Critical Exponents from Five-Loop Strong-Coupling $\phi^4$-Theory in $4 - \varepsilon$ Dimensions.

Hagen Kleinert
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

Verena Schulte-Frohlinde
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

With the help of strong-coupling theory, we calculate the critical exponents of O($N$)-symmetric $\phi^4$-theories in $4 - \varepsilon$ dimensions up to five loops with an accuracy comparable to that achieved by Borel-type resummation methods.

I. INTRODUCTION

Recently, one of us [1,2] has developed a new approach to critical exponents of field theories based on the strong-coupling limit of variational perturbation expansions [3,4]. This limit is relevant for critical phenomena if the renormalization constants are expressed in terms of the unrenormalized coupling constant. The theory was first applied successfully to O($N$)-symmetric $\phi^4$-theories in three dimensions yielding the three fundamental critical exponents $\nu, \eta, \omega$ with high accuracy.

The method has also been shown to work for perturbation expansions of these theories in $4 - \varepsilon$ dimensions [5], but here only two-loop expansions were treated, where all results can be written down explicitly. In this note we want to extend these calculations to five-loop levels.

II. RESUME OF STRONG-COUPLING THEORY

From model studies of perturbation expansions of the anharmonic oscillator we have learned that variational perturbation expansions possess good strong-coupling limits [8,9], with a speed of convergence governed by the convergence radius of the strong-coupling expansion [10,14]. This has enabled us to set up a simple algorithm [4] for deriving uniformly convergent approximations to functions of which one knows a few initial Taylor coefficients and an important scaling property: the functions approach a constant value with a given inverse power of the variable. The renormalized coupling constant $g$ and the critical exponents of a $\phi^4$-theory have precisely this property as a function of the bare coupling constant $g_B$. In $D = 4 - \varepsilon$ dimensions the approach is parametrized as follows

$$g(g_B) = g^* - \frac{\text{const}}{g_B^{\omega/\varepsilon}} + \ldots ,$$

(2.1)

where $g^*$ is the infrared-stable fixed point, and $\omega$ is called the critical exponent of the approach to scaling. This exponent is universal, governing the approach to scaling of every function of $F(g)$.

$$f(g_B) = F(g(g_B)) = F(g^*) + F'(g^*) \times \frac{\text{const}}{g_B} \equiv f^* + \frac{\text{const}'}{g_B^{\omega/\varepsilon}} .$$

(2.2)

This type of scaling behavior is observed experimentally in systems described by $\phi^4$-theories, and strong-coupling theory is designed to calculate $f(g^*)$ and $\omega$.

Let $f(g_B)$ be a function with this behavior and suppose that we know its first $L + 1$ expansion terms,

$$f_L(g_B) = \sum_{l=0}^{L} a_l g_B^l ,$$

(2.3)
More specifically than in Eq. (2.1), we assume that \( f(g_B) \) approaches its constant strong-coupling limit \( f^* \) in the form of an inverse power series

\[
 f_M(g_B) = \sum_{m=0}^{M} b_m(g_B^{-2/q})^m,
\]

with a finite convergence radius \( |q| \). Then the \( L \)th approximation to the value \( f^* \) is obtained from the strong-coupling formula (2.3)

\[
 f_L^* = \text{opt}_{g_B} \left[ \sum_{l=0}^{L} a_l v_l \hat{g}_B^l \right], \quad v_l \equiv \sum_{k=0}^{L-l} \binom{-ql/2}{k} (-1)^k.
\]

The quantities \( v_l \) are simply binomial expansions of \((1 - 1)^{-ql/2}\) up to the order \( L - l \). The expression in brackets has to be optimized in the variational parameter \( \hat{g}_B \). The optimum is the smoothest among all real extrema. If there are no real extrema, the turning points serve the same purpose.

The derivation of this rule is simple: We replace \( g_B \) in (2.3) trivially by \( \hat{g}_B \equiv g_B / \kappa^q \) with \( \kappa = 1 \). Then we rewrite, again trivially, \( \kappa^{-q} \) as \((K^2 + \kappa^2 - K^2)^{-q/2}\) with an arbitrary parameter \( K \). Each term is now expanded in powers of \( r = (\kappa^2 - K^2) / K^2 \) assuming \( r \) to be of the order \( g_B \). Then we take the limit \( g_B \to \infty \) at a fixed ratio \( \hat{g}_B \equiv g_B / K^3 \), so that \( K \to \infty \) like \( \hat{g}_B^{1/q} \) and \( r \to -1 \), yielding (2.3). Since the final result to all orders cannot depend on the arbitrary parameter \( K \), we expect the best result to any finite order to be optimal at an extremal value of \( K \), i.e., \( \hat{g}_B \).

