Some Sufficient Conditions for the Controllability of Wave Equations with Variable Coefficients

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Abstract

In this work, we present some easily verifiable sufficient conditions that guarantee the controllability of wave equations with non-constant coefficients. These conditions work as complements for those obtained in [3].

1 Introduction and the Main Results

Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^2$ boundary $\partial \Omega$. Let $a^{ij} \in C^1(\overline{\Omega})(i, j = 1, \cdots, n)$ such that $a^{ij} = a^{ji}$ and $A \triangleq (a^{ij})_{1 \leq i, j \leq n}$ is a uniformly positive definite matrix. Consider the following hyperbolic equation:

$$
\begin{align*}
\begin{cases}
y_{tt} - \sum_{i,j=1}^{n} (a^{ij} y_{x_i})_{x_j} = 0 & \text{in } (0, T) \times \Omega, \\
y = 0 & \text{on } (0, T) \times \partial \Omega, \\
y(0) = y_0, \ y_t(0) = y_1 & \text{on } \Omega.
\end{cases}
\end{align*}
$$

(1.1)

Here $(y_0, y_1) \in H^1_0(\Omega) \times L^2(\Omega)$. In order to establish the boundary observability estimate for the equation (1.1) by multiplier method or Carleman estimate, one needs the following conditions (see [2] for example):

Condition 1.1. There exists a function $d \in C^2(\overline{\Omega})$ such that

$$
\sum_{i,j=1}^{n} \left\{ \sum_{i',j'=1}^{n} \left[ 2a^{ij'} (a^{i'j} d_{x_{i'}})_{x_{j'}} - a^{ij'} a^{i'j'} d_{x_{i'}} d_{x_{j'}} \right] \right\} \xi_i \xi_j \geq \mu_0 \sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j,
$$

(1.2)

when $(x, \xi_1, \cdots, \xi_n) \in \overline{\Omega} \times \mathbb{R}^n$, and such that and

$$
|\nabla d| > 0 \quad \text{in } \overline{\Omega}.
$$

(1.3)

Remark 1.1. One can directly verify the following: The condition (1.2) is equivalent to that the matrix

$$
B = (b^{ij})_{1 \leq i, j \leq n} \triangleq \left( \sum_{i',j'=1}^{n} \left( a^{ij'} a^{i'j} d_{x_{i'}} d_{x_{j'}} + \frac{a^{ij'} a^{i'j} a^{i'i}}{2} - a^{ij} a^{i'j'} d_{x_{i'}} d_{x_{j'}} \right) \right)_{1 \leq i, j \leq n}
$$

(1.4)

is uniformly positive definite.

The function $d$ verifying (1.2) and (1.3) does not exist for some cases. This can be seen from the following example:

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Example 1.1. Let $\Omega = \{(x, y) : x^2 + y^2 \leq 2\}$. Let $(a^{ij})_{1 \leq i, j \leq 2} = \text{diag}(a^1, a^2)$ with $a^1(x, y) = a^2(x, y) = 1 + x^2 + y^2$. By an indirect proof based on the Geometric Control Condition given in \cite{1}, we can show that there is no such a function $d$ that satisfies \cite{12}.

Now, we study the existence of functions $d$ verifying \cite{12} for suitable $(a^{ij})_{1 \leq i, j \leq n}$. We will focus our studies on the special case where $A = (a^j)_{1 \leq i, j \leq n} = \text{diag}(a^1, \ldots, a^n)$, where $a^i \in C^1(\Omega)$. From Example 1.1, we see that even in this case, the above-mentioned functions $d$ may not exist. Thus, it is interesting to provide certain easily verifiable condition to ensure the existence of such functions $d$ in the case when $A$ is diagonal. The main results of this study are as follows:

Theorem 1.1. Let $A = \text{diag}(a^1, \ldots, a^n)$, with $a^i \in C^1(\Omega)$ $(1 \leq i \leq n)$, be positive uniformly definite over $\Omega$. If there exists $j \in \{1, \ldots, n\}$ such that all the terms of $(a^i_{x_i})_{1 \leq i \leq n, i \neq j}$ remain positive (or negative) over $\Omega$, then there is a function $d \in C^2(\Omega)$ verifying Condition \cite{13}.

