Ideal quantum gases in two dimensions

S. Viefers, F. Ravndal and T. Haugset

Institute of Physics
University of Oslo
N-0316 Oslo, Norway

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Abstract: Thermodynamic properties of non-relativistic bosons and fermions in two spatial dimensions and without interactions are derived. All the virial coefficients are the same except for the second, for which the signs are opposite. This results in the same specific heat for the two gases. Existing equations of state for the free anyon gas are also discussed and shown to break down at low temperatures or high densities.

1 Introduction

Physics of two-dimensional systems used to be of primarily academic interest in providing mathematically simpler versions of realistic problems in three dimensions. The fundamental discovery of Leinaas and Myrheim in 1977 [1] of the possibility for intermediate quantum statistics for identical particles in two dimensions interpolating between standard Bose-Einstein and Fermi-Dirac statistics therefore generated initially little general interest. Particles with this new statistics were later named anyons by Wilczek [2]. He showed that they could be modelled as carrying both a charge and a magnetic flux locked together in a very special way.

With the progress of modern microelectronics this situation changed [3]. The discovery of the integer quantum Hall effect [4] could be explained by the special quantum effects of electrons effectively confined between layers of different materials in planar transistors. Later it became clear that excitations in the fractional quantum Hall effect obeyed intermediate statistics [5]. This in turn suggested that the mechanism behind the new, high-temperature superconductivity seen in layers of copperoxide planes in different materials, could also be anyonic [6, 7, 8].
Even if anyons should not turn out to be of great practical importance in the coming years, two-dimensional systems in general will play an important role in microelectronics and material sciences. Since the third dimension is effectively frozen out only at very low temperatures, it is also obvious that quantum effects will govern the behaviour of these new systems.

Here we consider the thermodynamics of ideal quantum gases in two dimensions. Ordinary bosons and fermions are treated in the next section where their equation of state is derived. In Section 3 we review the basic properties of anyons and solve the simplest case of two anyons bound by a harmonic potential. The resulting energy spectrum is then used in Section 4 to calculate the second virial coefficient for anyons. Including some recent results for the higher virial coefficients, we then discuss in the last section their full equation of state.

\section{Statistical mechanics of bosons and fermions}

We consider \(N\) free and identical particles of mass \(m\) in a box of two-dimensional volume \(V\). Each particle is characterized by the momentum \(k\) and corresponding energy \(\varepsilon = k^2/2m\). When they are in thermal equilibrium at temperature \(T\) and chemical potential \(\mu\), their thermodynamics follows from the grand canonical partition function

\[\Xi = \prod_k [1 \pm e^{-\beta (\varepsilon_k - \mu)}]^{\pm 1},\]

where the upper signs are for Fermi-Dirac statistics and the lower ones for Bose-Einstein statistics [9]. Here \(\beta = 1/k_BT\) where \(k_B\) is the Boltzmann constant. In this ensemble the corresponding free energy density is simply the pressure \(P = \ln \Xi / \beta V\) which becomes

\[\beta P = \pm \frac{1}{V} \sum_k \ln [1 \pm e^{-\beta (\varepsilon_k - \mu)}].\]

The particle density \(\rho = N/V\) then follows from

\[\rho = \left(\frac{\partial P}{\partial \mu}\right)_\beta = \frac{1}{V} \sum_k \frac{1}{e^{\beta (\varepsilon_k - \mu)} + 1}.\]
Similarly, if \( E \) is the total energy of the particles, the energy density \( \mathcal{E} = E/V \) is given by the derivative

\[
\mathcal{E} = -\left( \frac{\partial}{\partial \beta} \beta P \right)_{\beta \mu} = \frac{1}{V} \sum_k \frac{\varepsilon_k}{e^{\beta (\varepsilon_k - \mu)} \pm 1}.
\] (4)

Since the lowest one-particle energy is here \( \varepsilon = 0 \), the chemical potential can not become negative for bosons. Their fugacity \( z = \exp(\beta \mu) \) will thus always be less than one. No such upper limit exits for fermions.

In two dimensions we can replace the momentum sum by the integral

\[
\sum_k \rightarrow V \int \frac{d^2 k}{(2\pi\hbar)^2}
\] (5)
in the thermodynamic limit where \( V \rightarrow \infty \). Writing \( kdk = m d\varepsilon \), we see that the particle density (3) is given by the integral

\[
\rho = \frac{m}{2\pi\hbar^2} \int_0^\infty d\varepsilon \frac{1}{e^{\beta (\varepsilon - \mu)} \pm 1} = \pm \frac{m}{2\pi\beta \hbar^2} \log (1 \pm z).
\] (6)

Introducing the thermal wavelength \( \Lambda = (2\pi \hbar^2 / m)^{1/2} \), we can write the result for the fugacity as

\[
z = \mp \left(1 - e^{\pm \rho \Lambda^2}\right).
\] (7)

The upper signs are still for fermions and the lower ones for bosons. For bosons \( z \) smoothly approaches the value one as the temperature goes to zero. Thus there is no Bose-Einstein condensation in two dimensions except at \( T = 0 \) when all the particles are in the ground state.

