (τοιοῦτο) things being $F$ (100c9 – d8), or the explanation of something being $F$.

4where $\mathbb{N}$ is the set of Natural Numbers.

5The formal definition of the convergence using the concept of a limit is: $\forall \varepsilon > 0$, there is a natural number $k_0$, such that $||F_i - F|| < \varepsilon$, whenever $n \geq k_0$. The $|| \cdot ||$ denotes the norm of the vector-topological space in concern. See also Rudin [24] for the mathematical definitions of the *limit* and the *least upper bound* ($l.u.b.$).

6The term appears in *Rep.* 509b – 511d, in 510b and in 511b, and shall be interpreted later in the paper. We do not give any translation, since any translation may lead to a specific interpretation. (In [30] *ad.loc.* is translated as ‘the principle that transcends assumption’.) For an extensive analysis of this passage and especially the concept and the status of this term we refer to Karasmanis [14], [15] and Benson [4].

7We shall see that this approach is compatible with the mathematics of Plato’s academy and Plato’s dialectic. For an elaborate and comprehensive exposition of Plato’s dialectic see Robinson [25] ch. 6, 7 and 10.
Symp. the above procedure is developed in the context of a particular Form, namely the Form of ‘Beauty’.

Furthermore, I shall argue and justify that the logics of the TMA and the TBA arguments do not contradict each other; instead they are consistent and mutually complementary. In this perspective it is necessary to apprehend this terminating \( F \)-Form, that is obtained as a (mathematical) limit of the above infinite sequence, in a somewhat different context than that of the various \( \{F_i\}_{i=0}^{\infty}, i \in \mathbb{N} \), in the sequence. This thesis is developed, properly analyzed and defended in the sequel.

2. Foundation of the Argument

Before proceeding, I shall briefly present the necessary historical background of the problem for developing and defending my arguments and my thesis.

In the following, the schematic letter ‘\( F \)’ shall serve as a dummy predicate for any predicate for which there is a Form. (It is typically used in place of predicate ‘large’ that appears in the TMA). We also note that we shall not deal with any issues related to the so called ‘Imperfection Argument’ (as entailed primarily in Rep. 523 – 525, or elsewhere in Platonic dialogues).

Gregory Vlastos, in his famous paper of 1954 [31], pointed out that the TMA is formally a non sequitur and he investigated the suppressed premises of the argument. There, he proposed the NI and SP axioms, asserting though that the only explicit premise of the TMA is the One-over-Many (OM) assumption.

\[
(OM_i.) \text{ If a number of things are all } F, \text{ there must be a single Form } F - ness, \\
in virtue of which we apprehend [them] as all } F.
\]

Vlastos, after the criticism of his first article (especially by Sellars [27]) and a long discussion, proposed in his second seminal article [32] a revised version of the premise-set for the TMA.

\^8 For example this argument does not posit a form even for every property-name; it posits a form for the predicate large but not a Form for the predicate man. And it supports that we can infer that is a Form of \( F \) only when we have a group that consists of imperfectly \( F \) things. Namely, the imperfection argument posits Forms both for restricted range of predicates and also a restricted range of groups. (For further details on the ‘Imperfection Argument’ we refer to G. Fine’s [10]).

\^9Sellars observed that Vlastos [31] in stating original NI and SP axioms had used the expression ‘\( F \)-ness’ as if it represented a proper name of a Form. Looked at in this way, SP and NI are defective, in that they contain free occurrences of the representative variable ‘\( F \)-ness’. According to Sellars the defect can be remedied with the aid of quantifiers. Thus, he proposed instead that ‘\( F \)-ness’ be taken to represent a quantifiable variable. This simple syntactic maneuver removed the remaining inconsistency. The TMA’s premises as Sellars formulated them in [27] are:
(OM$_{v2}$) If any of the set of things share a given character, then there exists a unique Form corresponding to that character; and each of these things has that character by participating in that Form.

(SP$_{v2}$) The Form corresponding to a given character itself has that character (Self-Predication).

(NI$_{v2}$) If anything has a given character by participating in a Form, it is not identical with that Form (Non-Identity).

There was (and still is) a long discussion on the different ways to express Self-Predication (SP) and Non-Identity (NI) premises-axioms. Here we shall adopt Cohen’s [6], [8] version of them, and we also allow for a ‘thing’ to be either a sensible particular or a Form. On this revised version SP and NI are compatible, but the three axioms together are not (see Cohen in [6], [8]). Furthermore, without the One-over-Many assumption the theory becomes incomplete. According to many authors, and in my point of view, Vlastos was mistaken in supposing that any version of OM with uniqueness quantifier would reintroduce the inconsistency.

To remedy this problem Cohen in [6], motivated by Sellars [27], replaced OM with One-Immediately-over-Many axiom (IOM) and employed more sophisticated logical machinery. Cohen’s reconstruction required quantifying over sets as well (without essentially refuting Vlastos’ [32] SP$_{v2}$ and NI$_{v2}$ new versions of these axioms). In this new setting Cohen [6] demonstrated the consistency of the TMA’s premises.

Although I do not refute Vlastos’ [31], pp.439 – 440 textual evidence that the TMA assumes only one Form $F$ per predicate-character, in this article we follow Geach [12], Cohen [6], Cohen and Keyt [7] and others in allowing for many Forms per predicate-character in the following context.

We adopt the fact that different things may belong to different levels (see also Cohen [6] p.468, note 31 and Section 3 of my article) and based on this, from one hand, we allow for many Forms

(OMs) If a number of things are all $F$, it follows that there is an $F$-ness in virtue of which they are all $F$.

(SPs) All $F$-nesses are $F$.

(NIs) If $x$ is $F$, then $x$ is not identical with any of the $F$-nesses by virtue of which it is $F$. (Actually Sellars’ original NI axiom is: If $x$ is $F$, then $x$ is not identical with the $F$-ness by virtue of which it is $F$. For a criticism on Sellars’ original version see Cohen and Keyt [7] at note 9).

10Actually we adopt the Accurately One over Many (AOM) assumption, since we allow for ‘things’ both sensible particulars and Forms. For details see G. Fine [10] ch. 14.

11This is due since Vlastos had been working in first-order logic with quantifiers ranging over particulars and Forms. For more information see also Cohen and Keyt [7] p.8 and note 18.
per predicate, but from the other hand we argue that these Forms create an increasing sequence that converges to a unique Form (per predicate).

We analyze and defend this thesis, providing the mathematical framework in which this infinite sequence of $F$ Forms is constructed and it converges-terminates in a unique $F$-Form (per predicate). Furthermore, we address possible problems that seem to arise from this approach. In particular, within this framework, we show that the logic of the TMA is consistent with Plato’s intention for the existence of a unique Form per predicate, that is the uniqueness thesis (see note 2). Plato for defending the uniqueness thesis introduced in Rep. 597c – d the famous Three-Bed-Argument, TBA. An analysis of the logic of the TBA (see Cohen [6], part VII) definitely calls for the existence of at most one Form per predicate; even though Plato meant TBA to defend uniqueness than ‘at most uniqueness’ thesis\(^{12}\). We will also discuss this in Section 3B.

Our mathematical interpretation is based on Plato’s comprehension of mathematics and particularly the concepts of apeiron, peras, limit, density and convergence as well as his approach on incommensurable magnitudes\(^{13}\). Moreover, our approach is consistent with Plato’s philosophy and his Dialectic theory. In supporting our thesis we provide and properly analyze concrete textual evidence from Plato’s dialogues, Plato’s commentators such as neoplatonists philosophers Plotinus and Proclus, as well as the bibliography developed in this area.

