RESOLUTION OF SINGULARITIES OF TORIC ORBIFOLDS AND EQUIVARIANT COBORDISM OF CONTACT TORIC MANIFOLDS

KOUSHIK BRAHMA, SOUMEN SARKAR, AND SUBHANKAR SAU

Abstract. Toric orbifolds are a generalization of simplicial projective toric varieties. In this paper, we show that there is a resolution of singularities of a toric orbifold. In a different category, the class of quasi-contact toric manifolds contains the class of good contact toric manifolds. We prove that a quasi-contact toric manifold is equivariantly a boundary. Moreover, we conclude that good contact toric manifolds and generalized lens spaces are equivariantly boundaries.

1. Introduction

Toric manifolds were introduced in the pioneering paper [5] by Davis and Januszkiewicz. These manifolds are topological generalization of smooth projective toric varieties. The paper [5] considered the standard action of the compact abelian torus $T^n$ on $\mathbb{C}^n$ as the local model to define toric manifolds. Briefly, a $2n$-dimensional smooth manifold with a locally standard $T^n$-action is called a toric manifold if the orbit space has the structure of an $n$-dimensional simple polytope. Moreover, [5, Section 7] initiated the notion of toric orbifolds generalizing toric manifolds. These orbifolds are explicitly studied in [17] with the name ‘quasitoric orbifolds’. Examples of toric orbifolds include simplicial projective toric varieties.

On the other hand, contact toric manifolds are odd-dimensional analogues of symplectic toric manifolds with Hamiltonian torus actions, see [1]. Lerman provided a complete classification of compact connected contact toric manifolds in [12, Theorem 2.18]. Motivated by the work of Davis and Januszkiewicz [5], Luo discussed the combinatorial construction of good contact toric manifolds and studied some of their topological properties in [14].

Recently, the construction of toric manifolds in [5] has been extended to introduce the notion of locally $k$-standard $T$-manifolds, in [19] where $T$ is a compact abelian torus. The paper [19] considers the invariant subset $\mathbb{C}^n \times (S^1)^k$ of $\mathbb{C}^{n+k}$ with respect to the standard $T^{n+k}$-action, and calls the restricted $T^{n+k}$-action on $\mathbb{C}^n \times (S^1)^k$ as the local model to define a locally $k$-standard $T$-manifold. In particular, when $k = 1$ we call this manifold a quasi-contact toric manifold. Briefly, a $(2n + 1)$-dimensional smooth compact $T^{n+1}$-manifold $N$ is called a quasi-contact toric manifold if each point of $N$ has an invariant neighbourhood which is diffeomorphic to an invariant open subset of $\mathbb{C}^n \times S^1$ such that the orbit space $N/T^{n+1}$ has the structure of an $n$-dimensional simple polytope. The article [19] shows that the category of quasi-contact toric manifolds contains all good contact toric manifolds.

Date: August 23, 2022.
2020 Mathematics Subject Classification. 14E15, 57R85, 52B11, 14M25, 57S12.
Key words and phrases. Resolution of singularity, toric orbifold, torus action, quasi-contact toric manifold, equivariant cobordism.
use the term ‘quasi-contact toric manifold’ as the topological counterpart of ‘good contact toric manifold’.

Resolution of singularity is a widely used tool in algebraic geometry nowadays, see [11,4]. One can trace back to Newton and Riemann for the idea of resolution of singularities of the curves. If \( X \) is a singular toric variety then there is a resolution of singularities of \( X \), see [6, Chapter 2]. In algebraic topology, [7] discusses the resolution of singularities of four dimensional toric orbifolds. We note that there are infinitely many toric orbifolds which are not toric varieties. For example, an equivariant connected sum of two weighted projective spaces is a toric orbifold but not a toric variety. In this article, we discuss the resolution of singularities for any toric orbifolds. Our technique is different than the proof of the resolution of singularities of a singular toric variety.

Lev Pontryagin introduced the notion of cobordism in [18]. Cobordism theory discusses about an equivalence on the same dimensional compact manifolds using the concept of the boundary of a manifold. Considering the disjoint union as addition, one can get an abelian group structure on these equivalence classes. If a manifold is a boundary of a manifold with boundary then it is called null-cobordant. The cobordism groups could be computed through homotopy theory using the Thom construction, see [21]. We have a complete information about oriented, non-oriented and complex cobordism rings. The notion of equivariant cobordism was introduced in early 1970’s, see [9, 10, 13]. The equivariant cobordism rings for some of the nontrivial groups can be found in [8]. However, they are not known for any nontrivial groups. One of the key reason is that in equivariant category, the Thom transversality theorem does not hold. Thus the equivariant cobordism theory cannot be reduced to the equivariant homotopy theory. In this article, we study the equivariant cobordism of quasi-contact toric manifolds, good contact toric manifolds and generalized lens spaces.

The paper is organized as follows. In Section 2 we recall the notion of a toric manifold and a toric orbifold following [5]. Then we recall the notion of blowup of a simple polytope as well as blowup of a toric orbifold following [2]. We discuss the orbifold singularity on any face in the orbit space of a toric orbifold. Then we apply the techniques of blowup of a toric orbifold for resolution of singularity and prove the following result.

Theorem A (Theorem 2.8). For a toric orbifold \( M(P, \lambda) \), there exists a resolution of singularities

\[
M(P^{(d)}, \lambda^{(d)}) \to \ldots \to M(P^{(1)}, \lambda^{(1)}) \to M(P, \lambda)
\]

such that \( M(P^{(d)}, \lambda^{(d)}) \) is a toric manifold, the toric orbifold \( M(P^{(i)}, \lambda^{(i)}) \) is obtained by a blowup of \( M(P^{(i-1)}, \lambda^{(i-1)}) \) for \( i = 1, 2, \ldots, d \) and the arrows \( \to \) indicate the associated blowups.

In Section 3 we recall the concept of quasi-contact toric manifolds and discuss some of their properties. The idea of the construction of these spaces is similar to the construction of toric manifolds introduced by Davis and Januszkiewicz [5]. Generalized lens spaces [20] and good contact toric manifolds are some well known examples of quasi-contact toric manifolds, see Example 3.5 and Example 3.6. Then we prove the following.

Theorem B (Theorem 3.8). Let \( N \) be a \((2n+1)\)-dimensional quasi-contact toric manifold. Then there is a smooth oriented \( T^{n+1} \)-manifold with boundary \( M \) such that \( \partial M \) is equivariantly diffeomorphic to \( N \).
As a consequence, one can obtain that good contact toric manifolds and generalized lens spaces are equivariantly cobordant to zero, and hence all the Stiefel-Whitney numbers of these manifolds are zero. We note that a lens space is a contact toric manifold.

