Frame-Independent Calculation of Spectral Indices from Inflation

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Abstract Spectral indices from models of inflation which incorporate a Generalized Einstein Theory (GET) gravity sector are calculated to first order in a slow-roll expansion. By quantizing a suitably-generalized measure of the intrinsic curvature perturbation, the spectral indices as calculated in the Jordan frame now match those as calculated following a conformal transformation, in the Einstein frame.

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1 Introduction

Over the past few years, a formalism has been developed for calculating the spectral index \( n_s \) of the primordial density perturbation, based on an expansion in inflationary “slow-roll” parameters.\(^1\) \(^2\) \(^3\) \(^4\) This formalism assumes that the gravitational portion of the action takes the canonical Einstein-Hilbert form. As demonstrated in \(^5\), this formalism may be applied to inflation models which incorporate a Generalized Einstein Theory (GET) gravitational action if use is made of a conformal transformation, which puts the action into the form of an Einstein-Hilbert gravitational sector with a minimally-coupled scalar field. However, the gauge-invariant measure of the intrinsic curvature perturbation upon which this formalism is based is not invariant with respect to such a conformal transformation. As discussed in section 3.2 of \(^5\), therefore, under certain initial conditions the spectral index as calculated in the Jordan frame differs from the spectral index as calculated following the conformal transformation, in the Einstein frame. In this Letter, we present a new means of calculating \( n_s \) for these GET models of inflation, the results of which are both gauge-invariant and frame-independent. By exploiting this new slow-roll expansion, the discrepancy in the spectral index between the two frames may be eliminated.

Throughout this Letter, we will assume that the background spacetime is that of a flat Friedmann-Robertson-Walker line element:

\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)d\vec{x}^2.
\]

(1)

The Hubble parameter is then \( H \equiv \dot{a}/a \), where overdots denote derivatives with respect to \( t \).

In section 2, we derive slow-roll expansions with which to calculate \( n_s \) for four distinct models of inflation. First to be treated is a model with an Einstein-Hilbert gravitational action and a minimally-coupled scalar field. Next, three closely-related GET models of inflation are considered: Induced-gravity Inflation (IgI), a theory with a nonminimally-coupled scalar field (NMSF), and a general scalar-tensor (GST) theory. In section 3, we evaluate \( n_s \) in both the Jordan and Einstein frames for the case of IgI, and show that the spectral indices now agree in the two frames. Concluding remarks follow in section 4.
2 Calculating the spectral index

The calculation of $n_s$ here follows closely the derivation by Stewart and Lyth for the case of a canonical gravity sector and a minimally-coupled scalar field [1]. Their treatment can be extended to GET models of inflation by quantizing a suitably-generalized measure of the intrinsic curvature perturbation. Here we turn to the gauge-invariant potential introduced by Bardeen [6], $\Phi_H^*$, which can be related to the intrinsic curvature perturbation for GETs (see [6] [7] [9]; henceforth we will drop the subscript “$H$”). Hwang has derived the field equations which $\Phi$ obeys in many specific GETs [7], and has further introduced variables [8] with which to cast these field equations into the same form as those studied in [1]. By defining new slow-roll parameters based on these variables, $n_s$ may be calculated unambiguously for GET models of inflation.

First we consider the same model as that treated in [1], Einstein gravity with a minimally-coupled scalar field ($\phi$), but by means of the potential $\Phi$. In analogy with [1] [4], we may define

$$\Phi \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \Phi_k(\eta) e^{i\vec{k}\cdot\vec{x}},$$

$$\langle \Phi_{\vec{k}}\Phi_{\vec{l}}^* \rangle \equiv \frac{2\pi^2}{k^3} P_{\Phi} \delta^3(\vec{k} - \vec{l}),$$

(2)

where $\eta$ is the conformal time, defined as $d\eta \equiv a^{-1}dt$. Introducing the variables $u$ and $z$ as

$$u \equiv \frac{\Phi}{\dot{\phi}}, \quad z \equiv \frac{H}{a\dot{\phi}},$$

(3)

we may construct a quantum operator $\hat{u}(x)$ as

$$\hat{u}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ u_k(\eta) \hat{a}_k e^{i\vec{k}\cdot\vec{x}} + u_k^*(\eta) \hat{a}_k^* e^{-i\vec{k}\cdot\vec{x}} \right],$$

