Abstract

Inspired by the algebraic expressions of the Hurwitz numbers, which are mainly expressed by the elements of the transform matrix from the Schur functions $s_\lambda$ to the homogeneous symmetric power sum function $p_\lambda$, and by the symplectic surgery theory of the relative GW-invariants [19], [14], using the elements of the transform matrix from the Jack function with two parameters [22] to the homogeneous symmetric power sum functions, we have constructed the Jack-Hurwitz numbers. As an application, we shall construct a series of newly genus-expanded cut-and-join differential operators, which can be thought of as the generalization of the Laplace-Beltrami operators and have the genus-expanded Jack function as their common eigenfunctions. Then we shall obtain some generating wave functions at the same level generated by the Jack-Hurwitz numbers, which can be expressed in terms of the newly cut-and-join differential operators. More important, our paper seems to reveal that there should exist a $(q, t)$-deformed relative Gromov-Witten theory.

Keyword: Jack-Hurwitz number; Jack function; genus expansion; cut-and-join operator; generating wave function

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1. Introduction

If we say that the differential operators, eigenfunctions and eigenvalues are the three musketeers playing the important roles in the mathematics and the physics, then the classical cut-and-join operators [8], [9], [10] and the Schur functions [22] are the pair of rangers who are playing a stage comedy of the love and the hate. They are flashing from the time to the time in the classical [13] or all kinds of refined (such as [5], [7]) Hurwitz Enumeration Problem and the integrable systems, the Hodge integral, the harmonic analysis in the Lie group, etc [8], [9], [10], [20], [27]. Meanwhile, the Hurwitz theory should be a great matchmaker to connect the theory of the Riemannian surface with the theory of the Harish-Chandra-Itzykson-Zuber integrals [11], [15]. The people pursues the topology/genus expansion property of a differential operator or a function, which is shown incisively and vividly in studying the limit of Harish-Chandra-Itzykson-Zuber integrals [26], and we will continue to adhere to this principle in this paper.

Inspired by the algebraic expressions of the Hurwitz numbers, which are mainly expressed by the elements of the transform matrix from the Schur functions $s_\lambda$ to the homogeneous symmetric power sum function $p_\lambda$, and by the symplectic surgery theory of the relative GW-invariants developed by A.M. Li and Y.B. Ruan [19], and by E. Ionel and T. Parker [14], using the elements of transform matrix from the Jack function $J_\lambda(q, t)$ with two parameters $(q, t)$ to the homogeneous symmetric power sum functions $p_\lambda$, we have constructed the Jack-Hurwitz numbers. As application, we obtain the newly genus expanded cut-and-join operators similar to [1], [24], [29], [30], which have the genus expanded Jack functions as the common eigenfunctions. The genus expanded cut-and-join operators are the generalization of the Laplace-Beltrami operators.

As a special case, let us see what have happened over the zonal polynomials [18], [28]. From the view of the geometry, it is well known that the zonal polynomials are eigenfunctions of the Laplace-Beltrami operator [12], [18]

\[
\Delta = \frac{1}{\det(g_{ij})} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial x_j},
\]

where $x_1, x_2, \cdots, x_N$ are coordinates of a point in a space with Riemannian metric differential form

\[
(ds)^2 = \sum_{i,j=1}^{N} g_{ij} dx_i dx_j
\]

\[
(g^{ij}) = (g_{ij})^{-1}
\]

Especially, let $GL(N, \mathbb{R}), O(N, \mathbb{R})$ be the real general linear group and the real orthogonal linear group, then we consider the homogenous space $M \cong GL(N, \mathbb{R})/O(N, \mathbb{R})$, which is isomorphic to the space of $N \times N$ real positive
definite symmetric matrices $X$ (by sending a nonsingular matrix $A$ to $AA'$) with congruence invariant Riemmanian metric \cite{18, 23}.

\begin{align*}
\left( ds \right)^2 &= \text{tr}(X^{-1}dXX^{-1}dX) .
\end{align*}

Then the Laplace-Beltrami operator concerned with the latent root $x_1, \ldots, x_N$ of $X$ is the following \cite{18}.

\begin{align*}
\Delta &= \sum_{i=1}^{N} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \sum_{i,j}^{N} x_i^2 (x_i - x_j)^{-1} \frac{\partial}{\partial x_j} - \sum_{i=1}^{N} \frac{1}{2} (N - 3) x_i \frac{\partial}{\partial x_i},
\end{align*}

where the last term will not change a homogeneous polynomial as eigenfunction but the eigenvalue, hence we leave this term out as \cite{4}, \cite{5}, \cite{6}.

\begin{align*}
D &= \sum_{i}^{N} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} x_i^2 (x_i - x_j)^{-1} \frac{\partial}{\partial x_j} - \sum_{i=1}^{N} \frac{1}{2} (N - 3) x_i \frac{\partial}{\partial x_i}.
\end{align*}

In the view of the physics, the zonal polynomials are the wave functions and the eigenvalues are the energy eigenvalues \cite{4}. To deal with the Schrodinger equation for the one-dimensional N-body problem, the physicist F. Calogero \cite{4} posed a similar Laplace-Beltrami operator as the formula (5) in 1969.

Introduce the time-variables or Miwa variables $p_1, p_2, \ldots$, (the power sum of the laten roots $x_1, \ldots, x_N$ of $X$), i.e., for any positive integer $i \geq 1$,

\begin{align*}
p_i &= \sum_{j=1}^{N} x_j^i ,
\end{align*}

then the Laplace-Beltrami operator (6) acting on the functions of the time-variables such as $f(p_1, p_2, \ldots)$ is equivalent to the classical cut-and-join operator with partition type $(1^d - 2^2)$ \cite{8}, \cite{20}, \cite{24} as

\begin{align*}
D &= \sum_{k,l \geq 1} \left[ klp_{k+l} \frac{\partial^2}{\partial p_k \partial p_l} + (k + l)p_k p_l \frac{\partial}{\partial p_{k+l}} \right] ,
\end{align*}

up a multiple $\frac{1}{2}$, which will again not change the eigenfunctions but eigenvalues.

Meanwhile, the zonal polynomial are closely concerned about the irreducible character of the general real linear group $GL(N, \mathbb{R})$ \cite{17}, \cite{22}.

In this paper, by defining the Jack-Hurwitz numbers, we will extend the above theory to the genus expanded cut-and-join operators and the genus expanded Jack functions such that the latter are the common eigenfunctions of the former. We have the following results (please check the notations in the rest of the paper):

\textbf{Definition 1.1.} (Definition 3.1) We define the connected genus $g \geq 0$ with degree $d > 0$ Jack-Hurwitz number over the closed connected Riemmann
surface $\Sigma^h$ with the ramification type $(\Delta_1, \cdots, \Delta_k)$ in terms of the element of the transformation matrix from the Jack functions to the power sum as

$$JH^g_d(\Delta_1, \cdots, \Delta_k) = \sum_{\lambda \vdash d} \left( \prod_{i=1}^k a_{\lambda}(\Delta_i)(q, t) \right) \frac{\dim \lambda}{\dim \lambda},$$

under the condition

$$\left(2 - 2h\right)d - \left(2 - 2g\right) = \sum_{i=1}^k (d - l(\Delta_i)),$$

otherwise, we define

$$JH^g_d(\Delta_1, \cdots, \Delta_k) = 0,$$

and once we take $d = 0$, no matter any conditions, it is convenient to always define

$$JH^g_0(\Delta_1, \cdots, \Delta_k) = 0.$$

**Theorem 1.2.** *(Theorem 3.3)*

1. Decrease the genus of the target surface directly, i.e., if $h \geq 1$, we have

$$JH^g_d(\Delta_1, \cdots, \Delta_k) = \sum_{\Delta \vdash d} JH^g_{h-1}(\Delta_1, \cdots, \Delta_k, \Delta, \Delta) z_{\Delta}(q, t),$$

2. (Associativity) Decrease the genus of the target surface and the numbers of the branched points by cutting it into two parts, i.e., for $h_1 + h_2 = h, h_1 \geq 0, h_2 \geq 0$, and for any positive integer $k > l \geq 1$, we have

$$JH^g_d(\Delta_1, \cdots, \Delta_k) = \sum_{\Delta} JH^{g_1}_{h_1}(\Delta_1, \cdots, \Delta_l, \Delta) z_{\Delta}(q, t) \times JH^{g_2}_{h_2}(\Delta, \Delta_{l+1}, \cdots, \Delta_k),$$

where the genus $g, g_1, g_2$ are determined by the formula (63).

