NORM INFLATION WITH INFINITE LOSS OF REGULARITY AT
GENERAL INITIAL DATA FOR NONLINEAR WAVE EQUATIONS IN
WIENER AMALGAM AND FOURIER AMALGAM SPACES

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ABSTRACT. We study the strong ill-posedness (norm inflation with infinite loss of regularity) for the nonlinear wave equation at every initial data in Wiener amalgam and Fourier amalgam spaces with negative regularity. In particular these spaces contain Fourier-Lebesgue, Sobolev and some modulation spaces. The equations are posed on \( \mathbb{R}^d \) and on torus \( \mathbb{T}^d \) and involve a smooth power nonlinearity. Our results are sharp with respect to well-posedness results of Bényi and Okoudjou (2009) and Cordero and Nicola (2009) in the Wiener amalgam and modulation space cases. In particular, we also complement norm inflation result of Christ, Colliander and Tao (2003) and Forlano and Okamoto (2020) by establishing infinite loss of regularity in the aforesaid spaces.

1. INTRODUCTION

We study strong ill-posedness for nonlinear wave (NLW) equations of the form

\[
\begin{cases}
\partial_t^2 u - \Delta u = \pm u^p(\nabla)\sigma - \rho \\
(u(0, \cdot), u_t(0, \cdot)) = (u_0, u_1)
\end{cases}
\]  

where \( (t, x) \in \mathbb{R} \times \mathcal{M}, \) \( \mathcal{M} = \mathbb{R}^d \) or \( \mathbb{T}^d, \) \( \sigma, \rho \in \mathbb{N}, \) \( \rho \in \mathbb{N} \cup \{0\} \) with \( \sigma \geq \max(\rho, 2), \) in Fourier amalgam and Wiener amalgam spaces. We recall:

Definition 1.1 (Hadamard’s well/ill posedness).

- The Cauchy problem (1.1) is called locally well-posed on \( \mathcal{X} = X_0 \times X_1 \) to \( Y \) if for every bounded set \( B \subset \mathcal{X}, \) there exist \( T > 0 \) and a Banach space \( X_T \hookrightarrow C([0, T], Y) \) such that (i) for all \( u_0 := (u_0, u_1) \in B, \) (1.1) has a unique solution \( u \in X_T \) with \( \mathcal{U} = (u(0, \cdot), u_t(0, \cdot)) = u_0 \) (ii) the solution map \( \mathcal{U} \mapsto u \) is continuous from \( (B, \|\cdot\|_\mathcal{X}) \) to \( C([0, T], Y) \).

- The Cauchy problem (1.1) is ill-posed if the solution map is not continuous.

- We say norm inflation (NI) occurs at \( u_0 \in \mathcal{X} \) in \( \mathcal{X} \) (in short \( NI_{\mathcal{X}}(u_0) \)) if given \( \epsilon > 0, \) there exist \( \widetilde{v}_0 \in \mathcal{X}, 0 < t < \epsilon \) with \( \|\mathcal{U} \mathcal{V}_0 - \widetilde{v}_0\|_{\mathcal{X}} < \epsilon \) such that for the solution \( v \) to (1.1) corresponding to the data \( \mathcal{V}_0 \) one has \( \|v(t)\|_{X_0} > \epsilon^{-1}. \)

- If \( NI_{\mathcal{X}}(0) \) occurs, we say (mere) norm inflation (at zero) occurs to (1.1) in \( \mathcal{X}. \)

- We say NI occurs with infinite loss of regularity at \( u_0 \in \mathcal{X} \) if for any given \( \theta \in \mathbb{R}, \epsilon > 0, \) there exist \( \mathcal{V}_0 \in \mathcal{X}, 0 < t < \epsilon \) with \( \|\mathcal{U} \mathcal{V}_0 - \mathcal{V}_0\|_{\mathcal{X}} < \epsilon \) such that for the solution \( v \) to (1.1) corresponding to the data \( \mathcal{V}_0 \) one has \( \|v(t)\|_{X_\theta} > \epsilon^{-1}. \)
For equivalent characterization via short-time Fourier transform (STFT) of $L^p_q(M)$, we have

$$\widehat{w}_{p,q}(M) = \left\{ f \in S'(M) : \| f \|_{\widehat{w}_{p,q}} = \left\| \| f(x) \|_{L^p_q(M)} \right\|_{L^p_q(M)} < \infty \right\},$$

where $Q_1 = (-\frac{1}{2}, \frac{1}{2})^d$, $F$ denotes the Fourier transform and $\langle \cdot \rangle^s = (1 + | \cdot |^2)^{s/2}$. Here $\hat{M}$ denotes the Pontryagin dual of $M$, i.e. $\hat{M} = \mathbb{R}^d$ if $M = \mathbb{R}^d$ and $\hat{M} = \mathbb{Z}^d$ if $M = \mathbb{T}^d$. See [27, Part II], among others, for details on space $S'(M)$. The Fourier-Lebesgue spaces $FL^p_q(M)$ are defined by

$$FL^p_q(M) = \left\{ f \in S'(M) : \| Ff \|_{L^p_q(\hat{M})} = \left\| \| \hat{f}(\xi) \|_{L^p_q(\hat{M})} \right\|_{L^p_q(\hat{M})} \right\}.$$

On the other hand, the modulation $M_{p,q}(M)$ and Wiener amalgam $W_{p,q}(M)$ spaces were introduced by Feichtinger in early 1980’s in [14]. To recall their definitions, let $\rho : \mathbb{R}^d \to [0, 1]$ be a smooth function satisfying $\rho(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$ and $\rho(\xi) = 0$ if $|\xi| \geq 1$. Let $\rho_n$ be a translation of $\rho$, that is, $\rho_n(\xi) = \rho(\xi - n), n \in \mathbb{Z}^d$ and denote $\sigma_n(\xi) = \frac{\rho_n(\xi)}{\sum_{\ell \in \mathbb{Z}^d} \rho(\ell)}$, $n \in \mathbb{Z}^d$. Then the frequency-uniform operators can be defined by

$$\Box_n = F^{-1} \sigma_n F.$$

Now the modulation $M_{p,q}^s(M)$ and and Wiener amalgam spaces $W_{p,q}^s(M)$, (with $1 \leq p, q \leq \infty, s \in \mathbb{R}$) are defined by the norms:

$$\| f \|_{M^s_{p,q}(M)} = \left\| \Box_n f \right\|_{L^p_q(M)} \langle n \rangle^s \|_{L^p_q(M)} \left\|_{L^p_q(M)} \right\|_{L^p_q(M)} \quad \text{and} \quad \| f \|_{W^s_{p,q}(M)} = \left\| \| f(x) \|_{L^p_q(M)} \right\|_{L^p_q(M)} \left\|_{L^p_q(M)} \right\|_{L^p_q(M)}.$$

See Remark 1.6 for equivalent characterization via short-time Fourier transform (STFT) of these spaces. It is known that

$$\widehat{W}_{p,q}^s(M) = \begin{cases} FL^p_q(M) & \text{if } p = q \\ M_{p,q}^s(M) & \text{if } p = 2 \\ H^s(M) = M_{2,2}^s(M) = W_{2,2}^s(M) & \text{if } p = 2 \\ FL^p_q(M) = M_{p,q}^s(M) = W_{p,q}^s(M) & \text{if } M = \mathbb{T}^d. \end{cases}$$

(See [25, Section 5] and [17, Proposition 11.3.1] for details.) In last two decades, modulation $M_{p,q}$ and Wiener amalgam $W_{p,q}$ spaces have been extensively studied in PDEs, see e.g. [2, 3, 5–8, 13, 24, 26, 29–31]. In particular, we have at least good local well-posedness theory for NLW (1.1) in these spaces. We summarize them in the following:

**Theorem A** (well-posedness). Let $1 \leq p, q \leq \infty, q' > (\sigma - 1)d$ where $q'$ is the Hölder conjugate of $q$, and let $Y_s = W_{p,q}^s(\mathbb{R}^d)$ or $M_{p,q}^s(\mathbb{R}^d)$ or $FL^p_q(\mathbb{R}^d)$.

1. [3, 4, 8, 13, 16, 24] NLW (1.1) is locally well-posed in $Y_s$ for $s > 0$.
2. [22, 28] NLW (1.1) is locally well-posed in $H^s(\mathbb{R}^d)$ for $s > s(\sigma, d), d > 2$, some $s(\sigma, d) > 0$.

