A Sharp Lieb–Thirring Inequality for Functional Difference Operators

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Abstract. We prove sharp Lieb–Thirring type inequalities for the eigenvalues of a class of one-dimensional functional difference operators associated to mirror curves. We furthermore prove that the bottom of the essential spectrum of these operators is a resonance state.

Key words: Lieb–Thirring inequality; functional difference operator; semigroup property

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To our friend and coauthor Leon Takhtajan on the occasion of his 70th birthday

1 Introduction

Let $P$ be the self-adjoint quantum mechanical momentum operator on $L^2(\mathbb{R})$, i.e., $P = -i \frac{d}{dx}$ and for $b > 0$ denote by $U(b)$ the Weyl operator $U(b) = \exp(-bP)$. By using the Fourier transform

$$\hat{\psi}(k) = (\mathcal{F}\psi)(k) = \int_{\mathbb{R}} e^{-2\pi ikx} \psi(x) \, dx$$

we can write the domain of $U(b)$ as

$$\text{dom}(U(b)) = \{ \psi \in L^2(\mathbb{R}) : e^{-2\pi bk} \hat{\psi}(k) \in L^2(\mathbb{R}) \}.$$ 

Equivalently, $\text{dom}(U(b))$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip $\{ z = x + iy \in \mathbb{C} : 0 < y < b \}$ such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leq y < b$ and there is a limit $\psi(x + ib - i0) = \lim_{\varepsilon \to 0^+} \psi(x + ib - i\varepsilon)$ in the sense of convergence in $L^2(\mathbb{R})$, which we will denote simply by $\psi(x + ib)$. The domain of the inverse operator $U(b)^{-1}$ can be characterised similarly.

For $b > 0$ we define the operator $W_0(b) = U(b) + U(b)^{-1} = 2 \cosh(bP)$ on the domain

$$\text{dom}(W_0(b)) = \{ \psi \in L^2(\mathbb{R}) : 2 \cosh(2\pi bk) \hat{\psi}(k) \in L^2(\mathbb{R}) \}.$$ 

The operator $W_0(b)$ is self-adjoint and unitarily equivalent to the multiplication operator $2 \cosh(2\pi bk)$ in Fourier space. Its spectrum is thus absolutely continuous covering the interval $[2, \infty)$ doubly.

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The full collection is available at https://www.emis.de/journals/SIGMA/Takhtajan.html
Let $V \geq 0$, $V \in L^1(\mathbb{R})$ now be a real-valued potential function. The scalar inequality $2 \cosh(2\pi bk) - 2 \geq (2\pi bk)^2$ implies the operator inequality

$$W_0(b) - 2 \geq -b^2 \frac{d^2}{dx^2}$$

(1.1)
on dom$(W_0(b))$. By Sobolev’s inequality, we can conclude that the operator

$$W_V(b) = W_0(b) - V$$

is symmetric and bounded from below on the common domain of $W_0(b)$ and $V$. We can thus consider its Friedrichs extension, which we continue to denote by $W_V(b)$. This operator acts as

$$(W_V(b)\psi)(x) = \psi(x + ib) + \psi(x - ib) - V(x)\psi(x).$$

Furthermore, by an application of Weyl’s theorem (in a version for quadratic forms) and Rellich’s lemma together with the fact that the form domain of $W_0(b)$ is continuously embedded in $H^1(\mathbb{R})$ (as discussed at the beginning of Section 4) the spectrum of $W_V(b)$ consists of essential spectrum $[2, \infty)$ and discrete finite-multiplicity eigenvalues below. Details of this argument in the similar case of a Schrödinger operator can be found in the upcoming book [2, Proposition 4.14].

We will show that the discrete spectrum satisfies a version of Lieb–Thirring inequalities for 1/2-Riesz means. When formulating the main result of the paper it is convenient to parametrise the eigenvalues (repeated with multiplicities) as $\lambda_j = -2\cos(\omega_j)$, where $\omega_j \in [0, \pi]$ for $\lambda_j \in [-2, 2]$ and $\omega_j \in [0, \pi]$ for $\lambda_j \leq -2$. Note that in the latter case $\lambda_j = -2\cosh(|\omega_j|)$.