2. Resolution of singularities of toric orbifolds

In this section, we discuss the constructive definition of toric manifolds and toric orbifolds following [5]. These spaces are even dimensional orbifolds and equipped with half-dimensional locally standard torus actions where the orbit spaces are simple polytopes. We also recall the notion of blowup of a toric orbifold along certain invariant suborbifolds. We discuss when an open convex bounded subset of $\mathbb{R}^n$ contains an integral point. We prove that there is a resolution of singularities of a toric orbifold.

A convex hull of finitely many points in $\mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 1}$ is called a convex polytope. If each vertex (zero dimensional face) in an $n$-dimensional convex polytope is the intersection of $n$ facets (codimension one faces), then the polytope is called a simple polytope. One can find the basic properties of simple polytopes in [22, 3]. For a simple polytope $P$ one can prove that there is a resolution of singularities of a toric orbifold.

We denote the standard $n$-dimensional torus by $\mathbb{T}^n$. Let $\lambda: F \to \mathbb{Z}^n$ be a characteristic function on $P$. We denote the vertex set of $P$ by $V(P) := \{b_1, \ldots, b_n\}$ and the facet set by $F(P) := \{F_1, \ldots, F_r\}$ throughout this paper.

**Definition 2.1.** Let $P$ be an $n$-dimensional simple polytope and $\lambda: F(P) \to \mathbb{Z}^n$ a function such that $\lambda(F_i)$ is primitive for $i \in \{1, \ldots, r\}$ and

\[(2.1) \quad \{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\} \text{ is linearly independent if } \bigcap_{j=1}^k F_{i_j} \neq \emptyset.\]

Then $\lambda$ is called an $R$-characteristic function on $P$, and the pair $(P, \lambda)$ is called an $R$-characteristic pair.

If the set $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$ in (2.1) spans a $k$-dimensional unimodular submodule of $\mathbb{Z}^n$, then $\lambda$ is called a characteristic function. In this case, the pair $(P, \lambda)$ is called a characteristic pair.

Observe that a characteristic function is an $R$-characteristic function. We denote $\lambda(F_i)$ by $\lambda_i$ and call it the $R$-characteristic vector or characteristic vector assigned to the facet $F_i$ according to the situation.

Next, we recall the construction of a toric orbifold from an $R$-characteristic pair $(P, \lambda)$. We denote the standard $n$-dimensional torus by $T^n$. Note that $\mathbb{Z}^n$ is the standard $n$-dimensional lattice in the Lie algebra of $T^n$. Also, each $\lambda_i \in \mathbb{Z}^n$ determines a line in $\mathbb{R}^n (= \mathbb{Z}^n \otimes \mathbb{R})$, whose image under the exponential map $\exp: \mathbb{R}^n \to T^n$ is a circle subgroup, denoted by $T_i$.

Let $F$ be a codimension-$k$ face of $P$ where $0 < k \leq n$ and $\hat{F}$ the relative interior of $F$. Then $F = \bigcap_{j=1}^k F_{i_j}$ for some unique facets $F_{i_1}, \ldots, F_{i_k}$ of $P$. Let $T_F := \langle T_{i_1}, \ldots, T_{i_k} \rangle$ be the $k$-dimensional subtorus of $T^n$ generated by $T_{i_1}, \ldots, T_{i_k}$. We define $T_P := 1 \in T^n$.

We consider the identification space $M(P, \lambda) := (T^n \times \hat{P})/ \sim$, where the equivalence relation $\sim$ is defined by

\[(t, x) \sim (s, y) \text{ if and only if } x = y \in \hat{F} \text{ and } t^{-1}s \in T_F.\]
We define \( \bar{p} \) where prim indicates the primitive vector of \( \alpha \in \mathbb{Z}^n \setminus \{0\} \). Then \( \bar{p} \) is an \( \mathcal{R} \)-characteristic function on \( \bar{P} \). The pair \((P, \lambda)\) and \((\bar{P}, \bar{\lambda})\) satisfy Definition 2.3. Thus \( M(\bar{P}, \bar{\lambda}) \) is a blowup of the toric orbifold \( M(P, \lambda) \) along the suborbifold \( \pi^{-1}(F) \).

\begin{equation}
\bar{\lambda}(\bar{F}_i) := \begin{cases} 
\lambda_i & \text{if } \bar{F}_i = F_i \cap \bar{P} \text{ for } F_i \in \mathcal{F}(P), \\
\text{prim}(\sum_{j=1}^{k} c_j \lambda_{ij}) & \text{if } \bar{F}_i = \bar{F}_{r+1} \text{ and } (c_1, \ldots, c_k) \in \mathcal{Q}(F)
\end{cases}
\end{equation}

where prim(\( \alpha \)) indicates the primitive vector of \( \alpha \in \mathbb{Z}^n \setminus \{0\} \). Then \( \bar{\lambda} \) is an \( \mathcal{R} \)-characteristic function on \( \bar{P} \). The pair \((P, \lambda)\) and \((\bar{P}, \bar{\lambda})\) satisfy Definition 2.3. Thus \( M(\bar{P}, \bar{\lambda}) \) is a blowup of the toric orbifold \( M(P, \lambda) \) along the suborbifold \( \pi^{-1}(F) \).
Let $(P, \lambda)$ be an $\mathcal{R}$-characteristic pair and $F$ a face as in Example 2.4. Let $\lambda_{ij} = \lambda(F_{ij})$ for all $j = 1, 2, \ldots, k$. Then the group

$$G_F(P, \lambda) := \frac{\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \rangle \otimes \mathbb{Z} \mathbb{R} \cap \mathbb{Z}^n}{\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \rangle}$$

is finite and abelian, where $\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \rangle$ is the submodule generated by $\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}\}$. This group measures the order of singularity of points in $\pi^{-1}(F) \subseteq M(P, \lambda)$. For simplicity, we may denote $G_F(P, \lambda)$ by $G_F$, whenever the context is clear. Note that if $F$ is a face of $F'$ then $|G_{F'}|$ divides $|G_F|$ and the group $G_F$ is trivial if $M(P, \lambda)$ is a toric manifold.

