S-duality of tree-level S-matrix elements
in D\textsubscript{3}-brane effective action

H. Babaei-Aghbolagh\textsuperscript{1} and Mohammad R. Garous\textsuperscript{2}

Department of Physics, Ferdowsi University of Mashhad,
P.O. Box 1436, Mashhad, Iran

Abstract

Recently it has been speculated that the S-matrix elements on the world volume of D\textsubscript{3}-branes should satisfy the ward identity corresponding to the S-duality transformations. For a single brane and in the low energy limit, this indicates that in the abelian Dirac-Born-Indeld and Chern-Simons actions the combination of contact terms and tree-level massless poles in an \(n\)-point function should satisfy the S-duality. In this paper we examine in details the S-matrix element of three gauge bosons and one two-form and the S-matrix element of six gauge bosons in favor of the above proposal.

Keywords: S-duality, S-matrix, Effective action

\textsuperscript{1}hossein.babaei66@gmail.com
\textsuperscript{2}garousi@um.ac.ir
1 Introduction

It is known that the type IIB superstring theory is invariant under S-duality [1, 2, 3, 4, 5, 6]. This symmetry should be carried by the S-matrix elements through the associated Ward identity [7]. That is, the S-matrix elements should be invariant under linear S-duality transformations on the external states and nonlinear S-duality transformation on the background fields. In particular, the S-matrix elements on the world volume of D$^3$-branes should be invariant under nonlinear S-duality transformations on the coupling constant and on the background R-R scalar field. This requires, in general, to combine the tree-level amplitudes with loop and non-perturbative amplitudes [8] - [27].

The world volume theory of a single D$^3$-brane at low energy is given by the Dirac-Born-Infeld-Chern-Simons (DBICS) action [28, 29, 30, 31]. This action is not invariant under the S-duality, however, its equations of motion and its energy-momentum tensor are invariant under the S-duality [32]. There are proposals for the $SL(2, R)$-covariant form of the D$^3$-brane action [33, 34]. In this paper, we would like to study the S-dual Ward identity of the S-matrix elements in the DBICS theory. This theory is not renormalizable, hence, the degree of momentum of an S-matrix element at tree-level is different from the degree of momentum at one loop-level, the degree of momentum at one loop-level is different from the degree of momentum at two loop-level, and so on. On the other hand, the standard S-duality transformations do not receive $\alpha'$ correction. As a result, the S-dual Ward identity in the DBICS theory can not relate the tree-level amplitudes to loop-level amplitudes. Instead, the Ward identity should leave invariant the tree-level amplitudes. It should relate the one loop-level amplitudes in this theory to the appropriate higher derivative corrections of the DBICS theory and to the appropriate non-perturbative amplitudes. Similarly for higher loop-level amplitudes. In this paper, we would like to show by explicit calculations that the combination of tree-level massless poles and contact terms in the DBICS theory satisfies the S-dual Ward identity.

The outline of the paper is as follows: We begin in section 2 by reviewing the low energy effective action of a single D$^3$-brane. In section 3, we calculate the scattering amplitude of three gauge bosons and one two-form in the presence of the background dilaton and R-R scalar fields. In subsection 3.1, we show that the amplitude satisfies the Ward identity corresponding to $SL(2, R)$ transformations. In section 4, we calculate the S-matrix element of six gauge bosons and show that the amplitude satisfies the S-dual Ward identity.

2 D-brane effective action

The bosonic massless fields on the world volume of D-branes are the world volume gauge boson and the transverse scalar fields. When the derivatives of the world volume field strengths and the derivative of bulk fields are small compare to $\alpha'$, the dynamics of the

1

1
D-branes of type II superstring theories is well-approximated by the effective world-volume field theory which consists of the Dirac-Born-Infeld (DBI) action and the Chern-Simons (CS) part. The DBI action describes the dynamics of the brane in the presence of NS-NS background fields, which can be found by requiring its consistency with the nonlinear T-duality \[28, 29\]. On the other hand, the CS part describes the coupling of D-branes to the R-R potentials \[30, 31\]. The DBICS action in the Einstein frame for single $D_3$-brane is\[1\]

\[
S_{D3} = -T_{D3} \int d^4x \sqrt{-\det(g_{ab} + e^{-\phi/2}B_{ab})} + T_{D3} \int [C^{(4)} + C^{(2)} \wedge B + \frac{1}{2} C^{(0)} \wedge B \wedge B]
\]

