Kepler’s Differential Equations

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Abstract

Although the differential calculus was invented by Newton, Kepler established his famous laws 70 years earlier by using the same idea, namely to find a path in a nonuniform field of force by small steps. It is generally not known that Kepler demonstrated the elliptic orbit to be composed of intelligeable differential pieces, in modern language, to result from a differential equation. Kepler was first to attribute planetary orbits to a force from the sun, rather than giving them a predetermined geometric shape. Even though neither the force was known nor its relation to motion, he could determine the differential equations of motion from observation. This is one of the most important achievements in the history of physics.

In contrast to Newtons *Principia* and Galileis *Dialogo* Kepler’s text is not easy to read, for various reasons. Therefore, in the present article, his results — most of them well known — are first presented in modern language. Then, in order to justify the claim, the full text of some relevant chapters of *Astronomia Nova* is presented. The translation from latin is by the present author, with apologies for its shortcomings.

1. Introduction

The three laws of Kepler are well known:

1. The orbits of the planets are ellipses with the sun in one of the focal points.
2. The radiusvector of the planet covers equal areas in equal times.
3. The squares of the revolution times are in the same relation as the cubes of the average distances.

Every student of physics learns also how to derive these laws from Newtons equation of motion and the gravitational law.
Less known is how Kepler discovered these laws, except that he found them when analysing Tycho de Brahe's observations of Mars. Almost unknown is that Kepler set up a complete system of differential equations of motion, i.e. he indicated the differential change of the planets coordinates $r$ and $\phi$ as a function of time. This he did even for two systems of polar coordinates, one with the sun at the origin, the other with the center of the ellipse at the origin. Moreover, he insisted that the latter system which is easier to integrate cannot be the ultimate solution because the center of the ellipse as an imaginary point is not accessible to the planet. The planet has to find its way by looking only to the sun and to the stars. The elliptical orbit is not predefined but results from the composition of differential pieces.

How important this concept is for the development of science will be clear to every physicist. For example, Einstein writes in an article about Newton [1]:

*Kepler’s laws give a complete answer to the question of how the planets move around the sun: elliptical form of the orbits, equal areas in equal times, relation between principal axis and revolution time. These rules however do not satisfy our need for causality. They are logically independent rules without any inner connection. The third law cannot be easily transferred to a central body different from the the sun (there is for example no relation between the revolution times of a planet around the sun and of a moon around its planet). Most important however, these laws refer to the motion as a whole and not to the question of how a state of motion is changed from its actual value to the one immediately following in time. They are, in our present language, integral laws and not differential laws.*

The differential law is the only form which satisfies the need for causality of a modern physicist. The clear conception of the differential law is one of the most important mental achievements of Newton. Not only the idea was required but also a mathematical formalism which, though existing in rudimentary form, had to be given a systematic shape. Newton found this in the form of the differential and integral calculus. The question of whether or not Leibniz independently of Newton invented the same mathematical methods may be left aside.

It is the purpose of the present paper to make Kepler’s contribution more clear, 400 years after the publication of *Astronomia Nova* [3], with the hope that the Einsteins of future generations will be better informed. Needless to say that Einstein himself was a great admirer of Kepler, as shown at many instances, e.g. in his article for the “Frankfurter Zeitung” [2] on the occasion of the 300th anniversary of Kepler’s death. He would have admired Kepler even more if he had known the following chapters of Kepler’s *Astronomia Nova*.

When dealing with an issue that is several hundred years old, one might of course ask, why was there no earlier notice? Probably there are several reasons. One of them is language. Not only at present hardly any scientist is familiar with Latin, also the mathematical language has evolved enormously since Kepler’s time. In *Astronomia Nova* there is no single equation, because the equal sign (=) was not yet invented, there is no abbreviation with letters, like $r$ for radius. Everything has to be explained by words. The only help comes from geometrical drawings. A second problem is, that Kepler had a wrong notion of forces, inherited from Aristoteles. According to Aristoteles the velocity of an object is proportional to the force applied to it and has the same direction. As is well known, it was Galilei who first questioned
this opinion, saying that a piece of lead of 100 pounds does not fall from a height of 100 cubits\(^1\) ten times as fast as a piece of 10 pounds. It was him who established by measurements the true relation between force and motion. His results, among them a very elegant, but mathematically incomplete argument about the centrifugal force, were not published until 1632, two years after Kepler’s death. The centrifugal force was not known to Kepler, neither the word nor the fact. The correct description of it is due to Huyghens (in 1673).

It is understandable that after Newton hardly anybody should have taken the pain to labour through Kepler’s *Astronomia Nova*. In this text, Kepler’s ideas about the laws of motion are mixed with his notion on forces. To separate them would unavoidably alter the text. Therefore the full text of the most relevant chapters including side remarks will be given below, with no attempt to condense it or even to change the sequence of words, if not necessary. It is not always easy to translate Kepler’s text into a modern language, partly because the concepts have changed. This is especially true for the concept of forces. Apparently Kepler distinguishes forces applying to dead matter (vis naturalis) and forces applying to living matter (vis animalis). The latter is translated here as “animated force”. Examples are given in Kepler’s text.

Readers who want to skip the speculations about forces can easily do so; they will have no problem to understand the subsequent arguments for differential equations. On the other hand, Kepler’s text is a fascinating document in itself. The fact that he published his ideas, even though he knew that they cannot be correct, probably reflects the same spirit as advocated in our times e.g. by Feynman, that theoreticians should not only talk about their achievements, but also indicate where they failed. The subsequent solution of the puzzle is then much more instructive.

It will not be difficult for a modern reader to condense the text, after having read it, into a more suitable form.

In a certain sense, Kepler’s differential equations are not just an attribute to an already existing beautiful theory, but they are a logical necessity right from the start. Kepler’s initial concept was, that planet coordinates should be referred to the sun, because it is the sun, the heart of the world, as he says, which turns the planets around. His first occupation with Tycho was to convert a table of Mars oppositions from the traditional reference to the center of the earths orbit into a table of Mars oppositions to the sun. Tycho saw that enterprise with a certain suspicion but he let Kepler do, because he and his longtime assistant Longomontanus had finished the Mars theory — even though not everything fitted exactly as they wished — and had started to work on the much more complicated motion of the moon.

Whenever a body moves in a field, where strength and direction of the force change continuously due to its very motion, then at a given moment only the immediately following piece of its trajectory can be predicted. Therefore already in the concept of the sun as a source of forces, a planets orbit as a solution of differential equations is inherent. That the elliptical orbit, in contrast to an eccentric circle, has in fact this property, to be the solution of a differential equation, must have given Kepler enormous satisfaction and additional confidence in the correctness of his result.

\(^1\)about 45m (note by the editor)
It is somewhat unfortunate and also unfair, that the historical reception has reduced Kepler’s work to the famous three laws. In reality the essential step from the traditional way of a purely geometric description of planetary orbits to a physical description was his idea of a force which comes from the sun. It is this concept which leads to differential equations. It may be compared to the introduction of the idea of a field instead of forces which act over long distances at the end of the 19th century. As the reader will see, the application of the idea was not immediate; Kepler started in the traditional way with a geometric concept. As Einstein phrased it: with one foot he was still in the medieval age. This is well known from Kepler’s biography, but it shows also up in his astronomical work. Yet it was him who made the first, essential step. The later transition to infinitesimally small increments looks from this point of view rather like a straightforward refinement. In our teaching there ought to be a corresponding change.

2. Kepler’s differential equations in modern form

Let a planet P (Fig.1) move around the sun S in an elliptical orbit centered at C. The sun

![Diagram](image)

Figure 1: The ellipse with the sun S in one of the focal points and a planet P which appears at an angle $\phi$ to the principle axis. In Kepler’s time the angle $\phi$ was measured from the aphelion A, nowadays it is measured from the perihelion. (see text)

S is located in one of the focal points of the ellipse. The so called “true anomaly” is the angle $\phi$ between the radiusvector PS and the principle axis of the ellipse (the “line of apsides”). The point Q is the point which corresponds to P on the circle centered on C with radius CA.
equal to the principle axis. The angle $\beta$ between the line QC and the principle axis is called the “eccentric anomaly” of the planet. The perpendicular drawn from S to the line QC or its extension intersects this line in B. For later reference, we mention here, that the angle SPC in which the eccentricity CS appears to the planet is referred to as “optical equation “ by Kepler ( the term “equation “ in astronomy denotes an angle, usually the angle between the real sun and a fictitious sun moving with constant angular velocity, as seen from the planet ).

Let $e= CS$ and $r=PS$, and $CA=1$.

The area $F$ covered by AQS is then the sum of the circular sector AQC and the triangle QCS

$$2F = \beta + e \sin \beta.$$ 

This equals to $\omega t$ according to Kepler’s second law, when $t$ is the time elapsed since the transition of the aphelion ( in A) and $\omega = 2\pi/T$, $T =$ time of revolution,

$$\omega t = \beta + e \sin \beta.$$ 

This is known as Kepler’s equation.

Because the area covered by the ellipse differs from the area covered by the circle by the constant amount $\sqrt{1-e^2}$, Kepler’s second law is also valid for the ellipse. In differential form it says

$$r^2 d\phi = \sqrt{1-e^2} \omega \, dt.$$ 

An important relation is $QB=PS =r$. From Fig.1 one reads

$$QB = QC + CB = 1 + e \cos \beta.$$ 

On the other hand, in the triangle PDS one has

$$(PS)^2 = r^2 = (DS)^2 + (PD)^2 = (e + \cos \beta)^2 + (1 - e^2) \sin^2 \beta = e^2 + 2e \cos \beta + 1 - e^2 \sin^2 \beta = 1 + 2e \cos \beta + e^2 \cos^2 \beta = (1 + e \cos \beta)^2,$$ 

$q.e.d.$

From $r = 1 + e \cos \beta$ follows

$$dr = e \, d(\cos \beta) = -e \sin \beta \, d \beta.$$ 

This is one of the differential equations which, together with Kepler’s second law, forms a complete system for $r(t), \beta(t)$. Kepler uses the integral

$$\int_{r_{\text{max}}}^{r} dr = e(1 - \cos \beta),$$
Figure 2: Geometric illustration of the *sinus versus β*. The *sinus versus β* is the sagitta in a circle with unit radius. It equals \((1 - \cos β)\).

for \((1 - \cos β)\) he writes *sinus versus β*. Another expression, in use both at his time and nowadays, is *sagitta*, the piece of an arrow which sticks out over the chord (Fig. 2).

