We investigate complex synchronization patterns such as cluster synchronization and partial amplitude death in networks of coupled Stuart-Landau oscillators with fractal connectivities. The study of fractal or self-similar topology is motivated by the network of neurons in the brain. This fractal property is well represented in hierarchical networks, for which we present three different models. In addition, we introduce an analytical eigensolution method and provide a comprehensive picture of the interplay of network topology and the corresponding network dynamics, thus allowing us to predict the dynamics of arbitrarily large hierarchical networks simply by analyzing small network motifs. We also show that oscillation death can be induced in these networks, even if the coupling is symmetric, contrary to previous understanding of oscillation death. Our results show that there is a direct correlation between topology and dynamics: Hierarchical networks exhibit the corresponding hierarchical dynamics. This helps bridging the gap between mesoscale motifs and macroscopic networks.
1. Introduction

In the last decades, synchronization and its control has sparked tremendous scientific interest in network science because of its wide applicability [1]. Examples of synchrony range from genetic oscillators [2] and population dynamics [3] via data mining [4] and power grid networks [5,6] to opinion formation [7]. While early research focused on in-phase (or zero-lag) synchronization, recently more complex synchronization patterns such as group or cluster [8–20], or partial synchronization [21–24] have moved towards the center of scientific interest – in theoretical studies as well as in experiments. Partial synchronization describes a state where a part of the network is in synchrony – this can be in-phase, cluster, or group synchronization – while other parts exhibit oscillation quenching, i.e., amplitude death or oscillation death [25–27], or oscillate incoherently. The spatial coexistence of coherent and incoherent dynamics in a network of identical elements is called a chimera state [28–39].

Amplitude death is associated with the stabilization of an already existing trivial steady state, while oscillation death is characterized by an inhomogeneous steady state which is induced by the coupling. Applications of amplitude death pertain mainly to controlling physical and chemical systems (e.g., coupled lasers) [40] and suppressing neuronal oscillations [41,42], while oscillation death has been suggested as a mechanism to generate heterogeneity in homogeneous systems, e.g., stem cell differentiation in morphogenesis [43]. So far oscillation death has been described as the result of symmetry breaking [27,44,45]. Here, we demonstrate that oscillation death can also occur in symmetric networks with symmetric coupling when the collective frequency of the oscillators tends to zero.

On the topological side, we focus on hierarchical networks exhibiting a fractal or self-similar structure [46–50], which is motivated by the intricate architecture found in neural networks: Diffusion Tensor Magnetic Resonance Imaging (DT-MRI) results show that neurons in the mammal brain are far from being linked homogeneously but are connected in a fractal manner, with fractal dimensions varying between 2.3 and 2.8, depending on local properties, on the subject, and on the noise reduction threshold [46,51–54]. Other examples of fractal networks include protein interaction networks [55] or biochemical reactions in the metabolism [56] and hyperlinks in the World Wide Web [57].

Ground-breaking work connecting the topological properties of a network with its dynamics has been done by Pecora et al. in introducing the master stability function for the completely synchronous state [58]. This method has been extended to time-delayed coupling [59–63] and group and cluster synchronization [9,13,18,64]. However, the master stability function considers the network topology on a global scale, while very little is understood about how the global dynamics is influenced by the network motifs, i.e., how the topology on a mesoscale level influences the dynamics of the whole network. A first step in this direction has been made by Do et al. establishing that certain mesoscale subgraphs are of crucial importance for the global dynamics of the network [65]. An analytical method for studying partial synchronization states in mesoscale motifs has been presented in [24]. Here, we show how, in hierarchical networks, we can analytically predict the global dynamics from the topology of the small motifs by extending the eigensolution concept suggested in [24].

As a model, we consider coupled Stuart-Landau oscillators, a paradigmatic normal form that naturally arises in an expansion of oscillator systems close to a Hopf bifurcation. Networks of Stuart–Landau oscillators have been used as oscillator models for neural networks [66]. Fairly simple topologies of coupled Stuart–Landau oscillators such as all-to-all networks [41,67–69] and regular arrays of Stuart–Landau oscillators [70] have been studied in several works. Here, we couple these oscillators in a more complex hierarchical topology. To this end, we introduce and compare three different models of hierarchical networks. We also explain, using the analytic eigensolution concept, the coupling induced transition between cluster state dynamics and oscillation death.
The paper is organized as follows: In Sec. 2, we introduce the Stuart-Landau oscillator. In Sec. 3, we elaborate on the eigensolution method used to find analytical solutions. We then present and briefly discuss the network topology of three different models in Sec. 5. Section 6 is organized into three subsections, each of which presents the numerical and analytical results for the three models, respectively. Finally, in Sec. 7, we provide a conclusion and discuss future directions of research.

2. Model

In this paper, we study the Stuart-Landau oscillator, a generic model for a system close to a Hopf bifurcation. The dynamics of the $i$th oscillator, $i \in \{1, \ldots, N\}$, is given by

$$\dot{z}_i = f(z_i) + \sigma e^{i\beta} \sum_{j=1}^{N} A_{ij} z_j,$$

$$f(z) = (\lambda + i\omega - |z|^2)z,$$

where $z \in \mathbb{C}$ and $\lambda, \omega, \sigma, \beta \in \mathbb{R}$, $\omega$ is the oscillator frequency. In the uncoupled oscillator ($\sigma = 0$), $\lambda$ is the bifurcation parameter: For $\lambda > 0$, a limit cycle of radius $\sqrt{\lambda}$ exists that is born in a supercritical Hopf bifurcation at $\lambda = 0$. The parameters $\sigma$ and $\beta$ are the coupling strength and coupling phase, respectively.

The oscillators are connected via an instantaneous coupling as given by the adjacency matrix $A_{ij}$. We normalize the adjacency matrix to unity row sum, i.e., $\sum_{j=1}^{N} A_{ij} = 1$, ensuring the existence of an invariant synchronization manifold. The three models specifying the topology of the network, and therefore $A$, are introduced and discussed in detail in Sec. 5.

