NONLINEAR TRAVELLING WAVES ON NON-EUCLIDEAN SPACES

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ABSTRACT. We study solutions of the form $v(t, x) = e^{i\lambda t} u(g(t)x)$, where $g(t)$ represents a one-parameter family of isometries, to nonlinear Schrödinger and Klein-Gordon equations on Riemannian manifolds, both compact and non-compact ones. The emphasis will be on the NLKG. Here $g(t)$ is generated by a Killing field $X$ and the case of interest is when $X$ has length $\leq 1$, which leads to hypoelliptic operators with loss of at least one derivative. In the compact case, we establish existence of travelling wave solutions via “energy” minimization methods and prove that at least compact isotropic manifolds have genuinely travelling waves. We establish certain sharp estimates on low dimensional spheres that improve results in [11] and carry out the subelliptic analysis for NLKG on spheres of higher dimensions utilizing their homogeneous coset space properties. These subelliptic phenomenon have no parallel in the setting of flat spaces. We also study related phenomenon on complete noncompact manifolds which have a certain radial symmetry using concentration-compactness type arguments. Lastly, we establish that small perturbations of the Killing field result in small perturbations of the resulting travelling waves.

1. Introduction, Setting and Notations

Let us consider a complete Riemannian manifold $(M, g)$, maybe with boundary. Let $X$ be a Killing field on the manifold, which flows by the one-parameter family of isometries $g(t)$ (if the manifold has a boundary, $X$ will be taken as tangent to it). We will, in the presence of a boundary, in general assume a Dirichlet boundary condition on $\partial M$, though Neumann boundary conditions can also be tackled by similar methods. The following is the nonlinear Schrödinger equation:

$$i\partial_t v + \Delta v = -K|v|^{p-1}v$$

and the following is the nonlinear Klein-Gordon equation:

$$\partial^2_{tt} v - \Delta v + m^2v = K|v|^{p-1}v$$

Our main thrust in this paper will be to study the nonlinear Klein-Gordon equation. However, for both the NLS and the NLKG, we will be looking for

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solutions of the form
\begin{equation}
\tag{1.3}
v(t, x) = e^{i\lambda t}u(g(t)x)
\end{equation}
with \(\lambda \in \mathbb{R}\). For understandable reasons, they are called travelling wave solutions, as opposed to ground/standing wave solutions, which are of the form \(v(t, x) = e^{i\lambda t}u(x)\). As an illustration, one might consider the setting in the Euclidean space, with \(g(t)x = x + tv\); this setting is somewhat well-known and easy enough to visualize. We also mention recent interest in standing wave solutions to (1.1) and (1.2) in [CM] and [MS].

1.1. Setting up the auxiliary equations and standing assumptions. If we differentiate (1.3), we get
\begin{equation}
\tag{1.4}
i\partial_t v = e^{i\lambda t}(\lambda u(g(t)x) + iXu(g(t)x))
\end{equation}
where, as mentioned before, \(X\) is a Killing field flowing by \(g(t)\). Thus, (1.1) holds iff
\begin{equation}
\tag{1.5}
-\Delta u + \lambda u - iXu = K|u|^{p-1}u
\end{equation}
Similarly, differentiating (1.2) gives us
\begin{equation}
\tag{1.6}
\partial^2_t v = e^{i\lambda t}(-\lambda^2 u(g(t)x) + 2i\lambda Xu(g(t)x) + X^2 u(g(t)x))
\end{equation}
Thus, (1.2) holds iff
\begin{equation}
\tag{1.7}
-\Delta u + (m^2 - \lambda^2)u + X^2 u + 2i\lambda X u = K|u|^{p-1}u
\end{equation}
We assume that the Killing field \(X\) is bounded, that is,
\begin{equation}
\tag{1.8}
\langle X, X \rangle \leq b^2 < \infty
\end{equation}
We already know that \(\Delta\) is self-adjoint, and (1.6) means that \(iX\) is a relatively bounded perturbation of \(\Delta\), which in turn means that \(-\Delta - iX\) is self-adjoint. Since the only eigenvalues of a self-adjoint operator will be real, we can assume
\begin{equation}
\tag{1.9}
\text{Spec}(-\Delta - iX) \subset [\alpha, \infty)
\end{equation}
As long as we are concentrating on compact manifolds, (1.6) is not an obstruction. We will also find the opportunity to say something about non-compact manifolds which have such bounded Killing fields (in general, non-compact manifolds do not have to have bounded Killing fields, hyperbolic spaces providing counterexamples: this can be realized by looking at the isometry group of the hyperbolic space and explicitly noting down the Killing fields.

Remark 1.1. Comparing (1.4) and (1.5) we can now justify our bias (as mentioned in the abstract) towards the NLKG, which is harder to study in the first place because of the presence of the second order operator \(X^2\).

\footnote{See [K], Chapter 4 for relevant definitions}
Depending on the length of $X$, $-\Delta + X^2$ may be elliptic, subelliptic\footnote{A self-adjoint second order differential operator $L$ is called subelliptic of order $\varepsilon$ ($0 < \varepsilon < 1$) at $x \in M$ if there is a neighbourhood $U$ of $x$ such that $||u||_{H^{\varepsilon}} \leq C(||Lu, u|| + ||u||^2) \forall u \in C_0^{\infty}(U)$, see [F] for more details.} or may even have negative eigenvalues. This is what makes the analysis interesting. It is worthwhile to mention at this point that from the point of view of standing wave solutions, the auxiliary equations for both NLS and NLKG look similar, so there is no difference in the respective analyses.

By a similar logic as above, we see that $\langle X, X \rangle \leq b^2 < 1$ implies that $-\Delta + X^2$ is a strongly elliptic nonnegative semidefinite self-adjoint operator and $2i\lambda X$ is a relatively bounded perturbation. Again we assume

\begin{equation}
\text{Spec}(-\Delta + X^2 + 2i\lambda X) \subset [\beta(\lambda), \infty)
\end{equation}

Finally, to fix notations, we define,

\begin{equation}
F_{\lambda,X}(u) = (-\Delta u - iXu + \lambda u, u)
\end{equation}

and

\begin{equation}
F_{m,\lambda,X}(u) = (-\Delta u + X^2 u + 2i\lambda Xu + (m^2 - \lambda^2)u, u)
\end{equation}

We also define

\begin{equation}
E_{\lambda,X}(u) = \frac{1}{2}(-\Delta u - iXu, u) - \frac{1}{p+1} \int_M |u|^{p+1}dV
\end{equation}

and

\begin{equation}
E_{\lambda,X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda Xu, u) - \frac{1}{p+1} \int_M |u|^{p+1}dM
\end{equation}

1.2. Outline of the paper. We now take the space to describe the overall outline of the paper. In Section 2, we establish the existence of certain constrained energy minimisers, i.e., we minimise the “energies” $E_{\lambda,X}(u)$ and $E_{\lambda,X}(u)$ subject to the “mass” $||u||_{L^2}$ being constant, and then use usual variational arguments to see that these constrained minimisers actually give solutions to (1.4) and (1.5). Our aim is, as discussed, to get “travelling wave” solutions, that is, solutions $u$ such that $Xu \neq 0$ identically. However, a solution to the constrained minimisation problem does not need to satisfy this; in fact, there can even be constant solutions. This concern is taken up in Section 3, where it is shown that on fairly general spaces and for at least a non-empty set of parameters $\lambda$ and $m$, we have honest travelling wave solutions to (1.4) and (1.5).

In Section 4, we extend the analysis on $S^2$ done in [TT] to a sphere of arbitrary dimension along absolutely similar lines of reasoning. We improve on an estimate on $S^2$ given in [TT] and show that our estimate is sharp. In Section 5, we look at how the existence of a contact structure interacts with our sub-Laplacian obtained in (1.5). We investigate, in particular, the sphere $S^7$ with reference to its contact structure.
So far in the paper, we have looked only at subsonic or sonic waves, in the sense that the Killing field $X$ has length less than or equal to 1. However, when $X$ has unrestricted length, we establish existence of constrained energy minimisers giving solutions to (1.4) and (1.5) in Section 6. Here we are very brief and only sketch the main lines of argument.

In Section 7, we look at (1.4) and (1.5) in the non-compact setting, albeit for nice spaces with radial symmetry and spaces that can be obtained from them by stitching. We establish constrained $F_{m,λ,X}$ minimisers and constrained energy minimisers in this setting. Let us note here that among the two, the latter is somewhat more analytically involved and requires the application of the concentration-compactness principle and an additional symmetry assumption on the manifold to work.

Finally in Section 8, we show that small perturbations of the Killing field (in the non-compact setting) results in small perturbations of the traveling waves themselves, and in Section 9, we make a few comments on the interaction of two power-type nonlinearities.

2. Existence of Energy minimisers

In [T1], it was proved that on compact $M$, with $λ > −α$ and $\text{Spec}(−Δ − iX) ⊂ [α, ∞)$

\[(2.1)\]

\[F_{λ,X}(u) \simeq ||u||_{H^1}^2 \forall u \in H^1_0(M)\]

where $H$ denotes the usual Sobolev spaces and a “0” in the subscript means, as usual, those elements which are compactly supported (the above fact is actually a restatement of elliptic regularity once it is known that $λ$ does not dip below the lowest possible eigenvalue of $−Δ − iX$).

