THE GEOMETRY OF BIELLIPTIC SURFACES IN \( \mathbb{P}^4 \)

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In 1988 Serrano [Ser], using Reider’s method, discovered a minimal bielliptic surface in \( \mathbb{P}^4 \). Actually he showed that there is a unique family of such surfaces and that they have degree 10 and sectional genus 6. It is easy to see that the only other smooth surfaces with these invariants are minimal abelian. There is a unique family of minimal abelian surfaces in \( \mathbb{P}^4 \); these arise as the smooth zero-schemes of sections of the Horrocks-Mumford bundle [HM]. The geometry of the Horrocks-Mumford surfaces (smooth or not) has been intensively studied (see e.g. [BHM]).

In this paper we shall, among other things, describe the geometry of the embedding of the minimal bielliptic surfaces. A consequence of this description will be the existence of smooth nonminimal bielliptic surfaces of degree 15 in \( \mathbb{P}^4 \).

The starting point of our investigations is the following consequence of Serrano’s result: The bielliptic surface of degree 10 has a fibration onto an elliptic curve whose fibres are all plane cubic curves. If \( V \) is the union of the planes of these plane cubics, then \( V \) has degree 5 and is the union of the trisecants to an elliptic quintic scroll in \( \mathbb{P}^4 \) with the scroll as its double surface. To see this assume that two members of the family of plane cubic curves span only a hyperplane, then the residual curve in that hyperplane section is a rational quartic curve, but that is impossible since it would dominate the elliptic base of the pencil. Thus in the dual space, the planes of \( V \) correspond to the lines of a smooth elliptic scroll. A first account of \( V \) was given by Segre [Seg].

In a recent paper Catanese and Ciliberto [CC] computed the cohomology of antiprincipal divisors on the symmetric products of an elliptic curve using Heisenberg invariants. This lead us to see that the minimal desingularization \( \tilde{V} \) of \( V \) contains 8 bielliptic surfaces. These are embedded in \( \mathbb{P}^4 \) via the map \( \tilde{V} \to V \). In order to understand this map well it is useful to interpret \( V \) as the set of all singular quadrics through a quintic elliptic normal curve \( E \) in \( \mathbb{P}^4 \). Blowing up the \( \mathbb{P}^2 \)-bundle \( \tilde{V} \) in a certain section we get a \( \mathbb{P}^1 \)-bundle \( \tilde{W} \) over \( S^2E \). It turns out that \( \tilde{W} \) is the natural desingularization of the secant variety \( W \) of the quintic curve \( E \). The threefold \( \tilde{W} \) contains 8 bielliptic surfaces blown up in 25 points. Via the map \( \tilde{W} \to W \) these surfaces are embedded as smooth surfaces of degree 15. In this way we can also find nonminimal abelian surfaces of degree 15 in \( \mathbb{P}^4 \) whose minimal model is isogeneous to a product. These surfaces are (5, 5)-linked to Horrocks-Mumford surfaces, their existence being known before. On the other hand the nonminimal bielliptic surfaces lie on a unique quintic, namely \( W \).

The above duality between \( V \) and the elliptic scroll can also be interpreted in
terms of the quadro-cubic Cremona transformation of Semple [Sem], [SR]. We explain how to construct the degree 15 surfaces with the help of this Cremona transformation.

In a subsequent paper we will study the syzygies and thus also the degenerations of abelian and bielliptic surfaces in \( \mathbb{P}^4 \). A consequence will be the existence of a family of smooth nonminimal abelian surfaces of degree 15 lying on only one quintic hypersurface.

We finally remark that the quintic elliptic scroll and the abelian and bielliptic surfaces of degree 10 and 15 are essentially the only smooth irregular surfaces known in \( \mathbb{P}^4 \) (all others can be derived via finite morphisms \( \mathbb{P}^4 \to \mathbb{P}^4 \)).

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§ 0. Heisenberg invariants on \( \mathbb{P}^2 \)

Here we collect some well-known facts about invariants of the Schrödinger representation of \( H_3 \), the Heisenberg group of level 3. Let \( x_0, x_1, x_2 \) be a basis of \( \text{H}^3(O_{\mathbb{P}^2}(1)) \) and consider the dual of the Schrödinger representation of \( H_3 \) on \( V = \text{H}^3(O_{\mathbb{P}^2}(1)) \) given by

\[
\begin{align*}
\sigma_3(x_i) &= x_{i-1} \\
\tau_3(x_i) &= \varepsilon_3^{-i} x_i \quad (\varepsilon_3 = e^{2\pi i/3})
\end{align*}
\]

where \( i \) is counted modulo 3 and \( \sigma_3 \) and \( \tau_3 \) generate \( H_3 \). Note that

\[
[\sigma_3, \tau_3] = \varepsilon_3^{-1} \cdot \text{id},
\]

hence \( H_3 \) is a central extension

\[
1 \to \mu_3 \to H_3 \to \mathbb{Z}_3 \times \mathbb{Z}_3 \to 1.
\]

The induced representation on \( \text{H}^c(O_{\mathbb{P}^2}(3)) \) decomposes into characters since \( \sigma_3 \) and \( \tau_3 \) commute on the third symmetric power of \( \text{H}^c(O_{\mathbb{P}^2}(1)) \). By \((a, b)\) we denote the character where \( \sigma_3 \) (resp. \( \tau_3 \)) acts by \( \varepsilon_3^a \) (resp. \( \varepsilon_3^b \)). Here again \( a, b \) have to be taken modulo 3. There is a pencil of invariant polynomials, called the Hesse pencil, spanned by

\[
x_0^3 + x_1^3 + x_2^3, \quad x_0 x_1 x_2
\]
and eight invariant polynomials corresponding to the eight non-trivial characters:

\[
F_{(1,0)}: \quad x_0^3 + \varepsilon_3 x_1^3 + \varepsilon_3^2 x_2^3
\]
\[
F_{(2,0)}: \quad x_0^3 + \varepsilon_3^2 x_1^3 + \varepsilon_3 x_2^3
\]
\[
F_{(0,1)}: \quad x_0 x_1^2 + x_1 x_2^2 + x_3 x_0^2
\]
\[
F_{(1,1)}: \quad x_0 x_1^2 + \varepsilon_3 x_1 x_2^2 + \varepsilon_3^2 x_2 x_0^2
\]
\[
F_{(2,1)}: \quad x_0 x_1^2 + \varepsilon_3^2 x_1 x_2^2 + \varepsilon_3 x_2 x_0^2
\]
\[
F_{(0,2)}: \quad x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0
\]
\[
F_{(1,2)}: \quad x_0^2 x_1 + \varepsilon_3 x_1^2 x_2 + \varepsilon_3^2 x_2^2 x_0
\]
\[
F_{(2,2)}: \quad x_0 x_1^2 + \varepsilon_3^2 x_1 x_2^2 + \varepsilon_3 x_2 x_0^2
\]

where \(F_{(a,b)}\) denotes the curve defined by the corresponding polynomial. On each smooth member of the Hesse pencil the group acts by translation of 3-torsion points. There are precisely four singular members, namely:

\[
T_{(0,1)}: \quad x_0 x_1 x_2
\]
\[
T_{(1,1)}: \quad (x_0 + \varepsilon_3^2 x_1 + \varepsilon_3 x_2)(x_0 + x_1 + \varepsilon_3 x_2)(x_0 + \varepsilon_3 x_1 + x_2)
\]
\[
T_{(1,0)}: \quad (x_0 + \varepsilon_3 x_1 + \varepsilon_3^2 x_2)(x_0 + \varepsilon_3^2 x_1 + \varepsilon_3 x_2)(x_0 + x_1 + x_2)
\]
\[
T_{(1,2)}: \quad (x_0 + x_1 + \varepsilon_3^2 x_2)(x_0 + \varepsilon_3 x_1 + \varepsilon_3 x_2)(x_0 + \varepsilon_3 x_1 + x_2)
\]

which equal \(x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_3\) for \(\lambda = \infty, -3\varepsilon_3^2, -3, -3\varepsilon_3\). For each \((i,j)\) the subgroup of order 3 which is generated by \(\sigma_i^3\) fixes the vertices of the triangle \(T(i,j)\).

Consider the involution

\[
\iota_3: \quad (x_0, x_1, x_2) \mapsto (x_0, x_2, x_1).
\]

This involution leaves each member of the Hesse pencil invariant. In fact choosing \((0,1,-1)\) as the origin it acts as \(x \mapsto -x\) on smooth members. The nontrivial characters come in pairs since \(\iota_3(F_{(a,b)}) = F_{(-a,-b)}\).

**Lemma 1.** The curves \(F_{(a,b)}\) are Fermat curves, and \(H_3\) acts on each of them with translation by a 3-torsion point and multiplication by \(\varepsilon_3\).

**Proof.** Let \(\eta^3 = \varepsilon_3\), \(\mu^3 = \frac{1}{9}\). Then

\[
x_0^3 + \varepsilon_3 x_1^3 + \varepsilon_3^2 x_2^3 = x_0^3 + (\eta x_1)^3 + (\eta^2 x_2)^3
\]
\[
x_0 x_1 + x_1^2 x_2 + x_2^2 x_0 = (\eta \mu(x_0 + \varepsilon_3 x_1 + \varepsilon_3 x_2))^3 + (\eta^2 \mu(x_0 + \varepsilon_3 x_1 + \varepsilon_3 x_2))^3
\]
\[
\quad + (\mu(x_0 + x_1 + x_2))^3
\]
\[
x_0^2 x_1 + \varepsilon_3 x_1^2 x_3 + \varepsilon_3^2 x_2^2 x_0 = (\eta \mu(x_0 + \varepsilon_3^2 x_1 + \varepsilon_3^2 x_2))^3 + (\eta^2 \mu(x_0 + \varepsilon_3 x_1 + x_2))^3
\]
\[
\quad + (\mu(x_0 + x_1 + \varepsilon_3 x_2))^3
\]
\[
x_0^2 x_1 + \varepsilon_3^2 x_1^2 x_2 + \varepsilon_3 x_2^2 x_0 = (\eta \mu(x_0 + \varepsilon_3 x_1 + x_2))^3 + (\eta^2 \mu(x_0 + \varepsilon_3 x_1 + \varepsilon_3 x_2))^3
\]
\[
\quad + (\mu(x_0 + x_1 + \varepsilon_3^2 x_2))^3.
\]
Since \( \iota_3(F_{(a,b)}) = F_{(-a,-b)} \) all curves are Fermat curves.

The Fermat curve \( F_{(a,b)} \) intersects each of the triangles \( T_{(i,j)} \), \((i,j) \neq \pm (a,b)\) in its vertices so \( H_3 \) has three subgroups of order 3 with 3 fixed points on \( F_{(a,b)} \). The fourth subgroup has no fixed points, hence \( H_3 \) acts as stated. \( \square \)

§ 1. THREEFOLDS CONTAINING BIELLIPTIC SURFACES

In this part we will construct a \( \mathbb{P}^2 \)-bundle over an elliptic curve \( E \), and a \( \mathbb{P}^1 \)-bundle over the symmetric product \( S^2 E \) of the elliptic curve containing bielliptic surfaces.

Choose a smooth element of the Hesse pencil
\[
E = E_\lambda = \{ x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0 \}
\]
where \( \lambda \neq \infty, -3, -3\varepsilon_3, -3\varepsilon_3^2 \). We choose the inflection point \( p_0 = (0, 1, -1) \) to be the origin of \( E \).

Let \( \xi_0, \xi_1, \xi_2 \) be a dual basis of \( x_0, x_1, x_2 \in V = \Gamma(\mathcal{O}_E(1)) \). The induced action of \( H_3 \) is given by
\[
\sigma_3(\xi_i) = \xi_{i+1}, \\
\tau_3(\xi_i) = \varepsilon_3^i \xi_i.
\]

Note that in this case (3) implies
\[
[\sigma_3, \tau_3] = \varepsilon_3 \cdot \text{id}.
\]

Next consider the line bundle \( \mathcal{O}_E(15p_0) \). Let \( y_0, \ldots, y_{14} \) be a basis of \( H^0(\mathcal{O}_E(15p_0)) \) such that \( H_{15} \), the Heisenberg group of level 15, acts in the standard way, i.e., by
\[
\sigma_{15}(y_i) = y_{i-1}, \\
\tau_{15}(y_i) = \varepsilon_{15}^{-1} y_i \quad (\varepsilon_{15} = e^{2\pi i/15}).
\]

From (5) it follows that
\[
[\sigma_{15}, \tau_{15}] = \varepsilon_{15}^{-10} \cdot \text{id} = \varepsilon_3 \cdot \text{id}.
\]
Hence identifying \( \sigma_{15}^5 \) with \( \sigma_3 \) and \( \tau_{15}^5 \) with \( \tau_3 \) we get an isomorphism of the subgroup of \( H_{15} \) generated by \( \sigma_{15}^5 \) and \( \tau_{15}^5 \) with \( H_3 \subset \text{SL}(V^\vee) \), where the latter inclusion is given by the Schrödinger representation. Since \( y_0, \ldots, y_{14} \) generate \( \mathcal{O}_E(15p_0) \) this gives an action of \( H_3 \) on the line bundle \( \mathcal{O}_E(15p_0) \) itself.

