Dynamo and the Adiabatic Invariant

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Abstract
The paper considers a dynamo generated by a shallow fluid layer in a celestial body (planet or star). This dynamo is based on the extra invariant for interacting magnetic Rossby waves.

Unified Astronomy Thesaurus concepts: Magnetohydrodynamics (1964); Geomagnetic fields (646); Solar dynamo (2001); Alfven waves (23)

1. Introduction

The magnetic field of a celestial body often crucially depends on the shallow fluid layer. In the case of the Sun, this layer is the tachocline (Spiegel & Zahn 1992), although its role is currently questioned (see review Charbonneau 2020). In the case of the Earth, researchers often assume the existence of a stably stratified layer at the top of the liquid iron outer core (the stratified ocean of the core in the terminology of Braginsky 1998). We consider the magnetohydrodynamics (MHD) of such a layer locally in the plane 3D layer tangent to the planet or star.

We linearize the dynamics on the background of a strong toroidal magnetic field \( B_0 \) and note the possibility to separate variables for an arbitrary stratification. This leads to the quadratic dispersion relation, characterized by two length scales:

\[ r = c_g / f \quad \text{and} \quad \ell = \sqrt{B_0 / \beta}. \tag{1} \]

The scale \( r \) is the Rossby radius of deformation; it characterizes the usual hydrodynamic Rossby waves without magnetism; \( f \) is the angular speed of local rotation; and \( c_g = \sqrt{g \ell} \) is the gravity wave speed, where the height \( \ell \) is determined by the eigenvalue problem resulting from the separation of variables. The scale \( \ell \) characterizes the strength of the background magnetic field \( B_0 \); and \( \beta \) is the latitudinal derivative of \( f \).

We notice that in each of the two cases (Earth and Sun), these scales have the same order of magnitude. If \( \ell = r \), the quadratic dispersion relation can be factorized, resulting in the two dispersion laws

\[ \omega_h = -|B_0| p, \tag{2a} \]
\[ \omega_m = \frac{|B_0| p r^2 k^2}{1 + r^2 k^2}. \tag{2b} \]

(Alfvén and magnetic Rossby waves; \( k = (p, q) \) is the wavevector \( k^2 = p^2 + q^2 \)).

We then see from Equation \( 2(b) \) that the long magnetic Rossby waves mostly carry magnetic energy, while in relatively short waves, the mechanical and magnetic energies have roughly the same magnitude. The following picture emerges. Some source (e.g., the arrival of light fluid bubbles from the inner core) excites relatively short waves \( \langle k \rangle \gtrsim 1 \). Interacting, they produce other waves. Most of the energy from the source is transferred toward large \( k \), but a small fraction of that energy gets into long waves \( \langle k \rangle \ll 1 \) and accumulates there. Most of the accumulated energy is magnetic. This picture is based on the domination of the zonal magnetic field \( B_0 \), but it will slowly decay (since \( B_0 \) slowly varies in space; only locally \( B_0 \) can be considered constant). In order for the dynamo to take place, there should be some energy supply specifically into the large-scale zonal magnetic field.

We use the adiabatic invariant (Balk et al. 1991; Balk 1991) to show that the energy mainly accumulates in the zonal magnetic field, and so the background field \( B_0 \) is maintained. This extra invariant was found for the usual hydrodynamic Rossby waves and plasma drift waves. However, since the dispersion law in Equation \( 2(b) \) differs from the Rossby dispersion law only by a Doppler shift, the extra invariant holds for the magnetic Rossby waves as well. It has an unusual spectral density

\[ \eta = \arctan \left( \frac{q + p \sqrt{3}}{rk^2} \right) - \arctan \left( \frac{q - p \sqrt{3}}{rk^2} \right). \tag{3} \]

the extra invariant is integral over \( k \) of the product of this function with the wave action spectrum (see Section 3). The function in Equation \( 3 \) is actually the only density that gives an independent extra invariant (in addition to the energy and momentum). We consider the buildup of magnetic field in the process resembling sunspot activity.

