Analytic Calculation of Neutrino Mass Eigenvalues

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Abstract

Implication of the neutrino oscillation search for the neutrino mass square difference and mixing are discussed. We have considered the effective majorana mass \(m_{ee}\), related for \(\beta\beta_{0v}\) decay. We find limits for neutrino mass eigen value \(m_i\) in the different neutrino mass spectrum, which explain the different neutrino data.

1 Introduction

The phenomenon of neutrino oscillation, impressive advance have been made to understand the phenomenology of neutrino oscillation through solar neutrino, atmospheric neutrino, reactor neutrino and accelerator neutrino experiment. The experimental research on the nature of neutrinos through terrestrial as well as extra-terrestrial approaches has finally confirmed the neutrino oscillation in atmospheric [1,2,3,4] solar [5,6,7,8,9], reactor [10, 11,12] and accelerator [13,14] neutrino sources, establishing that neutrinos have mass.
Furthermore, it is generally agreed that oscillations among three neutrino species are sufficient to explain the atmospheric, solar, reactor and accelerator neutrino puzzle. The neutrino oscillation experiments provide us with neutrino mass square differences, mixing angles and a possible hierarchy in the neutrino mass spectrum. The main physical goal in future experiment are the determination of the unknown parameter $\theta_{13}$. In particular, the observation of $\delta$ is quite interesting for the point of view that $\delta$ related to the origin of the matter in the universe. One of the most important parameter in neutrino physics is the magnitude of mixing angle $\theta_{13}$ and CP phase $\delta$. The oscillation data also suggest that the neutrinos may belong to either a normal hierarchy ($m_1 < m_2 < m_3$) or an inverted hierarchy ($m_3 < m_1 < m_2$). The data do not exclude the possibility that the mass of the light neutrino could be much larger than $\sqrt{\Delta m^2_{31}}$, which would imply the possible existence of a quasi-degenerate neutrino mass spectrum ($m_1 \approx m_2 \approx m_3$). On the other hand, the actual mass of neutrinos cannot be extracted from these data, only the study of tritium single $\beta$ decay and nuclear neutrino-less double beta decay together can provide sharpest limits on the mass and nature of neutrinos. Neutrino oscillations, which only depend on mass square difference, give no information about the absolute value of the neutrino mass squared eigenvalues. Hence, there are various possibilities of neutrino hierarchy spectrums consistent with solar and atmospheric neutrino oscillation data. The mass eigenstates with eigen values $m_i$ can be determined if the absolute value of effective mass of neutrino is exactly known. The current neutrino-less double beta decay experiments only provide the upper limit on effective Majorana neutrino mass $< m_{ee} >$ so that absolute scale of neutrino mass is not determined yet. Therefore, in the present work we have attempted to present a picture of neutrino mass spectrum in the case of normal, inverted and almost degenerate hierarchy of neutrino masses by taking some specific choices of effective mass.

In this paper, we will discuss the masses of the vacuum eigenstates $m_1, m_2$ and $m_3$ for different neutrino mass spectrum, namely normal mass hierarchy ($m_1 < m_2 < m_3$), inverted mass hierarchy ($m_3 < m_1 < m_2$) and almost degenerate spectrum ($m_1 \approx m_2 \approx m_3$). The present work is organized as follows. In Sec. 2, we outline the neutrino oscillation parameters. In Sec. 3, we have given the theoretical formalism to calculate mass eigenvalues $m_i$ for above mentioned hierarchies. In Sec. 4, we present the numerical results and Sec. 5 is devoted to the conclusions.
2 Mixing Angles and Neutrino Mass Squared Differences

The first evidence is the observation of zenith-angle dependence of atmospheric neutrino defect [15] dependent of the atmospheric neutrino $\nu_\mu \rightarrow \nu_\mu$ transition with the mass difference and the mixing as

$$\Delta_{31} = (1 - 2) \times 10^{-3} eV^2, \sin^2 2\theta_{23} = 1.0.$$  \hspace{1cm} (1)

The second evidence is the solar neutrino deficit [16], which is consistent with $\nu_\mu \rightarrow \nu_e/\nu_\tau$ transition. The SNO experiments [17] are consistent with the standard solar model [18] and strongly suggest the LMA solution.

