GLOBAL WELL-POSEDNESS AND POLYNOMIAL BOUNDS FOR THE DEFOCUSING \( L^2 \)-CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN \( \mathbb{R} \)

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Abstract. We prove global well-posedness for low regularity data for the one dimensional quintic defocusing nonlinear Schrödinger equation. Precisely we show that a unique and global solution exists for initial data in the Sobolev space \( H^s(\mathbb{R}) \) for any \( s > \frac{1}{3} \). This improves the result in [21], where global well-posedness was established for any \( s > \frac{4}{9} \). We use the I-method to take advantage of the conservation laws of the equation. The new ingredient in our proof is an interaction Morawetz estimate for the smoothed out solution \( Iu \). As a byproduct of our proof we also obtain that the \( H^s \) norm of the solution obeys polynomial-in-time bounds.

1. Introduction

In this paper we study the global well-posedness of the following initial value problem (IVP) for the \( L^2 \)-critical defocusing nonlinear Schrödinger equation (NLS):

\[
\begin{align*}
    iu_t + \Delta u - |u|^4 u &= 0, \\
    u(x, 0) &= u_0(x) \in H^s(\mathbb{R}), \quad x \in \mathbb{R}, \quad t \in \mathbb{R},
\end{align*}
\]

where \( H^s \) denotes the usual inhomogeneous Sobolev space of order \( s \).

We adopt the standard notion of local well-posedness, that is, we say that the IVP above is locally well-posed in \( H^s \) if for any initial data \( u_0 \in H^s \) there exists a positive time \( T = T(\|u_0\|_{H^s}) \) depending only on the norm of the initial data, such that a solution to the initial value problem exists on the time interval \([0, T]\), is unique in a certain Banach space of functions \( X \subset C([0, T], H^s_x) \), and the solution map from \( H^s \) to \( C([0, T], H^s_x) \) depends continuously on the initial data on the time interval \([0, T] \). If \( T \) can be taken arbitrarily large we say that the IVP is globally well-posed.

Local well-posedness for the initial value problem (1.1)-(1.2) in \( H^s \) for any \( s > 0 \) was established in [3], see also [4]. A local solution also exists for \( L^2 \) initial data [4], but the time of existence depends not only on the \( H^s \) norm of the initial data, but also on the profile of \( u_0 \). For more details on local existence see, for example, [2], [4] and [18].

Local in time solutions of (1.1)-(1.2) enjoy mass conservation

\[
\|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^n)}.
\]

Moreover, \( H^1 \) solutions enjoy conservation of the energy

\[
E(u)(t) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u(t)|^2 dx + \frac{1}{6} \int_{\mathbb{R}} |u(t)|^6 dx = E(u)(0),
\]

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which together with \([13.3]\) and the local theory immediately yields global in time well-posedness for \([1.1]-[1.2]\) with initial data in \(H^1\). On the other hand the \(L^2\) local well-posedness and the conservation of the \(L^2\) norm do not immediately imply global well-posedness as in the case of the finite energy data since in this case \(T = T(u_0)\). It is conjectured though that \([1.1]-[1.2]\) is globally well-posed for initial data in \(H^s\) for \(s \geq 0\).

Existence of global solutions to \([1.1]-[1.2]\) corresponding to initial data below the energy threshold was first obtained in \([6, 7]\) by using the method of “almost conservation laws”, or “I-method” (for a detailed description of this method see section \(3\) below). The gap between known local and global well-posedness was further filled out in \([21]\), where global well-posedness was obtained in \(H^s(\mathbb{R})\) with \(s > 4/9\). The question of global well-posedness below the energy space \(H^1\) we described above can also be formulated in \(\mathbb{R}^d\), where the equivalent \(L^2\) critical problem has nonlinearity \(|u|^{4/d}u\). For \(d = 2\) the I-method was also partially successful in \([8]\), and in \([13]\), where global well-posedness was obtained in \(H^s\) for \(s \geq 1/2\). This last paper is particularly interesting since it combines the I-method with a local in time Morawetz type estimate. In \([12]\) this idea was then extended to all dimensions \(d \geq 3\). Still none of these results reached the space \(L^2\). Recently in \([20]\) the authors proved global well-posedness in \(L^2\) and scattering for the \(L^2\)-critical NLS problem in all dimensions \(d \geq 3\), assuming spherically symmetric initial data. The proof relies upon a combination of several sophisticated tools among which compensated compactness and a frequency-localized Morawetz estimate. These tools seem not to be enough when \(d = 1, 2\) or when the data are no longer radial.

In this paper we only consider the case \(d = 1\) and we prove the following result:

**Theorem 1.1.** The initial value problem \([1.1]-[1.2]\) is globally well-posed in \(H^s(\mathbb{R})\), for any \(1 > s > \frac{4}{5}\). Moreover the solution satisfies

\[
\sup_{t \in [0,T]} \|u(t)\|_{H^s(\mathbb{R})} \leq C(1 + T)^{\frac{s(1-s)}{2(s+1)}}
\]

where the constant \(C\) depends only on \(s\) and \(\|u_0\|_{L^2}\).

We prove Theorem 1.1 by combining the I-method with an interaction Morawetz-type estimate for the smoothed out version \(Iu\) of the solution. Such a Morawetz estimate for an almost solution, that below we call “almost Morawetz”, is the main novelty of this paper. As mentioned above, the approach of combining the I-method with an interaction Morawetz estimate was described in \([13]\) where the \(L^2\)-critical NLS in 2d was treated; see also \([9]\). However in order to obtain global well-posedness for \([1.1]-[1.2]\) corresponding to initial data in \(H^s\) with \(s \leq 4/9\) it was not enough to obtain an interaction Morawetz estimate for the solution \(u\) itself (as in the case of \([9, 12, 13]\)).

Before giving an outline of the proof we say a few words about the above mentioned tools: the I-method and the “almost Morawetz” estimate.

The I-method was first introduced by Colliander et al (see, for example, \([4, 8, 9]\)). It is based on the almost conservation of a certain modified energy functional. The idea is to replace the conserved quantity \(E(u)\), which is no longer available for \(s < 1\), with an “almost conserved” variant \(E(Iu)\), where \(I\) is a smoothing operator of order \(1 - s\), which behaves like the identity for low frequencies and like a fractional integral operator for high frequencies. However \(Iu\) is not a solution to \([1.1]\) and hence one expects an energy increment. This increment is quantifying \(E(Iu)\) as an “almost conserved” energy. The key is to prove that on intervals of fixed length, where local well-posedness is satisfied, the increment of

\footnote{Note that global well-posedness for initial data in \(H^s\) with \(s > 4/9\) was established in \([21]\).}
the modified energy $E(Iu)$ decays with respect to a large parameter $N$ (for the precise definition of $I$ and $N$ we refer the reader to Section 3). This requires delicate estimates on the commutator between $I$ and the nonlinearity. When $d = 1$, hence the nonlinearity is algebraic, one can write the commutator explicitly using the Fourier transform, and control it by multi-linear analysis and bilinear estimates. The analysis above can be carried out in the $X^{s,b}$ spaces setting, where one can use the smoothing bilinear Strichartz estimate of Bourgain (see e.g. [1]) along with Strichartz estimates, to demonstrate the existence of global rough solutions (see [6, 7] and [21]).

We now turn to our second tool: the “almost Morawetz estimate”, that is an a priori interaction Morawetz-type estimate for the “approximate solution” $Iu$ to the initial value problem

$$iIu_t + \Delta Iu - I(|u|^4u) = 0,$$

$$Iu(x,0) = Iu_0(x) \in H^1(\mathbb{R}), \quad x \in \mathbb{R}, t \in \mathbb{R}.$$  

For the original problem (1.1) one can prove that solutions satisfy the following a priori bound (see [5])

$$\|u\|_{L_t^8L_x^\infty} \lesssim \sup_{t \in [0,T]} \|u\|_{H^1} \|u\|_{L^2}^7.$$  

For initial data below $H^1$ this estimate is not useful anymore since $u$ is not in $H^1$. We introduce the $I$-operator with the aim of getting an a priori estimate of the form

$$\|Iu\|_{L^8_tL^\infty_x} \lesssim \sup_{t \in [0,T]} \|Iu\|_{H^1} \|Iu\|_{L^2}^7 + \text{Error},$$

where the Error terms are negligible in some sense. To achieve this, we work with mixed Lebesgue spaces that we denote by $S_I(J)$ and define\(^2\) as

$$S_I(J) := \{ f \mid \sup_{(q,r) \text{ admissible}} \| (\partial_x)^{q} I f \|_{L^2_tL^r_x(J \times \mathbb{R})} < \infty \},$$

where the pair $(q, r)$ is said to be admissible if $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ and $2 \leq q, r \leq \infty$.

We show that inside the interval where the local solution exists, the error term is very small. The proof of this fact relies on harmonic analysis estimates of Coifman-Meyer type and is given in Section 4. Because of the fact that we work with local-in-time solutions we restrict the above a priori bounds to local intervals of the form $[t_0, t_1]$.

Now we give an outline of the proof to show how to combine our two tools. Fix a large value of time $T_0$. If $u$ is a solution to (1.1) in the time interval $[0, T_0]$, then $u^\lambda(x) = \frac{1}{\lambda^2} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is a solution to the same equation in $[0, \lambda^2 T_0]$. We choose the parameter $\lambda > 0$ so that $E(Iu^\lambda) = O(1)$. Using Strichartz estimates we show that if $J = [t_0, t_1]$ and $\|Iu^\lambda\|_{L^6_tL^6_x(J \times \mathbb{R}^n)} < \mu$, where $\mu$ is a small universal constant, then for $b$ close to $1/2$

$$\|u^\lambda\|_{S_I(J)} \lesssim \|Iu^\lambda(t_0)\|_{H^1},$$

and

$$\|u^\lambda\|_{X^{s,b}_I(J)} \lesssim \|Iu^\lambda(t_0)\|_{H^1},$$

\(^2\)See Section 2 for a definition of the operator $\langle \partial_x \rangle$.

