Some multidimensional integrable cases of nonholonomic rigid body dynamics *

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Abstract

In this paper we study the dynamics of the constrained $n$–dimensional rigid body (the Suslov problem). We give a review of known integrable cases in three dimensions and present their higher dimensional generalizations.

1 The Suslov problem

The equations of nonholonomic systems are not Hamiltonian. They are Hamiltonian with respect to the almost-Poisson brackets [5]. This is the reason that the integration theory of constrained mechanical systems is less developed than for unconstrained ones. However, in some solvable nonholonomic systems with an invariant measure, the phase space is foliated by invariant tori, placing these systems together with integrable Hamiltonian systems (see [6, 1, 9, 20, 14, 8]).

In this paper we are interested in the integrability of multidimensional generalizations of the Suslov nonholonomic rigid body problem.

Consider an $n$–dimensional rigid body motion around the fixed point $O = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$. The configuration space is the Lie group $SO(n)$. The matrix $g \in SO(n)$ maps the orthonormal frame $E_1, \ldots, E_n$ fixed in the space to the frame $e_1 = g \cdot E_1, \ldots, e_n = g \cdot E_n$ fixed in the body, $e_1 = (1, 0, \ldots, 0)^t, \ldots, e_n = (0, \ldots, 0, 1)^t$. For a motion $g(t) \in SO(n)$, the angular velocity and momentum (in body coordinates) are $\Omega(t) = g^{-1} \cdot g(t)$ and $M = J(\Omega)$ respectively. $J : so(n) \rightarrow so(n)^*$ is the inertia tensor and has the form: $M = J(\Omega) = I\Omega + \Omega I$, where $I$ is symmetric $n \times n$ matrix called mass tensor (see [8]). Here we identified $so(n)$ and $so(n)^*$ by the Killing scalar product.

Suppose that additional left–invariant constraints

$$\langle a^i, \Omega \rangle = 0, \quad i = 1, \ldots, r$$

(1)

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and potential force $V(\Gamma)$, where $\Gamma$ is a unit vector fixed in the space, are given. Let $D = \{\Omega \in so(n), \langle a^i, \Omega \rangle = 0, i = 1, \ldots, r\}$ and $\mathcal{D}$ be the corresponding left–invariant distribution of $T\text{SO}(n)$. The distribution $\mathcal{D}$ is nonintegrable (i.e., constraints are nonholonomic) if and only if $\mathcal{D}$ is not a subalgebra of $so(n)$.

The equations of motion of a rigid body subjected to the constraints $\mathbb{I}$ and the potential force $V(\Gamma)$ are

$$\dot{M} = [M, \Omega] + \frac{\partial V}{\partial \Gamma} \wedge \Gamma + \sum_{i=1}^{r} \lambda_i a^i, \quad \dot{\Gamma} = -\Omega \cdot \Gamma,$$

(2)

where Lagrange multipliers $\lambda_i$ are determined from the constraints $\mathbb{I}$. To describe a motion $g(t)$ on $\text{SO}(n)$ one should solve an additional equation $\dot{\Omega} = g^{-1} \cdot g$.

For $n = 3, r = 1$, the equations $\mathbb{I, 2}$ are equations of classical Suslov problem (see [19, 14, 1]). Therefore, the above system can be considered as a natural multidimensional generalization of the Suslov problem.

If $\mathcal{D}$ is an eigenspace of $J$ then the equations (2) take the form

$$J(\dot{\Omega}) = \text{pr}_D[J(\Omega), \Omega] + \text{pr}_D(\frac{\partial V}{\partial \Gamma} \wedge \Gamma), \quad \dot{\Gamma} = -\Omega \cdot \Gamma, \quad \Omega \in \mathcal{D},$$

(3)

Contrary to the general case (2), system (3) preserves the standard measure $d\Omega d\Gamma$ on $\mathcal{M} = D\{\Omega\} \times S^{n-1}\{\Gamma\}$. The energy $E = \frac{1}{2}[J(\Omega), \Omega] + V(\Gamma)$ is always an integral of (2), (3). If, besides the Hamiltonian function, we have $\dim \mathcal{D} + n - 4$ additional independent integrals, by the Jacobi theorem the solutions of (3) can be found by quadratures. Under the compactness conditions, the phase space $\mathcal{M}$ is foliated by invariant tori with quasi-periodic dynamic (after appropriate change of time), similarly as in the Liouville theorem [14, 1].