**Theorem 1.1.** Let $V \geq 0$ and let $V \in L^1(\mathbb{R})$. If $W_V(b) \geq -2$, then the discrete eigenvalues $\lambda_j = -2\cos(\omega_j) \in [-2, 2]$ (repeated with multiplicities) satisfy

$$\sum_{j \geq 1} \frac{\sin \omega_j}{\omega_j} \leq \frac{1}{2\pi b} \int_{\mathbb{R}} V(x) \, dx.$$ (1.2)

The constant in the inequality (1.2) is sharp in the sense that there is a potential $V$ such that (1.2) becomes equality.

**Remark 1.2.** Note that Theorem 1.1 does not allow to estimate eigenvalues below $-2$. In fact, from the proof of this theorem, the case of one eigenvalue below $-2$ could be included in the inequality (1.2). We expect that the inequality holds true for all eigenvalues below $-2$. However, the method we use in the proof prevents us from including all eigenvalues due to oscillating properties of the resolvent $(W_0(b) - \lambda)^{-1}$ for $\lambda < -2$.

Lieb–Thirring inequalities were first established for Schrödinger operators in [15]. For a one-dimensional Schrödinger operator $-\frac{d^2}{dx^2} - V$ on $L^2(\mathbb{R})$ with negative eigenvalues $\mu_1 \leq \mu_2 \leq \cdots < 0$, these bounds state that for any $\gamma \geq 1/2$ there is a constant $L_\gamma > 0$ such that

$$\sum_{j \geq 1} |\mu_j|^{\gamma} \leq L_\gamma \int_{\mathbb{R}} V(x)^{\gamma + 1/2} \, dx$$

(1.3)

for all $V \geq 0$, $V \in L^{\gamma + 1/2}(\mathbb{R})$. The condition $\gamma \geq 1/2$ is optimal. Inequality (1.1) implies that

$$\sum_{j \geq 1} |\lambda_j - 2|^{\gamma} \leq \frac{L_\gamma}{b} \int_{\mathbb{R}} V(x)^{\gamma + 1/2} \, dx$$

(1.4)

for all eigenvalues $\lambda_j \leq 2$ of $W_V(b)$. Under the additional assumption $W_V(b) \geq -2$, our bound (1.2) presents an improvement of (1.4) for $\gamma = 1/2$. This can be seen from the fact
that for $\gamma = 1/2$ the sharp constant in (1.3) is given by $L_{1/2} = 1/2$ [7] and from the strict inequality

$$|\lambda_j - 2| \leq |2 \cos \omega_j + 2|^{1/2} < \frac{\pi \sin \omega_j}{\omega_j}$$

for $\omega_j \in [0, \pi)$. The difference of the terms above vanishes as $\omega_j \to \pi$, implying that (1.4) is asymptotically optimal for small coupling. While the necessity of $\gamma \geq 1/2$ in the Lieb–Thirring inequality for Schrödinger operators does not allow us to conclude that (1.4) fails for $0 \leq \gamma < 1/2$, we will prove the following.

**Theorem 1.3.** Let $b > 0$. If $V \in L^1(\mathbb{R})$ with $\int_\mathbb{R} V \, dx > 0$, then $W_V(b)$ has at least one eigenvalue below 2. Furthermore, if $0 \leq \gamma < 1/2$, then there is no constant $L_\gamma$ such that (1.4) holds for all compactly supported $V$. This conclusion holds even under the assumption that $W_V(b) \geq -2$.

The study of different properties of functional difference operators $W_V(b)$ was considered before. In the case when $-V = V_0 = e^{2\pi bx}$ is an exponential function, the operator $W_{V_0}(b)$ first appeared in the study of the quantum Liouville model on the lattice [1] and plays an important role in the representation theory of the non-compact quantum group $SL_q(2, \mathbb{R})$. The spectral analysis of this operator was first studied in [9], see also [17]. In the case when $-V = 2 \cosh(2\pi bx)$ the spectrum of $W_V(b)$ is discrete and converges to $+\infty$. Its Weyl asymptotics were obtained in [13]. This result was extended to a class of growing potentials in [14]. More information on spectral properties of functional difference operators can be found in papers [4, 5, 10, 11, 16].

