Lie 3–algebra and super-affinization of split-octonions

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The purpose of this study is to extend the concept of a generalized Lie 3– algebra, known to the divisional algebra of the octonions $O$, to split-octonions $SO$, which is non-divisional. This is achieved through the unification of the product of both of the algebras in a single operation. Accordingly, a notational device is introduced to unify the product of both algebras. We verify that $SO$ is a Malcev algebra and we recalculate known relations for the structure constants in terms of the introduced structure tensor. Finally we construct the manifestly super-symmetric $\mathcal{N} = 1$ $SO$ affine super-algebra. An application of the split Lie 3–algebra for a Bagger and Lambert gauge theory is also discussed.

PACS numbers:

1. INTRODUCTION

The gauge-string correspondence \cite{38} is one of the most influential ideas in contemporary theoretical physics, particularly as a model for fundamental interactions and as a candidate for a unifying theory. The first specific example of such a correspondence, developed by Maldacena \cite{28}, has established a duality between the IIB string theory in the $AdS_5 \times S^5$ curved background and the $\mathcal{N} = 4$, 4–dimensional super Yang-Mills gauge theory (SYM). Another example is the ABJM model \cite{30}, which relates string theory in the $AdS_4 \times \mathbb{CP}^3$
background to the 3−dimensional $\mathcal{N} = 6$ Chern-Simons theory.

Recently, Bagger and Lambert [2–4] and also Gustavsson [14, 15] have proposed a super-symmetric action for a stack of $M$−branes which is a Chern-Simons model with a 3−algebra based gauge symmetry written as

$$\delta X^I = i \Gamma^I \Psi$$

$$\delta \Psi = \partial_\mu X^I \Gamma^\mu \Gamma^I \epsilon + i \kappa [X^I, X^J, X^K] \Gamma^{IJK} \epsilon. \quad (2)$$

In these transformations, we observe the 3−algebra that appears in the totally anti-symmetric bracket $[X^I, X^J, X^K]$. See also [22] for an example of application. From the 3−algebra one can define a metric which can be either positive definite or not. The original studies involve Euclidean positive definite metrics, but non-positive Lorentzian metrics were soon introduced [20, 31, 32] along with a non-anti-symmetric triple product [5] gauge theory. As the ABJM model is also a Chern-Simons theory, the introduction of a triple product in this context has also been studied [35, 36]. Further research involving triple algebras has been carried out in string theory [25] and in a more mathematical sense as a graded super-algebra [33].

These 3−algebras are not necessarily associative. Non-associative algebras can be divisional algebras, where if $a, b$ are in a divisional algebra, $\mathbb{A}$, $ab = 0$ implies that either $a = 0$ or $b = 0$. Kugo and Townsend [23] have shown that Lorentzian space-time spinors are associated to $\mathbb{A}$, based on the fact that $SO(n + 1, 1) = SL(2, \mathbb{A})$ to the four normed division algebras, and $n$ is related to the dimension of the field $\mathbb{A}$. Some research on the role of exceptional symmetries in physics has been carried out involving these non-associative algebras [10, 34, 37, 39]. Non-divisional algebras, where $ab = 0$ does not mean that either $a$ or $b$ is null, can also be constructed and they have applications in, for example, $M$−theory [24].

Finally, a non-associative algebra can be a Malcev algebra, an extension of a Lie algebra with non-zero Jacobian. An application of this fact is the construction of super-affine Lie algebras [16, 26]. This kind of construction is related to the current algebra, a subject which has several applications in physics, like the construction of a super-conformal manifestly $\mathcal{N} = 8$ super-symmetric hamiltonian [7], a non-linear sigma model [6], and a Lax pair for string theory in the Green and Schwarz formalism [17]. In this article we give a novel construction of an affine super-algebra for the split-octonion case.
This article is organized as follows: in section 2 the octonion and split-octonion algebras are discussed and a new formalism is introduced, and in section 3 a super-affinization of the $\mathbb{SO}$ algebra is constructed. The notion of a generalized split Lie 3–algebra is introduced in 4 and in section 5 a possible gauge theory constructed from the realization of this algebra is discussed.

2. THE SPLIT-OCTONION ALGEBRA

There are four division algebras: the real, complex, quaternionic and octonionic numbers [1]. From the complex, quaternionic and octonionic numbers, non-divisional, or split-algebras can be constructed. This means that, in these split-algebras, $ab = 0$ does not mean that either $a = 0$ or $b = 0$. One example of this is the split-octonion algebra [11]. To discuss the split-octonion algebra $\mathbb{SO}$ in conjunction with the octonion algebra $\mathbb{O}$ some notations are introduced. First, we assign $(S)\mathbb{O}$ to everything which is valid for both the algebras. Let us start with a description of the octonion algebra. A common notation to an octonion $a$ is written as $a = A_0 E_0 + A_\mu E_\mu$, where $E_0$ is the real component and $E_\mu$ are the imaginary components of the octonion. The Greek indices have rank $\mu = 1, \ldots, 7$ and $A_{i=0,\ldots,7}$ are real numbers. The $\mu$ index is of course summed in this notation. The products of the base elements obey the following multiplication table where $E_i = i$ was used. This

|   | 0 | 1   | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|-----|---|---|---|---|---|---|
| 0 | 0 | 1   | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | 1 | -0  | 4 | 7 | -2 | 6 | -5 | -3 |
| 2 | 2 | -4  | -0 | 5 | 1 | -3 | 7 | -6 |
| 3 | 3 | -7  | -5 | -0 | 6 | 2 | -4 | 1 |
| 4 | 4 | 2   | -1 | -6 | -0 | 7 | 3 | -5 |
| 5 | 5 | -6  | 3 | -2 | -7 | -0 | 1 | 4 |
| 6 | 6 | 5   | -7 | 4 | -3 | -1 | -0 | 2 |
| 7 | 7 | 3   | 6 | -1 | 5 | -4 | -2 | 0 |