To obtain polar coordinates centered on the sun we read from Fig.1

\[
 r \cos φ = e + \cos β.
\]

Replacing \(\cos β\) from equation (1) gives

\[
 r \cos φ = e + \frac{r - 1}{e} \\
 e r \cos φ = e^2 + r - 1 \\
 r (1 - e \cos φ) = 1 - e^2 \\
 \frac{1 - e^2}{r} = 1 - e \cos φ,
\]

or differentially

\[
 (1 - e^2) d \left( \frac{1}{r} \right) = -e d(\cos φ) \\
 d \left( \frac{1}{r} \right) = -\frac{e}{1 - e^2} d(\cos φ) \propto \sin φ \ d φ
\]

This is the relation referred to by Kepler when he says, \(\sin φ\) is the measure for the change of the apparent diameter of the sun, i.e. the angle under which the sun appears from the distance \(r\).
Figure 3: Construction of an eccentric circle (dashed line) on which a planet P turns around the sun in S, by the superposition of two circles, one with large radius SA around the center S, and a second one (the epicycle) with smaller radius (equal to the eccentricity CS) around the tip of the radius vector of the former circle (see text)(from *Astronomia Nova*).

It is the second differential equation which, together with the second law, completes the system $r(t), \phi(t)$. The general integral of this equation is

$$\frac{1}{r} = a + b \cos \phi,$$

with arbitrary constants $a$ and $b$, the equation of a cone section.

3. **Kepler’s oval hypothesis as a precursor to the ellipse.**

In order to understand the subsequent sections it is necessary to introduce Kepler’s idea how to modify a circular motion. He starts from a construction due to Appolonius, by which the
apparent nonuniform motion of the sun can be explained as a superposition of two uniform circular motions (Fig.3). In the later heliocentric system, a planet P is mounted on a small circle, the epicycle, the center A of which moves uniformly around a center S, the sun. If the motion of the planet on the epicycle is in opposite direction to the motion of the large circle, say clockwise, if the large circle moves anticlockwise, and both have the same time of revolution, the planet stays always in the same direction from the center of the epicycle (upwards in Fig.3), and so performs a circular motion, the center C of which is now displaced from the sun S by an amount equal to the radius of the epicycle (the epicycle is here thought to be fixed like a spoke to the large wheel). Kepler assumes that the reader understands this concept from the drawing. This is the traditional way how the motion of the earth around the sun (or in antiquity, the sun around the earth) was explained. The orbit of the planet is a circle, called the eccentric.

Now two observational facts on Mars called for a modification of this scheme. One is, that the motion in the eccentric is still not uniform, the other is, that the orbit is apparently not a perfect circle, but is narrower along the line of apsides. Kepler’s modification to the scheme of Appolonius was to give the planet a nonuniform motion, satisfying his second law, while the motion of the planet on the epicycle remains uniform. Since the motion on the orbit is slower near the aphelion, where the distance to the sun is largest, according to the second law, the motion of the epicycle is advanced in the orbit section following the aphelion. The planet deviates therefore from the circle to the inside. The lead of the epicycle “clock” diminishes as the planet approaches the perihelion; there it vanishes. In the subsequent section from perihelion to aphelion the lead is changed to a lack, again causing a deviation to the inside. Everyone will agree that this is an ingenious idea; there is no single free parameter. The mechanism of such a system however is totally unclear; it also contradicts Kepler’s own premises (that a circular motion around an imaginary point makes no sense). His excuse is funny enough (“Speedy dogs have blind offsprings “). As seen from Fig.4, the deviation from the circle at 90° from the apsides is around $e^2$, twice as much as in the final ellipse ($\sqrt{1-e^2} \approx 1 - e^2/2$). The difference, $e^2/2$, is three to four times the observational error, that is, it is significant. There are inconsistencies also in the time distribution. It is not necessary here to recall Kepler’s efforts to save his oval hypothesis, by modifications of his second law. This work cost him the entire year 1604, because the numerical integrations in steps of 1°, with five digit accuracy, for three different eccentricities, had all to be done by hand; logarithms were not yet invented. The discrepancy at 90°, alluded to above, plays a central role in the discovery of the ellipse, in chapter 56 of the Astronomia Nova, as will be seen below.

Besides a coordinate system fixed in space, Kepler likes to use a system in which the line joining the sun and the center of the epicycle has a fixed direction. The transformation between the two systems is, of course, only simple, if the motion of the epicycle center is known. It is not astonishing that in the course of the investigation this fictitious point, together with the whole epicycle, disappears into nothing.
Figure 4: Deviation from a circle in Kepler’s oval, at a position of the planet at 90° to the line of apsides. According to Kepler’s second law the planet has finished a quarter of the total revolution. The center of the epicycle in a circular path would be, according to the construction of fig.3, below the planet, by an amount $e$. The radiusvector of an epicycle running at constant speed is, in this position, advanced by an angle as indicated. Therefore the planet deviates from a circle by an amount $e^2$. 
4. Kepler’s axioms of planetary motion

It is significant for Kepler’s rigorous thinking that he formulates the foundation of his calculations as axioms, as he had already done in his work on optics. His axioms are:

1. The planets tend to rest at the place where they are positioned alone.

2. They are transported by a force which comes from the sun, from place to place along the ecliptic.

3. If the distance of a planet from the sun is unchanged, the planets orbit will be a circle.

4. If the same planet would move at different constant distances from the sun, the times of revolution would be in the ratio of the squares of the distances.

5. The force which is inherent to the planet is not sufficient to transport it from one place to another, since the planet has nor legs nor wings nor fins to support itself on the ether.

6. Inspite of this, the planets approach and regression from the sun is due to a force inherent in the planet.

All these axioms, says Kepler, are uncontradictory and are in agreement with nature, according to our present knowledge.

Maybe it is appropriate to add here a few comments before continuing. The third axiom sounds trivial: a circle is defined as a curve in which every point has the same distance to the center. The axiom is nontrivial, however, if one considers the third dimension. Then the axiom says, the planets orbits are planes. It was one of Kepler’s first actions to convince himself by several independent methods that the orbit of Mars is in fact, like the orbit of the earth, a plane around the sun, with a constant inclination to the ecliptic. This content of the axiom is however not immediately obvious. Had Kepler said: the orbits of the planets are planes in which also the sun is located, he would have said more and it would be clearer. This is an example of a certain lack of refurbishment which characterizes the whole of Astronomia Nova, certainly in part due to various difficulties in the edition, not to mention Kepler’s fragile health.

Also the forth axiom calls for a second thought. The situation contemplated here is somewhat unrealistic. Nobody can place a planet into a different circular orbit. What the axiom probably wants to say, is, that, if a planet in the apsides, where the distance from the sun is momentarily constant, would continue its orbit with the same speed v and at the same distance r, then its revolution time would be proportional to the square of the distance. The equality of the product $r \times v$ in the apsides is the empirical fact upon which later Kepler’s second law is based. He could have formulated the axiom as $r_1 \times v_1 = r_2 \times v_2$, where the indices refer to aphelion and perihelion. That is all he needs. The confusion in the literature about the difference between the constancy of $r \times v$ and of $r \times v_\perp$ (which is the second law) on the whole orbit, this confusion is in part Kepler’s own fault. His own very sophisticated investigations about this difference on a circular orbit will not be dealt with here, but they are most likely essential to his discovery of the ellipse, as treated in chapter 56 of the Astronomia Nova (see below).
The fifth and sixth axioms concern the forces. As will be seen, Kepler favours a kind of magnetic force, although he must admit, that the earth’s magnetic field would have the wrong direction. So he must leave the question open. One might wish that he had lived long enough to see that the gravitational force invented by himself to explain the tides as attraction by the moon is in fact all what is needed to explain planetary motion.

5. Arguments against circular orbits.

It is historically very interesting to see how the circular orbit, sacrosanct to the ancient Greeks, disappears in the course of time. Already Copernicus had given it up, in favour of another principle, the uniformity of motion. Both principles are wrong. Kepler started from the assumption that a force from the sun is responsible for the motion. It is remarkable that he could find the laws of motion knowing neither the force nor the relation between force and motion. In chapter 39 of *Astronomia Nova*, after presenting his axioms, he argues that a circular orbit is not compatible with these axioms and gives a preview of his later discovery of the elliptical orbit, leaving aside his own idea of an oval, that failed for various reasons. Apparently Kepler introduced this chapter 39 in order to familiarize the reader with the general concept before getting lost in the details of the calculations.

In the presentation of the text, we will, as for chapters 56 and 57 below, also include Kepler’s comments, presented in the original edition in small letters on the margin of the page. Here they are inserted in italics into the text, approximately at the same place as they appear in the original. Footnotes in the original, characterized by Kepler with a * in the text, are given as footnotes. A few footnotes by the present editor are characterized as such. The translation from Latin is not always straightforward. In case of doubt, the reader is asked to consult the original text.

A short comment will be added at the end of Kepler’s text. It may help the reader in a second reading.

After his presentation of the axioms, Kepler continues in chapter 39:

5.1 What the planet achieves by its motion, if the orbit due to the composition becomes a circle, i.e. how will the distances from the sun be obtained.

