An explicit simulation of arbitrarily-shaped pseudo-elastic hysteresis loops in shape-memory alloys

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Abstract. A new and explicit approach is proposed for accurately simulating pseudo-elastic hysteresis loops of any shape in shape-memory alloys (SMAs). For this purpose, new finite strain elastoplastic $J_2$-flow equations are established for modeling finite strain behavior of SMAs. A stress-strain loop of any given shape in each uniaxial loading-unloading cycle is derived exactly from these equations. Then, a new technique is further introduced toward integrating usual piecewise linear splines into a single smooth interpolating function in a unified form. With the proposed equations and technique, test data for SMA hysteresis loops of any shape may be automatically fitted with high accuracy based on a single-parameter identification.

1. Introduction

In the past decades, various constitutive models for simulating pseudo-elastic effects of shape memory alloys (SMAs) have been suggested from different standpoints. More details can be found in research articles [1, 2] and surveys [3, 4]. The modelling and engineering applications are summarized in [5]. Most recently, a straightforward approach toward simulating pseudo-elastic effects of SMAs has been proposed based on new elastoplastic $J_2$-flow constitutive equations at finite deformations. Details in this respect may be found in [6, 7] for early results and to [8, 9] for development. This new approach enables accurate simulation of finite strain data for SMA pseudo-elastic effects, thus bypassing tedious numerical procedures in treating a number of constitutive rate equations toward determining numerous unknown parameters.

A shape function with some parameters to be identified is still involved in the above approach. In this contribution, a new technique will further be introduced toward combining usual piecewise linear splines into a single smooth interpolating function in unified form, and then the latter will be used to replace the very mentioned shape function. As such, finite strain data for SMA hysteresis loops of any shape may be automatically fitted with high accuracy in the sense of identifying only a single parameter.

2. Explicit approach to pseudo-elastic hysteresis loops in SMA

2.1. New elastoplastic $J_2$-flow equations for modeling SMA pseudo-elastic effects

As demonstrated in [9], SMA pseudo-elastic effects at finite strain can be simulated with finite strain elastoplastic $J_2$-flow constitutive equations. Such elastoplastic equations are based on the additive separation of tensile $D$ into the elastic $D^e$ and plastic part $D^p$ parts, i.e.:
\[ D = D^e + D^p, \]  

(1)

In the above, the elastic part \( D^e \) is calculated by the self-consistent Eulerian rate equation below \([8, 9]\):

\[ D^e = \frac{1}{2G} \dot{t}^{log} + \frac{v}{E} (\text{tr} \dot{t}^{log}) I, \]

(2)

with the shear modulus \( G \), elastic modulus \( E \), and Poisson’s ratio \( v \). Here, \( I \) is the identity tensor, and \( \dot{t}^{log} \) is used to denote the co-rotational logarithmic rate of the Kirchhoff stress \( \tau = J \sigma \), where \( J \) and \( \sigma \) are the volumetric ratio and the Cauchy stress.

On the other side, a flow rule is established to prescribe the plastic part \( D^p \), as shown below \([6]\):

\[ D^p = \frac{\rho}{h} \left( (\tilde{\tau} - \alpha) \cdot \dot{\tau}^{log} \right) (\tilde{\tau} - \alpha), \]

(3)

In the above, the plastic indicator \( \rho \) takes binary values 1 and 0 in the loading and unloading cases, respectively, and \( \tilde{\tau} \) is used to designate the deviatoric part of the Kirchhoff stress \( \tau \); i.e. \( \tilde{\tau} = \tau - (\text{tr} \tau) I / 3 \); and, moreover, the back stress \( \alpha \) is specified by the following evolution equation:

\[ \dot{\alpha}^{log} = c D^p - r^{-1} \gamma \dot{\vartheta} \alpha, \]

(4)

where \( \dot{\alpha}^{log} \) is the co-rotational logarithmic rate of the back stress \( \alpha \); \( \vartheta \) is known as the dissipated work and determined from the following evolution equation:

\[ \dot{\vartheta} = (\tilde{\tau} - \alpha) : D^p, \]