(4)

where the creation and annihilation operators obey the usual commutation relations:

$$\left[ \hat{a}_{\vec{k}}, \hat{a}_{\vec{l}}^\dagger \right] = 0,$$

$$\left[ \hat{a}_{\vec{k}}, \hat{a}_{\vec{l}}^\dagger \right] = 0.$$

(5)

Denoting $d/d\eta$ by a prime, the mode functions $u_k$ then obey [3]:

$$u''_k + \left( k^2 - \frac{z''}{z} \right) u_k = 0$$

(6)
in a flat Friedmann universe. This equation is of exactly the same form as that presented by Mukhanov [10] for studying cosmological perturbations, and upon which the derivation by Stewart and Lyth [1] is based. Note, however, that the variable $z$ in equation (3) is the inverse of Mukhanov’s corresponding variable, $a\dot{\phi}/H$.

In order to solve equation (6) for $u_k$, we follow Stewart and Lyth [1] by introducing the slow-roll parameters:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \delta \equiv \frac{\ddot{\phi}}{H\dot{\phi}}.$$  \hspace{1cm} (7)

Then, from the definition of $z$ in equation (3), the $z''/z$ term in equation (6) may be written

$$\frac{z''}{z} = a^2 H^2 \left[ (2\epsilon + \delta) (1 + \epsilon + \delta) - \frac{1}{H} \left( \dot{\epsilon} + \dot{\delta} \right) \right].$$ \hspace{1cm} (8)

During inflation, $|\epsilon|, |\delta| \ll 1$. Furthermore, both $\dot{\epsilon}$ and $\dot{\delta}$ are second-order in $\epsilon$ and $\delta$; to first order, then, it is consistent to set $\dot{\epsilon} = \dot{\delta} = 0$, which we will do here. The conformal time $\eta$ thus takes the closed-form expression:

$$\eta = -\frac{1}{aH} \frac{1}{1 - \epsilon},$$ \hspace{1cm} (9)

so that equation (8) may be rewritten as:

$$\frac{z''}{z} = \frac{1}{\eta^2} \left( \nu^2 - \frac{1}{4} \right),$$ \hspace{1cm} (10)

with

$$\nu = \frac{1}{2} + \frac{2\epsilon + \delta}{1 - \epsilon}.$$ \hspace{1cm} (11)

Equation (8) is now simply Bessel’s equation; the mode functions $u_k$ may be written in terms of Hankel functions as

$$u_k(\eta) = (-\eta)^{1/2} \left[ A_k H^{(1)}_{\nu}(-k\eta) + B_k H^{(2)}_{\nu}(-k\eta) \right].$$ \hspace{1cm} (12)

Requiring that $u(x)$ behave as a free quantum field for $k/(aH) \gg 1$ (i.e., $u_k \to (2k)^{-1/2}e^{-ik\eta}$) sets:

$$B_k = 0, \quad A_k = \frac{\sqrt{\pi}}{2} \exp \left[ i \frac{\pi}{2} \left( \nu + \frac{1}{2} \right) \right].$$ \hspace{1cm} (13)

\footnote{Note that the slow-roll parameters $\epsilon$ and $\delta$ as defined in equation (7) are often rewritten, using the equations of motion for $H$ and $\phi$, in terms of either $dH/d\phi$ and higher derivatives (this is the so-called “Hubble slow-roll approximation” of [3]), or in terms of $dV/d\phi$ and higher derivatives (the “Potential slow-roll approximation” of [3]).}
In the opposite, long-wavelength limit \((k/(aH) \ll 1)\), the mode functions \(u_k\) thus behave as
\[
\langle 0|\hat{\Phi}_k\hat{\Phi}_\ell|0\rangle \propto |u_k|^2\delta^3(\vec{k} - \vec{\ell})^1,
\]
giving
\[
P^{1/2}_\Phi(k) \propto k^{3/2-\nu}.
\]