Let us now exercise with geometrical figures to see which laws are necessary to represent any planetary orbit. Let the orbit be a circle as believed hitherto, eccentric to the sun, the source of the force (Fig.5). The eccentric shall be CD, with center B, radius BC; the line of apsides shall be BC, the sun A and BA the eccentricity. The eccentric is divided into an arbitrary number of equal parts, starting from C on the line of apsides; their ends shall be connected to A. Therefore CA, DA, EA, FA, GA, HA will be the distances from the source of the end points of equal parts. Also (Fig.6) an epicycle $\gamma\delta$ shall be described around a center $\beta$ with radius $\beta\gamma$ equal to AB, and be divided into the same equal parts as the eccentric, starting from $\gamma$. The line $\beta\gamma$ shall be continued such that $\beta\alpha$ equals BC, and the point $\alpha$ shall be connected to the end points of the equal parts in the epicycle in the lines $\gamma\alpha, \delta\alpha, \epsilon\alpha, \zeta\alpha, \eta\alpha, \theta\alpha$. These lines will be equal to the distances drawn from A in the eccentric, respectively. This was already demonstrated in chapter 2 above. Now the arc $\delta\theta$ shall be described around the center $\alpha$ with
radius $\alpha \delta$; it intersects the diameter $\gamma \zeta$ in $\iota$; around the same center with radius $\alpha \epsilon$ the arc $\epsilon \lambda \eta$ intersects the diameter $\gamma \zeta$ in $\lambda$. The terminal points with equal distances from the aphelion $\gamma$ of the epicycle shall be connected in the lines $\delta \theta, \epsilon \eta$, which intersect the same diameter in $\kappa, \mu$, such that $\alpha \delta$ or $\alpha \iota$ is longer than $\alpha \kappa$, and $\alpha \epsilon$ or $\alpha \lambda$ longer than $\alpha \mu$.

**First mode: the planet itself runs the epicycle.**

If it were possible that the planet describes a perfect epicycle by an inherent force and at the same time its orbit becomes a perfect circle, then similar arcs would have to be completed simultaneously, in the eccenter and in the epicycle. Therefore, it is immediately obvious, by which means, by which measure the distances $\iota \alpha$ and $AD$ are made equal. Namely, since $\alpha \iota$ and $\alpha \theta$ are equal, the planet going from $\gamma$ to $\theta$ will necessarily and without special advice find the right distance $\alpha \theta$ equal to $AD$.

*The absurdities of this mode: first absurdity*

But besides that it seems to be in conflict with the fifth axiom if one says that the planet proceeds by an intrinsic force from place to place, there are many other absurdities involved. Let $AN$ be parallel to $BD$ and $AN$ be equal to $BD$. The epicycle around the center $N$ will go through $D$. Now, if $CD$ is on a perfect circle, the angles of the planet at the center $B$ of the eccenter and that of the epicycle center $N$ at the sun $A$ will be equal (by the equivalence demonstrated in chapter 2), such that the epicycle diameter $ND$, on which the planet sits in $D$, will remain parallel to $AB$ in space. Therefore the velocities of the epicycle center $N$ around the
sun in A and of the planet D around the center B of the eccenter will be the same; the motions will simultaneously intensify and relax, and, because of this intensification and relaxation is due to the larger or smaller distance of the planet to the sun, the center of the epicycle, which remains always

*Second absurdity*

in the same distance, would have to move slowly or fast because of the larger or shorter distance of the planet from the sun.

*Third absurdity*

And, while the force driving the planets is always faster than the planets, as shown in chapter 34, we would here have to assume one of the rays from the sun, AN, i.e. the line on which the center of the epicycle remains, to be

*Fourth absurdity*

sometimes late, sometimes early, again in contrast to what was said before, that a force in the same constant distance produces the same velocity.

*Fifth absurdity*

The planet however would have to be assumed to evolve from this imaginary ray AN in opposite directions unequally in equal times, in order that this ray itself becomes fast or slow.

*This last statement will be denied to be absurd below in chapter 49, while the other absurdities remain*

In this way we would be closer to the assumptions of the ancients but deviate enormously from our physical speculations, as shown in chapter 2. Also, my imagination is not sufficient to see a way in which this mode could be realized in nature.
Second mode, that the planet moves the eccenter.

It seems therefore simpler, if we look at the diameter ND of the epicycle which remains always parallel to itself. The planet will effect its motion inspecting not the epicycle but the center B of the eccenter and by keeping always the same distance from this center.

First absurdity

But in the beginning of this opus, in chapter 2, it was said, that it is totally absurd, that a planet (by whatever intelligence) figures itself a center and a distance from it, if the center is not marked by a special body. And also, if somebody says, the planet looks to the sun A and knows beforehand from memory, which distances from the sun it should keep in order to stay on a perfect eccenter, then this is even more remote. It also lacks the means to connect the perfectly circular orbit with the signs of increasing and decreasing diameter of the sun, and this for any whatsoever intelligence. These means are nothing else than to position the center B of the eccenter in a certain distance from the sun, what, as we already said, cannot be accomplished by an intelligence alone.

I do not deny, that one can imagine a center and a circle around it. But this I say, if this center exists only in the imagination, without indication of time or an external sign, then it is impossible to order in reality any mobile body into a perfectly circular orbit around it.

Second absurdity

In addition, if a planet would take its correct distances required for a circular orbit, from memory, it would take as well, like from the Prutenic or Alphonsinic tables, the equal arcs of the eccenter to be run in unequal times and by the external force of the sun, and so it would prescribe from memory, what the alien force of the sun should be effecting. All this is absurd.

Third absurdity

Especially since according to Aristoteles there is no science of the infinite; the infinite however is mixed in this intensification and relaxation.

But luckily, also the observations themselves do not support a perfect circle, as will be shown below is chapter 44. So these (seemingly) foolish speculations are not alone and are all the less subject to disparaging remarks.

Third mode, that the planet librates by an inherent force on the diameter of the epicycle.

It is therefore more likely that the planet itself does not care for the epicycle nor for the eccenter but the work which it effects, or to which it contributes, consists of a libration or balancing movement on the diameter $\gamma\zeta$ tending towards the sun.

By which measure does the planet measure the correct distance at any given time?

The planet cannot obtain the correct distances from a real epicycle.

To us the measure is is apparent from geometry and a drawing. Whenever the planet is promoted by the solar force to the line AD, we look for the angle CBD and make $\gamma\beta\delta$ the same.

And so we say, $\alpha\delta$ or the equal amount $\alpha\nu$ is the correct distance of the planet in D from the sun A. But this measure, available to us humans, we have already taken away from the planet by forcing it from the width of the epicycle into the narrowness of the diameter $\gamma\zeta$.

... nor by the arcs completed in the eccenter.

Namely in this inquiry it is easier to say what is not than what is. The planet, whenever it
is promoted by the sun to the lines from A to C, D, E, F, G, H, is supposed to assume the distances \( \gamma \alpha, \iota \alpha, \lambda \alpha, \zeta \alpha, \lambda \alpha, \iota \alpha \), respectively. Now, if its orbit is a perfect circle, then to equal pieces CD, DE, EF of the eccenter correspond unequal descents of the planet on the diameter, namely \( \gamma \iota, \iota \lambda, \lambda \zeta \), and furthermore in disturbed order, so that not the uppermost are smallest, and the lowest are largest, but the central ones \( \iota \lambda \) are largest, and the extremes \( \gamma \iota, \lambda \zeta \) are smaller, and the uppermost \( \gamma \iota \) a little smaller than the lowest \( \lambda \zeta \). Namely \( \gamma \kappa \) and \( \mu \zeta \) are equal, and \( \gamma \iota \) is smaller than \( \gamma \kappa \), but \( \lambda \zeta \) is larger than \( \mu \zeta \).

... nor by the time elapsed, nor by the angle at the sun, i.e. the true anomaly
And for the same reason \( \gamma \iota, \iota \lambda, \lambda \zeta \) are not proportional to the times spent in the equal arcs CD, DE, EF nor to the angles at the sun CAD, CAE, CAF. The time or the amount of time spent in equal arcs of the eccenter CD, DE, EF diminishes continuously from the uppermost to the lowest arc, the angles at the sun increase continuously, the librations \( \gamma \iota \) however increase towards the central one, \( \iota \lambda \).

Therefore, if the orbit of the planet is a perfect circle, the measure of the planets descent on the diameter \( \gamma \zeta \) is neither the time nor the space in the eccentric nor the angle at the sun. And these measures are repudiated by physical speculations as well.
... nor by an imaginated epicycle or eccentric
What, however, if we say the following? Even if the motion of the planet in an epicycle has problems, the libration could be such that similar distances from the sun are obtained as those obtained with a real epicycle?
First, we would attribute to the planets own force the knowledge of an imaginary epicycle and its effect on obtaining the distances from the sun. Also would we attribute the knowledge of future velocities caused by the common motion around the sun. Necessarily the same imaginary change of velocity in the motion of the imaginary epicycle would be required as with the real eccenter. This is even more unbelievable as the former concept, in which the motion of the planet was connected to the knowledge of the epicycle or eccentric. Therefore the counterarguments presented before can be repeated here; the concepts are almost identical.

below in chapter 57 the measure of this libration will be disclosed.
Yet in lack of a better concept, we must presently stop here. The more absurdities are involved, the easier any physicist will admit below in chapter 52, what the observations will confirm, that the orbit of a planet is not circular.

5.2 By which means or which measure does the planet learn its distance from the sun?
So far we dealt with the measure of this kind of libration. What remains to be done, is to inquire also about the measurement of this measure, i.e. its size or the motion in space. Namely it is not sufficient for the planet to know how far it should be from the sun, we should also require that the planet knows what to do in order to obtain the right distance.