**Remark 1.3.** We have noticed the following interesting facts (see Corollary (3.4) and Remark (3.5)):

$$JH^0_1((1), (1)) = \frac{1}{z_{(1)}(q, t)} = \frac{1 - t}{1 - q},$$

$$JH^0_1((1)) = \frac{1 - t}{1 - q},$$

thus we can determine $\frac{1 - t}{1 - q}$-curves passing through one point or two points. More important, the Jack-Hurwitz numbers seem to reveal that there should exist a two parameters $(q, t)$-deformation of the Gromov-Witten theory, i.e.,
there should exist \((q, t)\)-deformed relative Gromov-Witten moduli spaces and \((q, t)\)-deformed Gromov-Witten invariants, which is worth exploring.

**Definition 1.4.** (Definition 4.1) Following the idea of [29], for any partition \(\Delta \vdash d\), we define a genus expanded cut-and-join operator \(D(\Delta, h)\),

\[
D(\Delta, h) = \sum_{\Gamma, \Gamma' \vdash d} z_{\Gamma'}(q, t) h^{d+\ell(\Gamma')-\ell(\Delta)-\ell(\Gamma)} JH^3_d(\Gamma', \Delta, \Gamma) p_{\Gamma} \frac{\partial}{\partial p_{\Gamma'}},
\]

where genus \(g\) satisfies the formula (63):

\[
2d - (2 - 2g) = (d - \ell(\Gamma')) + (d - \ell(\Delta)) + (d - \ell(\Gamma)).
\]

**Theorem 1.5.** (Theorem 4.5) For any partition \(\Delta, \lambda \vdash d\), the following equality holds:

\[
D(\Delta, h) J_{\lambda}(x; q, t, h) = h^{d - \ell(\Delta)} a_{\lambda}(\Delta)(q, t) J_{\lambda}(x; q, t, h),
\]

i.e., the genus expanded Jack function \(J_{\lambda}(x; q, t, h)\) are the common eigenfunctions of the genus expanded cut-and-join operators \(D(\Delta, h)\) and \(h^{d - \ell(\Delta)} a_{\lambda}(\Delta)(q, t)\) are the corresponding eigenvalues.

**Theorem 1.6.** (Theorem 4.6) For a given \(d\), as the operators acting on the functions of the time-variables \(p = (p_1, p_2, \cdots)\), we have

\[
D(\Delta_1, h) D(\Delta_2, h) = \sum_{\Delta_3} h^{d - \ell(\Delta_1) + \ell(\Delta_3)} C_{\Delta_1, \Delta_2}^{\Delta_3} D(\Delta_3, h),
\]

where

\[
C_{\Delta_1, \Delta_2}^{\Delta_3} = z_{\Delta_3}(q, t) JH^3_d(\Delta_1, \Delta_2, \Delta_3)
\]

is a newly structure constants of the center subalgebra \(C(F[S_d])\) of the group algebra \(F[S_d]\) defined by the formula (128).

**Theorem 1.7.** (Theorem 5.7) For any \(i\), the generating wave functions satisfy

\[
\frac{\partial \Phi_h}{\partial u_i}(h|(u_1, \Delta_1), \cdots, (u_n, \Delta_n)|p^{(1)}, \cdots, p^{(k)}, p) = D(\Delta_i, h) \Phi_h(h|(u_1, \Delta_1), \cdots, (u_n, \Delta_n)|p^{(1)}, \cdots, p^{(k)}, p).
\]

**Theorem 1.8.** (Theorem 5.9)

\[
\Phi_h(h|(u_1, \Delta_1), \cdots, (u_n, \Delta_n)|p^{(1)}, \cdots, p^{(k)}, p) = \prod_{i=1}^{n} \exp(u_i D(\Delta_i, h)) \Phi_h(h|p^{(1)}, \cdots, p^{(k)}, p).
\]
In particular, if we take \( h = 0, k = 0, 1 \), we have
\[
\Phi_0 \{ h | (u_1, \Delta_1), \cdots, (u_n, \Delta_n) | p \} = \prod_{i=1}^n \exp(u_i D(\Delta_i, h)) \left( h^{-2d} \left( \frac{1 - t}{d!} \right) \right); \\
\Phi_0 \{ h | (u_1, \Delta_1), \cdots, (u_n, \Delta_n) | q, p \} = \prod_{i=1}^n \exp(u_i D(\Delta_i, h)) \left( \sum_{\Delta} h^{-2q(\Delta)} \frac{1}{z_{\Delta}(q, t) q_{\Delta} p_{\Delta}} \right).
\]

**Theorem 1.9.** (Theorem 6.1) For any partition \( \Delta \vdash d \), let \( C_\Delta \in C(C[S_d]) \) be the classical central element of \( C(C[S_d]) \), then \( \{ C_\Delta | \Delta \in \mathbb{P}_d \} \) also form a \( \mathbb{F} \)-basis of \( Z(\mathbb{F}[S_d]) \). By linearly extending, \( C(\mathbb{F}[S_d]) \) is a new commutative associative algebra under the following product,
\[
C_{\Delta_1} \bigotimes_{q, t} C_{\Delta_2} = C_{\Delta_1 \Delta_2} C_{\Delta_3},
\]
where
\[
C_{\Delta_1 \Delta_2} = z_{\Delta_3}(q, t) JH_d^g(\Delta_1, \Delta_2, \Delta_3)
\]
is the newly structure constant.

2. Notations and Preliminary

Throughout of the paper, we shall adopt the same notations as I. G. Macdonald’s book [22], those experts who are familiar with the book can skip this section.

2.1. **Partitions.** A partition [22], \( p_1 \) is a (finite or infinite) sequence
\[
\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r, \cdots),
\]
of non-negative integers in decreasing order:
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots,
\]
and containing only finitely many nonzero terms. The number of the nonzero terms is called the length of \( \lambda \), denoted by \( l(\lambda) \). The weight \( |\lambda| \) of \( \lambda \) is defined by
\[
|\lambda| := \sum_i \lambda_i.
\]

It is convenient to denote the partition by
\[
\lambda = (1^{m_1} 2^{m_2} \cdots r^{m_r} \cdots),
\]
where \( m_i = m_i(\lambda) := \#\{\lambda_j \in \lambda | \lambda_j = i\} \) is called the multiplicity of \( i \) in \( \lambda \).

Moreover we define \[^{22},\ p24\]

\[
 z_{\lambda} = \prod_{i \geq 1} i^{m_i} m_i!
\]

If \( |\lambda| = d \), we say that \( \lambda \) is a partition of \( d \), and denote it by \( \lambda \vdash d \). The set of all partitions of \( d \) is denoted by \( \mathbb{P}_d \), and the set of all partitions (including \( d = 0 \)) by \( \mathbb{P} \), which has a natural lexicographic ordering, \[^{22},\ p6\] i.e., for any two partition \( \lambda, \mu \in \mathbb{P} \) we call \( \lambda \geq \mu \) iff the first non-vanishing difference \( \lambda_i - \mu_i \) is nonnegative.

