I. INTRODUCTION

Plasma wakefield acceleration (PWA) technique [1–4] attracted many attention over the past several years and is envisione as one of the main options for the design of future collides and light sources [5–10].

Development of the PWA emerged into the rapid development of PWA theories from simple yet powerful analytical models [11–15] to complex and precise simulation codes [16–18]. Nonetheless, significant understanding of physics processes that underlies PWA technology has already been gained there is still an ongoing work on the theoretical side as well as extensive experimental developments. Several question such as optimal beam-loading [19], instability suppression and instability mechanisms [20–22] as well as acceptable tolerances [23] are still under active study.

One of the main difficulties that is common to all wakefield accelerators be it a plasma based or a structure based (dielectric loaded or corrugated structures [24–26]) machines is the beam break-up instability that directly affects the efficiency of the accelerator [27–29]. Many mechanisms and approaches including ion motion [21] BNS damping [20, 22] and other methods of resonance illumination has been investigated.

One of the approaches that stands aside is the transverse shaping of the driver beam that mainly includes injection of the flat beam (or the beam with the high ellipticity) [30–32] and dual driver injection [33]. It was predicted [33, 34] and recently experimentally demonstrated [35] that highly elliptic driver can suppress transverse wake in a structure based wakefield accelerating device. It turns out that asymmetric drivers are favorable and promising approach in the hollow channel plasma as well [36].

It has been recently realized that path towards a real machine unavoidably includes staging of the individual cells and this in turn results in tolerances in plasma cell alignments as well as on tolerances in plasma uniformity. The indirect evidence of the non-uniformity effects could be seen from the pioneering experiment on hollow channel plasma described in Refs.[37, 38].

Being motivated by the recent success on the flat beam application and the need of preliminary analysis on the transverse plasma non-uniformity, in this paper we develop a simple analytical model of the blow out plasma regime formed by the infinitely flat driver and investigate how local non-uniformity of the plasma density affects the problem. We demonstrate that in contrast to the similar cylindrically symmetric problem with the point driver considered in Refs.[13, 14] in the planar case the plasma has two regimes, namely a laminar flow and a turbulent flow. We demonstrate as well that in the case of a laminar flow and non-relativistic plasma electrons the transverse wake is absent even in the case of the transversely nonuniform plasma.

The model and results of the presented analysis may serve as a starting point for further investigation of the flat beam injection into the nonuniform plasma as well as may be useful for some basic parameter estimations for the ongoing experimental effort at the Argonne Wakefield Accelerator facility [39].

II. BASIC EQUATIONS

In this section we provide a brief overview of the the model that we utilize for the calculus and analysis.

A. General formulas

We start from the set of equations derived in Ref.[14]. We use the same convention as the Ref.[14] and we use dimensionless variables: time is normalized to $\omega_p^{-1}$, length to $k_p^{-1}$, velocities to the speed of light $c$, and momenta to $mc$. We also normalize fields to $mc\omega_p/e$, forces to $mc\omega_p$, potentials to $mc^2/e$, the charge density to $n_0e$, the plasma density to $n_0$, and the current density to $en_0c$. With $e$ being the elementary charge, $e > 0$.

Equation of motion for the plasma electrons could be written as

$$\frac{d\mathbf{p}}{dt} = \nabla \psi + \mathbf{z} \times \mathbf{B} - \mathbf{v} \times \mathbf{B}, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\gamma}. \quad (1)$$
Here \( \mathbf{p} \) is the momentum of the plasma electrons, \( \gamma = \sqrt{1 + \mathbf{p}^2} \) is the relativistic gamma factor of the plasma electrons, \( \mathbf{v} = \mathbf{p}/\gamma \) is the velocity and \( \psi = \phi - A_z \) is the pseudo potential that defines the wake field as

\[
E_z = \frac{\partial\psi}{\partial\xi}, \quad \mathbf{F}_\perp = -\nabla_\perp \psi, \tag{2}
\]

and \( \nabla = (\partial_x, \partial_y, -\partial_z) \). Here \( \mathbf{F}_\perp \) is the transverse part of the Lorentz force per unit charge of the test particle and \( \xi = t - z \).

