Abstract

We consider the exit event from a metastable state for the overdamped Langevin dynamics \( dX_t = -\nabla f(X_t)dt + \sqrt{h}dB_t \). Using tools from semiclassical analysis, we prove that, starting from the quasi stationary distribution within the state, the exit event can be modeled using a jump Markov process parametrized with the Eyring-Kramers formula, in the small temperature regime \( h \to 0 \). We provide in particular sharp asymptotic estimates on the exit distribution which demonstrate the importance of the prefactors in the Eyring-Kramers formula. Numerical experiments indicate that the geometric assumptions we need to perform our analysis are likely to be necessary. These results also hold starting from deterministic initial conditions within the well which are sufficiently small in energy. From a modeling viewpoint, this gives a rigorous justification of the transition state theory and the Eyring-Kramers formula, which are used to relate the overdamped Langevin dynamics (a continuous state space Markov dynamics) to kinetic Monte Carlo or Markov state models (discrete state space Markov dynamics). From a theoretical viewpoint, our analysis paves a new route to study the exit event from a metastable state for a stochastic process.
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Motivation and presentation of the results

In materials science, biology and chemistry, atomistic models are now used on a daily basis in order to predict the macroscopic properties from a microscopic description of matter. The basic ingredient is a potential energy function \( f : \mathbb{R}^d \to \mathbb{R} \) which associates to a set of coordinates of particles the energy of the system. In practice, \( d \) is very large, since the system contains many particles (from tens of thousands to millions).

Using this function \( f \), two types of models are built: continuous state space Markov models (stochastic differential equations), such as the Langevin or overdamped Langevin dynamics, and discrete state space Markov models (jump Markov processes). The objective of the analysis presented in this work is to make a rigorous link between these two types of approaches, and in particular to provide a justification of the use of Eyring-Kramers laws to parameterize jump Markov models, by studying the exit event from a metastable state for the overdamped Langevin dynamics.

Jump Markov models are used by practitioners for many reasons. From a modelling viewpoint, new insights can be gained by building such coarse-grained models, that are easier to handle than a large-dimensional stochastic differential equation. Form a numerical viewpoint, it is possible to simulate a jump Markov model over timescales which are much larger than the original Langevin dynamics. Moreover, there are many algorithms which use the underlying jump Markov model in order to accelerate the sampling of the original dynamics [67,70,71].

The section is organized as follows. First, the two models under consideration are introduced, namely the overdamped Langevin dynamics in Section 1.1 and the underlying jump Markov process in Section 1.2. Next, Section 1.3 is devoted to a review of the mathematical literature dealing with metastable processes and the exit event from a metastable state. In Section 1.4, the notion of quasi stationary distribution is reviewed. This is a crucial tool in our analysis, in order to connect the overdamped Langevin
dynamics to a jump Markov process. Then, in Section 1.5 our main result (Theorem 1) is stated. In Section 1.6 we generalize Theorem 1 in various directions and discuss the geometric assumptions used to state Theorem 1. Finally, in Section 1.7 we give an outline of the proof of Theorem 1 together with the general organization of the paper.

1.1 Overdamped Langevin dynamics and metastability

The continuous state space Markov model we consider in this work is the so-called overdamped Langevin dynamics in $\mathbb{R}^d$

$$dX_t = -\nabla f(X_t) dt + \sqrt{h} dB_t,$$

(1)

driven by the potential function $f : \mathbb{R}^d \to \mathbb{R}$. We assume in the following that $f$ is a $C^\infty$ Morse function (all the critical points are non degenerate). The parameter $h = 2k_B T > 0$ is proportional to the temperature $T$ and $(B_t)_{t \geq 0}$ is a standard $d$-dimensional Brownian motion. One henceforth assumes that

$$\exists h_0, \forall h < h_0, \int_{\mathbb{R}^d} e^{-\frac{2}{h} f(x)} dx < \infty.$$

The invariant probability measure of (1) is

$$e^{-\frac{2}{h} f(x)} dx \int_{\mathbb{R}^d} e^{-\frac{2}{h} f(y)} dy.$$

(2)

The basic observation which motivates the use of a jump Markov model to obtain a reduced description of the dynamics (1) is the following. In many practical cases of interest in biology, physics or chemistry, the dynamics (1) is metastable, meaning that the process $(X_t)_{t \geq 0}$ remains trapped for very long times in some regions (called metastable states). It is thus tempting to introduce an underlying jump process among these metastable states.

Let us consider a region $\Omega \subset \mathbb{R}^d$ and the associated exit event from $\Omega$. More precisely, let us introduce

$$\tau_\Omega = \inf\{t \geq 0 | X_t \notin \Omega\}$$

(3)

the first exit time from $\Omega$. The exit event from $\Omega$ is fully characterized by the couple of random variables $(\tau_\Omega, X_{\tau_\Omega})$. The focus of this work is the justification of a jump Markov process to model the exit event from the region $\Omega$, in the small temperature regime $h \to 0$.

1.2 From the potential function to a jump Markov process

The potential function $f$ can also be used to build a jump Markov process to describe the evolution of the system. Jump Markov models are continuous-time Markov processes with values in a discrete state space. In molecular dynamics such processes are known as kinetic Monte Carlo models [72] or Markov state models [7, 63, 64].
Kinetic Monte Carlo models  The basic requirement to build a kinetic Monte Carlo model is a discrete collection of states $D \subset \mathbb{N}$, with associated rates $k_{i,j} \geq 0$ for transitions from state $i$ to state $j$, where $(i,j) \in D \times D$ and $i \neq j$. The neighboring states of state $i$ are those states $j$ such that $k_{i,j} > 0$. The dynamics is then given by a jump Markov process $(Z_t)_{t \geq 0}$ with infinitesimal generator $L \in \mathbb{R}^{D \times D}$, where $L_{i,j} = k_{i,j}$ for $i \neq j$.

To be more precise, let us describe how to build the jump process $(Z_t)_{t \geq 0}$ by defining the residence times $(T_n)_{n \geq 0}$ and the subordinated Markov chain $(Y_n)_{n \geq 0}$. Starting at time 0 from a state $Y_0 \in D$, the model consists in iterating the following two steps over $n \geq 0$: given $Y_n$,

- first sample the residence time $T_n$ in $Y_n$ as an exponential random variable with parameter $\sum_{j \neq Y_n} k_{Y_n,j}$: $\forall i \in D$, $\forall t > 0$, \[ P(T_n > t | Y_n = i) = \exp \left( -\sum_{j \neq i} k_{i,j} t \right), \] (4)

- and then sample independently from $T_n$ the next visited state $Y_{n+1}$ using the following law:

  $\forall j \neq i$, $P(Y_{n+1} = j | Y_n = i) = \frac{k_{i,j}}{\sum_{j \neq i} k_{i,j}}$. \hspace{1cm} (5)

The continuous time Markov process $(Z_t)_{t \geq 0}$ is then defined as:

$\forall n \geq 0$, $\forall t \in \left[ \sum_{m=0}^{n-1} T_m, \sum_{m=0}^{n} T_m \right]$, $Z_t = Y_n$

with the convention $\sum_{m=-1}^{-1} T_m = 0$.

Transition rates and Eyring Kramers law  Starting from the potential function $f : \mathbb{R}^d \to \mathbb{R}$, one approach to build a kinetic Monte Carlo model is to consider a collection of disjoint subsets $(\Omega_k)_{k \in D}$ which form a partition of the space $\mathbb{R}^d$ and to set the transition rates $k_{i,j}$ by considering transitions between these subsets, see for example [9,24,72,73].

The concept of jump rate between two states is one of the fundamental notions in the modelling of materials. Many papers have been devoted to the rigorous evaluation of jump rates from a full-atom description. The most famous formula is probably the rate derived in the harmonic transition state theory [52, 69], which gives an explicit expression for the rate in terms of the underlying potential energy function: this is the Eyring-Kramers formula, that we now introduce.

Let us consider a subset $\Omega$ of $\mathbb{R}^d$, which should be thought as one of the subsets $(\Omega_k)_{k \in D}$ introduced above, say the state $k = 0$. If $\Omega$ is metastable (in a sense which will be made precise below), it seems sensible to model the exit event from $\Omega$ using a jump Markov model, as introduced in the previous paragraph. As explained above, this requires to define jump rates $(k_{0,j})$ from the state 0 to the neighboring states $j$. The aim of this paper is to prove that the rates associated with the dynamics (1) can be approximated using the Eyring-Kramers formula which writes:

$\forall n \geq 0$, $\forall t \in \left[ \sum_{m=0}^{n-1} T_m, \sum_{m=0}^{n} T_m \right]$, $Z_t = Y_n$

with the convention $\sum_{m=-1}^{-1} T_m = 0$. 

$k_{0,j} = A_{0,j} e^{-\frac{2}{\hbar}(f(z_j) - f(x_0))}$

(6)
where $A_{0,j} > 0$ is a prefactor, $x_0 = \arg\min_{x \in \Omega} f(x)$ is the global minimum of $f$ on $\Omega$ which is assumed to be unique and $z_j = \arg\min_{z \in \partial \Omega_j} f(z)$ where $\partial \Omega_j$ denotes the part of the boundary $\partial \Omega$ which connects the region $\Omega$ (numbered 0) with the neighboring region numbered $j$ (see Figure 1 for a schematic representation when $\Omega$ has 4 neighboring states).

The prefactors $A_{0,j}$ depend on the dynamics under consideration and on the potential function $f$ around $x_0$ and $z_j$. If $\Omega$ is the basin of attraction of $x_0$ for the gradient dynamics $\dot{x} = -\nabla f(x)$ so that the points $z_j$ are order one saddle points, the prefactor writes for the overdamped Langevin dynamics (1)

$$A_{0,j} = \frac{1}{2\pi} |\lambda^-(z_j)| \sqrt{\frac{\det(\text{Hess} f(x_0))}{\det(\text{Hess} f(z_j))}}$$

where $\lambda^-(z_j)$ is the negative eigenvalue of the Hessian of $f$ at $z_j$. The formulas (6)–(7) have been obtained by Kramers [44], but also by many authors previously, see the exhaustive review of the literature reported in [29]. We also refer to [29] for generalizations to the Langevin dynamics.

1.3 Review of the mathematical literature on the Eyring-Kramers formula

Let us give the main mathematical approaches to the study of the exit event from a domain for stochastic process in $\mathbb{R}^d$. See also [2] for a nice review.

Global approaches Some authors adopt a global approach: they look at the spectrum of the infinitesimal generator of the continuous space dynamics in the small temperature regime $h \to 0$. It can be shown that there are exactly $m$ small eigenvalues, $m$ being the number of local minima of $f$, and that these eigenvalues satisfy the Eyring-Kramers law (6), with an energy barrier $f(z_i) - f(x_i)$, $i = 1, \ldots, m$. Here, the saddle point $z_i$ attached to the local minimum $x_i$ is defined by (it is here implicitly assumed that the inf sup value is attained at a single saddle point $z_i$)

$$f(z_i) = \inf_{\gamma \in \mathcal{P}(x_i, B_i)} \sup_{t \in [0,1]} f(\gamma(t))$$
where \( \mathcal{P}(x_i, B_i) \) denotes the set of continuous paths from \([0, 1]\) to \(\mathbb{R}^d\) such that \(\gamma(0) = x_i\) and \(\gamma(1) \in B_i\) with \(B_i\) the union of small balls around local minima lower in energy than \(x_i\). For the dynamics (1), we refer for example to the work [33] based on semi-classical analysis results for Witten Laplacian and the articles [5, 6, 20] where a potential theoretic approach is adopted. In the latter results, a connection is made between the small eigenvalues and mean transition times between metastable states. Let us also mention the earlier results [40, 56]. These spectral approaches give the cascade of relevant time scales to reach from a local minimum another local minimum which is lower in energy. They do not give any information about the typical time scale to go from one local minimum to any other local minimum (say from the global minimum to the second lower minimum for example). These global approaches can be used to build jump Markov models using a Galerkin projection of the infinitesimal generator of \((X_t)_{t \geq 0}\) onto the first \(m\) eigenmodes, which gives an excellent approximation of the infinitesimal generator. This has been extensively investigated by Schütte and his collaborators [64], starting with the seminal work [63].

**Local approaches** In this work, we are interested in a local approach, namely the study of the exit event (exit time and exit point) from a fixed given metastable state \(\Omega\). The most famous approach to study the exit event is the large deviation theory [25]. It relies essentially on the study of slices of the process defined with a suitable sequence of increasing stopping times. In the theory of large deviations, the notion of rate functional is fundamental and gives the cost of deviating from a deterministic trajectory.

In the small temperature regime, large deviation results provide the exponential rates [6], but without the prefactors and without precise error bounds. It can also be proven that the exit time is exponentially distributed in this regime, see [13]. For the dynamics (1), a typical result on the exit point distribution is the following (see [25, Theorem 5.1]): for all \(\Omega'\) compactly embedded in \(\Omega\), for any \(\gamma > 0\), for any \(\delta > 0\), there exists \(\delta_0 \in (0, \delta]\) and \(h_0 > 0\) such that for all \(h < h_0\), for all \(x \in \Omega'\) such that \(f(x) < \min_{\partial \Omega} f\), and for all \(y \in \partial \Omega\),

\[
e^{-\frac{1}{\delta}(f(y) - f(z_1) + \gamma)} \leq \mathbb{P}^x(X_{\tau_{\Omega}} \in \mathcal{V}_{\delta_0}(y)) \leq e^{-\frac{1}{\delta}(f(y) - f(z_1) - \gamma)}
\]

where \(\mathcal{V}_{\delta_0}(y)\) is a \(\delta_0\)-neighborhood of \(y\) in \(\partial \Omega\).

The strength of large deviation theory is that it is very general: it applies to any dynamics (reversible or non reversible) and in a very general geometric setting, even though it may be difficult in such general cases to make explicit the rate functional, and thus to determine the exit rates. See for example [4] for a recent contribution to the non reversible case.

Many authors have developed partial differential approach to the same problem. We refer to [14] for a comprehensive review. In particular, formal approaches to study the exit time and the exit point distribution have been developed by Matkowsky, Schuss and collaborators in [55, 57, 61, 62] and by Maier and Stein in [50, 51], using formal expansions for singularly perturbed elliptic equations. Some of the results cited above actually consider more general dynamics than (1). Rigorous version of these derivations have been obtained in [15, 16, 21, 40, 42, 53, 54, 59].

**Rescaling in time and convergence to a jump process** Finally, some authors prove the convergence to a jump Markov process using a rescaling in time. See for
example [43] for a one-dimensional diffusion in a double well, and [26,54] for a similar problem in larger dimension. In [68], a rescaled in time diffusion process converges to a jump Markov process living on the global minima of the potential $V$, assuming they are separated by saddle points having the same heights.

In this work, we are interested in precise asymptotics of the distribution of $X_{t_\Omega}$. Our approach is local, justifies the Eyring-Kramers formula (6) with the prefactors and provides sharp error estimates (see (25)). It uses techniques developed in particular in the previous works [33,34,47,48]. Our analysis requires to combine various tools from semiclassical analysis to address new questions: sharp estimates on quasimodes far from the critical points for Witten Laplacians on manifolds with boundary, a precise analysis of the normal derivative on the boundary of the first eigenfunction of Witten Laplacians, and fine properties of the Agmon distance on manifolds with boundary.

Let us finally mention that a summary of the results of this work appeared in [18].

1.4 Quasi stationary distribution

The quasi stationary distribution is the cornerstone of our analysis. The quasi-stationary distribution can be seen as a local equilibrium for a metastable stochastic process when it is trapped in a metastable region. It is actually useful in order to make precise quantitatively what a metastable domain is. In all what follows, we focus on the overdamped Langevin dynamics (1) and a domain $\Omega \subset \mathbb{R}^d$. For generalizations to other processes, we refer to [11] and in particular to [58] for the existence of the quasi stationary distribution for the Langevin dynamics.

1.4.1 Definition and a first property of quasi stationary distributions

Let us first define the quasi stationary distribution.

**Definition 1.** Let $\Omega \subset \mathbb{R}^d$ and consider the dynamics (1). A quasi stationary distribution is a probability measure $\nu_h$ supported in $\Omega$ such that

$$\forall t \geq 0, \nu_h(A) = \frac{\int_{\Omega} P^x[A, t < \tau_\Omega] \nu_h(dx)}{\int_{\Omega} P^x[t < \tau_\Omega] \nu_h(dx)}.$$

Here and in the following, the superscript $x$ indicates that the stochastic process starts from $x \in \mathbb{R}^d$: $X_0 = x$. In words, if $X_0$ is distributed according to $\nu_h$, then $\forall t > 0$, $X_t$ is still distributed according to $\nu_h$ conditionally on $X_s \in \Omega$ for all $s \in (0,t)$. It is important to notice that $\nu_h$ is not the invariant measure (2) of the original process restricted to $\Omega$.

In all the following, we will consider that $\Omega$ is a bounded domain in $\mathbb{R}^d$. In this context, we have the following results from [46].

**Proposition 1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the dynamics (1). Then, there exists a probability measure $\nu_h$ with support in $\Omega$ such that, whatever the law of the initial condition $X_0$ with support in $\Omega$,

$$\lim_{t \to \infty} ||\text{Law}(X_t | t < \tau_\Omega) - \nu_h||_{TV} = 0.$$  \hfill (9)
Here and in the following, \( \text{Law}(X_t | t < \tau) \) denotes the law of \( X_t \) conditional to the event \( \{ t < \tau \} \). A corollary of this proposition is that the quasi stationary distribution \( \nu_h \) exists and is unique.

This proposition also explains why it is relevant to consider the quasi stationary distribution for a metastable domain. The domain \( \Omega \) is metastable if the convergence in (9) is much quicker than the exit from \( \Omega \). In the following of this paper, we will assume that \( \Omega \) is a metastable domain, and we will thus consider the exit event from \( \Omega \), assuming that \( X_0 \) is distributed according to the quasi stationary distribution \( \nu_h \).

### 1.4.2 An eigenvalue problem related to the quasi stationary distribution

In this section, a connection is made between the quasi stationary distribution and an eigenvalue problem for the infinitesimal generator of the dynamics (1)

\[
L_{f,h}^{(0)} = - \nabla f \cdot \nabla + \frac{h}{2} \Delta
\]  

(10)

with Dirichlet boundary conditions on \( \partial \Omega \). In the notation \( L_{f,h}^{(0)} \), the superscript \( (0) \) indicates that we consider an operator on functions, namely 0-forms. Here and in the following, we assume that the domain \( \Omega \) is a connected open bounded \( C^\infty \) domain in \( \mathbb{R}^d \).

The basic observation to define our functional framework is that the operator \( L_{f,h}^{(0)} \) is self-adjoint on the weighted \( L^2 \) space \( L^2_w(\Omega) = \{ u : \Omega \rightarrow \mathbb{R}, \int_{\Omega} u^2(x)e^{-\frac{2}{h}f(x)} \, dx < \infty \} \).

This gives a proper framework to introduce the Dirichlet realization \( L_{D,h}^{(0)}(\Omega) \) on \( \Omega \) of the operator \( L_{f,h}^{(0)} \):

**Proposition 2.** The Friedrichs extension associated with the quadratic form

\[
\phi \in C^\infty_c(\Omega) \mapsto \frac{h}{2} \int_{\Omega} |\nabla \phi|^2 e^{-\frac{2}{h}f(x)} dx,
\]

on \( L^2_w(\Omega) \), is denoted \( -L_{D,h}^{(0)}(\Omega) \). It is a non negative unbounded self adjoint operator on \( L^2_w(\Omega) \) with domain

\[
D \left( L_{D,h}^{(0)}(\Omega) \right) = H^1_{w,0}(\Omega) \cap H^2_w(\Omega)
\]

where \( H^1_{w,0}(\Omega) = \{ u \in H^1_w(\Omega), u = 0 \text{ on } \partial \Omega \} \).

**Proof.** The quadratic form \( \phi \in C^\infty_c(\Omega) \mapsto \frac{h}{2} \int_{\Omega} |\nabla \phi|^2 e^{-\frac{2}{h}f(x)} dx \) is symmetric, non negative and closable and its closure is the quadratic form \( Q : w \in H^1_{w,0}(\Omega) \mapsto \frac{h}{2} \int_{\Omega} |\nabla w|^2 e^{-\frac{2}{h}f(x)} dx \). Let \( -L_{D,h}^{(0)}(\Omega) \) be the self adjoint operator associated with \( Q \), which domains is

\[
D \left( -L_{D,h}^{(0)}(\Omega) \right) = \{ u \in H^1_{w,0}(\Omega), \exists b \in L^2_w(\Omega), \forall v \in L^2_w(\Omega), Q(u,v) = (b,v)_{L^2_w} \},
\]
and \(-L_{f,h}^{D,0}(\Omega)u = b\). This implies that in the sense of distribution, it holds
\(-h^2 \text{div} \left( e^{-\frac{\xi}{2}f} \nabla u \right) = b\) and from standard regularity results for elliptic operators, we get \(u \in H^2_w(\Omega)\). Therefore \(D \left( -L_{f,h}^{D,0}(\Omega) \right) = H^1_{w,0}(\Omega) \cap H^2_w(\Omega)\).

The compact injection \(H^1_{w}(\Omega) \subset L^2_w(\Omega)\) implies that the operator \(L_{f,h}^{D,0}(\Omega)\) has a compact resolvent. Consequently, its spectrum is purely discrete. Let us introduce \(\lambda_h > 0\) the smallest eigenvalue of \(-L_{f,h}^{D,0}(\Omega)\). One has the following proposition (see also \([46]\)), which follows from standard results for the first eigenfunction of an elliptic operator, see for example \([27]\).

**Proposition 3.** The smallest eigenvalue \(\lambda_h\) of \(-L_{f,h}^{D,0}(\Omega)\) is non degenerate and its associated eigenfunction \(u_h\) has a sign on \(\Omega\). Moreover \(u_h \in C^\infty(\overline{\Omega})\).

Without loss of generality, one can assume that:
\[
\begin{align*}
u_h &> 0 \text{ on } \Omega \quad \text{and} \quad \int_{\Omega} u_h^2(x) e^{-\frac{\xi}{2}f(x)} dx = 1. \quad (11)
\end{align*}
\]

The eigenvalue-eigenfunction couple \((\lambda_h, u_h)\) satisfies:
\[
\begin{align*}
-\lambda_h u_h &= \lambda_h u_h \quad \text{on } \Omega, \\
u_h &= 0 \quad \text{on } \partial \Omega. \quad (12)
\end{align*}
\]

The link between the quasi stationary distribution \(\nu_h\) (see Definition \([1]\)) and \(u_h\) is given by the following proposition (see for example \([46]\)):

**Proposition 4.** The unique quasi stationary distribution \(\nu_h\) associated with the dynamics \((1)\) and the domain \(\Omega\) is given by:
\[
\begin{align*}
\nu_h(dx) = \frac{u_h(x) e^{-\frac{\xi}{2}f(x)}}{\int_{\Omega} u_h(y) e^{-\frac{\xi}{2}f(y)} dy} dx, \quad (13)
\end{align*}
\]

where \(u_h\) is the eigenfunction associated with the smallest eigenvalue of \(-L_{f,h}^{D,0}(\Omega)\) (see Proposition \([3]\) which satisfies \((11)\).

**1.4.3 Back to the jump Markov process**

As explained in Section \([1.4.1]\) if the process remains for a sufficiently long time in the domain \(\Omega\), it is natural to consider the exit event starting from the quasi stationary distribution attached to \(\Omega\). The next proposition characterizes the law of this exit event.

**Proposition 5.** Let us consider the dynamics \((1)\) and the quasi stationary distribution \(\nu_h\) associated with the domain \(\Omega\). If \(X_0\) is distributed according to \(\nu_h\), the random variables \(\tau_\Omega\) and \(X_{\tau_\Omega}\) are independent. Furthermore \(\tau_\Omega\) is exponentially distributed with parameter \(\lambda_h\) and the law of \(X_{\tau_\Omega}\) has a density with respect to the Lebesgue measure on \(\partial \Omega\) given by
\[
\begin{align*}
z \in \partial \Omega \mapsto -\frac{h}{2\lambda_h} \frac{\partial_n u_h(z) e^{-\frac{\xi}{2}f(z)}}{\int_{\Omega} u_h(y) e^{-\frac{\xi}{2}f(y)} dy}, \quad (14)
\end{align*}
\]
where \( u_h \) is the eigenfunction associated with the smallest eigenvalue of \(-L^{D,(0)}_{f,h}(\Omega)\) (see Proposition 3) which satisfies (11).

Here and in the following, \( \partial_n = n \cdot \nabla \) stands for the normal derivative and \( n \) is the unit outward normal on \( \partial \Omega \).

This proposition shows that, starting from the quasi-stationary distribution in the domain \( \Omega \), the exit event can be modeled by a jump Markov process without any approximation. Indeed, using the notation of Section 1.2 let us consider that \( \Omega \subset \mathbb{R}^d \) is associated with the state 0. Let us assume that \( \Omega \) is surrounded by \( n \) neighboring states, associated with domains \((\Omega_i)_{i=1,...,n}\) (see Figure 1 for a schematic representation when \( n = 4 \)). Let us define the transition rates:

\[
\forall i \in \{1,...,n\}, \quad k_{0,i} = \frac{\mathbb{P}^{\nu h}(X_{\tau_\Omega} \in \partial \Omega \cap \Omega_i)}{\mathbb{E}^{\nu h}(\tau_\Omega)}.
\]

(15)

Then, by Proposition 5 the exit event is such that:

- The residence time \( \tau_\Omega \) is exponentially distributed with parameters \( \sum_{i=1}^n k_{0,i} \).
- The next visited state is independent of the residence time and is \( i \) with probability \( \frac{k_{0,i}}{\sum_{j=1}^n k_{0,j}} \).

This is exactly the two properties (4) and (5) which are required to define a transition using a jump Markov process. The quasi stationary distribution can thus be used to parameterize the underlying jump Markov process if the domains are metastable.

The question we would like to address in this work is now the following: what is the error introduced when one approximates the exact rates (15) using the Eyring-Kramers formula (6)–(7). From Proposition 5 since \( \mathbb{E}^{\nu h}(\tau_\Omega) = 1/\lambda_h \), one has the following formula for the exact rates:

\[
k_{0,i} = -\frac{h}{2} \int_{\partial \Omega \cap \partial \Omega_i} (\partial_n u_h)(z) e^{-\frac{2}{h}f(z)} \sigma(dz) \int_{\Omega} u_h(y) e^{-\frac{2}{h}f(y)} dy
\]

(16)

where \( \sigma \) denotes the Lebesgue measure on \( \partial \Omega \). We will be able to prove that in the small temperature regime \( h \to 0 \), the exact rates (16) can indeed by accurately approximated by the Eyring-Kramers formula (6) with explicit error bounds. The asymptotic analysis is done directly on the rates, and not only on the logarithm of the rates (which is the typical result obtained with the large deviation theory for example, see Section 1.3).

1.5 Statement of the main result

We state in this section the main result of this work (Theorem 1) on the asymptotic behavior of the normal derivative \( \partial_n u_h \) in the regime \( h \to 0 \), as well as its corollary on the exit point density and the accuracy of the approximation of the exit rates by the Eyring-Kramers formula.

This section is organized as follows. We introduce in Section 1.5.1 a crucial tool in our analysis, the Agmon distance. Then, in Section 1.5.2 we give the set of hypotheses which will be needed throughout this work. Finally, Section 1.5.3 is dedicated to the statement of our main result.
1.5.1 Agmon distance

Our results hold under some geometric assumptions which require to introduce the so-called Agmon distance. The objective of this section is to introduce this distance, which is particularly useful to quantify the decay of eigenfunctions away from critical points [35][66]. We introduce the Agmon distance in a general setting, namely for $\Omega$ a Riemannian manifold, but one could think of $\Omega$ as a $C^\infty$ connected open bounded subset of $\mathbb{R}^d$.

**Definition 2.** Let $\Omega$ be a $C^\infty$ oriented connected compact Riemannian manifold of dimension $d$ with boundary $\partial \Omega$ and $f : \Omega \to \mathbb{R}$ be $C^\infty$. Define $g : \Omega \to \mathbb{R}$ by

$$g(x) = |\nabla f(x)|, \quad \forall x \in \Omega,$$

$$g(x) = |\nabla_T f(x)|, \quad \forall x \in \partial \Omega,$$

(17)

where for any $x \in \partial \Omega$, $\nabla_T f(x)$ denotes the tangential gradient of the function $f$ on $\partial \Omega$. One defines the length $L$ of a Lipschitz curve $\gamma : I \to \Omega$, where $I \subset \mathbb{R}$ is an interval, by

$$L(\gamma, I) := \int_I g(\gamma(t)) |\gamma'(t)| \, dt \in [0, \infty].$$

Let us recall that the Rademacher theorem (see for example [23]) states that every Lipschitz function admits almost everywhere a derivative (which is then bounded by the Lipschitz constant). Therefore, if $I$ is bounded, then $L(\gamma, I) < \infty$. Let us now define the Agmon distance.

**Definition 3.** Let $g$ be the function introduced in Definition 2. The Agmon distance between $x \in \Omega$ and $y \in \Omega$ is defined by

$$d_a(x, y) = \inf_{\gamma \in \text{Lip}(x, y)} L(\gamma, (0, 1)),$$

(18)

where $\text{Lip}(x, y)$ is the set of curve $\gamma : [0, 1] \to \Omega$ which are Lipschitz with $\gamma(0) = x$, $\gamma(1) = y$.

The Agmon distance is obviously symmetric, non negative and satisfies the triangular inequality. It is a distance if the critical points of $f$ and $f|_{\partial \Omega}$ are isolated (see Proposition 31 below). Let us mention that in the case when $\Omega$ is a manifold without boundary, the Agmon distance introduced in Definition 3 coincides with the Agmon distance defined in [38, Appendix 2].

We will give in Section 3 more details about the Agmon distance we consider. In particular, it will be shown that the Agmon distance to the critical points of $f|_{\partial \Omega}$ coincides with the solution to the eikonal equation $|\nabla \Phi|^2 = |\nabla f|^2$ in neighborhoods of the critical points. This requires to use the tangential gradient of $f$ on $\partial \Omega$ in the definition of the Agmon distance (see (17)).

1.5.2 Notations and hypotheses

As already stated above, we assume that $\Omega$ is a connected open bounded $C^\infty$ domain of $\mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a $C^\infty$ function. We will need the following set of assumptions:

---

1Actually, as explained in Section 2, we will perform the analysis in a more general setting, namely when $\Omega$ is a $C^\infty$ oriented connected compact Riemannian manifold. In this introductory section, we stick to a simpler presentation, with $\Omega$ a subset of $\mathbb{R}^d$. 12
The function $f : \overline{\Omega} \to \mathbb{R}$ is a Morse function on $\Omega$ and the restriction of $f$ to the boundary of $\Omega$ denoted by $f|_{\partial\Omega}$, is a Morse function. The function $f$ does not have any critical point on $\partial\Omega$.

The function $f$ has a unique global minimum $x_0 \in \Omega$ in $\overline{\Omega}$:

$$\min_{\partial\Omega} f > \min f = \min_{\Omega} f = f(x_0).$$

The point $x_0$ is the unique critical point of $f$ in $\overline{\Omega}$. The function $f|_{\partial\Omega}$ has exactly $n \geq 1$ local minima denoted by $(z_i)_{i=1,\ldots,n}$ such that $f(z_1) \leq f(z_2) \leq \ldots \leq f(z_n)$.

\[ \partial_nf > 0 \text{ on } \partial\Omega. \]

In the following, $n_0 \in \{1, \ldots, n\}$ denotes the number of points in $\arg \min f|_{\partial\Omega}$:

$$f(z_1) = \ldots = f(z_{n_0}) < f(z_{n_0+1}) \leq \ldots \leq f(z_n).$$

We will need to define the basins of attraction of the local minima $z_i$ for the dynamics $\dot{x} = -\nabla_T f(x)$ in $\partial\Omega$, where, we recall, for any $x \in \partial\Omega$, $\nabla_T f(x)$ denotes the tangential gradient of $f$ on $\partial\Omega$.

**Definition 4.** Assume that [H1] holds. For each local minimum $z \in \partial\Omega$, one denotes by $B_z \subset \partial\Omega$ the open basin of attraction in $\partial\Omega$ of $z$ for the dynamics $\dot{x} = -\nabla_T f(x)$ in $\partial\Omega$. Additionally define $B^*_z := \partial\Omega \setminus B_z$.

From this definition, one obviously has that for each local minimum $z \in \partial\Omega$, for any $x \in B_z$, $f(x) \geq f(z)$.

As a consequence of the assumption [H1], the determinants of the Hessians of $f$ (resp. of $f|_{\partial\Omega}$) at the critical points of $f$ (resp. of $f|_{\partial\Omega}$) are non zero. These quantities appear in the prefactors of the Eyring-Kramers law (see Equation (25) below).

**Remark 1.** Let us recall how the Hessians are defined. Let $\phi : N \to \mathbb{R}$ be a $C^\infty$ function defined on a Riemannian $C^\infty$ manifold $N$ of dimension $d$. By standard results of Riemannian geometry, the Hessian $\text{Hess} \phi(x)$ of $\phi$ at a point $x \in N$ is defined as a bilinear symmetric form acting on vectors in the tangent space $T_xN$ as:

$$\forall X, Y \in \Gamma(TN), \text{Hess} \phi(X, Y) = \nabla_X d\phi(Y) \quad (19)$$

where $\nabla$ is the covariant derivative (Levi-Civita connection) and $d\phi$ is the differential of $\phi$. Then, $\det \text{Hess} \phi(x)$ is defined as the determinant of the bilinear form $\text{Hess} \phi(x)$ in any orthonormal basis of $T_xN$.

In practice, $\det \text{Hess} \phi(x)$ can be computed using a local chart as follows. Let us assume that $x_0$ is a critical point of $\phi$: $d_{x_0}\phi = 0$. Let us introduce $\psi : y \in U \mapsto \psi(y) \in V$ a local chart around $x_0$, where $U \subset \mathbb{R}^d$ is a neighborhood of $0$, $V \subset N$ is a neighborhood of $x_0$ and $\psi(0) = x_0$. Let us assume in addition that the vectors $(e_i)_{i=1,\ldots,d} := \left(\frac{\partial}{\partial y_i}(0)\right)_{i=1,\ldots,d}$ are orthonormal (thus defining an orthonormal basis of $T_{x_0}N$). Let us introduce the symmetric matrix $H$ associated with the second order differential of $\phi \circ \psi$ at point $0$: $\forall (u, v) \in \mathbb{R}^d \times \mathbb{R}^d$

$$D^2_0(\phi \circ \psi) \left( \sum_{i=1}^d u_i e_i, \sum_{i=1}^d v_i e_i \right) = u^T H v.$$
Then
\[ \text{det Hess } \phi(x_0) = \text{det } H. \]

This formula is only valid at a critical point and is a direct consequence of the definition \([19]\) of the Hessian and the explicit expression of the Levi Civita connection in the local chart \(\psi\):

\[
\nabla_X d\phi(Y)|_x = \sum_{i,j=1}^{d} \left( \frac{\partial^2 (\phi \circ \psi)}{\partial y_i \partial y_j} (y) - \sum_{k=1}^{d} \Gamma_{i,j}^k(\psi(y)) \frac{\partial (\phi \circ \psi)}{\partial y_k} (y) \right) Y_i X_j
\]

where \(x = \psi(y) \in V, \Gamma_{i,j}^k(x)\) are the Christoffel symbols of the connection \(\nabla\) associated with the basis \(((\partial_{y_j}(\psi^{-1}(x)))_{j=1,...,n}\) of \(T_x N\) and \((X_j)_{j=1,...,n}\) (respectively \((Y_j)_{j=1,...,n}\)) are the coordinates of \(X\) (respectively \(Y\)) in this basis.

### 1.5.3 Main result

In view of equations \([14]\) and \([16]\), we need to give an estimate of three quantities in order to analyze the exit point density and the asymptotic of the transition rates in the regime \(h \to 0\):

\[
\int_{\Sigma}(\partial_n u_h) e^{-\frac{\lambda}{2} f} \text{ for a subset } \Sigma \text{ of } \partial \Omega\int_{\Omega} u_h e^{-\frac{\lambda}{2} f}\text{ and } \lambda_h\text{, where, we recall } (\lambda_h, u_h)\text{ is defined by } [12].
\]

We will consider a subset \(\Sigma\) such that \(\Sigma \subset B_{z_i}\) for a local minimum \(z_i\) (see Definition \([4]\) for the definition of \(B_{z_i}\)). This is the objective of the next three results.

**Theorem 1.** Assume that \([H1]\), \([H2]\) and \([H3]\) hold. Moreover assume that

- \(\forall i \in \{1, \ldots, n\}, \inf_{z \in \Sigma_i} d_a(z, z_i) > \max[f(z_n) - f(z_i), f(z_i) - f(z_1)], \)
- \(\text{ where } \lambda_h\text{ is the eigenfunction associated with the smallest eigenvalue of } -L f^{(0))}_h(\Omega)\text{ (see Proposition }[8]\text{ which satisfies }[11]\text{ and }}
\]

\[
A_i(h) = -\frac{\text{det Hess } f(x_0)^{1/4} \partial_n f(z_i)}{\text{det Hess } f|_{\partial \Omega(z_i)}} 2\pi^{\frac{d-2}{2}} h^{\frac{d-6}{2}}
\]

**Remark 2.** As will become clear in the proof of Theorem \([4]\) it can actually be proven that for all \(i \in \{1, \ldots, n\}\), the residual \(r_i(h) = O(h)\) appearing in \([22]\) admits a full asymptotic expansion in \(h\): there exists a sequence \((b_{k,i})_{k \geq 0} \in \mathbb{R}^N\) such that for all \(N \in \mathbb{N}\), in the limit \(h \to 0\),

\[
r_i(h) = h \sum_{k=0}^{N} b_{k,i} h^k + O(h^{N+2}).
\]
We do not state our main result with this expansion since, for general domains \( \Omega \), the explicit computations of the sequence \((b_{k,i})_{k \geq 0}\) is not possible in practice. This remark also holds for all the residuals \(O(h)\) in the next results.

**Proposition 6.** Assume that \([H1]\), \([H2]\) and \([H3]\) hold. Then when \(h \to 0\)

\[
\int_{\Omega} u_h(x) e^{-\frac{2}{h} f(x)} dx = \frac{\pi^{\frac{d}{2}}}{(\det \text{Hess} f(x_0))^{\frac{1}{4}}} h^{\frac{d}{2}} e^{-\frac{h}{2} f(x_0)} \left(1 + O(h)\right),
\]

where \(u_h\) is the eigenfunction associated with the smallest eigenvalue of \(-L_{f,h}^{D,(0)}(\Omega)\) (see Proposition 3) which satisfies \([11]\).

**Proposition 7.** Assume that \([H1]\), \([H2]\), and \([H3]\) hold. Then, in the limit \(h \to 0\),

\[
\lambda_h = \frac{\sqrt{\det \text{Hess} f(x_0)}}{\sqrt{\pi h}} \sum_{i=1}^{n_0} \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess} f}(z_i)} e^{-\frac{2}{h} (f(z_i) - f(x_0))} \left(1 + O(h)\right), \tag{23}
\]

where \(\lambda_h\) is the smallest eigenvalue of \(-L_{f,h}^{D,(0)}(\Omega)\) (see Proposition 3).

Theorem 1 is the main contribution of this work. Actually Theorem 1 will be proven in a more general framework: namely when \(\overline{\Omega}\) is a \(C^\infty\) connected compact oriented Riemannian \(d\)-dimensional manifold with boundary \(\partial \Omega\). Theorem 1 Proposition 6 and Proposition 7 are respectively proved in Sections 4.5, 5.1.1 and 5.1.2. For the sake of completeness, we provide a proof of Proposition 7 in our specific setting, but this result actually holds under weaker geometric assumptions, see [17] or [33].

These results have the following consequence on the first exit point distribution and the estimate of the exact rates \((k_{0,i})\) using the Eyring-Kramers formula (see Section 1.4.3). We recall that \((X_t)_{t \geq 0}\) denotes the solution to \([1]\), \(\tau_\Omega\) is the exit time from the domain \(\Omega\) and \(\nu_h\) is the quasi stationary distribution associated with \((X_t)_{t \geq 0}\) and \(\Omega\).

**Corollary 8.** Under the hypotheses of Theorem 1 for \(i \in \{1, \ldots, n\}\) and for all open sets \(\Sigma_i \subset \partial \Omega\) containing \(z_i\) and such that \(\Sigma_i \subset B_{z_i}\), in the limit \(h \to 0\):

\[
\mathbb{P}^{\nu_h} [X_{\tau_\Omega} \in \Sigma_i] = \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess} f}(z_i)} \left(\sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess} f}(z_k)}\right)^{-1} e^{-\frac{2}{h} (f(z_i) - f(z_1))} \left(1 + O(h)\right). \tag{24}
\]

The hypotheses \([20]\) and \([21]\) are discussed in Section 1.6.2.

As a simple consequence of Corollary 8 we recover the well-known result that \((X_t)_{t \geq 0}\) leaves \(\Omega\) around the global minima of \(f\) on \(\partial \Omega\): for any collection of open sets \((\Sigma_j)_{1 \leq j \leq n_0}\) such that for all \(j \in \{1, \ldots, n_0\}\), \(\Sigma_j \subset B_{z_j}\) and \(z_j \in \Sigma_j\), in the limit \(h \to 0\), \(\mathbb{P}^{\nu_h} [X_{\tau_\Omega} \in \bigcup_{j=1}^{n_0} \Sigma_j] = 1 + O(h)\). Actually, this result can be proven with an exponentially small residual \((O(h))\) is replaced by \(O\left(e^{-c/h}\right)\) for some positive \(c\) in a more general setting (see for instance [25, 42, 59, 68]). Let us also refer to [17] where we discuss this result in a more general setting (for example \(f\) can have several critical points in \(\Omega\) and the assumptions \([20]\) and \([21]\) are not needed).
Corollary 9. Using the notation of Section 1.4.3, assume that $\Sigma_i$ is the common boundary between $\Omega$ and another domain $\Omega_i \subset \mathbb{R}^d$. Under the hypotheses of Theorem 1, the transition rate given by (15), to go from $\Omega$ to $\Omega_i$, satisfies, in the limit $h \to 0$,
\[ k_{0,i} = \frac{1}{\sqrt{\pi h}} \partial_n f(z_i) \frac{\sqrt{\text{det Hess} f(x_0)}}{\sqrt{\text{det Hess}_f(\partial \Omega)(z_i)}} e^{-\frac{\lambda}{2h}(f(z_i) - f(x_0))}(1 + O(h)). \] (25)

This corollary thus gives a justification of the Eyring-Kramers formula and the Transition State Theory to build Markov models. As stated in the assumptions, the exit rates are obtained assuming $\partial_n f > 0$ on $\partial \Omega$: the local minima $z_1, \ldots, z_n$ of $V$ on $\partial \Omega$ are therefore not saddle points of $f$ but so-called generalized saddle points (see [34,47]).

In a future work, we intend to extend these results to the case where the points $(z_i)_{1 \leq i \leq n}$ are saddle points of $f$, in which case we expect to prove the same result (25) for the exit rates, with a modified prefactor:
\[ k_{0,i} = \frac{1}{\pi} |\lambda^{-1}(z_j)| \frac{\sqrt{\text{det Hess} f(x_0)}}{\sqrt{\text{det Hess}_f(z_j)}} e^{-\frac{\lambda}{2h}(f(z_i) - f(x_0))}(1 + O(h)) \]
(this formula can be obtained using formal expansions on the exit time and Laplace’s method). Notice that the latter formula differs from (6)–(7) by a multiplicative factor $1/2$ since $\lambda h$ is the exit rate from $\Omega$ and not the transition rate to one of the neighboring state (see the remark on page 408 in [5] on this multiplicative factor 1/2 and the results on asymptotic exit times in [50] for example). This factor is due to the fact that once on the saddle point, the process has a probability one half to go back to $\Omega$, and a probability one half to effectively leave $\Omega$. This multiplicative factor does not have any influence on the law of the next visited state which only involves ratio of the rates $k_{0,i}$, see Section 1.4.3 and Equation (24).

1.6 Discussion and generalizations

As explained above, the interest of Theorem 1 is that it justifies the use of the Eyring-Kramers formula to model the exit event using a jump Markov model including the prefactors. It gives in particular the relative probability to leave $\Omega$ through each of the local minima $z_i$ of $f$ on the boundary $\partial \Omega$. Moreover, one obtains an estimate of the relative error on the exit probabilities (and not only on the logarithm of the exit probabilities as in (8)): it is of order $h$, see Equation (24).

In Section 1.6.1, we explain how this result can be generalized to a situation where the process $(X_t)_{t \geq 0}$ is assumed to start under another initial condition than the quasi stationary distribution. The importance of the geometric assumption (20)–(21) (resp. assumption (26)) to obtain the asymptotic result of Corollary 8 (resp. its generalization to deterministic initial conditions, see Corollary 10) is discussed in Section 1.6.2. Finally, in Section 1.6.3, we discuss extensions to less stringent conditions than (20)–(21). Moreover the exit through subsets of $\partial \Omega$ which do not necessarily contain one of the local minima $z_i$ of $f|_{\partial \Omega}$ is considered: this shows in particular the interest of estimating the prefactors in the asymptotic approximations of the exit rates.

1.6.1 Extension of the result to other initial conditions

The question we would like to address in this section is how to generalize Corollary 8 to a deterministic initial condition: $X_0 = x$ for $x \in \Omega$. 

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Corollary 10. Let us assume that all the hypotheses of Corollary 8 are satisfied, and that in addition there exists $i_0 \in \{2, \ldots, n\}$ such that

$$2(f(x_{i_0}) - f(z_1)) < f(z_1) - f(x_0).$$

Let $i \in \{1, \ldots, i_0\}$ and let $\alpha \in \mathbb{R}$ be such that

$$f(x_0) < \alpha < 2f(z_1) - f(z_i).$$

Then, for $i \in \{1, \ldots, n\}$ and for all open sets $\Sigma_i \subset \partial \Omega$ containing $z_i$ and such that $\Sigma_i \subset B_{z_i}$, we have uniformly in $x \in f^{-1}((\infty, \alpha]) \cap \Omega$, in the limit $h \to 0$:

$$\mathbb{P}^x[X_{\tau_h} \in \Sigma_i] = \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess}_f(x)}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess}_f(x)}} \right)^{-1} e^{-\frac{h}{2}(f(z_i) - f(z_1))}(1+O(h)).$$

(27)

Let us give a simple example to illustrate this result. In a situation where $n = 2$, this corollary shows that the estimates we have obtained on the probability to exit in a neighborhood of $z_2$ under the assumption $X_0 \sim \nu_0$ are still valid if $X_0 = x$ for $x \in f^{-1}((\infty, 2f(z_1) - f(z_2)) \cap \Omega$ under the assumption $f(z_1) - f(x_0) > 2(f(z_2) - f(z_1))$, which is stronger assumption than (21).

1.6.2 On the geometric assumptions (20), (21) and (20)

On the geometric assumption (20).

The question we would like to address is the following: is the assumption (20) necessary for the result on the exit point density (24) to hold?

In order to test this assumption numerically, we consider the following simple two-dimensional setting. The potential function is

$$f(x, y) = x^2 + y^2 - ax,$$

with $a \in (0, 1/9)$, and the domain $\Omega$ is defined by (see Figure 2):

$$\Omega = [0, 1]^2 \cup \{(x, y) \mid x^2 + (y - 1)^2 < 1\} \cup \{(x, y) \mid x^2 + (y + 1)^2 < 1\}.$$

The two local minima of $f$ on $\partial \Omega$ are $z_1 = (1, 0)$ and $z_2 = (-1, 0)$. Notice that $f(z_2) - f(z_1) = 2a > 0$. The potential $f$ has a unique critical point in $\Omega$, namely the global minimum $x_0 = (a/2, 0)$. Let us check that the assumptions of Theorem 3 are satisfied in this setting (i.e. for $a \in (0, \frac{1}{9})$). Indeed, the inequality $f(z_1) - f(x_0) > f(z_2) - f(z_1)$ is satisfied if and only if $1 - 3a + \frac{a^2}{3} > 0$ i.e. if and only if $a \notin (2(3 - \sqrt{8}), 2(3 + \sqrt{8}))$. Moreover, using Proposition 34, the inequality $d_a(z_1, B_{z_1}^c) > f(z_2) - f(z_1)$ is satisfied.

Finally, to check that the inequality $d_a(z_2, B_{z_2}^c) > f(z_2) - f(z_1)$ is satisfied we use Proposition 33 with $W = \{(x, y) \in \mathbb{R}^2, \|(x, y) - z_2\| \leq \frac{1}{4}\} \cap \overline{\Omega}$ and $W' = \{(x, y) \in \mathbb{R}^2, \|(x, y) - z_2\| \leq \frac{1}{3}\} \cap \overline{\Omega}$. In that case, one has $\alpha = \frac{2}{3}$ (where $\alpha$ is defined by (88)) and thus the inequality

$$\alpha \inf_{x \notin W} g(x) = \frac{1}{3} \min\left(\frac{2}{3}, 2\left(1 + \frac{2}{3}\right) - a\right) = \frac{1}{3} \min\left(\frac{2}{3}, \frac{2}{3} + a\right) > f(z_2) - f(z_1) = 2a$$

is satisfied if and only if $a < \frac{1}{9}$.
Let us consider the segment $\Sigma_2$ joining the two points $(-1, -1)$ and $(-1, 1)$. This subset of $\partial \Omega$ contains the highest saddle point $z_2$ and is included in $B_{z_2}$. From Theorem 1, we expect that, in the limit $h \to 0$,

$$P_{\nu_h}[X_{\tau_{\Omega}} \in \Sigma_2] = \exp \left( G \left( \frac{2}{h} \right) \right) (1 + O(h))$$

where

$$G(x) = \ln \left[ \frac{\partial_n f(z_2) \sqrt{\det \text{Hess}f|_{\partial \Omega}(z_1)}}{\partial_n f(z_1) \sqrt{\det \text{Hess}f|_{\partial \Omega}(z_2)}} - x (f(z_2) - f(z_1)) \right].$$

The function $G$ is compared to the numerically estimated function $F$ such that $F \left( \frac{2}{h} \right) = \ln \left( P_{\nu_h}[X_{\tau_{\Omega}} \in \Sigma_2] \right)$. In practice, the quasi stationary distribution $\nu_h$ is sampled using a Fleming-Viot particle system (the convergence diagnostics being a Gelman-Rubin statistics, see [3]) composed of $10^5$ particles. The probability $P_{\nu_h}(X_{\tau_{\Omega}} \in \Sigma_2)$ is estimated using a Monte Carlo procedure using $6 \times 10^5$ particles distributed according to the quasi stationary distribution $\nu_h$. The dynamics is discretized in time using an Euler-Maruyama scheme with a timestep $\Delta t$ which is made precise in the captions of the figures. On Figures 3 and 4, we observe an excellent agreement between the theory and the numerical results.

Now, the potential function $f$ is modified such that the assumption (20) is not satisfied anymore. More precisely, the potential function is

$$f(x, y) = (y^2 - 2 \ a(x))^3,$$

with $a(x) = a_1 x^2 + b_1 x + 0.5$ where $a_1$ and $b_1$ are chosen such that $a(-1 + \delta) = 0$, $a(1) = 1/4$ for $\delta = 0.05$. We have $f(z_1) = -1/8$ and $f(z_2) = -8(a(-1))^3 > 0 > f(z_1)$. Moreover, two 'corniches' (which are in the level set $f^{-1}(\{0\})$ of $f$, and on which $|\nabla f| = 0$) on the 'slopes of the hills' of the potential $f$ join the point $(-1 + \delta, 0)$ to $B^c_{z_2}$ (at the points $(1, 1/\sqrt{2}) \in B^c_{z_2}$ and $(1, 1/\sqrt{2}) \in B^c_{z_2}$) so that $\inf_{z \in B^c_{z_2}} d_a(z, z_2) < f(z_2) - f(z_1)$. 

Figure 2: The domain $\Omega$. 

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Indeed, in that case assumption (20) is not satisfied since

\[ \inf_{z \in B_{\epsilon_2}} d_a(z, z_2) \leq d_a \left( z_2, (1, 1/\sqrt{2}) \right) \]

\[ \leq d_a \left( z_2, (0, -1 + \delta) \right) + d_a \left( (0, -1 + \delta), (1, 1/\sqrt{2}) \right) \]

\[ = f(z_2) - f(0, -1 + \delta) + 0 \]

\[ = f(z_2) < f(z_2) - f(z_1). \]

Notice that the Hessians \((\text{Hess } f|_{\partial \Omega})(z_1)\) and \((\text{Hess } f|_{\partial \Omega})(z_2)\) are nonsingular. The functions \(f|_{\Omega}\) and \(f|_{\partial \Omega}\) are not Morse functions, but an arbitrarily small perturbation (which we neglect here) turns them into Morse functions. When comparing the numerically estimated probability \(P^\nu(X_{\tau_\Omega} \in \Sigma_2)\), with the theoretical asymptotic result in the limit \(h \to 0\), we observe a discrepancy on the prefactors, see Figure 5.

Therefore, it seems that assumption (20) is indeed required to get an accurate description of the dynamics by the jump Markov process using the Eyring-Kramers law to estimate the rates between the neighboring states.
On the geometric assumptions (21) and (26). To discuss the necessity of the assumptions (26) in Corollary 10 and (21) in Corollary 8 we consider a one-dimensional case, where the law of \( X_{t_0} \) when \( X_0 = x \) has an explicit expression. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( C^\infty \) and let \( z_1, z_2 \in \mathbb{R} \) such that \( z_1 < z_2 \). Let us assume that \( f'(z_1) < 0, f'(z_2) > 0, f(z_1) < f(z_2) \) and \( f \) has only one critical point in \((z_1, z_2)\) denoted by \( x_0 \). This implies in particular that \( f(x_0) = \min_{[z_1, z_2]} f < f(z_1) \). Moreover, let us assume that \( f''(x_0) > 0 \). Therefore, the hypotheses \([\text{H1}]-[\text{H2}]-[\text{H3}]\) hold. For \( x \in [z_1, z_2] \), let us denote by \( w_h(x) = \mathbb{P}_x[X_{t_1, z_2} = z_2] \). It is standard that using a Feynman-Kac formula, \( w_h \) solves the elliptic boundary value problem

\[
\frac{2}{h}g'' - g' f' = 0, \quad g(z_1) = 0, \quad g(z_2) = 1.
\]

Therefore, one has for \( x \in [z_1, z_2] \): \( w_h(x) = \frac{\int_{z_1}^{x} e^\frac{z}{h} f}{\int_{z_1}^{z_2} e^\frac{z}{h} f} \). Let \( x \in [z_1, z_2] \). Using Laplace’s method, if \( f(x) < f(z_1) \), one obtains in the limit \( h \to 0 \):

\[
\mathbb{P}_x^z[X_{t_1, z_2} = z_2] = -\frac{f'(z_2)}{f'(z_1)} e^{-\frac{2}{h}(f(z_2)-f(z_1))} (1 + O(h)),
\]

if \( f(x) = f(z_1) \), \( x \neq z_1 \), it holds in the limit \( h \to 0 \):

\[
\mathbb{P}_x^z[X_{t_1, z_2} = z_2] = f'(z_2) \left( \frac{1}{f'(x)} - \frac{1}{f'(z_1)} \right) e^{-\frac{2}{h}(f(z_2)-f(z_1))} (1 + O(h)),
\]

and if \( f(x) > f(z_1) \), it holds in the limit \( h \to 0 \):

\[
\mathbb{P}_x^z[X_{t_1, z_2} = z_2] = \frac{f'(z_2)}{f'(x)} e^{-\frac{2}{h}(f(z_2)-f(x))} (1 + O(h)).
\]

Therefore, in dimension one, the estimate (27) holds if and only if \( x \in f^{-1}((-\infty, f(z_1))) \). In accordance with Corollary 10, the asymptotic (27) only holds for initial conditions which are sufficiently low in energy. However, we observe that in this simple one-dimensional setting, the assumption (26) is not needed. We do not know if the result of Corollary 10 would hold in general without the assumption (26).

Figure 5: Logarithm of the probability \( \mathbb{P}_{\nu_h}(X_{t_1} \in \Sigma_2) \) as a function of \( \frac{2}{h} \); comparison of the theoretical result function (G) with the numerical result (function F, \( \Delta t = 2.10^{-3} \) and \( \Delta t = 5.10^{-4} \)).
Let us now discuss the assumption \( (21) \) in the framework of Theorem \( 1 \) and Corollary \( 8 \). From (13), one has:

\[
\mathbb{P}^{\nu_h}[X_{\tau(z_1,z_2)} = z_2] = \frac{\int_{z_1}^{z_2} u_h w_h e^{-\frac{2}{h} f}}{\int_{z_1}^{z_2} u_h e^{-\frac{2}{h} f}}.
\]

Using Lemma \[60\], Lemma \[18\] and \( (21) \), one has for some \( c > 0 \), for any \( \delta > 0 \) and for \( h \) small enough:

\[
u_h(x) = \frac{\chi(x)}{\sqrt{\int_{z_1}^{z_2} \chi^2 e^{-\frac{2}{h} f}}} (1 + \alpha_h) + r(x), \quad \text{for} \quad x \in \Omega
\]

with \( \alpha_h \in \mathbb{R}, \alpha_h = O(e^{-\frac{\alpha}{h}}), \int_{z_1}^{z_2} r^2 e^{-\frac{2}{h} f} = O(e^{-\frac{1}{h} (f(z_1) - f(x_0) - \delta)}) \) and where \( \chi \in C^\infty_c(z_1, z_2) \) is given by Lemma \[60\]. Therefore, one has:

\[
\mathbb{P}^{\nu_h}[X_{\tau(z_1,z_2)} = z_2] = \frac{1}{\int_{z_1}^{z_2} u_h e^{-\frac{2}{h} f}} \left[ \int_{z_1}^{z_2} \chi(x) f_{x_1} e^\frac{2}{h} (f(y) - f(x)) dy dx \right] \left( 1 + \alpha_h \right) + \int_{z_1}^{z_2} r w_h e^{-\frac{2}{h} f}.
\]

Using Proposition \[8\] and Laplace’s method, one gets for any \( \delta > 0 \), in the limit \( h \to 0 \):

\[
\mathbb{P}^{\nu_h}[X_{\tau(z_1,z_2)} = z_2] = -\frac{f(z_2)}{f(z_1)} e^{-\frac{2}{h} (f(z_2) - f(z_1))} (1 + O(\delta)) + O(e^{-\frac{1}{h} (f(z_2) - f(x_0) + f(z_1) - f(x_0) - \delta)}).
\]

Therefore, the result of Corollary \[8\] holds if

\[
2(f(z_1) - f(x_0)) > f(z_2) - f(z_1).
\]

This explicit computation in dimension one shows that the result of Corollary \( 1 \) indeed requires an assumption of the type: the height of the energy barrier to leave the well \( f(z_1) - f(x_0) \) is sufficiently large compared to the largest difference in energy of the saddle points \( f(z_2) - f(z_1) \). Notice that \( (28) \) differs from \( (21) \) by a multiplicative factor \( \frac{1}{\delta} \). We do not know if the result of Corollary \( 8 \) would hold in general under the weaker assumption \( (28) \). Finally, let us mention that when \( d = 1 \), \( (20) \) is always satisfied.

### 1.6.3 Extension of the result to more general subsets of \( \partial \Omega \)

It is actually possible to generalize the result of Theorem \( 1 \) and Corollary \( 8 \) to less stringent conditions than \( (20)-\( (21) \) and to more general subsets \( \Sigma \subset \partial \Omega \).

**Theorem 2.** Assume that \([H1]\), \([H2]\) and \([H3]\) hold. Assume that there exist \( k_0 \in \{1, \ldots, n\} \) and \( f^* \in \mathbb{R} \) such that \( f(z_{k_0}) \leq f^* \leq f(z_{k_0+1}) \) (with the convention \( f(z_{k_0+1}) = +\infty \) if \( k_0 = n \)),

\[
\begin{align*}
\forall i \in \{1, \ldots, k_0\}, \quad & \inf_{z \in B_{\delta_i}} d_\alpha(z, z_i) > \max[f^* - f(z_i), f(z_i) - f(z_1)], \\
\forall i \in \{k_0 + 1, \ldots, n\}, \quad & \inf_{z \in B_{\delta_i}} d_\alpha(z, z_i) > f^* - f(z_1),
\end{align*}
\]

and

\[
f(z_1) - f(x_0) > f^* - f(z_1).
\]

Let \( u_h \) be the eigenfunction associated with the smallest eigenvalue of \( -L^{D,(0)}_{f,h}(\Omega) \) (see Proposition \[3\] which satisfies \( (11) \).
1. For all \( i \in \{1, \ldots, k_0\} \) and for all smooth open set \( \Sigma_i \subset \partial \Omega \) containing \( z_i \), and such that \( \Sigma_i \subset B_{z_i} \), the limit (22) holds for \( \int_{\Sigma_i} (\partial_n u_h) e^{-\frac{2}{\pi \varepsilon} f} d\sigma \) and the limit (24) holds for \( \mathbb{P}^{\nu_h} [X_{\tau_2} \in \Sigma_i] \). Moreover, if \( f(z_{k_0+1}) > f(z_{k_0}) \), for all \( i \in \{k_0 + 1, \ldots, n\} \) and for all smooth open set \( \Sigma_i \subset \partial \Omega \) containing \( z_i \) and such that \( \Sigma_i \subset B_{z_i} \), there exist \( \varepsilon > 0 \) and \( h_0 > 0 \) such that in the limit \( h \to 0 \)

\[
\int_{\Sigma_i} (\partial_n u_h) e^{-\frac{2}{\pi \varepsilon} f} d\sigma = \left( \int_{\Sigma_{k_0}} (\partial_n u_h) e^{-\frac{2}{\pi \varepsilon} f} d\sigma \right) O \left( e^{-\frac{2}{\pi \varepsilon} f} \right),
\]

and

\[
\mathbb{P}^{\nu_h} [X_{\tau_2} \in \Sigma_i] = \mathbb{P}^{\nu_h} [X_{\tau_3} \in \Sigma_{k_0}] O \left( e^{-\frac{2}{\pi \varepsilon} f} \right)
\]

2. Let \( j_0 \in \{1, \ldots, k_0\} \) and \( \Sigma \subset \partial \Omega \) be a smooth open set such that \( \Sigma \subset B_{z_{j_0}} \) and \( \inf_{\Sigma} f = f^* \). Let \((B^*, p^*) \in \mathbb{R}_+^2 \times \mathbb{R} \) be such that

\[
\int_{\Sigma} (\partial_n f) e^{-\frac{2}{\pi \varepsilon} f} d\sigma = B^* \ h p^* \ e^{-\frac{2}{\pi \varepsilon} f^*} (1 + O(h)).
\]

Then, one obtains in the limit \( h \to 0 \)

\[
\int_{\Sigma} (\partial_n u_h) e^{-\frac{2}{\pi \varepsilon} f} d\sigma = -\frac{2B^* (\det \text{Hess}(f(x_0)))^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} h p^* - \frac{2}{4} e^{-\frac{2}{\pi \varepsilon} f} (2f^* - f(z_0)) (1 + O(h))
\]

and

\[
\mathbb{P}^{\nu_h} [X_{\tau_2} \in \Sigma] = \frac{B^*}{\pi^{\frac{1}{2}}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{(\det \text{Hess}(f_{\partial \Omega}(z_k)))^{-1}} \right)^{-1} h p^* - \frac{2}{4} e^{-\frac{2}{\pi \varepsilon} f} (f^* - f(z_1)) (1 + O(h)).
\]

In practice, the expansion (33) is given by Laplace’s method. Theorem 2 is a generalization of Theorem 1. Indeed, (29)-(30) is weaker than (20)-(21) ((20)-(21) implies (29)-(30) for \( k_0 = n \) and \( f^* = f(z_n) \)) and item 2 gives an asymptotic result on the exit probability through \( \Sigma \subset B_{z_{j_0}} \) even if \( z_{j_0} \notin \Sigma \).

As an illustration, let us state a corollary of this theorem, which demonstrates the importance of obtaining a precise asymptotic result including the prefactors. Let us consider a simple situation with only two local minima \( z_1 \) and \( z_2 \) on the boundary, with \( f(z_1) < f(z_2) \). Let us now compare the two exit probabilities (see Figure 6 for a schematic representation of the geometric setting):

- The probability to leave through \( \Sigma_2 \) such that \( \Sigma_2 \subset B_{z_2} \) and \( z_2 \in \Sigma_2 \);
- The probability to leave through \( \Sigma \) such that \( \Sigma \subset B_{z_1} \) and \( \inf_{\Sigma} f = f(z_2) \).

By classic results from the large deviation theory (see for example [8]) the probability to exit through \( \Sigma \) and \( \Sigma_2 \) both scale like a prefactor times \( e^{-\frac{2}{\pi \varepsilon} (f(z_2) - f(z_1))} \), the difference can only be read from the prefactors. Actually, using item 2 in Theorem 2 one obtains the existence of \( C > 0 \) such that in the limit \( h \to 0 \) (see Corollary 11 below),

\[
\frac{\mathbb{P}^{\nu_h} [X_{\tau_2} \in \Sigma]}{\mathbb{P}^{\nu_h} [X_{\tau_2} \in \Sigma_2]} \sim C \sqrt{h}.
\]
The probability to leave through $\Sigma_2$ (namely through the generalized saddle point $z_2$) is thus larger than through $\Sigma$, even though the two regions are at the same height. This result explains why the local minima of $f$ on the boundary (namely the generalized saddle points) play such an important role when studying the exit event. Let us now state the precise result.

**Corollary 11.** Assume [H1], [H2], [H3]. Assume that $f|_{\partial \Omega}$ has only two local minima $z_1$ and $z_2$ such that $f(z_1) < f(z_2)$ and,

- for $j \in \{1, 2\}$,
  \[
  \inf_{z \in B_{\epsilon_j}} d_\alpha(z, z_j) > f(z_2) - f(z_1),
  \]
- \[
  f(z_1) - f(x_0) > f(z_2) - f(z_1).
  \]

Let $\Sigma \subset \partial \Omega$ be a smooth open set such that $\Sigma \subset B_{z_2}$. Assume moreover that $\inf_{\Sigma} f = f(z_2)$ and that the infimum is attained at a single point $z^*$: $\inf_{\Sigma} f = f(z^*)$ (necessarily $z^* \in \partial \Sigma$). Finally, let us assume that $z^*$ is a non degenerate minimum of $f|_{\partial \Omega}$ and $\partial_{\nu(\partial \Sigma)} f|_{\partial \Sigma}(z^*) < 0$ where $\nu(\partial \Sigma)$ is the unit outward normal to $\partial \Sigma \subset \partial \Omega$. Then, one has the following asymptotic expansion of $\mathbb{P}_h^\omega [X_{\tau_1} \in \Sigma]$ in the limit $h \to 0$:

\[
\mathbb{P}_h^\omega [X_{\tau_1} \in \Sigma] = -\frac{\sqrt{h}}{2(\pi)^{\frac{d}{2}}} \frac{\partial_n f(z^*)}{\partial_n f(z^*) \sqrt{\det \text{Hess} f|_{\partial \Sigma}(z^*)}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess} f|_{\partial \Omega}(z_k)}} \right)^{-1} \times e^{-\frac{h}{2}(f(z_2) - f(z_1))}(1 + O(h)),
\]

with by convention, $\det \text{Hess} f|_{\partial \Sigma}(z^*) = 1$ if $d = 2$.

Corollaries 8, 10 and 11 imply the result (36) announced above.

**Remark 3.** By using Laplace’s method, one can check that the asymptotic results obtained in Corollaries 8, 10 and 11 on the law of $X_{\tau_1}$ can be obtained by writing that the density of $X_{\tau_1}$ with respect to the Lebesgue measure on $\partial \Omega$ is, in the limit $h \to 0$,

\[
z \in \partial \Omega \mapsto \frac{\partial_n f(z) e^{-\frac{h}{2}f(z)}}{\int_{\partial \Omega} \partial_n f e^{-\frac{h}{2}f} d\sigma}(1 + O(h)).
\]

This indeed yields exactly the same asymptotic results on the exit distribution. This is reminiscent of previous results obtained in [42, 59], where the authors proved that, starting from a deterministic initial condition in $\Omega$, $X_{\tau_1}$ has a density with respect to the Lebesgue measure on $\partial \Omega$ which satisfies, in the limit $h \to 0$, $z \in \partial \Omega \mapsto \frac{\partial_n f(z) e^{-\frac{h}{2}f(z)}}{\int_{\partial \Omega} \partial_n f e^{-\frac{h}{2}f} d\sigma} + o(1)$, which is a less precise estimate than (39).

