Mathematical Modeling of Boundary Stress State of Orthotropic Material

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Abstract. In the paper, the mathematical model for determination of stress and deformation fields inside and at the elastic region boundary of deformation of orthotropic materials under the biaxial loadings is developed. Within the proposed model, a mathematical model for determination of the material transition boundary from elastic into non-elastic deformation region is developed and a system of nonlinear equations for calculation of the values of its parameters is obtained. By using the continuous method of solution continuation with respect to the best parameter and Runge-Kutta method, the demarcation curves of absolutely elastic and non-elastic deformation regions for the pine tree are constructed. The analysis of the obtained curves is carried out and on its base it is shown that the proposed model, in contrast to the known mechanical theories of strength, makes possible to determine and substantiate the ultimate stress states of materials whose asymmetry coefficients of the boundaries of proportionality in some anisotropy directions are greater, and in others less than one.

1. Introduction
Nowadays, for modeling the boundary stress states of isotropic and anisotropic materials the implicit functions of two or more unknowns are generally used. These functions are determined by nonlinear algebraic equations of the second and higher orders. The equations with two unknowns are used to model biaxial boundary stress states in the planes of structural symmetry of the material and equations with three or more unknowns are used to solve the problem in the planar and more complex formulations. In the deformable solid mechanics, they are usually called criteria, or conditions of strength. For today, more than 40 such criteria have been proposed [1]. In particular, the criteria of Ashkenazi, Goldenblatt-Kopnov, Mises Hill, Zakharov and others belong to them.

However, their practical application is limited. They are mainly used only to approximate the results of experimental studies of the ultimate stress states of the mentioned materials [1].
These equations are not derived, they are phenomenological. They do not make possible to substantiate the peculiarities of deformability inside and on the boundary of the absolutely elastic region of deformation of solids with strong and weak asymmetries of strength characteristics [2, 3].

Therefore, the task of developing such mathematical model is actual, which, in contrast to existing phenomenological models, would make possible not only to approximate the data of empirical studies, but also to identify and justify the main factors of asymmetry characteristics of the ultimate stress state of the studied material.

The practical expediency of its solution is substantiated by the fact that today there are no models that would satisfactorily describe the ultimate stress states, the values of the asymmetry characteristics of which in some directions of structural symmetry of anisotropic materials are 2 ÷ 3 times larger, and in others - 2 ÷ 3 smaller per unit.

In addition, for today there are no technical solutions for the timely detection of the transition of these materials from absolutely elastic to non-elastic deformation region in the conditions of their processing and subsequent operation.

2. Mathematical model for determination of the absolutely elastic deformation region of orthotropic materials

In order to solve this problem, the mathematical model is synthesized [4], its components are:

- Assumption: in the extreme stress state the potential energy density of the material under biaxial loading \( \sigma_{11} \neq 0; \sigma_{22} \neq 0; \sigma_{12} = \sigma_{21} = \sigma_{33} = 0 \) is directly proportional to the relative bulk deformations of the material tested in the conditions of the uniaxial stretching (compressions) in the orthotropic directions, i.e.:

\[
\Pi(\sigma_{11}, \sigma_{22}) = k_1 \varepsilon_v(\sigma_{11}; 0) + k_2 \varepsilon_v(0; \sigma_{22}) + k_3;
\]

- The formula for determination of the potential energy density of the deformed material:

\[
\Pi(\sigma_{11}, \sigma_{22}) = 0.5 \sum_{i=1}^{2} \sigma_{ii} \varepsilon_v(\sigma_{11}; \sigma_{22})
\]

- The formula for determining the relative bulk deformation:

\[
\varepsilon_v(\sigma_{11}; \sigma_{22}) = \sum_{i=1}^{3} \varepsilon_{ii}(\sigma_{11}; \sigma_{22});
\]

- Hooke’s law for compressible orthotropic materials:

\[
\left\{
\begin{array}{l}
\varepsilon_{11} = L \left( \frac{1 - \mu_{23} \mu_{32}}{E_{11}} \sigma_{11} + \frac{\mu_{13} \mu_{32} - \mu_{12}}{\sqrt{E_{11} E_{22}}} \sigma_{22} \right) \\
\varepsilon_{22} = L \left( \frac{\mu_{23} \mu_{31} - \mu_{21}}{\sqrt{E_{11} E_{22}}} \sigma_{11} + \frac{1 - \mu_{13} \mu_{31}}{E_{22}} \sigma_{22} \right) \\
\varepsilon_{33} = L \left( \frac{\mu_{12} \mu_{21} - \mu_{31}}{\sqrt{E_{11} E_{33}}} \sigma_{11} + \frac{\mu_{31} \mu_{23} - \mu_{13}}{E_{22} E_{33}} \sigma_{22} \right) \\
\varepsilon_{12} = \varepsilon_{21} = \varepsilon_{23} = 0.
\end{array}
\right.
\]

