GAMES WITH NESTED CONSTRAINTS GIVEN BY A LEVEL STRUCTURE

FRANCISCO SÁNCHEZ-SÁNCHEZ AND MIGUEL VARGAS-VALENCIA*
Jalisco S/N, Valenciana, CP: 36240
CIMAT, A.C., Guanajuato, Gto, México

(Communicated by Ignacio García-Jurado)

Abstract. In this paper we propose new games that satisfy nested constraints given by a level structure of cooperation. This structure is defined by a family of partitions on the set of players. It is ordered in such a way that each partition is a refinement of the next one. We propose a value for these games by adapting the Shapley value. The value is characterized axiomatically. For this purpose, we introduce a new property called class balance contributions by generalizing other properties in the literature. Finally, we introduce a multilinear extension of our games and use it to obtain an expression for calculating the adapted Shapley value.

1. Introduction. There are practical situations where a set of agents have mechanisms to reach agreements. Usually those situations can be modeled as transferable utility cooperative games (TU games). One of the main problems is the distribution of payoffs between the agents. When a priori coalition agreements exist, such situations can be modeled using cooperative games with coalitional structures (that is, a partition of the set of players). Different extensions of the Shapley value for these games have been presented by Aumann and Dreze[2] and Owen[17]. However, sometimes the information of the cooperation structure provided by the coalitional structure is insufficient. Specifically, the coalitional structure may not describe the situations where cooperation arises on several levels (with multiple partitions of the set of players). For this case, Winter[20] introduced games with a level structure and presented an extension of the Owen value for this structure.

In the classical model of cooperative games, any set of players is a feasible coalition. Indeed, a cooperative game may be modeled as a real-valued function defined on the power set of players. But in some situations, cooperation between players can be restricted by exogenous conditions which prevent the formation of some of these coalitions. Some examples of restricted cooperation are presented in [12] and [14], for communication games; [5], for cooperative games under precedence constraints; [3], for games on convex geometries; and [11], for games with general coalitional structure.

In this paper we present a special restricted cooperation given by a level structure. To give an example, we will present the next situation with a political division.
structure. Consider the problem of allocating the payoff of each city for the maintenance of highways. The segments of highway are generally the responsibility of the entity in which they were built. In this way, the cooperation between the governmental entities can reduce maintenance costs in the entire highway system. Individual entities are encouraged to use federal funds on improving the efficiency and safety of this system. Then, some percent of the construction and maintenance costs of interstate highways in many countries have been paid mainly through fuel taxes, user fees, designated property taxes, and other taxes collected by the federal, state, and local governments. The costs of maintenance and percentages are different and the information is collected by each entity. For this reason, coalitions are only possible between cities within the same county, between counties of the same state and between different states.

A level structure (mentioned in the previous paragraph relating to maintenance of highway system taxes) describes a hierarchy of cooperation between the players as mentioned above, where cities are players. Winter[20] provides a model of games with this structure and proposes a solution called level structure value. This value is computed as the player’s expected marginal contributions to the coalitions formed by a sequential process. For any order that is consistent with the level structure, every player joins his predecessors in the order, with a uniform distribution over these orders. Nevertheless, the values of those marginal contributions can not be calculated from the previous situation. This requires information of coalitions that is missing. For example, when just one city joins its predecessor cities in an order consistent with the political division, we can obtain sets of counties, joining cities in a part of another county, or sets of states along with counties in another state joining some cities in another county. In this paper we focus on solving this problem with restricted information.

Here we present games with a coalitional structure similar to the one presented by Winter[20], but with an important conceptual difference. The cooperation among players is restricted by nested constraints given at the level structure. Thus the domain of the characteristic function is not the power set of players. In this structure, the set of players is classified into several levels according to a family of partitions of the set of players, \( \{P_0, P_1, ..., P_{p+1}\} \), where each partition is a refinement of the next one. Throughout the paper, \( P^0 \) denotes the partition of singletons. The last partition \( P^{p+1} \) stands for the partition of the grand coalition. These will be called trivial partitions. The elements of each partition will be called classes. Every class has a set of subclasses which is the set of elements in the previous partition contained in the class. In fact subclasses provide a partition of the class to which they belong. In our model, only subsets of players are considered feasible coalitions if they satisfy nested constraints in the following sense. A coalition \( S \) is the union of subclasses belonging to the same class. The games defined on the set of feasible coalitions will be called cooperative games with nested structures or simply nested games. The classical situation occurs exactly when the set of players only has the trivial partitions. In this way, we consider the nested constraints as a rule to form a coalition and we consider that only feasible coalitions can have a worth.