\(^3\)Here we were not able to use the $X^{s,b}$ spaces machinery. However the approach that we pursue creates no additional difficulties, since we can iterate our solutions in the intersection of the spaces $S_I$ and $X^{s,b}$, see Proposition 5.2 for details.
where
\[ X^{s,b}(J) = \{ f \mid \| f \|_{X^{s,b}(J \times \mathbb{R})} < \infty \}. \]

Moreover, in this same time interval where the problem is well-posed, we can prove the following “almost conservation law”, provided that \( s > 1/4 \).

\[
E^2(u^\lambda)(t_1) - E^2(u^\lambda)(t_0) \lesssim N^{-2}\| Iu^\lambda \|_{X^{s,b}(\{t_0,t_1\} \times \mathbb{R})}^{10} \lesssim N^{-2}\| Iu^\lambda(t_0) \|_{H^1}^{10} \lesssim N^{-2},
\]

where \( E^2(u^\lambda) \) denotes the second modified energy (for the definition see Section 3). Of course, for the arbitrarily large interval \([0, \lambda^2 T_0]\) we do not have
\[
\| Iu^\lambda \|_{L^6(L^5([0, \lambda^2 T_0]) \times \mathbb{R}^n)} \lesssim \mu.
\]

However, we can interpolate the a priori information that we have about the \( L^6 L^5 \) norm of \( Iu \) with the following a priori bound
\[
\| Iu^\lambda \|_{L^6(\{t_0, t_1\} \times \mathbb{R}^n)} \leq \frac{T}{T_1} \| Iu^\lambda \|_{L^\infty(\{t_0, t_1\} \times \mathbb{R}^n)} \leq T \| u^\lambda \|_{L^\infty(\{t_0, t_1\} \times \mathbb{R}^n)} = T \| u_0 \|_{L^2} = T \| u_0 \|_{L^2}.
\]

obtained using Hölder’s inequality, the definition of the \( I \), mass conservation and the fact that the problem is \( L^2 \)-critical. Then we get an \( L^6 L^5 \) bound valid for \( J = [t_0, t_1] \) and we use this bound to partition the arbitrarily large interval \([0, \lambda^2 T_0]\) into \( L \) intervals where the local theory uniformly applies. \( L = L(N, T) \) is finite and defines the number of the intervals in the partition that will make the Strichartz \( L^6 L^5 \) norm of \( Iu \) less than \( \mu \) in each interval. The next step is to prove that the second modified energy \( E^2(u^\lambda) \) is just an approximation of the first modified energy \( E^1(u^\lambda) = E(Iu^\lambda) \) in the sense that

\[
E^2(u^\lambda) \sim E^1(u^\lambda) + O\left(\frac{1}{N^\epsilon}\right),
\]

for \( s > 1/3 \) and \( N \gg 1 \). Since \( E(Iu^\lambda) \) controls the \( H^1 \) norm of \( Iu \) we have by (1.7)
\[
\| Iu^\lambda \|_{H^1} \lesssim LN^{-2}.
\]

To maintain the bound \( \| Iu^\lambda \|_{H^1} \lesssim 1 \) we must have that
\[
L(N, T) \sim N^2,
\]

and this condition will require \( s > 1/3 \). One should notice already that the restriction \( s > 1/3 \) appears in two separate parts of our proof: in (1.8) and in the last step recorded above. So this regularity is a threshold to the method, at least if one wants to use the second modified energy. For a more detailed proof the reader should check Section 5.

We conclude this introduction by announcing that work in progress of the fourth author in collaboration with J. Colliander and M. Grillakis, shows that a similar approach can be used in the \( L^2 \)-critical case when \( d = 2 \).

**Organization of the paper.** In Section 2 we introduce some notation and state important propositions that we will use throughout the paper. In Section 3 we review the \( I \) method, prove the local well-posedness theory for \( Iu \) and obtain an upper bound on the increment of the second modified energy. In Section 4 we prove the “almost Morawetz” inequality which is the heart of our argument. Finally in Section 5 we give the details of the proof of global well-posedness stated in Theorem 1.1.

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\(^4\)We use the second modified energy approach as in [7] and [21].
2. Notation and preliminaries

2.1. Notation. In what follows we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$. If $A \lesssim B$ and $B \lesssim A$ we say that $A \sim B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition $\langle a \rangle := 1 + |a|$ and $a \pm := a \pm \epsilon$ with $0 < \epsilon << 1$. The reader also has to be alert that we sometimes do not explicitly write down constants that depend on the $L^2$ norm of the solution. This is justified by the conservation of the $L^2$ norm.

2.2. Definition of spaces. We use $L^r_x(\mathbb{R})$ to denote the Lebesgue space of functions $f : \mathbb{R} \to \mathbb{C}$ whose norm
\[ \|f\|_{L^r_x} := \left( \int_{\mathbb{R}} |f(x)|^r \, dx \right)^{\frac{1}{r}} \]
is finite, with the usual modification in the case $r = \infty$. We also use the mixed space-time Lebesgue spaces $L^q_t L^r_x$ which are equipped with the norm
\[ \|u\|_{L^q_t L^r_x} := \left( \int_J \|u\|^q_{L^r_x} \, dt \right)^{\frac{1}{q}} \]
for any space-time slab $J \times \mathbb{R}$, with the usual modification when either $q$ or $r$ are infinity. When $q = r$ we abbreviate $L^q_t L^r_x$ by $L^q_t L^r_{t,x}$.

As usual, we define the Fourier transform of $f(x) \in L^1_x$ by
\[ \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) \, dx. \]
For an appropriate class of functions the following Fourier inversion formula holds:
\[ f(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} \hat{f}(\xi) \, d\xi. \]
We define the fractional differentiation operator $\|\partial_x\|^{\alpha}$ for any real $\alpha$ by
\[ \|\partial_x\|^{\alpha}u(\xi) := |\xi|^\alpha \hat{u}(\xi), \]
and analogously
\[ \langle \partial_x \rangle^{\alpha}u(\xi) := \langle \xi \rangle^\alpha \hat{u}(\xi). \]
The inhomogeneous Sobolev space $H^s$ is given via
\[ \|u\|_{H^s} = \|\langle \partial_x \rangle^s u\|_{L^2_x}, \]
while the homogeneous Sobolev space $\dot{H}^s$ is defined by
\[ \|u\|_{\dot{H}^s} = \|\langle \partial_x \rangle^s u\|_{L^2_x}. \]

Let $U(t)$ denote the solution operator to the linear Schrödinger equation
\[ iu_t + \Delta u = 0, \quad x \in \mathbb{R}, \]
that is
\[ U(t)u_0(x) = \int e^{2\pi i \xi x - (2\pi \xi)^2 t} \hat{u}_0(\xi) \, d\xi. \]
We denote by $X^{s,b} = X^{s,b}(\mathbb{R} \times \mathbb{R})$ the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{R})$ with respect to the following norm
\[ \|u\|_{X^{s,b}} = \|U(-t)u\|_{H_x^s L_t^2} = \|\langle \xi \rangle^s (\tau + 4\pi^2 \xi^2)^b \hat{u}(\xi, \tau)\|_{L^2_x L^2_t}, \]
where \( \tilde{u}(\xi, \tau) \) is the space-time Fourier Transform
\[
\tilde{u}(\xi, \tau) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i (\xi x + \tau t)} u(x, t) dx dt.
\]
Furthermore for a given time interval \( J \), we define
\[
\|f\|_{X^{s,b}(J)} = \inf_{g = f \text{ on } J} \|g\|_{X^{s,b}}.
\]
Often we will drop \( J \).

### 2.3. Some known estimates.

Now we recall a few known estimates that we shall need. First, we state the following Strichartz estimate [15, 17]. We recall that a pair of exponents \((q, r)\) is called admissible in \( \mathbb{R} \) if
\[
\frac{2}{q} + \frac{1}{r} = 1, \quad 2 \leq q, r \leq \infty.
\]

**Proposition 2.1.** Let \((q, r)\) and \((\tilde{q}, \tilde{r})\) be any two admissible pairs. Suppose that \( u \) is a solution to
\[
iu_t + \Delta u - G(x, t) = 0, \quad x \in J \times \mathbb{R},
\]
\[
u(x, 0) = u_0(x).
\]

Then we have the estimate
\[
\|u\|_{L^q_t L^r_x(J \times \mathbb{R})} \lesssim \|u_0\|_{L^2} + \|G\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(J \times \mathbb{R})}
\]
with the prime exponents denoting Hölder dual exponents.

Since
\[
\hat{U}(t)u_0(\xi) = e^{-2\pi i t u_0(\xi)}
\]
we have that
\[
\|U(t)u_0\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2}.
\]
Hence
\[
\|u\|_{L^\infty_t L^2_x} = \|U(t)U(-t)u\|_{L^\infty_t L^2_x} \lesssim \|U(-t)u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{1/2+}},
\]
where in the last inequality we applied the definition of the \( X^{s,b} \) spaces, the basic estimate \( \|u\|_{L^\infty} \leq \|\hat{u}\|_{L^1} \), and the Cauchy-Schwarz inequality. The estimate (2.3) combined with the Sobolev embedding theorem implies that
\[
\|u\|_{L^\infty_t L^\infty_x} \lesssim \|u\|_{X^{1/2+, 1/2+}}.
\]
The Strichartz estimate gives us
\[
\|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{0, 1/2+}}.
\]
If we interpolate equations (2.4) and (2.5) we get
\[
\|u\|_{L^p_t L^q_x} \lesssim \|u\|_{X^{\alpha_1(p), 1/2+}},
\]
with \( \alpha_1(p) = \left(\frac{1}{2} - \frac{3}{p}\right)^+ + 6 \leq p \leq \infty. \)
3. The I-method and the local well-posedness for the I-system

3.1. The I-operator and the hierarchy of energies. Let us define the operator $I$. For $s < 1$ and a parameter $N > 1$ let $m(\xi)$ be the following smooth monotone multiplier:

$$m(\xi) := \begin{cases} \frac{1}{N} & \text{if } |\xi| < N, \\ (\frac{|\xi|}{N})^{s-1} & \text{if } |\xi| > 2N. \end{cases}$$

We define the multiplier operator $I : H^s \to H^1$ by

$$I u(\xi) = m(\xi) \hat{u}(\xi).$$

The operator $I$ is smoothing of order $1 - s$ and we have that:

$$\|u\|_{X^s_{\infty, b_0}} \lesssim \|Iu\|_{X^s_{\infty, 1-b_0}} \lesssim N^{1-s}\|u\|_{X^s_{\infty, b_0}},$$

for any $s_0, b_0 \in \mathbb{R}$.

We set

$$E^1(u) = E(Iu),$$

where

$$E(u)(t) = \frac{1}{2} \int |u_x(t)|^2 dx + \frac{1}{6} \int |u(t)|^6 dx = E(u_0).$$

We call $E^1(u)$ the first modified energy. Since we base our approach on the analysis of a second modified energy, we collect some facts concerning the calculus of multilinear forms used to define the hierarchy, see, for example [21].

If $n \geq 2$ is an even integer we define a spatial multiplier of order $n$ to be the function $M_n(\xi_1, \xi_2, \ldots, \xi_n)$ on $\Gamma_n = \{(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n : \xi_1 + \xi_2 + \ldots + \xi_n = 0\}$ which we endow with the standard measure $\delta(\xi_1 + \xi_2 + \ldots + \xi_n)$. If $M_n$ is a multiplier of order $n$, $1 \leq j \leq n$ is an index and $l \geq 1$ is an even integer, the elongation $X^l_j(M_n)$ of $M_n$ is defined to be the multiplier of order $n + l$ given by

$$X^l_j(M_n)(\xi_1, \xi_2, \ldots, \xi_{n+l}) = M_n(\xi_1, \ldots, \xi_{j-1}, \xi_j + \ldots + \xi_{j+l}, \xi_{j+l+1}, \ldots, \xi_{n+l}).$$

Also if $M_n$ is a multiplier of order $n$ and $f_1, f_2, \ldots, f_n$ are functions on $\mathbb{R}$ we define

$$\Lambda_n(M_n; f_1, f_2, \ldots, f_n) = \int_{\Gamma_n} M_n(\xi_1, \xi_2, \ldots, \xi_n) \prod_{i=1}^{n} f_j(\xi_j),$$

and we adopt the notation $\Lambda_n(M_n; f) = \Lambda_n(M_n; f, f, \ldots, f, f)$. Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even $\xi_j$ indices, or of the odd $\xi_j$ indices.

If $f$ is a solution of (1.1) the following differentiation law holds for the multilinear forms $\Lambda_n(M_n; f)$:

$$\partial_t \Lambda_n(M_n) = i\Lambda_n(M_n \sum_{j=1}^{n} (-1)^j \xi_j^2) - i\Lambda_{n+4} \left( \sum_{j=1}^{n} (-1)^j X^1_j(M_n) \right).$$

Observe that in this notation the first modified energy (3.2) can be written as follows:

$$E^1(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x Iu|^2 dx + \frac{1}{6} \int_{\mathbb{R}} |Iu|^6 dx = \frac{1}{2} \Lambda_2(m_1 m_2) + \frac{1}{6} \Lambda_6(m_1 \ldots m_6)$$

where $m_j = m(\xi_j)$.

