Superfield continuous spin equations of motion

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ABSTRACT

We propose a description of supersymmetric continuous spin representations in 4D, \( N = 1 \) Minkowski superspace at the level of equations of motions. The usual continuous spin wave function is promoted to a chiral or a complex linear superfield which includes both the single-valued (span integer helicities) and the double-valued (span half-integer helicities) representations thus making their connection under supersymmetry manifest. The set of proposed superspace constraints for both superfield generate the expected Wigner’s conditions for both representations.

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1 Introduction

It is generally accepted that the elementary excitation in any fundamental theory are classified according to the symmetries of the vacuum. It follows that the free elementary particles are associated with the unitary and irreducible representations of relevant spacetime symmetry groups. For this reason, special attention is paid to the consideration of maximally symmetric spacetimes (Minkowski, de Sitter and anti-de Sitter). In the case of 4D Minkowski spacetime, Wigner [1], classified all such representations. The one particle representations are labeled by the mass and spin quantum numbers, which correspond to the eigenvalues of the two invariant Casimir operators (quadratic) $C_1 = P^m P_m$ and (quartic) $C_2 = W^m W_m$, where $P_m$ and $W_m$ are the momentum and Pauli-Lubanski vectors respectively. The one particle states inside the representations are labeled by the eigenvalues of the corresponding Cartan subalgebra (like the spin/helicity in the direction of motion).

Some of these representations appear in local field theories and string theories. These are the familiar finite size representations that describe massless particles with fixed integer or half-integer helicity and massive particles with integer or half-integer spin. A subset of them have been observed in nature. Other representations are the tachyonic particles which are characterized by negative eigenvalues of $C_1$. Their presence indicates instabilities and they were never observed. The rest, fall in the category of continuous spin representations (CSR) [2–12]. This type of representations are massless (vanishing $C_1$) and are characterized by a non vanishing eigenvalue of the second Casimir $C_2 = \mu^2$, where $\mu$ is a real, continuous parameter with dimensions of mass. There are two such representations, the single-valued one and the double-valued. The size of both of them is countable infinite and their spectrum includes all integer separated integer or half integer helicity states respectively with multiplicity one.

The infinite number of degrees of freedom per spacetime point was the reason why Wigner rejected the use of such representations, claiming that the heat capacity of a gas of continuous spin particles is infinite [13]. Further attempts to relate these representations with physical systems have also failed. In ref. [6, 7] it was shown that the free field description of these representations breaks causality or locality thus making impossible to construct a consistent quantum field theoretic description. Not surprising, these representations have been ignored. Yet, the same two exotic properties (presence of a continuous dimensionfull parameter and an infinite tower of massless helicities) are very appealing from the point of view of higher spin (gravity) theories. Consistent interacting higher spin theories require the presence of infinite massless particles with arbitrary high helicities [14] and a dimensionfull parameter to weight the higher number of derivatives required by the interactions. This dimensionfull parameter is usually identified with the radius of (A)dS spacetime. CSR naturally provide both these features, hence in principle it can be seen as a good candidate model for interacting higher spins in flat spacetime and even possibly bypassing some of the no-go theorems [12]. For this reason, recently there has been an increased interest in the study of CSR [15–36] investigating various kinematic (covariant on-shell, off-shell descriptions) and dynamic (interactions, scattering amplitudes) properties (for a review see [27]).

The object of this work is to study the continuous spin representations in the presence of supersymmetry. In ref. [8] it was shown that the supersymmetry charges are compatible (commute) with the transverse vector generators of $iso(3,1)$: $\Pi^i$, $i = 1, 2$ which define the CSR. Hence, it is expected that the single-valued CSR can be combined with the doubled-valued CSR in order to assemble supersymmetric continuous spin

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4An alternative terminology for such representations is “infinite spin”. This is a more appropriate name because it captures the essence of the spectrum of these representations (spin is not bounded) and avoids the use of the misleading term “continuous spin” (the helicity of the states is not continuous). Nevertheless, for historical reasons continuous spin is the prevailed nomenclature and that is the one we will use.
representation (sCSR) that include both integer and half-integer helicities. In this letter we aim to find a covariant, on-shell description of sCSR that makes the supersymmetry manifest. For that we use the 4D Minkowski, $\mathcal{N} = 1$ superspace formulation. We find that the usual Wigner wavefunction is elevated to a chiral or complex linear superfield. Its bosonic component correspond to two single valued CSR and its fermionic component corresponds to one double-valued CSR. We propose a set of covariant superspace constraints that such a wavefunction must satisfy on-shell in order to describe a sCSR as this is defined by the super-Poincaré algebra. By projecting into components we show that these constraints give back Wigner’s equations for the single and double valued CSR as expected. During the time of writing, new work appeared [37] which demonstrates the connection of the single valued CSR with the double valued CSR under on-shell supersymmetry. The authors showed that on-shell supersymmetry transformations map one CSR to the other. Our results generalize this for off-shell supersymmetry transformation since the superfields provide the necessary auxiliary fields to close the algebra of the transformations without the use of equations.

