Positive model theory of interpretations

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Abstract

We prove analogues of model theory results for $\mathcal{C} \rightarrow \mathcal{D}$ coherent functors, including variants of the omitting types theorem and some results on ultraproduct constructions. We introduce a distributive lattice valued invariant of $\mathcal{C} \rightarrow \text{Set}$ coherent functors that vanishes precisely on positively closed models, then we study its functorial properties.

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1 Introduction

One of the many arguments towards categorical logic is that the syntactic treatment of interpretations between theories yields complicated definitions which are hard to handle. On that basis it seems to be unclear why we would need to define maps between interpretations, how such a category could be proved to have 2-categorical limits and colimits (e.g. how should we glue theories together along interpretations) (see [BLV20]), or how natural model theoretic constructions (e.g. the formation of type spaces) are 2-functorial wrt. interpretations and their homomorphisms (discussed in Section 4). To
treat such questions we identify theories with algebraic structures (coherent categories) and their interpretations with the structure-preserving maps (coherent functors), following a simple idea explained in the Overview (based on [MR77]).

The second (related) argument is that this identification tells us that interpretations are the same as models in some exotic categories (in small coherent categories instead of Set), hence we can hope to prove model theoretic results for morphisms of theories. The third argument is that category theory has a considerable toolkit which can be applied once we translate logic problems to questions about coherent categories and coherent functors. There are many more. (Namely we can go further from the coherent category associated to the theory, and arrive at first to its pretopos completion ([MR77]), then to the classifying topos ([MR77],[Joh02]). Moreover from any of these syntactic presentations we can go to the semantics level: the accessible category of models ([AR94]). Each stage has its own theory and the transfer between these stages is also well-understood.) In this paper we will elaborate on the first three arguments.

2 Overview

We start with a quick summary of some chapters from the book [MR77], which will motivate our setting. In particular we describe how the study of $\mathcal{C} \to \text{Set}$ coherent functors (a special case of our investigations) equals many-sorted positive model theory, discussed e.g. in [Poi06], [YP07], [Bel14] and [Hay18].

Given an $L$-structure $M$, we can think of it as a collection of sets (the interpretation of $L$-formulas) and a collection of set-functions (the ones whose graph is a definable set). That is, an $L$-structure is a subcategory of Set, "parametrized" with formulas and function-like formulas between them.

However, if we take a homomorphism $M \to M'$ it will not preserve these definable sets in general, only those which correspond to positive existential formulas (the ones built up from atomic formulas, $\bot$, $\lor$, $\top$, $\land$ and $\exists$). This suggests that when we treat $M$ as a subcategory of Set (the functorial image of something), we should only care about definable sets and functions given by positive existential formulas.

When we look at this reduced subcategory we can see not only whether a positive existential formula is valid in $M$, i.e. whether it equals $[x_1 = x_1 \land \cdots \land x_k = x_k]^M$, but we can also see containment between various definable sets. (In other terms: we can see the validity of one implication but not the validity of a formula with nested implications.) This motivates the definition
of the coherent fragment $L^g_{\omega} \subseteq L_{\omega}$, which consists of formulas of the form $\forall \bar{x}(\varphi \to \psi)$ where $\varphi$ and $\psi$ are positive existential. We use the notation $\varphi \Rightarrow \psi$ and call it a coherent sequent.

This fragment is sufficiently rich: if we are given an arbitrary $L_{\omega}$ theory, we can replace it with an $L^g_{\omega}$-theory, whose models are the same, but the homomorphisms are the maps which were previously the elementary embeddings. (But after all this is what usually happens: either we take a simple (coherent) theory and then looking at homomorphisms or we take a theory with high complexity but then we only care about the elementary maps. This fragment captures both cases.) The construction is called Morleyization: we introduce a relation symbol for each formula $R \varphi \subseteq X^n$ and then write things like $R \varphi \land R \neg \varphi \Rightarrow \bot$ and $\top \Rightarrow R \varphi(\bar{x})$, etc. Using $\forall = \neg \exists \neg$ we ensured that for each $\varphi$: $\varphi^M = R^M_\varphi$, and then we just add $\top \Rightarrow R \psi(\bar{y})$ whenever $\psi$ is a formula in $T$.

The above mentioned parametrization is the so-called syntactic category: given a coherent theory $T \subseteq L^g_{\omega}$, we construct its syntactic category $C_T$ as follows: objects are positive existential $L$-formulas up to (allowed) change of free variables, and a morphism $[\varphi(\bar{x})] \to [\psi(\bar{y})]$ is given by a positive existential formula $\mu(\bar{x}, \bar{y})$ such that in every model $M$ of $T$: $\mu^M \subseteq \varphi^M \times \psi^M$ is the graph of a function (this has a syntactic characterization as well). We take a morphism to be such a $\mu$ up to $T$-provable equivalence.

This category will have some exactness properties, coming from the syntax, e.g. we have finite products: The product of $[\varphi(\bar{x})]$ and $[\psi(\bar{x}')]$ is $[\varphi(x) \land \psi(x')]$ where we change $x'$ to $x''$ if it was the same variable as $x$. (To verify this we would have to define the induced map with a formula, then prove its functionality, the commutativity of the triangles and uniqueness of the induced map: i.e. we would have to prove that certain formulas are $T$-provably equivalent.) Note that the interpretation of this conjunct in $M$ is the direct product of $\varphi^M$ and $\psi^M$, so $M$, as a functor preserves finite products. If we list all these properties we get: finite limits, pullback-stable finite unions and pullback-stable image factorizations. Categories having all of these are called coherent.

Then we get what we want: $M : C_T \to \textbf{Set}$ coherent functors are the same as $T$-models in $\textbf{Set}$ and $M \Rightarrow N$ natural transformations are the same as the model homomorphisms. Actually the situation is even better: given any coherent category $\mathcal{D}$ we can define what is an $L$-structure in $\mathcal{D}$: we choose an underlying object $A$, for function symbols we choose arrows $f^M : A^n \to A$ and for relation symbols subobjects $P^M \subseteq A^m$. It turns out that inductively we can define $\varphi^M$ for any positive existential $\varphi$: for $t_1 = t_2$ it is the equalizer of the interpretation of the terms, for $P(t_1, \ldots, t_m)$ it is a pullback, $\land$ and $\lor$
is given by intersection and union of subobjects, finally $\exists x \varphi$ is given by the image factorization of $\varphi^M \hookrightarrow A^{n+1} \xrightarrow{\pi} A^n$. Homomorphisms are expressed with commutative squares. (E.g. if $T$ is the theory of groups then these are group objects and their homomorphisms in $\mathcal{D}$.) It is still true that $\mathcal{D}$-models of $T$ correspond to $\mathcal{C}_T \rightarrow \mathcal{D}$ coherent functors and homomorphisms to natural transformations between those.

This gives two natural ideas: all results in (positive) model theory are just statements about $\mathcal{C} \rightarrow \text{Set}$ coherent functors. So if we could reprove these theorems in the categorical language, using just a few properties of $\text{Set}$, then we would get a theorem about $\mathcal{C} \rightarrow \mathcal{D}$ coherent functors for any coherent category $\mathcal{D}$ (at least if it satisfies a few extra categorical properties). The first possibility is to take $\mathcal{D}$ to be something interesting e.g. compact Hausdorff spaces, and then study group objects/ ZFC objects, etc. there, using model theory. The second possibility is to note that syntactic categories are also coherent and hence study $T \rightarrow T'$ interpretations = $\mathcal{C}_T \rightarrow \mathcal{C}_{T'}$ coherent functors this way.

Actually all (small) coherent categories are syntactic categories of (small, many-sorted, coherent) theories, as the whole process has an inverse: given a coherent category $\mathcal{C}$ we can form its canonical signature $L_{\mathcal{C}}$ whose sorts are the objects in $\mathcal{C}$ and it only has unary function symbols; the arrows of $\mathcal{C}$. (Alternatively we could also take an extended canonical signature, where we introduce relation symbols for subobjects.) Then the theory is just writing down everything we see (=identities, commutative triangles, finite limits, unions, effective epimorphisms (images)), as if we were in $\text{Set}$. E.g. if we see that the function symbol $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ is coming from a monomorphism $f : X \rightarrow Y$ of $\mathcal{C}$, then we write $\tilde{f}(x) = f(x') \Rightarrow x = x'$ as an axiom. The trick is, that when we interpret the two formulas in a coherent category we do it by universal constructions (equalizers in this case), and hence this sequence is valid in a model $M$ iff $\tilde{f}^M$ is a monomorphism (i.e. satisfies the defining universal property: injectivity makes no sense in general). So $Th(\mathcal{C})$-models in $\mathcal{D}$ are the same as $\mathcal{C} \rightarrow \mathcal{D}$ coherent functors (and homomorphisms are the same as natural transformations). From this $\mathcal{C}_{Th(\mathcal{C})} \simeq \mathcal{C}$ also follows and we can summarize these facts as "coherent categories, coherent functors and natural transformations are the same as many-sorted coherent theories, their interpretations (=models), and the homomorphisms between interpretations".

Finally note that the following are synonyms: coherent theory, $\aleph_0$-geometric theory, h-inductive theory (what we called a coherent sequent is what is called a simple h-inductive sentence in [Bel14]). An h-inductive theory is a set of finite conjunctions of simple h-inductive sentences, which is the same as a
set of simple $h$-inductive sentences).

We mentioned one possible use, namely that we can generalize model theory statements to make theorems about $C \to D$ coherent functors. The first three sections are mainly dealing with this approach:

- We recall the definition of elementary natural transformations (Lecture 18X, Definition 2. of [Lur18]), that correspond to those homomorphisms which reflect the validity of positive existential formulas. (Definition 3.2) In [Bel14] these are called immersions.

- Given a coherent functor $M : C \to D$ we study when a collection of subobjects $(N_x \to M_x)_{x \in C}$ gives rise to an elementary coherent subfunctor $N \Rightarrow M$. (Theorem 3.8)

- Following [Hay18] we define positively closed coherent functors (the positive analogue of existentially closed models). (Definition 3.16)

- Still following [Hay18] we define types (as prime filters over the subobject lattices $Sub_C(x)$), and prove variants of the omitting types theorem (for coherent functors with various codomains). (Theorem 4.26 and 5.6)

- It turns out that $\lambda$-pretoposes have filtered colimits of size $< \lambda$ which commute with finite limits. (Theorem 5.8 and 5.9)

- We repeat (and slightly extend) the definition of a $(\lambda, \kappa)$-coherent category and a $(\lambda, \kappa)$-pretopos from [Esp17]. These correspond to $L^3_{\lambda \kappa}$ theories. (Definition 5.12)

- We prove that a $(\lambda, \kappa)$-pretopos $D$ (satisfying two extra conditions: it is weakly Boolean and $\{0\} \subseteq Sub_D(1)$ is a $\kappa$-prime filter) is a rich-enough structure to perform $< \kappa$ ultraproduct constructions with $C \to D$ coherent functors (for any coherent category $C$). One might read it as ”infinite quantifier theories are so weird that the interpretations of finitary theories inside them are closed under ultraproducts”. (Theorem 5.13)

- The Loś-lemma holds. (Theorem 5.2)

- We define semantically complete coherent categories (any two models satisfy the same coherent sequents). (Definition 5.14) (We refer to this property as $M$ and $N$ are elementary equivalent. In [Bel14] the term ”$M$ and $N$ are companions” is used.)
• $\mathcal{C}$ is semantically complete iff it is weakly Boolean and 2-valued. (Theorem 6.22)

• If $\mathcal{C}$ is semantically complete then $\text{Coh}(\mathcal{C}, \text{Set})$ (i.e. $\mathcal{C} \to \text{Set}$ coherent functors and elementary natural transformations) has the joint embedding property. (Theorem 5.17)

There is an equally important second possibility: now that we have a clear algebraic understanding of interpretations between theories (coherent functors) and homomorphisms between interpretations (natural transformations) we can make use of the functoriality of certain model-theoretic constructions, and apply the toolkit of category theory:

• Given the subobject functor $\text{Sub}_\mathcal{C} : \mathcal{C}^{\text{op}} \to \text{DLat}$ we study its left Kan-extension along the Yoneda-embedding. Its restriction to coherent functors is denoted by $L_\mathcal{C} : \text{Coh}(\mathcal{C}, \text{Set}) \to \text{DLat}$. (Proposition 6.5)

• This construction is also functorial in $\mathcal{C}$, stated precisely in Definition 6.3.

• We give an explicit description for $LM = L_\mathcal{C}M$. $M$ is positively closed iff $LM = 2$. (Proposition 6.8 and 6.12)

• We give several equivalent conditions for a model $M$ to be positively closed. (Theorem 6.20) (1 $\iff$ 4 $\iff$ 5 can be found in [Bel14], for the one-sorted case, in the language of model theory.)

• The above left Kan-extension always preserves finite products of coherent functors. (Theorem 6.25)

3 Tarski-Vaught test

Notation 3.1. $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are coherent categories, $M, M', N$ are coherent functors, $\eta, \nu, \alpha$ are natural transformations.

If $\mathcal{C}$ is a pretopos and $\mathcal{E} = \text{Set}$ then the following is Definition 2. in Lecture 18X of [Lur18]:

Definition 3.2. We say that a natural transformation

$$\mathcal{C} \xrightarrow[N]{} \mathcal{E}$$

is

$\eta$
is elementary, if for each \( i : a \hookrightarrow x \) monomorphism in \( C \):

\[
\begin{array}{ccc}
N_{\alpha} & \xrightarrow{N_i} & N_{x} \\
\downarrow & & \downarrow \\
M_{\alpha} & \xrightarrow{M_i} & M_{x}
\end{array}
\]

is a pullback. The category of \( C \to \mathcal{E} \) coherent functors and elementary natural transformations is denoted by \( \text{Coh}_{e}(C, \mathcal{E}) \).

**Proposition 3.3.** If \( \eta : N \Rightarrow M \) is an elementary natural transformation, then for each \( x \in C \) the component \( \eta_x \) is monic.

**Proof.** Let \( r, s : A \to N(x) \) be maps for which \( \eta_xr = \eta_xs \). We have an induced dashed arrow in

\[
\begin{array}{ccc}
A & \xrightarrow{(\alpha, \beta)} & M(x) \\
\downarrow & & \downarrow \\
M(x) & \xrightarrow{M(\Delta)} & M(x) \times M(x) \\
\downarrow \eta_x \circ \alpha & & \downarrow \eta_x \times \eta_x \\
N(x) & \xrightarrow{N(\Delta)} & N(x) \times N(x)
\end{array}
\]

from which \( r = s \) follows. \( \square \)

**Definition 3.4.** Let \( M : C \to \mathcal{E} \) be a coherent functor and \( (\alpha_x : N(x) \hookrightarrow M(x))_{x \in C} \) be a collection of subobjects. We say that it satisfies the Tarski-Vaught criterion, if for each \( i : \varphi \to x \times y \) monomorphism of \( C \) the map \( f \circ j \) in

\[
\begin{array}{ccc}
M(\varphi) & \xrightarrow{M(i)} & M(x) \times M(y) \\
\downarrow & & \downarrow \\
s & \xrightarrow{\gamma} & M(x) \times N(y) \\
\downarrow \alpha_x \times 1 & \xrightarrow{\alpha_x \times \varphi} & \downarrow \gamma \\
q & \xrightarrow{j} & N(x) \times N(y) \\
\downarrow & & \downarrow \\
\uparrow \alpha_x \times 1 & \xrightarrow{p} \uparrow \\
q & \xrightarrow{f} & N(x) \times N(y) \\
\end{array}
\]

is surjective.
Proposition 3.5. If \((\alpha_x : N(x) \hookrightarrow M(x))_{x \in C}\) satisfies the Tarski-Vaught criterion then \(\alpha : N \Rightarrow M\) is a subfunctor.

Proof. We use the terminology of [MR77]. We have the following diagram:

\[
\begin{array}{c}
M(x) \xleftarrow{1 \times M(f)} M(x) \times M(y) \xrightarrow{\alpha_x \times 1} M(x) \\
\uparrow \downarrow \alpha_x \times 1 \downarrow \uparrow p_2 \\
s \xleftarrow{1 \times \alpha_y} N(x) \times M(y) \\
\downarrow \downarrow 1 \times \alpha_y \\
q \xleftarrow{1 \times \alpha_y} N(x) \times N(y)
\end{array}
\]

It would be enough to see that the subobject \(q \hookrightarrow N(x) \times N(y)\) is functional (from \(N(x)\) to \(N(y)\)), as in this case we had a commutative square

\[
\begin{array}{c}
\text{graph}(M(f)) \xleftarrow{1 \times \alpha_y} M(x) \times M(y) \\
\uparrow \downarrow \alpha_x \times 1 \\
\text{graph}(f') \xleftarrow{1 \times \alpha_y} N(x) \times N(y)
\end{array}
\]

from which the commutativity of

\[
\begin{array}{c}
M(x) \xrightarrow{M(f)} M(y) \\
\uparrow \downarrow \alpha_x \downarrow \uparrow \alpha_y \\
N(x) \xrightarrow{f'} N(y)
\end{array}
\]

follows.

