A LEMAÎTRE-TOLMAN-FRIEDMANN
UNIVERSE WITHOUT DARK ENERGY

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Abstract

We build a simple cosmological model by means of a parabolic
Lemaître-Tolman-Bondi (LTB) metric up to a redshift \( z \approx 0.4 \), an hy-
perbolic Friedmann metric for \( z \approx 0.4 \) up to the scale where dimming
galaxies are observed \( (z \approx 1.4) \) and a bulk spatially flat metric up to
the last scattering surface. Following Wiltshire, by taking into account
the different rate of clocks for an observer at the centre of a parabolic
LTB spacetime with respect to a one in the hyperbolic Friedmann met-
ic, an "apparent" negative deceleration parameter is perceived by the
observer at the centre of LTB, provided that all the regularity con-
ditions are imposed and the past null sections of the LTB and the
hyperbolic Friedmann metrics are identified. As a result, a first order
Hubble law emerges at low redshifts. A parameter \( K \) arises driving
the deceleration parameter perceived by the central LTB observer. Fi-
nally, we obtain that a negative value for the deceleration parameter
is compatible with the observed energy-density at our present epoch.

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1 Introduction

Supernovae type Ia (SNIa) observations of the past decade seem to indi-
cate an accelerating universe \([1,2]\). In the standard approach with the
Friedmann-Lemaître models (FRLW), an accelerating universe invokes the presence of a large amount of the so called dark energy. In the FRLW picture, this dark energy is given by the cosmological constant. The dark energy represents a puzzle and perhaps the biggest problem in the modern cosmology. In fact, a direct detection of a cosmological constant is still lacking. In the last decade, many attempts have been made (see [3, 19] and references therein) to obtain physically sensible models predicting a negative value for the deceleration parameter. Some authors (see for example [13, 14, 15, 17, 18, 19]) showed that inhomogeneities can generate an accelerating universe by using LTB metrics (see [20, 21, 22]), but several conditions must be imposed (see [3, 17]) in order to build regular physically viable models. In particular, in [16] it is shown that LTB metrics can mimic the distance-redshift relation of the FRLW models at least at the third order in a series expansion with respect to the redshift near the centre where the observer is located. More generally, in the LTB solutions, apparent acceleration in the redshift-distance relation seen by a central observer can be shown to coexist with a volume average deceleration on a spacelike hypersurface (see [23]). An accelerating universe can also be built by averaging inhomogeneities (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16]) by means of the techniques depicted in [5, 6, 7]. For a review on inhomogeneous cosmological models see [24, 25]. Particularly interesting is the idea developed in Wiltshire’s papers [9, 12]. In these papers the dimming of the distant galaxies is interpreted as a ”mirage” effect. This effect is due to the different rate of clocks located in averaged not expanding galaxies, where the metric is spatially flat, with respect to clocks in voids where the spatial curvature is negative. To a negative spatial curvature can be associated a positive quasilocal energy. This gravitational energy is not local, according to the strong equivalence principle. This approach seems to be very promising.

In this paper we adopt the point of view present in [9, 12]. In particular, we are interested in models that take into account the observed inhomogeneity of the universe at least up to a redshift $z \approx 0.1$ by means of LTB metrics. We apply the reasonings of [11], that, as claimed by the author, represent the first level step in solving the fitting problem (see [26]). We use for our local near universe a parabolic LTB metric, instead of a ”dressed” (see [7]) hyperbolic Friedmann metric. In particular, in the spirit of [11], we obtain a simple (crude!) model. Our starting point is the consideration that dimming galaxies are located at hight redshift ($z \sim 0.5 - 1.4$) [27]. Therefore, we build a model where, up to $z \approx 0.4$, the universe is filled with a parabolic LTB metric, for $z \geq 0.4$ with an hyperbolic Friedmann metric and for $z \geq 1.4$ with a bulk spatially flat Friedmann metric, according
to WMAP satellite data [28]. Similarly to [11], we neglect the coupling of the dynamics of the different scales depicted above. In this way, the SNIa dimming galaxies are in a region where the clocks are ticking slowly with respect to the ones of an observer located at the centre of a parabolic LTB metric. Within this crude model, firstly we obtain (see [29]) a linear Hubble law for small redshifts. Further, we show that a negative central deceleration parameter arises, provided that the necessary boundary and regularity conditions are imposed. Furthermore, we show that a negative central deceleration parameter emerges only for values of the density compatible with the ones actually observed (underdensity). Following Wiltshire, since the SNIa observations are made along the past light cone, we identify the null sections ($\theta, \phi = \text{const.}$) of the parabolic and the hyperbolic ”dimming” zone. It should be noticed that cosmological models with only two metrics can be exhaustively found in [30].

In section 2 we present the metrics composing the model together with the initial and regularity conditions. In section 3 we present our model. In section 4 we obtain the linear Hubble law for low redshifts. In section 5 regular solutions are discussed. In section 6 we obtain a distance-redshift relation together with the expression for the central deceleration parameter. Section 7 collects some final remarks and conclusions. Finally, the appendix is devoted to the study of the matching conditions.

