Compression of root systems and the $E$-sequence

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Abstract

We examine certain maps from root systems to vector spaces over finite fields. By choosing appropriate bases, the images of these maps can turn out to have nice combinatorial properties, which reflect the structure of the underlying root system. The main examples are $E_6$ and $E_7$.

1 Introduction

The primary goal of this paper is to provide a convenient way of visualising the root systems $E_6$ and $E_7$. There are two important relations on a root system that one might wish to have a good understanding of: the poset structure, in which $\alpha > \beta$ if $\alpha - \beta$ is a positive root, and the orthogonality structure, in which $\alpha \sim \beta$ if $\alpha$ and $\beta$ are orthogonal roots.

In our paper on cominuscule Schubert calculus, with Frank Sottile [8], we found that our examples required a good simultaneous understanding both these structures. This is easy enough to acquire for the root systems corresponding to the classical Lie groups. In $A_n$, for example, one can visualise the positive roots as the entries of an strictly upper triangular $(n + 1) \times (n + 1)$ matrix, where the $ij$ position represents the root $x_i - x_j$. Then $\alpha \geq \beta$, if and only if $\alpha$ is weakly right and weakly above $\beta$. Orthogonality is also straightforward in this picture: $\alpha$ and $\beta$ are non-orthogonal if there is some $i$ such that crossing out the $i^{th}$ row and the $i^{th}$ column succeeds in crossing out both $\alpha$ and $\beta$. See Figure 1.

In type $E$, it is less obvious how to draw such a concrete picture. Separately the two structures have been well studied in the contexts of minuscule posets [7, 9], and strongly regular graphs (see e.g. [1, 3, 4]). However, once one draws the Hasse diagram of the posets, the orthogonality structure suddenly becomes mysterious. Of course, one can always calculate which pairs of roots are orthogonal, but we would prefer a picture which allows us to do it instantly. Thus the main thrust of this paper is to get to Figures 4 and 6, which illustrate how one can simultaneously visualise $E_7$ and $E_6$ posets and orthogonality structures, at least restricted to certain strata.
of the root system. The restriction of these structures to the strata is exactly what is needed for the type $E$ examples in \cite{8}. With a little more work, one can use these figures to recover the partial order and orthogonality structures for the complete root system.

To reach these diagrams, we begin by considering certain maps from a root system to $\left(\mathbb{Z}/p\right)^m$, which are injective (or 2:1 if $p = 2$). Once we have some general observations about these maps, we give examples for $E_6$ and $E_7$ which are particularly nice. In these cases, we show that properties of the underlying root system are reflected in simple combinatorial structures on the target space, which is what allows us to produce diagrams in question. As the $E_7$ example is richer, we will discuss it before the $E_6$ example.

The idea of relating the $E_6$ and $E_7$ root systems to $\left(\mathbb{Z}/p\right)^m$ has appeared elsewhere. For example, Harris \cite{5} uses such an identification to describe the Galois group of the 27 lines on the cubic surface—one of the del Pezzo surfaces. The connection between del Pezzo surfaces and the exceptional Lie groups has been well established; we refer the reader to \cite{6}. One can also see such a relationship reflected in the well known identification of Weyl groups (see e.g. \cite{7}):

$$W(E_6) \cong SO(5; \mathbb{Z}/3) \cong O^-(6; \mathbb{Z}/2)$$
$$W(E_7) \cong \mathbb{Z}/2 \rtimes Sp(6; \mathbb{Z}/2).$$

These facts follow from the identifications outlined in this paper, and presumably have been proved in similar ways before before.
2 Compression of root systems

2.1 Simply laced root systems

Let $\Delta \subset \mathbb{R}^n$ be a simply laced root system, so that with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ we have $\langle \beta, \beta \rangle = 2$ for all $\beta \in \Delta$. We assume that $\Delta$ has full rank in $\mathbb{R}^n$. Let $\Lambda = \mathbb{Z}\Delta$ denote the lattice in $\mathbb{R}^n$ generated by $\Delta$.

Choose a basis of simple roots $\alpha_1, \ldots, \alpha_n \in \Delta$, for $\Lambda$. Let $\Delta^+$ denote the positive roots with respect to this basis, and $\Delta^-$ denote the negative roots. Recall that $\Delta^+$ is a partially ordered set, with $\beta > \beta'$ iff $\beta - \beta' \in \Delta^+$. Roots $\beta$ and $\beta'$ are comparable in the partial ordering iff $\langle \beta, \beta' \rangle > 0$.

For each $\beta \in \Lambda$, we define $\beta^i$ to be the coefficient of $\alpha_i$, when $\beta$ is expressed in the basis of the simple roots:

$$\beta = \sum_{i=1}^n \beta^i \alpha_i.$$ 

Let $\text{Dyn}$ denote the Dynkin diagram of $\Delta$. As $\Delta$ is simply laced, each component of $\text{Dyn}$ has type ADE. The vertices of $\text{Dyn}$ are denoted $v_1, \ldots, v_n$, and correspond (respectively) to the simple roots $\alpha_1, \ldots, \alpha_n$. When $\Delta$ is a simple root system (i.e. $\text{Dyn}$ has just one component), the affine Dynkin diagram $\widehat{\text{Dyn}}$ is obtained by adding a vertex $\hat{v}_n$ to $\text{Dyn}$, corresponding to the lowest root $\hat{\alpha}_n$ of $\Delta$.

2.2 Root systems over $\mathbb{Z}/p$

Let $p \geq 2$ be a positive integer. For reasons explained later in this section, we shall be mostly interested in the case where $p$ is a prime, or $p = 4$. Let $V$ be a finite rank free module over $\mathbb{Z}/p$, with a symmetric bilinear form $(\cdot|\cdot)$ taking values in $\mathbb{Z}/p$. Let $\Gamma = \{x \in V \setminus \{0\} \mid (x|x) = 2\}$.

Suppose that $\Gamma$ has subset $S = \{s_1, \ldots, s_n\}$ such that

$$\langle s_i, s_j \rangle = \langle \alpha_i, \alpha_j \rangle \pmod{p}, \quad \text{for all } i, j,$$

and if $p = 2$, assume $s_i \neq s_j$, for all $i \neq j$. (1)

Then we obtain a map $f : \Lambda \to V$ by extending the natural map $\alpha_i \mapsto s_i$ to a homomorphism of Abelian groups.

**Proposition 2.1.** If $\beta, \beta' \in \Lambda$ then

$$\langle f(\beta)|f(\beta') \rangle = \langle \beta, \beta' \rangle \pmod{p} \quad (2)$$

**Proof.** This is true for all pairs of simple roots, and both inner products are bilinear. \[\square\]

**Corollary 2.2.** Suppose $\beta \neq \beta' \in \Delta$. Then $\langle \beta, \beta' \rangle = 0 \iff \langle f(\beta)|f(\beta') \rangle = 0$.

**Proof.** Since $\beta, \beta'$ are roots of a simply laced root system, $\langle \beta, \beta' \rangle \in \{-1, 0, 1\}$, thus $\langle \beta, \beta' \rangle = 0 \iff \langle \beta, \beta' \rangle = 0 \pmod{p} \iff \langle f(\beta)|f(\beta') \rangle = 0$. \[\square\]
We now restrict the domain of $f$ to $\Delta$ if $p > 2$ and $\Delta^+$ if $p = 2$.

**Theorem 2.3.** If $p > 2$, the map $f : \Delta \rightarrow V$ is injective, and its image lies in $\Gamma$. If $p = 2$, the map $f : \Delta^+ \rightarrow V$ is injective, and its image lies in $\Gamma$.

