Mapping Self-Organized Criticality onto Criticality

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Abstract : We present a general conceptual framework for self-organized criticality (SOC), based on the recognition that it is nothing but the expression, "unfolded" in a suitable parameter space, of an underlying unstable dynamical critical point. More precisely, SOC is shown to result from the tuning of the order parameter to a vanishingly small, but positive value, thus ensuring that the corresponding control parameter lies exactly at its critical value for the underlying transition. This clarifies the role and nature of the very slow driving rate common to all systems exhibiting SOC. This mechanism is shown to apply to models of sandpiles, earthquakes, de-pinning, fractal growth and forest-fires, which have been proposed as examples of SOC.

Résumé : Nous proposons une stratégie générale pour identifier le mécanisme responsable des phénomènes critiques auto-organisés, basée sur l'idée qu'ils sont simplement la traduction, dans un espace de paramètres choisis, d'un point critique dynamique instable standart. La criticalité auto-organisé résulte du contrôle du paramètre d'ordre ajusté à une valeur positive tendant vers zéro, ce qui assure automatiquement que le paramètre de contrôle correspondant se cale exactement sur sa valeur critique de la transition de critique sous-jacente. Ce résultat explique le rôle particulier joué par le forçage infiniment lent qui est un caractère commun à tous les systèmes critiques auto-organisés. Nous appliquons ces idées aux modèles de tas de sable, aux modèles de tremblements de terre, de feux de forêts, aux transitions de décrochage et aux modèles de croissance fractale, qui ont été proposés comme autant d'exemples caractéristiques de la criticalité auto-organisée.
1 Introduction

Following a number of early insightful studies [1-6], the past decade has witnessed a clear acknowledgement that many natural phenomena must be described by power law statistics. Correspondingly, an intense activity has developed in order to understand the origin of these ubiquitous power law tails. This has led in particular to the concept of ‘self-organized criticality’ (SOC) [7-8], according to which certain dynamically driven spatially extended systems evolve spontaneously towards a critical globally stationary dynamical state with no characteristic time or length scales.

The fundamental idea underlying SOC is that, unlike phase transitions in equilibrium statistical physics, the critical state is reached without the need of fine-tuning a control parameter, i.e. the critical state is an attractor of the dynamics. To illustrate the basic ideas of SOC, Bak and co-workers used a cellular automaton inspired from the creation of avalanches in a pile of sand. In these models, ”sand” are added grain by grain on a lattice until a local slope becomes unstable, and an avalanche is initiated. In this way, the pile reaches a stationary critical state, characterized by a critical slope, in which additional grains of sand will fall off the pile via avalanches of all sizes from grain to lattice scale, distributed in size and lifetime according to a power law.

In spite of a large theoretical effort, the general conditions under which a physical system exhibits SOC are still largely unknown. Some facts have however been established:

- some systems qualify as SOC if their large scale evolution obeys a diffusion equation (possibly non-linear but with no characteristic time scale) which satisfies a global conservation law [9,10];

- There exists a class of systems which exhibit diffusion-like response but do not obey a global conservation law and nevertheless seem to exhibit SOC [11-14]. In these cases, the underlying mechanism for SOC is still not clear even if there seems to exist a deep relationship with the problem of synchronization of coupled threshold oscillators of relaxation occurring in some domain of the parameter space [8,15-18].

- More generally, a feedback mechanism must operate which, from the perspective of usual critical phenomena, describes the action of the order parameter onto the control parameter [19] and attracts the dynamics to a critical state.
This then suggests a mechanism for transforming usual ”unstable critical phase transitions” into SOC \[19\]. We call unstable those critical phenomena which are not self-organized.

In summary, the present theoretical understanding of SOC is rather fragmented with no real unifying perspective. Our goal here is to attempt to present a general theoretical framework for SOC, based on the recognition that it is nothing but the expression, ”unfolded” in a suitable parameter space, of an underlying genuine unstable critical point. This correspondence provides, what we believe is, the fundamental mechanism for SOC and more information on the relevant critical exponents can be obtained within this framework.

### 2 The Nature of Self-Organized Criticality

#### 2.1 General mechanism

The essence of our approach can be summarized in a few sentences. Consider a ”standard” unstable critical phase transition, such as the Ising ferromagnet or, analogously, bond percolation. Here, a spin, up or down, is assigned to each site with an exchange coupling constant $J$. Furthermore, one defines two nearest neighbour sites to be connected with a probability $p = 1 - e^{-2J/K_BT}$ if both have spin up. For zero external field, this defines a critical temperature $T_c$ or bond-density $\rho_c$ below which the order parameter $m_0$, the magnetization or the probability of an infinite cluster, is zero and behaves as $m_0 \propto (T_c - T)^\beta$ above. The transition is further characterized by a diverging correlation length $\xi \propto (T - T_c)^{-\nu}$ and susceptibility $\chi \propto (T - T_c)^{-\gamma}$ as $T_c$ is approached, hence quantifying the spatial fluctuations of the order parameter.

