Self-dual vortex-like configurations in SU(2) Yang-Mills Theory

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ABSTRACT

We show that there are solutions of the SU(2) Yang-Mills classical equations of motion in $\mathbb{R}^4$, which are self-dual and vortex-like (fluxons). The action density is concentrated along a thick two-dimensional wall (the world sheet of a straight infinite vortex line). The configurations are constructed from self-dual $R^2 \times T^2$ configurations.

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1 Introduction

The purpose of this paper is to show that there exist classical self-dual SU(2) Yang-Mills configurations in $R^4$ which are localized around a given 2-dimensional sheet. This sheet is the world sheet of a spatial flux-line (vortex or fluxon). Being self-dual the configuration is classically stable. The construction of the configurations proceeds by naturally embedding (by considering an infinite number of periods) a field configuration defined in $R^2 \times T^2$ ($T^n$ is the n-dimensional torus) into one in $R^4$.

Some years ago [1], our group studied the existence and properties of self-dual Yang-Mills configurations on the torus by numerical methods. Motivated by the Hamiltonian formulation, we considered the limit in which one of the sizes (time) of the torus goes to infinity, thus producing a well-defined $R \times T^3$ gauge field configuration which is localized in time. The configuration that we constructed was self-dual and carried topological charge $\frac{1}{2}$. Later on, configurations living on $R \times T^3$ with unit topological charge were found and studied [2].

There are two recent sets of works by other authors which have convinced us of the interest of looking into configurations which are living in $R^2 \times T^2$ and localized in the 2 large directions. The first is a series of papers [3] in which the authors were able to give an explicit analytic expression for a whole set of new caloron solution with non-trivial holonomy. These are self-dual Yang-Mills configurations in $R^3 \times T^1$ which are localized in the 3 non-periodic directions. Actually, the action density peaks at two points in $R^3$. If we naturally embed the configuration in $R^4$ the action density is concentrated around 2 lines, which are the world-lines of the so-called constituent monopoles. Thus, the case studied here ($R^2 \times T^2$) is missing to complete the picture. Of course, our numerical methods are a poor substitute of the analytic formulas. However, they can provide hints which could eventually lead to an analytic expression for these gauge field configurations. Furthermore, in the absence of analytic expressions our numerical method allows the determination, with high precision, of the different physical quantities and properties of these configurations.

A second group of papers [4] which motivated this work, consists on recent lattice gauge theory studies that give evidence in favour of the relevance of vortex-like configurations to explain the Confinement property of Yang-Mills theory. The presence of vortices in typical (generated by Monte Carlo simulations) Yang-Mills vacuum configurations is shown by performing the
maximal center projection. The resulting $Z_2$ degrees of freedom can account for the observed string tension, a behaviour described as *center dominance*. The authors (see Ref. [5]) argue that the location of the resulting $Z_2$ vortices signal the presence of underlying *center vortices*. These correspond to certain action density structures, which their center projection has allowed to pin down even without *cooling* [6]. In comparing with the more standard abelian projection, they argue that the abelian monopoles are located within the vortex walls, but their exact location has no special significance: they simply mimic the structure provided by the vortex wall. The crucial issue is then, what is the dimensionality of the relevant structures which are responsible for Confinement. The supporters of the fluxonic(vortex) scenario would say 2, as opposed to the unit dimensionality of monopole world-lines.

How do our configurations enter the picture? They provide candidates for the underlying action density structures: *center vortices*. As we will see, they indeed carry one unit of magnetic center flux [7]. Furthermore, the configurations, being self-dual, are solutions of the Euclidean equations of motion corresponding to a local minimum of the action. For example, they would emerge as structures seen by *cooling*. Even if this is not the case, the realization that there are vortex-like solutions of the classical Yang-Mills equations of motion is an important issue in itself.

The plan of the paper is as follows. In the next section we briefly review our method and present our results. Finally, in the last section, we discuss in further detail a few points concerning the way in which the configurations could appear in the Yang-Mills vacuum and the possible relation to Confinement.

