Research Article

Fixed Points of Integral Type Multivalued Contractive Mappings with \( w \)-Distance

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1. Introduction and Preliminaries

In 2002, Branciari [1] generalized the famous Banach contraction principle and proved the following fixed point theorem for the contractive mapping of the integral type.

**Theorem 1** (see [1]). Let \( T \) be a mapping from a complete metric space \((X,d)\) into itself satisfying

\[
\int_0^1 \varphi(t)dt \leq c \int_0^1 \phi(t)dt, \quad \forall x, y \in X,
\]

where \( c \in (0, 1) \) is a constant and \( \varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is Lebesgue integrable, summable in each compact subset of \( \mathbb{R}^+ \) and \( \int_0^1 \varphi(t)dt > 0 \) for each \( \varepsilon > 0 \). Then, \( T \) has a unique fixed point \( a \in X \) and \( \lim_{n \to \infty} T^n x = a \) for each \( x \in X \).

Using Hausdorff metric, Nadler [2] introduced the concept of multivalued contraction and proved a multivalued version of the well-known Banach contraction principle.

**Theorem 2** (see [2]). Let \((X,d)\) be a complete metric space and let \( T \) be a mapping from \( X \) into \( \text{CB}(X) \), where \( \text{CB}(X) \) is the family of all nonempty closed and bounded subsets of \( X \). Assume that there exists \( \phi \) such that

\[
H(T(x), T(y)) \leq cd(x, y), \quad \forall x, y \in X.
\]

Then, \( T \) has a fixed point.

In the past decades, various fixed point theorems concerning multivalued contractive mappings have been proved. Especially, Feng and Liu [3] generalized Theorem 2 and proved a few fixed point theorems for multivalued contractive mappings without Hausdorff metric.

**Theorem 3** (see [3]). Let \((X,d)\) be a complete metric space and \( T \) be a multivalued mapping from \( X \) into \( \text{CL}(X) \), where \( \text{CL}(X) \) is the family of all nonempty closed subsets of \( X \). Assume that

\[
\begin{align*}
(a_1) & \quad \text{There exist constants } b, c \in (0, 1) \text{ with } c < b \text{ such that, for any } x \in X, \text{ there is } y \in T(x) \text{ satisfying } \\
& \quad bd(x, y) \leq d(x, T(x)), \\
& \quad d(y, T(y)) \leq cd(x, y).
\end{align*}
\]

\[
(a_2) f: X \rightarrow \mathbb{R}^+ \text{ is lower semicontinuous, where } f(x) = d(x, T(x)), \forall x \in X.
\]

Then, \( T \) has a fixed point in \( X \).
In 1996, Kada et al. [4] introduced the concept of \( w \)-distance in a metric space and proved several fixed point theorems for single-valued contractive mappings under \( w \)-distance. Some other fixed point results concerning \( w \)-distance can be found in [5–9]. In 2007, Guran [5] deduced the following fixed point theorem, which is a generalization of Theorem 3.

**Theorem 4** (see [5]). Let \((X, d)\) be a complete metric space, \(T : X \longrightarrow \text{CL}(X)\) be a multivalued mapping, \(w : X \times X \longrightarrow \mathbb{R}^+\) be a \(w\)-distance on \(X\), and \(b \in (0, 1)\). Assume that

\[
(b_1) \text{ There exists } c \in (0, 1), \text{ with } c < b, \text{ such that, for any } x \in X, \text{ there is } y \in T(x) \text{ satisfying }
\]

\[
w(x, y) \leq w(x, T(x)), \quad w(y, T(y)) \leq bw(x, y).
\]

Then, \(T\) has a fixed point in \(X\).

Motivated by the results in [1, 3–5], we prove two fixed point results for multivalued contractive mappings of integral type with respect to \(w\)-distance in complete metric spaces. The results presented in this paper improve Theorems 2–4. Two examples with uncountably many points are included.

Throughout this paper, we denote by \(\mathbb{N}\) the set of positive integers, \(\mathbb{N}_0 = [0] \cup \mathbb{N}\), \(\mathbb{R} = (-\infty, +\infty)\), \(\mathbb{R}^+ = [0, +\infty)\), and \(\mathbb{R}^* = (0, +\infty)\).