Suppose now that it turns out to be natural for the system under consideration that, instead of controlling $T$, the ”operator” controls the order parameter $m_0$ and furthermore takes the limiting case of fixing it to a positive but arbitrary small value. The condition $m_0 \to 0^+$ is equivalent to $T \to T_c^-$. (Specifically, the scenario above comes more natural in the context of a ”zero strength” infinite cluster, where only one bond needs to be broken, than that of a ferromagnet.) In other words, the system is at the critical value of the unstable critical point and must therefore exhibit fluctuations at all scales in its response. This is nothing but the hallmark of the underlying unstable critical point. As will be shown explicitly in the following examples, this scenario applies most naturally to out-of-equilibrium driven systems.
More precisely, we shall argue that systems exhibiting SOC present a genuine critical transition when forced by a suitable control parameter, often in the form of a generalized force (torque for sandpile models, stress for earthquake models, force for depinning systems). Then, SOC appears as soon as one controls or drives the system via the order parameter of the critical transition (it turns out that in these systems, the order parameter is also the conjugate of the control parameter in the sense of mechanical Hamilton-Jacobi equations). The order parameter hence takes in general the form of a velocity or flux. The condition that the driving is at \( M \to 0^+ \) illuminates the special role played by the constraint of a very slow driving rate and common to all systems exhibiting SOC, as being the exact condition to control the order parameter at \( 0^+ \) which ensures the positioning at the exact critical value of the control parameter.

We shall now illustrate and develop this general idea, by a detailed discussion of four examples, the sandpiles, earthquake models, pinned-depinning lines or Charge-Density-Waves models, fractal growth processes and forest-fires.

2.2 The sandpile

2.2.1 Model of an unstable transition

Let us consider the cellular automaton sandpile model [7] and put it in a geometry inspired from experiments [20], namely that of a rotating cylinder. The cylinder axis is horizontal and is the same as the rotation axis. The cylinder is partially filled with "sand" presenting an initially flat horizontal interface below the axis. Suppose that the axis of the cylinder is held to a fixed frame by a torsion spring on which one can exert a controlled torque \( T \). If \( T = 0 \), the cylinder takes the position such that the surface of the sand is horizontal, i.e., the rotation angle \( \theta = 0 \). If one starts to exert a non-vanishing \( T \), the cylinder rotates up to an angle \( \theta \) such that the torque exerted by the tilted sandpile balances exactly the applied \( T \). Increasing \( T \), one finally reaches a critical value \( T_c \) at which the sandpile reaches its slope \( \theta_c \) of instability corresponding to the triggering of sand flow \( J \), whose magnitude increases with \( T > T_c \).

We witness a critical sliding transition, from a state of repose (\( J = 0 \) for \( T < T_c \)) to an active sliding state (\( J > 0 \) for \( T > T_c \)) corresponding to an average steady rotation of the cylinder at a non-zero average angular velocity \( < \frac{d\theta}{dt} > \). This rotation occurs, since increasing \( T \), the slope and therefore torque exerted by the sand can no more increase and balance the applied torque. One thus enters a dynamical regime in which the avalanches overlap and construct a non-zero fluctuating flow.
This critical transition is characterized by the way the average flow increases from zero above $T_c$ ($J \sim (T - T_c)^\beta$), as well as by the spatial and temporal correlations in the local burst of non-zero $J$ below $T_c$. The maximum size of these bursts when increasing $T$ by a small amount below $T_c$ (or by inserting a small perturbation such as a local grain-hole addition) allows one to define the correlation length $\xi$. Note that, from the constraint of grain conservation, the sand flux $J$ and the cylinder angular velocity $<\frac{d\theta}{dt}>$ are simply proportional and describe physically the same ordered response of the system to the increasing control parameter $T$ ($J \sim <\frac{d\theta}{dt}> = 0$ for $T < T_c$ and $J \sim <\frac{d\theta}{dt}> > 0$ for $T > T_c$). For our purpose, it is now more illuminating to speak of the order parameter in terms of $<\frac{d\theta}{dt}>$.

The above critical sliding transition occurs when controlling the torque $T$. However, suppose that one controls instead the angular velocity $\frac{d\theta}{dt}$ and impose $\frac{d\theta}{dt} = 0^+$, i.e. a vanishingly small positive value. In the language of mechanics, this corresponds to interchange the role of the conjugate variables $T$ and $\frac{d\theta}{dt}$ (defined by the fact that their product gives the mechanical power of the system). It is then clear that $T$ adjusts to its critical value $T_c$ and the response of the sandpile will just be that documented previously [7], in terms of power law distribution of avalanches.