Hence we can consider the natural action of \( H_3 \times H_3 \) on the rank 3 bundle \( W_E = \mathcal{O}_E(15p_0) \otimes V \). Let \( \Delta \) be the diagonal of \( H_3 \times H_3 \). Then \( \Delta \cong H_3 \) and
\[
\sigma_3(y_i \otimes x_j) = y_{i-5} \otimes x_{j-1}, \\
\tau_3(y_i \otimes x_j) = \varepsilon_{3}^{-i-j} y_i \otimes x_j.
\]

It follows from (2) and (6) that the centre of \( H_3 \) acts trivially on \( W_E \). Hence the quotient
\[
\mathcal{E}_E = W_E^\vee / \Delta
\]
is a rank 3 vector bundle over
\[ E/\mathbb{Z}_3 \times \mathbb{Z}_3 = E. \]

**Lemma 2.** (i) $\mathcal{E}_E$ is stable of degree -5.
(ii) $\text{det} \mathcal{E}_E = \mathcal{O}_E(-5p_0)$

*Proof.* (i) Since $\deg W_E = 45$, the degree of $\mathcal{E}_E$ is clearly -5. Now assume that $\mathcal{F} \subset \mathcal{E}_E^\vee$ is a subbundle of rank $r$ ($r = 1, 2$) and degree $d$ contradicting semistability, i.e., $d/r > 5/3$. Then $\mathcal{F}$ pulls back to a subbundle $\mathcal{F}' \subset W_E$ of degree $9d > 15r$. This implies that $\mathcal{F}' \otimes \mathcal{O}_E(-15p_0)$ and hence $W_E \otimes \mathcal{O}_E(-15p_0)$ has a nonconstant section, a contradiction.

(ii) $y_0y_5y_{10}$ is a section of $\text{det} W_E = \mathcal{O}_E(45p_0)$ which is invariant under the induced action of $\Delta$ on $\text{det} W_E$. It defines an invariant divisor on $E$ whose image in the quotient is a divisor linearly equivalent to $5p_0$. \qed

Let us now look at the corresponding action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on the trivial projective bundle $E \times \mathbb{P}^2 = E \times \mathbb{P}(V^\vee)$. Its quotient is a $\mathbb{P}^2$-bundle

\[ \mathbb{P}_E^2 = \mathbb{P}(\mathcal{E}_E) \]

where we use the geometric projective bundle. By the above lemma $\mathbb{P}_E^2$ is the unique indecomposable $\mathbb{P}^1$-bundle over $E$ with invariant $e = -1$ [Ha, V. theorem 2.15]. We consider the quotient map

\[ \pi: E \times \mathbb{P}^2 \to \mathbb{P}_E^2. \]

Clearly this map is unramified and we can use $\pi$ to compute the cohomology of line bundles on $\mathbb{P}_E^2$. This was done in [CC] for the dual bundle $\mathcal{E}_E^\vee$. We are particularly interested in line bundles numerically equivalent to the anticanonical bundle. The Picard group $\mathbb{P}_E^2$ is generated by the tautological bundle $\mathcal{O}_{\mathbb{P}_E^2}(1)$ and the pullback of the Picard group on $E$. The pullback of any line bundle on $\mathbb{P}_E^2$ to $E \times \mathbb{P}^2$ is the tensor product of a line bundle on $E$ and a line bundle on $\mathbb{P}^2$.

**Lemma 3 (Catanese, Ciliberto).** If $\mathcal{O}_{\mathbb{P}_E^2}(L)$ is numerically equivalent to the anticanonical bundle $\mathcal{O}_{\mathbb{P}_E^2}(-K)$ and $h^0(\mathcal{O}_{\mathbb{P}_E^2}(L)) > 0$ then either $L \equiv -K$ in which case $h^0(\mathcal{O}_{\mathbb{P}_E^2}(L)) = 2$ or $3L \equiv -3K$ and $L \neq -K$ in which case $h^0(\mathcal{O}_{\mathbb{P}_E^2}(L)) = 1$. Moreover there are 8 nonisomorphic bundles of the latter kind corresponding to the nontrivial characters of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

*Proof.* Let $L = -K + \rho$ where $\rho$ is the pullback of a degree 0 line bundle on $E$. Since $\pi$ is unramified $\pi^*(-K + \rho) = -K_{E \times \mathbb{P}_2} + \rho'$ where $\rho'$ also has degree 0. This bundle can only have sections when $\rho' = 0$. Since

\[ \mathcal{O}_{\mathbb{P}_E^2}(-K + \rho) \subset \pi_* \pi^* \mathcal{O}_{\mathbb{P}_E^2}(-K + \rho) = \pi_*(-K_{E \times \mathbb{P}_2}) \]

we are left to consider the decomposition

\[ \pi_*(-K_{E \times \mathbb{P}_2}) = \bigoplus_{\lambda \in (\mathbb{Z}_3 \times \mathbb{Z}_3)^\vee} (-K + L_{\lambda}) \]
where $L_{\mathcal{X}}$ is the torsion bundle associated to the character $\mathcal{X}$. I.e., $L_{\mathcal{X}}$ is a torsion bundle of degree 0 on $E$. Hence $L$ is of the form stated. The sections of $L$ are given by the sections of $\mathcal{O}_{E \times \mathbb{P}^2}(-K_{E \times \mathbb{P}^2})$ associated to the character $\mathcal{X}$. By what we have said in the previous paragraph the dimension of these sections is 2 if $\mathcal{X}$ is trivial and 1 otherwise.

**Lemma 4.** In the pencil $| - K|$ the singular members are four singular scrolls while the smooth members are abelian surfaces among which one is isomorphic to $E \times E$. The divisors $-K + L_{\mathcal{X}}$, $\mathcal{X}$ nontrivial, are smooth bielliptic surfaces.

**Proof.** The divisors $\pi^*(-K)$ and $\pi^*(-K + L_{\mathcal{X}})$ are $E \times E_{\mathcal{X}}$, where $E_{\mathcal{X}}$ is a member of the Hesse pencil and $E \times F_{(a,b)}$, respectively. On each of these surfaces the $\mathbb{Z}_3 \times \mathbb{Z}_3$-action is the one described in the previous paragraph. When $E_{\mathcal{X}}$ is smooth then the 9 base points of the Hesse pencil form a subgroup of the product, so the quotient is abelian. In particular when $\mathcal{X} = \lambda$ we get $E \times E/\mathbb{Z}_3 \times \mathbb{Z}_3$ where $\mathbb{Z}_3 \times \mathbb{Z}_3$ acts diagonally. It is easy to see that this quotient is again isomorphic to $E \times E$ (we shall soon discuss this in more detail). When $E_{\mathcal{X}}$ is a triangle then the surface upstairs is the union of three scrolls, whose quotient downstairs is irreducible since the group acts transitively on the edges of the triangles. Finally on $E \times F_{(a,b)}$ the group acts with translation on the first factor and with translation and multiplication on the second factor. Hence the quotient is bielliptic.

We want to describe the intersection of the abelian and bielliptic surfaces with the special abelian surface $A_0 \cong E \times E$ described above. By $A_K$ we’ll denote the general abelian surface in $| - K|$. Let $T_K$ be the singular scrolls in $| - K|$ and $B_{(a,b)}$ the bielliptic surfaces. Let us first consider the structure of the abelian and bielliptic surfaces. Each of them has an elliptic fibration over $E$ whose fibres are the plane cubic curves $E_{\mathcal{X}}$, $F_{(a,b)}$ respectively. For the abelian surfaces the elliptic fibration over $E_{\mathcal{X}}$ upstairs remains an elliptic fibration over $E_{\mathcal{X}}/\mathbb{Z}_3 \times \mathbb{Z}_3 = E_{\mathcal{X}}$ downstairs, the fibres being isomorphic to $E$. Upstairs the intersection of $E \times E$ and $E \times E_{\mathcal{X}}$ is 9 translates of the curve $E$ over the 9 base points of the Hesse pencil. On the quotient these translates are mapped to the same curve isomorphic to $E$, which in turn is a member of the fibration over $E_{\mathcal{X}}$ described above. For the bielliptic surfaces the elliptic fibration over $F_{(a,b)}$ upstairs is mapped to an elliptic fibration over $F_{(a,b)}/\mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathbb{P}^1$ downstairs, i.e., a pencil. The intersection upstairs with $E \times E$ is 9 translates by the group of $E$ over the 9 points of intersection $E_{\mathcal{X}} \cap F_{(a,b)}$ which are mapped to the same curve $A_0 \cap B_{(a,b)}$ downstairs. This is a member of the pencil over $\mathbb{P}^1$. The intersection $A_K \cap B_{(a,b)}$ downstairs is linearly equivalent to and different from $A_0 \cap B_{(a,b)}$, so $| - K|$ restricts to $B_{(a,b)}$ to give the pencil described above. In particular the three triple fibres of this pencil are the intersections $T_K \cap B_{(a,b)}$ for the scrolls $T_K$ coming from triangles $T_{(i,j)}$ with $(i,j) \neq \pm (a,b)$.

Our next aim is to describe the intersection of $A_K$, resp. of $B_{(a,b)}$ with $A_0$ more
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arithmetic. We look at the map

$$
\begin{pmatrix}
3 & 0 \\
-2 & -1
\end{pmatrix} : \begin{cases}
E \times E \to E \times E \\
(q_1, q_2) \mapsto (3q_1, -2q_1 - q_2)
\end{cases}
$$

The kernel of this map is the group $E^{(3)}$ of 3-torsion points of $E$ embedded diagonally into $E \times E$. Hence the above map induces an isomorphism

$$A_0 = (E \times E)/\mathbb{Z}_3 \times \mathbb{Z}_3 \cong E \times E.$$

Whenever we shall refer to $A_0$ as a product it will be via this isomorphism. Note that the curve \{$(q, -2q); q \in E$\} goes 9:1 onto the first factor and that \{$(0, -q); q \in E$\} is mapped isomorphically onto the second factor. Moreover the curve \{$(q, -5q); q \in E$\} goes 9:1 onto the diagonal and \{$(q, q); q \in E$\} is mapped 9:1 onto the antidiagonal of $E \times E$. Finally we consider the map given by $(-\frac{5}{2}, -\frac{1}{2})$ upstairs. One checks immediately that this induces an endomorphism downstairs, and that this endomorphism is $(0 3 \ 3 0)$, i.e., 3 times the standard involution interchanging the factors of $E \times E$.

**Lemma 5.** Let $\Delta_E$ be the diagonal in $A_0 = E \times E$.

(i) The curve $A_K \cap E \times E$ is

$$\{(q, r); 3r + 2q = 0\}$$

and $A_K \cap \Delta_E$ consists of the 25 points

$$\{(p, p); 5p = 0\}.$$

(ii) The curves $B_{(a, b)} \cap E \times E$ are

$$\{(q, r); 3r + 2q = -\tau_{(a, b)}\}$$

and $B_{(a, b)} \cap \Delta_E$ are the sets of points

$$\{(p, p); 5p = -\tau_{(a, b)}\},$$

where $0 \neq \tau_{(a, b)}, 3\tau_{(a, b)} = 0$.

**Proof.** (i) Upstairs $E \times E \cap E \times E = \{(q, \tau_3); q \in E, 3\tau_3 = 0\}$. The image of this set downstairs is \{(3q, -2q - \tau_3); q \in E\} which is the curve described. The second part follows immediately.

(ii) Similarly $E \times F_{(a, b)} \cap E \times E = \{(q, \tau_9); q \in E, 3\tau_9 = \tau_{(a, b)}\}$ where the $\tau_{(a, b)}$ are the 3-torsion points on $E$. Downstairs this is \{(3q, -2q - \tau_9); q \in E\} which gives the claim. 

