Quantitative regularity for parabolic De Giorgi classes

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Abstract

We deal with the De Giorgi Hölder regularity theory for parabolic equations with rough coefficients and parabolic De Giorgi classes which extend the notion of solution. We give a quantitative proof of the interior Hölder regularity estimate for both using De Giorgi method. Recently, the De Giorgi method initially introduced for elliptic equation has been extended to parabolic equation in a non quantitative way. Here we extend the method to the parabolic De Giorgi classes in a quantitative way. To this aim, we get a quantitative version of the non quantitative step of the method the parabolic intermediate value lemma, one of the two main tools of the De Giorgi method sometimes called “second lemma of De Giorgi”.

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1 Introduction

Let us first introduce the main results and a historical overview of the elliptic regularity theory of De Giorgi [8].

1.1 Main results

The idea of the paper is to give a quantitative proof of the parabolic De Giorgi interior Hölder regularity theorem so that it is possible to compute a lower bound of the Hölder coefficient for both solutions of the parabolic equation and functions in parabolic De Giorgi classes. Roughly speaking, the De Giorgi classes are sets of functions which satisfy energy estimates which contain enough information to get the Hölder continuity. We know that in particular a solution of the parabolic equation is a function of a De Giorgi class (see Proposition 2.4).

The parabolic equation we are interested in is the following

$$\partial_t u = \nabla_x \cdot (A \nabla_x u) + B \cdot \nabla_x u + g, \quad t \in (T_1, T_2), \quad x \in \Omega, \quad (1)$$

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where $T_1$ and $T_2$ are real numbers, $d$ is a positive integer, $\Omega$ is an open set of $\mathbb{R}^d$, $u$ is a real-valued function of $(t, x)$, $A = A(t, x)$ a $d \times d$ bounded measurable matrix and $A$ satisfies an ellipticity condition for two positive constants $\lambda, \Lambda$,

$$0 < \lambda I \leq A \leq \Lambda I, \quad (2)$$

and $g = g(t, x)$, $B = B(t, x)$ are bounded measurable coefficients, and satisfy,

$$\begin{cases} |B| \leq \Lambda, \\ g \in L^q((T_1, T_2) \times \Omega) \end{cases} \quad \text{where } q > \max(2, \frac{d+2}{d}). \quad (3)$$

We give the definition of weak solutions and parabolic De Giorgi classes in Definitions 2.1 and 2.3 and explain why a solution of a parabolic equation is in a De Giorgi class (see Proposition 2.4).

We define the parabolic cylinder $Q_r = (-r^2, 0) \times B_r$, where $B_r$ is the ball of radius $r$ centered at 0. Let us state the quantitative Hölder continuity theorem for the parabolic De Giorgi classes (see Definition 2.3).

**Theorem 1.1** (Interior Hölder continuity for parabolic De Giorgi classes). Let $u : Q_2 \to \mathbb{R}$ be a function in $DG^+(\gamma_1, \gamma_2, \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \gamma_3, p)$ where $1 \leq p \leq \frac{d+2}{d}$ and $\gamma_1, \gamma_2, \gamma_3 > 0$. Then $u \in C^\alpha(Q_1)$ with

$$\|u\|_{C^\alpha(Q_1)} \leq C \left(\|u\|_{L^2(Q_2)} + 1\right),$$

where $C$ and $\alpha$ depend only on $d, \gamma_1, \gamma_2, \gamma_3$ and $p$.

Since the solutions of the parabolic equation $[1]$ are in a De Giorgi class $DG^+(\gamma_1, \gamma_2, \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \gamma_3, p)$ we deduce the same result for the solutions.

**Corollary 1.2** (Interior Hölder continuity for weak solutions). Let $u : Q_2 \to \mathbb{R}$ be a solution of $[1]$ satisfying $[2]$ and $[3]$ such that $\|g\|_{L^1(Q_2)} \leq 1$. Then $u \in C^\alpha(Q_1)$ with

$$\|u\|_{C^\alpha(Q_1)} \leq C \left(\|u\|_{L^2(Q_2)} + 1\right),$$

where $C$ and $\alpha$ depend only on $d, \lambda$ and $\Lambda$.

**Remark 1.3.** Thanks to the scaling property of the equation and De Giorgi classes, Theorem 1.1 and Corollary 1.2 hold true for all $Q' = (s, T_2) \times \Omega'$ and $Q = (T_1, T_2) \times \Omega$ such that $\Omega' \subset \subset \Omega$ and $T_1 < s < T_2$, instead of $Q_1$ and $Q_2$ (see [32] page 16).

**Remark 1.4.** Corollary 1.2 is already proven in [32][20] in a non quantitative way. The proof is non quantitative because of a non quantitative step, the intermediate value lemma. Concerning Theorem 1.1 the interior Hölder continuity has already been studied in [25][11][22] with a different method than De Giorgi one which doesn’t involve a parabolic intermediate value lemma. Our purpose in this paper is to give a simple self-contained quantitative proof of this theorem so that we could investigate extensions to other equations, for example the kinetic Fokker-Planck equation. We also make the steps explicit so that it is possible to compute a lower bound for the Hölder exponent $\alpha$.

The main new result of this paper is the quantitative intermediate value lemma which allows to get a quantitative interior Hölder continuity theorem and to compute a lower bound for the Hölder exponent $\alpha$. Let us state this result.
Theorem 1.5 (Parabolic intermediate value lemma). Let $\gamma_1, \gamma_2, \gamma_3 > 0$ and $p$. Let $u$ be in $DG^+ (\gamma_1, \gamma_2, \gamma_3, p)$ such that $u \leq 1$ on $Q_3$. Let $Q_1 = (-2, -1) \times B_1$. Then for all $(k, l) \in \mathbb{R}^2$ such that $k < l \leq 1$, we have

$$(l - k)^2|\{u \leq k\} \cap Q_1||\{u \geq l\} \cap Q_1| \leq C|\{k < u < l\} \cap Q_2|^\frac{1}{p+2},$$

(4)

where $C$ depends only on $d, k, \gamma_1, \gamma_2, \gamma_3$ and $p$.

Remark 1.6. Theorem 1.5 is a step to obtain Hölder regularity with the De Giorgi method (see subsection 3.2). In the subsection 4.3, we will see that the intervals of time must be disjoint in the subsolution or $DG^+$ case because there exists counterexamples if they are not.

1.2 Historical overview

De Giorgi [8, 9] introduced techniques in 1957 to solve 19$^{th}$ Hilbert problem about the analytic regularity of local minimizers of an energy functional. In fact, these minimizers are solutions of quasilinear Euler-Lagrange equations. The idea of De Giorgi was to see quasilinear elliptic equations as linear elliptic equation with merely measurable coefficients. Thus he proved the Hölder regularity of solutions of elliptic equations with rough coefficients which was the last result to obtain to prove the analyticity since we can use Schauder estimates and a bootstrap argument to get the smoothness of the solutions. In 1958, Nash [30] got the result with different techniques for both elliptic and parabolic equations. Then, Moser [29] proved in 1960 the Hölder regularity with a different approach. These methods are now called the De Giorgi-Nash-Moser techniques.

In his paper [8], De Giorgi exhibited a class of functions that satisfy energy estimates and he showed that any function in this class is locally bounded and Hölder continuous. These classes of functions are called the De Giorgi classes. Ladyzhenskaya and Uralt’seva [26] extended this idea to linear parabolic equations with lower order terms and to quasilinear parabolic equations using a different method than De Giorgi [8]. They introduced the corresponding De Giorgi classes in the parabolic case and proved that Hölder estimate holds when $\pm u$ are both in a De Giorgi class. One can find more details in [25], in [12] and in Chapter 6 of [28].

There are extensions of the method in degenerate cases, like the $p-$Laplacian, by Ladyzhenskaya and Uralt’seva [27] in the elliptic case. Then DiBenedetto [10] covered the degenerate parabolic cases, see also DiBenedetto, Gianazza and Vespri [13, 14, 15].

Concerning nonlinear nonlocal time-dependent variational problems, Caffarelli and Vasseur [6] on the first hand and Caffarelli, Chan and Vasseur [4] on the second hand extended the method of De Giorgi to nonlocal parabolic equations and got a Hölder regularity result for solutions of problems with translation invariant kernels. Also Caffarelli, Soria, Vázquez [5] used the De Giorgi method to prove Hölder continuity of solutions of a porous medium equation with nonlocal diffusion effects. This kind of equation has also been studied earlier by Kassmann [24] using Moser’s techniques where he got local regularity results and by Kassmann and Felsinger [18] where they obtained a weak Harnack inequality.

Recently, Golse, Imbert, Mouhot, Silvestre and Vasseur proved the Hölder regularity and obtained Harnack inequalities for kinetic equations. More precisely, the Fokker-Planck kinetic equation with rough coefficients was studied by Golse, Imbert, Mouhot,
Vasseur [20] and provides the results for the Landau equation. Imbert and Silvestre [23] studied a class of kinetic integro-differential equations and deduced the results for the inhomogeneous Boltzmann equation without cut-off. The quantitative versions of the intermediate value lemmas in those cases are still an open question.

