Convergence of the logarithm of the characteristic polynomial of unitary Brownian motion to the Gaussian free field

Johannes Forkel, Isao Sauzedde

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Abstract

We prove that the real and imaginary parts of the logarithm of the characteristic polynomial of unitary Brownian motion converge to Gaussian free fields on the cylinder, as the matrix dimension goes to infinity. This is the natural dynamical analogue of the result for a fixed time by Hughes, Keating and O’Connell. Further it complements a result by Spohn on linear statistics of unitary Brownian motion, and a recent result by Bourgade and Falconet connecting the characteristic polynomial of unitary Brownian motion to a Gaussian multiplicative chaos measure. In the course of this research we also proved a Wick-type identity, which we include in this paper, as it might be of independent interest.

1 Introduction

As unitary Brownian motion preserves the Haar measure on the unitary group $U(n)$, to many results of Haar distributed unitary matrices there is a corresponding dynamical result for a unitary Brownian motion $U$ at equilibrium. This is in particular the case for some properties of the eigenvalues, whose dynamics have been studied first by Dyson, who computed a stochastic differential equation describing their evolution. In this paper, we intend to achieve such a transition from static to dynamic for the Hughes-Keating-O’Connell theorem on the large $n$ limit of the logarithm $\log p_n$ of the characteristic polynomial.

Characteristic polynomials of random matrices are fundamental objects in random matrix theory. They are closely related to the theory of log-correlated fields and to Gaussian multiplicative chaos. In the case of Haar-distributed matrices from the classical compact groups, there are also remarkable similarities between the statistics of the characteristic polynomial and those of the Riemann zeta function and other number-theoretic $L$-functions, which led to a number of very precise conjectures for those $L$-functions - see for a review.

The logarithm $\log |p_n|$ of the absolute value of the characteristic polynomial also enters the wide family of linear statistics of the eigenvalues $\lambda_1, \ldots, \lambda_n$, that is functions that can be expressed as $\sum_{i=1}^n f(\lambda_i)$. This family has received much attention already, both in the static and dynamical frameworks. These results allows, at an informal level, to identify the large $n$ limit of $\log |p_n|$ as the Gaussian free field. However, they assume too much regularity on $f$ to be applicable to $\log |p_n|$, and indeed, their conclusions do not hold for $\log |p_n|$, for the type of convergence they use is too strong.

In a recent paper, Bourgade and Falconet proved that $|p_n|^\alpha$, for certain $\alpha$ and when properly normalized, converge to a Gaussian multiplicative chaos measure associated to the Gaussian free field $h$ on the cylinder, i.e. informally the exponential of a multiple of $h$. This suggests again, but
as far as we know doesn’t actually imply, that \(\log|p_n|\) converges itself towards the Gaussian free field \(h\), in some appropriate function space. The goal of this paper is precisely to specify a Sobolev space in which we prove that this convergence holds. This is the natural dynamical version of the corresponding stationary result for Haar-distributed unitary matrices by Hughes, Keating and O’Connell [1], who proved that for any fixed time the logarithm of the characteristic polynomial converges to a a generalized Gaussian field on the unit circle.

In the last section, we state and prove an identity that allows to express the second moment of the trace of arbitrary products of a GUE matrix \(H\) and an independent CUE matrix \(U\) in terms of moments of \(U\) only. When the dimension \(n\) is large enough, the Diaconis-Shahshahani theorem on moments of traces of unitary matrices [13] allows to then compute this new expression explicitly as a polynomial in \(n\).

1.1 Context

We let \(U_n : [0, \infty) \to U(n)\) be a unitary Brownian motion started from Haar measure (for a precise definition see Section 2.1), and define its characteristic polynomial as

\[
p_n(t, \theta) := \det \left( I_n - e^{-idt} U_n(t) \right) = \prod_{k=1}^n \left( 1 - e^{i(\theta_k(t) - \theta)} \right), \quad (\theta, t) \in [0, 2\pi) \times [0, \infty),
\]

where \(0 \leq \theta_1(t) < \ldots < \theta_n(t) < 2\pi\) denote the eigenangles of unitary Brownian motion. We define its logarithm by

\[
\log p_n(t, \theta) := \sum_{k=1}^n \log(1 - e^{i(\theta_k(t) - \theta)}),
\]

with the branches on the RHS being the principal branches, such that

\[
\Im \log(1 - e^{i(\theta_k(t) - \theta)}) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right],
\]

with \(\Im 0 := \pi/2\).

Hughes, Keating and O’Connell proved that for any fixed time \(t \geq 0\), \(\log p_n(t, \cdot)\) converges to a generalized Gaussian field. Their result, reformulated to our setting, is as follows:

**Theorem 1.1** (Hughes, Keating, O’Connell [1]). For any \(\epsilon > 0\) and any fixed \(t \geq 0\), the sequence of random functions \((\log p_n(t, \cdot))_{n \in \mathbb{N}}\) converges in distribution in \(H^{-\epsilon}_0(S^1)\) to the generalized Gaussian field

\[
X(\theta) = \sum_{k=1}^{\infty} \frac{A_k}{\sqrt{k}} e^{ik\theta},
\]

where \(A_k\) is a complex Gaussian whose real and imaginary parts are independent centered Gaussians with variance \(1/(2k)\).

It is natural thus to assume that in the dynamic case, i.e. when considering \(\log p_n\) also as a function of \(t\), that the limit (in an appropriate function space) would be given by

\[
X(t, \theta) = \sum_{k=1}^{\infty} \frac{A_k(t)}{\sqrt{k}} e^{ik\theta}, \quad (1)
\]

where \(A_k(\cdot), k \in \mathbb{N}\), are independent complex Ornstein-Uhlenbeck processes started from their stationary distribution, i.e. (up to a linear time change) solutions to the SDEs

\[
dA_k(t) = -k A_k(t) dt + d \left( W_k(t) + i\tilde{W}_k(t) \right), \quad (2)
\]
with $A_k(0)$ being a complex Gaussian whose real and imaginary parts are independent Gaussians with variance $1/(2k)$, and $(W_k(t))_{t \geq 0}, (\dot{W}_k(t))_{t \geq 0}, k \in \mathbb{N}$, denoting real standard Brownian motions.

