ON GROUP GRADINGS ON PI-ALGEBRAS

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Abstract. We show that there exists a constant $K$ such that for any PI
algebra $W$ and any nondegenerate $G$-grading on $W$ where $G$ is any group
(possibly infinite), there exists an abelian subgroup $U$ of $G$ with
$[G : U] \leq \exp(W)^K$. A $G$-grading $W = \bigoplus_{g \in G} W_g$ is said to be nondegenerate
if $W_{g_1} W_{g_2} \cdots W_{g_r} \neq 0$ for any $r \geq 1$ and any $r$ tuple $(g_1, g_2, \ldots, g_r)$ in $G^r$.

1. Introduction

In the last two decades there were significant efforts to extend important results
in the theory of polynomial identities for (ordinary) associative algebras to $G$-graded
algebras, where $G$ is a finite group, and more generally to $H$-comodule algebras
where $H$ is a finite dimensional Hopf algebra. For instance Kemer’s representability
theorem and the solution of the Specht problem were established for $G$-graded
associative algebras over a field of characteristic zero (see [3], [18]). Recall that
Kemer’s representability theorem says that any associative PI-algebra over a field
$F$ of characteristic zero is PI-equivalent to the Grassmann envelope of a finite
dimensional $\mathbb{Z}_2$-graded algebra $A$ over some field extension $L$ of $F$ (see below the
precise statement). Another instance of these efforts is the proof of Amitsur’s
conjecture which was originally proved for ungraded associative algebras over $F$ by
Giambruno and Zaicev [10], and was extended to the context of $G$-graded algebras
by Giambruno, La Mattina and the first named author of this article (see [2],
[9]) and considerable more generally for $H$-comodule algebras by Gordienko [12].
Amitsur’s conjecture states that the sequence $c_n^{1/n}$, where $c_n = c_n(W)$ is the $n$th
term of the codimension sequence of $W$, has an integer limit (denoted by $\exp(W)$).

In [1] a different point of view was considered (in combining PI-theory and $G$-
gradings, still under the condition that $G$ is finite), namely asymptotic PI-theory
was applied in order to prove invariance of the order of the grading group on an
associative algebra whenever the grading is minimal regular (as conjectured by
Bahturin and Regev [1]). In fact, it is shown there that the order of the grading
group coincides with $\exp(W)$. Suppose now $G$ is arbitrary (i.e. not necessarily
finite). Our goal in this paper, roughly speaking, is to exploit the invariant $\exp(W)$
of the algebra $W$, in order to put a bound on the minimal index of an abelian
subgroup of $G$ whenever the algebra $W$ admits $G$-gradings satisfying certain natural
conditions. Let us remark here that most of our analysis is devoted to the case where

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the group $G$ is finite (a case where the results mentioned above can be applied) and then we pass to infinite groups.

In this paper we only consider fields which are of characteristic zero. Let $W$ be an associative PI-algebra over a field $F$. Suppose $W \cong \bigoplus_{g \in G} W_g$ is $G$-graded where $G$ is arbitrary. Suppose further, that the grading is nondegenerate, that is, for any positive integer $r$ and any tuple $(g_1, \ldots, g_r) \in G^{(r)}$, we have $W_{g_1}W_{g_2}\cdots W_{g_r} \neq 0$.

**Theorem 1.1. (Main theorem)**

There exists an integer $K$ such that for any PI-algebra $W$ and for any ”nondegenerate” $G$-grading on $W$ by any group $G$, there exists an abelian subgroup $U$ of $G$ with $[G:U] \leq \exp(W)^K$.

It is interesting to compare the theorem above with results of (1) Passman and Isaacs (2) Curzio, Longobardi, Maj and Robinson.

**Theorem 1.2 ([13]).** Let $G$ be any group and let $\mathbb{C}G$ be the group algebra over $\mathbb{C}$ (the field of complex numbers). If $\mathbb{C}G$ is a PI group algebra, then $G$ contains an abelian subgroup whose index is finite and is bounded by a function of $\exp(\mathbb{C}G)$.

We also note that there is a converse to the theorem above due to Kaplansky [15]. More precisely, if $G$ is abelian by finite, then $\mathbb{C}G$ is PI and its exponent is bounded by a function of the minimal index of an abelian subgroup. It follows that $\mathbb{C}G$ is PI if and only if $G$ is abelian by finite.

In order to state the second result recall that a group $G$ is said to be $n$-permutable, denoted by $P_n$, if for any $n$-tuple $(g_1, \ldots, g_n) \in G^{(n)}$ there exists a nontrivial permutation $\sigma \in \text{Sym}(n)$ such that

$$g_1g_2\cdots g_n = g_{\sigma(1)}g_{\sigma(2)}\cdots g_{\sigma(n)} \in G$$

Clearly, $P_n \subseteq P_{n+1}$. We denote by $P$ the family of all permutable groups.

**Theorem 1.3 ([8] [7]).** Let $G$ be any group. Then $G$ is permutable if and only if it is finite by abelian by finite. If $G$ is finitely generated, then it is permutable if and only if it is abelian by finite.

Clearly a group algebra $FG$ is nondegenerately graded (every homogeneous component has an invertible element) and hence Theorem 1.1 may be seen as a far reaching generalization of Theorem 1.2. In order to compare our result with Theorem 1.3 note that if a group $G$ grades nondegenerately a PI algebra $W$ then $G$ is permutable. In fact, if $W$ satisfies a multilinear polynomial identity of degree $n$ then $G$ is $n$-permutable (see section 3 below) and hence, if the group $G$ is finitely generated, already the permutability implies that $G$ is abelian by finite. Our main point here is the existence of a universal bound on the minimal index of an abelian subgroup $U$ in $G$ as a function of the exponent of the algebra. We emphasize that such a bound does not exists in terms of the permutability index. Indeed, it is known and not difficult to construct an infinite family of groups $G_k$, $k = 1, 2, \ldots$, all $n$-permutable for some fixed integer $n$, with no uniform bound on the minimal index of their abelian subgroups and hence no uniform bound on the PI exponent of the group algebras $FG_k$, $k = 1, 2, \ldots$. In section 4 we exhibit a family of finite groups $\{G_k\}_{k \in \mathbb{N}}$ with $\text{ord}(G_k) = p$, $p$ prime, for all $k$ (and hence $p + 1$-permutable), with $d_k = \min\{[G_k : U_k] \mid U_k \leq G_k \text{ abelian subgroup}\}$ and $\lim d_k = \infty$. As for
infinitely generated groups, it is known that \( n \)-permutability of \( G \) does not imply \( G \) being abelian by finite.

The main tools used in the proof of the main theorem are the representability theorem for \( G \)-graded algebras where \( G \) is a finite group \[3\] and Giambruno and Zaicev’s result on the exponent of \( W \) \[10\]. The representability theorem allows us to replace the \( G \)-graded algebra \( W \) by a finite dimensional \( G \)-graded algebra \( A \) (or the Grassmann envelope of a finite dimensional \( \mathbb{Z}_2 \times G \)-graded algebra \( A \)) whereas Giambruno and Zaicev’s result provides an interpretation of \( \exp(W) \) in terms of the dimension of a certain subalgebra of \( A \). The proof of Theorem 1.1 in case the group \( G \) is finite is presented in section 3. In section 4 we show how to pass from finite groups to arbitrary groups and by this we complete the proof of Theorem 1.1. In section 2 we recall some background on group gradings and PI theory needed for the proof of the main theorem and in the last section of the paper, section 5, we present (in addition to the family of \( n \)-permutable groups \( \{G_k\}_{k \in \mathbb{N}} \) mentioned in the previous paragraph), some examples which show the necessity of the conditions in the main theorem in the sense that (1) one cannot drop the nondegenerate condition on the gradings (2) one can get nondegenerate gradings on algebras with small exponent with arbitrary large finite abelian groups.