The approach to the limit of \( r \) is \( r = -1 + \kappa^2 / K^2 = -1 + O(g_B^{-2/q}) \). This implies the leading correction to \( f_L^* \) to be of the order \( g_B^{-2/q} \). Application of the theory to a function with the strong-coupling behavior (2.1) requires therefore a parameter \( q = 2 \varepsilon / \omega \) in formula (2.3). A systematic expansion in powers of \( K^2 \) leads to the strong-coupling expansion (2.4).

For \( L = 2 \) and \( 3 \) we have given in Ref. (3) analytic expressions for the strong-coupling limits (2.5). Setting \( \rho \equiv 1 + q/2 = 1 + \varepsilon / \omega \), the limits are for \( L = 2 \):

\[
 f_2^* = \text{opt}_{g_B} \left[ a_0 + a_1 \rho \hat{g}_B + a_2 \hat{g}_B^2 \right] = a_0 - \frac{1}{4} \frac{a_0^2}{a_2} \rho^2,
\]

and for \( L = 3 \):

\[
 f_3^* = \text{opt}_{g_B} \left[ a_0 + a_1 \rho(\rho + 1) \hat{g}_B + a_2(2\rho - 1) \hat{g}_B^2 + a_3 \hat{g}_B^3 \right]
 = a_0 - \frac{1}{3} \frac{a_0 a_2}{a_3} \left( 1 - \frac{2}{3} r \right) + \frac{2 a_0^3}{27 a_3^2} (1 - r),
\]

where \( r \equiv \sqrt{1 - 3a_1 a_2 / a_3^2} \) and \( a_1 \equiv 1 / a_1 \rho(\rho + 1) \) and \( a_2 \equiv a_2(2\rho - 1) \). The positive square root must be taken to connect \( g_B^2 \) smoothly to \( g_B^2 \) at small \( g_B \). If the square root is imaginary, the optimum is given by the unique turning point, which leads once more to the limit (2.7), but with \( r = 0 \).

The exponent \( \omega \) describing the approach to scaling can be determined from the expansion coefficients of an arbitrary function \( h(g_B) \) behaving like (2.3) as follows. Since \( h(g_B) \) goes to \( h^* \) in the strong-coupling limit, the logarithmic derivative \( s(g_B) \equiv h'(g_B) / h(g_B) \) must vanish at \( g_B = \infty \). If \( h(g_B) \) starts out as \( A_0 + A_1 g_B + \ldots \) or as \( A_1 g_B + A_2 g_B^2 + \ldots \), the logarithmic derivative is

\[
 s(g_B) = A_1' g_B + (2A_1' - A_1^2') g_B^2 + (A_1'^3 - 3A_1' A_1' A_2) g_B^3 + \ldots,
\]

where \( A_1' = A_1 / A_0 \), or

\[
 s(g_B) = 1 + \hat{A}_2 g_B + (2\hat{A}_3 - \hat{A}_2^2) g_B^2 + (\hat{A}_3^3 - 3\hat{A}_2 \hat{A}_3 - 3\hat{A}_4) g_B^3 + \ldots,
\]

where \( \hat{A}_i = A_i / A_1 \). The expansion coefficients on the right-hand sides may then be inserted into (2.6) or (2.7), whose left-hand sides have to vanish to ensure that \( h(g_B) \to h^* \).

Another formula for determining \( \omega \) is based on the fact that if the approach \( h(g_B) \to h^* \) is of the type (2.2), the function
\[ t(g_B) \equiv g_B^2 h''(g_B) = 2\Lambda_2 g_B + (-4\Lambda_2^2 + 6\Lambda_3)g_B^2 + (8\Lambda_2^3 - 18\Lambda_2\Lambda_3 + 12\Lambda_4)g_B^3 + \ldots \] (2.10)
must have the strong-coupling limit
\[ t(g_B) \rightarrow t^* = -\frac{\omega}{\varepsilon} - 1. \] (2.11)

### III. Renormalization Constants and Critical Exponents

Let us briefly recall the definitions of the $\phi^4$-theory in $D = 4 - \varepsilon$ dimensions whose five-loop expansions we want to evaluate. The bare euclidean action is

\[ A = \int d^D x \left\{ \frac{1}{2} [\partial \phi_B(x)]^2 + \frac{1}{2} m_B^2 \phi_B^2(x) + (4\pi)^2 \frac{\lambda_B}{4!} (\phi_B^2(x))^2 \right\}, \] (3.1)

where the field $\phi_B(x)$ is an $N$-dimensional vector, the action being $O(N)$-symmetric. The Ising model corresponds to $N = 1$, the superfluid phase transition by $N = 2$, and the classical Heisenberg magnet by $N = 3$, the critical behavior of dilute polymer solutions is described by $N = 0$.