Our main result proves precisely that (for a definition of the Sobolev spaces $H^s([0,T])$ and $H_0^s(S^1)$ see Section 2.2):

**Theorem 1.2** (Main Result). For any $s \in (0, \frac{1}{2})$, $\epsilon > s$ and $T > 0$, the sequence of random fields $(\log p_n(\cdot, \cdot))_{n \in \mathbb{N}}$ converges in distribution in the tensor product of Hilbert spaces $H^s([0,T]) \otimes H_0^s(S^1)$ to the generalized Gaussian field $X$ in $[7]$.

Since we consider unitary Brownian motion and Ornstein-Uhlenbeck processes in their equilibrium, they are all reversible, and can thus be defined for all $t \in \mathbb{R}$. In particular we can extend $p_n(t, \theta)$ and $X(t, \theta)$ to the infinite cylinder $C := \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$. By calculating the covariance functions of $\Re X$ and $\Im X$, one can see that both $\Re X$ and $\Im X$ are the Gaussian free field on $C$, i.e.

$$E(\Re X(t, \theta), \Re X(t', \theta')) = E(\Im X(t, \theta), \Im X(t', \theta')) = \pi \frac{1}{2} (\Delta_C)^{-1}(t, \theta, t', \theta') = \frac{1}{2} \log \frac{\max(e^{-t}, e^{-t'})}{|e^{-t+\theta} - e^{-t'+\theta}|},$$

where $\Delta_C = \partial_t^2 + \partial_\theta^2$ and $(-\Delta_C)^{-1}$ denotes the Green’s function. For an explicit calculation see Section 2.2 in [3]. Thus a direct consequence of Theorem 1.2 is that both $(\log p_n)_{n \in \mathbb{N}}$ and $(\Im \log p_n)_{n \in \mathbb{N}}$ converge to the Gaussian free field on the finite cylinder $[0, T] \times \mathbb{R}/2\pi \mathbb{Z}$.

**Remark 1.3.** This result shows that there is a trade-off between regularity in $\theta$ and regularity in $t$. We believe that the regularity we obtain is optimal, in the sense that for $s = 1/2$ or $\epsilon = s$, $X$ is almost surely not an element of the tensor product of $H^s([0,T]) \otimes H_0^s(S^1)$ anymore.

While the limiting field is rotationally invariant from an infinitesimal point of view, this is not the case for $\log p_n$ with finite $n$. In particular, one can exchange the regularity in the variable $t$ with the regularity in the variable $\theta$ for the limiting field, but for our proof of convergence to work, the Sobolev regularity $-\epsilon$ in the variable $\theta$ needs to be negative which is not the case for the Sobolev regularity $s$ in the variable $t$.

Just like in the stationary case, the Gaussian field $X$ can’t be defined pointwise as its variance at each point is infinite, but it can still be "exponentiated" to build a Gaussian multiplicative chaos (GMC) measure. When we let $h(t, \theta)$ denote the real part of $X(t, \theta)$, and denote by $h_\delta(t, \theta)$ a mollification of $h$, then for $\gamma \in (0, 2\sqrt{2})$ the random measures

$$e^{\gamma h(t, \theta)} d\theta dt := \lim_{\delta \to 0} e^{\gamma h_\delta(t, \theta)} - \frac{\Delta^2}{2} e^{\gamma h_\delta(t, \theta)} d\theta dt$$

exist and are non-trivial, where the limit is in probability w.r.t. the topology of weak convergence of measures on $\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$, see [13] for a self-contained proof of this fact. Bourgade and Falconet proved that exponentiating $\log p_n(t, \theta)$ in this way, and then taking the large $n$ limit, gives the same limiting measure as when first taking the large $n$ limit to obtain the Gaussian free field $h$, and then exponentiating it. Their result is the dynamical analogue to Webb’s result for fixed $t$ and the measures being on the unit circle [17], and its precise statement is as follows:

**Theorem 1.4** (Bourgade, Falconet [3]). For every $\gamma \in (0, 2\sqrt{2})$ it holds that

$$\lim_{n \to \infty} \frac{|p_n(t, \theta)|^\gamma}{E(|p_n(t, \theta)|^\gamma)} d\theta dt = e^{\gamma h(t, \theta)} d\theta dt,$$

where the convergence is in distribution in the space of Radon measures on the infinite cylinder $\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$, equipped with the topology of weak convergence.
Our main result complements this result in that it also shows the convergence of the underlying fields \(\log |p_n|\) to the Gaussian free field \(h\).

Further, our result is related to the below result by Spohn, which we also use in our proof. For real-valued functions \(f \in H_0^{3/2+\epsilon}(S^1, \mathbb{R})\), \(\epsilon > 0\), Spohn considered linear statistics of the eigenvalues \(e^{i\theta_1(t)}, ..., e^{i\theta_n(t)}\) of unitary Brownian motion (in fact he more generally considered interacting particles on the unit circle with different repulsion strengths):

\[
\xi_n(t, f) := \sum_{j=1}^{n} f(e^{i\theta_j(t)}), \quad (t, f) \in [0, \infty) \times H_0^{3/2+\epsilon}(S^1, \mathbb{R}).
\]

Since \(H_0^{-3/2-\epsilon}(S^1, \mathbb{R})\) is the dual space of \(H_0^{3/2+\epsilon}(S^1, \mathbb{R})\), one can consider \(\xi_n\) as a random continuous map \(t \mapsto \xi_n(t, \cdot) \in H_0^{-3/2-\epsilon}(S^1, \mathbb{R})\).