1.3 Contribution of this paper and comparison with existing result

The main contribution of this paper is the quantitative proof of the interior Hölder regularity result with De Giorgi method for parabolic De Giorgi classes and parabolic equations. So that we can compute explicitly the Hölder exponent, at least we can give an explicit lower bound. More precisely, there are two main new results. On one hand, we obtain a quantitative version of one key step of the proof, which was the last non-quantitative step in the parabolic De Giorgi method. This step is sometimes called second lemma of De Giorgi or intermediate value lemma. In the other hand, we extend the De Giorgi method for the parabolic De Giorgi classes. The Hölder continuity for these classes was already obtained in [25, 12] but with a different method than De Giorgi’s. Concerning the intermediate value lemma there are many quantitative versions in the elliptic case. De Giorgi [8, 9] obtained a quantitative version using an isoperimetric inequality argument, taken up by DiBenedetto [11] and Vasseur [32]. Recently, Hou and Niu [22] proved a quantitative version of this lemma using a Poincaré inequality. These versions are actually valid for any function in $H^1$. About parabolic equations, no quantitative version of this lemma seems to exist. One can find non-quantitative versions, for example in [32], a version obtained by contradiction with a compactness argument which works only for solutions of the parabolic equation. However, there exists a quantitative version of this lemma for nonlocal time-dependent integral operator [4] but it does not apply for local parabolic equations. Here we provide a new point of view which makes it possible to deal with the intermediate value lemma using only the energy estimate. So not only the proof is quantitative but it also works for the De Giorgi classes. Moreover, it gives another point of view by breaking the solution structure into sub and super-solution (resp. sub and super De Giorgi classes).

1.4 Aim and applications of the paper

In this paper, we investigate the De Giorgi method in order to provide a detailed self-contained proof which allows to deal with general assumptions where our aim would be to use this method for other equations. We focus on De Giorgi classes and De Giorgi method to be able to understand the structure and where the relevant information is contained to get the Hölder continuity. The De Giorgi classes make us understand how to get rid of the merely measurable coefficients so that it’s not a difficulty anymore. The De Giorgi method consists in two parts. In a first part we see that we can reduce the Hölder continuity theorem with steps which only use the scaling and linear structure of either the equation or the DG classes. So this part is likely to remain similar when we deal with other equations. The second part of the method consists in getting two lemmas called first and second lemma of De Giorgi in order to prove the reduced theorem. We explain how to extract the information from the energy estimate to get those two lemmas.
Moreover the proof is completely quantitative, we can compute explicitly the Hölder exponent, especially the new way of dealing with the intermediate value lemma which is quantitative gives hope to get this lemma for other cases also in a quantitative way. We give a proof which comes from the energy estimate and which is different from the elliptic case so that now we can deal with time dependent equation. We think for example that those techniques would apply to make the second lemma of De Giorgi quantitative for Hamilton-Jacobi equations studied in \cite{7,31} since the energy estimates for those equations are very similar to our case. Moreover, being able to compute explicitly the Hölder exponent can be useful for getting explicit rates. For example, it allows to study the behavior of solutions of quadratic reaction diffusion systems: Fellner, Morgan and Tang \cite{17} Theorem 1.1 got a polynomial bound of the solutions in a specific case and the exponent $\xi$ of the polynom depends on the Hölder exponent of a solution of a parabolic equation.

Also dealing with De Giorgi classes, allows to handle equations which are not included in the general case of the equation (1) with (2)-(3). For example, if the matrix $A$ is not necessarily bounded, then we cannot apply directly the result of the equation to get the Hölder regularity. But in some cases this matrix is explicit and even if it’s not bounded, we can get energy estimates which are relevant to define De Giorgi classes for this problem. For example, for this reaction-diffusion equation with self-diffusion

$$\partial_t u - \nabla_x \cdot (1 + u) \nabla_x u = u(1 - u),$$

where $u \geq 0$ and in $u \in L^3$, we can define the corresponding De Giorgi classes and get the Hölder continuity using the same techniques.

Our next purpose would be to apply these techniques to other equations to get Hölder regularity where the difficult part would be to understand what the “good” energy estimates which contains enough information are. As soon as we get the “good” energy estimate, our hope would be that the techniques in this paper would apply to conclude. For example, we would like to be able to treat the case of the following kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s.$$ 

But exhibiting the relevant De Giorgi classes in this case remains an open question (see subsection \ref{kinetic}). Being able to deal with De Giorgi classes for kinetic Fokker-Planck equation would then allow to handle matrices $A$ which are not necessarily bounded, in a kinetic framework (in a case where we have self-diffusion for example as mentionned previously in the parabolic framework).

### 1.5 Organisation of the paper

In Section 2, we give the notations and the definition that we use in this paper. In Section 3, we extend the steps of the De Giorgi method to get the Hölder regularity of parabolic De Giorgi classes, we prove Theorem \ref{parabolic} and deduce Corollary \ref{corollary}. In Section 4, we recall and simplify a proof of the intermediate value lemma in the elliptic case obtained in \cite{22} and prove Theorem \ref{elliptic} the parabolic case.
2 Notations and definitions

We give the notations that are used in this paper. Here in \( \mathbb{R}^d \), for \( r > 0 \) and \( x_0 \in \mathbb{R}^d \), \( B_r(x_0) \) is the ball of radius \( r \) center at \( x_0 \). \( B_r \) the ball of radius \( r \) of center \( 0 \). We define for \( r > 0 \) and \((t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d \) the parabolic cylinder \( Q_r(t_0, x_0) = (t_0 - r^2, t_0) \times B_r(x_0) \). We define as well the cylinder centered at \((0, 0)\) by \( Q_r = (-r^2, 0) \times B_r \) and the cylinder \( \overline{Q}_t = (-2, -1) \times B_1 \). For \( U \) an open bounded domain of \( \mathbb{R}^d \), we denote by \( C^\alpha(U) \) the space of Hölder continuous functions \( u \), with the norm

\[
\|u\|_{C^\alpha(U)} = \|u\|_{L^\infty(U)} + \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.
\]

We define the oscillation of a function \( u \) on a set \( E \) of \( \mathbb{R}^d \) by

\[
\text{osc}_E u = \sup_E u - \inf_E u.
\]

We define the positive (resp. negative) part of a function \( u \) by

\[
u_+ = \max(u, 0) \quad \text{(resp. } u_- = \max(-u, 0))\] .

For \( X = (t, x) \in \mathbb{R} \times \mathbb{R}^d \) with \( x = (x_1, \ldots, x_d) \), we define the norm \( \|X\| = \max(|t|, \|x\|_2) \) where \( \|x\|_2 = \left( \sum_{i=1}^d x_i^2 \right)^{\frac{1}{2}} \).

Let us introduce the notation for the measure of sets. Let \( E \) be a subset of \( \mathbb{R}^d \) or \( \mathbb{R}^{d+1} \), the measure of the set \( E \) is denoted by \( |E| \). For \( u : E \to \mathbb{R} \) and \( (a, b) \in \mathbb{R}^2 \), the sets \( \{u \geq a\} \cap E \), \( \{u \leq b\} \cap E \) and \( \{a < u < b\} \cap E \) will denote respectively \( \{y \in E, u(y) \geq a\} \), \( \{y \in E, u(y) \leq b\} \) and \( \{y \in E, a < u(y) < b\} \). This notation is used for the statements. In the proofs, we will use the following shorthand notations. The quantities \( |u|, |u|, |u|, \) \( |a < u < b, E| \) will denote respectively \( \{y \in E, u(y) \geq a\}, \{y \in E, u(y) \leq b\}, \{y \in E, a < u(y) < b\} \).

Let us give the definition of weak solution, sub-solution and super-solution of the parabolic equation \([1]\). Let \( T_1 < T_2 \) be real numbers and \( \Omega \) be an open set in \( \mathbb{R}^d \). Let \( Q = (T_1, T_2) \times \Omega \).

Definition 2.1 (Weak-solutions). We say that \( u \) is a weak subsolution (resp. weak supersolution) of \([1]\) satisfying \([2]\) and \([3]\), if \( u \in L^\infty((T_1, T_2); L^2(\Omega)) \) such that \( \nabla_x u \in L^2(Q) \) and \( \partial_t u \in L^2((T_1, T_2); H^{-1}(\Omega)) \), and for all \( \varphi \in C^\infty_c(Q) \) nonnegative we have

\[-\int_Q u \partial_t \varphi + \int_Q A \nabla_x u \cdot \nabla_x \varphi - \int_Q B \cdot \nabla_x u \varphi - \int_Q g \varphi \leq 0 \quad \text{(resp. } \geq 0)\] .

We say that \( u \) is a weak solution of \([1]\) if \( u \in L^\infty((T_1, T_2); L^2(\Omega)) \) such that \( \nabla_x u \in L^2(Q) \) and \( \partial_t u \in L^2((T_1, T_2); H^{-1}(\Omega)) \), and for all \( \varphi \in C^\infty_c(Q) \) we have

\[-\int_Q u \partial_t \varphi + \int_Q A \nabla_x u \cdot \nabla_x \varphi - \int_Q B \cdot \nabla_x u \varphi - \int_Q g \varphi = 0\] .

Remark 2.2. In what follows, we will drop the word weak for solutions, subsolutions and supersolutions but it will be implicitly assumed.