The large-scale behavior of \(\Phi\) is considered in \[7\] \[8\] \[9\]. Setting \(R_k \propto k^{-1}\Phi_k\) at this scale, where \(R\) is the intrinsic curvature perturbation, gives \(P^{1/2}_R \propto k^{-1}P^{1/2}_\Phi \propto k^{1/2-\nu}\). The spectral index is defined by \[1\] \[4\]
\[
n_s \equiv 1 + \frac{d\ln P_R}{d\ln k},
\]
so, from equation (16),
\[
n_s = 2 - 2\nu.
\]

To first order, \(\nu \simeq 1/2 + 2\epsilon + \delta\), so that equation (18) may be rewritten
\[
n_s \simeq 1 - 4\epsilon - 2\delta.
\]

This is the standard first-order result for Einstein gravity with a minimally-coupled scalar field. \[1\] \[3\] \[4\] Given \(|\epsilon|, |\delta| \ll 1\) during inflation, it is clear that such inflationary models generically predict density perturbation spectra which are close to the \(n_s = 1.00\) scale-invariant (Harrison-Zel’dovich) spectrum.

The foregoing derivation may now be repeated for the three GET models of inflation. First we consider Induced-gravity Inflation (IgI), the action for which may be written \[14\]
\[
S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\xi\phi^2R - \frac{1}{2}\phi,\mu\phi^\mu - V(\phi)\right],
\]
\[
V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2.
\]

Note that all three of the GET models considered here involve only a single scalar field, thereby avoiding the nonadiabatic “frictional damping” which arises in GETs which employ more than one scalar field. \[11\] For more on the calculation of spectral indices from inflationary models with several dynamical degrees of freedom, see \[12\]. Metric perturbations in string cosmologies with \(d\) spatial dimensions and \(n\) internal dimensions are considered in \[13\].
where \( \xi (>0) \) is the nonminimal coupling strength, and is related to the Brans-Dicke parameter \( \omega \) by \( \xi = (4\omega)^{-1} \). For IgI it is convenient to describe the equation of motion for \( \Phi \) in terms of the generalized variables \( u \) and \( z \):

\[
u 
\begin{align*}
  u &\equiv \frac{\phi^2}{H} \Phi, \\
  z &\equiv \frac{H}{a\phi} \left(1 + \frac{\dot{\phi}}{H\phi}\right),
\end{align*}
\]

(21)

where \( \Phi \) is the potential \( \Phi \) after a conformal transformation has been performed, which puts the action of equation (20) into the form of Einstein gravity with a (newly-defined) minimally-coupled scalar field; this conformal transformation will be considered below, in section 3. From equation (21) we may define \( \hat{u}(x) \) exactly as in equations (4) and (5). The new mode functions \( u_k \) then obey the same equation of motion as in equation (6), with \( z \) now given by equation (21).

In order to solve for \( u_k \), we again define the two slow-roll parameters \( \epsilon \) and \( \delta \) as in equation (7), and define a third slow-roll parameter:

\[
\alpha \equiv \frac{\dot{\phi}}{H\Phi}.
\]

(22)

In terms of \( \epsilon, \delta, \) and \( \alpha \), the \( z''/z \) term in equation (3) may be written:

\[
\frac{z''}{z} = a^2 H^2 \left[(2\epsilon + \delta) (1 + \epsilon + \delta) - \frac{1}{H} (\dot{\epsilon} + \dot{\delta})\right] - a^2 H \frac{\dot{\alpha}}{(1 + \alpha)} \left[1 + 2\epsilon + 2\delta - \frac{\ddot{\alpha}}{H\dot{\alpha}}\right].
\]

(23)

As demonstrated in [5], \( |\alpha| \ll 1 \) during inflation. Also, as for \( \dot{\epsilon} \) and \( \dot{\delta} \), \( \dot{\alpha} \) is second-order in the slow-roll parameters:

\[
\frac{\dot{\alpha}}{H} = \alpha (\delta + \epsilon - \alpha),
\]

