A definition of contraction of Lie algebra representations using direct limit

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Abstract. In this paper the frequently used procedures for contraction of Lie algebra representations which were introduced by İnönü and Wigner are reformulated using the notion of direct limit. A definition for contraction of Lie algebra representations based on this reformulation is given. The contractions of the skew-Hermitian irreducible representations of $so(3)$ to those of $iso(2)$ and of $iso(1,1)$ to those of Heisenberg Lie algebra are given as examples.

1. Introduction and preliminaries

In the early work of İnönü and Wigner [1] contraction of Lie algebra representations was demonstrated for the first time. Their examples roughly fall into two main procedures: convergence of differential operators and convergence of matrix elements. These two schemes are fundamentally different. In the first approach, a pointwise convergence of a family of differential operators acting on representation spaces of the Lie algebra that is being contracted defines a new well-defined differential operator which acts on a representation space of the contracted Lie algebra. In the second approach the matrix elements of a sequence of operators which act on representation spaces of the Lie algebra that is being contracted converge to matrix elements of an operator in a representation space of the contracted Lie algebra.

In the first approach the limiting procedure actually produces the limiting representation, while in the second approach there is no such construction but merely some identification that comes from the convergence of the matrix elements. In order to understand this difficulty we will review the contraction of $so(3)$ representations to those of $iso(2)$ as was given by İnönü and Wigner [1]. In that example, starting from a sequence of finite dimensional representations of $so(3)$ with increasing dimension, they obtained an infinite dimensional representation of $iso(2)$. For a fixed element of $so(3)$, as one runs over the sequence of the representations, one obtains a matrix with increasing size which converges, in some sense, to an infinite dimensional matrix corresponding to an element in an infinite dimensional representation of $iso(2)$. The type of convergence that İnönü and Wigner obtain was convergence of matrix elements. More precisely for each $j \in \mathbb{N}$ let $\rho_j$ be the $2j + 1$ dimensional irreducible representation of $so(3)$ with the canonical basis $\{|j,m\rangle|m = -j, -j + 1, \ldots, j\}$ and let $\eta_{\Xi}$ be the irreducible representation of...
iso(2) with casimir $\Xi^2$ and the canonical basis $\{|m\rangle | m \in \mathbb{Z}\}$. Then the formulas that İnönü and Wigner obtained are of the form

$$\lim_{j \to \infty} \langle j, m | \rho_j(J_z) | j, m \rangle = \langle m | \eta_{\Xi}(J_z) | m \rangle,$$

(1)

where $J_z$ is the generator of the rotations around the Z-axis, and similar expressions hold for all other generators of $so(3)$. Instead of finding intrinsic interpretation for the representation space and the operators that are obtained under the limit, one tends to make the following identifications

$$\lim_{j \to \infty} |j, m\rangle \longleftrightarrow |m\rangle$$

(2)

$$\lim_{j \to \infty} \rho_j(J_z) \longleftrightarrow \eta_{\Xi}(J_z)$$

(3)

The aim of this paper is to show that the natural mathematical notion which describes such limits is the direct limit [2] of inner product spaces. We will find the intrinsic properties of the sequence of representations that is being contracted and we will characterize the limit representation space and the limit operators rigourously. We will give a definition of contraction of Lie algebra representations; this definition can be applied in all classical examples of contraction of Lie algebra representations that are obtained via convergence of matrix elements as well as via convergence of differential operators. And as such, this formalism gives a unified approach to all contractions of Lie algebra representations.

Our paper is divided as follows: In the rest of this section we review the contraction of Lie algebras, and discuss the issue of contraction of their representations. We review briefly the notion of direct limit in the category of inner product spaces and give a relevant example. In section 2 we give a definition for contraction of Lie algebra representations. We also state and prove two propositions which show that the given definition is equivalent to the usual convergence of matrix elements which is a common way (e.g., [3]) to describe contraction of Lie algebra representations. We continue with a couple of basic examples of contraction of representations which are described in terms of our new formalism.

1.1. Contraction of Lie algebras
Contraction of Lie algebras is essentially a limit of Lie algebras. In 1951 such limits were considered for the first time by Segal [4]. A couple of years later İnönü and Wigner [1, 5] introduced the notion of contraction which was later generalized by Saletan [6]. We now give the formal definition for a contraction of Lie algebras using notations that are similar to those of Weimar-Woods [7].

**Definition 1** Let $U$ be a complex or real finite dimensional vector space. Let $\mathcal{G} = (U, [\cdot , \cdot ])$ be a Lie algebra with Lie product $[\cdot , \cdot ]$. For any $\epsilon \in (0, 1]$ let $t_\epsilon \in \text{Aut}(U)$ ($t_\epsilon$ is a linear invertible operator on $U$) and for every $X, Y \in U$ we define

$$[X, Y]_\epsilon = t_\epsilon^{-1}[t_\epsilon(X), t_\epsilon(Y)].$$

(4)

If the limit

$$[X, Y]_0 = \lim_{\epsilon \to 0^+} [X, Y]_\epsilon$$

(5)

exists for all $X, Y \in U$, then $[\cdot , \cdot ]_0$ is a Lie product on $U$ and the Lie algebra $\mathcal{G}_0 = (U, [\cdot , \cdot ]_0)$ is called the contraction of $\mathcal{G}$ by $t_\epsilon$ and we write $\mathcal{G} \overset{t_\epsilon}{\longrightarrow} \mathcal{G}_0$. 


