THE HODGE THEORY OF MAPS
LECTURES 4-5

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Abstract. These are the lecture notes from my two lectures 4 and 5 on the Hodge Theory of Maps delivered at the Hodge Theory Summer School at ICTP Trieste in June 2010. The lectures had a very informal flavor to them and, by choice, the notes reflect this fact. They are aimed at beginners, whatever that may mean. There are plenty of exercises and some references so you can start looking things up on your own. My little book [5] contains some of the notions discussed here, as well as some amplifications. These notes have appeared as a chapter in the book *Hodge Theory*, Princeton University Press, 2014, edited by E. Cattani, F. El Zein and P. Griffiths. Lecture notes from Lectures 1, 2 and 3 by L. Migliorini also appear in that book; the present notes are independent of them, but of course they are their natural continuation.

Contents

1. Lecture 4 1
   1.1. Sheaf cohomology and All That (A Minimalist Approach) 1
   1.2. The Intersection Cohomology Complex 11
   1.3. Verdier Duality 12
2. Lecture 5 15
   2.1. The Decomposition Theorem (DT) 15
   2.2. (Relative) Hard Lefschetz for Intersection Cohomology 17
References 20

1. Lecture 4

1.1. Sheaf cohomology and All That (A Minimalist Approach).
(1) We say that a sheaf of abelian groups $I$ on a topological space $X$ is *injective* if
the abelian-group-valued functor on sheaves $\text{Hom}(-, I)$ is exact.

See [4, 10, 13, 11].

Of course, the notion of injectivity makes sense in any abelian category, so we may speak of injective abelian groups, modules over a ring, etc.

(2) Exercise

[(a)]

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(a) Verify that for every sheaf \( F \), the functor \( \text{Hom}( -, F) \) is exact on one side (which one?), but, in general, not on the other.

(b) The injectivity of \( I \) is equivalent to the following: for every injection \( F \to G \) and every map \( F \to I \) there is a map \( G \to I \) making the “obvious” diagram (part of the exercise is to identify this diagram) commutative.

(c) A short exact sequence \( 0 \to I \to A \to B \to 0 \), with \( I \) injective, splits.

(d) If \( 0 \to A \to B \to C \to 0 \) is exact and \( A \) is injective, then \( B \) is injective if and only if \( C \) is.

(e) A vector space over a field \( k \) is an injective \( k \)-module.

(f) By reversing the arrows, you can define the notion of projectivity (for sheaves, modules over a ring, etc.). Show that free implies projective.

(3) It is a fact that every abelian group can be embedded into an injective abelian group. Obviously, this is true in the category of vector spaces!

(4) **Exercise**

Deduce from the embedding statement above that every sheaf \( F \) can be embedded into an injective sheaf. (Hint: consider the direct product sheaf \( \Pi_{x \in X} F_x \) on \( X \) and work stalk by stalk using 3.)

(5) By iteration of the embedding result established in Exercise 4, it is easy to show that given every sheaf \( F \), there is an *injective resolution of \( F \)*, i.e., a long exact sequence

\[
0 \longrightarrow F \overset{e}{\longrightarrow} I^0 \overset{d^0}{\longrightarrow} I^1 \overset{d^1}{\longrightarrow} I^2 \overset{d^2}{\longrightarrow} \cdots
\]

such that each \( I \) is injective.

(6) The resolution is not unique, but it is so in the homotopy category. Let us not worry about this; see [4] (part of the work to be done by the young (at heart) reader, is to dig out the relevant statement from the references given here!). Under suitable assumptions, usually automatically verified when working with algebraic varieties, the injective resolution can be chosen to be bounded, i.e., \( I^k = 0 \), for \( k \gg 0 \); see [13].

(7) Let \( f : X \to Y \) be a continuous map of topological spaces and \( F \) be a sheaf on \( X \). The *direct image sheaf* \( f_* F \) on \( Y \) is the sheaf

\[
Y \overset{\text{open}}{\supseteq} U \longmapsto F(f^{-1}(U)).
\]

You should check that the above definition yields a sheaf, not just a presheaf.

(8) A *complex of sheaves* \( K \) is a diagram of sheaves and maps of sheaves:

\[
\cdots \longrightarrow K^i \overset{d^i}{\longrightarrow} K^{i+1} \overset{d^{i+1}}{\longrightarrow} \cdots
\]

with \( d^2 = 0 \).

We have the *cohomology sheaves*

\[
\mathcal{H}^i(K) := \text{Ker} \ d^i / \text{Im} \ d^{i-1};
\]

recall that everything is first defined as a presheaf and you must take the associated sheaf; the only exception is the kernel: the kernel presheaf of a map of sheaves is automatically a sheaf (check this).
A *map of complexes* \( f : K \to L \) is a compatible system of maps \( f^i : K^i \to L^i \). Compatible means that the “obvious” diagrams are commutative.

There are the induced maps of sheaves \( H^i(f) : H^i(K) \to H^i(L) \) for all \( i \in \mathbb{Z} \).

A *quasi-isomorphism (qis)* \( f : K \to L \) is a map inducing isomorphisms on all cohomology sheaves.

The *translated complex* \( K[l] \) has \( (K[l])^i := K^{l+i} \) with the same differentials (up to the sign \((-1)^l\)).

Note that \( K[1] \) means moving the entries one step to the *left* and taking \(-d\).

An exact sequence of complexes is the “obvious” thing (make this explicit).

Later, I will mention *distinguished triangles*:

\[
K \to L \to M \to K[1].
\]

You can mentally replace this with a short exact sequence

\[
0 \to K \to L \to M \to 0
\]

and this turns out to be ok.

(9) The *direct image complex* \( Rf_*F \) associated with \((F, f)\) is “the” complex of sheaves on \( Y \)

\[
Rf_*F := f_*I,
\]

where \( F \to I \) is an injective resolution as above.

This is well defined up to unique isomorphism in the homotopy category. This is easy to verify (check it). For the basic definitions and a proof of this fact see [4] (note that there are no sheaves in this reference, the point is the use of the properties of injective objects).

(10) If \( C \) is a *bounded below* complex of sheaves on \( X \), i.e., with \( \mathcal{H}^i(K) = 0 \) for all \( i \ll 0 \) (and we assume this from now on), then \( C \) admits a *bounded below injective resolution*, i.e., a qis \( C \to I \), where each entry \( I^j \) is injective, and \( I \) is bounded below.

Again, this is well defined up to unique isomorphism in the homotopy category. \( Rf_* \) is a “derived functor.” However, this notion and the proof of this fact require plunging into the derived category, which we do not do in these notes. See [7].

(11) We can thus define the *derived direct image complex* of a bounded below complex of sheaves \( C \) on \( X \) by first choosing a bounded below injective resolution \( C \to I \) and then by setting

\[
Rf_*C := f_*I;
\]

this is a bounded below complex of sheaves on \( Y \).

(12) Define the *(hyper)cohomology groups of \((X \text{ with coefficients in}) \ C\)* as follows:

- Take the unique map \( c : X \to p \) (a point).
- Take the complex of global sections \( Rc_*C = c_*I = I(X) \).
- Set (the right-hand side, being the cohomology of a complex of abelian groups, is an abelian group)

\[
H^i(X, C) := H^i(I(X)).
\]
(13) **Exercise** (As mentioned earlier, from now on complexes are assumed to be bounded below.)

Use the homotopy statements to formulate and prove that these groups are well defined (typically, this means unique up to unique isomorphism; make this precise).