The proof method of Theorem 1.1 is similar to the proof of the sharp Lieb–Thirring inequality (1.3) for a one-dimensional Schrödinger operator in the case $\gamma = 1/2$ as presented in [6]. It relies on a property of convolutions of the resolvent kernels of the operator under consideration. Such a semigroup property was also recently established for Jacobi operators where it was again used to prove sharp Lieb–Thirring type inequalities [12]. With a different proof (not using the convolution property) the sharp inequalities for the Schrödinger operator and the Jacobi operator were first obtained in [7] and in [8], respectively. Despite formal similarity to the case of Jacobi operators, it is still surprising that the proof method works for functional difference operators $W_V(b)$. These operators could be considered as differential operators of infinite order since the symbol $\cosh(2\pi bk)$ can be written as an infinite Taylor series of symbols of even degree w.r.t. the variable $k$.

## 2 Free resolvent

Since $W_0(b) \geq 2$ we conclude that $W_0(b) - \lambda$ is an invertible operator for $\lambda < 2$. Let $\lambda = -2 \cos(\omega)$ with $\omega \in [0, \pi]$ if $\lambda \in [-2, 2]$ and $\omega \in i[0, \infty)$ if $\lambda < -2$. Then in Fourier space the inverse of $W_0(b) - \lambda$ is given by the multiplication operator $(2 \cosh(2\pi bk) + 2 \cos(\omega))^{-1}$.

Applying the inverse Fourier transform $\mathcal{F}^{-1}$ to $(2 \cosh(2\pi bk) + 2 \cos(\omega))^{-1}$ we find the kernel of the free resolvent $G_\lambda = (W_0(b) - \lambda)^{-1}$ that is

$$G_\lambda(x, y) = G_\lambda(x - y) = \frac{1}{2b \sinh(\frac{\omega}{2})(x - y)}.$$  \hspace{1cm} (2.1)

**Remark 2.1.** Note that $G_\lambda(x - y)$ is an even and positive kernel for $\omega \in [0, \pi]$ and it becomes oscillating if $\omega \in i(0, \infty)$. This fact is one of the reasons why we are able to study Lieb–Thirring inequalities only for the eigenvalues $\lambda_j \in [-2, 2]$. This interval contains all of the discrete spectrum if the potential $V$ is “small” enough. However, if $V$ generates eigenvalues lying in $(-\infty, -2)$, then the oscillating property of the Green’s function prevents us from obtaining the desired inequality for all eigenvalues.
Note that the value of $G_\lambda$ on the diagonal $x = y$ takes the form
\[
G_\lambda(0) = \frac{1}{2\pi b} \frac{\omega}{\sin \omega}
\]
and we can see the relation between the right-hand side of (2.2) and the expression in the left-hand side of (1.2). Due to our parameterisation of the spectral parameter, the convergence $\lambda \to 2^-$ implies $\omega \to \pi^-$ and thus
\[
G_\lambda(0) \sim \frac{1}{2b} \frac{1}{\sqrt{1 - \cos^2 \omega}} \sim \frac{1}{2b} \frac{1}{\sqrt{2 - \lambda}} \quad \text{as} \quad \lambda \to 2^-.
\]
If $\lambda \to -\infty$, then $\omega \to i\infty$ and
\[
G_\lambda(0) \sim \frac{1}{\pi b} |\lambda|^{-1} \log |\lambda|.
\]
In [17] L. Faddeev and L.A. Takhtajan studied the resolvent in a slightly different form
\[
G_\lambda(x, y) = \frac{\sigma}{\sinh \left( \frac{\pi x}{\sigma} \right)} \left( \frac{e^{-2\pi i\sigma(x-y)}}{1 - e^{-4\pi i\sigma(x-y)}} + \frac{e^{2\pi i\sigma(x-y)}}{1 - e^{4\pi i\sigma(x-y)}} \right),
\]
which coincides with (2.1) with $\sigma = i/2b$, $\lambda = 2 \cosh(2b\pi \kappa)$ and $\kappa = \frac{\omega - \pi}{2\pi b}$. It was pointed out that the free resolvent can be written using the analogues of the Jost solutions
\[
f_-(x, \kappa) = e^{-2\pi i\kappa} \quad \text{and} \quad f_+(x, \kappa) = e^{2\pi i\kappa}
\]
that appear in the theory of one-dimensional Schrödinger operators. Namely
\[
G_\lambda(x - y) = \frac{2\sigma}{\pi C(f_-, f_+) (\kappa)} \left( \frac{f_-(x, \kappa)f_+(y, \kappa)}{1 - e^{\pi i\sigma (x-y)}} + \frac{f_-(y, \kappa)f_+(x, \kappa)}{1 - e^{-\pi i\sigma (x-y)}} \right),
\]
where $\sigma' = -1/4$ and where $C(f, g)$ is the so-called Casorati determinant (a difference analogue of the Wronskian) of the solutions of the functional-difference equation
\[
C(f, g)(x, \kappa) = f(x + 2\sigma', \kappa)g(x, \kappa) - f(x, \kappa)g(x + 2\sigma', \kappa).
\]
For the Jost solutions $C(f_-, f_+)(x, \kappa) = 2 \sinh \left( \frac{\pi x}{\sigma} \right)$.