$$\Delta_{21} = 7 \times 10^{-5} eV^2, \sin^2 2\theta_{12} = 0.8.$$ \hspace{1cm} (2)

Solar neutrino experiments (Super-K, GALLEX, SAGE, SNO and GNO) show the neutrino oscillations, neutrino oscillation provide the most elegant explanation of all the data [19].

$$\Delta_{\text{solar}} = 7^{+5}_{-1.3} \times 10^{-5} eV^2,$$ \hspace{1cm} (3)

$$\tan^2 \theta_{\text{solar}} = 0.4^{+0.14}_{-0.1}.$$ \hspace{1cm} (4)

Atmospheric neutrino experiments (Kamiokande, Super-K) also show the neutrino oscillation. The most excellent fit to the all data [19].

$$\Delta_{\text{atmo}} = 2.0^{+1.0}_{-0.92} \times 10^{-3} eV^2,$$ \hspace{1cm} (5)

$$\sin^2 2\theta_{\text{atmo}} = 0.4^{+0.14}_{-0.10}.$$ \hspace{1cm} (6)

The CHOOZ reactor experiment [20] gives the upper bound of the third mixing angle $\theta_{13}$ as

$$\sin^2 \theta_{13} < 0.20 \hspace{1cm} \text{for} \hspace{1cm} |\Delta_{31}| = 2.0 \times 10^{-3} eV^2,$$ \hspace{1cm} (7)

$$\sin^2 \theta_{13} < 0.16 \hspace{1cm} \text{for} \hspace{1cm} |\Delta_{31}| = 2.5 \times 10^{-3} eV^2,$$ \hspace{1cm} (8)

$$\sin^2 \theta_{13} < 0.14 \hspace{1cm} \text{for} \hspace{1cm} |\Delta_{31}| = 3.0 \times 10^{-3} eV^2,$$ \hspace{1cm} (9)
at the 90% CL. The CP phase $\delta$ has not been constrained. The future neutrino experiments plan to measure the oscillation parameters precisely.

3 Effective Majorona Mass of Electron Neutrino

In the presence of three flavour neutrino mixing the electron neutrino is combination of mass eigenstate, $\nu_i$ with eigenvalue $m_i$

$$\nu_e = \sum U_{ei} \nu_{ij} \quad i = 1, 2, 3. \quad (10)$$

Here $U_{ei}$ are the elements of the mixing matrix, which relates the flavour states the the mass eigenstates. The $\beta\beta_0$ decay rate is determined by the effective Majorana mass of the electron neutrino $m_{ee}$. Under the assumption of three flavour neutrino mixing of neutrino, the effective Majorana neutrino mass $m_{ee}$ is

$$|m_{ee}| = |\sum |U_{ej}|^2 e^{i\phi_j} m_j|$$

$$= |c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 e^{i\phi_2} m_2 + s_{13}^2 e^{i\phi_3} m_3|. \quad (11)$$

Where $|U_{ej}| = 1, 2, 3$ are the absolute values of the elements of the first row of neutrino mixing matrix,

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} s_{23} s_{13} e^{i\delta} & c_{23} s_{13} \end{pmatrix},$$

here $c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij}$. The angle $\phi_2, \phi_3$ are the two majorana CP phase. The masses of the vacuum eigenstates are takes to be $m_1, m_2$ and $m_3$. On squaring eq(11),

$$|m_{ee}|^2 = A^2 m_1^2 + B^2 m_2^2 + C^2 m_3^2 + 2ABm_1m_2\cos \phi_2 + 2ACm_1m_3\cos \phi_3 + 2BCm_2m_3\cos(\phi_2 - \phi_3), \quad (12)$$