(14) The *direct image sheaves on* $Y$ *with respect to* $f$ *of the bounded below complex* $C$ *of sheaves on* $X$ *are*

$$R^if_*C := \mathcal{H}^i(Rf_*C) := \mathcal{H}^i(f_*I), \quad i \in \mathbb{Z}.$$  

These are well defined (see Exercise 13).

By boundedness, they are zero for $i \ll 0$ (depending on $C$).

If $C$ is a sheaf, then they are zero for $i < 0$.

(15) **Exercise**

Observe that if $C = F$ is a sheaf, then $R^0f_*F = f_*F$ (as defined earlier).

Prove that the sheaf $R^if_*C$ is the sheaf associated with the *presheaf*

$$U \mapsto H^i(f^{-1}(U), C). \quad (1)$$

(See [11].) This fact is very important in order to build an intuition for higher direct images. You should test it against the examples that come to your mind (including all those appearing in these notes).

Note that even if $C$ is a sheaf, then, in general, (1) above defines a presheaf. Give many examples of this fact.

Recall that while a presheaf and the associated sheaf can be very different, they have canonically isomorphic stalks! It follows that (1) can be used to try and determine the stalks of the higher direct image sheaves. Compute these stalks in many examples.

Remark that for every $y \in Y$ there is a natural map (it is called the base change map)

$$(R^if_*C)_y \rightarrow H^i(X_y, C|_{X_y}) \quad (2)$$

between the stalk of the direct image and the cohomology of the fiber $X_y := f^{-1}(y)$.

Give examples where this map is not an isomorphism/injective/surjective.

(16) **Exercise**

Given a sheaf $G$ on $Y$, the *pull-back* $f^*G$ is the sheaf associated with the *presheaf*

(the limit below is the direct limit over the directed set of open sets $W \subseteq Y$ containing $f(U)$):

$$U \mapsto \lim_{W \supseteq f(U)} G(W).$$

This presheaf is not a sheaf even when $f : X \rightarrow Y$ is the obvious map from a set with two elements to a set with one element (both with the discrete topology) and $G$ is constant.

The pull-back defined above should not be confused with the pull-back of a quasi-coherent sheaf with respect to a map of algebraic varieties. (This is discussed very well in [11]).

In [8], you will find a very beautiful discussion of the étale space associated with a presheaf, and hence with a sheaf. This is all done in the general context of sheaves of sets; it is very worthwhile to study sheaves of sets, i.e., sheaves without
the additional algebraic structures (sometimes the additional structure may hinder some of the basic principles).

The first, important surprise is that every map of sets yields a sheaf on the target: the sheaf of the local section of the map.

For example, a local homeomorphism, which can fail to be surjective (by way of contrast, the étale space of a sheaf of abelian groups on a space always surjects onto the space due to the obvious fact that we always have the zero section!) yields a sheaf on the target whose étale space is canonically isomorphic with the domain.

Ask yourself: can I view a 2:1 covering space as a sheaf? Yes, see above. Can I view the same covering as a sheaf of abelian groups? No, unless the covering is trivial (a sheaf of abelian groups always has the zero section!).

Whereas the definition of direct image \( f_* F \) is easy, the étale space of \( f_* F \) may bear very little resemblance to the one of \( F \). On the other hand, while the definition of \( f^* G \) is a bit more complicated, the étale space \( |f^* G| \) of \( f^* G \) is canonically isomorphic with the fiber product over \( Y \) of \( X \) with the étale space \( |G| \) of \( G \):

\[
|f^* G| = |G| \times_Y X.
\]

(17) It is a fact that if \( I \) on \( X \) is injective, then \( f_* I \) on \( Y \) is injective.

A nice proof of this fact uses the fact that the pull-back functor \( f^* \) on sheaves is the left adjoint to \( f_* \), i.e., (cf. [7])

\[
\text{Hom}(f^* F, G) = \text{Hom}(F, f_* G).
\]

(18) **Exercise**

Use the adjunction property in (17) to prove that \( I \) injective implies \( f_* I \) injective.

Observe that the converse does not hold.

Observe that if \( J \) is injective on \( Y \), then, in general, the pull-back \( f^* J \) is not injective on \( X \).

Find classes of maps \( f : X \to Y \) for which \( J \) injective on \( Y \) implies \( f^* J \) injective on \( X \).

(19) **Exercise**

Use that \( f_* \) preserves injectives to deduce that

\[
H^i(X, C) = H^i(Y, Rf_* C).
\]

(20) It is a fact that on a good space \( X \) the cohomology defined above with coefficients in the constant sheaf \( \mathbb{Z}_X \) is the same as the one defined by using singular and Cech cohomologies (see [14, 15]):

\[
H^i(X, \mathbb{Z}_X) = H^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}).
\]

(21) **Exercise** (For a different perspective on what follows, see Lecture 1.)

\( (a) \) Let \( j : \mathbb{R}^n - \{0\} \to \mathbb{R}^n \) be the open immersion. Determine the sheaves \( R^q j_* \mathbb{Z} \).

\( (b) \) (This is the very first occurrence of the decomposition theorem (DT) in these notes! Let \( X = Y = \mathbb{C}, X^* = Y^* = \mathbb{C}^* \), let \( f : \mathbb{C} \to \mathbb{C} \) be the holomorphic map \( z \mapsto z^2 \), and let \( g : \mathbb{C}^* \to \mathbb{C}^* \) be the restriction of \( f \) to \( \mathbb{C}^* := \mathbb{C} - \{0\} \).
Show that $R^i f_* \mathbb{Z}_X = 0$ for all $i > 0$. Ditto for $g$.

Show that there is a split short exact sequence of sheaves of vector spaces (if you use $\mathbb{Z}$-coefficients, there is no splitting)

$$0 \longrightarrow Q_Y \longrightarrow f_* Q_X \longrightarrow Q \longrightarrow 0$$

and determine the stalks of $Q$.

Ditto for $g$ and observe that what you obtain for $g$ is the restriction to the open set $Y^*$ of what you obtain for $f$ on $Y$. (This is a general fact that you may find in the literature as “the base change theorem holds for an open immersion.”)

The short exact sequence above, when restricted to $Y^*$, is one of locally constant sheaves (recall that a locally constant sheaf of abelian groups—you can guess the definition in the case of sheaves of sets with stalk a fixed set—with stalk a group $L$ is a sheaf that is locally isomorphic to the constant sheaf with stalk $L$) and the restriction $Q^*$ of $Q$ to $Y^*$ is the locally constant sheaf with stalk $\mathbb{Q}$ at a point $y \in Y^*$ endowed with the automorphism multiplication by $-1$ (explain what this must mean).

The locally constant sheaf $Q^*$ on $Y^*$ (and thus on the unit circle) is a good example of a nonconstant sheaf with stalk $\mathbb{Q}$ or $\mathbb{Z}$. Another good example is the sheaf of orientations of a nonorientable manifold: the stalk is a set given by two points; this is not a sheaf of groups! If the manifold is orientable, then the choice of an orientation turns the sheaf of sets into a locally constant sheaf of abelian groups with stalk $\mathbb{Z}/2\mathbb{Z}$.

(c) Show that on a good connected space $X$, a locally constant sheaf $L$ (we often call such an object a local system) yields a representation of the fundamental group $\pi_1(X, x)$ in the group $A(L_x)$ of automorphisms of the stalk $L_x$ at a pre-fixed point $x \in X$, and vice versa. (Hint: consider the quotient $(\tilde{X} \times L_x)/\pi_1(X, x)$ under a suitable action; here $\tilde{X}$ is a universal covering of $X$.)

