PERMUTATION-EQUIVARIANT
QUANTUM K-THEORY IV.
$\mathcal{D}_q$-MODULES

ALEXANDER GIVENTAL

Abstract. In Part II, we saw how permutation-equivariant quantum K-theory of a manifold with isolated fixed points of a torus action can be reduced via fixed point localization to permutation-equivariant quantum K-theory of the point. In Part III, we gave a complete description of permutation-equivariant quantum K-theory of the point by means of adelic characterization. Here we apply the adelic characterization to introduce the action on this theory of a certain group of $q$-difference operators. This action enables us to prove that toric $q$-hypergeometric functions represent K-theoretic GW-invariants of toric manifolds.

OVERRULED CONES AND $\mathcal{D}_q$-MODULES

In Part III, we gave the following adelic characterization of the big J-function $J_{pt}$ of the point target space. In the space $\mathcal{K}$ of “rational functions” of $q$ (consisting in fact of series in auxiliary variables with coefficients which are rational functions of $q$), let $\mathcal{L}$ denote the range of $J_{pt}$. We showed that an element $f \in \mathcal{K}$ lies in $\mathcal{L}$ if and only if Laurent series expansions $f_{(\zeta)}$ of $f$ near $q = \zeta^{-1}$ satisfy

(i) $f_{(1)} = (1 - q)e^{\tau/(1-q)} \times$ (power series in $q - 1$) for some $\tau \in \Lambda_+$,

(ii) when $\zeta \neq 1$ is a primitive $m$-th root of unity,

$$f_{(\zeta)}(q^{1/m}/\zeta) = \Psi^m(f_{(1)}/(1-q)) \times \text{(power series in } q - 1),$$

where $\Psi^m$ is the Adams operation extended from $\Lambda$ by $\Psi^m(q) = q^m$;

(iii) when $\zeta \neq 0, \infty$ is not a root of unity, $f_{(\zeta)}(q/\zeta)$ is a power series in $q - 1$.

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1For convergence purposes, we assume that the Adams operations $\Psi^k$ on $\Lambda$ with $k > 1$ increase certain filtration $\Lambda \supset \Lambda_+ \supset \Lambda_{++} \supset \cdots$, and that the domain of the J-function is $\Lambda_+$. 
Another way to phrase (i) is to say that $f_{(1)}$ lies in the range $\mathcal{L}^{fake}$ of the ordinary (or fake) J-function $J_{pt}^{ord}$ in the space $\hat{\mathcal{K}} = \Lambda((q - 1))$ of Laurent series in $q - 1$:

$$\mathcal{L}^{fake} = \bigcup_{\tau \in \Lambda_+} (1 - q)e^{\tau/(1-q)}\hat{K}_+, \quad \hat{K}_+ := \Lambda[[q - 1]].$$

The range $\mathcal{L}^{fake}$ is an example of an overruled cone: Its tangent spaces $T_\tau = e^{\tau/(1-q)}\hat{\mathcal{K}}_+$ are tangent to $\mathcal{L}^{fake}$ along the subspaces $(1 - q)T_\tau$ (which sweep $\mathcal{L}^{fake}$ as the parameter $\tau$ varies through $\Lambda_+$. As it will be explained shortly, this property leads to the invariance of $\mathcal{L}^{fake}$ to certain finite-difference operators.

Recall that in permutation-equivariant quantum K-theory, we work over a $\lambda$-algebra, a ring equipped with Adams homomorphisms $\Psi^m$, $m = 1, 2, \ldots$, $\Psi^1 = \text{Id}$, $\Psi^m\Psi^l = \Psi^{ml}$. Let us take $\Lambda := \Lambda_0[[\lambda, Q]]$ with $\Psi^m(\lambda) = \lambda^m$, $\Psi^m(Q) = Q^m$, where $\Lambda_0$ is any ground $\lambda$-algebra over $\mathbb{C}$.