**Proof.** We first suppose $p > 2$. Note that the fact that $f(\Delta) \subset \Gamma$ is clear from the fact that every $\beta \in \Delta$ satisfies $\langle \beta, \beta \rangle = 2$.

Now, suppose that $\beta, \beta' \in \Delta$, $f(\beta) = f(\beta')$. We show that $\beta = \beta'$.

For all $\gamma \in \Delta$ we have $(f(\beta)|f(\gamma)) = (f(\beta')|f(\gamma))$, so $\langle \beta, \gamma \rangle = \langle \beta', \gamma \rangle \pmod{p}$. In particular the set of roots perpendicular to $\beta$ and $\beta'$ are equal. implies $\beta$ and $\beta'$ belong to the same simple component of $\Delta$.

There are two cases: if the component is of type $A_2$, then it is trivial that we must have $\beta = \beta'$. If the component is not of type $A_2$, then the fact that $\beta$ and $\beta'$ have the same set of perpendicular roots implies that $\beta = \pm \beta'$. (In types $D$ and $E$, the roots perpendicular to any given root span an entire hyperplane, and in type $A$ it is easily checked.) However, for all $x \in \Gamma$, $x \neq -x$. Since $f(\beta) = f(\beta') \in \Gamma$, we cannot have $\beta' = -\beta$. Thus $\beta = \beta'$.

For $p = 2$, the fact that every $\beta \in \Delta^+$ satisfies $\langle \beta, \beta \rangle = 2$, implies that $f(\Delta^+) \subset \Gamma \cup \{ \overrightarrow{0} \}$. It is therefore enough to show that $f : \Delta^+ \cup \{ \overrightarrow{0} \} \rightarrow \Gamma \cup \{ \overrightarrow{0} \}$ is injective.

Suppose $\beta, \beta' \in \Delta^+ \cup \{ \overrightarrow{0} \}$, $f(\beta) = f(\beta')$. We show that $\beta = \beta'$.

As in the $p > 2$ case, for all $\gamma \in \Delta$, we have $\langle \beta, \gamma \rangle = \langle \beta', \gamma \rangle \pmod{2}$. Thus the sets $P(\beta)$ and $P(\beta')$, where

$$P(\beta) := \{ \gamma \in \Delta \mid \langle \beta, \gamma \rangle = 0 \} \cup \{ \pm \beta \},$$

coincide. This implies $\beta$ and $\beta'$ belong to the same simple component of $\Delta$. (In particular, if $f(\beta) = f(\beta') = \overrightarrow{0}$ then $\beta = \beta' = \overrightarrow{0}$.)

If this component is $A_1$ or $A_2$, it is trivial that $\beta = \beta'$.

If the component is $A_k$, $k \geq 4$, then $P(\beta)$ is a root system of type $A_{k-2} \times A_1$, where $\beta, \beta'$ are both in the $A_1$ component. If the component is $E_6$, $E_7$, or $E_8$, then $P(\beta)$ is a root system of type $A_5 \times A_1$, $D_6 \times A_1$ or $E_7 \times A_1$, where $\beta, \beta'$ are both in the $A_1$ component. Thus in both these cases $\beta = \beta'$.

However, if the component if $D_k$, $k \geq 3$, then $P(\beta)$ is a root system of type $D_{k-2} \times A_1 \times A_1$, where $\beta, \beta'$ are both in an $A_1$ component (a priori, not necessarily the same one). If $\beta, \beta'$ belong to the same $A_1$ component, then $\beta = \beta'$. So suppose they do not. We identify the roots of $D_k$ with the vectors $\{ e_i \pm e_j \mid i \neq j \} \subset \mathbb{R}^k$, where $e_1, \ldots, e_k$ is an orthonormal basis for $\mathbb{R}^k$. It is easy to see that if $\beta = e_i \pm e_j$, then $\beta' = e_i \mp e_j$. So $\overrightarrow{0} = f(\beta) - f(\beta') = f(2e_j)$. On the other hand, for all $k$, we have $\overrightarrow{0} = 2f(e_k - e_j) + f(2e_j) = f(2e_k)$. So $f(e_k \pm e_l) = f(e_k \mp e_l)$, for all $k \neq l$. But among these must be a pair of simple roots. We conclude that $f$ restricted to the simple roots is not injective, hence $S$ contains fewer than $n$ elements, a contradiction. \(\square\)
Remark 2.1. Although we will not have use for it here, if \( p \) is not a prime, one could also allow the possibility that \( V \) is not a free module. In this case Theorem 2.3 remains true provided \( f(\beta) \neq f(-\beta) \) for all \( \beta \in \Delta \). This will be the case whenever \( 2 \nmid p \) or when \( \text{Dyn} \) has no component of type \( A_1 \).

We now show that the most interesting cases are when \( p \) is a prime or \( p = 4 \).

Suppose \( p \geq 6 \) is not prime. Let \( p' \neq 2 \) be a proper divisor of \( p \). Let \( V' = V \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p' \). Let \( \rho : V \to V' \) denote the reduction modulo \( p' \) map. \( V' \) comes with a symmetric bilinear form \((\cdot|\cdot)'\), the reduction of \((\cdot|\cdot)\) modulo \( p' \).

Corollary 2.4. The composite map \( f' := \rho \circ f : \Delta \to V' \) is injective, and its image lies in \( \Gamma' = \{ x \in V' \mid (x|x)' = 2 \} \). Moreover \( (\beta, \beta') = (f'(\beta)|f'(\beta'))' \pmod{p'} \).

Proof. As \( p' \neq 2 \), this follows from the fact that \( \rho \) preserves inner products modulo \( p' \).}

2.3 Compression

The most interesting case of Theorem 2.3 occurs when rank \( m \) of \( V \) is smaller than the rank \( n \) of \( \Lambda \). If this is the case, we will call the map \( f \) a compression of the root system. Here we give a necessary and nearly sufficient condition for compression to be possible.

Let \( A \) be the Coxeter matrix of \( \Delta \), \( A_{ij} = \langle \alpha_i, \alpha_j \rangle \).

Proposition 2.5. If we have \( S \) as in equation (1), and \( m < n \), then \( p \) divides \( \det(A) \).

Proof. Let \( s \) be the \( m \times n \) matrix whose columns are are \( s_i \) in some basis, and let \( g \) be the \( m \times m \) matrix representing the bilinear form \((\cdot|\cdot)\) in the same basis. Then

\[
A_{ij} = (s_i|s_j) = (s^T g s)_{ij} \pmod{p}.
\]

If \( m < n \) then \( \det(s^T g s) = 0 \), so \( p | \det(A) \). \( \square \)

Conversely, if \( p \) is prime and \( p | \det(A) \), and \( A_p \) denotes the reduction of \( A \) modulo \( p \), then one can define \( V = (\mathbb{Z}/p)^n / \ker(A_p) \) and \( s_i \) is the image of the standard basis vector \( e_i \) under the natural map. This will satisfy (1), provided the \( s_i \) are all distinct and non-zero. The same construction works if \( p \) is not prime, though \( V \) will not necessarily be a free \( \mathbb{Z}/p \)-module.

In particular, we cannot hope for compression in \( E_8 \), a root system for which \( \det(A) = 1 \). For \( E_7 \), however, \( \det(A) = 2 \), and for \( E_6 \), \( \det(A) = 3 \). Thus we should expect compression of the \( E_7 \) and \( E_6 \) root systems to be possible, taking \( p = 2 \) or \( 3 \) respectively.
2.4 Structures on $V$

**Definition 2.2.** The **O-graph** of $V$ is the graph whose vertex set is $V$ and whose edges are pairs $(x, y)$, $x \neq y$ such that $(x|y) = 0$. The **N-graph** of $V$ is the complement of the O-graph, having vertex set $V$ and edges $(x, y)$ such that $(x|y) \neq 0$.