Let us give the definition of the parabolic De Giorgi sub-classes and super-classes.
Definition 2.3 (De Giorgi classes $DG^\pm(\gamma_1, \gamma_2, \gamma_3, p)$). Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$ and $T_1 < T_2$ two real numbers. For the positive parameters $\gamma_1, \gamma_2, \gamma_3$ and $1 \leq p \leq \frac{d+2}{d}$, we define the De Giorgi sub-class (resp. super-class) and denote by $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ (resp. $DG^- (\gamma_1, \gamma_2, \gamma_3, p)$) the set of function $u$ such that $u \in L^\infty ((T_1, T_2); L^2(\Omega))$ such that $\nabla_x u \in L^2(Q)$, which satisfies $\forall k \in \mathbb{R}$, $\forall (s,t) \in \mathbb{R}^2$ such that $T_1 \leq s < t \leq T_2$, $\forall 0 < r < R$ and $\forall x_0 \in \Omega$ such that $B_R(x_0) \subset \Omega$, we have the following inequality

$$
\int_{B_r(x_0)} (u-k)^2_\pm (t,x) dx + \gamma_1 \int_s^t \int_{B_R(x_0)} |\nabla_x (u-k)_\pm(\tau, x)|^2 dx d\tau 
\leq \int_{B_R(x_0)} (u-k)^2_\pm (s,x) dx + \frac{\gamma_2}{(R-r)^2} \int_s^t \int_{B_R(x_0)} (u-k)^2_\pm (\tau, x) dx d\tau 
+ \gamma_3 \left( \int_s^t \int_{B_R(x_0)} (u-k)^2_\pm (\tau, x) dx d\tau \right)^{1/p}.
$$

In fact we can prove that any weak subsolution (resp. supersolution) is in a De Giorgi sub-class $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ (super-class $DG^- (\gamma_1, \gamma_2, \gamma_3, p)$) for some parameters $\gamma_1, \gamma_2, \gamma_3$ and $p$. And any solution is in the intersection of a sub and super class $DG^+(\gamma_1, \gamma_2, \gamma_3, p) \cap DG^- (\gamma_1, \gamma_2, \gamma_3, p)$.

Proposition 2.4. Let $u$ be a sub-solution (resp. supersolution) of [1] satisfying [2] and [3]. Then there exist $\gamma_1, \gamma_2$ and $\gamma_3$ positive such that $u \in DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ (resp. $u \in DG^- (\gamma_1, \gamma_2, \gamma_3, p)$). Moreover if $u$ is a solution then there exist $\gamma_1, \gamma_2, \gamma_3$ and $p$ such that $u \in DG^+(\gamma_1, \gamma_2, \gamma_3, p) \cap DG^- (\gamma_1, \gamma_2, \gamma_3, p)$.

Proof. Here we deal with $u$ a subsolution of [1]. The case of supersolution is very similar and the case of the solution is a combination of both cases. It’s exactly deriving the energy estimates for the subsolution $(u-k)_+$ of [1] with the source term $g1_{(u-k)_+}$. Let us define $\varphi \in C^\infty_c (BR)$ such that $0 \leq \varphi \leq 1$, $|\nabla_x \varphi| \leq \frac{\epsilon}{R-r}$ and

$$\varphi = \begin{cases} 
1 & \text{in } B_r, \\
0 & \text{outside } B_R,
\end{cases}$$

and the sequence of functions

$$\psi_\epsilon(\tau) = \begin{cases} 
\frac{1}{\epsilon} (\tau - s) & \text{if } \tau \in (s, s + \epsilon) \\
1 & \text{if } \tau \in (s + \epsilon, t - \epsilon) \\
-\frac{1}{\epsilon} (\tau - t) & \text{if } \tau \in (t - \epsilon, t)
\end{cases}$$

The idea is to use the test function $(\tau, x) \rightarrow (u-k)_+ (\tau, x) \varphi^2(x)$ which is not with compact support in time by first using the function $(\tau, x) \rightarrow (u-k)_+ (\tau, x) \psi_\epsilon(\tau) \varphi^2(x)$ as a test function which is allowed by density arguments and then take $\epsilon \rightarrow 0$.

$$- \int_Q (u-k)_+ \partial_t [(u-k)_+ \psi_\epsilon \varphi^2] = \int_Q \partial_t (u-k)_+ (u-k)_+ \psi_\epsilon \varphi^2$$

$$= \frac{1}{2} \int_Q \partial_t (u-k)_+^2 \psi_\epsilon \varphi^2
= -\frac{1}{2} \int_Q (u-k)_+^2 \partial_t \psi_\epsilon \varphi^2

= -\frac{1}{2} \int_{s+\epsilon} \int_\Omega (u-k)_+^2 \varphi + \frac{1}{2\epsilon} \int_{t-\epsilon}^t \int_\Omega (u-k)_+^2 \varphi^2$$
By the Lesbesgue differentiation theorem when $\varepsilon \to 0$, we have
\[
-\int_Q (u - k)_+ \partial_t \left[ (u - k)_+ \psi_\varepsilon \varphi^2 \right] \to -\frac12 \int_{\Omega} (u - k)_+^2 (s, \cdot) \varphi^2 + \frac12 \int_{\Omega} (u - k)_+^2 (t, \cdot) \varphi^2. \quad (5)
\]

For the other terms, since there is no derivative in time, $\psi_\varepsilon$ will be a common factor for each term. By using the dominated convergence theorem, when $\varepsilon \to 0$, $\psi_\varepsilon \to 1$ almost everywhere so at the limit, the other terms will be
\[
I := \int_Q A \nabla_x (u - k)_+ \cdot \nabla_x \left[ (u - k)_+ \varphi^2 \mathbb{1}_{(s,t)} \right] - \int_Q B \cdot \nabla_x (u - k)_+ (u - k)_+^2 \mathbb{1}_{(s,t)} - \int_Q g (u - k)_+^2 \mathbb{1}_{(s,t)}.
\]

We then have by using a Young inequality and the fact that $\|g\|_{L^q} \leq 1$ and defining $p = \frac{q}{q-1}$,
\[
I \geq \lambda \int_s^t \int_{B_R} |\nabla_x (u - k)_+|^2 \varphi^2 - 2\Lambda \int_s^t \int_{B_R} |\nabla_x (u - k)_+| \varphi (u - k)_+ |\nabla_x \varphi| - \frac{\lambda}{4} \int_s^t \int_{B_R} |\nabla_x (u - k)_+|^2 \varphi^2 - \frac{4\Lambda^2}{\lambda} \int_s^t \int_{B_R} (u - k)_+^2 |\nabla_x \varphi|^2
\]
\[
\geq \lambda \int_s^t \int_{B_R} |\nabla_x (u - k)_+|^2 \varphi^2 - \frac{\lambda}{4} \int_s^t \int_{B_R} (u - k)_+^2 |\nabla_x \varphi|^2 - \frac{5\Lambda^2}{\lambda (R - r)^2} \int_s^t \int_{B_R} (u - k)_+^2 - \|g\|_{L^q} \left( \int_s^t \int_{B_R} (u - k)_+^\frac{q}{q-1} \varphi^2 \right)^\frac{q-1}{q}
\]
\[
\geq \frac{\lambda}{2} \int_s^t \int_{B_r} |\nabla_x (u - k)_+|^2 - \frac{5\Lambda^2}{\lambda (R - r)^2} \int_s^t \int_{B_R} (u - k)_+^2 - \|g\|_{L^q} \left( \int_s^t \int_{B_R} (u - k)_+^\frac{q}{q-1} \right)^\frac{q-1}{q}. \quad (6)
\]

Combining (5) and (6) we deduce that the subsolution $u$ is in the De Giorgi sub-class $DG^+ (\frac{\lambda}{2}, \frac{5\Lambda^2}{\lambda}, \|g\|_{L^q}, \frac{q}{q-1})$.

In Section 3 and 4, a universal constant will be a constant which only depends on $d, \gamma_1, \gamma_2, \gamma_3,$ and $p$.

### 3 De Giorgi method for parabolic De Giorgi classes

In this section, we are going to prove the interior Hölder continuity of functions in De Giorgi classes (Theorem 1.1) so in particular we deduce the result for weak-solutions of the parabolic equation (1) (see Corollary 1.2), thanks to Proposition 2.4. The idea is to reduce the interior Hölder continuity theorem using the oscillation of the solution. Step by step in Subsection 3.1 using the oscillation, we prove that it is enough to get a lowering of the maximum property to get the theorem. This lowering of the maximum property states that any function in a De Giorgi class smaller than 1 with enough mass below 0 is in fact far from 1 in a smaller cylinder. After reducing the theorem to the lowering of maximum property, we prove two tools called the first and the second lemma of De Giorgi which are the key ideas of the proof of this lowering of maximum property. In the end, Theorem 1.1 and Collorary 1.2 follow from the lowering of maximum property.
### 3.1 Reduction of the Hölder continuity theorem

In this subsection, we explain how to reduce the interior Hölder continuity theorem to the lowering of the maximum property. There are three steps to do it that we introduce in three lemmas. As the proof for weak-solutions [32, 21], this reduction only relies on “scaling and linearity properties” of the definition of the De Giorgi classes. More precisely, in the case where we do not have the last term is the definition of the De Giorgi class (for the equation it corresponds to the source term precisely, in the case where we do not have the last term is the definition of the De Giorgi classes.