Consider the algebra of finite-difference operators in $Q$. Such an operator is a non-commutative expression $D(Q, 1 - qQ\partial_Q, q^{\pm 1})$. Clearly, the space $\hat{\mathcal{K}}_+ = \Lambda[[q - 1]]$ (as well as $(1 - q)\hat{\mathcal{K}}_+$) is a $\mathcal{D}_q$-module. Consequently each ruling space $(1 - q)T_\tau = e^{\tau/(1-q)}(1 - q)\hat{K}_+$ is a $\mathcal{D}_q$-module too. Indeed,

$$qQ\partial_Qe^{\tau(Q)/(1-q)} = e^{\tau(Q)/(1-q)}e^{(\tau(qQ)-\tau(Q))/(1-q)},$$

where the second factor lies in $\hat{\mathcal{K}}_+$. Moreover, we have

**Proposition.** $e^{\lambda D(Q, 1 - qQ\partial_Q, q^{\pm 1})/(1-q)}\mathcal{L}^{fake} = \mathcal{L}^{fake}$.

**Proof.** The ruling space $(1 - q)T_\tau$ is a $\mathcal{D}_q$-module, and hence invariant under $D$. Therefore for $f \in (1 - q)T_\tau$, we have $Df/(1 - q) \in T_\tau$, i.e. the vector field defining the flow $t \mapsto e^{\lambda D/(1-q)}$ is tangent to $\mathcal{L}^{fake}$, and so the flow preserves $\mathcal{L}^{fake}$. It remains to take $t = 1$, which is possible thanks to $\lambda$-adic convergence.

**Remark.** Generally speaking, linear transformation $e^{\lambda D/(1-q)}$ does not preserve ruling spaces $(1 - q)T_\tau$, but transforms each of them into another such space. Indeed, preserving $\mathcal{L}^{fake}$, it transform tangent spaces $T_\tau$ into tangent spaces, and since it commutes with multiplication by $1 - q$, it also transforms ruling spaces $(1 - q)T_\tau$ into ruling spaces.

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2In terminology of S. Barannikov [1], this is a variation of semi-infinite Hodge structures: The flags $\cdots \subset (1 - q)T_\tau \subset T_\tau \subset (1 - q)^{-1}T_\tau \subset \cdots$ vary in compliance with “Griffiths’ transversality condition”.

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2In terminology of S. Barannikov [1], this is a variation of semi-infinite Hodge structures: The flags $\cdots \subset (1 - q)T_\tau \subset T_\tau \subset (1 - q)^{-1}T_\tau \subset \cdots$ vary in compliance with “Griffiths’ transversality condition”.
Likewise, cone $\mathcal{L} \subset \mathcal{K}$ is ruled by subspaces comparable to $(1 - q)\mathcal{K}_+$, namely by $(1 - q)L_\tau$, where $L_\tau := e^{\sum_{k>0} \psi_k(\tau)/k(1 - q^k)}\mathcal{K}_+$. However $L_\tau$ are not tangent to $\mathcal{L}$. Nonetheless the following result holds.

**Theorem.** The range $L$ of the big $J$-function $J_{pt}$ in the permutation-equivariant quantum $K$-theory of the point target space is preserved by operators of the form

$$e^{\sum_{k>0} \lambda_k \psi_k(D(1 - q^kQQ, q^{1/2}))}/k(1 - q^k).$$

**Remarks.** (1) The operator $D$ has constant coefficients, i.e. is independent of $Q$.

(2) Note that $\Psi_k(qQ\partial Q) = q^kQ\partial q^k = q^{kQ},$ and not $q^{kQ}\partial_q$ as in the exponent.

(3) The reader is invited to check that the theorem and its proof are extended without any changes to the case finite difference operators in several variables $Q_1, \ldots, Q_K$. We will use the theorem in this more general form in Part V.

**Proof.** Assuming that $(1 - q)f \in \mathcal{L}$, we use the adelic characterization of $\mathcal{L}$ to show that $(1 - q)g \in \mathcal{L}$, where

$$g(q) := e^{\sum_{k>0} \lambda_k \psi_k(D(1 - q^kQQ, q^{1/2}))/k(1 - q^k)}f(q).$$

First, this relationship between $g$ and $f$ also holds between $g^{(1)}$ and $f^{(1)}$ where however both sides need to be understood as Laurent series in $q - 1$. Since $f^{(1)} \in \mathcal{L}^{fake}$, Proposition implies that $g^{(1)} \in \mathcal{L}^{fake}$ too.

