Multi-particle Processes and Tamed Ultraviolet Divergences

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New approach to computing the amplitudes of multi-particle processes in renormalizable quantum field theories is presented. Its major feature is a separation of the renormalization from the computation. Within the suggested approach new computational rules are formulated. According to the new rules, the amplitudes under computation are expressed as a sum of effective Feynman amplitudes whose vertexes are the complete amplitudes of the processes involving not more than four particles, and the lines are the complete two-point functions. The new rules include prescriptions for computing the combinatorial factors by each amplitude. It is demonstrated that due to these prescriptions the combinatorial factors by the amplitudes that are divergent in the ultraviolet in four space-time dimensions vanish. Because of this, the computations within the new approach do not involve the ultraviolet renormalization. It is observed that the combinatorics of the new rules determines the dimension of the space-time.

Physics at the LHC is predominantly the physics of multi-particle processes. Substantial efforts have been invested in developing the technics for computing the amplitudes of such processes (see, e.g., [1] and [2]). One of the difficulties in organizing such computations is in keeping track of all the necessary renormalizations.

In this paper we present a new approach for computing the multi-particle amplitudes. Its major feature is a separation of the renormalization from the computation. The idea is that the renormalization is needed only for computing the basic processes involving not more than four particles. After computing the basic amplitudes, it should be possible to compute the multi-particle amplitudes combining the basic ones, and this combination should not involve any renormalization.

We demonstrate below that it is indeed possible to realize this program. Each effective Feynman amplitude enters the answer with a combinatorial factor, and the rules for computing these factors detailed below yield zero factors for the divergent diagrams. This is surprising, because it implies that the combinatorics involved here "knows" that the dimension of the space-time is four. Indeed, the diagrams disappearing from the answer are divergent in the ultraviolet (UV divergent) namely in dimension four.

Our starting point is the generating functional of the renormalized connected Green functions of the theory, \( W(J) \). We make the separation

\[
W(J) = -\frac{1}{2} JDJ + I(iDJ) + V(iDJ),
\]

where \( D \) is the matrix of renormalized propagators, \( I \) is the generating functional of three-particle and four-particle connected vertexes (i.e., it contains only terms of order \( O(J^3) \) and \( O(J^4) \)), and \( V \) is the generating functional of the renormalized connected vertexes involving more than four particles (i.e., its expansion in the sources starts from the fifth power, \( V = O(J^5) \)). Following [3] we call \( I \) the inaction functional (because it will be used to parametrize \( V \) and, despite that, it is not the action functional of the theory), and \( V \), the vertex functional.

It is demonstrated in [3] that if \( W \) is the generating functional of connected Green functions of a renormalizable theory with bare action involving only powers of the fields not exceeding the fourth power, the inaction and vertex functionals defined above are related to one another with the following equation:

\[
V = \Phi[I + V].
\]

We call it following [3] the inaction equation. Here \( \Phi \) is a transform acting on functionals of the fields:

\[
\Phi \equiv P(1 - PT^{-1} \log T \exp).
\]

Here the 'log' and 'exp' operations are inverse to one another, as well as \( T \) and \( T^{-1} \) operations. The last two operations are defined in the following way:

\[
T \equiv \exp \left( -\frac{1}{2} \frac{\delta}{\delta \phi} D \frac{\delta}{\delta \phi} \right), \quad T^{-1} \equiv \exp \left( \frac{1}{2} \frac{\delta}{\delta \phi} D \frac{\delta}{\delta \phi} \right).
\]

These operations acting on a product of functionals generate Feynman amplitudes with the lines corresponding to the propagators \( D \) and vertexes, to the functionals in the product (see [4] for an explanation of the functional methods involved here; we write the signs in the above formula and below for boson fields). The last operation
V = \sum_{k=2}^{\infty} V_n, \ V_n = O(I^n). \ \ \ \ (5)

For uniformity of notation, in the following we use \( V_1 \equiv I \). Our next task is to determine the \( n \)-th order in the expansion of \( \Phi[\sum_{l=1}^{\infty} V_l] \) in powers of an auxiliary parameter \( \lambda \) taking that each \( V_i \propto \lambda^i \).

Expanding the exponential and the logarithm in the definition of \( \Phi \), and using Eq. (2), we obtain that \( V_n \) is a sum of terms, each term is a product of \( V_i \) with \( i \) ranging from 1 to \( n-1 \). Notice that the term proportional to \( V_n \) does not appear in the expansion because of the presence of the identity operation in the brackets of the right hand side of Eq. (3).

