VERY SIMPLE REPRESENTATIONS: VARIATIONS ON A
THEME OF CLIFFORD

YURI G. ZARHIN

Abstract. We discuss a certain class of absolutely irreducible group representations that behave nicely under the restriction to normal subgroups and subalgebras. Interrelations with doubly transitive permutation groups and hyperelliptic jacobians are discussed.

1. Introduction

We start this paper with the following natural definition.

Definition 1.1. Let $V$ be a vector space over a field $k$, let $G$ be a group and $\rho: G \to \text{Aut}_k(V)$ a linear representation of $G$ in $V$. Suppose $R \subset \text{End}_k(V)$ is an $k$-subalgebra containing the identity operator $\text{Id}$. We say that $R$ is normal if

$$\rho(s)R\rho(s)^{-1} \subset R \quad \forall s \in G.$$

Examples 1.2. Clearly, $\text{End}_k(V)$ is normal. The algebra $k \cdot \text{Id}$ of scalars is also normal. If $H$ is a normal subgroup of $G$ then the image of the group algebra $k[H]$ in $\text{End}_k(V)$ is normal.

The following assertion is a straightforward generalization of well-known Clifford's theorem [1; 3, §49]; [13, §8.1]; [7, Ch. 6].

Lemma 1.3 (Lemma 7.4 of [15]). Let $G$ be a group, $k$ a field, $V$ a non-zero $k$-vector space of finite dimension $n$ and $\rho: G \to \text{Aut}_k(V)$ an irreducible representation. Let $R \subset \text{End}_k(V)$ be a normal subalgebra. Then:

(i) The faithful $R$-module $V$ is semisimple.

(ii) Either the $R$-module $V$ is isotypic or there exists a subgroup $G' \subset G$ of index $r$ dividing $n$ and a $G'$-module $V'$ of finite $k$-dimension $n/r$ such that $r > 1$ and the $G$-module $V$ is induced from $V'$.

The following notion was introduced by the author in [15] (see also [16]); it proved to be useful for the construction of abelian varieties with small endomorphism rings [15, 17].

Definition 1.4. Let $V$ be a non-zero finite-dimensional vector space over a field $k$, let $G$ be a group and $\rho: G \to \text{Aut}_k(V)$ a linear representation of $G$ in $V$. We say that the $G$-module $V$ is very simple if it enjoys the following property:

If $R \subset \text{End}_k(V)$ is a normal subalgebra then either $R = k \cdot \text{Id}$ or $R = \text{End}_k(V)$.

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Here is (obviously) an equivalent definition: if $R \subset \text{End}_k(V)$ is a normal subalgebra then either $\text{dim}_k(R) = 1$ or $\text{dim}_k(R) = (\text{dim}_k(V))^2$.

In this paper we prove that very simple representations over an algebraically closed field are exactly those absolutely irreducible representations that are not induced from a representation of a proper subgroup and do not split non-trivially into a tensor product of projective representations. This assertion remains valid for representations of perfect groups over finite fields. We also give a certain criterion that works for any ground field with trivial Brauer group.

The paper is organized as follows. In §2 we list basic properties of very simple representations. In §3 we discuss certain natural constructions of representations that are not very simple. The Section 4 contains the statement of main results about very simple representations and their proof. In the Section 5 we discuss interrelations between very simple representations and doubly transitive permutation groups. The last section contains applications to hyperelliptic Jacobians.

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2. VERY SIMPLE REPRESENTATIONS

Remarks 2.1. (i) Clearly, if $\text{dim}_k(V) = 1$ then the $G$-module $V$ is very simple. In other words, every one-dimensional representation is very simple.

(ii) Clearly, the $G$-module $V$ is very simple if and only if the corresponding $\rho(G)$-module $V$ is very simple.

(iii) Clearly, if $V$ is very simple then the corresponding algebra homomorphism $k[G] \rightarrow \text{End}_k(V)$ is surjective. Here $k[G]$ stands for the group algebra of $G$. In particular, a very simple module is absolutely simple.

(iv) If $G'$ is a subgroup of $G$ and the $G'$-module $V$ is very simple then the $G$-module $V$ is also very simple.

(v) Suppose $W$ is a one-dimensional $k$-vector space and

$$\kappa : G \rightarrow k^* = \text{Aut}_k(W)$$

is a one-dimensional representation of $G$. Then the $G$-module $V \otimes_k W$ is very simple if and only if the $G$-module $V$ is very simple. Indeed, there are the canonical $k$-algebra isomorphisms

$$\text{End}_k(V) = \text{End}_k(V) \otimes_k k = \text{End}_k(V) \otimes_k \text{End}_k(W) \cong \text{End}_k(V \otimes_k W),$$

which are isomorphisms of the corresponding $G$-modules.

(vi) Let $G'$ be a normal subgroup of $G$. If $V$ is a very simple $G$-module then either $\rho(G') \subset \text{Aut}_k(V)$ consists of scalars (i.e., lies in $k \cdot \text{Id}$) or the $G'$-module $V$ is absolutely simple. Indeed, let $R' \subset \text{End}_k(V)$ be the image of the natural homomorphism $k[G'] \rightarrow \text{End}_k(V)$. Clearly, $R'$ is normal. Hence either $R'$ consists of scalars and therefore $\rho(G') \subset R'$ consists of scalars or $R' = \text{End}_k(V)$ and therefore the $G'$-module $V$ is absolutely simple.

As an immediate corollary, we get the following assertion: if $\text{dim}_k(V) > 1$, the $G$-module $V$ is faithful very simple and $G'$ is a non-central normal subgroup of $G$ then the $G'$-module $V$ is absolutely simple; in particular,
**VERY SIMPLE REPRESENTATIONS**

$G'$ is non-abelian. In addition, if $k = F_2$ then the only abelian normal subgroup of $G$ is the trivial (one-element) subgroup.

