Supersymmetry, Spectrum and Fate of D0-Dp Systems with B-field

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Abstract

It has been shown that D0-Dp \((p = 2, 4, 6, 8)\) systems can be BPS in the presence of B-field even if they are not otherwise. We review the number of remaining supersymmetries, the open string ground state spectrum and the construction of the D0-Dp systems as solitonic solutions in the noncommutative super Yang-Mills theory. We derive the complete mass spectrum of the fluctuations to discuss the stability of the systems. The results are found to agree with the analysis in the string picture. In particular, we show that supersymmetry is enhanced in D0-D8 depending on the \(B\)-fields and it is consistent with the degeneracy of mass spectrum. We also derive potentials and discuss their implications for these systems.

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1 Introduction

D-branes in the background of constant $B$-field have many rich and interesting structures, one of which is that this system can be described by noncommutative theory [1, 2]. The relevance of noncommutative theory was first noticed in the Matrix model context [3], and the D-branes with $B$-field and noncommutative theories have been studied by many authors after the appearance of [4]. An important development in this direction is the construction of soliton solutions in scalar noncommutative field theory [5], and solitons in noncommutative scalar-gauge theory have been much studied [3]-[21]. In particular, a simple technique to construct exact solutions in various noncommutative gauge theories is introduced in ref. [16]. This solution generating technique has been used to construct many noncommutative soliton solutions which represent D-brane bound states [16, 20]. These solutions turned out to play a significant role in the study of tachyon condensation [22, 23]. The stability of these system thus constitutes an interesting subject.

In the absence of a $B$-field, the D0-D0, D0-D4 and D0-D8 systems in type IIA superstrings are supersymmetric (BPS), while the others are not. It has been discovered that the BPS conditions for the D0-D$p$ ($p = 2, 4, 6, 8$) systems are modified and those hitherto considered non-BPS can be BPS in the presence of the $B$-field [24, 25]. This can be checked by evaluating the ground state energy [26] or by examining Killing spinors [19, 27, 28]. The BPS D0-D4, for example, remains so only when the $B$-field is (anti-)self-dual and non-BPS D0-D6 becomes BPS for appropriate $B$. A new supersymmetric locus has also been found for a certain background for mysterious D0-D8 system. Stability around these BPS conditions was also partially analyzed in the string picture.

In view of these developments, it is interesting to explore the soliton realization of these brane systems and study their properties. In particular, we are interested in how the new supersymmetric locus mentioned above is different from others. The stability of the D0-D$p$ system observed in the string picture should also be reproduced in the framework of noncommutative theories. In this paper, we construct the noncommutative soliton solutions corresponding to the D0-D$p$ systems and consider the small fluctuations around the solutions. The solutions for D0-D2 and D0-D4 have been discussed in ref. [14]. Here we extend the analysis to all the D0-D$p$ systems, obtain the complete mass spectrum of the fluctuations and find that it is consistent with supersymmetry in the string consideration.
including its degeneracy. The mass spectrum also gives the precise stability conditions of
the brane systems, which also agree with those obtained in the string picture. Finally we
derive the potentials for possible tachyonic scalar modes and discuss the fate of the brane
systems.

This paper is organized as follows. In sect. 2.1, we describe the BPS conditions of
the D0-Dp system with B-field and the form of the conditions in the zero slope limit.
In sect. 2.2, we also analyze the zero point energy of the D0-Dp system [20, 24], and
then discuss the stability of this system in comparison with the BPS conditions obtained
in sect. 2.1. In sect. 3, we use noncommutative super Yang-Mills theory on the Dp-
branes with B-field to realize the D0-Dp system as a noncommutative soliton in the
theory [3, 8, 14, 16]. We then derive mass spectrum in the small fluctuations of this soliton
solution. In the process the gauge mode in the fluctuations is identified with the help of the
Gauss law. Details of the calculations are summarized in the appendix. We show that the
tachyonic modes are present (absent) when instability (stability) is obtained in the string
picture analysis in sect. 2. In one supersymmetric locus of the D0-D8 system, we show
that there is no massless spectrum but the degeneracy of the spectrum correctly reflects
the enhancement of supersymmetry of this system. In the other new supersymmetric
locus, there is massless spectrum but we find that no enhancement of supersymmetry is
possible. Finally in sect. 4, we derive potentials for the system.

2 D0-Dp with a Constant B-field in the String Picture

Let us consider the D0-Dp \((p = 2, 4, 6, 8)\) systems in Type IIA theory with a constant
B-field. The Dp-brane fills the directions \(x_0, \cdots, x_p\) and the B-field is block-diagonal and
lives in the directions \((x_1, x_2, \cdots, x_p, x_p)\):

\[
B = \text{diag}([B_1], \cdots, [B_{p/2}]) = \frac{\epsilon}{2\pi\alpha'} \text{diag}([b_1], \cdots, [b_{p/2}]), \tag{2.1}
\]

where \([B_i]\) and \([b_i] \ (i = 1, \cdots, p/2)\) are \(2 \times 2\) matrices

\[
[B_i] = \begin{pmatrix} 0 & -B_i \\ B_i & 0 \end{pmatrix} = \frac{\epsilon}{2\pi\alpha'} [b_i] = \frac{\epsilon}{2\pi\alpha'} \begin{pmatrix} 0 & -b_i \\ b_i & 0 \end{pmatrix}. \tag{2.2}
\]
The metric on the string worldsheet is written as $g_{ab} = \epsilon\delta_{ab}$ ($a, b = 1, \cdots, p$), $g_{00} = -1$. Here $\epsilon$ is a parameter to define the zero slope limit so as to give noncommutative theories [4, 26].

$$\alpha' \sim \epsilon^{1/2} \rightarrow 0, \quad B: \text{finite}, \quad b_i \sim \epsilon^{-1/2} \rightarrow \infty. \quad (2.3)$$

In this section we summarize the results on the BPS conditions of D0-Dp with the $B$-field and the ground state energies [24, 26]. The BPS conditions can be obtained by examining the constraints on the Killing spinor [27, 28]. In D0-D8, the supersymmetry is enlarged when the $B$-field satisfies a certain condition [29, 28].

### 2.1 BPS condition

Type IIA theory has 32 supercharges of opposite chirality, $Q_\alpha$ and $\tilde{Q}_\beta$ that originate from the left and right moving modes on the string worldsheet, respectively. In the presence of a D-brane, some linear combinations of the supersymmetries $\sum_\alpha \epsilon^\alpha Q_\alpha + \sum_\beta \bar{\epsilon}_\beta \tilde{Q}_\beta$ remain unbroken while the others become broken.

Putting

$$\tan \pi v_i = b_i, \quad -\frac{1}{2} < v_i < \frac{1}{2}, \quad (2.4)$$

we find that the BPS condition for unbroken supersymmetry is

$$\epsilon_\beta = (\Gamma_1 \cdots \Gamma_p e^{\pi v_1 \Gamma_{12} \cdots e^{\pi v_p/2 \Gamma_{p-1,p}}} )_{\beta\alpha} \epsilon^\alpha, \quad (2.5)$$

and $\bar{\epsilon}$ of opposite chirality is determined in terms of $\epsilon$. If (2.5) has a nontrivial solution, the corresponding supersymmetry remains unbroken. In fact, it happens when $v_1, \cdots, v_p/2$ in (2.4) satisfy certain conditions, which are summarized in what follows.