We recall the definition of the volume of a parallelepiped in an inner product space. Let $\{u_1, u_2, \ldots, u_k\}$ be an orthonormal basis in the $k$-dimensional real inner product space $V$. Let

$$C_V := \{ \sum_{i=1}^k r_i u_i \in V \mid 0 \leq r_i \leq \alpha_i, r_i \in \mathbb{R} \text{ and } 0 < \alpha_i \in \mathbb{R} \}.$$ 

Then $C_V$ is a $k$-dimensional parallelepiped and

$$\text{vol}(C_V) = \alpha_1 \alpha_2 \cdots \alpha_k.$$ 

For a face $F = \bigcap_{j=1}^k F_{ij}$, we consider the vector space $V_F = (\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \rangle \otimes \mathbb{Z} \mathbb{R})$. Let $\{v_1, \ldots, v_k\}$ be a basis of the lattice

$$L_F := (\langle \lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \rangle \otimes \mathbb{Z}) \cap \mathbb{Z}^n$$

in $V_F$. So $L_F = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k$. Define

$$C := \{ \sum_{i=1}^k r_i v_i \mid 0 \leq r_i < 1, r_i \in \mathbb{R} \}.$$ 

This $C$ is called a fundamental parallelepiped for the lattice $L_F$. It is well known that $\text{vol}(C)$ is independent of the bases of $L_F$. Let

$$C_F = \{ \sum_{j=1}^k r_j \lambda_{ij} \mid 0 \leq r_j < 1, r_j \in \mathbb{R} \}.$$ 

Then $|G_F|$ measures the volume of the $k$-dimensional parallelepiped $C_F \subset V_F$ made by the $k$ vectors $\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k}\}$, where $|G_F|$ is the order of the group $G_F$. Mathematically, one can write

$$\text{vol}(C_F) = |G_F| \times \text{vol}(C).$$ 

The following Minkowski theorem says when an open convex subset of $\mathbb{R}^n$ contains a non-zero lattice point.

**Proposition 2.5.** ([16] Theorem 5.2) Let $X$ be an open convex subset of a $k$-dimensional inner product space $V \subset \mathbb{R}^n$ and $C$ the fundamental parallelepiped for the lattice $V \cap \mathbb{Z}^n$. If $X$ is symmetric about origin and has volume greater than $2^k \text{vol}(C)$, then $X$ contains a point $b = (b_1, b_2, \ldots, b_n) \in V \cap \mathbb{Z}^n$ other than the origin.

**Lemma 2.6.** Let $\lambda$ be an $\mathcal{R}$-characteristic function on $n$-dimensional simple polytope $P$ and $F = \bigcap_{j=1}^k F_{ij}$ a face in $P$. If $|G_F| > 1$ then there exists a non-zero $\lambda_F \in \mathbb{Z}^n$ such that $\lambda_F = \sum_{j=1}^k c_j \lambda_{ij}$ with $|c_j| < 1$ and $c_j \in \mathbb{Q}$. Moreover, if $|G_{F'}| = 1$ for every face $F'$ properly
contains $F$ in $P$, then there exists a non-zero $\lambda_F \in \mathbb{Z}^n$ such that $\lambda_F = \sum_{j=1}^{k} c_j \lambda_{ij}$ where $0 < |c_j| < 1$ and $c_j \in \mathbb{Q}$ for $j = 1, 2, \ldots, k$.

**Proof.** The parallelepiped $C_F$ constructed in (2.4) is convex but not symmetric about the origin. Consider the union of $2^k$ many parallelepiped defined by

$$
\tilde{C}_F = \left\{ \sum_{j=1}^{k} r_j \lambda_{ij} \mid -1 < r_j < 1, r_j \in \mathbb{R} \right\}.
$$

This is an open parallelepiped which is convex and symmetric about the origin such that $\text{vol}(\tilde{C}_F) = 2^k \text{vol}(C_F) = 2^k |G_F| \text{vol}(C) > 2^k \text{vol}(C)$.

Then by Proposition 2.5 there exists a non-zero point $\lambda_F \in \mathbb{Z}^n \cap \tilde{C}_F$. Then there exist $c_j \in \mathbb{Q}$ with $|c_j| < 1$ for all $j \in \{1, 2, \ldots, k\}$ such that $\lambda_F = \sum_{j=1}^{k} c_j \lambda_{ij}$.

Now for the second part, without loss of generality, we may assume that $c_1 = 0$. Consider the face $F' := \cap_{j=2}^{k} F_{ij}$. Then $F'$ contains the face $F$ properly. By our assumption, we have $|G_{F'}| = 1$. Then $\{\lambda_{i2}, \lambda_{i3}, \ldots, \lambda_{ik}\}$ is a $\mathbb{Z}$-basis of the lattice $(\langle \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik} \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$.

Observe that this contradicts the existence of a non-zero lattice point $\lambda_F = \sum_{j=2}^{k} c_j \lambda_{ij}$ such that $|c_j| < 1$ for $j \in \{2, 3, \ldots, k\}$. Therefore $0 < |c_1| < 1$. Similarly, we can show $0 < |c_j| < 1$ for $j \in \{2, 3, \ldots, k\}$. \qed

**Remark 2.7.** Let $F = \bigcap_{j=1}^{k} F_{ij}$ be a face of $P$ and $\lambda$ an $\mathcal{R}$-characteristic function on $P$.

1. If $|G_F| = 1$, then

$$
(\langle \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik} \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n = \mathbb{Z}\lambda_{i1} + \mathbb{Z}\lambda_{i2} + \cdots + \mathbb{Z}\lambda_{ik},
$$

and there does not exist a point of $(\langle \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik} \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$ in the parallelepiped $C_F$ except the origin.

2. If $|G_F| > 1$ then the non-zero lattice points of $(\langle \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik} \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$ in the parallelepiped $C_F$ have a one-one correspondence to non-identity elements of the group $G_F$. So the total number of lattice points in the parallelepiped $C_F$ is $|G_F|$. Thus $\lambda_F$ obtained in the Lemma 2.4 is not unique if $|G_F| > 2$.

**Theorem 2.8.** For a toric orbifold $M(P, \lambda)$, there exists a resolution of singularities

$$
M(P^{(d)}, \lambda^{(d)}) \to \cdots \to M(P^{(1)}, \lambda^{(1)}) \to M(P, \lambda)
$$

such that $M(P^{(d)}, \lambda^{(d)})$ is a toric manifold, the toric orbifold $M(P^{(i)}, \lambda^{(i)})$ is a blowup of $M(P^{(i-1)}, \lambda^{(i-1)})$ for $i = 1, 2, \ldots, d$ and the arrows $\to$ indicate the associated blowups.