2. BACKGROUND AND SOME PRELIMINARY REDUCTIONS

We start by recalling some facts on \( G \)-graded algebras \( W \) over a field \( F \) of characteristic zero and their corresponding \( G \)-graded identities. We refer the reader to \[3\] for a detailed account on this topic.

**Remark 2.1.** In this section we consider only finite groups. Although some of the basic results in \( G \)-graded PI theory hold for arbitrary groups, one of our main tools, namely the “representability theorem” for \( G \)-graded PI algebras is false for infinite groups.

Let \( W \) be a PI-algebra over \( F \). Suppose \( W \) is \( G \)-graded where \( G \) is a finite group. Denote by \( I = \mathrm{Id}_G(W) \) the ideal of \( G \)-graded identities of \( W \). These are polynomials in the free \( G \)-graded algebra \( F\langle X_G \rangle \) over \( F \), that vanish upon any admissible evaluation on \( W \). Here, \( X_G = \bigcup_{g \in G} X_g \) and \( X_g \) is a set of countably many variables of degree \( g \). An evaluation on \( W \) is admissible if the variables from \( X_g \) are replaced only by elements of \( W_g \). The ideal \( I \) is a \( G \)-graded \( T \)-ideal, i.e. it is invariant under \( G \)-graded endomorphisms of \( F\langle X_G \rangle \).

We recall from \[3\] that the \( G \)-graded \( T \)-ideal \( I \) is generated by multilinear polynomials. Consequently, it remains invariant when passing to \( L \), any field extension of \( F \), in the sense that the \( G \)-graded ideal of identities of \( W_L \) over \( L \) is the span (over \( L \)) of the graded \( T \)-ideal of identities of \( W \) over \( F \).

The following observations play an important role in the sequel.

**Observation 2.2.** The condition “nondegenerate” \( G \)-grading on \( W \) can be easily translated into the language of \( G \)-graded polynomial identities. Indeed a \( G \)-grading on \( W \) is nondegenerate if and only if for any integer \( r \) and any tuple \((g_1, \ldots, g_r) \in G^r\), the \( G \)-graded multilinear monomial \( x_{g_1,1} \cdots x_{g_r,r} \) is a \( G \)-graded nonidentity of \( W \) (in short we say that \( \mathrm{Id}_G(W) \) contains no multilinear \( G \)-graded monomials). Consequently, if \( G \)-graded algebras \( W_1 \) and \( W_2 \) are \( G \)-graded PI-equivalent (i.e. have the same \( T \)-ideal of \( G \)-graded identities), then the grading on \( W_1 \) is nondegenerate if and only if the grading on \( W_2 \) is nondegenerate.
Observation 2.3. If $W_1, W_2$ are two $G$-graded algebras with $\text{Id}_G(W_1) = \text{Id}_G(W_2)$, then $\text{Id}(W_1) = \text{Id}(W_2)$ (the ungraded identities). In particular we have $\exp(W_1) = \exp(W_2)$. Indeed, this follows easily from the fact that a polynomial $p(x_1, \ldots, x_n)$ is an ungraded identity of an algebra $W$ with a $G$-grading if and only if the polynomial $p(\sum_{g \in G} x_{g,1}, \ldots, \sum_{g \in G} x_{g,n})$ is a graded identity of $W$ as a $G$-graded algebra.

As noted above, the “nondegeneracy” condition satisfied by a $G$-grading on $W$ depends only on the $T$-ideal of $G$-graded identities, hence we have that if the grading on a $G$-graded algebra $W$ over a field $F$ is nondegenerate, the same holds for the $G$-graded algebra $W_L = W \otimes_F L$. Similarly, the numerical invariant $\exp(W)$ of the algebra $W$ remains unchanged if we extend scalars.

Remark 2.4. The content of the main steps of the proof is to show that we can restrict our attention to more specific families of algebras. We emphasize however that these statements do not just say “it is sufficient to prove the main theorem for such and such family of algebras” but rather “if the constant $K$ is suitable for a certain family of algebras then the constant $f(K)$ is suitable for a larger family of PI-algebras”. It will follow from our proofs that $f$ is either the identity map or multiplication by 2.

Following the remark above it is convenient to adopt the following terminology.

Definition 2.5. We say that a family of PI-algebras $\mathcal{C}$ has the $\theta(K)$ property for a constant $K \geq 1$, if for any algebra $A \in \mathcal{C}$, and any nondegenerate $G$ grading on $A$ with a finite group $G$, there exists an abelian subgroup $H \leq G$ such that $[G : H] \leq \exp(A)^K$.

We may also restrict this property for classes of algebras with certain group grading (e.g. $G$-simple gradings).

The following lemma follows easily from our considerations above.

Lemma 2.6. Let $F$ be a field of characteristic zero, $L/F$ a field extension and $W$ a PI $F$-algebra. If there exists a constant $K$ such that $\theta(K)$ holds for $W_L = W \otimes_F L$, then $\theta(K)$ also holds for $W$.

Proof. Let $W$ be any PI-algebra over a field $F$ and suppose it is $G$-graded where $G$ is a finite group. Suppose further that the grading is nondegenerate. Extending scalars to $L$ we obtain a nondegenarate $G$-grading on $W_L$ and hence $G$ contains an abelian subgroup $U$ with $[G : U] \leq \exp(W_L)^K$. Since $\exp(W_L) = \exp(W)$ the result follows. \qed

Let us recall some terminology and some facts from Kemer’s theory extended to the context of $G$-graded algebras as they appear in [3].

Let $W$ be a $G$-graded algebra over $F$. Suppose that $W$ is PI (as an ungraded algebra). Kemer’s representability theorem for $G$-graded algebras says that there exists field extension $L/F$ and a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra $A$ over $L$ such that the Grassmann envelope $E(A)$ (with respect to the $\mathbb{Z}_2$-grading) yields a $G$-graded algebra which is $G$-graded PI-equivalent to $W_L$. In case the algebra $W$ is affine, or more generally in case it satisfies a Capelli polynomial, there exists a field extension $L/F$ such that the algebra $W_L$ is $G$-graded PI-equivalent to a finite dimensional $G$-graded algebra $A$ over $L$. This result will be used to reduce our discussion from infinite dimensional algebras to finite dimensional ones in case the
group $G$ is finite. By further extensions of scalars we assume, as we may by the previous lemma, that the field $L$ is algebraically closed.

Next, we would like to replace the finite dimensional $\mathbb{Z}_2 \times G$-graded algebra $A$ which appears in Kemer’s theorem in case $W$ is nonaffine (or the finite dimensional $G$-graded algebra $A$ in case $W$ is affine) with another finite dimensional algebra, which is PI-equivalent to $A$ and it is better understood.

Let $A$ be a finite dimensional $G$-graded algebra over $F$. Let $J = J(A)$ denote the Jacobson radical of $A$ and let $A/J$ be the corresponding semisimple quotient of $A$ by its radical. It is known that the algebra $A$ contains a $G$-graded subalgebra $A$, $G$-graded isomorphic to $A/J$, which supplements $J(A)$ (in $A$) as a $G$-graded $F$-vector space (see [6], [17]). Consider the decomposition of $A$ into the direct product of $G$-simple algebras $A_1 \times \ldots \times A_n$ where $n$ is a nonnegative integer (if $n = 0$ the algebra $A$ is nilpotent). Recall that the algebra $A$ is said to be ”full” if there is a permutation $\sigma \in Sym(n)$ of the $G$-simple components $A_1, \ldots, A_n$ such that $(the product in $A$) A_{\sigma(1)}J A_{\sigma(2)} \cdots J A_{\sigma(n)} \neq 0$. Here, we slightly abuse notation by identifying the algebra $A_i$ with its embedding in $A$.

**Proposition 2.7.** (see [3]) Any $G$-graded finite dimensional algebra $A$ is $G$-graded PI-equivalent to a direct product of $G$-graded ”full” algebras.

In fact the theorem as stated in [3] is stronger, namely, the full algebras can be assumed to be $G$-graded “basic” but since we won’t be using that stronger condition in this paper we don’t recall that definition here.