- **D0-D6**

For $p = 6$, the BPS condition is determined from (2.5) as

$$\pm v_1 \pm v_2 \pm v_3 = \pm \frac{1}{2}, \quad (2.6)$$

where the combination of the three $\pm$ signatures is arbitrary. By appropriate reversal of the coordinate axes, we can always arrange so that $v_1$, $v_2$, and $v_3$ (and hence $b_i$) are non-negative. For non-negative $(v_1, v_2, v_3)$ satisfying (2.6), there are four unbroken supercharges (referred to as 1/8 SUSY).
All the BPS conditions in (2.6) are not independent; independent relations are

\[ v_1 + v_2 + v_3 = \frac{1}{2}, \quad \text{(2.7)} \]
\[ -v_1 + v_2 + v_3 = \frac{1}{2}, \quad \text{and its permutations}. \quad \text{(2.8)} \]

It is convenient to parametrize these BPS conditions by the four parameters
\[ r_0 = \frac{1}{2} - (v_1 + v_2 + v_3) \]
and \[ r_i = r_0 + 2v_i \ (i = 1, 2, 3). \] In the zero slope limit (2.3), since \( v_i \) goes like
\[ v_i \to \frac{1}{2} \text{sign } b_i - \frac{1}{\pi b_i}, \quad \text{(2.9)} \]
the parameter \( r_0 \) behaves as
\[ r_0 \sim -1 + \frac{1}{\pi} \left( \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} \right). \quad \text{(2.10)} \]
We see that the BPS condition \( r_0 = 0 \) for (2.7) is not realized in the zero slope limit (2.3).

On the other hand, \( r_i \ (i = 1, 2, 3) \) tends to
\[ r_i \sim \frac{1}{\pi} \left( \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} \right) - \frac{2}{\pi b_i}, \quad \text{(2.11)} \]
which is consistent with the BPS conditions (2.8) in the zero slope limit.

- **D0-D2**
  For \( p = 2 \), the BPS condition for the unique parameter \( v_1 \) is
  \[ v_1 \to \frac{1}{2}, \quad \text{(2.12)} \]
  which is nothing but \( b_1 = \infty \), and this is consistent with the zero slope limit. When this
  is satisfied, the system possesses 16 unbroken supercharges (1/2 SUSY). At this point the
  system is essentially the D0-D0 system.

- **D0-D4**
  For \( p = 4 \), the independent BPS condition is
  \[ v_1 - v_2 = 0. \quad \text{(2.13)} \]

\[ ^1 \text{In Witten’s convention [24], the signs of } v_i \text{'s are arbitrary and the conditions can be written simply as (2.7). In our convention with } v_i \geq 0, \text{ the two cases (2.7) and (2.8) must be distinguished. Similar remarks apply to other cases.} \]
When this is satisfied, the system has *eight* unbroken supercharges (1/4 SUSY). Near the zero slope limit, the BPS condition (2.13) reduces to

\[
\frac{1}{b_1} - \frac{1}{b_2} = 0.
\]  

(2.14)

- **D0-D8**

For \( p = 8 \), the BPS condition is

\[
\pm v_1 \pm v_2 \pm v_3 \pm v_4 = \text{integer}. 
\]  

(2.15)

With the assumption \( v_i \geq 0 \), \((i = 1, 2, 3, 4)\), its independent branches are

\[
\begin{align*}
v_1 + v_2 + v_3 + v_4 &= 1, \\
-v_1 + v_2 + v_3 + v_4 &= 1 \text{ (and its permutations)}, \\
\end{align*}
\]  

(2.16)

and

\[
\begin{align*}
-v_1 - v_2 + v_3 + v_4 &= 0 \text{ (and its permutations)}, \\
-v_1 + v_2 + v_3 + v_4 &= 0 \text{ (and its permutations)}. \\
\end{align*}
\]  

(2.17)

(2.18)

(2.19)

The cases (2.16) and (2.17) are the new loci of supersymmetry found in ref. [24], and it is interesting to examine how the theory looks like there.

When one of the conditions (2.16)-(2.19) is satisfied, the system has *two* unbroken supercharges (1/16 SUSY). For the third case (2.18), say \(-v_1 - v_2 + v_3 + v_4 = 0\), supersymmetry may be enhanced if additional conditions are satisfied. If

\[
v_1 = v_3 \neq v_2 = v_4, \quad \text{or} \quad v_1 = v_4 \neq v_2 = v_3,
\]  

(2.20)

is obeyed, the number of the unbroken supercharges is raised to *four* (1/8 SUSY). If further

\[
v_1 = v_2 = v_3 = v_4,
\]  

(2.21)

six of the supersymmetries remain unbroken (3/16 SUSY) [23, 28].

There is no case that supersymmetry is enhanced for the new loci (2.16) and (2.17).

In the zero slope limit \((v_i \to \frac{1}{2})\), only (2.17) and (2.18) can be realized. These conditions may be parametrized by \( s_i = 1 - (v_1 + v_2 + v_3 + v_4) + 2v_i (i = 1, 2, 3, 4) \) and
\[ t_{ij} = -(v_1 + v_2 + v_3 + v_4) + 2v_i + 2v_j (i, j = 1, \cdots, 4; i > j). \]

Near the zero slope limit, they tend to
\[ s_1 \sim \frac{1}{\pi} \left( -\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} \right), \quad t_{12} \sim \frac{1}{\pi} \left( -\frac{1}{b_1} - \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} \right). \] (2.22)

These parameters are used in the following analysis.

### 2.2 Stability of the ground states

In this subsection we summarize the ground state energies of D0-Dp from the string and discuss the stability of the systems [14, 24]. To compute the ground state energies, we take the light-cone gauge as usual. Since we know that the zero point energies in the Ramond sector always vanish owing to the worldsheet supersymmetry, we only need to pay attention to the transverse coordinates \( X^1, \cdots, X^8 \) and their superpartners \( \psi^1, \cdots, \psi^8 \) in the Neveu-Schwarz sector. It is practical to utilize the complex coordinates and spinors, \( X_i = X_2^i - 1 + iX_2^{-i}, \bar{X}_i = X_2^i - 1 - iX_2^{-i}, \psi_i = \psi_2^i - 1 + i\psi_2^{-i}, \bar{\psi}_i = \psi_2^i - 1 - i\psi_2^{-i}. \)

From the mode expansion, the complex bosons \( X_i \) and fermions \( \psi_i (i = 1, \cdots, p/2) \) give zero point energy
\[ \text{(boson)} : \frac{1}{24} - \frac{v_i^2}{2}, \quad \text{(fermion)} : -\frac{1}{24} + \frac{1}{2} \left( \frac{1}{2} - v_i \right)^2. \] (2.23)

The zero point energy for \( X_j \) and \( \psi_j (j = p/2 + 1, \cdots, 4) \) satisfying the Dirichlet boundary conditions is obtained by substituting 1/2 to \( v_i \) in (2.22). The total zero point energy of the NS sector is thus
\[ E_0 = \frac{p - 4}{8} - \frac{1}{2} \sum_{i=1}^{p/2} v_i. \] (2.24)

Let us summarize the stability/instability of the D0-Dp system on the basis of (2.24).

- **D0-D2**

The zero point energy is \( E_0 = -\frac{1}{2}(1 + v_1) \). The BPS condition \( v_1 = \frac{1}{2} \) gives \( E_0 = -\frac{1}{2} \). Because \( \psi_i \) and \( \bar{\psi}_i \) have mode expansions
\[ \psi_i = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} e^{i(n+v_i)(\tau-\sigma)} \psi_{i,n+v_i}, \quad \bar{\psi}_i = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} e^{i(n-v_i)(\tau-\sigma)} \bar{\psi}_{i,n-v_i}, \] (2.25)

\( \psi_{i,v_i}^\dagger |E_0\rangle \) is massless, where \( \psi_{i,-v_i}^\dagger = \bar{\psi}_{i,-v_i}. \) Thus, in this supersymmetric case, a tower of massive states stands on this massless state after the GSO projection is applied.
On the other hand, when this system is not BPS (i.e. $0 \leq v_1 < 1/2$), the state $\psi_{1,v_1}^\dagger |E_0\rangle$ has a negative energy $E_1 = -\frac{1}{2}(\frac{1}{2} - v_1)$, which means that the system is unstable.