For the energy density (4) we similarly find in the thermodynamic limit

\[
\mathcal{E} = \frac{\beta}{\Lambda^2} \int_0^\infty d\varepsilon \frac{\varepsilon}{e^{\beta (\varepsilon - \mu)} \pm 1}.
\] (8)

A partial integration where the boundary term vanishes then gives

\[
\mathcal{E} = \pm \frac{1}{\Lambda^2} \int_0^\infty d\varepsilon \ln [1 \pm e^{-\beta (\varepsilon - \mu)}].
\] (9)

This is seen to equal the pressure (2) in the same limit. We thus have \( P = \mathcal{E} \). In three dimensions the corresponding result is the more well-known relation \( P = (2/3) \mathcal{E} \) for non-relativistic particles.
The integral (9) for the pressure can be expressed in a more compact way. Changing variable of integration to $t = 1 \pm z \exp(-\beta \varepsilon)$ we obtain

$$\beta P(\beta, \mu) = \mp \frac{1}{\Lambda^2} \text{Li}_2(1 \pm z)$$

where $\text{Li}_2(x)$ is the dilogarithmic function defined by [10]

$$\text{Li}_2(x) = -\int_1^x dt \frac{\log t}{t - 1}.$$  \hspace{1cm} (11)

When $0 < x \leq 2$ one can obtain a convergent series for the pressure by expanding the logarithm in (9) and integrating term by term. This expansion is thus valid at all temperatures for bosons and only at high temperatures for fermions.

With the explicit solution (7) for the fugacity we can now obtain a differential equation for the pressure which will give the equation of state. Forming the partial derivative

$$\left(\frac{\partial P}{\partial \rho}\right)_T = \left(\frac{\partial P}{\partial \mu}\right)_T \left(\frac{\partial \mu}{\partial \rho}\right)_T,$$

we see that the first derivative on the right-hand side is just the density (3) and the second is obtained from (7) as

$$\left(\frac{\partial \mu}{\partial \rho}\right)_T = \mp \frac{\Lambda^2}{\beta} \frac{1}{e^{\beta \rho \Lambda^2} - 1}.$$  \hspace{1cm} (12)

The equation of state is thus explicitly given by the integral

$$\beta P = \mp \int_0^\rho d\rho' \frac{\rho' \Lambda^2}{e^{\beta \rho' \Lambda^2} - 1},$$

as first obtained by Sen [11].

At high temperatures we can express the equation of state in terms of the virial coefficients. They can now be obtained by using the expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

which is convergent for $0 < |x| < 2\pi$ [10]. It defines the Bernoulli numbers $B_n$, giving $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$ etc. with $B_{2n+1} = 0$. We then obtain from (12) the virial expansion

$$\beta P = \sum_{\ell=1}^{\infty} A_\ell \rho^\ell \Lambda^{2(\ell-1)}$$

as first obtained by Sen [11].
where the dimensionless virial coefficients are simply given in terms of the Bernoulli numbers \[11, 12\],

\[ A_\ell = (\mp)^{\ell-1} \frac{B_{\ell-1}}{\ell!}. \]  

(15)

The lowest ones are \( A_2 = \pm 1/4, A_3 = 1/36, A_4 = 0 \), etc. and give for the equation of state at high temperatures or low densities

\[ \beta P = \rho \pm \frac{1}{4} \rho^2 \Lambda^2 + \frac{1}{36} \rho^3 \Lambda^4 - \frac{1}{3600} \rho^5 \Lambda^8 + \frac{1}{211680} \rho^7 \Lambda^{12} + \cdots. \]  

(16)

Notice the interesting fact that since all the odd Bernoulli numbers are zero after \( B_1 \), all the even virial coefficients are zero after \( A_2 \). Hence, the only difference between the pressures in the bosonic or fermionic gases comes from the second term in the expansion (16), i.e.

\[ P_F - P_B = \frac{1}{2\beta} \rho^2 \Lambda^2 = \frac{\hbar^2}{m} \pi \rho^2. \]  

(17)

The result does not depend on the convergence of the virial expansion. It follows directly from (10) when we make use of the special property of the dilogarithmic function \([10]\)

\[ \text{Li}_2(x) + \text{Li}_2(x^{-1}) = -\frac{1}{2} \log^2 x, \]  

(18)

where \( x = \exp(\rho \Lambda^2) \) in our case. The difference is just the pressure or energy density \( \mathcal{E}_0 = \hbar^2 \pi \rho^2 / m \) in the Fermi gas at zero temperature as shown in the Appendix. Since it is constant with respect to \( T \), we see that the specific heats of these ideal quantum gases are equal as pointed out by Aldrovandi \([13]\). It would be of interest to have a more direct understanding of this simple result.