It is worth mentioning that, in the statement of the main theorem, we don’t need the structural condition on $a^j_{x_j}$, where $j$ is the fixed index. Before carrying out the proof, we give two corollaries. The first one corresponds to the case $j = 1$:

Corollary 1.1. Let $A = \text{diag}(a^1, \ldots, a^n)$, with $a^i \in C^1(\Omega)$, $i = 1, \ldots, n$, be positive uniformly definite over $\Omega$. Suppose that

\[ a^k_{x_1} > 0 \ (\text{or } a^k_{x_1} < 0) \quad \text{over } \Omega, \quad \text{for } 2 \leq k \leq n, \tag{1.5} \]

then, there is a function $d \in C^2(\Omega)$ verifying Condition \cite{13}.

Corollary 1.2. Let $A = \text{diag}(a^1, a^2)$, with $a^1, a^2 \in C^1(\Omega)$, be positive uniformly definite over $\Omega$. Suppose that $a^1_{x_2}$ (or $a^2_{x_2}$) is either positive or negative over $\Omega$. Then there is a function $d \in C^2(\Omega)$ satisfying Condition \cite{13}.

2 Proof of Main Theorem:

Proof of Theorem 1.1 case 1. We first consider the following case:

\[ a^j_{x_j} < 0, \quad \text{uniformly over } x \in \Omega, \quad \text{for all } 1 \leq i \leq n \quad \text{with } i \neq j. \tag{2.1} \]

where $j$ is a fixed index. Let

\[ d \triangleq d(x) = e^{c(x+x_j)} + \sum_{1 \leq i \leq n, i \neq j} e^{\lambda x_i}, \quad x \in \Omega, \]

where $c > 0$ satisfies

\[ \min_{x \in \Omega} \{c + x_j\} \geq 1 + \max_{x \in \Omega} \sum_{1 \leq i \leq n, i \neq j} |x_i|, \tag{2.2} \]

and $\lambda > 0$ is a large number will be determined later. Using \cite{22}, one could check that the function $d(x)$ enjoys the following properties:

- For any $1 \leq i \leq n$,

\[ d_{x_ix_i} > 0, \quad d_{x_i} > 0, \quad \text{uniformly for } x \in \overline{\Omega}. \tag{2.3} \]

- For any $1 \leq i \leq n$,

\[ \lim_{\lambda \to +\infty} \frac{d_{x_i}}{d_{x_ix_i}} = 0 \quad \text{uniformly for } x \in \overline{\Omega}. \tag{2.4} \]

- For any $1 \leq i \leq n$ with $i \neq j$,

\[ \lim_{\lambda \to +\infty} \frac{d_{x_i}}{d_{x_j}} = 0, \quad \lim_{\lambda \to +\infty} \frac{d_{x_ix_i}}{d_{x_j}} = 0, \quad \text{uniformly for } x \in \overline{\Omega}. \tag{2.5} \]
From Remark 1.1 to prove \( d \) enjoys (1.2) for the case \( A = \text{diag} (a^1, \cdots, a^n) \), we only need to show the uniformly positivity of the following matrix:

\[
B = \frac{1}{2} \left( a^i a^l x_i d_{x_l} + a^i a^l x_l d_{x_i} \right)_{1 \leq i, l \leq n} + \text{diag} \left( (a^1)^2 d_{x_1 x_1} - \frac{1}{2} \sum_{k=1}^n a^k a^k x_k d_{x_k}, \cdots, (a^n)^2 d_{x_n x_n} - \frac{1}{2} \sum_{k=1}^n a^k a^k x_k d_{x_k} \right), \tag{2.6}
\]