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#### 3.1 Reduction of the Hölder continuity theorem

We define the universal constant

\[
\beta = \frac{1}{2 \left( \frac{26}{5.79} \right)^{\delta}} |Q_2| + 1
\]

smaller than 1, where \(\delta\) is the universal constant given in Lemma 3.8 and \(C\) is the constant of (18).

So the general idea is to get the first and second lemma of De Giorgi (Lemmas 3.8 and 3.13) for \(D^+(\gamma_1, \gamma_2, \gamma_3, p)\), to use those lemmas to deduce the lowering of maximum property for \(D^+(\gamma_1, \gamma_2, \beta \gamma_3, p)\), and then to deduce the interior Hölder continuity for \(D^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \beta \gamma_3, p)\) so for \(D^+(\gamma_1, \gamma_2, \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \gamma_3, p)\).

In fact, if \(u \in D^+(\gamma_1, \gamma_2, \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \gamma_3, p)\) then \(\beta u \in D^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \beta \gamma_3, p)\) so \(\beta u\) would be Hölder continuous and then \(u\) itself.

We first begin by proving that we can reduce the interior Hölder continuity of \(D^+(\gamma_1, \gamma_2, \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \gamma_3, p)\) to the lowering of maximum of \(D^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \beta \gamma_3, p)\).

#### Preliminary step: Reduction of the problem.

We prove step by step that one can reduce Theorem 1.1 to Lemma 3.6. Indeed, the Hölder continuity is a consequence of the following lemma.

**Lemma 3.1** (Traduction of the definition). Let \(u : Q_2 \to \mathbb{R}\) be a function in \(D^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap D^-(\gamma_1, \gamma_2, \beta \gamma_3, p)\) where \(\beta\) satisfies (7). Then \(u\) satisfies

\[
\forall (t_0, x_0) \in Q_1, \forall r \in \left(0, \frac{1}{2}\right), \quad \text{osc}_{B_r(t_0, x_0)} u \leq C r^\alpha \left(\|u\|_{L^2(Q_2)} + 1\right),
\]

where \(C\) and \(\alpha\) only depend on \(d, \gamma_1, \gamma_2, \gamma_3,\) and \(p\).

**Remark 3.2.** We can define the oscillation thanks to the \(L^2 - L^\infty\) estimate (Lemma 3.8).

We assume that Lemma 3.1 is true and prove Theorem 1.1.
Proof of Theorem 1.1. The function $\beta u$ is in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \beta \gamma_3, p)$. Let $X = (t, x) \in Q_1$ and $Y = (s, y) \in Q_1$. We define $Z = \frac{X + Y}{2}$, $X_1 = \frac{X + Z}{2}$ and $Y_1 = \frac{Y + Z}{2}$, $r = \|X - Y\|$. Using Lemma 3.1, we get

$$|u(X) - u(Z)| \leq \frac{\text{osc}}{Q_1(X)} u \leq C \left( \frac{\|X - Y\|}{4} \right)^\alpha \left( \|u\|_{L^2(Q_2)} + 1 \right),$$

$$|u(Z) - u(Y)| \leq \frac{\text{osc}}{Q_1(Y)} u \leq C \left( \frac{\|X - Y\|}{4} \right)^\alpha \left( \|u\|_{L^2(Q_2)} + 1 \right).$$

So by a triangular inequality, adding the last two inequalities, we deduce Theorem 1.1 for $\beta u$ and then for $u$.

Remark 3.3. Lemma 3.1 is just rewriting the interior Hölder regularity in terms of the oscillation. It doesn’t use the definition of De Giorgi classes.

We can deduce Corollary 1.2.

Proof of Corollary 1.2. It is a consequence of Theorem 1.1 and Proposition 2.4.

The previous lemma is a consequence of the following oscillation decrease. This version of the lemma is slightly different from the case without source term [32].

Lemma 3.4 (Local decrease of the oscillation). Let $u : Q_2 \to \mathbb{R}$ be a function in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \beta \gamma_3, p)$ where $\beta$ satisfies (7). Then there exists a constant $\theta \in \left( \frac{1}{2}, 1 \right)$ only depending on $d, \gamma_1, \gamma_2, \gamma_3,$ and $p$, such that

- if $\text{osc}_{Q_1} u \geq 2$, then $\text{osc}_{Q_{1/2}} u \leq \theta \text{osc}_{Q_1} u$,
- if $\text{osc}_{Q_1} u \leq 2$, then $\text{osc}_{Q_{1/2}} u \leq 2\theta$.

We assume that Lemma 3.4 is true and prove Lemma 3.1.

Proof of Lemma 3.1. Let us define for $n \in \mathbb{N} \setminus \{0\}$ a sequence of function in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \beta \gamma_3, p)$ where $\beta$ satisfies (7) (since $\frac{1}{4\theta} < 1$),

$$u_n(\tau, y) = \frac{2\theta^{1-n}}{\max(2, \text{osc}_{Q_{3/2}} u)} \left( t_0 + \frac{\tau}{2^n}, x_0 + \frac{y}{2^n} \right).$$

By induction let us prove that for all $n \in \mathbb{N} \setminus \{0\}$,

$$\text{osc}_{Q_{1/2}} u_n \leq 2\theta.$$  \hspace{1cm} (8)

Indeed for $n = 1$, we have [8] thanks to Lemma 3.4 since $\text{osc}_{Q_1} u_1 \leq 2$. Assuming that $\text{osc}_{Q_{1/2}} u_{n-1} \leq 2\theta$ and using Lemma 3.4, we distinguish two cases. If $\text{osc}_{Q_1} u_n \leq 2\theta$, we have [8]. If $\text{osc}_{Q_1} u_n \geq 2\theta$, we have

$$\text{osc}_{Q_{1/2}} u_n \leq \theta \text{osc}_{Q_1} u_n = \text{osc}_{Q_{1/2}} u_{n-1} \leq 2\theta,$$

...
and we deduce (8). So using (8) for \( n - 1 \) we have,

\[
\text{osc}_{\bar{Q}_n} u_n = \frac{1}{\theta_{Q_{1/2}}} \text{osc}_{\bar{Q}_{n-1}} u_{n-1} \leq 2.
\]

Thus we deduce by induction and using Lemmas 3.4 and 3.8 that for all \( n \geq 1 \),

\[
\text{osc}_{\bar{Q}_{2^n/2}(t_0, x_0)} u = \frac{\theta^n - 1}{2} \text{osc}_{\bar{Q}_{2^n/2}} u_n \leq (\theta^n - 1) C \left( \|u\|_{L^2(\bar{Q}_2)} + 1 \right).
\]

We choose \( \alpha \in (0, 1) \) such that \( \theta = \frac{1}{2^n} \). Let \( r \in \left(0, \frac{1}{2^n}\right) \). In particular there exists \( n \in \mathbb{N} \setminus \{0\} \) such that \( \frac{1}{2^n + 1} \leq r < \frac{1}{2^n} \). So we deduce that

\[
\text{osc}_{\bar{Q}_r(t_0, x_0)} u \leq \left(\frac{1}{2^n}\right)^\alpha C 2^{\alpha n} \left( \|u\|_{L^2(\bar{Q}_2)} + 1 \right) \leq r^\alpha C 4^n \left( \|u\|_{L^2(\bar{Q}_2)} + 1 \right).
\]

Remark 3.5. To prove Lemma 3.4 we only used the scaling properly of the definition of the De Giorgi classes.

The local decrease of the oscillation is a consequence of the following result.

**Lemma 3.6** (Lowering the maximum). *There exists a constant \( \mu \in (0, 1) \) which only depends on \( d, \gamma_1, \gamma_2, \gamma_3, \) and \( p \), such that for any function \( v : Q_2 \to \mathbb{R} \) in \( DG^+ (\gamma_1, \gamma_2, \beta \gamma_3, p) \) where \( \beta \) satisfies (7), if \( v \) verifies

\[
\left\{ \begin{array}{l}
v \leq 1 \text{ in } Q_2 \\
|\{v \leq 0\} \cap \bar{Q}_1| \geq \frac{|\bar{Q}_1|}{2},
\end{array} \right.
\]

then

\[
v \leq 1 - \mu \text{ in } Q_{1/2}.
\]

These cylinders are represented in Figure 1. We assume that Lemma 3.6 is true and prove Lemma 3.4.