Next, applying $\Psi^m$ to both sides, we find

$$\Psi^m(g^{(1)}) = e^{\sum_{l>0} \lambda_l \psi_l(D(1 - q^lQQ, q^{1/2}))/l(1 - q^l)}\Psi^m(f^{(1)}).$$

On the other hand, for an $m$-th primitive root of unity $\zeta$, taking into account that $\Psi^m(q) = q^m$ turns after the change $q \mapsto q^{1/m}\zeta$ into $q^l$, and that $q^mQ\partial_q$ turns after this change into $q^lQ\partial_q$, we find

$$g^{(1)}(q^{1/m}/\zeta) = e^\Delta e^{\sum_{l>0} \lambda_l \psi_l(D(1 - q^lQQ, q^{1/2}))/ml(1 - q^l)}f^{(1)}(q^{1/m}/\zeta),$$

where the finite-difference operator $\Delta$ has coefficients regular at $q = 1$. Here we factor off the terms regular at $q = 1$ using the fact that our operators have constant coefficients, and hence commute. Namely, $e^{A+B/(1 - q)}$, where $A$ and $B$ are regular at $q = 1$, can be rewritten as $e^A e^{B/(1 - q)}$.

We are given that $f^{(1)}(q^{1/m}/\zeta) = p \Psi^m(f^{(1)})$ where $p \in \widehat{\mathcal{K}}_+$. Since $[qQ\partial_q, Q] = (q - 1)QqQ\partial_q$ is divisible by $q - 1$, for any finite-difference
operator $B$, the commutator $\text{ad}_B(p) = [B, p]$ with the operator of multiplication by $p$ is divisible by $q - 1$. Therefore $e^{B/(1-q)} p = P e^{B/(1-q)}$, where $P = e^{\text{ad}_B/(1-q)}(p)$ is regular at $q = 1$. Thus, for some $P$ regular at $q = 1$ we have:

$$g(\zeta)(q^{1/m}/\zeta) = e^\Delta P e \sum_{d>0} \lambda^m q^m \lambda^m(d(1-q^{Q\partial Q}, q^{1/m})) / m(1-q^d) \Psi^m(f(1)).$$

Comparing this expression with $\Psi^m(g(1))$, take into account that $q^{\pm 1/m}$ coincides with $q^{\pm 1}$ modulo $q - 1$, and $1/(1 - q^{-lm}) - 1/m(1 - q^{-l})$ is regular at $q = 1$. Thus, again factoring off the terms regular at $q = 1$, we conclude that $g(\zeta)(q^{1/m}/\zeta)$ is obtained from $\Psi^m(g(1))$ by the application of an operator regular at $q = 1$.

From the explicit description of $L^{fake}$, we have $g(1) \in e^{\tau/(1-q)\hat{K}^+}$ for some $\tau$. Therefore $\Psi^m(g(1)) \in e^{\Psi^m(\tau)/m(1-q)\hat{K}^+}$. The latter is a $D_q$-module, and hence $g(\zeta)(q^{1/m}/\zeta) \in \Psi^m(g(1)) \hat{K}^+$ as required.

Finally, for $\zeta \neq 0, \infty$, which is not a root of unity, regularity of $g$ at $q = \zeta^{-1}$ is obvious whenever the same is true for $f$. □

### Γ-operators

**Lemma.** Let $l$ be a positive integer. Suppose that $\sum_d f_d Q^d$ represents a point on the cone $L \subset K$. Then the same is true about:

$$\sum_{d \geq 0} f_d Q^d \frac{ld-1}{r=0} (1 - \lambda q^{-r}), \sum_{d \geq 0} \frac{f_d Q^d}{\prod_{r=1}^{ld} (1 - \lambda q^r)}, \text{ and } \sum_{d \geq 0} f_d Q^d \frac{ld}{r=1} (1 - \lambda q^r).$$

