Magnetization and dynamically induced finite densities in three-dimensional Chern-Simons QED

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Abstract

In (2 + 1)-dimensional QED with a Chern-Simons term, we show that spontaneous magnetization occurs in the context of finite density vacua, which are the lowest Landau levels fully or half occupied by fermions. Charge condensation is shown to appear so as to complement the fermion anti-fermion condensate, which breaks the flavor U(2N) symmetry and causes fermion mass generation. The solutions to the Schwinger-Dyson gap equation show that the fermion self-energy contributes to the induction of a finite fermion density and/or fermion mass. The magnetization can be supported by charge condensation for theories with the Chern-Simons coefficient $\kappa = N e^2/2\pi$, and $\kappa = N e^2/4\pi$, under the Gauss law constraint. For $\kappa = N e^2/4\pi$, both the magnetic field and the fermion mass are simultaneously generated in the half-filled ground state, which breaks the U(2N) symmetry as well as the Lorentz symmetry.

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I. INTRODUCTION

Field theories in (2 + 1)-dimensional space-time have been intensively studied not only as a laboratory for (3 + 1)-dimensional field theory dynamics but also as effective theories at long distance in planar condensed matter physics [1]. In particular, some authors have investigated quantum electrodynamics in 2 + 1 dimensions (QED$_3$) in connection with the effective field theories for high-$T_c$ super-conductivity [2]. In this theory, there can be a topological, i.e. metric independent, gauge action, known as the Chern-Simons (CS) term, which connects a magnetic field $B$ with an electric charge density $e\langle\psi^\dagger\psi\rangle$ for the fermion field $\psi$. Because of this peculiar property, the CS term is essential in the field theoretical understanding of the fractional quantum Hall effect, which is based on the composite excitation of an electron and magnetic fluxes [3,4].

A theory whose gauge field action includes both the CS term and the Maxwell term was proposed by Deser, Jackiw, and Templeton [7] as an attempt to improve the infrared photon behavior in QED$_3$. In this theory (Chern-Simons QED$_3$) the coefficient $\kappa$ for the CS term gives the photon a gauge invariant mass which explicitly violates parity. Hosotani [8] showed that spontaneous magnetization occurs in Chern-Simons QED$_3$, because the gauge invariant photon mass $\kappa$ is completely screened out by parity-violating vacuum polarization effects. This also means that Lorentz symmetry is broken through the induced magnetic field and the resultant massless photon plays the role of the Nambu-Goldstone (NG) mode. In this theory, the Gauss law,

$$\kappa B = -e\langle\psi^\dagger\psi\rangle,$$

follows from the equation of motion. Thus the magnetized vacuum corresponds to the ground state with finite fermion density, $\langle\psi^\dagger\psi\rangle \neq 0$, that is, an occupied lowest Landau level.

In Ref. [8], however, the finite density vacuum configuration was set by hand as a state with a definite filling fraction $\nu$ that represents the degrees of occupation of fermions in the lowest Landau level. The Gauss law (1.1) tells us that spontaneous magnetization means a phase transition occurs between different vacua assigned to specific values of $\nu$. However, without any parameter connecting differently occupied vacua, it was not clear whether the condensate $\langle\psi^\dagger\psi\rangle$, which supports the magnetic field, is spontaneously induced or finely tuned.

In order to clarify the above situation, an external source term $\mu \psi^\dagger\psi$ was introduced in Ref. [9] to control the finite densities in the vacuum with a continuous parameter $\mu$. From the viewpoint of statistical mechanics, $\mu$ plays the role of the chemical potential. The parameter $\mu$ has to exceed the fermion mass $m$, or the energy of lowest Landau level, in order to induce a finite fermion density. We notice, however, from the field theoretical viewpoint, the chemical potential term $\mu \psi^\dagger\psi$ also plays the role of a term that explicitly breaks Lorentz

$^1$The application of gauge field theories to high-$T_c$ super-conductivity was originally proposed and studied in Ref. [3] in the context of the resonating valence bond state. The non-Fermi liquid behavior in the normal phase of the high-$T_c$ system has been studied with the help of effective gauge theories and their renormalization group analysis. See Ref. [3] for details.
symmetry. So long as the fermion mass $m$ remains constant, we cannot take the Lorentz symmetric limit $\mu \to 0$ while retaining non-zero finite density in the vacuum. Alternatively, however, if we take the symmetric limit such that $m \leq |\mu| \to 0$, the finite density vacuum might be spontaneously realized through the condensate $\langle \psi^\dagger \psi \rangle$. In other words, the finite density vacua are *spontaneously* realized if and only if the fermion bare mass becomes zero.

Recently, in Refs. [10,11] it was shown that a strong magnetic field enhances the condensate $\langle \bar{\psi} \psi \rangle$, or the dynamical generation of fermion mass, in $(2+1)$-dimensional four-fermion interaction models. Refs. [10,12] explained this effect as the dimensional reduction of the phase space for charged particles coupled to a strong magnetic field. In a strong magnetic field, the wave function for charged fermions is localized within a region whose size is given by the magnetic length: $l = 1/\sqrt{|eB|}$. The fermion in $2+1$ dimensions, therefore, behaves like that in $0+1$ dimension, while the photon field is charge neutral and propagates freely in $2+1$ dimensional space-time even in the presence of the background magnetic field. The condensate $\langle \bar{\psi} \psi \rangle$ is therefore easily formed much like in Bardeen-Cooper-Schrieffer (BCS) theory [10,12].

The Lagrangian of QED$_3$ with $N$ flavor massless four-component fermions has U$(2N)$ symmetry. When $N$ is smaller than its critical value $3 < N_c < 5$, the flavor U$(2N)$ symmetry is spontaneously broken by the condensate $\langle \bar{\psi} \psi \rangle$ and the fermion acquires a dynamically generated mass [15,16]. By means of the Schwinger-Dyson (SD) gap equation, Shpagin [17] showed that a fermion mass is dynamically generated and the U$(2N)$ symmetry is spontaneously broken irrespective of $N$ in QED$_3$ with an external magnetic field.

This result means that the magnetic field catalyzes the fermion mass generation and seems contradictory to the results of Refs. [8,9] at first sight, since the magnetic field is spontaneously induced only for massless fermions in Refs. [8,9]. If both results of Refs. [8,9] and Ref. [17] are true, the chemical potential $\mu \psi^\dagger \psi$ should affect the condensate $\langle \bar{\psi} \psi \rangle$ in the presence of a background magnetic field. It has obviously become an interesting question in Chern-Simons QED$_3$ to study the dynamical generation of both condensates, $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} \bar{\psi} \rangle$, or the fermion mass and the magnetic field, as a specific $(2+1)$-dimensional phenomenon.

In this paper, we will directly show that the spontaneous magnetization, or Lorentz symmetry breaking, occurs as the realization of the finite density vacuum in Chern-Simons QED$_3$. We will also examine the possibility that Lorentz symmetry is broken in a theory where both massive fermions and a magnetic field are spontaneously generated [18].

The paper is laid out as follows. In Section II we derive the effective action in the large $N$ limit that contains the screening of the $N$ flavors of fermions. We add to the Lagrangian both the fermion bare mass term $m \bar{\psi} \psi$ as an explicit breaking source term for the flavor U$(2N)$ symmetry, and the chemical potential term $\mu \bar{\psi} \psi$ for Lorentz symmetry. In Section III we directly calculate the condensates, $\langle \bar{\psi} \psi \rangle$ and $\langle \bar{\psi} \bar{\psi} \rangle$, as the large $N$ quantum correction of the fermion field in the presence of the background magnetic field. The effect of the explicit breaking parameters and the magnetic field $B$ are taken into account in the fermion

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2The enhancement of fermion mass generation by a magnetic field was found also in lattice QED$_3$ [13] and was applied to the effective field theories in planar condensed matter systems [14].
propagator by using the proper time method \[19\]. Thus we can see the dependence of \(m\) and \(\mu\) upon the condensates, \(\langle \bar{\psi}\psi \rangle\) and \(\langle \psi^\dagger\psi \rangle\), explicitly. It is found from the Gauss law (1.1) that the consistency condition for nonzero magnetic field places a restriction on the CS coefficient \(\kappa\) in the symmetric limit \(\mu \to 0, m \to 0\). In Section \[17\] we calculate the vacuum polarization tensor in the large \(N\) limit in a gauge invariant manner. We regularize the parity-violating part, or the induced CS term, in a manner consistent with the charge condensation. The improved photon propagator including the effect of vacuum polarization is also derived. In Section \[17\] we construct and solve the Schwinger-Dyson (SD) gap equation for the fermion self-energy that contains the dynamically induced fermion mass \(m_d\) and chemical potential \(\mu_d\). We investigate the solutions to the SD equation in the symmetric limit \(\mu \to 0, m \to 0\), to clarify whether or not the fermion mass and/or the magnetic field are dynamically generated.

Section \[18\] is devoted to the study of vacuum stability. We calculate the Cornwall-Jackiw-Tomboulis (CJT) potential \[20\] for the composite operators, \(\bar{\psi}\psi\) and \(\psi^\dagger\psi\), and the shift of zero-point energies for the fermions and the photon. In Section \[17\] we conclude our paper with a discussion of the relation between the results in Refs. \[8,9\], \[17\], and ours.

**II. LARGE \(N\) EFFECTIVE THEORY IN CHERN-SIMONS QED\(_3\)**

In this section we construct the large \(N\) effective action in Chern-Simons QED\(_3\), which includes long range screening of charged fermions. We use the metric diag\((g^{\mu\nu}) = (-1, 1, 1)\) throughout the paper.

**A. The model and the symmetries**

The Lagrangian density of Chern-Simons QED\(_3\) with massless fermion is

\[
\mathcal{L} = \bar{\psi} \gamma^\mu \left[i\partial_\mu + eA_\mu\right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\kappa}{2} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho - \frac{1}{2\xi} (\partial A)^2,
\]

where the third term is the Chern-Simons term which is topological (or metric independent) and the last term is a covariant gauge fixing term. From the Lagrangian (2.1) we obtain the equation of motion

\[
\partial_\nu F^{\nu\mu} - \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = -e\bar{\psi} \gamma^\mu \psi
\]

whose vacuum expectation value leads to the Gauss law constraint \(\kappa B = -e\langle \psi^\dagger\psi \rangle\) [cf. Eq. (1.1)], under a given magnetic field.

The smallest spinor representation of Lorentz group in \(2 + 1\) dimensions is provided by two-component spinors. A corresponding \(2 \times 2\) matrix representation of the Clifford algebra is expressed by the Pauli matrices

\[
\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2,
\]

which obey the anti-commutation relation: \(\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\). We have no other \(2 \times 2\) matrix which anti-commutes with all \(\gamma^\mu\). We, therefore, have no chiral symmetry that
would be broken by a mass term \( m \overline{\psi} \psi \) in two-component representation, but the discrete parity symmetry is broken by the same mass term. To construct a continuous group which resembles the chiral symmetry in \( 3 + 1 \) dimensions, it is necessary to enlarge the spinor representation to be the one with four-component spinors. Now we have a Clifford algebra given by three \( 4 \times 4 \) matrices, namely,

\[
\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.
\] (2.4)

We can then construct another two \( 4 \times 4 \) matrices

\[
\gamma^3 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^5 = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\] (2.5)

which anti-commute with \( \gamma^0, \gamma^1, \) and \( \gamma^2 \). Using \( \gamma^3 \) and \( \gamma^5 \), we can define a \( U(2) \) group with generators

\[
\tau_0 = I, \quad \tau_1 = \gamma^5, \quad \tau_2 = -i\gamma^3, \quad \text{and} \quad \tau_3 = \gamma^3\gamma^5.
\] (2.6)

We see that the Lagrangian density for a massless four-component fermion is invariant under this global \( U(2) \) symmetry. The appearance of mass term \( m \overline{\psi} \psi \) breaks the \( U(2) \) symmetry to a subgroup \( U(1) \times U(1) \) whose generators are \( \tau_0 \) and \( \tau_3 \). For the system of \( N \) flavor four-component fermions, global symmetry is extended to \( U(2N) \) symmetry which is nothing but the direct product of the \( U(2) \) symmetry and the flavor \( U(N) \) symmetry. Once again, the mass term \( m \overline{\psi} \psi \) breaks this \( U(2N) \) symmetry to its subgroup \( U(N) \times U(N) \).

In \( 2 + 1 \) dimensions, the inversion of two space axes is equivalent to a \( \pi \) rotation on the two-dimensional space so that parity transformation should be defined as an inversion of one spatial axis: \( P, (x, y) \rightarrow (x, y)_P = (-x, y) \). If we define the parity transformation of field operators, it should leave the Lagrangian of massless QED\(_3\) remain invariant. A specific construction is to transform a two-component spinor and the gauge field as follows;

\[
P\psi(t, x)P^{-1} = \sigma_1\psi(t, x_P), \quad P A^\mu(t, x)P^{-1} = (-1)^{\delta_{\mu\nu}} A^\mu(t, x_P).
\] (2.7)

A significant feature of the two-component Dirac spinor is the fact that mass term \( m \overline{\psi} \psi \) changes its sign under the parity transformation \( P \). As to the gauge field the CS term is odd under the above parity operation. It has been known that a two-component fermion induces a CS term with the same signature as the fermion mass through the one-loop vacuum polarization \[7,21\].

A convenient method to construct the parity even mass term for fermions in \( 2 + 1 \) dimensions is to use a four-component fermion in terms of two two-component fermions \( \psi_1 \) and \( \psi_2 \):

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\] (2.8)

Parity transformation for the four-component spinor \( \psi \) is defined as
\[ P\psi(t, x)P^{-1} = -i\gamma^3\gamma^1\psi(t, x_P). \] (2.9)

It includes the exchange of upper and lower two-component spinors, \( \psi_1 \rightarrow \psi_2, \psi_2 \rightarrow \psi_1 \), in addition to the \( P \) operation for each two-component spinor. Therefore it is easy to write the parity-conserving mass term:

\[ m\bar{\psi}\psi = m\bar{\psi}_1^\dagger\sigma_3\psi_1 - m\bar{\psi}_2^\dagger\sigma_3\psi_2, \] (2.10)

that breaks the \( U(2N) \) symmetry to the \( U(N) \otimes U(N) \), as well as the parity-violating mass term:

\[ m\bar{\psi}\tau_3\psi = m\bar{\psi}_1^\dagger\sigma_3\psi_1 + m\bar{\psi}_2^\dagger\sigma_3\psi_2, \] (2.11)

that is a singlet of the \( U(2N) \) group.

According to the theorem found by Vafa and Witten [22], it is energetically favorable that the \( U(2N) \) symmetry is broken to the vector-like global symmetry \( U(N) \otimes U(N) \) in the vector-like gauge theories such like QED_3. On the other hand, the theory of our interest contains the topological Chern-Simons term as well as the background magnetic field. Since the theory ceases to be vector-like in general, the stability of \( U(N) \otimes U(N) \) is not necessarily supported. In this paper we constrain ourselves to the case that the \( U(N) \otimes U(N) \) symmetry is unbroken, therefore the fermion mass is dynamically generated only as the parity-conserving mass term and the induced CS term due to the fermion mass is completely canceled.

**B. Effective action in the large \( N \) limit**

It is known that QED_3 is a super-renormalizable theory and its \( \beta \)-function for a dimensionless coupling constant, which is defined as the photon propagation in the large \( N \) limit, shows a nontrivial infrared fixed point [15]. On the fixed point the gauge coupling \( e \) scales like \( e \sim 1/\sqrt{N} \), which means that low energy effective theory is dominated by large \( N \) contributions. In other words we should not neglect the screening due to vacuum polarization at long distance. In the following, we construct the effective action in the large \( N \) limit, which rules infrared dynamics of Chern-Simons QED_3.

According to Ref. [15], we introduce a dimensionful coupling \( \alpha = Ne^2/4\pi \) and keep \( \alpha \) finite when \( N \) is taken to infinity so that radiative corrections can be added to the effective action successively in \( 1/N \) expansion. What we want to ask is whether or not a magnetic field is dynamically induced. On the ground state with the induced magnetic field, proper excitations are the fermions coupled with the magnetic field and the gauge field fluctuation around the magnetic field. Taking into account the above expected physics, we divide the gauge field \( A_\mu \) into a background field and a fluctuating field; \( A_\mu = \sqrt{N}(A_\mu^{\text{ext}} + A_\mu) \) with \( \sqrt{NA_\mu^{\text{ext}}}(x) = Bx_2\delta_{\mu 1} \). Notice that we have rescaled the gauge field by a factor \( \sqrt{N} \), which enables us to treat loop expansion as \( 1/N \) expansion. Subsequently, the magnetic field \( B \) must scale as \( \sqrt{N} \) in order that \( eB \) remains finite in the large \( N \) limit.