All the bulk fields in the action are pull-back onto the world-volume of D-brane, e.g.,

\[
g_{ab} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}
\]

where $X^a = \sigma^a$ in the static gauge, and the transverse scalar fields are all other components of $X^\mu$, i.e., $X^i = 2\pi \alpha' \Phi^i$. The abelian gauge field can be added to the action as $B \rightarrow \mathcal{F}$ where $\mathcal{F} = B + 2\pi \alpha' F$. This makes the action to be invariant under the B-field gauge transformation. This action is also invariant under the following R-R gauge transformations:

\[
\delta C = d\Lambda + H \wedge \Lambda
\]

where $C = C^{(4)} + C^{(2)}$, $\Lambda = \Lambda^{(3)} + \Lambda^{(1)}$ and $H$ is the field strength of $B$, i.e., $H = dB$.

It has been shown in \[32\] that the equations of motion and energy-momentum tensor of the above action are invariant under the $SL(2, \mathbb{R})$ transformation. However, the action itself is not invariant under this transformation. This is resulted from the fact that the gauge field in this action is the massless mode of the fundamental open string propagating on the world-volume of $D_3$-brane. Under the S-duality, the fundamental string transforms to the D-string whose massless mode is given by another gauge field, i.e., $\tilde{\mathcal{F}}$, which is the dual of the Born-Infeld gauge field $\mathcal{F}$. There are proposals for the $SL(2, \mathbb{R})$-covariant form of the $D_3$-brane action which include both Born-Infeld gauge field and its dual \[33, 34\]. A complex $SU(1,1)$-base $q_\alpha$ has been considered in \[34\]. By rotating this doublet at the same time that rotating the world volume and bulk fields, an $SU(1,1)$-covariant family of actions has been found in \[34\], i.e.,

\[
S_{D3} = -T_{D3} \int d^4x \sqrt{-\det\left(q_{ab} + \frac{q_\alpha F_{ab}}{q_\alpha M_{\alpha\beta} q_\beta} \right)^{1/2}} + \cdots
\]

where dots represent the Chern-Simons part which includes another doublet $\tilde{q}_\alpha$, and the matrix $M$, in the real $SL(2, \mathbb{R})$-base $q_\alpha$, appears in \[15\]. In this base, the DBI action \[1\]

\[1\] Our index convention is that the Greek letters ($\mu, \nu, \cdots$) are the indices of the space-time coordinates, the Latin letters ($a, d, c, \cdots$) are the world-volume indices and the letters ($i, j, k, \cdots$) are the normal bundle indices.
corresponds to \( q_{\alpha'} = (0, -1) \) and \( \tilde{q}_{\alpha'} = (1, 0) \). Under the S-duality, \( q_{\alpha'} \rightarrow \tilde{q}_{\alpha'} \) [34]. The DBI action then transforms under the S-duality to

\[
S_{D3} = -T_{D3} \int d^4x \sqrt{-\det \left( g_{ab} + \frac{C_{ab} + 2\pi \alpha' F_{ab}}{(e^{-\phi} + C_0^2 e^{\phi})^{1/2}} \right)} + \cdots
\]  

(5)

Using the actions (1) and (5), one may calculate various tree-level S-matrix elements. Their contact terms which are given by the appropriate terms in (1) and (5), are related to each other by the S-duality, and similarly the massless poles in (1) are related to the corresponding massless poles in the above theory by the S-duality. In this way one can calculate an \( SL(2, R) \)-covariant family of S-matrix elements.

In this paper, however, we are interested in studying to what extent the S-matrix elements in one of the above \( SL(2, R) \)-covariant family of actions are invariant under the \( SL(2, R) \) transformations. In particular, we are interested in studying the symmetry of the S-matrix elements in (1) which includes only one Born-Infeld field \( F \), under the S-duality transformation. Under the S-duality, the gauge field strength \( F \) in (1) transforms to the nonlinear combination of \( F \) [32], so it is nonsense to study the invariant of an S-matrix element with fix number of asymptotic gauge bosons. However, if the theory is invariant under the S-duality, one expects their S-matrix elements to satisfy the Ward identity corresponding to the S-duality which is a linear transformation on the asymptotic states [7].