To achieve this goal, we only need to show that all the leading principal minors of \( B \) are positive. In order to avoid the terrible expansion of the determinant, we shall make full use of the asymptotic behavior with respect to the parameter \( \lambda \). We denote by \( e_i \) the \( i \)-th standard basis of \( \mathbb{R}^n \) and by \( \{B_i\}_{i=1}^n \) the row vector of \( B \). It can be verified that, with a very large \( \lambda > 0 \), the matrix \( B \) is uniformly positive definite over \( \Omega \) if and only if all the leading principal minors of \( \tilde{B}(x, \lambda) := \begin{pmatrix} B_1 \\ \vdots \\ B_{j-1} \\ B_j \\ B_{j+1} \\ \vdots \\ B_n \end{pmatrix} \) \( \frac{d}{dx_j} \) is uniformly positive over \( \Omega \). This later condition is relatively easier to be verified because we could calculate the limit \( \tilde{B}(x, +\infty) = \lim_{\lambda \to +\infty} \tilde{B}(x, \lambda) \) and the condition (2.1) guarantees that all the leading principal minors of \( \tilde{B}(x, +\infty) \) are uniformly positive over \( \Omega \). Now we give the details of this:

By (2.6)

\[
B_j = \frac{1}{2} \left( a^j a^l x_j d_{x_l} + a^l a^j x_l d_{x_j} \right)_{1 \leq l \leq n} + \left( (a^j)^2 d_{x_j x_j} - \frac{1}{2} \sum_{k=1}^n a^k a^k x_k d_{x_k} \right) e_j \tag{2.7}
\]

Making use of (2.3), (2.4) and (2.5), we deduce

\[
\lim_{\lambda \to +\infty} \frac{B_j}{d_{x_j x_j}} = (a^j)^2 e_j \text{ uniformly for } x \in \Omega. \tag{2.8}
\]

In the same spirit, for \( 1 \leq i \leq n \) with \( i \neq j \), we have

\[
B_i = \frac{1}{2} \left( a^i a^l x_i d_{x_l} + a^l a^i x_l d_{x_i} \right)_{1 \leq l \leq n} + \left( (a^i)^2 d_{x_i x_i} - \frac{1}{2} \sum_{k=1}^n a^k a^k x_k d_{x_k} \right) e_i \tag{2.9}
\]

One could verify by using (2.4) and (2.5) that

\[
\lim_{\lambda \to +\infty} \frac{B_i}{d_{x_j}} = \frac{1}{2} a^i a^j e_j - \frac{1}{2} a^j a^i e_i \text{ uniformly for } x \in \Omega. \tag{2.10}
\]
By (2.18) and (2.20), we deduce that

\[
\begin{pmatrix}
\frac{B_1}{d_{x_1}} & \cdots & 0 & \frac{1}{2}a_1^d x_1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -\frac{1}{2}a_j^d x_j^{-1} & \frac{1}{2}a_j^{d-1} x_j^{-1} & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{2}a^d x_n^{-1} & \cdots & -\frac{1}{2}a^d x_n^{-1}
\end{pmatrix}
\]  

(2.11)

\[
\lim_{\lambda \to +\infty} \frac{B_1}{d_{x_1}} = \frac{B_1}{d_{x_1} + 1} \frac{B_1}{d_{x_n}} = 0 \quad \text{uniformly for } x \in \Omega.
\]

We deduce from the above formula and (2.3), (2.1) that all the leading principal minors of \( \hat{B}(x, \lambda) \) are uniformly positive with a large \( \lambda \). This complete the proof.  

**Proof of Theorem 1.1, case 2.** Here we discuss the case when

\[ a_{x_j}^d > 0, \quad \text{uniformly over } x \in \Omega, 1 \leq i \leq n, i \neq j, \]

(2.12)

where \( j \) is a fixed index. In this case, the proof is quite similar as above: we define a function

\[ d \equiv d(x) = e^{-\lambda(x_1 - c)} + \sum_{1 \leq i \leq n, i \neq j} e^{-\lambda x_i}, \quad x \in \Omega, \]

where \( c > 0 \) satisfies

\[
\max_{x \in \Omega} \{x - c\} + 1 \leq \min_{x \in \Omega} \sum_{1 \leq i \leq n, i \neq j} |x_i|, \quad (2.13)
\]

and \( \lambda > 0 \) is a large number will be determined later. Using (2.13), one could also check that the function \( d(x) \) enjoys the following properties:

- For any \( 1 \leq i \leq n \),
  \[ d_{x_i} < 0, \quad d_{x_{ii}} > 0, \quad \text{uniformly for } x \in \Omega. \]  
  (2.14)

- For any \( 1 \leq i \leq n \),
  \[ \lim_{\lambda \to +\infty} \frac{d_{x_i}}{d_{x_j}} = 0 \quad \text{uniformly for } x \in \Omega. \]  
  (2.15)