(d) Use the principle of analytic continuation and the monodromy theorem (cf. [12]) to prove that every local system on a simply connected space is constant (trivial representation).

(e) Give an example of a local system that is not semisimple.

(The relevant definitions are simple := irreducible := no nontrivial subobject; semisimple := direct sum of simples.)

(Hint: consider, for example, the standard $2 \times 2$ unipotent matrix.)

The matrix in the hint given above is the one of the Picard–Lefschetz transformation associated with the degeneration of a one-parameter family of elliptic plane cubic curves to a rational cubic curve with a node; in other words it is the monodromy of the associated nontrivial! fiber bundle over a punctured disk with fiber $S^1 \times S^1$.

(f) Given a fiber bundle, e.g., a smooth proper map (see the Ehresmann fibration lemma, e.g., in [16]) $f : X \to Y$, with fiber $F_y$, prove that the direct image sheaf is locally constant with typical stalk

$$(R^i f_* \mathbb{Z}_X)_y = H^i(X_y, \mathbb{Z}).$$
(g) Show that the Hopf bundle \( h : S^3 \rightarrow S^2 \), with fiber \( S^1 \), is not (isomorphic to) a trivial bundle. Though the bundle is not trivial, the local systems \( R^i h_* \mathbb{Z}_{S^3} \) are trivial on the simply connected \( S^2 \).

Do the same for \( k : S^1 \times S^3 \rightarrow S^2 \). Verify that you can turn the above into a proper holomorphic submersion of compact complex manifolds \( k : S \rightarrow \mathbb{C}P^1 \) (see the Hopf surface in [1]).

Show that the Deligne theorem (see Lecture 1) on the degeneration for smooth projective maps cannot hold for the Hopf map above. Deduce that this is an example of a map in complex geometry for which the decomposition theorem (DT) does not hold.

(h) Show that if a map \( f \) is proper and with finite fibers (e.g., a finite topological covering, a branched covering, the normalization of a complex space, for example of a curve, the embedding of a closed subvariety, etc.), then \( R^if_*F = 0 \) for every \( i > 0 \) and every sheaf \( F \).

Give explicit examples of finite maps and compute \( f_*\mathbb{Z} \) in those examples.

(22) Some examples of maps \( f : X \rightarrow Y \) to play with (some have already appeared above)

(a) \( f : (0, 1) \rightarrow [0, 1] \):
\[
f_*\mathbb{Z}_X = Rf_*\mathbb{Z}_X = \mathbb{Z}_Y;
\]
the base change map (2) is zero at \( 0 \in X \).

(b) \( f : \Delta^* \rightarrow \Delta \) (immersion of punctured unit disk into the unit disk in \( \mathbb{C} \)):
\[
f_*\mathbb{Z}_X = \mathbb{Z}_Y;
R^1f_*\mathbb{Z}_X = \mathbb{Z}_o = H^1(X, \mathbb{Z}_X), o \in \Delta \) the puncture;
there is a nonsplit exact sequence
\[
0 \rightarrow \mathbb{Z}_Y \rightarrow Rf_*\mathbb{Z}_X \rightarrow \mathbb{Z}_o[-1] \rightarrow 0.
\]

(c) \( f : \Delta \rightarrow \Delta, z \mapsto z^2 \):
\[
R^0f_* = f_*; R^2f_* = 0; f_* = Rf_*;
\]
the natural short exact sequence
\[
0 \rightarrow RY \rightarrow f_*RX \rightarrow Q(R) \rightarrow 0
\]
does not split for \( R = \mathbb{Z} \), but it splits if 2 is invertible in \( R \).

(d) \( f : \Delta^* \rightarrow \Delta^*, z \mapsto z^2 \):
\[
R^0f_* = f_*; R^2f_* = 0; f_* = Rf_*;
\]
the natural short exact sequence
\[
0 \rightarrow RY \rightarrow f_*RX \rightarrow Q(R) \rightarrow 0
\]
does not split for \( R = \mathbb{Z} \), but it splits if 2 is invertible in \( R \);
the stalk \( Q(R)_p \) at \( p := 1/4 \in \Delta^* \) (the target) is a rank-1 free \( R \)-module generated by the equivalence class \([1, -1]\) in \( R^2/R = (f_*RX)_p/(RY)_p \), modulo the equivalence relation \((a, b) \sim (a', b')\) if and only if \((a - a' = b - b')\); here \((a, b)\) is viewed as a constant \( R \)-valued function in the preimage of a small connected neighborhood of \( p \), this preimage being the disconnected union of two small connected neighborhoods of \( \pm 1/2 \in \Delta^* \) (the domain);
if we circuit once (e.g., counterclockwise) the origin of the target $\Delta^*$ starting at $1/4$ and returning to it, then the pair $(1, -1)$ is turned into the pair $(-1, 1)$; this is the monodromy representation on the stalk $\mathcal{Q}(\mathcal{R})_p$; we see that in order to split $\mathcal{R}_Y \to f_*\mathcal{R}_X$, or equivalently, $f_*\mathcal{R}_X \to \mathcal{Q}(\mathcal{R})$, we need to be able to divide by $2$.

In this example and in the previous one, the conclusion of the decomposition theorem (DT) (see Section 2.1) holds, provided we use coefficients in a field of characteristic $0$.

The DT already fails for integer and for $\mathbb{Z}/2\mathbb{Z}$ coefficients in these simple examples.

(e) $f : \mathbb{R} \to Y := \mathbb{R}/\sim$, where $Y$ is obtained by identifying $\pm 1 \in \mathbb{R}$ to one point $o$ (this can be visualized as the real curve $y^2 = x^2 - x^3$ inside $\mathbb{R}^2$, with $o$ the origin):

$f_*\mathcal{Z}_X = Rf_*\mathcal{Z}_X$; $(f_*\mathcal{Z}_X)_0 \simeq \mathbb{Z}^2$;

there is the natural nonsplit short exact sequence

$$0 \to \mathbb{Z}_Y \to f_*\mathcal{Z}_X \to \mathcal{Z}_o \to 0.$$  

Let $j : U := Y - \{o\} \to Y$ be the open immersion $j_*\mathcal{Z} = Rj_*\mathcal{Z}$; $(j_*\mathcal{Z})_0 \simeq \mathbb{Z}^4$;

there is the natural nonsplit short exact sequence

$$0 \to \mathbb{Z}_Y \to j_*\mathcal{Z}_U \to \mathcal{Z}_o^3 \to 0;$$

note that there is a natural nonsplit short exact sequence

$$0 \to f_*\mathcal{Z}_X \to j_*\mathcal{Z}_U \to \mathcal{Z}_o^2 \to 0. \quad (3)$$

(f) $f : \mathbb{C} \to Y := \mathbb{C}/\sim$, where $Y$ is obtained by identifying $\pm 1 \in \mathbb{C}$ to a point $o$ and let $j : U = Y - \{o\} \to Y$ (this can be visualized as the complex curve $y^2 = x^2 - x^3$ inside $\mathbb{C}^2$, with $o$ the origin):

this is analogous to the previous example, but it has an entirely different flavor:

$$Rf_*\mathcal{Z}_X = f_*\mathcal{Z} = j_*\mathcal{Z}_U. \quad (4)$$

This is another example where the DT holds (in fact here it holds with $\mathbb{Z}$-coefficients).

