Cluster nonequilibrium relaxation in Ising models observed with the Binder ratio

Yoshihiko Nonomura
International Center for Materials Nanoarchitectonics, National Institute for Materials Science, Tsukuba, Ibaraki 305-0044, Japan

Yusuke Tomita
College of Engineering, Shibaura Institute of Technology, Saitama 337-8570, Japan

The Binder ratios exhibit discrepancy from the Gaussian behavior of the magnetic cumulants, and their size independence at the critical point has been widely utilized in numerical studies of critical phenomena. In the present article we reformulate the nonequilibrium relaxation (NER) analysis in cluster algorithms using the (2,1)-Bindera ratio, and apply this scheme to the two- and three-dimensional Ising models. Although the stretched-exponential relaxation behavior at the critical point is not explicitly observed in this quantity, we find that there exists a logarithmic finite-size scaling formula which can be related with a similar formula recently derived in cluster NER of the correlation length, and that the formula enables precise evaluation of the critical point and the stretched-exponential relaxation exponent $\sigma$. Physical background of this novel behavior is explained by the simulation-time dependence of the distribution function of magnetization in two dimensions and temperature dependence of $\sigma$ obtained from magnetization in three dimensions.

PACS numbers: 05.10.Ln,64.60.Ht,75.40.Gb

I. INTRODUCTION

Finite-size corrections in physical quantities have been serious obstacles in numerical studies of critical phenomena. They result in poor scaling behaviors, and consequently in poor estimates of the critical point and critical exponents. Although a straightforward solution is to take correction terms into account, increase of fitting parameters also results in poor numerical estimation. Then, the Binder ratios defined as the homogeneous ratios of (magnetic) cumulants have been widely utilized in numerical studies of critical phenomena, because finite-size corrections in homogeneous ratios of cumulants of the same physical quantities are expected to be cancelled. Even though such cancellation may not be perfect, correction terms would start from much higher orders.

Recently we investigated early-time nonequilibrium relaxation (NER) in cluster algorithms numerically, and found that physical quantities show the stretched-exponential relaxation at the critical point, not the power-law one commonly observed in local-update algorithms. Although the fusion of cluster algorithms and NER resulted in much easier treatment of larger systems than conventional equilibrium simulations, resulting estimates of the critical point and critical exponents were comparable to previous studies in one or two decades ago with much smaller systems. The main reason is that we only know the empirical scaling form up to the leading order and finite-size corrections affect more seriously than in equilibrium simulations.

Then, we expect that the reformulation of the cluster NER using the Binder ratios would improve such a situation, while absence of the explicit size dependence in the Binder ratios in equilibrium may result in the absence of the stretched-exponential relaxation behavior. In the present article we analyze the (2,1)-Binder ratio defined by $B_{2,1} \equiv \langle m^2 \rangle / \langle m \rangle^2$, which behaves similarly to the commonly-used (4,2)-Binder ratio. Since step-by-step oscillation between positive and negative values of magnetization is observed in cluster algorithms, introduction of $B_{2,1}$ including $\langle |m| \rangle$ is natural in the present case. Quite recently, the present authors derived a phenomenological finite-size scaling formula of the size-independent quantity $\xi/L$ with the correlation length $\xi$, and a similar formula is also expected in $B_{2,1}$. Furthermore, we also observe simulation-time dependence of the distribution function of $|m|$ in order to clarify the origin of the stretched-exponential relaxation.

