NON-COLLISION SINGULARITIES IN THE PLANAR TWO-CENTER-TWO-BODY PROBLEM

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CONTENTS

1. Introduction 3
   1.1. Statement of the main result 3
   1.2. Motivations 4
   1.3. A glimpse of the 4-body problem 6
   1.4. Plan of the paper 6
2. Proof of the main theorem 6
   2.1. Idea of the proof 6
   2.2. Main ingredients 7
   2.3. Gerver map 8
   2.4. Asymptotic analysis, the local map 11
   2.5. Asymptotic analysis, the global map 12
   2.6. Admissible surfaces 12
   2.7. Construction of the singular orbit 14
3. Hyperbolicity of the Poincaré map 15
   3.1. Construction of invariant cones 15
   3.2. Expanding directions of the global map 18
   3.3. Checking transversality 24
4. $C^0$ estimates for global map 25
   4.1. Equations of motion in Delaunay coordinates 25
   4.2. A priori bounds 27
   4.3. Proof of Lemma 4.1 31
   4.4. Proof of Lemma 2.4 32
5. Derivatives of the Poincaré map 34
6. Variational equation 35

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Abstract. In this paper, we study a model of simplified four-body problem
called planar two-center-two-body problem. In the plane, we have two fixed
centers \( Q_1 = (-\chi, 0) \), \( Q_2 = (0, 0) \) of masses 1, and two moving bodies \( Q_3 \) and \( Q_4 \) of masses \( \mu \ll 1 \). They interact via Newtonian potential. \( Q_3 \) is captured by \( Q_2 \),
and \( Q_4 \) travels back and forth between two centers. Based on a model of Gerver,
we prove that there is a Cantor set of initial conditions which lead to solutions of
the Hamiltonian system whose velocities are accelerated to infinity within finite
time avoiding all early collisions. We consider this model as a simplified model
for the planar four-body problem case of the Painlevé conjecture.

1. Introduction

1.1. Statement of the main result. We study a two-center two-body problem.
Consider two fixed centers \( Q_1 \) and \( Q_2 \) of masses \( m_1 = m_2 = 1 \) located at distance
\( \chi \) from each other and two small particles \( Q_3 \) and \( Q_4 \) of masses \( m_3 = m_4 = \mu \ll 1 \).
\( Q_i \)s interact with each other via Newtonian potential. If we choose coordinates so
that \( Q_2 \) is at \((0,0)\) and \( Q_1 \) is at \((-\chi,0)\) then the Hamiltonian of this system can be
written as

\[
(1.1) \quad H = \frac{|P_3|^2}{2\mu} + \frac{|P_4|^2}{2\mu} - \frac{\mu}{|Q_3 - (-\chi,0)|} - \frac{\mu}{|Q_4 - (-\chi,0)|} - \frac{\mu^2}{|Q_3 - Q_4|}.
\]

We assume that the total energy of the system is zero.

We want to study singular solutions of this system, that is, the solutions which can
not be continued for all positive times. We will exhibit a rich variety of singular
solutions. Fix \( \varepsilon_0 < \chi \). Let \( \omega = \{\omega_j\}_{j=1}^\infty \) be a sequence of 3s and 4s.

Definition 1. We say that \((Q_3(t),Q_4(t))\) is a singular solution with symbolic
sequence \( \omega \) if there exists a positive increasing sequence \( \{t_j\}_{j=0}^\infty \) such that

- \( t^* = \lim_{j \to \infty} t_j < \infty \).
- \( |Q_3(t_j) - Q_2| \leq \varepsilon_0, \ |Q_4(t_j) - Q_2| \leq \varepsilon_0 \).
- If \( \omega_j = 4 \) then for \( t \in [t_{j-1}, t_j] \), \( |Q_3(t) - Q_2| \leq \varepsilon_0 \) and \( \{Q_4(t)\}_{t \in [t_{j-1}, t_j]} \)
winds around \( Q_1 \) exactly once.
- If \( \omega_j = 3 \) then for \( t \in [t_{j-1}, t_j] \), \( |Q_4(t) - Q_2| \leq \varepsilon_0 \) and \( \{Q_3(t)\}_{t \in [t_{j-1}, t_j]} \)
winds around \( Q_1 \) exactly once.
- \( |Q_1(t)| \to \infty \) as \( t \to t^* \).

During the time interval \([t_{j-1}, t_j]\) we refer to \( Q_{\omega_j} \) as the traveling particle and to
\( Q_{7-\omega_j} \) as the captured particle. Thus \( \omega_j \) prescribes which particle is the traveler
during the \( j \) trip.

We denote by \( \Sigma_\omega \) the set of initial conditions of singular orbits with symbolic se-
quence \( \omega \). Note that if \( \omega \) contains only finitely many 3s then there is a collision of
Q_3 and Q_2 at time t*. If ω contains only finitely many 4s then there is a collision of Q_1 and Q_2 at time t*. Otherwise at we have a collisionless singularity at t*.

**Theorem 1.** There exists µ_* ≪ 1 such that for µ < µ_* the set Σ_ω ≠ ∅.
Moreover there is an open set U in the phase space and a foliation of U by two-dimensional surfaces such that for any leaf S of our foliation Σ_ω ∩ S is a Cantor set.

**Remark 1.** By rescaling space and time variables we can assume that χ ≫ 1. In the proof we shall make this assumption and set ε_0 = 2.

**Remark 2.** It follows from the proof that the Cantor set described in Theorem 1 can be chosen to depend continuously on S. In other words Σ_ω contains a set which is local a product of a five dimensional disc and a Cantor set. The fact that on each surface we have a Cantor set follows from the fact that we have a freedom of choosing how many rotations the captured particle makes during j-th trip.

**Remark 3.** The construction presented in this paper also works for small nonzero energies. Namely, it is sufficient that the total energy is much smaller than the kinetic energies of the individual particles. The assumption that the total energy is zero is made to simplify notation since then the energies of Q_3 and Q_4 have the same absolute values.

**Remark 4.** One can ask if Theorem 1 holds for other choices of masses. The fact that the masses of the fixed centers Q_1 and Q_2 are the same is not essential and is made only for convenience. The assumption that Q_3 and Q_4 are light is important since it allows us to treat their interaction as a perturbation except during the close encounters of Q_3 and Q_4. The fact that the masses of Q_3 and Q_4 are equal allows us to use an explicit periodic solution of a certain limiting map (Gerver map) which is found in [G2]. It seems likely that the conclusion of Theorem 1 is valid if m_3 = µ, m_4 = cµ where c is a fixed constant close to 1 and µ is sufficiently small but we do not have a proof of that.

1.2. Motivations.

1.2.1. Non-collision singularity in N-body problem. Our work is motivated by the following fundamental problem in celestial mechanics. Describe the set of initial conditions of the Newtonian N-body problem leading to global solutions. The complement to this set splits into the initial conditions leading to the collision and non-collision singularities.

It is clear that the set of initial conditions leading to collisions is non-empty for all N > 1 and it is shown in [Sa1] that it has zero measure. Much less is known about the non-collision singularities. In particular the following basic problems are still open.

**Conjecture 1.** The set of non-collision singularities is non-empty for all N > 3.
Conjecture 2. The set of non-collision singularities has zero measure for all $N > 3$.

Conjecture 1 probably goes back to Poincaré who was motivated by King Oscar II prize problem about analytic representation of collisionless solutions of the $N$-body problem. It was explicitly mentioned in Painlevé’s lectures [Pa] where the author proved that for $N = 3$ there are no non-collision singularities. Soon after Painlevé, von Ziepel showed that if the system of $N$ bodies has a non-collision singularity then some particle should fly off to infinity in finite time. Thus non-collision singularities seem quite counterintuitive. However in [MM] Mather and McGehee constructed a system of four bodies on the line where the particles go to infinity in finite time after an infinite number of binary collisions (it was known since the work of Sundman [Su] that binary collisions can be regularized so that the solutions can be extended beyond the collisions). Since Mather-McGehee example had collisions it did not solve Conjecture 1 but it made it plausible. Conjecture 1 was proved independently by Xia [X] for the spacial five-body problem and by Gerver [G1] for a planar $N$ body problem where $N$ is sufficiently large. The problem still remains open for $N = 4$ and for small $N$ in the planar case. However in [G2] (see also [G3]) Gerver sketched a scenario which may lead to a non-collision singularity in the planar four-body problem. Gerver has not published the details of his construction due to a large amount of computations involved (it suffices to mention that even technically simpler large $N$ case took 68 pages in [G1]). The goal of this paper is to realize Gerver’s scenario in the simplified setting of two-center-two-body problem.

Conjecture 2 is mentioned by several authors, see e.g. [Sim Sa3 K]. It is known that the set of initial conditions leading to the collisions has zero measure [Sa1] and that the same is true for non-collisions singularities if $N = 4$. To obtain the complete solution of this conjecture one needs to understand better of the structure of the non-collision singularities and our paper is one step in this direction.

1.2.2. Well-posedness in other systems. Recently the question of global well-posedness in PDE attracted a lot of attention motivated in part by the Clay Prize problem about well-posedness of the Navier-Stokes equation. One approach to constructing a blowup solutions for PDEs is to find a fixed point of a suitable renormalization scheme and to prove the convergence towards this fixed point (see e.g. [LS]). The same scheme is also used to analyze two-center-two-body problem and so we hope that the techniques developed in this paper can be useful in constructing singular solutions in more complicated systems.

1.2.3. Poincaré’s second species solution. In his book [Po], Poincaré claimed the existence of the so-called second species solution in three-body problem, which are periodic orbits converging to collision chains as $\mu \to 0$. The concept of second species solution was generalized to the non-periodic case. In recent years significant progress was made in understanding second species solutions of both restricted [BM FNS] and full [BN] three-body problem. However the understanding of general second
species solutions generated by infinite aperiodic collision chains is still incomplete. Our result can be considered as a generalized version of second species solution. All masses are positive and there are infinitely many close encounters. Therefore the techniques developed in this paper can be useful in the study of the second species solutions.

1.3. A glimpse of the 4-body problem. Consider the same setting as in our main result but suppose that $Q_1$ and $Q_2$ are also free (not fixed). Then we can expect that during each encounter light particle transfers a fixed proportion of their energy and momentum to the heavy particle. The exponential growth of energy and momentum would cause $Q_1$ and $Q_2$ to go to infinity in finite time leading to a non-collision singularity.

A proof of this would however involve a significant amount of additional computation due to higher dimensionality of the full four-body problem. Indeed planar four-body problem has 16 dimensions since each particle has two position and two momentum coordinates. Removing the translation invariance we are left with 12 dimensions. Taking into account the rotation invariance leaves us with 10 dimensions. Energy conservation and taking a Poincaré section kills two more dimensions so we obtain a eight dimensional Poincaré map. We expect however that similarly to the problem at hand the Poincaré of the full four-body problem will have only two strongly expanding directions while other directions will be dominated by the most expanding ones. This would allow our strategy to extend to the full four-body problem leading to the complete solution of the Painlevé conjecture.

1.4. Plan of the paper. The paper is organized as follows. Section 2 and 3 constitute the main framework of the proof. In Section 2 we give a proof of the main Theorem 1 based on a careful study of the hyperbolicity of the properties of the Poincaré map. In Section 3 we summarize all later calculations and we prove the hyperbolicity results of Section 3. All the later sections provide calculations needed in Section 3. We define the local map to study the local interaction between $Q_3$ and $Q_4$ and global map to cover the time interval when $Q_4$ is traveling between $Q_1$ and $Q_2$. Sections 4, 6, 7 and 8 are devoted to the global map, while Sections 9, 10, and 12 study local map. Relatively short Sections 5 and 11 contain some technical results pertaining to both local and global maps. Finally, we have two appendices. In Appendix A, we include an introduction to the Delaunay coordinates for Kepler motion, which is used extensively in our calculation. In Appendix B, we summarize the information about Gerver’s model in [G2].

2. Proof of the main theorem

2.1. Idea of the proof. The proof of the Theorem 1 is based on studying the hyperbolicity of the Poincaré map. Our system has four degrees of freedom. We pick the zero energy surface and then consider a Poincaré section. The resulting Poincaré
map is six dimensional. In turns out that for orbits of interest (that is, the orbits where the captured particle rotates around \(Q_2\) and the traveler moves back and forth between \(Q_1\) and \(Q_2\)) there is an invariant cone family which consists of vectors close to a certain two dimensional subspace such that all vectors in the cone are strongly expanding. This expansion comes from the combination of shearing (there are long stretches then the motion of the light particles is well approximated by the Kepler motion and so the derivatives are almost upper triangular) and twisting caused by the close encounters between \(Q_4\) and \(Q_3\) and between \(Q_4\) and \(Q_1\). We restrict our attention to a two dimensional surface whose tangent space belong to the invariant cone and construct on such a surface a Cantor set of singular orbits as follows. The two parameters coming from the two dimensionality of the surface will be used to control the phase of the close encounter between the particles and their relative distance. The strong expansion will be used to ensure that the choices made at the next step will have a little effect on the parameters at the previous steps. This Cantor set construction based on the instability of near colliding orbits is also among the key ingredients of the singular orbit constructions in \([MM]\) and \([X]\).

2.2. Main ingredients. In this section we present the main steps in proving Theorem [1]. In Subsection 2.3 we describe a simplified model for constructing singular solutions given by Gerver \([G2]\). This model is based on the following simplifying assumptions:

- \(\mu = 0, \ \chi = \infty\).
- The particles do not interact except during a close encounter.
- Velocity exchange during close encounters can be modeled by an elastic collision.
- The action of \(Q_1\) on light particles can be ignored except that during the close encounters of the traveler particle with \(Q_1\) the angular momentum of the traveler with respect to \(Q_2\) can be changed arbitrarily.

The main conclusion of \([G2]\) is that the energy of the captured particle can be increased by a fixed factor while keeping the shape of its orbit unchanged. Gerver designs a procedure with two steps of collisions having the following properties:

- The incoming and outgoing asymptotes of the traveler are horizontal.
- The major axis of the captured particle remains vertical.
- After two steps of collisions, the elliptic orbit of the captured particle has the same eccentricity but smaller semimajor compared with the elliptic orbit before the first collision (see Fig 1 and 2).

For quantitative information, see Appendix [B].

Since the shape is unchanged after the two trips described above the procedure can be repeated. Then the kinetic energies of the particles grow exponentially and so the time needed for \(j\)-th trip is exponentially small. Thus the particles can make
in infinitely many trips in finite time leading to a singularity. Our goal therefore is to get rid of the above mentioned simplifying assumptions.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{angular_momentum_transfer_collision.png}
\caption{Angular momentum transfer collision}
\end{figure}

In Subsection 2.4 we study near collision of the light particles. This assumption that velocity exchange can be modeled by elastic collision is not very restrictive since both energy and momentum are conserved during the exchange and any change of velocities conserving energy and momentum amounts to rotating the relative velocity by some angle and so it can be effected by an elastic collision. In Subsection 2.5 we state a result saying that away from the close encounters we can disregard interaction between the light particles and the action of $Q_1$ to the particle which is captured by $Q_2$ can indeed be disregarded. In Subsection 2.6 we study the Poincaré map corresponding to one trip of one light particle around $Q_1$. After some technical preparations we present the main result of that section Lemma 2.7 which says that after this trip the angular momentum of the traveler particle indeed can change in an arbitrary way. Finally in Subsection 2.7 we show how to combine the above ingredients to construct a Cantor set of singular orbits.

2.3. Gerver map. Following [G2], we discuss in this section the limit case $\mu = 0, \chi = \infty$. We assume that $Q_3$ has elliptic motion and $Q_4$ has hyperbolic motion
with respect to the focus $Q_2$. Since $\mu = 0$, $Q_3$ and $Q_4$ do not interact unless they have exact collision. Since we assume that $Q_4$ just comes from the interaction from $Q_1$ located at $(-\infty, 0)$ and the new traveler particle is going to interact with $Q_1$ in the future, the slope of incoming asymptote $\theta^-_4$ of $Q_4$ and that of the outgoing asymptote $\bar{\theta}^+$ of the traveler particle should satisfy $\theta^- = 0$, $\bar{\theta}^+ = \pi$.

The Kepler motions of $Q_3$ and $Q_4$ has three first integrals $E_i$, $G_i$ and $g_i$ where $E_i$ denotes the energy, $G_i$ denotes the angular momentum and $g_i$ denotes the argument of periapsis. Since the total energy of the system is zero we have $E_4 = -E_3$. It turns out convenient to use eccentricities $e_i = \sqrt{1 + 2G_i^2E_i}$ instead of $G_i$ since the proof of Theorem 1 involves a renormalization transformation and $e_i$ are scaling invariant. The Gerver map describes the parameters of the elliptic orbit change during the interaction of $Q_3$ and $Q_4$. The orbits of $Q_3$ and $Q_4$ intersect in two points. We pick one of them. We use a discrete parameter $j \in \{1, 2\}$ to describe if the points meet at the first or at the second intersection (the intersection points will be numbered chronologically along the orbit of $Q_4$).

Since $Q_3$ and $Q_4$ only interact when they are at the same point the only effect of the interaction is to change their velocities. Any such change which satisfies energy and
momentum conservation can be described by an elastic collision. That is, velocities before and after the collision are related by

\[ v_3^+ = \frac{v_3^- + v_4^-}{2} + \left| \frac{v_3^- - v_4^-}{2} \right| n(\alpha), \quad v_4^+ = \frac{v_3^- + v_4^-}{2} - \left| \frac{v_3^- - v_4^-}{2} \right| n(\alpha), \]

where \( n(\alpha) \) is a unit vector making angle \( \alpha \) with \( v_3^- - v_4^- \).

With this in mind we proceed to define the Gerver map \( G(\bar{Q}) \) that the particle proceeds to move independently. Thus orbit simultaneously. At this point their velocities are changed by (2.1). After collision. has been explained above, and \( \omega \) will tell us which particle will be the traveler after the collision.

To define \( G \) we assume that \( Q_4 \) moves along the hyperbolic orbit with parameters \((-E_3, e_4, g_4)\) where \( g_4 \) is fixed by requiring that the incoming asymptote of \( Q_4 \) is horizontal. We assume that \( Q_3 \) and \( Q_4 \) arrive to the \( j \)-th intersection point of their orbit simultaneously. At this point their velocities are changed by (2.1). After that the particle proceed to move independently. Thus \( Q_3 \) moves on an orbit with parameters \((\bar{E}_3, \bar{e}_3, g_3)\), and \( Q_4 \) moves on an orbit with parameters \((\bar{E}_4, \bar{e}_4, \bar{g}_4)\).

If \( \omega = 4 \), we choose \( \alpha \) so that after the exchange \( Q_4 \) moves on hyperbolic orbit and \( \bar{\theta}_4 = \pi \) and let

\[ G_{e_4, j, 4}(E_3, e_3, g_3) = (\bar{E}_3, \bar{e}_3, \bar{g}_3). \]

If \( \omega = 3 \) we choose \( \alpha \) so that after the exchange \( Q_3 \) moves on hyperbolic orbit and \( \bar{\theta}_3 = \pi \) and let

\[ G_{e_4, j, 3}(E_3, e_3, g_3) = (\bar{E}_4, \bar{e}_4, \bar{g}_4). \]

In the following, to fix our notation, we always call the captured particle \( Q_3 \) and the traveler \( Q_4 \).

We will denote the ideal orbit parameters in Gerver’s paper [G2] of \( Q_3 \) and \( Q_4 \) before the first (respectively second) collision with \( * \) (respectively \( ** \)). Thus, for example, \( G_3^{**} \) will denote the angular momentum of \( Q_4 \) before the second collision. Moreover, the actual values after the first (respectively, after the second) collisions are denoted with a bar or double bar.

Note \( G \) has a skew product form

\[ \bar{e}_3 = f_e(e_3, g_3, e_4), \quad \bar{g}_3 = f_g(e_3, g_3, e_4), \quad \bar{E}_3 = E_3 f_E(e_3, g_3, e_4). \]

This skew product structure will be crucial in the proof of Theorem \([\text{G2}]\) since it will allow us to iterate \( G \) so that \( E_3 \) grows exponentially while \( e_3 \) and \( g_3 \) remains almost unchanged.

The following fact plays a key role in constructing singular solutions.

**Lemma 2.1.** \((\text{G2})\) There exist \((e_3^*, g_3^*)\), such that for sufficiently small \( \delta > 0 \) given \( \omega', \omega'' \in \{3, 4\} \), there exist \( \lambda_0 > 1 \) and functions \( e_4'(e_3, g_3) \), \( e_4''(e_3, g_3) \), defined in a small (depending on \( \delta \)) neighborhood of \((e_3^*, g_3^*)\), such that
(a) for $e_*^1, e_*^2$ given by $e'_1(e_3^1, g_3^1) = e_1^1$ and $e''_1(e_3^1, g_3^1) = e_1^2$, we have

$$ (e_3, g_3, E_3)^{**} = G_{e'_1, 1, \omega} (e_3, g_3, E_3)^{**}, \quad (e_3, -g_3, \lambda_0 E_3)^{*} = G_{e''_1, 2, \omega^*} (e_3, g_3, E_3)^{**}, $$

(b) If $(e_3, g_3)$ lie in a $\delta$ neighborhood of $(e_*^3, g_*^3)$, we have

$$ (\bar{e}_3, \bar{g}_3, \bar{E}_3) = G_{e'_1(e_3, g_3), 1, \omega} (e_3, g_3, E_3), \quad (\bar{e}_3, -\bar{g}_3, \bar{E}_3) = G_{e''_1(e_3, g_3), 2, \omega} (\bar{e}_3, \bar{g}_3, \bar{E}_3), $$

and

$$ \bar{e}_3 = e_3^*, \quad \bar{g}_3 = g_3^*, \quad \bar{E}_3 = \lambda (e_3, g_3) E_3 $$

where $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$.

Part (a) is the main result of the above lemma. It allows us to increase energy after two collisions without changing the shape of the orbit in the limit case $\mu = 0, \chi = \infty$. Part (b) is of a more technical nature. It allows us to fight the perturbation coming from the fact that $\mu > 0$ and $\chi < \infty$.

Lemma 2.1 is a slight restatement of the main result of [G2]. Namely part (a) is proven in Sections 3 and 4 of [G2] and part (b) is stated in Section 5 of [G2] (see equations (5-10)–(5-13)). The proof of part (b) proceeds by a routine numerical computation. For the reader’s convenience we review the proof of Lemma 2.1 in Appendix B explaining how the numerics is done.

Remark 5. In fact Gerver produces a one parameter family of the periodic solution. Namely one can take $e_3^*$ to be any number between 0 and $\frac{\sqrt{3}}{2}$ and $g_3^* = 0$. In the course of the proof of Theorem 1 we need to check several non-degeneracy conditions. This will be done numerically for $e_3^* = \frac{1}{2}$.

Remark 6. We try to minimize the use of numerics in our work. The use of numerics is always preceded by mathematical derivations. Readers can see that the numerics in this paper can also be done without using computer. We prefer to use the computer since computers are more reliable than humans when doing routine computations.

2.4. Asymptotic analysis, the local map. We assume that the two centers are at distance $\chi \gg 1$ and that $Q_3, Q_4$ have positive masses $0 < \mu \ll 1$. We also assume that $Q_3$ and $Q_4$ have initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4) \in \mathbb{R}^4 \times \mathbb{T}^2$ in the section $\{x_4(0) = -2, \dot{x}_4(0) > 0\}$ (the choice of this section in justified by Lemma 2.3 below). Here $\ell_3$ stands for the mean anomaly of $Q_3$, see Appendix A. We let particles move until one of the particles reach the surface $\{x_4 = -2, \dot{x}_4 < 0\}$ moving on hyperbolic orbit. We measure the final orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4)$. We call the mapping moving initial positions of the particles to their final positions the local map $\mathcal{L}$. In Fig. 3 of Section 3.2 the local map is to the right of the section $\{x = -2\}$.

Lemma 2.2. Suppose that the initial orbit parameters $(E_3, \ell_3, e_3, g_3, e_4, g_4)$ are such that the traveler particle(s) satisfy $\theta^- = O(\mu)$ and $\theta^+ = \pi + O(\mu)$ then the following
asymptotics holds uniformly
\[(\bar{E}_3, \bar{e}_3, \bar{g}_3) = \mathbf{G}_{e_4}(E_3, e_3) + o(1),\]
as \(\mu \to 0, \chi \to \infty\).

The lemma tells us Gerver map is a good approximation of the local map \(L\) for the real case \(0 < \mu \ll 1 \ll \chi < \infty\) for the orbits of interest. Lemma 2.2 will be proven in Section 10 where we also present some additional information about the local map (see Lemma 10.2).

We also need the following fact. Fix a small number \(\bar{\theta}\).

Lemma 2.3. Suppose the initial orbit parameters \((E_3, \ell_3, e_3, g_3, e_4, g_4)\) for the local map are such that \(E_3 = -\frac{1}{2} + O(\mu)\), and the incoming and outgoing asymptotes of \(Q_4\) satisfy \(\theta^- = O(\mu)\) and \(|\theta^+ - \pi| \leq \bar{\theta} \ll 1\). Then for \(\mu\) sufficiently small and \(\chi\) sufficiently large we have \(|Q_3| \leq 2 - \delta\) where \(\delta > 0\) is a constant independent of \(\mu\) and \(\chi\).

The proof of this lemma is also given in to Section 10.

2.5. Asymptotic analysis, the global map. As before we assume that the two centers are at distance \(\chi \gg 1\). We assume that initially \(Q_3\) moves on an elliptic orbit, \(Q_4\) moves on hyperbolic orbit and \(\{x_4(0) = -2, \dot{x}_4(0) < 0\}\). We assume that \(|y_4(0)| < C\) and after moving around \(Q_1\) it hits the surface \(\{x_4 = -2, \dot{x}_4 > 0\}\) so that \(|y_4| < C\). We call the mapping moving initial positions of the particles to their final positions the (pre) global map \(G\). In Section 2.6 we will slightly modify the definition of the global map but it will not change the essential features discussed here. In Fig 3 from Section 3.2 the global map is to the left of the section \(\{x = -2\}\). We let \((E_3, \ell_3, e_3, g_3, e_4, g_4)\) denote the initial orbit parameters measured in the section \(\{x_4 = -2, \dot{x}_4 < 0\}\) and \((\bar{E}_3, \bar{\ell}_3, \bar{e}_3, \bar{g}_3, \bar{e}_4, \bar{g}_4)\) denote the final orbit parameters measured in the section \(\{x_4 = -2, \dot{x}_4 > 0\}\). Fix a large constant \(C\).

Lemma 2.4. Assume that \(|y_4| < C\) holds both at initial and final moments. Then uniformly in \(\chi, \mu\) we have the following estimates
\[\begin{align*}
(a) & \ E_3 - E_3 = O(\mu), \quad \bar{G}_3 - G_3 = O(\mu), \quad \bar{g}_3 - g_3 = O(\mu), \\
(b) & \ \theta^+_4 = \pi + O(\mu), \quad \bar{\theta}^-_4 = O(\mu).
\end{align*}\]

The proof of this lemma is given in Section 4.

2.6. Admissible surfaces. Given a sequence \(\omega\) we need to construct orbits having singularity with symbolic sequence \(\omega\).

We will study the Poincaré map \(P = G \circ L\) to the surface \(\{x_4 = -2, \dot{x}_4 > 0\}\). It is a composition of the local and global maps defined in the previous sections.
Given $\delta$ consider open sets in the phase space defined by

$$U_1(\delta) = \left\{ \left| E_3 - \left(-\frac{1}{2}\right)\right|, |e_3 - e_3^*|, |\theta_4| < \delta, |e_4 - e_4^*| < \sqrt{\delta} \right\},$$

$$U_2(\delta) = \left\{ |E_3 - E_3^*|, |e_3 - e_3^*|, |\theta_4| < \delta, |e_4 - e_4^*| < \sqrt{\delta} \right\}.$$ 

We will also need the renormalization map $R$ defined as follows. Partition our section $\{x_4 = -2, \dot{x}_4 > 0\}$ into cubes of size $1/\sqrt{\chi}$ and on each cube we rescale the space and time so that

- in the center of the cube $Q_3$ has elliptic orbit with energy $-\frac{1}{2}$.
- the potential of the fixed centers is still $1/|Q_i - Q_j|$, $i = 1, 2$, $j = 3, 4$.

In addition we reflect the coordinates with respect to the $x$ axis. We define $\lambda = |E_3| > 1$ as the dilation rate where $E_3$ is the energy of $Q_3$ at the center of each cube. We push forward each cube to the section $\{x_4 = -2/\lambda, \dot{x}_4 > 0\}$. We include the piece of orbits from the section $\{x_4 = -2, \dot{x}_4 > 0\}$ to $\{x_4 = -2/\lambda, \dot{x}_4 > 0\}$ to the global map and apply the $R$ to the section $\{x_4 = -2/\lambda, \dot{x}_4 > 0\}$. So the locally constant map $R$ amounts to zooming in the configuration by multiplying by $\lambda$ and slowing down the velocity by dividing $\sqrt{\chi}$. This is then followed by a reflection. We have $R(\{x_4 = -2/\lambda, \dot{x}_4 > 0\}) = \{x_4 = -2, \dot{x}_4 > 0\}$, and

$$R(E_3, \ell_3, e_3, g_3, e_4, g_4) = (E_3/\lambda, \ell_3, e_3, -g_3, e_4, -g_4).$$

Note that the rescaling changes (for the orbits of interest, increases) the distance between the fixed centers by sending $\chi$ to $\lambda \chi$. Observe that at each step we have the freedom of choosing the centers of the cubes. We describe how this choice is made in the next section. In the following we give a proof of the main theorem based on the three lemmas, whose proofs are in the next section.

**Lemma 2.5.** There are cone families $K_1$ on $T_{U_1}(\mathbb{R}^4 \times \mathbb{T}^2)$ and $K_2$ on $T_{U_2}(\mathbb{R}^4 \times \mathbb{T}^2)$, each of which contains a two dimensional plane and a constant $c$ such that for all $x \in U_1(\delta)$ satisfying $P(x) \in U_2(\delta)$, and for all $x \in U_2(\delta)$ satisfying $R \circ P(x) \in U_1(\delta)$,

(a) $dP(K_1) \subset K_2$, $d(R \circ P)(K_2) \subset K_1$.

(b) If $v \in K_1$, then $||dP(v)|| \geq c\chi||v||$. If $v \in K_2$, then $||d(R \circ P)(v)|| \geq c\chi||v||$.

We call a $C^1$ surface $S_1 \subset U_1(\delta)$ (respectively $S_2 \subset U_2(\delta)$) admissible if $TS_1 \subset K_1$ (respectively $TS_2 \subset K_2$).

**Lemma 2.6.** (a) The vector $\dot{w} = \frac{\partial}{\partial x_3}$ is in $K_1$.

(b) Any plane $\Pi$ in $K_i$ the map projection map $\pi_{e_4, \ell_3} = (de_4, d\ell_3) : \Pi \rightarrow \mathbb{R}^2$ is one-to-one. In other words $(e_4, \ell_3)$ can be used as coordinates on admissible surfaces.

We call an admissible surface essential if $\pi_{e_4, \ell_3}$ is an $I \times \mathbb{T}^1$ for some interval $I$. In other words given $e_4 \in I$ we can prescribe $\ell_3$ arbitrarily.
We wish to construct a singular orbit in \( \tilde{\ell}_3 \) such that \( \mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_2(K\delta) \). Moreover if \( \text{dist}(\tilde{e}_4, \partial I) > 1/\chi \) then there is a neighborhood \( V(\tilde{e}_4) \) of \((\tilde{e}_4, \tilde{\ell}_3)\) such that \( \pi_{\tilde{e}_4, \tilde{\ell}_3} \circ \mathcal{P} \) maps \( V \) surjectively to 

\[ \{|e_4 - e^*_4| < K\delta\} \times \mathbb{T}^1. \]

(b) Given an essential admissible surface \( S_2 \subset U_2(\delta) \) and \( \tilde{e}_4 \in I(S_2) \) there exists \( \tilde{\ell}_3 \) such that \( \mathcal{R} \circ \mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_1(K\delta) \). Moreover if \( \text{dist}(\tilde{e}_4, \partial I) > 1/\chi \) then there is a neighborhood \( V(\tilde{e}_4) \) of \((\tilde{e}_4, \tilde{\ell}_3)\) such that \( \pi_{\tilde{e}_4, \tilde{\ell}_3} \circ \mathcal{R} \circ \mathcal{P} \) maps \( V \) surjectively to 

\[ \{|e_4 - e^*_4| < K\delta\} \times \mathbb{T}^1. \]

(c) For points in \( V(\tilde{e}_4) \) from parts (a) and (b), the particles avoid collisions before the next return and the minimal distance between the particles satisfies \( \mu \delta \leq d \leq \frac{K}{\delta} \).

Note that by Lemma 2.5 the diameter of \( V(\tilde{e}_4) \) is \( O(\delta/\chi) \).