**Theorem 1.5** (Spohn \[2\]). For any \(\epsilon > 0\), as \(n \to \infty\), \(\xi_n(t, f)\) converges to a stationary solution of the SDE

\[
\mathrm{d}\xi(t, f) = \xi(t, -\sqrt{-\partial_x^2}f) \, \mathrm{d}t + \mathrm{d}W(t, f'),
\]

where \( \mathrm{d}W \) is a white noise given by

\[
\mathbb{E}[\mathrm{d}W(t, f) \, \mathrm{d}W(s, g)] = 2\delta(t-s) \, \mathrm{d}s \, \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})g(e^{i\theta}) \, \mathrm{d}\theta,
\]

and where the convergence is in distribution in \(\mathcal{C}(\mathbb{R}, H^{-3/2-\epsilon}(S^1, \mathbb{R}))\), endowed with the topology of locally uniform convergence. The stationary distribution is given by a Gaussian with covariance

\[
\mathbb{E}(\xi(t, f)\xi(t, g)) = \sum_{k \neq 0} |k| f_k g_k.
\]

**Remark 1.6.** Here, \(\sqrt{-\partial_x^2}f\) is simply the function whose \(j\)th Fourier coefficient is \(|j|\) times the \(j\)th Fourier coefficient of \(f\).

We see that

\[
\log |p_n(t, x)| = \xi_n(t, f_x),
\]

where \(f_x(\theta) := \log |e^{i\theta} - e^{ix}| = -\Re \sum_{k=1}^{\infty} \frac{1}{k} e^{ik\theta} e^{-ikx} = -\sum_{k=1}^{\infty} \frac{1}{k} \cos(k(\theta - x)).\) Note that Theorem 1.5 can not be applied to obtain the limiting dynamics of \(\xi_n(t, f_x)\) for \(f_x, x \in [0, 2\pi)\), as these functions are not continuous and thus not in \(H^{3/2+}(S^1, \mathbb{R})\). However, since

\[
\sqrt{-\partial_x^2} f_x(\theta) = \sqrt{-\partial_x^2} f_x(\theta),
\]

\[
\mathbb{E}(W(f_x(t), W(f_y(t))) = 2 \frac{1}{2\pi} \int_0^{2\pi} f_x'(\theta)f_y'(\theta) \, \mathrm{d}\theta = \pi \delta(x-y),
\]

when we formally apply Theorem 1.5 to the functions \(f_x\), we expect that the dynamics of \(\tilde{h}(t, x) := \lim_{n \to \infty} \xi_n(t, f_x)\) should be given by the SPDE

\[
\mathrm{d}\tilde{h}(t, x) = -\sqrt{-\partial_x^2} \tilde{h}(t, x) \, \mathrm{d}t + \sqrt{\pi} \, \mathrm{d}W(t, \, \mathrm{d}x),
\]

where \(W( \mathrm{d}t, \, \mathrm{d}x)\) is a space-time white noise on the unit circle.

As was observed in Section 2.2 of \[3\], \(W( \mathrm{d}t, \, \mathrm{d}\theta)\) can be realized as \(\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{\sqrt{2\pi}} \, \mathrm{d}V_k(t) + \frac{\sin(k\theta)}{\sqrt{2\pi}} \, \mathrm{d}\tilde{V}_k(t)\), for independent real standard Brownian motions \(V_k, \tilde{V}_k, k \in \mathbb{N}\), which implies that

\[
\tilde{h}(t, \theta) = \sum_{k=1}^{\infty} B_k(t) \cos(k\theta) + \tilde{B}_k(t) \sin(k\theta),
\]
for independent real stationary Ornstein-Uhlenbeck processes \( B_k, C_k, k \in \mathbb{N} \), satisfying the SDEs
\[
\begin{align*}
    dB_k(t) &= -kB_k(t) \, dt + dV_k(t), \\
    d\tilde{B}_k(t) &= -k \tilde{B}_k(t) \, dt + d\tilde{V}_k(t).
\end{align*}
\]

One can see now that \( \tilde{h}(t, \theta) = \Re X(t, \theta) = h(t, \theta) \), which shows that our main result Theorem 1.2 confirms the limiting dynamics of \( p_n(t, \theta) \) expected from Spohn’s result.

To prove our main result we will need the following result from Bourgade and Falconet \[3, Corollary 3.2\]:

**Corollary 1.7** (Bourgade, Falconet). Let \( (z_1(t), \ldots, z_n(t))_{t \geq 0} \) denote the eigenvalue process of unitary Brownian motion, started at Haar measure, and denote \( \text{sgn}(x) = 1_{x>0} - 1_{x<0} \). For \( f, g \in H^1_0(S^1, \mathbb{R}) \), we have for every \( n \in \mathbb{N} \) and \( t \geq 0 \),
\[
\mathbb{E}
\left[
    \left|
        \sum_{j=1}^n f(z_j(0))
    \right| \left|
        \sum_{j=1}^n g(z_j(t))
    \right|
\right]
= \sum_{|k| \leq n-1} f_k g_{-k} \text{sgn}(k) e^{-|k|t} \frac{\sinh(k^2 t)}{\sinh(k \pi)} + \sum_{|k| \geq n} f_k g_{-k} e^{-\frac{k^2}{8} t} \frac{\sinh(k t)}{\sinh(k \pi)}
\]

## 2 Mathematical Preliminaries

### 2.1 Unitary Brownian motion

Brownian motion \((U_n(t))_{t \geq 0}\) on the unitary group \( U(n) \) is the diffusion governed by the stochastic differential equation
\[
dU_n(t) = \sqrt{2} dV_n(t) dB_n(t) - U_n(t) dt,
\]
with \((B_n(t))_{t \geq 0}\) denoting a Brownian motion on the space of skew-Hermitian matrices. That is
\[
B_n(t) = \sum_{k=1}^{n^2} X_k \tilde{B}^{(k)}(t),
\]
where \( \tilde{B}^{(k)} \), \( k = 1, \ldots, n^2 \), are independent one-dimensional standard Brownian motions, and where the matrices \( X_k \), \( k = 1, \ldots, n^2 \), are an orthonormal basis of the real vector space of skew-Hermitian matrices w.r.t. the scalar product \( \langle A, B \rangle := n \text{Tr}(AB^*) \). One such basis is given by the matrices \( \frac{1}{\sqrt{2n}} (E_{k,l} - E_{l,k}) \), \( \frac{1}{\sqrt{2n}} (E_{k,l} + E_{l,k}) \), \( 1 \leq k < l \leq n \), and \( \frac{1}{\sqrt{n}} E_{k,k}, 1 \leq k \leq n \).