Also, with $⟨X, X⟩ < 1$, $\text{Spec}(−Δ + X^2 + 2iλX) ⊂ [β(λ), ∞)$ and $m^2 > λ^2 − β(λ)$, we have

\[(2.2)\]

\[F_{m,λ,X}(u) \simeq ||u||_{H^1}^2 \forall u \in H^1_0(M)\]

\[(2.1)\] fact was then used to minimise $F_{λ,X}(u)$ over $H^1_0(M)$, subject to the constraint

\[(2.3)\]

\[\int_M |u|^{p+1}dM = \text{constant}\]

Similarly, \[(2.2)\] was used to minimise $F_{m,λ,X}(u)$ over $H^1_0(M)$, subject to the constraint \[(2.3)\], which would then give the solution to (1.5).

Here we take the alternative path of establishing energy minimisers.

Firstly, for NLS, we will try to minimise the “energy”

\[(2.4)\]

\[E_{λ,X}(u) = \frac{1}{2}(−Δu − iXu, u) − \frac{1}{p+1} \int_M |u|^{p+1}dV\]

subject to keeping the “mass” $Q(u) = ||u||^2_{L^2}$ fixed.

The reason for doing this, as discussed before, is the following:

Lemma 2.1. (Energy minimisers imply solutions)
• If $u \in H^1_0(M)$ minimises $E_{\lambda,X}(u)$, subject to $\|u\|^2_{L^2} = \beta$ (constant), then $u$ solves (1.4).

• If $u \in H^1_0(M)$ minimises $E_{\lambda,X}(u)$ subject to keeping the “mass” $\|u\|^2_{L^2} = \beta$ (constant), then $u$ solves (1.5).

Proof. On calculation, we can see that

\[
\frac{d}{d\tau} \bigg|_{\tau=0} E_{\lambda,X}(u + \tau v) = \text{Re}(-\Delta u - iXu - |u|^{p-1}u, v)
\]

Also,

\[
\frac{d}{d\tau} \bigg|_{\tau=0} Q(u + \tau v) = 2\text{Re}(u, v)
\]

So, if $u \in H^1(M)$ minimises $E_{\lambda,X}$ constrained by $Q(u) = \text{constant}$, then,

\[
(2.7) \quad \forall \in H^1(M), \text{Re}(u, v) = 0 \implies \text{Re}(-\Delta u - iXu - |u|^{p-1}u, v) = 0
\]

and so, there exists a $\lambda \in \mathbb{R}$ such that $\Delta u + iXu + |u|^{p-1}u = \lambda u$

which means, $-\Delta u + \lambda u - iXu = |u|^{p-1}u$.

Similarly, for the NLKG, we have

\[
\frac{d}{d\tau} \bigg|_{\tau=0} E_{\lambda,X}(u + \tau v) = \text{Re}(-\Delta u + 2i\lambda X u + X^2 u - |u|^{p-1}u, v)
\]

and

\[
\frac{d}{d\tau} \bigg|_{\tau=0} Q(u + \tau v) = 2\text{Re}(u, v)
\]

By a similar duality argument, we see that there exist constants $m, \lambda \in \mathbb{R}$ such that

\[
(2.10) \quad -\Delta u + Xu + 2i\lambda Xu + (m^2 - \lambda^2)u = |u|^{p-1}u
\]

So far we have argued that “mass” constrained “energy” minimisers would indeed give solutions to (1.4) and (1.5). Now we have to establish the existence of such constrained energy minimisers.

**Proposition 2.2. (Existence of energy minimisers)** If $p \in (1, 1 + 4/n)$ we can find minimisers for $E_{\lambda,X}$ in case of the NLS and $E_{\lambda,X}$ in case on the NLKG on compact manifolds, when the minimization is done in the class of $H^1_0(M)$ functions having constant $L^2$-norm. The respective minimisers then give solutions to (1.4) and (1.5).

Proof. For the NLS, let us define

\[
I_\beta = \inf\{E_{\lambda,X}|u \in H^1(M), Q(u) = \beta\}
\]

Let us recall the Gagliardo-Nirenberg inequality:

\[
(2.12) \quad \|u\|_{L^{p+1}} \leq C\|u\|_1^{\gamma}\|u\|_{H^1}^{\gamma}
\]
where $\gamma = \frac{n}{2} - \frac{m}{p+1}$ and hence $\gamma(p + 1) < 2$.

Now, we have,

$$F_{\lambda,X}(u) = \left(-\Delta u - iXu + \lambda u, u\right) \approx \frac{1}{2} \left(-\Delta u + \left(\frac{m^2 - \lambda^2}{2}\right)u\right)$$

$$= \frac{1}{2}(-\Delta u - iXu, u) - \frac{1}{p + 1} \int_M |u|^{p+1}dM + \frac{1}{p + 1} \int_M |u|^{p+1}dM + \frac{1}{2} \left(\lambda u, u\right)$$

$$= E_{\lambda,X}(u) + \frac{1}{p + 1} \int_M |u|^{p+1}dM + \frac{1}{2} \lambda Q(u)$$

$$\leq E_{\lambda,X}(u) + C Q(u)(p+1)\left(\frac{1 - \gamma}{2}\right) u_{H^1}^{(p+1)} + \frac{1}{2} \lambda Q(u)$$

This derivation implies two things:

If $Q(u) = \beta$ is constant, then $I_\beta > -\infty$. Also, since $\gamma(p + 1) < 2$, as $Q(u)$ remains fixed and $u_\nu$ is a sequence in $H^1(M)$ such that $E_{\lambda,X}(u_\nu) \to I_\beta$, then $||u||_{H^1}$ remains bounded. This is because, $F_{\lambda,X} \approx ||u||_{H^1}^2$.

Similarly, for NLKG, we have

$$||u||_{H^1}^2 \approx F_{m,\lambda,X}(u)$$

$$= (-\Delta u + X^2 u + 2i\lambda Xu + \left((m^2 - \lambda^2)u\right), u)$$

$$= (-\Delta u + X^2 u + 2i\lambda Xu, u) - \frac{2}{p + 1} \int_M |u|^{p+1}dM + \frac{2}{p + 1} \int_M |u|^{p+1}dM$$

$$+ \left((m^2 - \lambda^2)u, u\right)$$

$$= 2E_{\lambda,X}(u) + \frac{2}{p + 1} \int_M |u|^{p+1}dM + (m^2 - \lambda^2)Q(u)$$

$$\leq 2E_{\lambda,X}(u) + C Q(u)\left(\frac{p+1}{2}(1 - \gamma)\right) u_{H^1}^{(p+1)} + (m^2 - \lambda^2)Q(u)$$

$$= 2E_{\lambda,X}(u) + K ||u||_{H^1}^{(p+1)} + K'$$

where $K$ and $K'$ are constants. So, as before, if $u_\nu \in H^1(M)$ is a sequence satisfying $E_{\lambda,X} \leq I_\beta + \frac{1}{p_0}$, then $||u||_{H^1(M)}$ must be bounded. Also, $I_\beta > -\infty$.

So, in both cases, passing to a subsequence if need be,

$$u_\nu \to u$$

weak* in $H^1(M)$.

Now, by Rellich’s theorem, $u_\nu$ has a convergent subsequence, called $u_\nu$ again, converging in $L^2$-norm, and by the Gagliardo-Nirenberg inequality, in $L^{p+1}$-norm, and the $L^2$ limit is $u$.

So, by triangle inequality, $||u||_{L^2} = \beta$.

\footnote{For this inequality to work, we need to make the technical assumption $p \in (1, 1 + \frac{4}{n})$. Also see Remark 2.2 on page 6.}
Now to prove that \( u \) attains the infimum \( I_\beta \), that is,
\[
E_{\lambda,X}(u) = I_\beta
\]
It suffices to prove that \( E_{\lambda,X}(u_\nu) \to E_{\lambda,X}(u) \), and from our previous calculation, this reduces to proving that \( ||u_\nu||_{L^2}^2 \to ||u||_{L^2}^2 \) and \( ||u_\nu||_{L^{p+1}} \to ||u||_{L^{p+1}} \). These have already been established. A similar consideration goes through for the NLKG.

**Remark 2.3.** It is to be noted, however, that in [T1], existence of the solution was established for a larger range of \( p \), namely, \( p \in \left(1, \frac{n+2}{n-2}\right) \). The energy minimization method guarantees solutions for a smaller range, namely, \( p \in \left(1, 1 + \frac{4}{n}\right) \). This is the optimal range of \( p \) that makes the foregoing calculations with the Gagliardo-Nirenberg inequality work.

### 3. Nontriviality of solutions and a few other remarks

We must note that the mere existence of minimisers will not guarantee waves that are actually “travelling”. For example, when \( M \) is without boundary,
\[
u = [(m^2 - \lambda^2)/K]^{1/p}
\]
solves both (1.4) and (1.5) and it is natural to ask if this is a minimiser.

**3.1. Nontriviality on the sphere and torus: discussion.** This problem is discussed for the NLKG on \( S^n \) with \( \lambda = 0 \) and \( m > 0 \) in [T1]. Let us first sketch the main lines of argument as appear there:

**Step 1** Let, as before, \( u \in H^1(S^n) \) minimise \( F_{m,0,X}(u) \), subject to (2.3), so \( u \) solves
\[
-\Delta u + X^2 u + m^2 u = K_0 ||u||^{p-1} u
\]
First off, it is proved that if \( u \) is constant on each orbit of \( X \), or equivalently, \( Xu = 0 \), then \( u \) is actually constant.