By calculating the Feynman integrals, regularized via an expansion in $\varepsilon = 4 - D$ and arbitrary mass scale $\mu$, one obtains renormalized values of mass, coupling constant, and field related to the bare quantities by renormalization constants $Z_\phi, Z_m, Z_B$:

\[ m_B^2 = m^2 Z_m Z_\phi^{-1}, \quad \lambda_B = \lambda Z_\phi Z_m^{-2}, \quad \phi_B = \phi Z_\phi^{1/2}. \] (3.2)

Up to two loops, perturbation theory yields the following expansions in powers of the dimensionless reduced coupling constant $g_B \equiv \lambda_0 / \mu^\varepsilon$:

\[ g = g_B - \frac{N + 8}{3\varepsilon} g_B^2 + \left[ \frac{(N + 8)^2}{9\varepsilon^2} + \frac{3N + 14}{6\varepsilon} \right] g_B^3 + \ldots, \] (3.3)

\[ \frac{m_B^2}{m^2} = 1 - \frac{N + 2}{3} \frac{g_B^2}{\varepsilon} + \frac{N + 2}{9} \left[ \frac{N + 5}{\varepsilon^2} + \frac{5}{4\varepsilon} \right] g_B^4 + \ldots, \] (3.4)

\[ \frac{\phi_B^2}{\phi^2} = 1 + \frac{N + 2}{36} \frac{g_B^3}{\varepsilon} + \ldots. \] (3.5)

We refrain from writing down the lengthy five-loop expressions calculated in Ref. [6], since they can be downloaded from the internet [7]. We now set the scale parameter $\mu$ equal to the physical mass $m$ and consider all quantities as functions of $g_B = \lambda_B / m^\varepsilon$. In order to describe second-order phase transitions, we let $m_B^2$ go to zero like $\tau = \text{const} \times (T - T_c)$ as the temperature $T$ approaches the critical temperature $T_c$, and assume that also $m^2$ goes to zero, and thus $g_B$ to infinity. The latter assumption will be seen to be self-consistent after Eq. (4.10). Assuming the theory to scale as suggested by experiments, we now determine the value of the renormalized coupling constant $g$ in the strong-coupling limit $g_B \rightarrow \infty$, and the exponent $\omega$ of approach, assuming the behavior (2.1). First we apply formula (2.5) to the logarithmic derivative $s(g_B)$ of the function $g(g_B)$, which is determined by Eq. (2.9). Setting $s^* = 0$, determines the approximation $\omega_L$ to $\omega$.

The other critical exponents are found as follows from the experimental behavior of systems described by $\phi^4$-theories. We know that the ratios $m^2/m_B^2$ and $\phi^2/\phi_B^2$ have a limiting power behavior for small $m$:

\[ \frac{m^2}{m_B^2} \propto g_B^{-\eta_m/\varepsilon} \propto m^{\eta_m}, \quad \frac{\phi^2}{\phi_B^2} \propto g_B^{\eta/\varepsilon} \propto m^{-\eta}. \] (3.6)

The powers $\mu_m$ and $\eta$ can then be calculated from the strong-coupling limits of the logarithmic derivatives

\[ \eta_m(g_B) = -\varepsilon \frac{d}{d\log g_B} \log \frac{m^2}{m_B^2}, \quad \eta(g_B) = \varepsilon \frac{d}{d\log g_B} \log \frac{\phi^2}{\phi_B^2}. \] (3.7)

Inserting (3.4) and (3.5) on the right-hand sides yields the expansions
\[ \eta_m(g_B) = \frac{N+2}{3}g_B - \frac{N+2}{18}\left(5 + 2 \frac{N + 8}{\varepsilon}\right)g_B^2 + \ldots, \]  
\[ \eta(g_B) = \frac{N+2}{18}g_B^2 + \ldots. \] (3.8) (3.9)

When approaching the second-order phase transitions where the bare mass \( m_0^2 \) vanishes like \( r \equiv (T - T_c) \), the physical mass \( m^2 \) vanishes with a different power of \( r \). This power is obtained from the first equation in (3.6), which shows that \( m \propto r^{1/(2 - \eta_m)} \). In experiments one observes that the coherence length of fluctuations \( \xi = 1/m \) increases near \( T_c \) like \( r^{-\nu} \). Comparison with the previous equation shows that for the critical exponent \( \nu \) is equal to \( 1/(2 - \eta_m) \).

Similarly we see from the second equation in (3.6) that the scaling dimension \( D/2 - 1 \) of the free field \( \phi_0 \) for \( T \to T_c \) is changed in the strong-coupling limit to \( D/2 - 1 + \eta/2 \), the number \( \eta \) being the so-called anomalous dimension of the field. This implies a change in the large-distance behavior of the correlation functions \( \langle \phi(x)\phi(0) \rangle \) at \( T_c \) from the free-field behavior \( r^{-D+2} \) to \( r^{-D+2-\eta} \).