**Remark 2.1.** Unitary Brownian motion is usually defined using a different normalisation, i.e. satisfying the SDE \( d\hat{U}_n(t) = \hat{U}_n(t) \, dB_n(t) - \frac{1}{2} \hat{U}_n(t) \). With this normalisation the generator is given by one half times the Laplacian on \( U(n) \), which is the usual definition of Brownian motion on a Riemannian manifold. The relation between the two normalisations is \( \hat{U}_n(2t) = U_n(t) \).

In this paper we always consider unitary Brownian motion started from Haar measure on \( U(n) \), which is its stationary distribution. Thus \( U_n(t) \) is Haar distributed for all \( t \geq 0 \).

### 2.2 Sobolov spaces and their Tensor Product

Consider the space of square integrable \( \mathbb{C} \)-valued functions on the unit circle, with vanishing mean:
\[
L^2_0(S^1) = \left\{ f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta} : \sum_{k \in \mathbb{Z}} |f_k|^2 < \infty, f_0 = 0 \right\}.
\]

For \( s \geq 0 \), we define \( H^s_0(S^1) \) as the restriction of \( L^2_0(S^1) \) w.r.t. the functions for which the inner product
\[
\langle f, g \rangle_s = \sum_{k \in \mathbb{Z}} |k|^{2s} f_k \overline{g_k}
\]

is finite. For \( s \leq 0 \), we define \( H^s_0(S^1) \) as the completion of \( L^2_0(S^1) \) w.r.t. this scalar product. Note that \((H^s_0(S^1), \langle \cdot , \cdot \rangle_s)\) is a Hilbert space for all \( s \in \mathbb{R} \). For \( s \geq 0 \) it is a subspace of \( H^s_0(S^1) = L^2_0(S^1) \), i.e. the space of square-integrable functions with zero mean, while for \( s < 0 \), \( H^s_0(S^1) \) can be interpreted as the dual space of \( H^{-s}_0(S^1) \), i.e. as a space of generalized functions defined up to additive constant.

For \( T > 0 \), and \( s \in (0, 1) \), we define the fractional Sobolev space \( H^s([0, T]) \) as the subspace of \( L^2([0, T]) \), where the Slobodeckij inner product

\[
\langle f, g \rangle_s := \int_0^T f(t)g(t)\,dt + \int_0^T \int_0^T \frac{(f(t) - f(u))(g(t) - g(u))}{|t-u|^{1+2s}} \, du \, dt
\]

is finite. Note that \((H^s([0, T]), \langle \cdot , \cdot \rangle_s)\) is a Hilbert space for all \( s > 0 \).

**Remark 2.2.** For the fact that the fractional Sobolev spaces defined through Fourier series or through the Slobodeckij norm agree, the reader can consult e.g. [19].

For \( s > 0 \) and \( \epsilon > 0 \) we let \( H^s([0, T]) \otimes H^{-s}_0(S^1) \) denote the tensor product of Hilbert spaces \( H^s([0, T]) \) and \( H^{-s}_0(S^1) \). Since the inner product on that space is determined by

\[
\langle f \otimes h, g \otimes k \rangle_{s,-\epsilon} = \langle f, h \rangle_s \langle g, k \rangle_{-\epsilon} - \epsilon
\]

\[
= \int_0^T \langle f(t)h(t) \rangle_{s,-\epsilon} \, dt + \int_0^T \int_0^T \frac{(f(t) - f(u))(h(t) - h(u))}{|t-u|^{1+2s}} \, du \, dt \langle g, k \rangle_{-\epsilon},
\]

we obtain

\[
\langle F, G \rangle_{s,-\epsilon} = \int_0^T \langle F(t, \cdot), G(t, \cdot) \rangle_{s,-\epsilon} \, dt + \int_0^T \int_0^T \frac{(F(t, \cdot) - F(u, \cdot))(G(t, \cdot) - G(u, \cdot))}{|t-u|^{1+2s}} \, du \, dt,
\]

first when \( F \) and \( G \) are linear combinations of pure tensor products, and then for all \( F, G \in H^s([0, T]) \otimes H^{-s}_0(S^1) \) by density and continuity.

### 3 Proof of the main result Theorem 1.2

The proof strategy is as in the stationary case in [1]: we treat \((\log p_n)_{n \in \mathbb{N}}\) as a sequence in \( H^s([0, T]) \otimes H^{-s}_0(S^1) \), and show that if any of its subsequences has a limit then that limit has to be \( X \). We do this by showing that the finite-dimensional distributions of \((\log p_n)_{n \in \mathbb{N}}\), i.e. the distributions of finite sets of Fourier coefficients at a finite number of times, converge to those of \( X \). We then show that the set \((\log p_n)_{n \in \mathbb{N}}\) is tight in \( H^s([0, T]) \otimes H^{-s}_0(S^1) \). Since \( H^s([0, T]) \otimes H^{-s}_0(S^1) \) is complete and separable, Prokhorov’s theorem implies that the closure of \((\log p_n)_{n \in \mathbb{N}}\) is sequentially compact w.r.t. the topology of weak convergence. In particular this means that every subsequence of \((\log p_n)_{n \in \mathbb{N}}\) has a weak limit \( H^s([0, T]) \otimes H^{-s}_0(S^1) \). Since any such limit has to be \( X \) it follows that the whole sequence \((\log p_n)_{n \in \mathbb{N}}\) must converge weakly to \( X \).