(g) $f : S^3 \to S^3$, the famous Hopf $S^1$-bundle; it is a map of real algebraic varieties for which the conclusion of Deligne’s theorem Section 2.1(3) does not hold: we have the trivial local systems

$$R^0f_*\mathcal{Z}_X = R^1f_*\mathcal{Z}_X = \mathcal{Z}_Y, \quad R^i f_*\mathcal{Z}_X = 0 \forall i \geq 2$$

and a nonsplit (even if we replace $\mathbb{Z}$ with $\mathbb{Q}$) short exact sequence

$$0 \to \mathbb{Z}_Y \to Rf_*\mathcal{Z}_X \to \mathcal{Z}_Y[-1] \to 0$$

(n.b. if it did split, then the first Betti number $0 = b_1(S^3) = 1$).

(h) Consider the action of the group $\mathbb{Z}$ on $X' := \mathbb{C}^2 - \{(0, 0)\}$ given by $(z, w) \mapsto (2z, 2w)$.

There are no fixed points and the (punctured complex lines through the origin, $w = mz$, are preserved).
One shows that $X := X'/\mathbb{Z}$ is a compact complex surface (a Hopf surface, see [1]) endowed with a proper holomorphic submersion (i.e., with differential everywhere of maximal rank) $f : X \to Y = \mathbb{CP}^1$.

After dividing by the $\mathbb{Z}$-action, each line $w = mz$ turns into a compact Riemann surface of genus 1, which in turn is the fiber $f^{-1}(m)$. Of course, $m = \infty$ corresponds to the line $z = 0$.

If we take the unit 3-sphere in $\mathbb{C}^2$, then, $f|_{S^3} : S^3 \to \mathbb{CP}^1 = S^2$ is the Hopf bundle of the previous example.

There is a natural filtration of $Rf_*\mathbb{Z}_X$:

$$0 = K^{-1} \subseteq K^0 \subseteq K^1 \subseteq K^2 = Rf_*\mathbb{Z}_X$$

into subcomplexes with

$$K^0/K^{-1} = \mathbb{Z}_Y, \quad K^1/K^0 = \mathbb{Z}_Y^2, \quad K^2/K^1 = \mathbb{Z}_Y.$$

As in the previous example, we cannot have a splitting

$$Rf_*\mathbb{Z}_Y \simeq \mathbb{Z}_Y \oplus \mathbb{Z}_Y^2[-1] \oplus \mathbb{Z}_Y$$

(not even replacing $\mathbb{Z}$ with $\mathbb{Q}$) in view of the fact that this would imply that $1 = b_1(X) = 2$.

This is an example of a proper holomorphic submersion, where the fibers and the target are projective varieties, but for which the conclusion of Deligne’s theorem Section 2.1(3) does not hold.

(i) Let $C \subseteq \mathbb{CP}^2$ be a nonsingular complex algebraic curve (it is also a compact Riemann surface), let $U$ be the universal holomorphic line bundle on $\mathbb{CP}^2$ (the fiber at a point is naturally the complex line parametrized by the point), let $X$ be the complex surface total space of the line bundle $U|_C$, let $Y \subseteq \mathbb{C}^3$ be the affine cone over $C$; it is a singular surface with an isolated point at the vertex (origin) $o \in Y$.

The blow-up of $Y$ at the vertex coincides with $X$ (check this).

Let $f : X \to Y$ be the natural map (it contracts the zero section of $X$).

Let $j : U := Y - \{o\} \to Y$ be the open immersion.

We have the first (for us) example of the DT for a nonfinite map (for details see [6]):

$$Rf_*\mathbb{Q} \simeq \tau_{\leq 1}Rj_*\mathbb{Q}_U \oplus \mathbb{Q}_o[-2]$$

(given a complex $K$, its standard truncated subcomplex $\tau_{\leq i}K$ is the complex $L$ with $L^j = K^j$ for every $j < i$, $L^i := \text{Ker} d^i_K$, $K^j = 0$ for every $j > i$; its most important property is that it has the same cohomology sheaves $\mathcal{H}^j(L)$ as $K$ for every $j \leq i$ and $\mathcal{H}^j(L) = 0$ for every $j > i$).

The most important aspect of the splitting (5) is that the right-hand side does not contain the symbol $f$ denoting the map! This is in striking similarity with (4), another example of DT.

The relevant direct image sheaves for $f$ are

$$R^0f_*\mathbb{Q}_X = \mathbb{Q}_Y, \quad R^1f_*\mathbb{Q}_X = \mathbb{Q}_o^{2g}, \quad R^2f_*\mathbb{Q}_X = \mathbb{Q}_o.$$
(the map is proper, the proper base change theorem holds, see [12], or [13], so that the base change map (2) is an iso).

The relevant direct image sheaves for \( j \) are

\[
R_0 j_* Q_U = Q_Y, \quad R^1 j_* Q_U = Q_0^{2g}, \quad R^2 j_* Q_U = Q_0^{2g}, \quad R^3 j_* Q_U = Q_0;
\]

this requires a fair amount of work (as a by-product, you will appreciate the importance of the base change theorem for proper maps, which you cannot use here!):

- \( j_* Q_U = Q_Y \) is because \( U \) is connected;
- the computation on the higher \( R^i j_* Q_U \) boils down to determining the groups \( H^i(U, \mathbb{Q}_U) \) (see (1));
- on the other hand, \( U \to C \) is the \( \mathbb{C}^* \)-bundle of the line bundle \( U|_C \) and this calculation is carried out in [3] (in fact, it is carried out for the associated oriented \( S^1 \)-bundle) (be warned that [3] uses the Leray spectral sequence: this is a perfect chance to learn about it without being overwhelmed by the indices and by being shown very clearly how everything works; an alternative without spectral sequences is, for example, any textbook in algebraic topology covering the Wang sequence (i.e., the long exact sequence of an oriented \( S^1 \)-bundle; by the way, it can be recovered using the Leray spectral sequence!).

Note that if we replace \( \mathbb{Q} \) with \( \mathbb{Z} \) we lose the splitting (5) due to torsion phenomena.

Note that there is a nonsplit short exact sequence

\[
0 \to Q_Y \to \tau_{\leq 1} Rj_* Q_U \to Q_0^{2g} \to 0.
\]

A direct proof that this splitting cannot occur is a bit technical (omitted).

For us it is important to note that \( \tau_{\leq 1} Rj_* Q_U \) is the intersection complex of \( I_Y \) of \( Y \) (see Section 1.2) and intersection complexes \( I_Y \) never split nontrivially into a direct sum of complexes.

\( f : X = \mathbb{C} \times C \to Y \), where \( C \) is a compact Riemann surface as in the previous example, where \( Y \) is obtained from \( X \) by identifying \( \Gamma := \{0\} \times C \) to a point \( o \in Y \) and leaving the rest of \( X \) unchanged. Let \( U := Y - \{o\} = X - \Gamma \).

Note that \( \Gamma \) defines the trivial class in \( H^2(X, \mathbb{Z}) \), because you can send it to infinity!, i.e., view it as the boundary of \( \mathbb{R}_{\geq 0} \times C \).

The actual generator for \( H^2(X, \mathbb{Z}) = H^2(C, \mathbb{Z}) \) is given by the class of a complex line \( \mathbb{C} \times c, c \in C \).

You should contrast what is above with the previous example given by the total space of a line bundle with negative degree. Of course, here \( X \) is the total space of the trivial line bundle on \( C \).

The map \( f \) is not algebraic, not even holomorphic, in fact \( Y \) is not a complex space.