**Proof.** We use $q$-Gamma-function

$$\Gamma_q(x) := e^{\sum_{k \geq 0} x^k/k(1-q^k)} \sim \prod_{r=0}^{\infty} \frac{1}{1-xq^r}$$

for symbols of $q$-difference operators:

$$\frac{\Gamma_q^{-1}(\lambda q^{-Q\partial Q})}{\Gamma_q^{-1}(\lambda)} Q^d = Q^d \prod_{r=-\infty}^{0} (1 - \lambda q^r) = Q^d \prod_{r=0}^{ld-1} (1 - \lambda q^{-r}),$$

$$\frac{\Gamma_q^{-1}(\lambda q^{Q\partial Q})}{\Gamma_q^{-1}(\lambda)} Q^d = Q^d \prod_{r=-\infty}^{0} (1 - \lambda q^r) = Q^d \prod_{r=1}^{ld} (1 - \lambda q^r),$$

$$\frac{\Gamma_q^{-1}(\lambda)}{\Gamma_q^{-1}(\lambda q^{Q\partial Q})} Q^d = Q^d \prod_{r=1}^{ld} (1 - \lambda q^r)$$

respectively.

The result follows now from the theorem of the previous section. □
Application to fixed point localization

In Part II, we used fixed point localization to characterize the range (denote it $\mathcal{L}_X$) of the big J-function in permutation- (and torus-) equivariant quantum K-theory of $X = \mathbb{C}P^N$. Namely a vector-valued “rational function” $f(q) = \sum_{i=0}^{N} f^{(i)}(q) \phi_i$ represents a point of $\mathcal{L}_X$ if and only if its components pass two tests, (i) and (ii):

(i) When expanded as meromorphic functions with poles $q \neq 0, \infty$ only at roots of unity, $f^{(i)} \in \mathcal{L}$, i.e. represent values of the big J-function $J_{\text{pt}}$ in permutation-equivariant theory of the point target space;

(ii) Away from $q = 0, \infty$, and roots of unity, $f^{(i)}$ may have at most simple poles at $q = (\Lambda_j/\Lambda_i)^{1/m}$, $j \neq i$, $m = 1, 2, \ldots$, with the residues satisfying the recursion relations

$$\text{Res}_{q = (\Lambda_j/\Lambda_i)^{1/m}} \frac{f^{(i)}(q) dq}{q} = -\frac{Q^m}{C_{ij}(m)} f^{(j)}((\Lambda_j/\Lambda_i)^{1/m}),$$

where $C_{ij}(m)$ are explicitly described rational functions.

We even verified that the hypergeometric series

$$J^{(i)} = (1-q) \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^{d}(1-q^r) \prod_{j \neq i} \prod_{r=1}^{d}(1-q^r \Lambda_i/\Lambda_j)}$$

pass test (ii). Now we are ready for test (i). Indeed, we know from Part I (or from Part III) that

$$(1-q) \Gamma_q(Q) := (1-q)e^{\sum_{k>0} Q^k/(1-q^k)} = (1-q) \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^{d}(1-q^r)}$$

lies in $\mathcal{L}$. According to Lemma,

$$J^{(i)} = \prod_{j \neq i} \frac{\Gamma_{q^{-1}((\Lambda_j/\Lambda_i)-1)Q^d q}}{\Gamma_{q^{-1}((\Lambda_j/\Lambda_i)-1)}} (1-q) \Gamma_q(Q)$$

also lies in $\mathcal{L}$. Thus, we obtain

**Corollary 1.** The $K^0(\mathbb{C}P^N)$-valued function

$$J_{\mathbb{C}P^N} := \sum_{i=0}^{N} J^{(i)} \psi_i = (1-q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^{N} \prod_{r=1}^{d}(1-P \Lambda_j^{-1} q^r)},$$

where $P = O(-1)$ satisfies $\prod_{j=0}^{N} (1-P \Lambda_j^{-1}) = 0$, represents a value of the big J-function $J_{\mathbb{C}P^N}$.