Eqs.(1) have the following integral of motion

\[
\gamma - p_z - \psi = 1, \tag{3}
\]
as a consequence we have

\[
1 - v_z = \frac{1 + \psi}{\gamma}. \tag{4}
\]

In a quasi-static picture it is convenient to replace the derivative by time \( t \) with the derivative by \( \xi \). We use the fact that

\[
\frac{d\xi}{dt} = 1 - v_z, \tag{5}
\]

consequently for an arbitrary function \( f(\xi) \) we have.

\[
\frac{df}{dt} = \frac{d\xi}{dt} \frac{df}{d\xi} = (1 - v_z) \frac{df}{d\xi} = \frac{1 + \psi}{\gamma} \frac{df}{d\xi}. \tag{6}
\]

Since in a quasi-static picture momenta of the plasma electron is a function of \( \xi \) Eqs.(1) with Eq.(6) are reduced to

\[
\frac{d\mathbf{p}_\perp}{d\xi} = \frac{\gamma}{1 + \psi} \nabla_\perp \psi + \hat{\mathbf{z}} \times \mathbf{B}_\perp - \frac{B_z}{1 + \psi} \mathbf{p}_\perp \times \hat{\mathbf{z}}. \tag{7}
\]

Equation for the pseudo potential reads

\[
\Delta_\perp \psi = (1 - v_z)n_e - n_i(x), \tag{8}
\]

here \( n_e \) is the plasma electron density and \( n_i(x) \) is the ion density that depends on \( x \). In what follows, we will assume that

\[
n_i(x) = 1 + gx, \tag{9}
\]

with \( g \ll 1 \). Equations for the magnetic field are

\[
\Delta_\perp B_z = \hat{\mathbf{z}} \cdot (\nabla_\perp \times n_e \mathbf{v}_\perp), \tag{10}
\]
\[
\Delta_\perp \mathbf{B}_\perp = -\hat{\mathbf{z}} \times \nabla_\perp n_e \mathbf{v}_z - \hat{\mathbf{z}} \times \partial_\xi n_e \mathbf{v}_\perp. \tag{11}
\]

The final equation that complements equations for the fields is the continuity equation for the plasma given by

\[
\partial_\xi [n_e(1 - v_z)] + \nabla_\perp \cdot n_e \mathbf{v}_\perp = 0. \tag{12}
\]

B. 2D approximation

If one assumes that the charge density is uniform and the charge is distributed along the infinite line in \( y \) direction then the following simplification to the main set of equations could be made. Eq.(2) reduces to

\[
E_z = \frac{\partial\psi}{\partial\xi}, \quad F_x = -\frac{d\psi}{dx}, \quad F_y = 0. \tag{13}
\]

Equation for the pseudo potential (8) reads

\[
\frac{d^2\psi}{dx^2} = (1 - v_z)n_e - n_i, \tag{14}
\]

and continuity equation (12) reads

\[
\partial_\xi [n_e(1 - v_z)] + \frac{d}{dx} n_e v_x = 0. \tag{15}
\]

III. SHOCK WAVE

As discussed in details in Ref.[14] plasma electrons cross an infinitesimally thin layer in which the fields produced by the driver have a delta-function discontinuity

\[
\mathbf{E}_\perp = \mathbf{D}\delta(\xi), \\
\mathbf{B}_\perp = \hat{\mathbf{z}} \times \mathbf{D}\delta(\xi) \tag{16}
\]

with the transverse profile defined by the vector \( \mathbf{D} = (D_x, D_y) \).

To solve for the shock wave at \( \xi = 0 \), we assume that the plasma density in front of the moving driver has a linear gradient

\[
n_e^{(0)} = 1 + gx, \tag{17}
\]

where the uniform part of the density in the units that are introduced is 1, \( g \) is a constant, and \( x \) is the transverse coordinate. We will assume a small gradient,

\[
g \ll 1, \tag{18}
\]

and use the perturbation theory.