3 Three gauge bosons and one two-form amplitudes

Using the action (1), one can calculate the scattering amplitude of three gauge bosons and one two-form in the presence of the background dilaton and R-R scalar fields. The two-form can be either the antisymmetric B-field or the R-R two-form. When the two-form is the R-R two-form, the amplitude has only massless pole which is given by the following Feynman rule:

\[
\mathcal{A}(C^{(2)}_4) = V^a(F_1, F_2, F_3, A)G_{ab}(A)V^b(A, C^{(2)}_4)
\]  

(6)

where \( F_1, F_2, F_3 \) are the polarizations of the external gauge bosons, \( C^{(2)}_4 \) is the polarization of the external R-R two-form and \( A \) is the off-shell gauge field propagating between the two vertices. The vertices and the gauge boson propagator can be read from (1) to be

\[
V^a(F_1, F_2, F_3, A) \sim -e^{-2\phi_0} \left[ k \cdot F_1 \cdot F_2 \cdot F_3^a - \frac{1}{4} k \cdot F_1^a \text{Tr}(F_2 \cdot F_3) \right] + P(1, 2, 3)
\]

\[
V^b(A, C^{(2)}_4) \sim k_4 \cdot (*C^{(2)}_4)^b
\]

\[
G_{ab}(A) \sim -\frac{e^{\phi_0} \eta_{ab}}{k \cdot k}
\]  

(7)

3
where \((*C_4^{(2)})_{ab} = \frac{1}{2} \epsilon_{abcd} (C_4^{(2)})^{cd}\). In above equations, \(P(1,2,3)\) stands for the other five permutations of 1, 2, 3, and \(k\) is the off-shell momentum in the propagator which is \(k = -k_4 = k_1 + k_2 + k_3\). We have ignored the factors of \((2\pi\alpha')\) and \(T_{D_3}\).

Replacing (7) in the amplitude (6), one finds the following result:

\[
\mathcal{A}(C_4^{(2)}) \sim \frac{e^{-\phi_0}}{k_4 \cdot k_4} \left[ k_4 \cdot F_1 \cdot F_2 \cdot F_3 \cdot *C_4^{(2)} \cdot k_4 - \frac{1}{4} k_4 \cdot F_1 \cdot *C_4^{(2)} \cdot k_4 \cdot \text{Tr}(F_2 \cdot F_3) \right] + P(1,2,3) \quad (8)
\]

The amplitude satisfies the Ward identity corresponding to the abelian gauge symmetry and to the R-R gauge symmetry, e.g., if one replaces \(C_4^{(2)} \rightarrow k_4 \wedge \zeta_4^{(1)}\) where \(\zeta_4^{(1)}\) is an arbitrary one-form, the amplitude becomes zero.

When the two-form is the B-field, the amplitude has massless pole as well as contact term which are given by the following Feynman rule:

\[
\mathcal{A}(B_4) = V^a(F_1, F_2, F_3, A)G_{ab}(A)V^b(A, B_4) + V(F_1, F_2, F_3, B_4) \quad (9)
\]

where \(B_4\) is the polarization of the external B-field. The vertex \(V^b(A, B_4)\) and the contact term \(V(F_1, F_2, F_3, B_4)\) can be read from (11) to be

\[
V^b(A, B_4) \sim -e^{-\phi_0} k_4 \cdot B_4^b + C_0 k_4 \cdot B_4^b \quad (10)
\]

\[
V(F_1, F_2, F_3, B_4) \sim \frac{1}{2} e^{-2\phi_0} \left( \text{Tr}(B_4 \cdot F_1 \cdot F_2 \cdot F_3) - \frac{1}{4} \text{Tr}(B_4 \cdot F_1) \text{Tr}(F_2 \cdot F_3) \right) + P(1,2,3)
\]

Note that the dot means the contraction of the world volume indices, so even though \((k_4)_\mu B_4^{\mu} = 0\), the expression \(k_4 \cdot B_4^b\) in the first line is not zero. Replacing the vertices and propagator in (9), one finds the following result:

\[
\mathcal{A}(B_4) \sim -\frac{1}{2} e^{-2\phi_0} \left[ \text{Tr}(B_4 \cdot F_1 \cdot F_2 \cdot F_3) - \frac{1}{4} \text{Tr}(B_4 \cdot F_1) \text{Tr}(F_2 \cdot F_3) \right] + e^{-2\phi_0} \left[ k_4 \cdot F_1 \cdot F_2 \cdot F_3 \cdot B_4 \cdot k_4 - \frac{1}{4} k_4 \cdot F_1 \cdot B_4 \cdot k_4 \cdot \text{Tr}(F_2 \cdot F_3) \right] + C_0 \frac{e^{-\phi_0}}{k_4 \cdot k_4} \left[ k_4 \cdot F_1 \cdot F_2 \cdot F_3 \cdot B_4 \cdot k_4 - \frac{1}{4} k_4 \cdot F_1 \cdot B_4 \cdot k_4 \cdot \text{Tr}(F_2 \cdot F_3) \right] + P(1,2,3) \quad (11)
\]

One can easily check that the amplitude satisfies the Ward identity corresponding to the B-field gauge symmetry, i.e., if one replaces \(B_4^{ab} \rightarrow k_4^{a} \zeta_4^{b} - k_4^{b} \zeta_4^{a}\), the amplitude vanishes.

