Nonextensive statistics in stellar plasma and solar neutrinos

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Nonextensive and quantum uncertainty effects (related to the quasiparticles composing the stellar core) have strong influence on the nuclear rates and, of course, affect solar neutrino fluxes. Both effects do coexist and are due to the frequent collisions among the ions. The weakly nonextensive nature of the solar core is confirmed. The range of predictions for the neutrino fluxes is enlarged and the solar neutrino problem becomes less dramatic.

The stellar (like the solar) core is a weakly-nonideal plasma where: 1) mean Coulomb energy potential is not much smaller of the thermal kinetic energy; 2) Debye screening length $R_D \approx \alpha$ (interparticle distance) and Debye-Hückel conditions are only approximately verified; 3) it is not possible to separate individual and collective degrees of freedom; 4) inverse solar plasma frequency: $t_{pl} = \omega_{pl}^{-1} = \sqrt{m/4\pi ne^2} \approx 10^{-17}$ is of the same order of magnitude of the collision time $t_{coll} = \nu^{-1} = \langle n\sigma v \rangle$; 5) particles of plasma loose memory of the initial state only after many collisions, scattering process cannot be considered Markovian; 6) time needed to build up again the screening, after hard collisions, is not at all negligible.

The relation between energy and momentum implied by $\delta(\epsilon - p^2/2m)$ for free particles is no more valid. In dense media like stellar core plasma, for quasi-particles, we must use

$$
\delta g(\epsilon) = \frac{1}{\pi} \frac{g(\epsilon, p)}{((\epsilon - \epsilon_p - \Delta(\epsilon, \epsilon_p))^2 + g(\epsilon, \epsilon_p)^2)},
$$

where $\epsilon_p = p^2/2m$, $\Delta(\epsilon, \epsilon_p)$ and $g(\epsilon, p)$ are the real and imaginary parts of the one-particle retarded Green’s function self-energy.

For weakly non ideal plasmas one can show that approximately $\Delta \approx kT \Gamma/2$, where $\Gamma$ is the plasma parameter ($\Gamma = e^2/R_D kT$) and $g \propto h\nu$. At non zero value of $g$, a nonexponential tail appears in the distribution function $f_Q(p)$.

In fact, for large momenta, we have

$$
f_Q(p) = \int_{-\infty}^{\infty} f(\epsilon, p) \, \delta g(\epsilon) \, d\epsilon
= f_M(p) + \frac{1}{\pi} \int_{-\infty}^{[\min(\mu, \beta^{-1})]} \frac{g(\epsilon, p)}{(\epsilon - \epsilon_p)^2} \, d\epsilon
= f_M(p) + \frac{h\nu}{2\pi} \frac{kT}{\epsilon_p^2} e^{\mu/kT},
$$

$\mu$ being the chemical potential and $f_M(p)$ the Maxwellian distribution.

The value of the collision frequency $\nu$ is responsible of two different effects producing at high momenta important (although small) deviations of the Maxwellian distribution $f_M(p)$:

1. Quantum uncertainty effect: because of the frequent collisions, $f_M(p)$ can acquire a non-Maxwellian tail;

   q) Weak nonextensivity effect described by Tsallis statistics with entropic parameter $q$ due to long-range interactions and non-Markovian memory effects; when deviation is small ($q \approx 1$) the distribution can acquire an enhanced or a depleted tail, the correction of $f_M(p)$ being given by the factor $\exp \left[ -\frac{1-q}{2} (\epsilon_p^2)^2 \right]$.

Nuclear rates can be evaluated averaging the quasi-classical cross section $\sigma(\epsilon)$ over the momentum distribution, rather than the energy distribution, once we have substituted $\epsilon$ with $\epsilon_p$. A rigorous derivation of reaction rates within the $Q$ effect can be found in [8].