The plan of the paper is as follows. In section 2, we review the group theoretical description of CSR using the method of the eigenvalues of the Casimirs of the Poincaré algebra. Then we review the discussion of massless representations of the super-Poincaré algebra and we extend the discussion to the definition of sCSR. In section 3, we consider the superfield description of such representations and derive the superspace, covariant, equations of motion it must satisfy. In section 4, we discuss the component projection and recover the on-shell description of the single valued and double valued CSR.

2 Review of CSR and sCSR from the symmetry algebra viewpoint

For the definition and classification of the various representations, we are following the method of diagonalizing the Casimirs and the Cartan subalgebra generators of the stabilizer subgroup of the four dimensional Poincaré group and its $\mathcal{N} = 1$ supersymmetric extension.

2.1 4D Poincaré algebra

The Poincaré group is the group of isometries of Minkowski spacetime and it is generated by the set of rigid motions, i.e. translations ($\mathbb{P}_m$) and rotations ($\mathbb{J}_{mn}$) which satisfy the algebra:

\[
[\mathbb{J}_{mn}, \mathbb{J}_{rs}] = i\eta_{mr}\mathbb{J}_{ns} - i\eta_{ms}\mathbb{J}_{nr} + i\eta_{ns}\mathbb{J}_{mr} - i\eta_{nr}\mathbb{J}_{ms},
\]
\[
[\mathbb{J}_{mn}, \mathbb{P}_r] = i\eta_{mr}\mathbb{P}_n - i\eta_{nr}\mathbb{P}_m,
\]
\[
[\mathbb{P}_m, \mathbb{P}_n] = 0.
\]

The stabilizer subgroup (little group) is generated by the translations generator $\mathbb{P}_m$ and the Pauli-Lubanski vector $\mathbb{W}_m$, $\mathbb{W}^m = \frac{1}{2}\varepsilon^{mnr}\mathbb{J}_{nr}\mathbb{P}_s$ with the algebra

\[
[\mathbb{W}_m, \mathbb{P}_n] = 0, \quad [\mathbb{W}^m, \mathbb{W}^n] = i\varepsilon^{mnr}\mathbb{W}_r\mathbb{P}_s, \quad \mathbb{W}^m\mathbb{P}_m = 0,
\]
\[
[\mathbb{J}_{mn}, \mathbb{W}_r] = i\eta_{mr}\mathbb{W}_n - i\eta_{nr}\mathbb{W}_m.
\]

It is useful to keep in mind that when we consider for realizations of representations of the above algebra in terms of fields (wavefunctions) the abstract generators $\mathbb{J}_{mn}$, $\mathbb{P}_m$ are replaced by the corresponding operators $\mathcal{J}_{mn} = -ix_{[m}\partial_{n]} - i\mathcal{M}_{mn}$, $\mathcal{P}_m = -i\partial_m$ that generate the group action in the space of fields. Notice that $\mathbb{W}_m$, due to its structure, does not depend on the “orbital” part ($x_{[m}\mathbb{P}_n]$) of $\mathbb{J}_{mn}$ and only the “intrinsic”

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$^5$Its part connected to the identity

$^6$We follow the discussions in [3,38,18]
spin \((\mathcal{M}_{mn})\) part survives. The Casimirs of (1) are \(C_1 = \mathbb{P}^2\) and \(C_2 = \mathbb{W}^2\) and their eigenvalues label the representations. Additionally, \(\mathbb{P}_m\) is one of the Cartan subalgebra generators hence the one particle states have at least one additional label \(p_m\), the eigenvalue of \(\mathbb{P}_m\). The eigenvalue (label) \(p_m\) is restricted to the finite set of representative momenta \(k_m\), since all other possible momenta are generated by the action of various group elements (boosts). The different kinds of representative momenta are that of a massive particle at rest \(k_m = (−m, 0, 0, 0)\) (timelike), that of a massless particle \(k_m = (−E, 0, 0, E)\) (lightlike) or that of a tachyonic particle \(k_m = (m, 0, 0, 0)\) (spacelike).