The Lagrangian density (2.1) is divided into the gauge part and the fermion part; \( \mathcal{L} = \mathcal{L}_G + \mathcal{L}_F \),

\[
\mathcal{L}_G = -\frac{N}{4}F_{\mu\nu}F^{\mu\nu} - \frac{N\kappa}{2}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho - \frac{N}{2\xi}(\partial A)^2 - \frac{B^2}{2} - \sqrt{N}\kappa BA^0,
\]

\[
\mathcal{L}_F = \bar{\psi} \left[ i\gamma^\mu D_\mu(A^{\text{ext}} + A) - m + \mu\gamma^0 \right] \psi,
\] (2.12)
where $D_\mu[A] := \partial_\mu - ie\sqrt{N}A_\mu$ is a covariant derivative. The propagator for the fermion coupled to $A_\mu^{\text{ext}}$ is defined by

$$S(x, y) = -\left\langle x \left| \frac{1}{i\gamma^\mu D_\mu[A^{\text{ext}}] - m + \mu\gamma^0} \right| y \right\rangle.$$  \hspace{1cm} (2.13)

It is written in the proper time method \cite{19} such as

$$S(x, y) = \exp \left( \frac{ie}{2}\sqrt{N} (x - y)^\mu A_\mu^{\text{ext}}(x + y) \right) \bar{S}(x - y).$$  \hspace{1cm} (2.14)

Fourier transformation of $\bar{S}(x - y)$ is given by \cite{23}

$$\bar{S}(k) = i \int_0^\infty ds \exp \left[ -is \left( m^2 - k_\epsilon^2 + \frac{\tan(eBs)}{eBs}k^2 \right) \right]$$

$$\times \left\{ \left[ 1 + \gamma^\epsilon \gamma^2 \tan(eBs) \right] \left( m + \gamma^0 k_\epsilon \right) - (\gamma^1 k_1 + \gamma^2 k_2) \sec^2(eBs) \right\},$$  \hspace{1cm} (2.15)

where $k_\epsilon := k^0 + \mu + ie\text{sgn}(k^0)$ modifies the $ie$ prescription to be consistent with the shift of Hamiltonian by $\mu$ \cite{24}. In Eq. (2.13) the integration over $s$ is the only formal expression and, rigorously, we have to choose the sign of $s$ according to the sign convention of $k^0 + \mu$ so that the integral over $s$ converges. The photon propagator from $\mathcal{L}_G$ is

$$\Delta^{\mu\nu}(p) = \frac{1}{p^2 + \kappa^2} \left[ g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + i\kappa \frac{\epsilon^{\mu\nu\rho\gamma}}{p^2} \right] + \frac{\xi}{(p^2)^2},$$  \hspace{1cm} (2.16)

which tells us that the CS coefficient $\kappa$ behaves like a pole mass of the gauge field.

The large $N$ effective gauge action is obtained by integrating out the fermion field \cite{25}, that is,

$$\Gamma[A] = -i \ln \int [d\psi][d\bar{\psi}] \exp \left[ \int d^3x \mathcal{L} \right]$$

$$= \int d^3x \mathcal{L}_G - iN \text{Tr Ln} \left[ -i\gamma^\mu D_\mu[A^{\text{ext}} + A] + m - \mu\gamma^0 \right].$$  \hspace{1cm} (2.17)

To expand Tr Ln term with respect to the gauge field $A_\mu$, we will extract from it charge condensation, vacuum polarization, and nonlocal self-interactions of the gauge field. Specifically, it is

$$\Gamma[A] = -iN \text{Tr Ln} S^{-1} + N \int d^3x \left[ -\frac{B^2}{2N} - \frac{\kappa B}{\sqrt{N}} A^0(x) - \frac{1}{2} A_\mu(x) \Delta^{-1}_{\mu\nu}(-i\partial_\nu)A^\nu(x) \right]$$

$$+ N \int d^3x \left[ -\frac{e\langle \psi^\dagger \psi \rangle}{\sqrt{N}} A^0(x) + \frac{1}{2} \int d^3x A_\mu(x) \Pi_{\mu\nu}(-i\partial_\nu)A^\nu(x) \right] + \bar{\Gamma}[A],$$  \hspace{1cm} (2.18)

where $\langle \psi^\dagger \psi \rangle$ is the charge condensation induced by a magnetic field and obeys the Gauss law. $\bar{\Gamma}[A]$ denotes the nonlocal vertices of the gauge field. Vacuum polarization tensor $\Pi_{\mu\nu}$ is given by

$$\Pi_{\mu\nu}(p) = -Ne^2 \int \frac{d^3k}{i(2\pi)^3} \text{tr} \left[ \gamma_\mu \bar{S}(k) \gamma_\nu S(k - p) \right],$$  \hspace{1cm} (2.19)
which improves the kinetic term of the gauge field.

By introducing the improved photon propagator: \( D_{\mu\nu}(p) := \left[ \Delta_{\mu\nu}^{-1}(p) - \Pi_{\mu\nu}(p) \right]^{-1} \), the above effective action in a complicated form is simplified as

\[
\Gamma[A] = -iN \text{Tr} \ln S^{-1} - \Omega \frac{B^2}{2} - \frac{N}{2} \int d^3x A^\mu(x) D_{\mu\nu}^{-1}(-i\partial_x) A^\nu(x) + \tilde{\Gamma}[A],
\]

(2.20)

where \( \Omega \) is three-dimensional space-time volume. The linear terms for \( A_\mu \) in Eq. (2.18) have been canceled out by means of the Gauss law. Later in Section V we will study dynamical symmetry breaking through the analysis of SD gap equation for the fermion self-energy. It can be derived from the effective action (2.20). In Section IV vacuum polarization tensor \( \Pi_{\mu\nu}(p) \) is regularized in a gauge invariant manner, such as

\[
\Pi_{\mu\nu}(p) = (p_\mu p_\nu - p^2 g_{\mu\nu}) \Pi_e(p) + (p_\mu^+ p_\nu^- - p_\perp^2 g_{\mu\nu}) \Pi_\perp(p) - ip\rho \epsilon_{\mu\nu\rho} \Pi_o(p),
\]

(2.21)

where \( p_\perp = (0, p_1, p_2) \) and \( \text{diag}(g_{\mu\nu}) = (0, 1, 1) \). The parity-violating part \( \Pi_o(p) \) provides a quantum correction to the gauge invariant photon mass \( \kappa \) in the propagator \( D_{\mu\nu} \). It will be explained in Section II that the Gauss law forces the improved photon mass, \( \kappa - \Pi_o(0) \), to vanish in the presence of a magnetic field. Therefore our effective theory in Eq. (2.20) has the massless photon, whose propagator is given by \( D_{\mu\nu} \), as well as the low energy effective theory in QED\(_3\) with a magnetic field.

III. CONDENSATES AND CONFIGURATIONS OF VACUUM

In this section we explicitly calculate two condensates, \( \langle \bar{\psi}\psi \rangle \) and \( \langle \psi^\dagger\psi \rangle \), for fermions coupled with a magnetic field. We can find the possible configurations of vacuum (the lowest Landau level) by taking both condensates to the symmetric limit \( J_\pm := \mu \pm m \to 0 \). The condensates are calculated from the fermion propagator (2.15) which contains the explicit symmetry breaking parameters, \( \mu \) and \( m \). We shall find various patterns of symmetry breaking according to different approaches to the symmetric limit \( J_\pm \to 0 \). It is also clarified by applying the result of calculation to the Gauss law that possible values of the CS coefficient \( \kappa \) are restricted to \( |\kappa| = 0, \alpha, \) and \( 2\alpha \).

The propagator for \( N \) flavor fermions \( \psi_i \) \((i = 1, ..., N)\) coupled with a magnetic field is defined by

\[
S(x, y) \delta_{ij} := i \langle 0 | T\psi_i(x) \bar{\psi}_j(y) | 0 \rangle,
\]

(3.1)

where we assume the \( \text{U}(N) \otimes \text{U}(N) \) symmetry and extract the flavor indices as a diagonal metric \( \delta_{ij} \). \( S(x, y) \) in Eq. (3.1) is equivalent to the definition in Eq. (2.13). We introduce the functions, \( \mathcal{J} \) and \( \mathcal{J}_0 \), which are defined by the following equations:

\[
\mathcal{J}(m, \mu) := \lim_{y \to x} \text{tr} \langle 0 | T\psi_i(x) \bar{\psi}_i(y) | 0 \rangle = iN \text{tr} \ S(x, x), \tag{3.2}
\]

\[
\mathcal{J}_0(m, \mu) := \lim_{y \to x} \gamma^0 \langle 0 | T\psi_i(x) \bar{\psi}_i(y) | 0 \rangle = iN \text{tr} \gamma^0 S(x, x). \tag{3.3}
\]
The condensates, \( \langle \bar{\psi} \psi \rangle \) and \( \langle \psi^\dagger \psi \rangle \), are identified as \( J \) and \( J_0 \) in the symmetric limit \( J_{\pm} \to 0 \) such as

\[
\langle \bar{\psi} \psi \rangle = \lim_{J_{\pm} \to 0} J(m, \mu), \quad \langle \psi^\dagger \psi \rangle = \lim_{J_{\pm} \to 0} J_0(m, \mu).
\] (3.4)

If the condensate \( \langle \bar{\psi} \psi \rangle \) (\( \langle \psi^\dagger \psi \rangle \)) remains nonzero, the vacuum is realized as a non-singlet state of \( U(2N) \) symmetry (Lorentz symmetry). Precisely, the symmetry is spontaneously broken by the condensate though the \( U(2N) \) symmetry (Lorentz symmetry) is restored at the tree-level Lagrangian in the symmetric limit \( J_{\pm} \to 0 \). [9]

A. Calculations of condensates

Since the condensates will be computed in terms of the fermion propagator (2.13), a comment about the meaning of Wick rotation, \( k^0 = i\bar{k} \), in the presence of \( \mu \) should be noted. The improved \( i\epsilon \) prescription in Eq. (2.15) always forces poles of the fermion propagator on a complex \( k^0 \) plane to be located outside the contour which connects the real axis to the imaginary axis. Therefore the Euclidean propagator becomes

\[
\bar{S}(k_E) = \int_0^\infty ds \exp \left[ -s \left( m^2 + (\bar{k} - i\mu)^2 + \frac{\tanh \tau k_{\perp}^2}{\tau} \right) \right] \\
\times \left\{ 1 - i\gamma^1 \gamma^2 \text{sgn}(eB) \tanh \tau \left[ m + i\gamma^0 (\bar{k} - i\mu) \right] - \gamma_{\perp} k_{\perp} \text{sech}^2 \tau \right\}, \] (3.5)

by using \( \tau := |eB| s \). We will carry out analytic continuation \( \bar{k} \to \bar{k} - i\mu \) after integrating over a proper time \( s \). Hereafter the shifted momentum \( \bar{k} - i\mu \) in the proper time integration necessarily implies the above agreement even if we do not indicate it explicitly.

Now we calculate the condensates \( J \) and \( J_0 \). They are written in terms of the Euclidean propagator \( \bar{S}(k_E) \) as follows;

\[
J(m, \mu) = -N \int \frac{d^3 k_E}{(2\pi)^3} \text{tr} \bar{S}(k_E) \\
= -\frac{N}{2\pi^2} |eB| m \int_{-\infty}^{\infty} d\bar{k} \int_{1/\Lambda^2}^{\infty} ds \ e^{-s[(\bar{k} - i\mu)^2 + m^2]} \coth \tau, \] (3.6)

\[
J_0(m, \mu) = -N \int \frac{d^3 k_E}{(2\pi)^3} \text{tr} \gamma^0 \bar{S}(k_E) \\
= -\frac{N}{2\pi^2} |eB| i \int_{-\infty}^{\infty} d\bar{k} \ (\bar{k} - i\mu) \int_{0}^{\infty} ds \ e^{-s[(\bar{k} - i\mu)^2 + m^2]} \coth \tau, \] (3.7)

where the Gaussian integrals for \( k \) have been done and a cut-off scale \( \Lambda \) has been introduced to regularize the ultraviolet divergence in \( J(m, \mu) \). To extract finite density effects, we first decompose the integrands into Landau levels by using an identity:

\[
\coth \tau \equiv 1 + 2 \sum_{n=1}^{\infty} e^{-2n\tau}. \] (3.8)

Then we integrate out the proper time \( s \) and get
\[ J(m, \mu) = J(m, 0) - \frac{N}{2\pi^2} |eB| m \left[ \oint_{C_\mu} dz \frac{1}{z^2 + m^2} + 2 \sum_{n=1}^{\infty} \oint_{C_\mu} dz \frac{1}{z^2 + \mathcal{E}_n^2} \right], \quad (3.9) \]

\[ J_0(m, \mu) = -\frac{N}{2\pi^2} |eB| \left[ \oint_{C_\mu} dz \frac{iz}{z^2 + m^2} + 2 \sum_{n=1}^{\infty} \oint_{C_\mu} dz \frac{iz}{z^2 + \mathcal{E}_n^2} \right], \quad (3.10) \]

where \( \mathcal{E}_n := \sqrt{m^2 + 2n|eB|} \) is the \( n \)-th Landau level. The above integrations are carried out along a contour \( C_\mu \) on a complex \( z \)-plane depicted in Fig. 1. The contribution at zero density appears only in the condensate \( \langle \bar{\psi} \psi \rangle \) through \( J(m, 0) \) which is given by

\[ J(m, 0) = -\frac{N}{2\pi^3/2} |eB| m \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{1/2}} e^{-sm^2} \coth \tau. \quad (3.11) \]

Using contour integrals shown in Appendix A, we obtain the following results;

\[ J(m, \mu) = -\frac{N}{\pi^{3/2}} \Lambda m - \frac{N}{2\pi} |eB| \left[ 1 + \sqrt{2} ml \zeta \left( \frac{1}{2}, \frac{(ml)^2}{2} + 1 \right) \right] + \frac{N}{2\pi} |eB| \left[ \theta(|\mu| - m) + 2 \sum_{n=1}^{\infty} \frac{m}{\mathcal{E}_n} \theta(|\mu| - \mathcal{E}_n) \right], \quad (3.12) \]

\[ J_0(m, \mu) = \frac{N}{2\pi} |eB| \text{sgn}(\mu) \left[ \theta(|\mu| - m) + 2 \sum_{n=1}^{\infty} \theta(|\mu| - \mathcal{E}_n) \right], \quad (3.13) \]

where we have summed up the Landau levels with the generalized Riemann zeta function and the value of step function \( \theta(x) \) at \( x = 0 \) is chosen to be 1/2 at zero temperature limit of Fermi-Dirac distribution.

We notice that \( J_0 \) reduces to zero as we take the limit, \( B \to 0 \ (l \to \infty) \), in Eq. (3.13). The charge condensation \( J_0 \) is known to be unaffected by higher-order corrections \(^{[26]}\). Therefore the above one-loop result is exact as we shall make detailed comment later on in this section.

**B. Configurations of vacuum**

It should be noticed that we can read a filling factor \( \nu \) as a function of \( \mu \) from Eq. (3.13) \(^{[9,26,27]}\). Since the wave function for fermions in the lowest Landau level spreads out within a size of the magnetic length \( l = 1/\sqrt{|eB|} \), the number of states for an electron is estimated as \( 1/2\pi l^2 \) per unit area. We also know that \( \mu > 0 \ (\mu < 0) \) induces a finite density of fermion (anti-fermion) whose spin indicates the up- (down-) state in the lowest Landau level. \( \langle \psi^\dagger \psi \rangle \) is rewritten as

\[ \langle \psi^\dagger \psi \rangle = \frac{N}{2\pi l^2} (\nu_+ - \nu_-), \quad (3.14) \]

where \( \nu_+ \ (\nu_-) \) denotes the filling factor of fermion (anti-fermion) whose spin indicates the up- (down-) state. Comparing Eq. (3.13) with Eq. (3.14), we read

\[ \nu_\pm = \theta(\pm \mu) \theta(|\mu| - m). \quad (3.15) \]
Three different approaches to the symmetric limit are possible on a \((|\mu|, m)\) plane, namely, 

\[ |\mu| < m \to 0, \quad m < |\mu| \to 0, \quad \text{and} \quad |\mu| = m \to 0, \]

in which both condensates, \(\langle \bar{\psi} \psi \rangle\) and \(\langle \psi^\dagger \psi \rangle\), exhibit the different patterns of symmetry breaking (see Fig. 2) \[9\].

For \(|\mu| < m \to 0\), we approach to the origin of the \((|\mu|, m)\) plane along a line inside the upper area: \(|\mu| < m\). The condensates can be read as

\[ \langle \bar{\psi} \psi \rangle = -\frac{N}{2\pi l^2}, \quad \langle \psi^\dagger \psi \rangle = 0, \]

which shows spontaneous breaking of the U(2\(N\)) symmetry on an empty vacuum, that stands for the lowest Landau level with \(\nu_\pm = 0\). If we believe the Gauss law (1.1) naïvely, the generation of magnetic field is available only when charge condensation occurs. Since we have no charge condensation seen in Eq. (3.16), we expect no spontaneous magnetization and, subsequently, no spontaneous breaking of the Lorentz symmetry. However, we want to postpone our conclusion until we study the vacuum energy shift due to the magnetic field in Section II.

For \(m < |\mu| \to 0\), we choose a line inside the lower area, \(m < |\mu|\), to approach the symmetric limit. We get an opposite pattern of condensates to the former case, namely,

\[ \langle \bar{\psi} \psi \rangle = 0, \quad \langle \psi^\dagger \psi \rangle = \frac{N}{2\pi l^2} \text{sgn}(\mu), \]

which mean the vacuum, or the lowest Landau level, has been filled up as it is read from \((\nu_+, \nu_-) = (1, 0)\), or \((0, 1)\), and the U(2\(N\)) symmetry has been restored. This time we have obtained a finite density vacuum so that the magnetic field is supported by nonzero charge condensation. Therefore the magnetic field can be generated dynamically. If the gap equation for fermion self-energy favors a finite density solution, Lorentz symmetry is spontaneously broken through the induced magnetic field.