### 1.7 Strategy on the proof of Theorem 1

The aim of this section is to give an overview of the strategy for the proof of Theorem 1. In view of (22), we would like to identify the asymptotic behavior of the normal
derivative $\partial_n u_h$ on $\partial \Omega$ in the limit $h \to 0$. We recall that $(\lambda_h, u_h)$ are defined by the eigenvalue problem (12). By differentiating (12), we observe that $\nabla u_h$ satisfies

\begin{align}
\begin{cases}
  L_{f,h}^{(1)} \nabla u_h = -\lambda_h \nabla u_h & \text{on } \Omega, \\
  \nabla_T u_h = 0 & \text{on } \partial \Omega, \\
  \left(\frac{h}{2} \text{div} - \nabla f \right) \nabla u_h = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align}

where

$$L_{f,h}^{(1)} = \frac{h}{2} \Delta - \nabla f \cdot \nabla - \text{Hess } f$$

is an operator acting on 1-forms (namely on vector fields). Therefore $\nabla u_h$ is an eigen-1-form of the operator $-L_{f,h}^{(1)}$ with tangential Dirichlet boundary conditions (see (40)), associated with the small eigenvalue $\lambda_h$.

It is known (see for example [34]) that in our geometric setting, $-L_{f,h}^{D,(0)}(\Omega)$ admits exactly one eigenvalue smaller than $\frac{\sqrt{n}}{h}$, namely $\lambda_h$ with associated eigenfunction $u_h$ (this is because $f$ has only one local minimum in $\Omega$) and that $-L_{f,h}^{D,(1)}(\Omega)$ admits exactly $n$ eigenvalues smaller than $\frac{\sqrt{n}}{h}$ (where, we recall, $n$ is the number of local minima of $f$ on $\partial \Omega$). Actually, all these small eigenvalues are exponentially small in the regime $h \to 0$, the other eigenvalues being bounded from below by a constant in this regime. The idea is then to construct an appropriate basis (with so called quasi-modes, which are localized on the generalized saddle points $(z_i)_{i=1,...,n}$) of the eigenspace associated with small eigenvalues for $L_{f,h}^{D,(1)}(\Omega)$, and then to decompose $\nabla u_h$ along this basis.

The article is organized as follows. In Section 2 we introduce the general setting for the proof of our results, and the Gram-Schmidt procedure which allows, starting from a set of quasi-modes, to compute the projection of (an approximation of) $\nabla u_h$ along the quasi-modes. In order to to quantify the distance between the space spanned by these quasi-modes and the eigenspace of $L_{f,h}^{D,(1)}(\Omega)$ associated with small eigenvalues, we need to use so-called Agmon estimates. Section 3 is devoted to a presentation of the main properties of the Agmon distance which intervenes in these estimates. The most technical part is the effective construction of the quasi-modes using auxiliary simpler eigenvalue problems associated with each of the local minima $(z_i)_{i=1,...,n}$. This is explained in Section 4 which concludes the proof of Theorem 1. Finally, Section 5...
concludes the paper by providing the proofs of all the other results stated above, in particular Theorem 2.

2 General setting and strategy for the proof of Theorem 1

The general setting for proving the results presented in Section 1 will be the following: $\overline{\Omega}$ is a $C^\infty$ oriented connected compact Riemannian manifold of dimension $d$ with boundary $\partial \Omega$ and the function $f$ is a $C^\infty$ real valued function defined on $\overline{\Omega}$. One defines $\Omega := \overline{\Omega} \setminus \partial \Omega$. In particular, Theorem 1 will actually be proven in this framework. Notice that the assumptions [H1], [H2] and [H3] are still meaningful in this more general setting.

In order to use previous results from the literature on semi-classical analysis, we will transform the original problem (12) on $(\Omega, L)$ to an eigenvalue problem on the standard (non-weighted) Hilbert spaces $H^q(\Omega)$, by using a unitary transformation which relates the operator $L_{f,h}$ to the Witten Laplacians $\Delta_{f,h}$. This is explained in Section 2.1 together with some first well-known results on the spectrum of Witten Laplacians. Then, in Section 2.2, we explain what are the requirements on the quasi-modes we will build in order to obtain the estimate (22), see Proposition 17. Section 2.3 is finally devoted to the proof of Proposition 17.

2.1 Witten Laplacians

2.1.1 Notations for Sobolev spaces

For $p \in \{0, \ldots, d\}$, one denotes by $\Lambda^pC^\infty(\overline{\Omega})$ the space of $C^\infty$ $p$-forms on $\overline{\Omega}$. Moreover, $\Lambda^pC^\infty(\Omega)$ is the set of $C^\infty$ $p$-forms $v$ such that $tv = 0$ on $\partial \Omega$, where $t$ denotes the tangential trace on forms. Likewise, the set $\Lambda^pC^\infty(\Omega)$ is the set of $C^\infty$ $p$-forms $v$ such that $n v = 0$ on $\partial \Omega$, where $n$ denotes the normal trace on forms defined by: for all $w \in \Lambda^pC^\infty(\overline{\Omega})$, $nw := w|_{\partial \Omega} - tw$.

For $p \in \{0, \ldots, d\}$ and $q \in \mathbb{N}$, one denotes by $\Lambda^pH^q_w(\Omega)$ the weighted Sobolev spaces of $p$ forms with regularity index $q$, for the weight $e^{-\frac{2}{\hbar}f(x)}$ on $\Omega$: $v \in \Lambda^pH^q_w(\Omega)$ if and only if for all multi-index $\alpha$ with $|\alpha| \leq p$, the $\alpha$ derivative of $v$ is in $\Lambda^pL^2_w(\Omega)$ where $\Lambda^pL^2_w(\Omega)$ is the completion of the space $\Lambda^pC^\infty(\Omega)$ for the norm $w \in \Lambda^pC^\infty(\Omega) \mapsto \int_\Omega |w|^2 e^{-\frac{2}{\hbar}f}$.

See for example [65] for an introduction to Sobolev spaces on manifolds with boundaries. For $p \in \{0, \ldots, d\}$ and $q > \frac{1}{2}$, the set $\Lambda^pH^q_{w,T}(\Omega)$ is defined by $\Lambda^pH^q_{w,T}(\Omega) := \{ v \in \Lambda^pH^q_w(\Omega) \mid tv = 0 \text{ on } \partial \Omega \}$.

Notice that $\Lambda^pL^2_w(\Omega)$ is the space $\Lambda^pH^0_w(\Omega)$ and that $\Lambda^0H^1_{w,0}(\Omega)$ is the space $H^1_{w,0}(\Omega)$ than we introduced in Proposition 2 to define the domain of $L_{f,h}^{D,0}(\Omega)$. Likewise for $p \in \{0, \ldots, d\}$ and $q > \frac{1}{2}$, the set $\Lambda^pH^q_{w,T}(\Omega)$ is defined by $\Lambda^pH^q_{w,N}(\Omega) := \{ v \in \Lambda^pH^q_w(\Omega) \mid nw = 0 \text{ on } \partial \Omega \}$.

We will denote by $\| \cdot \|_{H^q_w}$ the norm on the weighted space $\Lambda^pH^q_w(\Omega)$. Moreover $\langle \cdot, \cdot \rangle_{L^2_w}$ denotes the scalar product in $\Lambda^pL^2_w(\Omega)$.

Finally, we will also use the same notation without the index $w$ to denote the standard Sobolev spaces defined with respect to the Lebesgue measure on $\Omega$.
2.1.2 Definition of Witten Laplacians

Let us first recall some basic properties of Witten Laplacians, as well as the link between those and the operators $L_p^{(p)}$ introduced above (see (10) and (41)). As already explained above, we will actually need in this article to work only with 0 and 1-forms ($p \in \{0, 1\}$).

For an introduction to the Hodge theory and the Hodge Laplacians on manifolds with boundary, one can refer to [65].

Denote by $d$ the exterior derivative on $\Omega$,

$$d^{(p)} : \Lambda^p C^\infty(\Omega) \to \Lambda^{p+1} C^\infty(\Omega),$$

and $(d^{(p)})^*$ its adjoint. The exterior derivative is 2 nilpotent,

$$d^{(p+1)} \circ d^{(p)} = 0,$$

and therefore $(d^{(p)})^* \circ (d^{(p+1)})^* = 0$. In all what follows, the superscript $(p)$ may be omitted when there is no ambiguity.

Let us now introduce the so called distorted exterior derivative

$$d^{(p)}_{f,h} := e^{-f} h d^{(p)} e^f = h d^{(p)} + df \wedge$$

and its formal adjoint

$$(d^{(p)}_{f,h})^* := e^f h (d^{(p)})^* e^{-f} = h (d^{(p)})^* + i \nabla f.$$

The distorted exterior derivative was firstly introduced by Witten in [74].

**Definition 5.** The Witten Laplacian is the non negative differential operator

$$\Delta^{(p)}_{f,h} := \left( d^{(p)}_{f,h} + (d^{(p)}_{f,h})^* \right)^2.$$

An equivalent expression of the Witten Laplacians is

$$\Delta^{(p)}_{f,h} = h^2 \Delta^{(p)}_H + |\nabla f|^2 + h \left( {\mathcal{L}} \nabla f + \nabla^* \nabla f \right),$$

where $L$ stands for the Lie derivative, $\nabla$ is the covariant derivative associated to the metric $g$ on $\Omega$ and $\Delta^{(p)}_H$ is the Hodge Laplacian acting on $p$-forms, defined by:

$$\Delta^{(p)}_H := \left( d^{(p)} + (d^{(p)})^* \right)^2.$$

We recall that $\Delta^{(p)}_H$ is a positive operator ( in $\mathbb{R}^d$, $\Delta^{(0)}_H = -\sum_{i=1}^d \partial^2_{x_i, x_i}$). The operator $L \nabla f + \nabla^* \nabla f$ is an operator of order 0 (namely a multiplicative operator). On 0-forms, namely on functions, the Witten Laplacian has the following expression

$$\Delta^{(0)}_{f,h} = h^2 \Delta^{(0)}_H + |\nabla f|^2 + h \Delta^{(0)}_H f.$$

Let us recall the complex structure associated with the Witten Laplacians defined on a manifold with boundary. This requires to use appropriate boundary conditions (see [34]).
Indeed, one can check that \( \Delta^p f, h \) on \( \Omega \) is the operator \( \Delta^D f, h (\Omega) \) with domain

\[
D \left( \Delta^D f, h (\Omega) \right) = \left\{ v \in \Lambda^p H^2(\Omega) \mid tv = 0, \ t d^* f, h v = 0 \right\}.
\]

The Neumann realization of \( \Delta^p f, h \) on \( \Omega \) is the operator \( \Delta^N f, h (\Omega) \) with domain

\[
D \left( \Delta^N f, h (\Omega) \right) = \left\{ v \in \Lambda^p H^2(\Omega) \mid nv = 0, \ n d f, h v = 0 \right\}.
\]

The operators \( \Delta^D f, h (\Omega) \) and \( \Delta^N f, h (\Omega) \) are both self-adjoint operators with compact resolvent.

We recall that \( t \) denotes the tangential trace on forms and that \( n \omega = \omega - tw \) is the normal trace. The proof of Proposition 12 can be found in [34, Section 2.4] and in [47, Section 4.2.3]. It is a generalization of what is stated in [65] for the Hodge Laplacians. One can check that the operator \( \Delta^D f, h (\Omega) \) is actually the Friedrichs extension associated to the quadratic form

\[
v \in \Lambda^p H^1(\Omega) \mapsto \left\| d f, h v \right\|^2_{L^2} + \left\| (d f, h)^* v \right\|^2_{L^2}.
\]

The following properties are easily checked

\[
d f, h \Delta f, h (\Omega) = \Delta f, h (\Omega) d f, h \quad \text{and} \quad d^* f, h \Delta f, h (\Omega) = \Delta f, h (\Omega) d^* f, h.
\]

Similar relations hold for \( \Delta^N f, h (\Omega) \). The relations (46) allow us to define the Dirichlet complex structure (see [10], [34] and [45]):

\[
\{0\} \rightarrow D \left( \Delta f, h (0) \right) \xrightarrow{d f, h} D \left( \Delta f, h (1) \right) \xrightarrow{d f, h} \cdots \xrightarrow{d f, h} D \left( \Delta f, h (d) \right) \rightarrow \{0\}
\]

\*

\[
\{0\} \xleftarrow{d^* f, h} D \left( \Delta f, h (0) \right) \xleftarrow{d^* f, h} D \left( \Delta f, h (1) \right) \xleftarrow{d^* f, h} \cdots \xleftarrow{d^* f, h} D \left( \Delta f, h (d) \right) \leftarrow \{0\}.
\]

One can define similarly the Neumann complex structure (see [47]).

One can relate the infinitesimal generator \( L_{f, h}^{(0)} \) of the diffusion (1) to the Witten Laplacian \( \Delta_{f, h}^{(0)} \) through the unitary transformation:

\[
\phi \in L^2_w(\Omega) \mapsto e^{-\frac{t}{\hbar}} \phi \in L^2(\Omega).
\]

Indeed, one can check that

\[
\Delta_{f, h}^{(0)} = \left. -2 \hbar e^{-\frac{t}{\hbar}} L_{f, h}^{(0)} (0) e^{\frac{t}{\hbar}} \right\|_{L^2_w(\Omega)}.
\]

Let us now generalize this to \( p \)-forms, using extensions of \( L_{f, h}^{(0)} \) to \( p \)-forms.

**Proposition 13.** The Friedrichs extension associated with the quadratic form

\[
v \in \Lambda^p C_0^\infty(\Omega) \mapsto \frac{h}{2} \left[ \left\| d f, h v \right\|^2_{L^2(\Omega)} + \left\| e^{\frac{2t}{\hbar}} (d f, h)^* v \right\|^2_{L^2(\Omega)} \right]
\]

on \( \Lambda^p L^2_w(\Omega) \), is denoted \( -L_{f, h}^{D, (p)} (\Omega) \). The operator \( -L_{f, h}^{D, (p)} (\Omega) \) is a positive unbounded self-adjoint operator on \( \Lambda^p L^2_w(\Omega) \). Besides, one has

\[
D \left( -L_{f, h}^{D, (p)} (\Omega) \right) = \left\{ v \in \Lambda^p H^2_w(\Omega) \mid tv = 0, \ t d^* \left( e^{-\frac{2t}{\hbar}} v \right) = 0 \right\}.
\]
For \( p = 0 \), the differential operator

\[
L_{f,h}^{(0)} = -\frac{h}{2} \Delta_{H}^{(0)} - \nabla f : \nabla
\]

is the infinitesimal generator \(^{(10)}\) of the overdamped Langevin dynamics \(^{(1)}\). For \( p = 1 \) one gets

\[
L_{f,h}^{(1)} = -\frac{h}{2} \Delta_{H}^{(1)} - \nabla f : \nabla - \text{Hess } f,
\]

where we recall \( \text{Hess } f \) is the Hessian of \( f \), see Remark \(^{(1)}\).

The generalisation of \(^{(47)}\) for \( p \)-forms is:

\[
\Delta_{f,h}^{D,(p)}(\Omega) = -2h e^{-\frac{f}{h}} \left(L_{f,h}^{D,(p)}(\Omega)\right) e^{-\frac{f}{h}}.
\]

The commutation relation \(^{(46)}\) writes on \( L_{f,h}^{D,(p)}(\Omega) \):

\[
L_{f,h}^{D,(p+1)}(\Omega) d = dL_{f,h}^{D,(p)}(\Omega) \text{ and } L_{f,h}^{D,(p-1)}(\Omega) d_{2f,h} = d_{2f,h} L_{f,h}^{D,(p)}(\Omega).
\]

Thanks to the relation \(^{(49)}\), the operators \( L_{f,h}^{D,(p)}(\Omega) \) and \( \Delta_{f,h}^{D,(p)}(\Omega) \) have the same spectral properties. In particular the operators \( L_{f,h}^{D,(p)}(\Omega) \) and \( \Delta_{f,h}^{D,(p)}(\Omega) \) both have compact resolvents, and thus a discrete spectrum. The generalization of Proposition \(^{(3)}\) is the following:

**Proposition 14.** The smallest eigenvalue of \(-L_{f,h}^{D,(0)}(\Omega)\), denoted by \( \lambda_{h} \), is positive and non degenerate. The associated eigenfunction \( u_{h} \) has sign on \( \Omega \). Moreover \( u_{h} \in C^{\infty}(\overline{\Omega}, \mathbb{R}) \).

Without loss of generality, one can assume that \( u_{h} \) satisfies \(^{(11)}\). The couple \((\lambda_{h}, u_{h})\) satisfies the following relation

\[
\begin{cases}
-L_{f,h}^{(0)} u_{h} = \lambda_{h} u_{h} \text{ on } \Omega, \\
u_{h} = 0 \text{ on } \partial \Omega.
\end{cases}
\]

Thanks to the relation \(^{(49)}\), the couple \((\mu_{h}, v_{h}) := \left(2h\lambda_{h}, u_{h} e^{-\frac{f}{h}}\right)\) is the first eigenvalue and eigenfunction of \( \Delta_{f,h}^{D,(0)}(\Omega) \). The couple \((\mu_{h}, v_{h})\) satisfies

\[
\begin{cases}
\Delta_{f,h}^{(0)} v_{h} = \mu_{h} v_{h} \text{ on } \Omega, \\
v_{h} = 0 \text{ on } \partial \Omega.
\end{cases}
\]

Moreover, \( v_{h} > 0 \) on \( \Omega \) and

\[
\int_{\Omega} v_{h}^{2}(x) \, dx = 1.
\]

The following lemma will be instrumental throughout this work.

**Lemma 15.** Let \((A, D(A))\) be a non negative self adjoint operator on a Hilbert Space \((\mathcal{H}, \|\cdot\|)\) with associated quadratic form \(q_{A}(u) = \langle u, Au \rangle\) with domain \(D(q_{A})\). Then for any \( u \in D(q_{A}) \) and \( b > 0 \)

\[
\left\| \pi_{b, +\infty}(A) u \right\|^{2} \leq \frac{q_{A}(u)}{b}
\]

where, for \( E \subset \mathbb{R} \) a Borel set, \( \pi_{E}(A) \) is the spectral projection of the operator \( A \) on \( E \).
Proposition 16. Under assumptions [H1], there are the local minima (\( f \) of index 1). A generalized critical point of index 1 for \( \Delta_{D,h} \), the number of local minima of \( f \) all \( \{ \) \( \) \( \) \( \) can actually be shown that, when \( h \) and \( \Delta \), the number of local minima of \( f \) and \( \Delta \) in the regime \( h \to 0 \) have already been studied in [34, Section 3]: when \( \nabla f \neq 0 \) on \( \partial \Omega \) and when \( f \) and \( f_{|\partial \Omega} \) are Morse functions, the dimension of \( \text{Ran} \left( \pi_{[0,h^{3/2}]}(\Delta_{f,h}^{D,(0)}(\Omega)) \right) \) (respectively \( \text{Ran} \left( \pi_{[0,h^{3/2}]}(\Delta_{f,h}^{D,(1)}(\Omega)) \right) \)) is equal to the number of local minima of \( f \) (respectively to the number of generalized critical points of index 1). A generalized critical point of index 1 for \( \Delta_{f,h}^{D,(1)}(\Omega) \) is either a local minimum of \( f \) on \( \partial \Omega \) such that \( \partial_h f(z_i) > 0 \) or a saddle point of index 1 of \( f \) inside \( \Omega \). In our setting, thanks to assumptions [H1], [H2] and [H3], there are \( n \) critical points of index 1, which are the local minima \( (z_i)_{i=1,...,n} \) of \( f_{|\partial \Omega} \).

Proposition 16. Under [H1], [H2], and [H3], there exists \( h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[ \dim \text{Ran} \pi_{[0,h^{3/2}]}(\Delta_{f,h}^{D,(0)}(\Omega)) = 1, \]

and

\[ \dim \text{Ran} \pi_{[0,h^{3/2}]}(\Delta_{f,h}^{D,(1)}(\Omega)) = n. \]

Proof. We refer to [34, Theorem 3.2.3] for the proof of this proposition. 

Let us denote by \( \mu_h^{(1),1} \leq \ldots \leq \mu_h^{(1),n} \) the eigenvalues of \( \Delta_{f,h}^{D,(1)}(\Omega) \) smaller than \( h^{3/2} \).

It can actually be shown that, when \( h \to 0 \), they are exponentially small: for \( j \in \{1,...,n\} \),

\[ \lim_{h \to 0} h \ln(\mu_h^{(1),j}) < 0. \]

Thanks to [49], similar results hold for \( L_{f,h}^{D,(p)}(\Omega) \): there exists \( h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[ \dim \text{Ran} \pi_{[0,\sqrt{h}]}(-L_{f,h}^{D,(0)}(\Omega)) = 1 \quad \text{and} \quad \dim \text{Ran} \pi_{[0,\sqrt{h}]}(-L_{f,h}^{D,(1)}(\Omega)) = n. \]

The spectral projection \( \pi_{[0,\sqrt{h}]}(-L_{f,h}^{D,(0)}(\Omega)) \) is the orthogonal projection in \( L^2_{\partial \Omega}(\Omega) \) on \( \text{span}(u_0) \) and thanks to the commutation property [50], we have the following crucial property:

\[ \nabla u_h \in \text{Ran} \pi_{[0,\sqrt{h}]}(-L_{f,h}^{D,(1)}(\Omega)). \]
For the ease of notation, for \( p \in \{1, 2\} \), we use in the following the notation:

\[
\pi_h^{(p)} := \pi_{(0, \Sigma)} \left( -L_{f,h}^{D,(p)}(\Omega) \right).
\]  

(54)

### 2.2 Statement of the assumptions required for the quasi-modes

#### 2.2.1 Statement of Proposition [17]

The next proposition gives the assumption we need on the quasi-modes \((\tilde{\psi}_i)_{i=1,\ldots,n}\) whose span approximates \(\text{Ran} \pi_h^{(1)}\), and \(\bar{u}\) whose span approximates \(\text{Ran} \pi_h^{(0)}\), in order to prove Theorem [1].

**Proposition 17.** Assume \([H1], [H2] \) and \([H3]\). As in the statement of Theorem [1], for all \(i \in \{1, \ldots, n\} \), \(\Sigma_i\) denotes an open set included in \(\partial \Omega\) containing \(z_i\) and such that \(\Sigma_i \subset B_{z_i}\).

Let us assume in addition that there exist \(n\) quasi-modes \((\tilde{\psi}_i)_{i=1,\ldots,n}\) and a family of quasi-modes \((\bar{u}_i = \bar{u}_\delta)_{\delta > 0}\) satisfying the following conditions:

1. For all \(i \in \{1, \ldots, n\}\), \(\tilde{\psi}_i \in \Lambda^1 H^1_{w,T}(\Omega)\) and \(\bar{u} \in \Lambda^0 H^1_{w,T}(\Omega)\). The function \(\bar{u}\) is non negative in \(\Omega\). Moreover, one assumes the following normalization: for all \(i \in \{1, \ldots, n\}\),

\[
\left\| \tilde{\psi}_i \right\|_{L^2_\omega} = \left\| \bar{u} \right\|_{L^2_\omega} = 1.
\]

2. (a) There exists \(\varepsilon_1 > 0\) such that for all \(i \in \{1, \ldots, n\}\), in the limit \(h \to 0\):

\[
\left\| \left(1 - \pi_h^{(1)} \right) \tilde{\psi}_i \right\|_{H^1_\omega}^2 = O \left(e^{-\frac{\varepsilon_1}{2}(\max(f(z_i)-f(z_0),f(z_i)-f(z_0)+\varepsilon_1))}\right).
\]

(55)

(b) For any \(\delta > 0\), in the limit \(h \to 0\):

\[
\left\| \nabla \bar{u} \right\|_{L^1_\omega}^2 = O \left(e^{-\frac{\varepsilon_1}{2}(f(z_i)-f(z_0)-\delta)}\right).
\]

3. There exists \(\varepsilon_0 > 0\) such that \(\forall (i, j) \in \{1, \ldots, n\}^2\) with \(i < j\), in the limit \(h \to 0\):

\[
\left\langle \tilde{\psi}_i, \tilde{\psi}_j \right\rangle_{L^2_\omega} = O \left(e^{-\frac{\varepsilon_1}{2}(f(z_i)-f(z_0)+\varepsilon_0)}\right).
\]

4. (a) There exist constants \((C_i)_{i=1,\ldots,n}\) and \(p\) which do not depend on \(h\) such that for all \(i \in \{1, \ldots, n\}\), in the limit \(h \to 0\):

\[
\left\langle \nabla \bar{u}, \tilde{\psi}_i \right\rangle_{L^2_\omega} = C_i \ h^p e^{-\frac{\varepsilon_1}{2}(f(z_i)-f(z_0))} (1 + O(h)).
\]

(b) There exist constants \((B_i)_{i=1,\ldots,n}\) and \(m\) which do not depend on \(h\) such that for all \((i, j) \in \{1, \ldots, n\}^2\), in the limit \(h \to 0\):

\[
\int_{\Sigma_i} \tilde{\psi}_j \cdot n \ e^{-\frac{\varepsilon_1}{2} f} d\sigma = \begin{cases} B_i \ h^m e^{-\frac{\varepsilon_1}{2} f(z_i)} (1 + O(h)) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

Then, for all \(i \in \{1, \ldots, n\}\), in the limit \(h \to 0\):

\[
\int_{\Sigma_i} (\partial_\nu u_h) e^{-\frac{\varepsilon_1}{2} f} d\sigma = C_i B_i \ h^{p+m} e^{-\frac{\varepsilon_1}{2}(f(z_i)-f(z_0))} (1 + O(h)),
\]

where \(u_h\) is the eigenfunction associated with the smallest eigenvalue of \(-L_{f,h}^{D,(0)}(\Omega)\) (see Proposition [14]) which satisfies [11].

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Let us comment on the assumptions made on the quasi-modes. Assumption 1 gives the proper functional setting and the normalization. Assumption 2 will be used to show that \( \text{span}(\tilde{\psi}_i, i = 1, \ldots, n) \) (respectively \( \text{span}(\tilde{u}) \)) is included in \( \text{Ran}(\pi_h^{(1)}) \) (respectively in \( \text{Ran}(\pi_h^{(0)}) = \text{span}(u_h) \)) up to exponentially small terms. Assumption 3 states the quasi-orthonormality of the quasi-modes \( \{\tilde{\psi}_i\}_{i=1,\ldots,n} \). Finally, Assumption 4(a) gives the components of the decomposition of \( \nabla \tilde{u} \) over \( \text{span}(\tilde{\psi}_i, i = 1, \ldots, n) \), and Assumption 4(b) is then used to find the asymptotic behavior of \( \int_{\Sigma} (\partial_n u_h) e^{-\frac{1}{h}f} d\sigma \), knowing those of \( \int_{\Sigma} \tilde{\psi}_j \cdot n e^{-\frac{1}{h}f} d\sigma \).

Theorem 1 is a consequence of this proposition. The construction of appropriate quasi-modes \( \tilde{\psi} \) and \( \{\tilde{\psi}_i\}_{i=1,\ldots,n} \) satisfying the requirements of Proposition 17 will be the focus of Section 4, where explicit values for the constants \( B_i, C_i, p \) and \( m \) will be given (see (31) and (35)).

### 2.2.2 Rewriting the hypotheses on \( \{\tilde{\psi}_i\}_{i\in\{1,\ldots,n\}} \) in Proposition 17

The quasi-modes \( \{\tilde{\psi}_i\}_{i\in\{1,\ldots,n\}} \) will be built using eigenforms of some Witten Laplacians. It will thus be more convenient to work in non weighted Sobolev spaces, and to actually consider the 1-forms (see (47)): for \( i \in \{1, \ldots, n\} \),

\[
\tilde{\phi}_i := e^{-\frac{1}{h}f} \tilde{\psi}_i \in \Lambda^1 H^1_T(\Omega).
\]

For further references, let us rewrite the hypotheses on the 1-forms \( \tilde{\psi}_j \) stated in Proposition 17 in terms of the 1-forms \( \tilde{\phi}_j \) defined by (56):

1. For all \( i \in \{1, \ldots, n\} \), \( \tilde{\phi}_i \in \Lambda^1 H^1_T(\Omega) \) and
   \[
   \|\tilde{\phi}_i\|_{L^2} = 1.
   \]

2. There exist \( \varepsilon_1 > 0 \) such that for all \( i \in \{1, \ldots, n\} \), in the limit \( h \to 0 \):
   \[
   \left\| \left( 1 - \pi_{[0,h]^{2}} \right) \left( \Delta_{T,h}^{(1)}(\Omega) \right) \tilde{\phi}_i \right\|_{H^1}^2 = O \left( e^{-\frac{1}{h}(\max|f(z_i)-f(z_j),f(z_i)-f(z_j)|+\varepsilon_1)} \right).
   \]

3. There exists \( \varepsilon_0 > 0 \) such that for all \( i, j \in \{1, \ldots, n\}^2 \), \( i < j \), in the limit \( h \to 0 \):
   \[
   \int_{\Omega} \tilde{\phi}_i(x) \cdot \tilde{\phi}_j(x) \, dx = O \left( e^{-\frac{1}{h}|f(z_j)-f(z_i)|+\varepsilon_0} \right).
   \]

4. (a) There exist constants \( (C_i)_{i=1,\ldots,n} \) and \( p \) which do not depend on \( h \) such that for all \( i \in \{1, \ldots, n\} \), in the limit \( h \to 0 \):
   \[
   \int_{\Omega} \nabla \tilde{u} \cdot \tilde{\phi}_i \; e^{-\frac{1}{h}f} = C_i \; h^p e^{-\frac{1}{h}(f(z_i)-f(x_0))} \left( 1 + O(h) \right).
   \]

(b) There exist constants \( (B_i)_{i=1,\ldots,n} \) and \( m \) which do not depend on \( h \) such that for all \( i, j \in \{1, \ldots, n\}^2 \), in the limit \( h \to 0 \):
   \[
   \int_{\Sigma_i} \tilde{\phi}_j \cdot n \; e^{-\frac{1}{h}f} d\sigma = \begin{cases} B_i \; h^m \; e^{-\frac{1}{h}f(z_i)} \left( 1 + O(h) \right) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
   \]
As mentioned above, the construction of quasi-modes \( \tilde{u} \) and \( (\tilde{\phi}_i)_{i=1,\ldots,n} \) satisfying those estimates will be the purpose of Section 4.

Let us comment on the equivalence between the first assumption here (namely (57)) and assumption 2(a) in Proposition 17 (namely (55)). This equivalence is a consequence of the following relation between the projectors which comes from (49):

\[
\pi_{(0,h^2)}^3 \left( \Delta_{f,h}^{D,1}(\Omega) \right) = e^{-\frac{1}{2}} f \frac{1}{\pi_h} e^{\frac{1}{2}} f.
\]

Indeed, using this relation, one has:

\[
\| e^{-\frac{1}{2}} f (1 - \pi_h^{(1)}) \tilde{\psi}_i \|_{H^1} = \left\| (1 - \pi_{(0,h^2)}^3 \left( \Delta_{f,h}^{D,1}(\Omega) \right)) \tilde{\phi}_i \right\|_{H^1}.
\]

Moreover, one easily checks that there exists \( C > 0 \) such that, for all \( h \in (0, 1) \) and for all \( u \in \Lambda^p H^1(\Omega) \),

\[
\| u \|_{H^1_+} \leq \frac{C}{h} \left\| u e^{-\frac{1}{2}} f \right\|_{H^1}.
\]

Therefore

\[
\| (1 - \pi_h^{(1)}) \tilde{\psi}_i \|_{H^1_+} \leq \frac{C}{h} \left\| (1 - \pi_{(0,h^2)}^3 \left( \Delta_{f,h}^{D,1}(\Omega) \right)) \tilde{\phi}_i \right\|_{H^1},
\]

which shows that (57) (with \( \varepsilon_1 \)) implies (55) (with \( \varepsilon_1/2 \)). A similar reasoning shows that (55) also implies (57), but this will not be used in the following.

### 2.3 Proof of Proposition 17

In all this section, we assume that the hypotheses [H1], [H2] and [H3] hold and we assume the existence of \( n + 1 \) quasi-modes \( (\tilde{u}, (\tilde{\psi}_j)_{j=1,\ldots,n}) \) satisfying the conditions of Proposition 17. In the following, \( \varepsilon \) denotes a positive constant independent of \( h \), smaller than \( \min(\varepsilon_1, \varepsilon_0) \), and whose precise value may vary (a finite number of times) from one occurrence to the other.

Let us start the proof with two preliminary lemmas relating \( \tilde{u} \) with \( u_h \) on the one hand, and span \( (\tilde{\psi}_j, j = 1, \ldots, n) \) with \( \text{Ran} \pi_h^{(1)} \) on the other hand.

**Lemma 18.** Let us assume that the assumptions of Proposition 17 hold. Then there exist \( c > 0 \) and \( h_0 > 0 \) such that for \( h \in (0, h_0) \),

\[
\left\| \pi_h^{(0)} \tilde{u} \right\|_{L^2_w} = 1 + O \left( e^{-\frac{c}{h}} \right).
\]

**Proof.** Since \( \tilde{u} \in \Lambda^0 H^1_{w,T} \), \( \left\| (1 - \pi_h^{(0)}) \tilde{u} \right\|_{L^2_w} \) is bounded from above by \( \sqrt{2} h^{1/4} \| \nabla \tilde{u} \|_{L^2_w} \) thanks to Lemma 15. In addition since \( \| \nabla \tilde{u} \|_{L^2_w}^2 = O \left( e^{-\frac{c}{h}} |f(z_1) - f(x_0)| \right) \) (see assumption 2(b) in Proposition 17), by taking \( \delta \in (0, f(z_1) - f(x_0)) \), one gets that \( \left\| \pi_h^{(0)} \tilde{u} \right\|_{L^2_w} = 1 + O \left( e^{-\frac{c}{h}} \right). \)

As a direct consequence of Lemma 18, one has that for \( h \) small enough \( \pi_h^{(0)} \tilde{u} \neq 0 \) and therefore (since moreover \( \tilde{u} \) is non negative in \( \Omega \)): \( \langle u_h, \pi_h \tilde{u} \rangle_{L^2_w} = \langle u_h, \tilde{u} \rangle_{L^2_w} \geq 0 \),

\[
u_h = \frac{\pi_h^{(0)} \tilde{u}}{\left\| \pi_h^{(0)} \tilde{u} \right\|_{L^2_w}}.
\]

Additionally, one has the following lemma concerning the 1-forms.
Lemma 19. Let us assume that the assumptions of Proposition 17 hold. Then there exists \( h_0 \) such that for all \( h \in (0, h_0) \),

\[
\text{span} \left( \pi_h(1) \tilde{\psi}_i, i = 1, \ldots, n \right) = \text{Ran} \pi_h(1).
\]

Proof. The determinant of the Gram matrix of the 1-forms \( \left( \pi_h(1) \tilde{\psi}_i \right)_{i=1,\ldots,n} \) is \( 1 + O(e^{-c/h}) \) thanks to the following identity

\[
\langle \pi_h(1) \tilde{\psi}_i, \pi_h(1) \tilde{\psi}_j \rangle_{L^2_\nu} = -\left\langle \left( \pi_h(1) - 1 \right) \tilde{\psi}_i, \left( \pi_h(1) - 1 \right) \tilde{\psi}_j \right\rangle_{L^2_\nu} + \langle \tilde{\psi}_i, \tilde{\psi}_j \rangle_{L^2_\nu} \tag{62}
\]

and the fact that, from assumptions 1, 2(a) and 3 in Proposition 17 there exist \( h_0 > 0, c > 0 \) such that for \( h \in (0, h_0) \),

\[
\langle \tilde{\psi}_i, \tilde{\psi}_j \rangle_{L^2_\nu} = (1 - \delta_{i,j})O(e^{-\frac{c}{h}}) + \delta_{i,j} \quad \text{and} \quad \left\| 1 - \pi_h(1) \right\|^2_{H^1_\nu} = O \left( e^{-\frac{c}{h}} \right).
\]

Moreover, from Proposition 16, \( \dim \text{Ran} \pi_h(1) = n \). This proves Lemma 19.

Thanks to Lemma 19, one can build on orthonormal basis \( (\psi_i)_{i=1,\ldots,n} \) of \( \text{Ran} \pi_h(1) \) using a Gram-Schmidt orthonormalization procedure on \( (\pi_h(1) \tilde{\psi}_i)_{i=1,\ldots,n} \). This will be done in Section 2.3.1 below. Then, since

\[
\nabla u_h \in \text{Ran} \left( \pi_h(1) \right) = \text{span} (\psi_j, j = 1, \ldots, n)
\]

(see (63)) and \( \left\| \psi_j \right\|_{L^2_\nu} = 1 \), one has

\[
\int_{\Sigma_k} (\partial_n u_h) e^{-\frac{\xi}{f}} \, d\sigma = \sum_{j=1}^n \langle \nabla u_h, \psi_j \rangle_{L^2_\nu} \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{\xi}{f}} \, d\sigma. \tag{63}
\]

The proof of Proposition 17 then consists in replacing (in the right-hand side of (63)) \( u_h \) by its expression (61) in terms of \( \tilde{u} \), and the \( (\psi_i)_{i=1,\ldots,n} \) by the \( (\tilde{\psi}_i)_{i=1,\ldots,n} \), and to use the assumptions of Proposition 17 to get an estimate of \( \int_{\Sigma_k} (\partial_n u_h) e^{-\frac{\xi}{f}} \, d\sigma \). Section 2.3.2 provides estimates on \( \langle \nabla u_h, \psi_j \rangle_{L^2_\nu} \). Section 2.3.3 provides estimates of \( \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{\xi}{f}} \). All these results are then gathered to conclude the proof of Proposition 17 in Section 2.3.4.

2.3.1 Gram-Schmidt orthonormalization

Using a Gram-Schmidt procedure, one obtains the following result.

Lemma 20. Let us assume that the assumptions of Proposition 17 hold. Then for all \( j \in \{1, \ldots, n\} \), there exist \( (\kappa_{ji})_{i=1,\ldots,j-1} \subset \mathbb{R}^{j-1} \) such that the 1-forms

\[
v_j := \pi_h(1) \left[ \tilde{\psi}_j + \sum_{i=1}^{j-1} \kappa_{ji} \tilde{\psi}_i \right], \tag{64}
\]

(with the convention \( \sum_{i=1}^0 = 0 \)) satisfy

- for all \( k \in \{1, \ldots, n\} \), \( \text{span} (v_i, i = 1, \ldots, k) = \text{span} \left( \pi_h(1) \tilde{\psi}_i, i = 1, \ldots, k \right) \),

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Let us assume that the assumptions of Proposition 17 hold. Then there exist Lemma 22.

This will be used extensively in the following.

Let us prove this lemma by induction. Concerning

Proof. Let us introduce the notation: for all

The result is then a consequence of assumption 3 in Proposition 17 and the identity [62].

Notice that since \( \pi_h^{(1)} \) is an \( L^2_w \)-projection, \( \left\langle \pi_h^{(1)} \tilde{\psi}_i, \pi_h^{(1)} \tilde{\psi}_j \right\rangle_{L^2_w} = \left\langle \pi_h^{(1)} \tilde{\psi}_i, \tilde{\psi}_j \right\rangle_{L^2_w} \). This will be used extensively in the following.

Lemma 22. Let us assume that the assumptions of Proposition 17 hold. Then there exist \( h_0 > 0 \) and \( c > 0 \) such that for all \( j \in \{1, \ldots, n\} \), \( i \in \{1, \ldots, j - 1\} \) and \( h \in (0, h_0) \)

\[
\kappa_{ji} = O \left( e^{-\frac{c}{h}(f(z_i) - f(z_j)) + e} \right) \quad \text{and} \quad Z_j = 1 + O \left( e^{-\frac{c}{h}} \right).
\]

Proof. Let us introduce the notation: for all \( i \in \{1, \ldots, n\} \),

\[
r_i := \left\| \left(1 - \pi_h^{(1)} \right) \tilde{\psi}_i \right\|^2_{L^2_w}\text{.}
\]

Let us prove this lemma by induction. Concerning \( \psi_1 \), one has from Lemma 20

\[
\psi_1 = \frac{v_1}{Z_1} \text{ with } v_1 = \pi_h^{(1)} \tilde{\psi}_1.
\]

Since \( \left\| \tilde{\psi}_1 \right\|_{L^2_w} = 1 \), one has \( Z_1 = \left\| \pi_h^{(1)} \tilde{\psi}_1 \right\|_{L^2_w} \leq 1 \). In addition, by Pythagorean’s Theorem and by assumption 2(a) in Proposition 17 on \( r_1 \), there exists \( c > 0 \) such that for \( h \) small enough

\[
Z_1^2 \geq 1 - \left\| \left(1 - \pi_h^{(1)} \right) \tilde{\psi}_1 \right\|^2_{L^2_w} \geq 1 - r_1 \geq 1 - e^{-\frac{c}{h}}.
\]

Thus \( Z_1 = 1 + O \left( e^{-\frac{c}{h}} \right) \). Now, concerning \( \psi_2 \), one has

\[
\psi_2 = \frac{v_2}{Z_2} \text{ with } v_2 = \pi_h^{(1)} \tilde{\psi}_2 - \left\langle \pi_h^{(1)} \tilde{\psi}_2, \tilde{\psi}_1 \right\rangle_{L^2_w} \psi_1.
\]
and thus $\kappa_{21} = -\frac{1}{Z_1} \left\langle \pi_h^{(1)} \tilde{\psi}_k, \tilde{\psi}_2 \right\rangle_{L^2_w} = O\left(e^{-\frac{1}{\pi}(f(z_2) - f(z_1) + \varepsilon)}\right)$ (by Lemma 21). Then one obtains that $Z_2 = 1 + O\left(e^{-\frac{1}{\pi}}\right)$ by a similar reasoning as the one we used above for $Z_1$.

In order to prove Lemma 22 by induction, let us now assume that for $k \in \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, k\}$, $l \in \{1, \ldots, j - 1\},$

$$\kappa_{jl} = O\left(e^{-\frac{1}{\pi}(f(z_j) - f(z_l) + \varepsilon)}\right)$$

and $Z_j = 1 + O\left(e^{-\frac{1}{\pi}}\right)$. One gets by the Gram-Schmidt procedure which defines the $(\psi_i)_{i=1, \ldots, n},$

$$\psi_{k+1} = \frac{v_{k+1}}{Z_{k+1}}$$

where, using the notation $\kappa_{ii} = 1,$

$$v_{k+1} = \pi_h^{(1)} \tilde{\psi}_{k+1} - \sum_{j=1}^{k} \left\langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \tilde{\psi}_j \right\rangle_{L^2_w} \psi_j$$

$$= \pi_h^{(1)} \tilde{\psi}_{k+1} - \sum_{j=1}^{k} \sum_{q=1}^{j} \frac{1}{Z_j} \langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \rangle_{L^2_w} \kappa_{jl} \kappa_{jq} \pi_h^{(1)} \tilde{\psi}_q$$

$$= \pi_h^{(1)} \tilde{\psi}_{k+1} - \sum_{q=1}^{k} \pi_h^{(1)} \tilde{\psi}_q \sum_{q=1}^{k} \sum_{l=1}^{j} \frac{1}{Z_j} \langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \rangle_{L^2_w} \kappa_{jl} \kappa_{jq}.$$

Then for $q \in \{1, \ldots, k\},$

$$\kappa_{(k+1)q} = -\sum_{j=q}^{k} \sum_{l=1}^{j} \frac{1}{Z_j} \langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \rangle_{L^2_w} \kappa_{jl} \kappa_{jq}. \quad (66)$$

Since $Z_j = 1 + O\left(e^{-\frac{1}{\pi}}\right),$ one gets $Z_j^{-1} = 1 + O\left(e^{-\frac{1}{\pi}}\right).$ Using Lemma 21 one obtains

$$\left\langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \right\rangle_{L^2_w} = O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_l) + \varepsilon)}\right),$$

since $l < k + 1$. Moreover one notices that $l \leq j$ and $q \leq j$. If $q < j$, $\kappa_{jl} \kappa_{jq} = O\left(e^{-\frac{1}{\pi}(f(z_j) - f(z_q) + \varepsilon)}\right),$ and thus

$$\left\langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \right\rangle_{L^2_w} \kappa_{jl} \kappa_{jq} = O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_l) + f(z_j) - f(z_q) + \varepsilon)}\right)$$

$$= O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_q) + \varepsilon)}\right).$$

If $l < j$ and if $q = j,$

$$\kappa_{jl} \kappa_{jq} = O\left(e^{-\frac{1}{\pi}(f(z_j) - f(z_l) + \varepsilon)}\right).$$

Since, if $l < j$ and $q = j,$ $f(z_q) = f(z_j) \geq f(z_l)$ one obtains that

$$\left\langle \pi_h^{(1)} \tilde{\psi}_{k+1}, \pi_h^{(1)} \tilde{\psi}_l \right\rangle_{L^2_w} \kappa_{jl} \kappa_{jq} = O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_l) + f(z_j) - f(z_q) + \varepsilon)}\right)$$

$$= O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_q) + \varepsilon)}\right) = O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_q) + \varepsilon)}\right).$$

If $l = q = j,$ $\kappa_{jl} \kappa_{jq} = 1.$ Thus one has: for $q \in \{1, \ldots, k\},$

$$\kappa_{(k+1)q} = O\left(e^{-\frac{1}{\pi}(f(z_{k+1}) - f(z_q) + \varepsilon)}\right).$$
Using (61)–(64)–(65)–(67), for all \( j \)

From Lemmata 18 and 22, one has

\[ \delta > 0 \]  

From (50), for any \( h \)

Proof. From (50), for any \( \phi \in H^1_{w,T}(\Omega) \) and \( v \in L^2_w(\Omega) \), it holds,

\[ \langle \nabla \pi_h^{(0)}, \pi_h^{(1)} \rangle_{L^2_w} = \langle \nabla \phi, \pi_h^{(1)} v \rangle_{L^2_w}. \]  

Using (61)–(64)–(65)–(67), for all \( j \in \{1, \ldots, n\} \), one has

\[ \langle \nabla u_h, \psi_j \rangle_{L^2_w} = C_j \int_{\Omega} e^{- \frac{1}{\lambda}(f(z_j) - f(x_0))} \, (1 + O(h)). \]  

2.3.2 Estimates on the interaction terms \( \langle \nabla u_h, \psi_j \rangle_{L^2_w} \)

Lemma 23. Let us assume that the assumptions of Proposition 17 hold. Then for \( j \in \{1, \ldots, n\} \), one has

\[ \langle \nabla u_h, \psi_j \rangle_{L^2_w} = C_j \int_{\Omega} \frac{e^{- \frac{1}{\lambda}(f(z_j) - f(x_0))}}{1 + O(h)). \]  

Therefore choosing \( \delta < \varepsilon \), there exists \( \varepsilon' > 0 \) such that

\[ \langle \nabla u_h, \psi_j \rangle_{L^2_w} = C_j \int_{\Omega} \frac{e^{- \frac{1}{\lambda}(f(z_j) - f(x_0))}}{1 + O(h))} + O \left( e^{- \frac{1}{\lambda}(f(z_j) - f(x_0)) + \varepsilon'} \right). \]  

This concludes the proof of Lemma 23.
2.3.3 Estimates on the boundary terms \( \left( \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{\pi}f} \, ds \right)_{(j,k) \in \{1, \ldots, n\}^2} \)

One denotes for \( k \in \{1, \ldots, n\} \), \( K_k := \max(f(z_k) - f(z_j), f(z_k) - f(z_1)) \geq 0. \)

**Lemma 24.** Let us assume that the assumptions of Proposition 17 hold. Then for all \( (j, k) \in \{1, \ldots, n\}^2 \), there exists \( \varepsilon > 0 \) such that it holds

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{\pi}f} \, ds = \begin{cases} O \left( e^{-\frac{1}{\pi}|f(z_j)+\varepsilon|} \right) & \text{if } k < j, \\
B_j h^m e^{-\frac{1}{\pi}|f(z_j)|} (1 + O(h)) + \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{\pi}|f(z_i)-f(z_j)+K_i+f(z_k)+\varepsilon|} \right) & \text{if } k = j, \\
O \left( e^{-\frac{1}{\pi}|f(z_j)+f(z_k)+\varepsilon|} \right) & \text{if } k > j.
\end{cases}
\]

**Proof.** Using (64)–(65) and writing \( \pi_h^{(1)} \tilde{\psi}_i = \tilde{\psi}_i + (\pi_h^{(1)} - 1) \tilde{\psi}_i \), one obtains that

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{\pi}f} \, ds = a_{jk} + b_{jk} + \sum_{i=1}^{j-1} (c_{jki} + d_{jki})
\]

with for \( (j, k) \in \{1, \ldots, n\}^2 \) and \( i \in \{1, \ldots, j - 1\} \),

\[
a_{jk} = Z_j^{-1} \int_{\Sigma_k} \tilde{\psi}_j \cdot n e^{-\frac{2}{\pi}f} \, ds, \quad b_{jk} = Z_j^{-1} \int_{\Sigma_k} (\pi_h^{(1)} - 1) \tilde{\psi}_j \cdot n e^{-\frac{2}{\pi}f} \, ds,
\]

\[
c_{jki} = Z_j^{-1} \kappa_{ji} \int_{\Sigma_k} \tilde{\psi}_i \cdot n e^{-\frac{2}{\pi}f} \, ds \quad \text{and} \quad d_{jki} = Z_j^{-1} \kappa_{ji} \int_{\Sigma_k} (\pi_h^{(1)} - 1) \tilde{\psi}_i \cdot n e^{-\frac{2}{\pi}f} \, ds.
\]

Using the trace theorem and assumption 2(a) in Proposition 17, one has, for some universal constant \( C, \)

\[
\int_{\Sigma_k} (\pi_h^{(1)} - 1) \tilde{\psi}_j \cdot n e^{-\frac{2}{\pi}f} \, ds \leq C \left\| (\pi_h^{(1)} - 1) \tilde{\psi}_j \right\|_{H^1} \sqrt{\int_{\Sigma_k} e^{-\frac{2}{\pi}f} \, ds} = O \left( e^{-\frac{1}{\pi}|K_j+f(z_k)+\varepsilon|} \right). \quad (69)
\]

If \( k = j \) and \( i \in \{1, \ldots, j - 1\} \), one gets, using (69), Lemma 22 and assumption 4(b) in Proposition 17,

\[
a_{jk} = B_j h^m e^{-\frac{1}{\pi}|f(z_j)|} (1 + O(h)), \quad b_{jk} = O \left( e^{-\frac{1}{\pi}|K_j+f(z_j)+\varepsilon|} \right),
\]

\[
c_{jki} = 0 \quad \text{and} \quad d_{jki} = O \left( e^{-\frac{1}{\pi}|f(z_j)-f(z_i)+K_i+f(z_k)+\varepsilon|} \right).
\]

If \( k \neq j \) and \( i \in \{1, \ldots, j - 1\} \), one gets using again (69), Lemma 22 and assumption 4(b) in Proposition 17,

\[
a_{jk} = 0, \quad b_{jk} = O \left( e^{-\frac{1}{\pi}|K_j+f(z_k)+\varepsilon|} \right),
\]

\[
c_{jki} = \begin{cases} O \left( e^{-\frac{1}{\pi}|f(z_j)+\varepsilon|} \right) & \text{if } k = i, \\
0 & \text{if } k \neq i, \end{cases} \quad \text{and} \quad d_{jki} = O \left( e^{-\frac{1}{\pi}|f(z_j)-f(z_i)+K_i+f(z_k)+\varepsilon|} \right).
\]

Notice that \( c_{jki} = 0 \) if \( j < k \) and that if \( j > k \) there exists only one \( i \) such that \( c_{jki} \neq 0 \), namely \( i = k \). This concludes the proof of the Lemma 24. \( \Box \)
2.3.4 Estimates on \( \left( \int_{\Sigma_k} (\partial_n u_h) e^{-\frac{r}{h}} d\sigma \right)_{k \in \{1, \ldots, n\}} \)

We are now in position to conclude the proof of Proposition 17 by proving that for \( k \in \{1, \ldots, n\} \), one has

\[
\int_{\Sigma_k} (\partial_n u_h) e^{-\frac{r}{h}} d\sigma = C_k B_k h^{p+m} e^{-\frac{1}{h}(2f(z_k)-f(x_0))} (1 + O(h)).
\]

Proof. Let us assume that the assumptions of Proposition 17 hold. Let us recall the decomposition (63):

\[
\int_{\Sigma_k} (\partial_n u_h) e^{-\frac{r}{h}} d\sigma = \sum_{j=1}^{n} \langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{r}{h}} d\sigma.
\]

Using Lemmas 23 and 24, we can now estimate the terms in the sum in the right-hand side. If \( j > k \), there exist \( \varepsilon > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{r}{h}} d\sigma = C_j h^p O \left( e^{-\frac{1}{h}(f(z_j)-f(x_0))} e^{-\frac{1}{h}|f(z_j)+\varepsilon|} \right) = C_j h^p O \left( e^{-\frac{1}{h}|2f(z_j)-f(x_0)+\varepsilon|} \right) = C_j h^p O \left( e^{-\frac{1}{h}|2f(z_k)-f(x_0)+\varepsilon|} \right).
\]

If \( j < k \), there exist \( \varepsilon > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_k} \psi_j \cdot n e^{-\frac{r}{h}} d\sigma = O \left( e^{-\frac{1}{h}|f(z_j)-f(x_0)+K_j+f(z_k)+\varepsilon|} \right)
\]

\[
+ \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h}|f(z_j)-f(x_0)+f(z_j)-f(z_i)+f(z_k)+\varepsilon|} \right) = O \left( e^{-\frac{1}{h}|f(z_j)-f(x_0)+f(z_j)-f(z_i)+f(z_k)+\varepsilon|} \right)
\]

\[
+ \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h}|f(z_j)-f(x_0)+f(z_j)-f(z_i)+f(z_n)-f(z_i)+f(z_k)+\varepsilon|} \right) = O \left( e^{-\frac{1}{h}|f(z_n)+f(z_k)-f(x_0)+\varepsilon|} \right)
\]

\[
+ \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h}|f(z_n)+f(z_k)-f(x_0)+2f(z_j)-f(z_i)+\varepsilon|} \right) = O \left( e^{-\frac{1}{h}|2f(z_k)-f(x_0)+\varepsilon|} \right).
\]

Finally if \( j = k \), \( \exists \varepsilon > 0 \) and \( \exists h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[
\langle \nabla u_h, \psi_k \rangle_{L^2_w} \int_{\Sigma_k} \psi_k \cdot n e^{-\frac{r}{h}} d\sigma = C_k B_k h^{p+m} e^{-\frac{1}{h}(2f(z_k)-f(x_0))} (1 + O(h)). \tag{70}
\]

From these estimates, for a fixed \( k \in \{1, \ldots, n\} \), the dominant term in the sum in the right-hand side of (63) is the one with index \( j = k \), namely (70). This concludes the proof of Proposition 17. \( \square \)
3 On the Agmon distance

In this section, we present the main properties of the Agmon distance introduced in Definition 3. The Agmon distance is useful to quantify the decay of eigenforms of some Witten Laplacians away from critical points of \( f \) and \( f|_{\partial \Omega} \). The Agmon distance on a domain without boundary has been extensively analyzed in many previous works (see in particular [35–37, 39]). The aim of this section is to generalize well-known results in the case without boundary to our context, namely for bounded domains. Indeed, to the best of our knowledge, this has not been done in the literature before in a comprehensive way.

For simplicity, all the proofs in this section are made for a bounded connected open \( C^\infty \) domain \( \Omega \subset \mathbb{R}^d \) (equipped with the geodesic Euclidean distance (85)) and for a \( C^\infty \) function \( f : \overline{\Omega} \to \mathbb{R} \). The generalization to a \( C^\infty \) compact connected Riemannian manifold of dimension \( d \) with boundary is straightforward. The notation \(|x - y|\) denotes the Euclidean distance between \( x \) and \( y \) in \( \mathbb{R}^d \). If one deals with a compact connected Riemannian manifold of dimension \( d \) with boundary, this distance has to be replaced by the geodesic distance on \( \overline{\Omega} \) for the initial metric and the scalar product between two vectors of \( \mathbb{R}^d \) has to be replaced by the one induced by the initial metric on the tangent space of \( \overline{\Omega} \).

This section is organized as follows. Section 3.1 is devoted to an equivalent definition of the Agmon distance, which will be crucial in the following. In Section 3.2, we then give a few useful properties of the Agmon distance. As already mentioned in Section 1.5.1, there is a link between the Agmon distance and the eikonal equation. This is explained in Section 3.3 and 3.5. This link is useful in order to build explicit curves realizing the Agmon distance, as explained in Section 3.4.

3.1 The set \( A(x, y) \) and an equivalent definition of the Agmon distance

In order to study the Agmon distance, it will be more convenient for technical reasons to restrict the class of curves appearing in the definition of the Agmon distance (see Definition 3).

**Definition 6.** For \( x, y \in \overline{\Omega} \), we denote by \( A(x, y) \) the set of curves \( \gamma : [0, 1] \to \overline{\Omega} \) which are Lipschitz with \( \gamma(0) = x \), \( \gamma(1) = y \) and such that the set \( \partial \{ t \in [0, 1] \mid \gamma(t) \in \partial \Omega \} \) is finite.

Here \( \partial \{ t \in [0, 1], \gamma(t) \in \partial \Omega \} \) denotes the boundary of the set \( \{ t \in [0, 1], \gamma(t) \in \partial \Omega \} \). The main result of this section is that, under assumption \([H3]\), the Agmon distance \( d_a \) satisfies (compare with (18)):

\[
\forall (x, y) \in \overline{\Omega}^2, \ d_a(x, y) = \inf_{\gamma \in A(x, y)} L(\gamma, (0, 1)). \tag{71}
\]

See Corollary 28 below.

The following lemma will be needed several times throughout this section. It motivates the use of the set \( A(x, y) \) appearing in Definition 6.

**Lemma 25.** Let \( x, y \in \overline{\Omega} \) and \( \gamma \in A(x, y) \). Then for any \( h : \overline{\Omega} \to \mathbb{R} \) which is \( C^1 \), one gets

\[
h(y) - h(x) = \int_{\{t \in [0, 1], \gamma(t) \in \Omega\}} (\nabla h)(\gamma) \cdot \gamma' + \int_{\{t \in [0, 1], \gamma(t) \in \partial \Omega\}} (\nabla T h)(\gamma) \cdot \gamma'. \tag{72}
\]
Here the notation \( \text{int} \) stands for the interior of a domain.

**Proof.** Since \( \gamma \) is Lipschitz, \( h \circ \gamma \) is Lipschitz and thus absolutely continuous. Therefore, one has:

\[
\begin{align*}
    h(y) - h(x) &= \int_0^1 \frac{d}{dt} (h \circ \gamma) \\
    &= \int_{\{t \in [0,1], \gamma(t) \in \Omega\}} \frac{d}{dt} (h \circ \gamma) + \int_{\text{int}\{t \in [0,1], \gamma(t) \in \partial \Omega\}} \frac{d}{dt} (h \circ \gamma) \\
    &\quad + \int_{\partial\{t \in [0,1], \gamma(t) \in \partial \Omega\}} \frac{d}{dt} (h \circ \gamma).
\end{align*}
\]

By definition of the set \( A(x, y) \) (see Definition 5), the set \( \partial\{t \in [0,1], \gamma(t) \in \partial \Omega\} \) has Lebesgue measure zero,

\[
\int_{\partial\{t \in [0,1], \gamma(t) \in \partial \Omega\}} \frac{d}{dt} (h \circ \gamma) = 0.
\]

The curve \( \gamma \) is continuous and thus the set \( \{t \in [0,1], \gamma(t) \in \Omega\} \) is open in \([0,1]\). Thus, using in addition that since \( \gamma \) is Lipschitz, it is almost everywhere differentiable (by the Rademacher Theorem), one has for almost every \( t \in [0,1] \):

\[
\frac{d}{dt} h(\gamma)(t) = \begin{cases} 
    \langle \nabla h \rangle (\gamma(t)) \cdot \frac{d}{dt} \gamma(t) & \text{a.e. on } \{t \in [0,1], \gamma(t) \in \Omega\} \\
    \langle \nabla_T h \rangle (\gamma(t)) \cdot \frac{d}{dt} \gamma(t) & \text{a.e. on } \text{int}\{t \in [0,1], \gamma(t) \in \partial \Omega\}.
\end{cases}
\]

This proves (72). \( \square \)

**Remark 4.** Notice that there exist Lipschitz curves \( \gamma \) such that \( \partial\{t \in [0,1], \gamma(t) \in \partial \Omega\} \) has a positive Lebesgue measure. Let us give an example. Consider \( \Omega = (0,1) \times (0,2) \) and the curve

\[
\gamma : t \in [0,1] \mapsto \left( t, \inf_{y \in K} |t - y| \right) \in [0,1]^2,
\]

where \( K \) is the Smith-Volterra-Cantor set in \([0,1]\), such that \( K \) is closed, has a positive Lebesgue measure and has an empty interior (see [60, Section 2.5]). Notice that the distance \( \inf_{y \in K} |t - y| \) to \( K \) is a Lipschitz function of \( t \in (0,1) \), so that \( \gamma \) is a Lipschitz function. The set \( K \) is closed and thus

\[
\{t \in [0,1], \gamma(t) \in \partial \Omega\} = \{t \in [0,1], \gamma(t) = 0\} = K.
\]

Therefore \( \partial\{t \in [0,1], \gamma(t) \in \partial \Omega\} = \overline{K} \setminus (\text{int}K) = K. \)

The following results will be useful to prove the equality (71) and to prove the existence of curves realizing the Agmon distance (see Section 3.4).  

**Proposition 26.** Assume that [H3] holds. Let \( \gamma : [0,1] \to \overline{\Omega} \) be a Lipschitz curve and assume that there exists a time \( t^* \in [0,1] \) such that \( \gamma(t^*) \in \partial \Omega \). Then there exists \( (a, b) \in [0,1]^2 \), with \( a \leq t^* \leq b \) and \( a < b \) such that for all \( (t_1, t_2) \in [0,1]^2 \), with \( a \leq t_1 < t_2 \leq b \), there exists a Lipschitz curve \( \gamma_{12} : [t_1, t_2] \to \overline{\Omega} \) satisfying

\[
\gamma_{12}(t_1) = \gamma(t_1) \text{ and } \gamma_{12}(t_2) = \gamma(t_2),
\]

where \( \gamma_{12} \) is the extending of \( \gamma \) into \([t_1, t_2]\). This extends the existence of curves realizing the Agmon distance to an entire interval. This is a consequence of the existence of a Lipschitz curve with a derivative outside the set \( \partial K \), which has a positive Lebesgue measure and contains an empty interior in \([0,1]^2\).
\[ L(\gamma, (t_1, t_2)) \geq L(\gamma_{12}, (t_1, t_2)) \]  \hspace{0.5cm} (73)

and, either \( \{ t \in [t_1, t_2], \gamma_{12}(t) \in \partial \Omega \} \) is empty, or its boundary \( \partial \{ t \in [t_1, t_2], \gamma_{12}(t) \in \partial \Omega \} \) consists of isolated points in \( \{ t \in [t_1, t_2], \gamma_{12}(t) \in \partial \Omega \} \). Moreover, if the following is satisfied:

\[ \exists (s_1, s_2, s_3) \in [t_1, t_2]^3, s_1 < s_2 < s_3, \gamma(s_1) \in \partial \Omega, \gamma(s_2) \in \Omega \text{ and } \gamma(s_3) \in \partial \Omega, \]  \hspace{0.5cm} (74)

then the inequality (73) is strict.

**Remark 5.** Notice that if \( t^* \in \partial \{ t \in [0, 1], \gamma(t) \in \partial \Omega \} \) is not isolated in \( \partial \{ t \in [0, 1], \gamma(t) \in \partial \Omega \} \), then there exists a neighborhood \( [t_1, t_2] \) of \( t^* \) in \( [0, 1] \) such that (74) is satisfied and thus the inequality (73) is strict. Therefore, if a Lipschitz curve \( \gamma \) realizes the infimum of \( L \) on \( \text{Lip}(x, y) \), then \( \partial \{ t \in [0, 1], \gamma(t) \in \partial \Omega \} \) is finite. This motivates the introduction of the set \( A(x, y) \).

**Proof.** Let \( t^* \in [0, 1] \) be such that \( \gamma(t^*) \in \partial \Omega \). The proof is divided into three steps.

**Step 1:** Introduction of a normal system of coordinates and definition of \( a \) and \( b \). Let us consider a neighborhood \( V_{\partial \Omega} \) of \( \gamma(t^*) \) in \( \partial \Omega \), and a smooth local system of coordinates in \( V_{\partial \Omega} \subset \partial \Omega \), denoted by \( x_T : V_{\partial \Omega} \to \mathbb{R}^{d-1} \). Let us now extend it to a tangential and normal system of coordinates around \( \gamma(t^*) \) in \( \bar{\Omega} \), denoted by \( \phi(x) = (x_T, x_N) \). The function \( \phi \) is defined from a neighborhood of \( \gamma(t^*) \) in \( \bar{\Omega} \) to \( \mathbb{R}^d \). Moreover, one has \( x_N(\cdot) \geq 0 \) and for all \( x, x_N(x) = 0 \) if and only if \( x \in \partial \Omega \). One can assume without loss of generality that \( \phi \) is defined on a neighborhood \( V_\alpha \) of \( \gamma(t^*) \) in \( \bar{\Omega} \) such that \( \phi(V_\alpha) = U \times [0, \alpha] \) for \( \alpha > 0 \), and \( U \subset \mathbb{R}^{d-1} \). For this normal system of coordinates, the metric tensor \( G \) is such that: \( \forall (x_T, x_N) \in U \times [0, \alpha] \),

\[ G(x_T, x_N) = \begin{pmatrix} \tilde{G}(x_T, x_N) & 0 \\ 0 & G_{NN}(x_T, x_N) \end{pmatrix}, \]

where \( \tilde{G}(x_T, x_N) \in \mathbb{R}^{(d-1) \times (d-1)} \) and \( G_{NN}(x_T, x_N) \in \mathbb{R} \) are smooth functions of \( (x_T, x_N) \). The existence of such a system of coordinates is a consequence of the collar theorem, see [65].

Under assumption [H3] (namely \( \partial_n f > 0 \) on \( \partial \Omega \)), there exist constants \( A > 1 \) and \( \varepsilon_1 > 0 \) such that for all \( x_N \in (0, \varepsilon_1] \) and for all \( x_T \in U \), (see [17] for the definition of \( g \))

\[ g(\phi^{-1}(x_T, x_N)) \geq Ag(\phi^{-1}(x_T, 0)). \]  \hspace{0.5cm} (75)

Since the local change of coordinates is smooth, for all \( \varepsilon \in (0, 1) \), there exists \( \eta > 0 \) such that for all \( x_N \in [0, \eta] \) and for all \( x_T \in U \), one has

\[ \tilde{G}(x_T, x_N) \geq (1 - \varepsilon)^2 \tilde{G}(x_T, 0). \]

Let us choose \( \varepsilon > 0 \) such that \( (1 - \varepsilon)A > 1 \). One can reduce \( V_\alpha \) such that \( 0 \leq x_N(x) \leq \min(\eta, \varepsilon_1) \) for all \( x \in V_\alpha \). By continuity of \( \gamma \), there exist \( (a, b) \in [0, 1]^2 \), with \( a \leq t^* \leq b \) and \( a < b \) such that for all \( t \in [a, b] \), \( \gamma(t) \in V_\alpha \). Let us introduce the components of \( \gamma \) in the normal system of coordinates: \( (\gamma_T(t), \gamma_N(t)) = \phi(\gamma(t)) \). Let us now define: for \( t \in [a, b] \),

\[ \tilde{\gamma}(t) := \phi^{-1}(\gamma_T(t), 0) \in \partial \Omega. \]
For almost every $t \in (a, b)$, if $\gamma(t) \in \partial \Omega$, $\gamma(t) = \phi^{-1}(\gamma_T(t), 0) = \tilde{\gamma}(t)$, $g(\gamma(t)) = g(\tilde{\gamma}(t))$ and

$$|\gamma'(t)|^2 = \left[(\gamma_T, \gamma_N)\right]^{Tr} G((\gamma_T, \gamma_N)) (\gamma_T, \gamma_N)'$$

$$= \left[\gamma_T'(t)\right]^{Tr} \tilde{G}((\gamma_T, \gamma_N)) \gamma_T'(t) + G_{NN}((\gamma_T, \gamma_N)) \gamma_N'(t)^2$$

$$\geq (1 - \varepsilon)^2 \left[\gamma_T'(t)\right]^{Tr} \tilde{G}((\gamma_T, \gamma_N)) \gamma_T'(t) = (1 - \varepsilon)^2 |\gamma'(t)|^2,$$

where the superscript $Tr$ stands for the transposition operator. For almost every $t \in (a, b)$, if $\gamma(t) \in \Omega$,

$$|\gamma'(t)|^2 = \left[(\gamma_T, \gamma_N)\right]^{Tr} G((\gamma_T, \gamma_N)) (\gamma_T, \gamma_N)'$$

$$= \left[\gamma_T'(t)\right]^{Tr} \tilde{G}((\gamma_T, \gamma_N)) \gamma_T'(t) + G_{NN}((\gamma_T, \gamma_N)) \gamma_N'(t)^2$$

$$\geq (1 - \varepsilon)^2 \left[\gamma_T'(t)\right]^{Tr} \tilde{G}((\gamma_T, \gamma_N)) \gamma_T'(t) = (1 - \varepsilon)^2 |\gamma'(t)|^2.$$

**Step 2: Definition of $\gamma_{12}$.** Let $(t_1, t_2) \in [0, 1]^2$, with $a \leq t_1 < t_2 \leq b$. Let us distinguish between two cases.