There is an analogous definition [7] for the case that the limit (5) is meaningful only on a sequence \(\{\epsilon_n\}_{n=0}^\infty\) which converges to zero when \(n \to \infty\):

**Definition 2** Let \(U\) be a complex or real vector space, \(G = (U, [\cdot, \cdot])\) a Lie algebra with Lie product \([\cdot, \cdot]\). For any \(n \in \mathbb{N}\) let \(t_n \in \text{Aut}(U)\) and for every \(X, Y \in U\) we define

\[
[X, Y]_n = t_n^{-1}([t_n(X), t_n(Y)]).
\]

If the limit

\[
[X, Y]_\infty = \lim_{n \to \infty} [X, Y]_n
\]

exists for all \(X, Y \in U\), then \([\cdot, \cdot]_\infty\) is a Lie product on \(U\) and the Lie algebra \(G_\infty = (U, [\cdot, \cdot]_\infty)\) is called the sequential contraction of \(G\) by \(t_n\) and we write \(G \xrightarrow{t_n} G_\infty\).

Specific examples of contractions of Lie algebras can be found in e.g., [1, 6, 8, 9].

1.2. Contraction of representations

Usually the notion of contraction of Lie algebra representations is used when \(G \xrightarrow{t(\epsilon)} G_0\) and there is some kind of limiting process which enables one to get a representation of \(G_0\) from a family of representations of \(G\). The first to consider the question of contraction of Lie algebra representations were Inönü and Wigner [1]; they described, using examples only, how to build faithful representations of \(G_0\) from a family of representations of \(G\). More specifically, for every \(\epsilon \in (0, 1]\) they had a faithful representation \(\rho_\epsilon\) of \(G\) such that when taking \(\epsilon\) to zero they obtained a faithful representation of \(G_0\).

After the pioneering work of Inönü and Wigner on contraction of Lie algebras representations, Saletan investigated contractions of finite dimensional representations of Lie algebras [6]. There have been several attempts to give a general procedure for the contraction of Lie algebras representations e.g., [3, 10, 11, 12, 13, 14], but up to now there does not seem to be a well accepted definition.

1.3. Direct limit of inner product spaces

Let \(I = (I, \prec)\) be a totally ordered set. Suppose that for every \(i \in I\) we are given an inner product space \((V_i, (\cdot, \cdot)_i)\) and for every \(i, j \in I\) such that \(i \prec j\) we have a linear map \(\varphi_{ij} : V_i \to V_j\) which preserves the inner product. If, in addition, for every \(i, j, k \in I\) such that \(i \prec j \prec k\) we have \(\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}\), and for all \(i \in I\) \(\varphi_{ii}\) is the identity operator on \(V_i\), we call such a collection of inner product spaces and linear maps a directed system of inner product spaces over \(I\). We will call it for short a directed system and denote it by \(\{V_i, \varphi_{ij}, I\}\).

Given a directed system \(\{V_i, \varphi_{ij}, I\}\) we denote its direct limit [2] by \(\lim_{\text{dir}} V_i\). We will use the following construction of the direct limit: let \(\{V_i, \varphi_{ij}, I\}\) be a directed system. Let \(X\) be the disjoint union of the \(V_i\) \(X = \bigsqcup_{i \in I} V_i\). We define an equivalence relation on \(X\) by: \(x \sim y\) if there exists \(k \in I\) such that \(i \prec k, j \prec k\) and \(\varphi_{ik}(x) = \varphi_{jk}(y)\). We denote the equivalence class of \(x\) by \([x]\) and the collection of the equivalence classes by \(V = X/\sim\). We give \(V\) a structure of inner product space by defining the multiplication of \([x]\) by a scalar \(\alpha\) to be \([\alpha x]\), the sum of \([x]\) and \([y]\) where \(x \in V_i\) and \(y \in V_j\) are equivalent (\(x \sim y\)) if there exists \(k \in I\) such that \(i \prec k, j \prec k\) and \(\varphi_{ik}(x) = \varphi_{jk}(y)\). We denote the equivalence class of \(x\) by \([x]\) and the collection of the equivalence classes by \(V = X/\sim\). We give \(V\) a structure of inner product space by defining the multiplication of \([x]\) by a scalar \(\alpha\) to be \([\alpha x]\), the sum of \([x]\) and \([y]\) where \(x \in V_i\) and \(y \in V_j\) is defined to be \([\varphi_{ik}(x) + \varphi_{jk}(y)]\) and the inner product \(\langle [x], [y]\rangle\) is given by \(\langle \varphi_{ik}(x), \varphi_{jk}(y)\rangle_{k}\) where \(k \in I\) is such that \(i \prec k, j \prec k\). \(V\) as constructed here is the direct limit of the directed system \(\{V_i, \varphi_{ij}, I\}\), i.e.

\[
V = X/\sim \xrightarrow{\text{dir}} V_i
\]
We remark that for each \( i \in I \) we have the natural morphism \( \varphi_i : V_i \rightarrow V \) which sends \( x \) to its equivalence class, \([x]\). This map is obviously linear and preserves the inner product and hence it is an embedding of \( V_i \) in \( V \). The maps \( \{\varphi_i\}_{i \in I} \) are compatible with the directed system in the following sense: For every \( i, j \in I \) such that \( i < j \), \( \varphi_i = \varphi_j \circ \varphi_{ij} \).