### 2 Numerical method and results

Our starting point is the consideration of $SU(2)$ Yang-Mills fields (we take $SU(2)$ for simplicity, although most statements remain valid, with the appropriate changes, for $SU(N)$) on the 4-dimensional torus (for a review see [8]). ’t Hooft [7] realized that when putting this fields on the torus, there are some new topological sectors, known as twist sectors, characterized by $2$-integer mod 2 vectors: $(\vec{m})_i = \frac{1}{2} \epsilon_{ijk} n_{jk}$ and $(\vec{k})_i = n_{0i}$. Thus, gauge fields (actually the bundles) can be classified into different sets labeled by these vectors and the ordinary topological charge. One can minimize the action functional within each set to obtain a classical solution (of the Euclidean
equations of motion). The absolute minimum within each sector is bounded by $8\pi^2$ times the absolute value of the topological charge, and only saturated by self-dual or anti-self-dual configurations. Our strategy to find the solutions is to consider the lattice formulation of Yang-Mills theory, and perform a minimization of the lattice action functional (Wilson action in our case). Twist sectors are easily implemented on the lattice $\mathbb{Z}^4$. Unfortunately, the set of lattice fields within each sector ($SU(2)^V$) is a connected set, and there is no natural splitting into topological charge sectors. Thus, what can be easily done with our methods is to find the configuration which minimizes the action for each twist sector without specifying the value of the topological charge. Particularly interesting are thus the so-called non-orthogonal twists ($\vec{k} \cdot \vec{m} \neq 0 \mod 2$) for which the topological charge is half integer. The resulting configuration is assured to be non-trivial ($F_{\mu\nu} \neq 0$). For the minimization procedure we use the naive cooling algorithm $[6]$. Since we are looking for the absolute minimum action configuration we do not have to be too sophisticated in this point. We refer the interested reader to Ref. $[1]$ for further details on the numerical technique. It is worthwhile to mention, that it takes only a few minutes in a standard workstation to generate one of the configurations used in this paper.

The specific characterization of this work compared to previous ones, is that we make the torus asymmetric in the different directions: 2 of the periods are taken much larger than the other two. Thus our lattices are of size $N_t \times N_s \times N_t \times N_s$ with the order of directions labeled 0, 1, 2, 3 and $N_t \gg N_s$. If a limiting $R^2 \times T^2$ configuration exists, we expect that our lattice configurations would tend to a well-defined limit as $N_t/N_s \to \infty$. Once this limit is approximately obtained, one must study the approach to the continuum limit as $a \equiv 1/N_s$ tends to zero. In this work, we have used $N_s = 4, 6$ and values of $N_t$ ranging from 12 to 24.

Concerning the value of the twist vectors $\vec{k}$ and $\vec{m}$, all non-orthogonal values ($\vec{k} \cdot \vec{m} \neq 0$) have been studied. This implies that our resulting configurations have topological charge $|Q| = \frac{1}{2}$, and the corresponding action is bounded from below by $4\pi^2$. Using the symmetries of the problem, one can reduce to 14 the total number of twist vector sets to study.

Now we will present our results. The first quantity which one obtains after the minimization process, is the value of the minimum lattice action $S_L$ for each twist and lattice size. By looking at this quantity alone one observes that the different sets of twist vectors can be classified into three classes. Class A is characterized by $m_2 = n_{13} = 1 \mod 2$. $S_L$ is very similar for all twist-
vectors within this class. We get $S_L/(4\pi^2)$ is 0.9597(1) for $N_s = 4$ $N_l = 16$, 20 and 0.982737(1) for $N_s = 6$ and $N_l = 24$. The number in parenthesis affects the last digit and represents the maximum variation obtained within the class. The fact that $S_L$ does not depend on $N_l$ to the $10^{-4}$ level (actually less than this, by comparing equal twist vectors) is in agreement with our expectation for a lattice configuration approximating an $R^2 \times T^2$ one. The fact that all components of the twist vectors, except for $m_2 = n_{13}$, become irrelevant in this limit, is also natural, since they involve one or both of the directions which tend to $R$. Now, we examine the approach of $S_L$ to the continuum limit. By using our $N_s = 4$ and $N_s = 6$ values in the formula $S_L = S_c + B a^2$, we determine the 2 parameters to be $B = -0.6634 (4\pi^2)$ and $S_c = 1.0011 (4\pi^2)$. Thus, the latter value equals the continuum action of a (anti-)self-dual configuration to 1 part in $10^3$. This discrepancy is of the size to be accounted for by $O(a^4)$ corrections. In summary, class A configurations behave according to our expectations, and provide a natural candidate for our $R^2 \times T^2$ configurations. The rest of our study will concentrate on them.

For completeness, we mention that the remaining choices of twist vectors (having $m_2 = 0 \bmod 0$) can be classified into 2 sets. Class B is made of the configurations with $n_{01} = n_{03} = n_{23}$ odd and $n_{12}$ even and all obtained by exchanging $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$. The rest is labeled class C. Again $S_L$ changes very little within each class. This time, however, the variation of $S_L$ with $N_l$ is comparable to that with $N_s$. This suggests that the configuration does not become localized in $t$ and $y$ as the corresponding periods go to infinity.