**Definition 1** (see [4]). Let \((X, d)\) be a metric space. A function \(w : X \times X \longrightarrow \mathbb{R}^+\) is called a \(w\)-distance in \(X\) if it satisfies the following:

\[
(w_1) \quad w(x, z) \leq w(x, y) + w(y, z), \quad \forall x, y, z \in X
\]

\[
(w_2) \quad \text{For each } x \in X, \text{ a mapping } w(x, \cdot) : X \longrightarrow \mathbb{R}^+ \text{ is lower semicontinuous}
\]

\[
(w_3) \quad \text{For any } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } w(z, x) \leq \delta \text{ and } w(z, y) \leq \delta \text{ imply } d(x, y) \leq \epsilon
\]

**Example 1.** Let \(X\) be a normed linear space with norm \(\|\cdot\|\), \(\alpha\) be a positive constant, and \(w : X \times X \longrightarrow \mathbb{R}^+\) be defined by

\[
w(x, y) = \|y\|^\alpha, \quad \forall x, y \in X.
\]

Then, \(w\) is a \(w\)-distance in \(X\).

**Proof.** Let \(x, y, z \in X\). It is clear that \((w_3)\) holds and

\[
w(x, z) = \|x\|^\alpha + \|z\|^\alpha = w(x, y) + w(y, z),
\]

which implies \((w_1)\). For each \(\epsilon > 0\), put \(\delta = (\epsilon/2)^\alpha\). If \(w(x, z) \leq \delta\) and \(w(z, y) \leq \delta\), it follows that

\[
d(x, y) = \|x - y\| \leq \|x\| + \|y\| = w^{1/\alpha}(x, z) + w^{1/\alpha}(z, y)
\]

\[
\leq \delta^{1/\alpha} + \delta^{1/\alpha} = \epsilon,
\]

which yields \((w_3)\). That is, \(w\) is a \(w\)-distance in \(X\). \(\square\)

**Example 2.** Let \(X = \mathbb{R}\) be endowed with the Euclidean metric \(d = |\cdot|\) and \(w : X \times X \longrightarrow \mathbb{R}^+\) be defined by

\[
BW(x, y) \leq w(x, T(x)), \quad w(y, T(y)) \leq bw(x, y).
\]

(\(b_2\) \(f_w : X \longrightarrow \mathbb{R}^+\) is lower semicontinuous, where \(f_w(x) = w(x, T(x))\), \(\forall x \in X\).

(11)

which implies \((w_1)\). For each \(\epsilon > 0\), put \(\delta = (\epsilon/2)^\alpha\). If \(w(x, z) \leq \delta\) and \(w(z, y) \leq \delta\), it is easy to see that

\[
d(x, y) = |x - y| \leq |z - x| + |z - y|
\]

\[
\leq \max(|a||z - x|, |az - y|) + \max(|a||z - y|, |ay - z|)
\]

\[
= w(x, z) + w(z, y)
\]

\[
\leq \delta + \delta = \epsilon,
\]

which yields \((w_3)\). That is, \(w\) is a \(w\)-distance in \(X\). Let \((X, d)\) be a metric space. For any \(u \in X\), \(D \subseteq X\), \(T : X \longrightarrow \text{CL}(X)\), and \(w : X \times X \longrightarrow \mathbb{R}^+\), put

\[
D(u, D) = \inf_{y \in D} d(u, y),
\]

\[
w(u, D) = \inf_{y \in D} w(u, y),
\]

(12)

\[
f(u) = d(u, T(u)), \quad f_w(u) = w(u, T(u)).
\]
A sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) is called an orbit of \( T \) if \( x_n \in T(x_{n-1}) \) for all \( n \in \mathbb{N} \).

**Definition 2.** Let \((X, d)\) be a metric space and \(T: X \to \text{CL}(X)\) be a multivalued mapping. A function \( g: X \to \mathbb{R}^+ \) is said to be

1. Lower semicontinuous in \( X \) if \( g(y) \leq \liminf_{n \to \infty} g(y_n) \) for each \( y \in X \) and \( \{y_n\}_{n \in \mathbb{N}} \subseteq X \) with \( \lim_{n \to \infty} y_n = y \).
2. \( T \)-orbitally lower semicontinuous at \( z \in X \) if \( g(z) \leq \liminf_{n \to \infty} g(x_n) \) for each orbit \( \{x_n\}_{n \in \mathbb{N}_0} \) of \( T \) with \( \lim_{n \to \infty} x_n = z \).
3. \( T \)-orbitally lower semicontinuous in \( X \) if it is \( T \)-orbitally lower semicontinuous at each \( z \in X \).

