Communication cost of classically simulating a quantum channel with subsequent rank-1 projective measurement

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A process of preparation, transmission and subsequent projective measurement of a qubit can be simulated by a classical model with only two bits of communication and some amount of shared randomness. However, no model for \( n \) qubits with a finite amount of classical communication is known at present. A lower bound for the communication cost can provide useful hints for a generalization. It is shown for example that the amount of communication must be greater than \( c2^n \), where \( c \simeq 0.01 \). The proof uses a quite elaborate theorem of communication complexity. Using a mathematical conjecture known as the “double cap conjecture”, we strengthen this result by presenting a geometrical and extremely simple derivation of the lower bound \( 2^n - 1 \). Only rank-1 projective measurements are involved in the derivation.

One of the main differences between quantum and classical physics lies in the concept of state. Whereas a classical state is associated with something that can be observed at least in principle, a quantum state is a mathematical object that provides mere information about the outcome probabilities of any conceivable measurement. This distinction is important and makes the following question nontrivial: How many bits of classical communication are necessary for simulating the communication of \( n \) qubits? Although the Hilbert space is continuous, however the full infinite information about the quantum state is not accessible in a single experimental realization. Thus, the goal of a classical simulation is much less than communicating the classical description of a quantum state. Its purpose is to reproduce the measurement outcomes, performed after the communication, in accordance with the quantum predictions. Indeed, it was shown in Ref. [1] that the communication of one qubit and any subsequent projective measurement can be simulated by a classical protocol using only 2.19 bits of communication on average. This result was improved in Ref. [2], where Toner and Bacon reported a protocol requiring just 2 bits of communication for each realization. Recently an alternative model was derived in Ref. [3]. All these protocols use some resource of shared randomness, that is, the sender and receiver share some set of random variables.

Classical models of quantum channels are important in quantum communication complexity [1] because they can establish a limit on the advantage that a quantum channel can provide on a classical channel for solving problems of distributed computing. Indeed, an optimal classical model could provide a natural measure of the power of quantum channels. However, no generalization to \( n \) qubits is known at present. A model for \( n \) qubits can be derived from a protocol reported in Ref. [3], however it requires a two-way classical communication. Furthermore, the amount of communication in each execution is not bounded and can be arbitrarily large, although its average is finite. Approximate models were reported in Ref. [2] and require an amount of communication growing linearly with \( n \). A lower bound for the communication cost of an exact simulation can turn out to be useful for finding an optimal exact protocol. First, it would lead to focus on attempts that satisfy the constraint. Second, the particular reasoning used for deriving a lower bound can suggest some general structure that the model should have, especially if the derivation is easily visualizable and does not require too much technicality. A lower bound was derived for example in Ref. [6], where Brassard et al. showed that the amount of communication cannot be smaller than \( c2^n \), where \( c \simeq 0.01 \). They considered the related problem of simulating quantum entanglement with classical communication, however the result can be easily adapted to the case of quantum channels. Their proof uses an elaborate theorem of communication complexity, which is not easily accessible without some technical knowledge.

In this article, we strengthen their result by deriving the lower bound \( 2^n - 1 \). The derivation is extremely simple and uses a geometry conjecture known as the “double cap conjecture”. Although this conjecture is a mathematical open problem, however there are some reasons supporting its plausibility. Unlike in Ref. [6], only rank-1 projective measurements are used in the derivation. This feature has a nontrivial consequence. Suppose that two parties perform local two-outcome measurements on a bipartite quantum state. In Ref. [2] it was shown that the quantum correlation of the outcomes can be classically simulated by using only two bits of communication for any dimension of the Hilbert space. However, these simulations do not correctly reproduce the marginal probabilities of the local outcomes. Using our result, we derive an exponential lower bound for the communication cost of reproducing the full probability distribution of the outcomes in the scenario of Ref. [2].

Let us state the double cap conjecture.

**Conjecture 1.** Let \( \mathcal{A} \) be the whole class of measurable subsets, \( M \), of the hypersphere \( \mathbb{S}^{d-1} = \{ \vec{x} \in \mathbb{R}^d : |\vec{x}| = 1 \} \), so that the sets \( M \) do not contain pairs of orthogonal vectors, that is,

\[
\vec{x}_1, \vec{x}_2 \in M \Rightarrow \vec{x}_1 \cdot \vec{x}_2 \neq 0.
\]

The supremum of volumes of such sets in \( \mathcal{A} \) is equal to
the volume of two opposite caps of angular width $\pi/2$.