(24)

so that to first order we may assume \( \dot{\epsilon} = \dot{\delta} = \dot{\alpha} = 0 \). From equation (24), this gives \( \alpha = \epsilon + \delta \) to first order for IgI. Taking \( \dot{\epsilon} = 0 \) means that the conformal time \( \eta \) is again given by equation (8), so that equation (23) again reduces to equation (11), with

\[
\nu = \frac{1}{2} + \frac{\epsilon + \alpha}{1 - \epsilon}.
\]

(25)

Proceeding as above in equations (12) to (17) we again arrive at \( n_s = 2 - 2\nu \), with \( \nu \) now given by equation (25). Approximating \( \nu \simeq 1/2 + \epsilon + \alpha \) to first order gives

\[
n_s \simeq 1 - 2\epsilon - 2\alpha.
\]

(26)
for IgI. As for the case of a minimally-coupled scalar field with an Einstein-Hilbert gravitational action, the spectral index for IgI thus remains close to the $n_s = 1.00$ scale-invariant spectrum.

Before evaluating $n_s$ for IgI (in section 3), we will next derive spectral indices for two other common GET models of inflation.

The next GET model of inflation to be considered is that of a nonminimally-coupled scalar field (NMSF), the action for which may be written (see, e.g., [16])

$$S = \int d^4x \sqrt{-g} \left[ \left(\frac{1 + \kappa^2 \xi \phi^2}{2\kappa^2}\right) R - \frac{1}{2} \phi_{\mu \nu} \phi^{\mu \nu} - V(\phi) \right],$$

where $V(\phi)$ can take a simple polynomial form, such as $V = \lambda \phi^4$, or can be of the Ginzburg-Landau form, as in equation (20). Here $\kappa^2 \equiv 8\pi G = 8\pi M_{pl}^{-2}$, where $M_{pl} \simeq 1.22 \times 10^{19}$ GeV is the present value of the Planck mass. The sign of $\xi$ in equation (27) is chosen to match that in [5], and is the opposite of Hwang’s choice in [8]. Following Hwang, we define the two quantities $F$ and $E$ as (noting the new sign of $\xi$):

$$F = 1 + \xi \phi^2, \quad E = 1 + \xi \phi^2(1 + 6\xi),$$

in terms of which the appropriate $u$ and $z$ variables may be written [8]

$$u = \frac{1}{\phi} \sqrt{\frac{F}{E}} \tilde{\Phi}, \quad z = \frac{H}{a \phi} \sqrt{\frac{F}{E}} \left(1 + \frac{1}{2} \frac{\dot{E}}{HE}\right).$$

With these definitions for $u$ and $z$, we again construct $\hat{u}(x)$ as in equations (11) and (13), with the mode functions $u_k$ obeying equation (14). Alongside $\epsilon$ and $\delta$ we now add the two slow-roll parameters

$$\beta = \frac{\dot{\hat{E}}}{2 HF}, \quad \gamma = \frac{\dot{E}}{2 HE},$$

with which the $z''/z$ term in equation (11) may be written

$$\frac{z''}{z} = a^2 H^2 \left[(2\epsilon + \delta + \gamma - \beta) (1 + \epsilon + \delta + \gamma - \beta) - \frac{1}{H} (\dot{\epsilon} + \dot{\delta} + \dot{\gamma} - \dot{\beta})\right] - a^2 H \left[\frac{\dot{\beta}}{(1 + \beta)} \left[1 + 2\epsilon + 2\delta + 2\gamma - 2\beta - \frac{\dot{\beta}}{H \beta}\right] \right].$$

Once again setting $\dot{\epsilon} = \dot{\delta} = \dot{\beta} = \dot{\gamma} = 0$ to first order in these four slow-roll parameters, equation (33) reduces to equation (11), with

$$\nu = \frac{1}{2} + \frac{2\epsilon + \delta + \gamma - \beta}{1 - \epsilon},$$

(32)
giving, to first order,
\[ n_s \simeq 1 - 4\epsilon - 2\delta - 2\gamma + 2\beta \] (33)
for the spectral index from an inflationary model with a nonminimally-coupled scalar field.