(vii) Suppose $F$ is a discrete valuation field with valuation ring $O_F$, maximal ideal $m_F$ and residue field $k = O_F/m_F$. Suppose $V_F$ is a finite-dimensional $F$-vector space and

$$\rho_F : G \to \text{Aut}_F(V_F)$$

is a $F$-linear representation of $G$. Suppose $T$ is a $G$-stable $O_F$-lattice in $V_F$ and the corresponding $k[\Gamma]$-module $T/m_FT$ is isomorphic to $V$. Assume that the $G$-module $V$ is very simple. Then the $G$-module $V_F$ is also very simple. In other words, a lifting of a very simple module is also very simple. (See [15], Remark 5.2(v).)

**Example 2.2.** Suppose $k = F_2, \dim_k(V) = 2, G = \text{Aut}_k(V) = \text{GL}_2(F_2)$. Then the faithful absolutely simple $G$-module $V$ is not very simple. Indeed, $G$ is isomorphic to the symmetric group $SS_3$ and therefore contains a non-central abelian normal subgroup isomorphic to the alternating group $A_3$. By Remark 2.1(vi), the $G$-module $V$ is not very simple.

Applying Remarks 2.1(ii) and 2.1(iv), we conclude that all two-dimensional representations over $F_2$ of any group are not very simple.

**Example 2.3.** Suppose $V$ is a finite-dimensional vector space over a finite field $k$ of characteristic $\ell$ and $G$ is a perfect subgroup of $\text{Aut}(V)$ enjoying the following properties:

(i) If $Z$ is the center of $G$ then the quotient $\Gamma := G/Z$ is a simple non-abelian group.

(ii) Every nontrivial projective representation of $\Gamma$ in characteristic $\ell$ has dimension $\geq \dim_k(V)$.

Then the $G$-module $V$ is very simple. See Cor. 5.4 in [15].

3. **Counterexamples**

Throughout this section, $k$ is a field, $V$ a non-zero finite-dimensional $k$-vector space and $\rho : G \to \text{Aut}_k(V)$ is a linear representation of $G$ in $V$.

**Example 3.1.** (i) Assume that there exist $k[\Gamma]$-modules $V_1$ and $V_2$ such that $\dim_k(V_1) > 1, \dim_k(V_2) > 1$ and the $G$-module $V$ is isomorphic to $V_1 \otimes_k V_2$. Then $V$ is not very simple. Indeed, the subalgebra

$$R = \text{End}_k(V_1) \otimes \text{Id}_{V_2} \subset \text{End}_k(V_1) \otimes_k \text{End}_k(V_2) = \text{End}_k(V_1 \otimes_k V_2) = \text{End}_k(V)$$

is normal but coincides neither with $k \cdot \text{Id}$ nor with $\text{End}_k(V)$. (Here $\text{Id}_{V_2}$ stands for the identity operator in $V_2$.) Clearly, the centralizer of $R$ in $\text{End}_k(V)$ coincides with $\text{Id}_{V_1} \otimes \text{End}_k(V_2)$ and is also normal. (Here $\text{Id}_{V_1}$ stands for the identity operator in $V_1$.)

(ii) Let $X \to G$ be a surjective group homomorphism. Assume that there exist $k[X]$-modules $V_1$ and $V_2$ such that $\dim_k(V_1) > 1, \dim_k(V_2) > 1$ and $V$, viewed as $X$-module, is isomorphic to $V_1 \otimes_k V_2$. Then the $X$-module $V$ is not very simple. Since $X$ and $G$ have the same image in $\text{Aut}_k(V)$, the $G$-module $V$ is also not very simple.
Definition 3.2. Let $V$ be a vector space over a field $k$, let $G$ be a group and $ho : G \to \text{Aut}_k(V)$ a linear representation of $G$ in $V$. Let

$$1 \to C \hookrightarrow X \xrightarrow{\pi} G \to 1$$

be a central extension of $G$, i.e., $C$ is a central subgroup of $G$ which coincides with the kernel of the surjective homomorphism $\pi : X \to G$. Suppose that the representation

$$X \xrightarrow{\pi} G \xrightarrow{\rho} \text{Aut}_k(V)$$

of $X$ is isomorphic to a tensor product $\rho_1 \otimes \rho_2 : X \to \text{Aut}_k(V_1 \otimes_k V_2)$ of two $k$-linear representations

$$\rho_1 : X \to \text{Aut}_k(V_1), \quad \rho_2 : X \to \text{Aut}_k(V_2)$$

with

$$\dim_k(V_1) > 1, \quad \dim_k(V_2) > 1.$$ 

Then we say that the $G$-module $V$ splits and call the triple $(X \xrightarrow{\pi} G; \rho_1, \rho_2)$ a splitting of the $G$-module $V$.

We say that $V$ splits projectively if both images $\rho_1(C) \subset \text{Aut}_k(V_1)$ and $\rho_2(C) \subset \text{Aut}_k(V_2)$ consist of scalars. In other words, one may view both $\rho_1$ and $\rho_2$ as projective representations of $G$. In this case we call $(X \xrightarrow{\pi} G; \rho_1, \rho_2)$ a projective splitting of the $G$-module $V$.

We call a splitting $(X \xrightarrow{\pi} G; \rho_1, \rho_2)$ absolutely simple if both $\rho_1$ and $\rho_2$ are absolutely irreducible representations of $X$.

Remarks 3.3. We keep the notations of Definition 3.2.

(i) Clearly,

$$\text{End}_X(V_1) \otimes_k \text{End}_X(V_2) \subset \text{End}_X(V_1 \otimes_k V_2) = \text{End}_X(V) = \text{End}_G(V).$$

This implies that if $\text{End}_G(V) = k$ then $\text{End}_X(V_1) = k$ and $\text{End}_X(V_2) = k$.

(ii) Suppose $W$ is a proper $X$-invariant subspace in $V_1$ (resp. in $V_2$). Then $W \otimes_k V_2$ (resp. $V_1 \otimes_k W$) is a proper $X$-invariant subspace in $V_1 \otimes_k V_2 = V$ and therefore the corresponding $X$-module $V$ is not simple. This implies that the $G$-module $V$ is also not simple.