**Step 2** Then, the metric on \( S^n \) is scaled, with \( S^n_r \) denoting the sphere with distance magnified by a factor of \( r \). Picking a point \( o \) on \( S^n \), it is observed that as \( r \to \infty \), \( S^n_r \) approaches flat Euclidean space \( \mathbb{R}^n \), whilst the Laplacian approaches the flat Laplacian. Now, if \( u^r \in H^1(S^n_r) \) denotes a minimiser of
\[
F^r_{m,0,X}(u) = ((-\Delta_r + X^2 + m^2)u, u)_{L^2(S^n_r)}
\]
subject to the constraint
\[
I^r_p(u) = \int_{S^n_r} |u|^{p+1} dV = \text{constant}
\]
(3.1)
and $u_r$ is a constant on each orbit of $X_r$, then $u_r$ is constant on $S^n_r$.
That means,
\[
F_{m,0,X}^r(u^r) = m^2|u^r|^2 A_n r^n
= m^2 A^{1/(p+1)} (A_n r^n)^{p/(p+1)}
\]
which is also the infimum of $F_{m,0,0}^r$, as $X^r u^r = 0$.

Step 3 Contradiction then comes from the fact that we know that for $n \geq 2$, there is a minimiser $u^0 \in H^1(\mathbb{R}^n)$ to $F_{m,0,0}^0(u) = ((-\Delta u + m^2)u, u)_{L^2(\mathbb{R}^n)}$ subject to (3.1) (see (3.1) below). However, in the above calculation, as $r \to \infty$, $F_{m,0,0}^r(u^r)$ blows up.

To complete the above discussion, we quote the following
Lemma 3.1. (Global constrained minimiser of $((-\Delta u + m^2)u, u)_{L^2(\mathbb{R}^n)}$)

Given
\[
(3.2) \quad n \geq 2, \quad p \in \left(1, \frac{n+2}{n-2}\right), \quad A \in (0, \infty)
\]
there is a minimiser $u^0 \in H^1(\mathbb{R}^n)$ to $F_m(u) = ((-\Delta + m^2)u, u)_{L^2(\mathbb{R}^n)}$ subject to the constraint $I_p^0(u^0) = A$.

Proof. For the proof, refer to [11], which is essentially a reworking of an argument in [10]. We just want to point out the following important fact: the proof also establishes that we can arrange so that the constrained minimiser $u^0$ is a radial function. \hfill \Box

We will use this scheme of proof to extend non-triviality of solutions to a larger class of manifolds, at least when $\lambda = 0$. Consider the $n$-dimensional torus given by $T = \mathbb{R}^n/(x \mapsto x + 1, y \mapsto y + 1)$.

We will now scale the metric up. Let $T_k$ denote the torus given by $\mathbb{R}^n/(x \mapsto x + k, y \mapsto y + k)$. Then, $\text{Vol}(T_k) = k^n$. We set $\varepsilon = 1/k$ and let $\Delta_{\varepsilon}$ denote the Laplacian on $T_k$. Also, given a Killing field $X$ on $T$ satisfying $\langle X, X \rangle < 1$, let $X_{\varepsilon}$ denote the corresponding Killing field on $T_k$.

Now, consider a minimiser $u^\varepsilon$ of $F_{m,0,X}^\varepsilon(u) = ((-\Delta_{\varepsilon} + X_{\varepsilon}^2 + m^2)u, u)_{T_k}$ subject to the constraint $\int_{T_k} |u|^{p+1} dV = A$. Now, we can prove that if for $\varepsilon > 0$, $X_{\varepsilon} u^\varepsilon = 0$, then $u^\varepsilon$ must be constant on $T_k$ (let us believe this for now; a proof of this fact is given in greater generality inside the proof of Proposition 3.3 below). Taking this for granted, we have
\[
X_{\varepsilon} u^\varepsilon = 0 \implies u^\varepsilon = \left(\frac{A}{k^n}\right)^{1/(p+1)}
\]
which makes
\[
F_{m,0,X}^\varepsilon(u) = m^2 A^{1/(p+1)} k^{np/(p+1)}
\]
This blows up as $k \to \infty$. Since $X_{\varepsilon} u^\varepsilon = 0$, this must also be the infimum of $F_{m,0,0}^r(u)$ subject to the constraint $\int_{T_k} |u|^{p+1} dV = A$. 

But, as \( k \to \infty \), we should have \( F_{m,0,0}^\varepsilon(u) \to F_{m,0,0}^0(u) \), which has a minimiser (the reason being, there is a big enough cube \( C \) in \( \mathbb{R}^n \) such that \( \int_C |u|^{p+1} = A - \delta \) very closely approximates \( A \). On this cube (with boundary), there is a minimiser belonging in \( H_0 \), with constraint \( \int_C |u|^{p+1} = A - \delta \) and \( F_{m,0,0}^\varepsilon(u) \) has a low value (obtained by restricting the corresponding constrained minimiser from \( \mathbb{R}^n \)), which contradicts the blow-up of \( F_{m,0,0}^\varepsilon(u) \) in the limit). This gives us a non-trivial travelling wave solution of the NLKG on the flat torus.

**Remark 3.2.** Observe that this method can at least be applied without change to all compact surfaces of genus higher than one, which have a locally hyperbolic geometry and also higher dimensional analogues. The existence of constrained minimisers of \( \langle (-\Delta + m^2)u, u \rangle_{L^2(\mathbb{H}^n)} \) in the hyperbolic space is already a known fact. We can just apply the same method outlined above, by increasing the length of the sides of the polygons from which these surfaces are obtained. The process is particularly easy to visualize in the ball model of the hyperbolic space.

### 3.2. General case

**Proposition 3.3.** (Travelling waves on isotropic manifolds) Given a compact isotropic manifold \( M \) of dimension \( n \geq 2 \), \( p \in (1, (n + 2)/(n - 2)] \), \( m > 0 \), \( K > 0 \) and a Killing field \( X \) such that \( \langle X, X \rangle \leq b^2 < 1 \), there exists a \( F_{m,0,X} - \)minimizing solution to the NLKG with \( Xu \neq 0 \).

**Proof.** Recall that isotropic manifolds are those Riemannian manifolds in which “the geometry is the same in every direction”, the formal definition being, given any \( p \in M \) and unit vectors \( v, w \in T_p(M) \), there exists \( \varphi \in \text{Isom}(M) \) such that \( \varphi(p) = p \) and \( d\varphi_p(v) = w \); in fact, the full power of “isotropicity” is not required here. It is enough to guarantee that given a tangent vector at a point, there exist \((n-1)\) isometries mapping that point to itself and mapping that tangent vector to \((n-1)\) linearly independent directions, that is, these \((n-1)\) vectors with the given tangent vector generate the tangent space at that point. This, of course, must happen for all points separately.

We start by observing the fact that \( Xu = 0 \Rightarrow u = \text{constant} \). Let

\[
F_{m,0,X}(u) = \langle (-\Delta + X^2 + m^2)u, u \rangle = ||\nabla u||_{L^2}^2 - ||Xu||_{L^2}^2 + m^2||u||_{L^2}^2
\]

and

\[
F_{m,0,0}(u) = \langle (-\Delta + m^2)u, u \rangle = ||\nabla u||_{L^2}^2 + m^2||u||_{L^2}^2
\]

Then, for all \( u \in H^1(M) \), we have

\[
F_{m,0,X}(u) \leq F_{m,0,0}(u)
\]

Now, if \( u \) is not travelling, that is, \( Xu = 0 \), then \( F_{m,0,X}(u) = F_{m,0,0}(u) \), which means that if \( u \in H^1(M) \) minimises \( F_{m,0,X} \) subject to (2.3), then
$u$ also minimises $F_{m,0,0}$ subject to $u$ also minimises $F_{m,0,0}$ subject to $u$. Now let us consider the function $v(x) = u(\phi(x))$ where $\phi \in \text{Isom}(M)$. We have $F_{m,0,0}(v) = F_{m,0,0}(u)$. But, since

\[ F_{m,0,X}(v) \geq F_{m,0,X}(u) = F_{m,0,0}(u) = F_{m,0,0}(v) \]

we have

\[ F_{m,0,X}(v) = F_{m,0,0}(v) \]

This gives, $Xv = 0$. Since this happens for all $\phi \in \text{Isom}(M)$, we have, by virtue of the isotropicity of the manifold the fact $u$ is constant.

Now, we cover the compact $M$ by finitely many geodesic neighbourhoods $U_1, U_2, \ldots, U_k$ around $x_1, x_2, \ldots, x_k$ respectively. We scale the metric on $M$ as $g_{ij}^r = rg_{ij}$, where $g_{ij}$ gives the metric on $M$. Then consider the partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. Let $u_r$ be the minimiser of $F_{m,0,X}^r$ appearing in the superscript because $F_{m,0,X}$ now varies as the metric varies with $r$ on $(M, r g_{ij})$. Also, let $F_{m,0,X}^{r,i}(u_r) = F_{m,0,X}^r(u_r \bigg|_{U_i})$. If $X_r u_r = 0 \forall r$, then $u_r$ is constant and $u_r$ minimises $F_{m,0,0}^r$ by the aforementioned arguments.

Then,

\[ F_{m,0,X}^r(u_r) = F_{m,0,0}^r(u_r) = \sum_{i=1}^{k} \phi_i F_{m,0,0}^{r,i}(u_r) \]

\[ \approx \sum_{i=1}^{k} \phi_i r^{p+1} F_{m,0,0}^{r,i}(u_r) = r^{p+1} \sum_{i=1}^{k} \phi_i F_{m,0,0}^{r,i}(u_r) \]

So, in the limit, as $r \to \infty$, $F_{m,0,0}^r$ blows up.