We consider equation for the vector \( \mathbf{D} \) that according to Ref.[14] reads

\[
\Delta_\perp \mathbf{D} = \frac{n_e^{(0)}}{70} \mathbf{D}. \tag{19}
\]

If we set \( g = 0 \) then due to the symmetry we have \( D_y^{(0)} = 0 \) and \( D_x^{(0)} = D_x^{(0)}(x, \xi) \). This immediately results in the equation for \( D_x^{(0)} \) in a form

\[
\frac{d^2}{dx^2} D_x^{(0)} = \frac{n_e^{(0)}}{70} D_x^{(0)}. \tag{20}
\]
With the unmodified plasma density and initial gamma set to the unity \((\nu_{e}^{(0)} = 1, \gamma_{0} = 1)\) for boundary conditions \(D_{x}(\pm\infty) = 0, D_{y}(x \to \pm 0) = \pm 2\pi \lambda\), where \(\lambda\) is the line charge density of the beam, we have

\[
D_{x}^{(0)} = \begin{cases} 
2\pi \lambda \exp(-x) & x > 0 \\
-2\pi \lambda \exp(x) & x < 0
\end{cases}.
\]

(21)

Now we consider \(n_{0}\) as given by Eq.(17). We apply perturbation theory and introduce anzats

\[
D_{x} = D_{x}^{(0)} + gD_{x}^{(1)},
\]

(22)

\[
D_{y} = D_{y}^{(0)} + gD_{y}^{(1)}.
\]

(23)

Substituting Eqs.(22) and Eq.(17) into the Eqs.(19), equating terms of the same order and assuming for simplicity \(x > 0\) we arrive at a set of equations for the corrections in the form

\[
\frac{d^{2}}{dx^{2}}D_{x}^{(1)} = D_{x}^{(1)} \pm 2\pi \lambda x \exp(\mp x),
\]

\[
\frac{d^{2}}{dx^{2}}D_{y}^{(1)} = D_{y}^{(1)}.
\]

(23)

Corrections should vanish both at the infinity and near the source \(D_{x,y}^{(1)}(x \to 0) = 0\) and \(D_{x,y}(\pm \infty) = 0\), consequently

\[
D_{x}^{(1)} = \mp \frac{\pi \lambda}{2} x(1 \pm x) \exp(\mp x),
\]

(24)

\[
D_{y}^{(1)} = 0.
\]

(24)

Combining Eq.(21) with Eq.(24) we finally arrive at

\[
D_{x} = \pm 2\pi \lambda \exp(\mp x) \mp \frac{g\pi \lambda}{2} x(1 \pm x) \exp(\mp x),
\]

(25)

\[
D_{y} = 0.
\]

(25)

Eq.(25) fully defines the shock electromagnetic wave produced by the infinitely long (in \(y\)) line that propagates along the \(z\)- axis.

**IV. SHAPE OF THE PLASMA BUBBLE**

### A. Ballistic approximation

As the first step in our considerations we neglect the effect of the plasma self-fields on the trajectories of the plasma electrons. This is a “ballistic” regime of plasma motion introduced in Ref.[13]; it assumes that the plasma electrons are moving with constant velocities.

We assume plasma electrons to be non-relativistic and \(v_{\parallel} \approx 1\), consequently for the upper half plane we get

\[
\frac{dx}{d\xi} \approx -2\pi \lambda \exp(-x_{0}) + \frac{g\pi \lambda}{2} x_{0}(1 + x_{0}) \exp(-x_{0}).
\]

(26)

**Figure 1.** Plasma flow in the ballistic approximation for the case of \(|\lambda| = 1/2\pi\) and \(g = 0\). Red line denotes trajectory with \(x_{0} = 0\) Eq.(32), and blue line denotes the envelope of the two stream flow given by Eq.(33), dashed line shows switching point for the plasma flow.