The outline of the present article is as follows. In section II, basic procedures of numerical calculations are summarized, which includes the finite-size scaling formula of $\xi/L$ mentioned above. In section III, detailed procedures of numerical calculations are exhibited through the analysis of the simulation-time dependence of the (2,1)-Binder ratio in the two-dimensional Ising model on a square lattice. We show that the above-mentioned finite-size scaling formula of $\xi/L$ also holds in $B_{2,1}$. In section IV, simulation-time dependence of the distribution function of magnetization is observed, and the origin of the stretched-exponential behavior of the critical cluster NER is clarified. In section V, similar analysis in the three-dimensional Ising model on a simple cubic lattice is described. In section VI, the relaxation exponent is directly evaluated from the early-time relaxation of magnetization, and compared with the results in sections III and V. The above descriptions are summarized in section VII.
II. FORMULATION

In the present article we investigate the two- and three-dimensional Ising models with the nearest-neighbor interaction on a square and simple cubic lattices, respectively,

\[ H = -J \sum_{\langle ij \rangle \in \text{n.n.}} S_i S_j, \quad S_i = \pm 1, \]  

with the Swendsen-Wang (SW) algorithm. We have already found that the stretched-exponential behavior is observed at the critical temperature \( T_c \) both in the decaying process from the perfectly-ordered state and the ordering process from the perfectly-disordered state, and that initial-time deviation in the decaying process is much larger than that in the ordering process. In the present article, we therefore concentrate on the ordering process from the perfectly-disordered state.

Early-time behavior of the absolute value of the magnetization at \( T_c \) is given by

\[ \langle |m(t,L)| \rangle \sim L^{-d/2} \exp(+c_m t^\sigma) \quad (0 < \sigma < 1), \]  

with the spatial dimension \( d \), a relaxation constant \( c_m \) possibly depending on observed quantities and the exponent \( \sigma \) independent of quantities. Here the explicit size dependence originates from the normalized random-walk growth of clusters. Similar behavior is also observed in the squared magnetization,

\[ \langle m^2(t,L) \rangle \sim L^{-d} \exp(+c_m^2 t^{2\sigma}), \]  

and the (2,1)-Binder ratio is therefore scaled as

\[ B_{2,1}(t,L) \equiv \frac{\langle m^2(t,L) \rangle / \langle |m(t,L)| \rangle^2}{\exp[ + (c_m^2 - c_m^2) t^{2\sigma} ]}. \]

Similarly, early-time behavior of the correlation length at \( T_c \) is expressed as

\[ \xi(t,L) \sim \exp(+c_\xi t^\sigma), \]  

while this quantity is scaled with \( L^{\beta/\nu} = L \) in equilibrium. Then, taking \( \rho \equiv 1/c_\xi \) and the size-independent quantity \( \xi(t,L)/L \) is scaled as

\[ \xi(t,L)/L \sim \exp(+\rho^{-1} t^\sigma - \ln L) \]

\[ \sim \exp[ +\rho^{-1} (t^\sigma - \ln L^\rho) ]. \]

This relation indicates that \( \xi(t,L)/L \) is scaled well for various system sizes with the scaling quantity \( t^\sigma - \ln L^\rho \) at least for the both ends of this quantity, and actually in the whole parameter region as shown numerically. Although this functional form of the scaling quantity resembles that of the nonequilibrium-to-equilibrium scaling observed in size-dependent quantities, origin of the exponent \( \rho \) is not the same, and such a scaling in \( \xi(t,L)/L \) may also hold in \( B_{2,1}(t,L) \). Note that \( \rho \) may depend on quantities because the coefficient \( c \) in Eq. (5) may depend on them, while the relaxation exponent \( \sigma \) is more fundamental and independent of quantities.

FIG. 1. (Color online) (a) Simulation-time dependence of the (2,1)-Binder ratio at the exact \( T_c \) for \( L = 1000 \) (crosses), 2000 (stars), 4000 (triangles) and 8000 (squares). No size dependence is observed at both ends of time evolution. (b) Semiempirical scaling plot of this quantity with \( \sigma \) = 0.3277(1) and \( \rho \) = 0.3168(3) based on the data inside of the dashed lines.

III. NUMERICAL RESULTS IN THE TWO-DIMENSIONAL ISING MODEL

In the present section, we explain the procedure to evaluate critical phenomena (especially the relaxation exponent \( \sigma \)) on the basis of the Binder ratio in the two-dimensional Ising model on a square lattice.