Inside each $U_i$, we can find an open ball $V_i$ around $x_i$ such that $V_i \cap U_j = \phi$ when $i \neq j$. At the limit, each of these $V_i$’s would look like the Euclidean space. Now we consider the radial functions $v_i^r = u_{r,i}^p(\text{dist}(a_i, x))$ which give constrained minimisers of $F_{m,0,0}^{r,i}(u)$ on each $V_i$ subject to $A_i = A/k$, where $a_i$ is an “origin” in each $V_i$. We have

\[ (3.3) \quad \int_{V_i} |v_i^r| = F_p^r(v_i^r) = A_i \]

and

\[ (3.4) \quad F_{m,0,0}^p(v_i^r) \to F_{m,0,0}^{0,i}(v_i^r) \]

where the last quantity represents the minimum value of $\langle (-\Delta + m^2)u, u \rangle_{L^2(\mathbb{R}^n)}$ subject to the constraint $\int_{\mathbb{R}^n} |u|^{p+1} = A_i$ and is a finite number from Lemma (3.1). Adding over $i$, this gives us a contradiction.

This means, for some $r$, there exists a minimiser $u^*_r$ of $F_{m,0,X}^r(u)$ subject to $\int_{(M, r g)} |u|^{p+1}$ such that $X_r u^*_r \neq 0$. Scaling back the metric appropriately, we get a minimiser $u^*$ on $(M, g)$ such that $\int_M |u|^{p+1}$ and $X u^* \neq 0$ \qed

**Remark 3.4.** As a verification, we inspect what happens to the scalar curvature as $r \to \infty$. We know that if the conformal change of metric is
given by $g' = u^{4/(n-2)}g$, then the corresponding change in scalar curvature is given by

$$R_{g'} = r^{-\frac{n+2}{4}} r^{\frac{n-2}{4}} R_g$$

Plugging in $u^{4/(n-2)} = r$, here we have

$$R_{g'} = \frac{1}{r} R_g$$

This gives some intuition behind the claim that a ball inside a curved manifold will “tend” towards the Euclidean space if the metric is scaled up sufficiently.

The next natural question is: what about the nontriviality of energy minimisers? Do they necessarily need to be travelling for some parameters? For example, like before,

$$u = \left( \frac{m^2 - \lambda^2}{K_0} \right)^{\frac{1}{p+1}}$$

satisfies the NLKG. The question is: is it an energy minimiser? We will not go into the details of this here. The crux of the discussion is: following previous lines of we can conclude that to attack this problem, one just needs to establish the following:

$$E_{0,0}(u) = \frac{1}{2} (-\Delta u, u) - \frac{1}{p+1} \int_M |u|^{p+1} dM$$

has a minimiser in $H^1(M)$ subject to $||u||_{L^2}^2$ being constant. When $M = \mathbb{R}^n$ or $\mathbb{H}^n$, this has already been established in \cite{CMMT}. This will lead to a corresponding version of the nontriviality statement for energy minimisers.

4. Subelliptic phenomenon on $S^n$

We will now extend the analysis done for $S^2$ in \cite{T1} to a sphere of dimension $n$.

4.1. Setting up the problem. Let $X_{ij}, i < j$, denote the vector field on $S^n$, which generates the rotation about the $ij$-plane leaving the complementary directions fixed. We will first see that the Laplacian of $S^n$ is given by $\Delta = \Sigma_{i<j} X_{ij}^2$.

$$X_{ij} = \frac{1}{\sqrt{2}} (E_{ij} - E_{ji})$$ (where $E_{ij}$ represents the matrix with an entry of 1 at the $ij$-th place and 0’s elsewhere) represent an orthonormal basis in $\frak{so}(n+1)$, with respect to the inner product $\langle A, B \rangle = \text{tr}(AB^t)$. Now, up to multiplication by a constant (which depends only on the dimension $n$), this is the Killing form on $\frak{so}(n+1)$, which gives a bi-invariant metric on $SO(n+1)$. Then the Casimir element $\Delta = \Sigma_{i<j} X_{ij}^2$ descends to the Laplacian on $S^n$, which can be seen as the coset space $SO(n+1)/SO(n)$.

So pick one of these $X_{ij}$'s, say without loss of generality, $X_{12}$, henceforth
called just $X$. It is to be noted that $\langle X, X \rangle < 1$ does not hold here, it equals 1 on the equator $\gamma$ around the $x$-axis. So $L_0$ is not elliptic, but it satisfies Hörmander’s condition for hypoellipticity with loss of one derivative (see [T2]; this also follows from the main theorem of [H]).

Also, by results in [T2] (Chapter XV), we one prove that $L_0 = \Delta - X^2$ is hypoelliptic with loss of a single derivative, which means the following:

\[ D\phi \in H^{s}_{\text{loc}} \Rightarrow u \in H^{s+1}_{\text{loc}} \]

Now, it is clear that $D(L_0) \subseteq H^1(S^n) \Rightarrow D((-L_0)^{1/2}) \subseteq H^{1/2}(S^n)$

and

\[ L_0 u, u \geq C ||u||_{H^{1/2}} \]

when $u$ is orthogonal to the constants, the justification of the last statement being:

\[ (-L_0)^{-1/2} : L^2 \to H^{1/2} \text{ when } \int u = 0 \text{ and if } (-L_0)^{1/2} u = v, \text{ then } \]

\[ ||(-L_0)^{-1/2} v||_{H^{1/2}} \leq C ||v|| \]

(we take this space to comment that there is no parallel of this phenomenon in the Euclidean setting. For example, if on $\mathbb{R}^n$ we select the Killing field $\frac{\partial}{\partial x_1}$, $\Delta - X^2$ is never hypoelliptic. This is also due to a result of Hörmander (for details, see [Y]). It seems that some curvature is necessary for subelliptic phenomena).

Now, if we let $L_\alpha = L_0 - i\alpha X$, we see clearly that $L_\alpha$ is self-adjoint. However, we need to establish the positive semidefiniteness of $L_\alpha$ for a certain range of $\alpha$ and find out what the range is if possible. In other words, to proceed, we must find an analogue of (4.2) for the operator $L_\alpha$. We actually have

**Lemma 4.1.** $L_\alpha = \Delta - X^2 - i\alpha X$ is positive semidefinite for $|\alpha| < n - 1$.

**Proof.** To start, we can do an eigenvector decomposition of $L^2(S^n)$ with respect to the self-adjoint $\Delta$ and since $X$ is Killing, it commutes with $\Delta$ and preserves its eigenspaces. Alternatively, we can use the well-known fact (see [Z], for instance) that the decomposition of $L^2(S^n)$ into irreducible components due to the natural $SO(n+1)$ action is the same as the eigenvalue decomposition of $L^2(S^n)$ with respect to $\Delta$. So, to establish the positive semidefiniteness of $L_\alpha$, we are content with looking at the eigenvalues of $X$ on each eigenspace of $\Delta$.

Let $V_k$ denote the space of degree $k$ harmonic polynomials defined on $\mathbb{R}^{n+1}$ restricted to $S^n$. It is known that all the eigenfunctions of the Laplacian on $\mathbb{R}^{n+1}$ restricted to $S^n$ is positive semidefinite for $|\alpha| < n - 1$. Just to be clear, a (pseudo)differential operator $D$ of order 2 defined on an open set $U$ is said to be hypoelliptic when for all distributions $\phi$,

\[ D\phi \in C^\infty \Rightarrow \phi \in C^\infty \]
$S^n$ are given by the members of $V_k$. The eigenvalue corresponding to $V_k$ is $k(k + n - 1)$. We know that $V_k$ is generated by polynomials of the form $P(x) = (c_1 x_1 + \ldots + c_{n+1} x_{n+1})^k$, where $c_i \in \mathbb{C}$ and $\Sigma c_i^2 = 0$. Now it is clear that the modulus of the eigenvalues of $X$ on $V_k$ are less than or equal to $k$. Also, calculation shows that $(c_1 x_1 + c_2 x_2)^k$ gives an eigenfunction of $X$ with eigenvalue $ik$. This finally implies that $L_\alpha$ is positive semidefinite with one dimensional kernel when $|\alpha| \leq n - 1$. □

Finally, for the existence result: to wit, let $q_*$ be the optimal number such that

$$D((-L_0)^{1/2}) \subset L^q(S^2), \forall q \in [2, q_*]$$

and hence

$$D((-L_0)^{1/2}) \hookrightarrow L^q(S^2) \text{ is compact}, \forall q \in [2, q_*]$$

Then we have

**Proposition 4.2. (Existence result on $S^n$)** With $X$ as above, assume

(4.3) \hspace{1cm} 2 < p + 1 < q_*

Also assume

(4.4) \hspace{1cm} |\lambda| < \frac{n - 1}{2}, m^2 > \lambda^2

Then, given $K > 0$, the equation

(4.5) \hspace{1cm} -L_2 \lambda u + (m^2 - \lambda^2)u = K|u|^{p-1}u

has a nonzero solution $u \in D((-L_0)^{1/2})$.

**Proof.** We know, under the above hypothesis,

(4.6) \hspace{1cm} F_{m, \lambda, X} \approx ||u||^2_{D((-L_0)^{1/2})}

For $p$ such that $D((-L_0)^{1/2}) \subset D^{p+1}(S^n)$, we can pick any positive number $\beta$ and minimise $F_{m, \lambda, X}$ under the constraint

(4.7) \hspace{1cm} \int_{S^n} |u|^{p+1}dS = \beta

The constrained minimiser will give a solution, as wanted. □

4.2. **What is the optimal $q_*$?** On $S^n$, $H^{1/2}$ embeds in $L^{\frac{2n}{n-1}}$. My mimicking the calculations in [\text{II}], we now try to see if this can be improved. By using the ellipticity of $\Delta$ away from $\gamma$,

(4.8) \hspace{1cm} u \in D((-L_0)^{1/2}) \Rightarrow \phi u \in H^1(S^n) \subseteq L^{\frac{2n}{n-2}}

by Sobolev embedding, where $\phi$ is any smooth function with support disjoint from $\gamma$.