Let us briefly conclude this section of the sandpile paradigm by describing how a specific model for the critical sliding instability can be implemented, knowing the initial rules of the sandpile cellular automaton. A given sand configuration is characterized by the set of column heights and slopes, the total torque just being the sum over all sites of the torque exerted by each column with respect to the cylinder axis of rotation. Then, a given grain configuration corresponds to a global torque $T$ (it is clear that many configuration have the same $T$ similarly to the finding that many micro-states correspond to the SOC macro-state, as found by [22] in a study of Abelian sandpile cellular automata). Then, a small increment of $T$ corresponds to a global increase of the local slopes possibly leading to local instabilities, i.e., avalanches. Starting from a small initial $T$, the avalanche regime is only transient. When one reaches and overpasses $T_c$, the continuous flow appears and the order parameter ("sand" flow) becomes positive, the avalanches being order parameter fluctuations. Thus, the size of the characteristic avalanches is the correlation length. Note that in contrast to a previous discussion of the connection between SOC and standard critical phenomena [21], our framework introduces naturally the current as the order parameter. The slope is just a dynamical variable which adjusts itself as a function of the external torque (control parameter).
2.2.2 Scaling laws

Viewed as a critical transition, one can define the corresponding power laws:

\[ J \sim (T - T_c)^\beta \]  \hspace{1cm} (1)

for the current flow (order parameter),

\[ \chi \sim |T_c - T|^{-\gamma} \]  \hspace{1cm} (2)

for the susceptibility or the response function to (i.e. current flow induced by) a small perturbation very close to the critical transition \[21\] and

\[ \xi \sim |T_c - T|^{-\nu} \]  \hspace{1cm} (3)

for the correlation length given by the linear size of the domain which is sensitive to a local perturbation. Note that we assume here that \( \gamma \) and \( \nu \) are the same on both sides of \( T_c \).

In general, one should expect these three exponents to be interdependent and obey a scaling relation expressing that the local flux induced by a perturbation is a function of the susceptibility and of the volume of the correlated domain. From here, we can infer the properties of SOC upon the system, i.e., when driving it with \( J \to 0^+ \), the system reacts by ”avalanches” distributed in size according to a power law

\[ P(s) \sim s^{-(1+\mu)} \]  \hspace{1cm} (4)

The maximum avalanche size is related to \( \xi \) by \( s_{\text{cutoff}} \sim \xi^D \) where \( D \) is the fractal dimension of the avalanches. Then, the flow caused by a local perturbation below \( T_c \) is simply the average size of avalanches, \( \chi \sim \int_{s_{\text{cutoff}}}^{s_{\text{max}}} sP(s)ds \sim s_{\text{cutoff}}^{1-\mu} \), yielding \( \mu = 1 - \frac{\nu}{\nu D} \). This expression has been derived previously and checked with numerical simulations \[21\] and yields \( \mu \simeq 0.1 \) in 2D and \( \mu \simeq 0 \) in 3D. In general, the avalanches are compact (\( D = d \), where \( d \) is the space dimension), showing that \( \mu \) can be determined from the properties of the critical transition (\( \gamma \) and \( \nu \)). The determination of the dynamical exponent \( z \), defined by the scaling of an avalanche duration \( t \sim \xi^z \), involves the exponent \( \beta \) in eq.(1). In the simplest picture where the dynamics is diffusive, the sand flux is proportional to the diffusion coefficient which is itself proportional to \( \frac{\xi^2}{t} \). This expression assumes that there is no renormalization of the microscopic transport coefficient entering in the definition of the diffusion coefficient and leads to \( z = 2 + \beta \nu \). This last expression is uncertain in general since the simple diffusion approximation is in question.
2.2.3 Non-conservative sandpile models

Our framework provides a simple and natural explanation for the observation by [11] that non-conservative sandpile models may be SOC. The authors present their model as a toy model for earthquakes. The local variable is the total force exerted on a site. In their initial model, the force on each site is increasing very slowly at a constant rate. When the force on a given site reaches a threshold, it is reset to zero while a fraction is redistributed on the nearest neighbors. The non-conservation stems from the fact that the total amount which is distributed is less than the initial value.