2 Initial and regularity conditions

The starting point of our simple model is the consideration that the dimming galaxies have been found at high redshifts ($z \sim 0.5-1.4$). Consequently, we assume that the universe with $z \simeq 0.5$ up to some units, is represented by an hyperbolic Friedmann metric with negative spatial curvature. In appropriate coordinates, the metric can be put in the form

$$ds_F^2 = -dt^2 + \varpi(t)^2[dt_F^2 + \sinh^2\eta_F d\Omega^2], \quad (1)$$

$$\Pi_i t = \frac{\Omega_i}{2(1 - \Omega_i)\varpi} (\sinh \xi - \xi),$$

$$\varpi(t) = \frac{\Omega_i\Pi_i}{2(1 - \Omega_i)} (\cosh \xi - 1),$$

$$\varpi(t)^2\Pi^2(1 - \Omega) = 1,$$

where $\Omega_i$ is an initial density parameter, $\Pi_i$ an initial Hubble constant and $\varpi_i$ an initial expansion factor to be specified. An observer in the portion of
universe given by (1) measures the observables by means of the comoving time \( t \). With respect to this time, an observer in (1) measures an Hubble flow with a time dependent Hubble constant \( \mathcal{H} \) given by

\[
\mathcal{H} = \frac{1}{a} \frac{da}{dt} = \frac{2 \mathcal{H}_i \sinh \xi}{\Omega_i (\cosh \xi - 1)} (1 - \Omega_i)^{\frac{3}{2}}.
\] (2)

In what follows we adopt the simplifying assumption that the dynamics (see [11]) of the pieces composing our model are independent. This is a "crude" assumption, but nevertheless it gives the possibility to study the role of inhomogeneities and structures in the distance-redshift relation. According to WMAP data [28], over the "dimming" zone, the universe can be modelled with a bulk Friedmann metric with zero spatial curvature, i.e.

\[
ds^2_B = -dt_B^2 + a_B(t_B)^2 (d\eta_B^2 + \eta_B^2 d\Omega^2),
\] (3)

where in (3) we have assumed a dust model. With (3), the Hubble flow \( H_B \) is given by

\[
H_B = \frac{1}{a_B} \frac{da_B}{dt_B} = \frac{2}{3t_B}.
\] (4)

Finally, we assume to live at the centre of a parabolic (vanishing spatial curvature) LTB spacetime up to a maximum of \( z \simeq 0.4 \). The metric is

\[
ds^2_{LTB} = -dT^2 + R^2_\eta d\eta^2 + R^2 d\Omega^2,
\] (5)

where, as usual:

\[
4\pi \rho = \frac{M_{\eta}(\eta)}{R_\eta R^2},
\] (6)

\[
R(T, \eta) = \left( \frac{9GM(\eta)}{2} \right)^{\frac{1}{3}} \left( T - Y_0(\eta) \right)^{\frac{2}{3}},
\] (7)

and subindices with comma denote partial derivative. The arbitrary function \( Y_0(\eta) \) is often called "bang function" and is usually interpreted as a big-bang singularity surface. The arbitrary function \( M(\eta) \) represents the gravitational mass inside a volume of radius \( \eta \) and \( \rho \) the local density.

Regularity conditions must be imposed to (5)-(7). First of all, the density \( \rho \) must be positive everywhere and finite at the centre \( \eta = 0 \) where the
observer is placed and \( R(T, \eta) \) must be vanishing at \( \eta = 0, \forall T \).
Mathematically (see [29, 31]):

\[
\rho(T, \eta \to 0) = \text{finite}, \quad \text{(8)}
\]

\[
R(T, \eta \to 0) \sim \eta f(t) , \quad R_{,\eta} > 0 \quad \forall \eta,t, \quad \text{(9)}
\]

\[
R(\eta,T) > 0 \quad \forall \ T \text{ and } \eta > 0 , \quad M(\eta \to 0) \sim \eta^3. \quad \text{(10)}
\]

Further, no trapped shell singularities must arise in the TB zone, i.e.

\[
\eta > 2GM(\eta). \quad \text{(11)}
\]

The absence of trapped shells is automatically satisfied near the centre, provided that conditions (9)-(10) are imposed. We also impose (see [17])
the condition

\[
Y_{0,\eta}(\eta = 0) = 0, \quad \text{(12)}
\]

to avoid potential problems at the centre, such as the "weak singularity" discussed in [17], although the severity of this problem is debated (see [32, 33]). In section 6 we show that the presence of this "weak singularity" does not affect the central deceleration parameter.

We consider now the initial conditions that we must impose to the metrics (1), (3) and (5). If we consider (see [11]) an early time such that \( \Omega_i \) is close to unity, all the three scales depicted above must be matched at that early time. As a result \( \bar{a}_i \approx a_{Bi} \).