The literature cited above, in which restricted cooperation is studied, propose adaptations of the value of Shapley to solve games in these new structures of cooperation. In them we can observe two different approaches of adaptation. For communication games and games with general coalitional structure, different ways

\[ ^1\text{An order is consistent with the level structure if players of the same class appear successively.} \]
are proposed to extend the characteristic function to the unfeasible coalitions in which the Shapley value can be calculated. On the other hand, the models for cooperative games under precedence constraints and games on convex geometries define processes in which only the worth of feasible coalitions are used. This avoids an artificial definition of the worth on unfeasible coalitions. In this paper we follow the second approach and give a solution for the new type of games defined in our model using only the worth of feasible coalitions. Specifically, we develop an adaptation of the Shapley value for these games. We study the structure of the feasible coalitions and obtain a value calculated with modified marginal contributions. Here a whole subclass joins or leaves the coalition. In the particular case that there is only one partition other than a trivial one, \((\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)\), our adaptations of the Shapley value matches the \textit{collective value} presented by Kamijo\cite{Kamijo10}.

There are well-known characterizations for values in games with level structures. They use generalizations or adaptations of \textit{balanced contributions principle} by Myerson\cite{Myerson13}. Calvo et al.\cite{Calvo14} provided a characterization of the \textit{level structure value} using the property of \textit{balanced contributions}. This property states that for any two subclasses that belong to the same class, the profit or deficit on the total payoff of the members in each subclass should be equal when the other subclass leaves the game. Gómez-Rúa and Vidal-Puga\cite{Gomez16} proposed another value for values in games with level structures; they characterize it with \textit{balanced per capita contributions property}. This property states that for any two subclasses that belong to the same class, the average amount that players in each subclass would gain or lose should be equal when the other subclass leaves the game. The average is taken over the cardinality of each subclass.

Now we will introduce a property for nested games by generalizing the \textit{collective balanced contributions property} of Kamijo\cite{Kamijo10}. We called this the \textit{class balanced contributions property}. This states that for any two subclasses that belong to the same class, the change in the payoff for each one player in each subclass should be equal when the other whole subclass leaves the game.

The \textit{multilinear extension} of Owen\cite[sec. 10]{Owen15} is an efficient tool for computing the Shapley value for games with large set of players. Owen et al.\cite{Owen16} modify this method to calculate the \textit{coalitional value} of Owen\cite{Owen17}. They propose a way to generalize this method for the level structure value. Here, we define the \textit{multilinear extension for nested games} and we use it to obtain another expression of the adapted Shapley value.

This paper is organized as follows: In Section 2, we introduce our model by describing the set of feasible coalitions. We show some algebraic properties of this set as a partially ordered set with the inclusion relation. Those properties are the principal tool for giving a representation of a nested game with respect to a basis of unanimity nested games. In Section 3, we determine a Shapley value for nested games by first defining it on the basis of unanimity nested games and then extending it linearly. To interpret the adaptation of the Shapley value, we will define a game for each class where its players will be the subclasses that comprise it. With the weighted Shapley value of these games, where the weights are the cardinalities of the subclasses, we define a distribution among the agents that make up such subclasses. Considering a weight vector to model asymmetry between agents, we can give an adapted weighted Shapley value for nested games. Finally, we define a \textit{multilinear extension} for the nested games. This extension allows us to show a different way to compute the adapted Shapley value.
2. Model formulation. In order to present our model of games under nested constraints of cooperation, we need to introduce some general aspects and notations. To describe the level structure, we consider multiple partitions of the set players \( N = \{1, 2, ..., n\} \). They have certain characteristics that we will introduce below.

Let \( P \) and \( Q \) be partitions of \( N \), we can say that \( P \) is a refinement of \( Q \), and is denoted by \( P \prec Q \), if for all \( X \in P \), there is \( Y \in Q \) such that \( X \subseteq Y \).

A level structure is a sequence \( P = (P_0, P_1, ..., P_{p+1}) \), with \( P_k \) a partition of \( N \), for all \( 0 \leq k \leq p+1 \).

We call \( P_k \) the \( k \)-th level of \( P \). We say that 0-level and \((p+1)\)-level are the trivial levels. When there are no different partitions of the trivial levels, we say that \( P \) is a trivial level structure.

We denote \( P_k = \{C_{k1}, C_{k2}, ..., C_{km_k}\} \) and we let \( M_k = \{1, 2, ..., m_k\} \). In this way, each element of \( P_k \) will be called a \( k \)-class. We will consider the set of indices of \( C_{kt} \) defined as follows:

\[
M_k = \{s \mid C_{s-1} \subseteq C_{kt}\}.
\]

Then, every class \( C_{kt} \) can be characterized as the joining of the elements in the partition \( P_{k-1} \) that compose it,

\[
C_{kt} = \bigcup_{s \in M_k} C_{s-1}.
\]

Now we are ready to introduce the model of games under nested constraints given by a level structure. Since this model only considers coalitions of players in sets of \((k-1)\)-classes that belong to the same \( k \)-class, only these coalitions will be allowed. The set containing these coalitions will be called the set of feasible coalitions of \( C_{kt} \), which is

\[
B_{kt} = \left\{ \bigcup_{s \in T} C_{s-1} \mid T \subseteq M_k \right\},
\]

for \( k = 1, ..., p+1 \) and \( t = 1, ..., m_k \). Then, the set of feasible coalitions on the level \( k \) is

\[
B_k = \bigcup_{t=1}^{m_k} B_{kt}.
\]

Therefore, the set of all feasible coalitions in a situation of cooperation with nested constraints is the joining of all feasible coalitions in the \((p+1)\)-levels,

\[
B_N = \bigcup_{k=1}^{p+1} B_k.
\]

We can take \((B_N, \sqsubseteq)\) as a partially ordered set (or poset). Moreover, \( B_N \) is a finite lattice with the usual join and meet operations\(^2\). It has some interesting properties. For this specify work the following properties are of paramount importance.