Do we have to attribute to the planets quasi a sense for the size of the sun?
Whoever is inclined, by this supposition of a perfectly circular orbit, to attribute an intelligence to the planet, regulating these librations, cannot say else than this intelligence must be observing the increasing and decreasing diameter of the sun, and by this means find out at a
given time the planets distance from the sun. Likewise the sailors cannot find out from the
ocean itself which distance they traversed in the waves since their path is not marked with any
borders. But by the time of navigation, if wind and waves have remained constant and the
ship never stopped, or by the direction of the wind and the different altitudes of the pole, or
by a combination of all these informations, or, if the gods like, by the motion of a system of
wheels which are dropped by means of fins into the waves, an instrument advocated by silly
mechanicians who transfer the quietness of the continents to the floods of the ocean. In the
same way, a planet cannot measure its location or space traversed towards the sun by itself,
since only the pure ether is in between without any signs, but it uses the time or its equivalent
in the same condition of forces, a possibility that was already denied above, or a mechanical
machinery, which is ridiculous (we assume planets to be round like the sun and the moon, also
it is likely that the whole ether field moves together with the planet), or finally some suitable
signs which vary with the distance of the planet from the sun, of which however none remains
except the variable apparent diameter of the sun.

In this way the planets would become geometers which measure their distance to the sun from
a single station, i.e. by the apparent diameter of the sun.

So we humans know that our distance to the sun is 229 times its diameter, when the diameter
is 30', and 222 when it is 31'. And also, if it were certain that this movement on the diameter
of the epicycle could not be effected by some material or magnetic force nor by a pure animated
force, but would be governed by an intelligence in the planet, nothing absurd would be stated.

There is something like an intelligence in the planets which respects the suns body.

That the sun is observed by the planets also in other respects, is apparent from the latitudes.
Namely, since the planets are seen to deviate to the sides from the central and royal road of this
force from the sun like in a torrent, as stated in chapter 38, they would describe minor circles
as seen from the earth or from the center of the world, parallel to some maximal circle, if they
did not respect the suns position intermediately, and would approach or recess in a straight
line through the center of the sun. But all planets describe maximal circles, which intersect
the ecliptic in locations opposite to the sun, as demonstrated for Mars in chapters 12, 13 and
14 above (see the margin of chapter 63). Therefore also the diameter \( \gamma \zeta \) of the libration is
oriented towards the sun and the latitudes respect the sun everywhere. Below in part V I will
transfer also this behaviour of the latitudes from the action of an intelligence to the action of
nature and of magnetic properties.

Possible objections to a sense for the suns body.

1. The small size.

But don’t tell me the diameter of the sun and its variation are so extremely small that they
cannot serve as a ruler. Namely for none of the planets the sun diameter vanishes completely.
At the earth it is thirty minutes, at Mars twenty, at Jupiter seven and at Saturn three, at Venus
however it is forty, at Mercury planely eighty and up to hundred and twenty. Not about the
smallness of this body one should complain, but about the inadequacy of the human senses for
the observation of such small quantities.

Look, how this whatsoever small body is able to move so remote bodies in a circle, as I demon-
strated for the upper planets. Everybody knows about the illumination of the whole world by
such a tiny body. It is then also believable that, whatever faculty the movers of a planet have
to observe the sun’s diameter, that this faculty is so much sharper than our eyes as their task
and eternal motion is more constant than our turbulent and confuse affairs.

2. The lack of sensual instruments.

Do you then attribute two eyes to each of the planets, Kepler? Not at all. Nor is it necessary.
Nor does one have to give them legs or wings to make them move. Solid orbits were already
excluded by Brahe. And our speculation does not exhaust all treasures of nature, so that it
could be scientifically stated how many senses exist. Yet admirable examples are at hand. Tell
me in terms of physics by which eyes the animated faculties of sublunar bodies observe the
position of the stars in the zodiac that in case of a harmonic disposition (what we call an
aspect) they jump up and enflame their work? Was it with her eyes that my mother sensed the
position of the stars, to know she was born in a configuration of Saturn, Jupiter, Mars, Venus
and Mercury in sextiles and trigons, and would therefore give birth to her children, especially
to me the first born, at those days where as many as possible of these aspects, especially of
Saturn and Jupiter, would reoccur or appear in quadratures, oppositions and conjunctions?
What I found true in all examples which occured to me up to the present day. But what do I
deal with these or other equally absurd matters if not for those who have exercised themselves
in nature more carefully than it is customary nowadays?

Consequently, the one whom we assume here to say that a planets orbit is a perfect circle, will
say with equal right, that the planet effects its libration by requiring that the sun diameter
appears approximately in the same inverse proportion to the maximum distance $\gamma \alpha$
as the
lines $\gamma \alpha, e \alpha, \zeta \alpha$ or the equivalent $\iota \alpha, \lambda \alpha, \zeta \alpha$, which correspond to equal arcs in the eccenter,
and by inspection of the sun diameter the correct distances in the neighbourhood of $\iota, \lambda, \zeta$ at
predetermined times will be obtained.

It should be known, however, that the increase of the sun diameter and the arc in the epicycle
do not well correspond to each other. Therefore the moving intelligence must have a very
good memory in accommodating equal increases of the sun diameter to inequal $\text{sinus versi}$ of the
epicycle arcs; more on this below in chapters 56, 57.

5.3. By which means a planet reaches the desired distance from the sun.

This shall be enough about the sign of the covered space. It remains that, as a third item, I
spend three words on the physical faculty to transport the planet. Whoever says the planet is
transported by an inherent force, says in no way something probable. This we declined in the
beginning. But also to the sun this force cannot be attributed in a simple way. What attracts
the planet will also have to repel it. This contradicts the simplicity of the solar body. But
whoever relates this transportation by some peculiar reasoning to a mutual agreement between
the sun and the planet, will alter the whole material of this chapter. In this respect a special
chapter, 57, will be devoted to the matter.

You can see, considerate and ingenious reader, that this opinion of a perfect eccentric circle
as a planets orbit involves many incredible things in the physical speculations. Not because it
gives the sun diameter as a sign to the planets mind — perhaps also the most correct theory
will do so — but because it attributes incredible things both to the mind and to the moving
agent.

2 Namely in chapter 57 the proportion will be slightly different
But we will have to learn, being really close, how to put these not yet perfect, but for the motion of the sun suitable speculations into numbers. It will be found useful for the more accurate invention of the truth, which is reserved for chapter 57, to have exercised ourselves beforehand.

Here ends chapter 39 of *Astronomia Nova*. Before continuing, a few remarks may be added which can help the reader in understanding the text. Kepler likes to use a coordinate system in which the line joining the sun and the center of the epicycle has always the same direction, upwards in the drawings. The purpose of the epicycle is to provide the distances between sun and planet, for a given time. It turns out in the course of the investigation, contained in chapters 42 - 55, that a more transparent and more correct calculation of the distances is obtained if one forgets about the epicycle and considers only the line joining the sun and the planet. When Kepler talks of a libration (from the Latin *libra* = balance) or balancing movement or upward and downward motion of the planet on a diameter of the epicycle which points to the sun, he means a motion on this line. That the line changes its direction in space is not relevant here. Newton later gave to this line the name of radiusvector (“the radius which carries” (the planet)). Kepler’s libration is a shortening or lengthening of this radiusvector, a kind of balancing movement around an equilibrium orbit which is a circle centered on the sun.

6. *Astronomia Nova*, chapter 56.

**Demonstration by the previous observations that the distances between Mars and sun should be taken quasi from the diameter of the epicycle.**

In chapter 46 the width of the moonlet to be cut off the semicircle was found to be 858 parts if the semidiameter of the circle has 100 000 parts, according to the theory of chapter 45. As I had seen rather clearly from two arguments presented in chapters 49, 50 and 55, that the width of this moonlet should be taken only half as much, i.e. 419 or better 432, and around 600 in the scale in which the semidiameter of Mars’ orbit is 152 350, I started to think, why and how a moonlet of this size could be cut off.

While I was anxiously deliberating, being aware that in chapter 45 planely nothing had been said and my triumph over Mars was futile, I came, perhaps by chance, across the *secans* of $5^\circ 18'$ which is the maximal optical equation. When I realized that it is 100 429, I woke up like from a dream and saw new light. I started to argue as follows. At mean longitudes the moonlet or the shortening of distances is maximal and has the size of the excess of the *secans* of the maximal optical equation over the radius 100 000. Ergo, if at mean longitudes instead of the *secans* the radius would be used, the effect would be what the observations suggest. And, referring to the drawing of chapter 40 (Fig.7),

I concluded in general, if instead of HA one would take HR, instead of VA really VR, and for EA now EB, and so everywhere, the same thing would happen at all points of the eccenter as what happens here at mean longitudes. And equivalently, in Fig. 6 instead of the line $\alpha \delta$ or $\alpha \iota$...
one would have to take $\alpha \kappa$ and instead of $\alpha \epsilon$ or $\alpha \lambda$ one would use $\alpha \mu$.

The reader is asked to recapitulate chapter 39. There he will find that it was argued already previously, what now the observations moreover confirm, that apparently the planet seems to perform a kind of libration quasi on the diameter of the epicycle which always points to the sun. He will also find that nothing was more contradicting this opinion than that at the time we were forced by the assumption of a perfectly circular orbit to make the uppermost librations $\gamma \iota$ different from the lowest $\lambda \zeta$, while they correspond to equal arcs in the eccenter, namely the former had to be shorter and the latter longer. Now, if the orbit is not circular and $\kappa \alpha$, $\mu \alpha$ are used instead of $\delta \alpha$, $\epsilon \alpha$, or $\iota \alpha$, $\lambda \alpha$, as was said, then it follows moreover that these librations are equal. So, what plagued us in chapter 39 for a long time, now turns into an argument that we found the truth 3.