We recall that

\[
\log(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}
\]

for \(|z| \leq 1\), where for \( z = 1 \) both sides equal \(-\infty\). By using the identity \( \log \det = \text{Tr} \log \) we see that the Fourier expansion of \( \log p_n \) w.r.t. the spacial variable \( \theta \) is given as follows:

\[
\log p_n(t, \theta) = -\sum_{k=1}^{\infty} \frac{\text{Tr}(U_k^n(t))}{k} e^{-ik\theta}.
\]
Lemma 3.1. Let \( \{(\log p_n)_k(t)\}_{k \geq 1} \) be the Fourier coefficients of \( (\log p_n)(t, \cdot) \). The process \( (t, k) \mapsto (\log p_n)_k(t) \) converges in finite-dimensional distributions towards the complex Ornstein-Uhlenbeck process \( (t, k) \mapsto A_k(t) \) defined in (3).

Proof. We prove convergence of the finite-dimensional distributions by showing that for any \( l \in \mathbb{N} \) and \( 0 \leq t_1 < t_2 < \ldots < t_l \leq T \), as \( n \to \infty \):

\[
\left( (\log p_n)_1(t_1), \ldots, (\log p_n)_1(t_2), \ldots, (\log p_n)_1(t_l), \ldots, (\log p_n)_l(t_1), \ldots, (\log p_n)_l(t_l) \right) \xrightarrow{d} \left( A_1(t_1), \ldots, A_1(t_2), \ldots, A_l(t_1), \ldots, A_l(t_l) \right).
\]

Let \( \epsilon_k : \theta \mapsto e^{ik\theta} \). Then, using the notations of Theorem 1.5, \( \log(p_n)_k(t) = \xi_n \left( t, \frac{s}{n} \right) \). Thus Spohn’s theorem, combined with the continuous mapping theorem with the appropriate continuous map \( C(\mathbb{R}, H^{-3/2-\epsilon}(S^1, \mathbb{R})) \to \mathbb{R}^l \), implies that

\[
\left( (\log p_n)_1(t_1), \ldots, (\log p_n)_1(t_2), \ldots, (\log p_n)_1(t_l), \ldots, (\log p_n)_l(t_1), \ldots, (\log p_n)_l(t_l) \right) \xrightarrow{d} \left( \xi(t_1, \frac{s_1}{n}), \ldots, \xi(t_1, \frac{s_2}{n}), \ldots, \xi(t_2, \frac{s_2}{n}), \ldots, \xi(t_l, \frac{s_2}{n}), \ldots, \xi(t_l, \frac{s_2}{n}) \right).
\]

Combining the real and imaginary part of \( \epsilon_k \), we obtain that the SDE for \( \xi(\cdot, \frac{s}{n}) \) reduces to

\[
d\xi \left( t, \frac{\epsilon_k}{k} \right) = -k\xi \left( t, \frac{\epsilon_k}{k} \right) \, dt + dB_k(t),
\]

where \( B_k \) is a complex Brownian motion, i.e. a process whose real and imaginary parts are independent standard Brownian motions. Besides, the Brownian motions \( (B_k)_{k \geq 0} \) are independent, so that \( (A_k)_{k \geq 1} \) and \( (\xi(\cdot, \frac{s}{n}))_{k \geq 1} \) are equal in distribution, which concludes the proof.

We proceed to show tightness of \( (\log p_n)_{n \in \mathbb{N}} \) in \( H^s([0, T]) \otimes H_0^{-\epsilon}(S^1) \), i.e. for every \( \delta > 0 \) we construct a compact \( K_\delta \subset H^s([0, T]) \otimes H_0^{-\epsilon}(S^1) \) for which

\[
\sup_{n \in \mathbb{N}} \mathbb{P} ((\log p_n) \in K_\delta) < \delta.
\]

We let \( 0 < s' < \epsilon' \) such that \( 0 < s < s' < \epsilon' < \epsilon \), and choose

\[
K_\delta = \left\{ F \in H^s([0, T]) \otimes H_0^{-\epsilon}(S^1) : ||F||_{s', -\epsilon'}^2 \leq C_\delta \right\},
\]

for a \( C_\delta \) depending on \( \delta \). By Lemma 3.2 below we see that \( K_\delta \) is compact in \( H^s([0, T]) \otimes H_0^{-\epsilon}(S^1) \), and by Lemma 3.3 below we see that \( \sup_{n \in \mathbb{N}} \mathbb{E} ( ||\log p_n||_{s', -\epsilon'}^2 ) < \infty \). Thus, when choosing \( C_\delta \) large enough, we see that

\[
\sup_{n \in \mathbb{N}} \mathbb{P} ((\log p_n) \in K_\delta) = \sup_{n \in \mathbb{N}} \mathbb{P} ( ||\log p_n||_{s', -\epsilon'}^2 > C_\delta ) \leq \frac{\sup_{n \in \mathbb{N}} \mathbb{E} ( ||\log p_n||_{s', -\epsilon'}^2 )}{C_\delta^2} < \delta,
\]

which shows tightness of \( \log p_n \) and thus together with Lemma 3.1 proves our Theorem 1.2.

Lemma 3.2. Let \( 0 < s < s' < \epsilon' < \epsilon \). Then, the inclusion of \( H^{s'}([0, T]) \otimes H_0^{-\epsilon'}(S^1) \) into \( H^s([0, T]) \otimes H_0^{-\epsilon}(S^1) \) is compact.