To the best of authors knowledge there is no local well-posedness result for (1.1) for in $\widehat{W}_{p,q}^s$ spaces (except the cases $p = 2$ or $p = q = 1, s = 0$). On the other hand, there is extensive literature on ill-posedness for (1.1). We summarize some of them which are most suitable in our context.
**Theorem B** (ill-posedness).

1. [12, Theorem 6] Norm inflation at zero occurs for (1.1) with nonlinearity $\pm |u|^2 u$ in $H^s(\mathbb{R})$ for $-1/2 < s < 0$.
2. [16, Theorem 1.7] NI at general initial data for (1.1) with nonlinearity $\pm u^\rho$ (the case $\rho = \sigma$ in our setting) occurs in $\tilde{w}_{s}^{p,q}(\mathbb{R}^d)$ ($1 \leq p, q \leq \infty$) for $s < 0$.
3. [20] NI with finite loss of regularity for (1.1) with nonlinearity $-u^{\rho}(\sigma > 3)$ in $H^s(\mathbb{R}^d)$ with some positive $s$ (precisely for $1 < s < d/2 - 2/(\sigma - 1), d \geq 3$).

The aim of this note is to complement positive results (Theorem A) by establishing strong ill-posedness for (1.1) in these spaces with negative regularity $s < 0$. We also complement ill-posedness results (Theorem B) by exhibiting infinite loss of regularity at general initial data. We now state our main result.

**Theorem 1.2.** Assume that $1 \leq p, q \leq \infty$, $s < 0$ and let

$$X_{s}^{p,q}(\mathcal{M}) = \begin{cases} \tilde{w}_{s}^{p,q}(\mathbb{R}^d) \text{ or } W_{s}^{2,q}(\mathbb{R}^d) & \text{for } \mathcal{M} = \mathbb{R}^d \quad \text{and} \quad X_{s}^{p,q}(\mathcal{M}) = X_{s}^{p,q}(\mathcal{M}) \times X_{s-1}^{p,q}(\mathcal{M}). 
\end{cases}$$

Then norm inflation with infinite loss of regularity occurs to (1.1) at each element in $X_{s}^{p,q}(\mathcal{M})$: For any $\theta \in \mathbb{R}$, $\varepsilon > 0$ and $\bar{u}_0 \in X_{s}^{p,q}(\mathcal{M})$ there exist $\tilde{u}_{0,\varepsilon} \in X_{s}^{p,q}(\mathcal{M})$ and $T > 0$ satisfying

$$\|\tilde{u}_0 - \tilde{u}_{0,\varepsilon}\|_{X_{s}^{p,q}} < \varepsilon, \quad 0 < T < \varepsilon$$

such that the corresponding smooth solution $u_\varepsilon$ to (1.1) with data $\tilde{u}_{0,\varepsilon}$ exists on $[0, T]$ and

$$\|u_\varepsilon(T)\|_{X_{s}^{p,q}} > \frac{1}{\varepsilon}.$$

In particular, for any $T > 0$, the solution map $X_{s}^{p,q} \times X_{s-1}^{p,q} \ni (u_0, u_1) \mapsto u \in C([0, T], X_{s}^{p,q})$ for (1.1) is discontinuous everywhere in $X_{s}^{p,q} \times X_{s-1}^{p,q}$ for all $\theta \in \mathbb{R}$.

For $q' > (\sigma - 1)d$, Theorem 1.2 is sharp in the sense that (1.1) is strongly ill-posed in $W_{s}^{2,q}(\mathbb{R}^d)$, $M_{s}^{2,q}(\mathbb{R}^d)$ for $s < 0$ while by Theorem A it is locally well-posed for $s \geq 0$. In fact, even more ill-posedness in $W_{s}^{2,q}(\mathbb{R}^d)$ is completely new as it is not covered by Theorem B. The particular case $X_{s}^{p,q} = \tilde{w}_{s}^{p,q}$ of Theorem 1.2 recover Theorem B (2) and further reveal that even worse situation occurs as the infinite loss of regularity (everywhere in $\tilde{w}_{s}^{p,q} \times \tilde{w}_{s-1}^{p,q}$) is present in all dimensions. Theorem 1.2 also complements Theorem B (3) by taking $\rho = \sigma$ in (1.1) by establishing infinite loss of regularity.

The circle of ideas to establish ill-posedness, via Fourier analytic approach, is originated from the abstract argument [1, Theorem 2 and Section 3] of Bejenaru and Tao in the context of quadratic nonlinear Schrödinger equation (NLS). In fact, the idea is to rewrite the solution of (1.1) as a power series expansion (Lemma 2.7) in terms of Picard iterations. It is then sufficient to establish the discontinuity for one Picard iterate to get the discontinuity of solution map at zero. This method is further developed by Iwabuchi and Ogawa [18] to establish stronger phenomena of NI for NLS. Later Kishimoto [19] establish NI for NLS with more general nonlinearity on the special domain $\mathbb{R}^{d_1} \times \mathbb{T}^{d_2}, d = d_1 + d_2$. The idea is to show that one term in the series exhibits instability and dominates all the other terms (Lemmata 3.3, 3.6 and 3.2, 3.5) after adding a perturbation $\tilde{\phi}_{0,N}$ to the data $\tilde{u}_0$, see (3.3). We shall notice that the existence time $T > 0$ is allowed to shrink for the purpose of establishing norm inflation while in [1] it is fixed and uniform with respect to the initial data. See [19, Section 2]

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1 Precisely this nonlinearity: $\sum_{j=1}^{n} \nu_j u^{\rho_j}(\tilde{u})^{\sigma_j - \rho_j}$ where $\nu_j \in \mathbb{C}$, $\sigma_j \in \mathbb{N}$, $\rho_j \in \mathbb{N} \cup \{0\}$ with $\sigma_j \geq \max(\rho_j, 2)$. 
for detail discussion on the approach. On the other hand, Oh [23] use power series expansion indexed by trees to establish NI at general initial data for cubic NLS on $\mathcal{M}$. Forlano and Okamoto [16] use this Fourier analytic approach, following presentation from [23], for NLW (1.1) (in the particular case $\rho = \sigma$) to establish norm inflation at general initial data.

In addition to these ideas, in the present paper, we fix the size of the support of perturbation $\tilde{\phi}_{0,N}^{\prime}$ in (3.3) of the initial data on the frequency side. This simplifies our analysis. Moreover, we choose this perturbation to be real valued and symmetric on the Fourier side i.e. $\mathcal{F}\tilde{\phi}_{0,N}^{\prime}(\cdot) = \mathcal{F}\tilde{\phi}_{0,N}^{\prime}$. This enables us to consider more general nonlinearity $w^{\rho}(\bar{u})^{\sigma-\rho}$ as compared to $u^{\sigma}$ in [16]. In order to get infinite loss of regularity (specifically, while showing the $X^{p,q}_{\theta}$-norm of the solution is arbitrarily large for all $\theta$), we restrict the Fourier transform of the solution at a particular frequency (say at $n = e_1$). This will allow us to compare two discrete Lebesgue norms with different weights i.e.

$$\|\langle \cdot \rangle^{\theta} f\|_{L^{(n = e_1)}} = 2^{(s-\theta)/2} \|\langle \cdot \rangle^{s} f\|_{L^{(n = e_1)}} \quad \text{for all } \theta \in \mathbb{R}$$

which eventually leads to infinite loss of regularity. In addition to this, it also allows us to use Plancherel Theorem to achieve the lower bound estimate for the second nontrivial Picard iterate in $W^{2,2}_{p,q}$-norm, see Lemma 3.6. We note that a similar idea was used by Kishimoto in [19, Appendix A] in the context of NLS to achieve norm inflation at zero with infinite loss of regularity in $H^{s}(\mathcal{M})$. Kishimoto [19] used modulation space $M^{2,1}$ to justify the convergence power series expansion while in the present paper we use Wiener algebra $\mathcal{F}L^{1}$ as in [16,23]. We employ these ideas together with the refinement of ideas used in [5,16,19,23] and properties of $\alpha^{p,q}_{\theta}, W^{p,q}_{s}$ to prove Theorem 1.2.