2.7. Construction of the singular orbit. Fix a number \( \varepsilon \) which is small but is much larger than both \( \mu \) and \( 1/\chi \). Let \( S_0 \) be an admissible surface such that the diameter of \( S_0 \) is much larger than \( 1/\chi \) and such that on \( S_0 \) we have

\[ |e_3 - \tilde{e}_3| < \varepsilon, \quad |g_3 - \tilde{g}_3| < \varepsilon, \]

where \((\tilde{e}_3, \tilde{g}_3)\) is close to \((e_3^*, g_3^*)\). For example, we can pick a point \( x \in U_1(\delta) \) and let \( \tilde{w} \) be a vector in \( K_1(x) \) such that \( \frac{\partial}{\partial e_3}(\tilde{w}) = 0 \). Then let

\[ S_0 = \{(E_3, \ell_3, e_3, g_3, e_4, g_4)(x) + a\tilde{w} + (0, b, 0, 0, 0)\}_{a \leq \varepsilon/K} \]

where \( K \) is a large constant.

We wish to construct a singular orbit in \( S_0 \). We define \( S_j \) inductively so that \( S_j \) is component of \( \mathcal{P}(S_{j-1}) \cap U_2(\delta) \) if \( j \) is odd and \( S_j \) is component of \( (\mathcal{R} \circ \mathcal{P})(S_{j-1}) \cap U_1(\delta) \) if \( j \) is even (we shall show below that such components exist). Let \( x = \lim_{j \rightarrow \infty} (\mathcal{R} \circ \mathcal{P})^{-j} S_2 \). We claim that \( x \) has singular orbit. Indeed by Lemma 2.1 the unscaled energy of \( Q_4 \) satisfies \( E(j) \geq (\lambda_0 - \tilde{\delta})^{3/2} \) where \( \tilde{\delta} \rightarrow 0 \) as \( \delta \rightarrow 0 \). Accordingly the velocity of \( Q_4 \) during the trip \( j \) is bounded from below by \( c\sqrt{E(j)} \geq c(\lambda_0 - \tilde{\delta})^{3/4} \).

Therefore \( t_{j+1} = t_j + O((\lambda_0 - \tilde{\delta})^{-3/4}) \) and so \( t_* = \lim_{j \rightarrow \infty} t_j < \infty \) as needed.

It remains to show that if we can find a component of \( \mathcal{P}(S_{2j}) \) inside \( U_2(\delta) \) and a component of \( (\mathcal{R} \circ \mathcal{P})(S_{2j+1}) \) inside \( U_1(\delta) \). Note that Lemma 2.7 allows to choose such components inside larger sets \( U_2(K\delta) \) and \( U_1(K\delta) \).

First note that by Lemma 2.4 on \( \mathcal{P}(S_{2j}) \cap U_1(K\delta) \) and on \( (\mathcal{R} \circ \mathcal{P}^2)(S_{2j}) \cap U_2(K\delta) \) we have \( \theta_4^- = O(\mu) \). Also by Lemma 2.7 \( e_4 \) can be prescribed arbitrarily. In other words we have a good control on the orbit of \( Q_4 \).

In order to control the orbit of \( Q_3 \) note that by Lemma 2.5(b) the preimage of \( S_{2j} \) has size \( O(1/\chi) \) and so by Lemmas 2.2, 2.4 and 2.6 given \( \varepsilon \) we have that \( e_3 \) and \( g_3 \) have oscillation less than \( \varepsilon \) on \( S_{2j} \). If \( \mu \) is small enough. Namely part (b) of Lemma 2.6 shows that \( e_3 \) and \( g_3 \) have oscillation \( O(1/\chi) \) on the preimage of \( S_{2j} \).
while Lemmas 2.2 and 2.4 show that the oscillations do not increase much after application of local and global map. Thus there exist \((\hat{e}_3, \hat{g}_3)\) such that on \(S_{2j}\) we have
\[
|e_3 - \hat{e}_3| < \varepsilon, \quad |g_3 - \hat{g}_3| < \varepsilon.
\]
Also due to rescaling defined in Section 2.6 and Lemma 2.4 we have
\[
|E_3 - (-\frac{1}{2})| = O\left(\frac{1}{\sqrt{x}} + \mu\right).
\]
Set
\[
S_{2j+1} = PV(e'(\hat{e}_3, \hat{g}_3)), \quad S_{2j+2} = (R \circ P)V(e''(\hat{e}_3, \hat{g}_3)).
\]
Then on \(S_{2j+1}\) we shall have
\[
|e_3 - e_3^\ast| < K\varepsilon, \quad |g_3 - g_3^\ast| < K\varepsilon \text{ and } |E_3 - E_3^\ast| < K\varepsilon
\]
while on \(S_{2j+2}\) we shall have
\[
|e_3 - e_3^\ast| < K^2\varepsilon, \quad |g_3 - g_3^\ast| < K^2\varepsilon \text{ and } |E_3 + \frac{1}{2}| < K(1/\sqrt{x} + \mu).
\]
Denote
\[
S_{2j+1} = \tilde{S}_{2j+1} \cap \{e_4 - e''(e_3^\ast, g_3^\ast) < \sqrt{\delta}\}, \quad S_{2j+2} = \tilde{S}_{2j+2} \cap \{e_4 - e'(e_3^\ast, g_3^\ast) < \sqrt{\delta}\}.
\]
Taking \(\varepsilon\) so small that \(K^2\varepsilon < \delta\) we get that \(S_{2j+1} \in U_2(\delta), S_{2j+2} \in U_1(\delta)\) as needed.
Finally we use the freedom to choose the appropriate partition in the definition of \(R\) to ensure that \(R\) is continuous on the preimage of \(V(e'(\hat{e}_3, \hat{g}_3))\) so that \(V(e'(\hat{e}_3, \hat{g}_3))\) is a smooth surface.

**Remark 7.** In fact we do not need to use exactly \(e'(\hat{e}_3, \hat{g}_3)\) and \(e''(\hat{e}_3, \hat{g}_3)\) in (2.2). Namely any \(V(e_4^\dagger)\) and \(V(e_4^\ddagger)\) would do provided that
\[
|e_4^\dagger - e_4'(\hat{e}_3, \hat{g}_3)| < \varepsilon, \quad |e_4^\ddagger - e_4''(\hat{e}_3, \hat{g}_3)| < \varepsilon.
\]
Different choices of \(e_4^\dagger\) and \(e_4^\ddagger\) allow us obtain different orbits. Since such freedom exists at each step of our construction we have a Cantor set of singular orbits with a given symbolic sequence \(\omega\).

3. Hyperbolicity of the Poincaré map

3.1. Construction of invariant cones. Here we derive Lemma 2.5, 2.6 and 2.7 from the asymptotics of the derivative of local and global maps.

**Lemma 3.1.** There exist continuous functions \(u(x), l(x)\) and \(B(x)\) such that if \(x \in U_\varepsilon(\delta)\) is such that \(L(x)\) satisfies \(\theta^- = O(\mu), |\theta^+ - \pi| \leq \hat{\theta} \ll 1\) where \(\theta\) is independent of \(\mu, \chi\), then we have
\[
dL(x) = \frac{1}{\mu} u(x) \otimes l(x) + B(x) + o(1).
\]
Moreover there exist a linear functional $\hat{l}_i$ and a vector $\hat{u}_i$ such that 
\begin{align*}
1 = \hat{l}_i + o(1), \quad u = \hat{u}_i + o(1), \quad B = \hat{B}_i + o(1), \text{ as } \delta, \mu, 1/\chi \to 0.
\end{align*}
This lemma is proven in Section 3.2.

We further define two new sets in the section \( \{ x_4 = -2, \, \dot{x}_4 > 0 \} \):
\begin{align*}
\hat{U}_1(\hat{\delta}) &= \{(e_3, g_3, E_3) - G_{e_3}(e_3^*, g_3^*, E_3^*) \mid < \hat{\delta}, \quad |\theta_1^+| < \hat{\delta} \text{ and } |G_3 + G_4 - (G_3^* + G_4^*)| < \hat{\delta} \}, \\
\hat{U}_2(\hat{\delta}) &= \{(e_3, g_3, E_3) - G_{e_3}(e_3^*, g_3^*, E_3^*) \mid < \hat{\delta}, \quad |\theta_1^+| < \hat{\delta} \text{ and } |G_3 + G_4 - (G_3^* + G_4^*)| < \hat{\delta} \}.
\end{align*}
Note that if $\hat{\delta} \geq \text{Const}$ then by Lemma 2.2 $\hat{U}_i(\hat{\delta})$ contains the part of $U_i$ consisting of the orbits which will have a close encounter with $Q_1$ during the next excursion around $Q_1$.

**Lemma 3.2.** Let $x$ and $y = G(x)$ be such that $|y(x)| \leq C, \, |y(y)| \leq C$ Then there exist linear functionals $l(x)$ and $\hat{l}(x)$ and vectorfields $\hat{u}(y)$ and $\bar{u}(y)$ such that 
\begin{align*}
\frac{dG(x)}{dx} = \chi^2 \bar{u}(y) \otimes \bar{l}(x) + \chi \bar{u}(y) \otimes \bar{l}(x) + O(\mu^2 \chi).
\end{align*}
Moreover there exist vector $w_j$ and linear functionals $\bar{l}_i$, $\bar{l}_i$ such that if $x \in \hat{U}_i(\hat{\delta})$ and $\hat{\delta} \to 0$ then 
\begin{align*}
\bar{l}(x) \to \bar{l}_i, \quad \bar{l}(x) \to \bar{l}_i.
\end{align*}
In addition, if $y \in \mathcal{R}^{-1}U_1(\delta)$ and $\delta \to 0$ then 
\begin{align*}
\text{span}(\bar{u}(y), \bar{u}(y)) \to \text{span}(w_1, \bar{w})
\end{align*}
and if $y \in U_2(\delta)$ and $\delta \to 0$ then 
\begin{align*}
\text{span}(\bar{u}(y), \bar{u}(y)) \to \text{span}(w_2, \bar{w})
\end{align*}
where $\bar{w} = \frac{\partial}{\partial x^3}$.

This lemma is proven in Section 3.2.

**Lemma 3.3.** The following non degeneracy conditions are satisfied.

(a1) span($\hat{u}_1, B\hat{l}_1(\bar{w})d\mathcal{R}w_2 - \hat{1}_1(d\mathcal{R}w_2)\bar{w}$) is transversal to Ker($\hat{l}_1$) \( \cap \) Ker($\hat{l}_1$).

(a2) de$_4$(span($d\mathcal{R}w_2, d\mathcal{R}\bar{w}$)) $\neq 0$.

(b1) span($\hat{u}_2, B\hat{l}_2(\bar{w})w_1 - \hat{l}_2(w_1)\bar{w}$) is transversal to Ker($\hat{l}_2$) \( \cap \) Ker($\hat{l}_2$).

(b2) de$_4$(w$_1$) $\neq 0$.

This lemma is proven in Section 3.3.

**Definition 2.** We now take $K_1$ to be the set of vectors which make an angle less than a small constant $\eta$ with span($d\mathcal{R}w_2, \bar{w}_2$), and $K_2$ to be the set of vectors which make an angle less than a small constant $\eta$ with span($w_1, \bar{w}_1$).
Proof of Lemma 2.5. Consider for example the case where \( x \in U_2(\delta) \). We claim that if \( \delta, \mu \) are small enough then \( dL(\text{span}(w, \tilde{\omega})) \) is transversal to \( \text{Ker}I_1 \cap \text{Ker}I_2 \). Indeed take \( \Gamma \) such that \( I(\Gamma) = 0 \). If \( \Gamma = aw_1 + \tilde{a}\tilde{w} \) then \( aI(w_1) + \tilde{a}\tilde{I}(\tilde{w}) = 0 \). It follows that the direction of \( \Gamma \) is close to the direction of \( \tilde{\Gamma} = \tilde{I}_2(\tilde{w})w_1 - \tilde{I}_2(w_1)\tilde{w} \). Next take \( \tilde{\Gamma} = bw + \tilde{b}\tilde{w} \) where \( bI(w_1) + \tilde{b}\tilde{I}(\tilde{w}) \neq 0 \). Then the direction of \( d\tilde{L}\tilde{\Gamma} \) is close to \( \tilde{\omega}_2 \) and the direction of \( dL(\Gamma) \) is close to \( B(\tilde{\Gamma}) \) so our claim follows.

Thus for any plane \( \Pi \) close to \( \text{span}(w_1, \tilde{\omega}) \) we have that \( dL(\Pi) \) is transversal to \( \text{Ker}I_1 \cap \text{Ker}I_2 \). Take any \( Y \in \mathcal{K}_2 \). Then either \( Y \) and \( w_1 \) are linearly independent or \( Y \) and \( \tilde{\omega} \) are linearly independent. Hence \( dL(\text{span}(Y, w_1)) \) or \( dL(\text{span}(Y, \tilde{\omega})) \) is transversal to \( \text{Ker}I_1 \cap \text{Ker}I_2 \). Accordingly either \( I_2(dL(Y)) \neq 0 \) or \( I_2(dL(\tilde{\omega})) \neq 0 \). If \( I_2(dL(Y)) \neq 0 \) then the direction of \( d(\overline{G} \circ L)(Y) \) is close to \( \tilde{u} \). If \( I_2(dL(\tilde{\omega})) \neq 0 \) then the direction of \( d(\overline{G} \circ L)(Y) \) is close to \( \tilde{u} \). In either case \( d(R \overline{G} \circ L)(Y) \in \mathcal{K}_1 \) and \( ||d(\overline{G} \circ L)(Y)|| \geq c\chi ||Y|| \). This completes the proof in the case \( x \in U_2(\delta) \). The case where \( x \in U_1(\delta) \) is similar.

Proof of Lemma 2.6. Part (a) follows from the definition of \( \mathcal{K}_i \). Also by part (b) of Lemma 3.3 the map \( \pi : \text{span}(w, \tilde{\omega}) \rightarrow \mathbb{R}^2 \) given by \( \pi(\Gamma) = (d\ell_3(\Gamma), de_4(\Gamma)) \) is invertible. Namely if \( \Gamma = aw + \tilde{a}\tilde{w} \) then
\[
a = \frac{de_4(\Gamma)}{de_4(w)}, \quad \tilde{a} = d\ell_3(\Gamma) - ad\ell_3(w).
\]
Accordingly \( \pi \) is invertible on planes close to \( \text{span}(w, \tilde{\omega}) \) proving our claim.

To prove Lemma 2.7 we need two auxiliary results.

**Sublemma 3.4.** Given \( \tilde{e}_4 \) there exists \( \tilde{\ell}_3 \) such that \( \mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) \in U_2(\delta) \).

The proof of this sublemma is postponed to Section 11.2.

**Sublemma 3.5.** Let \( \mathcal{F} \) be a map on \( \mathbb{R}^2 \) which fixes the origin and such that if \( \mathcal{F}(z) \in \mathbb{R}^2 \) then \( ||d\mathcal{F}(X)|| \geq \chi ||X|| \). Then for each \( a \) such that \( ||a|| < R \) there exists \( z \) such that \( ||z|| < R/\chi \) and \( \mathcal{F}(z) = a \).

**Proof.** Without the loss of generality we may assume that \( a = (r, 0) \). Let \( V(z) \) be the direction field defined by the condition that the direction of \( d\mathcal{F}(V(z)) \) is parallel to \( (1, 0) \). Let \( \gamma(t) \) be the integral curve of \( V \) passing through the origin and parameterized by the arclength. Then \( \mathcal{F}(\gamma(t)) \) has form \( (\sigma(t), 0) \) where \( \sigma(0) = 0 \) and \( ||\sigma(t)|| \geq \chi \) as long as \( ||\sigma|| < R \). Now the statement follows easily.

**Proof of Lemma 2.7.** (a) We claim that it suffices to show that for each \( (\tilde{e}_4, \tilde{\ell}_3) \) such that \( ||\tilde{e}_4 - \tilde{e}_4^*|| \leq \sqrt{\delta} \) there exist \( (\tilde{e}_4, \tilde{\ell}_3) \) such that
\[
(3.1) \quad \mathcal{P}(\tilde{e}_4, \tilde{\ell}_3) = (\tilde{e}_4, \tilde{\ell}_3).
\]
Indeed in that case Sublemma 4.9 from Section 4.3 says that the outgoing asymptote is almost horizontal. Therefore by Lemma 2.2 our orbit has \( (E_3, e_3, g_3) \) close
to $\mathbf{G}_{\hat{e}_4,2,4}(E_3(\hat{e}_4, \hat{\ell}_3), e_3(\hat{e}_4, \hat{\ell}_3), g_3(\hat{e}_4, \hat{\ell}_3))$. Next Lemma 2.4 shows that after the application of $\mathbf{G}$, $(E_3, e_3, g_3)$ change little and $\theta_4^{-}$ becomes $O(\mu)$ so that $\mathcal{P}(\hat{e}_4, \hat{\ell}_3) \in U_2(K \delta)$.

We will now prove (3.1). Our coordinates allow us to treat $\mathcal{P}$ as a map $\mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$. Due to Lemma 2.5 we can apply Sublemma 3.5 to the covering map $\tilde{\mathcal{P}} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\tilde{\chi} = c \chi$ obtaining (3.1). Part (b) of the lemma is similarly proven. Part (c) follows from Lemma 10.2 proven in Section 10. □

### 3.2. Expanding directions of the global map.

Estimating the derivative of the global map is the longest part of the paper. It occupies Sections 5–8.

It will be convenient to use the Delaunay coordinates $(L_3, \ell_3, G_3, g_3)$ for $Q_3$ and $(G_4, g_4)$ for $Q_4$. Delaunay coordinates are action-angle coordinates for the Kepler problem. We collect some facts about the Delaunay coordinates in Appendix A.

We divide the plane into several pieces by lines $x_4 = -2$ and $x_4 = -\chi/2$. Those lines cut the orbit of $Q_4$ into 4 pieces:

- $\{x_4 = -2, \dot{x}_4 < 0\} \to \{x_4 = -\chi/2, \dot{x}_4 < 0\}$. We call this piece (I).
- $\{x_4 = -\chi/2, \dot{x}_4 < 0\} \to \{x_4 = -\chi/2, \dot{x}_4 > 0\}$ turning around $Q_1$. We call it (III).
- $\{x_4 = -\chi/2, \dot{x}_4 > 0\} \to \{x_4 = -2, \dot{x}_4 > 0\}$. We call it (V)
- $\{x_4 = -2, \dot{x}_4 > 0\} \to \{x_4 = -2, \dot{x}_4 < 0\}$ turning around $Q_2$.

We composition of the first three pieces constitutes the global map. The last piece defines the local map. See Fig 3. Notice that when we define $\mathcal{R}$ in Section 2.6 after the second collision in Gerver’s construction, the global map sends $\{x_4 = -2, \dot{x}_4 < 0\}$ to $\{x_4 = -2/\lambda, \dot{x}_4 > 0\}$. Then $\mathcal{R}$ sends $\{x_4 = -2/\lambda, \dot{x}_4 > 0\}$ to $\{x_4 = -2, \dot{x}_4 > 0\}$ before applying local map. So without leading to confusion, when we are talking about sections after the second collision, we always talk about $\mathcal{R} \circ \mathbf{G}$ so that the section $\{x_4 = -2, \dot{x}_4 < 0\}$ is sent to $\{x_4 = -2, \dot{x}_4 > 0\}$.

![Figure 3. Poincaré sections](image-url)
The line \( x_4 = -\frac{x}{2} \) is convenient because if \( Q_4 \) is moving to the right of the line \( x_4 = -\frac{x}{2} \), its motion can be treated as a hyperbolic motion focused at \( Q_2 \) with perturbation caused by \( Q_1 \) and \( Q_3 \). If \( Q_4 \) is moving to the left of this line, its motion can be treated as a hyperbolic motion focused at \( Q_1 \) perturbed by \( Q_2 \) and \( Q_3 \).

Since we use different guiding centers to the left and right of the line of \( x_4 = -\frac{x}{2} \) we will need to change variables when \( Q_4 \) hits this line. This will give rise to two more matrices for the derivative of the global map: (\( II \)) will correspond to the change of coordinates from right to left and (\( IV \)) will correspond for the change of coordinates from left to right. Thus \( d\mathcal{G} = (V)(IV)(III)(II)(I) \). In turn, each of the matrices (\( II \)) and (\( IV \)) will be products of three matrices corresponding to changing one variable at a time. Thus we will have (\( II \)) = [(\( iii \))(\( ii \))\\( i \)) and (\( IV \)) = (\( ii' \))(\( ii' \))(\( i' \))].

The asymptotics of the above mentioned matrices is presented in the two propositions below.

To refer to a certain subblock of a matrix \( (\vec{z}) \), we use the following convention:

\[
(\vec{z}) = \begin{bmatrix}
(\vec{z})_{33} & (\vec{z})_{34} \\
(\vec{z})_{43} & (\vec{z})_{44}
\end{bmatrix}.
\]

Thus \( (\vec{z})_{33} \) is a \( 4 \times 4 \) matrix and \( (\vec{z})_{44} \) is a \( 2 \times 2 \) matrix. To refer to the \( (i, j) - th \) entry of a matrix \( (\vec{z}) \) (in the Delaunay coordinates mentioned above) we use \( (\vec{z})_{(i, j)} \).

For example, (\( I \))(1, 3) means the derivative of \( L_3 \) with respect to \( G_3 \) when the orbit moves between sections \( \{x_4 = -2\} \) and \( \{x_4 = -\frac{x}{2}\} \).

**Proposition 3.6.** Under the assumptions of Lemma 3.2 the matrices introduced above satisfy the following estimates.

\[
(I) = \begin{bmatrix}
1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\chi) & O(\mu\chi) & O(\mu\chi) & O(\mu\chi) \\
O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) \\
O(1) & O(\mu) & O(\mu) & O(\mu)
\end{bmatrix},
\]

\[
(i) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{\mathcal{G}_{4R}/k_R L_3}{k_R L_3^2 + G_{4R}^2} + O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & O(\frac{1}{\chi^2}) & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]
at the initial point and the final point. Moreover, the matrix of the renormalization map \( R \) has the form

\[
\begin{pmatrix}
1 + O(1/\chi) & O(1/\chi^2) & O(1/\chi^3) & O(1/\chi^4) & O(1/\chi^5) \\
O(1/\chi) & 1 + O(1/\chi) & O(1/\chi^2) & O(1/\chi^3) & O(1/\chi^4) \\
O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi^3) & O(1/\chi^4) \\
O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi^3) \\
O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi)
\end{pmatrix}
\]

where \( k_R = 1 + \mu \), \( L_3 = \tilde{L}_3 + O(\mu) = \tilde{L}_3 + O(\mu) = L_3 + O(\mu), G_3 = \tilde{G}_3 + O(\mu), \tilde{G}_3 = \tilde{G}_3 + O(\mu) \). Here \( L_3 \) and \( G_3 \) are the values of the Delaunay coordinates at the initial point and \( \tilde{L}_3 \) and \( \tilde{G}_3 \) are the values of the Delaunay coordinates at the final point. Moreover, the matrix of the renormalization map \( R \) has the form
Moreover, the Section 2.6 and the “−” appears due to the reflection.

**Proposition 3.7.** The $O(1)$ blocks in Proposition 3.6 can be written as a continuous function of $x$ and $y$ and an error which vanishes in the limit $\mu \to 0, \chi \to \infty$. Moreover the $O(1)$ blocks have the following limits for orbits of interest.

\[
(I)_{44} = \begin{bmatrix}
1 + \frac{\hat{L}_4}{2(L_4^2 + G_4^2)} & \frac{-\hat{L}_4}{2(L_4^2 + G_4^2)} \\
\frac{\hat{L}_4}{2(L_4^2 + G_4^2)} & 1 - \frac{\hat{L}_4}{2(L_4^2 + G_4^2)}
\end{bmatrix},
\quad
(V)_{44} = \begin{bmatrix}
1 - \frac{1/2\hat{L}_4}{L_4^2 + G_4^2} & -1/2\hat{L}_4 \\
\frac{1/2\hat{L}_4}{L_4^2 + G_4^2} & 1 + \frac{1/2\hat{L}_4}{L_4^2 + G_4^2}
\end{bmatrix},
\]

\[
(III)_{44} = \begin{bmatrix}
\frac{1}{2} & -\frac{L_4}{2} \\
\frac{3}{2} & \frac{1}{2}
\end{bmatrix}.
\]

In addition for map (I) we have

\[
((I)(5,1), (I)(6,1))^T = \begin{pmatrix}
-G_4\hat{L}_4 \\
2(L_4^2 + G_4^2)
\end{pmatrix}, \quad \begin{pmatrix}
-G_4\hat{L}_4^2 \\
2(L_4^2 + G_4^2)^2
\end{pmatrix}^T.
\]

Here and below the phrase after the first collision means that the initial orbit has parameters $(\frac{1}{2}, e_3^*, g_3^{**}) + o(1)$ for $Q_3$, $G_4$ satisfies $G_4 + G_3^{**} = G_3^{**} + G_4^{**} + o(1)$ and that at the final moment the angular momentum of $Q_4$ is close to $G_4^{**}$. The phrase after the second collision means that the initial orbit has parameters $(\frac{1}{2}, e_3^*, g_3^{**}) + o(1)$ for $Q_3$, $G_4$ satisfies $G_4 + G_3^{**} = G_3^{**} + G_4^{**} + o(1)$ and that at the final moment the angular momentum of $Q_4$ is close to $G_4^{**}$.

The estimates of $(I), (III), (V)$ from Proposition 3.6 are proven in Sections 4, 7. The estimates of $(II), (IV)$ are given in Section 8. Proposition 3.7 is proven in Section 6.2. Now we prove Lemma 3.2 based on the Proposition 3.7.

**Proof of Lemma 3.2**. $dG$ is a product of several matrices. We will divide the product into three groups. The following estimates are obtained from Proposition 3.6 by direct computation.

\[
(i)(I) = \begin{bmatrix}
1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\chi) & O(\mu \chi) & O(\mu \chi) & O(\mu \chi) & O(\mu \chi) \\
O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\mu) & O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) & O(\mu) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1)
\end{bmatrix}.
\]
\[
M = ([i'i'][i'i'](III)[iii')(ii')]
\]

\[
= \begin{bmatrix}
1 + O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi^2) \\
O(\chi) & O(1) & O(1) & O(1) & O(1) & O(1) \\
O(1/\chi) & O(1/\chi) & 1 + O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi^2) \\
O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi) & O(1/\chi^2) & O(1/\chi^2) \\
O(1) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\chi) \\
O(1/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & O(1) & O(\chi)
\end{bmatrix},
\]

\[
(V)(iii') = \begin{bmatrix}
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
O(\chi) & 1 + O(\mu) & O(\mu) & O(\mu) & O(1) \\
O(\mu^2\chi) & O(\mu) & O(1) & O(\mu) & O(\mu) \\
O(\mu^2\chi) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1) \\
O(\mu^2\chi) & O(\mu) & O(\mu) & O(\mu) & O(1)
\end{bmatrix}.
\]

We decompose (i)(I) and (V)(iii') as

\[
(i)(I) = \begin{bmatrix}
1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\
O(\chi) & O(\mu/\chi) & O(\mu/\chi) & O(\mu/\chi) & 0 & 0 \\
O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) & 0 & 0 \\
O(\mu) & O(\mu) & O(\mu) & O(1 + O(\mu)) & 1 + O(\mu) & 0 \\
O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)
\end{bmatrix} := [b][a]
\]

\[
(V)(iii') = \begin{bmatrix}
1 & 0 & 0 & 0 & O(\mu) & O(\mu) \\
0 & 1 & 0 & 0 & O(\mu) & O(\mu) \\
0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\
0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1)
\end{bmatrix}.
\]

\[
(V)(iii') = \begin{bmatrix}
O(\mu^2) & O(\mu) & O(\mu) & O(\mu) & 0 & 0 \\
0 & 1 & 0 & 0 & O(\mu) & O(\mu) \\
0 & 0 & 1 & 0 & O(\mu) & O(\mu) \\
0 & 0 & 0 & 1 & O(\mu) & O(\mu) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1) \\
O(1) & O(1) & O(1) & O(1) & O(1) & O(1)
\end{bmatrix} := [d][c]
\]
Note that $|d|$ and $|a|$ are bounded so they do not change the order of magnitude of the derivative growth. On the other hand, denoting $D = [c] M [b]$ we obtain

$$D = \begin{bmatrix}
O(\mu \chi) & O(\mu^2 \chi) & O(\mu^2 \chi) & O(\mu^2 \chi) & O(\mu) & O(\mu \chi) \\
O(\chi) & O(\mu \chi) & O(\mu \chi) & O(\mu \chi) & O(1) & O(\chi) \\
O(\mu \chi) & O(\mu^2 \chi) & O(\mu^2 \chi) & O(\mu^2 \chi) & O(\mu) & O(\mu \chi) \\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu \chi) \\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu \chi) \\
\end{bmatrix}.$$ 

Note that $D_{44} = M_{44}$. In particular

$$\frac{D(5,6)}{\chi^2} = \left( \frac{k_R}{L_3}, \frac{k_R}{L_3} \right) (III)_{44} \left( \frac{1}{L_3} \right) + o(1).$$

It follows that if $\chi$ is large and $\mu$ is small then $\frac{D(5,6)}{\chi^2}$ is uniformly bounded from above and below. Hence $D$ can be represented as

$$D = \chi^2 \bar{u}' \otimes \bar{I}' + \chi \bar{u} \otimes \bar{I} + O(\mu^2 \chi),$$

where

$$\bar{u}' = (O(\mu/\chi), O(1/\chi), O(\mu/\chi), O(\mu/\chi), 1, O(1/\chi))^T, \quad \bar{I}' = \begin{pmatrix} 0, 0, 0, 0, D(5,5)/D(5,6), 1 \end{pmatrix},$$

$$\bar{u} = (O(\mu), 1, O(\mu), O(\mu), 0)^T, \quad \bar{I} = (1, O(\mu), O(\mu), O(\mu), 0, 0)$$

and we have used the fact that $\frac{D(5,5)}{D(5,6)} = O\left( \frac{1}{\chi} \right)$. In the limit $\mu \to 0, \chi \to \infty$, we have

$$\bar{u}' = (0, 0, 0, 0, 1, 0)^T, \quad \bar{I}' = (0, 0, 0, 0, 1),$$

$$\bar{u} = (0, 1, 0, 0, 0, 0)^T, \quad \bar{I} = (1, 0, 0, 0, 0, 0)$$

This allows us to compute the limiting values of $\bar{I}$ and $\bar{I}$. Since $dG$ is obtained from $D$ by multiplying from the right and the left by bounded matrices we get

$$dG = \chi^2 \bar{u} \otimes \bar{I} + \chi \bar{u} \otimes \bar{I} + O(\mu^2 \chi),$$

where

$$\bar{u} = [d] \bar{u}', \quad \bar{u} = [d] \bar{u}', \quad \bar{I} = \bar{I}' [a], \quad \bar{I} = \bar{I}' [a].$$

Similarly Proposition 3.7 shows that as $\chi \to \infty, \mu \to 0 \ \bar{u} \to (0, 1, 0, 0, 0, 0)^T$ and it allows us to compute the limiting components of $\bar{u}$ except that we do not have the exact expression for $d\ell_3(\bar{u})$. However we do not need to know this component because we only interested in the span of $\bar{u}$ and $\bar{u}$ and $d\ell_3(\bar{u})$ can be suppressed by subtracting a suitable multiple of $\bar{u}$. It turns out that $\bar{I}$ has the same asymptotics as $\bar{I}'$, and $\bar{u}$ the same as $\bar{u}'$. The functional $\bar{I}$ is the limit of the sixth row of $[a]$, which is also the sixth row of $(i)(I)$. The vector $\bar{u}$ is the fifth column of $[d]$, which is also the fifth column of $(V)(ii'\ell')$. Thus the asymptotic parameters of $dG$ can be summarized as follows:
\[
\mathbf{I} = \left( -\frac{\tilde{G}_4 \tilde{L}_4}{L_4^2 + G_4^2}, 0, 0, 0, \frac{1}{L_4^2 + G_4^2}, -1 \right), \quad \mathbf{\tilde{I}} = (1, 0, 0, 0, 0, 0),
\]
\[
w = \left( 0, 0, 0, 0, 1, -\frac{\tilde{L}_4}{L_4^2 + G_4^2} \right)^T, \quad \mathbf{\tilde{w}} = (0, 1, 0, 0, 0)^T.
\]

3.3. Checking transversality. We study the local map numerically. The \(O(1/\mu)\) part of \(dL\) in Lemma 3.1 is

**Lemma 3.8.** The \(O(1/\mu)\) part of the matrix \(dL = \frac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)}\) is (using the notation of Lemma 3.1):

(a) for the first collision,
\[
l_1 = [\ast, \ast, \ast, \ast, 3.42, -2.54], \quad \hat{u}_1 = [-0.49, 2.71, 0.20, -0.63, -0.20, -0.64].
\]

(b) For the second collision:
\[
l_2 = [\ast, \ast, \ast, \ast, 3.44, -0.47], \quad \hat{u}_2 = [-1.00, 4.40, 0.53, -0.74, 0.34, -0.50].
\]

(c) If \(Q_3\) and \(Q_4\) switch roles after the collisions, the vectors \(\hat{u}_1\) and \(\hat{u}_2\) get a “−” sign. The computation is done using the choice of \(E_3^* = -\frac{1}{2}\) and \(\epsilon_3^* = \frac{1}{2}\), at Gerver’s collision points.

In the above lemma * denote the numbers whose precise numerical values are not needed in the proof.

To check the nondegeneracy condition, it is enough to know the following.