- **D0-D4**
  The zero point energy is $E_0 = -\frac{1}{2}(v_1 + v_2)$. If the BPS condition (2.13) is satisfied, we see $E_0 = -v_1$, which is negative. However, just as in the D0-D2 case, the ground state is the massless state $\psi_{i,v_i}^\dagger |E_0\rangle$ chosen by the GSO projection.

  If the BPS condition is not realized, the energy of the first excited state $\psi_{2,v_2}^\dagger |E_0\rangle$ is $\frac{1}{2}(v_2 - v_1)$ and that of $\psi_{1,v_1}^\dagger |E_0\rangle$ is $\frac{1}{2}(v_1 - v_2)$. Thus, the D0-D4 system always contains a tachyonic mode unless $v_1 - v_2 = 0$.

- **D0-D6**
  The zero point energy is $E_0 = \frac{1}{4} - \frac{1}{2}(v_1 + v_2 + v_3)$. The BPS condition is (2.7) or (2.8).

  Let us take the BPS conditions (2.8), say, $-v_1 + v_2 + v_3 = \frac{1}{2}$, and we have $E_0 = -v_1$. The first excited state $\psi_{1,v_1}^\dagger |E_0\rangle$ is massless which is kept by the GSO projection. Because each state $\psi_{i,v_i}^\dagger |E_0\rangle$ ($i = 1, 2, 3$) has energy $E_i = r_i/2$, the system contains a tachyonic mode when one of the $r_i$ is negative, while the system is stable if all $r_i$ are positive. Notice that there is no case that two of $r_i$ are simultaneously negative.

  On the other hand, for the BPS condition (2.7), we get $E_0 = 0$. Since the R sector contains a massless state, this massless state $|E_0\rangle$ in the NS sector should be kept by the GSO projection. The system is stable (unstable) if $r_0$ is positive (negative).

- **D0-D8**
  The zero point energy is $E_0 = \frac{1}{2}[1 - (v_1 + v_2 + v_3 + v_4)]$. D0-D8 has, up to the permutations of $\pm$ signatures, four branches of the BPS conditions, (2.16)-(2.19).

  First, at the new BPS locus (2.16), the system contains a unique massless state $|E_0\rangle$. Close to this point, the system is stable (unstable) if $1 - (v_1 + v_2 + v_3 + v_4) > 0$ ($< 0$).

  Second, at another new BPS locus (2.17), the stability of the system is the same as that of D0-D6. The system is unstable when one of the $s_i$ defined above (2.22) is negative, and is stable when all of $s_i$ are positive. There is no case in which two of $s_i$ are simultaneously negative. There is no enhancement of supersymmetry at these two loci.

  Third, for the BPS conditions (2.18), we find $E_0 = \frac{1}{2} - v_i - v_j$. This energy $E_0$ can be negative if $v_i$ and $v_j$ are sufficiently large. Thus, we should keep the four states $\psi_{k,v_k}^\dagger |E_0\rangle$ ($k = 1, \ldots, 4$) with energies $\frac{1}{2} - v_i$ ($i = 1, \ldots, 4$) in the GSO projection. Since
all these states have positive energies, the system is always stable for these BPS conditions. Though there is no massless state at this locus, supersymmetry is enhanced when additional conditions are satisfied and this is reflected in the mass spectrum. When (2.20) is satisfied, the spectrum is doubly degenerate (1/8 SUSY). If (2.21) is satisfied, the degeneracy is fourfold, reflecting 3/16 SUSY.

Finally, assuming the BPS conditions (2.19), for example, 
\[-v_1 + v_2 + v_3 + v_4 = 0,\]
we obtain 
\[E_0 = \frac{1}{2} - v_1,\]
which is positive. The state \(|E_0\rangle\) is kept by the GSO projection. (In particular, for \(v_1 \to \frac{1}{2}\), the BPS condition reduces to \(v_2 + v_3 + v_4 = \frac{1}{2}\) and the system becomes D0-D6, where we should keep the massless state \(|E_0\rangle\).) Since \(|E_0\rangle\) has a positive energy at this BPS locus, this system is stable around this BPS condition.

### 3 D0-Dp Solution and Mass Spectrum in the Solitonic Realization

In this section we concentrate on the zero slope limit (2.3), and consider the corresponding \((p + 1)\)-dimensional noncommutative \(U(1)\) gauge theory

\[
S = -\frac{1}{4g_{YM}^2 G_s / g_s} \int dt d^p x \sqrt{-G} G^{\mu \lambda} G^{\nu \sigma} F_{\mu \nu} \ast F_{\lambda \sigma},
\]

where \(g_s\) is the string coupling, \(g_{YM}^2 = (2\pi)^{p-2}(\alpha')(p-3)/2g_s\) and

\[
G_{ab} = g_{ab} - (2\pi \alpha')^2 (Bg^{-1}B)_{ab} \to \epsilon b^2 \delta_{ab},
\]

\[
G_s = g_s \left( \frac{\det(g + 2\pi \alpha' B)}{\det g} \right)^{\frac{1}{2}} \to g_s \prod_{i=1}^{p/2} b_i,
\]

in the zero slope limit. Though we should supplement (3.1) with fermionic terms when some supersymmetry is preserved, it is enough to consider only the bosonic terms for our purpose.

The noncommutativity of the space is manifested in the relation

\[
[x^{2i-1}, x^{2i}] = i\theta_i, \quad \theta_i = \frac{2\pi \alpha'}{\epsilon b_i} = \frac{1}{B_i}, \quad (i = 1, \cdots, p/2),
\]

where we assume \(b_i, \theta_i \geq 0\) as in sect. 2. Let us define complex coordinates

\[
z_j = \frac{1}{\sqrt{2}}(x^{2j-1} + i x^{2j}), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(x^{2j-1} - i x^{2j}),
\]

in the zero slope limit. Though we should supplement (3.1) with fermionic terms when some supersymmetry is preserved, it is enough to consider only the bosonic terms for our purpose.

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\]

in the zero slope limit. Though we should supplement (3.1) with fermionic terms when some supersymmetry is preserved, it is enough to consider only the bosonic terms for our purpose.
and creation/annihilation operators $a_i^\dagger = \bar{z}_i/\sqrt{\theta_i}$ and $a_i = z_i/\sqrt{\theta_i}$. Thus we use the operator formalism for the noncommutative theories. In the temporal $A_0 = 0$ gauge, we can rewrite (3.1) as

$$S = -\frac{\prod_{i=1}^{p/2}(2\pi \bar{b}_i)}{g_{\text{NYM}}^2} \int dt \left[ \sum_{i=1}^{p/2} \left( -\partial_t C_i \partial_t \bar{C}_i + \frac{1}{2} \left( [C_i, \bar{C}_i] + \frac{1}{b_i} \right)^2 \right) \right] + \sum_{i<j} \left( [C_i, \bar{C}_j] [C_j, \bar{C}_i] + [C_i, C_j] [\bar{C}_j, \bar{C}_i] \right), \quad (3.5)$$

where we have set $g_{\text{NYM}}^2 = g_{\text{YM}}^2 \prod_{i=1}^{p/2} b_i, \bar{b}_i = \epsilon b_i^2 \theta_i = 2\pi \alpha' b_i$ and

$$C_j = C_{z_j} = \frac{1}{\sqrt{\epsilon b_j}} \left( -iA_{z_j} + \frac{1}{\sqrt{\theta_j}} a_j^\dagger \right), \quad \bar{C}_j = C_j^\dagger = \bar{C}_{\bar{z}_j} = \frac{1}{\sqrt{\epsilon b_j}} \left( iA_{\bar{z}_j} + \frac{1}{\sqrt{\theta_j}} a_j \right). \quad (3.6)$$

In addition to the equations of motion, the gauge condition $A_0 = 0$ induces the Gauss law constraint

$$\sum_{i=1}^{p/2} \left( [C_i, \partial_t \bar{C}_i] + [\bar{C}_i, \partial_t C_i] \right) = 0. \quad (3.7)$$

On D0-Dp, we can construct exact solitonic solutions of (3.5) by applying solution generating technique [16, 14]. This technique enables us to find new solutions, which generally have nonzero soliton numbers, from a trivial solution by an almost gauge transformation with shift operator.

