Computation of maximal reachability submodules

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Abstract

A new and conceptually simple procedure is derived for the computation of the maximal reachability submodule of a given submodule of the state space of a linear discrete time system over a Noetherian ring $R$. The procedure is effective if $R$ is effective and if kernels and intersections can be computed. The procedure is compared with a rather different procedure by Assane et al. published recently.

1 Introduction

Let $A \in R^{n \times n}$, $B \in R^{n \times m}$ where for the moment $R$ is just a commutative ring. As usual, we associate to the pair $(A, B)$ the linear discrete time control processes

$$x_0, \quad x_1 = Ax_0 + Bu_0, \quad \cdots, \quad x_{k+1} = Ax_k + Bu_k, \quad \cdots \quad (1)$$

with states $x_k \in R^n$, inputs $u_k \in R^m$ and $k \in \mathbb{N}$.

A submodule $U$ of $R^n$ is called $(A, B)$-invariant if $AU \subseteq U + \text{im } B$. An $(A, B)$ invariant submodule $U$ is called reachable or reachability submodule if every state in $U$ can be reached from zero within $U$. The latter means:

$$\forall x \in U \ \exists r \in \mathbb{N}, \ u_0, \ldots, u_{r-1} \in R^m : \quad x_1 = Bu_0, \ldots, \ x_r = A^{r-1}Bu_0 + \ldots + Bu_{r-1} \in U \quad \text{and} \quad x_r = x.$$
It was shown (see e.g. [Hi, Theorem 2.15]) that this rather natural definition is equivalent to the definition of pre-controllability submodules in [CoPe] which is still more commonly known but less intuitive from a control point of view.

The zero-module is trivially \((A, B)\)-invariant and reachable. From the definitions it is clear that sums of \((A, B)\)-invariant or reachable submodules, respectively, are again \((A, B)\)-invariant or reachable. These facts imply that any submodule \(M\) of \(R^n\) contains a unique maximal \((A, B)\)-invariant submodule \(M^*\) and a unique maximal reachability submodule \(M^*_0\), where always \(M^*_0 \subseteq M^*\).

Maximal reachability submodules play an important role in the solutions to classical control problems such as disturbance decoupling. See [CoPe] and [AsPe] to give only two examples. It is therefore of practical importance to have methods at hand for the computation of generating systems of such modules. In [AsLaPe] for the first time a finite procedure was given for principal ideal domains and then strongly modified in [AsLaPe2] to work for Noetherian rings. The latter works as follows:

\(R\) is now supposed to be Noetherian.

**First step (precalculation):** \(S_0 := \text{im}\ B\)
and for \(k \geq 1\): \(S_k := \text{im}\ B + A(S_{k-1} \cap M)\).

This ascending sequence of modules stabilizes after finitely many steps and gives a submodule \(M_s\) which contains the image of \(B\). If \(M\) is represented as the kernel of some matrix \(C \in R^{n \times p}\), then \(M_s\) appears as the 'minimal \((C, A)\)-invariant submodule' containing the image of \(B\), see e.g. [AsLaPe2].

**Second step and main procedure:** \(W_0 := M_s \cap M \cap A^{-1}(\text{Im} B)\)
and for \(k \geq 1\): \(W_k := M_s \cap M \cap A^{-1}(W_{k-1} + \text{Im} B)\).

Once more, this gives an ascending sequence and an interesting proof in [AsLaPe2] shows that its limit is actually \(M^*_0\).

Of course - and the same is valid for the new procedure to be developed in this note - such a procedure can be realized in a concrete computation only if the ring \(R\) and all the occurring operations like \("A^{-1}\", \("\cap\") are effective in the sense of [CoCuSt, p.1].
2 New procedure via finite \((A, B)\)-cyclic submodules

Based on results from [BrSch] a quite different and conceptually simpler approach is possible. A submodule \(U\) of \(R^n\) is called \((A, B)\)-cyclic if for some \(u_k \in R^m\) and \(x_k\) from (1) with \(x_0 = 0\) one has

\[ U = \langle x_k : k \geq 0 \rangle. \]  

(2)

Thus an \((A, B)\)-cyclic submodule can be generated by the states of one single control process which begins with the zero-state.