We follow Cohen [6] without planning to raise any objections to $NI_{\nu 2}$ and $SP_{\nu 2}$ axioms. Essentially these axioms were accepted also by Cohen[6], Cohen and Keyt [7] and others. In the sequel we shall also see that a new form of OM axiom (based on a different construction of TMA), namely the IOM axiom, entails both $NI_{\nu 2}$ and $SP_{\nu 2}$ axioms. We note that it is not the purpose of this paper to discuss any issues regarding different interpretations of the SP axioms such as Broad Self Predication, Narrow Self Predication, or Pauline Predications\(^{14}\).
It is rather worth noting that the *Self-Predication* and *Self-Participation* are completely different concepts. That is, *Self-Predication (SP)* tell us that: $F$–ness is $F$, or in other words, is *predicated* as being $F$.

As for NI axiom we interpret it in the sense that:

*The Form by virtue of which a set of things are all (predicated as) $F$ is not itself a member of that set. Equivalently, nothing is $F$ by virtue of participating in itself.*

Henceforth, the NI axiom, if we adopt that participating in $F$–ness is supposed to explain being $F$, tells us that we can not explain a thing being $F$ by appealing to this very thing. In this sense NI should be more accurately phrased as Non-Self-Explanation (NSE) axiom.

We state the versions of SP and NI axioms (slightly modified from $NI_2$ and $SP_2$) that we adopt.

**Definition 1**

(SP): The Form by virtue of which things are (and are judged to be) $F$ is itself $F$.

That is: $F$–ness is $F$, $i \in \mathbb{N}$.

(NI): The Form by virtue of which a set of things are all $F$ is not itself a member of that set. Equivalently, nothing is $F$ by virtue of participating in itself, or nothing is explained being $F$ by appealing to this very $F$.

That is: $F$–ness does not participate in $F$–ness, $i \in \mathbb{N}$.

In the sequel, for constructing the TMA we adopt Cohen’s approach as developed in [6]. Our definitions are given in terms of a single undefined relational predicate, ‘participates in’. The schematic letter ‘$F$’, which shall serve as a dummy predicate, will play the role that ‘large’ plays as a sample predicate in the TMA. We note that almost universally it is assumed that Plato intended the TMA to hold for any predicate for which there is a Form; hence the letter ‘$F$’ is typically used to express this generality.

**Definitions 2:**

(D1) By an *$F$*-object (object for short), we mean any $F$-thing; anything that is, whether a particular-sensible thing (αἰσθητὸ) or a Form (ἰδέα), of which ‘$F$’ can be explained.

(D2) An $F$-particular (hereafter ‘particular’ for short) is an object in which nothing can participates in. That is a sensible thing (αἰσθητὸ) in Plato’s terminology.

(D3) A *Form* is an object that is not a particular.

Cohen’s analysis of the TMA clearly aims to exploit an analogy of the TMA with number theory. It suggests that Plato’s infinite regress of Forms (as in the TMA) is analogous to
the generation of the infinite sequence of Natural Numbers (\(\mathbb{N}\)) according to Peano axiomatic construction. It is crucial to state that Peano’s axiomatic foundation of \(\mathbb{N}\) (using the concept of successors) is compatible with Plato’s infinite sequence of Forms as well as with the theory of Numbers as developed in Plato’s academy.\(^{15}\)

Cohen’s TMA construction\(^{16}\) is in accordance and have counterparts with the standard Von Neumann set-theoretic construction, that corresponds to the definition of \(\mathbb{N}\) via Peano’s Postulates. The only premise that the TMA seems that does not have an immediate counterpart among the Peano’s Postulates is OM. Since OM’s role is to generate a new Form at each stage of the regress, its number-theoretic counterpart could be the successor function which generates the members of the infinite sequence of \(\mathbb{N}\). Thus, should OM be appropriately modified, it could be considered as operating for deriving the principle of Mathematical Induction. Specifically, if OM is to have a uniqueness quantifier, then it will need to be based on something stronger than Plato’s over relation. Cohen \([6]\) and Cohen and Keyt \([7]\), understanding this and using Plato’s over relation in the new version (IOM) of OM axiom, adequately define and justify the immediately-over function that corresponds to the successor function:

**Immediately-Over Function**

\[ y \text{ is immediately over } x \iff y \text{ is over } x, \text{ and there is no } z \text{ such that } y \text{ is over } z \text{ and } z \text{ is over } x. \]

It is worth mentioning that the immediately-over function is a function only with respect to a single sequence of F-Forms. That is, one F-Form is ‘immediately over’ another, if no third F-Form intervenes between the two. Cohen’s One-Immediately-Over-Many axiom, which entails OM-axiom, guarantees that every Form, in a certain sequence of Forms, has a unique successor.

**\(\text{IOM-axiom}\)** For any set of \(F\)’s, there is exactly one Form immediately over that set.

This axiom blocks self-participation, since it entails that Forms do not belong to the sets they are over; thus, the axiom NI is built in IOM-axiom. In addition, SP (as in Definition 1)

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\(^{15}\)For example, Plato considers that number 2 (as a Form) is not the result (or identical) of summation \(1 + 1\), neither the result of the division of a magnitude in two parts (\(Phaedo\) 101c); it is should be considered as the successor of 1. See also Cohen and Kyet \([7]\), Fowler \([11]\) and Taylor \([29]\) ch. 20.

\(^{16}\)For the development of Cohen’s construction see \([6]\) pp.461 − 467.

\(^{17}\)The immediately over definition can also be defined in terms of the over relation, the notion of the level of an object and the definition of the maximal set, as in Cohen \([6]\). Here we use an equivalent one as stated in R. Patterson in \([20]\), p. 54. The equivalence of the two definitions is also proved by R. Patterson in \([20]\).
is presupposed as well, because the values of the variables in the definition of *immediately-over* function have been restricted to objects that are \( F \).

Specifically, Cohen’s construction essentially forms an increasing sequence \( \{F - \text{ness}_i\}, i \in \mathbb{N} \), of \( F \)-objects. There, every object \( F - \text{ness}_i \), in short \( F_i \), defines a level \( i \). His construction is in accordance with Von Neumann set-theoretic one.\(^{19}\) Henceforth, the infinite sequence \( \{F_i\}_{i=0}^{\infty} \) becomes:

\[
F_0 \leftrightarrow P \\
F_1 \leftrightarrow P \cup \{F_0\} \\
F_2 \leftrightarrow P \cup \{F_0, F_1\} \\
F_3 \leftrightarrow P \cup \{F_0, F_1, F_2\} \\
\vdots
\]

In the above, \( P \) denotes the set of \( F \) particulars and \( \leftrightarrow \) denotes a one to one relation pairing a Form with the set of its participants.\(^{20}\) Now using the symbol \( \in \) to represent participation rather than set-membership, we obtain the infinite increasing sequence:

\[
(*) \quad P \in F_0 \in F_1 \in F_2 \in F_3 \in \ldots \in F_k \in \ldots
\]

In our thesis we will deviate (somewhat) from the strict \( \mathbb{N} \) analogy. Namely, *we adopt the above infinite sequence by equipping it with a certain topology*. Under this new perspective we establish our main result as given in the following claim. The proof of the claim is mainly presented in Section 3.

**Main Claim:**

(a) For every predicate \( F \) there exists a countable increasing sequence of \( F \)-objects, \( \{F_i\}_{i=0}^{\infty} \), as in \((*)\).

\(^{18}\)See note \((16)\)

\(^{19}\)Recall that in the Von Neumann construction each member of \( \mathbb{N} \) is a member of all its ‘descendants’. This fact and the Peano’s postulate, stating that ‘no two Natural Numbers have the same successor’, entail that no member of \( \mathbb{N} \) is its own successor. The Form-theoretic analogue of this is that no Form in the sequence participates in itself.

\(^{20}\)The symbol \( \leftrightarrow \) is used instead of the symbol ‘\( = \)’ to avoid any conceptual identification with the strict mathematical notion of equality or identity.
(b) The infinite countable increasing sequence \( \{F_i\}_{i=0}^{\infty} \) from part (a) converges to a unique \( F \)-object (as a limit) under a certain topology.

That is \( \lim_{i \to +\infty} F_i = F \).