For massless \((C_1 = 0)\) particles, \(\mathbb{P}_m\) will be identified with \(p_m = (−E, 0, 0, E)\). The orthogonality of \(\mathbb{W}_m\) with \(\mathbb{P}_m\) fixes the structure to the Pauli-Lubanski vector to be:

\[
\mathbb{W}_m = p_m W + \Pi_m
\]

where \(\Pi_m\) is the transverse vector \(\Pi_m = (0, \Pi_1, \Pi_2, 0)\) with \(\Pi_1 = E(\mathbb{J}_{23} + \mathbb{J}_{20})\), \(\Pi_2 = -E(\mathbb{J}_{13} + \mathbb{J}_{10})\) and \(W = \mathbb{J}_{12}\). According to (2) these elements satisfy the algebra

\[
[\Pi_1, \Pi_2] = 0 \ , \ i[\Pi_1, W] = \Pi_2 \ , \ i[\Pi_2, W] = -\Pi_1
\]

This is the algebra of group \(E2\) and describes the symmetries of the two dimensional euclidean plane perpendicular to the motion of the massless particle\(^7\). Instead of \(\Pi_i\) one can use the equivalent set of generator \(\Pi^\pm = \Pi_1 \pm i\Pi_2\), where the above algebra takes the simpler form:

\[
[\Pi^\pm, \Pi^\mp] = 0 \ , \ [\mathbb{J}_{12}, \Pi^\pm] = \pm \Pi^\pm
\]

The second Casimir takes the form

\[
C_2 = (\Pi_1)^2 + (\Pi_2)^2 = \Pi^+ \Pi^-
\]

Due to the structure of (4) there are two natural sets of eigenstates that one can use in order to describe the various representations. The first set includes the helicity states \(\langle \lambda | \rangle\), which are the eigenstates of \(\mathbb{J}_{12}\). They are labeled by a discrete integer or half-integer helicity and they are mixed under the nontrivial action of \(\Pi_i\). The second set includes the angle states, which are the eigenstates of \(\Pi_1\) and \(\Pi_2\). They are labeled by a continuous angle\(^8\) parameter and the action of \(\mathbb{J}_{12}\) results to a shift of this angle. The two sets of states are related through a Fourier transformation. The definition of helicity states is:

\[
\mathbb{J}_{12}|\lambda\rangle = \lambda |\lambda\rangle \ , \ C_2|\lambda\rangle = \mu^2|\lambda\rangle
\]

where the eigenvalue of \(C_2\) is a real positive number parametrized by the dimensionfull parameter \(\mu\). It is straightforward to show that the action of \(\Pi^\pm\) on a helicity state increases (or reduces) the helicity by one unit

\[
\mathbb{J}_{12}\Pi^\pm|\lambda\rangle = (\lambda \pm 1)\Pi^\pm|\lambda\rangle \Rightarrow \Pi^\pm|\lambda\rangle = \mu|\lambda \pm 1\rangle
\]

where the normalization of state \(\Pi^\pm|\lambda\rangle\) is fixed by the orthonormality of the helicity eigenstates and their Casimir eigenvalue. Hence by a repeated action of \(\Pi^\pm\) we can construct an infinite set of linearly independent states with integer separated helicity values

\[
(\Pi^\pm)^n|\lambda\rangle = \mu^n|\lambda \pm n\rangle \ , \ n \in \mathbb{N}
\]

\(^7\)\(\Pi_i\) are the two generators of translations along the two perpendicular directions and \(\mathbb{J}_{12}\) is the generator of rotation along the axis of motion.

\(^8\)This is the origin of the “continuous” spin terminology.
All of these states belong in the same representation because they have the same $C_2$ eigenvalue $\mu$

$$C_2 \left( \Pi^\pm \right)^n |\lambda\rangle = \mu^n \left( \Pi^\pm \right)^n |\lambda\rangle , \quad n = 0, 1, 2, ...$$

Furthermore, by doing a full rotation of the state $|\lambda\rangle$ we get $e^{2i\pi\lambda}|\lambda\rangle$, thus $\lambda$ is either $\pm N$ where $N$ is a non-negative integer (singled-valued representation) or $\pm N/2$ where $N$ is a positive half odd integer (doubled-valued representation). The conclusion is that the irreducible representation of E2 algebra (4) are classified by a dimensionfull, continuous, parameter $\mu$ and for $\mu \neq 0$ they are infinite dimensional. The single-valued representation includes all integer helicity states and the doubled-valued representation includes all the half odd integer helicity states. For the special case of $\mu = 0$, the action of generators $\Pi^\pm$ becomes trivial and does not lead to new states. Therefore, the infinite size representation collapses to a dimension two representation with the states $|\lambda\rangle, | - \lambda\rangle$. This special case corresponds to the description of massless particles with a fixed helicity.

On the other hand, the definition of the angle states is:

$$\Pi^\pm |\theta\rangle = \mu e^{\pm i\theta} |\theta\rangle , \quad C_2 |\theta\rangle = \mu^2 |\theta\rangle$$