(iii) Suppose that the $G$-module $V$ is absolutely simple and splits. It follows from (i) and (ii) that both $\rho_1$ and $\rho_2$ are also absolutely simple. In other words, every splitting of an absolutely simple module is absolutely simple.

Now the centrality of $C$ combined with the absolute irreducibility of $\rho_1$ and $\rho_2$ implies, thanks to Schur’s Lemma, that both images $\rho_1(C) \subset \text{Aut}_k(V_1)$ and $\rho_2(C) \subset \text{Aut}_k(V_2)$ consist of scalars. In other words, every splitting of an absolutely simple $G$-module is projective.

(iv) Suppose $G$ is a finite perfect group. We write $\gamma : \tilde{G} \to G$ for the universal central extension of $G$ \cite{14} Ch. 2, §9] also known as the representation group or the primitive central extension of $G$. It is known \cite{14} Ch. 2, Th. 9.18] that $\tilde{G}$ is also a finite perfect group and for each central extension $X \xrightarrow{\pi} G$ there exists a surjective homomorphism $\phi : \tilde{G} \to [X, X]$ to the derived subgroup $[X, X]$ of $X$ such that the composition

$$\tilde{G} \xrightarrow{\phi} [X, X] \subset X \xrightarrow{\pi} G$$

coincides with $\gamma : \tilde{G} \to G$. This implies that while checking whether the $G$-module $V$ admits a projective absolutely simple splitting, one may always
restrict oneself to the case of \( X = \tilde{G} \) and \( \pi = \gamma \), and deal exclusively with absolutely irreducible linear representations of \( \tilde{G} \) over \( k \).

(v) We refer the reader to [12] for a study of projective representations of arbitrary finite groups over not necessarily algebraically closed fields.

**Example 3.4.** Assume that there exists a subgroup \( G' \) in \( G \) of finite index \( m > 1 \) and a \( G' \)-module \( W \) such that the \( k[G]\)-module \( V \) is *induced* from the \( k[G']\)-module \( W \). Then \( V \) is not very simple. Indeed, one may view \( W \) as a \( G' \)-submodule of \( V \) such that \( V \) coincides with the direct sum \( \oplus_{\sigma \in G/G'} W \) and \( G \) permutes all \( \sigma W \)'s. We write \( \text{Pr}_{\sigma} : V \to \sigma W \subset V \) for the corresponding projection maps. Then \( R = \oplus_{\sigma \in G/G'} k \cdot \text{Pr}_{\sigma} \) is the algebra of all operators sending each \( \sigma W \) into itself and acting on each \( \sigma W \) as scalars. Clearly, \( R \) is normal but coincides neither with \( k \cdot \text{Id} \) nor with \( \text{End}_k(V) \).

Notice that if the \( G' \)-module \( V' \) is trivial (i.e., \( s(w) = w \) for all \( s \in G', w \in W \)) then the \( G \)-module \( V \) is not simple. Indeed, for any non-zero \( w \in W \) the vector

\[
v = \sum_{\sigma \in G/G'} \sigma(w) \in \oplus_{\sigma \in G/G'} \sigma W = V
\]

is a non-zero \( G \)-invariant element of \( V \). Since \( \dim_k(V) \geq m > 1 \), the \( G \)-module \( V \) is not simple.

Clearly, if \( G' = \{1\} \) is the trivial subgroup of \( G \) then every \( G' \)-module is trivial. This implies that if the conditions of Lemma 1.3 hold true then (in the notations of [13]) either the \( R \)-module \( V \) is isotypic or the \( G \)-module \( V \) is induced from a representation of a proper subgroup of finite index.

**Remark 3.5.** Suppose \( k'/k \) is a finite algebraic extension of fields. We write \( \text{Aut}(k'/k) \) for the group of \( k \)-linear automorphisms of the field \( k' \). It is well-known that \( \text{Aut}(k'/k) \) is finite and its order divides the degree \( [k' : k] \); the equality holds if and only if \( k'/k \) is Galois.

Suppose there exists a homomorphism \( \chi : G \to \text{Aut}(k'/k) \) enjoying the following property:

There exists a structure of \( k' \)-vector space on \( V \) such that

\[
\rho(s)(av) = (\chi(s)(a))v \quad \forall s \in G, a \in k', v \in V.
\]

We claim that if the \( G \)-module \( V \) is absolutely simple then \( k'/k \) is Galois and \( \chi \) is surjective. Indeed, let us consider the image \( \chi(G) \subset \text{Aut}(k'/k) \). We write \( k_0 \) for the subfield of \( \chi(G) \)-invariants in \( k' \). We have

\[
k \subset k_0 \subset k';
\]

the degree \( [k' : k_0] \) coincides with the order of \( \chi(G) \) and therefore divides the order of \( \text{Aut}(k'/k) \). Clearly, \( G \) acts on \( V \) by \( k_0 \)-linear automorphisms, i.e., \( k_0 \) commutes with the action of \( G \) on \( V \). Now the absolute irreducibility of \( V \) implies that \( k_0 = k \). This implies that \( [k' : k] = [k' : k_0] \). Since \( [k' : k] = [k' : k_0][k_0 : k] \), we conclude that \( [k' : k] = [k' : k_0] \) coincides with the order of \( \chi(G) \) and therefore divides the order of \( \text{Aut}(k'/k) \). This implies that \( [k' : k] \) coincides with the order of \( \text{Aut}(k'/k) \) and therefore \( k'/k \) is Galois. Since \( [k' : k] \) coincides with the order of \( \chi(G) \), the order of the group \( \text{Aut}(k'/k) \) must coincide with the order of its subgroup \( \chi(G) \) and therefore \( \chi(G) = \text{Aut}(k'/k) \), i.e., \( \chi \) is surjective.
Definition 3.6. We say that the $G$-module $V$ admits a twisted multiplication if there exist a nontrivial Galois extension $k'$ of $k$ and a surjective homomorphism $\chi : G \to \text{Gal}(k'/k)$ enjoying the following property:

There exists a structure of $k'$-vector space on $V$ such that

$$\rho(s)(av) = (\chi(s)(a))v \quad \forall s \in G, a \in k', v \in V.$$  

(Here $\text{Gal}(k'/k)$ stands for the Galois group of $k'/k$.) In other words, $G$ acts on $V$ by $k'$-semi-linear automorphisms.