$i = E_i, \ i = 0, 1, \ldots, 7$
table is, of course, a choice, as by changing the numbers corresponding to each element we change the table. For this choice, we can summarize the product of octonion base elements in the following set of laws:

\[ E_\mu E_\nu = -\delta_{\mu\nu} E_0 + c_{\mu\nu\kappa} E_\kappa, \quad E_0 E_\mu = E_\mu E_0 = E_\mu, \]
\[ E^2_0 = E_0, \quad \text{and, to } \mu \neq \nu, \quad E_\mu E_\nu = -E_\nu E_\mu. \quad (3) \]

The totally anti-symmetric tensor \( c_{\mu\nu\kappa} \) gives the structure constants of the algebra. To obtain an explicit realization we consider that the octonion basis also obeys

\[ E_\mu E_{\mu+1} = E_{\mu+3} \quad \text{and} \quad E_{\mu+7} = E_\mu. \quad (4) \]

Finally, using (3) we get the set of non-zero structure constants

\[ c_{124} = c_{235} = c_{346} = c_{457} = c_{561} = c_{672} = c_{713} = 1. \quad (5) \]

Octonion algebra has several sub-algebras. There are seven associative and non-commutative sub-algebras, isomorphic to the quaternion algebra, \( \mathbb{H} \sim \{ E_0, E_\mu, E_{\mu+1}, E_{\mu+3} \} \); seven associative and commutative sub-algebras, isomorphic to the complex number system \( \mathbb{C} \sim \{ E_0, E_\mu \} \) and one real associative and commutative algebra isomorphic to the real number system, so that \( \mathbb{R} \sim \{ E_0 \} \).

The structure constants of octonion algebra (3) constitute a totally anti-symmetric tensor, but in split-octonion algebra, the structure constants no longer have this property. To describe the situation, let us write the multiplication table where \( E_{i=0,\ldots,7} = i \), also. The split-octonion algebra \( \mathbb{SO} \) has the same number of sub-algebras as the octonion case, but the majority of them are no longer division algebras.

The four component sub-algebras are isomorphic to the split-quaternion algebra, so that \( \mathbb{SH} \sim \{ E_0, E_\mu, E_{\mu+1}, E_{\mu+3} \} \) with the exception of \( \mathbb{H} \sim \{ E_0, E_2, E_6, E_7 \} \). The sub-algebras of order two of the type \( \{ E_0, E_\mu \} \) are isomorphic either to complex or split-complex \( \mathbb{SC} \) number systems, so that \( \mu = \{1,3,4,5\} \) generates split-complex isomorphic algebras and \( \mu = \{2,6,7\} \) generates complex isomorphic algebras. The split cases do not generate totally anti-symmetric structure constants. As an example, in TableII we see that \( c_{124} = c_{142} = 1 \). We do not know a multiplication law like (3) for the split-octonion case. A proposal has appeared in [12], but it is too complicated for our purposes. In order to find something
TABLE II: Split-octonion Multiplication Table

|     | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 0   | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
| 1   | 1     | 0     | 4     | -7    | 2     | -6    | -5    | -3    |
| 2   | 2     | -4    | -0    | 5     | 1     | -3    | 7     | -6    |
| 3   | 3     | 7     | -5    | 0     | -6    | -2    | -4    | 1     |
| 4   | 4     | -2    | -1    | 6     | 0     | -7    | 3     | -5    |
| 5   | 5     | 6     | 3     | 2     | 7     | 0     | 1     | 4     |
| 6   | 6     | 5     | -7    | 4     | -3    | -1    | -0    | 2     |
| 7   | 7     | 3     | 6     | -1    | 5     | -4    | -2    | -0    |

\[ i = E_i, \ i = 0,1, \ldots, 7 \]

simpler, as executed in [20] we define a metric tensor \( \gamma_{\mu\nu} \) from the bilinear scalar product \( \text{Tr}(\cdot, \cdot) \), so that

\[ \gamma_{\mu\nu} = \text{Tr}(E_\mu, E_\nu). \quad (6) \]

For the octonion case, \( \gamma^{\mu\nu} = -\delta^{\mu\nu} \), and in the split-octonion case the non-zero elements come from the diagonal in the multiplication table and are

\[ \gamma^{00} = \gamma^{11} = -\gamma^{22} = \gamma^{33} = \gamma^{44} = \gamma^{55} = -\gamma^{66} = -\gamma^{77} = 1, \quad (7) \]

with \( \gamma^{\mu\nu} = 0 \) to \( \mu \neq \nu \) in both the cases. From the structure constants we write,

\[ c_{\mu\nu\kappa}E_\kappa = b_{\mu\nu}^\kappa E_\kappa = b_{\mu\nu}\gamma^{\lambda\kappa}E_\kappa, \quad (8) \]

where \( b_{\mu\nu\kappa} \) is a totally anti-symmetric tensor that we call the structure tensor, and \( \gamma^{\mu\nu} \) works as a diagonal metric tensor. The same letters for both the \( \mathbb{O} \) and \( \mathbb{SO} \) will be used, and will be differentiated explicitly in the text where necessary. With this choice, for the octonionic case, we have simply \( b_{\mu\nu\kappa} = -c_{\mu\nu\kappa} \), and for the split-octonionic case,

\[ b_{124} = b_{235} = b_{346} = b_{457} = b_{561} = -b_{672} = b_{713} = 1. \quad (9) \]

