STABLE BLOW-UP SOLUTIONS FOR THE SO(d)-EQUIVARIANT SUPERCritical YANG-MILLS HEAT FLOW

YEZHOU YI

ABSTRACT. We consider the $SO(d)$-equivariant Yang-Mills heat flow

$$
\partial_t u - \partial^2_r u - \frac{(d-3)}{r} \partial_r u + \frac{(d-2)}{r^2} u(1-u)(2-u) = 0
$$

in dimensions $d > 10$. We construct a family of $C^\infty$ solutions which blow up in finite time via concentration of a universal profile

$$
u(t, r) \sim Q\left(\frac{r}{\lambda(t)}\right),
$$

where $Q$ is a stationary state of the equation and the blow-up rates are quantized by

$$
\lambda(t) \sim c_u(T-t)^{\gamma}, \quad \gamma = \gamma(d) = \frac{d-4-\sqrt{(d-6)^2-12}}{2},
$$

Moreover, such solutions are in fact $(l-1)$-codimension stable under perturbation of the initial data.

KEYWORDS: Yang-Mills heat flow; Asymptotic behavior; $SO(d)$-equivariant; Stability.

MSC: 35B40, 35K15, 35K55

1. Introduction

We denote $E$ a principle fibre bundle over a $d$ dimensional Riemannian manifold $M$, with a semi-simple Lie group as structure group. Denoting $AdE$ the adjoint bundle to $E$, a smooth connection on $E$ is a smooth map from $M$ to $AdE \otimes T^* M$. Denoting $G$ as the Lie algebra of $G$, locally a connection $A$ is a $G$-valued 1-form on the coordinate patches $U_\alpha$ of $M$, as

$$
A_j : U_\alpha \rightarrow G
$$

with $A_j : U_\alpha \rightarrow G$. Denoting $D_A$ the covariant derivative with respect to $A$, the curvature $F_A$ of a connection $A$ is defined by $F_A = D_A \circ D_A$, locally it is $G$-valued 2-form $F_{j,k} dx^j dx^k$, where

$$
F_{j,k} := \partial_j A_k - \partial_k A_j + [A_j, A_k].
$$

The Yang-Mills functional $F$ is defined by

$$
F(A) = \int_M F_{j,k} F^{j,k} dvol_M,
$$

which is invariant under gauge transformations. The associated Euler-Lagrange equations read

$$
D^j F_{j,k} = 0, \quad (1.1)
$$

where $D_j =: \partial_j + [A_j, \cdot]$. Solutions of (1.1) are referred to as Yang-Mills connections. One way to find Yang-Mills connections is to study the $L^2$-gradient flow associated with $F$, i.e. the initial value problem

$$
\begin{cases}
\partial_t A_j(t, x) = -D^k F_{j,k}(t, x) \\
A_j(0, x) = A_{0j}(x)
\end{cases} \quad (1.2)
$$

for some initial connection $A_0$. Usually (1.2) is referred to as the Yang-Mills heat flow.

In this paper, we consider (1.2) under the situation $M = \mathbb{R}^d$, $G = SO(d)$, $E$ is the trivial bundle $\mathbb{R}^d \times SO(d)$, and we investigate connections given by

$$
A_j(x) = \frac{u(r)}{r^2} \sigma_j(x), \quad (1.3)
$$
where \( r = |x| \), \( u \) is a real-valued function on \([0, \infty)\), and \( \{\sigma_j\}_{j=1}^d \) are a basis for the Lie algebra \( so(d) \), given by

\[
(\sigma_j)_{\alpha}^\beta = \delta^\beta_\alpha x^\beta - \delta^\alpha_\beta x^\alpha, \quad \text{for } 1 \leq \alpha, \beta \leq d.
\]

Note that connections satisfying (1.3) are equivariant with respect to \( SO(d) \)-action, and are referred to as \( SO(d) \)-equivariant connections. In this case, (1.2) becomes

\[
\begin{align*}
\partial_t u - \partial_r^2 u - \frac{(d-3)}{r} \partial_r u + \frac{(d-2)}{r^2} u(1-u)(2-u) &= 0, \\
u(0, \cdot) &= u_0(\cdot)
\end{align*}
\]

We credit Dumitrascu [8] for the first derivation of equivariant supercritical Yang-Mills equation, the readers can also refer to Weinkove [28] for more details.

Let us briefly explain the meaning of energy supercritical. For any \( \lambda > 0 \), if \( u(t, r) \) is a solution of (1.4), then \( u(\frac{t}{\lambda}, \frac{r}{\lambda}) \) is also a solution. If we denote the energy functional of (1.4) as

\[
E(u(t)) := \frac{1}{2} \int_0^{+\infty} \left( |\partial_r u|^2 + \frac{(d-2)u^2(2-u)^2}{2r^2} \right) r^{d-3} dr,
\]

then we have for any radial function \( u_0 : \mathbb{R}^d \to \mathbb{R} \),

\[
E\left(u_0\left(\frac{r}{\lambda}\right)\right) = \lambda^{d-4} E(u_0(r)).
\]

Therefore, \( d < 4 \) corresponds to the energy subcritical cases, \( d = 4 \) corresponds to energy-critical case, and \( d \geq 5 \) corresponds to the energy supercritical cases.

Historically, there has been a lot of work devoted to the study of Yang-Mills heat flow. In the case \( d = 2 \) or 3, Råde [24] proved the flow of (1.2) exists for all time and converges to a Yang-Mills connection. In the case \( d = 4 \), global existence of solutions of (1.4) was established by Schlatter, Struwe and Tahvildar-Zadeh [26], and Waldron [27] for more general geometric situations of (1.2). In the case \( d \geq 5 \), solutions of (1.4) may blow up in finite time, see the works of Naito [23], Grotowski [15] and Gastel [9]. However, they did not give any structure of blow-up solutions while this paper manages to describe it. Weinkove [28] investigated the nature of singularities of Yang-Mills heat flow over a compact manifold and showed that under some assumptions of the blow-up rate, homothetically shrinking solitons appear as blow-up limits at singular points. Such objects correspond to self-similar solutions of Yang-Mills heat flow on the trivial bundle over \( \mathbb{R}^d \), which were also described explicitly in Section 4 in [28] for \( 5 \leq d \leq 9 \). On Weinkove’s self-similar blow-up solutions of (1.4), Donninger and Schörkhuber [7], Glogić and Schörkhuber [14] proved that these blow-up are stable when \( 5 \leq d \leq 9 \). For higher dimension cases \( d \geq 10 \), it is proved in Bizúi and Wasserman [2] Theorem 2 that for \( d \geq 10 \), there exists no self-similar blow-up solution of (1.4). It is then expected that type II blow-up solutions exist.

In this paper, we will construct blow up solutions of (1.4) for \( d > 10 \). In fact, high dimensional blow-up phenomenon has been widely studied for various types of partial differential equations. For the semilinear heat equation

\[
\partial_t u = \Delta u + |u|^{p-1}u
\]

with \( d \geq 11 \) and \( p > 1 + \frac{4}{d-4-2\sqrt{d-1}} \), Herrero and Velázquez [16] formally showed the existence of type II blow up with

\[
\|u(t)\|_{L^\infty} \sim \frac{1}{(T-t)^{\frac{2\alpha}{p-1}}}, \quad \text{as } l \in \mathbb{N}_+, \quad 2\alpha \ell > 1.
\]

The formal result was rigorously clarified by the works of Mizoguchi [22], Matano and Merle [20], Collot [4]. For energy supercritical nonlinear Schrödinger equation in dimensions \( d \geq 11 \), Merle, Raphaël and Rodnianski [21] constructed smooth blow-up solutions via a robust “modulation” method. For the energy critical focusing nonlinear Schrödinger equation in space dimensions \( d \geq 7 \), Jendrej [17] proved the existence of pure two bubbles, one of the bubbles develops at scale 1, whereas the length scale of the other converges to 0. Jendrej used energy-virial functional...
along with a well-designed approximation to operators which eliminates the unbounded part, such method has been adapted successfully in other dispersive equations, for example in energy critical wave equation when $d = 6$, see Jendrej [18]. For energy supercritical wave equation in $d \geq 11$, Collot [5] constructed blow-up with one bubble in the radial case, which also studied corresponding blow-up manifold and introduced an interesting Morawetz term to control local energy. For high dimensional harmonic heat flow, Ghoul, Ibrahim and Nguyen [12] showed one bubble blow-up exists for 1-corotational supercritical harmonic heat flow in $d \geq 7$. Also when $d = 7$, Ghoul [11] proved there exists the same blow-up structure with the blow-up rate as

$$\lambda(t) \simeq \frac{\sqrt{T - t}}{\log(T - t)}. \quad (1.5)$$

For energy supercritical wave maps with $d \geq 7$, one may refer to Ghoul, Ibrahim and Nguyen [10]. However, whether similar blow-up phenomenon happens remains an open problem for Yang-Mills heat flow.

Next we introduce the main result of this paper. Denoting $Q(r)$ as the ground state solution of $(1.4)$, i.e. it satisfies the equation

$$\begin{cases}
- \partial_r^2 Q - \frac{(d - 3)}{r} \partial_r Q + (d - 2) \frac{Q(1 - Q)(2 - Q)}{r^2} = 0, \\
Q(0) = \partial_r Q(0) = 0.
\end{cases} \quad (1.6)$$

In the author’s previous work [29], it shows that $(1.6)$ admits a solution which satisfies the asymptotics (when $d > 10$)

$$Q(r) = \begin{cases}
\frac{1}{2} r^2 + O(r^4) \text{ as } r \to 0 \\
1 - \alpha r^{-\gamma}(1 + O(r^{-2\gamma})) \text{ as } r \to \infty,
\end{cases} \quad (1.7)$$

where

$$\alpha > 0, \quad \gamma = \gamma(d) = \frac{d - 4 - \sqrt{(d - 6)^2 - 12}}{2}. \quad (1.8)$$

Note that when $d > 10$, $\gamma \in (1, 2)$.

Our goal is to study potential blow-up phenomenon of $(1.4)$, and our main result is the following.

**Theorem 1.1.** Let $d > 10$, $\gamma$ as in $(1.8)$, let $l$ be any positive integer, denoting

$$h := \left[ \frac{1}{2} \left( \frac{d - 2}{2} - \gamma \right) \right], \quad (1.9)$$

given $L \gg l$ a large integer and defining $k := L + h + 1$. Then there exists a smooth radial initial data $u_0$ such that the corresponding solution to $(1.4)$ has the decomposition

$$u(t, r) = Q \left( \frac{r}{\lambda(t)} \right) + q \left( t, \frac{r}{\lambda(t)} \right), \quad (1.10)$$

where

$$\lambda(t) = c(u_0)(T - t)^{\frac{1}{2}}(1 + o_{t \to T}(1)) \text{ with } c(u_0) > 0, \quad (1.11)$$

and

$$\lim_{t \to T} \|\nabla^\sigma q(t)\|_{L^2(r^{-2-\lambda d})} = 0 \text{ for all } \sigma \in [2h + 4, 2k]. \quad (1.12)$$

Moreover, the blow-up solution is $(l - 1)$-codimension stable.

**Remark 1.1.** Let us briefly explain the sense of $(l - 1)$-codimension stable. Our initial data is of the form

$$u_0 = Q_h(b(0)) + q_0, \quad (1.13)$$

where $Q_h$ is a deformation of $Q$ and $b = (b_1, \cdots, b_L)$ corresponds to possible unstable directions in a suitable neighborhood of $Q$. We will prove that for all $q_0 \in H^\sigma \cap H^{2k}$ small
enough, for all \((b_1(0), b_{i+1}(0), \ldots, b_L(0))\) small enough, there exists a choice of unstable directions \((b_2(0), \ldots, b_L(0))\) such that the solution of (1.4) with initial data (1.13) satisfies the conclusion of Theorem 1.1. This implies the constructed solution is \((l - 1)\)-codimension stable.

**Remark 1.2.** Again in view of Theorem 2 in Biz̆oń and Wasserman [2], one is supposed to find type II blow-up for all \(d \geq 10\). However, we consider only \(d > 10\) and exclude the limit case 

\[d = 10\] due to technical reasons, in \(d > 10\) cases we have \(1 < \gamma < 2, 0 < \delta := \frac{1}{2} \left(\frac{d - 2}{2} - \gamma\right) - \frac{1}{2} < 1\), and \(d - 2\gamma > 6\) which ensure involved estimates good enough. We conjecture that it is possible to construct type II blow-up solutions for the case \(d = 10\), with explicitly different blow-up rates than those in (1.11).

It appears that Yang-Mills heat flow share similar properties to harmonic heat flow in high dimensions, thus this paper has borrowed techniques from Ghoul et al. [12]. Also the preprint of Bensouilah, Duong and Ghoul [1] which appears after the preprint of this paper, has established similar blow-up results compared to Theorem 1.1 by different methods. Note that the main idea of this paper, in the author’s points of view, originates from Merle, Raphaël and Rodnianski [21] which seems astonishing since they studied blow-up for supercritical nonlinear Schrödinger equation. However, there are technical difficulties in order to get appropriate energy estimates which are supposed to decay well enough to close bootstrap after integration, especially when dealing with nonlinear term for energy estimates, those may be considered as the original part of this paper.

A closely related topic is the blow-up behavior for hyperbolic version of (1.4) as

\[
\partial_t^2 u - \partial_r^2 u - \frac{(d - 3)}{r} \partial_r u + \frac{(d - 2)}{r^2} u(1 - u)(2 - u) = 0,\tag{1.14}
\]

where we omit more general geometric backgound. Historically, the existence of blow-up for (1.1) was first proved by Cazenave, Shatah and Tahvildar-Zadeh in [3], they constructed singular traveling waves by using self-similar solutions. The self-similar blow-up for (1.14) is proved to be stable for all odd dimensions \(d \geq 5\), see Donninger [6] and Glogić [13]. Raphaël and Rodnianski [25] constructed stable one bubble blow-up when \(d = 4\), i.e. the energy-critical case. Also in dimension four, Jendrej [18] constructed two bubbles, and Krieger, Schlag, Tataru [19] showed the existence of a family of one bubble where the blow-up rates are a modification of the self-similar rate by a power of logarithm. We conjecture that one may establish similar results to Theorem 1.1 for (1.14) with exactly the same blow-up rates \(\lambda\).

This paper is organized as follows. In section 2, we show fundamental calculations about the linearized operator of (1.4) and establish coercivity properties which are crucial for both modulation estimates and energy estimates later. In section 3, we construct an approximate solution and estimate error terms. In section 4, one perturbs the modulation parameter equation near a set of explicit solutions (The author doubts that whether one can find explicitly non-polynomial exact solutions or first approximate solutions to further perturb with and succeed in closing the whole bootstrap argument). In later sections, it is a somewhat standard procedure in modulation methods, we decompose the solution, describe initial data, provide bootstrap assumptions, then under such assumptions, we estimate modulation parameters by taking \(L^2\) inner product of the “orthogonality” elements with the equation of the residue which also arises in previous decomposition, then derive the most crucial monotone estimates called energy estimates so that integrating it back helps us close bootstrap (since almost all bootstrap assumptions can be estimated through proper energy) and finally a contradictory argument relying on basic topological facts concludes the main Proposition, that is, Proposition 9.3.

**Notations.** Now we introduce some notations. Denoting \(\Lambda Q(y) := y \partial_y Q(y)\). By (1.7), we have

\[
\Lambda Q(y) = \begin{cases} 
  y^2 + O(y^4) & \text{as } y \to 0 \\
  \frac{\alpha \gamma}{y^\gamma} \left(1 + O\left(\frac{1}{y^{2\gamma}}\right)\right) & \text{as } y \to \infty
\end{cases}
\tag{1.15}
\]
Denoting the linearized operator \( \mathcal{L} := -\frac{\partial^2_y}{y} + \frac{(d-3)}{y^2} \partial_y + \frac{Z(y)}{y^4} \), where \( Z(y) := (d-2)f'(Q(y)) \) and \( f(u) := u(1-u)(2-u) \). Note that by substituting \( Q_A(r) := Q(\frac{r}{A}) \) into (1.6) in the variable \( y = \frac{r}{A} \) and acting on \( \partial_y |_{y=1} \), one gets \( \mathcal{L}(A\mathcal{Q}) = 0 \). Furthermore, denoting \( T_k := (-1)^k(\mathcal{L}^{-1})^k(A\mathcal{Q}) \), for \( 0 \leq k \leq L \).

For any two radial functions \( f_1 \) and \( f_2 \), denoting their inner product as

\[
\langle f_1, f_2 \rangle := \int_0^\infty f_1(y)f_2(y) y^{d-3} \, dy.
\]

For convenience, let \( f := \int_0^\infty f_1(y)y^{d-3} \, dy \).

Defining \( \chi \) as a smooth radial cut-off function such that \( \chi(y) = 1 \) for \( 0 \leq y \leq 1 \), \( \chi(y) = 0 \) for \( y \geq 2 \) and \( 0 < \chi(y) < 1 \) for \( 1 < y < 2 \). Then we denote \( \chi_M(y) := \chi(\frac{y}{M}) \).

For any smooth radial function \( g \), denoting \( g_{2k} := \mathcal{L}^k g \), \( g_{2k+1} := \mathcal{A} \mathcal{L}^k g \), for any \( k \in \mathbb{N} \). Denoting \( \mathcal{L}_\lambda := -\partial^2_y - \frac{(d-3)}{y^2} \partial_y + \frac{Z(y)}{y^4} \), etc, then we write \( g_{2k} := \mathcal{L}_\lambda^k g \), \( g_{2k+1} := \mathcal{A}_\lambda \mathcal{L}_\lambda^k g \).

For any \( b_1 > 0 \), we define \( B_0 := b_1^{-\frac{1}{2}} \), \( B_1 := B_0^{1+\eta} \), where \( 0 < \eta \ll 1 \) is to be choosen later.

We emphasize again that for \( \gamma \) as (1.8) and \( \delta \) as (1.9), we define \( \delta := \frac{1}{2}\left(\frac{d-2}{2} - \gamma \right) \), throughout this paper.

The notation \( A \lesssim B \) means that there exists a positive constant \( C \) such that \( A \leq CB \). And we denote \( A \simeq B \) if \( A \lesssim B \) and \( B \lesssim A \) are both valid.

### 2. ON THE LINEARIZED OPERATOR \( \mathcal{L} \)

In this section, we make preparations related to \( \mathcal{L} \). We shall omit some detailed proofs if they are obtained by direct computations.

#### 2.1. Decomposition, kernel and computation of the inverse of \( \mathcal{L} \).

**Lemma 2.1.** The operator \( \mathcal{L} \) factorizes as \( \mathcal{L} = \mathcal{A}^* \mathcal{A} \) with

\[
\mathcal{A} \omega := \left( -\partial_y + \frac{V(y)}{y} \right) \omega = -\Lambda Q \partial_y \left( \frac{\omega}{\Lambda Q} \right), \tag{2.1}
\]

\[
\mathcal{A}^* \omega := \left( \partial_y + \frac{d-3 + V(y)}{y} \right) \omega = \frac{1}{y^{(d-3)}\Lambda Q} \partial_y (y^{(d-3)}\Lambda Q \omega), \tag{2.2}
\]

where

\[
V(y) := \Lambda \ln(\Lambda Q) = \begin{cases} 2 + O(y^2) & \text{as } y \to 0 \\ -\gamma + O\left(\frac{1}{y^{2\gamma}}\right) & \text{as } y \to \infty \end{cases}. \tag{2.3}
\]

**Remark 2.1.** Note that

\[
[\mathcal{L}, \Lambda] = 2\mathcal{L} - \frac{\Lambda Z(y)}{y^2}. \tag{2.4}
\]

Denoting \( \widetilde{\mathcal{L}} := \mathcal{A}^* \mathcal{A} \). By Lemma 2.1,

\[
\widetilde{\mathcal{L}} = -\partial^2_y - \frac{(d-3)}{y^2} \partial_y + \frac{\tilde{Z}(y)}{y^2}, \tag{2.5}
\]

where

\[
\tilde{Z}(y) := (V + 1)^2 + (d-4)(V + 1) - \Lambda V. \tag{2.6}
\]

Next we find the other kernel of \( \mathcal{L} \). If \( \mathcal{L} \Gamma = 0 \), then \( \mathcal{A} \Gamma \) lies in the kernel of \( \mathcal{A}^* \). By (2.2), \( \mathcal{A} \Gamma \in \text{Span}\left\{\frac{1}{y^{d-3}\Lambda Q}\right\} \). By definition (2.1), we can impose that

\[
-\partial_y \Gamma + \frac{V(y)}{y} \Gamma = \frac{-1}{y^{d-3}\Lambda Q}. \]
It has a solution of the form
\[ \Gamma(y) = \Lambda Q(y) \int_1^y \frac{d\xi}{\xi^{d-3}(\Lambda Q(\xi))^2}, \]
and the asymptotics
\[ \Gamma(y) \simeq \begin{cases} \frac{c}{y^{d-2}} & \text{as } y \to 0 \\ \frac{c}{y^{d-4-\gamma}} & \text{as } y \to \infty \end{cases}. \tag{2.7} \]

Then we introduce calculations on the inverse of \( L \). By standard ODE theory, for any radial function \( g \), there exists a solution to \( L \omega = g \) written as
\[ \omega = -\Lambda Q \int_0^y g(x) \Lambda Q(x) x^{d-3} dx + \Lambda Q(y) \int_0^y g(x) \Gamma(x) x^{d-3} dx. \tag{2.8} \]
For the convenience of calculation, there is a two-step method to compute the above \( L^{-1} \) as follows.