### 3.1 S-dual Ward identity

To study the amplitudes (8) and (11) under the Ward identity corresponding to the global S-duality, one has to find the linear S-duality transformations on the external states and nonlinear S-duality transformations on the background fields. The S-duality transformation on the two-forms is a linear transformation, and on the gauge field is a nonlinear.
transformation\[32, 36, 37\]. The S-duality transformation on the two-forms and the linearized S-duality transformation on the gauge field are the following \[32, 36, 37\]:

\[
B \longrightarrow (\Lambda^{-1})^T B, \quad F \longrightarrow (\Lambda^{-1})^T F \quad \Lambda \in SL(2,R)
\] (12)

where the doublets \(B\) and \(F\) are

\[
B \equiv \begin{pmatrix} B^{(2)} \\ C \end{pmatrix}, \quad F \equiv \begin{pmatrix} e^{-\phi_0} F - C_0 \ast F \\ \ast F \end{pmatrix}
\] (13)

The S-duality transformation on the background fields \(\phi_0\) and \(C_0\) is \[32\]

\[
\mathcal{M}_0 \rightarrow \Lambda \mathcal{M}_0 \Lambda^T
\] (14)

where the matrix \(\mathcal{M}_0\) is

\[
\mathcal{M}_0 = e^{\phi_0} \begin{pmatrix} |\tau_0|^2 & C_0 \\ C_0 & 1 \end{pmatrix}
\] (15)

and \(\tau_0 = C_0 + ie^{-\phi_0}\). Since the dilaton and the R-R scalar appear only as the background fields in the amplitudes \(8\) and \(11\), we keep the nonlinear form of \(14\) in studying the S-dual Ward identity.

To demonstrate that the amplitudes \(8\) and \(11\) satisfy the S-dual Ward identity, we have to show that they are invariant under the above transformations. To this end, we consider the following terms:

\[
(*F^T) a^c \mathcal{M}_0 B_{cb} = e^{-\phi_0} F_a^c B_{cb} - (*F)_a^c C_{cb} - C_0 (*F)_a^c B_{cb}
\]

\[
(F^T_1) a^c \mathcal{M}_0 F_{2cb} = e^{-\phi_0} [(*F_1)_a^c (*F_2)_cb + F_1^a \ast F_{2cb}]
\] (16)

Using the transformations \(12\) and \(14\), one can easily verify that the left-hand sides are invariant under the S-duality. Using these S-duality invariant terms and using the following identity:

\[
\epsilon^{abcd} \epsilon^{efgh} = - \begin{vmatrix} \eta^{ae} & \eta^{af} & \eta^{ag} & \eta^{ah} \\ \eta^{be} & \eta^{bf} & \eta^{bg} & \eta^{bh} \\ \eta^{ce} & \eta^{cf} & \eta^{cg} & \eta^{ch} \\ \eta^{de} & \eta^{df} & \eta^{dg} & \eta^{dh} \end{vmatrix}
\] (17)

one can construct the following S-duality invariant contact term:

\[
\text{Tr}(F_1^T \mathcal{M}_0 F_2 \ast F_3^T \mathcal{M}_0 B_4) =
\]

\[
-e^{-2\phi_0} \left[ \text{Tr}(B_4 \ast F_1 \ast F_2) + \text{Tr}(B_4 \ast F_2 \ast F_1) - \frac{1}{2} \text{Tr}(B_4 \ast F_3) \text{Tr}(F_1 \ast F_2) \right]
\]

\[
-C_0 e^{-\phi_0} \left[ \text{Tr}(\ast B_4 \ast F_1 \ast F_2) + \text{Tr}(\ast B_4 \ast F_2 \ast F_1) - \frac{1}{2} \text{Tr}(\ast B_4 \ast F_3) \text{Tr}(F_1 \ast F_2) \right]
\]

\[
-e^{-\phi_0} \left[ \text{Tr}(\ast C_4 \ast F_1 \ast F_2) + \text{Tr}(\ast C_4 \ast F_2 \ast F_1) - \frac{1}{2} \text{Tr}(\ast C_4 \ast F_3) \text{Tr}(F_1 \ast F_2) \right]
\]