The DT cannot hold for \( f \): the relevant cohomology sheaves for \( Rf_* \) are

\[
f_* Q_X = Q_Y, \quad R^1 f_* Q_X = Q_0^{2g}, \quad R^2 f_* Q = Q_0;
\]

the relevant cohomology sheaves for \( \tau_{\leq 1} Rj_* Q_U \) are

\[
j_* Q_U = Q_Y, \quad R^1 j_* Q_U = Q_0^{2g+1};
\]
it follows that \((5)\), and hence the DT, do not hold in this case.
For more details and a discussion relating the first Chern classes of the trivial
and of the negative line bundle to the DT see \([6]\), which also explains (see also
\([5]\)) how to use Borel–Moore homology cycles to describe cohomology, as we
have suggested above.

1.2. The Intersection Cohomology Complex. We shall limit ourselves to define and
“calculate” the intersection complex \(I_X\) of a variety of dimension \(d\) with one isolated
singularity:

\[
Y = Y_{\text{reg}} \bigsqcup Y_{\text{sing}}, \quad U := Y_{\text{reg}}, \quad Y_{\text{sing}} = \{p\},
\]

\[U \xrightarrow{j} Y \xrightarrow{i} p.\]

This is done for ease of exposition only. Of course, the intersection cohomology complex
\(I_Y\), and its variants \(I_Y(L)\) with twisted coefficients, can be defined for any variety \(Y\),
regardless of the singularities.

(1) Recall that given a complex \(K\) the \(a\)th truncated complex \(\tau_{\leq a} K\) is the subcomplex
\(C\) with the following entries:

\[
C^b = K^b \quad \forall b < a, \quad C^a = \text{Ker} \ a^a, \quad C^b = 0 \quad \forall b > a.
\]

The single most important property is that

\[
H^b(\tau_{\leq a} K) = H^b(K) \quad \forall b \leq a, \quad \text{zero otherwise}.
\]

(2) Let \(Y\) be as above. Define the intersection cohomology complex (with coefficients
in \(\mathbb{Z}\), for example) as follows:

\[
I_Y := \tau_{\leq d-1} R j_* \mathbb{Z}_U.
\]

(3) Toy model

What follows is related to Section 1.1, Exercise 2222i.

Let \(Y \subseteq \mathbb{C}^3\) be the affine cone over an elliptic curve \(E \subseteq \mathbb{CP}^2\).
\(R^0 j_* \mathbb{Z}_U = \mathbb{Z}_Y\) (recall that we always have \(R^0 f_* = f_*\)).
As to the others we observe that \(U\) is the \(\mathbb{C}^*\)-bundle of the hyperplane line bundle
\(H\) on \(E\), i.e., the one induced by the hyperplane bundle on \(\mathbb{CP}^2\). By choosing a
metric, we get the unit sphere (here \(S^1\)) bundle \(U'\) over \(E\). Note that \(U'\) and \(U\)
have the same homotopy type. The bundle \(U' \to E\) is automatically an oriented
\(S^1\)-bundle. The associated Euler class \(e \in H^2(E, \mathbb{Z})\) is the first Chern class \(c_1(H)\).

(4) Exercise

(You will find all you need in \([3]\).) Use the spectral sequence for this oriented
bundle (here it is just the Wang sequence) to compute the groups

\[
H^i(U', \mathbb{Z}) = H^i(U, \mathbb{Z}).
\]

Answer: (Caution: the answer below is for \(\mathbb{Q}\)-coefficients only!; work this situation
out in the case of \(\mathbb{Z}\)-coefficients and keep track of the torsion.)

\[
H^0(U) = H^0(E), \quad H^1(U) = H^1(E), \quad H^2(U) = H^1(E), \quad H^3(U) = H^2(E).
\]
Deduce that, with $\mathbb{Q}$-coefficients (work out the $\mathbb{Z}$ case as well), we have that $I_Y$ has only two nonzero cohomology sheaves

$$\mathcal{H}^0(I_Y) = \mathbb{Q}_Y, \quad \mathcal{H}^1(I_Y) = H^1(E)_p \text{ (skyscraper at } p).$$

(5) Exercise

Compute $I_Y$ for $Y = \mathbb{C}^d$, with $p$ the origin.

$Answer$: $I_Y = \mathbb{Q}_Y$ (here $\mathbb{Z}$-coefficients are ok).

(6) The above result is general:

- if $Y$ is nonsingular, then $I_Y = \mathbb{Q}_Y$ ($\mathbb{Z}$ ok);
- if $Y$ is the quotient of a nonsingular variety by a finite group action, then $I_Y = \mathbb{Q}_Y$ ($\mathbb{Z}$ coefficients, KO!).

(7) Let $L$ be a local system on $U$. Define

$$I_Y(L) := \tau_{\leq d-1} Rj_* L.$$

Note that (this is a general fact)

$$\mathcal{H}^0(I_Y(L)) = j_* L.$$

(8) Useful notation: $j! L$ is the sheaf on $Y$ which agrees with $L$ on $U$ and has stalk zero at $p$.

(9) Exercise

[(a)]

(a) Let $C$ be a singular curve. Compute $I_C$.

$Answer$: let $f : \hat{C} \to C$ be the normalization. Then $I_C = f_* \mathbb{Z}_{\hat{C}}$.

(b) Let things be as in Section 1.1, Exercise 2121b. Let $L = (f_* \mathbb{Z}_X)|_{Y^*}$ and $M := \mathbb{Q}|_{Y^*}$. Compute

$$I_Y(L), \quad I_Y(M).$$

(c) Let $U$ be as in the toy model Exercise 3. Determine $\pi_1(U)$. Classify local systems of low ranks on $U$. Find some of their $I_Y$'s.

(d) Let $f : C \to D$ be a branched cover of nonsingular curves. Let $f^\circ : C^\circ \to D^\circ$ be the corresponding topological covering space, obtained by removing the branching points and their preimages.

Prove that $L := f^\circ_* \mathbb{Q}_C$ is semisimple ($\mathbb{Z}$-coefficients is KO!, even for the identity: $\mathbb{Z}$ is not a simple $\mathbb{Z}$-module!).

Determine $I_D(L)$ and describe its stalks. (Try the case when $C$ is replaced by a surface, threefold, etc.)

1.3. Verdier Duality. For ease of exposition, we work with rational coefficients.

(1) Let $M^m$ be an oriented manifold. We have Poincaré duality:

$$H^i(M, \mathbb{Q}) \cong H^{m-i}_c(M, \mathbb{Q}).$$

(6) Exercise

Find compact and noncompact examples of the failure of Poincaré duality for singular complex varieties.

(The easiest way to do this is to find nonmatching Betti numbers.)
(3) Verdier duality (which we do not define here; see [7]) is the culmination of a construction that achieves the following generalization of Poincaré duality to the case of complexes of sheaves on locally compact spaces.

Given a complex of sheaves $K$ on $Y$, its Verdier dual $K^*$ is a canonically defined complex on $Y$ such that for every open $U \subseteq Y$,

$$H^i(U, K^*) = H^{-i}_c(U, K)^*.$$  \hfill (7)

Note that $H^i_c(Y, K)$ is defined the same way as $H^i(Y, K)$, except that we take global sections with compact supports.