**Lemma 2.2.** Let \( g \in C^\infty_{rad} \), then \( L \omega = g \) exists a solution solved by
\[ A \omega = \frac{1}{y^{d-3} \Lambda Q} \int_0^y g(x) \Lambda Q(x) x^{d-3} dx, \]
\[ \omega = -\Lambda Q \int_0^y \frac{A \omega(x)}{\Lambda Q(x)} dx. \]
**Proof.** Acting \( A \) on (2.8) and making use of (2.1).

### 2.2. Coercivity of \( L \)

The proof of the following Hardy inequality is similar to Lemma B.1 in [21].

**Lemma 2.3.** Let \( \alpha > 0 \), \( \alpha \neq \frac{d-4}{2} \) and \( u \in D_{rad} := \{ u \in C^\infty_{c} \text{ with radial symmetry} \} \), then
\[ \int_1^\infty \frac{|\partial_y u|^2}{y^{2\alpha}} \geq \left( \frac{d-2(2\alpha+4)}{2} \right)^2 \int_1^\infty \frac{u^2}{y^{2\alpha+2d}} - C_{\alpha,d} u^2(1). \]
Then one can follow the road map in Appendix A in [12]. Firstly, we establish coercivity of \( A^* \) as follows.

**Lemma 2.4.** Let \( \alpha \geq 0 \). There exists \( C_\alpha > 0 \) such that for all \( u \in D_{rad} \), \( i = 0, 1, 2 \),
\[ \int \frac{|A^* u|^2}{y^{2i}(1+y^{2\alpha})} \geq C_\alpha \left( \int \frac{|\partial_y u|^2}{y^{2i}(1+y^{2\alpha})} + \int \frac{u^2}{y^{2i+2}(1+y^{2\alpha})} \right). \]

Denoting
\[ \Phi_M := \sum_{k=0}^L c_{k,M} \mathcal{L}^k(\chi_M \Lambda Q), \tag{2.9} \]
where
\[ c_{0,M} := 1, \quad c_{k,M} := (-1)^{k+1} \frac{\sum_{j=0}^{k-1} c_{j,M} \langle \mathcal{L}^j(\chi_M \Lambda Q), T_k \rangle}{\langle \chi_M \Lambda Q, \Lambda Q \rangle}, \quad 1 \leq k \leq L \tag{2.10} \]
Note that the choices of \( c_{k,M} \) are equivalent to
\[ \begin{align*}
\langle \Phi_M, \Lambda Q \rangle &= \langle \chi_M \Lambda Q, \Lambda Q \rangle \\
\langle \Phi_M, T_k \rangle &= 0 \quad \text{for } 1 \leq k \leq L.
\end{align*} \tag{2.11} \]
In particular,
\[ \langle \mathcal{L}^i T_k, \Phi_M \rangle = (-1)^k \langle \chi_M \Lambda Q, \Lambda Q \rangle \delta_{i,k}, \quad \text{for } 0 \leq i, k \leq L, \tag{2.12} \]
where
\[\delta_{i,k} := \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.\]

Then we establish coercivity of \(\mathcal{A}\) as follows.

**Lemma 2.5.** Let \(p \geq 0, i = 0, 1, 2\) and \(2i + 2p - (d - 2\gamma - 4) \neq 0\). Assuming in addition \(\langle u, \Phi_M \rangle = 0\), if \(2i + 2p > d - 2\gamma - 4\), then we have
\[
\int \frac{|\mathcal{A}u|^2}{y^{2i}(1 + y^{2p})} \geq \int \frac{|\partial_y u|^2}{y^{2i}(1 + y^{2p})} + \int \frac{u^2}{y^{2i+2}(1 + y^{2p})}.
\]

Next we are in place to establish coercivity of \(L\) as follows.

**Lemma 2.6.** Let \(k \in \mathbb{N}, i = 0, 1, 2\) and \(M = M(k)\) large enough. Then there exists \(c_{M,k} > 0\) such that for all \(u \in D_{\text{rad}}\) with \(\langle u, \Phi_M \rangle = 0\), if \(2i + 2k > d - 2\gamma - 6\), we have
\[
\int \frac{|L u|^2}{y^{2i}(1 + y^{2k})} \geq c_{M,k} \int \left( \frac{|\partial^2_y u|^2}{y^{2i}(1 + y^{2k})} + \frac{|\partial_y u|^2}{y^{2i+2}(1 + y^{2k+2})} + \frac{u^2}{y^{2i+2}(1 + y^{2k+2})} \right),
\]

and
\[
\int \frac{|L u|^2}{y^{2i}(1 + y^{2k})} \geq c_{M,k} \int \left( \frac{|\mathcal{A}u|^2}{y^{2i+2}(1 + y^{2k})} + \frac{u^2}{y^{2i+2}(1 + y^{2k+4})} \right).
\]

Finally we give the coercivity property of iterate of \(L\) as follows.

**Lemma 2.7.** Let \(k \in \mathbb{N}, M = M(k)\) large enough. Then there exists \(c_{M,k} > 0\) such that for any \(u \in D_{\text{rad}}\) with \(\langle u, L^m \Phi_M \rangle = 0\), if \(0 \leq m \leq k - h\), we have
\[
\mathcal{C}_{2k+2}(u) := \int |L^{k+1} u|^2 
\geq c_{M,k} \left( \sum_{j=0}^k \int \frac{|L^j u|^2}{y^{4(1 + y^{4(j-k)})}} + \int \frac{|L^j (\mathcal{A}^k u)|^2}{y^2} + \sum_{j=0}^{k-1} \int \frac{|\mathcal{A}(L^j u)|^2}{y^{6(1 + y^{4(k-j-1)})}} \right).
\]

**Remark 2.2.** We point out that in Lemma 2.7, when verifying the case for \(k = 0\), there is no need for orthogonal conditions, one just apply Lemma 2.4 and Lemma 2.5 to get
\[
\int |L u|^2 \geq \int |\mathcal{A} u|^2 \geq \int \frac{u^2}{y^4} + \int \frac{|\mathcal{A} u|^2}{y^2}.
\]

Also note that when \(d > 10, d - 2\gamma - 4 > 2\) holds, thus the assumption in Lemma 2.5 meets.

2.3. Leibniz rule for the iteration of \(L\). We introduce Leibniz rules for \(L^k\) and \(\mathcal{A} L^k\) as follows. One can prove it by induction on \(k\), a similar detailed proof is given in Lemma C.1 in [12].

**Lemma 2.8.** For any smooth radial function \(\phi, g\) and any \(k \in \mathbb{N}\), we have
\[
L^{k+1}(\phi g) = \sum_{m=0}^{k+1} g_{2m} \phi_{2k+2,2m} + \sum_{m=0}^{k} g_{2m+1} \phi_{2k+2,2m+1}; \tag{2.13}
\]
\[
\mathcal{A} L^k(\phi g) = \sum_{m=0}^{k} g_{2m+1} \phi_{2k+1,2m+1} + \sum_{m=0}^{k} g_{2m} \phi_{2k+1,2m}; \tag{2.14}
\]
where for \(k = 0\),
\[
\phi_{1,0} := -\partial_y \phi, \quad \phi_{1,1} := \phi, \\
\phi_{2,0} := -\partial_y^2 \phi - \frac{(d - 3 + 2V)}{y} \partial_y \phi, \quad \phi_{2,1} := 2 \partial_y \phi, \quad \phi_{2,2} := \phi,
\]
for $k \geq 1$,
\[
\begin{align*}
\phi_{2k+1,0} &:= -\partial_y \phi_{2k,0}, \\
\phi_{2k+1,2i} &:= -\partial_y \phi_{2k,2i} - \phi_{2k,2i-1}, \quad 1 \leq i \leq k, \\
\phi_{2k+1,2i+1} &:= \phi_{2k,2i} + \frac{(d - 3 + 2V)}{y} \phi_{2k,2i+1} - \partial_y \phi_{2k,2i+1}, \quad 0 \leq i \leq k - 1, \\
\phi_{2k+1,2k+1} &:= \phi_{2k,2k} = \phi, \\
\phi_{2k+2,0} &:= \partial_y \phi_{2k+1,0} + \frac{(d - 3 + 2V)}{y} \phi_{2k+1,0}, \\
\phi_{2k+2,2i} &:= \phi_{2k+1,2i-1} + \partial_y \phi_{2k+1,2i} + \frac{(d - 3 + 2V)}{y} \phi_{2k+1,2i}, \quad 1 \leq i \leq k, \\
\phi_{2k+2,2i+1} &:= \partial_y \phi_{2k+1,2i+1} - \phi_{2k+1,2i}, \quad 0 \leq i \leq k, \\
\phi_{2k+2,2k+2} &:= \phi_{2k+1,2k+1} = \phi.
\end{align*}
\]

3. Construction of an approximate solution

Denoting the self-similar change of variables as
\[
\omega(s, y) := u(t, r), \quad y := \frac{r}{\lambda(t)}, \quad s := s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)}.
\] (3.1)
We shall derive more information on the parameter $\lambda(t)$ later. Substituting (3.1) into (1.4), we get the renormalized flow
\[
\partial_s \omega - \partial_y^2 \omega - \frac{(d - 3)}{y} \partial_y \omega - \frac{\lambda}{\lambda} \Lambda \omega + \frac{(d - 2)}{y^2} \omega(1 - \omega)(2 - \omega) = 0.
\] (3.2)
In this section, we construct approximate solutions with respect to (3.2).

3.1. Definition and properties of degree and homogeneous admissible functions.

Firstly, we shall sum up function properties which we will encounter often and express them in a systematic and unified way.

**Definition 3.1.** Admissible function: we say $g \in C^\infty_{rad}$ is admissible of degree $(p_1, p_2) \in \mathbb{N} \times \mathbb{Z}$ if

(i) For $y$ close to 0, $g(y) = \sum_{k=p_1}^p c_k y^{2k+2} + O(y^{2\rho+4}).$

(ii) For $y \geq 1$, for all $k \in \mathbb{N}$, $|\partial_y^k g(y)| \lesssim y^{2p_2 - \gamma - k}.$

We abbreviate it as $g \sim (p_1, p_2).$

Under certain operations, the degree has the following properties. One just apply Lemma 2.2 and use induction method, we shall omit the details.

**Lemma 3.1.** Let $g$ be an admissible function of degree $(p_1, p_2) \in \mathbb{N} \times \mathbb{Z}$, then

(i) $\Lambda g \sim (p_1, p_2).$

(ii) $\mathcal{L} g \sim (p_1 - 1, p_2 - 1)$, for $p_1 \geq 1.$

(iii) $\mathcal{L}^{-1} g \sim (p_1 + 1, p_2 + 1)$.

(iv) $T_k \sim (k, k)$, for all $k \in \mathbb{N}$.

(v) $\Lambda T_k - (2k - \gamma) T_k \sim (k, k - 1)$, for all $k \in \mathbb{N}_+.$

**Definition 3.2.** Homogeneous admissible function: Let $L \gg 1$ be an integer and $m := (m_1, \ldots, m_L) \in \mathbb{N}^L, \quad b := (b_1, \ldots, b_L).$ We say that a radial function $g(b,y)$ is homogeneous of degree $(p_1, p_2, p_3) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{N}$, if it is a finite linear combination of monomials $\bar{g}(y) \prod_{k=1}^L b_k^{m_k}$ with $\bar{g}(y) \sim (p_1, p_2)$ and $\sum_{k=1}^L km_k = p_3.$ We abbreviate it as $g \sim (p_1, p_2, p_3).$
3.2. Estimate of the approximate profile and error terms.

**Proposition 3.1.** Let $d > 10$ and $L \gg 1$ be an integer. Then there exists a small enough universal constant $b^* > 0$ such that the following holds true. Let $b = (b_1, \ldots, b_L) : [s_0, s_1] \rightarrow (-b^*, b^*)^L$ be a $C^1$ map with a priori bound on $[s_0, s_1]$: \[0 < b_1 < b^*, \quad |b_k| \lesssim b_1^k, \text{ for } 2 \leq k \leq L.\] (3.3)

Then there exist profiles $S_1 = 0, S_k = S_k(b, y), 2 \leq k \leq L + 2$ such that

\[Q_{b(a)}(y) := Q(y) + \sum_{k=1}^{L} b_k(s)T_k(y) + \sum_{k=2}^{L+2} S_k(b, y) =: Q(y) + \Theta_{b(a)}(y)\] (3.4)

as an approximation to (3.2) satisfies

\[\partial_s Q_b - \partial_y^2 Q_b - \frac{(d-3)}{y} \partial_y Q_b + b_1 \Lambda Q_b + \frac{(d-2)}{y^2} f(Q_b) = \text{Mod}(t) + \Psi_b\] (3.5)

with the following properties.

(i) $\text{Mod}(t) = \sum_{k=1}^{L} [(b_k)_s + (2k - \gamma)b_1b_k - b_{k+1}][T_k + \sum_{j=k+1}^{L+2} \partial S_j \big|_{\partial b_k}].$

(ii) The tails $S_k$ satisfy

\[S_k \sim (k, k - 1, k), \text{ for } 2 \leq k \leq L + 2.\] (3.6)

\[\partial S_k \big|_{\partial b_m} = 0, \text{ for } 2 \leq k \leq m \leq L + 2.\] (3.7)

(iii) For all $0 \leq m \leq L,$

\[\int_{y \leq 2B_1} |\mathcal{L}^{h+m+1}\Psi_b|^2 + \int_{y \leq 2B_1} \frac{|\Psi_b|^2}{1 + y^{4(h+m+1)}} \lesssim b_1^{2m+2} C_1 \eta.\] (3.8)

For all $M \geq 1,$

\[\int_{y \leq 2M} |\mathcal{L}^{h+m+1}\Psi_b|^2 \lesssim M^2 b_1^{2L+6}.\] (3.9)

**Proof.** Defining the approximate solution to (3.2) as (3.4). In addition, assuming (3.3), $S_1 = 0$ and (3.7). Then we shall construct $S_k$ and verify (3.7) holds indeed. Applying (1.6) we get

\[
\partial_s Q_b - \partial_y^2 Q_b - \frac{(d-3)}{y} \partial_y Q_b + b_1 \Lambda Q_b + \frac{(d-2)}{y^2} f(Q_b) = \partial_s \Theta_b + \mathcal{L} \Theta_b + b_1 \Lambda Q + b_1 \Lambda \Theta_b \\
+ \frac{(d-2)}{y^2} [f(Q + \Theta_b) - f(Q) - f'(Q) \Theta_b] =: A_1 + A_2.
\]

Direct computation gives

\[
A_1 = \sum_{k=1}^{L} [(b_k)_s + (2k - \gamma)b_1b_k - b_{k+1}]T_k \\
+ \sum_{k=1}^{L} [b_1b_k(\Lambda T_k - (2k - \gamma)T_k) + b_1 \Lambda S_k] + b_1 \Lambda S_{L+1} + b_1 \Lambda S_{L+2} \\
+ \sum_{k=1}^{L+2} \mathcal{L} S_{k+1} + \sum_{k=2}^{L+2} \partial S_k.
\]
Note that
\[ \partial_s S_k = \sum_{j=1}^{L} [(b_j)_s + (2j - \gamma)b_1b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j} - \sum_{j=1}^{L} [(b_j)_s + (2j - \gamma)b_1b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}, \]

hence
\[ A_1 = \text{Mod}(t) + \sum_{k=1}^{L+1} [\mathcal{L}S_{k+1} + E_k] + E_{L+2}, \]

where for \( k = 1, \ldots, L, \)
\[ E_k := b_1b_k[\Lambda T_k - (2k - \gamma)T_k] + b_1\Lambda S_k - \sum_{j=1}^{k-1} [(2j - \gamma)b_1b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}, \]

for \( k = L + 1, L + 2, \)
\[ E_k := b_1\Lambda S_k - \sum_{j=1}^{L} [(2j - \gamma)b_1b_j - b_{j+1}] \frac{\partial S_k}{\partial b_j}. \]

For the expansion of \( A_2, \) by Taylor expansion with integral remainder, one gets
\[ A_2 = \frac{(d-2)}{y^2} \left[ \sum_{i=2}^{L+2} P_i + R_1 + R_2 \right], \]

where
\[ P_i := \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1=j} c_J \prod_{k=1}^{L} b^{j_k}_k T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \]
\[ R_1 := \sum_{j=2}^{L+2} \frac{f^{(j)}(Q)}{j!} \sum_{|J|_1=j} c_J \prod_{k=1}^{L} b^{j_k}_k T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \]
\[ R_2 := \frac{\Theta_b^{L+3}}{(L+2)!} \int_0^1 (1 - \tau)^{L+2} f^{(L+3)}(Q + \tau \Theta_b) \, d\tau, \]

with \( J = (i_1, \ldots, i_L, j_2, \ldots, j_{L+2}) \in \mathbb{N}^{2L+1} \) and
\[ |J|_1 = \sum_{k=1}^{L} i_k + \sum_{k=2}^{L+2} j_k, \quad |J|_2 = \sum_{k=1}^{L} k i_k + \sum_{k=2}^{L+2} k j_k. \]

Note that we take \( L \) large enough so that \( R_2 = 0. \) Thus
\[ \partial_s Q_b - \partial_s^2 Q_b - \frac{(d-3)}{y} \partial_s Q_b + b_1\Lambda Q_b + \frac{(d-2)}{y^2} f(Q_b) = \text{Mod}(t) + \Psi_b, \quad (3.10) \]

with
\[ \Psi_b := \sum_{k=1}^{L+1} [\mathcal{L}S_{k+1} + E_k + \frac{(d-2)}{y^2} P_{k+1}] + E_{L+2} + \frac{(d-2)}{y^2} R_1. \quad (3.11) \]

Motivated by (3.11), we define \( \{S_k\}_{k=1}^{L+2} \) as
\[
\begin{cases}
S_1 = 0 \\
S_k = \mathcal{L}^{-1} F_k 
\end{cases}
\]

with
\[ F_k := E_{k-1} + \frac{(d-2)}{y^2} P_k, \quad 2 \leq k \leq L + 2. \]
Then we aim at proving (3.6) and (3.7). Claim:

\[ F_k \sim (k - 1, k - 2, k) \quad \text{and} \quad \frac{\partial F_k}{\partial b_m} = 0, \quad 2 \leq k \leq m \leq L + 2. \quad (3.12) \]

One can prove it by induction.

When \( k = 2 \), note that by (1.7) and (iv) of Lemma 3.1,

\[ \frac{f^{(2)}(Q)}{y^2} T_1^2 \lesssim \begin{cases} y^6 \ll y^4 \text{ as } y \to 0 \\ y^{2-3\gamma} \ll y^{-\gamma} \text{ as } y \to \infty. \end{cases} \]

Combined with (v) of Lemma 3.1, we have

\[ F_2 = b_1^2 \left( \Delta T_1 - (2 - \gamma)T_1 + e\frac{f^{(2)}(Q)}{y^2} T_1^2 \right) \sim (1, 0, 2). \]

Then we show \( \leq k \implies k + 1 \), specifically speaking we need to prove

\[ F_{k+1} \sim (k, k - 1, k + 1) \quad \text{and} \quad \frac{\partial F_{k+1}}{b_m} = 0, \quad k + 1 \leq m. \quad (3.13) \]

By induction hypothesis, \( F_j \sim (j - 1, j - 2, j) \) and \( \frac{\partial F_j}{\partial b_m} = 0, \quad j \leq m \), then by (iii) of Lemma 3.1,

\[ S_j \sim (j, j - 1, j) \quad \text{and} \quad \frac{\partial S_j}{\partial b_m} = 0, \quad j \leq m, \quad \text{for any } 2 \leq j \leq k. \quad (3.14) \]

Let us estimate \( E_k \) and \( \frac{P_k}{y^3} \) separately. On \( E_k \), by (v) of Lemma 3.1, (3.3), (3.14) and (i) of Lemma 3.1, we have for the components of \( E_k \),

\[ b_1 b_k (\Delta T_k - (2k - \gamma)T_k) \sim (k - 1, k + 1), \]

\[ b_1 \Delta S_k \sim (k - 1, k + 1), \]

\[ \left[ (2j - \gamma) b_1 - \frac{b_{j+1}}{b_j} \right] \left( b_j \frac{\partial S_k}{\partial b_j} \right) \sim (k, k - 1, k + 1). \]

Hence \( E_k \sim (k - 1, k + 1) \). On \( \frac{P_{k+1}}{y^3} \), note that it is the finite linear combinations of terms of the form

\[ M_J := \frac{f^{(j)}(Q)}{y^2} \prod_{m=1}^{L} b_m^j T_m^l \prod_{m=2}^{L+2} S_m^l, \]

where \( J = (i_1, \ldots, i_L, j_2, \ldots, j_{L+2}), \quad |J|_1 = j, \quad |J|_2 = k + 1, \quad 2 \leq j \leq \min \{k + 1, L + 2 \} \). Note that by (1.7),

when \( y \to 0, \quad f^{(j)}(Q) \lesssim 1. \)

when \( y \to \infty, \quad f^{(j)}(Q) \lesssim \begin{cases} y^{-\gamma} \text{ for even } j \\ 1 \text{ for odd } j. \end{cases} \]

Combined with (3.3), (iv) of Lemma 3.1 and (3.14), we get

when \( y \to 0, \quad M_J \lesssim b_1^{k+1} y^{\sum_{m=1}^{(2m+2)} j_m + \sum_{m=2}^{L+2} (2m+2) j_m - 2} \ll b_1^{k+1} y^{2k+2}. \)

when \( y \to \infty, \quad M_J \lesssim \begin{cases} b_1^{k+1} y^{2k-\gamma(j+1)-2 \sum_{m=2}^{L+2} j_m} \text{ for even } j \ll b_1^{k+1} y^{2(k-1)-\gamma}. \end{cases} \]

Hence \( M_J \sim (k - 1, k + 1) \) and same holds for \( F_{k+1} \). By the definition of \( F_{k+1} \) and induction hypothesis, \( \frac{\partial F_{k+1}}{b_m} = 0 \) for \( 2 \leq k \leq m \leq L + 2 \) is easily verified. This completes the proof of (3.12). Then again by (iii) of Lemma 3.1, (3.6) and (3.7) hold true.

