Two-Loop $\Phi^4$-Diagrams from String Theory

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Abstract

Using the cutting and sewing procedure we show how to get Feynman diagrams, up to two-loop order, of $\Phi^4$-theory with an internal $SU(N)$ symmetry group, starting from tachyon amplitudes of the open bosonic string theory. In a properly defined field theory limit, we easily identify the corners of the string moduli space reproducing the correctly normalized field theory amplitudes expressed in the Schwinger parametrization.

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1 Introduction

Much interest has been devoted to the relation between string theory and field theory over the last years. It is well-known that any string theory reduces to an effective field theory in the so-called zero-slope limit where the inverse string tension $\alpha' \to 0$. The latter is a physical dimensional parameter acting as an ultraviolet cutoff in the integrals over loop momenta and so makes multiloop amplitudes free from ultraviolet divergences. This is a basic reason why a string theory can provide a consistent quantum theory of gravity, unified with non-abelian gauge theories.

Furthermore, string theory also manages to organize scattering amplitudes in a very compact form, which makes much easier to calculate non-abelian gauge theory amplitudes by starting from a string theory and performing the zero-slope limit, rather than using traditional field theory techniques. We would like here to add that the expression of string amplitudes is known explicitly, including also the measure of integration on moduli space, in the case of the bosonic open string for an arbitrary perturbative order [1].

These are the features of string theory that have led some authors to use it as an efficient tool to compute gluon amplitudes in Yang-Mills theory [2]÷[6] or in quantum gravity, where the improvement over traditional techniques was even more spectacular [7], and light has been shed on the perturbative relations between gravity and gauge theory [8]. We would like to add that these string-methods have also inspired some authors in developing very interesting techniques based on the world-line path-integrals [9].

The general procedure on which the derivation of field theory amplitudes from string amplitudes is based consists in starting, for example in the case of Yang-Mills amplitudes, from a given multiloop gluon string amplitude and in singling out different regions of the moduli space that, in the low-energy limit, reproduce different field theory diagrams. This program has been carried out at one-loop [10] [11] and, at this order, the five-gluon amplitude has been obtained for the first time [12].

Some attempts have been performed for extending to the two-loop case this procedure [11] [13] [14] which while, on the one hand, does not present many difficulties in
computing Yang-Mills vacuum diagrams, on the other hand becomes difficult to handle when treating amplitudes with external states, because of their complicated structure. In order to avoid the computational difficulties associated with this kind of multiloop Yang-Mills amplitudes, which are inessential for the understanding of the field theory limit, one can consider amplitudes involving scalar particles. In fact, differently from what happens in gauge theories, in string theory there is not a big conceptual difference between gluon and scalar diagrams. Therefore one can get from scalar amplitudes, with much less computational cost, the whole information about the corners of the moduli space reproducing the known field theoretical results: these regions are exactly the ones giving the correct field theory diagrams also in the case of gluon amplitudes.

The scalar particles one is referring are, of course, the tachyons of the bosonic string theory. So one can consider a slightly different zero-slope limit of the bosonic string in which only the lowest tachyonic state, with a mass satisfying $m^2 = -1/\alpha'$ is kept. In the case of tree and one-loop diagrams, this procedure is equivalent to take the zero-slope limit of an old pre-string dual model characterized by an arbitrary value of the intercept of the Regge trajectory. It was recognized the inconsistency of this model, but the field theory limit of tree and one-loop diagrams of this pre-string dual model was shown to lead to the Feynman diagrams of $\Phi^3$ theory [15].

In a previous paper [11], it has been explicitly shown that by performing the zero-slope limit as above explained, one correctly reproduces the Feynman diagrams of $\Phi^3$ theory, up to two-loop order. In this way it has been provided an algorithm generalizable to Yang-Mills theory in order to obtain all diagrams containing the three-gluon vertices.

This paper is the natural development of the program carried out in Ref. [11], its aim being to extend the conceptual scheme pursued in it, to two-loop amplitudes containing four-gluon vertices, whose very complicated structure, when external particles are involved, has not allowed to get meaningful results up to now. This means that, exploiting the above mentioned analogy between scalar and gluon amplitudes, one could start from string amplitudes involving tachyons, perform on them the field theory limit in which their mass is fixed and correctly identify the corners associated to the different
field theory diagrams of $\Phi^4$ theory. As observed in Ref. [11], this analysis may be a bit lengthy, since there are many corners of the string moduli space that contribute to the same field theory diagrams. But one can pursue an alternative and equivalent method, the *sewing and cutting* procedure, that also leads to the correct identification of field theory diagrams and that has been successfully applied to get two-loop $\Phi^3$ amplitudes in [11]. This is indeed the technique we use in this paper. We would like here to observe that, until now, four-vertices have been obtained as finite remainders of tachyon exchange [13] [16] and that one of the main peculiarities of our technique is that the identification of the right corner in the moduli space is independent on any handling tachyon exchange. Hence it is reasonable to think that the *sewing and cutting* procedure is easily extendible to consistent string theories.

Finally, another interesting motivation to analyze the field theory limit of scalar amplitudes is understanding the analogous limit in a string theory with a non-zero $B$-field. This limit yields non-commutative field theories and provides a hope that it would be possible to cure quantum field theory divergences [17] [18].

The paper is organised as follows.

In sect. 2 we present the operator formalism that we use for computing multiloop scalar amplitudes in the bosonic open string theory. We consider scalar particles having an internal $SU(N)$ symmetry group and perform a large $N$ limit of the relative amplitudes so that only planar diagrams are taken into account.

In sect. 3 we define the *sewing and cutting* procedure applying it to the four-tachyon tree amplitude. We define a proper field theory limit where cubic and quartic interactions are reproduced.

In sect. 4 we check the validity of this procedure by deriving from the two- and four-tachyon amplitudes at one-loop, respectively, the *tadpole* and the *candy* diagram of $\Phi^4$ theory.