The formation of $K^*$ is contravariantly functorial in $K$:

$$K \mapsto L, \quad K^* \leftarrow L^*,$$

and satisfies

$$K^{**} = K, \quad (K[l])^* = K^*[-l].$$

(4) Exercise

Recall the definition of the translation functor $[m]$ on complexes (see Section 1.1) and those of $H^i$ and $H^i_c$ and show directly that

$$H^i(Y, K[l]) = H^{i+l}(Y, K), \quad H^i_c(Y, K[l]) = H^{i+l}_c(Y, K).$$

(5) It is a fact that, for the oriented manifold $M^m$, the chosen orientation determines an isomorphism

$$\mathbb{Q}_Y^* = \mathbb{Q}_Y[m]$$

so that we get Poincaré duality. Verify this!; that is, verify that $(7) \implies (6)$

(do not take it for granted, you will see what duality means over a point!).

If $M$ is not oriented, then you get something else. See [3] (look for “densities”), see [13] (look for “sheaf of orientations”), see [12] (look for “Borel–Moore chains”), and the resulting complex of sheaves (see also [2]).

(6) One of the most important properties of $I_Y$ is its self-duality, which we express as follows (the translation by $d$ is for notational convenience): first set

$$\mathcal{I}_Y := I_Y[d]$$

(we have translated the complex $I_Y$, which had nonzero cohomology sheaves only in degrees $[0, d-1]$, to the left by $d$ units, so that the corresponding interval is now $[-d, -1]$); then we have that

$$\mathcal{I}_Y^* = \mathcal{I}_Y.$$

(7) Exercise

Use the toy model to verify that the equality holds (in that case) at the level of cohomology sheaves by verifying that (here $V$ is a “typical” neighborhood of $p$)

$$H^i(\mathcal{I}_V) = H^i(V, \mathcal{I}_V) = H^{-i}_c(V, \mathcal{I}_V)^*.$$  

(To do this, you will need to compute $H^i(V)[l]$ as you did $H^i(U)$; be careful though about using homotopy types and $H_c$!) You will find the following distinguished
triangle useful—recall we can view them as short exact sequences, and as such, yielding a long exact sequence of cohomology groups, with or without supports:

$$\mathcal{H}^0(I_Y) \rightarrow I_Y \rightarrow \mathcal{H}^1(I_Y)[-1] \rightarrow \cdots;$$

you will also find useful the long exact sequence

$$\cdots \rightarrow H^a_c(U) \rightarrow H^a_c(Y) \rightarrow H^a_c(p) \rightarrow H^{a+1}_c(U) \rightarrow \cdots.\tag{8}$$

(8) Define the intersection cohomology groups of $Y$ as

$$\text{IH}^i(Y) = H^i(Y, I_Y), \quad \text{IH}^i_c(Y) = H^i_c(Y, I_Y).$$

The original definition is more geometric and involves chains and boundaries, like in the early days of homology; see [2].

(9) Since $I^* = I_Y$, we get

$$H^i(Y, I_Y) = H^{-i}_c(Y, I_Y)^*.\tag{9}$$

Using $I_Y = I_Y[d]$, Verdier duality implies that

$$H^i(Y, I_Y) = H^{2n-i}_c(Y, I_Y)^*,$$

and we immediately deduce Poincaré duality for intersection cohomology groups on an arbitrarily singular complex algebraic variety (or complex space):

$$\text{IH}^i(Y, I_Y) = \text{IH}^{2d-i}_c(Y, I_Y)^*.\tag{10}$$

(10) Variant for twisted coefficients.

If $Y^o \subseteq Y_{\text{reg}} \subseteq Y$, $L$ is a local system on a nonempty open set $Y^o$ and $L^*$ is the dual local system, then we have $I_Y(L)$, its translated $I_Y(L)$, and we have a canonical isomorphism

$$I_Y(L)^* = I_Y(L^*).$$

There is the corresponding duality statement for the groups $\text{IH}^i(Y, I_Y(L))$, etc.:

$$\text{IH}^i(Y, I_Y(L^*)) = \text{IH}^{2d-i}_c(Y, I_Y(L))^*.\tag{11}$$

(11) Exercise

Define the dual local system $L^*$ of a local system $L$ as the sheaf of germs of sheaf maps $L \rightarrow \mathcal{Q}_Y$.

(a) Show that it is a local system and that there is a pairing (map of sheaves)

$$L \otimes_{\mathcal{Q}_Y} L^* \rightarrow \mathcal{Q}_Y,$$

inducing identifications

$$(L_y)^* = (L^*)_y.$$  

(Recall that the tensor product is defined by taking the sheaf associated with the presheaf tensor product (because of local constancy of all the players, in this case the presheaf is a sheaf): $U \mapsto L(U) \otimes_{\mathcal{Q}_U(U)} L^*(U)$).

(b) If $L$ is given by the representation $r : \pi_1(Y, y) \rightarrow A(L_y)$ (see Section 1.1, Exercise 2121c), find an expression for a representation associated with $L^*$. (Hint: inverse–transpose.)
(12) Verdier duality and $Rf_*$ for a proper map.
   It is a fact that if $f$ is proper, then
   $$(Rf_* C)^* = Rf_*(C^*).$$
   We apply this to $I_Y(L)^* = I_Y(L^*)$ and get
   $$(Rf_* I_Y(L))^* = Rf_* I_Y(L^*).$$
   In particular, $Rf_* I_Y$ is self-dual.

2. Lecture 5

2.1. The Decomposition Theorem (DT).
(1) Let $f : X \to Y$ be a proper map of algebraic varieties and $L$ be a semisimple
   (= direct sum of simples; simple = no nontrivial subobject) local system with
   $\mathbb{Q}$-coefficients (most of what follows fails with coefficients not in a field of characteristic 0)
   on a Zariski dense open set $X^o \subseteq X_{\text{reg}} \subseteq X$.
   Examples include
   • $X$ is nonsingular, $L = \mathbb{Q}_X$, then $I_X(L) = I_X = \mathbb{Q}_X$;
   • $X$ is singular, $L = \mathbb{Q}_{X_{\text{reg}}}$, then $I_X(L) = I_X$.
(2) Decomposition theorem
   The following statement is the deepest known fact concerning the homology of algebraic varieties.
   There is a splitting in the derived category of sheaves on $Y$:
   $$Rf_* I_X(L) \simeq \bigoplus_{b \in B} I_{Z_b}(L_b)[l_b],$$
   where
   • $B$ is a finite set of indices;
   • $Z_b \subseteq Y$ is a collection of locally closed nonsingular subvarieties;
   • $L_b$ is a semisimple local system on $Z_b$; and
   • $l_b \in \mathbb{Z}$.

   What does it mean to have a splitting in the derived category?
   Well, I did not define what a derived category is (and I will not). Still, we can deduce immediately from (8) that the intersection cohomology groups of the domain split into a direct sum of intersection cohomology groups on the target.
(3) The case where we take $I_X = I_X(L)$ is already important.
   Even if $X$ and $Y$ are smooth, we must deal with $I_Z$’s on $Y$, i.e., we cannot have a direct sum of shifted sheaves for example.

   Deligne’s theorem (1968), including the semisimplicity statement (1972) for proper smooth maps of smooth varieties (see Lectures 1 and 2) is a special case and it reads as follows:
   $$Rf_* \mathbb{Q}_X \simeq \bigoplus_{i \geq 0} R^i f_* \mathbb{Q}_X[-i], \quad I_Y(R^i f_* \mathbb{Q}_X) = R^i f_* \mathbb{Q}_X.$$
(4) **Exercise**

By using the self-duality of $I_Y$, the rule $(K[l])^* = K^*[-l]$, the DT above, and the fact that $I_T = I_T[\dim T]$, show that (8) can be rewritten in the following more symmetric form, where $r$ is a uniquely determined nonnegative integer:

$$Rf_*I_X \simeq \bigoplus_{i=-r}^r P^i[-i],$$

where each $P^i$ is a direct sum of some of the $I_Y$ appearing above, without translations $[-]!$, and

$$(P^i)^* = P^{-i} \quad \forall i \in \mathbb{Z}.$$  

Try this first in the case of smooth proper maps, where $Rf_*Q_X = \oplus R^i f_*Q_X[-i]$.