**Proof of Lemma 3.4**. We distinguish two cases: either \( \text{osc} u \geq 2 \) or \( \text{osc} u \leq 2 \). In the first case, we set \( v = \frac{2}{\text{osc}_{\bar{Q}_1}} u - \frac{\sup_{\bar{Q}_1} u + \inf_{\bar{Q}_1} u}{2} \), where the supremum and the infimum are taken in \( Q_1 \). So \( v \) is still in \( DG^+ (\gamma_1, \gamma_2, \beta \gamma_3, p) \cap DG^- (\gamma_1, \gamma_2, \beta \gamma_3, p) \) where \( \beta \) satisfies (7). Moreover \(-1 \leq v \leq 1 \) in \( B_1 \) and either \( v \) or \(-v \) satisfy (9). We deduce that

\[
\text{osc}_{B_{1/2}} v \leq 2 - \mu,
\]

and

\[
\text{osc}_{B_{1/2}} u \leq \left(1 - \frac{\mu}{2}\right) \text{osc}_{B_1} u.
\]

We deduce the result taking \( \theta = 1 - \frac{\mu}{2} \).
In the second case, we set $v = u - \frac{\sup u + \inf u}{2}$. The functions $v$ and $-v$ are still in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p) \cap DG^-(\gamma_1, \gamma_2, \beta \gamma_3, p)$ where $\beta$ satisfies (7). And either $v$ or $-v$ satisfies (9). So we have

$$\osc_{Q_{1/2}} u = \osc_{Q_{1/2}} v \leq 2 - \mu \leq 2 \left(1 - \frac{\mu}{2}\right).$$

We deduce the result taking $\theta = 1 - \frac{\mu}{2}$. \hfill \square

**Remark 3.7.** To prove Lemma 3.4 concerning the definition of the De Giorgi classes, we only use the fact that if $v$ is in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p)$ then $-v$ is in $DG^-(\gamma_1, \gamma_2, \beta \gamma_3, p)$ and reciprocally.

### 3.2 Lemmas of De Giorgi

In this subsection, we introduce the two lemmas of De Giorgi which strongly rely on the definition of the De Giorgi classes.

#### 3.2.1 First lemma of De Giorgi

Let us state the first lemma of De Giorgi which is a $L^2 - L^\infty$ estimate.

**Lemma 3.8** (First Lemma of De Giorgi: $L^2 - L^\infty$ estimate). There exists a positive constant $\delta$ which depends only on $d, \gamma_1, \gamma_2, \gamma_3$ and $p$ such that for any $u : Q_2 \to \mathbb{R}$ in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ the following implication holds true. If

$$\int_{Q_1} u_2^2 \leq \delta,$$
then we have
\[ u_+ \leq \frac{1}{2} \quad \text{in } Q_{1/2}. \]

**Remark 3.9.** By applying Lemma 3.8 to \( \frac{\sqrt{\delta u}}{\left(\|u\|_{L^2(Q_{1/2})} + 1\right)} \) we get the following inequality
\[ \|u_+\|_{L^\infty(Q_{1/2})} \leq C\left(\|u_+\|_{L^2(Q_{1})} + 1\right), \]
where \( C > 0 \) depends only on \( d, \gamma_1, \gamma_2, \gamma_3 \) and \( p \).

**Remark 3.10.** We can replace \( Q_{1/2} \) and \( Q_1 \) by respectively \( Q_{3/2} \) and \( Q_2 \) so that \( u \) is bounded in \( Q_{3/2} \) and the oscillation of \( u \) was well-defined in the previous lemmas.

**Remark 3.11.** By symmetry, we can get the same result for \( u_- \) and \( DG^- (\gamma_1, \gamma_2, \gamma_3, p) \) and deduce the result for \( u \) and \( DG^+ (\gamma_1, \gamma_2, \gamma_3, p) \cap DG^- (\gamma_1, \gamma_2, \gamma_3, p) \).

Before doing the proof let us introduce a lemma which will be useful for the proof.

**Lemma 3.12.** Let \( (V_k)_{k \leq 0} \) be a sequence of real numbers such that for all \( k \geq 1 \),
\[ V_k \leq C^k V_{k-1}^\alpha \quad (10) \]
where \( \alpha > 1 \). Then for \( V_0 < C^{-\frac{\alpha^2}{(\alpha-1)^2}} \), the sequence \( (V_k) \) converges to 0 when \( k \to \infty \).

**Proof.** By induction we have
\[ V_k \leq C^k (k-1)\alpha + \ldots + 2\alpha^{k-2} + \alpha^{k-1} V_0^\alpha = C^k S_k V_0^\alpha, \]
where \( S_k = \sum_{i=0}^k i\alpha^{k-i} \).

Let us prove that for all \( k \geq 1 \),
\[ S_k \leq \frac{\alpha^2}{(\alpha-1)^2} \alpha^{k-1}. \quad (11) \]
In fact,
\[ S_k = \alpha^{k-1} \sum_{i=0}^k i \left(\frac{1}{\alpha}\right)^{i-1}. \]
And we know that
\[ \sum_{i=0}^k X^i = \frac{1 - X^{k+1}}{1 - X}, \]
so by differentiating we get
\[ \sum_{i=0}^k iX^{i-1} = \frac{X^k(kX - (k + 1)) + 1}{(1 - X)^2}. \]
And since \( \alpha > 1 \), we deduce
\[ S_k \leq \alpha^{k-1} \frac{1}{(1 - \frac{1}{\alpha})^2}, \]
which gives (11). So we have
\[ V_k \leq C^\frac{\alpha^2}{(\alpha-1)^2} \alpha^{k-1} V_0^\alpha \leq \left( C^\frac{\alpha^2}{(\alpha-1)^2} V_0 \right)^\alpha. \]
And for \( V_0 < C^{-\frac{\alpha^2}{(\alpha-1)^2}} \), we deduce that \( V_k \to 0 \), when \( k \to \infty \). \( \square \)
The following proof already exists in [32, 20]. Our proof here is a bit different from [32] since it doesn’t use an interpolation inequality and from [20] since we use a Sobolev inequality instead of the $L^p$ gain of integrability relying on averaging lemmas [3] and we use the energy estimate in a different way.

Proof of Theorem 3.8. In this proof $C > 0$ will denote a constant which will only depend on $d, \gamma_1, \gamma_2, \gamma_3$, and $p$. We define

$$U_k = \int_{Q_{r_k}} (u - c_k)^2 dx dt,$$

where $r_k = \frac{1}{2}(1 + 2^{-k})$ and $c_k = \frac{1}{2}(1 - 2^{-k})$. We notice that $Q_{r_k}$ goes from $Q_1$ to $Q_{\frac{1}{2}}$ and $c_k$ from 0 to $\frac{1}{2}$. We would like to prove that $U_k$ satisfies the following induction formula

$$U_k \leq C^k U_{k-2}^\alpha,$$

where $C > 0$ is a universal constant and $\alpha > 1$ also. Defining $V_k = U_{2k}$, the sequence $(V_k)$ satisfy

$$V_k \leq C^{2k} V_{k-1}^\alpha,$$

and as in [20, Theorem 12], we deduce that $V_n = U_{2n}$ tends to 0 when $V_0 = U_0$ is small enough. Moreover we have $U_0 = \int_{Q_1} u^2_+ \text{ and } U_\infty = \int_{Q_{\frac{1}{2}}} (u - \frac{1}{2})^2_+ = 0$ and we deduce the result.

Let us prove the induction formula. Let us define the Sobolev exponent

$$\rho = \begin{cases} \frac{2d}{d-2} & \text{if } d > 2 \\ q & \text{if } d = 2, \quad \text{with } q \in (4, +\infty) \\ +\infty & \text{if } d = 1 \end{cases}$$

in the following Sobolev inequality, for almost every $t \in (-r_k^2, 0)$,

$$\|(u - c_k)_+(t, \cdot)\|_{L^p(B_{r_k})} \leq C(d)\|(u - c_k)_+(t, \cdot)\|_{H^1(B_{r_k})}, \quad (12)$$

where $C(d)$ is a constant which only depends on the dimension $d$ and which can be explicitly computed using [2]. Using an Hölder inequality, we have

$$U_k = \int_{Q_{r_k}} (u - c_k)^2_+ \leq \int_{-r_k^2}^{0} \left( \int_{B_{r_k}} (u - c_k)_+^\rho(t, \cdot) dx \right)^{\frac{2}{\rho}} \left| \{u(t, \cdot) \geq c_k\} \cap B_{r_k} \right|^{1 - \frac{2}{\rho}} dt. \quad (13)$$

Since $\{u(t, \cdot) \geq c_k\} = \{u(t, \cdot) \geq c_{k-1} + 2^{-k-1}\}$, we deduce that

$$\left| \{u(t, \cdot) \geq c_k\} \cap B_{r_k} \right|^{1 - \frac{2}{\rho}} \leq \left| \{u(t, \cdot) \geq c_{k-1} + 2^{-k-1}\} \cap B_{r_k} \right|^{1 - \frac{2}{\rho}} \leq \left( 2^{2k+2} \int_{B_{r_k}} (u - c_{k-1})^2_+(t, \cdot) \right)^{1 - \frac{2}{\rho}} \leq C^k \left( \sup_{t \in (-r_k^2, 0)} \int_{B_{r_k}} (u - c_{k-1})^2_+(t, \cdot) \right)^{1 - \frac{2}{\rho}}. \quad (14)$$
We can use the first part of the inequality defining the De Giorgi class (Definition 2.3) with \( s \) integrated in \((-r_{k-1}^2, -r_k^2)\) to bound the supremum and obtain in [14],

\[
|\{u(t, \cdot) \geq c_k\} \cap B_{r_k}|^{1 - \frac{2}{p}} \leq C^k \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})^2 dx + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^p \right)^{\frac{1}{p} - \frac{2}{p}} \\
\leq C^k \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})^2 dx + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} 1_{\{u \geq c_{k-1}\}} \right)^{1 - \frac{2}{p}} \\
+ C^k \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})^2 dx + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} 1_{\{u \geq c_{k-1}\}} \right)^{\frac{1}{p} \left( 1 - \frac{2}{p} \right)}.
\]