Obviously, if \( g \) is lower semicontinuous in \( X \), then \( g \) is \( T \)-orbitally lower semicontinuous in \( X \).

The following lemmas play important roles in this paper.

**Lemma 1** (see [10]). Let \( \varphi \in \Phi_2 \) and \( \{r_n\}_{n \in \mathbb{N}} \) be a nonnegative sequence. Then, \( \lim_{n \to \infty} \int_0^c \varphi(t)dt = 0 \) if and only if \( \lim_{n \to \infty} r_n = 0 \).

**Lemma 2** (see [4]). Let \( X \) be a metric space with metric \( d \) and let \( \omega \) be a \( \omega \)-distance in \( X \). Let \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) be sequences in \( X \), let \( (\alpha_n)_{n \in \mathbb{N}} \) and \( (\beta_n)_{n \in \mathbb{N}} \) be sequences in \( \mathbb{R}^+ \) converging to 0, and let \( x, y, z \in X \), then the following hold:

1. If \( w(x_n, y) \leq \alpha_n \) and \( w(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \). In particular, if \( w(x, y) \) and \( w(x, z) = 0 \), then \( y = z \).
2. If \( w(x_n, y_n) \leq \alpha_n \) and \( w(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( \{y_n\}_{n \in \mathbb{N}} \) converges to \( z \).
3. If \( w(x_n, y_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( n > m \), then \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence.
4. If \( w(x_n, y_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence.

**Lemma 3** (see [11]). Let \((X, d)\) be a metric space, \( w \) be a \( w \)-distance on \( X \), and \( D \subseteq \text{CL}(X) \). Suppose that there exists \( u \in X \) such that \( w(u, u) = 0 \). Then, \( w(u, D) = 0 \) if and only if \( u \in D \).

\[ 0 \leq \int_0^w(x_n, x_{n+1}) \varphi(t)dt \leq a \int_0^w(x_n, x_n) \varphi(t)dt \leq a^n \int_0^w(x_0, x_0) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0, \]

\[ 0 \leq \int_0^w(x_n, x_{n+1}) \varphi(t)dt \leq a \int_0^w(x_n, x_{n+1}) \varphi(t)dt \leq a^n \int_0^w(x_0, x_{n-1}) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0. \]

Letting \( n \to \infty \) in (16) and using Lemma 1, we infer that

\[ \lim_{n \to \infty} w(x_n, x_{n+1}) = 0, \]

\[ \lim_{n \to \infty} f_w(x_n) = 0. \]

### 2. Fixed Point Theorems

In this section, we establish fixed point theorems for multivalued contractive mappings (13) and (21), respectively.

**Theorem 5.** Let \((X, d)\) be a complete metric space, \( w \) be a \( w \)-distance in \( X \), and \( T: X \to \text{CL}(X) \) satisfy that for each \( x \in X \) there is \( y \in T(x) \) with

\[ b \int_0^w(x, T(x)) \varphi(t)dt \leq \int_0^w(x, T(x)) \varphi(t)dt, \]

\[ \int_0^w(y, T(y)) \varphi(t)dt \leq c \int_0^w(y, y) \varphi(t)dt, \]  

where \( b \) and \( c \) are constants in \((0, 1)\) with \( c < b \) and \( \varphi \in \Phi_2 \).

Then,

1. For each \( x_0 \in X \), there exists an orbit \( \{x_n\}_{n \in \mathbb{N}_0} \) of \( T \) such that \( \lim_{n \to \infty} x_n = u \) for some \( u \in X \).
2. \( f_w(u) = 0 \) if \( f_w: X \to \mathbb{R}^+ \) is \( T \)-orbitally lower semicontinuous at \( u \). Moreover, \( u \) is a fixed point of \( T \) if \( w(u, u) = 0 \).