The next ingredient we need is a result of Bahturin, Sehgal and Zaicev, which determines the $G$-graded structure of finite dimensional $G$-simple algebra over an algebraically closed field of characteristic zero.

Let $A$ be the algebra of $r \times r$-matrices over $F$ and let $G$ be any group (here, $G$ may be infinite). Fix an $r$-tuple $\alpha = (g_1, \ldots, g_r) \in G^{(r)}$. Consider the $G$-grading on $A$ given by

$$A_\alpha = \text{span}_F \{e_{i,j} : g = g_i^{-1}g_j\}.$$ 

One checks easily that this indeed determines a $G$-grading on $A$. Clearly, since the algebra $A$ is simple, it is $G$-simple as a $G$-graded algebra.

Next we present a different type of $G$-gradings on semisimple algebras which turn out to be $G$-simple. Let $H$ be any finite subgroup of $G$ and consider the group algebra $FH$. By Maschke’s theorem $FH$ is semisimple and of course $H$-simple (any nonzero homogeneous element is invertible). More generally, we may consider the algebra $FH$ as $G$-graded (put $A_g = 0$ for $g \in G \setminus H$) and as such it is $G$-simple. We can extend this construction even more by considering twisted group algebras $F^\alpha H$ as $G$-graded algebras, where $\alpha$ is a 2-cocycle in $Z^2(H, F^*)$ ($H$ acts trivially on $F$). Recall that $F^\alpha H = \text{span}_F \{U_h : h \in H\}$, $U_h U_{h_2} = \alpha(h_1, h_2)U_{h_1 h_2}$, for all $h_1, h_2 \in H$. We say that the basis $\{U_h : h \in H\}$ corresponds to the 2-cocycle $\alpha$.

**Remark 2.8.** In the sequel, whenever we say that $\{U_h : h \in H\}$ is a basis of $F^\alpha H$, we mean that the basis corresponds to the cocycle $\alpha$. One knows that in general an homogeneous basis of that kind corresponds to a cocycle $\alpha'$ cohomologous to $\alpha$.

Clearly, as in the group algebra $FH$, the nonzero homogeneous elements in $F^\alpha H$ are invertible and hence it is $G$-simple. We call this grading “fine” (i.e. every homogeneous component is of dimension $\leq 1$). In case the field $F$ is algebraically
closed of characteristic zero, we have that these two gradings (elementary and fine) are the building blocks of any $G$-grading on a finite dimensional algebra so that it is $G$-simple. This is a theorem of Bahturin, Sehgal and Zaicev.

**Theorem 2.9.** [9] Let $A$ be a finite dimensional $G$-graded simple algebra. Then there exists a finite subgroup $H$ of $G$, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of $H$ on $F$ is trivial, an integer $r$ and an $r$-tuple $(g_1, g_2, \ldots, g_r) \in G^r$ such that $A$ is $G$-graded isomorphic to $\Lambda = F^o H \otimes M_r(F)$ where $\Lambda_q = \text{span}_F \{U_h \otimes e_{i,j} \mid g = g_i^{-1} h g_j\}$. Here $U_h \in F^o H$ is a representative of $h \in H$ and $e_{i,j} \in M_r(F)$ is the $(i,j)$ elementary matrix.

In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity element of $A$ are homogeneous of degree $e \in G$.

The last ingredient we need is Regev, Giambruno and Zaicev’s PI-asymptotic theory. Let $W$ be an ordinary PI-algebra over an algebraically closed field $F$ of characteristic zero and let $\text{Id}(W)$ be its $T$-ideal of identities. Consider the $n!$-dimensional vector space

$$P_n = \text{span}_F \{x_{\sigma(1)} \cdots x_{\sigma(n)} : \sigma \in \text{Sym}(n)\}$$

and let $c_n(W) = \dim_F(P_n/P_n \cap \text{Id}(W))$ be the $n$-th term of the codimension sequence of the algebra $W$. It was proved by Regev in 72 [10] that the sequence $\{c_n(W)\}$ is exponentially bounded and conjectured by Amitsur that the limit $\lim_{n \to \infty} c_n^{1/n}$ exists (the exponent of $W$) and is a nonnegative integer. The conjecture was established by Giambruno and Zaicev in the late 90’s by showing that the limit coincides, roughly speaking, with the dimension of a certain subspace “attached” to $W$. Since we will need the precise statement in our proofs let us recall their result here.

Suppose $W$ is an affine algebra. It is known from Kemer’s representability theorem (for ungraded PI algebras; see also Prop. 2.7 above) that there exists a finite dimensional algebra $B$ which is the direct product of “full” algebras $B_1, \ldots, B_r$ with

$$\text{Id}(W) = \text{Id}(B) = \text{Id}(B_1 \times \ldots \times B_n) = \cap_i \text{Id}(B_i).$$

For each full algebra $B_i$ consider the semisimple subalgebra $\bar{B}_i$ contained in $B_i$ which supplements $J(B_i)$ and let $b_i = \dim_F(\bar{B}_i)$. Giambruno and Zaicev’s result for affine algebras may be stated by

$$\exp(W) = \exp(B) = \max_i \{b_i\}.$$  

Let us consider now arbitrary (i.e. not necessarily affine) PI-algebras. If $W$ is a PI-algebra, it follows from Kemer’s representability theorem that there exists a finite dimensional $\mathbb{Z}_2$-graded algebra $A = A_0 \bigoplus A_1$ such that $\text{Id}(W) = \text{Id}(E(A))$. Here $E = E_0 \bigoplus E_1$ denotes the infinite dimensional Grassmann algebra with the usual $\mathbb{Z}_2$- grading and $E(A) = (E_0 \otimes A_1) \bigoplus (E_1 \otimes A_1)$ is the Grassmann envelope of $A$ viewed as an ungraded algebra. Now, in view of Kemer’s theory for $G = \mathbb{Z}_2$-graded algebras, there exists a $\mathbb{Z}_2$-graded finite dimensional algebra $C$, $\mathbb{Z}_2$-PI-equivalent to $A$, which decomposes into the direct product $C_1 \times \ldots \times C_n$ of $\mathbb{Z}_2$-graded algebras which are “$\mathbb{Z}_2$-full” and we let $c_i$ be the dimension of the semisimple subalgebra of $C_i$ which supplements $J(C_i)$.

Giambruno and Zaicev’s result says that
exp(W) = \max \{c_1\}.

3. Proofs of main theorem-Finite groups

All groups considered in this section are finite. In particular, whenever we refer to the main theorem, we do that under the assumption that the group \( G \) is finite.

Let \( W \) be a PI-algebra over a field \( F \) of characteristic zero and let \( \exp(W) \) be its exponent. Suppose \( W \) is \( G \)-graded where \( G \) is a finite group.

**Proposition 3.1.** Suppose there exists an integer \( K \) such that \( \theta(K) \) holds for finite dimensional algebras over algebraically closed fields, then the main theorem holds with the same constant \( K \).

**Proof.** Let \( F \) be any field. Let \( W \) be an \( F \) PI-algebra, \( G \)-graded where \( G \) is finite and the \( G \)-grading is “nondegenerate”. We need to show there exists an abelian subgroup \( U \) of \( G \) such that \( \lceil G : U \rceil \leq \exp(W)^K \).

Let us consider first the case where \( W \) is affine. Applying [3] there exists a finite dimensional \( G \)-graded algebra \( B \) over a field extension \( L \) of \( F \) such that \( \text{Id}_G(W \otimes F L) = \text{Id}_G(B) \). Clearly, we may assume that \( L \) is algebraically closed by further extending the scalars if needed. Next, by Observations 2.2 and 2.3 we know that the \( \text{G}\)-grading on \( B \) is nondegenerate and \( \exp(W_L) = \exp(B) \). Finally, invoking Lemma 2.4 (“return from \( W_L \) to \( W \)”) we complete the proof of the proposition in case the algebra \( W \) is affine.