Now we can define a multiplication law which unifies the octonion and the split-octonion cases, namely,

\[ E_\mu E_\nu = \gamma_{\mu\nu} + b_{\mu\nu}^\kappa E_\kappa, \quad (10) \]
where, of course, \( b_{\mu\nu}^{\kappa} = \gamma^{\kappa\lambda} b_{\mu\nu\lambda} \). We also introduce a scalar product notation for the imaginary part of the \((\mathbb{S})\mathbb{O}\), so that

\[
\gamma^{\mu\nu} A_{\mu} B_{\mu} = A \cdot B \quad \text{and} \quad \gamma^{\mu\nu} A_{\mu} E_{\mu} = A \cdot E. \quad (11)
\]

We adopt a notation so that octonions and split-octonions are denoted by lower case Latin letters and their components by upper case Latin letters. Thus, two generic elements, \( a \) and \( b \), are written as \( a = A_0 + A \cdot E \) and \( b = B_0 + B \cdot E \), where the \( E_0 \) base component is superfluous, and thus omitted. We also introduce the notation where the contracted indices of the structure tensor become equal to their components, or,

\[
b_{\mu\nu}^{\kappa} A_{\kappa} = b_{\mu\nu A}. \quad (12)
\]

We stress that \( A \) in (12) has a very different meaning from the indices \( \mu \) and \( \nu \). \( A \) is a real valued parameter and the indexes are discrete; we can also say that \( A \), as an index, has neither the covariant nor the contravariant behaviors found for \( \mu \) and \( \nu \). On the other hand, as the notation keeps the anti-symmetry properties of the structure tensor \( b_{\mu\nu\lambda} \), it is a convenient way of doing the calculations. We describe several examples using the formalism.

### 2.1. Product of two elements

Two elements \( a = A_0 + A \cdot E \) and \( b = B_0 + B \cdot E \) such that \( a,b \in (\mathbb{S})\mathbb{O} \) can be multiplied and the result is,

\[
ab = A_0 B_0 + A \cdot B + A_0(B \cdot E) + B_0(A \cdot E) + b_{ABE}, \quad (13)
\]

where, of course, \( b_{ABE} = A_{\mu} B_{\nu} E_{\kappa} b^{\mu\nu\kappa} = A_{\mu} B_{\nu} E_{\kappa} \gamma^{\mu\nu'\rho} \gamma^{\rho\kappa} b_{\mu'\nu'\kappa'} \) and \( b_{ABE} \) mantains the anti-symmetry properties of the indices. In this notation the covariant properties of the \( b_3 \) tensor are, of course, not visible, as an example, \( b_{ABE} = b_{BE}^A \).

### 2.2. Properties of the structure tensor \( b_{\mu\nu\lambda} \),

The introduced notation is particularly useful to work with because the properties already known to the octonion structure constants can be rewritten for \( b_{\mu\nu\lambda} \). As an analogy to the
structure constant of octonions, which have the dual tensor $C_{\mu\nu\lambda\kappa}$, we define the $B_{\mu\nu\lambda\kappa}$ tensor, dual to $b_{\mu\nu\lambda}$ and expressed as,

$$B_{\mu\nu\rho\sigma} = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma\kappa\lambda} b^{\kappa\lambda}.$$  \hspace{1cm} (14)

For $\mathcal{O}$, we have the non-zero components of $B_{\mu\nu\kappa\lambda}$ as

$$B_{1275} = B_{1236} = B_{1435} = B_{2473} = B_{2465} = B_{3657} = 1,$$  \hspace{1cm} (15)

and for $\mathbb{S}\mathcal{O}$ we have the same non-zero components as $\mathcal{O}$, with the difference that all the values are given by $B_{\mu\nu\kappa\lambda} = -1$, with the exception of $B_{1435} = 1$. The dual structure tensor $B_{\mu\nu\kappa\lambda}$ has many useful properties. First, using the usual commutator $[\cdot, \cdot]$,

$$J_{\mu\nu\kappa} = [E_\mu, [E_\nu, E_\kappa]] + [E_\nu, [E_\kappa, E_\mu]] + [E_\kappa, [E_\mu, E_\nu]].$$  \hspace{1cm} (16)

The Jacobian (16) is identically zero for ordinary associative Lie-algebras, but for $(\mathbb{S})\mathcal{O}$ it is related to $B_4$ as

$$J_{\mu\nu\kappa} = -3B_{\mu\nu\kappa}E = b_{[\nu\kappa}^{\lambda}b_{\mu]\lambda}E.$$  \hspace{1cm} (17)

In (17), the indexes are anti-symmetrized with unit weight, as explained in the appendix. Identities involving the structure constants and the anti-symmetric $C_{\mu\nu\kappa\lambda}$ tensor, defined for the octonion algebra [13], can be adapted to the unified formalism defined above and are summarized in the appendix.

### 2.3. $(\mathbb{S})\mathcal{O}$ as Malcev algebras

A Malcev algebra is defined through a commutator, so that $[a, b] = \frac{1}{2}(ab - ba)$. It is known that $(\mathbb{S})\mathcal{O}$ are Malcev algebras [16], and we verify it with the developed formalism. For $x, y, z \in (\mathbb{S})\mathcal{O}$, the prescription to be satisfied is,

$$[x, x] = 0$$  \hspace{1cm} (18)

$$J (x, y, [x, z]) = [J(x, y, z), x]$$  \hspace{1cm} (19)

where $J(x, y, z)$ is the Jacobian,

$$J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y].$$  \hspace{1cm} (20)
Using (13) and the properties of the structure tensor, we obtain

\[
[[x, y], z] = b_{XY} b^\mu Z E \quad \text{and} \quad J(x, y, z) = 3B_{XYZE},
\]

which allows us to write,

\[
J(x, y, [z, x]) = -6 [b_{XYZ}(X \cdot E) - b_{XE}(X \cdot Z) - b_{ZE}(X \cdot Y)]
\]

which satisfies both sides of (19). As (18) is trivially satisfied, so \((S)\Omega\) is a Malcev algebra.