Also about the fact that the central parts $\kappa \mu$ are now larger than the outer parts $\gamma \kappa$, $\mu \zeta$, it will be said in chapter 57, that this is in agreement with nature, in contrast to what we could understand in chapter 39.

The remainder of chapter 56 is skipped here. It deals with a confirmation of the present hypothesis by observations at all possible longitudes.

3 The reader will notice that Kepler's conviction to have found the truth is here linked to the appearance of an unexpected symmetry (note by the editor).
7. Astronomia Nova, chapter 57.

By which principle of nature a planet is caused quasi to librate on a diameter of the epicycle.

It appears therefore from most certain observations that a planet's orbit in the ether is not a circle but an oval figure, and that it librates along the diameter of a small circle, as follows:

Definitions. What the circumferential distance and what the diametral distance shall be.

If a planet, after equal arcs in the eccentric center assumes the diametral distances $\gamma \alpha$, $\kappa \alpha$, $\mu \alpha$, $\zeta \alpha$ instead of the circumferential distances $\gamma \alpha$, $\delta \alpha$, $\epsilon \alpha$, $\zeta \alpha$, i.e. $\gamma \alpha$, $\iota \alpha$, $\lambda \alpha$, $\zeta \alpha$, then it is obvious that from the perfect semicircle a moonlet is cut off, the size of which at any location is the difference between the two quantities, say $\iota \kappa$, $\lambda \mu$. With this in mind, not due to a priori reasons, but due to observations, as I said, the physical speculations will proceed more correctly than so far.

Namely, this libration accommodates itself to the space covered in the eccentric, but not in a proportion of rational numbers, that the planets mind would associate equal parts $\gamma \kappa$, $\kappa \mu$, $\mu \zeta$ of the libration to equal arcs CD, DE, EF of the imperfect eccentric — the parts of the libration are in fact unequal —, but in a natural way, which is not based on the equality of the angles DBC, EBD, FBE, but on the strength of the constantly increasing angle DBC,EBC,FBC, which strength follows nearly the so called sine of the mathematicians. Then the ascent is gradually changed by continuous reduction into a descent which is more probable than a sudden change of the planets direction, what was said in chapter 39 to be in clear contradiction to the observations. Since moreover the size of the libration point to a natural process, its cause will also be natural, namely not a mind in the planet but a natural or perhaps corporal faculty. Now, since in chapter 39 we made for very good reasons the assumption that a planet cannot proceed from one place to another purely by an inherent force, we have to see if we can attribute the libration in part to a solar force.

Natural examples for this libration.

The oars.

This consideration brings us back to the oars introduced in chapter 39. Let there be a circular flow CDE, IGH (Fig.8)

and in it a navigator who turns his oar once in two revolutions of the planet by his own very constant force, such that in C the direction of the oar is perpendicular to the direction of the sun, and the oar is pointing in turn to the bow or the stern of the ship. In F the direction of the oar is towards the sun and in the other locations it is intermediate. The stream in DE will hit the oar such that the ship is directed towards A; in C the action is only very small since the oar is hardly inclined. The same is true in F, because the stream hits the oar directly. But in DE the action is strong because the oar by its inclination is well disposed to an approach to the sun. The opposite happens in the ascending semicircle. The stream comes in GH from the other side and directs the ship away from the sun. At the same time, ceteris paribus, in C the impact will be smaller than in F, because our stream is week in C and strong in F. Also

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4 The principle of this libration is proven to be natural.
5 which shall be the genuine and justifying measure of the libration, or the reason why the sinus versus of the eccentric anomaly measures this libration.
Figure 8: Illustration of two mechanisms which Kepler discusses in order to explain planetary orbits. The first mechanism assumes a circular flow around the sun into which the planet dips an oar to bring itself closer to or further from the sun. The second mechanism is based on a magnetic force. The planet is represented by a magnetic needle with constant direction, attracted or repelled by the sun. (from *Astronomia Nova*).

This serves us well because our libration followed equal spacings in the eccentric and the planet remained in the upper parts for longer time than in the lower parts.

*The defects of this example.*

This example teaches only the possibility of the matter. By itself it is less likely, because the restitution of the oar takes not the same time but twice as much as that of the stream, and because the faces of the planets as seen from the earth should be observed to change. The face of the moon however, which participates with the planets in the motion discussed here, does not change in its monthly evolution, but is always directed to the earth, from where its eccentricity is calculated. Add to this that the force of the stream is material (the water acts by its weight and material impact), but the force of the sun is immaterial. So, other comparisons for the planets have to be found. They are also not equipped with an oar, a physical instrument, to take the force of the weights (which the moving species of the sun does not have).
The example of the magnet.
But from this refutation another perhaps better suited example arises. As the stream so the oar. The stream is the immaterial species of a magnetic force in the sun. Maybe the oar has something in common with a magnet. What, if the planets are somehow gigantic round magnets? For the earth (one of the planets according Copernicus) there is no doubt. This was proven by William Gilbert.

The magnetic theory of Gilbert.
But let us briefly describe this force. The globe of a planet has two poles, one which follows the sun, the other which flees it. Let the axis defined by them be represented by a magnetic needle, the tip of which is attracted by the sun.

It seems that a magnetic disposition in a planet is the reason for this libration.
It will, however, be retained against its magnetic nature by the translation of the planet which keeps its axis always parallel to itself, except that in the course of centuries the axis points to different stars and causes thereby a progression of the aphelion. For both of them, in my opinion, a mind may be necessary, which is sufficiently well instructed for this motion by an animated faculty, a motion, not of the entire body from place to place (this motion was above in chapter 39 correctly taken away from a force inherent in the planet), but of its parts around a quasi quiescent center.

The example of the earth.
Let us examine in the earth an example of such an axis, following Copernicus. Namely, since the axis of the earth during the annual circuit around a center remains nearly equidistant to itself for all locations, summer and winter are caused. To the extent that very long centuries incline this axis, the stars are believed to proceed, and the equinoxes to recede.

What do we then hesitate, in order to save our ideas about eccentricities, to attribute to all the planets what we have seen in one of them (i.e. the earth) to be the case, according to our understanding of the precession of the equinoxes and of the suns rise and fall during its annual revolution?
Here Copernicus was mistaken, thinking a special principle would be necessary for the earth to librate annually from north to south, such that summer and winter result, and by the near equality of the libration and revolution times the small difference between sidereal and tropical year would result. All this results in fact only from the constant direction of the earth’s axis and does not need external reasons, except for the very slow precession of the equinoxes. So, also here no special advice is needed for the promotors of the planet to transport its body around the sun in a parallel site and simultaneously perform the libration. One depends on the other in a natural way. Only the progression of the aphelia remains to be reflected upon.

The reason why the libration is fastest in the middle.
Now, when the needle is in C and in F, there is no need for the planet to approach or recess, since the ends of the needle have the same distance from the sun, and the planet would direct the tip of the needle towards the sun, if not retained by that force which keeps its axis parallel.

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6The precession of the equinoxes is similar to the progression of the aphelia.
7For Copernicus the natural revolution of the planets around the sun was bound like on a wheel or like the motion of the moon around the earth; a third motion was therefore necessary to keep the earth’s axis parallel in space. (note by the editor).
When the planet leaves the point C, gradually the front will approach the sun, the tail will tend away from it. Gradually the planet starts to navigate towards the sun. Behind F gradually the tail approaches and the head leaves the sun. Gradually then the whole body, by natural hatred, flees from the sun. Near A, where the axis points directly to the sun, here the approach, there the flight is strongest. This is really what our assumptions, as deduced from observations, did require, where out of the parts of libration $\gamma \kappa$, $\kappa \mu, \mu \zeta$, corresponding to equal arcs in the eccenter, the middle parts, $\kappa \mu$ were very long, but the parts towards $\gamma, \zeta$ tiny. 

*The reason why the libration at top is slow, at bottom fast.*

But also this agrees, that the observations want $\gamma \kappa$ and $\mu \zeta$ to be equal, while their arcs $\gamma \delta, \epsilon \zeta$ or rather CD, EF in the eccenter are equal but are completed in unequal times, CD taking longer, such that the piece $\gamma \kappa$ of the libration is completed more slowly than the equal piece $\mu \zeta$.

Namely also the magnets approach each other more slowly from a large distance and faster from a small distance.

*The axis of force is retained parallel in a planet by a natural force.*

Also the force which keeps the magnetic axis parallel, disregarding the direction to the sun, can now be transferred from the occupation of a mind, as we thought a moment ago, to the work of nature.

*With an exception, however.*

Although there seems to be the obstacle that nature acts always the same way, while this retaining force is seen to be different at different times, like the inclination of the axis to the sun, to which this force has to be compared, is nearly vanishing at mean longitudes but is very strong at the aphelion and the perihelion. But what impedes this retaining force to be many times stronger than the inclination of the axis to the sun and to become hardly or not at all fatigated by such a weak adversary?

*The magnet as an example.*

Again let us take the magnet as an example. In a magnet manifestly two forces are mixed, one which directs to the pole, the other which aims at the iron. So, if the compass needle is directed to the pole, the iron is reached from the side; the compass deviates a little from the pole and is inclined towards the iron, due to some familiarity with the iron, but points predominantly to the pole.

*The reason why the magnet deviates somewhat from the pole.*

This happens according to Gilbert because the compass is deviated by continents of particularly large size; and so the reason for this declination is in the configuration of the land masses, of which, to the right or to the left, higher, larger and more powerful ones may be in the neighbourhood.

In this respect we can allow to both natural faculties to act on the same subject in an equalized manner and thus show the rather clear and by no means obscure reason for the translation of the aphelia in the moderated action of the two forces.