Here we omit the details for the proof of (3.8) and (3.9) since one needs only to use the degrees of the components of \( \Psi_b \) which are already known. This concludes the whole proof.
3.3. Localized approximation.

**Proposition 3.2.** Under the assumptions in Proposition 3.1, assuming in addition \(|(b_1)_r| \lesssim b_1^2\).

Defining the localized approximation of (3.2) as

$$\tilde{Q}(\mathbf{s}) := Q(y) + \sum_{k=1}^{L} b_k T_k + \sum_{k=2}^{L+2} \tilde{s}_k$$

with \(T_k := \chi_{B_k} T_k\), \(\tilde{s}_k := \chi_{B_k} S_k\) \quad (3.15)

Then \(\tilde{Q}\) satisfies the equation

$$\partial_s \tilde{Q} - \partial_y^2 \tilde{Q} - \frac{(d-3)}{y} \partial_y \tilde{Q} + b_1 \Lambda \tilde{Q} + \frac{(d-2)}{y^2} f(\tilde{Q}) = \bar{\Psi} + \chi_{B_1} \text{Mod}(t)$$

with \(\bar{\Psi}\) satisfying the following properties.

(i) For all \(0 \leq m \leq L - 1\),

$$\int |\mathcal{L}^{h+m+1} \bar{\Psi}|^2 + \int \frac{|\mathcal{A} \mathcal{L}^{h+m} \bar{\Psi}|^2}{1 + y^2} + \int \frac{|\mathcal{L}^{h+m} \bar{\Psi}|^2}{1 + y^4} + \int \frac{|\bar{\Psi}|^2}{1 + y^{4(h+m+1)}} \lesssim b_1^{2m+2+2(1-\delta)-C_L \eta},$$

and

$$\int |\mathcal{L}^{h+L+1} \bar{\Psi}|^2 + \int \frac{|\mathcal{A} \mathcal{L}^{h+L} \bar{\Psi}|^2}{1 + y^2} + \int \frac{|\mathcal{L}^{h+L} \bar{\Psi}|^2}{1 + y^4} + \int \frac{|\bar{\Psi}|^2}{1 + y^{4(h+L+1)}} \lesssim b_1^{2L+2+2(1-\delta)+\eta}.$$ \quad (3.17)

(ii) For all \(M \leq \frac{b_1}{2}\) and \(0 \leq m \leq L\),

$$\int_{y \leq 2M} |\mathcal{L}^{h+m+1} \bar{\Psi}|^2 \lesssim M^C b_1^{2L+6}.$$ \quad (3.18)

(iii) For all \(0 \leq m \leq L\),

$$\int_{y \leq 2B_0} |\mathcal{L}^{h+m+1} \bar{\Psi}|^2 + \int_{y \leq 2B_0} \frac{|\bar{\Psi}|^2}{1 + y^{4(h+m+1)}} \lesssim b_1^{2m+4+2(1-\delta)-C_L \eta}.$$ \quad (3.19)

**Proof.** Direct computation gives

$$\partial_s \tilde{Q} - \partial_y^2 \tilde{Q} - \frac{(d-3)}{y} \partial_y \tilde{Q} + b_1 \Lambda \tilde{Q} + \frac{(d-2)}{y^2} f(\tilde{Q})$$

\[= \chi_{B_1} \left[ \partial_s Q - \partial_y^2 Q - \frac{(d-3)}{y} \partial_y Q + b_1 \Lambda Q + \frac{(d-2)}{y^2} f(Q) \right] + \Theta_b \left[ \partial_s \chi_{B_1} - \left( \frac{(d-3)}{y^2} \partial_y \chi_{B_1} \right) + b_1 \Lambda \chi_{B_1} \right] - 2 \partial_y \chi_{B_1} \partial_y \Theta_b + b_1 (1 - \chi_{B_1}) \Lambda Q \]

\[+ \frac{(d-2)}{y^2} \left[ f(\tilde{Q}) - f(Q) - \chi_{B_1} (f(Q) - f(Q)) \right].\]

Then by (3.5),

$$\partial_s \tilde{Q} - \partial_y^2 \tilde{Q} - \frac{(d-3)}{y} \partial_y \tilde{Q} + b_1 \Lambda \tilde{Q} + \frac{(d-2)}{y^2} f(\tilde{Q}) =: \chi_{B_1} \text{Mod}(t) + \bar{\Psi},$$

where

$$\bar{\Psi} := \chi_{B_1} \Psi + \bar{\Psi}^{(1)} + \bar{\Psi}^{(2)} + \bar{\Psi}^{(3)},$$

$$\bar{\Psi}^{(1)} := b_1 (1 - \chi_{B_1}) \Lambda Q,$$

$$\bar{\Psi}^{(2)} := \frac{(d-2)}{y^2} \left[ f(\tilde{Q}) - f(Q) - \chi_{B_1} (f(Q) - f(Q)) \right],$$

$$\bar{\Psi}^{(3)} := \Theta_b \left[ \partial_s \chi_{B_1} - \left( \frac{(d-3)}{y^2} \partial_y \chi_{B_1} \right) + b_1 \Lambda \chi_{B_1} \right] - 2 \partial_y \chi_{B_1} \partial_y \Theta_b.$$
We only estimate the contribution of $\widetilde{\Psi}_b^{(2)}$ in (3.17)-(3.20). Note that in $\widetilde{\Psi}_b^{(2)}$, $y$ is supported in $B_1 \leq y \leq 2B_1$, thus its contribution to (3.19) and (3.20) hold trivially. Let us now estimate its contribution to (3.17) and (3.18). By Taylor expansion,

$$f(Q_b) - f(Q) = f'(Q)\Theta_b + \frac{f''(Q)}{2} \Theta_b^2 + \Theta_b^3,$$

with $B_1 \leq y \leq 2B_1$. (3.21)

Note that for $2 \leq k \leq L$, by (3.3) and (3.6) we see that $|S_k| \lesssim b_l^k y^{2(k-1)-\gamma}$. In comparison, $|b_k T_k| \lesssim b_l^k y^{2k-\gamma}$, which follows from (3.3) and (iv) of Lemma 3.1. Similarly, $|S_{L+1}| \lesssim b_l^{L+1} y^{2L-\gamma}$ and $|S_{L+2}| \lesssim b_l^{L+2} y^{2(L+1)-\gamma}$, in comparison, $|b_L T_L| \lesssim b_l^k y^{2k-\gamma}$. Therefore, the main order term of $\Theta_b$ is $\sum_{k=1}^L b_k T_k$, or say

$$|\Theta_b| \lesssim \sum_{k=1}^L b_k^k y^{2k-\gamma} 1_{B_1 \leq y \leq 2B_1}.$$

(3.22)

In particular, since $b_l^k y^{2k-\gamma} 1_{B_1 \leq y \leq 2B_1} \lesssim b_l^{2\gamma+(\gamma-k)}$, we have $|\Theta_b| \ll 1$. Substituting (3.22) into (3.21) and making use of (1.7), we get

$$\begin{cases}
|f(Q_b) - f(Q)| \lesssim |\Theta_b| \lesssim \sum_{k=1}^L b_k^k y^{2k-\gamma} 1_{B_1 \leq y \leq 2B_1} \\
|f(\widetilde{Q}_b) - f(Q)| \lesssim \chi_{B_1} |\Theta_b| \lesssim \sum_{k=1}^L b_k^k y^{2k-\gamma} 1_{B_1 \leq y \leq 2B_1}
\end{cases} \implies |\widetilde{\Psi}_b^{(2)}| \lesssim \sum_{k=1}^L b_k^k y^{2(k-1)-\gamma} 1_{B_1 \leq y \leq 2B_1}.$$

We further estimate that $|\widetilde{\Psi}_b^{(2)}| \lesssim \sum_{k=1}^L b_k^k B_1^{2(k-1)} y^{-\gamma} 1_{B_1 \leq y \leq 2B_1} = b_1 y^\gamma \sum_{k=1}^L b_1^{-\eta(k-1)} 1_{B_1 \leq y \leq 2B_1}$, then

$$\begin{align*}
\int |\mathcal{L}^{b_{h+m+1}} \widetilde{\Psi}_b^{(2)}|^2 &\lesssim b_1^2 \sum_{k=1}^L b_1^{-2(k-1)\eta} \int_{B_1 \leq y \leq 2B_1} |y^{-\gamma-2(h+m+1)}|^2 y^{d-3} \, dy \\
&\lesssim b_1^{2m+2+2(1+\eta)(1-\delta)} \sum_{k=1}^L b_1^{2m-2k+2)\eta}, \quad \text{for all } 0 \leq m \leq L.
\end{align*}
$$

This concludes the proof.

4. Linearization of $\{b_k\}_{k=1}$

Denoting $\{b_k\}_{k=1}$ as the solution of

$$\begin{cases}
(b_k^s + (2k-\gamma)b_k^s b_k^e - b_k^{s+1}) = 0 \\
b_k^{s+1} = b_k^{s+2} = \cdots = b_L^{s} = 0
\end{cases},$$

(4.1) where $b_L^{s+1} := 0$ and $L$ satisfying

$$\frac{\gamma}{2} < l \ll L$$

(4.2) is an integer to be chosen later. One can find a set of solution explicitly in the form $b_k^s = c_k s^{-k}$, with

$$\begin{align*}
c_1 &= \frac{l}{2l-\gamma}, \\
c_{k+1} &= \frac{\gamma(l-k)}{2l-\gamma} c_k, \quad 1 \leq k \leq l-1, \\
c_{L+1} &= c_{l+2} = \cdots = c_L = 0.
\end{align*}$$

(4.3)
Remark 4.1. Standard linearization of the system near \( \{b_k^e\}_{k=1}^L \) yields a Jacobian similar to the structure of \( A_t \). However, by further shaping the remainder as \( \frac{\partial b_k(s)}{\partial s} \), one finds advantages in calculation.

\[ (b_k)_s + (2\gamma - \beta)b_1b_k - b_{k+1} = \frac{1}{s^{k+1}}[s(U_k)_s - (A_tU)_k + O(|U|^2)], \]
\[ (b_l)_s - (2\gamma - \beta)b_1b_l = \frac{1}{s^{l+1}}[s(U_l)_s - (A_tU)_l + O(|U|^2)], \]

where \( A_t = (a_{i,j})_{1 \times 1} \) with
\[
\begin{cases}
  a_{1,1} = \frac{\gamma(1 - 1)}{2l - \gamma} - (2 - \gamma)c_1, \\
  a_{i,i} = \frac{\gamma(l - i)}{2l - \gamma}, \quad 2 \leq i \leq l, \\
  a_{i,i+1} = 1, \quad 1 \leq i \leq l - 1, \\
  a_{i,1} = -(2 - \gamma)c_1, \quad 2 \leq i \leq l, \\
  a_{i,j} = 0, \quad \text{otherwise}.
\end{cases}
\]

Moreover, \( A_t \) is diagonalizable with
\[ A_t = P_t^{-1}D_tP_t, \quad D_t = \text{diag}\left\{-1, \frac{2\gamma}{2l - \gamma}, \frac{3\gamma}{2l - \gamma}, \ldots, \frac{l\gamma}{2l - \gamma}\right\}. \]

**Remark 4.1.** Standard linearization of the system near \( \{b_k^e\}_{k=1}^L \) yields a Jacobian similar to the structure of \( A_t \). However, by further shaping the remainder as \( \frac{\partial b_k(s)}{\partial s} \), one finds advantages in calculation.

5. Decomposition of the Solution and Coercivity-Determined Estimates on the Remainder Term

5.1. Decomposition of the Solution. In this section, we shall use implicit function theorem to show the existence of decomposition
\[ u(t, r) = (\tilde{Q}_b + q)\left(t, \frac{r}{\lambda(t)}\right) \]

satisfying orthogonality conditions
\[ \langle q(s, y), \mathcal{L}^i\Phi_M \rangle = 0, \quad \text{for} \quad 0 \leq i \leq L. \]

Indeed, let \( u_0 \) be close to \( Q \) in some sense, then this closeness is propagated on a small time interval \([0, t_1]\). Defining the map
\[ \mathcal{T} : (t, \lambda, b_1, \ldots, b_L) \mapsto \left(u(t) - (\tilde{Q}_b)_{\lambda}, (\mathcal{L}^i\Phi_M)_{\lambda}\right)_{i=0, \ldots, L}. \]

Then we choose \( u_0 \) such that \( \mathcal{T} \) maps \( (0, \lambda, b_1^*, \ldots, b_L^* ) \) to the zero vector, for some \( \lambda^* \) close to 1 and \( b_i^* \) close to 0 for all \( 1 \leq i \leq L \). Note that by direct computation, the Jacobian of \( \mathcal{T} \) at \( t = 0, \lambda = 1, b = 0 \) is
\[ (-1)^{\frac{(1+L)!}{2}} \langle \lambda M \Delta Q, \lambda Q \rangle^{L+1} + \text{small correction}, \]

which is nonzero. Then by implicit function theorem, there exists unique functions \( \lambda = \lambda(t), \)
\[ b = b(t) \] such that
\[ \left( (u(t) - (\tilde{Q}_b)_{\lambda(t)}, (\mathcal{L}^i\Phi_M)_{\lambda(t)})_{i=0, \ldots, L} = 0 \text{ on some time interval } [0, t_1]. \]
5.2. **Coercivity-determined estimates on** $q$. Substituting (5.1) into (3.2) and making use of (5.2), we get the equation for the remainder term $q$ as

$$
\partial_s q - \frac{\lambda s}{\lambda} \Lambda q + \mathcal{L} q = -\vec{\Psi}_b - \overline{\text{Mod}} + \mathcal{H}(q) - \mathcal{N}(q) =: \mathcal{F},
$$

(5.3)

where

$$
\overline{\text{Mod}} := -\left(\frac{\lambda s}{\lambda} + b_1\right)\Lambda \tilde{\Psi}_b + \chi_B, \text{Mod}(t),
$$

(5.4)

$$
\mathcal{H}(q) := \frac{(d-2)}{y^2} [f'(Q) - f'(~\tilde{Q}_b)]q,
$$

(5.5)

$$
\mathcal{N}(q) := \frac{(d-2)}{y^2} [f(~\tilde{Q}_b + q) - f(~\tilde{Q}_b) - f'(~\tilde{Q}_b)q].
$$

(5.6)

In original variable, denoting $v(t, r) := q(s, y)$, then (5.3) becomes

$$
\partial_t v + \mathcal{L}_s v = \frac{1}{\lambda^2} \mathcal{F}_\lambda.
$$

(5.7)

Before we dig more into (5.3) or (5.7) for dynamic-determined estimates, which we shall call modulation estimates and energy estimates later. We make some preparations about estimates on $q$ which are coercivity-determined. For simplicity, denoting $\mathcal{E}_{2i} := \mathcal{E}_{2i}(q) = \int |\mathcal{L}^i q|^2$, for $i \in \mathbb{N}$.

**Lemma 5.1.** Recalling that $k := L + h + 1$, $L \gg l$ is a large integer and $h$ is defined as in (1.9), then we have the following coercivity-determined estimates.

(i) Near the origin $q$ has the Taylor expansion in the form

$$
q = \sum_{i=1}^{k} c_i T_{k-i} + r_q,
$$

(5.8)

with bounds

$$
|c_i| \lesssim \sqrt{\mathcal{E}_{2k}},
$$

(5.9)

$$
|\partial_y^j r_q| \lesssim y^{2k+1-\frac{j}{2}} |\ln y|^k \sqrt{\mathcal{E}_{2k}}, \quad 0 \leq j \leq 2k - 1, \quad y < 1.
$$

(5.10)

(ii) Pointwise bound near the origin: for $y < 1$,

$$
|q_{2i}| + |\partial_y^i q| \lesssim y^{-\frac{j}{2}+1} |\ln y|^k \sqrt{\mathcal{E}_{2k}} \quad \text{for} \ 0 \leq i \leq k - 1,
$$

(5.11)

$$
|q_{2i-1}| + |\partial_y^{2i-1} q| \lesssim y^{-\frac{j}{2}+2} |\ln y|^k \sqrt{\mathcal{E}_{2k}} \quad \text{for} \ 1 \leq i \leq k.
$$

(5.12)

(iii) Weighted bounds: for $1 \leq m \leq k$,

$$
\sum_{i=0}^{2m} \int \frac{|\partial_y^i q|^2}{1 + y^{4m-2i}} \lesssim \mathcal{E}_{2m}.
$$

(5.13)

Moreover, let $(i, j) \in \mathbb{N} \times \mathbb{N}_+$ with $2 \leq i + j \leq 2k$, then

$$
\int \frac{|\partial_y^i q|^2}{1 + y^{2j}} \lesssim \begin{cases}
\mathcal{E}_{2m}, & \text{for} \ i + j = 2m, \ 1 \leq m \leq k.
\end{cases}
$$

(5.14)

(iv) Pointwise bound far away: let $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $1 \leq i + j \leq 2k - 1$, then for $y \geq 1$,

$$
\frac{|\partial_y^i q|^2}{y^j} \lesssim \begin{cases}
\frac{1}{y^{|d-4|}} \mathcal{E}_{2m}, & \text{for} \ i + j = 2m, \ 1 \leq m \leq k.
\end{cases}
$$

(5.15)
Proof. This can be proved in a similar way as in Appendix B in [12], we shall still provide details since different asymptotics are involved.