The Ferrers diagram of a partition \( \lambda \) may be formally defined as the set of points \( s = (i, j) \in \mathbb{Z}^2 \) such that \( 1 \leq j \leq \lambda_i, 1 \leq i \leq l(\lambda) \). The conjugate of a partition \( \lambda \) is the partition \( \lambda' \) whose diagram is the transpose of the diagram \( \lambda \), i.e. the diagram obtained by reflection in the main diagonal. Hence \( \lambda'_i \) is the number of nodes in the \( i \)-th column of \( \lambda \), or equivalently

\[
 \lambda'_i = \#\{j : \lambda_j \geq i\}.
\]

We will also denote the Ferrers diagram by \( \lambda \). For each \( s = (i, j) \in \lambda \), we define the arm-length \( a(i, j) \), leg-length \( l(i, j) \) \[^{22},\ p337\] and hook length \( h(i, j) \) \[^{22},\ p10\] by

\[
 a(s) = a(i, j) = \lambda_i - j,
 l(s) = l(i, j) = \lambda'_j - i,
 h(s) = h(i, j) = a(i, j) + l(i, j) + 1.
\]

Let \( p = (p_1, p_2, p_3, \ldots,) \) be indeterminate variables or the homogeneous symmetric power sums (see, section(1)), which are called the time-variables or Miwa variables, and we assume that \( \Delta = (\delta_1, \cdots, \delta_n) \) is a partition with length \( n \). We denote

\[
 \Delta! := \prod_{i \geq 1} m_i(\Delta)!,
 p\Delta := p_{\delta_1} \cdots p_{\delta_n},
 \frac{\partial}{\partial p\Delta} := \frac{1}{\Delta!} \frac{\partial}{\partial p_{\delta_1}} \cdots \frac{\partial}{\partial p_{\delta_n}}.
\]

2.2. Symmetric function. The symmetric group \( S_N \) acts the ring \( \mathbb{Q}[x_1, \cdots, x_N] \) of polynomials in \( N \) independent variables \( x = (x_1, \cdots, x_N) \) with rational coefficients by permuting the variables, and a polynomial is symmetric if it is invariant under this action. The symmetric polynomials form a graded subring:

\[
 \Lambda_N = \mathbb{Q}[x_1, \cdots, x_N]^{S_N}
 = \bigoplus_{d \geq 0} \Lambda_N^d,
\]
where $\Lambda^d_N$ is consisted by the homogeneous symmetric polynomial of degree $d \geq 0$.

It is well-known that [22], p18, p63, in $\Lambda^d_N$, the basis consisted by the monomial symmetric functions $\{m_{\lambda}(x_1, \ldots, x_N)|\lambda \in \mathbb{P}_d\}$ form a $\mathbb{Z}$-basis, and the element symmetric polynomials $\{e_{\lambda}(x_1, \ldots, x_N)|\lambda \in \mathbb{P}_d\}$ is dual to the basis consisted by the complete symmetric polynomials $\{h_{\lambda}(x_1, \ldots, x_N)|\lambda \in \mathbb{P}_d\}$, the basis consisted by the power sum symmetric polynomials $\{p_{\lambda}(x_1, \ldots, x_N)|\lambda \in \mathbb{P}_d\}$ form an orthogonal basis of $\Lambda^d_N$, [22], p64, and the basis consisted by the Schur symmetric polynomials $\{s_{\lambda}(x_1, \ldots, x_N)|\lambda \in \mathbb{P}_d\}$ form an orthonormal basis of $\Lambda^d_N$. In the rest of the paper, we will neglect the variables $(x_1, \ldots, x_N)$, and write the above symmetric polynomial as $m_{\lambda}$, $e_{\lambda}$, $h_{\lambda}$, $p_{\lambda}$, $s_{\lambda}$.

Taking the inverse limit [22], p18, we have $\mathbb{Z}$-modules $\Lambda^d$:

$(41) \quad \Lambda^d = \lim \leftarrow N \Lambda^d_N.$

Let

$(42) \quad \Lambda = \oplus_{d \geq 0} \Lambda^d,$

so that $\Lambda$ is a free graded $\mathbb{Z}$-module generated by the monomial symmetric polynomial $m_{\lambda}$ for all partitions $\lambda$. Moreover the graded ring $\Lambda$ thus defined is called the ring of symmetric functions in countably many independent variables $x_1, x_2, \ldots$. For any commutative ring or field $A$, we define

$(43) \quad \Lambda_A \cong \Lambda \otimes_{\mathbb{Z}} A.$

Thus by the definition $(38)$, $\{p_\Delta|\Delta \in \mathbb{P}\}$ form a $\mathbb{Q}$-basis of $\Lambda_\mathbb{Q}$.

2.3. Macdonald functions with two parameters $(q, t)$. Let $q, t$ be independent variables, and let $\mathbb{F} = \mathbb{Q}(q, t)$ be the field of rational functions in $q$ and $t$, and let $\Lambda_\mathbb{F} \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$ denote the $\mathbb{F}$-algebra of symmetric functions with coefficients in $\mathbb{F}$. We define $(q, t)$-deformation of the formula $(32)$ by [22], p309, (2.1)

$(44) \quad z_{\lambda}(q, t) = z_{\lambda} \prod_{i=1}^{l(\lambda)} \frac{1 - q^\lambda_i}{1 - t^\lambda_i}.$

Then the Macdonald symmetric functions $P_{\lambda}(x; q, t)$ depending rationally on two parameters $(q, t)$ are characterized by the following two properties [21], [22], p322

(a) There exist $u_{\lambda \mu} \in \mathbb{F}$ such that

$(45) \quad P_{\lambda}(x; q, t) = m_{\lambda} + \sum_{\mu < \lambda} u_{\lambda \mu} m_{\mu},$

i.e., the transition matrix that expresses the Macdonald functions in terms of the monomial symmetric functions is strictly upper unitriangular.
(b) The Macdonald functions \( P_\lambda(x; q, t) \) are pairwise orthogonal relative to the scalar product defined by
\[
< p_\lambda, p_\mu >_{q, t} = \delta_{\lambda\mu} z_\lambda(q, t).
\]

2.4. The Jack function with two parameters \((q, t)\). For each partition \( \lambda \), we denote
\[
c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s) + l(s) + 1}),
\]
\[
c'_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s) + l(s)}),
\]
\[
j_\lambda(q, t) = c_\lambda(q, t)c'_\lambda(q, t).
\]

taking \( q = t^\alpha \), letting \( t \to 1 \), we define
\[
j_\lambda^{(\alpha)} = \lim_{t \to 1} \frac{j_\lambda(q, t)}{(1 - t)^{2\lambda}}
\]
\[
= \prod_{s \in \lambda} (\alpha a(s) + l(s) + 1)(\alpha(a(s) + 1) + l(s)).
\]

Remark 2.1. In generally, \( j_\lambda(q, t) \) is not perfect square except that \( q = t \), thus we could not use the square root of \( j_\lambda(q, t) \) to define the Jack-Hurwitz number because we hope that the Jack-Hurwitz numbers are the rational function of the parameters \( q, t \), (see the definition 3.1).

Definition 2.2. Now we can define the Jack functions with two parameter \((q, t)\) as
\[
J_\lambda(x; q, t) = c_\lambda(q, t)P_\lambda(x; q, t).
\]

Then we have the following orthogonal/cutting lemma, which is a key to our paper,

Lemma 2.3. Orthogonal/Cutting Lemma [22], p353, (8.7); p357. The Jack symmetric functions \( J_\lambda(x; q, t) \) have scalar product relative to the formula
\[
< J_\lambda(x; q, t), J_\mu(x; q, t) >_{q, t} = \delta_{\lambda\mu} j_\lambda(q, t).
\]

We list two important special cases of the Jack functions which are mostly related to our paper

1. When \( q = t \),
\[
J_\lambda(x; t, t) = s_\lambda \prod_{(i, j) \in \lambda} (1 - t^{hook(i, j)}),
\]
where \( s_\lambda \) is the Schur function.
(2) Taking \( q = t^\alpha \), and letting \( t \to 1 \), we arrive at the classical \( \alpha \)-symmetric functions, which were first defined by the statistician Henry Jack in 1969, \cite{10}, \cite{22}, p353, p381, (10.22),

\[
J_\lambda^{(\alpha)}(x) = \lim_{t \to 1} \frac{J_\lambda(x; t^\alpha, t)}{(1 - t)^{|\lambda|}},
\]

which are called by the classical Jack functions in the paper.

**Remark 2.4.** The classical Jack functions \( J_\lambda^{(\alpha)}(x) \) are the bridge to connect the Macdonald functions with the zonal functions \((\alpha = 2, O(N, \mathbb{R}))\), the Schur functions \((\alpha = 1, U(N, \mathbb{C}))\) and the symplectic zonal function \((\alpha = \frac{1}{2}, Sp(N, \mathbb{H}))\), thus we would like to define our invariants as Jack-Hurwitz to admire Henry Jack’s contributions.