To define the shift operator $S$, we prepare an ordering of the states

$$|n_1, \cdots, n_{p/2}\rangle = \prod_{i=1}^{p/2} \frac{1}{\sqrt{n_i!}} (a_i^\dagger)^{n_i} |0, \cdots, 0\rangle. \quad (3.8)$$

Two sets of $p/2$ non-negative integers, $\mathbf{m} = (m_1, \cdots, m_{p/2})$ and $\mathbf{n} = (n_1, \cdots, n_{p/2})$, for which we define $\bar{m}_j = \sum_{i=j}^{p/2} m_i$ and $\bar{n}_j = \sum_{i=j}^{p/2} n_i$ with $j = 1, \cdots, p/2$, are ordered by the following rules:

1. If $\bar{m}_j = \bar{n}_j$ for all $1 \leq j \leq \frac{p}{2}$, $\mathbf{m} = \mathbf{n}$.
2. If $\bar{m}_j = \bar{n}_j$ ($j = 1, \cdots, k - 1$) and $\bar{m}_k > \bar{n}_k$ for some $k$ ($1 \leq k \leq \frac{p}{2}$), $\mathbf{m} > \mathbf{n}$.
3. If $\bar{m}_j = \bar{n}_j$ ($j = 1, \cdots, k - 1$) and $\bar{m}_k < \bar{n}_k$ for some $k$ ($1 \leq k \leq \frac{p}{2}$), $\mathbf{m} < \mathbf{n}$.
We can order all the states by this rule. For example, in D0-D4, this orders the states as

\[
\begin{align*}
|0\rangle & = |0, 0\rangle, \\
|1\rangle & = |1, 0\rangle, |2\rangle = |0, 1\rangle, \\
|3\rangle & = |2, 0\rangle, |4\rangle = |1, 1\rangle, |5\rangle = |0, 2\rangle, \cdots, \\
|\frac{1}{2}(n_1 + n_2)^2 + \frac{1}{2}(n_1 + 3n_2)\rangle & = |n_1, n_2\rangle, \cdots.
\end{align*}
\]  

(3.9)

With these preparations, we define the shift operator

\[
S = \sum_{i=0}^{\infty} |i\rangle \langle i+1|,
\]  

(3.10)

which annihilates \(|0, \cdots, 0\rangle\) and satisfies

\[
SS^\dagger = 1, \quad S^\dagger S = 1 - P_0,
\]  

(3.11)

where \(P_0 = |0, \cdots, 0\rangle \langle 0, \cdots, 0|\) is a projection operator onto the vacuum.

As the trivial solution, we take

\[
C_i^{(0)} = \frac{1}{\sqrt{b_i}} a_i^\dagger,
\]  

(3.12)

which satisfies both the equation of motion and the Gauss law constraint. Using the shift operator (3.10), we can obtain another simple solution

\[
C_j^{(0)} = \frac{1}{\sqrt{b_j}} S^\dagger a_j^\dagger S.
\]  

(3.13)

It is also possible to use an arbitrary power of \(S\) to generate new solutions, but this does not essentially change the following results.

Let us investigate small fluctuations around the exact solution (3.13) represented by

\[
C_i = C_i^{(0)} + \delta C_i
\]  

\[
= C_i^{(0)} + P_0 A_i P_0 + P_0 W_i (1 - P_0) + (1 - P_0) \bar{T} P_0 + S^\dagger D_i S.
\]  

(3.14)

The mass matrices of the fluctuations obtained by substituting (3.14) to (3.5), and the complete list of the eigenvalues are relegated to the appendix. In what follows we discuss only the results and compare the mass spectrum with the analysis of the D0-Dp in terms
of string in sect. 2.2. In this comparison, we need, not the whole eigenvalues but only those of the lowest modes, and we examine when they become positive or negative. We also show that the degeneracy of the mass spectrum is in perfect agreement with the number of unbroken supersymmetries in D0-Dp, as discussed in sect. 2.

- **D0-D2**
  In this case [14], the lowest eigenvalue that we have to examine is that of the matrix element \( \langle 0 | T_1 S^1 | 0 \rangle \). The value turns out to be
  \[
  -\frac{1}{\bar{b}_1},
  \]  
  which is always negative and indicates that the system is always unstable.

- **D0-D4**
  The eigenvalues to be examined here are
  \[
  -\frac{1}{\bar{b}_1} + \frac{1}{\bar{b}_2}, \quad \frac{1}{\bar{b}_1} - \frac{1}{\bar{b}_2},
  \]  
  one of which is positive and the other is negative unless \( 1/\bar{b}_1 - 1/\bar{b}_2 = 0 \). Therefore this system is unstable except for the case where \( r_1 = 0 \) is satisfied. In this BPS case, other modes are all massive and we are left with two massless states from (3.16). This agrees with the fact that the system has *eight* unbroken supercharges.

- **D0-D6**
  The lowest eigenvalues to be considered are
  \[
  -\frac{1}{\bar{b}_1} + \frac{1}{\bar{b}_2} + \frac{1}{\bar{b}_3}, \quad \frac{1}{\bar{b}_1} - \frac{1}{\bar{b}_2} + \frac{1}{\bar{b}_3}, \quad \frac{1}{\bar{b}_1} + \frac{1}{\bar{b}_2} - \frac{1}{\bar{b}_3}.
  \]  
  When one of the BPS conditions (2.8), say, \( r_1 = 0 \) is satisfied, (3.17) becomes all non-negative. The (bosonic) mass spectrum contains only one massless mode in (3.17). This again agrees with the fact that this system has *four* unbroken supercharges.

Note that \( \bar{b}_i \) and \( b_i \) are proportional to each other, so that the BPS conditions in sect. 2.1 are directly translated into \( \bar{b}_i \).

It is noted in ref. [24] that the above field theoretical argument is reliable only in the small \( r_i \) region. The analysis for large \( r_i \) is beyond the scope of this paper.
all the eigenvalues in (3.17), not just one of them, must be positive for stability. Thus the analysis of stability/instability of D0-D6 based on the noncommutative field theory shows nice agreement with that in the string analysis in sect. 2.2.

**D0-D8**

The mass eigenvalues to be examined are

\[-\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}, \frac{1}{b_1} - \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}, \frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{b_3} + \frac{1}{b_4}, \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} - \frac{1}{b_4}. \tag{3.18}\]

When one of the new BPS conditions (2.17), say, \(s_1 = 0\) is satisfied, (3.18) gives

\[0, 2\left(\frac{1}{b_3} + \frac{1}{b_4}\right), 2\left(\frac{1}{b_2} + \frac{1}{b_4}\right), 2\left(\frac{1}{b_2} + \frac{1}{b_3}\right),\]

and one massless mode appears. This is consistent with the observation that D0-D8 contains two unbroken supercharges for \(s_1 = 0\). We note that here is one massless mode but we get no enhancement of supersymmetry. Similarly to D0-D6, we find that, close to \(s_1 = 0\), the stability is determined by the sign of \(s_1\) since other modes are all massive.

If one of the BPS conditions (2.18), for example, \(t_{12} = 0\) is satisfied, (3.18) reduces to

\[\frac{2}{b_2}, \frac{2}{b_1}, \frac{2}{b_4}, \frac{2}{b_3},\]

which are all positive and thus the system is stable around \(t_{12} = 0\). Moreover, if \(b_i\) satisfy \(b_1 = b_3 \neq b_2 = b_4\) (or \(b_1 = b_4 \neq b_2 = b_3\)) besides \(t_{12} = 0\), (3.20) contains two doubly degenerate eigenvalues, \(2/b_1\) and \(2/b_2\), consistent with the fact that the system has four unbroken supercharges. If \(b_i\) further satisfy the condition \(b_1 = b_2 = b_3 = b_4\), (3.20) indicates a fourfold degeneracy in the massive eigenvalues. This is due to the fact that the system has six supercharges.

