HOMOLOGY OPERATIONS IN SYMMETRIC HOMOLOGY

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ABSTRACT. Symmetric homology of a unital algebra $A$ over a commutative ground ring $k$ has been defined using derived functors and the symmetric bar construction of Fiedorowicz, in an analogous way as cyclic, dihedral or quaternionic homology has been defined. In this paper, it is found that the $HS_*(A)$ admits Dyer-Lashoff homology operations, and indeed, there is a Pontryagin product structure making $HS_*(A)$ into an associative commutative graded algebra. Some explicit computations are shown in low degree.

1. Introduction

Let $A$ be an associative, unital algebra over $k$, a commutative ground ring. In [1], we defined the symmetric homology of $A$ by $HS_*(A) := \text{Tor}^{\Delta S} (k, B_{*}^{sym} A)$, where $\Delta S$ is the category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms $[n] \rightarrow [m]$ are pairs $(\phi, g)$ such that $\phi \in \text{Mor}_\Delta ([n], [m])$ and $g \in \Sigma^{op}_{n+1}$ – that is, a non-decreasing set map paired with a permutation of the set $\{0, 1, \ldots, n\}$. We found a standard resolution of $k$ by projective $\Delta S^{op}$-modules, giving the result:

$$HS_*(A) = H_*(k[N(\Delta S)] \otimes_{\Delta S} B_{*}^{sym} A)$$

We also showed that including the emptyset as an initial object of our category also yields a complex that computes symmetric homology:

$$HS_*(A) = H_*(\mathcal{N}_+^* A) = H_*(k[N(\Delta S_+)] \otimes_{\Delta S_+} B_{*}^{sym} A) .$$

The category $\Delta S_+$ is permutative ([1], Prop. 4), which becomes crucial in the present paper.

First, we shall show how the permutative structure of $\Delta S_+$ implies that $\mathcal{N}_+^* A$ admits an $E_\infty$-algebra structure. This will be accomplished using various guises of the Barratt-Eccles operad [2], which are all "$E_\infty$" in a sense. We induce an operad-module [5] structure on the under-category $- \setminus \Delta S_+$ to one on the level of chains, using strong or lax symmetric monoidal functions [13]. Finally we may define operations at the level $k$-complexes, following May [7], which imply Dyer-Lashoff homology operations on $HS_*(A)$.

2. Definitions

We shall fix $S_n$ to be the symmetric group on the letters $\{1, 2, \ldots, n\}$, given by permutations $\sigma$ that act on the left of lists of size $n$, i.e., $\sigma(i_1, i_2, \ldots, i_n) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \ldots, i_{\sigma^{-1}(n)})$. Denote by $\Sigma_n$ the symmetric group on the letters $\{0, 1, \ldots, n-1\}$.

Recall, a category $\mathcal{C}$ is called symmetric monoidal category if it comes equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a unit object $e$, and a natural transformation (symmetry transformation) $s : \otimes \rightarrow \otimes \tau$ with $s^2 = \text{id}$, where $\tau : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is the functor $(A, B) \mapsto (B, A)$. Moreover, there are associativity, left unit and right unit transformations that must satisfy certain coherence relations. If these transformations are in fact identities (i.e., associativity of $\otimes$ is strict, and $e$ is a strict unit), then the category $\mathcal{C}$ is called permutative [1].

Note that coherence of the associativity relation of a symmetric monoidal category implies that the notation $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ (for objects $A_i$ of $\mathcal{C}$) is unambiguous. Indeed, for any two choices of grouping the expression, there is an isomorphism between the two results, and that isomorphism respects the unit and symmetry transformation structure.

Let $S$ denote the symmetric groupoid as category. Included in $S$ is the symmetric group on 0 letters. The objects of $S$ may be labeled by $\emptyset, \{1\}, \{1, 2\}, \ldots$, and the morphisms of $S$ are automorphisms, $\text{Aut}_S(n) := S_n$.

Presently, the definition of operad will be given in detail.

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Definition 1. Suppose \( C \) is a symmetric monoidal category, with unit \( e \) and symmetry transformation \( s \). An operad \( P \) in the category \( C \) is a functor \( P : S^\text{op} \to C \), with \( P(0) = e \), together with the following data:

1. Structure morphisms \( \gamma_{k,j_1,...,j_k} : P(k) \circ P(j_1) \circ \ldots \circ P(j_k) \to P(j) \), where \( j = \sum j_s \). For brevity, we denote these morphisms simply by \( \gamma \). These morphisms \( \gamma \) should satisfy the following associativity condition. The diagram below is commutative for all \( k \geq 0, j_s \geq 0, i_r \geq 0 \). Here, \( T \) is a map that permutes the components of the product in the specified way, using the symmetry transformation \( s \) of \( C \). Coherence of \( s \) guarantees that this is a well-defined map.

**Associativity:**

\[
\begin{array}{ccc}
P(k) & \circ & P(j_s) \circ P(i_r) \\
\downarrow \gamma \circ \text{id}^{\otimes j} & & \downarrow \text{id} \circ s^{\otimes k} \\
\downarrow & & \\
P(j) \circ P(i_r) & & P(j_s) \circ P(i_r) \\
\downarrow \gamma & & \\
P \left( \sum_{r=1}^{j} i_r \right) & & P \left( \sum_{r=1}^{j+s} i_r \right) \\
\end{array}
\]

2. A **Unit** morphism \( \eta : e \to P(1) \) making the following diagrams commute. Here, \( \ell : e \circ \text{id}_e \to \text{id}_e \) is the left unit natural isomorphism of \( C \), and \( r : \text{id}_e \circ e \to \text{id}_e \) is the right unit natural isomorphism. \( r^j \) is an *iterated* right unit map defined in the obvious way, by absorbing each object \( e \) of \( e^{\otimes j} \) one at a time.

**Left Unit Condition:**

\[
\begin{array}{ccc}
e \circ P(j) & \eta \circ \text{id} & \text{id}_e \circ \eta \circ \text{id} \\
\downarrow & & \downarrow \gamma \\
P(j) & & P(1) \circ P(j) \\
\downarrow \ell & & \\
P(j) & & \\
\end{array}
\]

**Right Unit Condition:**

\[
\begin{array}{ccc}
P(j) \circ e^{\otimes j} & \text{id} \circ \eta \circ \text{id} \circ e^{\otimes j} & \text{id} \circ \eta \circ \text{id} \circ e^{\otimes j} \\
\downarrow & & \downarrow \gamma \\
P(j) \circ e^{\otimes j} & & P(j) \circ e^{\otimes j} \\
\downarrow r^j & & \\
P(j) & & P(j) \circ e^{\otimes j} \\
\end{array}
\]

3. The right action of \( S_n \) on \( P(n) \) for each \( n \) must satisfy the following *equivariance conditions*. Both diagrams below are commutative for all \( k \geq 0, j_s \geq 0, (j = \sum j_s), \sigma \in S^\text{op}_k, \) and \( \tau_s \in S^\text{op}_j \). Here, \( T_\sigma \) is a morphism that permutes the components of the product in the specified way, using the symmetry transformation \( s \). The notation \( \sigma \{ j_1, \ldots, j_k \} \) represents the permutation of \( j \) letters which permutes the \( k \) blocks of letters (of sizes \( j_1, j_2, \ldots, j_k \)) according to \( \sigma \), and \( \tau_1 \oplus \ldots \oplus \tau_k \) denotes the image of \( (\tau_1, \ldots, \tau_k) \) under the evident inclusion \( S^\text{op}_{j_1} \times \ldots \times S^\text{op}_{j_k} \hookrightarrow S^\text{op}_j \).
Equivariance Condition A:

\[ \mathcal{P}(k) \odot \bigodot_{s=1}^{k} \mathcal{P}(j_s) \xrightarrow{id \odot T_{\sigma}} \mathcal{P}(k) \odot \bigodot_{s=1}^{k} \mathcal{P}(j_{\sigma^{-1}(s)}) \]

Equivariance Condition B:

\[ \mathcal{P}(k) \odot \bigodot_{s=1}^{k} \mathcal{P}(j_s) \xrightarrow{\gamma} \mathcal{P}(j) \]

Definition 2. For a symmetric monoidal category \( \mathcal{C} \) with product \( \odot \) and an operad \( \mathcal{P} \) over \( \mathcal{C} \), a \( \mathcal{P} \)-algebra structure on an object \( X \) in \( \mathcal{C} \) is defined by a family of maps \( \chi : \mathcal{P}(n) \odot_{S_n} X^\otimes n \to X \), which are compatible with the multiplication, unit maps, and equivariance conditions of \( \mathcal{P} \). Note, the symbol \( \odot_{S_n} \) denotes an internal equivariance condition. See [6], [10] for a more detailed definition.

If \( X \) is a \( \mathcal{P} \)-algebra, we will say that \( \mathcal{P} \) acts on \( X \).

Definition 3. Let \( \mathcal{P} \) be an operad over the symmetric monoidal category \( \mathcal{C} \), and let \( \mathcal{M} \) be a functor \( S^{op} \to \mathcal{C} \). A (left) \( \mathcal{P} \)-module structure on \( \mathcal{M} \) is a collection of structure maps, \( \mu : \mathcal{P}(n) \odot \mathcal{M}(j_1) \odot \ldots \odot \mathcal{M}(j_n) \to \mathcal{M}(j_1 + \ldots + j_n) \), satisfying the evident compatibility relations with the operad multiplication of \( \mathcal{P} \). For the precise definition, see [5].