*The reason for the motion of the aphelia.*

Let it be that this force to direct the axis towards the sun yields a little to the retaining force, in proportion of their strengths. In the semicircle of the aphelion like in C, the tip of the needle is inclined a little towards H, that means clockwise, the tail points away from the sun, winning somewhat over the retaining force. Therefore the aphelion will become retrograde. But in the
semicircle of the perihelion, like in F, the same tip will be inclined towards G, i.e. anticlockwise, again winning over the retaining force in the contrary direction. So the aphelion will become rectograde and fast. Because AF is shorter than AC, and the sun is closer to F than to C, also the force directing the magnetic axis towards the sun will be stronger in F than in C.

*Why the aphelia do not recess.*

The anticlockwise inclination at perihelion will therefore not only compensate the clockwise inclination at aphelion but will surpass it. So this is the reason why the apsides progress and do not recess. The aphelion which we found will therefore be equally valid in the true anomalies of 90° and 270°, where the axis of force is directed towards the sun, which is its natural direction. The motion of the aphelia will be in the form of a spiral, like the motion of the precession of the equinoxes, but for a different reason, as will be shown below in chapter 68. The parallel direction of the magnetic axis or the corresponding force, its guard, will not respect these or those stars, but only the location of the planets body, at any time. And, if you think it over, simply because this direction is more similar to quietness than to motion, in matter, and is with more right attributed to some disposition of the body than to some mind.

*It shall be agreed that a correctly positioned magnet will perform these librations.*

Let us then follow this similarity of a planets libration with the motion of a magnet more closely and this with a very fine geometric proof, to make it clear that magnets have the same motion as we learned for planets. Let DFA (Fig.9)

![Figure 9](astronomia_nova.jpg)

Figure 9: Illustration of how a planet with magnetic fibers in the direction AD is attracted or repelled by the sun located in the direction K (*from Astronomia Nova*).

be a round magnet or the body itself of Mars, DA the line in which the magnetic force acts, D the pole which aims at the sun, A the pole which flees it. First note, that in this speculation it amounts to the same if we consider the entire magnetic body or only one physical line of force, parallel to DA.
Hidden signs you offer, magnet, to the sailors
What wonder that erratics follow now your traces?

Since now this magnetic force is corporal, and dividable with the body, as proven by the Englishman Gilbert, B. Porta and others, certainly, because a globe consists of an infinite number of physical lines parallel to DA, whose force tends in one and the same world direction, about the individual lines separately the same judgement as to their way of motion will be made as for all of them joined together, and vice versa. Let therefore instead of the whole body and all its filaments the middle axis DA be proposed for argumentation. Let DA be bisected in B, and FBI drawn perpendicular to DA. If the planet is located such that BI points to the center of the sun, there will be no approach. The angles DBI and ABI are equal, and therefore of the same strength, this for approach, that for recession. This is the same as equilibrium in mechanics. So B, the center of Mars, will remain in one of the apsides, say the aphelion, farthest from the sun. Let us take some arc IC as a measure of the true anomaly and draw BC which points towards K. Let the planet be located in such a way that BC points towards the sun which is understood by K. First the measure of the planet’s strength of access shall be inquired. There will be approach, because the pole D which aims at the sun, is inclined towards the sun in K by the angle DBK. The pole A however flees the sun by the angle ABK. Because the strength of the angle is natural, it will be in the ratio of the balance. The ratio between DP and PA, where CP is the perpendicular to DA through C, will be the ratio of the balance. A balance suspended along the line BK with the arms remaining at the angle DBK will have the weights of the arm BD and the arm BA in the ratio DP to PA; in the same way as if he arms would be suspended in P and the weight of BA would correspond to PD, and the weight of BD to PA, then the line DA would be perpendicular to the balance line CP. See my "Optics" and do not trust experiments which are not done carefully. So, as DP to DA is the ratio of the strength of the angle ABC to that of the angle DBC. The force of recession is therefore measured by DP, that of access by PA. Subtract from PA the amount DP which gives AS. Ergo, SP is the measure of the force of access to the sun, corrected for the effect of recession, and this in proportion to AD as the maximum force. But as half of this, DB, measures the maximum force, also PB, half of PS, that means the sine CN of the true anomaly CBI, measures the pure force of access at this position of planet and sun. So the sine of the true anomaly is the measure of the strength of access of the planet to the sun at this location. And this is the measure of the increment of the force.

What is the measure of the space covered by the libration up to a given moment?

The measure of the space covered by these continuous increments of the force is very much different. The observations show that IH, the sinus versus of the arc GI, is the measure of the total oscillation, if GI is the eccentric anomaly corresponding to the true anomaly IC. This could have been also deduced from the previously indicated measure of the velocity CN, but now we brought the experience into agreement with the idea of the balance. Since the sine is the measure of the strength of every angle, the sum of sines will nearly equal the sum of the strengths or impressions of all equal parts of the circle, the common effect of which is the total
achieved libration. And the sum of sines IG (let the otherwise different anomalies IC and IG be equal to avoid confusion) is approximately in the same proportion to the sum of sines of the quadrant as IH, the sinus versus of this arc IG to IB, the sinus versus of the quadrant. I said approximately. Namely in the beginning, where the sinus versus is small and has small increments, it is only half the size of the sum of sines. Look here. Let the quadrant of 90° have 90 parts; the sum of 90 sines is 5,789,431. Already long ago I added them in order.

**What is the proportion of the sinus versus of any arc to the sum of sines of all previous degrees?**
The sum of the sines in the arc 1°, i.e. the first sine, is 1,745, and in the proportion to the previous sum, as 100,000 to 30. In contrast to this, the sinus versus of 1° is 15, which is half of 30.

The ratio is practically constant, with negligible error.
The reader should not be deterred by this unmathematical and wrong start. Before the amount of libration becomes significant, the two calculations have negligible difference. The sum of 15 sines which is 208,166 indicates 3,594. But the sinus versus of 15° is 3,407/100,000, a little less than the previous number. So the sum of 30 sines, which is 792,598 indicates by the law of proportions a libration of 13,691 in 100,000, but the sinus versus of 30° gives 13,397. And the sum of 60 sines, which is 2,908,017 indicates a little more than 50,000, while the sinus versus of 60° is 50,000.

The application of the demonstrated magnetic libration to the observed libration of a planet.
Since it is now demonstrated, if a magnet is positioned as we assume the planets to be positioned with respect to the sun, that its libration is measured by the sinus versus in what concerns the covered space. The planets are observed to librate by the same amount of the sinus versus of the eccentric anomaly. It is therefore very plausible that the planets are magnetic and dispositioned to the sun as we said.

The relation between the various sinus versi of the eccentric anomaly is the same as the relation between the sum of the sines of the corresponding true anomalies, very accurately.

Let us demonstrate now that I made no big mistake in assuming the arcs IC and IG to be identical. When I say, the arc IC is for the planet the measure of the true anomaly, I speak correctly, and then CN is the genuine measure of the strength which acts on the planet when the sun is on the line BK. If I say, however, IG is the measure of the eccentric anomaly, I speak incorrectly, because I use the circle of the planet to represent the eccenter. But since in the descendent semicircle of the eccenter a larger arc of the eccentric anomaly corresponds to a smaller arc of the true anomaly, i.e. IG corresponds to IC, we collect more sines in IG than in IC, and rightly so.

The later a planet is in any arc, the smaller the portions of the true anomaly have to be made, in order that their collected sines become the correct measure of the force emanated at this true anomaly.

Namely because the sine measures the strength, and the strength acts according to time and according to the distance to the sun (the closer the magnets the stronger they are), that means, to be brief, according to the arc IG, as many sines have to be included in IC as are found in IG.

Only in that respect we made an error, that we took these many sines larger than they are,
as GH is larger than CN. But this excess is in the first place by itself extremely small and
unnoticeable. Namely at the beginning of the quadrant the arcs IC and IG hardly differ, and
the sines are small, at the end of the quadrant, where the equation of the eccentric CG is at its
maximum, the sines differ very little.
Finally this error is to our advantage. The sum of the sines is always a little larger than the
sinus versus, to which we try to accomodate and reconcile the amounts of libratory and
magnetic deflections. Ergo, this our present error to accumulate larger instead of smaller sines,
is covered if we use the sinus versus instead of the sum of the sines. The sum of the sines is
not exactly the same as the sinus versus, but exceeds it, as far as the librations are concerned.
The defect in the proportion of the sinus versus and the sum of the sines is compensated by the
contrary error that we collected the larger sines of the eccentric anomaly instead of those of the
true anomaly.
So we brought the matter with very good reasoning to an end within the limits of sensitivity.
In conclusion, a planet, like a magnet, accesses and flees the sun by the law of balance, along an
imaginary diameter of the epicycle which tends towards the sun and the real and force diameter
DA of the planet points to mean longitudes, namely BD at our time to 29° tauri, BA to 29°
scorpii, since the aphelion is in 29° leonis.
The magnetic force in the planets is excited and activated by a similar force in the sun.
In this way the access by libration is achieved not by the action of a mind, but due to an
inherent and solitary magnetic force whose definition however depends on the foreign body of
the sun. The force by definition aims at the sun or flees from the sun. Although this connecting
force between magnets has to be mutual, I denied above in chapter 39 the force of the sun to
the planets to be attractive, or only to the extent that is required by the argument used. Here,
however, the force is assumed to be simultanouly attractive and, at another site, repulsive.
Also this is assumed, that the sun, like the virgin iron, is only aimed at but does not aim itself,
since its filaments were above supposed to be circular, those of the planets however, are here
taken to be straight.
The difficulty and imperfection of this example of the magnet
It is sufficient for me to have demonstrated the possibility of the matter in principle, by this
example of a magnet. Incidentally, about the matter itself I am in doubt. For what concerns
the earth, it is sure that its axis, by the constant direction of which the seasons are created, is
not suited for this libration and for the definition of the aphelion. Because the apogee of the
sun or the aphelion of the earth nowadays nearly coincides with the solstitials, and not with the
equinoxes, what would suit us. Nor does this axis remain in the same distance to the cardinal
points. But if this axis is unsuited, none in the whole body of the earth is suited, because there
is no line which remains at the same place, when the whole body turns around the previous
axis in its dayly untiring revolution.
About the mental principle of this libration. I hesitate to call it rational to avoid confusion with
a rational discourse.
But in fact, if no planely material or magnetic faculty can achieve the task given to each planet
separately, due to the lack of the proper means, that is to say a diameter in the body which
remains to itself always equidistant during translations, which is a defect that already appears
in one of them, namely the earth, then we have to call for a mind, which, as was said in chapter 39, gets to know the distances covered by looking at the increasing diameter of the sun; this mind is to direct the faculty of the planet, be it natural or animated, to accommodate the planet in a parallel site, such that it is driven by the force of the sun in the required way and librates with respect to the sun (Namely a mind alone without a faculty of inferior grade can by itself do nothing in a body). At the same time the mind is advised to make the times of periodic restitution and libration not just equal and so to transfer the apsides. The likelihood of these things was explained in chapter 39.