\(s \hookrightarrow N(x) \times M(y)\) is functional, as it is the equalizer of \(M(f) \circ \alpha_x \circ p_1\) and \(p_2 : N(x) \times M(y) \rightarrow M(y)\). We need to conclude the functionality of \(q\), which is, that \(Th(E)\) proves \(q(x, y) \land q(x, y') \Rightarrow y \approx y'\) and \(\top \Rightarrow \exists q(x, y)\). With diagrams it is verified as follows:
As the red map equalizes the blue arrows, the pink map equalizes the green ones. The second axiom follows, as in

\[
s \rightarrow N(x) \times M(y) \quad q \rightarrow N(x) \times N(y)
\]

if \( s \rightarrow N(x) \) is surjective then \( q \rightarrow N(x) \) is also, by our assumption.

**Proposition 3.6.** If the subfunctor \( \alpha : N \Rightarrow M \) satisfies the Tarski-Vaught test then it is elementary.

*Proof.* The map \( s \rightarrow p \) is the pullback of 1, hence it is an effective epi (iso). By assumption this implies that the dashed arrow is eff. epi, and since it is also monic, it is an isomorphism.
Definition 3.7. The collection \((\alpha_x : N(x) \hookrightarrow M(x))_{x \in C}\) preserves finite products if \(\alpha_1 : N(1) \hookrightarrow M(1) = 1_D\) is an isomorphism and there is an isomorphism \(\varphi\) making

\[
\begin{align*}
M(x) & \times M(y) & \xleftarrow{\cong} & M(x \times y) \\
\alpha_x \times \alpha_y & & & \alpha_{x \times y} \\
N(x) & \times N(y) & \xleftarrow{\varphi} & N(x \times y)
\end{align*}
\]

commute.

Theorem 3.8. Let \(M : C \to D\) be a coherent functor and let \((\alpha_x : N(x) \hookrightarrow M(x))_{x \in C}\) be a collection of subobjects. The following are equivalent:

1. \((\alpha_x)_{x \in C}\) satisfies the Tarski-Vaught criterion and preserves finite products,

2. it extends (uniquely) to a coherent elementary subfunctor \(\alpha : N \Rightarrow M\).

Proof. 1 \(\to\) 2. It remains to prove that \(N\) is coherent. As

\[
\begin{align*}
M(x) & \leftarrow M(x) \times M(y) & \xleftarrow{\cong} & M(x \times y) \\
\alpha_x & & & \alpha_{x \times y} \\
N(x) & \leftarrow N(x) \times N(y) & \xleftarrow{\varphi} & N(x \times y)
\end{align*}
\]

commutes and \(N(p_1)\) is uniquely determined, the preservation of finite products follows. As \(\alpha\) is elementary \(N(i)\) in

\[
\begin{align*}
M(a) & \xleftarrow{M(i)} M(x) & \xrightarrow{M(f)} & M(y) \\
\alpha_x & & & \alpha_{x \times y} \\
N(a) & \xleftarrow{N(i)} N(x) & \xrightarrow{N(f)} & N(y)
\end{align*}
\]

is an equalizer hence \(N\) preserves finite limits.

The preservation of unions is immediate, as unions are pullback-stable in \(D\). Finally let \(f : x \to y\) be an effective epimorphism in \(C\). In
every square is a pullback. By pullback-stability $s \to N(y)$ is an effective epi, then by assumption $N(f) : N(x) \to N(y)$ is also.

2 $\to$ 1. By the universal property of $M(x) \times M(y)$, the inner square in

commutes, hence $(\alpha_x)_{x \in \mathcal{C}}$ preserves products in the sense of Definition 3.7.

For the Tarski-Vaught criterion observe

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By the pullback-stability of effective epi-mono factorizations \( s \to N(\exists x \varphi) \xrightarrow{N(j)} N(y) \) is a factorization for \( s \to M(x) \times N(y) \to N(y) \).

The extra assumption on ”preserving finite products” is needed, e.g. the collection \( (N(x) = \emptyset \hookrightarrow M(x))_x \) trivially satisfies the Tarski-Vaught condition. The reason for this is that we only considered subobjects \( \varphi \hookrightarrow x \times y \) secretly assuming that if \( y = y_0 \times \ldots y_k \) then \( N \) will preserve this product. We can modify the definition to get:

**Definition 3.9.** Let \( \mathcal{C}, \mathcal{D} \) be coherent categories, \( M : \mathcal{C} \to \mathcal{D} \) be a coherent functor and \( (N_x \xrightarrow{\alpha_x} M_x)_{x \in \mathcal{C}} \) be a collection of subobjects. We say that it satisfies the Tarski-Vaught test in the stronger sense if for any subobject \( \varphi \hookrightarrow x_0 \times \ldots \times x_n \times y_0 \times \ldots y_m \) we have that \( f \circ j \) in

\[
\begin{array}{ccc}
M\varphi & \xrightarrow{\text{id} \times \alpha_0 \times \ldots \times \alpha_{m-1}} & Mx_0 \times \ldots \times Mx_{n-1} \times My_0 \times \ldots My_{m-1} \\
& \Downarrow{i} & \Downarrow{j} \\
\quad s \quad & Mx_0 \times \ldots \times Mx_{n-1} \times N y_0 \times \ldots N y_{m-1} & \to \quad N y_0 \times \ldots N y_{m-1} \\
& \Downarrow{\alpha_{x_0} \times \ldots \alpha_{x_{n-1}} \times i d} & \ \ \\
q & Nx_0 \times \ldots \times Nx_{n-1} \times N y_0 \times \ldots N y_{m-1} \\
\end{array}
\]

is effective epi (for \( n > 0, m \geq 0 \), where we define \( \prod_{\emptyset} N y_i \to \prod_{\emptyset} M y_i \) to be \( 1 \to 1 \)).

**Theorem 3.10.** Let \( M : \mathcal{C} \to \mathcal{D} \) be a coherent functor and \( (N x \xrightarrow{\alpha_x} M x)_{x \in \mathcal{C}} \) be a collection of subobjects. The following are equivalent:

1. It satisfies the Tarski-Vaught test in the stronger sense.
2. It extends uniquely to an elementary coherent subfunctor \( \alpha : N \Rightarrow M \).

**Proof.** 2 \( \Rightarrow \) 1 is clear from Theorem 3.8, as the component at \( x_0 \times \ldots \times x_{n-1} \) must be \( \alpha_{x_0} \times \ldots \times \alpha_{x_{n-1}} \). Assuming 1 we shall prove that the collection preserves products.

Terminal object: applying the condition to \( 1 \to 1 \) we get that in

\[
\begin{array}{ccc}
M(1) & \xrightarrow{\text{id}} & M(1) \times 1 \\
\downarrow{\alpha_1} & \quad & \downarrow{\alpha_1} \\
N(1) & \to & N(1) \times 1
\end{array}
\]
α₁ is effective epi, hence iso.

Binary products: taking $x \times y \xrightarrow{\Delta} (x \times y) \times x \times y$ yields that $q \to N(x \times y)$ in

$$
\begin{align*}
M(x \times y) & \xrightarrow{\cong} M(x \times y) \times Mx \times My \\
& \xrightarrow{\cong} Mx \times My \\
\downarrow & \downarrow & \downarrow \\
N(x \times y) & \xrightarrow{\cong} M(x \times y) \times Nx \times Ny \\
& \xrightarrow{\cong} Nx \times Ny \\
& \xrightarrow{\cong} q \\
\end{align*}
$$

is iso, and therefore $N(x \times y) \leq N(x \times y)$:

$$
\begin{align*}
M(x \times y) & \xrightarrow{\cong} M(x \times y) \times Mx \times My \\
\downarrow & \downarrow & \downarrow \\
N(x \times y) & \xrightarrow{\cong} N(x \times y) \times Nx \times Ny \\
\end{align*}
$$

The symmetric argument completes the proof.

In model theory one does not specify subsets of each definable set to describe a substructure, instead a single subset of the underlying set is used. As our setting is equivalent to the classical setup we should be able to do the same, via identifying the sorts among general objects (i.e. the objects $[x \approx x]$ if $C$ is a syntactic category). This tool will be used later as well.

**Definition 3.11.** Let $C$ be a coherent category. A filtration of $C$ is a set $S \subseteq Ob(C)$ together with a specified monomorphism $\iota_x : x \hookrightarrow s_{x,1} \times \ldots \times s_{x,n_x}$ (where $s_i \in S$) for every object $x$ of $C$, such that given any monomorphism $a \hookrightarrow s_1 \times \ldots \times s_n$ we have a commutative triangle
More precisely: \( \{s_1, \ldots, s_n\} = \{s'_{a',1}, \ldots, s'_{a',n}\} \) and the triangle is a square with the induced isomorphism on the right side.) Here we allow \( n_x = 0 \), i.e. to choose \( x \sim 1 \).

**Remark 3.12.** In other terms for each object \( a \) we choose finitely many objects \( s_1, \ldots, s_n \in S \) and maps \( \iota_{a,k} : a \to s_k \) satisfying the above conditions, that do not depend on which product object \( s_1 \times \cdots \times s_n \) we use.

**Proposition 3.13.** Every coherent category admits a filtration.

**Proof.** This property is invariant under equivalence, indeed if \( F : C \to D \) is an equivalence and \( (S, (\iota_x)_x) \) is a filtration on \( C \) then we can take \( S' = F[S] \) and if \( u \in D \) equals \( Fx \) for some \( x \in C \) then take \( \iota'_u : u \equiv F(\iota_x) \) otherwise take \( \iota'_u : u \equiv F(x) \xrightarrow{Fx} F(s_1) \times \cdots \times F(s_n) \) with an arbitrary isomorphism \( u \equiv Fx \). This gives a filtration on \( D \).

From [MR77] we know that every coherent category is the syntactic category of something. But then we can take \( S \) to be the set \( \{[x = x] : X \) is a sort\} and for any \( [\varphi(x_1, \ldots, x_n)] \) we take

\[
\iota_{[\varphi]} : [\varphi(x_1, \ldots, x_n)] = [\varphi(x_1, \ldots, x_n) \wedge x_1 \approx x'_1 \wedge \ldots \wedge x_n \approx x'_n] \\
\to [x'_1 \approx x'_1 \wedge \ldots \wedge x'_n \approx x'_n] = [x'_1 \approx x'_1] \times \cdots \times [x'_n \approx x'_n]
\]

(of course \( = \) is the induced isomorphism between the two product objects). In particular when \( \varphi \) is closed we take \( [\varphi] \xrightarrow{\varphi} [\top] \). The condition is satisfied, if \( [\varphi(\vec{x})] \xrightarrow{[\mu(\vec{x}, \vec{y})]} [\vec{y} \approx \vec{y}] \) is a monomorphism, then we can factor it through \( [\exists \vec{x}[\mu(\vec{x}, \vec{y})]] \) which is a canonical subobject of \( [\vec{y} \approx \vec{y}] \), as required. (This also works when \( \vec{y} \) is the empty sequence, i.e. when we are given a monomorphism \( [\varphi(x)] \xrightarrow{[\mu(x)]} [\top] \), which is the same as \( [\varphi(x)] \xrightarrow{[\varphi(x)]} [\top] \) as \( [\top] \) is terminal.)

**Proposition 3.14.** Let \( C \) be a coherent category and \( S \subseteq Ob(C) \), \( (\iota_x : x \leftrightarrow s_{x,1} \times \cdots \times s_{x,n_x})_{x \in C} \) be a filtration. Let \( N, M : C \to D \) be coherent functors and \( (\alpha_s :Ns \to Ms)_{s \in S} \) a collection of arrows in \( D \). If for any \( x \in C \) the dashed arrow in
exists then \( \alpha = (\alpha_x)_x \) is a natural transformation. If moreover these squares are pullbacks then \( \alpha \) is elementary.

**Proof.** In

\[
\begin{array}{c}
N_x \\ \alpha x \\ M x \\
\end{array} \quad \xrightarrow{Mt_x} \quad \begin{array}{c}
N s_{x,1} \times \cdots \times N s_{x,n_x} \\ \alpha_{s_{x,1}} \times \cdots \times \alpha_{s_{x,n_x}} \\
M s_{x,1} \times \cdots \times M s_{x,n_x} \\
\end{array}
\]

the commutativity of the rectangle and that of the right square make the left square commute (as \( M(\iota_x \times \iota_y) \) is mono). Taking the analogous diagram with the monomorphisms \( N(f) \) and \( M(f) \) (instead of their graphs), and using the pasting law for pullbacks yields the other half of the claim.

**Theorem 3.15.** Let \( C \) be coherent category with filtration \((\mathbb{S}, (\iota_x))_x\) and let \( M : C \to D \) be a coherent functor. For a collection of subobjects \( (\alpha_s : N(s) \hookrightarrow M(s))_{s \in \mathbb{S}} \) the following are equivalent:

1. \( (\alpha_s)_s \) satisfies the Tarski-Vaught condition, i.e. given \( \varphi \hookrightarrow (s_1 \times \cdots \times s_n) \times (s'_1 \times \cdots \times s'_m) \) the map \( f \circ j \) in

\[
\begin{array}{c}
M(\varphi) \\ M(s) \\
\end{array} \quad \xrightarrow{M(f)} \quad \begin{array}{c}
M s_1 \times \cdots \times M s_n \times M s'_1 \times \cdots \times M s'_m \\
\alpha_i \times \cdots \times \alpha_{s,n} \times \alpha'_{s',m} \\
\end{array}
\]

\[
\begin{array}{c}
p \\
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
M s_1 \times \cdots \times M s_n \times N s'_1 \times \cdots \times N s'_m \\
\alpha_i \times \cdots \times \alpha_{s,n} \times \alpha'_{s',m} \\
\end{array}
\]

\[
\begin{array}{c}
q \\
\end{array} \quad \xrightarrow{p} \quad \begin{array}{c}
N s_1 \times \cdots \times N s_n \times N s'_1 \times \cdots \times N s'_m \\
\end{array}
\]

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is surjective. (For \(n > 0, m \geq 0\). Note that if \(n = 0\) the condition trivially holds.)

2. There is a unique elementary coherent subfunctor \(\alpha : N \Rightarrow M\) with component \(\alpha_s\) at \(s \in S\). (Unique as a subobject of \(M\) in \(\mathcal{D}^C\).)

**Proof.** \(2 \Rightarrow 1\) is clear. It is also immediate that if \((\alpha_s)_s\) extends to a coherent elementary subfunctor then this extension is unique, and the components are calculated by the pullbacks

\[
\begin{array}{ccc}
Ma & \xrightarrow{M_{\alpha_s}} & Ms_{a,1} \times \cdots \times Ms_{a,n_a} \\
\alpha_a & & \alpha_{s_{a,1}} \times \cdots \times \alpha_{s_{a,n_a}} \\
Na & \xrightarrow{N_{\alpha_s}} & Ns_{a,1} \times \cdots \times Ns_{a,n_a}
\end{array}
\]

(when \(n_a = 0\) this means \(\alpha_a : Na = Ma \xrightarrow{id} Ma\).)

Now assume 1. Let \((\alpha_a)_a \in C\) be as above. The following diagram shows that our collection satisfies the Tarski-Vaught condition in the stronger sense.

\[
\begin{array}{ccc}
M\varphi & \xrightarrow{M_{\alpha_s}} & Ms_{a,1} \times \cdots \times Ms_{a,n_a} \\
\alpha_a & \downarrow & \alpha_{s_{a,1}} \times \cdots \times \alpha_{s_{a,n_a}} \\
N\varphi & \xrightarrow{N_{\alpha_s}} & Ns_{a,1} \times \cdots \times Ns_{a,n_a}
\end{array}
\]

Now we will discuss the categorical analogue of positively closed models, based on [Hay18].

**Definition 3.16.** Let \(C\) be coherent and let \(\mathcal{D}\) be \(|C|^+\)-geometric. The coherent functor \(N : C \rightarrow \mathcal{D}\) is positively closed if for each subobject \(\varphi \hookrightarrow x\) in \(C\) we have \(N(x) = N\varphi \cup \bigcup_{\psi \cap \varphi = \emptyset} N\psi\).

**Remark 3.17.** If \(C\) has filtration \((S, (t_x)_x)\) then it is enough to check the assumption for the subobjects \(\varphi \hookrightarrow s_0 \times \cdots \times s_{n-1}\) \((n \geq 0)\). (In the general case just take intersection with \(x \hookrightarrow s_{x,0} \times \cdots s_{x,n_{x-1}}\) and with its \(N\)-image.)
Proposition 3.18. If \( N : C \to D \) is positively closed then each natural transformation \( \alpha : N \Rightarrow M \) is elementary.

Proof. We have

\[
N(y) = N\varphi \sqcup \bigcup_{\varphi \cap \psi = \emptyset} N\psi \leq \alpha_y^*(M\varphi) \sqcup \bigcup_{\psi \cap \varphi = \emptyset} \alpha^*_\psi(M\psi) \leq N(y)
\]

and both summands on the left are contained in the corresponding one on the right from which \( N\varphi = \alpha^*_y(M\varphi) \) follows. \( \square \)

Remark 3.19. For \( C \to \text{Set} \) functors the converse is proved as part of Theorem 6.20.