The same condition for (7) it gives \( R(T_i, \eta) = \eta \bar{a}_i \). This condition, in terms of the bang function \( Y_0 \) reads:

\[
Y_0(\eta) = T_i - \frac{\pi^2 \eta^2}{3} \frac{\eta^2}{\sqrt{2GM}}. \quad \text{(13)}
\]

To study the condition (12) is more useful to write \( Y_0(\eta) \) as follows:

\[
Y_0(\eta) = T - \frac{\sqrt{2}}{3} \frac{R^2}{\sqrt{GM}}. \quad \text{(14)}
\]

In this section, to represent the dimming "zone", we used the hyperbolic Friedmann solution. This choice leads to simple computations, while an hyperpolic LTB spacetime generally does not allow to simple analytic expressions. For example, the volume average (15) for unbound LTB metrics is generally not at our disposal in an explicit workable form (see [3]). Further, note that the approximation used in this paper can be justified both
physically and mathematically. In the appendix we show that the continuity of the first fundamental form can be achieved on a non-comoving thin shell, ensuring that the spacetime is connected. Further, the Stephani metric (see [34]) could be used as a comoving thick shell located between the two transition zones composing our model (see the appendix). It is worth to be noticed that to calculate the distance-redshift relation only the conformal null sections of the metric come in action, since the astrophysical observations are performed along the past null cone.

3 The model

The further step in our study is to write the relations between the different scales composing the model. To this purpose, it is observed a broadly uniform Hubble law (see [12] and references therein). Therefore, following Wiltshire, we impose the equality of the Hubble flow of the three scales. Concerning the metric [5], we can calculate, in the spirit of the Buchert scheme [5], a volume average expansion up to some scale $\eta_D$, where dimming galaxies come in action. For the proper volume $V_D$ and the expansion $\theta$, we read:

$$V_D = 4\pi \int_0^{\eta_D} R_\eta R^2 d\eta, \quad \text{(15)}$$

$$<\theta>_D = \frac{d}{dT}(\ln V_D), \quad M_D = M(\eta_D), \quad \text{(16)}$$

$$a_D(T) = \left[\frac{(T - Y_0(\eta_D))}{(T_i - Y_0(\eta_D))}\right]^\frac{2}{3}. \quad \text{(17)}$$

As a result, for the averaged Hubble flow $H_{TB}$ in the LTB sector, we have

$$H_{TB} = \frac{<\theta>_D}{3} = \frac{2}{3[T - Y_0(\eta_D)]}. \quad \text{(18)}$$

After equating (2), (11) and (18), we get:

$$t_B = T - Y_0(\eta_D), \quad \text{(19)}$$

$$T(\xi) = \frac{\Omega_i}{3H_i(1 - \Omega_i)^{\frac{2}{3}}} \frac{(\cosh \xi - 1)^2}{\sinh \xi} + Y_0(\eta_D). \quad \text{(20)}$$

Apart from the constant in the right hand side, the expression (20) for the time delay is the one found in [11]. This is not a surprise because, after
averaging a parabolic LTB metric, we obtain an Hubble flow, equation (18), that, apart from the constant $Y_0(\eta_D)$, is the same of the bulk spatially flat metric. The constant $Y_0(\eta_D)$ takes into account the cut-off made in our model. Further, the term $Y_0(\eta_D)$ appears as a translational factor and thus does not enter in our analysis. We only mention the fact that such a constant can give a correction (positive if $Y_0(\eta_D) > 0$) to the age of the universe. For the lapse function $J(\xi) = \frac{dt}{dT}$, we get:

$$J(\xi) = \frac{3(1 + \cosh \xi)}{2(2 + \cosh \xi)}. \quad (21)$$

Formulas (20)-(21) describe the different rate of clocks between the parabolic TB observer and an hypothetical observer placed where the galaxies are dimming. Obviously, for the reasonings above, formula (21) is exactly the one found in [11], but is expressed in a different background.

If we want to describe the dimming of distant galaxies, we must to relate the metrics (1) and (5) on the past null cone, where the SNIa observations are performed. To this purpose, the radial null sections ($\theta, \phi = \text{const.}$) of (1) and (5) must be the same, i.e. $ds_{TB}^2 = J^2 ds_{TB}^2$. As a result, the following equations hold on inward radial null geodesics:

$$\frac{J}{a} dT = -d\eta_F, \quad (22)$$
$$R,\eta d\eta = -dT. \quad (23)$$

Therefore, our related metric in the LTB inhomogeneous spacetime is

$$ds_{TB}^2 = -dT^2 + a^2 J^2 d\eta_F^2 + R^2 d\Omega^2. \quad (24)$$

For a more complete discussion regarding the matching conditions see the appendix.