Suppose now that \( W \) is arbitrary (i.e., not necessarily affine). By [3] there exists a finite dimensional \( \mathbb{Z}_2 \times G \)-graded algebra \( C \cong \bigoplus_{(\varepsilon,g)\in\mathbb{Z}_2	imes G} C_{(\varepsilon,g)} \) over an (algebraically closed) field extension \( L \) of \( F \) such that \( W_L \) is \( G \)-PI-equivalent to \( E(C) = (E_0 \otimes C_0) \oplus (E_1 \otimes C_1) \) where \( C_0 = \bigoplus_{g \in G} C_{(0,g)} \) and \( C_1 = \bigoplus_{g \in G} C_{(1,g)} \).

The \( G \)-grading on \( E(C) \) is given by

\[
E(C)_g = (E_0 \otimes C_{(0,g)}) \oplus (E_1 \otimes C_{(1,g)}).
\]

We claim that the \( G \)-grading on \( C \) is nondegenerate (where \( C_g = C_{(0,g)} \oplus C_{(1,g)} \)).

To this end fix an \( n \)-th tuple \((g_1, \ldots, g_n) \in G^n \). By linearity we need to show that at least one of the \( 2^n \) monomials of the form

\[
x_{(\varepsilon_1,g_1)} x_{(\varepsilon_2,g_2)} \cdots x_{(\varepsilon_n,g_n)}
\]

is not in \( \text{Id}_{\mathbb{Z}_2 \times G}(C) \). Let us show that if this is not the case, then the monomial \( x_{g_1, \cdots, g_n} \) is a \( G \)-graded identity of \( E(C) \), contradicting the fact that the \( G \)-grading on \( E(C) \) and hence on \( W \) is nondegenerate. To see this consider the evaluation \( x_{g_i,i} = z_{0,i} \otimes a_{0,i} + z_{1,i} \otimes a_{1,i} \) for \( i = 1, \ldots, n \) where \( z_{\varepsilon,i} \in E_{\varepsilon} \) and \( a_{\varepsilon,i} \in C_{(\varepsilon,g_i)} \). This evaluation yields an expression with \( 2^n \) summands of the form

\[
z_{\varepsilon_1,1} z_{\varepsilon_2,2} \cdots z_{\varepsilon_n,n} \otimes a_{(\varepsilon_1,1)} a_{(\varepsilon_2,2)} \cdots a_{(\varepsilon_n,n)}
\]

which are all zero and the claim follows. Now let \( K \) be the constant determined in the proposition. It follows there exists an abelian subgroup \( U \) of \( G \) such that \( \lceil G : U \rceil \leq \exp(C)^K \). But \( \exp(C)^K \leq \exp_{\mathbb{Z}_2}(C)^K = \exp(W)^K \) and so \( \lceil G : U \rceil \leq \exp(W)^K \) as desired. \( \square \)

Our next step is to reduce to the case where the algebra is full as a \( G \)-graded algebra.
**Proposition 3.2.** Suppose there exists an integer \( K \) such that \( \theta(K) \) holds for any finite dimensional algebra \( A \) over \( F \) (algebraically closed field of characteristic zero) with full grading. Then the main theorem holds with the same constant \( K \).

**Proof.** By the preceding proposition we need to show \( \theta(K) \) holds when \( A \) runs over all finite dimensional \( G \)-graded algebras where the grading is nondegenerate. As explained in the previous section, we know that any finite dimensional algebra \( A \) with a \( G \)-grading is \( G \)-graded PI-equivalent to a direct product \( B_1 \times \cdots \times B_n \), some \( n \), of \( G \)-graded algebras which are \( G \)-full. Since any \( G \)-graded decomposition into a direct product of \( G \)-graded algebras is in particular a decomposition of ungraded algebras, applying Giambruno and Zaicev’s result (see the last two paragraphs of the previous section; for more details the reader is referred to [10]) we easily see that

\[
\exp(A) = \max_i \exp(B_i) \quad \text{and} \quad \exp(G) = \max_i \exp(G(B_i))
\]

but the maximum may be obtained by a different algebra \( B_1 \).

So the proposition will be proved if we show that the \( G \)-grading on \( B_i \) is nondegenerate for some \( i \). Suppose the contrary. Then for each \( i \) there is a \( G \)-graded multilinear monomial \( m_i \) in \( \text{Id}_G(B_i) \). Taking the product of these monomials (with disjoint sets of variables) we get that the \( G \)-grading on \( A \) is degenerate. Contradiction. \( \square \)

Our next step is to pass to the case where the algebra \( A \) is \( G \)-simple.

**Remark 3.3.** Clearly, any \( G \)-simple algebra is \( G \)-full. On the other hand the \( G \)-grading on a finite dimensional \( G \)-simple algebra may be degenerate. The characterization of the nondegenerate \( G \)-gradings making a finite dimensional algebra \( G \)-simple is presented in Lemma [3.5].

**Proposition 3.4.** Suppose there exists an integer \( K \) such that \( \theta(K) \) holds for any finite dimensional algebra \( A \) over \( F \) (algebraically closed field of characteristic zero) with \( G \)-simple grading. Then the main theorem holds with the same constant \( K \).

**Proof.** Applying the preceding reduction it is sufficient to prove that \( \theta(K) \) holds for finite dimensional \( G \)-graded algebras \( B \) which are \( G \)-full. Fix such an algebra \( B \) and consider the decomposition of \( B = \tilde{B} \oplus J(B) \) as a \( G \)-graded vector space where \( \tilde{B} \) is a \( G \)-graded semisimple subalgebra of \( B \) which supplements \( J(B) \). Let \( B \cong B_1 \times \cdots \times B_n \) be the decomposition of \( B \) into the direct product of \( G \)-simple algebras. Let us show there exists \( i_0 \) such that the \( G \)-grading on \( B_i \) is nondegenerate. If not, for every \( i \) the \( T \)-ideal of \( G \)-graded identities \( \text{Id}_G(B_i) \) contains a \( G \)-graded monomial \( m_i \). Assuming the sets of variables appearing in the different \( m_i \)'s are disjoint we consider their product \( f_0 = m_1 \cdots m_n \).

Since \( m_i \) is a \( G \)-graded identity of \( B_i \) for each \( i \), we have that \( f_0 \) is an identity of \( \tilde{B} = B_1 \times \cdots \times B_n \), and hence any evaluation of \( f_0 \) on \( B \) must be in \( J(B) \). Since the nilpotency index of \( J(B) \) is finite, taking the product of \( d = \text{nil}(J(B)) \) monomials as \( f_0 \) (with disjoint sets of variables) we get a multilinear monomial in \( \text{Id}_G(B) \) - contradiction. It follows that there is some \( i_0 \) such that the \( G \) grading on \( B_{i_0} \) is nondegenerate, and the result follows since \( \exp(B_{i_0}) \leq \exp(B) \).

\( \square \)

As promised in Remark [3.3] we characterize (in terms of Bahturin, Sehgal and Zaicev’s characterization of finite dimensional \( G \)-simple algebras) nondegenerate \( G \)-gradings on finite dimensional algebras \( A \) yielding a \( G \)-simple algebra. Recall that a \( G \)-grading on \( A \) is “strong” if for any \( g, h \in G \) we have \( A_g A_h = A_{gh} \).
Lemma 3.5. Let $A$ be a finite dimensional $G$-simple algebra. Then the following conditions are equivalent.

1. The $G$-grading on $A$ is nondegenerate.
2. $A_g \neq 0$, for every $g \in G$ and the grading is “strong”.
3. Let $F^aH \otimes M_r(F)$ be a presentation of the $G$-grading on $A$ (as given by Theorem 2.9) where $H$ is a finite subgroup of $G$ and $(g_1, \ldots, g_r) \in G^r$ is the $r$-tuple which determines the elementary grading on $M_r(F)$. Then every right coset of $H$ in $G$ is represented in the $r$-tuple.