The equality $(W_0(b) - \lambda)G(x - y) = \delta(x-y)$ could be interpreted as an equation of distributions. Since the functions $f_{\pm}(x, k)$ are Jost solutions, the distribution defined by $(W_0(b) - \lambda) \times G(x - y)$ is supported only at $x = y$, and its singular part coincides with the singular part of the distribution
\[
-\frac{2\sigma\sigma'}{\pi i C(f_-, f_+)(\kappa)} \left( \frac{f_-(x + 2\sigma', \kappa)f_+(y, \kappa) - f_-(y, \kappa)f_+(x + 2\sigma', \kappa)}{x - y - i0} + \frac{f_-(x - 2\sigma', \kappa)f_+(y, \kappa) - f_-(y, \kappa)f_+(x - 2\sigma', \kappa)}{x - y + i0} \right)
\]
in the neighbourhood of $x = y$. This singular part is equal to
\[
-\frac{2\sigma\sigma'}{\pi i} \left( \frac{1}{x - y - i0} - \frac{1}{x - y + i0} \right) = \delta(x-y),
\]
where the authors used the Sokhotski–Plemelj formula. This formula is similar to the respective formula for a Schrödinger operator when the Dirac $\delta$-function appears by differentiating a step function.
3 Proof of inequality (1.2)

3.1 Some auxiliary results

We first collect some results from [6] verbatim. Let $A$ be a compact operator on a Hilbert space $G$ and let us denote

$$\|A\|_n = \sum_{j=1}^{n} \sqrt{\lambda_j(A^*A)},$$

where $\lambda_j(A^*A)$ are the eigenvalues of $A^*A$ in decreasing order. Then by Ky Fan’s inequality (see for example [3, Lemma 4.2]) the functionals $\|\cdot\|_n$, $n = 1, 2, \ldots$, are norms and thus for any unitary operator $Y$ in $G$ we have

$$\|Y^*AY\|_n = \|A\|_n.$$

Definition 3.1. Let $A$, $B$ be two compact operators on $G$. We say that $A$ majorises $B$ or $B \prec A$, iff

$$\|B\|_n \leq \|A\|_n, \quad \text{for all } n \in \mathbb{N}.$$

Lemma 3.2. Let $A$ be a nonnegative compact operator acting in $G$, $\{Y(k)\}_{k \in \mathbb{R}}$ be a family of unitary operators on $G$, and let $g(k) \, dk$ be a probability measure on $\mathbb{R}$. Then the operator

$$B = \int_{\mathbb{R}} Y(k)^* AY(k) g(k) \, dk$$

is majorised by $A$.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n \leq \int_{\mathbb{R}} \|Y^*(k)AY(k)\|_n g(k) \, dk = \|A\|_n \int_{\mathbb{R}} g(k) \, dk = \|A\|_n. \quad \Box$$

Let $\lambda_j = -2 \cos \omega_j \leq 2$ be the eigenvalues of $W_0(b) - V$ with $V \geq 0$. In order to slightly simplify the notations it is convenient to write

$$\lambda_j = -2 \cos (\sqrt{\theta_j})$$

with $\theta_j \in (-\infty, \pi^2]$ and $\omega_j^2 = \theta_j$.