Deviations from the Maxwellian tail due to $Q$ and $q$ effects may lead to a strong increase or decrease of the nuclear rates in the solar core (solar models and solar neutrino problem are described in Ref.s [9–11]). $Q$ correction depends on the distribution $f(\epsilon, p)$ (Maxwell, Fermi, Tsallis,⋯), on the collision cross section $\sigma(\epsilon)$ and collision frequency $\nu$. Nuclear tunnelling rates are

$$
k_{ij}^Q = k_{ij}^M (1 + r_{ij}),
$$

where $k_{ij}^M$ and $r_{ij}$ are the free-particle cross section and the relative deviation in the $Q$ correction.
where \( r_{ij} \) is the ratio of the average \( Q \) power law part respect to the Maxwellian part. The results are \[6\] :

\[
(\text{pp}) : \quad r_{11}(0.0759 R_{\odot}) = 3.5 \cdot 10^{-3} ;
\]

\[
(^3\text{He}, ^3\text{He}) : \quad r_{33} = 4.5 \cdot 10^8 ;
\]

\[
(^3\text{He}, ^4\text{He}) : \quad r_{34} = 3 \cdot 10^9 ;
\]

\[
\frac{n_3}{n_{M3}} \approx 10^8 ; \quad \frac{n_{Be7}}{n_{MBe7}} = \frac{\Phi(\text{Be7})}{\Phi_{M}(\text{Be7})} = 1.50 ;
\]

\[
\frac{n_8}{n_{M8}} \approx 10^4
\]

\( L_{\odot} \) is conserved; neutrino pp flux \( \Phi(\text{pp}) \) is unchanged.

When long-range interactions are present and detailed correlations in space and time exist it is no longer true that the probability of a particle being in a state and the probability of a transition are statistically independent thus the two probabilities cannot be multiplied. The natural generalization of the Boltzmann-Gibbs statistics is the Tsallis nonextensive thermostatistics that we have already applied elsewhere \[1,2\].

Based on the generalized entropy form

\[
S_q = k \left(1 - \sum_i p_i^q\right) \left(\sum_i p_i = 1 , \quad q \text{ real}\right)
\]

Tsallis statistics uses conditional probabilities that, when \( q < 1 \) and \( q > 1 \), will respectively privilege the rare and the frequent events.

Among many applications, let us mention that from COBE data we have derived the distribution of peculiar velocities of clusters of spiral galaxies obtaining a remarkable fit (the function used was the \( q \)-generalized Maxwellian distribution essentially corresponding to an ideal classical gas) \[12\].

In the solar core the random electric total microfield can be decomposed in three main components: 1) slow varying, due to collective plasma oscillations, the particles see it as an almost constant external mean field over several collisions; 2) fast random, described by elastic diffusive cross section \( \sigma \approx 1/v \), the distribution remains Maxwellian; 3) short range two-body strong Coulomb effective interaction described by the ion sphere model with strict enforcement. The last is the component of our interest whose energy density can be expressed as \( \langle E^2 \rangle = (Fe/a^2)^2 \). We have found that the following two relations hold: \( F \simeq \alpha_1^{-2} \) and \( F^2 \simeq 3/\Gamma = 40 \).

The strong Coulomb cross section is \( \sigma_0 = 2 \pi \alpha_1^2 a^2 \) \[13\]. The quantity \( \alpha_1 \) depends on the ion-ion correlation function. We have found the analytical relation

\[
|1 - q| = \frac{2}{3} \frac{\sigma_1^2}{\sigma_1^2} = 12 \alpha_1^4 \Gamma^{-2} \ll 1 .
\]

The quantity \( \alpha_1 \) may be defined as

\[
\alpha_1 a = \int_0^R \int_0 R dR \int dt R^3 4 \pi n_i g(R, t) \times
\frac{\exp\left(-4 \pi n_i \int_0^R dR' \int_0 R' dR' g(R', t)\right)}{\int_0^R dR' \int_0 R' dR' g(R', t)},
\]

where \( P_{NN}(R) \) is the probability that nearest neighbor of ion \( i \) is at a distance \( R \) and \( g(R, t) \) is the correlation function (eventually time dependent). The value of \( \alpha_1 \) is within the range \( 0.4 < \alpha_1 < 0.89 \).