Given $\lambda > 0$, if $u$ solves (1.1), then scaling $u_{\lambda}(t,x) := \lambda^{\frac{2}{s-\tau}} u(\lambda t, \lambda x)$ also solves (1.1) with rescaled initial data $\lambda^{\frac{2}{s-\tau}}(u_{0}(\lambda x), u_{1}(\lambda x))$. This scaling leaves the homogeneous Sobolev $\dot{H}^{s}$ space invariant when $s = s_{c} = d/2 - 2/(\sigma - 1)$. The ill-posedness below the scaling critical regularity $s_{c}$ has been studied in [9,20,21]. We note that NI with finite loss of regularity at zero initial data for (1.1) is initiated by G. Lebeau in [20]², see Theorem B (3). While we initiate, to the best of the authors’ knowledge, NI with infinite loss of regularity at general initial data for (1.1) in the present paper. Our study of infinite loss of regularity for (1.1) is inspired form known NLS case in [10,11,19] while the inspiration of NI at general initial data comes from [16,23]. We also note that the details of proofs Lebeau in [20] does not seem to work for negative regularity $s < 0$. In [5,10,11], Carles and his collaborators have used geometric optics approach to establish infinite loss of regularity for NLS. Recently in [7], Bhimani and Haque have used the Fourier analytic approach to establish infinite loss of regularity for fractional Hartree and cubic NLS for some negative regularity (i.e for all $s < -\epsilon < 0$ for some $\epsilon > 0$). On the other hand Theorem 1.2 establishes infinite loss of regularity for NLW (1.1) for all $s < 0$. This is in strict contrast as compared to known NLS case results proved in [5,7,10] and Theorem 1.2 thus reveal new phenomena for NLW (1.1). We conclude our discussion with the following remarks.

Remark 1.3. In order to get the upper bound for each Picard iterate we must have $s \leq 0$ (Lemma 3.2) while we get the lower bound for one dominating Picard iterate for all $s \in \mathbb{R}$ (Lemma 3.3). The restriction $s < 0$ will be required at the final stage in the proof of Theorem 1.2. Our approach thus may not work for $s \geq 0$.

²This result is stronger than finite loss of regularity at zero as it is obatined with a single datum instead of a sequence of initial data converging to zero.
Remark 1.4. In [5, Theorem 1.6], Bhimani and Carles have established infinite loss of regularity, via geometric optics approach, for NLS in $M_{p,q}^{s}$ for all $1 \leq p, q \leq \infty$ and for some $s < 0$. This somewhat indicates that restriction on $p = 2$ in $M_{p,q}^{s}, W_{s}^{p,q}$ in Theorem 1.2 is just due to our approach and we believe that Theorem 1.2 is also true for any $1 \leq p \leq \infty$ in $M_{p,q}^{s}, W_{s}^{p,q}$. In fact, by taking $p = 2$, we can work on the Fourier side due to Plancherel theorem. This makes our analysis somewhat simple. On the other hand, for $p \neq 2$, we do not know how to deal with frequency-uniform decomposition operators $\Box_{n}$. Also taking Theorem A into account, the case $s = 0$ with $q' \leq (\sigma - 1)d$ in Theorem 1.2 remains open. We plan to address these issues in our future work.

Remark 1.5. Our method proof should be applicable to (1.1) on special domain $Z = \mathbb{R}^{d_{1}} \times \mathbb{T}^{d_{2}} (d_{1}, d_{2} \in \mathbb{N} \cup \{0\})$ with more general type non-linearity $\sum_{j=1}^{n} \nu_{j} u^{\rho_{j}}(u)^{q_{j} - \rho_{j}}$ where $\nu_{j} \in \mathbb{C}$, $\sigma_{j} \in \mathbb{N}$, $\rho_{j} \in \mathbb{N} \cup \{0\}$ with $\sigma_{j} \geq \max(\rho_{j}, 2)$; as in the NLS case [19].

Remark 1.6. The STFT of a $f \in S'(\mathcal{M})$ with respect to a window function $0 \neq g \in \mathcal{S}(\mathcal{M})$ is defined by

$$V_{g}f(x, y) = \int_{\mathcal{M}} f(t) \overline{T_{x}g(t)} e^{-2\pi i y t} dt, \quad (x, y) \in \mathcal{M} \times \hat{\mathcal{M}}$$

whenever the integral exists. Here, $T_{x}g(t) = g(t - x)$ is the translation operator on $\mathcal{M}$. It is known [29, Proposition 2.1], [14] that

$$\|f\|_{M_{p,q}^{s}} = \|V_{g}f(x, y)\|_{L^{p}(\mathcal{M})} \langle y \rangle^{s} \|_{L^{q}(\hat{\mathcal{M}})}$$
and

$$\|f\|_{W_{s}^{p,q}(\mathcal{M})} = \|V_{g}f(x, y)\|_{L^{q}(\hat{\mathcal{M}})} \langle y \rangle^{s} \|_{L^{p}(\mathcal{M})}.$$ The definition of the modulation space is independent of the choice of the particular window function, see [17, Proposition 11.3.2(c)].

2. Key Lemmas

For $u_{0}, u_{1} \in \mathcal{S}(\mathbb{R}^{d})$, (or in $C^{\infty}(\mathbb{T}^{d})$ in torus case) the wave propagator $S(t)(u_{0}, u_{1})$ is given by $S(t)(u_{0}, u_{1}) = \cos(t|\nabla|)u_{0} + \frac{\sin(t|\nabla|)}{|\nabla|} u_{1}$, in other words

$$\mathcal{F}S(t)(u_{0}, u_{1})(\xi) = \cos(t|\xi|)\mathcal{F}u_{0}(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \mathcal{F}u_{1}(\xi), \quad \xi \in \hat{\mathcal{M}}, \ t \in \mathbb{R}.$$

Let $\mathcal{X}_{s}^{p,q}(\mathcal{M}) = \hat{\mathcal{W}}_{s}^{p,q}(\mathcal{M}) \times \hat{\mathcal{W}}_{s-1}^{p,q}(\mathcal{M})$ with the norm

$$\|u_{0}\|_{\mathcal{X}_{s}^{p,q}} = \|u_{0}\|_{\hat{\mathcal{W}}_{s}^{p,q}} + \sqrt{2} \|u_{1}\|_{\hat{\mathcal{W}}_{s-1}^{p,q}} \quad (1 \leq p, q \leq \infty, s \in \mathbb{R}).$$

When $s = 0$, we write $\mathcal{X}^{p,q}(\mathcal{M}) = \mathcal{X}_{0}^{p,q}(\mathcal{M})$.

Lemma 2.1. Let $\mathcal{X}_{s}^{p,q}(\mathcal{M})$ be defined as above and $0 \leq t \leq 1$. Then $\|S(t)(u_{0})\|_{\hat{\mathcal{W}}_{s}^{p,q}} \leq \|u_{0}\|_{\mathcal{X}_{s}^{p,q}}$.

Proof. Since $|\cos(t|\xi|)| \leq 1$ and for $0 \leq t \leq 1$,

$$\frac{|\sin(t|\xi|)|}{|\xi|} \left(1 + |\xi|^{2}\right)^{1/2} \leq \begin{cases} t \left(1 + |\xi|^{2}\right)^{1/2} \leq \sqrt{2} & \text{if } |\xi| \leq 1 \\ \left(\frac{1 + |\xi|^{2}}{|\xi|}\right)^{1/2} \leq \sqrt{2} & \text{if } |\xi| \geq 1 \end{cases}$$

the result follows from the definition of $\hat{\mathcal{W}}_{s}^{p,q}(\mathcal{M})$. \qed
For \( f_1, \ldots, f_\sigma \in \mathcal{S}(\mathbb{R}^d) \) (or \( \in C^\infty(\mathbb{T}^d) \) in torus case), we define the multilinear operator \( \mathcal{H}_\sigma \) associated to the nonlinearity in (1.1) as follows

\[
\mathcal{H}_\sigma(f_1, \ldots, f_{2\sigma + 1}) = \prod_{\ell = 1}^{\rho} f_\ell \prod_{m = 1}^{\sigma} \tilde{f}_m.
\]

When \( f_1 = \cdots = f_\sigma = f \), we write \( \mathcal{H}_\sigma(f, f, \ldots, f) = \mathcal{H}_\sigma(f) \). We set

\[
\mathcal{N}_\sigma(u_1, \ldots, u_\sigma)(t) = \int_{0}^{t} \frac{\sin((t - \tau) |\nabla|)}{|\nabla|} \mathcal{H}_\sigma(u_1(\tau), \ldots, u_\sigma(\tau)) d\tau
\]

and write \( \mathcal{N}_\sigma(u_1, \ldots, u_\sigma) = \mathcal{N}_\sigma(u) \) for \( u = u_1 = \cdots = u_\sigma \). Recall that solution of (1.1) satisfies

\[
u(t) = S(t)(u_0, u_1) = \int_{0}^{t} \frac{\sin((t - \tau) |\nabla|)}{|\nabla|} \mathcal{H}_\sigma(u_\sigma(\tau)) d\tau = S(t)(u_0, u_1) \pm \mathcal{N}_\sigma(u)(t)
\]

**Definition 2.2** (Picard iteration). Let us set \( S_1[\tilde{u}_0](t) = S(t)(\tilde{u}_0) \) and

\[
S_k[\tilde{u}_0](t) = \sum_{k_1 + \cdots + k_\sigma = k} \int_{0}^{t} \frac{\sin((t - \tau) |\nabla|)}{|\nabla|} \mathcal{H}_\sigma(S_{k_1}[\tilde{u}_0], \ldots, S_{k_\sigma}[\tilde{u}_0])(\tau) d\tau \quad (k \geq 2).
\]

**Remark 2.3.** The empty sums in Definition 2.2 are considered as zeros. In view of this one can see that \( S_{(\sigma - 1)\ell + 2}[\tilde{u}_0] = S_{(\sigma - 1)\ell + 3}[\tilde{u}_0] = \cdots = S_{(\sigma - 1)\ell + 1}[\tilde{u}_0] = 0 \) for all \( \ell \in \mathbb{N} \cup \{0\} \).