Therefore, it is natural to call the system completely integrable if it can be integrated by the Jacobi theorem; or more generally (see [20]): if the trajectories of the system belong to invariant tori with the dynamic of the form

$$\dot{\varphi}_1 = \frac{\omega_1}{\Phi(\varphi_1, \ldots, \varphi_k)}, \ldots, \dot{\varphi}_k = \frac{\omega_k}{\Phi(\varphi_1, \ldots, \varphi_k)},$$

(4)

The flow of (1) in the new time $d\tau = \Phi^{-1} dt$ is quasi-periodic: $d\varphi_i/d\tau = \omega_i$. Moreover, for almost all frequencies $\omega_1, \ldots, \omega_k$, by smooth change of variables $\theta_i = \theta_i(\varphi)$ equations (1) can be reduced to the form $\dot{\theta}_i = \Omega_i = \omega_i/\Pi, i = 1, \ldots, k$, where $\Pi$ denotes the total measure of $T^k$ (see [1]).

Note that the integrability of equations (2) without the constraints (for example the multidimensional Euler [16], Lagrange [2] and Klebsh [18] cases) implies non-commutative integrability of unconstrained system. The phase space $T^*\text{SO}(n)$ is foliated on $d \leq \dim so(n)$ dimensional invariant isotropic tori with quasi-periodic dynamics (see [14, 1, 3]). In the nonholonomic case there is no Poisson structure. In order to precisely describe dynamic on the whole phase space $\mathcal{D}$ we have to solve kinematic equation $g^{-1}(t) \cdot \dot{g}(t) = \Omega(t)$. This problem, for the Fedorov–Kozlov integrable case, is studied by Zenkov and Bloch [21].
In the next section we give a review of the known integrable cases in three dimension. Then we present their natural higher dimensional generalizations. It appears that these systems are also completely integrable (section 3).

2 Three–dimensional integrable cases

For \( n = 3 \) we can identify the Lie algebra \( \mathfrak{so}(3), [\cdot, \cdot] \) with \( \mathbb{R}^3, \times \) via \( \Omega = (\Omega_1, \Omega_2, \Omega_3) = (-\Omega_{23}, \Omega_{13}, -\Omega_{12}) \). Then the equations of the motion (2) become:

\[
J\dot{\Omega} = J\Omega \times \Omega + \Gamma \times \frac{\partial V}{\partial \Gamma} + \lambda a, \quad \dot{\Gamma} = \Gamma \times \Omega, \quad \langle a, \Omega \rangle = 0,
\]

(5)

Without loss of generality we can take \( a = (0, 0, 1) \), i.e., the constraint is \( \Omega_3 = -\Omega_{12} = 0 \). Then \( D = \mathbb{R}^2\{\Omega_1, \Omega_2\} \). Geometrically, this means that only infinitesimal rotations in the planes \( e_1 \wedge e_3 \) and \( e_2 \wedge e_3 \) are allowed.

If \( V = 0 \), then the equations (5) form a closed system in variables \( \Omega_1, \Omega_2 \). If \( a \) is not an eigenvector of \( J \) then there is an asymptotic line \( l \) in \( D \). The angular velocity is moving along the energy ellipse \( E = \{E(\Omega) = h\} \) asymptotically to intersections points \( w_- \) and \( w_+ \) of \( E \) with \( l \) (see Suslov [19]). That system, obviously, has not an invariant measure. When \( a \) is an eigenvector of \( J \), then system (5) preserves the measure \( d\Omega d\Gamma \) and for \( V = 0 \) the solutions are very simple \( \Omega = \text{const} \).