It is shown in [BrSch] that \((A, B)\)-cyclic submodules are reachability submodules and that finitely generated reachability submodules are even finite \((A, B)\)-cyclic. The latter means that in addition to (2) one has \(x_k = 0\) for \(k > d\) and some \(d \in \mathbb{N}\).

The point is now that finite \((A, B)\)-cyclic submodules can be determined via the kernel of \([yE - A, -B]\) in \(R[y]^{n+m}\). If for \(f \in R[y]^n\), \(g \in R[y]^m\) one has \((yE - A)f =Bg\), then the coefficient vectors of \(f\) generate a finite \((A, B)\)-cyclic submodule and every finite \((A, B)\)-cyclic submodule \(U = \langle x_1, \ldots, x_d, 0, \ldots \rangle\) leads to a kernel element \([f \ g]\) with \(f = x_1 y^{d-1} + \ldots + x_d\) and \(g = u_0 y^d + \ldots + u_d\). Note that \(x_{d+1} = A_d x_d + B u_d = 0\). More details can be found in [BrSch].

For any \(f = x_1 y^{d-1} + \ldots + x_d \in R[y]^n\) let \(U_f := \langle x_1, \ldots, x_d \rangle\). Of course, \(U_f\) is contained in a given submodule \(M\) if and only if the coefficient vectors of \(f\) are from \(M\). Let \(\pi\) be the projection of \(R[y]^{n+m} = R[y]^n \oplus R[y]^m\) onto the first \(n\) components and let

\[ M := \text{Ker} [yE - A, -B] \cap (M[y] \times R[y]^m). \]  

(3)

Here \(M[y]\) is the submodule of \(R[y]^n\) of those polynomial vectors which have all their coefficient vectors from \(M\).

One arrives now at the following results:

**Observation.** (i) For every \(h \in M\) the submodule \(U_{\pi(h)}\) is a reachability submodule of \(M\) (true for any \(R\)).

(ii) Let \(R\) be Noetherian. For every reachability submodule \(U\) of \(M\) there is \(h \in M\) such that \(U = U_{\pi(h)}\).
Proposition. Let \( h_1, \ldots, h_s \) generate \( M \) as an \( R[y] \)-module, then the family of coefficient vectors of \( \pi(h_1), \ldots, \pi(h_s) \) generates \( M^*_0 \).

Proof of Observation. (i): By construction \( U_{\pi(h)} \) is finite \((A, B)\)-cyclic and thus by Proposition 1.5 in [BrSch] a reachability submodule.

(ii): Since \( R \) is Noetherian, \( U \) is finitely generated and reachable. By Proposition 1.7 in [BrSch] this implies that \( U \) is finite \((A, B)\)-cyclic. The foregoing discussion shows how to construct the desired \( h \in M \).

Proof of Proposition. Let \( f_1 = \pi(h_1), \ldots, f_s = \pi(h_s) \) and \( \tilde{M} = \sum_{i=1}^{s} U_{f_i} \).

We have to show \( \tilde{M} = M^*_0 \). \( M^*_0 \) is the sum of all reachability submodules of \( M \). Since \( R \) is Noetherian, all reachability submodules \( U \) of \( M \) are finitely generated. By part (ii) of the Observation such modules \( U \) can be represented as \( U = U_{\pi(h)} \) with some \( h \in M \). Since \( h = r_1 h_1 + \ldots + r_s h_s \) with some \( r_1, \ldots, r_s \in R[y] \), we obtain \( U \subseteq \tilde{M} \) for an arbitrary reachability submodule \( U \) of \( M \) and thus \( M^*_0 \subseteq \tilde{M} \).

The converse inclusion comes from the fact that by part (i) of the Observation \( U_{f_i} \) is a reachability submodule of \( M \) and therefore contained in \( M^*_0 \) for \( 1 \leq i \leq s \). The latter implies: \( \tilde{M} \subseteq M^*_0 \).