**Lemma 2.4** (Algebra property). The spaces \( \widehat{\mathcal{M}}^{p,q}(\mathcal{M}) \) is a pointwise \( \mathcal{F}L^1 \)-module with norm inequality

\[
\|fg\|_{\widehat{\mathcal{M}}^{p,q}} \leq \|f\|_{\mathcal{F}L^1} \|g\|_{\widehat{\mathcal{M}}^{p,q}} \quad (1 \leq p, q \leq \infty, s \in \mathbb{R}).
\]

In particular, \( \mathcal{F}L^1 \) is an algebra under pointwise multiplication, i.e. \( \|fg\|_{\mathcal{F}L^1} \leq \|f\|_{\mathcal{F}L^1} \|g\|_{\mathcal{F}L^1} \).

**Proof.** Follows from Young’s inequality. \( \square \)

**Lemma 2.5** (See [19]). Let \( \{b_k\}_{k = 1}^{\infty} \) be a sequence of nonnegative real numbers such that

\[
b_k \leq C \sum_{k_1 + \cdots + k_\sigma = k} b_{k_1} \cdots b_{k_\sigma} \quad \forall \ k \geq 2.
\]

Then we have \( b_k \leq b_1 C_0^{k - 1} \), for all \( k \geq 1 \), where \( C_0 = \frac{\sigma^2}{6} (C\sigma^2)^{1/(\sigma - 1)} b_1 \).

**Lemma 2.6.** For \( 0 \leq t \leq 1 \), for \( k \geq 1 \), one has

\[
\|S_k[\tilde{u}_0](t)\|_{\widehat{\mathcal{M}}^{p,q}} \leq C^{k^2} t^{\frac{k - 1}{\sigma - 1}} \|\tilde{u}_0\|_{x^{p,q}} \|\tilde{u}_0\|_{x^{p,q}}.
\]

**Proof.** Let \( \{a_k\} \) be a sequence of nonnegative real numbers such that

\[
a_1 = 1, \quad a_k = \frac{\sigma - 1}{2k - 1} \sum_{k_1 + \cdots + k_\sigma = k} a_{k_1} \cdots a_{k_\sigma} \quad \forall \ k \geq 2.
\]

where \( C > 1 \) to be chosen later. By Lemma 2.5 (2.5), we have \( a_k \leq c_k^k \) for some \( c_0 = c_0(\sigma) > 0 \). In view of this it is enough to prove: \( \|S_k[\tilde{u}_0](t)\|_{\widehat{\mathcal{M}}^{p,q}} \leq a_k t^{\frac{k - 1}{\sigma - 1}} \|\tilde{u}_0\|_{x^{p,q}} \|\tilde{u}_0\|_{x^{p,q}} \).

By Definition 2.2 and the fact \( |\sin \tau| \leq |\tau| \) together with Lemma 2.4, we have

\[
\|S_k[\tilde{u}_0](t)\|_{\widehat{\mathcal{M}}^{p,q}} \leq \sum_{k_1 + \cdots + k_\sigma = k} \int_{0}^{t} |t - \tau| \|S_{k_1}[\tilde{u}_0](\tau)\|_{\widehat{\mathcal{M}}^{p,q}} \prod_{\ell = 2}^{\sigma} \|S_{k_\ell}[\tilde{u}_0](\tau)\|_{x^{p,q}} d\tau.
\]
Therefore, by Lemma 2.1, we have
\[
\|S_\sigma[\tilde{u}_0](t)\|_{\tilde{\mathcal{W}}^{p,q}} \leq t \int_0^t \|S_1[\tilde{u}_0](\tau)\|_{\tilde{\mathcal{W}}^{p,q}} \prod_{\ell=2}^\sigma \|S_1[\tilde{u}_0](\tau)\|_{\mathcal{F}L^1} \, d\tau \leq t^2 \|\tilde{u}_0\|_{\mathcal{F}L^1}^{\sigma-1} \|\tilde{u}_0\|_{\mathcal{X}^{p,q}}.
\]
Since \(a_\sigma = 1\), the claim is true for \(k = \sigma\). Assume that the claim is true up to the level \(k - 1\). Then
\[
\|S_k[\tilde{u}_0](t)\|_{\tilde{\mathcal{W}}^{p,q}} \leq \sum_{k_1, \ldots, k_\sigma \geq 1 \atop k_1 + \cdots + k_\sigma = k} \|\tilde{u}_0\|_{\mathcal{F}L^1}^{k_1-1} \|\tilde{u}_0\|_{\mathcal{X}^{p,q}} \prod_{\ell=2}^\sigma \|\tilde{u}_0\|_{\mathcal{F}L^1}^{k_\ell} a_{k_1} \cdots a_{k_\sigma} t \int_0^t \tau^{\sigma-1} \, d\tau
\]
Hence, the claim is true at the level \(k\). This completes the proof. \(\square\)

**Lemma 2.7.** If \(0 < T \ll \min(1, M^{-\frac{\sigma-1}{\sigma}})\), then for any \(\tilde{u}_0 \in \mathcal{F}L^1\) with \(\|\tilde{u}_0\|_{\mathcal{F}L^1} \leq M\), there exists a unique solution \(u\) to integral equation (2.1) given by
\[
u = \sum_{k=1}^\infty S_k[\tilde{u}_0] = \sum_{\ell=0}^{\infty} S_{2\ell+1}[\tilde{u}_0] (2.2)
\]
which converges absolutely in \(C([0, T], \mathcal{F}L^1)\).

**Proof.** The proof goes in a similar line as the proof of Lemma 2.4 in [16]. Since the nonlinearity is different in our case, we shall briefly present the proof for the convenience of reader. Define
\[
\Psi(u)(t) = S(t)[\tilde{u}_0] \pm \int_0^t \sin((t-\tau)\nabla)\mathcal{H}_\sigma(u(\tau)) \, d\tau.
\]
Let \(0 < T \ll 1\). By Lemma 2.1 and following the proof of Lemma 2.6, we obtain
\[
\|\Psi(u)\|_{C([0, T], \mathcal{F}L^1)} \lesssim \|\tilde{u}_0\|_{\mathcal{F}L^1} + T^2 \|u\|_{C([0, T], \mathcal{F}L^1)}^{\sigma} \quad \text{and} \quad \|\Psi(u) - \Psi(v)\|_{C([0, T], \mathcal{F}L^1)} \lesssim T^2 \max\left(\|u\|_{C([0, T], \mathcal{F}L^1)}^{\sigma-1}, \|v\|_{C([0, T], \mathcal{F}L^1)}^{\sigma-1}\right) \|u - v\|_{C([0, T], \mathcal{F}L^1)}.
\]
Then considering the ball \(B_{2M}^T = \{\phi \in C([0, T], \mathcal{F}L^1) : \|\phi\|_{C([0, T], \mathcal{F}L^1)} \leq 2M\}\) with \(T^2, T^2M^{\sigma-1} \ll 1\), we find a unique fixed point of \(\Psi\) in \(B_{2M}^T\). Hence, the solution to (2.1). This proves the existence of unique solution. It is not hard to show that this solution is given by (2.2) (see for e.g. [16, Lemma 2.4] ). This completes the proof. \(\square\)