All these results are in agreement with those from the string analysis in sect. 2.

### 4 Scalar Potential from the Yang-Mills Action

In this section we evaluate the potentials in (3.5) including the quartic terms of the fluctuations. They enable us to discuss the ultimate fate of the brane system when the D0-Dp systems are unstable. We use the notations in sect. 3.

The quantity that we evaluate here is

\[V(T, D) = \sum_{i=1}^{p/2} V_{ii}(T_i, D_i) + \sum_{i<j} V_{ij}(T_i, T_j, D_i, D_j) + \sum_{i<j} V_{ij}(T_i, T_j) \tag{4.1}\]
where we decompose the sum of the squares of the field strengths as

\[ V_{ii}(T_i, D_i) = \text{Tr} \left( \left[ C_i, \bar{C}_i \right] + \frac{1}{b_i} \right)^2, \]

\[ V_{ij}(T_i, T_j, D_i, D_j) = 2 \text{Tr} \left[ C_i, C_j \right] \left[ \bar{C}_j, \bar{C}_i \right], \quad V_{ij}(T_i, T_j) = 2 \text{Tr} \left[ C_i, \bar{C}_j \right] \left[ C_j, \bar{C}_i \right]. \] (4.2)

In the action (3.3) the field \( A_i \) does not have the kinetic term. From the results in the appendix, all the elements of \( T_i \) except for \( \langle 0 | T_i S^\dagger | 0 \rangle \) and \( W_i \) have positive mass eigenvalues. Therefore, we are allowed to put \( A_i = W_i = 0 \) and \( \langle 0 | T_i S^\dagger | n \rangle = 0 \) unless \( n = 0 \) in (4.2). We then set \( T_i = \langle 0 | T_i S^\dagger | 0 \rangle \). Furthermore, for simplicity, we assume

\[ \langle m | D_i | n \rangle = 0 \quad \text{unless} \quad m_j = n_j + \delta_{ij}, \]

\[ [D_i, D_j] = [D_i, \bar{D}_j] = [\bar{D}_i, \bar{D}_j] = 0 \quad \text{if} \quad i \neq j, \]

\[ [D_i, a_j] = [\bar{D}_i, a_j^\dagger] = [\bar{D}_i, a_j] = 0 \quad \text{if} \quad i \neq j. \] (4.3)

The evaluation \( V_{ii}(T_i, D_i) \) is already done in [14]. The result is

\[ V_{ii}(T_i, D_i) = \left( |T_i|^2 - \frac{1}{b_i} \right)^2 + \left( |T_i|^2 - \langle e_i | D_i | 0 \rangle + \frac{1}{\sqrt{b_i}} \right)^2 + \sum_{n \neq 0} \left( \langle n | D_i | n - e_i \rangle + \sqrt{\frac{n_i}{b_i}} \right)^2 - \left( \langle n + e_i | D_i | n \rangle + \sqrt{\frac{n_i + 1}{b_i}} \right)^2, \] (4.4)

where we should put \( \langle n | D_i | n - e_i \rangle = 0 \) if \( n_i = 0 \). We can also calculate the cross terms \( V_{ij} \) and \( V_{ij} \) after imposing (4.3) to obtain

\[ V_{ij}(T_i, T_j, D_i, D_j) = 2 \left( \langle e_j | D_j | 0 \rangle + \frac{1}{\sqrt{b_j}} \right)^2 |T_i|^2 + 2 \left( \langle e_i | D_i | 0 \rangle + \frac{1}{\sqrt{b_i}} \right)^2 |T_j|^2, \] (4.5)

and

\[ V_{ij}(T_i, T_j) = 4 |T_i|^2 |T_j|^2. \] (4.6)

We can integrate out the \( D \) fields from (4.1) and get the effective potential \( V_{\text{eff}}(T) \) for the scalar field \( T_i \) [14]. Namely we set \( D_i \) to their stationary values and search for the minimum of \( V(T, D) \). Because the elements \( \langle n + e_i | D_i | n \rangle \) with \( n \neq 0 \) appear only in \( V_{ii} \), we should minimize \( V_{ii} \) with respect to them and get

\[ \langle n + e_i | D_i | n \rangle + \sqrt{\frac{n_i + 1}{b_i}} = \langle n | D_i | n - e_i \rangle + \sqrt{\frac{n_i}{b_i}} \] (4.7)
for $\mathbf{n} \neq \mathbf{0}$. By eliminating $\langle \mathbf{n} + \mathbf{e}_i | D_i | \mathbf{n} \rangle$, $(\mathbf{n} \neq \mathbf{0})$ by (1.7), we obtain

$$V(T, \langle \mathbf{e}_i | D_i | \mathbf{0} \rangle) = \frac{\rho}{2} \sum_{i=1}^{p/2} \left( |T_i|^2 - \frac{1}{b_i} \right)^2 + \sum_{i=1}^{p/2} \left( |T_i|^2 - \frac{1}{\sqrt{b_i}} \right)^2 + \frac{1}{2} \left( |T_i|^2 - \frac{1}{\sqrt{b_i}} \right)^2$$

$$+ 2 \sum_{i \neq j} \left( |\mathbf{e}_j | D_j | \mathbf{0} \rangle + \frac{1}{\sqrt{b_j}} |T_i|^2 + 2 \sum_{i \neq j} |T_i|^2 |T_j|^2. \right. \tag 4.8$$

Let us eliminate $\langle \mathbf{e}_i | D_i | \mathbf{0} \rangle$ in (4.8) and obtain $V_{\text{eff}}(T)$. According to the area to which $T_i$ belong, the values of $\langle \mathbf{e}_i | D_i | \mathbf{0} \rangle$ that minimize (4.8) are determined as follows:

1. If $T_i$ satisfy $|T_i|^2 - \sum_{i \neq j} |T_j|^2 + 1/b_i \geq 0$,

$$\left| \langle \mathbf{e}_i | D_i | \mathbf{0} \rangle + \frac{1}{\sqrt{b_i}} \right|^2 = |T_i|^2 - \sum_{i \neq j} |T_j|^2 + 1/b_i. \tag 4.9$$

2. If $|T_i|^2 - \sum_{i \neq j} |T_j|^2 + 1/b_i < 0$,

$$\left| \langle \mathbf{e}_i | D_i | \mathbf{0} \rangle + \frac{1}{\sqrt{b_i}} \right|^2 = 0. \tag 4.10$$

With the above preparations, we can obtain $V_{\text{eff}}(T)$ in each case.

- **D0-D2**

  Because $\tilde{b}_1 \geq 0$ and we have only $T_1$, we obtain [13]

  $$V_{\text{eff}}(T_1) = \left( |T_1|^2 - \frac{1}{\tilde{b}_1} \right)^2, \tag 4.11$$

  which is minimized at $|T_1| = 1/\sqrt{\tilde{b}_1}$ to give $V_{\text{min}} = 0$. Therefore, we obtain

  $$V_{\text{eff}}(0) - V_{\text{min}} = \frac{1}{\tilde{b}_1^2}. \tag 4.12$$

The minimum corresponds to pure D2-brane; the D0-brane dissolves completely into the D2-brane.