2. Waves in Shallow MHD

The MHD of stratified incompressible Boussinesq conducting fluid layer is described by the equations

\[ \nabla \times (v \cdot \nabla) v + f \times v = -\nabla \Phi + (B \cdot \nabla) B + \partial \hat{\rho} / \rho^*, \]
\[ \partial_t + v \cdot \nabla \rho = 0, \quad \nabla \cdot v = -\nabla \cdot B = 0, \]
\[ B_t + (v \cdot \nabla) B = (B \cdot \nabla) V, \]

written in the local Cartesian coordinates of the tangent plane layer (\( x \)-east, \( y \)-north, \( z \)-upward), with the gravitational and Coriolis forces included, \( \rho = [0, 0, -g] \) and \( f = [0, 0, f(y)] \), respectively. The fluid velocity \( V \), the magnetic field \( B \), the fluid density \( \rho \), and the divided by \( \rho^* \) pressure, \( \Phi \), are unknown functions of \( x, y, z \), and time \( t \); \( \partial \hat{\rho} / \rho^* \) is some value of almost constant density \( \rho \). Besides the thermodynamic pressure, \( \Phi \) also includes the magnetic pressure and the term resulting from the
centrifugal force. The magnetic field is normalized to have velocity units \((\mathbf{B}, \sqrt{\Omega_0 \mu})\) is the real magnetic field, where \(\mu\) is magnetic permeability); the magnetic diffusion and the viscous force are neglected.

The dynamics have the steady solution

\[
V_0 = 0, \quad B_0 = [B_0, 0, 0], \quad \varphi_0(z), \quad \Phi_0(z)
\]

\((\Phi_0 = -g \varphi_0/\varphi^*)\). Linearization on its background, \(V = V_0 + v, \quad B = B_0 + b, \quad \varphi = \varphi_0 + \rho, \quad \Phi = \Phi_0 + \varphi\), gives

\[
v^z - f v^x = -\dot{\varphi}_x + B_0 b^y,
\]

\[
v^y + f v^x = -\dot{\varphi}_y + B_0 b^z,
\]

\[
v^y = -\dot{\varphi}_z - g \rho / \varphi^* + B_0 b^x,
\]

\[
\rho v^z + \varphi = 0, \quad \nabla \cdot v = \nabla \cdot b = 0,
\]

\[
b^x = B_0 v^x, \quad b^y = B_0 v^y, \quad b^z = B_0 v^z
\]

(see Braginsky 1987, 1998); subscripts \(x, y, z, t\) denote partial derivatives, while superscripts \(x, y, z\) denote vector components.

Take \(\partial_x\) of Equation (4b) and subtract \(\partial_y\) of Equation (4a)

\[
(v^x - v^y)_{,t} + f (\rho_0/\varphi_0)_{,t} + \beta \varphi = B_0 (b^y - b^x)_{,t},
\]

here we also use Equation (4d); \(\beta = f'(y)\). Similarly, from the first two equations in Equation (4e),

\[
(b^x - b^y)_{,t} = B_0 (v^x - v^y)_{,t},
\]

Now let us use the quasi-geostrophic and hydrostatic approximations, common in geophysical fluid dynamics (e.g., Vallis 2006)

\[
v^x = -\dot{\varphi}_x / f, \quad v^y = \dot{\varphi}_y / f, \quad \dot{\varphi}_z = -g \rho / \varphi^*;
\]

then Equations 5(a) and (b) become a closed system on the pressure \(\varphi\) and the vertical component of the electric current density \(m = b^x - b^y\)

\[
\Delta \dot{\varphi} - \frac{f^2}{8} \left( \frac{\dot{\varphi}_x^2}{\varphi^*} + \frac{\dot{\varphi}_y^2}{\varphi^*} \right) \approx \beta \dot{\varphi}_x = f B_0 \dot{m}, \quad m = f^{-1} B_0 \Delta \dot{\varphi},
\]

\[
m = f^{-1} B_0 \Delta \varphi_x
\]

\((\Delta\) denotes the 2D Laplacian \(\partial_x^2 + \partial_y^2\)). These equations should be supplemented by the boundary condition of the vanishing normal velocity at the core–mantle boundary

\[
v^z = 0 \quad \Leftrightarrow \quad \dot{\varphi}_z = 0.
\]

The system described by Equation (6) conserves the following positive-definite \((\varphi_0 < 0)\) integral

\[
\int \left[ \frac{\Delta \dot{\varphi}^2}{f} - \frac{\dot{\varphi}_x^2}{\varphi^*} \right] dxdydz
\]

(the integration being over the fluid domain \(z < 0\)).