**Lemma 3.9.** Let \(x \in U_i(\delta)\) where \(\delta\) is small enough. If we take the directional derivative of the local map along a direction \(\Gamma_i \in \text{span}\{w_{3-i}, \tilde{w}\}\), such that
\[
\mathbf{\bar{l}}_i \cdot (dL \Gamma_i) = 0,
\]
then
\[
\lim_{\mu \to 0, \lambda \to \infty} \frac{\partial E_3^+}{\partial \Gamma_i} \neq 0,
\]
where \(E_3^+\) is the energy of \(Q_3\) after the close encounter with \(Q_4\).

The proofs of the two lemmas are postponed to Section 12.

Now we can check the nondegeneracy condition.

**Proof of Lemma 3.3.** We prove (b1) and (b2). The proofs of (a1) and (a2) are similar and are left to the reader.
To check (b2), $de_4$ we differentiate $e_4 = \sqrt{1 + (G_4/L_4)^2}$ to get $de_4 = \frac{1}{e_4} \left( \frac{G_4}{L_4^2} dG_4 - \frac{G_4^2}{L_4^3} dL_4 \right)$.

Thus (3.3) gives $de_4w = \frac{G_4}{L_4^2} \neq 0$ as claimed.

Next we check (b1) which is equivalent to the following condition

\begin{equation}
\text{det} \begin{pmatrix}
\ell_2(\hat{u}_2) & \ell_2(\hat{B}_2\Gamma') \\
\ell_2(\hat{u}_2) & \ell_2(\hat{B}_2\Gamma')
\end{pmatrix} \neq 0.
\end{equation}

where $\Gamma' = \ell_2(\hat{w})w_1 - \ell_2(w_1)\hat{w}$.

Let $\Gamma$ be a vector satisfying $\ell_2 \cdot (d\Gamma) = 0$ chosen as follows. $d\Gamma$ is a vector in $\text{span}\{\hat{u}_i, \hat{B}_i\Gamma_i\}$, so it can be represented as $d\Gamma_i = b\hat{u}_2 + b'\hat{B}_2\Gamma'$. Thus we can take $b = -\ell_2 \cdot \hat{B}_2\Gamma'$ and $b' = \ell_2 \cdot \hat{u}_2$ to ensure that $d\Gamma_i \in Ker\ell_2$. Note that we have $b' \neq 0$ by Lemma 3.8. Hence

$$\text{det} \begin{pmatrix}
\ell_2(\hat{u}_2) & \ell_2(\hat{B}_2\Gamma') \\
\ell_2(\hat{u}_2) & \ell_2(\hat{B}_2\Gamma')
\end{pmatrix} = \frac{1}{b'} \text{det} \begin{pmatrix}
\ell_2(\hat{u}_2) & \ell_2(d\Gamma) \\
\ell_2(\hat{u}_2) & \ell_2(d\Gamma')
\end{pmatrix} = \ell_2(d\Gamma).
$$

where the last equality holds since $\ell_2(d\Gamma) = 0$. By Lemma 3.8 $\ell_i = (1, 0, 0, 0, 0, 0)$. Therefore $\ell_2(d\Gamma) = \frac{\partial E_3}{\partial \Gamma}$ and so (b2) follows from Lemma 3.9. \(\square\)

4. \textit{C}^0 \textit{estimates for global map}

4.1. Equations of motion in Delaunay coordinates. We use Delaunay variables to describe the motion of $Q_3$ and $Q_4$ (for reader’s convenience we collect the basic properties of Delaunay variables in Appendix A). We have eight variables $(L_3, \ell_3, G_3, g_3)$ and $(L_4, \ell_4, G_4, g_4)$. We eliminate $L_4$ using the energy conservation and $\ell_4$ will play the role of independent variable.

After setting $v_{3,4} = P_{3,4}/\mu$ and dividing (1.1) by $\mu$ the Hamiltonian takes the form

\begin{equation}
H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{1}{|Q_4 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.
\end{equation}

When $Q_4$ is moving to the left of the section $\{x_4 = -\chi/2\}$, we consider the motion of $Q_3$ as elliptic motion with focus at $Q_2$, and $Q_4$ as hyperbolic motion with focus at $Q_1$, perturbed by other interactions. We can write the Hamiltonian in terms of Delaunay variables as

$$H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4|} - \frac{1}{|Q_3 - (-\chi, 0)|} - \frac{\mu}{|Q_3 - Q_4|}.$$

When $Q_4$ is moving to the right of the section $\{x_4 = -\chi/2\}$, we consider the motion of $Q_3$ as an elliptic motion with focus at $Q_2$, and that of $Q_4$ as a hyperbolic
motion with focus at $Q_2$ attracted by the pair $Q_2, Q_3$ which has mass $1 + \mu$ plus a perturbation. For $|Q_4| \geq 2$ we have the following Taylor expansion

$$\frac{\mu}{|Q_3 - Q_4|} = \frac{\mu}{|Q_4|} + \frac{\mu Q_3 \cdot Q_3}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right).$$

Hence the Hamiltonian takes form

$$H = \frac{v_3^2}{2} + \frac{v_4^2}{2} - \frac{1}{|Q_3|} - \frac{1 + \mu}{|Q_4|} - \frac{1}{|Q_3 - \langle \chi, 0 \rangle|} - \frac{1}{|Q_4 - \langle \chi, 0 \rangle|} - \frac{\mu Q_3 \cdot Q_4}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right).$$

In terms of the Delaunay variables we have

(4.2)

$$H_R = \frac{1}{2L_3^2} + \frac{(1 + \mu)^2}{2L_4^2} - \frac{1}{|Q_3 + \langle \chi, 0 \rangle|} - \frac{1}{|Q_4 + \langle \chi, 0 \rangle|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right).$$

We shall use the following notation. The coefficients of $\frac{1}{2L_4^2}$ in the Hamiltonian will be called $k_L = 1$ and $k_R = 1 + \mu$. The terms in the Hamiltonian containing $Q_4$ will be denoted by

(4.3) $V_R = -\frac{1}{|Q_4 + \langle \chi, 0 \rangle|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right)$, and $V_L = -\frac{1}{|Q_4|} - \frac{\mu}{|Q_3 - Q_4|}$.

Here subscripts L and R mean that the corresponding expressions are used when $Q_4$ is to the left (respectively to the right) of the line $Q = -\langle \chi, 0 \rangle$. Likewise for the terms containing $Q_3$ we define

(4.4) $U_R = -\frac{1}{|Q_3 + \langle \chi, 0 \rangle|} - \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right)$, $U_L = -\frac{1}{|Q_3 - \langle \chi, 0 \rangle|} - \frac{\mu}{|Q_3 - Q_4|}$.

The use of subscripts $R, L$ here is the same as above. Let us write down the full Hamiltonian equations with the subscripts $R$ and $L$ suppressed.

(4.5)

$$\begin{align*}
\dot{L}_3 &= -\frac{\partial Q_3}{\partial Q_3} \cdot \frac{\partial U}{\partial Q_3}, \\
\dot{G}_3 &= -\frac{\partial g_3}{\partial Q_3} \cdot \frac{\partial U}{\partial Q_3}, \\
\dot{L}_4 &= -\frac{\partial L_4}{\partial Q_4} \cdot \frac{\partial V}{\partial Q_4}, \\
\dot{G}_4 &= -\frac{\partial g_4}{\partial Q_4} \cdot \frac{\partial V}{\partial Q_4}.
\end{align*}$$
Next we use the energy conservation to eliminate $L_4$. We have
\begin{equation}
\frac{L_4^3}{R_k^2} = k_R L_3^2 \left( 1 - 3 L_3^2 \left( \frac{1}{|Q_3 + (\chi, 0)|} + \frac{1}{|Q_4 + (\chi, 0)|} \right) + \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} + O \left( \frac{\mu}{|Q_4|^3} \right) + O(1/\chi^2) \right) := k_R L_3^3 + W_R,
\end{equation}
\begin{equation}
\frac{L_4^3}{L_k^2} = k_L L_3^3 \left( 1 - 3 L_3^2 \left( \frac{1}{|Q_3 + (\chi, 0)|} + \frac{1}{|Q_4|} - \frac{\mu}{|Q_4 - Q_3|} + O(1/\chi^2) \right) \right) := k_L L_3^3 + W_L.
\end{equation}

We use $\ell_4$ as the independent variable. Dividing (4.5) by $\dot{\ell}_4$ and using (4.6) to eliminate $L_4$ we obtain
\begin{equation}
\begin{aligned}
\frac{dL_3}{d\ell_4} &= (kL_3 + W) \frac{\partial Q_3}{\partial L_3} \cdot \frac{\partial U}{\partial Q_3} \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right), \\
\frac{d\ell_3}{d\ell_4} &= -(kL_3 + W) \left( \frac{1}{L_3^2} + \frac{\partial U}{\partial Q_3} \right) \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right) + O \left( \frac{\mu}{|Q_4|^3} + 1/\chi^2 \right), \\
\frac{dG_3}{d\ell_4} &= (kL_3 + W) \frac{\partial Q_3}{\partial g_3} \cdot \frac{\partial U}{\partial Q_3} \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right), \\
\frac{dg_3}{d\ell_4} &= -(kL_3 + W) \frac{\partial Q_3}{\partial g_3} \cdot \frac{\partial U}{\partial Q_3} \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right), \\
\frac{dG_4}{d\ell_4} &= (kL_3 + W) \frac{\partial Q_4}{\partial G_4} \cdot \frac{\partial U}{\partial Q_4} \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right), \\
\frac{dg_4}{d\ell_4} &= -(kL_3 + W) \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial U}{\partial Q_4} \left( 1 + (kL_3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right).
\end{aligned}
\end{equation}

We shall use the following notation: $X = (L_3, \ell_3, G_3, g_3), Y = (G_4, g_4)$.

4.2. **A priori bounds.**

4.2.1. **Estimates of positions.** We have the following estimates for the positions.

**Lemma 4.1.** Given $C$ and $\delta$ there exists $C'$ such that if
\begin{equation}
|Q_3| < 2 - \delta, \quad |Q_4| < C
\end{equation}
then
(a) we have
\begin{equation}
\left| \frac{\partial Q_3}{\partial X} \right| < C';
\end{equation}
(b) when $Q_4$ is moving to the right of the section $\{x_4 = -\chi/2\}$ we have

\[
Q_4 \geq 2, \quad \text{if } |\ell^*_4| \leq |Q_4| \leq 2L_4(\ell_4^*)|\ell_4|, \quad \text{if } |\ell_4| \geq C,
\]

where $\ell^*_4$ is the value of $\ell_4$ restricted on $x_4 = -\chi/2$.

when $Q_4$ is moving to the left of the section $\{x_4 = -\chi/2\}$, we have

\[
|Q_4 - Q_1| \leq 2L_4(\ell_4^*)|\ell_4| + C'.
\]

The intuition behind this lemma is the following. Since the total energy of the system is zero and $Q_3$ and $Q_4$ interact only weakly with each other, then both particles have energies close to $1/2L_4^2(\ell^*_4)$ in absolute value. Since $Q_4$ spends most of the time away from $Q_1, Q_2$ and $Q_3$ most of its energy is kinetic energy so it moves with approximately constant speed. Since it makes a little progress in $y$ direction its velocity is almost horizontal most of the time. This explains (4.10), (4.11). To give the complete proof we have to use the Hamiltonian equations. See Section 4.3.

**Lemma 4.2.** If inequalities (4.8), (4.10), (4.11) are valid and in addition

\[
1/C \leq |L_3|, |L_4| \leq C, \quad |G_3|, |G_4| < C,
\]

then we have

\[
\frac{\partial Q_1}{\partial \ell_4} = O(1), \quad \frac{\partial Q_4}{\partial (L_4, G_4, g_4)} = O(\ell_4), \quad \frac{\partial Q_4}{\partial g_4} \cdot Q_4 = 0 \quad \text{and} \quad \frac{\partial Q_4}{\partial G_4} \cdot Q_4 = O(\ell_4)
\]

as $t \to \infty$.

**Proof.** This follows directly from Lemma A.2 in Appendix A.4. □

4.2.2. Estimates of potentials.

**Lemma 4.3.** Under the assumptions of Lemma 4.2 we have the following estimates for the potentials $U, V, W$:

(a) When $Q_4$ is moving to the right of the section $\{x_4 = -\chi/2\}$, we have

\[
V_R, U_R, W_R = O\left(\frac{1}{\chi} + \frac{\mu}{\ell_4^* + 1}\right).
\]

(b) When $Q_4$ is moving to the left of the section $\{x_4 = -\chi/2\}$, we have

\[
V_L, U_L, W_L = O\left(\frac{1}{\chi}\right).
\]

**Proof.** This follows directly from equations (4.3), (4.4) and (4.6) and (4.10) in Lemma 4.1. Our choice of the section $\{x_4 = -\chi/2\}$ excludes the collision between $Q_3$ and $Q_4$. So we put $\frac{\mu}{\ell_4^* + 1}$ to stress the fact that the denominator is bounded.
away from zero. We do the same thing in the following proofs without mentioning it any more. □

4.2.3. Estimates of gradients of potentials. To take partial derivatives w.r.t. Delaunay variables, we use the formulas
\[
\frac{\partial}{\partial X} = \frac{\partial Q_3}{\partial X} \cdot \frac{\partial}{\partial Q_3}, \quad \frac{\partial}{\partial Y} = \frac{\partial Q_4}{\partial Y} \cdot \frac{\partial}{\partial Q_4}.
\]

**Lemma 4.4.** Under the assumptions of Lemma 4.2 we have the following estimates for the gradients of the potentials
\[
\begin{align*}
\partial U_R & = O \left( \frac{1}{\chi^2} + \frac{\mu}{\ell_4^2 + 1} \right), \\
\partial V_R & = O \left( \frac{1}{\chi^2} + \frac{\mu}{|\ell_4|^3 + 1} \right), \\
\frac{\partial Q_4}{\partial (G_4, g_4)} & = O \left( \frac{1}{\chi^2} + \frac{\mu}{|\ell_4|^2 + 1} \right), \\
\frac{\partial V}{\partial (G_4, g_4)} & = O \left( \frac{1}{\chi^2} \right).
\end{align*}
\]

**Proof.** The estimates for the \( \frac{\partial}{\partial Q_3} \) terms are straightforward. Indeed, we only need to use the fact \( \frac{d}{dx} \frac{1}{|x|^k} = \frac{k}{|x|^{k+1}} \) together with the estimates in Lemma 4.1. The estimates of all \( \frac{\partial}{\partial (G_4, g_4)} \) terms are similar. We consider for instance \( \frac{\partial Q_4}{\partial G_4} \frac{\partial V_R}{\partial Q_4} \).

We have
\[
\frac{\partial Q_4}{\partial G_4} \frac{\partial V_R}{\partial Q_4} = \frac{\partial Q_4}{\partial G_4} \frac{Q_4 + (\chi, 0)}{|Q_4 + (\chi, 0)|^3} + O \left( \frac{\mu |\partial Q_4| |Q_4|^{-3}}{|G_4|} \right).
\]

The second term here is \( O(\mu/(\ell_4^2 + 1)) \) due to (4.10) and Lemma A.2(a). To handle the first term let \( \frac{\partial Q_4}{\partial G_4} = (a, b) \), \( Q_4 = (x, y) \). Note that equations (A.3), (A.4), (4.8), (4.10), and (4.12) show that \( x, \ell_4 \) are all comparable in the sense that the ratios between any two of these qualities are bounded from above and below. On the other hand Lemma A.2(a) tells us that \( ax + by = O(\ell_4) \). Since \( by = O(b) = O(\ell_4) \) we conclude that \( ax = O(\ell_4) \) and thus \( a = O(1) \). Thus the first term in (4.14) is
\[
\frac{\partial Q_4}{\partial G_4} \cdot Q_4 + a \chi \\
\frac{|Q_4 + (\chi, 0)|^3}{|Q_4 + (\chi, 0)|^3}.
\]

The numerator here is \( O(\chi) \) while the denominator is at least \( (\chi/2)^3 \). This completes the estimate of \( \frac{\partial Q_4}{\partial G_4} \frac{\partial V_R}{\partial Q_4} \). Other derivatives are similar. □

Plugging the above estimates into (4.7) we obtain the following.

**Lemma 4.5.** Under the assumptions of Lemma 4.2 we have the following estimates on the RHS of (4.7).
(a) When \(-\frac{\chi}{2} \leq x_4 \leq -2\) we have
\[
\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{\ell_4^2 + 1}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).
\]
(b) When \(Q_4\) is moving to the left of the section \(\{x = -\chi/2\}\), we have
\[
\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2}\right), \quad \frac{d\ell_3}{d\ell_4} = O(1).
\]

In Section 6 we will need the following bounds on the second derivatives.

**Lemma 4.6.** Under the assumptions of Lemma 4.2 we have the following estimates for the second derivatives.

\[
\begin{align*}
\frac{\partial^2 U_R}{\partial Q_3^2} &= O\left(\frac{1}{\chi^3} + \frac{\mu}{\ell_4^2 + 1}\right), \quad \frac{\partial^2 V_R}{\partial Q_3^2} = O\left(\frac{1}{\chi^3} + \frac{\mu}{\ell_4^2 + 1}\right), \quad \frac{\partial^2 (U_R, V_R)}{\partial Q_3 \partial Q_4} = O\left(\frac{\mu}{\ell_4^3 + 1}\right), \\
\frac{\partial^2 U_L}{\partial Q_3^2} &= O\left(\frac{1}{\chi^3}\right), \quad \frac{\partial^2 V_L}{\partial Q_3^2} = O\left(\frac{1}{\chi^3}\right), \quad \frac{\partial^2 (U_L, V_L)}{\partial Q_3 \partial Q_4} = O\left(\frac{1}{\chi^3}\right).
\end{align*}
\]

We omit the proof since it is again a direct computation.

### 4.3. Proof of Lemma 4.1

**Proof of Lemma 4.1.** Let \(\tau\) be the maximal time interval such that
\[
\begin{align*}
\frac{3}{4}|L_3(\ell_4^*)| &\leq |L_3| \leq \frac{4}{3}|L_3(\ell_4^*)|, \quad \frac{3}{4}|G_i(\ell_4^*)| \leq |G_i(\ell_4)| \leq \frac{4}{3}|G_i(\ell_4^*)|, \quad i = 3, 4,
\end{align*}
\]
on \([0, \tau]\) where \(\ell_4^*\) is the value \(\ell_4\) restricted on \(\{x_4 = -2\}\). (4.16) implies that \(e_4 = \sqrt{1 + G_3^2/L_3^2}\) is bounded. We always have we have \(|Q_4| \geq 2\) since \(Q_4\) is to the left of the section \(x_4 = -2\). Therefore (4.16) implies that \(L_4 = L_3 + O(\mu)\) in the right case and \(L_4 = L_3 + O(1/\chi)\) in the left case. Now formula (A.3) and Lemma A.1 allows us replace \(\sinh u, \cosh u\) by \((1 + o(1)) \frac{\ell_4^*}{e_4}\) as \(|\ell_4| \to \infty\). Thus
\[
|Q_4| = L_4 \sqrt{L_3^2 \cosh^2 u - e_4^2} + G_4^2 \sinh^2 u = L_4 \sqrt{(L_3^2 + G_3^2)(1 + o(1))^2 \frac{\ell_4^*}{e_4^2}} = L_4^2 (1 + o(1)) |\ell_4|.
\]

This proves estimate (4.10) for \(t \leq \min(\tau, \bar{\tau})\) where \(\bar{\tau}\) is the first time then \(x_4\) reaches \(-\frac{\chi}{2}\). Thus for \(t \leq \min(\tau, \bar{\tau})\) the assumptions of Lemma 4.5 are satisfied and hence
\[
\begin{align*}
\frac{dL_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4 - Q_3|^2}\right)
\end{align*}
\]
(note that to prove the estimates in Lemma 4.5 in the right case we do not need the assumption (4.11)). If we integrate (4.17) w.r.t. \(\ell_4\) on the interval of size \(O(\chi)\) we
find that the oscillations of \( L_3, G_4, G_3 \) are \( O(\mu) \). Therefore \( \bar{\tau} < \tau \) and we obtain the estimates of \( (4.10) \) up to the time \( \bar{\tau} \).

The analysis of the cases when \( Q_4 \) is to the left of the section \( \{x_4 = -\chi/2\} \) and then it travels back from \( \{x_4 = -\chi/2\} \) to \( \{x_4 = -2\} \) is similar once we establish the bounds on the angular momentum at the beginning of the corresponding pieces of the orbit. Let us show, for example, that at the moment then the orbit hits \( \{x_4 = -\frac{\chi}{2}\} \) for the first time, the angular momentum of \( Q_4 \) w.r.t. \( Q_1 \) is \( O(1) \).

Indeed we have already established that \( G_{4R} = -\frac{\chi v_{4y}}{2} - yv_{4x} = O(1) \). Also \( (4.16) \) shows that \( v = O(1) \) and so \( (4.8) \) implies that \( yv_{4x} = O(1) \). Accordingly \( \chi v_{4y} = -G_{4R} - yv_{4y} = O(1) \) and hence \( G_{4L} = G_{4R} + \chi v_{4y} = O(1) \) as claimed. The argument for the second time the orbit hits \( \{x_4 = -\frac{\chi}{2}\} \) is the same. This completes the proof of part (b).

To show part (a), we notice that \( \frac{\partial Q_3}{\partial X} \) depends on \( \ell_3, g_3 \) periodically according to equation \( (A.1) \). So part (a) follows since we have already obtained bounds on \( L_3 \) and \( G_3 \). \( \square \)

The next lemma gives more information about the \( Q_4 \) part of the orbit than Lemma \( 4.1 \). It justifies the assumptions of Lemma \( 4.2 \).

**Lemma 4.7.** Under the same hypothesis as Lemma \( 4.2 \), we have:

(a) when \( Q_4 \) is moving to the right of the section \( \{x = -\chi/2\} \), we have

\[
\tan g_4 = \text{sign}(u) \frac{G_4}{L_4} + O\left(\frac{\mu}{|\ell_4| + 1} + \frac{1}{\chi}\right), \quad |\ell_4|, \chi \to \infty.
\]

(b) when \( Q_4 \) is moving to the left of the section \( \{x = -\chi/2\} \), then \( G_4, g_4 = O(1/\chi) \) as \( \chi \to \infty \).

**Proof.** We prove part (b) first. From equation \( (A.5) \) we see that if \( \ell_4 \) is of order \( \chi \) and \( y = O(1) \) then \( G_4 \cos g_4 + \text{sign}(u)L_4 \sin g_4 = O(1/\chi) \). Integrating the estimates of Lemma \( 4.5 \) (b) we see that during the time \( x_4 \leq -\chi/2 \) we have

\[
(4.18) \quad G_4 = G^* + O(1/\chi), \quad L_4 = L^* + O(1/\chi), \quad g_4 = g^* + O(1/\chi)
\]

where \( (L^*, G^*, g^*) \) are the orbit parameters of \( Q_4 \) then it first hits \( \{x_4 = -\chi/2\} \). It follows that both

\[
G^* \cos g^* + L^* \sin g^* = O(1/\chi), \quad \text{and} \quad G^* \cos g^* - L^* \sin g^* = O(1/\chi).
\]

Since \( L^* \) is not too small this is only possible if \( G^* = O(1/\chi) \), \( g^* = O(1/\chi) \). Now part (b) follows from \( (4.18) \).

The proof of part (a) is similar. Consider for example the case when \( Q_4 \) moves to the right. Now \( (4.18) \) has to be replaced by

\[
(4.19) \quad G_4 = G^* + O\left(\frac{\mu}{|\ell_4| + 1} + \frac{1}{\chi}\right), \quad L_4 = L^* + O\left(\frac{\mu}{|\ell_4| + 1} + \frac{1}{\chi}\right), \quad g_4 = g^* + O\left(\frac{\mu}{|\ell_4| + 1} + \frac{1}{\chi}\right)
\]
(since we use part (a) of Lemma 4.5 rather than part (b)). As before we have

$$G^* \cos g^* + L^* \sin g^* = O(1/\chi).$$

Since $\cos g^*$ can not be too small (since otherwise $G^* \cos g^* - L^* \sin g^* \approx L^* \sin g$ would not be small) we can divide the last equation by $L^* \cos g$ to get $\tan g^* = \frac{G^*}{L^*} + O\left(\frac{1}{\chi}\right)$. Now part (a) follows from (4.19).

4.4. Proof of Lemma 2.4. We begin by demonstrating that the orbits satisfying the conditions of Lemma 2.4 satisfy the assumptions of Lemma 4.5.

Lemma 4.8. (a) Given $\delta, C$ there exist constants $\hat{C}, \mu_0$ such that for $\mu \leq \mu_0$ the following holds. Consider a time interval $[0, T]$ and an orbit satisfying the following conditions

(i) $x_4(t) \in (-\chi, -2)$ for $t \in (0, T)$, $x_4(0) = -2$, $x_4(T) = -\chi$.

(ii) $y_4(0) \leq C$, $y_4(T) \leq C$.

(iii) At time 0, $Q_3$ moves on an elliptic orbit which is completely contained in $\{x_3 \geq -(2 - \delta)\}$.

(iv) $1/C \leq |E_3| \leq C$, $|G_3| \leq C$, $|G_4| \leq C$.

Then $|y_4(t)| \leq \hat{C}$ for all $t \in [0, T]$.

(b) The result of part (a) remains valid if (i) is replaced by

(i') $x_4(t) < -2$ for $t \in (0, T)$, $x_4(0) = x_4(T) = -2$.

Proof. To prove part (a) we first establish a preliminary estimate showing that $Q_4$ travels roughly in the direction of $Q_1$.

Sublemma 4.9. Given $\theta > 0$ there exists $\mu_0, \chi_0$ such that the following holds for $\mu \leq \mu_0$, $\chi > \chi_0$. If the outgoing asymptote satisfies

$$|\pi - \theta_4^+(0)| > \tilde{\theta}$$

then $Q_4$ escapes from the two center system.

Proof. We consider the case $\theta_4^+(0) < \pi - \tilde{\theta}$, the other case is similar. If we disregard the influence of $Q_1$ and $Q_3$ then $Q_4$ would move on a hyperbolic orbit and its velocity would approach $(\sqrt{2E_4} \cos \theta_4^+(0), \sqrt{2E_4} \sin \theta_4^+(0))$. Accordingly given $R$ we can find $\bar{t}, \mu_0$ such that uniformly over all orbits satisfying (i)-(iv) and $\theta_4^+(0) < \pi - \tilde{\theta}$ we have for $\mu \leq \mu_0$

$$y_4(\bar{t}) > R, \quad v_{4y}(\bar{t}) > \sqrt{E_4} \sin \tilde{\theta}.$$

Let $\bar{\ell} = \inf\{t > \bar{t} : v_{4y} < \frac{E_3}{2} \sin \tilde{\theta}\}$. We shall show that $\bar{\ell} = \infty$ which implies the sublemma since for $t \in [\bar{t}, \bar{\ell}]$ we have

$$y_4(t) > R + (\tilde{\ell} - \bar{t}) \frac{E_4}{2} \sin \tilde{\theta}.$$
To see that \( \tilde{t} = \infty \) note that (4.21) implies that
\[
|\dot{v}_{4y}| \leq \frac{1}{(R + (\tilde{t} - \hat{t}) \sqrt{E_4} \sin \theta)^2}
\]
and so
\[
|v_{4y}(\hat{t}) - v_{4y}(\tilde{t})| \leq \int_{0}^{\infty} \frac{ds}{(R + s \sqrt{E_4} \sin \theta)^2} = \frac{2}{R \sqrt{E_4} \sin \theta}.
\]
Hence if \( R \) is sufficiently large we have \( v_{4y}(\hat{t}) \geq \frac{\sqrt{E_4}}{2} \sin \theta \) which is only possible if \( \hat{t} = \infty \).

We now consider the case \( |\pi - \theta^*| < \bar{\theta} \). Arguing as above we see that given \( R \), we can find for \( \mu \) small enough a time \( \hat{t} \) such that
\[
x_4(t) < -R, \quad v_{4x}(\hat{t}) < -\sqrt{E_4} \cos \bar{\theta}.
\]
Let \( \hat{t} \) be the first time after \( \bar{t} \) such that \( x_4 = -(\chi - R) \). Arguing as in Sublemma 4.9 we see that for \( t \in [\bar{t}, \hat{t}] \) we have \( |v_{4x}| \geq \frac{\sqrt{E_4}}{2} \cos \theta \). Hence the force from \( Q_2 \) and \( Q_3 \) is \( O(1/t^2) \) and the force from \( Q_1 \) is \( O(1/(t - t)^2) \). Accordingly \( v_4 \) remains \( O(1) \) so the energy of \( Q_4 \) remains bounded. Next if \( |y_4(\hat{t})| > R \) then the argument of Sublemma 4.9 shows that \( y_4(T) > R/2 \) giving a contradiction if \( R > 2C \). Accordingly we have for \( t \in [\bar{t}, T] \) that \( E_4 = O(1) \), \( y_4 = O(1) \) and \( G_{4L} = O(1) \). It remains to show that \( |y_4(t)| < \bar{C} \) for \( t \in [\bar{t}, \hat{t}] \). To this end let \( t^* \) be the first time when \( x_4 = -\chi/2 \).

We have that \( G_{4L}(t^*) = O(1) \) since \( G_{4L}(\hat{t}) = O(1) \) and for \( t \in [t^*, \hat{t}] \) we have \( G_{4R} = O(1/\chi) \). Likewise \( G_{4R} = O(1) \). Therefore \( \chi v_{4y}(t^*) = G_{4R} - G_{4L} = O(1) \) and so \( v_{4y}(t^*) = O(1/\chi) \). Since \( G_{4L}(t^*) = (\bar{C} v_{4y} - yv_{4x}) \) \( (t^*) \) we have \( y(t^*) = O(1) \). Next for \( t \in [t^*, \hat{t}] \) we have
\[
y_4(t) = y_4(t^*) + v_{4y}(t - t^*) + \int_{t^*}^{t} \int_{t^*}^{s} \ddot{y}_4(s) ds du.
\]
Note that
\[
\ddot{y}_4(s) = O \left( \frac{y}{(Q_4 - Q_1^3)} \right) = O \left( \frac{y}{(\bar{t} - s + R)^3} \right).
\]
Combining the last two estimates we get
\[
|y(t)| \leq C_1 + C_2 \sup_s \{ |y(s)| \} \int_{t^*}^{t} \int_{t^*}^{s} \frac{ds du}{(t - s + R)^3} \leq C + C \left( \frac{1}{R} + \frac{1}{\chi} \right) \sup_s \{ |y(s)| \}.
\]
We choose \( R \) large enough to get that \( |y| \) is bounded on \([t^*, \hat{t}]\). The argument for \([\bar{t}, \hat{t}]\) is the same except that the force from \( Q_3 \) is \( O \left( \frac{\mu y_4}{|Q_4|^3} \right) \). This completes the proof of part (a).

To prove part (b) we note that if \( |y_4(\hat{t})| > R^2 \) then \( Q_4 \) escapes by the argument of Sublemma 4.9. Hence \( |y_4(\hat{t})| < R^2 \). This implies (via already established part (a)
of the lemma) that $y$ is uniformly bounded on $[0, \hat{t}]$. The argument for $[\hat{t}, T]$ is the same with the roles of $Q_1$ and $Q_2$ interchanged. \qed

Proof of Lemma 2.4. Due to the previous lemma, we can use Lemma 4.5.

\[
\frac{dL_3}{d\ell_4}, \frac{dG_3}{d\ell_4}, \frac{dg_3}{d\ell_4}, \frac{dG_4}{d\ell_4}, \frac{dg_4}{d\ell_4} = O\left(\frac{1}{\chi^2} + \frac{\mu}{\ell_4^2 + 1}\right),
\]

For part (a) of Lemma 2.4, we integrate the equations of $dL_3/d\ell_4, dG_3/d\ell_4, dg_3/d\ell_4, dG_4/d\ell_4, dg_4/d\ell_4$ over time of order $\chi$ twice (since $Q_4$ first moves away from $Q_2$ and then comes back). Therefore we get

\[
O\left(2\int_2^\chi \left[\frac{\mu}{\ell_4^2 + 1} + \frac{1}{\chi^2}\right] d\ell_4\right) = O(\mu)
\]

estimate for the change of $L_3, G_3$ and $g_3$ proving part (a). Part (b) of Lemma 2.4 follows from Lemma 4.7. \qed

5. Derivatives of the Poincaré map

In computing $C^1$ asymptotics of both local and global maps we will need formulas for the derivatives of Poincaré maps between two sections. Here we give the formulas for such derivatives for the later reference.

Recall our use of notations. $X$ denotes $Q_3$ part of our system and $Y$ denotes $Q_4$ part. Thus

\[
X = (L_3, \ell_3, G_3, g_3), \quad Y = (G_4, g_4).
\]

$(X, Y)^i$ will denote the orbit parameters at the initial section and $(X, Y)^f$ will denote the orbit parameters at the final section. Likewise we denote by $\ell_4^i$ the initial “time” when $Q_4$ crosses some section, and by $\ell_4^f$ final “time” when $Q_4$ arrives at the next. We abbreviate the RHS of (4.7) as

\[
X' = U, \quad Y' = V.
\]

Here $'$ is the derivative w.r.t. $\ell_4$. We also denote $Z = (X, Y)$ and $W = (U, V)$ to simplify the notations further.