3. SUPER-SYMMETRIC AFFINIZATION

We can use the notation introduced above and the fact that \(S\Omega\) is a Malcev algebra to construct a super-current algebra. This kind of structure is useful in string theory to describe, for example, massive modes from a toroidal compactification \([21, 27]\). The algebra which describes the symmetry relating these modes is an affine Lie algebra, which can be obtained by affining a Lie algebra, or ever super-affining it in the case when there are fermions involved. In the case we are dealing with, there is already a super-affining procedure executed by \([16]\) for octonions, and it can be generalized for the split-octonion commutator algebra as follows.

For each generator, \(g_\mu\), of a Lie algebra, \(g\), with structure constants, \(f_{\mu\nu\lambda}\), we associate the fermionic super-field,

\[
\Psi_\mu(X) = \psi_\mu(x) + \theta \phi(x),
\]

where \(X = (x, \theta)\) denotes the super-space with \(\theta\) as a Grassmanian variable with \(\theta^2 = 0\). \(\psi(x)\) is a spin\(-\frac{1}{2}\) fermionic field, and \(\phi(x)\) a spin\(-1\) bosonic field. The super-affine \(\hat{g}\) is introduced through,

\[
\{\Psi_\mu(X), \Psi_\nu(Y)\} = f_{\mu\nu\lambda} \Psi_\lambda(Y) \delta(X, Y) + \kappa \text{tr}(g_\mu, g_\nu) D_Y \delta(X, Y),
\]

where \(\kappa \in \mathbb{R}\), \(\delta(X, Y) = (\theta - \eta) \delta(x - y)\), is a super-symmetric delta function with \(Y = (y, \eta)\) and \(D_Y = \partial_\eta + \eta \partial_y\) is a super-symmetric derivative. As non-associative algebras are not necessarily represented by matrices like Lie algebras \([9]\), we must redefine the trace that appears in (24). In the case where \(g_\mu = E_\mu \in (S)\Omega\), we have,

\[
\{\Psi_\mu(X), \Psi_\nu(Y)\} = f_{\mu\nu\lambda} \Psi_\lambda(Y) \delta(X, Y) + \kappa \Pi(E_\mu, E_\nu) D_Y \delta(X, Y),
\]
where $\Pi(E_\mu, E_\nu)$ is a projection over the identity in the composition law, in other words,

$$
\Pi(E_\mu, E_\nu) = \Re(E_\mu, E_\nu).
$$

(26)

So, with (23) on the left hand side of (25) we get,

$$
\\{\Psi_\mu(X), \Psi_\nu(Y)\} = \\{\psi_\mu(x), \psi_\nu(y)\} - \eta [\psi_\mu(x), \phi_\nu(y)] + \theta [\phi_\mu(x), \psi_\nu(y)] + \theta \eta [\phi_\mu(x), \phi_\nu(y)].
$$

(27)

Comparing the above result with the substitution of (23) on the right hand side of (25) in terms of the orders of $\eta$ and $\theta$ we obtain $\widehat{(\mathcal{S})\mathcal{O}}$ that is

$$
\\{\psi_\mu(x), \psi_\nu(y)\} = \kappa \delta_{\mu\nu} \delta(x - y)
$$

(28)

$$
[\psi_\mu(x), \phi_\nu(y)] = f_{\mu\nu\lambda} \psi_\lambda(y) \delta(x - y)
$$

(29)

$$
[\phi_\mu(x), \phi_\nu(y)] = \kappa \delta_{\mu\nu} \partial_y \delta(x - y) - f_{\mu\nu\lambda} \phi_\lambda(y) \delta(x - y)
$$

(30)

In the above equation, $\psi_\mu(x)$ and $\phi_\mu(x)$ are real fields and $\Psi_0(X)$ was associated with the $iE_0$ octonion. The super-affine $\widehat{(\mathcal{S})\mathcal{O}}$ algebra is also a Malcev super-algebra. Defining $\epsilon_x$ as 0 or 1 according to the bosonic or fermionic character of $x \in \widehat{(\mathcal{S})\mathcal{O}}$ we have the graded bracket,

$$
[x, y] = (-1)^{\epsilon_x \epsilon_y + 1} [y, x].
$$

(31)

The super-Jacobian is,

$$
J(x, y, z) = (-1)^{\epsilon_x \epsilon_z} [x, [y, z]] + (-1)^{\epsilon_y \epsilon_z} [z, [x, y]] + (-1)^{\epsilon_y \epsilon_x} [y, [z, x]].
$$

(32)

As $J(x, y, z)$ and $x$ satisfy (19) and (18), we have a Malcev algebra. So, we have characterized the super-symmetric affinization of the (split-)octonion algebra $\widehat{(\mathcal{S})\mathcal{O}}$, and shown that the proposed formalism unifies both the octonion algebras in the same formula.

### 4. The Generalized Split Lie 3–Algebra

Now, as executed by Yamazaki [40], who constructed a realization of the generalized Lie 3–algebra using $\mathcal{O}$, we have to define a 3–bracket. Accordingly, we define left (L) and right (R) operators, $L, R : (\mathcal{S})\mathcal{O} \to (\mathcal{S})\mathcal{O}$ which work as,

$$
L_a b = ab \quad \text{and} \quad R_a b = ba
$$

(33)
We also define the derivative operator \[29\],

\[
D_{a,b}x = \big( [L_a, L_b] + [R_a, R_b] + [L_a, R_b] \big) x
\]

\[
= \frac{1}{2} \big[ a(bx) - b(ax) + (xb)a - (xa)b + a(xb) - (ax)b \big].
\] (34)