In our previous studies, we evaluated the critical temperature \( T_c \) and critical exponents with the nonequilibrium-to-equilibrium scaling, where the initial-time stretched-exponential scaling form (such as Eq. (2)) and the finite-size scaling form in equilibrium (such as \( m_c(L) \sim L^{-\beta/\nu} \)) are coupled. Since such a scheme is essentially a four-parameter fitting \((T_c, \beta/\nu, \sigma, c_m)\), precision of the parameters was rather limited.

Here we propose a new scheme to evaluate such parameters more precisely. Since this scheme is based on a semiempirical scaling form following Eq. (4), we use the exact value, \( T_c = 2/\log(1 + \sqrt{2})|J/k_B| \approx 2.2691853 \ldots |J/k_B| \),
in order to avoid extra efforts for justification of the scaling form. Evaluation process of $T_c$ will be given in Section V for the three-dimensional case.

First, the $(2, 1)$-Binder ratio at $T_c$ is plotted versus simulation time in Fig. 1(a) for $L = 1000$ (6.4 x $10^5$ random number sequences (RNS) are averaged), 2000 (3.2 x $10^5$ RNS), 4000 (1.6 x $10^5$ RNS) and 8000 (0.8 x $10^5$ RNS). As expected, this quantity becomes size independent as the system approaches equilibrium. Moreover, it also seems size independent at the onset of simulations (origin of this behavior will be explained in the next section), and even its initial slope of time evolution looks vanishing, in spite of the expected stretched-exponential time dependence \( \text{4} \). Actually, this cancellation, \( c_{m^2} - \beta^2_m = 0 \), is specific to the Ising models. When classical vector spin models are simulated \( \text{3, 4} \) with the embedded-Ising-spin algorithm \( \text{7} \), such cancellation is not observed \( \text{4} \).

Although such size independence is suitable for evaluation of $T_c$, the exponent $\sigma$ does not appear explicitly in the expression of $B_{2,1}(t, L)$ anymore. Nevertheless, this quantity consists of \( \langle |m(t, L)| \rangle \) and \( \langle m^2(t, L) \rangle \), and they show the stretched-exponential relaxation as given in Eqs. \( \text{2} \) and \( \text{3} \). Then, the information of $\sigma$ would remain in $B_{2,1}(t, L)$ as a higher-order correction. In Fig. 1(b), this quantity is plotted versus a rescaled time $t^\sigma - \ln L^\rho$ with $\sigma = 0.3277(1)$ and $\rho = 0.3168(3)$. These exponents are evaluated so as to minimize the mutual residue of the data. The ranges of the data used for the fitting (represented by dashed lines in Fig. 1(b)) are also determined by minimizing the residue. Since the residue decreases monotonically as the upper range is increased, it is fixed so as to include all the data for $L = 8000$. Note that the estimate of $\sigma$ is not inconsistent with the expected value, $\sigma = 1/3$ \( \text{2} \).

IV. DISTRIBUTION FUNCTION OF MAGNETIZATION IN THE TWO-DIMENSIONAL ISING MODEL

In the present section, we analyze the distribution function of magnetization $P(|m|)$ in the two-dimensional Ising model at $T_c$ and clarify physical background of the stretched-exponential critical relaxation and simulation-time dependence of the $(2, 1)$-Binder ratio.

In Fig. 2, the distribution function of magnetization $P(|m|)$ is plotted versus the absolute value of magnetization $|m|$ at various Monte Carlo steps (MCS) for $L = 1000$. Data in the earlier stage (at 1, 2, 4, 8 and 16MCS) and in the later stage (at 16, 20, 24, 28, 50 and 100MCS) are shown in Figs. 2(a) and 2(b), respectively. This function is obtained from 6.4 x $10^5$ samples with different RNS and divided into 10,000 meshes for $0 \leq |m| \leq 1$. It is normalized as $\int_0^1 P(|m|) d|m| = 1$, and the data points in these figures are truncated for clear visualization. In the earlier stage, $P(|m|)$ has a dome-like shape with the peak at $|m| = 0$, and in the later stage it gradually approaches the equilibrium distribution with the peak around the critical magnetization $m_c(L)$.