The magnetic susceptibility is determined by the integrated correlation function \( \langle \phi_B(x)\phi_B(0) \rangle \). At zero coupling constant \( g_B \), this is proportional to \( 1/m_B^2 \propto r^{-1} \). The interaction changes this to \( m^{-2}\phi_B^2/\phi^2 \). This quantity has a temperature behavior \( m^{-2} = \tau^{-\nu(2-\eta)} \equiv \tau^{-\gamma} \), which defines the critical exponent \( \gamma = \nu(2 - \eta) \) governing the divergence of the susceptibility line. Using \( \nu = 1/(2 - \eta_m) \) and the expansions (3.8), (3.9), we obtain for \( \gamma(g_B) \) the perturbation expansion up to second order in \( g_B \):

\[ \gamma(g_B) = 1 + \frac{N+2}{6}g_B + \frac{N+2}{36}\left(N - 4 - 2\frac{N + 8}{\varepsilon}\right)g_B^2 + \ldots. \] (3.10)

**IV. EXPLICIT TWO-LOOP RESULTS**

Let us briefly recall the explicit results obtained in Ref. [5] from the above two-loop expansions. First we calculate the critical exponent \( \omega \) from the requirement that \( g(g_B) \) has a constant strong-coupling limit, implying the vanishing of (2.3) for \( g_B \to \infty \). We form the logarithmic derivative (2.9) of the expansion (3.3) up to the order \( g_B^2 \), and obtain from Eq. (2.6) the scaling condition

\[ 0 = 1 - \frac{1}{4} \frac{\dot{A}_2}{2A_3 - A_2^2} \rho^2. \] (4.1)

This fixes \( \rho = 1 + \varepsilon/\omega \) to

\[ \rho = \sqrt{8A_3/A_2^2 - 4}. \] (4.2)

Since \( \omega \) must be greater than zero, only the positive square root is physical. With the explicit coefficients \( A_1, A_2, A_3 \) of expansion (3.3), \( \rho \) becomes

\[ \rho = 2 \sqrt{1 + 3 \frac{3N + 14}{(N + 8)^2}\varepsilon}. \] (4.3)

The associated critical exponent \( \omega = \varepsilon/(\rho - 1) \) is plotted in Fig. 4. It has the \( \varepsilon \)-expansion

\[ \omega = \varepsilon - 3 \frac{3N + 14}{(N + 8)^2}\varepsilon^2 + \ldots, \] (4.4)

also shown in Fig. 1, which agrees with the first two terms obtained from renormalization group calculations [6].

From Eqs. (2.11), (2.10), and (2.4) we obtain for the critical exponent \( \omega \) a further equation

\[ -\frac{\omega}{\varepsilon} - 1 = -\frac{\rho}{\rho - 1} = -\frac{1}{2} \frac{\dot{A}_2^2}{3A_3 - 2A_2^2} \rho^2. \] (4.5)

which is solved by
\[ \rho = \frac{1}{2} + \sqrt{\frac{6A_3}{A_2^2} - \frac{15}{4}}, \]  

(4.6)

with the positive sign of the square root ensuring a positive \( \omega \). Inserting the coefficients of (3.3), this becomes

\[ \rho = \frac{1}{2} + \frac{3}{2} \sqrt{1 + 4 \frac{3N + 14}{(N + 8)^2} \varepsilon}. \]  

(4.7)

The associated critical exponent \( \omega = \varepsilon / (\rho - 1) \) has the same \( \varepsilon \)-expansion (4.4) as the previous approximation (4.3). When plotted in Fig. 1 the alternative approximation (4.7) is indistinguishable from the earlier one in (4.3) in the plot of Fig. 1.

\[ g^*(\varepsilon) \]

\[ \nu(\varepsilon) \]

\[ \omega^*(\varepsilon) \]

\[ \gamma(\varepsilon) \]

FIG. 1. Two-loop results of strong-coupling theory for critical exponents of the Ising universality class \( N = 1 \). The first figure shows the renormalized coupling at infinite bare coupling as a function of \( \varepsilon = 4 - D \) calculated from the first three perturbative expansion terms. The curve coincides with the \( \varepsilon \)-expansion up to the order \( \varepsilon^2 \). The dashed curve indicates the linear term. The other figures show the critical exponents \( \omega, \nu, \) and \( \gamma \). Dashed curves indicate linear and quadratic \( \varepsilon \)-expansions. The dots mark the values \( g^* \approx 0.48 \pm 0.003, \omega \approx 0.802 \pm 0.003, \nu = 0.630 \pm 0.002, \) and \( \gamma = 1.241 \pm 0.004 \) obtained from six-loop calculations [1].

Having determined \( \omega \), we can now calculate \( g^* \). Inserting the first two coefficients of the expansion (3.3) into (2.6), we obtain

\[ g^*_2 = a_0 - \frac{1}{4} \frac{a_1^2}{a_2} \rho^2. \]  

(4.8)

Together with (4.3), this yields

\[ g^*_2 = \frac{3}{N + 8} \varepsilon + \frac{9}{(N + 8)^3} \varepsilon^2, \]  

(4.9)

which is precisely the \( \varepsilon \)-expansion of \( g^* \) derived in two-loop renormalization group calculations.