As our two main examples involve $p = 2$ and $p = 3$, we consider some special inner products $(\cdot|\cdot)$ in these cases.

If $p = 2$, we let $V$ be an even dimensional vector space over $\mathbb{Z}/2$ with a symplectic form $(\cdot|\cdot)$. By symplectic form, we mean an (anti)symmetric non-degenerate bilinear form for which $(x|x) = 0$ for all $x$. Thus $\Gamma = V \setminus \{ \vec{0} \}$. We see that $S \subset V \setminus \{ \vec{0} \}$ satisfies the condition (1) iff the restriction of the N-graph to $S$ is isomorphic to $\text{Dyn}$. In this case, the associated map $f$ gives an injective map from $\Delta^+$ to $V \setminus \{ \vec{0} \}$.

If $p = 3$, we take $V$ to be an $m$-dimensional vector space over $\mathbb{Z}/3$, with the standard symmetric form $((x_1, \ldots, x_m) | (y_1, \ldots, y_m)) = \sum_{i=1}^{m} x_i y_i$. (3)

Note that

$$\Gamma = \{(x_1, \ldots, x_m) \mid \#\left\{ i \mid x_i \neq 0 \right\} = 2 \pmod{3}\}. \quad \text{(4)}$$

3 Application to type $E$

3.1 The $E$-sequence

Consider $\mathbb{R}^8$ with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Let $\alpha_1, \ldots, \alpha_8$ be the vectors

- $\alpha_1 = (1, -1, 0, 0, 0, 0, 0, 0)$
- $\alpha_2 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$
- $\alpha_3 = (0, 1, -1, 0, 0, 0, 0, 0)$
- $\alpha_4 = (0, 0, 1, -1, 0, 0, 0, 0)$
- $\alpha_5 = (0, 0, 0, 1, -1, 0, 0, 0)$
- $\alpha_6 = (0, 0, 0, 0, 1, -1, 0, 0)$
- $\alpha_7 = (0, 0, 0, 0, 0, 1, -1, 0)$
- $\alpha_8 = (0, 0, 0, 0, 0, 0, 1, -1)$

These are the simple roots of $E_8$, which span the $E_8$ lattice. They correspond to the vertices $v_1, \ldots, v_8$ of $\text{Dyn}(E_8)$, in the order shown on the right. This ordering of
simple roots of $E_8$ corresponds to the inclusion of groups below.

$$A_1 \longrightarrow D_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow E_5 \longrightarrow E_6 \longrightarrow E_7 \longrightarrow E_8$$

$$A_2 \times A_1 \longrightarrow A_4 \longrightarrow D_5$$

To obtain the root systems of $E_n$, $3 \leq n \leq 7$ we take the simple roots to be $\alpha_1, \ldots, \alpha_n$. These span the $E_n$ lattice. In general, the roots of $E_n$ are the Lattice vectors $\alpha \in \Lambda$ such that $\langle \alpha, \alpha \rangle = 2$.

The roots $\Delta(E_8)$ of $E_8$, are stratified as $\Delta(E_8) = \bigsqcup \Delta_s$. For $s \neq 2, 3$,

$$\Delta_s = \{ \beta \in \Delta(E_8) \mid \beta \geq \alpha_s, \text{ and } \beta \nless \alpha_t \text{ for all } t > s \}. \quad (5)$$

$\Delta_s$ are the roots of $E_s$ minus the roots of $E_{s-1}$. Equation (5) makes sense for $s = 2$, and $s = 3$; however it is convenient (and arguably correct) to put these into the same stratum:

$$\Delta_3 = \{ \pm \alpha_3, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \}.$$  

We also have a stratification of $\Delta^+(E_8) = \bigsqcup \Delta^+_s$ given by $\Delta^+_s = \Delta^+(E_8) \cap \Delta_s$. This stratification has the property that the roots of $E_n$, $3 \leq n \leq 8$ are precisely

$$\Delta(E_n) = \bigsqcup_{s \leq n} \Delta_s.$$  

For notational convenience, we define $s' = \max\{3, s + 1\}$, so that $\Delta_s$ and $\Delta_{s'}$ always denote consecutive strata.

For each stratum let $H_s$ denote the graph whose vertices are $\Delta^+_s$ and whose edges form the Hasse diagram of the poset structure on $\Delta^+$, restricted $\Delta^+_s$. Thus we have an edge joining $\beta$ and $\beta'$ if one of $\beta \pm \beta'$ is a simple root. These are shown in Figure 2.

Finally, it is worth noting the size of each stratum. The stratification $\Delta^+(E_8) = \bigsqcup \Delta^+_s$ has strata of sizes 1 ($s = 1$), 3 ($s = 3$), 6 ($s = 4$), 10 ($s = 5$), 16 ($s = 6$), 27 ($s = 7$) and 57 ($s = 8$).

### 3.2 A compression of $E_7$

We now take $\Delta$ to be the $E_7$ root system.

Let $F = (\mathbb{Z}/2 \times \mathbb{Z}/2, \oplus)$ denote the non-cyclic four element group. We denote the elements of this group $\{0, 1, 2, 3\}$, and the operation $a \oplus b$ is binary addition without carry. $F$ is a two dimensional vector space over $\mathbb{Z}/2$ and thus has a unique symplectic form:

$$(a|a') = \begin{cases} 0 & \text{if } a = 0, a' = 0 \text{ or } a = a' \\ 1 & \text{otherwise.} \end{cases}$$
We shall take $V = F^3$, and whenever possible we write a triple $(a, b, c) \in V$ simply as $abc$. We endow $V$ with the symplectic form

$$(abc|a'b'c') = (a|a') + (b|b') + (c|c').$$

We take as our subset $S \subset V$, the set $S = \{s_1, \ldots, s_7\}$, where

$s_1 = 100, \quad s_2 = 030, \quad s_3 = 300, \quad s_4 = 111, \quad s_5 = 003, \quad s_6 = 001, \quad s_7 = 033.$

**Proposition 3.1.** The restriction of the N-graph to $S$ is $\text{Dyn}(E_7)$. The natural homomorphism $f$ takes $\alpha_i$ to $s_i$.

**Proof.** This just needs to be checked. \hfill $\Box$

As a consequence of we obtain the following corollary of Theorem 2.3.

**Corollary 3.2.** The map $f : \Delta^+ \cup \{0\} \to V$ is a bijection.

**Proof.** It is an injection by Theorem 2.3. But $\#(\Delta^+ \cup \{0\}) = \#(V) = 64$, thus it is a bijection. \hfill $\Box$
3.3 Restriction to strata

Let $\Gamma$ denote the image of the stratum $\Delta_s^+$ under $f$. Here we show how natural structures on $\Delta_s^+$ are preserved under $f$, and are more palatable in $\Gamma$.

We define a new graph structure on $V$.

**Definition 3.1.** The **T-graph** is the graph with vertex set $V = F^3$, and $abc$ adjacent to $a'b'c'$, if exactly one of $\{a = a', b = b', c = c'\}$ holds.

Unlike the O-graph, the the T-graph has translation symmetries: for any $x \in V$, the map

$$y \mapsto x \oplus y$$

is an automorphism. It is a strongly regular graph. In particular every vertex has valence 27.