We have plotted the pressures of the gases as a function of temperature at fixed density in Fig.1. At high temperatures we can use the virial expansion (14). Since it only converges for \( \rho \Lambda^2 < 2\pi \), it breaks down at low temperatures. For the Fermi gas just above \( T = 0 \) we can instead use the Sommerfeld expansion \([9]\) derived in the Appendix. In two dimensions it is found to terminate after the second order. At slightly higher temperatures it smoothly matches up with the virial expansion. We can obtain the pressure of the Bose gas by just subtracting the constant term \( \mathcal{E}_0 \) from the Fermi pressure. In the Appendix we have also derived exact equations of state for these gases as explicit functions of the dimensionless quantity \( \rho \Lambda^2 \) combining the results in (7) and (10).
3 Quantum mechanics of anyons

The wavefunctions of bosons and fermions differ under the interchange of two or more particles. While the wavefunction for bosons is completely symmetric, the wavefunction for fermions is completely antisymmetric. An interchange of two particles described by the wavefunction $\psi(r_1, r_2)$ is shown in Fig.2a. Their relative vector is rotated by the angle $\Delta \phi = \pi$. The wavefunction is then transformed into $\psi(r_2, r_1)$. Since the two particles are indistinguishable, this wavefunction can only differ by a phase angle $\theta$ from the initial wavefunction,

$$\psi(r_2, r_1) = e^{i\theta} \psi(r_1, r_2).$$

(19)

Had the interchange taken place in the opposite direction as in Fig.2b, the final wavefunction would have been

$$\psi'(r_2, r_1) = e^{-i\theta} \psi(r_1, r_2)$$

(20)

since these two exchanges followed by each other are equivalent to no exchange.

In three dimensions we now see that this latter interchange is equivalent to the first since it can be obtained by an out-of-page rotation around the line connecting the two particles in their final position. Thus we have $\psi'(r_2, r_1) = \psi(r_2, r_1)$ which implies that $\exp(2i\theta) = 1$. This equation has the two solutions $\theta = 0 \mod 2\pi$ and $\theta = \pi \mod 2\pi$. In the first case the wavefunction is symmetric under the interchange and the particles are bosons, while in the other case it is antisymmetric and the particles are fermions.

In two dimensions the two interchanges cannot be related for particles and the parameter $\theta$ can have any value in the interval $0 \leq \theta < 2\pi$. We then have the possibility for intermediate statistics which interpolates between Bose-Einstein and Fermi-Dirac statistics. The wavefunction for the corresponding particles which are called anyons, will thus be neither completely symmetric nor antisymmetric. Needless to say, the statistical mechanics of anyon gases will be much different from the behaviour of bosons and fermions [14, 15].

A quantum mechanical description of $N$ anyons can be obtained by modifying the classical Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{N} m\dot{r}_i^2 - V(r_1, r_2, \cdots, r_N)$$

(21)

where the last term describes the potential energy of the particles and is symmetric in all its arguments. The quantum mechanical motion of this system is now given
by the integration of \( \exp (i \int dt L / \hbar) \) over all possible paths the system can take. When the motion corresponds just to an interchange of two particles, the net effect should be a change of the wavefunction by a phase as in (19). This effect can now be built into the above Lagrangian by adding a topological term

\[
L \to L_\theta = L + \frac{\theta}{\pi} \hbar \sum_{i<j} \dot{\phi}_{ij}.
\]  

(22)

This additional term is the sum of time derivatives of the relative angles of all pairs of particles in the system as illustrated in Fig.3. These angles obviously depend on the positions of the particles. When the angle \( \phi_{ij} \) changes by \( \Delta \phi_{ij} = \pi \), the corresponding change in the wavefunction is

\[
e^{i \frac{\theta}{\pi} \int_0^\pi d\phi} = e^{i \theta},
\]

as desired. From this modified Lagrangian we can obtain the canonical momenta and quantization can be performed by standard methods [15].