3. Analytical eigensolution

In [24] an analytical eigensolution approach was suggested to determine the in-phase synchronized, antiphase-synchronized, and amplitude death solutions of Eq. (2.1). Here, we will give a brief summary of the results obtained in [24]. While doing so, we will also generalize this eigensolution concept to cluster synchronization – states where all nodes oscillate with the same amplitude $r_0$ and the same frequency $\tilde{\omega}$ but are organized in equally sized clusters with a constant phase lag of $\frac{2\pi j}{N}$ ($j \in \{1, \ldots, N\}$ where $N$ is the total number of nodes) between consecutive nodes [60]. For $j > 0$, the number of clusters can be calculated as $M = \text{lcm}(j, N)/j$, where lcm stands for the least common multiple. $j = N$ corresponds to in-phase synchrony, while $j = N/2$ denotes the antiphase-synchronized state (for even $N$). In general, depending on $N$ and $j$, several states might exist which are characterized by the same number of clusters. Figure 1 shows a schematic view of all cluster states in a unidirectional ring configuration of four nodes. In panel (a), the nodes are in zero-lag synchronization. Panels (b) and (d) show two different splay states, i.e., $M = N = 4$, where the phase difference between subsequent nodes in panel (b) is $\pi/2$, i.e., $j = 1$, and in panel (d) $3\pi/2$, i.e., $j = 3$. Panel (c) depicts the anti-synchronized state.

The eigensolution concept is based on the idea that each eigenvector $v$ of the adjacency matrix, with components $v_{ij}$, fulfilling

$$|v_i| \in \{0, 1\}, \forall i = 1, \ldots, N,$$

corresponds to a solution, i.e., each component of $v$ is either zero or a complex root of unity. This seems to be a rather strong restriction on the eigenvector. However, the eigenvectors of all hierarchical network topologies considered in this paper fulfill this condition, and hence this method can directly be extended to all our models (described in Sec. 5). Furthermore, all eigenvectors of circulant adjacency matrices are known to have eigenvectors with components equal to complex roots of unity. Circulant matrices are of great current interest in the study of chimera states [35,37,71,72] because the corresponding topology is invariant under discrete rotations (dihedral symmetry group).
We use the ansatz $z_i = v_i \eta$ in Eq. (2.1), where $\eta$ denotes the eigenvalue of the adjacency matrix corresponding to the eigenvector $v$. Using Eq. (3.1), we can write $f(v_i z_\eta) = v_i f(z_\eta)$. Thus, Eqs. (2.1) decouples to

$$\dot{z}_\eta v_i = f(z_\eta) v_i + \sigma e^{i\beta} \eta z_\eta v_i.$$  

(3.2)

Dividing by $v_i$ and introducing

$$\tilde{\lambda} \equiv \sqrt{\lambda + \eta_r \sigma \cos \beta - \eta_c \sigma \sin \beta}, \quad \tilde{\omega} \equiv \omega + \eta_r \sigma \sin \beta + \eta_c \sigma \cos \beta.$$  

(3.3)

where $\eta_r, \eta_c$ are the real and complex parts of $\eta$, respectively, we obtain

$$\dot{z}_\eta = (\tilde{\lambda} + i\tilde{\omega} - |z_\eta|^2) z_\eta.$$  

(3.4)

This has the form of a decoupled Stuart-Landau oscillator $z_\eta$, the solution for which is known to be $z_\eta = \sqrt{\tilde{\lambda}} e^{i\tilde{\omega}t}$.

Substituting this back into $z_i = v_i z_\eta(t)$, the common amplitude and phase of the $i$th oscillator in a cluster state can be obtained as

$$r_i = r_o = |v_i| \sqrt{\lambda + \eta_r \sigma \cos \beta - \eta_c \sigma \sin \beta},$$  

(3.5)

and

$$\phi_i = \omega t = \tilde{\omega} t + \arg(v_i).$$  

(3.6)

Equations (3.5) and (3.6) can be interpreted as follows: For $|v_i| > 0$, the $i$th node oscillates with a phase shift of $\arg(v_i) - \arg(v_{i-1})$ with respect to the preceding node. Thus, if all components $v_i$ are real and non-zero, we obtain in-phase oscillations. If $v_i = 0$, the $i$th node undergoes amplitude death. For eigenvectors containing both zero and non-zero elements we obtain partial amplitude death.

4. Linear Stability Analysis

In the case of in-phase and anti-phase synchronization a master-stability ansatz [58] is possible (for details see [24]). In [24] a detailed stability analysis was given for in-phase and anti-phase synchronization and partial amplitude-death solution. In the following, we consider the stability of general cluster states as given in Eqs. (3.5) and (3.6). For these patterns, an analytic solution is only possible in the absence of partial amplitude death. We will focus here on this case. In the presence of partial amplitude death and cluster states, numerical methods should be used. Note that the analysis presented here is very similar to the one given in [60]. We include it here to increase the comprehensiveness and readability of our discussion.
We start by introducing polar coordinates \( z_j = r_j e^{i\phi_j} \) in the system of coupled oscillators given by Eqs. (2.1) and (2.2) yielding

\[
\dot{r}_i = (\lambda - r_i^2) r_i + \sigma \sum_j A_{ij} r_j \cos(\beta + \phi_j - \phi_i),
\]

\[
\dot{\phi}_i = \omega r_i + \sigma \sum_j A_{ij} r_j \sin(\beta + \phi_j - \phi_i). \tag{4.1}
\]

These equations need to be linearized around the solution given by Eqs. (3.5) and (3.6). For \( r_i = 0 \) the phase \( \phi_i \) of the oscillator is not defined. The use of polar coordinates therefore is restricted to solutions without dead oscillators, i.e. \( |v_i| = 1 \ \forall \ i \in \{1, \ldots, N\} \), corresponding to cluster states, respectively.