In sect. 5 the *sunset* and the *double-candy* diagrams are computed. Each field-theory diagram is obtained with its own right overall normalization.
2 Multiloop scalar amplitudes in the bosonic open string

The planar $h$-loop scattering amplitude of $M$ tachyons with momenta $p_1, \ldots, p_M$ is:

$$A^{(h)}_M(p_1, \ldots, p_M) = N^h \text{Tr} [\lambda^{a_1} \cdots \lambda^{a_M}] C_h \left[ 2g_s(2\alpha')^{d-2} \right]^M \times \int [dm]_h^M \prod_{i<j} \left[ \exp \left( \mathcal{G}^{(h)}(z_i, z_j) \right) \right]^{2\alpha' p_i \cdot p_j}$$

where $N^h \text{Tr}(\lambda^{a_1} \cdots \lambda^{a_M})$ is the appropriate $SU(N)$ Chan-Paton factor, with the $\lambda$’s being the generators of $SU(N)$ in the fundamental representation, normalized as

$$\text{Tr}(\lambda^a \lambda^b) = \frac{1}{2} \delta^{ab},$$

$g_s$ is the dimensionless string coupling constant, $C_h$ is a normalization factor given by:

$$C_h = \frac{1}{(2\pi)^{dh} g_s^{2h-2}} \frac{1}{(2\alpha')^{d/2}}$$

and $\mathcal{G}^{(h)}$ is the $h$-loop bosonic Green function

$$\mathcal{G}^{(h)}(z_i, z_j) = \log \mathcal{E}^{(h)}(z_i, z_j) - \frac{1}{2} \int_{z_i}^{z_j} \omega^\mu (2\pi \text{Im}\tau_{\mu\nu})^{-1} \int_{z_i}^{z_j} \omega^\nu,$$

with $\mathcal{E}^{(h)}(z_i, z_j)$ being the prime form, $\omega^\mu (\mu = 1, \cdots, h)$ the abelian differentials and $\tau_{\mu\nu}$ the period matrix. All these geometrical objects, which are peculiar of the open Riemann surface of genus $h$ on which the amplitude is defined, can be explicitly written in the Schottky parametrization of the surface itself, and their expressions can be found for example in Ref. [19]. The measure $[dm]_h^M$, when written in terms of the Schottky parameters, is given by:

$$[dm]_h^M = \frac{1}{dV_{abc}} \prod_{i=1}^M \frac{dz_i}{V_i(0)} \prod_{\mu=1}^h \left[ \frac{dk_\mu d\xi_\mu d\eta_\mu}{k_\mu^2(\xi_\mu - \eta_\mu)^2(1 - k_\mu)^2} \right] \times \left[ \text{det}(-i\tau_{\mu\nu}) \right]^{-d/2} \prod_{\alpha} \left[ \prod_{n=1}^\infty (1 - k_\alpha^n)^{-d} \prod_{n=2}^\infty (1 - k_\alpha^n)^2 \right]$$

(2.5)
where \( k_\mu \) are the multipliers, \( \xi_\mu \) and \( \eta_\mu \) the fixed points of the generators of the Schottky group and \( dV_{abc} \) is the projective invariant volume element:

\[
dV_{abc} = \frac{d\rho_ad\rho_b d\rho_c}{(\rho_a - \rho_b)(\rho_a - \rho_c)(\rho_b - \rho_c)} ,
\]

with \( \rho_a, \rho_b, \rho_c \) being any three of the \( M \) Koba-Nielsen variables \( z_i \), or of the \( 2h \) fixed points of the generators of the Schottky group, which can be fixed at will; finally, the primed product over \( \alpha \) denotes the product over the primary classes of elements of the Schottky group [19]. Furthermore, like in any planar open string amplitude, the Koba-Nielsen variables must be cyclically ordered along one of the boundaries of the world-sheet, for example according to:

\[
z_1 \geq z_2 \geq \cdots \geq z_M
\]  

and the ordering of Koba-Nielsen variables automatically prescribes the one of color indices.

The presence in (2.1) of terms involving the quantity \( V'_i(0) \) originates from the \( M \)-point \( h \)-loop vertex \( V_{M,h} \) of the operator formalism [1]. This can be regarded as a generating functional for scattering amplitudes among arbitrary states, at all orders in perturbation theory. In fact, by saturating the operator \( V_{M,h} \) with \( M \) states \( |\alpha_1 >, \ldots, |\alpha_M >, \) one obtains the corresponding amplitude:

\[
A^{(h)}(\alpha_1, \ldots, \alpha_M) = V_{M,h}|\alpha_1 > \ldots |\alpha_M > .
\]  

The explicit expression of \( V_{M,h} \) for planar diagrams of the open string depends on \( M \) real Koba-Nielsen variables \( z_i \) through \( M \) projective transformations \( V_i(\rho) \), which define local coordinate systems vanishing around each \( z_i \), i.e. such that:

\[
V_i(0) = z_i .
\]  

When \( V_{M,h} \) is saturated with \( M \) physical string states satisfying the mass-shell condition, the corresponding amplitude does not depend on the \( V_i \)'s. Such a dependence remains for off-shell string amplitudes and for these it is therefore necessary to make a
choice of the projective transformations $V_i$’s $[3] [11]$. It has been shown that a principle guiding such a choice is the requirement of projective invariance of off-shell string amplitudes $[20]$. Let us denote by $|p>$ a state representing a tachyon with momentum $p$. It is created by the vertex operator

$$V(z) = \mathcal{N}_t : e^{i\sqrt{2\alpha'} p \cdot X(z)} :$$

(2.9)

where colons denote the standard normal ordering on the modes of the open string coordinate $X(z)$ and $\mathcal{N}_t$ is a normalization factor $[11]$

$$\mathcal{N}_t = 2g_s(2\alpha')^{\frac{d-2}{2}}.$$ (2.10)

If we write, as usual

$$X^\mu(z) = \hat{q}^\mu - ij^\mu \log z + i \sum_{n \neq 0} \frac{\hat{a}_n^\mu}{n} z^{-n}$$ (2.11)

then the tachyon state is

$$|p> \equiv \lim_{z \to 0} V(z)|0> = \mathcal{N}_t e^{ip\cdot\hat{q}}|0>.$$ (2.12)

The tachyon state is on-shell if

$$p^2 = -m^2 = \frac{1}{\alpha'}.$$ (2.13)

The amplitude (2.1) is obtained by saturating $V_{M;h}$ on $M$ tachyon states defined in (2.12).

3 Tree diagrams from the sewing and cutting technique

3.1 Tree $\Phi^3$-diagrams

The planar tree scattering amplitude of $M$ on-shell bosonic open string tachyons with momenta $p_1, \ldots, p_M$ each satisfying the mass-shell condition $p^2 = -m^2 = \frac{1}{\alpha'}$ is obtained
Figure 1: Planar tree scattering amplitude

from the master equation in (2.1) considered for \( h = 0 \):

\[
A_M^{(0)}(p_1, \ldots, p_M) = \text{Tr} \left[ \lambda^{a_1} \cdots \lambda^{a_M} \right] C_0 N_{t}^M \int \prod_{i=1}^{M} \frac{dz_i}{dV_{abc}} \prod_{i<j} (z_i - z_j)^{2\alpha' p_i \cdot p_j}. \tag{3.1}
\]

The corresponding diagram is depicted in Fig. (1).