Thus, further we assume that \( a \) is an eigenvector of the inertia tensor \( J \). We can take \( J = \text{diag}(J_1, J_2, J_3) \). For the integrability, we need one additional integral, independent of the Hamiltonian function.

The known integrable cases are:

- The Lagrange case (see Kozlov [14]), when \( J_1 = J_2 \) and \( V = B_3 \Gamma_3 \). The integral is \( F = \langle J\Omega, \Gamma \rangle \).

- The Kharlamova case [11] where \( V(\Gamma) = B_1 \Gamma_1 + B_2 \Gamma_2 \). Then \( F = J_1 \Omega_1 B_1 + J_2 \Omega_2 B_2 \).

- The Klebsh–Tisserand case (Kozlov [14] [11]) with \( V(\Gamma) = \frac{c}{2} \langle J\Gamma, \Gamma \rangle \) and \( F = \frac{1}{h} \langle J\Omega, J\Omega \rangle - \frac{1}{h} \langle A \Gamma, \Gamma \rangle \), where \( A = cJ^{-1} \text{det} J \).

Let us note that in the first case the Lagrange multiplier is \( \lambda = 0 \), and this is actually an invariant subsystem of the usual Lagrange rigid body system. Also, Ziglin proved that if \( V(\Gamma) = B_3 \Gamma_3 \) than system (5) has a complementary meromorphic first integral in the complexified phase space \( \mathcal{M}_C = \mathbb{C}^2\{\Omega_1, \Omega_2\} \times S^2_\mathbb{C}\{\Gamma\} \) only in the Lagrange case \( J_1 = J_2 \) [22].

The Kharlamova and Klebsh–Tisserand cases can be uniquely described and generalized in the following way (see [7]). If \( J_1 \neq J_2 \) then the equations (5) of the Suslov problem are integrable for the potentials: \( V(\Gamma) = v_1(\Gamma_1, \Gamma_2^2 + \Gamma_3^2) + v_2(\Gamma_2, \Gamma_1^2 + \Gamma_3^2) \) where \( v_1 \) and \( v_2 \) are arbitrary functions of two variables. The corresponding third integral is: \( F = \frac{1}{h} \langle J\Omega, J\Omega \rangle + J_2v_1(\Gamma_1, \Gamma_2^2 + \Gamma_3^2) + J_1v_2(\Gamma_2, \Gamma_1^2 + \Gamma_3^2) \).
In addition, there are integrable cases of the Suslov problem, which include the gyroscopic force with the momentum $\epsilon \Gamma \times \Omega$ added to the right hand side of (4). Since the gyroscopic force is conservative, the Hamiltonian remains to be the first integral. As well as the usual Suslov problem, the system with the gyroscopic force is integrable for the potentials $V = 0$, $V(\Gamma) = B_1 \Gamma_1 + B_2 \Gamma_2$ and $V(\Gamma) = \frac{1}{2}(a_1 \gamma_1^2 + a_2 \gamma_2^2 + a_3 \gamma_3^2)$ (see [7]).

3 Multidimensional integrable cases

Without potential force, the equations (1), (2) represent a closed system in variable $\Omega$ ($\text{Euler–Poincare–Suslov}$ equations). We can consider Euler–Poincare–Suslov equations on other Lie algebras as well. They are reduced equations of nonholonomic systems on Lie groups with left–invariant kinetic energy and left–invariant constraints (see [14, 13]). Integrable examples can be found in [8, 10, 11].