- If the set $\{t \in [t_1, t_2], \gamma(t) \in \partial \Omega\}$ is non empty, let us consider $t_1^+ := \inf \{t \in [t_1, t_2], \gamma(t) \in \partial \Omega\}$ and $t_2^- := \sup \{t \in [t_1, t_2], \gamma(t) \in \partial \Omega\}$. The curve $\gamma_{12} : [t_1, t_2] \to \Omega$ is then defined by

$$\gamma_{12}(t) = \begin{cases} 
\gamma(t) & \text{if } t \in (t_1, t_1^+), \\
\tilde{\gamma}(t) & \text{if } t \in (t_1^+, t_2^-), \\
\gamma(t) & \text{if } t \in (t_2^-, t_2).
\end{cases}$$

Observe that for all $t \in (t_1, t_1^+) \cup (t_2^-, t_2)$, $\gamma(t) = \gamma_{12}(t)$, which implies $g(\gamma(t))|\gamma'(t)| = g(\gamma_{12}(t))|\tilde{\gamma}'_{12}(t)|$ almost everywhere in $(t_1, t_1^+) \cup (t_2^-, t_2)$. Moreover, using the fact that $A(1 - \varepsilon) > 1$, for almost every $t \in (t_1^+, t_2^-)$,

$$g(\gamma(t))|\gamma'(t)| \geq A(1 - \varepsilon)g(\phi^{-1}(\gamma_T(t), 0))|\tilde{\gamma}'_{12}(t)| > g(\tilde{\gamma})|\tilde{\gamma}'(t)| = g(\gamma_{12})|\gamma_{12}'(t)|.$$  (76)

Therefore (73) is satisfied.

- If the set $\{t \in [t_1, t_2], \gamma(t) \in \partial \Omega\}$ is empty, which means that $\forall t \in [t_1, t_2], \gamma(t) \in \Omega$, then simply defines the curve $\gamma_{12} : [t_1, t_2] \to \overline{\Omega}$ by $\gamma_{12} = \gamma$.

In both cases, the curve $\gamma_{12}$ is Lipschitz, $\gamma_{12}(t_j) = \gamma(t_j)$ for $j \in \{1, 2\}$ and (73) is satisfied. Moreover by construction of $\gamma_{12}$, the set $\partial \{t \in [t_1, t_2], \gamma_{12}(t) \in \partial \Omega\}$ consists of isolated points in $\{t \in [t_1, t_2], \gamma_{12}(t) \in \partial \Omega\}$, or is empty.

**Step 3: On the strict inequality in (73).** Assume that (74) holds and let us show that the inequality (73) is strict. Indeed, in that case $t_1^+ \leq s_1 < s_3 \leq t_2^-$ and by continuity of $\gamma$, there exists $(u_1, u_2) \in (s_1, s_3)^2$ such that $u_1 < s_2 < u_2$ and $\gamma([u_1, u_2]) \subset \Omega$. Thus, the inequality (76) holds almost everywhere on the open nonempty interval $(u_1, u_2)$ which implies that $L(\gamma, (u_1, u_2)) > L(\gamma_{12}, (u_1, u_2))$. This concludes the proof of Proposition 26.

A consequence of the previous proposition is the following result.
Proposition 27. Let $x, y \in \bar{\Omega}$ and assume that [H3] holds. For any Lipschitz curve $\gamma : [0, 1] \to \bar{\Omega}$ with $\gamma(0) = x$ and $\gamma(0) = y$, there exists $\gamma_1 \in A(x, y)$ such that $L(\gamma, (0, 1)) \geq L(\gamma_1, (0, 1))$.

Proof. The set $\partial \{t \in [0, 1], \gamma(t) \in \partial \Omega\}$ is closed, so its limit points are its non isolated points. Let us define $\text{Ad}(\gamma)$ as the set of limit points of $\partial \{t \in [0, 1], \gamma(t) \in \partial \Omega\}$. If $\text{Ad}(\gamma)$ is empty, then $\partial \{t \in [0, 1], \gamma(t) \in \partial \Omega\}$ is empty or consists of isolated points in $\partial \{t \in [0, 1], \gamma(t) \in \partial \Omega\}$ and since $\partial \{t \in [0, 1], \gamma(t) \in \partial \Omega\}$ is compact, this implies that $\gamma \in A(x, y)$ and Proposition 27 is thus proved by simply taking $\gamma_1 = \gamma$.

If $\text{Ad}(\gamma)$ is non empty, we will construct a curve $\gamma_1 \in A(x, y)$ such that

$$L(\gamma, (0, 1)) \geq L(\gamma_1, (0, 1)).$$

Without loss of generality, one can assume that 0 and 1 are not in $\text{Ad}(\gamma)$. Otherwise one could modify $\gamma$ in neighborhoods of 0 and 1 without increasing $L(\gamma, (0, 1))$ and without changing the end points using Proposition 26. To prove the result, we will show by induction on $N \geq 1$ the following property $\mathcal{P}_N$: for all $\{t_1, \ldots, t_N\} \subset \text{Ad}(\gamma)$, denote by $(a_j, b_j)_{j=1, \ldots, N}$ the open intervals given by Proposition 26 for each $t_i$; then, it is possible to change $\gamma$ to construct a Lipschitz curve $\gamma_1 : [0, 1] \to \bar{\Omega}$ with $\gamma_1(0) = x$ and $\gamma_1(0) = y$, such that $\gamma_1 = \gamma$ on $[0, 1] \setminus \left(\bigcup_{j=1}^N (a_j, b_j)\right)$ and $L(\gamma_1, (0, 1)) \geq L(\gamma_1, (0, 1))$. If $(a_{N+1}, b_{N+1}) \subset \bigcup_{j=1}^N (a_j, b_j)$, then $\mathcal{P}_{N+1}$ holds taking $\gamma_1$. Otherwise, there exist $K \in \mathbb{N}^*$ and $(q_1, \ldots, q_K, d_1, \ldots, d_K) \in [0, 1]^{2K}$ such that

$$(a_{N+1}, b_{N+1}) \cap \left[\bigcup_{j=1}^N (a_j, b_j)\right] = \bigcup_{i=1}^K [(q_i, d_i)],$$

with $0 < q_1 < d_1 < q_2 < d_2 < \ldots < q_K < d_K < 1$; the notation $[()$ and $])$ mean that the extremities can or not belong to the interval. In addition, for $i \in \{1, \ldots, K\}$, $q_i \in \{a_{N+1}\} \cup \{b_1, \ldots, b_N\}$ and $d_i \in \{b_{N+1}\} \cup \{a_1, \ldots, a_N\}$. Then applying Proposition 26 to $\gamma_1$ on each interval $(q_i, d_i) \subset (a_{N+1}, b_{N+1})$, one gets that it is possible to construct a Lipschitz curve $\gamma_2$ with $\gamma_2(0) = x$ and $\gamma_2(0) = y$, such that

$$L(\gamma_1, (0, 1)) \geq L(\gamma_2, (0, 1)), \text{ Ad}(\gamma_2) \cap \bigcup_{i=1}^K (q_i, d_i) = \emptyset.$$

If for some $k \in \{1, \ldots, K\}$ and $j \in \{1, \ldots, N\}$ $q_k = b_j$, then $q_k$ is isolated from the left in $\{t \in [0, 1], \gamma_1(t) \in \partial \Omega\}$ from the construction of $\gamma_1$ (see Proposition 26) and, by construction of $\gamma_2$, $q_k$ is also isolated from the right in $\{t \in [0, 1], \gamma_2(t) \in \partial \Omega\}$. Thus, since there exists $s \in [0, q_k]$ such that $\gamma_2 = \gamma_1$ on $[s, q_k]$, one has $q_k \notin \text{Ad}(\gamma_2)$. A similar reasoning holds for the points $d_k$. Thus

$$\text{Ad}(\gamma_2) \cap (a_{N+1}, b_{N+1}) \cap \left[\bigcup_{j=1}^N (a_j, b_j)\right] = \emptyset.$$
This proves that \( \text{Ad}(\gamma_2) \cap \bigcup_{j=1}^{N+1}(a_j, b_j) = \emptyset \) and thus \( P_{N+1} \).

By induction, we thus have proven \( P_N \) for all \( N \geq 1 \). Now, notice that by a compactness argument, there exist \( N \geq 0 \) and \( \{t_1, \ldots, t_N\} \subset \text{Ad}(\gamma) \) such that if one denotes by \((a_j, b_j)_{j=1, \ldots, N}\) the open intervals given by Proposition 26 for each \( t_i \), then
\[
\text{Ad}(\gamma) \subset \bigcup_{i=1}^{N}(a_i, b_i).
\]
Applying \( P_N \) yields the desired result.

A direct consequence of Proposition 27 is the following.

**Corollary 28.** Assume that \([H3]\) holds. Then the Agmon distance \( d_a \) introduced in Definition 3 satisfies \((71)\): for all \((x, y) \in \overline{\Omega}^2\)
\[
d_a(x, y) = \inf_{\gamma \in A(x, y)} L(\gamma, (0, 1)).
\]

In all what follows, we will often use the formula \((71)\) to study the property of the Agmon distance.

### 3.2 First properties of the Agmon distance

In this section, we aim at giving the basic properties of the Agmon distance.

#### 3.2.1 Upper bounds and topology of \((\overline{\Omega}, d_a)\)

**Proposition 29.** There exists a constant \( C \) such that for all \( x, y \in \overline{\Omega} \),
\[
d_a(x, y) \leq C|x - y|.\tag{77}
\]

For any fixed \( y \in \overline{\Omega} \), \( x \in \overline{\Omega} \mapsto d_a(x, y) \) is Lipschitz. Its gradient is well defined almost everywhere and satisfies for \( y \in \overline{\Omega} \) and for almost every \( x \in \Omega \),
\[
|\nabla_x d_a(x, y)| \leq |\nabla f(x)|.\tag{78}
\]

Moreover, if \([H3]\) holds, for all \( x, y \in \overline{\Omega} \), we have
\[
|f(x) - f(y)| \leq d_a(x, y).\tag{79}
\]

**Proof.** Let us first prove the inequality \((79)\). For any \( \gamma \in A(x, y) \), using Lemma 25 one has:
\[
|f(x) - f(y)| = \left| \int_0^1 \frac{d}{dt}(f \circ \gamma) \, dt \right|
= \left| \int_{\{t \in [0, 1], \gamma(t) \in \Omega\}} (\nabla f)(\gamma) \cdot \gamma' \, dt + \int_{\text{int}\{t \in [0, 1], \gamma(t) \in \partial \Omega\}} (\nabla_T f)(\gamma) \cdot \gamma' \, dt \right|
\leq \int_0^1 g(\gamma(t)) \left| \gamma'(t) \right| \, dt.
\]

Therefore \((79)\) is proved by taking the infimum over \( \gamma \in A(x, y) \) in the right-hand side (see Corollary 28). The second inequality \((77)\) is proven below in Lemma 30. Let us
conclude the proof, assuming it holds. Now let \( y \in \overline{\Omega} \) and \( x \in \Omega \). Since \( \Omega \) is open, there exists an open ball \( B \subset \Omega \) with center \( x \). Let \( z \in B \). For \( t \in [0, 1] \), the path \( \gamma(t) = tz + (1-t)x \) is included in \( B \). Then, one obtains
\[
|d_a(x, y) - d_a(z, y)| \leq d_a(x, z)
\]
\[
\leq |x - z| \int_0^1 g(tz + (1-t)x) \, dt
\]
\[
= |x - z| \int_0^1 |\nabla f|(x + t(z - x)) \, dt.
\]
Since \( f \) is smooth, up to considering a smaller ball \( B \) centered at \( x \), there exists a constant \( c > 0 \) such that for all \( z \in B \), for all \( t \in [0, 1] \),
\[
|\nabla f|(x + t(z - x)) \leq |\nabla f|(x) + c|x - z|.
\]
Thus, for all \( z \in B \),
\[
|d_a(x, y) - d_a(z, y)| \leq |x - z| \left(|\nabla f|(x) + c|x - z|\right).
\]
As a consequence, for any fixed \( y \in \overline{\Omega} \) and for almost \( x \in \Omega \) one gets \((78)\), by considering the limit \( z \to x \) in the previous inequality.

Let us now give the proof of \((77)\).

**Lemma 30.** The function \((x, y) \in \overline{\Omega} \times \overline{\Omega} \mapsto d_a(x, y)\) is bounded and satisfies
\[
\sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \frac{d_a(x, y)}{|x - y|} < \infty.
\]

**Proof.** Let us first prove by contradiction that \((x, y) \in \overline{\Omega} \times \overline{\Omega} \mapsto d_a(x, y)\) is bounded. Let us assume that there exists a sequence \((x_k, y_k)_{k \geq 1} \in \overline{\Omega} \times \overline{\Omega}\) such that for all \( k \geq 1 \),
\[
d_a(x_k, y_k) \geq k.
\]
Up to the extraction of a subsequence, it can be assumed that \(\lim_{k \to \infty}(x_k, y_k) = (x, y) \in \overline{\Omega} \times \overline{\Omega}\) (the convergence being for the Euclidean metric). Notice that
\[
d_a(x_k, y_k) \leq d_a(x_k, x) + d_a(x, y) + d_a(y, y).
\]
Let us consider \(d_a(x_k, x)\). If \( x \in \Omega \) then there exist an open ball \( B \subset \Omega \) centered on \( x \) and an integer \( N \) such that for all \( k \geq N \), \( x_k \in B \) and therefore \( \gamma(t) = tx_k + (1-t)x \in B \) for all \( t \in (0, 1) \). Then, by definition of the Agmon distance, for all \( k \geq 1 \),
\[
d_a(x_k, x) \leq \|g\|_{L^\infty(B)} |x - x_k|.
\]
If \( x \in \partial \Omega \) then there exist \( r > 0 \) and a \( C^\infty \) bijective map \( \phi : B(x, r) \to B(0, 1) \) such that \( \phi(x) = 0 \), \( \phi(B(x, r) \cap \partial \Omega) = Q_0 \) and \( \phi(B(x, r) \cap \overline{\Omega}) = Q_- \), where \( Q_0 := \{y = (y_1, \ldots, y_d), \ |y| \leq 1, \ y_d = 0\} \) and \( Q_- := \{y = (y_1, \ldots, y_d), \ |y| \leq 1, \ y_d \leq 0\} \). Moreover, there exists \( N \) such that for all \( k \geq N \), \( x_k \in B(x, r) \cap \overline{\Omega} \). Now, for any \( k \geq N \), let us consider
\[
\gamma(t) = \phi^{-1}((1-t)\phi(x_k) + t\phi(x)).
\]
Notice that $\gamma \in A(x_k, x)$ and satisfies $\gamma([0, 1]) \subset B(x, r)$. Moreover $c(t) = \phi(\gamma(t)) = (1 - t)\phi(x_k) + t\phi(x) \in Q_-$ for $t \in [0, 1]$. Then, one has:

$$d_a(x_k, x) \leq \int_0^1 g(\gamma)|\gamma'|,$$

$$\leq \|g\|_{L^\infty(B(x,r))} \int_0^1 |\gamma'|,$$

$$= \|g\|_{L^\infty(B(x,r))} \int_0^1 |\text{Jac}(\phi^{-1})(c)c'|,$$

$$\leq \|g\|_{L^\infty(B(x,r))} \|\text{Jac}(\phi^{-1})\|_{L^\infty(B(0,1))} |\phi(x_k) - \phi(x)|$$

and therefore, since $\phi$ is Lipschitz,

$$d_a(x_k, x) \leq C|x_k - x|,$$  \hspace{1cm} (83)

where $C$ is a constant independent of $k \geq N$. This shows that $d_a(x_k, x)$ is bounded. The same reasoning shows that $d_a(y, y_k)$ is bounded. This yields a contradiction, considering the inequality (82) and (81).

To show (80), one proceeds in the same way. Assume that there exists a sequence $(x_k, y_k) \in \Omega \times \Omega$ such that

$$d_a(x_k, y_k) \geq k |x_k - y_k|.$$  \hspace{1cm} (84)

Up to the extraction of a subsequence, it can be assumed that $\lim_{k \to \infty} (x_k, y_k) = (x, y) \in \Omega \times \Omega$. If $x \neq y$, then, for $k$ sufficiently large

$$\frac{d_a(x_k, y_k)}{|x_k - y_k|} \leq 2 \sup_{(x,y) \in \Omega} d_a(x, y) \frac{d_a(x, y)}{|x - y|} < \infty$$

which contradicts (84). If $x = y \in \Omega$, then, for all $k$ sufficiently large, the curve $\gamma(t) = tx_k + (1 - t)y_k$ is with values in $\Omega$ and therefore for all $k$ sufficiently large

$$d_a(x_k, y_k) \leq \|g\|_{L^\infty(\Omega)} |y_k - x_k|.$$  \hspace{1cm} (83)

This again leads to a contradiction when $k \to \infty$. Finally, if $x = y \in \partial \Omega$, using the same reasoning as above to prove (83), one has the existence of a constant $C$ such that for all $k$ sufficiently large,

$$d_a(x_k, y_k) \leq C|x_k - y_k|,$$

which again contradicts (84). This concludes the proof of Lemma 30.}

A consequence of the previous lemma is the following proposition.

**Proposition 31.** Assume that $[H1]$ holds. The space $(\Omega, d_a)$ is a compact separated metric space. Moreover the topology of the metric space $(\Omega, d_a)$ is equivalent to the topology induced by the Euclidean metric on $\Omega$.

**Proof.** It is clear that $d_a \geq 0$, and for all $(x, y, z) \in \Omega^3$, $d_a(x, y) \leq d_a(x, z) + d_a(z, y)$. Let us show that for any $(x, y) \in \Omega^2$, if $x \neq y$ then $d_a(x, y) > 0$. Let us denote by $d_e$ the geodesic distance on $\Omega$ for the Euclidean metric: for all $x, y \in \Omega$,

$$d_e(x, y) := \inf_{\gamma} \int_0^1 |\gamma'(t)| \, dt,$$  \hspace{1cm} (85)
where the infimum is taken over all the paths $\gamma : [0, 1] \rightarrow \overline{\Omega}$ which are Lipchitz with $\gamma(0) = x$ and $\gamma(1) = y$. Since $[H1]$ holds, the functions $f$ and $f|_{\partial \Omega}$ have a finite number of critical points, and, there exist $0 < r_1 < r_2 < d_e(x, y)$ such that the infimum of $g$ on the set $C(r_1, r_2) := \{ z \in \overline{\Omega}, r_1 < d_e(x, z) < r_2 \}$ is positive i.e. $c(r_1, r_2) := \inf_{C(r_1, r_2)} g > 0$. For any path $\gamma \in A(x, y)$, one has

$$\int_0^1 |\gamma'(t)| g(\gamma(t)) \, dt \geq c(r_1, r_2) r(C(r_1, r_2)),$$

where $r(C(r_1, r_2)) := \inf_{x \in C(r_1, r_2)} \sup_{y \in C(r_1, r_2)} d_e(x, y) > 0$. Then $d_a(x, y) > 0$. If $x = y$, it is clear that $d_a(x, y) = 0$ since $L(\gamma, (0, 1)) = 0$ where $\gamma(t) = x$ for all $t \in [0, 1]$. This shows that $(\overline{\Omega}, d_a)$ is separated.

The fact that $(\overline{\Omega}, d_a)$ is compact comes from the inequality (77) proved in Lemma 30. Indeed, since $(\overline{\Omega}, d_a)$ is a metric space, it is sufficient to prove the sequential compactness. Let $(x_n)_{n \geq 0}$ be a sequence in $\overline{\Omega}$. Since $\overline{\Omega}$ is compact for the Euclidean metric, one can extract a converging subsequence $(x'_n)_{n \geq 0}$ for the Euclidean metric. From (77), this subsequence is also converging for $d_a$, which ends the proof.

Let us finally prove the equivalence of the topologies on $\overline{\Omega}$. From Lemma 30 it is obvious that if a sequence $(x_n)_{n \geq 0}$ converges to $x$ in $\overline{\Omega}$ for the Euclidean metric, then $d_a(x_n, x)$ converges to 0. Conversely, let us assume that $(x_n)_{n \geq 0}$ is a sequence of $\overline{\Omega}$ such that $d_a(x_n, x)$ converges to 0, for a point $x \in \overline{\Omega}$. Since $\overline{\Omega}$ is compact for the Euclidean metric, it is enough to show that $x$ is the only limit point of the sequence to show that $(x_n)_{n \geq 0}$ converges to $x$ for the Euclidean metric. From Lemma 30, any limit point $y$ for the Euclidean metric is also an limit point for the Agmon distance, and thus, since $(\overline{\Omega}, d_a)$ is a separated space, $y = x$. This concludes the proof.

Notice that from the proof, it is obvious that the topology is separated as soon as $f$ and $f|_{\partial \Omega}$ have a finite number of critical points, which is a weaker assumption than $[H1]$.

Finally, the following lemma will be useful in the following.

**Lemma 32.** Assume that $[H3]$ holds. Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \rightarrow \overline{\Omega}$ a Lipchitz curve such that $\partial \{ t \in I, \gamma(t) \in \partial \Omega \}$ is finite and such that $x := \lim_{t \downarrow (\inf I)^+} \gamma(t)$ and $y := \lim_{t \uparrow (\sup I)^-} \gamma(t)$ exist. Then one has

$$d_a(x, y) \leq L(\gamma, I).$$

**Proof.** Let $(a, b) \in I^2$ with $a < b$ and define for $u \in [0, 1]$, $\gamma_{ab}(u) = \gamma(a + u(b - a))$. Then $\gamma_{ab} \in A(\gamma(a), \gamma(b))$. By definition of the Agmon distance (see Definition 3), $d_a(a, b) \leq L(\gamma_{ab}, (0, 1)) = L(\gamma, (a, b))$. Taking the limits $a \rightarrow (\inf I)^+$, $b \rightarrow (\sup I)^-$ and using the continuity of the Agmon distance, one obtains that $d_a(x, y) \leq L(\gamma, I)$. Lemma 32 is proved.

As a simple consequence of this lemma, we have the following simple remark.

**Remark 6.** Let $x^*$ be a local minimum of $f$ (from $[H3]$, $x^* \in \Omega$). Then, for all $x$ in the basin of attraction of $x^*$ for the dynamics

$$\gamma' = \begin{cases} -\nabla f(\gamma) & \text{in } \Omega \\ -\nabla_T f(\gamma) & \text{on } \partial \Omega \end{cases}$$

(86)
it holds \( x \in \Omega \) and
\[
d_{a}(x^{*}, x) = f(x) - f(x^{*}). \tag{87}
\]
Indeed, from (79), we already have \( d_{a}(x^{*}, x) \geq f(x) - f(x^{*}) \). To prove the reverse inequality, from Lemma \[32\] it is enough to exhibit a Lipschitz curve \( \gamma : I \to \overline{\Omega} \) such that \( \partial \{ t \in I, \gamma(t) \in \partial \Omega \} \) is finite, \( L(\gamma, I) = f(x) - f(x^{*}) \) and \( \lim_{t \to \gamma(\inf I)}^{+} \gamma(t) = x^{*} \) and \( \lim_{t \to \gamma(\sup I)}^{-} \gamma(t) = x \). Such a curve is given on the interval \( I = [0, \infty) \) by considering the solution to (86) with initial condition \( \gamma(0) = x \). Notice that if \( \exists t_{0} \) such that \( \gamma(t_{0}) \in \partial \Omega \), then \( \forall t \geq t_{0}, \gamma(t) \in \partial \Omega \). Thus, since \( \lim_{t \to +\infty} \gamma(t) = x^{*} \in \Omega \), one has: for all \( t \geq 0, \gamma(t) \in \Omega \). Therefore,
\[
f(x^{*}) - f(x) = \int_{0}^{+\infty} \frac{d}{dt} f(\gamma(t)) dt = \int_{0}^{+\infty} \nabla f(\gamma(t)) \cdot \gamma'(t) dt = -\int_{0}^{+\infty} |\nabla f(\gamma(t))|^{2} dt \]
\[
= -\int_{0}^{+\infty} |\gamma'(t)| dt = -\lim_{t \to +\infty} L(\gamma,(0,t)).
\]
This concludes the proof of (87).

Notice that for \( \varepsilon > 0 \) small enough, the set \( f^{-1}(\{ f(x^{*}), f(x^{*}) + \varepsilon \}) \cap B(x^{*},\varepsilon) \subset \Omega \) is a neighborhood of \( x^{*} \) which is included in the basin of attraction of \( x^{*} \) for the dynamics (86). Therefore (87) holds in a neighborhood of \( x^{*} \).

### 3.2.2 A lower bound on the Agmon distance

In this section, easily computable lower bounds on the Agmon distance are provided. This is in particular useful to check if the hypothesis \[20\] appearing in Theorem \[1\] is satisfied, see for example Section \[1.6.2\].

**Proposition 33.** Let \( z \in \overline{\Omega} \) and denote by \( \mathcal{W} \) and \( \mathcal{W}' \) two closed neighborhoods of \( z \) in \( \overline{\Omega} \) with \( \mathcal{W} \subset \mathcal{W}' \). Define
\[
\alpha := \inf \{ d_{e}(x, y), x \in \overline{\omega} \setminus \mathcal{W}', y \in \mathcal{W} \}, \tag{88}
\]
where \( d_{e} \) denotes the geodesic distance for the Euclidean metric, see (85). Assume that \( \alpha > 0 \) and that there exists \( K > 0 \) such that
\[
\inf_{x \in \mathcal{W} \setminus \mathcal{W}'} g(x) > \frac{K}{\alpha},
\]
where \( g \) has been introduced in Definition \[2\]. Then, for all set \( B \subset \overline{\omega} \) such that \( B \cap \mathcal{W}' = \emptyset \),
\[
\inf_{y \in B} d_{a}(z, y) > K,
\]
where \( d_{a} \) is the Agmon distance (see Definition \[3\]).

**Proof.** By assumption, there exists \( \varepsilon > 0 \) such that
\[
\inf_{x \in \mathcal{W} \setminus \mathcal{W}'} g(x) \geq \frac{K}{\alpha} + \varepsilon.
\]
Let \( y \in B \) and \( \gamma \in \text{Lip}(z, y) \). Let us define
\[
t_{2} = \inf \{ t \in [0,1], \gamma(t) \not\in \mathcal{W}' \}, \quad t_{1} = \sup \{ t \in [0,1], t < t_{2}, \gamma(t) \in \mathcal{W} \}.
\]
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Since $\gamma$ is continuous and $\alpha > 0$, it holds $0 < t_1 < t_2 < 1$ and one has $\gamma(t) \in W \setminus W$ for all $t \in [t_1, t_2]$, $\gamma(t_1) \in W = W$ and $\gamma(t_2) \in \Omega \setminus W$. Then, one has:

$$\int_0^1 g(\gamma(t))|\gamma'(t)| \, dt \geq \int_{t_1}^{t_2} g(\gamma(t))|\gamma'(t)| \, dt \geq \left(\frac{K}{\alpha} + \varepsilon\right) \int_{t_1}^{t_2} |\gamma'(t)| \, dt \geq \left(\frac{K}{\alpha} + \varepsilon\right) \alpha = K + \varepsilon \alpha.$$ 

Since $\varepsilon \alpha$ is independent of $y \in B$ and since $\varepsilon \alpha$ is also independent of the curve $\gamma$, one can take the infimum over $\gamma$ and $y \in B$. Thus $\inf_{y \in B} d_a(z, y) > K$.

We now give a simple sufficient condition for the hypotheses (20) to hold, in the case where $f|_{\partial \Omega}$ has only two local minima. This result is based on Proposition 49 that will be proven in Section 3.4.3 below and which shows that

$$d_a(z_1, z_2) > f(z_2) - f(z_1).$$

In particular, the condition stated in the following proposition has been used in Section 1.6.2 in order to check hypothesis (20).

**Proposition 34.** Assume that $[H1]$ and $[H3]$ hold and assume in addition that $f|_{\partial \Omega}$ has only two local minima $z_1$ and $z_2$ (with $f(z_1) \leq f(z_2)$) on $\partial \Omega$. Then, if $z_2$ is the only global minimum of $f|_{\partial \Omega}$ on $B_{z_1}^c$, one has

$$\inf_{z \in B_{z_1}^c} d_a(z_1, z) > f(z_2) - f(z_1).$$

**Proof.** Proposition 49 and the continuity of the Agmon distance ensure that there exist an open ball $B_2 \subset B_{z_1}^c$ centered at $z_2$, and $\varepsilon > 0$ such that for all $z \in B_2$

$$d_a(z_1, z) \geq f(z_2) - f(z_1) + \varepsilon.$$ 

Since $z_2$ is the only global minimum of $f|_{\partial \Omega}$ on $B_{z_1}^c$, there exists $\varepsilon' > 0$, such that for all $z \in B_{z_1}^c \setminus B_2$, $f(z) \geq f(z_2) + \varepsilon'$. In addition, from the inequality (79), for all $z \in B_{z_1}^c \setminus B_2$, it holds

$$d_a(z_1, z) \geq f(z) - f(z_1) \geq f(z_2) - f(z_1) + \varepsilon'.$$

Consequently $\inf_{z \in B_{z_1}^c} d_a(z_1, z) > f(z_2) - f(z_1)$.

### 3.3 Agmon distance near critical points of $f$ or $f|_{\partial \Omega}$ and eikonal equation

We will show that the Agmon distance $d_a$ locally solves the eikonal equation in a neighborhood of any critical point of $f|_{\partial \Omega}$ or $f$ (or equivalently, any point $x$ such that $g(x) = 0$, see (17)).
3.3.1 The Agmon distance near critical points of \( f \)

**Proposition 35.** Let us assume that [H1] holds. Let \( x^* \in \Omega \) be such that \( \nabla f(x^*) = 0 \). Let us denote by \((\mu_1, \ldots, \mu_d) \in (\mathbb{R})^d \) the eigenvalues of the Hessian of \( f \) at \( x^* \). Then there exist a neighborhood \( V^* \) of \( x^* \) in \( \Omega \) and a \( C^\infty \) function \( \Phi: V^* \to \mathbb{R} \) such that

\[
|\nabla \Phi|^2 = |\nabla f|^2, \quad \Phi(x_1, \ldots, x_d) = \sum_{i=1}^d |\mu_i| (x_i - x^*_i)^2 + O(|x - x^*|^3). \tag{89}
\]

Moreover, one has the following uniqueness result: if \( \tilde{\Phi} \) is a \( C^\infty \) real valued function defined on a neighborhood \( \tilde{V}^* \) of \( x^* \) satisfying (89), then \( \tilde{\Phi} = \Phi \) on \( V^* \cap \tilde{V}^* \).

Let us notice that \( \Phi(x^*) = 0 \). In addition, up to choosing a smaller neighborhood \( V^* \) of \( x^* \), one can assume that \( \Phi \) is positive on \( V^* \setminus \{x^*\} \). The point \( x^* \) is then a non degenerate minimum of \( \Phi \).

**Proof.** The proof is made in [31, Proposition 2.3.6] in the more general setting where \( |\nabla f|^2 \) is replaced in (89) by a smooth positive function \( W \) around a non degenerate minimum \( y^* \) of \( W \) such that \( W(y^*) = 0 \). Here \( W = |\nabla f|^2 \) and \( y^* = x^* \). This leads to \( \nabla W = 2 \text{Hess} f (\nabla f) \) and thus \( \text{Hess} W(x^*) = 2 (\text{Hess} f)^2(x^*) \) is a non degenerate matrix. Therefore \( x^* \) is indeed a non degenerate minimum of \( W = |\nabla f|^2 \).

**Proposition 36.** Let us assume that [H1] and [H3] hold. Let \( x^* \in \Omega \) be such that \( \nabla f(x^*) = 0 \). Then there exists a neighborhood \( U^* \) of \( x^* \) in \( \Omega \) such that for all \( x \in U^* \)

\[
d_a(x^*, x) = \Phi(x), \tag{90}
\]

where \( \Phi \) is the smooth solution of (89) and \( d_a \) is the Agmon distance.

**Proof.** Notice that hypothesis [H3] allows us to use Corollary 28. Let \( \Phi \) be a smooth solution of (89) on a neighborhood \( V^* \) of \( x^* \), as defined in Proposition 35 and such that \( \Phi \) is positive on \( V^* \setminus \{x^*\} \). There exists \( \varepsilon > 0 \) such that \( U^* := \Phi^{-1}(0, \varepsilon) \subset V^* \) is a neighborhood of \( x^* \) (consider for example \( \varepsilon = \inf \{\Phi(x), x \in V^*, B(x^*, r)\} \) > 0 where \( r > 0 \) is such that \( B(x^*, 2r) \subset V^* \)).

Let us first prove that for \( x \in U^* \), \( \Phi(x) \leq d_a(x, x^*) \). For \( x \in U^* \), one has \( \Phi(x) < \varepsilon \) and thus \( \Phi^{-1}(0, \Phi(x)) \subset U^* \). Let \( \gamma \) belong to \( A(x^*, x) \). Let us define the time \( t_0 := \inf \{ t \in [0, 1], \gamma(t) \notin \Phi^{-1}(0, \Phi(x)) \} \). By continuity of the curve \( \gamma \), one has \( t_0 > 0 \), \( \Phi(\gamma(t_0)) = \Phi(x) \) and for all \( t \in [0, t_0] \), \( \gamma(t) \in \Phi^{-1}(0, \Phi(x)) \subset U^* \). Thus, since the curve \( \gamma \) is Lipschitz and since for all \( t \in [0, t_0] \), \( \gamma(t) \in \Omega \), one has

\[
\Phi(x) = \int_0^{t_0} \frac{d}{dt} \Phi(\gamma(t)) \, dt = \int_0^{t_0} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \, dt
\leq \int_0^{t_0} |\nabla \Phi(\gamma(t))| |\gamma'(t)| \, dt
\leq \int_0^{t_0} |\nabla f(\gamma(t))| |\gamma'(t)| \, dt \leq \int_0^1 g(\gamma(t)) |\gamma'(t)| \, dt = L(\gamma, (0, 1)).
\]
Taking the infimum on the right-hand side over $\gamma \in A(x^*, x)$, one gets $\Phi(x) \leq d_a(x^*, x)$ for all $x \in U^*$. Let us now prove the reverse inequality: for $x \in U^*$, $\Phi(x) \geq d_a(x, x^*)$. For $x \in U^*$, let us define a curve $\gamma : \mathbb{R}_+ \to U^*$ by

$$\forall t \geq 0, \gamma'(t) = -\nabla \Phi(\gamma(t)) \text{ and } \gamma(0) = x.$$ 

Since the function $t \mapsto \Phi(\gamma(t))$ is decreasing, the curve $\gamma$ always belongs to $U^*$ and is defined on $\mathbb{R}_+$. Moreover $\gamma$ is $C^\infty$ and satisfies

$$\lim_{t \to +\infty} \gamma(t) = x^*.$$ 

Since $\gamma$ is with values in $U^* \subset \Omega$, one has

$$-\Phi(x) = \int_0^{+\infty} \frac{d}{dt} \Phi(\gamma(t)) \ dt = \int_0^{+\infty} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \ dt = -\int_0^{+\infty} |\nabla \Phi(\gamma(t))|^2 dt$$

$$= -\int_0^{+\infty} |\nabla \Phi(\gamma(t))| \cdot |\gamma'(t)| \ dt = -\int_0^{+\infty} g(\gamma(t)) \cdot |\gamma'(t)| \ dt = -\lim_{t \to +\infty} L(\gamma, (0, t)).$$

Thanks to Lemma 32,

$$d_a(x, x^*) \leq L(\gamma, (0, +\infty)) = \Phi(x).$$

Therefore $\Phi(x) = d_a(x^*, x)$ for all $x \in U^*$.

**Remark 7.** Let us mention a simple consequence of the previous proof that will be useful in the following. If $x^* \in \Omega$ is such that $\nabla f(x^*) = 0$, there exists a neighborhood $U^*$ of $x^*$ in $\Omega$ such that for all $x \in U^*$, there exists a $C^\infty$ curve $\gamma : \mathbb{R}_+ \to \Omega$ such that

$$d_a(x^*, x) = \int_0^{+\infty} |\nabla f(\gamma(t))| \cdot |\gamma'(t)| \ dt,$$

with $\gamma(0) = x$ and $\lim_{t \to +\infty} \gamma(t) = x^*$. The curve $\gamma$ is defined by

$$\gamma'(t) = -\nabla \Phi(\gamma(t)), \quad \gamma(0) = x,$$

where $\Phi$ solves (89). In addition $\{t \in [0, \infty), \gamma(t) \in \partial \Omega\}$ is empty.

### 3.3.2 The Agmon distance near critical points of $f|_{\partial \Omega}$

Let us first define the Agmon distance in the boundary $\partial \Omega$.

**Definition 7.** The Agmon distance between $x \in \partial \Omega$ and $y \in \partial \Omega$ in the boundary $\partial \Omega$ is defined by

$$d_{\partial \Omega}^a(x, y) = \inf_\gamma \int_0^1 |\nabla_T f(\gamma(t))| \cdot |\gamma'(t)| \ dt,$$

where the infimum is taken over all the paths $\gamma : [0, 1] \to \partial \Omega$ which are Lipschitz with $\gamma(0) = x$ and $\gamma(1) = y$.

Similarly to Remark 6, one has:

**Remark 8.** If $x^*$ is a local minimum of $f|_{\partial \Omega}$, one has $d_{\partial \Omega}^a(x^*, x) = f(x) - f(x^*)$ for all $x \in \partial \Omega$ which is in the basin of attraction of $x^*$ in $\partial \Omega$ for the gradient dynamics $\gamma' = -\nabla_T f(\gamma)$. 

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The next proposition is the equivalent of Proposition 36 for that Agmon distance in \( \partial \Omega \). Since \( \partial \Omega \) is a smooth manifold without boundary, the next result is a direct consequence of well known results from \([30], [19]\) and \([22]\).

**Proposition 37.** Let us assume that \([H1]\) holds. Let \( x^* \in \partial \Omega \) be such that \( \nabla_T f(x^*) = 0 \). Then there exists a neighborhood \( U^* \) of \( x^* \) in \( \partial \Omega \) such that \( y \mapsto d^R_\partial \Omega(x^*, y) \) is smooth on \( U^* \) and \( \forall x \in U^* \),

\[
\left| \nabla_T d^R_\partial \Omega(x^*, x) \right|^2 = |\nabla_T f(x)|^2. \tag{92}
\]

**Proof.** The boundary \( \partial \Omega \) is a \( C^\infty \) compact manifold and \( x^* \) is a non degenerate minimum of \( |\nabla_T f|^2 \). The proof is made in \([31]\) Proposition 2.3.6 in the more general setting where \( |\nabla_T f|^2 \) is replaced in \([92]\) and in \([91]\) by a smooth non negative function \( W \) around a non degenerate minimum \( y^* \) of \( W \) such that \( W(y^*) = 0 \). Here \( W = |\nabla_T f|^2 \) and \( y^* = x^* \). This leads to \( \nabla W = 2 \text{Hess}(f|_{\partial \Omega}) (\nabla_T f) \) and therefore \( x^* \) is a critical point of \( W = |\nabla_T f|^2 \) (which turns out to be a minimum). In addition, since \( \nabla_T f(x^*) = 0 \), one gets that \( \text{Hess}W(x^*) = 2 (\text{Hess}(f|_{\partial \Omega}))^2(x^*) \) which is a non degenerate matrix. \( \blacksquare \)

**Proposition 38.** Let us assume that \([H1]\) and \([H3]\) hold. Let \( x^* \in \partial \Omega \) be such that \( \nabla_T f(x^*) = 0 \). Then there exist a neighborhood \( V^* \) of \( x^* \) in \( \overline{\Omega} \) and a \( C^\infty \) function \( \Phi : V^* \to \mathbb{R} \) such that

\[
\begin{cases}
|\nabla \Phi|^2 = |\nabla f|^2 \text{ in } \Omega \cap V^*, \\
\Phi = d^R_\partial \Omega(x^*, \cdot) \text{ on } \partial \Omega \cap V^*, \\
\partial_n \Phi < 0 \text{ on } \partial \Omega \cap V^*. 
\end{cases} \tag{93}
\]

Moreover, one has the following uniqueness results: if \( \tilde{\Phi} \) is a \( C^\infty \) real valued function defined on a neighborhood \( \tilde{V}^* \) of \( x^* \) satisfying \([93]\), then \( \tilde{\Phi} = \Phi \) on \( \tilde{V}^* \cap V^* \).

Finally, up to choosing a smaller neighborhood \( V^\prime \) of \( x^* \), one can assume that \( \Phi \) is positive on \( V^\prime \setminus \{x^*\} \), so that \( x^* \) is a non degenerate minimum of \( \Phi \) on \( V^\prime \).

**Proof.** From Proposition 37 the function \( x \in \partial \Omega \mapsto d^R_\partial \Omega(x^*, x) \) is smooth near \( x^* \). Then, the result stated can be proven using the method of characteristics, see \([19]\) Theorem 1.5 or \([22]\) Section 3.2]. Let us mention that the proof crucially relies on the assumption \( \partial_n f(x^*) > 0 \). The fact that one can reduce \( V^\prime \) such that \( \Phi \) is positive on \( V^\prime \setminus \{x^*\} \) is a consequence of \( \partial_n \Phi < 0 \) on \( \partial \Omega \cap V^\prime \) together with the fact that \( x^* \) is the only minimum of \( d^R_\partial \Omega(x^*, \cdot) \) (which is positive on \( \partial \Omega \setminus \{x^*\} \)). \( \blacksquare \)

Let us state a simple corollary of Proposition 38 and Remark 8.

**Corollary 39.** Let us assume that \([H1]\) and \([H3]\) hold. Let \( x^* \in \partial \Omega \) be a local minimum of \( f|_{\partial \Omega} \). Then there exist a neighborhood \( V^* \) of \( x^* \) in \( \overline{\Omega} \) and a \( C^\infty \) function \( \Phi : V^* \to \mathbb{R} \) such that

\[
\begin{cases}
|\nabla \Phi|^2 = |\nabla f|^2 \text{ in } \Omega \cap V^*, \\
\Phi = f - f(x^*) \text{ on } \partial \Omega \cap V^*, \\
\partial_n \Phi < 0 \text{ on } \partial \Omega \cap V^*. 
\end{cases} \tag{94}
\]

Moreover, one has the following uniqueness results: if \( \tilde{\Phi} \) is a \( C^\infty \) real valued function defined on a neighborhood \( \tilde{V}^* \) of \( x^* \) satisfying \([93]\), then \( \tilde{\Phi} = \Phi \) on \( \tilde{V}^* \cap V^* \).
Finally, up to choosing a smaller neighborhood $V^*$ of $x^*$, one can assume that $\Phi$ is positive on $V^* \setminus \{x^*\}$, and that $\Phi - f > -f(x^*)$ in $V^* \cap (\partial \Omega)^c$. As a consequence,

$$\{x \in V^*, \Phi(x) = f(x) - f(x^*)\} \subset \partial \Omega. \quad (95)$$

Proof. All the statements but $(95)$ are direct consequences of Proposition 38 and the fact that $\partial_a^{\partial \Omega}(x^*, x) = f(x) - f(x^*)$, thanks to Remark 8. Now, notice that on $\partial \Omega \cap V^*$, $\Phi - f = -f(x^*)$ and $\partial_a(\Phi - f) < 0$ so that, up to choosing a smaller neighborhood $V^*$ of $x^*$, one can assume that $\Phi - f > -f(x^*)$ in $V^* \cap (\partial \Omega)^c$. This concludes the proof of $(95)$. \hfill \blacksquare

We are now in position to state the main result of this section.

**Proposition 40.** Let us assume that $[H1]$ and $[H3]$ hold. Let $x^* \in \partial \Omega$ be such that $\nabla_T f(x^*) = 0$. Then there exists a neighborhood $U^*$ of $x^*$ in $\overline{\Omega}$ such that for all $x \in U^*$

$$d_a(x, x^*) = \Phi(x),$$

where $\Phi$ solves $(H3)$ and $d_a$ is the Agmon distance.

Proof. Notice that hypothesis $[H3]$ allows us to use Corollary 28. The proof is similar to the one of Proposition 36. Let $x^* \in \partial \Omega$ be such that $\nabla_T f(x^*) = 0$. Let $\Phi$ be the smooth solution of $(H3)$ on a neighborhood $V^*$ of $x^*$ and such that $\Phi$ is positive on $V^* \setminus \{x^*\}$, as defined in Proposition 38. One chooses $\varepsilon > 0$ sufficiently small such that $U^* := \Phi^{-1}([0, \varepsilon)) \subset V^*$ is a neighborhood of $x^*$ in $\overline{\Omega}$.

Step 1. Let us first prove that for all $x \in U^*$, $\Phi(x) \leq d_a(x^*, x)$. For $x \in U^*$, one has $\Phi(x) < \varepsilon$ and thus $\Phi^{-1}([0, \Phi(x))) \subset U^*$. Let $\gamma$ belong to $A(x^*, x)$. Let us define the time $t_0 := \inf \{t \in [0, 1], \gamma(t) \notin \Phi^{-1}([0, \Phi(x)))\}$. By continuity of the curve $\gamma$, one has $t_0 > 0$, $\Phi(\gamma(t_0)) = \Phi(x)$ and for all $t \in [0, t_0)$, $\gamma(t) \in \Phi^{-1}([0, \Phi(x))) \subset U^*$. Thus, using Lemma 25, one obtains

$$\Phi(x) = \int_0^{t_0} \frac{d}{dt} \Phi \circ \gamma(t) \, dt = \int_0^{t_0} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \, dt \leq \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} \nabla T \Phi(\gamma(t)) \cdot \gamma'(t) \, dt + \int_{\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \, dt.$$

On the one hand,

$$\int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} \nabla T \Phi(\gamma(t)) \cdot \gamma'(t) \, dt = \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} \nabla T \partial_a^{\partial \Omega}(x^*, \gamma(t)) \cdot \gamma'(t) \, dt \leq \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} |\nabla T \partial_a^{\partial \Omega}(x^*, \gamma(t))||\gamma'(t)| \, dt \leq \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \partial \Omega\}} |\nabla f(\gamma(t))||\gamma'(t)| \, dt,$$

where one used the relations $(91)$ and $(92)$. On the other hand, using $(92)$, one obtains

$$\int_{\{t \in (0, t_0), \gamma(t) \in \Omega\}} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \, dt \leq \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \Omega\}} |\nabla \Phi(\gamma(t))||\gamma'(t)| \, dt \leq \int_{\text{int}\{t \in (0, t_0), \gamma(t) \in \Omega\}} |\nabla f(\gamma(t))||\gamma'(t)| \, dt.$$
Thus one gets
\[
\Phi(x) \leq \int_{\text{int}(t \in (0, t_0), \gamma(t) \in \partial \Omega)} |\nabla_T f(\gamma(t))| |\gamma'(t)| \, dt + \int_{\{t \in (0, t_0), \gamma(t) \in \Omega\}} |\nabla f(\gamma(t))| |\gamma'(t)| \, dt
\]
\[
= \int_0^{t_0} g(\gamma(t)) |\gamma'(t)| \, dt \leq \int_0^{1} g(\gamma(t)) |\gamma'(t)| \, dt = L(\gamma, (0, 1)).
\]
Taking the infimum on the right-hand side over \(\gamma \in A(x^*, x)\), one gets \(\Phi(x) \leq d_\alpha(x^*, x)\), for all \(x \in U^*\).

Step 2. Let us now prove the reverse inequality: \(\forall x \in U^*, d_\alpha(x, x^*) \leq \Phi(x)\) Let us define the following vector field on \(U^*\)
\[
X := \begin{cases} 
-\nabla \Phi \text{ in } \Omega \cap U^*, \\
-\nabla T \Phi \text{ on } \partial \Omega \cap U^*.
\end{cases}
\]
(96)

For \(x \in U^*\), let us define the curve \(\gamma\) by
\[
\forall t \geq 0, \gamma'(t) = X(\gamma(t)) \text{ and } \gamma(0) = x.
\]
(97)

Since the function \(t \mapsto \Phi(\gamma(t))\) is decreasing, the curve \(\gamma\) always belongs to \(U^*\) and is defined on \(\mathbb{R}_+\). If there exists a time \(t_0\) such that \(\gamma(t_0) \in \partial \Omega\), then, for all \(t \geq t_0\), \(\gamma(t) \in \partial \Omega\). The function \(\gamma\) is piecewise \(C^\infty\), continuous and satisfies
\[
\lim_{t \to +\infty} \gamma(t) = x^*.
\]

Let us define \(t_{\partial \Omega} = \inf \{t \in [0, +\infty), \gamma(t) \in \partial \Omega\} \in [0, \infty]\). One has
\[
-\Phi(x) = \int_0^{+\infty} \frac{d}{dt} \Phi \circ \gamma(t) \, dt
\]
\[
= \int_0^{t_{\partial \Omega}} \nabla \Phi(\gamma(t)) \cdot \gamma'(t) \, dt + \int_{t_{\partial \Omega}}^{+\infty} \nabla_T \Phi(\gamma(t)) \cdot \gamma'(t) \, dt
\]
\[
= -\left(\int_0^{t_{\partial \Omega}} |\nabla \Phi(\gamma(t))|^2 \, dt + \int_{t_{\partial \Omega}}^{+\infty} |\nabla_T \Phi(\gamma(t))|^2 \, dt\right)
\]
\[
= -\left(\int_0^{t_{\partial \Omega}} |\nabla \Phi(\gamma(t))| |\gamma'(t)| \, dt + \int_{t_{\partial \Omega}}^{+\infty} |\nabla_T \Phi(\gamma(t))| |\gamma'(t)| \, dt\right)
\]
\[
= -\int_0^{+\infty} g(\gamma(t)) |\gamma'(t)| \, dt = -\lim_{t \to +\infty} L(\gamma, (0, t)).
\]

Thanks to Lemma
\[
d_\alpha(x, x^*) \leq L(\gamma, (0, +\infty)) = \Phi(x).
\]

In conclusion, \(\Phi(x) = d_\alpha(x^*, x)\) for all \(x \in U^*\).

**Remark 9.** Let us mention a simple consequence of the previous proof that will be useful in the following. If \(x^* \in \partial \Omega\) is such that \(\nabla_T f(x^*) = 0\), there exists a neighborhood \(U^*\) of \(x^*\) such that for all \(x \in U^*\), there exists a piecewise \(C^\infty\) and continuous curve \(\gamma: \mathbb{R}_+ \to \overline{\Omega}\) such that
\[
d_\alpha(x^*, x) = \int_0^{+\infty} g(\gamma(t)) |\gamma'(t)| \, dt,
\]
with \(\gamma(0) = x\) and \(\lim_{t \to +\infty} \gamma(t) = x^*\). In addition \(\partial \{t \in [0, \infty), \gamma(t) \in \partial \Omega\}\) either consists of one point or is empty.
3.4 Curves realizing the Agmon distance

In this section, it is proven that for any two points \( x \in \Omega \) and \( y \in \Omega \), there exists a finite number of curves \((\gamma_i)_{i=1,\ldots,N}\) such that the sum of their lengths equals the Agmon distance \( d_a(x,y) \). The precise statement is given in the following theorem.

**Theorem 3.** Assume that \([H1]\) and \([H3]\) hold. Let \( x,y \in \Omega \). Then there exists a finite number of Lipschitz curves \((\gamma_j)_{j=1,\ldots,N}\) which are defined on possibly unbounded intervals \( I_j \subset \mathbb{R} \), with values in \( \Omega \), such that for all \( j \in \{1,\ldots,N\} \), the sets \( \partial \{ t \in I_j, \gamma_j(t) \in \partial \Omega \} \) are finite and

\[
d_a(x,y) = \sum_{j=1}^{N} L(\gamma_j, I_j).
\]

Additionally, by construction, the intervals \((I_j)_{j \in \{1,\ldots,N\}}\) are either \([0, +\infty)\), \((-\infty, 0]\) or \([0, 1]\). Moreover, if \( I_j = [0, +\infty) \) or \( I_j = (-\infty, 0] \), then \( \gamma_j \) is continuous and piecewise \( C^\infty \). If \( I_j = [0, 1] \), then \( \gamma_j \in A(\gamma_j(0), \gamma_j(1)) \). Finally the curves \((\gamma_1, I_1), \ldots, (\gamma_N, I_N))\) are ordered such that

\[
\lim_{t \to (\inf I_1)^+} \gamma_1(t) = x, \quad \lim_{t \to (\sup I_N)^-} \gamma_N(t) = y,
\]

and for all \( k \in \{1, \ldots, N - 1\} \),

\[
\lim_{t \to (\sup I_k)^-} \gamma_k(t) = \lim_{t \to (\inf I_{k+1})^+} \gamma_{k+1}(t).
\]

This section is entirely dedicated to the proof of Theorem 3. In the following, one denotes by

\[
\{x_1, \ldots, x_m\} = \{ x \in \Omega, g(x) = 0 \},
\]

where \( g \) is defined by (17) (there is a finite number of zeros of \( g \) thanks to \([H1]\)).

3.4.1 Preliminary results

Let us first consider the simple case when the curve realizing the Agmon distance does not meet zeros of \( g \).

**Lemma 41.** Assume that \([H1]\) and \([H3]\) hold. Let \((x,y) \in \Omega \times \Omega \). Let \((\gamma_n)_{n \geq 0} \in A(x,y)^\mathbb{N} \) be a minimizing sequence of curves for \( d_a(x,y) \): \( \lim_{n \to \infty} L(\gamma_n, (0,1)) = d_a(x,y) \). In addition, assume that for each \( k \in \{1, \ldots, m\} \), there exists a neighborhood \( V_k \) of \( x_k \) in \( \Omega \), such that:

\[
\forall n \in \mathbb{N}, \forall k \in \{1, \ldots, m\}, \text{Ran}(\gamma_n) \cap V_k = \emptyset.
\]

Then, there exists \( \gamma \in A(x,y) \) such that

\[
L(\gamma, (0,1)) = d_a(x,y).
\]

**Proof.** Let \( M \) be such that for all \( n \), \( L(\gamma_n, (0,1)) \leq M \) and let us define

\[
c := \inf_{\Omega \setminus (V_1 \cup \ldots \cup V_m)} g > 0.
\]
One defines for \( t \in [0,1] \), \( \phi_n(t) = \frac{L(\gamma_n,(0,t))}{L(\gamma_n,(0,1))} + t \). The map \( \phi_n \) is strictly increasing and continuous from \([0,1]\) to \([0,1]\). Therefore it admits an inverse. Setting \( \tilde{\gamma}_n(u) := \gamma_n \circ \phi_n^{-1}(u) \), one gets \( L(\gamma_n,(0,1)) = L(\tilde{\gamma}_n,(0,1)) \) and

\[
|\tilde{\gamma}_n'|(\phi_n(t)) = \frac{|\gamma_n'(t)|}{g(\gamma_n(t))} |\gamma_n'(t)| + 1 (L(\gamma_n,(0,1)) + 1) \\
\leq \frac{c}{c |\gamma_n'(t)| + 1} (L(\gamma_n,(0,1)) + 1) \\
\leq \frac{1}{c} (L(\gamma_n,(0,1)) + 1) \\
\leq \frac{1}{c} (M + 1).
\]

Thus, up to replacing \( \gamma_n \) by \( \tilde{\gamma}_n \), one may assume that the Lipschitz constants of \( \gamma_n \) are bounded uniformly in \( n \). In addition since for all \( t \in [0,1] \), \( \gamma_n(t) \in \overline{\Omega} \), the sequence \( (\gamma_n)_{n \geq 0} \) is relatively compact in \( C^0([0,1],\overline{\Omega}) \). Thus, up to the extraction of a subsequence, there exists a Lipschitz curve \( \gamma \) such that \( \lim_{n \to \infty} \gamma_n = \gamma \) uniformly on \([0,1]\). Moreover since \( (\gamma_n)_{n \geq 0} \) is bounded in \( H^1([0,1],\overline{\Omega}) \), up to the extraction of a subsequence, \( (\gamma_n)_{n \geq 0} \) converges weakly to \( \gamma \) in \( H^1([0,1],\overline{\Omega}) \). It is not difficult to see that for all \( t \in [0,1] \),

\[
\liminf_{n \to \infty} g(\gamma_n(t)) \geq g(\gamma(t)).
\]

Indeed, for \( t \in [0,1] \), there are two cases:

- if \( \gamma(t) \in \Omega \), then for \( n \) large enough, all the points \( \gamma_n(t) \) are in \( \Omega \) and thus \( \liminf_{n \to \infty} g(\gamma_n(t)) = \lim_{n \to \infty} g(\gamma_n(t)) = |\nabla f(\gamma(t))| = g(\gamma(t)) \),

- if \( \gamma(t) \in \partial \Omega \). Since \( N = \{ n, \gamma_n(t) \in \partial \Omega \} \cup \{ n, \gamma_n(t) \in \Omega \} \), one obtains that the set of limit points of \( (\gamma_n(t))_{n \geq 0} \) is included in \( \{|\nabla f(\gamma(t))|, |\nabla_T f(\gamma(t))|\} \). Therefore, from \([H3]\), one has: \( \liminf_{n \to \infty} g(\gamma_n(t)) \geq |\nabla_T f(\gamma(t))| = g(\gamma(t)) \).

Then, one obtains

\[
d_a(x,y) = \lim_{t \to \infty} \int_0^1 g(\gamma(t))|\gamma'(t)|dt \geq \liminf_{n \to \infty} \liminf_{p \to \infty} \int_0^1 g(\gamma_p(t))|\gamma'_n(t)|dt \\
\geq \liminf_{n \to \infty} \int_0^1 \liminf_{p \to \infty} g(\gamma_p(t))|\gamma'_n(t)|dt \\
\geq \liminf_{n \to \infty} \int_0^1 g(\gamma(t))|\gamma'_n(t)|dt \\
\geq \int_0^1 g(\gamma(t))|\gamma'(t)|dt.
\]

In the previous computation, one used Fatou Lemma and the lower semi continuity (for the weak convergence) of the convex functional

\[
h \in H^1([0,1],\overline{\Omega}) \mapsto \int_0^1 g(\gamma(t)) |h'(t)|dt.
\]

Since \([H3]\) holds, using Proposition \([27]\), there exits a curve \( \gamma_1 \in A(x,y) \) such that \( L(\gamma,(0,1)) \geq L(\gamma_1,(0,1)) \) and thus \( d_a(x,y) = L(\gamma_1,(0,1)) \).

\[\Box\]
Let us now introduce a sufficient condition so that a minimizing sequence of curves realizing the Agmon distance avoids a neighborhood of a zero of \( g \). For \( x \in \overline{\Omega} \), one introduces the following sets:

\[
\forall k \in \{1, \ldots, m\}, \quad A_k(x) := \{ z \in \overline{\Omega}, \quad d_a(x,z) = d_a(x,x_k) + d_a(x_k,z) \}. \tag{98}
\]

One notices that \( z \in A_k(x) \) if and only if \( x \in A_k(z) \).

**Proposition 42.** Assume that \([H1]\) and \([H3]\) hold. Let \((x,y) \in \overline{\Omega}^2\) and assume that there exists \( k \in \{1, \ldots, m\} \) such that \( y \notin A_k(x) \). If \((\gamma_n)_{n \geq 0} \in A(x,y)^\mathbb{N}\) is a minimizing sequence of curves for \( d_a(x,y) \), then there exists a neighborhood \( V_k \) of \( x_k \) in \( \overline{\Omega} \) and \( n_0 \in \mathbb{N} \), such that for all \( n \geq n_0 \),

\[
\text{Ran}(\gamma_n) \cap V_k = \emptyset. 
\]

**Proof.** If \( y \notin A_k(x) \), for a \( k \in \{1, \ldots, m\} \), then \( d_a(x,y) < d_a(x,x_k) + d_a(x_k,y) \) and thus \( y \neq x_k \) and \( x \neq x_k \). Let us define

\[
\varepsilon := d_a(x,x_k) + d_a(x_k,y) - d_a(x,y) > 0,
\]

and \( V_k := B_a \left( x_k, \min \left( \varepsilon, \frac{d_a(x_k,y)}{2} \right) \right) \) where

\[
\forall z \in \overline{\Omega}, \forall r > 0, B_a(z,r) := \left\{ u \in \overline{\Omega}, \quad d_a(z,u) < r \right\}. \tag{99}
\]

Notice that \( y \notin V_k \). We now prove Proposition 42 by contradiction. We assume that, up to the extraction of a subsequence, for all \( n \in \mathbb{N} \),

\[
\text{Ran}(\gamma_n) \cap V_k \neq \emptyset,
\]

and we define

\[
t_0^n := \inf \{ t \in [0,1], \quad \gamma_n(t) \in V_k \}, \quad t_1^n := \sup \{ t \in [0,1], \quad \gamma_n(t) \in V_k \}.
\]

We have for all \( n \in \mathbb{N} \), owing to the triangular inequality,

\[
L(\gamma_n, (0,t_0^n)) \geq d_a(x,x_k) - \frac{\varepsilon}{3}, \quad L(\gamma_n, (t_1^n, 1)) \geq d_a(x_k,y) - \frac{\varepsilon}{3}.
\]

Thus for all \( n \in \mathbb{N} \),

\[
L(\gamma_n(0,1)) \geq L(\gamma_n, (0,t_0^n)) + L(\gamma_n, (t_1^n, 1)) \geq d_a(x,x_k) + d_a(x_k,y) - \frac{2\varepsilon}{3} = d_a(x,y) + \frac{\varepsilon}{3}.
\]

This contradicts the fact that \( \lim_{n \to \infty} L(\gamma_n, (0,1)) = d_a(x,y) \).

A direct corollary of Proposition 42 and Lemma 41 is the following result:

**Corollary 43.** Assume that \([H1]\) and \([H3]\) hold. Let \( y \in \overline{\Omega} \) and assume that \( y \notin A_j(x) \) for all \( j \in \{1, \ldots, m\} \). Then there exists a curve \( \gamma \in A(x,y) \) such that

\[
d_a(x,y) = L(\gamma, (0,1)).
\]

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Notice that $y \notin A_j(x)$ for all $j \in \{1, \ldots, m\}$ implies in particular that $x$ and $y$ are not zeros of $g$. This corollary will be used below to build the curves $\gamma_j$ associated with intervals $I_j = [0, 1]$ in Theorem 3. The curves $\gamma_j$ associated with intervals $I_j = [0, +\infty)$ or $I_j = (-\infty, 0]$ will be built using the following lemma, which is a direct consequence of Remarks 7 and 9.

**Lemma 44.** Assume that \([H1] and [H3] hold. Let $k \in \{1, \ldots, m\}$. There exists a neighborhood $V_k$ of $x_k$ in $\overline{\Omega}$, such that for all $y \in V_k$, there exists a continuous and piecewise $C^\infty$ curve $\gamma$ defined on $(-\infty, 0]$ satisfying

\[
d_a(y, x_k) = L(\gamma, (-\infty, 0]), \quad \lim_{t \to -\infty} \gamma(t) = x_k, \quad \gamma(0) = y.
\]

In addition $\partial\{t \in (-\infty, 0], \gamma(t) \in \partial\Omega\}$ is either empty or a single point.

Before proving Theorem 3, we finally need two additional preliminary lemmas.

**Lemma 45.** Assume that \([H1] and [H3] hold. Let $u \in \overline{\Omega}$ and $w \in \overline{\Omega}$. For any $\delta > 0$ small enough, there exists $z_\delta$ such that $d_a(u, z_\delta) = \delta$ and $d_a(w, u) = d_a(w, z_\delta) + d_a(z_\delta, u)$.

**Proof.** Notice that $d_a(u, z_\delta) = \delta$ is equivalent to $z_\delta \in \partial B_a(u, \delta)$, where $B_a$ is defined by (99). We prove Lemma 45 by contradiction. Assume that there exists $\delta \in \left(0, \frac{d_a(u, w)}{2}\right)$ such that for all $z \in \partial B_a(u, \delta)$,

\[
d_a(w, u) < d_a(w, z) + d_a(z, u).
\]

By compactness of $\partial B_a(u, \delta)$, there exists $a_\delta > 0$ such that for all $z \in \partial B_a(u, \delta)$,

\[
d_a(w, u) + a_\delta \leq d_a(w, z) + d_a(z, u).
\]

Thus if $\gamma \in A(u, w)$, since there exists a time $t_\delta$ such that $\gamma(t_\delta) \in \partial B_a(u, \delta)$, one has

\[
L(\gamma, (0, 1)) = L(\gamma, (0, \gamma(t_\delta))) + L(\gamma, (\gamma(t_\delta), 0)) \\
\geq d_a(u, \gamma(t_\delta)) + d_a(\gamma(t_\delta), w) \\
\geq d_a(u, w) + a_\delta.
\]

This is impossible since by definition $d_a(u, w) = \inf_{\gamma \in A(u, w)} L(\gamma, (0, 1))$.

**Lemma 46.** Assume that \([H1] holds. Let $(x, y) \in \overline{\Omega}^2$ with $x \neq y$. Let us assume that there exists $j \in \{1, \ldots, m\}$ such that $y \in A_j(x)$. Then, there exist $N \in \mathbb{N}$ and a sequence $(b_j)_{j \in \{0, \ldots, N+1\}} \in \overline{\Omega}^{N+2}$, $b_0 = x$, $b_{N+1} = y$, $(b_j)_{j \in \{1, \ldots, N\}} \in \{x_1, \ldots, x_m\}^N$ (with the convention $\{x_1, \ldots, x_m\}^0 = \emptyset$) such that the following holds:

1. For all $i \in \{0, \ldots, N\}$, $b_i \neq b_{i+1}$ and

\[
d_a(x, y) = \sum_{i=0}^{N} d_a(b_i, b_{i+1}). \quad (100)
\]

2. For all $i \in \{0, \ldots, N\}$ and for all $z \in \{x, y, x_1, \ldots, x_m\} \setminus \{b_i, b_{i+1}\}$,

\[
d_a(b_i, b_{i+1}) < d_a(b_i, z) + d_a(z, b_{i+1}).
\]
Proof. Since \( x \neq y \), the following set

\[
E := \{(N, b), N \in \mathbb{N}, b = (b_j)_{j \in \{0, \ldots, N+1\}} \in \Omega^{N+2}, b_0 = x, b_{N+1} = y, \forall i \in \{0, \ldots, N\}, b_i \neq b_{i+1}, (b_j)_{j \in \{1, \ldots, N\}} \in \{x_1, \ldots, x_m\}^N, (100) \text{ holds}\}
\]

is not empty since by assumption it contains \((0, \{x, y\})\). For \((N, b) \in E\), one defines the cardinal of \((N, b)\) by the number of different critical points \(b\) contains. The cardinal of an element of \(E\) belongs to \(\{0, \ldots, m\}\).

Let us now consider an element \((N, b) \in E\) which is maximal for the cardinal. By construction, this element satisfies point 1 in Lemma 46. Let us now show that it also satisfies point 2 in Lemma 46. Notice that \(\{b_0, \ldots, b_{N+1}\} \subset \{x, y, x_1, \ldots, x_m\}\).

Let \(i \in \{0, \ldots, N\}\) and \(z \in \{x, y, x_1, \ldots, x_m\} \setminus \{b_i, b_{i+1}\}\). If \(z \in \{x, y, x_1, \ldots, x_m\} \setminus \{b_0, \ldots, b_{N+1}\}\), the equality \(d_a(b_i, b_{i+1}) = d_a(b_i, z) + d_a(z, b_{i+1})\) cannot hold since \(b\) has been chosen maximal in \(E\) for the cardinal. Thus, by the triangular inequality \(d_a(b_i, b_{i+1}) < d_a(b_i, z) + d_a(z, b_{i+1})\). If \(z \in \{b_0, \ldots, b_{N+1}\} \setminus \{b_i, b_{i+1}\}\), let us prove that \(d_a(b_i, b_{i+1}) < d_a(b_i, z) + d_a(z, b_{i+1})\) by contradiction. By the triangular inequality, if the previous inequality does not hold, one has \(d_a(b_i, b_{i+1}) = d_a(b_i, z) + d_a(z, b_{i+1})\) for some \(z \in \{b_0, \ldots, b_{N+1}\} \setminus \{b_i, b_{i+1}\}\). Let us denote by \(j_0 \in \{0, \ldots, i-1, i+2, \ldots, N+1\}\) the index such that \(z = b_{j_0}\). One has \(d_a(b_i, b_{i+1}) = d_a(b_i, b_{j_0}) + d_a(b_{j_0}, b_{i+1})\). Let us assume without loss of generality that \(j_0 < i\) (the case \(j_0 > i + 1\) is treated similarly). In this case, one has, using the triangular inequality:

\[
d_a(x, y) = \sum_{j=0}^{N} d_a(b_j, b_{j+1}) = \sum_{j=0}^{i-1} d_a(b_j, b_{j+1}) + d_a(b_i, b_{j_0}) + d_a(b_{j_0}, b_{i+1}) + \sum_{j=i+1}^{N} d_a(b_j, b_{j+1}) \geq d_a(x, y) + \sum_{j=j_0}^{i-1} d_a(b_j, b_{j+1}) + d_a(b_i, b_{j_0}).
\]

Thus, \(\sum_{j=j_0}^{i-1} d_a(b_j, b_{j+1}) + d_a(b_i, b_{j_0}) = 0\) and \(b_{j_0} = b_i\) which is in contradiction with \(z \notin \{b_i, b_{i+1}\}\). Therefore \(d_a(b_i, b_{i+1}) < d_a(b_i, b_{j_0}) + d_a(b_{j_0}, b_{i+1})\). This concludes the proof of Lemma 46.