where $A = c_3^2 c_2^2, B = c_3^2 s_2^2, C = s_3^2$.
4 Numerical Results

In table-1, table-3 and table-3, we list neutrino mass eigenvalue $m_i$ in case of different neutrino mass spectrum, using the best fit value given in ref[21]. We have varied $m_{ee}$ from 1-10 meV and 10-100 meV in case of normal and inverted hierarchy respectively. For degenerate case, $m_1 \approx m_2 \approx m_3$, and considering the analysis of the Sloan Digital Survey Data and WMAP data [22], Wilkinson Microwave Anisotropy Probe (WMAP) and 2 degree Field Galaxy Redshift Survey (2dFGRS) data [23], we have varied $m_{ee}$ from 100-600 meV. For normal hierarchy, $m_1$ varies from 0.0002eV - 0.01eV and $m_2$ varies from 0.009 eV- 0.013eV, while variation in $m_3$ is from 0.049-0.056 eV. The variation in $m_3$ is quite less in comparison to $m_2$ and $m_1$. In case of inverted hierarchy, $m_1$ and $m_2$ both vary from 0.01-0.263 eV while $m_3$ varies from 0.047-0.259 eV for considered values of $m_{ee}$. For 10-30 meV range of $m_{ee}$, $m_3$ values are higher than $m_1$ and $m_2$, while for 40-100 meV range of $m_{ee}$, $m_3$ value is lower than $m_1$ and $m_2$. The almost same variation for $m_1$ and $m_2$ also confirms the consideration of $m_1$, $m_2$, for inverted hierarchy. In case of AD hierarchy, all the $m_1$, $m_2$, $m_3$ values lie in the range from 0.26-1.74 eV. We compute the neutrino mass eigenvalue using eq (11). We have taken normal mass hierarchy $\Delta_{31} > 0$, inverted mass hierarchy $\Delta_{31} < 0$ and almost degenerated case. For simplicity, we have set the majorana phases $\phi = 0^\circ, 180^\circ$. 
| Mass Hierarchy | $m_\nu$ (eV) | Mass Eigenstate | Majorana Phases | Majorana Phases |
|---------------|-------------|----------------|-----------------|-----------------|
| Normal        | 0.010       | $m_1$          | $9.10 \times 10^{-2}$ | $2.62 \times 10^{-2}$ |
|               |             | $m_2$          | $1.33 \times 10^{-2}$ | $2.77 \times 10^{-2}$ |
|               |             | $m_3$          | $5.05 \times 10^{-2}$ | $5.60 \times 10^{-2}$ |
|               | 0.009       | $m_1$          | $8.9 \times 10^{-3}$  | $2.36 \times 10^{-2}$ |
|               |             | $m_2$          | $1.25 \times 10^{-2}$ | $2.52 \times 10^{-2}$ |
|               |             | $m_3$          | $5.03 \times 10^{-2}$ | $5.49 \times 10^{-2}$ |
|               | 0.008       | $m_1$          | $7.76 \times 10^{-4}$ | $2.11 \times 10^{-2}$ |
|               |             | $m_2$          | $1.17 \times 10^{-2}$ | $2.28 \times 10^{-2}$ |
|               |             | $m_3$          | $5.01 \times 10^{-2}$ | $5.38 \times 10^{-2}$ |
|               | 0.007       | $m_1$          | $6.61 \times 10^{-3}$ | $1.86 \times 10^{-2}$ |
|               |             | $m_2$          | $1.09 \times 10^{-2}$ | $2.05 \times 10^{-2}$ |
|               |             | $m_3$          | $4.99 \times 10^{-2}$ | $5.29 \times 10^{-2}$ |
|               | 0.006       | $m_1$          | $5.43 \times 10^{-3}$ | $1.61 \times 10^{-2}$ |
|               |             | $m_2$          | $1.03 \times 10^{-2}$ | $1.83 \times 10^{-2}$ |
|               |             | $m_3$          | $4.98 \times 10^{-2}$ | $5.20 \times 10^{-2}$ |
|               | 0.005       | $m_1$          | $4.21 \times 10^{-3}$ | $1.36 \times 10^{-2}$ |
|               |             | $m_2$          | $9.70 \times 10^{-3}$ | $1.62 \times 10^{-2}$ |
|               |             | $m_3$          | $4.97 \times 10^{-2}$ | $5.13 \times 10^{-2}$ |
|               | 0.004       | $m_1$          | $4.94 \times 10^{-4}$ | $1.13 \times 10^{-2}$ |
|               |             | $m_2$          | $9.22 \times 10^{-3}$ | $1.42 \times 10^{-2}$ |
|               |             | $m_3$          | $4.96 \times 10^{-2}$ | $5.08 \times 10^{-2}$ |
|               | 0.003       | $m_1$          | $1.61 \times 10^{-3}$ | $8.98 \times 10^{-2}$ |
|               |             | $m_2$          | $8.89 \times 10^{-3}$ | $1.25 \times 10^{-2}$ |
|               |             | $m_3$          | $4.95 \times 10^{-2}$ | $5.03 \times 10^{-2}$ |
|               | 0.002       | $m_1$          | $1.87 \times 10^{-4}$ | $6.84 \times 10^{-3}$ |
|               |             | $m_2$          | $8.74 \times 10^{-3}$ | $1.11 \times 10^{-2}$ |
|               |             | $m_3$          | $4.95 \times 10^{-2}$ | $4.99 \times 10^{-2}$ |
|               | 0.001       | $m_1$          | $1.35 \times 10^{-4}$ | $4.85 \times 10^{-2}$ |
|               |             | $m_2$          | $8.84 \times 10^{-3}$ | $9.99 \times 10^{-2}$ |
|               |             | $m_3$          | $4.95 \times 10^{-2}$ | $4.97 \times 10^{-2}$ |