We can prove that

(4.9) \hspace{1cm} D((-L_0)^{1/2}) = \{u \in L^2(S^n) : X_{ij} u \in L^2(S^n), X_{ij} \neq X\}
With that in place, we take a neighbourhood around the “north pole” and project it down to $\mathbb{R}^n$. This produces a compactly supported $u$ such that

$$u \in H^{1/2}(\mathbb{R}^n), \quad \partial_{x_i} u \in L^2(\mathbb{R}^n) \, \forall i \in \{2, 3, \ldots, n\}$$

where the tangent to the equator $\gamma$ gets mapped to the $x_1$-axis.

Now, observe that (4.10) implies, after Fourier transforming, $$(\xi_1^2 + \xi_2^4 + \ldots + \xi_n^4)^{1/4} \hat{u} \in L^2(\mathbb{R}^n) \quad \text{and} \quad \xi_i \hat{u} \in L^2(\mathbb{R}^n) \, \forall i \in \{2, 3, \ldots, n\}.$$ That means

$$\hat{f} = (\xi_1^2 + \xi_2^4 + \ldots + \xi_n^4)^{1/4} \hat{u} \in L^2(\mathbb{R}^n)$$

Labelling $u = k \ast f$, where

$$\hat{k} = (\xi_1^2 + \xi_2^4 + \ldots + \xi_n^4)^{-1/4}$$

we have that on $\mathbb{R}^n$, $\hat{k} \in S_{1/2,0}^{-1/2}(\mathbb{R}^n)$, where $S_{\rho,\delta}^m$ denotes the usual pseudo-differential symbol class. This means that

$$k \in C^\infty(\mathbb{R}^n \setminus 0)$$

Also, $k$ satisfies the anisotropic homogeneity

$$k(\delta^2 x_1, \delta x_2, \ldots, \delta x_n) = \delta^{-2} k(x_1, x_2, \ldots, x_n)$$

Define $\Omega_0 = \{(x_1, \ldots, x_n) : 1/2 \leq |x|^2 < 1\}$ and define $\Omega_j$ inductively as the image of $\Omega_{j-1}$ under the map

$$(x_1, x_2, \ldots, x_n) \mapsto (2^{-1} x_1, 2^{-1/2} x_2, \ldots, 2^{-1/2} x_n)$$

On calculation, we have

$$|k| \leq C 2^j \text{ on } \Omega_j$$

$$\text{Vol } \Omega_j = 2^{-((2n-1)/2)} \text{Vol } \Omega_{j-1} = C 2^{-((2n-1)/2)j}$$

Note that since $u$ has compact support, and $u = k \ast f$, in the preceding and ensuing calculations, we will say that $j \geq 0$ without any loss of generality. So then

$$\int |k| r^dV \leq C \sum_{j \geq 0} 2^j r^{-(2n-1)/2)j} < \infty$$

when $r < 2n-1$. Now by using interpolation, we have, $u \in L^r$, where $r < 2(2n - 1)$.

4.3. What about the boundary case $r = 2(2n - 1)$? Let us investigate this for the special case of $n = 2$. That is, our setting is now the sphere $S^2$.

We already have (also c.f. [T1])

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^2), \quad \forall q \in [2, 6)$$

Here we extend the above result up to $q = 6$ and also argue that this is actually sharp. Observe that in view of the inclusion $L^{p_1} \subset L^{q_1}$ when $q_1 \leq p_1$ on finite measure spaces, just proving the following is sufficient:
**Lemma 4.3. (Optimal embedding and sharpness)**

\[ \mathcal{D}((-L_0)^{1/2}) \hookrightarrow L^6(S^2) \]

Also, this embedding is sharp. That is,

\[ \mathcal{D}((-L_0)^{1/2}) \subset L^q(S^2) \implies q \leq 6 \]

**Proof.** We start by observing that

\[\mathcal{D}((-L_0)^{1/2}) = \{ u \in L^2(S^2) : Y u, Zu \in L^2(S^2) \}\]

Also, ellipticity away from the equator \( \gamma \) implies that

\[ u \in \mathcal{D}((-L_0)^{1/2}) \implies \varphi u \in H^1(S^2) \]

which embeds in \( L^\infty(S^2) \) (here \( \varphi \) is a smooth function with support outside \( \gamma \)).

We project a small neighbourhood around the north pole onto \( \mathbb{R}^2 \) such that \( \gamma \) gets mapped onto the x-axis. It is known that \( u \in H^{1/2}(\mathbb{R}^2) \) and \( \partial_y u \in L^2 \). Since \( u \) is compactly supported, \( \partial_y u \in L^2 \implies x \partial_y u \in L^2 \), which, coupled with the last fact, implies, \( y \partial_x u \in L^2 \).

We will use the first two pieces of data, namely, \( u \in H^{1/2}(\mathbb{R}^2) \) and \( \partial_y \sigma \). Observe that this means

\[ u \in H^{1/2}_{x}(L^2_y) \]

where the \( x \) and \( y \) subscripts denote \( x \) and \( y \) dependence respectively, and \( X(Y) \) means \( Y \)-valued functions with the mixed Lebesgue norm.

(For justification, see [LM], Section 2.1)

Also observe that the given data implies,

\[ u \in L^2_x(H^1_y) \]

This actually means

\[ \| u \|_{H^{1/2}_{x}(L^2_y)} < \infty \iff \| (1 + \eta^2)^{1/2} \hat{u}^y \|_{L^2_x} < \infty \]

where \( \hat{u}^y \) represents Fourier transform with respect to \( y \), that is, \( \hat{u}^y \) is now a function of \( x \) and \( \eta \). Now

\[ \| u \|_{H^{1/2}_{x}(L^2_y)} = \| (1 + \eta^2)^{1/2} \hat{u}^y \|_{L^2_x} \]

\[ = \| (1 + \eta^2)^{1/2} \hat{u}^y \|_{L^2_x} = \| (1 + \eta^2)^{1/2} \hat{u} \|_{L^2_x} \]

\[ < \infty \]

since \( (1 + \eta^2)^{1/2} \hat{u} \in L^2 \). This implies

\[ u \in H^{1/2}_{x}(L^2_y) \cap L^2_x(H^1_y) \]

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The author learnt this technique from a certain mathoverflow.net post.
Now we propose to use interpolation (LM volume II, Chapter 1 has a
detailed treatment of this sort of spaces and allied results). By our interpo-
lation result, we can say that

\[ u \in H^{1/\theta}_x (L^2_y) \cap L^2_x (H^{1-\theta}_y) \]

This is because, we can treat \( L^2 \) as \( H^0 \), or the zeroth Sobolev space. If we
solve \( 1/2\theta = 1 - \theta \), we get \( \theta = 2/3 \), which implies

\[ u \in H^{1/3}_x (H^{1/3}_y) \]

Now, when we use Sobolev embedding in one dimension, we know t hat
\( H^{1/3} \) embeds in \( L^6 \). That means,
\( u \in L^6_x (L^6_y) \), which implies, \( u \in L^6(\mathbb{R}^2) \).

We will now prove the next part of the lemma: that the estimate of \( u \in L^6 \)
as obtained above is sharp. To do this, let us emulate the scaling trick
as appears in Appendix A of [CMMT]. To restate what we have:
\( u \in H^{1/2}_x (\mathbb{R}^2) \), \( \partial_y u \in L^2(\mathbb{R}^2) \), and \( Zu \in L^2(S^2) \) which means \( (y\partial_x - x\partial_y)u \in L^2(\mathbb{R}^2) \). Since \( u \) has compact support, \( x\partial_y u \in L^2(\mathbb{R}^2) \), which implies
\( y\partial_x u \in L^2(\mathbb{R}^2) \).

Let us define

\[ u(r, \sigma, a, b, x, y) = r^\sigma u(r^a x, r^b y) \]

Then,

\[
||\partial_y u(r, \sigma, a, b)||^2_{L^2} = \int_{\mathbb{R}^2} |\partial_y u(r, \sigma, a, b, x, y)|^2 \, dx \, dy = \int_{\mathbb{R}^2} |\partial_y r^\sigma u(r^a x, r^b y)|^2 \, dx \, dy
\]

\[
= \int_{\mathbb{R}^2} |r^b \partial_z r^\sigma u(z_1, z_2)|^2 r^{-a} r^{-b} \, dz_1 \, dz_2
\]

\[
= r^{b+2\sigma-a} ||\partial_y u||^2_{L^2}
\]

Similarly, we can calculate,

\[
||y\partial_x u(r, \sigma, a, b)||^2_{L^2} = r^{2\sigma+a-3b} ||y\partial_x u||^2_{L^2}
\]

Now,

\[
\hat{u}(r, \sigma, a, b, \xi, \eta) \approx \int_{\mathbb{R}^2} u(r, \sigma, a, b, x, y) e^{-i(\xi x + \eta y)} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^2} r^\sigma u(r^a x, r^b y) e^{-i(\xi x + \eta y)} \, dx \, dy
\]

\[
= \int_{\mathbb{R}^2} r^\sigma u(z_1, z_2) e^{-i(\xi z_1 + \eta z_2)} r^{-a} r^{-b} \, dz_1 \, dz_2
\]

\[
= r^{-a-b} \hat{u}(r^{-a} \xi, r^{-b} \eta)
\]
Finally, to get a contradiction, we just have to take a sequence of \( \delta > \frac{p}{\delta} \)

We will want to compare this estimate with \( \|u\|_{H^{1/2}}^2 = \int_{\mathbb{R}^2} (\xi^2 + \eta^2)^{1/2} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \)
(by definition).