Lastly, we consider a general scalar-tensor theory (GST), the action for which is a generalization of the original Brans-Dicke theory [15]:
\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega(\phi)}{\phi} \phi_{\mu}\phi^{\mu} - V(\phi) \right]. \] (34)
Again following Hwang [8], we define \( w \equiv \omega + 3/2 \), and write \( u \) and \( z \) as
\[ u \equiv \frac{1}{\phi} \sqrt{\frac{\phi^3}{w}} \Phi, \quad z \equiv \frac{H}{a\phi} \sqrt{\frac{\phi}{w}} \left( 1 + \frac{1}{2} \frac{\dot{\phi}}{H\phi} \right), \] (35)
so that the mode functions \( u_k \) of \( \hat{u}(x) \) obey equation (6), with \( z \) now given by equation (35). To \( \epsilon \) and \( \delta \), as defined in equation (7), and \( \alpha \), as defined in equation (22), we now add the slow-roll parameter \( \zeta \)
\[ \zeta \equiv \frac{\dot{w}}{Hw}, \] (36)
with which the \( z''/z \) term in equation (8) may be written
\[ \frac{z''}{z} = a^2H^2 \left[ \left( 2\epsilon + \delta + \frac{1}{2}(\zeta - \alpha) \right) \left( 1 + \epsilon + \delta + \frac{1}{2}(\zeta - \alpha) \right) - \frac{1}{H} \left( \dot{\epsilon} + \dot{\delta} + \frac{1}{2}(\dot{\zeta} - \dot{\alpha}) \right) \right] - a^2H \frac{\dot{\alpha}}{2(1 + \alpha/2)} \left[ 1 + 2\epsilon + 2\delta + \zeta - \alpha - \frac{\dot{\alpha}}{H\dot{\alpha}} \right]. \] (37)
Taking \( \dot{\epsilon} = \dot{\delta} = \dot{\alpha} = \dot{\zeta} = 0 \) to first order, equation (37) reduces to equation (10), with
\[ \nu = \frac{1}{2} + \frac{2\epsilon + \delta + (\zeta - \alpha)/2}{1 - \epsilon}. \] (38)
To first order in the four slow-roll parameters, this yields the spectral index for a general scalar-tensor theory:
\[ n_s \simeq 1 - 4\epsilon - 2\delta - \zeta + \alpha, \] (39)
again close to the \( n_s = 1.00 \) scale-invariant spectrum.

The results for the spectral index from Induced-gravity Inflation (equation 26), from a theory with a nonminimally-coupled scalar field (equation 33), and from a general scalar-tensor theory...
(equation 39) all apply to the Jordan frame for these GETs; that is, for when the models are
specified with their explicit nonminimal $\phi R$ coupling, as in equations (21), (27), and (34). In
the next section, we demonstrate for the case of Induced-gravity Inflation that this Jordan-frame
formalism for $n_s$ yields the same results for the spectral index as those obtained in the Einstein
frame, after use has been made of a conformal transformation.

3 Spectral Index from IgI in Jordan and Einstein frames

We now compare the calculation of $n_s$ in the Jordan and Einstein frames of IgI; it was in the context
of this model that the discrepancy was first discussed, in section 3.2 of [5]. The action in equation
(20) yields the coupled field equations in the Jordan frame (for a flat Friedmann universe):

$$H^2 = \frac{1}{3\xi \phi^2} V(\phi) + \frac{1}{6\xi} \left( \frac{\dot{\phi}}{\phi} \right)^2 - 2H \left( \frac{\dot{\phi}}{\phi} \right),$$

$$\ddot{\phi} + 3H \dot{\phi} + \frac{\phi^2}{\phi} = \frac{1}{(1 + 6\xi)} \phi \left[ 4V(\phi) - \phi V'(\phi) \right],$$

(40)

where overdots again denote time derivatives, and primes denote $d/d\phi$. As in [5], we have assumed
that the classical background field $\phi$ is sufficiently homogenous, so that all spatial derivatives
become negligible.