Proof of (i). We claim: for $1 \leq m \leq k$, $q_{2k-2m}$ admits the following expansion and estimates near the origin:

$$q_{2k-2m} = \sum_{i=1}^{m} c_{i,m} T_{m-i} + r_{2m},$$

$$|c_{i,m}| \lesssim \sqrt{\delta_{2k}},$$

$$|\partial_{y} r_{2m}| \lesssim y^{2m+1-\frac{d}{4}} \ln y |\ln y|^{m} \sqrt{\delta_{2k}}, \text{ for } 0 \leq j \leq 2m - 1.$$  \hfill (5.16)

Then (i) immediately follows when one chooses $m = k$, and we focus on the proof of the claim. Defining the sequence $\{r_{i}\}_{i=0}^{2k}$ recursively as $r_{0} := q_{2k}$, and

$$r_{2i+1}(y) := \frac{1}{y^{d-3} \Lambda y} \int_{0}^{y} r_{2i} \Lambda y \cdot x^{d-3} \, dx,$$

$$r_{2i+2}(y) := -\Lambda y \int_{a}^{y} r_{2i+1} \Lambda y \, dx,$$

where we choose $a \in \left(\frac{1}{2}, 1\right)$ such that

$$|q_{2k-1}(a)|^{2} \lesssim \int_{y^{1}} \frac{1}{y^{d-3} \Lambda y} \lesssim \delta_{2k}, \text{ for } 1 \leq i \leq 2k.$$  \hfill (5.17)

Note that Lemma 2.7 justifies the last inequality, and the definition fulfills the relation $\mathcal{A} r_{2i+1} = r_{2i}, \mathcal{A} r_{2i+2} = r_{2i+1},$ for $0 \leq i \leq k - 1$. Next we prove (5.16) inductively. When $m = 1$, note that by $\mathcal{A} q_{2k-1} = q_{2k},$ we have

$$q_{2k-1} = \frac{1}{y^{d-3} \Lambda y} \int_{0}^{y} q_{2k} \Lambda y \cdot x^{d-3} \, dx + \frac{C}{y^{d-3} \Lambda y}.$$  \hfill (5.17)

From the facts that $\int \frac{|q_{2k}|^{2}}{y^{d-3} \Lambda y} \lesssim \delta_{2k}, \Lambda y(y) \sim y^{2}$ as $y \to 0$, and $\int_{y<1} \frac{|q_{2k}|^{2}}{y^{d-3} \Lambda y} = +\infty$, one deduces $C = 0.$ That is, $r_{1}(y) = q_{2k-1}(y).$ And by Hölder, we get

$$|r_{1}(y)| \lesssim \frac{1}{y^{d-3}} \left( \int_{0}^{y} |q_{2k}|^{2} x^{d-3} \, dx \right)^{\frac{1}{2}} \left( \int_{0}^{y} x^{4} \cdot x^{d-3} \, dx \right)^{\frac{1}{2}} \lesssim y^{-\frac{d}{2}+2} \sqrt{\delta_{2k}}, \text{ for } y < 1.$$  \hfill (5.17)

By the definition of $\{r_{i}\}$, $r_{2}(y) = -\Lambda y \int_{a}^{y} \frac{r_{1}}{\Lambda y} \, dx$, then straight calculation yields

$$|r_{2}(y)| \lesssim y^{2} \int_{a}^{y} \frac{x^{-\frac{d}{2}+2} \sqrt{\delta_{2k}}}{x^{2}} \, dx \lesssim y^{-\frac{d}{4}+3} \ln y \delta_{2k}, \text{ for } y < 1.$$  \hfill (5.17)

Note that $\mathcal{A} r_{2} = r_{1} = q_{2k-1},$ then $\mathcal{L} r_{2} = \mathcal{A} * q_{2k-1} = q_{2k} = \mathcal{L} q_{2k-2},$ and it tells that

$$q_{2k-2} = c \Lambda y + r_{2},$$  \hfill (5.17)

where we also used the facts that $\int \frac{|q_{2k-2}|^{2}}{y^{d-3} \Lambda y} \lesssim \delta_{2k}$ and $\int_{y<1} \frac{|r_{2}|^{2}}{y^{d-3} \Lambda y} = +\infty$. Then substituting the point $a$ into (5.17), we derive $|c| \lesssim \sqrt{\delta_{2k}},$ then it follows that

$$|q_{2k-2}| \lesssim y^{-\frac{d}{4}+3} \ln y \delta_{2k}, \text{ for } y < 1.$$  \hfill (5.17)

Next, applying the definition of $\mathcal{A}$ and asymptotics of $V$, we have

$$|\partial_{y} r_{2}| \lesssim |r_{1}| + \left| \frac{r_{2}}{y} \right| \lesssim y^{-\frac{d}{4}+2} \ln y \delta_{2k}, \text{ for } y < 1.$$  \hfill (5.17)
This concludes the proof when \( m = 1 \). Now assuming the cases for \( \leq m \) are true, and we prove that the case for \( m + 1 \) holds true. Note that one can readily verify the inequality

\[
\int_0^y | \ln x|^n \, dx \lesssim y | \ln y|^n,
\]

for all positive integers \( n \). Then it combined with induction hypothesis gives

\[
|r_{2m+1}| = \left| \frac{1}{y^{d-3}} \int_0^y r_{2m} \Lambda Q \cdot x^{d-3} \, dx \right| \lesssim \frac{1}{y^{d-1}} \sqrt{\mathcal{E}_{2k}} \int_0^y x^{2m+\frac{d}{2}} | \ln x|^m \, dx \lesssim y^{2m+2-\frac{d}{2}} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

Hence,

\[
|r_{2m+2}| = \left| -\Lambda Q \int_a^y \frac{r_{2m+1}}{\Lambda Q} \, dx \right| \lesssim y^2 \int_y^a \frac{x^{2m+2-\frac{d}{2}} \sqrt{x} \ln x}{x} \, dx \lesssim y^{2m+3-\frac{d}{2}} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

and we further estimate it into two cases. If \( 2m + 1 - \frac{d}{2} < 0 \), then

\[
|r_{2m+2}| \lesssim y^2 \cdot y^{2m+1-\frac{d}{2}} \sqrt{\mathcal{E}_{2k}} \int_a^y \frac{\ln x}{x} \, dx \lesssim y^{2m+3-\frac{d}{2}} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

If \( 2m + 1 - \frac{d}{2} \geq 0 \), note that using induction one can readily verify that

\[
\int_y^a x^{2m+1-\frac{d}{2}} \cdot \frac{\ln x}{x} \, dx \lesssim y^{2m+1-\frac{d}{2}} | \ln y|^m,
\]

then it immediately follows

\[
|r_{2m+2}| \lesssim y^{2m+3-\frac{d}{2}} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

Combining above two cases together, we get

\[
|r_{2m+2}| \lesssim y^{2m+3-\frac{d}{2}} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

Since \( \mathcal{A} r_{2m+2} = r_{2m+1} \), \( \mathcal{L} r_{2m+2} = r_{2m} \), one sees that

\[
\mathcal{L} q_{2k-2(m+1)} = q_{2k-2m} + \sum_{i=1}^m c_{i,m} T_{m-i} + r_{2m} = \sum_{i=1}^m -c_{i,m} \mathcal{L} T_{m+1-i} + \mathcal{L} r_{2m+2}.
\]

Due to the facts that \( \int_{y \leq 1} \frac{|q_{2k-2(m+1)}|^2}{y^2} \lesssim \mathcal{E}_{2k} \) and \( \int_{y < 1} \frac{\Gamma}{y^2} = +\infty \), we deduce

\[
q_{2k-2(m+1)} = \sum_{i=1}^m -c_{i,m} T_{m+1-i} + r_{2m+2} + \bar{c} \Lambda Q.
\]

Then substituting the point \( a \) into above expansion gives \( |\bar{c}| \lesssim \sqrt{\mathcal{E}_{2k}} \). And by induction hypothesis and direct computations, we have

\[
| \partial_x^j r_{2m+2} | \lesssim \sum_{i=0}^j \frac{|r_{2m+2-2i}|}{y^{i-1}} \lesssim \frac{\sqrt{\mathcal{E}_{2k}} \sum_{i=0}^j y^{2m+3-i-\frac{d}{2}} | \ln y|^m}{y^{i-1}} \lesssim y^{2m+3-\frac{d}{2} - j} | \ln y|^m \sqrt{\mathcal{E}_{2k}}.
\]

This concludes the proof of (5.16).

Proof of (ii). Using (5.16) for \( m = k - i \), we get

\[
|q_2| \lesssim \sum_{i=1}^m \sqrt{\mathcal{E}_{2k} y^{2+2(m-i)}} + \sqrt{\mathcal{E}_{2k} y^{2m+1-\frac{d}{2}}} | \ln y|^m \lesssim y^{-\frac{d}{2} + 3} | \ln y|^m \sqrt{\mathcal{E}_{2k}},
\]

for \( 0 \leq i \leq k - 1 \). And the second inequality follows from the relation \( \mathcal{A} q_{2i-2} = q_{2i-1} \), for \( 1 \leq i \leq k \).
Proof of (iii). Note that $|\partial_y q| \lesssim \sum_{j=0}^i \frac{|q_j|}{y^j}$, then by Lemma 2.7 and (ii), we have

$$
\sum_{i=0}^{2m} \int \frac{|\partial_y q|^2}{1 + y^{4m-2i}} \lesssim \mathcal{E}_{2m} + \sum_{i=1}^{2m-1} \int y < 1 |\partial_y^i q|^2 + \sum_{i=0}^{2m-1} \int_{y > 1} |\partial_y^i q|^2 \lesssim \mathcal{E}_{2m} + \mathcal{E}_{2k} \int_{y < 1} y^3 |\partial y|^{2k} dy + \sum_{i=0}^{2m-1} \sum_{j=0}^i \int_{y > 1} y^{4m-2j} \|q_j\|_2^2 \lesssim \mathcal{E}_{2m}.
$$

And consequently, one derives that if $i + j = 2m$ with $1 \leq m \leq k$,

$$
\int \frac{|\partial_y^i q|^2}{1 + y^{2j}} = \int \frac{|\partial_y^i q|^2}{1 + y^{4m-2i}} \lesssim \mathcal{E}_{2m}.
$$

If $i + j = 2m + 1$ with $1 \leq m < k - 1$,

$$
\int \frac{|\partial_y^i q|^2}{1 + y^{2j}} = \int \frac{|\partial_y^i q|^2}{1 + y^{6m+2-2i}} \lesssim \left( \int \frac{|\partial_y^i q|^2}{1 + y^{4m-2i}} \right)^{\frac{1}{2}} \left( \int \frac{|\partial_y^i q|^2}{1 + y^{4m+4-2i}} \right)^{\frac{1}{2}} \lesssim \sqrt{\mathcal{E}_{2m} \sqrt{\mathcal{E}_{2(m+1)}}}.
$$

Proof of (iv). By direct calculation and (iii), we derive that if $1 \leq i + j \leq 2k - 1$ and $y \geq 1$,

$$
\left| \frac{\partial_y q}{y^j} \right|^2 \lesssim \int y^+ \partial_x \left( \frac{(\partial_x q)^2}{x^{2j}} \right) dx
\lesssim \int y^+ \frac{|\partial_x q|^2}{x^{d-3} \cdot x^{2j}} dx + \int y^+ \frac{|\partial_x^{i+1} q|^2}{x^{2j+1} x^{d-3}} dx
\lesssim \frac{1}{y^{d-4}} \left( \int y^+ \frac{|\partial_x q|^2}{x^{2j+2}} + \int y^+ \frac{|\partial_x^{i+1} q|^2}{x^{2j}} \right)
\lesssim \frac{1}{y^{d-4}} \left\{ \begin{array}{ll}
\mathcal{E}_{2m}, & \text{for } i + j = 2m, \ 1 \leq m \leq k.
\end{array} \right.
\quad \sqrt{\mathcal{E}_{2m} \sqrt{\mathcal{E}_{2(m+1)}}}, \text{ for } i + j = 2m, \ 1 \leq m \leq k - 1.
$$

6. Description of initial data and bootstrap assumption

Assumptions on initial data are as follows.

**Definition 6.1.** Denoting $\mathcal{V} := P\mathcal{U}$. Let $s_0 \geq 1$, Assuming initially

(i) *(Initial unstable modes)*

$$
\mathcal{E}^{(1-\delta)} (\mathcal{V}_2(s_0), \cdots, \mathcal{V}_l(s_0)) \in \mathcal{B}_{l-1}(0,1).
$$

(ii) *(Initial stable modes)*

$$
\mathcal{E}^{(1-\delta)} \mathcal{V}_l(s_0) < 1, \ |b_k(s_0)| < s_0^{-\frac{k-l(2k-\gamma)}{2l-\gamma}} \text{ for } l + 1 \leq k \leq L.
$$

(iii) *(Initial energy)*

$$
\sum_{k=h+2}^k \mathcal{E}_{2k}(s_0) < s_0^{-\frac{10(l-1)}{2l-\gamma}}.
$$

(iv) *up to a fixed rescaling, we may assume*

$$
\lambda(s_0) = 1.
$$

Next, we set up the bootstrap assumption as follows.
Definition 6.2. Let $K \gg 1$ denote some large enough universal constant to be chosen later and $s \geq 1$. Defining $S_K(s)$ as the set of all $(b_1(s), \cdots, b_L(s), q(s))$ such that
(i) (Control of unstable modes)
\[ s \hat{\xi}^{(1-\delta)}(\mathcal{V}_2(s), \cdots, \mathcal{V}_l(s)) \in \mathcal{B}_{l-1}(0,1). \]  
(6.5)
(ii) (Control of stable modes)
\[ |s \hat{\xi}^{(1-\delta)} \mathcal{V}_l(s)| \leq 10, \quad |b_k(s)| \leq \frac{10}{s^k} \text{ for } l + 1 \leq k \leq L. \]  
(6.6)
(iii) (Control of highest order energy)
\[ \mathcal{E}_{2k}(s) \leq K s^{-[2L+2(1-\delta)(1+\eta)]}. \]  
(6.7)
(iv) (Control of lower order energy)
\[ \mathcal{E}_{2m}(s) \leq \begin{cases} K s^{-\frac{4m-d+2}{4m-d+2}}, & h + 2 \leq m \leq l + h. \\ K s^{-[2(m-h-1)+2(1-\delta)-K\eta]}, & l + h + 1 \leq m \leq k - 1. \end{cases} \]  
(6.8)
Remark 6.1. Note that we have
\begin{itemize}
\item[(i)] If $s_0$ is large enough, initial data $(b(s_0), q(s_0)) \in S_K(s_0)$.
\item[(ii)] If $(b(s), q(s)) \in S_K(s)$, then
\[ b_k(s) \simeq \frac{c_k}{s^k}, \quad 1 \leq k \leq l. \]
\end{itemize}
In particular,
\[ b_1(s) \simeq \frac{c_1}{s}, \quad |b_k(s)| \lesssim b_1^k(s), \quad 1 \leq k \leq L. \]
Thus the assumptions in Proposition 3.1 are justified.

7. Modulation estimates and improved bounds

Proposition 7.1. For $K \gg 1$ some universal large constant, assuming there is $s_0(K) \gg 1$ such that $(b(s), q(s)) \in S_K(s)$ on $s \in [s_0, s_1]$ for some $s_1 \geq s_0$. Then for $s \in [s_0, s_1]$, we have
\[ \sum_{k=1}^{L-1} |(b_k)_s + (2k - \gamma)b_1b_k - b_{k+1}| + |b_1 + \frac{\lambda_s}{\lambda}| \lesssim b_1^{L+1(1-\delta)(1+\eta)}, \]  
(7.1)
and
\[ |(b_L)_s + (2L - \gamma)b_1b_L| \lesssim \sqrt{\mathcal{E}_{2k}} K s^{2s} + b_1^{L+1(1-\delta)(1+\eta)}. \]  
(7.2)
Proof. The idea is to take inner product of (5.3) with $\mathcal{L}^k \Phi_M$ for $1 \leq k \leq L$, make use of orthogonal condition (5.2), apply H"older and coercivity property of $\mathcal{L}$.

Denoting $D(t) := |b_1 + \frac{\lambda_s}{\lambda}| + \sum_{k=1}^{L} |(b_k)_s + (2k - \gamma)b_1b_k - b_{k+1}|$. Now we take inner product of (5.3) with $\mathcal{L}^k \Phi_M$, and in view of (5.2) we have
\[ \langle \text{Mod}(t), \mathcal{L}^k \Phi_M \rangle = -\langle \mathcal{L}^k \tilde{\Phi}_b, \Phi_M \rangle - \langle \mathcal{L}^{k+1} q, \Phi_M \rangle - \langle -\frac{\lambda_s}{\lambda} \Lambda q - \mathcal{H}(q) + \mathcal{N}(q), \mathcal{L}^k \Phi_M \rangle. \]  
(7.3)
Then we estimate every term in (7.3). By (5.4),
\[ \langle \text{Mod}(t), \mathcal{L}^k \Phi_M \rangle = O(D(t)) \langle \Lambda \tilde{Q}_b, \mathcal{L}^k \Phi_M \rangle + \langle \text{Mod}(t), \mathcal{L}^k \Phi_M \rangle. \]  
(7.4)
By the degrees of $T_k$ and $S_k$, we see that
\[ \langle \Lambda \tilde{Q}_b, \mathcal{L}^k \Phi_M \rangle = \sum_{k=1}^{L} \langle b_k \Lambda T_k, \mathcal{L}^k \Phi_M \rangle + \sum_{k=2}^{L+2} \langle \Lambda S_k, \mathcal{L}^k \Phi_M \rangle. \]
where we also have used the fact that $|c_{j,M}| \lesssim M^2$, 0 $\leq j \leq L$, which can be verified by
induction on $j$. By (i) of Proposition 3.1 and (2.12),
\[
\langle \text{Mod}(t), \mathcal{L}^b \Phi_M \rangle = (-1)^L \langle \chi_M \Lambda Q, \Lambda Q \rangle [(b_L)_s + (2L - \gamma)b_L] + O\left( D(t) \sum_{k=1}^{L+2} \sum_{j=k+1}^{L+2} \langle \partial_{\Phi_k}, \mathcal{L}^b \Phi_M \rangle \right),
\]
where $\langle \partial_{\Phi_k}, \mathcal{L}^b \Phi_M \rangle \lesssim b_1^{-k} M^{d-2(2L+2-k)} \lesssim b_1 M^C$, for $k + 1 \leq j \leq L + 2$, $1 \leq k \leq L$.
Applying above estimates into (7.4), we get
\[
\langle \text{Mod}(t), \mathcal{L}^b \Phi_M \rangle = (-1)^L \langle \chi_M \Lambda Q, \Lambda Q \rangle [(b_L)_s + (2L - \gamma)b_L] + O(M^C b_1 D(t)).
\] (7.5)
Using (3.19) with $m = L - h - 1$ and Hölder, we have
\[
\langle \mathcal{L}^b \tilde{\Psi}_b, \Phi_M \rangle \lesssim \left( \int_{y \leq 2M} |\mathcal{L}^b \tilde{\Psi}_b|^2 \right)^{\frac{1}{2}} \left( \int_{y \leq 2M} |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^C b_1^{L+3}.
\] (7.6)
For the term $\langle \mathcal{L}^b \tilde{\Psi}_b, \Phi_M \rangle$, by Lemma 2.7,
\[
\mathcal{E}_{2k}(q) \gtrsim \int \frac{|\mathcal{L}^b \tilde{\Psi}_b|^2}{y^4(1 + y^{4(h-1)})} \gtrsim \int_{y \leq 2M} \frac{|\mathcal{L}^b \tilde{\Psi}_b|^2}{(1 + y^{4h})},
\]
then by Hölder it follows that
\[
|\langle \mathcal{L}^b \tilde{\Psi}_b, \Phi_M \rangle| \lesssim M^{2h} \left( \int_{y \leq 2M} \frac{|\mathcal{L}^b \tilde{\Psi}_b|^2}{(1 + y^{4h})} \right)^{\frac{1}{2}} \left( \int_{y \leq 2M} |\Phi_M|^2 \right)^{\frac{1}{2}} \lesssim M^{2h+\frac{d-2}{2}-\gamma} \sqrt{\mathcal{E}_{2k}(q)}.\] (7.7)
By triangle inequality,
\[
\langle -\frac{\lambda}{\lambda} \Lambda q, \mathcal{L}^b \Phi_M \rangle \lesssim (D(t) + b_1) \langle \Lambda q, \mathcal{L}^b \Phi_M \rangle.
\]
Note that by Hölder,
\[
\langle \Lambda q, \mathcal{L}^b \Phi_M \rangle \lesssim \left\| \frac{\partial_{\Phi_k} q}{y^2(1 + y^{2(k-2)+1})} \right\|_{L^2(y \leq 2M)} \|y^3(1 + y^{2(k-2)+1})\mathcal{L}^b \Phi_M\|_{L^2(y \leq 2M)} \lesssim M^C \sqrt{\mathcal{E}_{2k}},
\]
where we also have used the fact that by Lemma 2.7 and Lemma 2.6,
\[
\mathcal{E}_{2k}(q) \gtrsim \int \frac{|\mathcal{L}^b q|^2}{y^{1+1+y^{4(k-2)+2}}} \gtrsim \int \frac{|\partial_{\Phi_k} q|^2}{y^{4(k-2)+2}} + \int \frac{q^2}{y^{4(k-2)+2}}\] (7.8)
Therefore,
\[
\langle -\frac{\lambda}{\lambda} \Lambda q, \mathcal{L}^b \Phi_M \rangle \lesssim (D(t) + b_1) M^C \sqrt{\mathcal{E}_{2k}}.\] (7.9)
Again by Hölder and (7.8), we derive
\[
\langle -\mathcal{H}(q), \mathcal{L}^b \Phi_M \rangle \lesssim \langle \frac{q}{y^2} \mathcal{Q} \Theta_b + \Theta_b + (\Theta_b)^2, \mathcal{L}^b \Phi_M \rangle
\lesssim \left\| \frac{q}{y^2(1 + y^{2(k-2)+1})} \right\|_{L^2} \|y(1 + y^{2(k-2)+1})(\Theta_b + \Theta_b Q + \Theta_b^2)\mathcal{L}^b \Phi_M\|_{L^2(y \leq 2M)} \lesssim M^C b_1 \sqrt{\mathcal{E}_{2k}}.
\] (7.10)
By (7.8), we obtain
\[
\langle \mathcal{N}(q), \mathcal{L}^b \Phi_M \rangle \lesssim \langle \frac{q}{y^2}(|Q - 1| + |\Theta_b| + |q|), \mathcal{L}^b \Phi_M \rangle
\lesssim \int \frac{q^2}{y^{6(1+y^{4(k-2)+2})}} \|y^4(1 + y^{4(k-2)+2})(|Q - 1| + |\Theta_b| + |q|)\mathcal{L}^b \Phi_M\|_{L^\infty(y \leq 2M)}
\]
\[ (b_L)_s + (2L - \gamma)b_1b_L \leq M^C \varepsilon_{2k}. \]  

(7.11)

Substituting (7.5), (7.6), (7.7), (7.9), (7.10) and (7.11) into (7.3), we have

\[ |(b_L)_s + (2L - \gamma)b_1b_L| \leq \frac{\sqrt{\varepsilon_{2k}}}{\varepsilon_{2k}^{1/2}} + b_1^{L+3} + M^C b_1 D(t). \]  

(7.12)

Next we take inner product of (5.3) with \( \mathcal{L}^k \Phi_M \) for \( 1 \leq k \leq L - 1 \), we get

\[ \langle \text{Mod}(t), \mathcal{L}^k \Phi_M \rangle = -\langle \mathcal{L}^k \tilde{\Psi}_b, \Phi_M \rangle - \langle \frac{\lambda_s}{\lambda} \Lambda q - \mathcal{H}(q) + N(q), \mathcal{L}^k \Phi_M \rangle, \]

then similar to the derivation of (7.12), we have

\[ |(b_k)_s + (2k - \gamma)b_1b_k - b_{k+1}| \leq b_1^{L+3} + M^C b_1 (\sqrt{\varepsilon_{2k}} + D(t)). \]  

(7.13)

Then we take inner product of (5.3) with \( \Phi_M \), getting

\[ \langle \text{Mod}(t), \Phi_M \rangle = -\langle \tilde{\Psi}_b, \Phi_M \rangle - \langle \frac{\lambda_s}{\lambda} \Lambda q - \mathcal{H}(q) + N(q), \Phi_M \rangle. \]

Note that by above estimates,

\[ \langle \text{Mod}(t), \Phi_M \rangle = -\langle \frac{\lambda_s}{\lambda} + b_1 \rangle \langle \Lambda Q, \chi_M \Lambda Q \rangle + O(M^C b_1 D(t)). \]

Thus again similar to the derivation of (7.12), we see that

\[ \left| \frac{\lambda_s}{\lambda} + b_1 \right| \leq b_1^{L+3} + M^C b_1 (\sqrt{\varepsilon_{2k}} + D(t)). \]  

(7.14)

Now we sum up (7.12)-(7.14) and apply (6.7) to get

\[ D(t) \leq \frac{\sqrt{\varepsilon_{2k}}}{\varepsilon_{2k}^{1/2}} + b_1^{L+1+\left(1-\delta\right)+1+\frac{C_L}{2} \eta}, \]  

(7.15)

then substituting (7.15) back into (7.12)-(7.14), we get (7.1) and (7.2), this concludes the proof.