Also, \( \|u(r, \sigma, a, b)\|_{L^p}^p = r^{\sigma_p - a - b} \|u\|_{L^p}^p \)

Now, if \( 6 \) is not sharp, there will exist \( u \) satisfying such that \( u \in L^{6+\varepsilon} \), where \( \varepsilon > 0 \). By observing the above equations, we let \( \sigma = 1 \). Then we see that for \( a = 4 \) and \( b = 2 \) (and calling \( u(r, 1, 4, 2) = u_r \)), we have control on all three of \( \|\partial_y u_r\|_{L^2}, \|y\partial_x u_r\|_{L^2} \) and \( \|u_r\|_{H^{1/2}} \leq \|u\|_{H^{1/2}} \).

However, \( \|u_r\|_{L^{6+\varepsilon}} = r^\varepsilon \|u\|_{L^{6+\varepsilon}} \).
Clearly, as we let \( r \) increase, the left hand side increases, with a very fast decreasing support, since the support of \( u \) was compact to begin with.

Finally, to get a contradiction, we just have to take a sequence of \( u_r \) for fast increasing \( r \), with disjoint supports, and sum them up.

To be precise, we already have \( \|u_r\|_{L^{6+\varepsilon}} = Kr^\theta \), where \( K \) is a constant and \( \theta \in (0, 1) \). Define a new function \( u^* \) by \( u^* = \Sigma u_{r_n} \), where \( r_n \) is chosen such that \( 2^{n-1} \leq r_n^\theta < 2^n \) and such that all the \( u_{r_n} \) have disjoint support.
That way, we still preserve control over \( \|\partial_y u^*\|_{L^2}, \|y\partial_x u^*\|_{L^2} \) and \( \|u^*\|_{H^{1/2}} \), but the \( L^{6+\varepsilon} \)-norm of \( u^* \) blows up, contrary to our assumption.

4.4. Higher regularity for \( S^2 \). It has already been shown that
\[
 u \in \mathcal{D}((-L_0)^{1/2}) \Rightarrow u \in L^6
\]

Now if \( u \) is a solution of the NLKG, then we can do better. A specific case \((p = 3)\) has been worked out in [11] and it has been shown that \( u \) is then smooth. Now, if \( p \) is not an odd integer, we cannot expect a similar smoothness, because the nonlinearity of \([15]\) itself is then not smooth. However, we can expect higher Sobolev spaces and, in turn, higher \( L^r \) spaces for \( u \) when \( p \) is not an odd integer.

A demonstration: We calculate one such explicit case, namely, \( p = 4 \).
A word is in order regarding this choice. Firstly, as before, we have \( H = -L_{2n} + (m^2 - \lambda^2) \) and \( F(u) = K|u|^{p-1}u \). Let \( p < 6 \).
Then, \( F(u) \in L^{6/p} \).

By calculation, \((L^{6/p})^* = L^{\frac{6}{p-6}} \).
By using the Sobolev embedding theorem, we can find a \( \delta \) such that \( H^\delta \subseteq L^{\frac{6}{p-6}} \). On calculation, this happens when \( \delta > p/3 - 1 \). So, by duality
\[
 F(u) \in L^{6/p} \subseteq \bigcap_{\delta > p/3 - 1} H^{-\delta}
\]
which means
\[ u = P^{-1}(F(u)) \subseteq \bigcap_{\delta > p/3 - 1} H^{1 - \delta} \]

Note that we already know that \( u \in H^{1/2} \). So this bootstrapping process yields something better than what we started with only when \( \delta < 1/2 \), or equivalently, \( p < 9/2 \). So for an explicit demonstration we have chosen \( p = 4 \).

When \( p = 4 \), according to previous calculation,
\begin{equation}
(4.13) \quad u = P^{-1}(F(u)) \subseteq \bigcap_{\delta > 1/3} H^{1 - \delta} = \bigcap_{\varepsilon > 0} H^{2/3 - \varepsilon}
\end{equation}

Clearly, ellipticity away from \( \gamma \) means \( u \in H^1 \). So, choose neighbourhoods around the “north pole” of the 2-sphere in the following manner: \( U, V \) and \( W \) are open neighbourhoods such that \( V \subseteq \bar{V} \subseteq W \subseteq \bar{W} \subseteq U \). Also choose a smooth “bump function” \( \phi \) such that \( \text{supp}\phi \subseteq \bar{W} \) and \( \phi \equiv 1 \) on \( \bar{V} \). Now, \( \phi u \) satisfies NLKG inside \( V \), so \( u \in \bigcap_{\varepsilon > 0} H^{2/3 - \varepsilon} \).

Projecting down \( U \) on the plane, we also see that \( \partial_y u \in L^2 \) and \( u \) has compact support. This implies, by the interpolation procedure on mixed Sobolev spaces carried out before, \( u \in L^r \) when \( r < 10 \). This, in turn, implies that \( u \in \bigcap_{\varepsilon > 0} H^{4/5 - \varepsilon} \). This is a gain in regularity.

5. Positive semidefinite sub-Laplacian vis-a-vis contact structures

When \( \lambda = 0 \), we can extend the analysis done till now to a much larger class of manifolds to get an existential result for the NLKG. These are the class of so-called K-contact manifolds with an associated metric \( g \). To recall the definitions:

**Definition 5.1.** A contact manifold \((M^{2n+1}, \eta)\), with characteristic or Reeb vector field \( \xi \) has an associated Riemannian metric \( g \) if \( \eta(X) = g(X, \xi) \) and there exists a tensor field of type \((1, 1)\) such that \( \phi^2 = -I + \eta \otimes \xi \), \( d\eta(X, Y) = g(X, \phi Y) \). This is called a contact metric structure. A contact metric structure \((M, \phi, \xi, \eta, g)\) is called a K-contact structure if the Reeb vector field \( \xi \) is a Killing field (with respect to \( g \)).

Examples of K-contact manifolds abound in literature (see [1]). This includes in particular the Sasakian manifolds, and more particularly, the odd dimensional unit spheres (the word “unit” is important, for otherwise the metric from Euclidean space will not be an associated one).

Under this setting, let us consider the Reeb vector field, say \( X \), on \((M, \phi, X, \eta, g)\) and consider \( L_0 = \Delta - X^2 \), where \( \Delta \) is the Laplace-Beltrami operator on \( M \). Since \( \eta(X) = 1 \), we see that \( X \) has unit length throughout. It has been
shown by [BD], using a result of Radkevic, that $L_0$ is subelliptic of order $1/2$. So by a result of Kohn-Nirenberg, we can say that the sublaplacian $L_0$ is hypoelliptic with the loss of a single derivative (also from the subelliptic estimates it is clear that $L_0$ has discrete spectrum).

Now, we can either make the technical assumption that $L_\alpha = L_0 - i\alpha X$ is positive semidefinite, as is the case with the spheres (clear from spherical harmonics), or we can treat travelling wave solutions of NLKG with $\lambda = 0$. Under this assumption, using arguments quite similar to what has gone before, we can then give an existential statement of the NLKG on K-contact manifolds with an associated metric.

However, as we show below, in specific cases of manifolds with locally contact structures where the spectrum can be evaluated explicitly, we can say more.

5.1. **Estimates on $S^7$.** We start by noting that [11] already deals with the subelliptic phenomenon on $S^3$. To recap the salient features, the Laplacian on $S^3$ was written as

\[ \Delta = X^2 + Y^2 + Z^2 \]  

where $X, Y$ and $Z$ are the left-invariant vector fields on $SO(3)$ that generate $2\pi$ periodic rotations of $\mathbb{R}^3$ about the respective axes. Note that here we have

\[ \langle X, X \rangle = \langle Y, Y \rangle = \langle Z, Z \rangle = 1 \text{ on } S^3 \]

so a solution to the NLKG under these conditions can be called sonic wave solutions. Then, in line with the argument made for $S^2$, an existence result for the (1.5) is formulated under certain restrictions on $\lambda, m$ and $2 < p + 1 < q_*$.

Here we will try to give an analysis on $S^7$ along similar lines. Note that the above analysis on $S^3$ depends heavily on the fact that it can be given a Lie group structure. As is well known, $S^1$ and $S^3$ are the only such spheres. However, $S^7$ has many homogeneous space-like properties$^6$, one of them being: $S^7$ has a global orthonormal frame of Killing fields which do not commute. This allows us to write the Laplacian on $S^7$ as $\Delta = X_1^2 + X_2^2 + \ldots + X_7^2$ where $X_i$ generate the global orthonormal Killing frame.

In fact, the existence of global Killing frames is a stronger condition than being able to write the Laplacian as a sum of squares in local coordinates; for details, see [DN]. They show, among other things, that $S^1, S^3$ and $S^7$ are the only spheres that possess global orthonormal Killing frames.

So, our existence result is:

**Proposition 5.2. (Existence result on $S^7$)** With $X$ as above, assume

\[ 2 < p + 1 < q_* \]

$^6$The reason being, $S^7$ has a similar connection with the octonions as has $S^3$ with the quaternions
Also assume
\[ |\lambda| < \frac{1}{2} m^2 > \lambda^2 \]

Then, given \( K > 0 \), the equation
\[-L_2 \lambda u + (m^2 - \lambda^2)u = K|u|^{p-1}u \]
has a nonzero solution \( u \in D((-L_0)^{1/2}) \).