It remains, since we know already from observations the laws and size of this libration, by which the apparent diameter of the sun varies, — laws which we did not yet know in chapter 39 — that we find out if these laws are such that it is likely, that the planet can get to know them. The laws of the libration are that the sinus versus of the eccentric anomaly measures the part of the libration which has been completed.

The increases of the sun diameter are proportional to the sinus versus of the true anomaly. I state therefore in the beginning: given that, in agreement with observations, a planet after equal arcs in the eccentric is found in the points \( \gamma, \kappa, \mu, \zeta \) and not in \( \gamma, \iota, \lambda, \zeta \), then the correct measure of the increase of the sun diameter is the sinus versus of the true anomaly \( \sin \vartheta \); we also know that the sinus versus of the eccentric anomaly measures the libration.

Since therefore the mind of a planet, if there is one, realizes the space covered by the libration only by the increased diameter of the sun, as said in chapter 39, it will necessarily have to get knowledge of the sinus versus of the true anomaly, in order to increase accordingly the sun diameter.

The proof of this is here. Let the planet after equal arcs CD, DE, EF of the imperfect eccentric be in \( \gamma, \kappa, \mu, \zeta \). The point D and H shall be connected; the diameter CF is intersected in I. Since \( \delta \kappa \theta, \epsilon \mu \eta \) are straight lines, they intersect the epicycle in similar arcs as in the eccentric, by construction. So \( \gamma \zeta \) to \( \gamma \kappa \) will be in the same proportion as CF to CI; one is the measure for the other.

Because this is so, I claim, that consequently the increments of the diameter of the sun in \( \alpha \) as seen from \( \gamma, \kappa, \mu, \zeta \) will be accumulated by the same amount in which the sinus versus of the true anomaly increases. To prove this in general would be not appropriate. It can however easily seen to be true in general, if we prove it for the center and for the extremes.

In C the true anomaly is zero, the sinus versus is zero, and the sun, seen from \( \gamma \), appears smallest, such that also its increment is zero. In F the true anomaly is 180 \(^0\), the sinus versus equals the full diameter 200 000. And the sun, seen from \( \gamma \), appears largest, so its increment will have reached the total.

As to the true anomaly of 90 \(^0\): the perpendicular to CF in A intersects the eccentric in M; the tangent from \( \alpha \) to the epicycle contacts the epicycle in \( \nu \) (Fig.10).  

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8The sinus versus of the eccentric anomaly measures the libration of a planet.
The sinus versus of the true anomaly measures the increase of the sun diameter as seen by an observer in the planet.
Figure 10: Drawing by which Kepler proofs that the diameter $\alpha\xi$ of the sun appears midway between extremes, if the distance of the planet from the sun is $\alpha\alpha$, or else if the planet is located in position $\nu$ of the epicycle, where $\nu$ is the point in which the tangent from $\alpha$ touches the epicycle (from *Astronomia Nova*).

Because $\alpha\nu\beta$ is a right angle according to Euklid III.22 and MAB is a right angle by construction, and $\beta\nu$ and BA are equal by construction, and so $\beta\alpha$ and BM, the triangles are equal equal and congruent, therefore also the angles $\nu\beta\alpha$ and ABM are equal. The perpendicular to $\gamma\zeta$ from $\nu$ intersects $\gamma\zeta$ in $o$. Therefore, since $\nu\nu\beta$ is a right angle and also MAB, $\nu\beta\nu$ is equal to MBA and the triangles are similar, so $\nu\beta$ to $\beta\nu$ is as MB to BA and vice versa. Because $\nu\beta$, $\beta\gamma$, $\beta\zeta$ are equal and also MB, BC, BF, the joint of $\nu\beta$ and $\beta\nu$, i.e. $\gamma\nu$, relates to $\nu\zeta$ as the joint of MB and BA, i.e. CA, to AF. Since CA is the *sinus versus* of the eccentric anomaly CBM, and is understood to measure the corresponding part of the libration, $\gamma\nu$ will be this part. Ergo at the eccentric anomaly CBM, or the true anomaly $90^0$, the planet will be in $o$.

But the *sinus versus* of the true anomaly $90^0$, i.e. CAM, is half of the total diameter, i.e. 100 000. I claim that also the size of the diameter of the sun, seen from $o$, will be half way between the sizes seen from $\gamma$ and from $\zeta$, such that half of the increase is reached, when the planet is in $o$ below $\beta$.

Let the diameter of the sun be $\alpha\xi$, and the angles of vision $\xi\zeta\alpha$, $\xi\alpha\alpha$, $\xi\gamma\alpha$. Now AF and $\zeta\alpha$ are equal, also AC and $\alpha\gamma$. CA to AF is in the same relation as $\gamma\nu$ to $\alpha\zeta$. Ergo $\gamma\alpha$ to $\alpha\zeta$ is as $\gamma\nu$ to $\alpha\zeta$. But $\gamma\xi$ is unnoticeably different from $\gamma\alpha$ and so is $\gamma\zeta$ from $\xi\alpha$. Ergo $\gamma\xi$ to $\zeta\xi$ is practically as $\gamma\nu$ to $\alpha\zeta$. In the triangle $\gamma\xi\zeta$, the angle at $\xi$ is divided by the line $\xi\nu$ such that the basis $\gamma\zeta$ is divided in proportion of the sides $\gamma\xi$, $\zeta\xi$. Ergo by the inversion of Euklid VI.3, the angle $\gamma\zeta\xi$ is divided by the line $\xi\nu$ in two equal parts, and $\gamma\xi\nu$ is half of $\gamma\xi\zeta$, the full increment of the sun diameter, quod erat demonstrandum. So for the extremes and
the center it is demonstrated, that, if the diameter of the libration is divided by the planet in proportion of the $\sinus$ of the eccentric anomaly, then the diameter of the sun will increase in proportion to the $\sinus$ of the true anomaly.

This is apparent also from the following argument which is included for further evidence: Let the perpendicular to CF in B be BL, where L is the intersect with a circle around A with radius BC. Since CBL, the eccentric anomaly, is 90°, the $\sinus$ of it will be 100 000, half of the full diameter, therefore the libration $\gamma\beta$ will be half of the total $\gamma\zeta$, and the distance will be $\beta\alpha$. This is equal to AL, by construction, so the planet will be in L. And since AL equals BC or BM, BA is the common side and LBA like MAB are right angles, the triangles BMA and ALB are congruent. So also BL equals AM. But AM is the same as $\alpha\nu$, as above, and so is BL. But $\alpha\nu$, the hypothenuse of the triangle $\alpha\omega\nu$ is larger than $\alpha\omega$, ergo BL is larger than $\alpha\omega$, and AL larger than BL, therefore AL is much larger than $\alpha\omega$. The sun therefore appears smaller in the distance AL than in the distance $\alpha\omega$. The distance $\alpha\omega$ was already seen to be midway between the maximum and minimum, therefore in the distance AL the sun appears smaller than on average.

Even if in L half of the semicircle is absolved, less than half of the increment of the sun diameter is completed. Also because the true anomaly LAC is less than half, 90°. And this is what tormented us in chapter 39, as said in the preceding chapter 56. Namely if the planets orbit would be a perfect circle, the increase of the sun diameter would measure the increases of the $\sinus$ of the eccentric anomaly, the observation of which is more foreign to the mind than the observation of the true anomaly, as we heard already. See her from the contrary how convenient this measure is to the planet and how plausible.

The planet cannot have knowledge of the eccentric anomaly

If we would declare the measure of this libration, namely the $\sinus$ of the eccentric anomaly, as recommended by the observations, to be comprehensible to the planet, we would take away from its mind the means of the variable sun diameter, because this diameter does not accomodate itself to the $\sinus$ of the eccentric anomaly. The planets orbit is namely not a circle. And the mind of the planet would have to find out the parts of libration, or the spaces to be covered, without sign, what we already called absurd; he would also have to find the eccentric anomaly, that is the angle between two straight lines from the center of the eccentric, one through the aphelion, the other through the center of the planet. In Fig.8 it is the angle DBC (or the complement of KDB, where K is a straight line from D parallel to BC). If the mind perceives the angle KDB, it is necessary that it perceives the three points K, D, B. For D there is no doubt, because D is the center of its own sphere. About K, I have little doubt. BC and DK practically coincide due to the infinite distance of fixed stars into the same location among the stars, and the stars are real bodies.

This reasoning was not necessary in the natural mode discussed a little before. Therefore there is nothing wrong with the planet looking by some unknown sense to that fixed star which acts at the time as a host for the aphelion. Only for B it has to be denied that it can be sensed by the planet, because it is not endowed with a body.

Besides, if the reason disappears, why B should be inspected, also the effect goes away. But B has to be inspected, if a circle should be accomplished. The planetary orbits, however, are
not perfect circles, what was proven in chapter 42 from observations. Ergo the planets do not focus on B. And when B is quasi the center, it is later than the path CD. But if it is to be inspected by the planet, it has to be earlier.