Example 3.7. Let us assume that $V$ admits a twisted multiplication. Then the degree $[k':k]$ divides $\dim_k(V)$ and therefore $\dim_k(V) > 1$. Then the $G$-module $V$ is not very simple. Indeed, $k'$ is obviously normal but does coincide neither with $k \cdot \text{Id}$ nor with $\text{End}_k(V)$, since $k' \neq k$ and $\text{End}_k(V)$ is noncommutative.

Remark 3.8. Let us assume that either $k$ is algebraically closed or $G$ is perfect and $k$ is finite. Then $V$ never admits a twisted multiplication, because either every algebraic extension $k'/k$ is trivial or $G$ is perfect and every Galois group $\text{Gal}(k'/k)$ is abelian. In the latter case every homomorphism from perfect $G$ to abelian $\text{Gal}(k'/k)$ must be trivial.

4. Main Theorem

Theorem 4.1. Suppose the Brauer group of a field $k$ is trivial (e.g., $k$ is either finite or algebraically closed). Suppose $V$ is a non-zero finite-dimensional $k$-vector space and

$$\rho : G \to \text{Aut}_k(V)$$

is a linear representation of a group $G$ over $k$. Then the $G$-module $V$ is very simple if and only if all the following conditions hold:

(i) The $G$-module $V$ is absolutely simple;
(ii) The $G$-module $V$ does not admit a projective absolutely simple splitting;
(iii) The $G$-module $V$ is not induced from a representation of a proper subgroup of finite index in $G$;
(iv) The $G$-module $V$ does not admit a twisted multiplication.

Corollary 4.2. Let us assume that either $k$ is algebraically closed or $G$ is perfect and $k$ is finite. Suppose $V$ is a non-zero finite-dimensional $k$-vector space and

$$\rho : G \to \text{Aut}_k(V)$$

is a linear representation of a group $G$ over $k$. Then the $G$-module $V$ is very simple if and only if all the following conditions hold:

(i) The $G$-module $V$ is absolutely simple;
(ii) The $G$-module $V$ does not admit a projective absolutely simple splitting;
(iii) The $G$-module $V$ is not induced from a representation of a proper subgroup of finite index in $G$.

Proof of Corollary 4.2. Indeed, the Brauer group of $k$ is trivial. Now the proof follows readily from Theorem 4.1 combined with Remark 3.8.

Taking into account that every projective representation over $\mathbb{F}_2$ is, in fact, linear, we obtain the following assertion.
Corollary 4.3. Suppose $V$ is a non-zero finite-dimensional vector space over $F_2$ and

$$\rho : G \to \text{Aut}_{F_2}(V)$$

is a linear representation of a group $G$ over $F_2$. Then the $G$-module $V$ is very simple if and only if all the following conditions hold:

(i) The $G$-module $V$ is absolutely simple;

(ii) The $G$-module $V$ does not split into a tensor product

$$V \cong V_1 \otimes_{F_2} V$$

of two absolutely simple $G$-modules $V_1$ and $V_2$ with

$$\dim_{F_2}(V_1) > 1, \quad \dim_{F_2}(V_2) > 1;$$

(iii) The $G$-module $V$ is not induced from a representation of a proper subgroup of finite index in $G$.

Proof of Theorem 4.1. It follows from results of [3] that every very simple representation enjoys all the properties (i)-(iv). Now suppose that an absolutely irreducible representation

$$\rho : G \to \text{Aut}_k(V)$$

enjoys the properties (ii)-(iv). It follows from Remark 3.3(iii) that the $G$-module $V$ does not split.

Let $R \subset \text{End}_k(V)$ be a normal subalgebra. Since the $G$-module $V$ is not induced, it follows from Lemma 1.3 and Example 3.4 that the faithful $R$-submodule $V$ is isotypic. This means that there exist a simple $R$-module $W$, a positive integer $d$ and an isomorphism

$$\psi : V \cong W^d$$

of $R$-modules. The following arguments are inspired by another result of Clifford [1, Satz 17.5 on p. 567].

Let us put

$$V_1 = W, \quad V_2 = k^d.$$

The isomorphism $\psi$ gives rise to the isomorphism of $k$-vector spaces

$$V = W^d = W \otimes_k k^d = V_1 \otimes_k V_2.$$

We have

$$d \cdot \dim_k(W) = \dim_k(V)$$

Clearly, $\text{End}_R(V)$ is isomorphic to the matrix algebra $\text{Mat}_d(\text{End}_R(W))$ of size $d$ over $\text{End}_R(W)$.

Let us put

$$k' = \text{End}_R(W).$$

Since $W$ is simple, $k'$ is a finite-dimensional division algebra over $k$. Since the Brauer group of $k$ is trivial, $k'$ must be a field. Clearly, $k'$ is a finite algebraic extension of $k$.