We introduce small perturbations, \( \delta r_i \) and \( \delta \phi_i \), of the limit cycle solution \( z_i = v_i z \) in radius and phase

\[
r_i = r_i(1 + \delta r_i), \quad \phi_i = i \omega + \arg(v_i) + \delta \phi_i \tag{4.2}
\]

Inserting Eq. (4.2) into Eq. (4.1) and expanding all right hand sides around \( \delta r_i = 0, \delta \phi_i = 0 \) up to first order yields

\[
\delta \dot{r}_i = (\lambda - 3r_i^2) \delta r_i + \sigma \sum_{j=1}^{N} A_{ij} \left[ \cos(\beta + \arg(v_j) - \arg(v_i)) \delta r_j - r_0 \sin(\beta + \arg(v_j) - \arg(v_i)) \right] (\delta \phi_j - \delta \phi_i),
\]

\[
\delta \dot{\phi}_i = -\eta \sigma \sin \beta \delta r_i + \sigma \sum_{j=1}^{N} A_{ij} \left[ \sin(\beta + \arg(v_j) - \arg(v_i)) \delta r_j + \cos(\beta + \arg(v_j) - \arg(v_i)) \right] (\delta \phi_j - \delta \phi_i). \tag{4.3}
\]

We write this in a compact vector from:

\[
\dot{\zeta} = (Q + R)\zeta, \tag{4.4}
\]

where \( \zeta = (\delta r_1, \delta \phi_1, \ldots, \delta r_N, \delta \phi_N) \).

\( Q \) is a block diagonal matrix where the \( i \)th block \( Q_i \) is given by

\[
Q_i = \begin{pmatrix}
2r_0^2 + \sum_{j=0}^{N} A_{ij} \cos \theta_j + i \sum_{j=0}^{N} A_{ij} \sin \theta_j - \sum_{j=0}^{N} A_{ij} \cos \theta_j - i \sum_{j=0}^{N} A_{ij} \sin \theta_j
\end{pmatrix} \tag{4.5}
\]

with \( \theta_j = \beta + \arg(v_j) - \arg(v_i) \). \( R \) is also a block matrix. \( R_{ij} \), the block on position \( i,j \), reads

\[
R_{ij} = KA_{ij} \begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix}. \tag{4.6}
\]

In the case of in-phase and anti-phase synchronization a master-stability ansatz [58] is possible (for details see [24]), i.e., Eq. (4.4) can be block diagonalized. However, in the case of general cluster states as considered here, this is not feasible because \( Q_i \) depends on \( i \). Instead, we calculate the Floquet multiplier \( \mu \) directly as the eigenvalue of the matrix \( Q + R \). The Floquet exponent \( \Lambda \) can be obtained as \( \Lambda = \ln \mu \). The real part of \( \Lambda \) determines the stability of the considered state: For \( \text{Re}(\Lambda) < 0 \), the considered solution is stable, for \( \text{Re}(\Lambda) > 0 \) it is unstable.

5. Network topologies

Here we present the three different methods of constructing the hierarchical networks which are further studied in Sec. 6.
(a) 1D fractal

In this section, we elaborate on the method used to create the first model of a hierarchical network (1D fractal network) in a ring topology. This model was first introduced in [47] to study chimera states. The network is constructed by selecting a base pattern composed of ones and zeros. Then we iterate over this base $n$ times, substituting the base pattern in each iteration every time we encounter a one and a string of zeros of size $b$ every time we come across a zero, where $b$ is the length of the base. After $n - 1$ iterations we obtain the $n^{th}$ hierarchy level. We then have a string $S$ of size $b^n$. We use this string of ones and zeros as the first row of the adjacency matrix. Each following row of the matrix is obtained by shifting the previous row by one element to the right applying periodic boundary conditions. This results in a circulant matrix. Circulant matrices have well-known eigenvalues and eigenvectors, and are further discussed in Sec. 6(a).

As an example, let us consider a connectivity matrix generated from an initiation string (base) of size $b = 3$ with a base pattern (101). The base ($n = 1$), the string after the first iteration ($n = 2$), and after the second, in this case the final, iteration ($n = 3$) are shown in Fig. 2(a). The final string defines the connections of the first node to the other nodes of the network. The corresponding links are shown in purple (gray) in Fig. 2(b). The connections of all other nodes follow the same rules (not shown in Fig. 2(b) for the sake of clarity), in other words each node “sees” the same network; this is equivalent to the circulant property of the final network adjacency matrix. Thus, the adjacency matrix of our example is given by

\[
A = \begin{pmatrix}
1 & 0 & 1 & 0 & \ldots & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & \ldots & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 1 & 0 & 1 & 1
\end{pmatrix}.
\] (5.1)
Figure 3: Schematic representation of model 2 for $n = 2$. (a) Red (black) solid lines and blue (gray) dashed lines represent the links starting at the first and second node, respectively. The initial network motif given by the adjacency matrix $A_1$ is marked by the yellow (gray) dashed square. (b) Red (black) solid lines and blue (gray) dashed lines represent the links starting at the 13th and 12th node, respectively. The adjacency matrix is obtained by Eq. (5.2). For clarity, the links of the others nodes are not shown in panels (a) and (b).

(b) Modular fractal

In this section, we extend the 1D fractal network presented in Sec. 5(a) to a 2D hierarchy. Here, instead of a base string, we use an initial $b \times b$ base matrix $A_1$ of ones and zeros. The $b^n \times b^n$ adjacency matrix $A$ for the $n$th hierarchy level can then be formed by taking $n$ Kronecker products of the initial adjacency matrix $A_1$ with itself, i.e.,

$$A = A_1 \otimes \ldots \otimes A_1.$$  \hspace{1cm} (5.2)

This is essentially the 2D version of the procedure described in Sec. (a): We start with a $b \times b$ base matrix $A_1$ of size $m \times m$. If we encounter a non-zero element in the base, we substitute it with the element times the matrix $A_1$, whereas a zero is replaced by a zero matrix of size equal to the size of $A_1$. We repeat this substitution procedure $n$ times resulting in the adjacency matrix $A$ of size $m^{n-1} \times m^{n-1}$ given in Eq. (5.2). This matrix $A$ then defines the coupling topology, but it is no longer a circulant matrix, i.e., the ring topology is replaced by a modular topology. Note that by the method used to construct the adjacency matrix $A$, $A$ has a constant row sum if $A_1$ has a constant row sum.

In order to be able to apply the eigensolution method (Sec. 5), we choose the initial matrix $A_1$ to either be a circulant matrix (for example, the adjacency matrix from a network as described in Sec. 5(a)), or an adjacency matrix whose eigenvector components satisfy Eq. 3.1, such as the mesoscale motifs studied in [24].