At this point we want to return to the product $E \times \mathbb{P}^2$. Let $p, q$ be the projections onto $E$ and $\mathbb{P}^2$. Let $\mathcal{O}_E(15p_0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) = p^*\mathcal{O}_E(15p_0) \otimes q^*\mathcal{O}_{\mathbb{P}^2}(1)$. The centre of the diagonal $\Delta \subset H_3 \times H_3$ acts trivially on this line bundle which, therefore, descends to a line bundle $\mathcal{L}$ on $\mathbb{P}^2_E$. 

Proposition 6. (i) \( h^0(\mathcal{L}) = 5 \)
(ii) The following sections are invariant under \( \Delta \), hence define a basis of \( H^0(\mathcal{L}) \):

\[
\begin{align*}
  s_0 & = y_0 \otimes x_0 + y_5 \otimes x_1 + y_{10} \otimes x_2 \\
  s_1 & = y_3 \otimes x_0 + y_8 \otimes x_1 + y_{13} \otimes x_2 \\
  s_2 & = y_6 \otimes x_0 + y_{11} \otimes x_1 + y_1 \otimes x_2 \\
  s_3 & = y_9 \otimes x_0 + y_{14} \otimes x_1 + y_4 \otimes x_2 \\
  s_4 & = y_{12} \otimes x_0 + y_2 \otimes x_1 + y_7 \otimes x_2
\end{align*}
\]

Proof. (i) Clearly

\[
H^0(\mathcal{O}_E(15p_0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)) = H^0(\mathcal{O}_E(15p_0)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)).
\]

As an \( H_3 \)-module

\[
H^0(\mathcal{O}_E(15p_0)) = 5V^\vee.
\]

This can be seen by looking at the subspaces spanned by \((y_0, y_5, y_{10}), (y_3, y_8, y_{13}), (y_6, y_{11}, y_1), (y_9, y_{14}, y_4), (y_{12}, y_2, y_7)\). Hence as an \( H_3 \)-module

\[
H^0(\mathcal{O}_E(15p_0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)) = 5V^\vee \otimes V = 5(\bigoplus_{\chi \in \{\mathbb{Z}_3 \times \mathbb{Z}_3\}^\vee} V_{\chi}).
\]

I.e. we have 5 invariant sections, and thus \( h^0(\mathcal{L}) = 5 \).

(ii) It is straightforward to check that the \( s_i \) are invariant under \( H_3 \).

Next we consider the subgroup of \( H_{15} \) spanned by \( \sigma_{15}^3, \tau_{15}^3 \). From (5)

\[
[\sigma_{15}^3, \tau_{15}^3] = \varepsilon_{15}^{-9} \cdot \text{id} = \varepsilon_{5}^{-2} \cdot \text{id} \quad (\varepsilon_{5} = e^{2\pi i/5}).
\]

Hence mapping \( \sigma_{15}^3 \) to \( \sigma_5 \) and \( \tau_{15}^3 \) to \( \tau_5 \) we can identify this subgroup with \( H_5 \subset \text{SL}(\mathbb{C}^5) \), where this inclusion is given by the representation which arises from the Schrödinger representation of the Heisenberg group \( H_5 \) of level 5 by replacing \( \varepsilon \) by \( \varepsilon^2 \). Now let \( H_5 \) act on \( E \times \mathbb{P}^2 \) where the action on the second factor is trivial. Then \( H_5 \) acts on \( \mathcal{O}_E(15p_0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \) and it is straightforward to check that this action commutes with \( H_3 \). Hence we get an action of \( H_5 \) on \( \mathcal{L} \).

Proposition 7. The action of \( H_5 \) on \( H^0(\mathcal{L}) \) is given by

\[
\sigma_5(s_i) = s_{i-1}, \quad \tau_5(s_i) = \varepsilon_{5}^{-2i} s_i.
\]

Proof. Straightforward calculation. 

We have involutions on \( E \) (given by \( x \mapsto -x \)) and on \( \mathbb{P}^2 \) (given by \( \iota_3(x_i) = x_{-i} \)). Hence we have an involution \( \iota \) on \( E \times \mathbb{P}^2 \). This lifts to an involution on \( \mathcal{O}_E(15p_0) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \) where it acts on sections by

\[
\iota(y_i \otimes x_i) = y_{-i} \otimes x_{-i}.
\]

This involution does not commute with \( H_3 \), but we have an action of a semi-direct product \( H_3 \rtimes (\iota) \). In the quotient this defines an involution on \( \mathbb{P}_E^2 \) and on \( \mathcal{L} \). Note that on \( A_0 = E \times E \) this is given by \((-1, 0, 0 \cdot -1)\).
THE GEOMETRY OF BIELLIPTIC SURFACES IN $\mathbb{P}^4$

**Proposition 8.** $\iota$ acts on $H^0(\mathcal{L})$ by

$$\iota(s_i) = s_{-i}.$$ 

**Proof.** Immediately from (7). \[\square\]

Finally we remark that we really have an action of $(\mathbb{Z}_3 \times \mathbb{Z}_3)^2$ on $E \times \mathbb{P}^2$ and that $\mathbb{P}^2_E$ was constructed by taking the quotient with respect to the diagonal. Hence we have still got an action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $\mathbb{P}^2_E$ which on every fibre of $\mathbb{P}^2_E$ lifts to the Schrödinger representation of $H_3$.

Let $\Delta_E$ be the diagonal in $A_0 = E \times E$. We can consider the blow-up $\rho: U \to \mathbb{P}^2_E$ along $\Delta_E$. Since $\Delta_E$ is a section of $\mathbb{P}^2_E$ the variety $U$ has the structure of an $\Sigma^1$-bundle over $E$. Here $\Sigma^1$ denotes the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with $e = -1$.

**Lemma 9.** $U$ has the structure of a $\mathbb{P}^1$-bundle over $S^2E$.

**Proof.** Let $E_\Delta$ be the exceptional surface over $\Delta_E$ and $B = \mathcal{O}_{\mathbb{P}^2_E}(1)$. By $F$ we denote the class of a fibre of $\mathbb{P}^2_E$. For $\beta$ sufficiently large $|B - E_\Delta + \beta F|$ is base point free. This linear system maps each $\Sigma^1$ to a $\mathbb{P}^1$-bundle over a scroll over $E$, and it remains to determine this scroll. To do this we look at $A_0 = E \times E$. The map given by $|B - E_\Delta + \beta F|$ restricted to a curve $\{q\} \times E$ is nothing but projection of the plane cubic $E \subset \mathbb{P}^2$ from the point $q$. Hence we get an involution on $E \times E$ whose branch locus is the curve $\Delta' = \{(q,t) \in E \times E; 2t + q = 0\}$. $\Delta'$ is the image of $E \to E \times E$, $q \mapsto (\frac{2q}{\gamma})$. The isomorphism of $E \times E$ given by the matrix $\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ maps $\Delta'$ to the diagonal and the curves $\{q\} \times E$ to the translates of the antidiagonal $\{(q,-q); q \in E\}$. Under this isomorphism the above involution becomes the standard involution given by interchanging the factors. This proves that the scroll in question is indeed $S^2E$. We have indeed a (locally trivial) $\mathbb{P}^1$-bundle by [BPV, V.4.1] and the remark after this. \[\square\]

In view of the above lemma we shall change our notation and write

$$\mathbb{P}^1_{S^2E} = U.$$ 

The strict transform of $A_0 = E \times E$ under the map $\rho: \mathbb{P}^1_{S^2E} \to \mathbb{P}^2_E$ is again $E \times E$. The other abelian surfaces $A_K$ in $\mathbb{P}^2_E$ are blown up in the points

$$\Delta_E \cap (A_K \cap A_0) = \{q; 5q = 0\},$$

i.e., in 25 distinct points (cf. lemma 5). Similarly the surfaces $B_{(a,b)}$ are blown up in 25 distinct points by $\rho$. 
§ 2. **Two Quintic Hypersurfaces in \( \mathbb{P}^4 \)**

Let \( E \) be an elliptic normal curve of degree 5 in \( \mathbb{P}^4 \), embedded by the linear system \( |5p_0| \), where \( p_0 \in E \) is the origin which we have chosen before. We assume that \( E \) is invariant under the action of \( H_5 \) given by its Schrödinger representation. In this paragraph we will describe geometrically the following diagram

\[
\begin{array}{c}
\mathbb{P}^1_{S^2E} \\
\pi_1 \\
V_1 \\
p_1 \\
F \\
p_2 \\
V_2 \\
\pi_2 \\
\mathbb{P}^2_E \\
f_1
\end{array}
\]

Here \( V_1 \) and \( V_2 \) are quintic hypersurfaces in \( \mathbb{P}^4 \): \( V_1 \) is the secant variety to \( E \) and \( V_2 \) is ruled in an elliptic family of planes. \( V_2 \) is singular (set theoretically) along an elliptic quintic scroll whose trisecant variety it is. \( \mathbb{P}^1_{S^2E}, \mathbb{P}^2_E \) and \( \rho \) are as in § 1. Via \( \pi_1 \) and \( \pi_2 \) they are tied together via an incidence variety \( F \) consisting of pairs \((p, Q)\) where \( Q \) is a singular quadric containing \( E \) and \( p \in \text{Sing} Q \). The morphisms to \( V_1 \) and \( V_2 \) are projections to the first and second factor respectively. Hence both hypersurfaces can be described in terms of the 3-dimensional family of singular quadrics through \( E \). Furthermore one can make the identifications

\[
\mathbb{P}^1_{S^2E} = \{(p, (e_1, e_2))|e_1, e_2 \in E, p \in L_{(e_1, e_2)}\} = \{(p, W_p)|p \in V_1, W_p = H^0(I_{E \cup \{p\}}(2))\}
\]

where \( L_{(e_1, e_2)} \) is the secant line through \( e_1, e_2 \). The second projection for these incidence varieties is the map to \( S^2E \subset \mathbb{P}(H^0(I_E(2))^\vee) \). Thus \( \mathbb{P}^1_{S^2E} \) is the graph of the quadro cubic Cremona transformation restricted to the secant variety \( V_1 \) of \( E \). One can also show that \( V_2 \subset \mathbb{P}(H^0(I_E(2))) \) in this setting is the natural dual to \( S^2E \subset \mathbb{P}(H^0(I_E(2))^\vee) \). Since this will not be essential for our argument we will omit the proofs. Some of the results we collect here are also contained in [EL], [Hu], [d’Al].

Let \( E \) be as above. It is well known that \( h^2(I_E(2)) = 5 \) and that a basis for the space of quadrics through \( E \) is given by

\[
Q_i = x_i^2 + ax_{i+2}x_{i+3} - \frac{1}{a}x_{i+1}x_{i+4} \quad (i \in \mathbb{Z}_5).
\]

Here \( a \in \mathbb{C} \cup \{\infty\} \) and five such quadrics define a smooth elliptic curve if and only if \( a \) is not a vertex of the icosahedron, i.e., \( a \neq 0, \infty, \varepsilon_5^k(\varepsilon_3^2 + \varepsilon_5^2), \varepsilon_5^k(\varepsilon_5 + \varepsilon_5^2), k = 0, \ldots, 4 \) \( (\varepsilon_5 = e^{2\pi i/5}) \).

**Definition 10.** (i) For \( y \in \mathbb{P}^4 \) let \( M(y) \) be the symmetric 5×5-matrix

\[
M(y) := (y_{i+j}z_{i-j}) \quad 0 \leq i, j \leq 4
\]

where \( z \in \mathbb{P}^4 \), \( z_i = z_{-i} \) and \( z_0 = 2, z_1 = a, z_2 = -\frac{1}{a} \).

(ii) Let \( F \) denote the incidence variety

\[
F := \{(x, y) \in \mathbb{P}^4 \times \mathbb{P}^4; M(y)x = 0\}
\]
and let $V_1$ and $V_2$ denote the images under the first and second projections $p_1$ and $p_2$ of $F$ to the respective $\mathbb{P}^4$'s.