Table 1: Neutrino mass eigenvalue for normal hierarchy mass spectrum. Input value are given in ref[21]
\[ \begin{array}{|c|c|c|c|c|}
\hline
\text{Mass Hierarchy} & \text{\(m_\nu\)} & \text{Mass Eigenstate} & \text{Majorana Phases} & \text{Majorana Phases} \\
\hline
\text{Inverted} & 0.10 & m_1 & 1.04 \times 10^{-1} & 2.63 \times 10^{-2} \\
& & m_2 & 1.04 \times 10^{-1} & 2.63 \times 10^{-2} \\
& & m_3 & 9.18 \times 10^{-2} & 2.59 \times 10^{-2} \\
& 0.09 & m_1 & 9.33 \times 10^{-2} & 2.37 \times 10^{-2} \\
& & m_2 & 9.38 \times 10^{-2} & 2.37 \times 10^{-2} \\
& & m_3 & 7.98 \times 10^{-2} & 2.32 \times 10^{-2} \\
& 0.08 & m_1 & 8.29 \times 10^{-2} & 2.11 \times 10^{-2} \\
& & m_2 & 8.33 \times 10^{-2} & 2.11 \times 10^{-2} \\
& & m_3 & 6.73 \times 10^{-2} & 2.05 \times 10^{-2} \\
& 0.07 & m_1 & 7.24 \times 10^{-2} & 1.84 \times 10^{-2} \\
& & m_2 & 7.29 \times 10^{-2} & 1.85 \times 10^{-2} \\
& & m_3 & 5.38 \times 10^{-2} & 1.78 \times 10^{-2} \\
& 0.06 & m_1 & 6.18 \times 10^{-2} & 1.58 \times 10^{-2} \\
& & m_2 & 6.24 \times 10^{-2} & 1.58 \times 10^{-2} \\
& & m_3 & 3.85 \times 10^{-2} & 1.51 \times 10^{-2} \\
& 0.05 & m_1 & 5.11 \times 10^{-2} & 1.32 \times 10^{-2} \\
& & m_2 & 5.19 \times 10^{-2} & 1.32 \times 10^{-2} \\
& & m_3 & 1.65 \times 10^{-2} & 1.23 \times 10^{-2} \\
& 0.04 & m_1 & 4.11 \times 10^{-2} & 1.06 \times 10^{-1} \\
& & m_2 & 4.19 \times 10^{-2} & 1.06 \times 10^{-1} \\
& & m_3 & 2.56 \times 10^{-2} & 9.43 \times 10^{-2} \\
& 0.03 & m_1 & 3.09 \times 10^{-2} & 8.01 \times 10^{-2} \\
& & m_2 & 3.22 \times 10^{-2} & 8.10 \times 10^{-2} \\
& & m_3 & 3.71 \times 10^{-2} & 6.38 \times 10^{-2} \\
& 0.02 & m_1 & 2.07 \times 10^{-2} & 5.46 \times 10^{-2} \\
& & m_2 & 2.25 \times 10^{-2} & 5.51 \times 10^{-2} \\
& & m_3 & 4.38 \times 10^{-2} & 2.54 \times 10^{-2} \\
& 0.01 & m_1 & 1.01 \times 10^{-2} & 2.67 \times 10^{-2} \\
& & m_2 & 1.34 \times 10^{-2} & 2.81 \times 10^{-2} \\
& & m_3 & 4.73 \times 10^{-2} & 4.03 \times 10^{-2} \\
\hline
\end{array} \]