Now, we study the action density distribution for all class A configurations. The different twist vectors in the class give results which are consistent modulo space-time translations. The distribution is symmetric under the exchange $t \leftrightarrow y$ and $x \leftrightarrow z$. Another feature is that the action density distribution has a unique local maximum. Indeed, we use several interpolation methods to determine the location of the maximum with a precision of less than a tenth of the lattice spacing in each direction. Then, we make use of translation invariance to choose the origin of coordinates precisely at this point. With this choice, the action density distribution is even under a change of sign of any coordinate ($x_\mu \to -x_\mu$). In all subsequent expressions we write the value of the coordinates on the continuum taking $a = 1/N_s$ (The periods in $x$ and $z$ are set to 1).

To give an idea of the behavior of the action density distribution, we studied the energy profiles $E_\mu(x_\mu)$, which are the integrals of the action density over 3 of the 4 variables. $E_1 = E_3$ is a periodic function of its argument with
period $l_1 = l_3 = 1$. If we determine the first few Fourier coefficients and compare the data to the curve $S_L + 4\pi^2 B \cos(2\pi x) + 4\pi^2 C \cos(4\pi x)$, we obtain $B = 0.1612(3), C = 0.004(4)$ for $N_s = 4$ and $B = 0.16380(5), C = 0.00646(1)$ for $N_s = 6$. Errors represent again the variation within class A. Indeed, the first 2 Fourier coefficients describe the data to the 1\% level. A quadratic extrapolation to $a \to 0$ yields $B_c = 0.1659$. A much more crucial issue is the behaviour with respect to $t$ and $y$. For that purpose we studied $E_0 = E_2$. Indeed, this function shows a exponentially localized distribution around the origin. After the appropriate scaling is done, we show in Fig. 1 the lattice approximations to $\log(E_0(t))$. Data for $N_s = 4, 6$ are plotted together and compared with a straight line $D - wt$. Apart from the clear exponential fall off, one sees that the data for different $N_s$ compare nicely to within a few percent level. At the origin, for example, both data differ by 4\%. This agreement is striking given the large values of $a$ involved. The decay exponent $w$ was found to vary from 6.50(6) to 6.65(5) for $N_s = 4$ and 6 respectively. Errors reflect differences in fitting procedures, ranges, etc. We also looked to the joint $t$ and $y$ distribution. Our data suggests that the action density is indeed rotationally invariant in that plane, thus, depending only on $r = \sqrt{t^2 + y^2}$.

Hence, we have verified that we are actually dealing (to within the precision of our data) with a self-dual continuum configuration, which is periodic with unit period in the $x$ and $z$ directions and localized in the $t$ and $y$ directions. By considering more than one period in the $x, z$ directions we can obtain a new self-dual configuration with zero twist vectors (and hence, obtainable by using periodic boundary conditions on the lattice) and a larger size. The most important feature being that its action density distribution is concentrated around the $t = y = 0$ plane. This is the world sheet of our (thick) vortex. However, to verify that we are indeed dealing with vortices, we have to examine the behaviour of the Wilson loop around this object. The relevant loops, with our choice of coordinates, are the 0 2 loops. Hence, we evaluated square loops of increasing linear size $r$ centered at $t = y = 0$: $L(r, x, z) = \frac{1}{2} Tr(W_{02}(r \times r))$. The results are given in Fig. 2. We show the two extreme cases $x = z = 0$ and $x = z = \frac{1}{2}$; the rest of values give curves lying in between. Again data for $N_s = 4$ and $N_s = 6$ are plotted together to give an idea of the size of O(a) effects. One sees that $L(r)$ becomes 1 for small $r$, but tends to -1 as $r$ grows, as expected. It is, of course, this property that makes these configurations effective in disordered the Wilson loop and approaching Confinement, just as thin $Z_2$ fluxons in $Z_2$ gauge theories.
3 Considerations on Confinement

In the previous section, we have shown that there exist self-dual configurations corresponding to a vortex wall aligned along one plane. Taking this plane to be the $z - t$ plane (it could be any), we can interpret this wall as the world sheet of an infinitely long vortex line located around the $z$ axis. This is similar to the case of the abelian vortex solution found by Nielsen and Olesen \cite{NielsenOlesen}. Just as for that case, one can adiabatically deform the string to form a finite closed loop in 3-space. Similarly, the corresponding infinite cylinder in space-time can be changed into a 2-dimensional torus representing a vortex-antivortex loop, or (possibly) into a sphere representing the creation, propagation and annihilation of a vortex ring. However, our non-abelian vortex solution is not static as its abelian partner, but 

breathes at a fixed pace. The time-period, whose magnitude determines the thickness of the wall, can be any positive real number. Similarly, it is not constant but periodic in the $z$ direction (It is a breathing caterpillar). The fact that the periods in $z$ and $t$ are equal, as in our numerical example, is an unnecessary constraint. There are also solutions with different periods in both directions, as we explicitly verified numerically. We simply focused on the most symmetric case.