We set the $(d-1)$-dimensional volume of the sphere equal to one. According to conjecture 1, the maximum volume of $M$ is

$$V_d = \int_0^{\pi/4} \sin^{d-2} x \, dx \int_0^{\pi/2} \sin^{d-2} x \, dx. \quad (2)$$

For large $d$, the volume decreases exponentially as $2^{-\frac{d}{2}} \simeq 1.414^{-d}$. Indeed, this asymptotic behaviour is supported by the Frankl-Wilson theorem \[8\] which gives the upper bound $1.203^{-d}$ for the maximum volume. A result of Raigorodskii further lowers the upper bound to $1.225^{-d}$ \[8\], which is closer to the value given by the double cap conjecture. Besides these clues, it is also possible to give an intuitive reasoning in favour of conjecture 1. The reasoning is by construction. Suppose that we start with a set containing only a small region $\delta M_1$. This region is associated with a strip of forbidden points around a geodesic, that is, points that cannot be added to $M$ without breaking constraint \[1\]. We can also take the specular image on the opposite side since this does not increase the forbidden region. Then, we add another small region $\delta M_2$ and its opposite image to $M$. Thus, we have to add another strip of forbidden points (Fig. 1). If $d$ is greater than 2, it is better to make $\delta M_2$ as close as possible to $\delta M_1$, since this increases the overlap between the two strips and reduces the overall region of forbidden points (Fig. 1). In this way, we have more space for expanding the set $M$. The procedure is repeated and other small regions are added close to the previous ones. This reasoning suggests that the points of the maximum set satisfying constraint \[1\] are collected around some symmetry axis of the sphere, that is, the maximum set is the union of two opposite caps (Fig. 1) with angular width $\pi/2$. Notice that this reasoning does not work for $d = 2$, since the forbidden regions associated with two non-overlapping regions are always non-overlapping. Nevertheless the double cap sets (namely two arcs) are still maximal, although they are not the only ones.

Conjecture 1 has a natural generalization to vector spaces over the complex field. Let $C^{2N-1}$ be the set of unit vectors in a complex vector space $\mathbb{C}^N$, that is,

$$C^{2N-1} = \{ \mathbf{x} \in \mathbb{C}^N : |\mathbf{x}| = 1 \}.$$ 

There is a one-to-one correspondence between the elements in $C^{2N-1}$ and the points on a $(2N-1)$-dimensional sphere $S^{2N-1}$. Let us define the measure on $C^{2N-1}$ induced by the measure on $S^{2N-1}$.

**Conjecture 2.** Let $A$ be the whole class of measurable subsets, $M$, of $C^{2N-1}$ so that the sets $M$ do not contain pairs of orthogonal vectors. The supremum of volumes of such sets in $A$ is equal to the volume of a set of vectors $\mathbf{x}$ satisfying the condition $|\mathbf{x} \cdot \mathbf{s}|^2 > \frac{1}{2}$, where $\mathbf{s}$ is some unit vector.

Notice that the maximum set $M$ contains rays, that is, if $\mathbf{x} \in M$, then $\alpha \mathbf{x} \in M$ for any complex number $\alpha$ of modulus 1. In analogy with the real case, we concisely call the maximum set of conjecture 2 “double cap set”. The intuitive argument given in favour of conjecture 1 can be safely used for supporting conjecture 2. Also in this case, the two-dimensional case is special and the double cap set is not the only maximum set satisfying the constraint of the conjecture. Setting the $(2N-1)$-dimensional volume of the hypersphere equal to 1, the volume of the double caps in the complex case is

$$U_N = \int_0^{\pi/4} \cos x \sin^{2N-3} x \, dx \int_0^{\pi/2} \cos x \sin^{2N-3} x \, dx = 2^{1-N}. \quad (3)$$

The derivation of Eq. (3) is as follows. Let us denote by $\mathbf{y} \in \mathbb{C}^N$ the $2N$-dimensional real vector associated with a unit complex vector $\mathbf{x}$. In the real notation, the ‘complex’ double cap is given by any vector $\mathbf{y}$ such that

$$(\mathbf{y} \cdot \mathbf{s}_1)^2 + (\mathbf{y} \cdot \mathbf{s}_2)^2 > \frac{1}{2}, \quad (4)$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ are two suitable orthogonal unit vectors. The vector $\mathbf{y}$ can be written in the form

$$\mathbf{y} = \cos \theta \mathbf{u}_1 + \sin \theta \mathbf{u}_2, \quad (5)$$

where $\mathbf{u}_1$ is a unit vector of the two-dimensional subspace spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$, whereas $\mathbf{u}_2$ is a vector in the $(2N-2)$-dimensional orthogonal complement. By inequality \[4\] we have that the double caps are defined by the inequalities $0 \leq \theta < \pi/4$ and $3\pi/4 < \theta \leq \pi$. Thus, it is easy to realize that the double cap volume is

$$U_N = \frac{2}{W_{2N-1}} \int_0^{\pi/4} d\theta w_1(\cos \theta) w_{2N-3}(\sin \theta), \quad (6)$$

where $w_d(r)$ is the volume of a $d$-dimensional hypersphere of radius $r$ and $W_d \equiv w_d(1)$. This equation gives Eq. (3).