Such a state can be expanded in the complete basis of helicity states, hence we can write the ansatz

$$|\theta\rangle = \sum_{\lambda} f_\lambda(\theta) |\lambda\rangle$$

where $f_\lambda(\theta)$ are the expansion coefficients, which due to (11) must satisfy

$$f_\lambda(\theta) = e^{\pm i\theta} f_{\lambda \pm 1}(\theta)$$

This condition fixes uniquely the expansion coefficients, up to an overall normalization constant

$$f_\lambda(\theta) \sim e^{-i\lambda \theta}$$

hence the angle states are the Fourier dual states to the helicity states

$$|\theta\rangle \sim \sum_{\lambda} e^{-i\lambda \theta} |\lambda\rangle .$$

It is straightforward to see that under the rotation $e^{-i\alpha \hat{J}_{12}}$ the angle state $|\theta\rangle$ will result to the state $|\theta + \alpha\rangle$, thus giving to $\hat{J}_{12}$ the interpretation of translations in the $\theta$ sector. coordinate, the $\hat{J}_{12}$ covariant constraints

### 2.2 4D, $\mathcal{N} = 1$ super-Poincaré algebra

For the supersymmetric extension of the Poincaré algebra with only one supersymmetry \(^9\), we add to the list of Poincaré generators the four fermionic generators of supersymmetry $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$. Therefore, in addition to (1) we must consider the following:

$$[\hat{J}_{mn}, Q_\alpha] = i(\sigma_{mn})_{\alpha}^{\beta} Q_\beta , \quad [\hat{J}_{mn}, \bar{Q}^{\dot{\alpha}}] = i(\bar{\sigma}_{mn})^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}^{\dot{\beta}} ,$$

$$[\hat{P}_m, Q_\alpha] = 0 , \quad [\hat{P}_m, \bar{Q}_{\dot{\alpha}}] = 0 ,$$

$$\{ Q_\alpha, Q_\beta \} = 0 , \quad \{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0 , \quad \{ Q_\alpha, \bar{Q}_{\dot{\beta}} \} = -(\sigma^m)_{\alpha\dot{\alpha}} \hat{P}_m .$$

In this case the stabilizer subgroup must preserve both $P_m$ and $Q_\alpha$ and is straightforward to show that it is generated by $\hat{P}_m$ and $Z_m$, where

$$Z_m = \frac{1}{2} e^{mnrs} \hat{J}_{rs} \hat{P}_s + c(\sigma_m)^{\dot{\alpha}}_{\alpha}[Q_\alpha, \bar{Q}_{\dot{\alpha}}] .$$

\(^9\)We follow the discussions in [39, 40]
This is the supersymmetric version of the Pauli-Lubanski vector. It is important to realize that supersymmetry not only appears in the second term but also in the first term, through the generator of rotations. This can be seen by considering the superfield realization of the generator of rotations which takes the form $J_{mn} = -i \epsilon_{[m|n]} \partial_{n]} + i \theta^\beta (\sigma_{mn})^\alpha_\beta \partial_\alpha - i \theta^\beta (\sigma_{mn})^\beta_\alpha \partial_\alpha - i \mathcal{M}_{mn}$. Notice that this is not the same as the non-supersymmetric case, because the generator of rotations can also act on the fermionic directions of superspace. The parameter $c$ is a numerical coefficient and it’s value differs between the massive and massless case. This is because in the massless case, half the supersymmetry generators become trivial (vanish) and that qualitatively changes the structure of the algebra. To see this once again we identify $\mathbb{P}_m$ with $p_m = (-E, 0, 0, E)$. For this case, the supersymmetry algebra takes the form

$$\{Q_\alpha, \bar{Q}\bar{\alpha}\} = - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} E$$

and leads to

$$Q_2 = 0, \quad \bar{Q}_2 = 0.$$  

With these constraints taken into account, the value of $c$ is $c = -1/8$ and the algebra of the massless, supersymmetric Pauli-Lubanski vector $Z_m$ is:

$$[Z_m, \mathbb{P}_n] = 0, \quad [Z_m, Q_\alpha] = 0,$$

$$[Z^m, Z^n] = i \varepsilon^{mnr} Z_r \mathbb{P}_s, \quad Z^m \mathbb{P}_m = 0,$$

$$[J_{mn}, Z_r] = i \eta_{mr} Z_m - i \eta_{nr} Z_m.$$

An example of a qualitative difference between the above algebra and the corresponding algebra for massive particles is the commutativity of $Z_m$ with $Q_\alpha$. For massive representations this is no longer true and there is a non-trivial right hand side in $[Z_m, Q_\alpha]$.

The orthogonality of $Z_m$ with $\mathbb{P}_m$ fixes its structure to be the same as in the non-supersymmetric case:

$$Z_m = p_m Z + T_m$$

where $Z = J_{12} - \frac{1}{8E} (\hat{\sigma}_3)^{\alpha \bar{\alpha}} [Q_\alpha, \bar{Q}\bar{\alpha}]$ and $T_m$ is the transverse to $p_m$ vector as $\Pi_m$ in (3), $T_m = (0, T_1, T_2, 0)$ with $T_1 = E(J_{23} + J_{20}) - \frac{1}{8E} (\hat{\sigma}_1)^{\alpha \bar{\alpha}} [Q_\alpha, \bar{Q}\bar{\alpha}], \quad T_2 = -E(J_{13} + J_{10}) - \frac{1}{8E} (\hat{\sigma}_2)^{\alpha \bar{\alpha}} [Q_\alpha, \bar{Q}\bar{\alpha}]$. Using (19) the above expressions can be simplified