In the case of four on-shell tachyons Eq. (3.1) becomes

\[
A_4^{(0)}(p_1, p_2, p_3, p_4) = \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] C_0 N_{t}^4 \int_{0}^{1} dz (1 - z)^{2\alpha' p_2 \cdot p_3} z^{2\alpha' p_3 \cdot p_4}. \tag{3.2}
\]

where \( z_1, z_2 \) and \( z_4 \) have been respectively fixed at \( +\infty, 1 \) and 0.

In the limit \( \alpha' \to 0 \) the amplitude in Eq. (3.2) yields the tree level Feynman diagrams of scalar field theories: different regions in moduli space lead to different field theory diagrams. In fact, according to the corner of moduli space where the low-energy limit is performed, one can recover, for instance, \( \Phi^3 \)- or \( \Phi^4 \)-scalar diagrams.

In order to understand which regions in moduli space lead to the different field theory diagrams, one can use the so-called sewing and cutting procedure. This consists in starting from a string diagram and in cutting it in three-point vertices; next we fix the legs of each three-point vertex at \( +\infty, 1 \) and 0. Then we reconnect the diagram by inserting between two three-point vertices a suitable propagator acting as a well specified projective transformation. This is chosen in such a way that its fixed points are just the Koba-Nielsen variables of the two legs that have to be sewn. The geometric role of the propagator is to identify the local coordinate systems defined around the punctures to be sewn.

For illustrating in a simple case how this technique works, we show how the four-point tree diagrams of \( \Phi^3 \)-theory can be generated. The starting point is the four-tachyon tree
string diagram. This can be obtained by sewing two three-point vertices as shown in Fig. (2). We sew the leg corresponding to the point 0 in the vertex at the left hand in Fig. (2a) to the leg corresponding to the point \(+\infty\) in the one at the right hand through a propagator corresponding to the projective transformation

\[
S(z) = Az
\]  

(3.3)

which has 0 and \(+\infty\) as fixed points and the parameter \(A\), with \(0 \leq A \leq 1\), as multiplier. Performing the sewing means, in this procedure, to transform only the punctures of the three-point vertex at the right hand in Fig. (2a) through (3.3), hence the puncture \(z_3 = 1\) transforms into \(S(1) = A\) while the other two punctures remain unchanged.

![String diagram](image)

Figure 2: Sewing of two three vertices in the s-channel.

In general, after the sewing has been performed, the Koba-Nielsen variables become functions of the parameter \(A\) appearing in the projective transformation (3.3). It is possible to give a simple geometric interpretation to this parameter, if a correspondence is established between the projective transformation in Eq. (3.3) and the string propagator, written in terms of the operator \(e^{-\tau(L_0-1)}\). The latter indeed propagates an open string through imaginary time \(\tau\) and creates a strip of length \(\tau\). In fact the change of variable \(z = e^{-\tau}\) allows the string propagator to be written as

\[
\frac{1}{L_0-1} = \int_0^1 dz z L_0^{-2} = \int_0^\infty d\tau \exp \left( -\tau \alpha' \left[ p^2 + \frac{1}{\alpha'(N-1)} \right] \right)
\]  

(3.4)
and to establish the following relation between \( \tau \) and \( A \):

\[
\tau = -\log A. \tag{3.5}
\]

The multiplier \( A \) results to be therefore related to the length of the strip connecting two three-vertices.

On the other hand, since we want to reproduce \( \Phi^3 \)-theory diagrams we have to consider a low-energy limit of string amplitudes in which only tachyons propagate as intermediate states. This is achieved observing from (3.4) that the only surviving contribution in the limit \( \alpha' \to 0 \) with \( \tau \alpha' \) kept fixed is the one coming from the level \( N = 0 \), i.e. from tachyons with fixed mass given by \( m^2 = -\frac{1}{\alpha'} \). It is obvious that this also corresponds to the limit \( \tau \to \infty \) and hence, from (3.5), to \( A \to 0 \). From these considerations it seems natural to introduce the variable \( x = \tau \alpha' \) in terms of which the string propagator (3.4), reproduces, in the above mentioned limit, the scalar propagator

\[
\frac{1}{p^2 + m^2} = \int_0^\infty dx e^{-x(p^2 + m^2)}
\]

with \( x \) being interpreted as the Schwinger proper time.

From a geometrical point of view, one can imagine that the strip connecting the two three-vertices, in this field theory limit, becomes “very long and thin”, so that only the lightest states propagate.

Let us now rewrite the amplitude (3.2) in terms of the Schwinger parameter \( x \) or, equivalently, in terms of the multiplier \( A \):

\[
A_4^{(0)} = \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] \frac{\mathcal{N}_t^4}{g_s^2 (2\alpha')^{d/2}} \int_0^\infty dA \exp \left( 2\alpha' p_3 p_4 \log A \right)
\]

\[
= \frac{1}{8} \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] \frac{g_{s^3}}{[(p_1 + p_2)^2 + m^2]} \tag{3.6}
\]

where it has been used the well-known relation between \( g_s \) and \( g_{s^3} \) [11]:

\[
g_{s^3} = 16 g_s (2\alpha')^{\frac{d-6}{4}}. \tag{3.7}
\]

By performing the sum over inequivalent permutations and selecting those diagrams which contribute to the \( s \)-channel, we get
where

\[ G = \text{Tr} [\lambda^a \lambda^{a2} \lambda^{a3} \lambda^{a4}] + \text{Tr} [\lambda^a \lambda^{a2} \lambda^{a4} \lambda^{a3}] + \text{Tr} [\lambda^a \lambda^{a3} \lambda^{a4} \lambda^{a2}] + \text{Tr} [\lambda^a \lambda^{a4} \lambda^{a3} \lambda^{a2}] \]  

(3.9)

Taking into account the relation satisfied by the \( SU(N) \) generators \( \lambda^a \) (\( a = 1, \ldots, N^2 - 1 \)) in the \( N \)-dimensional fundamental representation:

\[ \{ \lambda^a, \lambda^b \} = \frac{1}{N} \delta_{ab} + d^{abc} \lambda^c \]

and the one satisfied by the totally symmetric tensor \( d^{abc} \)

\[ d^{abc} = 2 \text{Tr} \left[ \{ \lambda^a, \lambda^b \} \lambda^c \right], \]

one can get, in the large \( N \) limit

\[ G = \frac{1}{2} d^{a1a2l} d^{a3a4l} \]  