Much less obvious is the fact that it is also possible to depart from the periodic arrangement. To see this, just put back our solution in a torus with a size which is a multiple of the periods. Then, the Atiyah-Singer index theoremtell us that there are exactly $4N$ deformations of the solution that lead back to new self-dual solutions with equal action. The integer $N$ is the number of unit cells (equal to the number of $Q = \frac{1}{2}$ lumps, or to the total action divided by $4\pi^2$). This is just the right number of degrees of freedom required to move freely the lump centers away from the periodic arrangement. So it is quite plausible, that for small deformations a configuration looks like a picture of a 2-dimensional solid at finite temperature, with the ions displaced from the periodic array of equilibrium positions. One does not know, however, what could happen for large deformations.

In conclusion, the vortex solution that we have found has some features which are non-generic, artifacts of the method used to construct it. This is typical of the search of classical solutions, where normally the most symmetric ones are more easily found or constructed. We do not know if, in the same spirit, there exist solutions which cannot be constructed in terms of $Q = \frac{1}{2}$ lumps. For example, the new caloron solutions mentioned earlier are made of 2 lumps with $Q = \tau$ and $1 - \tau$ for any $0 \leq \tau \leq \frac{1}{2}$. This is an interesting
point which is harder to investigate with our method.

Finally, we would like to comment upon the possible relevance of these solutions for the QCD vacuum and Confinement. A good deal of strategies and ideas have been put forward to understand Confinement. However, it is important to realize that all the ideas are not necessarily conflicting. In the most popular approach \[11\], one maps the non-abelian problem into an abelian model (with monopoles) by means of the abelian projection, and uses the well-understood abelian dual-superconductor confinement mechanism to describe the non-abelian one. This is indeed very appealing and probably correct, however, although not frequently realized, this is not the whole story. Having abelian monopoles does not imply having Confinement, as the 4-dimensional compact abelian case shows. There is something special about the non abelian theory which makes the condensation work all the way to zero bare coupling. It is precisely the mechanism that drives the condensation that some of us are trying to understand. As an analogy, one can take the case of Superconductivity. Realizing that the Bose-Einstein condensation of charged particles underlies the phenomenon, does not eliminate the necessity to search for the microscopic mechanism that drives the condensation: BCS theory.

Our group has advocated that self-dual structures, like the configurations presented here, are indeed responsible for Confinement. Self-dual configurations are called multi-instanton in the literature, but this has not to be confused with an array of independent instantons. One may say, that when instantons are pushed together tightly, they \textit{merge} into new structures with special properties (see the comments on Ref. \[13\]). The vortex configuration presented here shows an example of this. For example, the topological charge per lump is \( \frac{1}{2} \) rather than 1. This may or may not be a generic feature, but it is very appealing to realize that the number of degrees of freedom of the moduli space equals the number of degrees of freedom of a liquid of \( Q = \frac{1}{2} \) lumps. Energy-entropy arguments of why these structures would become condensed, were given in our paper Ref. \[12\]. A crucial point raised by the advocates of the vortex mechanism \[9\], is what is the space-time dimensionality of the structures responsible for Confinement. This is an important question that the lattice community has to settle. However, there are self-dual structures of all dimensionality: from calorons(also called BPS monopoles or dyons) or their constituent monopoles, to the isotropic model of Ref. \[12\], passing through the vortex solutions of this paper. To complicate the picture, ordinary isolated instantons are also expected to be present, although their role
in Confinement is presumably limited.

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Figure 1: We plot the logarithm of the energy profile $\varepsilon_0(t)$ for our solution. Open circles and full squares correspond to lattice spacing values of $a = 0.1666$ and $0.25$ respectively. The straight line is given by $5.42 - 6.5t$. 

\[ \log(\log(\varepsilon_0(t))) \]

- **Circles** --> $N=6$
- **Squares** --> $N=4$
Figure 2: We plot the value of the trace of an $r \times r$ Wilson loop centered around the vortex $\mathcal{L}(r, x, z)$. The different points are labeled in the figure, with $\text{max}$ corresponding to $x = z = 0$ and $\text{min}$ to $x = z = \frac{1}{2}$.

Circles ---> $N_s=4$, max.
Ellipses ---> $N_s=6$, max.
Empty circles ---> $N_s=4$, min.
Empty ellipses ---> $N_s=6$, min.