$$Z = J_{12} - \frac{1}{8E} [Q_1, \bar{Q}_1],$$

$$T_1 = E(J_{23} + J_{20}), \quad T_2 = -E(J_{13} + J_{10}).$$

and their algebra is

$$[T_1, T_2] = 0, \quad i[T_1, Z] = T_2, \quad i[T_2, Z] = -T_1.$$  

Usually when describing the supersymmetric, massless representations (see e.g. [39], [40]) the case of continuous spin representations is considered as non-interesting or unworthy since it had no relation to supersymmetric generalization of conventional field theories. This is a main reason why such a case has not been discussed in details. Therefore both $T_1$ and $T_2$ are set to zero by hand like in non-supersymmetric case. However because the above algebra is identical to the non-supersymmetric (4) one, the entire discussion for continuous spin representations of the Poincaré algebra can be applied as to the super-Poincaré algebra by replacing $W$ and $\Pi_i$ with $Z$ and $T_i$ respectively plus the additional conditions (19). Therefore the definition of supersymmetric continuous spin representations with label $\mu$ is:

$$C_2|\mu, \varphi\rangle = \mu^2 |\mu, \varphi\rangle, \quad T^\pm |\mu, \varphi\rangle = \mu e^{\pm i \varphi} |\mu, \varphi\rangle, \quad Q_2 |\mu, \varphi\rangle = 0, \quad \bar{Q}_2 |\mu, \varphi\rangle = 0$$

where $T^\pm = T_1 \pm i T_2$ and $C_2 = T^+ T^-$.  

6
3 Superspace realization of supersymmetric continuous spin representations

The objective of this paper is to find a 4D, $N = 1$ Minkowski, superspace realization of sCSR in order to make the connection under supersymmetry between the singled-valued and the double-valued CSR, manifest. Therefore, the use of superfields is a natural choice and the question is to find the appropriate superfield and the necessary set of differential constraints required for the description of sCSR. These constraints have to be covariant under supersymmetry, so their nature does not change under a supersymmetry transformation. Hence, the constraints must be formulated in terms of the supersymmetry covariant derivatives $D_\alpha$ and $\bar{D}_{\dot{\alpha}}$. Therefore, we must express the various objects that participate in the discussion of section 2.2 in terms of the spinorial covariant derivatives and then impose the various diagonalization conditions.

3.1 From Hilbert space to superspace

As mentioned previously, once we consider the (super)field description of the various representations the abstract generators will be replaced by the familiar differential operators that describe the group action in the space of (super)fields:

$$\bar{J}_{mn} \rightarrow J_{mn} = -ix_{[m}\partial_{n]} + i\bar{\theta}^\beta (\sigma_{mn})^\dot{\alpha}_\beta \partial_\alpha - i\theta^\beta (\sigma_{mn})_\beta^\alpha \partial_\alpha - i\mathcal{M}_{mn} \ ,$$

$$Q_\alpha \rightarrow Q_\alpha = i\partial_\alpha + \frac{1}{2}\bar{\theta}^{\dagger}(\sigma^m)_{\alpha\dot{\alpha}} \partial_m \ ,$$

$$\bar{Q}_{\dot{\alpha}} \rightarrow \bar{Q}_{\dot{\alpha}} = i\partial_{\dot{\alpha}} + \frac{1}{2}\theta^{\dagger}(\sigma^m)_{\alpha\dot{\alpha}} \partial_m \ ,$$

$$\mathbb{P}_m \rightarrow \mathbb{P}_m = -i\partial_m = p_m \ .$$

The set of spinorial covariant derivatives with respect to supersymmetry are

$$D_\alpha = \partial_\alpha + \frac{1}{2}\bar{\theta}^{\dagger}(\sigma^m)_{\alpha\dot{\alpha}} \partial_m \ , \quad \bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + \frac{1}{2}\theta^{\dagger}(\sigma^m)_{\alpha\dot{\alpha}} \partial_m \ .$$

From the above, trivially the relation between $Q_\alpha$ and $D_\alpha$ can be written as:

$$Q_\alpha + iD_\alpha = 2i\partial_\alpha \ , \quad Q_{\dot{\alpha}} + i\bar{D}_{\dot{\alpha}} = 2i\partial_{\dot{\alpha}} \ .$$

which can be used to convert between $Q$s and $D$s. The constraints (19) can be written covariantly in the form $(\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m Q_\alpha = 0$, $(\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m \bar{Q}_{\dot{\alpha}} = 0$ and in superspace they are translated to:

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m Q_\alpha = 0 \Rightarrow \begin{cases} (\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m \partial_\alpha = 0 \ , \\ (\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m D_\alpha = 0 \Rightarrow [D^2, \bar{D}_{\dot{\alpha}}] = 0 \end{cases} \ (28)$$