In our paper, we shall care more about the relationships between the Jack functions and the power sum functions. Because the power sum symmetric functions \( \{p_\Delta | \Delta \in \mathbb{F}\} \) form an \( \mathbb{F} \)-basis of \( \Lambda_\mathbb{F} \), and hence we may express the Jack function \( J_\lambda(x; q, t) \) in terms of them, say

\[
J_\lambda(x; q, t) = \sum_{\Delta} a_\lambda(\Delta)(q, t)p_\Delta,
\]

where \( a_\lambda(\Delta)(q, t) \in \mathbb{F} \), especially, we denote \( dim\lambda = a_\lambda(1^{[\lambda]})(q, t) \), which is not generally identity and very important to our paper.

**Remark 2.5.** We believe that \( dim\lambda = a_\lambda(1^{[\lambda]})(q, t) \) is some kind of the normalized dimension of the irreducible representation, in fact, for the case \( q = t \), which is actually the normalized dimension of the irreducible complex representation symmetrical group \( S_{[\lambda]} \) corresponding to the partition \( \lambda \). For the classical Jack functions case, \( dim\lambda \) is identity.

Then we can restate the Orthogonal/Cutting Lemma as

**Lemma 2.6.** (1) Orthogonal/Cutting Lemma’ /The first orthogonal lemma

\[
\sum_{|\Delta| = |\lambda|} a_\lambda(\Delta)(q, t)z_\Delta(q, t)a_\mu(\Delta)(q, t) = \delta_{\lambda\mu}j_\lambda(q, t).
\]

(2) The second orthogonal lemma

\[
\sum_{|\lambda| = |\Delta_1| = |\Delta_2|} a_\lambda(\Delta_1)(q, t)\frac{1}{j_\lambda(q, t)}a_\lambda(\Delta_2)(q, t) = \delta_{\Delta_1, \Delta_2} \frac{1}{z_{\Delta_1}(q, t)}.
\]

**Remark 2.7.** For the orthogonality of the classical Jack functions, please refer to \cite{22}, p382, (10.31).
2.5. The initial values. In this section, we introduce some initial values of the differential equations (120).

If \( a \) is an indeterminate we denote by \((a; q)_\infty\) the formal infinite product

\[
(a; q)_\infty = \prod_{r=0}^{\infty} (1 - aq^r)
\]

regarded as a formal power series in \( a \) and \( q \). Let now \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) be two sequences of independent indeterminate, and define

\[
(58) \quad \Pi(x, y; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}
\]

Then we have \([22], p324, (4.13); p309, (2.6)\)

\[
(59) \quad \Pi(x, y; q, t) = \sum_{\Delta} j_\Delta(q, t)^{-1} J_\Delta(x; q, t) J_\Delta(y; q, t),
\]

and

\[
(60) \quad \Pi(x, y; q, t) = \sum_{\Delta} z_\Delta(q, t)^{-1} p_\Delta(x) p_\Delta(y),
\]

where \( J_\Delta(x; q, t) \) and \( J_\Delta(y; q, t) \) is the Jack functions with variables \( x \) and \( y \), respectively. In the Lemma (5.3), we will give out a geometric proof for the above last two equalities.

3. Jack-Hurwitz number

After a long preparation, we could define our Jack-Hurwitz number in this section.

Let \( \Sigma^h \) be a closed connected Riemann surface of genus \( h \geq 0 \), and \( \Sigma^g \) a closed connected Riemann surface of genus \( g \geq 0 \). For a general given points set \( \{q_1, \ldots, q_k\} \in \Sigma^h \), which is called the set of branch points, we call a holomorphic map \( f : \Sigma^g \to \Sigma^h \) a ramified covering of \( \Sigma^h \) of degree \( d \geq 0 \) by \( \Sigma^g \) with a ramification type \( (\Delta_1, \cdots, \Delta_k) \), if the preimages of \( f^{-1}(q_i) = \{p_{i,1}^1, \ldots, p_{i,l_i}^1\} \) with orders \( \Delta_i = (\delta_{i,1}, \cdots, \delta_{i,l_i}) \vdash d \) for \( i = 1, \cdots, k \), respectively, where \( l_i = l(\Delta_i) \).

**Definition 3.1.** We define the connected genus \( g \geq 0 \) with degree \( d > 0 \) Jack-Hurwitz number over the closed connected Riemann surface \( \Sigma^h \) with the ramification type \( (\Delta_1, \cdots, \Delta_k) \) in terms of the element of the transformation matrix from the Jack functions to the power sum as

\[
(62) \quad JH_h^{g,d}(\Delta_1, \cdots, \Delta_k) = \sum_{\lambda \vdash d} \frac{\dim^2 \lambda}{j_\lambda(q, t)} \prod_{i=1}^{k} \frac{a_\lambda(\Delta_i)(q, t)}{\dim \lambda} \frac{a_\lambda(\Delta_k)(q, t)}{\dim \lambda},
\]
under the condition

\[(2 - 2h)d - (2 - 2g) = \sum_{i=1}^{k} (d - l(\Delta_i)), \]

otherwise, we define (Conjecture, the formula \((62)\) should be zero once the last condition \((63)\) is not satisfied, but we do not know how to prove it directly. This conjecture does not influence our paper.)

\(JH_{g,d}^{h}(\Delta_1, \ldots, \Delta_k) = 0,\)

and once we take \(d = 0\), no matter any conditions, it is convenient to always define

\(JH_{g,0}^{h}(\Delta_1, \ldots, \Delta_k) = 0.\)

**Remark 3.2.** For the case \(k = 1, 2\), the above conjecture can be proved by the second orthogonal lemma (see the formula \((57)\)), referring to the lemma \((5.3)\).

By the Orthogonal/Cutting Lemma’, we can iteratively calculate the Jack-Hurwitz numbers by decreasing the genus or the numbers of the branched points.

**Theorem 3.3.**

1. **Decrease the genus of the target surface directly, i.e., if \(h \geq 1\), we have**

\[JH_{g,d}^{h}(\Delta_1, \ldots, \Delta_k) = \sum_{\Delta=\Delta} JH_{g-1}^{h}(\Delta_1, \ldots, \Delta_k, \Delta, \Delta)z_{\Delta}(q, t),\]

2. **(Associativity) Decrease the genus of the target surface and the numbers of the branched points by cutting it into two parts, i.e, for \(h_1 + h_2 = h, h_1 \geq 0, h_2 \geq 0\), and for any positive integer \(k > l \geq 1\), we have**

\[JH_{g,d}^{h}(\Delta_1, \ldots, \Delta_k) = \sum_{\Delta} JH_{g_1,d}^{h_1}(\Delta_1, \ldots, \Delta_l, \Delta)z_{\Delta}(q, t) \times JH_{g_2,d}^{h_2}(\Delta, \Delta_{l+1}, \ldots, \Delta_k).\]

where the genus \(g, g_1, g_2\) are determined by the formula \((63)\).
Proof.  (1) If \( h \geq 1 \), by the Orthogonal/Cutting lemma', we have

\[
\begin{align*}
JH^d_{h}(\Delta_1, \cdots, \Delta_k) &= \sum_{\lambda \vdash d} \left( \frac{\dim^2 \lambda}{j_\lambda(q,t)} \right)^{1-h} \frac{a_\lambda(\Delta_1)(q,t)}{\dim \lambda} \cdots \frac{a_\lambda(\Delta_k)(q,t)}{\dim \lambda} \\
&= \sum_{\lambda \vdash d} \left( \frac{\dim^2 \lambda}{j_\lambda(q,t)} \right)^{1-h} \frac{a_\lambda(\Delta_1)(q,t)}{\dim \lambda} \cdots \frac{a_\lambda(\Delta_k)(q,t)}{\dim \lambda} \\
& \quad \times \sum_{\Delta} \frac{a_\lambda(\Delta)(q,t)}{\dim \lambda} z_\Delta(q,t) a_\mu(\Delta)(q,t) \frac{\delta_{\lambda \mu}}{j_\lambda(q,t)} \\
&= \sum_{\Delta \vdash d} JH^d_{h-1}(\Delta_1, \cdots, \Delta_k, \Delta) z_\Delta(q,t).
\end{align*}
\]  

(68)