**Remark 7.1.** We remark that by (7.1) and (3.3), one has \(|(b_1)_s| \leq b_1^2\), this justifies the additional assumption in Proposition 3.2.

For the \( L \)-th equation in the \( b \)-system, one needs improved modulation estimate in order to close bootstrap concerning stable modes, and we establish it as follows.

**Proposition 7.2.** Under the assumptions in Proposition 7.1, we have for all \( s \in [s_0, s_1] \),

\[ (b_L)_s + (2L - \gamma)b_1b_L + (-1)^L \partial_s \left( \frac{\mathcal{L}^L q, \chi_{B_0} \Lambda Q}{\chi_{B_0} \Lambda Q, \Lambda Q} \right) \leq \frac{1}{b_0^{2s}} \left( \sqrt{\varepsilon_{2k}} + b_1^{L+1+\left(1-\delta\right)-C_L \eta} \right). \]  

(7.16)

For certainty, we assume \( L \gg 1 \) is an even integer.

**Proof.** Now we take inner product of (5.3) with \( \mathcal{L}^L (\chi_{B_0} \Lambda Q) \), and get

\[ \langle \chi_{B_0} \Lambda Q, \Lambda Q \rangle \left\{ \frac{d}{ds} \left[ \frac{\langle \mathcal{L}^L q, \chi_{B_0} \Lambda Q \rangle}{\chi_{B_0} \Lambda Q, \Lambda Q} \right] - \langle \mathcal{L}^L q, \chi_{B_0} \Lambda Q \rangle \frac{d}{ds} \left[ \frac{1}{\chi_{B_0} \Lambda Q, \Lambda Q} \right] \right\} \]

\[ = \langle \mathcal{L}^L q, \Lambda Q \partial_s \chi_{B_0} \rangle - \langle \mathcal{L}^{L+1} q, \chi_{B_0} \Lambda Q \rangle + \frac{\lambda_s}{\lambda} \langle \mathcal{L}^L \Lambda Q, \chi_{B_0} \Lambda Q \rangle \]

\[ - \langle \mathcal{L}^L \tilde{\Psi}_b, \chi_{B_0} \Lambda Q \rangle - \langle \mathcal{L}^L \text{Mod}(t), \chi_{B_0} \Lambda Q \rangle + \langle \mathcal{L}^L (\mathcal{H}(q) - \mathcal{N}(q)), \chi_{B_0} \Lambda Q \rangle. \]  

(7.17)

By Hölder and Lemma 2.7,

\[ |\langle \mathcal{L}^L q, \chi_{B_0} \Lambda Q \rangle| \lesssim \left( \int_{y \leq 2B_0} \frac{\mathcal{L}^L q^2}{y^4 + y^{4L+4}} \right)^{\frac{1}{2}} \| \chi_{B_0} \Lambda Q \|_{L^2} \| b_0^{2s} \|_{L^2} \leq b_0^{2s-\gamma+2b_0+2} \sqrt{\varepsilon_{2k}}, \]
where we also used the fact that \( B_0^{d-2-2\gamma} \lesssim \langle \Lambda Q, \chi B_0 \Lambda Q \rangle \lesssim B_0^{d-2-2\gamma} \). Note also that \(|\partial_s \chi B_0| = \frac{|y \partial_y b_0 \chi(\frac{y}{B_0})|}{2B_0 b_0} \lesssim 1 \) for \( y \leq 2B_0 b_0 \), then it follows

\[
\left| \langle \mathcal{L}^L q, \chi B_0 \Lambda Q \rangle \frac{d}{ds} \left[ \frac{1}{\langle \Lambda Q, \chi B_0 \Lambda Q \rangle} \right] \right| = \left| -\frac{\langle \mathcal{L}^L q, \chi B_0 \ Lambda Q \rangle}{\langle \Lambda Q, \chi B_0 \Lambda Q \rangle^2} \langle \Lambda Q, \Lambda Q \partial_s \chi B_0 \rangle \right|
\leq \frac{B_0^{d-2-\gamma+2h-2}}{B_0^{2(d-2)-4\gamma}} \varepsilon_{2k} b_1 \int_{B_0 \leq y \leq 2B_0} y^{-2\gamma+d-3} dy
\lesssim \frac{\varepsilon_{2k}}{B_0^2}.
\]

(7.18)

Again by Hölder and Lemma 2.7, we estimate

\[
|\langle \mathcal{L}^L q, \Lambda Q \partial_s \chi B_0 \rangle| \lesssim \left( \int \frac{|\mathcal{L}^L q|^2}{y^4 + y^{4h+4}} \right)^{\frac{1}{2}} \left( \int_{B_0 \leq y \leq 2B_0} (y^4 + y^{4h+4})|\Lambda Q|^2 \right)^{\frac{1}{2}} \left| \partial_s b_1 \right| b_1
\lesssim B_0^{d-2-\gamma+2h} \sqrt{\varepsilon_{2k}},
\]

(7.19)

and

\[
|\langle \mathcal{L}^{L+1} q, \chi B_0 \Lambda Q \rangle| \lesssim \left( \int \frac{|\mathcal{L}^{L+1} q|^2}{y^4 + y^{4h}} \right)^{\frac{1}{2}} \left( \int_{y \leq 2B_0} (y^4 + y^{4h})|\chi B_0 \Lambda Q|^2 \right)^{\frac{1}{2}}
\lesssim B_0^{d-2-\gamma+2h} \sqrt{\varepsilon_{2k}}.
\]

(7.20)

By Hölder, (7.1) and (7.8), we see that

\[
\left| \frac{\Lambda}{\chi} \langle \mathcal{L}^L q, \chi B_0 \Lambda Q \rangle \right| \lesssim b_1 \left( \int \frac{|\partial_y y|^2}{y^4 + y^{4(L+h)+2}} \right)^{\frac{1}{2}} \left( \int (y^6 + y^{4(L+h)+4}) |\mathcal{L}^L (\chi B_0 \Lambda Q)|^2 \right)^{\frac{1}{2}}
\lesssim B_0^{d-2-\gamma+2h} \sqrt{\varepsilon_{2k}}.
\]

(7.21)

By Hölder and (3.20) with \( m = L \), we have

\[
|\langle \mathcal{L}^L \tilde{q}_b, \chi B_0 \Lambda Q \rangle| \lesssim \left( \int_{y \leq 2B_0} \frac{|\tilde{q}_b|^2}{1 + y^{4(h+L+1)}} \right)^{\frac{1}{2}} \left( \int (1 + y^{4(h+L+1)}) |\mathcal{L}^L (\chi B_0 \Lambda Q)|^2 \right)^{\frac{1}{2}}
\lesssim B_0^{d-2-\gamma+2h} \varepsilon_{L+1}^{L+1-\gamma} - cL \eta.
\]

(7.22)

Next we estimate the term \( \langle \mathcal{L}^L (\mathcal{H}(q) - \mathcal{N}(q)), \chi B_0 \Lambda Q \rangle \). Considering under the condition \( y \leq 2B_0 \), we get \(|f'(\tilde{q}_b) - f'(Q)| \lesssim |Q \Theta_b + \Theta_b + \Theta_b^2|\), where \(|\Theta_b| \lesssim \sum_{k=1}^{L} b_k y^{2k-\gamma} + \sum_{k=2}^{L+2} b_k y^{2(k-1)-\gamma} \ll b_1^2\. Then \(|f'(\tilde{q}_b) - f'(Q)| \lesssim b_1^2 \ll 1\), and hence

\[
|\mathcal{H}(q)| \lesssim \frac{|q|}{y^2}. \quad \text{(a rough bound is enough here)}
\]

(7.23)

Note that by (iv) of Lemma 5.1, when \( 1 \leq y \leq 2B_0 \), \( |q| \lesssim y^{2L+2-2\delta-\gamma} \varepsilon_{2k} \approx b_1^{2+1-\delta-\gamma} \ll 1\), then

\[
|f(\tilde{q}_b + q) - f(\tilde{q}_b)| = |q|^2 (|\tilde{q}_b - 1| + |q|)
\lesssim q^2 (|Q - 1| + |\Theta_b| + |q|) \lesssim |q|^2.
\]

Therefore,

\[
|\mathcal{N}(q)| \lesssim \frac{|q|^2}{y^2} \lesssim \frac{|q|}{y^2}, \quad \text{for } y \leq 2B_0. \quad \text{(a rough bound is enough here)}
\]

(7.24)
By (7.23), (7.24), Hölder and (7.8), we see that
\[
|\langle \mathcal{L}^L(\mathcal{H}(q) - \mathcal{N}(q)), \chi_{B_0\Lambda Q} \rangle| \lesssim \int \frac{|q|}{y^2} |\mathcal{L}^L(\chi_{B_0\Lambda Q})| \\
\lesssim \left( \int \frac{|q|^2}{y^{6} + y^{4k}} \right)^{\frac{1}{2}} \left( \int (y^{2} + y^{4k-4}) |\mathcal{L}^L(\chi_{B_0\Lambda Q})|^2 \right)^{\frac{1}{2}} \\
\lesssim B_0^{d_{2k}^\gamma + 2h} \sqrt{\theta_{2k}}. \tag{7.25}
\]

It remains to estimate \( \langle \mathcal{L}^L \mathcal{M}od(t), \chi_{B_0\Lambda Q} \rangle \), direct computation gives
\[
\langle \mathcal{L}^L \mathcal{M}od(t), \chi_{B_0\Lambda Q} \rangle = (-1)^L \langle \Lambda Q, \chi_{B_0\Lambda Q} \rangle [(b_L)^2 + (2L - \gamma)b_1b_L] \\
- \left( \frac{\lambda_s}{\lambda} + b_1 \right) \langle \Lambda \Theta_b, \mathcal{L}^L(\chi_{B_0\Lambda Q}) \rangle \tag{7.26}
+ \left( \sum_{k=1}^{L} [(b_k)^2 + (2k - \gamma)b_1b_k - b_{k+1}] \sum_{j=k+1}^{L+2} \frac{\partial S_j}{\partial b_k}, \mathcal{L}^L(\chi_{B_0\Lambda Q}) \right).
\]

When \( y \leq 2B_0 \), we have \( b_1y^2 \lesssim 1 \), then
\[
\begin{aligned}
\sum_{k=1}^{L} |b_k\Lambda T_k| &\lesssim \sum_{k=1}^{L} b_k^2y^{2k-\gamma} \lesssim b_1y^{2-\gamma} \\
\sum_{k=2}^{L+2} |\Lambda S_k| &\lesssim \sum_{k=2}^{L+2} b_k^2y^{2(k-1)-\gamma} \lesssim b_1^2y^{2-\gamma}
\end{aligned}
\implies |\Lambda \Theta_b| \lesssim b_1y^{2-\gamma},
\]
and
\[
\sum_{j=k+1}^{L+2} \frac{\partial S_j}{\partial b_k} \lesssim \sum_{j=k+1}^{L+2} b_{j-k}^{-1}y^{2(j-1)-\gamma} \lesssim b_1y^{2k-\gamma}.
\]

Then by Proposition 7.1,
\[
\left| \frac{\lambda_s}{\lambda} + b_1 \right| |\langle \Lambda \Theta_b, \mathcal{L}^L(\chi_{B_0\Lambda Q}) \rangle| \lesssim b_1^{L+1+\delta(1+\eta)} \int_0^{2B_0} b_1y^{2-\gamma} \cdot y^{-\gamma-2L+d-3} \, dy \\
\lesssim b_1^{2L+1+(\delta)(1+\eta)} B_0^{d-2-2\gamma}, \tag{7.27}
\]
and
\[
\sum_{k=1}^{L} [(b_k)^2 + (2k - \gamma)b_1b_k - b_{k+1}] \left| \left\langle \sum_{j=k+1}^{L+2} \frac{\partial S_j}{\partial b_k}, \mathcal{L}^L(\chi_{B_0\Lambda Q}) \right\rangle \right| \\
\lesssim \left( \frac{\sqrt{\theta_{2k}}}{M^{2d}} + b_1^{L+1+\delta(1+\eta)} \right) \int_0^{2B_0} b_1y^{2L-\gamma} \cdot y^{-\gamma-2L+d-3} \, dy \\
\lesssim \left( \frac{\sqrt{\theta_{2k}}}{M^{2d}} + b_1^{L+1+\delta(1+\eta)} \right) b_1B_0^{d-2-2\gamma}. \tag{7.28}
\]

Now substituting (7.27) and (7.28) into (7.26), then gathering the estimates (7.18)-(7.22), (7.25) and (7.26) into (7.17) and dividing \((-1)^L\langle \Lambda Q, \chi_{B_0\Lambda Q} \rangle\), we would get (7.16), this concludes the proof.
8. Energy estimates

**Proposition 8.1.** Under the assumptions in Proposition 7.1, there are monotonicity formulas as follows. For high energy, we have
\[
\frac{d}{dt} \left\{ \frac{\mathcal{E}_{2k}}{\lambda^{4k-d+2}} [1 + O(b_1^{(1-\delta)})] \right\} \lesssim \frac{b_1}{\lambda^{4k-d+4}} \left[ b_1^{L+(1-\delta)(1+\eta)} \sqrt{\mathcal{E}_{2k}} + \frac{\mathcal{E}_{2k}}{M^{2\delta}} + v_1^{2L+2(1-\delta)(1+\eta)} \right].
\] (8.1)

And for lower energy, for all \( h+2 \leq m \leq k-1 \), we have
\[
\frac{d}{dt} \left\{ \frac{\mathcal{E}_{2m}}{\lambda^{4m-d+2}} [1 + O(b_1)] \right\} \lesssim \frac{b_1}{\lambda^{2m-d+4}} \left[ b_1^{m-h+1+(1-\delta)-C\eta} \sqrt{\mathcal{E}_{2m}} + b_1^{2(m-h+1)+2(1-\delta)-C\eta} \right].
\] (8.2)

**Proof.** Some aspects of this proof are parallel to the proof of Proposition 4.4 in [12], thus we shall omit some details. For simplicity, we shall only prove (8.1). For convenience, we abuse notation by abbreviating \( v_k^* \) as \( v_k \) for \( k \in \mathbb{N} \).

Firstly, we set up the energy identity. Acting \( \mathcal{L}^{k-1}_\lambda \) on equation (5.7), we have
\[
\partial_t v_{2k-2} + \mathcal{L}_{\lambda} v_{2k-2} = [\partial_t, \mathcal{L}^{k-1}_\lambda] v + \mathcal{L}^{k-1}_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right).
\] (8.3)

Then acting \( \mathcal{A}_\lambda \) on (8.3), we get
\[
\partial_t v_{2k-1} + \tilde{\mathcal{L}} v_{2k-1} = \frac{\partial V_\lambda}{r} v_{2k-2} + \mathcal{A}_\lambda [\partial_t, \mathcal{L}^{k-1}_\lambda] v + \mathcal{A}_\lambda \mathcal{L}^{k-1}_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right).
\] (8.4)

Making use of (8.3) and (8.4), we still have the energy identity as
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\mathcal{E}_{2k}}{\lambda^{4k-d+2}} + 2 \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-1} v_{2k-2} \right\}
\]
\[
= - \int |\tilde{\mathcal{L}} v_{2k-1}|^2 - \left( \frac{\lambda^s}{\lambda} + b_1 \right) \int \frac{(\Lambda \tilde{Z})_\lambda}{2\lambda^2 r^2} v_{2k-1}^2 - \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-2} \tilde{\mathcal{L}} v_{2k-1}
\]
\[
+ \int \frac{d}{dt} \left( \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-1} v_{2k-2} \right) + \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-1} \left[ [\partial_t, \mathcal{L}^{k-1}_\lambda] v + \mathcal{L}^{k-1}_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) \right]
\]
\[
+ \int \left( \tilde{\mathcal{L}} v_{2k-1} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-2} \right) \left( \frac{\partial V_\lambda}{r} v_{2k-2} + \mathcal{A}_\lambda [\partial_t, \mathcal{L}^{k-1}_\lambda] v + \mathcal{A}_\lambda \mathcal{L}^{k-1}_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) \right).
\] (8.5)

Now we estimate terms in (8.5). Note that by Lemma 2.7, we get
\[
\mathcal{E}_{2k}(q) \gtrsim \int \frac{|q_{2k-1}|^2}{y^2} + \sum_{j=0}^{k-1} \int y^4 (1 + y^4 k_{j-1}) + \sum_{j=0}^{k-2} \int y^4 (1 + y^4 (k-2-j)).
\] (8.6)

On the second term on the LHS of (8.5). Note that by (2.3),
\[
|\Lambda V(y)| \lesssim \begin{cases} y^2 & \text{as } y \to 0 \\ y^{-2} & \text{as } y \to \infty \end{cases} \implies |\Lambda V| \lesssim \frac{y^2}{1 + y^4}.
\]

Then by H"older and (8.6), we estimate
\[
\int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-1} v_{2k-2} = \frac{1}{\lambda^{4k-d+2}} \int \frac{b_1(\Lambda V)}{y} q_{2k-1} q_{2k-2}
\]
\[
\lesssim \frac{b_1}{\lambda^{4k-d+2}} \left( \int \frac{|q_{2k-1}|^2}{y^2} \right)^{\frac{1}{2}} \left( \int \frac{|q_{2k-2}|^2}{1 + y^4} \right)^{\frac{1}{2}} \lesssim \frac{b_1}{\lambda^{4k-d+2}} \mathcal{E}_{2k}(q).
\] (8.7)