For \(|\mu| = m \to 0\), the symmetric limit is taken along the boundary line of \(|\mu| = m\) on the \((|\mu|, m)\) plane. Both condensates remain finite and are given by

\[ \langle \bar{\psi} \psi \rangle = -\frac{N}{4\pi l^2}, \quad \langle \psi^\dagger \psi \rangle = \frac{N}{4\pi l^2} \text{sgn}(\mu), \]

which mean that the U(2\(N\)) symmetry is broken spontaneously in the half-filled vacuum with \((\nu_+, \nu_-) = (1/2, 0)\), or \((0, 1/2)\). As to the Lorentz symmetry, it might also be broken spontaneously through the induced finite density which supports the magnetic field through the Gauss law.

We notice that the two condensates always appear in such a complementary way as to keep the combination \(\langle \bar{\psi} \psi \rangle - |\langle \psi^\dagger \psi \rangle|\) as a nonzero constant: \(-N/2\pi l^2\). Moreover the above result depends only on the kinematics of fermions in the lowest Landau level. Even if we have no attractive interaction of photon exchange, the magnetic field alone can cause the nonzero condensate: \(\langle \bar{\psi} \psi \rangle - |\langle \psi^\dagger \psi \rangle|\). This is a specific phenomenon realized only in \((2 + 1)\)-dimensions, and, for \(\langle \bar{\psi} \psi \rangle\), the same discussion was already made in Ref. \[10\].

\[ \text{The restoration of U(2\(N\)) symmetry in a given finite density was also pointed out in Ref. \[10\] in (2 + 1)-dimensional Nambu-Jona-Lasinio model with a magnetic field.} \]
C. Consistency condition

In Chern-Simons QED$_3$, charge condensation $\langle \psi^\dagger \psi \rangle$ is necessarily connected with a magnetic field through the Gauss law (1.1), while Eqs. (3.16-3.18) tell us that the magnetic field keeps $\langle \psi^\dagger \psi \rangle$ as quantized values in a unit of $1/2\pi l^2$ even in the symmetric limit. Therefore, the Gauss law places a restriction on the CS coefficient $\kappa$ in order to retain the magnetic field in the symmetric limit. Substituting $\mathcal{J}_0(m, \mu)$, given by Eq. (3.13), for $\langle \psi^\dagger \psi \rangle$ in the Gauss law (1.1), we obtain in the case of $|\mu| < E_1$ the equation

$$B \left[ \kappa + 2\alpha \operatorname{sgn}(\mu eB) \theta(|\mu| - m) \right] = 0,$$

which is regarded as the self-consistency condition for a given magnetic field $B$. In order to keep the nonzero magnetic field in our system, $\kappa$ should be restricted to

$$\kappa = \begin{cases} 
0 & \text{for } |\mu| < m \to 0 \\
-\alpha \operatorname{sgn}(\mu eB) & \text{for } |\mu| = m \to 0 \\
-2\alpha \operatorname{sgn}(\mu eB) & \text{for } m < |\mu| \to 0
\end{cases}$$

If we start from the theories with $|\kappa| = 0$, $\alpha$, and $2\alpha$, which allow the penetration of nonzero magnetic field, then each symmetric limit $(|\mu|, m) \to (0, 0)$ is uniquely determined as $|\mu| < m \to 0$, $|\mu| = m \to 0$, or $m < |\mu| \to 0$, respectively.

The above constraint for $\kappa$ is also interpreted as follows. It will be shown in Section IV that, except for $|\mu| = E_n$ ($n = 1, 2, \ldots$), the charge condensation $\mathcal{J}_0(m, \mu)$ is related to the parity-violating part of vacuum polarization $\Pi_o(p)$ through the relation [26]:

$$-e\mathcal{J}_0(m, \mu) = B \Pi_o(0).$$

The constraint (3.19) is interpreted as $B [\kappa - \Pi_o(0)] = 0$ which means that a magnetic field applied to the system can penetrate it if and only if the effective photon mass, $\kappa_{\text{eff}} := \kappa - \Pi_o(0)$, becomes zero [8,28,29]. Otherwise, the system with $\kappa \neq \Pi_o(0)$ excludes a magnetic field irrespective of its origin. In this case we should set magnetic field zero and investigate dynamical generation of the parity-violating fermion mass as well as U(2N) symmetry breaking [8,11,4] in the theories with $\kappa = \Pi_o(0)$, we cannot exclude the possibility of spontaneous magnetization since we achieves the symmetric limit which allows the penetration of a given magnetic field in each theory [cf. Eq. (3.20)]. We notice that the above statements are not affected by any higher-order correction by virtue of the non-renormalization theorem of the induced CS coefficient $\Pi_o(0)$ (Coleman-Hill theorem [32]) even with finite density and magnetic field [26,33].

In Section V we investigate dynamical generation of fermion mass and magnetic field for theories with $|\kappa| = 0$, $\alpha$, and $2\alpha$. We shall confirm that the fermion mass $m_d$ and the chemical potential $\mu_d$ are dynamically generated in the symmetric limit $(|\mu|, m) \to (0, 0)$. In order to retain the consistency with a nonzero magnetic field, $m_d$ and $\mu_d$ should satisfy the same patterns as the explicit breaking parameters $m$ and $\mu$ do in Eq. (3.20) for each $\kappa$.

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It was pointed out in Ref. [31] that spontaneous breaking of the U(2N) symmetry turns out to be a first-order phase transition in the theories with $\kappa \neq \Pi_o(0)$.  

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4 It was pointed out in Ref. [31] that spontaneous breaking of the U(2N) symmetry turns out to be a first-order phase transition in the theories with $\kappa \neq \Pi_o(0)$.
IV. VACUUM POLARIZATION

This section is devoted to the gauge invariant derivation of the vacuum polarization tensor $\Pi_{\mu\nu}$ defined by Eq. (2.19), that is

$$\Pi_{\mu\nu}(p) = -Ne^2 \int \frac{d^3k}{i(2\pi)^3} \text{tr} \left[ \gamma_{\mu} \bar{S}(k)\gamma_{\nu} \bar{S}(k-p) \right].$$

(4.1)

It should be regularized so as to be gauge invariant such as

$$\Pi_{\mu\nu}(p) = (p_\mu p_\nu - p^2 g_{\mu\nu}) \Pi_e(p) + (p_\mu^{\perp} p_\nu^{\perp} - p_\mu^{\perp} g_{\mu\nu}) \Pi_\perp(p) - ip^\rho \epsilon_{\rho\mu\nu} \Pi_o(p).$$

(4.2)

As mentioned in Section III, effects of an explicit breaking parameter $\mu$, or equivalently a chemical potential, are included through an analytic continuation, $\bar{k} \rightarrow \bar{k} - i\mu$, of the third component of an Euclidean loop momentum. As far as we derive parity-conserving parts of vacuum polarization, they are obtained through the above analytic continuation from zero density results. In the following we divide the vacuum polarization tensor into zero density parts and finite density parts, that is,

$$\Pi_e(p) = \Pi_e^z(p) + \Pi_e^f(p),$$
$$\Pi_\perp(p) = \Pi_\perp^z(p) + \Pi_\perp^f(p),$$
$$\Pi_o(p) \equiv \Pi_o^f(p),$$

(4.3)

and derive them separately. In our setting of $N$ flavor four-component fermions the parity-violating part $\Pi_o(p)$ appears only as a finite density effect and is uniquely determined through a gauge invariant constraint between $\Pi_o(0)$ and charge condensation.

A. Gauge invariant regularization

The regularization of $\Pi_{\mu\nu}$ is worked out in the same gauge invariant manner as in the case without magnetic field. First we set $\mu = 0$ and mix two proper times of $\bar{S}(k)$ and $\bar{S}(k-p)$ by means of a parameter integration. Then we carry out the Gaussian integrals for spatial components of the loop momentum: $k$. We find that parity-conserving parts appear only as even terms for $k^0$ and include ultraviolet divergences, while a parity-violating part appears only as a linear term for $k^0$ and is finite.

As to the parity-conserving parts we see that the ultraviolet divergences appear as terms with the lower power of proper time $s$, namely, $s^{-3/2}$ in $2 + 1$ dimensions. Integrating out the proper time $s$ by part, we can subtract the ultraviolet divergences as boundary values of the integration over $s$. Then we find that remaining finite contributions satisfy the transverse form in Eq. (4.2). Zero density parts of $\Pi_e^z$ and $\Pi_\perp^z$ are determined as

5 Since in zero density a parity-violating part necessarily appears as an integration of an odd function of $k^0$ which is the zeroth component of the loop momentum, it vanishes after an integration over $k^0$. In our setting of fermions the parity violation due to the fermion bare mass $m$ is completely canceled out.
\[\Pi^\varphi(p) = \frac{\alpha}{2\sqrt{\pi}} \int_{-1}^{1} dv \int_{0}^{\infty} ds \int_{0}^{\infty} ds \, e^{-s[m^2 + \phi(p)]} \frac{\tau(cosh \tau v - v coth \tau sinh \tau v)}{sinh \tau},\]
\[\Pi^\perp(p) = \frac{\alpha}{2\sqrt{\pi}} \int_{-1}^{1} dv \int_{0}^{\infty} ds \, e^{-s[m^2 + \phi(p)]} \frac{2\tau(cosh \tau - cosh \tau v)}{sinh^3 \tau} - \Pi^\varphi(p),\]

where we have introduced a rescaled variable \(\tau := sl^{-2}\) and the function \(\phi(p)\) is defined by

\[\phi(p) := \frac{1 - v^2}{4} p_\parallel^2 + \frac{cosh \tau - cosh \tau v}{2\tau sinh \tau} p_\perp^2.\]

In order to derive the finite density contributions to the parity-conserving parts we only have to insert Gaussian integral:

\[\sqrt{\frac{s}{\pi}} \int_{-\infty}^{\infty} d\bar{k} e^{-s\bar{k}^2} = 1\]

into the expressions of \(\Pi^\varphi\) and \(\Pi^\perp\) in Eq. (4.4) and to perform analytic continuation \(\bar{k} \rightarrow \bar{k} - i\mu\).

As to the parity-violating part it is given as a combination of SO(2) invariant tensors under a given magnetic field, that is,

\[\Pi^{\text{odd}}_{\mu\nu}(p) = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} d\bar{k} (\bar{k} - i\mu) \int_{-1}^{1} dv \int_{0}^{\infty} ds \, e^{-s[(\bar{k} - i\mu)^2 + m^2 + \phi(p)]}\]
\[\times \left[ \frac{2\tau(cosh \tau v - 1)}{sinh^2 \tau} \right] \left[ p_\parallel (g_{\mu0} \epsilon^\perp_{\nu \rho} - g_{\nu0} \epsilon^\perp_{\mu \rho}) \right.\]
\[\left. + \frac{2\tau v sinh \tau v}{sinh \tau} p_\perp \epsilon^\perp_{\mu \nu} \right] \text{sgn}(eB),\]

which vanishes for \(\mu = 0\) after an integration over \(\bar{k}\). Even if we use an identity

\[p^\rho \epsilon_{\mu \nu \rho} \equiv p_\parallel (g_{\mu0} \epsilon^\perp_{\nu \rho} - g_{\nu0} \epsilon^\perp_{\mu \rho}) - p_\perp \epsilon^\perp_{\mu \nu}\]

in order to extract the gauge invariant tensor \(p^\rho \epsilon_{\mu \nu \rho}\), there remains an ambiguity in \(\Pi^\rho_{\mu}(p)\). However, in the presence of the magnetic field, there exists a gauge invariant constraint between \(\Pi^\rho_{\mu}(0)\) and charge condensation \(e\mathcal{J}^\rho_0(m, \mu) \mathbb{Z}\), namely,

\[e \frac{\partial \mathcal{J}^\rho_0(m, \mu)}{\partial B} = -\Pi^\rho_{\mu}(0).\]

By virtue of Eq. (1.9), \(\Pi^\rho_{\mu}(p)\) is uniquely determined from the result of Eq. (3.13).

Through the above procedures, the functions \(\Pi^\varphi\), \(\Pi^\perp\), and \(\Pi^\rho_{\mu}\) are determined as

\[\Pi^\varphi(p) = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} d\bar{k} \int_{-1}^{1} dv \int_{0}^{\infty} ds \, \left\{ e^{-s[(\bar{k} - i\mu)^2 + m^2 + \phi(p)]} - e^{-s[\bar{k}^2 + m^2 + \phi(p)]} \right\}\]
\[\times \frac{\tau(cosh \tau v - v coth \tau sinh \tau v)}{sinh \tau},\]
\[\Pi^\perp(p) = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} d\bar{k} \int_{1}^{1} dv \int_{0}^{\infty} ds \, \left\{ e^{-s[(\bar{k} - i\mu)^2 + m^2 + \phi(p)]} - e^{-s[\bar{k}^2 + m^2 + \phi(p)]} \right\}\]

...
\[ \times \frac{2\tau (\cosh \tau - \cosh \tau v)}{\sinh ^3 \tau} - \Pi_1^f(p), \tag{4.10} \]

\[ \Pi_1^f(p) = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} dk \, i(k - i\mu) \int_0^1 dv \int_0^{\infty} ds \, e^{-s[(k-i\mu)^2 + m^2 + \phi(p)]} \times \frac{2\tau (\cosh \tau \cosh \tau v - 1)}{\sinh ^2 \tau} \, \text{sgn}(eB). \]

As mentioned in Section IV, one cannot do the analytic continuation \( \tilde{k} \rightarrow \tilde{k} - i\mu \) before the proper time \( s \) is integrated out, and then the order of integrations should be kept in Eq. (4.10).

When the momentum \( p \) vanishes, it is possible to integrate out the parameter \( v \) and the proper time \( s \) before the integration over \( \tilde{k} \). Therefore we can look into how the analytic continuation works. The integrands in Eq. (4.10) are decomposed into the Landau levels by means of identities:

\[ \frac{\tau (\cosh \tau v - v \coth \tau \sinh \tau v)}{\sinh \tau} = \tau \sum_{n=0}^{\infty} \left[ 1 + (2n+1)v \right] e^{-(2n+1+v)\tau} \quad (v \rightarrow -v), \]

\[ \frac{2\tau (\cosh \tau - \cosh \tau v)}{\sinh ^2 \tau} = 4\tau \sum_{n=1}^{\infty} \left[ n^2 e^{-2n\tau} - (n+1) e^{-(2n+1+v)\tau} \right] + (v \rightarrow -v), \tag{4.11} \]

\[ \frac{2\tau (\cosh \tau \cosh \tau v - 1)}{\sinh ^2 \tau} = -2\tau \sum_{n=0}^{\infty} \left[ 2n e^{-2n\tau} - (2n+1) e^{-(2n+1+v)\tau} \right] + (v \rightarrow -v). \]

Inserting the above identities into Eq. (4.4) at zero momentum \( (p = 0) \), we acquire the following results for zero density parts after we integrate out the parameter \( v \) and the proper time \( s \):

\[ \Pi_1^f(0) = -2\alpha \left[ ml + 3\sqrt{2} \, \zeta \left( -\frac{1}{2}, \frac{(ml)^2}{2} + 1 \right) - \sqrt{\frac{3}{2}} (ml)^2 \, \zeta \left( \frac{1}{2}, \frac{(ml)^2}{2} + 1 \right) \right], \]

\[ \Pi_1^s(0) = 2\alpha \left[ ml + 3\sqrt{2} \, \zeta \left( -\frac{1}{2}, \frac{(ml)^2}{2} + 1 \right) + \frac{\sqrt{2}}{8} (ml)^4 \, \zeta \left( \frac{3}{2}, \frac{(ml)^2}{2} + 1 \right) \right], \tag{4.12} \]

where we have summed up contributions from the higher Landau levels by means of the generalized Riemann zeta function. Substituting Eq. (4.11) into Eq. (4.10) at zero momentum \( (p = 0) \), we also obtain the following expressions for the finite density parts:

\[ \Pi_1^f(0) = -\frac{\alpha}{\pi} \left\{ \int_{C_\mu} dz \, l^2 \ln l^2 (z^2 + m^2) + 2 \sum_{n=1}^{\infty} \left[ \int_{C_\mu} dz \, l^2 \ln l^2 (z^2 + E_n^2) + 2n \int_{C_\mu} dz \, \frac{1}{z^2 + E_n^2} \right] \right\}, \]

\[ \Pi_1^s(0) = \frac{8\alpha}{\pi} \sum_{n=1}^{\infty} \left[ -n \int_{C_\mu} dz \, \frac{1}{z^2 + E_n^2} + \frac{n^2}{l^2} \int_{C_\mu} dz \, \frac{1}{(z^2 + E_n^2)^2} \right] - \Pi_1^f(0), \tag{4.13} \]

\[ \Pi_1^s(0) = \frac{2\alpha}{\pi} \text{sgn}(eB) \left\{ \int_{C_\mu} dz \, \frac{iz}{z^2 + m^2} + 2 \sum_{n=1}^{\infty} \left[ \int_{C_\mu} dz \, \frac{iz}{z^2 + E_n^2} - 2n \int_{C_\mu} dz \, \frac{iz}{(z^2 + E_n^2)^2} \right] \right\}. \]

When \( |\mu| = E_n \), poles or branch points on a complex \( z \)-plane are located on the contour \( C_\mu \) and force us to define the integrations as their Cauchy’s principal values. Subsequently, we obtain (see also Appendix A)
Π_{f}(0) = -2\alpha \theta(|\mu| - m) + 2 \sum_{n=1}^{\infty} \left[ \theta(|\mu| - E_n) - \frac{n}{lE_n} \right] \theta(|\mu| - E_n) - \frac{n^2}{lE_n} \delta(|\mu| - E_n),

Π_{f}^\perp(0) = 4\alpha \sum_{n=1}^{\infty} \left[ 2n lE_n - \frac{n^2}{lE_n} \right] \theta(|\mu| - E_n) - \frac{n^2}{lE_n} \delta(|\mu| - E_n) - \Pi_{f}(0),

Π_{o}(0) = -2\alpha \text{sgn}(\mu eB) \left[ \theta(|\mu| - m) + 2 \sum_{n=1}^{\infty} \left[ \theta(|\mu| - E_n) - \frac{n}{lE_n} \right] \theta(|\mu| - E_n) - \frac{n^2}{lE_n} \delta(|\mu| - E_n) \right].