**Proof.** Let

$$
\mathcal{L} := \{ F \mid F \text{ is a face of } P \text{ and } |G_F(P, \lambda)| \neq 1 \}.
$$

Define a partial order $\leq$ on $\mathcal{L}$ by $F \leq F'$ if $F \subseteq F'$. Without loss of generality, let $F$ be a maximal element in the set $\mathcal{L}$ with respect to the partial order $\leq$.

Now consider the blowup of the polytope $P$ along the face $F$ and denote this blowup by $P^{(1)}$. If $\mathcal{F}(P) = \{F_1, \ldots, F_r\}$, then we have

$$
\mathcal{F}(P^{(1)}) = \{\bar{F}_1, \ldots, \bar{F}_r, \bar{F}_{r+1}\}
$$

and the associated blowups.
We can assume \( p \). Therefore, in this above process if a vertex of \( \bar{\mathcal{F}}_i \) is obtained by Lemma \ref{lem:resolution}. Then \((P^{(1)}, \lambda^{(1)})\) is an \( \mathcal{R} \)--characteristic pair.

There does not exist a face \( F' \) containing \( F \) such that \( |G_{F'}| \neq 1 \), by the definition of \( \mathcal{L} \). Thus we can assume \( 0 < |c_j| < 1 \) and \( c_j \in \mathbb{Q} \) for every \( j \in \{1, 2, \ldots, k\} \) by Lemma \ref{lem:resolution}. Let \( V(F) := \{b_1, b_2, \ldots, b_\ell\} \) and \( V(P) = \{b_1, b_2, \ldots, b_\ell, b_{\ell+1}, \ldots, b_m\} \).

Note that there is a homeomorphism from \( \bar{\mathcal{F}}_{r+1} \) to \( F \times \Delta^{k-1} \) preserving the face structure. Let \( V(\Delta^{k-1}) = \{u_1, u_2, \ldots, u_k\} \). We have \( P^{(1)} \subset P \) from Definition \ref{defn:resolution}.

Then
\[
V(P^{(1)}) := \{(b_p, u_s) \mid 1 \leq p \leq \ell; 1 \leq s \leq k\} \cup \{b_{\ell+1}, \ldots, b_m\}.
\]

We can assume \((b_p, u_s)\) is not a vertex of \( \bar{\mathcal{F}}_{i_0}\) for \( 1 \leq s \leq k \). For a vertex \( b_p \in V(F) \subset P \), let \( b_p = (\cap_{j=1}^n \bar{F}_{i_j}) \cap F = \cap_{j=1}^n F_{i_j} \). Then for all \( 1 \leq s \leq k \) and \( 1 \leq p \leq \ell \)
\[
(b_p, u_s) = \bigcap_{j=1}^n \bar{F}_{i_j} \bigcap_\ell \bar{F}_{r+1}.
\]

Now we calculate the singularity over each vertex in \( P^{(1)} \). Note that
\[
|G_{(b_p, u_s)}(P^{(1)}, \lambda^{(1)})| = |\det[\lambda_1, \ldots, \hat{\lambda}_s, \ldots, \lambda_n, \text{prim}(\sum_{j=1}^k c_j \lambda_j)]| = \left|\frac{|c_j|}{d_F}|G_{b_p}(P, \lambda)|<|G_{b_p}(P, \lambda)|\right|
\]
for \( 1 \leq p \leq \ell \) and \( 1 \leq s \leq k \) and \( d_F \) is a positive integer satisfying \( \lambda_F = d_F.\text{prim}(\lambda_F) \), and
\[
|G_{b_{\ell+i}}(P^{(1)}, \lambda^{(1)})| = |G_{b_{\ell+i}}(P, \lambda)| \text{ for } 1 \leq i \leq m - \ell.
\]

Therefore, in this above process if a vertex of \( P \) is not in the face \( F \) then the corresponding singularity remains same. Also corresponding to every vertex \( b \) in \( F \) we get exactly \( k \) many new vertices in \( \bar{\mathcal{F}}_{r+1} \) such that at each of them the singularity is strictly less than the singularity on \( b \in F \subset P \) in the given orbifold.

If \(|G_F| = 1\) for every faces of \( P^{(1)} \) then \( M(P^{(1)}, \lambda^{(1)}) \) is the desired resolution of singularities of \( M(P, \lambda) \). Otherwise we repeat the above process on \( M(P^{(1)}, \lambda^{(1)}) \) to obtain an \( \mathcal{R} \)-characteristic pair \((P^{(2)}, \lambda^{(2)})\) where \( P^{(2)} \) is a blowup of \( P^{(1)} \), and \( \lambda^{(2)} \) is defined similarly as in \((2.5)\) from \( \lambda^{(1)} \). If \(|G_F| = 1\) for every faces of \( P^{(2)} \) corresponding to the pair \((P^{(2)}, \lambda^{(2)})\) then we are done. Otherwise, we repeat the process. Since the order of \( G_F \) is finite for any face \( F \) of \( P \), this inductive process ends after a finite steps.

\( \square \)

**Example 2.9.** Consider the \( \mathcal{R} \)-characteristic function and the edge \( F \) of the triangular prism \( P_1 \) in Figure \ref{fig:prism}. We have the \( \mathbb{Z} \)-module generated by \( \{(1, 2, 0), (1, 0, 0)\} \). Then
\[
((1, 2, 0), (1, 0, 0)) \otimes \mathbb{R} \cap \mathbb{Z}^3 = \{(x, y, 0) | x, y \in \mathbb{Z}\} \cong \mathbb{Z}^2
\]
which has a basis \( \{(1, 0, 0), (0, 1, 0)\} \). In this case, \( \text{vol}(C_F) = 2 \). Thus for this edge \( F \), we get \(|G_F| = |\mathbb{Z}^2/(1, 2, 0), (0, 1, 0)| = 2 \). Since the faces containing \( F \) are facets of \( P_1 \), by
Lemma 2.6 there exists a non-zero $\lambda_F \in \mathcal{C}_F \cap \mathbb{Z}^3$. Here, the only non-zero lattice point in the parallelepiped $C_F$ is $\lambda_F = (1, 1, 0)$ which can be represented as

$$(1, 1, 0) = \frac{1}{2}(1, 0, 0) + \frac{1}{2}(1, 2, 0).$$

Also, similarly, one can calculate $|G_b| = 2$ for $b \in V(F)$ and $|G_b| = 1$ for $b \in V(P_1) - V(F)$. So, the maximal element in $\mathcal{L}$ for this case is $F$. Thus we blowup $P_1$ along the face $F$, and get the cube $P_1^{(1)}$ as in Figure 1(C). One can define $\lambda^{(1)}$ on $P_1^{(1)}$ as in (2.5). Then in the toric orbifold $M(P_1^{(1)}, \lambda^{(1)})$, we have $|G_E| = 1$ for every face $E$ of $P_1^{(1)}$. Thus $M(P_1^{(1)}, \lambda^{(1)})$ is a toric manifold, and it is a resolution of singularities of $M(P_1, \lambda)$. 