(3.10)

where a sum on \( l \) is understood. Hence

\[ A_4^{(0) s-channel} (p_1, p_2, p_3, p_4) = \frac{1}{16} \left[ \frac{g_5^2}{(p_1 + p_2)^2 + m^2} \right] d^{a1a2l} d^{a3a4l}. \]  

(3.11)

The result (3.11) can be also obtained by using standard techniques. Indeed, taking into account the equation

\[ \int_0^1 (1 - z)^{2\alpha' p_2 \cdot p_3} z^{2\alpha' p_3 \cdot p_4} = B [2\alpha' p_2 \cdot p_3 + 1, 2\alpha' p_3 \cdot p_4 + 1] \]  

(3.12)

with \( B \) being the Euler Beta function, the following amplitude is obtained in the limit \( \alpha' \rightarrow 0 \):

\[ A_4^{(0)} (p_1, p_2, p_3, p_4) = \text{Tr} [\lambda^a \lambda^{a2} \lambda^{a3} \lambda^{a4}] C_0 N_4 \frac{1}{\alpha'} \left[ \frac{1}{-s + m^2} + \frac{1}{-t + m^2} \right] \]  

(3.13)

The first term on the right hand reproduces the same result as in Eq. (3.11), including the overall factor. The second term, giving the amplitude in the \( t \)-channel, can be also obtained by the sewing and cutting procedure as it is shown in Fig. (3).
Figure 3: Sewing of two three vertices in the t-channel

In this case one has to sew the leg corresponding to $+\infty$ in the upper vertex in Fig. (3a) to the leg corresponding to 1 in the lower vertex through the transformation

$$S(z) = Az + 1 - A$$

having 1 and $+\infty$ as fixed points and which transforms the puncture $z_3 = 0$ of the upper vertex into $S(0) = 1 - A$. Therefore the amplitude (3.2), in the limit $A \to 0$, becomes:

$$A_4^{(0)}(p_1, \ldots, p_4) = \Tr [\lambda^{a_1} \cdots \lambda^{a_4}] C_0 N_t^4 \int_0^\epsilon dA \exp (2\alpha' p_2 \cdot p_3 \log A)$$

$$= C_0 N_t^4 \frac{1}{\alpha'} \left[ \frac{1}{-t + m^2} \right]$$

(3.14)

It is straightforward to show that the sum of the amplitudes (3.8) and (3.14) coincides with the four-point Green function in the field theory defined by the following action:

$$S = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{1}{2} m^2 \phi^a \phi^a - \frac{g_{abc}}{4} \partial^a \phi^b \phi^c \right]$$

(3.15)

3.2 Tree scalar $\Phi^4$-diagrams from String Amplitudes

The starting point is also in this case the amplitude (3.1) that, from (2.1), can be expressed also in terms of the Green functions $G^{(0)}(z_i, z_j)$, defined on the world-sheet in the following way:

$$G^{(0)}(z_i, z_j) = \log (z_i - z_j)$$

(3.16)
Our aim is now to consider a suitable limit of the string four-tachyon amplitude which can reproduce the diagram corresponding to the tree four-point vertex of $\Phi^4$-theory. With reference again to the Fig. (2), this diagram has to correspond to a limit in which the length of the tube connecting the two three-vertices composing the string diagram goes to zero in the limit $\alpha' \to 0$, i.e.

$$\tau = \frac{x}{\alpha'} = -\log A \to 0$$

This corresponds to the limit $A \to 1$, and hence $z \to 1$ or, equivalently, $z_3 \to z_2$.

In this limit the Green function $G^{(0)}(z_2, z_3)$ is divergent. We regularize it by introducing a cut-off $\epsilon$ on the world-sheet so that

$$\lim_{z_2 \to z_3} \log \left[ (z_2 - z_3) + \epsilon \right] = \log \epsilon$$

and

$$\lim_{\alpha' \to 0} \alpha' \log \epsilon = 0$$

We consider therefore the amplitude $A_4^{(0)}$ in Eq. (3.2) in the field theory limit defined by:

$$A = z = 1 - \epsilon, \quad \alpha' \to 0 \quad \text{and} \quad x = -\alpha' \log \epsilon \to 0.$$  \hspace{1cm} (3.17)

in which it reduces to

$$A_4^{(0)}(p_1, \ldots, p_4) = \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] C_0 N^4 \int_0^1 dz \epsilon^{2\alpha' p_2 \cdot p_1 \log \epsilon}$$

$$= 2^4 \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] g_s^2(2\alpha')^{d-4}. \hspace{1cm} (3.18)$$

The complete amplitude is obtained by performing the sum over non cyclic permutations, finally getting

$$A_4^{(0)\text{complete}} = 4g_s^2(2\alpha')^{d-4} \left\{ d^{a_1a_2a_3} d^{a_4a_1a_2} + d^{a_1a_3a_4} d^{a_2a_1a_4} + d^{a_1a_4a_2} d^{a_3a_2a_4} \right\}. \hspace{1cm} (3.19)$$

where a sum on repeated indices is understood.
We compare the result in (3.19) with the color ordered vertex generated by the following scalar field theory:

\[
\mathcal{L} = \text{Tr} \left[ \partial^\mu \phi \partial_\mu \phi + m^2 \phi^2 - \frac{g_{\phi^4}}{4!} \phi^4 \right]
\]  

(3.20)

obtaining the matching condition

\[
g_{\phi^4} = 4g_s(2\alpha')^{d/2-2}.
\]  

(3.21)

Eqs. (3.6) and (3.18) show that amplitudes of \(\Phi^3\)-theory are obtained at the boundaries of the integration region, while the ones of \(\Phi^4\)-theory get contributions from the whole integration region.

4 One-loop \(\Phi^4\)-diagrams from string amplitudes

4.1 Tadpole diagram

![Figure 4: Tadpole](image)

In this subsection we show how the tadpole diagram in \(\Phi^4\)-theory can be derived from string theory. The starting point will be, this time, the color ordered \(M\)-tachyon planar amplitude at \(h\) loops (2.1) specialized to the case \(M = 2\) and \(h = 1\). The projective invariance can be exploited fixing

\[
V_1'(0) = z_1 = 1, \quad \eta = 0, \quad \xi \to \infty
\]

so that the amplitude becomes

\[
A_2(p_1, p_2) = N\text{Tr} [\lambda^{a_1} \lambda^{a_2}] C_1 \left[ 2g_s(2\alpha')^{d/2} \right]^2 \int_0^1 \frac{dk}{k^2} \left[ -\frac{1}{2\pi} \log k \right]^{-\frac{d}{2}} \prod_{n=1}^\infty (1 - k^n)^{2-d} \times \int_k^1 \frac{dz}{z} \left[ \frac{\exp G(1)(1, z)}{\sqrt{z}} \right]^{2\alpha' p_1 \cdot p_2}.
\]  

(4.1)
We have imposed on the $V_i$’s ($i = 1, 2$) the condition $V_i'(0) = z_i$ \[3\] and have renamed $z_2 = z$.