For the metric (24), we can define a radial observed (by the central observer) Hubble flow with

$$H_{\text{obs}} = \frac{1}{a} \frac{da}{dT} = \frac{d}{dT} \left( \ln \frac{\pi}{J} \right). \quad (25)$$

It is in terms of (25) that we measure the Hubble flow. Concerning the luminosity-distance $d_L(z)$, for the metric (24) we get (see [15, 35, 36, 37, 38, 39]):

$$d_L = B_0(1 + z)^2, \quad B_0^2 = \frac{dS_0}{d\Omega_0} = R^2, \quad (26)$$

where $\Omega_0$ is the solid angle subtended by a bundle of null geodesics diverging from the observer and $S_0$ is the cross-sectional area of the bundle. We must
to integrate the relevant field equations for our purpose. First of all, by means of (1), (21) we can integrate equation (22) along the past null cone. We get:

\[ \eta_F = \xi_0 - \xi, \quad \xi \leq \xi_0, \tag{27} \]

where the subscript "0" denotes the actual time related to the central TB observer. Following Célérier [15], for the metric (24) we can express the observed redshift \( z \) in terms of the time parameter \( \xi \), i.e.:

\[ \frac{d\eta_F}{dz} = \frac{1}{(1 + z)} \frac{1}{\frac{a}{a_0}(\frac{T}{T_0})}. \tag{28} \]

and therefore, by means of (22) and integrating backward starting from \( z = 0 \), we read:

\[ 1 + z = \frac{J \ a_0}{J_0 \ a_0}. \tag{29} \]

Since the expression (29) has been evaluated along the past null cone, it is the same found in [11]. Nevertheless, since inhomogeneities have been taken into account in our model, we expect corrections with respect to the picture of the paper [11], in particular in the relation distance-redshift \( d_L(z) \). In evaluating the functions entering in the relation \( d_L(z) \), we need of the difference \( T_0 - T \), with \( T_0 \) the actual time. The equation (29) permit us to express \( \cosh \xi \) in terms of the redshift \( z \). As a result, after noticing that \( \Omega_0 = \frac{2}{1 + \cosh \xi_0} \) and with the help of (1) and (20), we obtain:

\[ T_0 - T = A \left[ \frac{(\cosh \xi_0 - 1)^{\frac{3}{2}}}{(1 + \cosh \xi_0)^{\frac{3}{2}}} - \frac{(\cosh \xi - 1)^{\frac{3}{2}}}{(1 + \cosh \xi)^{\frac{3}{2}}} \right], \]

\[ A = \frac{\Omega_0 (2 + \Omega_0)^2}{H_0 (1 - \Omega_0)^2 (2 + \Omega_0)^2} \frac{1}{(2 + \Omega_0)^2}, \tag{30} \]

where \( H_0 \) is the measured Hubble constant given by (25) and calculated at the present time \( \xi_0 \) (or \( T_0 \)), i.e.

\[ H_0 = 3 \frac{2 + \Omega_0^2}{(2 + \Omega_0)^2}. \tag{31} \]

Finally, along the past null geodesics we have \( M_{\eta} = M_T \frac{dT}{d\eta} \), and thus, thanks to (23), expression (6) becomes:

\[ \rho(T) = -\frac{M_T}{4\pi R(T)^2}. \tag{32} \]
Obviously, regularity of (32) for \( T \to T_0 \) requires that:

\[
M(T \to T_0) \sim (T_0 - T)^3 + o(1),
\]
\[
R(T \to T_0) \sim (T_0 - T) + o(1).
\]

4 Zeroth order solution: the linear Hubble law

To complete our model, we must integrate the equation (23). Obviously, because of the partial derivative of \( R \) in the left hand side, this equation cannot be integrated in this form. Nevertheless, the equation (23) can be easily integrated as follows. Firstly, we write:

\[
\left( \frac{dR(\eta, T(\eta))}{d\eta} \right) d\eta = \left( \frac{dR(T, \eta(T))}{dT} \right) dT = dR = R_{,\eta} \, d\eta + R_{,T} \, dT,
\]

and the equation (23) becomes

\[
\frac{dT}{(R_{,T} - 1)}.
\]

Further, from (7) we obtain \( R_{,T} = \sqrt{\frac{2GM}{R}} \), and as a result

\[
\frac{dT}{(\sqrt{\frac{2GM}{R}} - 1)}.
\]

We will explain the general strategy to integrate the equation (36) in the next section. First of all we are interested in the first order calculation of \( d_L(z) \). The first condition to impose is \( R > 2GM \), that is equivalent, near the centre, to the first of conditions (11) and rules out trapped shell singularities. It is worth to noticing that the conditions (33) near the centre are sufficient to satisfy the condition above mentioned. The zeroth order approximation for (36) emerges when \( R >> 2GM \). This extreme approximation it gives the correct first order of \( d_L(z) \). Consequently, after integrating with the appropriate boundary condition \( (R(T_0) = 0) \), we have:

\[
R(T, \eta(T)) = R(T) = T_0 - T + o(1).
\]