Note that by (2.3),
\[
\Lambda \tilde{Z} = 2 \Lambda V + (d-2) \Lambda V - \Lambda^2 V \lesssim \begin{cases} y^2 & \text{as } y \to 0 \\ y^{-2} & \text{as } y \to \infty \end{cases} \lesssim \frac{y^2}{1 + y^4}.
\]
Then by (7.1) and (8.6), we have
\[
\left(\frac{\lambda_s}{\lambda} + b_1\right) \int \frac{(\Lambda \tilde{Z})_\lambda}{\lambda^2 r^2} v_{2k-1}^2 \leq \left(\frac{\lambda_s}{\lambda} + b_1\right) \frac{1}{\lambda^{4k-d+4}} \int \frac{\Lambda \tilde{Z}}{y^2} q_{2k-1}^2 \leq \frac{b_1^{L+1+\gamma}(C)}{\lambda^{4k-d+4}} \int \frac{q_{2k-1}^2}{y^2} \leq \frac{b_1^{L+1+\gamma}(C)}{\lambda^{4k-d+4}} \mathcal{E}_{2k}(q). \tag{8.8}
\]
Again by (2.3) and (8.6), we get
\[
\left| \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-2} \tilde{Z} \lambda v_{2k-1} \right| \leq \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + \frac{1}{4} \int \frac{b_1^2 |(\Lambda V)_\lambda|^2}{\lambda^4 r^2} v_{2k-2}^2 = \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + \frac{b_1^2}{\lambda^{4k-d+4}} \int \frac{|\Lambda V(y)|^2}{y^2} q_{2k-2}^2 \leq \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-2}|^2 + \frac{C b_1^2}{\lambda^{4k-d+4}} \mathcal{E}_{2k}. \tag{8.9}
\]
Note that by (7.1), we derive
\[
\left| \frac{d}{dt} \left( \frac{b_1(\Lambda V)_\lambda}{\lambda^2} \right) \right| = \left| \frac{b_1}{\lambda} (\Lambda V)_\lambda - \frac{b_1}{\lambda^4} \lambda (\Lambda^2 V)_\lambda \right| \leq \frac{b_1^2}{\lambda^4} \left( |(\Lambda V)_\lambda| + |(\Lambda^2 V)_\lambda| \right) \tag{8.10}
\]
Then again by (2.3), Hölder and (8.6), we have
\[
\left| \int \frac{d}{dt} \left( \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} \right) v_{2k-1} v_{2k-2} \right| \leq \frac{b_1^2}{\lambda^{4k-d+4}} \int \frac{|\Lambda V| + |\Lambda^2 V|}{y} q_{2k-1} q_{2k-2} \leq \frac{b_1^2}{\lambda^{4k-d+4}} \left( \frac{|q_{2k-1}|^2}{y^2} \right)^{\frac{1}{2}} \left( \frac{|q_{2k-2}|^2}{y^4} \right)^{\frac{1}{2}} \leq \frac{b_1^2}{\lambda^{4k-d+4}} \mathcal{E}_{2k}. \tag{8.11}
\]
Note that by (7.1),
\[
\partial_t V_\lambda = -\frac{\lambda_s}{\lambda} V_\lambda \implies \left| \frac{\partial_t V_\lambda}{r} \right| \leq \frac{b_1}{\lambda^2} |(\Lambda V)_\lambda|. \tag{8.12}
\]
Then again using (2.3) and (8.6), we compute
\[
\left| \int \left( \frac{\tilde{Z}_\lambda v_{2k-1}}{\lambda^2 r} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-2} \right) \frac{\partial_t V_\lambda}{r} v_{2k-2} \right| \leq \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + C \int \left( \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} \right)^2 v_{2k-2}^2 = \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + C \frac{b_1^2}{\lambda^{4k-d+4}} \int \frac{|\Lambda V|^2}{y^2} q_{2k-2}^2 \leq \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + C \frac{b_1^2}{\lambda^{4k-d+4}} \int \frac{q_{2k-2}^2}{y^4} \leq \frac{1}{4} \int |\tilde{Z} \lambda v_{2k-1}|^2 + C \frac{b_1^2}{\lambda^{4k-d+4}} \mathcal{E}_{2k}(q). \tag{8.11}
\]
Similar to the estimate of (8.11), we have
\[
\left| \int \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-1} \left| \partial_t, \tilde{Z}_\lambda^k \right| v \right| + \left| \int \left( \frac{\tilde{Z} \lambda v_{2k-1}}{\lambda^2 r} + \frac{b_1(\Lambda V)_\lambda}{\lambda^2 r} v_{2k-2} \right) \partial_t \left| \partial_t, \tilde{Z}_\lambda^k \right| v \right| \leq C \left[ \int \frac{b_1^2}{\lambda^2} v_{2k-1}^2 + \frac{1}{\lambda^2} \left| \partial_t, \tilde{Z}_\lambda^k \right|^2 |(\Lambda V)_\lambda|^2 \right] + \int \left| \partial_t \left| \partial_t, \tilde{Z}_\lambda^k \right| v \right|^2
\]
(8.12) we claim that
\[
\int \frac{||\partial_t, \mathcal{L}^{-1}\mathcal{K}||^2}{\lambda^2(1+y^2)} + \int |\mathcal{A}[\partial_t, \mathcal{L}^{-1}\mathcal{K}]v|^2 \lesssim \frac{b^2_{1}}{\lambda^{4k-d+4}} \varepsilon_{2k}. \tag{8.13}
\]
We changed $y^4$ into $y^2$ so that the two integrals in (8.13) are of the same order. Let us prove this claim. Note that
\[
[\partial_t, \mathcal{L}^{-1}\mathcal{K}]g = \sum_{m=0}^{k-2} \mathcal{L}^m[\partial_t, \mathcal{L}]\mathcal{L}^{-k-m}g,
\]
for any $k \geq 2$ and any smooth radial function $g$. One can prove it by induction on $k$, we shall omit the details. Then
\[
[\partial_t, \mathcal{L}^{-1}\mathcal{K}]v = \sum_{m=0}^{k-2} \mathcal{L}^m \left( \frac{\partial_t Z_{\lambda}}{r^2} \mathcal{L}^{-k-2-m}v \right), \quad \text{with} \quad \frac{\partial_t Z_{\lambda}}{r^2} = -\frac{s_{\lambda}}{\lambda^{2}}. \tag{8.14}
\]
By (7.1), we deduce that
\[
\int \frac{||\partial_t, \mathcal{L}^{-1}\mathcal{K}||^2}{\lambda^2(1+y^2)} \lesssim \frac{b^2_{1}}{\lambda^{4k-d+4}} \sum_{m=0}^{k-2} \int \frac{1}{1+y^2} \left| \mathcal{L}^m \left( \frac{\Lambda Z}{y^2} \mathcal{L}^{-k-2-m}q \right) \right|^2.
\]
When $m = 0$, note that
\[
\left| \frac{\Lambda Z}{y^2} \right| = \left| \frac{(d-2)f'(Q)\Lambda Q}{y^2} \right| \lesssim \frac{1}{y^{2\gamma+2}} \lesssim \frac{1}{1+y^2}.
\]
Then by (8.6),
\[
\int \frac{1}{1+y^2} \left| \frac{\Lambda Z}{y^2} \mathcal{L}^{-k-2}q \right|^2 \lesssim \int \frac{q^2_{2k-4}}{1+y^{10}} \lesssim \varepsilon_{2k}(q).
\]
When $1 \leq m \leq k-2$, by (2.13) with $\phi = \frac{\Lambda Z}{y^2}$, $g = \mathcal{L}^{k-2-m}q$, we have
\[
\mathcal{L}^m \left( \frac{\Lambda Z}{y^2} \mathcal{L}^{-k-2-m}q \right) = \sum_{i=0}^{m} \mathcal{L}^{k-2-(m-i)} q \left( \frac{\Lambda Z}{y^2} \right)_{2m,2i} + \sum_{i=0}^{m-1} \mathcal{A} \mathcal{L}^{k-2-(m-i)} q \left( \frac{\Lambda Z}{y^2} \right)_{2m,2i+1} \lesssim \sum_{i=0}^{m} \frac{q^{2}_{2(k-2-(m-i))}}{1+y^{2\gamma+2+(m-i)}} + \sum_{i=0}^{m-1} \frac{q^{2}_{2(k-2-(m-i))+1}}{1+y^{2\gamma+2+(m-i)-1}},
\]
where we also used the fact that
\[
\left| \left( \frac{\Lambda Z}{y^2} \right)_{2m,i} \right| \lesssim \frac{1}{y^{2\gamma+2+2m-i}}, \quad \text{for} \ 0 \leq i \leq 2m.
\]
Then by (8.6), we see that
\[
\int \frac{1}{1+y^2} \left| \mathcal{L}^m \left( \frac{\Lambda Z}{y^2} \mathcal{L}^{-k-2-m}q \right) \right|^2 \lesssim \sum_{i=0}^{m} \int \frac{q^{2}_{2(k-2-(m-i))}}{1+y^{4\gamma+6+4(m-i)}} + \sum_{i=0}^{m-1} \int \frac{q^{2}_{2(k-2-(m-i))+1}}{1+y^{4\gamma+4+(m-i)}} \lesssim \sum_{i=0}^{m} \int \frac{q^{2}_{2(k-2-(m-i))}}{y^{4}(1+y^{4\gamma+6+4(m-i)})} + \sum_{i=0}^{m-1} \int \frac{q^{2}_{2(k-2-(m-i))+1}}{y^{6}(1+y^{4\gamma+4+(m-i)})} \lesssim \varepsilon_{2k}(q).
\]
This concludes the proof of (8.13).
Again by (2.3), Hölder and (6.6), we get
\[
\left| \int \frac{b_1(\Lambda \psi)}{\lambda^2 \gamma} v_{2k-1} \mathcal{L}_\lambda^{k-1} \left( \frac{1}{\lambda^2} F_\lambda \right) \right| = \frac{b_1}{\lambda^{4k-d+4}} \int \frac{\Lambda \psi}{\gamma} q_{2k-1} \mathcal{L}_\lambda^{k-1} F \\
\lesssim \frac{b_1}{\lambda^{4k-d+4}} \left( \int \frac{q_{2k-1}}{\gamma^2} \right)^{\frac{1}{2}} \left( \int \frac{|\mathcal{L}_\lambda^{k-1} F|^2}{1 + y^4} \right)^{\frac{1}{2}} \\
\lesssim \frac{b_1}{\lambda^{4k-d+4}} \sqrt{E_{2k}} \left( \int \frac{|\mathcal{L}_\lambda^{k-1} F|^2}{1 + y^4} \right)^{\frac{1}{2}}.
\] (8.15)

Similar to the estimate (8.15), we have
\[
\left| \int \frac{b_1(\Lambda \psi)}{\lambda^2 \gamma} v_{2k-2} \mathcal{L}_\lambda^{k-1} \left( \frac{1}{\lambda^2} F_\lambda \right) \right| \lesssim \frac{b_1}{\lambda^{4k-d+4}} \sqrt{E_{2k}} \left( \int \frac{|\mathcal{L}_\lambda^{k-1} F|^2}{1 + y^4} \right)^{\frac{1}{2}}.
\] (8.16)

On the term \( \int \mathcal{L}_\lambda v_{2k-1} \mathcal{L}_\lambda^{k-1} \left( \frac{1}{\lambda^2} F_\lambda \right) \). Denoting
\[
\xi_L := \frac{\langle \mathcal{L}_\lambda q, \psi \rangle}{\langle \psi, \Lambda Q \rangle} \mathcal{T}_L,
\]
we set up the decomposition
\[
\mathcal{F} := \partial_t \xi_L + \mathcal{F}_0 + \mathcal{F}_1,
\]
where \( \mathcal{F}_0 := -\widetilde{\psi}_b - \text{Mod} - \partial_t \xi_L \) and \( \mathcal{F}_1 := \mathcal{H}(q) - \mathcal{N}(q) \).

Then by Hölder, we obtain
\[
\int \mathcal{L}_\lambda v_{2k-1} \mathcal{L}_\lambda^{k-1} \left( \frac{1}{\lambda^2} F_\lambda \right) = \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}_\lambda q_{2k-1} \mathcal{L}_\lambda^{k-1} (\partial_t \xi_L + \mathcal{F}_0 + \mathcal{F}_1) \\
= \frac{1}{\lambda^{4k-d+4}} \left( \int \mathcal{L}_\lambda q_{2k-1} \mathcal{L}_\lambda^{k-1} (\partial_t \xi_L) + \int \mathcal{L}_\lambda q_{2k-1} \mathcal{L}_\lambda^{k-1} \mathcal{F}_0 + \int \mathcal{L}_\lambda q_{2k-1} \mathcal{L}_\lambda^{k-1} \mathcal{F}_1 \right) \\
\leq \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}_\lambda q \mathcal{L}_\lambda^{k-1} (\partial_t \xi_L) + \frac{1}{\lambda^{4k-d+4}} \left( \int |\mathcal{L}_\lambda q|^2 \right)^{\frac{1}{2}} \left( \int |\mathcal{L}_\lambda \mathcal{F}_0|^2 \right)^{\frac{1}{2}} \\
+ \frac{1}{8} \mathcal{L}_\lambda q_{2k-1}^2 + \frac{2}{\lambda^{4k-d+4}} \int |\mathcal{L}_\lambda \mathcal{L}_\lambda^{k-1} \mathcal{F}_1|^2 \\
\leq \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}_\lambda q \mathcal{L}_\lambda^{k-1} (\partial_t \xi_L) + \frac{1}{8} \int |\mathcal{L}_\lambda v_{2k-1}|^2 + \frac{C}{\lambda^{4k-d+4}} \left( \sqrt{E_{2k}} \|\mathcal{L}_\lambda \mathcal{F}_0\|_{L^2} + \|\mathcal{L}_\lambda^{k-1} \mathcal{F}_1\|_{L^2}^2 \right).
\] (8.17)

Now substituting (8.13) into (8.12), then gathering estimates (8.8)-(8.12) and (8.15)-(8.17) into (8.5), we have
\[
\frac{1}{2} \frac{d}{dt} \left\{ \frac{\mathcal{E}_{2k}}{\lambda^{4k-d+2}} (1 + O(b_1)) \right\} \leq -\frac{1}{8} \int |\mathcal{L}_\lambda v_{2k-1}|^2 + \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}_\lambda q \mathcal{L}_\lambda^{k-1} (\partial_t \xi_L) + \frac{C b_1^2}{\lambda^{4k-d+4}} \mathcal{E}_{2k} \\
+ \frac{C b_1}{\lambda^{4k-d+4}} \sqrt{\mathcal{E}_{2k}} \left[ \left( \int \frac{|\mathcal{L}_\lambda^{k-1} \mathcal{F}|^2}{1 + y^4} \right)^{\frac{1}{2}} + \left( \int \frac{|\mathcal{L}_\lambda^{k-1} \mathcal{F}|^2}{1 + y^3} \right)^{\frac{1}{2}} \right] \\
+ \frac{C}{\lambda^{4k-d+4}} \left( \sqrt{\mathcal{E}_{2k}} \|\mathcal{L}_\lambda \mathcal{F}_0\|_{L^2} + \|\mathcal{L}_\lambda^{k-1} \mathcal{F}_1\|_{L^2}^2 \right).
\] (8.18)

Note that in (8.18), integrals containing \( \mathcal{F} \) can be controlled by corresponding integrals containing \( \widetilde{\psi}_b, \text{Mod}, \mathcal{H}(q), \mathcal{N}(q) \). The integral containing \( \mathcal{F}_0 \) can be controlled by corresponding integral containing \( \widetilde{\psi}_b, \text{Mod} := \text{Mod} + \partial_t \xi_L \). The integral containing \( \mathcal{F}_1 \) can be controlled by corresponding integral containing \( \mathcal{H}(q), \mathcal{N}(q) \). Then we shall proceed to estimate terms in (8.18) with such further decompositions.
Next we estimate $\tilde{\Psi}_b$ term in (8.18). Applying (3.18), we see that
\[
\left( \int \frac{\left| \mathcal{L}^{k-1} \tilde{\Psi}_b \right|^2}{1 + y^2} \right)^{\frac{1}{2}} + \left( \int \frac{\left| \mathcal{L}^{k-1} \tilde{\Psi}_\theta \right|^2}{1 + y^4} \right)^{\frac{1}{2}} + \left\| \mathcal{L}^k \tilde{\Psi}_b \right\|_{L^2} \lesssim b_1^{L+1+(1-\delta)(1+\eta)}.
\] (8.19)

Then we estimate $\text{Mod}$ term in (8.18), we claim
\[
\left( \int \left| \mathcal{L}^{k-1} \text{Mod} \right|^2 \frac{1}{1 + y^2} \right)^{\frac{1}{2}} + \left( \int \left| \mathcal{L}^{k-1} \text{Mod} \right|^2 \frac{1}{1 + y^4} \right)^{\frac{1}{2}} \lesssim b_1^{(1-\delta)(1+\eta)} \left( \frac{\sqrt{\varepsilon_{2k}}}{M^{2k}} + b_1^{L+1+(1-\delta)(1+\eta)} \right).
\] (8.20)
It suffices to estimate the second term on the LHS, we shall omit the proof since it is a straight consequence of Proposition 7.1 and degrees of $T_i$ and $S_i$.

Next we estimate $\text{Mod}$ term in (8.18). Claim that
\[
\left( \int \left| \mathcal{L}^k \text{Mod} \right|^2 \right)^{\frac{1}{2}} \lesssim b_1 \left( \frac{\sqrt{\varepsilon_{2k}}}{M^{2k}} + b_1^{(1-\delta)} \sqrt{\varepsilon_{2k}} + b_1^{L+1+(1-\delta)(1+\eta) - cL} \right).
\] (8.21)

We further write
\[
\text{Mod} = -\left( \frac{\lambda}{\lambda} + b_1 \right) \Lambda \tilde{Q}_b + \sum_{i=1}^{L-1} \left[ (b_i)_s + (2i - \gamma) b_1 b_i - b_{i+1} \right] \tilde{T}_i
\]
\[+ \sum_{i=1}^{L} \left[ (b_i)_s + (2i - \gamma) b_1 b_i - b_{i+1} \right] \sum_{j=i+1}^{L+2} \Lambda \frac{\partial S_j}{\partial b_i}
\]
\[+ \left[ (b_L)_s + (2L - \gamma) b_1 b_L + \partial_s \left\{ \frac{\left\langle \mathcal{L} \tilde{q}, \chi_{B_0} \Lambda Q \right\rangle}{\Lambda Q, \chi_{B_0} \Lambda Q} \right\} \right] \tilde{T}_L + \left\langle \frac{\mathcal{L} \tilde{q}, \chi_{B_0} \Lambda Q}{\Lambda Q, \chi_{B_0} \Lambda Q} \right\rangle \partial_s \tilde{T}_L.
\]

Note that straight calculation yields
\[
\int \left| \mathcal{L}^k \Lambda \tilde{Q}_b \right|^2 + \sum_{i=1}^{L} \sum_{j=i+1}^{L+2} \left\| \mathcal{L}^k \left( \frac{\partial S_j}{\partial b_i} \right) \right\|^2 \lesssim b_1^2,
\]
\[
\sum_{i=1}^{L-1} \int \left| \mathcal{L}^k \tilde{T}_i \right|^2 \lesssim b_1^{2(2-\delta)(1+\eta)}, \quad \int \left| \mathcal{L}^k \tilde{T}_L \right|^2 \lesssim b_1^{2(1-\delta)(1+\eta)}.
\]
And by the proof of (7.18), it follows that
\[
\left| \frac{\left\langle \mathcal{L} \tilde{q}, \chi_{B_0} \Lambda Q \right\rangle}{\Lambda Q, \chi_{B_0} \Lambda Q} \right| \lesssim b_1^{-(1-\delta)} \sqrt{\varepsilon_{2k}}.
\] (8.22)

Also, in view of $|\partial_s \chi_{B_1}| = \left| \frac{(1+\eta) \gamma}{2 \gamma} \frac{\partial_{B_1} \chi_{B_1}}{\partial_{B_1}} \right| \lesssim 1_{B_1 \leq y \leq 2B_1} b_1$, we have
\[
\int \left| \mathcal{L}^k (\partial_s \tilde{T}_L) \right|^2 \lesssim b_1^2 \int_{B_1 \leq y \leq 2B_1} \frac{y^{d-3}}{y^{(d-L)+2}} dy \lesssim b_1^{2+2(1-\delta)(1+\eta)}.
\]

Then above estimates combined with Proposition 7.1 and Proposition 7.2 gives us (8.21).