Remark 1. By (2) we can ensure that each partially ordered subset \((B_{kt}, \sqsubseteq)\) and \((2^{M_k}, \sqsubseteq)\) are isomorphic. Indeed, every interval

\[
[S, T] = \{R \in B_N : S \subseteq R \subseteq T\}
\]

for \( S, T \in B_{kt}, S \subseteq T \), is Boole algebra.

\(^2\)The least upper bound of \( S \) and \( T \) in the poset is \( S \) join \( T \). Dually the greatest lower bound of \( S \) and \( T \) is \( S \) meet \( T \), [18, p. 285].
Remark 2. Let \( S, T \in \mathcal{B}_N \), if \( S \subseteq T \) and \( S \in \mathcal{B}^k_T \) but \( T \notin \mathcal{B}^k_T \) then
\[
[S, T] = [S, C^k_S] \cup [C^k_S, T].
\]
This means that \( [S, T] \) is not an atomic partially ordered set, due to the join of all coalitions covering \( S \) is \( C^k_S \subseteq T^3 \).

Below, we provide a situation of nested constraints of cooperation which we will use to exemplify several aspects throughout the paper. This has the political division structure mentioned in the introduction for the problem of cost allocation in the maintenance of highways.

Example 1. Let \( N = \{1, 2, 3, 4\} \) be a set of cities in a country. These cities are grouped into counties: \( C_1^1 = \{1\} \), \( C_2^1 = \{2, 3\} \) and \( C_3^1 = \{4\} \). Also, these cities come from different states \( C_1^2 = \{1, 2, 3\} \) and \( C_2^2 = \{4\} \). Furthermore, we denote the class of the country by \( C_1^3 = \{1, 2, 3, 4\} \). Then, the sets of indices that characterize the classes of the upper levels are:
\[
M_1^2 = \{1, 2\}, \quad M_2^2 = \{3\}, \quad M_3^2 = \{1, 2\}.
\]

Then, the set of feasible coalitions is
\[
\mathcal{B}_N = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}.
\]

The set \( \mathcal{B}_N \) defines the feasible coalitions for nested constraints of cooperation (hereafter referred to as nested structure), which allows us to give a definition for these types of games analogous to the TU games.

Definition 2.1. We call a cooperative nested game to the pair \((v, \mathcal{B}_N)\), where \( N = \{1, 2, \ldots, n\} \) is the set of players, \( \mathcal{B}_N \) is a nested structure and \( v: \mathcal{B}_N \to \mathbb{R} \) is a characteristic function such that \( v(\emptyset) = 0 \).

Throughout this paper, the function \( v \) is restricted on proper subsets of \( \mathcal{B}_N \), which are nested structures too. By abuse of the notation, we use the same letter to designate the function on \( \mathcal{B}_N \) and its restriction on the proper subsets.

Example 2. Let \( N = \{1, 2, 3, 4\} \) be the set of cities with nested structure presented in Example 1. The characteristic function which models maintenance cost of a highway system is represented in Table 1.

| \( S \) | \( \{1\} \) | \( \{2\} \) | \( \{3\} \) | \( \{4\} \) | \( \{2, 3\} \) | \( \{1, 2, 3\} \) | \( \{1, 2, 3, 4\} \) |
|---|---|---|---|---|---|---|---|
| \( v(S) \) | 1 | 1 | 0 | 1 | 2 | 4 | 6 |

Table 1. Characteristic function for maintenance cost of a highway system.

By \( G_{\mathcal{B}_N} \) we denote the vector space of all cooperative nested games on \( \mathcal{B}_N \). For \( T \in \mathcal{B}_N \) we define the nested unanimity game \( u_T \in G_{\mathcal{B}_N} \) by \( u_T(S) = 1 \) if \( S \supseteq T \) and \( u_T(S) = 0 \) otherwise. In this fashion, \( U_{\mathcal{B}_N} = \{u_T|T \in \mathcal{B}_N, T \neq \emptyset\} \) is a basis of \( G_{\mathcal{B}_N} \).