(2) By the definition of the Jack-Hurwitz (62) and the Orthogonal/Cutting Lemma', we have

\[
\begin{align*}
\text{RHS} &= \sum_{\Delta} \sum_{\lambda} \left( \frac{\dim^2 \lambda}{j_\lambda(q,t)} \right)^{1-h_1} \frac{a_\lambda(\Delta_1)(q,t)}{\dim \lambda} \cdots \frac{a_\lambda(\Delta_l)(q,t)}{\dim \lambda} a_\lambda(\Delta)(q,t) z_\Delta(q,t) \\
& \quad \times \sum_{\mu} \left( \frac{\dim^2 \mu}{j_\mu(q,t)} \right)^{1-h_2} \frac{a_\mu(\Delta_1)(q,t)}{\dim \mu} \cdots \frac{a_\mu(\Delta_{l+1})(q,t)}{\dim \mu} a_\mu(\Delta_k)(q,t) \\
&= \sum_{\lambda} \left( \frac{\dim^2 \lambda}{j_\lambda(q,t)} \right)^{2-h_1-h_2} \frac{a_\lambda(\Delta_1)(q,t)}{\dim \lambda} \cdots \frac{a_\lambda(\Delta_l)(q,t)}{\dim \lambda} a_\lambda(\Delta_{l+1})(q,t) \\
& \quad \times \sum_{\mu} \frac{a_\mu(\Delta_{l+1})(q,t)}{\dim \mu} \cdots \frac{a_\mu(\Delta_k)(q,t)}{\dim \mu} \\
&= LHS.
\end{align*}
\]

\( \square \)

Corollary 3.4. For any positive integer \( d \), let \((d) \vdash d \) be a partition of \( d \), then we have

\[
JH^0_d((d), (d)) = \frac{1}{z_{(d)}(q,t)} = \frac{1-t^d}{d(1-q^d)},
\]

especially, taking \( d = 1 \), we have

\[
JH^0_1((1), (1)) = \frac{1}{z_{(1)}(q,t)} = \frac{1-t}{1-q}.
\]

Proof. It follows from the Theorem (3.3) (2). \( \square \)
Remark 3.5. If there is only one branched point at the level one, we have

\[ JH^0_1((1)) = \frac{1-t}{1-q}, \]

following from \( J_1(x; q, t) = (1-t)P_1(x; q, t) = (1-t)p_1, \) [22], p323, (4.8), which does not conflict with the classical Hurwitz number, because in the special case, we have firstly to deal \( j((1)) \), and \( a((1))(q, t) \), i.e.,

\[ j((1)) = c((1))c'((1)) = (1-t)(1-q), \]

\[ j(1, 1) = \lim_{t \to 1} \frac{(1-t)^2}{j(t, t)} = 1. \]

More generally, if taking \( q = t^\alpha \), and letting \( t \to 1 \), the formula (71) would be understood as

\[ JH^0_1((1))(\alpha) = \lim_{t \to 1} \frac{1-t}{1-q} = \frac{1}{\alpha}, \]

where \( JH^0_1((1))(\alpha) \) is the specialization of the Jack-Hurwitz numbers to the classical Jack function \( J^\alpha_\lambda(x) \).

Remark 3.6. It is obvious that \( a_\lambda(\Delta)(q, t) = 0 \) unless \( |\Delta| = |\lambda| \) [22], p356. If \( |\Delta| > |\lambda| \), we could take \( a_\lambda(\Delta)(q, t) = 0 \), however, if \( |\Delta| < |\lambda| \), we could shift the weight of \( \Delta \) by adding \( |\lambda| - |\Delta| \)-numbers 1, i.e., we get a new partition \( \Delta' = 1^{m_1(\Delta)}(\lambda - |\Delta|)2^{m_2(\Delta)} \ldots \), if \( \Delta = 1^{m_1(\Delta)}2^{m_2(\Delta)} \ldots \), such that \( |\Delta'| = |\lambda| \), then we could define \( a_\lambda(\Delta)(q, t) := (\lambda - |\Delta| + m_1(\Delta))a_\lambda(\Delta')(q, t), \) [24], [30], then we would obtain a shift Jack-Hurwitz number, but in this paper, we shall always assume \( |\Delta| = |\lambda| \).

Remark 3.7.

1. The equation (63) is exact the Hurwitz formulas.
2. Take \( q = t \), and let \( t \to 1 \)

\[ J_\lambda^{(1)}(x) = \lim_{t \to 1} \frac{J_\lambda(x; t, t)}{(1-t)^{|\lambda|}}, \]

\[ = \lim_{t \to 1} \frac{s_\lambda \prod_{s \in \lambda} (1-t^{\text{hook}(s)})}{(1-t)^{|\lambda|}}, \]

\[ = s_\lambda \prod_{s \in \lambda} \text{hook}(s), \]

\[ = \sum_{\Delta} \left[ \frac{|\lambda|!}{\text{dim}(\lambda) z_\Delta} \chi_\lambda(\Delta) \right] p_\Delta, \]

where \( s_\lambda \) is the Schur function, and \( \text{dim}(\lambda) \), \( \chi(\Delta) \) are the dimensions and the character of the irreducible complex representation of
symmetric group $S_d$ associated with $\lambda$ and $\Delta$, respectively, [22], [24], [29]. Thus we have

$$j_\lambda(1, 1) = \left(\frac{\lambda!}{\dim(\lambda)}\right)^2,$$

(76)

$$a_\lambda(\Delta)(1, 1) = \left(\frac{\lambda!}{\dim(\lambda)}\right) \frac{\chi_\lambda(\Delta)}{z_\Delta},$$

(77)

$$a_\lambda(1^{\lambda})(1, 1) = \left(\frac{\lambda!}{\dim(\lambda)}\right) \frac{\chi_\lambda(1^{\lambda})}{z(1^{\lambda})} = 1.$$

(78)

Substitute the above data (76), (78) into (62), we return back to the classical Hurwitz number. [21], [29], p4, (8), it is exactly reason to make us to choose the Jack functions with two parameters as our objects, rather than the Macdonald functions. Moreover, we could obtain that, under the condition (63), the Jack-Hurwitz numbers are nonzero, since the classical Hurwitz numbers are nonzero due to the Riemannian existence theorem. The other reason for us to choose the Jack functions is that the Jack functions have better integrality than the Macdonald functions, for example, the normal of the Jack functions are the polynomial function in the parameters $q, t$, however, the normal of the Macdonald functions are the rational function of the parameters $q, t$, i.e., [22], p323 (4.11), p339 (6.19), p353. (8.7))

$$< J_\lambda(x; q, t), J_\lambda(x; q, t) >_{q,t} = c_\lambda(q, t)c'_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)}t^{l(s)+1})(1 - q^{a(s)+1}t^{l(s)}),$$

(79)

$$< P_\lambda(x; q, t), P_\lambda(x; q, t) >_{q,t} = \frac{c'_\lambda(q, t)}{c_\lambda(q, t)} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1}t^{l(s)}}{1 - q^{a(s)}t^{l(s)+1}}.$$

(80)

(3) Due to the theorem (5.3) (1), we mainly consider the special case $h = 0$ in the rest of the paper, and denote the Jack-Hurwitz numbers $JH_{g,d}^0(\Delta_1, \ldots, \Delta_k)$ by $JH_{g,d}^0(\Delta_1, \ldots, \Delta_k)$.

To define the disconnect Jack-Hurwitz number, we take the rule of the exponential-logarithmic correspondence, i.e., the disconnected Jack-Hurwitz numbers are defined by the product of all connected the Jack-Hurwitz numbers module their automorphism, and then sum all possible up, i.e., for any
integer \( g \in \mathbb{Z} \),

\[
JH^g_d(\Delta_1, \cdots, \Delta_k)_{\text{disconnected}} = \sum_{|\text{Aut}|} \prod_{i=1}^m JH^g_{d_i}(\Delta_{i_1}, \cdots, \Delta_{i_k}),
\]

where the sum satisfies the following conditions, for any positive integer \( m \geq 1 \),

\[
1 - g = \sum_{i} (1 - g_i), g_i \geq 0 \text{ for } i = 1, \cdots, m
\]

\[
d = \sum_{i} d_i, d_i \geq 0 \text{ for } i = 1, \cdots, m
\]

\[
\{\Delta_1, \cdots, \Delta_k\} = \bigcup_{i=1}^m \{\Delta_{i_1}, \cdots, \Delta_{i_k}\},
\]

and \(|\text{Aut}|\) is the cardinality of the automorphisms of the all connected components. To simply our notation, we shall denote the Jack-Hurwitz numbers by \( JH^g_d(\Delta_1, \cdots, \Delta_k) \) neglecting connectness or disconnectness.