With the help of (29), (30), we read:

\[
d_L(z) = \frac{z}{H_0} + o(z),
\]

that is the well known linear Hubble law for low redshifts. Therefore, the zeroth order of our model it gives the observed distance-redshift relation for \( z \ll 1 \).
5 Exact regular solutions

We can rewrite equation (36) as:

\[ R(T) = - \int_{T_0}^{T} \left( 1 - \sqrt{\frac{2GM}{R}} \right) dT. \]  

(39)

To integrate (39), we write \( \sqrt{\frac{2GM}{R}} = F(T) \), being \( F(T) \) a regular differentiable function. With the conditions (33) we must impose:

\[ F(T) \in (0, 1), \ R(T) > 0. \]  

(40)

In this way, after fixing an ansatz for \( F(T) \), we can integrate the equation (39), obtaining:

\[ R(T) = T_0 - T + \int_{T_0}^{T} F(T) dT. \]  

(41)

After solving the equation (41) for \( R(T) \), \( M(T) \) is given by

\[ M(T) = \frac{R}{2G} F(T)^2. \]  

(42)

Further, we impose the condition (12), that in terms of equation (14) seen as a function of \( T \) becomes:

\[ Y_{0,T}(T_0) = 0. \]  

(43)

Finally, the relation between \( \eta \) and \( T \) along the past null cone is obtained by inverting the equation (7) with the help of (13) i.e.

\[ \eta = \left( \frac{9GM}{2a_i^2} \right)^{\frac{1}{3}} \left[ R^{\frac{3}{2}} \sqrt{\frac{2}{9GM}} - T + T_i \right]^{\frac{2}{3}}. \]  

(44)

In the following (in particular for the distance-redshift relation (45)) it is essential the behaviour of \( F(T) \) (and \( R(T) \)) near the observer at the centre. As a result, any given expression for \( F(T) \) must have a taylor expansion near the centre fixed by the regularity conditions (40) and (43). Hence, we can take for \( F(T) \) a polynomial expression. Therefore, for our purposes, without loss of generality, we can take for \( T \leq T_0 \)

\[ F = \frac{H_0}{K} (T_0 - T) + \frac{H_0^2}{K^2} (T_0 - T)^2 + Q H_0^3 (T_0 - T)^3 \]  

(45)

\[ R = T_0 - T - \frac{H_0}{2K} (T_0 - T)^2 - \frac{H_0^2}{3K^2} (T_0 - T)^3 - \frac{Q}{4} H_0^3 (T_0 - T)^4, \]  

(46)
with $K$ and $Q$ adimensional constant. It is worth to be noticed that the first two terms in (45) are fixed by the condition (43). If we impose that the LTB scale is extended up to a maximum of $z \simeq 0.4$, we see that the conditions (40) are satisfied for $K > \frac{1}{2}$, with $Q$ of the same order of $K$. Further, note that the limit $z \simeq 0.4$ for the boundary of the TB metric only changes the allowed values for $K$ and $Q$. For example, for $z < 0.4$ $K > a > \frac{1}{2}$. Furthermore, note that if we impose the condition $Y_0 \geq 0$ for $T \to T_0$ with $Y_0(T_0) = 0$, we must have:

$$K = \frac{3}{2}H_0T_0,$$  \hspace{1cm} (47)

while, if we take $K < \frac{3}{2}H_0T_0$ (but positive), then $Y_0 \geq 0$ with $Y_0(T_0) \neq 0$. As we see in the next section, the positivity of $Y_0$, i.e. condition (47) implies a maximum possible value for the central deceleration parameter. It should be noticed that if we take a more general expression other than (45), in order to satisfy all the regularity conditions depicted above, the Taylor expansion of $F(T)$ near the centre must be equal to expression (45), at least for the first two terms. In the next section we show that are exactly these terms that enter in the expression for the central deceleration parameter. As a final remark for this section, note that in our model, thanks to the equation (42), the function $F(T)$ is related to $M(T)$ on the light null cone. Further, by means of the equation (44) we can (at least in principle) find $M = M(\eta)$, i.e. the dependence in terms of $\eta$. Consequently, with respect to our construction, the FLRW limit can be obtained by setting the particular expression for $F(T)$ on the past null cone such that, when expressed in terms of $\eta$ by means of the equation (44), we have $M(\eta) \sim \eta^3$ or $Y_0(\eta) = \text{constant}$.  