Suppose that we want to compute the derivative of the Poincaré map between the sections $S^i$ and $S^f$. Assume that on $S^i$ we have $\ell_4^i = \ell_4^i(Z^i)$ and on $S^f$ we have $\ell_4^f = \ell_4^f(Z^f)$. We want to compute the derivative $D$ of the Poincaré map along the orbit starting from $(Z^i_\ast, \ell_4^i)$ and ending at $(Z^f_\ast, \ell_4^f)$. We have $D = dF_3dF_2dF_1$ where $F_1$ is the Poincaré map between $S^i$ and $\{\ell_4 = \ell_4^i\}$, $F_2$ is the flow map between the times $\ell_4^i$ and $\ell_4^f$, and $F_3$ is the Poincaré map between $\{\ell_4 = \ell_4^f\}$ and $S^f$. We have $F_1 = \Phi(Z^i, \ell_4^i(Z^i), \ell_4^i)$ where $\Phi(Z, a, b)$ denotes the flow map starting from $Z$ at time $a$ and ending at time $b$. Since

\[
\frac{\partial \Phi}{\partial Z}(Z^i_\ast, \ell_4^i, \ell_4^i) = \text{Id}, \quad \frac{\partial \Phi}{\partial a} = -W
\]
we have \( dF_1 = \text{Id} - \mathcal{W}(\ell^i_4) \otimes \frac{D\ell^i_4}{DZ^i} \). Inverting the time we get

\[
dF_3 = \left( \text{Id} - \mathcal{W}(\ell^f_4) \otimes \frac{D\ell^f_4}{DZ^f} \right)^{-1}.
\]

Finally \( dF_2 = \frac{DZ(\ell^f_4)}{DZ(\ell^i_4)} \) is just the fundamental solution of the variational equation between the times \( \ell^i_4 \) and \( \ell^f_4 \). Thus we get

\[
(5.1) \quad \mathcal{D} = \left( \text{Id} - \mathcal{W}(\ell^f_4) \otimes \frac{D\ell^f_4}{DZ^f} \right)^{-1} \frac{DZ(\ell^f_4)}{DZ(\ell^i_4)} \left( \text{Id} - \mathcal{W}(\ell^i_4) \otimes \frac{D\ell^i_4}{DZ^i} \right).
\]

Next, we study the fundamental solution \( \frac{DZ(\ell^f_4)}{DZ(\ell^i_4)} \) of the variational equation. We consider \( Q_3 \) and \( Q_4 \) individually. The variational equation takes form

\[
\begin{align*}
\left( \frac{\partial X}{\partial X(\ell^i_4)} \right)' &= \frac{\partial U}{\partial X} \frac{\partial X}{\partial X(\ell^i_4)} + \frac{\partial U}{\partial Y} \frac{\partial Y}{\partial X(\ell^i_4)}, \\
\left( \frac{\partial Y}{\partial X(\ell^i_4)} \right)' &= \frac{\partial V}{\partial Y} \frac{\partial X}{\partial X(\ell^i_4)} + \frac{\partial V}{\partial X} \frac{\partial X}{\partial X(\ell^i_4)}, \\
\left( \frac{\partial Y}{\partial Y(\ell^i_4)} \right)' &= \frac{\partial V}{\partial Y} \frac{\partial Y}{\partial Y(\ell^i_4)} + \frac{\partial V}{\partial X} \frac{\partial X}{\partial Y(\ell^i_4)},
\end{align*}
\]

Using the Duhamel principle we see that the solution of the variational equation should satisfy

\[
\begin{align*}
\frac{\partial X(\ell^f_4)}{\partial X(\ell^i_4)} &= U(\ell^i_4, \ell^f_4) + \int_{\ell^i_4}^{\ell^f_4} U(\ell^i_4, \ell_4) \frac{\partial U}{\partial Y} \frac{\partial Y}{\partial X(\ell^i_4)} d\ell_4, \\
\frac{\partial Y(\ell^f_4)}{\partial X(\ell^i_4)} &= V(\ell^i_4, \ell^f_4) + \int_{\ell^i_4}^{\ell^f_4} V(\ell^i_4, \ell_4) \frac{\partial V}{\partial X} \frac{\partial X}{\partial Y(\ell^i_4)} d\ell_4,
\end{align*}
\]

where \( U \) and \( V \) denote the fundamental solutions of \( U' = \frac{\partial U}{\partial X} U \) and \( V' = \frac{\partial V}{\partial Y} V \) respectively.

6. Variational equation

The next step in the proof is the \( C^1 \) analysis of the global map. It occupies sections 6–8. We shall work under the assumptions of Lemma 3.2. In particular we will use the estimates of Section 4 and Appendix A.

The plan of the proof of Proposition 3.6 is the following. Matrices (I), (III) and (V) are treated in Sections 6 and 7. Namely, in Sections 6 we study the variational equation while in Section 7 we describe the contribution of the boundary terms. Finally in Section 8 we compute matrices (II) and (IV) which describe the change
of variables between the Delaunay coordinates with different centers which are used to the left and to the right of the line \( x = -\frac{\chi}{2} \).

6.1. Estimates of the coefficients.

**Lemma 6.1.** We have the following estimates for the RHS of the variational equation.

(a) When \( Q_4 \) is moving to the right of the section \( \{ x = -\chi/2 \} \), we have

\[
\frac{\partial U_R}{\partial X} \begin{bmatrix}
\frac{\partial U_R}{\partial X}
\end{bmatrix} = O \left( \begin{bmatrix}
\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \\
\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} & \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}
\end{bmatrix}
\right)
\]

In addition we have

\[
\frac{\partial V}{\partial Y} = -\frac{1}{\chi} \left[ \begin{array}{cc}
\xi L^4 \text{sign}(\dot{x}_4) & \xi L^3 \\
(G^2 + L^2)(1 - \xi)^3 & -\xi L^3 \\
-\xi L^3 & (1 - \xi)^3 \\
(G^2 + L^2)(1 - \xi)^3 & -\xi L^3 \text{sign}(\dot{x}_4)
\end{array} \right] + O \left( \frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2} \right),
\]

\[
\frac{\partial V}{\partial L_3} = -\frac{1}{\chi} \left( \begin{array}{cc}
-\xi G_4 L_3^4 \text{sign}(\dot{x}_4) & \xi G_4 L_3^4 \\
(L_4^2 + G_4^2)(1 - \xi)^3 & (L^2 + G_4^2)(1 - \xi)^3
\end{array} \right)^T + O \left( \frac{\mu}{\chi} + \frac{\mu}{|Q_4|^2} \right),
\]

where \( \xi = \frac{|Q_4|}{\chi} = \frac{|Q_4 - Q_2|}{\chi} \).
(b) When $Q_4$ is moving to the left of the section $x = -\chi/2$, we have

$$\frac{\partial U}{\partial X} \frac{\partial U}{\partial Y} = O \left( \begin{bmatrix} \frac{1}{\chi} & \frac{1}{\chi} & \frac{1}{\chi} & \frac{1}{\chi} & \mu & \mu \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \mu & \mu \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \mu & \mu \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \mu & \mu \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \mu & \mu \\ \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \frac{1}{\chi^2} & \mu & \mu \end{bmatrix} \right)$$

In addition we have

$$\frac{\partial V}{\partial Y} = -\frac{1}{\chi} \left[ \frac{\xi L^2 \text{sign}(\dot{x}_4)}{(1-\xi)^3} \quad \frac{\xi L^3}{(1-\xi)^3} \right] + O \left( \frac{\mu}{\chi} \right),$$

where $\xi = \frac{|Q_4 - Q_1|}{\chi}$.

Proof. (a) We estimate the four blocks of the derivative matrix separately.

- We begin with $\frac{\partial U_R}{\partial X}$ part. We consider first the partial derivatives of $\ell_3'$ since it is the largest component of $U$. Opening the brackets in the second line of (4.7) we get

$$(6.1) \quad \frac{d\ell_3}{d\ell_4} = -k + \frac{1}{L_3^3} \frac{W + k L_3^3 \frac{\partial Q_3}{\partial L_3} \cdot \frac{\partial U}{\partial Q_3} + k^2 L_3^3 \frac{\partial Q_3}{\partial L_4} \cdot \frac{\partial V}{\partial Q_3} + 2kW \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}}{(1-\xi)^3}.$$ 

Note that by (4.6)

$$(6.2) \quad W_R = k_R 3L_3^5 \left( \frac{1}{|Q_3 + (\chi,0)|} + \frac{1}{|Q_4 + (\chi,0)|} + \frac{\mu Q_4 \cdot Q_3}{|Q_4|^3} \right) + O \left( \frac{\mu}{|Q_4|^3} \right) = O \left( \frac{1}{\chi} + \frac{\mu}{|Q_4|^2} \right).$$

Observe that the RHS of (6.1) depends on $L_3$ in three ways. First, it contains several terms of the form $L_3^m$. Second, $Q_3$ depends on $L_3$ via (A.2). Third, $Q_4$ depends on $L_4$ via (A.5) and $L_4$ depends on $L_3$ via (4.6). In particular we need to consider the contribution to $\frac{\partial \ell_3}{\partial L_3 \partial \ell_4}$ coming from

$$\frac{\partial L_4}{\partial L_3} \frac{\partial}{\partial L_3 \partial \ell_4} = \frac{\partial L_4}{\partial L_3} \frac{\partial Q_4}{\partial L_4} \frac{\partial}{\partial \ell_4}.$$
By Lemma A.2 and equation (4.10) we have $\frac{\partial Q_4}{\partial L_4} = O(|Q_4|)$. Therefore the main contribution to (2,1) entry is $O \left( \frac{1}{\chi} + \frac{\mu}{|Q_4|^2} \right)$ and it comes from $\frac{\partial W_R}{\partial L_3} \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}$. Therefore the main contribution to (2,1) entry is $O \left( \frac{1}{\chi} + \frac{\mu}{|Q_4|^2} \right)$ and it comes from $\frac{\partial W_R}{\partial L_3} \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}$.

For the (2,2), (2,3), (2,4) entries, the computations are similar. We need to act $\frac{\partial}{\partial L_3}, \frac{\partial}{\partial G_3}, \frac{\partial}{\partial g_3}$ on (6.1). (4.6) and (6.2) show that the contribution coming from $\frac{\partial L_4}{\partial (\ell_3, G_3, g_3)} \frac{\partial Q_3}{\partial (\ell_3, G_3, g_3)}$ is $O \left( \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \right)$. It remains to consider the contribution coming from $\frac{\partial L_3}{\partial (\ell_3, G_3, g_3)} \frac{\partial Q_3}{\partial (\ell_3, G_3, g_3)}$.

Now the bound for (2,2), (2,3) and (2,4) entries follows directly from Lemmas 4.1, 4.3, 4.4, and 4.6.

Next, consider (1,1) entry. We need to estimate

$$\frac{\partial}{\partial L_3} \left( (kL_3^3 + W) \frac{\partial Q_3}{\partial \ell_3} \cdot \frac{\partial U}{\partial Q_3} \left( 1 + (kL_3^3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right) \right).$$

Using the Leibniz rule we see that the leading term comes from $\frac{\partial}{\partial L_3} \left( kL_3^3 \frac{\partial Q_3}{\partial \ell_3} \cdot \frac{\partial U}{\partial Q_3} \right)$ and it is of order $O \left( \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2} \right)$. The estimates for other entries of the $\frac{\partial U_R}{\partial X}$ part are similar to the (1,1) entry. This completes the analysis of $\frac{\partial U_R}{\partial X}$.

Next, we consider $\frac{\partial V_R}{\partial Y}$.

Using the Leibniz rule again we see that the main contribution to the derivatives of $V$ comes from differentiating

$$\begin{bmatrix}
L_3^3 \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \\
-L_3^3 \frac{\partial Q_4}{\partial G_4} \cdot \frac{\partial V}{\partial Q_4}
\end{bmatrix}$$

Consider the (5,5) entry. The main contribution to this entry comes from

$$\frac{\partial}{\partial G_4} \left( L_3^3 \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) = L_3^3 \left( \frac{\partial^2 Q_4}{\partial G_4 \partial g_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial^2 V}{\partial Q_4^2} \cdot \frac{\partial Q_4}{\partial G_4} \right).$$

By Lemmas 4.4 and 4.6 the first term is $|Q_4| \cdot O \left( \frac{1}{\chi^2} + \frac{\mu}{|Q_4|^3} \right) = O \left( \frac{1}{\chi} + \frac{\mu}{|Q_4|^2} \right)$ and the second term is $|Q_4|^2 \cdot O \left( \frac{1}{\chi^3} + \frac{\mu}{|Q_4|^4} \right) = O \left( \frac{1}{\chi} + \frac{\mu}{|Q_4|^2} \right)$. This gives the desired upper bound of the (5,5) entry. Notice that $O(1/\chi)$ term comes from
\[
L_3^3 \frac{\partial}{\partial G_4} \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial \tilde{V}}{\partial Q_4} \right)
\]

where \( \tilde{V} = -\frac{1}{|Q_4 + (\chi, 0)|} \). Thus we need to find the asymptotics of

\[
(6.3) \quad L_3^3 \frac{\partial}{\partial G_4} \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{(Q_4 + (\chi, 0))}{|Q_4 + (\chi, 0)|^3} \right).
\]

Let \( \frac{\partial Q_4}{\partial g_4} = (a, b) \). Arguing in the same way as in the estimation of (4.14) we see that \( a = O(1) \). Accordingly the numerator in (6.3) is \( O(\chi) \) so if we differentiate the denominator of (6.3) the resulting fraction will be of order \( O(\chi)O(\chi^{-3}) = O(\chi^{-2}) \). Hence \( O(1/\chi) \) term comes from

\[
\frac{\partial}{\partial G_4} \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{(Q_4 + (\chi, 0))}{|Q_4 + (\chi, 0)|^3} \right).
\]

The numerator here equals to

\[
\frac{\partial}{\partial G_4} \left( \frac{\partial Q_4}{\partial g_4} \cdot Q_4 \right) + \frac{\partial^2 Q_4}{\partial G_4 \partial g_4} \cdot (\chi, 0).
\]

The first term is \( O(\chi) \) due to Lemma A.2 so the main contribution comes from the second term. Using Lemma A.3 we see that (5, 5) entry equals to

\[
- \frac{L_3^3 L_4^4}{\sqrt{L_4^4 + G_4^2}} \frac{\chi \sinh u}{|Q_4 + (\chi, 0)|^3} + O \left( \frac{\mu}{|Q_4|^3} + \frac{\mu}{|Q_4|^2} \right).
\]

Recall that \( L_3 = L_4(1 + o(1)) \) (due to (4.6)) and \( \sinh u = \text{sign}(u) \frac{|\ell_4| L_4}{\sqrt{L_4^4 + G_4^2}} \) (due to (A.4)). Since Lemma 4.1 implies that \( |Q_4| = |\ell_4|/L_4^2(1 + o(1)) \) we obtain that \( O(1/\chi) \)-term in (5, 5) is asymptotic to

\[
- \frac{L_4^4 \text{sign}(u)}{L_4^2 + G_4^2} \frac{\chi |Q_4|}{(\chi - |Q_4|)^3}.
\]

Since \( u \) and \( \dot{x}_4 \) have opposite signs we obtain the asymptotics of \( O(1/\chi) \)-term claimed in part (a) of the Lemma 6.1. The analysis of other entries of \( \frac{\partial V_R}{\partial Y} \) is similar.

- Next, consider the \( \frac{\partial U_R}{\partial Y} \) term.

The analysis of (2, 5) entry is similar to the analysis of (2, 2) entry except that \( \frac{\partial}{\partial G_4} \left( k^2 L_3^3 \frac{\partial Q_4}{\partial L_4} \frac{\partial V}{\partial Q_4} \right) \) contains the term \( k^2 L_3^3 \frac{\partial^2 Q_4}{\partial L_4 \partial G_4} \frac{\partial V}{\partial Q_4} \) which is of order \( O(1/\chi) \).
due to Lemmas 4.6 and A.3 and this term provides the leading contribution for large $t$. The analysis of (2, 6) is similar to (2, 5).

The estimate of the remaining entries of $\frac{\partial U_R}{\partial Y}$ is similar to the analysis of (1, 1) entry.

- Thus to complete the proof of (a) it remains to consider $\frac{\partial V}{\partial X}$. We begin with (5, 1) entry. We need to act by $\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4}$ on

$$\left(kL_3^3 + W\right)\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \left(1 + (kL_3^3 + W)\frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}\right).$$

The leading term for the estimate of (5, 1) comes from

$$\left(\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4}\right) \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) = O\left(1 + \frac{\mu}{|Q_4|^2}\right) = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right).$$

Observe that $O(1/\chi)$ term here comes from $\frac{\partial}{\partial L_4} \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right)$ which can be analyzed in the same way as (5, 5) term. The analysis of (6, 1) is the same as of (5, 1).

The (5, 2) entry is equal to $\left(\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4}\right) \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) \Gamma$ where

$$\Gamma = kL_3^3 + W + k^2L_3^6 \frac{\partial Q_4}{\partial L_4} \frac{\partial V}{\partial Q_4} + 2kL_3^3W \frac{\partial Q_4}{\partial L_4} \frac{\partial V}{\partial Q_4} + W^2 \frac{\partial Q_4}{\partial L_4} \frac{\partial V}{\partial Q_4}.$$

Now the estimate of the (5, 2) entry follows from the following estimates

$$\Gamma = O(1), \quad \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right),$$

$$\left(\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4}\right) \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) = \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4} \left(\frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4}\right) = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right),$$

and

$$\left(\frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4}\right) \Gamma = O\left(\frac{1}{\chi^2} + \frac{\mu}{|Q_4|^2}\right).$$

The remaining entries of $\frac{\partial V}{\partial X}$ are similar to the (5, 2) entry. This completes the proof of part (a).

(b) The estimate of $\frac{\partial V_L}{\partial Y}$ and $\frac{\partial U_L}{\partial X}$ are the same as in part (a) however, now $|Q_4|$ is of order $\chi$ so $O(\mu/|Q_4|^2)$ is dominated by other terms. In addition to compute the leading part we need to use part (c) Lemma A.3 rather than part (b). Moreover, in
order to be able to use the formulas of that Lemma we need to shift the origin to $Q_1$. Therefore the coordinates of $Q_2$ become $(\chi, 0)$. Then we have

$$\frac{\partial V_L}{\partial Y} = L_3^3 \begin{bmatrix} \frac{\partial^2 Q_4}{\partial G \partial g} \cdot \frac{(-\chi, 0)}{|Q_4 - (\chi, 0)|^3} & \frac{\partial^2 Q_4}{\partial g^2} \cdot \frac{(-\chi, 0)}{|Q_4 - (\chi, 0)|^3} \\ -\frac{\partial^2 Q_4}{\partial G \partial g} \cdot \frac{(-\chi, 0)}{|Q_4 - (\chi, 0)|^3} & \frac{\partial^2 Q_4}{\partial G \partial g} \cdot \frac{(-\chi, 0)}{|Q_4 - (\chi, 0)|^3} \end{bmatrix} + O\left(\frac{\mu}{\chi}\right).$$

Now the asymptotic expression of $\frac{\partial V_L}{\partial Y}$ follows directly from Lemma A.3(c). We point out that the “−” sign in front of the matrices of $\frac{\partial V}{\partial Y}$ and $\frac{\partial V}{\partial L_3}$ comes from the fact that the new time $\ell_4$ that we are using satisfies $\frac{d\ell_4}{dt} = -\frac{1}{L_4^3} + o(1)$ as $\mu \to 0, \chi \to \infty$.

Next, we consider the $\frac{\partial U_L}{\partial Y}$ term. First consider (1, 5). We need to find $G_4$ derivative of

$$\left[\frac{\partial Q_3}{\partial \ell_3} \cdot \frac{\partial U}{\partial Q_3}\right] (kL_3^3 + W) \left(1 + (kL_3^3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}\right).$$

Differentiating the first factor we get using Lemma 4.6

$$\frac{\partial}{\partial G_4} \left(\frac{\partial Q_3}{\partial \ell_3} \cdot \frac{\partial U}{\partial Q_3}\right) = \frac{\partial^2 Q_4}{\partial L_4 \partial G_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial}{\partial G_4} \left(\frac{\partial V}{\partial Q_4}\right).$$

When we differentiate the product of the remaining factors then the main contribution comes from

$$\frac{\partial}{\partial G_4} \left(\frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}\right) = \frac{\partial^2 Q_4}{\partial L_4 \partial G_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial}{\partial G_4} \left(\frac{\partial V}{\partial Q_4}\right).$$

To bound the last expression we use Lemma A.3. Namely, the second derivative $\frac{\partial^2 Q_4}{\partial G_4 \partial L_4} = O(1) + \ell_4(0, 1)$, is almost vertical and $\frac{\partial V_L}{\partial Q_4} = \frac{Q_4}{|Q_4|^3} + \frac{\mu(Q_4 - Q_3)}{|Q_4 - Q_3|^3}$ is almost horizontal. This shows that $\frac{\partial^2 Q_4}{\partial G_4 \partial L_4} \frac{\partial V}{\partial Q_4} = \frac{1}{\chi^2}$. The main contribution to the second summand in (6.5) comes from $\frac{\partial}{\partial G_4} \left(\nabla \left(\frac{1}{|Q_4|}\right)\right)$. Using Lemma A.2, we get

$$\frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial}{\partial G_4} \left(\nabla \left(\frac{1}{|Q_4|}\right)\right) = (\ell_4(1, 0) + O(1)) \left(\frac{-\Id}{|Q_4|^3} + 3 \frac{Q_4 \otimes Q_4}{|Q_4|^5}\right) (\ell_4(0, 1) + O(1)) = \frac{1}{\chi^2}.$$

Since $\frac{\partial Q_3}{\partial \ell_3} \cdot \frac{\partial U}{\partial Q_3} = O(1/\chi^2)$ we get the required estimate for (1, 5) entry.
The estimates of other $\frac{\partial U_L}{\partial Y}$ terms are similar to the estimate of (1, 5) entry, except for (2, 5) and (2, 6) entries which are different because $\frac{d\ell_3}{d\ell_4}$ is larger than the other coordinates of $U$.

Now consider (2, 5). We need to compute

\[
- \frac{\partial}{\partial G_1} \left( (kL_3^3 + W) \left( \frac{1}{L_3^3} + \frac{\partial Q_3}{\partial L_3} \cdot \frac{\partial U}{\partial Q_3} \right) \left( 1 + (kL_3^3 + W) \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} \right) \right) 
\]

\[
= - \frac{\partial}{\partial G_1} \left( k + \frac{1}{L_3^3} W + kL_3^3 \frac{\partial Q_3}{\partial L_3} \cdot \frac{\partial U}{\partial Q_3} + k^2 L_3^3 \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} + 2kW \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{1}{\chi^3} \right) 
\]

\[
= 0 + \frac{1}{\chi^2} \frac{\mu}{\chi^2} + \frac{1}{\chi^2} \frac{\mu}{\chi^2} + 1 \frac{\mu}{\chi^3} + 1 \frac{\mu}{\chi^3} = O \left( \frac{1}{\chi^2} \right) 
\]

where the analysis of the leading terms is similar to (6.4), (6.5).

- Finally, we consider $\frac{\partial V_L}{\partial X}$. We begin with (5, 1). We need to compute

\[
\left[ \frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4} \right] \left( \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) \Gamma \right) 
\]

where

\[
\Gamma = kL_3^3 + W + k^2 L_3^3 \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} + 2kL_3^3 W \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4} + W^2 \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial V}{\partial Q_4}. 
\]

The main contribution to \[\left[ \frac{\partial}{\partial L_3} + \frac{\partial L_4}{\partial L_3} \frac{\partial}{\partial L_4} \right] \left( \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) \Gamma \right)\] comes from

\[
\frac{\partial L_4}{\partial L_3} \frac{\partial}{\partial L_4} \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) = \frac{\partial L_4}{\partial L_3} \frac{\partial^2 Q_4}{\partial L_3 \partial Q_4} \cdot \frac{\partial V}{\partial Q_4} + \frac{\partial L_4}{\partial g_4} \frac{\partial Q_4}{\partial L_4} \cdot \frac{\partial^2 V}{\partial Q_4^2} \frac{\partial Q_4}{\partial L_4}. 
\]

The two summands above can be estimated by $O(1/\chi^2)$ by the argument used to bound (6.5). Next a direct calculation shows that $\Gamma = O(1)$, \[\left[ \frac{\partial}{\partial L_3} + \frac{\partial L_4}{\partial L_3} \frac{\partial}{\partial L_4} \right] \Gamma = O(1)\] while $\left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) = O(1/\chi^2)$ by Lemma 4.4 This gives the required bound for the (5, 1) entry. The bound for the (6, 1) entry is similar.

Next, consider (5, 2). It equals to

\[
\left[ \frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4} \right] \left( \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) \Gamma \right) 
\]

The main contribution to \[\left[ \frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4} \right] \left( \frac{\partial Q_4}{\partial g_4} \cdot \frac{\partial V}{\partial Q_4} \right) \] comes from

\[
\frac{\partial}{\partial \ell_3} \left( \frac{\partial Q_4}{\partial g_4} \cdot \nabla \left( \frac{\mu}{Q_4 - Q_3} \right) \right)
\]

and it is of order $O \left( \frac{\mu}{\chi^2} \right)$. On the other hand the main contribution to \[\left[ \frac{\partial}{\partial \ell_3} + \frac{\partial L_4}{\partial \ell_3} \frac{\partial}{\partial L_4} \right] \Gamma\]
comes from $\frac{\partial W}{\partial \ell_3}$ and it is of order $O \left( \frac{1}{\chi^2} \right)$. Combining this with $C^0$ bounds mentioned used in the analysis of (5.1) we obtain the required estimate on the (5, 2) entry. The remaining entries of $\frac{\partial V_l}{\partial X}$ are similar to (5, 2).

6.2. Estimates of the solutions. We integrate the variational equations to get

$$\frac{\partial (X, Y)(\ell_1^i)}{\partial (X, Y)(\ell_1^j)}$$

in equation (5.1).

Recall that map (I) describes the transition between sections $\{x = -2\}$ and $\{x = -\frac{\chi}{2}\}$, map (III) describes the transition between sections $\{x = -\frac{\chi}{2}, \dot{x} < 0\}$ and $\{x = -\frac{\chi}{2}, \dot{x} > 0\}$, and map (V) describes the transition between sections $\{x = -\frac{\chi}{2}\}$, and $\{x = -2\}$.

Lemma 6.2. The following estimates are valid

(a) For maps (I) and (V),

\begin{equation}
\frac{\partial (X, Y)(\ell_1^i)}{\partial (X, Y)(\ell_1^j)} = \begin{bmatrix}
    1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
    O(1) & 1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
    O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) & O(\mu) & O(\mu) \\
    O(\mu) & O(\mu) & O(\mu) & 1 + O(\mu) & O(\mu) & O(\mu) \\
    O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\
    O(1) & O(\mu) & O(\mu) & O(\mu) & O(1) & O(1)
\end{bmatrix}.
\end{equation}

(b) For map (III),

\begin{equation}
\frac{\partial (X, Y)(\ell_1^i)}{\partial (X, Y)(\ell_1^j)} = \begin{bmatrix}
    1 + O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) \\
    O(1) & 1 + O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) \\
    O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1 + O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) \\
    O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & 1 + O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) \\
    O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(1) & O(1) \\
    O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(\frac{1}{\chi}) & O(1) & O(1)
\end{bmatrix}.
\end{equation}

(c) $\frac{\partial Y(\ell_1^i)}{\partial Y(\ell_1^j)}$ has the following limits as $\mu \to 0, \chi \to \infty$

Map (I):

$$\begin{bmatrix}
    1 + \frac{\tilde{L}_2}{2(L_2^3 + G_2^2)} & -\frac{\tilde{L}_2}{2L_4^3}
    \\
    \frac{\tilde{L}_2}{2(L_4^3 + G_2^2)^2} & 1 - \frac{\tilde{L}_2}{2(L_4^3 + G_2^2)^2}
\end{bmatrix},$$

Map (V):

$$\begin{bmatrix}
    1 - \frac{\tilde{L}_2}{2(L_2^3 + G_2^2)} & -\frac{\tilde{L}_2}{2L_4^3}
    \\
    \frac{\tilde{L}_2}{2(L_4^3 + G_2^2)^2} & 1 + \frac{\tilde{L}_2}{2(L_4^3 + G_2^2)^2}
\end{bmatrix},$$
Step 1. Keeping in mind the integrals

Proof. (a) We divide the proof into several steps.

In addition for map (I) we have \( \frac{\partial Y}{\partial L_3} \rightarrow \left( -\frac{\tilde{G}_4 L_4}{2(L_4^2 + G_4^2)}, -\frac{\tilde{G}_4 L_4^2}{2(L_4^2 + G_4^2)^2} \right)^T \).

Steps 1 and 2 imply that \( \delta L_3'(\ell_4) = O \left( \frac{\mu}{\ell_4^4 + 1} \right) \) proving the required bound for \( \delta L_3. \)

Step 4. Plugging the estimates of steps 2 and 3 into the equation for \( \delta L_3 \) we see that \( \langle \delta L_3, \delta g_3 \rangle(\ell_4) = 0 \) and hence \( \delta L_3, \delta G_3, \delta g_3)(\ell_4) = O(\mu) \) for all \( \ell_4 \in [\ell_4^1, \ell_4^2] \).

To prove (c) we need to find the asymptotics of \( V \). Consider map (I) first. \( V \) satisfies

\[
V' = \frac{\partial Y}{\partial Y} V.
\]
By already established part (a) $V = O(1)$ so the above equation can be rewritten as

$$V' = \frac{\xi L^2}{\chi(1 - \xi)^3} A V + O\left(\frac{\mu}{\ell_4^2 + 1} + \frac{\mu}{\chi}\right).$$

where $A = \begin{bmatrix} \frac{L^2}{(G^2 + L^2)} & \frac{L}{L^3} \\ \frac{L^2}{(G^2 + L^2)^2} & \frac{L^2}{(G^2 + L^2)^2} \end{bmatrix}$. Now Gronwall Lemma gives $V \approx \hat{V}$ where $\hat{V}$ is the fundamental solution of $\hat{V}' = \frac{\xi L^2}{\chi(1 - \xi)^3} A \hat{V}$. Using $\xi$ as the independent variable we get

$$\frac{d\hat{V}}{d\xi} = -\frac{\xi}{(1 - \xi)^3} A \hat{V}.$$ Note that $\xi(\ell_i^4) = o(1)$, $\xi(\ell_f^4) = \frac{1}{2} + o(1)$. Making a further time change $d\tau = \frac{\xi d\xi}{(1 - \xi)^3}$ we obtain the constant coefficient linear equation

$$\frac{d\hat{V}}{d\tau} = -A \hat{V}.$$ Observe that $\text{Tr}(A) = \text{det}(A) = 0$ and so $A^2 = 0$. Therefore

$$\hat{V}(\sigma, \tau) = \text{Id} - (\tau - \sigma) A.$$ Since $\tau = \frac{\xi^2}{2(1 - \xi)^2}$ we have $\tau(0) = 0$, $\tau\left(\frac{1}{2}\right) = \frac{1}{2}$. Plugging this into (6.10) we get the claimed asymptotics for map (I). The analysis of map (V) is similar. To analyze map (III) we split

$$\frac{\partial Y(\ell_f^4)}{\partial Y(\ell_i^4)} = \frac{\partial Y(\ell_f^m)}{\partial Y(\ell_i^4)} \frac{\partial Y(\ell_i^m)}{\partial Y(\ell_i^4)}$$

where $\ell_m^4 = \ell_i^4 + \ell_f^4$. Using the argument presented above we obtain

$$\frac{\partial Y(\ell_f^m)}{\partial Y(\ell_i^4)} = \begin{bmatrix} \frac{3}{2} & -\frac{L}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \frac{\partial Y(\ell_i^m)}{\partial Y(\ell_i^4)} = \begin{bmatrix} \frac{3}{2} & -\frac{L}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$ Multiplying the above matrices we obtain the required asymptotics for map (III).

Next using the same argument as in analysis of $\frac{\partial Y(\ell_f^4)}{\partial Y(\ell_i^4)}$ we obtain $\frac{\partial Y}{\partial L_3} \approx \mathbb{W}$ where

$$\mathbb{W}' = \frac{\xi L^2}{\chi(1 - \xi)^3} \left[ A \mathbb{W} + \left( \frac{GL}{L^2 + G^2}, \frac{GL^2}{(L^2 + G^2)^2} \right)^T \right].$$

In terms of the new time this equation reads

$$\frac{d\mathbb{W}}{d\tau} = -\left[ A \mathbb{W} + \left( \frac{GL}{L^2 + G^2}, \frac{GL^2}{(L^2 + G^2)^2} \right)^T \right].$$
Solving this equation using (6.10) and initial condition \((0,0)^T\), we obtain the asymptotics of \(\frac{\partial Y}{\partial L_3}\).

\[\Box\]

7. Boundary contributions and the proof of Proposition 3.6

According to (5.1) we need to work out the boundary contributions in order to complete the proof of Proposition 3.6.

7.1. Dependence of \(\ell_4\) on variables \((X,Y)\). To use the formula (5.1) we need to work out \((U,V)(\ell_4^t) \otimes \frac{\partial \ell_4}{\partial (X,Y)^t}\) and \((U,V)(\ell_4^f) \otimes \frac{\partial \ell_4}{\partial (X,Y)^f}\). Consider \(x_4\) component of \(Q_4\) (see equation (A.5)).

\[x_4 = \cos g_4(L_2^2 \sinh u_4 - e_4) - \sin g_4(L_4 G_4 \cosh u_4)\].