**Example 3.8.** By the Remark(3.5), for \( d \geq 1 \), \( JH^1_d - d((1^d)) = \frac{1}{d!} (1 - t)^d (1 - q)^d \), which is a disconnected Jack-Hurwitz number (with \( m = d \) same components).

### 4. Genus expanded cut-and-join operators/Jack functions with two parameters \((q, t)\)

In this section, we assume all partitions are the partitions of \( d \).

**Definition 4.1.** Following the idea of [29], for any partition \( \Delta \vdash d \), we define a genus expanded cut-and-join operator \( D(\Delta, z) \),

\[
D(\Delta, h) = \sum_{\Gamma, \Gamma' \vdash d} z^{\Gamma}(q, t) h^{d+t(\Gamma') - l(\Gamma') - l(\Gamma)} JH^g_d(\Gamma', \Delta, \Gamma) p_{\Gamma'} \frac{\partial}{\partial p_{\Gamma'}},
\]

where genus \( g \) satisfies the formula (63):

\[
2d - (2 - 2g) = (d - l(\Gamma')) + (d - l(\Delta)) + (d - l(\Gamma)).
\]

**Remark 4.2.**

1. Taking the connected Jack-Hurwitz number in the above formula (85), and \( \Delta = (1^{d-2}) \) and \( q = t \), letting \( t \to 1 \), we return back the classical cut-and-join operator [8], [9], [10], [24] or the Laplace-Beltrami operator [8] up a multiple \( \frac{1}{2} \), thus the above operator (85) is a generalization of the Laplace-Beltrami operator [8].
(2) As in [29], the power \(d + l(\Gamma') - l(\Delta) - l(\Gamma)\) of \(\hbar\) in the formula (85) is the "lost" genus after we have cut the target Riemann surface into two parts.

**Definition 4.3.** The genus expanded Jack functions \(J_\lambda(x; q, t, \hbar)(\lambda \vdash d)\) is defined by [29],[30]

\[
J_\lambda(x; q, t, \hbar) := \sum_{\Delta \vdash d} \hbar^{-d - l(\Delta)} a_\lambda(\Delta)(q, t)p_\Delta.
\]

**Remark 4.4.** (1) If we take \(\hbar = 1\), we return back to the Jack functions, thus we have extent the range of the definition of the Jack functions.

(2) If we take \(q = t\), and let \(t \to 1\), we return back to the genus expanded Schur functions, which were firstly appeared in [29].

**Theorem 4.5.** For any partition \(\Delta, \lambda \vdash d\), the following equality holds:

\[
D(\Delta, \hbar)J_\lambda(x; q, t, \hbar) = \hbar^{d - l(\Delta)} a_\lambda(\Delta)(q, t)J_\lambda(x; q, t, \hbar).
\]

i.e., the genus expanded Jack function \(J_\lambda(x; q, t, \hbar)\) are the common eigenfunctions of the genus expanded cut-and-join operators \(D(\Delta, \hbar)\) and \(\hbar^{d - l(\Delta)} a_\lambda(\Delta)(q, t)\) are the corresponding eigenvalues.

**Proof.** First of all, for any \(\Delta', \Gamma' \vdash d\), we notice that

\[
\frac{\partial}{\partial p_\Delta} p_{\Delta'} = \delta_{\Delta, \Delta'},
\]

\[
D(\Delta, \hbar)p_{\Gamma'} = \sum_{\Gamma \vdash d} z_{\Gamma'}(q, t) h^{d + l(\Gamma') - l(\Delta) - l(\Gamma)} JH^\circ_d(\Gamma', \Delta, \Gamma)p_{\Gamma}.
\]

Then the theorem follows from the Orthogonal/Cutting Lemma' and straightforward calculation, we skip it. \(\square\)

**Theorem 4.6.** For a given \(d\), as operators acting on the functions of the time-variables \(p = (p_1, p_2, \cdots)\), we have

\[
D(\Delta_1, \hbar)D(\Delta_2, \hbar) = \sum_{\Delta_3} \hbar^{d - l(\Delta_1) - l(\Delta_2) + l(\Delta_3)} C^\Delta_{\Delta_1 \Delta_2} D(\Delta_3, \hbar),
\]

where

\[
C^\Delta_{\Delta_1 \Delta_2} = z_{\Delta_3}(q, t) JH^\circ_d(\Delta_1, \Delta_2, \Delta_3)
\]

is a newly structure constants of the center subalgebra \(C(\mathbb{F}[S_d])\) of the group algebra \(\mathbb{F}[S_d]\) defined by the formula (128).
Proof. For any \( \Gamma' \vdash d \), by formula (90), we have

\[
D(\Delta_1, h) D(\Delta_2, h) p_{\Gamma'}
\]

\[
= D(\Delta_1, h) \sum_{\Gamma} z_{\Gamma'}(q, t) h^{d+\ell(\Gamma') - \ell(\Delta_2) - \ell(\Gamma)} JH_{d_1}^g (\Gamma', \Delta_2, \Gamma) p_{\Gamma'}
\]

\[
= \sum_{\Delta} h^{2d + \ell(\Gamma') - \ell(\Delta_1) - \ell(\Delta_2) - \ell(\Delta)} z_{\Gamma'}(q, t) \sum_{\Delta'} JH_{d_1}^g (\Gamma', \Delta_2, \Delta') z_{\Delta'}(q, t) JH_{d_2}^g (\Delta', \Delta_1, \Delta) p_{\Delta}
\]

\[
= \sum_{\Delta} h^{2d + \ell(\Gamma') - \ell(\Delta_1) - \ell(\Delta_2) - \ell(\Delta)} z_{\Gamma'}(q, t) \sum_{\Delta_3} JH_{d_1}^g (\Delta_1, \Delta_2, \Delta_3) z_{\Delta_3}(q, t) JH_{d_2}^g (\Delta_3, \Gamma') (\Delta, \Delta) p_{\Delta}
\]

\[
= \sum_{\Delta_3} h^{d - \ell(\Delta_1) - \ell(\Delta_2) + \ell(\Delta_3)} C_{\Delta_1, \Delta_2} \Delta_3 D(\Delta_3, h) p_{\Gamma'}
\]

which is equivalent to formula (91), and where \( g_1, g_2, g_3, g_4, g_5 \) are the genuses determined by the formula (63).

\[\square\]

Corollary 4.7. If we normalize the genus expanded cut-and-join operator \( D(\Delta, h) \) by a factor \( h^{-d+\ell(\Delta)} \),

\[
(94) \quad \hat{D}(\Delta, h) := h^{-d+\ell(\Delta)} D(\Delta, h),
\]

then for a given \( d \), as operators acting on the space of functions in the time-variables \( p = (p_1, p_2, \ldots) \), all genus expanded cut-and-join operators \( \hat{D}(\Delta, z) \) for \( \Delta \vdash d \) form a commutative associative algebra, which is denoted by \( D_d \),

\[
(95) \quad \hat{D}(\Delta_1, h) \hat{D}(\Delta_2, h) = \sum_{\Delta_3} C_{\Delta_1, \Delta_2}^{\Delta_3} \hat{D}(\Delta_3, h),
\]

i.e., we have an algebraic isomorphism:

\[
\hat{D}(\Delta, z) \cong Z(\mathbb{F}[[S_d]])
\]

\[
(96) \quad \hat{D}(\Delta, z) \mapsto C_\Delta.
\]

\[\square\]

Remark 4.8. It is worth to notice that to deal with the Schrodinger equation for the one-dimensional N-body problem, the physicist F. Calogero [1] has posed a similar Laplace-Beltrami operator as the formula (62) in 1969, it is a little challenge to find out the physics meaning of the genus expanded cut-and-join operators (63).
5. Generating wave functions and its differential equations