Using the flexibility propriety of alternative algebras, such as \((S)\Omega\), which says that the associator \((a,x,b) = (ax)b - a(bx)\) obeys,

\[
(a,x,b) = -(b,x,a),
\] (35)

we discover that \((34)\) is anti-symmetric in \(a\) and \(b\), namely \(D_{a,b}x = -D_{b,a}x\). Using

\[
b_{\kappa\rho\sigma} b^{\rho\sigma} = B_{\mu\nu\rho\sigma} - \gamma_{\mu\rho} \gamma_{\nu\sigma} + \gamma_{\mu\sigma} \gamma_{\nu\rho}
\]

and

\[
b_{\kappa\mu\nu} B_{\rho\sigma} = 3 \left( b_{\mu\rho\nu} \gamma_{\sigma} - b_{\nu\rho\mu} \gamma_{\sigma} \right)
\] (36)

one can write, for \(a,b,x \in (S)\Omega\),

\[
[L_a, L_b] x = \frac{1}{2} \big[ a(bx) - b(ax) \big]
\]

\[
= b_{ABX} + X_0 b_{ABE} - B_{ABXE} + (B \cdot X)(A \cdot E) - (A \cdot X)(B \cdot E)
\] (37)

\[
[R_a, R_b] x = \frac{1}{2} \big[ (xb)a - (xa)b \big]
\]

\[
= b_{BAX} + X_0 b_{BAE} - B_{ABXE} + (B \cdot X)(A \cdot E) - (A \cdot X)(B \cdot E)
\] (38)

\[
[L_a, R_b] x = \frac{1}{2} \big[ a(xx) - (ax)b \big]
\]

\[
= B_{ABXE}
\] (39)

one can write \((34)\) as,

\[
D_{a,b}x = 2 \big[ (B \cdot X)(A \cdot E) - (A \cdot X)(B \cdot E) \big] - B_{ABXE},
\] (40)

and using \((40)\) we can prove that

\[
D_{a,b}(xy) = (D_{a,b}x)y + x(D_{a,b})x.
\] (41)

The property \((41)\) is similar to the derivative of a product property of real analysis, and so \((34)\) is known as a derivative operator. This operator is used to define the 3-bracket product,

\[
[a, b, x] = D_{a,b}x,
\] (42)

where \(a, b, x \in (S)\Omega\). From \((42)\) we wish to construct a Lie 3-algebra, and we thus adopt the following definition \[8\]:

\[
D_{a,b}x = \big( [L_a, L_b] + [R_a, R_b] + [L_a, R_b] \big) x
\]

\[
= \frac{1}{2} \big[ a(bx) - b(ax) + (xb)a - (xa)b + a(xb) - (ax)b \big].
\] (34)
Definition  A generalized Lie 3−algebra is an algebra \( A \) endowed with a 3−product \([−,−,−]: A^3 \rightarrow A\) and a bilinear positive product \((−,−)\) whose \( a, b, c, x, y \in A \) satisfy,

1. Fundamental Identity

\[
[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], ] + [a, b, [x, y, c]]
\]

2. Metric Compatibility Condition

\[
([a, b, x], y) + (x, [a, b, y]) = 0
\]

3. Additional Symmetry Property

\[
([x, y, a], b) - (a, [x, y, b]) = 0.
\]

Yamazaki [40] has proved that the above definition is satisfied by \( O \), and our wish is to extend it to \( SO \). The property (41) is enough to prove that the fundamental identity of the definition is satisfied by (42) in both cases. Now we define a bilinear product, so that, for \( a, b \in (S)O \), we have,

\[
(a, b) = \text{Re}(a, \bar{b})
\]

\[
= \gamma^{\mu \nu} A_\mu B_\nu.
\]

where \( \text{Re}(a) \) picks the real part of \( a \in (S)O \) out and \( \bar{a} \) is the conjugate complex of \( a \). The properties,

\[
(ab, x) + (a, bx) = 0 \quad \text{and} \quad (ab, xy) - (ba, yx) = 0
\]

are enough to demonstrate the Metric Compatibility Condition and the Additional Symmetry Property of the definition. On the other hand, the positive definiteness required for the bilinear product is satisfied by \( O \), but not by \( S\Omega \), as this latter case does not have a positive definite metric tensor. So we define a Split Generalized Lie 3−algebra as having the very definition above with the only difference that \((−,−)\) is not positive definite.

5. A POSSIBLE GAUGE THEORY

Now we discuss several conjectures about a gauge theory based on the split Lie 3−algebra (42) and its symmetry group, which get its structure constants from the algebra

\[
[E_\mu, E_\nu, E_\lambda] = f_{\mu \nu \lambda}^\kappa E_\kappa.
\]
Gomis *et al.* [20] have discussed the conditions the structure constants $f_{\mu \nu \kappa \lambda}$ have to satisfy in order to construct a Lagrangian of a Bagger and Lambert (BL) gauge theory. These conditions are simply the fundamental identity (43) and totally anti-symmetrical to the indices. On the other hand Gustavsson [14] has briefly discussed the possibility of having non totally anti-symmetric structure constants. Yamazaki [40] has provided an example of such a theory [8] to the generalized Lie $3-$algebra in the case of octonions, whose structure constants were described as satisfying the relations

$$f_{\mu \nu \kappa \lambda} = -f_{\nu \mu \kappa \lambda} = -f_{\mu \nu \lambda \kappa} = f_{\kappa \lambda \mu \nu}. \quad (50)$$

As the derivation operator belongs to the group of auto-morphisms of $\mathbb{O}$, which is known to be the special Lie group $G_2$, this is naturally the gauge group of the theory. Some of the properties of the structure constants components are,

$$f_{0 \mu \nu \kappa} = 0 \quad (51)$$

$$f_{124 \mu} = f_{235 \mu} = f_{346 \mu} = f_{457 \mu} = f_{561 \mu} = f_{672 \mu} = f_{713 \mu} = 0. \quad (52)$$