Let us denote by $K_\lambda$ the Birman–Schwinger operator

$$K_\lambda = V^{1/2}G_\lambda V^{1/2}. \quad (3.1)$$

Let $\mu_j(K_\lambda)$ be the eigenvalues (in decreasing order) of the Birman–Schwinger operator $K_\lambda$ defined in $\lambda$. Then due to the Birman–Schwinger principle we have

$$1 = \mu_j(K_\lambda). \quad (3.2)$$

Let us define the operator

$$L_{\theta} := \frac{1}{G_{-2 \cos \sqrt{\theta}(0)}} K_{-2 \cos \sqrt{\theta}}.$$
where \( G_{-2\cos \sqrt{\theta}}(0) = \frac{1}{2\pi b \sin \sqrt{\theta}} \) is given in (2.2). Then from (3.2) we obtain

\[
\sum_{j \geq 1} \frac{1}{G_{\lambda_j}(0)} = \sum_{j \geq 1} \frac{1}{G_{\lambda_j}(0)} \mu_j(K_{\lambda_j}) = \sum_{j \geq 1} \mu_j(L_{\theta_j}).
\]

The integral kernel of the operator \( L_{\theta} \) is given by

\[
g_{\pi^2, \theta}(x) := \frac{\pi}{\sqrt{\theta}} \frac{\sinh \left( \frac{\sqrt{\theta}}{b} x \right)}{\sinh \left( \frac{\sqrt{\theta'}}{b} x \right)}.
\]

Consider a more general function

\[
g_{\varphi, \theta}(x) := \frac{\sqrt{\varphi}}{\sqrt{\theta}} \frac{\sinh \left( \frac{\sqrt{\theta}}{b} x \right)}{\sinh \left( \frac{\sqrt{\varphi}}{b} x \right)}.
\]

Since \( g_{\varphi, \theta}(0) = 1 \) its Fourier transform \( \hat{g}_{\varphi, \theta} = \mathcal{F}(g_{\varphi, \theta}) \) satisfies the equation

\[
\int_{\mathbb{R}} \hat{g}_{\varphi, \theta}(k) \, dk = 1.
\]

Moreover, for any \(-\infty < \theta < \varphi \) with \( 0 < \varphi < \pi^2 \) we have

\[
\hat{g}_{\varphi, \theta}(k) = \mathcal{F} \left( \frac{\sqrt{\varphi}}{\sqrt{\theta}} \frac{\sinh \left( \frac{\sqrt{\theta}}{b} x \right)}{\sinh \left( \frac{\sqrt{\varphi}}{b} x \right)} \right)(k) = \frac{2\pi \sin \left( \frac{\pi \sqrt{\theta}}{\sqrt{\varphi}} \right) b}{\sqrt{\theta} 2 \cosh \left( \frac{2\pi^2 b k}{\sqrt{\varphi}} \right) + 2 \cos \left( \frac{\pi \sqrt{\theta}}{\sqrt{\varphi}} \right)},
\]

and the right-hand side is positive. Thus \( \hat{g}_{\varphi, \theta} \, dk \) is a probability measure for such values.

Note also that importantly

\[
\frac{g_{\pi^2, \theta}(x)}{g_{\pi^2, \theta'}(x)} = \frac{\sqrt{\theta'}}{\sqrt{\theta}} \frac{\sinh \left( \frac{\sqrt{\theta}}{b} x \right)}{\sinh \left( \frac{\sqrt{\theta'}}{b} x \right)} = g_{\theta, \theta'}(x)
\]

and therefore

\[
(\hat{g}_{\pi^2, \theta'} * \hat{g}_{\theta', \theta})(k) = \hat{g}_{\pi^2, \theta}(k).
\]

This is the interesting convolution/semigroup property mentioned in the introduction. In the special case \(-\infty < \theta < 0 = \theta' \) analogous computations lead to the same result with \( \hat{g}_{0, \theta}(k) = \chi_{[-1,1]}(2\pi b k/\sqrt{\theta}) \pi b/\sqrt{|\theta|} \).

**Lemma 3.3 (monotonicity).** For \((\theta, \theta')\) such that \(-\infty < \theta \leq \theta'\) and \(0 \leq \theta' < \pi^2\) we have \( L_{\theta} < L_{\theta'} \).

