Gas-assisted Growth of Protoplanets in a Turbulent Medium

M. M. Rosenthal\textsuperscript{1,4}, R. A. Murray-Clay\textsuperscript{1}, H. B. Perets\textsuperscript{2}, and N. Wolansky\textsuperscript{3}

\textsuperscript{1}Department of Astronomy and Astrophysics, University of California, Santa Cruz, CA 95064, USA
\textsuperscript{2}Physics Department, Technion–Israel Institute of Technology, Haifa, 3200003, Israel
\textsuperscript{3}Computer Science Department, Columbia University, 500 West 120 Street, Room 450, New York, NY 10027, USA

\textbf{Abstract}

Pebble accretion is a promising process for decreasing growth timescales of planetary cores, allowing gas giants to form at wide orbital separations. However, nebular turbulence can reduce the efficiency of this gas-assisted growth. We present an order-of-magnitude model of pebble accretion that calculates the impact of turbulence on the average velocity of small bodies, the radius for binary capture, and the sizes of the small bodies that can be accreted. We also include the effect of turbulence on the particle scale height, which has been studied in previous works. We find that turbulence does not prevent rapid growth in the high-mass regime: the last doubling time to the critical mass to trigger runaway gas accretion \((M \sim 10 M_\oplus)\) is well within the disk lifetime, even for strong \((\alpha \gtrsim 10^{-2})\) turbulence. We find that, while the growth timescale is quite sensitive to the local properties of the protoplanetary disk, there are large regimes of parameter space over which large cores grow in less than the disk lifetime, if appropriately sized small bodies are present. Instead, the effects of turbulence are most pronounced for low planetary masses. For strong turbulence, the growth timescale is longer than the gas disk lifetime until the core reaches masses \(\gtrsim 10^{-2} - 10^{-1} M_\oplus\). A “flow isolation mass,” at which binary capture ceases, emerges naturally from our model framework. We comment that the dependence of this mass on orbital separation is similar to the semimajor axis distribution of solar system cores.

\textbf{Key words:} planets and satellites: formation – planets and satellites: gaseous planets – protoplanetary disks

1. Introduction

In the core accretion model of gas giant formation, the growth of a gas giant is constrained by two main factors: the growth timescale of the planet relative to the lifetime of the gas disk, and the amount of solid material available for a growing planet to accrete. Because the early stages of formation were not well-understood, many classical models of planet formation focus on later stages of growth, beginning with solid “planetesimals” of size \(\gtrsim \text{km}\). In these models, growth is too slow to produce gas giants at wide orbital separations. In contrast, close to the host star, there is insufficient material locally available to produce a solid core massive enough to grow a gas giant. While these processes can produce architectures similar to the solar system, they are not sufficient to explain the diverse system architectures that are observed around other stars. In particular, recent theoretical work has pointed to the possibility that accretion of “pebble-sized” bodies may be important in both determining the growth timescale of cores and providing a reservoir of solid material through radial drift (Ormel & Klahr 2010; Perets & Murray-Clay 2011; Lambrechts & Johansen 2012; Ormel & Kobayashi 2012; Lambrechts et al. 2014; Levison et al. 2015; Morbidelli et al. 2015; Ida et al. 2016; Visser & Ormel 2016; Xu et al. 2017).

In this paper, we will introduce an order-of-magnitude model of protoplanetary growth by pebble accretion, focusing on the regime in which the core is sufficiently massive that the gravity of the core is non-negligible. In what follows, we will use the term “protoplanet” to refer to cores in this regime. We will focus on incorporating the effects of local disk turbulence into the various length and velocity scales that set the growth timescale.

Before proceeding, we must briefly define a number of standard terms that will be used throughout this work. Within the context of a bottom-up formation model, the growth of gas giant planets proceeds by “core accretion,” i.e., a gas giant core grows until it reaches a large enough mass, \(M_{\text{crit}}\), that its atmospheric mass is comparable to its core mass. At this point, the core rapidly accretes gas from the nebula, culminating in a gas giant (see, e.g., Pollack et al. 1996). In this work, we will not address the physics that set \(M_{\text{crit}}\) (see, e.g., Rafikov 2006; Piso et al. 2015), but will instead consider growth timescales as a function of core mass. In the absence of some dissipative mechanism, the largest enhancement to the collision cross section comes from “gravitational focusing” by the large cores. Gravitational focusing refers to the effect by which large bodies can accrete material with impact parameters far outside their physical radius, through the influence of gravity. This effect is significant for particles with velocity dispersions smaller than the large-body escape velocity. In what follows, we will refer to models of core accretion where gravitational focusing is the largest enhancement to the accretion cross section as “canonical core accretion” or “planetaryesimal accretion.”

Two particular challenges to planetesimal accretion stem from the existence of directly imaged planets at wide orbital separations and “Super Earths” close to their host stars. The star HR 8799 has a system of four gas giants orbiting at \(a \approx 15–70\) au (Marois et al. 2008, 2010). These planets are likely formed in situ, as indicated by \(N\)-body integrations showing it is unlikely that this system was formed by scattering (Dodson-Robinson et al. 2009). Gravitational instability may be an alternative way to form the HR 8799 planets, as reviewed by Kratter & Lodato (2016). However, Kratter et al. (2010) argue that, if formed by gravitational instability, wide orbital separation gas giants should represent the low-mass tail of a distribution of stellar companions. Thus far, observations do

\textsuperscript{3} Corresponding author.
not clearly connect the population of wide-separation gas giants to the brown dwarf population (Bowler 2016). There exist a number of other wide-separation gas giants, but whether they formed in situ is less well-constrained. Super Earths are difficult to explain through local isolation mass, due to their large masses and proximity to the central star. At such small orbital separations, there is not enough material locally to grow such massive planets without causing the protoplanetary disk to become unstable to collapse (Schlichting 2014), indicating that radial drift of particles may be an important factor in planet formation. More massive particles can also move radially, due to Type I migration and/or gas dynamical friction (e.g., Grishin & Perets 2015).

These difficulties can be amended through a more detailed consideration of the interaction between the gas present in the disk and the material accreted by the growing cores. While the effect of gas drag on smaller (<0.1–1 km) planetesimals can be substantial (Rafikov 2004), even more striking is the effect of on smaller, mm–cm sized particles. For these bodies, gas drag can enhance accretion rates by dissipating the relative kinetic energy between the small bodies and growing cores during their interaction. Due to the sizes of bodies for which this is possible, this process is commonly referred to as “pebble accretion.” We will alternatively refer to it as “gas-assisted growth,” to highlight the idea that enhancements to growth come from the constructive effect of collisions between the small bodies being accreted and the gas particles, as well as to avoid confusion with the geological use of the term “pebble.” We also note that “pebbles” need not necessarily be particles of small sizes, but could also include “fluffy” aggregates of low density that have similar aerodynamic properties to rocky mm–cm sized particles. Because the term “pebble accretion” is well-established, we use these two terms interchangeably.

Models of gas-assisted growth in the context of planet formation find that gas drag acting on pebble-sized particles can lead to substantially higher growth rates than models that rely on growth via planetesimal accretion. For a wide range of disk parameters, massive \( M \gtrsim 10^{-3} M_\odot \) cores can accrete pebble-sized particles at impact parameters comparable to the core’s Hill radius (Ormel & Klahr 2010; Lambrechts & Johansen 2012). If particles of these sizes are present, cores accreting at these rates can easily grow a gas giant at wide orbital separation, as opposed to cores undergoing planetesimal accretion. The presence of these smaller pebbles is supported by observations of protoplanetary disks. Matching observations of the spectral energy distribution of disks requires a dust size distribution where most of the mass is in \( \sim \) mm-sized particles (D’Alessio et al. 2001), while sub-mm images of disks find solid surface densities in this size range that are comparable to the minimum-mass solar nebula (Andrews et al. 2009; Andrews 2015).

While these rapid growth timescales can solve some of the issues present in growing wide-separation gas giants, they present issues of their own. Chief among these is that pebble accretion is, in some respects, too efficient. Because the growth rate at large masses is so fast, the last doubling timescale to \( M_{\text{crit}} \) is extremely short in pebble accretion. Thus, gas-assisted growth seems to predict that growth of gas giants should be a ubiquitous phenomenon. Direct imaging surveys, however, place severe constraints on the existence of gas giants at wide orbital separation (e.g., Brandt et al. 2014; Chaumon et al. 2015; Bowler 2016; Galicher et al. 2016). Pebble accretion must therefore be inhibited, in some manner, from what current models naively predict.

One commonly neglected effect in models of the pebble accretion is the effect of turbulent gas velocities on planetary accretion efficiency. Turbulence can increase the velocity of the gaseous component of the disk. This, in turn, has a number of ramifications for gas-assisted growth: pebble velocities are now higher as well, accretion cross sections can shrink substantially, and the scale height of particles can increase. These effects can greatly decrease the efficiency of accretion.

The effects of turbulence on pebble accretion have been discussed in a number of different regimes. The majority of works that include turbulence discuss the growth of lower-mass planetesimals—in these models, the growing body is assumed to be of low enough mass that its gravity is negligible, i.e., these models discuss the effect of turbulence for accretion where the cross section is comparable to the body’s geometric cross section (e.g., Homann et al. 2016). Previous models of pebble accretion for higher-mass protoplanets generally neglect the effects of turbulence, or include it only by modifying the scale height of small bodies. A few works do modify the particle velocities or impact parameters due to the influence of turbulence. Ormel & Kobayashi (2012) examine growth over a large range of large body sizes, and include a turbulent component to the velocity through “turbulent stirring” on the random velocities of small bodies (Ormel & Kobayashi 2012). Guillot et al. (2014) include the effect of turbulence on the radial motion of small bodies. Chambers (2014) employ a methodology more similar to our own, using asymptotic expressions from Ormel & Cuzzi (2007) for the relative velocity between the small bodies and gas due to turbulence, and extending the Ormel & Klahr (2010) expressions for impact parameter and accreted particle sizes to include this turbulent component. In this formulation, however, the impact parameter and the sizes of small bodies that can be accreted are functions solely of the relative velocity between the small body and the core at infinity. Our approach, which separately calculates the parameters relevant to growth, as well as the velocity preceding and during the encounter between the small body and the core, can be more naturally extended to include turbulence, and captures facets of the problem not covered by the Chambers results. Furthermore, the focus of our study is distinct from those described above: these papers are concerned with holistically studying growth of planets at a few points in parameter space by including a wide variety of processes and modeling the problem numerically. Our methodology instead focuses on studying the effects of turbulence over a broad parameter space and understanding the conditions under which turbulence is important to pebble accretion.

With these considerations in mind, this paper presents an order-of-magnitude model of pebble accretion. We approach the problem in a different manner than past theories have, allowing separate changes to the different parameters that set the growth timescale, as opposed to grouping growth timescales into a few regimes. This allows us to more fully take into account the effects of turbulence than did previous studies, including the effects of turbulence on not just the particle scale height, but also the velocity dispersion of the small bodies as well as the impact parameters for accretion. This model can be applied over a wide range of parameter space to give results accurate to an order of magnitude, and can accurately describe the trends present in gas-assisted growth. We use this model to
discuss the overarching features of pebble accretion, as well as to investigate how turbulence modifies these features. We also discuss how pebble accretion operates in different regions of parameter space, particularly at wide orbital separations and low core masses. In these regimes, growth at intermediate masses may dominate the timescales for gas giant growth, which we will discuss in more detail in our Paper II (Rosenthal & Murray-Clay 2018).

In Section 2, we give an overview of how growth operates in the presence of nebular gas, and discuss broadly how we calculate the growth timescale in our model. In Section 3, we discuss our choices for modeling the velocities that enter into our calculation. Section 4 details how the length scales relevant to the accretion cross section are calculated. In Section 5, we give an overview of the output from our model, particularly discussing the broad features of pebble accretion as well as the effects of turbulence. We also give a detailed comparison between our modeling and other works on pebble accretion. Readers that are not concerned with the details of our model can find a summary of our algorithm in Appendix A, and may skip directly to Section 5 for our results. In Section 6, we discuss how gas-assisted growth operates when various parameters are adjusted. In Section 7, we note how the relatively simple assumptions on which our model is based lead naturally to a “flow isolation mass,” past which accretion of pebbles ceases. Finally, in Section 8, we summarize our results and discuss future extensions of our model.

2. Model Overview

2.1. Accretion in the Presence of Gas

We begin by discussing generally how we model growth of planets in the presence of nebular gas. The details of how specific quantities are calculated are deferred to subsequent sections.

Our calculation proceeds in an order-of-magnitude manner — i.e., the approximations made and the neglected effects mean that quoted values should be correct to within an order of magnitude. In what follows, we consider two types of bodies. The large bodies, or protoplanetary cores, are assumed to be massive enough that they are unaffected by gas drag and thus move at the local Keplerian orbital velocity. This constrains our cores to have radii \( r \geq 10 \text{ km} \), in which case their velocity will deviate from Keplerian by \( 10^{-3} \) of the gas velocity, at most, with a weak dependence on stellocentric distance for fiducial disk parameters. In practice, we rarely consider cores of such small size. We note also that, while these cores are insensitive to aerodynamical gas drag, large bodies with masses in the range \( 10^{22} \text{ g} < M < 10^{25} \text{ g} \) can still be affected by gas dynamical friction (Grishin & Perets 2016).

We do not include these effects here. The second type of particles considered are “small” bodies, which can be substantially affected by gas drag. The growth timescale in pebble accretion is strongly dependent on the size of the small body under consideration, unlike canonical core accretion where the size of the planetesimals enters only through its effect on the small body velocity dispersion. Thus, all calculations are performed as a function of small body radius, \( r_b \). Quantities of interest can later be averaged over size by assuming a size distribution for the small bodies. Note that quoted values of the growth timescale \( t_{\text{grow}} \) implicitly assume that all of the surface density is contained in particles of the given value of \( r_b \). For a size distribution where most of the mass is in the largest sizes of particles present (e.g., a Dohnanyi size distribution, Dohnanyi 1969), this value of \( t_{\text{grow}} \) is approximately equal to the growth timescale for a distribution of small body sizes where the maximum size present is the given value of \( r_b \). In this paper, we will not explicitly perform integrations over small body size; see our Paper II for examples of this process, as well as a discussion of the effects of altering the size distribution.

In gas-assisted growth, the interaction between the small body and the nebular gas substantially modifies the accretion process. As the small body approaches the core, gas drag will dissipate the kinetic energy of the small body relative to the large body. This loss of energy can cause small bodies on non-collisional trajectories to become bound to the core and eventually be accreted. This process is similar to the \( L^2 \) mechanism identified by Goldreich et al. (2002) for formation of Kuiper Belt binaries, with gas drag as the source of dissipation in the place of dynamical friction. Gas drag can also stop small bodies from accreting—if particles couple strongly to the gas as they flow around the core, then they will be unable to accrete.

Which of these processes occur(s) depends on the relative size of two different length scales: the stability radius, \( R_{\text{stab}} \), and the Bondi radius, \( R_b \). The stability radius is the smallest radius at which stable orbits by the small body about the large body are possible: outside of \( R_{\text{stab}} \), interactions between the small body and either the nebular gas or the central star will shear the small body away from the large body’s gravity. Within \( R_{\text{stab}} \), the small body can safely inspiral onto the core. The details of how \( R_{\text{stab}} \) is calculated are discussed in Section 4.1. The Bondi radius, on the other hand, is approximately the radius at which the escape velocity from the core is equal to the speed of sound in the gas:

\[
R_b = \frac{GM}{c_s^2},
\]

where \( M \) is the mass of the core and \( c_s = \sqrt{kT/\mu} \) is the isothermal sound speed of the gas. Here, \( k \) is Boltzmann’s constant and \( \mu \) is the mean molecular weight of the nebular gas. We consider the Bondi radius because it tells us roughly the length scale interior to which the the core can have a stable atmosphere, which has substantial effects on the flow pattern. For the lowest-mass cores we consider, the Bondi radius may be less than the physical radius of the core, \( R \). This occurs roughly at a core mass of:

\[
M_{\ast} \equiv \frac{\varepsilon_3^3}{G} \left( \frac{3}{4\pi G \rho_p} \right)^{1/2} \approx 2 \times 10^{-4} M_\ast \left( \frac{a}{30 \text{ au}} \right)^{-9/14} \left( \frac{\rho_p}{2 \text{ g cm}^{-3}} \right)^{-1/2},
\]

where \( \rho_p \) is the density of the protoplanet (e.g., Rafikov 2006), and we have used our fiducial disk parameters (see Section 5.1) for the expression in the second line. If \( R_b < R \), then the effects discussed below are unchanged, with \( R \) taking the place of \( R_b \). In what follows, we will discuss accretion for \( R_b < R_{\mu} \), where
particles that deplete their kinetic energy, will couple to the gas and the kinetic energy of the small body, which will then inspiral and accrete, at atmosphere of the nascent planet. This increased density may rapidly deplete radius will experience a rapid increase in gas density as they enter the around the core without accreting. Lower right panel: larger particles that do

\[ R_{stab} > R_b \]

particles can inspiral onto the core in a region where the gas

\[ KE < W \]

Figure 1. A graphical illustration of the energy regimes, used to determine whether small bodies are able to accrete. Upper left panel: here, \( R_{stab} > R_b \), so particles can inspiral onto the core in a region where the gas flow is not substantially altered by the core’s gravity. Particles that deplete their kinetic energy relative to the core via gas drag will inspiral onto the core and be captured. Upper right panel: if the particle is unable to deplete its kinetic energy during the interaction, then it will simply have its trajectory deflected before exiting \( R_{stab} \), and will not accrete. Lower left panel: here, \( R_b > R_{stab} \), so the gas flow is altered substantially when the small body is accreting. The gas will tend to flow around the atmospheric radius, so particles that have \( KE < W \), i.e., particles that deplete their kinetic energy, will couple to the gas and flow around the core without accreting. Lower right panel: larger particles that do not deplete their kinetic energy and instead penetrate into the atmospheric radius will experience a rapid increase in gas density as they enter the atmosphere of the nascent planet. This increased density may rapidly deplete the kinetic energy of the small body, which will then inspiral and accrete, similar to what occurs in the upper left panel.

\[ R_{stab} < R_b \]

\[ KE > W \]

\[ R_{H} \]

\[ R_{WS} \]

\[ R_{WS} > R_H \]

\[ R_{WS} < R_H \]

Figure 2. A cartoon illustration of the typical manner in which pebble accretion operates as the small body size is increased. The black circle represents the planet, while the blue circles depict incoming particles. The extent of the planet’s atmosphere is denoted by the gray-shaded region, and the yellow-shaded region shows the region where incoming particles can be accreted. Left panel: for small particles, \( R_{WS} = R_{stab} < R_b \), so the core’s gravitational sphere of influences lies inside its atmosphere. In this regime, particles couple to the local gas flow and proceed around the core without being accreted. Middle panel: for intermediate sizes of particles, \( R_{WS} > R_b \), meaning particles can be bound to the core in a region outside the core’s atmosphere. For these intermediate sizes of particles, the work done by gas drag exceeds the incoming kinetic energy of these small bodies, meaning that particles that pass interior to \( R_{WS} \) will be accreted. Right panel: finally, large particles will be so massive that their incoming kinetic energy is too large to be depleted by gas drag. These particles will not be accreted via pebble accretion, regardless of impact parameter. For the case shown here, \( R_{WS} \) has grown so much that \( R_{stab} = R_H \), but this is not always the case for particles with \( KE > W \).

\( R_{H} \) is the core’s Hill radius (see Section 4.1.1). We discuss accretion in the regime \( R_p > R_H \) in Section 7.

Given these considerations, we center our model around two main ideas about accretion in the presence of gas, which are summarized in Figure 1:

1. If the radius for stable orbits exceeds the Bondi radius, i.e., \( R_{stab} > R_b \), then the flow pattern of gas is not substantially altered in the region where particles can stably orbit the core. In this case, any small bodies that deplete their kinetic energy relative to the core within \( R_{stab} \) will inspiral onto the core and be accreted. On the other hand, any particles that are unable to dissipate their kinetic energy in this regime will pass out of \( R_{stab} \) and will not be accreted.

2. If instead the Bondi radius exceeds the stable orbit radius, i.e., \( R_b > R_{stab} \), then particles that are able to deplete their kinetic energy relative to the core will not accrete. This is due to the fact that, in this regime, well-coupled particles will tend to flow around the core’s atmosphere, which extends up to \( R_b \). If particles are instead unable to dissipate their kinetic energy, such that they penetrate into the atmospheric radius, the increase in density as the particle enters the growing planet’s atmosphere is taken to be so substantial that the particle will now be able to dissipate its kinetic energy and will accrete onto the core.

The first point is supported not only by order of magnitude considerations, but also by detailed numerical simulations of the growth of protoplanets in the presence of gas (e.g., Ormel & Klar 2010; Lambrechts & Johansen 2012). Analytic calculations also show that even small bodies several orders of magnitude larger than the sizes with which we will be concerned inspiral on times shorter than the disk lifetime. Thus, we are justified in neglecting this part of the accretion process (Perets & Murray-Clay 2011). We also neglect the possibility that the core’s envelope is periodically replenished by the protoplanetary disk; see Ormel et al. (2015) for a discussion of this possibility, and Alibert (2017) for an application of this replenishment to pebble accretion.