As an example, let us consider the case of the five node motif shown in Fig. 3(a) and investigated in [24]. Figure 3(b) shows the network topology of the final adjacency matrix $A$ for $n = 2$, i.e., $A = A_1 \otimes A_1$. As one can see, the topology is not an intuitive extrapolation from 1D to 2D, however, it retains a fractal structure. The normalized adjacency matrix $A_1$ of this motif is
Figure 4: Schematic view of model 3 for the $n_1 = 2$ hierarchy and with the initial motif given by Eq. (5.3) and depicted in Fig. 3(a). The red (gray) lines indicate the mean field or all-to-all coupling between motifs and the black lines mark the links inside the motif.

The resulting matrix $A$ is given explicitly in the Appendix A.3.

(c) Hierarchical

In this section, we present a hierarchical topology which is self-similar on every scale. In [73], synchronization with multiple delays has been investigated for this hierarchical network for the second level of the hierarchy ($n = 2$).

The network is created as follows: We start with a network motif given by the $m \times m$ matrix $A_1$. Then, the adjacency matrix $A$ after one iteration ($n = 2$) is given by

$$A = A_1 \otimes E_m + 1_m \otimes A_1$$

where $E_m$ is the $m \times m$ matrix with all entries equal to 1, and $1_m$ is the $m \times m$ identity matrix. $1_m \otimes A_1$ represents the direct coupling inside the motifs, whereas $A_1 \otimes E_m$ is the mean field (all-to-all) coupling between the motifs. The adjacency matrix for the $n^{th}$ level of the hierarchy is obtained by repeating the above procedure $n - 1$ times where we replace $A_1$ in Eq. (5.4) by the normalized adjacency matrix of the previous iteration step. Note that by the method used here to construct $A$, $A$ has a constant row sum if $A_1$ has a constant row sum.

As an example consider the case of the five node motif shown in Fig. 3(a) and investigated in [24]. Its normalised adjacency matrix is given by Eq. (5.3). Figure 3(a) depicts the final topology as calculated by Eq. (5.4). Clearly, the coupling between the motifs has the same structure as the coupling between the nodes inside one motif giving rise to a self-similar architecture.
6. Results

In this section, we study the dynamics for the network models discussed in Sec. 5. In particular, we discuss the application of the eigensolution method to hierarchical networks. We support our analytical results with numerical simulations.

(a) 1D fractal

This network has a circulant adjacency matrix, which has been introduced in Sec. 3. The general form of a circulant matrix is given by

\[
C = \begin{bmatrix}
  c_1 & c_N & \ldots & c_3 & c_2 \\
  c_2 & c_1 & c_N & \ldots & c_3 \\
  \vdots & c_2 & c_1 & \ddots & \vdots \\
  c_{N-1} & \ddots & \ddots & \ddots & c_N \\
  c_N & c_{N-1} & \ldots & c_2 & c_1
\end{bmatrix}.
\]

Its normalized eigenvectors [74] are given by

\[
v_j = \frac{1}{\sqrt{N}} (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T, \quad j = 1, \ldots, N,
\] (6.1)

where \(\omega = \exp\left(\frac{2\pi ij}{N}\right)\). The corresponding eigenvalues are given by

\[
\eta_j = c_1 + c_N \omega_j + c_{N-1} \omega_j^2 + \ldots + c_2 \omega_j^{n-1}, \quad j = 1, \ldots, N.
\] (6.2)

From Eq. (6.1) it follows that the eigenvectors of circulant matrices have components which are given by the roots of unity. Hence, they fulfill the property required by the eigensolution approach presented in Sec. 3, i.e., \(|v_i| \in \{0, 1\}\), where we defined \(v_i\) as the \(i\)th component of the eigenvector \(v\).

In accordance with the eigensolution method, substituting Eq. (6.1) into Eq. (3.6) yields the \(j\)th eigensolution for the \(k\)th oscillator, \(k = 1, \ldots, N\),

\[
z_k = v_k z(t) = \exp\left(\frac{2\pi ij}{N} \right)^k \sqrt{\lambda} \exp(i\tilde{\omega}t) = \sqrt{\lambda} \exp\left[i\tilde{\omega}t + \frac{2\pi ikj}{N}\right].
\] (6.3)

From Eq. (6.3) it follows that the \(j\)th eigensolution is a cluster state with a constant phase shift of \(2\pi j/N\) between neighboring nodes. As discussed in Sec. 3, the number of clusters is then given by \(\text{lcm}(j, N)/j\) [60]. The eigenvectors of circulant matrices as given by Eq. (6.1) do not have components equal to zero, and hence for non-degenerate eigenvalues, we do not find partial amplitude death eigensolutions in this hierarchical topology. For degenerate eigenvalues, we can obtain dead nodes by linear combinations of the corresponding eigenvectors. This is further discussed in the next subsection.

(i) Partial amplitude death in hierarchical networks

For model (a), partial amplitude death eigensolutions do not exist for non-degenerate eigenvalues. However, if the adjacency matrix of the network has degenerate eigenvalues \(\eta_1 = \eta_2\) or \(\eta_3 = \eta_4\) where \(i_1, i_2 \in \{1, \ldots, N\}\), we can find a new basis \(v_1^j\) and \(v_2^j\) of the subspace spanned by the corresponding eigenvectors \(v_1^j\) and \(v_2^j\) such that at least one of the entries of \(v_1^j\) or \(v_2^j\) is 0. We start by selecting a position \(m^*, m^* \in \{1, \ldots, N\}\), in the eigenvector which we wish
to convert to zero. We achieve this by constructing \( \tilde{v}^j \) as
\[
\tilde{v}^j = v^j l - v^j \tag{6.4}
\]
where \( l \) is given by
\[
l = \frac{v^j_m}{v^j_i}, \tag{6.5}
\]
and \( v^j_m \) denotes the \( (m^*) \)th element of the vector \( v^j \), \( i \in 1, 2 \ldots N \). For a circulant adjacency matrix, Eq. (6.5) reads
\[
l = \exp \left[ 2\pi i(j_2 - j_1) m^* / N \right]. \tag{6.6}
\]
Let \( B \) be the set of integers \( m \in \{1, \ldots, N\} \) for which the Eq. (6.4) holds true. We require that the set \( B \) always has at least one element, namely \( m^* \), the position we initially chose to be zero. Trivially, the cardinality of \( B \) equals the number of zeros in \( \tilde{v}^j \).