**Proof.** Let \( Y(k) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the unitary multiplication operator

\[
(Y(k)\psi)(x) = e^{-2\pi i k x} \psi(x)
\]

and let \( T \) be the projection onto \( V^{1/2} \), i.e.,

\[
(T\psi)(x) = V^{1/2}(x) \int_{\mathbb{R}} V^{1/2}(y) \psi(y) \, dy.
\]
Using \( Y(k') = Y(k' + x) \) and Lemma 3.2 we obtain

\[
L_\theta = \int_\mathbb{R} Y(k) \theta Y(k) \tilde{g}_{\eta^2, \phi}(k) \, dk
= \int_\mathbb{R} \int_\mathbb{R} Y(k) \theta Y(k) \tilde{g}_{\eta^2, \phi}(k - k') \, dk' \, dk
= \int_\mathbb{R} Y(k') \left( \int_\mathbb{R} Y(k') \theta Y(k) \tilde{g}_{\eta^2, \phi}(k') \, dk' \right) \tilde{g}_{\eta^2, \phi}(k) \, dk' \approx \lambda_0',
\]

where we have used that \( \tilde{g}_{\eta^2, \phi} \) is a probability measure. 

**Remark 3.4.** With a slight abuse of notations, Lemma 3.3 says that \( L_\lambda \sim L_{\lambda'} \) for any \( \lambda < 2 \) as long as \( \lambda' \leq \lambda \) and \( -2 \leq \lambda' < 2 \).

### 3.2 Proof of inequality (1.2)

We now enumerate the eigenvalues of the operator \( W_V(b) \) belonging to the interval \([-2, 2]\) such that \(-2 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) repeated with multiplicity. By using the monotonicity established in Lemma 3.3 we have a sequence of inequalities

\[
\frac{1}{G_{\lambda_1}(0)} = 2\pi b \sin \omega_1 \mu_1(L_{\theta_1}) \leq \mu_1(L_{\theta_2}),
\]

\[
\sum_{j=1}^{2} \frac{1}{G_{\lambda_j}(0)} = 2\pi b \sum_{j=1}^{2} \sin \omega_j \mu_j(L_{\theta_2}) \leq \sum_{j=1}^{2} \mu_j(L_{\theta_1}),
\]

\[
\sum_{j=1}^{3} \frac{1}{G_{\lambda_j}(0)} = 2\pi b \sum_{j=1}^{3} \sin \omega_j \mu_j(L_{\theta_3}) \leq \sum_{j=1}^{3} \mu_j(L_{\theta_1}), \quad \text{etc.}
\]

Note that we do not use any assumptions on the multiplicities of the eigenvalues, other than their finiteness. Furthermore, by Lemma 3.3 the same results also hold true if a single eigenvalue is below \(-2\). Continuing the above process and noting that the trace of \( L_{\theta} \) is \( \int_\mathbb{R} V \, dx \) for all \( \theta \), we finally obtain

\[
\sum_{j \geq 1} \frac{\sin \omega_j}{\omega_j} \leq \frac{1}{2\pi b} \int_\mathbb{R} V(x) \, dx.
\]

The proof is complete.

**Remark 3.5.** Note that \( \frac{2\cosh(2\pi b)}{b^2} \rightarrow (2\pi k)^2 \) tends to the symbol of the second derivative as \( b \rightarrow 0 \) and that \( W_{\mu^2 V(b)} \geq -2 \) for sufficiently small \( b \). We thus expect that it should be possible to recover the Lieb–Thirring inequality (1.3) for a Schrödinger operator with the sharp constant \( L_{1/2} = 1/2 \) from Theorem 1.1.

### 4 Sharpness of inequality (1.2)

Similarly to the case of Schrödinger operators, we aim to prove that the Lieb–Thirring inequality becomes an equality for Dirac-delta potentials. To this end let \( c > 0 \) and consider the potential \( V_c(x) = c\delta(x) \). To properly define \( W_{V_c}(b) \), we first note that the quadratic form \( \langle \psi, (W_0(b) - 2) \psi \rangle \) can be written as

\[
\langle \psi, (W_0(b) - 2) \psi \rangle = \int_\mathbb{R} \left| 2 \sinh(\pi b) \tilde{\psi}(k) \right|^2 \, dk = \int_\mathbb{R} |\psi(x + ib/2) - \psi(x - ib/2)|^2 \, dx. \tag{4.1}
\]
This can be seen by introducing the self-adjoint operator \( D(b) = U(b/2) - U(b/2)^{-1} = 2 \sinh \left( \frac{bP}{2} \right) \) and checking that \( D(b)^2 = W_0(b) - 2 \) either directly or by means of the identity \( \cosh(2\pi bk) - 1 = 2 \sinh(\pi bk)^2 \). The form domain of \( W_0(b) \) is thus \( \text{dom}(D(b)) = \text{dom}(W_0(b/2)) \subset H^1(\mathbb{R}) \) and on this domain Sobolev’s inequality yields that