The following is the analogue of Proposition 4.1. in [Hay18]:

Proposition 3.20. Let \( C \) be a coherent category with filtration \( (S, (\tau_x)_x) \), let \( D \) be \( |C|^+ \)-geometric and let \( M : C \to D \) be a coherent functor. Assume that we have subobjects \( \alpha_s : N(s) \hookrightarrow M(s) \) for \( s \in S \) such that for each \( \varphi \hookrightarrow s_0 \times \cdots \times s_{n-1} \times s'_0 \times \cdots \times s'_{m-1} \) in

\[
\begin{align*}
M\varphi & \to M s_0 \times \cdots \times M s_{n-1} \times M s'_0 \times \cdots \times M s'_{m-1} \\
N\varphi & \to N s_0 \times \cdots \times N s_{n-1} \times N s'_0 \times \cdots \times N s'_{m-1}
\end{align*}
\]

Then there is a unique positively closed coherent subfunctor \( \alpha : N \Rightarrow M \) with component \( \alpha_x \) at \( x \).

Proof. First we verify the Tarski-Vaught condition. We have to prove that the dashed arrow in
is an isomorphism. We have \( r \cap t = \emptyset \). By assumption there’s \( Ns_i' \times \ldots Ns_{m-1}' \leq r' \cup \bigcup_{\chi} M(\chi) \) hence \( Ns_i' \times \ldots Ns_{m-1}' = r \cup t \leq r' \cup t \). Taking intersection with \( r \) we get \( r \leq r' \). By Theorem 3.15 this implies that \((\alpha_s)\) extends (uniquely) to an elementary coherent subfunctor \( \alpha : N \Rightarrow M \). Our assumption with \( m = 0 \) gives that of Remark 3.17 which implies that \( N \) is positively closed. \( \square \)

4 Omitting types

**Definition 4.1.** \( \Sub_C : \mathcal{C} \to \mathbf{DLat}^{\mathcal{D}} \) is the functor which maps an object \( x \) to the distributive lattice \( \Sub(x) \) and an arrow \( f : x \to y \) to the homomorphism \( f^* : \Sub(y) \to \Sub(x) \). Given a coherent functor \( M : \mathcal{C} \to \mathcal{D} \) we have a natural transformation \( \Sub_M : \Sub_D \circ M \Rightarrow \Sub_C \) whose components are given by the homomorphisms \( M : \Sub(x) \to \Sub(Mx) \).

**Remark 4.2.** To see that \( \Sub_M \) is indeed a natural transformation we need the commutativity of

\[
\begin{array}{ccc}
\Sub_C(x) & \xrightarrow{f^*} & \Sub_C(y) \\
\downarrow M & & \downarrow M \\
\Sub_D(Mx) & \xleftarrow{(Mf)^*} & \Sub_D(My)
\end{array}
\]

which follows as \( M \) preserves pullbacks.

**Proposition 4.3.** For elementary \( \alpha \) the following commutes:

\[
\begin{tikzcd}
\mathcal{C} \ar{r}{\Sub_C} \ar{dr}{\Sub_D} & \mathbf{DLat}^{\mathcal{D}} \\
\mathcal{D} \ar{ur}{\Sub_N} \ar{ru}{\Sub_M} & 
\end{tikzcd}
\]

*Proof.* We need the commutativity of

\[
\begin{tikzcd}
\Sub_C(x) \\
\Sub_D(Mx) \ar{r}{\alpha^*_x} \ar{r}{M} & \Sub_D(Nx) \\
\ar{u}{N} \ar{u}{\alpha} \ar{u}{\alpha}
\end{tikzcd}
\]

which follows as \( \alpha \) is elementary. \( \square \)
Theorem 4.4 (Stone-duality for distributive lattices, see e.g. [Lom18]). The functor $\text{Spec} : \text{DLat}^{\text{op}} \to \text{Spec}$ which takes a distributive lattice $L$ to the spectral space of its prime filters, and a homomorphism $h : L \to L'$ to $\text{Spec}(h) : \text{Spec}(L') \to \text{Spec}(L)$, which is pulling back along $h$, is an equivalence of categories.

Remark 4.5. The basic opens (equivalently the compact opens) in $\text{Spec} L$ are of the form $[\varphi] = \{ p \in \text{Spec} L : \varphi \in p \}$ for some element $\varphi \in L$. Hence the closure of a prime filter $p$ is given as $\bigcap_{\psi \notin p} \{ r : \psi \notin r \}$, i.e. $q \in p$ if $q \leq p$.

Definition 4.6. We set $S_C = \text{Spec} \circ \text{Sub}_C$ and $S_M = \text{Spec} \circ \text{Sub}_M$. Elements of $S_C(x)$ are called $x$-types over $C$. An $x$-type $p$ is called somewhere dense if $\text{int}(p) \neq \emptyset$. An $x$-type $p$ is realized by $M : C \to D$ if $S^{-1}_M(p)$ contains a somewhere dense $Mx$-type, otherwise it is omitted.

Remark 4.7. If we start with the syntactic category of a 1-sorted coherent theory $C = C_T$ such that the single sort corresponds to the object $x \in C$, then restricting $S_C$ to the subcategory of finite powers of $x$ and morphisms whose coordinate functions are projections: $x^k \xrightarrow{(p_1, \ldots, p_n)} x^n$ yields the type space functor $\text{FinOrd}^{\text{op}} \to \text{Spec}$, studied e.g. in [Kam20].

Proposition 4.8. Every somewhere dense type is a maximal filter.

Proof. Assume $\emptyset \neq [\varphi] \subseteq \overline{\varphi}$. We claim $\varphi \in p$. If not, then $p \in S_C(x) \setminus [\varphi]$ and therefore $\overline{\varphi} \subseteq S_C(x) \setminus [\varphi]$ as the latter is closed. Then we have $[\varphi] \subseteq S_C(x) \setminus [\varphi]$ meaning $[\varphi] = \emptyset$. But then for each $p \leq p'$ we have $\varphi \in p'$ from which $p' \leq p$ follows. \[\square\]

Proposition 4.9. Take $p \in S_C(x)$ with $C \not\cong \ast$. The following are equivalent:

1. $\emptyset \neq [\varphi] \subseteq \overline{\varphi}$
2. $\psi \in p \Leftrightarrow \varphi \cap \psi \neq \emptyset$

Proof. 1 $\Rightarrow$ 2. In the previous proposition we have seen $\varphi \in p$, hence $\psi \in p$ implies $\varphi \cap \psi \in p$ and hence $\varphi \cap \psi \neq \emptyset$. Conversely, since $\varphi \cap \psi \neq \emptyset$ it is contained in a prime filter $q \in S_C(x)$. But then by Remark 4.5 we have $\psi, \varphi \in q \leq p$.

2 $\Rightarrow$ 1. As $C \not\cong \ast$ we have $\varphi \neq \emptyset$ and therefore $[\varphi] \neq \emptyset$. Assume $\varphi \in q \in S_C(x)$. Then $\psi \in q$ implies $q \ni \psi \cap \varphi \neq \emptyset$ and hence $\psi \in p$. \[\square\]

Proposition 4.10. Take a coherent functor $\not\cong \ast : C \xrightarrow{M} D$ and take $p \in S_C(x)$. The following are equivalent:
1. $p$ is realized by $M$

2. there is a subobject $a \in \text{Sub}_D(Mx)$ such that $\varphi \in p \iff M\varphi \cap a \neq \emptyset$ and $b_1, b_2 \leq a$, $b_1 \cap b_2 = \emptyset$ implies $b_1 = \emptyset$ or $b_2 = \emptyset$.

**Proof.** The type $p$ is realized by $M$ iff there is a subobject $a$ and a prime filter $u$ with $\emptyset \neq [a] \subseteq \pi$ (equivalently: a subobject $a$ such that $\{b : a \cap b \neq \emptyset\}$ is a prime filter, and this is equivalent to our condition on $\text{Sub}_D(a)$) and with $M^{-1}(u) = p$, i.e. with $\varphi \in p$ iff $M\varphi \in u$ iff $M\varphi \cap a \neq \emptyset$.

**Definition 4.11.** A coherent category $\mathcal{D}$ is weakly Boolean if for any pair of subobjects $a \hookrightarrow x \twoheadleftarrow b$ either $a \leq b$ or there is $\emptyset \neq u \leq a$ such that $u \cap b = \emptyset$.

**Definition 4.12.** An object $a \in \mathcal{C}$ is an atom if $|\text{Sub}\mathcal{C}(a)| = 2$. $\mathcal{C}$ is 2-valued if $1 \in \mathcal{C}$ is an atom.

**Definition 4.13.** We say that a type $p \in S_C(x)$ is strongly realized by $M : \mathcal{C} \to \mathcal{D}$ if the object $a$ of Proposition 4.10 is an atom. It is weakly omitted if it is not strongly realized. When $\mathcal{D}$ is weakly Boolean the two notions coincide.

**Proposition 4.14.** Let $M : \mathcal{C} \to \mathcal{D}$ be positively closed with $\mathcal{D}$ being $|\mathcal{C}|^{+}$-geometric and let $\mathcal{C}$ be 2-valued, i.e. $\text{Sub}_\mathcal{C}(1) \cong 2$. Assume that the set of somewhere dense points is dense in $S_D(Mx)$. Then every somewhere dense type $p \in S_C(x)$ is realized by $M$.

**Proof.** Assume $\emptyset \neq [\varphi] \subseteq \overline{p}$. Any coherent functor with 2-valued domain reflects the initial object. Indeed if $x \neq 0$ then $\exists! x \neq 0$ since any map to the initial object is an isomorphism. But then we must have $\exists! x = 1$ and hence $\exists! Mx = 1$ from which $Mx \neq \emptyset$ follows.

By Proposition 4.9 there’s

$$\psi \in p \iff \psi \cap \varphi \neq \emptyset \iff M\psi \cap M\varphi \neq \emptyset$$

as $M$ reflects the initial object. For $\psi \leq x$ we have $Mx = M\psi \sqcup \bigcup_{\psi' \cap \psi = \emptyset} M\psi'$. Take $u \in S_D(Mx)$ with $M\varphi \in u$. Then

$$\psi \in p \Rightarrow \forall \psi' : \psi' \cap \psi = \emptyset \rightarrow \psi' \notin p \Rightarrow M\psi' \cap M\varphi = \emptyset \Rightarrow M\varphi \cap \bigcup_{\psi' \cap \psi = \emptyset} M\psi' = \emptyset \Rightarrow M\varphi \leq M\psi \Rightarrow M\psi \in u$$

and

$$\psi \notin p \Rightarrow M\psi \cap M\varphi = \emptyset \Rightarrow \exists \psi' : \psi \cap \psi' = \emptyset \text{ and } M\varphi \cap M\psi' \neq \emptyset \Rightarrow \psi' \in p \Rightarrow M\psi' \in u \Rightarrow M\psi \notin u$$
meaning $M^{-1}(u) = p$ (and therefore $u \in S_M^{-1}(p)$). By assumption there is a somewhere dense prime filter $u \in [M \varphi] \neq \emptyset$ which completes the proof. \hfill \QED

**Proposition 4.15.** Let $f : x \to y$ be an arrow of $C$. Then $S_C(f)$ is an open map. If $f$ is mono/ effective epi then $S_C(f)$ is injective/ surjective.

**Proof.** Image-factorization $\exists f : \text{Sub}(x) \to \text{Sub}(y)$ is left adjoint to the pullback map $f^* : \text{Sub}(y) \to \text{Sub}(x)$, i.e. we have that $\exists f(r) \leq a$ iff $r \leq f^*(a)$. Hence we have $f^*(a) \land r \leq f^*(b)$ iff $\exists f(f^*a \land r) \leq b$ iff $a \land \exists f(r) \leq b$, where the last equivalence follows from Proposition 3.2.7. of [MR77]. By Theorem 8. of [Lom18] this implies that $S_C(f) = \text{Spec}(f^*)$ is open.

The second part follows from Theorem 6. of [Lom18], as if $f$ is monic then $f^*$ is surjective, while if $f$ is effective epi then $f^*$ is injective. \hfill \QED

**Definition 4.16.** Let $C$ be a coherent category and $\mathfrak{S}$ be a list of objects (typically a filtration but every element has $\omega$-many occurrences in the list). We write $S^\omega_C = S^\omega_C(\mathfrak{S})$ for the cofiltered limit $\lim_{X \in \mathfrak{S} \leq \omega} \text{Sub}_C(\prod_{s \in X} s)$ (i.e. the limit is taken over finite products from $\mathfrak{S}$ and the projection maps). Points of $S^\omega_C$ are called $\omega$-types over $C$ (even if $\mathfrak{S}$ is uncountable).

$S^\omega_C(\mathfrak{S})$ is the $\text{Spec}$-image of the cofiltered colimit $\text{colim}_{X \in \mathfrak{S} \leq \omega} \text{Sub}_C(\prod_{s \in X} s)$. In particular any coherent functor $M : C \to D$ yields an induced map

$$\text{colim}_{X \in \mathfrak{S} \leq \omega} \text{Sub}_C(\prod_{s \in X} s) \to \text{colim}_{X \in \mathfrak{S} \leq \omega} \text{Sub}_D(\prod_{s \in X} Ms)$$

whose $\text{Spec}$-image we call $S^\omega_M : S^\omega_D(M\mathfrak{S}) \to S^\omega_C(\mathfrak{S})$.

An $\omega$-type $q \in S^\omega_C(\mathfrak{S})$ is realized by $M : C \to D$ if there is a point $u \in S^\omega_D(M\mathfrak{S})$ with $S^\omega_M(u) = q$ and for which each projection $u_X \in S_D(\prod_{s \in X} Ms)$ is somewhere dense.

**Remark 4.17.** By Lemma 5.24.5. of [Sta21] $S^\omega_C$ is also the limit in $\text{Top}$.

**Proposition 4.18.** Let $* \not\subseteq C \overset{M}{\to} D$ be a coherent functor and take $q \in S^\omega_C(\mathfrak{S})$. The following are equivalent:

1. $q$ is realized by $M$

2. there are subobjects $b_X \hookrightarrow M(\prod_{s \in X} s)$ such that
   - $\exists (\pi_{i_1}, . . . , \pi_{i_k})(b_{\{s_{i_1}, . . . , s_{i_k}\}} \cap b_{\{s_{i_1}, . . . , s_{i_k}\}}) \neq \emptyset$
   - $a, a' \leq b_X, \land \exists a' = \emptyset \Rightarrow a = \emptyset$ or $a' = \emptyset$
   - $\varphi \in q_X \Leftrightarrow M\varphi \cap b_X \neq \emptyset$

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Proof. As elements of the limit correspond to compatible sequences, \( q \) is realized by \( M \) iff there’s \( u_X \in S_D(\prod_{s \in X} M s) \) with \( \emptyset \neq [b_X] \subseteq \mathfrak{u}_X \) such that
\[
S_M(\prod_{s \in X} u_X) = q_X \quad \text{and} \quad S_D((\pi_{i_1}, \ldots, \pi_{i_k}))(u_{\{s_1, \ldots, s_n\}}) = u_{\{s_1, \ldots, s_k\}}.
\]
By Proposition 4.10 the first two properties are equivalent to the last two bullet points while the third one means
\[
c \in u_{\{s_1, \ldots, s_k\}} \text{ iff } \text{c} \times \prod_{j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} M s_j \in u_{\{s_1, \ldots, s_n\}}
\]
i.e.
\[
c \cap b_{\{s_1, \ldots, s_k\}} \neq \emptyset \iff \text{c} \times \prod_{j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} M s_j \cap b_{\{s_1, \ldots, s_n\}} \neq \emptyset \iff
\]
\[
c \cap \exists (\pi_{i_1, \ldots, i_k}) b_{\{s_1, \ldots, s_n\}} \neq \emptyset
\]
Taking \( c = \exists (\pi_{i_1, \ldots, i_k}) b_{\{s_1, \ldots, s_n\}} \) implies the first bullet point. Conversely, assuming \( \exists (\pi_{i_1, \ldots, i_k}) b_{\{s_1, \ldots, s_n\}} \cap b_{\{s_1, \ldots, s_k\}} \neq \emptyset \) we conclude that \( \exists (\pi_{i_1, \ldots, i_k}) b_{\{s_1, \ldots, s_n\}} \in u_{\{s_1, \ldots, \}} \) and \( b_{\{s_1, \ldots, s_k\}} \times \prod_{j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} M s_j \in u_{\{s_1, \ldots, \}} \) from which the above equivalence follows.

Definition 4.19. A type \( q \in S_C^\omega(\mathfrak{S}) \) is strongly realized by \( M : C \rightarrow D \) if each \( b_X \) is an atom. (When \( D \) is weakly Boolean \( q \) is realized iff it is strongly realized.)

Proposition 4.20. Let \( \mathfrak{S} \) be countable and assume that for each \( s \in \mathfrak{S} \) the map \( s \rightarrow 1 \) is effective epi. Then for each finite \( X \subseteq \mathfrak{S} \) the projection map \( \pi_X : S_C^\omega(\mathfrak{S}) \rightarrow S_C(\prod_{s \in X} s) \) is an open surjection.