With these premises, let us consider the following scenario of quantum communication. Suppose that there...
are two parties, Alice and Bob. Alice prepares \( n \) qubits in a quantum state \(|\psi\rangle\), then she sends them to Bob, who finally performs a rank-1 projective measurement of the qubits. Let us denote by \(|\phi\rangle \langle \phi|\) the measured observable. Suppose now that Alice and Bob want to simulate this scenario using a classical channel. How many bits of communication are required by the simulation? For \( n = 1 \), the Toner-Bacon model shows that 2 bits of communication are sufficient. Using the double cap conjecture we will derive the lower bound \(-\log_2 V_N\), where \( N \equiv 2^n \) is the Hilbert space dimension. This result will be improved using the generalized conjecture 2, which gives the lower bound \( 2^n - 1 \).

Besides the communication resource in the classical simulation, the two parties are also allowed to share some common random variable \( X \). In other words, before the game begins, Alice and Bob receive an identical list of random values of \( X \) with probability distribution \( \rho(X) \). The variable \( X \) could be a real number, a vector or a set of vectors. No constraint on this shared resource is given. The classical protocol is as follows. Alice has a classical description of the state \(|\psi\rangle\) and generates an index \( k \) with probability distribution \( \rho(k|X,\psi) \). The index \( k \) takes \( R \) possible values. Then, she sends \( k \) to Bob. This requires \( \log_2 R \) bits of communication. Finally, Bob generates an event \(|\phi\rangle\) with probability \( P(|\phi\rangle|k,X) \). The protocol simulates the quantum channel and the subsequent measurement if

\[
\sum_k \int dX P(|\phi\rangle|k,X) \rho(k|X,\psi) \rho(X) = |\langle \phi|\psi \rangle|^2. \tag{7}
\]

The probability functions satisfy the constraints

\[
0 \leq P(|\phi\rangle|k,X) \leq 1,
\sum_k \rho(k|X,\psi) = 1, \quad \rho(k|X,\psi) \geq 0,
\int dX \rho(X) = 1, \quad \rho(X) \geq 0. \tag{8}
\]

This protocol has to satisfy the general property. Let \( \Omega_k(X) \) be the set of vectors \(|\psi\rangle\) such that \( \rho(k|X,\psi) \) is different from zero.

**Lemma 1.** For every value of \( k \) and \( X \), \( \Omega_k(X) \) does not contain any pair of orthogonal vectors.

A similar lemma was used for example in Refs. [11][11], where we proved that, in a Markov hidden variable theory, the number of continuous variables describing \( n \) qubits grows exponentially with \( n \). More precisely, we should say that the property stated by Lemma 1 holds apart from a zero-probability subset of values of \( X \). The physical meaning of lemma 1 is clear. All Bob knows about the state \(|\psi\rangle\) is contained in the values of the index \( k \) and \( X \). Given these values, Bob knows that \(|\psi\rangle\) is in a subset \( \Omega_k(X) \). But if \( \Omega_k(X) \) contains two orthogonal vectors, \(|\psi_1\rangle\) and \(|\psi_-\rangle\), and he wishes to measure the observable \(|\psi_1\rangle \langle \psi_1|\), he has no way to produce an outcome that is compatible with both the distinct states \(|\psi_1\rangle\) and \(|\psi_-\rangle\). Thus, \( \Omega_k(X) \) cannot contain pairs of orthogonal vectors. Here the formal proof.

**Proof by contradiction.** Suppose that there is a value \( l \) of \( k \), for some \( X \), such that \( \Omega_l(X) \) contains two orthogonal vectors, \(|\psi_1\rangle\) and \(|\psi_-\rangle\). Thus,

\[
\rho(l|X,\psi_n) \neq 0, \quad \text{for } n = \pm 1. \tag{9}
\]

From Eq. (7), we have that

\[
\sum_k \int dX P(\psi_n|k,X) \rho(k|X,\psi_n) \rho(X) = 1. \tag{10}
\]

Since \( \rho(l|X,\psi_n) \neq 0 \) for some \( X \) and the probability functions satisfy constraints (5), it is easy to realize that

\[
P(\psi_n|l, X) = 1 \quad \text{for } n = \pm 1. \tag{11}
\]