Here:

\[
L^{-1} = 1 + \mu_{22} \mu_{23} \mu_{31} + \mu_{21} \mu_{13} \mu_{32} - \mu_{13} \mu_{31} - \\
- \mu_{23} \mu_{32} - \mu_{21} \mu_{23};
\]

\( \sigma_{ii} \) - components of the stress tensor; \( \varepsilon_v(\sigma_{11}; \sigma_{22}) \) - dependences of the components of the strain tensor on the components of the stress tensor; \( \Pi(\sigma_{11}; \sigma_{22}) \) - dependence of the potential energy density of the deformed material on the components of the stress tensor; \( \mu_{ij} \) - Poisson’s ratios; \( E_{ii} \) - modules of elasticity; \( k_1, k_2 \) and \( k_3 \) - parameters depending on the signs of the stress tensor components \( \sigma_{ii} \).
Within the framework of the proposed model (1) - (5) the mathematical model for determining the region boundary of the absolutely elastic behavior of the orthotropic material under the biaxial loading conditions:

$$\Phi(\sigma_{11}, \sigma_{22}) = 0,$$

where

$$\Phi(\sigma_{11}, \sigma_{22}) = \frac{1 - \mu_{23}\mu_{32}}{2E_{11}} \sigma_{11}^2 + \left( \frac{\mu_{13}\mu_{32} + \mu_{23}\mu_{31}}{2\sqrt{E_{11}E_{22}}} - \mu_{12} + \mu_{21} \right) \sigma_{11}\sigma_{22} + \frac{1 - \mu_{13}\mu_{31}}{2E_{22}} \sigma_{22}^2 + \frac{D}{L} \sigma_{11} + \frac{E}{L} \sigma_{22}^2 + \frac{F}{L}.$$  

The values of D, E and F are calculated from the following system of the equations:

$$\begin{align*}
\phi_1(D,E) &= 0; \\
\phi_2(D,E) &= 0; \\
NF &= -a_{11}D - a_{22}E - b,
\end{align*}$$

where

$$\phi_1(D,E) = \left( \frac{3b - 2N}{N} a_{11} - b_{11} \right) D^2 + \left( \frac{b^2}{N} - c \right) D +$$

$$+ \left( \frac{a_{11}a_{22}}{N} - a_{12} \right) D^2 E + \left( q + a_{22}^2 - a_{11}^2 \right) DE^2 + 2,$$

$$\phi_2(D,E) = \left( \frac{ba_{22}}{N} - b_{22} \right) D E + \left( b_{11} - \frac{ba_{11}}{N} \right) E^2 + \left( a_{12} - \frac{a_{11}a_{22}}{N} \right) E^3,$$

$$a_{11} = \sum_{n=1}^{N} \tilde{\sigma}_{11}^{(n)}; \quad a_{22} = \sum_{n=1}^{N} \tilde{\sigma}_{22}^{(n)}; \quad a_{12} = \sum_{n=1}^{N} \tilde{\sigma}_{11}^{(n)} \tilde{\sigma}_{22}^{(n)};$$

$$b = \sum_{n=1}^{N} \Pi(\tilde{\sigma}_{11}^{(n)}, \tilde{\sigma}_{22}^{(n)}); \quad c = \sum_{n=1}^{N} \Pi^2(\tilde{\sigma}_{11}^{(n)}, \tilde{\sigma}_{22}^{(n)}),$$

$$b_{11} = \sum_{n=1}^{N} \tilde{\sigma}_{11}^{(n)} \Pi(\tilde{\sigma}_{11}^{(n)}, \tilde{\sigma}_{22}^{(n)}); \quad b_{22} = \sum_{n=1}^{N} \tilde{\sigma}_{22}^{(n)} \Pi(\tilde{\sigma}_{11}^{(n)}, \tilde{\sigma}_{22}^{(n)});$$

$$q = \sum_{n=1}^{N} \left( \tilde{\sigma}_{11}^{(n)} \right)^2 - \left( \tilde{\sigma}_{22}^{(n)} \right)^2;$$
3. Methods and algorithm for practical implementation of the mathematical model

According to the equation (6) and the system of equations (8), as well as the formulas (9) - (11) for determination and prediction of the boundary stress states of the orthotropic materials with the specified physical and mechanical characteristics, the following problems must be solved:

1) Solve the system of the equations (8);

2) Develop the algorithm for finding the roots of the nonlinear algebraic equation (6).