5
where the traces are over the world volume indices, and the following S-duality invariant massless pole:

\[
\frac{k_4 \cdot \mathcal{F}_4^T \mathcal{M}_0 \mathcal{F}_2 * \mathcal{F}_3^T \mathcal{M}_0 \mathcal{B}_4 \cdot k_4}{k_4 \cdot k_4} = \tag{19}
\]

\[
\frac{e^{-2\phi_0}}{k_4 \cdot k_4} \left[ -k_4 \cdot B_4 \cdot F_3 \cdot F_1 \cdot F_2 \cdot k_4 - k_4 \cdot B_4 \cdot F_3 \cdot F_2 \cdot F_1 \cdot k_4 + \frac{1}{2} k_4 \cdot B_4 \cdot F_3 \cdot k_4 \text{Tr}(F_1 \cdot F_2) \right] +
\]

\[
\frac{e^{-\phi_0}}{k_4 \cdot k_4} \left[ k_4 \cdot C_4^{(2)} \cdot F_3 \cdot F_2 \cdot F_1 \cdot k_4 + k_4 \cdot C_4^{(2)} \cdot F_1 \cdot F_2 \cdot F_3 \cdot k_4 - \frac{1}{2} k_4 \cdot C_4^{(2)} \cdot F_1 \cdot k_4 \text{Tr}(F_2 \cdot F_3) \right] -
\]

\[
\frac{1}{2} k_4 \cdot F_2 \cdot F_3 \cdot k_4 \text{Tr}(F_1 \cdot C_4^{(2)}) + \frac{1}{2} k_4 \cdot F_1 \cdot F_2 \cdot k_4 \text{Tr}(F_3 \cdot C_4^{(2)}) -
\]

\[
-k_4 \cdot F_1 \cdot F_2 \cdot C_4^{(2)} \cdot F_3 \cdot k_4 + k_4 \cdot F_2 \cdot F_3 \cdot C_4^{(2)} \cdot F_1 \cdot k_4 \right] +
\]

\[
\frac{C_0 e^{-\phi_0}}{k_4 \cdot k_4} \left[ k_4 \cdot B_4 \cdot F_3 \cdot F_2 \cdot F_1 \cdot k_4 + k_4 \cdot B_4 \cdot F_1 \cdot F_2 \cdot F_3 \cdot k_4 - \frac{1}{2} k_4 \cdot B_4 \cdot F_1 \cdot k_4 \text{Tr}(F_2 \cdot F_3) \right] -
\]

\[
-k_4 \cdot k_4 \text{Tr}(F_1 \cdot F_2 \cdot F_3 \cdot B_4) + \frac{1}{4} k_4 \cdot k_4 \text{Tr}(F_1 \cdot B_4) \text{Tr}(F_2 \cdot F_3) -
\]

\[
\frac{1}{2} k_4 \cdot F_2 \cdot F_3 \cdot k_4 \text{Tr}(F_1 \cdot B_4) + \frac{1}{2} k_4 \cdot F_1 \cdot F_2 \cdot k_4 \text{Tr}(F_3 \cdot B_4) -
\]

\[
-k_4 \cdot F_1 \cdot F_2 \cdot B_4 \cdot F_3 \cdot k_4 + k_4 \cdot F_2 \cdot F_3 \cdot B_4 \cdot F_1 \cdot k_4 \right]
\]

Note that the right-hand side of above equation has both massless poles and contact terms, i.e., the terms which have coefficient $k_4 \cdot k_4$ are contact terms. The contact terms have the same structure as the contact terms in the second and the third lines of \((18)\). On the other hand, the scattering amplitudes \((8)\) and \((11)\) do not have the contact terms that appear in the second and the third lines of \((18)\). Hence, the two S-duality invariants \((18)\) and \((19)\) should be combined with appropriate coefficients to cancel these undesirable contact terms, and produce the contact terms and the massless poles in the scattering amplitudes \((8)\) and \((11)\). In fact, one can write the combination of these amplitudes, i.e., $A = \mathcal{A}(C_4^{(2)}) + \mathcal{A}(B_4)$ as

\[
A \sim \frac{1}{4} \text{Tr}(\mathcal{F}_1^T \mathcal{M}_0 \mathcal{F}_2 * \mathcal{F}_3^T \mathcal{M}_0 \mathcal{B}_4) - \frac{1}{2} \frac{k_4 \cdot \mathcal{F}_1^T \mathcal{M}_0 \mathcal{F}_2 * \mathcal{F}_3^T \mathcal{M}_0 \mathcal{B}_4 \cdot k_4}{k_4 \cdot k_4} + P(1, 2, 3) \tag{20}
\]

Hence, the tree-level S-matrix element of three gauge bosons and one two-form satisfies the Ward identity corresponding to the gauge symmetries and to the S-duality. Note that there is no need to add loop or nonperturbative effects to satisfy the S-dual Ward identity. The loop amplitudes in the DBICS theory are in higher order of momentum, so the S-dual Ward identity does not allow to add them to the tree-level amplitudes.