We would like now to stress that if we want to reproduce diagrams of scalar field theories we have to ensure that only tachyon states propagate in the loops of string amplitudes. In fact this condition is fulfilled if small values of the multiplier $k$ are considered: indeed this parameter plays exactly the same role as the multiplier $A$ in the tree level amplitudes. Therefore an expansion in powers of $k$ is performed keeping the most divergent terms. In so doing we get

$$A_2^{(1)}(p_1, p_2) = \frac{N}{2} \frac{1}{(4\pi)^{d/2}} \frac{1}{(2\alpha')^{d/2}} \left[ 2g_s(2\alpha')^{d-2} \right] \int_0^1 dk \frac{1}{k^2} \left[ -\frac{1}{2} \log k \right]^{-\frac{d}{4}} \int_k^1 \frac{dz}{z} e^{2\alpha' G^{(1)}(1, z)}$$

(4.2)

where the Green function, in the limit we are considering, is

$$G^{(1)}(z_1, z_2) = \log(z_1 - z_2) - \frac{1}{2} \log z_1 z_2 + \frac{\log^2 z_1 / z_2}{2 \log k} + O(k)$$

(4.3)

Our aim is to identify the right limit to get the tadpole diagram in Fig. 4.

Starting from two three-vertices, we sew the leg 0 with the leg $+\infty$ according to the Fig. 5.

Such a sewing is performed by considering again the projective transformation $S(z) = Az$, which has $+\infty$ and 0 as fixed points and which transforms $z_2 = 1$ in the second vertex in Fig. (5a) in the multiplier $A$ getting the configuration shown in Fig. (5b).

The next step consists in performing a limit in which $z_2 \to z_1$, i.e. in which $A \to 1$ with $\alpha' \log(1 - A) \to 0$, as said before. In this limit we should get the tadpole diagram.
Indeed we have:

\[
A_2^{(1)}(p_1, p_2) = 2N \frac{1}{(4\pi)^{d/2}} g_s^2 (2\alpha')^{d/2} \int_0^1 \frac{dk}{k^2} \left[ -\frac{1}{2} \log k \right]^{-d/2} \int_k^1 \frac{dA}{A} e^{2\alpha' p_1 \cdot p_2 \log(1-A) - \frac{1}{2} \log A}
\]

\[
= \frac{2N}{(4\pi)^{d/2} 2\alpha'} g_s^2 \int_0^1 \frac{dk}{k^2} \left[ -\frac{1}{2} \log k \right]^{-d/2} + O(k)
\]

(4.4)

By defining:

\[
x = -\alpha' \log k
\]

with \(0 \leq x \leq +\infty\), we can rewrite (4.4) as follows:

\[
A_2^{(1)}(p_1, p_2) = \frac{N}{(4\pi)^{d/2}} \left[ 4g_s(2\alpha')^{d/2-2} \right] \int_0^\infty dx e^{-xm^2} x^{-d/2}
\]

\[
= \frac{N}{(4\pi)^{d/2}} \lambda_{\phi^4} \int_0^\infty dx e^{-xm^2} x^{-d/2}
\]

(4.5)

By using the matching condition established at the tree level (3.21) we get from string theory the tadpole diagram of \(\Phi^4\)-theory.

### 4.2 Candy diagram

![Candy diagram](image)

Figure 6: Candy diagram

Let us now derive the candy diagram from the four-tachyon one-loop amplitude:

\[
A_4^{(1)}(p_1, p_2, p_3, p_4) = \frac{N}{(4\pi)^{d/2}} \text{Tr} \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] \frac{1}{(2\alpha')^{d/2}} \left[ 2g_s(2\alpha')^{d/4} \right]^4 \times \int_0^1 \frac{dk}{k^2} \left[ -\frac{1}{2} \log k \right]^{-d/4} \int_k^1 \frac{dz_4}{z_4} \int_k^1 \frac{dz_3}{z_3} \int_k^1 \frac{dz_2}{z_2} \prod_{i<j=1}^4 \left[ \frac{\exp(\mathcal{G}(z_i, z_j))}{\sqrt{z_i z_j}} \right]^{2\alpha' p_i \cdot p_j}
\]

(4.6)

where we have performed the choice \(V_i'(0) = z_i\) and, in particular, we have fixed \(V_1'(0) = z_1 = 1\).
The diagram relative to this amplitude can be obtained by means of the sewing procedure illustrated in Fig. 7.

The four-particle vertices of the candy diagram can be generated by the corner of the moduli space where the Koba-Nielsen variables $z_1 \to z_2$ and $z_3 \to z_4$. This is performed by considering the limit in which the multipliers $B_i$ ($i = 1, 2$) $\to$ 1. We stress here that, in this limit, the Green functions $G(z_1, z_2)$ and $G(z_3, z_4)$ result to be divergent and we regularize them by introducing a cut-off $\epsilon$ on the world-sheet so that $B_i = 1 - \epsilon$. In this limit the length of the strips connecting the three-vertices become very short and in this way the four-particle vertices of the diagram in consideration are generated. Furthermore, in order to select in the loop only the lightest states, we also take the limit in which the multiplier $A \to 0$, and, after having performed both the limits, we send the cut-off to zero in all the regular expressions.

From these geometrical considerations that shed light on the different roles played by the multipliers $A$-like and $B$-like, we select the following corner of the moduli space reproducing the candy diagram of $\Phi^4$-theory:

$$A \to 0 \quad B_i = 1 - \epsilon \to 1$$

(4.7)

Let us now evaluate the amplitude (4.6) in the corner (4.7).

The first step consists in rewriting, in this region of the moduli space, the measure
and the integration region in the amplitude (4.6).