3.4.2 Proof of Theorem 3

Let us now prove Theorem 3. Recall that by assumption, the hypotheses [H1] and [H3] hold.

Proof. Let \(x, y \in \Omega\). If \(x = y\), then Theorem 3 is proved by taking the constant curve \(\gamma(t) = x\) for all \(t \in [0, 1]\). Let us deal with the case \(x \neq y\). If for all \(k \in \{1, \ldots, m\}\), \(y \notin A_k(x)\), Theorem 3 is a consequence of Corollary 43.

Let us now assume that there exists \(k \in \{1, \ldots, m\}\) such that \(y \in A_k(x)\). From Lemma 46, there exist \(N \in \mathbb{N}\) and a sequence \((b_j)_{j \in \{0, \ldots, N+1\}} \subset \Omega^{N+2}\) such that \(b_0 = x\),
\( b_{N+1} = y, (b_j)_{j \in \{1, \ldots, N\}} \subset \{x_1, \ldots, x_m\}^N \) (with the convention \( \{x_1, \ldots, x_m\}^0 = \emptyset \)) and for all \( k \in \{0, \ldots, N\}, \ b_k \neq b_{k+1} \) and

\[
d_{a}(x, y) = \sum_{k=0}^{N} d_{a}(b_k, b_{k+1}).
\] (101)

Let us deal with the case \( N \geq 2 \), the cases \( N = 0 \) and \( N = 1 \) are treated similarly. Let \( k \in \{1, \ldots, N-1\} \) and let us consider the term \( d_{a}(b_k, b_{k+1}) \) in (101) (the first term \( d_{a}(x, b_1) \) and the last term \( d_{a}(b_N, y) \) in the sum are treated in a similar way).

One can label the points \( \{x_1, \ldots, x_m\} \) such that \( b_k = x_1 \) and \( b_{k+1} = x_2 \). Point 2 in Lemma 46 implies that \( x_2 \notin A_j(x_1) \) for all \( j \in \{3, \ldots, m\} \). From Lemma 45, for any \( \delta > 0 \) there exists \( z_1 \in \partial B_a(x_1, \delta) \) such that \( d_{a}(x_1, x_2) = d_{a}(x_1, z_1) + d_{a}(z_1, x_2) \) (where \( B_a \) is defined by (100)). By taking \( \delta \) small enough, this implies that \( z_1 \notin A_1(x_2) \) and \( z_1 \notin \{x_1, \ldots, x_m\} \). Likewise, for any \( \delta > 0 \) there exists \( z_2 \in \partial B_a(x_2, \delta) \) such that \( d_{a}(z_1, x_2) = d_{a}(z_1, z_2) + d_{a}(z_2, x_2) \) and by taking \( \delta \) small enough, this implies that \( z_2 \notin A_2(z_1) \) and \( z_2 \notin \{x_1, \ldots, x_m\} \). Therefore one gets

\[
d_{a}(b_k, b_{k+1}) = d_{a}(x_1, x_2) = d_{a}(x_1, z_1) + d_{a}(z_1, z_2) + d_{a}(z_2, x_2).
\]

Taking \( \delta \) small enough and using Lemma 44, there exists a continuous and piecewise \( C^\infty \) curve \( \gamma_1 \) defined on \( (-\infty, 0] \) such that \( d_{a}(x_1, z_1) = L(\gamma_1, (-\infty, 0]) \), \( \lim_{t \to -\infty} \gamma_1(t) = x_1 \), \( \gamma_1(0) = z_1 \), and \( \partial \{t \in (-\infty, 0], \gamma_1(t) \in \partial \Omega\} \) is either empty or a single point. Similarly, there exists a continuous and piecewise \( C^\infty \) curve \( \gamma_2 \) defined on \( [0, +\infty) \) such that \( d_{a}(z_2, x_2) = L(\gamma_2, [0, +\infty)) \), \( \gamma_2(0) = z_2 \), \( \lim_{t \to +\infty} \gamma_2(t) = x_2 \) and \( \partial \{t \in [0, +\infty), \gamma_2(t) \in \partial \Omega\} \) is either empty or a single point. Let us show by contradiction that \( z_2 \notin A_j(z_1) \) for all \( j \in \{3, \ldots, m\} \). On the one hand, if \( z_2 \in A_j(z_1) \) for some \( j \in \{3, \ldots, m\} \), one has

\[
d_{a}(x_1, x_2) = d_{a}(x_1, z_1) + d_{a}(z_1, z_2) + d_{a}(z_2, x_2).
\]

On the other hand, \( x_1 \notin A_j(x_2) \), and thus

\[
d_{a}(x_1, x_2) < d_{a}(x_1, x_2) \leq d_{a}(x_1, z_1) + d_{a}(z_1, z_2) + d_{a}(z_2, x_2) \cdot
\]

This leads to a contradiction. Therefore \( z_2 \notin A_j(z_1) \) for all \( j \in \{3, \ldots, m\} \). One also has by a similar reasoning that \( z_2 \notin A_1(z_1) \). Indeed, If \( z_2 \in A_1(z_1) \), then one has on the one hand

\[
d_{a}(z_1, x_2) = d_{a}(z_1, z_2) + d_{a}(z_2, x_2) = d_{a}(z_1, x_1) + d_{a}(x_1, z_2) + d_{a}(z_2, x_2).
\]

On the other hand, since \( z_1 \notin A_1(x_2) \), one has

\[
d_{a}(z_1, x_2) < d_{a}(z_1, x_1) + d_{a}(x_1, x_2) \leq d_{a}(z_1, x_1) + d_{a}(x_1, z_2) + d_{a}(z_2, x_2) \cdot
\]

This leads to a contradiction. In conclusion \( z_2 \notin A_j(z_1) \) for all \( j \in \{1, \ldots, m\} \). Therefore, from Corollary 43, there exists a curve \( \gamma \in A(z_1, z_2) \) such that \( d_{a}(z_1, z_2) = L(\gamma, (0, 1)) \). In conclusion, we have built three curves \( \gamma, \gamma_1 \) and \( \gamma_2 \) such that

\[
d_{a}(b_k, b_{k+1}) = L(\gamma_1, (-\infty, 0]) + L(\gamma, (0, 1)) + L(\gamma_2, [0, +\infty)).
\]

A similar reasoning for all the terms in the sum in (101) concludes the proof of Theorem 3.
A consequence of Theorem 3 is the following.

**Lemma 47.** Let us assume that [H1] and [H3] hold. Let \((x, y) \in \overline{\Gamma}\). Let us denote by \(((\gamma_1, I_1), \ldots, (\gamma_N, I_N))\) the curves given by Theorem 3 ordered such that

\[
\lim_{t \to (\inf I_1)^+} \gamma_1(t) = x, \quad \lim_{t \to (\sup I_N)^-} \gamma_N(t) = y,
\]

and which realize the Agmon distance between \(x\) and \(y\). Let \(k_1 \leq k_2\) with \((k_1, k_2) \in \{1, \ldots, N\}^2\) and let \(t_1 \in I_{k_1}\) and \(t_2 \in I_{k_2}\). If \(k_1 = k_2\), \(t_1\) and \(t_2\) are chosen such that \(t_1 \leq t_2\). Then one has

\[
d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) = \begin{cases} 
L(\gamma_{k_1}, (t_1, \sup I_{k_1})) + \sum_{k=k_1+1}^{k_2-1} L(\gamma_k, I_k) + L(\gamma_{k_2}, (\inf I_{k_2}, t_2)) & \text{if } k_1 < k_2, \\
L(\gamma_{k_1}, (t_1, t_2)) & \text{if } k_1 = k_2,
\end{cases}
\]

where by convention, if \(k_2 = k_1 + 1\), \(\sum_{k=k_1+1}^{k_2-1} L(\gamma_k, I_k) = 0\). In addition the following equality holds

\[
d_a(x, y) = d_a(x, \gamma_{k_1}(t_1)) + d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) + d_a(\gamma_{k_2}(t_2), y).
\]

**Proof.** Let us make the proof for \(k_1 < k_2\) (the case \(k_1 = k_2\) is treated in a similar way). One has using the triangular inequality for the Agmon distance and Lemma 32

\[
d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) \leq L(\gamma_{k_1}, (t_1, \sup I_{k_1})) + \sum_{k=k_1+1}^{k_2-1} L(\gamma_k, I_k) + L(\gamma_{k_2}, (\inf I_{k_2}, t_2)).
\]

Let us now conclude the proof by contradiction. Assume that

\[
d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) < L(\gamma_{k_1}, (t_1, \sup I_{k_1})) + \sum_{k=k_1+1}^{k_2-1} L(\gamma_k, I_k) + L(\gamma_{k_2}, (\inf I_{k_2}, t_2)).
\]

Using the triangular inequality, one has

\[
d_a(x, y) \leq d_a(x, \gamma_{k_1}(t_1)) + d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) + d_a(\gamma_{k_2}(t_2), y) \\
< \sum_{k=1}^{k_1-1} L(\gamma_k, I_k) + L(\gamma_{k_1}, (\inf I_{k_1}, t_1)) + L(\gamma_{k_1}, (t_1, \sup I_{k_1})) \\
+ \sum_{k=k_1+1}^{k_2-1} L(\gamma_k, I_k) + L(\gamma_{k_2}, (\inf I_{k_2}, t_2)) + L(\gamma_{k_2}, (t_2, \sup I_{k_2})) \\
+ \sum_{k=k_2+1}^{N} L(\gamma_k, I_k) = d_a(x, y),
\]

where by convention, if \(k_1 = 1\), \(\sum_{k=1}^{k_1-1} L(\gamma_k, I_k) = 0\) and if \(k_2 = N\), \(\sum_{k=k_2+1}^{N} L(\gamma_k, I_k) = 0\). The last inequality is impossible and all the previous inequalities have to be equalities. This proves Lemma 47. 

\[\Box\]
### 3.4.3 On the equality in (79)

We end up this section with some results in case of equality in the inequality (79). We will prove in particular Proposition 49 which has been used in Section 3.2.2 above to give lower bounds on the Agmon distance.

**Corollary 48.** Let us assume that [H1] and [H3] hold. Let \( x, y \in \Omega \) with \( f(x) \leq f(y) \). Let us denote by \( ((\gamma_1, I_1), \ldots, (\gamma_N, I_N)) \) the curves given by Theorem 5 ordered such that

\[
\lim_{t \to (\inf I_i)^+} \gamma_i(t) = x, \quad \lim_{t \to (\sup I_N)^-} \gamma_N(t) = y,
\]

and which realize the Agmon distance between \( x \) and \( y \). If it holds:

\[
d_a(x, y) = f(y) - f(x),
\]

then for all \( i \in \{1, \ldots, N\} \), there exist measurable functions \( \lambda_i : I_i \to \mathbb{R}_+ \) such that for almost every \( t \) in \( \{t \in I_i, \, \gamma_i(t) \in \Omega\} \)

\[
\gamma_i'(t) = \lambda_i(t) \nabla f(\gamma_i(t)),
\]

and such that for almost every \( t \) in \( \operatorname{int} \{t \in I_i, \, \gamma_i(t) \in \partial \Omega\} \)

\[
\gamma_i'(t) = \lambda_i(t) \nabla_T f(\gamma_i(t)).
\]

Moreover, if \( I_i \) is not bounded (namely \( I_i = (-\infty, 0] \) or \( I_i = [0, +\infty) \)), \( \lambda_i(t) = 1 \) for almost every \( t \in I_i \), and if \( I_i = [0, 1], \lambda_i \in L^\infty([0, 1], \mathbb{R}_+) \).

**Proof.** Using Lemma 25 one gets using first the triangular inequality and then the Cauchy-Schwarz inequality

\[
\begin{align*}
f(y) - f(x) &= \sum_{k=1}^{N} \left( \int_{\{t \in I_k, \, \gamma_k(t) \in \Omega\}} (\nabla f)(\gamma_k) \cdot \gamma_k' + \int_{\operatorname{int} \{t \in I_k, \, \gamma_k(t) \in \partial \Omega\}} (\nabla_T f)(\gamma_k) \cdot \gamma_k' \right) \\
&\leq \sum_{k=1}^{N} \left( \int_{\{t \in I_k, \, \gamma_k(t) \in \Omega\}} |\nabla f(\gamma_k)||\gamma_k'| + \int_{\operatorname{int} \{t \in I_k, \, \gamma_k(t) \in \partial \Omega\}} |\nabla_T f(\gamma_k)||\gamma_k'| \right) \\
&= \sum_{k=1}^{N} L(\gamma_k, I_k) = d_a(x, y).
\end{align*}
\]

If \( d_a(x, y) = f(y) - f(x) \), then the previous inequality is necessarily an equality. This leads to the desired result applying the cases of equality in both the triangular inequality and the Cauchy-Schwarz inequalities. In particular, this gives, for \( k \in \{1, \ldots, N\} \), the existence of the nonnegative functions \( \lambda_k : I_k \to \mathbb{R}_+ \).

Assume now that \( I_i \) is not bounded. Using the construction of the curves \( (\gamma_k)_{k=1, \ldots, N} \), this implies that \( I_i \) is either \( (-\infty, 0] \) or \( [0, +\infty) \) and \( \gamma_i \) is constructed using the gradient flow of the eikonal solution near a critical point \( x^* \) of \( f \) or of \( f|_{\partial \Omega} \) (see Lemma 44 and Remarks 7 and 9). Let us assume that \( x^* \) is a critical point of \( f|_{\partial \Omega} \) and \( I_i = [0, +\infty) \) (the other cases are treated similarly). Let \( \Phi \) be the solution of (93) on the neighborhood \( V^* \) of \( x^* \) (see Proposition 38). By considering if necessary a smaller neighborhood \( V^\ast \), one can assume that \( V^\ast \cap \partial \Omega \subset U^\ast \), where \( U^\ast \) is the neighborhood of \( x^* \) in \( \partial \Omega \) introduced.
The set of curves \( \{ \) almost every \( \) Proposition \[37\], so that from \[38\] and \[32\], it holds on \( V^* \cap \partial \Omega \): \( |\nabla_T \Phi| = |\nabla_T f| \).

The curve \( \gamma_i \) satisfies by construction \( \lim_{t \to \infty} \gamma_i(t) = x^* \) and

\[
\gamma_i' = \begin{cases} 
- \nabla \Phi(\gamma_i) & \text{in } \Omega, \\
- \nabla_T \Phi(\gamma_i) & \text{on } \partial \Omega.
\end{cases}
\]

In addition, by the previous reasoning, one also has

\[
\gamma_i' = \begin{cases} 
\lambda_i \nabla f(\gamma_i) & \text{in } \Omega, \\
\lambda_i \nabla_T f(\gamma_i) & \text{on } \partial \Omega,
\end{cases}
\]

for some function \( \lambda_i : I_i \to \mathbb{R}_+ \). Taking the norm in the two previous equalities, using the fact that \( \Phi \) solves \[33\] together with the equality \[32\], one obtains that \( \lambda_i = 1 \). Let us finally consider the case \( I_i = [0, 1] \). Then, by construction, the curve \( \gamma_i \) does not meet any critical points of the functions \( f \) and \( f|_{\partial \Omega} \). This implies that \( \inf_{I_i} |\nabla f(\gamma_i)| > 0 \) and \( \inf_{I_i} |\nabla_T f(\gamma_i)| > 0 \), and thus, since \( \| \gamma_i' \|_{L^\infty} < \infty \), one concludes that \( \lambda_i \in L^\infty([0, 1], \mathbb{R}_+) \).

Let us define the notion of generalized integral curves.

**Definition 8.** Let \( D \subset \overline{\Omega} \) be a \( C^\infty \) domain and \( X \in C^\infty(D, \mathbb{R}) \). Let \( N \in \mathbb{N}^* \) and for \( i \in \{1, \ldots, N\} \), let \( I_i \subset \mathbb{R} \) be an interval and \( \gamma_i : I_i \to D \) be Lipschitz and which satisfy

\[
\lim_{t \to (\inf I_i)^+} \gamma_i(t) \in \overline{\Omega}, \quad \lim_{t \to (\sup I_i)^-} \gamma_N(t) \in \overline{\Omega},
\]

and for all \( k \in \{1, \ldots, N - 1\} \),

\[
\lim_{t \to (\sup I_k)^-} \gamma_k(t) = \lim_{t \to (\inf I_{k+1})^+} \gamma_{k+1}(t).
\]

The set of curves \( \{ \gamma_1, \ldots, \gamma_N \} \) is a generalized integral curve of \( \begin{cases} 
\nabla X & \text{in } D \cap \Omega, \\
\nabla_T X & \text{on } D \cap \partial \Omega
\end{cases} \) if for all \( i \in \{1, \ldots, N\} \), there exist measurable functions \( \lambda_i : I_i \to \mathbb{R}_+ \) such that for almost every \( t \) in \( \{ t \in I_i, \gamma_i(t) \in D \cap \Omega \} \): \( \gamma_i'(t) = \lambda_i(t) \nabla X(\gamma_i(t)) \), and such that for almost every \( t \) in \( \text{int} \{ t \in I_i, \gamma_i(t) \in \partial \Omega \cap D \} \): \( \gamma_i'(t) = \lambda_i(t) \nabla_T X(\gamma_i(t)) \).

The notion of generalized integral curve has been introduced in the case of manifolds without boundary in \[38\]. As introduced in Definition \[37\], the set of curves \( \{ \gamma_1, \ldots, \gamma_N \} \) given by Corollary \[48\] is a generalized integral curve of the vector field \( \begin{cases} 
\nabla f & \text{in } \Omega, \\
\nabla_T f & \text{on } \partial \Omega
\end{cases} \).

Let us mention that in the case when \( \Omega \) is a manifold without boundary, Corollary \[48\] is exactly \[38\] Lemma A2.2.

**Proposition 49.** Let us assume that \([H1]\) and \([H3]\) hold. Let us denote by \( \{z_1, \ldots, z_n\} \) the local minima of \( f|_{\partial \Omega} \) ordered such that \( f(z_1) < f(z_2) \leq \ldots \leq f(z_n) \). Then, for all \( i < j, (i, j) \in \{1, \ldots, n\}^2, \) one has

\[
d_{a}(z_i, z_j) > f(z_j) - f(z_i).
\]
Proof. From the inequality (79), one has \( d_a(z_i, z_j) \geq f(z_j) - f(z_i) \). Let us prove Proposition 49 by contradiction. Assume that \( d_a(z_i, z_j) = f(z_j) - f(z_i) \) for some \( i < j \). Denote by \((\gamma_1, I_1), \ldots, (\gamma_m, I_m)\) the curves given by Theorem 3 ordered such that

\[
\lim_{t \to (\inf I_1)} \gamma_1(t) = z_i, \quad \lim_{t \to (\sup I_m)} \gamma_m(t) = z_j,
\]

and which realize the Agmon distance between \( z_i \) and \( z_j \). Since \( d_a(z_i, z_j) = f(z_j) - f(z_i) \), from Corollary 48, for all \( i \in \{1, \ldots, m\} \), there exist measurable functions \( \lambda_i : I_i \to \mathbb{R}_+ \) such that for almost every \( t \) in \( t \in I_i \), \( \gamma_i(t) = \lambda_i(t) \nabla f (\gamma_i(t)) \), and such that for almost every \( t \) in \( \text{int} \{ t \in I_i \} \), \( \gamma_i(t) = \lambda_i(t) \nabla_T f (\gamma_i(t)) \) (the set of curves \( \{\gamma_1, \ldots, \gamma_m\} \) is a generalized integral curve of the vector field \( \nabla f \) in \( \Omega \), according to Definition 8). Let us recall that from Remark 9, \( I_1 = (-\infty, 0] \) and \( I_m = [0, +\infty) \) since \( z_i \) and \( z_j \) are critical points of \( f|_{\partial \Omega} \).

Step 1. Let us show that for all \( t \in (-\infty, 0] \), \( \gamma_1(t) \in \partial \Omega \). On the one hand, from Remark 9, \( \lim_{t \to -\infty} \gamma_1(t) = z_i \) and

\[
\gamma_1' = \begin{cases} \nabla \Phi(\gamma_1) \text{ in } \Omega \\ \nabla_T \Phi(\gamma_1) \text{ on } \partial \Omega, \end{cases}
\]

where \( \Phi \) solves (93). On the other hand, from Corollary 48 one has

\[
\gamma_1' = \begin{cases} \nabla f(\gamma_1) \text{ in } \Omega \\ \nabla_T f(\gamma_1) \text{ on } \partial \Omega. \end{cases}
\]

Then, for all \( t \geq 0 \), one has \( \frac{d}{dt} (f(\gamma_1(t)) - \Phi(\gamma_1(t))) = 0 \). Therefore there exists \( C > 0 \) such that for all \( t \in (-\infty, 0] \), \( \gamma_1(t) \in \{ x, f(x) = \Phi(x) = C \} \). Since \( \lim_{t \to -\infty} \gamma_1(t) = z_i \) and \( f(\Phi(z_i)) - f(z_i) = f(z_i) \), one gets that \( C = f(z_i) \) and thus for all \( t \in (-\infty, 0] \), \( \gamma_1(t) \in \{ x, f(x) = \Phi(x) = f(z_i) \} \). From Corollary 39 and Proposition 40, \( \gamma_1 \) lives in a neighborhood \( U^* \) of \( z_i \) such that (see Equation (95)):

\[
\{ x, f(x) - \Phi(x) = f(z_i) \} \subset \partial \Omega.
\]

We thus get that for all \( t \geq 0 \), \( \gamma_1(t) \in \partial \Omega \), and then \( \gamma_1'(t) = \nabla_T f(\gamma_1(t)) = \nabla_T \Phi(\gamma_1(t)) \).

Step 1 is proved.

Step 2. We are going to show that for all \( t \in I_k \), \( \gamma_k(t) \in \partial \Omega \). If it is not the case, from Step 1, there exist \( k \in \{2, \ldots, m\} \) and \( t_k \in I_k \) such that \( \gamma_k(t_k) \in \Omega \). Let us define the first time, denoted by \( t^* \), for which the curves \((\gamma_2, I_2), \ldots, (\gamma_m, I_m)\) leave \( \partial \Omega \). By construction of the curves \( \gamma_1, \ldots, \gamma_m \), there are two cases: either \( t^* \) is finite (and thus belongs to \( \text{int}(I_k) \) for \( k \in \{1, \ldots, m\} \)) or, there exist \( j \in \{1, \ldots, m-1\} \), \( s < 0 \) and \( z \in \partial \Omega \) such that \( g(z) = 0 \), \( \lim_{t \to +\infty} \gamma_j(t) = z \), \( \lim_{t \to -\infty} \gamma_{j+1}(t) = z \) and \( \gamma_{j+1}(-\infty, s) \subset \Omega \) for which we set \( t^* = -\infty \). Let us assume that \( t^* \) is finite and belongs to \( \text{int}(I_k) \) (the other case is treated similarly).

As in Step 1 of the proof of Proposition 26, let us now introduce a smooth tangential and normal system of coordinates around \( \gamma_k(t^*) \) in \( \Omega \), denoted by \( \phi(x) = (x_T, x_N) \). The function \( \phi \) is defined from a neighborhood of \( \gamma_k(t^*) \) in \( \Omega \) to \( \mathbb{R}^d \). Moreover, one has \( x_N \geq 0 \) and \( x_N(x) = 0 \) if and only if \( x \in \partial \Omega \). We may assume that the neighborhood \( V_\alpha \subset \mathbb{R}^d \) on which \( \phi \) is defined is such that \( \phi(V_\alpha) = U \times [0, \alpha] \) for \( \alpha > 0 \) and \( U \subset \mathbb{R}^{d-1} \).
Since $\partial_n f > 0$ on $\partial \Omega$, $\alpha > 0$ can be chosen small enough such that $\nabla f(x) \cdot n(x) > 0$ for all $x \in V_\alpha$ where $n(x) = -\frac{\nabla x_N(x)}{|\nabla x_N(x)|}$. Indeed, for $x \in \partial \Omega$, $n(x)$ is nothing but the unit outward normal to $\partial \Omega$.

Now, by continuity of the curve $\gamma_k$, there exists $\varepsilon > 0$ such that $[t^*, t^* + \varepsilon] \subset I_k$ and for all $t \in [t^*, t^* + \varepsilon]$, $\gamma_k(t) \in V_\alpha$. The mapping $t \in [t^*, t^* + \varepsilon] \mapsto x_N(\gamma_k(t))$ is Lipschitz and satisfies: for almost every $s \in (t^*, t^* + \varepsilon)$,

$$\frac{d}{ds} x_N(\gamma_k(s)) = -|\nabla x_N(\gamma_k(s))| \gamma_k'(s) \cdot n(\gamma_k(s)).$$

Then, for all $t \in [t^*, t^* + \varepsilon]$, one has:

$$\frac{d}{ds} x_N(\gamma_k(s)) = \begin{cases} 0 & \text{for a.e. } s \in \text{int } \{u \in (t^*, t), \gamma_k(u) \in \partial \Omega\} \\ -|\nabla x_N(\gamma_k(s))| \lambda_k(s) \nabla f(\gamma_k(s)) \cdot n(\gamma_k(s)) & \text{for a.e. } s \in \{u \in (t^*, t), \gamma_k(u) \in \Omega\}. \end{cases}$$

Since $\partial \{u \in (t^*, t), \gamma_k(u) \in \partial \Omega\}$ is of Lebesgue measure zero (see Theorem $3$) and since $\nabla f \cdot n > 0$ in $V_\alpha$, one has from Lemma $25$, for all $t \in [t^*, t^* + \varepsilon]$

$$x_N(\gamma_k(t)) = x_N(\gamma_k(t)) - x_N(\gamma_k(t^*)) = \int_{t^*}^t \frac{d}{ds} x_N(\gamma_k(s)) \ ds \leq 0.$$

This implies that for all $t \in [t^*, t^* + \varepsilon]$, $x_N(\gamma_k(t)) = 0$ and thus $\gamma_k(t) \in \partial \Omega$ for all $t \in [t^*, t^* + \varepsilon]$. This contradicts the definition of $t^*$. Step 2 is proved.

Step 3. From the last two steps, for all $t \in [0, +\infty)$, $\gamma_m(t) \in \partial \Omega$. From Corollary $48$, one has $\gamma_m'(t) = \nabla_T f(\gamma_m(t))$ for all $t \in [0, +\infty)$ and therefore, the map $t \in [0, +\infty) \mapsto f(\gamma_m(t))$ is increasing (indeed, one has $\frac{d}{dt} f(\gamma_m(t)) = |\nabla_T f(\gamma_m(t))|^2$).

This is impossible since $z_{\gamma}$ is a local minimum of $f|_{\partial \Omega}$. This concludes the proof of Proposition $49$ by contradiction.

### 3.5 Agmon distance in a neighborhood of the basin of attraction of a local minimum of $f|_{\partial \Omega}$ and eikonal equation

The aim of this section is to generalize the results of Section $3.3$ to relate the Agmon distance and the solution to an eikonal equation on a neighborhood of a basin of attraction $B_z$ (see Definition $4$) of a local minimum $z$ of $f|_{\partial \Omega}$. Let first introduce a solution to the eikonal equation $|\nabla \phi|^2 = |\nabla f|^2$ defined globally on a neighborhood of the boundary $\partial \Omega$.

**Proposition 50.** Let us assume that $[H3]$ holds. There exists a neighborhood of $\partial \Omega$ in $\overline{\Omega}$, denoted $V_{\partial \Omega}$, such that there exists $\Phi \in C^\infty(V_{\partial \Omega}, \mathbb{R})$ satisfying

$$\begin{cases} |\nabla \Phi|^2 = |\nabla f|^2 & \text{in } \Omega \cap V_{\partial \Omega} \\ \Phi = f & \text{on } \partial \Omega \\ \partial_n \Phi = -\partial_n f & \text{on } \partial \Omega. \end{cases} \quad (102)$$

Moreover, one has the following uniqueness results: if $\Phi$ is a $C^\infty$ real valued function defined on a neighborhood $\tilde{V}$ of $\partial \Omega$ satisfying $(102)$, then $\Phi = \Phi$ on $\tilde{V} \cap V_{\partial \Omega}$.

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Proof. Let \( z \in \partial \Omega \). Using \([19]\) or \([22]\), thanks to \([H3]\), there exists a neighborhood of \( z \) in \( \Omega \), denoted by \( \mathcal{V}_z \), such that there exists \( \Phi \in C^\infty(\mathcal{V}_z, \mathbb{R}) \) satisfying
\[
\begin{align*}
|\nabla \Phi|^2 &= |\nabla f|^2 \text{ in } \Omega \cap \mathcal{V}_z \\
\Phi &= f \text{ on } \partial \Omega \cap \mathcal{V}_z \\
\partial_n \Phi &= -\partial_n f \text{ on } \partial \Omega \cap \mathcal{V}_z.
\end{align*}
\]
Moreover, \( \mathcal{V}_z \) can be chosen such that the following uniqueness result holds: if a function \( \tilde{\Phi} \in C^\infty(\mathcal{V}_z, \mathbb{R}) \) satisfies the previous equalities, then \( \tilde{\Phi} = \Phi \) on \( \mathcal{V}_z \). Now, one concludes using the fact that \( \partial \Omega \) is compact and can thus be covered by a finite number of these neighborhoods \( (\mathcal{V}_z)_{z \in \partial \Omega} \). \( \blacksquare \)

Remark 10. Let us mention another standard approach to prove Proposition 50, using the notion of viscosity solutions. Let us recall some results from \([49, \text{Theorem 5.1}]\). For \((x, y) \in \Omega^2\), one defines
\[
\tilde{d}(x, y) := \inf_{T > 0, \gamma} \int_0^T |\nabla f(\gamma(t))| \, dt,
\]
where the infimum is taken over \( T > 0 \) and over Lipschitz curves \( \gamma : [0, T] \rightarrow \Omega \) which satisfy \( \gamma(0) = x, \gamma(T) = y, |\gamma'| \leq 1 \). Then \( v(x) := \inf \{ f(y) + \tilde{d}(x, y), y \in \partial \Omega \} \) is Lipschitz and is a viscosity solution of
\[
\begin{align*}
|\nabla v| &= |\nabla f| \text{ in } \Omega \\
v &= f \text{ on } \partial \Omega.
\end{align*}
\]
Let us notice that this implies \( |\partial_n v| = |\partial_n f| \). To prove Proposition 50 using this result, one has to show that \( v \) is \( C^\infty \) near \( \partial \Omega \) and \( \partial_n v = -\partial_n f \). This is a consequence of the characteristic method, see \([49, \text{Section 1.2}]\).

Remark 11. Let \( x^* \) be a local minimum of \( f|_{\partial \Omega} \) and let us denote by \( \tilde{\Phi} \) the solution to the eikonal equation \([94]\) introduced in Corollary \([39]\), defined on a neighborhood \( V^* \) of \( x^* \). Then, one has on \( V^* \cap V^\Omega \):
\[
\tilde{\Phi} = \Phi - f(x^*)
\]
where \( \Phi \) is the solution to \([102]\).

Let us now introduce the function \( f_- \) which will be used in the sequel.

Proposition 51. Assume that \([H3]\) holds. Let \( \Phi \in C^\infty(V^\Omega, \mathbb{R}) \) be the function introduced in Proposition 50. Let us define the function \( f_- \in C^\infty(V^\Omega, \mathbb{R}) \) by
\[
f_- = \frac{\Phi - f}{2}.
\]
Then, \( f_- = 0 \) on \( \partial \Omega \), and up to choosing a smaller neighborhood \( V^\Omega \) of \( \partial \Omega \), the function \( f_- \) is positive in \( V^\Omega \setminus \partial \Omega \) and \( |\nabla f_-| > 0 \) on \( V^\Omega \).

Proof. Since \( \partial_n (\Phi - f) = -2\partial_n f < 0 \) and \( \Phi = f \) on \( \partial \Omega \), then, up to choosing a smaller neighborhood \( V^\Omega \) of \( \partial \Omega \), one has \( \Phi > f \) on \( V^\Omega \setminus \partial \Omega \) and \( |\nabla (\Phi - f)| > 0 \) on \( V^\Omega \). \( \blacksquare \)
We are now in position to prove the main result of this section.

**Proposition 52.** Let us assume that [H1] and [H3] hold. Let $\Phi$ be the function given by Proposition 50. Denote by $z$ a local minimum of $f|_{\partial \Omega}$ and denote by $B_z \subset \partial \Omega$ the associated basin of attraction (see Definition 4). Besides, let $\Gamma_z \subset \partial \Omega$ be an open domain such that $\overline{\Gamma}_z \subset B_z$ and $z \in \Gamma_z$. Then there exists a neighborhood of $\overline{\Gamma}_z$ in $\overline{\Omega}$, denoted by $V_{\Gamma_z}$, such that $\partial V_{\Gamma_z} \cap \partial \Omega \subset B_z$ and for all $x \in V_{\Gamma_z}$,

$$d_a(x, z) = \Phi(x) - f(z).$$

Notice that in this proposition, $\Gamma_z$ can be chosen as large as we want in $B_z$.

**Proof.** Let $\Phi$ be the function given by Proposition 50. The proof is divided into three steps.

**Step 1.** Let us first define $V_{\Gamma_z}$. To this end let us denote by $f_-$ and $V_{\partial \Omega}$ respectively the function and the neighborhood of $\partial \Omega$ given by Proposition 51. For $\varepsilon > 0$ small enough one defines

$$V_\varepsilon = \{ y \in \Omega, \ 0 \leq f_-(y) \leq \varepsilon \} \subset V_{\partial \Omega}. \quad (104)$$

The parameter $\varepsilon > 0$ can be chosen such that there is no critical point of $f$ on $\partial V_\varepsilon \cap \Omega = \{ y \in \Omega, \ f_-(y) = \varepsilon \}$. The set $V_\varepsilon$ is a neighborhood of $\partial \Omega$ in $\overline{\Omega}$ (see Figure 7 for a schematic representation). Let us now fix such a $\varepsilon > 0$. Assumption [H3] together with the fact that $\partial_n \Phi < 0$ on $\partial \Omega$, imply that there exists a neighborhood $V_{\Gamma_z}$ of $\overline{\Gamma}_z$ in $\overline{\Omega}$, such that $V_{\Gamma_z} \subset V_\varepsilon$, $\partial V_{\Gamma_z} \cap \partial \Omega \subset B_z$ and

$$\partial_n \Phi > 0, \ \text{on } \partial V_{\Gamma_z} \cap \Omega.$$

The set $V_{\Gamma_z}$ is schematically represented on Figure 8.

![Figure 7: The set $V_\varepsilon$.](image)

**Step 2.** Let us first prove that for all $x \in V_{\Gamma_z}$, $d_a(x, z) \geq \Phi(x) - f(z)$. For $x \in V_{\Gamma_z}$, denote by $((\gamma_1, I_1), \ldots, (\gamma_N, I_N))$ the curves given by Theorem 3 ordered such that

$$\lim_{t \to (\sup I_1)^-} \gamma_1(t) = z, \quad \lim_{t \to (\sup I_N)^-} \gamma_N(t) = x,$$

and which realize the Agmon distance between $x$ and $z$. One has to deal with the two following cases:

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In the first case, since $\Phi$ is defined on $V_\varepsilon$ for all $j$

Using Lemma 25 and the fact that $|\nabla \Phi| > 0$ and such that there exists $\alpha > 0$ such that for all $t < t_1$, $\gamma_k(t) \in V_\varepsilon$

and thus

$$\Phi(x) - f(z) = \Phi(x) - \Phi(z) = \sum_{j=1}^{N} \int_{I_j} \frac{d}{dt} \Phi \circ \gamma_j(t) \, dt.$$ 

Using Lemma 25 and the fact that $|\nabla \Phi| = g$ on $\Omega \cap V_0\Omega$ and $|\nabla \gamma | = g$ on $\partial \Omega$, it holds, for all $j \in \{1, \ldots, N\}$,

$$\int_{I_j} \frac{d}{dt} \Phi \circ \gamma_j(t) \, dt \leq L(\gamma_j, I_j)$$

and thus

$$\Phi(x) - f(z) \leq \sum_{j=1}^{N} L(\gamma_j, I_j) = d_a(x, z).$$

Let us now consider the second case. Let us introduce $k_1 \in \{1, \ldots, N\}$ and $t_1 \in I_{k_1}$ such that for all $t < t_1$, $\gamma_{k_1}(t) \in V_\varepsilon$, for all $k \in \{1, \ldots, k_1 - 1\}$, for all $t \in I_k$, $\gamma_k(t) \in V_\varepsilon$

and such that there exists $\beta > 0$ such that for all $t \in (t_1, t_1 + \beta)$, $\gamma_{k_1}(t) \notin V_\varepsilon$. The couple $(k_1, t_1)$ thus represents the “first time” the curves $\gamma_1, \ldots, \gamma_N$ leave $V_\varepsilon$. Likewise, let us introduce $k_2 \in \{1, \ldots, N\}$ and $t_2 \in I_{k_2}$ such that for all $t > t_2$, $\gamma_{k_2}(t) \in V_\varepsilon$, for all $k \in \{k_2 + 1, \ldots, N\}$, for all $t \in I_k$, $\gamma_k(t) \in V_\varepsilon$

and such that there exists $\beta > 0$ such that for all $t \in [t_2 - \beta, t_2)$, $\gamma_{k_2}(t) \notin V_\varepsilon$. The couple $(k_2, t_2)$ thus represents the “last time” the curves $\gamma_1, \ldots, \gamma_N$ leave $\Omega \setminus V_\varepsilon$. From Step 1, there is no critical point of $f$ on $\partial V_\varepsilon \cap \Omega = \{y \in \Omega, f_-(y) = \varepsilon\}$. Therefore, by construction of the curves $(\gamma_k)_{k=1, \ldots, N}$, the times $t_1$ and $t_2$ are finite and belong respectively to int $I_{k_1}$ and int $I_{k_2}$. One has by continuity of $\gamma_{k_1}$ and $\gamma_{k_2}$, $f_-(\gamma_{k_1}(t_1)) = f_-(\gamma_{k_2}(t_2)) = \varepsilon$. Since $\Phi$ is defined on $V_\varepsilon$, using again Lemma 25 and the fact that $|\nabla \Phi| = g$ on $\Omega$ and $|\nabla \gamma | = g$, one has

$$|\Phi(\gamma_{k_1}(t_1)) - \Phi(z)| \leq \sum_{j=1}^{k_1-1} L(\gamma_j, I_j) + L(\gamma_{k_1}, (\inf I_{k_1}, t_1)).$$

In addition, using Lemma 47

$$\sum_{j=1}^{k_1-1} L(\gamma_j, I_j) + L(\gamma_{k_1}, (\inf I_{k_1}, t_1)) = d_a(z, \gamma_{k_1}(t_1)).$$

Figure 8: The set $V_{\Gamma_\varepsilon}$. 

1. either $\forall \; k \in \{1, \ldots, N\}, \forall t \in I_k$, $\gamma_k(t) \in V_\varepsilon$

2. or $\exists \; k \in \{1, \ldots, N\}$ and $\exists \; t \in I_k$, $\gamma_k(t) \in \Omega \setminus V_\varepsilon$.
Thus $|\Phi(\gamma_{k_1}(t_1)) - \Phi(z)| \leq d_a(z, \gamma_{k_1}(t_1))$. By similar arguments, one obtains $|\Phi(x) - \Phi(\gamma_{k_2}(t_2))| \leq d_a(\gamma_{k_2}(t_2), x)$. Thanks to the definition \((103)\) of $f_\gamma$ and using the fact that $f_\gamma(\gamma_{k_1}(t_1)) = f_\gamma(\gamma_{k_2}(t_2)) = \varepsilon$, one has $|f(\gamma_{k_2}(t_2)) - f(\gamma_{k_1}(t_1))| = |\Phi(\gamma_{k_2}(t_2)) - \Phi(\gamma_{k_1}(t_1))|$. In addition, using \((79)\) one obtains $d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) \geq |f(\gamma_{k_2}(t_2)) - f(\gamma_{k_1}(t_1))| = |\Phi(\gamma_{k_2}(t_2)) - \Phi(\gamma_{k_1}(t_1))|$. Using Lemma \(17\) and gathering these three last inequalities, one gets

$$d_a(x, z) = d_a(z, \gamma_{k_1}(t_1)) + d_a(\gamma_{k_1}(t_1), \gamma_{k_2}(t_2)) + d_a(\gamma_{k_2}(t_2), x)$$

$$\geq |\Phi(z) - \Phi(\gamma_{k_1}(t_1))| + |\Phi(\gamma_{k_2}(t_2)) - \Phi(\gamma_{k_1}(t_1))| + |\Phi(x) - \Phi(\gamma_{k_2}(t_2))|$$

$$\geq |\Phi(z) - \Phi(x)| + |\Phi(x) - \Phi(z)| = |\Phi(x) - f(z)|.$$

**Step 3.** Let us now show that for all $x \in V_{r_\gamma}$, $d_a(x, z) \leq \Phi(x) - f(z)$. The proof of this inequality is very similar to the second step in the proof of Proposition \(40\). For $x \in V_{r_\gamma}$, let $\gamma$ be defined by \((96)\)–\((97)\) (where $\Phi$ is defined by \((102)\)), with $\gamma(0) = x$. The function $\gamma$ is with values in $V_{r_\gamma}$ since $\partial_n \Phi > 0$ on $\partial V_{r_\gamma} \cap \Omega$ and $\partial V_{r_\gamma} \cap \partial \Omega \subset B_z$. Thus $\gamma$ is defined on $\mathbb{R}_+$. Thanks to the definition \((96)\) of the vector field $X$, if there exists a time $t_{\partial \Omega}$ such that $\gamma(t_{\partial \Omega})$ is in $\partial \Omega$, then, for all $t \geq t_{\partial \Omega}$, $\gamma(t) \in \partial \Omega$. The function $t \in \mathbb{R}_+ \mapsto \gamma(t)$ is continuous, piecewise $C^\infty$ and satisfies

$$\lim_{t \to +\infty} \gamma(t) = z.$$

Then, as in the second step of the proof of Proposition \(40\) one has

$$\Phi(x) - f(z) = L(\gamma, (0, \infty)).$$

Using Lemma \(32\) one obtains that $d_a(x, z) \leq L(\gamma, (0, \infty)) = \Phi(x) - f(z)$. This proves the inequality: for all $x \in V_{r_\gamma}$, $d_a(x, z) \leq \Phi(x) - f(z)$. This concludes the proof of Proposition \(52\).

The following corollary is similar to Corollary \(48\) in the sense that is deals with the case of equality between the Agmon distance and the function $\Phi$ introduced in Proposition \(50\). Corollary \(53\) will be needed in the proof of Proposition \(65\).

**Corollary 53.** Let us assume that \([H1]\) and \([H3]\) hold. Let $\Phi$ be the function introduced in Proposition \(50\) and, let $f_\gamma$ and $V_{\Omega}$ be respectively the function and the neighborhood of $\partial \Omega$ given by Proposition \(51\). Let $V_{\alpha}$ be defined by \((104)\) and where the parameter $\alpha > 0$ is chosen such $V_{\alpha} \subset V_{\Omega}$, there is no critical point of $f$ on $\partial V_{\alpha} \cap \Omega \cup \{w \in \Omega, f_\gamma(w) = \alpha\}$, $\partial_n f > 0$ on $\partial V_{\alpha} \cap \Omega$, $\partial_n f_\gamma < 0$ on $\partial V_{\alpha} \cap \Omega$ and $|\nabla \Phi| \neq 0$ in $V_{\alpha}$ (it is possible to choose such an $\alpha > 0$ since $\partial_{n} f = \partial_{n} \Phi = -\partial_{n} f < 0$ on $\partial \Omega = V_{\Omega}$). Let $x, y \in V_{\alpha}$ and denote by $((\gamma_1, I_1), \ldots, (\gamma_N, I_N))$ the curves given by Theorem \(3\) ordered such that $\lim_{t \to (\inf I_{1})^+} \gamma_1(t) = x$, $\lim_{t \to (\sup I_{N})^-} \gamma_N(t) = y$ and which realize the Agmon distance between $x$ and $y$. Let us assume that

$$\Phi(x) - \Phi(y) = d_a(x, y).$$

Then, for all $i \in \{1, \ldots, N\}$, $\Im \gamma_i \subset V_{\alpha}$ and there exist measurable functions $\lambda_i : I_i \to \mathbb{R}$ such that for almost every $t$ in $\{ t \in I_i, \gamma_i(t) \in \Omega \}$, one has $\gamma_i'(t) = -\lambda_i(t) \nabla \Phi(\gamma_i(t))$, and such that for almost every $t$ in $\{ t \in I_i, \gamma_i(t) \in \partial \Omega \}$, one has $\gamma_i'(t) = -\lambda_i(t) \nabla \Phi(\gamma_i(t))$. Moreover, if $I_i$ is not bounded (namely $I_i = (-\infty, 0]$ or $I_i = [0, +\infty)$), $\lambda_i(t) = 1$ for almost every $t \in I_i$, and if $I_i = [0, 1]$, $\lambda_i \in L^\infty([0, 1], \mathbb{R}_+)$. 69
Thus, for all $i$ if $V$ since one has by assumption and from Lemma 47:

all the previous inequalities are equalities and in particular, it holds:

\[ \Phi(x) - \Phi(y) = d_a(x, y) = d_a(x, \gamma_1(t_1)) + d_a(\gamma_1(t_1), \gamma_2(t_2)) + d_a(\gamma_2(t_2), y), \]

all the previous inequalities are equalities and in particular, it holds:

\[ d_a(\gamma_1(t_1), \gamma_2(t_2)) = \Phi(\gamma_1(t_1)) - \Phi(\gamma_2(t_2)) = f(\gamma_2(t_2)) - f(\gamma_1(t_1)) \geq 0. \]

Using Corollary 48 this implies that when restricting $\gamma_k$ to $I_k \cap [t_1, \infty)$ and $\gamma_k$ to $I_k \cap (-\infty, t_2]$ the set of curves $\{\gamma_1, ..., \gamma_k\}$ is a generalized integral curve of \( \nabla f \) in $\Omega$.

As in Step 2 of the proof of Proposition 49 let us introduce a smooth tangential and normal system of coordinates around $\gamma_k(t_1) \in \partial D$ in $D$, denoted by $\phi(x) = (xT, xN)$. The function $\phi$ is defined from a neighborhood of $\gamma_k(t_1)$ in $D$ to $\mathbb{R}^d$. Moreover, one has $x_N \geq 0$ and $x_N(x) = 0$ if and only if $x \in \partial D$. We may assume that the neighborhood $U_\beta \subset D$ on which $\phi$ is defined is such that $\phi(U_\beta) = U \times [0, \beta]$ for $\beta > 0$ and $U \subset \mathbb{R}^{d-1}$. Since $\partial_\phi f > 0$ on $\partial D$, $\beta > 0$ can be chosen small enough such that $\nabla f(x) \cdot n(x) > 0$ for all $x \in U_\beta$ where $n(x) = -\frac{\nabla f(x) \cdot \nabla x_N(x)}{\|
abla x_N(x)\|}$. Indeed, for $x \in \partial D$, $n(x)$ is nothing but the unit outward normal to $\partial D$. Now, by continuity of the curve $\gamma_k$, there exists $\mu > 0$ such that for all $t \in (t_1, t_1 + \mu]$, $\gamma_k(t) \in U_\beta$. The same considerations as in Step 2 of the proof of Proposition 49 can then be used to show that:

\[ x_N(\gamma(t)) \leq 0, \]

for all $t \in [t_1, t_1 + \mu]$ and thus $\gamma_k(t) \notin D$ for all $t \in [t_1, t_1 + \mu]$. This contradicts \( (105) \).

Thus, for all $i \in \{1, ..., N\}$, $\text{Im} \gamma_i \subset V_a$.

Then, the announced result follows by the same arguments as those used in the proof of Corollary 48 with $f$ replaced by $\Phi$ together with the fact that $\Phi$ satisfies $(102)$ on $V_a$ and for all $i \in \{1, ..., N\}$, $\text{Im} \gamma_i \subset V_a$. 

\[ \square \]
4 Construction of the quasi-modes and proof of Theorem 1

The aim of this section is to build the quasi-modes \( \tilde{u} \) and \( (\tilde{\phi}_i)_{i=1,...,n} \) satisfying the conditions stated in Section 2.2.2. Let us recall that \( \text{span}(\tilde{u}) \) (resp. \( \text{span}(\tilde{\phi}_i, i = 1, \ldots, n) \)) is intended to be a good approximation (in the sense made precise in items 1 and 2 in Proposition 17) of \( \text{Ran}(\pi_{[0,\sqrt{h}]}) \left( \mathcal{D}_{f,h} \right) \) (resp. \( \text{Ran}(\pi_{[0,\frac{h}{2}]} \left( \mathcal{D}_{f,h}^{(1)} \right) \)).

As recalled in Proposition 16, it is known that the dimension of \( \text{Ran}(\pi_{[0,h^{3/2}]}) \left( \mathcal{D}_{f,h} \right) \) is equal to the number of generalized critical points of index 1 (see [34, Section 3]) which are in our setting, thanks to assumptions [H1], [H2] and [H3], the local minima \((z_i)_{i=1,...,n}\) of \( f |_{\partial \Omega} \). In addition, it is known that the 1-forms in \( \text{Ran}(\pi_{[0,h^{3/2}]}) \left( \mathcal{D}_{f,h}^{(1)} \right) \) are localized in the limit \( h \to 0 \) in small neighborhoods of the local minima \((z_i)_{i=1,...,n}\).

For each local minimum \( z_i \), we construct an associated quasi-mode \( \tilde{\phi}_i \), using an auxiliary Witten Laplacian on 1-forms with mixed tangential-normal boundary conditions. This Witten Laplacian is defined on a domain \( \tilde{\Omega}_i \subset \Omega \) with suitable boundary conditions, so that its only small eigenvalue (namely in the interval \([0, h^{3/2}]\)) is 0, thanks to a complex property (see [34,47]). The associated eigenform is localized near \( z_i \), which can be proven thanks to Agmon estimates. Moreover, a precise estimate of this eigenform can be obtained thanks to a WKB expansion. The quasi-mode \( \tilde{\phi}_i \) is then this eigenform multiplied by a suitable cut-off function.

This section is organized as follows. In Section 4.1, we define a Witten Laplacian with mixed boundary conditions on a open domain \( \tilde{\Omega}_i \subset \Omega \) associated to each \( z_i, i \in \{1, \ldots, n\} \), and we study its spectrum. Section 4.2 is dedicated to the construction of the quasi-modes \( ((\tilde{\phi}_i)_{i=1,...,n}, \tilde{u}) \). In Section 4.3 we prove Agmon estimates on the eigenform associated with the smallest eigenvalue of the Witten Laplacian with mixed boundary conditions on \( \tilde{\Omega}_i \) and in Section 4.4 we compare this eigenform with a WKB approximation. We finally use this construction and these estimates to prove Theorem 1 in Section 4.5.

4.1 Geometric setting and definition of the Witten Laplacians with mixed boundary conditions

This section is organized as follows. In Section 4.1.1 we discuss some general results on traces of differential forms and we introduce the Witten Laplacians with mixed tangential Dirichlet boundary conditions and normal Dirichlet boundary conditions on manifolds with boundary. In Section 4.1.2 the domain \( \tilde{\Omega}_i \subset \Omega \) associated with each \( z_i, i \in \{1, \ldots, n\} \), is defined. Finally, Section 4.1.3 is dedicated to the study of the spectrum of the Witten Laplacian with mixed tangential Dirichlet boundary conditions and normal Dirichlet boundary conditions on \( \tilde{\Omega}_i \).

4.1.1 Trace estimates for differential forms and Witten Laplacians with mixed boundary conditions

In this section, we first discuss some general results on traces of differential forms. This is crucial to then build the Witten Laplacians with mixed boundary conditions. In the following, \( \tilde{\Omega} \) refers to any submanifold \( \tilde{\Omega} \) of \( \Omega \) with Lipschitz boundary. We will call such a submanifold a Lipschitz domain.
We first recall that for any Lipschitz domain $\hat{\Omega}$, the trace application
\[
\left\{ \begin{array}{l}
\Lambda^{p+1}H^1(\hat{\Omega}) \rightarrow \Lambda^{p+1}H^{\frac{1}{2}}(\partial\hat{\Omega}) \\
G \mapsto G|_{\partial\hat{\Omega}}
\end{array} \right.
\]
is a linear continuous and surjective application. We would like to present extensions of this result to less regular forms.

**Weak definition of traces**

For a Lipschitz domain $\hat{\Omega}$, let us introduce the functional spaces
\[
\Lambda^pH_d(\hat{\Omega}) := \left\{ u \in \Lambda^pL^2(\hat{\Omega}), \ du \in \Lambda^{p+1}L^2(\hat{\Omega}) \right\}
\]  
(106)

and
\[
\Lambda^pH_{d^*}(\hat{\Omega}) := \left\{ u \in \Lambda^pL^2(\hat{\Omega}), \ d^*u \in \Lambda^{p-1}L^2(\hat{\Omega}) \right\}
\]  
(107)
equipped with their natural graph norms. One recalls that for a differential form $f$ in $L^2(\partial\hat{\Omega})$, the tangential and normal components are defined as follows:
\[
f = tf + nf \quad \text{with} \quad tf = i_n(n^b \wedge f) \quad \text{and} \quad nf = n^b \wedge (i_n f),
\]  
(108)
where the superscript $b$ stands for the usual musical isomorphism: $n^b$ is the 1-form associated with the outgoing unit normal vector $n$. Moreover,
\[
\|f\|_{L^2(\partial\hat{\Omega})}^2 = \|tf\|_{L^2(\partial\hat{\Omega})}^2 + \|nf\|_{L^2(\partial\hat{\Omega})}^2 = \|n^b \wedge f\|_{L^2(\partial\hat{\Omega})}^2 + \|i_n f\|_{L^2(\partial\hat{\Omega})}^2.
\]
The Green formula for differential forms $(u,v) \in \Lambda^pH^1(\hat{\Omega}) \times \Lambda^{p+1}H^1(\hat{\Omega})$ writes
\[
\langle du, v \rangle_{L^2(\Omega)} - \langle u, d^*v \rangle_{L^2(\Omega)} = \int_{\partial\Omega} \langle n^b \wedge u, v \rangle_{T^2\Omega} d\sigma = \int_{\partial\Omega} \langle n^b \wedge u, n^b \wedge v \rangle_{T^2\Omega} d\sigma = \int_{\partial\Omega} \langle tu, i_nv \rangle_{T^2\Omega} d\sigma,
\]  
(109)
where we used the standard relation $(n^b \wedge)^* = i_n$.

Using this Green formula, the tangential (resp. normal) traces can be defined for forms in $\Lambda^pH_d(\hat{\Omega})$ (resp. $\Lambda^pH_{d^*}(\hat{\Omega})$) by duality. Indeed, for any $u \in \Lambda^pH_d(\hat{\Omega})$, $n^b \wedge u \in \Lambda^{p+1}H^{-\frac{1}{2}}(\partial\hat{\Omega})$ is defined by
\[
\forall g \in \Lambda^{p+1}H^{-\frac{1}{2}}(\partial\hat{\Omega}), \ \langle n^b \wedge u, g \rangle_{H^{-\frac{1}{2}}(\partial\hat{\Omega}), H^{\frac{1}{2}}(\partial\hat{\Omega})} = \langle du, G \rangle_{L^2(\hat{\Omega})} - \langle u, d^*G \rangle_{L^2(\hat{\Omega})},
\]  
(110)
where $G$ is any form in $\Lambda^{p+1}H^1(\hat{\Omega})$ whose trace in $\Lambda^{p+1}H^{\frac{1}{2}}(\partial\hat{\Omega})$ is $g$. This definition is independent of the chosen extension $G$. Similarly, for any $u \in \Lambda^pH_{d^*}(\hat{\Omega})$, $i_nu \in \Lambda^{p-1}H^{-\frac{1}{2}}(\partial\hat{\Omega})$ is defined by
\[
\forall g \in \Lambda^{p-1}H^{\frac{1}{2}}(\partial\hat{\Omega}), \ \langle i_nu, g \rangle_{H^{-\frac{1}{2}}(\partial\hat{\Omega}), H^{\frac{1}{2}}(\partial\hat{\Omega})} = \langle u, d^*G \rangle_{L^2(\hat{\Omega})} - \langle d^*u, G \rangle_{L^2(\hat{\Omega})},
\]  
(111)
where $G$ is any extension of $g$ in $\Lambda^{p-1}H^1(\hat{\Omega})$.

Let $\Gamma$ be any subset of $\partial\hat{\Omega}$. For $u \in \Lambda^pH_d(\hat{\Omega})$, we will write $tu|_{\Gamma} = 0$ if $n^b \wedge u|_{\Gamma} = 0$. If $u \in \Lambda^pH_d(\hat{\Omega})$ and $n^b \wedge u|_{\Gamma} \in \Lambda^{p+1}L^2(\Gamma)$, the tangential trace on $\Gamma$ is defined by
\[
u|_{\Gamma} := i_n(n^b \wedge u) \in \Lambda^pL^2(\Gamma), \quad \text{so that} \quad \|tu\|_{L^2(\Gamma)}^2 = \|n^b \wedge u\|_{L^2(\Gamma)}^2.
\]  
(112)
Similarly, for \( u \in \Lambda^p H^{d_\ast}_d(\hat{\Omega}) \), we will write \( nu|_\Gamma = 0 \) if \( i_n u|_\Gamma = 0 \). If \( u \in \Lambda^p H^{d_\ast}_d(\hat{\Omega}) \) and \( i_n u|_\Gamma \in \Gamma^{-1} L^2(\Gamma) \), the normal trace on \( \Gamma \) is defined by

\[
nu|_\Gamma := n^b \wedge (i_n u) \in \Lambda^p L^2(\Gamma), \quad \text{so that} \quad \|nu\|_{L^2(\Gamma)} = \|i_n u\|_{L^2(\Gamma)}. \quad (113)
\]

Lastly, if \( u \in \Lambda^p H_d(\hat{\Omega}) \cap \Lambda^p H^{d_\ast}_d(\hat{\Omega}) \) is such that \( n^b \wedge u|_\Gamma \in \Gamma^{-1} L^2(\Gamma) \) and \( i_n u \in \Gamma^{-1} L^2(\Gamma) \) then \( u \) admits a trace \( u|_\Gamma \) in \( L^2(\Gamma) \) defined by

\[
u|_\Gamma := tu|_\Gamma + nu|_\Gamma. \quad (114)
\]

This definition is compatible with \((108)\) and such a differential form satisfies

\[
\|u|_\Gamma\|_{L^2(\Gamma)}^2 = \|tu|_\Gamma\|_{L^2(\Gamma)}^2 + \|nu|_\Gamma\|_{L^2(\Gamma)}^2 = \|n^b \wedge u\|_{L^2(\Gamma)}^2 + \|i_n u\|_{L^2(\Gamma)}^2.
\]

All the above definitions coincide moreover with the usual ones when \( u \) belongs to \( \Lambda^p H^1(\hat{\Omega}) \).

Let us finally note for further references that if \( t \) are in \( L^2(\partial \hat{\Omega}) \), a direct consequence of the Green formula \((109)\) is the following: for every \( u, v \in \Lambda^p L^2(\hat{\Omega}) \) such that \( du, d^* u, d^* du, dv, d^* v \in \Lambda^p L^2(\hat{\Omega}) \) and \( n^b \wedge d^*_f h u, i_n d_f h u, n^b \wedge v, i_n v \in \Lambda^p L^2(\partial \hat{\Omega}), \)

\[
\langle (d_f h^*_f + d^*_f h d_f h) u, v \rangle_{L^2(\hat{\Omega})} = \langle d_f h u, d_f h v \rangle_{L^2(\hat{\Omega})} + \langle d^*_f h u, d^*_f h v \rangle_{L^2(\hat{\Omega})} + h \int_{\partial \hat{\Omega}} \langle n^b \wedge d^*_f h u, n^b \wedge (i_n v) \rangle_{T_\Sigma \hat{\Omega}} d\sigma - h \int_{\partial \hat{\Omega}} \langle n^b \wedge v, n^b \wedge (i_n d_f h u) \rangle_{T_\Sigma \hat{\Omega}} d\sigma. \quad (115)
\]

The Gaffney’s inequality

The following extension of Gaffney’s inequality (see \[65\]) will be useful in the sequel (we refer to Section \[2.1.1\] for the definitions of the Hilbert space \( \Lambda^p H^1_T(\hat{\Omega}) \) and \( \Lambda^p H^1_T(\hat{\Omega}) \)).

**Lemma 54.** Let \( \hat{\Omega} \) be a Lipschitz domain. The equality

\[
\left\{ u \in \Lambda^p L^2(\hat{\Omega}) \ s.t. \ du, d^* u \in L^2(\hat{\Omega}) \ and \ tu \equiv 0 \ on \ \partial \hat{\Omega} \right\} = \Lambda^p H^1_T(\hat{\Omega})
\]

holds algebraically and topologically, the functional space in the left-hand side being equipped with the norm associated with the scalar product

\[
Q(u, v) := \langle u, v \rangle_{L^2(\hat{\Omega})} + \langle du, dv \rangle_{L^2(\hat{\Omega})} + \langle d^* u, d^* v \rangle_{L^2(\hat{\Omega})}.
\]

In a similar way, the following equality holds algebraically and topologically:

\[
\left\{ u \in \Lambda^p L^2(\hat{\Omega}) \ s.t. \ du, d^* u \in L^2(\hat{\Omega}) \ and \ nu = 0 \ on \ \partial \hat{\Omega} \right\} = \Lambda^p H^1_N(\hat{\Omega}).
\]

Notice that in the definition of the functional spaces above, the equality \( tu = 0 \) and \( nu = 0 \) hold in the weak sense defined above (see \[112\] and \[113\]). A direct consequence of this lemma is that a differential form in \( \Lambda H^1_d(\hat{\Omega}) \cap \Lambda H^{d_\ast}_d(\hat{\Omega}) \) such that \( tu = 0 \) or \( nu = 0 \) on \( \partial \hat{\Omega} \) admits a trace in \( L^2(\partial \hat{\Omega}) \).

**Remark 12.** The statement of Gaffney’s inequality in \[65\] reads as follows (see indeed Corollary 2.1.6 and Theorem 2.1.7 there):

\[
\exists C > 0, \ \forall u \in \Lambda^p H^1_T(\hat{\Omega}) \cup \Lambda^p H^1_N(\hat{\Omega}), \quad \|u\|^2_{H^1(\hat{\Omega})} \leq CQ(u, u). \quad (116)
\]

Since it also holds that, for some \( C' > 0 \) and any \( u \in \Lambda^p H^1(\hat{\Omega}), \quad Q(u, u) \leq C'\|u\|^2_{H^1(\hat{\Omega})}, \)

the scalar products \( \langle \cdot, \cdot \rangle_{H^1} \) and \( Q(\cdot, \cdot) \) are then equivalent on both \( \Lambda^p H^1_T(\hat{\Omega}) \) and \( \Lambda^p H^1_N(\hat{\Omega}) \).

The above lemma can be seen as a generalization of this result.
Proof. We only prove the first equality in Lemma [54, the second one being similar. Let us define

$$H := \left\{ u \in \Lambda^p L^2(\hat{\Omega}) \text{ s.t. } du, d^*u \in \Lambda L^2(\hat{\Omega}) \text{ and } n^\flat \wedge u = 0 \text{ on } \partial\hat{\Omega} \right\}$$

which is a Hilbert space once equipped with the scalar product $Q$. From Gaffney’s inequality [10], $\Lambda^p H^1_\perp(\hat{\Omega})$ is a closed subset of $H$ and to conclude, we just have to show that $(\Lambda^p H^1_\perp(\hat{\Omega}))^\perp = \{0\}$, the orthogonal complement of $H$ being taken with respect to the norm inherited from $Q$. Consider then $u \in H$ such that for any $v \in \Lambda^p H^1_\perp(\hat{\Omega})$,

$$0 = Q(u, v) = \langle u, v \rangle_{L^2(\hat{\Omega})} + \langle du, dv \rangle_{L^2(\hat{\Omega})} + \langle d^*u, d^*v \rangle_{L^2(\hat{\Omega})}.$$

The above equality holds in particular for every $v \in D$ where

$$D = \left\{ v \in \Lambda^p H^2(\hat{\Omega}), \; tv|_{\partial\hat{\Omega}} = td^*v|_{\partial\hat{\Omega}} = 0 \right\}. \quad (117)$$

Fix such a $v$. Since $n^\flat \wedge u = 0$ on $\partial\hat{\Omega}$, applying (110) to $u$ and $dv \in \Lambda^{p+1}H^1(\hat{\Omega})$ then leads to

$$\langle du, dv \rangle_{L^2(\hat{\Omega})} = \langle u, d^*dv \rangle_{L^2(\hat{\Omega})}.$$

Applying also (111) to $u$ and $d^*v \in \Lambda^{p-1}H^1(\hat{\Omega})$ gives

$$\langle d^*u, d^*v \rangle_{L^2(\hat{\Omega})} = \langle u, dd^*v \rangle_{L^2(\hat{\Omega})} - \langle i_n u, d^*v|_{\partial\hat{\Omega}} \rangle_{H^{-\frac{1}{2}}(\partial\hat{\Omega}), H^{\frac{1}{2}}(\partial\hat{\Omega})}.$$

Since $\Lambda^p C^\infty(\hat{\Omega})$ is densely embedded in both $\Lambda^p H_d(\hat{\Omega})$ and $\Lambda^p H_{dp}(\hat{\Omega})$ (see for example [41, Proposition 3.1]), we have moreover for some sequence $(u_k)_{k \in \mathbb{N}}$ of $\Lambda^p C^\infty(\hat{\Omega})$ forms:

$$\langle i_n u, d^*v|_{\partial\hat{\Omega}} \rangle_{H^{-\frac{1}{2}}(\partial\hat{\Omega}), H^{\frac{1}{2}}(\partial\hat{\Omega})} = \lim_{k \to +\infty} \int_{\partial\hat{\Omega}} \langle i_n u_k, d^*v \rangle_{T^*\hat{\Omega}} d\sigma = \lim_{k \to +\infty} \int_{\partial\hat{\Omega}} \langle i_n u_k, n^\flat \wedge (i_n d^*v) \rangle_{T^*\hat{\Omega}} d\sigma = 0,$$

where the second equality is a consequence of $td^*v|_{\partial\hat{\Omega}} = 0$. It consequently follows

$$0 = \langle u, v \rangle_{L^2(\hat{\Omega})} + \langle u, d^*dv \rangle_{L^2(\hat{\Omega})} + \langle u, dd^*v \rangle_{L^2(\hat{\Omega})} = \langle u, (I + \Delta_H^{(p)}) v \rangle_{L^2(\hat{\Omega})}, \quad (118)$$

where $\Delta_H^{(p)}$ denotes the Hodge Laplacian on $\hat{\Omega}$ with domain $D$ defined by (117). Since the unbounded operator $(\Delta_H^{(p)}, D)$ is selfadjoint and nonnegative on $\Lambda^p L^2(\hat{\Omega})$, we have in particular $\text{Ran}(I + \Delta_H^{(p)}) = \Lambda^p L^2(\hat{\Omega})$ and we deduce from (118) that $u = 0$, which completes the proof.

The case of mixed normal-tangential Dirichlet boundary conditions

Let $\Gamma_T$ and $\Gamma_N$ be two disjoint open subsets of $\partial\hat{\Omega}$ such that $\Gamma_T \cup \Gamma_N = \partial\hat{\Omega}$. The objective of this section is to consider differential forms such that $tu = 0$ on $\Gamma_T$ and $nu = 0$ on $\Gamma_N$, and to state results on the existence of a trace in $L^2(\partial\hat{\Omega})$ for such differential forms, as well as subelliptic estimates.

In general, a trace in $L^2(\partial\hat{\Omega})$ does not exist in such a setting [8, 41]: one needs a geometric assumption, namely that $\Gamma_T$ and $\Gamma_N$ meet at an angle strictly smaller than
\(\pi\). This means that the angle between \(\Gamma_T\) and \(\Gamma_N\) measured in \(\hat{\Omega}\) is smaller than \(\pi\). More precisely, see \([8, 41]\), locally around any point \(x_0 \in \overline{\Gamma_T} \cap \overline{\Gamma_N}\), one requires that there exists a local system of coordinates \((x_1, x'')\) in \(\mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R}\) on a neighborhood \(V_0\) of \(x_0\), and two Lipschitz functions \(\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}\) and \(\psi : \mathbb{R}^{n-2} \rightarrow \mathbb{R}\) such that \(\hat{\Omega} \cap V_0 = \{x_n > \varphi(x_1, x'')\}\), \(\Gamma_T \cap V_0 = \{x_n = \varphi(x_1, x'')\) and \(x_1 > \psi(x'')\}\) and \(\Gamma_N \cap V_0 = \{x_n = \varphi(x_1, x'')\) and \(x_1 < \psi(x'')\}\) and

\[
\begin{align*}
\partial_{x_1}\varphi(x_1, x'') &\geq \kappa \text{ on } x_1 > \psi(x'') \\
\partial_{x_1}\varphi(x_1, x'') &\leq -\kappa \text{ on } x_1 < \psi(x'')
\end{align*}
\]

(119)

for some positive \(\kappa\). This is equivalent to the existence of a smooth vector field \(\theta\) on \(\partial \hat{\Omega}\) such that \((\theta, n) < 0\) on \(\Gamma_T\) and \((\theta, n) > 0\) on \(\Gamma_N\), which is one of the key ingredient of the proofs used in \([8, 41]\).