Figure 1.

Lemma 2.6 there exists a non-zero $\lambda_F \in \mathcal{C}_F \cap \mathbb{Z}^3$. Here, the only non-zero lattice point in the parallelepiped $C_F$ is $\lambda_F = (1, 1, 0)$ which can be represented as

$$(1, 1, 0) = \frac{1}{2}(1, 0, 0) + \frac{1}{2}(1, 2, 0).$$

Also, similarly, one can calculate $|G_b| = 2$ for $b \in V(F)$ and $|G_b| = 1$ for $b \in V(P_1) - V(F)$. So, the maximal element in $\mathcal{L}$ for this case is $F$. Thus we blowup $P_1$ along the face $F$, and get the cube $P_1^{(1)}$ as in Figure 1(C). One can define $\lambda^{(1)}$ on $P_1^{(1)}$ as in (2.5). Then in the toric orbifold $M(P_1^{(1)}, \lambda^{(1)})$, we have $|G_E| = 1$ for every face $E$ of $P_1^{(1)}$. Thus $M(P_1^{(1)}, \lambda^{(1)})$ is a toric manifold, and it is a resolution of singularities of $M(P_1, \lambda)$. 

3. Equivariant cobordism of quasi-contact toric manifolds

In this section, we recall the concept of quasi-contact toric manifolds and discuss some of their properties. Then we prove that any quasi-contact toric manifold is equivariantly the boundary of an oriented smooth manifold. In particular, good contact toric manifolds and generalized lens spaces are equivariantly cobordant to zero.

Consider the action $\alpha: T^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of $(n+1)$-dimensional torus $T^{n+1}$ on $\mathbb{C}^{n+1}$ defined by

$$\alpha((t_1, \ldots, t_n, t_{n+1}), (z_1, \ldots, z_n, z_{n+1})) = (t_1z_1, \ldots, t_nz_n, t_{n+1}z_{n+1}).$$
Then $\mathbb{C}^n \times S^1$ is a $T^{n+1}$-invariant subset of $\mathbb{C}^{n+1}$, and the orbit space $(\mathbb{C}^n \times S^1)/T^{n+1}$ is $\mathbb{R}^{2n}$. The restriction $\alpha|_{T^{n+1}\times(\mathbb{C}^n \times S^1)}$ is called the standard $T^{n+1}$-action on $\mathbb{C}^n \times S^1$.

**Definition 3.1.** A $(2n+1)$-dimensional smooth manifold $N$ with an effective $T^{n+1}$-action is called a quasi-contact toric manifold if the orbit space $N/T^{n+1}$ has the structure of an $n$-dimensional torus $T^n$.

Let $q: N \to Q$ be the orbit map where $Q$ is an $n$-dimensional simple polytope. Let $\mathcal{F}(Q) := \{E_1, \ldots, E_\ell\}$ be the set of facets of $Q$. Then each $N_j := q^{-1}(E_j)$ is a $(2n-1)$-dimensional $T^{n+1}$-invariant submanifold of $N$. Then, the isotropy subgroup of $N_j$ is a circle subgroup $T_j$ of $T^{n+1}$. The group $T_j$ is uniquely determined by a primitive vector $\eta_j \in \mathbb{Z}^{n+1}$ for $j = 1, 2, \ldots, \ell$; that is, we get a natural function

$$\eta: \{E_1, \ldots, E_\ell\} \to \mathbb{Z}^{n+1}$$

(3.1)

defined by $\eta(E_j) = \eta_j$.

We discuss the constructive definition of $(2n+1)$-dimensional quasi-contact toric manifolds on simple polytopes following [19]. Let $\mathcal{F}(Q) := \{E_1, \ldots, E_\ell\}$ be the set of facets of an $n$-dimensional simple polytope $Q$.

**Definition 3.2.** A function $\xi: \mathcal{F}(Q) \to \mathbb{Z}^{n+1}$ is called a hyper characteristic function if $\langle \xi_{j_1}, \ldots, \xi_{j_n} \rangle$ is a rank $n$ unimodular submodule of $\mathbb{Z}^{n+1}$ whenever $E_{j_1} \cap \cdots \cap E_{j_n} \neq \emptyset$ where $\xi_j := \xi(E_j)$ for $j = 1, \ldots, \ell$. The pair $(Q, \xi)$ is called a hyper characteristic pair.

Note that hyper characteristic function was defined on simplices in [20, Definition 2.1] and Definition 5.2 is the case $k = 1$ in [19, Definition 2.2].

Let $\xi$ be a hyper characteristic function on an $n$-dimensional simple polytope $Q$. For a point $p \in Q$, let $E_{j_1} \cap \cdots \cap E_{j_k}$ be the face of $Q$ containing $p$ in its relative interior. Then $\exp(\langle \xi_{j_1}, \ldots, \xi_{j_k} \rangle \otimes \mathbb{Z} \mathbb{R})$ is a $k$-dimensional subtorus of $T^{n+1}$. We denote this subgroup by $T_p$. If $p$ belongs to the relative interior of $Q$, we denote $T_p = 1$, the identity in $T^{n+1}$. We define the following identification space.

$$N(Q, \xi) := (T^{n+1} \times Q)/\sim'$$

(3.2)

where

$$(t, p) \sim' (s, q) \text{ if and only if } p = q \text{ and } t^{-1}s \in T_p.$$  

Here, $T^{n+1}$ acts on $N(Q, \xi)$ induced by the multiplication on the first factor of $T^{n+1} \times Q$.

**Proposition 3.3.** Let $(Q, \xi)$ be a hyper characteristic pair. Then the space $N(Q, \xi)$ in (3.2) is a quasi-contact toric manifold.

