To move or not to move:

Anomalous Spin-Hall effect on the Black Hole horizon

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Abstract

An “exotic photon” i.e. a mass and chargeless Carroll particle with anyonic spin, magnetic moment and “exotic” charges associated with the 2-parameter central extension of the 2-dimensional Carroll group moves on the horizon of a Kerr-Newman Black Hole consistently with the anomalous spin-Hall effect.

Key words: Anomalous Spin-Hall effect; motion on the horizon of a Black Hole; centrally extended Carroll particle;

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I. INTRODUCTION

The Anomalous Hall Effect (AHE) observed in ferromagnetic crystals in the absence of a magnetic field had been attributed to an anomalous current [1]. Some time later it was argued that spinning particles may behave similarly: this is the Spin-Hall effect [2]. Much more recently, a semiclassical explanation was proposed, using a Berry phase–extended framework [3]. The clue is the anomalous velocity relation

\[ \dot{x} = \frac{\partial \mathcal{E}(p)}{\partial p} - eE \times \Theta, \]  

where \( \mathcal{E}(p) \) is the band energy, \( E(x) \) the electric field, and \( \Theta(p) \) represents the Berry phase. The anomalous velocity term here is clearly the mechanical counterpart of the anomalous current. Choosing \( E \) in the \( x - y \) plane and \( \Theta \) perpendicular to it, eqn. (I.1) reduces to the “exotic” Galilean model based on a 2nd central extension of the planar Galilei group [4].

The “Carrollian” counterpart of Galilean physics, obtained by letting the velocity of light go to zero instead of going to infinity [5, 6] has attracted recent attention [7, 8]. The interest was moderated, though, by that a Carroll particle can not move [5–8].

The horizon of a Black Hole is, in particular, a Carroll manifold [9, 10] and therefore the “no-go theorem” above applies. In this Letter we show however that a clever extension [11] of the original Carroll model allows us to overcome the no-go theorem and an “exotic Carrollian photon” to be introduced below can move on the horizon of a Kerr-Newman Black Hole. In fact, it provides a novel version of the Spin-Hall effect.
In detail, we show that, taking advantage of the double central extension of the Carroll group in 2 space dimensions [11–13] endows a Carroll particle with two central charges: an “exotic” and “magnetic” one, $\kappa_{exo}$ and $\kappa_{mag}$, respectively. Such a particle can theoretically be mass and chargeless, carry anyonic spin $\chi$ and magnetic moment $\mu$ and can be coupled to an electromagnetic field through a spin-orbit (Stern-Gerlach) term $H = \mu \chi B$ (III.1). Assuming that such particles we shall call “exotic photons” do indeed exist, we show that they do move for example, on the horizon of a Kerr-Newman Black Hole, namely exhibiting a dual version of the Anomalous Hall Effect. Our clue is that $E^* = \nabla B$ behaves as an effective electric field, $e^* = \mu \chi$ as an effective electric charge, and one of the extension parameters behaves as a sort of “dual mass”. See [9, 11, 14] for details of the underlying theory.

II. CARROLL STRUCTURE OF THE KERR-NEWMAN HORIZON

A Kerr-Newman Black Hole characterized by its mass $M$, angular momentum $J$, and charge $Q$ can be described by using the Eddington-like coordinates $(u, r, \theta, \phi)$ [15]. In these coordinates, the metric

$$g = -\frac{\Delta}{\Sigma} \left( du + \frac{\Sigma}{\Delta} dr - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left( adu - (r^2 + a^2) d\phi \right)^2 + \Sigma d\theta^2 + \frac{\Sigma}{\Delta} dr^2 \quad (II.1a)$$

$$\Sigma = r^2 + a^2 \cos \theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad (II.1b)$$

where $a = J/M$, and its inverse are regular on the horizon.