(iii) Let $M'(x)$ be the $5 \times 5$-matrix defined by

$$M'(x)^t y = M(y)^t x.$$  

**Remark 11.** This set-up was also considered in [A] in the case of a general point $z \in \mathbb{P}^4$. Here we have chosen a special point, namely one that lies on the conic section invariant under the icosahedral group $A_5$ on the Bring plane $z_i = z_{-i}$; $i = 0, \ldots, 4$ (see [BHM]). This conic can be identified naturally with the modular curve of level 5, $X(5)$, which is in 1:1-correspondence with $H_5$-invariantly embedded elliptic quintics in $\mathbb{P}^4$. Under this identification $z$ corresponds to the curve $E$ we have started with. The matrix $M$ was first considered by Moore.

**Proposition 12.** (i) The set of quadrics $\{xM(y)^t x; y \in \mathbb{P}^4\}$ is $\mathbb{P}(H^c(I_E(2)))$.

(ii) $F$ can be identified with the incidence variety of pairs $(p, Q)$ where $Q$ is a singular quadric through $E$ and $p \in \text{Sing } Q$.

(iii) $V_1$ and $V_2$ are Heisenberg invariant quintic hypersurfaces in $\mathbb{P}^4$.

**Proof.** (i) It is easily checked that

$$2Q_{3i}(x) = xM(e_i)^t x \quad (i = 0, \ldots, 4).$$

(ii) This follows immediately since $F$ is given by $M(y)^t x = 0$.

(iii) Since $M(y)^t x = 0$ is equivalent to $M'(x)^t y = 0$ it follows that $V_1$ and $V_2$ are given by the quintic equations $\det M'(x) = 0$ resp. $\det M(y) = 0$. Because both $M'(x)$ and $M(y)$ are invariant under $H_5$, up to an even number of permutations of rows and columns, these equations are $H_5$-invariant. Since $H_5$ has no characters on $H^c(O_{24}(n)), n < 5$ it follows that $V_1$ and $V_2$ are in fact reduced of degree 5. 

**Remark 13.** Since the general singular quadric through $E$ has rank 4, it follows that the projection $F \to V_2$ is generically finite and hence $F$ is also of dimension 3.

**Corollary 14.** (i) $V_1$ is the locus of singular points of the singular quadrics through $E$.

(ii) $V_1 = \text{Sec } E$.

**Proof.** (i) is obvious.

(ii) If $p \in \text{Sec } E \setminus E$ then projection to $\mathbb{P}^3$ from $p$ maps $E$ to a nodal quintic curve in $\mathbb{P}^3$, which always lies on a quadric surface. Hence $p$ lies on a quadric cone through $E$. This implies $\text{Sec } E \subset V_1$ and since both hypersurfaces have degree 5, the claim follows.

**Corollary 15.** $V_2$ is the discriminant locus of the family of quadrics through $E$. 


Proof. Clear.

The mapping $p_1$ (resp. $p_2$) is a “small resolution” of $V_1$ (resp. $V_2$), i.e., a (singular) point where $M'(x)$ (resp. $M(y)$) has rank 3 is replaced by a $\mathbb{P}^1$. One of our aims is to describe the rank 3 loci. For $M(y)$ the corresponding $\mathbb{P}^1$s yield the locus of the singular lines. We shall come back to this later. For $M'(x)$ it is simpler.

Proposition 16. (i) The quintic hypersurface $V_1 = \text{Sec } E$ is singular precisely at $E$ where it has multiplicity 3.
(ii) The curve $E$ is exactly the locus where rank $M'(x) = 3$.

Proof. We shall first prove that the multiplicity of Sec $E$ along $E$ is three. This was already known to Segre [Seg], [Sem]. Here we reproduce his proof. We consider a point $p \in E$ and choose a general line $l$ through $p$. We can assume that $l$ meets Sec $E$ transversally at a finite number of smooth points outside $p$. Since secants and tangents of $E$ do not meet outside $E$ (see [Hu, Lemma IV.11]) every such point of intersection lies on a unique secant or tangent of $E$. On the other hand projection from a general line $l$ maps $E$ to a plane curve of degree 4 which, by the genus formula, must have 2 nodes. Hence $l$ intersects Sec $E$ in precisely 2 points (counted properly) outside $p$, and it follows that the multiplicity of Sec $E$ along $E$ is 3.

Now assume that a singularity $x$ of Sec $E$ exists outside $E$. Let $l$ be a line through $x$ which meets $E$ in a point $p$, but is neither a secant nor a tangent line of $E$. (Such a line exists since $x$ lies on at most one secant or tangent — see above). Using [Hu, Proposition IV.4.6] we can also assume that $l$ is not a singular line of a rank 3 quadric through $E$. It follows that $l$ is not contained in Sec $E$. The latter would only be possible if projection from $l$ defines a 2:1 map onto a conic, but this implies that $l$ is the vertex of a rank 3 quadric. Since the multiplicity of Sec $E$ along $E$ is 3 and since $x$ was assumed to be singular, it follows that the intersection of $l$ and Sec $E$ consists precisely of the two points $x$ and $p$. Projection from $l$ now gives a curve of degree 4 and genus 1 in $\mathbb{P}^2$ with exactly one singular point, given by the unique secant or tangent of $E$ through $x$. On the other hand, we can project from $x$ first. In this case $E$ is mapped to a quintic curve $E'$ in $\mathbb{P}^3$ with one singularity and arithmetic genus 2, which lies on a unique quadric surface $Q' \subset \mathbb{P}^3$. We have two possibilities
(1) $Q'$ is a smooth quadric. In this case $E'$ is a divisor on $Q'$ of bidegree $(2,3)$. Then projection from a general point on $E'$ (which corresponds to a general choice of the point $p \in E$) projects $E'$ to a plane quartic with 2 different singularities, a contradiction to what we have found above.
(2) $Q'$ is a quadric cone. In this case $E'$ contains the vertex of this cone as a smooth point and meets every rulings of $Q'$ in 2 points outside the vertex. Again projection from a general point on $E'$ gives a quartic plane curve with two different singularities and we have arrived at the same contradiction as above.

It follows that Sec $E$ has no singularities outside $E$.

(ii) The locus where rank $M'(x)$ has rank $\leq 3$ is contained in $\text{Sing } V_1$. On the other hand if $p \in E$, then projection from $p$ gives a smooth quartic elliptic curve in $\mathbb{P}^3$. 

which lies on a pencil of quadrics. Hence $E$ lies precisely on a pencil of quadric cones with vertex $p$. It follows that

$$M(y)^t p = M'(p)^t y$$

for $y$ in some (linear) $\mathbb{P}^1$, and $M'(p)$ has rank exactly 3. \hfill \Box

Remark 17. One shows easily that

$$M'(x) = \left( \frac{\partial Q_{ij}}{\partial x_i} \right)_{0 \leq i,j \leq 4}.$$

We consider the natural desingularization

$$\tilde{V}_1 := \{ (p, \{ e_1, e_2 \}) \in \mathbb{P}^4 \times S^2 E; \ p \in L_{(e_1, e_2)} \}$$

of $V_1 = \text{Sec}(E)$, where $L_{(e_1, e_2)}$ is the secant line through $e_1, e_2$. Projection onto $S^2 E$ gives $\tilde{V}_1$ a structure of a $\mathbb{P}^1$-bundle over the surface $S^2 E$. Let $\pi_1$ denote projection to the first factor. Then $\pi_1$ contracts the divisor

$$D_1 := \{ (p, \{ e_1, e_2 \}); \ p \in \{ e_1, e_2 \} \}.$$

We have an isomorphism

$$\psi_1 : \begin{cases} D_1 \cong E \times E \\ (p, \{ e_1, e_2 \}) \mapsto (p, e_1 + e_2) \end{cases}.$$

Next we consider the natural composition

$$\tilde{V}_1 \to S^2 E \to E$$

where the map $S^2 E \to E$ maps $\{ e_1, e_2 \}$ to $e_1 + e_2$. The fibre of this map over a point $e \in E$ is the surface

$$\{ (p, \{ e_1, e_2 \}); \ e_1 + e_2 = e, \ p \in L_{(e_1, e_2)} \}.$$

This is a ruled surface over the curve

$$E/\kappa \cong \mathbb{P}^1$$

where $\kappa$ is the involution on $E$ given by $\kappa(q) = -q + e$. Via $\pi_1$ this is a smooth, rational ruled surface in $\mathbb{P}^4$, i.e., a cubic scroll. As an abstract surface this is $\mathbb{P}^2$ blown up in a point, or equivalently the Hirzebruch surface $\Sigma^1$. In this way $\tilde{V}_1$ acquires the structure of a $\Sigma^1$-fibration over $E$. We denote the fibre of this fibration over a point $e \in E$ by $\Sigma^1_e$. We shall often identify $\Sigma^1_e$ with $\pi_1(\Sigma^1_e)$. Thus we can write

$$\tilde{V}_1 = \{ (p, e); \ p \in \Sigma^1_e \subset V_1 \}.$$

We will use this notation in the sequel.

**Proposition 18.** The map $\pi_1$ defines an isomorphism from $\tilde{V}_1 \setminus D_1$ with $\text{Sec} E \setminus E$. It contracts $D_1$ to the curve $E$ and its differential has rank 2 at every point of $D_1$. 
Proof. Since every point on $\text{Sec } E \setminus E$ lies on a unique secant or tangent of $E$ the map from $\tilde{V}_1 \setminus D_1$ to $\text{Sec } E \setminus E$ is bijective. Since both are smooth, it is an isomorphism. We have already seen that $\pi_1$ contracts $D_1$ to the curve $E$. Hence the differential of $\pi_1$ along $D_1$ has rank at most 2. On the other hand consider the fibres $\Sigma^1_e$ of the map $\tilde{V}_1 \to E$. Via the map $\pi_1$ they are embedded into $\mathbb{P}^4$, and hence the differential of $\pi_1$ has rank at least 2 at every point of $\tilde{V}_1$. 

Now we return to the cubic scroll $\Sigma^1_e \subset \mathbb{P}^4$. Since $\Sigma^1_e$ is the degeneration locus of a $2 \times 3$ matrix with linear coefficients, it follows that there is a $\mathbb{P}^2$ of quadrics containing $\Sigma^1_e$. All of these quadrics are singular. Geometrically they arise as follows: Projection from $p \in \Sigma^1_e$ maps $\Sigma^1_e$ to a quadric in $\mathbb{P}^3$. Then take the cone over the quadric in $\mathbb{P}^3$. Note that the quadric surface in $\mathbb{P}^3$ is singular if and only if $p$ is on the exceptional line in $\Sigma^1_e$. In this case the corresponding quadric hypersurface is singular along the exceptional line in $\Sigma^1_e$. Finally it follows easily from $\mathcal{V}_1 = \text{Sec } E$ that every singular quadric through $E$ arises in the way described above.

We define

$$\tilde{V}_2 := \{(Q,e); \ e \in E, \ Q_e \text{ is a quadric through } \Sigma^1_e\}.$$ 

Via the obvious map $\tilde{V}_2 \to E$ this carries the structure of a $\mathbb{P}^2$-bundle. For $p \in \Sigma^1_e$ we denote by $Q_e = Q_e(p)$ the unique quadric through $\Sigma^1_e$ which is singular at $p$. This enables us to define the following maps:

$$f_1: \begin{cases} \tilde{V}_1 \to \mathcal{F} \\ (p, e) \mapsto (p, Q_e) \end{cases}$$

where $(p, e)$ stands for the point $p \in \Sigma^1_e$, and

$$f_2: \begin{cases} \tilde{V}_1 \to \tilde{V}_2 \\ (p, e) \mapsto (Q_e, e). \end{cases}$$

In this way we get a commutative diagram

$$\begin{CD} 
\tilde{V}_1 @>{f_2}>> \tilde{V}_2 \\
@V{\pi_1}VV @V{\pi_2}VV \\
V_1 @>{p_1}>> \mathcal{F} @>{p_2}>> V_2.
\end{CD} \tag{8}$$

Moreover it follows from our geometric discussion that $f_2$ contracts precisely the divisor

$$X := \{(p, e); \ p \in \text{exceptional line in } \Sigma^1_e\}.$$ 

In other words $f_2$ is the blowing down map from the $\Sigma^1$-bundle $\tilde{V}_1$ to the $\mathbb{P}^2$-bundle $\tilde{V}_2$. Furthermore $X$ is an elliptic ruled surface and $\pi_1(X)$ is the locus of singular lines.

We now return to the divisor $D_1 \cong E \times E$ in $\tilde{V}_1$. 

**Lemma 19.** (i) $f_1$ is an isomorphism outside $D_1$.
(ii) $f_1(p, e) = f_1(p', e')$ if and only if $p' = p \in E$ and $e + e' = -p$.