We have

$$\text{End}_k(V) \supset \text{End}_R(V) = \text{Mat}_d(k') \supset k'.$$

In particular,

$$k \subset k' \subset \text{End}_k(V).$$
Clearly, $\text{End}_R(V) \subset \text{End}_k(V)$ is stable under the adjoint action of $G$. This induces a homomorphism

$$\alpha : G \to \text{Aut}_k(\text{End}_R(V)) = \text{Aut}_k(\text{Mat}_d(k')) = \text{Aut}_k(V_2)/k^* = \text{PGL}(V_2).$$

Since $k'$ is the center of $\text{Mat}_d(k')$, it is stable under the conjugate action of $G$. Thus we get a homomorphism $\chi : G \to \text{Aut}(k'/k)$ such that

$$\chi(s)(a) = \alpha(s)(a) = \rho(s)u\rho(s)^{-1} \quad \forall s \in G, a \in k'.$$

I claim that the absolute irreducibility of $V$ implies that $k'/k$ is Galois and $\chi$ is surjective. Indeed, the inclusion $k' \subset \text{End}_k(V)$ provides $V$ with a natural structure of $k'$-vector space and it is clear that

$$\rho(s)(av) = (\chi(s)(a)v) \quad \forall s \in G, a \in k', v \in V.$$

It follows from Remark that $k'/k$ is Galois and $\chi : G \to \text{Aut}(k'/k) = \text{Gal}(k'/k)$ is surjective. Since $V$ does not admit a twisted multiplication, $k' = k$.

This implies that $\text{End}_R(V) = \text{Mat}_d(k)$ and one may rewrite $\alpha$ as

$$\alpha : G \to \text{Aut}_k(\text{Mat}_d(k)) = \text{Aut}(\text{End}_k(V_2)) = \text{Aut}_k(V_2)/k^* = \text{PGL}(V_2).$$

It follows from the Jacobson density theorem that $R = \text{End}_k(W) = \text{End}_k(V_1)$. The adjoint action of $G$ on $R$ gives rise to a homomorphism

$$\beta : G \to \text{Aut}_k(\text{End}_k(W)) = \text{Aut}_k(\text{End}_k(V_1)) = \text{Aut}_k(V_1)/k^* = \text{PGL}(V_1).$$

Notice that

$$R = \text{End}_k(V_1) = \text{End}_k(V_1) \otimes \text{Id}_{V_2} \subset \text{End}_k(V_1) \otimes_k \text{End}_k(V_2) = \text{End}_k(V).$$

Clearly, there exists a central extension $\pi : X \to G$ such that one may lift projective representations $\alpha$ and $\beta$ to linear representations

$$\rho'_2 : X \to \text{Aut}_k(V_2), \quad \rho_1 : X \to \text{Aut}_k(V_1).$$

respectively. For instance, one may take as $X$ the subgroup of $G \times \text{Aut}_k(V_1) \times \text{Aut}_k(V_2)$ which consists of all triples $(g, u_1, u_2)$ such that $\alpha(g)$ coincides with the image of $u_2$ in $\text{Aut}_k(V_2)/k^*$ and $\beta(g)$ coincides with the image of $u_1$ in $\text{Aut}_k(V_1)/k^*$. In this case the homomorphisms $\pi, \rho_1, \rho'_2$ are just the corresponding projection maps

$$(g, u_1, u_2) \mapsto g; \quad (g, u_1, u_2) \mapsto u_1; \quad (g, u_1, u_2) \mapsto u_2.$$

Now I am going to check that the tensor product $\rho_1 \otimes \rho'_2$ coincides with the composition

$$\rho \pi : X \to G \to \text{Aut}_k(V)$$

up to a twist by a linear character of $X$.

In order to do that, notice that if $x \in X$ and $g = \pi(x) \in G$ then the conjugation by $\rho(g)$ in $\text{End}_k(V) = \text{End}_k(V_1) \otimes_k V_2$ leaves stable $R = \text{End}_k(V_1) \otimes \text{Id}_{V_2}$ and coincides on $R$ with the conjugation by $\rho_1(x) \otimes \text{Id}_{V_2}$ (by the definition of $\beta$ and $\rho_1$). Since the centralizer of $\text{End}_k(V_1) \otimes \text{Id}_{V_2}$ in

$$\text{End}_k(V) = \text{End}_k(V_1) \otimes_k \text{End}_k(V_2)$$

coinsides with $\text{Id}_{V_1} \otimes \text{End}_k(V_2)$, there exists $u \in \text{Aut}_k(V_2)$ such that

$$\rho(g) = \rho_1(x) \otimes u.$$
Since the conjugation by $\rho(g)$ leaves stable the centralizer of $R$, i.e. $\text{Id}_V \otimes \text{End}_k(V)$ and coincides on it with the conjugation by $\text{Id}_V \otimes \rho'_2(x)$ (by the definition of $\alpha$ and $\rho'_2$), there exists a non-zero constant $\lambda = \lambda(x) \in k^*$ such that $u = \lambda \rho'_2(x)$. This implies that for each $x \in X$ there exists a non-zero constant $\lambda = \lambda(x)$ such that
$$\rho \pi(x) = \rho_1(x) \otimes u = \lambda \cdot \rho_1(x) \otimes \rho'_2(x).$$

Since both
$$\rho \pi : X \to \text{Aut}_k(V), \quad \rho_1 \otimes \rho'_2 : X \to \text{Aut}_k(V),$$
are group homomorphisms, one may easily check that the map
$$X \to k^*, \quad x \mapsto \lambda = \lambda(x)$$
is a group homomorphism (linear character). Let us define $\rho_2$ as the twist
$$\rho_2 : X \to \text{Aut}_k(V), \quad \rho_2(x) = \lambda(x) \cdot \rho'_2(x) \quad \forall x \in X.$$Clearly, $\rho_2$ is a linear representation of $X$ and
$$\rho \pi = \rho_1 \otimes \rho_2.$$Since the $G$-module $V$ does not split, either $\dim_k(V_1) = 1$ or $\dim_k(V_2) = 1$. If $\dim_k(V_1) = 1$ then $R = \text{End}_k(W) = \text{End}_k(V_1) = k$ consists of scalars. If $\dim_k(V_2) = 1$ then $d = \dim_k(V_2) = 1$, i.e., $V = W$ and $R = \text{End}_k(W) = \text{End}_k(V)$. □

5. Doubly transitive permutation groups

Let $B$ be a finite set consisting of $n \geq 3$ elements. We write $\text{Perm}(B)$ for the group of all permutations of $B$. A choice of ordering on $B$ gives rise to an isomorphism
$$\text{Perm}(B) \cong S_n.$$Let $G$ be a subgroup of $\text{Perm}(B)$. For each $b \in B$ we write $G_b$ for the stabilizer of $b$ in $G$; it is a subgroup of $G$.