The system (Equation (6)) allows separation of variables

\[
\phi = Z(z) \tilde{\phi}(x, y, t), \quad m = Z(z) \tilde{m}(x, y, t),
\]

\[
\frac{g}{f^2} \Delta \tilde{\phi}_x + \beta \tilde{\phi}_x - f B_0 \tilde{m}_x = \frac{1}{Z} \left( \frac{Z'}{\varphi_0^*} \right)'
\]

Denoting the separation constant by \(1/\hbar\), we find

\[
\Delta \tilde{\phi}_x - \left( \frac{f^2}{gh} \right) \tilde{\phi}_x + \beta \tilde{\phi}_x = f B_0 \tilde{m}_x,
\]

\[
m = f^{-1} B_0 \Delta \varphi_x
\]

\((\varphi_0^* = \varphi^* \equiv \hbar = \frac{1}{h})\). Exponential density profile:

\[
\varphi_0 = \varphi^*[1 - \exp(z/H)],
\]

where \(H\) is the effective depth, and the parameter \(\epsilon < 1\) controls the total density variation. Linear density profile:

\[
\varphi_0 = \varphi^*[1 - \epsilon (H + z)/H]
\]

\((z = -\infty\) is replaced by \(z = -H, -H < z < 0\). In this case, \(Z'' + (\epsilon / H) Z = 0, \quad Z'(0) = Z(-H) = 0, \quad Z'(+\infty) = 1\), and

\[
\omega^2(k^2 + r^2) - \beta \varphi = B_0^2 p^2 k^2;
\]

the Rossby radius \(r\) and the wavevector \(k = (p, q)\) are introduced in Section 1.

Model examples. Exponential density profile:

\[
\varphi_0 = \varphi^*[1 - \epsilon (z/H)]
\]

with some function \(\chi(\zeta)\) that monotonically increases from 0 at \(\zeta = -\infty\) to 1 at \(\zeta = 0\). Then \(h = O(h/H)\).

If the magnetic field is negligible, we have the usual hydrodynamic Rossby waves, \(\omega \propto -\beta / (k^2 + r^2)\). When \(h = \infty\), the relation in Equation (9) coincides with the one derived by Hide (1966) for the 2D dynamics of the Taylor–Proudman columns extending through the bulk of the liquid core and constrained by the core–mantle boundary. The dispersion relation in Equation (9) was obtained for waves in “shallow water” MHD (Gilman 2000) by Zaqarashvili et al. (2007), in that case, \(h\) is the depth of the shallow layer, and \(g\) is the reduced gravity. For large \(k\), the relation in Equation (9) gives the Alfvén waves, \(\omega \approx \pm B_0 k\).

Let us estimate the two scales in Equation (1) for the Earth and Sun. Earth: \(f \approx 7 \times 10^{-5} \text{s}^{-1}, \beta \approx 4 \times 10^{-11} \text{m}^{-1} \text{s}^{-1}\) (at latitude 30°). The toroidal magnetic field has no signature on the surface of the earth, and so \(B_0\) is somewhat uncertain; we assume \(B_0 \approx 0.3 \text{m} \text{s}^{-1}\), while \(c_g \approx 7 \text{m} \text{s}^{-1}\). Then \(r \approx \ell \approx 90 \text{km}\).

Sun: For the solar tachocline, the situation is less clear. The assumptions made to derive the system of Equation (6) can be unsatisfactory. The Boussinesq approximation can fail. The geostrophic balance for the Sun is “marginal” (Gilman & Dikpati 2014). Moreover, the parameters of the tachocline significantly vary across it. Even parameters \(f, \beta\) are less obvious, because of the Sun’s differential rotation. We assume \(f \approx 3 \times 10^{-6} \text{s}^{-1}, \beta \approx 10^{-14} \text{m}^{-1} \text{s}^{-1}\), \(B_0 \approx c_g \approx 300 \text{m} \text{s}^{-1}\) (see Schecter et al. 2001). Then \(r \approx 3 \times 10^6 \text{m}\) and \(\ell \approx 2 \times 10^6 \text{m}\).