The (outer) horizon $\mathcal{H}$ of a Kerr-Newman black hole is the constant hypersurface defined by $\Delta = 0$. Note that the seemingly problematic $dr^2$ terms in (II.1a) containing $\Delta$ in their denominator cancel one another out. Then we consider the 2 + 1 dimensional structure [9, 10] whose ingredients are the induced metric and a vector,

$$\tilde{g} = g_{\Delta=0} = \frac{\sin^2 \theta}{\Sigma} \left( adu - (r^2 + a^2) d\phi \right)^2 + \Sigma d\theta^2, \quad (II.2a)$$

$$\xi = \partial_u + \Omega_H \partial_\phi \quad \text{where} \quad \Omega_H = \frac{a}{r^2 + a^2}, \quad (II.2b)$$

respectively. $\Omega_H$ is the angular velocity of the horizon. The restricted metric (II.2a) is singular as made manifest by the coordinate change $(\theta, \phi, u) \mapsto (\theta, \varphi = \phi - \Omega_H u, u)$, which
leads to the metric,
\[ \tilde{g} = \frac{(r^2 + a^2)}{\Sigma} \sin^2 \theta \, d\varphi^2 + \Sigma d\theta^2 \quad \text{&} \quad \xi = \partial_u, \] (II.3)

The kernel is generated by the vector \( \xi, \tilde{g}(\xi) = 0 \). Thus we have a degenerate metric and a vector field in its kernel, allowing us to conclude that the horizon of a Kerr-Newman black hole carries a Carroll structure \((S^2 \times \mathbb{R}, \tilde{g}, \xi)\) as introduced in [7].

The degenerate “metric” \( \tilde{g} \) carries the geometric information of the \( S^2 \) part of the black hole, while \( \xi \) generates the \( \mathbb{R} \) part.

III. MOTION ON THE KERR-NEWMAN HORIZON

Geodesic motion induced on a Carroll submanifold is necessarily massless [7]. A massless Carroll particle can indeed stay fixed on the horizon, but cannot move [7, 14]. However the horizon is a \( 2 + 1 \) dimensional Carroll manifold, therefore the particle may carry two more parameters associated with the double central extension [11–14]. It may carry also anyonic spin.

An “exotic photon” with no mass and charge, \( m = 0 \) and \( e = 0 \), but with nonvanishing magnetic moment \( \mu \) and anyonic spin \( \chi \) can be coupled to the electromagnetic field through a spin-field term \( \mathcal{H} = \mu \chi \nabla B \) where \( B \) is the magnetic field on the horizon. The PB relations for an uncharged but massive particle with double extension are,
\[
\{x_i, p_j\} = \frac{m}{m^*} \delta_{ij}, \quad \{x_i, x_j\} = \frac{\kappa_{exo}}{mm^*} \epsilon_{ij}, \quad \{p_i, p_j\} = \frac{m}{m^*} \kappa_{mag} \epsilon_{ij}, \quad (III.1)
\]

where \( m^* = m \left(1 - \frac{\kappa_{exo}\kappa_{mag}}{m^2}\right) \) is an effective mass. Then the Hamilton equations for the purely spin-orbit Hamiltonian \( \mathcal{H} \) with no kinetic term [11, 14] can be presented as
\[
\dot{x}^i = -\mu \chi \frac{\kappa_{exo}}{m^2 - \kappa_{exo}\kappa_{mag}} \epsilon^{ij} \partial_j B, \quad \dot{p}_i = \frac{\mu \chi m^2}{m^2 - \kappa_{exo}\kappa_{mag}} \partial_i B, \quad (III.2)
\]

where the dot denotes the derivative w.r.t. the Carrollian time coordinate defined by \( \xi \), see (II.2b). Note that the two equations are uncoupled and only the first one is relevant for dynamics. Letting \( m \to 0 \) in (III.2), \( \kappa_{exo} \) drops out, yielding the extended dynamics [11, 14]
\[
\kappa_{mag} \dot{x}^i = \mu \chi \epsilon^{ij} \partial_j B, \quad \dot{p}_i = 0, \quad (III.3)
\]
which describes an Anomalous Spin-Hall Effect with $\kappa_{\text{mag}}$ behaving as a “dual mass”. The gradient of the magnetic field behaves as an effective electric field $E^*$ and $e^* = \mu \chi$ as an effective electric charge. Eqn. (III.3) shows a sort of “duality” with the usual AHE.