**Remark.** Note that all summands with $d > 0$ are reduced rational functions of $q$, and so the Laurent polynomial part of $J_{\mathbb{C}P^N}$ consists of the dilaton shift term $1-q$ only. This means that $J_{\mathbb{C}P^N}$ represents the
value of the big J-function $J_{\mathbb{C}P^N}(t)$ at the input $t = 0$. Hence it is the small J-function (not only in permutation-equivariant but also) in the ordinary quantum K-theory of $\mathbb{C}P^N$. In this capacity it was computed in [4] by ad hoc methods.

One can derive this way many other applications. To begin with, consider quantum K-theory on the target $E$ which is the total space of a vector bundle $E \to X$. To make the theory formally well-defined, one equips $E$ with the fiberwise scaling action of a circle, $T'$, and defines correlators by localization to fixed points $E^{T'} = X$ (the zero section of $E$). This results in systematic twisting of virtual structure sheaves on the moduli spaces $X_{g,n,d}$ as follows:

$$O_{g,n,d}(E) := \frac{O_{g,n,d}(X)}{\text{Euler}^K_{T'}(E_{g,n,d})}, \quad E_{g,n,d} = (\text{ft}_{n+1})_* \text{ev}^*_n(E),$$

where the $T'$-equivariant K-theoretic Euler class of a bundle $V$ is defined by

$$\text{Euler}^K_{T'}(V) := \text{tr}_{\lambda \in T'} \left( \sum_k (-1)^k \wedge^k V^* \right).$$

The division is possible in the sense that the $T'$-equivariant Euler class is invertible over the field of fractions of the group ring of $T'$. The elements $E_{g,n,d} \in K^0(X_{g,n,d})$ are invariant under permutations of the marked points. (In fact [2, 3], for $d \neq 0$, $E_{g,n,d} = \text{ft}^* E_{g,0,d}$ where $\text{ft} : X_{g,n,d} \to X_{g,n,d}$ forgets all marked points.) Thus, we obtain a well-defined permutation-equivariant quantum K-theory of $E$.

**Corollary 2.** Let $X = \mathbb{C}P^N$, and $E = \bigoplus_{j=1}^M O(-l_j)$. Then the following q-hypergeometric series

$$I_E := (1 - q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^N \prod_{r=1}^d (1 - P\lambda_j^{-1} q^r)} \prod_{j=1}^M \prod_{r=1}^{l_j - 1} (1 - \lambda P^{-l_j} q^{-r})$$

represents a value of the big J-function in the permutation-equivariant quantum K-theory of $E$.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on the fibers of $E$ as multiplication by $\lambda^{-1}$. The K-theoretic Poincaré pairing on $X$ is twisted into $(a, b)_E = \chi(X; ab/ \text{Euler}^K_{T'}(E)).$
Example. Let \( X = \mathbb{C}P^1, \ E = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). In \( I_E \), pass to the non-equivariant limit \( \Lambda_0 = \Lambda_1 = 1 \):
\[
I_E = (1 - q) + (1 - \lambda P^{-1})^2 \times
(1 - q) \sum_{d > 0} Q^d \rho^2 (1 - \lambda P^{-1}q^{-1})^2 \cdots (1 - \lambda P^{-1}q^{1-d})^2
\]

The factor \((1 - \lambda P^{-1})^2\), equal to Euler\( K_T'\), reflects the fact that the part with \( d > 0 \) is a push-forward from \( \mathbb{C}P^1 \) to \( E \). In the second non-equivariant limit, \( \lambda = 1 \), it would turn into 0 (since \((1 - P^{-1})^2 = 0\) in \( K^0(\mathbb{C}P^1) \)). However, what the part with \( d > 0 \) is push-forward of, survives in this limit:
\[
(1 - q) \sum_{d > 0} Q^d \rho^2 (1 - Pq)^2 (1 - P^2)^2 \cdots (1 - P^d)^2
\]

This example is usually used to extract information about "local" contributions of a rational curve \( \mathbb{C}P^1 \) lying in a Calabi-Yau 3-fold with the normal bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \).