The above results show that these zero components come either from a product that involves the $E_0$ component or from the seven associative sub-algebras. The zero and non-zero components are the same both to octonion and split-octonion algebras. On the other hand, the non-zero structure constants can be decomposed into a totally anti-symmetric and a pairwise antisymmetric part which satisfy (50). Namely, we have,

$$f_{\mu \nu \kappa \lambda} = t_{\mu \nu \kappa \lambda} + (\delta_{\mu \kappa} \delta_{\nu \lambda} - \delta_{\mu \lambda} \delta_{\nu \kappa}) p_{\mu \nu \mu \nu}, \quad (53)$$

where $t_{\mu \nu \kappa \lambda}$ is totally anti-symmetric and $p_{\mu \nu \mu \nu}$ satisfies (50). For the octonion case, we have,

$$f_{1257} = -f_{1236} = -f_{2347} = f_{3415} = f_{4526} = f_{5637} = -f_{6712} = 1 \quad (54)$$

$$f_{\mu \nu \mu \nu} = -2 \quad (55)$$

and for the split-octonion case we have

$$f_{1257} = -f_{1236} = -f_{2347} = -f_{3415} = f_{4526} = f_{5637} = -f_{6712} = -1 \quad (56)$$

$$f_{\mu \nu \mu \nu} = 2, \quad \text{if} \quad E_\mu^2 \neq E_\nu^2 \quad \text{and} \quad (57)$$

$$f_{\mu \nu \mu \nu} = -2, \quad \text{if} \quad E_\mu^2 = E_\nu^2. \quad (58)$$
The non-zero components have some curious features. The \( t_{\mu \nu \kappa \lambda} \) components correspond to the cosets of the associative sub-algebras. For example, the coset of the \( \{ \pm E_0, \pm E_1, \pm E_2, \pm E_4 \} \) subgroup is \( \{ \pm E_3, \pm E_5, \pm E_6, \pm E_7 \} \), and we have \( f_{5637} \) as a non-zero component. The same occurs with the sub-algebras generated by \( \{ E_0, E_\mu \} \). The cosets of these sub-algebras generate the other non-zero components of the structure constants.

Far from curiosities, we can say that these results present a feature of the structure constants of the algebra that has not been described up until this point. Of course, the details about the structure constants are relevant to the technical construction of the gauge theory. For the \( SO \) case, the Lie 3—algebra has the non-compact \( G_2 \) as its group of automorphisms \[12, 18, 19\], and so this is the gauge group of the theory. The \( M2 \)-brane theories dual to these gauge theories are not known, but if the conjecture of correspondence is correct, the split-octonion case is a realizations to the non-compact \( G_2 \) gauge group in the same sense that \( G_2 \) is a realization of the octonion case. Studies in this direction are currently being developed.

**Appendix**

Here we give useful identities obeyed by the structure tensors \( b_{\nu \rho \kappa} \) and \( B_{\mu \nu \kappa \lambda} \). These relations were calculated based on former identities involving the structure constants of octonions summarized by Gunaydin and Ketov \[13\]. The square brackets denote an anti-symmetrized product. As an example, the anti-symmetrized product of \( U_i \) and \( V_{n-i} \) is given by,

\[
U_{[a_1...a_i} V_{a_{i+1}...a_n]} = \frac{1}{n!} \sum_{\sigma(a_1...a_n)} \text{sign}(\sigma) U_{\sigma(a_1...a_i} V_{a_{i+1}...a_n)}
\]

where \( \sigma(a_1...a_n) \) gives all the \( n! \) permutations of the \( n \) indexes and \( \text{sign}(\sigma) \) gives a positive sign to an even number of permutations and a negative sign to an odd number of
permutations. Now we write the identities,

\[ b^2 = -42 \]  
(60)

\[ B^2 = 168 \]  
(61)

\[ b_{\mu\kappa\lambda} b_{\nu}^{\kappa\lambda} = -6\gamma_{\mu\nu} \]  
(62)

\[ b_{\rho|\mu\nu} b_{\lambda|\rho\eta} + b_{\sigma|\mu\nu} b_{\lambda|\rho\eta} + b_{\eta|\mu\nu} b_{\lambda|\sigma\rho} = B_{\rho\eta|\mu\nu \gamma\lambda|\sigma} + B_{\eta\sigma|\mu\nu \gamma\lambda|\rho} + B_{\sigma\rho|\mu\nu \gamma\lambda|\eta} - \]  
\[ -\gamma_{\mu\nu}\gamma_{\rho\eta}\gamma_{\tau\mu} - \gamma_{\alpha\nu}\gamma_{\tau\rho}\gamma_{\eta\mu} - \gamma_{\tau\nu}\gamma_{\eta\rho}\gamma_{\mu\eta} \]  
(63)

\[ b_{\kappa\mu\rho} b_{\kappa\nu\lambda}^\rho + b_{\kappa\mu\lambda} b_{\kappa\nu\rho}^\lambda = \gamma_{\mu\rho}\gamma_{\nu\lambda} - \gamma_{\mu\lambda}\gamma_{\nu\rho} - 2\gamma_{\mu\nu}\gamma_{\rho\lambda} \]  
(64)

\[ b_{\kappa\mu\nu} b_{\kappa\rho\sigma} = B_{\mu\nu\rho\sigma} - \gamma_{\mu\rho}\gamma_{\nu\sigma} + \gamma_{\mu\sigma}\gamma_{\nu\rho} \]  
(65)

\[ b_{\kappa\lambda\rho} B_{\mu\nu}^{\kappa\lambda\mu} = -4b_{\rho\mu\nu} \]  
(66)

\[ B_{\mu\kappa\lambda\eta} B_{\nu}^{\kappa\lambda\eta} = 24\gamma_{\mu\nu} \]  
(67)