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5One can actually informally define a hierarchy of modified energies for different nonlinear dispersive equations, see [6, 8, 9].
Now we define the second modified energy
\[ E^2(u) = -\frac{1}{2} \Lambda_2(m_1\xi_1m_2\xi_2) + \frac{1}{6} \Lambda_6(M_6(\xi_1, \xi_2, ..., \xi_6)), \]
where \( M_6(k_1, k_2, ..., k_6) \) is the following multiplier:
\[ M_6(\xi_1, \xi_2, ..., \xi_6) = \frac{1}{3} \xi_1^2 - \frac{1}{2} \xi_2^2 + \frac{1}{3} \xi_3^2 - m_2^2 \xi_2^2 + m_3^2 \xi_3^2 - m_4^2 \xi_4^2 - m_5^2 \xi_5^2 - m_6^2 \xi_6^2. \]

We remark that the zero set of the denominator corresponds to the resonant set for a six-waves interaction. Also \( M_6 \) contains more “cancellations” than the multiplier \( m_1...m_6 \) that appears in \( E^1 \).

The differentiation rules together with the fundamental theorem of calculus implies the following Lemma, which will be used to prove that \( E^2 \) is almost conserved.

**Lemma 3.1.** Let \( u \) be an \( H^1 \) solution to \((1.1)\). Then for any \( T \in \mathbb{R} \) and \( \delta > 0 \) we have
\[ E^2(u(T + \delta)) - E^2(u(T)) = \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t)) dt, \]
with
\[ M_{10} = -\frac{10}{960} \sum \{ M_6(\xi_{abcd}, \xi_f, \xi_g, \xi_b, \xi_i, \xi_j) - M_6(\xi_a, \xi_{bcdef}, \xi_g, \xi_h, \xi_i, \xi_j) + M_6(\xi_a, \xi_b, \xi_{defg}, \xi_h, \xi_i, \xi_j) - M_6(\xi_a, \xi_b, \xi_c, \xi_{degh}, \xi_i, \xi_j) + M_6(\xi_a, \xi_b, \xi_c, \xi_d, \xi_{efgh}, \xi_j) - M_6(\xi_a, \xi_b, \xi_c, \xi_d, \xi_e, \xi_{fghij}) \}, \]
where the summation runs over all permutations \( \{a, c, e, g, i\} = \{1, 3, 5, 7, 9\} \) and \( \{b, d, f, h, j\} = \{2, 4, 6, 8, 10\} \). Furthermore if \( |\xi_j| \ll N \) for all \( j \) then the multiplier \( M_{10} \) vanishes.

As it was observed in [21] one has

**Proposition 3.2.** The multiplier \( M_6 \) defined in \((3.4)\) is bounded on its domain of definition.

### 3.2. Modified Local Well-Posedness

In this subsection we shall prove a local well-posedness result for the modified solution \( Iu \) and some \textit{a priori} estimates for it.

Let \( J = [t_0, t_1] \) be an interval of time. We denote by \( Z_I(J) \) the following space:
\[ Z_I(J) = S_I(J) \cap X_I^{1,b}(J) \]
where \( b = 1/2^+ \) and
\[ S_I(J) = \{ f \mid \sup_{(q,r) \text{ admissible}} \| \partial_x^r I f \|_{L^q_x L^r_t(J \times \mathbb{R})} < \infty \}, \]
\[ X_I^{1,b}(J) = \{ f \mid \| I f \|_{X_I^{1,b}(J \times \mathbb{R})} < \infty \}. \]

**Proposition 3.3.** Let \( s > 0 \). and consider the IVP
\[ iu_{tt} + \Delta u - I(|u|^4 u) = 0, \]
\[ Iu(x, t_0) = Iu_0(x) \in H^1(\mathbb{R}), \quad x \in \mathbb{R}, \ t \in \mathbb{R}. \]

Then for any \( u_0 \in H^s \) there exists a time interval \( J = [t_0, t_0 + \delta] \), \( \delta = \delta(\|Iu_0\|_{H^s}) \) and there exists a unique \( u \in Z_I(J) \), solution to \((3.6)\) and \((3.7)\). Moreover there is continuity with respect to the initial data.
This remark combined with (3.9) implies that using the Strichartz estimate in Proposition 2.1, for any pair of admissible exponents (\(q, r\)), we obtain
\[
|\partial_x Iu|_{L^q_t L^r_x(J \times \mathbb{R})} \leq \delta^{1/6} \|Iu\|_{H^1}.
\]

\[\Box\]

**Proposition 3.4.** Let \(s > 0\). If \(u\) is a solution to the IVP (3.6)-(3.7) on the interval \(J = [t_0, t_1]\), which satisfies the following a priori bound
\[
\|Iu\|_{L^6_t L^6_x(J \times \mathbb{R})}^6 < \mu,
\]
where \(\mu\) is a small universal constant, then
\[
\|u\|_{Z_t(J)} \lesssim \|Iu_0\|_{H^1}.
\]

**Proof.** We start by obtaining a control of the Strichartz norms. Applying \(\langle \partial_x \rangle\) to (3.6) and using the Strichartz estimate in Proposition 2.1 for any pair of admissible exponents \((q, r)\) we obtain
\[
\|\langle \partial_x \rangle Iu\|_{L^q_t L^r_x} \lesssim \|Iu_0\|_{H^1} + \|\langle \partial_x \rangle I(|u|^4 u)\|_{L^q_t L^r_x}.
\]

Now we notice that the multiplier \(\langle \partial_x \rangle I\) has a symbol which is increasing as a function of \(|\xi|\) for any \(s > 0\). Using this fact one can modify the proof of the Leibniz rule for fractional derivatives and prove its validity for \(\langle \partial_x \rangle I\). See also Principle A.5 in the appendix of [18]. This remark combined with (3.9) implies that
\[
\|\langle \partial_x \rangle Iu\|_{L^q_t L^r_x} \lesssim \|Iu_0\|_{H^1} + \|\langle \partial_x \rangle I(|u|^4 u)\|_{L^q_t L^r_x}
\]
\[
\lesssim \|Iu_0\|_{H^1} + \|u\|_{Z_t(J)} \|u\|_{L^6_t L^6_x}^4
\]
(3.10)

where to obtain (3.10) we used Hölder’s inequality and the definition of \(Z_t(J)\).

In order to obtain an upper bound on \(\|u\|_{L^6_t L^6_x}\) we perform a Littlewood-Paley decomposition along the following lines. We note that a similar approach was used in [9], Lemma 3.1. We write
\[
u = u_{N_0} + \sum_{j=1}^{\infty} u_{N_j},
\]
where \(u_{N_0}\) has spatial frequency support for \(\langle \xi \rangle \leq N\), while \(u_{N_j}\) is such that its spatial Fourier transform is supported for \(\langle \xi \rangle \sim N_j = 2^j h_j\) with \(h_j \gtrsim \log N\) and \(j = 1, 2, 3, \ldots\). Let \(\epsilon > 0\). The triangle inequality applied on (3.11) gives
\[
\|u\|_{L^6_t L^6_x} \leq \|u_{N_0}\|_{L^6_t L^6_x} + \sum_{j=1}^{\infty} \|u_{N_j}\|_{L^6_t L^6_x}
\]
\[
= \|Iu_{N_0}\|_{L^6_t L^6_x} + \sum_{j=1}^{\infty} \|u_{N_j}\|_{L^6_t L^6_x} \|u_{N_j}\|_{L^6_t L^6_x}^{1-\epsilon}.
\]
(3.12)
On the other hand, by using the definition of the operator $I$, the definition of the $u_{N_j}'s$ and the Marcinkiewicz multiplier theorem we observe the following:

\begin{align}
(3.13) & \quad \|Iu_{N_0}\|_{L_t^6L_x^6} \lesssim \|Iu\|_{L_t^6L_x^6}, \\
(3.14) & \quad \|u_{N_j}\|_{L_t^6L_x^6} \lesssim N_j^{1-s}N^{s-1}\|Iu_{N_j}\|_{L_t^6L_x^6}, \quad j = 1, 2, 3, \ldots \\
(3.15) & \quad \|u_{N_j}\|_{L_t^6L_x^6} \lesssim N_j^{s-1}N^{s-1}\|\langle \partial_x \rangle Iu_{N_j}\|_{L_t^6L_x^6}, \quad j = 1, 2, 3, \ldots
\end{align}

Now we use the estimates (3.13) - (3.15) to obtain the following upper bound on (3.12)

\[ \|u\|_{L_t^6L_x^6} \lesssim \|Iu\|_{L_t^6L_x^6} + \sum_{j=1}^{\infty} N_j^{-s+\epsilon}N^{-s+\epsilon}\|Iu_{N_j}\|_{L_t^6L_x^6}^{s} \|\langle \partial_x \rangle Iu_{N_j}\|_{L_t^6L_x^6}^{1-s}, \]

which after noticing that $N^{-s+\epsilon} \leq 1$ implies

\[ \|u\|_{L_t^6L_x^6} \lesssim \|Iu\|_{L_t^6L_x^6} + \sum_{j=1}^{\infty} N_j^{-s+\epsilon}N^{-s+\epsilon}\|Iu_{N_j}\|_{L_t^6L_x^6}^{s} \|\langle \partial_x \rangle Iu_{N_j}\|_{L_t^6L_x^6}^{1-s}. \]

Now using the cheap Littlewood-Paley inequality

\[ \sup_j \|u_{N_j}\|_{L^p} \lesssim \|u\|_{L^p} \]

for $1 \leq p \leq \infty$, and summing (3.16), we have that for any $s > \epsilon$

\[ \|u\|_{L_t^6L_x^6} \lesssim \|Iu\|_{L_t^6L_x^6} + \|Iu\|_{L_t^6L_x^6}^{s} \|\langle \partial_x \rangle Iu\|_{L_t^6L_x^6}^{s}, \]

which combined with (3.10) implies

\[ \|\langle \partial_x \rangle Iu\|_{L_t^6L_x^6} \lesssim \|Iu_0\|_{H_x^s} + \|u\|_{Z_{s}(J)} \|Iu\|_{L_t^6L_x^6}^{4-s} + \|u\|_{Z_{s}(J)} \|Iu\|_{L_t^6L_x^6}^{5-s}. \]

Now we shall obtain a control of the $X^{s,b}$ norm. We use Duhamel’s formula and the theory of $X^{s,b}$ spaces (for details, see, [14, 16]) to obtain

\[ \|\langle \partial_x \rangle Iu\|_{X^{0,\frac{s}{2}+}} \lesssim \|Iu_0\|_{H_x^s} + \|\langle \partial_x \rangle I(|u|^4u)\|_{X^{0,\frac{s}{2}+}}. \]

However interpolating between

\[ \|u\|_{L_t^6L_x^6} \lesssim \|u\|_{X^{0,\frac{s}{2}+}} \]

and

\[ \|u\|_{L_t^2L_x^2} \lesssim \|u\|_{X^{0,0}} \]

we obtain

\[ \|u\|_{L_t^6L_x^6} \lesssim \|u\|_{X^{0,0}} \]

which by duality gives

\[ \|u\|_{X^{0,-\frac{s}{2}+}} \lesssim \|u\|_{L_t^6L_x^6}. \]