**Proof.** Now, we show (c1). Let \( x_0 \) be an arbitrary point in \( X \) and \( a = (c/b) \). It follows from (13) that there exists \( x_1 \in T(x_0) \) such that

\[ b \int_0^w(x_0, x_1) \varphi(t)dt \leq \int_0^w(x_0, x_1) \varphi(t)dt, \]

\[ \int_0^w(x_1, T(x_1)) \varphi(t)dt \leq c \int_0^w(x_1, x_1) \varphi(t)dt. \]

Continuing this process, we choose easily a sequence \( \{x_n\}_{n \in \mathbb{N}_0} \) in \( X \) satisfying

\[ x_{n+1} \in T(x_n), \]

\[ b \int_0^w(x_n, x_{n+1}) \varphi(t)dt \leq \int_0^w(x_n, x_{n+1}) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0, \]

\[ \int_0^w(x_{n+1}, T(x_{n+1})) \varphi(t)dt \leq c \int_0^w(x_{n+1}, x_{n+1}) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0. \]

Next, we claim that \( \{x_n\}_{n \in \mathbb{N}_0} \) is a Cauchy sequence. It follows from (15) and \( \varphi \in \Phi_2 \) that

\[ \varphi(t)dt \leq \int_0^w(x_0, x_0) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0, \]

\[ \varphi(t)dt \leq \int_0^w(x_0, x_{n-1}) \varphi(t)dt, \quad \forall n \in \mathbb{N}_0. \]

\[ \lim_{n \to \infty} w(x_n, x_{n+1}) = 0, \]

\[ \lim_{n \to \infty} f_w(x_n) = 0. \]
Making use of (16), \((w_t)\) and \(\varphi \in \Phi_2\), we conclude that
\[
\int_0^t w(x_t, x_{t-}) \varphi(t) dt \leq \sum_{i=0}^{\infty} \int_0^t w(x_{t-i}, x_{t-i+1}) \varphi(t) dt \leq \sum_{i=0}^{\infty} a_i \int_0^t w(x_{t-i}, x_{t-i+1}) \varphi(t) dt \leq \sum_{i=0}^{\infty} \alpha^i \int_0^t w(x_{t-i}, x_{t-i+1}) \varphi(t) dt
\]
which together with Lemmas 1 and 2 yields that \(\{x_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence. Completeness of \(X\) implies that there exists some \(u \in X\) such that
\[
\lim_{n \to \infty} x_n = u.
\]
Finally, we show \((c_2)\). Since \(f_w : X \to \mathbb{R}_+\) is \(T\)-orbitally lower semicontinuous at \(u\), it follows from (17) that
\[
0 \leq f_w(u) \leq \liminf_{n \to \infty} f_w(x_n) = 0,
\]
which means that \(f_w(u) = 0\). Thus, Lemma 3 and \(w(u, u) = 0\) yield that \(u \in T(u)\). This completes the proof.

**Theorem 6.** Let \((X, d)\) be a complete metric space, \(w\) be a \(w\)-distance in \(X\), and \(T : X \to \text{CL}(X)\) satisfy that for each \((x, y) \in X \times T(x)\), there exists \(z \in T(y)\) with
\[
\int_0^{w(x, y)} \varphi(t) dt \leq c \int_0^{w(x, z)} \varphi(t) dt,
\]
where \(c\) is a constant in \((0, 1)\) and \(\varphi \in \Phi_2\). Then, \((c_1)\) and \((c_2)\) hold.

**Proof.** Now, we show \((c_1)\). Let \((x_0, x_1)\) be an arbitrary point in \(X \times T(x_0)\). It follows from (21) that there exists \(x_2 \in T(x_1)\) such that
\[
\int_0^{w(x_1, x_2)} \varphi(t) dt \leq c \int_0^{w(x_1, x_2)} \varphi(t) dt.
\]
Continuing this process, we construct a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) satisfying
\[
x_{n+1} \in T(x_n), \quad \forall n \in \mathbb{N},
\]
\[
\int_0^{w(x_n, x_{n+1})} \varphi(t) dt \leq c \int_0^{w(x_n, x_{n+1})} \varphi(t) dt, \quad \forall n \in \mathbb{N}.
\]
Next, we claim that \(\{x_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence. It follows from (23) and \(\varphi \in \Phi_2\) that
\[
0 \leq \int_0^{w(x_n, x_{n+1})} \varphi(t) dt \leq c^n \int_0^{w(x_n, x_{n+1})} \varphi(t) dt, \quad \forall n \in \mathbb{N}.
\]
Letting \(n \to \infty\) in (24) and using Lemma 1, we infer that
\[
\lim_{n \to \infty} w(x_n, x_{n+1}) = 0,
\]
which together with (23) implies that
\[
\lim_{n \to \infty} f_w(x_n) = 0.
\]
Combining (25) and (26), we conclude that
\[
\lim_{n \to \infty} f_w(x_n) = 0.
\]

The rest of the proof is similar to that of Theorem 5 and is omitted. This completes the proof.