Then we estimate $\mathcal{H}(q)$ term in (8.18). Claim
\[
\int \left( \frac{\mathcal{L} \mathcal{L}^{k-1} \mathcal{H}(q)}{1 + y^2} \right)^{\frac{1}{2}} + \int \left( \frac{\mathcal{L} \mathcal{L}^{k-1} \mathcal{H}(q)}{1 + y^4} \right)^{\frac{1}{2}} + \int \left| \mathcal{L} \mathcal{L}^{k-1} \mathcal{H}(q) \right|^2 \lesssim b_1^2 \varepsilon_{2k}.
\] (8.23)

It suffices to estimate the third term on the LHS. Denoting
\[
\mathcal{H}(q) =: \phi q, \quad \text{where } \phi := \frac{-3(d-2)}{y^2} \tilde{\Theta}_b [2(Q-1) + \tilde{\Theta}_b] \text{ and } \tilde{\Theta}_b := \chi_{B_1} \Theta_b.
\]
Using (2.14), we get
\[
\mathcal{A}L^{k-1}H(q) = \sum_{m=0}^{k-1} q_{2m+1} \phi_{2k-1, 2m+1} + q_{2m} \phi_{2k-1, 2m}.
\]
Note that by direct computation,
\[
| (Q - 1) \tilde{\Theta}_b | = \left| \left( \sum_{i=1}^{L} \chi_{B_i} b_i T_i + \sum_{i=2}^{L+2} \chi_{B_i} S_i \right) (Q - 1) \right| \lesssim 1_{y \leq 2B_1} \left( \sum_{i=1}^{L} b_i y^{2i-\gamma} + \sum_{i=2}^{L+2} b_i y^{2(i-1)-\gamma} \right) y^\gamma \lesssim 1_{y \leq 2B_1} b_1 y^{2-\gamma},
\]
where we also used the fact that
\[
(b_1 y^2)^N y^{-\gamma} \lesssim b_1^{\frac{1}{2} - \eta(N - \frac{3}{2})} \ll 1, \text{ for any integer } N \geq 1 \text{ and any } y \leq 2B_1.
\]
Then in general,
\[
| \phi_{k, i} | \lesssim 1_{y \leq 2B_1} \frac{b_1}{1 + y^{\gamma + k-1}}.
\]
Hence combined with (8.6), we derive
\[
\int | \mathcal{A}L^{k-1}H(q) |^2 \lesssim \sum_{m=0}^{k-1} b_1^2 \left( \int_{y \leq 2B_1} \frac{|q_{2m+1}|^2}{1 + y^{2\gamma + 4(k-m)-\delta}} + \int \frac{|q_{2m}|^2}{1 + y^{2\gamma + 4(k-m)-2}} \right)
\]
\[
\lesssim b_1^2 \left( \sum_{m=0}^{k-2} \int \frac{|q_{2m+1}|^2}{y^\delta (1 + y^{4(k-2-m)})} + \int \frac{|q_{2k-1}|^2}{y^2} + \sum_{m=0}^{k-1} \int \frac{|q_{2m}|^2}{y^{2\gamma + 4(k-1-m)}} \right)
\]
\[
\lesssim b_1^2 \mathcal{E}_{2k}(q).
\]
This concludes the proof of (8.23).

Next we estimate \(N(q)\) term in (8.18). Claim
\[
\int | \mathcal{A}L^{k-1}N(q) |^2 \lesssim b_1^{2L+1 + 2(1-\delta)(1+\eta)}
\]  
(8.24)
and
\[
\int \frac{| \mathcal{A}L^{k-1}N(q) |^2}{1 + y^2} + \int \frac{| \mathcal{L}^{k-1}N(q) |^2}{1 + y^4} \lesssim b_1^{2L+2 + 2(1-\delta)(1+\eta)}.
\]  
(8.25)
We shall only prove (8.24) since the proof of (8.25) is similar. Let us estimate the integral for \(y \leq 1\) and \(y \geq 1\) separately.

When \(y < 1\). Rewriting
\[
N(q) = \frac{q^2}{y^2} \phi, \text{ where } \phi := \phi_1 + \phi_2, \ \phi_1 := (d-2)(3Q_b + q), \ \phi_2 := -3(d-2).
\]
By (i) of Lemma 5.1, we get
\[
\frac{q^2}{y^2} = \frac{1}{y^2} \left( \sum_{i=0}^{k-1} c_i T_i(y) + r_q(y) \right)^2 = \sum_{i=0}^{k-1} \tilde{c}_i y^{2i+2} + \tilde{r}_q
\]
with
\[
| \tilde{c}_i | \lesssim \mathcal{E}_{2k}, \ | \tilde{c}_i \tilde{r}_q | \lesssim y^{2k+1-\frac{d}{2}-j} | \ln y|^k \mathcal{E}_{2k}.
\]
By (1.7), Proposition 3.1, (i) of Lemma 5.1 and (iii) of Definition 6.2, we have
\[
\phi_1 = \sum_{i=0}^{k-1} \tilde{c}_i y^{2i+2} + \tilde{r}_q
\]
with
\[
| \tilde{c}_i | \lesssim 1, \ | \tilde{c}_i \tilde{r}_q | \lesssim y^{2k+1-\frac{d}{2}-j} | \ln y|^k.
\]
It immediately follows
\[ \mathcal{N}(q) = \sum_{i=0}^{k-1} \widehat{c}_i y^{2i+2} + \widehat{r}_q \]
with
\[ |\widehat{c}_i| \lesssim \mathcal{E}_{2k} \text{ and } |\partial_y^j \widehat{r}_q| \lesssim y^{2k+1-\frac{j}{2}} \ln y |y|^k \mathcal{E}_{2k}. \]
Hence,
\[
|\mathcal{A} \mathcal{L}^{k-1} \mathcal{N}(q)| = |\mathcal{A} \mathcal{L}^{k-1} (\sum_{i=0}^{k-1} \widehat{c}_i y^{2i+2}) + \mathcal{A} \mathcal{L}^{k-1} \widehat{r}_q| \\
\lesssim \sum_{i=0}^{k-1} |\widehat{c}_i| y^3 + \sum_{i=0}^{2k-1} |\partial_y^j \widehat{r}_q| \lesssim y^{-\frac{j}{2}+2} \ln y |y|^k \mathcal{E}_{2k}.
\]
Then by (iii) of Definition 6.2, we estimate
\[
\int_{y<1} |\mathcal{A} \mathcal{L}^{k-1} \mathcal{N}(q)|^2 \lesssim \mathcal{E}_{2k}^2 \int_{y<1} y |y|^{2k} dy \lesssim \mathcal{E}_{2k}^2 \lesssim b_1^{4[L+(1-\delta)(1+\eta)]}. \tag{8.26}
\]
When \( y \geq 1 \), rewriting
\[ \mathcal{N}(q) = Z^2 \phi, \text{ where } Z := \frac{q}{y}, \phi := (d-2)[3(\tilde{Q}_b-1) + q], \]
By Leibniz rule, we get
\[
\int_{y \geq 1} |\mathcal{A} \mathcal{L}^{k-1} \mathcal{N}(q)|^2 \lesssim \sum_{k=0}^{2k-1} \sum_{y \geq 1} \frac{|\partial_y^k \mathcal{N}(q)|^2}{y^{4k-2k-2}} \\
\lesssim \sum_{k=0}^{2k-1} \sum_{i=0}^{k} \int_{y \geq 1} \frac{|\partial_y^i Z^2|^2 |\partial_y^{k-i} \phi|^2}{y^{4k-2k-2}} \\
\lesssim \sum_{k=0}^{2k-1} \sum_{i=0}^{k} \int_{y \geq 1} \frac{|\partial_y^m Z|^2 |\partial_y^{-m} Z|^2 |\partial_y^{k-i} \phi|^2}{y^{4k-2k-2}}.
\]
Then we focus on proving that for \( 0 \leq k \leq 2k-1, 0 \leq i \leq k, 0 \leq m \leq i, \)
\[ A_{k,i,m} := \int_{y \geq 1} \frac{|\partial_y^m Z|^2 |\partial_y^{-m} Z|^2 |\partial_y^{k-i} \phi|^2}{y^{4k-2k-2}} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)}, \tag{8.27}
\]
which would conclude the proof of (8.24). We shall split the proof into three cases as the following three paragraphs.

When \( k = 0 \). In this case, \( k = i = m = 0 \). Note that \( \phi \) is bounded as \( y \to \infty \), then
\[ A_{0,0,0} = \int_{y \geq 1} \frac{|q|^4 |\phi|^2}{y^{4k+5-d}} dy \lesssim \int_{y \geq 1} \frac{|q|^4}{y^{4k+5-d}} dy + \int_{y > B_0} \frac{|q|^4}{y^{4k+5-d}} dy.
\]
By (iv) of Lemma 5.1, Definition 6.2 and recall that \( d = 4\ell + 4\delta + 2\gamma + 2 \), we have
\[
\int_{1 \leq y \leq B_0} \frac{|q|^4}{y^{4k+5-d}} dy \lesssim \left\| \frac{|y^{-d-4}|q|^2}{y^{2(2k-1)}} \right\|_{L^\infty(y \geq 1)} \left\| \frac{|y^{-d-4}|q|^2}{y^{2(2\ell+2\delta+3)}} \right\|_{L^\infty(y \geq 1)} \int_{1 \leq y \leq B_0} y^{4\ell-4\delta-2\gamma+5} dy \\
\lesssim \mathcal{E}_{2k} \mathcal{E}_{2(l+\delta+2)} B_0^{4\ell-4\delta-2\gamma+6} \lesssim b_1^{1+\gamma-K\eta}+2L+2(1-\delta)(1+\eta). \tag{8.28}
\]
Similarly,
\[
\int_{y > B_0} \frac{|q|^4}{y^{4k+5-d}} dy \lesssim \left\| \frac{|y^{-d-4}|q|^2}{y^{2(2k-2\ell-1)}} \right\|_{L^\infty(y \geq 1)} \left\| \frac{|y^{-d-4}|q|^2}{y^{2(2\ell+2\delta+1)}} \right\|_{L^\infty(y \geq 1)} \int_{y > B_0} y^{-4\delta-2\gamma+1} dy \\
\lesssim \mathcal{E}_{2(k-\ell)} \mathcal{E}_{2(l+\ell+1)} B_0^{4\ell-4\delta-2\gamma+2} \lesssim b_1^{2L+2(1-\delta)(1+\eta)+(1+\gamma)-2(K+1-\delta)\eta}. \tag{8.29}
\]
When \( k \geq 1 \) and \( i = k \). By Leibniz rule,

\[
|\partial^n Z|^2 \lesssim \sum_{j=0}^n \frac{|\partial_j q|^2}{y^{2n-2n-2j}} \quad \text{for all} \quad n \in \mathbb{N}. \quad \Rightarrow \quad A_{k,k,m} \lesssim \sum_{j=0}^m \sum_{l=0}^{k-m} \int_{y \geq 1} \frac{|\partial_j q|^2 |\partial_l q|^2}{y^{4k-2j-2l+2}}.
\]

Direct computation implies

\[
B_{j,l} := \int_{y \geq 1} \frac{|\partial_j q|^2 |\partial_l q|^2}{y^{4k-2j-2l+2}} \, dy
\]

\[
= \int_{1 \leq y \leq B_0} \frac{(y^{d-4} |\partial_j q|^2)(y^{d-4} |\partial_l q|^2)}{y^{4k-2j-2l+4}} \, dy + \int_{y > B_0} \frac{(y^{d-4} |\partial_j q|^2)(y^{d-4} |\partial_l q|^2)}{y^{4k-2j-2l+4}} \, dy
\]

\[
\lesssim \left\| \frac{(y^{d-4} |\partial_j q|^2)(y^{d-4} |\partial_l q|^2)}{y^{4k-2j-2l+4}} \right\|_{L^\infty(y \geq 1)} b_1^{2\delta + \gamma - 4} + \left\| \frac{(y^{d-4} |\partial_j q|^2)(y^{d-4} |\partial_l q|^2)}{y^{4k-2j-2l+4}} \right\|_{L^\infty(y \geq 1)} b_1^{2\delta + \gamma - 4}
\]

\[
=: B_{j,l,j_1,j_2} b_1^{2\delta + \gamma - 4} + B_{j,l,j_1,j_2} b_1^{2\delta + \gamma - 1},
\]

where \( J_1 + J_2 = 2k + 2a + 3, J_3 + J_4 = 2k + 2h \), and those \( J \) are to be determined. For the first right hand side term in (8.30), when \( l \geq 3 \), we choose \( J_1 = 2k - 2l + 3, J_2 = 2h + 2l \). Then by (iv) of Lemma 5.1 and Definition 6.2, we see that

\[
B_{j,l,j_1,j_2} \lesssim \left\| \frac{y^{d-4} |\partial_j q|^2}{y^{2J_1-2}} \right\|_{L^\infty(y \geq 1)} \left\| \frac{y^{d-4} |\partial_l q|^2}{y^{2J_2-2l}} \right\|_{L^\infty(y \geq 1)}
\]

\[
\lesssim \delta J_{l+1} \sqrt{\delta J_{l} \delta J_{l+2}} b_1^{2L-1+3(1-\delta)+4-\frac{1}{2}K\eta + \frac{2}{2\gamma}(2\delta - 2\gamma)} \lesssim b_1^{2L+4(1-\delta)+3-\frac{1}{2}K\eta},
\]

where in the last inequality we used the fact that \( \frac{l}{2\gamma} > \frac{1}{2} \). When \( l = 1 \) or 2, we choose \( J_1 = 2k - 1, J_2 = 2h + 4 \), then similarly one gets for \( l = 2 \),

\[
B_{j,l,j_1,j_2} \lesssim \delta J_{l+1} \sqrt{\delta J_{l} \delta J_{l+2}} \lesssim b_1^{2L+2(1-\delta)(1+\eta)+5-2\delta - \frac{1}{2}K\eta},
\]

and for \( l = 1 \), there holds

\[
B_{j,l,j_1,j_2} \lesssim \delta J_{l+1} \sqrt{\delta J_{l} \delta J_{l+2}} \lesssim b_1^{2L+2(1-\delta)(1+\eta)+5-2\delta - \frac{1}{2}K\eta}.
\]

Hence

\[
B_{j,l,j_1,j_2} \lesssim \left\{ \begin{array}{ll} b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{2}{2\gamma}(4K+2(1-\delta))} & \text{for } l \geq 3, \\
 b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{1}{2}K\eta} & \text{for } l = 2, \\
 b_1^{2L+1+2(1-\delta)(1+\eta)+\gamma K\eta} & \text{for } l = 1. \end{array} \right.
\]

For the second right hand side term in (8.30), we similarly have

\[
B_{j,l,j_1,j_2} \lesssim \delta J_{l+1} \sqrt{\delta J_{l} \delta J_{l+2}} \lesssim b_1^{2L+4(1-\delta)(1+\eta)+\frac{2}{2\gamma}(4K+2(1-\delta))}.
\]

where we choose \( J_3 = 2k - 2l, J_4 = 2h + 2l \). Thus

\[
B_{j,l,j_1,j_2} b_1^{2\delta + \gamma - 1} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{1}{2}K\eta}.
\]

Therefore, injecting the estimates (8.31) and (8.32) into (8.30), we derive the estimate of \( B_{j,l} \) as

\[
B_{j,l} \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{1}{2}K\eta},
\]

which holds for any positive integer \( l \).
When $k \geq 1$ and $i \leq k - 1$. Again by Leibniz rule, we further write

$$A_{k,i,m} \lesssim \sum_{j=0}^{m} \sum_{l=0}^{i-m} \int_{y \geq 1} |\partial_y^j q| |\partial_y^l q| |\partial_y^{k-i} \phi|^2 \frac{1}{y^{4k-2j-2l+2-(k-i)}}.$$  

We shall need pointwise estimates of $\partial_y^n \phi$, for $n \in \mathbb{N}_+$. Note that by degrees of $T_k$ and $S_k$, we derive

$$|\partial_y^n \tilde{Q}_b| = |\partial_y^n \left( Q + \sum_{k=1}^{L} \chi_{B_k} b_k T_k + \sum_{k=2}^{L+2} \chi_{B_k} S_k \right) | \lesssim \frac{1}{y^{\gamma+n}} + \sum_{k=1}^{L} b_k^\eta y^{2k} \lesssim b_1^{-\eta(L+1)}.$$  

By (iv) of Lemma 5.1 and Definition 6.2, we have for $1 \leq y \leq B_0$,

$$|\partial_y^n q|^2 \lesssim y^{2(2k-1-n)} \frac{1}{y^{2k-1-n}} \lesssim y^{2(2k-1-n)-(d-4)} e_{2k} \lesssim b_1^{\eta+2(1-\delta)\eta},$$  

for $y \geq B_0$,

$$|\partial_y^n q|^2 \lesssim y^{2(2k+L+2n+1-n)} \frac{1}{y^{2k+L+2n+1-n}} \lesssim y^{2(2k+L+2n+1-4) e_{2k+2L+2} \lesssim y^{4L+4(1-\delta)+b_1^{\eta+2(1-\delta)+K\eta}}.$$  

Thus

$$|\partial_y^n \phi|^2 \lesssim |\partial_y^n \tilde{Q}_b|^2 + |\partial_y^n q|^2 \lesssim \begin{cases} b_1^{-2(L+1)\eta}, & \text{when } 1 \leq y \leq B_0. \\ b_1^{-CL,Kn\eta+\gamma+2(n(y_1 y_2)^{2L+2(1-\delta)})}, & \text{when } y \geq B_0. \end{cases}$$  

Then similar to the proof of (8.33), we compute

$$A_{k,i,m} \lesssim b_1^{-CL,Kn \eta} \sum_{j=0}^{m} \sum_{l=0}^{i-m} \left( \int_{1 \leq y \leq B_0} y^{2k-2j-2l+2\gamma} y^{d-3} dy \right) + b_1^{\gamma+\alpha} \int_{y \geq B_0} y^{2k-2j-2l+2\gamma} y^{d-3} dy \lesssim b_1^{2L+1+2(1-\delta)(1+\eta)+\frac{2}{3}-C_{L,K,\delta,\eta}},$$  

(8.34)

where $\alpha := k - i + 2l + 2(1 - \delta)$.

In view of (8.28)-(8.34), we conclude the proof of (8.27), then by (8.26) and (8.27), we complete the proof of (8.24).

It remains to estimate the integral $\frac{1}{\chi_{y_{k+1}}} \int \mathcal{L}^k q \mathcal{L}^k (\partial_s \xi_L)$ in (8.18). Let us further write

$$\frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}^k q \mathcal{L}^k (\partial_s \xi_L) = \frac{1}{\lambda^{4k-d+4}} \left( \int \mathcal{L}^k q \mathcal{L}^k \xi_L - \frac{1}{2} \int |\mathcal{L}^k \xi_L|^2 \right) + \frac{(4k-d+4) \lambda_k}{\lambda^{4k-d+4}} \left( \int \mathcal{L}^k q \mathcal{L}^k \xi_L - \frac{1}{2} \int |\mathcal{L}^k \xi_L|^2 \right) - \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}^k (\partial_s q - \partial_s \xi_L) \mathcal{L}^k \xi_L.$$  

(8.35)

Recall the proof of (8.21), we deduce that

$$\int |\mathcal{L}^k \xi_L|^2 \lesssim b_1^{2(1-\delta)\eta} e_{2k}.$$  

(8.36)
Then by H"older, we get
\[ \left| \int \mathcal{L}^k q \mathcal{L}^k \xi_L \right| \lesssim b_1^{1+(\delta)\eta} \mathcal{E}_{2k}. \]

Thus
\[ \frac{d}{ds} \frac{1}{\lambda^{4k-d+4}} \left( \int \mathcal{L}^k q \mathcal{L}^k \xi_L - \frac{1}{2} \int |\mathcal{L}^k \xi_L|^2 \right) = \frac{d}{dt} \left\{ \frac{\mathcal{E}_{2k}}{\lambda^{4k-d+2}} O(b_1^{1+(\delta)\eta}) \right\}, \tag{8.37} \]
and
\[ \left| \frac{(4k-d+4) \lambda}{\lambda^{4k-d+4}} \left( \int \mathcal{L}^k q \mathcal{L}^k \xi_L - \frac{1}{2} \int |\mathcal{L}^k \xi_L|^2 \right) \right| \lesssim b_1^{1+(\delta)\eta} \mathcal{E}_{2k}. \tag{8.38} \]

On the third line on the RHS of (8.35). By (5.3), we further write
\[ \int \mathcal{L}^k (\partial_q \xi - \partial_s \xi_L) \mathcal{L}^k \xi_L = \frac{\lambda}{\lambda} \int \Lambda q \mathcal{L}^k \xi_L - \int \mathcal{L}^k q \mathcal{L}^{k+1} \xi_L + \int \mathcal{L}^k (-\overline{\Psi}_b - \overline{Mod} + \mathcal{H}(q) - N(q)) \mathcal{L}^k \xi_L. \tag{8.39} \]

By H"older, (7.8) and (8.22), we have
\[ \left| \frac{\lambda}{\lambda} \int \Lambda q \mathcal{L}^k \xi_L \right| \\lesssim b_1 \left( \frac{|\partial_q q|^2}{1 + y^{4k-2}} \right) \left( \int y^2 (1 + y^{4k-2}) \left| \mathcal{L}^2 k \left( \frac{\langle L q, \chi_{B_c} \Lambda Q \rangle}{\langle \chi_{B_C}, \Lambda Q \rangle} \right) (1 - \chi_B) T_L \right| \right)^2 \lesssim b_1^{1+(\delta)\eta} \mathcal{E}_{2k}. \tag{8.40} \]

Again by (8.22),
\[ \int |\mathcal{L}^{k+1} \xi_L|^2 \lesssim \left( \frac{\langle L q, \chi_{B_c} \Lambda Q \rangle}{\langle \chi_{B_c}, \Lambda Q \rangle} \right)^2 \int |\mathcal{L}^{k+1} ((1 - \chi_B) T_L)|^2 \lesssim b_1^{2+2(\delta)\eta} \mathcal{E}_{2k}, \]
combined with H"older, we see that
\[ \left| \int \mathcal{L}^k q \mathcal{L}^{k+1} \xi_L \right| \lesssim b_1^{1+(\delta)\eta} \mathcal{E}_{2k}. \tag{8.41} \]

By H"older, (8.19), (8.21) and (8.36), we get
\[ \left| \int \mathcal{L}^k (-\overline{\Psi}_b - \overline{Mod}) \mathcal{L}^k \xi_L \right| \lesssim \left( \int |\mathcal{L}^k (-\overline{\Psi}_b + \overline{Mod})|^2 \right) \left( \int |\mathcal{L}^k \xi_L|^2 \right)^{\frac{1}{2}} \lesssim b_1^{(1+1)+(1+1+\delta)(1+\eta)} \mathcal{E}_{2k} + b_1^{1+(\delta)\eta} \mathcal{E}_{2k}. \tag{8.42} \]

Similarly, by H"older, (8.23), (8.25) and (8.36), we obtain
\[ \left| \int \mathcal{L}^k (\mathcal{H}(q) - N(q)) \mathcal{L}^k \xi_L \right| \lesssim \left( \int |\mathcal{L}^{k-1} (\mathcal{H}(q) - N(q))|^2 \right) \left( \int (1 + y^4) |\mathcal{L}^{k+1} \xi_L|^2 \right)^{\frac{1}{2}} \lesssim b_1^{1+\eta(1+\delta)} \mathcal{E}_{2k} + b_1^{1+(\delta)\eta} \mathcal{E}_{2k} + b_1^{1+\eta(1+\delta)+L+1+(1-\delta)(1+\eta)} \mathcal{E}_{2k}. \tag{8.43} \]

Substituting (8.40)-(8.43) into (8.39), and then taking (8.37)-(8.39) into (8.35), we derive
\[ \frac{1}{\lambda^{4k-d+4}} \int \mathcal{L}^k q \mathcal{L}^k (\partial_s \xi_L) = \frac{d}{dt} \left\{ \frac{\mathcal{E}_{2k}}{\lambda^{4k-d+2}} O(b_1^{1+(\delta)\eta}) \right\} \]
\[ + \frac{b_1}{\lambda^{4k-d+4}} O(b_1^{1+(\delta)\eta} \mathcal{E}_{2k} + b_1^{1+(\delta)\eta} b_1^{L+1+(1-\delta)(1+\eta)} \mathcal{E}_{2k}). \tag{8.44} \]

Now substituting (6.7) in the first term in the second line of (8.44), then collecting estimates (8.19)-(8.21), (8.23)-(8.25) and (8.44) into (8.18), we get the desired estimate (8.1), this concludes the proof.
9. Improved bootstrap, transverse crossing and conclusion

Next we prove improved bootstrap estimates as follows.

