The Non-integrability of a Silnikov Equation *

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Abstract Based on the Lie group theory, the one-parameter Lie group admitted by a Silnikov equation (see [1]) is studied. The result reveals that the Silnikov equation accepts no global analytical non-trivial one-parameter Lie group. In this sense, the Silnikov equation is not integrable in quadrature.

Key words Silnikov equation, Lie group, Infinitesimal generator

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1 Introduction

In the references [1, 2], Keying Guan proposed and studied the dynamical system described by

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y, z) = y \\
\frac{dy}{dt} &= Q(x, y, z) = z \\
\frac{dz}{dt} &= R(x, y, z) = x^3 - a^2 x - y - bz,
\end{align*}
\]

where \(a, b\) are both positive constants, and he found system (1) is an important particular case of Silnikov equation. Based on both qualitative method and numerical tests, he found system (1) has a series of interesting phenomena. For example, this system may have "faint attractor (this term is suggested by the author)", spatial limit closed orbits with different rotation numbers and bifurcation of the limit closed orbits. In [2], Guan and Beiye Feng studied the period-doubling cascades of the spatial limit closed orbit of system (1), and so on. It is revealed that the system is a very rich source and an ideal model of three-dimensional autonomous ordinary differential system in the research of the bifurcation and chaos theory.

In this manuscript, we proceed to study the three-dimensional system using Lie group theory. Professor Guan (the author of [1]) has noticed that the system may not accept any global analytical

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one-parameter Lie group excepting for a trivial Lie group, and in this sense, one can conclude strictly that the system is not integrable in quadrature. But he mentioned that he has not given a detailed proof (see http://blog.sciencenet.cn/u/guanky).

In the fact, system (1) is corresponding to a third-order ordinary differential equation

\[ a^2y = y^3 - y''' - y' - by'', \]  \( \text{(2)} \)

where \( y' \) is the derivative about variable \( y \) with respect to variable \( t \). Since the variable \( t \) is not appeared explicit in equation (2), it admits a trivial one-parameter Lie group \( t \rightarrow t + c \). Let \( z(y) = y' \), the above equation can turn to the following second order equation

\[ z^2 z'' + zz'^2 + z + a^2y - y^3 + b'z = 0. \]  \( \text{(3)} \)

The one-parameter Lie group admitted by (3) is same as that admitted by (2) excepting for the trivial one-parameter Lie group admitted by (2). In this manuscript, we consider system (3) using Lie group theory, and find the fact that the system accepts no global analytical one-parameter Lie group.

2 The One-parameter Lie group admitted by the system

For the sake of convenience, we write equation (3) as

\[ x^2 \ddot{x} + x\dot{x}^2 + x + a^2t - t^3 + b\dot{x} = 0. \]  \( \text{(4)} \)

We suppose that \( V = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x} \) is the infinitesimal generator of Lie group admitted by equation (4). Therefore,

\[ V^{(2)} = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x} + \eta^{(1)}(t, x, \dot{x}) \frac{\partial}{\partial \dot{x}} + \eta^{(2)}(t, x, \dot{x}, \ddot{x}) \frac{\partial}{\partial \ddot{x}} \]

be the 2th-extended infinitesimal generator, where

\[ \eta^{(1)}(t, x, \dot{x}) = \eta_t + (\eta_x - \xi_t)\dot{x} - \xi_x \dot{x}^2, \]

\[ \eta^{(2)}(t, x, \dot{x}, \ddot{x}) = \eta_{tt} + (2\eta_{tx} - \xi_{tt})\dot{x} + (\eta_{xx} - 2\xi_{tx})\dot{x}^2 - \xi_{xx} \dot{x}^3 + (\eta_x - 2\xi_t)\ddot{x} - 3\xi_x \dot{x}. \]