For fixed \(x_4 = -\chi/2\) or \(-2\), we can solve for \(\ell_4\) as a function of \(L_4, G_4, g_4\). From the calculations in the Appendix A.2 Lemma A.2 and the implicit function theorem, we get

for the section \(x_4 = -\chi/2\),

\[\left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4}\right)\bigg|_{x_4=-\chi/2} = (O(\chi), O(1), O(1)),\]

for the section \(x_4 = -2\),

\[\left(\frac{\partial \ell_4}{\partial L_4}, \frac{\partial \ell_4}{\partial G_4}, \frac{\partial \ell_4}{\partial g_4}\right)\bigg|_{x_4=-2} = (O(1), O(1), O(1)).\]

Using equation (4.6) which relates \(L_4\) to \(L_3\), we obtain for the section \(\{x_4 = -\chi/2\}\),

\[
\frac{\partial \ell_4}{\partial (X,Y)}\bigg|_{x_4=-\chi/2} = (O(\chi), O(1/\chi), O(1/\chi), O(1/\chi), O(1), O(1)),
\]

\[(U,V)\bigg|_{x_4=-\chi/2} = (O(1/\chi^2), O(1), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2))^T,\]

For the section \(\{x_4 = -2\}\),

\[
\frac{\partial \ell_4}{\partial L_3}\bigg|_{x_4=-2} = (O(1), O(\mu), O(\mu), O(\mu), O(1), O(1)),
\]

\[(U,V)\bigg|_{x_4=-2} = (O(\mu), O(1), O(\mu), O(\mu), O(\mu), O(\mu))^T.\]

The matrix \((U,V) \otimes \frac{\partial \ell_4}{\partial (X,Y)}\bigg|_{x_4=-\chi/2}\) has rank 1 and the only nonzero eigenvalue is \(O(1/\chi)\), and \((U,V) \otimes \frac{\partial \ell_4}{\partial (X,Y)}\bigg|_{x_4=-2}\) has rank 1 and the only nonzero eigenvalue is \(O(\mu)\). So the inversion appearing in (5.1) is valid.
7.2. **Asymptotics of matrices** \((I), (III), (V)\) **from the Proposition 3.6**. Here we complete the computations of matrices \((I), (III)\) and \((V)\).

**The boundary contribution to** \((I)\). In this case, \(\ell_i^4\) stands for the section \(\{x_4 = -2\}\) and \(\ell_f^4\) stands for the section \(\{x_4 = -\chi/2\}\). So we use equation (7.2) to form \((U, V)(\ell_i^4) \otimes \frac{\partial \ell_i^4}{\partial (X, Y)}\) in equation (5.1) and equation (7.1) to form \((U, V)(\ell_f^4) \otimes \frac{\partial \ell_f^4}{\partial (X, Y)}\)

\[
\left(\begin{array}{cccc}
1 + O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) \\
O(\chi) & 1 + O(1/\chi) & O(1/\chi) & O(1/\chi) \\
O(1/\chi) & O(1/\chi^3) & 1 + O(1/\chi^3) & O(1/\chi^3) \\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & 1 + O(1/\chi^3)
\end{array}\right)
\]

We have

\[
\left(\text{Id} - (U, V)(\ell_i^4) \otimes \frac{\partial \ell_i^4}{\partial (X, Y)}\right)^{-1}
\]

\[
= \begin{pmatrix}
1 + O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & O(1/\chi^3) \\
O(\chi) & 1 + O(1/\chi) & O(1/\chi) & O(1/\chi) \\
O(1/\chi) & O(1/\chi^3) & 1 + O(1/\chi^3) & O(1/\chi^3) \\
O(1/\chi) & O(1/\chi^3) & O(1/\chi^3) & 1 + O(1/\chi^3)
\end{pmatrix}
\]

Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix \((I)\) stated in Proposition 3.6.

**The boundary contribution to** \((III)\)

This time we use equation (7.1) to form both \((U, V)(\ell_i^4) \otimes \frac{\partial \ell_i^4}{\partial (X, Y)}\) and \((U, V)(\ell_f^4) \otimes \frac{\partial \ell_f^4}{\partial (X, Y)}\) in equation (5.1).

The matrix \(\left(\text{Id} - (U, V)(\ell_i^4) \otimes \frac{\partial \ell_i^4}{\partial (X, Y)}\right)^{-1}\) has the same form as (7.3). Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix \((III)\) stated in Proposition 3.6.

**The boundary contribution to** \((V)\)

This time we use equation (7.1) to form \((U, V)(\ell_i^4) \otimes \frac{\partial \ell_i^4}{\partial (X, Y)}\) and equation (7.2) to form \((U, V)(\ell_f^4) \otimes \frac{\partial \ell_f^4}{\partial (X, Y)}\) in equation (5.1).
The matrix \(\left(\text{Id} - (U,Y)(\ell_4^I) \otimes \frac{\partial \ell_4}{\partial (X,Y)^I}\right)^{-1}\) has the form
\[
\begin{bmatrix}
1 + O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu) \\
O(1) & 1 + O(\mu) & O(\mu) & O(\mu) & O(1) & O(1) \\
O(\mu) & O(\mu^2) & 1 + O(\mu^2) & O(\mu^2) & O(\mu) & O(\mu) \\
O(\mu) & O(\mu^2) & O(\mu^2) & 1 + O(\mu^2) & O(\mu) & O(\mu) \\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & 1 + O(\mu) & O(\mu) \\
O(\mu) & O(\mu^2) & O(\mu^2) & O(\mu^2) & O(\mu) & 1 + O(\mu)
\end{bmatrix}
\]

Now we use equation (5.1) and Lemma 6.2 to obtain the asymptotics of the matrix \((V)\) stated in Proposition 3.6.

8. Switching foci

Recall that we treat the motion of \(Q_1\) as a Kepler motion focused at \(Q_2\) and treat it as a Kepler motion focused at \(Q_1\) when it is moving to the left of the section \(\{x = -\chi/2\}\). Therefore, we need to make a change of coordinates when \(Q_4\) crosses the section \(\{x_4 = -\chi/2\}\). These are described by the matrices \((II)\) and \((IV)\). Under this coordinate change the \(Q_3\) part of the Delaunay variables does not change. The change of \(G_4\) is given by the difference of angular momentums w.r.t. different reference points \((Q_1\) or \(Q_2\)). To handle it we introduce an auxiliary variable \(v_{4y}\)-the \(y\) component of the velocity of \(Q_4\). Relating \(g_4\) with respect to the different reference points to \(v_{4y}\) we complete the computation.

8.1. From the right to the left. We have \((II) = \frac{\partial (L_3, \ell_3, G_3, G_{4L}, g_{4L})}{\partial (L_3, \ell_3, G_3, G_{4R}, g_{4R})} |_{x_4 = \chi/2} = (iii)(ii)(i)\) where matrices \((i)\), \((ii)\) and \((iii)\) correspond to the following coordinate changes.

\[
(G, g)_{4R} \left(\frac{\chi}{2}\right) \xrightarrow{(i)} (G, v_y)_{4R} \left(\frac{\chi}{2}\right) \xrightarrow{(ii)} (G, v_y)_{4L} \left(\frac{\chi}{2}\right) \xrightarrow{(iii)} (G, g)_{4L} \left(\frac{\chi}{2}\right).
\]

Computation of matrices \((i)\) and \((iii)\) in Proposition 3.6. \((i)\) is given by the relation

\[
v_{4y} = \frac{1}{L_{4R}} \sinh u_{4R} \sin g_{4R} - \frac{G_{4R}}{L_{4R}^2} \cos g_{4R} \cosh u_{4R}
\]

where last relation follows from (4.6). Recall that by Lemma 4.7, \(g_{4R} = \arctan \frac{G_{4R}}{L_{4R}} + O(1/\chi)\). In addition \([8.1]\) below and the fact that \(G_{4R}\) and \(G_{4L}\) are \(O(1)\) implies \(v_{4y} = O(1/\chi)\). Now the asymptotics of \((i)\) is obtained by a direct computation. We
compute \( \frac{dv_4}{dL_3} \); the other derivatives are similar but easier. We have \( \frac{dv_4}{dL_3} = \frac{dv_4}{dL_4} \frac{dL_4}{dL_3} \).

The second term is \( k_R + O(1/\chi) \). On the other hand

\[
dv_4 = \frac{\partial}{\partial L_4} \left( \frac{1}{L_4} \sin u_4R \sin g_4R - \frac{G_4R}{L_4^2} \cos g_4R \cosh u_4R \right) \frac{1}{1 - e_4R \cosh u_4R} + v_4R \frac{\partial g_4R}{\partial L_4} \cosh u_4R + \frac{\partial v_4R}{\partial L_4} \frac{\partial L_4}{\partial L_3}.
\]

The main contribution comes from the first term which equals

\[-\frac{G_4R}{L_4R(L_4^2 + G_4^2)} + O(1/\chi).\]

The second term is \( O(1/\chi) \) since \( v_4R = O(1/\chi) \). Next rewriting

\[
v_4 = \frac{1}{L_4} \tanh u_4R \sin g_4R - \frac{G_4R}{L_4^2} \cos g_4R \frac{1}{(1/\cosh u_4R) - e_4R}
\]

we see that

\[
\frac{\partial v_4}{\partial L_4} \frac{\partial L_4}{\partial L_3} = O(1/\chi^2) \times O(\chi) = O(1/\chi)
\]

since \( \frac{\partial L_4}{\partial L_3} = O(\chi) \) by (7.1),

(ii) is given by

\[(8.1) \quad G_L = G_R - \chi v_4,
\]

which comes from the simple relation \( v_4 \times Q_4 = v_4 \times (Q_4 - Q_1) + v_4 \times Q_1 \). Here \( G_4R \) and \( v_4 \) are independent variables so the computation of the derivative of (ii) is straightforward.

To compute the derivative of (iii) we use the relation

\[
v_4 = \frac{1}{L_4} \sinh u_4L \sin g_4L - \frac{G_4L}{L_4^2} \cos g_4L \cosh u_4L \frac{1}{1 - e_4L \cosh u_4L}
\]

where \( u_4 < 0 \). Arguing the same way as for (i) and using the fact that by Lemma 4.7, \( G_L, g_L = O(1/\chi) \), \( -\sinh u_4L, \cosh u_4L \simeq \frac{\ell_4L}{e_4L} \) we obtain

\[
\delta v_4L = \frac{\delta G_4L}{k_R^2 L_3^2} + \frac{\delta g_4L}{k_R L_3} + \text{HOT}
\]

Hence

\[
\delta g_4R = -\frac{\delta G_4L}{k_R L_3} + k_R L_3 \delta v_4 + \text{HOT} = -\frac{\delta G_4L/k_R + \chi \delta v_4}{k_R L_3} + \text{HOT}
\]

completing the proof of the lemma. □
8.2. From the left to the right. At this step we need to compute

\[(IV) = \left. \frac{\partial (L_3, L_3, G_3, g_3, G_{4L}, g_{4L})}{\partial (L_3, L_3, G_3, g_3, G_{4L}, g_{4L})} \right|_{x_4 = \chi/2} = (iii')(ii')(i')\]

where the matrices \((iii'), (ii')\) and \((i')\) correspond to the following changes of variables.

\[(G, g)_L \left( \frac{\chi}{2} \right) ^{(i')}(G, v_{4y})_L \left( \frac{\chi}{2} \right) ^{(ii')}(G, v_{4y})_R \left( \frac{\chi}{2} \right) ^{(ii')(i')}(G, g)_R \left( \frac{\chi}{2} \right).\]

Computation of matrices \((iii')(ii')(i')\) in Proposition 3.6. \((i')\) is given by

\[v_{4y} = \frac{1}{L_{4L}} \sinh u_{4L} \sin g_{4L} - \frac{G_{4L}}{L_{4L}^2} \cos g_{4L} \cosh u_{4L} \frac{1}{1 - e_{4L} \cosh u_{4L}} < 0.\]

Here \(u_L > 0\) and \(G_{4L}, g_{4L} = O(1/\chi)\). \((ii')\) is given by

\[G_R = G_L + \chi v_{4yL}.\]

Now the analysis is similar to Subsection 8.1. In particular the main contribution to \([(ii')(i')]_{4L}\) comes from

\[\frac{\partial (G_{4L}, v_{4y})}{\partial (G_{4L}, v_{4y})} = \frac{\partial (G_{4L}, v_{4y})}{\partial (G_{4L}, v_{4y})} \frac{\partial (G_{4L}, g_{4L})}{\partial (G_{4L}, g_{4L})} = \left[ \begin{array}{cc} 1 & \chi \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ \frac{1}{L_3} & O \left( \frac{1}{\chi} \right) \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ \frac{1}{L_3} & O \left( \frac{1}{\chi} \right) \end{array} \right].\]

The analysis of 43 part is similar.

\((iii')\) is given by

\[G_R = G_R, \quad v_{4y} = \frac{1}{L_{4R}} \sinh u_{4R} \sin g_{4R} - \frac{G_{4R}}{L_{4R}^2} \cos g_{4R} \cosh u_{4R} \frac{1}{1 - e_{4R} \cosh u_{4R}} < 0.\]

Here \(u_{4R} < 0\), and by Lemma 4.7, \(\tan g_{4R} = -\frac{G_{4R}}{L_{4R}} + O(1/\chi)\). To get the asymptotics of the derivative we first show that similarly to Subsection 8.1 we have

\[dv_{4y} = \left( -\frac{G_{4R}}{L_3 (k_R^2 L_3^2 + G_{4R}^2)} + O(1/\chi), O(1/\chi^2), O(1/\chi^2), O(1/\chi^2), \frac{1}{k_R^2 L_3^2 + G_{4R}^2}, \frac{1}{k_R L_3} \right)\]

and then take the inverse. \(\square\)

9. Approaching close encounter

In this paper we choose to separate local and global maps by section \(\{x_4 = -2\}\). We could have used instead \(\{x_4 = -10\}\), or \(\{x_4 = -100\}\). Our first goal is to show that the arbitrariness of this choice does not change the asymptotics of derivative of the local map (we have already seen in Sections 8.2 and \ref{sect:8.2} that it does not in change the asymptotics of the derivative of the global map).
We choose the section \( \{|Q_3 - Q_4| = \mu^\kappa\}, \quad 1/3 < \kappa < 1/2 \). Outside the section the orbits are treated as perturbed Kepler motions and inside the section the orbits are treated as two body scattering. We shall estimate the errors of this approximation. We break the orbit into three pieces: from \( \{x_4 = -2, \dot{x}_4 > 0\} \) to \( \{|Q_3 - Q_4| = \mu^\kappa\}\), \( \{|Q_3 - Q_4| = \mu^\kappa\}\) to \( \{|Q_3 - Q_4| = \mu^\kappa\}\) and from \( \{|Q_3 - Q_4| = \mu^\kappa\}\) to \( \{x_4 = -2, \dot{x}_4 > 0\}\).

In this section we consider the two pieces of orbit outside the section \( \{|Q_3 - Q_4| = \mu^\kappa\}\). The Hamiltonian that we use is (4.1). Then we convert the Cartesian coordinates to Delaunay coordinates. The resulting Hamiltonian is

\[
H_L = -\frac{1}{2L_3^2} + \frac{1}{2L_4^2} - \frac{1}{|Q_4 + (\chi,0)|} - \frac{1}{|Q_3 + (\chi,0)|} - \frac{\mu}{|Q_3 - Q_4|},
\]

The difference with the Hamiltonian (4.2) is that we do not do the Taylor expansion to the potential \( \frac{1}{|Q_3 - Q_4|} \).

The next lemma and the remark after it tell us that we can neglect those two pieces.

**Lemma 9.1.** Consider the orbits satisfying the conditions of Lemma 3.1. For the pieces of orbit from \( x_4 = -2, \dot{x}_4 > 0 \) to \( |Q_3 - Q_4| = \mu^\kappa \) and from \( |Q_3 - Q_4| = \mu^\kappa \) to \( x_4 = -2, \dot{x}_4 > 0 \), the derivative matrices have the following form in Delaunay coordinates

\[
\frac{\partial(X,Y)^-}{\partial(x,y)^-} = \left[ \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ O(1) & O(1) & O(1) & O(1) & O(1) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] + O(\mu^{1-2\kappa}+1/\chi^3).
\]

**Proof.** The proof follows the plan in Section 5. We first consider the integration of the variational equation. We treat the orbit as Kepler motions perturbed by \( Q_1 \) and interaction between \( Q_3 \) and \( Q_4 \). Consider first the perturbation coming from the interaction of \( Q_3 \) and \( Q_4 \). The contribution of this interaction to the variational equation is of order \( \frac{\mu}{|Q_3 - Q_4|^3} \). If we integrate the variational equation along an orbit such that \( |Q_3 - Q_4| \) goes from \( -2 \) to \( \mu^\kappa \), then the contribution has the order

(9.1)

\[
O \left( \int_{-2}^\mu \frac{\mu^\kappa}{|t|^3} dt \right) = O(\mu^{1-2\kappa}).
\]

Similar consideration shows that the perturbation from \( Q_1 \) is \( O(1/\chi^3) \).

On the other hand absence of perturbation, all Delaunay variables except \( \ell_3 \) are constants of motion. The (2, 1) entry is also \( o(1) \) following from the same estimate as the (2, 1) entry of the matrix in Lemma 6.1. After integrating over time \( O(1) \), the solutions to the variational equations have the form

\[
\text{Id} + O(\mu^{1-2\kappa}+1/\chi^3).
\]
Next we compute the boundary contributions. The analysis is the same as Section 7.

The derivative is given by formula (5.1). We need to work out \((\mathcal{U}, \mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)}\) and \((\mathcal{U}, \mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)^T}\). In both cases we have

\[
(\mathcal{U}, \mathcal{V}) = (0, 1, 0, 0, 0, 0) + O(\mu^{-2})
\]

For the section \(\{x_4 = -2\}\), we use (7.2). For the section \(\{|Q_3 - Q_4| = \mu^\kappa\}\), we have

\[
\frac{\partial \ell_4}{\partial (X,Y)} = - \left( \frac{\partial |Q_3 - Q_4|}{\partial \ell_4} \right)^{-1} \frac{\partial |Q_3 - Q_4|}{\partial (X,Y)} = - \frac{(Q_3 - Q_4) \cdot \frac{\partial (Q_3 - Q_4)}{\partial \ell_4}}{(Q_3 - Q_4) \cdot \frac{\partial (Q_3 - Q_4)}{\partial \ell_4}}
\]

We will prove in Lemma 10.2(c) below that the angle formed by \(Q_3 - Q_4\) and \(v_3 - v_4\) is \(O(\mu^{-1})\) (the proof of Lemma 10.2 does not rely on section 9). Thus in (9.2) we can replace \(Q_3 - Q_4\) by \(v_3 - v_4\) making \(O(\mu^{-1})\) mistake. Hence

\[
\frac{\partial \ell_4}{\partial (X,Y)} = \frac{(v_3 - v_4) \cdot \frac{\partial (Q_3 - Q_4)}{\partial (X,Y)}}{(v_3 - v_4) \cdot \frac{\partial (Q_3 - Q_4)}{\partial \ell_4}} + O(\mu^{-1})
\]

Note that \(\frac{\partial Q_4}{\partial \ell_4}\) is parallel to \(v_4\). Using the information about \(v_3\) and \(v_4\) from Appendix B.1 we see that \(\langle v_3, v_4 \rangle \neq \langle v_4, v_4 \rangle\). Therefore the denominator in (9.2) is bounded away from zero and so

\[
\frac{\partial \ell_4}{\partial (X,Y)} = (O(1), O(1), O(1), O(1), O(1), O(1)).
\]

We also need to make sure the second component \(\frac{\partial \ell_4}{\partial \ell_3}\) is not close to 1, so that \(\text{Id} - (\mathcal{U}, \mathcal{V})(\ell_4^f) \otimes \frac{\partial \ell_4^f}{\partial (X,Y)}\) is invertible when \(|Q_3 - Q_4| = \mu^\kappa\) serves as the final section. In fact, due to (4.7) \(\frac{\partial \ell_4}{\partial \ell_3} \approx -1\). Using formula (5.1), we get the asymptotics stated in the lemma. \(\square\)

**Remark 8.** Using the explicit value of the vectors \(\hat{I}_2, \hat{I}_3, w, \hat{w}\) in equations (3.3), we find that in the limit \(\mu \to 0, \chi \to \infty\)

\[
\left( \frac{\partial (X,Y)}{\partial (X,Y)(-2)} \right) \text{span}\{w, \hat{w}\} = \text{span}\{w, \hat{w}\}
\]

and

\[
\hat{I}_2 \left( \frac{\partial (X,Y)(-2)}{\partial (X,Y)^+} \right) = \hat{I}_2, \quad \hat{I}_3 \left( \frac{\partial (X,Y)(-2)}{\partial (X,Y)^+} \right) = \hat{I}_3
\]

This tells us that we can neglect the derivative matrices corresponding to the pieces of orbit from \(x_4 = -2, \dot{x}_4 > 0\) to \(|Q_3^- - Q_4^-| = \mu^\kappa\) and from \(|Q_3^+ - Q_4^+| = \mu^\kappa\) to
\[ x_4 = -2, \dot{x}_4 > 0. \]

We thus can identify \( d\mathcal{L} \) with
\[
\frac{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)}{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)} + O(\mu^{1-2\kappa})
\]
where \((L_3, \ell_3, G_3, g_3, G_4, g_4)^\pm\) denote the Delaunay variables measured on the section \( \{|Q_3^\pm - Q_4^\pm| = \mu^\kappa\} \).

10. \( C^0 \) estimate for the local map

In Sections 10 and 12 we consider the piece of orbit from \(|Q_3^- - Q_4^-| = \mu^\kappa\) to \(|Q_3^+ - Q_4^+| = \mu^\kappa\). Because of Remark 8, we simply write \( d\mathcal{L} \) to stand for the derivative for this piece.

10.1. Justifying Gerver’s asymptotics. It is convenient to use the coordinates of relative motion and the motion of mass center. We define
\[
(10.1) \quad v_\pm = v_3 \pm v_4, \quad Q_\pm = \frac{Q_3 \pm Q_4}{2}.
\]

Here ”-“ refers to the relative motion and ”+“ refers to the center of mass motion.

To study the relative motion, we make the following rescaling:
\[
(10.2) \quad q_- := Q_- / \mu, \quad \tau := t / \mu \text{ and } v_- \text{ remains unchanged.}
\]

In this way, we zoom in the picture of \(Q_3\) and \(Q_4\) by a factor \(1/\mu\).

Then we have the following lemma.

**Lemma 10.1.** (a) Inside the sphere \(|Q_-| = \mu^\kappa, 1/3 < \kappa < 1/2\), the motion of the center of mass is a Kepler motion focused at \(Q_2\) perturbed by \(O(\mu^{2\kappa})\).

\[
(10.3) \quad \dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa}).
\]

(b) In the rescaled variables, the relative motion is a Kepler motion focused at the origin perturbed by \(O(\mu^{1+2\kappa})\).

\[
(10.4) \quad \frac{dq_-}{d\tau} = \frac{v_-}{2}, \quad \frac{dv_-}{d\tau} = \frac{q_-}{2|q_-|^3} + O(\mu^{1+2\kappa}).
\]

**Proof.** Note that (10.1) preserves the symplectic form.

\[
\text{The Hamiltonian becomes}
\]
\[
(10.5) \quad H = \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{1}{|Q_+ + Q_-|} - \frac{1}{|Q_+ - Q_-|} - \frac{1}{|Q_+ + Q_- + (\chi, 0)|} - \frac{1}{|Q_+ - Q_- + (\chi, 0)|}
\]
\[
= \frac{|v_-|^2}{4} - \frac{\mu}{2|Q_-|} + \frac{|v_+|^2}{4} - \frac{2}{|Q_+|} + \frac{|Q_+|^2}{2|Q_+|^3} - \frac{3|Q_+ - Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}) + O(1/\chi),
\]
where the $O(\mu^{3\kappa})$ includes the $|Q_-|^3$ and higher order terms. In the following, we drop $O(1/\chi)$ terms since $1/\chi \ll \mu$. So the Hamiltonian equations for the motion of the mass center part are

$$
\dot{Q}_+ = \frac{v_+}{2}, \quad \dot{v}_+ = -\frac{2Q_+}{|Q_+|^3} + O(\mu^{2\kappa})
$$

proving part (a) of the lemma.

Next, we study the relative motion. From equation (10.5), we get the equations of motion for the center of mass

$$
\dot{Q}_- = \frac{v_-}{2}, \quad \dot{v}_- = -\frac{\mu Q_-}{2|Q_-|^3} - \frac{Q_-}{|Q_+|^3} + \frac{3|Q_+ \cdot Q_-|Q_+}{|Q_+|^5} + O(\mu^{2\kappa}),
$$

as $\mu \to 0$, where $O(\mu^{2\kappa})$ includes quadratic and higher order terms of $|Q_-|$. After making the rescaling according to (10.2) the equations for the relative motion part become

$$
\frac{d|Q_-|}{d\tau} = \frac{v_-}{2}, \quad \frac{d\mu}{d\tau} = \frac{q_-}{2|q_-|^3} + \frac{\mu^2 q_-}{|Q_+|^3} - \frac{3\mu^2 Q_+ \cdot q_- |Q_+|}{|Q_+|^5} + O(\mu^{1+2\kappa}). \quad \square
$$

Lemma 10.1 implies the following $C^0$ estimate.

Let $v_{3,4}^-, Q_{3,4}^-$ be the velocities and positions measured at the time when the orbit of the system enters $|Q_3 - Q_4| = \mu^\kappa$ and $v_{3,4}^+, Q_{3,4}^+$ be the velocities and positions measured at the time when the orbit of the system exits $|Q_3 - Q_4| = \mu^\kappa$. Recall that we assume that $1/3 < \kappa < 1/2$.

**Lemma 10.2.** (a) We have the following equations

$$
\begin{align*}
\begin{cases}
    v_3^+ &= \frac{1}{2} R(\alpha)(v_3^- + v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\
    v_4^+ &= \frac{1}{2} R(\alpha)(v_3^- - v_4^-) + \frac{1}{2}(v_3^- + v_4^-) + O(\mu^{(1-2\kappa)/3} + \mu^{3\kappa-1}), \\
    Q_3^+ + Q_4^+ &= Q_3^- + Q_4^- + O(\mu^\kappa), \\
    |Q_3^- - Q_4^-| &= \mu^\kappa, \quad |Q_3^+ - Q_4^+| = \mu^\kappa,
\end{cases}
\end{align*}
$$

where $R(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$.

$$
\alpha = \pi + 2\arctan\left(\frac{G_{in}}{\mu \mathcal{L}_{in}}\right), \quad \text{and} \quad \frac{1}{4\mathcal{L}_{in}^2} = \frac{v_2^2}{4} - \frac{\mu}{2|Q_-|^3}, \quad G_{in} = 2v_- \times Q_-.
$$

(b) We have $\mathcal{L}_{in} = O(1)$. If $\alpha$ is bounded away from 0 and $2\pi$ by an angle independent of $\mu$ then $G_{in} = O(\mu)$ and the closest distance between $Q_3$ and $Q_4$ is bounded away from zero by $\delta \mu$ and from above by $\mu/\delta$ for some $\delta > 0$ independent of $\mu$.

(c) If $\alpha$ is bounded away from $\pi$ by an angle independent of $\mu$. Also, the angle formed by $Q_-$ and $v_-$ is $O(\mu^{1-\kappa})$. 

(d) The time interval during which the orbit stays in the sphere $|Q_-| = \mu^\kappa$ is 
\[ \Delta t = \mu \Delta \tau = O(\mu^\kappa). \]

**Remark 9.** Part (d) is very intuitive. The radius of the sphere $|Q_-| = \mu^\kappa$ is $\mu^\kappa$. The relative velocity is $O(1)$ and it gets larger when $Q_-$ gets closer to the origin. So the total time for the relative motion to stay inside the sphere is $O(\mu^\kappa)$.

**Proof.** In the proof, we omit the subscript in standing for the variables inside the sphere $|Q_-| = \mu^\kappa$ without leading to confusion.

The idea of the proof is to treat the relative motion as a perturbation of Kepler motion and then approximate the relative velocities by their asymptotic values for the Kepler motion.

Fix a small number $\delta_1$. Below we derive several estimates valid for the first $\delta_1$ units of time the orbit spends in the set $|Q_-| \leq \mu^k$. We then show that $\Delta t \ll \delta_1$. It will be convenient to measure time from the orbit enters the set $|Q_-| < \mu^k$.

Using the formula in the Appendix A.1, we decompose the Hamiltonian (10.5) as $H = H_{rel} + \mathfrak{h}(Q_+, v_+)$ where 
\[ H_{rel} = \frac{\mu^2}{4L^2} + \frac{|Q_-|^2}{2|Q_+|^2} - \frac{|Q_+ \cdot Q_-|^2}{2|Q_+|^5} + O(\mu^{3\kappa}), \quad \text{as } \mu \to 0, \]

and $\mathfrak{h}$ depends only on $Q_+$ and $v_+$.

Note that $H$ is preserved and $\dot{\mathfrak{h}} = O(1)$ which implies that $\frac{L}{\mu}$ is $O(1)$ and moreover that ratio does not change much for $t \in [0, \delta_1]$. Using the identity $\frac{\mu^2}{4L^2} = \frac{v^2}{4} - \frac{\mu}{2|Q_-|}$ we see that initially $\frac{L}{\mu}$ is uniformly bounded from below for the orbits from Lemma 2.2. Thus there is a constant $\delta_2$ such that for $t \in [0, \delta_1]$ we have $\delta_2 \mu \leq L(t) \leq \frac{\mu}{\delta_2}$.

Expressing the Cartesian variables via Delaunay variables (c.f. equation (A.3) in Section A.2) we have up to rotation by angle $g$

\[ q_1 = \frac{1}{\mu}L^2 \cosh u = O(\mu^\kappa), \quad q_2 = \frac{1}{\mu}LG \sinh u = O(\mu^\kappa) \]

where $u - e \sinh u = \ell$. We have

\[ G = \frac{\partial H}{\partial Q_-} \frac{\partial Q_-}{\partial g} = O(|Q_-|^2) = O(\mu^{2\kappa}). \]

Since $G = 2v_- \times Q_-$ we conclude that $G(0) = O(\mu^\kappa)$ and hence $G(t) = O(\mu^\kappa)$ for all $t \in [0, \delta_1]$. This shows that $e = O(\mu^{\kappa-1})$. Now equation (10.9) shows that $\cosh u = O(\mu^{\kappa-1})$ and so $\ell = O(\mu^{\kappa-1})$. Next

\[ \ell = -\frac{\partial H}{\partial L} = -\frac{\mu^2}{2L^3} \frac{\partial H_{rel}}{\partial Q_-} \frac{\partial Q_-}{\partial L} = -\frac{\mu^2}{2L^3} + O(\mu^\kappa)O(\mu^{\kappa-1}) = -\frac{\mu^2}{2L^3} + O(\mu^{2\kappa-1}). \]
Since the leading term here is at least \( \frac{\delta^3}{2\mu^2} \) while \( \ell = O(\mu^{\kappa-1}) \) we obtain part (d) of the lemma. In particular the estimates derived above are valid for the time the orbit spends in \( |Q_-| \leq \mu^\kappa \). Next

\[
\dot{g} = -\frac{\partial H}{\partial G} = -\frac{\partial H}{\partial Q_-} \frac{\partial Q_-}{\partial G} = O(\mu^\kappa)O(\mu^{\kappa-1}) = O(\mu^{2\kappa-1}).
\]

Integrating over time \( \Delta t = O(\mu^\kappa) \) we get

\[
|g^+ - g^-| = O(\mu^{3\kappa-1}).
\]

Therefore \( g \) and \( \arctan \frac{G}{L} \) change by \( O(\mu^{3\kappa-1}) \).

We are now ready to derive the first two equations of (10.7). Let us denote till the end of the proof \( \phi = \arctan \frac{G}{L} \), \( \gamma = \frac{1}{2} - \kappa \). Recall (see (A.3)) that

\[
(10.12) \quad p_1 = \tilde{p}_1 \cos g + \tilde{p}_2 \sin g, \quad p_2 = -\tilde{p}_1 \sin g + \tilde{p}_2 \cos g
\]

where

\[
\tilde{p}_1 = \frac{\mu G}{L} \frac{\sinh u}{1 - e \cosh u}, \quad \tilde{p}_2 = \frac{\mu G}{L^2} \frac{\cosh u}{1 - e \cosh u}.
\]

Consider two cases.

(I) \( G \leq \mu^\kappa + \gamma \). In this case on the boundary of the sphere \( |Q_-| = \mu^\kappa \) we have

\[
\ell > \delta_3 \mu^{-\gamma}
\]

for some constant \( \delta_3 \). Thus

\[
\frac{p_2}{p_1} = \frac{\mu G}{L^2} \frac{\cosh u \cos g + \mu}{L} \frac{\sinh u \sin g}{L} = \frac{G}{L} \pm \tan g + O(\mu^{-2\gamma}) = \tan(g \pm \phi) + O(\mu^{-2\gamma}).
\]

where the plus sign is taken if \( u > 0 \) and the minus sign is taken if \( u < 0 \). Since \( \arctan \) is globally Lipschitz, this completes the proof in case (I).