It is worth noting that the above \( F \)-Form, as a limit of the infinite sequence, could be identified with the unique \( F \)-Form, the \( F \)-itself for a certain predicate. Henceforth, it is in accordance with Plato’s intention for supporting the uniqueness thesis (see the previous discussion and note \[21\] ad.loc.). Furthermore, since the least upper bound is attained (as being a mathematical limit), part (b) of our Claim overrules A. Codes’ thesis for the role of TMA as raising an objection to the theory of judgement (see Section 1 ad.loc.).

The convergence to this limit Form should be understood in the concept of Plato’s academy mathematics and it must be comprehended in a mathematical framework. We note that Plato had a good knowledge of Eudoxus’s method of exhaustion for approximating lengths and areas (see Taylor \[29\], ch.20, ‘Forms and Numbers’ and Anapolitanos \[2\]). Plato had also used the technique of anthuphairesis (\( \dot{\alpha} \nu \delta \omega \phi \alpha \dot{\zeta} \phi \sigma \pi \zeta \)) throughout his work.\[22\] In Section 4 (based primarily on Philebus) we shall provide adequate evidence that Plato apprehended the two different notions of infinite, namely the countable and uncountable ones, both in relation to commensurable and incommensurable magnitudes and the concepts of density and the continuum (see also Karasmanis \[16\]). In this sense, it is highly probable that Plato had also a good grasp on irrational numbers.

3. The Defense of the Main Claim and Related Issues

In this section I defend and eventually provide an adequate proof of my thesis as it was stated in the Main Claim.

I am convinced that Cohen’s TMA version, as presented in the previous section, is closer to Plato’s apprehension of the theory of Forms and to his inclination to think that while the One-over-Many axiom yields exactly one Form for the set under consideration at each step, that principle is consistent with there being more than one Forms over the set with which we start. More precisely, over the set of \( F \)-things just one Form appears or comes into view, even though it turns out they will appear more in the process (by repetitive applications of the TMA on the new sets). Our analysis will set the mathematical framework of the above construction,

\[21\] In Plato’s language is referred also as ‘F-itself’, ‘\( \dot{\alpha} \nu \delta \omega \phi \alpha \dot{\zeta} \phi \sigma \pi \zeta \)’ \( F \) (see Phaedo 74b3 – 4, 100b5 – 7 et.al).
\[22\] For the technique of anthuphairesis consult Fowler \[11\], pp.322-328, Anapolitanos \[2\] and primarily Negropontis \[19\]. These matters are also studied for the needs of our article in Section 4 ad.loc..
providing essentially a proof of the Main Claim. In addition, within our framework, we address, analyze and finally resolve some problems regarding the logic of the TMA and its relation to Plato’s theory of Forms.

It is important to emphasize that if one considers the first time a set \( S_0 \) of \( F \)-things, exactly one \( F \)-Form \( F_0 \) shall appear immediately over that set, hence there is exactly one \( F \)-form. In the second step, where the TMA is applied on \( S_0 \cup F_0 \) (the set appeared in the first step and the Form \( F_0 \)) again one and only one \( F \)-Form \( F_1 \) shall appear immediately over this new second set. This process continues up to infinitum, thus creating an infinite sequence of \( F \)-objects; the sequence of Forms \( \{ F_i \}_{i=0}^{\infty} \).

Next I present the crucial arguments that defend the Main Claim. In subsection A I prove part (a) of the Main Claim and in subsection B the part (b) of it.

A. At this point we have to clarify what is involved in the claim that a Form can be ‘over’ its participants. It is clear that Plato thought of Forms as being on a higher ontological level than the sensible particulars participated in them. Towards this direction there is a strong textual evidence in Plato’s work: cf., e.g., Rep 515d, 477ff.; Phdo. 74a, 78d ff.; Tim. 28a, 49e; Symp. 210a-212b., et passim.

As we have already seen, the TMA seems to extend this notion by assuming, in general, that a Form is on a higher level than its participants, either these are sensible particulars or Forms. That is, each new Form that appears in each application of the TMA, see (\( \star \)), is in a higher ‘level’23 within the Platonic Realm of Forms, than its predecessor and hence its participants. Mathematically, according to (\( \star \)), it is formed an increasing infinite sequence of \( F \)-objects, \( \{ F_i \}_{i=0}^{\infty} \).

I defend the above in the grounds that it provides a precise formulation of the logical structure implicit in Plato’s arguments. The key is hiding in how Plato interpreters the One-over-Many principle. There is no indication that Plato himself ever tried to restrict the One-over-Many principle in the way to generate one Form for each predicate but no more than one. We can support this thesis by recalling Phaedo’s doctrine of the homonymy of Forms and their particulars as well as the interpretation of the famous formulation in Rep. 596a6 – a7:

‘We are in the habit of assuming one Form for each set of many things to which we assign the same name’.

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23We have to stress that these ‘levels’ do not denote different degrees of existence, since this should be incompatible with Plato’s ontological dualism theory.
According to 596a, if for a set of many things ‘we give the same name’, then on this set the One-over-Many principle could be applied. Based on this and in relation to the TMA, Cohen in [6] p.474, argues, and I agree with him, that it seems inevitable that Plato would ultimately include Forms in sets to which One-over-Many principle is applicable. Here we must also note that the wording of Republic 596a does not commit Plato to the existence of a Form to every predicate. For Plato may well use the word ‘name’ (ὀνόμα) not for every predicate, but for every name that denotes a property or, as we might say, for every property-name in the sense of true-correct names as expressed in G. Fine [10], p.304 – 305, notes 44 and 46.

Additionally, it is worth mentioning that Plato’s dialectic, as presented in Rep. 509d – 511e, as well as in 532d – 535d, is in the general line of thought of part (a) of the Main Claim. In particular, Plato in the analogy of Simile Divided Line advances the hierarchical progressing model of levels that leads to the anupotheton arxēn, claiming that the whole procedure is done in the realm of noēsis (νόησις) and precisely in the section of epistēmē (ἐπιστήμη), via the exercise of dialectic method.

Analogous arguments are advanced in the seminal work of Proclus’ Commentary on Plato’s Parmenides (translated by G. Morrow) and particularly in 879.15 – 28 and more emphatically in 881.23 – 34.

It is worth mentioning that in Symposium the concept of ‘Rungs’ of ‘Diotima’s ladder’ and Plato’s analysis on this support the model of increasing sequence of Forms. There, the procedure is presented in a detailed and vivid manner where the limit Form is the one of ‘Beauty’. Of course there the corresponding convergent sequence \(\{F_i\}\) consists of F-objects that are not Forms. This fact is not of a major importance since in this paper the references to Symp.

24 See note 10.
25 For example, in Politicus 262a – e Plato denies that there are forms corresponding to every general term. To know what Forms there are, we need to known not only what words a language contains, but what the genuine divisions in nature are.
26 This argument is presented clearly in Cratylus 386ff where he seems to use ‘name’ in a more restricted way. According to Crat., ‘n’ counts as a true-correct name only if it denotes a real property or kind and reveals the outlines of the essence (οὐσία). G. Fine in [10], pp.112 – 113, 304 – 305, 315 provides an extensive analysis and a thorough exposition of these notions, and on what Plato meant by the word ‘name’ (ὀνόμα).
27 For an analysis of these passages from Republic and their relation to Plato’s dialectic consult Karasmanis [13], [15], Benson [4], J. Annas [1] ch 10,11, as well as Robinson [25] ch. 6, 7, 10.
28 We quote: ‘...And from there in turn he will be chasing after unities of unity, and his problems will extend to infinity, until, coming up against the very boundaries of intellect, he will behold in them the distinctive creation of the Forms, in the self-created, the supremely simple, the eternal...’.
are primarily aiming in clarifying and analyzing the procedure itself as well as the concepts of density and convergence.