\[
|\psi(0)|^2 \leq \varepsilon \int_{\mathbb{R}} |\psi'(x)|^2 \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}} |\psi(x)|^2 \, dx \leq \frac{\varepsilon}{b^2} \int_{\mathbb{R}} |2 \sinh(\pi bk)\hat{\psi}(k)|^2 \, dk + \frac{1}{\varepsilon} \int_{\mathbb{R}} |\psi(x)|^2 \, dx
\]

for any choice of \( \varepsilon > 0 \). The KLMN theorem thus allows us to define \( W_0(b) - V_c \). As a rank one perturbation of the operator \( W_0(b) \) the potential \( V_c \) generates no more than one eigenvalue below the continuous spectrum \([2, \infty)\).

In Fourier space the eigenequation \((W_0(b) - c\delta)\psi_c = \lambda \psi_c\) becomes

\[
2 \cosh(2\pi bk)\hat{\psi}_c(k) - c\psi_c(0) = \lambda \hat{\psi}_c(k)
\]

by means of the formal identity \( \mathcal{F}(\delta\psi_c) = \psi_c(0) \). Writing again \( \lambda = -2\cos \omega \) we obtain

\[
\hat{\psi}_c(k) = \frac{c\psi_c(0)}{2 \cosh(2\pi bk) + 2 \cos \omega}
\]

and therefore

\[
\psi_c(x) = c\psi_c(0)G_{-2\cos \omega}(x) = \frac{c\psi_c(0)}{2b \sin \omega \sinh \left( \frac{\omega}{b} x \right)}. \tag{4.3}
\]

Of course we could have seen this immediately by using the equation for the Green’s function \((W_0(b) + 2\cos \omega)G_{-2\cos \omega}(x) = \delta(x)\).

Letting \( x \to 0 \) in (4.3) we find

\[
1 = \frac{c}{2b \sin \omega \pi}
\]

or equivalently

\[
\frac{\sin \omega}{\omega} = \frac{c}{2\pi b}. \tag{4.4}
\]

Since \( \frac{\sin \sqrt{\theta}}{\sqrt{\theta}} \) is a monotone decreasing function of \( \theta = \omega^2 \in (-\infty, \pi^2] \) that takes all values in \([0, \infty)\), for any \( c > 0 \) there is a unique solution \( \omega_c \) to (4.4) and vice versa. If \( c/(2\pi b) < 1 \) then \( \omega_c \in (0, \pi) \) and otherwise \( \omega_c \in i[0, \infty) \). Since \( \int V_c \, dx = c \), the identity (4.4) can be rewritten as

\[
\frac{\sin \omega}{\omega} = \frac{1}{2\pi b} \int_{\mathbb{R}} V_c(x) \, dx
\]

showing that the Lieb–Thirring inequality is satisfied for potentials \(-c\delta\) with a single eigenvalue that can be placed anywhere in \((-\infty, 2)\) by choosing \( c > 0 \) suitably.

**Remark 4.1.** If we choose the normalising constant \( \psi(0) > 0 \) then the eigenfunction defined in (4.3)

\[
\psi_c(x) = \frac{c\psi(0)}{2b \sin \omega_c \sinh \left( \frac{\omega_c}{b} x \right)}
\]
is positive assuming that the coupling constant \( c \) is small enough satisfying the inequality \( c/(2\pi b) \leq 1 \) and thus \( \omega_c \in [0, \pi) \). Note that if \( c/(2\pi b) = 1 \) then \( \omega_c = 0 \) and

\[
\psi_c(x) = \frac{\pi \psi(0)x}{b \sinh \left( \frac{x}{b} \right)} > 0.
\]

However, if the coupling constant \( c > 2\pi b \) then \( \omega_c \in i(0, \infty) \) and hence

\[
\psi_c(x) = \frac{c \psi(0)}{2b \sinh |\omega_c|} \sin \left( \frac{\omega_c |x|}{b} \right)
\]

is an oscillating function and in particular has an infinite number of zeros. This contradicts a possible conjecture that the eigenfunction for the lowest eigenvalue is strictly positive.

