Conformal invariance in three-dimensional rotating turbulence

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We examine three–dimensional turbulent flows in the presence of solid-body rotation and helical forcing in the framework of stochastic Schramm-Löwner evolution curves (SLE). The data stems from a run on a grid of 1536\textsuperscript{3} points, with Reynolds and Rossby numbers of respectively 5100 and 0.06. We average the parallel component of the vorticity in the direction parallel to that of rotation, and examine the resulting $\langle \omega_z \rangle$ field for scaling properties of its zero-value contours. We find for the first time for three-dimensional fluid turbulence evidence of nodal curves being conformal invariant, belonging to a SLE class with associated Brownian diffusivity $\kappa = 3.6 \pm 0.1$. SLE behavior is related to the self-similarity of the direct cascade of energy to small scales in this flow, and to the partial bi-dimensionalization of the flow because of rotation. We recover the value of $\kappa$ with a heuristic argument and show that this value is consistent with several non-trivial SLE predictions.

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Self-similarity in physics is a common phenomenon, with identical properties of a system when considered at different scales. Rugged coast lines, fractals, traffic in computer networks, growth processes, geometrical properties of interfaces, phase transitions in critical phenomena such as in the Ising model for spontaneous magnetization, classical and quantum field theory, often display power-law scaling of some variable and such scaling exponents have been the object of intense investigations resulting in the finding of broad classes of universality.

A property stronger than scale invariance is conformal invariance, under transformations that preserve angles with rescaling that depends on position; it is difficult to test, since it implies the need to investigate the scaling of multi-point high-order correlation functions. However, recent developments by Schramm in particular (see e.g. [1] and references therein) allow in some cases for a statistical characterisation of conformal invariance. Such scaling laws can be related to Brownian motion (which is scale invariant, and conformal in two dimensions) in what is now named Schramm-Löwner evolution (SLE), with as sole parameter the diffusivity $\kappa$ associated with this Brownian motion. In this approach, the driving of the Löwner equation (Eq. 2 below) is stochastic, with a conformal map allowing to go from static (fixed time) two-dimensional (2D) paths in the complex plane $\mathbb{C}$ to “dynamic” one-dimensional (1D) motions. In other words, it allows one to describe paths in $\mathbb{C}$ by a succession (convolution) of conformal maps obeying a differential equation. Schramm’s theorem (see, e.g., [1]) states that if and only if the driving is Brownian is the measure of the 2D paths conformally invariant.

Two-dimensional turbulence differs in many ways from the three-dimensional (3D) case because of the presence of an extra invariant in the absence of viscosity, the enstrophy $S = \langle |\nabla \times \mathbf{u}|^2/2 \rangle$, leading to an inverse cascade of energy $E = \langle |\mathbf{u}|^2/2 \rangle$ to large scales [2], with $\mathbf{u}$ the velocity field. It was shown in [3] that this inverse cascade, which is known to lack intermittency and is self-similar, can be viewed in the framework of conformal invariance when examining zero-vorticity lines; it belongs to the universality class with $\kappa = 8/3$ (the enstrophy cascade to small scales, however, is not SLE [3]). These results stem from direct numerical simulations (DNS) on grids of up to 16,384\textsuperscript{2} points, with forcing at intermediate wavenumber, $k_F/k_{\text{min}} \approx 100$, with $k_{\text{min}} = 2\pi/L_0$, $L_0$ being the size of the vessel.

In the case of 3D Navier-Stokes (NS) incompressible flows at high Reynolds numbers, the cascade of energy to small scales is not self-similar, because of the presence of strong vorticity gradients. Only one time scale is present, the eddy turn-over time $\tau_{\text{NL}} \sim l/\nu$, with $\nu$ the velocity at scale $l$, and dimensional analysis gives an energy spectrum $E(k) \propto k^{-5/3}$ that is quite close to observed spectra in the atmosphere or in laboratory experiments. However, when introducing solid body rotation $\Omega$ with inertial time $\tau_{\Omega} \sim 1/\Omega$, $E(k)$ steepens and its spectral index can be recovered by taking into account the weakening of nonlinear interactions due to the inertial waves [4]. In this case, self-similarity and Gaussianity in the 3D direct energy cascade was found recently both in the laboratory [5, 6] and in DNS [7, 8], more clearly so in the presence of helicity, i.e., velocity-vorticity correlations [9].