Solution to the equations above gives electron trajectories

\[
x=x_{0} - 2\pi \lambda \xi \exp(-x_{0}) + \frac{g\pi \lambda \xi}{2} x_{0}(1 + x_{0}) \exp(-x_{0}).
\]

(27)

Fist we consider the case of \(g = 0\) and plot electron trajectories in Fig.1 setting for the simplicity \(\lambda = 1/2\pi\). From Fig.1 we observe that plasma flow has two regimes: laminar flow, when trajectories do not cross and two stream flow. Switching point \(\xi_{sw}\) could be found from the following considerations. We consider the upper half plane \((x > 0)\). Two stream flow appears when the most inner electron trajectory crosses the next closest trajectory. This condition could be expressed as

\[
-2\pi \lambda \xi_{sw} = \delta x_{0} - 2\pi \lambda \xi_{sw} \exp(-\delta x_{0}).
\]

(28)

Here \(\delta x_{0}\) is the distance between starting points of the two trajectories. Solving for \(\xi_{sw}\) and assuming \(\lambda < 0\) we get

\[
\xi_{sw} = \frac{1}{2\pi|\lambda|} \frac{\delta x_{0}}{1 - \exp(-\delta x_{0})}.
\]

(29)

Proceeding with the limit \(\delta x_{0} \to 0\) we finally have

\[
\xi_{sw} = \frac{1}{2\pi|\lambda|}.
\]

(30)

Before the switching point bubble has a linear "triangular" shape and the boundary is simply defined by the most inner trajectory

\[
x_{b} = 2\pi|\lambda|\xi, \quad \xi < \xi_{sw}.
\]

(31)

Bubble boundary for the two stream flow (for \(\xi \geq \xi_{sw}\)) could be found as an envelope for the plasma electron
trajectories from the condition
\[ \frac{dx}{dx_0} = 0, \]  
and reads
\[ x_b = 1 + \ln \left( 2\pi|\lambda|\xi \right), \quad \xi \geq \xi_{sw}. \]  

Next we account for the plasma gradient (now \( g \neq 0 \)). Plasma flow still have two regimes and the switching point now should be found from the the condition
\[ 2\pi|\lambda|\xi_{sw} = \delta x_0 + 2\pi|\lambda|\xi_{sw} \exp(-\delta x_0) - \frac{g\pi|\lambda|\xi_{sw}}{2} \delta x_0 (1 + \delta x_0) \exp(-\delta x_0). \]  
Solving for \( \xi_{sw} \) gives
\[ \xi_{sw} = \frac{\delta x_0}{2\pi|\lambda| - \exp(-\delta x_0) + \frac{g}{2}\delta x_0 (1 + \delta x_0) \exp(-\delta x_0)}. \]  
Proceeding with the limit \( \delta x_0 \to 0 \) we have
\[ \xi_{sw} = \frac{1}{2\pi|\lambda|}. \]  

We note that in the case of a small plasma gradient, the switching point remains at the same location as in the case of the uniform plasma and the first part of the plasma boundary remains the same and given by Eq.(32). It is worth to mention that if \( \lambda \ll 1 \) then switching point \( \xi_{sw} \to \infty \) and plasma flow is laminar. We note that in the case \( g \neq 0 \) boundary for the two stream part of the flow could not be expressed in a simple form as Eq.(32) becomes a transcendental equation. However, the flow could be easily found numerically.

### B. Small charge regime

Now we switch to a different approximation and do not neglect shielding anymore. First we derive general expression for the force that acts on the plasma electrons. Form Eq.(14) with Eq.(13) in the case of the non-relativistic plasma electrons (\( v_z \ll 1 \)) we get
\[ \frac{dF_x}{dx} = n_i - n_e. \]  
Integrating equation above we get
\[ F_x(x) = \int_{0}^{x} (n_i - n_e) d\tilde{x} + F_x(0). \]  
At rest plasma is electrically neutral so the total charge of the electrons and ions should be the same in both lower \((x < 0)\) and upper \((x > 0)\) half planes. As far as the charge is conserved one may write
\[ \int_{0}^{\infty} (n_i - n_e) d\tilde{x} = 0, \]  
consequently from Eq.(38) we have \( F_x(\infty) = F_x(0) \). Lorentz force \( F_x \) have to vanish at the infinity, consequently we conclude that \( F_x(0) = 0 \). With this observation one may rewrite Eq.(38) as
\[ F_x(x) = \int_{0}^{x} (n_i - n_e) d\tilde{x}. \]  