**Proof.** (i) This follows from proposition 16 (i).
(ii) If $f_1(p, e) = f_1(p', e')$ then clearly $p = p'$ by construction of the map $f_1$. Now $Q_e = Q_e'$ means that $Q_e$ is a singular quadric with vertex $p \in E$ containing both $\Sigma'_e$ and $\Sigma'_e$. $\Sigma'_e$ and $\Sigma'_e'$ are determined by the families of planes in $Q_e$. A plane intersects $E$ in two points besides $p$, defining a line in the ruling of the scroll. If $L_{(e_1, e_2)} \subset \Sigma'_e$ and $L_{(e'_1, e'_2)} \subset \Sigma'_e'$ then $e_1, e_2, e'_1, e'_2$ and $p$ are contained in a $\mathbb{P}^3$, hence $$e_1 + e_2 + e'_1 + e'_2 + p = 0.$$ So $$e + e' = -p.$$ The converse is analogous. □

From this lemma it follows that $f_1$ restricts to $D_1 \cong E \times E$ as the quotient map to $E \times E/\iota'$, where $\iota'$ is the involution $\iota'(p, e) = (p, -p - e)$. The curve $\Delta' := \{(2e, e); e \in E\}$ is pointwise fixed under $\iota'$, while $\iota'$ acts as the standard involution on the curve $(\Delta')^- := \{(0, e); e \in E\}$.

Consider the change of coordinates (compare the proof of lemma 9):

$$\psi_2: \begin{cases} E \times E \to E \times E \\ (p, e) \mapsto (p + e, -e). \end{cases}$$

This maps $\Delta'$ to the diagonal $\Delta = \{(e, e); e \in E\}$ and $(\Delta')^-$ to the antidiagonal $\Delta^- = \{(e, -e); e \in E\}$. Moreover $\iota'$ becomes the involution $\iota$ interchanging the two factors. From now on we shall identify $D_1$ with $E \times E$ via the isomorphism $\psi := \psi_2 \circ \psi_1$. Finally we denote by $\tilde{\Delta}$ the image of the diagonal $\Delta$ in $S^2E = E \times E/\iota$.

**Proposition 20.** The exceptional divisor $X \subset \Sigma^-_E$ intersects $D_1 = E \times E$ in the diagonal $\Delta$.

**Proof.** The involution $\iota$ (resp. $\iota'$) is induced by a switching of cubic scrolls in a singular quadric. A fixed scroll is precisely the unique scroll containing $E$ in a rank 3 quadric. □

**Corollary 21.** $\mathcal{F}$ is singular along an elliptic scroll $S^2E$. The scroll $f_1(X)$ intersects $S^2E$ along $\tilde{\Delta}$.

**Proposition 22.** The singular scroll $\pi_1(X)$ has degree 15. The curve $\Delta$ is mapped 4:1 to $E$ by $\pi_1$.

**Proof.** For the first part see [Hu, prop. IV.4.7]. The second statement follows since $\Delta' \to E$ is given by $(-2e, e) \mapsto -2e$. It also follows since the pencil of quadrics with vertex $p \in E$ contains 4 rank 3 quadrics (see the proof of proposition 16). □

We now turn our attention to the quintic hypersurface $V_2$. 
Proposition 23. Restricted to $D_1 = E \times E$ the blowing down map $f_2$ is an isomorphism.

Proof. Fix some $e \in E$. Then the exceptional line in $\Sigma^1_e$ and the curve $E$ intersect transversally. $\square$

Proposition 24. (i) The quintic hypersurface $V_2$ is ruled by an elliptic family of planes.
(ii) The map $\pi_2$ restricted to $D_1 \subset \tilde{V}_2$ maps $D_1 \cong E \times E$ surjectively 2:1 onto a quintic elliptic scroll $S^2E$. The scroll $S^2E$ parametrizes those quadrics which are singular at a point of $E$. The quintic hypersurface $V_2$ is the trisecant scroll of $S^2E$. It is singular exactly at $S^2E$ (set theoretically).
(iii) Via $\pi_2$ the diagonal $\Delta \subset D_1 \cong E \times E$ is mapped to a degree 10 curve $\tilde{\Delta}$ in $\mathbb{P}^4$. The curve $\tilde{\Delta}$ parametrizes the rank 3 quadrics through $E$.
(iv) The map $\pi_2$ gives an isomorphism of $\tilde{V}_2 \setminus D_1$ with $V_2 \setminus S^2E$.
(v) The rank of the differential of $\pi_2$ is 3 everywhere with the exception of $\Delta$ where it is 2.

Proof. (i) The fibre of $\tilde{V}_2$ over a point $e \in E$ is mapped to the net of quadrics through the scroll $\Sigma^1_e$.
(ii) We have already seen that $f_1$ restricted to $D_1 \cong E \times E$ factors through $S^2E$. Hence using diagram (8) the same must be true for $\pi_2$. The map from $S^2E$ to $\mathbb{P}^4$ given by $\pi_2$ is injective, which means that the image has degree at least 5. The ruling of $S^2E$ over a point $p \in E$ is mapped to the pencil of quadrics through $E$ which are singular at $p$. Now intersect $V_2$ with a general plane. Since $V_2$ is singular on the image of $S^2E$, this intersection is a plane curve with at least 5 singular points.
Since the map from $\tilde{V}_2 \setminus D_1$ to $V_2 \setminus S^2E$ is bijective, this curve dominates the elliptic base curve of $\tilde{V}_2$, and therefore, by the genus formula, it cannot have more than 5 singular points. Thus $\pi_2(S^2E)$ has degree 5 and, by the same argument, $V_2$ has no singularities outside $\pi_2(S^2E)$. If $C_0$ is a section of $S^2E$ with $C_0^2 = 1$ and $F$ is a fibre, then the map from $S^2E$ to $\mathbb{P}^4$ is given by the linear system $|C_0 + 2F|$. In fact, by $H_5$–invariance, the map is given by the complete linear system, in which case it is well known to be an embedding. We therefore identify $S^2E$ with its image. It remains to show that $V_2$ is the trisecant scroll of $S^2E$. Now, the curve $C_0$ moves in an elliptic family on $S^2E$, so each member is a plane cubic curve. Thus the planes of these curves are part of the trisecant scroll of $S^2E$. Since $S^2E$ is the singular part of $V_2$, each such trisecant is contained in $V_2$ by Bezout. But the planes of the trisecant scroll cannot dominate the elliptic base curve, hence these planes must coincide with the elliptic family of planes of $V_2$.
(iii) The curve $\Delta$ is the branch locus of the map $E \times E \rightarrow S^2E \subset \mathbb{P}^4$. It is well known that this is mapped to a curve $\Delta$ of degree 10 in $\mathbb{P}^4$ (in fact the class of $\Delta$ on $S^2E$ is $4C_0 - 2F$ and the assertion follows from $(4C_0 - 2F)(C_0 + 2F) = 10$). The assertion that $\tilde{\Delta}$ parametrizes the rank 3 quadrics through $E$ follows from the description of the map $p_2$ and proposition 20.
(iv) We have already seen that the map from $\tilde{V}_2 \setminus D_1$ to $V_2 \setminus S^2E$ is bijective. Since both sets are smooth, the claim follows.
(v) By (iv) the rank of $d\pi_2$ is 3 outside $D_1$. Since the fibres of $V_2$ are mapped to planes in $\mathbb{P}^4$, it follows that the rank of the differential is at least 2 everywhere. Since $\Delta$ is the branch locus of the map $E \times E \to S^2E$ the rank of $d\pi_2$ cannot be 3 along $\Delta$. It remains to prove that the rank of the differential is 3 on $D_1 \setminus \Delta$. Let $x$ be a point on $D_1 \setminus \Delta$ and let $E_x$ be the elliptic curve through $x$ which is mapped to a ruling of $S^2E$. The differential of $\pi_2$ restricted to $E_x$ is 2 at $x$. Hence it is enough to see that the ruling $L_x = \pi_2(E_x)$ and the plane $\mathbb{P}_x^2$ which is the image of the fibre of $V_2$ containing $x$ meet transversally. For this it is enough to show that $L_x$ is not contained in $\mathbb{P}_x^2$. But the intersection of $\mathbb{P}_x^2$ with $S^2E$ is a smooth plane cubic and does not contain a line.

Remark 25. A general symmetric $5 \times 5$ matrix with linear coefficients has rank 3 along a curve of degree 20.

We are now in a position to connect the geometric approach of this paragraph with the abstract approach from § 1. To do this, recall the sections $s_0, \ldots, s_4$ of $\mathcal{L}$ from proposition 6.

Proposition 26. There is an isomorphism $\tilde{V}_2 \cong \mathbb{P}_E^2$ such that the map $\pi_2$ is given by $s_0, \ldots, s_4$.

Proof. The argument has two parts. First we identify $\tilde{V}_2$ with $\mathbb{P}(N_E(-2))$, where $N_E(-2)$ is the twisted normal bundle of the elliptic curve $E \subset \mathbb{P}^4$. Afterwards we show that $\mathbb{P}(N_E(-2)) \cong \mathbb{P}_E^2$ and in fact also the existence of an $H_5$-isomorphism between $\mathcal{O}_{\mathbb{P}(N_E(-2))}(1)$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$.

We use the basis of $H^\circ(I_E(2))$ given by

$$Q_i = x_i^2 + ax_{i+2}x_{i+3} - \frac{1}{a}x_{i+1}x_{i+4} \quad (i \in \mathbb{Z}_5).$$

The natural map

$$H^\circ(I_E(2)) \otimes \mathcal{O}_E \xrightarrow{\cdot a} N^*_E(2)$$

is surjective and there is an exact sequence

$$0 \to K \xrightarrow{\beta} H^\circ(I_E(2)) \otimes \mathcal{O}_E(-1) \xrightarrow{A} H^\circ(I_E(2)) \otimes \mathcal{O}_E \xrightarrow{\cdot a} N^*_E(2) \to 0$$

with

$$A = \begin{pmatrix}
0 & ax_4 & -x_3 & x_2 & -ax_1 \\
-ax_4 & 0 & ax_2 & -x_1 & x_0 \\
x_3 & -ax_2 & 0 & ax_0 & -x_4 \\
-x_2 & x_1 & -ax_0 & 0 & ax_3 \\
a x_1 & -x_0 & x_4 & -ax_3 & 0
\end{pmatrix}$$

(see [Hu, p. 68]). Dualising this sequence we get

$$H^\circ(I_E(2))^\vee \otimes \mathcal{O}_E \xrightarrow{(A)^{-1} = A(1)} H^\circ(I_E(2))^\vee \otimes \mathcal{O}_E(1) \xrightarrow{\beta} K^* \xrightarrow{0} 0.$$

Hence

$$K^* \cong N^*_E(3),$$
i.e.,

\[ K \cong N_E(-3). \]

We want now to describe the map

\[
\mathbb{P}(N_E(-2)) \hookrightarrow \mathbb{P}(H^\circ(I_E(2))) \times E \\
\downarrow \\
\mathbb{P}(H^\circ(I_E(2)))
\]

where the horizontal map is given by the inclusion

\[ N_E(-2) \overset{\beta(1)}{\hookrightarrow} H^\circ(I_E(2)) \otimes O_E. \]

We first want to identify the subbundle

\[ P(N_E(-2)) = \{(p, Q) ; p \in E, Q \in \operatorname{Im} \beta(1)|_p \} \subset P(H^\circ(I_E(2)) \times E. \]

**Claim 1.** \( Q \in \operatorname{Im} \beta(1)|_p \) if and only if \( Q \) contains the unique cubic scroll containing the secant \( L_{(o,-p)} \).

**Proof of the claim.** Consider the matrix

\[
M' = \left( \frac{\partial Q_{3k}}{\partial x_i} \right)_{i,j} \quad (i, j \in \mathbb{Z}_5)
\]

from remark 17. By proposition 16 (ii) this has rank 3 on \( E \). One easily checks that the entries of \( A^tM' \) are all elements of \( H^\circ(I_E(2)) \). Since \( A \) has rank 2 on \( E \) the sequence

\[
H^\circ(I_E(2)) \otimes O(-1) \overset{tM'}{\rightarrow} H^\circ(I_E(2)) \otimes O_E \overset{A}{\rightarrow} H^\circ(I_E(2)) \otimes O_E(1)
\]

is exact. Therefore

\[ \operatorname{Im} \beta(1) = \operatorname{Im} tM'. \]

Since there is a net of quadrics through a cubic scroll it suffices to show that any quadric in the image of \( tM'(p) \) contains the scroll. For this it suffices to show that the secant lines \( L_{(o,-p)} \) and \( L_{(\eta_5,-\eta_5,-p)} \), where \( \eta_5 \) is a non-zero 5-torsion point, are contained in \( Q \) (recall that the cubic scroll in question is the union of all secants \( L_{(q,r)} \) with \( q + r = -p \)). If \( Q \) contains two secants in the scroll it must contain the scroll by Bezout.