Since rotating flows tend to become quasi-2D (but not strictly 2D, as our results will confirm) when strong rotation is imposed, the question thus arises as to whether SLE can be identified in such flows. To this end, we examine the large data set produced in a run of rotating helical turbulence on a grid of 1536\textsuperscript{3} points, with $L_0 = 2\pi$ and forcing at $k_F=7$; an inverse cascade of energy to large scales (with constant negative flux) is observed, but with too little extent in wavenumber to allow for a SLE analysis similar to that performed in [3] for the 2D NS inverse cascade. We concentrate instead on the direct energy
The rotation is imposed in the vertical (z) direction, \( F \) and \( P \), and the total helicity order Runge-Kutta. Note that in 3D, besides energy, the de-aliasing rule, and the temporal scheme is a second-order Runge-Kutta. The direction along the trajectories is as trajectories in the 2D plane that keep positive field. We integrated the 3D NS equations in the rotating frame for an incompressible flow. The rotation for an inverse cascade a Reynolds number \( Re = U_0^2 \pi / [\nu k_F] \approx 5100 \) (with \( \nu \) the viscosity), and the Rossby number \( Ro = U_0^2 k_F / [2 \pi \Omega] \approx 0.06; U_0 \approx 1 \) is the r.m.s. velocity. We integrated the 3D NS equations in the rotating frame for an incompressible flow (\( \nabla \cdot \mathbf{u} = 0 \)); with \( \omega = \nabla \times \mathbf{u} \) the vorticity, they read:

\[
\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + 2 \Omega \times \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{F}; \tag{1}
\]

\( \mathcal{P} \) is the total pressure modified by the centrifugal term, and \( \mathbf{F} \) is a helical Arnold-Beltrami-Childress forcing. The rotation is imposed in the vertical (z) direction, with \( \Omega = 9 \pi/2 \). The code is fully parallelized, uses the 2-axis, we explore iso-contours of zero field. We have shown numerically for this flow the existence of scale invariance for the direct energy cascade and the Gaussianity of the velocity in [9] (see Figs. 7 and 8), also examining anisotropy at different times using a \( SO(2) \times \mathbb{R} \) decomposition (see Figs. 2 and 3 in [9] for the actual scaling ranges). We now probe the conformal invariance of these 2D curves viewed as paths in the upper complex plane; the paths are encoded in a “driving function” \( \xi(\tau) \) obtained through the chordal Löwner equation below, with \( g_{\alpha}(\zeta) \) (\( \zeta \in \mathbb{C} \)) a conformal map (see, e.g., [12]):

\[
\partial_\tau g_{\tau}(\zeta) = \frac{2}{g_{\tau}(\zeta) - \xi(\tau)}; \tag{2}
\]

\( \xi(\tau) \) is the unknown 1D real continuous stochastic driving function for the path. In order to estimate \( \xi(\tau) \) numerically, we use the zipper algorithm (ZA) with vertical slits [14]. Then \( g_{\alpha,\delta \tau}(\zeta) = a + \sqrt{|(\zeta - a)^2 + 4 \delta \tau|} \) conformally maps the upper plane minus the vertical slit in \( \mathbb{C}, [(a, 0); (a, 2 \sqrt{\tau})] \), into the upper plane: ZA gradually zips the whole path onto the x-axis using the composition of functions \( g_{\alpha,\delta \tau}(\zeta) \) for different \( \delta \tau \). We thus transform the erratic nodal line in the plane (inset in Fig. 1 described below) into an unknown motion along the real axis, \( \xi(\tau) \).

To test for conformal invariance, we therefore must ask: Is \( \xi(\tau) \) a Gaussian process? Does it correspond to a Brownian motion? And if so, what is its diffusivity? To answer the first question, one can use the classical Kolmogorov-Smirnov (KS) test, and check (i) whether its \( pKS \) value is above a given threshold for a wide range of driving times \( \tau \), and (ii) whether the steps in this motion are independent. When both tests are favorable, we then consider the scaling of the variance of \( \xi(\tau) \). If the scaling is reasonably linear with \( \tau \), we will conclude that the set of driving functions likely stems from a Brownian process, and hence that the vorticity isolines obtained as indicated above are likely to be conformally invariant. The linear scaling also gives us the diffusivity \( \kappa \) which describes entirely the statistics of the SLE process.