Remark 3.6. Note that in general (i.e. the algebra $A$ is not necessarily $G$-simple) the first two conditions are not equivalent. For instance, the $\mathbb{Z}_2$-grading on the infinite dimensional Grassmann algebra is nondegenerate but not strong as $E_1E_1 \subset E_0$ (or $E_0E_0 \subset E_0$ in case the algebra $E$ is assumed to have no identity element).

Proof. Note first that since $A$ is assumed to be finite dimensional $G$-simple, each of the conditions implies that $G$ is finite. As for the 3rd condition of the lemma we replace (as we may) the given presentation with another so that the $r$-tuple has the following form

$$(g(1,1) \cdot \cdot \cdot g(1,d_1), g(2,1), \cdot \cdot \cdot g(2,d_2), \cdot \cdot \cdot g(s,1), \cdot \cdot \cdot g(s,d_s))$$

where

- $r = d_1 + \cdot \cdot \cdot + d_s$.
- $g_{i,1} = g_{i,2} = \ldots = g_{i,d_i}$ (denoted by $z_i$), and for $i \neq k$ the elements $g_{i,j}, g_{k,l}$ represent different right $H$-cosets in $G$.
- $g_{i,j} = e$ for $j = 1, \ldots, d_i$.

(2) $\rightarrow$ (1) : It is clear that if all homogeneous components are nonzero and the $G$-grading is strong, then it is nondegenerate.

(1) $\rightarrow$ (3) : Suppose (3) does not hold. We claim there exists a multilinear monomial of degree at most $r$ which is a $G$-graded identity of $F^aH \otimes M_r(F) = \text{span}_F \{U_h \otimes e_{i,j} : h \in H, 1 \leq i, j \leq r\}$.

It is convenient to view the matrices in $M_r(F)$ as $s \times s$ block matrices corresponding to the decomposition $d_1 + \cdot \cdot \cdot + d_s = r$. More precisely, let $D_k = d_1 + \cdot \cdot \cdot + d_k$ and decompose $M_r(F) = \bigoplus_{i,j=1}^{D_i} M_{[i,j]}$ into the direct sum of vector spaces $M_{[i,j]} = \text{span}\{e_{k,l} : D_{i-1} < k \leq D_i, D_j-1 < l \leq D_j\}$. Note that $M_{[i,j]}$ are submatrices supported on a single block of size $d_i \times d_j$. This decomposition is natural in the sense that $(F^aH \otimes M_r(F))_g$ is the direct sum of the vector spaces $U_h \otimes M_{[i,j]}$ such that $z_i^{-1}hz_j = g$.

For a fixed index $i \in \{1, \ldots, s\}$ and an element $g \in G$, consider the equation $hz_j = z_ig$. It has a solution if and only if $Hz_ig$ has a representative in $(z_1, \ldots, z_s)$. It follows that if $U_h \otimes B$ is homogeneous of degree $g$ and $Hz_ig$ has no representative in $(z_1, \ldots, z_s)$, then the $i$-th row of blocks in $B$ must be zero.

Consider the multilinear monomial

$$x_{w_1,1}x_{w_2,2} \cdot \cdot \cdot x_{w_n,n}$$

where $x_{w_i,i}$ is homogeneous of degree $w_i \in G$. We will show there exist $w_i \in G, i = 1, \ldots, n$ so that the monomial above is a $G$-graded identity.

Note first that such a monomial (being multilinear) is a $G$-graded identity if and only if it is zero on graded assignments of the form $x_{w_i,i} = U_{h_i} \otimes A_i$ which span
that if for any such homogeneous assignment, the $B_i = A_1 A_2 \cdots A_i$ is zero, then $A$ must be zero (since each of its blocks rows is zero).

Following the argument above we choose $w_i \in G$ such that for each $i$ the right coset $H z_i w_1 \cdots w_i$ (i.e. the right coset of $H$ represented by $z_i$ times the homogeneous degree of $U_{h_1} \cdots U_{h_i} \otimes B_i$) has no representative in $(z_1, ..., z_s)$. Now, from the assumption, there is some $z \in G$ such that $Hz$ has no representative in $(z_1, ..., z_s)$. Thus, choosing $w_i = (z_i w_1 \cdots w_{i-1})^{-1} z$ we obtain the required result.

$(3) \rightarrow (2)$: Suppose that all right $H$-cosets are represented in the tuple $(g_1, \ldots, g_r)$. To show that the grading is strong, it is enough to show that any basis element $U_k \otimes e_{i,j}$ can be written as a product in $A_{w_1} A_{w_2}$ where $w_1 \cdot w_2 = g_i^{-1} h g_j$. Indeed, since each right coset has a representative, we can find $k$ such that $g_k \in H g_i w_1 = H g_j w_i^{-1}$. Letting $h_1 = g_i w_1 g_k^{-1}$ and $h_2 = g_k w_i g_j^{-1}$, we get that $a = U_{h_1} \otimes e_{i,k}$, $b = U_{h_2} \otimes e_{k,j}$ are in $A_{w_1}, A_{w_2}$ respectively and $a \cdot b = \alpha(h_1, h_2) U_h \otimes e_{i,j}$. □

We are ready to reduce the proof of the main theorem ($G$ finite) to the case where the algebra $W$ is isomorphic to a twisted group algebras $F^\alpha G$.

**Proposition 3.7.** Suppose there exists an integer $K$ such that $\theta(K)$ holds for any twisted group algebra over algebraically closed field of characteristic zero. Then the main theorem holds with the same constant $K$.

**Proof.** By the preceding proposition it is sufficient to show the constant $K$ is suitable for all $G$-simple algebras. Let $A = F^\alpha H \otimes M_r(F)$ be a $G$-graded $G$-simple algebra and suppose the grading is nondegenerate. We need to show there exists an abelian subgroup $U$ of $G$ with $[G : U] \leq \exp(A)^K$. By assumption, there is a subgroup $U$ of $H$ such that $[H : U] \leq \exp(F^\alpha H)^K$. But by the lemma we have that $[G : H] \leq r$ and hence we have

$$[G : U] = [G : H][H : U] \leq r \exp(F^\alpha H)^K \leq r^2 \exp(F^\alpha H)^K \leq (r^2 \exp(F^\alpha H))^K$$

since $K \geq 1$. But from Giambruno Zaicev’s theorem [10] we know that $r^2 \exp(F^\alpha H) = \exp(A)$ and the result follows. □

Our next task is to reduce from twisted group algebras to group algebras.

**Remark 3.8.** Such result would follow easily if we knew that $\exp(FH) \leq \exp(F^\alpha H)$ for any $\alpha \in Z^2(H, F^*)$. However such a statement is false in general as the following example shows.

**Example 3.9.** Let $A_4$ be the alternating group of order 12 and consider the group algebra $FA_4$ and the twisted group algebra $F^\alpha A_4$ where $\alpha$ is the (nontrivial) 2-cocycle which corresponds to the double cover $\hat{A}_4$ of $A_4$. It is well known that the group $A_4$ has an irreducible representation of degree 3 (and 3 nonequivalent linear representations) hence $\exp(A_4) = 9$. On the other hand the twisted group algebra $F^\alpha A_4$ admits no representation of degree 1 (one knows that any twisted group algebra $F^\beta H$ admits a representation of degree 1 if and only if the cocycle $\beta$ is trivial) and hence, by a dimension count, we have $\exp(F^\alpha A_4) = 4$.

**Proposition 3.10.** Suppose there exists an integer $K$ such that $\theta(K)$ hold for any finite dimensional group algebra $FG$ over $F$ (algebraically closed field of characteristic zero). Then the main theorem holds with the constant $2K$.
Proof. By the preceding proposition it is sufficient to show the constant $2K$ is suitable for all twisted group algebras $F^\gamma G$. By Maschke’s theorem, the twisted group algebra $F^\gamma G$ is semisimple and so we have $F^\gamma G \cong M_{d_1}(F) \times \cdots \times M_{d_s}(F)$ where $d_1 \leq d_2 \leq \cdots \leq d_s$. Of course the number of simple components $s$ and the degrees $d_i$ depend on the cocycle $\gamma$ and so we write $D_\gamma = d_s$.