\[ b_{\kappa\mu\rho} b_{\kappa\nu\sigma} = -\frac{1}{2}b_{\kappa\mu\lambda} b_{\kappa\rho\sigma} + \frac{1}{2}B_{\mu\nu\rho\sigma} + \frac{1}{2}(\gamma_{\mu\rho}\gamma_{\nu\sigma} + \gamma_{\mu\sigma}\gamma_{\nu\rho}) - \gamma_{\mu\nu}\gamma_{\rho\sigma} \]  
(68)

\[ B_{\mu\kappa\lambda\gamma} B_{\eta\sigma}^{\kappa\lambda\eta} = 4(\gamma_{\mu\sigma}\gamma_{\nu\tau} - \gamma_{\mu\tau}\gamma_{\nu\sigma}) - 2B_{\mu\nu\sigma\tau} \]  
(69)

\[ 2b_{\kappa|\mu|\nu} B_{\rho|\eta|\sigma}^{\kappa\nu|\rho|\sigma} = b_{\kappa|\mu\nu} B_{\rho|\eta|\sigma}^{\kappa\nu|\rho|\sigma} - b_{\kappa\eta|\mu} B_{\rho|\nu|\sigma}^{\kappa|\nu|\rho|\sigma} \]  
(70)

\[ b_{\kappa\nu|\mu\rho} B_{\kappa\nu|\mu\rho} = b_{\kappa\sigma|\mu\rho} B_{\kappa|\nu|\mu\rho|\sigma} = b_{\sigma|\mu\nu\rho} \gamma_{\rho|\sigma} - b_{\tau|\mu\nu\rho} \gamma_{\rho|\tau} \]  
(71)

\[ b_{\kappa|\mu\nu} B_{\kappa|\rho\sigma}^{\mu\nu|\rho\sigma} = -2(b_{\rho|\mu\nu} \gamma_{\rho|\sigma} - b_{\eta|\mu\nu} \gamma_{\eta|\sigma}) \]  
(72)

\[ b_{\kappa|\rho\eta} B_{\mu\nu|\tau}^{\kappa|\rho\eta} = 3(b_{\rho|\mu\nu} \gamma_{\rho|\eta} - b_{\eta|\mu\nu} \gamma_{\eta|\rho}) \]  
(73)

\[ b_{\kappa|\rho\eta} B_{\mu\nu|\tau}^{\kappa|\rho\eta} = 2B_{\mu\nu\rho|\gamma|\tau} \]  
(74)

\[ b_{\mu\nu\rho|\gamma|\eta} - b_{\mu\nu|\rho|\eta} = \frac{3}{4}(b_{\sigma|\mu\nu} \gamma_{\rho|\eta} - b_{\eta|\mu\nu} \gamma_{\rho|\eta}) \]  
(75)

\[ B_{\kappa|\mu\nu\rho} B_{\kappa|\eta|\mu\nu\rho}^{\kappa|\eta|\mu\nu\rho} = \gamma_{\mu\gamma}\gamma_{\rho|\tau} + \gamma_{\alpha\nu}\gamma_{\tau|\rho}\gamma_{\eta\mu} + \gamma_{\tau\nu}\gamma_{\eta\rho}\gamma_{\mu\sigma} - \]  
\[ -3(\gamma_{\eta\rho}\gamma_{\sigma|\tau}) + B_{\sigma|\eta\mu\nu\rho\gamma}\gamma_{\rho|\sigma\tau} + B_{\tau|\eta\mu\nu\rho\gamma}\gamma_{\rho|\sigma\tau} \]  
(76)

\[ B_{\kappa|\mu\nu\rho} B_{\kappa|\rho\eta|\mu\nu\rho}^{\kappa|\rho\eta|\mu\nu\rho} = b_{\sigma|\mu\nu|\rho\eta} - 2B_{\sigma|\mu\nu\rho\eta|\eta} - 2(b_{\sigma|\mu\nu} \gamma_{\rho|\eta} - b_{\rho|\mu\nu} \gamma_{\rho|\eta}) \]  
(77)

**Acknowledgments** Sergio Giardino thanks CNPq for its financial support under grant number 152191/2008-9.

1. J. C. Baez. “The octonions”. *Bull. Am. Math. Soc.*, 39:145–205, (2002) math/0105155.
2. J. Bagger and N. Lambert. “Modeling multiple M2’s”. *Phys. Rev.*, D75:045020, (2007) [hep-th/0611108].
3. J. Bagger and N. Lambert. “Comments on multiple $M2$–branes”. *JHEP*, **02**:105, (2008) arXiv:0712.3738[hep-th].

4. J. Bagger and N. Lambert. “Gauge Symmetry and Supersymmetry of Multiple M2-Brances”. *Phys. Rev.*, **D77**:065008, (2008)arXiv:0711.0955[hep-th].

5. J. Bagger and N. Lambert. “3–algebras and $N = 6$ Chern-Simons gauge theories”. *Phys. Rev.*, **D79**:025002, (2009) arXiv:0807.0163[hep-th].

6. R. Benichou and J. Troost. "The conformal current algebra on supergroups with applications to the spectrum and integrability". *JHEP*, **1004**:121, (2010) arXiv:1002.3712 [hep-th]. *

Temporary entry *.

7. H. L. Carrion, M. Rojas, and F. Toppan. "Division algebras and extended $N = 2$, $N = 4$, $N = 8$ superKdVs". *J.Phys.A*, **A36**:3809–3820, (2003) nlin/0109002.

8. S. A. Cherkis and C. Saemann. “Multiple M2-branes and Generalized 3-Lie algebras”. *Phys. Rev.*, **D78**:066019, (2008) arXiv:0807.0808[hep-th].

9. J. Daboul and R. Delbourgo. "Matrix representation of octonions and generalizations". *J.Math.Phys.*, 40:4134–4150, (1999) hep-th/9906065.