(15)

where we used that \((u - c_{k-1})_+ \leq (u - c_{k-1})^2 + 1_{\{u \geq c_{k+1}\}}\) to get the last bound. Since we have

\[
\int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} 1_{\{u \geq c_{k-1}\}} = |\{u \geq c_{k-1}\} \cap Q_{r_{k-1}}| \\
\leq |\{u(t, \cdot) \geq c_{k-2} + 2^{-k}\} \cap Q_{r_{k-1}}| \\
\leq 2^{2k} \int_{Q_{r_{k-1}}} (u - c_{k-2})_+^{2} \\
\leq 2^{2k} U_{k-2},
\]

we deduce using [15],

\[
|\{u(t, \cdot) \geq c_k\} \cap B_{r_k}|^{1 - \frac{2}{p}} \leq C^k \left( (U_{k-1} + U_{k-2})^{1 - \frac{2}{p}} + (U_{k-1} + U_{k-2})^{\frac{1}{p} \left( 1 - \frac{2}{p} \right)} \right) \\
\leq C^k \left( U_{k-2}^{1 - \frac{2}{p}} + U_{k-2}^{\frac{1}{p} \left( 1 - \frac{2}{p} \right)} \right)
\]

(16)

We notice that the last bound is independent of the variable \( t \) so it remains to bound

\[
\int_{-r_{k-1}^2}^{0} \left( \int_{B_{r_k}} (u - c_k)_+^p (t, \cdot) dx \right)^{\frac{2}{p}} dt \text{ in } (13).
\]

Using the Sobolev inequality (12) and the second part of the inequality defining the De Giorgi class (Definition 2.3) with \( s \) integrated in \((-r_{k-1}^2, -r_k^2)\), we deduce

\[
\int_{-r_{k-1}^2}^{0} \left( \int_{B_{r_k}} (u - c_k)_+^p (t, \cdot) dx \right)^{\frac{2}{p}} \leq C \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_k}} (u - c_k)_+^2 (t, \cdot) dx + 2^{2k} \int_{B_{r_{k-1}}} \nabla_x (u - c_k)_+^2 \right) \\
\leq C \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^p \right)^{\frac{1}{p}} \\
\leq C \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} 1_{\{u \geq c_{k-1}\}} \right) \\
+ C \left( \int_{-r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 + \int_{r_{k-1}^2}^{0} \int_{B_{r_{k-1}}} 1_{\{u \geq c_{k-1}\}} \right)^{\frac{1}{p}} \\
\leq C^k \left( U_{k-2} + U_{k-2}^{\frac{2}{p}} \right).
\]

(17)
By definition $U_k$ is non-increasing so assuming that $U_0 < 1$, we have $U_k < 1$ for every $k \geq 0$. Combining (16) and (15) and assuming $U_0 < 1$, we deduce that $U_k$ satisfies the formula

$$U_k \leq C^k \left( U_{k-2} + U_{k} \frac{1}{2} \right) \left( U_{k-2}^{1-\frac{2}{p}} + U_{k}^{1-\frac{2}{p}(1-\frac{2}{p})} \right) \leq C^k U_{k-2}^\alpha,$$

with $\alpha = \frac{1}{p} \left( 2 - \frac{2}{p} \right) > 1$ which ends the proof using Lemma 3.12 choosing $\delta < C^{-\frac{2}{\alpha-1}}$. \hfill \Box

### 3.2.2 Second lemma of De Giorgi

To prove the result of lowering of maximum (Lemma 3.6) we need also the so-called second lemma of De Giorgi, the intermediate value lemma.

**Lemma 3.13** (Second lemma of De Giorgi: Intermediate value lemma). *Let $u$ be in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ such that $u \leq 1$ on $Q_2$. Let $Q_1 = (-2, -1) \times B_1$. Then we have*

$$|\{f \leq 0\} \cap Q_1||\{f \geq \frac{1}{2}\} \cap Q_1| \leq C|\{0 < f < \frac{1}{2}\} \cap Q_2|^{\frac{1}{p+2}},$$

where $C$ only depends on $d, \gamma_1, \gamma_2, \gamma_3$, and $p$.

**Proof.** We apply Theorem 1.5 with $k = 0$ and $l = \frac{1}{2}$. \hfill \Box

### 3.3 Proof of the lowering of the maximum lemma

Now we can prove Lemma 3.6 using the first and the second lemma of De Giorgi.

**Proof of Lemma 3.6.** We introduce a sequence of function $v_k$ in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$,

$$\begin{cases}
  v_0 = v \\
  v_k = 2 \left( v_{k-1} - \frac{1}{2} \right).
\end{cases}$$

Here $v$ is a function in $DG^+(\gamma_1, \gamma_2, \beta \gamma_3, p)$ where $\beta$ satisfies (7) and the functions $v_k$ are in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ (it will be explained at the end of the proof why the sequence $v_k$ remains in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$). More precisely, we have $v_k = 2^k \left( v - (1 - 2^{-k}) \right)$. So that the sets $\{0 < v_k < \frac{1}{2}\} = \{1 - \frac{1}{2^k} < v < 1 - \frac{1}{2^{k+1}}\}$ are disjoints and the sequence $v_k$ still satisfies (9).

If $\int_{Q_1} (v)^2 \leq \delta$ then by Lemma 3.8, $v \leq \frac{1}{2}$ and we have the result. By the same arguments, if $\int_{Q_1} (v)^2 \leq \delta$ then by Lemma 3.8, $v \leq \frac{3}{4}$ and we have the result.

If not, we consider $k_0 \geq 1$ an index such that $\int_{Q_1} (v_k)^2 > \delta$, for any $0 \leq k \leq k_0$. We have the following inequalities for any $0 \leq k \leq k_0 - 1$,

$$|\{v_k \geq \frac{1}{2}\} \cap Q_1| = |\{v_{k+1} \geq 0\} \cap Q_1| \geq \int_{Q_1} (v_{k+1})^2 > \delta,$$

and

$$|\{v_k \leq 0\} \cap Q_1| \geq |\{v \leq 0\} \cap Q_1| \geq \frac{|Q_1|}{2}.$$
So by the intermediate value lemma (Lemma 3.13),

\[ |\{0 < v_k < \frac{1}{2}\} \cap Q_2| \geq \left( \frac{\delta}{C} \frac{|Q_1|}{2} \right)^6. \]

By summing all the intermediate measure and using the fact that the sets are disjoints we have,

\[ |Q_2| \geq \sum_{k=1}^{k_0} |\{0 < v_k < \frac{1}{2}\} \cap Q_2| \geq k_0 \left( \frac{\delta}{C} \frac{|Q_1|}{2} \right)^6. \]

So \( k_0 \) is bounded such that

\[ k_0 \leq \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2|, \]

and necessarily, there exists \( k \leq \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2| + 1 \) such that \( \int_{Q_1} (v_k)^2 \leq \delta \) so by Lemma 3.8 we have \( (v_k)_+ \leq \frac{1}{2} \) in \( Q_{1/2} \) so that

\[ v \leq 1 - \frac{1}{2^{k+1}} \leq 1 - \frac{1}{2 \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2| + 2} \] in \( Q_{1/2} \),

and we choose \( \mu = \frac{1}{2 \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2| + 2} \). So in the end we deal only with the sequence until a universal index, so choosing \( \beta = \frac{1}{2 \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2| + 1} \), for all \( k \leq \left( \frac{2C}{\delta |Q_1|} \right)^6 |Q_2| + 1, v_k \) is in \( DG^+(\gamma_1, \gamma_2, \gamma_3, p) \).

\[ \square \]

4 Intermediate value lemma

In this section, we deal with intermediate value lemmas for functions in \( H^1 \) and for functions in \( DG^+(\gamma_1, \gamma_2, \gamma_3, p) \). We first recall the lemma in the \( H^1 \) case since we use it in the proof of the \( DG^+(\gamma_1, \gamma_2, \gamma_3, p) \) case. Then we give the proof of Theorem 1.5, the intermediate value lemma for functions in the De Giorgi class \( DG^+(\gamma_1, \gamma_2, \gamma_3, p) \).

4.1 Functions in \( H^1 \)

We give a simpler proof of [22, Theorem 2.9], about an intermediate value lemma for functions which are bounded in the Sobolev space \( H^1 \). This lemma in an alternative version of the De Giorgi isoperimetric inequality [32, Lemma 10]. As we previously saw, it is a crucial tool in the De Giorgi proof of the Hölder regularity for solutions of elliptic equations.