6 Central deceleration parameter and observed density

With the help of (29), (30) we can express $M(T), R(T)$ in terms of the measured redshift $z$. By taking the expressions (45), (46) and after a Taylor expansion near the centre ($z = 0$), we get:

$$d_L = \frac{z}{H_0} +$$ \hspace{1cm} (48)

$$+ \frac{z^2}{4H_0 K(2 + \Omega_0^2)} \left[ 5K\Omega_0^4 - 2\Omega_0^4 + 4K\Omega_0^2 - 8\Omega_0^2 + 2K\Omega_0^2 - 8 + 16K \right] + o(z^2).$$
The central deceleration parameter $q_0$ is given by (see [15, 17, 40])

$$q_0 = -H_0 \frac{d^2}{dz^2} (d_L(z = 0)) + 1,$$  \(49\)

that with \(48\) becomes:

$$q_0 = \left[ -3K\Omega_0^4 + 2\Omega_0^4 - 4K\Omega_0^3 + 8\Omega_0^2 + 6K\Omega_0^2 + 8 - 8K \right] 2K(2 + \Omega_0^2)^2.$$  \(50\)

If we do not consider the flat Friedmann metric, we could estimate the bulk central deceleration parameter $q_{0B}$ at early times by setting $\Omega_0 = 1$ in \(50\). As a result:

$$q_{0B} = -\frac{1}{2} + \frac{1}{K}.$$  \(51\)

Note that for $K \to 1$ it follows that $q_{0B} \simeq \frac{1}{2}$. By taking the asymptotic limit $T_0 \to \infty$ ($\Omega_0 \to 0$) we have

$$q_{0\infty} = -1 + \frac{1}{K}.$$  \(52\)

From \(52\) we see that our model is consistent with an accelerating universe provided that $K > 1$. For very large values of $K$ we obtain $q_{0\infty} \simeq -1$. If we put in \(50\) an estimate value for $\Omega_0$ as, for example, $\Omega_0 \leq \frac{1}{10}$, we obtain that for $K \to 1^+$ again $q_0 < 0$. More generally, $q_0 < 0$ at the times for which:

$$K > \frac{(2\Omega_0^4 + 8\Omega_0^2 + 8)}{(3\Omega_0^4 + 4\Omega_0^3 - 6\Omega_0^2 + 8)}.$$  \(53\)

For example, for $\Omega_0 \simeq \frac{1}{10}$, $K > 1.0017$. Therefore, also for the actual universe our model admits a negative central deceleration parameter. By taking the limit value for $K$ given by \(47\), ($\Omega_0(T_0) = 0$) we obtain a maximum negative value for $q_{0\infty}$ i.e. $q_{0\infty} = -\frac{1}{7}$. As a final step, we consider the local density given by \(32\). It is a simple matter to see that, with expressions \(45\), \(46\), by taking the limit $T \to T_0$, we obtain:

$$\rho_0 = \rho(T_0) = \frac{3H_0^2}{8\pi G K^2}.$$  \(54\)

Expression \(54\) represents an interesting result: an actual underdensity is in agreement with a negative value for $q_0$. This is in agreement with the actual estimation predicted by Friedmann models with $\rho_c = \frac{3H_0^2}{8\pi G}$ and $\frac{\rho}{\rho_c} < 1$. 12
Finally, note that our calculations can be easily changed if we do not impose the condition (12). We must only make the following substitutions in the equation (45): $K \rightarrow a$, $K^2 \rightarrow b$, being $(a, b) > 0$ (the weak singularity is ruled out when $a^2 = b$). As a result, since the $q_0$ parameter only involves the second order in $T_0 - T$ in the expression (46), the central deceleration parameter remains unaffected and so also expression (54) does not change. The so-called weak singularity comes in action only at the third order in $z$ in the redshift-distance relation.

7 Conclusions

Following Wiltshire papers, we build a model for the universe without dark energy, by taking into account the observed inhomogeneous universe for low redshifts by means of a LTB metric. When the clocks of the observer at the centre of a parabolic LTB spacetime are related to the ones placed in an hyperbolic Friedmann metric where are located the dimming galaxies, an apparent negative value for the central deceleration parameter arises, provided that all the regularity and boundary conditions are imposed. We attempted to introduce a model in which both a description of the low redshift irregularities and of the large scale homogeneity where present. Since a coupling between the dynamics of the different scales composing the universe is neglected, the model is a "crude" approximation. Nevertheless, our model allows for a negative value of the central deceleration parameter that is in agreement with the observed underdensity for the actual universe [28]. With respect to the Friedmannian picture of [11], the use of a LTB parabolic metric to describe our nearby universe, permit us the introduction of a dimensionless parameter $K$ whose allowed values are in agreement with an accelerating universe. In [11] the deceleration parameter runs to zero asymptotically from positive values.

In any case, we have also shown that the so-called weak singularity comes in action only at the third order in $z$ in the expression (48), and as a result the parameter $q_0$ does not "feel" such a hypothetical singularity. Physically, at the scale where dimming galaxies come in action, we adopt an hyperbolic Friedmann metric. In fact, with such a metric is associated a positive gravitational energy that is the "source" of the slowing rate of the clocks when compared with the central (parabolic) LTB observer. According to WMAP satellite data [28] and CMBR [41] over the "dimming" zone, we adopt a spatially flat Friedmann metric. As a final remark we cite the recent paper [42] where, by means of an investigation of new sperimental
data, it seems that we live in a universe with a deceleration parameter near to \( q_0 \simeq 0^- \), i.e. with a slowing down of the cosmic acceleration. These results seem to be in disagreement with the standard LCDM model and could encourage the point of view of our work.