Every nested game is a linear combination of nested unanimity games,
\[
v = \sum_{T \in \mathcal{B}_N} \Delta_T(v) u_T \quad \text{and} \quad v(S) = \sum_{T \subseteq S \subseteq \mathcal{B}_N} \Delta_T(v), \quad \forall S \in \mathcal{B}_N.
\]

\(^3\)R cover \( S \) if \( S \subset R \) and not exist \( Q \in \mathcal{B}_N \) such that \( S \subset Q \subset R \).
Using the Möbius inversion formula [18, p. 303], we have that
\[ \Delta_T(v) = \sum_{R \subseteq T} \mu(R, T) v(R), \quad \forall T \in \mathcal{B}_N, \]
where \( \mu \) is the Möbius function. Hence by Remarks 1 and 2, while applying techniques for computing Möbius function [18, Example 3.8.3, Corollary 3.9.5], \( \mu(R, T) \) takes the form
\[ \mu(R, T) = \begin{cases} (-1)^{\ell(R, T)} & \text{if } R, T \in B^k_i \\ 0 & \text{otherwise,} \end{cases} \]
where \( \ell(R, T) := |\{C^{k-1}_r : C^{k-1}_r \subseteq T \setminus R\}|. \) Then
\[ \Delta_T(v) = \sum_{R \subseteq T} (-1)^{\ell(R, T)} v(R), \quad \forall T \in \mathcal{B}_N. \] (3)

Following [7], we call \( \Delta_T(v) \) the **dividend** of \( T \) in the game \( v \).

Now, we are going to define **games of k-classes**\(^4\). The main idea on these games is to observe the \((k-1)\)-class as agents by using the sets of indices in (1).

**Definition 2.2.** Let \((v, \mathcal{B}_N)\) be a nested game. The **game of classes** in \( C^k_s \) of \((v, \mathcal{B}_N)\) is a game \((M^k_s, v_{C^k_s})\) defined by
\[ v_{C^k_s}(R) = v\left( \bigcup_{s \in R} C^{k-1}_s \right), \]
for each \( R \subseteq M^k_s \).

If there is no ambiguity, we will denote the game \((M^k_s, v_{C^k_s})\) only with its characteristic function \( v_{C^k_s} \).

**Remark 3.** By definition, we can note that if \( C^k_s \subseteq C^{k+1}_s \), then the index \( t \in M^{k+1}_s \) is a player in \( v_{C^{k+1}_s} \) and \( M^k_t \) is the grand coalition in \( v_{C^k_t} \). In fact,
\[ v(C^k_s) = v_{C^k_t}(M^k_t) = v_{C^{k+1}_s}(\{t\}). \]

**Example 3.** The games of classes for the nested game \((v, \mathcal{B}_N)\) of Example 2 are \((M^2_s, v_{C^2_s}), (M^2_s, v_{C^2_s})\) and \((M^3_s, v_{C^3_s})\). They have characteristic functions presented in Tables 2 and 3, respectively:

| \(R\) | \(\{1\}\) | \(\{2\}\) | \(\{1, 2\}\) |
|-------|-------|-------|-------|
| \(v_{C^2_s}(R)\) | 1 | 2 | 4 |
| \(R\) | \(\{3\}\) |
| \(v_{C^3_s}(R)\) | 1 |

**Table 2.** Games of classes for counties as players.

| \(R\) | \(\{1\}\) | \(\{2\}\) | \(\{1, 2\}\) |
|-------|-------|-------|-------|
| \(v_{C^2_s}(R)\) | 4 | 1 | 6 |

**Table 3.** Game of classes for states as players.

\(^4\)This game is referred to as a quotient game in [17].
Given a fixed player \( i \in N \), we know that it belongs to a class for each level. These classes are involved in different games of classes in which player \( i \) has a role. We will denote the set of all games of classes in which \( i \) is present by

\[
J_i := \{ v_{C_i^k} \mid i \in C_i^k \}.
\]

In order to simplify the notation for the rest of the paper, we will usually denote the index of the \( k \)-class containing player \( i \) as \([i]_k\), and we will write it simply \([i]\) when no confusion arises, that is, the player \( i \) is in class \( C_{[i]}^k \), for each \( k \).

3. **Adapted Shapley value.** A solution or value for cooperative nested games is a mapping \( f \) which assigns to every game \((v,B_N)\) a vector \( f(v,B_N) \in \mathbb{R}^n \), where its entries represent payoff for each player in \( N \). We will introduce a value for nested games that is an adaptation of the Shapley value.

We provide a value similar to the Shapley value in the sense that the games in the basis \( U_{B_N} \) are assigned the same value as the Shapley value assigned to unanimity classic games

\[
\varphi_i(u_T,B_N) = \begin{cases} 
\frac{1}{|T|} & \text{if} \ i \in T, \\
0 & \text{otherwise}.
\end{cases}
\] (4)

A linear value that satisfies (4) for every game \( v \in G_{B_N} \) is

\[
\varphi_i(v,B_N) = \sum_{S \in B_k} \gamma_S (v(S) - v(S_{-i})),
\] (5)

where \( \Delta_S(v) \) are the dividends of \( v \). Hence from our formula for the dividends, we have that equation (5) which takes the form

\[
\varphi_i(v,B_N) = \sum_{S \in B_k} \gamma_S (v(S) - v(S_{-i})),
\]

where \( S_{-i} := S \setminus C_{[i]}^{k-1} \), for all \( S \in B_k \) and

\[
\gamma_S = \sum_{T \supset S \atop S,T \in B_k^i} \frac{(-1)^{(S,T)}}{|T|}, \quad S \neq \emptyset.
\]

This expression of \( \varphi \) allows us to observe the similarity with the weighted Shapley value\(^5\) for the games in \( J_i \) if we take each \( C_i^{k-1} \subset C_i^k \) as a player through \((M_i^k,v_{C_i^k})\). Because it is useful to provide an initial interpretation of \( \varphi \) and characterize it, we will give the following expression,

\[
\varphi_i(v,B_N) = v(i) + \sum_{w \in J_i} S h_{[i]}^\lambda(w) - w([i]) \frac{1}{\lambda_{[i]}}
\] (6)

where every weight \( (\lambda_j) \) is the cardinality of the class represented by \( j \) in the game of classes \( w \). It is straightforward to check that (6) is linear and satisfies (4).