Our second point invokes the classical solution of flow around an obstacle: for \( R_b < R_H \), the flow of the nebular gas is subsonic, meaning the core’s atmosphere is approximately incompressible. See Ormel (2013) Figure 5(A) for an example of this flow pattern in the context of a planet embedded in a protoplanetary disk. We note here that more detailed simulations of the flow show structure that may produce circulation into \( R_b \) (Ormel 2013; Fung et al. 2015), which we assume we can neglect for the level of accuracy we desire in this model. In addition, we do not explicitly calculate the work done on particles interior to \( R_b \) — our assumption that particles with \( KE > W \) will always be accreted for \( R_{stab} < R_b \) will eventually be violated for large enough particle sizes. See Inaba & Ikoma (2003) for an in-depth discussion of accretion in this regime.

While the criteria above are used throughout parameter space, in order to better illustrate how we model gas-assisted growth, we also provide a simplified “sketch” of how accretion in our model operates — one that accurately describes pebble
accretion over a large amount of parameter space. Figure 2 shows a “cartoon” of gas-assisted growth for a fixed core mass and semimajor axis in the disk, with small body radius increasing from left to right. Each panel shows the core’s Bondi radius, $R_b$, as well as the two radii that are used to determine $R_{\text{stab}}$—the Hill radius $R_H$ (see Section 4.1.1), which is the distance past which the stellar gravity will pull particles off the core, and the radius $R_{WS}$ (see Section 4.1.2), beyond which gas drag pulls particles off the core. Because smaller particles have a higher ratio of surface area to volume and therefore experience larger gas drag accelerations, $R_{WS}$ is smaller for smaller particles. In the far left panel, the particles have low mass, meaning gas drag can easily pull them off of the core, and $R_{WS}$ lies inside the core’s atmosphere. Because these particles have low mass, they easily dissipate their kinetic energy during the encounter with the core, meaning they are in the regime in the lower left-hand panel of Figure 1 and do not accrete. As particle size grows, $R_{WS}$ increases as well, until it exceeds $R_b$, the scale of the core’s atmosphere; i.e., there now exists a region exterior to the core’s atmosphere where particles can stably orbit the core. Because these particles are still of relatively low mass, they are able to dissipate their kinetic energy through gas drag. We therefore fall into the upper left-hand panel of Figure 1, which signals the onset of pebble accretion (middle panel of Figure 2). Finally, as particle size continues to increase, we eventually reach a point where the particles are so massive that they no longer dissipate their kinetic energy. As particle size increases, $R_{WS}$ will continue to increase, meaning we clearly still have $R_{\text{stab}} > R_b$. These particles are therefore in the upper right-hand panel of Figure 1 and will not accrete (right-hand panel of Figure 2).

Given this formalism, for the purposes of discussing whether a given small body will accrete, we simply need to compare the magnitude of the kinetic energy of the particle relative to the core and the work done on the particle during its encounter with the core. The kinetic energy of the particle relative to the core before the encounter is

$$KE = \frac{1}{2} m v_{\infty}^2,$$

(3)

while we take the work done by gas drag to be simply

$$W = 2 R_{\text{acc}} F_D(v_{\text{enc}}),$$

(4)

where $R_{\text{acc}} = \max (R_{\text{stab}}, R_b)$, and $F_D(v_{\text{enc}})$ is the drag force on a small body moving at a velocity $v_{\text{enc}}$, which is the relative velocity between the small body and the large body during the encounter. Section 3.6 gives our calculation of $v_{\text{enc}}$ and a discussion of why $R_{\text{acc}}$ is used for determining the work done by gas drag.

Our consideration of the ranges of particle sizes that can be accreted is an important aspect to our modeling that is often omitted in other works. See Section 5.5 for a detailed discussion.

Given the uncertainties in the size distribution of small bodies present in protoplanetary disks, understanding the extent of particle radii for which gas-assisted growth is possible is an important facet of studying the role of pebble accretion in planet formation. Furthermore, an important effect of nebular turbulence is to change the range of small body sizes that can be accreted (see Section 6.1), which makes a detailed consideration of the small body sizes where pebble accretion can operate an important aspect of our model.

In summary, a particle will be able to accrete if one of the following criteria are met:

1. $R_{\text{stab}} > R_b$ and $2 F_D(v_{\text{enc}}) R_{\text{acc}} > \frac{1}{2} m v_{\infty}^2$
2. $R_{\text{stab}} < R_b$ and $2 F_D(v_{\text{enc}}) R_{\text{acc}} < \frac{1}{2} m v_{\infty}^2$

Small bodies that do not satisfy either of these criteria will not be able to accrete, i.e., we set $t_{\text{grow}} = \infty$ for these particles. We emphasize that setting $t_{\text{grow}} = \infty$ refers only to the timescale for growth by pebble accretion: it is possible these particles could still be accreted via less-efficient processes, such as capture by gravitational focusing unassisted by gas drag or by collisions unaffected by gravity (see Equation (128)). In other words, having $t_{\text{grow}} = \infty$ does not necessarily imply that growth actually halts. See Section 5.5 for further discussion. While our model could, in principle, be extended to include these effects, in this work we are concerned primarily with gas-assisted growth; therefore, we do not explicitly include these other timescales.

2.2. Growth Timescale for Protoplanets

We will now elaborate upon how the growth timescales for cores are computed. In order to calculate the growth timescale for the large bodies for a given core mass $M$, we use the usual expression (see, e.g., Goldreich et al. 2004, hereafter GLS)

$$t_{\text{grow}} \equiv \left( \frac{1}{M} \frac{dM}{dt} \right)^{-1}.$$

(5)

The rate at which small bodies encounter the core is given by $n_{\text{acc}} v_{\infty}$, where $n$ is the volumetric number density of small bodies of a given size, $\sigma_{\text{acc}}$ is the cross section for accretion of the small body by the large body, and $v_{\infty}$ is the velocity at which small bodies encounter the large body. Note that $v_{\infty}$ is not necessarily the velocity of the small body during its encounter with the core, because this encounter can change the relative velocity; $v_{\infty}$ is the relative velocity between the two bodies at large separations. The number density of solids is simply $n = \rho_s / m = f_s \Sigma / (2 m H_p)$ where $\rho_s$ is the mass density of the small bodies, $\Sigma$ is the surface density of the gaseous component of the disk, $f_s$ is the solid-to-gas mass ratio, $m$ is the mass of the small body, and $H_p$ is the scale height of solids in the disk. Because each accretion of a small body increases the mass of the large body by $m$, the growth timescale is given by

$$t_{\text{grow}} = \frac{M H_p}{2 f_s \Sigma v_{\infty} R_{\text{acc}} H_{\text{acc}}},$$

(6)

where we have decomposed $\sigma_{\text{acc}}$ into the product of length scales in the plane of the disk and perpendicular to it: $\sigma_{\text{acc}} = (2 R_{\text{acc}}) (2 H_{\text{acc}})$. Thus, the aim of our calculation is to
determine the quantities \( R_{\text{acc}}, H_{\text{acc}}, H_p \) and \( v_{\infty} \). Once these quantities are known, we can immediately determine the growth timescale. The role each of these quantities plays in the growth timescale is illustrated graphically in Figure 3.

In the next few sections, we will expound upon how each of the above quantities are calculated in the context of pebble accretion. Our algorithm is summarized in Appendix A.

3. Velocity of the Small Bodies through the Disk

As small bodies move through the disk, their velocities are affected by drag from the nebular gas, as well as gravitational interactions with both the central star and the core. In order to calculate \( t_{\text{grow}} \), we need to understand what sets \( v_{\infty} \), the velocity of a small body relative to the large body, and \( v_{\text{enc}} \), the velocity of the small body during its encounter with the core. Calculating these velocities requires us to treat not only the gas drag force on the small bodies, but also to understand how drag from the laminar and turbulent components of the gas velocity each contribute to the velocity of the small bodies. Furthermore, the gas velocity influences the size of both \( R_{\text{acc}} \) and \( H_{\text{acc}} \) in ways that can have substantial impact on \( t_{\text{grow}} \).

We begin by reviewing our choices for modeling gas drag regimes and introduce the stopping time to parameterize the coupling between the small bodies and the gas (Section 3.1). The gas is taken have both a bulk, laminar component that is independent of time, and a fluctuating, turbulent component that time averages to zero. Both of these components can have an effect on the small bodies’ velocity, and we discuss each separately (Sections 3.2 and 3.3). These two components can be combined to give the average velocity between the small body and the large one due to gas drag, \( v_{\text{pk}} \) (Section 3.4). Gas drag is not the only source of relative velocity in the disk—the shear present in the disk due to the dependence of the Keplerian orbital frequency on semimajor axis can also affect these relative velocities (Section 3.5). Gas drag can also have a strong effect on the relative velocity between the small body and the core during their encounter, \( v_{\text{enc}} \), as can the gravitational force from the core (Section 3.6). These sections synthesize results from many works, which we present in detail, in order to clearly lay out the framework and assumptions upon which our model is based. Readers who only wish to review our choices for modeling the velocity may consult the summary of our calculation in Appendix A.

3.1. Gas Drag and Stopping Time

Gas drag is generally treated by breaking the drag force, \( F_D \), into a number of different regimes. We summarize our choices below; for a more in-depth discussion, see Batchelor (2000). First, we distinguish between the “diffuse regime,” which applies for particles with \( r_s < 9 \lambda / 4 \), and the “fluid regime,” \( r_s > 9 \lambda / 4 \). Here, \( r_s \) is the radius of the small body and \( \lambda \) is the mean free path of the gas particles. In the diffuse, non-supersonic, regime, the drag force on the particle is given by the Epstein drag law

\[
F_D = \frac{4}{3} \pi \rho_g v_{\text{th}} v_{\text{rel}} r_s^2,
\]

where \( \rho_g = \Sigma / (2 H_p) \) is the density of the gas, \( v_{\text{th}} = \sqrt{8 \pi c_s} \) is the thermal velocity of the gas, and \( v_{\text{rel}} \) is the relative velocity between the gas and the object. For a particle in the fluid regime, we must consider an additional parameter—the Reynolds number of the particle, given by \( Re = 2 r_s v_{\text{rel}} / (0.5 v_{\text{th}} \lambda) \). For \( Re < 1 \), the particle is in the Stokes drag regime, and the drag force is given by

\[
F_D = 3 \pi \rho_g v_{\text{th}} v_{\text{rel}} \lambda r_s,
\]

where \( \lambda \) is the mean free path of the gas particles. For \( Re > 1 \), the particle is in the Ram pressure regime, and

\[
F_D = \frac{1}{2} \pi \rho g r_s^2 v_{\text{rel}}^2.
\]

To mitigate discontinuities in the drag force, we use a smoothed drag force law in the fluid regime given by Cheng (2009):

\[
F_D = \frac{1}{2} C_D(Re) \pi r_s^2 \rho_g v_{\text{rel}}^2,
\]

where

\[
C_D(Re) = \frac{24}{Re} (1 + 0.27 Re)^{0.43} + 0.47 [1 - \exp(-0.04 Re^{0.38})].
\]

As \( Re \to 0 \), we have \( C_D \to 24 / Re \), so \( F_D \) reduces to the Stokes drag law given in (8). As \( Re \to \infty \), \( C_D \to 0.47 \), in which case \( F_D \) becomes the Ram drag force given in (9) (with a slightly different prefactor).

We can define a timescale from the drag force, known as the “stopping time” of the particle

\[
t_s \equiv \frac{m v_{\text{rel}}}{F_D}.
\]

The Epstein and Stokes drag laws are both linear in velocity—in these linear drag regimes, it is straightforward to calculate, from Equations (7) and (8), for spherical particles of uniform

\footnote{It is easy to see that we can generally neglect the supersonic regime \( (v_{\text{rel}} > c_s) \), because \( v_{\text{rel}} \lesssim v_{\text{gas}} \approx n_p c_s \) (see Section 3.2 for a discussion of the notation), we have \( c_s < n_p c_s \approx H_p / a > 1 \), where \( H_p \approx c_s / \Omega \) is the gas scale height and \( \Omega \) is the local Keplerian orbital frequency. Thus, for the supersonic case, the protoplanetary disk has an aspect ratio greater than 1, in strong opposition to observations of protoplanetary disks.}
In these regimes, the stopping time of the particle is a function only of the properties of the particle and the gas; in particular, it is independent of the particle’s velocity. Hence, \( t_s \) is often used as a parameterization of the particle’s size, \( r_p \), in terms of how well the small body is coupled to the gas flow (e.g., Chiang & Youdin 2010). If the drag law is instead quadratic in velocity, as in Equation (10), we numerically solve for the stopping time using Equations (18), (23), and (25). In practice, we solve these equations iteratively to calculate a self-consistent solution.

### 3.2. Laminar Velocity of Small Particles

Due to the internal pressure of the gas, the gaseous component of the protoplanetary disk will move at a sub-Keplerian orbital velocity. Weidenschilling (1977) gives the difference in velocity \( \Delta v \) as \( \Delta v \approx \eta_{\text{gas}} \), where \( \eta_{\text{gas}} \) is the local Keplerian orbital velocity, \( \eta_{\text{gas}} = a \Omega \), and \( \eta \) is a measure of the local gas pressure support, with approximate value \( \eta \approx c_s^2 / (2v_k^2) \). Due to this sub-Keplerian rotation, small bodies experience a “headwind” from the gas, which produces a drag force on the small bodies. If we use a polar coordinate system such that \( \hat{r} \) denotes the direction pointing away from the central star and \( \hat{\phi} \) denotes the direction of the disk’s rotation, then the drag force causes the small bodies to move with a sub-Keplerian velocity in the \( \hat{\phi} \) direction, and to drift in the \(-\hat{r}\) direction. In the above notation, the particle acquires a laminar velocity relative to the gas, given by

\[
v_{\text{r, gas}} = -2\eta_{\text{gas}} \left( \frac{\tau_s}{1 + \tau_s^2} \right),
\]

\[
v_{\text{r, gas}} = -\eta_{\text{gas}} \left( \frac{1}{1 + \tau_s^2} \right).
\]

where \( \tau_s = t_s \Omega \). See Nakagawa et al. (1986) for further details.\(^7\)

Because the laminar gas velocity relative to Keplerian is simply \( v_{\text{gas, r}} = -\eta_{\text{gas}} \hat{\phi} \), the velocity of the particle relative to Keplerian is

\[
v_{\text{r, k}} = -2\eta_{\text{gas}} \left( \frac{\tau_s}{1 + \tau_s^2} \right).
\]

\(^7\) There will also be a turbulent component to \( v_{\text{r, k}} \), which stems from the inward diffusion of the gas due to the turbulent viscosity \( \nu = \alpha_c H_k \) (see Section 3.3 for a discussion of the notation). Guillot et al. (2014) give this velocity as \( v_{\text{r, k}} = v_{\text{th}} / (1 + \tau_s^2) \), where \( v_{\text{th}} \approx \alpha_c H_k / \rho_k \) is the radial velocity of the gas due to turbulence. Following Guillot et al., the turbulent component \( v_{\text{r, k}} \) dominates for \( \tau_s < \tau_{\text{th}} = \alpha_c^2 / (2\eta_{\text{gas}}) \approx \alpha \). Lambrechts & Johansen (2014) show that, for the parameters they consider, this velocity is negligible compared to \( v_{\text{th}} \) because the diffusive length scale \( \ell \) is always less than the global scale of the disk \( a \). However, their expression for \( \ell \) can be rewritten as \( \ell / \tau_{\text{th}} \approx \alpha (H_k / a)^2 / (2\eta_{\text{gas}}) \approx \alpha / \tau_{\text{th}} \), which leads to the same conclusion as above for the size at which turbulence dominates. For our purposes, we neglect this effect because the small particles for which \( v_{\text{r, k}} \) move at velocities comparable to the fluctuating turbulent velocity \( v_{\text{th}} \), which dominates: \( v_{\text{r, k}} / \lambda \sim \sqrt{\nu} H_k / \rho_k \ll 1 \).

It is straightforward to show that the magnitude of the laminar component of the particle’s velocity is

\[
v_{\text{r, k}} = -\eta_{\text{gas}} \left( \frac{\tau_s}{1 + \tau_s^2} \right),
\]

relative to the gas, and

\[
v_{\text{r, k}} = -\eta_{\text{gas}} \left( \frac{1}{1 + \tau_s^2} \right),
\]

relative to Keplerian.

### 3.3. Turbulent Velocity of Small Particles

In order to describe the strength of the turbulence in the disk, we use the standard Shakura-Sunyaev \( \alpha \) parameterization of the effective kinematic viscosity. We employ \( \alpha \) simply as a convenient parameterization of the strength of turbulence; for our purposes, \( \alpha \) is fundamentally a local quantity and not necessarily connected with the accretion rate onto the star. In terms of \( \alpha \), the effective kinematic viscosity of the turbulent gas, \( \nu_{\text{th}} \), is given by (Shakura & Sunyaev 1973):

\[
\nu_{\text{th}} = \alpha \kappa H_k,
\]

where \( H_k = c_k / \Omega \) is the scale height of the gas. If we write the viscosity as the product of the local turbulent velocity and the largest-scale turbulent eddies, \( \nu = v_t \lambda \), and have \( \lambda_t \approx \nu_t / \Omega \), then \( \nu_t = \sqrt{\nu} \lambda_t \).

We use the Kolmogorov theory of turbulence to determine the turbulent energy spectrum. As described in Cuzzi & Hogan (2003), per the Kolmogorov theory, turbulence exists over a range of scales or “eddies,” which are characterized by their wave number \( k = 1 / \ell \). The largest-scale eddies, which occur on the length scale \( L_c \), are the scale on which energy is supplied by the turbulence; these large-scale eddies “turn over” and transfer their energy to smaller-scale eddies until energy is finally dissipated by kinematic viscosity on some smallest scale \( \eta \). If we assume that the rate of energy transfer between scales is independent of eddy size, we can show that the energy spectrum of the turbulence is given by \( E(k) \propto k^{-5/3} \), where \( E(k) \) has units of energy per mass per wavenumber. The velocity associated with \( k \) is then \( v(k) = (2kE(k))^{1/2} \propto k^{-1/3} \) and the overturn time for an eddy of wavenumber \( k \) is \( t_k = 1/(kv(k)) \propto k^{-2/3} \). Thus, the larger-scale eddies move more slowly and contain more of the turbulent kinetic energy. The size of the smallest-scale eddies can be determined by setting the rate of energy loss from molecular viscosity equal to the eddy turnover time, which gives \( \eta = (\nu^2 / c_k) \). In our expression for \( \nu_t \) in terms of \( \eta \) and \( \nu_t \), we have defined the Reynolds number of the turbulence, \( Re_t = \nu_t / \nu \). In terms of \( \alpha \), we have \( Re_t = \alpha \kappa H_k / (\nu_t \lambda) \).
Qualitatively, the behavior of a small body in response to an eddy of wavenumber $k$ depends on the ratio of the particle’s stopping time to the eddy turnover time $t_k$. Particles with $t_s < t_k$ will come to equilibrium with the eddy before it turns over, and will follow the large-scale motion of the eddy. On the other hand, particles with $t_s > t_k$ will not come to equilibrium with the eddy before it turns over, and will therefore only receive a small perturbative kick from the eddy. To characterize the small body’s response to the turbulence, most authors use the Stokes number, $St \equiv t_s / t_k$, where $t_k$ is the overturn time of the largest eddies. We make the usual assumption that $t_k = \Omega^{-1}$, in which case $St = \tau_s$, the parameter used in describing the particle’s interaction with the laminar gas flow. This permits the use of one parameter, which we will hereafter refer to as $St$, to characterize the particle’s interaction with both the laminar and turbulent components of the nebular gas flow. See Youdin & Lithwick (2007) for a more in-depth discussion of the effect of varying $t_k$.