Particularly, Plato in *Symp.* 210e, among others, states ‘...passing from view to view of beautiful things, in the right and regular ascent...’, noting also that the ascension to the final Rung, corresponding to the ultimate Form (of ‘Beauty’) itself, has to be done in a ‘correct and orderly succession’ (ἐφεξῆς ὁρθῶς τὰ καλὰ). This is even more clear in *Symp.* 211b–c where the nature of the ascending procedure to the true-Form of Beauty is analyzed. From this passage we hold on the phrase ‘ὡσπερ ἐπαναβασμοῖς χρώμενον’, ‘as on the rungs of a ladder’, and the use of the word ‘ἐπανιών’, ‘that ascends’. According to Plato this describes the ‘right approach’ (ὁρθῶς) for ‘almost being able to lay hold of the final true F-Form’ (σχεδὸν ἄν τι ἦπικεφτο τοῦ τέλος), which constitutes also the ultimate goal and the conclusion of the whole procedure. There, each *rung of the ladder* defines a level (in an analogy to (⋆)), where each level is higher and contains the preceding ones. According to Vlastos the whole procedure moves ‘closer step by step to the Beauty itself’. We have to state that nothing prevents us from assuming that the same model holds analogously for all Forms (predicates) and is not restricted only to the Form of ‘Beauty’.

The ascending procedure described in *Symp.* could be considered as analogous to the abstract one presented in *Rep.* 511b, where we hold the phrase ‘οἷον ἐπιβιβάσεις’. This ascending procedure it is a fundamental one within the Platonic dialectic (see also note 27).

It is convincing that the previous analysis strongly supports the concept of *degrees of hierarchy* among the plurality of Forms $F_i$, $i \in \mathbb{N}$, that appear by repetitive applications of TMA. Thus, the sequence $\{F_i\}_{i=0}^{\infty}$ (in (⋆)) is justified as increasing.

**B.** I establish the second part of the main Claim. Thus, I show that the infinite regress of F-Forms $\{F_i\}_{i=0}^{\infty}$ (for a particular predicate) converges to the unique (terminating) F-Form, the so called F-*itself* ($\chi\alpha\nu\theta\iota\beta\nu$ ‘$F$’) for the predicate in concern. Furthermore, I address the various questions that arise regarding the topological framework of this convergence, as well as the nature of the limit-Form $F$ and the way it should be understood in relation to the various $F_i$, $i \in \mathbb{N}$, of the sequence in concern.

I will argue that the TMA model adopted in this article, based in Cohen’s analysis, is also compatible with Plato’s intention that *One-over-Many* principle (in its *One-Immediately-over-Many* version) yields to a uniqueness thesis. In this framework we shall defend that the logics of the TMA and the TBA are not inconsistent, but consistent and rather mutually complementary. Thus, the *uniqueness* thesis, that Plato intended to support by introducing the TBA, should be
apprehended in the context that the unique $F$-Form is the limit of the convergent sequence $\{F_i\}_{i=0}^{\infty}$; that is, $\lim_{i \to +\infty} F_i = F^{29}$.

We have to inform the reader that a different approach on this subject, claiming that the logics of the TMA and the TBA are rather inconsistent, is presented in G. Fine [10], pp. 235–238. In defending her thesis, G. Fine also argues that, in this particular case, Plato consistently rejects the NI axiom.

We stress that Cohen, while examining the consistency and logic of the IOM axiom, discovered that in order for his IOM axiom to be consistent the set theory his formalization presupposes cannot include the Principle of Abstraction$^{30}$.

\[
(\exists \alpha)(\forall x)(x \in \alpha \leftrightarrow FX).
\]

That is, for any predicate $F$, there is a set $\alpha$ consisting of all and only objects to which that predicate applies$^{31}$.

Cohen’s problem was the existence of a universal set of $F$’s; that is a highest level set containing all the $F$-objects in all (lower) levels. For, if there were such a set (the universal set of $F$’s), the increasing sequence in $(\star)$, that corresponds to the Von Neumann set theoretic construction of $\mathbb{N}$, it would contain a maximal element (and hence should not be an infinite one). Of course this cannot happen, since it contradicts the IOM axiom and the fact that the set $\mathbb{N}$ does not have a maximal element.

Cohen addressed this problem and he stated that if someone wants to retain the Principle of Abstraction, the IOM should have been somewhat altered (see Cohen [6], note 33).

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$^{29}$This $F$ is the unique Form, as defined in Phaedo, the ‘F itself’ (74a11 – 12), the ‘F without qualification’ (74bff), the ‘F’ that it can never seem non-’F” (74c1 – 3), see also note [3]. It is the Form as the final stage of the ascending procedure described above. In Symp. 211c this $F$ is further understood as the unchangeable end, the goal, the conclusion of the ascending procedure, ‘αὐτὸ τελευτῶν ὃ ἔστι’, tangent to the very essence of $F$-ness. Similar terminology and way of apprehending this ‘$F$-itself’ is encountered in many Platonic dialogues, such as Phaedo et.al. For example in Phaedo 101c it is described as the termination of the ascending procedure to the one Form which is ‘adequate’, ‘ἐξερχόμενον ἐξ ἑξήντα’, The ascending procedure is also developed elaborately in an abstract manner in Rep. 509e – 511d and shall be discussed in the sequel. See also Karasmanis [14] for the analysis of hypothetical method in this passage.

$^{30}$It is out of the scope of this article to argue about the intrinsic of the Principle of Abstraction per se. For the validity and its difficulties see Quine [23], pp. 134 – 136, 249, 300.

$^{31}$This principle should be considered in relation to the axiom schema of comprehension of Zermelo-Fraenkel set theory. The interested reader should also study the nature of this axiom schema and the possible problems arising by an improper use of it. For details and an analysis we refer to Cori and Lascar [9] pp.112 – 113.
Here we follow a different approach, offering a solution to the problem and retaining the original version of IOM axiom and the Principle of Abstraction. This is done by claiming that the increasing infinite sequence \( \{F_i\}_{i=0}^{\infty} \) converges within a certain topology; hence there exists a (mathematical) limit of this sequence. This limit as the attainable least upper bound of the increasing sequence could be considered as the the ‘highest level set’ containing all the \( F \)-objects in all (lower) levels; thus the Principle of Abstraction is retained. We believe that our approach is closer to Plato’s theory of Forms and specifically to his intention for accepting the uniqueness thesis. In the sequel we support this thesis.

Plato in Rep. 597c – d presents the Three Bed Argument and he applies the One-over-Many principle to eventually prove the uniqueness of the Form (the ‘bed’ in the particular case). But as analyzed by Cohen [6] and G. Fine [10], the application of the TMA on the TBA what does really proves is that there is at most one Form of ‘bed’. It is remarkable that in order to conclude that there is exactly one Form of ‘bed’ we must show that the sequence as constructed, using repeatedly the TMA, eventually stops. This cannot be done, since the TMA produces infinite sequence. Thus, the logic of the TMA together with that of the TBA lead to the conclusion that there are none Forms. Of course this was not the intention of Plato since it contradicts his Ontological theory.

Here we provide adequate evidence for establishing the proof of part (b) of the main Claim. This also entails that that the Principle of Abstraction and the IOM are retained, and the existence of the infinite sequence is proved to be genuine without leading to any contradiction. In addition, within this context, A. Codes’ thesis that the TMA raises an objection to the theory of judgement (see Section 1 ad.loc) is overruled. For doing this we argue that the \( F \)-Form, the ‘\( F \)-itself’, that appears as the limit-terminating point of the above increasing sequence must not be committed to the Non-Identity (NI) or Non-Self-Explanatory (NSE) axiom.

Indeed, form one hand this \( F \)-Form clearly satisfies Self-Predication axiom (since it is predicated as being \( F \)). But from the other hand, it is the limit of the increasing sequence of \( \{F_i\}_{i=0}^{\infty} \) and hence there are no further \( F \)-Forms beyond this particular \( F \)-Form, in contrast with the rest \( F_i, i \in \mathbb{N}, F \)-Forms in the sequence (\( \star \)). These arguments lead to the conclusion that the limit \( F \)-Form should be comprehended as self explained. Hence it could not satisfy the Non-Identity (or NSE) axiom.