We now turn to the critical exponents. Taking the expansion (3.8) for \( \nu \) to infinite \( g_B \), we obtain from formula (2.6) the limiting value

\[ \eta_m = \frac{\varepsilon}{4} \frac{N}{N + 8 + 5\varepsilon/2 \rho^2}. \]  

(4.10)

This is certainly positive, so that the first equation (3.6) ensures that with \( m_B^2 \) also \( m^2 \) goes to zero, a necessary condition for the self-consistency of the theory.
The corresponding \( \nu = 1/(2 - \eta_m) \) is plotted in Fig. 1. With the approximation (4.3) for \( \rho \), we find for \( \nu \) the \( \varepsilon \)-expansion

\[
\nu = \frac{1}{2} + \frac{1}{4N+8}\varepsilon + \frac{(N+2)(N+3)(N+20)}{8(N+8)^3}\varepsilon^2 + \ldots, \tag{4.11}
\]

which is also shown in Fig. 1, and agrees with the renormalization group result to this order in \( \varepsilon \).

As a third independent critical exponent we calculate \( \gamma = (2 - \eta)/(2 - \eta_m) \) by inserting the coefficients of the expansion (3.10) into formula (2.6) and obtain

\[
\gamma = 1 + \frac{\varepsilon}{2N+8} + \frac{1}{4} \frac{(N+2)(N^2+22N+52)}{(N+8)^3}\varepsilon^2 + \ldots, \tag{4.12}
\]

plotted in Fig. 1. This has an \( \varepsilon \)-expansion

\[
\gamma = 1 + \frac{\varepsilon}{2N+8} + \frac{1}{4} \frac{(N+2)(N^2+22N+52)}{(N+8)^3}\varepsilon^2 + \ldots, \tag{4.13}
\]

also shown in Fig. 1, and agreeing with renormalization group results to this order. The critical exponent \( \eta = 2 - \gamma/\nu \) has the \( \varepsilon \)-expansion

\[
\eta = \frac{(N+2)}{2(N+8)^2} + \ldots .
\]

The above results demonstrate that variational strong-coupling theory can successfully be applied to \( \phi^4 \)-theories in \( D = 4 - \varepsilon \) dimensions and yields resummed expressions for the \( \varepsilon \)-dependence of all critical exponents. Their \( \varepsilon \)-expansions agree with those obtained from renormalization group calculations.

In order to achieve better accuracies we shall now apply the method to five-loop expansions.

**V. EXTENSION TO FIVE LOOPS**

The above calculations are again extended to five-loops using the power series for the critical exponents of Ref. [6,7]. In a first step, we determine the parameter \( \rho = 1 + \varepsilon/\omega \) for which the logarithmic derivative of \( g(g_B) \) approaches zero for \( g_B \to \infty \) [see Eq. (2.9)]. We therefore insert the coefficients of the power series of \( s(g_B) \equiv g_B g'(g_B)/g(g_B) \) into Eq. (2.5) and determine \( \rho = 1 + \varepsilon/\omega \) for \( L = 2, 3, 4, 5 \), to make \( s^*_L = 0 \). The resulting \( \varepsilon \)-expansions for the approach-to-scaling parameter \( \omega \) reproduce the well-known \( \varepsilon \)-expansions in [6] up to the corresponding order. In Figure 2, the approximations \( \omega_L \) are plotted against the number of loops \( L \) for \( \varepsilon = 1 \).

Apparently, the five loop results are still some distance away from a constant \( L \to \infty \)-limit. The slow approach to the limit calls for a suitable extrapolation method. The general convergence behavior in the limit \( L \to \infty \) was determined in [1] to be of the general form

\[
f^*(L) \approx f^* + \text{const} \times e^{-c L^{1-\omega}}. \tag{5.1}
\]

We therefore plot the approximations \( s_L \) for a given \( \omega \) near the expected critical exponent against \( L \). To exploit this knowledge and fit the points by the theoretical curve (5.1) to determine the limit \( s^* \). Then \( \omega \) is varied, and the plots are repeated until \( s^* \) is zero. The resulting \( \omega \) is the desired critical exponent, and the associated plots are shown in Fig. 3. Since the optimal variational parameter \( g_B \) comes from minima and turning points for even and odd approximants in alternate order, the points are best fitted by two different curves.

In order to determine the common constant \( c \) one plots even and odd approximations \( s_L \) directly against the variable \( x_L = e^{-c L^{1-\omega}} \). The constant \( c \) is then used to fit straight lines through even and odd approximations which cross at zero \( x_L \). This procedure is shown in Fig. 3, and yields the curve shown in Fig. 4. The resulting \( \omega \)-values are listed in Table I. They will now be used to derive the strong-coupling limits for the exponents \( \nu \), \( \gamma \) and \( \eta \).
FIG. 2. Critical exponent of approach to scaling $\omega$ calculated from $s_L^* = 0$, plotted against the order of approximation $L$.