**Proof.** Let $Z_Q$ be the moment angle manifold corresponding to $Q$, see [3, Section 6.2]. Then $Z_Q$ is a smooth manifold and there is a smooth $T^m$-action on $Z_Q$ where $m$ is the number of facets of $Q$, see [3, Corollary 6.2.5]. The space $N(Q, \xi)$ has a manifold structure
which satisfy condition (1) and (2) in Definition 3.1 see [19, Proposition 2.3]. If rank of \( \langle \xi_1, \ldots, \xi_\ell \rangle \) is \( n \), then \( N(Q, \xi) \) is equivariantly homeomorphic to \( M(Q, \lambda_\xi) \times S^1 \) for some toric manifold \( M(Q, \lambda_\xi) \), see [19, Proposition 2.6]. If rank of \( \langle \xi_1, \ldots, \xi_\ell \rangle \) is \( n + 1 \), then \( N(Q, \xi) \) is equivariantly homeomorphic to \( Z_Q / T_\xi \) for some \( (m - n - 1) \)-dimensional subgroup \( T_\xi \) of \( T^m \), see [19, Proposition 2.7]. Also, \( M(Q, \lambda_\xi) \) is equivariantly homeomorphic to \( Z_Q / T_\lambda_\xi \) for some some \( (m - n) \)-dimensional subgroup \( T_\lambda_\xi \) of \( T^m \). Therefore, \( N(Q, \xi) \) has a smooth structure such that the \( T^{n+1} \)-action \( N(Q, \xi) \) is smooth. \( \square \)

Note that the function \( \eta \) defined in (3.1) satisfies Definition 3.2 see the explanation in [19, Subsection 2.1]. Also the orientation of a quasi-contact toric manifold \( N \) can be induced from an orientation of \( Q \) and \( T^{n+1} \).

**Proposition 3.4.** [19] Let \( N \) be a quasi-contact toric manifold over the \( n \)-dimensional simple polytope \( Q \), and \( \eta \) a function as defined in (3.1). Then \( N \) is equivariantly diffemorphic to \( N(Q, \eta) \).

**Proof.** This follows by similar arguments in the proofs of Lemma 1.4 and Proposition 1.8 in [5] and Proposition 3.3. \( \square \)

**Example 3.5** (Generalized lens spaces). Let \( \Delta^n \) be the \( n \)-dimensional simplex and \( \xi \) a hyper characteristic function on it. We assume that the rank of \( \langle \xi(E) \mid E \in \mathcal{F}(\Delta^n) \rangle \subseteq Z^{n+1} \) is \( n + 1 \). The article [20] shows that \( N(\Delta^n, \xi) \) is equivariantly homeomorphic (hence diffeomorphic) to the orbit space \( S^{2n+1} / G_\xi \) for some free action of a finite group \( G_\xi \). This space is called a *generalized lens space* in [20]. In particular, if \( \{ \xi(E) \mid E \in \mathcal{F}(\Delta^n) \} \) forms a basis of \( Z^{n+1} \), then \( N(\Delta^n, \xi) \) is homeomorphic to \( S^{2n+1} \).

Consider an integer \( p > 1 \) and \( n \) integers \( q_1, \ldots, q_n \) such that \( \gcd(p, q_i) = 1 \) for all \( i = 1, \ldots, n \). Then \( \mathbb{Z}_p \) acts freely on \( S^{2n+1} \) by the following

\[
g(z_0, z_1, \ldots, z_n) = (g q_1 z_0, g q_2 z_1, \ldots, g q_n z_n).
\]

The lens space \( L(p; q_1, \ldots, q_n) \) is defined to be the orbit space \( S^{2n+1} / \mathbb{Z}_p \). The paper [20] showed that there is a hyper characteristic function \( \xi \) on \( \Delta^n \) such that \( L(p; q_1, \ldots, q_n) \) is equivariantly diffeomorphic to \( N(\Delta^n, \xi) \). \( \square \)

**Example 3.6** (Good contact toric manifolds). Luo [14, Chapter 2] discussed the construction of good contact toric manifolds which are compact connected contact toric manifolds studied in [12, Section 2]. We briefly, recall the construction. Let \( Q \) be an \( n \)-dimensional simple lattice polytope embedded in \( \mathbb{R}^{n+1} \setminus \{0\} \). Consider the cone \( C(Q) \) on \( Q \) with apex \( 0 \in \mathbb{R}^{n+1} \) and the set \( \{ \tilde{E} \mid E \in \mathcal{F}(Q) \} \) of facets of \( C(Q) \) where \( \tilde{E} := C(E) \setminus \{0\} \). Let \( \xi(E) \) be the primitive outward normal vector on \( \tilde{E} \). This defines a function \( \xi : \mathcal{F}(Q) \to \mathbb{Z}^{n+1} \).

Since the facets of \( C(Q) \setminus \{0\} \) intersects transversely, the function \( \xi \) satisfies Definition 3.2. Then the space \( N(Q, \xi) \) is \( T^{n+1} \)-equivariantly homeomorphic to a good contact toric manifold whose moment cone is \( C(Q) \). Moreover, a good contact toric manifold can be obtained in this way. For details, we refer [12] and [14]. \( \square \)

**Lemma 3.7.** Let \( Q \) be an \( n \)-dimensional simple polytope and

\[
\xi : \mathcal{F}(Q) \to \mathbb{Z}^{n+1}
\]
a hyper characteristic function. Then there exists \( a = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1} \) such that \( \{ a, \xi(E_{j_1}), \ldots, \xi(E_{j_m}) \} \) is a linearly independent subset in \( \mathbb{Z}^{n+1} \) whenever \( E_{j_1} \cap \cdots \cap E_{j_m} \) is a vertex of \( Q \).