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m \bar{Q}_{\dot{\alpha}} = 0 \Rightarrow \begin{cases} (\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m \partial_{\dot{\alpha}} = 0 \ , \\ (\bar{\sigma}^m)^{\dot{\alpha}\alpha}p_m \bar{D}_{\dot{\alpha}} = 0 \Rightarrow [\bar{D}^2, D_\alpha] = 0 \end{cases} \ (29)$$

The expression for the supersymmetric Pauli-Lubanski vector $Z_m$ is:

$$Z^m \rightarrow Z^m = -\frac{i}{2}\varepsilon^{mnrps} \mathcal{M}_{nrps} + \frac{1}{8}(\bar{\sigma}^m)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}] \ .$$

In the last one, it is interesting to observe how the $\theta$-derivatives ($\partial_\alpha$ and $\partial_{\dot{\alpha}}$) originating from (27) combined with their constraints (28,29) cancel the theta dependent part of $J_{mn}$ leaving only the usual (internal) Poincaré part. Therefore, according to the decomposition (21) we find the following for $Z$ and $T_i$

$$Z = -i\mathcal{M}_{12} + \frac{1}{8\varepsilon}(\bar{\sigma}^3)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}] \ ,$$

$$T_1 = -iE(\mathcal{M}_{23} + \mathcal{M}_{20}) + \frac{1}{8\varepsilon}(\bar{\sigma}^1)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}] \ ,$$

$$T_2 = iE(\mathcal{M}_{13} + \mathcal{M}_{10}) + \frac{1}{8\varepsilon}(\bar{\sigma}^2)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}] \ .$$
Once again, we can use the constraints \((\tilde{\sigma}^\alpha)^\alpha p_m D_\alpha = 0 \) to simplify the above expressions (only the \( D_1 \) and \( \tilde{D}_1 \) parts survive):

\[
Z = -i\mathcal{M}_{12} + \frac{1}{8\pi^2} [D_1, \tilde{D}_1] , \quad T_1 = -iE(\mathcal{M}_{23} + \mathcal{M}_{20}) , \quad T_2 = iE(\mathcal{M}_{13} + \mathcal{M}_{10}) .
\]  

(32)

Notice that the \( T_i \) found above do not seem to be aware of the presence of supersymmetry and match precisely the non-supersymmetry discussion. The only contribution of supersymmetry in the definition of sCSR seem to be the D-constraints (28, 29).

### 3.2 Superfield description of sCSR

Looking back to the wavefunction description of CSR, there are two clues that provide some guidance. The first one is the infinite size of the representations with all integer separated helicities participating in the spectrum of the theory. That means that we can not describe CSR with a finite collection of tensor fields and one should consider the countable infinite set of increasing rank bosonic (or fermionic) tensor fields. The second clue is that the action of rotations on the angle states gives a shift in the angle parameter. This indicates that the intrinsic spin generator \( \mathcal{M}_{mn} \) can be interpreted as the derivation with respect to an appropriate “internal” coordinate not related to spacetime. Both of these features suggest that one should introduce an auxiliary coordinate \( \xi_m \) and consider the generating “functions” \( \phi(\xi, x) \). Morally, an expansion in terms of \( \xi_m \) will generate an infinite list of spacetime fields with all possible ranks and the \( \xi \)-orbital angular momentum generator \( \xi_{[m\pi_n]}^{\alpha\alpha} \) will correspond to the intrinsic spin generator \( -i\mathcal{M}_{mn} \) and thus giving to all these fields the appropriate helicity value. The role of this auxiliary coordinate is to provide a mechanism in order to group the infinite set of components in to a multiplet with the correct bookkeeping for their helicities in order to match the spectrum of CSR. This approach turned out successful and provides the correct (Wigner’s) covariant conditions for the field description of CSR.

All these features remain present in the case of sCSR, thus it is natural to follow a similar path. For these reasons we consider an expand version of superspace by introducing the auxiliary coordinate \( \xi_m \) such that the action of internal spin generator \( -i\mathcal{M}_{mn} \) on standard superspace superfield tensors is reproduced by the action of \( -i\xi_m \frac{\partial}{\partial \xi_m} \) on the extended superspace, rank zero (scalar) superfield \( \Phi(\xi, x, \theta, \tilde{\theta}) \). Notice that the extension of superspace takes place only in the bosonic sector, in order to keep the same number of superchargers. Therefore \( T_i \) can be written as:

\[
T_1 = -i\xi_2 p^m \frac{\partial}{\partial \xi_m} + i \frac{\partial}{\partial \xi^2} p^m \xi_m , \quad (33)
\]

\[
T_2 = i\xi_1 p^m \frac{\partial}{\partial \xi_m} - i \frac{\partial}{\partial \xi^1} p^m \xi_m
\]