To simplify the procedure for solving the first problem, it should be noted, that since the third equation of the system (21) is linear, and the first two are independent of the parameter F, the problem of finding solutions of this system is equivalent to the problem of finding solutions of the system:

\[
\begin{align*}
\varphi_1(D,E) &= 0; \\
\varphi_2(D,E) &= 0,
\end{align*}
\]  

(12)

The solution of the system of the equations (12) is based on a multi-step method [5], the algorithm of its practical implementation is to substantiate the initial approximate solution \((D_0, E_0)\) of the problem and its further refinement by using the following iterative formulas:

\[
y^{(k)} = X^{(k)} - \frac{2}{3} J^{-1}(X^{(k)}) G(X^{(k)});
\]

(13)

\[
X^{(k+1)} = X^{(k)} - \left( I - \frac{3}{2} \left( 3 J(y^{(k)}) - J(X^{(k)}) \right)^{-1} \right) \cdot \left( 3 J(y^{(k)}) - J^{-1}(X^{(k)}) \right) F G(X^{(k)}),
\]

(14)

where I – unity matrix; \(X = (D,E)^T\) – vector of variables; \(G(X) = (\varphi_1(X), \varphi_2(X))^T\) – vector-function; \(J(X)\) – Jacobi matrix.

For determination of the approximate solution of the system (12), assume that the results of the experimental studies of orthotropic material on uniaxial tension and compression in the directions of anisotropy are known.

In particular, consider that under the uniaxial tension and compression conditions along one of the anisotropy directions \(\sigma_{11} = \sigma_{11}^+\) and \(\sigma_{11} = \sigma_{11}^-\), respectively, and along the other direction - \(\sigma_{22} = \sigma_{22}^+\) and \(\sigma_{22} = \sigma_{22}^-\).

Then, since according to the problem condition a pair of numbers \((\sigma_{11}^+, 0; \sigma_{22}^+), (0; \sigma_{22}^-)\) are approximate solutions of the equation (6), then \(D_0\) and \(E_0\) are approximate solutions of the following systems of equations:

\[
\begin{align*}
\frac{1 - \mu_{23} \mu_{32}}{2E_{11}} \sigma_{11}^+ + \frac{D_0}{L} \frac{\sigma_{11}^+ + F}{L} &= 0; \\
\frac{1 - \mu_{23} \mu_{32}}{2E_{11}} \sigma_{11}^- + \frac{D_0}{L} \frac{\sigma_{11}^- + F}{L} &= 0,
\end{align*}
\]

(15)

\[
\begin{align*}
\frac{1 - \mu_{13} \mu_{31}}{2E_{22}} \sigma_{22}^+ + \frac{E_0}{L} \frac{\sigma_{22}^+ + F}{L} &= 0; \\
\frac{1 - \mu_{13} \mu_{31}}{2E_{22}} \sigma_{22}^- + \frac{E_0}{L} \frac{\sigma_{22}^- + F}{L} &= 0,
\end{align*}
\]

(16)

From here:

\[
D_0 = -\frac{(1 - \mu_{23} \mu_{32})}{2E_{11}} \frac{L (\sigma_{11}^+ + \sigma_{11}^-)}{L}.
\]

(17)
\[ E_0 = \frac{(1 - \mu_3 \mu_{31})}{2E_2} L(\sigma_{zz} + \sigma_{zz}^-). \]

So, the initial approximate solution of the system of equations (15) depends only on the physical and mechanical characteristics of the studied material.