Now we prove Eq. (4.9) that is the gauge invariant constraint between the charge condensation and the parity-violating part of vacuum polarization and is essential to determine the Π_{o}(p) uniquely. We see the following relation is obtained from the definition of the fermion propagator [cf. Eq. (2.13)];

$$\frac{\partial}{\partial B} S(x, y) = \int d^3z S(x, z) \sqrt{N} e \gamma^\mu \frac{\partial A_\mu^{\text{ext}}(z)}{\partial B} S(z, y),$$

$$= e \int d^3z z_2 S(x, z) \gamma^1 S(z, y),$$

where we have used the explicit form of the gauge potential $A_\mu^{\text{ext}}(z)$ in the second line. Applying the above equation to the definition of charge condensation in Eq. (3.3), we obtain

$$e \frac{\partial}{\partial B} J_0(m, \mu) = iNe^2 \int d^3z z_2 \text{tr} \left[ \gamma^0 S(x, z) \gamma^1 S(z, x) \right]$$

$$= \int d^3p \delta^3(p) \left( x_2 - i \frac{\partial}{\partial p_2} \right) \Pi_{f1}(p) = -\Pi_{o}(0).$$

(4.16)

We see that in Eq. (4.14) delta function singularities appear only in the higher Landau levels, $E_n (n \geq 1)$, which have gap energies owing to the magnetic field $B$. In order to take the symmetric limit $J_\pm \to 0$, it is enough to constrain the region of parameters, $J_\pm$, to be on the lowest Landau level. For $|\mu| < E_1$ we notice that Eq. (4.19) reduces to

$$-eJ_0(m, \mu) = B \Pi_{o}(0),

(4.17)

which has been used in Section [4] to derive the consistency condition for the Chern-Simons coefficient $\kappa$ in the magnetic field.

B. The improved photon propagator

In Section [5] we shall investigate dynamical symmetry breaking by means of the Schwinger-Dyson gap equation for the fermion self-energy in the lowest Landau level. According to the planer property of our model, the radiative effect of an improved photon propagator is saturated in the infrared region of its momentum, which will be shown in the following.

Recall that the improved photon propagator $D_{\mu\nu}$, which is deduced from the effective action $\Gamma[A]$ in Eq. (2.20), is given by

$$D_{\mu\nu}^{-1}(p) = \Delta_{\mu\nu}^{-1}(p) - \Pi_{\mu\nu}(p).$$

(4.18)
The tensor form of \( D_{\mu\nu} \) becomes

\[
D_{\mu\nu}(p) = D_e(p) \left[ g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + D_p(p) \left[ g_{\mu\nu}^+ \frac{p^\mu p^\nu}{p^2} \right] + D_o(p) i\epsilon_{\mu\nu}\rho p^\rho + \xi \frac{p_\mu p_\nu}{(p^2)^2},
\]

(4.19)

with a covariant gauge fixing \( \xi \). The functions \( D_e, D_p, \) and \( D_o \) are determined as

\[
D_e(p) G(p) = 1 + \Pi_e(p) + \frac{p^2}{p^2} \Pi_p(p),
\]

\[
D_p(p) G(p) = \frac{p^2}{p^2} \Pi_p(p),
\]

\[
D_o(p) G(p) = \frac{\kappa - \Pi_o(p)}{p^2},
\]

(4.20)

where we have introduced the function \( G(p) \) which is related with a determinant of \( D^{-1} \) and is defined by

\[
G(p) := \left[ 1 + \Pi_e(p) \right] \left\{ p^2 \left[ 1 + \Pi_e(p) \right] + p^2 \Pi_p(p) \right\} \\
+ \left[ \kappa - \Pi_o(p) \right]^2 \\
\equiv -\xi (p^2)^{-2} \det D^{-1}(p).
\]

(4.21)

The consistency condition, \( \kappa = \Pi_o(0) \), means that the function \( G(p) \) reduces to zero as \( p^\mu \to 0 \) in the limit \( J_{\pm} \to 0 \), where symmetries are restored in the tree-level Lagrangian. Therefore a photon becomes a massless propagating mode which is regarded as a Nambu-Goldstone (NG) mode due to the spontaneous breaking of Lorentz symmetry [8].

Under a given magnetic field, we have shown that there exist three possible theories with the quantized CS coefficient \( |\kappa| = 0, \alpha, \) and \( 2\alpha \), each of which has a massless photon. As far as we are concerned with the infrared behavior of the photon, it is enough to approximate the vacuum polarization functions with its values at zero momentum. From Eq. (4.12) and Eq. (4.14) the parity-conserving parts \( \Pi_e \) and \( \Pi_p \) are read as

\[
\Pi_e(0) = -6\sqrt{2} \zeta(-1/2) \alpha l,
\]

\[
\Pi_p(0) = 3\sqrt{2} \zeta(-1/2) \alpha l,
\]

(4.22)

which are irrespective of approaches to the symmetric limit \( J_{\pm} \to 0 \). The improved photon propagator behaves, for all of the theories with \( |\kappa| = 0, \alpha, \) and \( 2\alpha \), like

\[
D_{\mu\nu}(p) \simeq \frac{1}{p^2[1 + c\alpha l]} \left[ g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + \xi \frac{p_\mu p_\nu}{(p^2)^2}
\]

\[
+ \frac{c}{p^2[1 + c\alpha l]} \left\{ 2 p^2[1 + c\alpha l] - c p^4 \right\}
\]

(4.23)

with a positive constant \( c = -6\sqrt{2} \zeta(-1/2) \simeq 1.76397 \).
C. Expansions in infrared and ultraviolet regions of momentum

In Section VI we shall investigate stability of vacua which are realized as possible solutions of SD gap equation, through the calculation of vacuum energy. For this purpose we need to know the asymptotic behavior of vacuum polarization functions both in infrared and ultraviolet regions of momentum.

First we separate the functions into two parts according to $p^2 < (l^2 - 1)$ or $p^2 > (l^2 - 1)$ such as

$$
\Pi_i(p) = \Pi_i^< (p) \theta(1 - (lp)^2) + \Pi_i^> (p) \theta((lp)^2 - 1),
$$

where $i = e, \perp, \text{and } o$. The functions $\Pi_i^< (p)$ are estimated by expanding them with respect to $(lp)^2$. In rigorous sense the expansion around $p^2 = 0$ is possible only if the functions are analytic around $p^2 = 0$. Indeed at zero momentum the functions $\Pi_i^< (p)$ show singularities of delta functions when $|\mu| = \mathcal{E}_n$ ($n \geq 1$) as read from Eq. (4.14). However, in our present study, since the parameter $\mu$ is nothing but the probe to pick up the spontaneous breaking of Lorentz symmetry and finally we take the limit $J_\pm \to 0$, we only have to consider the case of $|\mu| < \mathcal{E}_1$, which is free from the delta function singularities at the zero momentum. It is also the reason why we believe $\Pi_o(0)$ is unaffected by higher-order corrections even when $|\mu| = m$. In the higher momentum $p^2 > l^{-2}$, the functions $\Pi_i^> (p)$ are expanded with respect to $(lp)^{-2}$ by means of the asymptotic expansion. Since the magnetic field can be treated as a weak field in comparison with the momentum, the expansion with $(lp)^{-2}$ matches the one with respect to the magnetic field.

Now we shall provide a formal expansion for each case. In the lower momentum ($\text{strong field regime}$) we can formally expand exponential factors in Eq. (4.4) and Eq. (4.10) with respect to $(lp)^2$. By using $x := (lp)^2$, they are given by

$$
\Pi_i^< (p) = \alpha_l \left[ L_e^{(0)} + L_e^{(1)} x + L_e^{(2)} x^2 + \cdots \right],
\Pi_\perp^< (p) = \alpha_l \left[ L_\perp^{(0)} + L_\perp^{(1)} x + L_\perp^{(2)} x^2 + \cdots \right],
\Pi_o^< (p) = \alpha \left[ L_o^{(0)} + L_o^{(1)} x + L_o^{(2)} x^2 + \cdots \right].
$$

All coefficients $L_e^{(n)}$, $L_\perp^{(n)}$, and $L_o^{(n)}$ ($n = 0, 1, \cdots$) are dimensionless and show the delta function singularities in $|\mu| = \mathcal{E}_n$ ($n \geq 1$), which means breakdown of analyticity around $p^2 = 0$. However, as far as we consider the symmetric limit $J_\pm \to 0$, such singularities do not appear in any coefficient. Actually, our vacuum is realized in the symmetric limit as the lowest Landau level which is free from the delta function singularities. The first order coefficients are given by

$$
L_e^{(0)} = c, \quad L_\perp^{(0)} = -\frac{c}{2}, \quad \text{and} \quad L_o^{(0)} = \bar{c},
$$

where, for convenience sake, we have introduced $\bar{c} := \kappa/\alpha$ which becomes $\bar{c} = 0, \pm 1, \text{and } \pm 2$ in the symmetric limit.

In the higher momentum ($\text{weak field regime}$) the functions $\Pi_i^> (p)$ are systematically expanded as a power series of $1/x$ as shown in Appendix B. They are formally given as an asymptotic series in the symmetric limit such as
\[
\Pi_e^>(p) = \frac{\alpha l}{\sqrt{x}} \left[ H_e^{(0)} + H_e^{(1)} \frac{1}{x} + H_e^{(2)} \frac{1}{x^2} + \cdots \right],
\]
\[
\Pi_\perp^>(p) = \frac{\alpha l}{\sqrt{x}} \left[ H_\perp^{(0)} + H_\perp^{(1)} \frac{1}{x} + H_\perp^{(2)} \frac{1}{x^2} + \cdots \right],
\]
\[
\Pi_o^>(p) = \alpha \left[ H_o^{(0)} + H_o^{(1)} \frac{1}{x} + H_o^{(2)} \frac{1}{x^2} + \cdots \right].
\]

(4.27)

According to the even or odd character under the parity, we find that coefficients \(H_e^{(2n+1)}\), \(H_\perp^{(2n+1)}\), and \(H_o^{(2n)}\) vanish for all nonnegative integers \(n\). We approximate the vacuum polarization functions as their asymptotic expansions up to \(O(x^{-1})\) and the coefficients are given by (see Appendix B):

\[
H_e^{(0)} = \lim_{J_\pm \to 0} \left[ \theta(m - |\mu|)I(p; m) + \theta(|\mu| - m) \theta\left[p^2 - 4(\mu^2 - m^2)\right]I(p;\mu) \right],
\]
\[
H_o^{(0)} = \lim_{J_\pm \to 0} \sgn(\kappa) \theta(|\mu| - m) \theta\left[p^2 - 4(\mu^2 - m^2)\right] \frac{4\left[p^2 - 2(\mu^2 - m^2)\right]}{p\sqrt{p^2 - 4(\mu^2 - m^2)}},
\]
\[
H_\perp^{(0)} \equiv 0,
\]

(4.28)

where the function \(I(p; u)\) is given by Eq. (B.13) in Appendix B and we use \(\sgn(\kappa) \equiv -\sgn(\mu eB)\), which obeys the Gauss law constraint [cf. Eq. (3.20)].

We notice that \(H_e^{(0)}\) coincides with the coefficient calculated in the absence of the magnetic field. This expansion in the weak field regime turns out to match the one around \(B = 0\) (or \(l \to \infty\)). The coefficients are determined in the symmetric limit such as (see Appendices B and C for details)

\[
H_e^{(0)} = \frac{\pi}{2}, \quad H_\perp^{(0)} = 0, \quad \text{and} \quad H_o^{(1)} = 2\bar{c}.
\]

(4.29)

We will show in Section V that the coefficient \(H_o^{(1)}\), which is proportional to CS coefficient \(\kappa\), essentially contributes to a linear term of the magnetic field \(B\) in the shift of photon zero-point energy.

### V. SCHWINGER-DYSON GAP EQUATION

In this section we study the Schwinger-Dyson gap equation for the fermion self-energy and show that dynamical breaking of both U(2\(N\)) and Lorentz symmetries are realized as one of nontrivial solutions to the SD equation. As confirmed in Section III B, both condensates corresponding to the spontaneous breaking of U(2\(N\)) and Lorentz symmetries necessarily appear in such a complementary manner as to keep \(\langle \bar{\psi}\psi \rangle - |\langle \psi^\dagger \psi \rangle|\) at a nonzero constant. It is a specific feature of \((2 + 1)\)-dimensional physics due to the kinematics of fermions in the magnetic field and this feature is also reflected in the structure of the SD equation.

The fermion self-energy contains, as its dynamical variables, a scalar component \(m_d\) and a \(\gamma^0\) component \(\mu_d\), which correspond to the dynamically induced fermion mass and chemical

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6We use \(p\) as \(p \equiv \sqrt{p^2}\) except for the usage of \(p\) as an argument of functions.
potential, respectively. These variables, $m_d$ and $\mu_d$, are determined as solutions to the SD gap equation in the same self-consistent manner as the Hartree-Fock equation for energy gap in BCS theory. The analysis of condensates in Section III B implies that it is convenient to use the proper combination $J_\pm = \mu \pm m$ instead of the explicit breaking parameters, $\mu$ and $m$, in order to manifest the spontaneous breaking of symmetries. Subsequently, we are led to combinations $\omega_\pm := (\mu_d \pm m_d)l$ [18]. This seems quite natural assignment once we recognize that $-\omega_-$ ($\omega_+$) corresponds to the energy gap of fermions on the lowest Landau level relative to the chemical potential (or Fermi energy) when $\mu_d > 0$ ($\mu_d < 0$). In fact the coupled SD equations for $m_d$ and $\mu_d$ can be reduced to two decoupled equations. One is the gap equation for $\omega_+$ and the other is that for $\omega_-$ as shown later. This is another specific feature of (2 + 1)-dimensional physics as that in condensates.

A. Schwinger-Dyson equation in the ladder approximation

We start our argument with construction of the SD equation in the ladder (bare vertex) approximation. In Section II we have derived the 1/$N$ leading effective theory in which a magnetic field couples to fermions as a background field. Our aim of gap equation analysis is to confirm realization of spontaneously magnetized vacuum supported by the condensate $\langle \psi^\dagger \psi \rangle \neq 0$. As in BCS theory, what sorts of excitations appear in a given system depends on the ground state which should be realized as one of solutions to the gap equation for self-energies of the excitations.

In our present study the proper excitation is nothing but the fermion coupled with a magnetic field. Therefore we must construct the SD equation based on the bare fermion propagator $S$ defined by Eq. (2.13) which should be recognized as the one for the proper excitation on the magnetized vacuum. As to the photon propagator, it has already been derived from the 1/$N$ leading effective action (2.20) and has shown the massless photon behavior through the Gauss law as given by Eq. (4.23). Thus the SD equation has the same contents as the one in QED$_3$ with a magnetic field [17] apart from the fact that the fermion self-energy contains a $\gamma^0$ component other than a scalar component.

The SD equation is given by the recurrent form of

$$G(x, y) = S(x, y) - ie^2 \int d^3 z d^3 t S(x, z) \gamma^\mu G(z, t) \gamma^\nu G(t, y) D_{\mu\nu}(z - t),$$  \hspace{1cm} (5.1)$$

(see Fig. 3) where $G$ denotes a full fermion propagator which should be consistently determined through the SD equation. We assume that the full propagator $G$ also has the same form:

$$G(x, y) = \exp \left( i e \sqrt{N} (x - y)^\mu A^\text{ext}_\mu (x + y) \right) \tilde{G}(x - y)$$ \hspace{1cm} (5.2)$$

as $S$ in Eq. (2.14). We substitute the propagators into Eq. (5.1) and perform the Fourier transform. After integrating out some of spatial coordinates, we obtain

$$\tilde{G}(p) = \tilde{S}(p) - \frac{ie^2}{(2\pi)^5} \int d^3 k d^2 q_\perp d^2 R_\perp e^{-i R_\perp q_\perp} \times \tilde{S} \left( p^0, p_\perp + \frac{e B R_\perp}{2} \right) \gamma^\mu \tilde{G}(k) \gamma^\nu \tilde{G} \left( p^0, p_\perp - \frac{e B R_\perp}{2} \right)$$

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\[ \times D_{\mu\nu} \left( p^0 - k^0, p_\perp - k_\perp + eB \tilde{R}_\perp \right), \]  

(5.3)

where \( \tilde{R}_\perp := (0, R_1, R_2) \) and \( \tilde{R}_\perp := (0, -R_2, R_1) \).