Due to their interaction with the turbulent gas, small bodies will move relative to inertial space with some non-zero root-mean-square (rms) velocity, $v_{pi,t}$. The rms velocity can be calculated through order-of-magnitude means, as follows (Youdin & Lithwick 2007): For $St \gg 1$, we have $t_s \gg t_k$; in this regime, particles receive many uncorrelated “kicks” from the largest-scale eddies over a single stopping time, causing the particle to experience a random walk in velocity. Because the particle’s velocity damps out over a time $t_s$, the particle receives approximately $N \sim t_s / t_k \sim St$ kicks of amplitude $v \sim v_s t_k / t_s = v_s / St$, resulting in a velocity $v_{pi,t} \sim v_s / \sqrt{St}$. On the other hand, for $St \ll 1$, we expect $v_{pi,t} \sim v_s$. The simplest expression that has the correct behavior in each of these limits is given by

$$v_{pi,t} = \frac{v_s}{\sqrt{1 + St}}.$$  (21)

In order to better calculate the velocities of smaller particles, i.e., particles with $St < 1$, we employ expressions from Ormel & Cuzzi (2007, hereafter OC07), who give closed-form equations for the rms velocity of solid particles suspended in a turbulent medium. By following the methodology of Voelk et al. (1980), but also using results that take into account the finite inner scale of eddies at $k_{ij} = 1 / \bar{j}$, OC07 arrive at an analytic expression for the rms inertial space velocity of particles suspended in a turbulent medium

$$v_{pi,t}^2 = \nu_s^2 \left(1 - \frac{St^2(1 - Re_s^{-1})}{(St + 1)(St + Re_s^{-1})} \right).$$  (22)

Relative to the gas, the velocity is:

$$v_{pg,t}^2 = \nu_s^2 \left( \frac{St^2(1 - Re_s^{-1})}{(St + 1)(St + Re_s^{-1})} \right).$$  (23)

Equation (23) was originally derived by Cuzzi & Hogan (2003) for particles with $St \ll 1$. The results of numerical calculations and comparison with simulations, however, lead OC07 to argue that Equations (22) and (23) hold to order unity for particles of arbitrary Stokes number. We are therefore justified in applying this equation to determine the rms turbulent velocity of arbitrary-sized accreting particles, with one caveat. We first note that, for large Stokes numbers, we have $v_{pg,t} \rightarrow \nu_s/Re_s^{1/4}$. This cannot be completely correct because, in the limit $St \rightarrow \infty$, we expect for the particles to be so massive that they are completely unaffected by turbulence, and thus $v_{pi,t} \rightarrow 0$. In addition, we note that, for $St \ll Re_s$, as long as we do not have $St < 1$, Equation (22) reduces to (21). Thus, we can use (21) to determine the velocity of larger $St$ particles; in practice, we use (21) for $St \geq 10$. This form should connect smoothly to the more precise form given in Equation (22) as long as $Re_s \gg 10$. Using the fiducial values given in Section 5.1, the Reynolds number of the turbulence is given by

$$Re_s = 4.07 \times 10^{10} \alpha \left( \frac{a}{\text{au}} \right)^{-1}.$$  (24)

Thus, unless we are at extremely large orbital separations with weak turbulence, we can model the full range of Stokes numbers in this manner with little error.

3.4. Combining Velocities and Changing Frames

We have previously discussed how to calculate the laminar and turbulent components of the particle’s velocity. However, in order to calculate the total rms velocity, we need to understand how to combine these two components. Furthermore, calculation of drag forces requires knowledge of the velocity of the particle relative to the gas, whereas calculation of $v_{pi}$ will require the velocity of the particle relative to the local Keplerian velocity, because this is the velocity at which small bodies approach the core. Therefore, we also need to understand how to convert our velocities from one frame to another. The methodology necessary for our calculation is summarized here; for a more detailed derivation, see Appendix B.

We let $\delta v$ represent the “fluctuating” or turbulent component of the gas velocity and $v$ represent the laminar component—i.e., if we write $v = \bar{v} + \delta v$, then $\langle \delta v \rangle = 0$, where $\langle \ldots \rangle$ denotes ensemble averaging. We can then show that

$$v^2 = \bar{v}^2 + \langle \delta v^2 \rangle,$$  (25)

i.e., the turbulent and laminar components combine in quadrature.

If the subscripts $p$, $g$, and $k$ denote the velocity of the small bodies, the gas, and the local Keplerian velocity, respectively, then

$$v_{pk} = \bar{v}_{pg} + \bar{v}_{kg}.$$  (26)

Thus, the laminar component of the particle’s velocity can be changed from one frame to another in the usual manner (i.e.,
the $\hat{\phi}$ component is altered by $v_p \hat{\phi}$, while the $\hat{r}$ component is unchanged, independent of the turbulent velocity.

For the turbulent component of the velocity, one can show

$$\langle \delta v_{k}^2 \rangle = \langle \delta v_{p}^2 \rangle + \langle \delta v_{pg}^2 \rangle,$$

which is given in a number of works (e.g., Csanady 1963; Cuzzi et al. 1993, and OC07), and is usually derived by considering the Fourier components of the turbulent velocities in frequency space. An alternate derivation is given in Appendix B.

Neglecting the effects from the fact that the Keplerian velocity is not truly an inertial reference frame (see Youdin & Lithwick (2007) for a discussion of non-inertial effects), Equations (25)–(27) fully specify how to calculate all the velocities relevant to the problem from the input given by Equations (14), (15), and (22).

3.5. $v_{\text{shear}}$

Because the Keplerian velocity of the disk varies as $v_k \propto r^{1/2}$, bodies that are separated substantially in the radial direction will move relative to one another in the azimuthal direction, even in the absence of other effects. For our purposes, we approximate this “shear” velocity as

$$v_{\text{shear}} = \Omega r,$$

where $r$ is the separation between the two bodies in the radial direction. For small bodies encountering growing protoplanets, we have $r \approx R_{\text{acc}}$. If this velocity is larger than the drift-dispersion velocity of small bodies, $v_{pk}$, then $v_{\text{shear}}$ will set $v_x$, the velocity at which small bodies encounter cores; i.e., we set

$$v_x = \max(v_{pk}, v_{\text{shear}}).$$

We will refer to particles with $v_x = v_{pk}$ as “drift-dispersion dominated,” and particles with $v_x = v_{\text{shear}}$ as being “shear-dominated.”

3.6. $v_{\text{enc}}$ and Calculating Work

The relevant velocity for calculating the drag force remains to be determined. Note that this velocity is relative to the gas (in contrast to $v_x$, which is relative to the local Keplerian velocity) and cannot be set by shear, because the gas and the particles will have the same shear velocity. For particles that approach the core with velocities faster than the local circular orbit velocity about the large body, i.e., for particles with $v_x > v_{\text{orbit}} = \sqrt{GM/r_{\text{acc}}}$, the dominant effect of the gravitational force from the core will be to give the particle a small “kick” in the direction perpendicular to its motion. By using the impulse approximation, one can show that the magnitude of the resultant perpendicular velocity is of order $v_{\text{kick}} = GM/\left(4R_{\text{acc}}v_x \right)$ (see, e.g., Binney & Tremaine 2008). This effect is illustrated in Figure 4 (solid curve). As also shown in Figure 4, particles that approach the core with velocities such that $v_x < v_{\text{orbit}}$ will experience a substantial change in their total velocity during their interaction with the core (dashed curve). During the encounter, the magnitude of the particle’s velocity when the particle is a distance $r$ from the core is approximately $|v| \approx \sqrt{GM/r}$ (see the lower right-hand panel of Figure 4). If the magnitude of the work done on the particle can be expressed as $W(r) = F_D(r)r$, then for a particle in a linear drag regime, we have $W(r) \propto r^{1/2}$, i.e., the majority of the work done during encounter occurs when the particle is at the largest scales. If the particle is in a quadratic (Ram) drag regime, then $F_D$ is independent of $r$, so the work done at all scales is approximately equal. In either regime we are therefore justified in approximating the work done during the encounter as $W = 2F_D(v_{\text{enc}})R_{\text{acc}}$, where $v_{\text{enc}} = v_{\text{orbit}}(R_{\text{acc}}) = \sqrt{GM/R_{\text{acc}}}$.

Finally, if the particle’s drift velocity relative to the gas, $v_{pg}$, is larger than the velocity from the gravitational influence of the core, then $v_{pg}$ will set the velocity during the encounter.

In summary, we define the relevant velocity for the work calculation, $v_{\text{enc}}$, to be

$$v_{\text{enc}} = \begin{cases} \max(v_{\text{orbit}}, v_{pg}), & v_x < v_{\text{orbit}} \\ \max(v_{\text{kick}}, v_{pg}), & v_x > v_{\text{orbit}}. \end{cases}$$

4. Calculation of Accretion Cross Section

We now turn to how the particle velocities discussed in the previous section are used to calculate the growth timescale of the core. We will discuss how the length and width of the accretion cross section, $R_{\text{acc}}$ and $H_{\text{acc}}$, are calculated, as well as how the scale height of the small bodies, $H_p$, is determined.
4.1. Determining \( R_{\text{acc}} \)

The capture radius, \( R_{\text{acc}} \), is the radius interior to which a small body will accrete if certain energy criteria are met, as discussed in Section 2.1. The scale \( R_{\text{acc}} \) is also used to determine both the incoming kinetic energy of a small body and the work done by gas drag during the encounter. In this work, we assume that particles that cannot dissipate their kinetic energy at \( R_{\text{acc}} \) will not be able to accrete on a smaller length scale. To demonstrate this is generally the case, we note that, for a general impact parameter \( b \), we have

\[
\frac{KE}{W} = \frac{mv_{\infty}^2}{4F_D(v_{\text{enc}})b}. \tag{31}
\]

If the small body’s velocity is shear-dominated, i.e., \( v_{\infty} = v_{\text{shear}} \), then it can be shown analytically that the only particles that may have \( KE/W > 1 \) are those shearing into \( R_{\text{H}} \). This case, particles with substantially smaller impact parameters will not penetrate into \( R_{\text{H}} \) due to the nature of three-body trajectories (see, e.g., Petit & Henon 1986). If instead \( v_{\infty} = v_k \), then we have

\[
\frac{KE}{W} \propto \frac{1}{F_D(v_k)b} \propto (v_{\text{enc}},b)^{-1}, \quad \text{Epstein, Stokes, Ram}. \tag{32}
\]

The maximal possible scaling of \( v_{\text{enc}} \) with impact parameter is \( v_{\text{enc}} = v_k \propto b^{-1} \). Therefore, Equation (32) implies that \( KE/W \) is always minimized at large \( b \) when gas drag is in the Epstein or Stokes regime, and we conclude that the energy criterion at the largest possible accretion scale is sufficient. The only case for which gas-assisted growth may be ruled out by the energy criterion at a large scale and yet possible at a smaller impact parameter is in the Ram pressure drag regime. We do not include this possibility in our calculation of the accretion rate. This case only holds over a small amount of parameter space, as it requires particles to have both \( r_e > 9\lambda/4 \) and \( Re \gg 1 \). Thus, our modeling may rule out accretion of large particles in the inner disk (where the Ram regime is most important) that would be able to accrete at smaller scales.

To determine \( R_{\text{acc}} \), we note that capture is, in principle, possible either within the planet’s atmosphere, where the gas density increases substantially, or within the radius at which a small body can stably orbit the core. Thus, the capture radius is set by the relative size of the atmospheric radius, which extends up to \( R_p \) (for \( R_p < R_{\text{H}} \); see Section 7 for \( R_p > R_{\text{H}} \)), and the stability radius, \( R_{\text{stab}} \):

\[
R_{\text{acc}} = \max(R_{\text{stab}}, R_p). \tag{33}
\]

The stability radius is determined by requiring that the force on the small body is dominated by the gravity of the core. We also consider two other forces that can disrupt accretion, which leads to two different length scales that can set \( R_{\text{stab}} \).

4.1.1. The Hill Radius

The first scale is set by demanding that the small body not be sheared off by the gravity from the host star; this leads to the traditional measure of stability, the Hill radius, where the gravitational acceleration from the core is balanced by the tidal gravity from the star (Hill 1878). For \( M_\star \gg M \), where \( M_\star \) is the mass of the central star,

\[
R_H \approx a \left( \frac{M}{3M_\star} \right)^{1/3}, \tag{34}
\]

where \( a \) is the semimajor axis of the core’s orbit.

The fact that accretion can occur for impact parameters of order \( R_H \) is one of the main enhancements to the growth rate that come from pebble accretion. In the absence of gas, the maximal impact parameter for accretion is smaller than \( R_H \) by a factor \( (R/R_H)^{1/2} \), where \( R \) is the radius of the planet (see Appendix C). Both Lambrechts & Johansen (2012, hereafter LJ12) and Ormel & Klahr (2010, hereafter OK10) find regimes where \( R_{\text{acc}} \sim R_H \) in their numerical calculations of pebble accretion. In LJ12’s framework, accretion at \( R_H \) occurs for core masses greater than \( M_f = \sqrt[3]{1/3} v_{\text{gas}}/(G\Omega) \) and particle sizes \( St \gtrsim 1 \).

In OK10’s framework, impact parameters comparable to \( R_H \) occur in what they refer to as the “three-body regime,” which occurs for \( St > 1 \), and \( St > \zeta_\text{i} = v_{\text{vk}}/v_H \), where \( v_H \equiv R_{\text{H}} \Omega \) is the Hill velocity. In this regime, OK10 give the impact parameter as \( b_{\text{3b}}/R_H = 1.7p^{1/2} + 1.0/St \), where \( p = R/R_H \). This agrees generally with our results that \( R_{\text{acc}} \approx R_H \) in the Hill accretion regime.

The 1/\( St \) dependence given by OK10 appears to encapsulate the chaotic nature of trajectories for particles that shear into the Hill sphere: often particles will orbit the core many times before exiting the Hill sphere. This effect can cause particles that would not accrete according to our energy criterion to be captured, because our modeling only includes particles that dissipate their energy over one orbital crossing. If we use the result from Goldreich et al. (2002), who studied binary capture of Kuiper Belt objects by dynamical friction, that the probability of capture, \( P_{\text{cap}} \), for a particle that shears into the Hill radius is equal to the fraction of energy it dissipates over one orbit, then this leads to an effective accretion rate of

\[
\frac{dM}{dt} \sim R_H v_H \Sigma_p W \frac{W}{KE}. \tag{35}
\]

For high-mass cores, the gravitational force from the core dominates the velocity of the small body during the encounter, and the encounter velocity is \( v_{\text{enc}} \sim v_H \). This in turn implies that \( W/KE \propto 1/\text{St} \). For 2D accretion (\( H_\star < R_{\text{acc}} \)), which is the regime modeled by OK10, our expression therefore agrees with their results. For low-mass cores, however, the particle-gas relative velocity will instead dominate the velocity during the encounter, which will lead to scaling that differs from 1/\( St \) in this regime.

In summary, particles that have \( R_{\text{stab}} = R_H \) and \( v_{\infty} = v_H \) accrete with timescales given by

\[
t_{\text{grow}}' = \frac{t_{\text{grow}}}{\min(1, W/KE)}, \tag{36}
\]

where \( t_{\text{grow}} \) is given by Equation (6). In what follows, Equation (36) is used to modify \( t_{\text{grow}} \) for particles that shear into \( R_H \).

\[\footnote{LJ12 do not explicitly detail the Stokes number range where accretion at \( R_H \) occurs—because they run their simulations only for particles of size \( St = 10^{-2}, 10^{-1}, 10^0 \), they merely note that only particles with \( St = 10^{-2} \) appear to accrete at an impact parameter less than the Hill radius (see below). Following Xu et al. (2017), we take their regime where \( R_{\text{acc}} = R_H \) to hold only for \( St \gtrsim 1 \).} \]
4.1.2. The WISH Radius and Shearing Radius

In the presence of gas, even if the gravity from the core dominates over the tidal gravity from the star, it is possible that the relative acceleration of the core and the small body will be strong enough to strip the small body away from the core. In this regime, \( R_{\text{stab}} \) is set not by the Hill radius, but by what Perets & Murray-Clay refer to as the wind-shearing (WISH) radius (Perets & Murray-Clay 2011). At the WISH radius, the differential acceleration between the large and the small body due to gas drag is balanced by gravitational acceleration. That is,

\[
R_{\text{WS}}' = \sqrt{\frac{G(M + m)}{\Delta a_{\text{WS}}}},
\]

where \( m \) is the mass of the small body, and \( \Delta a_{\text{WS}} \) is the differential acceleration between the small and the large body, given by

\[
\Delta a_{\text{WS}} = \left| \frac{F_D(M)}{M} - \frac{F_D(m)}{m} \right|,
\]

where \( F_D \) is the force exerted on the particle due to gas drag. For the mass ratios considered here, the first term in Equation (38) is generally negligible. To determine the relevant velocity for calculating the drag force \( F_D \), we note that particles that are accreted by the core will have their velocity modified by the core’s gravity, increasing their velocity relative to the gas during the encounter. The most extreme case occurs when the velocity of particles match the local Keplerian velocity at which the core is moving. These particles temporarily experience the full gas velocity with respect to the core, which is a combination of the sub-Keplerian and turbulent velocities, as well as the Keplerian shear in the disk. We can encompass this behavior in the limit \( M \gg m \) by writing \( R_{\text{WS}}' \) as

\[
R_{W} \approx \sqrt{\frac{G M m}{F_D(\max(v_{\text{gas}}, v_{\text{shear}}))}},
\]

where \( v_{\text{gas}} = \sqrt{\frac{\eta^2 v_e^2 + \alpha c_s^2}{2}} \) is the rms velocity of the nebular gas; we use the prime to differentiate this radius from the definition of \( R_{\text{WS}} \) used below. The case \( v_{\text{gas}} > v_{\text{shear}} \) is discussed in detail by Perets & Murray-Clay (2011). In the three drag regimes discussed in Section 3.1, Perets & Murray-Clay (2011) give \( R_{\text{WS}} \) as

\[
R_{\text{WS}} = (GM\rho_c)^{1/2} \times \begin{cases} 
\left( \frac{1}{\rho_c v_{\text{gas}}} \right)^{1/2} r_s^{1/2}, & \text{Epstein} \\
\left( \frac{4}{9} \frac{\sigma}{\mu v_{\text{gas}}} \right)^{1/2} r_s, & \text{Stokes} \\
\left( \frac{1}{0.165} \frac{1}{\rho_c v_{\text{gas}}} \right)^{1/2} r_s^{1/2}, & \text{Ram}
\end{cases}
\]

where \( \rho_c \) is the internal density of the small body. We will use the term WISH radius and the symbol \( R_{\text{WS}} \) to refer solely to the case \( v_{\text{gas}} > v_{\text{shear}} \). For the case \( v_{\text{shear}} > v_{\text{gas}} \), on the other hand, the right-hand side of Equation (39) now depends on the impact parameter. In this regime, we will refer to the impact parameter as the “shearing radius,” \( R_{\text{shear}} \). In general, \( R_{\text{shear}} \) is determined by numerically solving the equation

\[
R_{\text{shear}} = \sqrt{\frac{G M m}{F_D(R_{\text{shear}}^2)}}.
\]

For a particle in a linear drag regime, the particle’s Stokes number is independent of velocity, which allows us to solve analytically for \( R_{\text{shear}} \):

\[
R_{\text{shear}} = (3St)^{1/3} R_H.
\]

In Section 5.5, we will compare our results to OK10 and LJ12 in the laminar regime—for now, we translate their impact parameters into our notation.

The WISH radius, in the laminar regime, to the effective accretion radius, \( r_{\text{sd}} \), used by LJ12, as well as the settling radius, \( b_{\text{set}} \), used by OK10. LJ12 give the effective accretion radius as \( r_d = (t_f/t_s)^{1/2} r_b \). Here, \( t_f = r_b / v_{\text{inj}} \) is the drift time across the “Bondi radius”—\( r_b = GM/v_{\text{inj}}^2 \). As LJ12 note, \( r_d \) is equal to the WISH radius, as can be seen from the fact that

\[
r_d = \left( t_f/v_{\text{inj}}^2 \right) = R_{\text{WS}}.
\]

LJ12 also give a radius equal to \( R_{\text{shear}} \) (up to a factor of \( 3^{1/3} \)) for particles with \( St = 10^{-2} \) and \( M > M_\star \).

In their “settling” regime, OK10 determine the impact parameter \( b_{\text{set}} = R_{\text{acc}}/R_H \) by solving the cubic equation

\[
b_{\text{set}}^3 + \frac{2}{3} R_{\text{WS}}^2 b_{\text{set}} - 8 St = 0.
\]

If the cubic term in the above equation is negligible, then we get the solution

\[
b_{\text{set}} = \sqrt[3]{12 St R_{\text{WS}}^2}.
\]

In terms of \( \zeta_w \), we can rewrite \( R_{\text{WS}} \) as

\[
R_{\text{WS}} = \sqrt[3]{\left( \frac{St}{\zeta_w} \right)^{1/2} R_H},
\]

so \( b_{\text{set}} = 2 R_{\text{WS}} \) in the laminar regime. If, on the other hand, the middle term is negligible, the solution is

\[
b_{\text{set}} = 2 St^{1/3},
\]

which is \( \sim R_{\text{shear}} \). One can verify by inspection that the solution to Equation (44) is approximately the minimum of Equations (45) and (47).

4.1.3. Combining Length Scales

Once \( R_H, R_{\text{WS}}, \) and \( R_{\text{shear}} \) are calculated, we take the stability radius to simply be the smallest of these three radii:

\[
R_{\text{stab}} = \min(R_{\text{WS}}, R_{\text{shear}}, R_H).
\]

We now have the pieces necessary to determine \( R_{\text{acc}} \). We first compute \( R_{\text{WS}} \) and \( R_{\text{shear}} \), the minimum of these two length scales determines the scale on which gas drag will pull small bodies off the core before they can be accreted. We then calculate \( R_H \), the scale on which the stellar gravity disrupts accretion, and calculate \( R_{\text{stab}} \) by taking in the minimum of these

---

11 As LJ12 note, their definition of the Bondi radius differs from the definition used in other works (including this one), which uses \( c_s \) in the place of \( v_{\text{inj}} \).
scales. Finally, we take $R_{\text{acc}}$ to be:

$$R_{\text{acc}} = \max(R_{\text{sub}}, R_\text{g}).$$

(49)

Note that accretion within $R_{\text{acc}}$ only occurs when additional energy criteria are met, as discussed in Section 2.1.

### 4.2. Determining $H_{\text{acc}}$

We now turn to the height of the accretion rectangle. The height of this rectangle will be the minimum of the dust scale height and the capture radius, as any particle outside the capture radius will not be able to accrete on to the core:

$$H_{\text{acc}} = \min(R_{\text{acc}}, H_\text{p}).$$

(50)

In general, the solid particles will tend to settle to the midplane of the disk, due to the vertical influence of gravity and the lack of pressure support. However, several processes will tend to oppose this settling, giving the particles some finite vertical extent. In our model, we include two different processes that drive particles vertically—the first is turbulent diffusion due to the isotropy of the small body’s turbulent velocities.