Each term

\[
D_i(w) := \frac{S h_{[i]}^\lambda(w) - w([i])}{\lambda_{[i]}}
\]

\(^5\)The weighted Shapley value was introduced by Kalai and Samet[9]. By \( Sh^\lambda(w) \) we denote this value for a game \( w \) and a weight vector \( \lambda \).
has the following interpretation: The game of classes is solved with a weighted
Shapley value \( (Sh^\lambda) \), where the weights represent the asymmetry of size in the
participant classes. The difference \( Sh^\lambda(w) - w([i]) \) represents the excess or defect
of the class between the allocated value and the quantity that can be obtained
by itself \( (w([i]) = v(C^i)) \). Finally, an egalitarian distribution of that difference is
made among the members of the same class. The distribution in equal shares of that
amount provides a solution suggesting a “collective” support in each class. In fact,
expression (6) of \( \varphi \) arises as a generalization for the well-known \textit{collective value}
presented by Kamijo[10] for coalitional structures. If we take \( \Psi = (P^0, P^1, P^2) \)
which only has one partition different from the trivial partitions, we obtain the
simplified expression
\[
\varphi_i(v, B_N) = Sh_i(v) + \frac{Sh^\lambda_i(v_{C^i}) - v(C^i)}{|C^i|}, \quad \forall i \in N,
\]
and that is equivalent to the \textit{collective value}.

As a next step we will show some interesting properties that the value \( \varphi \) satisfies.
For this we should consider \( B_N \) restricted to the coalition \( S \),
\[
B_S := \{ T \in B_N \mid T \subseteq S \}.
\]
If we restrict the nested game to a class \( C^i \), we have the value \( \varphi \) for a player \( i \in C^i \).
It is given by
\[
\varphi_i(v, B_{C^i}) = v(i) + \sum_{k=1}^{l} D_i(v_{C^k}).
\]

We continue in this fashion obtaining the next proposition.

\textbf{Proposition 1}. Let \( C^i \subseteq C^{i+1}_s \). Given player \( i \in C^i \), then the value in the
restricted structure \( B_{C^{i+1}} \) is
\[
\varphi_i(v, B_{C^{i+1}}) = \varphi_i(v, B_{C^i}) + D_i(v_{C^{i+1}}).
\]

\textbf{Example 4}. To calculate the payoff which assigns solution (6) for player 2 of the
game presented in Example 2 we have
\[
v((\{2\}) = 1, \quad D_2(C^1_2) = \frac{1}{2}, \quad D_2(C^2_2) = \frac{1}{3}, \quad D_2(C^3_2) = \frac{1}{4}.
\]
Furthermore, the final allocation of the value is
\[
\varphi(v, B_N) = \left( 1 + \frac{1}{3} + \frac{1}{4}, \frac{1}{2} + \frac{1}{3}, \frac{1}{2} + \frac{1}{4}, \frac{1}{3} + \frac{1}{4}, \frac{1}{3} + \frac{1}{4}, \frac{1}{4} \right) = \left( \frac{19}{12}, \frac{25}{12}, \frac{13}{12}, \frac{5}{4} \right).
\]

\textit{Characterization}. Now we will provide an axiomatic characterization for the value
\( \varphi \). We introduce the property that describes efficiency and another property which
extends the \textit{balanced contributions property}.

A value \( f \) is \textit{efficient} if \( \sum_{i \in N} f_i(v, B) = v(N) \) for each \( v \in G_{B_N} \).

\textbf{Definition 3.1}. A value \( f \) satisfies the \textit{Class Balanced Contributions} (CBC) if and
only if, for all \( C^i_s \subseteq C^{i+1}_s \) (\( l = 0, ..., p \)) we have
\[
f_i(v, B_{C^{i+1}}) - f_i(v, B_{C^{i+1}\setminus C^i}) = f_j(v, B_{C^{i+1}}) - f_j(v, B_{C^{i+1}\setminus C^i}),
\]
for every \( i \in C^i, \ j \in C^j \) and all \( v \in G_{B_N} \).
CBC requires that given two classes $C_i^l$ and $C_s^l$ in the same upper class $C_{r+1}^l$, the change in payoff measured by the value of each player in $C_i^l$, if $C_s^l$ leaves the game, must be equal to the change in payoff measured by the value to each player in $C_s^l$ if $C_i^l$ leaves the game.

**Remark 4.** If we have only the trivial levels $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1)$, then $\mathcal{B}_N = 2^N$. Then the CBC property matches with the following property, defined by Myerson[13].