Hence (3.19) implies

\[ \|Iu\|_{X^{1,\frac{s}{2}+}} \lesssim \|Iu_0\|_{H_x^s} + \|\langle \partial_x \rangle I(|u|^4u)\|_{L_t^6L_x^6}. \]
As we noticed earlier when obtaining an upper bound on the Strichartz norms, the Leibnitz rule for fractional derivatives can be proved for $\langle \partial_x \rangle I$. Therefore after applying the Leibnitz rule to (3.20) we obtain:

$$
\|Iu\|_{X^1, \frac{1}{2}^+} \lesssim \|Iu_0\|_{H^1_x} + \|\langle \partial_x \rangle Iu\|_{L^6_t L^5_x} \|u^4\|_{L^4_t L^3_x} \triangleq (3.21)
$$

where to obtain (3.21) we used Hölder’s inequality and the definition of $Z_I(J)$. An upper bound on $\|u\|_{L^6_t L^6_x}$ is given by (3.17). In order to obtain an upper bound on $\|u\|_{L^6_t L^6_x}$ we proceed as follows. First, we perform a dyadic decomposition and write $E_{3.3}$. In order to further bound the right-hand side of (3.22), first we notice that 

$$
E\bigg(\sum_{j=1}^\infty N_j \lesssim N_s^{-1} \|\langle \partial_x \rangle^{1-\delta} Iu_N\|_{L^6_t L^6_x}, \ j=1, 2, 3, ...
$$

In order to further bound the right-hand side of (3.22), first we notice that $N_s^{-1} \leq 1$. Then we apply the cheap Littlewood-Paley inequality as before and sum (3.22) to obtain for any $s > \delta$

$$
\|u\|_{L^6_t L^6_x} \lesssim \|\langle \partial_x \rangle^{1-\delta} Iu\|_{L^6_t L^6_x} + \|u\|_{X^1, \frac{1}{2}^+} \triangleq (3.23)
$$

However

$$
\|\langle \partial_x \rangle^{1-\delta} Iu\|_{L^6_t L^6_x} \leq \|Iu\|_{X^1, \frac{1}{2}^+} \triangleq (3.24)
$$

which follows from interpolating

$$
\|u\|_{L^6_t L^6_x} \leq \|u\|_{X^{0, \frac{1}{2}^+}}
$$

\[\] and

$$
\|u\|_{L^\infty_t L^\infty_x} \leq \|u\|_{X^{0, \frac{1}{2}^+}}.
$$

Now we combine (3.23) and (3.24) to obtain

$$
\|u\|_{L^6_t L^6_x} \lesssim \|Iu\|_{X^1, \frac{1}{2}^+} \triangleq (3.25)
$$

By applying the inequalities obtained in (3.17) and (3.25) to bound the right-hand side of (3.21), we obtain

$$
\|Iu\|_{X^1, \frac{1}{2}^+} \lesssim \|Iu_0\|_{H^1_x} + \|u\|_{Z_I(J)} \|Iu\|_{L^6_t L^5_x} + \|u\|_{Z_I(J)} \|Iu\|_{L^6_t L^5_x}. \triangleq (3.26)
$$

The desired bound (3.8) follows from (3.18) and (3.26).

3.3 $E^2(u)$ is a small perturbation of $E^1(u)$. Now we shall prove that the second energy $E^2(u)$ is a small perturbation of the first energy $E^1(u)$.
Decomposition remark. Our approach to prove that the second energy is a small perturbation of the first energy as well as to prove a decay for the increment of the second energy is based on obtaining certain multilinear estimates in appropriate functional spaces which are $L^2$-based. Hence, whenever we perform a Littlewood-Paley decomposition of a function we shall assume that the Fourier transforms of the Littlewood-Paley pieces are positive. Moreover, we will ignore the presence of conjugates. At the end we will always keep a factor of order $N_{\text{max}}^{-c}$, where $N_{\text{max}}$ is the largest frequency of the different pieces, in order to perform the summations. The details for both the Propositions 3.5 and 3.6 can be found in [21], but we choose to present the main points of the argument to make the paper as self-contained as possible.

Proposition 3.5. Assume that $u$ solves (1.1) with $s > 1/3$. Then

$$E^2(u) = E^1(u) + O(N^{-c})\|Iu\|_{H^1}^6.$$  

Moreover if $\|Iu\|_{H^1} = O(1)$ then $\|\partial_x Iu\|_{L^2}^2 \lesssim E^2(u)$.

Proof. By the definition of the first and second modified energy we have

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{6}\Lambda_6(M_6) = E^1(u) + \frac{1}{6}\Lambda_6(M_6 - \prod_{i=1}^6 m_i).$$

Hence it suffices to prove the following pointwise in time estimate

$$|\Lambda_6(M_6 - \prod_{i=1}^6 m_i)|\|Iu(\cdot, t)\|_{H^1}^6 \lesssim O(N^{-c})\|Iu(\cdot, t)\|_{H^1}^6.$$  

Combining the Decomposition Remark with the fact that $M_6$ is bounded (by Proposition 3.2) and that $m$ is bounded (by its definition) it is enough to show that

$$\int_{\Gamma_6} \prod_{j=1}^6 \hat{u}(\xi_j) \lesssim \frac{1}{N^c}\|Iu(\cdot, t)\|_{H^1}^6.$$  

Towards this aim, we use again a dyadic decomposition and one can easily check that it suffices to show the following:

$$\int_{\Gamma_6} \prod_{j=1}^6 \hat{u}_{N_j}(\xi_j) \lesssim \frac{1}{N^c}(N_1 \cdots N_6)^{0-} \prod_{j=1}^6 \|Iu_{N_j}(\cdot, t)\|_{H^1}^6$$  

where we recall that $u_{N_j}$ is supported around $\langle \xi \rangle \sim N_j$.

We can rearrange the sizes of the frequencies so that $N_1^* \geq N_2^* \geq N_3^* \geq N_4^* \geq N_5^* \geq N_6^*$ and for simplicity set $u_{N_j} = u_j$ and $u_{N_j^*} = u_j^*$.

We may assume that $N_1^* \gtrsim N$, otherwise $M_6 - \prod_{i=1}^6 m_i \equiv 0$ and (3.28) follows trivially. We have that

$$\int_{\Gamma_6} \prod_{j=1}^6 \hat{u}_j(\xi_j) \lesssim \frac{1}{(N_1^*)^c} \int (N_1^*)^c u_1^6 \prod_{j=2}^6 u_j^6 \lesssim \frac{1}{N^c} \|\nabla^c u_1^*\|_{L^6}^6 \prod_{j=2}^6 \|u_j^*\|_{L^6}^6$$  

by reversing Plancherel and applying Hölder’s inequality. Moreover by Sobolev embedding

$$\|\nabla^c u_1^*\|_{L^6}^6 \lesssim \|u_1^*\|_{H^{1/3+}_2}^6.$$
and thus
\[
\int_{\Gamma_6} \prod_{j=1}^{6} \hat{u}_j(\xi_j) \lesssim \frac{1}{N^\epsilon} \|u^*_j\|_{H^1_{x,t}^{6/3+}}^6.
\]
In addition for \( s > 1/3 \) we have
\[
\|u^*_j\|_{H^1_{x,t}^{6/3+}} \lesssim \|Iu^*_j\|_{H^1_{x,t}^{6/3+}} \lesssim \|Iu^*_j\|_{H^1_{x,t}^1}.
\]
Thus (3.29) follows.

**Remark.** As mentioned in the introduction, we obtain global well-posedness for \( s > 1/3 \), which is exactly the restriction on \( s \) of the previous proposition. So this regularity is a threshold to the method, at least if one wants to use the second modified energy.

### 3.4. An upper bound on the increment of \( E^2(u) \). In Lemma 3.1 we proved that an increment of the second energy can be expressed as
\[
E^2(u(T + \delta)) - E^2(u(T)) = \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t)) dt.
\]
Hence in order to control the increment of the second energy we shall find an upper bound on the \( \Lambda_{10} \) form, which we do in the following proposition.

**Proposition 3.6.** For any Schwartz function \( u \), and any \( \delta \sim 1 \), we have that
\[
\left| \int_0^\delta \Lambda_{10}(M_{10}; u(t)) \right| \lesssim N^{-2} \|Iu\|_{X^{1,1/2+}}^{10},
\]
for \( s > 1/4 \).

**Proof.** We observe that \( M_{10} \) is bounded as an elongation of the bounded multiplier \( M_6 \). We perform a dyadic decomposition as in Proposition 3.5 and we borrow the same notation. Since we are integrating over \( \Gamma_{10} \), we have that \( N^*_1 \sim N^*_2 \), and we may assume that \( N^*_1 \gtrsim N \) otherwise \( M_{10} \equiv 0 \). Thus we consider \( N^*_1 \sim N^*_2 \gtrsim N \).

Since \( \frac{1}{m(N^*_1)(N^*_2)} \lesssim \frac{1}{N^2} \) we have
\[
\left| \int_0^\delta \int M_{10} \prod_{j=1}^{10} \hat{u}_j \right| \lesssim \frac{(N^*_1)^0}{N^2} \|Iu\|_{X^{1,1/2+}}^2 \|u\|_{X^{1/2,0-1/2+}}^4 \|u\|_{X^{1/2,0-1/2+}}^8 \|u\|_{X^{1/2,0-1/2+}}^{10} \prod_{j=3} \|u^*_j\|_{L^2_{x,t}^{1/2}}^{1/2} \]
\[
(3.30)
\]
\[
\lesssim \frac{(N^*_1)^0}{N^2} \|Iu\|_{X^{1,1/2+}}^2 \|u\|_{X^{1/2,0-1/2+}}^8 \|u\|_{X^{1/2,0-1/2+}}^{10} \]
\[
(3.31)
\]
where in order to obtain (3.30) we use (2.5) and to obtain (3.31) we use the fact that for \( s > 1/4 \) the following inequality holds
\[
\|u\|_{X^{1/4,0-1/2+}} = \|u\|_{X^{1/4,0-1/2+}} \lesssim \|Iu\|_{X^{1,1/2+}}.
\]
\[\square\]
4. The almost Morawetz estimate

In this section we shall prove an interaction Morawetz estimate for the smoothed out solution \( Iu \), hence the name “almost Morawetz” estimate. Our estimate reads as follows:

**Theorem 4.1.** Let \( u \in \mathcal{S} \) be a solution to the NLS

\[
iu_t + \Delta u = |u|^4 u, \quad (x, t) \in \mathbb{R} \times [0, T].
\]

Then, there exists a convex function \( a : \mathbb{R}^4 \to \mathbb{R} \) with \( \nabla a \in L^\infty(\mathbb{R}^4) \), such that

\[
\|Iu\|_{L^8_t L^8_x([0,T] \times \mathbb{R})} \lesssim \sup_{[0,T]} \|Iu\|_{\dot{H}^1_x},
\]

\[
+ \left| \int_0^T \int_{\mathbb{R}^4} \nabla a(x_1, x_2, x_3, x_4) \cdot \{N_{\text{bad}}, Iu_1 Iu_2 Iu_3 Iu_4\}_p dx_1 dx_2 dx_3 dx_4 dt \right|,
\]

where

\[
N_{\text{bad}} = \sum_{k=1}^4 \left( I(N_k(u)) - N_k(Iu) \right) 
\quad \prod_{j=1, j \neq k}^4 Iu_j,
\]

\[
N(f) = |f|^4 f, \quad N_k(f) = N(f_k), \quad \text{and} \quad \{\cdot\}_p \text{ is the momentum bracket defined by}
\]

\[
\{f, g\}_p = \Re(f \nabla g - g \nabla f).
\]

Moreover in a time interval \( J = [t_0, t_1] \) where the solution \( u \) belongs to the space \( Z_1(J) \) we have that

\[
\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^4} \nabla a(x_1, x_2, x_3, x_4) \cdot \{N_{\text{bad}}, Iu_1 Iu_2 Iu_3 Iu_4\}_p dx_1 dx_2 dx_3 dx_4 dt \right| \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^{12}.
\]

**Remark 4.2.** In all of our arguments we will work with smooth (Schwartz) solutions. This will simplify the calculations and will enable us to justify the steps in the subsequent proofs. Then the local well-posedness theory and the perturbation theory that have been established (see, for example, \([4]\)) for this problem can be applied to approximate the \( H^s \) solutions by smooth solutions.