Proof. Writing \( \mathfrak{S} = (s_i)_{i < \omega} \) yields a final subdiagram \( s_0 \leftarrow s_0 \times s_1 \leftarrow \ldots \). As effective epimorphisms are pullback-stable each projection \( s_0 \times \cdots \times s_{n-1} \leftarrow s_0 \times \cdots \times s_n \) is effective epi, then by Proposition 4.15 each arrow \( S_C(s_0 \times \cdots \times s_{n-1}) \leftarrow S_C(s_0 \times \cdots \times s_n) \) is surjective.

In \( \mathbf{Set} \) the transfinite composition of surjections is surjective, hence the same holds in \( \mathbf{Top} \) as the forgetful functor preserves limits. By Remark 4.17 it follows that each \( \pi_{\{s_0, \ldots, s_n\}} \) is onto. Hence each \( \pi_X \) is onto.

By Lemma 5.14.2. of [Sta21] the preimages \( \{\pi_{\{s_0, \ldots, s_n\}}^{-1}(V) : n \in \mathbb{N}, V \subseteq S_C(\prod_{i \leq n} s_i) \text{ open}\} \) form a basis of the limit. As images distribute over unions it suffices to prove that these are mapped to opens by \( \pi_{\{s_0, \ldots, s_k\}} \)'s. When \( k = n \) we get back \( V \) by surjectivity. If \( k \geq n \) it follows from \( S_C(\prod_{i \leq k} s_i) \leftarrow S_C(\prod_{i \leq n} s_i) \) being continuous. If \( k \leq n \) it follows from \( S_C(\prod_{i \leq k} s_i) \leftarrow S_C(\prod_{i \leq n} s_i) \) being open. \( \square \)
Now we rewrite Lemma 4.2. of [Hay18] with coherent categories. The following is Theorem 4.2. in [Ish75]:

**Theorem 4.21 (Isbell).** Any compact sober space (hence every spectral space) is a Baire-space, i.e. a countable intersection of dense open sets is dense.

**Definition 4.22.** Fix a countable coherent category $C$ with filtration $S$ such that for each $s \in S$ the map $s \to 1$ is effective epi. Let $\bar{S} = (s_i)_{i<\omega}$ be the list of objects from $S$ where every element appears $\omega$-many times. We will write $\pi_K$ for $\pi_{\{s_0, \ldots, s_{K-1}\}} : S_C(\bar{S}) \to S_C(\prod_{n<K} s_n)$.

Take $\varphi \mapsto s_{i_0} \times \ldots \times s_{i_{k-1}} \times s_{j_0} \times \ldots \times s_{j_{l-1}}$ with $k > 0, l \geq 0$, and a function $f : k \to \omega$ such that $s_{i_n}$ and $s_{f(n)}$ are copies of the same element of $S$ (we will refer to this as the sort of $s_{i_n}$ is the same as the sort of $s_{f(n)}$). Given $g : l \to \omega$ such that the sort of $s_{j_n}$ coincides with that of $s_{g(n)}$ we denote by $\varphi_{f,g} \mapsto s_0 \times \ldots \times s_{K-1}$ the pullback of $\varphi$ along $\langle \pi_{f(0)}, \ldots, \pi_{f(k-1)}, \pi_{g(0)}, \ldots, \pi_{g(l-1)} \rangle : \prod_{n<K} s_n \to s_{i_0} \times \ldots \times s_{i_{k-1}} \times s_{j_0} \times \ldots \times s_{j_{l-1}}$ for some $K > \max\{f(n), g(m) : n < k, m < l\}$. Let $O_{\varphi}^f$ be the following open set:

$$O_{\varphi}^f = \bigcup_{g : l \to \omega} \pi_K^{-1}( [\varphi_{f,g}^*]) \cup \bigcup_{\chi \mapsto s_{i_0} \times \ldots \times s_{i_{k-1}} \times s_{j_0} \times \ldots \times s_{j_{l-1}}} \pi_K^{-1}( [\chi_{f,h}^*])$$

Finally we set $G = \bigcap_{\varphi} \bigcap_{f} O_{\varphi}^f \subseteq S_C^\omega(\bar{S})$.

**Remark 4.23.** The above definition (hopefully) expresses the following: we fix an $\omega$-sequence of variables $x_i$ such that every sort appears infinitely many times. Then for any formula $\varphi(x, y)$ and any identification of $x$ with some variables from the sequence $(f x)$, either we can identify $y$ with some other variables $(g y)$ such that $\varphi(f x, g y)$ is in our $\omega$-type or there’s $\chi(x, z)$ and an identification $h$ such that $\chi(f x, h z)$ is in our $\omega$-type and this is $T$-provably an obstruction for $\varphi$ being present, i.e. $T \vdash \exists y \varphi(f x, y) \land \exists z \chi(f x, z) \leftrightarrow \bot$.

**Proposition 4.24.** $G$ is comeagre in $S_C^\omega(\bar{S})$.

**Proof.** We have to show that each $O_{\varphi}^f$ is dense. It suffices to prove that for any non-empty basic open $\emptyset \neq [\psi] \subseteq S_C(s_0 \times \ldots s_{K-1})$:

$$\pi_K^{-1}( [\psi]) \cap O_{\varphi}^f \neq \emptyset.$$

We can assume $K > \max\{f(n) : n < k\}$ (otherwise take the preimage of $V$ in the sequence). Let $g : l \to [K, L] \subseteq \omega$ be an injective function with $\text{sort}(s_{j_n}) = \text{sort}(s_{g(n)})$. If $\psi \times s_K \times \ldots \times s_{L-1} \cap \varphi_{f,g}^* = \emptyset$ then $\pi_K^{-1}( [\psi]) \subseteq O_{\varphi}^f$ (as
we have to prove that $\Delta$ is $\prod c$ this map. Then $(\phi \chi)\rightarrow \prod N(s_{j_0}) \times \cdots \times N(s_{j_{l-1}})$ is effective epi for each $s \in S$ (e.g. because $D$ is 2-valued). Then there is a (unique) positively closed coherent subfunctor $\alpha : N \Rightarrow M$ with component $N(s) \Rightarrow M(s)$ at $s \in S$.

**Proof.** By pullback-stability we have

$$N(s) \times N(s') = \bigcup_{\text{sort}(s_i) = \text{sort}(s)} c_i \times \bigcup_{\text{sort}(s_j) = \text{sort}(s')} c_j = (M(s) \times \bigcup_j c_j) \cap (\bigcup_i c_i \times M(s')) = \bigcup_{i,j} c_i \times c_j$$

and more generally $N(s_{j_0}) \times \cdots \times N(s_{j_k}) = \bigcup_{i_1 \cdots i_k : \text{sort}(s_{i_n}) = \text{sort}(s_{j_n})} c_{i_1} \times \cdots \times c_{i_k}$ for $k \geq 1$.

By Proposition 3.20 we have to prove

$$N(s_{j_0}) \times \cdots \times N(s_{j_{l-1}}) \leq \exists M(\varphi) \cap N(s_i') \times N(s_j') \cup \bigcup_{\chi \rightarrow \varphi \cap \exists \psi = \emptyset} M(\chi)$$

for $\varphi \leftarrow s_{i_0} \times \cdots \times s_{i_{k-1}} \times s_{j_0} \times \cdots \times s_{j_{l-1}}$ and $k, l \geq 0$.

First assume $k > 0, l \geq 0$. Fix $\varphi$ and $f : k \rightarrow \omega$ such that $\text{sort}(s_{i_n}) = \text{sort}(s_{j_n})$. Since $q \in G$ either we have $g : l \rightarrow \omega$ with $\text{sort}(s_{j_n}) = \text{sort}(s_{g(n)})$ such that $\varphi_{f,g}^* \in q_k$ or there is $\chi \leftarrow s_{i_0} \times \cdots \times s_{i_{k-1}} \times s_{i_0} \times \cdots \times s_{i_{l-1}}$ with $(\exists \pi \chi)^{f} \cap (\exists \pi \varphi)^{f} = \emptyset$ and there is $h : r \rightarrow \omega$ with $\text{sort}(s_{i_n}) = \text{sort}(s_{h(n)})$ such that $\chi_{f,h}^* \in q_l$. In the first case $b_k \cap M(\varphi_{f,g}^*) = \emptyset$ hence $b_k \leq M(\varphi_{f,g}^*)$ as $K$ is an atom. Therefore $\prod_{n=m=f(n)} c_m \leq 3_{M(\varphi_{f,g}^*) \cap N(s_{i_n}')} \times N(s_{i_0}')$ hence $c_{f(0)} \times \cdots \times c_{f(k-1)} \leq 3_{M(\varphi \cap N(s_{i_n}')}}$ $\cap N(s_{i_0}')) \cap N(s_{i_0}')$. Similarly we get that in the second case $\prod_{n=m=f(n)} c_m \leq 3_{M(\varphi_{f,h}^*) \cap N(s_{i_n}')} = M(\varphi_{f,h}^*)$. We have a projection map $\prod_{n=m=f(n)} s_m \leftarrow s_{f(0)} \times \cdots \times s_{f(k-1)}$. Let $\bar{\chi}$ be the pullback of $\exists \pi \chi$ along this map. Then $c_{f(0)} \times \cdots \times c_{f(k-1)} \leq M(\bar{\chi} \cap \Delta)$ and $(\bar{\chi} \cap \Delta) \cap \exists \pi \varphi = \emptyset$ (where $\Delta$ is $\prod_{n=m=f(n)} s_m \leftarrow s_{f(0)} \times \cdots \times s_{f(k-1)}$).

Now assume $k = 0, l \geq 0$ and take $\varphi \leftarrow s_{j_0} \times \cdots \times s_{j_{l-1}}$. We have to prove that

$$1 \leq \exists (M \varphi \cap N s_j') \cup \bigcup_{\chi \rightarrow \varphi \cap \exists \psi = \emptyset} M \chi$$

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As $Ns \times Ns_j \to 1$ is effective epi (for any $s$) it is enough to prove that the pullback of the inequality along this map holds, i.e.

$$Ns \times Ns_j \leq Ns \times (M \varphi \cap Ns_j) \cup \bigcup_{\chi \to 1 \chi \cap (s \times s_j)} M(\chi \times s \times s_j)$$

We have $\chi \cap \exists \exists \varphi = \emptyset$ iff $\neg \varphi \cap \emptyset = \emptyset$ (using $\exists f(a \cap f^{-1}b) = \exists f(a \cap b)$, and therefore this inequality follows by

$$Ns \times Ns_j \leq (M(s \times \varphi) \cap Ns \times Ns_j) \cup \bigcup_{\chi \to s \times s_j} M\chi$$

\[ \square \]

**Theorem 4.26.** Let $\mathcal{C}$ be a countable coherent category with filtration $\mathcal{S}$ such that for each $s \in \mathcal{S}$: $s \to 1$ is effective epi, and let $\mathcal{D}$ be 2-valued $\aleph_1$-geometric. Let $E_X \subseteq S_\mathcal{C}(\prod_{s \in X} s)$ be meagre subsets (for $X \subseteq \mathcal{S}$ finite). Then there is a comeagre set $A \subseteq S_\mathcal{C}(\mathcal{S})$ such that for any $q \in A$ is strongly realized by some $M : \mathcal{C} \to \mathcal{D}$ coherent functor then $M$ has a positively closed coherent subfunctor $N$ which weakly omits each type from each $E_X$.

**Proof.** Take $A = G \setminus \bigcup_X \pi_X^{-1}(E_X)$. As the preimage of a meagre set at an open map is meagre we have that $A$ is comeagre. Assume that $q \in A$ is strongly realized by $M : \mathcal{C} \to \mathcal{D}$, i.e. there are atomic subobjects $c_i \leq M(s_i)$ such that for $\varphi : \prod_{s_i \in X} s_i$ we have $\prod_{c_i \in X} c_i \cap M(\varphi) \neq \emptyset \Leftrightarrow \prod_{c_i \in X} c_i \leq M(\varphi) \Leftrightarrow \varphi \in q_X$ and let $(\mathcal{C}(s_i))_{s \in S}$ be as before. By the previous proposition it extends to a positively closed coherent subfunctor $N : \mathcal{C} \to \mathcal{D}$.

Assume that $N$ realizes an $X = \{s_{i_0}, \ldots, s_{i_m-1}\}$-type $p \in E_X$. That is, there is an atom $a \leq \prod_{r \in m} N(s_{i_r})$ such that $a \cap N\varphi \neq \emptyset$ iff $\varphi \in p$. As $\prod_{r \in m} N(s_{i_r}) = \bigcup_{\text{sort}(s_{i_r}) = \text{sort}(s_{i_r})} c_{j_0} \times \cdots \times c_{j_{m-1}}$ there is $j_0, \ldots, j_{m-1}$ such that $a \cap c_{j_0} \times \cdots \times c_{j_{m-1}} \neq \emptyset$. Then the two atoms must coincide hence $p = \pi_{\{s_{j_0}, \ldots, s_{j_{m-1}}\}}(q)$ which is a contradiction. \[ \square \]

One of our next goals is to give examples of pairs $(\mathcal{C}, \mathcal{D})$ such that small sets of $\mathcal{C}$-types can be omitted by $\mathcal{D}$-models. Since in the $\text{Set}$-case the realization of $\omega$-types is guaranteed by the compactness theorem, we study ultraproducts.

## 5 Ultraproducts

**Definition 5.1.** Let $\mathcal{D}$ be a category with products and filtered colimits. Let $(D_i)_{i \in I}$ be a set of objects and $U \subseteq \mathcal{P}(I)$ be an ultrafilter. The ultraproduct of
\(D_i \text{ wrt. } U \) is \(\prod_{i \in I} D_i \backslash U = \text{colim}_{J \in U} \prod_{i \in J} D_i\). (The colimit is taken over the projection maps. The diagram is filtered as \(U\) is closed under intersections.)

The following is the generalization of Theorem 2.1.1 in [Lur]:

**Theorem 5.2** (Łoś-lemma). Let \(\kappa\) and \(\lambda\) be regular cardinals such that \(\gamma < \kappa \Rightarrow 2^\gamma < \lambda\) and let \(\mathcal{D}\) be a \(\lambda\)-geometric category such that

1. it has \(<\kappa\) products,
2. for any object \(x \in \text{Sub}_D(x)\) the \(<\kappa\) meets distribute over binary unions,
3. if \((x_i \xrightarrow{f_i} y_i)_{i < \gamma < \kappa}\) are effective epimorphisms then \(\prod_i x_i \xrightarrow{\prod_i f_i} \prod_i y_i\) is an effective epimorphism,
4. the \(<\kappa\)-product of non-initial objects is non-initial,
5. it has filtered colimits of size \(<\lambda\), which commute with finite limits.
6. it is weakly Boolean

Then \(\text{Coh}(C, \mathcal{D})\) is closed under \(<\kappa\) ultraproducts in \(\mathcal{D}^C\) for any coherent category \(C\).

Moreover given \(A_i \hookrightarrow X_i \hookleftarrow B_i\) in \(\mathcal{D}\):

\[\prod_i A_i \backslash U \subseteq \prod_i B_i \backslash U \text{ iff } \{i : A_i \subseteq B_i\} \in U.\]

**Proof.** We have to prove that \(\prod_i F_i \backslash U\) is coherent when each \(F_i : C \to \mathcal{D}\) is. \(\prod J F_i\) preserves finite limits, the initial object (in a coherent category any map \(x \to \emptyset\) is an isomorphism as the subobject \(\emptyset \hookrightarrow \emptyset\) is pullback-stable, in particular \(\prod J \emptyset \xmapsto{\pi_i} \emptyset\) is iso), and effective epimorphisms, by assumption. As filtered colimits commute with finite limits and all colimits, \(\prod_i F_i \backslash U\) preserves finite limits, the initial object and effective epimorphisms.

It remains to prove that \(\prod_i F_i \backslash U\) preserves binary unions \(a \cup b \hookrightarrow x\), i.e. that the induced map \(f\) in

\[\begin{array}{cccc}
\prod_i F_i(a) \backslash U & \longrightarrow & \prod_i F_i(a) \cup \prod_i F_i(b) \backslash U & \longrightarrow & \prod_i F_i(b) \backslash U \\
\downarrow f & & \downarrow & & \downarrow \\
\prod_i F_i(a \cup b) \backslash U & \longrightarrow & \prod_i F_i(x) \backslash U
\end{array}\]

is an effective epimorphism.
is surjective. Take $\emptyset \neq v \hookrightarrow \prod_i F_i(a \cup b)/U$. For some ultra-set $J \in U$: $v$ intersects the image of the cocone map $\prod_J F_i(a \cup b) \rightarrow \prod_I F_i(a \cup b)/U$ as pullbacks and filtered colimits commute. Let $\emptyset \neq w \hookrightarrow \prod_J F_i(a) \cup F_i(b)$ be a non-empty subobject in the preimage of this intersection. As

$\prod_J F_i(a) \cup F_i(b) = \bigcap J F_0(x) \times \cdots \times (F_i(a) \cup F_i(b)) \times F_{i+1}(x) \times \cdots$ =

$\bigcup_{\varepsilon(i) \in \{a,b\}} \bigcap J F_0(x) \times \cdots \times F_i(\varepsilon(i)) \times F_{i+1}(x) \times \cdots = \bigcup J F_i(\varepsilon(i))$

(using that in $\text{Sub}_D(\prod_J F_i(x))$ the $< \kappa$ meets distribute over binary unions), for some choice-function $\varepsilon$: $w' = w \cap \prod_J F_i(\varepsilon(i))$ is non-empty. But for some ultra-set $J' \subseteq J$: $\varepsilon(i)$ is constant, let’s say it equals $a$. The commutativity of

implies that the image of $f$ intersects $v$, i.e. it intersects every non-empty subobject of the codomain. As $D$ was weakly Boolean, we get that $f$ is surjective.