Let $\{U_g\}_{g \in G}, \{V_g\}_{g \in G}, \{W_g\}_{g \in G}$ be the respective bases of $FG, F^\alpha G, F^{-\alpha} G$. Define $\psi : FG \to F^\alpha G \otimes F^{-\alpha} G$ by setting $\psi(U_g) = V_g \otimes W_g$. Clearly, this is an injective homomorphism, hence there is an embedding of $M_{D_\lambda}(F)$ in $F^\alpha G \otimes F^{-\alpha} G$. But this tensor product is isomorphic to a product of matrix algebra, where the largest is $M_{D_\alpha \cdot D_{\alpha-1}}(F)$, so it follows that $D_1 \leq D_\alpha \cdot D_{\alpha-1}$.

The map $\eta : F^\alpha G \to F^{-\beta} G$, given by $V_g \mapsto W_g^{-1}$ is an anti-isomorphism of algebras and since matrix algebras are anti-isomorphic to themselves we obtain $D_\alpha = D_{\alpha-1}$ and so $D_1 \leq D_{2\alpha}$. Now, by assumption there is an abelian subgroup $U$ of $G$ such that $[G : U] \leq \exp(FG)^K = D_1^K \leq D_{2\alpha}^K = \exp(F^\alpha G)^{2K}$. This completes the proof of the proposition.

The passage from group algebras to twisted group algebras is the only step in which we increase the constant $K$. While the constant $2K$ is sufficient, it may not be necessary. Indeed, the bound $D_\alpha \times D_\alpha$ on $D_1$ ($D_\alpha$ is the largest degree of an irreducible representation of $F^\alpha G$) can be slightly improved by replacing one factor $D_\alpha$ by the smallest degree of an irreducible representation of that algebra. Recall that for the bound in the last proposition we used an embedding of $FG$ inside $F^\alpha G \otimes F^{-\alpha} G$. In the next lemma we investigate embeddings of $FG$ in algebras of the form $F^\alpha G \otimes M_k(F)$. Note that any such embedding implies that $D_1 \leq D_\alpha \cdot k$.

As the next proposition shows, such an embedding exists if and only if $k$ is at least the smallest degree of a representation of $F^\alpha G$.

**Proposition 3.11.** Let $F^\alpha G$ and $F^\beta G$ be twisted group algebras of $G$ over $F$ with 2-cocycles $\alpha$ and $\beta$. As usual, $F$ is an algebraically closed field of characteristic zero. Let $M_r(F)$ be the algebra of $r \times r$-matrices over $F$ and let $F^\beta G \otimes M_r(F)$ be a $G$-simple algebra where the $r$-tuple which provides the elementary grading is trivial (i.e. $(g_1, \ldots, g_r) = (e, \ldots, e)$). Then the twisted group algebra $F^\alpha G$ may be embedded as a $G$-graded algebra in $F^\beta G \otimes M_r(F)$ if and only if the twisted group algebra $F^{\alpha \beta^{-1}} G$ has a nontrivial representation into $M_r(F)$. In particular the group algebra $FG$ can be embedded in $F^{\alpha \beta^{-1}} G \otimes M_r(F)$ if and only if $F^\alpha G$ can be represented in $M_r(F)$.

**Proof.** We first note that since $F^\alpha G$ is $G$-simple, any nonzero $G$-graded map into $F^\beta G \otimes M_r(F)$ is necessarily a $G$-graded embedding.

Let $\{U_g\}_{g \in G}, \{V_g\}_{g \in G}$ be bases for $F^\alpha G, F^\beta G$ respectively. If $\psi : F^\alpha G \to F^\beta G \otimes M_r(F)$ is a nonzero $G$-graded map, then for every $g \in G$ there is some $0 \neq a_g \in M_r(F)$ such that $\psi(U_g) = V_g \otimes a_g$. Since $\psi$ is a homomorphism we get that

$$\psi(U_g U_h) = \alpha(g, h) \psi(U_{gh}) = \alpha(g, h) V_{gh} \otimes a_{gh}$$

$$\psi(U_g) \psi(U_h) = (V_g \otimes a_g)(V_h \otimes a_h) = \beta(g, h) V_{gh} \otimes a_g a_h$$
For any finite group \( G \), any finite group has an abelian subgroup following result of D. Gluck (see [11]) in which he bounds the minimal index of an abelian subgroup \( U \) in \( G \) in terms of the maximal character degree of \( G \). We emphasize that the proof uses the classification of finite simple groups.

**Theorem 3.12.** (D. Gluck) There exists a constant \( m \) with the following property. For any finite group \( G \) there exists an abelian subgroup \( U \) of \( G \) such that \( [G : U] \leq b(G)^m \), where \( b(G) \) is the largest irreducible character degree of \( G \).

We can now complete the proof of the main theorem for finite groups.

We note that by Giambruno and Zaicev’s result \( \exp(FG) = b(G)^2 \) and hence, any finite group has an abelian subgroup \( U \) with \( [G : U] \leq b(G)^m = \exp(FG)^{m/2} \). Combining with our results above, we see that if a PI-algebra \( W \) admits a nondegenerate \( G \)-grading where \( G \) is a finite group, then there is an abelian subgroup \( U \) with \( [G : U] \leq \exp(W)^{2m^2} = \exp(W)^m \). In particular, taking \( K = m \) where \( m \) is determined by the theorem above, will do.

4. Proofs of Main Theorem—Infinite Groups

In this section we prove the main theorem for arbitrary groups. Let us sketch briefly the structure of our proof. In the preceding section we proved the main theorem for arbitrary finite groups. Our first step in this section is to prove the main theorem for groups which are finitely generated and residually finite. Next, we pass to finitely generated groups (not necessarily residually finite) by the following argument. Any group \( G \) which grades nondegenerately a PI algebra is permutable and hence being finitely generated, it is abelian by finite by [13] and hence residually finite. Finally we show how to pass from finitely generated groups to arbitrary groups. We emphasize that the constant \( K \) (which appears in the main theorem) remains unchanged when passing from finite groups to arbitrary group.

**Proposition 4.1.** Suppose the main theorem holds for arbitrary finite groups with the constant \( K \), that is, for any finite group \( G \) and any PI algebra \( W \) which is nondegenerately \( G \)-graded, there exists an abelian subgroup \( U \subseteq G \) with \( [G : U] \leq \exp(W)^K \). Then the main theorem holds for finitely generated residually finite groups with the same constant \( K \).

**Proof.** Since \( G \) is finitely generated, by Hall’s theorem [13] there are finitely many subgroups of index \( \leq \exp(A)^K \). Denoting these groups by \( U_1, ..., U_n \), we wish to show that one of them is abelian. Suppose the contrary, hence we can find \( g_i, h_i \in U_i \) such that \( e \neq [g_i, h_i] \) for any \( i = 1, ..., n \). Let \( N \) be a normal subgroup of finite index which does not contain any of the \([g_i, h_i]\).