**Proposition 9.1.** Given initial data as in Definition 6.1, assuming for some large universal constant $K$ there is $s_0(K) \gg 1$ such that $(b(s), q(s)) \in S_K(s)$ on $s \in [s_0, s_1]$ for some $s_1 \geq s_0$. Then for all $s \in [s_0, s_1]$, we have

\[ |V_1(s)| \leq s^{-\frac{\eta(1-\delta)}{2}}, \]

\[ |b_k(s)| \lesssim s^{-(k+\eta(1-\delta))}, \quad \text{for } l + 1 \leq k \leq L, \]

\[ \epsilon_{2m}^2 \leq \left\{ \begin{array}{ll}
\frac{1}{2} K s^{-\frac{(4m-2d+2)}{2l-\gamma}}, & h + 2 \leq m \leq l + h, \\
\frac{1}{2} K s^{-\frac{2(1-\delta) - K\eta}{2}}, & l + h + 1 \leq m \leq k - 1,
\end{array} \right. \]

\[ \epsilon_{2k}^2 \leq \frac{1}{2} K s^{-\frac{2L+2(1-\delta)(1+\eta)}}. \]

**Proof.** In order to make use of Proposition 8.1, let us firstly estimate $\lambda$ in the variable $s$. By Lemma 4.1 and Definition 6.2, we have

\[ b_1(s) = \frac{l}{2l - \gamma s} + \frac{U_1}{s} = \frac{l}{2l - \gamma s} + O \left( \frac{1}{s^{\frac{1}{2}} + \frac{1}{2} (1-\delta)} \right), \]

combined with (7.1), it follows that

\[ -\frac{\lambda(s)}{\lambda} = b_1 + O(b_1 L + 1 + (1-\delta)(1+\eta)) = \frac{l}{2l - \gamma s} + O \left( \frac{1}{s^{\frac{1}{2}} + \frac{1}{2} (1-\delta)} \right). \]

Or say

\[ |\partial_s \ln \left( s^{\frac{l}{\gamma}} \lambda \right) | \lesssim \frac{1}{s^{\frac{1}{2}} + \frac{1}{2} (1-\delta)}. \]

Integrating (9.5) from $s_0$ to $s$ yields

\[ e^{-s_0 \frac{\eta(1-\delta)}{2}} \lesssim \frac{\lambda(s) s^{\frac{l}{\gamma}}}{s_0^{\frac{l}{\gamma}}} \lesssim e^{s_0 \frac{\eta(1-\delta)}{2}}, \]

hence

\[ \lambda(s) \simeq \left( \frac{s_0}{s} \right)^{\frac{\eta}{\gamma}}. \]

Improved bound for $\epsilon_{2k}^2$ : Integrating (8.1) from $s_0$ to $s$, we get

\[ \frac{\epsilon_{2k}^2(s)}{\lambda(s)^{4k-d+2}} (1 + O(b_1(s)^{\eta(1-\delta)})) \lesssim \epsilon_{2k}(s_0) (1 + O(b_1(s_0)^{\eta(1-\delta)})) + \int_{s_0}^{s} \frac{b_1(\tau)}{\lambda(\tau)^{4k-d+4}} \left( b_1(\tau)^{L+1+2(1-\delta)(1+\eta)} \sqrt{\epsilon_{2k}(\tau)} \right) d\tau + \frac{\epsilon_{2k}(\tau)}{M^{2\delta}} + b_1(\tau)^{2L+2(1-\delta)(1+\eta)}, \]

where we used the assumption $\lambda(s_0) = 1$. Then applying (9.6), $b_1(s) \simeq \frac{l}{s}$, Definition 6.1 and Definition 6.2, and for convenience discarding $\frac{1}{\lambda(\tau)^{\gamma}}$ in the integral (note that $\frac{1}{\lambda(\tau)^{\gamma}} \lesssim \left( \frac{s_0}{s} \right)^{\frac{\eta}{\gamma}} \lesssim 1$), we get

\[ \epsilon_{2k}(s) \lesssim s_0^{-\frac{1}{2\gamma} (4L+4(1-\delta)-2\gamma)} \left( \sqrt{K} + \frac{K}{M^{2\delta}} + 1 \right) s^{-\frac{1}{2\gamma} (4k-d+2)} \int_{s_0}^{s} \tau^{-(2L+2(1-\delta)(1+\eta))} + \frac{1}{\lambda^{(\gamma)}(4k-d+2)} d\tau. \]
\[
\lesssim s_0^{-3L+\gamma} s^{-(2L-\gamma)} + \left(\sqrt{K} + \frac{K}{M^{2\gamma} + 1}\right) s^{-(2L+2(1-\delta)(1+\eta))} \leq \frac{K}{2} s^{-(2L+(1-\delta)(1+\eta))},
\]

where in the second inequality we used the fact that
\[
\frac{l}{2l-\gamma} (4k-d+2) - 2L - 2(1-\delta)(1+\eta)
\]
\[
= 2 \left(\frac{2l}{2l-\gamma} - 1\right) L + 2 \left(\frac{2l}{2l-\gamma} - (1+\eta)\right) (1-\delta) - \frac{2l}{2l-\gamma} \gamma > 0.
\]

Improved bound for \( E_{2m} \) with \( h + 2 \leq m \leq l + h \): Similarly, one integrates (8.2) from \( s_0 \) to \( s \), then applying (9.6), \( b_1(s) \simeq \frac{s}{s} \), Definition 6.1 and Definition 6.2, and discarding \( \frac{1}{r} \) in the integral, after some direct computations which we shall omit, one gets
\[
E_{2m}(s) \leq \frac{K}{2} s^{-\frac{l}{2l-\gamma} (4m-d+2)}. \]
For the same reason, we omit the proof of improved bound for \( E_{2m} \) with \( l + h + 1 \leq m \leq |k| - 1 \).

Improved bound for \( b_k \) with \( l + 1 \leq k \leq L \): We aim to prove (9.2) by induction on \( k \). When \( k = L \), denoting
\[
\tilde{b}_L := b_L + \frac{L^L q_1, \chi_B, \Lambda, Q}{\langle \chi_B, \Lambda, Q \rangle},
\]

Note that by (8.22) and (6.7), we have
\[
\left| \frac{\mathcal{L}q_1, \chi_B, \Lambda, Q}{\langle \chi_B, \Lambda, Q \rangle} \right| \lesssim b_1^{L+\eta(1-\delta)},
\]

Hence \( |\tilde{b}_L| \lesssim b_1^L \). Straight calculation gives
\[
\frac{d}{ds} \left( \tilde{b}_L(s) \right) = \frac{1}{\lambda^{2\gamma}} \left[ (\tilde{b}_L)_s + (2L-\gamma) b_1 \tilde{b}_L - \lambda \right] \left( \frac{\lambda}{\lambda} + b_1 \right) \tilde{b}_L.
\]

Note that by (7.16), (6.7) and (9.7), we see that
\[
|\tilde{b}_L)_s + (2L-\gamma) b_1 \tilde{b}_L| \leq \left| (\tilde{b}_L)_s + \partial_s \frac{L^L q_1, \chi_B, \Lambda, Q}{\langle \chi_B, \Lambda, Q \rangle} + (2L-\gamma) b_1 \tilde{b}_L + \left( b_1 + (2L-\gamma) b_1 \right) \frac{L^L q_1, \chi_B, \Lambda, Q}{\langle \chi_B, \Lambda, Q \rangle} \right| \lesssim \tilde{b}_L^{L+1+\eta(1-\delta)}
\]
\[
\lesssim b_1^{L+1+\eta(1-\delta)}
\]

Then by (7.1), we get
\[
\frac{d}{ds} \left( \tilde{b}_L(s) \right) \lesssim \frac{b_1^{L+\eta(1-\delta)}}{\lambda^{2\gamma}}.
\]

Then integrating (9.8) from \( s_0 \) to \( s \), applying (9.6), Definition 6.1, \( b_1(s) \simeq \frac{s}{s} \) and the fact that
\[-(L + 1 + \eta(1-\delta)) + (2L-\gamma) \frac{l}{2l-\gamma} = \frac{l_7}{2l-\gamma} \left( L - 1 - \frac{1}{2} \left( 1 - \frac{1}{l} \right) \gamma \right) - \eta(1-\delta) - 1 > -1, \]

we see that
\[
\tilde{b}_L(s) \lesssim \lambda(s)^{2L-\gamma} \tilde{b}_L(s_0) + \lambda(s)^{2L-\gamma} \int_{s_0}^{s} b_1(\tau)^{L+\eta(1-\delta)} \lambda(\tau)^{2\gamma} \tilde{b}_L(d\tau)
\]
\[
\lesssim s^{-(L-\gamma)} \cdot s_0^{\left[ \frac{L+\gamma}{2L-\gamma} - (1-\delta) \right]} + s^{-(L+\eta(1-\delta))} \cdot s_0^{\left[ \frac{L+\gamma}{2L-\gamma} - (1-\delta) \right]} + s^{L-\eta(1-\delta)}
\]
\[
\lesssim s^{L-\eta(1-\delta)}.
\]
where in the last inequality, one may assume \( s_1 \) is a fixed multiplier of \( s_0 \). Thus \( |b_L| \lesssim \tilde{b}_L| + \frac{(|\mathcal{L}_s b_s|_{\mathcal{A}^0(Q)} + |\mathcal{L}_s b_s|_{\mathcal{A}^0(Q)})}{\langle \chi_{V_0} \rangle_{\mathcal{A}^0(Q)}} \lesssim s^{-\gamma(1-\delta)} \). Assuming (9.2) holds for \( k+1 \), we aim to show it holds true for \( k \).

By induction hypothesis and (7.1), we estimate
\[
\frac{d}{ds} b_L(s) = \frac{1}{\lambda^{2k-\gamma}} \left( (b_k)_s + (2k - \gamma) b_1 b_k - b_{k+1} - (2k - \gamma) \left( \frac{\lambda_s}{\lambda} + b_1 \right) b_k + b_{k+1} \right) \\
\lesssim \frac{1}{\lambda^{2k-\gamma}} \left( b_{k+1+1}(1-\delta)(1+\eta) + b_1^{k+1+1+1-\delta}(1+\eta) \right) \lesssim \frac{b_{k+1+1+1-\delta}(1+\eta)}{\lambda^{2k-\gamma}}. \tag{9.9}
\]

Similarly, integrating (9.9) from \( s_0 \) to \( s \), then applying (9.6), Definition 6.1 and the fact that
\[-(k+1+\eta(1-\delta)) + \frac{l}{2l-\gamma} (2k-\gamma) = \frac{\gamma}{2l-\gamma} (k-l) - \eta(1-\delta) - 1 > -1,\]
we obtain
\[
|b_k(s)| \lesssim \lambda(s)^{2k-\gamma} b_k(s_0) + \lambda(s)^{2k-\gamma} \int_{s_0}^{s} \frac{b_1(\tau)^{k+1+1+\eta(1-\delta)}}{\lambda(\tau)^{2k-\gamma}} d\tau \\
\lesssim s^{-(k-\frac{d}{2})} s_0^{-(5k-2\gamma)} + s^{-\frac{l}{2l-\gamma}(2k-\gamma)} \int_{s_0}^{s} \tau^{-(k+1+\eta(1-\delta)) + \frac{l}{2l-\gamma}(2k-\gamma)} d\tau \\
\lesssim s^{-(k-\eta(1-\delta))} \cdot s_0^{\gamma(1-\delta)} \cdot s_0^{-(5k-2\gamma)} + s^{-(k-\eta(1-\delta))} \\
\lesssim s^{-k-\eta(1-\delta)},
\]
where again in the last inequality, one may assume \( s_1 \) is a fixed multiplier of \( s_0 \), this concludes the proof of (9.2).

Improved bound for \( \mathcal{V}_1(s) \) : By direct computation, for any \( 1 \leq k \leq l \),
\[
s(\mathcal{V}_k)_s = \sum_{j=1}^{l-1} (P_1)_{k,j} [s(\mathcal{U}_j)_s - (\mathcal{A}_i \mathcal{U}_j)] + (P_1)_{k,l} [s(\mathcal{U}_l)_s - (\mathcal{A}_i \mathcal{U}_l)] + (D_1 \mathcal{V})_k.
\]

Note that by Lemma 4.1, Proposition 7.1, Definition 6.2 and (9.2), we get
\[
|s(\mathcal{U}_j)_s - (\mathcal{A}_i \mathcal{U}_j)| \lesssim s^{j+1} |(b_j)_s + (2j - \gamma) b_1 b_j - b_{j+1}| + O(|\mathcal{U}|^2) \\
\lesssim s^{-\eta(1-\delta)},
\]
\[
|s(\mathcal{U}_l)_s - (\mathcal{A}_i \mathcal{U}_l)| \lesssim s^{l+1} |(b_l)_s + (2l - \gamma) b_1 b_l - b_{l+1}| + s^{l+1} b_{l+1} + O(|\mathcal{U}|^2) \\
\lesssim s^{-\eta(1-\delta)}.
\]

It immediately follows that
\[
s \mathcal{V}_s = D_1 \mathcal{V} + O(s^{-\eta(1-\delta)}).
\tag{9.10}
\]

In particular,
\[
|(s \mathcal{V}_1)_s| \lesssim s^{-\eta(1-\delta)}. \tag{9.11}
\]

Integrating (9.11) from \( s_0 \) to \( s \), then applying Definition 6.1, we see that
\[
\left( \frac{s_0}{s} \right)^{1-\frac{2l(1-\delta)}{2}} - C s^{\frac{\eta(1-\delta)}{2}} \leq \frac{s^{\frac{\eta(1-\delta)}{2}} \mathcal{V}_1(s)}{\left( \frac{s_0}{s} \right)^{1-\frac{2l(1-\delta)}{2}}} \leq \left( \frac{s_0}{s} \right)^{1-\frac{2l(1-\delta)}{2}} + C s^{\frac{\eta(1-\delta)}{2}}
\]
for \( s_0 < s \leq s_1 \). This concludes the proof of (9.1).

Next we describe the reduction to a finite-dimensional problem and transverse crossing property.

**Proposition 9.2.** There exists \( K_1 \geq 1 \) such that for any \( K \geq K_1 \), there exists \( s_{0,1}(K) > 1 \) such that for all \( s_0 \geq s_{0,1}(K) \) the following holds. Given initial data at \( s = s_0 \) as in Definition 6.1, if \((b(s), q(s)) \in \mathcal{S}_K(s)\) for all \( s \in [s_0, s_1] \) with \((b(s), q(s)) \in \partial \mathcal{S}_K(s_1)\) for some \( s_1 \geq s_0 \), then (i) Reduction to a finite-dimensional problem:
\[
s_{f}^{\frac{\eta(1-\delta)}{2}} (\mathcal{V}_2(s_1), \ldots, \mathcal{V}_i(s_1)) \in \partial \mathcal{B}_{f-1}(0,1). \tag{9.12}
\]
Transverse crossing:
\[
\frac{d}{ds} \sum_{i=2}^{l} s^{\frac{n(1-\delta)}{2}} V_i(s)^2 > 0.
\] (9.13)

**Proof.** (i) is a direct consequence of Proposition 9.1. Let us prove (ii). Note that by (9.10),
\[
s(V_i) = \frac{i \gamma}{2l-\gamma} V_i + O(s^{-\eta(1-\delta)}), \quad \text{for } 2 \leq i \leq l.
\]

Combined with Definition 6.2, we have
\[
\frac{d}{ds} \sum_{i=2}^{l} s^{\frac{n(1-\delta)}{2}} V_i(s)^2 = 2s^{\eta(1-\delta)-1} \sum_{i=2}^{l} \left( \frac{\eta(1-\delta)}{2} V_i^2 + s(V_i) V_i \right)
\]
\[
= 2s^{\eta(1-\delta)-1} \left\{ \sum_{i=2}^{l} \left( \frac{i \gamma}{2l-\gamma} + \frac{\eta(1-\delta)}{2} \right) V_i^2 + O(s^{-\eta(1-\delta)}) \right\}
\]
\[
\geq C \sum_{i=2}^{l} s^{\frac{n(1-\delta)}{2}} V_i(s)^2 \quad \text{for } s = s_1
\]
\[
\geq O(s^{-\eta(1-\delta)-1}) \geq \frac{1}{s} > 0.
\]

Then we are in place to show the existence of solutions to (3.2) that are trapped in \( S_K(s) \) for any large \( s \).

**Proposition 9.3.** There exists \( K_2 \geq 1 \) such that for any \( K \geq K_2 \), there exists \( s_0(K) > 1 \) such that for all \( s_0 \geq s_{0,2} \), there exists initial data satisfying Definition 6.1 such that \((b(s), q(s)) \in S_K(s) \) for all \( s \geq s_0 \).

**Proof.** Let us argue by contradiction. If not, defining
\[
s_* := \sup \{ s \mid s \geq s_0 \text{ such that } (b(s), q(s)) \in S_K(s) \}.
\]
as the exit time, then we have for any initial data satisfying Definition 6.1, \( s_* < +\infty \). Defining the map
\[
\Xi : B_{l-1}(0,1) \rightarrow \partial B_{l-1}(0,1)
\]
\[
s_0 \rightarrow (V_2(s_0), \ldots, V_l(s_0)) \mapsto s_* \frac{n(1-\delta)}{2} (V_2(s_*), \ldots, V_l(s_*)).
\]

By (i) of Proposition 9.2, \( \Xi \) is well defined. By (ii) of Proposition 9.2, \( \Xi \) restricted on \( \partial B_{l-1}(0,1) \) is the identity map. Then \( \Xi \) is a continuous map and an identity on the boundary of a ball which is not possible in view of the Brouwer’s fixed point theorem, this concludes the proof.

Finally we finish the proof of Theorem 1.1.

Expression of \( \lambda \) in original time variable. Recall by the proof of (9.6), we get
\[
-\lambda_t \lambda = c(u_0) \lambda^\frac{2(n-1)}{n} (1 + o(1)),
\]
or say
\[
\partial_t \left( \lambda^\frac{2(n-1)}{n} \right) = -c(u_0)(1 + o(1)).
\]

Then integrating from \( t \) to \( T \), we see that
\[
\lambda(t) = c(u_0)(1 + o(1))(T - t)^\frac{1}{2}.
\]

On the smallness of Sobolev norms of \( q \). By (iii) of Lemma 5.1 and Definition 6.2, we have
\[
\int |\partial_q^m q|^2 \lesssim \epsilon_2^m \rightarrow 0, \quad \text{as } s \rightarrow \infty, \quad \text{for } h + 2 \leq m \leq k.
\]

This concludes the whole proof of Theorem 1.1.
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Yezhou Yi
School of Mathematics and Statistics
Ningbo University, Ningbo, 315211, Zhejiang, P.R. China
Email address: yiyezhou@nbu.edu.cn