For these reasons I deny, that the *sinus versus* of the eccentric anomaly gives to the planet the measure of its libration, not because it is not the measure, but because, although it is the measure, it cannot be observed by the planet. But if we give to the planet the increasing and decreasing diameter of the sun as measure or a support by which it finds the correct and by themselves imperceptible distances during the libration, and state the true anomaly, in Fig.8 the angle DAC or rather KDA, to be the rule and measure which the planet should observe in order to vary this sun diameter, according to the proof given a moment ago, then we are on the right track.

The planet can know the true anomaly.

Namely both signs are perceptible: as far as the part of the libration is concerned, the growing and decreasing diameter of the sun, for the measurement of the angle three points endowed with bodies. Namely in A is the sun itself, In D the planet and in K the stars near the aphelion.

Perhaps we have to say (what we considered already in chapter 39 above, in case the forces of nature be not sufficient to govern the celestial motions) that the planets have to be given a sense for the light of the stars and the sun such that they can estimate this angle of true anomaly from the rays which concur in the center of their bodies.

If a planet has a sense for the true anomaly, it does not estimate the angle, but the sine of the angle.

Only one difficulty has to be removed. Why is not the angle itself a measure for the task of the planet which is to increase the sun diameter on approach, but instead of the angle its *sinus versus*? And by which means does the planet perceive the sine of the true anomaly? Does it proceed like humans by geometric reasoning? So far, however, in administrating celestial motions, no task had to be given to planet which could not be dealt with by a divine instinct, prevailing since the very begin of the world up to the present day, beyond any calculational effort. Let us repeat, as we just said, that the sine of the true anomaly indicates the strength of the angle KDA, of which Aristoteles talks in his mechanics, and also we, in this same chapter shortly before. Namely two arms at an obtuse angle are easier to direct than when at right angles, and this in proportion to the sines. And conversely, two arms joined under an acute angle are easier to force to a straight line, where their ends join, than when at a right angle. Repeat the proof with the premisses given above.

So in one way there is nothing absurd if we say (in our human way of understanding) given that planets have a sense for the strength of an angle, that they get to know the sines of the angles. But why does the sine measure the natural strength of an angle? No wonder we are back to the principles of nature. Let there be, as previously, certain fibers in the planet, in which a magnetic force sits along a line which points to the sun. Let also that be attributed, not to the nature of the body, as before, but to an animated faculty or one which reigns over it.

9Shortly before the sine of the eccentric anomaly (or the corresponding true anomaly) indicated the strength of libration, but the *sinus versus* of the eccentric anomaly indicated the achieved libration; here the sine of the true anomaly indicates the velocity at which the sun diameter increases, the *sinus versus* of the true anomaly however indicates the total achieved increase by all preceding velocities.
from inside, that it directs this magnetic axis always to the same stars while it is seized by the sun, except that in the course of centuries the axis is slightly deviated. So here arises a conflict between the animated and the natural faculties, and the animated one wins. Nothing else was said in chapter 34, when we argued that the planets naturally like to rest but are moved by the external force of the sun.

Or take a more suitable example. The natural weight of the human arms points downwards to the center of the earth. The animated faculty of a banner carrier, however, enables him to erect the banner over his head and to swing it around. Here the animated faculty wins over the natural weight and it would win forever if the man’s body with all its faculties would not be mortal.

With these assumptions a planet will be able to understand and perceive the strengths of angles due to a struggle between the animated faculty which is prepared to retain the magnetic axis and the magnetic force in direction of the sun.

This view seems also to be confirmed by the example of the moon, which is certain to be accelerated when in line with sun and earth, maybe by this strength of angles.

Finally, the conclusion will be this:

The type of celestial motions, if supported by a mind.

A planet in the aphelion is not inclined towards the sun but is moved forward according to the distance AC; following this promotion it gets to the angle KDA. In proportion to the strength of this angle the planet itself increases the diameter of the sun, approaching the sun. By this approach it diminishes the distance, which becomes AD. Due to the minor distance the planet moves faster. Faster also changes the angle KDA, and faster the planet increases the semidiameter of the sun (ceteris paribus). In this way a perennial circulation is achieved, not by intervals, as we assumed in our reasoning and in the calculations, but completely continuously.

Comparison of the mental and the magnetic principle.

I said this so far under the condition that, if necessary, we have to resort to a mind, if the libration about which we know from observations cannot be achieved by some magnetic force in the planets. Incidentally, if one wants to compare this natural and that mental motion, the former stands per se and needs nothing, the latter, however it is endowed with an animated faculty to move the body, seems to confirm the former and to require its help. Namely in the first place a mind by itself cannot do anything in a body. It is necessary to add a faculty to execute its projects in the librating planet. This faculty will be either animated or natural and magnetic. Animated it cannot be. Namely an animated faculty cannot transport its body from place to place (as required in this libration) without the help of another body. So the faculty will be magnetic, i.e. a natural consense between the bodies of the sun and the planet. So the mind calls to nature and to magnets for help. Thus the mind, when half the way through with its work which consists in increasing or decreasing the diameter of the sun according to the size of the true anomaly, has completed in the upper half the longer part $\gamma_0$ of the libration and in the lower half the smaller part $\alpha_\zeta$. Neither $\gamma_0$ nor $\alpha_\zeta$ correspond to the time spent. For $\gamma_0$ more time is needed than what its excess over $\alpha_\zeta$ would require. Also the time intervals do not continuously increase from $\zeta$ towards $\gamma$; for $\gamma\kappa$ less time is needed than for $\mu\zeta$. But the works of a mind use to be constant.
Therefore we had to endow the mind with an animated and a magnetic faculty and to install a struggle between them to remind him of his duty, what neither the equality of the time spent or that of the space covered could have done. So again we asked for help from nature.

On the other side, all these modifications are in fact inherent to the work of the external magnetic force of the sun in conjunction with a magnetic force in the planet, as explained above. If these magnetic forces do all the work by themselves, what is the direction of a mind needed for?

Also, if we were uncertain about a magnetic force inside the planets, contemplating the axis of the earth which is not on the suns line of apsides, this difficulty is in fact common to both explanations. Namely, assuming a mind, we are nevertheless forced to admit an axis such as we want it for the earth, by means of which the mind learns about the strength of the angle, or its $\sinus$ $versus$. Against that idea we are vehemently urged by probability to ascribe this libration of the planets, which follows beyond any doubt the laws of nature, to ascribe it totally to nature, however it may manifest itself in the planets.

And this so much that I do not know if I have made this sensual comprehension of the sun and the stars which I reluctantly accept myself and attribute it to a mind in the planet, if I made it sufficiently plausible to a philosophically minded reader.

In addition, there seems to be even in the methods which we attributed to a mind — the most reliable ones of all — a certain geometric uncertainty of which I do not know if it is not repudiated by God himself, who so far was always found to proceed in a mathematically demonstrable way. Namely, if a planet in order to approach the sun, partly by an inherent force, comes degree by degree of the solar force closer to the sun (as it does) and if these different degrees reciprocally strengthen its own force of approach by increasing the angle which is assumed to determine the law of approach, or else the sun diameter, the degree of approach will be given to some extent by the approach itself, and will be, in the intention of the planet, simultaneously beforehand and afterwards. Since the approach is by unequal pieces, it will need a measure. The result will then be given not by a direct calculation, but quasi by the regula falsi, since both forces influence each other and develop themselves at the same time in the same revolution of the bodies.

Here we may stop with Kepler’s text. The remainder of chapter 57 is devoted to ideas on precession.

Since the last paragraph really deals with the question that the integration of differential equations may be complicated, in words which are written decades before the invention of the calculus, the text will be repeated here in the original Latin version for the benefit of readers who will appreciate. It is also a fine example of Kepler’s literary style.
Accedit et hoc, quod in ipsis etiam modis, quos Menti praescr ipsimus, omnium, qui possunt esse, probatissimis, implicari videtur quaedam incertitudo Geometrica; quae nescio an non a Deo ipso repudietur, qui hactenus semper demonstrativa via progressus esse deprehenditur. Nam si Planeta prout ad Solem, partim vi insita approinquavert, in alium et alium gradum virtutis ex Sole adventitiae venit (ut quidem venit) et si diversi gradus, reciproce ipsius etiam Planetae vim appropinquandi intendunt, dum angulum augent, qui Regula ponitur appropinquationis, seu auctionis diametri Solis: Nisus Planetae proprius, denique sibi ipsi fiet ex parte mensura, et in intentione Planetae, simul prius et posterius; cum sit per partes inaequales, et ob hoc ipsum mensura indiguerit. Quo pacto non demonstrativa, sed quasi per regulam falsi, dabitur exploratio temperandarum virium utriusque virtutis, ut eodem tempore sese expediant, eodem corporis circumactu.

8. A concluding comment.

Perhaps it is not totally inappropriate to add here a final remark. It looks like it might be possible to set up differential equations of motion without knowledge of the forces. This, however, would be a wrong conclusion. What Kepler was missing is the connection between force and acceleration. Acceleration is a quantity not immediately obvious for an observer. Its relation to force could only be established by quantitative measurements — more specifically by the measurements of Galilei. In the generalization of Galilei’s results a new difficulty appears, or rather becomes manifest: acceleration, and also velocity, are quantities which cannot even be precisely defined without the transition to the infinitely small. This difficulty was removed by Newton.

From a solution of Kepler’s differential equations it is, of course, possible to calculate the acceleration at each point. The acceleration will be found to be always directed towards the sun and to be proportional to $1/r^2$. This is the most elegant way to derive the law of gravitation. It was done so by Newton. Whether Newton considered first the easier derivation from Kepler’s third law, is irrelevant. Given the connection between force and acceleration, things are so intimately connected, that one method is inconceivable without the other.

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