A value $f$ satisfies Balanced Contributions if only if

$$f_i(v, 2^N) - f_i(v, 2^N \setminus \{j\}) = f_j(v, 2^N) - f_j(v, 2^N \setminus \{i\}),$$

for all $i, j \in N$ and all nested game $v \in G_2^N$.

**Remark 5.** When there is only one different level from the trivial levels $\mathfrak{P} = (\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2)$, the CBC property coincides with the following property presented by Kamijo[10].

A value $f$ satisfy Collective Balanced Contributions if only if

$$f_i(v, \mathcal{B}_N) - f_i(v, \mathcal{B}_N \setminus C_i^1) = f_j(v, \mathcal{B}_N) - f_j(v, \mathcal{B}_N \setminus C_j^1),$$

for every $i \in C_i^1$, $j \in C_j^1$, $i \neq j$, and for all $v \in G_2\mathcal{B}_N$.

**Proposition 2.** $\varphi$ is efficient and satisfies CBC.

**Proof.** It is straightforward to check that $\varphi$ is efficient. Now we will show that $\varphi$ satisfies CBC: Let $(v, \mathcal{B}_N) \in G_2\mathcal{B}_N$, $i, j \in N$ and $C_i^l, C_j^l \subset C_{r+1}^l$, for fixed $r$. By definition of $\varphi$, we note that if $l = 0$ then $\varphi$ is the Shapley value over $C_r^1$ and clearly it satisfies CBC. Now if $l \geq 1$, by Proposition 1, we have

$$D_i(v_{C_{r+1}^l}) = \varphi_i(v, \mathcal{B}_{C_{r+1}^l}) - \varphi_i(v, \mathcal{B}_{C_{r+1}^l \setminus C_i^1}).$$

We obtain a similar expression for player $j$ but we must verify that $D_i(v_{C_{r+1}^l}) = D_j(v_{C_{r+1}^l})$. Calculating this expression we obtain

$$\frac{Sh^\lambda_{[i]}(v_{C_{r+1}^l}) - Sh^\lambda_{[i]}(v_{C_{r+1}^l \setminus C_i^1})}{\lambda_{[i]}} = \frac{Sh^\lambda_{[j]}(v_{C_{r+1}^l}) - Sh^\lambda_{[j]}(v_{C_{r+1}^l \setminus C_j^1})}{\lambda_{[j]}},$$

which was proved by Hart and Mas-Colell[8], p. 604. Thus we conclude that $\varphi$ satisfies the property of CBC.

Next we present a characterization of $\varphi$ using the mentioned properties in Proposition 2.

**Theorem 3.2.** There exists a unique value for cooperative games with nested structure that satisfies the properties of efficiency and CBC. That value is $\varphi$.

**Proof.** As we showed in Proposition 2, we know that $\varphi$ is efficient and satisfies CBC. Next, we fix a nontrivial $\mathcal{B}_N$. There is at least a nontrivial level. Let $f$ and $g$ be two efficient solutions satisfying CBC. We will show that $i \in N$, $f_i(v, \mathcal{B}_N) = g_i(v, \mathcal{B}_N)$ for any $v \in G_2\mathcal{B}_N$.

We will prove the result using induction on the number of levels that define $\mathcal{B}_N$. Consider the level $k = 1$. Since the lower level classes are singletons, we have that the feasible coalitions in $\mathcal{B}_{C_i^1}$ are

$$B_{[i]}^1 = 2^{C_i^1}. $$
Then the game in the first level, \((v, B_{C_1})\), is a classical TU game and CBC is equivalent to the property of balanced contributions defined by Myerson[13]. Therefore we can conclude that

\[
f_i(v, B_{C_1}) = g_i(v, B_{C_1}) = Sh_i(C_1, v).
\]

Now we assume that the result holds for levels \(k, k \leq l \leq p\). This means

\[
f_i(v, B_{C_k}) = g_i(v, B_{C_k}),
\]

for every class \(C^k \in B^k \subseteq B_N\), with \(1 \leq k \leq l\), and we proceed to prove it for level \(l + 1\).

Let us use induction on the cardinality of set of indices of \(C^{l+1}_{r}\), denoted by \(M^{l+1}_{r}\). Consider \(i \in C^l_{[i]} \subseteq C^{l+1}_{r}\). First assume that \(|M^{l+1}_{r}| = 1\), i.e. the only subclass that \(C^{l+1}\) contains is \(C^l_{[i]}\) and feasible coalitions of nested structures \(B_{C^{l+1}_{r}}\) and \(B_{C^{l}_{[i]}}\) are equal. Hence

\[
(v, B_{C^{l+1}_{r}}) = (v, B_{C^{l}_{[i]}}),
\]

thus by the induction hypothesis we conclude that

\[
f_i(v, B_{C^{l+1}_{r}}) = f_i(v, B_{C^{l}_{[i]}}) = g_i(v, B_{C^{l}_{[i]}}) = g_i(v, B_{C^{l+1}_{r}}).
\]

Assuming the result holds for \(|M^{l+1}_{r}| = m\), we will prove that the result holds for \(|M^{l+1}_{r}| = m + 1\).