We have also calculated the scattering amplitude of one gauge boson, two transverse scalars and one two-form. Using the fact that the transverse scalar in the abelian theory is
invariant under the S-duality, we have found that the amplitude satisfies the S-dual Ward identity.

4 Six gauge bosons amplitude

In this section we are going to show that the scattering amplitudes of only gauge bosons also satisfy the S-dual Ward identity. The first non-zero scattering amplitude of gauge bosons in the DBICS theory is the S-matrix element of four gauge bosons which has only contact terms. It has been shown in \[35\] that this amplitude can be written in manifestly S-dual invariant form as

\[
\mathcal{A} \sim \text{Tr}(\mathcal{F}_1^T \mathcal{M}_0 \mathcal{F}_2 \mathcal{F}_3^T \mathcal{M}_0 \mathcal{F}_4) + \text{Tr}(\mathcal{F}_1^T \mathcal{M}_0 \mathcal{F}_3 \mathcal{F}_2^T \mathcal{M}_0 \mathcal{F}_4) + \text{Tr}(\mathcal{F}_1^T \mathcal{M}_0 \mathcal{F}_4 \mathcal{F}_2^T \mathcal{M}_0 \mathcal{F}_3)
\]

All higher point functions have both contact terms as well as massless pole.

The S-matrix element of six gauge bosons in which we are interested in this section, is given by the following Feynman rule:

\[
\mathcal{A} = \sum_{i=1}^{10} A_i + V(F_1, F_2, F_3, F_4, F_5, F_6)
\]

where the massless poles are given as

\[
\begin{align*}
A_1 &= V^a(F_1, F_2, F_3, A) G_{ab}(A) V^b(A, F_4, F_5, F_6) \\
A_2 &= V^a(F_1, F_2, F_4, A) G_{ab}(A) V^b(A, F_3, F_5, F_6) \\
A_3 &= V^a(F_1, F_2, F_5, A) G_{ab}(A) V^b(A, F_3, F_4, F_6) \\
A_4 &= V^a(F_1, F_2, F_6, A) G_{ab}(A) V^b(A, F_3, F_4, F_5) \\
A_5 &= V^a(F_1, F_3, F_4, A) G_{ab}(A) V^b(A, F_2, F_5, F_6) \\
A_6 &= V^a(F_1, F_3, F_5, A) G_{ab}(A) V^b(A, F_2, F_4, F_6) \\
A_7 &= V^a(F_1, F_3, F_6, A) G_{ab}(A) V^b(A, F_2, F_4, F_5) \\
A_8 &= V^a(F_1, F_4, F_5, A) G_{ab}(A) V^b(A, F_2, F_3, F_6) \\
A_9 &= V^a(F_1, F_4, F_6, A) G_{ab}(A) V^b(A, F_2, F_3, F_5) \\
A_{10} &= V^a(F_1, F_5, F_6, A) G_{ab}(A) V^b(A, F_2, F_3, F_4)
\end{align*}
\]

where the vertex and propagator appear in (7), and the contact term can be read from (1) to be

\[
V(F_1, F_2, F_3, F_4, F_5, F_6) \sim \frac{e^{-3i\phi}}{12} \left[ \text{Tr}(F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6) - \frac{3}{8} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4 \cdot F_5 \cdot F_6) \\
+ \frac{1}{32} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4) \text{Tr}(F_5 \cdot F_6) \right] + P(1, 2, 3, 4, 5, 6)
\]

(23)
where \( P(1, 2, 3, 4, 5, 6) \) stands for all other permutations of 1, 2, 3, 4, 5, 6. Note that the action (11) has the coupling \( C_0 \wedge F \wedge F \), however, for constant background field \( C_0 \) it does not produce the vertex \( V(C_0, F_1, A)^6 \), so there is no other term in the amplitude (21).