**Proof.** Let \( b \) be a vertex of \( Q \). Then \( b = E_{j_1} \cap \cdots \cap E_{j_m} \) for some unique facets \( E_{j_1}, \ldots, E_{j_m} \). Let \( \mathbb{Z}_b \) be the submodule of \( \mathbb{Z}^{n+1} \) generated by \( \xi_{j_1}, \ldots, \xi_{j_m} \). So the rank of \( \mathbb{Z}_b \) is \( n \) for any vertex \( b \in V(Q) \). Since there are only finitely many vertices in \( Q \) we have \( \mathbb{Z}^{n+1} \setminus \bigcup_{b \in V(Q)} \mathbb{Z}_b \) is non-empty. If possible, let

\[
Z^{n+1} = \bigcup_{b \in V(Q)} \mathbb{Z}_b.
\]

Let \( V_b := \mathbb{Z}_b \otimes_{\mathbb{Z}} \mathbb{R} \). Then \( V_b \) is an \( n \)-dimensional linear subspace of \( \mathbb{R}^{n+1} \). From (3.3) it follows that

\[
\mathbb{R}^{n+1} = \mathbb{Z}^{n+1} \otimes_{\mathbb{Z}} \mathbb{R} = \bigcup_{b \in V(Q)} (\mathbb{Z}_b \otimes_{\mathbb{Z}} \mathbb{R}) = \bigcup_{b \in V(Q)} V_b,
\]

which is a contradiction as \( V(Q) \) is a finite set. Let \( a \in \mathbb{Z}^{n+1} \setminus \bigcup_{b \in V(Q)} \mathbb{Z}_b \) be primitive. Then \( a \) is the desired vector of the lemma. \( \square \)

**Theorem 3.8.** Let \( N \) be a \((2n + 1)\)-dimensional quasi-contact toric manifold. Then there is a smooth oriented \( T^{n+1} \)-manifold with boundary \( M \) such that \( \partial M \) is equivariantly diffeomorphic to \( N \).

**Proof.** There exist a hyper characteristic pair \((Q, \xi)\) such that \( N \) is equivariantly diffeomorphic to \( N(Q, \xi) \) by Proposition 3.4. Let \( \mathcal{F}(Q) := \{ E_1, \ldots, E_l \} \) and \( I = [0, 1] \). Then \( Q \times I \) is an \((n+1)\)-dimensional simple polytope and \( \mathcal{F}(Q \times I) := \{ E_1 \times I, \ldots, E_l \times I, Q \times \{0\}, Q \times \{1\} \} \). Let \( a \in \mathbb{Z}^{n+1} \) satisfies Lemma 3.7. Define a map \( \lambda : \mathcal{F}(Q \times I) \to \mathbb{Z}^{n+1} \) by

\[
\lambda(F) = \begin{cases} 
\xi(E_i) & \text{if } F = E_i \times I \text{ for } i = 1, 2, \ldots, l \\
\mathbb{R} & \text{if } F = Q \times \{0\}, Q \times \{1\}.
\end{cases}
\]

Then \( \lambda \) is an \( R \)-characteristic function on \( P := Q \times I \), and \( M(P, \lambda) \) is a toric orbifold. Theorem 2.8 gives a resolution of singularities

\[
M(P^{(d)}, \lambda^{(d)}) \to \cdots \to M(P^{(1)}, \lambda^{(1)}) \to M(P, \lambda)
\]

for \( M(P, \lambda) \), where \( M(P^{(d)}, \lambda^{(d)}) \) is a toric manifold and the arrows represent the associated blowups.

Note that for any face \( E \) of \( Q \), the codimension of \( E \) in \( Q \) is same as the codimension of the face \( E \times I \) in \( P \). Moreover, if

\[
E = \bigcap_{s=1}^{k} E_{j_s}
\]

for some unique facets \( E_{j_1}, E_{j_2}, \ldots, E_{j_k} \) of \( Q \), then \( E_{j_1} \times I, E_{j_2} \times I, \ldots, E_{j_k} \times I \) are facets of \( P \) and

\[
E \times I = \bigcap_{s=1}^{k} (E_{j_s} \times I).
\]

Now \( \lambda(E_{j_s} \times I) = \xi(E_{j_s}) \) and \( \{ \xi(E_{j_1}), \xi(E_{j_2}), \ldots, \xi(E_{j_k}) \} \) is a direct summand of \( \mathbb{Z}^{n+1} \). Then we have

\[
|G_{E \times I}(P, \lambda)| = 1
\]
for any face \( E \times I \) of \( P \) where \( E \) is a face of \( Q \). Therefore, if \( |G_F(P, \lambda)| \neq 1 \) for a face \( F \) of \( P \), then either \( F \subseteq Q \times \{0\} \) or \( F \subseteq Q \times \{1\} \). Thus we have \( |G_{(E \times I) \cap P(j)}(P^{(j)}, \lambda^{(j)})| = 1 \) for every \( j = 1, \ldots, d \). Therefore, the blowups in the resolution (3.5) need to take only on the faces arising from \( Q \times \{0\} \) or \( Q \times \{1\} \subseteq P \). Since \( P \) is a simple polytope, one can consider the necessary blowups in the resolution (3.5) such that \( Q \times \left[ \frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} \right] \subseteq P^{(d)} \), for some \( \delta > 0 \). Let

\[
\tilde{P} := \iota^{-1}(Q \times [0, 1/2]) \subseteq P^{(d)}
\]

where \( \iota \) is the inclusion map

\[
\iota: P^{(d)} \to P.
\]

Then \( \tilde{P} \) is a simple polytope. Let

\[
\pi': M(P^{(d)}, \lambda^{(d)}) = (P^{(d)} \times T^{n+1})/\sim \to P^{(d)}
\]

be the quotient map. We prove that \((\pi')^{-1}(\tilde{P})\) is a \( T^{n+1} \)-manifold with boundary, where the boundary is \( N(Q, \xi) \).

Let \( \alpha \) be a point in \( \tilde{P} \) and \( \alpha \not\in \{(x, \frac{1}{2}) : x \in Q\} \). Since \( M(P^{(d)}, \lambda^{(d)}) \) is a smooth manifold, then there exists a neighbourhood \( U_0 \) of \( \alpha \) in \( \tilde{P} \) such that \( U_0 \) does not intersect \( \{(x, \frac{1}{2}) : x \in Q\} \). So \((\pi')^{-1}(U_0)\) is an open subset of the smooth manifold \( M(P^{(d)}, \lambda^{(d)}) \).

Now if \( \alpha = (x, \frac{1}{2}) \in \tilde{P} \) for some \( x \in Q \), then consider the tubular neighbourhood \( Q \times \left[ \frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} \right] \) of \( Q \times \{\frac{1}{2}\} \) in \( \tilde{P} \). Note that \( \{(x, \frac{1}{2}) : x \in Q\} = Q \times \{\frac{1}{2}\} \) which is nothing but \( Q \). Then

\[
(Q \times \left( \frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} \right) \times T^{n+1})/\sim = \left( (Q \times \left\{ \frac{1}{2} \right\} \times T^{n+1})/\sim \right) \times \left( \frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} \right).
\]

The characteristic function on \( P^{(d)} \) induces a hyper characteristic function on the facets of \( Q \times \left\{ \frac{1}{2} \right\} \) which is same as the hyper characteristic function on the facets of \( Q \). This implies that \( \left( (Q \times \left\{ \frac{1}{2} \right\} \times T^{n+1})/\sim \right) \) is equivariantly diffeomorphic to \( N(Q, \xi) \). Thus the manifold with boundary \((\pi')^{-1}(\tilde{P})\) satisfies our claim. Hence a quasi-contact toric manifold is equivariantly cobordant to zero.