In the sequel we present how Plato understands and explains the above thesis. For doing this we study the framework within which he comprehends, in my point of view, the convergence of \( \{F\}_{i=0}^{\infty} \) to the unique \( F \) (for each predicate).
Of course in order to talk about convergence and limits we must assume a certain topology. This topology should be the one closer to Plato’s understanding of mathematical concepts. Plato, as we mentioned in Section 2 and as we shall see in Section 4, had a good grasp on the Eudoxus’ exhaustion method and the technique of anthuphairesis, as well as an apprehension of the concepts of peras, apeiron, density and continuum and their relation to commensurable and incommensurable magnitudes.

The topology we consider, and hence the convergence we are referring to, is also in accordance with Plato’s dialectic process (more specifically as presented and analyzed in Rep. 509c – 511e and 532d – 535a) as well as his apprehension of the fundamental concept of the anupotheton arxēn (ἀνυπόθετον ἀρχὴν) in Republic, but applied here for each particular predicate. In Rep. 510b Plato states clearly that the highest rung of the ladder is not reached until the entire domain of epistēmē has been exhausted via the dialectic process. This principle, established by an exhaustive scrutiny, should be understood as higher than all premises-hypotheses, ‘ὑποθέσεις’. It is higher, in the sense that contrary to them it has an axiomatic status (playing the role of a system of axioms), it is non-hypothetical, it is situated in the highest point of the intelligible world (in Republic’s Simile Divided Line) and it does not require derivation (see also Karasmanis [14], [15] and Benson [4], p.190).

We emphasize that mathematically the anupotheton arxēn is the ultimate-final unique Form apprehended as the mathematical limit of the infinite increasing sequence under the presupposed topology. It must be noted that this Form is comprehended not as a transcendental ontological mystery but in the mathematical sense of the least upper bound of the increasing sequence of Forms (see (⋆) and part (a) of the main Claim) that it is eventually attained; hence it becomes a limit Form. Apart from Republic, it should be apprehended as the one (‘μονοειδὲς’ in Phaedo’s language) that should be parallelized with the highest-terminating rung of Diotima’s ladder, which is tangent to the very essence of F-ness.

Within this framework, this limit-Form could also be conceived as not committed to the Non-Identity (NI) axiom. This is due for being the F-Form, the anupotheton arxēn that in addition to the above gives an account to all the lower level $F_i, i \in \mathbb{N}$, F-Forms, but itself does not require derivation. In this sense, there do not exist further F-Forms in higher order level(s)

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32 As argued above even though it is not committed to NI axiom, it is predicated as being F and thus it satisfies the SP axiom.

33 see Symp. 211b – c where that F is the Form of ‘Beauty’.
to be depended on, or any Forms for providing an explanation to it. Thus, the Form in discussion, though is predicated as \( F \), should be regarded as self-explanatory. In an abstract and an ontological level this Final stage (this particular limit of the sequence) should be considered as the unchangeable unique ‘Is’, ‘ἔἶναι’, predicated as \( F \) and apprehended as the terminating point-Form of the increasing infinite sequence-process, the process of \( \text{becoming-gignesthai} \) (‘γίγνεσθαι’). This infinite process, using Plato’s terminology, leads to the unconditional, immutable, objective, unchangeable, perfect and unique ‘Is’.

In Section 4 we present and analyze, mainly within the context of \( \text{Philebus 23c – 27c} \), the mathematical framework of this process. It is the process \textit{per se} which is compactly phrased at \( 26d7 - 9 \) as \textit{genesis eis ouσίαν} - ‘γένεσις εἰς οὐσίαν’. This procedure it can be parallelized to the dialectic one, that under the cause of \textit{noesis} exhausts the entire domain of knowledge-\( \text{epistēmē} \) in order to arrive and terminate to the purest mental state, the unique \( F \)-Form. This Form, as stated earlier, is non-hypothetical (‘ἀνυπόθετον ἀρχὴν’ using \textit{Republic’s} terminology) is not committed to NI (better NSE) and has an axiomatic status.

Summarizing, we conclude that the increasing infinite sequence of \( F \)-Forms derived by repetitive applications of the TMA (for each predicate) converges eventually to a (mathematical) limit (a terminating point) which is unique (as a limit of a sequence). This is in accordance to Plato’s intention in supporting the \textit{Uniqueness of a Form} per predicate. Furthermore, our thesis is consistent with the \textit{Principle of Abstraction} in Logic as analyzed earlier in this paper.

At this point we provide further textual evidence supporting the concept of \textit{convergence} (as stated earlier) strengthening our arguments in part (b) of the main Claim.

We mention that the notion of the \textit{limit}, as the terminating point of this infinite but convergent sequence, as well as the whole theory established above, it is also present in some form or as parts of it in many Platonic texts (of the middle and late period), such as \textit{Symp.}, \textit{Phaedrus}, \textit{Rep.},

\[34\] Strictly, this should be better considered as modal, stating: ‘that there are not required further \( F \)-Forms in higher order level to be depended on, or needed for providing an explanation to it.’ But no harm is done here by simplifying it as existential, since the TMA shows existence of \( F \)-things. Furthermore, in a mathematical-logic language, this unique Form is not a derivation, or a theorem, but it has the status of an axiom, or of a system of axioms, and thus it does not require a proof. Moreover, due to its status, all the information of the system can be retrieved from it .

\[35\] In \textit{Symp. 211a} Plato characterizes this ‘Is’ as: ‘... ever-existent and neither comes to be nor perishes, neither waxes (growths) nor wanes (declines, decreased)’...’. In \textit{211b} is characterized as unchangeable and is affected by nothing. Further, in \textit{211c} this ‘Is’ is revealed at the end of the ascending procedure, characterized as the very essence of the \( F \)-ness.

\[36\] For an analysis of this passage, as well as the crucial phrase ‘γένεσις εἰς οὐσίαν’ we refer also to Section 4.
Philebus, Epistle 7 et.al. Its distinct status, primarily in Philebus, shall be analyzed in Section 4. In addition, the notion of the limit was adopted and studied by Neoplatonists philosophers and Plato’s commentators such as Proclus and Plotinus.

More specifically, Plato in Symp. 210e – 211a characterizes the limit Form $F$, in relation to the termination of the infinite sequence, as being revealed ‘abruptly, suddenly’-’ἐξαίφνης’. In addition he states that it exist unconditionally and is ‘the perfect thing, the wondrous and beautiful in nature’- ‘θαυμαστὸν τὴν φύσιν καλὸν’. Furthermore, in Plato’s Epistle 7.341c we encounter related notions that emphasize the concept of the upper bound of the infinite sequence that it is revealed suddenly-’συζῆν ἐξαίφνης’. Analogous approach is encountered also in Plotinus Enneads 43.17.

The statement that the terminating $F$-Form can be achieved as a limit of the increasing sequence in the most perfect manner is furthermore emphasized in the process described in Symp. 211a – e. There, it is characterized (in 211e) as ‘the divine beauty itself, in its unique form’- ’θεῖον καλὸν μονοειδὲς’ and in Symp. 212a as the tangent-contact to the truth-’τοῦ ἀληθοῦς ἔφαπτομένῳ’. In Symp. 211c9 is called the ‘terminating point of the ascending procedure’- ‘αὐτὸ τελευτῶν ὃ ἔστι’.

Similar ideas about the nature of this ‘limit’, the terminating point-Form, as being the contact approach and intercourse with the truth is evident throughout the Platonic corpus.

Very emphatically, the Neoplatonist philosopher Proclus in his work Commentary on Plato’s Parmenides 881,23 – 33 analyzes the concepts of infinite (apeiron) regress that arrives to a terminating mental point/peras-’νοερὸν πέρας’, via the process of the intellect/noēsis-’νόησις’.