Following Ref. [17], we assume the strong magnetic field \( m (m_d) \ll l^{-1} \), so that Eq. (5.3) can be simplified owing to decoupling of the higher Landau levels \( \mathcal{E}_n (n \geq 1) \). The bare fermion propagator \( \tilde{S}(k) \) in Eq. (2.13) can be decomposed into the Landau level poles [24]:

\[ \tilde{S}(k) = \exp \left( -\frac{k^2_\perp}{|eB|} \right) \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eB, k)}{m^2 + 2|eB|n - k^2_\perp}. \]  

(5.4)

The function \( D_n \) is given by

\[ D_n(eB, k) = (m + k_\gamma^0) \left[ L^0_n \left( \frac{2k^2_\perp}{|eB|} \right) 2\Lambda_+ - L^0_{n-1} \left( \frac{2k^2_\perp}{|eB|} \right) 2\Lambda_- \right] + \frac{4k_\gamma^0 L^1_{n-1} \left( \frac{2k^2_\perp}{|eB|} \right)}{2}, \]  

(5.5)

where \( L^\alpha_n(x) \) are the generalized Laguerre polynomials and \( \Lambda_{\pm} \) denote the projection operators to spin states:

\[ \Lambda_{\pm} := \frac{1}{2} \mp i\gamma^1 \gamma^2 \text{sgn}(eB). \]  

(5.6)

We can easily see in Eq. (5.4) that under the strong magnetic field, \( m \ll l^{-1} \), the lowest Landau level dominates and all other higher Landau levels are negligible. Therefore, in SD equation (5.1), we approximate the Fourier transform of \( \tilde{S} \) with their lowest Landau level contributions, that is

\[ \tilde{S}(k) \simeq e^{-l^2 k^2_\perp} \frac{1}{m - \gamma^0 k_\gamma^0} 2\Lambda_. \]  

(5.7)

We notice that fermions on the lowest Landau level essentially behave like \((0+1)\)-dimensional objects. It is natural to write down the full fermion propagator also in the following \((0+1)\)-dimensional form:

\[ \tilde{G}(k) \simeq e^{-l^2 k^2_\perp} \frac{1}{m - \gamma^0 k_\gamma^0} 2\Lambda_. \]  

(5.8)

where \( \tilde{g}(k^0) \) is a matrix commutative with \( \Lambda_+ \).

Substituting \( \tilde{S} \) and \( \tilde{G} \) into Eq. (5.3), we carry out \( q_\perp, R_\perp \) integration so that Eq. (5.3) becomes simplified \((0+1)\)-dimensional form [17]:

\[ \tilde{D}(p^0) = m - \gamma^0 (p^0 + \mu) - \frac{ie^2}{(2\pi)^3} \int_{-\infty}^{\infty} dk^0 \gamma^0 \tilde{g}(k^0) \gamma^0 \tilde{D}(p^0 - k^0), \]  

(5.9)

where the function \( \tilde{D} \) is defined by

\[ \tilde{D}(p^0) := -\int d^2 p_\perp e^{-l^2 p^2_\perp/2} D_{00}(p^0, p_\perp). \]  

(5.10)
Note that since we assume there is a magnetic field, the Gauss law forces the effective CS coefficient at zero momentum to be zero. The photon in the magnetic field, therefore, behaves as a massless mode in infrared region no matter how we take the symmetric limit \( J_\pm \to 0 \). According to Eq. (4.23), the one dimensional photon propagator \( \tilde{D} \) is represented by the integral exponential function which has a logarithmic behavior in the infrared region of momentum such as

\[
e^{2l} \frac{e^2}{(2\pi)^3} \tilde{D}(p) \approx -\frac{\alpha_0}{\pi} \ln |lp|, \tag{5.11}
\]

where \( \alpha_0 := \alpha l/N (1 + c \alpha l) \) and \( p := -ip^0 \). The constant \( c \) denotes parity even vacuum polarization effect, \( \Pi_e(0) \equiv c \alpha l \), and is given by \( c = -6\sqrt{2} \zeta(-1/2) \approx 1.76397 \). Thus only the infrared momentum region is relevant in Eq. (5.10).

B. Rearrangement of dynamical variables

The matrix function \( \tilde{g} \) is written in an SO(2) invariant form;

\[
\tilde{g}^{-1}(p^0) = B(p^0) + \hat{A}(p^0) p^0 - \gamma^0 \left[ \hat{B}(p^0) + A(p^0) p^0 + i\epsilon \text{sgn}(p^0) \right], \tag{5.12}
\]

where the functions \( A, B, \hat{A}, \) and \( \hat{B} \) are even functions of \( p^0 \). The self-energy part is divided into scalar and \( \gamma^0 \) components as the functions \( B \) and \( \hat{B} \), respectively. Since \( p^0 \) is SO(2) invariant in itself, \( \hat{A}(p^0) p^0 \) should be involved in \( \tilde{g}^{-1} \) as an odd function part of the scalar component. We can set the functions \( A(p^0) \) and \( B(p^0) \) to be positive definite without loss of generality. Following the physical implication mentioned in the beginning, we rearrange the functions \( A, B, \hat{A}, \) and \( \hat{B} \) as

\[
A_\pm(p^0) := A(p^0) \pm \hat{A}(p^0), \\
B_\pm(p^0) := \hat{B}(p^0) \pm B(p^0). \tag{5.13}
\]

If we recognize that the functions \( \pm B_\pm/A_\pm \) are identified with the relative energy of the proper excitation to the Fermi energy, it seems natural to assume that \( B_\pm/A_\pm \) have a definite sign irrespective of its argument. Then we can perform the Wick rotation \( k^0 = ik \) in the SD equation without any ambiguity. Actually \( \tilde{g}(k^0) \) is decomposed into two propagators which correspond to the proper excitations in the lowest Landau level;

\[
\tilde{g}(k^0) = \frac{1 - \gamma^0}{2} \left[ \frac{1}{B_+(k^0) + A_+(k^0) k^0 + i\epsilon \text{sgn}(k^0)} \right] \\
- \frac{1 + \gamma^0}{2} \left[ \frac{1}{B_-(k^0) + A_-(k^0) k^0 + i\epsilon \text{sgn}(k^0)} \right], \tag{5.14}
\]

whose poles are located on a complex \( k^0 \) plane at

\footnote{We use \( k, p \) instead of \( \bar{k}, \bar{p} \) only in this section to simplify the mathematical formulas.}
\[ k^0 = \begin{cases} -|B_+/A_+| + i\epsilon & \text{for } B_+/A_+ > 0 \\ |B_-/A_-| - i\epsilon & \text{for } B_-/A_- < 0 \end{cases} \quad (5.15) \]

Thus we can perform the Wick rotation without any residual contributions from poles.

Substituting Eq. (5.12) into the SD equation (5.9) and performing the Wick rotation \( k^0 = ik, \ p^0 = ip \), we obtain two sets of coupled integral equations \cite{18}

\[
B_\pm(p) - J_\pm = \frac{e^2}{(2\pi)^3} \int_{-\infty}^{\infty} dk \frac{B_\pm(k)}{A_\pm^2(k) k^2 + B_\pm^2(k)} \tilde{D}(p - k), \quad (5.16)
\]

\[
p[1 - A_\pm(p)] = \frac{e^2}{(2\pi)^3} \int_{-\infty}^{\infty} dk \frac{A_\pm(k) k}{A_\pm^2(k) k^2 + B_\pm^2(k)} \tilde{D}(p - k), \quad (5.17)
\]

which are divided into the coupled equations for \((A_+, B_+)\) and those for \((A_-, B_-)\) by virtue of the specific feature of \((2 + 1)\)-dimensions \cite{cf. Eq. (5.14)}. It is obvious that the above equations have trivial solutions \(B_\pm(p) \equiv 0\) in the symmetric limit \(J_\pm \to 0\).

Firstly we determine \(A_\pm(p)\) for the trivial solutions, \(B_\pm(p) \equiv 0\), as well as for the nontrivial solutions. We differentiate both sides of Eq. (5.17) with respect to \(p\) and obtain

\[
1 - A_\pm(p) - p A'_\pm(p) = -\frac{\alpha_0}{\pi l} \int_{-\infty}^{\infty} dk \frac{A_\pm(k) k}{A_\pm^2(k) k^2 + B_\pm^2(k)} \frac{1}{p - k}, \quad (5.18)
\]

where we have used the asymptotic form of \(\tilde{D}\) in Eq. (5.11). According to massless photon behavior in the infrared region, the integral in Eq. (5.18) shows infrared divergence if we naively set \(B_\pm(p) \equiv 0\). Besides a spurious photon mass and a gauge fixing parameter do not play any role to regularize this infrared divergence because of the logarithmic behavior of \(\tilde{D}(p)\). We therefore need to leave \(B_\pm\) in the integrand as a cut-off to regularize the infrared divergence even for the trivial solutions.

Let us suppose that \(A_\pm(p)\) are constant in almost all of the momentum regions except for \(p \approx 0\), while \(p A'_\pm(p)\) are negligible even in \(p \approx 0\). For the large momentum \(p\) the integral in Eq. (5.18) vanishes and we get \(A_\pm(\infty) = 1\). For \(p \approx 0\) we approximate the fermion propagator as

\[
\frac{1}{A_\pm^2(k) k^2 + B_\pm^2(k)} \approx \frac{1}{k^2 + B_\pm^2(0)}, \quad (5.19)
\]

which obeys our assumption. Thus Eq. (5.18) reduces to

\[
1 - A_\pm(0) \approx \frac{\alpha_0}{\pi l} \int_{-\infty}^{\infty} dk \frac{A_\pm(k)}{k^2 + B_\pm^2(0)}, \quad (5.20)
\]

and we obtain \cite{17}

\[
A_\pm(0) \approx \left[ 1 + \frac{\alpha_0}{|lB_\pm(0)|} \right]^{-1}. \quad (5.21)
\]

Although \(A_\pm(0)\) reduce to zero for the trivial solutions \(B_\pm(p) \equiv 0\), \(A_\pm(p) \approx 1\) are satisfied except for the momentum around \(p \approx 0\) and are consistent with the requirement of Ward-Takahashi identity under the ladder (bare vertex) approximation.
Now we solve Eq. (5.16) and find the nontrivial solutions \( B_\pm(0) \neq 0 \) in the symmetric limit \( J_\pm \to 0 \). Since the integral in Eq. (5.16) is dominated at the infrared region by the logarithmic behavior of the photon propagator in Eq. (5.11), the approximation in Eq. (5.19) is also valid and Eq. (5.16) is simplified as

\[
B_\pm(0) = -\frac{\alpha_0}{\pi l} \int_{-\infty}^{\infty} dk \frac{B_\pm(k)}{k^2 + B_\pm^2(0)} \ln|lk|.
\]

(5.22)

If the above gap equation has nontrivial solutions, they should satisfy \(|lB_\pm(0)| \ll 1\) so as to be consistent with the lowest Landau level dominance. The smallness of the solutions leads to the result that the dominant contribution comes from the infrared region, \( k \approx 0 \), in the integral in Eq. (5.22). Thus we can replace \( B_\pm(k) \) with \( B_\pm(0) \) and obtain the gap equation:

\[
\omega_\pm = -\frac{\alpha_0}{\pi} \int_{-\infty}^{\infty} ds \frac{\omega_\pm}{s^2 + \omega_\pm^2} \ln|s|,
\]

(5.23)

where we introduce dimensionless variables \( \omega_\pm := lB_\pm(0) \). This equation has the nontrivial solutions \( \pm \omega_s \) given by \( \omega_s = -\alpha_0 \ln \omega_s \), as well as the trivial one \[17\]. Note that the nontrivial solution \( \omega_s \) satisfies the condition \( \omega_s \ll 1 \), since \( \alpha_0 < 1 \) for any \( e^2 \) and \( N \). If \( \omega_\pm \) appear as one of the nontrivial solutions \( \pm \omega_s \), \( A_\pm(0) \) are determined as

\[
A_\pm(0) \approx \left[ 1 - \frac{1}{\ln \omega_s} \right]^{-1} \approx 1,
\]

(5.24)

which is consistent with the Ward-Takahashi identity, \( A_\pm(p) \equiv 1 \), due to the smallness of the nontrivial solution \( \omega_s \).

C. Classification of the nontrivial solutions

The obtained gap equations are the same for all three theories assigned to \( |\kappa| = 0, \alpha, 2\alpha \). On the other hand, in order to maintain the self-consistent magnetic field for each \( \kappa \), the dynamical mass \( m_d \) and the chemical potential \( \mu_d \) have to obey the same relation as the explicit breaking parameters \( m \) and \( \mu \) satisfy in Eq. (3.20) (see Fig. 2). In fact \( \omega_\pm \) do not necessarily choose the nontrivial solutions \( \pm \omega_s \). Consistent solutions are automatically assigned to each \( \kappa \) such as

\[
(\omega_+, \omega_-) = \begin{cases} 
(\omega_s, -\omega_s) & , \kappa = 0 \\
(\omega_s, 0) \text{ or } (0, -\omega_s) & , |\kappa| = \alpha \\
(\omega_s, \omega_s) & , |\kappa| = 2\alpha
\end{cases},
\]

(5.25)

where the replacement of \( \omega_s \) with \( -\omega_s \) provides another set of nontrivial solutions. It is intriguing that the solutions for \( |\kappa| = \alpha \) appear as combinations of a nontrivial solution and a trivial one. The dynamical variables, \( m_d \) and \( \mu_d \), which are defined by \( m_d := (\omega_+ - \omega_-)/2l \) and \( \mu_d := (\omega_+ + \omega_-)/2l \), respectively, are determined as

\[
(m_d, |\mu_d|) = \begin{cases} 
(\omega_s/l, 0) & \text{for } \kappa = 0 \\
(\omega_s/2l, \omega_s/2l) & \text{for } |\kappa| = \alpha \\
(0, \omega_s/l) & \text{for } |\kappa| = 2\alpha
\end{cases}.
\]

(5.26)
For \( \kappa = 0 \), fermions acquire their dynamical mass, while the vacuum becomes empty \( (\mu_d = 0) \). The fermion self-energy changes entirely into the dynamical mass and reproduces the result in Ref. [17]. For \( |\kappa| = 2\alpha \), the self-energy is used to occupy the lowest Landau level fully with fermions. Then there is no mass generation and it supports dynamically the result in Ref. [8]. Noteworthy case is \( |\kappa| = \alpha \) where the self-energy is shared by \( m_d \) and \( \mu_d \). The vacuum is realized as the lowest Landau level half-filled by massive fermions. This solution is a new one which spontaneously breaks the \( U(2N) \) symmetry as well as the Lorentz symmetry at the same time [18].

VI. SPONTANEOUS MAGNETIZATION AND FERMION MASS GENERATION

It still remains unknown whether or not the nontrivial solutions to SD gap equation as well as the self-consistent magnetic field are energetically more favorable than a trivial solution without any magnetic field. In order to solve the above problem, we have to investigate the effective potential which is obtained as the vacuum energy shift owing to the spontaneous breaking of Lorentz symmetry and/or \( U(2N) \) symmetry. Since the solutions, \( m_d \) and \( \mu_d \), are connected with the self-consistent magnetic field \( B \), the effective potential depends only on \( B \) and is given as a function \( V(B) \). It is composed of four parts including the Maxwell energy:

\[
V(B) = V_{\text{CJT}}(B) + V_F(B) + V_P(B) + \frac{B^2}{2}. \tag{6.1}
\]

\( V_{\text{CJT}} \) denotes the CJT potential [20] which gives the energy difference between a nontrivial vacuum and a trivial one under the presence of the magnetic field. \( V_F \) (\( V_P \)) corresponds to the shift of fermion (photon) zero-point energy due to the magnetic field in the symmetric limit.

In the following we derive the potentials, \( V_{\text{CJT}}, V_F, \) and \( V_P \), separately and investigate stability of vacuum in the large \( N \) limit for each value of \( \kappa \). We employ strong coupling expansion with respect to \( 1/\alpha l \sim \sqrt{B/e^3} \) in order to estimate the lowest order contribution to \( V_P \) based on the expansion of vacuum polarization functions in two regimes of momentum, which is shown in Section IV.