Using the vertex and the propagator in (7), one can easily calculate the massless poles in (23), e.g., \( A_1 \) becomes

\[
A_1 = -\frac{e^{-3\phi_0}}{k \cdot k} \left[ k \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6 \cdot k - \frac{1}{4} \text{Tr}(F_1 \cdot F_2)k \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6 \cdot k \right. \\
- \frac{1}{4} k \cdot F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot k \text{Tr}(F_5 \cdot F_6) + \frac{1}{16} k \cdot F_3 \cdot F_4 \cdot k \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_5 \cdot F_6) \left. \right] + \cdots
\]

where \( k = k_1 + k_2 + k_3 = -(k_4 + k_5 + k_6) \), and dots represent all other terms resulting from the permutations of the labels in the vertex (7). Similar relations can be found for all other \( A_2, \ldots, A_{10} \). Since the result is in terms of field strength of the external polarization vectors, i.e., \( F_i \), the amplitude (21) satisfies the Ward identity corresponding to the abelian gauge transformation. In the next subsection, we show that it also satisfies the Ward identity corresponding to the S-duality.

### 4.1 S-dual Ward identity

The contact terms (23) and the massless poles (24) show that the amplitude (21) is independent of the background R-R scalar field \( C_0 \). Therefore, to study the S-dual Ward identity of the amplitude (21), one can set \( C_0 = 0 \) in the S-duality transformations (12) and (14). To simplify this study, we choose the \( SL(2, R) \) matrix \( \Lambda \) to be

\[
\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(25)

This simplifies the S-duality transformations (12) and (14) to

\[
F \rightarrow e^{-\phi_0} \ast F \quad ; \quad e^{-\phi_0} \rightarrow e^{\phi_0}
\]

(26)

Using the identity (17), one finds the following transformations:

\[
e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6) \rightarrow -e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdot F_5 \cdot F_6)
\]

\[
e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4 \cdot F_5 \cdot F_6) \rightarrow -e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4 \cdot F_5 \cdot F_6)
\]

\[
e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4) \text{Tr}(F_5 \cdot F_6) \rightarrow -e^{-3\phi_0} \text{Tr}(F_1 \cdot F_2) \text{Tr}(F_3 \cdot F_4) \text{Tr}(F_5 \cdot F_6)
\]

(27)

As a result, the contact term (23) is antisymmetric under the S-duality transformations (26), i.e.,

\[
V(F_1, F_2, F_3, F_4, F_5, F_6) \rightarrow -V(F_1, F_2, F_3, F_4, F_5, F_6)
\]

(28)
This is consistent with the fact that the DBI action is not invariant under the S-duality. Because of the above transformation, one expects the massless poles transform to the massless poles and twice the contact term $V(F_1, F_2, F_3, F_4, F_5, F_6)$.

The scattering amplitude (21) transforms under the S-duality (26) to

$$\tilde{A} = \sum_{i=1}^{10} *A_i - V(F_1, F_2, F_3, F_4, F_5, F_6)$$

(29)

where $*A_1$ is the same as (24) in which $F_i$ is replaced by $*F_i$, and similarly for all other $*A_2, \cdots, *A_{10}$. One may write the $*F_i$ in the amplitude (29) as $(*F_i)_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}_i$ and then use the identity (17) to remove the six volume forms, i.e., rewrite the amplitude in term of various contractions of $F_i$, and then compare the result with the original amplitude (21). In this approach, in order to use the identity (17), there are different ways to pair two volume forms. Different pairings give different expressions for the amplitude (29) in terms of $F_i$. However, they all must be identical. In fact, the expression $N_{*}F_1_{*}F_2_{*}F_3_{*}F_4_{*}F_5_{*}F_6_{*}M$ where $N, M$ are two arbitrary vectors, can be written in 15 different expressions. These identities result from 15 different paring of volume forms. Using these identities, we have found that the amplitudes (21) and (29) are identical.

Alternatively, one may use the matrix form of field strengths $F$ and $*F$

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad *F = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

(30)

to write the amplitudes (21) and (29) in terms of electric field $\vec{E}$ and magnetic field $\vec{B}$. In this way also we have found that the amplitude (21) is invariant under the S-duality (26), i.e.,

$$\tilde{A} = A$$

(31)

This ends our illustration of the fact that the S-matrix element of six gauge bosons satisfies the Ward identity corresponding to the S-duality. We have also calculated the S-matrix element of two gauge bosons and four transverse scalars and found that the amplitude satisfies the Ward identity corresponding to the S-duality transformations (26).

Acknowledgments: H.B-A would like to thank D. Mahdavian Yekta for useful discussions.