For $St > \alpha$, the scale height of the particles due to their interaction with turbulence is given by (Youdin & Lithwick 2007):

$$H_t = \sqrt[\alpha]{St} H_\text{g}.$$  

(51)

We can understand this expression by separately considering larger particles with $St \gg 1$ and small, well-coupled particles with $St \ll 1$. For the larger particles, as discussed in Section 3.3, we can think of the particle’s velocity as resulting from the large number of uncorrelated kicks it receives from the eddies. In this case, we have $v_{\text{pk},t} \sim v_{\text{f}}/\sqrt{St}$, which gives a scale height of

$$H_t \sim \frac{v_{\text{f}}}{\Omega \sqrt{St}} \sim \sqrt[\alpha]{St} H_\text{g},$$

(52)

since $v_{\text{f}} = \sqrt{\Omega} H_\text{g} \Omega$.

As described in Youdin & Lithwick (2007), for $St \ll 1$ we expect the particles to be well-coupled to the gas, and thus expect the particles’ diffusion coefficient to be approximately equal to the gas value. We can express the turbulent diffusion coefficient of the gas as $D_g = \alpha_c c_s H_\text{g}$. In what follows, we take $\alpha_c \approx \alpha$ unless otherwise stated. While $\alpha$ and $\alpha_c$ can differ, because $\alpha$ enters our calculation mainly as a parameterization of the turbulent gas velocity, we are more interested in the relative size of $\alpha_c$ and $v_c$, the vertical turbulent gas velocity. Magnetohydrodynamic simulations find that these two values are similar to order of magnitude (e.g., Xu et al. 2017). Tighter coupled particles should settle to the midplane at approximately their terminal velocity, so their time to settle to the midplane from a height $z$ is approximately $t_{\text{sett}} \sim z/v_{\text{term}}$. At terminal velocity, the drag force balances the vertical component of gravity, $F_{g,z} \sim (GMm/r^2)(z/r) \sim m\Omega^2 z$. Setting $F_{\text{drag}} \sim m\Omega^2 z$ implies that $v_{\text{term}} \sim St \Omega z$, which gives $t_{\text{sett}} \sim 1/(St \Omega)$. Finally, setting $t_{\text{sett}}$ equal to the particle diffusion time $H_\text{p}^2/D_g$ gives Equation (51). Carballido et al. (2006) derive (51) for $St \gg 1$ by more rigorous means, whereas Dubrulle et al. (1995) do the same for $St \ll 1$.

For $St > \alpha$, we have $H_t > H_\text{g}$, which is clearly incorrect; turbulence cannot drive particles to heights larger than the extent of the gas disk. We therefore cap $H_t$ at $H_\text{g}$. Dubrulle et al. (1995) derive (51) as the scale height of the dust-to-gas ratio $\rho_d/\rho_g$ and note that this height $h$ is related to the dust scale height by the equation

$$H_t = h \left[ 1 + \left( \frac{h}{H_\text{g}} \right)^{2} \right]^{-1/2},$$

(53)

which has the expected behavior that $H_t \rightarrow H_\text{g}$ as $St \rightarrow \infty$. Equation (53) can be used in the place of Equation (51), though this does not substantially change the results. We employ (51) in order to maintain relative simplicity in our analytic results.

Turbulence’s modification to the particle scale height is discussed in a number of works, including OK10 and LJ12, though it is often not explicitly included in calculations of growth rate. Instead, authors generally note that, for $H_t > R_{\text{acc}}$, the growth timescale is increased by a factor of $H_t/R_{\text{acc}}$. We also note that Equation (53) can be written as

$$H_t = H_\text{g} \sqrt[\alpha]{\Omega + St},$$

(54)

which is employed by Ormel & Kobayashi (2012) for the dust scale height in the presence of turbulence. Equation (54) agrees with Equation (28) of Youdin & Lithwick (2007) without their factor of $\xi^{-1/2}$, where $\xi \equiv 1 + St/(1 + St)$ for the values chosen in this paper.

Even in the absence of strong turbulence, the Kelvin–Helmholtz shear instability prevents small bodies from settling too close to the midplane. For small particles, which are well-coupled to the gas flow, this leads to a scale height of

$$H_{\text{KH}} = H_\text{g} \frac{2\eta_k}{\Omega},$$

(55)

(e.g., Lee et al. 2010). In order to extend this scale height to include larger particles, we assume that, even when large bodies dominate the mass distribution, a population of small bodies exists that is sufficient to drive the Kelvin–Helmholtz shear instability—and therefore turbulence—close to the midplane. We expect the turbulent velocity in this case to be comparable to the vertical shear rate, which is $\sim \eta_k$. In this case, we can make arguments analogous to those preceding Equation (52), leading to a scale height $H_{\text{KH}} \sim \eta_k/\sqrt{\Omega} \sqrt{St}$. Thus, we can describe $H_{\text{KH}}$ over the full range of particle sizes using the expression

$$H_{\text{KH}} = \frac{H_\text{g}^2}{a} \min(1, St^{-1/2}).$$

(56)

If only large bodies are present, they may need to settle further in order to excite the Kelvin–Helmholtz instability, which would change the dependence on $St$. We leave a self-consistent calculation to another work. Furthermore, by employing this expression for larger particle sizes, we are neglecting the possibility of dynamical stirring of the large bodies. Mutual scatterings and/or the gravitational force of the core can drive small particles vertically, and may be more important than interactions with the gas for determining the scale height of larger particles, which are decoupled from the gas.
Finally, we take the solid scale height to simply be the maximum of these two heights:

\[
H_p = \max (H_s, H_{KH}).
\]  

(57)

The possibility of 3D accretion \((H_p > R_{acc})\) in the non-turbulent regime is generally neglected in works on pebble accretion, but we find that it has non-trivial effects on the growth timescale.

5. Overview of Results

Having detailed how our model operates, we can now present its output. For reference, we also provide a table detailing the symbols we use (Table 1). In this section, we broadly discuss our results in order to introduce how gas-assisted growth operates in the presence of turbulence.

5.1. Model Parameters

We begin by providing the values of the fiducial parameters used for describing the protoplanetary disk, which are necessary to provide concrete numerical results. A discussion of the effects of varying some of these parameters is given in Section 6.

We assume a surface density profile of

\[
\Sigma(a) = 500 \text{ g cm}^{-2} \left( \frac{a}{\text{au}} \right)^{-1},
\]

(58)

where \(a\) is the semimajor axis in the disk. This profile is chosen to be consistent with measurements of the surface density in solids of size 0.1–1 mm, taken from sub-mm observations of protoplanetary disks (Andrews et al. 2009; Andrews 2015), which probe these particle sizes. In order to calculate the surface density profile of solid bodies, \(\Sigma_{p\ast}\), we assume a constant dust-to-gas ratio of \(f_s \equiv \Sigma_{p\ast}/\Sigma = 1/100\). The semi-major axis dependence of the temperature profile is taken from Chiang & Youdin (2010)

\[
T = T_0 \left( \frac{a}{\text{au}} \right)^{-3/7}.
\]

(59)

For most purposes, we will take the prefactor to be \(T_0 = 200\) K, which would place the Earth outside of the water ice line during its formation (e.g., Powell et al. 2017). We also note this prefactor is consistent with the assumption that the disk is heated primarily by irradiation from a central star with luminosity \(L_* \sim 3 L_\odot\) (e.g., Ida et al. 2016), which is appropriate for a solar mass star of age 1 Myr (Tognelli et al. 2011). The effects of varying \(T_0\) are discussed in Section 6.2.
From here, we can then calculate the isothermal sound speed and scale height of the disk in the usual manner: \(c_s = \sqrt{kT/\mu}\) and \(H_s = c_s/\Omega\). Here, \(\mu\) is the mean molecular weight of particles in the disk; we take \(\mu = 2.35 m_H \approx 3.93 \times 10^{-24} \text{ g}\), which assumes a disk composed of 70% \(H_2\) and 30% \(He\) by mass. We can also calculate the gas density \(\rho_g = \Sigma/(2H_s)\), and the mean free path of gas particles, \(\lambda = \mu/\rho_g c_s\), where \(\sigma\) is the neutral collision cross section, which we take to be \(\sigma \sim (3 \text{ Å})^2 \sim 10^{-15} \text{ cm}^2\). In order to convert from Stokes number, which is the relevant parameter for calculating \(t_{\text{grow}}\), to physical size, we need to specify the internal density of the pebbles, \(\rho_s\). We take this density to be \(\rho_s = 2 \text{ g cm}^{-3}\) unless stated otherwise, which is appropriate for rocky bodies. Note, however, that the solids in protoplanetary disks may be fluffly, i.e., their densities may be low, which can have important ramifications for planet formation (see, e.g., Kataoka et al. 2013). While we consider the effect of varying \(M_s\) in Section 6.4, unless otherwise noted, all values quoted in the paper are for a solar mass star, \(M_s = M_\odot\). As noted in Section 2.1, the radius of the large body needs to be \(\gtrsim 10 \text{ km}\), which corresponds to \(M \sim 10^{-3} m_\odot\) for a density of \(2 \text{ g cm}^{-3}\). In practice, we do not consider cores that are close to this lower limit.

A summary of our fiducial disk parameters is given in Table 2.

### Table 2
Fiducial Disk Parameters

| Parameter                | Symbol | Formula/value |
|--------------------------|--------|---------------|
| Solid-to-gas ratio       | \(f_s\) | 0.01          |
| Mean molecular weight of gas particles | \(\mu\) | \(2.35 m_H \approx 3.93 \times 10^{-24} \text{ g}\) |
| Neutral collision cross section | \(\sigma\) | \(10^{-15} \text{ cm}^2\) |
| Gas surface density      | \(\Sigma\) | \(\Sigma(\alpha) = 500 \left(\frac{\alpha}{0.01}\right)^{-1} \text{ g cm}^{-2}\) |
| Protoplanetary disk temperature | \(T\) | \(T(\alpha) = 200 \left(\frac{\alpha}{0.01}\right)^{-1/7} \text{ K}\) |
| Isothermal sound speed   | \(c_s\) | \(c_s = \sqrt{kT/\mu}\) |
| Average thermal velocity | \(v_{th}\) | \(v_{th} = \sqrt{\frac{2kT}{\mu}}\) |
| Gas scale height         | \(H_g\) | \(H_g = \frac{c_s}{\Omega}\) |
| Gas density              | \(\rho_g\) | \(\rho_g = \Sigma/(2H_s)\) |
| Gas mean free path       | \(\lambda\) | \(\lambda = \mu/\rho_g c_s\) |

Note. A value indicates that the quantity is a constant input parameter to the code, whereas a formula indicates that the quantity is calculated in the prescribed manner from input parameters and constants.

For the purpose of understanding the broad features of pebble accretion, we present a typical output of our model in Figure 5. Figure 5 plots the growth timescale (Equation (6)) as a function of the small body radius, \(r_s\), for a core of mass \(M = 10^{-1} M_\odot\) at \(a = 1 \text{ au}\). The growth timescales for several \(\alpha\) values, ranging from purely laminar flow to extremely strong turbulence, are shown. The timescale is set to \(\infty\) for particles that do not satisfy the energy criteria discussed in Section 2.1, so the plotted range for each value of \(\alpha\) indicates the range of small body radii the core can accrete via pebble accretion. The Stokes numbers corresponding to the given values of radius are shown for reference (the laminar drift velocity is used to calculate \(St\) for particles not in a linear drag regime).

We begin by examining the laminar case, where \(\alpha = 0\). For large values of \(r_s\), particles shear into the Hill radius. However, growth is slow because particles are large and cannot fully dissipate their kinetic energy relative to the gas, which increases \(t_{\text{grow}}\) by a factor of \(KE/W\). Once particles are small enough that \(KE \sim W\), they can accrete on rapid timescales. In terms of the small body’s Stokes number, we can generally write the criterion for particles to be able to deplete their kinetic energy as

\[
St < \frac{4v_{\text{enc}}R_{\text{acc}}\Omega}{v_\infty^2},
\]

(60)

For particles of this size, gravity dominates over gas drag effects—therefore, \(R_{\text{H}}\) determines \(R_{\text{acc}}\), as opposed to \(R_{\text{WS}}\). The large size of these particles also means that their velocity dispersion and drift velocity are low, so the approach velocity is set by shear, i.e., \(v_{\text{enc}} = v_\infty\). Using these values to calculate \(v_{\text{enc}}\) and plugging the result into Equation (60) gives

\[
St < 4\sqrt{3},
\]

(61)

which will be a general upper limit on Stokes numbers that the core is able to accrete rapidly. Note that, if the small body is not in a linear drag regime, then the Stokes number is no longer independent of velocity. In this case, the Stokes number calculated for the small body moving through the disk (which is relative to \(v_{\infty}\)) will not necessarily be the same as the Stokes number during the encounter with the core (which is relative to \(v_{\text{enc}}\)).

Growth for \(R_{\text{acc}} = R_{\text{H}}\) and \(v_{\text{enc}} = v_\infty\) is extremely rapid, in comparison to the planetesimal accretion timescale. As small body size continues to decrease, however, gas drag
considerations eventually overtake three-body effects, and the shear radius shrinks below the Hill radius. This causes the growth timescale to increase for \( r_s \lesssim 20 \) cm. From this point, the shear radius continues to shrink with small body size, slowing accretion. The first kink in the slope around \( r_s \sim 15 \) cm comes from small bodies changing from the fluid regime to the diffuse regime, while the second kink around \( r_s \sim 0.2 \) cm stems from \( R_{\text{shear}} \) shrinking below \( H_p \), which dilutes the density of small bodies, in turn slowing accretion. Eventually, the shear radius shrinks below the Bondi radius, which signals the point at which particles are so small that they will couple to the gas and flow around \( R_p \). This causes growth to cut off for small values of \( r_s \).

As increasingly strong turbulence is taken into account, the picture becomes more complicated. For the \( \alpha = 10^{-1} \) case, growth for large particle sizes is quite similar to the laminar case, because the growth parameters are solely determined by the core’s mass for high \( St \). As the particle size shrinks, we still find a small range of radii where growth occurs at \( R_{\text{acc}} = R_H \), \( v \equiv v_{\text{fr}} \). Rather rapidly however, several new features emerge that were not previously present. First, the strong turbulence greatly increases the particle scale height. As particle size decreases, this scale height increases until it becomes larger than \( R_H \), slowing down growth. As particle size decreases, for \( St < 1 \), the drift-dispersion velocity of particles increases as they couple more strongly to the gas. Due to the rapid turbulent velocity of the gas, \( v_{\text{fr}} \) actually overtakes \( v_H \) around \( r_s = 100 \) cm, slightly mitigating the increase in growth timescale. The \( \text{WISH} \) radius is also decreased due to the strong turbulence, which makes it easier to pull particles off the core. Because of this, the radius at which \( R_{\text{WS}} < R_H \) is much higher than in the laminar case, and growth is again inhibited around \( r_s = 50 \) cm as the \( \text{WISH} \) radius begins determining \( R_{\text{acc}} \). Another kink in the slope appears around \( r_s = 20 \) cm as the scale height of particles becomes so large that \( H_p = H_p \), at which point the particle scale height stops increasing. The final kink occurs at \( r_s \approx 15 \) cm, and is again due to the particles switching from the fluid to the diffuse regime. Growth again stops when \( R_{\text{WS}} < R_H \), but because of the decreased values of \( \text{WISH} \) radius relative to the laminar case, this occurs at a larger value of \( r_s \). Similar features occur for the smaller values of \( \alpha \), but due to the weaker turbulence, these features occur at smaller values of \( r_s \). The only exception is the domination of the turbulent rms velocity over \( v_{\text{fr}} \), which only occurs in the \( \alpha = 10^{-1} \) case for these values of parameters. As can be seen in the figure, for lower values of \( \alpha \), wider ranges of small body sizes can accrete at what appears to be a minimal value of timescale, which we discuss in the next section.

5.3. \( t_{\text{Hill}} \)

The floor of the growth timescale in Figure 5 corresponds to a minimum value of \( t_{\text{grow}} \), which cores reach when they can accrete over their entire Hill sphere. The appearance of this minimum growth timescale, or maximal growth rate, is a consistent thread throughout the parameter space probed by our model.

We refer to this timescale as the “Hill timescale,” or \( t_{\text{Hill}} \). This timescale is reached when all aspects of the accretion processes are set by the core’s gravity, i.e., when we have \( R_{\text{acc}} = R_H > H_p \) and \( v_c = v_H \). From Equations (6) and (50), it is easy to see that, if \( H_p < R_{\text{acc}} \), then \( t_{\text{grow}} \) is independent of \( H_p \). Using these values for our timescale parameters in (6) gives the value of \( t_{\text{Hill}} \) as

\[
t_{\text{Hill}} = \frac{M}{2f_0 \Sigma R_H^2 \Omega}.
\] (62)

A similar regime is identified in Section 3.2 of Lambrechts & Johansen (2012); they refer to it as “Hill Accretion.” However, they only use this term to differentiate between the drift-dominated and shear-dominated regimes, which roughly corresponds to where \( R_{\text{acc}} = R_H \) for their model. Within their Hill regime, the growth timescale is not necessarily equal to \( t_{\text{Hill}} \). See Section 5.5 for a more in-depth comparison of the two models.

In general, \( t_{\text{Hill}} \) represents an extremely rapid timescale for growth. Scaled to fiducial values, the minimum timescale can be expressed as

\[
t_{\text{Hill}} \approx 4300 \text{ years} \left( \frac{a}{\text{au}} \right)^{1/2} \left( \frac{M}{M_\odot} \right)^{1/3}.
\] (63)

This represents a significant enhancement over the timescale achievable via canonical core accretion, as can be seen by comparing Equations (63) and (130).

This is the substantial decrease in growth timescale promised by gas-assisted growth. The source of this decrease can be mostly attributed to an increase in \( R_{\text{acc}} \). As previously stated, the minimum timescale indicates accretion in a regime such that essentially all particles that encounter the Hill radius of the growing planet are accreted. By contrast, in the absence of gas, the trajectories of particles that enter the Hill radius are chaotic, and whether a collision occurs is extremely sensitive to the value of the small body’s impact parameter; see, for example, Petit & Henon (1986). For this reason, encounters with the core in this regime are often treated probabilistically, with the collision rate of small bodies equal to the product of the rate at which small bodies encounter the radius and the probability that a particle inside the Hill radius will accrete. Performing such a calculation leads to an effective impact parameter for accretion in the so-called “three-body regime” that is, to order of magnitude, equal to the geometric mean of the Hill radius and the planetary radius—\( b \sim \sqrt{R \times R_p} \); see GLS and Ormel & Klahr (2010), Equation (5). From comparison with Equation (129), we see that the maximum increase in accretion rate that can be provided by pebble accretion is of order \( R_H/R \).

To see why \( t_{\text{Hill}} \) is generally the fastest rate of growth possible, we first note that because \( R_{\text{stab}} = \min (R_H, R_{\text{WS}}, R_{\text{shear}}) \), it is clear that the maximal accretion cross section is of order \( \sigma_{\text{gas,max}} \sim R_H^2 \). Given that turbulence can increase the rms velocity of small bodies to values substantially larger than \( v_H \), however, it is conceivable that strong turbulence could increase the growth rate by increasing \( v_c \), i.e., by increasing the rate at which small bodies encounter the core. However, increasing the velocity of small bodies drives them vertically as well as horizontally, which decreases the density of small bodies and slows accretion. This trade-off rules out accretion on timescales faster than \( t_{\text{Hill}} \) which we demonstrate below, to order of magnitude. We first note that

\[
H_{KH} = \frac{2\eta_k}{\Omega} \max(1, St^{-1/2}) \gtrsim \frac{v_{p,k} t_{\text{Hill}}}{\Omega}.
\] (64)
We also make the approximation that
\[ v_{pk} = \max(v_{pk,f}, v_{pk,l}), \]
(66)
rather than the quadrature of \( v_{pk,f} \) and \( v_{pk,l} \). We separately consider “2D accretion,” defined by \( H_p < R_{\text{acc}} \), and “3D accretion,” where \( H_p > R_{\text{acc}} \).

For 2D accretion, we can write \( t_{\text{grow}} \) as
\[
\begin{align*}
  t_{\text{grow}} &= \frac{M}{2\Sigma_p R_{H}v_{\infty}} \\
  &= t_{\text{Hill}} \left( \frac{R_{H} \Omega}{v_{\infty}} \right),
\end{align*}
\]
(68)
where \( t_{\text{Hill}} \equiv M/(2\Sigma_p R_{H}^2 \Omega) \) and \( \Sigma_p \equiv f_p \Sigma \). Thus, \( t_{\text{grow}} < t_{\text{Hill}} \) requires \( v_{\infty} > v_{pH} \) while still having \( R_{H} > H_p \). However, we see from Equations (64)–(66) that \( R_{H} > H_p \) implies \( v_{pH} > v_{pk} \), so \( v_{\infty} > v_{pH} \). Thus, the timescale for growth cannot drop below \( t_{\text{Hill}} \) for 2D accretion.