Since in the three–dimensional case, only infinitesimal rotations in the planes $e_1 \wedge e_3$ and $e_2 \wedge e_3$ are allowed, Fedorov and Kozlov suggested that multidimensional analogue of Suslov’s conditions can be define in the following way: only infinitesimal rotations in the planes $e_1 \wedge e_n$, ..., $e_{n-1} \wedge e_n$, i.e., in the planes containing the vector $e_n$ are allowed [8]. Therefore, we have the following constraints imposed to the components of the angular velocity matrix:

$$\Omega_{ij} = 0, \quad i, j \leq n - 1. \quad (6)$$

Then

$$\mathcal{M} = \mathbb{R}^{n-1}\{\Omega_{1n}, \Omega_{2n}, \ldots, \Omega_{n-1,n}\} \times S^{n-1}\{\Gamma\}. \quad (7)$$

Really, for $V = 0$, the system has the same behavior as three–dimensional system. If $D$ is an eigen space of the inertia operator $J$ then the solutions are simply $\Omega = \text{const}$. In general, all trajectories lying on the energy ellipsoid $E = \frac{1}{2}(J, \Omega) = h$ are double asymptotic: they tend to the diametrically opposed points $w_+$ and $w_-$ as $t \to \pm \infty$ (see [3]).

With the choice of constraints (6) we shall see that three–dimensional integrable cases with invariant measure have their multidimensional analogues.

Suppose that the mass tensor is diagonal $I = \text{diag}(I_1, I_2, \ldots, I_n)$. Then $D$ is an invariant subspace of $J$. The orthogonal complement of $D$ in $so(n)$ is $so(n-1)$. Since $(so(n), so(n-1))$ is a symmetric pair ($[so(n-1), so(n-1)] \subset so(n-1)$, $[so(n-1), D] \subset D$, $[D, D] \subset so(n-1)$) the equations (3) take the following simple form:

$$J(\dot{\Omega}) = pr_D(\frac{\partial V}{\partial \Gamma} \wedge \Gamma), \quad \dot{\Gamma} = -\Omega \Gamma, \quad \Omega \in D,$$

or coordinately:

$$((I_i + I_n)\dot{\Omega}_m = \frac{\partial V}{\partial \Gamma_i} - \Gamma_i \frac{\partial V}{\partial \Gamma_n},$$

$$\dot{\Gamma}_i = -\Gamma_i \Omega_{in}, \quad i = 1, 2, \ldots, n - 1,$$

$$\dot{\Gamma}_n = \Gamma_1 \Omega_{1n} + \Gamma_2 \Omega_{2n} + \ldots + \Gamma_{n-1} \Omega_{n-1,n}. \quad (8)$$
3.1 The Lagrange case

Let the mass tensor be of the form $I = \text{diag}(I_1, \ldots, I_1, I_n)$, the rigid body be placed in a homogeneous gravitational field and the center of mass be on the axes of the dynamical symmetry $e_n$. Then the potential is of the form $V(\Gamma) = B_n \Gamma_n$ and system (8) can be seen as an invariant subsystem of the unconstrained $n$–dimensional rigid body motion:

\begin{align*}
(I_i + I_j)\dot{\Omega}_{ij} &= (I_i - I_j) \sum_k \Omega_{ik} \Omega_{kj} = 0, \quad i, j \leq n - 1, \\
(I_1 + I_n)\dot{\Omega}_{in} &= (I_1 - I_n) \sum_k \Omega_{1k} \Omega_{kn} - B_n \Gamma_n, \\
\dot{\Gamma}_i &= -\sum_k \Omega_{ik} \Gamma_k, \quad i = 1, \ldots, n.
\end{align*}

The Hamiltonian equations (9) represent the natural generalization of the three–dimensional Lagrange rigid body system. Their integrability is proved by Beljaev [2]. However, this does not give us directly the integrability on the invariant subspace $\Omega_{ij} = 0, \quad i, j \leq n - 1$. In order to prove the integrability of the Suslov problem we shall use the symplectic reduction, instead of verifying independence of Beljaev’s integrals on the invariant subspace.