We notice that if the two stream regime does not develop then the shape of the bubble is always defined by the most inner trajectory of plasma electron that starts at \( x_0 = 0 \). Force that acts on this electron from the side of the plasma ions could be found from Eq.(39) with \( n_e = 0 \) (inside the bubble there are no electrons). For the unperturbed case of \( g = 0 \) we get familiar expression of the ion focusing force
\[ F_x^i = x, \quad x \leq x_b. \]  

We note that \( F_x \) was defined as a force per unit charge, consequently the force that acts on the plasma electron is recovered by multiplying \( F_x \) by the charge of the plasma electron. In our notations it is simply \(-1\).

Using initial conditions \( x_0 = 0 \) and \( V_0 = -2\pi\lambda \) one can arrive at the expression for the bubble shape in the form
\[ x_b = 2\pi|\lambda| \sin(\xi). \]  
The range of validity for the formula above could be established from the condition of the crossing of the two closest trajectories and following the same steps as in the ballistic approximation one can deduce equation for the switching point in the form
\[ \sin(\xi_{sw}) = \frac{1}{2\pi|\lambda|}. \]  
For this equation to have no real solutions for the \( \xi_{sw} \) the inequality \( \sin(\xi_{sw}) > 1 \) should hold, consequently
\[ |\lambda| < \frac{1}{2\pi}. \]  
Condition above sets the upper limit on the line charge density for the plasma flow to be laminar and described by Eq.(41).

Next we notice that if the condition (43) significantly fulfilled then small perturbation to the ion and electron densities as well as to the ion force should not induce two stream regime and the plasma flow should remain laminar. Consequently, one may account for the ion density gradient
\[ n_i = 1 + gx, \]  

with the help of Eq.(39) we get
\[ F^i_x = x + g \frac{x^2}{2}, \quad x \leq x_b. \] (45)

As far as bubble boundary is given by the most inner electron trajectory equation for the bubble boundary reads
\[ \frac{d^2x_b}{d\xi^2} = -x_b - g \frac{x_b^2}{2}, \] (46)

with the initial conditions \( x_b(0) = 0 \) and \( x'_b(0) = 2\pi\lambda \).
Equation (46) could be solved using perturbation method and solution up to the terms of the order \( O(g^2) \) reads
\[ x_b = 2\pi|\lambda| \sin(\xi) - \frac{8\pi^2\lambda^2}{3} g \left[ \sin\left(\frac{\xi}{2}\right) \right]^4. \] (47)

From Eq.(42) we observe that in the case of the laminar flow and no plasma gradient bubble shape is universal and scales linearly with the charge. However, in the case of the plasma gradient as follows from Eq.(47) bubble shape always depends on the charge. Introducing normalized variable \( x^n_b = x_b/(2\pi|\lambda|) \) we arrive at the final bubble shape in the form
\[ x^n_b = \sin(\xi) - \frac{4\pi|\lambda|}{3} \left[ \sin\left(\frac{\xi}{2}\right) \right]^4. \] (48)

We conclude that transverse plasma gradient results in "bending" of the bubble towards less dense plasma. Qualitatively this asymmetry is proportional to the product of the driver charge density and the strength of the plasma gradient.

V. PLASMA FLOW AND PLASMA DENSITY

Immediately behind the driver, at \( \xi = 0^+ \), the plasma density \( n_0 \) is given by Eq.(17).

If we assume that electron trajectories are known then from the continuity of the plasma flow we conclude that
\[ n_e(x, y, \xi) dS = n_0(x_0, y_0) dS_0 \] (49)

from which it follows that
\[ n_e(x, y, \xi) = n_0(x_0, y_0) \frac{dS_0}{dS}. \] (50)

Accounting for the translation symmetry in \( y \) we simply have
\[ n_e(x, \xi) = n_0(x_0) \frac{dx_0}{dx}. \] (51)

or, substituting \( n_0 \) from Eq.(17), we have
\[ n_e(x, \xi) = (1 + g x_0) \frac{dx_0}{dx}. \] (52)