We now consider the 3D accretion regime when the Hill radius sets the geometry, i.e., \( H_p > R_{\text{acc}} = R_{H} \). In this case, we can write the growth timescale as
\[
\begin{align*}
  t_{\text{grow}} &= \frac{MH_p}{2\Sigma_p R_{H}v_{\infty}} = t_{\text{Hill}} \left( \frac{H_{p} \Omega}{v_{\infty}} \right).
\end{align*}
\]
(69)
If the particle is shear-dominated, then \( t_{\text{grow}} > t_{\text{Hill}} \) because \( R_{H} < H_p \). Furthermore, because Equations (64)–(66) imply \( H_{p} \Omega \gtrsim v_{pk} \), if the particle is drift-dispersion dominated then we again have \( t_{\text{grow}} > t_{\text{Hill}} \). Thus, in the 3D accretion regime, \( t_{\text{Hill}} \) also represents the smallest possible timescale.

5.4. Gas-mitigated Growth

While growth at \( t_{\text{Hill}} \) is extremely rapid, it is important to realize that, in practice, the minimum timescale can only be achieved for a certain range of small body sizes. Maximal accretion efficiency occurs for particles where the dominant effect of the gas is to dissipate particles’ kinetic energy as they encounter the growing core, and all other aspects of the accretion process are set by the gravitational influence of the core. When the mass of small or large bodies is low enough that gas determines aspects of the accretion process, the timescale for growth will increase—in many cases, quite substantially.

To more thoroughly examine what causes the timescale to increase above \( t_{\text{Hill}} \) as the interaction between the gas and the small bodies increases in importance, we plot the four quantities that go into the calculation of \( t_{\text{grow}} \), along with timescale itself, in Figure 6. As \( \alpha \) increases, the gas-particle interaction will dominate over the gravitational interaction between the small body and the core. For low values of \( \alpha \), we have \( R_{\text{WS}} > R_{H} > H_p \) and \( v_{\infty} = v_{\text{shear}} \) (as expected for \( R_{H} > H_{p} \)), so the particle is in the 2D accretion regime and is able to accrete at \( t_{\text{Hill}} \). As \( \alpha \) increases, the particle scale height begins to increase, until eventually \( H_p > R_{H} \). At this point, accretion becomes less efficient and the timescale correspondingly increases. There are several other kinks in the timescale graph, which are caused by, in order of increasing \( \alpha \):

1. \( R_{\text{WS}} \) decreasing to the point that \( R_{\text{WS}} < R_{H} \).
2. \( v_{\infty} \) being set by dispersion instead of shear, i.e., switching to the drift-dispersion dominated regime.
3. Reaching \( H_p = R_{H} \), at which point \( H_p \) stops increasing.

For large \( \alpha \) values, the growth timescale is almost an order of magnitude larger than \( t_{\text{Hill}} \), indicating a substantial slowdown in growth rate.
5.5. Correspondence with Previous Models of Pebble Accretion

In this section, we compare our model to the analytic results from OK10 and Johansen & Lambrechts (2017, hereafter JL17)\(^\text{12}\) in the laminar (\(\alpha = 0\) regime), as well as the extension of the OK10 model by Chambers (2014) to include turbulence. We discuss why our framework is more useful for incorporating the results of turbulence.

In order to facilitate comparison between the various models, we begin by highlighting some important features that emerge in our modeling of the growth rate. Figure 7 shows a plot of \(P_{\text{col}} \equiv 2 \rho_{\text{acc}} v_{\text{sc}}\) as a function of \(r_s\) and \(M\) for a core at \(a = 30\) au with \(\alpha = 0\). Several important features, along with their analytic formulae, are marked on the plot. We discuss these features in detail below. In calculating the analytic expressions for these features, we implicitly assume a linear drag regime so that we can employ the analytic expressions for \(R_{\text{WS}}\) and \(R_{\text{shear}}\) (see Appendix A).

In this regime, we have \(R_{\text{acc}} = R_{\text{WS}}, v_{\text{sc}} \approx v_{\text{gas}},\) and \(v_{\text{enc}} = v_{\text{kick}},\) which leads to a maximum size of

\[
St = 12 \left( \frac{v_H}{v_{\text{gas}}} \right)^3. \tag{71}
\]

However, as particle size continues to increase, we see that accretion actually resumes once particles become large enough. This is caused by these large particles decoupling from the gas, which lowers \(v_{\text{sc}} = v_{\text{pk}}\) and raises \(v_{\text{enc}} = v_{\text{pg}}\) enough to overcome the increased mass of these particles. If we set \(R_{\text{acc}} = R_{\text{H}},\ v_{\text{pg}} \approx v_{\text{gas}},\) and approximate \(v_{\text{pk}}\) by its Taylor Expansion at \(\infty\) in the laminar regime, \(v_{\text{pk}} \approx 2v_{\text{gas}}/St,\) then we can derive an approximate expression for the Stokes number at which accretion commences again:

\[
St = \frac{v_{\text{gas}}}{v_H}. \tag{72}
\]

Note that, as particle size continues to increase, the energy criteria are only satisfied for a small range of particle sizes. However accretion continues probabilistically in this regime, in accordance with Equation (36).

The gap in particle size where accretion is not possible disappears once the core surpasses a certain mass. If particles that have their encounter velocity dominated by the particle-gas relative velocity, \(v_{\text{enc}} = v_{\text{pg}},\) become available for accretion, then the range of particle sizes the core can accrete will be extended because these particles can dissipate more of their kinetic energy due to their larger encounter velocities. See Section 6.1 for more discussion of this effect. For these particles to be available, the previously derived upper limit on particle size, \(St = 12v_H^3/v_{\text{gas}}^3,\) must occur after the transition where \(v_{\text{pg}} = v_{\text{kick}}.\) Using the Taylor expansion of \(v_{\text{pg}}\) about 0 in the laminar regime, \(v_{\text{pg}} \approx 2v_{\text{gas}}St,\) we see that the latter

\[St < \min \left( \frac{3}{2} \frac{v_{\text{gas}}^3}{c_s^4}, 9 \frac{v_H^6}{c_s^6} \right). \tag{70}\]

In the bottom of the plot, we see that accretion shuts off for particles that pass a certain maximum size at low core masses.

\[\text{Figure 7.} \text{ A plot of} \ P_{\text{col}} = 2 \rho_{\text{acc}} v_{\text{sc}} \text{ as a function of} \ r_s \text{ and} \ M \text{ for a core at} \ a = 30 \text{ au with} \ \alpha = 0. \text{ Several important features, along with their formulae, are marked on the plot. See the text for a discussion of what causes these features to emerge.}\]
transformation occurs at \( \text{St} \approx (3/4)^{1/3}(v_H/v_{\text{gas}}) \). Therefore, the mass scale where this transition occurs is given by

\[
\frac{v_H}{v_{\text{gas}}} = 48^{-1/3}.
\] (73)

Finally, at high core masses, the transition where \( KE = W \) occurs at a fixed Stokes number of

\[
\text{St} = 4\sqrt[3]{3},
\] (74)

which is noted on the plot as well (see Section 5.2 for further discussion).

We now contrast the output from our model in the laminar regime to that of the analytic models of OK10 and JL17. Figure 8 below shows plots of both \( P_{\text{col}} = 2R_{\text{acc}}v_{\text{esc}} \) and \( t_{\text{grow}} \) for a core at 30 au, using our model, OK10’s, and JL17’s (for a discussion of how the parameters of the protoplanetary disk are modeled, see Section 5.1). We plot \( P_{\text{col}} \) to disentangle the effects of particle scale height on the growth timescale, as we discuss below.

In all three figures, we see the same broad features: while pebble accretion operates for a broad range of particle sizes, there exists an “optimal” range near \( \text{St} = 1 \) (\( r_p \approx 10 \text{ cm} \)) at which accretion reaches its maximal possible rate, and the growth timescale becomes comparable to \( t_{\text{Hill}} = M/(2R_{\text{Hill}}^2\Omega_{\text{Hill}}) \). In all three models, optimum accretion at \( \text{St} \approx 1 \) does not begin until the core has reached a “large enough” mass. Furthermore, as the core mass increases, the core’s growth rate grows as well, in large part due to the increasing size of the core’s Hill radius. The growth timescale, however, increases as the core grows, because the core’s growth rate scales with \( M \) to a power less than one.

In our and OK10’s modeling, we see two other notable features: first, there exists a gap in the range of particle sizes that can be accreted for low core masses. These particles cannot accrete in our framework because their incoming kinetic energies are too high. Second, on the right-hand side of the plots (\( r_p \gtrsim 100 \text{ cm} \)), particles have \( R_{\text{Hill}} < R_{\text{gas}} \), \( R_{\text{Hill}} \) (i.e., \( R_{\text{acc}} = R_{\text{Hill}} \)), but accretion is inefficient. This is due to the fact that these particles cannot fully dissipate their kinetic energy during a single orbital crossing; however, because of the chaotic nature of trajectories for particles with impact parameter \( \gtrsim R_{\text{Hill}} \), their growth timescale is increased by a factor of \( KE/W \) (see Section 4.1.1 for more detail).

As can be verified from the plots, in the regions of parameter space where our model predicts that pebble accretion should operate, our results agree (within factors of a few) with both of these analytic models. The main disagreements are:

1. The mass scale above which accretion occurs on timescales comparable to \( t_{\text{Hill}} \) is not exactly the same in the models. As discussed previously, our model has a transition in behavior past \( v_H/v_{\text{gas}} \approx 48^{-1/3} \). In JL17’s model, the parameters that set \( t_{\text{grow}} \) change for a mass \( M > M_c \), where \( M_c = (1/3)v_{\text{gas}}^3/(G\Omega) \). This is equivalent to \( v_H/v_{\text{gas}} = (1/3)^{1/3} \), which is clearly comparable to the cut
2. As discussed above, in our modeling, accretion shuts off for particles below a certain radius, which stems from the fact that, once \( R_{\text{lab}} \) shrinks below \( R_0 \), we expect particles to flow around the core’s atmosphere (see Figure 1, particularly the lower left-hand panel). The regions where this effect occurs are denoted by red hatching. In these regions, we expect growth by all mechanisms to be inhibited, e.g., capture by gravitational focusing or even physical collisions with the core will not occur. (In contrast, in the white regions of the plot, we still expect capture mechanisms other than gas-assisted growth to operate.) Because this process is set by the modification of the flow by the core’s gravity, we would not expect OK10’s integrations to capture it, because they assume a constant headwind velocity in their integrations. LJ12’s simulations do not appear to reach high enough core masses with small enough particle sizes to capture this effect.\(^{14}\) This effect does, however, appear to be consistent with the results of Ormel (2013), who find minimal accretion for small particle sizes at high core masses (see their Figure 12).\(^{15}\)

3. JL17 define a “weak coupling” regime, where the stopping time of the particle exceeds the time to pass the protoplanet, \( t_p = GM/(v_{\text{gas}} + v_{H})^2 \). For \( v_{\text{gas}} \gg v_{H} \), this criterion is equivalent to \( St > 3v_{H}^3/v_{\text{gas}} \), which is in line with our and OK10’s criterion \( St > 12v_{H}^3/v_{\text{gas}}^2 \). For \( v_{H} \gg v_{\text{gas}} \), the passing time criterion reduces to \( St > 3 \), which is again similar to our \( St > 4^{1/3} \) limit. For particle sizes exceeding this critical Stokes number, JL17 exponentially smooth \( R_{\text{acc}} \) to zero, which differs from our and OK10’s modeling.

4. In the lower right-hand region, OK10’s results and ours diverge. In this regime, particles that shear into \( R_{\text{H}} \) cannot fully dissipate their kinetic energy over one orbital crossing. This makes them accrete probabilistically, in accordance with Equation (36); i.e., instead of shutting off accretion because \( W < KE \) in this regime, we increase the growth timescale by a factor \( KE/W \). For particles with \( v_{\text{acc}} \sim v_{H} \), which holds for larger cores, this increases the growth timescale by a factor of \( St \), which agrees with OK10’s expression for \( R_{\text{acc}} \) in this regime: \( R_{\text{acc}} = R_{H}/St \). However, for the low-mass cores in the bottom right-hand corner of the plot, the particle-gas relative velocity dominates over the core’s gravity, changing the Stokes number dependence.

Thus, while our results are generally in order-of-magnitude agreement with other modeling in the laminar regime, there are few notable regions where our results diverge. It can be shown analytically that the agreement between our model and the other two essentially stems from the fact that, to order of magnitude, many of the transitions between length and velocity regimes occur in the same region of parameter space. For example, the transitions where \( R_{\text{H}} = R_{\text{WS}} \) and \( v_{\text{ac}}, t = v_{H} \) both occur for \( St \approx St_{\text{acc}} \). Therefore, in the laminar case, there are essentially fewer regimes that need to be covered.

When turbulence is included, however, this is no longer the case. As an example, Figure 9 compares our modeling to the pebble accretion modeling of Chambers (2014). The expressions used in Chambers are analogous to the OK10 expressions, with the replacement of OK10’s \( \zeta_c \equiv \eta v_{\text{k}}/\nu_p \) parameter with \( v_{\text{acc}}/v_{\text{gas}} \), where \( v_{\text{gas}} = \max(v_{H}, \sqrt{\alpha c_e}) \). In our comparison, we neglect the exponential smoothing term they carry over from OK10, as well as terms involving eccentricity and inclination, as they are not included in our model.\(^{16}\)

As can be seen from the figure, for low core masses and small body sizes, our models are in order-of-magnitude agreement. Again, analytic calculations can be used to demonstrate that, for low core masses and small particle sizes, the expressions between the two models agree to order of magnitude. For example, the cutoff in Stokes number employed by Chambers—\( St_{\text{crit}} = 12(v_{H}/v_{\text{gas}})^3 \) is replicated in our model for small cores and small particle radii. However, as can also be seen in the figure, there are numerous regimes covered in our modeling that are not captured by extending the OK10 expressions. A prominent example is the feature in the lower right-hand corner, where low-mass cores can accrete “large” particles (~10^{-3}–10^3 cm) on rapid timescales. This feature results from the fact that, for low-core masses, the velocities of these larger particles are set by their interactions with the nebular gas. Because these large particles are not well-coupled to the gas flow, they have low kinetic energies relative to the core, but high velocities relative to the gas, allowing a certain range of particle radii to dissipate an amount of energy during the encounter that is comparable to their kinetic energy. The shape of this feature depends on the transitions where \( KE \sim W \) as well as where \( R_{\text{WS}} \sim R_{\text{H}} \) and \( v_{\text{ac}} \sim v_{H} \). In the presence of turbulence, these transitions no longer need occur at similar regions of parameter space, leading to a more complex model of pebble accretion that is not well-captured by extending the OK10 expressions.

5.6. Comparison to Xu et al. (2017)

We close this section by comparing the results of our model to numerical simulations of pebble accretion in the presence of turbulence. Xu et al. (2017) performed magnetohydrodynamic (MHD) simulations of gas-assisted growth of turbulence due to the magnetorotational instability (MRI). These simulations provide an excellent point of comparison for our order-of-magnitude model, because they use much more detailed physics but apply over a narrower range of parameter space.

\(^{14}\) LJ12’s 1e-1.0.01 simulation, which is the highest core mass and smallest particle size they simulate, is actually marginal in the sense that \( R_{\text{acc}} = R_0 \) in our notation. While there is a reduction in accretion in this run, it is unclear whether accretion is actually substantially reduced in the way we would expect in our modeling. It is also unclear whether LJ12’s resolution is fine enough to resolve the core’s atmosphere and therefore the modification of the flow pattern.

\(^{15}\) In the bottom panel of Figure 12 of Ormel (2013), almost all particle streamlines are unable to accrete for \( St = 10^{-3} \) (in our notation). For \( St = 10^{-3} \), however, particles over a range of impact parameters are able to accrete. For the parameters used by Ormel (2013), the particle size where \( R_{\text{lab}} (= R_{\text{WS}} \text{ in this region of parameter space}) \) is less than \( R_0 \) is \( St \approx 3 \times 10^{-4} \).

\(^{16}\) The Chambers (2014) modeling of the velocity of small bodies induced by turbulence appears to use expressions for the rms particle—particle relative velocity, as opposed to the velocity of small bodies relative to the local Keplerian orbital velocity. This may be an error in the text, as other expressions used by these authors to model the relative velocity between a small body and an embryo (e.g., their Equation (8)), are expressions for a particle relative to the local Keplerian velocity. In particular, their expression tends toward 0 for small Stokes number particles, whereas we would expect that small, well-coupled particles move relative to the embryo at velocities comparable to the rms turbulent gas velocity (assuming the embryos are decoupled from the gas, which is a good approximation given that \( M_{\text{emb}} \geq 5 \times 10^{-6} M_\odot \) in their work). This appears to underestimate the incoming velocities of small bodies for strong turbulence (@ \( \alpha \geq 0 \)). Therefore, our comparison with Chambers (2014) uses the expression \( v_{\text{lab}} = \alpha c_e/(1 + St) \) in place of their Equation (9).
As we will show below, our model agrees with the Xu et al. results to order of magnitude, as we would expect. Xu et al. perform six sets of simulations; they use two different values of core mass, at three different levels of turbulence for each core. The core masses are parameterized in terms of the “thermal mass,” which Xu et al. give as

\[
\frac{M}{M_\odot} = \left( \frac{a}{30 \text{ au}} \right)^{3/4}.
\]

Note that this is the same as the flow isolation mass (see Section 7) for the values used in their work. The simulations are performed at core masses of \( M_\odot \approx \frac{160}{31} \) and \( M_\odot \approx \frac{310}{31} \). The three levels of turbulence are achieved by performing a pure hydrodynamic simulation, a non-ideal MHD simulation with ambipolar diffusion, and an ideal MHD simulation. Thus far in our work, we have assumed that \( \alpha \approx \alpha_z \), where \( \alpha_z \) is the diffusion coefficient for the particle scale height, but Xu et al. are able to separately calculate \( \alpha \) and \( \alpha_z \). We also use their value for \( v_z \) to calculate our \( \alpha \) value, as we use \( \alpha \) to parameterize the turbulent gas velocity. For the pure hydrodynamic run, they have \( v_z = \alpha_z = 0 \). For the ambipolar diffusion run, they give \( \left( \frac{v_z}{c_s} \right)^2 = 3.0 \times 10^{-4} \) and \( \alpha_z = 7.8 \times 10^{-4} \), whereas they give \( \left( \frac{v_z}{c_s} \right)^2 = 1.21 \times 10^{-2} \) and \( \alpha_z = 4.4 \times 10^{-3} \) for the ideal MHD simulation.

A comparison between our analytic model and their simulations is shown in Figure 10. This figure plots \( k_{\text{abs}} \equiv M/\dot{M}_{\text{Hill}} \) as a function of the particle Stokes number \( St \). Here, \( M = M/\dot{M}_{\text{grow}} \) is the growth rate of the core, and \( \dot{M}_{\text{Hill}} = M/\dot{M}_{\text{Hill}} \) is the growth rate for Hill accretion. The solid lines depict the model presented in this paper, while the data points show the results from the Xu et al. paper. For the purposes of matching their model, we have used a temperature profile consistent with their results, as well as their values of \( \alpha_z \) for the calculation of \( H_p \) (as opposed to using \( \alpha \)), and set the scale height in the laminar case to be \( H_{p,\text{lam}} = 0.01 H_q \) to be consistent with their choice.

As can be seen in Figure 10, our model nicely achieves the intended goal of reproducing the trends found by the numeric results, as well as being accurate to within a factor of two for all of the values found in the numeric simulations. It is worth noting here that the simulations are carried out at quite large values of core mass, and serve mostly to confirm our prediction that the efficiency of pebble accretion is not hugely reduced by increasing turbulence. It would be interesting to investigate in future simulations whether the drop-off in efficiency for lower core masses predicted in our model (see Section 6.1) is reproduced in the numeric simulations.

6. Exploration of Parameter Space

In this section, we give an overview of the effects of varying a few of the most important parameters in our model. In the first three sections, we discuss varying parameters related to...
separations, only a small range of particle sizes, if any, accrete on timescales comparable to \( t_{\text{Hill}} \).

To illustrate how core mass and orbital separation affect the growth timescale, we plot \( t_{\text{grow}} \) as a function of both \( r_h \) and \( M \) for a core at \( a = 5 \) au in Figure 11, and for a core at \( a = 50 \) au in Figure 12. To begin, we note that growth is generally slower at wide orbital separation, because the dynamical time \( t_{\text{dyn}} \sim \Omega^{-1} \) is larger and the solid surface density is smaller. For low core mass, there is also a gap in the particle sizes that can be accreted. This gap is small at low turbulence, but as \( \alpha \) increases, the width of this gap increases as well. This effect is enhanced at wide orbital separation, where the gap extends to higher core masses and is overall wider. Thus, low-mass cores can often only accrete at small pebble radii. These small particles are accreted on slow timescales that are \( \gg t_{\text{disk}} \), as small pebbles can only be captured at low impact parameters and have low densities because they can be easily lofted by turbulence. Thus, we see that growth is much slower at low core mass and wide orbital separation, particularly as the strength of turbulence is increased. If there exists a population of \( \gg 10 \) m “boulders,” then we see from the figures that there exists a narrow range of these larger particle sizes for which accretion can proceed on rapid timescales, even if accretion of pebble-sized particles is slow.