FIG. 3. Extrapolation of the solutions of the equation $s_L^* = 0$ to $L \to \infty$ with the help of the theoretically expected large-$L$ behavior. The $\omega$ where $s_L^*$ goes to zero for $L \to \infty$ determines the critical exponent $\omega = 2/q$. The best extrapolating function is written on top of the figure.
FIG. 4. Same plot as in Fig. 3, but against the variable $x_L = e^{-cL^{1-\omega}}$. The parameter $c$ is fixed by requiring the straight lines to cross on the vertical axis. When this intercept lies at the origin, we have found the critical exponent $\omega$, written on the top of each plot. For comparison, we also show a direct plot against $L$ in Fig. 3.

A. Exponents $\nu$

For the calculation of the critical exponent $\nu$, we proceed in two different ways. This will give us an idea of the systematic error of the method. First we find the five-loop expansions for $\nu(g_B)$ using the relation $\nu(g_B) = 1/[2 - \eta_m(g_B)]$. From this we calculate their strong-coupling approximation $\nu_L$ for $L = 2, 3, 4, 5$. After extrapolating these to infinite $L$, we obtain the numbers listed for different universality classes $O(N)$ in Table I under the heading (I). The corresponding extrapolation fits are plotted in Figure 5 and 6. The resulting values for the critical exponent $\nu(\infty)$ are indicated by horizontal lines in Fig. 6.

The second way proceeds by calculating the strong-coupling values of $\eta_m(g_B)$ for $L = 2, 3, 4, 5$. After extrapolating these to infinite $L$, the critical exponent $\nu$ is found from $\nu = 1/(2 - \eta_m)$. The results are listed in Table I under the heading (II). The table shows in parentheses the $L = 5$ approximation for each quantity, from which we see the extrapolation distance of this value from the infinite-$L$ limit.

By repeating all calculations for a slightly different $\omega$-value, we deduce the dependence of our results on the critical exponent $\omega$ used in the resummation process:

$$
\Delta \nu = \begin{cases} 
-0.0900 \times (\omega - 0.8035) \\
-0.1375 \times (\omega - 0.7998) \\
-0.1853 \times (\omega - 0.7948) \\
-0.2271 \times (\omega - 0.7908)
\end{cases} \quad \text{for} \quad \begin{cases} 
N = 0 \\
N = 1 \\
N = 2 \\
N = 3
\end{cases}.
$$

(5.2)
\[ \nu, N = 0 \]
\[ x = e^{-0.9375} L^{0.19655} \]

\[ \nu, N = 1 \]
\[ x = e^{-3.94274} L^{0.2052} \]

\[ \nu, N = 2 \]
\[ x = e^{-3.94274} L^{0.2052} \]

\[ \nu, N = 3 \]
\[ x = e^{-3.94274} L^{0.2052} \]

FIG. 5. Critical exponent \( \nu_L(I) \) obtained from variational perturbation theory plotted as a function of \( x_L \). Requiring the lines to cross at \( x_L = 0 \) determines the parameter \( c \) in \( x_L \). See in the text.

B. Exponents \( \eta \) and \( \gamma \)

The calculation of the critical exponent \( \eta \) is difficult in all resummation schemes since the power series of \( \eta(g_B) \) starts out with \( g_B^2 \), so that there is one approximation less than for \( \nu \). The three points approximation \( \eta_3, \eta_4, \eta_5 \) we obtain from the five-loop expansions are not sufficient to carry out the above extrapolation procedure. The exponent is therefore calculated from the strong-coupling limit of the power series for \( \tilde{\eta}(g_B) \equiv \eta_m(g_B) + \eta(g_B) \) which supplies us with the combination of critical exponents \( 2 - 1/\nu + \eta \). After finding \( \tilde{\eta}^* \) we subtract from this \( 2 - 1/\nu \) and obtain the desired \( \eta \). If we use \( \nu(1) \) of Table I in this subtraction, we obtain \( \eta \)-values listed as \( \eta(1) \) in Table I. From \( \nu(2) \) we get \( \eta(2) \). The fits leading to the strong-coupling limits of \( \tilde{\eta}(g_B) \) are shown in Figures 7 and 8. As before, the limiting values for \( L \rightarrow \infty \) are indicated by horizontal lines. The fitted extrapolation function is displayed on top of each figure.

An independent strong-coupling calculation for the critical exponent \( \eta \) may be obtained by resumming the series expansion for the critical exponent of the susceptibility \( \gamma = \nu(2 - \eta) \). The extrapolation plots for this exponent are shown in Figs. 9 and 10. The resulting value for \( \gamma \) is also contained in Table I. As in all entries, we have listed the fifth-order approximations in parentheses to illustrate the extrapolation distance to infinite order \( L \).