One main advantage of the approach via (3) is that one can (for appropriate rings \( R \)) compute the kernel of \([yE - A, -B]\) once for all independently of \( M \). This gives us a first result a module which is of use not only for determining \( M^*_0 \), see e.g. [BrSch]. In order to determine \( M^*_0 \) for some specific \( M \) it remains to calculate an intersection of two modules and after that one merely truncates the results and extracts the coefficient vectors.. Explicit calculation is - of course - only possible over an effective Noetherian ring with an effective method to determine the kernel and intersection in (3). Examples of such rings are \( \mathbb{Z}, \mathbb{Q}[t_1, \ldots, t_n], \mathbb{F}[t_1, t_n] \) where \( \mathbb{F} \) is a finite field. The determination of \( \text{Ker} [yE - A, -B] \) can then be done with the help of Gröbner basis calculations as indicated in [BrSch]. A standard technique also via Gröbner bases for the computation of the intersections of modules is (e.g.) described in [AdLou]. In both cases any generating system would do as well. Several current software packages for symbolic computation can be utilized to perform explicit calculations.

A sound comparison of the different procedures for the computation of maximal reachability submodules requires a detailed investigation of their complexities. This remains as a future task.
The following two examples are over $\mathbb{Q}[t]$ and $\mathbb{Q}[t,w]$. Computations have been done combining the well-known packages Macaulay2 and MapleV Release 5.1.

**Examples**

(A) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -t \\ t & 0 \\ 0 & t \end{bmatrix}$ and $M = \text{im} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ as in Example 1 of [AsLaPe2].

To determine $M^*_0$ we first obtain

$$\text{Ker} [yE - A, -B] = \text{im} \begin{bmatrix} t & -t - y \\ -t & -ty \\ -t & 0 \\ t & -y^2 \\ -y & 0 \end{bmatrix}$$

This leads to $M = hR[y]$ with $h = t[t, -t, -t, t, -y]$, which in turn leads to with $f = \pi(h) = t[t, -t, -t]$. There is only one coefficient vector to be extracted from $f$ (viewed as a polynomial vector in the variable $y$). Therefore the final result is: $M^*_0 = fR$. By [AsLaPe2] we know $M^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix} R$ and thus $M^*_0 \subseteq M^*$.

This example is interesting also since here the classical Wonham-algorithm to determine $M^*$ does not converge and up to now no general finite procedure is known. For principal ideal domains, however, a procedure has been developed in [AsLaLoPe].

(B) In the second example we start with matrices from [AsLaPe2], Example 4.3, where a system with two incommensurable delays is investigated.

Let

$$A = \begin{bmatrix} w^4 & t & 0 \\ x^3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} t & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } M = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & w \\ 0 & 1 \end{bmatrix}.$$

Here Macaulay2 computes

$$\text{Ker} [yE - A, -B] = \text{im} \begin{bmatrix} 0 & -t + y \\ 0 & w^4 \\ -t & -ty + y^2 \\ -ty & 0 \\ -ty & (-w^4 t + t^4) - t^3 y - ty^2 + y^3 \end{bmatrix}.$$
which leads to $M = hR[y]$ with

$$h = \left[ t^2 - ty, -w^4t, -w^3t, -w^3 + ty - y^2, (w^4t^2 - t^5) + (-w^3t + t^4)y \right].$$

Now $\pi(h) = x_1y + x_2$ where $x_1 = [t, -t, 0, 0]$ and $x_2 = [t^2, -w^4t, -w^3t]$ and according to the Proposition we obtain as final result: $M_0^* = \langle x_1, x_2 \rangle$ (compare with $R_2^*$ in [AsLaPe2, 4.3]). Note that by the new procedure we automatically get $M_0^*$ represented as an $(A, B)$-cyclic subspace. In more complex examples one obtains $M_0^*$ as a sum of $(A, B)$-cyclic modules. For reasons of space I do not give an example for this.

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