For determination of the roots of the nonlinear equation (6) the method of continuous continuation by the best parameter is used [6, 7], its idea is to reduce the problem of finding solutions of the nonlinear algebraic equation of the second order (6) to the integration of the Cauchy problem:

\[ \begin{align*}
    \frac{d\sigma_{11}}{d\lambda} &= \frac{f_{11}}{\sqrt{f_{11}^2 + f_{22}^2}}, \quad \sigma_{11}(0) = \sigma_{11}^0; \\
    \frac{d\sigma_{22}}{d\lambda} &= \pm \frac{f_{22}}{\sqrt{f_{11}^2 + f_{22}^2}}, \quad \sigma_{22}(0) = 0,
\end{align*} \tag{18} \]

where

\[ f_{11} = \frac{\partial\phi(\sigma_{11}, \sigma_{22})}{\partial \sigma_{11}}; \quad f_{22} = \frac{\partial\phi(\sigma_{11}, \sigma_{22})}{\partial \sigma_{22}}; \tag{19} \]

\[ d\lambda^2 = d\sigma_{11}^2 + d\sigma_{22}^2 \tag{20} \]

The independent variables \( \sigma_{11} \) and \( \sigma_{22} \) are functions of the parameter \( \lambda \), which is calculated from the point \( (\sigma_{11}^0, 0) \) along the tangent to the curve (6).

Thus, the algorithm for numerical implementation of the mathematical model (6) - (7) consists of:

1) Determining the model parameters \( D \) and \( E \) from the system of equations (12). For solving this problem, it is proposed to use a multi-step method, namely iterative formulas (13) - (14). The initial approximation of the solution of the system (12) is taken as numerical values \( D_0 \) and \( E_0 \) of the parameters \( D \) and \( E \) determined from the formulas (17);

2) Determining the numerical value of the model parameter \( F \) from the third equation of the system (8);

3) Drawing the curve (6) using the method of continuous continuation by the best parameter which consists in integration of the Cauchy problem (18) - (20) by the numerical Runge-Kutta method of the fourth order with the given integration step.

4. Results of mathematical modeling of boundary biaxial stressed states of wood and their analysis

In order to solve this problem, numerical experiments have been performed to determine the ultimate stresses in wood of different species.

In particular, in the MATLABR2012b environment the above proposed algorithm for studying the mathematical model (6) - (7) has been implemented and according to the Table 1 the delimitation curves of absolutely elastic and non-elastic deformation regions for pine tree with biaxial stress states in the planes of structural symmetry have been constructed.

**Table 1.** Physico-mechanical characteristics of wood at the temperature \( T = 20 ^\circ C \) and relative humidity \( W = 12\% \) [8]: in the numerator - values of the tensile stresses, and in the denominator – values of the compression stresses

| Wood species | Modules of elasticity, MPa | Boundaries of proportionality, MPa | Poisson’s ratios |
|--------------|---------------------------|-----------------------------------|-----------------|
|              | \( E_{uu} \) | \( E_{rr} \) | \( E_{tt} \) | \( \sigma_{uu} \) | \( \sigma_{rr} \) | \( \sigma_{tt} \) | \( \mu_{uu} \) | \( \mu_{rr} \) | \( \mu_{tt} \) |
| Pine         | 11300 | 1740 | 900 | 102 | 10 | 7.8 | 0.078 | 0.306 | 0.463 |
|              | 46.3 | 12 | 14 | 0.045 | 0.504 | 0.527 |

The results are graphically shown in Fig. 1.
The analysis of the obtained research results has been carried out and based on it, it is established that the characteristics of the drawn curves meet the basic heuristic requirements for the construction of known mechanical strength theories [1], according to them the curves of the ultimate stress state for anisotropic materials must:

1) cover the beginning of the Cartesian coordinate system;
2) be smooth and convex.

**Figure 1.** Delimitation curves of elastic and non-elastic areas of deformation of pine wood at the temperature \( T = 20 \, ^\circ\text{C} \), relative humidity \( W = 12\% \) and biaxial stress state in radial-axial (a), axial-tangential (b and radial-tangential (c) in planes of structural symmetry.
Conclusions
The new mathematical model for determining the elastic behavior region of anisotropic materials under the biaxial loadings has been developed, which, unlike known models, makes possible not only to describe satisfactorily, but also to substantiate the geometry of this region for materials, in which the asymmetry coefficients of the proportionality boundaries are different in different directions of anisotropy.

The adaptation of numerical methods, such as the method of continuous continuation by the best parameter, Runge-Kutta method and multi-step methods for construction of curves of delimitation of biaxial absolutely elastic and inelastic areas of pine deformation in planes of structural symmetry has been carried out.

The comparative analysis of the numerical and experimental data has been performed and it has been shown that they correlate well with each other. Thus, the adequacy of the proposed mathematical model and the correctness of the assumptions underlying this model have been confirmed.

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