**Remark 1.** Theorem 5 generalizes Theorems 2–4. The below example demonstrates that Theorem 5 generalizes indeed Theorems 2 and 3.

**Example 3.** Let \(X = [0, (5/4)]\) be endowed with the Euclidean metric \(d = |·|\), \(u = 0\), \(w : X \times X \to \mathbb{R}_+\), \(T : X \to \text{CL}(X)\), and \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) be defined by
\[
w(x, y) = y, \quad \forall x, y \in X,
\]
\[
T(x) = \begin{cases} \frac{3}{4} x^2, & x \in [0, 1) \cup \left[1, \frac{5}{4}\right], \\ \frac{3}{4} x, & x = 1, \end{cases}
\]
\[
c = \frac{15}{17},
\]
\[
b = \frac{16}{17}
\]
\[
\varphi(t) = 1, \quad \forall t \in \mathbb{R}_+.
\]
It is clear that \(w(u, u) = 0\), \(\varphi \in \Phi_2\), and
\[
f_w(x) = w(x, T(x)) = \frac{3}{4} x^2, \quad \forall x \in X,
\]
is continuous in \(X\). It follows that \(f_w(x)\) is lower semicontinuous in \(X\). In order to verify (13), for each \(x \in X\), we have to consider the following cases.

**Case 1.** \(x \in [0, 1) \cup (1, (2/\sqrt{3})) \cup ((2/\sqrt{3}), (5/4)]\). Clearly, there exists \(y = (3/4)x^2 \in T(x) = [(3/4)x^2]\) satisfying
\[
b \int_0^{w(x, y)} \varphi(t) dt = \frac{16}{17} \frac{3}{4} x^2 \leq \frac{3}{4} x^2 = \int_0^{w(x, T(x))} \varphi(t) dt,
\]
\[
\int_0^{w(y, T(y))} \varphi(t) dt = \frac{27}{64} x^4 \leq \frac{15}{17} \frac{3}{4} x^2 = c \int_0^{w(x, y)} \varphi(t) dt.
\]
Case 2. \( x = 1 \). It is clear that there exists \( y = (3/4) \in T(x) = \{ (3/4), (4/5) \} \) such that

\[
\begin{align*}
&b \int_0^{w(x,y)} \varphi(t)dt = \frac{16}{17} \cdot \frac{3}{4} = \frac{3}{4} = \int_0^{w(x,T(x))} \varphi(t)dt, \\
&\quad \int_0^{w(y,T(y))} \varphi(t)dt = \frac{27}{64} \cdot \frac{15}{17} + \frac{1}{4} = c \int_0^{w(x,y)} \varphi(t)dt. 
\end{align*}
\]

(31)

Case 3. \( x = (2/\sqrt{3}) \). It follows that there exists \( y = 1 \in T(x) = \{ (3/4)x^2 \} \) such that

\[
\begin{align*}
&b \int_0^{w(x,y)} \varphi(t)dt = \frac{16}{17} \cdot \frac{3}{4} = \frac{3}{4} = \int_0^{w(x,T(x))} \varphi(t)dt, \\
&\quad \int_0^{w(y,T(y))} \varphi(t)dt = \frac{3}{4} \leq \frac{15}{17} \cdot 1 = c \int_0^{w(x,y)} \varphi(t)dt. 
\end{align*}
\]

(32)

Hence, (13) holds. That is, the conditions of Theorem 5 are satisfied. It follows from Theorem 5 that \( T \) has a fixed point \( 0 \in X \).

However, we cannot invoke Theorem 3 to show that the mapping \( T \) has a fixed point in \( X \). Suppose that the conditions of Theorem 3 are satisfied. Take \( x_* = 1 \). For \( y_* \in Tx_* = \{ (3/4), (4/5) \} \), we consider two possible cases as follows:

Case 1: \( y_* = (3/4) \). It follows that

\[
\frac{21}{64} = d \left( \frac{3}{4}, \frac{27}{64} \right) = d(y_*, T(y_*)) \leq cd(x_*, y_*) = \frac{1}{4} c, \tag{33}
\]

which implies that \( c \geq (21/16) > 1 \), which is impossible because \( c \in (0,1) \).

Case 2: \( y_* = (4/5) \). It is easy to see that

\[
\frac{8}{25} = d \left( \frac{4}{5}, \frac{12}{25} \right) = d(y_*, T(y_*)) \leq cd(x_*, y_*) = \frac{1}{5} c, \tag{34}
\]

which implies that \( c \geq (8/5) > 1 \), which is a contradiction because \( c \in (0,1) \).