In this way we can describe anyons as ordinary particles in two dimensions with a special, topological interaction. Bosons obey symmetric statistics and prefer to be in the same quantum state. Fermions can be considered as bosons with a statistical interaction of strength \( \alpha \equiv \theta / \pi = 1 \). This results in an effective repulsion between fermions which is usually explained by the Pauli principle. Similarly, one can describe bosons as fermions with an additional, statistical interaction with \( \alpha = 1 \). The effect of this topological interaction for arbitrary values of the parameter \( \alpha \) in systems of many anyons is not yet known.

Essentially the only known system where exact results can be derived is the case of two anyons in a harmonic oscillator potential

\[
V(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2} m \omega^2 (\mathbf{r}_1^2 + \mathbf{r}_2^2).
\]  

(23)

This is equivalent to the case of two anyons in a uniform magnetic field, see e.g. [16]. Introducing the center-of-mass coordinate \( \mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2) \) and the relative coordinate \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = (r \cos \phi, r \sin \phi) \), the Lagrangian (22) becomes

\[
L_\theta = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} M \omega^2 \mathbf{R}^2 + \frac{1}{2} \mu (r^2 + r^2 \dot{\phi}^2 + \omega^2 r^2) + \alpha \hbar \dot{\phi}.
\]  

(24)

We have here introduced the total mass \( M = 2m \) and the reduced mass \( \mu = m / 2 \). The statistical parameter enters only in the relative motion described by the last term in the Lagrangian. The center-of-mass motion is described by the first part which is just an ordinary, two-dimensional harmonic oscillator. In order to quantize the system we need the canonical momentum \( p_r = m \dot{r} \) and

\[
p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} + \alpha \hbar.
\]  

(25)
They allow the construction of the Hamiltonian function of the system. In the quantum description the relative motion is governed by the operator

\[ H = -\frac{\hbar^2}{2\mu} \left[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial}{\partial \phi} - i\alpha \right)^2 \right] + \frac{1}{2} \mu \omega^2 r^2. \] (26)

The Schrödinger equation is separable, and the corresponding wave functions can be written as

\[ \psi(r, \phi) = e^{im\phi} R(r), \] (27)

where the angular quantum number \( m \) is an even number if the particles with the statistical interaction are bosons and an odd number if they are fermions. For a given angular momentum the radial Schrödinger equation then simplifies to

\[ \left( -\frac{\hbar^2}{2\mu} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} (m - \alpha)^2 \right] + \frac{1}{2} \mu \omega^2 r^2 \right) R(r) = ER(r) \] (28)

which has exactly the same form as for an ordinary, two-dimensional harmonic oscillator with an angular momentum \( \ell = m - \alpha \). The eigenvalues are thus

\[ E_{nm} = \hbar \omega [2n + |m - \alpha| + 1], \] (29)

where \( n = 0, 1, 2, \ldots \) is the radial quantum number. If the particles are bosons, the angular momentum takes the values \( m = 0, \pm 2, \pm 4, \ldots \). The quantized energy levels as functions of the statistical parameter are shown in Fig.4. They are seen to repeat themselves for other values of \( \alpha \) with period 2 although the energy of each individual state with definite quantum numbers \( (n, m) \) is not periodic.

From Fig.4 we see that the energy levels fall into two separate classes with opposite slopes. Within each class the degeneracy increases linearly with the number of the level counted from below. For more than two anyons we only know the lowest energy levels from numerical calculations [17]. Again one finds that they can be grouped into different classes depending on their slopes as functions of \( \alpha \). This can be explained by crude, semi-classical considerations [18]. In the case of \( N \) anyons, exact expressions for the ground state and some excited states have been found, see for example [16]. Apart from that, few exact results are known for systems of more than two anyons.
4 The second virial coefficient

Gases of fermions or bosons at very high temperatures behave as classical, ideal gases, independent of the quantum statistics. We expect to see the same behaviour in a gas of anyons. At lower temperatures quantum effects come into play depending on the statistics. They result in a non-zero value for the second virial coefficient, which for a gas confined to a volume $V$, can be calculated from\[9\]

$$B_2 = \lim_{V \to \infty} \frac{V}{2} \left( 1 - \frac{Z_2}{Z_1^2} \right)$$

where $Z_1$ and $Z_2$ are the partition functions of one and two particles respectively. While $Z_1$ is independent of statistics and thus the same for all free particles, $Z_2$ for anyons is more difficult to derive. It was first calculated by Arovas et al. [19] using Feynman’s path integral formalism [20]. Since we know all the energy levels (29) of the two-anyon system, it can also be directly obtained from [21]

$$Z_2 = Z_1 \sum_{nm} e^{-\beta E_{nm}} = Z_1 \sum_{n=0}^{\infty} e^{-\beta \hbar \omega (2n+1)} \sum_{s=-\infty}^{\infty} e^{-\beta \hbar \omega |2s-\alpha|}$$

where we have written the angular momentum $m = 2s$ when the anyons are described in terms of bosons. The factor $Z_1$ in front gives the contribution from the center-of-mass motion. From (24) we see that it equals the partition function for one particle in a two-dimensional harmonic oscillator potential, i.e.