Each term of this expansion is marked with a double partition of \( n \): first time because of the expansion of the logarithm, and the second time, of the exponential of Eq. (3). It is explained in [3] that the inaction equation (2) can be used to determine \( V \) as a power series in \( I \), and the expansion starts from the second power. Here we give the rules to compute the terms of this expansion,

\[
V = \sum_{k=2}^{\infty} V_n, \ V_n = O(I^n). \ \ \ \ (5)
\]

FIG. 1. a: The tree corresponding to the double partition \( t = \{2, 2, 1, \{1\}\} \): dark edges are related to the expansion of the logarithm and the light ones, to the expansion of the exponential of Eq. (3). b: A complete tree extending the tree on the left.

A tree obtained from another tree with shuffling the branches does not give a separate contribution. To avoid double counting, we will always place “heavier” branches to the left from the lighter ones, like on Fig. 1.

In this way we obtain

\[
V_n = \sum_{t \in T_2[n], t \neq \{(n)\}} Z_2(t)PT^{-1}L(t) \prod_{i=1}^{TV[i][i]} T. \ \ \ \ (6)
\]

Here \( n > 1 \), summation runs over the set \( T_2[n] \) of all double partitions of \( n \) excluding the partition \( \{(n)\} \), \( Z_2(t) \) is a rational coefficient we detail later (its subscript reminds that it is related to a double partition of an integer), \( L(t) \) is the length of the first partition (for the partition from Fig. 1, \( L(t) = 2 \)), and \( t[i][j] \) is the \( i \)-th element of the partition (for the partition from Fig. 1, \( t[[1]] = \{2, 2, 1\} \), and \( t[[2]] = \{1\} \)).

The last unexplained object in the right hand side of Eq. (6) is \( V_L \) under the product, where \( L \) is a list of integers. It is

\[
V_L = \prod_{i \in L} V_i^{p(i, L)}, \ \ \ \ (7)
\]

where the product is over the distinct elements of the list \( L \), and \( p(i, L) \) is the number of times the integer \( i \) enters the list \( L \).

The coefficient \( Z_2(t) \) in Eq. (6) is

\[
Z_2(t) = (-1)^{L(t)}(L(t) - 1)! \prod_{i=1}^{L(t)} z_{t[i][i]}; \frac{1}{z_L} = \prod_{i \in L} p(i, L)!. \ \ \ \ (8)
\]

Here \( L(t) \) is the length of the double partition \( t \), the product in the denominator runs over the distinct sub-partitions \( s \) of \( t \), \( p(s, t) \) is the number of times \( s \) enters \( t \), and the notations in the definition of \( z_L \) are the same as in Eq. (7).

We now point out that both \( T \)-product and \( T^{-1} \)-product in Eq. (6) generate lines of the Feynman amplitudes, and there is a partial cancellation between the amplitudes which lines come from these two \( T \)-products. The outcome of these partial cancellation is that the lines appear only between \( V_i[i][i] \) and \( V_i[j][j] \) with \( i \neq j \).

The outcome of these considerations is the following formula for \( V_n \):

\[
V_n = \sum_{t \in T_2[n], t \neq \{(n)\}} Z_2(t)PT_l \prod_{i=1}^{L(t)} V_i[i][i]. \ \ \ \ (10)
\]

FIG. 1. a: The tree corresponding to the double partition \( t = \{2, 2, 1, \{1\}\} \): dark edges are related to the expansion of the logarithm and the light ones, to the expansion of the exponential of Eq. (3). b: A complete tree extending the tree on the left.
Here the subsequent application of the two operations $T$, $T^{-1}$ in Eq. (6) has been replaced with the application of the single operation $T_t$ depending on the double partition (or two-level tree) $t$.

Now we can use Eq. (10) recursively to determine $V_n$ in terms of $V_1 \equiv I$. This is possible because there are products of $V_i$ with $i < n$ in the right hand side of this equation, and for each $V_i$ with $i > 1$ we can again use the representation of Eq. (10).

The outcome of this recursion is the following representation:

$$V_n = \sum_{t \in T[n]} Z(t)T_tI^n. \tag{11}$$

The summation here runs over the set $T[n]$ of complete trees of successive partitions of the integer $n$. By complete we mean that at the bottom of the partition there are only units, and there is no possibility of further partitioning. Each iteration adds two levels of the partition as it was above with the first iteration representing $V_1$. An example of a complete tree extending the one on the left part of Fig. 1 see on the right part of Fig. 1. It marks a term in the sum representing $V_6$.