In the course of this paper, it shall become necessary to induce structures up from small categories to simplicial sets, then to simplicial \( k \)-modules, and finally to \( k \)-complexes. Each of these categories is symmetric monoidal. For notational convenience, all operads, operad-algebras, and operad-modules will carry a subscript denoting the ambient category over which the structure is defined:

| Category            | Sym. Mon. Product | Notation   |
|---------------------|-------------------|------------|
| Small categories    | \textbf{Cat}      | \( \mathcal{P}_{\text{cat}} \) |
| Simplicial sets     | \textbf{SimpSet}  | \( \mathcal{P}_{\text{ss}} \) |
| Simplicial \( k \)-modules | \textbf{k-SimpMod} | \( \mathcal{P}_{\text{sm}} \) |
| \( k \)-complexes    | \textbf{k-Complexes} | \( \mathcal{P}_{\text{ch}} \) |
Remark 4. The notation \( \hat{\otimes} \), appearing in Richter [13], is useful for indicating degree-wise tensoring of graded modules: \( (A \otimes B)_n := A_n \otimes_k B_n \), as opposed to the standard tensor product (over \( k \)) of complexes: \( (A \otimes B)_n := \bigoplus_{p+q=n} A_p \otimes_k B_q \).

Furthermore, we are interested in certain functors from one category to the next in the list. These functors preserve the symmetric monoidal structure in a sense we will make precise in Section 3 – hence, it will follow that these functors send operads to operads, operad-modules to operad-modules, and operad-algebras to operad-algebras.

\[
\begin{array}{c}
\text{(Cat, \times)} \\
\downarrow N \quad \text{(Nerve of categories)} \\
\text{(SimpSet, \times)} \\
\downarrow k[-] \quad \text{(k-linearization)} \\
\text{(k-SimpMod, \hat{\otimes})} \\
\downarrow N \quad \text{(Normalization functor)} \\
\text{(k-Complexes, \otimes)}
\end{array}
\]

Remark 5. Note, the normalization functor \( N \) is one direction of the Dold-Kan correspondence between simplicial modules and complexes [13].

Remark 6. The ultimate goal of the paper is to construct an \( E_\infty \) structure [7] on the chain complex associated with \( Y+A \), i.e. an action by an \( E_\infty \)-operad. While we could define the notion of \( E_\infty \)-operad over general categories, it would require extra structure on the ambient symmetric monoidal category \( (\mathcal{C}, \otimes) \), which the examples above possess. To avoid needless technicalities, we shall instead define versions of the Barratt-Eccles operad over each of our ambient categories, and take for granted that they are all “\( E_\infty \)-operads” in this general sense. For a very clear treatment of \( E_\infty \)-operads and algebras in the category of chain complexes, see [6].

Definition 7. Let \( \mathcal{D}_{cat} \) denote the operad in the category \( \text{Cat} \), where \( \mathcal{D}_{cat}(m) \) is the category \( ES_m \). That is, the objects of \( \mathcal{D}_{cat}(m) \) are the elements of \( S_m \), and for each pair of objects, \( \sigma, \tau \), there is a unique morphism \( \tau \sigma^{-1} \) from \( \sigma \) to \( \tau \). The structure map \( \delta \) in \( \mathcal{D}_{cat} \) is a functor defined on objects by:

\[
\delta : \mathcal{D}_{cat}(m) \times \mathcal{D}_{cat}(k_1) \times \ldots \times \mathcal{D}_{cat}(k_m) \longrightarrow \mathcal{D}_{cat}(k), \quad \text{where } k = \sum k_i
\]

\[
(\sigma, \tau_1, \ldots, \tau_m) \mapsto \sigma\{k_1, \ldots, k_m\}(\tau_1 \oplus \ldots \oplus \tau_m)
\]

The action of \( S_m^{op} \) on objects of \( \mathcal{D}_{cat}(m) \) is given by right multiplication of group elements.

Remark 8. We are following the notation of May for our operad \( \mathcal{D}_{cat} \) (See [9], Lemmas 4.3, 4.8). May’s notation for \( \mathcal{D}_{cat}(m) \) is \( \Sigma_m \), and he defines the related operad \( \mathcal{D} \) over the category of spaces, as the geometric realization of the nerve of \( \Sigma \). The nerve of \( \mathcal{D}_{cat} \) is generally known in the literature as the Barratt-Eccles operad (See [2], where the notation for \( N\mathcal{D}_{cat} \) is \( \Gamma \)).

We end this section with one very nice result about permutative categories.

Proposition 9. If \( \mathcal{C} \) is a permutative category, then \( \mathcal{C} \) admits the structure of a \( \mathcal{D}_{cat} \)-algebra.

Proof. The structure map is a family of functors, \( \theta : \mathcal{D}_{cat}(m) \times \mathcal{C}^m \rightarrow \mathcal{C} \), defined on objects and morphisms as in Diagram[1]. Note, \( T \tau \sigma^{-1} \) is the map that permutes the components according to the permutation \( \tau \sigma^{-1} \).
using the symmetry transformation of $\mathcal{E}$.

\[
\begin{align*}
(s, C_1, \ldots, C_m) & \xrightarrow{\theta} C_{\sigma^{-1}(1)} \odot \ldots \odot C_{\sigma^{-1}(m)} \\
\tau s^{-1} \times f_1 \times \ldots \times f_m & \xrightarrow{\theta} D_{\sigma^{-1}(1)} \odot \ldots \odot D_{\sigma^{-1}(m)} \\
(\tau, D_1, \ldots, D_m) & \xrightarrow{\theta} D_{\tau^{-1}(1)} \odot \ldots \odot D_{\tau^{-1}(m)}
\end{align*}
\]

(1)

We just need to verify that $\theta$ satisfies the expected equivariance condition. This condition is easily verified on objects, as $\theta(s\tau, C_1, \ldots, C_m) = C_{(s\tau)^{-1}(1)} \odot \ldots \odot C_{(s\tau)^{-1}(m)} = C_{\tau^{-1}(s^{-1}(1))} \odot \ldots \odot C_{\tau^{-1}(s^{-1}(m))} = \theta(s, C_{\tau^{-1}(1)}, \ldots, C_{\tau^{-1}(m)})$. The verification on morphisms is similarly straightforward.

3. Operad-Module Structure

In order to best define the $E_\infty$ structure of $\mathcal{Y}^+ A$, we will begin with an operad-module structure over the category of small categories, then induce this structure up to the category of $k$-complexes.

Recall, $\Delta \mathcal{S}_+$ is the category containing $\Delta \mathcal{S}$ as full subcategory and containing an initial object $[-1]$, representing the empty set. As noted in Section 1 of [1], $\Delta \mathcal{S}_+$ is isomorphic to the category of non-commutative sets, $\mathcal{F}(as)$. The objects of $\mathcal{F}(as)$ are the sets $0, 1, 2, \ldots$, and the morphisms are set maps together with total orderings of each preimage set $[1, 1, 1]$.

Definition 10. Define for each $m \geq 0$, a category, $\mathcal{K}_{cat}(m) := [m - 1] \setminus \Delta \mathcal{S}_+ = m \setminus \mathcal{F}(as)$.

Identifying $\Delta \mathcal{S}_+$ with $\mathcal{F}(as)$, we see that the morphism $(\phi, g)$ of $\Delta \mathcal{S}_+$ consists of the set map $\phi$, precomposed with $g^{-1}$ in order to indicate the total ordering on all preimage sets. Thus, precomposition with symmetric group elements defines a right $S_m$-action on objects of $m \setminus \mathcal{F}(as)$. When writing morphisms of $\mathcal{F}(as)$, we may avoid confusion by writing the automorphisms as elements of the symmetric group rather than its opposite group, with the understanding that $g \in Mor\mathcal{F}(as)$ corresponds to $g^{-1} \in Mor\Delta \mathcal{S}_+$. In this way, $\mathcal{K}_{cat}$ becomes a functor $S \rightarrow \text{Cat}$.

\[
\mathcal{K}_{cat}(m) \times S_m \rightarrow \mathcal{K}_{cat}(m)
\]

\[
(\phi, g).h := (\phi, gh)
\]

Let $m, j_1, j_2, \ldots, j_m \geq 0$, and let $j = \sum j_s$. We shall define a family of functors,

\[
\mu = \mu_{m, j_1, \ldots, j_m} : \mathcal{K}_{cat}(m) \times \prod_{s=1}^{m} \mathcal{K}_{cat}(j_s) \rightarrow \mathcal{K}_{cat}(j).
\]

(2)

For compactness and clarity, some notations are in order. Let $\underline{m} = \{1, 2, \ldots, m\}$ as ordered list, along with a left $S_m$ action, $\tau_{\underline{m}} := \{\tau^{-1}(1), \ldots, \tau^{-1}(m)\}$. Then for any permutation, $\tau_{\underline{m}}$, of the ordered list $\underline{m}$, and any list of $m$ numbers, $\{j_1, \ldots, j_m\}$, the set $\{j_{\tau^{-1}(1)}, \ldots, j_{\tau^{-1}(m)}\}$ will be denoted $j_{\tau_{\underline{m}}}$.

Furthermore, if $f_1, f_2, \ldots, f_m \in Mor\mathcal{F}(as)$, denote $f_{\underline{m}} := f_{\tau^{-1}(1)} \odot \ldots \odot f_{\tau^{-1}(m)}$. If $g_1, g_2, \ldots, g_m \in Mor\mathcal{F}(as)$ (and each $g_i$ is composable with $f_i$), then denote $(g_{\tau_{\underline{m}}} f_{\underline{m}}) := g_{\tau^{-1}(1)} f_{\tau^{-1}(1)} \odot \ldots \odot g_{\tau^{-1}(m)} f_{\tau^{-1}(m)}$.