The ordering of the Koba-Nielsen variables determines the integration regions of the multipliers $A$ and $B_i$ in terms of which the whole amplitude is expressed, after the sewing. More precisely, in the limits (4.7), one gets:

$$
\int_0^1 \frac{dk}{k^2} \int_0^1 \frac{dz_2}{z_2} \int_k^{z_2} \frac{dz_3}{z_3} \int_k^{z_3} \frac{dz_4}{z_4} \simeq \int_0^1 \frac{dk}{k^2} \int_0^1 dB_1 \int_k^{B_1} \frac{dA}{A} \int_k^{B_2} dA + O(k)
$$

(4.8)

For this diagram, the proper times associated to the single propagators in the loop, are identified with the Schwinger parameters

$$
t_1 = -\alpha' \log k / A \quad \quad t_2 = -\alpha' \log A
$$

(4.9)

where $k$ has to be understood as the proper time of the whole loop.

The Green functions defined in (4.3), in this limit, simplify as follows:

$$
2\alpha' G(z_1, z_2) = 2\alpha' \log \epsilon ,
$$

$$
2\alpha' G(z_3, z_4) = 2\alpha' \log \epsilon ,
$$

$$
2\alpha' G(z_1, z_3) = 2\alpha' G(z_1, z_4) = 2\alpha' G(z_2, z_3) = 2\alpha' G(z_2, z_4) ,
$$

$$
2\alpha' G(z_1, z_3) = -\alpha' \log A - \frac{(\alpha' \log A)^2}{-\alpha' \log k} .
$$

(4.10)

In particular the Green function $2\alpha' G(z_1, z_3)$, when written in terms of the Schwinger parameters, becomes

$$
2\alpha' G(z_1, z_3) = t_2 - \frac{t_2^2}{t_1 + t_2} .
$$

(4.11)

By expressing the full amplitude in terms of $t_1$ and $t_2$ one gets:

$$
A^{(1)}_4(p_1, p_2, p_3, p_4) = \frac{N}{(4\pi)^{d/2} 2} d^{a_1a_2} d^{a_3a_4} \left[ 2^d g_s^4 (2\alpha')^{d-4} \right] \int_0^\infty dt_1 \int_0^\infty dt_2 (t_1 + t_2)^{-d/2} e^{-m^2(t_1 + t_2)} e^{-2\alpha' (p_1 + p_2)^2} \left[ t_2 - \frac{t_2^2}{t_1 + t_2} \right]
$$

(4.12)

Once again we have the right result in field theory by using the matching condition (3.21).
5 Two-loop $\Phi^4$-diagrams from string amplitudes

5.1 Sunset-diagram

In this subsection we study the field theory limit of the two-tachyon two-loop amplitude. It is obtained through the general expression (2.1) considered for $M = h = 2$:

$$A_2^{(2)}(p_1,p_2) = N^2 Tr [\lambda^a \lambda^b] C_2 N_0^2 \int [dm]^2 \left[ \exp \frac{G^{(2)}(z_1, z_2)}{\sqrt{V_1'(0) V_2'(0)}} \right]^{2 \alpha' p_1 \cdot p_2}, \quad (5.1)$$

where the two-loop expressions for $V_i'(0)$ are given by [11]:

$$(V_i'(0))^{-1} = \left| \frac{1}{z_i - \rho_a} - \frac{1}{z_i - \rho_b} \right| \quad (5.2)$$

with $\rho_a$ and $\rho_b$ depending on the position of $z_i$ and being the two fixed points on the left and on the right hand of $z_i$.

For this amplitude we would like to select the regions of moduli space that, in the field theory limit, reproduce the scalar diagrams of $\Phi^4$ theory, in particular in this section we consider the sunset-diagram, depicted in Fig. (8).

![Sunset diagram](image)

Figure 8: Sunset diagram

Also in this case, the first step consists in making a limit that could select the tachyon particles in the internal string loop. This is achieved considering the amplitude in the limit of small multipliers $k_\mu$ and keeping the most divergent contribution.

In this limit the various expressions appearing in the amplitude take a very simple form. In particular, the Green functions become [11]
\[
G^{(2)}(z_i, z_j) = \log(z_i - z_j) + \frac{\log^2 T_{ij} \log k_2 + \log^2 U_{ij} \log k_1 - 2 \log T_{ij} \log U_{ij} \log S}{2(\log k_1 \log k_2 - \log^2 S)} \tag{5.3}
\]

with
\[
S = \frac{(\eta_1 - \eta_2)(\xi_1 - \xi_2)}{(\xi_1 - \eta_2)(\eta_1 - \xi_2)} \quad T_{ij} = \frac{(z_j - \eta_1)(z_i - \xi_1)}{(z_j - \xi_1)(z_i - \eta_1)} \quad U_{ij} = \frac{(z_j - \eta_2)(z_i - \xi_2)}{(z_i - \eta_2)(z_j - \xi_2)} \tag{5.4}
\]

The measure, once used the projective invariance to fix \(z_1 = 1\), \(\xi_2 = +\infty\) and \(\eta_2 = 0\), becomes:
\[
[dm]^2 = \frac{dz_2}{\prod_{i=1}^2 V_i'(0)} \prod_{\mu=1}^2 \frac{dk_\mu}{k_\mu^2} \frac{d\xi_1 d\eta_1}{(\xi_1 - \eta_1)^2} [\det (-i\tau_{\mu\nu})]^{-d/2} \tag{5.5}
\]

where the period matrix \(\tau_{\mu\nu}\), in the limit of small multipliers, is given by:
\[
\det(-i\tau_{\mu\nu}) = \frac{1}{4\pi^2} \left[\log k_1 \log k_2 - \log^2 S\right]. \tag{5.6}
\]

The second step consists in selecting the regions of moduli that, in the field theory limit, reproduce the sunset-diagram of \(\Phi^4\) scalar theory. We cut the two-loop string diagram in all possible three-vertices, as shown in Fig. (5.14). Again, by using the projective invariance, we fix the legs of the all three vertices at \(+\infty, 1, 0\) and we sew them using suitable projective transformation having as fixed points the values of the legs we are sewing.

In the case we are here considering we only need the following transformation
\[
S(z) = A z \quad \text{and} \quad S(z) = B_i z
\]
with \(i = 1, 2\).

At the end of this procedure we reach the configuration depicted in Fig. (9b), i.e.:
\[
\begin{align*}
z_1 &= 1 & z_2 &= AB_1 B_2 & \xi_2 &= +\infty \\
\eta_2 &= 0 & \xi_1 &= B_1 & \eta_1 &= AB_1
\end{align*} \tag{5.7}
\]
From (5.7) we see that the limits to take into account for having the quartic vertices of the sunset, are the ones in which \( z_2 \to \eta_1 \) and \( z_1 \to \xi_1 \). They are simply realized considering a configuration in which the \( B_i \)'s are close to 1, which, in our regularization scheme, means to introduce a cut-off on the world sheet, such that \( B_i = 1 - \varepsilon \).