**Example 1** Let \( I \) be the totally ordered set which is \( \mathbb{N} \) with the usual order of natural numbers. Let \( \{V_i, \varphi_{ij}, I\} \) be the directed system of inner product spaces over \( I \) which is defined as follows: For every \( n \in \mathbb{N} \), \( V_n = C^n \) with the standard inner product. For \( m \leq n \) we have \( \varphi_{mn}((x_1, x_2, \ldots, x_m)) = (x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0) \). In this case the direct limit is the inner product space of all infinite sequences with finite numbers of non zero entries and with the standard inner product.

In terms of the above construction the elements of the direct limit are given by

\[
[(x_1, x_2, \ldots, x_m)] = \{(x_1, x_2, \ldots, x_m), (x_1, x_2, \ldots, x_m, 0), (x_1, x_2, \ldots, x_m, 0, 0), \ldots\}
\]

where \( x_m \neq 0 \). The inner product of \( [(x_1, x_2, \ldots, x_m)], [(y_1, y_2, \ldots, y_n)] \in V \) where \( m \leq n \) is given by \( \langle [(x_1, x_2, \ldots, x_m)], [(y_1, y_2, \ldots, y_n)] \rangle = \sum_{i=1}^{\infty} x_i y_i \). We identify \( [(x_1, x_2, \ldots, x_m)] \) with \( (x_1, x_2, \ldots, x_m, 0, 0, \ldots) \).

2. Contraction of representations as direct limit

The set up in the continuous case is as follows. Let \( I_1 \) denote the totally ordered set which consists of the interval \((0, 1]\) and the order relation \( < \) defined by: \( x < y \iff x \geq y \). Let \( \mathcal{G} = (U, [\cdot, \cdot]) \) be a Lie algebra and suppose that \( \mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0 \). Given a directed system of inner product spaces indexed by \( I_1 \), \( \{V_i, \varphi_{ij}, I_1\} \) and a family of representations of \( \mathcal{G} \) on the same inner product spaces, \( \{(\rho_i, V_i)\}_{i \in I_1} \), for every \( X \in U \) and \([v_{\epsilon_1}] \in V \) such that \( v_{\epsilon_1} \in V_{\epsilon_1} \), we define the formal expression \( \rho_0(X)[v_{\epsilon_1}] \) to be: \( \lim_{\epsilon \rightarrow 0^+} [\rho(t_{\epsilon}(X))\varphi_{t_{\epsilon}t} v_{\epsilon_1}] \) where \( \lim_{\epsilon \rightarrow 0^+} \) is the direct limit as constructed in the preceding section.

We note that \( \rho_0(X)[v_{\epsilon_1}] \) does not depend on the representative \( v_{\epsilon_1} \), because if \([v_{\epsilon_2}] = [v_{\epsilon_1}] \) and \( v_{\epsilon_2} \in V_{\epsilon_2} \) then without loss of generality we can assume that \( \epsilon_1 \geq \epsilon_2 \) so \( v_{\epsilon_2} = \varphi_{t_{\epsilon_2}t_{\epsilon_1}}(v_{\epsilon_1}) \Rightarrow [\rho(t_{\epsilon}(X))\varphi_{t_{\epsilon}t} v_{\epsilon_1}] = [\rho(t_{\epsilon}(X))\varphi_{t_{\epsilon_2}t} v_{\epsilon_2}] = [\rho(t_{\epsilon}(X))\varphi_{t_{\epsilon_1}t} v_{\epsilon_1}] \). \( V \) is an inner product space and hence also a metric space so the notion of limit of elements of \( V \) is defined. Now we are ready to give two equivalent definitions.

**Definition 3** Let \( \mathcal{G} = (U, [\cdot, \cdot]) \) be a Lie algebra such that \( \mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0 \). Let \( \{V_i, \varphi_{ij}, I_1\} \) be a directed system of inner product spaces and \( \{(\rho_i, V_i)\}_{i \in I_1} \) a family of representations of \( \mathcal{G} \). A representation \( \eta : \mathcal{G}_0 \rightarrow \text{gl}(W) \) is called a contraction of the family of representations \( \{(\rho_i, V_i)\}_{i \in I_1} \) with respect to the contraction \( \mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0 \) if

(i) For every \([v] \in V \), \( X \in U \) the limit \( \rho_0(X)[v] \) exists

(ii) There exists a linear invertible transformation \( K : V \rightarrow W \) such that for every \([v] \in V \), \( X \in U \) we have \( \rho_0(X)[v] = K^{-1} \eta(g)K([v]) \)

and we denote it by \( \rho_0 \xrightarrow{t(\epsilon)} \eta \).