(5)

and the three hardening quantities \( r, c \) and \( \gamma \) are the yield strength, the Prager modulus, and the hysteresis modulus. Here, each of them relies on both the dissipated work \( \vartheta \) and the magnitude of the back stress \( \alpha \), i.e.

\[ r = r(\vartheta, \xi), \quad \xi = \sqrt{1.5 \text{tr} \alpha^2}, \]

(6)

Finally, the plastic modulus \( h \) is given by

\[ h = \frac{2}{3} cr^2 + \frac{4}{9} r^3 \gamma - \frac{4}{9} \gamma r^2 \tilde{\tau}^\alpha \xi + \frac{1}{9} r^2 \xi^{-1} (1.5 c \tilde{r}^{-1} - \gamma \xi) \Lambda, \]

(7)

where

\[ r' = \frac{\partial r}{\partial \vartheta}, \quad \tilde{r}' = \frac{\partial r}{\partial \xi}, \quad \Lambda = 1.5 r^{-1} (\tilde{\tau} - \alpha) : \alpha, \]

(8)

Details may be found in \([9]\).

2.2. Strain-stress functions for loading and unloading cases

A pseudo-elastic hysteresis loop is composed of a stress-strain curve at loading and a stress-strain curve at unloading, which form a flag-like loop with an initial straight line part on the left and a straight line part up on the right and with an upper curvilinear part and a lower curvilinear part in between. Of them, the slope of either of the two straight line parts is the elastic modulus \( E \), specified by the starting point at loading and the initial yield point. Furthermore, a strain-stress function is used to prescribe the upper curvilinear part with the initial yield point \( (r_0, \tau_0/E) \), as shown below:
\[ h = p(\tau), \quad p(r_0) = \frac{r_0}{E}, \]  
\[ \text{and the lower part finishing at the origin by} \]
\[ h = p(b\tau + r_0) - p(r_0) + (1-b)\frac{\tau}{E}, \]
\[ \text{As will be explained slightly later, the function } p(\tau) \text{ may be given to accurately fit any given test data for the upper curvilinear part of a hysteresis loop, without involving any adjustable parameters, and, moreover, the single parameter } b \text{ may be found to accurately fit data for the lower curvilinear part.} \]

2.3. Nonlinear hardening quantities in explicit forms

In the sense of exactly, automatically reproducing any given hysteresis loop, the three hardening quantities \( r, c \) and \( \gamma \) may be determined explicitly from the shape function \( p(\tau) \), etc. Results are given as follows [9]:

\[ r = \frac{r_0}{b+1} + \frac{b-1}{b+1} \xi(1-e^{-\lambda \beta \xi}) + \frac{b r_0}{b+1} e^{-\beta \xi}, \]
\[ c = \frac{2}{3} \frac{K(r,-\xi \tilde{\tau}') - r \tilde{\tau}'}{1-r'^2}, \]
\[ \gamma = 1.5 \xi^{-1} \tilde{\tau}' + \frac{K(r,-\xi \tilde{\tau}') - K(r,\Lambda)}{\Lambda + \xi \tilde{\tau}'}, \]

where \( \lambda \) and \( \beta \) are fairly large dimensionless parameters characterizing the localized effect at \( \xi=0 \) and \( \Theta=0 \), respectively; and \( K(\tau, \Lambda) \) is the plastic slope [9] given below:

\[ K(r, \Lambda) = \left[ \frac{r+\Lambda}{r+\Lambda} \right] K(r+|\Lambda|) + \left( 1 - \frac{r+\Lambda}{r+\Lambda} \right) K^+(|\Lambda|-r), \]
\[ K^+(\tau) = \frac{b^{-1}}{p'(b\tau + r_0) - \tau^{-1}}, \]

In the above, \( [x] = (x+|x|)/2 \).

As explained in [9], for each loading-unloading cycle, the uniaxial stress-strain curves derived from the elastoplastic \( J_2 \)-flow equations (1)-(8) with the hardening quantities given in equations (11)-(15) automatically form a hysteresis loop, and the upper loading part and the lower unloading part represented by equations (9)-(10) can be exactly reproduced. As such, the proposed elastoplastic equations with the hardening quantities given above can explicitly, accurately simulate any given shape of SMA pseudo-elastic hysteresis loop in a loading-unloading cycle. This may be done by choosing the two proper functions in equations (9)-(10) accurately fitting test data. This will be done below.