37A similar terminology is used in Phaedrus 250b.
38Epistle 7341c ‘...but, as a result of continued application to the subject itself and communion therewith, it is brought to birth in the soul on a sudden, as light...’.
39In Republic 398a this perfect form is characterized as ‘divine and holy’-’τερόν καὶ θαυμαστὸν’.
40For an analysis of the passage 201d – 212c in Symposium we recommend Taylor 29, ch 9, section 8 and Vlastos 34.
41This idea is present and analyzed in many Platonic works and is also frequent in Aristotle and the Neoplatonists. The process of arriving to this terminating point that is tangential to the true-Form is analogous to the one described earlier in Symp. 210eff, Rep. 490b, as well as in Rep. 509c – 511d in the context of Plato’s dialectic theory. (For details see J. Annas 1 and for a meticulous analysis of Rep. 509c – 511d see Karasmanis 15 and Benson 1).
42We quote:’...And from there in turn he will see other more comprehensive unities, and he will be chasing after unities of unity, and his problems will extend to infinity, until, coming up against the very boundaries of the intellect, he will behold in them the distinctive creation of Forms, in the self-created, the supremely simple, the eternal...’. A similar line of thought is present also in Plotinus, Enneads 2.4.15, 15 – 16.
The term ‘νοερὸν πέρας’ should be apprehended as analogous to ‘ἀνυπόθετον ἀρχὴν’ of Republic. It is the termination, the limit of the dialectic process. This process occurs in the ἐπιστήμη-epistēmē (the upper part of νόησις-noēsis) of the ‘Simile Divided Line’. In addition, Proclus in *Commentary of the First Book of Euclid’s Elements* is arguing, using the concepts of peras and apeiron (in the line of thought of Plato’s *Philebus*), in order to establish the convergence apprehended via the notion of the *limit*. This approach shall be discussed in some extend in the next section, where we shall compare and cross-examine it with the one that appears in Plato’s *Philebus*.

In the next section we study, in some extend, the mathematical concepts and the general framework involved and required for establishing our main Claim. Our analysis is based primarily on Plato’s *Philebus* and the key notions of peras, apeiron, density and continuum in relation to commensurable and incommensurable magnitudes.

4. The Mathematical framework of the Argument

I would like to start by arguing that Plato apprehended the structure of incommensurable magnitudes in a way that these magnitudes correspond in a unique and well defined manner to the modern concept of the Dedekind cut. For the precise definition of the Dedekind cut the reader should consult Rudin.

Furthermore, it is important to note that Plato captured the notions of density of magnitudes and the complete continuum in the sense that they include incommensurable cuts (see Karasmanis [16] p.394).

It seems that Plato considers incommensurability as an essential feature of magnitudes. This can be most vividly observed in *Philebus* 23c – 27e, as well as *Theaetetus* 147d4 – 148b4, where Plato’s approach to the continuum is developed. In *Philebus* Plato makes an effort to explore the relation between continuum, infinite divisibility and incommensurability in contrast with commensurable things that are capable of appearing in ratios and proportions.

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43 For further clarification and analysis of these terms we recommend Karasmanis [14], [15] section III ([cf. note 3 *ad loc.*]) and Benson [4].

44 This is more clear in the dialogues *Theaetetus*, *Philebus*; for details see Taylor [29], ch. 20, Fowler [11], and Anapolitanos [2].

45 In Rudin [24] the Real numbers are constructed as Dedekind cuts in a unique and well defined manner which follows the line of thought of Eudoxus’ exhaustion method. It is important to note that square roots of non square numbers, (see *Theaet. 147d4 – 148b4*), as well as the incommensurable magnitudes could be obtained via the technique of anthuphaireisis, ‘ἀνθυφαίρεσις’. Anthuphaireisis entails also the concept of ‘cut’ (see Taylor [29], ch. 20) and is presented formally and in great detail in the 10th book of the Elements of Euclid. See also note 60.
We study the above mathematical concepts and the notions of peras and apeiron based primarily on the passage 23c – 27c of Philebus.

We have to note that in 16b – 19a Plato says that ‘the things that are ever said to be’ (ἀεὶ λεγόμενα) are made up of one and many, with peras and apeiron inherent (σύμφυτον) in them, but in the second passage Socrates asserts that peras and apeiron are two different general classes of things.

In 23ff Plato makes a fourfold division: peras (πέρας), apeiron (ἄπειρον), mixed (μεικτόν) and cause (αἰτία). That is how he examines these four categories-classes one by one.

In 23c10 – d1 the class of mixed is assumed as a combination of peras and apeiron. Karasmanis in [16] examines systematically the characteristics of peras and apeiron and attempts to answer a plethora of questions regarding their nature, their status in the fourfold division and their relation with the notions of commensurability and incommensurability.

The class of ‘cause’ (αἰτία) is explained in 23d6 – 7 as the cause of the existence of the third class, that of the mixed. According to Plato this is the cause of the combination (σύμμειξις) of the other two classes, the peras and the apeiron.

In 23e and 25d11 – e2 Socrates states that he should investigate the mechanism (αἰτία) that the separated peras and apeiron are mixed together, explaining how are becoming a unity. Plato in 24c – d explains more accurately what he means by ‘apeiron’ and ‘more and less’, ‘τὸ πλέον καὶ τὸ ἔλαττον’. He says that wherever the ‘more and less’ are present they exclude any definite quantity, poson (ποσόν). This passage says that the essential characteristic of the ‘more and less’ is the absence of any definite quantity, for the presence of definite quantity and measure (μέτρον) in the place where the ‘more and the less’ is present will abolish the ‘more and the less’. It seems that according to Plato the notions ‘more and less’ and ‘definite quantity’ are mutually excluding.

In 24d the notion of continuum is further analyzed. Specifically in 24d4 – 5 Plato uses the expression ‘προχωρεῖ γὰρ καὶ οὐ μένει’ translating as ‘goes on without pause’ or ‘progressing and never stationary’; henceforth the notion of a continuous motion is advanced. In the same line of thought Plato at 31a says that:

46 See note 13.
47 We must note that this is indeed a problematic passage where the term ‘ἀεὶ λεγόμενα’ are most probably the ideas-Forms. For a discussion and the various interpretations of the term apeiron in this passage, we refer to Karasmanis [16] p.390, notes 8, 9. Analogous to this approach, for the so called ontos on-’翁τως δν’, is advanced by Proclus in The Elements of Theology, 89.
‘apeiron in itself does not have and will never have any precisely marked beginning, middle or end’.

Thus, we could assert that what is characterized by the ‘more and less’ should be continuous and, therefore, the characteristics that Plato attributes to the apeiron point to continuous magnitudes. Such an apeiron as continuous is infinitely divisible; this last property is the main characteristic of magnitudes. The same idea is also clear in 24e – 25a. In passages 25b and 25e he talks about the class of peras where he obviously relates that to the class of commensurable magnitudes (analogously one could relate it to the countable set of rational numbers). Thus, it seems that Plato suggests that the main characteristic of his peras is commensurability.

We find further evidence for this claim in two other passages from the Philebus:

(a) ‘That of equal and double, and whatever puts an end to opposites being at odds with each other, and by the introduction of number that makes them commensurate-summetra (σύμμετρα) and harmonious’, 25d11 – e2.

(b) ‘Again, in the case of extremes of cold and heat its advent removes what is far too much and apeiron and produces what is measured-emmetron (ἐμμέτρον) and commensurable (summetron)’, 26a7 – 9.

If peras is what makes things commensurate, then apeiron must be the source of incommensurability. I think that Plato here is using the term commensurate-summetron in a rather technical, mathematical sense. Now, if we agree that Plato says in 24e7 – 25a2[48] and in 25a6 – 7[49] that apeiron admits opposite characteristics to those of peras, then we have to conclude that incommensurability is a further and very important characteristic of apeiron. It seems then that Plato relates discontinuity to commensurability and probably (ex silentio) continuity to incommensurability.