- **D0-D4**

  Without loss of generality, we can assume $\tilde{b}_1 \geq \tilde{b}_2 \geq 0$. There are three regions of $T_i$ to be considered separately:

  **Region I:** $-1/\tilde{b}_1 \leq |T_1|^2 - |T_2|^2 \leq 1/\tilde{b}_2$

  $$V_{\text{eff}}(T_1, T_2) = 8|T_1|^2 |T_2|^2 + 2 \left( \frac{1}{b_2} - \frac{1}{b_1} \right) |T_1|^2 + 2 \left( \frac{1}{b_1} - \frac{1}{b_2} \right) |T_2|^2 + \frac{1}{b_1^2} + \frac{1}{b_2^2}. \tag 4.13$$
Region II: $|T_1|^2 - |T_2|^2 < -1/\bar{b}_1$

$$V_{\text{eff}}(T_1, T_2) = |T_1|^4 + 6|T_1|^2|T_2|^2 + |T_2|^4 + \frac{2}{\bar{b}_2}|T_1|^2 - \frac{2}{\bar{b}_2}|T_2|^2 + \frac{2}{\bar{b}_1^2} + \frac{1}{\bar{b}_2^2}. \quad (4.14)$$

Region III: $|T_1|^2 - |T_2|^2 > 1/\bar{b}_2$

$$V_{\text{eff}}(T_1, T_2) = |T_1|^4 + 6|T_1|^2|T_2|^2 + |T_2|^4 - \frac{2}{\bar{b}_1}|T_1|^2 + \frac{2}{\bar{b}_1}|T_2|^2 + \frac{1}{\bar{b}_1^2} + \frac{2}{\bar{b}_2^2}. \quad (4.15)$$

In Region I, because the coefficient of $|T_1|^2$ is positive, the local minimum there is realized at $|T_1|^2 = 0$ and $|T_2|^2 = 1/\bar{b}_1$. In Region II, $V_{\text{eff}}(T)$ is a uniformly increasing function of $|T_1|^2$. Therefore, the local minimum is at $|T_1|^2 = 0$ again, and $(4.14)$ becomes

$$V_{\text{eff}}(0, T_2) = \left(|T_2|^2 - \frac{1}{\bar{b}_2} \right)^2 + \frac{2}{\bar{b}_1^2}. \quad (4.16)$$

In summary, if $\bar{b}_1 \geq \bar{b}_2$, the minimum of $V_{\text{eff}}(T)$ is $V_{\text{min}} = 2/\bar{b}_1^2$.

The discussion in the case $\bar{b}_1 \leq \bar{b}_2$ is similar to the above. We thus obtain

$$V_{\text{eff}}(0, 0) - V_{\text{min}} = \left|\frac{1}{\bar{b}_1^2} - \frac{1}{\bar{b}_2^2} \right|, \quad (4.17)$$

which vanishes when the BPS condition $(2.14)$ is satisfied.

A couple of remarks are in order. First, as long as $\bar{b}_1 \neq \bar{b}_2$, the coefficient of either $|T_1|^2$ or $|T_2|^2$ in $V_{\text{eff}}(T)$ is negative around $T_i = 0$. Therefore, the one-soliton state $C_i^{(0)}$ is stable only if $\bar{b}_1 = \bar{b}_2$. Second, it is clear from the potential $(4.1)$ that it has a global minimum at the trivial solution $(3.12)$, which corresponds to pure D4-brane. The reason why we do not see this in our potential is that we have restricted the form of $D_i$ within $(4.3)$ for simplicity. The obtained minimum of $V_{\text{eff}}(T)$ with $|T_1|^2 = 0$ and $|T_2|^2 = 1/\bar{b}_1$ for $\bar{b}_1 \geq \bar{b}_2$ is certainly one of the minima but is not a global minimum of the full potential. This implies that the D0-brane could decay to this point but would further decay towards D4 when the system is not BPS. Even so, it is interesting to explicitly calculate the potential and see that it has various branches where we find local minima. It is left for a future study to examine whether there is a monotonously descending way to connect $T_i = 0$ and another minimum if we put off the condition $(4.3)$.

- **D0-D6**

Suppose that $\bar{b}_1 \geq \bar{b}_2 \geq \bar{b}_3$ and it is easy to show that the potential has minimum at
$T_1 = T_2 = 0$, as in the D0-D4. The potential then becomes

$$V_{\text{eff}}(T_3) = \begin{cases} 
-|T_3|^4 + 2 \left( \frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{b_3} + \frac{1}{b_4} \right) |T_3|^2 + \frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} + \frac{1}{b_4^2} & (|T_3|^2 \leq \frac{1}{b_1}) \\
2 \left( \frac{1}{b_2} - \frac{1}{b_3} \right) |T_3|^2 + \frac{2}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_4^2} & (\frac{1}{b_1} \leq |T_3|^2 \leq \frac{1}{b_2}) \\
(\frac{|T_3|^2 - \frac{1}{b_3}}{2})^2 + 2 \left( \frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_4^2} \right) & (|T_3|^2 \geq \frac{1}{b_2})
\end{cases} \quad (4.18)$$

For $1/b_1 + 1/b_2 - 1/b_3 > 0$, the point $T_i = 0$ is a local minimum of $V_{\text{eff}}(T)$, so that the D0-D6 is stable. On the other hand, if $1/b_1 + 1/b_2 - 1/b_3 < 0$, the vacuum with $T_1 = T_2 = T_3 = 0$ is unstable and decays into other local minimum. One possible minimum is the one from the potential (4.18), with

$$V_{\text{eff}}(0) - V_{\text{min}} = -\frac{1}{b_1^2} - \frac{1}{b_2^2} + \frac{1}{b_3^2}. \quad (4.19)$$

There also exists a minimum in the original potential (4.1) at the trivial solution (3.12). So if this D0-D6 is unstable, we again expect that it may decay into the above solution but that would further decay to pure D6.

**D0-D8**

With $b_1 \geq b_2 \geq b_3 \geq b_4$, the potential is again minimum at $T_1 = T_2 = T_3 = 0$ and reduces to

$$V_{\text{eff}}(T_4) = \begin{cases} 
-2|T_4|^4 + 2 \left( \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} - \frac{1}{b_4} \right) |T_4|^2 + \frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} + \frac{1}{b_4^2} & (|T_4|^2 \leq \frac{1}{b_1}) \\
-|T_4|^4 + 2 \left( \frac{1}{b_2} + \frac{1}{b_3} - \frac{1}{b_4} \right) |T_4|^2 + \frac{2}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} + \frac{1}{b_4^2} & (\frac{1}{b_1} \leq |T_4|^2 \leq \frac{1}{b_2}) \\
2 \left( \frac{1}{b_3} - \frac{1}{b_4} \right) |T_4|^2 + \frac{2}{b_1^2} + \frac{2}{b_2^2} + \frac{1}{b_3^2} + \frac{1}{b_4^2} & (\frac{1}{b_2} \leq |T_4|^2 \leq \frac{1}{b_3}) \\
(\frac{|T_4|^2 - \frac{1}{b_4}}{2})^2 + 2 \left( \frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} \right) & (|T_4|^2 \geq \frac{1}{b_3})
\end{cases} \quad (4.20)$$

For $1/b_1 + 1/b_2 + 1/b_3 - 1/b_4 > 0$ close to the new supersymmetric locus (2.17), the point $T_i = 0$ is a local minimum of $V_{\text{eff}}(T)$, so that the D0-D6 is stable. On the other hand, if $1/b_1 + 1/b_2 + 1/b_3 - 1/b_4 < 0$, the vacuum with $T_1 = T_2 = T_3 = T_4 = 0$ is unstable and decays into other local minima. The analysis is then similar to the D0-D6 case.

At the supersymmetric loci (2.18), the system is always stable.

In the course of writing this paper, ref. [30] came to our attention where the spectrum of the D0-D4 system is discussed. A brief discussion of other cases is also given. We thank the author for pointing this out.