Now \( \operatorname{Im} tM'(p) \) is spanned by the elements

\[
M'_i(p) \begin{pmatrix} Q_0 \\ \vdots \\ Q_4 \end{pmatrix}
\]

where \( M'_i(p) \) is the \( i \)-th row of \( M' \) evaluated at \( p \). The origin has coordinates \((0,a,-1,1,-a)\) and we can take \( \eta_5 \) to be \((a,-1,1,-a,0)\). If \( p \) has coordinates \((x_0, \ldots, x_4)\) then \(-p\) has coordinates \((x_0, x_4, x_3, x_2, x_1)\) and \(-p - \eta_5\) has coordinates \((x_0, x_4, x_3, x_2, x_1)\).
(x_4, x_3, x_2, x_1, x_0). Evaluating the quadrics in Im'M'(p) on the secant lines one gets quadrics in the coordinates x_i which vanish on E. This proves the claim.

This shows that \( \tilde{V}_2 = \mathbb{P}(N_E(-2)) \) and that the map to \( \mathbb{P}^4 \) is given by \( \mathcal{O}_{\mathbb{P}(N_E(-2))}(1) \).

By [Hu, proposition V.1.2] the twisted normal bundle \( N_E(-2) \) is indecomposable with \( c_1(N_E(-2)) = \mathcal{O}_E(-1) \). By Atiyah’s classification [At] \( N_E(-2) \cong \mathcal{E}_E \) and in particular \( \mathbb{P}(N_E(-2)) \cong \mathbb{P}_E^2 \). Both bundles \( N_E(-2) \) and \( \mathcal{E}_E \) come with an \( H_5 \)-action which covers the same action on \( E \). Since \( N_E(-2) \) resp. \( \mathcal{E}_E \) are stable, and hence simple, the two \( H_5 \)-actions on \( N_E(-2) \) and \( \mathcal{E}_E \) differ at most by a character. But since the induced actions on the respective determinants coincide, this character must be trivial. By construction \( L = \mathcal{O}_{\mathbb{P}(\mathcal{E}_E)}(1) \) (As a check note that the representation of \( H_5 \) on both \( H^0(L) \) (see proposition 7) and on \( H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E}_E(2))}(1)) = H^0(I_E(2)) \) are in each case derived from the Schrödinger representation by replacing \( \varepsilon \) by \( \varepsilon^2 \)).

In any case the above argument shows that we have an \( H_5 \)-isomorphism between \( \mathcal{O}_{\mathbb{P}(N_E(-2))}(1) \) and \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) and we are done. \( \square \)

**Remark 27.** We have seen in proposition 24 (ii) that \( A_0 = E \times E \) is mapped 2:1 by \( \pi_2 \) onto an elliptic quintic scroll. Since the abelian surfaces \( A_K \) and the bielliptic surfaces \( B_{(a,b)} \) are numerically equivalent to \( A_0 \) on \( V_2 \) these surfaces must be mapped to surfaces in \( \mathbb{P}^4 \) of degree 10.

**Proposition 28.** There is an isomorphism \( \tilde{V}_1 \cong \mathbb{P}_{32E}^1 \) such that \( f_2 \) is identified with \( \rho \).

**Proof.** \( f_2: \tilde{V}_1 \to \tilde{V}_2 \) is the blow up of \( \tilde{V}_2 \) in the diagonal of \( D_1 \) after we have identified \( D_1 \) with \( E \times E \) via the isomorphism \( \psi \). Recall also that the map \( \pi_2 \) is bijective outside \( D_1 \) and that \( \pi_2 \) restricted to \( E \times E \) is a 2:1 branched covering onto its image whose branch locus is the diagonal of \( E \times E \). The map \( \rho: \mathbb{P}_{32E}^1 \to \mathbb{P}_E^2 \) is the blow up of \( \mathbb{P}_E^2 \) along the diagonal of \( A_0 = E \times E \). In view of our identification of \( \pi_2 \) with the map given by \( s_0, \ldots, s_4 \), it is enough to prove the following: The map \( (s_0: \ldots: s_4) \) restricted to \( A_0 = E \times E \) is a 2:1 branched covering with branch locus the diagonal. But this is easy to see. Recall that the curve \( \{(q, q) \in E \times E \} \subseteq E \times \mathbb{P}^2 \) is mapped 9:1 to the antidiagonal \( \{(q, -q); q \in E \} \) in \( A_0 \). By construction of \( L \) this shows that the degree of \( L \) restricted to the antidiagonal, and hence all its translates, is 2. Moreover the degree of \( L \) restricted to \( A_0 \) is 10. It is well known that then \( A_0 \) is mapped 2:1 onto a quintic elliptic scroll with branch locus the diagonal (e.g. see [HL]). \( \square \)

We are now ready to prove that the maps \( \pi_1 \) and \( \pi_2 \) give rise to abelian and bielliptic surfaces of degree 15, resp. 10. Before we do this, we recall from lemma 5 that

\[
A_K \cap A_0 = \{(q, r) \in E \times E; 3r + 2q = 0\}
\]

resp.

\[
B_{(a,b)} \cap A_0 = \{(q, r) \in E \times E; 3r + 2q = -\tau(a,b)\}.
\]

We set

\[
E_K = A_K \cap A_0, \quad E_{(a,b)} = B_{(a,b)} \cap A_0.
\]
Moreover, we consider the following curves on $E \times E$:

$$\Delta_p^- = \{(e, -e + p), \ e \in E\}.$$ 

Under the quotient map $E \times E \to S^2 E$ these curves are mapped to the rulings of the $\mathbb{P}^1$-bundle $S^2 E$.

**Lemma 29.** The curves $E_K$, resp. $E_{(a,b)}$, intersect the curves $\Delta_p^-$ transversally in one point.

**Proof.** A point $(e, -e + p)$ lies on $E_K$ if and only if $-e + 3p = 0$, i.e., $e = 3p$. Both curves are elliptic curves. Two curves on an abelian surface which do not coincide, meet transversally. The claim for the curve $E_{(a,b)}$ is proved in exactly the same way. \hfill \Box 

**Theorem 30.** (i) Let $A_K$ be a smooth element different from $A_0$ in the pencil $|−K|$ on $\mathbb{P}^2_E \cong \widetilde{V}_2$. Then $\pi_2$ embeds $A_K$ as a smooth abelian surface of degree 10.

(ii) The bielliptic surfaces $B_{(a,b)}$ are also embedded as surfaces of degree 10 by the map $\pi_2$.

**Proof.** (i) By proposition 24 the map $\pi_2$ is an isomorphism outside $D_1 \cong A_0$. Hence it is sufficient to consider the intersection $E_K = A_K \cap A_0$. We first claim that $\pi_2$ restricted to $E_K$ is injective. The map $\pi_2$ identifies points $(q, r)$ and $(r, q)$. Assume that two such points lie on $E_K$. This implies that

$$3r + 2q = 0, \quad 3q + 2r = 0.$$ 

Subtracting these two equations from each other gives $q = r$, and hence $(q, r) = (r, q)$. Finally we have to check that the differential of $\pi_2$ restricted to $A_K$ is injective at the 25 points $E_K \cap \Delta$. For this recall that the kernel of $d\pi_2$ along $\Delta$ is given by the directions defined by the curves $\Delta_p^-$. The claim follows, therefore, from lemma 29. The degree of the embedded surfaces is 10 by remark 27.

(ii) The same proof goes through for the surfaces $B_{(a,b)}$. \hfill \Box 

**Remarks 31.** (i) The pencil $|−K|$ contains 4 singular elements corresponding to the 4 triangles in the Hesse pencil. These surfaces are mapped to translation scrolls of quintic elliptic curves where the translation parameter is a non-zero 3-torsion point.

(ii) All abelian surfaces $A_K$ are isogeneous to a product. Hence this construction does not give the general abelian surface in $\mathbb{P}^4$. On the other hand we get all minimal bielliptic abelian surfaces in $\mathbb{P}^4$ in this way (up to a change of coordinates).

(iii) The abelian surfaces in $\mathbb{P}^4$ which are of the form $E \times F / \mathbb{Z}_3 \times \mathbb{Z}_3$ were studied by Barth and Moore in [BM]. By their work the pencils which we have constructed above, are tangents to the rational sextic curve $C_6$ in the space of Horrocks-Mumford surfaces which parametrizes Horrocks-Mumford surfaces which are double structures on elliptic quintic scrolls.

(iv) The involution $\iota$ of proposition 8 induces the involution $x \mapsto −x$ on the surfaces $A_K$. The surfaces $B_{(a,b)}$ are identified pairwise, more precisely $\iota B_{(a,b)} = B_{(−a,−b)}$. This follows since the 8 characters $F_{(a,b)}$ are identified in this way by the Heisenberg involution on $\mathbb{P}^2$. 
Theorem 32. (i) Let $A_K$ be a smooth element different from $A_0$ in the pencil $|-K|$ on $\mathbb{P}_E^1 \cong \tilde{V}_2$. Then the map $\pi_1$ embeds $A_K$ as a smooth non-minimal abelian surface of degree 15 in $\mathbb{P}^4$.

(ii) The surfaces $\tilde{B}_{(a,b)}$ are embedded by $\pi_1$ as smooth bielliptic surfaces of degree 15.

Proof. (i) Again it is enough to look at the intersection of $\tilde{A}_K$ with $A_0$ on $\mathbb{P}_E^1 \cong \tilde{V}_1$. The curves $\Delta_p$ are contracted by $\pi_1$. These are the only tangent directions which are in the kernel of the differential of $\pi_1$. Hence our claim follows again from lemma 29. The double point formula reads
\[ d^2 = 10d + 5HK + K^2 - e. \]
In our case $HK = 25$, $K^2 = -25$ and $e = 25$. This leads to
\[ d(d - 10) = 75 \]
and the only positive solution is $d = 15$.

(ii) The claim about the bielliptic surfaces can be proved in exactly the same way. □

Remark 33. The degree of the surfaces $\tilde{A}_K$, resp. $\tilde{B}_{(a,b)}$, can also be computed by studying the linear system which maps $\tilde{V}_1$ to $\mathbb{P}^4$. We shall come back to this.

We shall now turn our attention to the quintic hypersurfaces which contain the surfaces $A_K$, $B_{(a,b)}$, $\tilde{A}_K$ and $\tilde{B}_{(a,b)}$.

Proposition 34. The bielliptic surfaces $B_{(a,b)}$ lie on a unique quintic hypersurface, namely $V_2$.