Proof. From the Kondrachov embedding theorem, the inclusion \( \iota_1 \) of \( H^{s'}([0, T]) \) into \( H^s([0, T]) \) is compact, as well as the inclusion \( \iota_2 \) from \( H_0^{s'}(S^1) \) into \( H_0^s(S^1) \). Then the dual operator \( \iota_2^* : H_0^{-\epsilon'}(S^1) \to H_0^{-\epsilon}(S^1) \) is also compact. On Hilbert spaces, the tensor product of two compact operators is also compact (see e.g. [20]), so that \( \iota_1 \otimes \iota_2^* \) is compact indeed.

\footnote{In [20], the result is stated for endomorphisms, but this extra assumption is not used in the proof.}
Lemma 3.3. For all \( s \in (0, \frac{1}{2}) \) and all \( \epsilon > s \), it holds that \( \sup_{n \in \mathbb{N}} \mathbb{E} \left( || \log p_n ||^2_{\epsilon, \epsilon} \right) < \infty \).

Proof: We see that

\[
\mathbb{E} \left( || \log p_n ||^2_{\epsilon, \epsilon} \right) = \mathbb{E} \left( \int_0^T || \log p_n (\cdot, t) ||^2_{\epsilon} dt \right) + \mathbb{E} \left( \int_0^T \int_0^T \frac{|| \log p_n (\cdot, t) - \log p_n (\cdot, r) ||^2_{\epsilon}}{|t - r|^{2s + 1}} dr dt \right).
\]

For the first summand it holds that (with \( k \wedge n \) denoting \( \min \{k, n\} \))

\[
\mathbb{E} \left( \int_0^T || \log p_n (\cdot, t) ||^2_{\epsilon} dt \right) = \int_0^T \mathbb{E} \left( \sum_{k=1}^{\infty} k^{-2-2\epsilon} \frac{\text{Tr}(U_n(t)^k)}{k^2} \right) dt = T \sum_{k=1}^{\infty} k^{-2-2\epsilon} \mathbb{E} \left( |\text{Tr}(U_n(0)^k)|^2 \right)
\]

\[
< T \sum_{k=1}^{\infty} k^{-1-2\epsilon} < \infty.
\]

For the second summand it holds that:

\[
\mathbb{E} \left( \int_0^T \int_0^T \frac{|| \log p_n (\cdot, t) - \log p_n (\cdot, r) ||^2_{\epsilon}}{|t - r|^{2s + 1}} dr dt \right)
\]

\[
= \sum_{k=1}^{\infty} k^{-2-2\epsilon} \int_0^T \int_0^T \mathbb{E} \left( \frac{|| \text{Tr}(U_n(t)^k) - U_n(r)^k ||^2}{|t - r|^{2s + 1}} \right) dr dt
\]

\[
\leq CT \sum_{k=1}^{\infty} k^{-2-2\epsilon} \int_0^T \mathbb{E} \left( \frac{|| \text{Tr}(U_n(t)^k) - U_n(0)^k ||^2}{t^{2s + 1}} \right) dt
\]

\[
\leq CT \sum_{k=1}^{\infty} k^{-2-2\epsilon} \int_0^{k-1} \mathbb{E} \left( \frac{|| \text{Tr}(U_n(t)^k) - U_n(0)^k ||^2}{t^{2s + 1}} \right) dt + CT \sum_{k=1}^{\infty} k^{-2-2\epsilon} \int_{k-1}^{\infty} \frac{4\mathbb{E} \left( || \text{Tr}(U_n(0)^k) ||^2 \right)}{t^{2s + 1}} dt.
\]

For the second summand in (3) we get

\[
\int_{k-1}^{\infty} \frac{4\mathbb{E} \left( || \text{Tr}(U_n(0)^k) ||^2 \right)}{t^{2s + 1}} dt = 8s(n \wedge k) k^{2s},
\]

which is sufficient since \( \sum_{k=1}^{\infty} k^{-2-2\epsilon + 1 + 2s} \) is finite as soon as \( s < \epsilon \).

For the first sum in (3) we use Corollary 1.7 which implies that for all \( k \geq 1 \)

\[
\mathbb{E} \left( \text{Tr}(U_n(t)^k) \text{Tr}(U_n(0)^k) \right) = \mathbb{E} \left( \text{Tr}(U_n(0)^k) \right) = 1_{k < n} e^{-kt} \sinh \left( \frac{k^2 t}{n} \right) \sinh \left( \frac{2t}{n} \right) + 1_{k \geq n} e^{-\frac{k^2 t}{n}} \sinh (kt) \sinh \left( \frac{2t}{n} \right)
\]

\[
= e^{-\frac{k(k \wedge n)}{n}} \sinh \left( \frac{k(k \wedge n)}{n} \right) \sinh \left( \frac{2t}{n} \right),
\]

with \( k \wedge n := \max\{k, n\} \). Using this, and the fact that \( \sinh x \geq x \) and \( 1/\sinh x \geq 1/x - x/6 \) for
all $x > 0$, we see that for $t < k^{-1}$:

$$
\mathbb{E} \left[ \left| \operatorname{Tr}(U^k_n(t) - U^k_n(0))^2 \right| \right] = \mathbb{E} \left[ \left| \operatorname{Tr}(U^k_n(0))^2 \right| \right] + \mathbb{E} \left[ \left| \operatorname{Tr}(U^k_n(t))^2 \right| \right] - 2\mathbb{E} \left[ \operatorname{Tr}(U^k_n(t)) \operatorname{Tr}(U^k_n(0)) \right] \\
= 2(k \land n) - 2e^{-\frac{k(k \land n)t}{n}} \sinh \left( \frac{k(k \land n)t}{n} \right) / \sinh \left( \frac{kt}{n} \right) \\
\leq 2(k \land n) - 2e^{-\frac{k(k \land n)t}{n}} \left( \frac{k(k \land n)}{n} \right)^{-1} - \frac{kt}{6n} \\
= 2(k \land n) - 2e^{-\frac{k(k \land n)t}{n}} \left( k \land n - \frac{k^2t^2(k \land n)}{6n^2} \right) \\
= 2e^{-\frac{k(k \land n)t}{n}} \frac{k^2t^2(k \land n)}{6n^2} + 2(k \land n)(1 - e^{-\frac{k(k \land n)t}{n}}) \\
\leq 2k^3t^2 + 2(k \land n) \frac{k(k \land n)t}{n} \\
\leq 4k^2t.
$$