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Appendix

A Mass matrix and mass eigenvalues on D0-Dp

By substituting (3.14) to (3.5), and evaluating it up to the second order in the fluctuations, we obtain the mass matrix. For this sake, we define

\[ W_i(n) = \langle 0 | W_i S^\dagger | n \rangle, \quad T_i(n) = \langle 0 | T_i S^\dagger | n \rangle. \] (A.1)

Let \( e_j \) be \((p/2)\)-dimensional unit vectors. The contribution from each term is as follows:

- **quadratic terms in** \( \text{Tr} \left( \frac{1}{b_i}([C_i, \bar{C}_j] + \frac{1}{b_i})^2 \right) \):

\[
\frac{1}{b_i} \text{Tr}(W_i W_i - T_i T_i) + \text{Tr}(W_i C_j^{(0)} - T_i C_i^{(0)})(C_j^{(0)} W_i - C_i^{(0)} T_i) = \sum_{n_1=0}^\infty \cdots \sum_{n_{p/2}=0}^\infty \left\{ \frac{1}{b_i} |W_i(n)|^2 - \frac{1}{b_i} |T_i(n)|^2 
+ \sqrt{\frac{n_i}{b_i}} W_i(n - e_i) - \sqrt{\frac{n_i+1}{b_i}} T_i(n + e_i) \right\}^2, \] (A.2)

where we have dropped the terms \( \frac{1}{2} \left( \frac{1}{b_i} \right)^2 \) and \( \frac{1}{2} \text{Tr}(a_i^\dagger \bar{D}_i + [D_i, a_i])^2 \), which are not necessary for the analysis of the mass spectrum.

- **quadratic terms in** \( \text{Tr}([C_i, \bar{C}_j][C_j, \bar{C}_i]) \):

\[
\text{Tr}((W_i \bar{C}_j^{(0)} - T_j C_i^{(0)})(C_j^{(0)} W_i - C_i^{(0)} T_j) + (C_i^{(0)} W_j - \bar{C}_j^{(0)} T_i)(W_j C_i^{(0)} - T_i \bar{C}_j^{(0)})) = \sum_{n_1=0}^\infty \cdots \sum_{n_{p/2}=0}^\infty \left\{ \sqrt{\frac{n_i}{b_i}} W_i(n - e_i) - \sqrt{\frac{n_i+1}{b_i}} T_i(n + e_i) \right\}^2 
+ \sqrt{\frac{n_j}{b_j}} W_j(n - e_j) - \sqrt{\frac{n_j+1}{b_j}} T_j(n + e_j) \right\}^2. \] (A.3)
• quadratic terms in $\text{Tr}([C_i, C_j][C_j, C_i])$:

$$\text{Tr}((W_i\bar{C}_j^{(0)} - W_j\bar{C}_i^{(0)})(\bar{C}_j^{(0)}\bar{W}_i - \bar{C}_i^{(0)}\bar{W}_j) + (C_i^{(0)}\bar{T}_j - C_j^{(0)}\bar{T}_i)(T_j\bar{C}_i^{(0)} - T_i\bar{C}_j^{(0)})) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_{p/2}=0}^{\infty} \left\{ \left| \sqrt{\frac{n_j+1}{b_j}} W_j(n + e_j) - \sqrt{\frac{n_i+1}{b_i}} W_j(n + e_i) \right|^2 + \left| \sqrt{\frac{n_i}{b_i}} T_j(n - e_i) - \sqrt{\frac{n_j}{b_j}} T_i(n - e_j) \right|^2 \right\}.$$ (A.4)

By assembling the terms (A.2)-(A.4), we obtain the quadratic terms of the fluctuations in the Lagrangian. Defining a $p$-dimensional column vector

$$\mathcal{V}(n) = (W_1(n - e_1), \cdots, W_{p/2}(n - e_{p/2}), T_1(n + e_1), \cdots, T_{p/2}(n + e_{p/2}))^t$$ (A.5)

and a $p \times p$ symmetric matrix, whose upper diagonal elements are given by

$$[\mathcal{M}(n)]_{ij} = \begin{cases} \frac{n_i+1}{b_i} + \sum_{k \neq i}^{p/2} \frac{n_k+1}{b_k} & (1 \leq i = j \leq p/2) \\ \frac{n_{i-p/2}}{b_{i-p/2}} + \sum_{k \neq i-p/2}^{p/2} \frac{2n_k+1}{b_k} & (p/2 + 1 \leq i = j \leq p) \\ -\sqrt{\frac{n_i}{b_i}b_j} & (1 \leq i < j \leq p/2) \\ -\sqrt{\frac{n_{i-p/2+1}(n_{j-p/2+1})}{b_{i-p/2}b_{j-p/2}}} & (p/2 + 1 \leq i < j \leq p) \\ -\sqrt{\frac{n_i(n_{i-p/2+1})}{b_i(b_{j-p/2})}} & (1 \leq i \leq p/2, p/2 + 1 \leq j \leq p) \end{cases},$$ (A.6)

we can see that the sum of the quadratic terms are written as

$$(\text{sum of the quadratic terms}) = \sum_{n_1, \cdots, n_{p/2}=-1}^{\infty} \mathcal{V}(n)^t \mathcal{M}(n) \mathcal{V}(n).$$ (A.7)

Here and in what follows, we should set $W_i(m) = T_i(m) = 0$ if $m_j < 0$ for some $j$.

Let us diagonalize the mass matrix $\mathcal{M}(n)$ to obtain the mass eigenvalues. Before going into the details, we should note here that from the Gauss law constraint (3.7) the quantity

$$\sum_{i=1}^{p/2} \left( \sqrt{\frac{n_i}{b_i}} W_i(n - e_i) + \sqrt{\frac{n_i+1}{b_i}} T_i(n + e_i) \right)$$ (A.8)

is time-independent and thus does not propagate. This fact is important in the following discussions.
A.1 D0-D2

In D0-D2 \[\text{(A.4)}\], the quadratic terms are

\[-\frac{1}{b_1} |T_1(0)|^2 + 0 \cdot |T_1(1)|^2 \]

\[+ \sum_{n_1=1}^{\infty} \left( \bar{W}_1(n_1 - 1), \bar{T}_1(n_1 + 1) \right) \left( \begin{array}{cc}
\frac{n_1+1}{b_1} & -\sqrt{ \frac{n_1(n_1+1)}{b_1} } \\
-\frac{n_1}{b_1} & \frac{n_1}{b_1} 
\end{array} \right) \left( \begin{array}{c}
W_1(n_1 - 1) \\
T_1(n_1 + 1)
\end{array} \right) \]

\[= -\frac{1}{b_1} |T_1(0)|^2 + 0 \cdot |T_1(1)|^2 \]

\[+ \sum_{n_1=1}^{\infty} \frac{2n_1 + 1}{b_1} \left| \sqrt{ \frac{n_1}{2n_1 + 1} } W_1(n_1 - 1) + \sqrt{ \frac{n_1 + 1}{2n_1 + 1} } T_1(n_1 + 1) \right|^2 \]

\[+ \sum_{n_1=1}^{\infty} \frac{2n_1 + 1}{b_1} \left| \sqrt{ \frac{n_1 + 1}{2n_1 + 1} } W_1(n_1 - 1) - \sqrt{ \frac{n_1}{2n_1 + 1} } T_1(n_1 + 1) \right|^2. \quad (A.9)\]

Though the second and third terms in \[(A.9)\] contain zero eigenvalues, they correspond to the unphysical modes \[(A.8)\]. Therefore, the physical mass spectrum is \((2n_1 + 1/b_1)\) \((n_1 = -1, 1, 2, \cdots)\) without any degeneracy. Note that \(n_1 = 0\) is absent in this spectrum.