Invoking the “inflationary attractor” assumption [8] [17] yields the approximate Jordan-frame
field equations [3]:

$$H^2 \simeq \frac{1}{3\xi \phi^2} V(\phi),$$

$$3H \dot{\phi} \simeq \frac{1}{(1 + 6\xi)} \phi \left[ 4V(\phi) - \phi V'(\phi) \right].$$

(41)

Integration gives the closed-form solutions during the inflationary epoch:

$$\phi(t) = \phi_o \pm \sqrt{\frac{4\lambda \xi}{3(1 + 6\xi)^2}} v^2 t,$$

$$\frac{a(t)}{a_o} = \left( \frac{\phi(t)}{\phi_o} \right)^{(1+6\xi)/4\xi} \exp \left[ \frac{(1 + 6\xi)}{8\xi v^2} \left( \phi_o^2 - \phi^2(t) \right) \right],$$

(42)

where $\phi_o$ and $a_o$ are values at the beginning of inflation. In the solution of $\phi(t)$, the $+$ corresponds to
the “new inflation” initial conditions ($\phi_o \ll v$), and the $-$ corresponds to “chaotic inflation” initial
conditions \((\phi_o \gg v)\). For early times, then, under new inflation initial conditions the expansion is predominantly quasi-power-law \((a(t) \propto t^{(1+6\xi)/4\xi})\), whereas under chaotic inflation initial conditions the expansion is quasi-de Sitter \((a(t) \propto \exp(\phi_o \sqrt{\lambda/3\xi} \, t))\). As demonstrated in \(\mathcal{F}\), only the power-law expansion case is affected by the discrepancy in \(n_s\).

Working with the field equations in equation (41) (appropriate for a first-order analysis) and the inflationary solutions for \(\phi(t)\) and \(a(t)\) in equation (42), we may evaluate the Jordan-frame slow-roll parameters \((\epsilon, \delta, \text{and} \alpha)\) for the new inflation scenario as:

\[
\begin{align*}
\epsilon &= \frac{4\xi}{1 + 6\xi}, \quad \delta = 0, \\
\alpha &= \frac{2\xi}{1 + 6\xi} \left(1 + \frac{\epsilon}{\alpha}\right).
\end{align*}
\]

The equation for \(\alpha\) may be solved to give either \(\alpha = \epsilon\) or \(\alpha = -\epsilon/2\); yet for the new inflation initial conditions, \(\epsilon > 0\), and \(\dot{\phi} > 0\), so only the solution \(\alpha = \epsilon\) may be chosen. (Of course, we arrive at the same result by considering that for IgI under either initial conditions, \(\alpha = \epsilon + \delta\) to first order, and, from equation (42), \(\delta = 0\) for IgI to first order.) From equation (25), this yields

\[
\nu = \frac{1}{2} + \frac{2\epsilon}{1 - \epsilon} = \frac{1}{2} + \frac{8\xi}{1 + 2\xi},
\]

or

\[
n_s = 2 - 2\nu = 1 - \frac{16\xi}{1 + 2\xi}
\]

for the spectral index from IgI, as calculated in the Jordan frame. Note that by using the new slow-roll expansion, based on the variables \(u\) and \(z\) in equation (21), this result for \(n_s\) differs from that calculated in the Jordan frame using the Einstein-frame formalism of \(\mathcal{F}\) \(\mathcal{I}\). (Compare equation (43) with equation (47) in \(\mathcal{F}\) or equation (27) in \(\mathcal{I}\).)

To demonstrate the frame-independence of this result for \(n_s\), we may compare equation (45) with a calculation in the Einstein frame. The action in equation (20) may be written in the Einstein frame if we make the following conformal transformation (see, e.g., \(\mathcal{F}\) \(\mathcal{I}\)):

\[
\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \Omega^2(x) = \kappa^2 \xi \phi^2,
\]

\(9\)
where quantities in the Einstein frame are marked by a tilde. As above in section 2, \( \kappa^2 \equiv 8\pi M_{pl}^{-2} \).