**Lemma 4.1** (Intermediate value lemma in \( H^1 \)). Let \( u \in H^1(B_R) \). Then for all \( (k, l) \in \mathbb{R}^2 \) such that \( k \leq l \), we have

\[ (l-k) |\{u \leq k\} \cap B_R| \times |\{u \geq l\} \cap B_R| \leq R |B_R| |\{k < u < l\} \cap B_R| \left( \frac{1}{2} \right) \sqrt{\int_{B_R} |\nabla (u-k)_+|^2 \, dx}. \]

\[ (19) \]
Proof. We will use the shorthand notations $|u \leq k|$, $|u \geq l|$ and $|k < u < l|$ for the measures of the sets $\{x \in B_R, u(x) \leq k\}$, $\{x \in B_R, u(x) \geq l\}$ and $\{x \in B_R, k < u(x) < l\}$. We define the following truncated function
\[
v(x) = \begin{cases} 
0 & \text{if } u(x) \leq k, \\
u(x) - k & \text{if } k < u(x) < l, \\
l - k & \text{if } u(x) \geq l.
\end{cases}
\] (20)

By Stampacchia theorem in [19, Theorem 7.8] or [16], we have $v \in H^1(B_R)$. By Poincaré inequality since $v \in W^{1,1}(B_R)$, see for example [11, Theorem 3.2], we have
\[
\int_{B_R} |v(x) - \bar{v}| \, dx \leq R \int_{B_R} |\nabla v(x)| \, dx,
\] (21)
where $\bar{v} = \frac{1}{|B_R|} \int_{B_R} v(x) \, dx$. The sets $\{x \in B_R, v(x) = 0\}$ and $\{x \in B_R, v(x) = l - k\}$ are respectively denoted by $\{v = 0\}$ and $\{v = l - k\}$ and their measures by $|v = 0|$ and $|v = l - k|$. We have the following inequalities
\[
\frac{(l - k)}{|B_R|} |v = 0||v = l - k| \leq \int_{\{v = 0\}} \bar{v} \, dx \leq \int_{\{v = 0\}} |v(x) - \bar{v}| \, dx \leq \int_{B_R} |v(x) - \bar{v}| \, dx,
\] (22)
and by Cauchy-Schwarz inequality
\[
\int_{B_R} |\nabla v(x)| \, dx = \int_{\{k < u < l\}} |\nabla v(x)| \, dx \leq \sqrt{\int_{B_R} |\nabla (u - k)_+(x)|^2 \, dx} |\{k < u < l\} \cap B_R|^{1/2}.
\] (23)

Using (21), (22) and (23) and the equalities $|v = 0| = |\{u \leq k\} \cap B_R|$ and $|v = l - k| = |\{u \geq l\} \cap B_R|$, we deduce [19]. \qed

4.2 Functions in $DG^+$

In this section, we prove Theorem 1.5. The proof of this theorem deeply uses the definition of the De Giorgi class. We can see the inequality of Definition 2.3 as two inequalities. The second one
\[
\gamma_1 \int_s^t \int_{B_r(x_0)} |\nabla_x (u - k)_{\pm}(\tau, x)|^2 \, dx \, d\tau \\
\leq \int_{B_R(x_0)} (u - k)^2_{\pm}(s, x) \, dx + \frac{\gamma_2}{(R - r)^2} \int_s^t \int_{B_R(x_0)} (u - k)^2_{\pm}(\tau, x) \, dx \, d\tau \\
+ \gamma_3 \left( \int_s^t \int_{B_R(x_0)} (u - k)^p_{\pm}(\tau, x) \, dx \, d\tau \right)^{1/p},
\]
contains the information which quantify the fact that there is no jump in the space variable $x$. In fact, it helps us to bound the norm of the gradient of a function by a universal constant since $(u - k)_{\pm}$ is bounded and to get an intermediate value lemma in $H^1$ which only depends on the measures and universal constant. So we first get the following lemma.
Lemma 4.2 (Universal bound of the $L^2$ of the gradient). Let $u : Q_2 \to \mathbb{R}$ be a function in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ such that $u \leq 1$ on $Q_{\frac{5}{4}}$. Then there exists a constant $\bar{C} > 0$ such that for all $-2 < s < t < 0$, we have
\[
\int_s^t \int_{B_{\frac{5}{4}}} |\nabla_x (u - k)_+|^2 \, dx \, d\tau \leq \bar{C},
\]
where $\bar{C}$ only depends on $d, k, \gamma_1, \gamma_2, \gamma_3$ and $p$.

Proof of Lemma 4.2. We use Definition 2.3 for $r = \frac{5}{4}$, $R = \frac{3}{2}$, $x_0 = 0$ and we deduce
\[
\int_s^t \int_{B_{\frac{5}{4}}} |\nabla_x (u - k)_+|^2 \, dx \, d\tau \leq \frac{(1-k)^2}{\gamma_1}|B_{\frac{3}{2}}| + 32\frac{\gamma_2}{\gamma_1}|B_{\frac{3}{2}}|(1-k)^2 + 2\frac{\gamma_2}{\gamma_1}|\bar{B}_{\frac{3}{2}}|^\frac{1}{p} (1-k).
\]

The second lemma is a first step for the proof of Theorem 1.5. It gives “almost” an intermediate value lemma with an error which is small for close times. We will see that the first inequality of Definition 2.3
\[
\int_{B_{s}(x_0)} (u - k)_+^2(t, x) \, dx
\]
\[
\leq \int_{B_{R}(x_0)} (u - k)_+^2(s, x) \, dx + \frac{\gamma_2}{(R - r)^2} \int_s^t \int_{B_{R}(x_0)} (u - k)_+^2(\tau, x) \, dx \, d\tau
\]
\[
+ \gamma_3 \left( \int_s^t \int_{B_{R}(x_0)} (u - k)_+^p(\tau, x) \, dx \, d\tau \right)^{1/p}
\]
contains the information which quantify the fact that $u$ cannot do an increasing jump in time in term of measures. In fact, $(u - k)_+$ is bounded so the previous inequality becomes
\[
\int_{B_{s}(x_0)} (u - k)_+^2(t, x) \, dx \leq \int_{B_{R}(x_0)} (u - k)_+^2(s, x) \, dx + C(t - s)^{1/p}.
\]

Lemma 4.3 (A key inequality for close times). Let $u : Q_2 \to \mathbb{R}$ be a function in $DG^+(\gamma_1, \gamma_2, \gamma_3, p)$ such that $u \leq 1$ on $Q_{\frac{5}{4}}$. Then for all $(k, l) \in \mathbb{R}^2$ such that $k < l \leq 1$ and for all $(t_1, t_2, \tau) \in (-2, 0)^3$ such that $-2 < t_1 < \tau < t_2 < 0$, we have
\[
(l-k)^2|u \geq l, (\tau, t_2) \times B_1 | u \leq k, (t_1, \tau) \times B_1 | \leq C |k < u < l, (t_1, \tau) \times B_2| + \frac{1}{2} + C(t_2 - t_1)^{2+\frac{1}{p}},
\]
where $C$ only depends on $d, k, \gamma_1, \gamma_2, \gamma_3$ and $p$.

Proof of Lemma 4.3. In this proof, let $C > 0$ be a constant which only depends on $d, k, \gamma_1, \gamma_2, \gamma_3$ and $p$ which will change from line to line. Thanks to the definition of De Giorgi classes, we have
\[
\int_{B_1} (u - k)_+^2(t, x) \, dx
\]
\[
\leq \int_{B_{\frac{5}{4}}} (u - k)_+^2(s, x) \, dx + 16\gamma_2 \int_s^t \int_{B_{\frac{5}{4}}} (u - k)_+^2(\tau, x) \, dx \, d\tau
\]
\[
+ \gamma_3 \left( \int_s^t \int_{B_{\frac{5}{4}}} (u - k)_+^p(\tau, x) \, dx \, d\tau \right)^{1/p}.
\]

(24)
First, we bound the left hand side from below
\[ \int_{B_1} (u - k)^2 (t, x) dx \geq \int_{\{y \in B_1 : u(t,y) \geq l\}} (l - k)^2(t, x) dx \geq (l - k)^2 \{u(.,.) \geq l\} \cap B_1. \] (25)
Second, we bound from above each term of the right hand side. The first term gives
\[ \int_{B_{\frac{5}{4}}} (u - k)^2(s,x) dx \leq \int_{\{y \in B_{\frac{5}{4}} : k < u(s,y) < l\}} (u - k)^2(s,x) dx + \int_{\{y \in B_{\frac{5}{4}} : u(s,y) \geq l\}} (u - k)^2(s,x) dx \leq C \left( |\{k < u(.,.) < l\} \cap B_{\frac{5}{4}}| + |\{u(.,.) \geq l\} \cap B_{\frac{5}{4}}| \right). \]
(26)
The second term gives
\[ 16\gamma_2 \int_s^t \int_{B_{\frac{5}{4}}} (u - k)^2(\tau, x) dx d\tau \leq C(t - s). \] (27)
And the third term gives
\[ \gamma_3 \left( \int_s^t \int_{B_{\frac{5}{4}}} (u - k)^p(\tau, x) dx d\tau \right)^{1/p} \leq C(t - s)^\frac{1}{p}. \] (28)
So combining (24), (25), (26), (27) and (28), we deduce
\[ (l - k)^2 \{u(.,.) \geq l\} \cap B_1 \leq C \left( |\{k < u(.,.) < l\} \cap B_{\frac{5}{4}}| + |\{u(.,.) \geq l\} \cap B_{\frac{5}{4}}| \right) + C(t - s) + C(t - s)^\frac{1}{p}. \]
Multiplying the last inequality by |\{u(.,.) \leq k\} \cap B_{\frac{5}{4}}| and using the fact that
\[ |\{u(.,.) \geq l\} \cap B_{\frac{5}{4}}| \{u(.,.) \leq k\} \cap B_{\frac{5}{4}}| \leq C |\{k < u(.,.) < l\} \cap B_{\frac{5}{4}}| \int_{B_{\frac{5}{4}}} |\nabla_x (u - k)^+|^2(s,x) dx, \]
thanks to Lemma 19 (since \(u(.,.) \in H^1(B_{\frac{5}{4}})\) by Fubini’s theorem), we get
\[ (l - k)^2 \{u(.,.) \geq l\} \cap B_1 \{u(.,.) \leq k\} \cap B_1 \leq C \left( |\{k < u(.,.) < l\} \cap B_{\frac{5}{4}}| + |\{k < u(.,.) < l\} \cap B_{\frac{5}{4}}| \int_{B_{\frac{5}{4}}} |\nabla_x (u - k)^+|^2(s,x) dx \right) + C(t - s) + C(t - s)^\frac{1}{p}. \]
We integrate the latter over \(s \in [t_1, \tau]\) and \(t \in [\tau, t_2]\) with \(-2 \leq t_1 < \tau < t_2 \leq 0\) and obtain using Lemma 12
\[ (l - k)^2 |u \geq l, (\tau, t_2) \times B_1| \{u \leq k, (t_1, \tau) \times B_1\} \leq C \left( |\{k < u < l \cap (t_1, \tau) \times B_{\frac{5}{4}}| + |\{k < u < l \cap (t_1, \tau) \times B_{\frac{5}{4}}| \frac{1}{2} \sqrt{C} \right) + C(t_2 - t_1)^3 + C(t_2 - t_1)^{2 + \frac{1}{2}}. \] (29)
Simplifying (29), we have