Let \(\Gamma\) be any subset of \(\partial \hat{\Omega}\). According to \([41\), Proposition 3.1], the space

\[
\left\{ u \in L^pC^\infty(\hat{\Omega}), \ u \equiv 0 \text{ in a neighborhood of } \partial \hat{\Omega} \setminus \Gamma \right\}
\]

is densely embedded in both

\[
L^pH_{d, \Gamma}(\hat{\Omega}) := \left\{ u \in L^pH_d(\hat{\Omega}), \ \text{supp}(n^b \wedge u) \subset \overline{\Gamma} \right\}
\]

and

\[
L^pH_{d^*, \Gamma}(\hat{\Omega}) := \left\{ u \in L^pH_{d^*}(\hat{\Omega}), \ \text{supp}(i_n u) \subset \overline{\Gamma} \right\}.
\]

In addition, according to \([41\), Theorem 3.4], for \((u, v) \in L^pH_{d}(\hat{\Omega}) \times L^{p+1}H_{d^*}(\hat{\Omega})\) satisfying the trace conditions

\[
i_n v \in L^pL^2(\partial \hat{\Omega}), \ \text{supp}i_n v \subset \Gamma \quad \text{and} \quad n^b \wedge u \in L^{p+1}L^2(\Gamma),
\]

or

\[
n^b \wedge u \in L^{p+1}L^2(\partial \hat{\Omega}), \ \text{supp}(n^b \wedge u) \subset \Gamma \quad \text{and} \quad i_n v \in L^pL^2(\Gamma),
\]

one has the following Green formula (compare with (109)):

\[
\langle du, v \rangle_{L^2(\hat{\Omega})} - \langle u, d^* v \rangle_{L^2(\hat{\Omega})} = \int_{\Gamma} \langle \n^b \wedge u, n^b \wedge (i_n v) \rangle_{T^\perp T^\perp} d\sigma = \int_{\Gamma} \langle i_n(n^b \wedge u), i_n v \rangle_{T^\perp T^\perp} d\sigma.
\]

(120)

One is now ready to state the following proposition implied by Theorems 1.1 and 1.2 of \([41]\) (see also Theorems 4.1 and 4.2 of \([28]\)).

**Proposition 55.** Let us assume that \(\hat{\Omega}\) is a Lipschitz domain. Let \(\Gamma_T\) and \(\Gamma_N\) be two disjoint Lipschitz open subsets of \(\partial \hat{\Omega}\) such that \(\Gamma_T \cup \Gamma_N = \partial \hat{\Omega}\) and such that \(\Gamma_T\) and \(\Gamma_N\) meet at an angle strictly smaller than \(\pi\). Then, the following results hold:

(i) Let \(u\) be a differential form such that

\[
u \in L^pL^2(\hat{\Omega}), \ du \in L^2(\hat{\Omega}), d^* u \in L^2(\hat{\Omega}), \ t u|_{\Gamma_T} = 0 \quad \text{and} \quad n u|_{\Gamma_N} = 0.
\]

Then \(u\) satisfies

\[
u \in L^pH_{d^*}(\hat{\Omega}) \quad \text{and} \quad i_n u, n^b \wedge u \in L^pL^2(\partial \hat{\Omega})
\]

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as well as the subelliptic estimate:

\[ \|u\|_{H^2(\hat{\Omega})} + \|u|_{\partial \hat{\Omega}}\|_{L^2(\partial \hat{\Omega})} \leq C \left( \|u\|_{L^2(\hat{\Omega})} + \|du\|_{L^2(\hat{\Omega})} + \|d^* u\|_{L^2(\hat{\Omega})} \right), \]  

(121)

where \( u|_{\partial \hat{\Omega}} \) is defined by (114).

(ii) The unbounded operators \( d_T^{(p)}(\hat{\Omega}) \) and \( \delta_N^{(p)}(\hat{\Omega}) \) on \( \Lambda^p L^2(\hat{\Omega}) \) defined by

\[ d_T^{(p)}(\hat{\Omega}) = d_{f,h}^{(p)} \text{ with domain } D \left( d_T^{(p)}(\hat{\Omega}) \right) = \left\{ u \in \Lambda^p L^2(\hat{\Omega}), \ d_{f,h}u \in \Lambda^{p+1} L^2(\hat{\Omega}), \ tu|_{\Gamma_T} = 0 \right\}, \]

and

\[ \delta_N^{(p)}(\hat{\Omega}) = \left( d_{f,h}^{(p)} \right)^* \text{ with domain } D \left( \delta_N^{(p)}(\hat{\Omega}) \right) = \left\{ u \in \Lambda^p L^2(\hat{\Omega}), \ d_{f,h}^*u \in \Lambda^{p-1} L^2(\hat{\Omega}), \ nu|_{\Gamma_N} = 0 \right\}, \]

are closed, densely defined, and adjoint one of each other.

Note that in the point (i) of Proposition 55, \( d \) and \( d^* \) can be replaced by \( d_{f,h} \) and \( d_{f,h}^* \) owing to the relations \( d_{f,h} = hd + df \wedge \) and \( d_{f,h}^* = hd^* + idf \). Moreover, the point (ii) is actually proven in [28,41] for \( d \) and \( d^* \) but remains true for \( d_{f,h} \) and \( d_{f,h}^* \) since \( (df \wedge)^* = idf \) on \( L^2(\hat{\Omega}) \).

The mixed Witten Laplacian \( \Delta_{f,h}^M(\hat{\Omega}) \)

We are now in position to define the mixed Witten Laplacian \( \Delta_{f,h}^M(\hat{\Omega}) \) (the upper-script \( M \) stands for mixed boundary conditions) with tangential Dirichlet boundary conditions on \( \Gamma_T \) and normal Dirichlet boundary conditions on \( \Gamma_N \) (see [28,41] for more results on such operators). The operator \( \Delta_{f,h}^M(\hat{\Omega}) \) on \( L^2(\hat{\Omega}) \) is defined by

\[ \Delta_{f,h}^M(\hat{\Omega}) := d_T^{(p-1)}(\hat{\Omega}) \circ \delta_N^{(p)}(\hat{\Omega}) + \delta_N^{(p+1)}(\hat{\Omega}) \circ d_T^{(p)}(\hat{\Omega}), \]  

(122)

in the sense of composition of unbounded operators, where \( d_T \) and \( \delta_N \) have been introduced in Proposition 55. Noticing that for any \( u \in \Lambda^p H^1(\hat{\Omega}) \) such that \( tu|_{\Gamma_T} = 0 \), one has \( td_{f,h}u|_{\Gamma_T} = 0 \), which is well known for \( u \in \Lambda^p H^1(\hat{\Omega}) \) and can be checked here using (110). Likewise, one has \( td_{f,h}u|_{\Gamma_T} = 0 \) for \( u \in \Lambda^p H^2(\hat{\Omega}) \) such that \( tu|_{\Gamma_T} = 0 \). This implies in particular

\[ \begin{cases} \text{Im} \ d_T \subset \text{Ker} \ d_T & \text{and} & d_T^2 = 0 \\ \text{Im} \ \delta_N \subset \text{Ker} \ \delta_N & \text{and} & \delta_N^2 = 0 \end{cases} \]  

(123)

(since \( d_{f,h}d_{f,h} = 0 \) in the distributional sense). Owing to this last relation and to Proposition 55, a result due to Gaffney (see e.g. the proof of [28, Propositions 2.3 and 2.4]) states that \( \Delta_{f,h}^M(\hat{\Omega}) \) is a densely defined nonnegative selfadjoint operator on \( L^2(\hat{\Omega}) \) (with domain \( \{124\} \) see below).

The domain \( D \left( Q_{f,h}^M(\hat{\Omega}) \right) \) of the closed quadratic form \( Q_{f,h}^M(\hat{\Omega}) \) associated with \( \Delta_{f,h}^M(\hat{\Omega}) \) is given by

\[ D \left( Q_{f,h}^M(\hat{\Omega}) \right) = D \left( d_T^{(p)}(\hat{\Omega}) \right) \cap D \left( \delta_N^{(p)}(\hat{\Omega}) \right) = \left\{ v \in \Lambda^p L^2(\hat{\Omega}), \ dv \in L^2(\hat{\Omega}), \ d^* v \in L^2(\hat{\Omega}), \ tv|_{\Gamma_T} = 0 \text{ and } nv|_{\Gamma_N} = 0 \right\}, \]

and for any \( u, v \in D \left( Q_{f,h}^M(\hat{\Omega}) \right) \),

\[ Q_{f,h}^M(\hat{\Omega})(u, v) = \langle d_T u, d_T v \rangle_{L^2} + \langle \delta_N u, \delta_N v \rangle_{L^2}. \]
This is proven in [28, Theorem 2.8].

The domain \( D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) \) is explicitly given by:

\[
D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) = \left\{ u \in L^2(\Omega) \text{ s.t. } df_h u, d_h^* df_h u, d_h df_h u, d_h d_h^* u \in L^2(\Omega) \right. \\
\left. \text{ and } tu|_{\Gamma_T} = 0, \; td_h^* u|_{\Gamma_T} = 0, \; n u|_{\Gamma_N} = 0, \; nd_h u|_{\Gamma_N} = 0 \right\}. \tag{124}
\]

The traces \( td_h^* u \) and \( nd_h u \) are a-priori defined in \( H^{-\frac{1}{2}}(\Omega) \) but actually belong to \( L^2(\Omega) \). Indeed, we have \( nd_h u|_{\Gamma_N} = 0 \) by definition of \( D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) \) and \( td_h u|_{\Gamma_T} = 0 \) by (110), so \( df_h u \) is in \( D \left( \Delta_{f,h}^{M,(p+1)}(\Omega) \right) \) and therefore has a trace in \( L^2(\Omega) \) according to Proposition~55. This argument also holds for \( d_h^* u \in D \left( \Delta_{f,h}^{M,(p-1)}(\Omega) \right) \).

We end up this section with the following lemma which will be frequently used in the sequel.

**Lemma 56.** Under the assumptions of Proposition~55, the following formula holds: for any \( u \in D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) \),

\[
Q_{f,h}^{M,(p)}(\Omega)(u,u) = \|df_h u\|_{L^2(\Omega)}^2 + \|d_h^* u\|_{L^2(\Omega)}^2 \\
= h^2 \|du\|_{L^2(\Omega)}^2 + h^2 \|d^* u\|_{L^2(\Omega)}^2 \\
+ \|\nabla f\|_{L^2(\Omega)}^2 + h \langle (\mathcal{L}_f + \mathcal{L}_f^*) u, u \rangle_{L^2(\Omega)} \\
- h \left( \int_{\Gamma_T} - \int_{\Gamma_N} \right) \langle u, u \rangle_{T^2 \Omega} \partial_n f d\sigma, \tag{125}
\]

where \( \mathcal{L} \) stands for the Lie derivative.

Notice that the boundary integral terms are well defined since \( u|_{\partial \Omega} \in L^2(\Omega) \) thanks to point (i) in Proposition~55.

**Proof.** For \( u \in D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) \), one first gets by straightforward computations,

\[
\|df_h u\|_{L^2(\Omega)}^2 + \|d_h^* u\|_{L^2(\Omega)}^2 = h^2 \|du\|_{L^2(\Omega)}^2 + h^2 \|d^* u\|_{L^2(\Omega)}^2 \\
+ \|\nabla f\|_{L^2(\Omega)}^2 + h \langle (df \land u, du)_{L^2(\Omega)} + \langle du, df \land u \rangle \\
+ \langle d^* u, i\nabla f u \rangle_{L^2(\Omega)} + \langle i\nabla f u, d^* u \rangle \\
= h^2 \|du\|_{L^2(\Omega)}^2 + h^2 \|d^* u\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2 \\
+ h \langle (\mathcal{L}_f + \mathcal{L}_f^*) u, u \rangle_{L^2(\Omega)} + h \langle (df \land u, du) \rangle_{L^2(\Omega)} \\
- \langle d^* (df \land u), u \rangle_{L^2(\Omega)} - \langle di\nabla f u, u \rangle_{L^2(\Omega)} - \langle i\nabla f u, d^* u \rangle_{L^2(\Omega)}),
\]

where the last equality holds thanks to the the relations \((d\varphi \land)^* = i\nabla \varphi\),

\[
\mathcal{L}_f = d \circ i\nabla \varphi + i\nabla \varphi \circ d \quad \text{and} \quad i\nabla \varphi (d\varphi \land u) + d\varphi \land (i\nabla \varphi u) = |\nabla \varphi|^2 u.
\]

To get the boundary integral terms in (125) one uses [120], which gives here, since \( u \in D \left( \Delta_{f,h}^{M,(p)}(\Omega) \right) \) and \( df \land u, i\nabla f u \in \Lambda H_d(\Omega) \cap \Lambda H_{d^*}(\Omega) \):

\[
\langle df \land u, du \rangle_{L^2(\Omega)} - \langle d^* (df \land u), u \rangle_{L^2(\Omega)} = \int_{\Gamma_N} \langle n^y \land u, n^y \land i\nabla f u \rangle_{T^2 \Omega} d\sigma \tag{126}
\]
and

\[ (i\nabla f u, d^* u)_{L^2(\Omega)} - (d i\nabla f u, u)_{L^2(\Omega)} = -\int_{\Gamma_T} (n^b \wedge i\nabla f u, n^b \wedge i_n u)_{T^*_2 \Omega} d\sigma. \]  \hspace{1cm} (127)

Now, note that by Lemma 54, \( u \) is a \( \Lambda^p H^1 \) form outside \( \overline{\Gamma_T} \cap \overline{\Gamma_N} \) and then admits a boundary trace defined a.e. on \( \partial \Omega \) and belonging to \( L^2_{loc}(\partial \Omega \setminus (\overline{\Gamma_T} \cap \overline{\Gamma_N})) \). But this trace has to be \( u|_{\partial \Omega} \) as defined by \( (\ref{boundary_trace}) \) and is hence a \( L^2(\partial \Omega) \) form owing to the first point of Proposition 55. The proof follows easily: looking at \( (\ref{boundary_trace}) \) for example, one gets, defining \( \Gamma_N^\varepsilon := \{ x \in \Gamma_N, \, \phi N(x, \partial \Gamma_N) > \varepsilon \} \),

\[
\int_{\Gamma_N^\varepsilon} (n^b \wedge u, n^b \wedge i_n (df \wedge u))_{T^*_2 \Omega} d\sigma = \lim_{\varepsilon \to 0^+} \int_{\Gamma_N^\varepsilon} (n^b \wedge u, n^b \wedge i_n (df \wedge u))_{T^*_2 \Omega} d\sigma \\
= \lim_{\varepsilon \to 0^+} \int_{\Gamma_N^\varepsilon} \langle u, i_n (df \wedge u) \rangle_{T^*_2 \Omega} d\sigma \\
= \lim_{\varepsilon \to 0^+} \int_{\Gamma_N^\varepsilon} \langle u, i_n (df \wedge u) \rangle_{T^*_2 \Omega} d\sigma \\
= \lim_{\varepsilon \to 0^+} \int_{\Gamma_N^\varepsilon} (\partial_n f \langle u, u \rangle_{T^*_2 \Omega} - \langle u, df \wedge i_n u \rangle_{T^*_2 \Omega}) d\sigma \\
= \int_{\Gamma_N} \partial_n f \langle u, u \rangle_{T^*_2 \Omega} d\sigma,
\]

where we used the usual trace properties for \( H^1 \) forms on \( \Gamma_N^\varepsilon \), the fact that \( i_n u = 0 \) at the second to last line and the Lebesgue dominated convergence theorem at the last line.

4.1.2 Construction of the domain \( \hat{\Omega} \)

In this section we assume \([H1],[H2]\) and \([H3]\). Let us consider \( z_i \in \{ z_1, \ldots, z_n \} \) a local minimum of \( f|_{\partial \Omega} \). The objective of this section is to build the domain \( \hat{\Omega} \) on which the Witten Laplacian with mixed tangential-normal boundary conditions will be defined.

This auxiliary operator will be a Witten Laplacian associated with tangential and normal Dirichlet boundary conditions, such that \( z_i \) remains the only generalized critical point. Let us recall that \( x_0 \in \Omega \) is the minimum of \( f \) on \( \Omega \). Let \( \Omega_0 \) be a small smooth open neighborhood of \( x_0 \) such that the \( \partial_i f < 0 \) on \( \Gamma_0 = \partial \Omega_0 \), \( n \) being the outward normal derivative to \( \Omega \setminus \Omega_0 \). Let \( \Gamma_1,i \) denote a subset of \( B_{z_i} \), as large as we want in \( B_{z_i} \). The basic idea is to define \( \hat{\Omega} = \Omega \setminus \overline{\Gamma_0} \) and to consider a Witten Laplacian on \( \hat{\Omega} \), with tangential zero boundary conditions on \( \Gamma_0 \cup \Gamma_{1,i} \) and with normal zero boundary conditions on \( \partial \Omega \setminus \overline{\Gamma_{1,i}} \). This would indeed yield an operator on a domain \( \hat{\Omega} \) with a single generalized critical point, namely \( z_i \). There is however a technical difficulty in this approach, related to the fact that differential forms with mixed normal and tangential Dirichlet boundary conditions are singular at the boundary between the domains where tangential and normal boundary conditions are applied, as explained in Section 4.1.1.

With the previous construction, \( \Gamma_{1,i} \) and \( \partial \Omega \setminus \overline{\Gamma_{1,i}} \) meet at an angle \( \pi \). We therefore need to define a domain \( \hat{\Omega}_1 \), strictly included in \( \Omega \setminus \overline{\Gamma_0} \), with boundary \( \partial \hat{\Omega}_1 = \Gamma_0 \cup \Gamma_{1,i} \cup \Gamma_{2,i} \) where \( \Gamma_0 = \partial \Omega_0 \) as defined above, \( \Gamma_{1,i} \cap \Gamma_{2,i} = \emptyset \), \( \Gamma_{1,i} \subset B_{z_i} \) is as large as we want in \( B_{z_i} \) and \( \Gamma_{2,i} \) meets \( \Gamma_{1,i} \) at an angle strictly smaller than \( \pi \) (see \( (\ref{boundary_trace}) \) above for a proper definition). We will then consider a Witten Laplacian with tangential zero boundary
conditions on $\Gamma_0 \cap \Gamma_{1,i}$ and normal zero boundary conditions on $\Gamma_{2,i}$. Moreover, in order not to introduce new generalized critical point on $\Gamma_{2,i}$, we would like to keep the property $\partial_n f > 0$ on $\Gamma_{2,i}$ (where $n$ denotes the outward normal derivative to $\Omega$). The aim of this section is indeed to define such a domain $\hat{\Omega}$.

A system of coordinates on a neighborhood of $\partial \Omega$.

Let us consider the function $f_-$ defined on a neighborhood $V_{\partial \Omega}$ of $\partial \Omega$, as introduced in Proposition 51. Recall that $f_-(x) = 0$ for $x \in \partial \Omega$ and that $V_{\partial \Omega}$ can be chosen such that $f_- > 0$ on $V_{\partial \Omega} \setminus \partial \Omega$ and $|\nabla f_-| \neq 0$ on $V_{\partial \Omega}$. Let us now consider $\varepsilon > 0$ such that

$$V_\varepsilon = \{y \in \Omega, 0 \leq f_-(y) \leq \varepsilon\} \subset V_{\partial \Omega}.$$  

For any $x \in V_\varepsilon$, the dynamics

$$\begin{cases} \gamma_x'(t) = -\frac{\nabla f_-}{|\nabla f_-|^2}(\gamma_x(t)) \\ \gamma_x(0) = x \end{cases}$$  

is such that $\gamma_x(t_x) \in \partial \Omega$, where

$$t_x = \inf\{t, \gamma_x(t) \not\in \text{int} V_\varepsilon\}. $$

This is indeed a consequence of the fact that $\frac{d}{dt} f_-(\gamma_x(t)) = -1 < 0$ on $[0, t_x)$.

The application

$$\Gamma : \begin{cases} V_\varepsilon \to \partial \Omega \times [-\varepsilon, 0] \\ x \mapsto (\gamma_x(t_x), -t_x) \end{cases}$$

defines a $C^\infty$ diffeomorphism. The inverse application of $\Gamma$ is $(x', x_d) \in \partial \Omega \times [-\varepsilon, 0] \mapsto \gamma_{x'}(x_d)$.

**Definition 9.** Let us assume that the hypothesis [H3] holds. Let us define the following system of coordinates for $x \in V_\varepsilon$:

$$\forall x \in V_\varepsilon, (x'(x), x_d(x)) = (\gamma_x(t_x), -t_x) \in \partial \Omega \times [-\varepsilon, 0].$$

Notice that, by construction (since $\frac{d}{dt} f_-(\gamma_x(t)) = -1$),

$$x_d(x) = -f_-(x).$$

Thus, in this system of coordinates, $\{x_d = 0\} = \partial \Omega$ and $\{x_d < 0\} = \partial \Omega \cap V_\varepsilon$.

We will sometimes need to use a local system of coordinates in $\partial \Omega$, that we will then denote by the same notation $x'$. By using the same procedure as above, $(x', x_d)$ then defines a local system of coordinates. Let us make this precise. For $y \in \partial \Omega$, let us consider $x' : V_y \to \mathbb{R}^{d-1}$ a smooth local system of coordinates in $\partial \Omega$, in a neighborhood $V_y \subset \partial \Omega$ of $y$. These coordinates are then extended in a neighborhood of $V_y$ in $\Omega$, as constant along the integral curves of $\gamma'(t) = \frac{\nabla f_-}{|\nabla f_-|^2}(\gamma(t))$, for $t \in [0, \varepsilon]$. The function $x \mapsto (x', x_d)$ (where, we recall, $x_d(x) = -f_-(x)$) thus defines a smooth system of coordinates in a neighborhood $W_y$ of $y$ in $\overline{\Omega}$. In this system of coordinates, the tensor metric $G$ writes:

$$G(x', x_d) = G_{dd}(x', x_d) \, dx_d^2 + \sum_{i,j=1}^{d-1} G_{ij}(x', x_d) \, dx_i dx_j.$$
where \( x' = (x_1, \ldots, x_{d-1}) \). In particular if \( \psi : V_y \to \mathbb{R} \) is a Lipschitz function which only depends on \( x' \), it holds a.e. on \( V_y \):

\[
|\nabla \psi(x',x_d)| = |\nabla(\psi|_{\Sigma_d})(x')|,
\]

(130)

where \( \forall a > 0, \Sigma_a := \{ x \in V_y, x_d(x) = a \} \) is endowed with the Riemannian structure induced by the Riemannian structure in \( \Omega \).

Definitions of the functions \( \Psi_1, f_{+,i} \) and \( f_{-,i} \).

**Definition 10.** Let us assume that the hypotheses \([H1]\) and \([H3]\) hold. Let us consider \( z_i \) a local minimum of \( f_i|_{\Omega} \) as introduced in hypothesis \([H2]\). Let us define on \( \Omega \) the following Lipschitz functions

\[
\Psi_i(x) := d_a(x,z_i), \quad f_{+,i} := \frac{\Psi_i + f - f(z_i)}{2} \quad \text{and} \quad f_{-,i} := \frac{\Psi_i - (f - f(z_i))}{2}.
\]

Owing to \( \Psi_i(x) = d_a(x,z_i) \geq |f(x) - f(z_i)| \) for all \( x \in \Omega \), the functions \( f_{\pm,i} \) are non negative and

\[
f = f(z_i) + f_{+,i} - f_{-,i} \quad \text{and} \quad \Psi_i = f_{+,i} + f_{-,i} \quad \text{on} \quad \Omega.
\]

Let \( \Gamma_{1,i} \subset B_{z_i} \) be an open smooth \( d - 1 \) dimensional manifold with boundary such that \( z_i \in \Gamma_{1,i} \) and \( \Gamma_{1,i} \subset B_{z_i} \). From Proposition 52, there exists a neighborhood of \( \Gamma_{1,i} \) in \( \bar{\Omega} \), denoted \( V_{\Gamma_{1,i}} \), such that \( \partial V_{\Gamma_{1,i}} \cap \partial \Omega \subset B_{z_i} \) and for all \( x \in V_{\Gamma_{1,i}} \),

\[
\Psi_i(x) = \Phi(x) - f(z_i).
\]

where \( \Phi \) is the solution to the eikonal equation in a neighborhood of the boundary (see Proposition 50). Notice that it implies that on \( V_{\Gamma_{1,i}} \), the function \( f_{-,i} \) coincides with the function \( f_{-} \) defined in Proposition 51 on \( \partial \Omega \cap V_{\Gamma_{1,i}} \). Moreover, it implies that the functions \( f_{\pm,i} \) are \( C^\infty \) on \( V_{\Gamma_{1,i}} \) and one has:

on \( V_{\Gamma_{1,i}} \cap \partial \Omega \), \( f_{+,i} = f - f(z_i), \quad f_{-,i} = 0, \quad \partial_n f_{+,i} = 0, \quad \text{and} \quad \partial_n f_{-,i} = -\partial_n f \),

where \( n \) is the unit outward normal to \( \Omega \). Therefore, as in Proposition 51, up to choosing a smaller neighborhood \( V_{\Gamma_{1,i}} \) of \( \Gamma_{1,i} \) in \( \bar{\Omega} \), the function \( f_{-} \) is positive on \( V_{\Gamma_{1,i}} \setminus \partial \Omega \) and such that

\[
|\nabla f_{-,i}| \neq 0 \quad \text{in} \quad V_{\Gamma_{1,i}}.
\]

(131)

Besides, since \( |\nabla \Psi_i| = |\nabla f| \) in \( V_{\Gamma_{1,i}} \), one has

\[
\nabla f_{+,i} \cdot \nabla f_{-,i} = 0 \quad \text{in} \quad V_{\Gamma_{1,i}},
\]

(132)

and thus

\[
|\nabla \Psi_i|^2 = |\nabla f|^2 = |\nabla f_{+,i}|^2 + |\nabla f_{-,i}|^2 \quad \text{in} \quad V_{\Gamma_{1,i}}.
\]

(133)

In the following, we will assume in addition that \( V_{\Gamma_{1,i}} \) is sufficiently small so that the system of coordinates \( (x',x_d) \) introduced in Definition 9 is well defined on \( V_{\Gamma_{1,i}} \). A consequence of (132) is that \( \frac{d}{dt} f_{+,i}(\gamma_x(t)) = 0 \), where \( \gamma_x \) satisfies (128). Thus, in the system of coordinates \( (x',x_d) \), the functions \( f_{+,i}, \Psi_i \) and \( f \) write:

\[
f_{+,i}(x',x_d) = f_{+,i}(x',0), \quad \Psi_i(x',x_d) = f_{+,i}(x',0) - x_d
\]

and

\[
f(x',x_d) = f(z_i) + f_{+,i}(x',0) + x_d.
\]
Notice that by construction
\[ \forall x \in V_{1,i}, \ |\nabla f_{+,i}(x)| = 0 \iff x'(x) = x'(z_i). \quad (133) \]
Indeed, \( f_{+,i}(x', x_d) = f_{+,i}(x', 0) \) and \( x' \mapsto f_{+,i}(x', 0) = f(x', 0) - f(z_i) \) has a single critical point at \( x'(z_i) \).

Strongly stable domain in \( B_{2z_i} \). In order to build an appropriate domain \( \Omega_i \), we will need to define \( \Gamma_{1,i} \subset B_{z_i} \) as a strongly stable domain, as defined now.

**Definition 11.** A smooth open set \( A \subset \partial \Omega \) is called strongly stable if
\[ \forall \sigma \in \partial A, \quad \langle \nabla f|_{\partial \Omega}(\sigma), n_\sigma(A) \rangle_{\partial \Omega, \partial \Omega} > 0, \]
where \( n_\sigma(A) \in T_\sigma \partial \Omega \) denotes the outward normal to \( A \) at \( \sigma \in \partial A \).

Notice that \( \nabla f|_{\partial \Omega} = \nabla_T f = \nabla f_{+,i} \) (this is due to the fact that on \( B_{z_i} \), one has \( f - f(z_i) = \Psi_i \) and thus \( \nabla_T f = \nabla_T \Psi_i \)). Thus, the strong stability condition appearing in Definition 11 is equivalent to
\[ \forall \sigma \in \partial A, \quad \partial n_\sigma(A) f_{+,i}(\sigma) > 0. \]
The name "stable" is justified by the following: if \( A \subset \partial \Omega \) is strongly stable, then for any curve satisfying for all \( t > 0 \), \( \gamma'(t) = -\nabla f|_{\partial \Omega}(\gamma(t)) \) with \( \gamma(0) \in \overline{A} \), one has for all \( t \geq 0 \), \( \gamma(t) \in \overline{A} \).

The following proposition will be needed to get the existence of an arbitrary large and strongly stable domain in \( B_{z_i} \).

**Proposition 57.** Let us assume that the hypotheses [H1] and [H2] hold. For all compact sets \( K \subset B_{z_i} \), there exists a \( C^\infty \) open domain \( A \) which is strongly stable in the sense of Definition 11, simply connected and such that \( K \subset A \) and \( \overline{A} \subset B_{z_i} \).

**Proof.** For the ease of notation, we drop the subscript \( i \) in the proof. One will first construct the set \( A \). Then it will be proven that \( A \) has the stated properties. For \( a > 0 \), let us define
\[ L_a := f|^{-1}_{\partial \Omega}([f(z), f(z) + a]) \cap B_z. \]
For a fixed \( a > 0 \) small enough \( L_a \) is a \( C^\infty \) simply connected open set (which contains \( z \)) with boundary the level set \( f|^{-1}_{\partial \Omega}([f(z) + a]) \). The domain \( L_a \) is \( C^\infty \) since \( f \) is \( C^\infty \).

Let us define for \( x \in B_z \) the curves \( \gamma_x \) by
\[ \gamma_x'(t) = \nabla f|_{\partial \Omega}(\gamma_x(t)), \quad \gamma_x(0) = x. \]
For any \( x \in \partial L_a \), for all \( t > 0 \), \( \gamma_x(-t) \in L_a \) since \( t \geq 0 \mapsto f|_{\partial \Omega}(\gamma_x(-t)) \) is decreasing \( \left( \frac{df}{dt}|_{\partial \Omega}(\gamma_x(-t)) = -\nabla f|_{\partial \Omega}(\gamma_x(-t)) \right|^2 \) and \( f|_{\partial \Omega}(\gamma_x(0)) = a \). Let us now define for \( T > 0 \)
\[ A_T := \{ \gamma_x(t), \ x \in \partial L_a, t \in [0,T) \} \cup L_a \subset B_z. \]
One clearly has \( A_T \subset A_{T'} \) if \( T < T' \). One claims that \( A_T \) is a \( C^\infty \) simply connected open set which satisfies
\[ \forall \sigma \in \partial A_T, \quad \partial n_\sigma(A_T) f|_{\partial \Omega}(\sigma) > 0. \]
Let us first prove that $A_T$ is $C^\infty$. One has $\partial A_T = \{ \gamma_x(T), \ x \in \partial L_a \}$. The boundary of $A_T$ is thus a $C^\infty$ homotopy of $\partial L_a$ where the homotopy function is 

$$H(t, x) = \gamma_x(t).$$

Additionally since this homotopy is with values in $B_z$ and since $L_a$ is simply connected (because $L_a$ can be asymptotically retracted on $z$ in the sense that for all $x \in L_a$, $\lim_{t \to -\infty} H(t, x) = z$), $A_T$ is simply connected. Let us prove that $A_T$ is open. Let $x_0 \in A_T \setminus \overline{\Lambda_a}$. There exists a time $t_0 \in (0, T)$ such that $\gamma_{x_0}(t_0) \in L_a$. Let us define $\varepsilon_0 = d_{\partial \Omega}(\gamma_{x_0}(t_0), \partial L_a)/2 > 0$. Since the mapping $y \mapsto \gamma_y(0)$ is $C^\infty$, there exists $\varepsilon_1 > 0$ such that if $d_{\partial \Omega}(x, y) \leq \varepsilon_1$ then $d_{\partial \Omega}(\gamma(x), \gamma(y)) \leq \varepsilon_0/2$ and thus $\gamma_y(t_0) \in L_a$. Moreover, since $B_z \setminus \overline{\Lambda_a}$ is open, it can be assumed, taking maybe $\varepsilon_1 > 0$ smaller, that $B_{\partial \Omega}(x_0, \varepsilon_1) \subset B_z \setminus \overline{\Lambda_a}$. Then, by continuity $\forall y \in B_{\partial \Omega}(x_0, \varepsilon_1)$, there exists $t_0(y) \in (0, t_0) \subset (0, T)$ such that $\gamma_y(t_0(y)) \in \partial L_a$, which implies that $y \in A_T \setminus \Lambda_a$. Thus $A_T \setminus \Lambda_a$ is open. In addition, since $L_a$ is open and since $\overline{\Lambda_a} \subset A_T$, one has that $\text{int}(A_T) = \text{int}(A_T \setminus \overline{\Lambda_a}) \cup \overline{\Lambda_a} = (A_T \setminus \overline{\Lambda_a}) \cup \overline{\Lambda_a} = A_T$. Therefore the set $A_T$ is open.

Let us now prove that $A_T$ is strongly stable (see Definition 11). By construction, $A_T$ is stable for the dynamics $\gamma' = -\nabla f|_{\partial \Omega}(\gamma)$ and thus one has

$$\forall \sigma \in \partial A_T, \quad \partial_{n_\sigma}(A_T) f|_{\partial \Omega}(\sigma) \geq 0.$$ 

Let us defined now the function

$$\Upsilon : x \in B_z \setminus \overline{\Lambda_a} \mapsto (x', t) \in \partial L_a \times \mathbb{R}_+ \ s.t \ \gamma_{x'}(t) = x.$$

Notice that $\Upsilon$ is a $C^\infty$ diffeomorphism from $B_z$ onto its range, and let us denote $F := \Upsilon^{-1}$ its inverse function ($F(x', t) = \gamma_{x'}(t)$). Assume that there exists $x \in A_T$ such that $\partial_{n_\sigma}(A_T) f|_{\partial \Omega}(x) = 0$ and let $(x', T) = \Upsilon(x)$. This implies that $\nabla f|_{\partial \Omega}(x) \in T_x \partial A_T$ and thus $\partial_t F(x', T) \in T_x \partial A_T$. Furthermore $\text{ran}(d_{x'} F(\cdot, T)) = T_x \partial A_T$ and thus $d_{(x', T)} F$ is not invertible which contradicts the fact that $F$ is a diffeomorphism. It remains to prove that for any compact set $K \subset B_z$, there exists $T > 0$ such that $K \subset A_T$. One has

$$B_z = \bigcup_{T > 0} A_T.$$

Indeed, if $x \in \overline{\Lambda_a}$, $x \in A_T$ for all $T > 0$ and if $x \in B_z \setminus \overline{\Lambda_a}$, $\lim_{t \to -\infty} \gamma_x(t) = z$ and thus there exists $s > 0$ such that $\gamma_x(-s) \in \partial L_a$ which implies that $x \in A_s$. Let $K \subset B_z$ be a compact set. Then $K \subset \bigcup_{T > 0} A_T$ and thus by compactness there exists a sequence $(T_j)_{j=1,...,N} \subset \mathbb{R}$, with $0 < T_1 < \ldots < T_m$ such that $K \subset \bigcup_{j=1}^m A_{T_j} = A_{T_m}$. This concludes the proof.

Construction of the domain $\hat{\Omega}_i$. In this section, we introduce the domain $\hat{\Omega}_i$ (associated with $z_i$) on which the auxiliary Witten Laplacian with mixed tangential-normal Dirichlet boundary conditions is constructed.

**Proposition 58.** Let us assume that the hypotheses [H1], [H2] and [H3] hold. Let us fix a neighborhood $\Omega_0$ of $x_0$ (the global minimum of $f$ in $\hat{\Omega}$) such that

$$\partial_n f < 0 \text{ on } \Gamma_0 := \partial \Omega_0$$

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where $n$ denotes the outward normal to $\Omega \setminus \Omega_0$ on $\Gamma_0$. Let us consider a critical point $z_i$ of $f|_{\partial \Omega}$. Then there exists a smooth open subset $\Gamma_{1,i}$ of $B_{z_i}$ containing $z_i$ and arbitrarily large in $B_{z_i}$, a neighborhood $V_{\Gamma_{1,i}}$ of $\Gamma_{1,i}$ in $\overline{\Omega}$ such that $V_{\Gamma_{1,i}} \cap \partial \Omega \subset B_{z_i}$ and a Lipschitz subset $\hat{\Omega}_i$ of $\Omega \setminus \Omega_0$ which are such that the following properties are satisfied:

1. Following Proposition 52,

\[ \forall x \in V_{\Gamma_{1,i}}, d_a(x, z_i) = \Phi(x) - f(z_i) \]

where $\Phi$ is the solution of the eikonal equation \((102)\);

2. The system of coordinates $(x', x_d)$ is defined on $V_{\Gamma_{1,i}}$, see Definition 7;

3. $\partial \hat{\Omega}_i$ is composed of two connected components: $\Gamma_0$ and $\Gamma_{1,i} \cup \Gamma_{2,i}$;

4. $\Gamma_{1,i}$ and $\Gamma_{2,i}$ meet at an angle smaller than $\pi$, see $(119)$ for a precise definition;

5. \begin{align*}
\forall x \in \partial \hat{\Omega}_i \setminus \Gamma_0, \quad &\partial_n f(x) > 0 \quad (134) \\
\text{where } n \text{ is the outward normal derivative of } \hat{\Omega}_i;
\end{align*}

6. and

\begin{align*}
\forall x \in \Gamma_{2,i} \cap V_{\Gamma_{1,i}}, \quad &\partial_n f_+(x) > 0. \\
&\text{(135)}
\end{align*}

7. Moreover, for all $\delta > 0$, $\hat{\Omega}_i$ (and $\Omega_0$) can be chosen such that

\[ \sup\{d_e(x, y), x \in \Gamma_{2,i}, y \in B_{z_i}^-\} \leq \delta \quad (136) \]

and

\[ \sup\{d_e(x_0, x), x \in \Gamma_0\} \leq \delta \quad (137) \]

where, we recall, $d_e$ denotes the geodesic distance for the Euclidean metric on $\overline{\Omega}$.

We refer to Figure 9 for a schematic representation of $\hat{\Omega}_i$.

**Proof.** The domain $\hat{\Omega}_i \subset \Omega$ is built as follows. First, let us fix a neighborhood $\Omega_0$ of $x_0$ such that $(137)$ is satisfied and

\[ \partial_n f < 0 \text{ on } \Gamma_0 = \partial \Omega_0 \]

where $n$ denotes the outward normal to $\Omega \setminus \Omega_0$ on $\Gamma_0$ (this can be done for example by considering $\Omega_0 = \{x, f(x) < f(x_0) + \eta\}$ for some positive $\eta$). Second, let us consider a smooth subset $\Gamma_{1,i}$ of $B_{z_i}$ which may be as large as we want in $B_{z_i}$, and which is strongly stable (see Proposition 57 for the existence of such a set):

\[ \langle \nabla f|_{\partial \Omega}, n(\Gamma_{1,i})\rangle_{\partial \Omega} > 0 \text{ on } \partial \Gamma_{1,i}, \quad (138) \]

where $n(\Gamma_{1,i})$ denotes the outward normal derivative of $\Gamma_{1,i}$.

Once $\Gamma_{1,i}$ is fixed, the existence of a neighborhood $V_{\Gamma_{1,i}}$ of $\Gamma_{1,i}$ in $\overline{\Omega}$ such that $V_{\Gamma_{1,i}} \cap \partial \Omega \subset B_{z_i}$ and such that items 1 and 2 are fulfilled is a direct consequence of Proposition 52.
Let us now consider the system of coordinates \((x', x_d)\) introduced in Definition \(9\).

Let \(V_{\partial \Gamma_{1,i}} \subset \partial \Omega\) denote a neighborhood of \(\partial \Gamma_{1,i}\) in \(\partial \Omega\) and

\[ V^+_{\partial \Gamma_{1,i}} = V_{\partial \Gamma_{1,i}} \cap \Gamma_{1,i}. \]

The domain \(\hat{\Omega}_i\) is then defined as follows:

\[ \hat{\Omega}_i = \Omega \setminus (\Gamma_0 \cup \{ x' \in V^+_{\partial \Gamma_{1,i}} \text{ such that } x_d(x) \in [\varphi(x'), 0]\}) \]

where \(\varphi : V^+_{\partial \Gamma_{1,i}} \to \mathbb{R}_+\) is a smooth function such that

\[ \exists \varepsilon > 0, \forall x' \in \partial \Gamma_{1,i}, \varphi(x') \geq \varepsilon \]

and \(\hat{\Omega}_i\) is a connected Lipschitz subset of \(\Omega\). In the following, we denote by \(\Gamma_{2,i} = \partial \hat{\Omega}_i \setminus (\Gamma_{1,i} \cup \Gamma_0)\) (so that item 3 is true), see Figure 10 for a schematic representation.

For each point \(z \in \partial \Gamma_{1,i}\), there is a small neighborhood \(V\) of \(z\) such that

\[ V \cap \Gamma_{2,i} \subset \{ x' = (x', x_d), x' \in \partial \Gamma_{1,i} \text{ and } x_d(x) \in (-\eta, 0)\}, \]

for some \(\eta \in (0, \varepsilon)\). By choosing \(\Gamma_{1,i}\) sufficiently large in \(B_{z_1}\) and \(\varphi\) such that \(\max \varphi\) is sufficiently small, \(\text{(136)}\) is satisfied. This concludes the verification of item 7.

For each point \(y \in \partial \Gamma_{1,i}\), it is possible to construct locally a normal system of coordinate \(x' = x_T = (x_{T,1}, x_{T,2}, \ldots, x_{T,d-1})\) in a neighborhood \(V_y\) of \(y\) in \(\partial \Omega\), such that \(\Gamma_{1,i} \cap V_y = \{ x \in V_y, x_T,1(x) \leq 0 \}\), \(V^+_{\partial \Gamma_{1,i}} \cap V_y = \{ x \in V_y, x_T,1(x) \geq 0 \}\) and \(\partial \Gamma_{1,i} \cap V_y = \{ x \in V_y, x_T,1(x) = 0 \}\). As explained after Definition \(9\) by extending this system of coordinate inside \(\Omega\) as constant along the curve associated with the vector field \(\nabla f|_{V_y} \to (x'(x), x_d(x))\) then defines a local system of coordinates in a neighborhood \(W_y\) of \(y\) in \(\Omega\). For all \(x \in \partial \Gamma_{1,i}\), the vector \(n_z(\Gamma_1) = \frac{\nabla x_T,1(x)}{\nabla x_T,1(x)}\) is the outward normal vector to \(\Gamma_{1,i}\) on \(\partial \Gamma_{1,i}\). By a compactness argument, one gets that \(\partial \Gamma_{1,i} \subset \sup_{k=1} V_{y_k}\) for a finite number of points \(y_k \in \partial \Gamma_{1,i}\). See Figure 11 for a schematic representation of the function \(\varphi\) in this system of coordinates.

Let us now look at the boundary of \(\hat{\Omega}_i\) in a neighborhood of \(\partial \Gamma_{1,i}\) (see Figure 10). For \(\sigma \in \partial \hat{\Omega}_i\), let us denote by \(n_z(\hat{\Omega}_i)\) the unit outward normal to \(\hat{\Omega}_i\). Let us show that

\[ \lim_{\sigma \to z} n_z(\hat{\Omega}_i) = n_2(\Gamma_{1,i}), \]

where the limit is taken for \(\sigma \in \Gamma_{2,i}\). Let us prove \(\text{(140)}\). For any point \(z \in \partial \Gamma_{1,i}\), there is a small neighborhood \(V\) of \(z\) in \(\Omega\) such that the system of coordinates \((x_T, x_d)\) introduced above is well defined. In this system of coordinates,

\[ \partial \hat{\Omega}_i \cap V \cap \Gamma_{2,i} \subset \{ x \in V, x_T,1(x) = 0 \text{ and } x_d(x) \in [-\varphi(x'(x_T(x))), 0]\}. \]

Moreover, the outward normal to \(\hat{\Omega}_i\) on this subset is \(n(\hat{\Omega}_i) = \frac{\nabla x_T,1}{\nabla x_T,1}\) and thus, by construction, for all \(z \in \partial \Gamma_{1,i}\), \(\text{(140)}\) holds.

As a consequence of \(\text{(140)}\), the two submanifolds \(\Gamma_{1,i}\) and \(\Gamma_{2,i}\) meet at an angle smaller than \(\pi\) (see \(\text{(119)}\) and Figure 12). This shows that item 4 is satisfied. Moreover, using \(\text{(136)}\) and the fact that \(\nabla f|_{\partial \Omega} = \nabla f_+\) on \(\partial \Omega\), one has: for all \(z \in \partial \Gamma_{1,i}\),

\[ \lim_{\sigma \to z} \nabla f_+(\sigma) \cdot n_\sigma(\hat{\Omega}_i) = \lim_{\sigma \to z} \nabla f(\sigma) \cdot n_\sigma(\hat{\Omega}_i) = \nabla f(z) \cdot n_2(\Gamma_{1,i}) > 0 \]
where the limits are taken for $\sigma \in \Gamma_{2,i}$. Thus, up to choosing $\varphi$ with max $\varphi$ sufficiently small, it is possible to build $\tilde{\Omega}_i$ such that (see Figure 10)

$$\forall x \in \Gamma_{2,i} \text{ such that } x'(x) \in \partial \Gamma_{1,i}, \partial_n f_+(x) > 0$$

where $n$ here denotes the outward normal to $\tilde{\Omega}_i$. This implies item 6. It also implies (since $\forall x \in \Gamma_{2,i} \text{ such that } x'(x) \in \partial \Gamma_{1,i}, \partial_n f_+(x) - \partial_n f_-(x) = \partial_n f_{+i}(x)$):

$$\forall x \in \Gamma_{2,i} \text{ such that } x'(x) \in \partial \Gamma_{1,i}, \partial_n f(x) > 0. \quad (141)$$

Finally, by using (141) and since $\partial_n f > 0$ on $\partial \Omega$, up to choosing $\varphi$ with max $\varphi$ sufficiently small, it is possible to build $\tilde{\Omega}_i$ such that (see Figure 10)

$$\forall x \in \partial \tilde{\Omega}_i, \partial_n f(x) > 0$$

where $n$ again denotes the outward normal to $\tilde{\Omega}_i$. This is item 5, and this concludes the proof of Proposition 58.

Definition 12. Let us assume that the hypotheses [H1], [H2] and [H3] hold. Let us consider a critical point $z_i$ of $f|_{\partial \Omega}$. In the following, we denote by $S_{M,i} := \{\hat{\Omega}_i, \Gamma_0, \Gamma_{1,i}, \Gamma_{2,i}, V_{\Gamma_{1,i}}\}$ an ensemble of sets satisfying the requirements of Proposition 58.

In the following, in order to reduce the amount of notation, the index $i$ will sometimes be omitted. Thus, we will denote

$$z = z_i, \Gamma_1 = \Gamma_{1,i}, \Gamma_2 = \Gamma_{2,i}, \hat{\Omega} = \hat{\Omega}_i, V_{\Gamma_1} = V_{\Gamma_{1,i}}, \Psi = \Psi_i, f_+ = f_{+i} \text{ and } f_- = f_{-i}.$$ 

We shall warn the reader whenever the index $i$ is dropped.

4.1.3 On the spectrum of the Witten Laplacian $\Delta^M_{f,h}(\hat{\Omega}_i)$

Throughout this section, one assumes [H1], [H2] and [H3]. In this section, we introduce a Witten Laplacian with mixed tangential and normal Dirichlet boundary conditions, associated with the critical point $z_i$. Let $S_{M,i} := \{\hat{\Omega}_i, \Gamma_0, \Gamma_{1,i}, \Gamma_{2,i}, V_{\Gamma_{1,i}}\}$ be an ensemble of sets associated with $z_i$, see Definition 12.

Let us now consider the Witten Laplacian $\Delta^M_{f,h}$ on $\hat{\Omega}_i$ with zero tangential boundary conditions on

$$\Gamma_T = \Gamma_0 \cup \Gamma_{1,i}$$

and zero normal boundary conditions on

$$\Gamma_N = \Gamma_{2,i}$$

as defined at the end of Section 4.1.1 (see (122)–(124)). The main result of this section concerns the spectrum of the operator $\Delta^M_{f,h}(\hat{\Omega}_i)$.

Proposition 59. Let us assume that the hypotheses [H1], [H2] and [H3] hold. Let $\Delta^M_{f,h}(\hat{\Omega}_i)$ be the unbounded nonnegative selfadjoint operator on $L^2(\hat{\Omega}_i)$ defined by (122) and with domain (124) with $\Gamma_T = \Gamma_1 \cup \Gamma_0$ and $\Gamma_N = \Gamma_2$. One has:

(i) The operator $\Delta^M_{f,h}(\hat{\Omega}_i)$ has compact resolvent.
\( \Gamma_{1,i} \) in \( \partial \Omega \)

Figure 9: The ensemble of sets \( S_{M,i} = \{ \hat{\Omega}_i, \Gamma_0, \Gamma_{1,i}, \Gamma_{2,i}, V_{\Gamma_{1,i}} \} \) associated with a critical point \( z_i \) of \( f|_{\partial \Omega} \).
(ii) For any eigenvalue $\lambda_p$ of $\Delta^{M,(p)}_{f,h}(\hat{\Omega}_i)$ and associated eigenform $u_p \in D\left(\Delta^{M,(p)}_{f,h}(\hat{\Omega}_i)\right)$, one has $d_{f,h}u_p \in D\left(\Delta^{M,(p+1)}_{f,h}\right)$ and $d^*_{f,h}u_p \in D\left(\Delta^{M,(p-1)}_{f,h}\right)$, with

$$d_{f,h}\Delta^{M,(p)}_{f,h}u_p = \Delta^{M,(p+1)}_{f,h}d_{f,h}u_p = \lambda_p d_{f,h}u_p$$

and

$$d^*_{f,h}\Delta^{M,(p)}_{f,h}(\hat{\Omega}_i)u_p = \Delta^{M,(p-1)}_{f,h}(\hat{\Omega}_i)d^*_{f,h}u_p = \lambda_p d^*_{f,h}u_p.$$  

If in addition $\lambda_p \neq 0$, either $d_{f,h}u_p$ or $d^*_{f,h}u_p$ is nonzero.
(iii) There exist $c > 0$ and $h_0 > 0$ such that for any $p \in \{0, \ldots, n\}$ and $h \in (0, h_0)$,
\[
\dim \text{Ran } \pi_{[0, ch^2]} \left( \Delta_{f,h}^{M,(p)}(\hat{\Omega}) \right) = \delta_{1,p} \quad \text{and} \quad \text{Sp} \left( \Delta_{f,h}^{M,(1)}(\hat{\Omega}) \right) \cap [0, ch^2) = \{0\}.
\]

**Proof.** Since the critical point $z_i$ is fixed, for the ease of notation, we drop the subscript $i$ in the proof.

The point $(i)$ follows from the compactness of the embedding $H^\frac{1}{2}(\hat{\Omega}) \hookrightarrow L^2(\hat{\Omega})$ (since additionally $D \left( Q_{f,h}^{M}(\hat{\Omega}) \right) \hookrightarrow H^\frac{1}{2}(\hat{\Omega})$ is continuous according to Proposition 55). The point $(ii)$ is then a straightforward consequence of the characterization of the domain of $\Delta_{f,h}^{M,(p)}(\hat{\Omega})$ together with [123]. The statement in the case $\lambda_p \neq 0$ follows from $0 \neq \lambda_p \langle u_p, u_p \rangle_{L^2(\hat{\Omega})} = \langle \Delta_{f,h}^{M,(p)}(\hat{\Omega})u_p, u_p \rangle_{L^2(\hat{\Omega})} = \langle df, hu_p, df, hu_p \rangle_{L^2(\hat{\Omega})} + \langle d^f, hu_p, d^f, hu_p \rangle_{L^2(\hat{\Omega})}$.

Let us now give the proof of the last point $(iii)$, which is a consequence of $(ii)$ together with ideas from [34, 47], since $z_i$ is the only generalized critical point of $f$ in $\hat{\Omega}$ for $\Delta_{f,h}^{M,(1)}(\hat{\Omega})$. According to the last part of $(ii)$, it suffices to prove that for some $c > 0$, one has for any $p \in \{0, \ldots, n\}$ and $h$ small enough,
\[
\dim \text{Ran } \pi_{[0, ch^2]} \left( \Delta_{f,h}^{M,(p)}(\hat{\Omega}) \right) = \delta_{1,p}.
\]

Pick up $u \in D \left( Q_{f,h}^{M,(p)}(\hat{\Omega}) \right)$. From the Green formula (125) and from the fact that $\mathcal{L}_f + \mathcal{L}_{\nabla f}$ is a $0$th order differential operator, one has that there exists $C_0 > 0$ such that for all $u \in D \left( Q_{f,h}^{M,(p)}(\hat{\Omega}) \right)$ and all smooth cut-off function $\chi$ supported in $\hat{\Omega}$ (whose support avoids $\partial \Omega$):
\[
\| df, h\chi u \|_{L^2(\Omega)}^2 + \| d^\ast, h\chi u \|_{L^2(\Omega)}^2 \geq \left( \inf_{\supp \chi} \| \nabla f \| - hC_0 \right) \| \chi u \|^2.
\]
Thus, since $f$ has no critical point in $\hat{\Omega}$, there exists some $C > 0$ independent of $u \in D \left( Q_{f,h}^{M,(p)}(\hat{\Omega}) \right)$ such that for any smooth cut-off function $\chi$ supported in $\hat{\Omega}$ (whose support avoids $\partial \Omega$) and $h$ small enough,
\[
Q_{f,h}^{M,(p)}(\hat{\Omega})(\chi u) \geq C\| \chi u \|^2.
\]

Note in addition that owing to $\partial_n f > 0$ on $\Gamma_2$ and $\partial_n f < 0$ on $\Gamma_0$, the boundary terms in the Green formula (125) are non negative, for any smooth cut-off function $\chi$ supported in a neighborhood of any point in $\Gamma_2 \cup \Gamma_0$ (whose support avoids some neighborhood of $\Gamma_1$). Thus, the same considerations show that for $h$ small enough, for such functions $\chi$, taking maybe $C$ smaller, $\forall u \in D \left( Q_{f,h}^{M,(p)}(\hat{\Omega}) \right)$
\[
Q_{f,h}^{M,(p)}(\hat{\Omega})(\chi u) \geq C\| \chi u \|^2.
\]

According to the analysis done in [34, Section 3.4], the same estimate also holds for $\chi$ supported in a sufficiently small neighborhood of some point $x \neq z$, $x \in \Gamma_1$ (whose support avoids a neighborhood of $\{z\} \cup \partial \Gamma_1$). This is related to the fact that $\Gamma_1$ does not contain any generalized critical point of $f$ in the tangential sense except $z$. Let us now show that such an estimate is also valid near $\partial \Gamma_1$. In order to prove it, one recalls that
\[
f = f(z) + f_+ - f_- \ a.e \ on \ \Omega \ \text{and} \ \| \nabla f \|^2 = \| \nabla f_+ \|^2 + \| \nabla f_- \|^2 \ a.e \ near \ \partial \Gamma_1,
\]

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where \( f_\pm \) are smooth and satisfy the following relations on \( B_2 \):

\[
f_+ = f - f(z), \quad f_- = 0, \quad \partial_n f_+ = 0 \text{ and } \partial_n f_- = -\partial_n f.
\]

Hence, for any \( \chi \) supported in a sufficiently small neighborhood of \( \partial \Gamma_1 \), one deduces from the relation \( Q^{M,(p)}_{f,h}(\chi u) \geq 0 \), the Green formula (125), and the fact that \( \mathcal{L}_-\nabla f_+ + \mathcal{L}_+\nabla f_- \) is a \( 0^\text{th} \) order differential operator, that there exists \( C_1 > 0 \) independent of \( u \in D\left( Q^{M,(p)}_{f,h}(\Omega) \right) \) such that:

\[
h \left( \int_{\Gamma_1} - \int_{\Gamma_2} \langle \chi u, \chi u \rangle_{T_\sigma^2 \Omega} \partial_n f_- \, d\sigma \right) \geq -h^2 \| d\chi u \|^2_{L^2(\Omega)} - h^2 \| d^* \chi u \|^2_{L^2(\Omega)}
\]

\[
- \| \nabla f_- \| \chi u \|^2_{L^2(\Omega)} - C_1 h\| \chi u \|^2_{L^2(\Omega)}.
\]

Using again the Green formula (125), the relation \( f - f(z) = f_+ - f_- \) with \( \partial_n f_+ = 0 \) on \( \Gamma_1 \), and the fact that \( \mathcal{L}_-\nabla f_+ + \mathcal{L}_+\nabla f_- \) is a \( 0^\text{th} \) order differential operator then leads to the existence of \( C_2 > 0 \) independent of \( u \in D\left( Q^{M,(p)}_{f,h}(\Omega) \right) \) s.t.

\[
Q^{M,(p)}_{f,h}(\chi u) \geq \| \nabla f_+ \chi u \|^2_{L^2(\Omega)} - C_2 h\| \chi u \|^2_{L^2(\Omega)} + h \int_{\Gamma_2} \langle \chi u | \chi u \rangle_{T_\sigma^2 \Omega} \partial_n f_+ \, d\sigma.
\]

Since \( f_+ \) has no critical point around \( \partial \Gamma_1 \) (see (133)), one has then for \( h \) small enough, taking maybe \( C \) smaller:

\[
Q^{M,(p)}_{f,h}(\chi u) \geq C\| \chi u \|^2_{L^2(\Omega)} + h \int_{\Gamma_2} \langle \chi u | \chi u \rangle_{T_\sigma^2 \Omega} \partial_n f_+ \, d\sigma. \tag{142}
\]

Let us recall that due to our construction of \( \Gamma_2 \) near \( \partial \Gamma_1 \), one has (see (135) in Proposition 58):

\[
\partial_n f_+(\sigma) > 0 \text{ for } \sigma \in \Gamma_2 \text{ sufficiently close to } \partial \Gamma_1.
\]

This implies that for \( \chi \) supported near \( \partial \Gamma_1 \) with sufficiently small support and \( h \) small enough:

\[
Q^{M,(p)}_{f,h}(\chi u) \geq C\| \chi u \|^2_{L^2(\Omega)}.
\]

Lastly, since \( z \) is a generalized critical point with index 1 in the tangential sense, it follows from [24, Proposition 4.3.2] that for \( \chi \) supported in a neighborhood of \( z \) and \( h \) small enough, the spectrum of the Friedrichs extension associated with the quadratic form

\[
\left\{ v \in \Lambda^p H^1(\supp \chi); \, \langle v | \Gamma_1 = v | \partial \supp \chi \partial \Gamma_1 = 0 \right\} \ni v \mapsto \| d_{f,h} v \|^2_{L^2} + \| d^*_{f,h} v \|^2_{L^2}
\]

does not meet \([0,h^{\frac{3}{2}}]\) if \( p \neq 1 \), and consists of exactly one eigenvalue in \([0,h^{\frac{3}{2}}]\) which is actually exponentially small – i.e. of the form \( O(e^{-\frac{C}{h^2}}) \) – if \( p = 1 \). Denote by \( \psi_1 \) some normalized eigenvalue associated with this exponentially small eigenvalue in this last case.

Using the IMS localization formula (see for example [12] for a proof)

\[
\forall (\chi_k)_{k \in \{1,...,K\}} \in \left( C^\infty(\overline{\Omega}) \right)^K \text{ s.t. } \sum_{k=1}^{K} \chi^2_k = 1
\]

\[
Q^{M,(p)}_{f,h}(\chi_k u) = \sum_{k=1}^{K} \left( Q^{M,(p)}_{f,h}(\chi_k u) - h^2 \| \nabla \chi_k u \|^2_{L^2(\Omega)} \right) \tag{143}
\]

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and the previous analysis then shows that choosing \( \chi_1 \in C^\infty(\Omega) \) supported in a neighborhood of \( z \) with \( \chi_1 = 1 \) near \( z \), one gets for some \( C, C' > 0 \) independent of \( u \in D\left(\mathcal{Q}_{f,h}^{M,(p)}(\Omega)\right) \) and \( h \) small enough:

\[
\mathcal{Q}_{f,h}^{M,(p)}(\Omega)(u) \geq \mathcal{Q}_{f,h}^{M,(p)}(\Omega)(\chi_1 u) + C\|(1 - \chi_1^2)^{1/2} u\|_{L^2(\Omega)}^2 - C'h^2 \|u\|_{L^2(\Omega)}^2.
\]

If \( p \neq 1 \), one deduces immediately from (144) that for \( h \) small enough,

\[
\mathcal{Q}_{f,h}^{M,(p)}(\Omega)(u) \geq \frac{h}{2} \|\chi_1 u\|_{L^2(\Omega)}^2 + C\|(1 - \chi_1^2)^{1/2} u\|_{L^2(\Omega)}^2 - C'h^2 \|u\|_{L^2(\Omega)}^2,
\]

and then that for some \( c > 0 \) and \( h \) small enough:

\[
\mathcal{Q}_{f,h}^{M,(p)}(\Omega)(u) \geq c h^{\frac{3}{2}} \|u\|_{L^2(\Omega)}^2.
\]

If \( p = 1 \), one obtains from (144) the same conclusion for any \( u \) such that \( \int_{\Omega} u \chi_1 \psi_1 = 0 \) and therefore \( \Delta^{M,(p)}(\Omega) \) has no eigenvalue in \([0, c h^2]\) if \( p \neq 1 \) and at most one if \( p = 1 \). To end up the proof, it is sufficient to remark that \( \psi_1 \), the extension of \( \psi_1 \) to \( \tilde{\Omega} \) by \( 0 \) outside \( \text{supp} \chi_1 \), belongs to \( D\left(\mathcal{Q}_{f,h}^{M,(1)}(\Omega)\right) \) and satisfies for \( h \) small:

\[
\mathcal{Q}_{f,h}^{M,(1)}(\Omega)(\tilde{\psi}_1) = \|df,h \tilde{\psi}_1\|_{L^2(\Omega)}^2 + \|df,h \tilde{\psi}_1\|_{L^2(\Omega)}^2 = O(e^{-\frac{C_3}{h^3}}) < c h^{\frac{3}{2}}.
\]

Following Proposition [59], let us introduce an \( L^2 \)-normalized eigenform \( u_{h,i}^{(1)} \) of \( \Delta^{M,(1)}(\Omega_i) \) associated with the eigenvalue 0:

\[
\Delta^{M,(1)}(\Omega_i) u_{h,i}^{(1)} = 0 \text{ in } \Omega_i \text{ and } \left\|u_{h,i}^{(1)}\right\|_{L^2(\Omega_i)} = 1.
\]

The quasi-mode \( \tilde{\phi}_i \) will be built using a suitable truncation of \( u_{h,i}^{(1)} \). Notice that thanks to item (iii) in Proposition [59], \( u_{h,i}^{(1)} \) is unique up to a multiplication by \( \pm 1 \): this multiplicative constant will be fixed in Proposition [64] below.

### 4.2 Definition of the quasi-modes

Throughout this section, one assumes [H1], [H2] and [H3]. In this section, we construct the function \( \tilde{u} \) and a family of 1-forms \((\tilde{\phi}_i)_{i=1,...,n}\) which will satisfy the estimates stated in Section 2.2.2. For each \( i \in \{1, \ldots, n\} \), the 1-form \( \tilde{\phi}_i \) will be constructed by a suitable truncation of an eigenfunction \( u_{h,i} \) associated with the eigenvalue 0 of the mixed Witten Laplacian attached with \( z_i \in \{z_1, \ldots, z_n\} \), as defined in Section 4.1.3.

We recall that \( \Sigma_i \) is an open set included in \( \partial \Omega \) containing \( z_i \) which is such that \( \Sigma_i \subset B_{z_i} \) (see Definition [4]).

#### 4.2.1 Definition of the quasi-mode \( \tilde{u} \)

**Definition 13.** Let us consider the global minimum \( x_0 \) introduced in the hypothesis [H2]. Let \( \chi \in C^\infty_c(\Omega) \) such that \( \{x \in \Omega | \chi(x) = 1\} \) is a neighborhood of \( x_0 \) and such that \( 0 \leq \chi \leq 1 \) (in particular \( \chi(x_0) = 1 \)). The quasi-mode \( \tilde{u} \) is defined by

\[
\tilde{u} := \frac{\chi}{\sqrt{\int_{\Omega} \chi^2 e^{-2\chi}}},
\]

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The function \( \tilde{u} \) belongs to \( C_c^\infty(\Omega) \) and therefore \( \tilde{u} \in H^1_0 \left( e^{-\frac{2}{\pi} f(x) dx} \right) \). The function \( \chi \) will be chosen such that \( \text{supp}(|\nabla \chi|) \) is as close as needed to \( \partial \Omega \), as will be made precise in Section 4.5.

Let us first prove that \( \tilde{u} \) satisfies item 2(b) in Proposition 17.

**Lemma 60.** Let us assume that the hypotheses \([H1]\) and \([H2]\) hold. There exist \( h_0 > 0 \) and \( C > 0 \), for any \( \delta > 0 \) there exists \( \chi \in C_c^\infty(\Omega) \) such that the set \( \{ x \in \Omega | \chi(x) = 1 \} \) is a neighborhood of \( x_0 \), \( 0 \leq \chi \leq 1 \) and for all \( h \in (0, h_0) \)

\[
\int_{\Omega} |\nabla \tilde{u}(x)|^2 e^{-\frac{2f(x)}{\pi} dx} \leq C h^{-1/2} e^{-\frac{2(f(x_0) - f(x_0) - \delta)}{h}},
\]

where \( \tilde{u} \) is defined in Definition 13.

**Proof.** There exists a constant \( C \) such that

\[
\int_{\Omega} |\nabla \tilde{u}(x)|^2 e^{-\frac{2f(x)}{\pi} dx} \leq C \int_{\text{supp} \nabla \chi} e^{-\frac{2f(x)}{\pi} dx}.
\]

Since \( \text{supp} \nabla \chi \) can be chosen arbitrarily close to \( \partial \Omega \) and since \( z_1 \) is the minimum of \( V \) on \( \partial \Omega \), by continuity of \( f \), for any \( \delta > 0 \) there exists \( \chi \in C_c^\infty(\Omega) \) such that \( \{ x \in \Omega | \chi(x) = 1 \} \) is a neighborhood of \( x_0 \), \( 0 \leq \chi \leq 1 \) and

\[
\int_{\text{supp} \nabla \chi} e^{-\frac{2f(x)}{\pi} dx} \leq C e^{-\frac{2(f(z_1) + 2\delta)}{h}}.
\]

Moreover, since \( x_0 \) is the global minimum of \( f \) in \( \Omega \), one gets, using Laplace’s method

\[
\int_{\Omega} \chi^2(y) e^{-2f(x_0) dy} = \frac{(\pi h)^{\frac{d}{2}}}{\sqrt{\det \text{Hess} f(x_0)}} e^{-2f(x_0)} (1 + O(h)).
\]

This yields the desired estimate. \( \blacksquare \)

Notice that item 2(b) in Proposition 17 is a direct consequence of Lemma 60.

### 4.2.2 Definition of the quasi-mode \( \tilde{\phi}_i \) attached to \( z_i \)

Let \( z_i \) be a local minimum of \( f|_{\partial \Omega} \). Let \( S_{M,i} := \{ \Omega_i, \Gamma_0, \Gamma_{1,i}, \Gamma_{2,i}, V_{\Gamma_{1,i}} \} \) be an ensemble of sets associated with \( z_i \), see Definition 12. Thanks to Propositions 57 and 58, the set \( \Gamma_{1,i} \) can be taken such that \( \Sigma_i \subset \Gamma_{1,i} \). We recall that Section 4.1 was dedicated to the construction of a domain \( \Omega_i \subset \Omega \) and a mixed Witten Laplacian \( \Delta_{M,(1)}(\Omega_i) \) (see (122)) associated with this ensemble of sets \( S_{M,i} \). Proposition 59 gives the spectral properties of the operator \( \Delta_{M,(1)}(\Omega_i) \). In the following, we consider a normalized eigenform \( u^{(1)}_{h,i} \in D \left( \Delta_{M,(1)}(\Omega_i) \right) \) associated with the first eigenvalue 0, i.e. such that (145) holds

The quasi-mode \( \tilde{\phi}_i \) is defined as the following truncation of \( u^{(1)}_{h,i} \).