For the second part first assume $J = \{i : A_i \subseteq B_i\} \in U$. Then we get a commutative triangle

For the second part first assume $J = \{i : A_i \subseteq B_i\} \in U$. Then we get a commutative triangle
for each \( U \ni J' \subseteq J \). As such ultrasets with reversed inclusion form a cofinal subdiagram of all ultrasets (with reversed inclusion), this induces the required map \( \prod_I A_i/U \to \prod_I B_i/U \). Conversely assume \( \prod_I A_i/U \subseteq \prod_I B_i/U \). If \{i : A_i \subseteq B_i\} \notin U then \( J = \{i : \exists \emptyset \neq v_i \hookrightarrow B_i : v_i \cap A_i \cap B_i = \emptyset\} \in U \) (as \( D \) is weakly Boolean). For each \( U \ni J' \subseteq J \) we get

\[
\emptyset \neq \prod_{J'} v_i \leftarrow \prod_{J'} B_i
\]

therefore (cofinality + every \( \bullet \to \emptyset \) map is iso + filtered colimits commute with pullbacks):

\[
\emptyset \neq \prod_J v_i/U \leftarrow \prod_J B_i/U = \prod_I B_i/U
\]

contradiction. \( \square \)

**Remark 5.3.** In the list of assumptions we could change 4. to

4′. \( \{0\} \subseteq \text{Sub}_D(1) \) is a \( \kappa \)-prime ideal.

Indeed, given objects \((x_i)_{i<\gamma<\kappa}\) we can take the effective epi-mono factorizations \( x_i \twoheadrightarrow u_i \hookrightarrow 1 \), and as \( x_i \neq 0 \), also \( u_i \neq 0 \). By 3. map \( \prod_i x_i \to \prod_i u_i \) is effective epi and the codomain is \( \bigcap_i u_i \) which is not 0 by our assumption.

**Theorem 5.4.** Let \( D \) be as above. The diagonal map \( \Delta : M \Rightarrow M^I/U \) is an elementary natural transformation.

**Proof.** It is a natural transformation because

\[
\begin{array}{ccc}
Mx & \xrightarrow{Mf} & M y \\
\downarrow & & \downarrow \\
Mx^J & \xrightarrow{Mf^J} & My^J \\
(\psi_J)_x & & (\psi_J)_y \\
Mx^I/U & \xrightarrow{Mf^I/U} & My^I/U
\end{array}
\]

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commutes. Note that the maps $\Delta_x$ are monomorphisms as filtered colimits commute with monomorphisms:

\[
\begin{array}{cccc}
M_x & \rightarrow & M_x & \rightarrow & \ldots & \rightarrow & M_x \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_x^J & \rightarrow & M_x^J & \rightarrow & \ldots & \rightarrow & M_x^J/U
\end{array}
\]

Given a monomorphism $a \hookrightarrow x$ assume that the induced map in

\[
\begin{array}{cccc}
Ma & \rightarrow & Mx \\
\downarrow & & \downarrow \\
Ma^J/U & \rightarrow & Mx^J/U
\end{array}
\]

is not effective epi. As $D$ is weakly Boolean there is a subobject $\emptyset \neq v \hookrightarrow R$ which is disjoint from $Ma$.

There is $J \in U$ such that $v^*$, the pullback of $v \rightarrow Ma^J/U$ along the cocone map $Ma^J \rightarrow Ma^J/U$ is non-empty:

\[
\begin{array}{cccc}
\emptyset & \rightarrow & Ma^J \\
\downarrow & & \downarrow \\
v & \rightarrow & Ma^J/U \\
\downarrow & & \downarrow \\
Mx & \rightarrow & Mx^J/U
\end{array}
\]

(And here the image of $v^* \rightarrow v \rightarrow Mx^J/U$ is disjoint from $Ma \xrightarrow{\Delta_x} Ma^J/U$.)

From this we get a diagram

\[
\begin{array}{cccc}
v^* & \rightarrow & Ma^J & \rightarrow & Ma^J/U \\
\downarrow & & \downarrow & & \downarrow \\
Mx & \rightarrow & Mx^J & \rightarrow & Mx^J/U
\end{array}
\]
where the rectangle and the right square commutes. If the equalizer of the two $v^* \to Mx^J$ maps is non-empty we replace $v^*$ with that. Otherwise there should be $J \supseteq J' \in U$ such that the equalizer of the two maps post-composed with the projection is non-empty. This is because filtered colimits and equalizers commute:

\[ \emptyset \xrightarrow{eq} v^* \xrightarrow{\delta} Mx^J \]

\[ \emptyset \xrightarrow{eq} v^* \xrightarrow{\delta} Mx^{J'} \]

\[ \ldots \]

\[ \emptyset \xrightarrow{eq} v^* \xrightarrow{\delta} Mx^J/U \]

So now we have a commutative outer square in

\[ \emptyset \neq v^{**} \xrightarrow{e} v^* \xrightarrow{\delta} Mx \]

\[ v^* \xrightarrow{\delta} Ma \xrightarrow{pb} Mx^{J'} \]

\[ Ma^{J'} \xrightarrow{\delta} Mx^{J'/U} \]

for some $\emptyset \neq v^{**} \hookrightarrow v^*$ and $J \supseteq J' \in U$. But as the inner square is a pullback we get the induced map. Therefore the images of $v^* \to Mx \to Mx^{J'} \to Mx^J/U$ and $Ma \xrightarrow{\delta} Ma^{J'} \to Mx^{J'} \to Mx^J/U$ are not disjoint which is a contradiction. \qed

As usual Loš-lemma implies compactness:

**Theorem 5.5.** Let $\mathcal{D}$ be as above. Let $L$ be a signature with $|L| \cdot \aleph_0 < \kappa$ and $T \subseteq L^{\omega_1}_{\omega}$ be a theory. If each finite subset $S \subseteq T$ has a model in $\mathcal{D}$ then $T$ has a model in $\mathcal{D}$.

**Proof.** Let $\mathcal{C}_S$ be the syntactic category of $S$ (objects are coherent $L$-formulas up to change of free variables, morphisms are $S$-provable equivalence classes of $S$-provably functional formulas, see [MR77]). For $S \subseteq S'$ we have a coherent functor $\mathcal{C}_S \to \mathcal{C}_{S'}$ mapping $[\varphi(x)] \xrightarrow{[\mu(x,y)]} [\psi(y)]$ to $[\varphi(x)] \xrightarrow{[\mu(x,y)]} [\psi(y)]$ (which
is well-defined as \( S \vdash \mu \iff \mu' \) implies \( S' \vdash \mu \iff \mu' \). For each finite \( S \subseteq T \) fix \( M_S : \mathcal{C} = \mathcal{C}_0 \rightarrow \mathcal{C}_S \rightarrow \mathcal{D} \).

Let \( U \subseteq \mathcal{P}([T]^{<\omega}) \) be an ultrafilter extending \( \{\hat{S} = \{S' \in [T]^{<\omega} : S \subseteq S'\} : S \in [T]^{<\omega}\} \). Then by Theorem 5.2 \( M = \prod_{[T]^{<\omega}} M_S/U : \mathcal{C} \rightarrow \mathcal{D} \) is a coherent functor with

\[
M([\varphi(\vec{x}, \vec{z}) \land \vec{y} \approx \vec{y}]) \subseteq M([\psi(\vec{y}, \vec{z}) \land \vec{x} \approx \vec{x}]) \quad \text{for each } \varphi(\vec{x}, \vec{z}) \Rightarrow \psi(\vec{y}, \vec{z}) \in T
\]

(More precisely the subobject above is represented by \( [\varphi(\vec{x}, \vec{z}) \land \vec{y} \approx \vec{y}] \subseteq [\psi(\vec{y}, \vec{z}) \land \vec{x} \approx \vec{x}] \)) for each \( \varphi(\vec{x}, \vec{z}) \Rightarrow \psi(\vec{y}, \vec{z}) \in T \).

Hence \( M \) factors through \( \mathcal{C}_T \).

**Theorem 5.6.** Assume the following:

- \( \mathcal{C} \) is countable with a filtration \( S \) such that for each \( s \in S \): \( s \rightarrow 1 \) is effective epi.
- \( \mathcal{D} \) is \((2^{\aleph_0})^+\)-geometric, weakly Boolean, 2-valued, in which atoms are dense. It has all countable products. It has filtered colimits of size \( \leq 2^{\aleph_0} \) which commute with finite limits.
- There is a \( \mathcal{C} \rightarrow \mathcal{D} \) conservative coherent functor.

Then given meagre subsets \( E_X \subseteq S_\mathcal{C}(\prod_{s \in X} s) \) for \( X \subseteq S \) finite, there is a \( \mathcal{C} \rightarrow \mathcal{D} \) positively closed coherent functor which omits each type from each \( E_X \).

**Proof.** First we check that the conditions of Theorem 5.2 are satisfied. Assume that each \( f_i : x_i \rightarrow y_i \) is effective epi and that \( \prod_i f_i \) is not. As \( \mathcal{D} \) is weakly Boolean with dense atoms there is an atom \( a \) which is disjoint from the image of \( \prod_i f_i \). Taking atoms in each preimage and taking their product shows \( a \) to lie in the image of \( \prod_i f_i \).

We also have to prove \( \bigcap_{i<\omega} a_{0,i} \cup a_{1,i} = \bigcup_{i \in \omega} \bigcap_j a_{x(i),j} \). The \( \geq \) direction is obvious, if this is a proper subobject we get an atom which is disjoint from it, this yields contradiction.

By Theorem 4.26 and by Theorem 4.21 it suffices to prove that every \( \omega \)-type \( p \in S_\mathcal{C}(\tilde{S}) \) is (strongly) realized by a \( \mathcal{C} \rightarrow \mathcal{D} \) coherent functor.

Let \( L \) be the extended canonical signature \( L_\mathcal{C} \) of \( \mathcal{C} \) plus countably many relation symbols \( (R_i \leftrightharpoons s_i)_{i<\omega} \) and \( T \) be the theory

\[
Th(C) + \{ R_0(x_0) \land \ldots R_{k-1}(x_{k-1}) \Rightarrow \varphi(x_0, \ldots, x_{k-1}) : \varphi \in \pi_{\{i_0, \ldots, i_{k-1}\}}(q) \} + \\
\{ R_0'(x'_0) \land \ldots \land R_{l-1}'(x'_{l-1}) \land \psi(x'_0, \ldots, x'_{l-1}) \Rightarrow \bot : \psi \notin \pi_{\{j_0, \ldots, j_{l-1}\}}(q) \}
\]

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We have to prove that $T$ has a model in $\mathcal{D}$ where each $R_i$ is interpreted as an atomic subobject.

By the previous theorem it suffices to prove that every finite $T_0 \subseteq T$ has such a model $M_{T_0}$ in $\mathcal{D}$, as in this case a countable ultraproduct of those is a model of $T$. In this ultraproduct the interpretation of $R_i$ is the filtered colimit of countable products of $R_i^{M_{T_0}}$’s, which is an atom.

Given $Th(C)$ plus finitely many formulas from the rest of the axioms mentioning $(\varphi_i)_{i<m}$ and $(\psi_i)_{i<n}$, adding extra variables (taking pullbacks along the projections) we may assume that these formulas contain the same free variables (are all subobjects of the same finite product of $s_i$’s). As $\bigcap \varphi_i \in \pi_q(q)$, $\bigvee \psi_j \not\in \pi_q(q)$ we get $T \nvdash \bigwedge \varphi_i \Rightarrow \bigvee \psi_i$ and by assumption there is a model $M : C \to \mathcal{D}$ in which $M(\bigwedge \varphi_i) \not\subseteq M(\bigvee \psi_i)$ and as $\mathcal{D}$ is weakly Boolean with dense atoms there is an atom in $M(\bigwedge \varphi_i) \setminus M(\bigvee \psi_i) \subseteq M(s)$ and its projections interpret the $R_i$’s which appear in the finitely many formulas.

Our next goal is to see that (most of) the conditions of Theorem 5.2 are satisfied when $\mathcal{D}$ is the pretopos completion of the syntactic category of some $L_\lambda\kappa$-theory. We start with the observation:

**Proposition 5.7.** Let $\mathcal{D}$ be an exact category (e.g. a pretopos) and $Y \xrightarrow{(q_1,q_2)} X \times X$ be a (not necessarily monic) map. If it satisfies the axioms of internal equivalence relations, namely:

1. (reflexivity) There is a map $r' : X \to Y$ with $q_1r' = q_2r' = 1_X$.
2. (symmetry) There is an effective epimorphism $s' : Y \to Y$ such that $(q_1,q_2) \circ s' = (q_2,q_1)$.
3. (transitivity) Let

$$
\begin{array}{ccc}
Y \times_X Y & \xrightarrow{q_2} & Y \\
\downarrow q_1' & & \downarrow q_1 \\
Y & \xrightarrow{q_2} & X
\end{array}
$$

be a pullback diagram. There is an effective epimorphism $t' : Y \times_X Y \to Y$ such that $(q_1q_1',q_2q_2') = (q_1,q_2) \circ t'$.

then the coequalizer of $f$ and $g$ exists.

**Proof.** We will prove that in the image factorization $\langle q_1,q_2 \rangle : Y \xrightarrow{p} R \xleftarrow{(f,g)} X \times X$ the subobject $R$ is an internal equivalence relation on $X$. For reflexivity
we can take $r = pr'$. Using that effective epi-mono factorizations are unique we get a map:

![Diagram 1]

proving symmetry. Finally we have

![Diagram 2]

and using the uniqueness of image factorizations again, we get

![Diagram 3]

proving transitivity. Therefore the coequalizer of $f$ and $g$ exists. Since $p$ is (effective) epi, this is also the coequalizer of $q_1$ and $q_2$.  

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**Theorem 5.8.** Let $\mathcal{D}$ be a $\lambda$-pretopos. Then it has filtered colimits of size $< \lambda$, which are pullback-stable. Moreover any $\lambda$-geometric functor between $\lambda$-pretoposes preserves filtered colimits of size $< \lambda$.

**Proof.** Let $d_\bullet : I \to \mathcal{D}$ be a filtered diagram. We will prove that the pair

$$\bigsqcup_i f_i \xleftarrow{k} g_j \xrightarrow{p_{f,g}} \bigsqcup_i d_i$$

satisfies the axioms of Proposition 5.7, and hence its coequalizer exists. Here we take the coproducts over the pullback diagrams

$$p_{f, g} \leftarrow \bigsqcup_i d_i \leftarrow_{d_j} \bigsqcup_i f_i$$

**Reflexivity:** Take $\bigsqcup_i d_i \xrightarrow{1_{d_i}} \bigsqcup_i f_i \xleftarrow{k} g_j p_{f,g}(d_f, d_g)$ which maps $d_i$ to the pullback of $(1_{d_i}, 1_{d_i})$.

**Symmetry:** Take

$$\bigsqcup_i d_i \leftarrow_{d_f} \bigsqcup_i f_i \xrightarrow{1_{d_i}} \bigsqcup_i d_i$$

which maps $p_{b}(d_f, d_g)$ to $p_{b}(d_g, d_f)$ identically (or more precisely with the induced isomorphism). This is an isomorphism, in particular an effective epi.