It follows from Remark 1 in [3] that Theorem 3 extends Theorem 2. Thus, Theorem 2 is not applicable in proving the existence of fixed points for the multivalued contractive mapping \( T \) in \( X \).

Remark 2. The following example shows that Theorem 6 is different from Theorem 2.

**Example 4.** Let \( X = \mathbb{R}^+ \) be endowed with the Euclidean metric \( d = | \cdot | \), \( w: X \times X \to \mathbb{R}^+ \), \( T: X \to \text{CL}(X) \), and \( \varphi: \mathbb{R}^+ \to \mathbb{R}^+ \) be defined by

\[
\begin{align*}
&w(x, y) = y, \quad \forall x, y \in X, \\
&T(x) = [0, x], \quad \forall x \in X, \tag{35} \\
&\varphi(t) = t^{-(1/2)}, \quad \forall t \in \mathbb{R}^+.
\end{align*}
\]

Put \( c \in ((1/12), 1) \). Clearly, \( \varphi \in \Phi_2, w(0, 0) = 0, (X, d) \) is a complete metric space, and

\[
f_w(x) = w(x, T(x)) = \inf \{ t: t \in [0, x] \} = 0, \quad \forall x \in X, \tag{36}
\]

is continuous in \( X \).

For each \( (x, y) \in X \times T(x) \), there exists \( z = (y/4) \in T(y) \) with

\[
\int_0^{w(x, y)} \varphi(t)dt = \sqrt[4]{c} \leq 2c\sqrt[4]{y} = c \int_0^{w(x,y)} \varphi(t)dt, \tag{37}
\]

that is, (21) holds. It follows from Theorem 6 that \( T \) has a fixed point \( 0 \in X \).

However, Theorem 2 is useless in proving the existence of fixed points for the mapping \( T \) in \( X \). Suppose that (2) holds. Take \( x_* = 1 \) and \( y_* = 2 \). It follows that

\[
H(T(x_*), T(y_*)) = H(T(1), T(2)) = 1 \leq c = cd(1, 2) = cd(x_*, y_*), \tag{38}
\]

which is impossible because \( c \in [0, 1) \).

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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**References**

[1] A. Branciari, “A fixed point theorem for mappings satisfying a general contractive con of integral type,” *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, pp. 531–536, 2002.

[2] S. Nadler Jr., “Multi-valued contraction mappings,” *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[3] Y. Feng and S. Liu, “Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings,” *Journal of Mathematical Analysis and Applications*, vol. 317, no. 1, pp. 103–112, 2006.

[4] O. Kada, T. Suzuki, and W. Takahashi, “Nonconvex minimization theorems and fixed point theorems in complete metric spaces,” *Mathematica Japonica*, vol. 44, pp. 381–391, 1996.

[5] L. Guran, “Fixed points for multivalued operators with respect to a \( w \)-distance on metric spaces,” *Carpathian Journal of Mathematics*, vol. 23, pp. 89–92, 2007.
[6] M. Imdad and F. Rouzkard, “Fixed point theorems in ordered metric spaces via $w$-distances,” *Fixed Point Theory and Applications*, vol. 2012, no. 1, p. 17, 2012.

[7] H. Lakzian, H. Aydi, and B. E. Rhoades, “Fixed points for ($\phi, \psi, p$)-weakly contractive mappings in metric spaces with $w$-distance,” *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 6777–6782, 2013.

[8] H. Lakzian, V. Rakočević, and H. Aydi, “Extensions of Kannan contraction via $w$-distances,” *Aequationes Mathematicae*, vol. 93, no. 6, pp. 1231–1244, 2019.

[9] R. Zuhra, M. S. M. Noorani, and F. Shaddad, “Contraction mapping principle in partially ordered quasi metric space concerning to $w$-distances,” *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 2, pp. 699–712, 2017.

[10] Z. Liu, X. Li, S. M. Kang, and S. Y. Cho, “Fixed point theorems for mappings satisfying contractive conditions of integral type and applications,” *Fixed Point Theory and Applications*, vol. 2011, no. 1, p. 18, 2011.

[11] L.-J. Lin and W.-S. Du, “Some equivalent formulations of the generalized Ekeland’s variational principle and their applications,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 1, pp. 187–199, 2007.