References

[1] A. Font, L. E. Ibanez, D. Lust and F. Quevedo, Phys. Lett. B 249, 35 (1990).
[2] S. J. Rey, Phys. Rev. D 43, 526 (1991).
[3] A. Sen, Int. J. Mod. Phys. A 9, 3707 (1994) [arXiv:hep-th/9402002].
[4] A. Sen, Phys. Lett. B 329, 217 (1994) [arXiv:hep-th/9402032].
[5] J. H. Schwarz, [arXiv:hep-th/9307121].
[6] C. M. Hull and P. K. Townsend, Nucl. Phys. B 438, 109 (1995) [arXiv:hep-th/9410167].
[7] M. R. Garousi, JHEP 1111, 016 (2011) [arXiv:1106.1714 [hep-th]].
[8] M. B. Green and M. Gutperle, Nucl. Phys. B 498, 195 (1997) [arXiv:hep-th/9701093].
[9] M. B. Green and P. Vanhove, Phys. Lett. B 408, 122 (1997) [arXiv:hep-th/9704145].
[10] M. B. Green, M. Gutperle and P. Vanhove, Phys. Lett. B 409, 177 (1997) [arXiv:hep-th/9706175].
[11] E. Kiritsis and B. Pioline, Nucl. Phys. B 508, 509 (1997) [arXiv:hep-th/9707018].
[12] M. B. Green, M. Gutperle and H. h. Kwon, Phys. Lett. B 421, 149 (1998) [arXiv:hep-th/9710151].
[13] B. Pioline, Phys. Lett. B 431, 73 (1998) [arXiv:hep-th/9804023].
[14] M. B. Green and S. Sethi, Phys. Rev. D 59, 046006 (1999) [arXiv:hep-th/9808061].
[15] M. B. Green, H. h. Kwon and P. Vanhove, Phys. Rev. D 61, 104010 (2000) [arXiv:hep-th/9910055].
[16] N. A. Obers and B. Pioline, Class. Quant. Grav. 17, 1215 (2000) [arXiv:hep-th/9910115].
[17] A. Sinha, JHEP 0208, 017 (2002) [arXiv:hep-th/0207070].
[18] N. Berkovits, JHEP 0409, 047 (2004) [arXiv:hep-th/0406055].
[19] E. D’Hoker and D. H. Phong, Nucl. Phys. B 715, 3 (2005) [arXiv:hep-th/0501197].
[20] E. D’Hoker, M. Gutperle and D. H. Phong, Nucl. Phys. B 722, 81 (2005) [arXiv:hep-th/0503180].
[21] M. B. Green and P. Vanhove, JHEP 0601, 093 (2006) [arXiv:hep-th/0510027].
[22] M. B. Green, J. G. Russo and P. Vanhove, JHEP 0702, 099 (2007) [arXiv:hep-th/0610299].
[23] A. Basu, Phys. Rev. D 77, 106003 (2008) [arXiv:0708.2950 [hep-th]].
[24] A. Basu, Phys. Rev. D 77, 106004 (2008) [arXiv:0712.1252 [hep-th]].

[25] C. P. Bachas, P. Bain and M. B. Green, JHEP 9905, 011 (1999) [arXiv:hep-th/9903210].

[26] A. Basu, JHEP 0809, 124 (2008) [arXiv:0808.2060 [hep-th]].

[27] M. R. Garousi, Phys. Lett. B 701, 465 (2011) [arXiv:1103.3121 [hep-th]].

[28] R. G. Leigh, Mod. Phys. Lett. A 4, 2767 (1989).

[29] C. Bachas, Phys. Lett. B 374, 37 (1996) [arXiv:hep-th/9511043].

[30] J. Polchinski, Phys. Rev. Lett. 75, 4724 (1995) [arXiv:hep-th/9510017].

[31] M. R. Douglas, [arXiv:hep-th/9512077].

[32] G. W. Gibbons and D. A. Rasheed, Phys. Lett. B 365, 46 (1996) [arXiv:hep-th/9509141].

[33] M. Cederwall and A. Westerberg, JHEP 9802 (1998) 004 [arXiv:hep-th/9710007].

[34] E. A. Bergshoeff, M. de Roo, S. F. Kerstan, T. Ortin and F. Ricciioni, JHEP 0702 (2007) 007 [arXiv:hep-th/0611036].

[35] M. R. Garousi, Phys. Rev. D 84, 126019 (2011) [arXiv:1108.4782 [hep-th]].

[36] A. A. Tseytlin, Nucl. Phys. B 469, 51 (1996) [arXiv:hep-th/9602064].

[37] M. B. Green and M. Gutperle, Phys. Lett. B 377, 28 (1996) [arXiv:hep-th/9602077].