**Definition 4** Let \( \mathcal{G} = (U, [\cdot, \cdot]) \) be a Lie algebra such that \( \mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0 \). Let \( \{V_i, \varphi_{ij}, I_1\} \) be a directed system of inner product spaces and \( \{(\rho_i, V_i)\}_{i \in I_1} \) a family of representations of \( \mathcal{G} \). If

(i) For every \([v] \in V \), \( X \in U \) the limit \( \rho_0(X)[v] \) exists

(ii) The map \( \rho_0 : \mathcal{G}_0 \rightarrow \text{gl}(V) \) is a Lie algebra homomorphism

we call \( \rho_0 \) a contraction of the family of representations \( \{(\rho_i, V_i)\}_{i \in I_1} \) with respect to the contraction \( \mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0 \) and we denote it by \( \rho_0 \xrightarrow{t(\epsilon)} \rho_0 \).
Definitions 3 and 4 are obviously equivalent, however in most cases the representation that is obtained under the limit is already known and therefore it is possible to avoid checking condition (ii) of definition 4. Definition 4 stresses the fact that the family of representations of $G$ along with the associated directed system possess all the data on the limit representation space.

Similarly, in the sequential case, let $I_2$ denote the totally ordered set which consists of the natural numbers, $\mathbb{N}$ and the usual order of natural numbers, i.e. $m < n \iff m \leq n$. Let $G = (U, [\cdot, \cdot])$ be a Lie algebra and suppose that $G \xrightarrow{l_n} \mathcal{G}_\infty$. Whenever we are given a directed system of inner product spaces indexed by $I_2$, $\{V_i, \varphi_{ij}, I_2\}$ and a sequence of representations of $G$, $\{(\rho_i, V_i)\}_{i \in I_2}$, on the same inner product spaces of the directed system, for every $X \in U$ and $[v_m] \in V$ such that $v_m \in V_m$, we define the formal expression $\rho_\infty(X)[v_m]$ to be:

$$\lim_{n \to \infty} [\rho_n(t_n(X))\varphi_{mn}(v_m)].$$

**Definition 5** Let $G = (U, [\cdot, \cdot])$ be a Lie algebra such that $G \xrightarrow{l_n} \mathcal{G}_\infty$. Let $\{V_i, \varphi_{ij}, I_2\}$ be a directed system of inner product spaces and $\{(\rho_i, V_i)\}_{i \in I_2}$ a family of representations of $G$. A representation $\eta : \mathcal{G}_\infty \to gl(W)$ is called a contraction of the sequence of representations $\{(\rho_i, V_i)\}_{i \in I_2}$ with respect to the contraction $G \xrightarrow{l_n} \mathcal{G}_\infty$ if

(i) For every $[v] \in V$, $X \in U$ the limit $\rho_\infty(X)[v]$ exists

(ii) There exists a linear invertible transformation $K : V \to W$ such that for every $[v] \in V$, $X \in U$ we have $\rho_\infty(X)[v] = K^{-1}\eta(X)K([v])$

and we denote it by $\rho \xrightarrow{l_n} \eta$.

The relation between the above definitions and the convergence of matrix elements in given in the following two simple propositions.

**Proposition 1** In the notations of the definitions above when $\rho \xrightarrow{l_n} \rho_0$, for every $v_1 \in V_{\epsilon_1}$, $v_2 \in V_{\epsilon_2}$ and $X \in U$

$$\langle v_1, \rho_0(X)[v_2] \rangle = \lim_{\epsilon \to 0^+} \langle \varphi_{\epsilon^1_1}(v_1), \rho_\epsilon(t_\epsilon(X))\varphi_{\epsilon_2^2}(v_2) \rangle_\epsilon. \quad (9)$$

**Proof 1** The proof simply follows from the definition of the inner product of the direct limit:

$$\langle v_1, \rho_0(X)[v_2] \rangle = \left\langle v_1, \lim_{\epsilon \to 0^+} [\rho_\epsilon(t_\epsilon(X))\varphi_{\epsilon_2^2}(v_2)] \right\rangle = \lim_{\epsilon \to 0^+} \langle v_1, [\rho_\epsilon(t_\epsilon(X))\varphi_{\epsilon_2^2}(v_2)] \rangle = \lim_{\epsilon \to 0^+} \langle \varphi_{\epsilon_1^1}(v_1), \rho_\epsilon(t_\epsilon(X))\varphi_{\epsilon_2^2}(v_2) \rangle_\epsilon \quad \square$$

We note that if in addition $K$ is unitary then we have

$$\langle K[v_1], \eta(X)K[v_2] \rangle_W = \langle v_1, \rho_0(g)[v_2] \rangle = \lim_{\epsilon \to 0^+} \langle \varphi_{\epsilon_1^1}(v_1), \rho_\epsilon(t_\epsilon(X))\varphi_{\epsilon_2^2}(v_2) \rangle_\epsilon \quad (11)$$

where $\langle \cdot, \cdot \rangle_W$ stands for the inner product on $W$.

**Proposition 2** Let $G = (U, [\cdot, \cdot])$ be a Lie algebra such that $G \xrightarrow{l_n} \mathcal{G}_0$. Let $\{V, \varphi_{ij}, I_1\}$ be a directed system of inner product spaces and $\{(\rho_i, V_i)\}_{i \in I_1}$ a family of representations of $G$.