$$Z_1 = \left( \sum_{n=0}^{\infty} e^{-\beta \hbar \omega (n+\frac{1}{2})} \right)^2 = \frac{1}{4 \sinh^2(\beta \hbar \omega / 2)}$$

In (31) the double sum is just the product of two such geometric series and gives similarly

$$Z_2 = Z_1 \frac{\cosh \beta \hbar \omega (1 - \alpha)}{2 \sinh^2(\beta \hbar \omega)}$$

for two anyons in a harmonic oscillator potential.

The infinite volume limit $V \to \infty$ in (30) is replaced by the zero slope limit $\omega \to 0$ for particles confined instead by a harmonic potential. In this limit (32) approaches $Z_1 \to 1/(\beta \hbar \omega)^2$ which should be compared with the result for a free particle in a box of volume $V$,

$$Z_1 = V \int \frac{d^2k}{(2\pi \hbar)^2} e^{-\beta k^2/2m} = \frac{V}{\Lambda^2}$$

9
where \( \Lambda \) again is the thermal wavelength. Comparing these two results, we have the relation

\[
V = \left( \frac{\Lambda}{\beta \hbar \omega} \right)^2
\]

(35)

In addition, in the standard formula \([30]\) we must make the replacement \( V/2 \to V \) when the particles are confined by an oscillator potential as here \([22]\). Expanding the last term to second order in \( \omega \), we obtain

\[
2 \frac{Z_2}{Z_1} = \left[ 1 + \frac{1}{2} (\beta \hbar \omega)^2 (1 - \alpha)^2 + \cdots \right] \left[ 1 - \frac{1}{4} (\beta \hbar \omega)^2 + \cdots \right]
\]

\[
= 1 - \left[ \frac{1}{4} - \frac{1}{2} (1 - \alpha)^2 \right] (\beta \hbar \omega)^2 + \cdots
\]

The second virial coefficient for anyons can thus be written as \( B_2 = A_2 \Lambda^2 \) where its dimensionless value is

\[
A_2 = \frac{1}{4} - \frac{1}{2} (1 - \alpha)^2.
\]

(36)

For ordinary bosons which have \( \alpha = 0 \), we recover the result \( A_2 = -1/4 \) from Section 2, while \( \alpha = 1 \) gives the standard fermion result \( A_2 = 1/4 \). \( A_2 \) is plotted as a function of the statistical parameter in Fig.5a where it is seen to interpolate smoothly between the fermionic and bosonic values. The cusps at the bosonic points are found to disappear when there is an additional interaction potential between the anyons depending on the separation as \( 1/r^2 \) \([23]\).

5 Equation of state for anyons

In order to calculate the higher virial coefficients \( B_3, B_4, \ldots \), we need the partition functions for \( N = 3, 4, \ldots \) anyons. But the energy levels and the corresponding degeneracies for these systems are not known. Instead of such exact calculations one has to rely on perturbation theory or numerical methods. Following the latter approach, Myrheim and Olaussen \([24]\) have used Monte Carlo simulations of the three-anyon problem which have enabled them to extract the third virial coefficient. In this way they have obtained the simple result

\[
A_3 = \frac{1}{36} + \frac{1}{12 \pi^2} \sin^2 \theta.
\]

(37)
Corrections are expected to go as $\sin^4 \theta$, but are absent or so small that they cannot be determined with the present level of accuracy in the simulations. Higher virial coefficients will be very difficult to obtain by this method. $A_3$ is plotted in Fig.5b and is seen to take the common value $A_3 = 1/36$ derived previously for fermions and bosons.

The Monte-Carlo result is in agreement with analytic calculations where the grand partition function has been obtained perturbatively to first \[25\] and second \[26\] order in the statistical angle. These results have more recently been confirmed in the Chern-Simons formulation of anyons \[15\] to the same order in perturbation theory \[27\]. Extracting the lowest virial coefficients and plotting the resulting pressure as function of temperature at fixed density, one finds as expected a curve in between the fermion and boson lines in Fig.1 as long as $\rho \Lambda^2 < 1$ \[12\]. It has been shown that the relation $E = P$ between energy density and pressure is valid for anyons as it is for bosons and fermions \[28\]. One can thus also obtain the specific heat for anyons at high temperatures \[12\]. Since the virial expansion cannot be used at low temperatures, one obtains no information about the thermodynamics of anyons in this regime where quantum effects dominate.