\[(l-k)^2|u \geq l, (\tau, t_2) \times B_1|/|u \leq k, (t_1, \tau) \times B_1| \leq C|k < u < l, (t_1, \tau) \times B_2|^{\frac{1}{2}}
+ C(t_2 - t_1)^{2 + \frac{1}{p}}, \tag{30}\]

which ends the proof.

Now let us prove Theorem 1.5. The idea of the proof is to understand that the “error”
term \((t_2 - t_1)^{2 + \frac{1}{p}}\) in Lemma 4.3 is negligible compared to the other terms when \(t_2 - t_1\) is small and when the intervals are well-chosen.

**Proof of Theorem 1.5** Using that \(|u \leq k, (t_1, \tau) \times B_1| = |u \leq l, (t_1, \tau) \times B_1| - |k < u < l, (t_1, \tau) \times B_1|\), we deduce from Lemma 4.3

\[(l-k)^2|u \geq l, (\tau, t_2) \times B_1|/|u \leq l, (t_1, \tau) \times B_1| \leq C|k < u < l, Q_2|^{\frac{1}{2}} + C(t_2 - t_1)^{2 + \frac{1}{p}}.\]

We discretize the time interval. Let \(n \in \mathbb{N} \setminus \{0\}, \alpha_n = \frac{1}{n}, \overline{T} = -1\) and \(t_k = k\alpha_n\). Necessarily by the pigeonhole principle, there exists \(i \in [1, n]\) such that

\[|u \leq k, (t_{i-1}, t_i) \times B_1| \geq \frac{|u \leq k, Q_1|}{n}, \tag{31}\]

and there exists \(j \in [n, 2n - 1]\) such that

\[|u \geq l, (t_j, t_{j+1}) \times B_1| \geq \frac{|u \geq l, Q_1|}{n}. \tag{32}\]

But since we would like adjacent intervals of time, we relax the inequalities (31) and (32) and we still have

\[|u < l, (t_{i-1}, t_i) \times B_1| \geq \frac{|u \leq k, Q_1|}{2n}, \tag{33}\]

and

\[|u \geq l, (t_j, t_{j+1}) \times B_1| \geq \frac{|u \geq l, Q_1|}{2n}. \tag{34}\]

We distinguish two cases, either there exists \(m \in [i, 2n - 1]\) such that \(m + 1\) does not satisfy (33) (i.e., (33) is false for \(i = m + 1\)), or for all \(m \in [i, 2n - 1]\), \(m + 1\) does satisfy (33). In the first case, letting \(p\) be the first integer \(m\) satisfying “\(m + 1\) does not satisfy (33),” we have

\[|u < l, (t_p, t_{p+1}) \times B_1| < \frac{|u \leq k, Q_1|}{2n},\]

so

\[|u \geq l, (t_p, t_{p+1}) \times B_1| \geq |B_1|\alpha_n - \frac{|u \leq k, Q_1|}{2n} \geq \frac{|u \leq l, Q_1|}{2n}\]

and

\[|u < l, (t_{p-1}, t_p) \times B_1| \geq \frac{|u \leq k, Q_1|}{2n}.\]

In the second case, let \(p = j\). Then in all cases, using Lemma 4.3 we have,

\[(l-k)^2\frac{|u \leq k, Q_1|}{2n} \leq \frac{|u \geq l, (t_{p-1}, t_p) \times B_1|}{2n} \leq C_1|k < u < l, Q_2|^{\frac{1}{2}} + C_2\left(\frac{2}{n}\right)^{2 + \frac{1}{p}}.\]
Thus, we have

$$(l - k)^2 |u \leq k, Q_1|u \geq l, \bar{Q}_1| \leq Cn^2 |k < u < l, Q_2|^{\frac{1}{2}} + Cn^{-\frac{1}{2}}.$$ 

So necessarily $|k < u < l, Q_2| > 0$. And taking $n$ such that $Cn^{-\frac{1}{2}} \leq C \frac{n^2 |k < u < l, Q_2|^{\frac{1}{2}}}{2}$, for example $n = \left\lfloor \frac{2}{|k < u < l, Q_2|^{\frac{1}{2}}} \right\rfloor + 1$, we get

$$(l - k)^2 |u \leq k, Q_1|u \geq l, \bar{Q}_1| \leq C|k < u < l, Q_2|^{\frac{1}{2}}.$$ 

This achieves the proof of the theorem.

4.3 Remarks and counterexamples

We remark that Theorem [1.5] is false for subsolutions if we replace $Q_1$ by $Q_1^{-1}$. For example, the function

$$f(t, x) = \begin{cases} 1 & \text{for } x + 2t < -2 \\ 0 & \text{for } x + 2t \geq -2, \end{cases}$$

is a subsolution of (1) in $Q_2$ but does not satisfy Theorem [1.5] for $k = 0$ and $l = 1$ with $Q_1$ instead of $\bar{Q}_1$. In fact, the intermediate value lemma does not allow increasing jump in time. In the solution case of for function in $DG^+ \cap DG^-$, we can obtain the same inequality with $Q_1$ instead of $\bar{Q}_1$ in (4).

4.3.1 Extension to kinetic equations?

Let us consider the following kinetic Fokker-Planck equation of [20],

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s, \quad (t, v, x) \in Q_2, \quad (35)$$

where $Q_R = (-R^2, 0) \times B_R \times B_{R^3}$ is a kinetic cylinder. We define $\bar{Q}_1 = (-2, -1) \times B_1 \times B_1$. In dimension $d = 1$, considering the following subsolution

$$f(t, x, v) = \begin{cases} 1 & \text{for } x + 2t < -2 \\ 0 & \text{for } x + 2t \geq -2, \end{cases}$$

we notice that it does not satisfy an inequality of the form

$$|\{f \leq 0\} \cap \bar{Q}_1|^{\alpha} \{f \geq \frac{1}{2}\} \cap Q_1|^{\beta} \leq C|\{k < f < l\} \cap Q_2|^{\gamma}, \quad (36)$$

for some constants $\alpha, \beta, \gamma$ and $C$ which do not depend on the $f$. In fact, for some parameters $c > 1$ (to have a subsolution) and $a \in \mathbb{R}$,

$$f_{a,c}(t, x, v) = \begin{cases} 1 & \text{for } x + ct < a \\ 0 & \text{for } x + ct \geq a, \end{cases}$$

is also a subsolution of (35). Drawing many lines of discontinuity $x + ct = a$, we notice that to find a valid intermediate value inequality, we must consider two cylinders which cannot be both crossed by the same line of discontinuity $x + ct = a$. More precisely, we must
have a “gap” in time between the two cylinders of the same size (or at least not smaller) than the two cylinders. Let us change the definition of $Q^1$ by $Q^1 = (-3, -2) \times B_1 \times B_1$. The two domains $Q_1$ and $Q_1$ are never both crossed by the same line of discontinuity $x + ct = a$. That is why this intermediate value inequality seems to be more accurate,

$$\text{Card}\{f \leq 0\} \cap Q_1\cap \{f \geq \frac{1}{2}\} \cap Q_1 \leq C\text{Card}\{k < f < l\} \cap Q_R\cap.$$

In fact, the local energy estimate usually used for this equation (see for example [20, Lemma 11]) is too weak to be able to define kinetic De Giorgi classes in this way. We are losing information especially in the variable $x$ and subsolution are not bounded in $H^1$ so we cannot use directly the proof of the parabolic case. An idea then could be to keep the term $\partial_t f + v \cdot \nabla_x f$ as one block and to understand what would be the “good way” to integrate the equation and this term should makes the gap appear.

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