**Definition 14.** Let us assume that the hypotheses \([H1]\), \([H2]\) and \([H3]\) hold. Let \( \chi_i \in C_c^\infty (\Omega) \) be such that:

1. \( \chi_i \in C_c^\infty \left( \hat{\Omega}_i \cup \Gamma_{1,i} \right) \) (and thus \( \chi_i = 0 \) on a neighborhood of \( \Gamma_{2,i} \cup \Gamma_0 \) and on a neighborhood of \( \partial \Omega \setminus \Gamma_{1,i} \)).
Figure 13: The support of $\chi_i$ on $\partial \Omega$.

2. $\chi_i = 1$ on a neighborhood of $\Sigma_i$ in $\bar{\Omega}_i$,

3. $0 \leq \chi_i \leq 1$.

One defines $\mathcal{V}_i := \{x \in \Omega | \chi_i(x) = 1\}$. The quasi-mode $\tilde{\phi}_i$ is defined on $\Omega$ by:

$$
\tilde{\phi}_i := \frac{\chi_i u_{h,i}^{(1)}}{\sqrt{\int_{\Omega} |\chi_i(x) u_{h,i}^{(1)}(x)|^2 \, dx}}.
$$

Figure 14: The set $\mathcal{V}_i = \{x \in \Omega | \chi_i(x) = 1\}$ and, in gray, the support of $\nabla \chi_i$. 

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The support of $\chi_1$ on $\partial \Omega$ is represented on Figure 13 and the support of $\chi_i$ in $\Omega$ is represented in Figure 14. The function $\chi_i$ will be chosen such that the supp$(|\nabla \chi_i|)$ is as close as needed from $B_{z_i}^c \subset \partial \Omega$ or from $x_0$, as will be made precise at the end of Section 4.5. This is possible thanks to item 7 in Proposition 58.

Using Lemma 54 and the fact that $t \chi_i u_{h,i}^{(1)} = 0$ (on $\partial \Omega$), one easily shows that
\[
\tilde{\phi}_i \in L^1 H^1_0(\Omega).
\]

Using now standard elliptic regularity results, one can check that $\tilde{\phi}_i$ is actually a $C_c^\infty (\Omega \cup \Gamma_{1,i})$ function.

We will show in Section 4.5 that the family of forms $(\tilde{u}, \tilde{\phi}_1, \ldots, \tilde{\phi}_n)$ satisfy the estimates stated in Sections 2.2.1 and 2.2.2. This requires some preliminary results on the eigenforms $(u_{h,i}^{(1)})_{i \in \{1, \ldots, n\}}$ that are provided in Section 4.3 and 4.4.

### 4.3 Agmon estimates on $u_{h,i}^{(1)}$

Throughout this section, one assumes $[H1]$, $[H2]$ and $[H3]$. In all this section, we consider, for a fixed critical point $z_i$, an ensemble of sets $S_{M,i}$ associated with $z_i$ (see Definition 12) and an $L^2$-normalized eigenform $u_{h,i}^{(1)}$ of $\Delta_{f,h}^{M,i}(\Omega_i)$ associated with the eingenvalue $\lambda_h$, as introduced at the end of Section 4.1.3.

The aim of this section is to prove the following proposition.

**Proposition 61.** Let us assume that the hypotheses $[H1]$, $[H2]$ and $[H3]$ hold. Any $L^2$-normalized eigenform $u_{h,i}^{(1)}$ of $\Delta_{f,h}^{M,i}(\Omega_i)$ associated with the eigenvalue $0$ satisfies:

\[
\exists N \in \mathbb{N}, \left\| e^{\frac{\Psi}{h}} u_{h,i}^{(1)} \right\|_{L^2(\Omega_i)}^2 + \left\| d\left( e^{\frac{\Psi}{h}} u_{h,i}^{(1)} \right) \right\|_{L^2(\Omega_i)}^2 + \left\| d^* \left( e^{\frac{\Psi}{h}} u_{h,i}^{(1)} \right) \right\|_{L^2(\Omega_i)}^2 = O(h^{-N})
\]

where, we recall, $\Psi_{1}(x) = d_a(x, z_i)$ (see Definition 10).

For the ease of notation, we drop the subscript $i$ in the remaining of this section.

The proof is inspired by the first part of the proof of [34 Proposition 4.3.2] where the authors consider a Witten Laplacian with mixed tangential – full Dirichlet boundary conditions in a local system of coordinates in a neighborhood of $z$. The proof actually only requires that $u_{h}^{(1)}$ is an eigenform of $\Delta_{f,h}^{M,i}(\Omega_i)$ associated with an eigenvalue $\lambda_h = O(h)$. It crucially relies on the following Agmon-type energy equality.

**Lemma 62.** Let us assume that the hypotheses $[H1]$, $[H2]$ and $[H3]$ hold. Let $\varphi$ be a real-valued Lipschitz function on $\overline{\Omega}$. Then, for any $u \in D \left( \mathcal{Q}_{f,h}^{M,i}(\Omega_i) \right)$, one has:

\[
\mathcal{Q}_{f,h}^{M,i}(\Omega_i)(u, e^{\frac{\varphi}{h}} u) = h^2 \left\| d e^{\frac{\varphi}{h}} u \right\|_{L^2(\Omega_i)}^2 + h^2 \left\| d^* e^{\frac{\varphi}{h}} u \right\|_{L^2(\Omega_i)}^2 + (|\nabla f|^2 - |\nabla \varphi|^2 + h L \nabla \varphi + h L \nabla \varphi f) e^{\frac{\varphi}{h}} u, e^{\frac{\varphi}{h}} u \right\|_{L^2(\Omega_i)}^2 + h \left( - \int_{\Gamma_0 \cup \Gamma_1} + \int_{\Gamma_2} \right) \langle u, u \rangle_{T_{2,\Omega} e^{\frac{\varphi}{h}} \partial_n f} \ d\sigma.
\]

Moreover, when $u \in D \left( \Delta_{f,h}^{M,i}(\Omega_i) \right)$, the left-hand side equals $\langle e^{\frac{\varphi}{h}} \Delta_{f,h}^{M,i}(\Omega_i) u, u \rangle_{L^2(\Omega_i)}$.

**Proof.** This result is standard for manifolds without boundary or for bounded manifolds and quadratic forms with full normal or tangential boundary conditions (see e.g. [19, 34, 47]). We extend it here to our setting.
Note first that $u \in D \left( Q_{f,h}^{M,(1)}(\Omega) \right)$ implies $e^{2\hat{z}} u \in D \left( Q_{f,h}^{M,(1)}(\Omega) \right)$, since for $u \in D \left( Q_{f,h}^{M,(1)}(\Omega) \right)$, $n^\flat \wedge e^{2\hat{z}} u = e^{2\hat{z}} n^\flat \wedge u$ and $i_u e^{2\hat{z}} u = e^{2\hat{z}} i_u u$. One then gets by straightforward computations:

$$Q_{f,h}^{M,(1)}(\Omega)(u, e^{2\hat{z}} u) = \langle f_{h,u}, f_{h}(e^{2\hat{z}} u) \rangle + \langle d_{f_{h,u}}^*, d_{f_{h}}(e^{2\hat{z}} u) \rangle$$

$$= \langle e^{\hat{z}} f_{h,u}, f_{h}(e^{\hat{z}} u) \rangle + \langle e^{\hat{z}} d_{f_{h,u}}^*, d_{f_{h}}(e^{\hat{z}} u) \rangle - \langle e^{\hat{z}} d_{f_{h,u}}^*, d_{f_{h}}(e^{\hat{z}} u) \rangle + \langle d_{f_{h,u}}^*, d_{f_{h}}(e^{\hat{z}} u) \rangle$$

and hence to

$$Q_{f,h}^{M,(1)}(\Omega)(u, e^{2\hat{z}} u) = Q_{f,h}^{M,(1)}(\Omega)(\tilde{u}) - \langle |\nabla \varphi|^2 \tilde{u}, \tilde{u} \rangle - \langle d_{f} \wedge \tilde{u}, d_{f_{h}} \tilde{u} \rangle$$

$$+ \langle d_{f_{h} \tilde{u}}, d_{f} \wedge \tilde{u} \rangle + \langle i_{\nabla \varphi} \tilde{u}, d_{f_{h}}^* \tilde{u} \rangle - \langle d_{f_{h} \tilde{u}}, i_{\nabla \varphi} \tilde{u} \rangle$$

and hence to

$$Q_{f,h}^{M,(1)}(\Omega)(u, e^{2\hat{z}} u) = Q_{f,h}^{M,(1)}(\Omega)(\tilde{u}) - \langle |\nabla \varphi|^2 \tilde{u}, \tilde{u} \rangle .$$

One concludes by applying Lemma 56.

**Proof.** (of Proposition 61) Following the proof of Proposition 4.3.2, one proves the result in two steps. First, the Agmon estimate along $\Gamma_1 \subset \partial \Omega$ is proven by applying Lemma 62 with a function $\varphi$ close to $f_+ = \Psi$. The Agmon estimate in $\Omega$ is then obtained using again Lemma 62 with $\varphi$ close to $\Psi$, and the Agmon estimate along $\Gamma_1$.

In order to separate the analysis along $\Gamma_1$ and elsewhere, one introduces two smooth cut-off functions $\chi_0$ and $\chi_1$ on $\overline{\Omega}$ such that:

$$\chi_1 := \sqrt{1 - \chi_0^2}, \quad \chi_0 = 0 \text{ on } V_1' \cap \partial \Omega \subset V_1' \subset \overline{\Omega}, \text{ for a set } V_1' \subset \overline{\Omega} \text{ such that for some } \varepsilon > 0, (\text{see Figure } 15)$$

(i) $\overline{(V_1 + B(0, \varepsilon)) \cap \Omega} \subset V_1'$,

(ii) $\Gamma_1^\prime := V_1' \cap \partial \Omega$ is smooth and $(\Gamma_1 + B(0, \varepsilon)) \cap \partial \Omega \subset \Gamma_1'$ and $(\Gamma_1 + B(0, \varepsilon)) \cap \partial \Omega \subset B_2$,

(iii) $\Psi = d_{\partial}(z, \cdot)$ is a smooth solution to the following eikonal equation in $V_{1}'$ (see Proposition 52):

$$\begin{cases}
|\nabla \Psi|^2 = |\nabla f|^2 \text{ in } V_{1}' \\
\Psi = f - f(z) \text{ on } \Gamma_1' \\
\partial_{\nu} \Psi = -\partial_{\nu} f \text{ on } \Gamma_1'.
\end{cases}$$

It is possible to choose $V_{1}'$ such that all the properties stated previously on $V_{1}$ also hold on $V_{1}'$ (in particular (131), (132) and the properties stated in Proposition 58). We recall that one has by (78):

$$|\nabla \Psi|^2 \leq |\nabla f|^2 \text{ a.e. in } \Omega.$$
Thus
\[
|\nabla f_\pm| = \left|\nabla \left(\frac{\Psi \pm (f - f(z))}{2}\right)\right| \leq |\nabla f| \text{ a.e. in } \Omega.
\]

Thanks to the relations \(f - f(z) = f_+ - f_-\) and \(\Psi = f_+ + f_-\), together with the equality \(|\nabla \Psi|^2 = |\nabla f|^2\) a.e in \(V_{\Gamma_1}'\), one has
\[
\nabla f_- \cdot \nabla f_+ = 0 \quad \text{a.e. in } V_{\Gamma_1}', \quad |\nabla f|^2 = |\nabla f_+|^2 + |\nabla f_-|^2 \quad \text{a.e. in } V_{\Gamma_1}'.
\]

Let now \(u_h^{(1)} \in D\left(\Delta_{M_0}^{(1)}(\Omega)\right)\) satisfy
\[
\Delta_{M_0}^{(1)}(\Omega) u_h^{(1)} = 0 \quad \text{and} \quad \left\|u_h^{(1)}\right\|_{L^2(\Omega)} = 1.
\]

Step 1: Agmon estimate in \(\Gamma_1\).

In this part, we are going to prove the estimate (147) with \(\Psi\) replaced by \(f_+\) namely:
\[
\left\|e^{\frac{f_-}{h}} u_h^{(1)}\right\|_{L^2(\tilde{\Omega})} + \left\|d(e^{\frac{f_-}{h}} u_h^{(1)})\right\|_{L^2(\tilde{\Omega})} + \left\|d^*(e^{\frac{f_-}{h}} u_h^{(1)})\right\|_{L^2(\tilde{\Omega})} = O(h^{-N_0})
\]
for some integer \(N_0\). By the trace result (121), this will give the estimate \(\left\|e^{\frac{f-f(z)}{h}} u_h^{(1)}\right\|_{L^2(\Gamma_1)} = O(h^{-N_0})\) which is the first step to prove (147).

To get these results, the idea is to apply Lemma 62 to a convenient \(\varphi\) comparable with \(f_+\) and such that \(|\nabla \varphi| \leq |\nabla f_+|\). This kind of estimate is classic and the ideas
behind the computations presented below, which follow [34, 47], originate from the article [35] where similar estimates were obtained in the case of manifolds without boundary. The presence of a boundary leads here to some technicalities which can somehow hide the reasoning and we refer for example to [19, 30] for a presentation of semi-classical Agmon estimates in manifolds without boundary. We recall from the work [35] that if one just wants to get an error of the form $O(e^{-\varepsilon_0 h})$ with $\varepsilon_0 > 0$ arbitrarily small, the choice $\varphi = (1 - \varepsilon) f_+$ is sufficient, but it does not yield an error of the form $O(h^{-N})$. To get such an error term, a good choice for $\varphi$ is (149).

Let $\varphi : \Omega \to \mathbb{R}$ be the following Lipschitz function:

\[
\varphi = \begin{cases} 
  f_+ - Ch \ln \frac{f_+}{h} & \text{if } f_+ > Ch, \\
  f_+ - Ch \ln C & \text{if } f_+ \leq Ch,
\end{cases}
\]

(149)

for some constant $C > 1$ that will be fixed at the end of this step. Define the associated level sets:

\[\Omega_- = \{ x \in \Omega \text{ s.t. } f_+(x) \leq Ch \} \quad \text{and} \quad \Omega_+ = \hat{\Omega} \setminus \Omega_- .\]

Then $\nabla \varphi = \nabla f_+$ a.e. in $\Omega_-$ and

\[\nabla \varphi = \nabla f_+ \left( 1 - \frac{Ch}{f_+} \right) \quad \text{a.e. in } \Omega_+ .\]

This implies in particular the two following inequalities valid a.e. on $\Omega_+$ and that will be used in the sequel:

\[|\nabla f_+|^2 - |\nabla \varphi|^2 = |\nabla f_+|^2 \left( 2Ch - \frac{C^2 h^2}{f_+^2} \right) \geq Ch \frac{|\nabla f_+|^2}{f_+} \quad \text{on } \Omega_+ \]

(150)

\[|\nabla f|^2 - |\nabla \varphi|^2 \geq |\nabla f|^2 - |\nabla f_+|^2 \left( 1 - \frac{Ch}{f_+} \right) \geq Ch \frac{|\nabla f|^2}{f_+} \quad \text{on } \Omega_+ .\]

(151)

The last inequality in (151) is a consequence of the inequality $|\nabla f_+|^2 \leq |\nabla f|^2$. This implies in particular that $|\nabla \varphi| \leq |\nabla f|$ a.e. in $\hat{\Omega}$.

Note lastly that there exists a constant $K > 0$ depending on $f_+$ and $f$, such that

\[\frac{|\nabla f|^2}{f_+} \geq K \text{ in } \hat{\Omega} \quad \text{and} \quad \frac{|\nabla f_+|^2}{f_+} \geq K \text{ in } V'_1 ,\]

(152)

the last inequality being a consequence of the facts that $f_+(x', x_d) = f_+(x', 0)$ and $x' \mapsto f(x', 0) = f(z) + f_+(x', 0)$ is a Morse function with $z$ as only critical point.

Now, using the fact that $\Delta^{M,1}_f(\hat{\Omega}) u^{(1)}_h = 0$ and the IMS localisation formula (143), one gets

\[0 = Q_{f,h}^{M,1}(\hat{\Omega})(u^{(1)}_h, e^{2\pi i} u^{(1)}_h) = \sum_{k \in \{0, 1\}} Q_{f,h}^{M,1}(\hat{\Omega})(\chi_k u^{(1)}_h, e^{2\pi i} \chi_k u^{(1)}_h) - h^2 \left\| \nabla \chi_k e^{\pi i} u^{(1)}_h \right\|^2_{L^2(\hat{\Omega})} .\]

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Setting $\tilde{u}^{(1)}_h := e^{\chi_h}u^{(1)}_h$ and applying (148) to $\chi_h u^{(1)}_h$, $k \in \{0,1\}$, one obtains:

$$C_1 h^2 \sum_{k \in \{0,1\}} \left\| \chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 = C_1 h^2 \left\| u^{(1)}_h \right\|_{L^2(\Omega)}^2$$

$$\geq \sum_{k \in \{0,1\}} \left[ \left\| h d\chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + \left\| h d^* \chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + \langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \chi_k \tilde{u}^{(1)}_h, \chi_k \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} \right]$$

$$+ h \langle \mathcal{L}_f + \mathcal{L}_f \rangle \chi_k \tilde{u}^{(1)}_h, \chi_k \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} + h \left( \int_{\Gamma_2} - \int_{\Gamma_1} \right) \langle \chi_0 \tilde{u}^{(1)}_h, \chi_0 \tilde{u}^{(1)}_h \rangle \partial_n f \, ds,$$

where $C_1 = \max(\|\nabla \chi_0\|_{\infty}^2, \|\nabla \chi_1\|_{\infty}^2)$. Note that one has used that $\chi_0 = 0$ on $\Gamma_0$, $\chi_1 = 0$ on $\Gamma_1$ and

$$\left( - \int_{\Gamma_0} + \int_{\Gamma_1} \right) \langle \chi_1 \tilde{u}^{(1)}_h, \chi_1 \tilde{u}^{(1)}_h \rangle \partial_n f \, ds \geq 0,$$

which follows from $\partial_n f > 0$ on $\Gamma_2$ and from $\partial_n f < 0$ on $\Gamma_0$.

Now, since $\mathcal{L}_f + \mathcal{L}_f$ is a $0^{th}$ order differential operator, and $| \tilde{u}^{(1)}_h(x) | \leq e^{\mathcal{C}} | u^{(1)}(x) |$ a.e. on $\Omega_-$, one obtains that for some constants $C_2$ (independent of $C$) and $C_3(C)$ depending on $C$,

$$C_3(C) h \geq \sum_{k \in \{0,1\}} \left[ \left\| h d\chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + \left\| h d^* \chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + \langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \chi_k \tilde{u}^{(1)}_h, \chi_k \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} \right]$$

$$- C_2 h \left\| \chi_k \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + h \left( \int_{\Gamma_2} - \int_{\Gamma_1} \right) \langle \chi_0 \tilde{u}^{(1)}_h, \chi_0 \tilde{u}^{(1)}_h \rangle \partial_n f \, ds.$$

(153)

Let us first consider the case $k = 1$. Using $|\nabla \varphi| \leq |\nabla f|$ and (151)–(152), one gets:

$$\langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \chi_1 \tilde{u}^{(1)}_h, \chi_1 \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} - C_2 h \left\| \chi_1 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2$$

$$\geq \langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \chi_1 \tilde{u}^{(1)}_h, \chi_1 \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} - C_2 h \left\| \chi_1 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2$$

$$\geq \left( \left( C h \frac{|\nabla f|^2}{f_+} - C_2 h \right) \chi_1 \tilde{u}^{(1)}_h, \chi_1 \tilde{u}^{(1)}_h \right)_{L^2(\Omega)}$$

$$\geq (KC - C_2) h \left\| \chi_1 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2.$$

(154)

Let us then consider the case $k = 0$. In this case, one deduces from $\text{supp} \chi_0 \subset \mathcal{V}_\Gamma'$, where $|\nabla f|^2 = |\nabla f_+|^2 + |\nabla f_-|^2$, from $|\nabla \varphi|^2 = |\nabla f_+|^2$ on $\Omega_-$, and from (150)–(152) the inequality:

$$\langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \chi_0 \tilde{u}^{(1)}_h, \chi_0 \tilde{u}^{(1)}_h \rangle_{L^2(\Omega)} - C_2 h \left\| \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2$$

$$= \left\| \nabla f - \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + \langle |\nabla f_+|^2 - |\nabla \varphi|^2 \rangle \chi_0 \tilde{u}^{(1)}_h, \chi_0 \tilde{u}^{(1)}_h \rangle_{L^2(\Omega_-)}$$

$$+ \langle |\nabla f_+|^2 - |\nabla \varphi|^2 \rangle C_2 h \chi_0 \tilde{u}^{(1)}_h, \chi_0 \tilde{u}^{(1)}_h \rangle_{L^2(\Omega_-)}$$

$$\geq \left\| \nabla f - \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 + (KC - C_2) h \left\| \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega_-)}^2$$

$$\geq (1 + 2C_4(C) h) \left\| \nabla f - \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega)}^2 - (KC - C_2) h \left\| \chi_0 \tilde{u}^{(1)}_h \right\|_{L^2(\Omega_-)}^2,$$

(155)
where $C_4(C) := \frac{KC - C_2}{2\|\nabla f\|^2_{L^\infty(V_1')}}$ (see (131)) and $C$ has been chosen large enough to ensure that $KC - C_2 > 0$.

In order to get a lower bound for the boundary term in (153), one uses the fact that the mixed Witten Laplacian $\Delta^{M(1)}_{f,h}(\tilde{\Omega})$ associated with $\tilde{f} = -\tilde{\chi}_0 f_-$ where $\tilde{\chi}_0 \in C^\infty([0,1])$, $\tilde{\chi}_0 = 1$ on supp $\chi_0$, supp $\tilde{\chi}_0 \subset (V_1' + B(0,\alpha)) \cap \Omega$ for $\alpha > 0$ such that $f_-$ is smooth on supp $\chi_0$ and $\tilde{h} = \frac{h}{1 + C_4(C)h}$ is nonnegative. Starting from the inequality $(1 + C_4(C)h)Q^{M(1)}_{f,h}(\tilde{\Omega})(\chi_0 \tilde{u}_h^{(1)}, \chi_0 \tilde{u}_h^{(1)}) \geq 0$ and then applying Lemma 56 to $\chi_0 \tilde{u}_h^{(1)} \in D\left(Q^{M(1)}_{f,h}(\tilde{\Omega})\right)$ lead to (since $\chi_0 = 0$ on $\Gamma_0$):

$$h \left( \int_{\Gamma_1} - \int_{\Gamma_2} \langle \chi_0 \tilde{u}_h^{(1)}, \chi_0 \tilde{u}_h^{(1)} \rangle \partial_n f_- d\sigma \geq \frac{h^2}{1 + C_4(C)h} \left( \|d\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 + \|d^* \chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 \right) + (KC - C_2)h \|\chi_1 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2 \right.$$

$$- hC_5 \|\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2 \left. + \frac{h^2}{1 + C_4(C)h} \left( \|d\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 + \|d^* \chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 \right) - hC_5 \|\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2 \right) - hC_5 \|\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2 \right).$$

In the last computation one has used that $1 \geq \frac{h}{1 + C_4(C)h}$. It follows from (135) that $\partial_n f_+ > 0$ on supp $\chi_0 \cap \Gamma_2$. Then, since $|\tilde{u}_h^{(1)}(x)| \leq e^{C}|u_h^{(1)}(x)|$ a.e. on $\Omega_-$, there exists $C_6(C, C_2, K)$ such that

$$C_6h \geq \sum_{k \in \{0, 1\}} \frac{C_6h^3}{1 + C_4h} \left( \|d\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 + \|d^* \chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 \right) + (KC - C_2)h \|\chi_1 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2$$

$$+ C_4h \|\nabla f_- \chi_0 \tilde{u}_h^{(1)}\|_{L^2(\tilde{\Omega})}^2 - (KC - C_2)h \|\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2$$

$$- hC_5 \|\chi_0 \tilde{u}_h^{(1)}\|_{L^2(\Omega_2)}^2 + h \int_{\Gamma_2} \langle \chi_0 u \chi_0 u \rangle \partial_n f_+ d\sigma.$$
Step 2: Agmon estimate in \( \hat{\Omega} \).

One follows the same approach as in step 1 but with the function

\[
\varphi = \begin{cases} 
\Psi - Ch \ln \frac{\Psi}{h}, & \text{if } \Psi > Ch, \\
\Psi - Ch \ln C, & \text{if } \Psi \leq Ch, 
\end{cases}
\]

where the constant \( C > 1 \) will be fixed later on, and with the associated level sets:

\[
\Omega_+ = \left\{ x \in \hat{\Omega} \text{ s.t. } \Psi(x) \leq Ch \right\} \quad \text{and} \quad \Omega_- = \hat{\Omega} \setminus \Omega_+.
\]

Applying formula (148) then leads to (note that \( |\tilde{u}_h^{(1)}| \leq C^2 |u_h| \) on \( \Omega_- \), \( \partial_h f < 0 \) on \( \Gamma_0 \) and \( \partial_n f > 0 \) on \( \Gamma_2 \)):

\[
C_2(C)h \left( 1 + \int_{\Gamma_1} \langle \tilde{u}_h^{(1)}, \tilde{u}_h^{(1)} \rangle_{T^2 \Omega} \, d\sigma \right) \geq \left\| h d\tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)} + \left\| h d^* \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)}
\]

\[
+ (\langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \tilde{u}_h^{(1)}, \tilde{u}_h^{(1)} \rangle_{L^2(\Omega)} - C_1 h \left\| \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega_+)}),
\]

where \( \tilde{u}_h := e^{\frac{\Psi}{h}} u_h^{(1)} \), the constant \( C_1 \) is independent of \( C \), whereas \( C_2 \) is a constant depending on \( C \). Besides, due to the relations

\[
\Psi = f - f(z) \text{ on } \Gamma_1 \text{ and } e^{\frac{\Psi}{h}} \leq e^{\frac{\Psi}{C}} \text{ on } \hat{\Omega},
\]

the trace estimate obtained in (159) implies

\[
\int_{\Gamma_1} \langle \tilde{u}_h^{(1)}, \tilde{u}_h^{(1)} \rangle_{T^2 \Omega} \, d\sigma = O(h^{-2N_0}).
\]

Injecting (161) in (160) then gives

\[
\left\| h d\tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)} + \left\| h d^* \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)}
\]

\[
+ (\langle |\nabla f|^2 - |\nabla \varphi|^2 \rangle \tilde{u}_h^{(1)}, \tilde{u}_h^{(1)} \rangle_{L^2(\Omega)} - C_1 h \left\| \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega_+)} = O(h^{1-2N_0}).
\]

Since moreover \( |\nabla \Psi|^2 \leq |\nabla f|^2 \) (see (78)) and \( f \) has no critical point in \( \hat{\Omega} \), one gets:

\[
|\nabla f|^2 - |\nabla \varphi|^2 \geq |\nabla f|^2 - |\nabla \Psi|^2 \left( 1 - \frac{Ch}{\Psi} \right) \geq Ch \frac{|\nabla f|^2}{\Psi} \geq CC_3 h \text{ on } \Omega_+
\]

where \( C_3 > 0 \) is independent of \( C \). Since \( |\nabla f|^2 \geq |\nabla \Psi|^2 \) a.e. on \( \Omega_- \), adding the term \( (CC_3 - C_1)h \left\| \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega_-)} \) to (162) leads to

\[
\left\| h d\tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)} + \left\| h d^* \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega)} + (CC_3 - C_1)h \left\| \tilde{u}_h^{(1)} \right\|^2_{L^2(\Omega_+)} = O(h^{1-2N_0})
\]

Now, taking \( C > \frac{C_2}{CC_3} \) gives, since \( \varphi - \Psi \geq -C_4 h \ln \frac{1}{h} \), there exists \( N_1 > 0 \) such that:

\[
\left\| e^{\frac{\Psi}{h}} u_h^{(1)} \right\|^2_{L^2(\Omega)} + \left\| d(e^{\frac{\Psi}{h}} u_h^{(1)}) \right\|^2_{L^2(\Omega)} + \left\| d^* (e^{\frac{\Psi}{h}} u_h^{(1)}) \right\|^2_{L^2(\Omega)} = O(h^{-N_1}).
\]

This concludes the proof of (147).
4.4 Comparison of the eigenform $u_{h,i}^{(1)}$ and its WKB approximation

Throughout this section, one assumes [H1], [H2] and [H3]. In all this section, we consider, for a fixed critical point $z_i$, an ensemble of sets $S_{M,i}$ associated with $z_i$ (see Definition 12) and an $L^2$-normalized eigenform $u_{h,i}^{(1)}$ of $\Delta^{M,(1)}_{f,h}(\bar{\Omega})$ associated with the eigenvalue 0, as introduced at the end of Section 4.1.3. For the ease of notation, we drop the subscript $i$ in all this section.

4.4.1 Construction of the WKB expansion of $u_h^{(1)}$

Let $z$ be a local minimum of $f|_{\partial \Omega}$. Before going through a rigorous construction of a WKB expansion $u_{z,wkb}^{(1)}$ of $u_h^{(1)}$ in a neighborhood of $z$, let us explain formally how we proceed. Let us recall that the 1-form $u_h^{(1)}$ satisfies:

$$
\begin{align*}
\Delta_{f,h}^{(1)} u_h^{(1)} &= 0 \text{ in } \bar{\Omega}, \\
\mathbf{t} u_h^{(1)} &= 0 \text{ and } \mathbf{t} d_{f,h} u_h^{(1)} = 0 \text{ on } \Gamma_1,
\end{align*}
$$

plus additional boundary conditions on $\Gamma_0 \cup \Gamma_2$ that we do not recall since the objective is to approximate $u_h^{(1)}$ in a neighborhood of $z$ in $\bar{\Omega}$. The behavior of $u_h^{(1)}$ in a neighborhood of $z$ exhibited in Proposition 61 suggests to take $u_{z,wkb}^{(1)}$ of the form $u_{z,wkb}^{(1)}(x,h) = a^{(1)}(x,h) e^{-\frac{d_a(x,z)}{h}}$ where $a^{(1)}$ is expanded in powers of $h$: $a^{(1)}(x,h) = \sum_{k \geq 0} a_k^{(1)}(x) h^k$ and to look for 1-forms $(a_k^{(1)})_{k \geq 0}$ so that $u_{z,wkb}^{(1)}$ is a nontrivial 1-form satisfying (compare with (164)):

$$
\begin{align*}
\Delta_{f,h}^{(1)} u_{z,wkb}^{(1)} &= O(h^\infty) e^{-\frac{d_a(x,z)}{h}} \text{ in } \bar{\Omega}, \\
\mathbf{t} u_{z,wkb}^{(1)} &= 0 \text{ and } \mathbf{t} d_{f,h} u_{z,wkb}^{(1)} = 0 \text{ on } \Gamma_1,
\end{align*}
$$

where the meaning of $O(h^\infty)$ is formally $h^s O(h^\infty) = o_h(1)$ for any $s \in \mathbb{R}$. The boundary conditions in (165) ensures that when cutting suitably a solution to $a^{(1)}$ near $\Gamma_1$, the resulting 1-form belongs to the form domain of $\Delta_{f,h}^{M,(1)}(\bar{\Omega})$ (this is needed if one wants to approximate $u_h^{(1)}$ on $\partial \bar{\Omega}$). Instead of directly trying to solve (165), the construction of $u_{z,wkb}^{(1)}$ can be simply done as follows (see [34] Section 4.2). Using the complex property, one considers $u_{z,wkb}^{(1)} = d_{f,h} u_{z,wkb}^{(0)}$ where the function $u_{z,wkb}^{(0)} = a^{(0)}(\cdot,h) e^{-\frac{d_a(x,z)}{h}}$ where $a^{(0)}(x,h) = \sum_{k \geq 0} a_k^{(0)}(x) h^k$ for a non trivial family of functions $(a_k)_{k \geq 0}$ such that:

$$
\begin{align*}
\Delta_{f,h}^{(0)} u_{z,wkb}^{(0)} &= O(h^\infty) e^{-\frac{d_a(x,z)}{h}} \text{ in } \bar{\Omega}, \\
u_{z,wkb}^{(0)} &= e^{-\frac{1}{h} f} \text{ on } \Gamma_1.
\end{align*}
$$

This implies the boundary condition: $a^{(0)} = 1$ on $\Gamma_1$. Then, if $u_{z,wkb}^{(0)} = a^{(0)} e^{-\frac{d_a(x,z)}{h}}$ is a solution of (166), we set:

$$
u_{z,wkb}^{(1)} = d_{f,h} u_{z,wkb}^{(0)}.
$$

One can easily check that the 1-form $u_{z,wkb}^{(1)}$ then satisfies (165) and the extra boundary condition $\mathbf{t} d_{f,h} u_{z,wkb}^{(1)} = O(h^\infty) e^{-\frac{f(x,z)}{h}}$ on $\Gamma_1$. Indeed, it holds:

$$
d_{f,h} u_{z,wkb}^{(0)} = e^{-\frac{d_a(x,z)}{h}} \left( d(f - d_a(x,z)) a^{(0)} + h d a^{(0)} \right),
$$
which implies \( t u^{(1)}_{z, w_{k b}} = 0 \) since \( a^{(0)}(0) = 1 \) and
\[
f - d_a(\cdot, z) = f(z) \text{ on } \Gamma_1.
\] (167)

In addition, one has
\[
d^*_{f, h} d_f h u^{(0)}_{z, w_{k b}} = \Delta h u^{(0)}_{z, w_{k b}} = O(h^\infty) e^{-\frac{d_a(x, z)}{h}}
\]
which implies \( t d^*_{f, h} u^{(1)}_{z, w_{k b}} = O(h^\infty) e^{-\frac{d_a(x, z)}{h}} \) and
\[
\Delta^{(1)} f, h d_f h u^{(0)}_{z, w_{k b}} = d_f h \Delta^{(0)} f, h u^{(0)}_{z, w_{k b}} = O(h^\infty) e^{-\frac{d_a(x, z)}{h}}.
\]

Thus, the 1-form \( u^{(1)}_{z, w_{k b}} \) satisfies (165).

Expanding in powers of \( h \) the function \( e^{\frac{d_a(x, z)}{h}} \), \( u^{(0)}_{z, w_{k b}} \) is a solution of (166) if it holds:
\[
|\nabla d_a(x, z)| = |\nabla f(x)|, \text{ for } x \in \hat{\Omega},
\] (168)
which is satisfied at least in a neighborhood of \( z \) (see Proposition 52) and if \( (a_k^{(0)})_{k \geq 0} \) satisfies the following transport equations, defined recursively by:
\[
\begin{cases}
(\Delta \Phi - \Delta f + 2 \nabla \Phi \cdot \nabla) a_0^{(0)} = 0 & \text{in } \hat{\Omega} \\
(\Delta \Phi - \Delta f + 2 \nabla \Phi \cdot \nabla) a_{k+1}^{(0)} = \Delta a_k^{(0)} & \text{in } \hat{\Omega}, \forall k \geq 0,
\end{cases}
\] (169)
with boundary conditions
\[
\begin{cases}
a_0^{(0)} = 1 & \text{on } \Gamma_1 \\
a_{k}^{(0)} = 0 & \text{on } \Gamma_1, \forall k \geq 1.
\end{cases}
\] (170)

For a fixed \( k \), the transport equation can be solved locally around each \( z \in \partial \Omega \) thanks to the condition \( \partial_n \Phi = -\partial_n f < 0 \) on \( \partial \Omega \) and thus on a neighborhood of \( \partial \Omega \) (independent

A preliminary construction.

Let \( \Phi \) be the solution of the eikonal equation (102) on a neighborhood \( V_{\partial \Omega} \) of the boundary \( \partial \Omega \). Let us introduce the formal transport operator
\[
T := \Delta \Phi - \Delta f + 2 \nabla \Phi \cdot \nabla.
\]

Let us consider the solutions to the following transport equations, defined recursively by
\[
\begin{cases}
T a_0 = 0 & \text{in } V_{\partial \Omega} \\
T a_{k+1} = \Delta a_k & \text{in } V_{\partial \Omega}, \forall k \geq 0,
\end{cases}
\] (170)
with boundary conditions
\[
\begin{cases}
a_0 = 1 & \text{on } \partial \Omega \\
a_{k} = 0 & \text{on } \partial \Omega, \forall k \geq 1.
\end{cases}
\] (171)

For a fixed \( k \), the transport equation can be solved locally around each \( z \in \partial \Omega \) thanks to the condition \( \partial_n \Phi = -\partial_n f < 0 \) on \( \partial \Omega \) and thus on a neighborhood of \( \partial \Omega \) (independent
of \( k \) using a compactness argument. Therefore, up to choosing a smaller neighborhood \( V_{\partial}\Omega \) of \( \partial\Omega \) in \( \Omega \), there exists a unique sequence of \( C^\infty(V_{\partial}\Omega) \) functions \((a_k)_{k \geq 0}\) solution to (170)-(171).

There exists a function \( a = a(x,h) \) (called a resummation of the formal symbol \( \sum_{k=0}^{+\infty} a_k h^k \) \( C^\infty \) and uniformly bounded together with all its derivatives such that

\[
a(x,h) = 1 \text{ on } \partial\Omega \quad \text{and} \quad a(x,h) \sim \sum_{k=0}^{+\infty} a_k(x) h^k.
\]

This means that for \( a - \sum_{k=0}^{+\infty} a_k h^k \) is \( O(h^\infty) \) in the following sense: for all compact \( K \) in \( V_{\partial}\Omega \), for all \( \alpha \in \mathbb{N}^d \), for all \( N \in \mathbb{N} \),

\[
\left\| \partial^\alpha_x \left( a - \sum_{k=0}^{N} a_k(x) h^k \right) \right\|_{L^\infty(K)} \leq C_{K,\alpha,N} h^{N+1}.
\]

(172)

Such a construction is standard and can be found in [19] or in [34], where it is done using a Borel summation. Moreover \( a \) is unique up to a term of order \( O(h^\infty) \). Let us now define on \( V_{\partial}\Omega \):

\[
\begin{align*}
    &u^{(0)}_{wkb}(x,h) := a(x,h) e^{-\frac{\Phi}{h}}.  \\
\end{align*}
\]

By construction of the sequence \((a_k)_{k \geq 0}\), the function \( u^{(0)}_{wkb} \) solves

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \Delta^{(0)}_{f,h} u^{(0)}_{wkb} = O(h^\infty) e^{-\frac{\Phi}{h}} \quad \text{in } V_{\partial}\Omega,  \\
        u^{(0)}_{wkb} = e^{-\frac{\Phi}{h}} = e^{-\frac{f}{h}} \quad \text{on } \partial\Omega,
    \end{array} \right.
\end{align*}
\]

where \( O(h^\infty) \) is defined in (172). Indeed, using (44), \( |\nabla f|^2 = |\nabla \Phi|^2 \) on \( V_{\partial}\Omega \), and the equations (170) satisfied by \((a_k)_{k \geq 0}\).

\[
e^\frac{\Phi}{h} \Delta^{(0)}_{f,h} u^{(0)}_{wkb} = -h^2 \Delta a(x,h) + h[a(x,h)\Delta \Phi + 2\nabla \Phi \cdot \nabla a(x,h)] - a(x,h)|\nabla \Phi|^2
\]

\[
+ a(x,h)|\nabla f|^2 - ha(x,h)\Delta f
\]

\[
\sim h Ta_0 + h^2 \sum_{k=0}^{+\infty} h^k(Ta_{k+1} - \Delta a_k)
\]

\[
= O(h^\infty).
\]

In addition, it holds \( u^{(0)}_{wkb} = e^{-\frac{\Phi}{h}} \) on \( \partial\Omega \) since \( a(x,h) = 1 \) on \( \partial\Omega \). Let us now define on \( V_{\partial}\Omega \):

\[
\begin{align*}
    &u^{(1)}_{wkb} := d_{f,h} u^{(0)}_{wkb}.  \\
\end{align*}
\]

The 1-form \( u^{(1)}_{wkb} \) satisfies:

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \Delta^{(1)}_{f,h} u^{(1)}_{wkb} = O(h^\infty)e^{-\frac{\Phi}{h}} \quad \text{in } V_{\partial}\Omega,  \\
        t u^{(1)}_{wkb} = 0 \quad \text{on } \partial\Omega,  \\
        t d_{f,h} u^{(1)}_{wkb} = O(h^\infty)e^{-\frac{\Phi}{h}} \quad \text{on } \partial\Omega,
    \end{array} \right.
\end{align*}
\]

(173)

where \( O(h^\infty) \) is defined in (172). Indeed, one has \( t u^{(1)}_{wkb} = t d_{f,h} u^{(0)}_{wkb} = d_{f,h} u^{(1)}_{wkb} = d_{f,h} t \left( a(x,h)e^{-\frac{\Phi}{h}} \right) = d_{f,h} e^{-\frac{f}{h}} = 0 \) since \( a(x,h) = 1 \) and \( \Phi = f \) on \( \partial\Omega \).
Moreover \( t d^*_{f,h} u^{(1)}_{wkb} = t d^*_{f,h} d_f h u^{(0)}_{wkb} = t \Delta^{(0)}_{f,h} u^{(0)}_{wkb} = O(h^\infty) e^{-\frac{\Phi}{h}} \). Finally, \( \Delta^{(1)}_{f,h} u^{(1)}_{wkb} = \Delta^{(1)}_{f,h} d_f h u^{(0)}_{wkb} = d_f h \Delta^{(0)}_{f,h} u^{(0)}_{wkb} = (hd + df \wedge) O(h^\infty) e^{-\frac{\Phi}{h}} = O(h^\infty) e^{-\frac{\Phi}{h}} \).

WKB expansion of \( u^{(1)}_h \).

Let \( z \) be local minimum of \( f|_{\Omega} \). Let us now define the WKB expansion of \( u^{(1)}_h \) on \( V_{\Omega} \) by:

\[
u^{(1)}_{z,wkb} := e^{\frac{f(z)}{h}} u^{(1)}_{wkb} = e^{\frac{f(z)}{h}} d_f h u^{(0)}_{wkb} = d_f h \left( a(\cdot, h) e^{-\frac{\Phi-f(z)}{h}} \right).
\]

(174)

One recalls (see Proposition [52]) that for any smooth open domain \( \Gamma \) such that \( \Gamma \subset \Gamma_1 \) and \( z \in \Gamma \), there exists a neighborhood of \( \Gamma \) in \( \Omega \), denoted by \( V_{\Gamma} \subset V_{\Omega} \cap (\Gamma_1 \cup \Omega) \), such that for all \( x \in V_{\Gamma} \),

\[
\Psi(x) = d_a(x, z) = \Phi(x) - f(z).
\]

Lemma 63. Let us assume that the hypotheses \([H1]\) and \([H3]\) hold. Let us consider \( z \) a local minimum of \( f|_{\Omega} \) as introduced in hypothesis \([H2]\). The 1-form \( \nu^{(1)}_{z,wkb} \) satisfies

\[
\begin{aligned}
\Delta^{(1)}_{f,h} u^{(1)}_{z,wkb} &= O(h^\infty) e^{-\frac{\Phi-f(z)}{h}} \text{ in } V_{\Omega}, \\
\nu^{(1)}_{z,wkb} &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

(175)

where \( O(h^\infty) \) is defined in \([172]\). For any \( \chi \in C^\infty_c(V_{\Gamma}) \) such that \( \chi = 1 \) on a neighborhood of \( z \), it holds: in the limit \( h \to 0 \),

\[
\int_{\Omega} |\chi(x) \nu^{(1)}_{z,wkb}(x)|^2 dx = C^2_{z,wkb} h^{\frac{d+1}{2}} (1 + O(h)),
\]

(176)

where

\[
C_{z,wkb} := \pi^{\frac{d+1}{2}} \frac{\sqrt{2 \partial_{h} f(z)}}{(\det \text{Hess} f|_{\Omega}(z))^\frac{1}{4}}.
\]

(177)

Furthermore, there exists \( C > 0 \) such that for \( h \) small enough,

\[
\|
\chi \nu^{(1)}_{z,wkb}
\|_{H^1(\Omega)} \leq Ch^{-1},
\]

(178)

and \( \Phi^{M,(1)}(\Omega)\chi \nu^{(1)}_{z,wkb} = O(h^\infty) \).

Proof. Equation \((175)\) is easily obtained from \((173)\). Let us now prove \((176)\) and \((177)\). Notice that one can write (using \((42)\))

\[
\nu^{(1)}_{z,wkb} = d_f h \left[ e^{-\frac{\Phi-f(z)}{h}} \sum_{k=0}^{+\infty} a_k h^k \right]
\]

\[
= e^{\frac{f(z)}{h}} (hd) e^{\frac{f(z)}{h}} \left[ e^{-\frac{\Phi-f(z)}{h}} \sum_{k=0}^{+\infty} a_k h^k \right]
\]

\[
= e^{-\frac{\Phi-f(z)}{h}} e^{\frac{f(z)}{h}} (hd) e^{\frac{f(z)}{h}} \left[ \sum_{k=0}^{+\infty} a_k h^k \right]
\]

\[
= e^{-\frac{\Phi-f(z)}{h}} (d(f - \Phi) \wedge a_0)
\]

\[
+ e^{-\frac{\Phi-f(z)}{h}} \left[ hd \sum_{k=0}^{+\infty} a_k h^k + d(f - \Phi) \wedge \sum_{k=0}^{+\infty} a_k h^k \right].
\]

(179)
Recall that the function $\chi$ is supported in $V_T$ and the function $x \mapsto \Phi(x) - f(z)$ has a unique minimum on $V_T$ which is $z$ since $\Phi(x) - f(z) = d_a(x, z) \geq 0$ on $V_T$. Therefore, in the limit $h \to 0$:

$$
\sqrt{\int_{\Omega} \left| \chi(x) u^{(1)}_{z, wkb}(x) \right|^2 \, dx} = \sqrt{\int_{\Omega} \left| e^{-\phi(x) / h} \chi \left( f - \Phi \right) \wedge a_0 \right|^2 (1 + O(h))}.
$$

Additionally, since $\chi(z) = 1$ and $|d(f - \Phi)(z)|^2 = |\nabla (f - \Phi)(z)|^2 = |\nabla_T(f - \Phi)(z)|^2 + (2\partial_n f(z))^2 = (2\partial_n f(z))^2$, one gets using Laplace’s method

$$
\int_{\Omega} \left| e^{-\phi(z) / h} \chi d(f - \Phi) \wedge a_0 \right|^2 = (\pi h)^{\frac{d+1}{4}} \frac{2\partial_n f(z)}{\pi \sqrt{\det \text{Hess} f|_{\partial\Omega}(z)}} (1 + O(h)). \tag{180}
$$

Let us give more details on how to obtain (180). Recall that on $\text{supp}(\chi)$, $\Phi - f(z) = f_+ + f_- - \Phi = f - \Psi - f(z) = -2f_-$, and, on $\text{supp}(\chi) \cap \partial\Omega$, $\partial_n f = -\partial_n f_- = |\nabla f_-|$. Thus, using the coordinate set introduced in Definition 9 and the co-area formula $dx = |\nabla f_-| \, d\eta$ (see for example [1]),

$$
\int_{\Omega} \left| e^{-\phi(z) / h} \chi d(f - \Phi)(x) \wedge a_0(x) \right|^2 \, dx = 4 \int_{-\alpha}^{0} e^{-2\pi h} \int_{\Omega} e^{-2\pi h} \chi^2 a_0^2 \nabla f_- \, d\sigma_{\Omega} \, d\eta
$$

where $\Sigma_\eta = \{ x, f_-(x) = -\eta \}$, $\sigma_{\Sigma_\eta}$ is the Lebesgue measure on $\Sigma_\eta$. In the last equality, $j(x', \eta)$ is the Jacobian of the parametrization of $\Sigma_\eta$ by $x' \in \partial\Omega$. Using the Laplace formula, for any $\eta \in [-\eta_0, 0]$ with $\eta_0 > 0$ sufficiently small so that $\chi^2(z, \eta) \neq 0$ for all $\eta \in [-\eta_0, 0]$, one has

$$
\int_{\partial\Omega} e^{-2\pi h} \chi^2 (x', \eta) a_0^2 (x', \eta) |\nabla f_-|(x', \eta) j(x', \eta) \, d\sigma_{\partial\Omega}(x')
$$

where $O(h)$ is a function of $\eta$ and $h$ with $L^\infty$ norm in $\eta \in [0, \eta_0]$ bounded from above by a constant times $h$ (thanks to the regularity of the involved terms), for sufficiently small $h$. Thus, using again Laplace’s method:

$$
\int_{\partial\Omega} \left| e^{-\phi(z) / h} \chi d(f - \Phi) \wedge a_0 \right|^2
$$

$$
= 4 \int_{-\alpha}^{0} e^{-2\pi h} \frac{d+1}{4} (\text{detHess} f_+(z))^{-1/2} \chi^2(z, \eta) a_0^2(z, \eta) |\nabla f_-|(z, \eta) j(z, \eta) \, d\eta (1 + O(h))
$$

$$
= 2h(\pi h)^{\frac{d+1}{4}} (\text{detHess} f_+(z))^{-1/2} \chi^2(z, 0) a_0^2(z, 0) |\nabla f_-|(z, 0) j(z, 0) (1 + O(h))
$$

Since $\chi(z, 0) = 1$, $a_0(z, 0) = 1$, $\text{Hess} f_+(z) = \text{Hess} f|_{\partial\Omega}(z)$ and $j(z, 0) = 1$, this concludes the proof of (180), and thus of (176)-(177).

Now, writing

$$
u^{(1)}_{z, wkb} = e^{-\phi(z) / h} [d(f - \Phi) \wedge a(\cdot, h) + h \, da(\cdot, h)],
$$

and noticing that $\Phi - f(z) \geq 0$ on $\text{supp} \chi$, one has $\| \chi \nu^{(1)}_{z, wkb} \|_{H^1(\Omega)} \leq Ch^{-1}$. 

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It remains to prove the last statement. Using the fact that \( \text{supp}[\Delta^{(1)}_{f,h}, \chi] \subset \text{supp} \chi \), \( \Phi - f(z) = \Psi \geq c' > 0 \) on \( \text{supp} \nabla \chi \) and \((175)\), one gets
\[
\Delta^{(1)}_{f,h}(\chi u_{z,wkb}^{(1)}) = \chi \Delta^{(1)}_{f,h}(u_{z,wkb}^{(1)}) + |\Delta^{(1)}_{f,h}, \chi| u_{z,wkb}^{(1)} = O(h^{\infty}) + O(e^{-\frac{c}{T}}) = O(h^{\infty}).
\]
Therefore, from the Cauchy-Schwartz inequality, one has \( \langle \chi u_{z,wkb}^{(1)} \Delta^{(1)}_{f,h}(\chi u_{z,wkb}^{(1)}) \rangle_{L^2(\Omega)} = O(h^{\infty}) \). The fact that \( Q_{f,h}^{M,\Omega}(\Omega) \chi u_{z,wkb}^{(1)} = O(h^{\infty}) \) then follows from an integration by parts and the boundary conditions in \((175)\). □

### 4.4.2 A first estimate of the accuracy of the WKB approximation

Recall that \( z \in \{z_1, \ldots, z_n\} \) is a local minimum of \( f|\partial \Omega \) and that \( u_{z,wkb}^{(1)} \) is a \( L^2 \)-normalized eigenform of \( \Delta^{M,\Omega} f_{1}(\Omega) \) associated with the eigenvalue \( 0 \). The objective of this section is to prove that \( u_{h}^{(1)} \) is accurately approximated by the function \( u_{z,wkb}^{(1)} \) defined in the previous section. The computations below are inspired by those made in \([34, \text{Chapter 4}]\) where the authors were adapting \([30,35]\) to manifolds with boundary. The novelty is that we compare the two \( 1 \)-forms on a neighborhood of \( B_z \), instead of a neighborhood of \( z \).

Take two smooth open sets \( \Gamma_{St} \subset \Gamma'_{St} \subset \Gamma_1 \) which are strongly stable (see Definition \[1\] and Proposition \[57\]) and such that, for some positive \( \varepsilon \), \( (\Gamma_{St} + B(0, \varepsilon)) \cap \partial \Omega \subset \Gamma_{St} \) and \( (\Gamma'_{St} + B(0, \varepsilon)) \cap \partial \Omega \subset \Gamma_1 \), see Figure \((15)\). The fact that \( \Gamma_{St} \) and \( \Gamma'_{St} \) are stable and thus that \( V_{\Gamma_{St}} \) and \( V'_{\Gamma_{St}} \) are stable under the dynamics \((183)\)-see below- will actually be needed only to get refined estimates in Section \[4.4.3\]. Let us now consider the system of coordinate \((x', x_d)\) (see Definition \[9\]) which is well defined on \( V_{\Gamma_{St}} \) by assumption (see item 2 in Proposition \[58\]). Let us introduce the Lipschitz sets \( V_{\Gamma_{St}} \) and \( V'_{\Gamma_{St}} \)
\[
V_{\Gamma_{St}} = \{(x', x_d) \in \Gamma_{St} \times (-a, 0)\} \quad \text{and} \quad V'_{\Gamma_{St}} = \{(x', x_d) \in \Gamma'_{St} \times (-a', 0)\}
\]
where \( 0 < a < a' \) are small enough so that \( V_{\Gamma_{St}} \subset V'_{\Gamma_{St}} \subset V_{\Gamma_1} \). By construction, there exists \( \varepsilon > 0 \) such that \( V_{\Gamma_{St}} + B(0, \varepsilon) \subset V'_{\Gamma_{St}} \) and \( V'_{\Gamma_{St}} + B(0, \varepsilon) \subset V_{\Gamma_1} \cap (\hat{\Omega} \cup \Gamma_1) \) (see again Figure \[(15)\] for a schematic representation of these sets). In addition \( V_{\Gamma_{St}} \cap \Gamma_1 = \Gamma_{St} \) and \( V'_{\Gamma_{St}} \cap \Gamma_1 = \Gamma'_{St} \). Moreover, \( a \) and \( a' \) can be chosen sufficiently small so that the sets \( V_{\Gamma_{St}} \) and \( V'_{\Gamma_{St}} \) are stable under the dynamics
\[
x'(t) = \begin{cases} -\nabla \Phi(x(t)) & \text{on } \Omega \\ -\nabla_T \Phi(x(t)) & \text{on } \partial \Omega. \end{cases}
\]
This stability is a consequence of two facts. First, for \( x(t) \) solution to \((183)\), \( \frac{d}{dt} f_-(x(t))' = -|\nabla f_-(x(t))|^2 \) (since \( \nabla \Phi \cdot \nabla f_- = |\nabla f_-|^2 \) on \( V_{\Gamma_1} \), thanks to \((132)\), and \( \nabla_T \Phi \cdot \nabla f_- = 0 \) in \( \partial \Omega \) so that \( \forall t \geq 0, x_d(x(0)) \leq x_d(x(t)) \leq 0 \). Second, by construction, for sufficiently small \( a \) and \( a' \),
\[
\forall x \in \partial V_{\Gamma_{St}} \text{ such that } x'(x) \in \partial \Gamma_{St}, \nabla \Phi(x) \cdot n_x(V_{\Gamma_{St}}) > 0
\]
(where \( n(V_{\Gamma_{St}}) \) is the unit outward normal to \( V_{\Gamma_{St}} \)). Indeed for any \( z \in \partial \Gamma_{St}, \lim_{\sigma \to z} n_\sigma(V_{\Gamma_{St}}) = n_z(\Gamma_{St}) \) (where the limit is taken for \( \sigma \in \partial V_{\Gamma_{St}} \) with \( x'(\sigma) \in \partial \Gamma_{St} \), see \((140)\) for
a proof, and since \( \Gamma_{St} \) is chosen strongly stable, for \( z \in \partial \Gamma_{St} \), \( \nabla \Phi(z) \cdot n_z(V_{\Gamma_{St}}) = (\nabla f_+ + \nabla f_-) \cdot n_z(\Gamma_{St}) = \nabla f_+ \cdot n_z(\Gamma_{St}) = \nabla f|_{\partial \Omega} \cdot n_z(\Gamma_{St}) > 0 \). The argument is of course the same for \( V_{\Gamma_{St}} \).

Let us now introduce two smooth cut-off functions \( 0 \leq \chi \leq \eta \in C^\infty_c(\bar{\Omega} \cup \Gamma_1) \) satisfying

\[
\chi = 1 \quad \text{in a neighborhood of } \overline{V_{\Gamma_{St}}}, \quad \text{supp } \chi \subset V_{\Gamma_{St}} \tag{184}
\]

and \( \eta = 1 \) in a neighborhood of \( \overline{V_{\Gamma_{St}'}}, \quad \text{supp } \eta \subset V_{\Gamma_1} \cap (\bar{\Omega} \cup \Gamma_1). \tag{185} \)

Notice that by construction, \( \eta = 0 \) on \( \Gamma_2 \). In the following, we moreover assume that \( \chi \) and \( \eta \) are tensor products in the system of coordinates \((x', x_d)\) (this will actually be needed only to get refined estimates in Section 4.4.3):

\[
\chi(x', x_d) = \chi_1(x')\chi_d(x_d) \quad \text{and} \quad \eta(x', x_d) = \eta_1(x')\eta_d(x_d). \tag{186}
\]

Let \( \kappa \in \{ \chi, \eta \} \). Owing to Lemma 54, the 1-form \( \kappa_h^{(1)} \) belongs to \( \Lambda^1 H^1_f(\bar{\Omega}) \). The first a priori estimate on \( \kappa(u_h^{(1)} - c(h)u_{z,wkb}) \) is the following:

**Proposition 64.** Let us assume that the hypotheses \([H1], [H2] \) and \([H3] \) hold. For \( \kappa \in \{ \chi, \eta \} \), one has

\[
\left\| \kappa(u_h^{(1)} - c_z(h)u_{z,wkb}^{(1)}) \right\|_{H^1(\bar{\Omega})} = O(h^{\infty}) \tag{187}
\]

where

\[
c_z(h)^{-1} = (u_h^{(1)} , \chi z,wkb^{(1)})_{L^2(\Omega)}. \tag{188}
\]

The 1-form \( u_h^{(1)} \) can be chosen such that \( c_z(h) > 0 \). Additionally, when \( h \to 0 \)

\[
c_z(h) = C_z,wkb^{-1} h^{-\frac{d+1}{4}} \left(1 + O(h^{\infty})\right), \tag{189}
\]

where \( C_z,wkb \) is defined by (177).

Notice that \( |c_z(h)|^{-1} \) is equivalent (in the limit \( h \to 0 \)) to \( \| \kappa u_{z,wkb}^{(1)} \|_{L^2(\Omega)} \) (see (176)), and can thus be simply understood as a normalizing factor.

**Proof.** Let us first consider the case \( \kappa = \chi \), the other case is considered at the end of the proof. One defines \( k(h) := (u_h^{(1)} , \chi z,wkb^{(1)})_{L^2(\Omega)} \in \mathbb{R} \). If \( k(h) < 0 \), then one changes \( u_h^{(1)} \) to \( -u_h^{(1)} \) so that one can suppose without loss of generality that

\[
k(h) \geq 0.
\]

For \( h \) small enough, one has (from Proposition 59, item (iii))

\[
\pi_{[0,ch^{3/2}])(\Delta_{f,h}^{M(1)}(\hat{\Omega}))(\chi z,wkb^{(1)}) = k(h)u_h^{(1)}.
\]

Le us define

\[
\alpha_h := \chi(u_{z,wkb}^{(1)} - k(h)u_h^{(1)}).
\]

Thus, the following identity holds for \( h \) small enough

\[
\alpha_h = k(h)(1 - \chi) u_h^{(1)} + \pi_{[h^{3/2},+\infty]}(\Delta_{f,h}^{M(1)}(\hat{\Omega}))(\chi z,wkb^{(1)}).
\]

Notice that, using Cauchy-Schwarz inequality and Lemma 63, there exist \( C > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \)

\[
|k(h)| \leq Ch^{\frac{d+1}{4}}.
\]

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Therefore, using Lemma 15, Proposition 61 and Lemma 63 we get
\[ \| \alpha_h \|_{L^2(\Omega)}^2 \leq 2k(h)^2 \| (1 - \chi) u_h^{(1)} \|_{L^2(\Omega)}^2 + 2\| \pi_{\left[ ch^{3/2}, +\infty \right]} \left( \Delta_{f,h}^{M,(1)}(\Omega) \right) (\chi u_{z,wkb}^{(1)}) \|_{L^2(\Omega)}^2 \]
\[ \leq C h^{d+1}\| (1 - \chi) u_h^{(1)} \|_{L^2(\Omega)}^2 + C h^{-3/2} Q_{f,h}^{M,(1)}(\Omega)(\chi u_{z,wkb}^{(1)}) \]
\[ \leq C h^{d+1} h^{-N_0} e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}} + C h^{-3/2} Q_{f,h}^{M,(1)}(\Omega)(\chi u_{z,wkb}^{(1)}) \]
\[ = O(h^{\infty}), \]
with \( c := \inf_{\text{supp}(1 - \chi)} \Psi > 0 \) (since \( \chi = 1 \) near \( z \)) and the integer \( N_0 \) is given by Proposition 61. Moreover, since \( d_{f,h} = h d + df \wedge \) and \( d_{f,h}^* = h d^* + i \nabla f \), one obtains using the triangular inequality, the Gaffney inequality (116) (since Proposition 61). Moreover, since \( C \), considering \( C_1 \) is given by (177) in Lemma 63. Therefore, since \( Q_{f,h}^{M,(1)}(\Omega)(\chi u_{z,wkb}^{(1)}) = O(e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}}) \) (from the Agmon estimate (147)) and \( Q_{f,h}^{M,(1)}(\Omega)(\chi u_{z,wkb}^{(1)}) = O(h^{\infty}) \),
\[ \| \alpha_h \|_{H^1(\Omega)} \leq C(\| d\alpha_h \|_{L^2(\Omega)}^2 + \| d^* \alpha_h \|_{L^2(\Omega)}^2 + \| \alpha_h \|_{L^2(\Omega)}^2) \]
\[ \leq C h^{-2} \left( Q_{f,h}^{M,(1)}(\Omega)(\alpha_h) + \| \alpha_h \|_{L^2(\Omega)}^2 \right) \]
\[ = O(h^{\infty}). \]
Moreover since \( \| \chi u_{h}^{(1)} \|_{L^2(\Omega)} = 1 + O(e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}}) \) (from the Agmon estimate (147)), by considering \( \| \chi (u_{z,wkb}^{(1)} - k(h) u_{h}^{(1)}) \|_{L^2(\Omega)} = O(h^{\infty}) \), one gets:
\[ k(h)^2 = \frac{\| \chi u_{z,wkb}^{(1)} \|_{L^2(\Omega)}^2}{\frac{\| \chi u_{h}^{(1)} \|_{L^2(\Omega)}^2 - 2}{c z h}} + O(h^{\infty}) \]
\[ = \frac{C_{z,wkb}^2 h^{d+1} + O(h^{\infty})}{1 + O(e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}})}, \]
with \( C_{z,wkb} \) is given by (177) in Lemma 63. Therefore, since \( k(h) \geq 0, k(h) = C_{z,wkb} h^{d+1} (1 + O(h^{\infty})) \). This concludes the proof of (187) for \( \kappa = \chi \), by choosing \( c_z(h) := k(h)^{-1} \).

Let us now deal with the case \( \kappa = \eta \). There exists \( c > 0 \) such that, for \( h \) sufficiently small,
\[ \| \eta(u_{z,wkb}^{(1)} - k(h) u_{h}^{(1)}) \|_{H^1(\Omega)} \leq \| \alpha_h \|_{H^1(\Omega)} + \| (\eta - \chi)(u_{z,wkb}^{(1)} - k(h) u_{h}^{(1)}) \|_{H^1(\Omega)} \]
\[ \leq O(h^{\infty}) + \| (\eta - \chi) u_{z,wkb}^{(1)} \|_{H^1(\Omega)} + |k(h)| \| (\eta - \chi) u_{h}^{(1)} \|_{H^1(\Omega)} \]
\[ \leq O(h^{\infty}) + e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}}. \]
The last inequality is the consequence of two facts. First, \( \| (\eta - \chi) u_{h}^{(1)} \|_{H^1(\Omega)} = e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}} \) thanks to Proposition 61 and (116) together with the fact that \( \chi = \eta \) near \( z \). Second, a direct computation shows that
\[ \| (\eta - \chi) u_{z,wkb}^{(1)} \|_{H^1(\Omega)} \leq C h^{-1} e^{-\frac{\| \text{supp}(\eta - \chi) \|_{L^2}}{h}} \leq e^{-\frac{\| \alpha_h \|_{L^2(\Omega)}^2}{2}}. \]
This concludes the proof of Proposition 64.

The estimate we obtained in Proposition 64 is sufficient to get the result of Theorem 1. The more precise estimates on Section 4.4.3 are only needed to prove Theorem 2.
4.4.3 A more accurate comparison on the WKB approximation

The objective of this section is to combine the techniques used to obtain the Agmon estimates of Proposition 61 and the first estimate of the accuracy of the WKB approximation of Proposition 64 in order to obtain a more precise estimate of the latter.

Let us start with estimates which are simple consequences of Proposition 61 and Proposition 64. Notice that, for \( \kappa \in \{ \chi, \eta \} \), one obviously gets from Proposition 61 the following relation in \( \Lambda^1 H^1(\Omega) \):

\[
\exists N_0 \in \mathbb{N}, e^\Phi \kappa (u_h^{(1)} - c_z(h)z_{wkb}^{(1)}) = O(h^{-N_0}).
\]  

(190)

For the term involving \( z_{wkb}^{(1)} \), this is due to \( \Psi(x) = \Phi(x) - f(z) \) on supp \( \kappa \) and the estimate \([189]\) on \( c_z(h) \). Let us now set

\[
w_h := \kappa(u_h^{(1)} - c_z(h)z_{wkb}^{(1)}).
\]  

(191)

The 1-form \( w_h \) is in \( C^\infty_c(\bar{\Omega} \cup \Gamma_1) \) and satisfies in \( \Omega \):

\[
\Delta_{f,h} w_h = \kappa \Delta_{f,h} (u_h^{(1)} - c_z(h)z_{wkb}^{(1)}) + [\Delta_{f,h}, \kappa](u_h^{(1)} - c_z(h)z_{wkb}^{(1)})
\]

\[
= -c_z(h)\kappa \Delta_{f,h} z_{wkb}^{(1)} + [\Delta_{f,h}, \kappa](u_h^{(1)} - c_z(h)z_{wkb}^{(1)})
\]

\[
= (r_1 + r_1')e^{-\frac{\Phi}{\kappa}},
\]  

(192)

where, owing to \([175]\) and \([189]\):

\[
r_1 := -e^\Phi c_z(h)\kappa \Delta_{f,h} z_{wkb}^{(1)} = O(h^\infty)
\]  

(193)

in \( \Lambda^1 L^2(\Omega) \) and, from \([190]\):

\[
r_1' := e^\Phi [\Delta_{f,h}, \kappa](u_h^{(1)} - c_z(h)z_{wkb}^{(1)}) = O(h^{-N_0}) \text{ in } \Lambda^1 L^2(\Omega)
\]

supp \( r_1' \subset \text{supp } \nabla \kappa \).  

(194)

Additionally, one gets similarly on the boundary \( \Gamma_1 \):

\[
tw_h|_{\Gamma_1} = 0 \text{ and } td_{f,h} w_h|_{\Gamma_1} = (r_2 + r_2')e^{-\frac{\Phi}{\kappa}} = (r_2 + r_2')e^{-\frac{f(z)}{\kappa}},
\]

where owing to \([175]\) and \([189]\):

\[
r_2 := te^\Phi \kappa d_{f,h}^h (u_h^{(1)} - c_z(h)z_{wkb}^{(1)})|_{\Gamma_1} = -te^\Phi \kappa c_z(h) d_{f,h}^h z_{wkb}^{(1)}|_{\Gamma_1} = O(h^\infty)
\]  

(195)

in \( L^2(\partial \Omega) \) and

\[
r_2' := te^\Phi h i \nabla \kappa (u_h^{(1)} - c_z(h)z_{wkb}^{(1)})|_{\Gamma_1} = O(h^{-N_0}) \text{ in } L^2(\partial \Omega) \text{ with supr } r_2' \subset \Gamma_1 \cap \text{supp } \nabla \kappa
\]  

(196)

We are now in position to prove the following proposition.

**Proposition 65.** Let us assume that the hypotheses \([H1]\), \([H2]\) and \([H3]\) hold. One has the following estimate in the limit \( h \to 0 \):

\[
\left\| e^\Phi (u_h^{(1)} - c_z(h)z_{wkb}^{(1)}) \right\|_{H^1(\Gamma_{St})} = O(h^\infty).
\]  

(197)

where \( c_z(h) \) is defined by \([188]\) and where, we recall, \( \Psi(x) = d_a(x, z) \) and \( V_{St} \) is defined by \([182]\).  

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Proof. As for the proof of Proposition 61 one first proves an estimate along the boundary $\Gamma_1$ before propagating it in $V_{\Gamma St}$.