We prove the above almost Morawetz estimate inspired by the idea of the proof of the interaction Morawetz estimate for the defocusing nonlinear cubic Schrödinger equation on \( \mathbb{R}^3 \), \([9]\) and the one that recently appeared in \([5]\). However we establish a Morawetz estimate for the almost solution, i.e for \( Iu \) itself, which is the main novelty of our approach. In order to make our presentation complete, first we recall a general approach of obtaining interaction Morawetz estimates \([10]\). Then we present our derivation of the interaction Morawetz estimate for the almost solution \( Iu \).

4.1. Towards interaction Morawetz estimates. In this subsection we follow \([10]\). Note that here we work in general dimension \( d \).

Let \( u \in \mathcal{S} \) be a solution to the NLS

\[
iu_t + \Delta u = \mathcal{N}(u), \quad (x, t) \in \mathbb{R}^d \times [0, T].
\]

We say that \( \mathcal{N} \) corresponds to a defocusing potential \( G \) (meaning \( G \) positive) if

\[
\{\mathcal{N}, u\}_p = \partial_j G.
\]
Here we are denoting by \( \{f, g\}_p \) the vector whose components are given by
\[
\{f, g\}_p^j = \Re(f \partial_j g - g \partial_j f).
\]
For example, in the case when \( \mathcal{N}(u) = |u|^4 u \) we have that \( \{\mathcal{N}, u\}_p^j = -\partial_j G \), where \( G = \frac{2}{3} |u|^6 \).

Now let us define the momentum density via
\[(4.5) \quad T_{0j} = 2\Im(\bar{u} \partial_j u)\]
and the linearized momentum current
\[(4.6) \quad L_{jk} = -\partial_j \partial_k (|u|^2) + 4\Re(\bar{\partial_j u} \partial_k u).\]

An easy computation shows that
\[(4.7) \quad \partial_t T_{0j} + \partial_k L_{jk} = 2\{\mathcal{N}, u\}_p^j ,\]
where we adopt Einstein’s summation convention. Thus by integrating in space we have that in the case when \( \mathcal{N} \) corresponds to a potential, then the total momentum is conserved in time,
\[
\int_{\mathbb{R}} T_{0j}(x, t) dx = C.
\]

Finally, if \( a : \mathbb{R}^d \to \mathbb{R} \) is convex then we define the Morawetz action associated to \( u \) by the formula
\[(4.8) \quad M_a(t) = 2 \int_{\mathbb{R}^d} \nabla a(x) \cdot \Im(\bar{u}(x) \nabla u(x)) dx.\]

We now recall a classical result. The first step in the proof of the estimate (4.2) is obtained by a slight modification of the argument in the following Proposition, in the case when the forcing term \( \mathcal{N} \) does not correspond to a defocusing potential. We will state this result in the form of a corollary.

**Proposition 4.3.** Let \( a : \mathbb{R}^d \to \mathbb{R} \) be a convex function and \( u \) be a smooth solution to equation (4.4) on \( \mathbb{R}^d \times [0, T] \) with a defocusing potential \( G \). Then, the following inequality holds
\[(4.9) \quad \int_0^T \int_{\mathbb{R}^d} (-\Delta a)|u(x, t)|^2 dx dt \leq \sup_{t \in [0, T]} |M_a(t)|.\]

**Proof.** Without loss of generality, we can assume that \( a \) is smooth. Then, a standard approximation argument concludes the proof for the general case.

According to (4.5) the Morawetz action can be written as
\[
M_a(t) = \int_{\mathbb{R}^d} \partial_j a T_{0j}.
\]
Then thanks to (4.7),
\[
\partial_t M_a(t) = \int_{\mathbb{R}^d} \partial_j a \partial_t T_{0j} \\
= \int_{\mathbb{R}^d} \partial_j a (-\partial_k L_{jk} + 2\{N, u\}_p) \\
= \int_{\mathbb{R}^d} \partial_j a (-\partial_k L_{jk} - 2\partial_j G) \\
= \int_{\mathbb{R}^d} (\partial_j \partial_k a) L_{jk} dx + 2 \int_{\mathbb{R}^d} \Delta a G dx,
\]
(4.10)
where in the last equality we used integration by parts. Now (4.11) combined with the definition of \(L_{jk}\) (4.6) implies
\[
\partial_t M_a(t) = \int_{\mathbb{R}^d} (\Delta a) |a|^2 dx + 2 \int_{\mathbb{R}^d} \Delta a G dx + 4 \int_{\mathbb{R}^d} (\partial_j \partial_k a) \Re(\partial_j u \partial_k u) dx.
\]
(4.11)
Since \(\partial_j \partial_k a\) is weakly convex we have that
\[
4(\partial_j \partial_k a) \Re(\partial_j u \partial_k u) \geq 0
\]
and the trace of the Hessian of \(\partial_j \partial_k a\), which is \(\Delta a\), is positive. Thus
\[
-\int_{\mathbb{R}^d} (\Delta a) |a|^2 dx \leq \partial_t M_a(t).
\]
Hence by the fundamental theorem of calculus we have that
\[
\int_0^T \int_{\mathbb{R}^d} (\Delta a) |u(x, t)|^2 dx dt \lesssim \sup_{t \in [0, T]} |M_a(t)|.
\]
(4.12)

In the case of a solution to an equation with a nonlinearity which is not associated to a defocusing potential, we immediately obtain the following corollary.

**Corollary 4.4.** Let \(a : \mathbb{R}^d \to \mathbb{R}\) be convex and \(v\) be a smooth solution to the equation
\[
iv_t + \Delta v = \tilde{N}, \quad (x, t) \in \mathbb{R}^d \times [0, T].
\]
(4.13)
Then, the following inequality holds
\[
\int_0^T \int_{\mathbb{R}^d} (\Delta a) |v(x, t)|^2 dx dt + 2 \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{\tilde{N}, v\}_p dx dt \lesssim \sup_{t \in [0, T]} |M_a(t)|,
\]
where \(M_a(t)\) is the Morawetz action corresponding to \(v\).

Now we are ready to derive the main inequality in obtaining interaction Morawetz estimates. Let \(u_k, k = 1, 2\), be solutions to (4.4) with nonlinearity \(\tilde{N}_k\) in \(d_k\)–spatial dimensions. Assume \(\tilde{N}_k\) has a defocusing potential \(G_k\). Define the tensor product \(u := (u_1 \otimes u_2)(x, t)\) for \(x\) in \(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\) by the formula
\[
(u_1 \otimes u_2)(x, t) = u_1(x_1, t) u_2(x_2, t).
\]
Then, it can be easily verified that $u_1 \otimes u_2$ solves (4.4) with forcing term $N = \tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1$. Since
\[
\{\tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1, u_1 \otimes u_2\}_p = \left\{\{\tilde{N}_1, u_1\}_p \otimes |u_2|^2, \{\tilde{N}_2, u_2\}_p \otimes |u_1|^2\right\}
\]
we have the important fact that the tensor product of defocusing semilinear Schrödinger equations is also defocusing in the sense that
\[
\{\tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1, u_1 \otimes u_2\}_p = -\nabla G,
\]
where $\nabla = (\nabla_{x_1}, \nabla_{x_2})$ and $G = G_1 \otimes |u_2|^2 + G_2 \otimes |u_1|^2$, thus $G \geq 0$. Since $u_1 \otimes u_2$ solves (4.4) and obeys momentum conservation with a defocusing potential, we can apply Proposition 4.3 to obtain
\[
\int_0^T \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} (-\Delta \Delta a)|u_1 \otimes u_2|^2(x, t)dxdt \lesssim \sup_{t \in [0, T]} |M_a^\otimes(t)|,
\]
where $\Delta = \Delta_{x_1} + \Delta_{x_2}$ is the $d_1 + d_2$ Laplacian, $a$ is any real-valued convex function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and $M_a^\otimes(t)$ is the Morawetz action that corresponds to $u_1 \otimes u_2$.

Clearly, this argument can be generalized to an arbitrary number of solutions $u_k$ to (4.4) with nonlinearity $\tilde{N}_k$ with a defocusing potential $G_k$. Indeed one obtains
\[
\int_0^T \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_k}} (-\Delta \Delta a)|v|^2(x, t)dxdt \lesssim \sup_{t \in [0, T]} |M_a^\otimes(t)|,
\]
with $x := (x_1, \ldots, x_k)$, $v(x) := \bigotimes_{i=1}^k u_i(x_i)$, $\tilde{N} := \sum_{i=1}^k \tilde{N}_i \otimes j \neq i u_j$, and $M_a^\otimes(t)$ the Morawetz action corresponding to $v$.

Moreover, in the case when $\tilde{N}_k$ does not have a defocusing potential, then according to Corollary 4.4 we get
\[
\int_0^T \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_k}} (-\Delta \Delta a)|v|^2(x, t)dxdt + 2 \int_0^T \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_k}} \nabla a \cdot \{\tilde{N}, v\}_p dxdt \lesssim \sup_{t \in [0, T]} |M_a^\otimes(t)|.
\]
We will use (4.17) in the proof of Theorem 4.1.

4.2 Interaction Morawetz estimates. Estimate (4.16) turns out to be very useful, as a careful choice of the function $a$ allows one to obtain bounds on a particular Lebesgue norm of a solution to equation (4.4) with defocusing potential. For example, let $u$ be a solution to (4.4) with a defocusing potential, in dimension $d = 3$. Let $k = 2$, $d_1 = d_2 = 3$, $u_1 = u_2 = u$, and choose
\[
a(x) = a(x_1, x_2) = |x_1 - x_2|.
\]
Then an easy calculation shows that $-\Delta \Delta a = C\delta(|x_1 - x_2|)$, and equation (4.16) gives
\[
\int_0^T \int_{\mathbb{R}^3} |u(x, t)|^4dx \lesssim \sup_{t \in [0, T]} |M_a^\otimes(t)|.
\]
It can be shown using Hardy’s inequality (for details see [2]) that when $d = 3$
\[
\sup_{t \in [0, T]} |M_a^\otimes(t)| \lesssim \sup_{t \in [0, T]} \|u(t)\|^2_{H^{1/2}}
\]
and thus
\[
\|u(x, t)\|^4_{L^4} \lesssim \sup_{t \in [0, T]} \|u(t)\|^2_{H^{1/2}}.
\]
which is the interaction Morawetz estimate that appears in [9].