In Eq. (11), the factors $Z(t)$ are the products of $Z_2(t')$ over all two-level sub-trees $t' \subset t$:

$$Z(t) \equiv \prod_{t' \subset t, t' \in T_2} Z_2(t'). \tag{12}$$

For example, the complete tree on the right part of Fig. 1 contains three two-level sub-trees: the one on the left part of Fig. 1, and two trees on a lower level representing two ways of the double partitioning of 2. Corresponing factorization of $Z(t)$ for this tree is $(1/2)(-1/2)(1/2)$. To give a definition, a two level sub-tree $t' \subset t$ is a sub-tree of $t$ with a root on an even level of $t$ consisting of all the edges of $t$ located at the two subsequent levels and descending from the root of $t'$.

Because of lack of space we do not give explicit definition of the operation $T_t$ for the case of complete trees. We only point out that it involves the operation $P$ and the operations $T_{t' \subset t}$.

Each term in the sum of Eq. (11) is a sum of Feynman amplitudes $A(g)$ corresponding to graphs $g = (V,L)$, where $V$ is the set of vertexes and $L$ is the set of edges of the graph $g$:

$$T_tI^n = \sum_{g \in G_n} C[t,g]A(g). \tag{13}$$

Here $G_n$ is the set graphs with the number of vertexes $|V| = n$, and the number of edges $|L|$ restricted in a certain way (see below). No line of $g \in G_n$ can start and end on the same vertex. No vertex of $g \in G_n$ has more than 4 incident edges.

The amplitudes $A(g)$ are defined as follows:

$$A(g) \equiv \left[ \prod_{l \in L} \left( \frac{\delta}{\delta \phi_{l1}} D \frac{\delta}{\delta \phi_{l2}} \right) \prod_{v \in V} I(\phi_v) \right]_{\phi_v = \phi}. \tag{14}$$

Here the first product runs over the edges of $g$, $l_1$ and $l_2$ are the vertexes joined by the edge $l$; the second product runs over the vertexes of $g$; the field argument of each $I$ is labelled with a vertex; after taking all the variational derivatives all the fields are set to the common value $\phi$.

Because of the involvement of the operation $P$ in the definition of $T_t$ of Eq. (11), the expansion of $A(g)$ in powers of the fields should start from the fifth power. This is achieved by restricting the number of edges of $g \in G_n$. The concrete form of this last restriction depends on the presence of the three-point vertexes in $I$. To simplify the presentation, we assume from this point on that the interaction functional $I$ contains only four-point vertexes. In this case, the number of edges should not exceed $2n - 3$.

Substituting representation Eq. (13) in Eq. (11) and changing the order of summation, we finally obtain

$$V_n = \sum_{g \in G_n} C(g)A(g). \tag{15}$$

The combinatorial factor $C(g)$ is defined as follows:

$$C(g) \equiv \sum_{t \in T[n]} Z(t)C[t,g]. \tag{16}$$

Here the summation runs over the set $T[n]$ of all the complete trees of partitions of the integer $n$. We now discuss the combinatorial factor $C[t,g]$ of Eq. (13). Like the conventional symmetry factor (see, e.g., [5]), $C[t,g]$ is inverse proportional to the size of the finite set of automorphisms of $g$, $|\text{Aut}(g)|$. But it is not proportional to $n!$, which is the number of ways to label the $n$ vertexes of $g$. Because the graphs $g$ appear in our case via the action of the operation $T_t$ depending on the complete tree $t$, not all of the possible labelings of the vertexes of $g$ contribute to $C[t,g]$. The number of the contributing labelings is $N(t,g) \leq n!$. As a result,

$$C[t,g] = \frac{N(t,g)}{|\text{Aut}(g)|}. \tag{17}$$

where $N(t,g)$ is the number of contributing labelings of the graph $g$ depending on $t$.

To define $N(t,g)$ of Eq. (17), we consider labelings of the vertexes of $g$ with the terminating vertexes of the complete tree $t$ (the number of terminating vertexes of $t$ coincides with the number of vertexes of $g$). Now we give the criteria for determination of the labelings that do not contribute to $N(t,g)$.

