The semi-infinite $q$-boson system with boundary interaction

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*Joint work with Erdal Emsiz
Previous work on the $q$-boson system (without boundary):

- N.M. Bogoliubov, R.K. Bullough, G.D. Pang, *Exact solution of a $q$-boson hopping model*, Phys. Rev. B, 1993.

- N.M. Bogoliubov, A.G. Izergin, A.N. Kitanine, *Correlation functions for a strongly correlated boson system*, Nucl. Phys. B, 1998.

- N.V. Tsilevich, *The quantum inverse scattering method for the $q$-boson model and symmetric functions*, Funct. Anal. Appl., 2006.

- C. Korff, *Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra*, CMP, 2013.

- J. F. van Diejen & E. Emsiz, *Diagonalization of the infinite $q$-boson system*, 2013.

Talk based on the preprint:

- J. F. van Diejen & E. Emsiz, *The semi-infinite $q$-boson system with boundary interaction*, 2013.
1. $q$-Boson system with boundary interaction

**Fock space**

$$\mathcal{F} := \bigoplus_{n \geq 0} \mathcal{F}(\Lambda_n)$$

with $\mathcal{F}(\Lambda_n) := \{ f : \Lambda_n \to \mathbb{C} \}$ where

$$\Lambda_n := \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}$$
A deformed $q$-boson field algebra
Parameters: $q \in (0, 1)$ and $c \in \mathbb{R}$

Action of generators $\beta_k^*, \beta_k, q^{N_k}$ ($k \in \mathbb{N}$) on $f \in \mathcal{F}(\Lambda_n)$:

$$(\beta_k f)(\lambda) := f(\beta_k^* \lambda) \quad (\lambda \in \Lambda_{n-1}),$$

$$(\beta_k^* f)(\lambda) := \begin{cases} [m_k(\lambda)](1 - c\delta_k q^{m_0(\lambda)-1})f(\beta_k \lambda) & \text{if } m_k(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases} \quad (\lambda \in \Lambda_{n+1})$$

$$(q^{N_k} f)(\lambda) := q^{m_k(\lambda)}f(\lambda) \quad (\lambda \in \Lambda_n),$$

Notation:

$\delta_k$ is 1 if $k = 0$ and zero otherwise.

$\beta_k^* \lambda$ adds a component with value $k$ to $\lambda$

$\beta_k \lambda$ deletes a component with value $k$ from $\lambda$

$m_k(\lambda)$ counts $\lambda_j$, $1 \leq j \leq n$ such that $\lambda_j = k$.

$$[m] := \frac{1 - q^m}{1 - q} = 1 + q + \cdots + q^{m-1}$$
Commutation relations

\[ \beta_k q^{N_k} = q^{N_k+1} \beta_k, \quad \beta^*_k q^{N_k} = q^{N_k-1} \beta^*_k, \]
\[ \beta_k \beta^*_k = [N_k + 1](1 - c\delta_k q^{N_0}), \quad \beta_k \beta^*_k - q\beta^*_k \beta_k = 1 - c\delta_k q^{2N_0} \]

and

\[ [\beta_l, \beta_k] = [\beta^*_l, \beta^*_k] = [q^{N_l}, q^{N_k}] = [q^{N_l}, \beta_k] = [q^{N_l}, \beta^*_k] = [\beta_l, \beta^*_k] = 0 \]

for \( l \neq k \).

Here:

\[ q^{N_k+r} = q^r q^{N_k} \quad (r \in \mathbb{Z}) \]

and

\[ [N_k + r] = \frac{1 - q^{N_k+r}}{1 - q} \]
Physical interpretation

— Let $|\lambda\rangle$ be characteristic function supported on $\lambda \in \Lambda_n$.

— The state $|\lambda\rangle$ encodes a configuration of $n$ particles on $\mathbb{N}$ with $m_k(\lambda)$ particles at site $k \in \mathbb{N}$.

— The operators $\beta^*_k$, $\beta_k$ and $N_k$ are the particle creation, annihilation, and number operators at the site $k$:

$$
\beta_k |\lambda\rangle = \begin{cases}
|\beta_k \lambda\rangle & \text{if } m_k(\lambda) > 0 \\
0 & \text{otherwise}
\end{cases}, \quad \beta^*_k |\lambda\rangle = [m_k(\lambda) + 1](1 - c \delta_k q^{m_0(\lambda)})|\beta^*_k \lambda\rangle
$$

and $q^{N_k} |\lambda\rangle = q^{m_k(\lambda)} |\lambda\rangle$. 
Example

\[ \lambda = (5, 5, 4, 4, 4, 2, 2, 1) \]

\[ \beta_2^* \lambda = (5, 5, 4, 4, 4, 2, 2, 2, 1) \]

\[ \beta_2 \lambda = (5, 5, 4, 4, 4, 2, 2, 2, 1) \]
Hamiltonian

\[ H = H_{q,a,c} = a[N_0] + \sum_{k \in \mathbb{N}} (\beta_k^* \beta_{k+1} + \beta_{k+1}^* \beta_k) \]

where \( a \in \mathbb{R} \).