Then, $P(|m|)$ is scaled with system sizes. In Fig. 3(a), it is scaled in equilibrium (actually at $t_{eq} = 200$ MCS) for $L = 1000$, 2000, 4000 and 8000. Here $P(|m|)L^{-\beta/\nu}$ is scaled with $|m|L^{\beta/\nu}$ \( \text{10} \) with the exact critical exponent $\beta/\nu = 1/8$, and this behavior is consistent with the fact that the peak of $P(|m|)$ in equilibrium corresponds to the critical magnetization, $m_c(L) \sim L^{-\beta/\nu}$. This behavior also holds during the relaxation process. In Fig. 3(b), $P(|m|)$ with a plateau-like distribution is scaled with the same formula for $L = 1000$ (at 23MCS), 2000 (at 29MCS), 4000 (at 36MCS) and 8000 (at 44MCS). Although similar scaling behavior is also expected to be observed at other moments, it is generally difficult to take corresponding configurations for different system sizes.

Next, we consider the scaling behavior at the onset of relaxation. Although similar scaling analyses become further difficult, a square-$|m|$ plot of $P(|m|; t)$ (Fig. 4(a) for
$L = 1000$) results in the Gaussian distribution function,
\[ P(|m|; t, L) \sim \exp \left[ -\left( \frac{|m|}{m_G(t, L)} \right)^2 \right]. \quad (7) \]
That is, the early-time relaxation behavior is described by the Gaussian distribution, and the size-independent initial value of $B_{2,1}(t = 0)$ is derived from the Gaussian integrals $\langle \cdots \rangle_G$,
\[ B_{2,1}(t = 0) \equiv \langle m^2(t = 0) \rangle_G = \frac{m_G^2}{(m_G/\sqrt{\pi})^2} = \frac{\pi}{2}, \quad (8) \]
and numerical data of $P(|m|; t, L)$ for various system sizes and Monte Carlo steps are fitted with Eq. (7) to evaluate $m_G(t, L)$ as shown in Fig. 3(b). Here $m_G(t, L) L$ is plotted in a semi-log scale versus $t^\sigma$ with $\sigma = 1/3$, and such linear behavior suggests the following scaling form,
\[ m_G(t, L) \sim L^{-1} \exp \left( C t^\sigma \right), \quad \sigma = 1/3. \quad (9) \]

We can obtain various insights from the above results. First, we find that the stretched-exponential relaxation of magnetization $m_G(t, L)$ originates from the simulation-time dependence of the width of the Gaussian distribution of magnetization, which suggests that the scaling behavior is fundamental in the SW algorithm. Second, the data in Fig. 4(b) reveal that the width of the Gaussian distribution function shrinks with $t$ in a semi-log scale, which is much faster than the scaling of the peak value of $P(|m|; t, L)$ in equilibrium (Fig. 3(a)) or the width of it at the moment with a plateau-like distribution (Fig. 3(b)) with $\sim L^{-\beta/\nu}$.

The power $-1$ in Eq. (9) would be identified with $-d/2$, which characterizes the random-walk growth of magnetization as seen in Eq. (2). Third, the data for each system size in Fig. 4(b) are plotted up to the limit where the distribution function is described well with the Gaussian formula, and this limit coincides with that of early-time independence of $B_{2,1}(t, L)$ displayed in Fig. 4(a).
The regions between II and IV may correspond to the almost takes a constant value again, $\sigma_4$. For temperatures in all the sizes and temperatures, the relaxation exponent $\sigma$ shows that the critical temperature $T_c$ increases as $\sigma \approx \rho \approx 1/6$ (e.g. at $T = 4.511519J/k_B$). A similar relation also exists in two dimensions, where such convergence of the two exponents occurs at $\sigma \approx \rho \approx 1/3$.