**Transitivity:** By (the infinitary analogue of) Proposition 8, Lecture 7. in [Lur18] we have that

$$\bigsqcup_i d_i \leftarrow_{d_f} \bigsqcup_i f_i \xrightarrow{1_{d_i}} \bigsqcup_i d_i$$

is a pullback.
In other terms for each zig-zag \((f', g', f, g)\) we take

\[
\begin{align*}
\text{We shall define a map } t \text{ from this coproduct to } \\
\bigcup_{i \leftrightarrow k} i \hookrightarrow_{j} p_b(d_f, d_g). \text{ We will do this by just extending the above diagram to } \\
&\text{and then take the induced map. The commutativity condition in the axiom is precisely the commutativity of the two triangles in which } t \text{ appears. To make sure that } t \text{ is surjective we just have to assume that zig-zags of the form } \\
&\bullet \rightarrow \bullet = \bullet = \bullet \leftarrow \bullet \text{ were chosen to be completed with identities.} \\
&\text{This proves that the coequalizer } \\
\bigcup_{i \leftrightarrow k} i \hookrightarrow_{j} p_b(d_f, d_g) &\xrightarrow{\cup p_f,g} \bigcup_i d_i &\xrightarrow{\text{}} c
\end{align*}
\]
exists. To prove that \((\varphi|_{d_i})_i\) is the colimit cocone we need that given \((\psi_i : d_i \to c')_i\) they form a cocone over \(d\), iff \(\bigsqcup_i \psi_i\) coequalizes the above pair of maps, i.e. iff for any span \(i \xrightarrow{f} k \xleftarrow{g} j\) we have that the left half of

\[
\begin{array}{ccc}
  p\text{b}(d_f, d_g) & \xrightarrow{\psi_i} & c' \\
  \downarrow \psi_j & & \downarrow \psi_j \\
  d_j & \xrightarrow{d_g} & d_a
\end{array}
\]

commutes. \(\Rightarrow\): In this case the left square is the gluing of two commutative squares. \(\Leftarrow\): Take \(g = 1_k\).

This coequalizer is clearly preserved by any \(\lambda\)-geometric functor between \(\lambda\)-pretoposes.

We are left to show that filtered colimits are pullback-stable. When we form pullbacks along \(\alpha\) in

\[
\begin{array}{ccc}
  p\text{b}(d_f, d_g)^* & \xrightarrow{\psi_j} & d^*_i \\
  \downarrow \psi_j & & \downarrow \psi_j \\
  d_j^* & \xrightarrow{d_g^*} & d^*_a
\end{array}
\]

the blue squares are pullbacks by the pasting law, and if we form pullback along the orange map we get that the top face of the cube is a pullback. Using this, plus the distributive law Proposition 8, Lecture 7 of \([Lur18]\) again, we identify the pullback of our coequalizer as
So we have to check that the first row is a coequalizer. We know that the image factorization \( \langle P, Q \rangle : A \xrightarrow{\psi} B \xrightarrow{(F,G)} X \times X \) yields the kernel pair of \( \Phi \), and we would like to see that the image factorization of \( \langle P', Q' \rangle \) yields the kernel pair of \( \Phi' \). This can be checked on

\[
A' = \bigsqcup_{i \to k \leftarrow j} pb(d^*_f, d^*_g) = X' = \bigsqcup_i d^*_i \xrightarrow{\phi' = \bigsqcup \phi^*_i} c'
\]

\[
A = \bigsqcup_{i \to k \leftarrow j} p(d_f, d_g) = X = \bigsqcup_i d_i \xrightarrow{\phi = \bigsqcup \phi_i} c
\]

Indeed, by the pasting law again, every face of the right inner cube is a pullback. The square with two dashed arrows commutes, as \( (F, G) \) is monic. Then this square is also a pullback and hence \( \tilde{\psi} \) is effective epi as we wished.

\[\text{Theorem 5.9. Let } D \text{ be a } \lambda\text{-pretopos. Then filtered colimits of size } < \lambda \text{ commute with finite limits.}\]

\[\text{Proof. Let } J \text{ be the Grothendieck-topology on } D \text{ generated by the } < \lambda \text{ effective epimorphic families. This topology is subcanonical and the Yoneda-embedding}\]
\( y : \mathcal{D} \to Sh(\mathcal{D}, J) \) is a (fully faithful) \( \lambda \)-geometric functor (see e.g. Proposition 6.1.4. and 6.1.6. of [Kam18]). By the above theorem it preserves and reflects finite limits and filtered colimits of size \(< \lambda \). But in \( Sh(\mathcal{D}, J) \) finite limits commute with filtered colimits. \( \square \)

We recall the following definition from [Mak90]:

**Definition 5.10.** A category \( \mathcal{D} \) is \( \kappa \)-regular if it is regular, has \(< \kappa \)-limits and the transfinite cocomposition of \(< \kappa \)-many effective epimorphisms is an effective epi, i.e. given an ordinal \( \alpha < \kappa \) and a limit-preserving functor \( F : \alpha^{op} \to \mathcal{D} \), such that each \( i \to j \) arrow in \( \alpha \) is mapped to an effective epi \( Fi \leftarrow Fj \), the limit cone \((Fi \leftarrow F\alpha)_{i<\alpha}\) consists of effective epis.

\( \mathcal{D} \) is \( \kappa \)-exact if it is \( \kappa \)-regular and Barr-exact (i.e. internal equivalence relations have a quotient).

**Proposition 5.11.** In a \( \kappa \)-regular category the product of \(< \kappa \)-many effective epimorphisms is an effective epi.

**Proof.** The following is a limit diagram:

\[
\begin{array}{ccc}
  & & x_0 \times x_1 \times x_2 \times \ldots \\
  & f_0 \times f_1 \times f_2 \times \ldots \downarrow & 1 \times f_1 \times f_2 \times \ldots \downarrow \\
 f_0 \times 1 \times 1 \times \ldots \downarrow & & 1 \times 1 \times f_2 \times \ldots \\
 y_0 \times y_1 \times y_2 \times \ldots & & x_0 \times y_1 \times y_2 \times \ldots & & x_0 \times x_1 \times y_2 \times \ldots \\
\end{array}
\]

and the horizontal maps are effective epimorphisms by pullback-stability. \( \square \)

We repeat Definition 2.1.2. of [Esp17] but with allowing different cardinals for the \( \exists \) and for the \( \lor \) properties.

**Definition 5.12.** Fix regular cardinals \( \lambda \geq \kappa \geq \aleph_0 \).

Let \( P \) be a rooted tree such that the degree of any vertex is \(< \lambda \) and the length of any branch is \(< \kappa \). Call these \((\lambda, \kappa)\)-trees. We say that a diagram \( F : P^{op} \to \mathcal{D} \) is proper if the 1-neighbourhood of any vertex is mapped to a jointly epimorphic family and \( F \) restricted to any branch is limit-preserving. We say that \( F \) is completely proper if it is proper and the transfinite cocomposition of the branches form an effective epimorphic family on the root. \( \mathcal{D} \) is \((\lambda, \kappa)\)-regular if for any \((\lambda, \kappa)\)-tree \( T \), any proper diagram \( F : T^{op} \to \mathcal{D} \) is completely proper.

Let \( \mathcal{D} \) be a \( \lambda \)-geometric category (resp. a \( \lambda \)-pretopos). It is \((\lambda, \kappa)\)-coherent (resp. it is a \((\lambda, \kappa)\)-pretopos) if it has \(< \kappa \)-limits, for any object \( x \) in \( \text{Sub}_\mathcal{D}(x) \) the \(< \kappa \) meets distribute over \(< \lambda \) joins, and it is \((\lambda, \kappa)\)-regular.
We proved:

**Theorem 5.13.** Let $\lambda \geq \kappa \geq \aleph_0$ be regular cardinals, such that $\gamma < \kappa \Rightarrow 2^\gamma < \lambda$. Let $\mathcal{D}$ be a weakly Boolean $(\lambda, \kappa)$-pretopos, in which $\{0\} \subseteq \text{Sub}_\mathcal{D}(1)$ is a $\kappa$-prime ideal: i.e. if the intersection of $< \kappa$-many subobjects of $1$ is $0$ then one of them was $0$.

Then for any coherent category $\mathcal{C}$, the set $\text{Coh}(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathcal{D}^\mathcal{C}$ is closed under $< \kappa$-ultraproducts. Moreover given subobjects $A_i \hookrightarrow X_i \leftarrow B_i$ in $\mathcal{D}$, we have $\prod_i A_i \mathcal{U} \leq \prod_i B_i \mathcal{U}$ iff $\{i : A_i \leq B_i\} \in \mathcal{U}$.

**Definition 5.14.** Let $M, N : \mathcal{C} \to \mathcal{D}$ be coherent functors. We say that $M$ and $N$ are elementary equivalent if for any subobjects $a \hookrightarrow x \leftarrow b$ in $\mathcal{C}$: $Ma \subseteq Mb$ iff $Na \subseteq Nb$. $\mathcal{C} \not\cong \ast$ is semantically complete if any two $\mathcal{C} \to \text{Set}$ coherent functors are elementary equivalent.

**Proposition 5.15.** Let $\mathcal{C}$ be Boolean and $M, N : \mathcal{C} \to \mathcal{D}$ be coherent functors. If there is an elementary natural transformation $\alpha : M \Rightarrow N$ then $M$ and $N$ are elementary equivalent.

**Proof.** Given any $x \in \mathcal{C}$, we have $Nx = \emptyset$ iff $Mx = \emptyset$ as

$$
\begin{array}{ccc}
M(\exists_i x) & \to & M1 \\
\approx & & \approx \\
N(\exists_i x) & \to & N1
\end{array}
$$

is a pullback. But as $\mathcal{C}$ is Boolean we can use $Na \subseteq Nb$ iff $N(a \cap \neg b) = \emptyset$. 

Our next goal is to show that if $\mathcal{C}$ is semantically complete then $\text{Coh}_\mathcal{C}(\mathcal{C}, \text{Set})$ has the joint embedding property.

**Theorem 5.16.** Let $M : \mathcal{C} \to \text{Set}$ be a coherent functor and $U \subseteq \mathcal{P}(I)$ be a $\kappa$-regular ultrafilter ($\kappa \geq |\mathcal{C}| \cdot \aleph_0$). Then $M/\mathcal{U}$ is $\kappa^+$-universal, i.e. given $N : \mathcal{C} \to \text{Set}$ which is elementary equivalent to $M$ and $|N| = \sum_{x \in \mathcal{C}} |Nx| \leq \kappa$, there is an elementary natural transformation $\alpha : N \Rightarrow M$.

**Proof.** Fix a filtration $\langle \mathcal{S} \subseteq \text{Ob}(\mathcal{C}), (x \hookrightarrow s_{x,1} \times \ldots s_{x,n_x})_{x \in \mathcal{C}} \rangle$. The (positive) elementary diagram $\Delta_x(N)$ of $N$ (wrt. $\mathcal{S}$) is the set of tuples

$$
\varphi(s_1 \times \ldots \times s_m, (u_i)_{i \in I_1}, (\bar{a}_{e,1}, \ldots \bar{a}_{e,n_e})_e) = \\
\{s_1 \times \ldots \times s_m, (u_i \hookrightarrow \bar{s})_{i \leq k}, (\bar{a}_{e,i} \in \mathcal{S})_{e \in \{*, c\}, 1 \leq i \leq n_e} \} 
$$

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with \( s_i \in S, \bar{a}_{s,i} \in Nu_1^{(1)} \cap \cdots \cap Nu_k^{(k)} \), where \( Nu^* = Nu \) and \( Nu^c = N\bar{s} \setminus Nu \). (I.e. we fix finitely many definable subsets in \( N\bar{s} \) and note the position of finitely many tuples in the resulting partition.) Clearly \( |\Delta_c(N)| = |N| \cdot |C| \cdot \aleph_0 \leq \kappa \).

Let \( E \subseteq U \) be a collection of ultrasets such that \(|E| = \kappa \) and for all \( i \in I \): \( \{J \in E : i \in J\} \) is finite. Fix an injection \( f : \Delta_c(N) \rightarrow E \). Then \( \gamma(i) = \{ \varphi \in \Delta_c(N) : i \in f(\varphi) \} \) is finite. Given a fixed \( \varphi \), the set \( \{ i \in I : \varphi \in \gamma(i) \} = f \varphi \) is an ultraset.

If \( Ns = \emptyset \) for some \( s \), then \( \alpha_s = \emptyset : \emptyset \rightarrow \emptyset \). Fix \( i \in I \). We concatenate all tuples in \( \gamma(i) \) to get just one long sequence of elements \( a_1, \ldots, a_t \in Ns_1 \times \cdots \times Ns_t \). Of course \( a_i = a_j \) can happen, so to remember this we introduce a subobject \( N\delta \subseteq Ns_1 \times \cdots \times Ns_t \) which is the corresponding diagonal. Now we pull back all the prescribed subsets to get subsets of \( Ns_1 \times \cdots \times Ns_t \), e.g. if \( (a_1, a_3) \in Nu \subseteq Ns_1 \times Ns_3 \) is in \( \gamma(i) \) then we replace it with \( (a_1, a_2, a_3, \ldots) \in Nu \times Ns_2 \times Ns_4 \times \ldots \), and similarly if \( (a_1, a_3) \not\in Nu \subseteq Ns_1 \times Ns_3 \) is in \( \gamma(i) \) then we replace it with \( (a_1, a_2, a_3, \ldots) \not\in Nu \times Ns_2 \times Ns_4 \times \ldots \). So we only have to care about the position of one sequence wrt. finitely many definable subsets of \( N\bar{s} \), let’s say it lies in \( Nu_1 = N\delta, Nu_2, \ldots Nu_r \) and it is in the complement of \( Nu_1, \ldots Nu_r \).

The existence of our sequence \( \bar{a} \) proves \( Nu_1 \cap \cdots \cap Nu_r \not\subseteq Nv_1 \cup \cdots \cup Nv_r \), hence \( Mu_1 \cap \cdots \cap Mu_r \not\subseteq Mv_1 \cup \cdots \cup Mv_r \). Take any \( \bar{b} \in Mu_1 \cap \cdots \cap Mu_r \cap M\bar{v}_1 \cap \cdots \cap M\bar{v}_r \). We define \( b_j = (\alpha_{s_j}(a_j))_i \). As any \( \varphi \in \Delta_c(N) \) appears in ultra-many \( \gamma(i) \)’s, we have defined \( (\alpha_{s_j}(a_j))_i \) for ultra-many \( i \)’s and it was defined for each \( a \in N_s \) (for each \( s \)).

By Proposition 3.14 we shall check that given any \( x \in C \) the dashed arrow in

\[
\begin{array}{c}
N_x & \xleftarrow{N_{ix}} & Ns_{x,1} \times \cdots \times Ns_{x,ux} \\
M^I \! U \! x & \xrightarrow{M^I \! U \! ix} & M^I \! U \! s_{x,1} \times \cdots \times M^I \! U \! s_{x,ux} \\
\end{array}
\]

exists and the square is a pullback. If some \( Ns_{x,i} = \emptyset \) then there’s nothing to check. Otherwise it follows from the fact that for any \( \bar{a} \in N\bar{s} \), the prescription \( a \in Nx \) or \( a \in Nx^c \) appears in ultra-many \( \gamma(i) \)’s.

As a corollary we got:

**Theorem 5.17.** If \( C \) is semantically complete then \( \text{Coh}_c(C, \text{Set}) \) has the joint embedding property. (When \( C \) is Boolean the converse also holds.)
6 Measuring positive closedness

Recall the definition of the type-space functor $S_C : C \xrightarrow{\text{Sub}_C} \text{DLat}^{\text{op}} \xrightarrow{S_{\text{Spec}}} \text{Spec}$ associated to a coherent category $C$ (cf. Definition 4.6). Given a model (i.e. a coherent functor) $M : C \to \text{Set}$, there is a natural transformation

$$
\begin{array}{ccc}
C & \xrightarrow{M} & \text{Set} \\
\downarrow{S_C} & \searrow{tp} & \searrow{U} \\
\text{Spec} & \xrightarrow{} & \text{Spec}
\end{array}
$$

whose component $tp_x : M(x) \Rightarrow \overline{S_C}(x)$ maps $a \in M(x)$ to $tp(a/\emptyset) := S_{M,x}(\hat{a})$ (where $U$ is the forgetful functor, $\overline{S_C}$ is $U \circ S_C$, $S_{M,x}$ maps an ultrafilter of $	ext{Sub}_{\text{Set}}(Mx)$ to its preimage along $M : \text{Sub}_C(x) \to \text{Sub}_{\text{Set}}(Mx)$, finally $\hat{a}$ is the principal ultrafilter on $a$).

To check that $tp$ is a natural transformation we need $S_{M,y}(\hat{M}(f)(\hat{a})) = (f^*)^{-1}(S_{M,x}(\hat{a}))$ which is $M^{-1}(f^*)^{-1}(\hat{a}) = (f^*)^{-1}M^{-1}(\hat{a})$ which follows by the commutativity of

$$
\begin{array}{ccc}
\text{Sub}_C(x) & \xrightarrow{f^*} & \text{Sub}_C(y) \\
\downarrow{M} & & \downarrow{M} \\
\text{Sub}_{\text{Set}}(Mx) & \xleftarrow{(Mf)^*} & \text{Sub}_{\text{Set}}(My)
\end{array}
$$

Note that for an elementary natural transformation $\alpha : M \Rightarrow N$ the following commutes:

$$
\begin{array}{ccc}
Mx & \xrightarrow{\hat{\alpha}} & S_{\text{Set}}(Mx) \\
\downarrow{\alpha_x} & & \downarrow{(\alpha_x)^{-1}} \\
Nx & \xrightarrow{\hat{\alpha}} & S_{\text{Set}}(Nx)
\end{array}
\quad
\begin{array}{ccc}
\text{Spec}(x) & \xrightarrow{M^{-1}} & S_{\text{Set}}(Mx) \\
\downarrow{\alpha} & & \downarrow{M^{-1}} \\
\text{Spec}(x) & \xrightarrow{N^{-1}} & S_{\text{Set}}(Nx)
\end{array}
$$

which means $tp_N \circ \alpha = tp_M$.