Consider the Lagrange rigid body system on the whole phase space $T^*SO(n)$. It is invariant with respect to the right Hamiltonian action of the group $SO(n-1)$ (rotations which fix symmetry axes $e_n$). The corresponding moment map $\Phi : T^*SO(n) \to so(n)^*$ is given by functions $M_{ij}, \quad i, j \leq n - 1$ considered as the left invariant functions on $T^*SO(n)$. One can easily prove:

**Lemma 3.1** The reduced system on $\Phi^{-1}(0)/SO(n-1) = T^*(SO(n)/SO(n-1)) = T^*S^{n-1}$ is the spherical pendulum.

The spherical pendulum is a completely integrable system. Therefore, the invariant submanifold $\Phi^{-1}(0) \subset T^*SO(n)$ of the Lagrange system, as well as the rest of the phase space, is almost everywhere foliated by invariant tori with quasi–periodic dynamics. (More about the relationships between integrability and reductions one can find in [23] [24].) The Suslov problem on $D \subset TSO(n)$, after Legendre transformation, coincides with the Lagrange system on $\Phi^{-1}(0)$. Thus we obtain:

**Theorem 3.1** The Suslov nonholonomic rigid body problem with constraints (6), the mass tensor of the form $I = \text{diag}(I_1, \ldots, I_1, I_n)$ and the center of the mass on the axes of the dynamical symmetry $e_n$, in a presence of a homogeneous gravitational field (the multidimensional Lagrange case) is completely integrable.

Note that the unconstrained system on $T^*SO(n)$ is invariant with respect to the left Hamiltonian action of the group $SO(n-1)$ (rotations which fix $\Gamma$), for any inertia tensor $J$ and potential $V(\Gamma)$. In this sense, the Lagrange system admits two different $SO(n-1)$ reductions. For $n = 3$ this corresponds to the
fact that two Eulerian angles, the angle of precession \( \psi \) and the angle of pure rotations \( \varphi \), are cyclic variables. Elimination of the variable \( \varphi \) with the zero value of the conjugate momentum \( p_\varphi = 0 \) gives the spherical pendulum on \( S^2 \) (for example, see [4]).

### 3.2 The Kharlamova case

The natural multidimensional generalization of the Kharlamova case can be defined as the constrained rigid body motion in a gravitational field with the center of mass laying in the plane spanned by \( e_1, e_2, \ldots, e_{n-1} \). Then the potential force is of the form

\[
V = B_1 \Gamma_1 + B_2 \Gamma_2 + \ldots + B_{n-1} \Gamma_{n-1}
\]

and equations (8) become:

\[
(I_i + I_n) \dot{\Omega}_{in} = B_i \Gamma_n, \\
\dot{\Gamma}_i = -\Gamma_n \Omega_{in}, \quad i = 1, 2, \ldots, n - 1, \\
\dot{\Gamma}_n = \Gamma_1 \Omega_{1n} + \Gamma_2 \Omega_{2n} + \ldots + \Gamma_{n-1} \Omega_{n-1,n}.
\]  

(10)

Note that the singular points of (10) are given with \( \Gamma_n = \dot{\Gamma}_n = 0 \). Also, equations (10) have integrals

\[
F_{i,j}(\Omega) = \frac{I_i + I_n}{B_i} \Omega_{in} - \frac{I_j + I_n}{B_j} \Omega_{jn}, \quad i, j = 1, 2, \ldots, n - 1.
\]

Therefore, it is natural to introduce new coordinates \( \omega_1, \ldots, \omega_{n-1}, \gamma_1, \ldots, \gamma_n \) in the following way:

\[
\omega_1 = \frac{I_1 + I_n}{B_1} \Omega_{1n}, \\
\omega_i = \frac{I_i + I_n}{B_i} \Omega_{in} - \frac{I_1 + I_n}{B_1} \Omega_{1n}, \quad i = 2, \ldots, n - 1, \\
\gamma_1 = -\frac{I_1 + I_n}{B_1} \Gamma_1, \\
\gamma_i = -\frac{I_i + I_n}{B_i} \Gamma_i + \frac{I_1 + I_n}{B_1} \Gamma_1, \quad i = 2, \ldots, n - 1, \\
\gamma_n = \Gamma_n.
\]