### 3. The proof of Theorem 1.2

We first prove NI with infinite loss of regularity at general data in \(\overline{\mathcal{F}L^1}(\mathcal{M}) \cap \mathcal{X}^{p,q}_s(\mathcal{M})\). Subsequently, for general data in \(\mathcal{X}^{p,q}_s(\mathcal{M})\) we use the density of \(\overline{\mathcal{F}L^1}(\mathcal{M}) \cap \mathcal{X}^{p,q}_s(\mathcal{M})\) in \(\mathcal{X}^{p,q}_s(\mathcal{M})\) (\(s < 0\)). So let us begin with \(\tilde{u}_0 \in \overline{\mathcal{F}L^1}(\mathcal{M}) \cap \mathcal{X}^{p,q}_s(\mathcal{M})\). Let \(N, R \gg 1\), \(Q = [-1, 1]^d\) and \(e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d\). Set \(\Sigma_N = \{\pm Ne_1, \pm 2Ne_1\}\) and
\[
\mathcal{F}\phi_{0,N} = R\chi_{\Omega_N} \quad \text{with} \quad \Omega_N = \bigcup_{\eta \in \Sigma_N} (\eta + Q).
\]
Note that $\mathcal{F}\phi_{0,N} = \mathcal{F}\phi_{0,N}(-\cdot)$ by the symmetry of the set $Q$ and $\Sigma$ and
\[
\|\phi_{0,N}\|_{\tilde{\mathcal{F}}_{p,q}} = R\|\chi_{n+Q_{1}}\tilde{\phi}_{0,N}\|_{L^{p}}\langle n \rangle^{s} \ll = R\left(\sum_{n \in \mathbb{Z}^{d}} \| (n+Q_{1}) \cap \Omega_{N} \|^{2} \langle n \rangle^{s}\right)^{1/q} \sim RN^{s}. \tag{3.2}
\]
We take
\[
\tilde{\phi}_{0,N} = (\phi_{0,N}, 0) \quad \text{and} \quad \tilde{u}_{0,N} = \tilde{u}_{0} + \tilde{\phi}_{0,N}. \tag{3.3}
\]

**Lemma 3.1** (See Lemma 3.6 in [19]). Let $\tilde{\phi}_{0,N}$ be given by (3.3). Then there exists $C > 0$ such that for all $k \in \mathbb{N}$, we have
\[
\|\text{supp } \mathcal{F} S_{k}[\tilde{\phi}_{0,N}](t) \| \leq C^{k}, \quad \forall t \geq 0.
\]

**Proof.** Note that $\text{supp } \mathcal{F} S_{1}[\tilde{\phi}_{0,N}](t) \subseteq \text{supp } \mathcal{F} u_{0}$; which is contained in 4 cubes with volumes $2^{d}$. Hence, $\|\text{supp } \mathcal{F} S_{1}[\tilde{\phi}_{0,N}](t) \| \leq 2^{d}4, \quad \forall t \geq 0.$ Thus, it is enough to prove the following claim: $\text{supp } \mathcal{F} S_{k}[\tilde{\phi}_{0,N}](t)$ is contained in $4c_{d}^{k-1}$ number of cubes with volume $2^{d}$. Clearly the claim is true for $k = 1$. Assume that the claim is true up to $k - 1$ stage. Then
\[
\text{supp } \mathcal{F} S_{k}[\tilde{\phi}_{0,N}](t) \subseteq \sum_{k_{1},\ldots,k_{2\sigma+1} \geq 1} \sum_{k_{1}+\cdots+k_{2\sigma+1} = k} \text{supp } v_{k}(t)
\]
where $v_{k}$ is either $\mathcal{F} S_{k}[\tilde{\phi}_{0,N}]$ or $\mathcal{F} S_{k}[\tilde{\phi}_{0,N}]$. Using induction we conclude that the set in RHS is contained in
\[
d^{2\sigma} \prod_{k_{1}+\cdots+k_{2\sigma+1} = k} 4c_{d}^{k_{1}-1} = 4^{2\sigma}d^{2\sigma}c_{d}^{k-2\sigma-1} = 4(4d)^{2\sigma}c_{d}^{k-2\sigma-1}
\]
number of cubes with volume $2^{d}$. Set $c_{d} = 4d$ and $C = c_{d} = 4d$ to conclude. \hfill \square

### 3.1. Estimates in $\hat{\phi}_{p,q}^{\sigma}$. The next result is the analogue of [19, Lemma 3.7].

**Lemma 3.2.** Let $\tilde{u}_{0,N}$ be given by (3.3), $s \leq 0$, $1 \leq p, q \leq \infty$ and $0 \leq t \leq 1$. Then there exists $C > 0$ independent of $R, N, t$ such that followings hold:

1. $\|\tilde{u}_{0,N} - \tilde{u}_{0}\|_{\tilde{\mathcal{F}}_{p,q}} \leq RN^{s}$
2. $\|S_{1}[\tilde{u}_{0,N}](t)\|_{\tilde{\mathcal{F}}_{p,q}} \leq 1 + RN^{s}$
3. $\|S_{2\sigma+1}[\tilde{u}_{0,N}](t) - S_{2\sigma+1}[\tilde{\phi}_{0,N}](t)\|_{\tilde{\mathcal{F}}_{p,q}} \leq t^{2}R^{\sigma-1}$
4. $\|S_{k}[\tilde{u}_{0,N}](t)\|_{\tilde{\mathcal{F}}_{p,q}} \leq C^{k}R^{k}\frac{t^{2\sigma-1}}{\sigma-1}$.

**Proof.** (1) follows from (3.2). By Lemma 2.1 and (3.2) we have $\|S_{1}[\tilde{\phi}_{0,N}](t)\|_{\tilde{\mathcal{F}}_{p,q}} \leq RN^{s}$. Then (2) follows by triangular inequality. By Lemma 2.6 (with $p = q = \infty$, $s = 0$) and (3.2), we obtain
\[
\|S_{k}[\tilde{\phi}_{0,N}](t)\|_{\tilde{\mathcal{F}}_{p,q}} \leq \sup_{\xi \in \mathcal{M}} \|\mathcal{F} S_{k}[\tilde{\phi}_{0,N}](t, \xi)\| \text{ supp } \mathcal{F} S_{k}[\tilde{\phi}_{0,N}](t)^{1/p} \|\langle n \rangle^{s}\|_{\ell^{q}(n+Q_{1}) \cap \text{supp } \mathcal{F} S_{k}[\tilde{u}_{0}](t)}
\leq C^{k}R^{k}\frac{t^{2\sigma-1}}{\sigma-1} \|\langle n \rangle^{s}\|_{\ell^{q}(n+Q_{1}) \cap \text{supp } \mathcal{F} S_{k}[\tilde{u}_{0}](t)}.
\]
where $|A|_{\mu, q}$ denotes the $\hat{\mathcal{M}}$-measure of the set $A$. Since $s \leq 0$, for any bounded set $D \subseteq \mathbb{Z}^{d}$, we have $\|\langle n \rangle^{s}\|_{\ell^{q}(n \in D)} \leq \|\langle n \rangle^{s}\|_{\ell^{q}(n \in B_{D})}$ where $B_{D} \subseteq \mathbb{R}^{d}$ is the minimal ball centred at the
origin with \( |D| \leq |B_D| \). In view of this and Lemma 3.1, we obtain
\[
\| \langle n \rangle^s \|_{\ell^q(\text{supp } S_k[\tilde{u}_0](t))} \leq \| \langle n \rangle^s \|_{\ell^q(\{ |n| \leq C^{k/q} \})} \leq C^{k/q}.
\]
Therefore
\[
\| S_k[\tilde{\phi}_{0,N}](t) \|_{\dot{\omega}^{p,q}} \leq C^k R^k t^{2^{k-1}/\sigma - 1}. \tag{3.4}
\]
Now observe that
\[
I_k(t) := S_k[\tilde{u}_0,N](t) - S_k[\tilde{\phi}_{0,N}](t) = \sum_{k_1, \ldots, k_\sigma \geq 1 \atop k_1 + \cdots + k_\sigma = k} \mathcal{N}(S_k[\tilde{u}_0 + \tilde{\phi}_{0,N}], \ldots, S_{k_\sigma}[\tilde{u}_0 + \tilde{\phi}_{0,N}]) - \mathcal{N}(S_k[\tilde{\phi}_{0,N}], \ldots, S_{k_\sigma}[\tilde{\phi}_{0,N}])
\]
where \( C = \{ (\tilde{u}_0, \tilde{\phi}_{0,N}) \setminus \{ (\tilde{\phi}_{0,N}, \ldots, \tilde{\phi}_{0,N}) \} \). Observe that \( C \) has at least one coordinate as \( \tilde{u}_0 \). Using Lemma 2.6 it follows that
\[
\| I_k(t) \|_{\dot{\omega}^{p,q}} \leq \sum_{k_1, \ldots, k_\sigma \geq 1 \atop k_1 + \cdots + k_\sigma = k} \| \mathcal{N}(S_k[\tilde{u}_0], \ldots, S_{k_\sigma}[\tilde{u}_0]) \|_{\mathcal{F}L^1} \cdot \| \tilde{\phi}_{0,N} \|_{\mathcal{F}L^1} \cdot \prod_{j=2}^{\sigma} \| S_{k_j}[\tilde{v}_j] \|_{\mathcal{F}L^1}
\leq (2^\sigma - 1) \| \tilde{\phi}_{0,N} \|_{\mathcal{F}L^1} \cdot \prod_{j=2}^{\sigma} \| S_{k_j}[\tilde{v}_j] \|_{\mathcal{F}L^1} \cdot \int_0^t \tau^{2^{k-1}/\sigma - 1} d\tau \sum_{k_1, \ldots, k_\sigma \geq 1 \atop k_1 + \cdots + k_\sigma = k} a_{k_1} \cdots a_{k_\sigma}
\leq 2^{\sigma+2} a_k t^{2^{k-1}/\sigma - 1} R^{k-1} \| \tilde{u}_0 \|_{\mathcal{F}L^1} \leq C^k t^{2^{k-1}/\sigma - 1} R^{k-1} \| \tilde{u}_0 \|_{\mathcal{F}L^1}
\]
as \( R \gg 1 \). Note that (3) is the particular case \( k = \sigma \) and (4) follows using the above and (3.4).