In the following, assume morphisms $f_i$ and $g_i$, for $1 \leq i \leq m$ have specified sources and targets: $j_i \overset{f_i}{\rightarrow} p_i \overset{g_i}{\rightarrow} q_i$. Now, using the above notation, define $\mu$ on objects by,

\[
\mu(s, f_1, f_2, \ldots, f_m) := \sigma\{p_1, p_2, \ldots, p_m\}(f_1 \odot f_2 \odot \ldots \odot f_m) = \sigma\{p_{\underline{m}}\} f_{\underline{m}} \odot
\]

(3)
Define \( \mu \) on morphisms by:

\[
\begin{align*}
(\sigma, f_1, \ldots, f_m) & \quad \xrightarrow{\mu} \quad \sigma(p_m) f_m^\circ \\
\tau \sigma^{-1} \times g_1 \times \ldots \times g_m & \quad \downarrow \quad \quad \downarrow (\tau \sigma^{-1})(g_m) g_m^\circ \\
(\tau, g_1 f_1, \ldots, g_m f_m) & \quad \xrightarrow{\mu} \quad \tau(g_m) (g_m f_m)^\circ
\end{align*}
\]

(4)

It is useful to note three properties of the block permutations and symmetric monoidal product in the category \( \coprod_{m \geq 0} \mathcal{K}_{\text{cat}}(m) \).

**Proposition 11.** 1. For \( \sigma, \tau \in S_m \), and non-negative \( p_1, p_2, \ldots, p_m \),

\[
(\sigma \tau)(p_m) = \sigma(p_m) \tau(p_m).
\]

2. For \( \sigma \in S_m \), and morphisms \( g_i \in \text{Mor}_{\mathcal{J}(\text{cat})}(p_i, q_i) \), \( 1 \leq i \leq m \),

\[
\sigma(g_m) g_m^\circ = g_m^\circ \sigma(p_m).
\]

3. For morphisms \( j_i : p_i \to p_i, \) \( 1 \leq i \leq m \),

\[
(g_m f_m)^\circ = g_m^\circ f_m^\circ.
\]

**Proof.** Property 1 is a standard composition property of block permutations. See [6], p. 41, for example. Property 2 expresses the fact that it does not matter whether we apply the morphisms \( g_i \) to blocks first, then permute those blocks, or permute the blocks first, then apply \( g_i \) to the corresponding permuted block. Finally, property 3 is a result of functoriality of the product \( \circ \). \( \square \)

Using these properties, it is straightforward to verify that \( \mu \) is a functor. Our goal for the remainder of this section is to prove:

**Theorem 12.** The family of functors \( \mu \) defines a \( \mathcal{D}_{\text{cat}} \)-module structure on \( \mathcal{K}_{\text{cat}} \).

The proof will follow from a technical lemma. First, let \( M \) be a monoid with unit, 1. Let \( \mathcal{X}_+^* M := N(\mathcal{J} \setminus \Delta S_+) \times \Delta S_+ B_{\mathcal{J}(\text{cat})}^m M \). This construction is analogous to the construction \( \mathcal{X}_+^* \) of [1], Section 4, only using \( \Delta S_+ \) in place of \( \Delta S \).

**Lemma 13.** \( \mathcal{X}_+^* M \) is the nerve of a permutative category.

**Proof.** Consider a category \( \mathcal{J} M \) whose objects are the elements of \( \coprod_{n \geq 0} M^n \), where \( M^0 \) is understood to be the set consisting of the empty tuple, \( \{()\} \). Morphisms of \( \mathcal{J} M \) consist of the morphisms of \( \Delta S_+ \), reinterpreted as follows: A morphism \( f : [p] \to [q] \) in \( \Delta S \) will be considered a morphism \( (m_0, m_1, \ldots, m_p) \to f_i(m_0, m_1, \ldots, m_p) \in M^{p+1} \). The unique morphism \( \iota_n \) will be considered a morphism \( () \to (1, 1, \ldots, 1) \in M^{n+1} \). The nerve of \( \mathcal{J} M \) consists of chains,

\[
(m_0, \ldots, m_n) \xrightarrow{f_1} f_1(m_0, \ldots, m_n) \xrightarrow{f_2} \ldots \xrightarrow{f_i} f_i \ldots f_1(m_0, \ldots, m_n)
\]

Such a chain can be rewritten uniquely as an element of \( M^n \) together with a chain in \( N \Delta S \).

\[
\left( [n] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \ldots \xrightarrow{f_i} [n_i], (m_0, \ldots, m_n) \right)
\]

which in turn is uniquely identified with an element of \( \mathcal{X}_+^* M \):

\[
\left( [n] \xrightarrow{\iota} [n] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \ldots \xrightarrow{f_i} [n_i], (m_0, \ldots, m_n) \right)
\]

Clearly, since any element of \( \mathcal{X}_+^* M \) may be written so that the first morphism of the chain component is the identity (cf. [1], Remark. 14), \( N(\mathcal{J} M) \) can be identified with \( \mathcal{X}_+^* M \).

Now, we show that \( \mathcal{J} M \) is permutative. Define the product on objects:

\[
(m_0, \ldots, m_p) \circ (n_0, \ldots, n_q) := (m_0, \ldots, m_p, n_0, \ldots, n_q).
\]

Now, by [1], Prop. 4, \( \Delta S_+ \) is permutative. Use the product of \( \Delta S_+ \) to define products of morphisms in \( \mathcal{J} M \). Associativity is strict, since it is induced by the associativity of \( \circ \) in \( \Delta S_+ \). There is also a strict
unit, the empty tuple, \( () \). The symmetry transformation (i.e., \( s : \circ \to \circ \mathcal{T} \)) is defined on objects by block transposition:

\[
s_{p,q} : (m_0, \ldots, m_p) \oplus (n_0, \ldots, n_q) \to (n_0, \ldots, n_q) \oplus (m_0, \ldots, m_p)
\]

\[
(m_0, \ldots, m_p, n_0, \ldots, n_q) \mapsto (n_0, \ldots, n_q, m_0, \ldots, m_p).
\]

\[\square\]

Let \( X = \{x_i\}_{i \geq 1} \) be a countable set of formal algebraically-independent indeterminates, and \( J(X_+) \) the free monoid on the set \( X_+ := X \cup \{1\} \). The category \( \mathcal{T}J(X_+) \) is permutative, so by Prop \( \text{9} \) \( \mathcal{T}J(X_+) \) admits the structure of \( \mathcal{D}_{\text{cat}} \)-algebra, with structure map \( \theta \). We may identify:

\[
\prod_{m \geq 0} (\mathcal{K}_{\text{cat}}(m) \times S_m X^m) = \mathcal{T}J(X_+),
\]

via the map \( (f, x_i, x_{i2}, \ldots, x_{im}) \mapsto f(x_i, x_{i2}, \ldots, x_{im}) \). The fact that \( J(X_+) \) is the free monoid on \( X_+ \) ensures that there is a map in the inverse direction, well-defined up to action of the symmetric group \( S_m \).

Note, the action of \( S_m \) on \( X^m \) is by permutation of the components \( \{x_i, x_{i2}, \ldots, x_{im}\} \).

Next, let \( m, j_1, j_2, \ldots, j_m \geq 0 \), and let \( j = \sum j_s \). Furthermore, let

\[
X_s = (x_{j_1} + j_1 + \ldots + j_{s-1} + 1, \ldots, x_{j_1} + j_2 + \ldots + j_s).
\]

We shall define a functor \( \alpha = \alpha_{m, j_1, \ldots, j_m} : \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m \mathcal{K}_{\text{cat}}(j_s) \to \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m (\mathcal{K}_{\text{cat}}(j_s) \times S_{j_s} X^{j_s}) \).

This functor \( \alpha \) is defined on objects and morphisms according to the following diagram:

\[
\begin{array}{ccc}
(\sigma, f_1, f_2, \ldots, f_m) & \xrightarrow{\alpha} & (\sigma, \prod_{s=1}^m (f_s, X_s)) \\
\tau \sigma^{-1} \times g_1 \times \ldots \times g_m & \downarrow & \tau \sigma^{-1} \times (g_1 \times i_{d_1}) \times \ldots \times (g_m \times i_{d_m}) \\
(\tau, g_1 f_1, g_2 f_2, \ldots, g_m f_m) & \xrightarrow{\alpha} & (\tau, \prod_{s=1}^m (g_s f_s, X_s))
\end{array}
\]

Let \( \text{inc} \) be the inclusion of categories:

\[
\text{inc} : \mathcal{D}_{\text{cat}}(m) \times \prod_{s=1}^m (\mathcal{K}_{\text{cat}}(j_s) \times S_{j_s} X^{j_s}) \to \mathcal{D}_{\text{cat}}(m) \times \left[ \prod_{i \geq 0} (\mathcal{K}_{\text{cat}}(i) \times S_i X^i) \right]^m,
\]

induced by the evident inclusion of for each \( s, \mathcal{K}_{\text{cat}}(j_s) \times S_{j_s} X^{j_s} \hookrightarrow \prod_{i \geq 0} (\mathcal{K}_{\text{cat}}(i) \times S_i X^i) \). Let \( \alpha_0 \) be the functor \( \mathcal{K}_{\text{cat}}(j) \to \mathcal{K}_{\text{cat}}(j) \times S_j X^j \), defined by \( \alpha_0(f) = (f, x_1, x_2, \ldots, x_j) \), and \( \text{inc}_0 \) be the inclusion \( \mathcal{K}_{\text{cat}}(j) \times S_j X^j \hookrightarrow \prod_{i \geq 0} (\mathcal{K}_{\text{cat}}(i) \times S_i X^i) \).
Next, consider the following diagram. The top row is the map $\mu$ of Eq. (3), and the bottom row is the operad-algebra structure map for $\mathcal{J}J(X_+)$. 

\[
\begin{array}{ccc}
\mathcal{P}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{X}_{\text{cat}}(j_s) & \xrightarrow{\mu} & \mathcal{X}_{\text{cat}}(j) \\
\downarrow \alpha & & \downarrow \alpha_0 \\
\mathcal{P}_{\text{cat}}(m) \times \prod_{s=1}^{m} \left( \mathcal{X}_{\text{cat}}(j_s) \times S_{j_s} X^{j_s} \right) & \xrightarrow{\theta} & \prod_{i \geq 0} \left( \mathcal{X}_{\text{cat}}(i) \times S_i X^{i} \right) \\
\downarrow \text{inc} & & \downarrow \text{inc}_0 \\
\mathcal{P}_{\text{cat}}(m) \times \left[ \prod_{i \geq 0} \left( \mathcal{X}_{\text{cat}}(i) \times S_i X^{i} \right) \right]^m & \xrightarrow{\theta} & \left[ \prod_{i \geq 0} \left( \mathcal{X}_{\text{cat}}(i) \times S_i X^{i} \right) \right]^m 
\end{array}
\]