Suppose that $\eta : \mathcal{G}_0 \to gl(W)$ is a representation such that $W$ has a countable orthonormal basis. Assume that for every $\epsilon \in I_1$ there is a linear transformation that preserves the inner product, $\tau_\epsilon : V_{\epsilon} \to W$ such that:

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(i) For every $\epsilon_1 \leq \epsilon_2$, $\tau_{\epsilon_2} = \tau_{\epsilon_1} \circ \varphi_{\epsilon_2 \epsilon_1}$
(ii) For every $w \in W$ there is some $\bar{\epsilon} \in I_1$ and some $v \in V_{\bar{\epsilon}}$, such that $\tau_{\bar{\epsilon}}(v) = w$
(iii) For every $X \in U$, $v_1, v_2 \in V_{\bar{\epsilon}}$

\[
\lim_{\epsilon \to 0^+} \langle \varphi_{\epsilon_0}(v_1), \rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)\rangle = \langle \tau_{\epsilon_0}(v_1), \eta(X)\tau_{\epsilon_0}(v_2)\rangle_W
\]

(iv) For every $X \in U$, every $\epsilon_0 \in I$ and every $v \in V_{\epsilon_0}$

\[
\dim \text{span} \left\{ \bigcup_{\epsilon \leq \epsilon_0} \tau_{\epsilon}\rho(\epsilon(X))\varphi_{\epsilon_0}(v) \right\} < \infty
\]

Then $(\eta, W)$ is a contraction of the family of representations $\{\rho(\epsilon), V_\epsilon\}_{\epsilon \in I_1}$ with respect to the contraction $\mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0$

**Proof 2** Let $V$ be the direct limit of $\{V_\epsilon, \varphi_{\epsilon t_\epsilon}, I_1\}$. We define a map $K : V \longrightarrow W$ by $K([v_\epsilon]) = \tau_{\epsilon_0}(v_\epsilon)$ for $v_\epsilon \in V_{\epsilon_0}$. It easily follows that $K$ is a well defined isometry. We will now show that the limit defining $\rho_0(X)$ exists and equals $K^{-1}\eta(X)K$.

For any $X \in U$, $[v_1], [v_2] \in V$ such that $v_1, v_2 \in V_{\epsilon_0}$ we have:

\[
\lim_{\epsilon \to 0^+} \langle [v_1], [\rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)] \rangle = \lim_{\epsilon \to 0^+} \langle \varphi_{\epsilon_0}(v_1), \rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)\rangle_{\epsilon}
\]

\[
= \langle \tau_{\epsilon_0}(v_1), \eta(X)\tau_{\epsilon_0}(v_2)\rangle_W = \langle K([v_1]), \eta(X)K([v_2])\rangle_W
\]

And since $K^{-1}$ is an isometry, we get

\[
\lim_{\epsilon \to 0^+} \langle [v_1], [\rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)] \rangle = \langle [v_1], K^{-1}\eta(X)K([v_2]) \rangle
\]

which means that $\{[\rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)]\}_{\epsilon \in [0,1]}$ weakly converges to $K^{-1}\eta(X)K([v_2])$ when $\epsilon$ goes to zero. From (iv) and the fact that weak convergence in finite dimensional spaces is equivalent to convergence in norm it follows that for every $X \in U$ and $v \in V_{\epsilon_0}$ $\{[\rho(\epsilon_t(X))\varphi_{\epsilon_0}(v_2)]\}_{\epsilon \in [0,1]}$ converges in norm to $K^{-1}\eta(X)K([v_2])$ as $\epsilon$ goes to zero. Hence for every $[v] \in V$, $X \in U$ the limit $\rho_0(X)[v]$ exists and we have $\rho_0(X)[v] = K^{-1}\eta(X)K([v])$. So we have proved that $\rho_t \xrightarrow{t(\epsilon)} \eta$.

3. Examples

In this section we demonstrate our definition of contraction of Lie algebra representations in the continuous and the discrete case.

The contraction of $so(3)$ to $iso(2)$

This contraction is given e.g., in [1, 3, 11]. We will formulate it in terms of the given definition while trying to keep some of the notations of [1]. The basis $\{J_x, J_y, J_z\}$ along with the commutation relations $[J_i, J_j] = \epsilon_{ijk}J_k$ define the Lie algebra $so(3)$. The basis $\{P_x, P_y, P_z\}$ along with the commutation relations $[L_z, P_x]_{iso(2)} = -P_y$, $[L_z, P_y]_{iso(2)} = P_x$ define the Lie algebra $iso(2)$. The contraction transformation $t_\epsilon(J_x) = \epsilon J_x$, $t_\epsilon(J_y) = \epsilon J_y$, $t_\epsilon(J_z) = J_z$ leads to the commutation relations $[J_z, J_x]_0 = J_y$, $[J_z, J_y]_0 = -J_x$ and hence it is isomorphic to $iso(2)$. 