The dependence on the value of \( \omega \) is of the same order of magnitude as for \( \nu \):

\[
\Delta \gamma = \begin{cases} 
-0.1500 \times (\omega - 0.8035) \\
-0.2237 \times (\omega - 0.7998) \\
-0.3147 \times (\omega - 0.7948) \\
-0.4014 \times (\omega - 0.7908) 
\end{cases}
\quad \text{for} \quad \begin{cases} 
N = 0 \\
N = 1 \\
N = 2 \\
N = 3 
\end{cases}.
\]  

(5.3)

C. Comparison with Previous Results and Experiments

In Table I we have added to our results also those obtained by other methods. Since an extensive Table has been published before (Table IV in Ref. [2]), we confine ourselves here only to results of the resummation of the \( \varepsilon \)-expansion by Guida and Zinn-Justin in [14], and those from three-dimensional variational perturbation theory to sixth order for
ω and to order 7 for ν and η in Ref. [2]. The difference between ν (I) and ν (II), and η (I) and η (II) is considerably smaller than the typical errors in the other references.

The results of our strong-coupling theory agree very well with those obtained from Borel-type resummation although we do not make use of the known large-order behavior. For a good test of the reliability of our results we compare our results with experiments. The most precise experimental values are available from specific heat measurements performed on superfluid helium near the λ-point at zero gravity in the space shuttle in 1992, which are reported in Ref. [2]. There one finds for the essential exponent α = 2 – 3ν the value

$$\alpha = -0.01285 \pm 0.00038,$$

(5.4)

corresponding to

$$\nu = 0.67095 \pm 0.00013.$$  

(5.5)

Our resummation results in Table 1 imply a value

$$\nu_{\text{ours}} = 0.6697 \pm 0.0013,$$  

(5.6)

corresponding to

$$\alpha_{\text{ours}} = -0.0091 \pm 0.0039.$$  

(5.7)

This agrees satisfactorily with the experimental result. In Fig. 11 we have compared our result with other experiments and various theoretical determinations.

D. Conclusion

Application of strong-coupling theory to fix-loop perturbation expansions of O(N)-symmetric φ^4-theories in 4-ε dimensions yield satisfactory values for all critical exponents.

[1] H. Kleinert, Phys. Rev. D 57, 2264 (1998) (www.physik.fu-berlin.de/~kleinert/257); Addendum: ibid. D 58, 1077 (1998) (cond-mat/9803268).
[2] H. Kleinert, Seven Loop Critical Exponents from Strong-Coupling φ^4-Theory in Three Dimensions, FU-Berlin preprint 1998 (hep-th/9812197).
[3] H. Kleinert, Phys. Lett. A 173, 332 (1993) (www.physik.fu-berlin.de/~kleinert/kleine_re3.html#213).
[4] Details of strong-coupling theory are found in Chapter 5 of the textbook H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, World Scientific Publishing Co., Singapore 1995, second extended edition, pp. 1–850 (www.physik.fu-berlin.de/~kleinert/ kleiner.html#b5).
[5] H. Kleinert, Phys. Lett. B 434, 74 (1998) (cond-mat/9801167).
[6] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin, Phys. Lett. B 272, 39 (1991) (hep-th/9503230); H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B 342, 284 (1995) (cond-mat/9503035).
[7] Address: http://www.physik.fu-berlin.de/~kleinert/b8/programs.html.
[8] W. Janke and H. Kleinert, Phys. Lett. A 199, 287 (1995) (quant-ph/9502018).
[9] W. Janke and H. Kleinert, Phys. Rev. Lett. 75, 2787 (1995) (quant-ph/9502019).

This paper contains references to earlier, less powerful calculations of strong-coupling expansion coefficients from weak-coupling perturbation theory.
[10] H. Kleinert and W. Janke, Phys. Lett. A 206, 283 (1995) (quant-ph/9509003).

Note that the convergence behavior of expansions for the anharmonic oscillator is different from field theoretic ones considered here since the corrections to the strong-coupling expansion (2.1) are not all of the form 1/g^Nω^B with integer N. There are also daughter corrections 1/g^Nω^B with ω^B \neq ω. These will be neglected, this being equivalent to the neglect of confluent singularities at the infrared-stable fixed point in the renormalization group approach discussed by B.G. Nickel, Phyica A 177, 189 (1991); A. Pelissetto and E. Vicari (University of Pisa preprint IFUP-TH 52/97).
[11] H. Kleinert, Phys. Lett. A 207, 133 (1995) (quant-ph/9507003).
FIG. 6. Same plot as in Fig. 3, but against $L$. The fit-function is written on top of the figure.
FIG. 7. Determination of the critical exponent \( \eta \) from the strong-coupling limit of \( \eta_m + \eta \) plotted as a function of \( x_L \). Requiring the lines to cross at \( x_L = 0 \) determines the parameter \( c \) in \( x_L \). See in the text.