(5)

I claim that this diagram commutes. Let $w := (\sigma, f_1, \ldots, f_m) \in \mathcal{P}_{\text{cat}}(m) \times \prod_{s=1}^{m} \mathcal{X}_{\text{cat}}(j_s)$ be arbitrary. Following the left-hand column of the diagram, we obtain the element

\[
\alpha(w) = \left( \sigma, \prod_{s=1}^{m} (f_s, X_s) \right).
\]

(6)

It is important to note that the list of all $x_s$ that occur in expression (6) is exactly \{\(x_1, x_2, \ldots, x_j\)\} with no repeats, up to permutations by $S_{j_1} \times \ldots \times S_{j_m}$. Thus, after applying $\theta$, the result is an element of the form $\theta \alpha(w) = (F, x_1, x_2, \ldots, x_j)$, up to permutations by $S_j$. So, $\theta \alpha(w)$ is in the image of $\alpha_0$, say $\theta \alpha(w) = \alpha_0(v)$. All that remains is to show that $\mu(w) = v$. Let us examine closely what the morphism $F$ in the formula for $\theta \alpha(w)$ must be. $\alpha(w)$ is identified with the element $(\sigma, \prod_{s=1}^{m} f_s(X_s))$ of $\mathcal{P}_{\text{cat}}(m) \times [\mathcal{J}J(X_+)]^m$, and $\theta$ sends this element to $\theta \alpha(w) = \bigodot_{s=1}^{m} f_{\sigma^{-1}(s)}(X_{\sigma^{-1}(s)})$. Now $\theta \alpha(w)$ is interpreted in $\mathcal{X}_{\text{cat}}(j) \times S_j X^{j}$ as follows: Begin with the tuple $(x_1, x_2, \ldots, x_j)$. This tuple is divided into blocks of sizes $j_1, \ldots, j_m$, $(X_1, \ldots, X_m)$. Apply $f_1 \odot \ldots \odot f_m$ to obtain $f_1(X_1) \odot \ldots \odot f_m(X_m)$. Finally, apply the block permutation $\sigma \{p_1, \ldots, p_m\}$ to obtain the correct order in the result (see Fig. 1). This shows that $F = \sigma \{p_1, \ldots, p_m\}(f_1 \odot \ldots \odot f_m)$, as required.

We now proceed with the proof of Thm. 12.

Now that we have the diagram (5), it is straightforward to show that $\mu$ satisfies the associativity condition for an operad-module structure map. Essentially, associativity is induced by the associativity condition of the algebra structure map $\theta$. All that remains is to verify the unit and equivariance conditions.

It is trivial to verify the left unit condition (and there is no corresponding right unit condition in an operad-module structure). Note, the unit object of $\mathcal{P}_{\text{cat}}(1)$ is the identity element of $S_1$.

Finally, specify the right-action of $\rho \in S_j$ on $\mathcal{X}_{\text{cat}}(j)$ as precomposition by $\rho$. That is, $f \rho := f \rho$ for $f \in \mathcal{J}J \setminus \mathcal{F}(as)$. For clarity, we include the routine check that verifies the equivariance condition on the level of objects. Let $f_i \in \mathcal{X}_{\text{cat}}(j_i)$ (for $1 \leq i \leq m$) have specified sources and targets, $i \overset{f_i}{\rightarrow} p_i$. The following diagrams commute by the properties stated in Prop. 11.
\[
\begin{align*}
\bullet x_1 & \quad \bullet y_1 \\
\bullet x_2 & \quad \bullet y_{p_1} \\
\vdots & \quad \vdots \\
\bullet x_{j_1} & \quad \bullet y_{p_1+1} \\
\bullet x_{j_1+1} & \quad \bullet y_{p_1+1} \\
\vdots & \quad \vdots \\
\bullet x_{j_2} & \quad \bullet y_{p_2} \\
\vdots & \quad \vdots \\
\bullet x_{j_{m-1}+1} & \quad \bullet y_{p_{m-1}+1} \\
\vdots & \quad \vdots \\
\bullet x_{j_m} & \quad \bullet y_{p_m}
\end{align*}
\]
\[
\begin{align*}
\bullet f_1 & \quad \bullet f_1(X_1) \\
\bullet f_2 & \quad \bullet f_2(X_2) \\
\vdots & \quad \vdots \\
\bullet f_m & \quad \bullet f_m(X_m)
\end{align*}
\]

\[f_1 \circ \ldots \circ f_m \quad \sigma \{p_1, \ldots, p_m\}\]

\textbf{Figure 1.} \(\theta_\alpha(\sigma, f_1, \ldots, f_m)\), interpreted as an object of \(\mathcal{K}_{\text{cat}}(j) \times S_j X^j\)

\begin{align*}
\text{Equivariance A:} \\
(\sigma, f_1, \ldots, f_m) & \xrightarrow{\text{id} \times T_\tau} (\sigma, f_{\tau^{-1}(1)}, \ldots, f_{\tau^{-1}(m)}) \\
(\sigma \tau, f_1, \ldots, f_m) & \xrightarrow{\mu} \sigma \{p_{\tau m}\} f_{\tau m} \\
(\sigma \tau) \{p_m\} f_{\tau m} & \xrightarrow{\tau(\{j_m\})} \sigma \{p_{\tau m}\} \tau \{p_m\} f_{\tau m} \\
\end{align*}
Equivariance B:

\[
(\sigma, f_1, \ldots, f_m) \xrightarrow{\mu} \sigma(p_m) f_m^\otimes
\]

\[
\text{id} \times \tau_1 \times \ldots \times \tau_m \downarrow \quad \quad \tau_1 \otimes \ldots \otimes \tau_m \downarrow
\]

\[
(\sigma, f_1 \tau_1, \ldots, f_m \tau_m) \xrightarrow{\mu} \sigma(p_m) (f_m \tau_m)^\otimes \quad \quad \sigma(p_m) f_m^\otimes \tau_m^\otimes
\]

Remark 14. It turns out that \(\mathcal{K}_{cat}\) is in fact a pseudo-operad. Recall from [6] that a pseudo-operad is a ‘non-unitary’ operad. That is, there are multiplication maps that satisfy operad associativity, and actions by the symmetric groups that satisfy operad equivariance conditions, but there is no requirement concerning a left or right unit map. The multiplication is defined as the composition:

\[
\mathcal{K}_{cat}(m) \times \prod_{s=1}^{m} \mathcal{K}_{cat}(j_s) \xrightarrow{\pi \times \text{id}'} \mathcal{D}_{cat}(m) \times \prod_{s=1}^{m} \mathcal{K}_{cat}(j_s) \xrightarrow{\mu} \mathcal{K}_{cat}(j_1 + \ldots + j_m),
\]

where \(\pi : \mathcal{K}_{cat}(m) \rightarrow \mathcal{D}_{cat}(m)\) is the projection functor defined as isolating the group element (automorphism) of an \(\mathcal{F}(as)\) morphism: \(\pi(\phi, g) := g\). Indeed, \(\pi\) defines a covariant isomorphism of the subcategory \(\text{Aut}(\mathcal{D} \backslash \mathcal{F}(as))\) onto \(\mathcal{D}_{cat}(m)\).

4. Inducing the Operad-Module Structure to the Level of Chain Complexes

Now that we have a \(\mathcal{D}_{cat}\)-module structure on \(\mathcal{K}_{cat}\), we shall proceed in steps to induce this structure to an analogous operad-module structure on the level of \(k\)-complexes. First, we shall require the definition of lax symmetric monoidal functor. The following appears in [14], as well as [12].

Definition 15. Let \(\mathcal{C}\), resp. \(\mathcal{C}'\), be a symmetric monoidal category with multiplication \(\odot\), resp. \(\boxdot\). Denote the associativity maps in \(\mathcal{C}\), resp. \(\mathcal{C}'\), by \(a\), resp. \(a'\), and the commutation maps by \(s\), resp. \(s'\). A functor \(F : \mathcal{C} \rightarrow \mathcal{C}'\) is a lax symmetric monoidal functor if there are natural transformations \(f : FA \boxdot FB \rightarrow F(A \odot B)\) such that the following diagrams are commutative:

\[
FA \boxdot (FB \boxdot FC) \xrightarrow{id \boxdot f} FA \boxdot F(B \odot C) \xrightarrow{f} F(A \odot (B \odot C))
\]

\[
(FA \boxdot FB) \boxdot FC \xrightarrow{f \boxdot id} F(A \odot B) \boxdot FC \xrightarrow{f} F((A \odot B) \odot C)
\]

\[
FA \boxdot FB \xrightarrow{f} F(A \odot B)
\]

\[
FB \boxdot FA \xrightarrow{f} F(B \odot A)
\]

If the transformation \(f\) is a natural isomorphism, then the functor \(F\) is called strong symmetric monoidal.