Analogously, in the case $d = 1$, let $u$ be a solution to (4.4) with a defocusing potential, and let $k = 4, d_1 = \ldots = d_4 = 1$, and $u_1 = \ldots = u_4 = u$. Then (4.16) reads

$$
\int_0^T \int_{\mathbb{R}^4} (\Delta \Delta a) \prod_{i=1}^{4} |u(x_i, t)|^2 dx_1 dx_2 dx_3 dx_4 dt \lesssim \sup_{t \in [0, T]} |M_a^{\otimes 4}(t)|.
$$

(4.18)

In order to proceed as in the case $d = 3$ and obtain a bound on a Lebesgue norm of $u$, we need to choose an appropriate function $a$, and get an upper bound on the right hand side. An elegant idea to obtain the desired bound can be found in [5]. Precisely, one performs an orthonormal change of variables $z = Ax$ with $A$ orthonormal matrix, and writes (4.18) with respect to the $z$-variable. Notice that $\Delta_z = \Delta_x$ and also the orthonormal change of variables leaves invariant the inner product which appears in the Morawetz action on the right hand side of (4.18). Choosing the convex function $a(z)$ to be $a(z) = (z_2^2 + z_3^2 + z_4^2)^{1/2}$ (hence $-\Delta_z \Delta_z a(z) = 4\pi \delta(z_2, z_3, z_4)$), it is possible to estimate quickly the right hand side, and then going back to the $x$-variable one obtains the following estimate

$$
\|u\|_{L^8_t L^\infty_x} \lesssim \sup_{t \in [0, T]} \|u\|_{H^1} \|u\|_{L^2}^{7/2}.
$$

(4.19)

For the details we refer the reader to [5]. We will use the same orthonormal transformation and the same choice of function $a$ in the proof of our almost Morawetz estimate.

4.3. Almost Morawetz estimate. Proof of Theorem 4.1

Recall that $u \in \mathcal{S}$ is a solution to the NLS

$$
iu_t + \Delta u = \mathcal{N}(u), \quad (x, t) \in \mathbb{R} \times [0, T],
$$

(4.20)

with $\mathcal{N}(u) = |u|^4 u$. Let us set

$$
IU(x, t) = I \otimes I \otimes I \otimes I(u(x_1, t) \otimes u(x_2, t) \otimes u(x_3, t) \otimes u(x_4, t)) = \prod_{j=1}^{4} Iu(x_j, t).
$$

If $u$ solves (4.20) for $d = 1$, then $IU$ solves (4.20) on $\mathbb{R}^4 \times [0, T]$, with right hand side $\mathcal{N}_I$ given by

$$
\mathcal{N}_I = \sum_{k=1}^{4} (I(N_k) \prod_{j=1, j \neq k}^{4} Iu_j).
$$

Here and henceforth we set $u_k = u(x_k, t)$, and $N_k = \mathcal{N}(u_k)$. Hence, according to (4.17), we have the following estimate:

$$
\int_0^T \int_{\mathbb{R}^4} (\Delta \Delta a)|IU|^2(x, t) dx dt + 2 \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\mathcal{N}_I, IU\}_p dx dt \lesssim \sup_{t \in [0, T]} |M_a^I(t)|,
$$

(4.21)

with $M_a^I(t)$ the Morawetz action associated to $IU$, and $a : \mathbb{R}^4 \to \mathbb{R}$ convex function.

Now let us decompose,

$$
\mathcal{N}_I = \mathcal{N}_{good} + \mathcal{N}_{bad},
$$

where
where

\[ N_{\text{good}} = \sum_{k=1}^{4} (N(Iu_k) \prod_{j=1,j\neq k}^{4} Iu_j), \]

\[ N_{\text{bad}} = \sum_{k=1}^{4} (I(N_k) - N(Iu_k)) \prod_{j=1,j\neq k}^{4} Iu_j. \]

The first summand \( N_{\text{good}} \) creates a defocusing potential and thus

\[ \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{N_{\text{good}}, IU\}_p dx dt \geq 0. \]

Therefore, (4.21) yields,

(4.22) \[ \int_0^T \int_{\mathbb{R}^4} (-\Delta \Delta a) |IU|^2(x,t) dx dt \lesssim \sup_{[0,T]} |M_a^I(t)| + \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{N_{\text{bad}}, IU\}_p dx dt \right|. \]

After performing a change of variables as in Subsection 4.2, and using the same weight function \( a \) we obtain the following estimate:

(4.23) \[ \|IU\|_{L_{\infty}^8 L_{\infty}^2}^8 \lesssim \sup_{[0,T]} \|IU\|_{H^1} \|IU\|_{L^2}^7 + \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{N_{\text{bad}}, Iu_1 Iu_2 Iu_3 Iu_4\}_p dx_1 dx_2 dx_3 dx_4 dt \right|. \]

Notice that the dot product on the right hand side is left invariant under the change of variables. Also, \( |\nabla_z a(z)| = 1 \) and hence, since the matrix \( A \) is orthonormal, \( |\nabla_x a(x)| = 1 \). Thus we immediately obtain the \( L^\infty \) bound on \( \partial_x a, |\partial_x a(x)| \leq 1, i = 1, 2, 3, 4, \) that we strongly use in the following calculations.

This concludes the proof of the first part of our Theorem. We now proceed to prove the estimate (4.23), which is the core of this theorem.

We restrict on a time interval \( J = [t_0, t_1] \) on which the solution \( u \) belongs to the space \( Z_I(J) \). We wish to compute the dot product under the sign of integral in (4.23). First we observe that \( \nabla a \) is real valued, thus

\[ \nabla a \cdot \Re(f \nabla g - g \nabla f) = \Re(\nabla a \cdot (f \nabla g - g \nabla f)). \]

Hence, the desired dot product equals

(4.24) \[ \Re \left( \sum_{i=1}^{4} \partial_x a \left( N_{\text{bad}} \partial_x Iu_1 Iu_2 Iu_3 Iu_4 - \left( \prod_{j=1}^{4} Iu_j \right) \partial_x N_{\text{bad}} \right) \right). \]
We start by computing the first summand. Using the definition of $\mathcal{N}_{\text{bad}}$, and the fact that $\partial_{x_1}$ acts only on $Iu_1$ we obtain the following,

\[
\mathcal{N}_{\text{bad}} \partial_{x_1} (Iu_1 Iu_2 Iu_3 Iu_4) - \left( \prod_{j=1}^{4} Iu_j \right) \partial_{x_1} \mathcal{N}_{\text{bad}}
\]

\[
= \left( \sum_{k=1}^{4} (I(N_k) - N(Iu_k)) \prod_{h=1, h \neq k}^{4} Iu_h \right) (\partial_{x_1} Iu_1)(Iu_2 Iu_3 Iu_4)
\]

\[
- \left( \prod_{j=1}^{4} Iu_j \right) \partial_{x_1} \left( \sum_{k=1}^{4} (I(N_k) - N(Iu_k)) \prod_{h=1, h \neq k}^{4} Iu_h \right)
\]

\[
= \left( \sum_{k=1}^{4} (I(N_k) - N(Iu_k)) \prod_{h=1, h \neq k}^{4} Iu_h \right) (\partial_{x_1} Iu_1)(Iu_2 Iu_3 Iu_4)
\]

\[
- \prod_{j=1}^{4} Iu_j \left( \sum_{k=2}^{4} (I(N_k) - N(Iu_k)) \partial_{x_1}(\prod_{h=2, h \neq k}^{4} Iu_h) \right)
\]

\[
- \prod_{j=1}^{4} Iu_j \left( \sum_{k=2}^{4} (I(N_k) - N(Iu_k)) (\partial_{x_1} Iu_1)(\prod_{h=2, h \neq k}^{4} Iu_h) \right)
\]

\[
= (I(N_1) - N(Iu_1))(\partial_{x_1} Iu_1)(Iu_2 Iu_3 Iu_4)^2
\]

\[
+ \left( \sum_{k=2}^{4} (I(N_k) - N(Iu_k)) \prod_{h=1, h \neq k}^{4} Iu_h \right) (\partial_{x_1} Iu_1)(Iu_2 Iu_3 Iu_4)
\]

\[
- \prod_{j=1}^{4} Iu_j \left( \partial_{x_1}(I(N_1) - N(Iu_1)) \prod_{k=2}^{4} Iu_k \right)
\]

\[
- \prod_{j=1}^{4} Iu_j \left( \sum_{k=2}^{4} (I(N_k) - N(Iu_k)) (\partial_{x_1} Iu_1)(\prod_{h=2, h \neq k}^{4} Iu_h) \right)
\]

Now notice that the 2nd and 4th term in the last expression cancel each other, hence

\[
\mathcal{N}_{\text{bad}}(\partial_{x_1} (Iu_1 Iu_2 Iu_3 Iu_4)) - \left( \prod_{j=1}^{4} Iu_j \right) \partial_{x_1} \mathcal{N}_{\text{bad}}
\]
\[
= (I(N_1) - N(Iu_1))(\partial_{x_1} Iu_1)(Iu_2 Iu_3 Iu_4)^2
- \prod_{j=1}^4 Iu_j \left( \partial_{x_1} (I(N_1) - N(Iu_1)) \prod_{k=2}^4 Iu_k \right)
= [(I(N_1) - N(Iu_1))\partial_{x_1} Iu_1 - \partial_{x_1} (I(N_1) - N(Iu_1)) Iu_1] (Iu_2 Iu_3 Iu_4)^2.
\]

Hence the first summand in (4.24) is given by,
\[
\Re \left( \partial_{x_1} a \left[ (I(N_1) - N(Iu_1))\partial_{x_1} Iu_1 - \partial_{x_1} (I(N_1) - N(Iu_1)) Iu_1 \right] (Iu_2 Iu_3 Iu_4)^2 \right).
\]

Analogously one can see that the \( i \)th summand, \( i = 1, 2, 3, 4 \) is given by:
\[
\Re \left( \partial_{x_i} a \left[ (I(N_i) - N(Iu_i))\partial_{x_i} Iu_i - \partial_{x_i} (I(N_i) - N(Iu_i)) Iu_i \right] \prod_{j=1, j \neq i}^4 Iu_j \right)^2.
\]

Thus, our error term
\[
E = \int_{t_0}^{t_1} \int_{\mathbb{R}^4} \nabla a \cdot \left( N_{bad}(\nabla(Iu_1 Iu_2 Iu_3 Iu_4)) - \prod_{j=1}^4 Iu_j \nabla N_{bad} \right)
\]
reduces to
\[
E = \Re \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^4} \sum_{k=1}^4 \{ \partial_{x_k} a \left[ (I(N_k) - N(Iu_k))\partial_{x_k} Iu_k - \partial_{x_k} (I(N_k) - N(Iu_k)) Iu_k \right] \right.
\]
\[
\times \left( \prod_{j=1, j \neq k}^4 Iu_j \right)^2 dx_1 dx_2 dx_3 dx_4 dt).
\]

Hence, by symmetry,
\[
(4.25) \quad |E| \lesssim |E|,
\]
where
\[
E = \int_{t_0}^{t_1} \int_{\mathbb{R}^4} \{ \partial_{x_1} a \left[ (I(N_1) - N(Iu_1))\partial_{x_1} Iu_1 - \partial_{x_1} (I(N_1) - N(Iu_1)) Iu_1 \right] \right.
\]
\[
\times \left( \prod_{j=2}^4 Iu_j \right)^2 dx_1 dx_2 dx_3 dx_4 dt.
\]

We have,
\[
(4.26) \quad |E| \leq E_1 + E_2
\]
where
\[
E_1 = \int_{t_0}^{t_1} \int_{\mathbb{R}^4} |\partial_{x_1} a||I(N_1) - N(Iu_1)||\partial_{x_1} Iu_1|| \prod_{j=2}^4 |Iu_j|^2 dx_1 dx_2 dx_3 dx_4 dt
\]
and
\[ E_2 = \int_{t_0}^{t_1} \int_{\mathbb{R}^4} |\partial_{x^1} a| |\partial_{x^1} (I(N_1) - N(Iu_1))| |Iu_1| \prod_{j=2}^{4} |Iu_j|^2 dx_1 dx_2 dx_3 dx_4 dt. \]