(11)

Then system (10) get the simple form

\[
\frac{d\omega_1}{dt} = \gamma_n, \\
\frac{d\omega_i}{dt} = 0, \quad i = 2, \ldots, n - 1, \\
\frac{\gamma_i}{dt} = \gamma_n \omega_i, \quad i = 1, 2, \ldots, n - 1, \\
\frac{\gamma_n}{dt} = -\left( \frac{B_1}{I_1 + I_n} \right)^2 \gamma_1 \omega_1 - \sum_{i=2}^{n-1} \left( \frac{B_i}{I_i + I_n} \right)^2 (\gamma_1 + \gamma_i)(\omega_1 + \omega_i).
\]  

(12)
Now, we can easily integrate the above system. We have $\omega_2 = \omega_0^2, \ldots, \omega_{n-1} = \omega_0^{n-1}$. By substitution of the first equation $\dot{\omega}_1 = \gamma_n$ to the equations with $\gamma_i$-s we get the trajectory as a function of the variable $\omega_1$:

\[
\begin{align*}
\gamma_1 - \gamma_0^0 &= \frac{1}{2}((\omega_1)^2 - (\omega_0^0)^2), \\
\gamma_i - \gamma_i^0 &= \omega_i(\omega_1 - \omega_0^0), \quad i = 2, \ldots, n-1,
\end{align*}
\]

where $\gamma_0^0, \ldots, \gamma_{n-1}^0$ are initial conditions.

Since $\langle \Gamma, \Gamma \rangle = 1$, we have

\[
\gamma_n^2 = \Gamma_n^2 = 1 - \sum_{i=1}^{n-1} \Gamma_i^2 = 1 - \left( \frac{B_1}{I_1 + I_n} \right)^2 \gamma_1^2 - \sum_{i=2}^{n-1} \left( \frac{B_i}{I_i + I_n} \right)^2 (\gamma_1 + \gamma_i)^2 = P_4(\omega_1),
\]

where $P_4$ is a four degree polynomial depending on initial conditions as parameters. Finally, to get the trajectory as a function of the time $t$ one should solve the elliptic integral

\[
t - t_0 = \int_{\omega_1^0}^{\omega_1} \frac{d\omega_1}{\pm \sqrt{P_4(\omega_1)}}.
\]

The curve $(\gamma_1(\omega_1), \ldots, \gamma_{n-1}(\omega_1))$ lies within the ellipsoid

\[
\Sigma = \left\{ \left( \frac{B_1}{I_1 + I_n} \right)^2 \gamma_1^2 + \sum_{i=2}^{n-1} \left( \frac{B_i}{I_i + I_n} \right)^2 (\gamma_1 + \gamma_i)^2 = 1 \right\}.
\]

On $\Sigma$ we have $\gamma_n = \Gamma_n = 0$. Therefore the variable $\omega_1$ varies in the interval $[\xi_1, \xi_2]$ where $\xi_1$ and $\xi_2$ are adjacent roots of the polynomial $P_4$ between which it takes positive values and $(\gamma_1(\xi_i), \ldots, \gamma_{n-1}(\xi_i)) \in \Sigma$. Thus, the trajectory is either asymptotic or periodic with period

\[
T = 2 \int_{\xi_1}^{\xi_2} \frac{d\omega_1}{\sqrt{P_4(\omega_1)}}.
\]

We can summarize the above considerations in the following theorem.

**Theorem 3.2** The phase space $\mathcal{L}$ of the multidimensional Kharlamova system is almost everywhere foliated by invariant circles. The periodic motions can be expressed as elliptic functions of time $t$.