Observe that the nerve functor \(N : \text{Cat} \rightarrow \text{SimpSet}\) is strong symmetric monoidal with associated natural isomorphism, \(S_n : N\mathcal{A}_{cat} \times N\mathcal{B}_{cat} \rightarrow N(\mathcal{A}_{cat} \times \mathcal{B}_{cat})\), defined on \(n\)-chains via:

\[
\left( A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} A_n, B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} \ldots \xrightarrow{g_n} B_n \right) \xrightarrow{S_n} (A_0, B_0) \xrightarrow{f_1 \times g_1} \ldots \xrightarrow{f_n \times g_n} (A_n, B_n)
\]

The \(k\)-linearization functor, \(k[-] : \text{SimpMod} \rightarrow k\text{-SimpMod}\) is also strong symmetric monoidal, with associated natural isomorphism, \(k[\mathcal{A}_{ss}] \boxtimes k[\mathcal{B}_{ss}] \rightarrow k[\mathcal{A}_{ss} \times \mathcal{B}_{ss}]\), defined degree-wise:

\[
k[\mathcal{A}_{ss}]_n \boxtimes k[\mathcal{B}_{ss}]_n \xrightarrow{\boxtimes} k[\mathcal{A}_{ss} \times \mathcal{B}_{ss}]_n = k[(\mathcal{A}_{ss} \times \mathcal{B}_{ss})_n].
\]

Finally, the normalization functor, \(N : k\text{-SimpMod} \rightarrow k\text{-Complexes}\) is lax symmetric monoidal, with associated natural transformation being the Eilenberg-Zilber shuffle map (see [13]).

\[
Sh : N\mathcal{A}_{sm} \otimes N\mathcal{B}_{sm} \rightarrow N(\mathcal{A}_{sm} \otimes \mathcal{B}_{sm})
\]
Now, define the versions of $\mathcal{D}_{\text{cat}}$ and $\kappa_{\text{cat}}$ over the various symmetric monoidal categories we are considering:

\begin{align*}
\mathcal{D}_{\text{ss}} &= N\mathcal{D}_{\text{cat}} & \kappa_{\text{ss}} &= N\kappa_{\text{cat}} \\
\mathcal{D}_{\text{sm}} &= k[\mathcal{D}_{\text{ss}}] & \kappa_{\text{sm}} &= k[\kappa_{\text{ss}}] \\
\mathcal{D}_{\text{ch}} &= N\mathcal{D}_{\text{sm}} & \kappa_{\text{ch}} &= N\kappa_{\text{sm}}
\end{align*}

Remark 16. Observe that for each $m \geq 0$, $\mathcal{D}_{\text{ch}}$ is a contractible $k$-complex. Thus $\mathcal{D}_{\text{ch}}$ is an $E_{\infty}$-operad in the sense of May \cite{May1972}. By analogy, we would like to say that the operads $\mathcal{D}_{\text{cat}}$, $\mathcal{D}_{\text{ss}}$ and $\mathcal{D}_{\text{sm}}$ are $E_{\infty}$ in their ambient categories, however a precise definition of $E_{\infty}$ over general categories does not yet exist.

Lemma 17. Let $(\mathcal{C}, \odot, e)$ and $(\mathcal{C}', \sqcup, e')$ be symmetric monoidal categories, and $F : \mathcal{C} \to \mathcal{C}'$ a lax symmetric monoidal functor with associated natural transformation $f$. Moreover, suppose that $e' = Fe$.

1. If $\mathcal{P}$ is an operad over $\mathcal{C}$, then $F\mathcal{P}$ is an operad over $\mathcal{C}'$.
2. If $\mathcal{P}$ is an operad and $\mathcal{M}$ is a $\mathcal{P}$-module over $\mathcal{C}$, then $F\mathcal{M}$ is an $F\mathcal{P}$-module over $\mathcal{C}'$.
3. If $\mathcal{P}$ is an operad over $\mathcal{C}$ and $Z \in \text{Obj} \mathcal{C}$ is a $\mathcal{P}$-algebra, then $FZ$ is an $F\mathcal{P}$-algebra over $\mathcal{C}'$.

Proof. Note that properties (9) and (10) imply that the associativity transformation $a'$ and symmetry transformation $s'$ of $\mathcal{C}'$ can be induced by the associativity transformation $a$ and symmetry transformation $s$ of $\mathcal{C}$. That is, all symmetric monoidal structure is carried by $F$ from $\mathcal{C}$ to $\mathcal{C}'$.

Denote by $f^m$ the natural transformation induced by $f$ on $m + 1$ components:

\[ f^m : FA_0 \square FA_1 \square \cdots \square FA_m \to FA_0 \odot A_1 \odot \cdots \odot A_m. \]

Technically, we should write parentheses to indicate associativity in the source and target of $f^m$, but property (7) and the coherence of the associativity maps in symmetric monoidal categories makes this unnecessary. Let $\mathcal{P}$ have structure map $\gamma$. Define the structure map $\gamma'$ for $F\mathcal{P}$:

\[ \gamma' := F\gamma \circ f^m : F\mathcal{P}(m) \square F\mathcal{P}(j_1) \square \cdots \square F\mathcal{P}(j_m) \to F\mathcal{P}(j_1 + \cdots + j_m). \]

If $\mathcal{P}$ has unit map $\eta : e \to \mathcal{P}(1)$, then the unit map of $F\mathcal{P}$ will be defined by $\eta' := F\eta : e' \to F\mathcal{P}(1)$.

Now, verifying that the proposed structure on $F\mathcal{P}$ defines an operad is straightforward but tedious, and so the required diagrams are omitted.

Assertion 2 and 3 are proved similarly.

Corollary 18. The structure map $\mu$ of Thm. 12 induces a structure map $\mu_*$ on the level of chain complexes, making $\kappa_{\text{ch}}$ into a $\mathcal{D}_{\text{ch}}$-module.

\[ \mu_* : \mathcal{D}_{\text{ch}}(m) \otimes \kappa_{\text{ch}}(j_1) \otimes \cdots \otimes \kappa_{\text{ch}}(j_m) \to \kappa_{\text{ch}}(j_1 + \cdots + j_m). \]

Proof. Since $\kappa_{\text{cat}}$ is a $\mathcal{D}_{\text{cat}}$-module, and the functors $N$, $k[-]$ and $N$ are symmetric monoidal (the first two in the strong sense, the third in the lax sense), this result follows immediately from Lemma 17.

5. $E_{\infty}$-Algebra Structure

In this section we use the operad-module structure defined in the previous section to induce a related operad-algebra structure.

Definition 19. Suppose $(\mathcal{C}, \odot)$ is a cocomplete symmetric monoidal category. We say $\odot$ distributes over colimits, or $\mathcal{C}$ is a distributive symmetric monoidal category, if the natural map $\text{colim}_{i \in I} (B \odot C_i) \to B \odot \text{colim}_{i \in I} C_i$ is an isomorphism.

Remark 20. All of the ambient categories we consider in this paper, $\text{Cat}$, $\text{SimpSet}$, $k$-$\text{SimpMod}$, and $k$-Complexes, are cocomplete and distributive.

Lemma 21. Suppose $(\mathcal{C}, \odot)$ is a cocomplete distributive symmetric monoidal category, $\mathcal{P}$ is an operad over $\mathcal{C}$, $\mathcal{L}$ is a left $\mathcal{P}$-module, and $Z \in \text{Obj} \mathcal{C}$. Then

\[ \mathcal{L}(Z) := \prod_{m \geq 0} \mathcal{L}(m) \otimes_{\text{sym}} Z^\otimes m \]

admits the structure of a $\mathcal{P}$-algebra.
Remark 22. The notation $\mathcal{L}(Z)$ appears in Kapranov and Manin [3]. The concept is also present in [6] as the Schur functor of an operad (cf. Def 1.24), $\mathcal{S}_\mathcal{P}(Z)$.

Proof. What we are looking for is an equivariant map $\mathcal{P}(\mathcal{L}(Z)) \to \mathcal{L}(Z)$, that is, a map

$$
\prod_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} \left[ \prod_{m \geq 0} \mathcal{L}(m) \otimes_{S_m} Z^\otimes m \right] \otimes^n \to \prod_{m \geq 0} \mathcal{L}(m) \otimes_{S_m} Z^\otimes m,
$$
satisfying the required associativity conditions for an operad-algebra structure.

Observe that the equivariant product $\circ_{S_m}$ may be expressed as the coequalizer of the maps corresponding to the $S_m$-action. That is,

$$
\mathcal{L}(m) \otimes_{S_m} Z^\otimes m = \text{coequalizer} \left\{ \sigma^{-1} \circ \sigma \right\},
$$

where $\sigma^{-1} \circ \sigma : \mathcal{L}(m) \otimes Z^\otimes m \to \mathcal{L}(m) \otimes Z^\otimes m$ is given by right action of $\sigma^{-1}$ on $\mathcal{L}(m)$ and by permutation of the factors of $Z^\otimes m$ by $\sigma$ (See [6], formula (1.11)). Thus, $\mathcal{L}(Z)$ may be expressed as a (small) colimit. Since we presuppose that $\circ$ distributes over all small colimits, it suffices to fix a non-negative integer $n$ as well as $n$ non-negative integers $m_1, m_2, \ldots, m_n$, and examine the following diagram:

$$
\begin{array}{ccc}
\mathcal{P}(n) \otimes \left[ \mathcal{L}(m_1) \otimes Z^\otimes m_1 \right] \otimes \ldots \otimes \left[ \mathcal{L}(m_n) \otimes Z^\otimes m_n \right] & \xrightarrow{T} & \mathcal{L}(m_1 + \ldots + m_n) \otimes (Z^\otimes m_1 \otimes \ldots \otimes Z^\otimes m_n) \\
\mathcal{P}(n) \otimes (\mathcal{L}(m_1) \circ \ldots \circ \mathcal{L}(m_n)) \otimes (Z^\otimes m_1 \otimes \ldots \otimes Z^\otimes m_n) & \xrightarrow{\mu \circ \text{id}} & \mathcal{L}(m_1 + \ldots + m_n) \otimes (Z^\otimes m_1 \otimes \ldots \otimes Z^\otimes m_n) \\
& \xrightarrow{a^*} & \mathcal{L}(m_1 + \ldots + m_n) \otimes Z^\otimes (m_1 + \ldots + m_n)
\end{array}
$$