**Definition 5.1.** For any given genus \( h \geq 0 \), degree \( d \geq 0 \) and partitions \( \Delta_1, \ldots, \Delta_n \vdash d \), the generating wave functions at the level \( d \) is defined by

\[
(97) \Phi_h \{ h|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)| \p(1), \ldots, \p(k) \} = \sum_{t_1, \ldots, t_n \geq 0} h^{2g-2} J H_h^{g,d}(\Delta_1, \ldots, \Delta_n, \Gamma_1, \ldots, \Gamma_k, \Gamma) \left[ \prod_{j=1}^n \frac{u_j^{l_j}}{l_j!} \right] \left[ \prod_{i=1}^k \frac{p_{\Gamma_i}^{(i)}}{\dim \lambda} \right] \Phi_0
\]

where \( h, u_1, \ldots, u_n \) are indeterminate variables, \( \p(1), \ldots, \p(k), p \) are the series of (possibly different) time-variables, and \( J(\lambda, \p(i)) \) is the Jack function but having the different series of time-variables, and \( 2g - 2 \) is determined by the formula (98):

\[
(98) (2 - 2h)d - (2\cdot 2g) = \sum_{j=1}^n l_j(d - l(\Delta_j)) + \sum_{j=1}^k (d - l(\Gamma_j)) + (d - l(\Gamma)).
\]

**Remark 5.2.** For the classical Jack functions, a special generating functions appear in [5] arxiv.2004.07824, p2 (2).

**Lemma 5.3.** Applying the Corollary (3.4) and the Example (3.8), we have the following special initial values (see the equations (99)):

\[
\Phi_0\{ h||p \} = \sum_{\lambda} \sum_{\Delta} h^{-d-l(\Delta)} \frac{\dim \lambda}{\j(\lambda, q, t)} \frac{a_{\lambda}(\Delta)(q, t)}{\dim \lambda} p_{\Delta}
\]

\[
(99) = \sum_{\lambda} \frac{\dim \lambda}{\j(\lambda, q, t)} J(\lambda, q, t, h)
\]

\[
(100) = h^{-2d} \frac{p^d_1}{d!} \frac{(1 - t)^d}{(1 - q)^d}
\]

\[
\Phi_0\{ h||p(1) \} = \sum_{\lambda} \sum_{\Delta_1, \Delta_2} h^{-l(\Delta_1) - l(\Delta_2)} \frac{\dim \lambda}{\j(\lambda, q, t)} \frac{a_{\lambda}(\Delta_1)(q, t)}{\dim \lambda} \frac{a_{\lambda}(\Delta_2)(q, t)}{\dim \lambda} p_{\Delta_1} p_{\Delta_2}
\]

\[
(101) = \sum_{\lambda} h^{2d} \frac{1}{\j(\lambda, q, t)} J(\lambda, q, t, h)(p(1)) J(\lambda, q, t, h)(p)
\]

\[
(102) = \sum_{\Delta} h^{-2l(\Delta)} \frac{1}{z_{\Delta}(q, t)} p_{\Delta}(1) p_{\Delta}
\]
where $J_\lambda(q, t, h)(p^{(1)})$ and $J_\lambda(q, t, h)(p)$ to denote the genus expanded Jack function $J_\lambda(x; q, t, h), J_{\lambda}(y; q, t, h)$, but with the comprehensible different time-variables series $p^{(1)}, p$. 

**Proof.** The formula (99), (100) and (101) follow from the definition of Jack-Hurwitz number and Remark (3.5). The formula (102) can be deduced from the Corollary (3.4) and the restricted condition (63) of the nonzero of the Jack-Hurwitz numbers. We can also prove the above equality by the second orthogonal lemma (the formula (57)) as

$$
\Phi_0\{h||p\} = \sum_\lambda \sum_\Delta h^{-d-\ell(\Delta)} \frac{dim^2 \lambda}{j_\lambda(q, t)} \frac{a_\lambda(\Delta)(q, t)}{dim \lambda} p_\Delta
$$

(103)

$$
= \sum_\Delta h^{-d-\ell(\Delta)} p_\Delta \sum_\lambda \frac{dim^2 \lambda}{j_\lambda(q, t)} \frac{a_\lambda(\Delta)(q, t) a_\lambda(1^d)}{dim \lambda}
$$

(104)

$$
= \sum_\Delta h^{-d-\ell(\Delta)} p_\Delta \delta_{\Delta, 1^d} \frac{1}{z_\Delta(q, t)}
$$

(105)

$$
= h^{-2d p_1^d \frac{(1-t)^d}{d! (1-q)^d}}
$$

$$
\Phi_0\{h||p^{(1)}\} = \sum_\lambda \sum_{\Delta_1, \Delta_2} h^{-\ell(\Delta_1) - \ell(\Delta_2)} \frac{dim^2 \lambda}{j_\lambda(q, t)} \frac{a_\lambda(\Delta_1)(q, t) a_\lambda(\Delta_2)(q, t)}{dim \lambda} p^{(1)}_{\Delta_1} p_{\Delta_2}
$$

(106)

$$
= \sum_{\Delta_1, \Delta_2} h^{-\ell(\Delta_1) - \ell(\Delta_2)} p^{(1)}_{\Delta_1} p_{\Delta_2} \sum_\lambda \frac{1}{j_\lambda(q, t)} a_\lambda(\Delta_1)(q, t) a_\lambda(\Delta_2)(q, t)
$$

(107)

$$
= \sum_{\Delta_1, \Delta_2} h^{-\ell(\Delta_1) - \ell(\Delta_2)} p^{(1)}_{\Delta_1} p_{\Delta_2} \delta_{\Delta_1, 1^d} \frac{1}{z_\Delta(q, t)}
$$

(108)

$$
= \sum_\Delta h^{-2\ell(\Delta)} \frac{1}{z_\Delta(q, t)} p^{(1)}_{\Delta} p_\Delta.
$$

**Remark 5.4.**

(1) The formula (104), (107) have proven our conjecture for the case $k = 1, 2$.

(2) For the case of the classical Jack functions, taking $q = t^\alpha$, and letting $t \to 1$, the formula (100) and (102) should be understood as (see also [2], p6,

$$
\Phi_0\{h||p\}(\alpha) = \lim_{t \to 1} h^{-2d p_1^d \frac{(1-t)^d}{d! (1-q)^d}}
$$

(109)

$$
= h^{-2d p_1^d \frac{1}{d! \alpha^d}};
$$
\begin{align}
\Phi_0\{h\|p^{(1)}, p\}(\alpha) &= \lim_{t \to 1} \sum_{\Delta} h^{-2\alpha(\Delta)} \frac{1}{z_\Delta(q,t)} p^{(1)}_\Delta p_{\Delta} \\
&= \sum_{\Delta} h^{-2\alpha(\Delta)} \frac{1}{z_\Delta(\alpha(\Delta))} p^{(1)}_\Delta p_{\Delta}.
\end{align}

(3) If we take $\hbar = 1$, then the calculation from the formula (101) to the formula (102) is equivalent to the calculation from the formula (60) to the formula (61) at the level $d$, thus we give out a geometric proof for the calculation from the formula (60) to the formula (61).

(4) The formula (100) can be regarded as the genus $g$, the degree $d$ and parameters $q, t$-expansions of hook formula [22], [24], and the formula (102) as the genus $g$, the degree $d$ and parameter $q, t$-expansions of the refined Cauchy-Littlewood identity [3], [22], [24].

Example 5.5. Let us check the formula (99), (100) or the formula (109) for the case of the Jack functions at the level 2, we have
\begin{align}
J_{(2)}(p_{12}, p_2; q, t) &= \frac{1}{2}(1 + q)(1 - t)^2 p_{12} + \frac{1}{2}(1 - q)(1 - t^2) p_2; \\
J_{(12)}(p_{12}, p_2; q, t) &= \frac{1}{2}(1 - t)(1 - t^2) p_{12} - \frac{1}{2}(1 - t)(1 - t^2) p_2; \\
j_{(2)}(q, t) &= (1 - t)(1 - qt)(1 - q)(1 - q^2); \\
j_{(12)}(q, t) &= (1 - t)(1 - t^2)(1 - qt)(1 - q);
\end{align}

Thus it is easy to check the formula (109) is right.