Particle hopping and interaction at boundary:
II. Eigenfunctions

Action of $H$ in the $n$-particle subspace

Proposition (Action on $\mathcal{F}(\Lambda_n)$)

$$(Hf)(\lambda) = a[m_0(\lambda)] + \sum_{1 \leq j \leq n, \lambda - e_j \in \Lambda_n} [m_{\lambda_j}(\lambda)]f(\lambda - e_j)$$

$$+ \sum_{1 \leq j \leq n, \lambda + e_j \in \Lambda_n} (1 - c \delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]f(\lambda + e_j)$$

\[\begin{array}{c}
\text{Diagram of } \lambda_1 \text{ and } \lambda_2
\end{array}\]
Bethe Ansatz wave function

\[ \phi_{\xi}(\lambda) := \sum_{\sigma \in S_n} \sum_{\epsilon \in \{\pm 1\}^n} C(\epsilon \xi_\sigma) e^{i\langle \lambda, \epsilon \xi_\sigma \rangle} \]

with

\[ C(\xi) := \prod_{1 \leq j \leq n} \frac{1 - a e^{-i\xi_j} + c e^{-2i\xi_j}}{1 - e^{-2i\xi_j}} \times \prod_{1 \leq j < k \leq n} \frac{1 - q e^{-i(\xi_j + \xi_k)}}{1 - e^{-i(\xi_j + \xi_k)}} \left( \frac{1 - q e^{-i(\xi_j - \xi_k)}}{1 - e^{-i(\xi_j - \xi_k)}} \right) \]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R}^n \) and \( \epsilon \xi_\sigma := (\epsilon_1 \xi_{\sigma_1}, \epsilon_2 \xi_{\sigma_2}, \ldots, \epsilon_n \xi_{\sigma_n}) \).
**Proposition (Eigenfunctions)**

\[ H\phi_\xi = \hat{\varepsilon}(\xi)\phi_\xi \quad \text{with} \quad \hat{\varepsilon}(\xi) := 2 \sum_{j=1}^{n} \cos(\xi_j) \]

**Proof**

—For fixed \( \lambda \in \Lambda_n \), the wave function \( \phi_\xi(\lambda) \) is a Laurent polynomial in \( e^{i\xi_1}, \ldots, e^{i\xi_n} \) known as the Macdonald spherical function of \( BC \)-type of weight \( \lambda \).

—The eigenvalue equation follows from the following recurrence formula for the Macdonald spherical function (Pieri formula):

\[
2\phi_\xi(\lambda) \sum_{j=1}^{n} \cos(i\xi_j) = a[m_0(\lambda)]\phi_\xi(\lambda) + \sum_{\substack{1 \leq j \leq n \\ \lambda - e_j \in \Lambda_n}} [m_{\lambda_j}(\lambda)]\phi_\xi(\lambda - e_j) \\
+ \sum_{\substack{1 \leq j \leq n \\ \lambda + e_j \in \Lambda_n}} (1 - c\delta_{\lambda_j} q^{m_0(\lambda)-1})[m_{\lambda_j}(\lambda)]\phi_\xi(\lambda + e_j)
\]
Orthogonality of the Macdonald spherical functions

When choosing $a$ and $c$ such that the roots $r_1, r_2$ of the polynomial $r^2 - ar + c$ belong to $(-1, 1)$:

$$a = r_1 + r_2 \text{ and } c = r_1 r_2 \text{ with } r_1, r_2 \in (-1, 1).$$

Then:

$$\frac{1}{(2\pi)^n} \int_A \phi_\xi(\lambda) \overline{\phi_\xi(\mu)} \frac{d\xi}{|C(\xi)|^2} = \begin{cases} \mathcal{N}_n(\lambda) & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$A := \{\xi \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0\},$$

and

$$\mathcal{N}_n(\lambda) = (c; q)_{m_0(\lambda)} \prod_{k \in \mathbb{N}} [m_k(\lambda)]!$$

with

$$(c; q)_m := (1 - c)(1 - qc) \cdots (1 - cq^{m-1})$$

and

$$[m]! := \frac{(q; q)_m}{(q; q)_1} = [m][m-1] \cdots [2][1].$$
Diagonalization
The Hilbert space isomorphism between \( \ell^2(\Lambda_n) \) and \( L^2(A, d\xi) \) given by the Fourier transform

\[
(Ff)(\xi) := \sum_{\lambda \in \Lambda_n} f(\lambda) \overline{\Psi_\xi(\lambda)} \quad (f \in \ell^2(\Lambda_n))
\]

with inversion formula

\[
(F^{-1}\hat{f})(\lambda) = \frac{1}{(2\pi)^n} \int_A \hat{f}(\xi) \Psi_\xi(\lambda) d\xi \quad (\hat{f} \in L^2(A, d\xi)),
\]

where

\[
\Psi_\xi(\lambda) := i^n |C(\xi)|^{-1} \mathcal{N}_n(\lambda)^{-1/2} \phi_\xi(\lambda),
\]

diagonalizes the self-adjoint Hamiltonian

\[
H := \mathcal{N}_n^{-1/2} H \mathcal{N}_n^{1/2}
\]

in \( \ell^2(\Lambda_n) \).
Theorem (vD & E. 2013)

\[ H = F^{-1} \circ \varepsilon \circ F, \]

where \( \varepsilon \) denotes the real multiplication operator by the eigenvalue

\[ (\hat{\varepsilon} \hat{f})(\xi) = \hat{\varepsilon}(\xi) \hat{f}(\xi) \quad (\hat{f} \in L^2(A, d\xi)). \]