**Definition 3.2.** For $v \in V$, the **link** on $v$ in the T-graph, denoted $L(v)$, is the restriction of the T-graph to the vertices adjacent to $v$. The **antilink** on $v$ in the T-graph, denoted $L_c(v)$ is the restriction of the T-graph of the vertices non-adjacent to $v$ (not including $v$).

**Lemma 3.3.** The image of the largest stratum $\Gamma^+_s$ is the vertex set of $L(\vec{0})$.

In other words, $\Gamma^+_s$ is the set of $abc \in V$ such that exactly one of $\{a = 0, b = 0, c = 0\}$ holds.

Note that this result is not independent of the choice of $S$ for the images of the simple roots. We have chosen $S$ quite carefully, in part to make this lemma hold. It is possible, (and not difficult) to check this result on each of the 27 roots of $\Delta_s^+$; however, since a symmetry argument is available, we present it here.

**Proof.** We know that $f(\alpha_7) = 033 \in \Gamma_s^+$, thus it suffices to show that $\Gamma_s^+$ is invariant under the following symmetries:

$$
\begin{align*}
(a, b, c) &\mapsto (c, b, a) \\
(a, b, c) &\mapsto (a, c, b) \\
(a, b, c) &\mapsto ([1 \rightarrow 2 \rightarrow 3 \rightarrow 1] \cdot a, b, c) \\
(a, b, c) &\mapsto ([3 \leftrightarrow 2] \cdot a, b, c)
\end{align*}
$$

These symmetries come from a Dynkin diagram construction, which we first describe for any $E_n$. A similar construction can also be used for types $A$ and $D$. Let $D = \text{Dyn}(E_n)$. We decorate each vertex of $D$ with the corresponding simple root in $\Delta$.

Choose a vertex $v_i \in D$, where $i \notin \{1, 2, 8\}$. If we delete the edge $(v_i, v_{i+1})$ from $D$, the diagram breaks up into two components $D', D''$ where $D'$ is the component containing $v_1$. If $i = n$, $D''$ will be empty. Note that $D'$ is a sub-Dynkin diagram of $D$, and hence corresponds to a sub-root system $\Delta' \subset \Delta$.

We apply the following construction to obtain a new Dynkin diagram $\tilde{D}$:
1. Add to $D'$ the affine vertex $\hat{v}$, to form the affine Dynkin diagram $\hat{D}'$. This vertex is decorated with the lowest root $\hat{\alpha} \in \Delta'$.

2. For every vertex $\hat{D}'$, replace the root which decorates the vertex by its negative.

3. Delete the vertex $v_i$.

4. If $D''$ is not empty, reattach it by forming an edge $(\hat{v}, v_{i+1})$. The result is $\tilde{D}$.

The underlying graph $\tilde{D}$ is isomorphic to $D$, but under this isomorphism, the roots decorating the vertices have changed. The roots decorating $\tilde{D}$ correspond to a new system of simple roots for $\Delta$. Thus this process corresponds gives an automorphism of $\Delta$, and hence to an automorphism of $\Gamma$.

Returning now to the $E_7$ case, we note that for any root $\beta$, $\beta^7$ is preserved modulo 2 under each of these automorphisms. Thus each automorphism restricts to an automorphism of $\Delta_7$, and hence of $\Gamma^+_7 = f(\Delta^+_7) = f(\Delta_7)$. The symmetries are the automorphisms of $\Gamma^+_7$ given by the construction above, using vertices $v_7, v_6, v_4, v_3$ respectively. It is sufficient to check this on the images of the simple roots, and this is easily done. See Figure 3.

The following construction is useful for relating the other strata to $\Delta^+_7$. Put $z_7 = \overrightarrow{0}$, $\zeta_s = \sum_{i=s}^7 \alpha_i$, and $z_s = f(\zeta_s)$ for $s = 1, 3, 4, 5, 6$. If $\beta \in \Delta^+_s$, define $\tilde{\beta} = \beta + \zeta_s$. Note that $\tilde{\beta} \in \Delta^+_7$.

**Theorem 3.4.** On $\Gamma^+_s$, the image of any stratum, the T-graph agrees with the O-graph. In particular, for $\beta, \beta' \in \Delta^+_s$, $\langle \beta, \beta' \rangle = 0$ if and only if $f(\beta)$ and $f(\beta')$ agree in exactly one coordinate.
Proof. We assume $\beta \neq \beta'$, since the result is trivial otherwise. Put $f(\beta) = abc$, $f(\beta') = a'b'c'$.

We have $\langle \beta, \beta' \rangle = (a|a') + (b|b') + (c|c') \pmod{2}$ Thus $\langle \beta, \beta' \rangle = 0 \iff$ an odd number of \{(a|a'), (b|b'), (c|c')\} are non-zero.

We first show the theorem is true for the stratum $\Delta_7^+$. By Lemma 3.3, exactly one of $\{a, b, c\}$ and exactly one of $\{a', b', c'\}$ is zero. Suppose $a = 0, a' = 0$. Then we have $\langle \beta, \beta' \rangle = 0 \iff b \neq b'$ and $c \neq c' \iff abc$ and $a'b'c'$ agree in exactly one coordinate. Suppose $a = 0, b' = 0$. Then $\langle \beta, \beta' \rangle = 0 \iff c = c' \iff abc$ and $a'b'c'$ agree in exactly one coordinate. The remaining cases are the same as these two by symmetry.

To show the statement for other strata $\Delta_s^+$, we consider $\tilde{\beta}$ and $\tilde{\beta}'$. As $\beta, \beta'$ belong to the same stratum, we have $\langle \tilde{\beta}, \tilde{\beta}' \rangle = \langle \beta, \beta' \rangle$. Also, $f(\tilde{\beta}) = f(\beta) \oplus z_s$ is the image of $f(\beta)$ under an automorphism of the T-graph; thus $f(\tilde{\beta})$ is adjacent to $f(\tilde{\beta}')$ $\iff$ $f(\tilde{\beta})$ is adjacent to $f(\beta)$.

\[ \square \]

In general, all of the strata can be described in terms of links in the T-graph.

**Theorem 3.5.** We have the following identifications:

1. $\Gamma_7^+ = \mathcal{L}(z_s) \cap \left( \bigcap_{7 \geq t > s} \mathcal{L}^c(z_t) \right)$.

2. $\Gamma_7^+ \oplus z_s = \mathcal{L}(z_7) \cap \left( \bigcap_{7 > t \geq s} \mathcal{L}^c(z_t) \right)$.

**Proof.** We already know this is true for $s = 7$, so assume $s \leq 6$.

Let us calculate $\mathcal{L}(z_t) \setminus \mathcal{L}(\overrightarrow{0})$. First note that by Theorem 3.4

\[ \mathcal{L}(\overrightarrow{0}) \setminus \mathcal{L}(z_t) = \{x \in \Gamma_7^+ \mid (x|z_t) = 1\} \tag{8} \]

We have $y \in \mathcal{L}(z_t) \setminus \mathcal{L}(\overrightarrow{0}) \iff y \oplus z_t \in \mathcal{L}(z_t \oplus z_t) \setminus \mathcal{L}(\overrightarrow{0} \oplus z_t) = \Gamma_7^+ \setminus \mathcal{L}(z_t) \iff y \oplus z_t \in \Gamma_7^+$ and $(y \oplus z_t|z_t) = 1$, by (8). Now since $z_t \in \Gamma_7^+$, $(y|z_t) = (y \oplus z_t|z_t) = 1$ implies that $y \oplus z_t \in \Gamma_7^+$ if and only if $y \notin \Gamma_7^+$. Thus we see that

\[ \mathcal{L}(z_t) \setminus \mathcal{L}(\overrightarrow{0}) = \{y \in V \mid y \notin \Gamma_7^+, (y|z_t) = 1\}. \tag{9} \]