Furthermore, in order to select scalars in the other sewn legs, we also perform the limit \( A \to 0 \).

In this corner of moduli space we have the following ordering:

\[
\xi_2 = +\infty \gg z_1 = 1 > \xi_1 \gg \eta_1 > z_2 \gg \eta_2 = 0.
\]

and let us now evaluate the amplitude in this corner.

The Green function \( G(z_1, z_2) \), defined in (5.3), takes the following form:

\[
2\alpha' G(z_1, z_2) \simeq 2\alpha' \log(1 - B_1B_2A) + \left[ \alpha' \log(\varepsilon^2) + \alpha' \log A \right]^2 \log k_2 + \left( \alpha' \log A \right)^2 \log k_1 - 2 \left[ \alpha' \log(\varepsilon^2) + \alpha' \log A \right] \left( \alpha' \log A \right)^2 \\
\frac{1}{(\alpha' \log k_1) (\alpha' \log k_2) - (\alpha' \log A)^2} \left( \alpha' \log A \right)^2 \\
\frac{1}{(-\alpha' \log k_1) (-\alpha' \log k_2) - (\alpha' \log A)^2} \left( \alpha' \log A \right)^3
\]

(5.8)

the local coordinates (5.2) become

\[
(V_1'(0))^{-1} = \left| \frac{1}{z_1 - \xi_2} - \frac{1}{z_1 - \xi_1} \right| = \frac{1}{\varepsilon}
\]

Figure 9: Sewing for the sunset diagram
\[ (V'_2(0))^{-1} = \left| \frac{1}{z_2 - \eta_2} - \frac{1}{z_2 - \eta_1} \right| = \frac{1}{A\epsilon} \] (5.9)

while the measure results to be:

\[ [dm]^2 = \prod_{i=1}^{2} dB_i \frac{dk_i}{k_i^2} \frac{dA}{\epsilon^2} (4\pi^2)^{d/2} \left[ \log k_1 \log k_2 - \log^2 A \right]^2 \] (5.10)

In our field theory limit we integrate \( A \)-like variables on the lower boundary of the integration region and \( B \)-like variables in the whole integration region of the multipliers, i.e. \([0, 1]\), as already done in the tree level case.

The Schwinger parameters of the field theory diagram are related, as explained in the previous sections, to the proper times of the propagators used in the sewing. In this case we get:

\[ t_1 = -\alpha' \log A \quad t_i = -\alpha' \log \frac{k_1}{A} \] (5.11)

with \( i = 2, 3 \). Expressing the whole amplitudes in terms of these parameters, we rewrite the exponential involving the Green function as:

\[
\left[ \exp G^{(2)}(z_1, z_2) \right]^{2\alpha' p_1 \cdot p_2} \left[ V'_1(0) V'_2(0) \right] = \exp \left\{ -p_1^2 \left[ \frac{t_1}{t_2 t_3 + t_1 t_2 + t_1 t_3} \right] \right\} \] (5.12)

and the measure

\[ [dm]^2 = (\alpha')^{d/2-3} \prod_{i=1}^{3} dt_i \prod_{i=1}^{2} d B_i \left[ t_1 t_2 + t_1 t_3 + t_2 t_3 \right]^{-d/2} e^{-m^2(t_1+t_2+t_3)} e^{-2\alpha' \log \epsilon} \] (5.13)

In the expression of the measure it appears an explicit dependence on the cut-off \( \epsilon \), that disappears in our regularization scheme. Finally, by inserting these expressions in the amplitude (5.1) and writing explicitly the values of the normalization constants we get:

\[
A^{(2)} \left( p_1, p_2 \right) = \frac{N^2}{(4\pi)^d} \left[ 4g_s^2 \left( 2\alpha' \right)^{d/2-2} \right]^2 \int_0^{+\infty} \prod_{i=1}^{3} dt_i \left[ t_1 t_2 + t_1 t_3 + t_2 t_3 \right]^{-d/2} e^{-m^2(t_1+t_2+t_3)} e^{-2\alpha' \log \epsilon} \]
\times \exp \left\{ -p_1^2 \left[ \frac{t_1 t_2 t_3}{t_2 t_3 + t_1 t_2 + t_1 t_3} \right] \right\} \] (5.14)
This expression, once the matching condition (3.21) is used, is coincident, including the overall factor, with the corresponding one obtained in field theory starting from the action (3.20).

5.2 Double candy diagram

In this subsection we show how to get the double-candy diagram of $\Phi^4$ theory, Fig. (10), starting from the two-loop four-tachyon amplitude in bosonic string theory.

The starting point is again the master formula (2.1) with $M = 4$ and $h = 2$:

$$A_4^{(2)}(p_1p_2p_3p_4) = N^2 Tr \left[ \lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_4} \right] C_2 N_0^4 \int [dm]^4 \prod_{i<j} \left[ \frac{\exp \mathcal{G}^{(2)}(z_i, z_j)}{V_i'(0) V_j'(0)} \right]^{2a'_{ij} p_i \cdot p_j}$$

(5.15)

where the expressions for $V_i'(0)$ are given by (5.2).

As in the case of the sunset-diagram, we expand the previous amplitude for small values of the multipliers $k_\mu$, keeping the most divergent contribution that is the one corresponding to the tachyon state and again the Green functions reduce to the form given in (5.3).

The measure, once used the projective invariance to fix $z_4 = 1$, $\xi_2 = +\infty$ and $\eta_2 = 0$, becomes:

$$[dm]^4_2 = \prod_{i=1}^3 \frac{dz_i}{V_i'(0)} \prod_{\mu=1}^2 \frac{dk_\mu}{k_\mu^2} \frac{d\xi_1 d\eta_1}{(\xi_1 - \eta_1)^2} \left[ det (-i\tau_{\mu\nu}) \right]^{-d/2}$$

(5.16)

where the period matrix $\tau_{\mu\nu}$ in the limit of small multipliers is given by Eq. (5.6).
Let us now identify the corner of the moduli space that, in the field theory limit, reproduces the two-loop candy diagram, according to our procedure. In Fig. (11) it is shown the final configuration that we reach applying the sewing procedure with the following projective transformations:

\[ S_i = B_i z \quad \hat{S}_1 = A_1 z \quad \hat{S}_2 = A_2 z \quad (5.17) \]

with \( i = 1, 2, 3. \)

Once the sewing procedure is completed, the Koba-Nielsen variables and the moduli of the surface, are expressed in terms of the multiplier of the transformations. For the sewing configuration shown in Fig. (5.1) we get the following correspondence among multipliers, Koba-Nielsen variables and moduli:

\[ \xi_2 = \infty \quad \eta_2 = 0 \quad \xi_1 = B_1 B_2 A_1 \quad \eta_1 = (1 - A_2) A_1 B_1 \]

\[ z_1 = A_1 B_1 \quad z_2 = [1 - A_2 (1 - B_3)] A_1 B_1 \quad z_4 = 1 \quad z_3 = B_1 \quad (5.18) \]

Again, if we want to obtain the four-particle vertices peculiar of the two-loop candy diagram, we have to take in consideration the corner of the Koba-Nielsen variables characterized by \( z_1 \to z_2, z_3 \to z_4 \) and by the modulo \( \xi_1 \to 1 \). This configuration is achieved considering the limits in which \( B_i \to 1 \) and introducing the suitable regularizators, when necessary.