Interestingly, in both cases (Earth and Sun), the scales \(\ell\) and \(r\) are similar. If these scales are exactly equal

\[
c_g^2 / f^2 = |B_0| / \beta,
\]

the dispersion relation in Equation (9) can be factorized, and its two roots are exactly the frequencies in Equation (2). They represent a west-propagating Alfvén wave and an east-propagating magnetic Rossby wave (east is defined in the
direction of rotation, north is $\pi/2$ counterclockwise from the east, and so $\beta > 0$. Figure 1 shows the exact smaller frequency defined by the dispersion relation in Equation (9).

3. Dynamo Action

3.1. Longer Waves Have Mostly Magnetic Energy

Consider the energy share of each quantity $v^x$, $v^y$, $v^z$, $\rho$, $b^x$, $b^y$, $b^z$

$$
\begin{bmatrix}
E_{v^x} & E_{v^y} & E_{v^z} \\
E_{b^x} & E_{b^y} & E_{b^z}
\end{bmatrix}
\equiv \begin{array}{c}
m \int_{-\infty}^{0} \left( |v^x|^2 \quad |v^y|^2 \quad |v^z|^2 \right) dz, \\

E_{b^x} \quad E_{b^y} \quad E_{b^z},
\end{array}
$$

One can check that the integral

$$
E = \int (E_{b^x} + E_{b^y} + E_{b^z} + E_{v^x} + E_{v^y} + E_{v^z}) dx dy
$$
is conserved by the system described by Equation (4) under the boundary condition in Equation 6(c). This conservation holds because the vector $B_0$ has no $z$ component; otherwise the energy integral would need to be extended outside of the fluid domain.

We can express all variables via $\phi$ and $m$. The quasi-geostrophy gives $v^x$ and $v^y$; the hydrostacy gives $\rho$. From the first formula in Equation 4(d), $v^z = -\rho_0/\phi_0$. From the first formula in Equation 4(d) and the third in Equation 4(e),

$$(b^z + B_0 \rho_0 / \phi_0) = 0 \implies b^z = -B_0 \rho_0 / \phi_0.$$

This is the linearized version of the magnetic analog of Ertel’s theorem (Hide 1983), saying that the quantity $B \cdot \nabla \phi$—in the fully nonlinear dynamics—is constant along the lines of fluid motion. Using the last formula in Equation 4(d) and the definition of $m$, we find $b^x$ and $b^y$

$$
\Delta b^x = -m_y - b^x_{b^y}, \quad \Delta b^y = m_x - b^y_{b^x}.
$$

Take a single wave

$$
\phi = Z(z) e^{i(px + qy - \omega t)}.
$$

According to Equation 6(a) and to the separation of variables,

$$
m = \frac{\omega (k^2 + r^2) + \beta p}{f B_0 \rho} Z e^{i(px + qy - \omega t)}.
$$

Now we find all the quantities in that wave (the right-hand sides in the following equations are to be multiplied by $\exp [i(px + qy - \omega t)]$)

$$
v^x = -\frac{i q Z}{f}, \quad v^y = \frac{i p Z}{f}, \quad \rho = -\frac{\phi^* Z'}{g},
$$

$$
v^z = \frac{-i \omega Z'}{g \phi_0 / \phi^*}, \quad b^z = \frac{i B_0 Z'}{g \phi_0 / \phi^*},
$$

$$
b^x = \frac{1}{k^2} \left[ \frac{i q \omega (k^2 + r^2 - \beta p)}{f B_0 p} - p^2 B_0 - \frac{q B_0 Z'}{gh} \right],
$$

$$
b^y = \frac{1}{k^2} \left[ \frac{-i \omega (k^2 + r^2 - \beta p)}{f B_0 p} - pq B_0 - \frac{q B_0 Z'}{gh} \right].
$$

(Equations 12(c), (d) use Equation 8(c)).