The set $\Gamma_7^+$ is characterized by

\[ y \in \Gamma_7^+ \iff (y|z_i) = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{if } i > s. \end{cases} \]
Thus, using (3) we see that
\[
\Gamma^+_s = \{ y \in \Gamma \mid y \notin \Gamma_7, \ (y \mid z_s) = 1, \ (y \mid z_t) = 0 \text{ for all } t > s \}
\]
\[
= \left( \mathcal{L}(z_s) \setminus \mathcal{L}(\overrightarrow{0}) \right) \setminus \left( \{ \overrightarrow{0} \} \cup \bigcup_{t=s+1}^6 \mathcal{L}(z_t) \setminus \mathcal{L}(\overrightarrow{0}) \right)
\]
\[
= \mathcal{L}(z_s) \cap \left( \bigcap_{7 \geq t > s} \mathcal{L}^c(z_t) \right).
\]

On the other hand, using (3),
\[
\Gamma^+_s \oplus z_s = \{ x \in \Gamma^+_7 \mid (x \oplus z_s \mid z_s) = 1, \ (x \oplus z_s \mid z_t) = 0 \text{ for all } t > s \}
\]
\[
= \{ x \in \Gamma^+_7 \mid (x \mid z_t) = 1, \text{ for all } t \geq s \}
\]
\[
= \bigcap_{7 > t \geq s} \mathcal{L}(\overrightarrow{0}) \setminus \mathcal{L}(z_t)
\]
\[
= \mathcal{L}(z_7) \cap \left( \bigcap_{7 \geq t > s} \mathcal{L}^c(z_t) \right).
\]

3.4 Partial ordering

We now show how one can recover the partial ordering on \( \Delta^+_s \) from \( \Gamma^+_s \).

Lemma 3.6. Let \( x_1, x_2 \in \Gamma^+_7 \). If \( x_1 \) and \( x_2 \) are orthogonal, there exists a unique vector \( x_3 \in \Gamma^+_7 \) such that \( \{ x_1, x_2, x_3 \} \) are pairwise orthogonal, and moreover, \( x_1 \oplus x_2 \oplus x_3 = \overrightarrow{0} \). Conversely if \( x_1 \oplus x_2 \in \Gamma^+_7 \) then \( x_1 \) and \( x_2 \) are orthogonal.

In light of Theorem 3.4, this is quite easy to show for our preferred choice of \( S \). Nevertheless, this result is true for any \( S \) satisfying (3), and so we give a more general proof.

Proof. Let \( \beta_1 = f^{-1}(x_1) \in \Delta^+_7 \) and \( \beta_2 = f^{-1}(x_2) \in \Delta^+_7 \). View \( \beta_1 \) and \( \beta_2 \) as roots in the \( E_8 \) root system. Assume that \( x_1 \) and \( x_2 \) are orthogonal; hence \( \langle \beta_1, \beta_2 \rangle = 0 \). Throughout the proof we use the fact that the sum of two roots is a root if their inner product is negative.

To begin, for any \( \beta \in \Delta^+_7 \), we have \( \langle \alpha_8, \beta \rangle = -1 \) so \( \alpha_8 + \beta \) is a root of \( E_8 \). Similarly, \( \langle \alpha_8 + \beta_1, \beta_2 \rangle = -1 \) so \( \alpha_8 + \beta_1 + \beta_2 \) is a root of \( E_8 \).

To show existence let \( \hat{\alpha}_8 \) denote the affine (lowest) root of \( E_8 \). Then \( \langle \alpha_8 + \beta_1 + \beta_2, \alpha_8 + \hat{\alpha}_8 \rangle = -1 \) so \( \hat{\alpha}_8 + 2\alpha_8 + \beta_1 + \beta_2 \) is a root. Let
\[
\beta_3 = -(\hat{\alpha}_8 + 2\alpha_8 + \beta_1 + \beta_2),
\]
and \( x_3 = f(\beta_3) \). Note that \( \beta_3^8 = 0, \beta_3^7 = 1 \), so \( \beta_3 \in \Delta^+_7 \), hence \( x_3 \in \Gamma^+_7 \). And we can explicitly check \( \langle \beta_3, \beta_1 \rangle = \langle \beta_3, \beta_2 \rangle = 0 \), so \( \{ x_1, x_2, x_3 \} \) are pairwise orthogonal.
Finally, note that the affine root of $E_8$ has the property that $\langle \hat{\alpha}_8, \gamma \rangle = 0$ for all roots $\gamma \in \Delta(E_7)$. Thus, for all $\gamma \in \Delta(E_7)$, we have

\[
(x_1 \oplus x_2 \oplus x_3|f(\gamma)) = \langle \beta_1 + \beta_2 + \beta_3, \gamma \rangle \pmod{2}
\]

\[
= \langle \hat{\alpha}_8 - 2\alpha_8, \gamma \rangle \pmod{2}
\]

\[
= 0.
\]

Thus $x_1 \oplus x_2 \oplus x_3 = \vec{0}$.

For uniqueness, let $x_3$ be any vector orthogonal to $x_1$ and $x_2$, and let $\beta_3 = f^{-1}(x_3) \in \Delta^+_7$. $\langle \alpha_8 + \beta_1 + \beta_2, \alpha_8 + \beta_3 \rangle = -1$, so $\gamma = -(2\alpha_8 + \beta_1 + \beta_2 + \beta_3)$ is a root of $E_8$. But $(\gamma)^8 = -2$, and the only root of $E_8$ with this property is the affine root $\hat{\alpha}_8$. We conclude that (10) must hold.

For the converse, note that if $x_1$ and $x_2$ are not orthogonal, then $\beta_1 - \beta_2$ is a root not in $\Delta_7$, hence $x_1 \oplus x_2 \not\in \Gamma^+_7$. \hfill \Box

**Corollary 3.7.** Suppose $x, y \in \Gamma^+_s$. Then $(x|y) = 0$ if and only if $x \oplus y \in \Gamma^+_7$

**Proof.** Note that $x \oplus z_s, y \oplus z_s \in \Gamma^+_7$. Thus $(x|y) = 0 \iff (x \oplus z_s|y \oplus z_s) = 0 \iff x \oplus y = x \oplus z_s \oplus y \oplus z_s \in \Gamma^+_7$, by Lemma 3.6. \hfill \Box

**Theorem 3.8.** Let $\alpha \not\in \Delta^+_s$ be a positive root. Let $\beta \in \Delta^+_s$. Then $f(\beta) \oplus f(\alpha) \in \Gamma^+_s$ if and only if either $\beta + \alpha \in \Delta^+_s$ or $\beta - \alpha \in \Delta^+_s$.