Note that decomposing the terms of this series into two summands: with poles at roots of unity, and with poles at 0 or \( \infty \), we obtain non-zero Laurent polynomials in each degree \( d \). They form the input \( t = \sum_{d > 0} t_d(q, q^{-1})Q^d \) of the big J-function whose value \( J_E(t) \) is given by the series.

Finally, note that though the input is non-trivial, it is defined over the \( \lambda \)-algebra \( \mathbb{Q}[\![Q]\!] \). This means that, although we are talking about permutation-equivariant quantum K-theory, the hypergeometric functions here, and in Corollary 2 in general, represent symmetrized K-theoretic GW-invariant, i.e. \( S_n \)-invariant part of the sheaf cohomology.

Similarly, one can introduce K-theoretic GW-invariants of the super-bundle \( \Pi E \) (which is obtained from \( E \to X \) by the "parity change" \( \Pi \) of the fibers) by redefining the virtual structure sheaves as
\[
\mathcal{O}_{g,n,d}^{\text{virt}}(\Pi E) := \mathcal{O}_{g,n,d}(X) \mathcal{E}l_{\mathcal{L}} K_T'(E_{g,n,d}).
\]

When genus-0 correlators of this theory have non-equivariant limits (e.g. when \( E \) is a positive line bundle, and \( d > 0 \)), the limits coincide with the appropriate correlators of the submanifold \( Y \subset X \) given by a holomorphic section of \( \Pi E \).

Corollary 3. Let \( X = \mathbb{C}P^N \), and \( E = \oplus_{j=1}^M \mathcal{O}(l_j) \). Then the following \( q \)-hypergeometric series
\[
I_{\Pi E} = (1 - q) \sum_{d \geq 0} \frac{Q^d}{\prod_{j=0}^{N} \prod_{r=1}^{d_j} (1 - \lambda P_j^{-1}q^r)} \prod_{j=1}^{M} \prod_{r=-\infty}^{l_j}(1 - \lambda P_j q^r)
\]
represents a value of the big J-function in the permutation-equivariant quantum K-theory of $E$.

Here $\lambda \in T' = \mathbb{C}^\times$ acts on fibers of $E$ as multiplication by $\lambda$. The Poincaré pairing is twisted into $(a, b)_{\Pi E} = \chi(X; ab \text{Euler}^K_T(E))$.

**Example.** When all $l_j > 0$, it is safe pass to the non-equivariant limit $\Lambda_j = 1$ and $\lambda = 1$:

$$I_{\Pi E} = (1 - q) \sum_{q \geq 0} Q^d \prod_{j=1}^M \prod_{r=1}^{l_j d} \left( 1 - P^r q^j \right) \prod_{d=1}^N \left( 1 - P q^d \right)^{N+1},$$

which represents a value of the big J-function of $Y \subset \mathbb{C}P^N$, pushed-forward from $K^0(Y)$ to $K^0(\mathbb{C}P^N)$. Here $Y$ is a codimension-$M$ complete intersection given by equations of degrees $l_j$. Taking in account the degeneration of the Euler class in this limit, one may assume that $P$ satisfies the relation $(1 - P)^{N+1-M} = 0$.

When $\sum_j l_j^2 \leq N + 1$, the Laurent polynomial part of this series is $1 - q$, i.e. the corresponding input $t$ of the J-function vanishes. In this case the series represents the small J-function of the ordinary quantum K-theory on $Y$. This result was obtained in [5] in a different way: based on the adelic characterization of the whole theory, but without the use of fixed point localization. As we have seen here, when $t \neq 0$, the series still represents the value $J_Y(t)$ in the symmetrized quantum K-theory of $Y$.

In Part V these results will be carried over to all toric manifolds $X$, toric bundles $E \to X$, or toric super-bundles $\Pi E$. In fact, the intention to find a home for toric $q$-hypergeometric functions with non-zero Laurent polynomial part was one of the motivations for developing the permutation-equivariant version of quantum K-theory.

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