Furthermore, considering also the limit \( A_i \to 0 \), we select scalar particles in the other internal legs.

The corner of moduli space reproducing the \( \Phi^4 \) scalar diagram illustrated in Fig. 5.1 is

\[ A_i \to 0 \quad B_i = 1 - \epsilon \quad (5.19) \]

Let us now evaluate the amplitude \((5.15)\) in this corner.
The Green functions (5.3) in the limit (5.19) take the simple form:

\[
\alpha' G(z_1, z_3) = \alpha' G(z_1, z_4) = \alpha' G(z_2, z_3) = \alpha' G(z_2, z_4),
\]
\[
\alpha' G(z_1, z_3) = -\frac{1}{2} (-\alpha' \log A_2)^2 - \frac{1}{2} (-\alpha' \log k_1)^2,
\]
\[
\alpha' G(z_1, z_2) \simeq \alpha' \log (A_1 A_2 \epsilon),
\]
\[
\alpha' G(z_3, z_4) \simeq \alpha' \log(\epsilon) \simeq 0. \quad (5.20)
\]

A similar manipulation can be done for the local coordinates \(V'_i(0)\) that in the limit (5.19) become:

\[
(V'_1(0))^{-1} = \left| \frac{1}{z_1 - \eta_1} - \frac{1}{z_1 - \xi_1} \right| \simeq \frac{1}{A_1 A_2} \simeq \frac{1}{A_1 A_2}
\]
\[
(V'_2(0))^{-1} = \left| \frac{1}{z_2 - \eta_1} - \frac{1}{z_2 - \xi_2} \right| \simeq \frac{1}{A_1 A_2}
\]
\[
(V'_3(0))^{-1} = \left| \frac{1}{z_3 - \xi_1} - \frac{1}{z_3 - \xi_2} \right| = 1 \quad (V'_4(0))^{-1} = \left| \frac{1}{z_4 - \xi_1} - \frac{1}{z_4 - \xi_2} \right| = 1 \quad (5.21)
\]

and for the measure:

\[
[dm]^2 = \left[ \prod_{i=1}^{2} \frac{dA_i}{A_i} \right]^3 \prod_{i=1}^{2} dB_i \prod_{i=1}^{2} \frac{dk_i}{k_i^2} (\log k_1 \log k_2)^{-d/2} \left[ \frac{1}{\epsilon^2} \right]. \quad (5.22)
\]

As regards the integration region, we observe that the sewing procedure determines an ordering of the Koba-Nielsen variables and of the fixed points. In the case in consideration here we get in fact the following ordering:

\[
\xi_2 = +\infty \gg z_4 = 1 \geq z_3 = B_1 \geq z_1 = A_1 B_1 \gg \xi_1 = B_2 A_1 B_1 \geq \eta_2 = 0
\]

and

\[
z_1 \geq z_2 = [1 - A_2(1 - B_3)] A_1 B_1 > \eta_1 = (1 - A_2) A_1 B_1.
\]

In the field theory limit (5.19) we integrate the multipliers \(B\)-like between 0 and 1 and the multipliers \(A\)-like, between 0 and \(\delta\) being \(\delta\), a positive infinitesimal quantity.
The Schwinger parameters in this case are related to the $A_i$'s by the following relations \[11\]:

\[
t_{i+2} = -\alpha' \log A_i \quad t_1 = -\alpha' \log \frac{k_1}{A_2} \quad t_2 = -\alpha' \log \frac{k_2}{A_1}
\]

with $i = 1, 2$.

Rewriting the Green functions in terms of the Schwinger parameters:

\[
\prod_{i<j=1}^{4} \left[ \exp \frac{G(2)(z_i, z_j)}{\sqrt{V_i'(0)V_j'(0)}} \right]^{2\alpha' \cdot p_i \cdot p_j} = \exp \left\{ -(p_1 + p_2)^2 \left[ \frac{t_1 t_4}{t_1 + t_4} + \frac{t_2 t_3}{t_2 + t_3} \right] \right\}
\]

and the measure

\[
\int [dm]^4_2 = (\alpha')^{d-4} (2\pi)^d \int_0^{+\infty} \prod_{i=1}^{4} \int_0 t_i \prod_{i=1}^{3} dB_i e^{-m^2(t_1 + t_2 + t_3 + t_4)} (t_2 + t_3)^{-d/2} (t_4 + t_1)^{-d/2}.
\]

we get

\[
A_4^{(2)}(p_1 \cdots p_4) = \frac{N^2}{(4\pi)^2} \sum_{a_1 a_2 t} \frac{24 g_s^2 (2\alpha')^{d-4}}{2^5} \prod_{i=1}^{4} \int_0^{+\infty} \prod_{i=1}^{3} dB_i e^{-m^2(t_1 + t_2 + t_3 + t_4)} (t_1 + t_4)^{-d/2} (t_2 + t_3)^{-d/2} e^{-(p_1 + p_2)^2 \left[ \frac{t_1 t_2 + t_2 t_3 + t_3 t_1}{t_1 + t_2 + t_3} \right]}
\]

where a sum over inequivalent permutations of the external particles has been done analogously as in the one-loop candy-diagram case.

Now using the matching condition (3.21), we get the same result, including the overall factor, as the one obtained in field theory.

In conclusion, we have used the sewing and cutting procedure in order to show how $\Phi^4$-theory diagrams can be reproduced from string amplitudes, up to two loop-order. The whole information so obtained can be in principle extendible to Yang-Mills diagrams involving quartic interactions.
Figure 11: Sewing of the two-loop candy diagram

\[ \eta = (1 - A) A B \]

\[ z = [1 - A(1 - B)] A B \]

\[ \xi = B A B \]

Figure 12: Sewing configuration for the two-loop candy diagram

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