Now, suppose that we work at fixed global force, i.e. we impose that the sum over all sites of the local forces is constant. Increasing this global force to a larger value may or may not trigger readjustments. The point again is that there exists a critical value for the global force above which the avalanches never stop, characterizing a non-zero average velocity $v$. Again, assuring $v \to 0^+$ places the system at the critical point by adjusting the global force to its critical value. It is then clear why the condition of conservation is not crucial for the appearance of SOC: SOC is seen to rely fundamentally on the existence of an underlying sliding critical point, which does not need conservation, as well-documented in studies of the critical dynamics of unstable second order phase transitions [23].

2.3 Earthquakes and tectonics

It has been argued by several authors [24,25] that the earthquake phenomenology in geology is the signature of SOC. Let us thus consider a model elastic tectonic plate [26,27], scaled down in the laboratory to perform a mechanical thought experiment, namely a shear deformation imposed at its border for instance. The simplest situation is where a shear force $F$ is imposed on two opposite borders (the other two being free), say, by a spring set-up. (One could similarly consider a compressive experiment as done in triaxial tests [28], without any change in our discussion.) As the applied force $F$ increases, the plate (which can contain pre-existing damage such as cracks and faults) starts to deform increasing the internal damage [29]. For sufficiently low $F$, after some transient during which the system deforms and adjusts itself to the applied force, the system becomes static and nothing happens: the strain rate or velocity of deformation becomes zero (here we are neglecting any additional creep or ductile behavior). As $F$ increases, the transient becomes longer and longer since larger and larger plastic-like deformations will develop within the plate. There exists
a critical plasticity threshold $F_c$ at which the plate becomes globally "plastic", in the sense that it starts to flow with a non-zero strain rate $\frac{d\varepsilon}{dt}$ under fixed $F$. As $F$ increases above $F_c$, the shear strain rate $\frac{d\varepsilon}{dt}$ increases. In models and in laboratory experiments, this plastic transition from a brittle to a ductile behavior is critical in the usual sense. $F$ is the control parameter and $\frac{d\varepsilon}{dt}$ qualifies as the order parameter ($\frac{d\varepsilon}{dt} = 0$ for $F < F_c$ and $\frac{d\varepsilon}{dt} > 0$ for $F > F_c$).

Instead of controlling the force exerted on the system, let us apply instead a constant (very small) shear rate at the border of the plate. We thus recover the "natural" boundary condition for earthquakes and plate tectonic deformations (the typical relative plate velocity is of order 1 cm/year compared to the fast rupture velocity during an earthquake of order 2 km/sec.). Such a situation has been studied in many works showing the existence of SOC, both in the shape of power law earthquake size distribution and in the fractal fault geometry selected by the earthquake dynamics. It now becomes clear that this "natural" condition is nothing but driving the plate at the critical point $F = F_c$ by controlling the order parameter $\frac{d\varepsilon}{dt}$ to a very small value, thus ensuring the critical properties of the systems. Note that SOC will appear if the corresponding transition is critical. This may not be the case for some specific models such as the spring-block models [30].

### 2.4 Depinning transitions

Consider an elastic line lying within a random system of pinning impurities or asperities and pulled by a force density per unit length or at its two ends in a direction perpendicular to its average direction. This can be a stretched or directed polymer with electric charges at its ends which are submitted to an electric field [31, 32], a vortex line in a superconductor of type II with a superficial electric current which in the presence of a magnetic field will create a Lorentz force on the two ends of the vortex line at the sample borders [33], an interface created by a fluid displacing another fluid in a porous medium [34] or a magnetic domain wall in the presence of quenched disorder. The electric or magnetic field (the field $E$ from now on) plays the role of the control parameter. For a given (small) $E$ and some initial conditions, the overdamped dynamics will ensure the relaxation of the line in a well-defined line configuration. When increasing $E$ from some small value by some small increment, the line may stay fixed or may readjust itself via a sequence of localized sliding events into another static conformation. As $E$ increases, it has been well-documented that there is a critical value $E_c$ above which the line begins to move. The point $E = E_c$.
is a pinned-depinned critical point very similar to usual dynamical critical points, endowed with all the corresponding properties. The order parameter is the average velocity $v$ of the line and scales as $v \sim (E - E_c)^\beta$.

Suppose now that instead of controlling $E$, one drives the line ends at a constant vanishingly small velocity, i.e., one controls the conjugate to the field, namely the order parameter. By the same arguments as above, the line is then automatically at its critical pinned-depinned transition point. As a consequence, long-range spatial correlations and large fluctuations appear, reflected in the distribution of burst-like sliding events occurring along the line.