Let $k$ be a field. We write $k^B$ for the $n$-dimensional $k$-vector space of maps $h : B \to k$. The space $k^B$ is provided with a natural action of $\text{Perm}(B)$ defined as follows. Each $s \in \text{Perm}(B)$ sends a map $h : B \to k$ into $sh : b \mapsto h(s^{-1}(b))$. The permutation module $k^B$ contains the $\text{Perm}(B)$-stable hyperplane
$$(k^B)^0 = \{ h : B \to k \mid \sum_{b \in B} h(b) = 0 \}$$and the $\text{Perm}(B)$-invariant line $k \cdot 1_B$ where $1_B$ is the constant function $1$. The quotient $k^B/(k^B)^0$ is a trivial $1$-dimensional $\text{Perm}(B)$-module.

Clearly, $(k^B)^0$ contains $k \cdot 1_B$ if and only if $\text{char}(k)$ divides $n$. If this is not the case then there is a $\text{Perm}(B)$-invariant splitting
$$k^B = (k^B)^0 \oplus k \cdot 1_B.$$Clearly, $k^B$ and $(F^B)^0$ carry natural structures of $G$-modules.

Now, let us consider the case of $k = F_2$. If $n$ is even then let us define the $G$-module
$$Q_B := (F_2^B)^0/ (F_2 \cdot 1_B).$$If $n$ is odd then let us put
$$Q_B := (F_2^B)^0.$$
Remark 5.1. Clearly, $\dim_{F_2}(Q_B) = n - 1$ if $n$ is odd and $\dim_{F_2}(Q_B) = n - 2$ if $n$ is even. In both cases $\dim_{F_2}(Q_B) \geq 2$. One may easily check that $Q_B$ is a faithful $G$-module if $n \neq 4$.

The $G$-module $Q_B$ is called the heart over the field $F_2$ of the group $G$ acting on the set $B$. The aim of this section is to find out when the $G$-module $Q_B$ is very simple. It follows from Example 2.2 that if $Q_B$ is very simple then $\dim_{F_2}(Q_B) > 2$ and therefore $n \geq 5$.

Remark 5.2. Assume that $n$ is odd. Then one may easily check that $\text{End}_G(Q_B) = F_2$ if and only if $G$ is doubly transitive. This implies that if $n$ is odd and the $G$-module $Q_B$ is absolutely simple then $G$ is doubly transitive. This implies that if $n$ is odd and the $G$-module $Q_B$ is very simple then $G$ is doubly transitive.

Remark 5.3. Let us assume that $n \geq 5$ is even, the $G$-module $Q_B$ is absolutely simple but $G$ is not transitive. Let us present $B$ as a disjoint union of two non-empty $G$-invariant subsets $B_1$ and $B_2$. Suppose each $B_i$ contains, at least, 2 elements. Without loss of generality we may assume that $(B_2) \geq 2$ and therefore $(B_2) \geq 3$.

There is an embedding of $G$-modules

$$\kappa : F_2^{B_1} \hookrightarrow (F_2^B)^0, \quad h \mapsto \kappa(h)$$

defined as follows.

$$\kappa(h)(b_1) = h(b_1) \quad \forall b_1 \in B_1; \quad \kappa(h)(b_2) = \sum_{b \in B_1} h(b) \quad \forall b_2 \in B_2.$$

Suppose $1_B \in \kappa(F_2^{B_1})$. Then both $B_1$ and $B_2$ consist of odd number of elements and therefore

$$(B_1) \geq 3, \quad (B_2) \geq 3.$$ 

Clearly,

$$2 \leq (B_1) - 1 = \dim_{F_2}(\kappa(F_2^{B_1})) = n - (B_2) - 1 \leq n - 4 < \dim_{F_2}(Q_B).$$

Therefore the $G$-module $Q_B$ is not simple. Contradiction.

Now suppose $1_B$ does not lie in $\kappa(F_2^{B_1})$. Then

$$1 \leq (B_1) = \dim_{F_2}(\kappa(F_2^{B_1})) = (B_1) = n - (B_2) \leq n - 3 < \dim_{F_2}(Q_B).$$

Therefore the $G$-module $Q_B$ is not simple. Contradiction.

This implies that either $B_1$ or $B_2$ is a singleton.

We conclude that if $n$ is even, the $G$-module $Q_B$ is simple but $G$ is not transitive then $B$ is the disjoint union of two $G$ orbits of cardinality $n - 1$ and 1 respectively. In other words, there exists $b \in B$ such that $G = G_b$ and the action of $G$ on $B \setminus \{b\}$ is transitive. Notice that if we denote $B \setminus b$ by $B'$ then the $G$-modules $Q_B$ and $Q_{B'}$ are isomorphic [17, Remark 2.5 on p. 95] (see also [10, Hilfsatz 3b]). Applying Remark 5.2 to $B'$, we conclude that the action of $G$ on $B'$ is doubly transitive.