Proof. We consider the elliptic quintic scroll $S^2E \subset V_2 \subset \mathbb{P}^4$. Recall that $S^2E$ is the quotient of $E \times E$ by the involution which interchanges the two factors. Let $C_{p_0}$ be the section of $S^2E$ which is the image of $\{p_0\} \times E$, resp. $E \times \{p_0\}$ in $S^2E$ where $p_0$ is the origin of $E$ which we have chosen before. Note that the normal bundle of $C_{p_0}$ in $S^2E$ is the degree 1 line bundle which is given by the origin. Let $F_{p_0}$ be the fibre over the origin of the map $S^2E \rightarrow E$, $\{q_1, q_2\} \mapsto q_1 + q_2$. Moreover let $H$ be the hyperplane section of $S^2E \subset \mathbb{P}^4$. It follows immediately from our choice of the line bundle $L$ in § 1 that
\[ H \sim C_{p_0} + 2F_{p_0}. \]
Next we consider the intersection $C_{(a,b)} = B_{(a,b)} \cap S^2E$. The curve $C_{(a,b)}$ is by lemma 29 a section of $S^2E$. Now $E \times E \rightarrow S^2E$ is ramified along the diagonal and maps the curve $E_{(a,b)}$ isomorphically to $C_{(a,b)}$ so, combining with lemma 5(ii), the intersection of $C_{(a,b)}$ with the diagonal is twice the set $\{(p, p); 5p = -\tau_{(a,b)}\}$. Thus $C_{(a,b)} \cong C_{p_0} + 12F$. In fact the intersection with the diagonal goes by the map $S^2E \rightarrow E$ to $\{4p; 5p = -\tau_{(a,b)}\}$, which summed up is $\tau_{(a,b)}$. Therefore
\[ C_{(a,b)} \sim C_{p_0} + 11F_{p_0} + F_{(a,b)} \]
where $F_{(a,b)}$ is the fibre over the 3-torsion point $\tau_{(a,b)}$. Finally recall that

$$K \sim -2C_{p_0} + F_{p_0}. \tag{11}$$

Let $Q$ be a quintic containing $B_{(a,b)}$. We first claim that $Q$ must contain $S^2E$. In order to see this look at the exact sequence

$$0 \to \mathcal{O}_{S^2E}(5H - C_{(a,b)}) \to \mathcal{O}_{S^2E}(5H) \to \mathcal{O}_{C_{(a,b)}}(5H) \to 0.$$ 

It follows from formulas (9), (10), and (11) that

$$5H - C_{(a,b)} \sim 4C_{p_0} - F_{p_0} - F_{(a,b)} \sim -2K + (F_{p_0} - F_{(a,b)}). \tag{12}$$

It is well known that $h^0(\mathcal{O}_{S^2E}(-2K + F_p - F_q)) = 0$ unless $2p = 2q$ (cf. [CC]). Hence $Q$ must contain $S^2E$. We next claim that $Q = V_2$. In order to see this, consider a plane on $V_2$, i.e., a trisecant plane of $S^2E$. Both $S^2E$ and $B_{(a,b)}$ intersect such a plane in different irreducible cubic curves. Since $Q$ has degree 5 it must contain this plane and hence $V_2$. By reasons of degree this implies $Q = V_2$. $\square$

**Remark 35.** The surfaces $A_K$ lie on 3 independent quintics. This follows e.g. from the fact that $A_K$ is the zero-scheme of a section $s$ of the Horrocks-Mumford bundle $F$ (where we normalize $F$ such that $c_1(F) = 5$, $c_2(F) = 10$). In other words there is an exact sequence

$$0 \to \mathcal{O}_{F_s} \to F \to I_{A_K}(5) \to 0.$$ 

The claim follows from $h^s(F) = 4$ (see [HM]).

Note that if we replace $B_{(a,b)}$ by a surface $A_K$ in the proof of proposition 34 we obtain $-2K$ in formula (11). Since $h^s(\mathcal{O}_{S^2E}(-2K)) = 2$, this gives rise to two more elements in $H^0(\mathcal{O}_{S^2E}(-2K))$ which vanish along $C_K = A_K \cap S^2E$. These can be lifted to $H^s_5$-invariant quintics in $\mathbb{P}^4$. It is then easy to check that any $H^s_5$-invariant quintic which contains $C_K$ must contain $A_K$. (Look at the intersection of the quintic with the cubic curves on $A_K$. Unless the quintic contains these curves, this would split up into two $H^s_5$-orbits of length 9 and 6 resp., a contradiction.) In this way one can also prove the existence of 3 independent quintics through the surfaces $A_K$.

We now turn our attention to the degree 15 surfaces.

**Proposition 36.** (i) The non-minimal abelian surfaces $\tilde{A}_K$ lie on exactly three quintic hypersurfaces. They are linked (5,5) to translation scrolls.

(ii) The non-minimal bielliptic surfaces $\tilde{B}_{(a,b)}$ lie on a unique quintic hypersurface, namely $V_1$.

**Proof.** (i) Let $\tilde{H}_1$ be the hyperplane section on $\mathbb{P}^1_{S^2E}$ given by the map $\pi_1: \mathbb{P}^1_{S^2E} \to V_1 \subset \mathbb{P}^4$. Let $C_{p_0}$, resp. $F_{p_0}$ be the fibres over the curves $C_{p_0}$, resp. $F_{p_0}$ in $S^2E$ with respect to the map $\mathbb{P}^1_{S^2E} \to S^2E$. The classes $H_1$, $C_{p_0}$ and $F_{p_0}$ generate the Neron-Severi group of $\mathbb{P}^1_{S^2E}$. Under $\pi_1$ the surface $F_{p_0}$ is mapped isomorphically to a cubic scroll, while $C_{p_0}$ is mapped birationally to the cone over an elliptic curve of degree 4 in $\mathbb{P}^4$ with vertex in the origin $p_0$ of the elliptic curve in $\mathbb{P}^4$ for which $V_1$ is the secant variety. Thus we get the following intersection numbers: $H_1^3 = 5$, $H_1^2C_{p_0} = 4$, $H_1^2F_{p_0} = 3$, $H_1C_{p_0}F_{p_0} = H_1C_{p_0}^2 = 1$ and $C_{p_0}^2F_{p_0} = C_{p_0}F_{p_0}^2 = H_1F_{p_0}^2 = F_{p_0}^3 = C_{p_0}^3 = 0.$
The exceptional divisor $X = E_\Delta$ is a section of the $\mathbb{P}^1$-bundle $\mathbb{P}^1_{S^2E}$. Under $\pi_1$ it is mapped birationally onto a ruled surface of degree 15. From this information it is easy to compute that numerically $X \equiv H_1 - 2C_{p_0} + 6F_{p_0}$. It is also straightforward to check that for the canonical divisor $K \equiv -2H_1 + C_{p_0} + 2F_{p_0}$. In fact we claim that these equalities are also true with respect to linear equivalence. It is enough to check this on a section of the composite projection $\mathbb{P}^1_{S^2E} \to S^2E \to E$. We consider the curve $D = X \cap D_1 = X \cap (E \times E)$. On $E \times E$ this is the diagonal by proposition 20. Using the projection $\mathbb{P}^1_{S^2E} \to S^2E$ we can identify the section $X$ with $S^2E$. Then by [Hu, lemma IV.4.4] we have $D \sim C_{p_0} + 12F_{p_0}$ on $X$. Since $X$ is mapped to a ruled surface of degree 15 we have $\hat{H}_1|_X \equiv C_{p_0} + 7F_{p_0}$. Since $D$ is mapped by multiplication with $-2$ four to one onto the elliptic curve $E$ and since $E$ is embedded by $|5p_0|$ we have in fact that this equality also holds with respect to linear equivalence. We also note that $X$ restricted to $D$ is trivial. This follows since $X$ restricted to $A_0$ is $D$ and since the normal bundle of $D$ in $A_0$ is trivial. Hence in order to check that $X \sim H_1 - 2C_{p_0} + 6F_{p_0}$ it is enough to prove that the restriction of $\hat{H}_1 - 2C_{p_0} + 6F_{p_0}$ to $D$ is trivial. This follows from

$$(-C_{p_0} + 13F_{p_0})(C_{p_0} + 12F_{p_0}) \sim 0$$

which has to be read as an equality of divisors on $D \cong E$. Next we want to prove that

$$K \sim -2\hat{H}_1 + C_{p_0} + 2F_{p_0}.$$  

We know that $(K + X)|_D \sim K_X|_D \sim -25p_0$. The first is the adjunction formula, the second follows from

$$(-2C_{p_0} + F_{p_0})(C_{p_0} + 12F_{p_0}) \sim -25p_0.$$  

Since $X|_D \sim 0$ it is now enough to show that $-2\hat{H}_1 + C_{p_0} + 2F_{p_0}$ restricted to $D$ is linearly equivalent to $-25p_0$. This follows from

$$(-C_{p_0} - 12F_{p_0})(C_{p_0} + 12F_{p_0}) \sim -25p_0.$$  

Hence we have proved that

$$\hat{X} \sim \hat{H}_1 - 2C_{p_0} + 6F_{p_0}, \quad K \sim -2\hat{H}_1 + C_{p_0} + 2F_{p_0}.$$  

Since $A_K$ is anticanonical on $\mathbb{P}^2_E$ we get that $\hat{A}_K \sim -K + X \sim 3\hat{H}_1 - 3C_{p_0} + 4F_{p_0}$. Furthermore $-K \sim D_1$, and $D_1$ is contracted under $\pi_1$ to the curve $E$. Since twice the anticanonical divisor on $S^2E$ moves in a pencil it follows that also $4C_{p_0} - 2F_{p_0}$ moves in a pencil on $\mathbb{P}^1_{S^2E}$. Note that

$$5\hat{H}_1 \sim -K + A_K + (4C_{p_0} + 2F_{p_0}).$$  

The members of the pencil $(4C_{p_0} - 2F_{p_0})$ on $\mathbb{P}^1_{S^2E}$ are mapped to the translation scrolls of $E$. Take such a translation scroll (which is not a quintic elliptic scroll). Then it is a Horrocks-Mumford surface (cf. [Hu2], [BHM]) and hence lies on three quintics of which $V_1$ is one. We can choose a pencil of quintics through such a scroll which does not contain $V_1$. All these quintics contain $E$. They cut out a pencil of residual surfaces and it follows from (14) that this is just the pencil formed by the
surfaces $\tilde{A}_K$. Hence every such surface is linked $(5,5)$ to a translation scroll $S$. Now consider the well known liaison sequence \((\text{cf. } [PS])\):

\[ 0 \to I_{\mathcal{N}_s \tilde{A}_K(5)} \to I_{\tilde{A}_K(5)} \to \omega_S \to 0. \]

We have $h^0(I_{\mathcal{N}_s \tilde{A}_K(5)}) = 2$, $h^1(I_{\mathcal{N}_s \tilde{A}_K(5)}) = 0$. Since $S$ is a a Horrocks-Mumford surface $\omega_S = \mathcal{O}_S$ and hence $h^0(\omega_S) = 1$. It follows that $h^0(I_{\tilde{A}_K(5)}) = 3$.

(ii) Now consider a bielliptic surface $\tilde{B}_{(a,b)}$. If $\tilde{B}_{(a,b)}$ lies on two quintics, it would be linked to a surface $T$ in the numerical equivalence class of $4\bar{C}_{p_0} - 2\bar{F}_{p_0}$. Since $3\tilde{B}_{(a,b)}$ is linearly equivalent to $3\tilde{A}_K$, we must have that $3T$ is linearly equivalent to $3(4\bar{C}_{p_0} - 2\bar{F}_{p_0})$ while $T$ is not linearly equivalent to $4\bar{C}_{p_0} - 2\bar{F}_{p_0}$. But in this numerical equivalence class the only effective divisors are the pencil $(4\bar{C}_{p_0} - 2\bar{F}_{p_0})$ and the three divisors $4\bar{C}_p - \bar{F}_{p_0} - \bar{F}_r$ where $\tau$ is a non-trivial 2-torsion point.

\textbf{Remark 37.} At this point we would like to say a few more words about liaison. As said before, the space $\Gamma(F)$ of sections of the Horrocks-Mumford bundle has dimension 4. The three-dimensional space $\mathbb{P}T = \mathbb{P}(\Gamma(F))$ parametrizes the Horrocks-Mumford (HM) surfaces $X_s = \{s = 0\}$ where $0 \neq s \in \Gamma(F)$. The space of Heisenberg invariant quintics is related to $\Gamma(F)$ via the isomorphism $\Lambda^2\Gamma(F) \cong \Gamma_{H_5}(\mathcal{O}_{p^+}(5))$ given by the natural map $\Lambda^2\Gamma(F) \to \Gamma(\Lambda^2F) = \Gamma(\mathcal{O}_{p^+}(5))$. Set

\[ \mathbb{P}^2_{H_5} = \mathbb{P}(\Gamma_{H_5}(\mathcal{O}_{p^+}(5))) \cong \mathbb{P}(\Lambda^2\Gamma(F)). \]

In $\mathbb{P}^2_{H_5}$ we consider the Plücker quadric $G = G(1,3)$ of decomposable tensors. If $X_{s_0} = \{s_0 = 0\}$ is a HM-surface, then

\[ \Gamma(I_{X_{s_0}}(5)) = \{s_0 \land s; s \in \Gamma(F)\}. \]

This defines a $\mathbb{P}^2$ of decomposable tensors. In $G = G(1,3)$ this is an $\alpha$-plane, i.e., a plane of lines through one point. In this way we get a bijection between $\mathbb{P}T$ and the set of all $\alpha$-planes in $G$. Now consider a line in an $\alpha$-plane, i.e., a pencil of quintics spanned by quintics of the form $s_0 \land s_1$ and $s_0 \land s_2$. This give rise to a complete intersection

\[ Y_1 \cap Y_2 = X_{s_0} \cup X' \]

where $X'$ is of degree 15. By the liaison sequence which we have already used before, we find that $h^0(I_{X_{s_0}}(5)) = 3$. The space of quintics is spanned by $s_0 \land s_1$, $s_0 \land s_2$ and $s_1 \land s_2$. To see that $s_1 \land s_2$ is contained in this space consider $X' \setminus X_{s_0}$. At these points $s_0$ does not vanish and $s_1$ and $s_2$ are linearly dependent of $s_0$. It follows that $s_1 \land s_2 = 0$. Hence all quintics through $X'$ are in particular $H_5$-invariant and the space of these quintics is a $\beta$-plane in $G$, i.e., a $\mathbb{P}^2$ of lines which lie in a fixed plane in $\mathbb{P}^3$.