Once the core exceeds a certain critical mass, the gap disappears and accretion proceeds much more rapidly, allowing pebbles to accrete on timescales comparable to \( t_{\text{Hill}} \). To emphasize this effect, in Figure 13 we plot the range of particle sizes that can accrete on timescales within a factor of two of \( t_{\text{Hill}} \). Each panel depicts a different core mass, while different hatching patterns depict different levels of turbulence. As expected from the above discussion, for low core mass, only close-in cores with low levels of nebular turbulence can accrete particles on timescales comparable to \( t_{\text{Hill}} \). As core mass increases, however, particles farther out in the disk can be accreted with growth timescales \( \sim t_{\text{Hill}} \), even when turbulence is strong.

We shall now discuss more quantitatively what sets the range of particle sizes that can be accreted, as well as how that range is affected by changing the core mass and orbital separation. For low-mass cores accreting small particles, we expect \( K_{\text{WS}} \) to set the impact parameter. Because the particles are so small, and therefore well-coupled to the gas, we will have \( v_{\text{pk}} \approx v_{\text{gas}} \) and \( v_{\text{pg}} \ll v_{\text{kick}} \). Thus, in this regime, we expect \( \alpha_{\infty} \approx \alpha_{\text{gas}} \) (because shear is negligible for low-mass cores) and \( \alpha_{\text{enc}} \approx \alpha_{\text{kick}} \) (because \( v_{\text{gas}} < v_{\text{orbit}} \) in this regime). Plugging these values into our expressions for \( KE \) and \( W \), we can show that having \( KE < W \) requires

\[
St < 12\frac{v_H^3}{v_{\text{gas}}^3},
\]

which was given previously in Equation (71). This is one facet that makes accretion less efficient for low-mass cores, high turbulence, and wide orbital separation—because \( v_H/\sqrt{\alpha_{\text{gas}}} \propto M^{1/3}a^{-1/2}/\alpha^{-1/2} \) in the laminar regime (i.e., approximating \( v_{\text{gas}} \approx \alpha_{\text{gas}} \)) and \( \propto M^{1/3}a^{-1/2}/\alpha^{-1/2} \) in the turbulent regime \( (v_{\text{gas}} \approx \sqrt{\alpha_{\text{gas}}}) \), decreasing core mass, increasing \( \alpha \), or increasing orbital distance will all make the limit on \( St \) more stringent.

As small body radius increases, \( v_{\text{kick}} \) decreases due to the increasing size of the core’s WISH radius, while \( v_{\text{pg}} \) rises as particles decouple from the gas. Eventually, we reach the point

Figure 10. A comparison of the growth rate \( k_{\text{abs}} \), from our model and the MHD simulations of Xu et al. (2017), plotted as a function of the Stokes number \( St \). Results are given for three different values of \( \alpha \), representing different levels of turbulence. The solid lines show the output from our model, while the data points show the Xu et al. results.
where \( v_{pg} > v_{kick} \), meaning that now \( v_{enc} = v_{pg} \). In this regime, the dependence of the energy on small body radius becomes more complex: throughout a large amount of parameter space, \( KE/W \) increases with particle size, which encapsulates the fact that it is more difficult for heavier particles to dissipate their kinetic energy. If \( v_{enc} = v_{pg} \), however, the work done during the encounter increases strongly with small body radius, which actually makes it easier for large particles to be accreted. Furthermore, if \( v_{S} = \infty \), then the incoming \( KE \) of particles can also decrease with small body radius, making it even easier for heavier particles to be accreted.

This effect is illustrated in Figure 14, which plots the ratio \( KE/W \) as a function of \( r_s \) for different core masses. The two different panels show cores at different orbital separations. For the low-mass cores in the right-hand panel, we see that once \( r_s \) increases past a certain value, the qualitative behavior of \( KE/W \) changes—instead of monotonically increasing, the slope flattens out or even decreases. However, for the very low-mass cores, this behavior is inconsequential because particles are ruled out from accreting before we reach a large enough particle size that \( v_{enc} = v_{pg} \) and this more complex behavior starts. For larger core masses, however, this is no longer the case, and the range of particle sizes that can be accreted is greatly extended. This is what causes the gap seen in the range of accreted particle sizes to disappear—once particles with \( v_{enc} = v_{pg} \) become available, accretion generally continues until the limit \( St = 4\sqrt{3} \) discussed in Equation (61) is reached. From comparison of the two panels, we also see that this effect is much more prominent at wide orbital separations—at 1 au, none of the cores exhibit the change in slope seen at 50 au.

Thus, we see that growth changes qualitatively once particles with \( v_{pg} > v_{kick} \) can be accreted. The critical value of mass where this occurs can be approximated by calculating the mass at which \( v_{pg} > v_{kick} \) is reached before the critical Stokes number given in Equation (76) is reached. If we keep the approximations that led to Equation (76) but take \( v_{pg} \approx 2v_{gas}St \) in the laminar regime and \( v_{pg} \approx v_{gas} \sqrt{St} \) in the turbulent regime, we can solve for the Stokes number past which \( v_{pg} > v_{kick} \). Doing so and setting the resulting Stokes number equal to (76) yields a mass limit that can, in both regimes, be approximated by

\[
\frac{v_{L}}{v_{gas}} \approx 48^{-1/3}. \tag{77}
\]

This inefficiency of pebble accretion at lower core masses is also identified by Visser & Ormel (2016), who calculate the growth timescale of planetesimals by numerically integrating the pebble equation of motion in the presence of two different laminar gas flow patterns. Visser & Ormel find that pebble accretion is only faster than gravitational focusing (which they refer to as the “Safronov regime”) once the growing planetesimal exceeds a certain radius, \( R_{PA} \). They also find that this critical radius increases further out in the disk, which is also consistent with our results. It can be seen from our analytic expressions, however, that while these effects are present in our
model even in the laminar case, they are strongly amplified by the presence of turbulence.

6.2. Effects of Varying $T_0$

In this section, we consider the effects of varying the prefactor to the temperature profile, $T_0$. While changing the disk temperature will affect the properties of the small bodies, such as number density and composition, by changing what volatile species are present in solids at a given location, we neglect such effects in what follows. The consequences of changing the temperature profile are complex and depend on the local disk parameters as well as the core mass and strength of turbulence. Nevertheless, in general, increasing the temperature is detrimental to the accretion rate when gas-dominated processes set the scales relevant to growth. For growth at $H_{\text{Hill}}$, however, none of the scales depend on temperature, so the most rapid growth timescales are unaffected by increasing the disk temperature.

The primary effect of varying $T_0$ on growth timescales is due to the dependence of $c_s$ on $T$: $c_s \propto T^{1/2}$. Because $H_{\text{KH}} \propto c_s^2$, and $H_r \propto c_s$, increasing $T_0$ will increase $H_p$. Similarly, $v_{\text{pk},t} \propto c_s^2$ and $v_{\text{pk},t} \propto c_s$, so increasing $T_0$ will also increase $v_{\infty}$ in the drift-dominated regime. Finally, from inspection of Equation (40), we see that $T_0$ affects $R_{\text{WS}}$ by changing $\rho_0 \left(\propto c_s^{-3}\right)$, $v_{\text{th}}$, and $v_{\text{gas}}$. By inspecting the scaling of $R_{\text{WS}}$ one can show that $R_{\text{WS}}$ is a decreasing function of $T_0$.

While some of the above effects increase $t_{\text{grow}}$, while other decrease it, the general effect of increasing $T_0$ is to slow down growth. To see this, we can make the same approximations described in Equations (64)–(66). As in Section 5.3, we consider the 2D and 3D regimes separately.

In the 2D regime where $H_p < R_{\text{acc}}$, $H_{\text{acc}} = R_{\text{acc}}$, so we have $t_{\text{grow}} \propto (R_{\text{acc}} v_{\infty})^{-1}$. Having $R_{\text{acc}} > H_p$ implies that $v_{\text{shear}} = R_{\text{acc}} \Omega > v_{\text{pk}}$, which in turn implies that $R_{\text{shear}} < R_{\text{WS}}$. Thus, we expect that $t_{\text{grow}}$ is independent of $T_0$ in this regime.

For 3D accretion, we have $H_p > R_{\text{acc}}$, which implies that $v_{\infty} = v_{\text{pk}}$. Therefore we have $t_{\text{grow}} \propto (R_{\text{WS}}^2 \Omega)^{-1}$, which is an increasing function of $T_0$. We therefore expect that (approximately) higher $T_0$ should lead to slower growth. An example of the difference caused by increasing $T_0$ is shown in Figure 15.

A secondary effect of increasing $T_0$ is that $t_r$ and $S_t$, which depend on $\rho_0$, $v_{\text{th}}$, and $\lambda$, are also affected. However, inspection of Equation (13) shows that the effects of $T_0$ on $t_r$ cancel out in the Epstein regime. Therefore, $T_0$ only affects $S_t$ in the fluid regime. This substantially diminishes the importance of this dependence, because the small body sizes that grow the most efficiently are in the Epstein regime for a large amount of parameter space. For particles that are in the fluid regime, higher values of $T_0$ will shift values of $S_t$ to lower values of $r_s$.

Because of the approximate cancellation between $v_{\infty}$ and $H_p$ in the 3D regime, the maximal increase in timescale that can be provided by increasing $T_0$ is generally of order $t_{\text{hot}}/t_{\text{cold}} \sim R_{\text{acc,cold}}^2/R_{\text{acc,hot}}^2$, where hot and cold denote the

Figure 12. Growth timescale as a function of core mass, for $a = 50$ au. The red-hatched region denotes where growth is completely shut off in our modeling, whereas the white regions show places where gas-assisted growth will not operate, but the core can still grow by other means (e.g., gravitational focusing).
Higher and lower values of $T_0$ respectively. Because $RH$ generally represents an upper limit on $R_{\text{acc}}$ (ignoring accretion in the $R_{\text{acc}} = R_\text{g}$ regime), the maximal increase is of order $(R_{\text{WS,cold}}/R_{\text{WS,hot}})^2$, which can be of order $(c_s,\text{hot}/c_s,\text{cold})^3 = (T_{0,\text{hot}}/T_{0,\text{cold}})^{3/2}$ in the laminar Stokes and Ram regimes. In practice, this maximal value is rarely reached because it requires satisfying $r_s > 9\lambda/4$ to be in the fluid regime, while simultaneously having $St < v_{\text{gas}}^2/(3\nu)$ in order to have $R_{\text{WS}} < R_{\text{shear}}$. In practice, therefore, the increase in growth timescale generally goes as $T_{0,\text{hot}}/T_{0,\text{cold}}$. Because accretion timescales at $t_{\text{Hill}}$ are not dependent on temperature, increasing $T_0$ does not affect the rapid growth rates supplied by gas-assisted growth.

Changing $T_0$ will also have an effect on the range of particle sizes that the growing core is able to accrete. For small particle sizes, the accretion cutoff is determined by the point where $R_b = R_{\text{WS}}$. Because $R_b \propto c_s^{-2}$, which is stronger than the dependence of $R_{\text{WS}}$ on $c_s$ regardless of drag regime, increasing $T_0$ will decrease the size of the Bondi radius relative to the WISH radius. This, in turn, will allow the core to accrete smaller-sized bodies. For larger particle sizes, the effect of increasing $T_0$ is not as clear-cut. For particles accreting at $t_{\text{Hill}}$, the only effect of increasing $T_0$ is to increase the drag force on accreting small bodies, which allow the core to accrete larger sizes. In other regimes, however, increasing $T_0$ can also increase the approach velocity of small bodies. In these cases, the increase in kinetic energy of accreting particles outweighs the larger amount of work done, and the maximal size of small bodies accreted is reduced.
6.3. Growth in a Gas-depleted Disk

In this section, we explore growth in a disk where the gas density has been reduced by a factor of 100, but the solid surface density is unchanged; i.e., we have \( \Sigma = \Sigma_{g,0}(a/\text{au})^{-1} \), where \( \Sigma_{g,0} = \Sigma_0/100 \), and \( \Sigma_0 = 500 \text{ g cm}^{-2} \) is the prefactor employed elsewhere in this paper for the gas surface density. However, we keep \( \Sigma_{r,0} = 5 \text{ g cm}^{-2} \). We note that this may affect some of the expressions used in this work, which implicitly assume \( \rho_s \gg \rho_g \) where \( \rho_g \) is the volumetric mass density of the small bodies. We neglect any such effects in what follows. These choices for gas and solid densities are made to emulate the conditions in the disk when the gas component of the disk is in the process of photoevaporating, which is an important stage in some theories of planet formation (e.g., Lee & Chiang 2016; Freilikh & Murray-Clay 2017). As we shall show below, the predominant effect of reducing the gas surface density is to shift the range of small body sizes where accretion occurs to lower values.

An example of growth in a depleted disk is shown in Figure 16. As can be seen in the figure, the predominant effect of changing \( \Sigma \) is to shift growth down to lower values of \( r_s \). This is due to the fact that the quantities that go into calculating \( t_{\text{grow}} \), even the energy criteria, are functions of \( r_s \) through their dependence on \( S_t \) alone. In the Epstein regime, \( t_s \propto r_s^2/\rho_s \), and in the Stokes regime, \( t_s \propto r_s^3/\rho_s \), so the radius corresponding to a given Stokes number decreases when the surface density, and correspondingly the volumetric density, are decreased. This is what causes the shift to lower radii seen in Figure 16. Other than this shift, however, there is essentially no change in the growth timescale. In other words, when the timescale is viewed as a function of Stokes number, i.e., if we consider \( t_{\text{grow}}(St) \) as opposed to \( t_{\text{grow}}(r_s) \), then this function is independent of \( \Sigma \).

The sole caveat is that, for particles not in a linear drag regime, the Stokes number of a particle is now dependent on the particle’s velocity, which means that the Stokes numbers used for calculating different quantities might not be the same. For example, we can express the WISH radius as \( R_{\text{WISH}} = \sqrt{GMSt/(v_{\text{gas}})\Omega} \), but the \( St \) value in that expression is defined with respect to \( v_{\text{rel}} = v_{\text{gas}} \), meaning it will not be the same as the Stokes number for, e.g., \( v_{\text{rel}} \). In this case, the simple argument that all quantities are solely dependent on \( S_t \) breaks down. In practice, if we use, e.g., the Stokes number defined with respect to laminar drift velocities for comparison purposes, the discrepancy between the two surface densities is minor. Furthermore, if we write the critical radius dividing the fluid and diffuse drag regimes, \( r_s = 9\lambda/4 \), in terms of Stokes number, it is straightforward to show that the particle is in the fluid regime for

\[
St_{\text{crit}} > 7.4 \times 10^{-2}(a/\text{au})^{23/7}\left(\frac{\Sigma_{g,0}}{500 \text{ g cm}^{-2}}\right)^{-2}.
\] (78)

Because of the strong scaling of \( St_{\text{crit}} \) with \( a \), the differences between the two surface densities disappear entirely for \( a > 5 \text{ au} \).

Thus, in general, the range of small body sizes where accretion is effective will shift to lower values as the gaseous component of the disk dissipates. The subsequent effects of such a shift are quite sensitive to the underlying size distribution of the small bodies. A particularly salient issue here is whether radius or Stokes number controls the processes that produce the size distribution, as could be the case if, e.g.,
6.4. Effects of Varying Stellar Mass

In this section, we discuss the dynamical effect of varying \( M_* \). Clearly, higher-mass stars will have a higher overall temperature, and should on average have higher surface densities as well. Because we have discussed those effects in previous sections, in this section we consider only the effect of varying \( M_* \), leaving the other properties of the disk unchanged.

Changing \( M_* \) has an effect on the growth process solely though its effect on the local Keplerian orbital frequency \( \Omega \) and the size of the Hill radius \( R_H \). Higher-mass stars have lower dynamical times, which tends to speed up growth. On the other hand, in the presence of more massive stars, the Hill radius of a growing planet will shrink as it becomes more difficult to hold on to accreting material, which clearly inhibits growth. The interplay between these two factors will determine the net effect of varying the stellar mass—which effect dominates, and therefore whether changing stellar mass is beneficial or detrimental to growth, depends on the other input parameters.

An example is shown in Figure 17. The figure shows a plot of \( t_{\text{grow}} \) versus \( r_s \) for five different stellar masses; the panels shows the growth timescale for two different levels of turbulence. Many of the main features of varying \( M_* \) are visible in the figure. For the \( \alpha = 0 \) case, small particles (\( r_s \lesssim 0.1 \) mm) are accreted much more rapidly by the more massive stars. In this regime, particles have \( R_{\text{acc}} = R_{\text{WS}} \) and accrete in 3D, with scale height \( H_p = H_{\text{KH}} \). Thus, the primary effect here of varying stellar mass is to change \( \eta_{\text{WS}} \) and \( \rho_s \) along with \( \Omega \). In the Epstein regime, however, these effects on both \( R_{\text{WS}} \) and \( S_t \) cancel out, and because \( H_{\text{KH}} \sim 2\eta_{\text{WS}}/\Omega \), the increase in growth rate in this regime scales roughly as \( \Omega \sim M_*^{1/2} \).

Eventually, the particle size increases to the point that \( R_{\text{shear}} < R_{\text{WS}} \), which also implies that particles are now shear-dominated. Because this change roughly corresponds to where we shift from 3D to 2D in the lamination regime, the effect of increasing stellar mass in this regime is actually to \textit{increase} the growth timescale, \( t_{\text{grow}} \propto (R_{\text{shear}}^2 \Omega)^{-1} \sim S_t^{-3/2} R_H^{-2} \Omega^{-1} \sim M_*^{1/6} \), because \( S_t \) is independent of \( M_* \) in the Epstein regime. This is what causes the overlap in \( t_{\text{grow}} \) curves as \( r_s \) increases. Due to the fact that changing \( M_* \) can switch the regime that determines one of our growth parameters (e.g., shear- versus dispersion-dominated) the maximal change in growth rate for two stellar masses \( M_{*,1} \) and \( M_{*,2} \) in this regime can be complicated. In general, however, the change is of order \( \Omega_1/\Omega_2 = (M_{*,1}/M_{*,2})^{1/2} \), as can be seen in Figure 17.

As particle size increases, the cores reach the point where they can accrete particles at \( R_{\text{acc}} = R_H \). As can be seen in Figure 17, the value of \( r_s \) where this transition occurs is a decreasing function of \( M_* \) due to the decreased size of the Hill radius at higher stellar mass. Figure 17 also shows that increasing stellar mass initially decreases the growth timescale, but further increasing the mass of the star actually increases the growth timescale. This is due to the fact that decreasing \( M_* \) increases \( H_{\text{KH}} \) affecting whether the largest particles are accreting in the 2D or 3D regime. Low-mass stars will accrete in 3D, in which case increasing \( M_* \) will decrease the growth timescale by bringing the particle density down and the rate of shear up. Once the stellar mass becomes sufficiently large, however, we will have \( R_H > H_{\text{KH}} \), and the core will accrete in 2D. In this case, increasing \( M_* \) will slow down growth by decreasing the size of the Hill radius.

As turbulence increases, the difference between the various stellar masses for small particle size disappears. This effect
occurs due to a balance between the decreased growth rate from decreasing encounter rate and an increased growth rate due to an increased particle density, because we generally accrete in 3D in this regime. If particles are drift-dispersion dominated, then \( v_{sc} = v_{pk} \approx v_{gas} \) and \( R_{acc} = R_{WS} \). When \( \alpha = 0 \), \( R_{WS} \) is no longer independent of \( M_\star \). If \( \alpha > \eta \), then \( v_{gas} \) is approximately constant with respect to varying \( \Omega \), meaning that the quantity in the denominator of \( t_{grow} - R_{WS} v_{pk} \propto M_\star^{-1/2} \). Furthermore, if \( \eta_{\text{shear}} > v_{pk} \), then \( R_{acc} = R_{\eta_{\text{shear}}} \), and the denominator of \( t_{grow} \) is \( \propto R^3_{\eta_{\text{shear}}} \propto \Omega^{-1} \propto M_\star^{-1/2} \). For sufficiently strong turbulence that \( H_{\text{turb}} > H_{\text{KH}} \), we have \( H_0 \propto \Omega^{-1} \propto M_\star^{-1/2} \). Thus, these effects approximately cancel out for 3D accretion in the strong turbulence regime, causing the growth curves to merge.

For high core masses, the transition on the right side of the graphs where \( t_{grow} \) increases as a function of \( r_c \) occurs at essentially the same value of small body radius, independent of \( M_\star \). As noted in Equation (61), the energy criterion \( KE < W \) can be rewritten as \( St < 4\sqrt{3} \) for particles in this growth regime. For particles in the Epstein regime the Stokes number is independent of \( M_\star \), causing the cutoff radius to be approximately the same for all stellar masses. The timescale in this regime is weakly dependent on \( M_\star \)—these large particles accrete at \( t_{\text{Hill}} \), with an increase in growth timescale \( \propto St \), which again is essentially independent of \( M_\star \). Thus, the growth timescale goes as \( t_{grow} \propto (R_{H VH}^{-1}) \propto M_\star^{1/6} \).

In the laminar regime, the cutoff at small radii is also independent of \( r_c \). This cutoff occurs when \( R_{WS} < R_{\rho} \), and because, as previously noted, \( R_{WS} \) is independent of \( M_\star \) in the laminar regime, this cutoff is independent of \( M_\star \) as well. As turbulence increases, \( R_{WS} \) becomes a decreasing function of \( M_\star \), causing the cutoff in growth to shift to higher values of \( r_c \).