One can extract the ”avalanche” distribution from numeric simulations on an overdamped elastic string in a random medium pulled by a force per unit length [31]. Using the fact that the typical transverse fluctuation over a scale $z < \xi$ along the line is of order $z$ implies that the characteristic jump size for a portion of the line of length $z$ is proportional to $z$. Then, the average transverse motion over a correlation length is proportional to $\int_1^\xi z P(z) dz \sim \xi^{1-\mu}$, where $P(z) \sim z^{-(1+\mu)}$ is the distribution of sliding events in the corresponding SOC problem. It seems reasonable to assume that this average transverse motion occurs over a time scale proportional to $\xi$ (it is easy to correct for any other scaling), which leads to an average velocity just above threshold scaling as $J \sim (E - E_c)^\beta \sim \xi^{-\mu}$, giving the prediction $\mu = \frac{2}{\nu} \simeq 0.25$ using the results $\beta \simeq 0.25$ and $\nu \simeq 1$ [31].

The same type of reasoning applies to a charged-density-wave (say an elastic line pulled in a direction parallel to its average direction) which is well-known to exhibit a depinning critical transition at some critical value of the driving field [35]. Again, driving the CDW at a constant very small velocity creates a SOC system with a power law distribution of sliding events.

\section*{2.5 Fractal growth processes}

It has been proposed that fractal growth processes, such as diffusion-limited-aggregation (DLA), fluid imbibition as exemplified for instance by invasion percolation, dendritic growth, dielectric breakdown, rupture in random media, constitute another class of systems exhibiting SOC [36-38].

These growth problems seem however to belong to a different class of phenomena than sandpile models since e.g. the internal geometrical structure of an aggregate is quenched (and only its perimeter is active) whereas the critical state of a sandpile is continuously rearranging under the action of external forcing.
Here, we want to point out that a stronger similarity emerges when using the proposed framework and the fractal self-organization of these growth processes can be linked to the existence of an underlying critical point. We shall illustrate our ideas on the annealed dielectric breakdown model in a cylindrical geometry (best suited to define a steady state regime), which is known to be equivalent to DLA [39]. The base of the cylinder is fixed at potential 0 and its other end is fixed at some non-zero value $V$. The rate of growth from the base is assumed to be a stochastic process controlled by a probability proportional to the electric field on the base growth surface. The DLA model is recovered in the limit where the growth is done quasi-statically. This situation corresponds to fixing the growth rate (equivalently the particle flux in the DLA model) $J \rightarrow 0^+$. 

Let us now consider the case where we impose a constant average potential gradient $-\frac{dV}{dz}$, i.e. electric field $E$, along the cylinder axis. Furthermore, let us assume that dielectric breakdown, i.e. growth on a site, can only occur above a certain threshold, which is a quenched random variable distributed according to a given distribution. This condition ensures that the growth problem now becomes similar to a pinned-depinned transition. For those sites above their threshold, the rate of growth is again assumed to be a stochastic process controlled by a probability proportional to the electric field. If the applied electric field is very small, a few sites will break down and the growth cluster will evolve until all sites are below their threshold. This is the bound state. Increasing the applied electric field above a certain threshold $E_c$ (a function of the threshold distribution), the system begins to grow in an unbound way at a finite rate which increases as $E$ gets larger. The finite growth velocity is determined by the details of the dynamical breakdown processes (which we do not describe here). Note that a finite velocity corresponds to a finite particle flux in the DLA model. For the critical transition, the control (resp. order) parameter is the electric field or particle concentration gradient (resp. the growth velocity or particle flux). In the regime above $E_c$, the growing clusters do not exhibit self-similarity at all scales, but a finite correlation length appears which is a decreasing function of the growth rate. In the language of fluid interfaces, pushed with an average velocity $c$, the relevant dimensionless number is the ratio of the Bernoulli velocity pressure $\rho v^2$ to the Laplace pressure $\frac{\sigma}{b}$ (similar to a Bond number), where $\rho$ is the pushing fluid density, $\sigma$ is the surface tension and $b$ the Hele-Shaw cell thickness or the interface radius of curvature.

In summary, the fractal structure of DLA clusters can be viewed as the result of
the quasi-static regime, i.e. the control of the order parameter (growth rate) of the corresponding critical transition at an infinitesimal (positive) value.

Similar mappings can be defined for the invasion percolation problem and rupture problems. In this respect, it is interesting to realize that the quasi-static rupture models extensively studied in the statistical literature [40] are characterized by the rupture of elements one by one, i.e. by controlling the rate of rupture (order parameter) to a vanishingly small value. Truly dynamical models of rupture [41] are in this sense beyond the unstable critical point, separating the stable finite damage phase from the fully rupture phase.