**Open problem.** Assume that the discrete spectrum \( \sigma_d(W_V(b)) \) of the operator \( W_V(b) \) satisfies the property \( \sigma_d(W_V(b)) \subset [-2, 2) \). Is it true that the eigenfunction corresponding to the lowest eigenvalue could be chosen strictly positive?

## 5 Necessity of \( \gamma \geq 1/2 \)

The following argument is similar to that presented in the upcoming book [2, Propositions 4.41 and 4.42] for the case of a Schrödinger operator. For \( \varepsilon > 0 \) let \( \psi_{\varepsilon}(x) = 1/\cosh(2\varepsilon x/b) \). If \( \varepsilon \) is sufficiently small, say \( \varepsilon \leq \varepsilon_0 \), then \( \psi_{\varepsilon} \in \text{dom}(W_0(b)) \). Using (4.1) we compute that

\[
\langle \psi_{\varepsilon}, (W_0(b) - 2)\psi_{\varepsilon} \rangle = \frac{b \sin^2 \varepsilon}{2\varepsilon} \int_{\mathbb{R}} \left| \frac{2 \sinh x}{\cos^2 \varepsilon \cosh^2 x + \sin^2 \varepsilon \sinh^2 x} \right|^2 dx \leq C b \varepsilon
\]

(5.1)

for a constant \( C > 0 \) independent of \( \varepsilon \leq \varepsilon_0 \). For any potential \( V \in L^1(\mathbb{R}) \) it holds that \( \langle \psi_{\varepsilon}, V\psi_{\varepsilon} \rangle \to \int_{\mathbb{R}} V dx \) as \( \varepsilon \to 0 \) by dominated convergence and thus for sufficiently small \( \varepsilon \)

\[
\langle \psi_{\varepsilon}, (W_V(b) - 2)\psi_{\varepsilon} \rangle < 0.
\]

By the min-max principle this proves the first part of Theorem 1.3.

For the second assertion of the theorem we choose more specifically the compactly supported potential \( V(x) = c \chi_{[-1/2, 1/2]}(x/b) \). By Sobolev’s inequality \( W_V(b) \geq -2 \) for sufficiently small \( c \leq c_0 \) such that all the discrete eigenvalues of \( W_V(b) \) are contained in \([-2, 2)\). Furthermore \( \|\psi_{\varepsilon}\|^2 = b/\varepsilon \) and, since \( \tanh x \geq x/2 \) for \( 0 \leq x \leq 1 \),

\[
\langle \psi_{\varepsilon}, V\psi_{\varepsilon} \rangle = cb \int_{-1/2}^{1/2} |\cosh(2\varepsilon x)|^{-2} dx = \frac{cb \tanh \varepsilon}{\varepsilon} \geq \frac{1}{2} cb
\]

(5.2)

for \( \varepsilon \leq 1 \). We now choose \( \varepsilon = c \delta \). If \( \delta \leq \min(\varepsilon_0/c_0, 1/c_0) \) such that \( \varepsilon \leq \min(\varepsilon_0, 1) \), then (5.1) and (5.2) both hold and

\[
\frac{\langle \psi_{\varepsilon}, (W_V(b) - 2)\psi_{\varepsilon} \rangle}{\|\psi_{\varepsilon}\|^2} \leq C \delta^2 - \frac{1}{2} c \varepsilon = c^2 \delta \left( C \delta - \frac{1}{2} \right).
\]

Choosing \( \delta < \min(\varepsilon_0/c_0, 1/c_0, 1/2C) \) we can conclude by the min-max principle that \( W_V(b) - 2 \) has a negative eigenvalue \( |\lambda_1| \leq -c^2 \delta \left( \frac{1}{2} - C \delta \right) \). If a Lieb–Thirring inequality (1.4) were to hold for \( \gamma < 1/2 \) then for some finite \( L_\gamma \)

\[
e^{2 \gamma \delta \left( \frac{1}{2} - C \delta \right)} \leq \frac{L_\gamma}{b} \int_{\mathbb{R}} V(x)^{\gamma+\frac{1}{2}} dx = L_\gamma e^{\gamma+\frac{1}{2}},
\]

which is clearly a contradiction if \( c \to 0 \).
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