For low core masses, the low-end cutoff still has the same dependence on \( M_\star \). However, growth will now cut off at the limit given by Equation (76), which is an increasing function of \( M_\star \) in both the laminar and turbulent regimes.

7. Flow Isolation Mass

In this section, we discuss accretion in the regime \( R_b > R_H \). In general, the core’s atmosphere extends up to a scale \( R_{\text{atm}} = \min(R_b, R_H) \), as once \( R_b > R_H \), \( R_{\text{atm}} \) is limited by gravitational effects from the central star, as opposed to the core’s ability to bind nebular gas. Thus, for core masses large enough that \( R_b > R_H \), the energy criteria discussed in Section 2.1 are the same with \( R_b \rightarrow R_H \).

Figure 11 shows the emergence of a feature for high core masses where only larger sizes of particles can accrete. This occurs when the core reaches a \( M_{\text{flow}} \) such that \( R_b = R_H \). As discussed in Section 2.1, in the regime \( R_b > R_H \), small bodies with \( R_{\text{stab}} < R_b \) will not be able to accrete if they dissipate their kinetic energy during their interaction with the core—because the gas will flow around the core’s approximately incompressible static atmosphere, particles that couple to the gas flow near the core will be pulled away without accreting. For lower-mass cores, only particles that are small enough that \( R_{WS} < R_b \) are restricted from accreting in this manner, i.e., this consideration dictates the lower limit on the particle size that can be accreted. Once the core’s mass is large enough that \( R_b > R_H \), however, we now have \( R_{\text{atm}} = R_b \), and therefore \( R_{\text{stab}} \lesssim R_{\text{atm}} \) for all particle sizes. Thus, any particles that dissipate their kinetic energy relative to the core will be pulled around by the gas flow without accreting. Because \( R_{\text{atm}} = R_H \), we expect the gas flow to be around \( R_H \) instead of \( R_b \) as it was in the lower-mass case. See Figure 18 for an illustration.

We note that the gas will be accelerated by the core’s gravity as it passes through \( R_b \); interior to \( R_b \), the local orbital velocity exceeds the sound speed, which means the flow can accelerate to supersonic velocities. In this case, it is less clear that the core’s atmosphere will act as an incompressible obstacle. However, there still should exist a scale \( r \) past which the flow can no longer penetrate the core’s atmosphere. To see this, we can compare the incoming kinetic energy of the flow to the binding energy of the core’s atmosphere. In a time \( \Delta t \), the mass of gas entering into the length scale \( r \) is of order \( \rho_{\text{neb}} v_{\text{app}} \Delta t r^3 \), where \( \rho_{\text{neb}} \) is the volumetric mass density of the nebula and \( v_{\text{app}} \) is the velocity of the incoming gas relative to the core. Because \( R_b \gtrsim R_H \) implies that \( v_H < c_s < v_{\text{gas}} \), we have \( v_{\text{app}} \gtrsim v_H \). The timescale for gas to enter \( r \) is \( \Delta t \sim \rho_{\text{neb}} v_{\text{app}} r^3 / \rho_{\text{atm}} v_{\text{esc}}^2 r^3 \). Thus, the ratio of the incoming kinetic energy to the binding energy is

\[
\frac{\rho_{\text{neb}} v_{\text{app}} r^3}{\rho_{\text{atm}} v_{\text{esc}}^2 r^3} \lesssim \frac{\rho_{\text{neb}} r^3}{3 \rho_{\text{atm}} R_H^2},
\]

where \( v_{\text{app}} \ll v_H \) and \( v_H^2 / v_{\text{esc}}^2 = r/(3R_H) \). The quantity above is clearly \( \ll 1 \) for \( r \lesssim R_H \), particularly because, close to the planet, we expect \( \rho_{\text{atm}} \gg \rho_{\text{neb}} \). Thus, there exists a scale \( r \lesssim R_H \) where the incoming kinetic energy of the gas is much less than the binding energy of the core’s atmosphere.

The core reaching \( M_{\text{flow}} \) signals a rapid cutoff in the accretion of pebbles: for masses just below \( M_{\text{flow}} \), there will always exist a wide range of particle sizes that dissipate their kinetic energy during interaction with the core. Once \( M > M_{\text{flow}} \), accretion of these particle sizes, which generally represent the most rapid accretion rates, suddenly shuts off. We can demonstrate analytically that a broad range of particle sizes satisfying \( KE < W \) will be present for \( M = M_{\text{flow}} \). To begin, we first show that, at this mass scale, we never have \( R_{\text{stab}} = R_{WS} \). First, \( R_b = R_H \) implies that \( v_{\text{app}} = c_s / \sqrt{3} \). The
analytic criterion for the relative sizes of $R_b$, $R_{WS}$, and $R_{\text{shear}}$ for $R_b = R_H$, can therefore be summarized as:

$$R_{\text{shear}} < R_{WS}: \quad St > \frac{1}{\sqrt{3}} \left( \frac{v_{\text{gas}}}{c_s} \right)^3$$

$$R_{WS} > R_b: \quad St > \frac{1}{\sqrt{3}} \left( \frac{v_{\text{gas}}}{c_s} \right).$$

Because $v_{\text{gas}}/c_s \ll 1$, the above implies that $R_{\text{stab}} = R_{\text{shear}}$ occurs prior to accretion commencing for $R_{WS} > R_b$. Thus, accretion commences for $R_{\text{shear}} > R_b$, which occurs at a Stokes number of

$$St > 9 \left( \frac{v_H}{c_s} \right) = \frac{1}{3}.$$  \hspace{1cm} (80)

On the other hand, accretion will cease when $KE > W$. Because these particles are clearly shear-dominated, we will simply have our criterion given in Equation (61) for particles that shear into $R_H$

$$St < 4\sqrt{3}.$$  \hspace{1cm} (81)

Thus, any particle in the range $1/3 < St < 4\sqrt{3}$ will be accreted for masses just below $M_{\text{flow}}$, but for $M > M_{\text{flow}}$ particles in this range will no longer be accreted, i.e., $M > M_{\text{flow}}$ represents a general point throughout parameter space where accretion of small bodies through pebble accretion cuts off.

Solving $R_b = R_H$ for $M$ gives $M_{\text{flow}}$ as

$$M_{\text{flow}} = \frac{1}{\sqrt{3}} \frac{c_s^3}{G\Omega}. \hspace{1cm} (82)$$

Plugging in fiducial values of our parameters gives

$$M_{\text{flow}} = 4.4 M_\oplus \left( \frac{T_0}{200 \text{ K}} \right)^{3/2} \left( \frac{a}{\text{au}} \right)^{6/7} \left( \frac{M_\oplus}{M_{\odot}} \right)^{-1/2}.$$  \hspace{1cm} (83)

For the temperature profile used in this work, the value of flow isolation mass is markedly similar to the distribution of solar system cores. Figure 19 shows the flow isolation mass, scaled down by a factor of four (dashed line), both plotted as a function of semimajor axis. Also shown are the masses of the cores of the solar system planets. Values for the four giant planets are taken from Guillot (2005). For the terrestrial planets, the total mass is shown. For the gas giants, the mass of the core is used, whereas the total mass in solids is plotted for the ice giants.

The flow isolation mass may be most relevant in the context of the ice giant planets, which were previously thought to form by growing to their local isolation mass, though note that the ice giants are thought to have migrated substantially from the location of their initial formation (reviewed, e.g., in Morbidelli...
et al. 2008). In the context of pebble accretion, however, planets are not limited by locally available material due to radial drift of solids. Flow isolation mass could provide a plausible mechanism that sets the mass of these planets in the context of pebble accretion (see Frelikh & Murray-Clay 2017 for a more in-depth discussion).

A similar mass is identified by Lambrechts & Johansen (2014), who refer to it as the “pebble isolation mass.” We strongly emphasize, however, that the existence of the pebble isolation mass is based on different physics than our flow isolation mass: the pebble isolation mass is based on the gravity of the planet opening a gap in the pebble disk, as opposed to the planet altering the local flow of nebular gas. Lambrechts & Johansen (2014) calculate the pebble isolation mass by identifying the point where gravitational perturbations from the growing core’s gravity render the gas velocity immediately outside of the planet’s orbit “super-Keplerian,” which tends to push pebbles outward rather than bring them in. Lambrechts & Johansen perform numerical simulations at 5 au in order to determine the pebble isolation mass at this orbital separation. They then analytically calculate the dependence of $M_{\text{flow}}$ on the disk aspect ratio, $H/a$, and determine that $M_{\text{flow}} \propto (H/a)^3$. Combining these results, Lambrechts & Johansen give the pebble isolation mass as

$$M_{\text{flow}} \approx 20 M_\odot \left( \frac{a}{5 \text{ au}} \right)^{3/4}.$$  \hspace{1cm} (84)

Despite the different physics used to calculate these mass scales, the flow and pebble isolation masses occur at roughly the same value—to illustrate this, we first note that because $H/a \propto c_s / (\Omega a)$, we have $(H/a)^3 \propto T^{3/2} a^{3/2}$. Thus, our scaling agrees with their result. Furthermore, if we use a temperature in agreement with their choice, $T \approx 270$ K ($a/\text{au}$)$^{1/2}$, then our criterion for pebble isolation mass, i.e., that $R_b = R_H$, gives

$$M_{\text{flow}} = 23.2 \left( \frac{a}{5 \text{ au}} \right)^{3/4} M_\odot,$$ \hspace{1cm} (85)

in rough agreement with their result. Finally, we note that the flow isolation mass is similar in scale to the “thermal mass” discussed in Section 5.6, past which the core’s Hill radius exceeds the disk scale height (Lin & Papaloizou 1993).

8. Summary and Conclusions

In this paper, we have presented an order-of-magnitude model of pebble accretion that accounts for the effects of turbulence on a variety of timescale parameters. We calculate the growth timescale for a planet as

$$t_{\text{grow}} = \frac{M R_p}{2 \sum v_{\text{c}} R_{\text{acc}} H_{\text{acc}}}.$$ \hspace{1cm} (86)

We calculate $v_{\text{c}}$, $R_{\text{acc}}$, $H_{\text{acc}}$, and $R_p$ separately, allowing a number of different physical processes to set each of these velocity and length scales. Our model uses the wind-shearing radius of Perets & Murray-Clay (2011) to take into account the effects of gas drag on the stability of small bodies during the accretion process, which can set $R_{\text{acc}}$ instead of $R_H$ or $R_s$. We also use the approximate formulae presented by Ormel & Cuzzi (2007) for the rms turbulent velocity of small bodies in a turbulent medium to incorporate the effects of turbulence into $v_{\text{c}}$. An incoming small body has its incoming kinetic energy compared to the work done by gas drag during the encounter, which determines the range of small body sizes that the core can accrete. The resulting model gives the growth timescale as well as a variety of other important parameters ($v_{\text{ke}}, H_{\text{ke}}, F_p, \ldots$). Due to its relative simplicity, our model can be applied over a large range of parameter space; it can also be coupled with other physical processes, such as planetary migration.

Studying the output of our model reveals many important aspects of protoplanetary growth via pebble accretion in the presence of turbulence. Once protoplanets reach a large enough size (ranging from $10^{-4}$ to $10^{-1} M_\odot$, depending on the strength of turbulence, as well as the location in the disk and the sizes of pebbles available), growth timescales become far shorter than the lifetime of the gaseous protoplanetary disk. This results remains true even for extremely strong ($\alpha \geq 10^{-2}$) turbulence. These enhanced growth rates are more than substantial enough to allow the formation of gas giants at large orbital separations, where planetesimal accretion is inefficient, provided the cores can first reach this high mass.

Of equal importance however, are the regions where pebble accretion is not as efficient. We find that turbulence can substantially lower the growth rate at low core masses. For lower core masses and stronger turbulence, a smaller range of particle sizes are able to accrete at $t_{\text{grow}}$, and it becomes easier for turbulence to entirely cut off gas-assisted growth for more massive particles. These effects are exacerbated at wide orbital separations, where the detrimental effects of the gas on the accretion process are more substantial. Thus, when studying the growth in pebble accretion, it is important to consider not just the maximal accretion efficiency where the core accretes over its Hill radius, but also what sizes of small bodies are available and how these small bodies are affected by their interactions with the gas. These effects can have considerable ramifications for the predictions of any theory of planet formation by pebble accretion.

While we have used fiducial values of disk parameters in order to provide concrete numeric results in this paper, our model is quite flexible with regard to the disk parameters used. We briefly discussed the effects of modifying a few of these parameters, namely the temperature and surface density profiles, as well as the stellar mass. While these effects are complicated, we can briefly summarize them as follows:

1. For higher temperatures, accretion generally slows down. The timescale for the most rapid accretion, where particles accrete over the entirety of the core’s Hill sphere, is unaffected by disk temperature, though the range of particle sizes that can accrete at this rate may shrink in hotter disks.

2. Depleting the gas surface density shifts the correspondence between Stokes number and small body radius, essentially shifting the curve of $t_{\text{grow}} (\tau)$ to smaller values of particle size. The scale of $t_{\text{grow}}$ is unchanged.

3. In a laminar disk, increasing stellar mass makes accretion of smaller particle radii, which accrete at $R_{\text{acc}} = R_{\text{WS}}$, more efficient due to the increased gas density and shear rate. In a turbulence disk, the growth rate for small particle radii is insensitive to stellar mass. For larger particle sizes where $R_{\text{acc}} = R_H$, the effect of increasing stellar mass is not as clear-cut, but the overall effect is much less significant than in the small-radius case.
Finally, we identified a natural upper limit to core mass in the context of pebble accretion: the “flow isolation mass.” Past this mass, the Bondi radius of the core will exceed its Hill radius, in which case the region where the core’s gravity alters the gas flow exceeds the radius for stable orbits about the core, regardless of the small body size being accreted. In this regime, the normal mechanism for pebble accretion will not be able to operate, because particles that dissipate their kinetic energy relative to the core will follow the flow of gas and be pulled around the core without accreting. This upper limit may set the critical mass that triggers runaway gas accretion because, once cores reach this limit, their accretion luminosity will drop, allowing them to accrete substantially more gas from the surrounding nebula. We note that the value of the flow isolation mass as a function of semimajor axis is quite similar to the distribution of the cores of solar system planets. While this is an intriguing possibility, further study is needed to determine the importance of this mass scale.

An interesting and important extension to our model would be to consider lower-mass cores, for which the effects of gas are more pronounced. This would require modeling not just the effects of the rms random velocity of the small bodies, but also the particle–particle relative velocity between the small body and the core. Expressions for the turbulence-induced relative velocity are given in, e.g., OC07, but these formulae are much more complex than Equation (22). Another effect that may be more important for lower masses is consideration of the full probability density function of the particle–particle relative velocity. In this work, we have used only the rms value for velocity, but considering particles in the low- and high-velocity tails of the distribution can have important effects, as has been noted for early stages of growth (Windmark et al. 2012). Our preliminary investigations of gas-assisted growth at planetesimal sizes show that many novel features appear in this regime that are not present in the higher-mass case—for example, the range of particle sizes that planetesimals can accrete efficiently can be much narrower than the range for protoplanets. This effect could lead to stratification in the composition of planetesimals, which could be observed in our solar system. Furthermore, for low masses, the actual collision velocity between the core and small body can be smaller than their initial relative velocity, due to the inspiral of the particle, which can have important ramifications for whether a given collision results in growth, bouncing, or fragmentation.

There are a large number of applications for our model to address issues in planet formation. In Paper II we present one such application: namely, we apply our theory to the question of formation of gas giants at wide orbital separation. We are particularly interested in how the strength of turbulence can help place restrictions on when gas giant formation is possible, which may help us understand why large-separation gas giants are so uncommon even though pebble accretion timescales are so rapid.

The authors wish to thank Diana Powell, Renata Frelikh, and John McCann for their useful discussion, and Eugene Chiang for his thoughtful suggestions on the manuscript. We also thank the anonymous referee for their helpful comments, which improved the quality of the manuscript. M.M.R. and R.A.M.C. acknowledge support from NSF CAREER grant No. AST-1555385. H.B.P. is supported by the Israel science foundation I-CORE grant 1829/12 and the MINERVA center for life under extreme planetary conditions.

Appendix A

Summary of Calculation Algorithm

In this appendix, we summarize the recipe for calculating the growth timescale.

Our model takes in five input parameters: $M_s$, $a$, $M$, $r_s$, and $\alpha$. Once these parameters are specified, we can calculate the parameters of the protoplanetary disk at the given orbital separation (see Table 2). We then need to calculate the Stokes number of the small body $St = t_e\Omega$. Particles with $r_s < 9\lambda/4$ (which applies over most of parameter space) are in the Epstein regime, which allows us to immediately calculate their stopping time:

$$t_{e,eps} = \frac{\rho_g r_s}{\rho_p v_{th}}.$$  

(87)

For $r_s > 9\lambda/4$, the stopping time is calculated numerically. We begin by setting $v_{pg} = v_{gas} = \sqrt{\alpha \sigma_s^2 + \eta^2 v_p^2}$. We then calculate the drag force on the particle using

$$F_D = \frac{1}{2}C_D(Re)\pi r_s^2 \rho_g v_p v_{pg},$$  

(88)

where

$$C_D(Re) = \frac{24}{Re} (1 + 0.27Re^{0.43}) + 0.47[1 - \exp(-0.04Re^{0.38})].$$  

(89)

and $Re = 4r_sv_{pg}/(v_{th}\lambda)$. The stopping time is then given by

$$t_s = \frac{mv_{pg}}{F_D},$$  

(90)

where $m$ is the mass of the small body. Using this stopping time, we can recalculate $v_{pg}$. The relevant equations are

$$v_{pg,\ell} = v_{pg} St_4 \sqrt{\frac{4 + St^2}{1 + St^2}}$$  

(91)

and

$$v_{pg,\ell} = \begin{cases} \sqrt{\alpha c_s \left( \frac{St^2 (1 - Re_j^{-1/2})}{(St' + 1)(St' + Re_j^{-1/2})} \right)^{1/2}}, & St' < 10 \\ \sqrt{\alpha c_s \left( \frac{St'}{1 + St'} \right)^{1/2}}, & St' \geq 10. \end{cases}$$  

(92)

Here, $Re_j \equiv \alpha c_s H_g/(v_{th}\lambda)$ and $St' \equiv t_e/t_{eddy}$, where

$$t_{eddy} = \sqrt{\frac{\Omega}{1 + \left( \frac{\eta v_{pg}}{v_{pg}\ell} \right)^2}}.$$  

(93)

The total velocity relative to the gas is

$$v_{pg} = \sqrt{v_{pg,\ell}^2 + v_{pg,\ell}'}.$$  

(94)

Using this new velocity, we can recalculate $F_D$ and obtain a new value of $t_e$. We then iterate this process until we obtain the desired accuracy for $St$.  

30
Once $St$ is known, we calculate the length scales needed to determine the growth timescale. We first calculate $R_{\text{stab}}$:

$$R_{\text{stab}} = \min(R_{WS}, R_{\text{shear}}, R_H).$$

(95)

Here, $R_H$ is the planet’s Hill radius,

$$R_H = \left(\frac{M}{3M_*}\right)^{1/3},$$

(96)

and $R_{WS}$ is the planet’s WISH radius,

$$R_{WS} = \sqrt[3]{\frac{GMm}{F_D(V_{\text{gas}})}},$$

(97)

For a particle in a linear drag regime, there is a simple analytic expression for $R_{WS}$:

$$R_{WS} = R_H\sqrt{3St}\left(\frac{v_H}{v_{\text{gas}}}\right).$$

(98)

$R_{\text{shear}}$ is the shearing radius, which is the solution to the equation

$$R_{\text{shear}} = \sqrt{\frac{GMm}{F_D(R_{\text{shear}}\Omega)}}.$$

(99)

In nonlinear drag regimes, we solve this equation numerically. For a particle in a linear drag regime, the above equation has the analytic solution

$$R_{\text{shear}} = R_H(3St)^{1/3}.$$  

(100)

Using $R_H$ and the planet’s Bondi radius

$$R_b = \frac{GM}{c_s^2},$$

(101)

we calculate $R_{\text{atm}}$:

$$R_{\text{atm}} = \min(R_b, R_H),$$

(102)

which in turn tells us the impact parameter for accretion:

$$R_{\text{acc}} = \max(R_{\text{stab}}, R_{\text{atm}}).$$

(103)

The scale height of the small bodies is determined by the Kelvin–Helmholtz scale height

$$H_{\text{KH}} = \frac{2\nu_s}{\Omega} \min(1, St^{-1/2}),$$

(104)

and turbulent scale height

$$H_t = H_s \min\left(\sqrt[3]{\frac{\alpha}{St}}, 1\right).$$

(105)

We take $H_p$ to be:

$$H_p = \max(H_t, H_{\text{KH}}).$$

(106)

Using $R_{\text{acc}}$ and $H_p$, we can determine $H_{\text{acc}}$:

$$H_{\text{acc}} = \min(R_{\text{acc}}, H_p).$$

(107)

We now calculate the approach velocity of the small bodies, $\nu_{\infty}$. The laminar and turbulent components of the velocity due the particle’s interactions with the gas are given by

$$v_{pk,t} = \nu_{pk,t} \frac{\sqrt{1 + 4St^2}}{1 + St^2}.$$

(108)

and

$$v_{pk,t} = \begin{cases} \sqrt{\alpha} c s \left(1 - \frac{St'^2(1 - Re^{-1/2})}{(St' + 1)(St' + Re^{-1/2})}\right)^{1/2}, & St' < 10 \\ \sqrt{\alpha} c s \frac{1}{1 + St'}, & St' \geq 10 \end{cases}$$

(109)

where again $St' = \nu/\nu_{\text{eddy}}$, with $\nu_{\text{eddy}}$ given by Equation (93). The total velocity is again given by

$$v_{pk} = \sqrt{v_{pk,t}^2 + v_{pk,t}'^2},$$

(110)

The value of $\nu_{\infty}$ is then given by

$$\nu_{\infty} = \max(v_{pk}, v_{\text{shear}}),$$

(111)

where $v_{\text{shear}} = R_{\text{acc}}\Omega$.