2.6 The Forest-Fire Model

The self-organized critical forest-fire model introduced by Drossel & Schwabel [13,42] is another example of the proposed mechanism. The model is defined on a d-dimensional lattice, where each site is either empty, tree or fire. The lattice is updated synchronously according to the following four rules: 1) A trees will grow on an empty site with probability $p$, 2) Fire becomes empty 3) Fire spreads to n.n trees and 4) A tree catches fire spontaneously with probability $f$. The existence of a critical point in the limit $f/p \to 0$ is expected, since the average number of trees destroyed by a lightning is $\langle s \rangle = (f/p)^{-1} (1 - \rho_t)/\rho_t$ ($\rho_t$ is the mean tree density) provided $f \ll p \ll 1/T(s)$ [42], where $T(s)$ is the average time it takes to burn a tree cluster of size $s$. This separation of time scales is again nothing but a condition of slow driving common for SOC-models. Furthermore, $\rho_t < 1$ in order for $\langle s \rangle$ to diverge. This then suggests $\rho_t$ as the control parameter for the critical transition in agreement with the proposed framework. Changing the first rule of the model allows one to tune $\rho_t$ to its critical value: Each time a tree is burned down a new tree is put at random thus keeping $\rho_t$ fixed to a controlled value. The order parameter is then the density of fires $\rho_f$, the condition $f/p \to 0$ above corresponding to $\rho_f = 0^+$. In the case of $f = 0$, the forest-fire can only propagate on dynamically connected tree clusters, thus defining a critical tree density below which forest-fire will finally extinguish itself and burn forever above. It is then clear that the parameter $f$ plays the role of an external field thus modifying the transition and explaining the systematic deviations from a pure power law seen in simulations [42,43].

3 Concluding remarks
1. We have proposed a general conceptual framework for self-organized criticality, which consists in mapping SOC onto unstable critical points, controlled by driving the corresponding order parameter at an infinitesimal value.

Our present theory clarifies and extends the previously recognized mechanism of a feedback of the order parameter on the control parameter as being essential for SOC [19].

The mapping between SOC and usual critical transitions offers a novel but natural route to study further the properties associated to SOC, namely by characterizing the critical properties of the underlying critical point itself. In a way, we have only displaced the search for the underlying mechanism for SOC to that of the appearance of unstable critical points. However, we feel that this is a significant improvement for two reasons: 1) some of the underlying unstable critical points are known and have been documented per se. Their knowledge can thus bring new light on the SOC models. In particular, the present framework illuminates the physical meaning of the slow driving common to all systems exhibiting SOC. 2) Even if the underlying critical point is new, its study can probably be quite efficient by employing the large toolbox developed in the last twenty or more years in this field.

2. **Supercritical bifurcations**: Note that our framework applies directly to supercritical and Hopf bifurcations, when considering the situation where the order parameter is controlled at the value $0^+$. This situation can be modelled analytically by writing a general (possibly complex) Landau-Ginzburg equation for the order parameter fluctuations, conditioned to have a vanishingly small average order parameter. This is left for the future.

3. **Renormalization group and fixed-scale transformation**: Controlling the order parameter $J \rightarrow 0^+$ of an unstable critical point does not allow for a standard renormalization group procedure. Indeed, in this situation, there is no control parameter and the critical exponents cannot be obtained by the standard procedure in terms of the derivatives of the renormalization group transformation of the control parameters under scale transformation. Thus, the exponents related to the distance from the critical point do not exist. Let us stress that this is not due to a special attractive property of the critical point, as claimed in [44], but results solely from the special driving conditions which ensures the exact positioning of the dynamics on the unstable critical point. In this situation, a
generalized renormalization procedure should reflect the exact positioning on a critical point, i.e. should give the signature of an attractive critical point. This is indeed born out in the fixed scale transformation procedure introduced by [44]. We believe that finding a general renormalization theory for systems right at their critical point could provide a new class of tools for general critical phenomena. This could be related to conformal invariance theory [45].

4. **Relation with Goldstone modes** : Our approach allows us to clarify the proposal [46] that SOC stems from the non-linear dynamics of Goldstone modes, however by returning the logic of Obukhov’s argument : criticality in SOC is not the effect of interaction of Goldstone gapless modes as claimed; the gapless modes result rather from the underlying unstable critical point, stabilized by the special driving condition. Let us recall that, in the long-wavelength limit, Goldstone modes reduce to a homogeneous displacement of the sample or to a uniform rotation of the whole spin system. Since a SOC system is right at the critical point of a standard unstable critical phase transition characterized by a spontaneous symmetry breaking, the avalanches can be viewed as nothing but the Goldstone fluctuations (i.e. large scale displacements) attempting to restore the broken symmetry. In the case of a discrete symmetry breaking, the avalanches correspond to droplet fluctuations [47].