To ensure the existence of the eigensolution, we require that the original assumption made while finding the eigensolutions still holds for the new eigenvector, i.e., the following condition is satisfied:
\[
|v^j_i|^2 \in \{0, 1\} \tag{6.7}
\]
for \( \forall i \in 1 \ldots N \).

While this condition is fulfilled for the eigenvectors calculated according to Eq. (6.1), it is not automatically fulfilled for a linear combination of these eigenvectors. We can rewrite the conditions for the existence of an eigensolution by substituting Eqs. (6.1) and (6.4) into Eq. (6.7):
\[
| \exp[2\pi i(j_2) k / N] - l \exp[2\pi i(j_1) k / N] |^2 = c. \tag{6.8}
\]
for \( \forall k \in \{1, \ldots, N\} \) such that \( k \notin B \), i.e., for all nonzero entries of the eigenvector \( \tilde{v}^j \). The normalization constant is \( c \). Substituting for \( l \) from Eq. (6.6) into Eq. (6.8), we obtain
\[
\exp \left[ \frac{2\pi i(j_2 - j_1)(m^* - k)}{N} \right] = c. \tag{6.9}
\]
If this condition holds \( \forall k \notin B \), an eigensolution exists for the corresponding values of \( j_1, j_2 \), which represents a state where all nodes \( m \in B \) are amplitude dead.

As an example, we consider the four-node ring network shown in Fig. 5(a) and investigated in [24]. For this network, \( \eta_1 = \eta_3 = 0 \) holds. The corresponding normalized eigenvectors are \( v^1 = 0.5(1, i, -1, -i) \) and \( v^2 = 0.5(1, -i, 1, -i) \). For \( m^* = 2 \), \( l \) is calculated to be \(-1\) and \( \tilde{v}^j = \{1, 0, -1, 0\} \). The set of all indices for which \( \tilde{v}^j = 0 \) is \( B = \{2, 4\} \). For all \( k \notin B \), i.e., \( k = 1, 3 \), Eq. (6.9) is satisfied. Therefore, a solution of the form \( \tilde{v}^j \) exists, where nodes 1 and 3 are antiphase-synchronized with respect to each other (marked by blue and red color in Fig. 5(a)), and nodes 2 and 4 are amplitude dead (the green nodes in Fig. 5(a)).

The stability of the solution is found by using the linear stability analysis for partial amplitude death states suggested in [24]. The value of the real part of the largest Floquet exponent for this network is positive throughout the parameter space (as seen in Fig. 5(b)). Numerically, we have investigated networks up to 800 nodes, and all partial amplitude death solutions we have found are unstable.

(ii) Oscillation death

In this section, we show that oscillation death can arise as a cluster state with vanishing common frequency, even for symmetric coupling. For the sake of simplicity, let us consider a network with base \((0110)\) and \( n = 2 \), i.e., \( N = 16 \). We investigate the \( j = 4 \)-cluster state with number of clusters given by \( M = \text{lcm}(4, 16) / 4 = 4 \).

For the considered network, Fig. 6 depicts the stable regions for the 4-cluster state in green (light gray), for in-phase synchrony in red (intermediate gray), and for complete amplitude death in blue (dark gray). In the white region we obtain solutions that do not fall into any of the
categories we study, i.e., in-phase synchronization, cluster synchronization, or partial amplitude death. The solutions in the white region include patterns with non-equal radii and/or non-equal frequencies. For \( \lambda = 3, \sigma = -2 \), a 4-cluster state is observed, as predicted by the stability plot in Fig. 6(a) (corresponding parameter values are marked by a green star). The corresponding space-time plots are shown in Fig. 6(b), respectively, and show that the cluster state has a nonvanishing frequency, i.e., \( \tilde{\omega} \neq 0 \). For a schematic representation of this state see Fig. 6(c). For \( \lambda = 3, \sigma = -1 \) (marked by a red triangle in Fig. 6(a)), we also observe that the 4-cluster state is stable, however, here the common frequency as given by Eq. (3.5), is zero as can be seen in the corresponding space-time plot (Fig. 6(d)). Thus, for these parameters we obtain oscillation death. A schematic figure of this state is shown in Fig. 6(e).

In addition, it is also possible to obtain a mixed-death state consisting of coexisting partial oscillation death and partial amplitude death. Oscillation death occurs as a consequence of vanishing oscillation frequency for pairs of nodes whose corresponding eigenvector components are non-zero and symmetric about zero, whereas the eigenvector component corresponding to nodes that are in the amplitude death state is exactly zero. This state can be found for motifs in [24]: We consider a motif or network showing amplitude death, e.g., the motif shown in Fig. 5(a), and vary the coupling strength such that the frequency of the oscillating nodes becomes zero, i.e., we change \( \sigma \) in Eq. (3.3) such that \( \tilde{\omega} = 0 \). This means that the oscillating nodes stop oscillating with a non-zero radius given by \( r_0 \) (see Eq. (3.3)), while the nodes which where amplitude-dead from the beginning remain at the origin (recall that in Eq. (3.3), \( |v_0| = 0 \) corresponds to the nodes undergoing amplitude death). Note that the stability analysis for oscillation death is the same as that for cluster states, since oscillation death arises here as a cluster state with \( \tilde{\omega} = 0 \).