As discussed above, each complete tree defines a set of two-level sub-trees. The criteria are related to this set. To check them, for each two-level sub-tree $t' \subset t$, consider its completion inside $t$, $t'_c \subset t$ ($t'_c$ consists of all the edges of $t$ descended from the root of $t'$). It defines a subgraph $g' \subset g$ as the set of vertexes labeled by the terminating vertexes of $t'_c$, and all the edges of $g$ joining these vertexes. A labeling does not contribute to $N(t,g)$ if there is a two-level sub-tree whose subgraph $g'$ contains too many lines: $|L(g')| > 2n' - 3$, where $n'$ is the number of vertexes in $g'$. This is the first criterion.
The second and the last criterion involves a two-level partition of the vertexes of the above \( g' \) related to a two-level sub-tree \( t' \subset t \). To define this partition, consider an edge \( l \in t' \) on the first level of \( t' \). It defines a subset of the vertexes \( V'_t \subset V(g') \) labeled by the terminating vertexes of \( t'_l \) descending from the lower end point of the edge \( l \). These subsets form a partition of the vertexes of \( g' \). In turn, consider an edge \( e_l \in t' \) originating from the lower end point of an edge \( l \). As above, it defines a subset of vertexes \( V'_e \subset V_t \). A labeling does not contribute to \( N(t,g) \) if there is a two-level sub-tree containing a first level edge \( l \) and second level edges \( e_l \neq e'_l \) attached to \( l \) such that the subgraph \( g' \) contains edges joining vertexes from \( V'_e \) to vertexes from \( V'_{e'} \).

In the leading order of the perturbation theory under description, there are two graphs in \( G_2 \): with no lines and with a single line joining the two vertexes; due to the above rules, only the graph with a single line contributes, and its combinatorial factor is 1/2.

FIG. 2. Graphs and trees of the third order; numbers on the graph panes are \( |Aut(g)| \) of Eq. (17), on the tree panes, the \( Z(t) \) factors of Eq. (16).

In the third order, there are 7 graphs and 7 complete trees shown on the panes of Fig. 2 along with the components of the 7-dimensional “vectors” \( |Aut(g)| \) and \( Z(t) \).

\[ N(t,g) \text{ of Eq. (17) is in this case the following 7 by 7 matrix:} \]

\[
N(t,g) = \begin{pmatrix}
6 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 2 & 0 & 0 & 0 & 0 & 0 \\
6 & 4 & 4 & 2 & 2 & 2 & 0 \\
6 & 4 & 4 & 2 & 2 & 2 & 0 \\
6 & 6 & 4 & 6 & 4 & 4 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6
\end{pmatrix}.
\]

Here, for example, the first row corresponds to the tree on the first tree pane of Fig. 2, and the first column, to the graph on the first graph pane of Fig. 2.

In the fourth order, there are 33 trees and 35 graphs. Because of lack of space, we do not give explicitly the values of \( Z(t) \), \( |Aut(g)| \), and \( N(t,g) \) for this case. But the ultimate result for \( V \) can be described as follows.

\[ V \text{ is given with the following formula:} \]

\[
V = \sum_{g \in G_4} \frac{A(g)}{|Aut(g)|},
\]

where the amplitude \( A(g) \) is defined in Eq. (14) and \( |Aut(g)| \) is the size of the group of automorphisms of the graph \( g \). The summation runs over the set of graphs \( G_4 \) defined as follows.

Any graph \( g \in G_4 \) is connected, has the number of vertexes \( |V(g)| \) and the number of edges \( |L(g)| \) restricted by the inequality \( 2|V(g)| - 3 \geq |L(g)| \). On top of that, the scaling dimension of any two-particle irreducible sub-graph \( g' \subset g \in G_4 \) is negative at the space-time dimension 4 and propagator scaling dimension -2.

A scaling dimension of a two-particle irreducible sub-graph \( g' \) at space-time dimension \( d \) and scaling dimension of the propagator -2 is \( d(l(g')) - 2|L(g')| \), where \( l(g') \) is the number of loops of \( g' \) and \( |L(g')| \) is its number of edges. The subscript on \( G_4 \) is the space-time dimension. We stress that namely \( G_4 \) appears in Eq. (18), which singles out 4 as the preferable dimension of the space-time.

We have checked the representation of Eq. (18) up to the fourth order of the perturbation theory. Checking it in the higher orders is under way.

It is remarkable in Eq. (18) that not only the disconnected graphs have not appeared in it, but also all the connected graphs UV divergent namely in 4 space-time dimensions have completely canceled out. (There are altogether 14 such graphs in the fourth order, and one such graph in the third order, which is graph No. 6 on Fig. 2.)

We stress again that the computation of the combinatorial coefficients is a strictly combinatorial matter, and does not use any properties of the Feynman integrals.

In conclusion, we have given above new rules for computing the multi-particle amplitudes as series in products of the basic amplitudes involving not more than four particles. Ultraviolet divergences do not appear in the expansion. As a by-product, we have found a combinatorial determination of the space-time dimension.

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