Instead of relying on the virial expansion, one can try to use the pressure as function of fugacity in the discussion of the equation of state. To first order in $\theta$, this is found to be \[25, 27\]

$$\beta P = \frac{1}{\Lambda^2} \left[ Li_2(1-z) - \frac{\theta}{\pi} \ln^2(1-z) \right]. \quad (38)$$

With the density $\rho = z \partial / \partial z(\beta P)$ we can plot the pressure as function of temperature with the fugacity as a parameter. The result is shown in Fig.6 for two small values of the statistical angle. We see that when the temperature gets sufficiently small, the resulting pressure is no longer well-defined \[25\]. When $\theta$ becomes smaller, this happens at a corresponding lower temperatures. A similar breakdown of perturbation theory is also seen using the analytical results \[26, 27\] for the pressure valid to second order in $\theta$. It can be traced back to a violation of $d\rho/dz \geq 0$ which follows from general statistical mechanics for systems in thermodynamic equilibrium \[25\]. This inequality is derived in the Appendix. Since it is here perturbatively violated, one can even get negative particle densities for sufficiently low temperatures. Higher order perturbative results will probably not give much more understanding of the physical phenomena causing this unexpected behaviour at low temperatures in the anyon gas.

When the statistical angle $\theta$ is small, anyons can be described as bosons with a hard-core repulsion. Such a system is known to become a superfluid at low temperatures \[9\]. Similarly, anyons with $\theta \sim 2\pi$ behave as fermions with a weak, attractive
potential. This gives rise to a pairing force between the fermions so they effectively become bosons which condense. If the particles are charged, we then have a superconducting system. These heuristic arguments are actually supported by approximative and non-perturbative calculations of anyons at very low temperatures and are probably the reason for the breakdown of the perturbative equation of state in the same region.

6 Discussion and conclusion

Until the discovery of fractional statistics, the thermodynamics of ideal quantum gases in two dimensions was completely understood although a few detailed properties have first been unravelled during the last couple of years. But these gases are in general characterized by a statistical angle which describe anyons interpolating between ordinary bosons and fermions. With the recent calculations of the first few virial coefficients we have today a fairly good knowledge of the high-temperature behaviour of the ideal anyon gas. There are also many indications and arguments for the gas undergoing a transition into a superfluid phase at low temperatures. However, the details of this transition and the physical properties of the new phase are still to a large extent unknown.

Anyons have many very beautiful properties and there is every reason to believe that in the coming years many of the present problems will be much better understood or even solved. These results for two-dimensional quantum gases might even some time in the future have practical consequences with the ever increasing proliferation of new materials and structures in solid state physics.

We would like to to thank Professor S. Ouvry for several useful suggestions and comments.
7 Appendix

For fermions at zero temperature the chemical potential is just the Fermi energy $\varepsilon_F$. As in three dimensions it follows directly from the particle density

$$\rho = \frac{m}{2\pi\hbar^2} \int_0^{\varepsilon_F} d\varepsilon$$

which gives $\varepsilon_F = 2\pi\rho\hbar^2/m$. The corresponding energy density is then

$$\mathcal{E}_0 = \frac{m}{2\pi\hbar^2} \frac{1}{2} \varepsilon_F^2 = \frac{\hbar^2}{m\pi\rho^2}$$

(39)

which also will be the pressure in the fermion gas at zero temperature.

At temperatures just above zero we can use the Sommerfeld expansion [9]. In the calculation of the energy, we then find it to terminate after just two terms.

$$\mathcal{E}_F = \frac{m}{2\pi\hbar^2} \left[ \frac{1}{2} \mu^2 + \frac{\pi^2}{6} (kT)^2 \right]$$

(40)

Similarly, we find no power corrections to the chemical potential, i.e. $\mu = \varepsilon_F$ as long as $k_B T \ll \varepsilon_F$. This is also in agreement with the exact result (7) which gives

$$\mu = \varepsilon_F + k_B T \ln \left( 1 - e^{-\rho\Lambda^2} \right)$$

(41)

For the low-temperature energy density of the fermion gas we thus have

$$\mathcal{E}_F = \mathcal{E}_0 \left[ 1 + \frac{\pi^2}{3} \left( \frac{1}{\rho\Lambda^2} \right)^2 \right]$$

(42)

where $\Lambda = (2\pi\beta\hbar^2/m)^{1/2}$ is the thermal wavelength. Since we have from (17) that $\mathcal{E}_F - \mathcal{E}_B = \mathcal{E}_0$, we see that the last term in (12) is the bosonic energy density $\mathcal{E}_B$ at low temperatures.