A.2 D0-D4

In D0-D4, the quadratic terms are

\[\left( -\frac{1}{b_1} + \frac{1}{b_2} \right) |T_1(0, 0)|^2 + \left( \frac{1}{b_1} - \frac{1}{b_2} \right) |T_2(0, 0)|^2 \]

\[+ \left( T_1(1, 0) \right)^\dagger \left( \frac{1}{b_2} + \sqrt{ \frac{1}{b_1 b_2} } \right) \left( \begin{array}{c}
T_1(1, 0)
\end{array} \right) \]

\[+ \sum_{n_2=1}^{\infty} \left( -\frac{1}{b_1} + \frac{2n_2 + 1}{b_2} \right) |T_1(0, n_1)|^2 + \sum_{n_2=1}^{\infty} \left( \frac{2n_1 + 1}{b_1} - \frac{1}{b_2} \right) |T_2(n_1, 0)|^2 \]

\[+ \sum_{n_1=1}^{\infty} \left( W_1(n_1 - 1, 0) \right)^\dagger \left( \begin{array}{ccc}
\frac{n_1+1}{b_1} & -\sqrt{ \frac{n_1(n_1+1)}{b_1} } \\
-\frac{n_1}{b_1} & \frac{n_1}{b_1} 
\end{array} \right) \left( \begin{array}{c}
W_1(n_1 - 1, 0) \\
T_1(n_1 + 1, 0)
\end{array} \right) \]

\[+ \sum_{n_2=1}^{\infty} \left( W_2(0, n_2 - 1) \right)^\dagger \left( \begin{array}{ccc}
\frac{1}{b_1} & -\sqrt{ \frac{n_2(n_2+1)}{b_1} } \\
-\frac{n_2}{b_1 b_2} & \frac{n_2}{b_1 b_2}
\end{array} \right) \left( \begin{array}{c}
W_2(0, n_2 - 1) \\
T_1(n_2 + 1)
\end{array} \right) \]

\[+ \sum_{n_2=1}^{\infty} \left( W_2(0, n_2 - 1) \right)^\dagger \left( \begin{array}{ccc}
\frac{1}{b_1} & -\sqrt{ \frac{n_2(n_2+1)}{b_1} } \\
-\frac{n_2}{b_1 b_2} & \frac{n_2}{b_1 b_2}
\end{array} \right) \left( \begin{array}{c}
W_2(0, n_2 - 1) \\
T_1(n_2 + 1)
\end{array} \right) \]
\[ + \sum_{n_1,n_2=1}^{\infty} \mathcal{V}(n)^\dagger \mathcal{M}(n) \mathcal{V}(n), \quad (A.10) \]

where \( \mathcal{V} \) and \( \mathcal{M} \) are defined in (A.5) and (A.6) with \( p = 4 \), respectively. We again find that the above matrices include zero eigenvalues. However, just as in the D0-D2 case, we can show that the zero eigenvalues correspond to the unphysical gauge modes (A.8). This is also true for D0-D6 and D0-D8, and we will not comment on the zero eigenvalues in what follows.

Omitting these unphysical zero eigenvalues, the physical mass spectrum is

\[ \frac{2n_1 + 1}{b_1} + \frac{2n_2 + 1}{b_2}, \quad (A.11) \]

where \( n_1, n_2 = -1, 0, 1, 2, \cdots \) except the case in which \( n_1 \) and \( n_2 \) are \(-1\) simultaneously. The degeneracy of the mass eigenvalues varies depending on the size of mass matrices in (A.10). It is summarized in table 1, where “*” means any positive integer.

| \( (n_1, n_2) \) | mass eigenvalue | degeneracy |
|-----------------|-----------------|------------|
| \((-1, 0)\)     | \(-1/b_1 + 1/b_2\)  | 1          |
| \((0, -1)\)     | \(1/b_1 - 1/b_2\)  | 1          |
| \((0, 0)\)      | \(1/b_1 + 1/b_2\)  | 1          |
| \((-1, *)\)     | \(-1/b_1 + (2n_2 + 1)/b_2\) | 1 |
| \((*, -1)\)     | \((2n_1 + 1)/b_1 - 1/b_2\) | 1 |
| \((0, *)\)      | \(1/b_1 + (2n_2 + 1)/b_2\) | 2 |
| \((*, 0)\)      | \((2n_1 + 1)/b_1 + 1/b_2\) | 2 |
| \((*, *)\)      | \((2n_1 + 1)/b_1 + (2n_2 + 1)/b_2\) | 3 |

Table 1: Degeneracy of mass eigenvalues in D0-D4

A.3 D0-D6

Here (and for the following D0-D8) we do not give the explicit mass matrices but simply present the mass eigenvalues:

\[ \frac{2n_1 + 1}{b_1} + \frac{2n_2 + 1}{b_2} + \frac{2n_3 + 1}{b_3}, \quad n_i = -1, 0, 1, 2, \cdots, \quad (A.12) \]

where any two of \( n_i \) cannot be simultaneously \(-1\).
The degeneracies are shown in table 2. The eigenvalues are symmetric under the permutation of $i$, so the degeneracy, for example, of the eigenvalues with $(n_1, n_3, n_2)$, is the same as that with $(n_1, n_2, n_3)$. Therefore, we show only those for $n_1 \leq n_2 \leq n_3$.

| $(n_1, n_2, n_3)$ | mass eigenvalue | degeneracy |
|-------------------|-----------------|------------|
| $(-1, 0, 0)$      | $-1/b_1 + 1/b_2 + 1/b_3$ | 1          |
| $(-1, 0, *)$      | $-1/b_1 + 1/b_2 + (2n_3 + 1)/b_3$ | 1          |
| $(-1, *, *)$      | $-1/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3$ | 1          |
| $(0, 0, 0)$      | $1/b_1 + 1/b_2 + 1/b_3$ | 2          |
| $(0, 0, *)$      | $1/b_1 + 1/b_2 + (2n_3 + 1)/b_3$ | 3          |
| $(0, *, *)$      | $1/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3$ | 4          |
| $(*, *, *)$      | $(2n_1 + 1)/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3$ | 5          |

Table 2: Degeneracy of mass eigenvalues in D0-D6

A.4 D0-D8

The mass eigenvalues are

$$\frac{2n_1 + 1}{b_1} + \frac{2n_2 + 1}{b_2} + \frac{2n_3 + 1}{b_3} + \frac{2n_4 + 1}{b_4}, \quad n_i = -1, 0, 1, 2, \ldots, \quad (A.13)$$

where any two of $n_i$ cannot be simultaneously $-1$. The degeneracies with $n_1 \leq n_2 \leq n_3 \leq n_4$ are given in table 3.

| $(n_1, n_2, n_3, n_4)$ | mass eigenvalue | degeneracy |
|------------------------|-----------------|------------|
| $(-1, 0, 0, 0)$        | $-1/b_1 + 1/b_2 + 1/b_3 + 1/b_4$ | 1          |
| $(-1, 0, 0, *)$        | $-1/b_1 + 1/b_2 + 1/b_3 + (2n_4 + 1)/b_4$ | 1          |
| $(-1, 0, *, *)$        | $-1/b_1 + 1/b_2 + (2n_3 + 1)/b_3 + (2n_4 + 1)/b_4$ | 1          |
| $(-1, *, *, *)$        | $-1/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3 + (2n_4 + 1)/b_4$ | 1          |
| $(0, 0, 0, 0)$        | $1/b_1 + 1/b_2 + 1/b_3 + 1/b_4$ | 3          |
| $(0, 0, 0, *)$        | $-1/b_1 + 1/b_2 + 1/b_3 + (2n_4 + 1)/b_4$ | 4          |
| $(0, 0, *, *)$        | $1/b_1 + 1/b_2 + (2n_3 + 1)/b_3 + (2n_4 + 1)/b_4$ | 5          |
| $(0, *, *, *)$        | $-1/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3 + (2n_4 + 1)/b_4$ | 6          |
| $(*, *, *, *)$        | $(2n_1 + 1)/b_1 + (2n_2 + 1)/b_2 + (2n_3 + 1)/b_3 + (2n_4 + 1)/b_4$ | 7          |

Table 3: Degeneracy of mass eigenvalues in D0-D8
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