**Proof.** We apply Hörmander's sum of squares (see [H]) on \( L_0 = \Delta - X_1^2 \) where \( X = X_1 \) without loss of generality, to conclude that it is hypoelliptic with the loss of one derivative. We quickly write down the conclusions:
\[ D(L_0) \subset H^{1/2}(S^7) \]
\[ -(L_0 u, u) \geq C||u||^2_{H^{1/2}} \text{ if } \int_{S^7} udV = 0 \]
\[ -(L_0 u, u) \geq C_\alpha||u||^2_{H^{1/2}} \text{ if } \int_{S^7} udV = 0 \]

Also, positive semidefiniteness of \( L_\alpha \) for \( \alpha < 1 \) follows from spherical harmonics and the particular form of \( X \). Following the line of argument in the proof of Proposition (4.2), this is all we needed to establish. \( \square \)

**Remark 5.3.** Note that the above proposition is not a special case of Proposition (4.2), because of the very different natures of the sub-Laplacians and merits its own different proof.

### 5.2. Optimal exponent \( q^* \)

For us, the problem is again to determine how high \( q^* \) can be. In this case the Sobolev Embedding yields,
\[ H^{1/2}(S^7) \subseteq L^{7/3}(S^7) \]

However, we have the following

**Lemma 5.4.**
\[ H^{1/2}(S^7) \hookrightarrow L^{8/3} \]

Also, this is optimal. That is,
\[ H^{1/2}(S^7) \hookrightarrow L^q \implies q \leq 8/3 \]

**Proof.** We observe that we can localize an open set of \( S^7 \) to an open set in \( \mathbb{H}^7 \) with coordinates \((p, q, t)\) such that
\[ (1-L_0)^{-1/2}u(z) = k(z, \cdot) * u(z) \]

plus lower order terms, where \( \hat{k} \in C^\infty(\mathbb{R}^7 \setminus 0) \) satisfies the following anisotropic homegeneity:
\[ \hat{k}(sx, sy, s^2 \tau) = s^{-1} \hat{k}(x, y, \tau) \]

which means,
\[ k(\lambda p, \lambda q, \lambda^2 t) = \lambda^{-7} k(p, q, t) \]

where \( p \) and \( q \) refer to pairs of real numbers and \((p, q, t) \in \mathbb{H}^7 \). For details on these kinds of calculations, see [T3 and FS]. A brief comment on the
heuristics of this approach: just as a symplectic manifold can be modelled locally by $\mathbb{R}^{2n}$ with the usual symplectic structure, the Heisenberg group also forms a very convenient local model for manifolds with contact structure. Very briefly, a Heisenberg group $H^n$ as a $C^\infty$ manifold is $\mathbb{R}^{2n+1}$. If we denote points in $H^n$ by $(p_j, q_j, t_j)$ with $t \in \mathbb{R}, p_j, q_j \in \mathbb{R}^n$ then define the (non-commutative) group operation on $H^n$ by

$$(p_1, q_1, t_1). (p_2, q_2, t_2) = (p_1 + p_2, q_1 + q_2, t_1 + t_2 + \frac{1}{2} p_1 \cdot q_2 - \frac{1}{2} p_2 \cdot q_1)$$

See also [P].

We now want to see what (5.3) implies. Let us assume without loss of generality that $k(p, q, t) = 1$ when $|p|^2 + |q|^2 + t^2 = 1$

If we let $\Omega_0 = \{(p, q, t) : 1 \leq k(p, q, t) \leq 8\}$ and define $\Omega_k$ inductively as the image of $\Omega_{k-1}$ under the map

$$(p, q, t) \mapsto (2^{-1} p, 2^{-1} q, 2^{-2} t)$$

we see that

$\text{Vol } \Omega_k = 2^{-8} \text{Vol } \Omega_{k-1} = C 2^{-8k}$

Hence, with $B = \cup_{k \geq 0} \Omega_k$, on calculation, we have

$$\int_B |k(p, q, t)|^p dV \leq C \sum_{k \geq 0} 2^{7pk} 2^{-8k} < \infty$$

which means that

$$k \in L^p_{\text{loc}}(\mathbb{H}^7) \text{ for } p < \frac{8}{7}$$

By interpolation we find that $k * u \in L^q_{\text{loc}}(\mathbb{H}^7)$ for $q < \frac{8}{3}$.

We will prove that the above estimate of $p < \frac{8}{7}$ is optimal. The derivation will employ a similar scaling trick as used before, though the calculations will be simpler.

Observe that the integration here takes place on a Heisenberg group, but we note that the ordinary Lebesgue measure on $\mathbb{R}^{2n+1}$ gives the Haar measure on $\mathbb{H}^n$, and the Heisenberg group is unimodular. So we can transform coordinates and repeat similar calculations that we did for $\mathbb{R}^2$.

Define

$$k(r, \sigma, p, q, t) = r k(r^\sigma p, r^\sigma q, r^{2\sigma} t)$$

The right hand side is equal to $r.r^{-7\sigma} k(p, q, t) = r^{(1-7\sigma)} k(p, q, t)$ by anisotropic homogeneity. Also $k(r, \sigma, p, q, t) \in C^\infty(\mathbb{H}^7 \setminus 0)$ for all $r, \sigma$.

Now we calculate the local $L^p$-norm of $k(r, \sigma)$ in two different ways:
Firstly,
\[ ||k(r, \sigma)||_{B(0,r)}^p = \int_{B(0,r)} |k_r^r|^p dV \]
\[ = \int_{B(0,r)} r^{p-7\sigma} |k(p, q, t)|^p dV \]
\[ = r^{(1-7\sigma)p} ||k||_{B(0,r)}^p \]
the second step coming from the anisotropic homogeneity.

Again
\[ ||k(r, \sigma)||_{B(0,r)}^p = \int_{B(0,r)} |rk(r^\sigma p, r^\sigma q, r^\sigma t)|^p dV \]
\[ = \int_{B(0,r')} r^p |k(p', q', t')|^p r^{-8\sigma}, r' > r \]
\[ = r^{p-8\sigma} ||k||_{B(0,r')}^p \]
the second step coming from a change of variables.

Comparing and we see that \((1 - 7\sigma)p > p - 8\sigma\), meaning \(p < \frac{8}{7}\). This also takes care of the endpoint case of \(8/7\).

**Remark 5.5.** One observes that similar arguments can be used to show that the estimate of \(p < 4/3\) in Lemma 4.3 of [T1] is sharp.

### 6. Energy minimisers in the supersonic case

In this short section, we will make a few quick remarks about the energy minimiser scheme in the supersonic case for the NLKG. This means, we are letting \(X\) be an arbitrary Killing field on a compact \(n\)-dimensional Riemannian manifold and looking for solutions to (1.5) of the type (1.3). Now, as has been seen in [T1], straightforward minimization of \(F_{m, \lambda, X}(u)\) over \(H^1(M)\) is not possible. However, we can minimise \(F_{m, \lambda, X}(u)\) keeping it restricted to the space

\[ V_\mu = \{ u \in H^1_0(M) : Xu = i\mu u \} \]

By Schur’s lemma, there are countably many \(\mu\) such that

\[ L^2(M) = \bigoplus V_\mu \]

where \(\tilde{V}_\mu\) is the closure of \(V_\mu\) in \(L^2(M)\). We will talk about the energy minimization scheme, which means that we will try to minimise

\[ \mathcal{E}_{\lambda, X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda Xu, u) - \frac{1}{p+1} \int_M |u|^{p+1} dM \]
subject to \( Q(u) = ||u||_{L^2}^2 = \beta \). The minimization is done over \( V_\mu \) where \( \mu \) is such chosen that \( V_\mu \) is non-trivial ((6.2) tells us that there are plenty of choices for \( \mu \)). But this is equivalent to minimising

\[
E'_m,\lambda,X(u) = \frac{1}{2}(-\Delta u - \mu^2 u - 2\lambda \mu - \lambda^2 u, u) - \frac{1}{p+1} \int_M |u|^{p+1} dM
\]

which is equivalent to minimising

\[
E'_\mu(u) = \frac{1}{2}||\nabla u||_{L^2}^2 - \frac{1}{p+1} \int_M |u|^{p+1} dM
\]

when \( ||u||_{L^2}^2 = \beta \) has been held fixed.

However, this is now equivalent to the minimization problem for standing waves, which has been treated in [CMMT]. Here also, we can proceed along similar lines. The only change is we have to make sure at every step that we are still within \( V_\mu \), which goes through without any problems. Also, the fact that constrained energy minimisers give solutions to the supersonic NLKG is realized by arguments parallel to the ones made in Section 2 about the \( \mathcal{E}_{\lambda,X} \) constrained minimisers (Lemma (2.1)). Clearly, similar considerations also hold for the NLS.

7. Some results in the noncompact setting

In this section we will try to repeat the analysis of section 1 in the case of non-compact manifolds \( M \) which are of the form \( N \times [0, \infty) \), where \( N \) is compact, and \( M \) has the product metric \( g = dr^2 + \phi(r)g_N \), where of course \( \phi \) is smooth. Spaces of the form \( N \times \mathbb{R} \) can be dealt with by stitching these together. If \( X \) is a Killing field on \( N \), we also denote the induced Killing field on \( M \) by \( X \). For now, we will consider only those \( M \) which have bounded geometry and those \( \phi \) such that \( \langle X, X \rangle \leq b^2 < 1 \). Clearly, there are plenty of such manifolds.