Similarly, we have that

\[
\sum_k \int dX P(\psi_{-n}|k,X) \rho(k|X,\psi_n) \rho(X), \tag{12}
\]

which implies that

\[
P(\psi_{-n}|l, X) = 0 \quad \text{for } n = \pm 1, \tag{13}
\]

but this equation is in contradiction with Eq. (11). The lemma is proved. \( \square \)

Since the classical model has to work for any \(|\psi\rangle\), we have that the union \( \cup_k \Omega_k(X) \) contains every vector of the Hilbert space for any \( X \). Thus, if \( \mathcal{V}[\Omega_k(X)] \) is the volume of the set \( \Omega_k(X) \), then

\[
\sum_k \mathcal{V}[\Omega_k(X)] \geq 1. \tag{14}
\]

Notice that this equation and lemma 1 hold for both real and complex Hilbert spaces.

At this point, let us state the main theorem.

**Theorem 1.** If conjecture 1 is true, then a process of preparation, transmission and subsequent rank-1 projective measurement of \( n \) qubits cannot be simulated with an amount of communication smaller than \(-\log_2 V_N\), where \( N \equiv 2^n \). If conjecture 2 is also correct, then the lower bound is increased to \( 2^n - 1 \).

**Proof.** The proof is trivial. Using conjecture 1, lemma 1 (adapted to the case of a real Hilbert space) and Eq. (14), we have that

\[
RV_N \geq 1, \tag{15}
\]

where \( R \) is the number of values that the index \( k \) can take. Thus, the minimal number of bits is \( \log_2 R = -\log_2 V_N \). Similarly, by conjecture 2 we have that the lower bound is \(-\log_2 U_N = 2^n - 1 \). \( \square \)

Notice that the lower bounds are not just on the average number of bits, but on the minimal number of bits communicated in each single execution of the simulation. Apart from \( N = 2 \), the volume \( V_N \) is always strictly larger than \( U_N \), thus the lower bound \( 2^n - 1 \) is stronger than \(-\log_2 V_N \). For large \( N \), \( V_N \) is well-approximated by the formula

\[
V_N \approx 2^{-\frac{N}{2}} \frac{\sqrt{2\pi N}}{N}. \tag{16}
\]
Even if the double cap conjecture was false, it is possible to prove a slightly weaker theorem. A result in Ref. [9] implies that the maximum volume on a hypersphere under constraint (11) must be smaller than $(\theta + \epsilon)^{-N}$ for each $\epsilon > 0$ and all sufficiently large $N$, with $\theta \equiv (2/\sqrt{3})^{\sqrt{2}} \simeq 1.225$. Thus, using the same proof of theorem 1, we have the following.

**Theorem 2.** A process of preparation, transmission and subsequent rank-1 projective measurement of $n$ qubits cannot be simulated with an amount of communication smaller than $2^n \log_2(\theta + \epsilon)$ for each $\epsilon > 0$ and all sufficiently large $n$.

This theorem establishes the asymptotic lower bound $0.293 \times 2^n$ for the communication cost.

Taking for granted the intuitive double cap conjecture and its ‘complex’ generalization, the proved theorem 1 is extremely simple and has a geometric interpretation. Apart from this advantage, it strengthens the result in Ref. [6] in two ways. First, it gives a stronger lower bound for the communication cost. Second, the derivation uses only rank-1 projective measurements. This last feature has a nontrivial consequence. Let us consider the scenario discussed in Ref. [7]. Two parties, Alice and Bob, share a bipartite quantum state and perform local two-outcome measurements. Each local outcome, $s_a$ and $s_b$, is a bit taking values $\pm 1$. In Ref. [7] it was shown that the correlation $\langle s_a s_b \rangle$ can be reproduced by a classical simulation with only 2 bits of communication regardless of the dimension of the Hilbert space. However, the reported models does not reproduce the correct marginal distributions of $s_a$ and $s_b$. Indeed, Theorem 1 (but the weaker theorem 2 would be sufficient) implies that an exact reproduction of the full probability distribution requires an exponentially growing amount of communication. It is known that any classical protocol that simulates $n$ qubits can be converted into a classical protocol simulating a quantum channel of $n$ qubits with a negligible increase of communication. A general conversion method is reported for example in Ref. [8].

In conclusion, using a plausible mathematical conjecture, we have derived the lower bound $2^n - 1 - n$ for the classical communication cost of simulating the quantum communication of $n$ qubits with subsequent rank-1 projective measurement. Our proof is simple and can provide useful hints for finding optimal one-way protocols that simulate quantum channels.

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