There are no corrections to these results involving higher orders in $1/\rho\Lambda^2$ which is small at low temperatures. Instead, there will be exponential corrections of the order $\exp(-\rho\Lambda^2)$. One way to see this, is to use the following property of the dilogarithmic function [10]

$$\text{Li}_2(x) + \text{Li}_2(1-x) = \frac{\pi^2}{6} - \ln x \ln(1-x)$$

(43)
For the energy density (10) of the boson gas we then have
\[ E_B = \frac{1}{\beta \Lambda^2} \text{Li}_2 \left( e^{-\rho \Lambda^2} \right) \]
\[ = \frac{E_0}{(\rho \Lambda^2)^2} \left( \frac{\pi^2}{3} + 2 \left[ \rho \Lambda^2 \ln \left( 1 - e^{-\rho \Lambda^2} \right) - \text{Li}_2 \left( 1 - e^{-\rho \Lambda^2} \right) \right] \right). \]

This provides an exact and explicit result for the energy density of a two-dimensional gas of free bosons valid at all temperatures. Adding the constant (39), we then get the corresponding exact result for the fermion gas.

We now want to show that for a gas in thermodynamic equilibrium the derivative of the density \( \rho \) with respect to the fugacity \( z = \exp (\beta \mu) \) is always positive. From the grand canonical partition function (9)
\[ \Xi = \sum_{N=0}^{\infty} z^N Z_N \]
where \( Z_N \) is the canonical partition function for exactly \( N \) particles, we obtain the average number of particles
\[ \langle N \rangle = \frac{\partial \ln \Xi}{\partial \ln z} = \frac{1}{\Xi} \sum_{N=0}^{\infty} N z^N Z_N. \]
The density is \( \rho = \langle N \rangle / V \) where \( V \) is the volume of the system. Taking the derivative with respect to the fugacity at constant volume, yields
\[ V \frac{\partial \rho}{\partial z} = \frac{1}{\Xi} \sum_{N=0}^{\infty} N^2 z^{N-1} Z_N - \frac{1}{\Xi^2} \sum_{N=0}^{\infty} N z^N Z_N \sum_{N'=0}^{\infty} N' z^{N'-1} Z_{N'}. \]
Combining the two terms, we get
\[ \frac{\partial \rho}{\partial z} = \frac{1}{z V} \sum_{N,N'} z^{N+N'} Z_N Z_{N'} (N^2 - NN'). \]
Because of the symmetry between \( N \) and \( N' \) we can rewrite this as
\[ \frac{\partial \rho}{\partial z} = \frac{1}{2z V} \sum_{N,N'} z^{N+N'} Z_N Z_{N'} \left( \frac{1}{2} [N^2 + N'^2] - NN' \right) \]
\[ = \frac{1}{2z V} \sum_{N,N'} z^{N+N'} Z_N Z_{N'} (N - N')^2 \]
which is always a positive quantity.
To lowest order in perturbation theory we had the result (38) for the pressure in the anyon gas as function of the fugacity. The corresponding density is

$$\rho = \frac{\partial P}{\partial \mu} = -\frac{1}{\Lambda^2} \ln(1 - z) \left[ 1 - \frac{\theta}{\pi} \frac{2z}{1 - z} \right].$$

(48)

The last term is seen to give a violation of the above bound at low temperatures where the fugacity approaches one.

References

[1] J. M. Leinaas and J. Myrheim, “On the Theory of Identical Particles,” Nuovo Cimento 37B, 1-23 (1977).

[2] F. Wilczek, “Magnetic Flux Angular Momentum and Statistics,” Phys. Rev. Lett. 48, 1144-1146 (1982). A more pedagogic presentation is given by R. Mackenzie and F. Wilczek, “Peculiar spin and statistics in two space dimensions,” Int. J. Mod. Phys. A3, 2827 - 2853, (1988).

[3] For a lucid introduction to these new phenomena, see for instance L.J. Challis, “Physics in less than three dimensions,” Contemp. Phys. 33, 111 - 127 (1992).

[4] K. von Klitzing, “The Quantized Hall Effect,” Rev. Mod. Phys. 58, 519-531 (1986). More recent theoretical ideas are given by R.E. Prange and S.M. Girvin (eds.), The Quantum Hall Effect (Springer Verlag, Berlin, 1990).

[5] D. Arovas, J. R. Schrieffer and F. Wilczek, “Fractional Statistics and the Quantum Hall Effect,” Phys. Rev. Lett. 53, 722-723 (1984).

[6] R.B. Laughlin, “The Relationship between High-Temperature Superconductivity and the Fractional Quantum Hall Effect,” Phys. Rev. Lett. 60, 2677-2680 (1988); “Superconducting Ground State of Noninteracting Particles Obeying Fractional Statistics,” Science 242, 525-533 (1988).