(II) \( G > \mu^\kappa + \gamma \). In this case \( \frac{G}{L} \gg 1 \) and so it suffices to show that \( \frac{p_2}{p_1} \) (or \( \frac{p_1}{p_2} \)) changes little during the time the orbit is inside the sphere. Consider first the case where \( |g^-| > \frac{\pi}{4} \) so \( \sin g \) is bounded from below. Then

\[
\frac{p_2}{p_1} = \cot g + O(\mu^{1-(\kappa+\gamma)})
\]

proving the claim of part (a) in that case. The case \( |g^-| \leq \frac{\pi}{4} \) is similar but we need to consider \( \frac{p_1}{p_2} \). This completes the proof in case (II).

Combining equation (10.3) and Lemma 10.1(c) we obtain

\[
(10.13) \quad Q^+_\pm = Q^-_\pm + O(\mu^\kappa).
\]

We also have \( Q^+_\pm = Q^-_\pm + O(\mu^\kappa) \) according to the definition of the sections \( \{ |Q^\pm| = \mu^\kappa \} \). This proves the last two equations in (10.7). Plugging (10.13) into (10.3) we see that

\[
v^+_\pm = v^-_\pm + O(\mu^\kappa).
\]
This completes the proof of part (a).

The first claim of part (b) has already been established. The estimate of $G$ follows from the formula for $\alpha$. The estimate of the closest distance follows from the fact that if $\alpha$ is bounded away from 0 and $2\pi$ then the $Q_-$ orbit of $Q_-(t)$ is a small perturbation of Kepler motion and for Kepler motion the closest distance is of order $G$. We integrate the $\dot G$ equation (10.10) over time $O(\mu^\kappa)$ to get the total variation $\Delta G$ is at most $\mu^{3\kappa}$, which is much smaller than $\mu$. So $G$ is bounded away from 0 by a quantity of order $O(\mu)$.

Finally part (c) follows since we know $G = \mu^\kappa|v_-| \sin \angle(v_-, Q_-) = O(\mu)$. □

Proof of Lemma 2.2. Letting $\mu = 0$ in the first two equations of (10.7) we obtain the equations of elastic collisions. Namely, both the kinetic energy conservation
\[
|v_3^+|^2 + |v_4^+|^2 = |v_3^-|^2 + |v_4^-|^2
\]
and momentum conservation
\[
v_3^+ + v_4^+ = v_3^- + v_4^-
\]
laws hold. On the other hand, the Gerver’s map $G$ in Lemma 2.2 is also defined through elastic collisions. The assumption $\bar \theta^+ = \pi + O(\mu)$ implies that $\alpha$ in (10.7) is $\mu$ close to its value in Gerver’s case. As a result, Lemma 10.2 says actually the same thing as Lemma 2.2 up to a change of variables going from Cartesian coordinates to the set of variables $E_3, \ell_3, G_3, g_3, G_4, g_4$. □

10.2. Proof of Lemma 2.3.

Proof. We follow the same argument as in the proof of Lemma 2.2 to get that the orbit of $Q_3$ is a small (of order $O(\bar \theta)$) deformation of Gerver’s $Q_3$ ellipse. So we only need to prove this lemma in Gerver’s setting. Since the $Q_3$ ellipse has semimajor 1 in Gerver’s case, the distance from the apogee to the focus is strictly less than 2. Therefore we can find some $\delta > 0$ such that $|Q_3| \leq 2 - 2\delta$ in the Gerver case. The rest of the proof is similar to the proof of Lemma 4.1. □

11. Consequences of $C^0$ estimates

Here we obtain corollaries $C^0$ estimates for the local end global maps. Namely, in subsection 11.1 we show that the orbits we construct are collision free. In subsection 11.2 we show that the angular momentum can be prescribed freely during the consecutive iterations of the inductive scheme, that is, we prove Sublemma 3.4.

11.1. Avoiding collisions. Here we exclude the possibility of collisions. The possible collisions may occur for the pair $Q_3, Q_4$ and the pair $Q_1, Q_4$. The fact that there is no collision between $Q_4$ and $Q_1$ is a consequence of the following result.
Lemma 11.1. If an orbit satisfies the conditions of Lemma 4.1 and there is a collision between $Q_4$ and $Q_1$ then we have $\bar{G}_4 + G_4 = O(\mu)$ where $\bar{G}_4$ denotes the angular momentum of $Q_4$ after the application of the global map.

Proof. Since we are concerned with the orbit of $Q_4$ it is convenient to use $L_4$ instead of $L_3$. $L_4$ satisfies the equation $L_4' = -\frac{dt}{d\ell_4} \frac{\partial V_L}{\partial \ell_4} = O(1/\chi^2)$, where $\partial \mu$ means the $d/d\ell_4$ derivative. We write the equations of motion as $Y' = V$, where $Y = (L_4, G_4, g_4)$ and $V$ is the RHS of the Hamiltonian equations (4.5).

We run the orbit coming to a collision backward so that we can compare it to the orbits exiting collision. We shall use the subscript $in$ to refer to the orbit coming to collision with time direction reversed the subscript $out$ for the orbit exiting collision. We have

$$ (Y_{in} - Y_{out})' = O \left( \left\| \frac{\partial V}{\partial Y} \right\| \frac{1}{|Q_4 - Q_3|^2} \right) $$

where the last term comes from the $\frac{\mu}{|Q_4 - Q_3|}$ term in the potential $V_L$. We integrate this estimate for $\ell_4$ starting from the collision and ending when the outgoing orbit hits the section $\{x_4 = -\chi/2\}$. The initial condition is $Y_{in} - Y_{out} = 0$ since $L_4, G_4, g_4$ assume the same values before and after the $Q_4-Q_1$ collision. Next, we have

$$ \int_{\ell_4}^{\ell_4'} \frac{\partial V}{\partial Y} d\ell_4 = O(1), \quad \int_{\ell_4}^{\ell_4'} O \left( \frac{\mu}{|Q_4 - Q_3|^2} \right) d\ell_4 = O(\mu/\chi) $$

and the Gronwall Lemma imply that

$$ (Y_{in}(\ell_4') - Y_{out}(\ell_4')) = O(\mu/\chi). $$

Next we estimate the angular momentum of $Q_4$ w.r.t. $Q_2$. We have

$$ G_{4R} = G_{4L} + v_4 \times (-\chi, 0) = G_{4L} + v_{4y} \chi, $$

where $v_{4y}$ is the $y$ component of the velocity of $Q_4$ at the time the orbit hits the section $\{x_4 = -\chi/2\}$. Using the equation (A.5) in the Appendix A.2 and Lemma 4.7 we see that for the orbits of interest

$$ v_{4y} = \frac{k}{L_4^2} (L_4 \sin g_4 - G_4 \cos g_4) + O \left( \frac{1}{\chi^2} \right). $$

Now (11.1) shows that $v_{4y,in} - v_{4y,out} = O(\mu/\chi)$. Hence (11.2) implies that

$$ G_{4R,in} - G_{4R,out} = O(\mu). $$
Finally the proof of Lemma 4.1 shows that the angular momentum of $Q_4$ w.r.t. $Q_2$ changes by $O(\mu)$ during the time the orbits moves from the section $\{x_4 = -\chi/2\}$ to the section $\{x_4 = -2\}$. □

Now we exclude the possibility of collisions between $Q_3$ and $Q_4$. Note that $Q_3$ and $Q_4$ have two potential collision points corresponding to two intersections of the ellipse of $Q_3$ and the branch of the hyperbola utilized by $Q_4$. See Fig 1 and 2 in Section 2.3. Now it follows from Lemma 10.2(b) that $Q_3$ and $Q_4$ do not collide near the intersection where they have the close encounter. We need also to rule out the collision near the second intersection point. This was done by Gerver in [G2]. Namely he shows that the time for $Q_3$ and $Q_4$ to move from one crossing point to the other are different. As a result, if $Q_3$ and $Q_4$ come to the correct intersection points nearly simultaneously, they do not collide at the wrong points. To see that the travel times are different recall that by second Kepler’s law the area swiped by the moving body in unit time is a constant for the two-body problem. In terms of Delaunay coordinates, this fact is given by the equation $\ell = \pm \frac{1}{L_3}$ where $-$ is for hyperbolic motion and $+$ for elliptic. In our case, we have $L_3 \approx L_4$ when $\mu \ll 1, \chi \gg 1$. Therefore in order to collide $Q_3$ and $Q_4$ must swipe nearly the same area within the unit time. We see from Fig 1 and Fig 2, the area swiped by $Q_4$ is a proper subset of that by $Q_3$ between the two crossing points. Therefore the travel time for $Q_4$ is shorter.

11.2. Choosing angular momentum.

Proof of the Sublemma 3.3 The proof consists of three steps.

Step 1. We show that there exists an orbit satisfying the assumptions of Lemma 4.1. The proof of Lemma 11.1 shows that to prove this it suffices demonstrate the existence of the orbit where $Q_4$ collides with $Q_1$ since for the collision orbit $Q_4$ nearly returns back to its initial position. Next Sublemma 4.9 shows that if after the application of the local map we have $\theta_4^+(0) = \pi + \bar{\theta}$ then the orbit hits the line $x_4 = -\chi$ so that its $y_4$ coordinate is a large positive number and if $\theta_4^+(0) = \pi - \bar{\theta}$ then the orbit hits the line $x_4 = -\chi$ so that its $y_4$ coordinate is a large negative number. Therefore due to the Intermediate Value Theorem it suffices to show that our curve contains points $x_1, x_2$ such that $\theta_4^+(x_1) = \pi - \bar{\theta}$, $\theta_4^+(x_2) = \pi + \bar{\theta}$. We have the expression $\theta_4^+ = g_4 - \arctan \frac{G_4}{L_4}$. Then we use Lemma 3.1 and the numerics in Lemma 3.8 to show that $\frac{\partial \theta_4^+}{\partial \ell_3} = c(x)/\mu$ with $c(x) \neq 0$ continuous w.r.t. $x$ in a neighbourhood of Gerver’s collision point in Lemma 3.1. We choose $\bar{\theta} \ll 1$ but independent of $\mu$ such that the assumption of Lemma 3.1 and Sublemma 4.9 is satisfied.
Step 2. We show that there exists an orbit such that $e_4(\mathcal{P}(\ell_3, \tilde{e}_4))$ is close to $e_4^*$. To this end we use the strong expansion of the Poincaré map. Namely, if we fix $e_4 = \tilde{e}_4$, then $\mathcal{P}$ becomes a function of one variable $\ell_3$. By working out the vectors and functionals of Lemma 3.1 and Lemma 3.2 given by (3.3), we see that $\frac{\partial}{\partial \ell_3}$ does not lie in the $\text{Ker} I_\ell$. Therefore $d\mathbb{L}(x) \frac{\partial}{\partial \ell_3} = \frac{c(x)}{\mu} u_i(x) + O(1)$, where $c$ depends on $x$ continuously. Next, Lemma 3.3 shows that $\tilde{I}_i(\mathbb{L}(x)) \cdot u_i(x) \neq 0$ and that $\tilde{I}_i$ contains nonzero $\partial/\partial e_4$ component. Therefore the projection of $\mathcal{P} = G \circ \mathbb{L}$ to the $e_4$ component, i.e. $\tilde{e}_4(\ell_3, \tilde{e}_4) : T^1 \to \mathbb{R}^1$ as a function of $\ell_3$ with $e_4 = \tilde{e}_4$ fixed is strongly expanding. Namely, its derivative is bounded from below by $\frac{\hat{c}_\chi^2}{\mu}$ provided that the assumptions of Lemma 4.1 are satisfied (for the orbits of interest this will always be the case according to Lemma 4.8). However, the map $\tilde{e}_4(\ell_3, \tilde{e}_4)$ is not injective. We study $G_4(\ell_3, \tilde{e}_4)$ instead of $\tilde{e}_4(\ell_3, \tilde{e}_4)$ using the relation $e = \sqrt{1 + 2(G/L)^2}$. We have the same strong expansion for $G_4(\ell_3, \tilde{e}_4)$ since our estimates of the $d\mathbb{L}_\ell$, $dG$ are done using $G_4$ instead of $e_4$. Thus it follows from the strong expansion of the map $G_4(\ell_3, \tilde{e}_4)$ that a $R$-neighborhood of $G_4^*$ (corresponding to $e_4^*$) is covered if $\ell_3$ varies in a $\frac{R\mu}{\hat{c}_\chi^2}$-neighborhood. Taking $R$ large we can ensure that $G_4$ changes from a large negative number to a large positive number. Then we use the intermediate value theorem to find $\ell_3$ such that $|G_4 - G_4^*| < \sqrt{\delta}$, hence $|\tilde{e}_4 - e_4^*| < \sqrt{\delta}$.

Step 3. We show that for the orbit just constructed $\mathcal{P}(\ell_3, \tilde{e}_4) \in U_2(\delta)$. Since $e_4$ changes substantially $Q_4$ must pass close to $Q_1$ and hence $\mathbb{L}(\tilde{e}_4, \tilde{I}_3)$ must have $\theta_4^*$ small. Therefore by Lemma 2.2 $\mathbb{L}(\tilde{e}_4, \tilde{I}_3)$ has $(E_3, e_3, g_3)$ close to $G_{\tilde{e}_4, 2.4}(E_3(\tilde{e}_4, \tilde{I}_3), e_3(\tilde{e}_4, \tilde{I}_3), g_3(\tilde{e}_4, \tilde{I}_3))$. It follows that

$$|E_3 - E_3^*| < K\delta, \quad |e_3 - e_3^*| < K\delta, \quad |g_3 - g_3^*| < K\delta.$$  

Next Lemma 2.4 shows that after the application of $G$, $(E_3, e_3, g_3)$ change little and $\theta_4^*$ becomes $O(\mu)$. \hfill $\square$

12. Derivative of the local map

12.1. Justifying the asymptotics. Here we give the proof of Lemma 3.1. Our goal is to show that the main contribution to the derivative comes from differentiating the main term in Lemma 10.2.

Proof of Lemma 3.1. Since the transformation from Delaunay to Cartesian variables is symplectic and the norms of the transformation matrices are independent of $\mu$, it is sufficient to prove the lemma in terms of Cartesian coordinates. To go to the coordinates system used in Lemma 3.1, we only need to multiply the Cartesian derivative matrix by $O(1)$ matrices, namely, by $\frac{\partial(L_3, \ell_3, G_3, g_3, G_4, g_4)^+}{\partial(Q_3, v_3, Q_4, v_4)^+}$ on the left.
and by \( \frac{\partial (Q_3, v_3, Q_4, v_4)}{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)} \) on the right. This does not change the form of the \( dL \) stated in Lemma 3.1.

As before we use the formula (5.1). We need to consider the integration of the variational equations and also the boundary contribution. The proof is organized as follows. The main part of the proof is the study of the relative motion part, while controlling the motion of the mass center is easier.

For the relative motion part, we use the Delaunay variables \((L, \ell, G, g)\). Using \( \ell \) as the time variable we get from (10.5) that the equations for relative motion take the following form (recall that the scale for \( \ell \) is \( O(\mu \kappa - 1) \)):

\[
\begin{align*}
\frac{\partial L}{\partial \ell} &= -2\mu^{-2}L^3 \frac{\partial H}{\partial \ell} \left( 1 - 2\mu^{-2}L^3 \frac{\partial H}{\partial L} + \ldots \right) = O(\mu^{2+\kappa}), \\
\frac{\partial G}{\partial \ell} &= -2\mu^{-2}L^3 \frac{\partial H}{\partial g} \left( 1 - 2\mu^{-2}L^3 \frac{\partial H}{\partial L} + \ldots \right) = O(\mu^{1+2\kappa}), \\
\frac{\partial g}{\partial \ell} &= 2\mu^{-2}L^3 \frac{\partial H}{\partial G} \left( 1 - 2\mu^{-2}L^3 \frac{\partial H}{\partial L} + \ldots \right) = O(\mu^{2\kappa}),
\end{align*}
\]

where \( \ldots \) denote the lower order terms. The estimates of the last two equations follow from (10.10) and (10.11) while the estimate of the first equation is similar to the last two.

Then we analyze the variational equations.

\[
\begin{align*}
\delta \frac{dL}{d\ell} = &\ O \left( \begin{pmatrix}
\mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{1+2\kappa} \\
\mu^{1+\kappa} & \mu^{2\kappa} & \mu^{1+2\kappa} \\
\mu^{2\kappa-1} & \mu^{2\kappa-1} & \mu^{2\kappa}
\end{pmatrix} \begin{pmatrix}
\delta L \\
\delta G \\
\delta g
\end{pmatrix}\right) + O \left( \begin{pmatrix}
\mu^{2+\kappa} & 0 \\
\mu^{1+2\kappa} & 0 \\
\mu^{2\kappa} & 0
\end{pmatrix} \begin{pmatrix}
\delta Q_+ \\
\delta v_+
\end{pmatrix}\right).
\end{align*}
\]

In the following, we first set \( \delta Q_+ = 0 \) and work with the fundamental solution of the homogeneous equation. Then we will prove that \( \delta Q_+ \) is negligible.

Introducing \( \delta \mathcal{G} = \frac{\delta g}{\mu^{3\kappa-2}} \) we need the asymptotics of the fundamental solution of

\[
\begin{align*}
\delta \frac{dL}{d\mathcal{G}} = &\ O \left( \begin{pmatrix}
\mu^{1+\kappa} & \mu^{1+\kappa} & \mu^{5\kappa-1} \\
\mu^{1+\kappa} & \mu^{2\kappa} & \mu^{5\kappa-1} \\
\mu^{1-\kappa} & \mu^{1-\kappa} & \mu^{2\kappa}
\end{pmatrix} \begin{pmatrix}
\delta L \\
\delta G \\
\delta \mathcal{G}
\end{pmatrix}\right).
\end{align*}
\]

Integrating this equation over time \( \mu^{\kappa-1} \) we see that the fundamental solution is \( O(1) \). Now arguing the same way as in section 6.2 we see that the fundamental
inside the sphere

Next, we compute the boundary contribution. In terms of the Delaunay variables

\[ \frac{\partial \ell}{\partial (L, G, g)} = \left( \frac{\partial |Q_-|}{\partial \ell} \right)^{-1} \frac{\partial |Q_-|}{\partial (L, G, g)} = (O(\mu^{\kappa-1}), O(\mu^{\kappa-2}), 0). \]

Indeed, due to (10.9) we have \( \frac{\partial |Q_-|}{\partial g} = 0, \frac{\partial |Q_-|}{\partial \ell} = O(\mu), \frac{\partial |Q_-|}{\partial \ell} = O(\mu) \) and \( \frac{\partial |Q_-|}{\partial G} = O(\mu^{\kappa-1}) \). Combining this with (12.1) we get

\[ \frac{\partial \ell, \partial G, \partial g}{\partial (L, G, g)} \otimes \frac{\partial \ell}{\partial (L, G, g)} = O \left( \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix} \right). \]

Using (5.1) we obtain the derivative matrix

\[ \frac{\partial (L, G, g)}{\partial (L, G, g)} = \left( \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix} \right)^{-1} \]

\[ \times \left( \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix} \right) \left( \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix} \right) \]

\[ = \text{Id} + O \left( \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & 0 \\ \mu^{3\kappa} & \mu^{3\kappa-1} & 0 \\ \mu^{3\kappa-1} & \mu^{3\kappa-2} & 0 \end{bmatrix} \right) \]

We are now ready to compute the relative motion part of the derivative of the Poincaré map. For the space variables, we are only interested in the angle \( \Theta := \arctan \left( \frac{q_2}{q_1} \right) \) since the length \(|(q_1, q_2)| \) is fixed when restricted on the sphere.

We split the derivative matrix as follows:

\[ \frac{\partial (\Theta, v_-)}{\partial (\Theta, v_-)} = \frac{\partial (\Theta, v_-)}{\partial (L, G, g)} \frac{\partial (L, G, g)}{\partial (L, G, g)} \]

\[ = \text{Id} + \frac{\partial (\Theta, v_-)}{\partial (L, G, g)} \]
NON-COLLISION SINGULARITIES IN THE PLANAR TWO-CENTER-TWO-BODY PROBLEM

\[ \frac{\partial (\Theta_-, v_-)^+}{\partial (\mathcal{L}, G, g)^+} \frac{\partial (\mathcal{L}, G, g)^-}{\partial (\Theta_-, v_-)^-} + \frac{\partial (\Theta_-, v_-)^+}{\partial (\mathcal{L}, G, g)^+} P \frac{\partial (\mathcal{L}, G, g)^-}{\partial (\Theta_-, v_-)^-} = I + II. \]

Using equations (10.9) and (10.12) we obtain

\[ (12.10) \quad \frac{\partial (\Theta_-, v_-)^+}{\partial (\mathcal{L}, G, g)^+} = O \left( \begin{bmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{bmatrix} \right). \]

Next, we consider the first term in (12.9).

\[ (12.11) \quad I = \frac{\partial (\Theta_-, v_-)^+}{\partial \mathcal{L}^+} \otimes \frac{\partial \mathcal{L}^-}{\partial (\Theta_-, v_-)^-} + \frac{\partial (\Theta_-, v_-)^+}{\partial G^+} \otimes \frac{\partial G^-}{\partial (\Theta_-, v_-)^-}. \]

Using the expressions

\[ \frac{1}{4\mathcal{L}^2} = \frac{v_-^2}{4} - \frac{\mu}{2|Q_-|}, \quad G = v_- \times Q_- = |v_-| \cdot |Q_-| \sin \angle(v_-, Q_-) \]

we see that

\[ (12.12) \quad \frac{\partial \mathcal{L}^-}{\partial (\Theta_-, v_-)^-} = O(1), \quad \frac{\partial G^-}{\partial (\Theta_-, v_-)^-} = (O(\mu^\kappa), O(\mu^\kappa)). \]

Next, we have \( \frac{\partial (\Theta_-, v_-)^+}{\partial g^+} = (O(1), O(1)) \) from equations (10.9) and (10.12). To obtain the derivatives of \( g \) we use the fact that

\[ \frac{p_2}{p_1} = \frac{\sin g \sinh u \pm \frac{G}{\mu L} \cos g \cosh u}{\cos g \sinh u \mp \frac{G}{\mu L} \sin g \cosh u} = \frac{\tan g \pm \frac{G}{\mu L} \tan g}{1 \mp \frac{G}{\mu L} \tan g} + e^{-2|u|}E(G/\mu L, g, u), \]

where \( E \) is a smooth function satisfying \( \frac{\partial E}{\partial g} = O(1) \) as \( \ell \to \infty \). Therefore we get

\[ g = \arctan \left( \frac{p_2}{p_1} - e^{-2|u|}E(G/\mu L, g) \right) \mp \arctan \frac{G}{\mu L} \text{ as } \ell \to \infty. \]

We choose the + when considering the incoming orbit parameters. Thus

\[ \frac{\partial g}{\partial (\Theta_-, v_-)} \left( 1 + O(e^{-2|u|}) \right) = \frac{\partial \arctan \frac{p_2}{p_1}}{\partial (\Theta_-, v_-)} + \frac{\partial \arctan \frac{G}{\mu L}}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial (\Theta_-, v_-)} + \frac{\partial \arctan \frac{G}{\mu L}}{\partial G} \frac{\partial G}{\partial (\Theta_-, v_-)} + O(e^{-2|u|}). \]

Hence

\[ (12.13) \quad \frac{\partial g}{\partial (\Theta_-, v_-)} = O \left( \frac{1}{\mu} \right) \frac{\partial G}{\partial (\Theta_-, v_-)} + O(1), \]
where the $1/\mu$ comes from $\frac{\partial \arctan \frac{G}{\mu L}}{\partial G}$ and all other terms are $O(1)$ or even smaller. Therefore

\[(12.14)\]

\[
I = \frac{1}{\mu} \left( \frac{\partial (\Theta^-, v^-)}{\partial G^+} + \mu \frac{\partial \arctan \frac{G^-}{\mu L}}{\partial g^+} \right) \otimes \frac{\partial G^-}{\partial (\Theta^-, v^-)} + \left( \frac{\partial (\Theta^-, v^-)}{\partial G^+} \otimes \frac{\partial \Theta^-}{\partial (\Theta^-, v^-)} + \frac{\partial \arctan \frac{G^-}{\mu L}}{\partial g^+} \otimes \frac{\partial \Theta^-}{\partial (\Theta^-, v^-)} \right).
\]

Since the expression in parenthesis of the first term is $O(1)$, $I$ has the rate of growth required in Lemma 3.1.

Now we study the second term in (12.9)

\[(12.15)\]

\[
II = O \left( \begin{bmatrix} 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \\ 1 & \mu^{-1} & 1 \end{bmatrix} \begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa-1} & \mu^{2\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa-1} \\ \mu^{3\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa-1} \end{bmatrix} \right) \frac{\partial (\mathcal{L}, G, g)^-}{\partial (\Theta^-, v^-)}
\]

\[
= O \left( \begin{bmatrix} \mu^{3\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa-1} \\ \mu^{3\kappa-2} & \mu^{3\kappa-2} & \mu^{3\kappa-2} \\ \mu^{3\kappa-1} & \mu^{3\kappa-1} & \mu^{3\kappa-1} \end{bmatrix} \right) \frac{\partial \mathcal{L}^-}{\partial (\Theta^-, v^-)}
\]

\[
= O \left( \begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix} \right) \otimes \frac{\partial G^-}{\partial (\Theta^-, v^-)} + O \left( \begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix} \right) \otimes \frac{\partial g^-}{\partial (\Theta^-, v^-)}
\]

where we use the assumption that $\kappa < 1/2$, which implies $\mu^{2\kappa} < \mu^{3\kappa-1}$ and $\mu^{2\kappa-1} < \mu^{3\kappa-2}$. The first summand in (12.15) is $O(\mu^{3\kappa-1})$. Therefore (12.13) implies that

\[(12.16)\]

\[
II = \frac{1}{\mu} \begin{bmatrix} \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \\ \mu^{3\kappa-1} \end{bmatrix} \otimes \frac{\partial G^-}{\partial (\Theta^-, v^-)} + O(\mu^{3\kappa-1}).
\]
Now we combine (12.14) and (12.16) to get (12.17)

\[
\frac{\partial Q^+}{\partial Q^-} = \frac{1}{\mu} \left( \mu \frac{\partial \Theta^+}{\partial \Theta^-} + \mu \frac{\partial \Theta^-}{\partial \Theta^-} + O(\mu^{3\kappa-1}) \right) \right)
\]

(12.17) has the structure stated in the lemma. In (12.17), we use the variable \( \Theta^- \) for the relative position \( Q^- \) and we have \( \frac{\partial G^-}{\partial Q^-} = O(\mu^\kappa) \). To get back to \( Q^- \), i.e. to obtain \( \frac{\partial(Q_-, v_-)^+}{\partial(Q_-, v_-)^-} \), we use \( Q^- = \mu^\kappa \cos \Theta_-, \sin \Theta_- \). So we have the estimate \( \frac{\partial Q^+}{\partial Q^-} = O(\mu^\kappa) \frac{\partial \Theta^+}{\partial \Theta^-} + O(\mu^{\kappa-1}) \). To get \( \frac{\partial \Theta^+}{\partial Q^-} \), we use the transformation from polar coordinates to Cartesian, \( \frac{\partial r^-_\nu}{\partial Q^-} = \frac{\partial r^-_\nu}{\partial Q^-} + \frac{\partial r^-_\nu}{\partial Q^-} \), where \( r^-_\nu = |Q^-| = \mu^\kappa \), therefore we have \( \frac{\partial r^-_\nu}{\partial Q^-} = 0 \) and \( \frac{\partial r^-_\nu}{\partial Q^-} = 0 \) since in the expression

\[
\frac{1}{4L^2} = \frac{v^2}{4} - \frac{\mu}{2L^2}, \text{ the angle } \Theta^- \text{ plays no role. Finally, we have } \frac{\partial \arctan \frac{p^-_\nu}{\mu} L^2}{\partial Q^-} = 0.
\]

So we get

\[
\frac{\partial(Q_-, v_-)^+}{\partial(Q_-, v_-)^-} = \frac{1}{\mu} \left( O(\mu^\kappa)_{1 \times 2} , O(1)_{1 \times 2} \right) \times \left( O(1)_{1 \times 2} , O(\mu^\kappa)_{1 \times 2} \right) + O(1)_{4 \times 4} + O(\mu^{3\kappa-1}).
\]

It remains to show that other entries of the derivative matrix are \( O(1) \). Consider the following decomposition

\[
\frac{\partial(Q_-, v_-, Q_+, v_+)^+}{\partial(Q_-, v_-, Q_+, v_+)^-} = \frac{\partial(Q_-, v_-, Q_+, v_+)^+}{\partial(L, G, g, Q_+, v_+)(\ell^f)} \frac{\partial(L, G, g, Q_+, v_+)(\ell^f)}{\partial(L, G, g, Q_+, v_+)(\ell^f)} \frac{\partial(L, G, g, Q_+, v_+)(\ell^f)}{\partial(L, G, g, Q_+, v_+)(\ell^f)} := \begin{bmatrix} M & 0 & A & 0 \\ 0 & \text{Id} & B & \text{Id} \\ C & D & E & F \\ A' & 0 & B' & \text{Id} \end{bmatrix} N \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} MACA'N + MADD'B'N & MAD \\ (BC + E)A'N + (BD + F)B'N & BD + F \end{bmatrix}
\]
We have already computed \( M, A, C, A', \) and \( N \) (see (12.10), (12.7), (12.12), (12.13)), where \( C \) is (12.5) and \( ACA' = \text{Id} + P \) is (12.8). We still need to compute \( B, B', D, E, F, \) and \( \delta A' \).

From the Hamiltonian (10.5), we have
\[
\dot{\ell} = -\frac{1}{2\mu L^3} + O(\mu^2\kappa). \tag{12.19}
\]

We need to supplement (12.1) and (12.2) by the following equations.
\[
\begin{align*}
(12.19) & \quad \frac{dQ^+}{d\ell} + \frac{dv_+}{d\ell} = -\frac{v_+}{2}(2\mu L^3)(1 + O(\mu^{2\kappa+1})) = O(\mu), \\
& \quad \frac{dv_+}{d\ell} = \left(\frac{2Q^+}{Q^+}\right)^2 (2\mu L^3)(1 + O(\mu^{2\kappa+1})) = O(\mu).
\end{align*}
\]

It follows from (12.6) and (12.19) that
\[
(12.21) \quad B, B' = O\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}\right) \otimes O\left(\begin{bmatrix} [\mu^{\kappa-1}, \mu^{2\kappa-2}, 0] \end{bmatrix}\right).
\]

Next, we obtain
\[
(12.22) \quad D = O\left(\begin{bmatrix} \mu^{2\kappa} & \mu^{2\kappa} \\ \mu^{3\kappa-1} & \mu^{3\kappa-1} \end{bmatrix}\right), \quad E = O\left(\begin{bmatrix} \mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \\ \mu^{\kappa} & \mu^{3\kappa-1} & \mu^{3\kappa} \end{bmatrix}\right),
\]
\[
(12.23) \quad F = \text{Id} + O\left(\begin{bmatrix} \mu^{3\kappa} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{3\kappa} \end{bmatrix}\right).
\]

It is a straightforward computation that \( CA' \) dominates \( DB' \), so \( ADB' \) provides a small correction to the \( P \) in \( ACA' = \text{Id} + P \) in (12.8). Therefore \( MACA'N + MADB'N \) in (12.18) has the same structure as \( MACA'N \) obtained in (12.14) and (12.15). Next (12.21), (12.22), (12.23) give
\[
BD + F = \text{Id} + O\left(\begin{bmatrix} \mu^{5\kappa-1} & \mu^{\kappa} \\ \mu^{\kappa} & \mu^{5\kappa-1} \end{bmatrix}\right).
\]

Accordingly
\[
(12.24) \quad (BC + E)A'N + (BD + F)B'N = \frac{1}{\mu} [O(\mu^{2\mu})]_{1\times 4} \otimes \frac{\partial G^-}{\partial (\theta, v)} + O(\mu^\kappa).
\]

Finally, we have \( MAD = [O(\mu^{3\kappa-1})]_{3\times 2} \).