I refer to 26a to stress that according to Socrates, the perfection (in the art of music) or harmony and moderation (in the case of temperature) can be achieved by properly combining-mixing the opposite directions of apeiron (see Karasmanis [16], p.391 – 392) and finding the limit-peras[50]. This limit-peras within this context is restated (and identified) in 26d8 – 9 as the

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[48] We quote: ‘All things which appear to us to become more or less, or to admit of emphatic and gentle and excessive and the like, are to be put in the class of the infinite as their unity...’.

[49] We quote: ‘... and the things which do not admit of more and less and the like, but do admit of all that is opposed to them...’.

[50] We quote 26a2 – 4: ‘...And in the acute and the grave, the quick and the slow, which are unlimited, the addition of these same elements creates a limit-peras and establishes the whole art of music in all its perfection, does it not?...’.
emmetron-measured and summetron-commensurable. In 26b Plato generalizes the assertion that the limit leads to perfection and that is the way to arrive to the realm of Forms (ἰδέες).

Now peras gives not just any determination of apeiron but the right one according to specific case of art. When Plato speaks about apeiron in the framework of mixed and the mechanism of mixis, it seems that he broadens its meaning. In 26d7 − 9 Plato states:

‘...And as to the third class, understand that I mean every offspring of these two [peras and apeiron] which comes into being (γένεσις εἰς οὐσίαν), to a stable and immutable essence, as a result of the limits-measures (μέτρων) created by the cooperation of the the peras....’

Thus, according to the above passage and to 23d6 − 7 (commented earlier) peras is drown onto apeiron via the class of the cause (αἰτία) to create the third class, the class of mixed (μεικτὸν) according to a specific process. Summarizing all of the above we conclude:

(✠) The incommensurability of apeiron can be approximated and eventually be a limit in a specific manner; that is, by imposing peras, which makes things commensurate, on it.

We could further conclude that Plato considers the class of mixed in relation to the unchangeable perfect ‘Is (εἶναι) which is formed by imposing, with the aid of the class of peras, a limit, a due measure on the class of apeiron via a specific process. This process that results to the ‘Is’, is characterized in Plato’s language (26d8) as ‘genesis eis ousian’-‘γένεσις εἰς οὐσίαν’.

Plato clearly supports the above thesis in other parts of Philebus such as 27d6 − 10:

‘for that class is not formed by mixture of any two things, but of all the things which belong to the apeiron, bound by the peras; and therefore this victorious life (ὁ νικηφόρος βίος) would rightly (ὀρθῶς) be considered a part of this class.’

For an in depth comprehension of this thesis the translation of the phrase ‘γένεσις εἰς οὐσίαν’ in 26d8 is crucial. It is translated as ‘coming into being’, that entails a continuous process, a creative procedure; it is a becoming that generates and eventually gives rise to something

See the notes and notes.

We quote from 26b..‘all the beauties of our world arise by mixture of the apeiron with the peras’, and he continues to state that ‘for many glorious beauties of the soul this goddess,... beholding the violence and universal wickedness which prevailed, since there was no limit-peras (see note) of pleasures or of indulgence in them, established law and order, which contains a limit-peras (see note)...’.

Here the word ‘μέτρων’ (measure) should be comprehended as ‘due measure or limit’-’right proportion’.

Here peras should be comprehended rather as ‘that which limits or has limits’. See also Philebus 30a.

See also the discussion in Section 3 ad.loc..
stable via the proper-correct way in due measure. In this sense, it leads to the ideal-optimum ‘victorious life’ which belongs to the class of mixed\textsuperscript{56}. At \textit{Philebus} 55a3, the \textit{genesis} as the becoming, the generation, is presented as the opposite of ‘destruction’-\textit{φθορά}. The word \textit{genesis} appears often in Plato. Particularly, in \textit{Phaedrus} 245e2 – 5 is presented as the source of motion and origin, where the self-motion is comprehended as the essence, ‘\textit{οὐσία}’. The other term ousia-‘\textit{οὐσία}’ is central in Plato’s philosophy. It primarily denotes the essence, the true substance, the stable and true being, the immutable reality, the \textit{εἶναι}\textsuperscript{57}. We have to mention that in \textit{Rep.}534a2 – 4 Plato claims that the two terms-concepts, \textit{genesis} and ousia have different ontological and epistemological status. He emphasizes that \textit{doxa} (δόξα) is concerned and is dealing with \textit{genesis}, while \textit{noēsis}-\textit{νόησις} is the one that deals with the essence-ousia\textsuperscript{58}. Analogous interpretation is advanced also in Plato’s \textit{Tim.} 29c4, \textit{cf.} \textit{Sophist} 232c7 – 9.

There is a passage in Proclus’ \textit{Commentary of the First Book of Euclid’s Elements} which seems to advance similar ideas, regarding \textit{peras} and \textit{apeiron} in relation to \textit{commensurable} and \textit{incommensurable} magnitudes\textsuperscript{59}, and hence it is strengthening our conclusion (\textit{✠}). Although Proclus does not refer to Plato at all, I find it highly probable that he has in mind our passages on \textit{peras} and \textit{apeiron} in the \textit{Philebus}, as we have analyzed them above.

‘Mathematicals are the offspring of the limit-peratos (\textit{πέρατος}) and the unlimited-apeirian (\textit{ἀπειρίαν}), but not of the primary principles alone, nor of the hidden intelligible causes, but also of secondary principles that proceed from them... .This is why in these orders of being there are ratios-logoi (\textit{λόγοι}) proceeding to infinity-apeiron (\textit{ἄπειρον}), but controlled by the principle of the limit-peratos. For number, beginning with unity, is capable of indefinite increase, yet any number you choose is finite; magnitudes-megethôn (\textit{μεγεθῶν}) likewise are divisible without end, yet the magnitudes distinguished from one another are all bounded, and the actual parts of a whole are limited. If there were no infinity-apeirias (\textit{ἀπειρίας}), all magnitudes would be commensurable and there would be

\textsuperscript{56}Earlier in 27d1 – 2 this ‘victorious life’ is characterized as the mixed life of pleasure-\textit{ḥedonē} and prudence-\textit{phronēsis}. See also Taylor [29], ch 16.

\textsuperscript{57}In contrast to ‘\textit{οὐσίαν}’ as the ‘\textit{εἶναι}’, the ‘\textit{Is}’, the stable being and the immutable reality, Plato uses in \textit{Theaet.}185c9 – 10 the term ‘\textit{μὴ εἶναι}’- ‘\textit{μὴ ἔειναι}’.

\textsuperscript{58}For details about the status of these terms we refer to Karasmanis [15] pp.148 – 149 and p.156 section III, J. Annas [1] ch. 10, 11, and Adams comments in [30] \textit{ad loc}.

\textsuperscript{59}Though that in matters of divisibility Proclus does not see the possibility of an infinite divisibility that does not involve incommensurability, something that Plato probably observed. For an in depth analysis of the notions of incommensurability and infinite divisibility in Proclus see Anapolitanos and Demis [3].
nothing inexpressible-arrēton (ἀρρήτον) or irrational-alogon (ἄλογον) features that are thought to distinguish geometry from arithmetic.’ (6. 7-22; transl. by Morrow 1970)

Proclus in his works had also systematized material from Platonic dialogues. Specifically, in his work *The Elements of Theology* (86.16 – 20, 24 – 26) he clearly distinguishes the two types of infinity (ἄπειρον) that most probably correspond to the ones that are suggested in *Philebus* and are analyzed in the sequel. Analogous approach was advanced by Plotinus in *Ennead* 2.4.15 – 16.