Let \(|M^{l+1}_{r}| = m + 1 > 1\), then there exists \(j \in C^l_{[j]} \subseteq C^{l+1}_{r}\), such as \(C^l_{[i]} \neq C^l_{[j]}\). Due to the CBC property, we have

\[
h_i(v, B_{C^{l+1}_{r}}) - h_j(v, B_{C^{l+1}_{r}}) = h_i(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}) - h_j(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}),
\]

for \(h = f, g\). Thus by induction hypothesis, we have

\[
f_i(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}) = g_i(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}),
\]

\[
f_j(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}) = g_j(v, B_{C^{l+1}_{r}} \setminus C^l_{[i]}).
\]

Therefore

\[
f_i(v, B_{C^{l+1}_{r}}) - g_i(v, B_{C^{l+1}_{r}}) = f_j(v, B_{C^{l+1}_{r}}) - g_j(v, B_{C^{l+1}_{r}}).
\]

We may now use efficiency to conclude that if we fix \(i\) and take summation over \(j \in C^{l+1}_{r}\), then we have

\[
f_i(v, B_{C^{l+1}_{r}}) - g_i(v, B_{C^{l+1}_{r}}) = 0.
\]

Thus we have proved that \(f_i(v, B_{C^{l+1}_{r}}) = g_i(v, B_{C^{l+1}_{r}})\) for all \(i \in C^{l+1}_{r}\).

Finally, applying the induction hypothesis on levels, we conclude that \(f_i(v, B_{C^k}) = g_i(v, B_{C^k})\) for each \(k \leq p + 1\). Specifically if we take \(k = p + 1\), we obtain

\[
f_i(v, B_N) = g_i(v, B_N)
\]

for all \(i \in N\). \(\square\)

**Proposition 3.** (Kamijo[10], Theorem 2) Let \(\mathcal{B} = (\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2)\) be a level structure. The collective value (3) is a unique efficient solution satisfying CBC.
Kamijo[10] obtained the characterization of the collective value by efficiency, balance contributions and collective balance contributions properties. Then, proof of the above result is straightforward by Remark 4, Remark 5 and Theorem 3.2. Moreover, this work provides a different approach of the result due to the domain of characteristic functions in nested games is $\mathcal{B}_N$. We thus conclude that the collective value only uses the information of coalitions that satisfy the nested constraints when the level structure is $\mathfrak{P} = (\mathfrak{P}^0, \mathfrak{P}^1, \mathfrak{P}^2)$.

**Remark 6.** If we introduce a weight vector to model certain asymmetry between the players, then a generalization of the weighted Shapley value can be obtained analogously to the adaptation of the Shapley value. To obtain a solution that takes into account this asymmetry, we propose the value:

$$\varphi^\omega_i(v, B_N) = v(i) + \omega_i \sum_{w \in J_i} \frac{Sh^\omega_i(w) - w([i])}{\omega_i},$$

where $\omega_j$ is the sum of weights of the members of class what $j$ represents in the game of classes $w$.

Its characterization is analogous to the Theorem 3.2 by slightly adapting the CBC property. A value $f$ satisfies $\omega$-Class Balanced Contributions ($\omega$-CBC) if and only if, for all $C^l_i, C^l_s \subset C^{l+1}_i (l = 0, \ldots, p)$ we have

$$\frac{f_i(v, B_{C^{l+1}_i}) - f_i(v, B_{C^{l+1}_i \setminus C^l_i})}{\omega_i} = \frac{f_j(v, B_{C^{l+1}_s}) - f_j(v, B_{C^{l+1}_s \setminus C^l_i})}{\omega_j},$$

for every $i \in C^l_i, j \in C^l_s$ and all $v \in G_{B_N}$.

Thus, $\omega$-CBC requires that the contributions of classes are balanced in proportion to the weights of each player.

4. The multilinear extension of nested games. As mentioned Owen et al.[16], the multilinear extension (MLE) has been shown to be an effective tool for computing the Shapley value for TU games with large players’ set. For this reason they modify the method of MLE to calculate the value of Owen[17] for games with coalition structure. Moreover they propose a way to generalize this method for the level structure value of Winter[20]. In this section, we modify MLE to calculate the value $\varphi$ for nested games.

In accordance with the work of Owen[15, sec. 10] we define the multilinear extension of a nested game using the linear decomposition presented in the Section 2 as follows

$$f(v)[q_1, \ldots, q_n] = \sum_{S \in B_N} \prod_{j \in S} q_j \Delta_S(v),$$

where $\Delta_S(v)$ for all $S \in B_N$ are the dividends defined in (3).

It is possible to calculate the value $\varphi$ for a nested game using this extension. The following result shows a new way to do it.