Equation (12) determines the energy shares; we write them for the magnetic Rossby wave, when $\omega$ is given by Equation 2(b), and Equation (11) holds

$$
E_v^x = \frac{q^2}{f^2}, \quad E_v^y = \frac{p^2}{f^2}, \quad E_v^z = \frac{1}{e_v^2},
$$

$$
E_b^x = \frac{\beta p^2 k^4}{f^4 (k^2 + r^2)^2} \Lambda, \quad E_b^y = \frac{\beta p^2 k^4}{f^4 \Lambda},
$$

$$
E_b^z = \frac{1}{f^2 k^4} [q^2 (k^2 + r^2)^2 + p^2 (\beta f)^2],
$$

$$
E_b^r = \frac{1}{f^2 k^4} [p^2 (k^2 + r^2)^2 + p^2 q^2 (\beta f)^2],
$$

where all right-hand sides should be multiplied by $(1/2) \int_{-\infty}^{0} Z^2 dz$, and

$$
\Lambda = h^2 \int \left( \frac{Z'}{(\phi_0 / \phi^*)} \right)^2 dz / \int Z^2 dz.
$$

Equation (13) shows that the content of magnetic energy $E_b = E_b^x + E_b^y + E_b^z$ is bigger in longer waves. This is because of $k^4$ in the denominators of Equations 13(c), (d). More on this in Section 3.3.) We assume that sources generate relatively short waves. Interacting between each other, they produce longer waves. Most of the energy is carried by long waves. The energy accumulation in long waves is a very general fact, occurring in a majority of physical systems. It is independent of cascades: Even if the energy cascade exists, it can be directed toward the smaller scales. For example, in the weakly nonlinear system of gravity waves (Zakharov 1985), the energy does cascade toward the small scales, but the long waves carry most of the energy. Specifically in the system of slow magnetic waves, the energy accumulation in long waves was indicated (Balk 2014) by the infrared divergence of the corresponding Kolmogorov–Zakharov spectrum.

3.2. The Energy Accumulation in the Zonal Magnetic Field

However, this transfer of the small–scale kinetic energy into the large-scale magnetic energy is insufficient for the dynamo. The above approach requires dominating the zonal magnetic field $B_0$ but it will gradually decay (since $B_0$ slowly varies in space). There should be some mechanism of energy supply specifically into the large-scale zonal magnetic field. This mechanism—described below—is due to the extra invariant.
The main interactions in an arbitrary wave system are due to the resonances

\[ p_1 = p_2 + p_3, \quad q_1 = q_2 + q_3, \quad \omega_1 = \omega_2 + \omega_3, \]  

(14)

where \( \omega = \omega(k) \) is the frequency of the wave with wavevector \( k = (\omega_1, q_1, \ldots) \) (i = 1, 2, 3).

A dispersion law \( \omega(k) \) is said to be degenerative (Zakharov 
& Schulman 1980) if there exists an independent function \( \eta(k) \) conserved in the resonance interactions. Whenever vectors \( \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \) satisfy Equation (14), we also have \( \eta(\mathbf{k}_1) = \eta(\mathbf{k}_2) + \eta(\mathbf{k}_3) \); the function \( \eta(k) \) should be linearly independent of the functions \( p, q, \) and \( \omega(k) \)—otherwise the \( \eta \) equation is a mere linear combination of the resonance equations.

In a weakly nonlinear system, the conservation of the function \( \eta(k) \) in resonance interactions (Equation (14)) implies the adiabatic conservation of the integral

\[ I = \int \eta(k) \frac{E_k}{\omega(k)} \]  

(15)

The existence of the extra invariant is independent of the form of nonlinearity, as long as the system is Hamiltonian (which is usually the case for a physical system). This is proved by showing cancellation of the small denominator (Zakharov 
& Schulman 1988).

The conservation is adiabatic in the sense that it holds approximately over long time. If the wave amplitudes have order \( \epsilon \rightarrow 0 \), then \( I = O(\epsilon^2) \), but \( \Delta I = I(t) - I(0) \) is at least \( O(\epsilon^3) \) over time intervals \( O(\epsilon^{-1}) \).

Any wave system conserves obvious invariants of the momentum and energy

\[ P_x = \int \frac{P}{\omega} E_k dk, \quad P_y = \int \frac{q}{\omega} E_k dk, \quad E = \int E_k dk. \]

The conservation of these integrals are implied by the resonance relations (Equation (14)). For the Rossby wave system, the y-momentum \( P_y \) is not a real physical invariant (see Balk 
& Yoshikawa 2009), since the function \( q/\omega \) is singular (as \( p \rightarrow 0 \)). The linear combination of the x-momentum and the energy determines the enstrophy

\[ F = \tau^2 (|B_0|^2 P_x - E) = \int \frac{E_k}{k^2} dk. \]

The conservation of the energy \( E \) and enstrophy \( F \) implies that most of the energy from the source should be transferred toward large \( k \), and most of the enstrophy toward small \( k \).