The converse is also true:

**Proposition 6.1.** $\alpha : M \Rightarrow N$ is elementary iff $tp_N \circ \alpha = tp_M$.
Proof. If $\alpha$ is not elementary then there is a subobject $a \hookrightarrow x$ such that $\alpha^*_a(N(a \hookrightarrow x)) \setminus M(a \hookrightarrow x) \neq \emptyset$. If $s$ is an element of it, then $N^{-1}(\alpha_s^*)^{-1}(\bar{s}) \neq M^{-1}(\bar{s})$.

We would like to classify all natural transformations $M \Rightarrow S_C$.

**Definition 6.2.** Define $\text{Coh}_e/\text{Set}$ to be the strict 2-category whose objects are $C \to \text{Set}$ coherent functors, the 1-cells are pairs $(F : C \to D, \eta_F : M \Rightarrow NF)$ where $F$ is a coherent functor and $\eta_F$ is an arbitrary natural transformation, and the 2-cells are elementary natural transformations $\alpha : F \Rightarrow G$ making

$$
\begin{array}{ccc}
C & \overset{\alpha}{\longrightarrow} & D \\
\downarrow F & & \downarrow G \\
\eta_F & \downarrow \downarrow \eta_G & \downarrow N \\
\eta_F \circ F & = & \eta_G \circ G
\end{array}
$$

commutative. We have an evident forgetful functor $\text{Coh}_e/\text{Set} \to \text{Coh}_e$.

**Definition 6.3.** We define a 2-functor $L : \text{Coh}_e/\text{Set} \to \text{DLat}$ as follows: Given a 2-cell as pictured above, we take $LM$ to be the colimit

$$
\text{colim}((\int M)^{op} \to C^{op} \to \text{DLat})
$$

and $L\eta_F = L\eta_G$ to be the map induced by

$$
\begin{array}{ccc}
\text{Sub}_C(x)^a & \overset{\alpha^*_a}{\longrightarrow} & \text{Sub}_D(Gx)^{\eta_G,s} \\
\downarrow F & & \downarrow G \\
\text{Sub}_D(Fx)^{\eta_F,s} & \overset{\alpha^*_a}{\longrightarrow} & \text{Sub}_D(Gx)^{\eta_G,s} \\
\text{colim}((\int N)^{op} \to D^{op} \to \text{DLat}) & \overset{\text{Sub}_D}{\longrightarrow} & \text{DLat}
\end{array}
$$

**Remark 6.4.** $LM$ is the weighted colimit $\text{colim}^M \text{Sub}_C = \text{Lan}_Y \text{Sub}_C(M)$.

**Proposition 6.5.** $L$ restricted to the fiber over $C$ is

$$L_C := \text{Coh}(C, \text{Set}) \to \text{Set}^C \xrightarrow{\text{Lan}_Y \text{Sub}_C} \text{DLat}$$
Proof. Given \( \eta : M \Rightarrow N \) we see it as the 1-cell \((1_C, \eta)\). As \( \text{Lan}_Y \text{Sub}_C \) preserves colimits, \( \text{Lan}_Y \text{Sub}_C(C(x', -) \xrightarrow{f} C(x, -)) \) is \( \text{Sub}_C(x') \xrightarrow{f} \text{Sub}_C(x) \), and since for each \((x, a) \in \int M\) we have a commutative square

\[
\begin{array}{ccc}
C(x, -) & \xrightarrow{1_x \mapsto a} & M \\
\downarrow \scriptstyle{\sim} & & \downarrow \scriptstyle{\eta} \\
C(x, -) & \xrightarrow{1_x \mapsto \eta_y(a)} & N
\end{array}
\]

we get that \( \text{Sub}_C(x)^a \to L_C M \xrightarrow{\text{Lan}_Y \text{Sub}_C \eta} L_C N \) and \( \text{Sub}_C(x)^a \to L_C M \xrightarrow{L_C \eta} L_C N \) both equal \( \text{Sub}_C(x)^a \xrightarrow{id} \text{Sub}_C(x)^{\eta_x(a)} \to L_C N \) and therefore \( L_C(\eta) = L((1_C, \eta)) = \text{Lan}_Y \text{Sub}_C(\eta) \).

\[ \square \]

**Proposition 6.6.** Given a coherent category \( C \) we have isomorphisms

\[
\text{DLat}(LM, K) \cong \text{Nat}(M, \text{DLat}(\text{Sub}_C(-), K))
\]

natural in \( M \in \text{Set}^C \) and \( K \in \text{DLat} \). Therefore \( L_C \) preserves filtered colimits (since \( \text{Coh}(C, \text{Set}) \hookrightarrow \text{Set}^C \) preserves them and \( \text{Lan}_Y \text{Sub}_C \) is a left adjoint).

Proof. This can be found e.g. as Theorem I.5.2. of [MM92].

\[ \square \]

**Remark 6.7.** Taking \( K = 2 \) we get an isomorphism \( \text{Spec}(LM) \cong \text{Nat}(M, \overline{S_C}) \), using the natural isomorphism \( \text{Spec} \cong \text{DLat}(-, 2) \). Given a prime filter \( p = (LM \xrightarrow{P} 2)^{-1}(1) \) the corresponding natural transformation \( \widehat{p} : M \Rightarrow \overline{S_C} \) is defined as \( \widehat{p}_x(a) = (\text{Sub}_C(x)^a \to LM \xrightarrow{P} 2)^{-1}(1) \).

In particular if \( p \subseteq q \in \text{Spec}(LM) \) are prime filters then \( \widehat{p}_x(a) \subseteq \widehat{q}_x(a) \) for all \( x \in C \) and \( a \in Mx \).

**Proposition 6.8.** \( LM \) is computed as follows: its underlying set is

\[
\bigsqcup_{a \in Mx} \text{Sub}_C(x)/\sim
\]

where \( u \mapsto x^a \sim v \mapsto y^b \) iff there exists \( \varphi \mapsto x \times y \) such that \((a, b) \in M \varphi\) and \( \varphi \cap u \times y = \varphi \cap x \times v \).

Note that \( u \mapsto x^a \sim u \times y \mapsto x \times y^{(a, b)} \) for any \( y \in C \) and \( b \in M y \) therefore any two elements of \( LM \) have representatives in a common \( \text{Sub}(x)^a \). The lattice operations are defined through these representatives.

Given \((F, \eta_F)\) as above, \( \text{Lan}_N \text{Sub}_D(y) \xrightarrow{\sim} \bigsqcup_{b \in M y} \text{Sub}_D(y)/\sim \) takes the equivalence class of \( u \mapsto x^a \) to that of \( F(u \mapsto x)^{\eta_F, a} \).

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Proof. As $M$ preserves finite limits $(\int M)^{op}$ is filtered, hence $u \leftrightarrow x^a \sim v \leftrightarrow y^b$ iff there is $w \leftrightarrow z, c \in Mz$ and arrows $f : z \to x, g : z \to y$ with $Mf(c) = a$ and $Mg(c) = b$, such that $w = f^*u = g^*v$. Taking the effective epi-mono factorization of $(f, g) : z \to x \times y$ and using that if $h$ is effective epi then $h^*$ is injective, completes the argument. \hfill \Box

**Remark 6.9.** We can repeat the construction in the language of model theory. Let $T \subseteq \Sigma_{\omega_1}^\omega$ be a coherent theory and $M$ a model of $T$. $LM$ is the following distributive lattice: its underlying set is $\{\langle \psi(x_1, \ldots, x_k), (a_i \in M_X) \rangle \}/\sim$ where $\psi$ is a positive existential $\Sigma$-formula, $\vec{a}$ is a tuple of elements having the appropriate sorts, and $\langle \psi(x_1, \ldots, x_k), (a_i \in M_X) \rangle \sim \langle \chi(y_1, \ldots, y_l), (b_i \in M_Y) \rangle$ iff there’s a positive existential $\varphi(x_1, \ldots, x_k, y_1, \ldots, y_l)$ such that $T \models \varphi \land \psi \iff \varphi \land \chi$ and $M \models \varphi(a_1, \ldots, a_k, b_1, \ldots, b_l)$. Here $\vec{x}$ and $\vec{y}$ are disjoint (so if $y_i = x_j$ then we replace it with a fresh variable $y'$). $\langle \psi(x_1, \ldots, x_k), (a_i \in M_X) \rangle \land \langle \chi(y_1, \ldots, y_l), (b_i \in M_Y) \rangle = \langle \psi \land \chi(x_1, \ldots, x_k, y_1, \ldots, y_l), (a_i, \ldots, a_k, b_1, \ldots, b_l) \rangle$ and similarly for $\vee$. $(\vec{x}$ and $\vec{y}$ are still disjoint, the top element is $[\top]$, the bottom element is $[\bot]$. $\langle \psi(\vec{x}, \vec{a}) \rangle \sim [\top]$ iff $M \models \psi(\vec{a})$.

**Example 6.10.** Given a distributive lattice $K$, every coherent functor $K \to \text{Set}$ factors as $K \overset{P}{\to} 2 \to \text{Set}$, hence yields a prime filter $p = P^{-1}(1) \in \text{Spec}(K)$. Then $LP = K/p = K/\sim_p$ where $a \sim_p b$ iff $\exists x \in p : x \cap a = x \cap b$.

In particular $L_K$ does not preserve ultraproducts if $K$ has an infinite quotient $K/p$. Given $(p_i \in \text{Spec}(K))_{i \in I}$ the prime filter corresponding to the ultraproduct structure (evaluation) is $p' = \prod_i p_i/\mathcal{U}$ given by $a \in p'$ iff $\{i : a \in p_i\} \in \mathcal{U}$. Clearly $\prod_i K/p_i/\mathcal{U} \cong K/p'$ does not hold in general, e.g. $(K/p)^{\mathcal{U}} \not\cong K/p$.

**Remark 6.11.** $LM \not\cong *$ for any $M : \mathcal{C} \to \text{Set}$ coherent functor.

**Proposition 6.12.** $M : \mathcal{C} \to \text{Set}$ is positively closed iff $LM = 2$.

Proof. $\Rightarrow$ We claim that $u \leftrightarrow x^a \sim v \leftrightarrow y^b$ iff $(a \in Mu \leftrightarrow b \in Mv)$. This is enough as $x \leftrightarrow x^a \not\sim \emptyset \leftrightarrow x^a$ proves $|LM| \geq 2$ and since in any triple $u \leftrightarrow x^a$, $v \leftrightarrow y^b$, $w \leftrightarrow z^c$ two members are equivalent we get $|LM| \leq 2$.

To prove the ‘only if’ direction observe $a \in Mu \Rightarrow (a, b) \in M(\varphi \cap u \times y) = M(\varphi \cap x \times v) \Rightarrow b \in Mv$. To prove the converse first assume $a \in Mu$ and $b \in Mv$. Then $\varphi = u \times v$ proves $u \leftrightarrow x^a$ and $v \leftrightarrow y^b$ to be equivalent. If $a \not\in Mu$ and $b \not\in Mv$ then $(a, b) \not\in M(u \times y \cup x \times v)$ and since $M$ was positively closed there is $\varphi \leftrightarrow x \times y$ such that $\varphi \cap (u \times y \cup x \times v) = \emptyset$ and $(a, b) \in M\varphi$. 44
\( \Leftarrow \) Take \( u \leftrightarrow x \) with \( a \not\in Mu \). By the above argument \( u \leftrightarrow x^a \) cannot be equivalent to \( x \leftrightarrow x^a \) hence it must be equivalent to \( \emptyset \leftrightarrow x^a \) as \( LM \) has two elements. That is, we have \( \varphi \leftrightarrow x \times x \) with \( (a, a) \in M\varphi \) and with \( \varphi \cap u \times x = \emptyset \), therefore \( \exists \pi \varphi \cap u = \emptyset \) and \( a \in M(\exists \pi \varphi) \).

**Corollary 6.13.** \( M : C \rightarrow \text{Set} \) is positively closed iff \( \text{tp}_M \) is the unique natural transformation \( M \Rightarrow \overline{SC} \).

**Proposition 6.14.** Let

\[
\begin{array}{c}
C \\
\downarrow F \\
\downarrow N \\
\downarrow \Set
\end{array}
\]

be a 1-cell in \( \text{Coh}_c/\text{Set} \). If \( F \) is conservative, full wrt. subobjects, and \( \eta \) is elementary then \( L((F, \eta)) \) is injective. If \( F \) is full wrt. subobjects, finitely covers its codomain \( (i.e. \ for \ y \in D \ there's \ a \ finite \ effective \ epimorphic \ family \ (\varphi x_i \rightarrow x)_{i<n}) \), and \( \eta \) is pointwise surjective then \( L((F, \eta)) \) is surjective \( (if \ N \ is \ \kappa\text{-geometric then it is enough to assume that } F \ \kappa\text{-covers its codomain}) \).

**Proof.** Injectivity: \( [(u \leftrightarrow x)^a] \) is mapped to \( [(Fu \overset{Fi}{\longrightarrow} Fx)^{\eta a}] \). Assume \( (Fu \overset{Fi}{\longrightarrow} Fx)^{\eta a} \sim (Fu \overset{Fj}{\longrightarrow} Fy)^{\eta b} \) that is, there’s \( \chi \leftrightarrow Fx \times Fy \) with \( \chi \cap (Fx \times Fv) = \chi \cap (Fu \times Fy) \) and with \( (\eta x(a), \eta y(b)) \in N\chi \). As \( F \) is full wrt. subobjects, \( \chi = Fw \), as \( F \) is conservative \( w \cap (x \times v) = w \cap (u \times y) \) and as \( \eta \) is elementary \( (a, b) \in w \).

Surjectivity: As \( F \) is full wrt. subobjects and \( \eta \) is pointwise surjective it is enough to prove that any \( (\chi \leftrightarrow z)^a \) is equivalent to some \( (\chi' \leftrightarrow Fx)^d \). The effective epimorphic family \( (h_i : Fx_i \rightarrow z)_i \) is mapped to a covering by \( N \), hence for some \( i \) there’s \( d' \in NFx_i \) with \( Nh_i(d') = a \). Then \( (h_i^*\chi \leftrightarrow Fx_i)^{d'} \) is equivalent to our original element.

We make use of Theorem 2.9. of [CL01]:

**Theorem 6.15.** A distributive lattice \( K \) has Krull-dimension \( \leq n \) iff for any \( x_1, \ldots x_{n+1} \) there are \( a_1, \ldots a_{n+1} \) such that

\[
a_1 \land x_1 = 0, \quad a_2 \land x_2 \leq a_1 \lor x_1, \quad \ldots \quad a_{n+1} \land x_{n+1} \leq a_n \lor x_n, \quad a_{n+1} \lor x_{n+1} = 1
\]
Remark 6.16. By Proposition 6.12 if \( C \) is Boolean then for every \( M : C \to \text{Set} \) coherent functor \( LM = 2 \) is a Boolean algebra. By the above theorem if \( K \) is the filtered colimit of the distributive lattices \( K_i \), and each has Krull-dimension \( \dim(K_i) \leq n \) then \( \dim(K) \leq n \). Hence if \( \dim(\text{Sub}_C(x)) \leq n \) for each \( x \in C \) then \( \dim(LM) \leq n \).

Proposition 6.17. \( \{1\} \subseteq LM \) is a prime filter. The corresponding natural transformation is \( \text{tp}_M \).

Proof. We need that \( u \lor v \sim x \leftrightarrow x^a \) implies \( u \sim x \) or \( v \sim x \). Indeed, \( u \lor v \sim x \) implies \( a \in M(u) \lor M(v) \) from which \( a \in M(u) \) or \( a \in M(v) \).

By Remark 6.7 the \( x \)-component of the corresponding natural transformation takes \( a \in Mx \) to \( \{u \hookrightarrow x : u \hookrightarrow x^a \sim x \leftrightarrow x^a\} = \{u \hookrightarrow x : Mu \ni a\} \).

Corollary 6.18. \( \dim(LM) = 0 \Rightarrow LM = 2 \).

Corollary 6.19. Given a coherent functor \( M : C \to \text{Set} \) and a natural transformation \( \tau : M \Rightarrow \mathcal{S}_C \) we have \( \text{tp}_{M,x}(a) \subseteq \tau_x(a) \) for all \( x \in C \) and \( a \in Mx \).

(By Remark 6.7.)

Theorem 6.20. Let \( M : C \to \text{Set} \) be a coherent functor. The following are equivalent:

1. \( M \) is positively closed, i.e. for all \( u \hookrightarrow x \in C \) and \( a \in Mx \setminus Mu \) there is \( v \hookrightarrow x \) such that \( u \cap v = \emptyset \) and \( a \in Mv \).