Step 1. Comparison in $\Gamma_1$. Let us consider $w_h$ defined by (191) and the cut-off function $\kappa = \eta$ defined in (185). Like in the first step of the proof of Proposition 61, we are going to prove an estimate of the form (197) with $\Psi$ replaced by $f_+$. More precisely, we want to show that

$$\left\| \frac{f_+}{h} w_h \right\|_{H^1(V_{\Gamma St})} = \left\| \frac{f_+}{h} (u^{(1)}_h - c_z(h)u^{(1)}_{z,wkb}) \right\|_{H^1(V_{\Gamma St})} = O(h^\infty),$$

which implies in particular the following estimate along the boundary, since $f_+ = f - f(z)$ in $\Gamma_1$,

$$\left\| \frac{f_+}{h} (u^{(1)}_h - c_z(h)u^{(1)}_{z,wkb}) \right\|_{H^{1/2}(\Gamma_{St})} = O(h^\infty).$$

In the following, we denote (see Figure 15 for a schematic representation of the set $V_\eta$) $V_\eta = \text{supp} \eta$.

In the system of coordinates $(x', x_d)$, $x \in V_\eta$ if and only if $x'(x) \in \text{supp} \eta_1$ and $x_d(x) \in \text{supp} \eta_d$. We recall that $V_\eta$ is a compact set of $\Omega \cup \Gamma_1$. As for the proof of Proposition 61, we introduce the sets

$$\Omega_- = \{ x \in V_\eta \text{ s.t. } f_+(x) \leq C h \} \text{ and } \Omega_+ = V_\eta \setminus \Omega_-,$$

and define the Lipschitz function $\varphi : V_\eta \to \mathbb{R}$ by

$$\varphi = \begin{cases} f_+ - C h \ln \frac{f_+}{h} & \text{if } f_+ > C h, \\ f_+ - C h \ln C & \text{if } f_+ \leq C h, \end{cases}$$

for some constant $C > 1$ that will be fixed at the end of this step. Notice for further purposes that

$$\lim_{h \to 0} \| \varphi - f_+ \|_{L^\infty(V_\eta)} = 0. \quad (200)$$

We recall that in the system of coordinates $(x', x_d)$, $\varphi$ is independent of $x_d$.

The reasoning below is based on [35], see also [30] p. 49–52 for a presentation in the case without boundary. According to (198), we want to get an error of the form $O(h^N)$ with $N$ arbitrary. We are not going to work with the above phase function $\varphi$ as we did in the proof of Proposition 61, but with a phase function $\varphi_N$ also depending on some arbitrary $N \in \mathbb{N}$. Let us define

$$\tilde{w}_h = e^{\varphi_N} w_h = e^{\varphi_N} \eta (u^{(1)}_h - c_z(h)u^{(1)}_{z,wkb}).$$

Combining the integration by parts formula (148) (with $u = w_h$ and $\varphi = \varphi_N$) with the Green formula (115) (with $u = w_h$ and $v = e^{\varphi_N} w_h$) leads to the estimate

$$\left\| e^{\varphi_N} \Delta_{f,h}^{(1)} w_h \right\|_{L^2(V_\eta)} \left\| \tilde{w}_h \right\|_{L^2(V_\eta)} + \frac{e^{\varphi_N}}{h} \left\| d_{f,h}^* w_h \right\|_{L^2(C_\Gamma)} \left\| \tilde{w}_h \right\|_{L^2(C_\Gamma)} \geq \left\| h d \tilde{w}_h \right\|_{L^2(V_\eta)}^2 + \left\| h d_{f,h}^* \tilde{w}_h \right\|_{L^2(V_\eta)}^2 - h \int_{C_\Gamma} \langle \tilde{w}_h, \tilde{w}_h \rangle_{T^*_\partial h} \delta_{\partial h} f d\sigma$$

$$+ \left( |\nabla f|^2 - |\nabla \varphi_N|^2 + h L_{\nabla f} + h L_{\nabla f}^* \right) \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)}. \quad (201)$$

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Let us explain formally how the function $\varphi_N$ is chosen. Roughly speaking, using similar arguments as in the proof of Proposition 61, it is natural to choose $\varphi_N = \varphi + Nh \ln \frac{1}{h}$ and to try to prove that the left-hand side of (201) is bounded from above by $O(h^{-N_1}) \| \tilde{w}_h \|_{H^1(V_h)}$ for some $N_1 \in \mathbb{N}$ independent of $N$. This would indeed lead to an estimate of the form $\| \tilde{w}_h \|_{H^1(V_h)} = O(h^{-N_1})$ (for some maybe larger $N_1$) and finally to the desired estimate on $w_h$ since $\| \tilde{w}_h \|_{H^1(V_h)} \simeq h^{-N} \| e^{\frac{\tau}{\Omega}} w_h \|_{H^1(V_h)}$. To get this upper bound, a trace theorem and (192)–(196) yield the following estimate from (201):

$$\| e^{\frac{\varphi_N - \psi}{h}} \|_{L^\infty(V_h)} O(h^\infty) + \| e^{\frac{\varphi_N - \psi}{h}} \|_{L^\infty(\text{supp}(\eta_h))} O(h^{-N_1}) \geq \| \tilde{w}_h \|_{H^1(V_h)} \simeq h^{-N} \| e^{\frac{\tau}{\Omega}} w_h \|_{H^1(V_h)}$$

for some $N_1 \in \mathbb{N}$ independent of $N$. It can be checked that $\| e^{\frac{\varphi_N - \psi}{h}} \|_{L^\infty(V_h)} = \| e^{\frac{\varphi_N - \psi}{h}} \|_{L^\infty(\text{supp}(\eta_h))} = O(h^{-N})$ so that the first term is well controlled, but the second one is of order $O(h^{-N_1-N})$. These relations suggest a choice of $\varphi_N$ satisfying $\varphi_N \leq f_+ - \Psi$ on $\text{supp} \eta$ so that $\| e^{\frac{\varphi_N - \psi}{h}} \|_{L^\infty(\text{supp}(\eta_h))} = O(1)$. This would yield the desired estimate $\| e^{\frac{\tau}{\Omega}} w_h \|_{H^1(V_h)} = O(h^{N-N_1})$.

Let us now enter the rigorous proof. The above considerations (see also [30] p. 49–52) lead to define, for any $N \in \mathbb{N}$,

$$\varphi_N = \min \left\{ \varphi + Nh \ln \frac{1}{h}, \psi \right\},$$

(202)

where the Lipschitz function $\psi : V_\eta \to \mathbb{R}$ is defined by the following relation, for some $\varepsilon \in (0, 1)$ that will be specified below:

$$\psi(x', x_d) = \psi(x', 0) = \min \left\{ \varphi(y', 0) + (1 - \varepsilon) d_{a}^{\Omega}(x', y'), \ y' \in \text{supp} \eta_1 \right\}.$$  

(203)

Here, $d_{a}^{\Omega}(x', y')$ denotes the Agmon distance associated with $f|_{\partial \Omega}$ between $x'$ and $y'$ along the boundary (see Definition 7), i.e. the distance induced by the metric $|\nabla (f|_{\partial \Omega})|^2 ds^2$, where $ds^2$ denotes the restriction of the Euclidean metric to the boundary $\partial \Omega$.

Step 1-a: Preliminary estimates on $\varphi_N$.

Let us first show that there exists $\varepsilon \in (0, 1)$ such that for any $h \in (0, h_0(N, \varepsilon))$ with $h_0 = h_0(N, \varepsilon)$ small enough,

$$\varphi_N = \varphi + Nh \ln \frac{1}{h} < \psi \quad \text{in} \quad V_\eta \cap \left\{ x' \in \overline{\Gamma_{St}} \right\}.$$  

(204)

The proof of (204) is as follows. From (79) applied to $d_{a}^{\Omega}$,

$$\forall (x', y') \in \overline{\Gamma_{St}} \times \text{supp} \eta_1, \quad f_+(x', 0) < f_+(y', 0) + d_{a}^{\Omega}(x', y').$$  

(205)

The inequality above is strict since if $f_+(x', 0) = f_+(y', 0) + d_{a}^{\Omega}(x', y')$ for some $(x', y') \in \overline{\Gamma_{St}} \times \text{supp} \eta_1$, then there exists a generalized integral curve (in the sense of Definition 8) of $-\nabla (f|_{\partial \Omega}) = -\nabla f_+$ joining $x' \in \overline{\Gamma_{St}}$ to $y' \in \text{supp} \eta_1$ (this is a consequence of Corollary 18 applied to the Agmon distance $d_{a}^{\Omega}$ on $\partial \Omega$ rather than the Agmon distance $d_a$ in $\Omega$). But since $\Gamma_{St}$ is strongly stable, any integral curve of $-\nabla f_+$ remains in $\overline{\Gamma_{St}}$, and thus cannot reach $y'$ which is not in $\overline{\Gamma_{St}}$ (see (185)).

From the strict inequality (205), there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0)$,

$$\forall (x', y') \in \overline{\Gamma_{St}} \times \text{supp} \eta_1, \quad f_+(x', 0) \leq f_+(y', 0) + (1 - \varepsilon) d_{a}^{\Omega}(x', y'),$$  

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and thus, considering the limit $h \to 0$ (see (200)) and the infimum over $y' \in \text{supp } \nabla \eta_1$ of the right-hand side, there exists $\varepsilon > 0$ used to define $\psi$ (see (203)) and such that, for sufficiently small $h$,
\[ \forall x' \in \overline{\Gamma_{\mathcal{St}}}, \quad f_+(x', 0) < \psi(x', 0). \]

Moreover, since $\lim_{h \to 0} \| \varphi + Nh \ln \frac{1}{h} - f_+ \|_{L^\infty(V_\eta)} = 0$ (thanks to (200)), one obtains for $h$ small enough, $\forall x' \in \overline{\Gamma_{\mathcal{St}}}, \varphi(x', 0) + Nh \ln \frac{1}{h} < \psi(x', 0)$, and by definition of $\varphi_N$, $\varphi_N(x', 0) = \varphi(x', 0) + Nh \ln \frac{1}{h}$ which leads to (204) for $x = (x', 0)$, with $x' \in \overline{\Gamma_{\mathcal{St}}}$. The fact that $\varphi_N$ and $\psi$ does not depend on $x_d$ in the system of coordinates $(x', x_d)$ concludes the proof of (204).

Let us now prove that\[ \exists M < \frac{1}{1 - \varepsilon}, \forall x \in V_\eta, |\nabla \psi(x)| \leq M (1 - \varepsilon) |\nabla f_+(x)|. \tag{206} \]
The triangular inequality applied to $d^\partial\Omega_a$ leads to the relation (since $\psi(x', x_d)$ does not depend on $x_d$)
\[ \forall x, y \in V_\eta, |\psi(x', x_d) - \psi(y', y_d)| \leq (1 - \varepsilon)d^\partial\Omega_a(x', y'). \tag{207} \]
where we denote $(x', x_d)$ (resp. $(y', y_d)$) the coordinates of $x$ (resp. $y$) in the system of coordinates (129). Let us first show that (207) implies that for a.e. $x' \in V_\eta \cap \partial \Omega$, $|\nabla (\psi|\partial \Omega)(x')| \leq (1 - \varepsilon) \left| \nabla (f|\partial \Omega)(x') \right| = (1 - \varepsilon) \left| \nabla (f_+|\partial \Omega)(x') \right|$. \tag{208}

Indeed, let us consider a local parametrization in $\mathbb{R}^{d-1}$ of a neighborhood in $\partial \Omega$ of a point $x' \in \partial \Omega$. In this local chart, let us consider $y_a = x' + \alpha \frac{\nabla (\psi|\partial \Omega)}{|\nabla (\psi|\partial \Omega)|}(x')$. One has, in the limit $\alpha \to 0$, $\psi(y'_a, 0) - \psi(x', 0) = \alpha |\nabla (\psi|\partial \Omega)(x')| + o(\alpha)$ and likewise using the inequality (78) applied to $d^\partial\Omega_a$ (see also [19, p. 53]) $d^\partial\Omega_a(x', y'_a) - d^\partial\Omega_a(x', x') \leq \alpha |\nabla (f|\partial \Omega)(x')| + o(\alpha)$.

By considering the limit $\alpha \to 0$, one thus deduces (208) from (207). Now, one can check that, uniformly in $x' \in V_\eta \cap \partial \Omega$,
\[ \lim_{x_d \to 0} |\nabla \psi(x', x_d)| = |\nabla (\psi|\partial \Omega)(x')| \quad \text{and} \quad \lim_{x_d \to 0} |\nabla f_+(x', x_d)| = |\nabla (f_+|\partial \Omega)(x')|. \tag{209} \]

Indeed, using the fact that $\psi$ does not depend on $x_d$, one first has almost everywhere (see (130)) $|\nabla \psi(x', x_d)| = |\nabla (\psi|\Sigma_{x_d})(x')|$ where $\forall a > 0$, $\Sigma_a = \{ x \in V_\eta, x_d(x) = a \}$ is endowed with the Riemannian structure induced by the Riemannian structure in $\Omega$. Now, let us consider the smooth diffeomorphism $\Gamma_{x_d} : \Sigma_{x_d} \to \partial \Omega$ such that for all $x = (x', x_d) \in \Sigma_{x_d}$, $\Gamma_{x_d}(x) = (x', 0) \in \partial \Omega$. The result (209) on $\psi$ is then a consequence of the fact that $\psi|\partial \Omega \circ \Gamma_{x_d} = \psi|\Sigma_{x_d}$ and $\lim_{x_d \to 0} \| \Gamma_{x_d} - \text{Id} \|_{W^{1, \infty}(\Sigma_{x_d})} = 0$ so that the Jacobian associated to the change of metric from $\Sigma_{x_d}$ to $\partial \Omega$ converges to Id, uniformly on $\Sigma_{x_d}$. The same reasoning show that (209) also holds for $f_+$ since $f_+$ does not depend on $x_d$.

By combining (208) and (209), one obtains (206) for some $M > 1$. Moreover, $M$ can be chosen as close to 1 as needed, up to modifying $\eta$ (and thus $V_{\mathcal{St}} \subset V_{\mathcal{St}} \subset V_\eta$) such that for all $x \in V_\eta$, $\| \Gamma_{x_d} - \text{Id} \|_{W^{1, \infty}(\Sigma_{x_d})}$ is as close to 0 as needed.
Let us finally mention the following inequalities, valid for $h \in (0, h_0)$ with $h_0 = h_0(N, \varepsilon) > 0$ small enough:

$$\varphi_N \leq f_+ + Nh \ln \frac{1}{h} \leq \Psi + Nh \ln \frac{1}{h} \quad \text{in } V_\eta \quad \text{(210)}$$

and (195),

$$\varphi_N = \psi \leq \varphi \leq f_+ \leq \Psi \quad \text{in } V_\eta \cap \{x' \in \supp \nabla \eta \}$$

and since $\Psi = f_+ + f_- > f_+$ on $\{x_d \in \supp \eta_d'\}$, one has

$$\varphi_N \leq f_+ + Nh \ln \frac{1}{h} \leq \Psi \quad \text{in } V_\eta \cap \{x_d \in \supp \eta_d'\}. \quad \text{(212)}$$

Step 1b: Proof of (198).

We are now ready to prove (198). Controlling the left-hand side of (201) using the relations (192)–(196) gives

$$\left\| (r_1 + r_1') e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(V_\eta)} \| \tilde{w}_h \|_{L^2(V_\eta)} + \left\| (r_2 + r_2') e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(\Gamma_1)} \| \tilde{w}_h \|_{L^2(\Gamma_1)} \geq \| hd\tilde{w}_h \|_{L^2(V_\eta)}^2 + \| hd^* \tilde{w}_h \|_{L^2(V_\eta)}^2 - h \int_{\Gamma_1} \langle \tilde{w}_h, \tilde{w}_h \rangle \Omega \tilde{\tau} \Omega \partial_{\tilde{h}} f d\sigma + \langle (|\nabla f|^2 - |\nabla \varphi N|^2 + h L \nabla f + h L \nabla \varphi N) \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)},$$

where, since $\varphi_N - \Psi \leq Nh \ln \frac{1}{h}$ (by (210)) and $r_i = O(h^\infty)$ for $i \in \{1, 2\}$ (by (193) and (195)),

$$\left\| r_1 e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(V_\eta)} + \left\| r_2 e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(\Gamma_1)} = O(h^\infty),$$

and, since $\varphi_N \leq \Psi$ on $\supp \nabla \eta$ (by (211)–(212)) and $supp r_i' \subset supp \nabla \eta$ for $i \in \{1, 2\}$ (by (194) and (196)),

$$\left\| r_1' e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(V_\eta)} + \left\| r_2' e^{\frac{\varphi_N - \varphi}{h}} \right\|_{L^2(\Gamma_1)} = O(h^{-N_0}).$$

This leads to the existence of $C_1 = C_1(N) > 0$ such that for $h$ small enough:

$$C_1 h^{-N_0} \| \tilde{w}_h \|_{H^1(V_\eta)} \geq \| hd\tilde{w}_h \|_{L^2(V_\eta)}^2 + \| hd^* \tilde{w}_h \|_{L^2(V_\eta)}^2 - h \int_{\Gamma_1} \langle \tilde{w}_h, \tilde{w}_h \rangle \Omega \tilde{\tau} \Omega \partial_{\tilde{h}} f d\sigma + \langle (|\nabla f|^2 - |\nabla \varphi N|^2 + h L \nabla f + h L \nabla \varphi N) \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)}.$$ 

Since $\varphi_N \leq \varphi + Nh \ln \frac{1}{h}$, $\varphi \leq C h$ on $\Omega_-$ and $\| w_h \|_{H^1(\Omega_-)} = O(h^\infty)$ (see (187))

$$\| \tilde{w}_h \|_{L^2(\Omega_-)} \leq e^{C h^{-N}} \| w_h \|_{L^2(\Omega_-)} \leq C_2(C, N). \quad \text{(213)}$$

Thus, since $\mathcal{L} \nabla f + \mathcal{L}^e \nabla \varphi N$ is a 0th order differential operator, we get the existence of $C_3 > 0$ independent of $(C, N)$ and of $C_4 = C_4(C, N)$ such that:

$$C_4 h^{-N_0} \| \tilde{w}_h \|_{H^1(V_\eta)} \geq \| hd\tilde{w}_h \|_{L^2(V_\eta)}^2 + \| hd^* \tilde{w}_h \|_{L^2(V_\eta)}^2 - h \int_{\Gamma_1} \langle \tilde{w}_h, \tilde{w}_h \rangle \Omega \tilde{\tau} \Omega \partial_{\tilde{h}} f d\sigma + \langle (|\nabla f|^2 - |\nabla \varphi N|^2 + C_3 h \tilde{w}_h^2) \tilde{w}_h, \tilde{w}_h \rangle_{L^2(\Omega_+)}.$$ 

Moreover, by definition of $\varphi_N$, a.e. in $\Omega_+$, $\nabla \varphi_N = \nabla \psi 1_{\{\varphi_N = \psi\}} + \nabla f_+(1 - \frac{C h}{f_+}) 1_{\{\varphi_N < \psi\}}$. Now,
• On \( \{ \varphi_N = \psi \} \), since by (204) \( \{ \varphi_N = \psi \} \) avoids a neighborhood of \( \{(z, x_d), x_d \in \text{supp } \eta_d \} = \{ x \in V_\eta, |\nabla f_+(x)| = 0 \} \) (see (133)), we get
  \[
  |\nabla f_+|^2 - |\nabla \varphi_N|^2 \geq (1 - M^2(1 - \epsilon)^2) |\nabla f_+|^2 \geq c_\epsilon > 0,
  \]
  where (206) have been used;

• On \( \{ \varphi_N < \psi \} \cap \Omega_+ \), we get like in the proof of Proposition 61 (see (150) and (152)),
  \[
  |\nabla f_+|^2 - |\nabla \varphi_N|^2 \geq KCh.
  \]

Choosing \( C > \max(1, \frac{C_8}{K}) \), we obtain that for \( h \) small enough:

\[
C_4(h^{-N_0} \| \tilde{w}_h \|_{H^1(V_\eta)} + 1) \geq \|hd\tilde{w}_h\|_{L^2(V_\eta)}^2 + \|hd^*\tilde{w}_h\|_{L^2(V_\eta)}^2 - h \int_{\Gamma_1} (\tilde{w}_h, \tilde{w}_h)_{T \Omega} \partial_n f \, d\sigma
\]

\[
+ \langle |\nabla f_-| \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)} + (KC - C_3)h \left( \|\tilde{w}_h\|_{L^2(V_\eta)}^2 - \|\tilde{w}_h\|_{L^2(\Omega_-)}^2 \right)
\]

(214)

We can now control from below the r.h.s. of the above estimate exactly as we did at the end of the first step of Proposition 61 defining \( C_5(C) := \frac{KC - C_3}{h^2 \|f - \tilde{t}\|_{L^\infty(V_\eta)}} \) (see (131)), one gets the inequality

\[
(KC - C_3)h \|\tilde{w}_h\|_{L^2(V_\eta)}^2 + \langle |\nabla f_-| \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)} \geq (1 + 2C_5h) \langle |\nabla f_-| \tilde{w}_h, \tilde{w}_h \rangle_{L^2(V_\eta)}
\]

(215)

and from Lemma 56 applied with \( u = \tilde{w}_h \), \( f = -\tilde{t}f_- \) where \( \tilde{t} \in C^\infty(\bar{\Omega}, [0, 1]) \), \( \tilde{t} = 1 \) on \( \text{supp } \eta \), \( \text{supp } \tilde{t} \subset (\text{supp } \eta + B(0, \alpha)) \cap \Omega \) for \( \alpha > 0 \) such that \( f_- \) is smooth on \( \text{supp } \tilde{t} \) and \( \chi_{1 + C_5h} \) instead of \( h \), one gets the following lower bound:

\[
-h \int_{\Gamma_1} (\tilde{w}_h, \tilde{w}_h) \partial_n f \, d\sigma = h \int_{\Gamma_1} (\tilde{w}_h, \tilde{w}_h) \partial_n f_- \, d\sigma
\]

\[
\geq -(1 + C_5h) \left( |\nabla f_-| \tilde{w}_h \right)_{L^2(V_\eta)}^2
\]

\[
- \frac{h^2}{1 + C_5h} \left( \|d\tilde{w}_h\|_{L^2(V_\eta)}^2 + \|d^*\tilde{w}_h\|_{L^2(V_\eta)}^2 \right) - C_6h \|\tilde{w}_h\|_{L^2(V_\eta)}^2
\]

(216)

where \( C_6 \) is some positive constant independent of \( C \) (it only depends on \( f_- \)). Injecting the estimates (215) and (216) in (214) then leads to:

\[
C_4(h^{-N_0} \| \tilde{w}_h \|_{H^1(V_\eta)} + 1) \geq \frac{C_5h^3}{1 + C_5h} \left( \|d\tilde{w}_h\|_{L^2(V_\eta)}^2 + \|d^*\tilde{w}_h\|_{L^2(V_\eta)}^2 \right)
\]

\[
+ C_5h \|\nabla f_-| \tilde{w}_h \|_{L^2(V_\eta)}^2 - (KC - C_3)h \|\tilde{w}_h\|_{L^2(\Omega_-)}^2 - C_6h \|\tilde{w}_h\|_{L^2(V_\eta)}^2.
\]

Then, since \( |\nabla f_-| \geq c > 0 \) on \( \bar{V}_\eta \) (see (131)), \( \lim_{C \to \infty} C_5(C) = +\infty \). Therefore, since \( C_6 \) is independent of \( C \), one can choose \( C \) such that \( c^2C_5 - C_6 > 0 \), which implies, remembering also \( \|\tilde{w}_h\|_{L^2(\Omega_-)} \leq C_2(C, N) \) (see (213)), the existence of a constant \( C_7 > 0 \) and a constant \( h_0 > 0 \) such that, for every \( h \in (0, h_0) \),

\[
\|\tilde{w}_h\|_{L^2(V_\eta)}^2 + \|d\tilde{w}_h\|_{L^2(V_\eta)}^2 + \|d^*\tilde{w}_h\|_{L^2(V_\eta)}^2 \leq \frac{C_7}{h^3} (h^{-N_0} \| \tilde{w}_h \|_{H^1(V_\eta)} + 1).
\]

According to Gaffney’s inequality (116), this finally leads to the existence of a positive constant \( C_8 \) such that

\[
\|\tilde{w}_h\|_{H^1(V_\eta)} \leq C_8h^{-N_0 - \delta}.
\]
Moreover, according to \(204\), we have \(\varphi_N = \varphi + Nh \ln \frac{1}{h} \) in \(V_\eta \cap \left\{ x' \in \Gamma_{St}^\prime \right\} \) and then
\[
\varphi_N - Nh \ln \frac{1}{h} - f_+ \geq -C_9 h \ln \frac{1}{h} \quad \text{(with a constant } C_9 \text{ independent of } N) \quad \text{in } V_\eta \cap \left\{ x' \in \Gamma_{St}^\prime \right\}.
\]
Therefore, there exists \(N_1\) independent of \(N\) such that for \(h\) small enough,
\[
\left\| \frac{f_+}{h} \right\|_{H^1(V_{\Gamma_{St}}^\prime)} \leq C_N h^{N-N_1},
\]
from which \(198\) and \(199\) follow since \(N\) is arbitrary.

Step 2: Comparison in \(V_{\Gamma_{St}}^\prime\).

We work now with the cut-off function \(\chi\) defined in \(184\). Recall that \(\eta \chi = \chi\). Similarly as in the previous step, let us define the sets
\[
\Omega_- = \left\{ x \in V_{\Gamma_{St}}^\prime \text{ s.t. } \Psi(x) \leq Ch \right\} \quad \text{and} \quad \Omega_+ = V_{\Gamma_{St}}^\prime \setminus \Omega_-,
\]
and the function
\[
\varphi_N = \min \left\{ \varphi + Nh \ln \frac{1}{h}, \psi \right\},
\]
where \(\varphi\) and \(\psi\) are respectively defined by
\[
\varphi = \begin{cases} 
\Psi - Ch \ln \frac{\Psi}{h} & \text{if } \Psi > Ch \\
\Psi - Ch \ln C & \text{if } \Psi \leq Ch,
\end{cases}
\]
and
\[
\psi(x) = \min \left\{ \varphi(y) + (1 - \varepsilon)d_a(x, y), \quad y \in \text{supp } \nabla \chi \right\}.
\]
The constant \(C > 1\) will be chosen at the end of the proof. Following the proof of \(204\), there exists \(\varepsilon \in (0, 1)\) such that for any \(h \in (0, h_0)\) with \(h_0 = h_0(N, \varepsilon)\) small enough,
\[
\varphi_N = \varphi + Nh \ln \frac{1}{h} < \psi \quad \text{in } V_{\Gamma_{St}}^\prime. \quad (217)
\]
Indeed, using the fact that \(\Psi(x) = d_a(x, z)\) and a triangular inequality,
\[
\forall (x, y) \in V_{\Gamma_{St}}^\prime \times \text{supp } \nabla \chi, \quad \Psi(x) < \Psi(y) + d_a(x, y).
\]
The inequality is strict since if \(\Psi(x) = \Psi(y) + d_a(x, y)\) for some \((x, y) \in V_{\Gamma_{St}}^\prime \times \text{supp } \nabla \chi\), then \(\Phi(x) - \Phi(y) = d_a(x, y)\) and from Corollary \(53\) up to modifying \(\eta\) such that \(V_\eta \subset V_a\) (see Corollary \(53\) for the definition of \(V_a\)) there exists a generalized integral curve (in the sense of Definition \(8\)) of
\[
\left\{ \begin{array}{l}
-\nabla \Phi \text{ on } V_a \cap \Omega \\
-\nabla T \Phi \text{ on } \partial \Omega
\end{array} \right\} \quad \text{joining } x \in V_{\Gamma_{St}}^\prime \text{ to } y \notin V_{\Gamma_{St}}^\prime. \quad \text{This contradicts the fact that } V_{\Gamma_{St}}^\prime \text{ is stable for } (183). \quad \text{The end of the proof of } (217) \quad \text{then follows exactly the same lines as the proof of } (204).
\]
Moreover, owing to the properties of \(d_a\), one has analogously to \(208\) the following estimate valid a.e. in \(V_{\Gamma_{St}}^\prime\):
\[
|\nabla \psi| \leq (1 - \varepsilon) |\nabla f| = (1 - \varepsilon) |\nabla \Psi|. \quad (218)
\]
Let us finally mention the following inequalities, valid for $h \in (0, h_0)$ with $h_0 = h_0(N, \varepsilon) > 0$ small enough:

$$\varphi_N \leq \Psi + Nh \ln \frac{1}{h} \quad \text{in } V^*_\text{St}$$  \hspace{1cm} (219)

and $\varphi_N = \psi \leq \varphi \leq \Psi$ on supp $\nabla \chi$.  \hspace{1cm} (220)

We are now in position to prove (197). Let us define

$$\tilde{w}_h = e^{\frac{\varphi_N}{h}} w_h = e^{\frac{\varphi_N}{h}} \chi (u^{(1)}_h - c_z(h) u^{(1)}_{z,\text{wkb}}).$$

Using the relations (192)–(196) and the integration by parts formulae (148) and (115), there exists $C_1 > 0$ (only depending on $f$) such that

$$\left| \left| (r_1 + r_1') e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(V^*_\text{St})} \left|\left| \tilde{w}_h \right|\right|_{L^2(V^*_\text{St})} + \left| \left| (r_2 + r_2') e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(\Gamma_1)} \left|\left| \tilde{w}_h \right|\right|_{L^2(\Gamma_1)}$$

$$+ C_1 h \int_{\Gamma_1} (\tilde{w}_h \tilde{\varphi}_h) \partial^*_\Omega \sigma \geq \| h d\tilde{w}_h \|^2_{L^2(V^*_\text{St})} + \| h d^\ast \tilde{w}_h \|^2_{L^2(V^*_\text{St})}$$

$$+ \left( \| \nabla f \|^2 - | \nabla \varphi_N |^2 - C_1 h \| \tilde{w}_h \|^2_{L^2(\Omega_\ast)} \right),$$  \hspace{1cm} (221)

where we have used the fact that almost everywhere on $\Omega_\ast$, $| \nabla \varphi_N |$ is either equal to $| \nabla \Psi | = | \nabla f |$ or to $| \nabla \Psi | \leq (1 - \varepsilon) | \nabla f |$. Moreover, since $\varphi_N - \Psi \leq Nh \ln \frac{1}{h}$ (see (219), one has from (193) and (195)

$$\left| \left| r_1 e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(V^*_\text{St})} + \left| \left| r_2 e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(\Gamma_1)} = O(h^\infty)$$

and, since $\varphi_N \leq \Psi$ on supp $\nabla \chi$ (see (220)), one gets from (194) and (196)

$$\left| \left| r_1 e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(V^*_\text{St})} + \left| \left| r_2 e^{\frac{\varphi_N - \Psi}{h}} \right|\right|_{L^2(\Gamma_1)} = O(h^{-N_0}).$$

Additionally, since $\Psi = f - f(\varepsilon)$ on $\Gamma_1$ and $\varphi_N - \Psi \leq Nh \ln \frac{1}{h}$ (see (219), we deduce from the relation (199) obtained in the first step the following estimate:

$$\left| \left| \tilde{w}_h \right|\right|_{L^2(\Gamma_1)} = \left| \left| e^{\frac{\varphi_N}{h}} \chi (u^{(1)}_h - c_z(h) u^{(1)}_{z,\text{wkb}}) \right|\right|_{L^2(\Gamma_1)} = O(h^\infty).$$

Consequently, using in addition the relation

$$\left| \left| \tilde{w}_h \right|\right|_{L^2(\Omega_\ast)} \leq C h^{-N} \left| \left| w_h \right|\right|_{L^2(\Omega_\ast)} \leq C_2(C, N),$$

(since $\varphi_N \leq \varphi + Nh \ln \frac{1}{h}, \varphi \leq Ch$ on $\Omega_\ast$ and $\left| \left| w_h \right|\right|_{L^2(V^*_\text{St})} = O(h^\infty)$ from (187)), we deduce from (221) the existence of some positive constant $C_3 = C_3(C, C_1, N)$ such that

$$C_3 \left( h^{-N_0} \left| \left| \tilde{w}_h \right|\right|_{L^2(V^*_\text{St})} + 1 \right) \geq \| h d\tilde{w}_h \|^2_{L^2(V^*_\text{St})} + \| h d^\ast \tilde{w}_h \|^2_{L^2(V^*_\text{St})}$$

$$+ \left( \| \nabla f \|^2 - | \nabla \varphi_N |^2 - C_1 h \| \tilde{w}_h \|^2_{L^2(\Omega_\ast)} \right).$$  \hspace{1cm} (222)

Lastly, one has a.e. in $\Omega_+, \nabla \varphi_N = \nabla \psi I_{\{ \varphi_N = \psi \}} + \nabla \psi (1 - \frac{Ch}{\varphi_N}) I_{\{ \varphi_N < \psi \}}$, and thus

- on $\{ \varphi_N = \psi \}$, from (218),

$$| \nabla f |^2 - | \nabla \varphi_N |^2 \geq (2\varepsilon - \varepsilon^2) | \nabla f |^2 \geq c_\varepsilon > 0,$$
• on \( \{ \varphi_N < \psi \} \cap \Omega_+ \), there exists \( C_4 > 0 \) independent of \( C \) such that,
\[
|\nabla f|^2 - |\nabla \varphi_N|^2 \geq C_4 h |\nabla f|^2 \geq C_4 Ch.
\]

Taking \( C > \frac{C_4}{C_2} \) and adding \( (CC_4 - C_1)h \| \tilde{w}_h \|^2_{\mathcal{L}_2(V_{St})} \) to (222) then leads to
\[
C_5 \left( h^{-N_0} \| \tilde{w}_h \|^2_{\mathcal{L}_2(V_{St})} + 1 \right) \geq \| hd\tilde{w}_h \|^2_{\mathcal{L}_2(V_{St})} + \| hd^*\tilde{w}_h \|^2_{\mathcal{L}_2(V_{St})} + (CC_4 - C_1)h \| \tilde{w}_h \|^2_{\mathcal{L}_2(V_{St})},
\]
for a constant \( C_5 \) depending on \( C \) and \( N \). Using Gaffney’s inequality (116), we consequently get the existence of \( C_6 > 0 \) such that
\[
\| \tilde{w}_h \|_{\mathcal{H}_1(V_{St})} \leq C_6 h^{-N_0 - \frac{3}{2}}.
\]

Now, since \( \varphi_N - N h \ln \frac{1}{h} - \Psi \geq -C_7 h \ln \frac{1}{h} \) in \( V_{St} \) (with a constant \( C_7 \) independent of \( N \), from the definition of \( \varphi \) and the fact that \( \varphi_N - N h \ln \frac{1}{h} = \varphi \) in \( V_{St} \), see (217)), we also get the existence of \( N_2 \) independent of \( N \) such that for \( h \) small enough,
\[
\| e^{\varphi} w_h \|_{\mathcal{H}_1(V_{St})} \leq C_N h^{N - N_2},
\]
for some constant \( C_N > 0 \), which concludes the proof of Proposition 65.

\[
\boxed{4.5 \quad \text{Proof of Theorem 1}}
\]

The aim of this section is to conclude the proof of Theorem 1 by checking that the function \( u \) and the family of 1-forms \( (\phi_i)_{i = 1, \ldots, n} \) introduced in Section 4.2 satisfy the estimates appearing in Proposition 17 rewritten in the flat space (see Section 2.2.2). In all this section, we assume in addition to the hypotheses \( \textbf{[H1]}, \textbf{[H2]} \) and \( \textbf{[H3]} \), that (20) and (21) hold.

From Sections 4.2.1 and 4.2.2 it only remains to prove (57), (58), (59) and (60). Let us start with a lemma about the normalisation term appearing in (146).

Lemma 66. \textit{Let us assume that the hypotheses \( \textbf{[H1]}, \textbf{[H2]} \) and \( \textbf{[H3]} \) hold. Let us define \( \Theta_i := \sqrt{\int_{\Omega} \chi_i(x) u_{h,i}^{(1)}(x)^2 \, dx} \). There exist \( c > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \),
\[
\Theta_i^2 = 1 + O \left( e^{-\frac{c}{h}} \right).
\]}

\textbf{Proof.} On the one hand, one has the upper bound
\[
\| \chi_i u_{h,i}^{(1)} \|_{\mathcal{L}^2(\Omega)} = \| \chi_i u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)} \leq \| u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)} = 1.
\]
On the other hand, the triangular inequality yields the lower bound
\[
\| \chi_i u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)} \geq \| u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)} - \| (1 - \chi_i) u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)} = 1 - \| (1 - \chi_i) u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)}.
\]
Thanks to Proposition 64, there exist \( N \in \mathbb{N} \) and \( c > 0 \) independent of \( h \) such that
\[
\| (1 - \chi_i) u_{h,i}^{(1)} \|_{\mathcal{L}^2(\tilde{\Omega}_i)}^2 = \int_{\tilde{\Omega}_i} \left| (1 - \chi_i(x)) u_{h,i}^{(1)}(x) e^{\frac{1}{h}d_a(x,z_i)} e^{\frac{c}{h}d_a(x,z_i)} \right|^2 \, dx \
\leq \int_{\tilde{\Omega}_i \setminus \tilde{V}_i} \left| u_{h,i}^{(1)}(x) e^{\frac{1}{h}d_a(x,z_i)} e^{\frac{c}{h}d_a(x,z_i)} \right|^2 \, dx \
\leq C h^{-N} e^{-\inf_{\tilde{\Omega}_i \setminus \tilde{V}_i} \frac{2}{h}d_a(\cdot,z_i)} \leq C e^{-\frac{c}{h}},
\]

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where, we recall \( \mathcal{V}_i = \{ x \in \Omega, \chi_i = 1 \} \). This concludes the proof of Lemma 66.

We are now in position to check the estimates stated in Section 2.2.2.

**Step 1. Study of the term \( \| (1 - \pi_{[0, h^2]}^c) (\Delta f_{i,h}^{D,1}) \| \tilde{\phi}_i \|_{H^1(\Omega)} \).**

We recall that from (116), \( \tilde{\phi}_i \) belongs to \( \Lambda^1 H^1(\Omega) \) and then we get from Lemma 15 that there exist \( c > 0 \) and \( h_0 > 0 \) such that for all \( h \in (0, h_0) \),

\[
\| (1 - \pi_{[0, h^2]}^c) (\Delta f_{i,h}^{D,1}) \| \tilde{\phi}_i \|_{L^2(\Omega)}^2 \leq ch^{-3/2} \left( \| df_{i,h} \tilde{\phi}_i \|_{L^2(\Omega)}^2 + \| df_{i,h} \tilde{\phi}_i \|_{L^2(\Omega)}^2 \right).
\]

Moreover, from Proposition 59 (items (ii) et (iii))

\[
d_{f,h} \tilde{\phi}_i = \Theta_i^{-1} \left( \chi_i d_{f,h} u_{h,i}^{(1)} + h d\chi_i \wedge u_{h,i}^{(1)} \right) = \Theta_i^{-1} h d\chi_i \wedge u_{h,i}^{(1)},
\]

and

\[
d_{f,h}^* \tilde{\phi}_i = \Theta_i^{-1} \left( \chi_i d_{f,h}^* u_{h,i}^{(1)} - h u_{h,i}^{(1)} \cdot \nabla \chi_i \right) = -\Theta_i^{-1} h u_{h,i}^{(1)} \cdot \nabla \chi_i.
\]

As a consequence, using Lemma 66 and Proposition 61, one gets for some \( N \in \mathbb{N} \) and for some \( c > 0 \) which may change from one occurrence to another,

\[
\| (1 - \pi_{[0, h^2]}^c) (\Delta f_{i,h}^{D,1}) \| \tilde{\phi}_i \|_{L^2(\Omega)}^2 \leq ch^{-3/2} \left( \| \chi_i d_{f,h} u_{h,i}^{(1)} \|_{L^2(\Omega)}^2 + \| h u_{h,i}^{(1)} \cdot \nabla \chi_i \|_{L^2(\Omega)}^2 \right)
\]

\[
\leq ch^{1/2} \int_{\text{supp} \nabla \chi_i} \left( u_{h,i}^{(1)}(x) e^{\frac{1}{2} d_a(x,z_i)} e^{-\delta d_a(x,z_i)} \right)^2 dx
\]

\[
\leq ch^{1/2 - N} e^{-\inf_{\text{supp} \nabla \chi_i} \frac{\delta}{2} d_a(\cdot, z_i)}.
\]

(224)

The function \( \chi_i \) can be chosen such that the set \( \{ x \in \Omega_i, \nabla \chi_i(x) \neq 0 \} \) is as close as one wants to \( \Gamma_{2,i} \), and to \( \Gamma_0 \) (see Figure 14). Therefore by continuity of the Agmon distance, using (136)–(137), for any \( \delta > 0 \), one can choose \( \chi_i \) satisfying the three conditions stated in Definition 14 and such that

\[
\inf_{z \in \text{supp} \nabla \chi_i} d_a(z, z_i) \geq \min \left( d_a(x_0, z_i), \inf_{z \in B_{\epsilon_i}^c} d_a(z, z_i) \right) - \delta.
\]

(225)

From (20), there exists \( r > 0 \) such that

\[
\inf_{z \in B_{\epsilon_i}^c} d_a(z, z_i) \geq \max [f(z_n) - f(z_i), f(z_i) - f(z_1)] + r.
\]

In addition, using (21), there exists \( r' > 0 \) such that

\[
d_a(z_i, x_0) \geq f(z_i) - f(x_0) \geq f(z_1) - f(x_0) \geq f(z_n) - f(z_1) + r'.
\]

Therefore, choosing \( \chi_i \) such that \( \delta < \min(r, r') \), there exists \( \epsilon' > 0 \) such that

\[
\inf_{z \in \text{supp} \nabla \chi_i} d_a(z, z_i) \geq \max [f(z_n) - f(z_i), f(z_i) - f(z_1)] + \epsilon'.
\]

(226)
Using the estimate (226) in (224), there exist $\varepsilon_1 > 0$, $c > 0$, $N \in \mathbb{N}$ and $h_0 > 0$, such that for every $h \in (0, h_0)$

$$
\left\| \left(1 - \pi_{[0, h^2]} \right) \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right\|_{L^2(\Omega)}^2 \leq c h^{-N} e^{-\frac{2}{\pi} (\max |f(z_n) - f(z_i), f(z_i) - f(z_j)| + \varepsilon) + \varepsilon}
\leq e^{-\frac{2}{\pi} (\max |f(z_n) - f(z_i), f(z_i) - f(z_j)| + \varepsilon)}.
$$

(227)

This last inequality leads to the desired estimate in the $L^2(\Omega)$-norm. In order to get the same upper bound in the $H^1(\Omega)$-norm, notice now that one has

$$
\left(1 - \pi_{[0, h^2]} \right) \left( \Delta^{D,1}_{f,h} (\Omega) \right) d_{f,h} \tilde{\phi}_i = d_{f,h} \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i
\leq \text{hd} \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i + df \wedge \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i.
$$

Therefore it holds

$$
\left\| df \wedge \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 \leq Ce^{-\frac{2}{\pi} (K_i + \varepsilon)}.
$$

Moreover, using (225) and (224) there exist $\varepsilon > 0$, $C > 0$ and $h_0 > 0$, such that for all $h \in (0, h_0)$,

$$
\left\| \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) d_{f,h} \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 \leq \left\| df \wedge \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 = \Theta_1 \left\| d_{f,h} \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 \leq Ce^{-\frac{2}{\pi} (K_i + \varepsilon)}.
$$

Thus one gets: there exist $\varepsilon > 0$, $C > 0$ and $h_0 > 0$, such that for all $h \in (0, h_0)$,

$$
h^2 \left\| d \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 \leq Ce^{-\frac{2}{\pi} (K_i + \varepsilon)}.
$$

Similarly, there exist $\varepsilon > 0$, $C > 0$ and $h_0 > 0$, such that for all $h \in (0, h_0)$

$$
h^2 \left\| d^* \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i \right\|_{L^2(\Omega)}^2 \leq Ce^{-\frac{2}{\pi} (K_i + \varepsilon)}.
$$

As a consequence, using (116) there exist $\varepsilon > 0$, $C > 0$ and $h_0 > 0$, such that for all $h \in (0, h_0)$

$$
\left\| \left(1 - \pi_{[0, h^2]} \left( \Delta^{D,1}_{f,h} (\Omega) \right) \right) \tilde{\phi}_i \right\|_{H^1(\Omega)}^2 \leq Ce^{-\frac{2}{\pi} (K_i + \varepsilon)}.
$$

This concludes the proof of (57).

Step 2. Study of the terms $\int_{\Omega} \tilde{\phi}_i \cdot \tilde{\phi}_j$ for $(i, j) \in \{1, \ldots, n\}^2$.

Let $(i, j) \in \{1, \ldots, n\}^2$. Let us assume without loss of generality that $i < j$, so that $f(z_i) \leq f(z_j)$. From Proposition 19 one has the inequality $d_a(z_i, z_j) > f(z_j) - f(z_i)$.

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Now, according to Proposition 61 and Lemma 66 and to the triangular inequality for $d_a$, there exist $\varepsilon > 0$, $N \in \mathbb{N}$ and $h_0 > 0$ such that for all $h \in (0, h_0)$,

$$\left| \int_{\Omega} \tilde{\phi}_i(x) \cdot \tilde{\phi}_j(x) \, dx \right| \leq e^{-\frac{d_a(x_i, x_j)}{h}} \int_{\text{supp} \chi_i} e^{-\frac{d_a(x_i, x_j)}{h}} |\tilde{\phi}_i(x)| \, dx \leq \Theta_i^{-1} \Theta_j^{-1} \left\| \chi_i u_{h, 1}^{(1)} \right\|_{L^2(\Omega_i)} \left\| \chi_j u_{h, 1}^{(1)} \right\|_{L^2(\Omega_j)} e^{-\frac{d_a(x_i, x_j)}{h}} \leq C h^{-N} e^{-\frac{1}{4}(f(z_j) - f(z_i) + \varepsilon)}.$$  

This concludes the proof of (58).

Step 3. Study of the terms $\int_{\Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{h}}$ for $(i, j) \in \{1, \ldots, n\}^2$.

By construction, for all $(i, j) \in \{1, \ldots, n\}^2$ such that $i \neq j$, one has

$$\int_{\Sigma_i} \tilde{\phi}_j \cdot n \, e^{-\frac{f}{h}} = 0.$$  

Now, let us compute the term $\int_{\Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{h}}$. Let $u_{z_i, \text{wkb}}^{(1)}$ be the WKB expansion defined by (174). Following the beginning of Section 4.4.2, let us consider

1. a neighborhood $V_{\Gamma_{1, i}}$ of $\Sigma_i$ in $\overline{\Omega}$, which is stable under the dynamics (183) and such that, for some $\varepsilon > 0$, $V_{\Gamma_{1, i}} + B(0, \varepsilon) \subset V_{\Gamma_{1, i}} \cap (\Gamma_{1, i} \cup \Omega_i)$

2. and a cut-off function $\chi_{\text{wkb}, i} \in C^\infty(\Omega_i \cup \Gamma_{1, i})$ with $\chi_{\text{wkb}, i} \equiv 1$ on a neighborhood of $V_{\Gamma_{1, i}}$ and such that $\text{supp} \chi_{\text{wkb}, i} \subset V_{\Gamma_{1, i}} \cap (\Omega_i \cup \Gamma_{1, i})$.

Using Proposition 64, there exists $c_{z_i}(h) \in \mathbb{R}^*_+$ such that

$$\left\| \chi_{\text{wkb}, i} \left( u_{h, 1}^{(1)} - c_{z_i}(h) u_{z_i, \text{wkb}}^{(1)} \right) \right\|_{H^1(\Omega_i)} = O(h^\infty).$$

Let us now introduce

$$\tilde{\phi}_{z_i, \text{wkb}} := c_{z_i}(h) \chi_{\text{wkb}, i} u_{z_i, \text{wkb}}^{(1)}$$  

so that

$$\int_{\Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{h}} = \int_{\Sigma_i} \phi_{z_i, \text{wkb}} \cdot n \, e^{-\frac{f}{h}} + \int_{\Sigma_i} \left( \tilde{\phi}_i - \phi_{z_i, \text{wkb}} \right) \cdot n \, e^{-\frac{f}{h}}.$$  

Let us first deal with the term $\int_{\Sigma_i} \phi_{z_i, \text{wkb}} \cdot n \, e^{-\frac{f}{h}}$. Using Laplace’s method (the computation is similar to (181)), one gets when $h \to 0$ (since $\Phi = f$ and $\partial_n \Phi = -\partial_n f$ on $\partial \Omega$)

$$\int_{\Sigma_i} \chi_{\text{wkb}, i} u_{z_i, \text{wkb}}^{(1)} \cdot n \, e^{-\frac{f}{h}} = \int_{\Sigma_i} e^{-\frac{\Phi - f(z_i)}{h}} a_0 \partial_n (f - \Phi) e^{-\frac{f}{h}} (1 + O(h)) = 2 e^{-\frac{f(z_i)}{h}} \int_{\Sigma_i} e^{-\frac{2f}{h}} a_0 \partial_n f (1 + O(h)) = \frac{2 \partial_n f(z_i) \pi \frac{d-1}{2}}{\sqrt{\det \text{Hess} f |_{\partial \Omega}(z_i)}} h \frac{d-1}{2} e^{-\frac{f}{h}}(1 + O(h)).$$

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Then one obtains when \( h \to 0 \)
\[
\int_{\Sigma_i} \tilde{\phi}_{zi,wkb} \cdot n e^{-\frac{t}{h}}f = c_{zi}(h) \frac{2 \partial_n f(z_i) \pi^{\frac{d-1}{2}}}{\sqrt{\det \text{Hess} f|_{\partial \Omega}(z_i)}} h^\frac{d-1}{2} e^{-\frac{t}{h}}f(z_i)(1 + O(h)).
\]

We recall from Proposition \([64]\) that in the limit \( h \to 0 \):
\[
c_{zi}(h) = C_{zi,wkb}^{-1} h^{-\frac{d+1}{2}} (1 + O(h^\infty)),
\]
where the constant \( C_{zi,wkb} \) is defined by \([177]\). Therefore, in the limit \( h \to 0 \)
\[
\int_{\Sigma_i} \tilde{\phi}_{zi,wkb} \cdot n e^{-\frac{t}{h}}f = \frac{\pi^{\frac{d-1}{2}}}{(\det \text{Hess} f|_{\partial \Omega}(z_i))^{1/4}} h^{-\frac{d-3}{2}} e^{-\frac{t}{h}}f(z_i)(1 + O(h)).
\]

Let us now deal with the term \( \int_{\Sigma_i} (\tilde{\phi}_i - \tilde{\phi}_{zi,wkb}) \cdot n e^{-\frac{t}{h}}f \). One obtains using Lemmata \([63]\) and \([66]\) that there exist \( C > 0, h_0 > 0 \) and \( \eta > 0 \) such that for all \( h \in (0, h_0) \)
\[
\left| \int_{\Sigma_i} (\tilde{\phi}_i - \tilde{\phi}_{zi,wkb}) \cdot n e^{-\frac{t}{h}}f \right| = \left| \int_{\Sigma_i} \left( \frac{u^{(1)}_{h,i}}{\Theta_i} - c_{zi}(h) u^{(1)}_{zi,wkb} \right) \cdot n e^{-\frac{t}{h}}f \right|
\leq \frac{e^{-\frac{t}{h}}f(z_i)}{\Theta_i} \int_{\Sigma_i} \left| \left( \frac{u^{(1)}_{h,i}}{\Theta_i} - c_{zi}(h) u^{(1)}_{zi,wkb} \right) \cdot n \right|
+ e^{-\frac{t}{h}}f(z_i) \left\| \Theta_i - 1 \right\|_{C_{zi}(h)} \left| \int_{\Sigma_i} u^{(1)}_{zi,wkb} \cdot n \right|
\leq C e^{-\frac{t}{h}}f(z_i) \left\| \chi_{wkb,i} \left( \frac{u^{(1)}_{h,i}}{\Theta_i} - c_{zi}(h) u^{(1)}_{zi,wkb} \right) \right\|_{H^1(\Omega_i)}
+ C e^{-\frac{t}{h}}f(z_i) e^{-\frac{t}{h}}h^{-\frac{d+5}{4}} \left\| \chi_{wkb,i} u^{(1)}_{zi,wkb} \right\|_{H^1(\Omega_i)}.
\]

Therefore, one obtains using Proposition \([64]\) and \([178]\)
\[
e^{-\frac{t}{h}}f(z_i) \int_{\Sigma_i} (\tilde{\phi}_i - \tilde{\phi}_{zi,wkb}) \cdot n e^{-\frac{t}{h}}f = O(h^\infty) + C e^{-\frac{t}{h}}h^{-\frac{d+5}{4}} = O(h^\infty).
\]

In conclusion, we have when \( h \to 0 \)
\[
\int_{\Sigma_i} \tilde{\phi}_i \cdot n e^{-\frac{t}{h}}f = \int_{\Sigma_i} \tilde{\phi}_{zi,wkb} \cdot n e^{-\frac{t}{h}}f (1 + O(h^\infty)),
\]
which gives the expected estimate
\[
\int_{\Sigma_i} \tilde{\phi}_j \cdot n e^{-\frac{t}{h}}f = \begin{cases} B_i h^m e^{-\frac{t}{h}}f(z_i) (1 + O(h)) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
\] (230)

where
\[
m = \frac{d-3}{4} \quad \text{and} \quad B_i = \frac{\pi^{\frac{d-1}{2}} \sqrt{2 \partial_n f(z_i)}}{(\det \text{Hess} f|_{\partial \Omega}(z_i))^{1/4}}.
\] (231)

This concludes the proof of \([60]\).

Step 4. Study of the term \( \int_{\Omega} \nabla \tilde{u} \cdot \tilde{\phi}_i e^{-\frac{t}{h}}f \).
First one has the equality by Definition \[13\]

\[
\int_{\Omega} \nabla \tilde{u} \cdot \tilde{\phi}_i \, e^{-\frac{f}{2}} = \int_{\Omega} \nabla \chi \cdot \tilde{\phi}_i \, e^{-\frac{f}{2}}
\]

where \(\nabla \chi \cdot \tilde{\phi}_i = i \nabla \chi \tilde{\phi}_i = \tilde{\phi}_i (\nabla \chi)\). The denominator of the right-hand side being easily computed thanks to Laplace’s method:

\[
\sqrt{\int_{\Omega} \chi^2 e^{-\frac{2}{2}f}} = \frac{(\pi h)^{d/2}}{(\det \text{Hess} f(x_0))^{1/2}} e^{-\frac{f(x_0)}{h}} (1 + O(h)).
\]

Using an integration by parts and the fact that \(d^*(u_{h_i}^{(1)} e^{-f/h}) = 0\) in \(L^2(\Omega_i)\) (see Proposition \[59\] items (ii) and (iii)) which is valid for all \(h\) small enough, one obtains

\[
\int_{\Omega} \nabla \chi \cdot \tilde{\phi}_i \, e^{-\frac{f}{2}} = -\int_{\Omega} \nabla (1 - \chi) \cdot \chi_i \frac{u_{h_i}^{(1)}}{\Theta_i} \, e^{-\frac{f}{2}}
\]

\[
= \int_{\Omega} (1 - \chi) \nabla \chi_i \cdot \frac{u_{h_i}^{(1)}}{\Theta_i} \, e^{-\frac{f}{2}} - \int_{\partial \chi_i} (1 - \chi) \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}}.
\]

Using the fact that \(\chi = 0\) on \(\partial \Omega\), one then obtains:

\[
\int_{\partial \Omega} (1 - \chi) \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}} = \int_{\partial \Omega \cap \supp \chi_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}} = \int_{\Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}} + \int_{(\partial \Omega \cap \supp \chi_i) \setminus \Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}}.
\]

Using (230), in the limit \(h \to 0\):

\[
\int_{\Omega} \nabla \chi \cdot \tilde{\phi}_i \, e^{-\frac{f}{2}} = -B_i h^m \, e^{-\frac{f(z_i)}{2}} (1 + O(h))
\]

\[
- \int_{(\partial \Omega \cap \supp \chi_i) \setminus \Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}} + \int_{\Omega} (1 - \chi) \nabla \chi_i \cdot \frac{u_{h_i}^{(1)}}{\Theta_i} \, e^{-\frac{f}{2}}.
\]

Let us now prove that the two last terms in (232) are negligible compared to the first one.

On the compact set \((\partial \Omega \cap \supp \chi_i) \setminus \Sigma_i\) one has \(f(z) > f(z_i)\) since \(z_i \in \Sigma_i\) is the only global minimum of \(f\) on \(B_{z_i}\) and \(\supp \chi_i \cap \partial \Omega \subset \Gamma_i, i \subset B_{z_i}\). Then, using Proposition \[61\] and \[116\], there exist \(\varepsilon > 0, h_0 > 0, C > 0\) and \(N \in \mathbb{N}\) such that for all \(h \in (0, h_0)\)

\[
\left| \int_{(\partial \Omega \cap \supp \chi_i) \setminus \Sigma_i} \tilde{\phi}_i \cdot n \, e^{-\frac{f}{2}} \right| \leq C e^{-\frac{f(z_i) + \varepsilon}{h}} \left\| \tilde{\phi}_i \cdot n \right\|_{H^1(\Omega_i)}
\]

\[
\left| \int_{\Omega} (1 - \chi) \nabla \chi_i \cdot \frac{u_{h_i}^{(1)}}{\Theta_i} \, e^{-\frac{f}{2}} \right| \leq C e^{-\frac{f(z_i) + \varepsilon}{h}} \left\| \chi_i u_{h_i}^{(1)} \right\|_{H^1(\Omega_i)}
\]

\[
\leq C e^{-\frac{f(z_i) + \varepsilon}{h}} h^{-N} \leq C e^{-\frac{f(z_i) + \varepsilon/2}{h}}.
\]
Thus, using the Cauchy-Schwarz inequality and Proposition 61, there exists $c > 0$ such that

$$\inf_{\text{supp}(1 - \chi_i) \nabla \chi_i} (d_a(\cdot, z_i) + f) \geq \inf_{B_{\epsilon z_i}} (d_a(\cdot, z_i) + f) - \delta.$$ 

Therefore, taking $\chi_i$ such that $\delta < r/2$, one has, when $h \to 0$

$$\int_{\text{supp}(1 - \chi_i) \nabla \chi_i} \left| (1 - \chi_i) \nabla \chi_i \cdot u_{h,i}^{(1)} e^{-\frac{1}{\sqrt{h}}} \right| \leq C \left\| u_{h,i}^{(1)} e^{-\frac{d_a(z_i) + 1}{\sqrt{h}}} \right\|_{L^2(\Omega)} e^{-\frac{1}{\sqrt{h}}} \inf_{\text{supp}(1 - \chi_i) \nabla \chi_i} (d_a(\cdot, z_i) + f)$$

$$\leq C h^{-N} e^{-\frac{1}{\sqrt{h}} \inf_{B_{\epsilon z_i}} (d_a(\cdot, z_i) + f - \delta)}.$$ (234)

Besides, from assumption (20)

$$\inf_{z \in B_{\epsilon z_i}} [d_a(z, z_i) + f(z)] > f(z_i).$$

Indeed, the inequality (20) implies that there exists $r > 0$ such that for all $z \in B_{\epsilon z_i}^c$, $d_a(z, z_i) \geq f(z_i) - f(z_1) + r$ and therefore for all $z \in B_{\epsilon z_i}$ one obtains

$$d_a(z, z_i) + f(z) \geq f(z_i) + (f(z) - f(z_1)) + r \geq f(z_i) + r.$$  

Therefore, taking $\chi_i$ such that $\delta < r/2$, one has, when $h \to 0$

$$\int_{\text{supp}(1 - \chi_i) \nabla \chi_i} \left| (1 - \chi_i) \nabla \chi_i \cdot u_{h,i}^{(1)} e^{-\frac{1}{\sqrt{h}}} \right| = O \left( e^{-\frac{f(z_i) + r}{\sqrt{h}}} \right)$$

for some constant $c > 0$.

In conclusion, in the limit $h \to 0$,

$$\int_{\Omega} \nabla \tilde{u} \cdot \phi_i e^{-\frac{1}{\sqrt{h}} f} dx = C_i h^p e^{-\frac{1}{\sqrt{h}} (f(z_i) - f(x_0)) (1 + O(h))},$$

with

$$C_i = -B_i \left( \frac{\det \text{Hess} f(x_0))^{1/4}}{\pi^{1/4}} \right) = -\frac{1}{\pi^{1/4}} \sqrt{2D h f(z_i) (\det \text{Hess} f(x_0))^{1/4}} (\det \text{Hess} f(z_i))^{1/4}$$

and $p = m - \frac{d}{4} = -\frac{3}{4}$.

where $B_i$ and $m$ have both been defined in (231). This concludes the proof of (59), and thus the proof of Theorem 1.

5 Consequences and generalizations of Theorem 1

5.1 Proofs of Proposition 6, Proposition 7, Corollary 8 and Corollary 10

5.1.1 Proof of Proposition 6

Assume that [H1], [H2] and [H3] hold. From Lemma 18 and since the function $\tilde{u}$ is non negative in $\Omega$, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$

$$u_h = \frac{\pi h(0) \tilde{u}}{\|\pi h(0) \tilde{u}\|_{L^2}},$$

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where \( u_h \) is the eigenfunction associated with the smallest eigenvalue of \(-L_{f,h}^{(D),(0)}(\Omega)\) (see Proposition 14) which satisfies (11) and \( \tilde{u} \) is introduced in Definition 13. Then, there exists \( h_0 > 0 \) such that for all \( h \in (0, h_0), \)

\[
\int_\Omega u_h e^{-\frac{\pi}{h} f} = \int_\Omega \frac{\bar{\pi}_h(0) \tilde{u}}{\| \bar{\pi}_h(0) \tilde{u} \|_{L^2_w}} e^{-\frac{\pi}{h} f} = \int_\Omega \left[ \tilde{u} + \left( \bar{\pi}_h(0) - 1 \right) \tilde{u} \right] e^{-\frac{\pi}{h} f}.
\]

From the definition of \( \chi \) (see Definition 13) and using Laplace’s method, one obtains (in the limit \( h \to 0) \)

\[
\int_\Omega \chi e^{-\frac{\pi}{h} f} = \frac{e^{-\frac{\pi}{h} f(x_0)}}{\sqrt{\det \text{Hess}(f(x_0))}} (1 + O(h))
\]

and likewise

\[
\int_\Omega \chi^2 e^{-\frac{\pi}{h} f} = \frac{e^{-\frac{\pi}{h} f(x_0)}}{\sqrt{\det \text{Hess}(f(x_0))}} (1 + O(h))
\]

In addition, using Lemma 18 one has \( \| \bar{\pi}_h(0) \tilde{u} \|_{L^2_w} = 1 + O \left( e^{-\frac{\pi}{h}} \right) \). Therefore, it holds when \( h \to 0, \)

\[
\frac{1}{\| \bar{\pi}_h(0) \tilde{u} \|_{L^2_w}} \int_\Omega \tilde{u} e^{-\frac{\pi}{h} f} = \frac{h^\frac{d}{4} \pi_x^\frac{d}{2}}{(\det \text{Hess}(f(x_0)))^{1/4}} e^{-\frac{\pi}{h} f(x_0)} (1 + O(h)).
\]

Moreover, from Lemma 18 there exist \( c > 0, h_0 > 0 \) and \( C > 0 \) such that for \( h \in (0, h_0) \)

\[
\frac{1}{\| \bar{\pi}_h(0) \tilde{u} \|_{L^2_w}} \left[ \int_\Omega (\bar{\pi}_h(0) - 1) \tilde{u} e^{-\frac{\pi}{h} f} \right] \leq C \left[ (1 - \bar{\pi}_h(0)) \tilde{u} \right]_{L^2_w} \sqrt{\int_\Omega e^{-\frac{\pi}{h} f}} \leq Ce^{-\frac{\pi}{h}} e^{-\frac{\pi}{h} f(x_0)}
\]

Thus, one has when \( h \to 0, \)

\[
\int_\Omega u_h e^{-\frac{\pi}{h} f} = \frac{\pi_x^\frac{d}{2}}{(\det \text{Hess}(f(x_0)))^{1/4}} h^\frac{d}{4} e^{-\frac{\pi}{h} f(x_0)} (1 + O(h)).
\]

This proves Proposition 6.

5.1.2 Proof of Proposition 7

The aim of this section is to prove (23). To this end, we first state in Proposition 67 some estimates that the quasi-modes constructed in Section 4.2 satisfy under hypotheses [H1], [H2] and [H3]. Let us emphasize that these estimates are weaker than those obtained in Section 4.5 where in addition to [H1]-[H2]-[H3], also the hypotheses (20) and (21) were assumed. Then, we prove that the estimates of Proposition 67 imply (23).

Proposition 67. Let us assume that the hypotheses [H1], [H2] and [H3] hold. Then there exist \( n + 1 \) quasi-modes \( (\tilde{\psi}_i)_{i=1,...,n}, \tilde{u} \) which satisfy the following estimates:

1. \( \forall i \in \{1,\ldots,n\}, \tilde{\psi}_i \in \Lambda^1 H^1_{w,T}(\Omega) \) and \( \tilde{u} \in \Lambda^0 H^1_{w,T}(\Omega) \). The function \( \tilde{u} \) is nonnegative in \( \Omega \). Moreover \( \forall i \in \{1,\ldots,n\}, \| \tilde{\psi}_i \|_{L^2_w} = \| \tilde{u} \|_{L^2_w} = 1. \)
2. (a) There exists $\varepsilon_1 > 0$, for all $i \in \{1, \ldots, n\}$, in the limit $h \to 0$:
\[
\left\| (1 - \pi_h^{(1)}) \tilde{\psi}_i \right\|^2_{H^1_w} = O \left( e^{-\frac{c}{h}} \right).
\]

(b) For any $\delta > 0$, $\| \nabla \tilde{u} \|^2_{L^2_w} = O \left( e^{-\frac{c}{(f(z_1) - f(x_0) - \delta)} } \right)$. 

3. There exists $\varepsilon_0 > 0$ such that $\forall (i, j) \in \{1, \ldots, n\}^2$, $i < j$, in the limit $h \to 0$:
\[
\left\langle \tilde{\psi}_i, \tilde{\psi}_j \right\rangle_{L^2_w} = O \left( e^{-\frac{c}{h}} \right).
\]

4. There exist $\varepsilon_0 > 0$, such that for all $i \in \{1, \ldots, n\}$, in the limit $h \to 0$:
\[
\left\langle \nabla \tilde{u}, \tilde{\psi}_i \right\rangle_{L^2_w} = C_i h^p e^{-\frac{c}{h}(f(z_1) - f(x_0))} (1 + O(h)) + O \left( e^{-\frac{c}{h}(f(z_1) - f(x_0) + \varepsilon_0)} \right),
\]
where the constants $p$ and $(C_i)_{i=1,\ldots,n}$ are given by \[(235)\].

Proof. Thanks to the hypotheses \([H1], [H2]\) and \([H3]\), one can introduce the $n+1$ quasi-modes $((\tilde{\phi}_i)_{i=1,\ldots,n}, \tilde{\psi})$ built in Section 4.2. Recall that $\tilde{\psi}_i = e^{\frac{i}{h}f(z_1)} \phi_i$ for $i \in \{1, \ldots, n\}$.

Then, one easily obtains that $((\tilde{\psi}_i)_{i=1,\ldots,n}, \tilde{\psi})$ satisfy the estimates stated in Proposition 67, following exactly the computations made on $((\tilde{\phi}_i)_{i=1,\ldots,n}, \tilde{\psi})$ in Section 4.2. 2(a) follows from \[(224)\], 2(b) is a consequence of Lemma 60, 3 follows from \[(228)\] and 4 is a consequence of \[(232), (233), (234)\] (in \[(234)\], one uses that for $\delta > 0$ small enough, there exists $c > 0$ such that $\inf_{B_{\delta/2}} (d_{\alpha}(., z_i) + f - \delta) \geq f(z_1) + c$).

Let us now prove that the estimates stated in Proposition 67 imply \[(23)\], which will conclude the proof of Proposition 7.

Proof. From \[(51)\] together with the assumption $\| u_h \|_{L^2_w} = 1$, it holds
\[
\lambda_h = -\langle L^{D,(0)}_{f,h} (\Omega) u_h, u_h \rangle_{L^2_w} = \frac{h}{2} \| \nabla u_h \|^2_{L^2_w},
\]
where $u_h$ is the eigenfunction associated with the smallest eigenvalue of $-L^{D,(0)}_{f,h} (\Omega)$ (see Proposition 14). Recall that $\nabla u_h \in \text{Ran} \pi_h^{(1)}$ (see \[(53)\]).

In addition, let us recall that from items 1, 2(a) and 3 in Proposition 67 and using the proof of Lemma 19 there exists $h_0$ such that for all $h \in (0, h_0)$,
\[
\text{span} \left( \pi_h^{(1)} \tilde{\psi}_i, i = 1, \ldots, n \right) = \text{Ran} \pi_h^{(1)}.
\]

Let us denote by $(\psi_i)_{i=1,\ldots,n}$ the orthonormal basis of $\text{Ran} \pi_h^{(1)}$ resulting from the Gram-Schmidt orthonormalisation procedure applied to the set $(\pi_h^{(1)} \tilde{\psi}_i)_{i=1,\ldots,n}$ (see Lemma 20) so that
\[
\| \nabla u_h \|^2_{L^2_w} = \sum_{j=1}^n |\langle \nabla u_h, \psi_j \rangle_{L^2_w}|^2.
\]

We now want to estimate the terms $\langle \nabla u_h, \psi_j \rangle_{L^2_w}$. 