Upshot:

–The spectrum of \( H \) in the \( n \)-particle subspace \( \ell^2(\Lambda_n) \) is absolutely continuous and bounded:

\[ \sigma(H|_{\ell^2(\Lambda_n)}) = [-2n, 2n]. \]
Impenetrable boson limit: hard phase model

For $a = c = q = 0$ our $n$-particle Hamiltonian reduces to that of a system of impenetrable bosons:

$$(H_0 f)(\lambda) = \sum_{1 \leq j \leq n, \varepsilon = \pm 1} f(\lambda + \varepsilon e_j).$$
*q*-boson dynamics

\[
(e^{itH} f)(\lambda) = \frac{1}{(2\pi)^n} \int_A e^{it\hat{\epsilon}(\xi)} \hat{f}(\xi) \Psi_\xi(\lambda) d\xi \quad \hat{f} = Ff
\]

*Question*: Behavior for large times \( t \to \pm \infty \)?
The stationary $S$-matrix comparing the dynamics of the deformed $q$-bosons with that of the impenetrable bosons is read-off upon rewriting wave function as:

$$
\Psi_\xi(\lambda) = \mathcal{N}_n(\lambda)^{-1} \sum_{\sigma \in S_n} \text{sign}(\epsilon \sigma) \hat{S}(\epsilon \xi_\sigma) 1/2 e^{i\langle \rho + \lambda, \epsilon \xi_\sigma \rangle}.
$$

Here $\rho := (n - 1, \ldots, 2, 1, 0)$, $\text{sign}(\epsilon \sigma) := \epsilon_1 \cdot \cdots \epsilon_n \text{sign}(\sigma)$, and

$$
\hat{S}(\xi) := \prod_{1 \leq j < k \leq n} s(\xi_j - \xi_k) s(\xi_j + \xi_k) \prod_{1 \leq j \leq n} s_0(\xi_j),
$$

where

$$
s(x) := \frac{1 - qe^{-ix}}{1 - qe^{ix}} \quad \text{with} \quad s(x)^{1/2} = \frac{1 - qe^{-ix}}{|1 - qe^{ix}|}
$$

and

$$
s_0(x) := \frac{1 - ae^{-ix} + ce^{-2ix}}{1 - ae^{ix} + ce^{2ix}} \quad \text{with} \quad s_0(x)^{1/2} = \frac{1 - ae^{-ix} + ce^{-2ix}}{|1 - ae^{ix} + ce^{2ix}|}.
$$
Wave- and scattering operators

Theorem (vD & E 2013)

The operator limits
\[ \Omega^\pm := s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \]

converge in the strongly in the \( \ell^2(\Lambda_n) \)-norm and the corresponding wave operators \( \Omega^\pm \) are given by unitary operators in \( \ell^2(\Lambda_n) \) of the form
\[ \Omega^\pm = F^{-1} \circ \hat{S}^{\mp1/2} \circ F_0. \]

Consequently, the scattering operator comparing the dynamics of \( H \) and \( H_0 \) is given by the unitary operator
\[ S := (\Omega^+)^{-1} \Omega^- = F_0^{-1} \circ \hat{S} \circ F_0, \]

where
\[ (\hat{S} \hat{f})(\xi) := \hat{S}(\epsilon_\xi \xi_\sigma_\xi) \hat{f}(\xi) \quad (\hat{f} \in L^2(\Lambda, d\xi)). \]

Here the sign-configuration \( \epsilon_\xi \) and the permutation \( \sigma_\xi \) must be chosen such that the components of the velocity vector \( \nabla \hat{\epsilon}(\epsilon_\xi \xi_\sigma_\xi) \) are all positive and ordered from large to small, where \( \nabla \hat{\epsilon}(\xi) = (-2 \sin(\xi_1), \ldots, -2 \sin(\xi_n)) \).

Proof By means of a stationary phase analysis using the plane wave expansion of the eigenfunctions.
Supplementary References

Properties of Macdonald spherical functions:

- I.G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, 2003.

Scattering of discrete lattice models on $\Lambda_n$:

- S.N.M. Ruijsenaars, *Factorized weight functions vs. factorized scattering*, CMP 2002.

- J. F. van Diejen, *Scattering theory of discrete (pseudo) Laplacians on a Weyl chamber*, Amer. J. Math. 2005.
Thank you