Since \(|\partial_{x^1} a| \leq 1\), after applying Fubini’s theorem we have

\[ E_1 \leq \left( \int_{t_0}^{t_1} \int_{\mathbb{R}} |I(N_1) - N(Iu_1)| |\partial_{x^1} Iu_1| dx_1 dt \right) \|Iu\|^6_{L^\infty_t L^2_x}, \]

and

\[ E_2 \leq \left( \int_{t_0}^{t_1} \int_{\mathbb{R}} |\partial_{x^1} (I(N_1) - N(Iu_1))| |Iu_1| dx_1 dt \right) \|Iu\|^6_{L^\infty_t L^2_x}. \]

Since the pair \((\infty, 2)\) is admissible, we then obtain (rename \(x_1 = x\)):

\[ E_1 \leq \left( \int_{t_0}^{t_1} \int_{\mathbb{R}} |I(N) - N(Iu)| |\partial_{x} Iu| dx \ dt \right) \|u\|^6_{Z_1(J)}, \]

and

\[ E_2 \leq \left( \int_{t_0}^{t_1} \int_{\mathbb{R}} |\partial_{x} (I(N) - N(Iu))| |Iu| dx \ dt \right) \|u\|^6_{Z_1(J)}. \]

Therefore,

\[ E_1 \leq \|I(N) - N(Iu)\|_{L^1_t L^2_x} \|\partial_{x} Iu\|_{L^\infty_t L^2_x} \|u\|^6_{Z_1(J)}, \]

and

\[ E_2 \leq \|\partial_{x} (I(N) - N(Iu))\|_{L^1_t L^2_x} \|Iu\|_{L^\infty_t L^2_x} \|u\|^6_{Z_1(J)}. \]

Again, since \((\infty, 2)\) is admissible we obtain:

\[ E_1 \leq \|I(N) - N(Iu)\|_{L^1_t L^2_x} \|u\|^7_{Z_1(J)}, \]

and

\[ E_2 \leq \|\partial_{x} (I(N) - N(Iu))\|_{L^1_t L^2_x} \|u\|^7_{Z_1(J)}. \]

Therefore, from (4.20) and the bounds above, we deduce that

\[ |E| \leq \left( \|I(N) - N(Iu)\|_{L^1_t L^2_x} + \|\partial_{x} (I(N) - N(Iu))\|_{L^1_t L^2_x} \right) \|u\|^7_{Z_1(J)}. \]

We proceed to estimate \(\|\partial_{x} (I(N) - N(Iu))\|_{L^1_t L^2_x}\), which is the hardest of the two terms. Toward this aim, let us observe that since \(N(u) = \|u\|^p u\) with \(p = 4\), we will be able to work on the Fourier side to estimate the commutator \(I(N) - N(Iu)\).

We compute\(^6\)

\[ \partial_{x} (I(N) - N(Iu))(\xi) = \int_{\xi_1 + \cdots + \xi_5} i\xi [m(\xi) - m(\xi_1) \cdots m(\xi_5)] \hat{u}(\xi_1) \cdots \hat{u}(\xi_5) d\xi_1 \cdots d\xi_5. \]

\(^6\)We ignore complex conjugates, since our computations are not effected by conjugation.
We decompose $u$ into a sum of dyadic pieces localized around $N_j$ in the usual way. In all the estimates that follow we obtain a factor of order $N_{\text{max}}^{-\epsilon}$ in order to be able to perform the summations at the end. We omit this technical detail. Then,

$$
(4.28) \quad \| \partial_x (I(N) - N(Iu)) \|_{L^1_t L^2_x} = \| \partial_x (\widehat{I(N)} - \widehat{N(Iu)}) \|_{L^1_t L^2_x}
$$

$$
\leq \sum_{N_1, \ldots, N_5} \| \int_{\xi = \xi_1 + \ldots + \xi_5; |\xi| \sim N_i} \xi [m(\xi) - m(\xi_1) \cdots m(\xi_5)] \hat{u}_1 \cdots \hat{u}_5 d\xi_1 \cdots d\xi_5 \|_{L^1_t L^2_x}
$$

$$
= \sum_{N_1, \ldots, N_5} \| \int_{\xi = \xi_1 + \ldots + \xi_5; |\xi| \sim N_i} \xi [m(\xi) - m(\xi_1) \cdots m(\xi_5)] \hat{Iu}_1 \cdots \hat{Iu}_5 d\xi_1 \cdots d\xi_5 \|_{L^1_t L^2_x}.
$$

Without loss of generality, we can assume that the $N_j$’s are rearranged so that

$$
N_1 \geq \ldots \geq N_5.
$$

Set,

$$
\sigma(\xi_1, \ldots, \xi_5) = (\xi_1 + \ldots + \xi_5) \frac{[m(\xi_1 + \ldots + \xi_5) - m(\xi_1) \cdots m(\xi_5)]}{m(\xi_1) \cdots m(\xi_5)}.
$$

Then,

$$
\sigma(\xi_1, \ldots, \xi_5) = \sum_{j=1}^6 \sigma_j(\xi_1, \ldots, \xi_5),
$$

with

$$
\sigma_j(\xi_1, \ldots, \xi_5) = \chi_j(\xi_1, \ldots, \xi_5) \sigma(\xi_1, \ldots, \xi_5),
$$

where $\chi_j(\xi_1, \ldots, \xi_5)$ are smooth characteristic functions of the sets $\Omega_j$ defined as follows:

- $\Omega_1 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1 \ll N \}$.
- $\Omega_2 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1 \geq N \gg N_2 \}$.
- $\Omega_3 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1 \geq N_2 \gg N \gg N_3 \}$.
- $\Omega_4 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1 \geq N_2 \geq N_3 \gg N \gg N_4 \}$.
- $\Omega_5 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1 \geq N_2 \geq N_3 \geq N \gg N_5 \}$.
- $\Omega_6 = \{ |\xi_i| \sim N_i, i = 1, \ldots, 5; N_1, \ldots, N_5 \gg N \}$.

Hence, from (4.28) we get,

$$
(4.29) \quad \| \partial_x (I(N) - N(Iu)) \|_{L^1_t L^2_x}
$$

$$
\lesssim \sum_{N_1, \ldots, N_5} \sum_{j=1}^6 \| \int_{\xi = \xi_1 + \ldots + \xi_5} \sigma_j(\xi_1, \ldots, \xi_5) \hat{Iu}_1 \cdots \hat{Iu}_5 d\xi_1 \cdots d\xi_5 \|_{L^1_t L^2_x} = \sum_{N_1, \ldots, N_5} \sum_{j=1}^6 L_j.
$$

We proceed to analyze the contribution of each of the integrals $L_j$.

**Contribution of $L_1$.** Since $\sigma_1$ is identically zero, $L_1$ gives no contribution to the sum above.

**Contribution of $L_2$.** We have,

$$
\| \int_{\xi = \xi_1 + \ldots + \xi_5} \sigma_2(\xi_1 + \ldots + \xi_5) \hat{Iu}_1 \cdots \hat{Iu}_5 d\xi_1 \cdots d\xi_5 \|_{L^1_t L^2_x}
$$
We have used the fact that

\[ \frac{N}{\xi_1 \xi_2} \sigma_2(\xi_1, \ldots, \xi_5) \langle \hat{\partial_x} \rangle Iu_1 \langle \hat{\partial_x} \rangle Iu_2 \ldots \hat{I}u_5 \, d\xi_1 \ldots d\xi_5 \|L^1_{t \xi} L^2 \]

\[ \lesssim \frac{1}{N} \| \langle \partial_x \rangle Iu_1 \| L^2_t L^{10}_x \| \langle \partial_x \rangle Iu_2 \| L^2_t L^{10}_x \prod_{j=3}^5 \| Iu_j \| L^2_t L^{10}_x \]

where in the last line we used the Coifman-Meyer multiplier theorem, and Hölder in time.

The application of the multiplier theorem is justified by the fact that the symbol

\[ a_2(\xi_1, \ldots, \xi_5) = \frac{N}{\xi_1 \xi_2} \sigma_2(\xi_1, \ldots, \xi_5) \]

is of order zero. The \( L^\infty \) bound follows after an application of the mean value theorem. Indeed,

\[ |a_2(\xi_1, \ldots, \xi_5)| \leq \frac{N}{N_1 N_2} |\xi_1 + \ldots + \xi_5| \frac{\|\nabla_{\xi_1} m(\xi_1)(\xi_2 + \ldots + \xi_5)\|}{m(\xi_1)} \lesssim N \frac{N_1}{N_1 N_2} \frac{N_2}{N_1} \lesssim 1. \]

**Contribution of \( L^3_\xi \).** We have,

\[ \| \int_{\xi=\xi_1+\ldots+\xi_5} \frac{N}{\xi_1 \xi_2} \sigma_3(\xi_1, \ldots, \xi_5) \langle \hat{\partial_x} \rangle Iu_1 \langle \hat{\partial_x} \rangle Iu_2 \ldots \hat{I}u_5 \, d\xi_1 \ldots d\xi_5 \|L^1_t L^2_{\xi} \]

\[ = \frac{1}{N} \| \int_{\xi=\xi_1+\ldots+\xi_5} \frac{N}{\xi_1 \xi_2} \sigma_3(\xi_1, \ldots, \xi_5) \langle \hat{\partial_x} \rangle Iu_1 \langle \hat{\partial_x} \rangle Iu_2 \ldots \hat{I}u_5 \, d\xi_1 \ldots d\xi_5 \|L^1_t L^2_{\xi} \]

\[ \lesssim \frac{1}{N} \| \langle \partial_x \rangle Iu_1 \| L^2_t L^{10}_x \| \langle \partial_x \rangle Iu_2 \| L^2_t L^{10}_x \prod_{j=3}^5 \| Iu_j \| L^2_t L^{10}_x \]

where in the last line we used the Coifman-Meyer multiplier theorem, and Hölder in time.

The application of the multiplier theorem is justified by the fact that the symbol

\[ a_3(\xi_1, \ldots, \xi_5) = \frac{N}{\xi_1 \xi_2} \sigma_3(\xi_1, \ldots, \xi_5) \]

is of order zero. The \( L^\infty \) bound follows from the following chain of inequalities,

\[ |a_3(\xi_1, \ldots, \xi_5)| \lesssim \frac{N}{N_1 N_2} |\xi_1 + \ldots + \xi_5| \|m(\xi_1 + \ldots + \xi_5)\| \frac{m(\xi_1) m(\xi_2)}{m(\xi_1)} + 1 \]

\[ \lesssim \frac{N}{N_1 N_2} \left( \frac{N_1}{m(N_2)} + N_1 \right) \lesssim 1. \]

We have used the fact that \(|\xi|m(\xi)| is monotone increasing and thus

\[ |(\xi_1 + \ldots + \xi_5)m(\xi_1 + \ldots + \xi_5)| \lesssim N_1 m(\xi_1). \]

It is now evident what is the contribution of the remaining cases.