Let $\tilde{A}_K$ be one of our non-minimal degree 15 abelian surfaces. By proposition 36 the surface $\tilde{A}_K$ is linked to a translation scroll. Hence it defines a $\beta$-plane in $G$. For every line in this plane there exists exactly one $\alpha$-plane intersecting the $\beta$-plane in this line. Hence every such line gives rise to liaison with an HM-surface. In this
way $\tilde{A}_K$ is linked to a 2-dimensional family of HM-surfaces. This 2-dimensional family is parametrized by a linear $\mathbb{P}^2$ in $\mathbb{P}^4$. Since the singular HM-surfaces form an irreducible surface of degree 10 [BM] in $\mathbb{P}^4$, it follows in particular that $\tilde{A}_K$ is linked to smooth abelian surfaces.

We want to conclude this paragraph with a short discussion of the 6-secants of the surfaces $A_K$ and $B_{(a,b)}$. The 6-secants of the surfaces $A_K$ are exactly the 25 Horrocks-Mumford lines [HM].

The 6-secant formula from [L] shows that the surfaces $B_{(a,b)}$ either also have 25 6-secants or infinitely many. In fact there are exactly 25 6-secants and we shall now describe them. First note that every 6-secant of $B_{(a,b)}$ must lie in one of the planes of $V_2$ (it must be contained in $V_2$ by reasons of degree and since the base of the bundle $\mathbb{P}^2_\mathbb{P}^4$ is elliptic, it must be in one of the fibres). Now fix a point $e \in E$ and let $f = f_2|\Sigma^1_e$ be the blowing down map $\Sigma^1_e \to \mathbb{P}^2_e$. If we interpret $\Sigma^1_e$ as a cubic scroll in $\mathbb{P}^4$, then $\Sigma^1_e$ consists of the secants of the elliptic quintic curve $E$ joining points $p$ and $q$ with $p + q = e$. We denote this curve by $E_e \subset \Sigma^1_e$. The map $f: \Sigma^1_e \to \mathbb{P}^2_e \subset \mathbb{P}^4$ is given by the linear system $|X_e + l|$ where $l$ are the fibres of the $\mathbb{P}^1$-bundle $\Sigma^1_e$ and $X_e$ is the exceptional line. By what we have said before, we have $l \cap E_e = \{p, q\}$ with $p + q = e$. One also computes easily that $X_e \cap E_e = \{-2e\}$ where we consider $-2e$ as a point on $E_e$. It follows that the map $f$ is given by the linear system $(-2e) + e + p_0 \sim 5p_0 - 2e_0$ on $E_e$ where $2e_0 = e$. The composite $\pi_2 \circ f_2$ maps both curves $E_e$ and $\iota' E_e \subset E \times E$ onto the same plane cubic curve in $\mathbb{P}^4$. When considering the change of coordinates $\psi_2$ followed by the involution interchanging the factors on $E \times E$, $E_e$ is mapped to the first factor, while $\iota' E_e$ is mapped to the second factor. Therefore, by lemma 5 (ii), the intersection $B_{(a,b)} \cap \iota'(E_e)$ consists of the points $\{(p, -p - e); 2p = e - \tau_{(a,b)}\}$. These have the same image on the scroll as the points $p$ on $X_e$ with $2p = e - \tau_{(a,b)}$. These are the points $p_i = e_0 + \tau_{(a,b)} + \tau_i$, where the $\tau_i$, $i = 0, 1, 2, 3$, are the 2-torsion points with $\tau_0 = p_0$. The points $\tau_i$, $i = 1, 2, 3$, are collinear if and only if $\sum_{i=1}^3 p_i + 2e_0 = p_0$ which is equivalent to $5e_0 = p_0$, i.e., $e_0$ is a 5-torsion point on $E$. In this case we also get $e_0 = -4e_0 = -2e \in X_e \cap E_e$. Hence for $e$ a 5-torsion point on $E$ the line through these 3 points intersects $B_{(a,b)}$ in addition in 3 points on its plane cubic in $\mathbb{P}^2_e$, hence is a 6-secant. The above discussion also shows that there are only finitely many 6-secants, and hence we have found them all.

§ 3. Cremona transformations

In this paragraph we want to explain how the abelian, resp. bielliptic surfaces of degree 15 can be constructed via Cremona transformations.

It is known that the quadrics through an elliptic quintic curve $E \subset \mathbb{P}^4$ define a Cremona transformation $\Phi: \mathbb{P}^4 \dasharrow \mathbb{P}^4$ [Sem]. Via $\Phi$ the secant variety of $E$ is mapped to a quintic elliptic scroll $S^2 E$ in $\mathbb{P}^4$. The exceptional locus of $\Phi$ is mapped to the trisecant variety of this scroll. The cubics through the quintic elliptic scroll $S^2 E$ define the inverse of $\Phi$. Under $\Phi^{-1}$ the trisecant variety of $S^2 E$ is mapped to
E, while the exceptional divisor is mapped to the secant variety of E. We refer to [Sem] for details on the geometry of this transformation.

We consider the scroll $S^2E$ whose trisecant variety equals $V_2$. On $S^2E$ there exist three elliptic 2-sections $E_i$, $i = 1, 2, 3$, such that $S^2E$ is the translation scroll of $E_i$ defined by a non-zero 2-torsion point $p_i$. Let $\tilde{V}_2 \cong \mathbb{P}^2_E$ be the desingularization of $V_2$. Then the curves $\Delta_i = \{(e + p_i, e); e \in E\} \subset A_0 \cong E \times E$ are mapped 2:1 to the elliptic curves $E_i$. Let $\Delta_E \subset A_0 \cong E \times E$ be the diagonal. Let $\bar{H}_2$ be the line bundle on $\tilde{V}_2$ given by the map $\pi_2: \tilde{V}_2 \to V_2 \subset \mathbb{P}^4$.

Next consider the isomorphism

$$\phi_i: E \times \mathbb{P}^2 \to E \times \mathbb{P}^2 \quad (e, x) \mapsto (e + p_i, x).$$

This commutes with the diagonal action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on $E \times \mathbb{P}^2$ and hence induces an isomorphism

$$\bar{\phi}_i: \bar{V}_2 \to \bar{V}_2$$

which maps $\Delta_E$ to $\Delta_i$. Note also that $\bar{\phi}_i$ maps the surfaces $A_K$, resp. $B_{(a, b)}$ to themselves. Recall that $\bar{V}_1$ is the blow-up of $\bar{V}_2$ along $\Delta_E$. Let $\bar{V}_1(i)$ be the blow-up of $\bar{V}_2$ along $\Delta_i$. Then $\phi_i$ induces an isomorphism

$$\bar{\phi}_i: \bar{V}_1 \to \bar{V}_1(i)$$

such that the diagram

$$\begin{array}{ccc}
\bar{V}_1 & \xrightarrow{\bar{\phi}_i} & \bar{V}_1(i) \\
\downarrow{\rho} & & \downarrow{\rho(i)} \\
\bar{V}_2 & \xrightarrow{\bar{\phi}_i} & \bar{V}_2
\end{array}$$

commutes where the vertical maps are the blowing down maps. $X$ is the exceptional locus of $\rho$. Let $X_i$ be the exceptional loci of the maps $\rho^{(i)}$. By abuse of notation we denote the pullback of $\bar{H}_2$ by $\rho$ also by $\bar{H}_2$. We denote the pullback of $\bar{H}_2$ by $\rho^{(i)}$ by $\bar{H}_2^{(i)}$. The Cremona transformation defined by the quadrics through $E_i$ gives rise to the linear system $|2\bar{H}_2^{(i)} - X_i|$ on $\bar{V}_1(i)$. Note that

$$(\bar{\phi}_i)^*(2\bar{H}_2^{(i)} - X_i) \sim \rho^*(\bar{\phi}_i)^*(2\bar{H}_2) - X \sim 2\bar{H}_2 - X.$$  

The latter follows since $\bar{H}_2$ restricted to $\Delta_E$ has degree 10 and this implies that translation by a 2-torsion point leaves the linear equivalence class invariant.

**Proposition 38.** $\bar{H}_1 \sim 2\bar{H}_2 - X$.

**Proof.** We first claim that $\bar{H}_1 \equiv 2\bar{H}_2 - X$. The Néron-Severi group of $\bar{V}_1$ is generated by $\bar{H}_2$, $X$ and $\Sigma^1$ where $\Sigma^1$ denotes the class of a fibre of $\bar{V}_1 \to E$. Restriction to such a fibre implies immediately that $\bar{H}_1 \equiv \alpha \Sigma^1 + 2\bar{H}_2 - X$. To compute $\alpha$ we use $\bar{H}_2^2 = 5$. Since $(\Sigma^1)^2 = 0$ this implies

$$(15) \quad 5 = 9\alpha + (2\bar{H}_2 - X)^3.$$
Now $\bar{H}_2 \cdot X = 0$, and $\bar{H}_2 \cdot X^2 = -10$, since $X$ is blown down to the diagonal $\Delta_E \subset A_0$ and $\Delta_E \cdot \bar{H}_2 = 10$. On the other hand $X^3 = -25$ from our computations in the proof of proposition 36, thus $\alpha = 0$ as claimed. In order to prove the proposition it is now enough to consider the restriction of $\bar{H}_1$, resp. $2\bar{H}_2 - X$ to the section $D$ which we have already used in the proof of proposition 36. Via $\rho$ the curves $D$ and $\Delta_E$ are identified. We know that $\bar{H}_1$ restricted to $D$ is linearly equivalent to $20p_0$. On the other hand $\bar{H}_2$ restricted to $\Delta_E$ is linearly equivalent to $10p_0$ (this can be seen e.g. by using proposition 20 and the construction of the line bundle $L$). We also have already seen in the proof of proposition 36 that the restriction of $X$ to $D$ is trivial. This proves the proposition.

**Corollary 39.** The map $\pi_1: \tilde{V}_1 \rightarrow V_1$ is given by the complete linear system $|\bar{H}_1| = |2\bar{H}_2 - X|$.

**Proof.** We have to show that the (affine) dimension of the linear system $|2\bar{H}_2 - X|$ is five. We consider the exact sequence

$$0 \rightarrow O_{\tilde{V}_1}(2\bar{H}_2 - A_0 - X) \rightarrow O_{\tilde{V}_1}(2\bar{H}_2 - X) \rightarrow O_{A_0}(2\bar{H}_2 - \Delta_E) \rightarrow 0.$$

Since $(2H_2 - A_0 - X) \cdot H_2 \cdot \Sigma^1 = -1$ it follows that $h^0(O_{\tilde{V}_1}(2\bar{H}_2 - A_0 - X)) = 0$. Hence we have the inclusion

$$0 \rightarrow H^0(O_{\tilde{V}_1}(2\bar{H}_2 - X)) \rightarrow H^0(O_{A_0}(2\bar{H}_2 - \Delta_E)).$$

The linear system $|2\bar{H}_2 - \Delta_E|$ restricted to $A_0$ has degree 0 on the curves $\{(p, -p + e); p \in E\}$ and degree 5 on the curves $\{(e, p); p \in E\}$. It follows that $h^0(O_{A_0}(2\bar{H}_2 - \Delta_E)) = 5$ and this proves the corollary.

**Corollary 40.** The hyperplane bundle of $\hat{A}_K$ is of the form $2H' - \sum_{i=1}^{25} E_i$ where $H'$ is a polarization of type $(1,5)$ on the minimal model of $\hat{A}_K$.

**Proof.** Immediately from proposition 38.

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