**Results:** We now apply the procedure to the 15363 DNS data. After performing the average (either in \( x \) or in \( y \)), fifteen temporal snapshots are analyzed, separated by approximately one eddy turn-over time. The resulting dataset has in excess of \( 3.5 \times 10^7 \) points for each averaging direction. In Fig. 1 is given a snapshot of the flow. The slice in the bottom shows the same field component when vertically averaged, \( \langle \omega_3 \rangle_z \) (bottom slice). The flow displays features of both
Figure 2 summarizes the analysis, for \( \tau \) as a function of the angle \( \theta \), as illustrated by the sketch at the bottom. Inset: Mean gyration radius \( D \) of nodal lines as a function of the number of pixels \( N \). Dashed lines: theoretical predictions for \( \kappa_S = 3.6 \); open and filled circles as in Fig. 2.

Finally, we confirm the scaling we found for \( \kappa \) by examining some of the predictions on statistical properties of nodal lines of \( \bar{\omega}_z \) that can be made using the SLE framework (see 11). A classical one concerns the fractal dimension of the nodal lines, but less trivial features predicted by SLE include for instance the so-called “winding angle” or the gyration radius. The winding angle prediction states that the probability \( P_{\text{left}} \) of a SLE line to leave a point \( z_0 = \rho e^{i\theta} \) in \( \mathbb{C} \) to its left depends only on \( \kappa \) and \( \theta \), following a known expression 11. Figure 3 shows the results obtained from our datasets as a function of \( \theta \), as well as the mean gyration radius of the nodal lines as a function of their length in pixels in the top-right inset. In both cases, the SLE predictions for \( \kappa = 3.6 \), given with the dashed lines, appears convincing.

We thus conclude that our analysis identifies conformal invariance for nodal lines of the vertical component of the vorticity field when averaging parallel to the direction of rotation, and fails to identify such invariance.
in its transverse average. For the parallel-averaged vertical vorticity, the associated diffusivity is $\kappa \approx 3.6 \pm 0.1$. Moreover, SLE predictions for this value of $\kappa$ agree well with our results. It is also important to remark that our analysis could fail to reject the hypothesis of Gaussianity if data were insufficiently resolved; this is not surprising since it is hard to distinguish SLE behavior from something close to SLE. In spite of these limitations, the data analyzed here up to the spatial resolution considered is found to be consistent with SLE behavior.

Discussion: Rotating helical turbulence may be perhaps the first documented case presenting SLE scaling for three-dimensional flows undergoing a direct cascade of energy and of helicity to small scales, when properly averaged in the direction of rotation. Conformal invariance is a strong local property and allows determination of a series of scaling laws, as exemplified in [3, 10] for 2D NS and other related 2D cases such as surface quasi-geostrophic (SQG) flows, and as found here as well. SLE obtains convincingly for the vertical component of the geostrophic (SQG) flows, and as found here as well. SLE of a series of scaling laws, as exemplified in [3, 10] for 2D NS and other related 2D cases such as surface quasi-geostrophic (SQG) flows, and as found here as well. SLE is found to be consistent with SLE behavior.

Hence, the value of $\kappa$ is quite sensitive to $e$ or $a_S$ [16]. For 2D NS, $a_S=1/3$ and $\kappa=8/3$, as found in [3] (with dual value $\kappa^* = 6$). For rotating helical turbulence, $a_S=3/4$, using a phenomenological model based on three assumptions [7, 9]: wave-modulated energy spectrum; domination of the helicity cascade to small scales; and maximal helicity. The first hypothesis allows to write that the transfer of energy to small scale is slowed down in the proportion $\tau_{NL}/\tau_{TL}$; the second one stems from the fact that the energy undergoing an inverse cascade to large scale, little energy is left to feed the small scales, whereas helicity only possesses a small-scale cascade and thus is the determining factor in this direct cascade. These two concepts lead to $e + h = 4$, with helicity spectrum $H(k) \sim k^{-h}$. The third assumption gives $h = e - 1$ and thus $\zeta_p = 3p/4$, a value reported experimentally as well [6, 10]. From Eq. [3], we then obtain $\kappa = 4$, close to the value we find given the statistics.

The connection between SLE and statistical properties of turbulence allows one to look at such flows with a new eye, and to build bridges between fluid dynamics and other research areas in mathematics, condensed matter, percolation, and quantum field theory. Other three-dimensional flows may be studied with the same tools when the flow is self-similar and symmetries allow for a reduction of dimensionality. As an example, we leave for future work an investigation of SLE properties in the inverse cascade of rotating turbulence.

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