**Comment.** I call the integral \( F \) the enstrophy just because it is a linear combination of the x-momentum and the energy (similar to the usual system of hydrodynamic Rossby waves, where the enstrophy is also a linear combination of the x-momentum and energy). The conservation of \( F \) follows from the resonance relations in Equation (14) because functions \( p \) and \( \omega \) are trivially conserved in the triad interactions of Equation (14). If the energy transfer is local in scales, then unlike the 2D hydrodynamics, here, the energy cascades toward small scales, while \( F \) cascades toward large scales.

The Rossby wave dispersion law (Equation 2(b)) is degenerative (Balk et al. 1991; Balk 1991), with the function \( \eta(k) \) given in Equation (3). This is actually the only independent function conserved in the interactions in Equation (14). The conservation of the corresponding integral

\[ \eta(k) \frac{E_k}{\omega(k)} \]  

in Equation (15) implies the energy accumulation in the region \( p \ll q \) (Balk 2005). We will now explain this fact and show the energy accumulation in the large-scale zonal magnetic field. Consider a modification of \( \eta \)

\[ \tilde{\eta}(k) = |B_0| \eta(k) + 2\sqrt{3} r [\omega(k) - |B_0| p]. \]

(16)

Obviously, \( \tilde{\eta} \) is also conserved in the resonance interactions (it is a linear combination of conserved functions), but it has better asymptotic properties

\[ \tilde{\eta}(k) \sim |B_0| 8\sqrt{3} \times \begin{cases} p^3 \frac{t}{q^2(1 + r^2q^2)}, & p \rightarrow 0, \\
-5^5 \frac{p^3(q^2 + 5p^2)}{5^5k^5}, & k \rightarrow \infty,
\end{cases} \]

(17)

so that \( \tilde{\eta} \) rapidly decreases as \( p \rightarrow 0 \) or \( k \rightarrow \infty \) (faster than \( \omega, p, \) or \( \eta \) alone). Interestingly, in Equation (17), the first asymptotics holds for all values of \( q \), and the second in all directions.

The energy parcel \( E_k dk \) always carries with it the enstrophy parcel \( \tilde{E}_k dk \) and the parcel \( |B_0| \tilde{\eta}(k) \omega(k) E_k dk \) of the extra invariant \( I = \int \frac{\tilde{\eta}(k)}{\omega(k)} E_k dk \), with the latter being similar to the integral in Equation (15). Figure 2 shows the ratio of the extra invariant parcel to the energy parcel

\[ G(p, q) = \frac{\tilde{\eta}(k)}{\omega(k)} E_k dk \]  

(18)

Figure 2 and the asymptotics in Equation (17) imply the energy accumulation in region \( p \ll q \). Indeed, if the energy is accumulated at large scales away from the region \( p \ll q \), then a huge amount of the extra invariant would be required. This can be seen in Figure 2, which shows large values of the function in Equation (18) away from the \( q \)-axis, when \( rk \ll 1 \). But the extra invariant generation is limited by the source.

Now, according to Equation (13), the accumulated energy is mostly \( F_p \).

**3.3. Estimating Energy Shares in Long Waves**

To see the latter more accurately, let us first note that the second term on the right in Equations 13(c), (d) is negligible.