In relation to the above, Plato in *Philebus* 27e speaks about pleasure states that it is *apeiron* both in ‘quantity and degree’, ‘καὶ πλήθει καὶ τὸ μᾶλλον’. Karasmanis in [16] attempts to provide an explanation to this, even though he admits that this passage is rather problematic and the possible conclusions drawn from it are not absolutely certain. In spite of this, he goes on providing evidence that is highly probable that *apeiron* in quantity (πλήθος) is something that is infinity by addition; that is, this *apeiron* is not continuous but discrete. Furthermore, the *apeiron* in degree (τὸ μᾶλλον) does not have definite quantities; therefore, it is continuous or infinite by division.

Now we discuss in some extend the concepts of commensurability and incommensurability in relation to the terms-classes of *peras* and *apeiron*.

The *incommensurability* appears as special case of of infinite divisibility. Plato in *Theaet.* 147d – 148e presents topics from the theory of incommensurability (see Karasmanis [16] note 16 and Fowler [11]).

To prove that two magnitudes are incommensurable, Greek mathematics use the technique of anthuphairesis (ἄνθυφαίρεσις) which is used mainly to find the greatest common divisor between two numbers. The technique of anthuphairesis is presented in Euclid’s Elements as Proposition X.2 (see also Karasmanis [16] note 21). It is important to state that an *infinite* anthuphairetic process of reciprocal subtraction between two magnitudes shows that these magnitudes are *incommensurable*. Moreover, a *finite-terminating* anthuphairetic process shows that these magnitudes are *commensurable*.

It is evident that Plato apprehended these results. Specifically, the construction of incommensurable magnitudes in *Theaet.* 147d – 148e, as well as the philosophical aspect of anthuphairetic process in Plato’s dialogues, such as *Sophist* 264b9 – 268d5 (see Negrepontis [19]), point to Plato’s deep grasp of the technique of anthuphairesis and his intention to apply it systematically in a

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60For a formal and extensive presentation of the technique of anthuphairesis and the presentation of the proofs of the results regarding commensurable and incommensurable magnitudes the reader should consult Fowler [11] (1987 chs 2 and 5), Anapolitanos [2], Knorr [17] (1975, chs 2, 4, 7), and Sinnige [28] (1986, pp. 73 – 80).
general philosophical framework. Thus, taking into account all of the above as well as the interpretation given for the phrase ‘quantity and degree’ in 27e, one could rather safely concludes that Plato comprehended the two types of infinity, namely the countable and uncountable ones.\footnote{I believe that we should view this just as Plato’s attempt to discern between two types of infinity. In modern mathematical terms, \textit{apeiron in quantity} corresponds to countably infinity (with cardinality $\mathbb{N}_0$); and \textit{apeiron in degree} corresponds to uncountably infinity (with cardinality $2^\mathbb{N}_0$). For further analysis see Karasmanis \cite{10}, Anapolitanos \cite{2} and Anapolitanos and Demis \cite{3}.}

In this way we see that we have two kinds of infinite divisibility that are strongly related to magnitudes:

1. \textit{the Zenonian infinite divisibility} that results to the Aristotelian continuum, which it points to what we nowadays call the set of rational numbers (see Karasmanis note 4 in \cite{16}). This is a characteristic of commensurability-noting that \textit{commensurable} magnitudes have finite anthuphairesis-and discrete infinity. It could be considered analogous to the type of infinite that characterizes the set of rational numbers, $\mathbb{Q}$.

2. \textit{Anthuphairetic infinite divisibility} which produces the \textit{incommensurable} and makes the \textit{continuum dense}, thus generating magnitudes. It could be considered analogous to the type of infinity that characterizes the irrational numbers $\mathbb{\bar{Q}}$. We note that by making the continuum dense we obtain the \textit{Real Line} $\mathbb{R}$.

Henceforth, supplementing the Aristotelian continuum-which is characterized by Zenonian infinite divisibility and corresponds to commensurables-with the incommensurables- which are obtained via the anthuphairetic infinite divisibility- we obtain all magnitudes. The density of the continuum and hence the construction of the \textit{Real Line} could be written now (using modern mathematical terminology) as $\mathbb{R} = \mathbb{Q} \cup \mathbb{\bar{Q}}$. Recall also that $\mathbb{Q}$ has cardinality $\mathbb{N}_0$ and is a countable set, but $\mathbb{\bar{Q}}$ as well as $\mathbb{R}$ have cardinality $2^{\mathbb{N}_0}$ and hence are uncountable sets.

Summarizing, we state the following correspondences that clarify the terms-classes \textit{peras}, \textit{apeiron}:

1. \textit{peras} $\equiv$ \textit{commensurability} $\equiv$ \textit{finite anthuphairetic process}.

2. \textit{apeiron} $\equiv$ \textit{incommensurability} $\equiv$ \textit{infinite anthuphairetic process}.

Here we must note that the concepts of infinity as described above, as well as the construction of real numbers (and in particular irrational numbers, which correspond to incommensurable magnitudes).

\footnote{where $\mathbb{\bar{S}}$ denotes the complement of a set $S$.}
magnitudes) impose a certain natural topology on the Real Line. Within the framework of this topology the various mathematical notions, especially the concepts of density and convergence, should be understood. This framework provides the appropriate meaning of the various mathematical notions involved in the formulation of the main Claim and additionally it is the only possible one consistent with the Mathematics of Plato’s Academy.

Furthermore, the concept of convergence (as apprehended in this Section) was used to argue in Section 3B that the logics of the TMA and the TBA arguments are not contradictory but they are rather consistent.

Of course one could naturally ask: Did Plato apprehended all these results? I believe, as the above analysis showed, that yes, it is highly probable. We have to state that Plato indeed had a thorough knowledge of the mathematics of his era and especially the philosophical and foundational problems of it (see Karasmanis [14] and Fowler [11]). Our analysis leads us to suppose that Plato had a strong grasp on approaching magnitudes in terms of incommensurability rather than the Zenonian infinite divisibility. Additionally, and in relation to the above, it is not unreasonable to suppose that Plato comprehended the modern mathematical notion of density and through it the approach of the continuum as the closure (in the topology described earlier) of the rational numbers in Real numbers, $\mathbb{Q} = \mathbb{R}$.

5. Conclusion

In this section, summarizing our results, we state the main conclusions of our article.

The increasing infinite sequence of Forms $\{F_i\}_{i=0}^\infty$ constructed by repetitive applications of the TMA shall be understood as a convergent one. Precisely, it converges to the unique $F$-Form for a particular predicate in the framework of the topology developed in Section 4. That

\[ \text{Every element of the Real Line, and in particular every irrational number, is the limit of a sequence of rational numbers. That is, } \forall x \in \mathbb{R}, \text{ there is a sequence of rational numbers } \{r_i\}_{i=0}^\infty, \text{ such that } \lim_{i \to +\infty} r_i = x. \]

For a formal proof of the theorem see Rudin [24].
is \( \lim_{i \to +\infty} F_i = F \). In a philosophical level, we could claim that this limit \( F \)-Form should be apprehended as the constant-unchangeable ‘Is’, ‘εἶναι’. As we have seen in Sections 3, 4, the convergence of the sequence \( \{F_i\}_{i=0}^{\infty} \) to \( F \) could apprehended as corresponding to the procedure called \textit{genesis eis ousian} - ‘γένεσις εἰς οὐσίαν’, applied for each predication-Form.

Furthermore, we studied the apparent contradictory aspects of:

1. The appearances of a plurality of Forms (for each predication) that is generated by repetitive applications of the TMA (adopting Cohen’s, [3] thesis).
2. The intention of Plato for assuming a unique Form per Predicate in supporting the Uniqueness thesis (as this is entailed in the TBA et.al).

In this direction, we provided adequate arguments defending the thesis that the logics of the TMA and the TBA are not contradictory, but rather consistent and mutually complementary. In doing this, we have also supported that the final-limit \( F \)-Form is not committed to NI axiom. Thus, we showed that the \textit{Uniqueness thesis} is safeguarded and the \textit{Abstraction Principle} in logic is retained.



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Note: The original texts as well as their translations (unless stated otherwise) have been taken form the electronic database *Thesaurus Linguae Graecae*.