[7] Y.H. Chen, B.I. Halperin, F. Wilczek and E. Witten, “On Anyon Superconductivity,” Int. J. Mod. Phys. B3, 1001-1067 (1989).

[8] F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific, Singapore, 1990).
[9] See, for example, D.A. McQuarrie, *Statistical Mechanics* (Harper & Row, New York, 1976), K. Huang, *Statistical Mechanics* (John Wiley & Sons, New York, 1988)

[10] M. Abramowitz and I.S. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[11] D. Sen, “Quantum and Statistical Mechanics of Anyons,” Nucl. Phys. B360, 397-408 (1991).

[12] S. Viefers, *Statistical Mechanics of Anyons in the Mean Field Approximation*, Cand. scient. thesis, University of Oslo, 1993.

[13] R. Aldrovandi, “Two-Dimensional Quantum Gas,” Fort. d. Physik. 40, 631-649 (1992).

[14] Introductions to the physics of anyons can be found in G.S. Canright and S.M. Girvin, “Fractional Statistics: Quantum Possibilities in Two Dimensions,” Science 247, 1197 - 1205 (1990); I.J.R. Aitchison and N.E. Mavromatos, “Anyons,” Contemp. Phys. 32, 219 - 233 (1991); S. Forte, Rev. Mod. Phys., “Quantum mechanics and field theory with fractional spin and statistics,” 64, 193 - 236 (1992).

[15] A. Lerda, *Anyons* (Springer Verlag, Berlin, 1992).

[16] M. D. Johnson and G. S. Canright, “Anyons in a magnetic field,” Phys. Rev. B41, 6870 - 6881 (1990)

[17] M. Sporre, J. J. M. Verbaarschot and I. Zahed, “Numerical Solution of the Three-Anyon Problem,” Phys. Rev. Lett. 67, 1813-1816 (1991)

[18] J.Aa. Ruud and F. Ravndal, “Systematics of the N-Anyon Spectrum,” Phys. Lett. B291, 137-141 (1992).

[19] D. Arovas, J.R. Schrieffer, F. Wilczek and A. Zee, “Statistical Mechanics of Anyons,” Nucl. Phys. B251, 117-126 (1985)

[20] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[21] A. Comtet, Y. Georgelin and S. Ouvry, “Statistical Aspects of the Anyon Model,” J. Phys. A22, 3917 - 3925 (1989).

[22] K. Olaussen, “On the harmonic oscillator regularization of partition functions,” (Theoretical Physics Seminar in Trondheim, No. 13, 1992).
[23] D. Loss and Y. Fu, “Second Virial Coefficient of an Interacting Anyon Gas,” Phys. Rev. Lett. 67, 294-297 (1991).

[24] J. Myrheim and K. Olaussen, “The Third Virial Coefficient of Free Anyons,” Phys. Lett. B299, 267-272 (1993).

[25] A. Comtet, J. McCabe and S. Ouvry, Perturbative Equation of State for a Gas of Anyons, Phys. Lett. B260, 372-376 (1991).

[26] A. Dasnières de Veigy and S. Ouvry, “Perturbative Anyon Gas,” Nucl. Phys. B388, 715-755 (1992).

[27] M.A. Valle Basagoiti, “Pressure in Chern-Simons Field Theory to Three-Loop Order,” Phys. Lett. B306, 307-311 (1993); R. Emparan and M. A. Valle Basagoiti, “Three-Loop Calculation of the Anyonic Full Cluster Expansion,” Mod. Phys. Lett. A8, 3291-3299 (1993).

[28] T. Haugset and F. Ravndal, “Scale Anomalies in Nonrelativistic Field Theories in 2+1 Dimensions,” Phys. Rev. D49, 4299-4301 (1994).
Figure captions:

Figure 1: Equations of state for free bosons and fermions in two dimensions as functions of \(1/\rho \Lambda^2 \propto T/\rho\).

Figure 2: The exchange of two anyons generates a phase-factor which depends on the direction of the corresponding rotation in their center of mass.

Figure 3: Definition of the angle \(\phi_{ij}\) in the positions of two anyons.

Figure 4: Spectrum of two anyons in a harmonic oscillator potential.

Figure 5: The second a) and third b) virial coefficient as a function of the statistical parameter \(\alpha = \theta/\pi\).

Figure 6: The equation of state to first order in \(\theta\) for \(\theta = 0, 0.05, 0.1\), with \(\theta\) increasing from below. The temperature at which the approximation breaks down, is seen to increase with \(\theta\).
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