We now have enough information to calculate the growth timescale $t_{\text{grow}}$:

$$t_{\text{grow}} = \frac{MH_p}{2f_s \Sigma \nu_{\infty} R_{\text{acc}} H_{\text{acc}}}.$$  

(112)

In order to determine whether a particle can accrete, we calculate its incoming kinetic energy

$$KE = \frac{1}{2} m\nu_{\infty}^2,$$

(113)

and the work done by gas drag

$$W = 2F_D(\nu_{\text{enc}})R_{\text{acc}}.$$  

(114)

Here, $\nu_{\text{enc}}$ is the velocity of the particle during its interaction with the core

$$\nu_{\text{enc}} = \begin{cases} \max(\nu_{\text{orbit}}, \nu_{pk}), & \nu_{\infty} < \nu_{\text{orbit}} \\ \max(\nu_{\text{kick}}, \nu_{pk}), & \nu_{\infty} > \nu_{\text{orbit}} \end{cases},$$

(115)

where $\nu_{\text{orbit}} = \sqrt{GM/R_{\text{acc}}}$, and $\nu_{\text{kick}} = GM/(R_{\text{acc}}\nu_{\infty})$. Particles can accrete if

1. $R_{\text{stab}} > R_b$ and $W > KE$
2. $R_{\text{stab}} < R_b$ and $W < KE$
3. $R_{\text{stab}} = R_H$ and $\nu_{\infty} = v_H$.

If the particle does not fall into any of these regimes, then we set $t_{\text{grow}} = \infty$. In case 3, the growth timescale is given by

$$t_{\text{grow}} = \frac{t_{\text{grow}}'}{\min(1, W/KE)},$$

(116)

where $t_{\text{grow}}'$ is given by Equation (112).

**Appendix B**  
**Derivation of Velocity Formulae in Different Frames**

In this appendix, we give detailed derivations of the equations given in Section 3.4. In what follows, we assume that the turbulence is ergodic, i.e., that time averaging the system is equivalent to ensemble averaging, and that the turbulence is a stationary process.

The methods for calculating turbulent velocity in, e.g., Voelk et al. (1980) begin by decomposing the total velocity $v$ into time-averaged and fluctuating components, a technique known
as “Reynolds averaging.” (See Cuzzi et al. 1993, Appendix A). That is, we take \( \mathbf{v} = \mathbf{\bar{v}} + \Delta \mathbf{v} \), such that \( \langle \Delta \mathbf{v} \rangle = 0 \), where \( \langle \ldots \rangle \) denotes ensemble averaging. Here, \( \mathbf{\bar{v}} \) is associated with the laminar component of velocity, and \( \Delta \mathbf{v} \) is associated with the turbulent velocity. We can use this same decomposition to determine how to combine the laminar and turbulent components, as well as how to compute the velocity after changing reference frames.

We first note that decomposing the velocity as above, taking the dot product of each side of the equation and time averaging, gives

\[
\langle \mathbf{v} \cdot \mathbf{v} \rangle = \langle \mathbf{\bar{v}} \cdot \mathbf{\bar{v}} \rangle + 2 \langle \mathbf{\bar{v}} \cdot \Delta \mathbf{v} \rangle \tag{117}
\]

\[
\langle \mathbf{\bar{v}} \cdot \mathbf{\bar{v}} \rangle = \langle \mathbf{\bar{v}} \cdot \mathbf{\bar{v}} \rangle + \langle \mathbf{\bar{v}} \cdot \Delta \mathbf{v} \rangle \tag{118}
\]

which is the same as Equation (25) and is used to combine the turbulent and laminar components of the velocity, which are calculated separately.

For the purposes of this problem, we are concerned with velocities relative to two frames: velocities relative to the total gas velocity are needed for the calculation of drag forces, while velocities relative to the local Keplerian velocity are needed to determine the rate at which small bodies encounter large ones. If the subscripts \( p, g, k \) denote the velocity of the small bodies, the gas, and the Keplerian velocity, respectively, then we may write

\[
\mathbf{v}_{pk} = \mathbf{v}_{pg} + \mathbf{v}_{kg}. \tag{119}
\]

Reynolds averaging (119) gives

\[
\mathbf{\bar{v}}_{pk} = \mathbf{\bar{v}}_{pg} + \mathbf{\bar{v}}_{kg}. \tag{120}
\]

Thus, the laminar component of the particle’s velocity can be changed from one frame to another in the usual manner, independent of the turbulent velocity, as is done in the main text (see Equation (26)).

To compute how the turbulent velocity changes between frames, we require the equation of motion for the grains. Following Voelk et al. (1980), for a particle in a linear drag regime \((F_D \propto v)\), we can write

\[
\frac{d \mathbf{v}_{pk}}{dt} = \mathbf{a}_s - \mathbf{v}_{pg}, \tag{121}
\]

where \( \mathbf{a}_s \) is the acceleration due to forces other than gas drag, such as gravity. Reynolds averaging and subtracting the result from (121) gives

\[
\frac{d}{dt} \langle \mathbf{\delta v}_{pk} \rangle = -\langle \mathbf{\delta v}_{pg} \rangle. \tag{122}
\]

where we have assumed that \( \mathbf{a}_s \) only varies on large spatial scales, so \( \langle \mathbf{a}_s \rangle = 0 \). Subtracting Equations (120) from (119) and rearranging slightly gives

\[
\langle \mathbf{\delta v}_{kg} \rangle = \langle \mathbf{\delta v}_{pk} \rangle - \langle \mathbf{\delta v}_{pg} \rangle. \tag{123}
\]

Taking the dot product and time averaging gives

\[
\langle \langle \mathbf{\delta v}_{kg} \rangle \rangle = \langle \mathbf{\delta v}_{pk} \rangle^2 + \langle \mathbf{\delta v}_{pg} \rangle^2 - 2 \langle \mathbf{\delta v}_{pk} \cdot \mathbf{\delta v}_{pg} \rangle. \tag{124}
\]

Plugging Equation (122) into the last term on the right-hand side gives

\[
\langle \mathbf{\delta v}_{kg} \rangle^2 = \langle \mathbf{\delta v}_{pk} \rangle^2 + \langle \mathbf{\delta v}_{pg} \rangle^2 - 2 \langle \mathbf{\delta v}_{pk} \cdot \mathbf{\delta v}_{pg} \rangle + 2 \langle \mathbf{\delta v}_{pg} \rangle - 2 \langle \mathbf{\delta v}_{pg} \rangle. \tag{125}
\]

\[
= \langle \mathbf{\delta v}_{pk}^2 \rangle + \langle \mathbf{\delta v}_{pg}^2 \rangle + \langle \mathbf{\delta v}_{pg} \rangle - 2 \langle \mathbf{\delta v}_{pk} \cdot \mathbf{\delta v}_{pg} \rangle. \tag{126}
\]

For a stationary process, the last term will be zero, so we have

\[
\langle \mathbf{\delta v}_{kg}^2 \rangle = \langle \mathbf{\delta v}_{pk}^2 \rangle + \langle \mathbf{\delta v}_{pg}^2 \rangle, \tag{127}
\]

which is used in our model to convert the turbulent component of the velocity from the frame relative to the gas to the frame relative to Keplerian (cf. Equation (27)).

### Appendix C

#### Canonical Core Accretion Timescale

In this appendix, we give a short summary of how growth proceeds for cores accreting particles for which the gas drag force is negligible. See Goldreich et al. (2004) for a more in-depth review of these processes. Because small bodies in this regime cannot dissipate their kinetic energy through gas drag, in order for accretion to occur, the impact parameter of the incoming particle must be small enough that it collides with the core. The maximum impact parameter at which a particle will be gravitationally focused into a collision with the core is given by

\[
R_{\text{focus}} = R \left( 1 + \frac{v_{\text{esc}}^2}{v_{\text{circ}}^2} \right)^{1/2}, \tag{128}
\]

where \( R \) is the radius of the core and \( v_{\text{esc}} = \sqrt{2GM/R} \) is the escape velocity from the core. The scale height of small bodies is given by the vertical component of their velocity dispersion, \( H_v \sim v_v / \Omega \).

While there are a number of gas-free growth regimes, and therefore timescales, we confine our attention to the regime where the velocity dispersion of the small bodies is approximately the Hill velocity, \( v_v = R_{\text{Hill}} \Omega \). Here, \( R_{\text{Hill}} \) is the core’s Hill radius (see Section 4), and \( \Omega \) is the local Keplerian orbital frequency. This regime gives the highest possible growth rate without invoking some external mechanism to bring the velocity dispersion below the Hill velocity, because interactions with the core will drive bodies up to the Hill velocity. In this regime, the core can accrete over the entirety of its Hill radius, so \( \sigma \approx 4R_{\text{focus}} \). If we set \( v_{\text{circ}} \approx v_v \approx v_H \) and note that \( v_{\text{esc}} / v_H \sim (R_{\text{Hill}} / R)^{1/2} \), then we see that \( t_{\text{GF}} \) is of order

\[
t_{\text{GF, } v=\text{Hill}} \sim \frac{M}{2f_{\Sigma} R_{\text{Hill}} \Omega} \left( \frac{R_{\text{Hill}}}{R} \right). \tag{129}
\]

Scaled to our fiducial values (see Section 5.1), the timescale is given by

\[
t_{\text{GF}} \approx 7 \times 10^5 \text{ years} \left( \frac{a}{a_{\text{au}}} \right)^{3/2} \left( \frac{M}{M_\odot} \right)^{1/3}. \tag{130}
\]

If there are \( \sim \text{km} \) sized objects available, then cores may grow by gravitational focusing in addition to gas-assisted growth. Furthermore, if the gravitational focusing timescale is shorter than the gas-assisted growth timescale for a given set of parameters, then gravitational focusing will be the dominant mechanism of growth, which can cause cores to grow in regimes where gas-assisted growth is slow.
Appendix D
Details of Turbulent Velocity Calculation

In this appendix, we describe some details of the calculation of the velocity of small bodies due to turbulence. The results of Voelk et al. (1980) and all work derived from these results, particularly Equation (22), rest on the assumption that there is a well-defined stopping time for the particle, independent of the particle’s velocity. Thus, Equation (22) may not hold when the particle enters the Ram pressure gas drag regime. However, Equation (21) holds whenever the particle’s stopping time is large enough that it receives many “kicks” from the largest-scale eddies before a stopping time has elapsed. In other words, Equation (21) holds when $St > 1$. For $Re > 1$, the particle will be in the Ram pressure regime, which is quadratic in velocity, so we need only be concerned about the validity of our approximation if we have $Re > 1$ before $St > 1$.

Figure 20 shows a plot of the Reynolds number of particles as a function of semimajor axis and Stokes number. We have restricted the figure to show $St < 1$, because we expect Equation (21) to hold to reasonable accuracy for larger values of $St$. In our plot, we indicate the region where $Re > 25$, which we take as the approximate region where accuracy of our model is in question. The plot is for $\alpha = 10^{-1}$, which is the most restrictive case; for lower values of $\alpha$, the region where $St < 1$ and $Re > 25$ shrinks.

The laminar velocity also affects the turbulent velocity of the particle. This effect can be qualitatively understood by considering the fact that a laminar component to the particle’s velocity decreases the amount of time that the particle interacts with a turbulent eddy of a given wave number $k$. The original Voelk et al. (1980) result is dependent on the value of what they call $k^*$, which is the divide between two kinds of eddies: those that are large enough for the particle to come into equilibrium with the eddy, and those that either decay or are traversed by the particle before they have a substantial frictional effect. To order of magnitude, the relative velocity between the particle and the gas is simply equal to the velocity of the eddy to which the particle is marginally coupled, i.e., the velocity of the eddy with wavenumber $k^*$. Because the presence of a laminar component to the velocity will affect which eddies the small bodies can drift through over a stopping time, introducing a laminar velocity will change the value of $k^*$, which in turn has important effects on the rms turbulent velocity as a function of particle size. OC07 therefore refer to the effects of a laminar component of velocity as “eddy-crossing effects.”

OC07 neglect the effect of a laminar component on the particle velocity, noting that it is only important in the weakly turbulent regime for small Stokes numbers. As we are interested here in varying the strength of turbulence and determining its effect on planetary growth rates, the weakly turbulent regime is pertinent.

Youdin & Lithwick (2007) note that, in the regime where eddy-crossing effects are non-negligible, we can approximate these effects by using an effective large-eddy turnover time, $t_{edd}'$, given by

$$t_{edd}' = \frac{t_L}{\sqrt{1 + (v_{rel}/v_L)^2}}, \quad (131)$$

where $v_{rel}$ is the laminar component of the particle’s velocity, measured relative to the rms gas velocity.

In conclusion, we calculate the turbulent component of the small body’s velocity by combining Equations (21) and (22), with the large-eddy turnover time $t_{edd}$ modified by Equation (131).

References

Agnor, C. B., Canup, R. M., & Levison, H. F. 1999, Icar, 142, 219
Alibert, Y. 2017, arXiv:1705.06008
Andrews, S. M. 2015, PASP, 127, 961
Andrews, S. M., Wilner, D. J., Hughes, A. M., Qi, C., & Dullemend, C. P. 2009, ApJ, 700, 1502
Batchelor, G. 2000, An Introduction to Fluid Dynamics, First Cambridge Mathematical Library (Cambridge: Cambridge Univ. Press)
Benz, W., Slattery, W. L., & Cameron, A. G. W. 1988, Icar, 74, 516
Binney, J., & Tremaine, S. 2008, Galactic Dynamics (Princeton, NJ: Princeton Univ. Press)
Birnstiel, T., Klahr, H., & Ercolano, B. 2012, A&A, 539, A148
Blum, J., & Wurm, G. 2008, ARA&A, 46, 21
Bottke, W. F., Vokrouhlický, D., Marchi, S., et al. 2015, Sci, 348, 321
Bowler, B. P. 2016, PASP, 128, 102001
Brandt, T. D., McElwain, M. W., Turner, E. L., et al. 2014, ApJ, 794, 159
Canup, R. M. 2012, Sci, 338, 1052
Carballido, A., Fromang, S., & Papaloizou, J. 2006, MNRAS, 373, 1633
Chambers, J. E. 2014, Icar, 233, 83
Chauvin, G., Vigan, A., Bonnefoy, M., et al. 2015, A&A, 573, A127
Cheng, N. S. 2009, Powder Tech., 189, 395
Chiang, E., & Youdin, A. N. 2010, AREPS, 38, 493
Csanady, G. T. 1963, JAtS, 20, 201
Cuzzi, J. N., Dobrovolskis, A. R., & Champney, J. M. 1993, Icar, 106, 127
Cuzzi, J. N., & Hogan, R. C. 2003, Icar, 164, 127
D’Alessio, P., Calvet, N., & Hartmann, L. 2001, ApJ, 553, 321
Dodson-Robinson, S. E., Veras, D., Ford, E. B., & Beichman, C. A. 2009, ApJ, 707, 79
Dohnanyi, J. S. 1969, JGR, 74, 2531
Doubile, B., Mortill, G., & Sterzik, M. 1995, Icar, 114, 237
Freilich, R., & Murray-Clay, R. A. 2017, AJ, 154, 98
Fung, J., Artymowicz, P., & Wu, Y. 2015, ApJ, 811, 101
Galicher, R., Marois, C., Macintosh, B., et al. 2016, A&A, 594, A63
Goldreich, P., Lithwick, Y., & Sari, R. 2002, Nat, 420, 643

ORCID iDs

M. M. Rosenthal https://orcid.org/0000-0003-3938-3099
R. A. Murray-Clay https://orcid.org/0000-0001-5061-0462
H. B. Perets https://orcid.org/0000-0002-5004-199X
Goldreich, P., Lithwick, Y., & Sari, R. 2004, ARA&A, 42, 549
Grishin, E., & Perets, H. B. 2015, ApJ, 811, 54
Grishin, E., & Perets, H. B. 2016, ApJ, 820, 106
Guillot, T. 2005, AREPS, 33, 493
Guillot, T., Ida, S., & Ormel, C. W. 2014, A&A, 572, A72
Hill, G. W. 1878, AnJM, 1, 5
Homann, H., Guillot, T., Bec, J., et al. 2016, A&A, 589, A129
Ida, S., Guillot, T., & Morbidelli, A. 2016, A&A, 591, A72
Inaba, S., & Ikoma, M. 2003, A&A, 410, 711
Johansen, A., & Lambrechts, M. 2017, AREPS, 45, 359
Kataoka, A., Tanaka, H., Okuzumi, S., & Wada, K. 2013, A&A, 557, L4
Kleine, T., Touboul, M., Bourdon, B., et al. 2009, GeCoA, 73, 5150
Kratter, K., & Lodato, G. 2016, ARA&A, 54, 271
Kratter, K. M., Murray-Clay, R. A., & Youdin, A. N. 2010, ApJ, 710, 1375
Lambrechts, M., & Johansen, A. 2012, A&A, 544, A32
Lambrechts, M., & Johansen, A. 2014, A&A, 572, A107
Lambrechts, M., Johansen, A., & Morbidelli, A. 2014, A&A, 572, A35
Lee, A. T., Chiang, E., Asay-Davis, X., & Barranco, J. 2010, ApJ, 718, 1367
Lee, E. J., & Chiang, E. 2016, ApJ, 817, 90
Leinhardt, Z. M., & Stewart, S. T. 2012, ApJ, 745, 79
Levison, H. F., Kretke, K. A., & Duncan, M. J. 2015, Natur, 524, 322
Lin, D. N. C., & Papaloizou, J. C. B. 1993, in Protostars and Planets III, ed. E. H. Levy & J. I. Lunine (Tucson, AZ: Univ. Arizona Press), 749
Markiewicz, W. J., Mizuno, H., & Voelk, H. J. 1991, A&A, 242, 286
Marois, C., Macintosh, B., Barman, T., et al. 2008, Sci, 322, 1348
Marois, C., Zuckerman, B., Konopacky, Q. M., Macintosh, B., & Barman, T. 2010, Natur, 468, 1080
Morbidelli, A., Lambrechts, M., Jacobson, S., & Bitsch, B. 2015, Icar, 258, 418
Morbidelli, A., Levison, H. F., & Gomes, R. 2008, in The Solar System Beyond Neptune, ed. M. A. Barucci et al. (Tucson, AZ: Univ. Arizona Press), 275
Nakagawa, Y., Sekiya, M., & Hayashi, C. 1986, Icar, 67, 375
Ormel, C. W. 2013, MNRAS, 428, 3526
Ormel, C. W., & Cuzzi, J. N. 2007, A&A, 466, 413
Ormel, C. W., & Klahr, H. H. 2010, A&A, 520, A43
Ormel, C. W., & Kobayashi, H. 2012, ApJ, 747, 115
Ormel, C. W., Shi, J.-M., & Kuiper, R. 2015, MNRAS, 447, 3512
Perets, H. B., & Murray-Clay, R. A. 2011, ApJ, 733, 56
Petit, J.-M., & Henon, M. 1986, Icar, 66, 536
Piso, A.-M. A., Youdin, A. N., & Murray-Clay, R. A. 2015, ApJ, 800, 82
Pollack, J. B., Hubickyj, O., Bodenheimer, P., et al. 1996, Icar, 124, 62
Powell, D., Murray-Clay, R., & Schlichting, H. E. 2017, ApJ, 840, 93
Rafikov, R. R. 2004, AJ, 128, 1348
Rafikov, R. R. 2006, ApJ, 648, 666
Rosenthal, M. M., & Murray-Clay, R. A. 2018, ApJ, submitted
Schlichting, H. E. 2014, ApJL, 795, L15
Shakura, N. I., & Sunyaev, R. A. 1973, A&A, 24, 337
Tognelli, E., Prada Moroni, P. G., & Degl'Innocenti, S. 2011, A&A, 533, A109
Visser, R. G., & Ormel, C. W. 2016, A&A, 586, A66
Voelk, H. J., Jones, F. C., Morfill, G. E., & Roeser, S. 1980, A&A, 85, 316
Weidenschilling, S. J. 1977, MNRAS, 180, 57
Windmark, F., Bärsch, T., Ormel, C. W., & Dullemond, C. P. 2012, A&A, 544, L16
Xu, Z., Bai, X.-N., & Murray-Clay, R. A. 2017, ApJ, 847, 52
Yin, Q., Jacobsen, S. B., Yamashita, K., et al. 2002, Natur, 418, 949
Youdin, A. N., & Lithwick, Y. 2007, Icar, 192, 588