In the proof of next the lemma we follow the strategy of Proposition 3.4 in [16]. Although Proposition 3.4 in [16] considers a different nonlinearity \((w^\sigma)\), the symmetry about the origin of the real valued function \( F u_0 \) allows us to cover our choice of nonlinearity \((w^\rho \tilde{u}^{\sigma-\rho})\). We have presented the proof in detail as it will be used in the proof of the similar estimates in \( W^{s,q}_s(\mathbb{R}^d) \) spaces (Lemma 3.6).

**Lemma 3.3.** Let \( \tilde{\phi}_{0,N} \) be given by (3.3), \( s \in \mathbb{R}, 1 \leq p, q \leq \infty, N^{-1/2} < T < 1 \). Then we have
\[
\| S_\sigma[\tilde{\phi}_{0,N}](T) \|_{\dot{\omega}^{p,q}} \geq \left\| \chi_{n+Q_1}(\xi) \mathcal{F} S_\sigma[\tilde{\phi}_{0,N}](T)(\xi) \right\|_{L^p} \| \langle n \rangle^s \|_{\ell^q(\{ n = e_1 \})} \geq R^T T^2.
\]

**Proof.** We shall first briefly gives the guideline of the proof. In order to establish the required lower estimate for \( \| S_\sigma[\tilde{\phi}_{0,N}](T) \|_{\dot{\omega}^{p,q}} \), we first write \( \mathcal{F} S_\sigma[\tilde{\phi}_{0,N}](T)(\xi) \) (with \( \xi \in Q_2 \)) in terms of a double sum (see (3.5)). We shall naturally arrive summation over \( \mathcal{A} \) by the choice of data (3.3) with large \( N \). And then over \( \mathcal{B} \) by applying suitable trigonometric identities for cosine functions. Further we will divide the terms under summation into two category: one collection of good terms (say \( I_0 \)) which helps us to get the lower bound and the other collection of bad terms (\( I_1 \)) which none the less will have some upper bound. Subsequently we will choose the time \( T \) so that the good terms dominate over the bad terms to achieve the required estimate.
We now produce the details of the proof. Note that \( \overline{F_0}(-\cdot) = F_0 \). Set

\[
\Gamma_\xi = \left\{ (\xi_1, \ldots, \xi_{2\sigma + 1}) \in \mathbb{R}^{\sigma d} : \sum_{\ell = 1}^{\sigma} \xi_\ell = \xi \right\}
\]

and \( d\Gamma_\xi \) denote the \((\sigma - 1)\)-dimensional Lebesgue measure on the hyperplane \( \Gamma_\xi \). Note that for \( \xi \in Q_2 \), using \( N \gg 1 \) we have

\[
\mathcal{F}_{S_\sigma}[\tilde{\phi}_{0,N}(\cdot)](T)(\xi) = \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} \left[ \left( \frac{1}{|\xi|} \cos(t \cdot |\xi|) \mathcal{F}_0(\cdot) \right) * \left( \frac{1}{|m+\rho+1|} \cos(t \cdot |\xi|) \overline{F_0}(-\cdot) \right) \right](\xi)dt
\]

\[
= \int_0^T \frac{\sin((T-t)|\xi|)}{|\xi|} \int_{\Gamma_\xi} \prod_{\ell = 1}^{\sigma} \cos(t |\xi_\ell|) \mathbf{1}_{\eta + Q_2}(\xi_\ell) d\Gamma_\xi dt
\]

\[
= R^\sigma \frac{\sum_{A} \sum_{B} T \sin((T-t)|\xi|)}{|\xi|} \int_{\Gamma_\xi} \prod_{\ell = 1}^{\sigma} \cos(t |\xi_\ell|) \mathbf{1}_{\eta + Q_2}(\xi_\ell) d\Gamma_\xi dt
\]

(3.5)

where the sums are taken over

\[
A = \left\{ (\eta_1, \ldots, \eta_\sigma) \in \Sigma^\sigma : \sum_{\ell = 1}^{\sigma} \eta_\ell = 0 \right\}, \quad B = \{(\varepsilon_1, \ldots, \varepsilon_\sigma) \in \{\pm 1\}^\sigma \}
\]

respectively. For \( \eta = (\eta_1, \ldots, \eta_\sigma) \in \Sigma^\sigma \) set

\[
B_0(\eta) = \left\{ (\varepsilon_1, \ldots, \varepsilon_\sigma) \in \{\pm 1\}^\sigma : \sum_{\ell = 1}^{\sigma} \varepsilon_\ell |\eta_\ell| = 0 \right\}, \quad B_1(\eta) = B \setminus B_0(\eta).
\]

Then splitting the inner sum in (3.5) over \( B_0(\eta) \) and \( B_1(\eta) \) we write

\[
\mathcal{F}_{S_\sigma}[\tilde{\phi}_{0,N}(\cdot)](T)(\xi) = \frac{R^\sigma}{4^\sigma} \sum_{A} \left( I_0(\eta, T, \xi) + I_1(\eta, T, \xi) \right).
\]

Note that for each \( \eta \in A \) the set \( B_0(\eta) \) is non empty. This is because for \( \eta \in A \), \( \sum_{\ell = 1}^{\sigma} \eta_\ell = 0 \) and then \( (\frac{m_1}{|m_1|}, \ldots, \frac{m_\sigma}{|m_\sigma|}) \) \( \in B_0(\eta) \) (here \( (\eta_\ell)_1 \) denotes the first coordinate of \( \eta_\ell \)). For a fixed \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_\sigma) \in B_0(\eta) \) and \( \xi_\ell \in \eta_\ell + Q_2 \) one has (using triangular inequality)

\[
\left| \sum_{\ell = 1}^{\sigma} \varepsilon_\ell |\xi_\ell| \right| \leq \sum_{\ell = 1}^{\sigma} |\varepsilon_\ell| |\eta_\ell| + \sum_{\ell = 1}^{\sigma} |\xi_\ell - \eta_\ell| \leq 1.
\]

Therefore, for \( 0 \leq t \leq T \ll 1 \), we have \( \cos (t \sum_{\ell = 1}^{\sigma} \varepsilon_\ell |\xi_\ell|) \gg \frac{1}{2} \). On the other hand, we have

\[
\frac{\sin((T-t)|\xi|)}{|\xi|} \gg T - t
\]

provided \( 0 \leq t < T \ll 1 \) and \( \xi \in Q_2 \). Hence, for \( \xi \in Q_2 \), we have

\[
I_0(\eta, T, \xi) \gg \sum_{\varepsilon \in B_0} \int_0^T (T-t) \int_{\Gamma_\xi} \prod_{\ell = 1}^{\sigma} \mathbf{1}_{\eta + Q_2}(\xi_\ell) d\Gamma_\xi dt \gg T^2 \mathbf{1}_{Q_2}(\xi)
\]
as $1_{\alpha+Q_2} \ast 1_{\beta+Q_2} \geq c_d 1_{\alpha+\beta+Q_2}$ for $\alpha, \beta \in \mathbb{R}^d$. Therefore, for $\xi \in Q_2$, $N \gg 1$ and $0 < T \ll 1$, we have

$$\frac{R^\sigma}{4^\sigma} \sum_A I_0(\eta, T, \xi) \geq T^2 R^\sigma 1_{Q_2}(\xi). \quad (3.6)$$