V. NUMERICAL RESULTS IN THE THREE-DIMENSIONAL ISING MODEL

In the present section, we evaluate critical phenomena of the three-dimensional Ising model on a simple cubic lattice by the scheme explained in Section III. Here we vary the temperature around the critical region, and show that the critical temperature $T_c$ can be evaluated very accurately from the temperature dependence of the relaxation exponent $\sigma(T)$, which is of interest in itself.

Temperature dependence of the relaxation exponents $\sigma(T)$ and $\rho(T)$ evaluated from the scaling plot similarly to the one in Fig. 4(b) is displayed in Fig. 5. Some examples of the fitting of $B_{2,1}(t, L)$ versus $t^\rho - \ln L^\sigma$ for $L = 140, 200, 280$ and $400$ $(4.0 \times 10^3$ RNS are averaged in all the sizes and temperatures) are given in Figs. 6(a)-(d) at $T = 4.511518J/k_B$, $4.511520J/k_B$, $4.511522J/k_B$ and $4.511524J/k_B$, respectively. The range of the temperature shown in Fig. 5 is consistent with an estimate of the critical temperature based on the conventional NER analysis using the $6648 \times 6648 \times 6656$ cluster, $J/k_BT_c = 0.221654(5)$ or $T_c = 4.511522(10)/J/k_B$.

The temperature region shown in Fig. 5 is divided into five subregions. For $T < 4.511512J/k_B$ (region I), the relaxation exponent $\sigma(T)$ increases as $T$ decreases. For $4.511512J/k_B \leq T \leq 4.511519J/k_B$ (region II), the exponent $\sigma(T)$ almost takes a constant value $\sigma = 0.178(2)$. For $4.511519J/k_B < T < 4.511523J/k_B$ (region III), $\sigma(T)$ exhibits a sharp temperature dependence. For $4.511523J/k_B \leq T \leq 4.511531J/k_B$ (region IV), $\sigma(T)$ almost takes a constant value again, $\sigma = 0.107(4)$. For $T > 4.511531J/k_B$ (region V), $\sigma(T)$ is much smaller.

Physical interpretation of these results is as follows: The regions between II and IV may correspond to the critical one, and the critical temperature can be estimated as $T_c = 4.511521(2)J/k_B$. Rapidly-changing value of $\sigma(T)$ indicates large fluctuations around $T_c$, and the above estimate of $T_c$ is identified with the region III. The estimate $\sigma = 0.178(2)$ in the region II (just below $T_c$) would stand for the true relaxation exponent. On the other hand, the estimate $\sigma = 0.107(4)$ in the region IV (just above $T_c$) might be a fictitious one in finite systems, which signals the stretched-exponential relaxation still holds in this region. Large discrepancies of $\sigma(T)$ in the regions I and V simply tell that these regions are off-critical and the stretched-exponential relaxation does not hold anymore. In the region I, the system is in the ordered phase and the magnetization grows exponentially, and $\sigma(T)$ is expected to approach unity as $T$ decreases. In the region V, the system is in the paramagnetic phase and no magnetic long-range order is stable.

Note that the exponents $\sigma(T)$ and $\rho(T)$ are not so different around $T_c$, and they almost coincide with each other in the region II. Actually, their relation in sizes is reversed in the regions II and IV, and they seem to converge at $\sigma \approx \rho \approx 1/6$ (e.g. at $T = 4.511519J/k_B$). A similar relation also exists in two dimensions, where such convergence of the two exponents occurs at $\sigma \approx \rho \approx 1/3$.