As we know (ref.[3]), Equation (4) will accept the above symmetry group if and only if

\[ V^{(2)}(\ddot{x} - f(t, x, \dot{x})) = 0 \text{ when } \ddot{x} = f(t, x, \dot{x}), \]  \( \text{(5)} \)

where \( f(t, x, \dot{x}) = \frac{t^3 - a^2t - x - \dot{x}^2 - b\dot{x}}{x^2} \).
Substituting \( \eta^{(1)}(t, x, \dot{x}), \eta^{(2)}(t, x, \dot{x}) \) and \( \ddot{x} = \frac{t^3 - a^2t - x - \dot{x}^2 - bx}{x^2} \) to (5), we have

\[
\xi\frac{a^2 - 3t^2}{x^2} + \frac{2(t^3 - a^2t - x - \dot{x}^2 - bx) + (1 + \dot{x}^2 + b\dot{x})x}{x^3} + [\eta_t + (\eta_x - \xi_t)\dot{x} - \xi_xx]b + 2\dot{x} \\
+ [\eta_{tt} + (2\eta_{tx} - \xi_{tt})\dot{x} + (\eta_{txx} - 2\xi_{tx})\dot{x}^2 - \xi_{txx}x^3 \\
+ (\eta_x - 2\xi_t)\frac{t^3 - a^2t - x - \dot{x}^2 - bx}{x^2} - 3\xi_xx\frac{t^3 - a^2t - x - \dot{x}^2 - bx}{x^2}] = 0
\]

After straightforward computing, it appears

\[
\dot{x}(-bx\eta + b(\eta_x - \xi_t)x^2 - bx^2(\eta_x - 2\xi_t) + 2\eta_t x^2 + (2\eta_{tx} - \xi_{tt})x^3 - 3\xi_xx(t^3 - a^2t - x)) \\
+ \dot{x}^2(2b\xi_x x^2 - \eta x + 2(\eta_x - \xi_t)x^2 + (\eta_{xx} - 2\xi_{tx})x^3 - (\eta_x - 2\xi_t)x^2) \\
+ \dot{x}^3(-2\xi_xx^2 - \xi_{xx}x^3 + 3\xi_x x^2) \\
+ \xi(a^2 - 3t^2)x + 2\eta(t^3 - a^2t - x) + \eta_x + \eta_t x^3 + (\eta_x - 2\xi_t)(t^3 - a^2t - x) + b\eta_x x^2 \\
= 0.
\]

The resulting determining equations for \( \xi \) and \( \eta \) are given by

\[
- bx\eta + b(\eta_x - \xi_t)x^2 - bx^2(\eta_x - 2\xi_t) + 2\eta_t x^2 + (2\eta_{tx} - \xi_{tt})x^3 - 3\xi_xx(t^3 - a^2t - x) = 0, \\
2b\xi_x x^2 - \eta x + 2(\eta_x - \xi_t)x^2 + (\eta_{xx} - 2\xi_{tx})x^3 - (\eta_x - 2\xi_t)x^2 = 0, \\
-2\xi_xx^2 - \xi_{xx}x^3 + 3\xi_x x^2 = 0, \\
\xi(a^2 - 3t^2)x + 2\eta(t^3 - a^2t - x) + \eta_x + \eta_t x^3 + (\eta_x - 2\xi_t)(t^3 - a^2t - x) + b\eta_x x^2 = 0.
\]

Form the above third equation of (6), one sees that

\[
\xi = \frac{1}{2} f_1(t) x^2 + f_2(t),
\]

where \( f_1(t) \) and \( f_2(t) \) are analytical functions. Then, we consider the second equation of (6). After simplifying the equation, we are led to

\[
-\eta + 2\eta_x x + \eta_x x + (\eta_{xx} - 2\xi_{tt})x^2 = 0.
\]

Substituting \( \xi \) to the above equation, we have

\[
\eta_{xx} x^2 + \eta_x x - \eta = 2f_1(t)x^3 - 2bf_1(t)x^2.
\]

Without lost of generality, we can let \( \eta = g_1(t)x^3 + g_2(t)x^2 + g_3(t)x + g_4(t), \) where \( g_i(t), i = 1, 2, 3, 4 \) are all analytical functions. According to the above equation, one can obtain easily

\[
g_1(t) = \frac{1}{4}f_1(t),
\]
\[
g_2(t) = -\frac{2b}{3}f_1(t),
\]
\[
g_4(t) = 0.
\]
Thus far, \( \eta = \frac{1}{4} f_1'(t)x^3 - \frac{2b}{3} f_1(t)x^2 + g_3(t)x. \) (8)