These estimates of the matrix (12.18) are enough to conclude the Lemma. To summarize, we get the resulting derivative estimate as
\[
(12.25) \quad (12.18) = \frac{1}{\mu} (\mu^\kappa_{1\times 2}, 1_{1\times 2}, \mu^{2\kappa}_{1\times 4}) \otimes (1_{1\times 2}, \mu^\kappa_{1\times 2}, 0_{1\times 4}) + \left[ \begin{array}{cc} 0_{4\times 4} & \mu^{3\kappa-1} \\ \mu^\kappa & \text{Id} + \mu^\kappa \end{array} \right]. \]

\( \Box \)
The above proof actually gives us more information. Below we use the Delaunay variables \((L_3, \ell_3, G_3, g_3, G_4, g_4)^\pm\) as the orbit parameters outside the sphere \(|Q_-| = \mu^e\) and add a subscript \(in\) to the Delaunay variables inside the sphere. We relate \(C^0\) estimates of Lemma 10.2 to the \(C^1\) estimates obtained above. Namely consider the following equation which is obtained by discarding the \(O(\mu^{3e-1})\) and \(O(\mu^e)\) errors in (10.7)

\[
(12.26) \quad Q^\pm_+ = 0, \quad v^\pm_+ = R(\alpha)v^- = R(\alpha/2 + g)(|v^-|, 0), \quad Q^\pm_+ = Q^\mp_+, \quad v^\pm_+ = v^\mp_+.
\]

**Corollary 1.** The derivative of the local map has the following form

\[
(12.27) \quad dL = \frac{1}{\mu}(\hat{u} + O(\mu^{3e-1})) \otimes 1 + \hat{B} + O(\mu^{3e-1}),
\]

where \(\hat{u}, 1\) and \(\hat{B}\) are computed by discarding the \(O(\mu^{3e-1})\) and \(O(\mu^e)\) errors in (12.26). In particular,

\[
\hat{u} = \frac{\partial Q_3, v_3, Q_4, v_4}{\partial (Q_3, v_3, Q_4, v_4)} \left( \mu \frac{\partial \alpha}{\partial G_{in}} \right),
\]

\[
1 = \frac{\partial G_{in}}{\partial (Q_3, v_3, Q_4, v_4)} - \frac{\partial (Q_3, v_3, Q_4, v_4)}{\partial (L_3, \ell_3, G_3, g_3, G_4, g_4)}.
\]

**Proof.** The derivative matrix of (12.26) is block diagonalized. We get identity for the derivative of the motion of the mass center part, which agrees with the entry \(BD + F\) in (12.18) in the limit \(\mu \to 0\). Thus we only need to focus on relative motion part.

Our computation of (12.14) is based on formula (10.9), where the velocity is written as

\[
v^+_+ = R(\alpha/2 + g)(1/L, 0) + O(e^{-2|u|}), \quad |u| \to \infty.
\]

Next, we have \(1/L = |v^-| + o(1)\) as \(\mu \to 0\). Also in (12.14), we have \(\frac{\partial \Theta^\pm}{\partial v^-} = O(1)\), which implies \(\frac{\partial Q^\pm}{\partial v^-} = O(\mu^e)\) as \(\mu \to 0\) since \(Q^\pm = \mu^e(\cos \Theta^\pm, \cos \Theta^\pm)\).

Differentiating (12.26) we get

\[
(12.28) \quad \frac{\partial v^\pm}{\partial v^-} = \frac{\partial v^\pm}{\partial \alpha} \frac{\partial \alpha}{\partial G_{in}} \otimes \frac{\partial G_{in}}{\partial v^-} + \frac{\partial v^\pm}{\partial \alpha} \frac{\partial \alpha}{\partial L_{in}} \otimes \frac{\partial L_{in}}{\partial v^-} + \frac{\partial v^\pm}{\partial \alpha} \frac{\partial \alpha}{\partial L_{in}} \otimes \frac{\partial L_{in}}{\partial v^-},
\]

where in the last summand we use \(\partial / \partial \) to stand for the partial derivative w.r.t. the explicit dependence on \(v^-\). Notice that \(\frac{\partial \alpha}{\partial G_{in}} = O(1/\mu)\) and \(\frac{\partial \alpha}{\partial L_{in}} = O(1)\). Thus (12.28) has the form of \(\frac{1}{\mu} \hat{u} \otimes 1 + \hat{B}\) which gives (12.27) after changing variables to Delaunay variables outside the sphere.
Notice that \( g = \alpha/2 + \arctan \frac{p_2^-}{p_1} \) and \( v^- = (p_1^-, p_2^-) \). We express

\[
\frac{\partial v^+}{\partial v^-} = \frac{\partial v^+}{\partial \arctan \frac{p_2^-}{p_1}} + \frac{\partial \arctan \frac{p_2^-}{p_1}}{\partial v^-} + \frac{\partial v^+}{\partial |v^-|} \frac{\partial |v^-|}{\partial v^-}.
\]

Then we identify \( \frac{1}{\mu} \left( \frac{\partial (\Theta^+, v^+)}{\partial G^+} + \mu \frac{\partial \arctan G^-}{\partial G^-} \right) \) in (12.14) with

\[
\frac{\partial (\Theta^+, v^+)}{\partial G^+} + \frac{\partial \arctan \frac{p_2^-}{p_1}}{\partial G^-}
\]

We thus have shown that the formal derivative in (12.26) agrees with the derivative obtained in the proof of Lemma 3.1. \( \square \)

**Corollary 2.** If we take derivative along a direction \( \gamma(s) : (-\varepsilon, \varepsilon) \to \mathbb{R}^6 \) be a curve such that \( \Gamma = \gamma'(0) = O(1) \) and \( \frac{\partial G^-}{\partial s} = O(\mu) \) then in equations

\[
\begin{align*}
|v_3^+|^2 + |v_4^+|^2 &= |v_3^-|^2 + |v_4^-|^2 + o(1), \\
v_3^+ + v_4^+ &= v_3^- + v_4^- + o(1), \\
Q_3^+ + Q_4^+ &= Q_3^- + Q_4^- + o(1),
\end{align*}
\]

obtained from equation (10.7), the \( o(1) \) terms are small in the \( C^1 \) sense. Namely, we can drop the \( o(1) \) terms when we take the derivative \( \frac{d}{ds} \).

**Proof.** For the motion of the mass center, it follows from Corollary 1 that

\[
\frac{\partial (Q^+, v^+)}{\partial (Q^-, v^-, Q^+, v^+)} = \frac{1}{\mu} \frac{\partial (Q^+, v^+)}{\partial \alpha} \otimes 1 + (0_{4 \times 4}, \text{Id}_{4 \times 4}) + o(1). \]

We already obtained that \( \frac{\partial (Q^+, v^+)}{\partial \alpha} = O(\mu^{2\kappa}) \) (see equation (12.24)). Due to Corollary 1 our assumption that \( \frac{\partial G^-}{\partial s} = O(\mu) \) implies that

\[
(12.29) \quad \Gamma = O(\mu)
\]

which suppresses the \( 1/\mu \) term. This proves the last two identities of the Corollary.
To derive the first equation we use the fact that the Hamiltonian \(10.5\) is preserved. Namely we use the fact that RHS \(10.5\) is the same in \(+\) and \(−\) variables. It is enough to show \(\frac{d}{dx}(|v_+|^2 - |v_-|^2) = o(1)\) since we already have the required estimate for the velocity of the mass center. In \(10.5\), the terms involving only \(Q_+, v_+\) are handled using the result of the previous paragraph. The term \(-\frac{\mu}{|Q_-|}\) vanishes when taking derivative since \(|Q_-| = \mu^\kappa\) is constant. All the remaining terms have \(Q_-\) to the power 2 or higher. We have \(\frac{\partial Q_-}{\partial s} = O(1)\) since \(\Gamma = O(1)\). We also have \(\frac{\partial Q_+}{\partial s} = O(1)\) due to \(12.29\). Therefore after taking the \(s\) derivative, any term involving \(Q_-\) is of order \(O(\mu^\kappa)\). This completes the proof of the energy conservation part.

12.2. Proof of the Lemma 3.8. In this section we work out the \(O(1/\mu)\) term in the Local map.

Proof. The proof is done numerically.

Before collision, \(1 = \frac{\partial G_{in}}{\partial -}\). According to Corollary 1 we can differentiate the asymptotic expression of Lemma 10.2. We have

\[
\left(\frac{\partial G_{in}}{\partial -}, \frac{\partial G_{in}}{\partial g_4}\right) = (v_3^- - v_4^-) \times \left(\frac{\partial}{\partial G_4}, \frac{\partial}{\partial g_4}\right) Q_4 + (v_3^- - v_4^-) \times \left(\frac{\partial Q_4}{\partial \ell_4}, \frac{\partial \ell_4^-}{\partial \ell_4}, \frac{\partial \ell_4^-}{\partial G_4}, \frac{\partial \ell_4^-}{\partial g_4}\right) + O(\mu^\kappa).
\]

We need to eliminate \(\ell_4\) using the relation \(|Q_3 - Q_4| = \mu^\kappa\).

\[
\left(\frac{\partial \ell_4^-}{\partial G_4}, \frac{\partial \ell_4^-}{\partial g_4}\right) = \left(\frac{\partial |Q_3 - Q_4|}{\partial \ell_4^-}\right)^{-1} \left(\frac{\partial |Q_3 - Q_4|}{\partial G_4}, \frac{\partial |Q_3 - Q_4|}{\partial g_4}\right)
\]

\[
= \frac{(Q_3 - Q_4) \cdot \frac{\partial Q_4}{\partial \ell_4^-}}{(Q_3 - Q_4) \cdot \frac{\partial Q_4}{\partial G_4}} = \frac{(v_3^- - v_4^-) \cdot \frac{\partial Q_4}{\partial g_4}}{(v_3^- - v_4^-) \cdot \frac{\partial Q_4}{\partial \ell_4^-}} + O(\mu^{1-\kappa}).
\]

Here we replaced \(Q_3^-, Q_4^-\) by \(v_3^- - v_4^-\) using the fact that the two vectors form an angle of order \(O(\mu^{1-\kappa})\) (see Lemma 10.2(c)). Therefore

\[
\left(\frac{\partial G_{in}}{\partial G_4}, \frac{\partial G_{in}}{\partial g_4}\right) = (v_3^- - v_4^-) \times \left(\frac{\partial}{\partial G_4}, \frac{\partial}{\partial g_4}\right) Q_4
\]

\[
- (v_3^- - v_4^-) \times \left(\frac{\partial Q_4}{\partial \ell_4^-} \left(\frac{v_3^- - v_4^-) \cdot (\frac{\partial Q_4}{\partial G_4}, \frac{\partial Q_4}{\partial g_4})}{(v_3^- - v_4^-) \cdot \frac{\partial Q_4}{\partial \ell_4^-}}\right) + O(\mu^\kappa).
\]

We use Mathematica and the data in the Appendix B.2 to work out \(\frac{\partial G_{in}}{\partial -}\).
After collision, \( \dot{u} = \frac{\partial - \partial \alpha}{\partial \alpha} \). In equation (10.7), we let \( \mu \to 0 \). Applying the implicit function theorem to (10.7) with \( \mu = 0 \) we obtain

\[
\left( \frac{\partial (Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial (X^+, Y^+)} + \frac{\partial (Q_3^+, v_3^+, Q_4^+, v_4^+)}{\partial (X^+, Y^+)} \right) \cdot \frac{\partial (X^+, Y^+)}{\partial \alpha} = \frac{1}{2} \left( 0, 0, R \left( \frac{\pi}{2} + \alpha \right) (v_3^- - v_4^+) \right) 0, 0, -R \left( \frac{\pi}{2} + \alpha \right) (v_3^- - v_4^+) T
\]

where \( R(\pi/2 + \alpha) = \frac{dR(\alpha)}{d\alpha} \) and \( \frac{\partial \ell_4^+}{\partial (X^+, Y^+)} \) is given by (9.2). Again we use Mathematica to work out the \( \frac{\partial - \partial \alpha}{\partial \alpha} \).

To obtain a symbolic sequence with any order of symbols 3, 4 as claimed in the main theorem, we notice that the only difference is that the outgoing relative velocity changes sign \((v_3^+ - v_4^+) \to -(v_3^+ - v_4^+))\). So we only need to send \( \dot{u} \to -\dot{u} \). \( \square \)

12.3. Proof of the Lemma 3.9. Recall that Lemmas 3.1 and 3.2 give the following form for the derivatives of local map and global maps

\[
dL_u = \frac{1}{\mu} u \otimes 1 + B + O(\mu^\alpha),
\]

\[
dG = \chi^2 \tilde{u} \otimes \tilde{1} + \chi \tilde{u} \otimes \tilde{1} + O(\mu^2 \chi),
\]

where we suppress the subscript \( i \) standing for the first or second collision. Moreover, in the limit \( \chi \to \infty, \mu \to 0 \),

\[
\text{span}\{\tilde{u}, \tilde{u}\} \to \text{span}\{w_i, \tilde{w}\}, \quad 1 \to \tilde{1}_i, \tilde{1} \to \tilde{1}_i, \quad \tilde{I}_i \to \tilde{I}_i, \quad i = 1, 2.
\]

We first prove an abstract lemma that reduces the study of the local map of the \( \mu > 0 \) case to \( \mu = 0 \) case.

Lemma 12.1. Suppose the vector \( \tilde{\Gamma}_\mu \in \text{span}\{\tilde{u}, \tilde{u}\} \) satisfies \( \tilde{I}(dL\tilde{\Gamma}_\mu) = 0 \) and \( ||\tilde{\Gamma}_\mu||_\infty = 1 \). Then the following limits exist

\[
\Gamma_i = \lim_{\mu \to 0} \tilde{\Gamma}_\mu \text{ and } \lim_{\mu \to 0} dL\tilde{\Gamma}_\mu = \Delta_i.
\]

Moreover we have \( \tilde{I}_i(\Delta_i) = 0 \) and \( I_i(\Gamma_i) = 0 \).

Proof. Denote \( \Gamma_\mu' = I(\bar{u})\tilde{u} - I(\bar{u})\tilde{u} \in \text{Ker} I \) and let \( v_\mu \) be a vector in \( \text{span}(\bar{u}, \bar{u}) \) such that \( v_\mu \to v \) as \( \mu \to 0 \) and \( \tilde{I}(v_\mu) = 0 \). Suppose that

\[
\tilde{\Gamma}_\mu = a_\mu v_\mu + b_\mu \Gamma_\mu'
\]

then

\[
dL(\tilde{\Gamma}_\mu) = \frac{a_\mu}{\mu} I(v_\mu) u + a_\mu B(v_\mu) + b_\mu B \Gamma_\mu'.
\]

(12.30)
So \( \tilde{I}(dL(\tilde{\Gamma}_\mu)) = 0 \) implies that
\[
(12.31) \quad a_\mu = -\mu \frac{b_\mu \tilde{I}(B\Gamma_\mu')}{\tilde{I}(v_\mu) \tilde{I}(u) + \mu B(v_\mu)}.
\]
Therefore \( a_\mu = O(\mu) \) and hence \( \tilde{\Gamma}_\mu = b_\mu \Gamma_\mu + o(1) \). It follows that as \( \mu \to 0 \) \( \Gamma_\mu \)
approaches the vector in \( \text{Ker} I \) with \( \ell^\infty \) norm 1. Now the remaining statements of
the lemma follow from equations \( (12.30) \) and \( (12.31) \). \( \square \)

To compute the numerical values it is more convenient for us to work with polar
coordinates. We need the following quantities.

\( \psi \): polar angle, related to \( u \) by 
\[
\tan \psi = \sqrt{1 + e^2} \tan \frac{u}{2}
\]
for ellipse;

\( E \): energy; \( e \): eccentricity, \( e = \sqrt{1 + 2EM^2} \);

\( G \): angular momentum, \( g \): argument of periapsis.

The subscripts 3, 4 stand for \( Q_3 \) or \( Q_4 \). The superscript \( \pm \) refers to before or after
collision. Recall that all quantities are evaluated on the sphere
\( |Q_3 - Q_4| = \mu \kappa \). We
choose the positive \( y \) axis as the axis \( \psi = 0 \).

We have the formula 
\[
r = \frac{G^2}{1 - e \cos \psi}
\]
for conic sections in which the perigee lies on
the axis \( \psi = \pi \). In our case we have
\[
(12.32) \quad \left\{ \begin{array}{l}
\begin{array}{l}
\frac{G^2}{1 - e^2} \frac{(G^2_3)^2}{1 - e^3 \sin (\psi^+_3 + g^+_3)} + o(1), \\
\frac{G^2}{1 - e^2} \frac{(G^2_4)^2}{1 - e^3 \sin (\psi^+_4 - g^+_4)} + o(1).
\end{array}
\end{array}
\right.
\]
\( o(1) \) terms are small when \( \mu \to 0 \) (recall that we always assume that \( \chi \gg 1/\mu \)).

**Lemma 12.2.** Under the assumptions of Corollary 2 we have
\[
\frac{dr^+_3}{ds} = \frac{dr^+_4}{ds} + o(1), \quad \frac{dr^-_3}{ds} = \frac{dr^-_4}{ds} + o(1), \quad \frac{d\psi^+_3}{ds} = \frac{d\psi^+_4}{ds} + o(1), \quad \frac{d\psi^-_3}{ds} = \frac{d\psi^-_4}{ds} + o(1).
\]
Moreover in \( (12.32) \) the \( o(1) \) terms are also \( C^1 \) small when taking the \( s \) derivative.

**Proof.** To prove the statement about \( (12.32) \), we use the Hamiltonian \( (4.1) \). The \( r_{3,4} \)
obey the Hamiltonian system \( (4.1) \). The estimate \( (9.1) \) shows the \( \frac{-\mu}{|Q_3 - Q_4|} \) gives
small perturbation to the variational equations. The two \( O(1/\chi) \) terms in \( (4.1) \) are
also small. This shows that the perturbations to Kepler motion is \( C^1 \) small.

Next we consider the derivatives \( \frac{\partial r_{3,4}}{\partial s} \). We consider first the case of “-”. From
the condition \( |\tilde{r}_3 - \tilde{r}_4| = \mu^\kappa \), for the Poincaré section we get
\[
(\tilde{r}_3 - \tilde{r}_4) \cdot \frac{d}{ds}(\tilde{r}_3 - \tilde{r}_4) = 0.
\]
This implies \((\vec{r}_3 - \vec{r}_4) \perp \frac{d}{ds}(\vec{r}_3 - \vec{r}_4)\).

We also know the angular momentum for the relative motion is

\[ G_{in} = (\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times (\vec{r}_3 - \vec{r}_4) = O(\mu), \]

which implies \(\dot{\vec{r}}_3 - \dot{\vec{r}}_4\) is almost parallel to \(\vec{r}_3 - \vec{r}_4\). The condition \(\frac{\partial G_{in}^-}{\partial s} = O(\mu)\) reads

\[ \left( \frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) \times (\vec{r}_3 - \vec{r}_4) + (\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times \left( \frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) = O(\mu). \]

Since the first term is \(O(\mu^{\kappa})\) due to our choice of the Poincare section we see that

\[ (\dot{\vec{r}}_3 - \dot{\vec{r}}_4) \times \left( \frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) = o(1). \]

Since \(\frac{d}{ds}(\vec{r}_3 - \vec{r}_4)\) is almost perpendicular to \((\dot{\vec{r}}_3 - \dot{\vec{r}}_4)\) by the analysis presented above
we get \(\frac{d}{ds}(\vec{r}_3 - \vec{r}_4) = o(1)\). Taking the radial and angular part of this vector identity
and using that \(r_4 = r_3 + o(1), \psi_4 = \psi_3 + o(1)\) we get "-" part of the lemma.

To repeat the above argument for "+" variables, we first need to establish \(\frac{\partial G_{in}^-}{\partial s} = O(\mu)\). Indeed, using equations (12.8) and (12.18) we get

\[
\frac{\partial G_{in}^+}{\partial \psi} = \frac{\partial G_{in}^+}{\partial (\mathcal{L}, G_{in}, g, Q_+, v_+)} \frac{\partial (\mathcal{L}, G_{in}, g, Q_+, v_+)}{\partial \psi} = O(\mu^{3\kappa}, 1, \mu^{3\kappa}, \mu_{1x2}^{3\kappa}, \mu_{1x2}^{3\kappa}), O(1, 1, 1, 1), O(1) = O(\mu).
\]

It remains to show \(\left( \frac{d}{ds}(\vec{r}_3 - \vec{r}_4) \right) = O(1)\) in the "+" case. Since we know it is
truethe "+" case follows, because the directional derivative of the
local map \(dL\Gamma\) is bounded due to our choice of \(\Gamma\).

We are now ready to describe the computation of Lemma 3.9. The reader may
notice that the computations in the proofs of Lemmas 3.9 and 2.1 are quite similar.
Note however that Lemma 3.9 describes the subleading term for the derivative of the
Gerver map. By contrast the leading term can not be understood in terms of
the Gerver map since it comes from the possibility of varying the closest distance
between \(Q_3\) and \(Q_4\) and this distance is assumed to be zero in Gerver’s model.

We will use the following set of equations which follows from (12.26).

\[
E_3^- + E_4^- = E_3^+ + E_4^+,
\]

\[
G_3^- + G_4^- = G_3^+ + G_4^+,
\]
The formula
\[ \frac{e_3^+}{G_3} \cos(\psi_3^+ + g_3^+) + \frac{e_4^+}{G_4} \cos(\psi_4^+ - g_4^+) = \frac{e_3^-}{G_3} \cos(\psi_3^- + g_3^-) + \frac{e_4^-}{G_4} \cos(\psi_4^- - g_4^-), \]

(12.35)

\[ \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi_3^+ + g_3^+)} = \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)}, \]

(12.36)

\[ \psi_3^+ = \psi_3^- \]

(12.37)

\[ \frac{(G_4^+)^2}{1 - e_4^+ \sin(\psi_4^+ + g_4^+)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi_4^- + g_4^-)}, \]

(12.38)

\[ \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi_3^- + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi_4^- + g_4^-)}, \]

(12.39)

\[ \psi_4^- = \psi_3^- \]

(12.40)

\[ \psi_4^+ = \psi_3^+ \]

(12.41)

In the above equations we have dropped \( o(1) \) terms for brevity. We would like to emphasize that the above approximations hold not only in \( C^0 \) sense but also in \( C^1 \) sense when we take the derivatives along the directions satisfying the conditions of Corollary 2 (12.33) is the approximate conservation of the energy, (12.34) is the approximate conservation of the angular momentum and (12.35) follows from the approximate momentum conservation (see the derivation of (B.2) in Appendix B.3). The possibility of differentiating these equations is justified in Corollary 2. The remaining equations reflect the fact that \( Q_3^\pm \) and \( Q_4^\pm \) are all close to each other. The possibility of differentiating these equations is justified by Lemma 12.2.

We set the total energy to be zero. So we get \( E_4^\pm = -E_3^\pm \). This eliminates \( E_4^\pm \). Then we also eliminate \( \psi_4^\pm \) by setting them to be equal \( \psi_3^\pm \).

**Proof of the Lemma 3.9.** Recall that we need to compute \( dE_3^\pm(\text{d}L\Gamma) \) where \( \Gamma \in Ker1 \cap span\{w, \tilde{w}\} \). (3.3) tells us that in in Delaunay coordinates we have

\[ w = (0, 1, 0, 0, 0, 0), \quad \tilde{w} = (0, 0, 0, 0, 1, a) \quad \text{where} \quad a = \frac{-L_4^-}{(L_4^+)^2 + (G_4^-)^2}. \]

(12.42)

The formula \( \tan \frac{\psi}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2} \) which relates \( \psi \) to \( u \) through \( u \) shows that (12.42) also holds if we use \( (L_3, \psi_3, G_3, g_3, G_4, g_4) \) as coordinates. Hence \( \Gamma \) has the form \( (0, 1, 0, 0, c, ca) \). To find the constant \( c \) we use (12.39).

Note that the expression \( dE_3^\pm(\text{d}L\Gamma) \) does not involve \( d\psi_3^\pm \). Therefore we can eliminate \( \psi_3^\pm \) from consideration by setting \( \psi_3^+ = \psi_3^- = \psi \) (see (12.37)). Let \( L \) denote the projection of our map to \( (L_3, G_3, g_3, G_4, g_4) \) variables. Thus we need to find \( dE_3^\pm(\text{d}L\Gamma) \). To this end write the remaining equations (12.34), (12.35), (12.36),...
and \((12.38)\) formally as \(F(Z^+, Z^-) = 0\), where in \(Z^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)\) and \(Z^- = (E_3^-, \psi, G_3^-, g_3^-, G_4^-, g_4^-)\).

We have

\[
\frac{\partial F}{\partial Z^+} dL \Gamma + \frac{\partial F}{\partial Z^-} \Gamma = 0.
\]

However, \(\frac{\partial F}{\partial Z^-}\) is not invertible since \(F\) involves only four equations of \(F\) while \(Z^+\) has 5 variables. To resolve this problem we use that by definition of \(\Gamma\) we have

\[
\bar{l} \cdot \frac{\partial Z}{\partial \psi} = 0,
\]

where \(\bar{l} = (G_4 + L_4, \frac{1}{(L_4^2 + G_4^2)^{1/2}}, 0, 0, \frac{1}{(L_4^2 + G_4^2)^{1/2}}, 1)\) by (3.3). Thus we get

\[
\left[ \frac{\bar{l}}{\partial Z^+} \right] dL \Gamma = - \left[ \frac{\partial F}{\partial Z^+} \right]^{-1} \left[ \frac{\partial F}{\partial Z^-} \Gamma \right]
\]

and so

\[
dL \Gamma = - \left[ \frac{\bar{l}}{\partial Z^+} \right]^{-1} \left[ \frac{\partial F}{\partial Z^-} \Gamma \right].
\]

We use computer to complete the computation. We only need the entry \(\frac{\partial E_3^+}{\partial \psi}\) to prove Lemma 3.9. It turns out this number is 1.855 for the first collision and −1.608 for the second collision. Both are nonzero as needed. □

**Appendix A. Delaunay coordinates**

### A.1. Elliptic motion

The material of this section could be found in [Al]. Consider the two-body problem with Hamiltonian

\[
H(P, Q) = \frac{|P|^2}{2m} - \frac{k}{|Q|}, \quad (P, Q) \in \mathbb{R}^4.
\]

This system is integrable in the Liouville-Arnold sense when \(H < 0\). So we can introduce the action-angle variables \((L, \ell, G, g)\) in which the Hamiltonian can be written as

\[
H(L, \ell, G, g) = -\frac{mk^2}{2L^2}, \quad (L, \ell, G, g) \in T^*\mathbb{T}^2.
\]

The Hamiltonian equations are

\[
\dot{L} = \dot{G} = \dot{g} = 0, \quad \dot{\ell} = \frac{mk^2}{L^3}.
\]

We introduce the following notation \(E\)-energy, \(M\)-angular momentum, \(e\)-eccentricity, \(a\)-semimajor axis, \(b\)-semiminor axis. Then we have the following relations which explain the physical and geometrical meaning of the Delaunay coordinates.

\[
a = \frac{L^2}{mk}, \quad b = \frac{LG}{mk}, \quad E = -\frac{k}{2a}, \quad M = G, \quad e = \sqrt{1 - \left(\frac{G}{L}\right)^2}.
\]
Moreover, \( g \) is the argument of periapsis and \( \ell \) is called the mean anomaly, and \( \ell \) can be related to the polar angle \( \psi \) through the equations

\[
\tan \frac{\psi}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}, \quad u - e \sin u = \ell.
\]

We also have the Kepler’s law \( \frac{a^3}{T^2} = \frac{1}{(2\pi)^2} \) which relates the semimajor axis \( a \) and the period \( T \) of the ellipse.

Denoting particle’s position by \((q_1, q_2)\) and its momentum \((p_1, p_2)\) we have the following formulas in case \( g = 0 \).

\[
\begin{align*}
q_1 &= a (\cos u - e), \\
q_2 &= a \sqrt{1 - e^2} \sin u,
\end{align*}
\]

\[
\begin{align*}
p_1 &= -\sqrt{mka^{-1/2}} \sin u \frac{\sin u}{1 - e \cos u}, \\
p_2 &= \sqrt{mka^{-1/2}} \frac{\sqrt{1 - e^2} \cos u}{1 - e \cos u},
\end{align*}
\]

where \( u \) and \( \ell \) are related by \( u - e \sin u = \ell \).

Expressing \( e \) and \( a \) in terms of Delaunay coordinates we obtain the following

\[
q_1 = \frac{L^2}{mk} \left( \cos u - \sqrt{1 - \frac{G^2}{L^2}} \right), \\
q_2 = \frac{LG}{mk} \sin u.
\]

\[
p_1 = -\frac{mk}{L} \sin u \frac{\sin u}{1 - \sqrt{1 - \frac{G^2}{L^2} \cos u}}, \\
p_2 = \frac{mk}{L^2} \frac{G \cos u}{1 - \sqrt{1 - \frac{G^2}{L^2} \cos u}}.
\]

Here \( g \) does not enter because the argument of perihelion is chosen to be zero. In general case, we need to rotate the \((q_1, q_2)\) and \((p_1, p_2)\) using the matrix

\[
\begin{bmatrix}
\cos g & -\sin g \\
\sin g & \cos g
\end{bmatrix}.
\]

Notice that the equation [A.1] describes an ellipse with one focus at the origin and the other focus on the negative \( x \)-axis. We want to be consistent with [G2], i.e. we want \( g = \pi/2 \) to correspond to the “vertical” ellipse with one focus at the origin and the other focus on the positive \( y \)-axis (see Appendix [B.2]). Therefore we rotate the picture clockwise. So we use the Delaunay coordinates which are related to the Cartesian ones through the equation

\[
\begin{align*}
q_1 &= \frac{1}{mk} \left( L^2 \left( \cos u - \sqrt{1 - \frac{G^2}{L^2}} \right) \cos g + LG \sin u \sin g \right), \\
q_2 &= \frac{1}{mk} \left( -L^2 \left( \cos u - \sqrt{1 - \frac{G^2}{L^2}} \right) \sin g + LG \sin u \cos g \right).
\end{align*}
\]
A.2. Hyperbolic motion. The above formulas can also be used to describe hyperbolic motion, where we need to replace “\( \sin \rightarrow \sinh, \cos \rightarrow \cosh \)” (c.f. [A] [F]). Namely, we have

\[
q_1 = \frac{L^2}{mk} \left( \cosh u - \sqrt{1 + \frac{G^2}{L^2}} \right), \quad q_2 = \frac{LG}{mk} \sinh u,
\]

\[
p_1 = -\frac{mk}{L} \frac{\sinh u}{\sqrt{1 + \frac{G^2}{L^2} \cosh u}}, \quad p_2 = -\frac{mk}{L^2} \frac{G \cosh u}{\sqrt{1 + \frac{G^2}{L^2} \cosh u}}.
\]

where \( u \) and \( l \) are related by

\[
u - e \sinh u = \ell, \quad \text{where } e = \sqrt{1 + \left( \frac{G}{L} \right)^2}.
\]

This hyperbola is symmetric w.r.t. the \( x \)-axis, opens to the right and the particle moves clockwise on it when \( u \) increases (\( \ell \) decreases). When the particle moves to the right of \( x = -\frac{\chi}{2} \) line we have a hyperbola opening to the left and the particle moves anti-clockwise. To achieve this we first reflect \((q_1, q_2)\) around the \( y \)-axis, then rotate it by an angle \( g \). If we restrict \( |g| < \pi/2 \), then the particle moves anti-clockwise on the hyperbola as \( u \) increases (\( \ell \) decreases) due to the reflection. Thus we have

\[
q_1 = -\frac{1}{mk} \left( \cos gL^2 (\cosh u - e) + \sin gLG \sinh u \right), \quad q_2 = \frac{1}{mk} \left( -\sin gL^2 (\cosh u - e) + \cos gLG \sinh u \right),
\]

\[
P = \frac{mk}{1 - e \cosh u} \left( \frac{1}{L} \sinh u \cos g + \frac{G}{L^2} \sin g \cosh u, \frac{1}{L} \sinh u \sin g - \frac{G}{L^2} \cos g \cosh u \right).
\]

If the incoming asymptote is horizontal, then the particle comes from the left, and as \( u \) tends to \(-\infty\), the \( y \)-coordinate is bounded and \( x \)-coordinate is negative. In this case we have \( \tan g = -\frac{G}{L}, \quad g \in (-\pi/2, 0) \).

If the outgoing asymptote is horizontal, then the particle escapes to the left, and as \( u \) tends to \(+\infty\), the \( y \)-coordinate is bounded and \( x \)-coordinate is negative. In this case we have \( \tan g = +\frac{G}{L}, \quad g \in (0, \pi/2) \).

When the particle \( Q_4 \) is moving to the left of the section \( \{x = -\chi/2\} \), we treat the motion as hyperbolic motion focused at \( Q_1 \). We move the origin to \( Q_1 \). The hyperbola opens to the right. The orbit has the following parametrization

\[
q_1 = \frac{1}{mk} \left( \cos gL^2 (\cosh u - e) - \sin gLG \sinh u \right), \quad q_2 = -\frac{1}{mk} \left( \sin gL^2 (\cosh u - e) + \cos gLG \sinh u \right).
\]

\( (A.6) \)
A.3. Large $\ell$ asymptotics: auxiliary results. In the remaining part of Appendix A we study the first and second order derivatives of $Q$ w.r.t. the hyperbolic Delaunay variables $(L, \ell, G, g)$. These computations are used in our proof. The next lemma allows us to simplify the computations. Since the hyperbolic motion approaches a linear motion, this lemma shows that, we can replace $u$ by $\ln(\mp \ell/e)$ when taking first and second order derivatives.

**Lemma A.1.** Let $u$ be the function of $\ell, G$ and $L$ given by (A.4). Then we can approximate $u$ by $\ln(\mp \ell/e)$ in the following sense.

\[
\begin{align*}
    u &\mp \ln \frac{\mp \ell}{e} = O(\ln |\ell|/\ell), \quad \frac{\partial u}{\partial \ell} = \pm 1/\ell + O(1/\ell^2), \\
    \left( \frac{\partial}{\partial L}, \frac{\partial}{\partial G} \right)(u \pm \ln e) &= O(1/|\ell|), \\
    \left( \frac{\partial}{\partial L}, \frac{\partial}{\partial G} \right)^2(u \pm \ln e) &= O(1/|\ell|),
\end{align*}
\]

Here the first sign is taken if $u > 0$ and the second sign is taken then $u < 0$. The estimates above are uniform as long as $|G| \leq K$, $1/K \leq L \leq K$, $\ell > \ell_0$ and the implied constants in $O(\cdot)$ depend only on $K$ and $\ell_0$.