**Contribution of \( L^4_\xi \).**

\[ \| \int_{\xi=\xi_1+\ldots+\xi_5} \sigma_4(\xi_1 + \ldots + \xi_5) \langle \hat{\partial_x} \rangle Iu_1 \langle \hat{\partial_x} \rangle Iu_2 \ldots \hat{I}u_5 \, d\xi_1 \ldots d\xi_5 \|L^1_t L^2_{\xi} \]

\[ \lesssim \frac{1}{N^2} \| \langle \partial_x \rangle Iu_1 \| L^2_t L^{10}_x \| \langle \partial_x \rangle Iu_2 \| L^2_t L^{10}_x \| \langle \partial_x \rangle Iu_3 \| L^2_t L^{10}_x \prod_{j=4}^5 \| Iu_j \| L^2_t L^{10}_x \]

where in this case the symbol to which we apply the multiplier theorem is:
We proceed here to determine the bound on ∂r. Recall that hence, (4.31)

\[ |N \xi_1 \xi_2 - \xi_2 (\xi_1 + \ldots + \xi_5) [m(\xi_1 + \ldots + \xi_5) - m(\xi_1)]| \]

where in this case the symbol to which we apply the multiplier theorem is:

\[ a_5(\xi_1, \ldots, \xi_5) = \frac{N^3}{\xi_1 \xi_2 \xi_3 \xi_4} \sigma(\xi_1, \ldots, \xi_5). \]

**Contribution of L_6.**

\[ \| \int_{\xi=\xi_1+\ldots+\xi_5} \sigma_6(\xi_1 + \ldots + \xi_5) \hat{I} u_1 \ldots \hat{I} u_5 d\xi_1 \ldots d\xi_5 \|_{L^1_x L^2_t} \lesssim \frac{1}{N^3} \prod_{j=1}^{5} \| \langle \partial_x \rangle I u_j \|_{L^6_t L^3_x}, \]

where in this case the symbol to which we apply the multiplier theorem is:

\[ a_6(\xi_1, \ldots, \xi_5) = \frac{N^4}{\xi_1 \xi_2 \xi_3 \xi_4 \xi_5} \sigma_6(\xi_1, \ldots, \xi_5). \]

In all the cases above, we proved the \( L^\infty \) bound for the symbols \( a_i(\xi_1, \ldots, \xi_5), i = 2, \ldots, 6. \) We proceed here to determine the bound on \( \partial_\xi a_2. \) The remaining bounds are left to the reader. Recall that

\[ a_2(\xi_1, \ldots, \xi_5) = \frac{N}{\xi_1 \xi_2} \sigma(\xi_1, \ldots, \xi_5) \chi_2(\xi_1, \ldots, \xi_5). \]

Hence,

\[ |\partial_\xi a_2| \lesssim |\partial_\xi (\frac{N}{\xi_1 \xi_2} \sigma(\xi_1, \ldots, \xi_5) \chi_2(\xi_1, \ldots, \xi_5))| + \frac{N}{\xi_1 \xi_2} \sigma(\xi_1, \ldots, \xi_5)|. \]

The bound on the second summand, follows as the \( L^\infty \) bound on \( a_2. \) We proceed to bound the first summand, that in turn is bounded by the sum of the following two terms:

\[ (4.30) \]

\[ |N \xi_1 \xi_2 - \xi_2 (\xi_1 + \ldots + \xi_5) [m(\xi_1 + \ldots + \xi_5) - m(\xi_1)]| \]

\[ (4.31) \]

\[ |N \frac{\xi_1 + \ldots + \xi_5}{\xi_1 \xi_2} [\frac{\partial_\xi m(\xi_1 + \ldots + \xi_5)}{m(\xi_1)} - m(\xi_1 + \ldots + \xi_5) \frac{\partial_\xi m(\xi_1)}{m^2(\xi_1)}]|. \]

Again, an application of the mean value theorem gives that

\[ (4.30) \lesssim \frac{N}{N_1 N_2} \frac{N_2}{N_1} \lesssim 1. \]
As for (4.31), it is easy to see that
\[ |(\xi_1 + \ldots + \xi_5) \frac{\partial_{\xi_1} m(\xi_1 + \ldots + \xi_5)}{m(\xi_1)}| \lesssim 1 \]
while, using also the monotonicity of $|\xi| m(\xi)$ in the form
\[ |(\xi_1 + \ldots + \xi_5)m(\xi_1 + \ldots + \xi_5)| \lesssim N_1 m(\xi_1) \]
one gets
\[ |(\xi_1 + \ldots + \xi_5)m(\xi_1 + \ldots + \xi_5) \frac{\partial_{\xi_1} m(\xi_1)}{m^2(\xi_1)}| \lesssim 1. \]
Thus, (4.31) \(\lesssim 1\),
and we obtain the desired bound on $\partial_{\xi_1}a_2$.

Finally, since the pair $(5, 10)$ is admissible, we obtain that in all of the cases above
\[
\left\| \int_{\xi_1 = \xi_1 + \ldots + \xi_5} \sigma_i(\xi_1 + \ldots + \xi_5) \widehat{Iu_1} \ldots \widehat{Iu_5} \xi_1 \ldots d\xi_5 \right\|_{L^1_t L^2_x} \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^5.
\]
Therefore, we deduce from (4.29) that
\[
\left\| \partial_x (I(\mathcal{N}) - \mathcal{N}(Iu)) \right\|_{L^1_t L^2_x} \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^5.
\]
Analogously,
\[
\left\| I(\mathcal{N}) - \mathcal{N}(Iu) \right\|_{L^1_t L^2_x} \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^5.
\]
Hence, in view of (4.27) we obtain the following estimate for the error term,
\[
|E| \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^{12}.
\]
Thus, (4.25) implies
\[
\left| \int_{t_0}^{t_1} \int_{\mathbf{R}^4} \nabla a \cdot \left( N_{bad}(\nabla (Iu_1 Iu_2 Iu_3 Iu_4)) - \prod_{j=1}^{4} Iu_j \nabla N_{bad} \right) \right| \lesssim \frac{1}{N} \|u\|_{Z_1(J)}^{12},
\]
which concludes the proof.

5. PROOF OF THEOREM 1.1

Suppose that $u(x, t)$ is a global in time solution to (1.1) with initial data $u_0 \in C^\infty_0(\mathbb{R}^n)$. Set $u^\lambda(x) = \frac{1}{\lambda^2} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$. We choose the parameter $\lambda$ so that $\|Iu_0^\lambda\|_{H^1} = O(1)$, that is
\[ \lambda \sim N^{\frac{1}{12}}. \]
Next, let us pick a time $T_0$ arbitrarily large, and let us define
\[ S := \{0 < t < \lambda^2 T_0 : \|Iu^\lambda\|_{L^6_t L^2_x([0, t] \times \mathbb{R})} \leq K T_0^\frac{1}{4} N^{\frac{1}{4}} \}, \]
with $K$ a constant to be chosen later. We claim that $S$ is the whole interval $[0, \lambda^2 T_0]$. Indeed, assume by contradiction that it is not so, then since

$$\|Iu^\lambda\|_{L^6_t L^6_x([0,T] \times \mathbb{R})}$$

is a continuous function of time, there exists a time $T \in [0, \lambda^2 T_0]$ such that

(5.1) $$\|Iu^\lambda\|_{L^6_t L^6_x([0,T] \times \mathbb{R})} > KT \frac{1}{18} N^\frac{1}{3}$$

(5.2) $$\|Iu^\lambda\|_{L^6_t L^6_x([0,T] \times \mathbb{R})} \leq 2KT \frac{1}{18} N^\frac{1}{3}.$$

We now split the interval $[0,T]$ into subintervals $J_k$, $k = 1, ..., L$ in such a way that

$$\|Iu^\lambda\|_{L^6_t L^6_x(J_k \times \mathbb{R})} \leq \mu,$$

with $\mu$ as in Proposition 3.4. This is possible because of (5.2). The number $L$ of possible subintervals must satisfy

(5.3) $$L \sim \frac{(2KT \frac{1}{18} N^\frac{1}{3})^6}{\mu} \sim \frac{(2K)^6 T \frac{1}{18} N^\frac{1}{3}}{\mu}.$$

From Proposition 3.4 and Propositions 3.5 and 3.6 we know that, for any $1/3 < s < 1$

$$\sup_{[0,T]} E(Iu^\lambda(t)) \lesssim E(Iu^\lambda_0) + \frac{L}{N^2}$$

and by our choice of $\lambda$, $E(Iu^\lambda_0) \lesssim 1$. Note that if we restrict to $s > 1/3$ we can apply the previous Propositions. Hence, in order to guarantee that

$$E(Iu^\lambda) \lesssim 1$$

holds for all $t \in [0,T]$ we need to require that

$$L \lesssim N^2.$$

Since $T \leq \lambda^2 T_0$, according to (5.3), this is fulfilled as long as

(5.4) $$\frac{(2K)^6 (\lambda^2 T_0) \frac{1}{18} N^\frac{1}{3}}{\mu} \sim N^2.$$

From our choice of $\lambda$, the expression (5.4) implies that

$$T_0 \frac{1}{18} (2K)^6 \sim N^\frac{4}{3} - \frac{2(1-s)}{3s} = N^{-2(3s-1) \frac{3}{3s}}.$$

Thus if $s > 1/3$, we have that $N$ is a large number for $T_0$ large.

Now recall the a priori estimate (4.23)

$$\|Iu\|_{L^8_t L^8_x} \lesssim \sup_{0,T} \|Iu\|_{H^1} \|Iu\|_{L^2} + \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{N_{bad}, Iu_1 Iu_2 Iu_3 Iu_4\} p dx_1 dx_2 dx_3 dx_4 dt \right|.$$

Set

$$Error(t) := \int_{\mathbb{R}^4} \nabla a \cdot \{N_{bad}, Iu_1 Iu_2 Iu_3 Iu_4\} p dx_1 dx_2 dx_3 dx_4.$$
By Theorem 4.1 and Proposition 3.4 on each interval $J_k$ we have that
$$\left| \int_{J_k} \text{Error}(t)dt \right| \lesssim \frac{1}{N} \|u\|_{L_2}^2 \lesssim \frac{1}{N} \|Iu^\lambda(t_0)\|_{L_2}^2 \lesssim \frac{1}{N}.$$

Summing all the $J_k$’s we have that
$$\left| \int_0^T \text{Error}(t)dt \right| \lesssim L \frac{1}{N} \sim \frac{N^2}{N} \sim N.$$
Therefore,
$$\|Iu^\lambda\|^2_{L_2} \lesssim \sup_{t \in [0,T]} \|Iu^\lambda\|_{H^1} \|Iu^\lambda\|^2_{L_2} + \left| \int_0^T \text{Error}(t)dt \right| \lesssim 1 + N \sim N,$$
which implies
\begin{equation}
\|Iu^\lambda\|^2_{L_2} \lesssim N^{\frac{1}{2}}.
\end{equation}

On the other hand H"older inequality in time together with the definition of the $I$ operator and conservation of mass gives
\begin{equation}
\|Iu^\lambda\|^2_{L_2} \lesssim T^{\frac{1}{2}} \|Iu^\lambda\|_{L^\infty} \|Iu^\lambda\|^2_{L_2} \lesssim T^{\frac{1}{2}} \|u^\lambda\|_{L_2} = T^{\frac{1}{2}} \|u_0\|_{L_2} \sim T^{\frac{1}{2}}.
\end{equation}

Interpolation between (5.5) and (5.6) gives that
$$\|Iu^\lambda\|^2_{L_2} \lesssim C T^{\frac{1}{8}} N^{\frac{1}{2}}.$$ 

This estimate contradicts (5.1) for an appropriate choice of $K$. Hence $S = [0, \lambda^2 T_0]$, and $T_0$ can be chosen arbitrarily large. In addition, we have also proved that for $s > 1/3$
$$\|Iu^\lambda(\lambda^2 T_0)\|_{H^s} = O(1).$$

Then,
$$\|u(T_0)\|_{H^s} \lesssim \|u(T_0)\|_{L^2} + \|u_0\|_{L^2} = \|u_0\|_{L^2} + \lambda^s \|Iu^\lambda(\lambda^2 T_0)\|_{H^s} \lesssim \lambda^s \|u_0\|_{L^2} \lesssim N^{1-s} \lesssim T_0^{\frac{s}{(3s-1)}}.$$ 

Since $T_0$ is arbitrarily large, the a priori bound on the $H^s$ norm concludes the global well-posedness of the the Cauchy problem (1.1) + (1.2).

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