The definition of sCSR is:

\[
T_1 \Phi(x, \xi, \theta, \tilde{\theta}) \sim \mu \Phi(x, \xi, \theta, \tilde{\theta}) \Rightarrow \begin{cases} p^m \xi_m \Phi(x, \xi, \theta, \tilde{\theta}) = 0 , \\ p^m \frac{\partial}{\partial \xi_m} \Phi(x, \xi, \theta, \tilde{\theta}) = i\mu \Phi(x, \xi, \theta, \tilde{\theta}) \end{cases} \quad (34)
\]

\[
C_2 \Phi(x, \xi, \theta, \tilde{\theta}) = \mu^2 \Phi(x, \xi, \theta, \tilde{\theta}) \Rightarrow \xi^m \xi_m \Phi(x, \xi, \theta, \tilde{\theta}) = \Phi(x, \xi, \theta, \tilde{\theta}) \quad (35)
\]

\[
Q_2 \Phi(x, \xi, \theta, \tilde{\theta}) = 0 \Rightarrow [D^2, \tilde{D}_\alpha] \Phi(x, \xi, \theta, \tilde{\theta}) = 0 \quad (36)
\]

\[
\tilde{Q}_2 \Phi(x, \xi, \theta, \tilde{\theta}) = 0 \Rightarrow [\tilde{D}^2, D_\alpha] \Phi(x, \xi, \theta, \tilde{\theta}) = 0 \quad (37)
\]

The first three equations are Wigner’s conditions for CSR. The last two are the additional supersymmetric constraints in order to describe sCSR. Equations (36, 37) are solved by either a chiral superfield \( \Phi (\tilde{D}_\alpha \Phi = 0) \) with the equation of motion \( D^2 \Phi = 0 \) or by a complex linear superfield \( \Sigma (\tilde{D}^2 \Sigma = 0) \) with the equation of motion \( D_\alpha \Sigma = 0 \).

\(^{10}\pi_m \) is the conjugate variable to \( \xi_m \) such that \([\xi^m, \pi_n] = i\delta^m_n \)
4 Components discussion and the recovery of the single and double valued CSR

For the case of the chiral superfield description, eq (34,35,36,37) take the form 11:

\[ \xi^{\alpha\dot{\alpha}} \tilde{D}_\alpha D_\dot{\alpha} \Phi = 0, \]
\[ \tilde{D}^{\dot{\alpha}} D^\alpha \frac{\partial}{\partial \xi_{\alpha\dot{\alpha}}} \Phi = -i\mu \Phi, \]
\[ \xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \Phi = \frac{1}{2} \Phi, \]
\[ \tilde{D}_\alpha \Phi = 0, \]
\[ D^2 \Phi = 0. \]

By projecting these superspace equations into equations for the component fields of \( \Phi \) we find that the lowest component \( \phi = \Phi|_{\theta=0=\bar{\theta}} \) describes the singled valued CSR

\[ \xi^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi = 0, \partial^{\alpha\dot{\alpha}} \frac{\partial}{\partial \xi_{\alpha\dot{\alpha}}} \phi = -\mu \phi, \xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \phi = \frac{1}{2} \phi, \Box \phi = 0 \] (43)

and the lowest fermionic component \( \psi_{\alpha} = D_\alpha \Phi|_{\theta=0=\bar{\theta}} \) describes the doubled-valued CSR:

\[ \xi^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \psi_{\beta} = 0, \partial^{\alpha\dot{\alpha}} \frac{\partial}{\partial \xi_{\alpha\dot{\alpha}}} \psi_{\beta} = -\mu \psi_{\beta}, \xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \psi_{\beta} = \frac{1}{2} \psi_{\beta}, \partial^{\alpha\dot{\alpha}} \psi_{\alpha} = 0 \] (44)

Similarly, the complex linear superfield description of sCSR takes the form:

\[ \xi^{\alpha\dot{\alpha}} D_\alpha \tilde{D}_{\dot{\alpha}} \Sigma = 0, \]
\[ D^\alpha \tilde{D}^{\dot{\alpha}} \frac{\partial}{\partial \xi_{\alpha\dot{\alpha}}} \Sigma = -i\mu \Sigma, \]
\[ \xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \Sigma = \frac{1}{2} \Sigma, \]
\[ \tilde{D}^2 \Sigma = 0, \]
\[ D_\alpha \Sigma = 0. \]

and it is straightforward to show that the components \( \varphi = \Sigma|_{\theta=0=\bar{\theta}} \) and \( \lambda = D_\alpha \tilde{D}_{\dot{\alpha}}|_{\theta=0=\bar{\theta}} \) satisfy the same (43,44) conditions and thus provide a description of integer and half-integer CSR. Of course this alternative description of sCSR exist due to the well-known duality between chiral and complex linear superfields which flips eq. (41,42) with (48,49).