A. Vacuum energy shift due to fermions: \( V_F \)

The vacuum energy in the large \( N \) limit is given by the first term of effective action in Eq. (2.20). It depends on the explicit breaking parameters \( (m, \mu) \) as well as the magnetic field \( B \) such as

\[
\Omega E_1(m, \mu; B) := iN \text{Tr} \text{Ln} S^{-1}, \tag{6.2}
\]

where \( \Omega \) denotes the three-dimensional space-time volume and \( S^{-1} \) is the inverse of a bare fermion propagator. It is nothing but the zero-point energy for fermions coupled with the magnetic field \( B \). The shift of zero-point energy for fermions due to the magnetic field is determined in the symmetric limit as
\[ V_F(B) := \lim_{\epsilon \to 0} \left[ E_1(m, \mu; B) - E_1(m, \mu; B \to 0) \right]. \quad (6.3) \]

It is convenient to use the condensate \( J_0 \) to derive the finite density part of \( E_1 (\mu \neq 0) \). From the definition (6.2), we notice
\[
\begin{align*}
\frac{\partial E_1(m, \mu; B)}{\partial \mu} &= -\Omega^{-1} iN \text{Tr} [\gamma^0 S] \\
&= -J_0(m, \mu),
\end{align*}
\]
where \( J_0 \) is the condensate calculated in Section [II] and given by Eq. (3.13). Integrating both sides over \( \mu \), we obtain another form of \( E_1 \);
\[
E_1(m, \mu; B) = E_1(m, 0; B) - \int_0^\mu d\mu' J_0(m, \mu'; B). \quad (6.5)
\]

As to the computation of zero density part, the proper time method is used. We see from the definition (6.2) that
\[
\Omega E_1(m, 0; B) = iN \text{Tr} \ln [m - i\nabla] = iN \frac{2}{\text{Tr} \ln [m^2 + \nabla^2]},
\]
where \( \nabla := \gamma^\mu D_\mu [A^{\text{ext}}] \) and we have used the relation \( \gamma^5 \nabla \gamma^5 = -\nabla \). The symbol \( \text{Ln} \) denotes a logarithmic function whose argument is given by operators, and \( \text{Tr} \) means a trace over space-time coordinates as well as over spinor indices. The \( \text{Tr} \ln \) term is rewritten with an integration over the proper time;
\[
\text{Tr} \ln [m^2 + \nabla^2] = -\int d^3x \int_{1/\Lambda^2}^\infty \frac{ds}{s} \text{tr} \langle x | e^{-is[m^2+\nabla^2]} | x \rangle.
\]

Eqs. (2.13-2.15) lead to the following relation
\[
\langle x | e^{-is[m^2+\nabla^2]} | x \rangle = \int \frac{d^3k}{(2\pi)^3} \exp \left[ -is \left( m^2 - k_0^2 + \frac{\tan(eBs)}{eBs} k^2 \right) \right] \left[ 1 + \gamma^1 \gamma^2 \tan(eBs) \right]. \quad (6.8)
\]
Applying this relation to Eqs. (6.6-6.7), we obtain, after Wick rotation and Gaussian integration,
\[
E_1(m, 0; B) = \frac{N}{4\pi^{3/2}} |eB| \int_{1/\Lambda^2}^\infty \frac{ds}{s^{3/2}} e^{-sm^2} \coth(|eB|s)
\]
\[
= -\frac{N}{2\pi^{3/2}} \Lambda m^2 - \frac{N}{2\pi} |eB|^{3/2} \left[ ml + 2\sqrt{2} \zeta \left( -\frac{1}{2}, \frac{(ml)^2}{2} + 1 \right) \right]. \quad (6.9)
\]
where the function \( \zeta(z, q) \) denotes the generalized Riemann zeta function which has appeared in Section [II]. The finite density part is calculated from the condensate \( J_0 \) in Eq. (3.13) as
\[
\int_0^\mu d\mu' J_0(m, \mu'; B) = \frac{N}{2\pi} |eB|^{3/2} \left[ l(|\mu| - m) \theta(|\mu| - m) + 2 \sum_{n=1}^\infty l(|\mu| - \mathcal{E}_n) \theta(|\mu| - \mathcal{E}_n) \right].
\]
\[ (6.10) \]
Thus the explicit formula for $E_1$ is given by

$$E_1(m, \mu; B) = -\frac{N}{2\pi^{3/2}} \Lambda m^2 - \frac{N}{2\pi} |eB|^{3/2} \left[ ml + 2\sqrt{2} \zeta \left( -\frac{1}{2}, \left(\frac{ml}{2}\right)^2 + 1 \right) \right.

+ l(\mu) - m)\theta(\mu - m) + 2 \sum_{n=1}^{\infty} l(\mu - E_n)\theta(\mu - E_n) \right].$$

(6.11)

The fermion vacuum energy in the absence of the magnetic field is obtained by taking the weak field limit in Eq. (6.11). We find

$$E_1(m, \mu; B \to 0) = -\frac{N}{2\pi^{3/2}} \Lambda m^2 + \frac{N}{3\pi} m^3,$$

(6.12)

where we have used the asymptotic formula for the zeta function $\zeta(z, q)$,

$$\zeta(z, q) \sim \frac{1}{(z - 1)q^{z-1}} \left[ 1 + \frac{z - 1}{2q} + \cdots \right], \quad (q \to \infty).$$

(6.13)

We notice that the ultraviolet divergence which appears in $B \neq 0$ can be completely canceled out by another ultraviolet divergence in $B = 0$.

After taking the symmetric limit, $V_F$ is determined as an ultraviolet finite function of $B$, that is,

$$V_F(B) = -\frac{N}{4\pi} |eB|^{3/2} 4\sqrt{2} \zeta(-1/2),$$

(6.14)

which is irrespective of the values of $\kappa$, or approaches to the symmetric limit.

**B. Vacuum energy shift due to the nontrivial solution: $V_{CJT}$**

The Schwinger-Dyson gap equation is derived from the effective action for composite operators, $\bar{\psi}\psi$ and $\psi^\dagger \psi$. It is given by a functional $\Gamma[G]$ of the full fermion propagator $G$ which is defined by

$$G(x, y) \delta_{ij} := i \langle 0 | T \psi_i(x) \bar{\psi}_j(y) | 0 \rangle.$$  

(6.15)

The equation which leads to stationary points of $\Gamma[G]$ is equivalent to the SD gap equation. The effective action $\Gamma[G]$, or Cornwall-Jackiw-Tomboulis action, is given by [20]

$$N \Gamma[G] = -iN \text{Tr} \left[ \ln G^{-1} + S^{-1}G - 1 \right] + N \Gamma_2[G],$$

(6.16)

where $\Gamma_2[G]$ denotes all of the 2PI (two-particle irreducible) bubble diagrams composed of the full propagator $G$. The large $N$ contributions to $\Gamma_2[G]$ are given by a gauge invariant two-loop diagram (see Fig. 4);

$$N \Gamma_2[G] = \frac{Ne^2}{2} \text{Tr} [G\gamma^\mu G\gamma^\nu] D_{\mu\nu},$$

(6.17)
where the photon propagator $D_{\mu\nu}$ involves the large $N$ screening of fermions and has already been given in Section IV. We see that a stationary point for $\Gamma[G]$ actually provides the SD gap equation

$$\frac{\delta \Gamma[G]}{\delta iG} \equiv G^{-1} - S^{-1} + \frac{\delta \Gamma_2[G]}{\delta iG} = 0,$$

which has been solved in Section V and shows the nontrivial solutions given by Eq. (5.26) as well as a trivial one.

What we want to know is stability of the vacuum which is realized as a nontrivial solution $(m_d, \mu_d)$ for each $\kappa$. To see this we only have to investigate which is energetically more favorable between the nontrivial solution and the trivial one. The vacuum energy due to the dynamical solutions $(m_d, \mu_d)$ is provided by the CJT action as its value on the stationary point;

$$\Omega_E(m_d, \mu_d; B) := -N \Gamma[G_{sol}]
= iN \text{Tr} \left[ \ln G_{sol}^{-1} + \frac{1}{2} \left( S^{-1} - G_{sol}^{-1} \right) G_{sol} \right],$$

where $G_{sol}$ means the solutions to the SD equation (6.18) and we have eliminated the 2PI contribution by means of Eq. (6.18). Thus the vacuum energy shift due to the nontrivial solution is determined as

$$V_{\text{CJT}}(B) := \lim_{J_{\pm} \to 0} \left[ E(m_d, \mu_d; B) - E(0, 0; B) \right].$$

To estimate $V_{\text{CJT}}$ from the solutions obtained in Section V, we adopt the following procedure as an approximation, that is the replacement

$$\lim_{J_{\pm} \to 0} G_{sol} \sim S \big| (m, \mu) \to (m_d, \mu_d),$$

where $S$ denotes the bare fermion propagator defined by Eq. (2.13). All of the Landau levels contribute to $V_{\text{CJT}}$. However, the lowest Landau level dominates due to smallness of the solutions, $m_d \ll 1$ and $\mu_d \ll 1$. Under the replacement (6.21) we see

$$\lim_{J_{\pm} \to 0} iN \text{Tr} \ln G_{sol}^{-1} = \Omega E_1(m_d, \mu_d; B),$$

where the energy function $E_1$ is the same as the shift of fermion zero-point energy. We also find that the second term in Eq. (6.19) becomes

$$\lim_{J_{\pm} \to 0} \frac{iN}{2} \text{Tr} \ln \left[ (\mu_d \gamma^0 - m_d) G_{sol} \right] = \frac{\Omega}{2} \left[ \mu_d J_0(m_d, \mu_d) - m_d J(m_d, \mu_d) \right],$$

where the functions $J_0$ and $J$ are the same condensates calculated in Section III except for the replacement $(m, \mu) \to (m_d, \mu_d)$. Using Eqs. (3.12, 3.13) and Eq. (6.11) we obtain

$$\lim_{J_{\pm} \to 0} E(m_d, \mu_d; B) = -\frac{N}{4\pi} |eB|^{3/2} \left[ m_d l + l(|\mu_d| - m_d) \theta(|\mu_d| - m_d)
- \sqrt{2} (m_d l)^2 \zeta \left( \frac{1}{2}, \frac{(m_d l)^2}{2} + 1 \right) + 4 \sqrt{2} \zeta \left( -\frac{1}{2}, \frac{(m_d l)^2}{2} + 1 \right) \right],$$

(6.24)
where only the lowest Landau level contributes to the finite density part due to smallness of the solutions, while the zero density parts appear as a summation over all the Landau levels. Notice that the ultraviolet divergences in $E_1$ and $J$ are completely canceled out in the whole of $E$.

Thus $V_{\text{CJT}}$ is determined in such a form which depends on the nontrivial solutions in Eq. (5.26) as

$$V_{\text{CJT}}(B) = -\frac{N}{4\pi} |eB|^{3/2} \left[ l \max \{m_d, |\mu_d| \} + O((m_d l)^4) \right],$$

(6.25)

which shows that the vacuum energy is shifted to be negative. Therefore it is confirmed that dynamically generated solutions are energetically favorable for each $\kappa$ in the self-consistent magnetic field $B$.

C. Vacuum energy shift due to photon: $V_P$

$V_P$ is derived from the effective action (2.20) as the next-to-leading-order contribution in $1/N$, that is,

$$V_P(B) := \lim_{J_{\pm} \to 0} \left( -\frac{1}{2\Omega} \right) i \text{Tr} \ln D^{-1} - (B \to 0)$$

$$= \lim_{J_{\pm} \to 0} \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \ln \left[ \hat{D}(p) D^{-1}(p) \right],$$

(6.26)

where we have introduced $\hat{D}_{\mu\nu}(p)$ which is defined as the photon propagator $D_{\mu\nu}(p)$ in $B \to 0$, and used matrix notation for space-time indices. The symbol $\text{tr}$ thereby means the trace for space-time indices. We notice that $V_P$ is the shift of photon zero-point energy due to the magnetic field.

In order to estimate the integral in Eq. (6.26), we have to expand $V_P$ into a power series with respect to the magnetic field $B$. However we have supposed the strong magnetic field which supports the lowest Landau level dominance and therefore guarantees the analysis of the SD gap equation in Section V. In fact we are led to the small solutions, $m_d (|\mu_d|) \ll l^{-1}$, in comparison with the magnetic field. Hence we need another scale which can be taken to be larger than the magnetic field and provides the small parameter for the expansion.

We have indeed such a scale, that is, the gauge coupling $\alpha$ to which the Chern-Simons coefficient $\kappa$ is related through the Gauss law. The parameter $r := 1/\alpha l$ can be small by choosing a sufficiently large $\alpha$ with the magnetic length $l$ fixed finite. This results in a hierarchy of three scales, $m_d (|\mu_d|) \ll l^{-1} \ll \alpha$, which should be achieved for our present study and the solutions $(m_d, \mu_d)$ actually allow this situation. Thus we recognize that the expansion of $V_P(B)$ is a kind of strong coupling expansion with respect to $1/\alpha$. It is enough to pick up and estimate the lowest order contribution of the expansion with respect to $r$ in order to investigate stability of vacuum.

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8 Hereafter we use the propagator $D (\hat{D})$ where the symmetric limit $J_{\pm} \to 0$ has been taken.
To do the above procedure it is necessary at first to expand an integrand of Eq. (6.26) in such a consistent way that matches the expansion with respect to $1/\alpha l$. In the following we apply an expansion of $D$ around $\hat{D}$ and try to estimate the lowest order contribution to $V_P$. The analysis is based on the expansion of vacuum polarization in two regimes of momentum in Section III.

The propagators $D$ and $\hat{D}$ are given by

$$D^{-1}_{\mu\nu}(p) := \Delta^{-1}_{\mu\nu}(p) - \Pi_{\mu\nu}(p),$$

$$\hat{D}^{-1}_{\mu\nu}(p) := \Delta^{-1}_{\mu\nu}(p) - \hat{\Pi}_{\mu\nu}(p),$$

(6.27) where $\hat{\Pi}_{\mu\nu}(p)$ denotes the vacuum polarization in the limit $B \to 0$ and is expressed as

$$\hat{\Pi}_{\mu\nu}(p) = (p_{\mu}p_{\nu} - p^2 g_{\mu\nu}) \hat{\Pi}(p),$$

(6.28) which is written in the gauge invariant form even in the presence of the finite density $\mu \neq 0$. The function $\hat{\Pi}(p)$ is determined as

$$\hat{\Pi}(p) = \lim_{J_{\pm} \to 0} \frac{\alpha}{p} \left[ \theta(m - |\mu|)I(p; m) + \theta(|\mu| - m) \theta\left[p^2 - 4(\mu^2 - m^2)\right]I(p; \mu) \right],$$

(6.29) with the function $I(p; u)$ given by Eq. (B15). From Eq. (6.27) $D$ is rewritten in the form of expansion around $\hat{D}$ such as

$$D^{-1}_{\mu\nu}(p) = \hat{D}^{-1}_{\mu\nu}(p) - \delta \Pi_{\mu\nu}(p),$$

$$\delta \Pi_{\mu\nu}(p) := \Pi_{\mu\nu}(p) - \hat{\Pi}_{\mu\nu}(p).$$

(6.30) Thus we obtain the following expansion

$$\text{tr} \ln \left[ \hat{D}(p)D^{-1}(p) \right] \equiv \text{tr} \ln \left[ 1 - \hat{D}(p) \delta \Pi(p) \right] = -\sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[ \hat{D}(p) \delta \Pi(p) \right]^n. \quad (6.31)$$

Each term in the above summation is estimated through the decomposition of a rank $2 \times 3 \times 3$ matrix $\hat{D} \delta \Pi$ into a $2 \times 2$ unit matrix and Pauli matrices. It is expressed as

$$\text{tr} \left[ \hat{D}(p) \delta \Pi(p) \right]^n = 2 \sum_{m=0}^{[n/2]} nC_{2m} \frac{\left[R(p)\right]^{n-2m} \left[S^2(p)\right]^m}{\left[Q(p)\right]^n}, \quad (6.32)$$

where $[n/2]$ means the integer part of $n/2$ and the functions $Q$, $R$, and $S = (S_1, S_2, S_3)$ are determined as

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9 The parity-violating part vanishes in the limit $B \to 0$ as shown in Eq. (4.10) due to our setting of fermions.
\[ Q(p) = p^2 \left[ 1 + \tilde{\Pi}(p) \right]^2 + \kappa^2, \]
\[ R(p) = -p^2 \left[ 1 + \tilde{\Pi}(p) \right] \left[ \delta \Pi_e(p) + \frac{p^2}{2p^2} \delta \Pi_{\perp}(p) \right] + \kappa \delta \Pi_o(p), \]
\[ S_1(p) = - \left[ 1 + \tilde{\Pi}(p) \right] \frac{p^2}{2} \delta \Pi_{\perp}(p), \]
\[ S_2(p) = \frac{p^2}{2p} \kappa \delta \Pi_{\perp}(p), \]
\[ S_3(p) = -ip \left\{ \left[ 1 + \tilde{\Pi}(p) \right] \delta \Pi_o(p) + \kappa \left[ \delta \Pi_e(p) + \frac{p^2}{2p^2} \delta \Pi_{\perp}(p) \right] \right\}. \]

The deviations \( \delta \Pi_e \), \( \delta \Pi_{\perp} \), and \( \delta \Pi_o \) are the counter parts for \( \Pi_e \), \( \Pi_{\perp} \), and \( \Pi_o \) in Eq. (6.33), respectively.

In Section IV we have expanded the vacuum polarization functions in each regime of momentum. The energy shift \( V_p \) is divided into two parts according to two regimes of the loop momentum. After the Wick rotation, it is given by

\[ V_p(B) = \sum_{n=1}^{\infty} \sum_{m=0}^{n/2} \frac{n C_{2m}}{n} \left[ V_{nm}^< (B) + V_{nm}^> (B) \right], \]

where \( \theta \) denotes the polar angle for the Euclidean momentum \( p^\mu \). We apply the (asymptotic) expansion for the vacuum polarization functions in the strong (weak) field regime of momentum to \( V_{nm}^< \) (\( V_{nm}^> \)).