From the form of the potential, \( V(\phi) \), in equation (20), we may set \( \kappa^2 = (\xi v^2)^{-1} \) for I\( g \)I. If we further define a new scalar field \( \varphi \) and its potential \( U \) by

\[
\frac{d\varphi}{d\phi} \equiv \sqrt{1 + \frac{6}{\kappa^2 \xi^2} \frac{1}{\phi^2}}; \quad U \equiv \frac{1}{(\kappa^2 \xi^2 \phi^2)^2} V(\phi),
\]

then the action in this frame becomes

\[
S = \int d^4 \tilde{x} \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2 \varphi, \mu \varphi^\mu} - U(\varphi) \right],
\]

(48)

giving the familiar field equations:

\[
\dot{\tilde{H}}^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 + U(\varphi) \right],
\]

\[
\frac{d^2 \varphi}{dt^2} + 3\tilde{H} \frac{d\varphi}{dt} + \frac{dU}{d\varphi} = 0,
\]

(49)

where \( d\tilde{t} = \Omega(x)dt \), \( d\tilde{x} = d\vec{x} \), \( \tilde{a}(\tilde{t}) = \Omega(x)a(t) \), and \( \tilde{H} = \tilde{a}^{-1} \frac{d\tilde{a}}{d\tilde{t}} \).

Under new inflation conditions, \( \Omega \propto \phi \propto t \); given \( a(t) \propto t^p \), with \( p = (1+6\xi)/4\xi \), the transformed scale factor thus becomes \( \tilde{a}(\tilde{t}) \propto \tilde{t}^{\tilde{p}} \), with \( \tilde{p} = (p+1)/2 = (1+10\xi)/8\xi \). From equations (50) and (49), the slow-roll parameters \( \tilde{\epsilon} \) and \( \tilde{\delta} \) may be evaluated as:

\[
\tilde{\epsilon} = -\tilde{\delta} = \frac{1}{\tilde{p}} = \frac{8\xi}{1 + 10\xi}.
\]

(50)

Having made the conformal transformation of equation (46), our theory is now in the form of an Einstein-Hilbert gravitational action with a minimally-coupled scalar field; the appropriate \( \nu \) from section 2 is therefore that given in equation (11), which becomes

\[
\tilde{\nu} = \frac{1}{2} + \frac{\tilde{\epsilon}}{1 - \tilde{\epsilon}} = \frac{1}{2} + \frac{8\xi}{1 + 2\xi},
\]

(51)

giving, from equation (13),

\[
\tilde{n}_s = 1 - \frac{16\xi}{1 + 2\xi}.
\]

(52)

Comparing equations (45) and (52) it is clear that \( n_s = \tilde{n}_s \); the spectral index as calculated in the Jordan frame matches the spectral index as calculated in the Einstein frame.
4 Conclusion

By developing an expansion in slow-roll parameters appropriate to the complicated equation of motion for the gauge-invariant potential $\Phi$, we have extended the usual Einstein-frame formalism for calculating the spectral index into a form which may be used for GET models of inflation. The analysis has been conducted to first order only in the slow-roll parameters. It could be continued to second order by following Stewart and Lyth’s original derivation [1], that is, by treating the spectrum $P_\Phi$ as adiabatic in the slowly-varying $\epsilon, \delta, \alpha, \beta, \gamma, \text{and} \zeta$. Yet, because of the more complicated field equations for $\phi(t)$ and $a(t)$ in the Jordan frame, as compared with the corresponding field equations in the Einstein frame (compare, e.g., equation (40) with (49)), an expansion to second order in the Jordan-frame parameters would be exceedingly difficult to evaluate. Instead, one should exploit the frame-independent nature of this formalism, and evaluate the spectral index for such GETs in the Einstein frame, as done in [5].

The import of this work has been to remove the ambiguity, as discussed in [5], which formerly plagued the evaluation of spectral indices for GET models of inflation. By seizing upon a generalization of the usual Einstein-frame slow-roll expansion for $n_s$, which reduces to this expansion following a conformal transformation, we have developed a self-consistent formalism with which to calculate $n_s$ for GETs.

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