\( \square \)

**Example 3.9.** Consider the pentagon \( Q \) and the hyper characteristic function

\[
\xi: \mathcal{F}(Q) \to \mathbb{Z}^3
\]

as in Figure 2(A). Then we have the quasi-contact toric manifold \( N(Q, \xi) \). There exists a point \( a \in \mathbb{Z}^3 \) which satisfies Lemma 3.7 for this pair \((Q, \xi)\). In this case one can take \( a = (1, 2, 0) \). Then using (3.4) we can define an \( R \)-characteristic function \( \lambda \) on the pentagonal prism \( P := Q \times I \), see Figure 2(B). Thus, we have the toric orbifold \( M(P, \lambda) \).

In the simple polytope \( P \), the only faces \( F \) with \( |G_F| \neq 1 \) are two edges \( F_0 \) and \( F_1 \) which are coloured with blue in Figure 2(B) and the 4 vertices of these edges. Using the techniques of the proof of Theorem 2.8, first we blowup \( P \) along the face \( F_0 \) and get the simple polytope \( P' \) as in 2(C). We denote the corresponding blowup of toric orbifold \( M(P, \lambda) \) by \( M(P', \lambda') \). Note that after this blowup the only face \( F \) in the polytope \( P' \) with \( |G_F| \neq 1 \) is \( F_1 \) and the two vertices of \( F_1 \), see figure 2(C).

Note that \( Q \times \left\{ \frac{1}{2} \right\} \subseteq P' \subseteq P \) which is denoted by the dotted pentagon filled with yellow in Figure 2(B) and 2(C). Also \( Q \times \left\{ \frac{1}{2} \right\} \) can be identified with \( Q \). Let \( \tilde{P} \) denotes the lower
Figure 2. Procedure of constructing a manifold whose boundary is a given quasi-contact toric manifold.

portion of this pentagon $Q \times \{ \frac{1}{2} \}$ in $P'$, that is $\bar{P} = (Q \times \{ 0, \frac{1}{2} \}) \cap P'$. Then $N(Q, \xi)$ is equivariantly the boundary of the oriented smooth manifold $(\pi')^{-1}(\bar{P})$ with boundary for this example. 

\[ \text{Corollary 3.10. Any good contact toric manifold is equivariantly the boundary of an oriented smooth manifold.} \]

\[ \text{Proof. This follows from Example 3.6 and Theorem 3.8.} \]

\[ \text{Corollary 3.11. Any generalized lens space is equivariantly the boundary of an oriented smooth manifold.} \]

\[ \text{Proof. This follows from Example 3.5 and Theorem 3.8.} \]

We note that the conclusion of Corollary 3.11 can be obtained from [8]. However, our proof is more geometric and explicit. We also note that the article [20] gave partial answer to Corollary 3.11 under some number theoretic sufficient conditions.

\[ \text{Remark 3.12. Using Theorem 3.8 and [15] Theorem 4.9, we get that all Stiefel-Whitney numbers of quasi-contact toric manifolds are zero.} \]

Acknowledgments. The authors would like to thank Dong Youp Suh and Jongbaek Song for helpful discussion. The first author thanks ‘IIT Madras’ for PhD fellowship. The second author thanks ‘International office IIT Madras’ and Science and Engineering Research Board India for research grants. The third author thanks to IIT Madras for PDEF fellowship and IMSc for PDF fellowship.
References

[1] C. P. Boyer and K. Galicki. A note on toric contact geometry. *J. Geom. Phys.*, 35(4):288–298, 2000.

[2] K. Brahma, S. Sarkar, and S. Sau. Torsion in the cohomology of blowups of quasitoric orbifolds. *Topology Appl.*, 295:Paper No. 107666, 23, 2021.

[3] V. M. Buchstaber and T. E. Panov. *Toric topology*, volume 204 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[4] S. D. Cutkosky. *Resolution of singularities*, volume 63 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.

[5] M. W. Davis and T. Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.*, 62(2):417–451, 1991.

[6] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[7] S. Ganguli and M. Poddar. Blowdowns and McKay correspondence on four dimensional quasitoric orbifolds. *Osaka J. Math.*, 50(2):397–415, 2013.

[8] B. Hanke. Geometric versus homotopy theoretic equivariant bordism. *Math. Ann.*, 332(3):677–696, 2005.

[9] E. C. Hook. Equivariant cobordism and duality. *Trans. Amer. Math. Soc.*, 178:241–258, 1973.

[10] E. C. Hook. Equivariant cobordism and homotopy type. *Duke Math. J.*, 40:805–814, 1973.

[11] J. Kollár. *Lectures on resolution of singularities*, volume 166 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2007.

[12] E. Lerman. Contact toric manifolds. *J. Symplectic Geom.*, 1(4):785–828, 2003.

[13] P. Löffler. Equivariant unitary cobordism and classifying spaces. In *Proceedings of the International Symposium on Topology and its Applications (Budva, 1972)*, pages 158–160, 1973.

[14] S. Luo. Cohomology of contact toric manifolds and Hard Lefschetz Property of Hamiltonian GKM manifolds. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Cornell University.

[15] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[16] C. D. Olds, A. Lax, and G. P. Davidoff. *The geometry of numbers*, volume 41 of *Anneli Lax New Mathematical Library*. Mathematical Association of America, Washington, DC, 2000. Appendix I by Peter D. Lax.

[17] M. Poddar and S. Sarkar. On quasitoric orbifolds. *Osaka J. Math.*, 47(4):1055–1076, 2010.

[18] L. S. Pontryagin. Characteristic cycles on differentiable manifolds. *Mat. Sbornik N. S.*, 21(63):233–284, 1947.

[19] S. Sarkar and J. Song. Equivariant cohomological rigidity of certain $T$–manifolds. *Algebr. Geom. Topol.*, 21(7):3601–3622, 2021.

[20] S. Sarkar and D. Y. Suh. A new construction of lens spaces. *Topology Appl.*, 240:1–20, 2018.

[21] R. Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[22] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Email address: koushikbrahma95@gmail.com

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Email address: soumen@iitm.ac.in

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Email address: subhankarsau18@gmail.com