In the strong field regime \( (p < l^{-1}) \), we notice from the results of Eq. (4.26) that

\[ \delta \Pi_e = \frac{\alpha l}{\sqrt{x}} \left[ -\frac{\pi}{2} + \sqrt{x} L_e(x) \right], \quad \delta \Pi_{\perp} = \alpha l \, L_{\perp}(x), \quad \delta \Pi_o = \alpha L_o(x), \]

where \( L_e, L_{\perp}, \) and \( L_o \) are expressed as analytic functions. Substituting the above results into Eq. (6.33), \( Q \), \( R \), and \( S^2 \) are written in such a factorized form as

\[ Q = \alpha^2 \left[ \left( \frac{\pi}{2} + r \sqrt{x} \right)^2 + \tilde{c}^2 \right], \quad R = \alpha^2 f \left( \sqrt{x}; r \right), \quad S^2 = \alpha^4 g \left( \sqrt{x}; r \right), \]

where analytic functions \( f(u; r) \) and \( g(u; r) \) depend on the small parameter \( r := 1/\alpha l \). Then \( V_{nm}^< \) becomes

\[ V_{nm}^< (B) = -\frac{1}{2\pi^2 l^3} \int_0^1 d\eta \int_0^1 du \frac{u^2 \left[ f(u; r) \right]^{n-2m} \left[ g(u; r) \right]^m}{(ru + \frac{\pi}{2})^2 + \tilde{c}^2}. \]

We notice that a factor \(-1/2\pi^2 l^3 \) supplies \( r^3 \). If an integral over \( u \) diverges by going to \( r = 0 \), it lowers the power of \( r \) to be less than three. Since in the limit \( r \to 0 \) both of \( f \) and
In Eq. (6.37) converge and the lowest order is determined as $O(r^3)$ which is the same order as $V_F$ and $V_{C,RT}$.

In the weak field regime ($p > l^{-1}$) we find from the results of Eq. (4.29) that

$$
\delta \Pi_e^e = \frac{\alpha l}{x^{5/2}} H_e \left( \frac{1}{x} \right), \quad \delta \Pi_\bot = \frac{\alpha l}{x^{5/2}} H_\bot \left( \frac{1}{x} \right), \quad \delta \Pi_o = \frac{\alpha}{x} H_o \left( \frac{1}{x} \right),
$$

where $H_e$, $H_\bot$, and $H_o$ are also expressed as analytic functions. Substituting the above results into Eqs. (6.33), $Q$, $R$, and $S^2$ are written in such a factorized form as

$$
Q = \alpha^2 x \left[ (r + \frac{\pi}{2} \sqrt{x})^2 + \frac{c^2}{x} \right], \quad R = \frac{\alpha^2}{x} F \left( \frac{1}{\sqrt{x}}; r \right), \quad S^2 = \frac{\alpha^4}{x} G \left( \frac{1}{\sqrt{x}}; r \right),
$$

by using analytic functions $F(u; r)$ and $G(u; r)$. Then we obtain the following formula for $V_{nm}^>:

$$
V_{nm}^> (B) = -\frac{1}{2 \pi^2 l^3} \int_0^1 \frac{d\eta}{2 \pi^2 l^3} \int_0^1 \frac{du}{2 \pi^2 l^3} \frac{u^{2(n-1)} [F(u; r)]^{n-2m} [G(u; r)]^m}{u^{2m} \left( r + \frac{\pi}{2} u \right)^{2} + c^2 u^2}. \tag{6.40}
$$

The convergence at $r = 0$ is not so trivial this time because of the factor $u^{4(n-1)/2m}$. However $G(u; r)$ supplies another factor $u^2$ when $r = 0$, which can be seen in the explicit form of it. Therefore we obtain an identity $G(u; 0) = u^2 \tilde{G}(u)$ with the analytic function $\tilde{G}(u)$. Inserting the above identity into Eq. (6.40), we find in the limit of $r \to 0$

$$
V_{nm}^> (B) \propto -\frac{1}{l^3} \int_0^1 \frac{d\eta}{2 \pi^2 l^3} \int_0^1 \frac{du}{2 \pi^2 l^3} \frac{u^{2(n-2)} [F(u; 0)]^{n-2m} [\tilde{G}(u)]^m}{(r + \frac{\pi}{2} u)^{2} + c^2 u^2}. \tag{6.41}
$$

The lowest order contribution would appear in the above integrals if they show divergence. The integral shows divergence only when $n = 1$, namely, $V_{10}^>$. In order to extract the lowest order contribution, it is therefore sufficient to estimate the coefficient of the most singular part of the integral in $V_{10}^>$:

$$
V_{10}^> (B) = -\frac{1}{2 \pi^2 l^3} \int_0^1 \frac{d\eta}{2 \pi^2 l^3} \int_0^1 \frac{du}{2 \pi^2 l^3} \frac{F(u; r)}{(r + \frac{\pi}{2} u)^{2} + c^2 u^2}. \tag{6.42}
$$

From the definition in Eq. (6.33) the function $F(u; r)$ is given by

$$
F(u; r) = c \tilde{H}_o(u^2) - u \left( r + \frac{\pi}{2} u \right) \left[ H_e(u^2) + \frac{\eta^2}{2} H_\bot(u^2) \right], \tag{6.43}
$$

where we see that the second term supplies a factor $u^2$ at $r = 0$ so that it never contributes to the lowest order. Thus it is clarified that the lowest order contribution is determined only by $H_o(0)$ such as

$$
V_{10}^> (B) = -\frac{1}{2 \pi^2 l^3} \left[ \frac{1}{r} H_o(0) \frac{d}{dr} \left( \frac{2c}{2r + \pi} \right) + \text{const.} + O(\ln r) \right], \tag{6.44}
$$
This means that even if we take all of the large $N$ contributions in vacuum polarization into account, the lowest order contribution to $V_P$ is determined only by $H_o(0)$. Therefore the above result is exact in the large $N$ limit. The lowest coefficient $H_o(0)$ has already estimated as $H_o(0) = 2\bar{c}$ which leads the coefficient of $|eB|$ to be negative definite.

Thus the vacuum energy shift $V_P$ up to $O(B^{3/2})$ is obtained as $V_{10}^{<}$ in Eq. (6.44). Specifically, it is given by

$$V_P(B) = -\frac{|\kappa|}{\pi^2} |eB| \arctan \left( \frac{2|\kappa|}{\pi \alpha} \right) + O(B^{3/2}). \quad (6.45)$$

It is obvious that the negative linear term for $|eB|$ is owing to the absence of topological photon mass in $B \neq 0$ as well as the presence of it in $B = 0$ as pointed out by Hosotani [8]. If we set $\bar{c} = \pm 2$, the above result completely matches Ref. [8].

### D. Stability of vacuum

Now we are in a position to discuss the stability of vacuum, or a possibility of spontaneous magnetization. Recall that the entire effect of vacuum energy shift is given by Eq. (6.1). We notice that the negative linear term for $B$ appears only in theories with nonzero $\kappa$, in which the magnetic field is possibly supported by the charge condensation $e\langle \psi^\dagger \psi \rangle$. Therefore, in the theories with $|\kappa| = \alpha$ and $2\alpha$, the potential has its stable stationary point at $B \neq 0$, so the spontaneous magnetization occurs and Lorentz symmetry is dynamically broken.

In the theory with $|\kappa| = 2\alpha$, the SD gap equation tells us that the fermion self-energy is exhausted to induce a charge density on vacuum so that there is no generation of fermion mass. Then the vacuum appears as a fully-filled magnetized vacuum [8]. Alternatively, in the theory with $|\kappa| = \alpha$, the SD gap equation shows not only the charge condensation but also the dynamical fermion mass, whose stability under the background magnetic field is supported by the CJT potential. Moreover the entire energy shift shows that the background field itself is also induced dynamically. Thus we are led to the half-filled vacuum in which both of the U$(2N)$ symmetry and the Lorentz symmetry are spontaneously broken [18].

As to the theory with $\kappa = 0$, that is QED$_3$ with $N$ four-component fermions, the situation becomes rather subtle because there is no Chern-Simons term, or the negative linear term in $B$. The lowest order term in $V_P$ is provided as a $B^{3/2}$ term, so all of higher-order corrections in the expansion of $\Pi_{\mu\nu}$ contribute to its coefficient. However, in the entire energy shift $V$, the $B^{3/2}$ term is saturated by that of $V_F$ with a factor $N$. Therefore at least in the large $N$ limit the entire potential has a positive $B^{3/2}$ term and results in no spontaneous magnetization.

We can see this more explicitly in the formula for $V_P$. Through an identity $\text{Tr} \ln \equiv \ln \text{Det}$, it is rewritten as

$$V_P(B) = \lim_{\lambda \to 0} \frac{1}{2} \int d^3p \frac{1}{i(2\pi)^3} \ln \left[ \frac{[1 + \Pi_{\perp}(p)][1 + \Pi_{\perp}(p) + (p^2/p^2_\perp)\Pi_{\perp}(p)]}{[1 + \Pi(p)]^2} \right], \quad (6.46)$$

in QED$_3$ [cf. Eq. (4.21)]. Since in QED$_3$ a photon field is always massless whether a magnetic field is turned on or off, there is no essential difference due to the magnetic field in the weak
field regime of momentum\[10\]

We therefore cut off the integral at \( p = l^{-1} \) and approximate the vacuum polarization functions as their values at \( p = 0 \). Then we obtain

\[
V_P(B) = \frac{1}{4\pi^2 l^3} \int_0^1 d\eta \int_0^1 du \, u^2 \ln \left[ \frac{[r + c][r + c - c\eta^2/2]}{[r + \pi/2u]^2} \right].
\]

(6.47)

We see that the integrand becomes negative around \( u = 0 \) due to the \( 1/p \) behavior of \( \hat{\Pi}(p) \) so that \( V_P \) is led to be negative. The integrals can be performed analytically and \( V_P \) is obtained as

\[
V_P(B) = \frac{1}{12\pi^2} |eB|^{3/2} K \left( \frac{2}{\alpha} \sqrt{|eB|} \right),
\]

(6.48)

with the function \( K(x) \) defined by

\[
K(x) := -\frac{8}{3} - \frac{2}{x^3} \left[ \ln(x + 1) - x + \frac{x^2}{3} - \frac{x^3}{3} \right] + \ln \left( \frac{x + b}{x + 1} \right) + \ln \left( \frac{x + b/2}{x + 1} \right)
+ 2 \sqrt{\frac{2(x + b)}{b}} \arccoth \sqrt{\frac{2(x + b)}{b}},
\]

(6.49)

where we have introduced \( b := 2c/\pi \approx 1.1298 \).

\( K(x) \) shows monotonically increasing behavior and becomes negative definite in \( x \geq 0 \) for \( b < 4/3 \), that is \( K(0) \leq K(x) < 0 \) as shown in Fig. 3. Hence, if an inequality \( |K(0)| < 2\pi c N \) is satisfied, then the \( B^{3/2} \) term in the entire energy shift \( V(B) \) becomes positive semi-definite and shows monotonically increasing behavior which leads us to a non-magnetized vacuum. If we suppose large \( N \), the above inequality is trivially satisfied, which only means that the \( B^{3/2} \) term is dominated by the large \( N \) contribution of \( V_P \). However the parameter \( c \) (or \( b \)), which is determined by the kinematics of fermion in the magnetic field, satisfies

\[
\frac{|K(0)|}{2\pi c} = \frac{1}{2\pi c} \left( -\frac{8}{3} + 2\sqrt{2} \arccoth \sqrt{2} + \ln(2c^2/\pi^2) \right) \approx 0.0572877.
\]

(6.50)

Therefore, even if \( N = 1 \), the spontaneous magnetization or equivalently the spontaneous Lorentz symmetry breaking does not occur in QED\(_3\) with \( N \) four-component fermions.

**VII. CONCLUSION**

In this paper we have investigated dynamical symmetry breaking in Chern-Simons QED\(_3\) associated with the realization of finite density vacua (or occupied lowest Landau levels).

Through the explicit estimation of condensates \( \langle \bar{\psi} \psi \rangle \) and \( \langle \psi^\dagger \psi \rangle \), we have clarified that in the presence of the Chern-Simons interaction the magnetic field is necessarily connected

\[10\] This situation changes drastically in Chern-Simons QED\(_3\). If \( B = 0 \), a photon has a large topological mass of the order of \( \alpha (> l^{-1}) \) without any screening. Therefore the photon behaves as a massive field even in the weak field regime of momentum \( (p > l^{-1}) \) when \( B = 0 \).
to the charge condensation $e\langle \bar{\psi}\psi \rangle$ through the Gauss law in such a way as to restrict the possible values of the CS coefficient $\kappa$ to $\pm Ne^2/2\pi$, $\pm Ne^2/4\pi$, and 0 [8,28,29]. In other words, the magnetic field can penetrate the system only if the photon effectively becomes massless [8]. We also have found that both condensates $\langle \bar{\psi}\psi \rangle$ and $\langle \bar{\psi}\gamma^\mu\gamma^5\psi \rangle$ always appear so as to complement each other and to keep the combination $\langle \bar{\psi}\psi \rangle - |\langle \bar{\psi}\gamma^5\psi \rangle|$ constant. The symmetry breaking patterns, or the vacuum configurations, are displayed in Table. I.

In the theory with $|\kappa| = Ne^2/2\pi$, the vacuum is realized as a fully-filled lowest Landau level $[(\nu_+,\nu_-) = (1,0)$, or $(0,1)]$ which is a singlet of the flavor U(2$N$) group. Only the Lorentz symmetry is spontaneously broken by the induced magnetic field together with charge condensation, as pointed out in Refs. [8,9]. Alternatively, in the theory with $\kappa = 0$, or QED$_3$ with $N$ four-component fermions, the vacuum is realized as an empty lowest Landau level $[\nu_+ = 0]$ even in the presence of the magnetic field. The U(2$N$) symmetry is spontaneously broken, while the magnetic field is not accompanied by charge condensation [10,17]. As to the theory with $|\kappa| = Ne^2/4\pi$, both condensates coexist in the half-filled lowest Landau level $[(\nu_+,\nu_-) = (1/2,0)$, or $(0,1/2)]$. This vacuum allows the spontaneous breaking of both symmetries, that is, the U(2$N$) symmetry and the Lorentz symmetry [18].

The above situation suggests that in the fermion self-energy the scalar component $m_d$ and the $\gamma^0$ component $\mu_d$ also complement each other. We have also attempted to find the dynamical solutions which correspond to the possible configurations of the above three classes through the analysis of the Schwinger-Dyson gap equation.

In the theory with $|\kappa| = Ne^2/2\pi$, the realization of a fully-filled vacuum is supported by the solution $(m_d,|\mu_d|) = (0,\omega_s/l)$. The fermion self-energy has only a $\gamma^0$ component, the originally massless fermions remain massless and there is no generation of fermion mass. This solution reproduces and verifies the result of Ref. [8], that is, vanishing fermion mass and broken Lorentz symmetry. Thus, the magnetic field does not necessarily lead to mass generation if the vacuum is fully-filled by fermions. This is in sharp contrast to the results of Ref. [17], which are based on the empty vacuum and correspond to the theory with $\kappa = 0$. In this case the solution to the gap equation has $(m_d,|\mu_d|) = (\omega_s/l,0)$. The fermion self-energy is saturated by the dynamically generated fermion mass, which then results in an empty vacuum.

In the case $|\kappa| = Ne^2/4\pi$, we have found a novel solution $(m_d,|\mu_d|) = (\omega_s/2l,\omega_s/2l)$ which causes spontaneous magnetization as well as the dynamical generation of fermion mass [18]. The scalar component $m_d$ and the $\gamma^0$ component $\mu_d$ are comparable to each other. The fermion self-energy is shared, half filling the lowest Landau level and half supplying fermions their dynamical mass. In the resultant half-filled vacuum the U(2$N$) symmetry and the Lorentz symmetry are simultaneously broken through the dynamically generated fermion mass and the induced magnetic field [18].

We have investigated the question of vacuum stability for each class through the explicit calculation of the energy shift $V(B)$ due to the magnetic field $B$. We find that a negative linear term in $B$ always appears for $|\kappa| = Ne^2/2\pi$ and $Ne^2/4\pi$, leading to a nonzero magnetic field at the stationary point. Moreover, the estimated coefficient for the above term is exact in the large $N$ limit and matches the former result in Ref. [8].

For $\kappa = 0$, or QED$_3$ with $N$ four-component fermions, the lowest term in the energy shift behaves as $B^{3/2}$. The fermions and the photon make opposite contributions. Since the energy shift due to the fermions is the leading one at large $N$ and is positive, the entire
energy shift becomes positive at least as $N \to \infty$. However, our rough estimate has shown that even if $N = 1$ the entire shift is positive due to the kinematics of fermions in the magnetic field. Therefore we can conclude that spontaneous magnetization does not occur in QED$_3$ with $N$ four-component fermions.

It is an open problem whether or not the magnetized vacua found at zero temperature are maintained also at finite temperature, although the Lorentz symmetry is explicitly broken by the heat bath. The stability of the magnetized vacua at finite temperature was confirmed in Ref. [37] in the same context as Ref. [8]. On the other hand, Ref. [38] found that in $2 + 1$ dimensions and in the external magnetic field the condensate $\langle \bar{\psi} \psi \rangle$ is unstable at finite temperature $T$ due to nonanalyticity at the origin in the $(m, T)$ plane. Thus one needs to take into account the possibility of dynamical generation of fermion mass as well as spontaneous magnetization also in the finite temperature. Another problem is to extend our study to non-Abelian gauge theories, namely, Chern-Simons QCD$_3$, whose intimate relationship with the frustrated Heisenberg antiferromagnets is studied in Ref. [39].