The nuclear rates can be written

\[
k_{ij}^q = k_{ij}^M \exp\left(-(1-q)/2 \gamma \right),
\]
with $\gamma = (E_{\text{Gamow}}/4kT)^{2/3}$.

Using $q = 0.99$ (for all particles) we obtain:

$\Phi(pp) = 62.2 \cdot 10^9$ cm$^{-2}$ s$^{-1}$,

$\Phi(7Be) = 2.87 \cdot 10^9$ cm$^{-2}$ s$^{-1}$,

$\Phi(N, O) = 0.21 \cdot 10^9$ cm$^{-2}$ s$^{-1}$,

$\Phi(B) = 1.65 \cdot 10^6$ cm$^{-2}$ s$^{-1}$

(SSM: 5.15 $\cdot$ 10$^6$, exper.: 2.45 $\pm$ 0.08 $\cdot$ 10$^6$),

Gallium = 100 SNU

(SSM: 129 $\pm$ 7, exper. (Gallex): 77.5 $\pm$ 7.7),

Chlorine = 2.84 SNU

(SSM: 7.7 $\pm$ 1.1, exper.: 2.65 $\pm$ 0.23).

The quantum uncertainty $Q$ and nonextensive $q$ effects produce corrections to the standard nuclear rates that can be expressed as

$$k_{ij} = k^M_{ij} \left[ \exp \left( -\frac{1 - q}{2} \gamma_{ij} \right) + r_{ij} \exp \left( -\frac{1 - q}{2} \gamma^*_{ij} \right) \right],$$

both effects cannot be neglected due to the value of the collision frequency $\nu$.

The $Q$ effect is effective at higher momenta than $q$-effect $\gamma^* \approx 3\gamma$. Defining

$$A = \frac{\Phi(Be^7)/\Phi^M(Be^7)}{\Phi(B)/\Phi^M(B)}, \quad B = \frac{\Phi(B)}{\Phi^M(B)},$$

$$C = \frac{\Phi(Be^7)}{\Phi^M(Be^7)}; \quad \delta_{ij} = \frac{1 - q_{ij}}{2}$$

(the index $M$ means that the flux is calculated using the Maxwellian distribution) and $k_{e7} = x k^M_{17}, k_{e7} = y k_{17}$, we have derived the following constraints for the solar neutrino fluxes

1) $A = \frac{C}{B} = \frac{e^{\delta_{17}\gamma_{17}}}{r_{17}} e^{-\delta_{17}\gamma_{17}} \ll r_{17} e^{-\delta_{17}\gamma_{17}}$,

2) $\frac{y}{x} A = 1$,

3) $\frac{n_{Be^7}}{n_{Be^7}^M} = \frac{n_{3}}{n_{M3}} \left( e^{-\delta_{34}\gamma_{34}} + r_{34} e^{-\delta_{34}\gamma_{34}} \right) \left[ 1 + \frac{1}{2x} \left( e^{-\delta_{17}\gamma_{17}} + r_{17} e^{-\delta_{17}\gamma_{17}} \right) \right]^{-1}$.

A reasonable evaluation of $\alpha_1$ gives $\alpha_1 = 0.55$; with $\Gamma = 0.072$ (solar core) we have $q = 0.989$ ($\delta_{ij} = 0.005$) for all components.

Assuming $n_{3}/n_{3}^M \simeq 3 \cdot 10^{-3}$, we obtain

$$\frac{\Phi(Be^7)}{\Phi^M(Be^7)} = \frac{1}{50}, \quad \frac{\Phi(B)}{\Phi^M(B)} \approx 1/2,$$

Gallium = 81 SNU,

Chlorine = 2.8 SNU;
\( \Phi(pp) \) and luminosity are practically unchanged respect to SSM value. The CNO reactions are strongly enhanced by the \( Q \) effect but the \( e^{-\delta\gamma} \) factor strongly reduces it.

The assumption concerning the value of \( n_3 \) is within the constraints actually imposed by helioseismology because in the region \( r/R_\odot < 0.2 \) the value of \( n_3 \) can be assumed within a large range of variability \([1,0]\).

We thank S. Turck-Chièze and V. Savchenko for comments and advices.

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