An interesting observation is that due to (40,47) the solutions must be searched in the space of distributions. This is a characteristic property of CSR and sCSR. In addition, the coordinate \( \xi_{\alpha\dot{\alpha}} \) can not be written as the product of two twistors because that will make it a lightlike coordinate \( (\xi_{\alpha\dot{\alpha}} \neq \omega_{\alpha} \bar{\omega}_{\dot{\alpha}}) \). However one can introduce two sets of twistors \( \omega_{\alpha}^I \) with \( I = 1,2 \). Then one can decompose \( \xi_{\alpha\dot{\alpha}} \) in the following manner:

\[ \xi_{\alpha\dot{\alpha}} = \omega_{\alpha}^I \bar{\omega}_{\dot{\alpha}}^I \varepsilon_{IJ} \] (50)

This decomposition makes contact with the description in [36,37] where two twistors \( \pi_{\alpha} \) and \( \rho_{\alpha} \) and their conjugates where used for the description of CSR. The correspondence is \( \omega_{\alpha}^I = \{\pi_{\alpha}, \rho_{\alpha}\} \) and thus it will relate the component fields found here with the ones used in the BRST description done in [36].

11 Convert the vector index to spinorial indices \( \xi^{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma^m)^{\alpha\dot{\alpha}} \xi_m \) and \( \frac{\partial}{\partial \xi_{\alpha\dot{\alpha}}} = (\sigma^m)_{\alpha\dot{\alpha}} \frac{\partial}{\partial \xi_m} \).
5 Summary and discussion

In this work we define the supersymmetric continuous spin representation (sCSR) (eq. 24). To find a superfield description of it we extended standard 4D, $\mathcal{N} = 1$ superspace with the addition of an auxiliary, commuting, coordinate $\xi_{\alpha\dot{\alpha}}$ in order to construct generating superfunction that group together the countable infinite number of supersymmetric multiplets of increasing superhelicity that appear in the spectrum of sCSR. These are the supersymmetric extension of Wigner’s wavefunctions used for the description of CSR. We find two descriptions. The first is based on a a chiral superfield $\Phi(\xi, x, \tau, \bar{\theta})$ ($\overline{D}_{\dot{\alpha}}\Phi = 0$) and the proposed set of covariant equations of motion it must satisfy is:

$$\xi^{\alpha\dot{\alpha}} D_{\alpha} D_{\dot{\alpha}} \Phi = 0 ,$$
$$\overline{D}^{\dot{\alpha}} D^{\alpha} \frac{\partial}{\partial \xi^{\alpha\dot{\alpha}}} \Phi = -i\mu \Phi ,$$
$$\xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \Phi = \frac{1}{2} \Phi ,$$
$$D_{\alpha}^{\alpha} \Phi = 0 .$$

The second description is dual to the first one and uses a complex linear superfield $\Sigma(\xi, x, \tau, \bar{\theta})$ ($\overline{D}^{2} \Sigma = 0$) and it must satisfy the following equations:

$$\xi^{\alpha\dot{\alpha}} D_{\alpha} \overline{D}_{\dot{\alpha}} \Sigma = 0 ,$$
$$D^{\alpha} \overline{D}^{\dot{\alpha}} \frac{\partial}{\partial \xi^{\alpha\dot{\alpha}}} \Sigma = -i\mu \Sigma ,$$
$$\xi^{\alpha\dot{\alpha}} \xi_{\alpha\dot{\alpha}} \Sigma = \frac{1}{2} \Sigma ,$$
$$D_{\alpha} \Sigma = 0 .$$

The projected components $\phi(\xi, x) = \Phi|_{\theta = 0 = \bar{\theta}}$ or $\psi(\xi, x) = \Sigma|_{\theta = 0 = \bar{\theta}}$ are the Wigner’s wavefunctions that describe the single valued continuous spin representation (spans integer helicities) Similarly the components $\psi_{\alpha}(\xi, x) = D_{\alpha} \Phi|_{\theta = 0 = \bar{\theta}}$ or $\lambda(\xi, x) = D_{\alpha} \Sigma|_{\theta = 0 = \bar{\theta}}$ are Wigner’s wavefunctions for the description of the doubled valued continuous spin representation (spans half odd integer helicities). Additionally, there are auxiliary fields $F(\xi, x) = D^{2} \Phi|_{\theta = 0 = \bar{\theta}}$ or $\rho_{\alpha}(\xi, x) = D_{\alpha} \Sigma|_{\theta = 0 = \bar{\theta}}$ which vanish automatically by the equations of motion. Nevertheless, these fields appear in the supersymmetry transformation of the components $\phi(\xi, x), \psi_{\alpha}(\xi, x)$ or $\psi(\xi, x), \lambda_{\alpha}(\xi, x)$ as dictated by the chiral or complex linear superfields respectively. These supersymmetry transformations are the off-shell completion of the supersymmetry transformations between the integer and half-integer CSR in [37].

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