The gauge sector of Chern-Simons QED$_3$ is dual to the Abelian Chern-Simons-Higgs theory, where the kinetic term for the Higgs field provides the gauge boson mass term through unitary gauge fixing [40]. On the other hand, the $(2 + 1)$-dimensional Thirring model can be reformulated as an Abelian Higgs theory with the realization of a massive composite gauge boson through the introduction of a spurious Higgs boson [41]. It may, therefore, also be interesting to study another infrared sensitive theory;

$$\mathcal{L}' = \bar{\psi} \gamma^\mu \left[ i \partial_\mu + e A_\mu \right] \psi - \frac{1}{2G} A_\mu A^\mu - \frac{\kappa}{2} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho,$$

(7.1)

whose dynamical gauge boson might play an important role in planar condensed matter systems.

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11Some authors [34,36] claim that spontaneous magnetization occurs in QED$_3$. Their analysis is, however, based on two-component fermions and assumes a heavy fermion mass which breaks parity explicitly.
APPENDIX A: CONTOUR INTEGRALS

In this appendix we provide formulas for contour integrals which appear in Section III and Section IV. They are given for an arbitrary positive constant $M$ and an integer $n$ by

$$I^{(n)}(M) := \frac{1}{\pi} \oint_{C_{\mu}} dz \frac{1}{(z^2 + M^2)^n},$$

$$I^{(n)}(-M) := \frac{1}{\pi} \oint_{C_{\mu}} dz \frac{iz}{(z^2 + M^2)^n},$$

$$I^{(0)}(M) := \frac{1}{\pi} \oint_{C_{\mu}} dz \ln |z^2 + M^2|,$$

where the contour $C_{\mu}$ is a rectangle which connects two trajectories $(-R - i\mu, +R - i\mu)$ and $(-R, +R)$, and finally we take the limit $R \to \infty$ (see Fig. 1).

When $|\mu| = M$, poles or branch points of the integrand are put just on the contour $C_{\mu}$. We therefore replace the integrals in $|\mu| = M$ as their Cauchy’s principal values. Notice that integrals with $n \geq 2$ show divergence at $|\mu| = M$ and become ill-defined. To avoid these ambiguities we define them as $n$-th $M$ derivatives of the integrals $I^{(1)}_\pm$, namely,

$$I^{(n)}_\pm (M) = \frac{1}{(n-1)!} \left( -\frac{1}{2M} \frac{\partial}{\partial M} \right)^{n-1} I^{(1)}_\pm (M).$$

The integrals $I^{(1)}_\pm$ are calculated by means of the residue theorem. When $|\mu| > M$, the poles $z = \pm iM$ of the integrand are located inside the contour $C_{\mu}$. Thus we obtain the following results

$$I^{(1)}_+ (M) = -\frac{1}{M} \theta(|\mu| - M),$$

$$I^{(1)}_- (M) = -\text{sgn}(\mu) \theta(|\mu| - M),$$

where the step functions take a value 1/2 due to the redefinition as their Cauchy’s principal value at $|\mu| = M$ and they match the zero temperature limit of Fermi-Dirac distribution function.

By use of Eq. (A4) the integrals $I^{(2)}_\pm$ are determined as

$$I^{(2)}_+ (M) = -\frac{1}{2M^3} \theta(|\mu| - M) - \frac{1}{2M^2} \delta(|\mu| - M),$$

$$I^{(2)}_- (M) = -\frac{1}{2M} \text{sgn}(\mu) \delta(|\mu| - M),$$

which show delta function singularities at $|\mu| = M$ as mentioned above. They cannot be neglected because they appear as the zero temperature limit of the $M$ derivative of Fermi-Dirac distribution function.

As to the integral $I^{(0)}$, its integrand does not have any pole but branch points at $z = \pm iM$. When $|\mu| > M$, the branch points are located inside the contour $C_{\mu}$ therefore the integral is replaced with the one integrated along a minimal contour that rounds the branch cut $|\text{Im}z| \geq M$ on the imaginary axis. Thus we obtain

$$I^{(0)}(M) = 2(|\mu| - M) \theta(|\mu| - M).$$
APPENDIX B: ASYMPTOTIC EXPANSION

In the following we provide the asymptotic expansion for the function \( I(X) \) which is given by the integral

\[
I(X) := \int_0^\infty d\tau \, G(\tau) \, e^{-X\Phi(\tau)}.
\]  
(B1)

We suppose that the function \( \Phi(\tau) \) is analytic at \( \tau = 0 \), as well as the function \( G(\tau) \), and an odd function for \( \tau \) which monotonically increases in \( \tau \geq 0 \). For convenience, we normalize \( \Phi \) as

\[
\left. \frac{d}{d\tau} \Phi(\tau) \right|_{\tau=0} = 1.
\]  
(B2)

Our aim is to expand \( I(X) \) into the asymptotic series for the large \( X \), that is

\[
I(X) \sim \sum_{n=0}^\infty \frac{I_n}{X^n}, \quad (X \to \infty).
\]  
(B3)

We notice that there exists the inverse function \( f(y) := \Phi^{-1}(y) \) which is also an odd function of \( y \) and monotonically increases in \( y \geq 0 \) from the definition of \( \Phi(\tau) \). We exchange the variable of integration with \( y = \Phi(\tau) \) in Eq. (B1) and obtain

\[
I(X) = \int_0^\infty dy \frac{df(y)}{dy} G(f(y)) e^{-Xy}.
\]  
(B4)

Because of an exponential factor \( e^{-Xy} \), the neighborhood of \( y = 0 \) dominantly contributes to the integral for the large \( X \). It is therefore allowed in Eq. (B4) to expand \( F(y) := \frac{df(y)}{dy} G(f(y)) \) into a power series of \( y \). Thus the asymptotic series of \( I(X) \) is obtained as

\[
I(X) \sim \frac{1}{X} \sum_{n=0}^\infty \frac{F(n)}{X^n}, \quad (X \to \infty),
\]  
(B6)

where \( F(n) \) denotes the \( n \)-th derivative of \( F(y) \) at \( y = 0 \). The coefficients \( F(n) \) are determined by derivatives of \( G \) and \( f \), and the series up to \( O(X^{-4}) \) becomes

\[
I(X) \sim \frac{G(0)}{X} + \frac{G(1)}{X^2} + \frac{G(2) + f(3)G(0)}{X^3} + \frac{G(3) + 3f(3)G(1)}{X^4} + \cdots, \quad (X \to \infty).
\]  
(B7)

where \( G(n) \) (\( f(n) \)) denotes the \( n \)-th derivative of \( G \) (\( f \)) at \( \tau = 0 \) (\( y = 0 \)).

Now we apply the above expansion to the vacuum polarization functions given in Section IV. The functions \( \Pi_e, \Pi_\perp, \) and \( \Pi_o \) are written in the same form as Eq. (B1), that is,
\[
\Pi_e(p) = \frac{\alpha l^2}{\pi} \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \int_0^{\infty} d\tau \, G_e(\tau) \, e^{-X(\bar{k} - i\mu)} \Phi(\tau),
\]
\[
\Pi_\perp(p) = \frac{\alpha l^2}{\pi} \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \int_0^{\infty} d\tau \, G_\perp(\tau) \, e^{-X(\bar{k} - i\mu)} \Phi(\tau),
\]
\[
\Pi_o(p) = \frac{\alpha l^2}{\pi} \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \, i(\bar{k} - i\mu) \int_0^{\infty} d\tau \, G_o(\tau) \, e^{-X(\bar{k} - i\mu)} \Phi(\tau),
\]
where we define the functions \(X(\bar{k})\) and \(\Phi(\tau)\) by
\[
X(\bar{k}) := l^2 \left[ \bar{k}^2 + M_v^2 \right], \quad M_v := \sqrt{m^2 + \frac{1 - v^2}{4} p^2},
\]
\[
\Phi(\tau) := \tau \left\{ 1 + \frac{x}{X(\bar{k} - i\mu)} \left[ \frac{\cosh \tau - \cosh \tau v}{2 \sinh \tau} - \frac{1 - v^2}{4} \right] \cos^2 \theta \right\},
\]
where \(x := (lp)^2\) and \(\theta\) denotes the polar angle for the Euclidean momentum \(p^\mu\). The functions \(G_e, G_\perp,\) and \(G_o\) are read from Eq. (4.10) as
\[
G_e(\tau) = \frac{\tau (\cosh \tau v - v \coth \tau \sinh \tau v)}{\sinh \tau},
\]
\[
G_\perp(\tau) = \frac{2\tau (\cosh \tau - \cosh \tau v)}{\sinh^3 \tau} - G_e(p),
\]
\[
G_o(\tau) = \frac{2\tau (\cosh \tau \cosh \tau v - 1)}{\sinh^2 \tau} \text{sgn}(eB),
\]
all of which are analytic at \(\tau = 0\).

We notice that \(\Phi(\tau)\) is a monotonically increasing odd function of \(\tau\) and satisfies the normalization condition of Eq. (B2). Therefore the vacuum polarization functions are expanded into the same form as the asymptotic series in Eq. (B7) in the weak field regime \((l^{-2} \ll p^2)\);
\[
\Pi_e(p) = \frac{\alpha l^2}{\pi} \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \left[ \frac{G_e^{(0)}}{X(\bar{k} - i\mu)} + \frac{G_e^{(1)}}{X^2(\bar{k} - i\mu)} + O(X^{-3}) \right],
\]
\[
\Pi_\perp(p) = \frac{\alpha l^2}{\pi} \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \left[ \frac{G_\perp^{(0)}}{X(\bar{k} - i\mu)} + \frac{G_\perp^{(1)}}{X^2(\bar{k} - i\mu)} + O(X^{-3}) \right],
\]
\[
\Pi_o(p) = \frac{\alpha l^2}{\pi} \text{sgn}(eB) \int_0^1 dv \int_{-\infty}^{\infty} d\bar{k} \, i(\bar{k} - i\mu) \left[ \frac{G_o^{(0)}}{X(\bar{k} - i\mu)} + \frac{G_o^{(1)}}{X^2(\bar{k} - i\mu)} + O(X^{-3}) \right].
\]
Since the only dimensionful parameter is the momentum \(p\) after taking the symmetric limit \(J_\pm \to 0\), the \(X^{-n}\) terms in the above equations are identified with the \(n\)-th terms of the asymptotic series in Eq. (B27), whose coefficients up to \(O(X^{-1})\) are given by
\[
H_e^{(0)} = \lim_{J_\pm \to 0} p \int_0^1 dv \left( 1 - v^2 \right) \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} dz \frac{1}{z^2 + M_v^2} + \oint_{C_\mu} dz \frac{1}{z^2 + M_v^2} \right],
\]
\[
H_o^{(1)} = \lim_{J_\pm \to 0} p^2 \text{sgn}(eB) \int_0^1 dv \left( 1 + v^2 \right) \frac{1}{\pi} \oint_{C_\mu} dz \frac{iz}{[z^2 + M_v^2]^2},
\]
\[
H_\perp^{(0)} \equiv 0.
\]
The other coefficients up to $O(X^{-1})$ become trivially zero due to even or odd property of the functions $G_e$, $G_\perp$, and $G_o$ under the parity. The integrals over $z$ are carried out as the contour integrals given in Appendix A and are given by

$$H_e^{(0)} = \lim_{J \to 0} p \int_0^1 dv \frac{1 - v^2}{M_v} \left[ 1 - \theta(\vert \mu \vert - M_v) \right],$$

$$H_o^{(1)} = -\lim_{J \to 0} p^2 \text{sgn} (\mu e B) \int_0^1 dv \frac{1 + v^2}{2M_v} \delta(\vert \mu \vert - M_v),$$

which have appeared in Section IV and the integrals over parameter $v$ can be calculated by using formulas in Appendix C. Thus we obtain

$$H_e^{(0)} = \lim_{J \to 0} \left[ \theta(m - \vert \mu \vert) I(p; m) + \theta(\vert \mu \vert - m) \theta \left[ p^2 - 4(\mu^2 - m^2) \right] I(p; \mu) \right],$$

$$H_o^{(0)} = -\lim_{J \to 0} \text{sgn} (\mu e B) \theta(\vert \mu \vert - m) \theta \left[ p^2 - 4(\mu^2 - m^2) \right] \frac{4[p^2 - 2(\mu^2 - m^2)]}{p\sqrt{p^2 - 4(\mu^2 - m^2)}}$$

with the function $I(p; u)$ which is defined as

$$I(p; u) := \frac{2|u|\sqrt{p^2 - 4(u^2 - m^2)}}{p^2} + \left(1 - \frac{4m^2}{p^2}\right) \arctan \left(\frac{\sqrt{p^2 - 4(u^2 - m^2)}}{2|u|}\right).$$

**APPENDIX C: FEYNMAN INTEGRALS AT FINITE DENSITIES**

In this paper we have used the following integral formulas in calculations of loop diagrams;

$$J_n^0(p) := \frac{p}{2} \int_0^1 \frac{dv}{M_v^n} = I_n(m),$$

$$J_n(p) := \frac{p}{2} \int_0^1 \frac{dv}{M_v^n} \theta(\vert \mu \vert - M_v)$$

$$= \theta(\vert \mu \vert - m) \left\{ I_n(m) - \theta \left[ p^2 - 4(\mu^2 - m^2) \right] I_n(\mu) \right\},$$

$$J_n(p) := \frac{p}{2} \int_0^1 \frac{dv}{M_v^n} \delta(\vert \mu \vert - M_v)$$

$$= \theta(\vert \mu \vert - m) \theta \left[ p^2 - 4(\mu^2 - m^2) \right] \bar{I}_n(\mu),$$

where $n$ is an arbitrary integer and the functions $I_n$ and $\bar{I}_n$ are given by

$$I_n(u) := \frac{\sqrt{C(p) - u^2}}{|C(p)|^{n/2}} {}_2F_1 \left( \frac{1}{2}, \frac{n-1}{2}, \frac{3}{2}, 1 - \frac{u^2}{C(p)} \right),$$

$$\bar{I}_n(u) := \frac{1}{|u|^{n-1}\sqrt{C(p) - u^2}},$$

with the hypergeometric function ${}_2F_1(a, b; c; z)$ and
\[ C(p) := m^2 + \frac{p^2}{4}. \]  

(C6)

For instance the vacuum polarization function \( \hat{\Pi}(p) \) in Eq. (6.28) is given by

\[
\hat{\Pi}(p) = \alpha \int_0^1 dv \frac{1-v^2}{M_v} [1 - \theta(|\mu| - M_v)] \\
= \frac{8\alpha}{p^3} \left\{ J_{-1}^0(p) - m^2 J_1^0(p) - [J_{-1}(p) - m^2 J_1(p)] \right\} \\
= \frac{8\alpha}{p^3} \left\{ \theta(m - |\mu|) \left[ I_{-1}(m) - m^2 I_1(m) \right] \\
+ \theta(|\mu| - m) \theta[p^2 - 4(\mu^2 - m^2)] \left[ I_{-1}(\mu) - m^2 I_1(\mu) \right] \right\}. \tag{C7}
\]

The functions \( I_{-1} \) and \( I_1 \) are calculated as

\[
I_{-1}(u) = \frac{|u|\sqrt{C(p) - u^2}}{2} + \frac{C(p)}{2} \arctan \left( \frac{\sqrt{C(p) - u^2}}{|u|} \right), \tag{C8}
\]

\[
I_1(u) = \arctan \left( \frac{\sqrt{C(p) - u^2}}{|u|} \right). \tag{C9}
\]

Finally we obtain

\[
\hat{\Pi}(p) = \frac{\alpha}{p} \left\{ \theta(m - |\mu|) I(p; m) + \theta(|\mu| - m) \theta[p^2 - 4(\mu^2 - m^2)] I(p; \mu) \right\}, \tag{C10}
\]

where we define the function \( I(p; u) \) as

\[
I(p; u) := \frac{8}{p^2} \left[ I_{-1}(u) - m^2 I_1(u) \right] \\
= \frac{2|u|\sqrt{p^2 - 4(u^2 - m^2)}}{p^2} + \left( 1 - \frac{4m^2}{p^2} \right) \arctan \left( \frac{\sqrt{p^2 - 4(u^2 - m^2)}}{2|u|} \right). \tag{C11}
\]
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FIG. 1. The contour of integration, $C_\mu$, on the complex $z$-plane (for $\mu > 0$).

FIG. 2. Possible approaches to the symmetric limit on the $(|\mu|, m)$-plane: I. $|\mu| < m \to 0$, II. $m < |\mu| \to 0$, and III. $|\mu| = m \to 0$. 

\[ \langle \overline{\psi} \psi \rangle \neq 0 \]
\[ \langle \psi^+ \psi \rangle = 0 \]
FIG. 3. The Schwinger-Dyson equation. The lines with blob mean the full propagators.

FIG. 4. The two-particle irreducible (2PI) diagram at the leading order in $1/N$ expansion.
FIG. 5. The $b$ dependence of the function $K(x)$. 
| $\kappa$ | $\langle \bar{\psi} \psi \rangle$ | $|\langle \psi^\dagger \psi \rangle|$ | $m_d$ | $|\mu_d|$ |
|-----|-----------------|-----------------|--------|--------|
| 0   | $-N/2\pi l^2$  | 0               | $\omega_s/l$ | 0      |
| $\alpha$ | $-N/4\pi l^2$  | $N/4\pi l^2$   | $\omega_s/2l$ | $\omega_s/2l$ |
| $2\alpha$ | 0               | $N/2\pi l^2$   | 0      | $\omega_s/l$ |

**TABLE I.** Condensates and solutions to the Schwinger-Dyson equation for each $\kappa$. 