FIG. 8. The same as above, plotted against the order of approximation \( L \) and the fit-function written on top of the figure.
FIG. 9. Critical exponent $\gamma$ obtained from variational perturbation theory plotted as a function of $x_L$. Requiring the lines to cross at $x_L = 0$ determines the parameter $c$ in $x_L$. See in the text.

$N = 0$
$\gamma = 1.1503$

$N = 0$
$\gamma L = 1.1576 - 22.3144e^{-5.7223 L^{0.19655}}$

$N = 1$
$\gamma = 1.2349$

$N = 1$
$\gamma L = 1.2349 - 17.1802e^{-5.0149 L^{0.2092}}$

$N = 2$
$\gamma = 1.3105$

$N = 2$
$\gamma L = 1.3105 - 13.7081e^{-4.4763 L^{0.2052}}$

$N = 3$
$\gamma = 1.3830$

$N = 3$
$\gamma L = 1.3831 - 11.6646e^{-4.0803 L^{0.20915}}$

FIG. 10. Same plot as in Fig. (9), but against $L$. The fit-function is written on top of the figure.
FIG. 11. Critical exponent $\nu$ in comparison with experimental data and results from other resummation schemes.

| $N$ | VPT, $D = 4 - \varepsilon$ | Borel-Res. (GZ) | VPT, $D = 3$ | MN, $D = 3$ |
|-----|-----------------------------|------------------|---------------|---------------|
| 0   | $0.80345(0.7448)$          | $0.828 \pm 0.023$ | $0.810$       |               |
| 1   | $0.7998(0.7485)$           | $0.814 \pm 0.018$ | $0.805$       |               |
| 2   | $0.7948(0.7530)$           | $0.802 \pm 0.018$ | $0.800$       |               |
| 3   | $0.7908(0.7580)$           | $0.794 \pm 0.018$ | $0.797$       |               |

$\nu^{-\infty}(\nu)$ (I) $\eta^{-\infty}(\nu)$ (II) $\nu^{-\infty}(\nu)$ (I) $\eta^{-\infty}(\nu)$ (II)

| $N$ | $0.5874(0.5809)$ | $0.5878(0.5832)$ | $0.5875 \pm 0.0018$ | $0.5883$ |
|-----|------------------|------------------|---------------------|---------|
| 1   | $0.6292(0.6171)$ | $0.6294(0.6222)$ | $0.6293 \pm 0.0026$ | $0.6305$|
| 2   | $0.6697(0.6509)$ | $0.6692(0.6597)$ | $0.6685 \pm 0.0040$ | $0.6710$|
| 3   | $0.7081(0.6821)$ | $0.7063(0.6951)$ | $0.7050 \pm 0.0055$ | $0.7075$|

$\eta^{-\infty}(\eta)$ (I) $\eta^{-\infty}(\eta)$ (II)

| $N$ | $0.0316(0.0234)$ | $0.0305(0.0234)$ | $0.0300 \pm 0.0060$ | $0.03215$|
|-----|------------------|------------------|---------------------|---------|
| 1   | $0.0373(0.0308)$ | $0.0367(0.0308)$ | $0.0360 \pm 0.0060$ | $0.03370$|
| 2   | $0.0396(0.0365)$ | $0.0396(0.0365)$ | $0.0385 \pm 0.0065$ | $0.03480$|
| 3   | $0.0367(0.0409)$ | $0.0402(0.0409)$ | $0.0380 \pm 0.0060$ | $0.03447$|

$\gamma^{-\infty}(\gamma)$

| $N$ | $1.1576(1.1503)$ | $1.1575 \pm 0.0050$ | $1.161$ | $1.1569 \pm 0.0004$ |
|-----|------------------|---------------------|--------|---------------------|
| 1   | $1.2349(1.2194)$ | $1.2360 \pm 0.0040$ | $1.241$ | $1.2378 \pm 0.0006$ |
| 2   | $1.31045(1.2846)$ | $1.3120 \pm 0.0085$ | $1.318$ | $1.3178 \pm 0.0010$ |
| 3   | $1.3830(1.3452)$ | $1.3830 \pm 0.0135$ | $1.390$ | $1.3926 \pm 0.0010$ |

TABLE I. Critical exponents of five-loop strong-coupling theory and comparison with the results from Borel-type resummation of Refs. [14] (GZ) and [18] (MN), and from variational perturbation theory of Ref. [2]. The parentheses behind each number show the five-loop approximation to see the extrapolation distance. The two values for $\nu$ come once from a resummation of the series for $\nu$ itself (I), once from the series for $\nu^{-1}$ (II). The two values for $\eta$ come from subtracting once the value $\nu(I)$ and once the value $\nu(II)$.