(b) Modular fractal

Recall that the adjacency matrix for the \( n^{th} \) level of the hierarchy is given by \( A = A_1 \odot \ldots \odot A_1 \), where \( A_1 \) is our initial motif. The eigenvectors of \( A \) are then given by the Kronecker product of

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Figure 5: Example of partial amplitude death. (a) Four node network motif colored according to the predicted eigensolution. Green (light grey) marks nodes which exhibit amplitude death. The red (intermediate grey) and the blue node (dark grey) exhibit antiphase-synchronized oscillations with respect to each other. (b) Largest real part of the Floquet exponents \( \Lambda \) for the motif and the eigensolution shown in (a).
Figure 6: Eigensolutions of the 1D fractal model. (a) Linear stability analysis. CIS stands for complete in-phase synchronization. Parameter values where this state is stable are marked by red (intermediate gray) shading. Blue (dark gray) shading denotes the stability region of the complete amplitude death (CAD) state, i.e., a state where all nodes are in the trivial steady state. Green (light gray) shading denotes the stable region of a $j = 4$ cluster state. (b) Space-time plot for $\sigma = -2$ (marked by the blue star in panel (a)). The corresponding state is schematically depicted in (c). (d) Space-time plot for $\sigma = -1$ (marked by the red triangle in panel (a)). The corresponding state is schematically depicted in (e). Other parameters: base $(0110)$, $n = 2$, $N = 16$, $j = 4$, $\lambda = 3$, $\omega = 2$, $\beta = 0$.

combinations of $n$ eigenvectors of $A_1$, i.e., by

$$v = v^i \otimes v^j \otimes \ldots \otimes v^k \quad i, j, k \in \{1, \ldots , m\}$$

(6.10)
where $v^i, v^j, \ldots, v^k$ are chosen from the eigenvectors of $A_1$. Refer to Sec. (a) in Appendix A for the proof.

Here, we consider the five-node motif shown in Fig. 3(a), for $n = 3$ and the eigenvector $v \otimes v$, where $v = (1, 1, 0, -1, -1)$. As seen in Fig. 7, the network dynamics is nested or self-similar on every scale because we choose $i = j = k$ in Eq. 6.10. In other words, since we choose the same eigenvector three times, we obtain a three-fold hierarchy in the dynamics. On the largest scale, the corresponding nodes of the first two groups of 25 nodes each behave in an identical fashion, and are antiphase-synchronized with the corresponding nodes of the last two groups of 25 nodes. The middle group is amplitude dead. Within each group, the first two motifs are in synchrony with respect to each other, while the last two are in antiphase-synchronization with respect to the first two, and the middle motif is dead. And finally, within a motif, the dynamics follows the same pattern, i.e., the first two motifs are antiphase-synchronized with respect to the last two nodes, and the middle node is dead. Schematically, the dynamics of the motif is shown in Fig. 7(a).

If we choose the initial matrix $A_1$ to be a circulant matrix obtained from a hierarchical network as created in Sec. 6(a), we obtain a 2D modular hierarchical structure: The initial $b^{n_1} \times b^{n_1}$ matrix $A_1$ has a fractal topology with a repeating base, and the final matrix $A$ has the same coupling structure with respect to these modules $A_1$. Thus we have two parameters that determine the hierarchy: $n_1$ that decides the level of hierarchy in $A_1$, and $n$ that sets the level of multi-hierarchy in $A$.

The $n$th level of the hierarchy can then be created by taking the $n$-fold Kronecker product of this matrix with itself. Thus, the total number of nodes are $N = b^{n_1 \cdot n}$. The adjacency matrix $A$, when $A_1$ is circulant, is a block circulant matrix [75].

Let us consider first the case $n = 2$. We choose the $o$th and the $p$th eigenvector of our initial $m \times m$ matrix $A_1$. According to Eq. (6.1) and Eq. (6.10), the components of the final eigenvector are then given by:

$$v_{j \cdot m+k} = (v^o \otimes v^p)_{j \cdot m+k} = \exp \left(\frac{2\pi j o}{m}\right) \cdot \exp \left(\frac{2\pi k p}{m}\right) = \exp \left(\frac{2\pi (j o + k p)}{m}\right)$$

(6.11)

where $k, j \in \{1, \ldots, m\}$. Thus, by the eigensolution method, the nodes in each group are in a cluster state with a constant phase shift of $\exp(2\pi i p/m)$ between neighboring nodes, and the...
corresponding nodes of the groups are in a cluster state with a constant phase shift of \(\exp(2\pi i o/m)\) between neighboring groups. Hence, we obtain hierarchical dynamics: a cluster state of cluster states.

The components of the final eigenvector for \(n = 3\) are then given by

\[
v_j \cdot m^2 + k \cdot m + l = (v^0 \otimes v^p \otimes v^q)_{j \cdot m^2 + k \cdot m + l} = \exp\left(\frac{2\pi i j o}{m}\right) \exp\left(\frac{2\pi i k p}{m}\right) \exp\left(\frac{2\pi i l q}{m}\right)
\]

where we use the \(o\)th, \(p\)th, and \(q\)th eigenvector of \(A_1\).

As an example, we now consider two networks with the same base \((101)\) but with interchanged values of \(n, n_1\) to demonstrate the direct dependence of the dynamics on the chosen hierarchy and multi-hierarchy.

In Fig. 8, we use the base \((101)\) with \(n_1 = 3, n = 2\) respectively. There are \(N = (3^3)^2 = 729\) nodes. We chose to consider the state corresponding to \(o = 0\) and \(p = 17\) in Eq. (6.11). Since \(n = 2\), we have a two-hierarchy cluster state, with 27 groups of 27 nodes each: Corresponding nodes in each group (every 27th node) are fully synchronized (Fig. 8 (b)) because \(o = 0\). Consecutive nodes in each group are phase shifted with respect to each other with a constant phase of \(\phi = \frac{2\pi}{27}\) as a result of \(p = 17\) (Fig. 8 (c)).

In Fig. 9, we consider the base \((101)\) with \(n_1 = 2, n = 3\). Since \(n = 3\), we anticipate a three-level hierarchy of cluster states: cluster state of cluster states of cluster states. We choose \(o = 0, p = 5\), and \(q = 8\) in Eq. (6.12). There are \(N = (3^2)^3 = 729\) nodes with 9 groups of 81 nodes each. Each group consists of 9 subgroups of 9 nodes each. On the smallest scale, nodes in each subgroup are in a splay state, i.e., neighboring nodes are phase shifted with respect to each other with a constant phase of \(\phi = 2\pi/9\) (Fig. 9(d)) because \(q = 8\). Each subgroup within a group is also in a cluster state, i.e., every 9th node has a constant phase shift of \(\phi = 2\pi\cdot 5/9\) with respect to the previous (Fig. 9(c)), this is due to \(p = 5\). Finally, on the large scale, every group is synchronized meaning that the phase lag between every 81st node is zero (Fig. 9(b)) corresponding to \(o = 0\).