This may help to get used to the change of indexing scheme as you go from $I_Y$ to $I_Y = I_Y[\dim]$. 

(5) **Exercise**

(a) Go back to all the examples we met earlier and determine, in the cases where the DT is applicable, the summands appearing on the left of (8).

(b) (See [6].) Let $f : X \to C$ be a proper algebraic map with connected fibers, $X$ a nonsingular algebraic surface, $C$ a nonsingular algebraic curve.

Let $C^o$ be the set of regular values, $\Sigma := C \setminus C^o$ (it is a fact that it is finite).

Let $f^o : X^o \to C^o$ and $j : C^o \to C$ be the obvious maps. Deligne’s theorem applies to $f^o$ and is a statement on $C^o$; show that it takes the following form:

$$Rf^o_*Q_{X^o} \simeq Q_{C^o} \oplus R^i f^o_*Q_{X^o}[-1] \oplus Q_{C^o}[-2].$$

Show that the DT on $C$ must take the form (let $R^1 := R^1 f^o_*Q_{X^o}$)

$$Rf_*Q_X \simeq Q_C \oplus j_* R^1[-1] \oplus Q_C[-2] \oplus V_{\Sigma}[-2],$$

where $V_{\Sigma}$ is the skyscraper sheaf on the finite set $\Sigma$ with stalk at each $\sigma \in \Sigma$ a vector space $V_{\sigma}$ of rank equal to the number of irreducible components of $f^{-1}(\sigma)$ minus 1.

Find a more canonical description of $V_{\sigma}$ as a quotient of $H^2(f^{-1}(\sigma))$.

Note that this splitting contains quite a lot of information. Extract it:

- The only feature of $f^{-1}(\sigma)$ that contributes to $H^*(X)$ is its number of irreducible components; if this is 1, there is no contribution, no matter how singular (including multiplicities) the fiber is.

- Let $c \in C$, let $\Delta$ be a small disk around $c$, let $\eta \in \Delta^*$ be a regular value. We have the bundle $f^* : X_{\Delta^*} \to \Delta^*$ with typical fiber $X_{\eta} := f^{-1}(\eta)$.

We have the (local) monodromy for this bundle; i.e., $R^i$ is a local system; i.e., $\pi_1(\Delta^*) = \mathbb{Z}$ acts on $H^i(X_{\eta})$.

Denote by $R^1 \pi_1 \subseteq R^1_{\eta}$ the invariants of this (local) action.

Show the following general fact: for local systems $L$ on a good connected space $Z$ and for a point $z \in Z$ we have that the invariants of the local system $L_z^{\pi_1(Z)} = H^0(Z, L)$.
Let \( X_c := f^{-1}(c) \) be the central fiber; there are the natural restriction maps
\[
H^1(X_\eta) \supseteq H^1(X_\eta) \overset{f_{\ast}}{\to} H^1(f^{-1}(\Delta)) \xrightarrow{\sim} H^1(X_c).
\]
Use the DT above to deduce that \( r \) is surjective—this is the celebrated local invariant cycle theorem: all local invariant classes come from \( X_\Delta \); it comes for free from the DT.

Finally observe, that in this case, we indeed have \( Rf_\ast Q_X \simeq \bigoplus Rf_\ast Q[-i] \) (but you should view this as a coincidence due to the low dimension).

(c) Write down the DT for a projective bundle over a smooth variety.
(d) Ditto for the blowing up of a nonsingular subvariety of a nonsingular variety.
(e) Let \( Y \) be a threefold with an isolated singularity at \( p \in Y \). Let \( f : X \to Y \) be a resolution of the singularities of \( Y \): \( X \) is nonsingular, \( f \) is proper and it is an isomorphism over \( Y \setminus \{ p \} \).

(i) Assume \( \dim f^{-1}(p) = 2 \); show, using the symmetries expressed by Exercise 4, that the DT takes the form
\[
Rf_{\ast} Q_X = I_Y \oplus V_p[-2] \oplus W_p[-4],
\]
where \( V_p \simeq W_p^* \) are skyscraper sheaves with dual stalks.
Hint: use \( H_4(X_\eta) \neq 0 \) (why is this true?) to infer, using that \( H^4(I_X) = 0 \), that one must have a summand contributing to \( R^4f_\ast Q \).
Deduce that the irreducible components of top dimension 2 of \( X_p \) yield linearly independent cohomology classes in \( H^2(X) \).

(ii) Assume \( \dim f^{-1}(p) \leq 1 \). Show that we must have
\[
Rf_{\ast} Q_X = I_Y.
\]

Note that this is remarkable and highlights a general principle: the proper algebraic maps are restricted by the fact that the topology of \( Y \), impersonated by \( I_Y \), restricts the topology of \( X \).

As we have seen in our examples to play with at the end of Section 1.1, there are no such general restriction in other geometries, e.g., proper \( C^\infty \) maps, proper real algebraic maps, proper holomorphic maps.

2.2. (Relative) Hard Lefschetz for Intersection Cohomology.

(1) Let \( f : X \to Y \) be a projective smooth map of nonsingular varieties and \( \ell \in H^2(X, \mathbb{Q}) \) be the first Chern class of a line bundle on \( X \) which is ample (Hermitian positive) on every \( X_y \).

We have the iterated cup product map (how do you make this precise?)
\[
\ell^i : R^j f_\ast Q_X \longrightarrow R^{j+2i} f_\ast Q_X.
\]

For every fiber \( X_y := f^{-1}(y) \), we have the hard Lefschetz theorem ([9]) for the iterated cup product action of \( \ell_y \in H^*(X_y, \mathbb{Q}) \); let \( d = \dim X_y \).

The hard Lefschetz theorem on the fibers of the smooth proper map \( f \) implies at once that we have the isomorphisms of sheaves
\[
\ell^i : R^{d-i} f_\ast Q_X \xrightarrow{\sim} R^{d+i} f_\ast Q_X
\]
and we view this fact as the relative hard Lefschetz theorem for smooth proper maps.

(2) In an earlier exercise, you were asked to find examples of the failure of Poincaré duality. It was suggested you find examples of (necessarily singular) complex projective varieties of complex dimension $d$ for which one does not have the symmetry predicted by Poincaré duality: $b_{d-i} = b_{d+i}$, for every $i \in \mathbb{Z}$. Since the conclusion of the hard Lefschetz theorem yields the same symmetry for the Betti numbers, we see that for these same examples, the conclusion of the hard Lefschetz theorem does not hold.

If the hard Lefschetz theorem does not hold for singular projective varieties, the sheaf-theoretic counterpart (9) cannot hold (why?) for an arbitrary proper map, even if the domain and target are nonsingular and the map is surjective (this is due to the singularities of the fibers).

In short, the relative hard Lefschetz does not hold if formulated in terms of an isomorphism between direct image sheaves.