VI. COMPARISON OF THE RELAXATION EXPONENT $\sigma$ DIRECTLY OBTAINED FROM STRETCHED-EXPONENTIAL RELAXATION

In the previous section, a large temperature dependence of the “relaxation exponent” $\sigma(T)$ is observed in the region III in Fig. 5 in the three-dimensional Ising model. Then, it is interesting to investigate whether this large fluctuation is actually observed in physical quantities or a fictitious behavior specific to the analysis based on the Binder ratio. For this purpose, direct observation of the exponent $\sigma$ defined in Eq. (2) is straightforward.

A. Two-dimensional Ising model at $T_c$

Consequence of the above observation in the two-dimensional Ising model at $T_c$ for $L = 8000$ is displayed in Fig. 7. Here relaxation data of $\langle |m(t)| \rangle$ from 1 to $t$ MCS are fitted with Eq. (2), and the estimate of $\sigma(t)$ is plotted versus $t$ used for the fitting. For larger $t$, the stretched-exponential relaxation has already been saturated (it is signaled by the drop of $B_{2,1}(t, L)$ from the Gaussian value $\pi/2$ in Fig. 1(a)), and the fitting based on Eq. (2) becomes poor (indicated by large error bars) and the estimate of $\sigma(t)$ deviates rapidly as $t$ increases.

For $L = 8000$, $\sigma(t)$ monotonically decreases as $t$ increases up to $t = 26$, which is the upper limit of the Gaussian behavior of the distribution function of $|m|$ as shown in Fig. 3(b). The residue for fitting of the data between 1 to 26 MCS takes minimum and we have $\sigma = 0.330(5)$, which is both consistent with the estimate from $B_{2,1}(t, L)$, $\sigma = 0.3277(1)$ and the expected value $\sigma = 1/3$. For $27 \lesssim t \lesssim 40$, $\sigma(t)$ still weaves around
FIG. 6. (Color online) Semi-empirical scaling plot of the $(2,1)$-Binder ratio based on the data inside of the dashed lines for $L = 140$ (stars), 200 (triangles), 280 (squares) and 400 (circles) at (a) $T = 4.511518 \, \text{J/k}_B$, (b) $4.511520 \, \text{J/k}_B$, (c) $4.511522 \, \text{J/k}_B$ and (d) $4.511524 \, \text{J/k}_B$. The estimates of $\sigma(T)$ and $\rho(T)$ at each temperature are exhibited in each figure.

σ = 1/3, which means that the simulation-time dependence (2) at $T_c$ with $\sigma \approx 1/3$ is optimal and that small discrepancy from it can be absorbed into that formula.

B. Three-dimensional Ising model around $T_c$

Then, similar analysis is also possible in the three-dimensional Ising model, even though maximum linear size is smaller and statistical and systematic errors are larger than that in the two-dimensional one. Moreover, in order to understand nontrivial temperature dependence of $\sigma(T)$ displayed in Fig. 6, we should consider both $\langle |m| \rangle$ and $\langle m^2 \rangle$ which consist of $2^B 1_{B,1} \equiv \langle m^2 \rangle / \langle |m| \rangle^2$.

Although the fitting of $\langle |m| \rangle$ in three dimensions is still possible, that of $\langle m^2 \rangle$ is difficult. Since this quantity is nothing but the magnetic susceptibility, its fluctuating behavior around $T_c$ is beyond the accuracy of the present data. Then, instead of the fitting based on Eqs. (2) and (3), we take logarithm of both sides of them,

$$\log \langle |m(t, L)| \rangle = c_m t^\sigma + A_m,$$

$$\log \langle m^2(t, L) \rangle = c_{m^2} t^\sigma + A_{m^2}. \quad (11)$$
After these trivial transformations, all the data points turn to contribute to the fitting equally, and the process of fitting becomes more stable. Then, the fitting of $\log(\langle m^2 \rangle)$ becomes possible. On the other hand, the data in the initial several MCS including larger statistical errors tend to contribute too much in this logarithmic fitting, and such data should be eliminated. Here we take a systematic approach to delete the early-time data one by one until the residue of the fitting is minimized. In addition, we take a larger system size $L = 560$ ($2.0 \times 10^4$ RNS are averaged) for this analysis, because data points included in the Gaussian region at $T_c$ increases as the system size increases, as shown in Figs. 5(a) and 5(b).