In summary, we observe that \(n = 2\) shows a two-level hierarchy in the dynamics, whereas \(n = 3\) shows a three-level hierarchy. Analogously, for an \(n\)-level multi-hierarchy of the adjacency matrix, the dynamics is given by \(n\) nested cluster states.
(c) Hierarchical

In this model, the topology is self-similar on each scale. The creation of this network is elaborated upon in Sec. 5(c). The eigenvector of the adjacency matrix $A$ of the $n$th hierarchy can be written as $n$ Kronecker products of the original eigenvectors of initial the motif $A_1$, i.e.,

$$v = v^i \otimes v^j \otimes \ldots \otimes v^k \quad i,j,k \in \{1, \ldots m\}$$  \hspace{1cm} (6.13)

where $v^i, v^j, \ldots, v^k$ are chosen from the set of eigenvectors of $A_1$. However, in contrast to the modular fractal model, here we have the additional requirement that the sum of all elements of the last eigenvector in the Kronecker product has to equal zero, i.e., $\sum_{j=0}^{m} v^k_j = 0$ in Eq. (6.13). For details see Sec. (b) in Appendix A, where we derive Eq. (6.13). This condition seems to be rather strict. However, it is fulfilled for all the eigenvectors of all motifs discussed in [24] which are all generic, normalized motifs of up to five nodes. Once the eigensolutions are established their stability can be studied as in [24]. In special cases, if all the eigenvectors in the Kronecker product are the same, hierarchical dynamics is obtained.

As an example, we investigate the initial motif $A_1$ described by Eq. (5.3) for a hierarchy $n = 3$. The number of nodes is given by $5^n = 125$. We consider the eigensolution given by $v \otimes v \otimes v$, where $v = (1, 1, 0, -1, -1)$ is an eigenvector of $f A_1$. The motif is shown in Fig. 7(a), where
Figure 10: Non-eigensolution dynamics in the hierarchical model. (a), (b) and (c): initial motifs. Colors denote the eigensolutions of the motif (which do not directly translate to the larger network). The green node shows amplitude death, the red and blue nodes are antiphase-synchronized with respect to each other. (d), (e), and (f) are space-time plots for the networks built from the initial motifs in (a), (b), and (c), respectively. Other parameters: \( n = 2 \), \( N = 5^2 = 25 \), \( \sigma = -5 \), \( \beta = 0 \), \( \lambda = 1 \), \( \omega = 2 \).

the color scheme indicates the state corresponding to \( v = (1, 1, 0, -1 - 1) \). We observe that the dynamics is identical to the dynamics shown in Fig. 7(b)) which we have obtained for the modular fractal model. This is due to the fact that in spite of very different topologies, the eigenvectors of the networks, and thus the dynamics, are identical.

We also study solutions that do not correspond to eigensolutions. We do so for the three different motifs as shown in Fig. 10(a), (b), and (c), respectively, and for a hierarchy of \( n = 2 \). The space-time plots are shown in Fig. 10(d), (e), and (f), respectively. We observe that these solutions have as well a hierarchy in their dynamics. The nodes of the middle motif and the middle node of the remaining motifs are either phase shifted or have different radii than the remaining nodes or both. This is a result of the structure of the motifs: The middle nodes of all three motifs have a different connectivity than the other nodes. Additionally, in Fig. 10(f), the first two motifs have a different amplitude than the last two, which is reflected in the horizontal asymmetry in the topology about the central node. Thus, the correlation between network topology and dynamics is not limited to eigensolutions.

7. Conclusion

In this paper we have presented three different models of networks with hierarchical or fractal connectivities, and studied them analytically using an eigensolution concept. The eigensolution
concept was developed in [24] to describe synchronization, anti-synchronization, and partial amplitude death. Here, we extend this concept to cluster states and to larger hierarchical networks created from basic motifs. In combination with hierarchical topologies this leads to complex synchronization patterns such as cluster synchronization, partial synchronization, oscillation death, partial amplitude death and nested dynamics. In particular, we observe that the fractal nature of the network translates to fractal synchronization patterns. Understanding these synchronization patterns helps to bridge the gap between relatively simple, by now well understood states like in-phase synchronization and much more complex synchronization patterns like chimera states.

The first model we have considered has a one-dimensional fractal topology with a circulant adjacency matrix, and the eigensolutions are cluster states. For networks with circulant adjacency matrix, we have mathematically derived the conditions for the existence of partial amplitude death eigensolutions and calculated their stability. We have also shown that oscillation death, possibly coexisting with partial amplitude death, arises in such networks as a special cluster state with zero-frequency. This is in contrast to previous work, where oscillation death was observed as a result of symmetry breaking. Therefore, we establish here a second mechanism leading to oscillation death.

The second model has a two-dimensional modular fractal topology, and the third model is a direct extension of mesoscale motifs to larger self-similar hierarchical networks. In both the second and the third model we see the direct influence of topology on dynamics, i.e., for a hierarchical topology, we obtain hierarchical dynamics. Although the second and the third model yield similar dynamics, they have vastly different topologies that resemble very different natural systems and hence, have different applications. The second model has a fractal hierarchical topology whereas the third model has a self-similar hierarchical topology. In addition, while the second model is applicable to networks created from the mesoscale motifs in [24] as well as to all networks with circulant adjacency matrices, the third model is relevant only for the former.