Next, we consider the first equation of (6). Substituting (7) and (8) to the first equation of (6), one has

\[
2 f_1''(t)x^4 - \frac{15}{4} f_1'(t)x^3 + \left(\frac{2b^2 + 9}{3} f_1(t) + 4g_3'(t) - f_2''(t)\right)x^2 \\
+ (-3f_1(t)(t^3 - a^2t) - bg_3(t) + bf_2'(t))x = 0.
\] (9)

The left side of (9) is a polynomial on \( x \). So, we can obtain that \( f_1(t) \) is a constant function \( c_1 \) and

\[
\frac{2b^2 + 9}{3} f_1(t) + 4g_3'(t) - f_2''(t) = 0,
\] (10)

\[
-3f_1(t)(t^3 - a^2t) - bg_3(t) + bf_2'(t) = 0.
\]

After integrating the first equation of (10) and multiplying \( b \), it appears

\[
4bg_3(t) - bf_2'(t) = \frac{(2b^2 + 9)bc_1}{3}t + c_2b,
\] (11)

where \( c_2 \) is an integrating constant. Adding the second equation of (10) to (11), we have

\[
g_3(t) = \frac{c_1(t^3 - a^2t)}{b} - \frac{(2b^2 + 9)c_1t}{9} + \frac{c_2}{3}.
\]

Based on (10), we can obtain

\[
f_2(t) = \frac{c_1t^4}{b} - \frac{2c_1a^2}{b}t^2 - \frac{(2b^2 + 9)c_1t^2}{18} + \frac{c_2t}{3} + c_3,
\]

where \( c_3 \) is an integrating constant. So,

\[
\xi = \frac{1}{2} c_1x^2 + \frac{c_1t^4}{b} - \frac{2c_1a^2}{b}t^2 - \frac{(2b^2 + 9)c_1t^2}{18} + \frac{c_2t}{3} + c_3,
\]

\[
\eta = -\frac{2b}{3} c_1x^2 + \left(\frac{c_1(t^3 - a^2t)}{b} - \frac{(2b^2 + 9)c_1t}{9} + \frac{c_2}{3}\right)x.
\]

Substituting \( \xi \) and \( \eta \) to the fourth equation of (3) and computing straightway, we can obtain \( c_1 = c_2 = c_3 = 0 \). It reveals \( \xi(t, x) = 0, \eta(t, x) = 0 \). Consequently, equation (3) or (4) admits a one-parameter Lie group with infinitesimal generator

\[
V = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x},
\]

where \( \xi(t, x) = 0, \eta(t, x) = 0 \). That is, equation (3) admits no global analytical one-parameter Lie group. Equation (2) admits no global analytical one-parameter Lie group excepting for a trivial Lie group \( t \to t + c \), and it is not integrable in quadrature.

Clearly, when \( b = 0 \), (2) is the special case

\[
a^2y = y^3 - y''' - y'.
\] (12)
The above discussion can also indicate (12) not admitting any global analytical one-parameter Lie group excepting for a trivial Lie group \( t \rightarrow t + c \), and it is not integrable in quadrature.

On the general Silnikov equation mentioned in [1], it can be studied with the similar way. Relevant results will appear in later manuscript.

3 Conclusions

In this manuscript, we have proved that the Silnikov equation is not integrable in quadrature based on Lie group theory. So far, we have seen the Silnikov equation possessing some dynamical characters and particularity. It is believed that the Silnikov equation still needs to be explored using other methods.

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References

[1] Keying Guan, Non-trivial Local Attractors of a Three-dimensional Dynamical System. arXiv.org, http://arxiv.org/abs/1311.6202(2013).

[2] Keying Guan, Beiye Feng, Period-doubling Cascades of a Silnikov Equation. arXiv.org, http://arxiv.org/abs/1312.2043(2013).

[3] George W. Bluman, Stephen c. Anco, Symmetry and Integration Methods for Differential Equations. New York: Springer-Verlag, 2004.