(3) Recall the symmetric form of the DT (see Section 2.1, Exercise 4):

$$Rf_*\mathcal{I}_X \simeq \bigoplus_{i=-r}^{r} P^i[-i].$$

It is a formality to show that given a map $f : X \to Y$ and a cohomology class $\ell \in H^2(X, \mathbb{Q})$ we get iterated cup product maps

$$\ell^i : P^j \to P^{j+2i}.$$ 

The relative hard Lefschetz theorem (RHL) is the statement that if $f$ is proper and if $\ell$ is the first Chern class of an ample line bundle on $X$, or at least ample on every fiber of $f$, then we have that the iterated cup product maps

$$\ell^i : P^{-i} \to P^i$$  \hspace{1cm} (10)

are isomorphisms for every $i \geq 0$.

In other words, the conclusion of the RHL (9) for smooth proper maps, expressed as an isomorphism of direct image sheaves, remains valid for arbitrary proper maps provided

- we push forward $\mathcal{I}_X$, i.e., we form $Rf_*\mathcal{I}_X$, vs. $Rf_*\mathbb{Q}_X$ for which nothing so clean holds in general; and
- we consider the complexes $P^i$, instead of the direct image sheaves.

In the interest of perspective, let me add that the $P^i$ are the so-called perverse direct image complexes of $\mathcal{I}_X$ with respect to $f$ and are special perverse sheaves on $Y$. The circle of ideas is now closed:

RHL is a statement about the perverse direct image complexes of $Rf_*\mathcal{I}_X$!

Note that Verdier duality shows that $P^{-i} = (P^i)^*$. Verdier duality holds in general, outside of the realm of algebraic geometry and holds, for example for the Hopf surface map $h : S \to \mathbb{CP}^1$. In the context of complex geometry, the RHL, $\ell^i : P^{-i} \simeq P^i$, is a considerably deeper statement than Poincaré duality.
Exercise

(a) Make the statement of the RHL explicit in the example of a map from a
surface to a curve (see Section 2.1, Exercise 55b).

(b) Ditto for Section 2.1, Exercise 55e5(e)i. (Hint: in this case you get \( \ell : V_p \cong W_p \).)

Interpret geometrically, i.e., in terms of intersection theory, the isomorphism
\( i : V_p \cong W_p^* \) (PD) and \( \ell : V_p \cong W_p \) (RHL).

(Answer: (see [6]) let \( D_k \) be the fundamental classes of the exceptional divisors
(which are the surfaces in \( X \) contracted to \( p \)); interpret \( W_p \) as (equivalence
classes of) topological 2-cycles \( w \); then \( i \) sends \( D_k \) to the linear map sending
\( w \) to \( D_k \cdot w \in H^6(X, \mathbb{Q}) \cong \mathbb{Q} \); the map should be viewed as the operation
of intersecting with a hyperplane section \( H \) on \( X \) and it sends \( D_k \) to the
2-cycle \( D_k \cap H \). Now you can word out the conclusions of PD and RHL and
appreciate them.)

(5) The hard Lefschetz theorem on the intersection cohomology groups \( \text{IH}(Y, \mathbb{Q}) \) of
a projective variety \( X \) of dimension \( d \). Let us apply RHL to the proper map
\( X \to \text{point} \):

let \( \ell \) be the first Chern class of an ample line bundle on \( X \) of dimension \( d \), then

\[ \ell^i : \text{IH}^{d-i}(X, \mathbb{Q}) \xrightarrow{\cong} \text{IH}^{d+i}(X, \mathbb{Q}). \]

(6) Hodge–Lefschetz package for intersection cohomology.

Let \( X \) be a projective variety. Then the statements (see [9] for these statements)
of the two (hard and hyperplane section) Lefschetz theorems, of the primitive
Lefschetz decomposition, of the Hodge decomposition and of the Hodge–Riemann bilinear relations hold for the rational intersection cohomology group of \( \text{IH}(X, \mathbb{Q}) \).

(7) Exercise (Compare what follows with the first part of Lecture 3.)

Let \( f : X \to Y \) be a resolution of the singularities of a projective surface with
isolated singularities (for simplicity only; after you solve this, you may want to
tackle the case when the singularities are not isolated).

Show that the DT takes the form

\[ Rf_* \mathbb{Q}_X[2] = \mathcal{I}_Y \oplus V_\Sigma, \]

where \( \Sigma \) is the set of singularities of \( Y \) and \( V_\Sigma \) is the skyscraper with fiber \( V_\sigma = H^2(X_\sigma) \) (here \( X_\sigma := f^{-1}(\sigma) \)).

Deduce that the fundamental classes \( E_i \) of the curves given by the irreducible
components in the fibers are linearly independent.

Use Poincaré duality to deduce that the intersection form (cup product) matrix
\( |E_i \cdot E_j| \) on these classes is nondegenerate.

(Grauert proved a general theorem, valid in the analytic context and for an
analytic germ \( (Y, o) \) that even shows that this form is negative definite.)

Show that the contribution \( \text{IH}^*(Y) \) to \( H^*(X) \) can be viewed as the space orthogonal,
with respect to the cup product, to the span of the \( E_i \)'s.
Deduce that $\text{IH}^j(Y)$ sits inside $H^*(X,\mathbb{Q})$ compatibly with the Hodge decomposition of $H^*(X,\mathbb{C})$, i.e., $\text{IH}^j(Y,\mathbb{Q})$ inherits a pure Hodge structure of weight $j$.

References

[1] W. Barth, C. Peters, A. Van de Ven. *Compact complex surfaces, Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) 4. Springer, Berlin, 1984. 7, 9
[2] A. Borel et al. *Intersection cohomology*, volume 50 of *Progress in Mathematics*. Birkhäuser, Boston-Basel-Stuttgart, 1984. 13, 14
[3] R. Bott, L. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer, New York, 1982. 10, 11, 13
[4] H. Cartan, S. Eilenberg. *Homological algebra*. Princeton University Press, Princeton, 1999. 1, 2, 3
[5] M. A. de Cataldo. *The Hodge theory of projective manifolds*. Imperial College Press, London, 2007. 1, 11
[6] M. A. de Cataldo, L. Migliorini. Intersection forms, topology of algebraic maps and motivic decompositions for resolutions of threefolds. In *Algebraic Cycles and Motives*, I, volume no. 343 of the London Math. Soc., pages 102–137. Cambridge University Press, Cambridge, UK, 2007. 9, 11, 16, 19
[7] S. Gelfand, Y. I. Manin. *Methods of homological algebra*, second edition. Springer Monographs in Mathematics. Springer, Berlin, 2003. 3, 5, 13
[8] R. Godement. *Topologie algébrique et théorie des faisceaux*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, XIII. Actualités Scientifiques et Industrielles, No. 1252. Hermann, Paris, 1973. 4
[9] P. Griffiths, J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience, New York, 1978. 17, 19
[10] A. Grothendieck. Sur quelques points d’algèbre homologique. The Tohoku Mathematical Journal, Second Series 9:119–221, 1957. 1
[11] R. Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, New York, 1977. 1, 4
[12] B. Iversen. *Cohomology of sheaves*. Universitext, Springer, Berlin, 1986. 6, 10, 13
[13] M. Kashiwara, P. Schapira. *Sheaves on manifolds*. With a chapter in French by Christian Houzel. Corrected reprint of the 1990 original. Volume 292 of the *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, 1994. 1, 2, 10, 13
[14] D. Mumford. *Algebraic geometry I: complex projective varieties*. Reprint of the 1976 edition. Classics in Mathematics. Springer, Berlin, 1995. 5
[15] E. Spanier. *Algebraic topology*. Corrected reprint of the 1966 original. Springer, New York, 1994. 5
[16] C. Voisin, *Hodge theory and complex algebraic geometry, I, II*, volumes 76 and 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, English edition, 2007. Translated from the French by Leila Schneps. 6