- For any \( 1 \leq i \leq n \) with \( i \neq j \),
  \[ \lim_{\lambda \to +\infty} \frac{d_{x_i}}{d_{x_j}} = 0, \quad \lim_{\lambda \to +\infty} \frac{d_{x_i x_i}}{d_{x_j}} = 0, \quad \text{uniformly for } x \in \Omega. \]  
  (2.16)

As before, we deduce from (2.14), (2.15) and (2.16) that the matrix \( B \) is uniformly positive definite if and only if all the leading principal minors of the matrix \( \hat{B}(x, \lambda) := \)

\[
\begin{pmatrix}
-\frac{B_1}{d_{x_1}} & \cdots & 0 & \frac{1}{2}a_1^d x_1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -\frac{1}{2}a_j^d x_j^{-1} & \frac{1}{2}a_j^{d-1} x_j^{-1} & \cdots & 0 \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \frac{1}{2}a^d x_n^{-1} & \cdots & -\frac{1}{2}a^d x_n^{-1}
\end{pmatrix}
\]

is
uniformly positive over \( \Omega \) when \( \lambda \) is large enough. By (2.6)

\[
B_j = \frac{1}{2} \left( a^j a^i_{x_j} d_{x_1} + a^j a^i_{x_i} d_{x_j} \right)_{1 \leq i \leq n} + \left( (a^j)^2 d_{x_j, x_j} - \frac{1}{2} \sum_{k=1}^{n} a^k a^j_{x_k} d_{x_k} \right) e_j
\]  
(2.17)

Making use of (2.15) and (2.16), we deduce

\[
\lim_{\lambda \to +\infty} \frac{B_j}{d_{x_j, x_j}} = (a^j)^2 e_j \quad \text{uniformly for } x \in \Omega.
\]  
(2.18)

In the same spirit, for \( 1 \leq i \leq n \) with \( i \neq j \), we have

\[
B_i = \frac{1}{2} \left( a^i a^i_{x_i} d_{x_1} + a^i a^i_{x_i} d_{x_i} \right)_{1 \leq i \leq n} + \left( (a^i)^2 d_{x_i, x_i} - \frac{1}{2} \sum_{k=1}^{n} a^k a^i_{x_k} d_{x_k} \right) e_i
\]  
(2.19)

One could verify by using (2.4) and (2.5) that

\[
\lim_{\lambda \to +\infty} \frac{B_i}{d_{x_i, x_i}} = \frac{1}{2} a^i a^i_{x_i} e_j e_j - \frac{1}{2} a^i a^i_{x_i} e_i \quad \text{uniformly for any } x \in \Omega.
\]  
(2.20)

By (2.18) and (2.20), we deduce that

\[
\lim_{\lambda \to +\infty} \frac{B_i}{d_{x_i, x_i}} = \frac{1}{2} a^i a^i_{x_i} = 0 \quad \text{uniformly for } x \in \Omega.
\]  
(2.21)

The above formula implies

\[
\lim_{\lambda \to +\infty} \frac{B_i}{d_{x_i, x_i}} = \frac{1}{2} a^i a^i_{x_i} = 0 \quad \text{uniformly for } x \in \Omega.
\]  
(2.22)

3 Examples and Comments

There have been a lot of conditions to ensure the existence of the function \( d \). In (3) (see also (4)), the author provides a sectional curvature condition to guarantee the existence of functions...
This condition is that the sign of the sectional curvature function \( k \) for the Riemannian manifold, with a metric \( A^{-1} = (a^{ij})_{1 \leq i,j \leq n} \), is either positive or negative over \( \Omega \).

In this section, we will compare the condition in Theorem 1.1 with the above-mentioned condition given in [3]. Then, we will see some advantage can be taken from the condition in Theorem 1.1. First, [3] needs the \( C^\infty \)-regularity for coefficients \( a^{ij} \); while our Theorem 1.1 only needs the \( C^1 \)-regularity for coefficients. Second (more important), there are many cases which can be solved by our Theorem 1.1 but cannot be solved by the sectional curvature condition provided in [3]. Here, we present an example to explain the second advantage above-mentioned.