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For every non negative integer \( j \), we denote by \((\rho^j, V_j)\) the \( 2j + 1 \) dimensional irreducible representation of \( so(3) \). \( V_j \) has the orthonormal basis \( \{ |\frac{j}{m}| m \in \{-j, -j + 1, \ldots, j\} \} \) on which the representation is given by

\[
\rho^j(J_z)|\frac{j}{m}\rangle = im|\frac{j}{m}\rangle \tag{14}
\]

\[
\rho^j(J_x)|\frac{j}{m}\rangle = -\frac{1}{2}\sqrt{(j-m)(j+m+1)}|\frac{j}{m+1}\rangle + \frac{1}{2}\sqrt{(j-m+1)(j+m)}|\frac{j}{m-1}\rangle \tag{15}
\]

\[
\rho^j(J_y)|\frac{j}{m}\rangle = \frac{i}{2}\sqrt{(j-m)(j+m+1)}|\frac{j}{m+1}\rangle + \frac{i}{2}\sqrt{(j-m+1)(j+m)}|\frac{j}{m-1}\rangle \tag{16}
\]

For every positive number \( \Xi \), \( iso(2) \) has an irreducible infinite dimensional representation, which we denote by \((\eta_\Xi, W_\Xi)\). In fact these are all the infinite dimensional irreducible representations of \( iso(2) \) that are integrable to a unitary representation of \( ISO(2) \). \( W_\Xi \) has an orthonormal basis \( \{ |\Xi_m| m \in \{0, \pm 1, \pm 2, \ldots\} \} \) on which the representation is given by

\[
\eta^\Xi(L_z)|\Xi_m\rangle = im|\Xi_m\rangle \tag{17}
\]

\[
\eta^\Xi(P_x)|\Xi_m\rangle = i\frac{\Xi}{m+1} + i\frac{\Xi}{m-1} \tag{18}
\]

\[
\eta^\Xi(P_y)|\Xi_m\rangle = -i\frac{\Xi}{m+1} + i\frac{\Xi}{m-1} \tag{19}
\]

We fix a positive number \( \Xi_0 \) and define a monotonous sequence of positive numbers that converges to zero by \( \epsilon_j = \frac{\Xi_0}{j} \). We take the directed system of inner product spaces to be \( \{V_n, \varphi_{nm}, I_2\} \), where for \( j \leq k \), \( \varphi_{jk}(|\frac{j}{m}\rangle) = |\frac{k}{m}\rangle \). We take the sequence of representations \( \{ (\rho^j, V_j) \}_{j \in I_2} \). One can show that the set \( B = \{ |\frac{m}{|m|}\rangle | m = 0, \pm 1, \pm 2, \ldots\} \) forms a basis for \( V \rightarrow V \).

For every \( |\frac{m}{|m|}\rangle \in B \) we have

\[
\rho_\infty(J_z)|\frac{m}{|m|}\rangle = \lim_{j \rightarrow \infty} [\rho^j((t_\epsilon(J_z))|\frac{j}{m}\rangle) = \lim_{j \rightarrow \infty} [im|m|] = \tag{20}
\]

\[
\lim_{j \rightarrow \infty} [im|m|] = im|m| \]

\[
\rho_\infty(J_x)|\frac{m}{|m|}\rangle = \lim_{j \rightarrow \infty} [\rho^j((t_\epsilon(J_x))|\frac{j}{m}\rangle) = \tag{21}
\]

\[
\lim_{j \rightarrow \infty} \left[ -\frac{\Xi_0}{2j} \sqrt{(j-m)(j+m+1)}|\frac{j}{m+1}\rangle + \frac{\Xi_0}{2j} \sqrt{(j-m+1)(j+m)}|\frac{j}{m-1}\rangle \right] \]

\[
\lim_{j \rightarrow \infty} \left\{ -\frac{\Xi_0}{2j} \sqrt{(j-m)(j+m+1)}|\frac{|m+1|}{m+1}\rangle + \frac{\Xi_0}{2j} \sqrt{(j-m+1)(j+m)}|\frac{|m-1|}{m-1}\rangle \right\} = \tag{22}
\]

\[
-\frac{\Xi_0}{2j} |\frac{|m+1|}{m+1}\rangle + \frac{\Xi_0}{2j} |\frac{|m-1|}{m-1}\rangle \]

\[
\rho_\infty(J_y)|\frac{m}{|m|}\rangle = \frac{i\Xi_0}{2j} |\frac{|m+1|}{m+1}\rangle + \frac{i\Xi_0}{2j} |\frac{|m-1|}{m-1}\rangle \]

In the above we used the fact that \( |\frac{m}{|m|}\rangle = |\frac{|m|}{m}\rangle \) and this holds since \( |\frac{m}{|m|}\rangle = \left\{ |\frac{|m|}{m}\rangle | j \geq |m| \right\} \).

We define \( K : V \rightarrow W_\Xi \_0 \) on \( B \) by \( K(\frac{m}{|m|}) = \Xi_0 \frac{m}{|m|} \) and we extend it by linearity to all \( V \). \( K \) is
obviously one to one and onto, and for every $\|\rho_i\|^j \in B$ we also have:

$$\rho_\infty(J_x)(\|\rho_i\|^j) = K^{-1}\eta_\Xi(\psi(J_x))K(\|\rho_i\|^j)$$  \hspace{1cm} (23)$$

$$\rho_\infty(J_y)(\|\rho_i\|^j) = K^{-1}\eta_\Xi(\psi(J_y))K(\|\rho_i\|^j)$$  \hspace{1cm} (24)$$

$$\rho_\infty(J_y)(\|\rho_i\|^j) = K^{-1}\eta_\Xi(\psi(J_y))K(\|\rho_i\|^j)$$  \hspace{1cm} (25)$$