A further development of this paper could be to consider, as in [10, 12], the dynamics by means of the full Buchert [5] formalism by taking into account the inhomogeneous structure of the nearby observed universe together with the backreaction or to use the full covariant machine given in [6]. Not a simple task!

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APPENDIX

We now study the matching problem between a parabolic LTB solution and an hyperbolic Friedmann solution on a regular comoving surface \( S \). Since of the spherical symmetry of both metrics, we can use spherical symmetric coordinates \( \xi^\alpha \) on \( S \). Therefore, if we denote with \( \psi^\alpha_+ \) the coordinates of the Friedmann metric and with \( \psi^\alpha_- \) the ones of the LTB metric, we have (see [35]) \( \xi^\alpha = \psi^\alpha_+ = \psi^\alpha_- \) and, as a result, we can take \( \xi^0 = t = T, \xi^2 = \theta, \xi^3 = \phi \). In this way the comoving surface \( S \) is given by:

\[
S_- = \eta - S_0 = 0 \quad , \quad S_+ = \eta_F - S_0 = 0 ,
\]

where \( S_0 \) denotes the boundary of the LTB metric. The continuity of the first fundamental form (see [43, 44]) on \( S \), i.e. \( ds^2_S = g_{\alpha\beta} d\xi^\alpha d\xi^\beta (\alpha = (0,2,3)) \) leads to \( ds^2_- = ds^2_+ \), i.e.

\[
(R)|_S = \mp[\sinh \eta_F]|_S. \tag{56}
\]

For the unit normals \( n^-_\mu \) and \( n^+_\mu \) we have

\[
n^-_\mu = R_\eta \delta^1_\mu \quad , \quad n^+_\mu = \overline{a}\delta^3_\mu . \tag{57}
\]

For the second fundamental form \( K_{\alpha\beta} = (n_{\alpha\beta})|_S \) we have \( K_{\alpha\beta} = \partial_\xi^\alpha \partial_\xi^\beta K_{\mu\nu} \). The continuity condition on \( S \), i.e. \( K^+_{\alpha\beta} = K^-_{\alpha\beta} \), becomes

\[
1 = (\cosh \eta_F)|_S. \tag{58}
\]
From equations (56) and (58) it is evident that is not formally possible to match an hyperbolic Friedmann metric with a parabolic LTB one on the comoving surface (55).

If we take for the outer metric (+) an unbound comoving LTB line element

$$ds^2_{TB} = -d\tilde{T}^2 + \frac{\tilde{R}^2}{f^2} d\eta^2 + \tilde{R}^2 d\Omega^2,$$

with \( f^2 > 1 \), instead of (58) we have

$$ (R)|_S = (\tilde{R})|_S, \ T = \tilde{T} + \ \text{constant}, \ (f)|_S = 1. \quad (60)$$

As a result, for the lapse function \( J \) we have

$$J = \frac{dT}{d\tilde{T}} = 1.$$

Remember that, since astrophysical observations seem to show a broadly uniform "Hubble flow" (see [12]), we must perform the matching according to this astrophysical evidence. As a result, the equation (21) must be satisfied for the lapse function (\( J \neq 1 \)). Hence, as an example, we can think to match on a non-comoving thin shell. To do this, we can take for the shell:

$$\eta = \eta_s(T(\tau)), \quad \eta_F = \eta_{fs}(t(\tau)), \quad (61)$$

where \( \tau \) is the proper time on the shell. Since we have \( T = T(\tau), \ t = t(\tau) \), the continuity of the pull-back of the metric on the non-comoving shell \( S \) it gives (on \( S \)):

$$\frac{dT}{d\tau} = A, \quad (62)$$

$$\frac{dt}{d\tau} = B, \quad (63)$$

$$\pi \sinh \eta_{fs} = R, \quad (64)$$

$$\frac{dt}{dT} = J, \quad (65)$$

$$A = \sqrt{1 + \tilde{R}_s^2 \dot{\eta}_s^2}, \quad (66)$$

$$B = \sqrt{1 + \tilde{a}^2 \dot{\eta}_{fs}^2}, \quad (67)$$

where dot is the derivative of the proper time on the shell (all the expressions are calculated on the surface (61)). The system (62)–(67) can be integrated in different ways. As an example, from the equation (62) we have

$$d\tau = \sqrt{1 - \left( \frac{d\eta_s}{dT} \right)^2 \tilde{R}_s^2 dT}. \quad (68)$$
Thanks to (68), equation (63) becomes:

\[ J^2 - 1 = -\left(\frac{d\eta_s}{dT}\right)^2 R_{\eta}^2 + \xi^2 \left(\frac{d\eta_{fs}}{dT}\right)^2. \]  

(69)

The equation (64) permit us to express \( \sinh \eta_{fs} \) in terms of \( (\eta_s, T) \), and thus we can use equation (69) to resolve for \( \eta_s \). Finally, we can integrate equation (68) to obtain the relation \( \tau = H(T) \).