Example 5.6. Let us check the formula (99), (100) or the formula (109) for the case of the classical Jack functions at the level 2, we have
\begin{align}
J_{(2)}^{(\alpha)}(x) &= p_1^2 + \alpha p_2; \\
J_{(12)}^{(\alpha)}(x) &= p_1^2 - p_2; \\
j_{(2)}^{(\alpha)} &= 2\alpha^2(\alpha + 1); \\
j_{(12)}^{(\alpha)} &= 2\alpha(\alpha + 1).
\end{align}

Thus it is easy to check the formula (109) is right.

Theorem 5.7. For any $i$, we have
\begin{align}
\frac{\partial \Phi_h\{h||(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}}{\partial u_i} &= D(\Delta_i, h)\Phi_h\{h||(u_1, \Delta_1), \ldots, (u_n, \Delta_n)|p^{(1)}, \ldots, p^{(k)}, p\}.
\end{align}
Proof. Firstly, we have

$$\frac{\partial \Phi_g}{\partial u_i} \{h| (u_1, \Delta_1), \ldots, (u_n, \Delta_n)| p^{(1)}, \ldots, p^{(k)}, p \}$$

\[= \sum_{l_1, \ldots, l_n \geq 0} \sum_{\Gamma_1, \ldots, \Gamma_n, \Gamma} h^{2g^i(d) + 2\Delta_1 + \cdots + \Delta_n + \cdots + \Delta_n, \cdots + \Delta_n, \Gamma_1, \cdots, \Gamma_n, \Gamma} \]

\[\sum \left( \frac{(u_j)^{l_j}}{(l_j - 1)!} \prod_{j=1, j \neq i}^{n} \frac{(u_j)^{l_j}}{l_j!} \right) \prod_{j=1}^{k} p_{\Gamma_j} \right] p_{\Gamma}.\]

We can write RHS of equation (120) as

\[\text{RHS} = \sum_{l_1 \geq l_1, \ldots, l_n \geq 0} \sum_{\Gamma_1, \ldots, \Gamma_n, \Gamma} h^{2g_+^i - 2} \times JH_d^{g_+^i}(\Delta_1, \cdots, \Delta_1, \cdots, \Delta_i, \cdots, \Delta_i, \cdots, \Delta_n, \cdots, \Delta_n, \Gamma_1, \cdots, \Gamma_n, \Gamma) \]

\[\times \left( \frac{(u_j)^{l_j}}{(l_j - 1)!} \prod_{j=1, j \neq i}^{n} \frac{(u_j)^{l_j}}{l_j!} \right) \prod_{j=1}^{k} p_{\Gamma_j} \] \[D(\Delta_i, h)p_{\Gamma'}, \]

where \(\tilde{l}_i\) means that we omit \(l_i\), and \(2g_+^i - 2\) is also determined by the formula (121):

\[(2-2h)d - (2-2g_+^i) = \sum_{j=1}^{n} l_j(d - l(\Delta_j)) - (d - l(\Delta_i)) + \sum_{j=1}^{k} (d - l(\Gamma_j)) + (d - l(\Gamma')) ,\]

Moreover we have the following facts:

Fact (1) : By Theorem (4.5),

$$D(\Delta_i, h)p_{\Gamma'} = z_{\Gamma'}(q, t) \sum_{\Gamma} h^{d + l(\Gamma') - l(\Delta_i) - l(\Gamma)} JH_d^{g_+^i}(\Gamma', \Delta_i, \Gamma)p_{\Gamma}.$$  

Fact (2) : \(JH_d^{g_+^i}(\Gamma', \Delta_i, \Gamma) \neq 0\) only if

\[(122) \quad 2d - (2 - 2g_+^i) = (d - l(\Gamma')) + (d - l(\Delta_i)) + (d - l(\Gamma));\]

Fact (3) : By the formula (98), (121), (122), we have

$$g = g_+^i + g_+^i + l(\Gamma') - 1.$$  

Then the theorem follows from the Orthogonal/Cutting Lemma’ (2.6). \qed
Remark 5.8. (1) We notice that the same phenomenon of "genus lost" appears as \[29, 30\], i.e., we "lost" the genus after we have executed the cutting surgery:

\[
(2g - 2) - (2g_i^+ - 2)
= (2g_i^+ - 2) + 2l(\Gamma')
= d + l(\Gamma') - l(\Delta) - l(\Gamma).
\]

(123)

(2) It is because that the generating function \(\Phi_h\) satisfies the equation (120), we would like to call it by the generating wave function at the level \(d\). It is obvious that we should add all degree \(d\) and all genus \(g\) to the infinity to get the classical wave functions, but we would have to deal with the countable infinite variables, which we will deal it in the next paper.

Theorem 5.9.

\[
\Phi_h\{h|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)\}|p^{(1)}, \ldots, p^{(k)}, p\}
= \prod_{i=1}^{n} \exp(u_i D(\Delta_i, h)) \Phi_h\{h||p^{(1)}, \ldots, p^{(k)}, p\}.
\]

(124)

In particular, if we take \(h = 0, k = 0, 1\), we have

\[
\Phi_0\{h|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)\}|p\}
= \prod_{i=1}^{n} \exp(u_i D(\Delta_i, h)) (h^{-2d} p^{d} \frac{1}{(1-t)^d})
\]

(125)

\[
\Phi_0\{h|(u_1, \Delta_1), \ldots, (u_n, \Delta_n)\}|q, p\}
= \prod_{i=1}^{n} \exp(u_i D(\Delta_i, h)) \left[ \sum_{\Delta} h^{-2l(\Delta)} \frac{1}{z_\Delta(q, t)} q\Delta p_\Delta \right].
\]

(126)

Proof. It follows from Theorem (5.7) and the initial values (100), (102). \(\Box\)

6. A NEWLY COMMUTATIVE ASSOCIATIVE STRUCTURE OVER THE CENTRAL SUBALGEBRA \(C(\mathbb{F}[S_d])\)

In this section, we apply the Jack-Hurwitz numbers to define a newly commutative associative product over \(C(\mathbb{F}[S_d])\).

For any given nonnegative integer \(d\), let \(C(\mathbb{C}[S_d])\) be the classical central subalgebra of the complex group algebra \(\mathbb{C}[S_d]\), and let \(\mathbb{F} = \mathbb{Q}(q, t)\) be the function fields of parameters \(q, t\) with the rational coefficients, then if we replace \(\mathbb{C}\) by \(\mathbb{F}\), we obtain \(C(\mathbb{F}[S_d])\).
Theorem 6.1. For any partition $\Delta \vdash d$, let $C_\Delta \in C(C[S_d])$ be the classical central element of $C(C[S_d])$, then $\{C_\Delta \mid \Delta \in \mathbb{P}_d\}$ also form a $\mathbb{F}$-basis of $Z(\mathbb{F}[S_d])$. By linearly extending, $C(\mathbb{F}[S_d])$ is a newly commutative associative algebra under the following product,

\begin{equation}
C_{\Delta_1} \boxtimes_{q,t} C_{\Delta_2} = C_{\Delta_3}^{\Delta_1 \Delta_2},
\end{equation}

where

\begin{equation}
C_{\Delta_3}^{\Delta_1 \Delta_2} = z_{\Delta_3}(q,t) JH_d^\beta(\Delta_1, \Delta_2, \Delta_3)
\end{equation}

is a newly structure constant.

Proof. It follows by the commutativity (Definition (62)) and associativity (Theorem (3.3) (2) for the case $h = 0$) of the Jack-Hurwitz numbers, we skip it.

Remark 6.2. Take $q = t$, and let $t \to 1$, we return back to the classical central subalgebra $C(C[S_d])$. It is an interesting problem to set up the relationships between the newly commutative associative algebra to the representation theory of the Lie groups.

Remark 6.3. All of the results in this paper can be obviously specialized to the classical Jack functions, and we would rather to call the specialized Jack-Hurwitz theory by the classical Jack-Hurwitz theory. But we would remind that we have to be careful to deal with the special case, due to that during taking $q = t^\alpha$, and taking the limit $t \to 1$, we have to divided something as (Remark (3.7) (2)), rather then simply take $t = 1$.

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