### 3.3 The Klebsh–Tisserand case

By multidimensional Klebsh–Tisserand case we mean the nonholonomic rigid body motion in the presence of the quadratic potential $V(\Gamma) = \frac{1}{2}(B_1\Gamma_1^2 + \ldots + B_n\Gamma_n^2)$:

\[
\begin{align*}
(I_1 + I_n)\dot{\Omega}_n &= (B_1 - B_n)\Gamma_1 \Gamma_n, \\
\dot{\Gamma}_i &= -\Gamma_n \Omega_{in}, \quad i = 1, 2, \ldots, n-1, \\
\dot{\Gamma}_n &= \Gamma_1 \Omega_{1n} + \Gamma_2 \Omega_{2n} + \ldots + \Gamma_{n-1} \Omega_{n-1,n}.
\end{align*}
\]
The system (14) is similar to the Fedorov–Kozlov integrable case [8]. We have integrals
\[ F_i(\Omega, \Gamma) = (B_i - B_n)\Gamma_i^2 + (I_i + I_n)\Omega_n^2 \quad i = 1, 2, \ldots, n - 1. \]

Note that the Hamiltonian function can be expressed in terms of \( F_i \)-s:
\[ E = \frac{1}{2}(nB_n + F_1 + \ldots + F_{n-1}). \]

The phase space (7) of multidimensional Klebsh-Tisserand system is foliated by invariant varieties
\[ T_c = \{ F_i = c_1, \ldots, F_{n-1} = c_{n-1} \}. \]

Suppose that \( B_i > B_n, i = 1, \ldots, n - 1 \). Then the functions \( F_i \) are positive and \( S^1_{c_i} = \{ F_i = c_i \} \) are circles in the planes \( \mathbb{R}^2\{\Omega_{in}, \Gamma_i\}, i = 1, 2, \ldots, n - 1 \). The invariant surface \( T_c \) is a two–covering of
\[ S^1_{c_1} \times \ldots \times S^1_{c_{n-1}} \cap \{ \Gamma_1^2 + \ldots + \Gamma_{n-1}^2 \leq 1 \} \]
determined by the function \( \Gamma_n = \pm \sqrt{1 - (\Gamma_1^2 + \ldots + \Gamma_{n-1}^2)}. \) The branching points of the covering correspond to the zeros of \( \Gamma_n \). In particular, if constants \( c_i \) satisfy inequality
\[ \frac{c_1}{B_i - B_n} + \ldots + \frac{c_{n-1}}{B_{n-1} - B_n} < 1, \quad (15) \]
then \( \Gamma_n \neq 0 \) and \( T_c \) is the union of two disjoint \((n - 1)\)-dimensional tori (or less dimensional tori if some of \( c_i \)-s are equal to zero).

The motion on \( T_c \) can be solved in term of the new time \( d\tau = \Gamma_n dt \). Define the angular variables \( \varphi_i \) on circles \( S^1_{c_i} \) by putting
\[ \Omega_{in} = \sqrt{\frac{c_i}{I_i + I_n}} \sin \varphi_i, \quad \Gamma_i = \sqrt{\frac{c_i}{B_i - B_n}} \cos \varphi_i. \]

Using (14), we obtain the following quasi-periodic dynamics on \( T_c \):
\[ \frac{d\varphi_i}{d\tau} = \omega_i = \frac{B_i - B_n}{I_i + I_n}, \quad i = 1, 2, \ldots, n - 1. \]

The frequencies \( \omega_i \) depend only of \( I_i \) and \( B_i \). If the trajectories are periodic on one torus, they are periodic on the rest of the tori as well. Also, in the original time, system (14) takes the form \( \dot{\varphi}_i = \omega_i/\Gamma_n^{-1} \).

**Theorem 3.3** If \( c_1, \ldots, c_{n-1} \) satisfy the inequality (15) then the invariant variety \( T_c \) is the disjoint union of two invariant tori. The multidimensional Klebsh-Tisserand system (14) on \( T_c \) is completely integrable.

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