Remark 5.4. Suppose $n = 2m$ is even, $G$ is transitive but not doubly transitive. Assume also the $G$-module $Q_B$ is very simple. Then $n \geq 5$ and $Q_B$ is absolutely simple. According to [10, Satz 11], this implies that $m$ is odd and there exists a subgroup $H \subset G$ of index 2 such that $B$ can be presented as a disjoint union of two $H$-invariant subsets $B_1$ and $B_2$ of cardinality $m$. 
Since \( n \geq 5 \), we conclude that \( m \geq 3 \). There is an embedding of \( H \)-modules
\[
\kappa : F_2^{B_1} \to (F_2^B)^0, \quad h \mapsto \kappa(h)
\]
defined as follows.
\[
k(\kappa(h)) = \kappa(h) = h(b_1) \quad \forall b_1 \in B_1; \quad \kappa(h) = \sum_{b \in B_1} h(b) \quad \forall b_2 \in B_2.
\]

Clearly, \( 1_B \in \kappa(F_2^{B_1}) \). We have
\[
2 = m - 1 = \dim_{F_2}(\kappa(F_2^{B_1})) = m - 1 < 2m - 2 = n - 2 = \dim_{F_2}(Q_B).
\]
Therefore the \( H \)-module \( Q_B \) is not simple. Since \( H \) is obviously normal in \( G \) and the \( G \)-module \( Q_B \) is very simple, we conclude that \( H \) acts on \( Q_B \) via scalars. Since \( F_2 = \{1\} \), \( H \) acts on \( Q_B \) trivially. But this contradicts to the faithfulness of the \( G \)-module \( Q_B \).

We conclude that if \( n \geq 5 \) is even, \( G \) is transitive and the \( G \)-module \( Q_B \) is very simple then \( G \) must be doubly transitive.

To summarize, we arrive to the following conclusion.

**Theorem 5.5.** Suppose that \( n \geq 3 \) is an integer, \( B \) is an \( n \)-element set, \( G \subset \text{Perm}(B) \) is a permutation group. Suppose that the \( G \)-module \( Q_B \) is very simple. Then \( n \geq 5 \) and one of the following two conditions holds:

(i) \( G \) acts doubly transitively on \( B \);
(ii) \( n \) is even, there exists a \( G \)-invariant element \( b \in B \) and \( G \) acts doubly transitively on \( B' := B \setminus \{b\} \). In addition, the \( G \)-modules \( Q_B \) and \( Q_{B'} \) are isomorphic.

**Example 5.6.** Suppose that there exist a positive integer \( m > 2 \) and an odd power prime \( q \) such that \( n = \frac{q^m - 1}{q - 1} \) and one may identify \( B \) with the \((m - 1)\)-dimensional projective space \( P^{m-1}(F_q) \) over \( F_q \) in such a way that \( G \) contains \( L_m(q) = \text{PSL}_m(F_q) \). Then the \( G \)-module \( Q_B \) is very simple.

Indeed, in light of Remark 2.3(ii), we may assume that \( G = L_m(q) \); in particular \( G \) is a simple non-abelian group acting doubly transitively on \( B = P^{m-1}(F_q) \).

Assume that \((m, q) \neq (4, 3)\). It follows from a result of Guralnick [5] that every nontrivial projective representation of \( G = L_m(q) \) in characteristic 2 has dimension \( \geq \dim_{F_2}(Q_B) \) (see [13] Remark 4.4). It follows from Example 2.3 that the \( G \)-module \( Q_B \) is very simple.

So, we may assume that \( m = 4, q = 3 \). We have \( n = \#(B) = 40 \) and \( \dim_{F_2}(Q_B) = 38 \). It is known [11] that the \( G \)-module \( Q_B \) is absolutely simple.

According to the Atlas [2] pp. 68-69, \( G = L_4(3) \) has two conjugacy classes of maximal subgroups of index 40. All other maximal subgroups have index greater than 40. Therefore all proper subgroups of \( G \) have index greater than 39 > 38 and therefore \( Q_B \) is not induced from a representation of a proper subgroup.

It follows from the Table on p. 165 of [9] that all absolutely irreducible representations of \( G \) in characteristic 2 have dimension which is not a strict divisor of 38. Applying Corollary 4.3 we conclude that \( Q_B \) is very simple.

**Theorem 5.7.** Suppose that \( n \geq 5 \) is an integer, \( B \) is a set consisting of \( n \) elements. Suppose \( G \subset \text{Perm}(B) \) is one of the known doubly transitive permutation groups (listed in [11] [4]). Then the \( G \)-module \( Q_B \) is very simple if and only if one of the following conditions holds:
(i) $G$ is isomorphic either to the full symmetric group $SS_n$, or to the alternating group $A_n$;
(ii) There exist a positive integer $m > 2$ and an odd power prime $q$ such that $n = \frac{q^m - 1}{q - 1}$ and one may identify $B$ with the $(m - 1)$-dimensional projective space $\mathbf{P}^{m-1}(\mathbb{F}_q)$ over $\mathbb{F}_q$ in such a way that $G$ contains $L_m(q) = \text{PSL}_m(\mathbb{F}_q)$;
(iii) $q = n - 1$ is a power of 2 and one may identify $B$ with the projective line $\mathbf{P}^1(\mathbb{F}_q)$ in such a way that $G$ contains $L_2(\mathbb{F}_q) = \text{PSL}_2(\mathbb{F}_q)$;
(iv) There exists a positive integer $d \geq 2$ such that $q := 2^d, n = q^3 + 1$ and $G$ contains a subgroup isomorphic to the projective special unitary group $U_3(q) = \text{PSU}(3, \mathbb{F}_q)$;
(v) There exists a positive integer $d \geq 2$ such that $q = 2^{2d+1}, n = q^2 + 1$ and $G$ contains a subgroup isomorphic to the Suzuki group $Sz(q)$;
(vi) $n = 11$ and $G$ is isomorphic either to $L_2(11)$ or to the Mathieu group $M_{11}$;
(vii) $n = 12$ and $G$ is isomorphic either to $M_{11}$ or to the Mathieu group $M_{12}$.