**Proposition 4.** Let $(v, B_N), i \in N$ and $f(v)$ the multilinear extension of $v$. Then the value $\varphi$ is given by

$$\varphi_i(v, B_N) = \int_0^1 \frac{\partial f(v)}{\partial q_t}[t, \ldots, t] dt.$$
Proof. Calculating the partial derivative of \( f(v) \) with respect to \( q_i \) we have

\[
\frac{\partial f(v)}{\partial q_i} [q_1, ..., q_n] = \sum_{S \ni i \in B_N} \prod_{j \in S, j \neq i} q_j \Delta_S(v).
\]

Now, evaluating this derivative in the main diagonal \([t, ..., t] \) and integrating with respect to \( t \), we obtain

\[
\int_0^1 \frac{\partial f(v)}{\partial q_i} [t, ..., t]dt = \int_0^1 \sum_{S \ni i \in B_N} t^{|S|-1} \Delta_S(v) dt
= \sum_{S \ni i \in B_N} \frac{\Delta_S(v)}{|S|} = \varphi(v, B_N).
\]

5. Conclusions and future work. The main conclusion of this paper is that the model of games under nested constraints gives rise to a special type of games with domain of feasible coalitions in a new structure composed of Boolean algebras. Due to the properties of such structure, the algebraic extension of the Shapley value which we use for this type of games allowed us to characterize it and obtain a closed expression for that value.

There are a number of directions for future work, in particular for the advancement of applications in the study of more complex situations. On the one hand, it would be interesting to study this model with a communication structure (graph) and propose new values. As a first step in this direction, the study of two-level communication structure models have been presented by Brink et al.[19]. In fact, since the characterization of the values presented here are made with the property of CBC, this suggests a clear pathway to propose an adapted Myerson value for n-level communication structure. On the other hand, another way is to study different solutions for the multilinear extension of the nested games. Hence, it would also be interesting to propose a model for games with fuzzy coalitions under nested structure of cooperation constraints. It would also be interesting to propose adaptations and characterize different solution concepts of Shapley value for games that arise in our model. The power index Banzhaf would be adapted for simple games as the generalization presented by Álvarez-Mozos and Tejada[1]. Other general concepts of solution as the core and the nucleolus would be studied.

Acknowledgments. The authors want to thank our anonymous referees for their helpful comments. The authors acknowledge support from research grant 167924 from CONACyT.

REFERENCES

[1] M. Álvarez-Mozos and O. Tejada, Parallel characterizations of a generalized shapley value and a generalized banzhaf value for cooperative games with level structure of cooperation, Decision Support Systems, 52 (2011), 21–27.
[2] R. J. Aumann and J. H. Dreze, Cooperative games with coalition structures, International Journal of Game Theory, 3 (1974), 217–237.
[3] J. M. Bilbao and P. H. Edelman, The shapley value on convex geometries, Discrete Applied Mathematics, 103 (2000), 33–40.
[4] E. Calvo, J. J. Lasaga and E. Winter, The principle of balanced contributions and hierarchies of cooperation, Mathematical Social Sciences, 31 (1996), 171–182.
[5] U. Faigle and W. Kern, The shapley value for cooperative games under precedence constraints, *International Journal of Game Theory*, 21 (1992), 249–266.

[6] M. Gómez-Rúa and J. Vidal-Puga, Balanced per capita contributions and level structure of cooperation, *Top*, 19 (2011), 167–176.

[7] J. C. Harsanyi, A simplified bargaining model for the n-person cooperative game, *Part of the Theory and Decision Library book series*, 28 (1960), 44–70.

[8] S. Hart and A. Mas-Colell, Potential, value, and consistency, *Econometrica: Journal of the Econometric Society*, 57 (1989), 589–614.

[9] E. Kalai and D. Samet, On weighted shapley values, *International Journal of Game Theory*, 16 (1987), 205–222.

[10] Y. Kamijo, The collective value: A new solution for games with coalition structures, *Top*, 21 (2013), 572–589.

[11] G. Koshevoy and D. Talman, Solution concepts for games with general coalitional structure, *Mathematical Social Sciences*, 68 (2014), 19–30.

[12] R. B. Myerson, Graphs and cooperation in games, *Mathematics of Operations Research*, 2 (1977), 225–229.

[13] R. B. Myerson, Conference structures and fair allocation rules, *International Journal of Game Theory*, 9 (1980), 169–182.

[14] G. Owen, Values of graph-restricted games, *SIAM Journal on Algebraic Discrete Methods*, 7 (1986), 210–220.

[15] G. Owen, Multilinear extensions of games, in *The Shapley value: essays in honor of Lloyd S. Shapley* (ed. A. E. Roth), Cambridge University Press, 1988, chapter 10, 139–151.

[16] G. Owen, E. Winter et al., The multilinear extension and the coalition structure value, *Games and Economic Behavior*, 4 (1992), 582–587.

[17] G. Owen, Values of games with a priori unions, in *Mathematical Economics and Game Theory*, Springer, 141 (1977), 76–88.

[18] R. Stanley, *Enumerative Combinatorics*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986.

[19] R. van den Brink, A. Khemelnitskaya and G. van der Laan, An owen-type value for games with two-level communication structure, *Annals of Operations Research*, 243 (2016), 179–198.

[20] E. Winter, A value for cooperative games with levels structure of cooperation, *International Journal of Game Theory*, 18 (1989), 227–240.

Received December 2016; revised November 2017.

E-mail address: mvargas@cimat.mx
E-mail address: sanfco@cimat.mx