2.4. Combining linear splines into a single smooth function

Let a data set for stress and strain be given for the upper curvilinear part of a pseudo-elastic hysteresis loop, namely:

\[ (r_0, r_0/E), \quad (\tau_1, h_1), \quad \ldots \quad (\tau_N, h_N) \]
The strain-stress function \( h=p(\tau) \) fitting these data and, in the meantime, producing the straight line part from the origin \((0,0)\) to the initial yield point \((r_0, \frac{r_0}{E})\) is given as follows:

\[
h = \omega_0 \tau E^{-1} + \omega_1 S_1 + \cdots + \omega_N S_N ,
\]

where \( S_1, \cdots, S_N \) are \( N \) piecewise linear splines fitting the data set given by equation (16), namely,

\[
S_k = h_{k-1} + \frac{h_k - h_{k-1}}{\tau_k - \tau_{k-1}}(\tau - \tau_{k-1}), \quad \tau_{k-1} \leq \tau \leq \tau_k , \quad k = 1, 2, \ldots, N ,
\]

and \( \omega_0, \omega_1, \cdots, \omega_N \) are smooth functions of localized property below:

\[
\omega_0 = \frac{1}{2} \left( \tanh 10(\tau + r_0) - \tanh 10(\tau - r_0) \right),
\]

\[
\omega_N = \frac{1}{2} \left( \tanh 10(\tau - r_N) + 1 \right),
\]

\[
\omega_k = \frac{1}{2} \left( \tanh 10(\tau - \tau_{k-1}) - \tanh 10(\tau - \tau_k) \right), \quad k = 1, 2, \ldots, N-1 ,
\]

Each such localized function \( \omega_k \) is introduced to display the following properties: it actually takes the constant value \( 1 \) within the interval \((\tau_{k-1}, \tau_k)\) and nearly vanishes outside this interval. Thus, the strain-stress function given in equations (17)-(21) is the expected smooth function that automatically, accurately reproduces any given data set given by equation (16) and, in so doing, no adjustable parameters need be identified.

3. Results and discussion
Experimental data for a pseudo-elastic loop at temperature \(-40^\circ\) are presented for Ti-Ni alloy with the as quenched condition in [10]. Accurate simulation results are depicted in figure 1 with the initial yield stress \( r_0 = 277\text{MPa} \) and elastic modulus \( E=27.5\text{GPa} \), as well as the parameter value \( b = 0.26 \).

![Figure 1](image-url)
Another tensile experimental case of Cu-Al-Be alloy in [11] has been considered for the stress-strain response in a pseudo-elastic cycle at 20°. Again, good agreements are achieved with the following parameter values: $r_0 = 63$ MPa, $E = 62.9$ GPa and $b = 0.54$, as shown in figure 2.

![Figure 2. Simulation results of test data in [8] for the pseudo-elastic hysteresis loop of a Cu-Al-Be alloy (test data in solid dots, axial Hencky strain in percent and axial stress in MPa)](image)

In figures 1-2, stress-strain data for the loading-unloading procedure is explicitly fitted by the proposed single-variable shape functions, bypassing time-consuming processes in dealing with the nonlinear constitutive rate equations incorporating a number of unknown parameters. By means of the above shape functions, each nonlinear hardening quantity of the proposed model may be derived in an explicit form, as detailed in section 2.3.

4. Conclusion

A new and explicit approach is proposed for the purpose of accurately simulating SMA pseudo-elastic hysteresis loops of any given shape. Toward this objective, new elastoplastic J2-flow equations are established for modeling finite strain behaviors of SMAs, in conjunction with a new technique for combining a given set of linear splines into a single smooth function. With the new approach proposed, accurate simulation results are achieved merely with a single parameter.

It is expected that the proposed approach may be extended to treat SMA pseudo-elastic effects under multiple loading-unloading cycles. Results will be reported elsewhere.

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