Proof. The fact that all the $G$-modules $Q_B$ arises from 5.7(i)-(vii) are very simple was proven in [13] (cases (i), (iii), (v), (vi), (vii)), [12] (case (iv)) and in the present paper (Example 5.6 case (ii)). On the other hand, the paper [11] (complemented by [8]) contains the list of doubly transitive $G \subset \text{Perm}(B)$ with absolutely simple $Q_B$. In addition to the cases 5.7(i)-(vii), the $G$-module $Q_B$ is absolutely simple only if one of the following conditions holds:

(a) There exists an odd power prime $q$ and a positive integer $d$ such that $n = q^d$ and one may identify $B$ with the affine space $\mathbf{F}_q^d$ in such a way that $G$ is contained in $AGL(d, \mathbb{F}_q)$ and contains the group $\mathbb{F}_q^d$ of translations. Here $AGL(d, \mathbb{F}_q)$ is the group of permutations of $\mathbb{F}_q^d$ generated by the group $A GL(d, \mathbb{F}_q)$ of affine transformations and the Frobenius automorphism;
(b) There exists an odd power prime $q$ such that $n = q + 1$ and one may identify $B$ with the projective line $\mathbf{P}^1(\mathbb{F}_q)$ in such a way that $G$ becomes a 3-transitive subgroup of $\text{PGL}(2, \mathbb{F}_q)$. Here $\text{PGL}(2, \mathbb{F}_q)$ is the group of permutations of $\mathbf{P}^1(\mathbb{F}_q)$ generated by $\text{PGL}(2, \mathbb{F}_q)$ and the Frobenius automorphism.

In the case (a) the group $\mathbb{F}_q^d$ of translations is a proper normal abelian subgroup of $G$. It follows from Remark 2.1(vi) that $Q_B$ is not very simple.

In the case (b) let us consider the intersection $G' := G \cap \text{PSL}(2, \mathbb{F}_q)$. Clearly, $G'$ is a normal subgroup of $G$. Since the $\text{PSL}(2, \mathbb{F}_q)$-module $Q_B$ is not absolutely simple [11], the $G'$-module is also not absolutely simple. By Remark 2.1(vi), $G'$ acts on $Q_B$ by scalars. Since $\mathbb{F}_q^2 = \{1\}$ and the $G$-module $Q_B$ is faithful, $G' = \{1\}$. Since $\text{PSL}(2, \mathbb{F}_q)$ is a subgroup of index 2 in $\text{PGL}(2, \mathbb{F}_q)$, the intersection $H := G \cap \text{PGL}(2, \mathbb{F}_q)$ is either a normal subgroup of order 2 in $G$ or trivial (one-element subgroup). In the latter case $G$ is isomorphic to a subgroup of the cyclic quotient $\text{PGL}(2, \mathbb{F}_q)/\text{PGL}(2, \mathbb{F}_q)$ and therefore is commutative which contradicts the absolute simplicity of the $G$-module $Q_B$. In the former case, $H$ is an abelian normal subgroup of $G$ and it follows from Remark 2.1(vi) that $Q_B$ is not very simple.

6. Applications to hyperelliptic Jacobians

Throughout this section we assume that $K$ is a field of prime characteristic $p$ different from 2. We fix its algebraic closure $K_a$ and write $\text{Gal}(K)$ for the absolute
Remark 6.1. Clearly, Gal(\(R\))/\(K\)) is transitive if and only if the polynomial \(f(x)\) is irreducible. It is also clear that the following conditions are equivalent:

(i) There exists a root \(\alpha \in K\) and an irreducible polynomial \(f_1(x) \in K[x]\) of degree \(n - 1\) and without multiple roots such that

\[ f(x) = (x - \alpha)f_1(x); \]

(ii) There exists a Gal(\(f\))-invariant element \(\alpha \in \mathcal{R}_f\) such that Gal(\(f\)) acts transitively on \(\mathcal{R}_f \setminus \{\alpha\}\).

Let \(C_f\) be the hyperelliptic curve \(y^2 = f(x)\). Its genus \(g\) is \(\frac{n-1}{2}\) if \(n\) is odd and \(\frac{n-2}{2}\) if \(n\) is even. Let \(J(C_f)\) be the jacobian of \(C_f\); it is a \(g\)-dimensional abelian variety defined over \(K\). Let \(J(C_f)_2\) be the kernel of multiplication by 2 in \(J(C_f)(K)\); it is \(2g\)-dimensional \(\mathbb{F}_2\)-vector space provided with the natural action

\[ \text{Gal}(K) \rightarrow \text{Aut}_{\mathbb{F}_2}(J(C_f)_2) \]

of Gal(\(K\)). It is well-known (see for instance [1]) that the homomorphism Gal(\(K\)) \(\rightarrow\) Aut_{\mathbb{F}_2}(\(J(C_f)_2\)) factors through the canonical surjection Gal(\(K\)) \(\rightarrow\) Gal(\(K(\mathcal{R}_f)/K\)) = Gal(\(f\)) and the Gal(\(f\))-modules \(J(C_f)_2\) and \(Q_{2\mathcal{R}_f}\) are isomorphic. It follows easily that the Gal(\(K\))-module \(J(C_f)_2\) is very simple if and only if the Gal(\(f\))-module \(Q_{2\mathcal{R}_f}\) is very simple. Combining Theorem 5.5 and Remark 6.1 we obtain the following statement.

**Theorem 6.2.** Suppose the Gal(\(K\))-module \(J(C_f)_2\) is very simple. Then one of the following conditions holds:

(i) The polynomial \(f(x) \in K[x]\) is irreducible and its Galois group Gal(\(f\)) acts doubly transitively on \(\mathcal{R}_f\);

(ii) \(n\) is even, there exists a root \(\alpha \in K\) of \(f\) and an irreducible polynomial \(f_1(x) \in K[x]\) of degree \(n - 1\) and without multiple roots such that

\[ f(x) = (x - \alpha)f_1(x). \]

In addition, the Galois group Gal(\(f_1\)) of \(f_1\) acts doubly transitively on \(\mathcal{R}_{f_1} = \mathcal{R}_f \setminus \{\alpha\}\).

**Remark 6.3.** In the case (ii) the hyperelliptic curves \(C_f\) and \(C_{f_1} : y^2 = f_1(x)\) are birationally isomorphic over \(K\).

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Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

E-mail address: zarhin@math.psu.edu