Note that for $\varepsilon \in B_1(\eta)$ and $\xi_\ell \in \eta_\ell + Q_2$ one has $|\sum_{\ell=1}^\sigma \varepsilon_\ell |\xi_\ell| | \leq N$, this together with

$$\sum_{\ell=1}^\sigma \varepsilon_\ell |\xi_\ell| = \sum_{\ell=1}^\sigma \varepsilon_\ell |\eta_\ell| + \sum_{\ell=1}^\sigma \varepsilon_\ell (|\xi_\ell| - |\eta_\ell|) = \sum_{\ell=1}^\sigma \varepsilon_\ell |\eta_\ell| + O(1)$$

implies

$$\left| \sum_{\ell=1}^\sigma \varepsilon_\ell |\xi_\ell| \right| \sim N. \quad (3.7)$$

Therefore using this in

$$I_1(\eta, T, \xi) = \sum_{\varepsilon \in B_1(\eta)} \frac{1}{2|\xi|} \int_0^T \int_{\Gamma_\xi} \left[ \sin \left( T |\xi| - t \left( |\xi| + \sum_{\ell=1}^\sigma \varepsilon_\ell |\xi_\ell| \right) \right) + \sin \left( T |\xi| - t \left( |\xi| - \sum_{\ell=1}^\sigma \varepsilon_\ell |\xi_\ell| \right) \right) \right] dt$$

$$\times \prod_{\ell=1}^\sigma 1_{\eta_\ell + Q_2}(\xi_\ell) d\Gamma_\xi$$

we get $|I_1(\eta, T, \xi)| \leq N^{-1} 1_{2^{\sigma-1}Q_2}(\xi)$ for $\xi \in \frac{3}{4} e_1 + Q_\frac{1}{2}$ as $1_{\alpha+Q_2} \ast 1_{\beta+Q_2} \leq c_d 1_{\alpha+\beta+2Q_2}$. Hence for $\xi \in \frac{3}{4} e_1 + Q_\frac{1}{2}$,

$$\frac{R^\sigma}{4^\sigma} \left| \sum_A I_1(\eta, T, \xi) \right| \leq N^{-1} R^\sigma 1_{2^{\sigma-1}Q_2}(\xi). \quad (3.8)$$

Therefore using (3.6) we have for $\xi \in \frac{3}{4} e_1 + Q_\frac{1}{2} \subset Q_2$

$$|F_{\sigma}[\phi_{0,N}](T, \xi)| \geq T^2 R^\sigma \quad (3.9)$$

provided $T^2 \gg N^{-1}$ and $0 < T \ll 1$. Thus we conclude

$$\|S_{\sigma}[\phi_{0,N}](T)\|_{\ell^p_{\xi}} \geq \left\| \lambda_{\alpha+Q_4}(\xi)F_{\sigma}[\phi_{0,N}](T)(\xi) \right\|_{\ell^p_{\xi}} \|\tilde{\eta}_{n(\kappa=1)}\|_{\ell^q(\kappa=1)} \geq T^2 R^\sigma \quad (3.10)$$

if $N^{-1/2} < T \ll 1$. \qed
3.2. Estimates in $W_{s}^{2,q}$.

Lemma 3.4 (inclusion). Let $p,q,q_{1},q_{2} \in [1, \infty]$ and $s \in \mathbb{R}$. Then (1) $\|f\|_{W_{s}^{2,q}} \leq \|f\|_{\tilde{W}_{s}^{2,q}}$ if $q \leq 2 \ (2) \|f\|_{W_{s}^{p,q_{1}}} \leq \|f\|_{W_{s}^{p,q_{2}}}$ if $q_{1} \geq q_{2}$.

Proof. (1) is a consequence of Minkowski inequality and Plancherel theorem whereas (2) follows from the fact that $\ell^{p_{2}} \hookrightarrow \ell^{q_{1}}$ if $q_{1} \geq q_{2}$. \hfill \Box

Lemma 3.5. Let $\tilde{u}_{0,N}$ be given by (3.3), $s \leq 0, 1 \leq q \leq \infty$ and $0 \leq t \leq 1$. Then there exists $C > 0$ independent of $R, N, t$ such that the followings hold:

(1) $\|\tilde{u}_{0,N} - \tilde{u}_{0}\|_{W_{s}^{2,q}} \leq R N^{s}$, where $W_{s}^{2,q} = W_{s}^{2,q} \times W_{s}^{2,q}$

(2) $\|S_{1}[\tilde{u}_{0,N}] (t)\|_{W_{s}^{2,q}} \leq 1 + R N^{s}$

(3) $\|S_{\sigma}[\tilde{u}_{0,N}] (t) - S_{\sigma}[\tilde{\phi}_{0,N}] (t)\|_{W_{s}^{2,q}} \leq t^{2} R^{q - 1}$

(4) $\|S_{k}[\tilde{u}_{0,N}] (t)\|_{W_{s}^{2,q}} \leq C^{k} R^{k} t^{2 k - 1}$.

Proof. By Lemma 3.4, we have

$$\|\tilde{u}_{0,N} - \tilde{u}_{0}\|_{W_{s}^{2,q}} \leq \left\{ \begin{array}{ll}
\|\tilde{u}_{0,N} - \tilde{u}_{0}\|_{\tilde{W}_{s}^{2,q}} \leq R N^{s} & \text{for } q \in [1, 2] \\
\|\tilde{u}_{0,N} - \tilde{u}_{0}\|_{W_{s}^{2,q}} \leq R N^{s} & \text{for } q \in (2, \infty)
\end{array} \right.$$  

using Lemma 3.2 (1). Similarly the other estimates also follow from Lemma 3.2. \hfill \Box

Lemma 3.6. Let $\tilde{\phi}_{0,N}$ be given by (3.3), $s \in \mathbb{R}, 1 \leq q \leq \infty, N^{-1/2} < T \ll 1$. Then we have

$$\|S_{\sigma}[\tilde{\phi}_{0,N}] (T)\|_{W_{s}^{2,q}} \geq \left\| \square_{n} S_{\sigma}[\tilde{\phi}_{0,N}] (T) (1 + |n|) \right\|_{\ell^{q(\sigma, \varepsilon_{1})}} \geq R^{q} T^{2}.$$  

Proof. Note that using Plancherel theorem and (3.9) we have

$$\|S_{\sigma}[\tilde{\phi}_{0,N}] (T)\|_{W_{s}^{2,q}} \geq \left\| F^{-1} s_{\sigma} F S_{\sigma}[\tilde{\phi}_{0,N}] (T) (1 + |n|)^{\sigma} \right\|_{\ell^{q(\sigma, \varepsilon_{1})}} \geq R^{q} T^{2}.$$  

This completes the proof. \hfill \Box

3.3. Proof of main result.

Proof of Theorem 1.2. We first consider the case when $X_{s}^{p,q}(\mathcal{M}) = \tilde{W}_{s}^{p,q}(\mathcal{M})$. In view of the comment at the beginning of this section, it is enough to prove NI with infinite loss of regularity at $\tilde{u}_{0} \in \overline{F \mathcal{L}^{1}}(\mathcal{M}) \cap X_{s}^{p,q}(\mathcal{M})$. By Lemma 2.7, we have the existence of a unique solution to (1.1) with initial condition given by (3.3) in $\mathcal{F} \mathcal{L}^{1}(\mathcal{M})$ up to time $T$ whenever ($\|\tilde{u}_{0}\|_{\overline{F \mathcal{L}^{1}}} + R^{2} = T \ll 1$ which is implied by (0) $R^{2} \ll T \ll 1$ as $R \gg 1$. In view of Lemma 3.2 and if $R T^{\frac{2}{\sigma - 1}} < 1$ (or (0)), $\sum_{\ell=2}^{\infty} \|S_{(\sigma - 1) \ell + 1}[\tilde{u}_{0,N}] (T)\|_{\tilde{W}_{s}^{\sigma,q}}$ can be bounded by the sum of the geometric series. Specifically, we have

$$\sum_{\ell=2}^{\infty} \|S_{(\sigma - 1) \ell + 1}[\tilde{u}_{0,N}] (T)\|_{\tilde{W}_{s}^{\sigma,q}} \leq R^{2} \sum_{\ell=2}^{\infty} (C R^{1} (\sigma - 1) \ell T^{2\ell} \leq R^{2\sigma - 1} T^{4}. \ (3.11)$$