10. M. J. Duff and S. Ferrara. “Black hole entropy and quantum information”. *Lect. Notes Phys.*, **755**:93–114, (2008) hep-th/0612036.

11. R. Foot and G. C. Joshi. “Nonstandard signature of space-time, superstrings and the split composition algebras”. *Lett. Math. Phys.*, **19**:65, (1990).

12. M. Gogberashvili. “Rotations in the space of split octonions”. arXiv:0808.2496[math-ph].

13. M. Gunaydin and S. V. Ketov. “Seven-sphere and the exceptional $N = 7$ and $N = 8$ superconformal algebras”. *Nucl. Phys.*, **B467**:215–246, (1996) hep-th/9601072.

14. A. Gustavsson. “Selfdual strings and loop space Nahm equations”. *JHEP*, **04**:083, (2008) arXiv:0802.3456[hep-th].

15. A. Gustavsson. “Algebraic structures on parallel $M2$–branes”. *Nucl. Phys.*, **B811**:66–76, (2009) arXiv:0709.1260[hep-th].

16. M. R. H. L. Carrion and F. Toppan. “An $N = 8$ superaffine Malcev algebra and its $N = 8$ Sugawara”. *Phys. Lett.*, **A291**:95–102, (2001) hep-th/0105313.

17. W. D. L. III and B. C. Vallilo. "Integrability of the Gauged Linear Sigma Model for $AdS_5 \times S^5$". *JHEP*, **0911**:007, (2009) arXiv:0804.4507[hep-th].

18. V. J. Beckers, V. Hussin and P. Winternitz. “Nonlinear equations with superposition formulas..."
and the exceptional group \(G_2\). 1. Complex and real forms of \(G_2\) and their maximal subalgebras”.

19. V. J. Beckers, V. Hussin and P. Winternitz. “Nonlinear equations with superposition formulas and the exceptional group \(G_2\). 2. Classification of the equations”.

20. G. M. J. Gomis and J. G. Russo. “Bagger-Lambert theory for general Lie algebras”. \textit{JHEP}, \textbf{06}:075, (2008) arXiv:0805.1012[hep-th].

21. M. Kaku. “Strings, Conformal Fields, and M-Theory”. Springer; 2nd edition (2000).

22. C. Krishnan and C. Maccaferri. “Membranes on Calibrations”. \textit{JHEP}, \textbf{07}:005, (2008) arXiv:0805.3125[hep-th].

23. T. Kugo and P. K. Townsend. “Supersymmetry and the Division Algebras”. \textit{Nucl. Phys.}, \textbf{B221}:357, (1983).

24. Z. Kuznetsova and F. Toppan. “Superalgebras of (split-)division algebras and the split octonionic \(M\)–theory in (6,5)–signature”. (2006) hep-th/0610122.

25. K. Lee and J.-H. Park. “3–algebra for supermembrane and 2–algebra for superstring”. \textit{JHEP}, \textbf{04}:012, (2009) arXiv:0902.2417[hep-th].

26. H. Lin. ”Kac-Moody Extensions of 3–Algebras and \(M2\)–branes”. \textit{JHEP}, \textbf{0807}:136, (2008) arXiv:0805.4003 [hep-th].

27. D. Lust and S. Theisen. “Lectures on String Theory”. Lecture Notes in Physics vol. 346, Springer (1989).

28. J. M. Maldacena. “The large \(N\) limit of superconformal field theories and supergravity”. \textit{Adv. Theor. Math. Phys.}, \textbf{2}:231–252, (1998), hep-th/9711200.

29. Y. Nambu. “Generalized hamiltonian dynamics”. \textit{Phys. Rev.}, \textbf{D7}:2405–2414, (1973).

30. D. L. J. O. Aharony, O. Bergman and J. M. Maldacena. “\(N=6\) superconformal Chern-Simons-matter theories, \(M2\)–branes and their gravity duals”. \textit{JHEP}, \textbf{10}:091, (2008) arXiv:0806.1218[hep-th].

31. J. M. F.-O. P. De Medeiros and E. Mendez-Escobar. “Lorentzian Lie 3–algebras and their Bagger-Lambert moduli space”. \textit{JHEP}, \textbf{07}:111, (2008) arXiv:0805.4363[hep-th].

32. J. M. F.-O. P. de Medeiros and E. Mendez-Escobar. “Metric Lie 3–algebras in Bagger-Lambert theory”. \textit{JHEP}, \textbf{08}:045, (2008)arXiv:0806.3242[hep-th].

33. J. Palmkvist. “3–algebras, triple systems and 3–graded Lie superalgebras”. \textit{J. Phys.}, \textbf{A43}:015205, (2010) arXiv:0905.2468[hep-th].
34. P. Ramond. “Exceptional groups and physics”. (2003) hep-th/0301050.

35. M. M. Sheikh-Jabbari. “A new 3–algebra representation for the $\mathcal{N} = 6 \text{ su}(N) \times \text{ su}(N)$ superconformal Chern-Simons theory. *JHEP*, 12:111, (2008) arXiv:0810.3782[hep-th].

36. S. Terashima and F. Yagi. “M5–brane Solution in ABJM Theory and 3–algebra. *JHEP*, 12:059, (2009) arXiv:0909.3101[hep-th].

37. F. Toppan. “Exceptional structures in mathematics and physics and the role of the octonions”. (2003) hep-th/0312023.

38. E. Witten. “Anti-de Sitter space and holography”. *Adv. Theor. Math. Phys.*, 2:253–291, (1998), hep-th/9802150.

39. E. Witten. “Quest for unification”. (2002) hep-ph/0207124.

40. M. Yamazaki. “Octonions, $G_2$ and generalized Lie 3–algebras”. *Phys. Lett.*, B670:215–219, (2008) arXiv:0809.1650[hep-th].