**Proof.** We see from formula (A.4) that $\sinh u \simeq \cosh u = -\frac{\ell}{e} + O(\ln |\ell|)$ when $u > 0$ and $\sinh u \simeq -\cosh u \simeq -\frac{\ell}{e} + O(\ln |\ell|)$ when $u < 0$ and $|u|$ large enough. This proves $C^0$ estimate.

Now we consider the first order derivatives. We assume that $u > 0$ to fix the notation. Differentiating (A.4) with respect to $\ell$ we get

\[
\frac{\partial u}{\partial \ell} - e \cosh u \frac{\partial u}{\partial \ell} = 1, \quad \frac{\partial u}{\partial \ell} = 1/\ell + O(1/\ell^2).
\]

Next, we differentiate (A.4) with respect to $L$ to obtain

\[
\frac{\partial u}{\partial L} - e \cosh u \frac{\partial u}{\partial L} = 0.
\]

Therefore,

\[
\frac{\partial u}{\partial L} = \frac{\sinh u}{1 - e \cosh u} \frac{\partial e}{\partial L} = -\frac{1}{e} \frac{\partial e}{\partial L} + O(e^{-|u|}) = -\frac{\partial}{\partial L} \ln(e) + O(1/|\ell|).
\]

The same argument holds for $\frac{\partial}{\partial G}$: This proves $C^1$ part of the Lemma.

Now we consider second order derivatives. We take $\frac{\partial^2}{\partial L^2}$ as example. Combining

\[
\frac{\partial^2 u}{\partial L^2} - \frac{\partial^2 e}{\partial L^2} \sinh u - 2 \cosh u \frac{\partial e}{\partial L} \frac{\partial u}{\partial L} - e \cosh u \left( \frac{\partial^2 u}{\partial L^2} - e \sinh u \left( \frac{\partial u}{\partial L} \right)^2 \right) = 0.
\]
with $C^1$ estimate proven above we get
\[
\frac{\partial^2 u}{\partial L^2} = -\frac{1}{e} \frac{\partial^2 e}{\partial L^2} - \frac{2}{e \partial L} \frac{\partial e}{\partial L} + \left( \frac{\partial u}{\partial L} \right)^2 + O\left( \frac{1}{\ell} \right)
\]
\[
= -\frac{1}{e} \frac{\partial^2 e}{\partial L^2} + \left( \frac{1}{e} \frac{\partial e}{\partial L} \right)^2 + O\left( \frac{1}{\ell} \right) = \frac{\partial^2 e}{\partial L^2} \ln e + O\left( \frac{1}{\ell} \right).
\]
This concludes the $C^2$ part of the lemma. 

In the estimate of the derivatives presented in the next two subsections we shall often use the following facts. Let $f = \ln e$.

(A.7) \( f_G = \frac{G}{L^2 + G^2}, \quad f_L = -\frac{G^2}{L(L^2 + G^2)} \),

(A.8) \( (f)_{GG} = \frac{L^2 - G^2}{(L^2 + G^2)^2}, \quad (f)_{LG} = -\frac{2GL}{(L^2 + G^2)^2} \).

A.4. First order derivatives. In the following computations, we assume for simplicity that $m = k = 1$. To get the general case we only need to divide positions by $mk$.

**Lemma A.2.** Under the same conditions as in Lemma A.1 we have the following result for the first order derivatives

(a) \[ \left| \frac{\partial Q_4}{\partial L_4} \right| = O(1), \quad \left| \frac{\partial Q_4}{\partial (L_4, G_4, g_4)} \right| = O(\ell), \quad \frac{\partial Q_4}{\partial g_4} \cdot Q_4 = 0. \]

In addition
\[ \frac{\partial Q_4}{\partial G_4} \cdot Q_4 = O_{C^2(L,G,g)}(\ell). \]

(b) If in addition we have \[ \left| g \mp \arctan \frac{G}{L} \right| \leq C/\ell \] where $-$ sign is taken for $u > 0$ and $+$ sign is taken for $u < 0$ then we have the following bounds for (A.5)
\[ \frac{\partial Q_4}{\partial G} = \sinh u \left( 0, \sqrt{L^2 + G^2} \right) + O(1), \quad \frac{\partial Q_4}{\partial L} = - \sinh u \left( 2\sqrt{L^2 + G^2}, \frac{GL}{\sqrt{L^2 + G^2}} \right) + O(1). \]

(c) If in addition to the conditions of Lemma A.1 we have $G, g = O(1/\chi)$ and $\ell = O(\chi)$, then we have the following bounds for (A.6)
\[ \frac{\partial Q_4}{\partial G} = \sinh u(0,1) + O(1), \quad \frac{\partial Q_4}{\partial L} = \sinh u(2,0) + O(1). \]

**Remark 10.** The assumptions of the lemma and the next lemma hold in our situation due to Lemma 4.7.
Proof. We consider only the case $u > 0$. We have

$$Q_4 = O(1) - \sinh u (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG), \text{ as } \ell \to \infty.$$  

Now the first three estimates of part (a) follow easily. Next

$$\frac{\partial Q_4}{\partial G} = -(\cosh u)u'_G (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) - \sinh u (\sin gL, -\cos gL) + O(1).$$

Using Lemma A.1, we obtain

$$Q_4 \cdot \frac{\partial Q_4}{\partial G} = \frac{1}{2} (\sinh u)^2 (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG) + O(1).$$

We prove (b) in the $+$ case, the $-$ case being similar. Assume first that $g$ is exactly equal to $\arctan \frac{G}{L}$. Using (A.9) and (A.7) we obtain

$$\frac{\partial Q_4}{\partial G} = (\cosh u) f_G (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG)$$

$$= \sinh u \left( \frac{G}{L^2 + G^2} \left( \frac{L^3 + LG^2}{\sqrt{L^2 + G^2}} \cdot 0 \right) - \left( \frac{GL}{\sqrt{L^2 + G^2}^2} - \frac{L^2}{\sqrt{L^2 + G^2}} \right) \right) + O(1)$$

$$= \sinh u \left( 0, \frac{L^2}{\sqrt{L^2 + G^2}} \right) + O(1).$$

(A.10)

$$\frac{\partial Q_4}{\partial L} = (\cosh u) f_L (\cos gL^2 + \sin gLG, \sin gL^2 - \cos gLG)$$

$$= - \sinh u \left( \frac{G^2/L}{L^2 + G^2} \left( \frac{L^3 + LG^2}{\sqrt{L^2 + G^2}} \cdot 0 \right) + \left( \frac{2L^2 + G^2}{\sqrt{L^2 + G^2}^2} \frac{GL}{\sqrt{L^2 + G^2}} \right) \right) + O(1)$$

$$= - \sinh u \left( 2\sqrt{L^2 + G^2}, \frac{GL}{\sqrt{L^2 + G^2}} \right) + O(1).$$

This proves (b) under the assumption $g = \arctan \frac{G}{L}$. If $\left| g - \arctan \frac{G}{L} \right| < \frac{C}{\ell}$ then we get an additional $O(1)$ error in the above computation which does not change the final result.

Part (c) follows from part (b) since both $g$ and $\arctan \frac{G}{L}$ are $O(1/\ell)$. \qed
A.5. Second order derivatives. The following estimates of the second order derivatives are used in integrating the variational equation.

Lemma A.3. We have the following information for the second order derivatives of $Q_4$ w.r.t. the Delaunay variables.

(a) Under the conditions of Lemma A.2(a) we have

\[
\frac{\partial^2 Q_4}{\partial g_4^2} = -Q_4, \quad \frac{\partial^2 Q_4}{\partial g_4 \partial G_4} = \left( \frac{\partial}{\partial G_4}, \frac{\partial}{\partial g_4} \right) \left( \frac{\partial |Q_4|^2}{\partial g_4} \right) = (0, 0), \quad \frac{\partial^2 Q_4}{\partial G_4^2} = O(\ell).
\]

In addition $\frac{\partial^2 Q_4}{\partial L_4^2} = O(\ell)$.

(b) Under the conditions of Lemma A.2(b) we have we have

\[
\frac{\partial^2 Q_4}{\partial G_4^2} = \frac{L^2}{(L^2 + G^2)^{3/2}} \left( L \cosh u, -2G \sinh u \right) + O(1),
\]
\[
\frac{\partial^2 Q_4}{\partial g_4 \partial G_4} = \left( -\frac{L^2 \sinh u}{\sqrt{L^2 + G^2}}, 0 \right) + O(1),
\]
\[
\frac{\partial^2 Q_4}{\partial g_4 \partial L_4} = \left( \frac{GL \sinh u}{\sqrt{L^2 + G^2}}, -2\sqrt{L^2 + G^2} \cosh u \right) + O(1),
\]
\[
\frac{\partial^2 Q_4}{\partial G \partial L} = \frac{L}{(L^2 + G^2)^{3/2}} \left( -LG \cosh u, (L^2 + 3G^2) \sinh u \right) + O(1).
\]

(c) Under the conditions of Lemma A.2(c) we have

\[
\frac{\partial^2 Q_4}{\partial G_4^2} = -\cosh u (1, 0) + O(1), \quad \frac{\partial^2 Q_4}{\partial g \partial G} = -L \sinh u (1, 0) + O(1),
\]
\[
\frac{\partial^2 Q_4}{\partial g \partial L} = L \sinh u (0, 2) + O(1), \quad \frac{\partial^2 Q_4}{\partial G \partial L} = \cosh u (0, 1) + O(1).
\]

Proof. The estimate $\frac{\partial^2 Q_4}{\partial G_4^2} = O(\ell)$ follows immediately from Lemma A.2. The estimate $\frac{\partial^2 Q_4}{\partial L_4^2} = O(\ell)$ follows immediately from (A.5) (or (A.6)).

The estimates of the derivatives involving $g_4$ are relatively easy since the dependence of $Q_4$ on $g_4$ is through a rotation. We consider $\frac{\partial^2 Q_4}{\partial L_4 \partial g_4}$, for example, the other derivatives are similar. Differentiating (A.10) with respect to $g$ and using (A.7) we
get

\[ \frac{\partial^2 Q_4}{\partial L_4 \partial g_4} = \cosh u f_L (-L^2 \sin g + LG \cos g, L^2 \cos g + LG \sin g) \]
\[- \sinh u (-2L \sin g + G \cos g, 2L \cos g + G \sin G) + O(1) \]
\[= - \sinh u \frac{G^2}{L(L^2 + G^2)} \left( \frac{-L^2 G + L^2 G}{\sqrt{L^2 + G^2}}, \frac{L^3 + LG^2}{\sqrt{L^2 + G^2}} \right) - \sinh u \left( \frac{-2LG + LG}{\sqrt{L^2 + G^2}}, \frac{2L^2 + G^2}{\sqrt{L^2 + G^2}} \right) + O(1) \]
\[= - \sinh u \left( \frac{G^2}{\sqrt{L^2 + G^2}}, - \sinh u \left( \frac{L^2 + LG^2}{\sqrt{L^2 + G^2}} \right) + O(1) \right) \]
\[= \sinh u \left( \frac{LG}{\sqrt{L^2 + G^2}}, -2 \sqrt{L^2 + G^2} \right) + O(1). \]

Next, we compute \( \frac{\partial^2 Q_4}{\partial G_4 \partial L_4} \) and \( \frac{\partial^2 Q_4}{\partial G_4^2} \). We consider only the case \( u > 0 \) and take the + sign. The other cases are similar.

As in the proof of Lemma A.2 it suffices to consider the case \( g = \arctan \frac{G}{L} \). Differentiating the expression for \( \frac{\partial Q_4}{\partial G_4} \) and using Lemma A.1 (A.7) and (A.8) we obtain

\[ \frac{\partial^2 Q_4}{\partial G_4^2} = -L (\sinh u (\ln e)_G)^2 - \cosh u (\ln e)_G (\cos gL + \sin gG, \sin gL - \cos gG) \]
\[+ 2L \cosh u (\ln e)_G (\sin g, - \cos g) + O(1) \]
\[= L \sinh u \left( \frac{L^2 - 2G^2}{(L^2 + G^2)^2} \right) \left( \frac{L^2}{(L^2 + G^2)^{1/2}}, \frac{G}{(L^2 + G^2)^{1/2}}, 0 \right) \]
\[+ 2L \sinh u \frac{G}{L^2 + G^2} \left( \frac{G}{(L^2 + G^2)^{1/2}}, \frac{L}{(L^2 + G^2)^{1/2}} \right) + O(1) \]
\[= \frac{L^2}{(L^2 + G^2)^{3/2}} \sinh u (L, -2G) + O(1) \]

proving the estimate for \( \frac{\partial^2 Q_4}{\partial G_4^2} \). Next,

\[ \frac{\partial^2 Q_4}{\partial G_4 \partial L_4} = - (\sinh u)_G (\cos gL^2 + \sin g LG, \cos gL^2 - \cos gLG) - (\sinh u)_L (\sin gL, - \cos gL) \]
\[ - (\sinh u)_G (2 \cos gL + \sin gG, 2 \sin gL - \cos gG) - \sinh u (\sin g, - \cos g) + O(1) \]
\[ -\sinh u \ln e - \cosh u \ln e \ln e - \cosh u \ln e \ln e + \sinh u \ln e \ln e = -\frac{L^2 + G^2}{(L^2 + G^2)^{1/2}} \ln e \ln e - \frac{G L}{(L^2 + G^2)^{1/2}} \ln e \ln e \]

\[ = L \frac{L^2 + G^2}{(L^2 + G^2)^{3/2}} \sinh u \ln e \ln e + \cosh u \ln e \ln e \frac{L^2 + G^2}{(L^2 + G^2)^{1/2}} (-LG, L^2 + 3G^2) + O(1). \]

Part (c) follows from part (b) as in Lemma \text{A.2} \quad \square

\textbf{Appendix B. Gerver’s mechanism}

\textbf{B.1. Gerver’s result in [G2].} We summarize the result of [G2] in the following table. Recall that the Gerver scenario deals with the limiting case \( \chi \to \infty, \mu \to 0. \) Accordingly \( Q_1 \) disappears at infinity and there is no interaction between \( Q_3 \) and \( Q_4. \) Hence both particles perform Kepler motions. The shape of each Kepler orbit is characterized by energy, angular momentum and the argument of periapsis. In Gerver’s scenario, the incoming and outgoing asymptotes of the hyperbola are always horizontal and the semimajor of the ellipse is always vertical. So we only need to describe on the energy and angular momentum.

| 1st collision | @\((-\varepsilon_0 \varepsilon_1, \varepsilon_0 + \varepsilon_1)\) | 2nd collision | @\((\varepsilon_0, 0)\) |
|---------------|------------------|---------------|------------------|
| \( Q_3 \)    | \(-1/2\)       | \( Q_3 \)    | \( 1/2 \to -\frac{\varepsilon_1}{2\varepsilon_0} \)   |
| \( Q_4 \)    | \(1/2\)        | \( Q_4 \)    | \( 1/2 \to \frac{\varepsilon_1}{2\varepsilon_0} \)    |
| energy       | \( \varepsilon_1 \to -\varepsilon_0 \) | angular momentum | \( p_1 \to -p_2 \)  |
| eccentricity | \( \varepsilon_0 \to \varepsilon_1 \) | \( -\varepsilon_0 \) | \( \sqrt{2}\varepsilon_0 \)   |
| semimajor    | \( 1 \)        | semimajor     | \( 1 \to (\frac{\varepsilon_0}{\varepsilon_1})^2 \)  |
| semiminor    | \( \varepsilon_1 \to \varepsilon_0 \) | semiminor      | \( \frac{\varepsilon_0}{\varepsilon_1} \) \to \( \sqrt{2}\varepsilon_0 \to \sqrt{2}\varepsilon_1 \) |

Here

\[ p_{1,2} = \frac{-Y \pm \sqrt{Y^2 + 4(X + R)}}{2}, \quad R = \sqrt{X^2 + Y^2}. \]

and \((X, Y)\) stands for the point where collision occurs (the parenthesis after @ in the table). We will call the two points the Gerver’s collision points.

In the above table \( \varepsilon_0 \) is a free parameter and \( \varepsilon_1 = \sqrt{1 - \varepsilon_0^2}. \)

At the collision points, the velocities of the particles are the following.

For the first collision,

\[ v_3^- = \left( \frac{-\varepsilon_0^2}{\varepsilon_0 \varepsilon_1 + 1}, \frac{-\varepsilon_0}{\varepsilon_0 \varepsilon_1 + 1} \right), \quad v_4^- = \left( \frac{1}{R_{p_1}}, \frac{1}{R_{p_1}} \right). \]
\[ v_3^+ = \left( \frac{-\varepsilon_0^2}{\varepsilon_0 \varepsilon_1 + 1}, \frac{\varepsilon_1}{\varepsilon_0 \varepsilon_1 + 1} \right), \quad v_4^+ = \left( -1 + \frac{Y}{R \rho_2}, \frac{-1}{R \rho_2} \right). \]

For the second collision,

\[ v_3^- = \left( \frac{-\varepsilon_1}{\varepsilon_0}, \frac{-1}{\varepsilon_0} \right), \quad v_4^- = \left( 1, \frac{\sqrt{2}}{\varepsilon_0} \right), \quad v_3^+ = \left( 1, \frac{-1}{\varepsilon_0} \right), \quad v_4^+ = \left( \frac{-\varepsilon_1}{\varepsilon_0}, \frac{\sqrt{2}}{\varepsilon_0} \right). \]

**B.2. Numerical information for a particularly chosen \( \varepsilon_0 = 1/2. \)** For the first collision \( \varepsilon_3 : \frac{1}{2} \to \frac{\sqrt{3}}{2}. \)

We want to figure out the Delaunay coordinates \( (L, u, G, g) \) for both \( Q_3 \) and \( Q_4. \)

(Here we replace \( \ell \) by \( u \) for convenience.) The first collision point is

\[ (X, Y) = (-\varepsilon_0 \varepsilon_1, \varepsilon_0 + \varepsilon_1) = \left( -\frac{\sqrt{3}}{4}, \frac{1 + \sqrt{3}}{2} \right). \]

Before collision

\[ (L, u, G, g)_3^- = \left( 1, -\frac{5\pi}{6}, \frac{\sqrt{3}}{2}, \pi/2 \right), \quad (L, u, G, g)_4^- = (1, 1.40034, p_1, -\arctan p_1), \]

\[ v_3^- = \left( \frac{-3}{\sqrt{3} + 4}, \frac{-2}{\sqrt{3} + 4} \right) \approx -0.523, 0.349), \quad v_4^- = \left( 1 - \frac{2(1 + \sqrt{3})}{(4 + \sqrt{3})p_1}, \frac{4}{(4 + \sqrt{3})p_1} \right) \approx (-0.805, 1.322), \]

where

\[ p_1 = \frac{-Y + \sqrt{Y^2 + 4(X + R)}}{2} = \frac{-\varepsilon_0 + \varepsilon_1 + \sqrt{5 + 2\varepsilon_0 \varepsilon_1}}{2} = 0.52798125. \]

After collision

\[ (L, u, G, g)_3^+ = \left( 1, \frac{2\pi}{3}, -\frac{1}{2}, \pi/2 \right), \quad (L, u, G, g)_4^+ = (1, 0.515747, -p_2, -\arctan p_2), \]

\[ v_3^+ = \left( \frac{1}{\sqrt{3} + 4}, \frac{2\sqrt{3}}{\sqrt{3} + 4} \right) \approx 0.174, 0.604), \quad v_4^+ = \left( -1 + \frac{2(1 + \sqrt{3})}{(4 + \sqrt{3})p_2}, \frac{-4}{(4 + \sqrt{3})p_2} \right) \approx (-1.503, 0.368) \]

where

\[ p_2 = \frac{-Y - \sqrt{Y^2 + 4(X + R)}}{2} = \frac{-\varepsilon_0 + \varepsilon_1 - \sqrt{5 + 2\varepsilon_0 \varepsilon_1}}{2} = -1.894006654. \]

For the second collision \( \varepsilon_3 : \frac{\sqrt{3}}{2} \to \frac{1}{2}. \)

The collision point is \( (X, Y) = (\varepsilon_0^2, 0) = \left( \frac{1}{4}, 0 \right). \)

Before collision

\[ (L, u, G, g)_3^- = \left( 1, -\frac{\pi}{6}, -\frac{1}{2}, \pi/2 \right), \quad (L, u, G, g)_4^- = \left( 1, 0.20273, \sqrt{2}/2, -\arctan \frac{\sqrt{2}}{2} \right), \]
After collision
\[
(L, u, G, g)_3^+ = \left(\frac{1}{\sqrt{3}}, \frac{\pi}{3}, -\frac{1}{2}, -\pi/2\right), \quad (L, u, G, g)_4^+ = \left(1, 0.45815, \sqrt{2}/2, \arctan \frac{\sqrt{6}}{2}\right),
\]
\[
v_3^- = (-\sqrt{3}, -2), \quad v_4^- = (1, 2\sqrt{2}).
\]

**B.3. Control the shape of the ellipse.** As it was mentioned before Lemma 2.1 was stated by Gerver in [G2]. There is a detailed proof of part (a) of our Lemma 2.1 in [G2]. However since no details of the proof of part (b) were given in [G2] we go other main steps here for the reader’s convenience even though computations are quite straightforward.

**Proof of Lemma 2.1.** Recall that Gerver’s map depends on a free parameter \(e_4\) (or equivalently \(G_4\)). In the computations below however it is more convenient to use the polar angle \(\psi\) of the intersection point as the free parameter. It is easy to see that as \(G_4\) changes from large negative to large positive value the point of intersection cover the whole orbit of \(Q_3\) so it can be used as the free parameter. Our goal is to show that by changing the angles \(\psi_1\) and \(\psi_2\) of the first and second collision we can prescribe the values of \(\bar{e}_3\) and \(\bar{g}_3\) arbitrarily. Due to the Implicit Function Theorem it suffices to show that
\[
\det \begin{bmatrix}
\frac{\partial \bar{e}_3}{\partial \psi_1} & \frac{\partial \bar{g}_3}{\partial \psi_1} \\
\frac{\partial \bar{e}_3}{\partial \psi_2} & \frac{\partial \bar{g}_3}{\partial \psi_2}
\end{bmatrix} \neq 0.
\]

To this end we use the following set of equations
\[
(B.1) \quad G_3^+ + G_4^+ = G_3^- + G_4^-,
\]
\[
(B.2) \quad \frac{e_3^+}{G_3^+} \cos(\psi + g_3^+) + \frac{e_4^+}{G_4^+} \cos(\psi - g_4^-) = \frac{e_3^-}{G_3^-} \cos(\psi + g_3^-) + \frac{e_4^-}{G_4^-} \cos(\psi - g_4^-),
\]
\[
(B.3) \quad \frac{(G_3^+)^2}{1 - e_3^+ \sin(\psi + g_3^+)} = \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi + g_3^-)},
\]
\[
(B.4) \quad \frac{(G_4^+)^2}{1 - e_4^+ \sin(\psi + g_4^+)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi - g_4^-)},
\]
\[
(B.5) \quad g_4^+ = \arctan \frac{G_4^+}{L_4^+}.
\]

Here \(e_3, e_4\) and \(L_4\) are functions of the other variables according to the formulas of Appendix A.
(B.1)–(B.5) are obtained as follows. (B.1) is the angular momentum conservation, (B.3) means that the position of $Q_3$ does not change during the collision, (B.4) means that $Q_3$ and $Q_4$ are at the same point immediately after the collision and (B.5) says that after the collision the outgoing asymptote of $Q_4$ is horizontal.

It remains to derive (B.2). Represent the position vector as $\vec{r} = r \hat{e}_r$. Then the velocity is $\dot{\vec{r}} = \dot{r} \hat{e}_r + r \dot{\psi} \hat{e}_\psi$. The momentum conservation gives

$$(\dot{\vec{r}}_3)^- + (\dot{\vec{r}}_4)^- = (\dot{\vec{r}}_3)^+ + (\dot{\vec{r}}_4)^+.$$  

Taking the angular component of the velocity we get

$$(\dot{r}_3^- \dot{\psi}_3^- + r_4^- \dot{\psi}_4^-) = r_3^+ \dot{\psi}_3^+ + r_4^+ \dot{\psi}_4^+.$$  

In our notation the polar representation of the ellipse takes form

$$r = G_2^1 - e \sin(\psi + g).$$

Differentiating this equation we obtain the following relation for the radial component of the Kepler motion

$$\dot{r} = \frac{G^2}{(1 - e \sin(\psi + g))^2} e \cos(\psi + g) \dot{\psi} = \frac{r^2}{G^2} e \cos(\psi + g) \frac{G}{r} = \frac{e}{G} \cos(\psi + g).$$

Plugging this into (B.6) we obtain (B.2).

We can write (B.1)–(B.5) in the form

$$F(\vec{Z}^- - \tilde{\vec{Z}}, \vec{Z}^+) = 0$$

where $\vec{Z}^- = (E_3^-, G_3^-, g_3^-, \psi)$, $\vec{Z}^+ = (E_3^+, G_3^+, g_3^+, G_4^+, g_4^+)$, and $\tilde{\vec{Z}} = (G_4^-, g_4^-)$ are considered as functions $\vec{Z}^-$. By the Implicit Function Theorem we have

$$\frac{\partial \tilde{\vec{Z}}^+}{\partial \vec{Z}^-} = -\left( \frac{\partial F}{\partial \vec{Z}^+} \right)^{-1} \left( \frac{\partial F}{\partial \vec{Z}^-} + \frac{\partial F}{\partial \tilde{\vec{Z}}} \frac{\partial \tilde{\vec{Z}}}{\partial \vec{Z}^-} \right).$$

Thus to complete the computation we need to know $\frac{\partial \tilde{\vec{Z}}}{\partial \vec{Z}^-}$. In order to compute this expression we use the equations

$$(B.7) \quad g_4^- = - \arctan \frac{G_4^-}{L_4^-}$$

which means that the incoming asymptote of $Q_4$ is horizontal and

$$(B.8) \quad \frac{(G_3^-)^2}{1 - e_3^- \sin(\psi + g_3^-)} = \frac{(G_4^-)^2}{1 - e_4^- \sin(\psi - g_4^-)},$$

which means that $Q_3$ and $Q_4$ are at the same place immediately before the collision. Writing these equations as $I(\vec{Z}^-, \tilde{\vec{Z}}) = 0$ we get by the Implicit Function Theorem

$$\frac{\partial \tilde{\vec{Z}}}{\partial \vec{Z}^-} = -\left( \frac{\partial I}{\partial \tilde{\vec{Z}}} \right)^{-1} \frac{\partial I}{\partial \vec{Z}^-}.$$
so that the required derivative equals to

\[
\frac{\partial Z^+}{\partial Z^-} = - \left( \frac{\partial F}{\partial Z^+} \right)^{-1} \left( \frac{\partial F}{\partial Z^-} - \frac{\partial F}{\partial Z^+} \left( \frac{\partial I}{\partial Z} \right)^{-1} \frac{\partial I}{\partial Z^-} \right).
\]

Combining (B.9) with the formula

\[
de_3 = -\frac{2G_3E_3dG_3 + G_3^2dE_3}{\sqrt{1 - 2G_3^2E_3}}
\]

which follows from the relation \( e_3 = \sqrt{1 - 2G_3^2E_3} \) we obtain the two entries \( \frac{\partial \bar{e}_3}{\partial \psi_2} = -0.158494 \) and \( \frac{\partial \bar{g}_3}{\partial \psi_2} = 0.369599 \) The meanings of these two entries are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the second collision.

We need more work to figure out the two entries \( \frac{\partial \bar{e}_3}{\partial \psi_1} \) and \( \frac{\partial \bar{g}_3}{\partial \psi_1} \), which are the changes of the eccentricity and argument of periapsis after the second collision if we vary the phase of the first collision. We describe the computation of the first entry, the second one is similar. We use the relation

\[
\frac{\partial \bar{e}_3}{\partial \psi_1} = \frac{\partial \bar{e}_3}{\partial \bar{E}_1^+} \frac{\partial \bar{E}_1^+}{\partial \psi_1} + \frac{\partial \bar{e}_3}{\partial \bar{G}_1^+} \frac{\partial \bar{G}_1^+}{\partial \psi_1} + \frac{\partial \bar{e}_3}{\partial \bar{g}_1^+} \frac{\partial \bar{g}_1^+}{\partial \psi_1}.
\]

Now \( \left( \frac{\partial \bar{E}_1^+}{\partial \psi_1}, \frac{\partial \bar{G}_1^+}{\partial \psi_1}, \frac{\partial \bar{g}_1^+}{\partial \psi_1} \right) \) is computed using (B.9) and the data for the first collision. Noticing that the quantities \( E_3, G_3, g_3 \) after the first collision is the same as those before the second collision, we replace \( \left( \frac{\partial \bar{E}_3}{\partial E_3^+}, \frac{\partial \bar{e}_3}{\partial G_3^+}, \frac{\partial \bar{g}_3}{\partial g_3^+} \right) \) by \( \left( \frac{\partial \bar{E}_3}{\partial E_3^-}, \frac{\partial \bar{e}_3}{\partial G_3^-}, \frac{\partial \bar{g}_3}{\partial g_3^-} \right) \) and compute it using (B.9) and the data for the second collision. It turns out that the resulting matrix is

\[
\begin{bmatrix}
\frac{\partial \bar{e}_3}{\partial \psi_1} & \frac{\partial \bar{g}_3}{\partial \psi_1} \\
\frac{\partial \bar{e}_3}{\partial \psi_2} & \frac{\partial \bar{g}_3}{\partial \psi_2}
\end{bmatrix} = \begin{bmatrix}
0.620725 & 2.9253 \\
-0.158494 & 0
\end{bmatrix},
\]

which is obviously nondegenerate.

\[ \Box \]

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NON-COLLISION SINGULARITIES IN THE PLANAR TWO-CENTER-TWO-BODY PROBLEM

References

[A] V.I. Arnold, Mathematical methods in classical mechanics. Springer, (1989).
[Al] A. Albouy, Lectures on the two-body problem, Classical and Celestial Mechanics: The Recife Lectures (H. Cabral and F. Diacu, Eds.), Princeton University Press, Princeton, NJ, (2002).
[BM] S. Bolotin, R.S. MacKay, Nonplanar second species periodic and chaotic trajectories for the circular restricted three-body problem, Celestial Mech Dyn Astr 94, (2006), 433–449.
[BN] S. Bolotin, P. Negrini, Variational approach to second species periodic solutions of Poincaré of the 3 body problem, [arXiv:1104.2288]
[F] L. Floria, a simple derivation of the hyperbolic Delaunay variables, The Astronomical journal, 110, No 2, (1995), 940–942.
[FNS] J. Font, A. Nunes, C. Simo, Consecutive quasi-collisions in the planar circular RTBP, Nonlinearity 15, (2002), 115–142.
[G1] J. Gerver, The existence of pseudocollisions in the plane, J. Differential Eq. 89 (1991) 1–68.
[G2] J. Gerver, Noncollision singularity: Do four bodies suffice? Experiment Math. 12, (2003), 187–198.
[G3] J. Gerver, Noncollision singularities in the n-body problem, in Dynamical systems. Part I, 57–86, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2003.
[K] O. Knill, [http://www.math.harvard.edu/~knill/seminars/intr/index.html]
[LL] L. Landau, Lifschitz, Mechanics. Third Edition: Volume 1 (Course of Theoretical Physics).
[LS] D. Li, Ya. G, Sinai Blowups of complex-valued solutions for some hydrodynamic models, Regul. Chaotic Dyn. 15 (2010) 521–531.
[MM] J. Mather, R. McGehee, Solutions of the collinear four body problem which become unbounded in finite time, Dynamical Systems, Theory and Applications (J. Moser, ed.), Lecture Notes in Physics 38, Springer-Verlag, Berlin, (1975), 573–597.
[Pa] P. Painlevé, Lecons sur la theorie analytique des equations differentielles, Hermann, Paris, 1897.
[Po] H. Poincaré, New methods of celestial mechanics. Translated from the French. History of Modern Physics and Astronomy, 13. American Institute of Physics, New York, 1993.
[Sa1] D. Saari, Improbability of collisions in Newtonian gravitational systems. Trans. Amer. Math. Soc. 162 (1971), 267–271; erratum, ibid. 168 (1972), 521.
[Sa2] D. Saari, A global existence theorem for the four-body problem of Newtonian mechanics. J. Differential Equations 26 (1977) 80–111.
[Sa3] D. Saari, Collisions, rings, and other Newtonian N-body problems. CBMS Regional Conference Series in Mathematics, 104., AMS, Providence, RI, 2005
[Sim] B. Simon, Fifteen problems in mathematical physics. Perspectives in mathematics, 423–454, Birkhauser, Basel, 1984.
[Su] K. F. Sundman, Nouvelles recherches sur le probleme des trois corps, Acta. Soc. Sci. Fennicae 35 (1909), 3–27.
[X] Z. Xia, the existence of noncollision singularity in Newtonian systems, Annals of Mathematics, second series, Vol 135, Issue 3, (1992), 411–468.