where $\psi$ is the Lie algebra isomorphism between the contracted Lie algebra and $iso(2)$ that is defined by $\psi(J_x) = L_x$, $\psi(J_y) = P_y$ and $\psi(J_z) = P_x$. Hence $(\eta_\Xi, W_\Xi)$ is the contraction of the sequence of representations $\{(\rho^j, V^j\})_{j \in I_B}$ with respect to the sequential contraction $so(3) \xrightarrow{t_{(s)}} iso(2)$. Proposition 1 guarantees that we also have the following convergence of matrix elements

$$\langle \|\rho_i\|^j \|\rho_\infty(J_x)\|\|\rho_i\|^j \rangle = \lim_{j \to \infty} \langle \|\rho_i\|^j | \rho^j(t_{(s)}(J_x)) | \|\rho_i\|^j \rangle = i \rho_{(s)} = \frac{\rho_{(s)} \Xi 0}{2} = \frac{\Xi 0}{2}$$  \hspace{1cm} (26)$$

$$\langle \|\rho_i\|^j \|\rho_\infty(J_y)\|\|\rho_i\|^j \rangle = \lim_{j \to \infty} \langle \|\rho_i\|^j | \rho^j(t_{(s)}(J_y)) | \|\rho_i\|^j \rangle = \frac{\rho_{(s)} \Xi 0}{2} = \frac{\Xi 0}{2}$$  \hspace{1cm} (27)$$

$$\langle \|\rho_i\|^j \|\rho_\infty(J_z)\|\|\rho_i\|^j \rangle = \lim_{j \to \infty} \langle \|\rho_i\|^j | \rho^j(t_{(s)}(J_z)) | \|\rho_i\|^j \rangle = \frac{\rho_{(s)} \Xi 0}{2} = \frac{\Xi 0}{2}$$  \hspace{1cm} (28)$$

$$\rho_\infty(J_x) = K\eta_\Xi(J_x)K^{-1}$$  \hspace{1cm} (29)$$

We note we could have proven this contraction using proposition 2 by defining $\tau_j(\|\rho_i\|^j) = \rho_{(s)}$.}

**The contraction of $iso(1, 1)$ to to Heisenberg Lie algebra**

The basis $\{X_1, X_2, X_3\}$ along with the commutation relations $[X_3, X_1] = X_1$, $[X_3, X_2] = -X_2$ define the Lie algebra $iso(1, 1)$. The basis $\{a, a^\dagger, 1\}$ along with the commutation relations $[a, a^\dagger] = 1$ define the Lie algebra of the Heisenberg group. We denote this Lie algebra by $\mathfrak{h}$. The contraction transformation $t_{(s)}(X_1) = 2aX_1$, $t_{(s)}(X_1 + X_2) = X_1 + X_2$, $t_{(s)}(X_3) = \epsilon X_3$ leads to the commutation relations $[X_3, X_1 + X_2]_0 = X_1$ and hence it is isomorphic to $\mathfrak{h}$ and the isomorphism $\psi$ is given by $\psi(X_1) = 1$, $\psi(X_1 + X_2) = a^\dagger$, $\psi(X_3) = a$.

In this example we realize the representations by differential operators on certain functions spaces which seem more suitable for the contraction of these representations. Let $L^2_c(\mathbb{R}, dx)$ denote the inner product space of smooth compactly supported functions that are square integrable relative to the inner product $(f, g) = \int f(x) \overline{g(x)} dx$ Using the Mackey machine [15], it is easy to show that for every $a, b \in \mathbb{R}$ such that $a^2 + b^2 \neq 0$ the map

$$\rho(a, b) : iso(1, 1) \rightarrow gl(L^2_c(\mathbb{R}, dx))$$  \hspace{1cm} (30)$$
\[ \rho_{(a,b)}(X_1) = iae^x \]
\[ \rho_{(a,b)}(X_2) = ibe^{-x} \]
\[ \rho_{(a,b)}(X_3) = \frac{d}{dx} \]

is an irreducible representation of \( \text{iso}(1,1) \) which is a differential of a unitary irreducible representation of \( \text{ISO}(1,1) \). In fact these are all such representation of \( \text{iso}(1,1) \) that are infinite dimensional.

It is well known that for any \( A \in \mathbb{R}^* \), the maps
\[ \eta_A : \mathfrak{h} \rightarrow g(l(L^2_{c,\infty}(\mathbb{R}, dx))) \]
\[ \eta_A(\mathbb{1}) = iA \]
\[ \eta_A(a^\dagger) = iAx \]
\[ \eta_A(a) = \frac{d}{dx} \]

define an irreducible representation of \( \mathfrak{h} \) and in fact they exhaust all irreducible representations of \( \mathfrak{h} \) that are differentials of unitary representations of the Heisenberg Lie group.

We fix a non zero real number \( A_0 \). We now construct the directed system and the family of representations of \( \text{iso}(1,1) \) that contracts to \( \eta_{A_0} \). For every \( \epsilon \in I_1 \) let \( V_\epsilon \) be the inner product space \( L^2_{c,\infty}(\mathbb{R}, dx) \) and we define a function \( \psi_\epsilon : \mathbb{R} \rightarrow \mathbb{R} \) by \( \psi_\epsilon(x) = \epsilon x \). We note that \( P_\epsilon : L^2_{c,\infty}(\mathbb{R}, dx) \rightarrow V_\epsilon \) which is given by \( P_\epsilon(f) = f \circ \psi_\epsilon \) is an isomorphism. Its inverse \( P_\epsilon^{-1} \) is given by \( P_\epsilon^{-1}(f) = f \circ \psi_\epsilon^{-1} \).