![Figure 2. Extra invariant per energy, i.e., the function in Equation (18). The color bar shows the values of \( \ln[G(p, q)] \). Each of the two extreme areas (where \( G \ll e^{-12} \) or \( G \gg e^4 \)) is shown by a single color.](image-url)
Indeed, these equations can be written in the form

\[
\begin{align*}
E_b^x &= \frac{q^2 f}{k^2} \left( \frac{k^2 + r^{-2}}{k^2} \right)^2 \left[ 1 + \left( \frac{p^2}{k^2 + r^{-2}} \right)^2 \right], \\
E_b^y &= \frac{p^2 f}{k^2} \left( \frac{k^2 + r^{-2}}{k^2} \right)^2 \left[ 1 + \left( \frac{q^2}{k^2 + r^{-2}} \right)^2 \right],
\end{align*}
\]

where \( \mathcal{L}' = q^{-1} \) is the length scale in the y-direction. The applicability of the \( \beta \)-plane approximation requires \( \beta \mathcal{L}' \ll f \), and so the bracketed expressions here are roughly equal to 1. Thus,

\[
\frac{E_v^x}{E_b^x} \sim \frac{E_v^y}{E_b^y} \sim \frac{E_v^z}{E_b^z} = \left( \frac{r^2 k^2}{1 + r^2 k^2} \right)^2 \sim \begin{cases} 1, & rk \gg 1, \\
(rk)^4, & rk \ll 1. \end{cases}
\]

(Strictly speaking, when \( rk \gg 1 \), the value of expression \( r^2 k^2/(1 + r^2 k^2) \) changes from 1/2 to 1, but we disregard this.) From the third formula in Equation 13(a) and the relation in Equation (11),

\[
\frac{E_b^y}{E_b^x + E_b^y} \sim \left( \frac{rk}{1 + r^2 k^2} \right)^2 \sim \begin{cases} (rk)^2, & rk \gg 1, \\
1/4, & rk \sim 1, \\
(rk)^2, & rk \ll 1. \end{cases}
\]

These asymptotics show that, in short waves \((rk \gg 1)\), the magnetic and mechanical energies are of the same order, but long waves \((rk \ll 1)\) mostly carry magnetic energy. Now,

\[
\begin{align*}
\frac{E_b^y}{E_b^x} &\sim \left( \frac{p^2}{q} \right)^2, \\
\frac{E_b^x}{E_b^y} &\sim \left( \frac{p}{q} \right)^2 \left( \frac{r^2 k^2}{1 + r^2 k^2} \right)^2 \left( \frac{\beta \mathcal{L}'}{f} \right)^2.
\end{align*}
\]

For the density profile in Equation 10(c), the distance \( \Lambda \) has the magnitude of the effective depth \( H \lesssim \mathcal{L} \), and so \( \beta \Lambda \lesssim f \). Therefore, Equation (19) shows the dominance of \( E_b^y \) in the large-scale magnetic energy.

**Latitudinal dependence.** Let us note that Equation (11) is dependent on the latitude \( \vartheta \), since

\[ f = 2 \Omega_0 \sin \vartheta, \quad \beta = (2 \Omega_0 / R_0) \cos \vartheta \]

\( (\Omega_0 \) is the angular speed of the Earth’s rotation); \( f \to 0 \) as \( \vartheta \to 0 \), and \( \beta \to 0 \) as \( \vartheta \to \pi/2 \). So, Equation (11)—which we can refer to as the scale resonance—is realized at some latitude \( \vartheta \) (provided \( c_e \) does not strongly depend on the latitude, as \( c_e \) is not directly related to the rotation or magnetic field). Because of the energy accumulation in the large-scale zonal magnetic field, the magnitude \( B_0 \) will increase, and therefore Equation (11) will be no longer satisfied at the latitude \( \vartheta \). However, the scale resonance in Equation (11) will be satisfied at a new lower latitude \( \vartheta = \vartheta_0 \), and so the magnetic field will still be maintained. If the energy supply is large enough, \( B_0 \) will still increase. The latitude responsible for the generation of the magnetic field will gradually decrease. This resembles sunspot activity; a similar phenomenon should take place in the Earth.

In this process, the basic zonal magnetic field is buildup to

\[ |B_0| \approx \frac{c_e^2}{2R_0 \Omega_0} \sin^2 \vartheta. \]

As the resonance latitude (where the scale resonance in Equation (11) is realized) changes, say from \( 35^\circ \) to \( 15^\circ \), the Alfvén speed \( B_0 \) increases about 6 times (for the Earth, from \( 0.25 \text{ m s}^{-1} \) to \( 1.5 \text{ m s}^{-1} \); for the Sun, from \( 75 \) to \( 450 \text{ m s}^{-1} \)), provided \( c_e \) remains constant.