**Proof.** Certainly if one of $\beta \pm \alpha \in \Delta^+_s$, then $f(\beta) \oplus f(\alpha) \in \Gamma^+_s$. Suppose that neither $\beta + \alpha$ nor $\beta - \alpha$ is in $\Delta^+_s$. Note that neither can be in $\Delta^-_s$ either. If one of $\beta \pm \alpha$ is a root, then it belongs to some stratum other than $\Delta^+_s$, so $f(\beta) \oplus f(\alpha) \not\in \Gamma^+_s$. On the other hand if neither is a root then $\langle \beta, \alpha \rangle = 0$. Suppose that $f(\beta) \oplus f(\alpha) \in \Gamma^+_s$. Then

\[
(f(\beta) \oplus f(\alpha)|f(\beta)) = (f(\alpha)|f(\beta))
\]

\[
= \langle \beta, \alpha \rangle \pmod{2}
\]

\[
= 0.
\]

By Corollary 3.7, we conclude $f(\alpha) \in \Gamma^+_7$, a contradiction. \hfill \Box

One can visualise $\Gamma^+_7$ as the squares one sees looking at the corner of a $3 \times 3 \times 3$ cube. The elements are arranged as shown in Figure 4. We can recover the Hasse diagram $H_7$ of the poset structure on $\Delta^+_7$ in this picture, as follows. For each simple root $\alpha_i$, $i = 1, \ldots, 6$, we draw an edge joining $x$ and $y$ if $y = x \oplus f(\alpha_i)$. By Theorem 3.8 we will draw such an edge if and only if the corresponding roots in $\Gamma^+_7$ are related by addition a simple root, which is exactly how the Hasse diagram is constructed. A similar procedure also works on the smaller strata.

In this picture, orthogonality is easy to determine as well. By Theorem 3.7, this is determined by the links in the T-graph. For any $x \in \Gamma^+_7$, the set of $y \in \Gamma^+_7$ orthogonal
to \( x \) can be described as follows: if \( y \) is on the same face of the \( 3 \times 3 \times 3 \) cube as \( x \), then \( y \) is not in the same row or column as \( x \); if \( y \) is on a different face from \( x \) then \( y \) is in the same extended row/column as \( x \). Figure 4 shows \( \mathcal{L}(021) \cap \Gamma_7^+ \), which is set of root images orthogonal to 021.

![Figure 4: Orthogonality and partial order in \( \Gamma_7^+ \).](image)

More generally, one can compare \( \alpha \in \Delta_s^+ \) and \( \beta \in \Delta_t^+ \) by considering \( \tilde{\alpha} \), and \( \tilde{\beta} \).

**Proposition 3.9.** Suppose \( \alpha \in \Delta_s^+ \) and \( \beta \in \Delta_t^+ \), and \( s \leq t \). Then \( \alpha < \beta \) if and only if \( \tilde{\alpha} < \tilde{\beta} \). If \( s = t \) then \( \alpha \) is orthogonal to \( \beta \) if and only if \( \tilde{\alpha} \) is orthogonal to \( \tilde{\beta} \). If \( s < t \) then \( \alpha \) is orthogonal to \( \beta \) if and only if \( \tilde{\beta} \) is orthogonal to both \( \tilde{\alpha} \) and \( \zeta_s \), or to neither.

**Proof.** For the first statement, the ‘if’ direction is clear, as \( \tilde{\beta} - \tilde{\alpha} > \beta - \alpha \). Conversely, if \( \beta > \alpha \) then \( \beta^i \geq 1 \) for \( i = s', \ldots, t \), hence \( \beta - \alpha = \beta - \alpha - \sum_{i=s'}^{t} \alpha_i \) is still positive.

The statements about orthogonality follow from the following calculation:

\[
(f(\alpha) | f(\beta)) = \begin{cases} (f(\tilde{\alpha}) | f(\tilde{\beta})) & \text{if } s = t \\ (f(\tilde{\alpha}) | f(\tilde{\beta})) \oplus (z_s | f(\tilde{\beta})) & \text{if } s < t. \end{cases}
\]

3.5 **Action of the Weyl group of** \( E_6 \) **on** \( \Gamma_7^+ \)

The O-graph restricted to \( \Gamma_7^+ \) is a well known object; its complement is the Schläfli graph (see e.g. [1, 3, 4] for alternate descriptions), which describes the incidence
relations of the 27 lines on a cubic surface. It is well known that the full automorphism
group of the Schl"afli graph is the Weyl group of $E_6$. Many of these automorphisms
are manifest from our description.

If $\phi_1, \phi_2, \phi_3$ are automorphisms of $F$, then
\[ (a_1, a_2, a_3) \mapsto (\phi_1(a_1), \phi_2(a_2), \phi_3(a_3)) \] (11)
is manifestly an automorphism of the Schl"afli graph. If $\pi \in S_3$ is a permutation of
$\{1, 2, 3\}$ then we have the automorphism
\[ (a_1, a_2, a_3) \mapsto (a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}). \] (12)

If $\alpha \in \Delta(E_6)$, then the action of the reflection $r_\alpha$ on $\Gamma_7^+$ is given by
\[ r_\alpha(f(\beta)) = f(r_\alpha(\beta)) = \begin{cases} 
\beta \pm \alpha & \text{if } \langle \alpha, \beta \rangle = \mp 1 \\
\beta & \text{if } \langle \alpha, \beta \rangle = 0.
\end{cases} \]

Using Theorem 3.8, we see that for $x \in \Gamma_7^+$,
\[ r_\alpha(x) = \begin{cases} 
x \oplus f(\alpha) & \text{if } x \oplus f(\alpha) \in \Gamma_7^+ \\
x & \text{otherwise}.
\end{cases} \] (13)

Each $r_\alpha$ swaps six pairs $x \leftrightarrow x \oplus f(\alpha)$ and the restriction of the O-graph to these
12 vertices is a union of two $K_6$ graphs, which are maximal cliques. These pairs are
known as Schl"afli double sixes—there are 36 in total, each arising in this way for some
unique $\alpha \in \Delta^+(E_6)$.

From (13), it is easy to verify that the automorphisms (11) are generated by
reflections in the roots $\alpha_1, \alpha_3$ (generating all possible $\phi_1$); $\hat{\alpha}_6, \alpha_2$ (for $\phi_2$); $\alpha_6, \alpha_5$ (for
$\phi_3$); whereas the automorphism (12) corresponds to $S_3$ symmetry of the affine Dynkin
diagram $\hat{\text{Dyn}}(E_6)$. These alone do not generate the Weyl group of $E_6$; however,
together with $r_{\alpha_4}$ they do, since this extended list includes all reflections in simple
roots.

### 3.6 Order ideals

**Definition 3.3.** If $(Y, \leq)$ is a poset, an order ideal in $Y$ is a subset $J \subset Y$ such
that if $x \in J$, and $y \leq x$ then $y \in J$. The set of all order ideals in $Y$ is denoted $\mathcal{J}(Y)$
and is itself a poset, ordered by inclusion.

It is a remarkable fact that the posets $\Delta_s^+$ are related by such a construction:
there is an isomorphism
\[ \mathcal{J}(\Delta_s^+) \cong \begin{cases} 
\Delta_s^+ & \text{if } s = 3, 4, 5, 6 \\
\Delta_s^+ \setminus \{-\hat{\alpha}_8\} & \text{if } s = 7.
\end{cases} \] (14)

We refer the reader to [14] for an explanation of this phenomenon. Here we will explore
some interesting relationships between this isomorphism and our compression map $f$. 15
Definition 3.4. If $P$ and $R$ are graphs, an open map from $P$ to $R$ is a function $h : \text{vert}(P) \to \text{vert}(R)$ such that

1. $h$ is a homomorphism of graphs, i.e. if $(u, u') \in \text{edge}(P)$, then $(h(u), h(u')) \in \text{edge}(R)$;

2. $h$ is locally surjective, i.e. for every $v \in \text{vert}(P)$, $h$ maps the neighbours of $v$ surjectively to the neighbours of $h(v)$.

Equivalently, $h$ induces an open map on the topological spaces of the graphs.

Proposition 3.10. For $3 \leq s \leq 7$, there is a unique function

$$h_s : \Delta_s^+ \to \{1, \ldots, s\}$$

such that $h_s(\alpha_s) = s$, and $\beta \mapsto \psi_{h_s(\beta)}$ is an open map of graphs from $H_s$ to $\text{Dyn}(E_s)$. For $s = 8$ no such function exists.