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Using 2(b) in Proposition 6 and using the proof of Lemma 18, one has that for \( h \) small enough and \( \pi_h^{(0)} \bar{u} \neq 0 \) and therefore, since moreover \( \bar{u} \) is non negative in \( \Omega \), \( u_h = \frac{\pi_h^{(0)} \bar{u}}{\| \pi_h^{(0)} \bar{u} \|_{L^2_\omega}} \). Thus one has (see (68), for \( j \in \{1, \ldots, n\}, \)

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_\omega} = \frac{Z_j^{-1}}{\| \pi_h^{(0)} \bar{u} \|_{L^2_\omega}} \left[ \langle \nabla \bar{u}, \psi_j \rangle_{L^2_\omega} + \langle \nabla \bar{u}, \left( \pi_h^{(1)} - 1 \right) \bar{v}_j \rangle_{L^2_\omega} \right]
+ \frac{Z_j^{-1}}{\| \pi_h^{(0)} \bar{u} \|_{L^2_\omega}} \sum_{i=1}^{j-1} \kappa_{ji} \left[ \langle \nabla \bar{u}, \psi_i \rangle_{L^2_\omega} + \langle \nabla \bar{u}, \left( \pi_h^{(1)} - 1 \right) \bar{v}_i \rangle_{L^2_\omega} \right]
\]

where \( (\kappa_{ji})_{1 \leq i < j \leq n} \) and \( (Z_j)_{1 \leq j \leq n} \) are defined in Lemma 20.

Now, using the items 1, 2(a) and 3 of Proposition 67 and the proof of Lemma 22, one obtains that there exist \( \varepsilon_0 > 0 \) and \( h_0 > 0 \) such that \( \forall h \in (0, h_0), \forall (i, j) \in \{1, \ldots, n\}^2 \), it holds

\[
\kappa_{ji} = O \left( e^{-\frac{\varepsilon_0}{\pi}} \right) \quad \text{and} \quad Z_i = 1 + O \left( e^{-\frac{\varepsilon_0}{\pi}} \right), \tag{238}
\]

Injecting (238) and the estimates 2 and 4 of Proposition 67 into (68) leads to the existence of \( \varepsilon' > 0 \) and \( h_0 > 0 \) such that \( \forall h \in (0, h_0), \)

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_\omega} = C_i h^p e^{-\frac{\varepsilon'}{\pi} (f(z_j) - f(x_0))} (1 + O(h)) + O \left( e^{-\frac{\varepsilon'}{\pi} (f(z_j) - f(x_0) + \varepsilon')} \right), \tag{239}
\]

where the constants \( p \) and \( (C_i)_{i=1, \ldots, n} \) are given by (235). Using (239) in (236) and (237), there exists \( \varepsilon' > 0 \) such that in the limit \( h \to 0 \)

\[
\lambda_h = \frac{h}{2} \sum_{j=1}^{n} C_i^2 h^{2p} e^{-\frac{\varepsilon'}{\pi} (f(z_j) - f(x_0))} (1 + O(h)) + O \left( e^{-\frac{\varepsilon'}{\pi} (f(z_j) - f(x_0) + \varepsilon')} \right).
\]

Therefore, the estimate (23) holds and Proposition 7 is proved.

### 5.1.3 Proof of Corollary 8

According to (14) one has for \( i \in \{1, \ldots, n\} \):

\[
\mathbb{P}^\nu_x \left[ X_{\Sigma i} \in \Sigma_i \right] = - \frac{h}{2\lambda_h} \int_{\Omega} \left( \partial_n u_h \right)(z) e^{-\frac{\varepsilon}{\pi} f(z) \sigma(dz) \int_{\Omega} u_h(y) e^{-\frac{\varepsilon}{\pi} f(y)dy}. \right.
\]

Let us assume that [H1], [H2] and [H3] together with the inequalities (20) and (21) hold. From Proposition 6 and Proposition 7, one obtains when \( h \to 0 \)

\[
\lambda_h \frac{2}{h} \int_{\Omega} u_h e^{-\frac{\varepsilon}{\pi} f} dx = 2\pi^{\frac{d-2}{2}} \left( \det \text{Hess} f(x_0) \right)^{\frac{1}{2}} h^{\frac{d-4}{2}} \times \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess} f(z_k)}} e^{-\frac{\varepsilon}{\pi} (2f(z_k) - f(x_0))} (1 + O(h)).
\]

Then, using in addition Theorem 4 to estimate \( \int_{\Sigma_i} \left( \partial_n u_h \right) e^{-\frac{\varepsilon}{\pi} f} d\sigma \ (i \in \{1, \ldots, n\}) \), one proves Corollary 8.
5.1.4 Proof of Corollary 10

Before starting the proof of Corollary 10 let us notice that under the assumptions stated in Corollary 8 for all \( i \in \{1, \ldots, n\} \) and for any test function \( F \in C^\infty(\partial \Omega) \) satisfying \( \text{supp} \ F \subset B_{z_i} \) and \( z_i \in \text{int} (\text{supp} \ F) \), when \( h \to 0 \),

\[
\mathbb{E}^\nu_h[F(X_{\tau_h})] = \frac{\partial_n f(z_i)}{\sqrt{\text{det} \, \text{Hess} f|_{\partial \Omega}(z_i)}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\text{det} \, \text{Hess} f|_{\partial \Omega}(z_k)}} \right)^{-1} e^{-\frac{2}{h}(f(z_i)-f(z_1))(F(z_i)+O(h))}.
\]

(240)

The strategy for the proof of Corollary 10 is to first extend (240) to a deterministic initial condition, and then to deduce the result of Corollary 10.

Let us first notice that we can assume without loss of generality (up to increasing \( \alpha \) if \( \alpha \) is smaller than \( f(x_0) + f(z_i) - f(z_1) \), see (26)) that

\[
f(x_0) + f(z_i) - f(z_1) < \alpha < 2f(z_1) - f(z_i) \quad (241)
\]

For such an \( \alpha \), let us define \( K_\alpha := f^{-1}((-\infty, \alpha]) \cap \Omega \).

We would like to show that (240) holds when \( X_0 = x \in f^{-1}((-\infty, \alpha]) \cap \Omega \), for any test function \( F \in C^\infty(\partial \Omega) \) satisfying \( \text{supp} \ F \subset B_{z_i} \) and \( z_i \in \text{int} (\text{supp} \ F) \).

Let us introduce the principal eigenfunction \( \tilde{u}_h \) of \( L_{f,h}^{(0)}(\Omega) \):

\[
\begin{cases}
-L_{f,h}^{(0)} \tilde{u}_h = \lambda_h \tilde{u}_h \text{ on } \Omega, \\
\tilde{u}_h = 0 \text{ on } \partial \Omega,
\end{cases}
\]

(242)

with \( \tilde{u}_h > 0 \) on \( \Omega \) and normalized such that

\[
\int_{\Omega} \tilde{u}_h^2 \, dx = 1.
\]

(243)

Notice that \( u_h \) solution to (12) only differs from \( \tilde{u}_h \) by a multiplicative constant so that, from Proposition 4

\[
\nu_h(dx) = Z_h(\Omega)^{-1} \tilde{u}_h(x)e^{-\frac{2}{h}f(x)}dx,
\]

(244)

where, for any set \( O \subset \Omega \),

\[
Z_h(O) = \int_O \tilde{u}_h \, e^{-\frac{2}{h}f}.
\]

For \( F \in C^\infty(\partial \Omega) \), let us define \( w_h(x) = \mathbb{E}^\nu[F(X_{\tau_h})] \) for all \( x \in \overline{\Omega} \). The function \( w_h \) is such that: \( \forall h > 0 \) and \( x \in \Omega \),

\[
|w_h(x)| \leq \|F\|_{L^\infty}.
\]

Moreover, a standard Feynman-Kac formula shows that \( w_h \) satisfies

\[
\begin{cases}
L_{f,h}^{(0)} w_h = 0 \text{ on } \Omega, \\
w_h = F \text{ on } \partial \Omega,
\end{cases}
\]

(245)

where, we recall, the differential operator \( L_{f,h}^{(0)} \) is defined by (10). Our objective is to compare \( w_h(x) \) with \( \mathbb{E}^\nu[F(X_{\tau_h})] \).
By the Markov property, using (244), we have

$$
E^{\nu_h} [F(\tau_{\Omega})] = \left( \int_{\Omega} \tilde{\nu}_h e^{-\frac{r}{2} f} \right)^{-1} \left( \int_{\Omega} w_h \tilde{\nu}_h e^{-\frac{r}{2} f} \right) = Z_h^{-1}(\Omega) \left( \int_{\Omega \setminus K_\alpha} w_h \tilde{\nu}_h e^{-\frac{r}{2} f} \right) + Z_h^{-1}(\Omega) \left( \int_{K_\alpha} w_h \tilde{\nu}_h e^{-\frac{r}{2} f} \right). \tag{246}
$$

In order to estimate the first term in (246), we need a leveling property for $\tilde{\nu}_h$, which is stated in [15, Theorem 2.4].

**Lemma 68.** Let us assume that [H1], [H2] and [H3] hold and let us consider $\tilde{\nu}_h$ the principal eigenfunction of $L_{f,h}^{D,(0)}(\Omega)$ (see (242)) with normalization (243). Then, for any compact set $K \subset \Omega$,

$$
\lim_{h \to 0} \left\| \tilde{\nu}_h - \left( \int_{\Omega} dx \right)^{-1/2} \right\|_{L^\infty(K)} = 0.
$$

Notice that the reason why we consider a smooth test function $F$ rather than $1_\Sigma$ is that we would like to apply the results in [15].

A direct consequence of Lemma 68 is the following limit,

$$
\lim_{h \to 0} h \ln \left( Z_h(\Omega) \right) = -2f(x_0). \tag{247}
$$

Indeed, from the normalization of $\tilde{\nu}_h$, we get $Z_h(\Omega) \leq e^{-\frac{r}{2} f(x_0)}$, and from Lemma 68 we have, for $h$ small enough and for $r > 0$ such that the open ball $B(x_0, 2r)$ is included in $\Omega$,

$$
Z_h(\Omega) \geq Z_h(B(x_0, r)) \geq \frac{1}{2} \left( \int_{\Omega} dx \right)^{-1/2} \int_{B(x_0, r)} e^{-\frac{2r}{2}}.
$$

Since $\lim_{h \to 0} h \ln \left( \int_{B(x_0, r)} e^{-\frac{2r}{2}} dx \right) = -2f(x_0)$, we get (247).

Now, for the first term in (246), using (247), we have for any $\delta > 0$, for $h$ small enough,

$$
Z_h(\Omega)^{-1} \left| \int_{\Omega \setminus K_\alpha} w_h \tilde{\nu}_h e^{-\frac{r}{2} f} \right| \leq \| F \|_{L^\infty} e^{\frac{\delta}{2}} e^{-\frac{2}{2} (\inf_{\Omega \setminus K_\alpha} f - f(x_0))} = \| F \|_{L^\infty} e^{\frac{\delta}{2}} e^{-\frac{2}{2} (\alpha - f(x_0))}
$$

and thus, thanks to (241), by choosing $\delta$ small enough, there exists $c > 0$ such that, for all $h$ small enough,

$$
Z_h(\Omega)^{-1} \left| \int_{\Omega \setminus K_\alpha} w_h \tilde{\nu}_h e^{-\frac{r}{2} f} \right| \leq \| F \|_{L^\infty} e^{\frac{\delta}{2}} e^{-\frac{2}{2} (f(z_1) - f(z_1) + c)}. \tag{248}
$$

In order to estimate the second term in (246), we need a leveling property for $w_h$.

**Lemma 69.** Let us assume that [H1], [H2] and [H3] hold, as well as (241). Let us consider $w_h$ solution to (245). Then there exists $C > 0$ such that for any $\delta > 0$, for any $h$ small enough, for all $x, y \in K_\alpha$,

$$
|w_h(x) - w_h(y)| \leq C e^{\frac{\delta}{2}} e^{-\frac{2}{2} (f(z_1) - \alpha)}.
$$

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Proof. From [21, Theorem 1], it is known that for any $\delta > 0$, for any $h$ small enough and for all $x, y \in K_\alpha$,
\[ |w_h(x) - w_h(y)| \leq C e^{\frac{\delta}{\pi}} e^{-\frac{2}{\pi} V_\Omega(K_\alpha)}, \]
where $V_\Omega(K_\alpha)$ is defined by,
\[ V_\Omega(K_\alpha) = \inf_{x \in K_\alpha} \inf_{T > 0} \inf_{\gamma \in \text{Abs}(T, x, \partial \Omega)} \frac{1}{4} \int_0^T |\gamma + \nabla f(\gamma)|^2 dt \]
where $\text{Abs}(T, x, \partial \Omega)$ is the set of absolutely continuous functions $\gamma : [0, T] \to \overline{\Omega}$ satisfying $\gamma(0) = x$ and $\gamma(T) \in \partial \Omega$. For any $\gamma \in \text{Abs}(T, x, \partial \Omega)$, we have
\[ \int_0^T |\gamma + \nabla f(\gamma)|^2 dt - \int_0^T |\gamma - \nabla f(\gamma)|^2 dt = 4 \int_0^T \gamma \cdot \nabla f(\gamma) dt = 4 (f(\gamma(T)) - f(x)), \]
and therefore, it holds
\[ \int_0^T |\gamma + \nabla f(\gamma)|^2 dt \geq 4 (f(\gamma(T)) - f(x)) \geq 4 (f(z_1) - f(x)). \]
Finally we obtain
\[ V_\Omega(K_\alpha) \geq f(z_1) - \max_{x \in K_\alpha} f(x) = f(z_1) - \alpha. \]
This concludes the proof of Lemma 69.

We are now in position to estimate the second term in (246). Using Lemma 69, we get, for any $\delta > 0$, in the limit $h \to 0$, uniformly in $y_0 \in K_\alpha$,
\[ Z_h^{-1}(\Omega) \left( \int_{K_\alpha} w_h \tilde{u}_h e^{-\frac{2}{\pi} f} dx \right) = w_h(y_0) \frac{Z_h(K_\alpha)}{Z_h(\Omega)} + O \left( e^{\frac{\delta}{\pi}} e^{-\frac{2}{\pi} (f(z_1) - \alpha)} \right) \frac{Z_h(K_\alpha)}{Z_h(\Omega)}. \]
Therefore, by choosing $\delta > 0$ small enough, thanks to (241), there exists $c > 0$ such that, in the limit $h \to 0$,
\[ Z_h^{-1}(\Omega) \left( \int_{K_\alpha} w_h \tilde{u}_h e^{-\frac{2}{\pi} f} dx \right) = w_h(y_0) \frac{Z_h(K_\alpha)}{Z_h(\Omega)} + O \left( e^{-\frac{2}{\pi} f(z_1) - f(z_1) + c} \right) \frac{Z_h(K_\alpha)}{Z_h(\Omega)}. \]
In addition we have
\[ \frac{Z_h(K_\alpha)}{Z_h(\Omega)} = 1 + O \left( e^{-\frac{2}{\pi}} \right) \]
for some $c > 0$ independent of $h$. Indeed
\[ \frac{Z_h(K_\alpha)}{Z_h(\Omega)} = 1 - \frac{Z_h(\Omega \setminus K_\alpha)}{Z_h(\Omega)} \]
and using (247), we get for any $\delta > 0$,
\[ \frac{Z_h(\Omega \setminus K_\alpha)}{Z_h(\Omega)} \leq e^{\frac{\delta}{\pi}} e^{-\frac{2}{\pi} (\min_{x \in K_\alpha} f - f(z_0))} = O \left( e^{-\frac{2}{\pi}} \right), \]
for some $c > 0$ independent of $h$ by choosing $\delta$ small enough. Gathering the results (248), (249), (250) in (246), there exists $c > 0$ independent of $h$ such that, in the limit $h \to 0$, it holds: uniformly in $y_0 \in K_\alpha$,
\[ \mathbb{P}^n[F(X_{\tau_0})] = w_h(y_0) \left( 1 + O \left( e^{-\frac{2}{\pi}} \right) \right) + O \left( e^{-\frac{2}{\pi} f(z_1) - f(z_1) + c} \right), \]
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Let $i \in \{1, \ldots, n\}$ and let us assume that $\text{supp} F \subset B_{z_i}$ and $z_i \in \text{int} (\text{supp} F)$. Then, combining the last estimate with (240) implies that uniformly in $x \in f^{-1}((-\infty, a]) \cap \Omega$, in the limit $h \to 0$:

$$
\mathbb{E}^x[F(X_{\tau_n})] = \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess}f_{\partial \Omega}(z_i)}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess}f_{\partial \Omega}(z_k)}} \right)^{-1} e^{-\frac{\pi}{8}(f(z_i)-f(z_1))(F(z_i)+O(h))}.
$$

(251)

Let $\Sigma_i \subset \partial \Omega$ containing $z_i$ and such that $\bar{\Sigma}_i \subset B_{z_i}$. Then, there exist $F, G \in C^\infty(\partial \Omega)$ such that $\text{supp} F \subset B_{z_i}$, $z_i \in \text{int} (\text{supp} F)$, $\text{supp} G \subset B_{z_i}$, $z_i \in \text{int} (\text{supp} G)$, $F \leq 1_{\Sigma_i} \leq G$ and $F(z_i) = G(z_i) = 1$. From the inequality

$$
\mathbb{E}^x[F(X_{\tau_n})] \leq \mathbb{P}^x[F_{\tau_n} \in \Sigma_i] \leq \mathbb{E}^x[G(X_{\tau_n})],
$$

together with (251) applied to $F$ and $G$, one gets in the limit $h \to 0$:

$$
\mathbb{P}^x[F_{\tau_n} \in \Sigma_i] = \frac{\partial_n f(z_i)}{\sqrt{\det \text{Hess}f_{\partial \Omega}(z_i)}} \left( \sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \text{Hess}f_{\partial \Omega}(z_k)}} \right)^{-1} e^{-\frac{\pi}{8}(f(z_i)-f(z_1))(1+O(h))}.
$$

This concludes the proof of Corollary 10.

5.2 Proofs of Theorem 2 and Corollary 11

In this section, we prove Theorem 2. The proof is similar to the one made for Theorem 1: the estimates of Proposition 17 and the construction of the quasi-mode associated with $z_{j_0}$ are modified. The proof of Theorem 2 is organized as follows. In Section 5.2.1, we give the estimates required for the $n+1$ quasi-modes. Then, in Section 5.2.2, we prove that these estimates imply Theorem 2. In Section 5.2.3, the construction of the quasi-modes is given and we check that they satisfy the estimates stated in Section 5.2.1.

5.2.1 Statement of the assumptions required for the quasi-modes

For the ease of notation, for $p \in \{0, 1\}$, the orthogonal projector $\pi_{[0, 2\pi]}(-L^D_{f,h}(p)(\Omega))$ is still denoted by $\pi^{(p)}_{h}$, see [54].

The next proposition gives the assumption we need on the quasi-modes $(\tilde{\psi}_i)_{i=1,\ldots,n}$, whose span approximates $\text{Ran} \pi^{(1)}_{h}$, and $\tilde{u}$ whose span approximates $\text{Ran} \pi^{(0)}_{h}$, in order to prove Theorem 2. It is the equivalent of Proposition 17 in the more general setting of Theorem 2.

**Proposition 70.** Let us assume that the hypotheses [H1], [H2] and [H3] hold. Let $\Sigma_i$ denotes an open set included in $\partial \Omega$ containing $z_i$ ($i \in \{1, \ldots, n\}$) and such that $\bar{\Sigma}_i \subset B_{z_i}$. Let $k_0 \in \{1, \ldots, n\}$ and $f^*$ such that

$$
f(z_{k_0}) \leq f^* \leq f(z_{k_0+1}),
$$

(with the convention $f(z_{k_0+1}) = +\infty$ if $k_0 = n$). Finally, let $\Sigma \subset \partial \Omega$ be a smooth open set such that $\Sigma \subset B_{z_{j_0}}$, for some $j_0 \in \{1, \ldots, k_0\}$ and $\inf_{\Sigma} f = f^*$.

Let us assume that there exist $n$ quasi-modes $(\tilde{\psi}_i)_{i=1,\ldots,n}$ and a family of quasi-modes $(\tilde{u} = \tilde{u}_\delta)_{\delta>0}$ satisfying the following conditions:
1. \( \forall i \in \{1, \ldots, n\}, \hat{\psi}_i \in \Lambda^1 H_{w,T}^1(\Omega) \) and \( \tilde{u} \in \Lambda^0 H_{w,T}^1(\Omega) \). The function \( \tilde{u} \) is non-negative in \( \Omega \). Moreover, \( \forall i \in \{1, \ldots, n\}, \left\| \hat{\psi}_i \right\|_{L^2_w} = \left\| \tilde{u} \right\|_{L^2_w} = 1. 

2. (a) There exists \( \varepsilon_1 > 0 \) such that for all \( i \in \{1, \ldots, k_0\} \), it holds in the limit \( h \to 0 \):

\[
\left\| \left( 1 - \pi_h^{(1)} \right) \hat{\psi}_i \right\|_{H^1_w}^2 = O \left( e^{-\frac{2}{h} (f^*(z_i) - f(z_1) + \varepsilon_1)} \right),
\]

and for all \( i \in \{k_0 + 1, \ldots, n\} \),

\[
\left\| \left( 1 - \pi_h^{(1)} \right) \hat{\psi}_i \right\|_{H^1_w}^2 = O \left( e^{-\frac{2}{h} (f^*(z_i) + \varepsilon_1)} \right).
\]

(b) For any \( \delta > 0 \),

\[
\left\| \nabla \tilde{u} \right\|_{L^2_w}^2 \equiv O \left( e^{-\frac{2}{h} (f(z_1) - f(x_0) - \delta)} \right).
\]

3. There exists \( \varepsilon_0 > 0 \), \( \forall (i, j) \in \{1, \ldots, n\}^2 \), if \( i < j \leq k_0 \) in the limit \( h \to 0 \):

\[
\langle \hat{\psi}_i, \tilde{\psi}_j \rangle_{L^2_w} = O \left( e^{-\frac{2}{h} (f(z_i) - f(z_i) + \varepsilon_0)} \right),
\]

and if \( k_0 < j < i < j \), in the limit \( h \to 0 \):

\[
\langle \hat{\psi}_i, \tilde{\psi}_j \rangle_{L^2_w} = O \left( e^{-\frac{2}{h} (f^*(z_1) + \varepsilon_0)} \right).
\]

4. (a) There exists \( \varepsilon' > 0 \) and there exist constants \((C_i)_{i=1,\ldots,n}\) and \( p \) which do not depend on \( h \) such that for \( i \in \{1, \ldots, k_0\} \), in the limit \( h \to 0 \):

\[
\left\| \nabla \tilde{u}, \hat{\psi}_i \right\|_{L^2_w} = C_i \ h^p e^{-\frac{2}{h} f(z_i)} \ (1 + O(h)),
\]

and for \( i \in \{k_0 + 1, \ldots, n\} \), in the limit \( h \to 0 \):

\[
\left\| \nabla \tilde{u}, \hat{\psi}_i \right\|_{L^2_w} = C_i \ h^p e^{-\frac{2}{h} f(z_i)} \ (1 + O(h)) + O \left( e^{-\frac{2}{h} (f^*(z_1) + \varepsilon)} \right).
\]

(b) There exist constants \((B_i)_{i=1,\ldots,n}\) and \( m \) which do not depend on \( h \) such that for all \( (i, j) \in \{1, \ldots, n\}^2 \), in the limit \( h \to 0 \):

\[
\int_{\Sigma_i} \tilde{\psi}_j \cdot n e^{-\frac{2}{h} f} d\sigma = \begin{cases} B_i h^m e^{-\frac{2}{h} f(z_i)} (1 + O(h)) & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

(c) There exist \( C^* \) and \( p^* \) independent of \( h \) such that for all \( i \in \{1, \ldots, n\} \), in the limit \( h \to 0 \):

\[
\int_{\Sigma} \tilde{\psi}_i \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{i,j_0} C^* h^{p^*} e^{-\frac{1}{h} (2 f^*(z_{j_0}) - f(z_j))] (1 + O(h)).
\]

Let \( u_h \) be the eigenfunction associated with the smallest eigenvalue of \( -L_{f,h}^{(D)}(0) \) (see Proposition [4]) which satisfies [11]. Then, one has:

- For all \( i \in \{1, \ldots, k_0\} \), in the limit \( h \to 0 \)

\[
\int_{\Sigma_i} (\partial_{n} u_h) e^{-\frac{2}{h} f} d\sigma = C_i B_i h^{p+m} e^{-\frac{2}{h} (f(z_i) - f(x_0))] (1 + O(h)).
\]

Moreover, if \( f(z_{k_0}) < f(z_{k_0+1}) \), there exists \( \varepsilon > 0 \) such that for all \( i \in \{k_0 + 1, \ldots, n\} \) in the limit \( h \to 0 \)

\[
\int_{\Sigma_i} (\partial_{n} u_h) e^{-\frac{2}{h} f} d\sigma = \left( \int_{\Sigma_{k_0}} (\partial_{n} u_h) e^{-\frac{2}{h} f} d\sigma \right) O \left( e^{-\frac{2}{h}} \right).
\]
• In the limit $h \to 0$:
\[
\int_{\Sigma} (\partial_h u_h) e^{-\frac{2}{h}f} d\sigma = C\cdot C_{j_0} h^n + p e^{-\frac{2}{h}(2f^* - f(x_0))} (1 + O(h)).
\]

The estimates (22), (31) and (34) in Theorem 2 are a consequence of this proposition and of the construction of the appropriate quasi-modes $(\tilde{\psi}_i)_{i=1,\ldots,n}$, see Section 5.2.3 which will show that $(B_i)_{i=1,\ldots,n}$, $m$, $(C_i)_{i=1,\ldots,n}$, $p$ are given by (231)-(235) and $C^*$, $q^*$ are given by Lemma 77. Moreover, the remainder estimates (24), (32) and (35) in Theorem 2 are a consequence of the asymptotics (22), (31) and (34) together with Proposition 6, Proposition 7 and (14).

5.2.2 Proof of Proposition 70

The proof of Proposition 70 follows closely the same steps as the proof of Proposition 17. We only highlight the main differences. In all this section, we assume that the hypotheses [H1], [H2] and [H3] hold. Let $f^* \in \mathbb{R}$, $k_0 \in \{1, \ldots, n\}$, $j_0 \in \{1, \ldots, k_0\}$, $(\Sigma_i)_{i \in \{1, \ldots, n\}}$ and $\Sigma$ be as stated in Proposition 70. In addition, let us assume the existence of $n + 1$ quasi-modes $(\tilde{u}_i, (\tilde{\psi}_i)_{i=1,\ldots,n})$ satisfying all the conditions of Proposition 70. In the following, $\varepsilon$ denotes a positive constant independent of $h$, smaller than $\min(\varepsilon_1, \varepsilon_0, \varepsilon')$, and whose precise value may vary (a finite number of times) from one occurrence to the other.

Let us recall a result relating $\tilde{u}$ with $u_h$ on the one hand, and $\text{span}(\tilde{\psi}_j, j = 1, \ldots, n)$ with $\text{Ran} \pi_h^{(1)}$ on the other hand. The following lemma is a direct consequence of Lemma 18 Lemma 19 and the assumptions 1, 2 and 3 of Proposition 70.

**Lemma 71.** Let us assume that the assumptions of Proposition 70 hold. Then, there exist $c > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,
\[
\left\| \pi_h^{(0)} \tilde{u} \right\|_{L^2}^2 = 1 + O \left( e^{-\frac{c}{h}} \right).
\]
In addition, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$,
\[
\text{span}(\pi_h^{(1)} \tilde{\psi}_i, i = 1, \ldots, n) = \text{Ran} \pi_h^{(1)}.
\]

A direct consequence of Lemma 71 and the fact that $\tilde{u}$ is non negative in $\Omega$ is that it holds for $h$ small enough:
\[
u_h = \frac{\pi_h^{(0)} \tilde{u}}{\left\| \pi_h^{(0)} \tilde{u} \right\|_{L^2}^2}. \tag{252}
\]

Let us denote by $(\psi_i)_{i=1,\ldots,n}$ the orthonormal basis of $\text{Ran} \pi_h^{(1)}$ resulting from the Gram-Schmidt orthonormalization procedure on the set $(\pi_h^{(1)} \tilde{\psi}_i)_{i=1,\ldots,n}$ (see Lemma 20). Then, since $\nabla u_h \in \text{Ran} \left( \pi_h^{(1)} \right)$ is span $(\psi_j, j = 1, \ldots, n)$ (see (53)) and $\langle \psi_j, \psi_i \rangle_{L^2_\Sigma} = \delta_{i,j}$, one has for any $\Gamma \subset \partial\Omega$:
\[
\int_{\Gamma} (\partial_h u_h) e^{-\frac{2}{h}f} d\sigma = \sum_{j=1}^n \langle \nabla u_h, \psi_j \rangle_{L^2_\Sigma} \int_{\Gamma} \psi_j \cdot n e^{-\frac{2}{h}f} d\sigma. \tag{253}
\]

Let $(\kappa_{ij})_{(i,j) \in \{1,\ldots,n\}^2, i < j}$ and $(Z_{ij})_{(i,j) \in \{1,\ldots,n\}}$ be the matrix and vector obtained through the Gram-Schmidt orthonormalization procedure, see Lemma 20.
Now, to prove Proposition 70, the strategy consists in proving precise estimates when $h \to 0$, of the terms

$$\kappa_{ji}, \ Z_j, \ \langle \nabla u_h, \psi_j \rangle_{L^2_w} \text{ and } \left( \int_\Gamma \psi_j \cdot n \ e^{-\frac{\theta}{h}f} \ d\sigma \right)_{\Gamma \in \{\Sigma, \Sigma_1, \ldots, \Sigma_n\}}$$

for $(i, j) \in \{1, \ldots, n\}^2$, $i < j$. Then, they will be used to obtain a precise estimate of (253) when $h \to 0$. This is the purpose of the next steps.

Step 1. Estimates on the terms $(\kappa_{ji})_{(i, j) \in \{1, \ldots, n\}^2, i < j}$ and of $(Z_i)_{i \in \{1, \ldots, n\}}$.

**Lemma 72.** Let us assume that the assumptions of Proposition 70 hold. Then, there exist $\varepsilon > 0$ and $h_0 > 0$ such that for all $(i, j) \in \{1, \ldots, n\}^2$ with $i < j$ and all $h \in (0, h_0)$, if $j \leq k_0$:

$$\left\langle \pi_h^{(1)} \tilde{\psi}_i, \pi_h^{(1)} \tilde{\psi}_j \right\rangle_{L^2_w} = O \left( e^{-\frac{1}{h}(f(z_j)-f(z_i)+\varepsilon)} \right),$$

and if $j > k_0$:

$$\left\langle \pi_h^{(1)} \tilde{\psi}_i, \pi_h^{(1)} \tilde{\psi}_j \right\rangle_{L^2_w} = O \left( e^{-\frac{1}{h}(f^*-f(z_i)+\varepsilon)} \right).$$

**Proof.** The proof follows the same lines as the proof of Lemma 21. If $i < j$ and $j \leq k_0$, from assumption 2(a) in Proposition 70 and since $f^* \geq f(z_1)$, one gets

$$\left\langle (1 - \pi_h^{(1)}) \tilde{\psi}_i, (1 - \pi_h^{(1)}) \tilde{\psi}_j \right\rangle_{L^2_w} \leq \|(1 - \pi_h^{(1)}) \tilde{\psi}_i\|_{L^2_w} \|(1 - \pi_h^{(1)}) \tilde{\psi}_j\|_{L^2_w} \leq O \left( e^{-\frac{1}{h}(f^*-f(z_i)+f(z_j)-f(z_1)+\varepsilon)} \right) = O \left( e^{-\frac{1}{h}(f(z_j)-f(z_i)+\varepsilon)} \right).$$

If $i < j$ and $k_0 < j$, from assumptions 1 and 2(a) in Proposition 70, one gets

$$\left\langle (1 - \pi_h^{(1)}) \tilde{\psi}_i, \pi_h^{(1)} \tilde{\psi}_j \right\rangle_{L^2_w} \leq \|\tilde{\psi}_i\|_{L^2_w} \|\pi_h^{(1)} \tilde{\psi}_j\|_{L^2_w} \leq O \left( e^{-\frac{1}{h}(f^*-f(z_i)+\varepsilon)} \right).$$

Lemma 72 is then a consequence of (62) together with assumption 3 in Proposition 70.

**Lemma 73.** Let us assume that the assumptions of Proposition 70 hold. Then, there exist $\varepsilon > 0$ and $h_0 > 0$ such that for all $(i, j) \in \{1, \ldots, n\}^2$ with $i < j$ and all $h \in (0, h_0)$, if $j \leq k_0$:

$$\kappa_{ji} = O \left( e^{-\frac{1}{h}(f(z_j)-f(z_i)+\varepsilon)} \right),$$

and if $j > k_0$:

$$\kappa_{ji} = O \left( e^{-\frac{1}{h}(f^*-f(z_i)+\varepsilon)} \right).$$

In addition, there exist $c > 0$ and $h_0 > 0$ such that for all $j \in \{1, \ldots, n\}$ and $h \in (0, h_0)$,

$$Z_j = 1 + O \left( e^{-\frac{1}{h}} \right).$$

**Proof.** If $i < j$ and $j \leq k_0$, the estimates on $\kappa_{ji}$ and $Z_j$ are proved by induction as in the proof of Lemma 22. Let us now deal with the case $i < j$ and $k_0 < j$. For $j = k_0 + 1$, it follows from (66) that for all $i < k_0 + 1$,

$$\kappa_{(k_0+1)i} = \sum_{k=i}^{k_0} \sum_{l=1}^{k} \frac{1}{w^2} \left\langle \pi_h^{(1)} \tilde{\psi}_{k_0+1}, \pi_h^{(1)} \tilde{\psi}_l \right\rangle_{L^2_w} \kappa_{kl} \kappa_{ki},$$

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where we use the notation $\kappa_{ii} = 1$ for every $i \in \{1, \ldots, n\}$. Since $1 \leq k \leq k_0$, one has $Z_k^{-1} = 1 + O\left(e^{-\frac{1}{k}}\right)$. In addition, since $1 \leq l \leq k \leq k_0$ and $1 \leq i \leq k \leq k_0$, one has $\kappa_{kl}\kappa_{ki} = O(1)$. From Lemma 72, one has for $1 \leq l < k_0 + 1$, $\langle \pi_h^{(1)} \bar{\psi}_{k_0+1}, \pi_h^{(1)} \bar{\psi}_l \rangle_{L^2_h} = O\left(e^{-\frac{1}{k}(J^{*} - f(z_1)+\varepsilon)}\right)$. Therefore, one obtains for all $i < k_0 + 1$,

$$\kappa_{(k_0+1)i} = O\left(e^{-\frac{1}{k}(J^{*} - f(z_1)+\varepsilon)}\right).$$

The fact that $Z_{k_0+1} = 1 + O\left(e^{-\frac{1}{k}}\right)$, comes from the fact that the terms $\langle \kappa_{(k_0+1)i} \rangle_{i \in \{1, \ldots, k_0\}}$ are exponentially small and the fact that $\left\| \pi_h^{(1)} \bar{\psi}_{k_0+1} \right\|_{L^2_h} = 1 + O\left(e^{-\frac{1}{k}}\right)$. In order to prove by induction the estimates on $\kappa_{ji}$ for $i < j$ and $j > k_0$, let us now assume that for some $k \in \{k_0+1, \ldots, n\}$ and for all $j \in \{k_0+1, \ldots, k\}$, $i \in \{1, \ldots, j-1\}$,

$$\kappa_{ji} = O\left(e^{-\frac{1}{k}(J^{*} - f(z_1)+\varepsilon)}\right) \quad \text{and} \quad Z_j = 1 + O\left(e^{-\frac{1}{k}}\right).$$

It follows from (66), for $q \in \{1, \ldots, k\}$,

$$\kappa_{(k+1)q} = \sum_{j=q}^{k} \sum_{l=1}^{j} \frac{1}{2Z_j^2} \langle \pi_h^{(1)} \bar{\psi}_{k+1}, \pi_h^{(1)} \bar{\psi}_l \rangle_{L^2_h} \kappa_{jl}\kappa_{jq},$$

where we used the notation $\kappa_{ii} = 1$. Since $1 \leq j \leq k$, one has $Z_j^{-1} = 1 + O\left(e^{-\frac{1}{k}}\right)$. In addition, since $1 \leq l \leq j \leq k$ and $1 \leq q \leq j \leq k$, one has $\kappa_{jl}\kappa_{jq} = O(1)$. From Lemma 72, one has for $1 \leq l < k + 1$ and $k > k_0$, $\langle \pi_h^{(1)} \bar{\psi}_{k+1}, \pi_h^{(1)} \bar{\psi}_l \rangle_{L^2_h} = O\left(e^{-\frac{1}{k}(J^{*} - f(z_1)+\varepsilon)}\right)$. Therefore, one obtains for all $1 \leq q < k + 1$,

$$\kappa_{(k+1)q} = O\left(e^{-\frac{1}{k}(J^{*} - f(z_1)+\varepsilon)}\right).$$

The fact that $Z_{k+1} = 1 + O\left(e^{-\frac{1}{k}}\right)$, comes from the fact that the $(\kappa_{(k+1)q})_{q \in \{1, \ldots, k\}}$ are exponentially small and the fact that $\left\| \pi_h^{(1)} \bar{\psi}_{k+1} \right\|_{L^2_h} = 1 + O\left(e^{-\frac{1}{k}}\right)$. This concludes the proof by induction.

\textbf{Step 2.} Estimates on the interaction terms $\langle \nabla u_h, \psi_j \rangle_{L^2_h}$ for $j \in \{1, \ldots, n\}$.

\textbf{Lemma 74.} Let us assume that the assumptions of Proposition 70 hold. Then, for $j \in \{1, \ldots, k_0\}$, in the limit $h \to 0$:

$$\langle \nabla u_h, \psi_j \rangle_{L^2_h} = C_j h^p e^{-\frac{1}{k}(f(z_j)-f(x_0))} (1 + O(h)),$$

and for $j \in \{k_0 + 1, \ldots, n\}$, there exists $\varepsilon' > 0$ such that in the limit $h \to 0$:

$$\langle \nabla u_h, \psi_j \rangle_{L^2_h} = C_j h^p e^{-\frac{1}{k}(f(z_j)-f(x_0))} (1 + O(h)) + O\left(e^{-\frac{1}{k}(J^{*} - f(x_0)+\varepsilon')}\right).$$

\textbf{Proof.} For $j \in \{1, \ldots, k_0\}$, the proof of the estimate of $\langle \nabla u_h, \psi_j \rangle_{L^2_h}$ is exactly the same as for Lemma 23. Let $j \in \{k_0 + 1, \ldots, n\}$. Using (67)–(61)–(64)–(65), one has

$$\langle \nabla u_h, \psi_j \rangle_{L^2_h} = a_j + b_j + c_j,$$

where $a_j$, $b_j$ and $c_j$ are defined by

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Lemma 75. With the convention $\sum_{\kappa_0}^{k_0} \delta_1 + \sum_{\kappa_0}^{k_0} \delta_1 \cdot O \left( \sum_{\kappa_0}^{k_0} \delta_1 \cdot O \right)$,

Therefore, choosing $\delta \in (0, \varepsilon)$, there exists $\varepsilon' > 0$ such that for $\varepsilon > 0$

This concludes the proof of Lemma 74.

Step 3. Estimates on the boundary terms $\left( \int_{\Gamma} \psi_j \cdot n e^{-\frac{2}{h}} f d\sigma \right)_{j \in \{1, \ldots, n\}, \Gamma \in \{\Sigma, \Sigma_1, \ldots, \Sigma_n\}}$.

Lemma 75. Let us assume that the assumptions of Proposition 70 hold. Then, there exists $\varepsilon > 0$ such that in the limit $h \to 0$ one has:

- If $k \in \{1, \ldots, k_0\}$ and $j \in \{1, \ldots, k_0\}$,

- If $k \in \{1, \ldots, k_0\}$ and $j \in \{k_0 + 1, \ldots, n\}$,

- If $k \in \{k_0 + 1, \ldots, n\}$ and $j \in \{1, \ldots, k_0\}$,

$$
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h}} f d\sigma = \delta_{j,k} B_k h^m e^{-\frac{2}{h} f(z_k)} (1 + O(h)) + O \left( e^{-\frac{2}{h} (2f(z_k) - f(z_j)) + \varepsilon} \right)
\quad + 1_{j > k} O \left( e^{-\frac{2}{h} f(z_k) + \varepsilon} \right).
$$

$$
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O \left( e^{-\frac{2}{h} (2f(z_{k_0}) - f(z_j)) + \varepsilon} \right).
$$

$$
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O \left( e^{-\frac{2}{h} (2f(z_{k_0}) - f(z_j)) + \varepsilon} \right).
$$

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If \( k \in \{k_0 + 1, \ldots, n\} \) and \( j \in \{k_0 + 1, \ldots, n\}\),

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j,k} O \left( h^m e^{-\frac{1}{h} f(z_k)} \right) + O \left( e^{-\frac{1}{h} (f(z_{k_0}) + \varepsilon)} \right).
\]

**Proof.** For all \( (j, k) \in \{1, \ldots, n\}^2 \), using (64)–(65) and Lemmata 73 and 71 together with assumption 4(b) in Proposition 70, one has in the limit \( h \to 0 \):

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = Z_j^{-1} \left[ \int_{\Sigma_k} \tilde{\psi}_j \cdot n e^{-\frac{2}{h} f} d\sigma + \int_{\Sigma_k} (\pi_h^{(1)} - 1) \tilde{\psi}_j \cdot n e^{-\frac{2}{h} f} d\sigma \right]
\]

\[
+ Z_j^{-1} \sum_{i=1}^{j-1} \kappa_{ij} \left( \int_{\Sigma_k} \tilde{\psi}_i \cdot n e^{-\frac{2}{h} f} d\sigma + \int_{\Sigma_k} (\pi_h^{(1)} - 1) \tilde{\psi}_i \cdot n e^{-\frac{2}{h} f} d\sigma \right)
\]

\[
= \delta_{j,k} B_h h^m e^{-\frac{1}{h} f(z_k)} (1 + O(h)) + \frac{1}{h} \left\| \left( 1 - \pi_h^{(1)} \right) \tilde{\psi}_j \right\|_{H^0_{\text{loc}}} O \left( e^{-\frac{1}{h} f(z_k)} \right)
\]

\[
+ \sum_{i=1}^{j-1} \kappa_{ij} \delta_{i,k} O \left( h^m e^{-\frac{1}{h} f(z_k)} \right) + \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h} (f(z_i) - f(z_k) + f(z_k) + \varepsilon)} \right).
\]

(254)

Let us now deal separately with the two cases \( k \in \{1, \ldots, k_0\} \) and \( k \in \{k_0 + 1, \ldots, n\} \).

In the following, we use assumption 2(a) in Proposition 70 and Lemma 73 to estimate (254).

Case 1: \( k \in \{1, \ldots, k_0\} \). If \( j \in \{1, \ldots, k_0\} \), from (254), one gets in the limit \( h \to 0 \):

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j,k} B_h h^m e^{-\frac{1}{h} f(z_k)} (1 + O(h)) + O \left( e^{-\frac{1}{h} (f(z_k) + \varepsilon)} \right)
\]

\[
+ \sum_{i=1}^{j-1} \delta_{i,k} O \left( e^{-\frac{1}{h} (f(z_i) + \varepsilon)} \right) + \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h} (f(z_i) - f(z_k) + f(z_k) + \varepsilon)} \right).
\]

Since \( f^* \geq f(z_k) \) for all \( k \in \{1, \ldots, k_0\} \) and since there exists \( i < j \) such that \( \delta_{i,k} = 1 \) if and only if \( j > k \), there exists \( \varepsilon > 0 \) such that for all \( (k, j) \in \{1, \ldots, k_0\}^2 \) and for all \( h \) small enough,

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j,k} B_h h^m e^{-\frac{1}{h} f(z_k)} (1 + O(h)) + O \left( e^{-\frac{1}{h} (2f(z_k) + \varepsilon)} \right) + 1_{j > k} O \left( e^{-\frac{1}{h} (f(z_k) + \varepsilon)} \right).
\]

If \( j \in \{k_0 + 1, \ldots, n\} \), from (254), one gets

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O \left( e^{-\frac{1}{h} (f^* - f(z_k) + \varepsilon)} \right) + \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h} (f(z_i) + \varepsilon)} \right).
\]

Case 2: \( k \in \{k_0 + 1, \ldots, n\} \). If \( j \in \{1, \ldots, k_0\} \), from (254) and since \( f(z_k) \geq f^* \geq f(z_{k_0}) \), one has

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O \left( e^{-\frac{1}{h} (f^* - f(z_k) + \varepsilon)} \right) + \sum_{i=1}^{j-1} O \left( e^{-\frac{1}{h} (f^* + f(z_k) + f(z_k) - 2f(z_k) + \varepsilon)} \right)
\]

\[
= O \left( e^{-\frac{1}{h} (2f(z_{k_0}) + f(z_k) + \varepsilon)} \right).
\]

If \( j \in \{k_0 + 1, \ldots, n\} \), from (254), one gets

\[
\int_{\Sigma_k} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j,k} O \left( h^m e^{-\frac{1}{h} f(z_k)} \right) + O \left( e^{-\frac{1}{h} (f(z_k) + \varepsilon)} \right),
\]

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which leads to the desired estimate since \( f(z_k) \geq f(z_{k+1}) \). This concludes the proof of Lemma 75.

**Lemma 76.** Let us assume that the assumptions of Proposition 70 hold. Then, for \( j \in \{1, \ldots, k_0\} \) one has when \( h \to 0 \):

\[
\int_{\Sigma} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j_0,j} C^* h^q e^{-\frac{1}{h} (2f^*-f(z_{j0}))} (1 + O(h)) + O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + \epsilon)} \right)
\]

and for \( j \in \{k_0 + 1, \ldots, n\} \) one has \( \int_{\Sigma} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + \epsilon)} \right) \).

**Proof.** Let \( j \in \{1, \ldots, n\} \). Using (64)–(65) and Lemmata 73 and 71 together with assumption 4(e) in Proposition 70 one has

\[
\int_{\Sigma} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = Z_j^{-1} \left[ \int_{\Sigma} \tilde{\psi}_j \cdot n e^{-\frac{2}{h} f} d\sigma + \sum_{i=1}^{j-1} k_{ji} \left( \int_{\Sigma} \tilde{\psi}_i \cdot n e^{-\frac{2}{h} f} d\sigma + \int_{\Sigma} (\pi_h^{(1)} - 1) \tilde{\psi}_j \cdot n e^{-\frac{2}{h} f} d\sigma \right) \right]
\]

\[
= \delta_{j_0,j} C^* h^q e^{-\frac{1}{h} (2f^*-f(z_{j0}))} (1 + O(h)) + \frac{1}{h} \left\| (1 - \pi_h^{(1)}) \tilde{\psi}_j \right\|_{H^1_{\tilde{\omega}}} O\left( e^{-\frac{1}{h} f} \right)
\]

\[
+ \sum_{i=1}^{j-1} k_{ji} O\left( h^q e^{-\frac{1}{h} (2f^*-f(z_{j0}))} \right) + \frac{k_{ji}}{h} \left\| (1 - \pi_h^{(1)}) \tilde{\psi}_i \right\|_{H^1_{\tilde{\omega}}} O\left( e^{-\frac{1}{h} f} \right).
\]

Let us first deal with the case \( j \in \{1, \ldots, k_0\} \). Using assumption 2(a) in Proposition 70 and Lemma 73 one gets from (255)

\[
\int_{\Sigma} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = \delta_{j_0,j} C^* h^q e^{-\frac{1}{h} (2f^*-f(z_{j0}))} (1 + O(h)) + O\left( e^{-\frac{1}{h} (f^*-f(z_j) + \epsilon + f^*)} \right)
\]

\[
+ \sum_{i=1}^{j-1} O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + f^*-f(z_{j0}) + \epsilon)} \right)
\]

\[
= \delta_{j_0,j} C^* h^q e^{-\frac{1}{h} (2f^*-f(z_{j0}))} (1 + O(h)) + O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + \epsilon)} \right).
\]

Let us now deal with the case \( j \in \{k_0 + 1, \ldots, n\} \). In that case, one obtains from (255), assumption 2(a) in Proposition 70, Lemma 73 together with the fact that \( f^* \geq f(z_i) \) for all \( i \in \{1, \ldots, k_0\} \) and \( f^* \geq f(z_{k0}) \geq f(z_{j0}) \),

\[
\int_{\Sigma} \psi_j \cdot n e^{-\frac{2}{h} f} d\sigma = O\left( e^{-\frac{1}{h} (f^*-f(z_j) + \epsilon + f^*)} \right) + \sum_{i=1}^{k_0} \delta_{j_0,i} O\left( e^{-\frac{1}{h} (f^*-f(z_j) + 2f^*-f(z_{j0}) + \epsilon)} \right)
\]

\[
+ \sum_{i=1}^{k_0} O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + f^*-f(z_{j0}) + \epsilon)} \right) + \sum_{i=k_0+1}^{j-1} O\left( e^{-\frac{1}{h} (2f^*-2f(z_j) + \epsilon + f^*)} \right)
\]

\[
= O\left( e^{-\frac{1}{h} (2f^*-f(z_j) + \epsilon)} \right).
\]

This concludes the proof of Lemma 76.
Step 4. Estimates on the boundary terms \( \left( \int_{\Gamma} (\partial_n u_h) e^{-\frac{2}{h} f} \, d\sigma \right)_{g \in \{\Sigma, \Sigma_1, ..., \Sigma_n\}} \).

We are now in position to conclude the proof of Proposition 70.

**Proof.** Let us assume that the assumptions of Proposition 70 hold. The proof is divided into two cases.

**Case 1:** \( \Gamma = \Sigma_k \) in (253) for some \( k \in \{1, \ldots, n\} \). If \( k \in \{1, \ldots, k_0\} \), from Lemmata 74 and 75 and the fact that \( f^* \geq f(z_{k_0}) \geq f(z_k) \), one obtains that there exists \( \varepsilon > 0 \) such that for all \( j \in \{1, \ldots, n\} \), in the limit \( h \to 0 \)

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_h} \psi_j \cdot n e^{-\frac{2}{h} f} \, d\sigma = \delta_{jk} B_k C_k h^{m+p} e^{-\frac{1}{n}(2f(z_k) - f(x_0))}(1 + O(h))
+ O \left( e^{-\frac{1}{n}(2f(z_k) - f(x_0) + \varepsilon)} \right).
\]

Therefore, from (253), one gets for all \( k \in \{1, \ldots, k_0\} \), in the limit \( h \to 0 \)

\[
\int_{\Sigma_h} (\partial_n u_h) e^{-\frac{2}{h} f} \, d\sigma = B_k C_k h^{m+p} e^{-\frac{1}{n}(2f(z_k) - f(x_0))}(1 + O(h)).
\]

If \( k \in \{k_0 + 1, \ldots, n\} \). From Lemmata 74 and 75 one has for \( j \in \{1, \ldots, k_0\} \):

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_h} \psi_j \cdot n e^{-\frac{2}{h} f} \, d\sigma = O \left( h^p e^{-\frac{1}{n}(2f(z_{k_0}) - f(x_0) + \varepsilon)} \right).
\]

and for \( j \in \{k_0 + 1, \ldots, n\} \) (since \( f(z_j) \geq f(z_{k_0+1}) \) and \( f(z_{k_0}) \leq f^* \leq (z_{k_0+1}) \)):

\[
\langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_h} \psi_j \cdot n e^{-\frac{2}{h} f} \, d\sigma = \delta_{jk} O \left( h^p e^{-\frac{1}{n}(2f(z_{k_0}) - f(x_0))} \right) + O \left( e^{-\frac{1}{n}(2f(z_{k_0}) - f(x_0) + \varepsilon)} \right).
\]

Therefore, if one assumes that \( f(z_{k_0+1}) \geq f(z_{k_0}) \), from (253), one gets for all \( k \in \{k_0 + 1, \ldots, n\} \) and for all \( h \) small enough

\[
\int_{\Sigma_h} (\partial_n u_h) e^{-\frac{2}{h} f} \, d\sigma = O \left( e^{-\frac{1}{n}(2f(z_{k_0}) - f(x_0) + \varepsilon)} \right).
\]

**Case 2:** \( \Gamma = \Sigma \) in (253). From (253) and using Lemma 76 and Lemma 74 one has

\[
\int_{\Sigma} \partial_n u_h e^{-\frac{2}{h} f} \, d\sigma = C_{j_0} C^h q^p e^{-\frac{1}{n}(2f^* - f(x_0))}(1 + O(h)) + \sum_{j=1, j \neq j_0}^{k_0} O \left( h^p e^{-\frac{1}{n}(f(x_0) + 2f^* + \varepsilon)} \right)
+ \sum_{j=k_0+1}^{n} O \left( h^p e^{-\frac{1}{n}(f(z_j) - f(x_0) + 2f^* - f(z_1) + \varepsilon)} \right) + O \left( e^{-\frac{1}{n}(f^* - f(x_0) + 2f^* - f(z_1) + \varepsilon)} \right)
= C_{j_0} C^h q^p e^{-\frac{1}{n}(2f^* - f(x_0))}(1 + O(h)),
\]

which is the desired result. Proposition 70 is proved.
5.2.3 Construction of the quasi-modes which satisfy the estimates of Proposition 70

In this section, we first construct the quasi-modes \((\tilde{\psi}_i)_{i \in \{1, \ldots, n\}}\) and the family of quasi-modes \((\tilde{\varphi}_i = \tilde{u}_i)_{i \in \mathbb{N}}\). Then, we prove that they satisfy the estimates stated in Proposition 70. In all this section, one assumes that the hypotheses [H1], [H2] and [H3] hold. Let \((\Sigma_i)_{i \in \{1, \ldots, n\}}\) and \(\Sigma\) be as in Proposition 70.

Construction of the quasi-modes.

The \(n + 1\) quasi-modes \(((\tilde{\psi}_i)_{i \in \{1, \ldots, n\}}, \tilde{u})\) are constructed as in Section 4.2 except \(\tilde{\psi}_{j_0}\) (where we recall that \(j_0 \in \{1, \ldots, n\}\) is such that \(\Sigma \subset B_{\tilde{z}_{j_0}}\)). Recall that for all \(j \in \{1, \ldots, n\}\), \(\tilde{\psi}_j\) is defined as:

\[
\tilde{\psi}_j = e^{\frac{\pi}{j} \delta} \phi_j \in \Lambda^1 H^1_{w,T}(\Omega).
\]  (256)

for a well chosen \(\tilde{\phi}_j \in \Lambda^1 H^1_{w}(\Omega)\).

Thanks to the hypotheses [H1], [H2] and [H3], one can introduce the \(n\) quasi-modes \(((\tilde{\phi}_i)_{i \in \{1, \ldots, n\}} \setminus \{j_0\}), \tilde{u})\) built in Section 4.2 (see Definitions 13 and 14).

The construction of \(\tilde{\phi}_{j_0}\) requires to take into account the set \(\Sigma\) in addition to the set \(\Sigma_{j_0}\) when defining the cut off function \(\chi_{j_0}\) in Definition 14. Let us make this precise. Let \(S_{M,j_0} := \{\hat{\Omega}_{j_0}, \Gamma_0, \Gamma_{1,j_0}, \Gamma_{2,j_0}, V_{\Gamma_{1,j_0}}\}\) be an ensemble of sets associated with \(z_{j_0}\), see Definition 12. Thanks to Proposition 70, the set \(\Gamma_{1,j_0}\) can be taken such that

\[
\Sigma_{j_0} \cup \tilde{\Sigma} \subset \Gamma_{1,j_0}.
\]

We recall that Section 4.1 was dedicated to the construction of a domain \(\hat{\Omega}_{j_0} \subset \Omega\) and a mixed Witten Laplacian \(\Delta_{f,h}^{M,(1)}(\hat{\Omega}_{j_0})\) (see (122)) associated with this ensemble of sets \(S_{M,j_0}\). Proposition 59 gives the spectral properties of the operator \(\Delta_{f,h}^{M,(1)}(\hat{\Omega}_{j_0})\). In the following, we consider a normalized eigenform \(u_{h,j_0}^{(1)} \in D(\Delta_{f,h}^{M,(1)}(\hat{\Omega}_{j_0}))\) associated with the first eigenvalue 0, i.e. such that

\[
\Delta_{f,h}^{M,(1)}(\hat{\Omega}_{j_0}) u_{h,j_0}^{(1)} = 0 \text{ in } \hat{\Omega}_{j_0} \text{ and } \left\| u_{h,j_0}^{(1)} \right\|_{L^2(\hat{\Omega}_{j_0})} = 1.
\]

The quasi-mode \(\tilde{\phi}_{j_0}\) is then defined as the following truncation of \(u_{h,j_0}^{(1)}\).

\[\text{Definition 15. Let us assume that the hypotheses [H1], [H2] and [H3] hold. Let } \chi_{j_0} \in C^\infty(\bar{\Omega}) \text{ be such that:}\]

1. \(\chi_{j_0} \in C^\infty_c(\hat{\Omega}_{j_0} \cup \Gamma_{1,j_0})\) (and thus \(\chi_{j_0} = 0\) on \(\Gamma_{2,j_0} \cup \Gamma_0\) and \(\chi_{j_0} = 0\) on a neighborhood of \(\partial \Omega \setminus \Gamma_{1,j_0}\)),

2. \(\chi_{j_0} = 1\) on a neighborhood of \(\Sigma_{j_0} \cup \Sigma\) in \(\overline{\Omega_{j_0}}\),

3. \(0 \leq \chi_{j_0} \leq 1\).

The quasi-mode \(\tilde{\phi}_{j_0}\) is defined on \(\Omega\) by:

\[
\tilde{\phi}_{j_0} := \sqrt{\int_{\Omega} \frac{\chi_{j_0} u_{h,j_0}^{(1)}(x)}{\left| \chi_{j_0}(x) u_{h,j_0}^{(1)}(x) \right|^2} \, dx}.
\]  (257)
The quasi-modes satisfy the estimates stated in proposition \([H3]\).

Using in addition to \([H1]\)-\([H2]\)-\([H3]\) the hypotheses \([29]\) and \([30]\), one easily obtains that \(v_1, v_2, \ldots, v_n\) satisfy the estimates 1, 2, 3, 4(a) and 4(b) stated in Proposition \([H3]\) following exactly the computations made on \(v_i = e^{ih/2}u_i\) in Section 4.5. 2(a) follows from \([224]\)-\([225]\)-\([29]\)-\([30]\), 2(b) is proven in Lemma \([60]\), 3 follows from \([228]\)-\([29]\), 4(b) is proven in Step 3 in Section 4.5 and 4(a) is a consequence of \([232]\)-\([233]\)-\([234]\)-\([29]\). In particular, one gets that the constants \((B_i)_{i=1, \ldots, n}, (C_i)_{i=1, \ldots, n}\) and \(p\) in Proposition \([H3]\) are given by \([231]\)-\([233]\).

The following lemma deals with the assumption 4(c) in Proposition \([H3]\) which requires to use Proposition \([65]\).

**Lemma 77.** Let us assume that the hypotheses \([H1]\), \([H2]\) and \([H3]\) hold. Let \(j \in \{1, \ldots, n\}\). Then, when \(h \to 0\), one has

\[
\int_{\Sigma} \tilde{\psi}_j \cdot n \ e^{-\frac{i}{h} f} \, d\sigma = \delta_{j_0, j} \frac{B^* \sqrt{2}}{\pi} \sqrt{\det \text{Hess} f(z_{j_0})} \ e^{-\frac{4}{h} f^*(f(z_{j_0}))} \left(1 + O(h)\right),
\]

where \(B^* \) and \(p^* \) are defined by \([33]\).

**Proof.** By construction, if \(j \neq j_0\) then \(\tilde{\psi}_j \equiv 0\) on \(B_{z_{j_0}}\). Let us deal with the case \(j = j_0\). Using \([256]\), one has

\[
\int_{\Sigma} \tilde{\psi}_{j_0} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma = \int_{\Sigma} \delta_{j_0} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma. \tag{258}
\]

Let \(u^{(1)}_{z_{j_0}, wkb}\) be the WKB expansion defined by \([174]\). Following the beginning of Section 4.4.2, let us consider

1. a neighborhood \(V_{\Gamma_{z_{j_0}}} \) of \(\Sigma\) in \(\bar{\Omega}\), which is stable under the dynamics \([183]\) and such that, for some \(\varepsilon > 0\), \(V_{\Gamma_{z_{j_0}}} + B(0, \varepsilon) \subset V_{\Gamma_{1, j_0}} \cap (\bar{\Omega}_{j_0} \cup \Gamma_{1, j_0})\)

2. and a cut-off function \(\chi_{wkb, j_0} \in C^{\infty}(\bar{\Omega}_{j_0} \cup \Gamma_{1, j_0})\) with \(\chi_{wkb, j_0} \equiv 1\) on a neighborhood of \(V_{\Gamma_{z_{j_0}}}\) such that \(\text{supp} \chi_{wkb, j_0} \subset V_{\Gamma_{1, j_0}} \cap (\bar{\Omega}_{j_0} \cup \Gamma_{1, j_0})\).

Using Proposition \([65]\), there exists \(c_{z_{j_0}}(h) \in \mathbb{R}^+\) such that

\[
\left\| e^{rac{i}{h} f_{*}(\cdot, z_{j_0})} \left( u^{(1)}_{\chi_{h, j_0} - c_{z_{j_0}} (h) u^{(1)}_{z_{j_0}, wkb}} \right) \right\|_{H^1(V_{\Gamma_{z_{j_0}}})} = O(h^\infty).
\]

Let us now introduce

\[
\tilde{\phi}_{z_{j_0}, wkb} := c_{z_{j_0}}(h) \chi_{wkb, j_0} u^{(1)}_{z_{j_0}, wkb}
\]

so that

\[
\int_{\Sigma} \tilde{\phi}_{j_0} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma = \int_{\Sigma} \tilde{\phi}_{z_{j_0}, wkb} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma + \int_{\Sigma} \left( \tilde{\phi}_{j_0} - \tilde{\phi}_{z_{j_0}, wkb} \right) \cdot n \ e^{-\frac{i}{h} f} \, d\sigma. \tag{259}
\]

Let us first deal with the term \(\int_{\Sigma} \tilde{\phi}_{z_{j_0}, wkb} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma\) in \([259]\). Using \([33]\), one has (since \(\Phi = f, \partial_n \Phi = -\partial_n f\) and \(a_0 = 1\) on \(\partial \Omega\), see \([179]\)) when \(h \to 0\),

\[
\int_{\Sigma} \tilde{\phi}_{z_{j_0}, wkb} \cdot n \ e^{-\frac{i}{h} f} \, d\sigma = c_{z_{j_0}}(h) \int_{\Sigma} \chi_{wkb, j_0} u^{(1)}_{z_{j_0}, wkb} \cdot n \ e^{-\frac{i}{h} f} = 2 c_{z_{j_0}}(h) \int_{\Sigma} \partial_n f \ e^{-\frac{i}{h} (2f - f(z_{j_0}))} (1 + O(h)) = 2 c_{z_{j_0}}(h) B^* h^p e^{-\frac{i}{h} (2f - f(z_{j_0}))} (1 + O(h)).
\]

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Then using (189), one obtains in the limit $h \to 0$:

$$\int_\Sigma \tilde{\phi}_{j_0, wkb} \cdot n \ e^{-\frac{1}{\hbar} f} d\sigma = \frac{B^* \sqrt{2} \ (\det \text{Hess} f|_{\partial \Omega}(z_{j_0}))^{\frac{1}{4}}} {\pi^{\frac{d-1}{2}}} \ e^{-\frac{\hbar}{4}(2f^* - f(z_{j_0}))} (1 + O(h)).$$

(260)

Let us now estimate the term $\int_\Sigma (\tilde{\phi}_{j_0} - \tilde{\phi}_{z_{j_0},wkb}) \cdot n \ e^{-\frac{1}{\hbar} f}$ in (259). Since $d_a(\cdot, z_{j_0}) = f - f(z_{j_0}) = \Phi - f(z_{j_0})$ on $\Sigma$, one obtains using Lemma 66 there exist $C > 0$, $h_0 > 0$ and $\eta > 0$ such that for all $h \in (0, h_0)$,

$$\left| \int_\Sigma (\tilde{\phi}_{j_0} - \tilde{\phi}_{z_{j_0},wkb}) \cdot n \ e^{-\frac{1}{\hbar} f} d\sigma \right| \leq C e^{-\frac{\hbar}{4}(2f^* - f(z_{j_0}))} \left\| \int_\Sigma \left( u_{j_0}(1) - c_{z_{j_0}}(h) u_{z_{j_0},wkb}(1) \right) e^\frac{d_a(\cdot, z_{j_0})}{\hbar} \right\|_{H^1(\Omega_{j_0})}.$$

Since it holds $u_{z_{j_0},wkb} e^\frac{\Phi(\cdot, z_{j_0})}{\hbar} = d_f(\Phi - f(z_{j_0}), h a(\cdot, h) = h d_a(\cdot, h) + \nabla(f - \Phi) \land a(\cdot, h)$ (see (174)), there exists $C > 0$ such that for all $h$ small enough,

$$\left\| X_{wkb,j_0} u_{z_{j_0},wkb} e^\frac{\Phi(\cdot, z_{j_0})}{\hbar} \right\|_{H^1(\Omega_{j_0})} \leq C.$$ 

Then, one obtains using Proposition 66 and (189):

$$e^{-\frac{1}{\hbar}(2f^* - f(z_{j_0}))} \left| \int_\Sigma (\tilde{\phi}_{j_0} - \tilde{\phi}_{z_{j_0},wkb}) \cdot n \ e^{-\frac{1}{\hbar} f} \right| = O(h^{\infty}) + C e^{-\frac{\hbar}{4}} (2f^* - f(z_{j_0})) = O(h^{\infty}).$$

(261)

Injecting the estimates (260)–(261) in (259) and using (258) imply that in the limit $h \to 0$:

$$\int_\Sigma \tilde{\psi}_{j_0} \cdot n \ e^{-\frac{1}{\hbar} f} d\sigma = \frac{B^* \sqrt{2} \ (\det \text{Hess} f|_{\partial \Omega}(z_{j_0}))^{\frac{1}{4}}} {\pi^{\frac{d-1}{2}}} \ e^{-\frac{\hbar}{4}(2f^* - f(z_{j_0}))} (1 + O(h)).$$

This proves Lemma 77.

In conclusion, the $n$ quasi-modes $(\tilde{\psi}_{i})_{i=1,\ldots,n}$ and the family of quasi-modes $(u_{i})_{i=1,\ldots,n}$ satisfy all the conditions of Proposition 70. This concludes the proof of Theorem 2.

5.2.4 Proof of Corollary 11

Let us assume that the hypotheses [H1]–[H2]–[H3] hold and let us assume that $f|_{\partial \Omega}$ has only two local minima $z_1$ and $z_2$ such that $f(z_1) < f(z_2)$. Let $\Sigma \subset \partial \Omega$ be a smooth open set such that $\Sigma \subset B_{z_1}$ and $f^* := \inf_{\Sigma} f$. 

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In addition, let us assume that \([37]\) and \([38]\) hold and let us assume that \(f^* = f(z_2)\). Then, the inequalities \([37]\) and \([38]\) are exactly \([29]\) and \([30]\) (in the case \(n = 2\) with \(j_0 = 1\) and \(k_0 = 2\)). Therefore, \([35]\) holds. It remains to compute the prefactor in \([35]\). To this end, we need the constants \(B^*\) and \(p^*\) in \([33]\). Let us assume that there is only one minimizer \(z^*\) of \(f\) on \(\Sigma\). This implies that \(z^* \in \partial \Sigma\) since \(z_1\) is the only critical point of \(f|_{\partial \Omega}\) in \(B_{z_1}\). Furthermore, we assume that \(z^*\) is a non degenerate minimum of \(f|_{\partial \Sigma}\) with \(\partial_{n(\partial \Sigma)} f(z^*) < 0\) where \(n(\partial \Sigma)\) is the unit outward normal to \(\partial \Sigma \subset \partial \Omega\). Then, using Laplace’s method, in the limit \(h \to 0\): 
\[
\int_{\Sigma} \partial_n f e^{-\frac{2}{h} f} d\sigma = -\frac{\partial_n f(z^*) \left(\pi h\right)^{\frac{d}{2}}}{2\pi \partial_{n(\partial \Sigma)} f(z^*) \sqrt{\det \text{Hess}_{f|_{\partial \Sigma}}(z^*)}} e^{-\frac{2}{h} f^* \left(1 + O(h)\right)},
\]
with by convention, \(\det \text{Hess}_{f|_{\partial \Sigma}}(z^*) = 1\) if \(d = 2\). This specifies the constants \(B^*\) and \(p^*\) appearing in \([33]\). This ends the proof of Corollary 11.

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