In this diagram, $T$ is the evident shuffling of components so that the components $\mathcal{L}(m_i)$ are grouped together, $\mu$ is the operad-module structure map for $\mathcal{L}$, and $a^*$ stands for the various associativity maps that are required to obtain the final form. This composition defines a family of maps

$$
\eta : \mathcal{P}(n) \otimes \bigcirc_{i=1}^{n} \mathcal{L}(m_i) \otimes Z^\otimes m_i \to \mathcal{L}(m_1 + \ldots + m_n) \otimes Z^\otimes (m_1 + \ldots + m_n).
$$

The maps $\eta$ pass to $S_m$-equivalence classes, producing a family of maps:

$$
\overline{\eta} : \mathcal{P}(n) \otimes \bigcirc_{i=1}^{n} \mathcal{L}(m_i) \otimes_{S_m} Z^\otimes m_i \to \mathcal{L}(m_1 + \ldots + m_n) \otimes_{S_{m_1} \times \ldots \times S_{m_n}} Z^\otimes (m_1 + \ldots + m_n).
$$

Now suppose $M$ is a right $(S_{m_1 + \ldots + m_n})$-object, and $N$ is a left $(S_{m_1 + \ldots + m_n})$-object. Let $p$ be the projection map, $M \otimes_{S_{m_1} \times \ldots \times S_{m_n}} \xrightarrow{p} M \otimes_{S_{m_1 + \ldots + m_n}} N$. Define the family of maps, $\chi := p \circ \overline{\eta}$,

$$
\chi : \mathcal{P}(n) \otimes \bigcirc_{i=1}^{n} \mathcal{L}(m_i) \otimes_{S_m} Z^\otimes m_i \to \mathcal{L}(m_1 + \ldots + m_n) \otimes_{S_{m_1 + \ldots + m_n}} Z^\otimes (m_1 + \ldots + m_n).
$$

At this point, we have a structure map $\chi$ defined for each $n$: $\chi : \mathcal{P}(n) \otimes (\mathcal{L}(Z))^\otimes n \to \mathcal{L}(Z)$. Since $\mathcal{L}$ is a left $\mathcal{P}$-module, $\chi$ is compatible with the structure maps of $\mathcal{P}$. Equivarance follows from external equivariance conditions on $\mathcal{L}$ as $\mathcal{P}$-module, together with the internal equivariance relations present in $\mathcal{L}(m_1 + \ldots + m_n) \otimes_{S_{m_1 + \ldots + m_n}} Z^\otimes (m_1 + \ldots + m_n)$, inducing the required operad-algebra structure map

$$
\chi : \mathcal{P}(n) \otimes_{S_n} (\mathcal{L}(Z))^\otimes n \to \mathcal{L}(Z).
$$

□
Now we return to \( \mathcal{F}_+ \mathcal{A} = k[N(\Delta S_+)] \otimes_{\Delta S_+} B_*^{sym} A \), the \( k \)-simplicial module that computes \( HS_*(A) \). Observe, we may identify:
\[
k[N(\Delta S_+)] \otimes_{\Delta S_+} B_*^{sym} A = \bigoplus_{n \geq 0} \mathcal{K}_{\text{sm}}(n) \otimes_{S_n} A^{\otimes n} = \mathcal{K}_{\text{sm}}(A).
\]
(Note, \( A \) is regarded as a trivial simplicial object, with all faces and degeneracies being identities.)

**Lemma 23.** \( \mathcal{K}_{\text{sm}}(A) \) has the structure of an \( E_\infty \)-algebra over the category of simplicial \( k \)-modules,
\[
\chi : D_{\text{sm}}( \mathcal{K}_{\text{sm}}(A) ) \longrightarrow \mathcal{K}_{\text{sm}}(A).
\]

**Proof.** This follows immediately from Lemma 21 and the fact that \( k \text{-SimpMod} \) is cocomplete and distributive.

**Remark 24.** The fact that \( \mathcal{K}_{\text{cat}} \) is a pseudo-operad (See Remark 14) implies that \( \mathcal{K}_{\text{sm}} \) and \( \mathcal{K}_{\text{sm}} \) are also pseudo-operads (c.f. Lemma 17). Now, the properties of the Schur functor do not depend on existence of a right unit map for \( \mathcal{K}_{\text{sm}} \) (c.f. [5], p. ??), so we could conclude immediately that \( S_{\mathcal{K}_{\text{sm}}}(A) = \mathcal{K}_{\text{sm}}(A) \) is a ‘pseudo-operad’-algebra over \( \mathcal{K}_{\text{sm}} \). However it is rather tricky to prove the \( \mathcal{K}_{\text{cat}} \) is a pseudo-operad.

**Lemma 25.** The \( D_{\text{sm}} \)-algebra structure on \( \mathcal{K}_{\text{sm}}(A) \) induces a quotient \( D_{\text{sm}} \)-algebra structure on \( \mathcal{K}_+ \mathcal{A} \),
\[
\overline{\chi} : D_{\text{sm}}( \mathcal{K}_+ \mathcal{A} ) \longrightarrow \mathcal{K}_+ \mathcal{A}
\]

In other words, \( \mathcal{K}_+ \mathcal{A} \) is an \( E_\infty \)-algebra over the category of simplicial \( k \)-modules.

**Proof.** We must verify that the structure map \( \chi \) from Lemma 23 is well-defined on equivalence classes in \( \mathcal{K}_+ \mathcal{A} \). Let \( \overline{\chi} \) be defined by applying \( \chi \) to a representative, so that we obtain a map
\[
\overline{\chi} : D_{\text{sm}}(n) \otimes_{S_n} (\mathcal{K}_+ \mathcal{A})^{\otimes n} \longrightarrow \mathcal{K}_+ \mathcal{A}.
\]
It suffices to check \( \overline{\chi} \) is well-defined on 0-chains. Let \( f_i, g_i \), for \( 1 \leq i \leq n \), be morphisms of \( \mathcal{F}(\text{as}) \) with specified sources and targets, \( m_i \xrightarrow{f_i} p_i \xrightarrow{g_i} q_i \). Let \( V_i \) be a simple tensor of \( A^{\otimes m_i} \), that is, \( V_i := a_1 \otimes a_2 \otimes \ldots \otimes a_{m_i} \), for some \( a_i \in A \). Let \( \sigma \in S_n \cdot D_{\text{sm}}(n) \otimes_{S_n} (\mathcal{K}_+ \mathcal{A})^{\otimes n} \) is generated by 0-chains of the form:
\[
\sigma \otimes (g_1 f_1 \otimes V_1) \otimes \ldots \otimes (g_n f_n \otimes V_n).
\]
The map \( \overline{\chi} \) sends the element (9) to \( \sigma \{ q_n \} (g_2 f_2) \otimes V_n^{\otimes} \), where \( V_n^{\otimes} := V_1^{\otimes} \otimes \ldots \otimes V_n^{\otimes} \in A^{\otimes (m_1 + \ldots + m_n)} \).

The map \( \overline{\chi} \) is equivalent under \( \text{Mor}(\mathcal{F}(\text{as})) \)-equivariance to:
\[
\sigma \otimes (g_1 f_1(V_1)) \otimes \ldots \otimes (g_n f_n(V_n))
\]
and \( \overline{\chi} \) sends this element to:
\[
\sigma \{ q_n \} (g_2 f_2) \otimes [f_n(V_n)]^{\otimes} = \sigma \{ q_n \} (g_2 f_2) \otimes [f_n(V_n)]^{\otimes} \approx \sigma \{ q_n \} (g_2 f_2) \otimes V_n^{\otimes} = \sigma \{ q_n \} (g_2 f_2) \otimes V_n^{\otimes}.
\]
This proves \( \overline{\chi} \) is well defined, and so \( \mathcal{K}_+ \mathcal{A} \) admits the structure of a \( D_{\text{sm}} \)-algebra.

**Theorem 26.** The \( D_{\text{sm}} \)-algebra structure on \( \mathcal{K}_+ \mathcal{A} \) induces a \( D_{\text{ch}} \)-algebra structure on \( N\mathcal{K}_+ \mathcal{A} \) (as \( k \)-complex).

**Proof.** Again since the normalization functor \( N \) is lax symmetric monoidal, the operad-algebra structure map of \( \mathcal{K}_+ \mathcal{A} \) induces an operad algebra structure map over \( k \)-complexes:
\[
\overline{\chi} : D_{\text{ch}}(n) \otimes_{S_n} (N\mathcal{K}_+ \mathcal{A})^{\otimes n} \longrightarrow N\mathcal{K}_+ \mathcal{A}.
\]

\[ \square \]

6. Homology Operations

Recall, for a commutative ring \( k \) and a cyclic group \( \pi \) of order \( p \), there is a periodic resolution of \( k \) by free \( k\pi \)-modules (cf [17], [1]):

**Definition 27.** Let \( \tau \) be a generator of \( \pi = C_p \), let \( N = 1 + \tau + \ldots + \tau^{p-1} \), and let \( W_i \) be the free \( k\pi \)-module on the generator \( e_i \), for each \( i \geq 0 \). Then \( (W_+, d) \) is a free resolution of \( k \), where \( d \) is defined by:
\[
\begin{align*}
d(e_{2i+1}) &= (\tau - 1)e_{2i} \\
d(e_{2i}) &= Ne_{2i-1}
\end{align*}
\]
In what follows, we shall specialize to $p$ prime and to $k = \mathbb{Z}/p\mathbb{Z}$ (as a ring). Let $\pi = C_p$ (as group), and denote by $W_*$ the standard resolution of $k$ by $k$-modules, as in Definition 27. Recall, $\mathcal{D}_{ch}(p)$ is a contractible $k$-complex on which $S_p$ acts freely. Embed $\pi \hookrightarrow S_p$ by $\tau \mapsto (1, p, p-1, \ldots, 2)$. Clearly $\pi$ acts freely on $\mathcal{D}_{ch}(p)$ as well. Thus, there exists a homotopy equivalence $\xi : W_* \rightarrow \mathcal{D}_{ch}(p)$.