We note that if plasma flow is laminar (electron trajectories do not cross) then force that is associated with the plasma electrons could be easily found with the help of Eq.(39) and with Eq.(52) reads
\[ F^e_x = - \int_0^x (1 + g x_0) \frac{dx_0}{dx} dx = -x_0(x) - g x_0(x^2) \frac{2}{2}. \] (53)

Interestingly this leads to a closed form solution for the electron trajectories as the equation of motion for each plasma electron with the initial conditions \( x(0) = x_0 \) and \( x'(0) = -D_x(x_0) \) now reads
\[ \frac{d^2x}{dx^2} = -x - g \frac{x^2}{2} + x_0 + g \frac{x^2}{2}. \] (54)

Equation above could be again solved using perturbation approach.
\[ x = x^{(0)} + gx^{(1)} + O(g^2). \] (55)

With Eq.(25) equation for the leading order \( x^{(0)} \) reads
\[ \frac{d^2x^{(0)}}{d\xi^2} = -x^{(0)} + x_0, \]
\[ x^{(0)}(0) = x_0, \] (56)
\[ \left. \frac{dx^{(0)}}{d\xi} \right|_{\xi=0} = \pm 2\pi|\lambda| \exp(\mp x_0). \]

Here the upper sign is for the case \( x > 0 \) and lower sign is for case \( x < 0 \). Solution to this equation reads
\[ x^{(0)} = x_0 \pm 2\pi|\lambda| \exp(\mp x_0) \sin(\xi). \] (57)

Next to the leading order \( x^{(1)} \) could be found by substituting Eq.(55) with Eq.(25) into the Eq.(54) and equating terms of the same order. After some algebra we have
\[ \frac{d^2x^{(1)}}{d\xi^2} = -x^{(1)} - \left( \frac{x^{(0)}}{2} \right)^2 + \frac{x^2}{2}, \]
\[ x^{(1)}(0) = 0, \]
\[ \left. \frac{dx^{(1)}}{d\xi} \right|_{\xi=0} = \mp \frac{\pi|\lambda|}{2} x_0 (1 \pm x_0) \exp(\mp x_0). \] (58)
After substituting Eq. (57) into the Eq. (58) expression for the \( x^{(1)} \) is found to be
\[
x^{(1)} = -\frac{8\pi^2|\lambda|^2}{3} \exp(-2x_0) \left[ \sin\left(\frac{\xi}{2}\right)^4 \right]
\]
\[
\pm \pi|x_0\xi \exp(\mp x_0) \cos(\xi)
\]
\[
\mp \frac{\pi|x_0|^2}{2} \exp(\mp x_0) [3 \pm x_0 \sin(\xi)].
\]

Combining Eq. (57) and Eq. (59) with the help of the Eq. (55) we plot electron trajectories in Fig. 2 (left panel) for the case \( g = 0.2 \) (moderately in-homogeneous plasma density) and \( |\lambda| = 0.6/2\pi \). As expected, the flow is laminar and modification to the bubble shape is very small (of the order \( \sim \lambda^2 g \)) as dictated by the Eq. (47).

To examine the total plasma density \( n(x, \xi) = n_i(x, \xi) - n_e(x, \xi) \) we compare several cases \( n(x, \pi/4), n(x, \pi/8) \) and \( n(x, \pi/2) \) in Fig. 3. The plot is produced with the help of the Eq. (52). As far as the flow is laminar the solution to the Eq. (55) with Eq. (57) and Eq. (59) is unique with respect to \( x_0 \) for a given \( x \) and could be found numerically. Consequently, one may derive explicit expression for the plasma density in terms of \( x \) and using this numerical solution compute \( n(x_0(x), \xi) \). We do not provide final expressions as they are bulky but could be easily produced once needed.

As could be seen from the Fig. 3 in contrast to the cylindrical symmetric case (see Ref. [13]) plasma density does not have a singularity at the plasma boundary, however it has a jump that corresponds to the jump in the electron density. We observe as well, that as expected jump in the electron density increases towards the bubble wall and is maximal at the bubble boundary.