Coupling to the gravitational field amounts to replacing the derivative on $p_i$ in (III.2) by a covariant derivative [11]. However, minimal coupling does not change the velocity equation, which is indeed the only relevant one for the poor Carrollian dynamics.

Turning now to the Kerr-Newman Black Hole, the electromagnetic tensor induced on the horizon is, $\tilde{F} = \frac{aQr(r^2 + a^2)}{(r^2 + a^2 \cos^2 \theta)^2} \sin \theta \, d\theta \wedge d\varphi$, which translates to having a magnetic field $B$ on the horizon,

$$ B = \frac{2aQr(r^2 + a^2)}{(r^2 + a^2 \cos^2 \theta)^3} \cos \theta, $$

(III.4)
whose non-vanishing requires non-zero charge $Q$ and angular momentum $J$ (since $a = J/M$).

Using (comoving) angular coordinates $(\theta, \varphi, u)$ we see that the electric field induced on the horizon vanishes. The radial component would survive, but disappears in the 2+1 restriction. The gradient of $B$ is tangent to the longitudinal great circles $\varphi = \text{const.}$ By (III.3), the motion is governed by,

$$ \dot{x}^\theta = 0, \quad \kappa_{\text{mag}} \dot{x}^\varphi = \left(2aQr + \mu \chi\right) \frac{(r_+^2 + a^2)(r_+^2 - 5a^2 \cos^2 \theta)}{(r_+^2 + a^2 \cos^2 \theta)^4} \sin \theta. \quad \text{(III.5)} $$

Thus our “exotic photon” performs azimuthal circular motion on the horizon, depicted in fig.1. Consistently with the Hall behavior, the motion is perpendicular to the (longitudinal) effective electric field $E^* = \nabla B$, whereas $e^* = \mu \chi$ plays the role of an effective electric charge. The direction of the rotation depends on the sign of $aQ\mu \chi/\kappa_{\text{mag}}$ and is thus correlated with the sign of the Black Hole angular momentum $J$ and its charge $Q$. The angular velocity goes smoothly to zero as we approach the poles and depends on the radius of the horizon roughly as $r_+^{-3}$, implying that the rotation would be more important for smaller black holes.

The rotation we have just found, although reminiscent of the frame-dragging by a rotating black hole, is however unrelat ed to it: frame-dragging is hidden in the coordinates, which are comoving with the horizon.

### IV. CONCLUSION

Eqn. (III.3) says that our “exotic photon” i.e. a Carroll particle with no mass and charge but with non-vanishing magnetic moment, anyonic spin and “exotic” extension parame-
FIG. 1: On the horizon of a Kerr-Newman Black Hole the velocity field (III.5) is perpendicular to the axis of rotation and obeys the Anomalous Hall law with an effective electric field $E^* = \nabla B$ which is tangent to the longitudinal great circles with $e^* = \mu \chi$ playing the role of an effective electric charge. The arrows indicate the directions and norms.

atters, coupled to a position-dependent magnetic field moves according to the AHE velocity-momentum relation (I.1) with the kinetic term turned off: it has a purely anomalous velocity. The gradient $E^* = \nabla B$ of the magnetic field behaves as an effective electric field and $e^* = \mu \chi$ is an effective electric charge. The “magnetic” extension parameter $\kappa_{mag}$ behaves in turn as a sort of “dual mass”: we have an Anyonic Anomalous Spin-Hall effect which is sort of “dual” to the usual one. The horizon of a Kerr-Newman Black Hole realizes these conditions: its magnetic field $B$ in (III.4) induces anomalous Hall motion for our “exotic photon”. But do such particles indeed exist? With no direct experimental proof we can only recall Dirac’s phrases about his magnetic monopole [16]:

“This new development . . . is merely a generalisation of the possibilities . . . Under these circumstances one would be surprised if Nature had made no use of it.”
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