2. \( LM = 2 \),

3. \( LM \) is a Boolean algebra,

4. For all \( x \in C \) and \( a \in Mx \) the prime filter \( \text{tp}_{M,x}(a) \) is maximal,

5. For any coherent functor \( N : C \to \text{Set} \) each \( M \Rightarrow N \) natural transformation is elementary.

Proof. 1 \( \iff \) 2 \( \iff \) 3 \( \Rightarrow \) 5 was already proved. 4 \( \Rightarrow \) 2 follows by Corollary 6.19. We are left to show 5 \( \Rightarrow \) 4. Basically we repeat the proof of Theorem 8.2.4. of [Hod93].

Let \( M : C \to \text{Set} \) be a model and \( p = \{\varphi \hookrightarrow x : a \in M\varphi\} \) be the \( x \)-type realized by \( a \in Mx \). Let \( q \supseteq p \) be a maximal filter. By Proposition 6.1 it is enough to prove that there is a model \( N : C \to \text{Set} \) and a natural transformation \( \alpha : M \Rightarrow N \) such that \( q = \{\psi \hookrightarrow x : \alpha_x(a) \in N\psi\} \).

Let \( L_C \) be the extended canonical language of \( C \) and add constant symbols \( (c_y)_{y \in M^y} \) for each \( y \in C \). Let \( L \) be the resulting signature. \( T \subseteq L^\omega_{\omega} \) is
defined to be \( \text{Th}(\mathcal{C}) + \text{Diag}(M) + (R_\psi(c_a))_{\psi \in \varphi} \) where \( \text{Diag}(M) \) is the diagram of \( M \), i.e. it is the set of all closed atomic formulas \( \varphi(c_{a_1}, \ldots, c_{a_n}) \) for which \( M \models \varphi(a_1, \ldots, a_n) \). It is easy to prove that \( T \) is consistent; a model of \( T \) yields a coherent functor \( N : \mathcal{C} \to \text{Set} \) together with specified elements \( (c_b^N \in N y)_{y \in \mathcal{C}, b \in M y} \) such that \((\alpha_y : b \mapsto c_b^N)_{y \in \mathcal{C}} \) is a natural transformation with the required property.

Assume \( T \) to be inconsistent. Then (using compactness and that \( q \) is closed under finite meets) there are finitely many sentences in \( \text{Diag}(M) \) and there is \( x \leftrightarrow \psi \in q \) such that

\[
\text{Th}(\mathcal{C}) \models \varphi_1(c_{a_1}, \ldots, c_{a_{\lambda_1}}) \land \cdots \land \varphi_k(c_{a_{k_1}}, \ldots, c_{a_{k_\lambda_k}}) \land R_\psi(c_a) \Rightarrow \bot
\]

Since the new constant symbols do not appear in \( \text{Th}(\mathcal{C}) \), we can replace them by free variables \( r_1, \ldots, r_n, s \) and get

\[
\text{Th}(\mathcal{C}) \models \exists r_1, \ldots, r_n[\varphi_1(r_1, \ldots, r_n, s) \land \cdots \land \varphi_k(r_1, \ldots, r_n, s)] \land R_\psi(s) \Rightarrow \bot
\]

The left conjunct corresponds to a subobject \( \chi \xrightarrow{\leftarrow} x \) which is in \( p \) (as \( M \) makes this formula valid when we interpret \( s \) to be \( a \)), and since the right conjunct corresponds to \( \psi \), we get \( q \ni \chi \cap \psi = \emptyset \) which is a contradiction.

**Theorem 6.21.** For any coherent category \( \mathcal{C} \not\equiv * \) there is a positively closed model \( N : \mathcal{C} \to \text{Set} \).

**Proof.** Let \( M : \mathcal{C} \to \text{Set} \) be any model. Take all pairs \( \varphi_i = (u_i \xleftarrow{} x_i, a_i \in M x_i \setminus M u_i)_{i < \kappa} \). We will define a \( \kappa \)-chain of models and homomorphisms, whose colimit exists by [AR94]. \( M_0 = M \), in limit steps we take colimits. Given \( M_i \), if there is an \( N \) with a natural transformation \( M_i \xrightarrow{\alpha} N \) such that \( \alpha_{x_i}(a_i) \in Nu_i \)

then we choose one, otherwise take \( M_i \xrightarrow{id} M_i = M_{i+1} \). \( M_\kappa = M' \). Now we do the same with \( M' \) to get \( M'' \), etc. \( N \) can be taken to be the colimit of \( M \Rightarrow M' \Rightarrow M'' \Rightarrow \ldots. \)

Given \( N \xrightarrow{\beta} N' \), if it is not elementary, then for some \( u \xleftarrow{} x \) in \( \mathcal{C} \), there is an element \( a \in \beta_x^{-1}(N' u) \setminus Nu \). This is contained in some \( M'^{m \ldots}x \), and therefore also in \( M'^{m \ldots}u \subseteq Nu \) by construction.

**Theorem 6.22.** A coherent category \( \mathcal{C} \) is semantically complete if and only if it is weakly Boolean and 2-valued (i.e. \(|\text{Sub}_{\mathcal{C}}(1)| = 2\)).

**Proof.** \( \Rightarrow. \) By the completeness theorem there is a \( \mathcal{C} \to \text{Set}^I \) conservative coherent functor. Given \( a \xrightarrow{1} 1 \) and a model \( M : \mathcal{C} \to \text{Set} \), either \( Ma = \emptyset \) or \( Ma = \ast \), and by our assumption this does not depend on the choice of \( M \). So \( a = 0 \) or \( a = 1 \) and \(|\text{Sub}_{\mathcal{C}}(1)| = 2\).
Now pick a pair of subobjects \( a \hookrightarrow x \hookleftarrow b \). By Theorem 6.21 there is a positively closed model \( M : \mathcal{C} \to \text{Set} \). By assumption if \( a \not\leq b \) then \( Ma \not\subseteqMb \) and hence there is an element \( p \in Ma \setminus Mb \). As \( M \) is positively closed there is \( v \hookrightarrow x \) such that \( v \cap b = \emptyset \) and \( p \in Mv \). Then we can choose \( u = a \cap v \).

\( \Leftarrow \). As \(|\text{Sub}_{\mathcal{C}}(1)| = 2\) any coherent functor \( \mathcal{C} \to \text{Set} \) reflects the initial object. Then our condition on the subobject lattices guarantees that either \( Ma \leq Mb \) for every model \( M \) or \( Ma \not\leq Mb \) for every model \( M \).

**Remark 6.23.** The 2-functoriality of \( L \) implies that given any coherent functor \( F : \mathcal{C} \to \mathcal{D} \) we have a natural transformation

\[
\begin{array}{ccc}
\text{Coh}(\mathcal{C}, \text{Set}) & \xrightarrow{L_C} & \text{DLat} \\
\downarrow F^* & & \phi_F \\
\text{Coh}(\mathcal{D}, \text{Set}) & \xleftarrow{L_D} & \\
\end{array}
\]

with \( \phi_{F,M} = L((F, id)) : L_C(MF) \to L_D(M) \). By Proposition 6.14 when \( F \) is a Morita-equivalence \( \phi_F \) is an isomorphism.

Under certain assumptions on \( F \) we also have:

**Proposition 6.24.** Let \( \mathcal{C} \) be a coherent category, \( \tilde{\mathcal{C}} \) be \( \kappa \)-geometric and let \( \varphi : \mathcal{C} \to \tilde{\mathcal{C}} \) be a coherent functor which is fully faithful, full wrt. regular subobjects and for each \( y \in \tilde{\mathcal{C}} \) there are maps \( (h_i : \varphi(x_i) \to y)_{i < \lambda < \kappa} \) with \( y = \bigcup_{i < \lambda} \exists h_i \varphi(x_i) \). Then we have the following natural isomorphisms:

\[
\begin{array}{ccc}
\text{Coh}(\mathcal{C}, \text{Set}) & \xrightarrow{\varphi^*} & \text{Set}^\mathcal{C} \\
\downarrow \cong & & \downarrow \cong \\
\text{Geom}_\kappa(\tilde{\mathcal{C}}, \text{Set}) & \xleftarrow{\text{Lan}_{\varphi}} & \text{Set}^{\tilde{\mathcal{C}}} \\
\end{array}
\]

with \( \psi_{\mathcal{C}(x, -)} \) is \( \text{Sub}_\mathcal{C}(x) \xrightarrow{\cong} \text{Sub}_{\tilde{\mathcal{C}}}(\varphi x) \).

In particular under these assumptions \( \varphi^* : \text{Geom}_\kappa(\tilde{\mathcal{C}}, \text{Set}) \to \text{Coh}(\mathcal{C}, \text{Set}) \) is fully faithful.
Proof. We have

\[
\begin{array}{ccc}
\text{Coh}(\mathcal{C}, \text{Set}) & \xrightarrow{\varphi^*} & \text{Set}^\mathcal{C} \\
\uparrow \varphi^* & & \uparrow \varphi^* \\
\text{Geom}_\kappa(\tilde{\mathcal{C}}, \text{Set}) & \xrightarrow{\text{Lan}_{\text{Sub}_\mathcal{C}}} & \text{DLat} \\
\end{array}
\]

Since \(\varphi\) is fully faithful, \(\text{Lan}_{\varphi}\) is also fully faithful and hence \(\varphi^* \circ \text{Lan}_{\varphi} \cong 1_{\text{Set}^\mathcal{C}}\). Therefore it is enough to see that the full subcategory \(\text{Geom}_\kappa(\tilde{\mathcal{C}}, \text{Set})\) lies in the essential image of \(\text{Lan}_{\varphi}\). That is, given a \(\kappa\)-geometric functor \(M : \tilde{\mathcal{C}} \rightarrow \text{Set}\) we should prove that it is the colimit of representables of the form \(\tilde{\mathcal{C}}(\varphi x, -)\).

Write \(M \varphi = \text{colim}_{(x, -) \in \mathcal{C}} M(x, -)\). Then we can define a natural transformation \(\text{colim}_{(x, -) \in \mathcal{C}} \tilde{\mathcal{C}}(\varphi x, -) \Rightarrow M\) induced by the natural transformations \(\tilde{\mathcal{C}}(\varphi x_i, -) \Rightarrow M\) which map \(1_{\varphi x_i}\) to \(a_i \in M(\varphi x_i)\) at the object \((x_i, a_i \in M(\varphi x_i))\) of the indexing category. We would like to prove that this is an isomorphism.

To see that it is pointwise mono assume that we are given \(f, g : x \rightarrow y\) with \(Mf(a) = Mg(a)\) for some \(a \in M(\varphi x)\). We should see that there is \(h : z \rightarrow x\) and \(c \in M\varphi z\) such that \(M\varphi h(c) = a\) and \(f \circ \varphi h = g \circ \varphi h\) (this is sufficient as \((\int M \varphi)^{op}\) is filtered). As \(\varphi\) is full wrt. regular subobjects the equalizer of \(f\) and \(g\) is of the form \(\varphi h\) (and we can take \(c = a\)).

To see that it is pointwise epi pick \(a \in M(y)\) and a jointly covering family \((h_i : \varphi x_i \rightarrow y)_{i<\lambda<\kappa}\). As \(M\) is \(\kappa\)-geometric \((Mh_i)_{i<\lambda}\) is jointly covering so there is \(i < \lambda\) and \(b \in M\varphi x_i\) with \(Mh_i(b) = a\). Hence \(a\) is in the image of the \(y\)-component of \(\tilde{\mathcal{C}}(\varphi x_i, -) \Rightarrow M\) which map \(1_{\varphi x_i}\) to \(b\).

Finally, as \(\varphi\) is bijective on objects (conservative and full wrt. subobjects) the induced map between the colimits

\[
\begin{array}{ccc}
\mathcal{C}(x, -) & \xrightarrow{\varphi'^*} & \mathcal{C}((\varphi x)', -) \\
\uparrow & & \uparrow \\
\text{Lan}_{\varphi^{*}}M & \xrightarrow{\text{Lan}_{\varphi^{*}}M} & \text{Lan}_{\varphi^{*}}M \\
\end{array}
\]

is an isomorphism. \(\Box\)
Theorem 6.25. Given a coherent category $C$ the functor $\text{Lan}_Y \text{Sub}_C$ preserves finite products of coherent functors.

**Proof.** Let $\varphi : C \to \tilde{C}$ be the pretopos completion of the coherent category $C$. Then $\varphi^*$ satisfies the conditions of Proposition 6.24 we have the above 2-cells. Since $\varphi^*$ is an equivalence (hence essentially surjective), $\text{Lan}_\varphi$ preserves finite products by Example A4.1.10. of [Joh02], and $\Psi$ is iso, it is enough to prove that $\text{Lan}_Y \text{Sub}_{\tilde{C}}$ preserves finite products of $\kappa$-geometric functors.

As $\tilde{C}$ is a pretopos it has finite disjoint coproducts, and $\text{Sub}_{\tilde{C}} : \tilde{C}^{\text{op}} \to \text{DLat}$ maps them to products. Then post-composing with the forgetful functor $\text{Sub}_{\tilde{C}} : \tilde{C}^{\text{op}} \to \text{Set}$ yields a finite product preserving functor whose left Kan-extension along the Yoneda-embedding $\text{Lan}_Y \text{Sub}_{\tilde{C}}$ is also finite product preserving as finite products form a sound doctrine in the sense of [Adá+02]. But if $M : \tilde{C} \to \text{Set}$ is a lex functor then $\text{Lan}_Y \text{Sub}_{\tilde{C}}(M)$ is the underlying set of $\text{Lan}_Y \text{Sub}_{\tilde{C}}(M)$ as $(\int M)^{\text{op}}$ is filtered and the forgetful functor $U : \text{DLat} \to \text{Set}$ preserves filtered colimits. As it reflects all limits it follows that $\text{Lan}_Y \text{Sub}_{\tilde{C}}$ preserves finite products of lex functors.

**Question 6.26.** Can we find conditions on $C$ which guarantee that $\text{Lan}_Y \text{Sub}_C$ preserves $< \kappa$-products of coherent functors? As it preserves filtered colimits in this case $L_C$ would preserve $< \kappa$-ultraproducts. We have seen in Example 6.10 that this is not the case in general.

**References**

[AR94] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. Vol. 189. London Mathematical Society Lecture Note Series. London: Cambridge University Press, 1994.

[Adá+02] Jiří Adámek et al. “A classification of accessible categories”. In: *Journal of Pure and Applied Algebra* 175 (2002), pp. 7–30.

[Bel14] Mohammed Belkasmi. “Positive model theory and amalgamations”. In: *Notre Dame Journal of Formal Logic* 55.2 (2014), pp. 205–230.

[BLV20] John Bourke, Stephen Lack, and Lukáš Vokřínek. *Adjoint functor theorems for homotopically enriched categories*. 2020. URL: https://arxiv.org/abs/2006.07843.

[CL01] Thierry Coquand and Henri Lombardi. “Hidden constructions in abstract algebra Krull Dimension of distributive lattices and commutative rings”. In: *Commutative ring theory and applications* (2001).
[Esp17] Christian Espíndola. *Infinitary first-order categorical logic*. 2017. URL: https://arxiv.org/abs/1701.01301.

[Hay18] Levon Haykazyan. “Spaces of Types in Positive Model Theory”. In: *Journal of Symbolic Logic* 84 (2018), pp. 833–848.

[Hod93] Wilfrid Hodges. *Model Theory*. Cambridge University Press, 1993.

[Isb75] John R. Isbell. “Function spaces and adjoints”. In: *Mathematica Scandinavica* 36 (1975), pp. 317–339.

[Joh02] Peter Johnstone. *Sketches of an Elephant - A Topos Theory Compendium*. Oxford University Press, 2002.

[Kam18] Mark Kamsma. *Classifying Topoi and Model Theory (master’s thesis)*. 2018. URL: https://markkamsma.nl/ajax/download.php?file=math/mark-kamsma-classifying-topoi-and-model-theory.pdf.

[Kam20] Mark Kamsma. *Type space functors and interpretations in positive logic*. 2020. URL: https://arxiv.org/abs/2005.03376.

[Lom18] Henri Lombardi. “Spectral Spaces Versus Distributive Lattices: A Dictionary”. In: *Advances in Rings, Modules and Factorizations. Rings and Factorizations*. Springer Proceedings in Mathematics and Statistics 321 (2018), pp. 223–245.

[Lur18] Jacob Lurie. *Lecture notes in Categorical Logic*. 2018. URL: https://www.math.ias.edu/~lurie/278x.html.

[Lur] Jacob Lurie. *Ultracategories*. URL: https://people.math.harvard.edu/~lurie/papers/Conceptual.pdf.

[MM92] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. New York, NY: Springer, 1992.

[Mak90] Michael Makkai. “A theorem on Barr-exact categories with an infinitary generalization”. In: *Annals of Pure and Applied Logic* 47 (1990), pp. 225–268.

[MR77] Michel Makkai and Gonzalo E. Reyes. *First Order Categorical Logic*. Berlin, Heidelberg: Springer, 1977.

[Poi06] Bruno Poizat. “Univers positifs”. In: *The Journal of Symbolic Logic* 71 (2006), pp. 969–976.

[Sta21] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2021.
[YP07] Itaï Ben Yaacov and Bruno Poizat. “Fondements de la logique positive”. In: The Journal of Symbolic Logic 72 (2007), pp. 1141–1162.