Note that

$$\|u_{N}(T)\|_{\tilde{W}_{s}^{\sigma,q}} \geq \|\chi_{n+Q_{1}} F u_{N}(T)\|_{L_{p} \langle n \rangle^{\sigma}} \geq \|\chi_{n+Q_{1}} F u_{N}(T)\|_{\ell^{q(\sigma, \varepsilon_{1})}} \sim_{\varepsilon_{1}} \|\chi_{n+Q_{1}} F u_{N}(T)\|_{L_{p} \langle n \rangle^{\sigma}} \geq \|\chi_{n+Q_{1}} F u_{N}(T)\|_{\ell^{q(\sigma, \varepsilon_{1})}}$$
therefore by Lemma 2.7 and triangle inequality, we obtain

\[ \|u_N(T)\|_{\bar{w}^p,q} \geq \|\chi_{n+Q_1}\mathcal{F}_\sigma[u_{0,N}](T)\|_{L^p}\langle n\rangle^s\|_{\ell^q(n=e_1)} - c\left(\|\chi_{n+Q_1}\mathcal{F}_1[u_{0,N}](T)\|_{L^p}\langle n\rangle^s\|_{\ell^q(n=e_1)} \right) \]

\[ + \sum_{\ell=2}^{\infty} \|\chi_{n+Q_1}\mathcal{F}_2\sigma\ell[u_{0,N}](T)\|_{L^p}\langle n\rangle^s\|_{\ell^q(n=e_1)} \]

\[ \geq \|\chi_{n+Q_1}\mathcal{F}_\sigma[u_{0,N}](T)\|_{L^p}\langle n\rangle^s\|_{\ell^q(n=e_1)} - c\|S_1[\bar{u}_{0,N}](T)\|_{\bar{w}^p,q} - c\sum_{\ell=2}^{\infty} S_{(\sigma-1)\ell+1}[\bar{u}_{0,N}](T)\|_{\bar{w}^p,q}. \]

Assume \( m \in \mathbb{N} \) be given. In order to ensure \( \|u(T)\|_{\bar{w}^p,q} \geq \|S_\sigma[\bar{u}_{0,N}](T)\|_{\bar{w}^p,q} \gg m \) we rely on the conditions

\[ \|\chi_{n+Q_1}\mathcal{F}_\sigma[u_{0,N}](T)\|_{L^p}\langle n\rangle^s\|_{\ell^q(n=e_1)} \geq \begin{cases} \|S_1[\bar{u}_{0,N}](T)\|_{\bar{w}^p,q}, \\ \Sigma_{\ell=2}^{\infty} S_{(\sigma-1)\ell+1}[\bar{u}_{0,N}](T)\|_{\bar{w}^p,q}, \\ m. \end{cases} \] _{(3.12)}

To achieve (3.12)-(3.14), we rely on Lemmata 3.2, 3.3. To use Lemma 3.3, we impose (i) \( N^{-1/2} \ll T \ll 1 \). In view of Lemmata 3.2, 3.3, and (3.11), to prove (3.13) it is sufficient to have: (ii) (a) \( R^\sigma T^2 \gg R^{2\sigma-1} T^4 \iff R^{\sigma-1} T^2 \ll 1 \) (\( \Leftrightarrow 0 \)) and (ii) (b) \( T^2 R^{\sigma-1} \ll R^\sigma T^2 \iff R \gg 1 \). To achieve (3.14) we impose (iii) \( R^\sigma T^2 \gg m \) (along with (ii) (b)). To ensure \( \|\bar{u}_{0,N} - \bar{u}_0\|_{X^p,q} < 1/m \), in view of (3.2) we impose (iv) \( RN^s < 1/m \). At the end (iii) (along with (ii) (b)) imply (3.12) using Lemma 3.2.

**Case** \( -\frac{1}{\sigma-1} \leq s < 0 \).

Set \( R = N^{-s-\delta}, T = N^{\frac{\sigma-1}{2}(s+\frac{\delta}{2})} \) with \( 0 < \delta < 1 \) satisfying \( \frac{\sigma+1}{2}\delta < -s \). Note that with \( N \gg 1 \) we have \( R \gg 1 \) and

\[ N^{\frac{1}{2}T^{-1}} \sim N^{\frac{1}{2}-\frac{\sigma-1}{2}(s+\frac{\delta}{2})} = N^{-\frac{\sigma-1}{2}(s+\frac{\delta}{2}+\frac{1}{2})} \ll 1 \] as \( s + \frac{1}{\sigma-1} + \frac{\delta}{2} > s + \frac{1}{\sigma-1} \geq 0 \)

\[ T \sim N^{\frac{\sigma-1}{2}(s+\frac{\delta}{2})} \ll 1 \] as \( s + \frac{\delta}{2} < -\sigma\delta/2 < 0 \)

\[ R^{\sigma-1} T^2 \sim N^{-(\sigma-1)(s+\delta)+\frac{\sigma-1}{2}(s+\frac{\delta}{2})} = N^{-(\sigma-1)\frac{s}{2}} \ll 1 \] as \( -(\sigma-1)\frac{\delta}{2} < 0 \)

\[ RN^s \sim N^{-(s+\delta)+s} = N^{-\delta} \ll \frac{1}{m} \] as \( -\delta < 0 \)

\[ R^\sigma T^2 \sim N^{-\sigma(s+\delta)+\sigma-1)(s+\delta/2)} = N^{-s-(\sigma+1)\delta/2} \gg m \] as \( -s - (\sigma + 1)\delta/2 > 0 \).

**Case** \( s < -\frac{1}{\sigma-1} \).

Set \( R = N^{\frac{1}{\sigma-1}-\delta}, T = N^{\frac{1}{2}+\frac{\sigma-1}{4}\delta} \) with \( 0 < \delta \ll 1 \) satisfying \( (\sigma + 1)\delta < \frac{2}{\sigma-1} \) and \( (\sigma - 1)\delta < \frac{1}{2} \).
Note that with $N \gg 1$ we have $R \gg 1$ and

\[ N^{-1/2}T^{-1} \sim N^{-1/2+1/2-(\sigma-1)\delta/4} = N^{-(\sigma-1)\delta/4} \ll 1 \quad \text{as } (\sigma-1)\delta/4 > 0 \]

\[ T \sim N^{-1/2+(\sigma-1)\delta/4} \ll 1 \quad \text{as } -1/2 + (\sigma-1)\delta/4 < 0 \]

\[ R^\sigma T^2 \sim N^{1-(\sigma-1)\delta-1+(\sigma-1)\delta/2} = N^{-(\sigma-1)\delta/2} \ll 1 \quad \text{as } - (\sigma-1)\delta/2 < 0 \]

\[ R^N \sim N^{1/(\sigma-1)-\delta+s} \ll \frac{1}{m} \quad \text{as } 1/(\sigma-1) + s < 0 \]

\[ R^\sigma T^2 \sim N^{\sigma(\frac{1}{\sigma-1}-\delta)-1+(\sigma-1)\delta/2} = N^{\frac{1}{\sigma-1}-(\sigma+1)\delta/2} \gg m \quad \text{as } 1/(\sigma-1) - (\sigma+1)\delta/2 > 0. \]

Thus with both the cases the conditions (i)-(iv) are satisfied and hence we are done with the case $X^p_s = \hat{\omega}^{p,q}_s$.

For the case $X^p_s = W^{2,q}_s$ we use same argument as above: Note that using Lemmata 3.5, 3.6.

\[
\|u_N(T)\|_{W^{2,q}_s} \\
\geq \left\|\Box_n u_N(T) \langle n \rangle^\sigma \|_{L^2} \right\|_{L^2} \sim \theta, s \left\|\Box_n u_N(T) \langle n \rangle^\sigma \|_{L^2} \right\|_{L^2} \\
\geq \left\|\Box_n S_\sigma [u_{0,N}(T) \langle n \rangle^\sigma \|_{L^2} \right\|_{L^2} - c \|S_1[u_{0,N}](T)\|_{W^{2,q}_s} - c \sum_{\ell=2}^{\infty} \|S_{(\sigma-1)\ell+1}[\tilde{u}_{0,N}](T)\|_{W^{2,q}_s} \\
\geq \left\|\Box_n S_\sigma [u_{0,N}(T) \langle n \rangle^\sigma \|_{L^2} \right\|_{L^2} > m. \]

and $\|\tilde{u}_{0,N} - \tilde{u}_0\|_{X_{s}^{2,q}} < 1/m$ provided we choose $R, N, T$ as in the case of $\hat{\omega}^{p,q}_s$. \hfill \Box

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