The only proof we know of this fact is to check it case by case, which is straightforward but unenlightening.

Proposition 3.11. The isomorphism (14) is canonical, and given by $\psi_s : J(\Delta_s^+) \to \Delta_s^+$, where

$$\psi_s(J) = \alpha_s' + \sum_{\beta \in J} \alpha_{h_s(\beta)}$$

Proof. It is clear that any isomorphism (14) must be of the form $\psi(J) = \alpha_s' + \sum_{\beta \in J} \alpha_{h(\beta)}$ for some function $h : \Delta_s^+ \to \{1, \ldots, s\}$. Since the $\alpha_s$ is the minimal element of $\Delta_s^+$ and $\alpha_s'$ and $\alpha_{s'} + \alpha_s$ are the two smallest elements of $\Delta_s^+$, we must have $h(\alpha_s) = s$. In light of Proposition 3.10 it suffices to show that $h$ must induce an open map from $H_s$ to $\text{Dyn}(E_s)$.

Suppose $\beta \triangleright \beta'$, is an edge of $H_s$, and let $i = h(\beta)$, $i' = h(\beta')$. We show that $(v_i, v_{i'})$ is an edge in the Dynkin diagram, i.e. $\langle \alpha_i, \alpha_{i'} \rangle = -1$. Consider the order ideals $J = \{\gamma \in \Delta_s^+ | \gamma \leq \beta'\}$, $J' = J \setminus \{\beta\}$ and $J'' = J \setminus \{\beta, \beta'\}$. We have $\psi(J) \triangleright \psi(J') \triangleright \psi(J'')$, where $\psi(J) - \alpha_i = \psi(J') = \psi(J'')\alpha_{i'}$. Hence $\langle \psi(J) - \alpha_i, \alpha_{i'} \rangle = 1$. However, note that $\psi(J) - \alpha_i$ is not a root. If it were then there would two order ideals between $J$ and $J''$, namely $J'$ and $\psi^{-1}(\psi(J) - \alpha_{i'})$, which is impossible if $\beta \triangleright \beta'$. Thus $\langle \psi(J), \alpha_{i'} \rangle \leq 0$. We conclude that $\langle \alpha_i, \alpha_{i'} \rangle = \langle \psi(J), \alpha_{i'} \rangle - \langle \psi(J) - \alpha_i, \alpha_{i'} \rangle \leq -1$.

For local surjectivity, suppose $\beta \in \Delta_s^+$, and let $i = h(\beta)$. Let $(v_i, v_j)$ be an edge in the Dynkin diagram. We show that there exists an edge $(\beta, \gamma) \in H_s$ such that $h(\gamma) = j$.

For every $J \in \mathcal{J}(\Delta_s^+)$, let $J' = J \setminus \{\beta\}$, and define

$$\mathcal{J}_\beta = \{J \in \mathcal{J}(\Delta_s^+) \mid J' \in \mathcal{J}(\Delta_s^+)\}.$$
and note that this set is non-empty. Choose some $J \in J_\beta$. We have $\psi(J) - \psi(J') = \alpha_i$, thus $\langle \psi(J), \alpha_j \rangle - \langle \psi(J'), \alpha_j \rangle = -1$. Thus either $\langle \psi(J), \alpha_j \rangle < 0$ or $\langle \psi(J'), \alpha_j \rangle > 0$.

In the first case, let $J_{\text{max}} = \{\beta' \in \Delta^+_s \mid \beta' \not\reln \beta\}$ denote the maximal element of $J_\beta$. Note that $\langle \psi(J_{\text{max}}), \alpha_j \rangle \leq \langle \psi(J), \alpha_j \rangle < 0$, and thus $\psi(J_{\text{max}}) + \alpha_j$ is a root. Let $K = \psi^{-1}(\psi(J_{\text{max}}) + \alpha_j)$. We have $K \setminus J = \{\gamma\}$, where $h(\gamma) = j$. Finally, by the definition of $J_{\text{max}}$, we must have $\gamma \succ \beta$.

In the second case, let $J_{\text{min}} = \{\beta' \in \Delta^+_s \mid \beta' \leq \beta\}$ denote the maximal element of $J_\beta$. We have $\langle \psi(J_{\text{min}}), \alpha_j \rangle \geq \langle \psi(J), \alpha_j \rangle > 0$, and thus $\psi(J_{\text{min}}) - \alpha_j$ is a root. Letting, $K = \psi^{-1}(\psi(J_{\text{min}}) - \alpha_j)$, we have $J \setminus K = \{\gamma\}$, where $h(\gamma) = j$, and $\gamma \prec \beta$.

Figure 5 shows the map $h_7$ pictured on the corner of the $3 \times 3 \times 3$ cube. There is a striking symmetry in this picture. In particular if we impose the equivalence relation $1 \sim 6$ and $3 \sim 5$ on $\{1, \ldots, 7\}$, the numbers have full $S_3$ symmetry. This equivalence relation is the one that comes from the involution on the affine Dynkin diagram $\hat{\text{Dyn}}(E_7)$. Furthermore the $S_3$ symmetry is exactly broken by the rule that 5s and 6s are connected to a 7 by a path in $H_7$ on the same face of the cube, whereas 1s and 3s are not.

![Figure 5: The map $h_7$.](image)

To understand this symmetry, let $\mu : \Delta^+_7 \to \Delta^+_7$, denote the automorphism from the proof of Lemma 3.3 corresponding to $v_7 \in \text{Dyn}(E_7)$, and let $\rho : \Delta^+_7 \to \Delta^+_7$ denote the automorphism corresponding to $v_6 \in \text{Dyn}(E_7)$. Explicitly, $\mu$ is given by the formula:

$$\mu(\beta) = -\alpha_7 - \sum_{k=1}^{7} \beta^k \varepsilon(\alpha_k),$$
where $\varepsilon$ is the involution on $\widehat{\text{Dyn}}(E_7)$, taking $\alpha_7 \leftrightarrow \hat{\alpha}_7$, $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_3 \leftrightarrow \alpha_5$, and fixing $\alpha_2$, $\alpha_4$. Under the identification of $\Delta^+_7$ with $\Gamma^+_7$, $\mu$ and $\rho$ are more simply described by $\mu(abc) = cba$, and $\rho(abc) = bca$.

The involution $\mu$ induces an involution $\tilde{\mu}$ on order ideals:

\[ J \leftrightarrow \tilde{\mu}(J) = \{ \mu(\beta) \mid \beta \not\in J \}. \]

Note that the root $\hat{\alpha}_7 = f^{-1}(303) \in \Delta^+_7$, which is mutually orthogonal to $\alpha_7$ and $\hat{\alpha}_7$, is a fixed point of $\mu$. We use $\hat{\alpha}_7$ to partition $\mathcal{J}(\Delta^+_7)$ into four disjoint sets:

\[ \mathcal{J}_0 = \{0\} \]
\[ \mathcal{J}_1 = \{ J \neq 0 \mid \hat{\alpha}_7 \not\in J \} \]
\[ \mathcal{J}_2 = \{ J \neq \Delta^+_7 \mid \hat{\alpha}_7 \in J \} \]
\[ \mathcal{J}_3 = \{ \Delta^+_7 \} \]

In fact $\tilde{\mu}$ gives a bijection between $\mathcal{J}_i$ and $\mathcal{J}_{3-i}$.