In Figs. 8(a)-(f), relaxation exponent $\sigma(t)$ estimated with the above procedure (each data point is obtained from the data from $t_0(t)$ to $t$-MCS, where $t_0(t)$ gives the minimum residue) is plotted versus $t$ based on Eqs. (10) (circles) and (11) (squares) at (a) $T = 4.511514J/k_B$, (b) $4.511518J/k_B$, (c) $4.511520J/k_B$, (d) $4.511522J/k_B$, (e) $4.511524J/k_B$, and (f) $4.511528J/k_B$ with the dashed line corresponding to $\sigma = 1/6$. Each pair covers the regions II, III and IV in Fig. 5 respectively. Apparently, behavior of $\sigma$ obtained from $\langle |m| \rangle$ in Figs. 8(a)-(d) is similar to that in Fig. 7 which indicates that the regions II and III are included in the critical one. The exponent obtained from $\langle m^2 \rangle$ behaves similarly to that from $\langle |m| \rangle$ in the region III, while the former exceeds the latter in the region II. Although these behaviors may not seem consistent with that in Fig. 5 it actually is. When almost similar functions are divided with each other, the quotient may have little parameter dependence and therefore it may be too sensitive for a small change of conditions. Fictitious rapid change of $\sigma(T)$ in the region III in Fig. 5 can be explained with this mechanism. On the other hand, discrepancy of $\sigma(t)$ in the region II rather stabilizes the fitting of $B_{2,1}(t, L)$ in this region and results in small change of $\sigma(T)$ in Fig. 5. The estimate $\sigma = 0.178(2)$ based on $B_{2,1}(t, L)$ in this region is slightly larger than $\sigma \approx 1/6$, which is consistent with a larger estimate of $\sigma(t)$ from $\langle m^2 \rangle$ than that from $\langle |m| \rangle$ there. In the region IV, $\sigma(t)$ from $\langle |m| \rangle$ monotonically increases as $t$ decreases, which means that the critical relaxation characterized by a specific value of $\sigma$ is not observed in this region, even though the stretched-exponential relaxation formula still looks plausible there. The exponent $\sigma(t)$ from $\langle m^2 \rangle$ also behaves similarly but smaller than that from $\langle |m| \rangle$ for any $t$, which results in $\sigma = 0.107(4)$ based on $B_{2,1}(t, L)$, which is fairly smaller than $\sigma \approx 1/6$.

VII. SUMMARY AND DISCUSSION

In the present article we analyze the early-time critical relaxation of the $(2, 1)$-Binder ratio $B_{2,1}$ in the two- and three-dimensional Ising models simulated with the Swendsen-Wang algorithm. In addition to the well-known size independence in equilibrium, this quantity also shows size independence at the onset of relaxation when simulations are started from the perfectly-disordered state in the Ising models. Recently a size-
independent quantity $\xi(t,L)/L$ was shown to be scaled by $t^\sigma - \ln L^\rho$ with the stretched-exponential critical relaxation exponent $\sigma$ and the supplemental exponent $\rho$ related with the coefficient of power in the stretched-exponential function. When $B_{2.1}(t,L)$ is assumed to be scaled with the same scaling quantity $t^\sigma - \ln L^\rho$ at the exact critical temperature in the two-dimensional Ising model, we have $\sigma = 0.3277(1)$ and $\rho = 0.3168(3)$, which is consistent with the expected value $\sigma = 1/3$ and satisfies $\sigma \approx \rho$. In the similar analysis in the three-dimensional Ising model, the critical temperature is identified with the region where $\sigma(T)$ seems to change rapidly as $T_c = 4.511521(2)J/k_B$, which is consistent with previous numerical studies. The relaxation exponent almost takes a constant value $\sigma = 0.178(2)$ just below $T_c$, which is slightly larger than $\sigma \approx \rho \approx 1/6$ directly evaluated from the critical relaxation formula of magnetization.