For each \( \epsilon \in I_1 \) and \((a,b) \in \mathbb{R} \times \mathbb{R}^*\) we intertwine \( \rho_{(a,b)} \) with \( P_\epsilon \) to get the equivalent representation \( \rho_{(a,b,\epsilon)} = P_\epsilon \circ \rho_{(a,b)} \circ P_{\epsilon}^{-1} \) given by:
\[ \rho_{(a,b,\epsilon)} : \text{iso}(1,1) \rightarrow gl(V_\epsilon) \]
\[ \rho_{(a,b,\epsilon)}(X_1) = iae^{\epsilon x} \]
\[ \rho_{(a,b,\epsilon)}(X_2) = ibe^{-\epsilon x} \]
\[ \rho_{(a,b,\epsilon)}(X_3) = \frac{1}{\epsilon} \frac{d}{dx} \]

For \( \epsilon_i, \epsilon_j \in I_1 \) such that \( \epsilon_i \geq \epsilon_j \) we define the linear isometry \( \varphi_{\epsilon_i,\epsilon_j} : V_{\epsilon_j} \rightarrow V_{\epsilon_i} \) to be the identity map. These maps are obviously compatible. Since all the inner product spaces \( V_\epsilon \) are \( L^2_{c,\infty}(\mathbb{R}, dx) \) and all the transition maps are the identity maps, obviously the direct limit of this directed system is naturally isomorphic to \( L^2_{c,\infty}(\mathbb{R}, dx) \) and we will address \( L^2_{c,\infty}(\mathbb{R}, dx) \) as the direct limit in this case. Essentially this means that instead of working with the equivalence class \([f(x)]\) we can just work with \( f(x) \). Let \( a(\epsilon) = \frac{A_0}{2\epsilon}, b(\epsilon) = -\frac{A_0}{2\epsilon} \). For every \( f \in V = L^2_{c,\infty}(\mathbb{R}, dx) \) and every \( x \in \mathbb{R} \) we note that
\[ \rho_0(X_1)(f)(x) = \lim_{\epsilon \rightarrow 0^+} \rho_{(a(\epsilon), b(\epsilon), \epsilon)}(t_\epsilon(X_1))(f)(x) = \]
\[ \lim_{\epsilon \rightarrow 0^+} iA_0 e^{\epsilon x}(f)(x) = iA_0(f)(x) = \eta_{A_0}(\psi(X_1))(f)(x) \]
\[ \rho_0(X_1 + X_2)(f)(x) = \lim_{\epsilon \rightarrow 0^+} \rho_{(a(\epsilon), b(\epsilon), \epsilon)}(t_\epsilon(X_1 + X_2))(f)(x) = \]
\[ \lim_{\epsilon \rightarrow 0^+} \frac{A_0}{\epsilon} \sinh(\epsilon x)(f)(x) = iA_0 x(f)(x) = \eta_{A_0}(\psi(X_1 + X_2))(f)(x) \]
\[ \rho_0(X_\delta)(f)(x) = \lim_{\epsilon \to 0^+} \rho_{(a(\epsilon),b(\epsilon),\epsilon)}(t_\epsilon(X_\delta))(f)(x) = \] 

Since \( f \) is compactly supported then the pointwise convergences appearing above also imply convergence in \( L^2_c(\mathbb{R}, dx) \) and hence \( \eta_{A_0} \) is the contraction of the family of representations \( \{ (\rho_{(a(\epsilon),b(\epsilon),\epsilon)}, V_\epsilon) \}_{\epsilon \in I_1} \) with respect to the contraction \( \text{iso}(1,1) \xrightarrow{t_{\epsilon}} \mathfrak{h} \) where the identity map takes the role of the intertwiner \( K \) that appears in definition 4.

4. Discussion
We have shown how the notion of direct limit naturally appears in the theory of contraction of Lie algebra representations. The definition that was given enabled us to give intrinsic meaning for the limit representation space as a direct limit space and to the limit operators as operators on that direct limit space. An essential ingredient in our construction, that was implicit in the work of İnönü and Wigner [1], is the family of compatible embeddings.

We have demonstrated how to find these embeddings and the associated directed system in the two major procedures for obtaining contraction of Lie algebra representations: convergence of matrix elements and convergence of differential operators. We have shown that these procedures are essentially equivalent. In particular, if one obtains contraction of representations by convergence of differential operators then by proposition 1 the convergence of the matrix elements is assured.

In all examples, known to the authors, in which contraction of Lie algebra representations is given by convergence of matrix elements the conditions of proposition 2 hold and hence it is also contraction of representations according to definition given above. In all other examples known to the authors, where contraction of representations is obtained by convergence of differential operators, the implicit directed system could be found as in the second example above.

Acknowledgments
AM is grateful to Prof. Weimar-Woods for a helpful discussion.
The research of the 2nd author was supported by the center of excellence of the Israel Science Foundation grant no. 1691/10.
JLB thanks the Department of Physics, Technion, for its warm hospitality and support during visits while this work was being carried out, and the FRAP-PSC-CUNY for some support.

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