In Fig. 4 we plot plasma density close to the end of the bubble \( \xi = 3\pi/4 \) where the witness beam is placed to maximize acceleration rate. One can observe that modest initial seed in plasma density gradient \( g = 0.2 \) results in a sensible imbalance of the total plasma density at the bubble boundary. This in turn may result in the asymmetry of the witness self wake and consequently may affect the emittance of the witness beam.

\section{VI. \textbf{Wakefield}}

It was established in Sec. IV B that in the small charge regime a flat driver does not produce any transverse wake even in the case of the non-homogeneous plasma. The force is purely focusing inside the bubble and is given by Eq. (45). Combining first and second formula in Eq. (13) we get
\[
\frac{dE_z}{dx} = -\frac{dF_x}{d\xi}.
\]

The expression above is widely known as Panofsky-Wenzel theorem (see Ref. [40, 41]) From Eq. (60) we immediately conclude that \( E_z \) is constant inside the bubble and depends on \( \xi \) only.

Substitution of Eq. (38) into Eq. (60) gives
\[
\frac{dE_z}{dx} = -\frac{dF^0_x}{d\xi},
\]

with \( F^0_x \) given by Eq. (53). Expanding the \( \xi \) derivative on the right hand side and integrating over \( x \) we get for the \( E_z \) inside the bubble
\[
E_z = \int_{x_b} dx_0 \frac{dx}{d\xi} dx + g \int_{x_b} dx_0 \frac{dx}{d\xi} dx_0.
\]

Interestingly both integrals in the expression above could be evaluated explicitly. Indeed, switching from integration over \( x \) to integration over \( x_0 \) we get
\[
E_z = \int_{0}^{\infty} dx_0 \frac{dx(x_0)}{d\xi} dx_0 + g \int_{0}^{\infty} dx_0 \frac{dx(x_0)}{d\xi} dx_0.
\]
Figure 5. Longitudinal wake potential per unit length $W_z = -E_z/\lambda$ for the case $|\lambda| = 0.6/2\pi$, $g = 0.2$ (blue line) and $g = 0$ (red dashed line).

with $x(x_0)$ defined through Eq.(55) and $x^{(0)}$ and $x^{(1)}$ given by Eq.(57) and Eq.(59) respectively.

After some algebra we finally get

$$E_z = -2\pi \lambda \cos(\xi) - g\pi \lambda \left[ \frac{\cos(\xi)}{2} + \sin(\xi) \left[ \frac{2\pi \lambda}{3} - \frac{2}{3} \pi \lambda \cos(\xi) - \xi \right] \right].$$

We observer that if the plasma is homogeneous than expression for the electric field is exactly the same as in the case of the linear plasma response despite the bubble formation. Local density gradient, however leads to the slight (proportional to the magnitude of the density gradient $g$) nonliterary in the wakefield (see Fig.5).

We would like to reiterate two important conclusions. In the case of a laminar plasma flow an infinitely flat driver does not produce any transverse wake even if the local plasma density fluctuates from the equilibrium. In the laminar regime of the plasma flow in the flat case the longitudinal component of the electric field $E_z$ generated by the driver is constant inside the bubble and in the case of a homogeneous plasma has the same harmonic from as in the case of the linear plasma response.

VII. CONCLUSION

In this paper we have suggested a simple model of the flat bubble formation by the flat driver beam. Using suggested model and ballistic approximation introduced in Ref.[13] we have demonstrates that in the case of the flat drive two regimes (laminar and turbulent) could develop. The switching point where one type of the flow switches to another depends solely on the line charge density of the driver.

We have investigated laminar plasma flow regime that is generate by the flat electron driver. It was demonstrated that a small perturbation to the plasma density results in “bending” of the bubble towards lower plasma gradient. Despite such a modification to the bubble shape and symmetry breaking, in the case of the non-relativistic plasma flow the wake generated by the flat driver does not have any deflecting components. Focus- ing force of the ion column has the same gradient as the initial gradient of the plasma density. It is worth mentioning that this gradient is independent of the longitudinal coordinate $\xi$ and consequently should not affect the emittance of the accelerated beam.

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