**4. Discussion**

The possible connection between dynamo and the extra invariant was suggested in an earlier paper (Balk 2014). However, that paper considered only the dynamics of long waves; they do have the dispersion in Equation 2(b)—even without the relation \( r = \ell \) (the scales \( r \) and \( \ell \) are introduced in Section 1), but it was required that \( rk \ll 1 \). In this situation, the extra invariant implies the energy accumulation only in the sector \( 60^\circ < \theta < 90^\circ \) (\( \theta \) is the polar angle of the wavevector \( k \)). This is insufficient for dynamo (the energy accumulation fails to ensure the maintenance of the background zonal magnetic field \( B_0 \)). The paper (Balk 2014) pointed out that if the generated amount of the extra invariant \( \tilde{I} \) is small enough, then the energy would accumulate near the axis \( q \) (\( \theta = 90^\circ \)), ensuring the maintenance of \( B_0 \). However, I do not see any physical reason to expect the small generation of the extra invariant inside the Earth or Sun. (That paper only noted the application of the extra invariant to a different problem of the nuclear fusion, namely, to the emergence of a zonal flow in the drift wave dynamics. There, we indeed can control the sources and sinks, ensuring the small generation of the extra invariant.) The present paper considers magnetic Rossby waves of any wavelength (compared to the length \( r \)). We have noted in the Introduction that in the two cases of the Earth and Sun, the scales in Equation (1) have the same order; if these scales are exactly equal, \( r = \ell \), then the quadratic dispersion relation in Equation (9) can be factorized, producing exactly the dispersion in Equation 2(b) even for shorter waves (when \( rk \gg 1 \)). So, it is unnecessary to assume the small amount of the extra invariant \( \tilde{I} \); the dynamics always lead to energy accumulation in the zonal magnetic field.

Let us stress the following circumstance. Only linear dispersion relations have been derived in the present paper. However, this allows us to make conclusions about weakly nonlinear dynamics (Zakharov & Schulman 1980). In particular, the conservation of the function \( \eta(k) \) in the resonance interactions of Equation (14) implies the adiabatic conservation of the integral in Equation (15) in the weakly nonlinear dynamics. The invariant in Equation (15) is exactly conserved by the wave kinetic equation, which accounts for the triad resonance interactions. The role of nonresonance interactions decreases as interactions become more nonresonant. So, the dynamics conserve the extra quantity \( I \) approximately over long time, i.e., adiabatically.

Unfortunately, I do not have a simple physical mechanism explaining why the MHD lead to energy transfer into the zonal magnetic field. I do not even know the physical meaning of the extra conservation, let alone the corresponding continuous symmetry (F. van Heerden 2007, private communication).

Let us mention what happens if the background magnetic field has both toroidal and poloidal components:
\( B_0 = (B_0^x, B_0^y, 0) \). In this situation, the separation of variables is still possible; it leads to the dispersion relation

\[
\omega^2 (k^2 + r^{-2}) + \beta \nu \omega = (B_0^x p + B_0^y q)^2 k^2 \tag{20}
\]

(Hide 1966; Zaqarashvili et al. 2007). However, this more general dispersion relation does not allow factorization. Moreover, in the limit of long waves, Equation (20) does not produce the Rossby dispersion law, and the extra conservation fails. Even more, if \( B_0^y \) cannot be neglected compared to \( B_0^x \), the dispersion relation in Equation (20) shows that the \( \omega^2 \) term is essential (without the \( \omega^2 \) term, \( p \to 0 \Rightarrow \omega \to \infty \)).

To summarize, the present paper is based on the observation that the two length scales \( r \) and \( \ell \) (introduced in Equation (1)) have the same order of magnitude for both the Earth and Sun. The considered dynamo mechanism can be expressed in two statements:

1. The linearization on the background of the strong zonal magnetic field \( B_0 \) (Section 2) shows that longer waves carry more magnetic energy (Section 3.1), and so, under the assumption of energy accumulation in long waves, small-scale kinetic energy is transferred into large-scale magnetic energy.

2. The presence of the extra invariant shows that the transferred energy is mainly accumulated as zonal magnetic energy \( (p \ll q) \), and so the background magnetic field \( B_0 \) is maintained (Section 3.2). This accumulation also supports the first statement (Section 3.3).

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