Thus we see that when $s < 1/2$, the first sum in (3) is bounded by

$$
T \sum_{k=1}^{\infty} k^{-1-2e+2s},
$$

which is finite for $s < \epsilon$. This finishes the proof. \qed

4 A Wick-type identity

In this section we consider expectations of the form

$$
\mathbb{E} \left[ \operatorname{Tr} (H^{\sigma_1} H^{\sigma_2} \cdots H^{\sigma_j}) \operatorname{Tr} (H^{\sigma_1} H^{\sigma_2} \cdots H^{\sigma_j}) \right],
$$

where $\sigma_1, \ldots, \sigma_j \in \mathbb{Z}$, $U \in U(n)$ is Haar-distributed and independent from $H$, which is a GUE(n) matrix, i.e., $H_{ii} \sim \mathcal{N}(0,1)$ for $i = 1, \ldots, n$, and $\Re H_{ij} = \Re H_{ji} \sim \mathcal{N}(0,1/2)$, $\Im H_{ij} = -\Im H_{ji} \sim \mathcal{N}(0,1/2)$ for $1 \leq i < j \leq n$, with entries being independent up to the Hermitian symmetry.

Let $C_{2j} = \{ \pi \in S_{2j} : \pi^2 = \text{Id}, \forall l \in \{1, \ldots, 2j\}, \pi(l) \neq l \}$, i.e., $C_{2j}$ is the set of pairings on $\{1, \ldots, 2j\}$. Then we see that

$$
\mathbb{E} \left[ \operatorname{Tr} (H^{\sigma_1} H^{\sigma_2} \cdots H^{\sigma_j}) \operatorname{Tr} (H^{\sigma_1} H^{\sigma_2} \cdots H^{\sigma_j}) \right] = \sum_{i_1, \ldots, i_2j, i_1, \ldots, i_j} \mathbb{E} \left[ H_{i_1i_2} (U^{\sigma_1})_{i_2i_3} \cdots H_{i_{2j-1}i_{2j}} (U^{\sigma_j})_{i_{2j}i_1} H_{i_1i_2} (U^{\sigma_1})_{i_3i_4} \cdots H_{i_{2j-1}i_{2j}} (U^{\sigma_j})_{i_{2j}i_1} \right]
$$

$$
= \sum_{i_1, \ldots, i_{2j}} \mathbb{E} \left[ H_{i_1i_2} H_{i_3i_4} \cdots H_{i_{2j-1}i_{2j}} H_{i_{2j+1}i_{2j+2}} H_{i_{2j+3}i_{2j+4}} \cdots H_{i_{3j-1}i_{3j}} \right]
$$

$$
\times \mathbb{E} \left[ (U^{\sigma_1})_{i_2i_3} (U^{\sigma_2})_{i_4i_5} \cdots (U^{\sigma_j})_{i_{2j-1}i_{2j+2}} (U^{\sigma_1})_{i_{2j+1}i_{2j+3}} (U^{\sigma_2})_{i_{2j+3}i_{2j+4}} \cdots (U^{\sigma_1})_{i_{3j-1}i_{3j+2}} \right].
$$

4.1 Wick-type identity
The condition \((i_{2l-1}, i_{2l}) = (i_{2\pi(l)}, i_{2\pi(l)-1})\) \(\forall l \in \{1, \ldots, 2j\}\) \(\forall i_1, \ldots, i_{4j} \in \{1, \ldots, n\}\) allows to define a map \(\pi \mapsto \tilde{\pi}\) from \(C_{2j}\) to \(C_{4j}\) by the formula
\[
\tilde{\pi}(2l-1) = 2\pi(l), \quad \tilde{\pi}(2l) = 2\pi(l) - 1, \quad l = 1, \ldots, 2j.
\]

Further we define the pairing \(\rho \in C_{4j}\) as
\[
\rho := (23)(45) \cdots (2j, 1)(2j+1, 2j+2)(2j+3, 2j+6) \cdots (2j+2l-1, 2j+2l+2) \cdots (4j-1, 2j+2).
\]

See Example 4.2 for a list of the pairings \(\pi, \tilde{\pi}\) and \(\rho\), for \(j = 2\), and Figure 4 for their depiction.

Note that \(\rho\) and all pairings \(\tilde{\pi}\) pair even numbers with odd numbers, thus \(\tilde{\pi}\rho\) maps even numbers to even numbers and odd numbers to odd numbers. Using the pairing \(\tilde{\pi}\), the even numbers \(i_2, i_4, \ldots i_{4j}\) determine all the odd ones. Thus we see that

\[
\sum_{i_1, \ldots, i_{4j}} \sum_{\pi \in C_{2j}} \prod_{l=1}^{2j} (U^{\sigma_l})_{i_{2l-1} i_{2l}}
\]

\[
	imes E \left[ (U^{\sigma_1})_{i_2 i_3} (U^{\sigma_2})_{i_4 i_5} \cdots (U^{\sigma_j})_{i_{2j-1} i_{2j}} (U^{\sigma_j})_{i_{2j+1} i_{2j+2}} \cdots (U^{\sigma_{j+1}})_{i_{4j-1} i_{4j+1}} \right]
\]

\[
= \sum_{\pi \in C_{2j}} \sum_{i_1, \ldots, i_{4j}} \prod_{l=1}^{2j} (U^{\sigma_l})_{i_{2l-1} i_{2l}}
\]

\[
	imes E \left[ (U^{\sigma_1})_{i_2 i_{\hat{\sigma}_1}} (U^{\sigma_2})_{i_4 i_{\hat{\sigma}_2}} \cdots (U^{\sigma_j})_{i_{2j-1} i_{\hat{\sigma}_j}} (U^{\sigma_{j+1}})_{i_{4j-1} i_{\hat{\sigma}_{j+1}}} \right]
\]

\[
= \sum_{\pi \in C_{2j}} E \left( \sum_{i_2, i_4, \ldots, i_{4j}} 2^j \prod_{l=1}^{2j} (U^{\sigma_l})_{i_{2l-1} i_{2l}} \right),
\]

where
\[
\hat{\sigma}_l = \begin{cases} 
\sigma_l, & l = 1, 2, \ldots, j, \\
-\sigma_{l-j-1}, & l = j + 2, \ldots, 2j, \\
-\sigma_j, & l = j + 1.
\end{cases}
\]