Observe that the complex $N^{\mathcal{V}_+^+} A$ computes $H S_*(A)$, since it is defined as the quotient of $\mathcal{V}_+^+ A$ by degeneracies. By Thm. 26, $N^{\mathcal{V}_+^+} A$, has the structure of $E_\infty$-algebra, so by results of May, if $x$ is an element of $H_*(N^{\mathcal{V}_+^+} A) = H S_*(A)$, then $e_i \otimes x^{op}$ is a well-defined element of $H_*(W \otimes_{k\pi} (N^{\mathcal{V}_+^+} A)^{op})$. We then use the homotopy equivalence $\xi : W_* \rightarrow \mathcal{D}_{ch}(p)$ and $\mathcal{D}_{ch}$-algebra structure of $N^{\mathcal{V}_+^+} A$ to produce the following map. Define $\kappa$ as the composition:

$$H_* \left( W \otimes_{k\pi} (N^{\mathcal{V}_+^+} A)^{op} \right) H_* \left( \mathcal{D}_{ch}(p) \otimes_{k\pi} (N^{\mathcal{V}_+^+} A)^{op} \right) H_* (N^{\mathcal{V}_+^+} A)$$

This gives a way of defining homology (Dyer-Lashoff) operations on $H S_*(A)$. Following definition 2.2 of [7], first define the maps $D_i$. For $x \in H S_q(A)$ and $i \geq 0$, define

$$D_i(x) := \kappa \left( e_i \otimes x^{op} \right) \in H S_{pq+i}(A).$$

**Definition 28.** If $p = 2$, define:

$$P_s : H S_q(A) \rightarrow H S_{q+s}(A)$$

$$P_s(x) = \begin{cases} 0 & \text{if } s < q \\ D_{s-q}(x) & \text{if } s \geq q \end{cases}$$

If $p > 2$ (i.e., an odd prime), let

$$\nu(q) = (-1)^{s+\frac{a(q-1)(p-1)}{2}} \left[ \left( \frac{p-1}{2} \right)^q \right],$$

and define:

$$P_s : H S_q(A) \rightarrow H S_{q+2s(p-1)}(A)$$

$$P_s(x) = \begin{cases} 0 & \text{if } 2s < q \\ \nu(q)D_{2s-q}(p-1)(x) & \text{if } 2s \geq q \end{cases}$$

$$\beta P_s : H S_q(A) \rightarrow H S_{q+2s(p-1)-1}(A)$$

$$\beta P_s(x) = \begin{cases} 0 & \text{if } 2s \leq q \\ \nu(q)D_{2s-q}(p-1)-1(x) & \text{if } 2s > q \end{cases}$$

Note, the definition of $\nu(q)$ given here differs from that given in [7] by the sign $(-1)^s$ in order that all constants be collected into the term $\nu(q)$.

7. **Algebra Structure of $H S_*(A)$**

As a corollary of Thm Y-operad algebra-complexes, we obtain a well-defined multiplication on the graded $k$-module, $\bigoplus_{i \geq 0} H S_i(A)$.

**Theorem 29.** $H S_*(A)$ admits a Pontryagin product, giving it the structure of associative, graded commutative algebra.

**Proof.** Let $[x]$ and $[y]$ be homology classes in $H S_*(A)$. Choose any 0-chain $c \in \mathcal{D}_{ch}(2)$ corresponding to the generator of $1 \in H_0(\mathcal{D}_{ch}(2)) = k$. Now, the product $[x] \cdot [y]$ is defined to be the image of $c \otimes x \otimes y$ under the composite:

$$\mathcal{D}_{ch}(2) \otimes N^{\mathcal{V}_+^+} A \otimes N^{\mathcal{V}_+^+} A \rightarrow \mathcal{D}_{ch}(2) \otimes_{k[S_2]} \left( N^{\mathcal{V}_+^+} A \otimes N^{\mathcal{V}_+^+} A \right) \xrightarrow{\bar{x}} N^{\mathcal{V}_+^+} A$$

The choice of $c$ does not matter, since $\mathcal{D}_{ch}(2)$ is contractible. Indeed, since each $\mathcal{D}_{ch}(p)$ is contractible, Thm. 26 shows that $N^{\mathcal{V}_+^+} A$ is a homotopy-everything complex, analogous to the homotopy-everything spaces of Boardman and Vogt [3]. Thus, the product defined here is associative and commutative in the graded sense on the level of homology (see also May [8], p. 3). Note, in what follows, the product map will simply be denoted $\bar{x}$. □
Remark 30. The Pontryagin product of Thm. 29 is directly related to the algebra structure on the complexes $\text{Sym}_*(p)$ of [1], Section 8.5. Indeed, if $A$ has augmentation ideal $I$ which is free and has countable rank over $k$, and $I^2 = 0$, then by Cor. 76, the spectral sequence of Cor. 83 would collapse at the $E_1$ stage, giving:

$$HS_n(A) \cong \bigoplus_{p \geq 0} \bigoplus_{u \in X^{p+1}/\Sigma_{p+1}} H_n(E_u \otimes G_u \text{Sym}_*(p); k).$$

The product structure of $\text{Sym}_*(p)$ may be viewed as a restriction of the algebra structure of $HS_*(A)$ to the free orbits.

Corollary 31. Let $A$ be a unital associative $k$-algebra. If the ideal generated by the commutator sub-module is equal to the entire algebra (i.e. $([A, A]) = A$), then $HS_*(A)$ is trivial in all degrees.

Proof. Thm. 103 of [1] gives $HS_0(A) = A/([A, A])$, so $HS_0(A)$ is trivial. Now for any $x \in HS_q(A)$, we have $x = 1 \cdot x = 0 \cdot x$, using the Pontryagin product of Thm. 29.

Remark 32. It was pointed out in [1], Remark 105, that symmetric homology fails to preserve Morita equivalence based on the fact that $HS_0(M_n(A)) = M_n(A)/([M_n(A), M_n(A)]) = 0$, for $n > 1$. Corollary 31 shows the failure in a big way: $HS_*(M_n(A))$ is trivial for $n > 1$.

Proposition 33. The product of Thm. 29 when restricted to $HS_0(A) \otimes HS_0(A) \rightarrow HS_0(A)$ is the standard algebra multiplication map $A/([A, A]) \otimes A/([A, A]) \rightarrow A/([A, A])$.

Proof. Examine the partial chain complex $0 \leftarrow N\mathcal{Y}_0^+ A \xrightarrow{\partial_1} N\mathcal{Y}_1^+ A$. It is straightforward to verify that $\partial_1$ collapses the module of 0-chains to those of the form $([0] \rightarrow [0]) \otimes x$ (for $x \in A$). The product of such an element with another typical element, $([0] \rightarrow [0]) \otimes y$ yields $([1] \rightarrow [1]) \otimes (x \otimes y) \approx ([0] \rightarrow [0]) \otimes xy$.

Note that by working with the partial complex of [1], Thm. 102, an explicit formula for the product $HS_0(A) \otimes HS_1(A) \rightarrow HS_1(A)$ can be determined. For convenience, Thm. 102 is displayed below:

Theorem 34. (Thm. 102 of [1]) $HS_i(A)$ for $i = 0, 1$ may be computed as the degree 0 and degree 1 homology groups of the following (partial) chain complex:

$$0 \rightarrow A \xrightarrow{\partial_1} A \otimes A \otimes A \xrightarrow{\partial_2} (A \otimes A \otimes A \otimes A) \otimes A,$$

where

$$\partial_1 : a \otimes b \otimes c \mapsto abc - cba,$$

$$\partial_2 : \begin{cases} a \otimes b \otimes c \otimes d & \mapsto ab \otimes c \otimes d + d \otimes ca \otimes b + bca \otimes 1 \otimes d + d \otimes bc \otimes a, \\ a & \mapsto 1 \otimes a \otimes 1. \end{cases}$$

Proposition 35. For a unital associative algebra $A$ over commutative ground ring $k$, $HS_1(A)$ is a left $HS_0(A)$-module, via the map $A/([A, A]) \otimes HS_1(A) \rightarrow HS_1(A)$ given by,

$$[a] \otimes [x \otimes y \otimes z] \mapsto [ax \otimes y \otimes z] - [x \otimes ya \otimes z] + [x \otimes y \otimes az].$$

Here, elements of $HS_1(A)$ are represented as equivalence classes of elements in $A \otimes A \otimes A$, via complex 12. Moreover, there is a right module structure, $HS_1(A) \otimes HS_0(A) \rightarrow HS_1(A)$ given by,

$$[x \otimes y \otimes z] \otimes [a] \mapsto [xa \otimes y \otimes z] - [x \otimes ay \otimes z] + [x \otimes y \otimes za],$$

and the two actions are equal.

Remark 36. This module structure was first discovered on the chain level before the Pontryagin product was discovered. Below is the explicit derivation using Thm. 29.

Proof. Our goal will be to set up an explicit equivalence between the partial complex 12 and the standard complex $\mathcal{P}_* A$, at least up to degree 1, then use the shuffle map to find an explicit formula for the product in the standard complex, and finally to transfer the result back to complex 12. Below is a diagram of what must be done. Denote complex 12 by $\mathcal{P}_* A$, and let $\mathcal{P}_*$ be the partial projective resolution underlying the
complex (II, complex (68)). As in [1], $\mathcal{Y}_+^*$ will denote the projective resolution of $k$ underlying the complex $\mathcal{Y}_+ A$.