In two dimensions, we also analyze the distribution function of magnetization $P(|m|)$ at the exact critical temperature. In equilibrium and at the simulation time with a flat distribution, $P(|m|, L)L^{-\beta/\nu}$ is scaled with $|m|L^\beta/\nu$, which is consistent with the finite-size scaling of the critical magnetization, $m_c(L) \sim L^{-\beta/\nu}$. In the early-time relaxation, the distribution function has the Gaussian form, $P(|m|; t, L) \sim \exp\left[-|m|^2 / m_c^2(t, L)\right]$ with $m_c(t, L) \sim L^{-d/2} \exp(Ct^\sigma)$. This behavior is consistent with the Gaussian onset value $B_{2.1}(t = 0) = \pi/2$, which is almost unchanged until the Gaussian distribution breaks down. The exponent $\sigma$ is directly related with simulation-time dependence of the width of the Gaussian distribution, and its size dependence is proportional to $L^{-d/2}$, which shrinks much faster than that of the equilibrium distribution.

In addition, the exponent $\sigma$ is also evaluated directly from the stretched-exponential relaxation of magnetization at the critical temperature. In two dimensions at the exact $T_c$, the residue of the fitting based on the stretched-exponential relaxation formula takes minimum when all the data within the Gaussian region is used, and we have $\sigma = 0.330(5)$. In three dimensions around the estimated $T_c$ from fluctuating behavior of $B_{2.1}$, we evaluate $\sigma$ both from $\langle|m|\rangle$ and $\langle m^2 \rangle$ in order to compare with the estimate obtained from $B_{2.1} \equiv \langle m^2 \rangle / \langle|m|\rangle^2$. In the temperature region identified with $T_c$, $\sigma$ obtained from the both magnetizations are consistent with $\sigma \approx 1/6$, which clarifies that the fluctuating behavior of $\sigma(T)$ in this region is a fictitious one owing to too little parameter dependence of $B_{2.1}$. In the region just below $T_c$, $\langle|m|\rangle$ similarly gives $\sigma \approx 1/6$ while $\langle m^2 \rangle$ larger $\sigma(T)$, which results in a slightly larger but stable estimate of $\sigma$. In the region just above $T_c$, convergent stretched-exponentially relaxation in magnetizations cannot be observed anymore, which results in a large discrepancy of $\sigma(T)$ from $\sigma \approx 1/6$.

ACKNOWLEDGMENTS

The random-number generator MT19937\textsuperscript{12} was used for numerical calculations. Most calculations were performed on the Numerical Materials Simulator at National Institute for Materials Science. This study was supported by JSPS KAKENHI Grant Number JP16K05493.

[1] K. Binder, Z. Phys. B 43, 119 (1981).
[2] Y. Nonomura, J. Phys. Soc. Jpn. 83, 113001 (2014).
[3] Y. Nonomura and Y. Tomita, Phys. Rev. E 92, 062121 (2015).
[4] Y. Nonomura and Y. Tomita, Phys. Rev. E 93, 012101 (2016).
[5] As a recent review, Y. Ozeki and N. Ito, J. Phys. A 40, R149 (2007).
[6] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 58, 86 (1987).
[7] U. Wolff, Phys. Rev. Lett. 62, 361 (1989), Nucl. Phys. B 322, 759 (1989).
[8] Y. Tomita and Y. Nonomura, in preparation.
[9] Y. Nonomura and Y. Tomita, in preparation.
[10] Y. Tomita, Y. Okabe, and C.-K. Hu, Phys. Rev. E 60, 2716 (1999).
[11] N. Ito, Pramana J. Phys. 64, 871 (2005).
[12] M. Matsumoto and T. Nishimura, ACM TOMACS 8, 3 (1998). Further information is available from the Mersenne Twister Home Page, currently maintained by M. Matsumoto.