5. **Relation with singular diffusion** : The singular diffusion property of continuous equations obtained by taking the hydrodynamic limit of SOC models [48] derives straightforwardly from our framework, since it is the direct signature of the precise localization at an unstable critical point. Thus, all theories of SOC in terms of singular diffusion [48,49] are only the expression that the governing equation is that of a system sitting precisely at a critical point. Let us recall that even more generally, singular diffusion occurs on the approach to any supercritical bifurcation, the best-known example being the Rayleigh-Bénard instability. In this case, the order parameter is the average convection velocity and the fluctuations are associated to streaks or patches of non-zero velocity occurring below the critical Rayleigh number $R_c$ at which global convection starts off. The spatial diffusion coefficient $D(R)$ diverges as $D(R) \sim (R_c - R)^{-\frac{\gamma}{2}}$ and reflects the existence of very large velocity fluctuations. A scaling argument can be written to get this powerlaw [50]. Similarly, the singular diffusion found in SOC models can also be derived from similar scaling reasoning [51], showing
the common origin of the singular diffusion.

6. Our proposed framework is not in contradiction with the recent recognition by several authors [8,15-18] that SOC is deeply connected to the mechanism of partial synchronization of relaxation oscillators with thresholds. Under a slow driving of the order parameter, each threshold element (for instance a column in the sandpile) taken in isolation undergoes periodic oscillations of relaxation. The complete problem corresponds to describe the result of the coupling between these oscillations, in terms of a competition between synchronization and desynchronization effects. In other words, the dynamics of relaxation oscillators describes the detailed response of the critical point under the slow driving of the order parameter.
References

[1] V. Pareto, Cours d’économie politique. Reprinted as a volume of *Oeuvres Complètes* (Droz, Geneva, 1896-1965).

[2] B. V. Gnedenko, A.N. Kolmogorov, *Limit distributions for sum of independent random variables*, Addison Wesley, Reading, MA, (1954).

[3] P. Lévy, Théorie de l’addition des variables aléatoires (Gauthier Villars, Paris,1937-1954).

[4] E. Montroll, M. Shlesinger, ‘On the Wonderful world of Random Walks’, in Nonequilibrium phenomena II, From stochastic to hydrodynamics, Studies in statistical mechanics XI (J.L. Lebowitz and E.W. Montroll, eds) (North Holland, Amsterdam, 1984).

[5] B.B. Mandelbrot, ‘The Fractal Geometry of Nature’ (Freeman, San Francisco, 1983).

[6] E. Fama, Management Science 11, 404 (1965).

[7] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987); Phys. Rev. A 38, 364 (1988).

[8] D. Sornette, Les phénomènes critiques auto-organisés, Images de la Physique 1993, édition du CNRS.

[9] Hwa T. and Kardar M., Phys.Rev.Lett. 62 , 1813 (1989)

[10] Grinstein G., Lee D.-H and Sachdev S., Phys.Rev.Lett. 64 , 1927 (1990); Grinstein G. et Lee D.-L., Phys.Rev.Lett. 66, 177 (1991)

[11] Z. Olami, H.J.S. Feder and K. Christensen, Phys.Rev.Lett. 68, 177 (1991); K. Christensen and Z. Olami, Phys.Rev.A46, 1829 (1992)

[12] P. Bak, K. Chen and M. Creutz, Nature (London) 342, 780 (1989); P. Alstrom and J. Leao, Phys. Rev. E49, R2507 (1994)

[13] B. Drossel and F. Schwabl, Phys.Rev.Lett. 69, 1629 (1992)

[14] Bak P. and K. Sneppen, Phys. Rev.Lett. 71, 4083 (1993); Flyvbjerg H., K. Sneppen and P. Bak, Phys. Rev.Lett. 71, 4087 (1993); Ray T.S. and N. Jan, Phys. Rev.Lett. 72, 4045 (1994)

[15] K. Christensen, Self-organization in models of sandpiles, earthquakes and flashing fireflies, PhD Thesis, (Nov. 1992)

[16] A. Corral, C.J. Pérez, A. Díaz-Guilera and A. Arenas, Self-organized criticality and synchronization in a lattice model of integrate-and-fire oscillators, preprint (1994)
[17] A.A. Middleton and C. Tang, Self-organized criticality in non-conserved systems, preprint (1994)

[18] S. Bottani, Pulse-coupled relaxation oscillators: from biological synchronization to self-organized criticality, preprint (1994)

[19] D. Sornette, J. Phys. I France 2, 2065 (1992); N. Fraysse, A. Sornette and D. Sornette, J. Phys. I France 3, 1377 (1993)

[20] J. Rajchenbach, Phys. Rev. Lett. 65, 2221 (1990)