The last set of chain maps are equivalences between the normalized complexes. Now, the product of chains can be found by traversing the following diagram counter-clockwise:

$$
\begin{array}{cccc}
\mathcal{P}_{ch}(2) \otimes (N_\varepsilon A)^{\otimes 2} & \longrightarrow & N_\varepsilon A \\
\text{id} \otimes (N F_\varepsilon)^{\otimes 2} & & \longleftarrow & \text{id} \otimes (N G_\varepsilon)^{\otimes 2} \\
\mathcal{P}_{ch}(2) \otimes (N \mathcal{Y}_+^* A)^{\otimes 2} & \longrightarrow & N \mathcal{Y}_+^* A \\
\chi & & \longleftarrow & \chi \\
\rightleftharpoons & & \rightleftharpoons & \\
N (\mathcal{P}_{ch}(2) \otimes (\mathcal{Y}_+^* A)^{\otimes 2}) & & \\
\end{array}
$$

(13)

The tedious part is writing down the explicit partial chain equivalences, $F_\alpha$ and $G_\alpha$. Fix an integer $n$. The top row of diagram (14) is the partial projective resolution $\mathcal{P}_\varepsilon$ at the object $[n]$, and the bottom row is the standard projective resolution $\mathcal{Y}_+^*$ at the object $[n]$. The 0-modules in degree $-1$ are omitted.

$$
\begin{array}{cccc}
k [\text{Mor}_{\Delta^+ S} ([n], [0])] & \longleftarrow & k [\text{Mor}_{\Delta^+ S} ([n], [2])] & \longleftarrow & k [\text{Mor}_{\Delta^+ S} ([n], [3])] \\
& & \longleftarrow & k [\text{Mor}_{\Delta^+ S} ([n], [0])] \\
F_0 & \longleftarrow & F_1 & \longleftarrow & F_2 \\
G_0 & \longleftarrow & G_1 & \longleftarrow & G_2 \\
\end{array}
$$

(14)

Set $F_0 : k [\text{Mor}_{\Delta^+ S} ([n], [0])] \longrightarrow k [N ([n] \setminus \Delta^+ S_0)]$ as the identity (that is, the morphism $\phi : [n] \rightarrow [0]$ gets sent to itself, regarded as a 0-chain in nerve of the under-category). For any integer $m$, let $0_m$ be the unique order-preserving $\Delta^+ S_+$ morphism $[m] \rightarrow [0]$. Set $G_0 : k [N ([n] \setminus \Delta^+ S_0)] \longrightarrow k [\text{Mor}_{\Delta^+ S} ([n], [0])]$ as the map $\phi \mapsto 0_m \phi$, where $[m]$ is the target of $\phi$.

Now, $G_0 F_0 = \text{id}$. To ensure $F_0 G_0 \simeq \text{id}$, we introduce the homotopy map $h_0 : k [N ([n] \setminus \Delta^+ S_0)] \longrightarrow k [N ([n] \setminus \Delta^+ S_1)]$, sending $\phi$ to the 1-chain $-(\phi, 0_m)$.

The map $F_1 : k [\text{Mor}_{\Delta^+ S} ([n], [2])] \longrightarrow k [N ([n] \setminus \Delta^+ S_1)]$ is given by $F_1 (\phi) := (\phi, z_0 z_1 z_2) - (\phi, z_2 z_1 z_0)$, where the morphisms $z_0 z_1 z_2$ and $z_2 z_1 z_0$ are written in tensor notation [1]. Briefly, $z_0 z_1 z_2$ is simply $0_2$, while $z_2 z_1 z_0$ is the map $[2] \rightarrow [0]$ that reverses the order of the source.

The map $G_1$ is a bit trickier. Consider a 1-chain $(\phi, \psi)$ in $N ([n] \setminus \Delta^+ S_+)$. Decompose each morphism into a morphism of $\Delta^+$ and an isomorphism (which can be done uniquely [1]): $\phi = \beta \circ h, \psi = \alpha \circ g$. Now, $h \in \Sigma_{n+1}$. For each $n$, let $\mathcal{G}_{n+1}$ be a chosen maximal connected directed tree whose vertices are the permutations of $n+1$ letters, and whose edges are labeled by a $\Delta^+$-morphism $[2] \rightarrow [0]$ that re-orders the permutation at the tail to obtain the permutation at the head. See Figure 2. For clarity, we use $a, b, c, d$ in place of the formal variables $z_0, z_1, z_2, z_3$ in the diagram.

The initial choices involved in constructing $\mathcal{G}_{n+1}$ do not matter when one passes to the level of homology since the difference of any two paths would be a 1-cycle of $\mathcal{P}_\varepsilon$, hence also a boundary. Then define $G_1$ by:

$$
G_1 (\beta \circ h, \alpha \circ g) := \text{Sum of signed edge labels in the path from } h \text{ to } hg^a.
$$
It should be obvious that both $F_1$ and $G_1$ are compatible with the differentials. What is not obvious is the fact that $F_1 G_1 \simeq id$ and $G_1 F_1 \simeq id$. For $\phi : [n] \to [2]$, decompose $\phi = \beta \circ h$. Then

$$G_1 F_1(\phi) = G_1(\beta \circ h, 0_2 \circ id) - G_1(\beta \circ h, 0_2 \circ (0, 2))$$

$$= [\text{Path from } h \text{ to } h] - [\text{Path from } h \text{ to } h(0, 2)]$$

$$= -[\text{Path from } h \text{ to } h(0, 2)]$$

where $(0, 2) \in \Sigma_{2+1}$ is just the permutation that reverses the order of the three symbols $0, 1, 2$. Now, since $G_0 F_0 = id$, we are free to define a homotopy map $H_0 : k [\text{Mor}_{\Delta S} ([n], [0])] \to k [\text{Mor}_{\Delta S} ([n], [2])]$ as $H_0 = 0$. Thus, what we need is a map $H_1 : \mathcal{P}_1 \to \mathcal{P}_2$ so that $\phi - G_1 F_1(\phi) = \partial_2 H_1(\phi)$. Now, by exactness of complex $(12)$, an appropriate choice of the homotopy map $H_1$ exists if and only if $\phi - G_1 F_1(\phi)$ is a 1-cycle. This is certainly true, since $G_1 F_1(\phi)$ is nothing more than a decomposition of $\phi$ into a series of morphisms, each of which is a 3-block permutation of the $n + 1$ letters. In a similar manner, it can be deduced that $F_1 G_1 \simeq id$. 

**Figure 2.** One possible choice of $\mathcal{G}_{3+1}$
We now have all of the ingredients we need to complete the computation. To avoid confusion with the indeterminates, we shall use bold letters for elements of the algebra $A$. Any indeterminates of the form $z_i$ occur as part of the tensor notation for $\Delta S$-morphisms. Let $a \in N\mathcal{P}_0A$ and $x \otimes y \otimes z \in N\mathcal{P}_1$. Since we are free to choose any 0-chain of $\mathcal{D}_{ch}$, let $c = \text{id}_S$.

\[
\begin{align*}
\text{id} \otimes (a) \otimes (x \otimes y \otimes z) \\
\downarrow \text{id} \otimes (N\mathcal{P}_1)^{\otimes 2} \\
\text{id} \otimes [(z_0) \otimes (a)] \otimes [(z_0 \otimes z_1 \otimes z_2, z_0 z_1 z_2) \otimes (x \otimes y \otimes z)] \\
- \text{id} \otimes [(z_0) \otimes (a)] \otimes [(z_0 \otimes z_1 \otimes z_2, z_2 z_1 z_1) \otimes (x \otimes y \otimes z)] \\
\downarrow \text{sh} \\
(id, id) \otimes [(z_0, z_0) \otimes (a)] \otimes [(z_0 \otimes z_1 \otimes z_2, z_0 z_1 z_2) \otimes (x \otimes y \otimes z)] \\
- (id, id) \otimes [(z_0, z_0) \otimes (a)] \otimes [(z_0 \otimes z_1 \otimes z_2, z_2 z_1 z_0) \otimes (x \otimes y \otimes z)] \\
\downarrow \nabla \\
[(z_0 \otimes z_1 \otimes z_2 \otimes z_3, z_0 \otimes z_1 z_2 z_3) - (z_0 \otimes z_1 \otimes z_2 \otimes z_3, z_0 \otimes z_3 z_2 z_1)] \otimes (a \otimes x \otimes y \otimes z) \\
\downarrow N\mathcal{G}_s \\
[(\text{Path from } z_0 z_1 z_2 z_3 \to z_0 z_1 z_2 z_3) - (\text{Path from } z_0 z_1 z_2 z_3 \to z_0 z_3 z_2 z_1)] \otimes (a \otimes x \otimes y \otimes z) \\
- (z_0 z_1 z_2 \otimes z_3 \otimes 1 + z_3 z_0 \otimes z_1 \otimes z_2 + z_2 z_1 \otimes z_3 \otimes z_0) \otimes (a \otimes x \otimes y \otimes z)
\end{align*}
\]

Now, in $HS_1(A)$, the last element may be expressed

\begin{equation}
(15) \quad \text{[axy} \otimes z \otimes 1] + [za \otimes x \otimes y] + [yx \otimes z \otimes a]
\end{equation}

It is a fun exercise to show that $[15]$ is equivalent to $[xa \otimes y \otimes z] - [x \otimes ya \otimes z] + [x \otimes y \otimes za]$.

The product $HS_1(A) \otimes HS_0(A) \to HS_1(A)$ can be found explicitly in a similar manner. The fact that the two products agree follows from the fact that their difference is a boundary in $\mathcal{P}_A$. 

Using GAP, we have made the following explicit computations of the $HS_0(A)$-module structure on $HS_1(A)$ for some $\mathbb{Z}$-algebras, $A$.

| $A$ | $HS_1(A \mid \mathbb{Z})$ | $HS_0(A)$-module structure |
|-----|----------------|-----------------------------|
| $\mathbb{Z}[t]/(t^2)$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[t]/(t^3)$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ and $t^2u = 0$ |
| $\mathbb{Z}[t]/(t^4)$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_2]$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_3]$ | 0 | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_4]$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | Generated by $u$ with $2u = 0$ |
| $\mathbb{Z}[C_5]$ | 0 | |

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