Note that $\mathcal{J}_1$ can be identified with the order ideals in $\Gamma^+_7 \cap \mathcal{L}^c(033)$, whereas $\mathcal{J}_2$ can be identified with the order ideals in $\Gamma^+_7 \cap \mathcal{L}^c(330)$. Each order ideal in the latter is a $120^\circ$ rotation of an order ideal in the former. Thus we also have a bijection $\tilde{\rho} : \mathcal{J}_1 \to \mathcal{J}_2$ defined by

\[ \tilde{\rho}(J) = \{ \rho(\beta) \mid \beta \in J \setminus \{ \alpha_7 \} \} \cup \{ \beta \mid \beta \leq \hat{\alpha}_7 \}. \]

Since $\mathcal{J}(\Delta^+_7) \cong \Delta^+_8 \setminus \{ -\hat{\alpha}_8 \}$, we must have corresponding structures on $\Delta^+_8 \setminus \{ -\hat{\alpha}_8 \}$, which are related by $\psi_7$. Indeed we have an involution $\nu$ on this set, defined by

\[ \nu(\beta) = -\beta - 2\alpha_8 - \hat{\alpha}_8, \]

where $\psi_7 \circ \tilde{\mu} = \nu \circ \psi_7$. The image of each $\mathcal{J}_i$ is simply described as

\[ \psi_7(\mathcal{J}_i) = \{ \beta \in \Delta_8 \mid \beta^7 = i \}. \]

And we have a map $\sigma : \psi_7(\mathcal{J}_1) \to \psi_7(\mathcal{J}_2)$ given by

\[ \sigma(\beta) = -\alpha_8 - \mu(\beta - \alpha_8), \]

where $\psi_7 \circ \tilde{\rho} = \sigma \circ \psi_7$.

We can now explain the symmetry seen in Figure 3.

Suppose $J, J' \in \mathcal{J}(\Delta^+_7)$, with $J = J' \cup \{ \beta \}$, and let $h_7(\beta) = i$ so that $\psi_7(J) - \psi_7(J') = \alpha_i$. Then $\tilde{\mu}(J') = \mu(J) \cup \mu(\beta)$, and $\psi_7(\tilde{\mu}(J')) - \psi_7(\mu(J)) = \nu(\psi_7(J')) - \nu(\psi_7(J)) = \alpha_i$. Thus $h_7(\mu(\beta)) = i = h_7(\beta)$. This explains the reflectional symmetry.

To see the near rotational symmetry, we consider $J, J' \in \mathcal{J}_1$, with $J = J' \cup \beta$. Then $\tilde{\rho}(J) = \tilde{\rho}(J') \cup \{ \rho(\beta) \}$. We see that $\psi_7(\tilde{\rho}(J)) - \psi_7(\tilde{\rho}(J')) = \alpha_i$ for some $i$, whereas $\psi_7(\tilde{\rho}(J)) - \psi_7(\rho(J')) = \sigma(\psi_7(J)) - \sigma(\psi_7(J')) = \varepsilon(\alpha_i)$. Thus $h(\beta)$ and $h(\rho(\beta))$ will be reflections of each other in $\text{Dyn}(E_7)$. 
3.7 A compression of $E_6$

The discussion in Sections 3.3 and 3.4 gives a description of the strata $\Delta_s$ for all $s \leq 7$, but it is not the most symmetrical one for $s \leq 6$. For $s \leq 5$, it is easy to obtain a nice description of the strata, as they are subsets of the $D_5$ root system. For $s = 6$, we can obtain a pleasant description by working over $\mathbb{Z}/3$.

Let $\Delta$ be the $E_6$ root system. Let $V = (\mathbb{Z}/3)^5$, with the standard symmetric form \((\mathbb{Z}/3)^5\). Let $S = \{s_1, \ldots, s_6\}$, where

- $s_1 = 12000$
- $s_2 = 00012$
- $s_3 = 01200$
- $s_4 = 00120$
- $s_5 = 00011$
- $s_6 = 11111$

![Diagram](image)

Proposition 3.12. For $S$ as above the relations \((\mathbb{Z}/3)\) hold.

As a consequence of we obtain the following corollary of Theorem 2.3, we have

**Corollary 3.13.** The map $f : \Delta \to \Gamma$ is a bijection.

**Proof.** It is an injection by Theorem 2.3. To show it is a bijection, we must calculate the size of $\Gamma$. The vectors in $\Gamma$ have either 2 or 5 non-zero coordinates, each of which can be $\pm 1$. Thus there are $2^2 \binom{5}{2} + 2^5 \binom{5}{5} = 72$ elements in $\Gamma$. But $\#(\Delta) = 72$, so $f$ is a bijection. \(\square\)

Let $\Gamma_s = f(\Delta_s)$, and $\Gamma_s^+ = f(\Delta_s^+)$ denote the images of the strata under $f$.

**Theorem 3.14.** The image of the top stratum, $\Gamma_6^+$, is the set of vectors in $V$ with all coordinates non-zero, and an even number of coordinates equal to 2:

$$\Gamma_6^+ = \{(x_1, \ldots, x_5) \in V \mid \prod_{i=1}^{5} x_i = 1\}$$

**Proof.** The argument is parallel to the proof of Lemma 3.3. One can check that the automorphisms corresponding to Dynkin diagram vertices $v_3, v_4, v_5$ give permutations of the coordinates which generate the symmetric group $S_5$, hence $S_5$ acts on the coordinates of $\Gamma_6^+$. Furthermore, the automorphism corresponding to $v_6$ is $(x_1, x_2, x_3, x_4, x_5) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5)$. Applying these automorphisms to $11111 = f(\alpha_6)$, we see that all 16 elements of $\Gamma_6^+$ are indeed of the correct form. \(\square\)

**Definition 3.5.** The **T-graph** for $V$ is the graph whose vertex set is $V$, and $x = (x_1, \ldots, x_5)$ is adjacent to $y = (y_1, \ldots, y_5)$ if $x_i = y_i$ for exactly one $i$.

**Theorem 3.15.** The T-graph and the O-graph agree when restricted to the image of any stratum $f(\Delta_s^+)$.  

The proof is analogous to that of Theorem 3.4.
**Theorem 3.16.** Let $\alpha$ be a positive root. Let $\beta \in \Delta^+_s$. Then $f(\beta) + f(\alpha) \in \Gamma^+_s$ if and only if $\beta + \alpha \in \Delta^+_s$.

**Proof.** Certainly if $\beta + \alpha \in \Delta^+_s$, then $f(\beta) + f(\alpha) \in \Gamma^+_s$. Suppose that $\beta + \alpha \notin \Delta^+_s$. If $\beta = \alpha$, then $f(\beta) + f(\alpha) = f(-\beta)$ is the image of a negative root, so $f(\beta) + f(\alpha) \notin \Gamma^+_s$. Otherwise, since $\beta + \alpha$ is not a root, we must have $\langle \beta, \alpha \rangle \in \{0, 1\}$. In this case, $\langle \beta + \alpha, \beta + \alpha \rangle = 4 + 2\langle \beta, \alpha \rangle \not\equiv 2 \pmod{3}$, so in fact $f(\beta) + f(\alpha) \notin \Gamma$.

As we did with $\Delta^+_7$, we can recover the partial order structure on $\Delta^+_6$ (and the smaller strata) using Theorem 3.16. If the elements of $\Gamma^+_6$ are arranged as shown in Figure 6, we join $x$ and $y$ if $x - y$ is a simple root. In light of Theorem 3.15, orthogonality is also easily determined in this picture. Figure 6 shows the example of $\mathcal{L}(11122) \cap \Gamma^+_6$, which gives the set of root images orthogonal to 11122.

![Figure 6: Orthogonality and partial order in $\Gamma^+_6$.](image)

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