The work presented here is of particular interest for neuroscience where recently a lot of emphasis has been put on the relation between structural connectivity and functional connectivity in the brain [76–79]. Evidence from empirical studies suggests that the presence of a direct anatomical connection between two brain areas is associated with stronger functional interactions between these two areas [76,80–82]. Our results support these empirical results through theoretical investigation. In addition, they can give valuable insight because they provide a completely analytical framework while employing a complex hierarchical structure that mimics the hierarchical nature of neurons in the brain [47,51–53]. The fractal or self-similar hierarchical organization of neural networks is studied in [83–86]. The advantage of this theoretical study is that it allows for investigating the interplay of dynamics and topology on every scale, from the smallest to the largest structural level as well as the investigation of dynamics of each individual node. It is therefore a powerful complement to experimental work, which, due to its challenging nature, is often limited to mean-field approximations [87], and to theoretical work representing neural dynamics in terms of overall statistics, i.e., representing entire cortical regions as one node [87–90]. Besides applying it to neuroscience, our work can also be used to study the functional dynamics of metabolic networks, which also have been shown to display a hierarchical topology as in our third model [56].

This correlation between dynamics and topology is not limited to the eigensolutions. We have shown that synchronization patterns corresponding to more general solutions can be predicted in hierarchical networks from the knowledge of motif topology. The study of these solutions and the extension of the eigensolution concept to time-delayed networks would be an interesting topic of further research.

Authors’ Contributions. S. Krishnagopal introduced the models and performed the analysis and simulations under the supervision of J. Lehnert and E. Schöll. The analytical results in this paper are based on the eigensolution technique developed by W. Poel under the supervision of A. Zakharova and E. Schöll [24]. All authors contributed to writing the manuscript.
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8. Appendix A

The proofs from Sec. 6 are given here.

(a) A.1 : Eigenvectors of networks with modular fractal topology

Here we will discuss how the eigenvectors of the second model can be calculated from the initial motif, i.e., we derive Eq. (6.10). Let us consider the initial matrix

\[
A
\]

Here we will discuss how the eigenvectors of the second model can be calculated from the initial matrix. Let us consider the initial matrix $A$. Then, $A_1v^i = \lambda_i v^i$ and $A_1v^j = \lambda_j v^j$ holds. In order for $v^i \otimes v^j$, where $i, j \in \{1, \ldots, m\}$, to be an eigenvector of the above matrix, the following equation has to be satisfied:

\[
(A_1 \otimes A_1 - \lambda)(v^i \otimes v^j) = 0. \tag{8.1}
\]

We make use of the following property of the Kronecker product: $(H \otimes J)(J \otimes K) = HJ \otimes IK$, where $H, I, J$, and $K$ are arbitrary matrices of appropriate dimensions. Using in addition $A_1v^k = \lambda_k v^k$, $k \in \{i, j\}$, Eq. (8.1) evaluates to:

\[
\lambda_i\lambda_j v^i \otimes v^j - \lambda v^i \otimes v^j = 0, \quad \text{for } \lambda = \lambda_j \lambda_i. \tag{8.2}
\]

Thus, $v^i \otimes v^j$ is an eigenvector of $A$. The corresponding eigenvalue is given by $\lambda = \lambda_j \lambda_i$.

The adjacency matrix for the $n$th hierarchy is obtained by repeating the procedure above $n - 1$ times and replacing $A_1$ with the matrix of the previous iteration each time. We, thus, obtain

\[
v = v^i \otimes v^j \otimes \ldots \otimes v^k, \quad \text{where } i, j, k \in \{1, \ldots, m\}, \tag{8.3}
\]

which is equivalent to Eq. (6.10).
For the five-node motif shown in Fig. 3(a) [24] and described by the adjacency matrix (c) A.3: Example of a modular fractal adjacency matrix Eq. (6.13), the final adjacency matrix is given by $A_{m^2} = A_1 \otimes E_m + 1_m \otimes A_1$ where $E_m$ is a matrix of size $m \times m$ where all the entries are 1, and $1_m$ is the $m \times m$ identity matrix. $v^i$, and $v^j$ are eigenvectors of $A_1$: $A_1v^i = \lambda_i v^i$, $A_1v^j = \lambda_j v^j$. In order for $v^i \otimes v^j$, where $i, j \in \{1, \ldots m\}$, to be an eigenvector of the above matrix, the following equation has to be satisfied:

\[(A_1 \otimes E_m + 1_m \otimes A_1 - \lambda)(v^i \otimes v^j) = 0. \]  

(8.4)

Using the property of the Kronecker product $(H \otimes I)(J \otimes K) = HJ \otimes IK$, where $H, I, J$, and $K$ are arbitrary matrices of appropriate dimensions, and $A_1v^k = \lambda_k v^k, k \in \{i, j\}$, Eq. (8.4) evaluates to

\[A_1v^i \otimes \left(\sum_{i=1}^m v^j_i\right) - \lambda_i v^i \otimes v^j = 0, \]

(8.5)

where $a_m$ is a vector of ones of size $m$. If $\sum_{i=1}^m v^j_i = 0$, this reduces to

\[\lambda_j v^i \otimes v^j - \lambda v^i \otimes v^j = 0 \quad \text{for} \quad \lambda = \lambda_j. \]  

(8.6)

Thus, $v^i \otimes v^j$ is an eigenvector of $A$. The corresponding eigenvalue is given by $\lambda = \lambda_j$. The adjacency matrix $A$ for the $n^{th}$ level of the hierarchy is obtained by repeating the above $n - 1$ times and replacing $A_1$ with the matrix of the previous iteration each time. Doing so we obtain Eq. (6.13).

(c) A.3: Example of a modular fractal adjacency matrix

For the five-node motif shown in Fig. 3(a) [24] and described by the adjacency matrix $A_1$ (Eq. (5.3)), the final adjacency matrix $A$ for $n = 2$, i.e., $A = A_1 \otimes A_1$ is given explicitly by

\[A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & \ldots & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \ldots & 0 & \frac{1}{12} & \frac{12} {12} & 0 & \frac{1}{12} \\ 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \ldots & 0 & \frac{1}{12} & \frac{12} {12} & 0 & \frac{1}{12} \\ 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \ldots & 0 & \frac{1}{12} & \frac{12} {12} & 0 & \frac{1}{12} \\ 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} & \frac{1}{9} & \ldots & 0 & \frac{1}{9} & \frac{1}{9} & 0 & \frac{1}{9} \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} & \ldots & 0 & \frac{1}{12} & \frac{12} {12} & 0 & \frac{1}{12} \end{pmatrix}. \]  

(8.7)