By repeatedly applying \(\tilde{\pi}\rho\) to \(\{2, 4, \ldots, 4j\}\), we get a partition of \(\{2, 4, \ldots, 4j\}\) into orbits. The set of these orbits we denote by \(\mathcal{O}_{\tilde{\pi}\rho}\). We see that

\[
\sum_{\pi \in C_{2j}} E \left( \sum_{i_2, i_4, \ldots, i_{4j}} 2^j \prod_{l=1}^{2j} (U^{\sigma_l})_{i_{2l-1} i_{2l}} \right)
\]

\[
= \sum_{\pi \in C_{2j}} E \left( \prod_{i_2, i_4, \ldots, i_{4j}} \prod_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} \prod_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} (U^{\sigma_\omega})_{i_{2l-1} i_{2l}} \right)
\]

\[
= \sum_{\pi \in C_{2j}} E \left( \prod_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} \text{Tr} \left( \prod_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} (U^{\sigma_\omega}) \right) \right)
\]

\[
= \sum_{\pi \in C_{2j}} E \left( \prod_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} \text{Tr} \left( U^{\sum_{\omega \in \mathcal{O}_{\tilde{\pi}\rho}} \sigma_\omega} \right) \right).
\]

Putting together (4), (5), (6) and (7), we have proven the following proposition:
Proposition 4.1. Let $H$ be an $n \times n$ matrix from the GUE($n$), and let $U \in U(n)$ be independent and Haar-distributed. Then for $j \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_j \in \mathbb{N}$ it holds that

$$E \left( \text{Tr} (H^{\sigma_1} U \sigma_1 H^{\sigma_2} U \sigma_2 \ldots H^{\sigma_j} U \sigma_j) \text{Tr} (H^{\sigma_1} U \sigma_1 H^{\sigma_2} U \sigma_2 \ldots H^{\sigma_j} U \sigma_j) \right)$$

$$= \sum_{\pi \in \mathcal{C}_j} E \left( \prod_{o \in \mathcal{O}_{\pi}} \text{Tr} \left( U^{\sum_{o \in \mathcal{O}_{\pi}} \delta_o} \right) \right).$$

Example 4.2. For $j = 2$ we see that $\rho = (14)(23)(58)(67)$, and (see Figure 1)

$$\pi = (12)(34), \quad \tilde{\pi} = (14)(23)(58)(67), \quad \tilde{\pi}\rho = (2)(4)(6)(8),$$

$$\tilde{\pi} = (14)(23), \quad \tilde{\pi} = (16)(25)(38)(47), \quad \tilde{\pi}\rho = (28)(46),$$

and that $\tilde{\sigma}_2 = \sigma_1, \tilde{\sigma}_4 = \sigma_2, \tilde{\sigma}_6 = -\sigma_2$ and $\tilde{\sigma}_8 = -\sigma_1$. Thus from Lemma 4.1 it follows that

$$E \left( \text{Tr} (H^{\sigma_1} U \sigma_1 H^{\sigma_2} U \sigma_2) \text{Tr} (H^{\sigma_1} U \sigma_1 H^{\sigma_2} U \sigma_2) \right)$$

$$= E \left( \text{Tr} U^{\sigma_1} \text{Tr} U^{\sigma_2} \text{Tr} U^{\sigma_1} \text{Tr} U^{\sigma_2} \right)$$

$$+ E \left( \text{Tr} U^{\sigma_1-\sigma_2} \text{Tr} U^{\sigma_2-\sigma_2} \right)$$

$$+ E \left( \text{Tr} U^{\sigma_1-\sigma_2} \text{Tr} U^{\sigma_2-\sigma_2} \right)$$

$$= \begin{cases} 2\sigma_1^2 + n^2 + n^2 & \sigma_1 = \sigma_2, \\ \sigma_1 \sigma_2 + n^2 + |\sigma_1 - \sigma_2| & \sigma_1 \neq \sigma_2, \end{cases}$$

where the last equality holds for large enough $n$ by Theorem 4.3.

Figure 1: The pairing $\rho$ is in black, the three pairings $\tilde{\pi}$ in $\tilde{\mathcal{C}}_8$ are in red.

Theorem 4.3. (Diaconis, Shahshahani [13]) Let $U$ be a Haar-distributed random matrix in $U(n)$ and let $Z_1, \ldots, Z_k$ be i.i.d. standard complex Gaussian random variables. Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ with $a_j, b_j \in \mathbb{N}$, and let $n \in \mathbb{N}$ be such that

$$\max \left\{ \sum_{j=1}^k j a_j, \sum_{j=1}^k j b_j \right\} \leq n.$$

then

$$E \left( \prod_{j=1}^k (\text{Tr}(U^j))^{a_j} (\text{Tr}(U^j))^{b_j} \right) = \delta_{ab} \prod_{j=1}^k j^{a_j} a_j! = E \left( \prod_{j=1}^k (\sqrt{\lambda} Z_j)^{a_j} (\sqrt{\lambda} Z_j)^{b_j} \right).$$
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