For the unit normals we have:

\[ \eta^-_\mu = [-\dot{\eta}_sR_{,\eta}, R_{,\eta}A, 0, 0], \]

(70)

\[ \eta^+\mu = [-\dot{\eta}_{fs}\alpha, \pi B, 0, 0]. \]

(71)

The equations for the continuity of the extrinsic curvature are:

\[ \dot{\eta}_sR_{,\eta}RR_{,T} + RA = \dot{\eta}_{fs}\alpha^2 \xi \sinh \eta_{fs} + \]

\[ \pi B \sinh \eta_{fs} \cosh \eta_{fs}, \]

\[ \dot{\eta}_sR_{,\eta}\ddot{T} - R_{,\eta}\ddot{\eta}_sA + \]

\[ \dot{\eta}_s^2 R_{,\eta}R_{,\eta}T - R_{,\eta,\eta}\dot{\eta}_s^2 A = \]

\[ \dot{\eta}_{fs}\alpha^2 \xi - \pi B \ddot{\eta}_{fs} + \dot{\eta}_{fs}^2 \alpha^2 \xi. \]

(72)

Conditions (72) seem to be incompatible with the equations (62)-(65). In particular, we have no sufficient number of functions to satisfy the conditions (72).

Concerning the matching between the hyperbolic and the parabolic Friedmann solutions, for the first fundamental form we have the equations (62)-(67) with \( R_{,\eta} \to a_B \) in (66), \( R \to a_B \) in (64) and \( J = \frac{dt}{dT} \) instead of the equation (65) (with the same \( J \)) and, obviously, with different non-comoving surfaces and proper time on the shell. Hence, also for the Friedmann metrics all the reasonings after equation (67) are also valid. As a result, at least with respect to the intrinsic curvature, the matching conditions can be satisfied. It is worth to be noticed that the continuity of the metric on the non-comoving thin shell is the minimum requirement ensuring that the whole spacetime is connected.

We briefly discuss another possibility. We can take the spherically simmetric
perfect fluid Stephani metric (see \[34\]):

\[
\begin{align*}
ds^2 &= -D^2 d\tau^2 + \frac{Y^2}{V^2} \left[ dr^2 + r^2 d\Omega^2 \right], \\
V &= 1 + \frac{1}{4} k(\tau) r^2, \\
D(\tau, r) &= F(\tau) \frac{Y(\tau)}{V} \frac{d}{d\tau} \left( \frac{V}{Y} \right), \\
k(\tau) &= Y^2 \left[ C^2(\tau) - \frac{1}{F^2(\tau)} \right],
\end{align*}
\]

being \(F, C, Y\) free functions. The metric (73) can be used to model a co-moving thick shell (see \[45\]) between the LTB and the hyperbolic Friedmann zone and so also between the hyperbolic and the parabolic Friedmann metrics. Differently from the thin shell discussed above, the metric (73) has, at least in principle, a sufficient number of arbitrary functions to perform the matching by imposing the continuity of the first and the second fundamental form, avoiding surface-layer matter, although in practice this can result not easy. However, in this case the calculations of this article could be changed. In any case, if this thick shell is "small" with respect to the glued regions, one may believe that the deviation from the calculations performed in this paper remains "small". In particular, if we denote with \(\eta_1, \eta_2\) the boundary of the thick shell between the parabolic and the hyperbolic metric, then if \(\eta_2 - \eta_1\) is "small", we expect a "small" deviation on the redshift-distance relation. Obviously, we could use more general metrics for the thick shell than the Stephani one. In a future work we shall consider the model of this paper with the introduction of a comoving thick shell.

As a further remark for this appendix, it should be stressed that, from astrophysical data, emerges the necessity of more scales to describe the whole universe we observe. If this is the case, the observed broadly uniform Hubble flow imposes the equality of the (spatial averaged) Hubble flow of the different metrics composing the universe we observe. As shown in this paper (see equation (20), the equality of the Hubble flow leads to a clock delay effect. If this reasoning is correct, as a consequence, the time flow cannot be chosen globally uniform.

In this paper, the matching is performed by equating the conformally related null sections with conformal factor given by \(J\) (the lapse function). It is worth to be noticed that for the calculation of the distance-redshift relation, only the null geodesics come in action. Generally, since only null geodesics can probe the cosmological scales under consideration, it seems to
be reasonable to impose, at least, the matching between the null sections. In this way we have a crude but sufficient approximation to explore the role of the observed inhomogeneities on the distance-redshift relation. As a final remark, note that, with the introduction of a LTB parabolic metric, we have shown that the calculation performed in [11] are compatible with an accelerating universe, while in [11] the use of only Friedmann spacetimes leads to a universe with $q \to 0^+$ at late times.

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