Define an induced \( G/N \) grading on \( A \) by setting \( A_{gN} = \bigoplus_{h \in N} A_{gh} \). Clearly, the induced \( G/N \)-grading on \( A \) is nondegenerate, thus by the main theorem there is
some $U \leq G$ such that $[G : U] \leq \exp(A)^K$ and $U/N$ is abelian. By the construction, $U = U_i$ for some $i$, and we get that $[g_i, h_i] \in N$ - a contradiction. \qed

Assume now that $G$ is finitely generated and grades nondegenerately a PI algebra $A$. Let us show that $G$ must be permutable. Indeed, let $f = \sum c_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n), n}$ be a nonzero ordinary identity of $A$ and assume that $c_{id} = 1$. Fix a tuple $g_1, \ldots, g_n \in G$ and consider the graded identity

$$\tilde{f} = f(x_{g_1,1}, \ldots, x_{g_n,n}) = \sum_{h \in G} f_h(x_{g_1,1}, \ldots, x_{g_n,n})$$

where $f_h$ is the $h$ homogenous part of $\tilde{f}$. Since $\tilde{f}$ is a graded identity, its homogenous parts are also graded identities. Letting $g = g_1 \cdots g_n$, the polynomial $f_g$ contains the monomial $x_{g_1,1} \cdots x_{g_n,n}$ (with coefficient 1). Since the grading is nondegenerate, $f_g$ is not a monomial and therefore has another monomial with nonzero coefficient corresponding to some permutation $\sigma \neq id$, hence $g_1 \cdots g_n = g_{\sigma(1)} \cdots g_{\sigma(n)}$. This can be done for any tuple of length $n$, so it follows that $G$ is $n$-permutable.

As explained above, applying permutability we can pass from finitely generated residually finite groups to arbitrary finitely generated groups. For future reference we put it in the next proposition.

**Proposition 4.2.** Suppose the main theorem holds for finitely generated residually finite groups with the constant $K$. Then the main theorem holds for arbitrary finitely generated groups with the same constant $K$.

The main theorem now follows from the following proposition.

**Proposition 4.3.** Let $d$ be a positive integer and let $G$ be any group. Suppose that any finitely generated subgroup $H$ of $G$ contains an abelian subgroup $U_H$ with $[H : U_H] \leq d$. Then there exists an abelian subgroup $U$ of $G$ with $[G : U] \leq d$.

**Remark 4.4.** The proposition above generalizes a statement which appears in [14] but the proof is basically the same (see Lemma 3.5 and the proof of Theorem II). We believe the result of the proposition is well known, however we were unable to find an appropriate reference in the literature.

**Proof.** Let $A \leq F \leq G$. We say that $(F, A)$ is a pair if $F$ is f.g., $A$ is abelian and $[F : A] \leq d$. We write $(F, A) \leq (F_1, A_1)$ for two pairs if $F \cap A_1 = A$. Note in particular that we have $[F_1 : A_1] \geq [F : A]$.

A pair $(F, A)$ is called good if whenever $F \leq F_1 \leq G$ with $F_1$ finitely generated, there is a pair $(F_1, A_1)$ with $(F, A) \leq (F_1, A_1)$. By the assumption of the proposition, we get that $(e, e)$ is a good pair.

1. We first claim that if $(F, A)$ is good and $F \leq H \leq G$ with $H$ finitely generated, we can find $B \leq H$ such that $(H, B)$ is a good pair and $(F, A) \leq (H, B)$.

Since $(F, A)$ is a good pair, there are pairs $(H, B_i)$ with $(F, A) \leq (H, B_i)$, and by Hall’s theorem, only finitely many such pairs. Suppose by negation that none of them are good pairs, thus we can find $H_i \leq F_i \leq G$ f.g. such that there are no abelian subgroups $A_i$ with $(H_i, B_i) \leq (F_i, A_i)$. The group $K = \langle F_1, \ldots, F_n \rangle$ is f.g. so there is some abelian subgroup $A_K \leq K$ of index $\leq d$ such that $(F, A) \leq (K, A_K)$. Clearly, there is some $i$ such that $A_K \cap H = B_i$, but then $(H, B_i) \leq (F_i, F_i \cap A_K)$ - contradiction.
(2) Let \((F,A)\) be a good pair with \(s = [F : A]\) maximal, and note that if 
\((F,A) \leq (H,B)\) are good pairs, then we must have \([F : A] = [H : B]\). 
We claim also that \([G : C_G(B)] \leq d\). Fix some \(g \in G\), then \((H,g)\) has an 
abelian subgroup \(C\) such that \((H,B) \leq ((H,g),C)\) are good pairs. 
Let \(g_1,\ldots,g_s\) left coset representatives of \(A\) in \(F\), then these are also rep-
resentatives of all the cosets of \(C\) in \((H,g)\). In particular \(g \in g_iC\) for some \(i\). Since \(B \leq C\) are abelians, we get that \(g_iC \leq g_iC_G(B)\). It 
follows that \(g_1,\ldots,g_s\) are also coset representatives for \(C_G(B)\) in \(G\), hence 
\([G : C_G(B)] \leq s \leq d\).

(3) Assume now that \((F,A)\) is a good pair with \([F : A] = s\) and \([G : C_G(A)]\) 
maximal. Define 

\[ J = \langle B \mid (H,B) \geq (F,A) \text{ is a good pair} \rangle. \]

We claim that \(J\) is abelian and \([G : J] \leq d\).

Suppose that \(a \in B_1, b \in B_2\) with \((H_i,B_i) \geq (F,A)\) are good pairs. Since 
\(A \leq B_1\), we have that \(C_G(B_1) \leq C_G(A)\), but from the maximality of 
\([G : C_G(A)]\), it follows that there is an equality. Similarly, we have that 
\(C_G(B_2) = C_G(A)\) and since \(B_2\) is abelian we get that \(b_2 \in B_2 \subseteq C_G(B_2) = 
C_G(B_1)\), so that \(b_1,b_2\) commute. It follows that \(J\) is abelian.

Suppose now that \([G : J] > d\), hence we can find \(g_0,\ldots,g_d\) different coset 
representatives of \(J\) in \(G\). The group \(F_1 = \langle F,g_1,\ldots,g_d \rangle\) is finitely 
generated, thus, we can find \(A_1 \leq F_1\) such that \((F_1,A_1)\) is a good pair bigger 
than \((F,A)\), and in particular \([F_1 : A_1] \leq d\). But this means that there are 
some \(0 \leq i < j \leq d\) with \(g_i^{-1}g_j \in A_1 \subseteq J\) which is a contradiction. Thus, 
\([G : J] \leq d\) and we are done.

\(\square\)

5. Some examples

Let \(G\) be a finitely generated group grading nondegenerately a PI algebra \(A\), and 
thus it is \(n\) permutable for some \(n \in \mathbb{N}\). While \(G\) must be abelian by finite, the 
minimal index of an abelian subgroup cannot be bounded by a function of the per-
mutability index. Indeed, if there was such a function \(f(n)\), then given an arbitrary 
\(n\)-permutable group \(H\), its finitely generated subgroup would be \(n\)-permutable as 
well. By the assumption, each such subgroup has an abelian subgroup of index 
\(\leq f(n)\), and hence by Proposition \([\text{4.3}]\) so would \(G\). Thus, by Theorem \([\text{4.3}]\) every 
finitely generated group grading nondegenerately a PI algebra \(A\) is actually abelian by finite, which is false.

Let us give a concrete example, i.e., a family of \((finite) n\)-permutable groups 
\(\{G_k\}_{k \in \mathbb{N}}\), with \(d_k = \min\{[G_k : U_k] \mid U_k \text{ abelian subgroup}\}\) and \(\lim d_k = \infty\).

\textbf{Example 5.1.} Let \(G = C_p^{2n}\) for some \(n\) and let \(\alpha \in Z^2(G, \mathbb{C}^*)\) be a nontrivial two 
cocycle. It is well known that up to a coboundary \(\alpha\) takes values which are roots of 
unity, and for \(G\) above, the values must be \(p\)-roots of unity. Thus, we may consider 
it as a cocycle in \(Z^2(G, \mathbb{C}_p)\) which corresponds to a central extension 

\[ 1 \to C_p \to H \to G \to 1. \]

Note that \([H,H]\) is the kernel of the projection \(H \to G\) above (since the cocy-
cole is nontrivial), and in addition \([H,H] \leq Z(H)\). By Lemma (3.2) in \([3]\), since 
\([H,H] = p\), we have that the group \(H\) is \(p + 1\) permutable.