[21] C. Tang and P. Bak, Phys. Rev. Lett. 60, 2347 (1988)

[22] Dhar D. et Ramaswamy R., Phys. Rev. Lett. 63, 1659 (1989); Dhar D., Phys. Rev. Lett. 64, 1613 (1990)

[23] Hohenberg P.C. and Halperin B.I., Rev. Mod. Phys. 49, 435 (1977)

[24] Sornette, A. and D. Sornette, Europhys. Lett. 9, 197 (1989); Sornette D., Self-organized criticality in plate tectonics, in Proceeding of the NATO ASI, Spontaneous formation of space-time structures and criticality, eds. by Riste T. and Sherrington D., Geilo, Norway 2-12 April 1991 (Kluwer Academic Press, 1991), p.57.

[25] Bak P. and Tang C., J. Geophys. Res. 94, 15635 (1989); K. Ito and M. Matsuzaki, J. Geophys. Res. 95, 6853 (1990)

[26] Cowie P., C. Vanneste and D. Sornette, J. Geophys. Res. 98, 21809 (1993); Miltenberger P., D. Sornette and C. Vanneste, Phys. Rev. Lett. 71, 3604 (1993); Sornette D., P. Miltenberger and C. Vanneste, Pageoph 142, 491 (1994).

[27] A. Cochard and R. Madariaga, Pageoph 142, 419 (1994)

[28] Evesque P., J. Phys. France 51, 2515 (1990); D. Sornette, A. Sornette and P. Evesque, Frustration and disorder in granular media and tectonic blocks: implications for earthquake complexity, Nonlinear Processes in Geophysics, in press (1994)

[29] Lockner D.A., J.D. Byerlee, V. Kuksenko, A. Ponomarev and A. Sidorin, Nature 350, 39 (1991)

[30] Burridge R. and Knopoff L., Bull. Seismol. Soc. Am. 57, 341 (1967); Carlson J.M. and Langer J.S., Phys. Rev. Lett. 62, 2632 (1989); Schmittbuhl J., Vilotte J.-P. and Roux S., Europhys. Lett. 21, 375 (1993)

[31] M. Dong, M.C. Marchetti, A.A. Middleton and V. Vinokur, Phys. Rev. Lett. 70, 662 (1993)

[32] M. Mézard, J. Phys. France 51, 1831 (1990)

[33] C. Tang, S. Feng and L. Golubovic, Phys. Rev. Lett. 72, 1264 (1994)

[34] Horváth V.K., F. Family and T. Vicsek, J. Phys. A24, L25 (1991)

[35] D.S. Fisher, Phys. Rev. B31, 1396 (1985)
[36] P. Alstrom, Phys.Rev. A 38, 4905 (1988); Phys.Rev. A 41, 7049 (1990)
[37] Bak P. and K. Chen, Physica D 38, 5 (1989)
[38] I.M. Jánosi and A. Czirók, Fractals 2, 153 (1994)
[39] L. Niemeyer, L. Pietronero and H.J. Wiesmann, Phys.Rev.Lett. 22, 1033 (1984); J. Kertész, Dielectric breakdown and single crack models, in ref.[40], p. 261
[40] H.J. Herrmann and Roux S., eds., Statistical models for the fracture of disordered media (Elsevier, Amsterdam, 1990)
[41] D. Sornette and Vanneste C., Phys.Rev.Lett. 68, 612 (1992); D. Sornette, C. Vanneste and L. Knopoff, Phys.Rev.A 45, 8351 (1992); C. Vanneste and D. Sornette, J.Phys.I France 2, 1621 (1992)
[42] S. Clar, B. Drossel & F. Schwabel, Phys.Rev.E 50, 1009 (1994)
[43] K. Christensen, H. Flyvbjerg & Z. Olami, Phys.Rev.Lett. 71, 2737 (1993)
[44] L. Pietronero, A. Vespignani and S. Zapperi, Phys.Rev.Lett. 72, 1690 (1994)
[45] J.L. Cardy, Physica A 140, 219 (1986)
[46] S. Obukhov, Phys.Rev.Lett. 65, 1395 (1990)
[47] Huse D. and Fisher D.S., Phys.Rev.B 35, 6841 (1987)
[48] Carlson J.M., Chayes J.T., Grannan E.R. an Swindle G.H., Phys.Rev.Lett. 65, 2547 (1990)
[49] Bántay P. and I.M. Jánosi, Phys. Rev. Lett. 68, 2058 (1992)
[50] D. Sornette, J.Phys.I France 4, 209 (1994)
[51] Kadanoff L.P., A.B. Chhabra, A.J. Kolan, M.J. Feigenbaum, I. Procaccia, Phys. Rev. A 45, 6095 (1992).