**Example 3.1.** Let \( A = \text{diag}(a^1, a^2) \), where \( a^1, a^2 \in C^\infty(\Omega) \). Suppose that \( a^2_{x_j} < 0 \) over \( \Omega \). By Theorem 1.1 or Corollary 1.2 there is a function \( d \in C^2(\Omega) \) verifying Condition 1.1. However, by making use of the sectional curvature condition provided in [3], we cannot imply the existence of the above-mentioned \( d \). In fact, after some computation, one can see that the sectional curvature given by the metric \( A^{-1} \) is as follows:

\[
k = \frac{1}{4(a^1a^2)^2} \left[ a^2_{x_1}a^2_{x_1} + a^1(a^2_{x_1})^2 - 2a^1a^2_{x_1}a^2_{x_1} \right].
\]

(3.1)

From (3.1), one can construct many such \( a^i \), \( i = 1, 2 \), with the property that \( a^2_{x_1} < 0 \) over \( \Omega \), such that the corresponding \( k \) changes its sign over \( \Omega \).

Here, we provide one of them as follows: Let \( \Omega = \{(x_1, x_2) : (x_1 - 2)^2 + x_2^2 < 3/2\} \subset \mathbb{R}^2 \). Let \( a^1 = e^{\mu_1 x_1} \) and \( a^2 = e^{-\mu_2 x_1^2} \), where \( \mu_1 \) and \( \mu_2 \) satisfy

\[
\mu_1 > 0; \quad \mu_2 > 0; \quad \mu_1 + 2\mu_2 < 2; \quad 3\mu_1 + 18\mu_2 > 2.
\]

(3.2)

Clearly, (3.2) has solutions.

In this case, it is clear that \( a^2_{x_1} < 0 \) over \( \Omega \) because \( x_1 > 0 \) over \( \Omega \). From (3.1), we see

\[
4(a^1 a^2)^2 k = -2\mu_1\mu_2 x_1 e^{\mu_1 x_1 - 2\mu_2 x_1^2} + 4\mu_2^2 x_1 e^{\mu_1 x_1 - 2\mu_2 x_1^2} + 4\mu_2 e^{\mu_1 x_1 - 2\mu_2 x_1^2} - 8\mu_2 x_1^2 e^{\mu_1 x_1 - 2\mu_2 x_1^2} = -2\mu_2 e^{\mu_1 x_1 - 2\mu_2 x_1^2} (\mu_1 x_1 + 2\mu_2 x_1^2 - 2).
\]

From (3.2), it follows that

\[
(\mu_1 x_1 + 2\mu_2 x_1^2 - 2) \big|_{x_1 = 1} < 0
\]

and

\[
(\mu_1 x_1 + 2\mu_2 x_1^2 - 2) \big|_{x_1 = 3} > 0.
\]

Hence, \( k > 0 \) in the set \( \Omega \cap \{(x_1, x_2) : x_1 = 1\}; \) while \( k < 0 \) in the set \( \Omega \cap \{(x_1, x_2) : x_1 = 3\}. \)

From these, we conclude that \( k \) changes its sign over \( \Omega \). Therefore, the method in [3] does not work for the current case.

The next two examples are taken from [3] for which the existence can be ensured by either the sectional curvature condition provided in [3] or our Theorem 1.1.

**Example 3.2.** Let \( A = (a^{ij})_{1 \leq i,j \leq 2} = \text{diag}(e^{x_1^2+y^2}, e^{x_2^2+y^2}) \). One can directly check that

\[
a^2_{x_1} = 3y^2 e^{x_1^2+y^2} > 0.
\]

Then, according to Theorem 1.2, there is a \( d \) satisfying (1.2) and (1.3).

**Example 3.3.** Let \( A = (a^{ij})_{1 \leq i,j \leq 2} = \text{diag}(e^{x+y}, e^{x+y}) \). One can easily check that \( a^2_{x_1} = e^{x+y} > 0 \). Then, by Theorem 1.3 there exists a \( d \) satisfying (1.2) and (1.3).

**Remark 3.1.** The sectional curvature condition provided in [3] works better than our Theorem 1.1 when \( a^3 \) is not of diagonal form. For instance, the Example 3.2 in [3].

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