Let \(B = \mathbb{C}^\alpha G\) be the corresponding twisted group algebra with basis \(\{U_g\}_{g \in G}\). If
$A \leq H$ is an abelian group of minimal index, we may assume that $[H,H] \leq A$, and thus consider $\tilde{A} = A/[H,H] \leq G$. Clearly, the group $A$ is abelian if and only if $[U_{g_1},U_{g_2}] = 0$ for any $g_1,g_2 \in \tilde{A}$.

For $g,h \in G$ set $\mu(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)}$, namely the scalar satisfying $U_g U_h = \mu(g,h) U_h U_g$.

It is easily seen that $\mu : G \times G \to \mathbb{C}^\times$ is a bicharacter, i.e. $\mu(g_1 g_2,h) = \mu(g_1,h) \mu(g_2,h)$ and $\mu(g,h_1 h_2) = \mu(g,h_1) \mu(g,h_2)$. Under these notation we have that $\mu(g_1, g_2) = 1$ for any $g_1, g_2 \in \tilde{A}$.

Identifying $C_p$ with the additive group of the field $F_p$ with $p$ elements, we see that $\mu$ is a bilinear map. If $\text{dim}_{F_p}(A) > \frac{1}{2}\text{dim}_{F_p}(G)$, or equivalently $[G : \tilde{A}] < p^n$, then there is some $e \neq u \in \tilde{A}$ such that $\mu(u,g) = 1$ for all $g \in G$. Thus, if $\mu$ is nondegenerate, i.e. for any $e \neq h \in G$ there is some $g \in G$ such that $\mu(h,g) \neq 1$, then $[H : A] = [G : \tilde{A}] \geq p^n$.

Note that to say that $\mu$ is nondegenerate is equivalent to say that $U_g$ is in the center of the twisted group algebra if and only if $g = e$. This is in particular true for matrix algebra where the center is one dimension.

Thus, we are left to find twisted group algebras with groups $C_p^{2n}$ (for $p$ fixed) which are isomorphic to matrix algebras.

Fix a prime $p$ and let $\sigma, \tau$ be generators for $C_p \times C_p$. Let $B$ be the twisted group algebra $B = \bigoplus_{0 \leq i,j \leq p-1} \mathbb{C} U_{\sigma^i \tau^j}$ where the multiplication is defined by

$$U_{\sigma^i \tau^j} U_{\sigma^l \tau^j} = U_{\sigma^{i+l} \tau^j}, \quad U_{\sigma^i} U_{\tau^j} = \zeta U_{\sigma} U_{\tau}$$

and $\zeta$ is a primitive $p$-root of unity. It is well known that $B \cong M_p(\mathbb{C})$, and therefore $\bigotimes_{1}^n B$ is a twisted group algebra with the group $C_p^{2n}$ and we are done by noting that tensor product of matrix algebras is again a matrix algebra. We remark here that the function $\mu$ defined above plays a central role in the theory of twisted crossed product and their polynomial identities (see [1]).

**Remark 5.2.** Let $\alpha_n \in Z^2(C_p^{2n}, \langle \zeta \rangle)$ be the 2-cocycles constructed in the previous example, and let $H_n$ be the central extensions defined by these cocycles. The last example shows that the group algebra $\mathbb{C}H_n$ has an irreducible representation of degree $p^n$. Kaplansky's theorem [2] states that if a group has an abelian subgroup of index $m$, then all of its irreducible representations are finite with degree at most $m$, thus providing another proof that the minimal index of an abelian subgroup of $H_n$ goes to infinity.

Next we provide some examples/counter examples to statements that are related to the main theorem.

**Example 5.3.** Let $F$ be an algebraically closed field of characteristic zero. For any finite abelian group $G$, the group algebra $FG$ is isomorphic to a product of $|G|$ copies of $F$. In particular, we get that $\exp(FG) = 1$. Hence, we cannot hope to get an inequality of the form $|G| \leq \exp(A)^K$ for any constant $K$.

More generally, given an $H$-graded algebra $A$ with a nondegenerate grading, the algebra $B = FG \otimes A$ has a natural $G \times H$ grading which is also nondegenerate. In addition we have that $\exp(B) = \exp(A)$. While the grading group is of course larger, the index of the largest abelian group remains the same.

**Example 5.4.** Suppose we omit the requirement that $Id_G(A)$ has no $G$-graded monomials and only assume that $A$ satisfies monomials of high degrees (as a function of $\text{dim}(A)$ or the cardinality of $G$). We show that the consequence of the main
that each element of $s$ choice of the tuple $\leq x$ the largest abelian subgroup tends to infinity.

Consider the algebras $A_m$ of upper triangular matrices $m \times m$ where the diagonal matrices consist only of scalar matrices. Note that by Giambruno and Zaicev’s result (see last paragraph of section 2) we have $\exp(A) = 1$. Let $G$ be a group of order $n$ and assume that $m = n^2 + 1$. Let $s' = (g_1, ..., g_n) \in G^n$ be a tuple such that each element of $G$ appears in $s'$ exactly once and let $s \in G^{(n^2+1)}$ be $n$ copies of $s'$ with additional $g_1$ at the end. Consider the algebra $A_m$ with the elementary grading corresponding to the tuple $s$. We claim that $A_m$ has no graded multilinear monomial identities of degree $\leq n$.

Fix $1 \leq i \leq n^2+1-n$ and $h \in G$. We first note that by the definition of the grading we have that $e_{i,j}$ is homogeneous of degree $s_i^{-1}s_j$ for each $1 \leq i < j \leq n^2$. By the choice of the tuple $s$, the elements $\{s_i^{-1}s_{i+1}, s_i^{-1}s_{i+2}, ..., s_i^{-1}s_{i+n}\}$ are all distinct, and therefore we can choose $j = j(i, h)$ such that $i < j \leq i + n$ and $e_{i,j} \in A_k$ for every $i \leq n^2+1-n$ and $h \in G$.

Let $x_{h_1,1} \cdots x_{h_n,n}$ be any multilinear monomial for some $h_1, ..., h_n \in G$.

Set $i_1 = 1$. Given $i_k$, define $i_{k+1}$ to be $j(i_k, h_k)$ so that $e_{i_k,i_{k+1}}$ is homogeneous of degree $h_k$ and $i_k < i_{k+1} \leq i_k + n$. It is now easy to see by induction that $i_k \leq 1 + (k - 1)n \leq 1 + n^2$ for all $1 \leq k \leq n$ so that $e_{i_1,i_2} \cdots e_{i_n,i_{n+1}}$ is well defined as an element of $A$ and it is a non zero evaluation of $x_{h_1,1} \cdots x_{h_n,n}$.

For a finite group $G$, denote by $\gamma(G)$ the smallest index of an abelian subgroup in $G$. Let $G_n$ be any sequence of groups where $\gamma(G_n)$ goes to infinity with $n$. By the argument, the algebras $B_n = A_{|G_n|^{2+1}}$ have $G_n$ gradings such that

- $\dim(B_n)$ and $|G_n|$ goes to infinity.
- $B_n$ has no multilinear monomial identities of degrees smaller then $|G_n|$.
- $\exp(B_n) = 1$.

Example 5.5. Suppose we have a sequence of algebras $A_n$ with $d_n = \exp(A_n)$ monotonically increasing (i.e. to infinity). Can we necessarily find groups $G_n$ and nondegenerate $G_n$-gradings such that the index of any abelian subgroups $U_n$ of $G_n$ tends to infinity?

The answer is negative as the algebras of upper triangular matrices show. More precisely, let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices, which have exponent $\exp(UT_n(F)) = n$. By a theorem of Valenti and Zaicev [19], every $G$-grading on $UT_n(F)$ is isomorphic to an elementary grading. Unless the grading is trivial, the grading cannot be nondegenerate since $UT_n(F)_{g}$ contains only upper triangular matrices with zero on the diagonal for every $e \neq g \in G$, so $x_{g,1} \cdots x_{g,n}$ is an identity. We conclude that the only nondegenerate grading is with the trivial group, so in particular there are no nondegenerate grading such that the index of the largest abelian subgroup tends to infinity.

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