Table 2: Neutrino mass eigenvalue for inverted hierarchy mass spectrum. Input value are given in ref[21]
| Mass Hierarchy       | $m_\nu$ (eV) | Mass Eigenstate | Majorana Phases $\phi_2 = 0^\circ, \phi_3 = 180^\circ$ | Majorana Phases $\phi_3 = 0^\circ, \phi_2 = 180^\circ$ |
|----------------------|-------------|----------------|-------------------------------------------------|-------------------------------------------------|
| Almost degenerate    | 0.6         | $m_1$          | 1.58                                            | 1.74                                            |
|                      |             | $m_2$          | 1.58                                            | 1.74                                            |
|                      |             | $m_3$          | 1.58                                            | 1.74                                            |
|                      | 0.5         | $m_1$          | 1.32                                            | 1.45                                            |
|                      |             | $m_2$          | 1.32                                            | 1.45                                            |
|                      |             | $m_3$          | 1.32                                            | 1.45                                            |
|                      | 0.4         | $m_1$          | 1.06                                            | 1.16                                            |
|                      |             | $m_2$          | 1.06                                            | 1.16                                            |
|                      |             | $m_3$          | 1.06                                            | 1.16                                            |
|                      | 0.3         | $m_1$          | 0.79                                            | 0.87                                            |
|                      |             | $m_2$          | 0.79                                            | 0.87                                            |
|                      |             | $m_3$          | 0.79                                            | 0.87                                            |
|                      | 0.2         | $m_1$          | 0.53                                            | 0.58                                            |
|                      |             | $m_2$          | 0.53                                            | 0.58                                            |
|                      |             | $m_3$          | 0.53                                            | 0.58                                            |
|                      | 0.1         | $m_3$          | 0.26                                            | 0.29                                            |
|                      |             | $m_2$          | 0.26                                            | 0.29                                            |
|                      |             | $m_3$          | 0.26                                            | 0.29                                            |

Table 3: Neutrino mass eigenvalue for degenerated mass spectrum. Input values are given in ref[21]

5 Conclusions

Future and present search for neutrinoless double beta decay purpose at probing lepton number violation and the Majorona nature of neutrinos with remarkable precession. Several experimental programs is currently under discussion. We find the mass eigenvalue $m_i$ in case of normal hierarchy, inverted mass hierarchy and almost degenerate neutrino mass spectrum. By taking $m_{ee} = (0.010 - 0.001)$ eV as reference value, for normal mass hierarchy spectrum. We have predicted that $m_1$ varies from 0.0002eV - 0.01eV and $m_2$ varies from 0.009 eV - 0.013eV, while variation in $m_3$ is from 0.049-0.056 eV. In case of inverted hierarchy, $m_1$ and $m_2$ both vary from 0.01-0.263 eV while $m_3$ varies from 0.047-0.259 eV for considered values of $m_{ee}$. For 10-30 meV range of $m_{ee}$, $m_3$ values are higher than $m_1$ and $m_2$, while for 40-100 meV
range of $m_{ee}, m_3$ value is lower than $m_1$ and $m_2$. The almost same variation for $m_1$ and $m_2$ also confirms the consideration of $m_1, m_2, m_3$ for inverted hierarchy. In case of AD hierarchy, all the $m_1, m_2, m_3$ values lie in the range from 0.26-1.74 eV. We have calculated the mass eigenvalue $m_1,m_2$ and $m_3$ for all the three mass spectrum considered by taking some specific choice of effective neutrino mass depending on the type of mass spectrum. The ordering of mass states depends on choice of $m_{ee}$, hence precise determination of $m_{ee}$ from single beta decay experiment (Tritium beta decay) and future neutrino-less double beta decay experiments will make the picture of mass spectrum clear.

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