7.1. \( F_{m,\lambda,X} \) minimisers. We know (see [CMMT], for example) that in general one cannot expect global minimisers of \( F_{m,\lambda,X} \) on \( H^1(M) \) when \( M \) is non-compact, even if it has rotational symmetry. However, we can minimise \( F_{m,\lambda,X} \) on the class of radial functions which are in \( H^1(M) \), that is, we will try to minimise \( F_{m,\lambda,X} \) over

\[
H^1_r(M) = \{ u \in H^1(M) : u \text{ is a radial function} \}
\]

A word is in order regarding what is meant by a radial function. Here it means those functions which are dependent only on the variable \( r \) running over \([0, \infty)\) of the space \( N \times [0, \infty) \), i.e., when \( x \in N \), we are considering
only those functions $f$ for which $f(x, r) = \phi(r)$.

To do this, we first need a lemma:

**Lemma 7.1.** On a non-compact complete manifold $M$ of dimension $n$ of bounded geometry and having radial symmetry in the sense as described at the beginning of this section, if $f \in H^1_r(M)$, then $f$ vanishes at infinity.

**Proof.** It will be clear that this statement is akin to a similar statement in [S]. First of all, we can assert that $f \in H^1_r(M) \Rightarrow f \in C(M \setminus U)$, where $U$ is a neighbourhood of the origin, let's say, without loss of generality, a ball of radius 1. We also see immediately that it suffices to consider only those functions $f$ such that $f$ has some derivatives with respect to the radial variable. This is in analogy with the following: when we have a continuous function $g$ on $\mathbb{R}$, we can approximate $g$ (in the uniform norm) by smooth $g_n$ on $[-n, n]$ (by the Stone-Weierstrass theorem, for example) and then paste these $g_n$ together by a partition of unity subordinate to the open cover $\{(-n, n)\}$.

Now, if $f$ does not vanish at infinity, then, there exists an $\varepsilon > 0$ such that no matter what compact set in $M$ we select, $f$ attains a value greater than $\varepsilon$ outside this compact set. By scaling, if necessary, we can use $\varepsilon = 1$. Let $q_k$ be a sequence of points satisfying the following:

(a) all the $q_k$ fall on the same straight line from the origin, in other words, they differ only in their $r$-coordinates
(b) $\text{dist}(q_k, q_{k+1}) > 1$ for all $k$
(c) $f(q_k) > 1$ for all $k$
(d) somewhere in between $q_k$ and $q_{k+1}$, $f$ drops below $1/2$

Now then, subdivide $M$ into circular discs (with center at the origin) around each $q_k$ and call these discs $D_k$. $D_k$ are chosen such that $f$ falls below $1/2$ somewhere inside each $D_k$ and also $D_k \cap D_{k+1} = \emptyset$ for all $k$.

Clearly,

$$\int_{D_k} |\nabla f| \geq \frac{1}{2} C$$

where the constant $C$ is explained below. Since this is happening for all $k$, this will contradict the fact that $f \in H^1_r(M)$. \hfill \Box

Here, we have assumed a lower bound $C$ on the function $A(r)$. To recall what $A(r)$ is, the volume form on $M$ is given by $dM = A(r) dr dN$.

To give some alternative criteria under which we can force $f$ to vanish at infinity, we refer to Lemma 2.1.1 from [MT], which says the following:

**Lemma 7.2.** Assume that $A(r)$ satisfies either

$$\int_{|r| \geq 1} \frac{dr}{A(r)} < \infty$$

or

$$\lim_{|r| \to \infty} A(r) = \infty, \quad \text{and} \quad \sup_{|r| \geq 1} \left| \frac{A'(r)}{A(r)} \right| < \infty$$
Then
\[ f \in H^1_r(M) \Rightarrow f|_{M_1} \in C(M_1) \] and
\[ \lim_{|r| \to \infty} |f(r)| = 0 \]
where \( M_1 \) consists of all the points having \( r \)-coordinates \( \geq 1 \).

Now, we have

**Proposition 7.3.** On a non-compact manifold \( M \) as described above, we can minimise \( F_{m,\lambda,X}(u) \) in the class of functions \( H^1_r(M) \) subject to (3.1), giving travelling wave solutions to the nonlinear Klein-Gordon equations. A similar statement is possible for the nonlinear Schrödinger with a simpler proof.

**Proof.** We already know, under suitable spectral assumptions,

\[ F_{m,\lambda,X} \approx \|u\|_{H^1(M)} \quad (7.1) \]

We also have, \( H^1_r(M) \hookrightarrow L^q(U) \) compactly, \( q \in \left[2, \frac{2n}{n-2}\right) \), where \( \bar{U} \) is compact in \( M \). Also, by the lemma just proved,

\[ u \in H^1_r(M) \Rightarrow u \text{ vanishes at infinity} \]

So,

\[ u \in H^1_r(M) \Rightarrow u \in L^\infty(M \setminus U) \]

Also, \( u \in L^2(M) \). This means, by interpolation,

\[ u \in L^q(M \setminus U) \text{ for all } q \in [2, \infty] \]

We also have,

\[ \int_{M \setminus U} |u|^q dM \leq \|u\|_{L^\infty(M \setminus U)}^{q-2} \int_{M \setminus U} |u|^2 dM \quad (7.2) \]

\[ \leq \|u\|_{L^\infty(M \setminus U)}^{q-2} \|u\|_{H^1(M)}^2 \quad (7.3) \]

and

\[ u \in H^1_r(M) \Rightarrow u \in L^q(M) \forall q \in \left[2, \frac{2n}{n-2}\right) \Rightarrow u \in L^{p+1}(M) \forall p \in \left(1, \frac{n+2}{n-2}\right) \]

As usual, let

\[ I_\beta = \inf \{ F_{m,\lambda,X}(u) : u \in H^1_r(M) \} \]

Clearly, \( I_\beta > 0 \), because of (7.1) and (8.1). Now, take a sequence \( u_\nu \in H^1_r(M) \), \( \|u_\nu\|_{L^{p+1}} = \beta \), such that \( F_{m,\lambda,X}(u_\nu) \leq I_\beta + 1/\nu \).

Passing to a subsequence if necessary and without changing the notation, \( u_\nu \rightharpoonup u \in H^1_r(M) \) weakly, which implies, by Rellich compactness,

\[ u_\nu \longrightarrow u \text{ in } L^{p+1}(U) - \text{ norm for all relatively compact } U \quad (7.4) \]
Also, using (8.1), using (7.4) with very large \( U \)'s and the fact that \( u, u_\nu \) vanish at infinity (this is Lemma 7.1), we have

\[
(7.5) \quad u_\nu \to u \text{ in } L^{p+1}(M \setminus U) - \text{norm}
\]

meaning, finally that

\[
||u||_{L^{p+1}}^{p+1} = \beta
\]

So a minimiser is obtained. The fact that a constrained minimiser will give a solution is realized in the usual way. \( \square \)

7.2. Existence of energy minimisers. We take the space to write about constrained energy minimisers on the types of spaces \( M = N \times [0, \infty) \), as mentioned at the beginning of this section. We will be very brief and only outline the salient features of this derivation. To be exact, we are trying to minimise the energy

\[
E_{\lambda,X}(u) = \frac{1}{2}(-\Delta u + X^2 u + 2i\lambda Xu, u) - \frac{1}{p+1} \int_M |u|^{p+1} dM
\]

subject to \( ||u||_{L^2}^2 = \beta \) (constant), the minimization being done over \( H^1_r(M) \), the radial functions on \( M \) (radial in the sense made clear at the beginning of this section), and \( p \in (1, 1+4/n) \).

As before,

\[
I_\beta = \inf \{ E_{\lambda,X}(u) : u \in H^1_r(M), ||u||_{L^2}^2 = \beta \}
\]

Arguing as before with the Gagliardo-Nirenberg inequality, we can conclude that \( I_\beta > -\infty \). Taking a sequence \( u_\nu \) such that

\[
E_{\lambda,X}(u_\nu) < I_\beta + 1/\nu
\]

and labeling as before the weak* limit of \( u_\nu \) in \( H^1_r(M) \) as \( u \), we can see, by previous discussion, that establishing \( u \) as the constrained energy minimiser amounts to establishing three things:

- \( u_\nu \to u \) in \( L^2 \)-norm
- \( u_\nu \to u \) in \( L^{p+1} \)-norm
- \( u_\nu \to u \) in \( H^1 \)-norm or equivalently, \( \nabla u_\nu \to \nabla u \) in \( L^2 \)-norm

Now, the second bullet point will be achieved by a variant of the method used to establish (7.4) and (8.11) as outlined in the analysis for the \( F_{m,\lambda,X} \) minimisers earlier in this section. We will say more on the third bullet point later. Let us talk about the first bullet point now.

This requires the techniques of concentration-compactness, as laid out in [L]. We take the space to give a formal statement of this. The statement was originally made in the setting of the Euclidean space, but as noted in [CMMT], the concentration-compactness principle and most of the subsidiary results generalize to manifolds of bounded geometry with no changes at all. We will state the reformulated version as appears there.
Proposition 7.4. Let $M$ be a Riemannian manifold. Fix $\beta \in (0, \infty)$. Let \( \{u_\nu\} \in L^{p+1}(M) \) be a sequence satisfying \( \int_M |u_\nu|^{p+1}dV = \beta \). Then, after extracting a subsequence, one of the following three cases holds:

(i) Vanishing: If $B_R(y) = \{x \in M : d(x, y) \leq R\}$ is the closed $R$-ball around $y$, then for all $R \in (0, \infty)$,
\[
\lim_{\nu \to \infty} \sup_{y \in M} \int_{B_R(y)} |u_\nu|^{p+1}dV = 0
\]

(ii) Concentration: There exists a sequence of points $\{y_\nu\} \subset M$ with the property that for each $\varepsilon > 0$, there exists $R(\varepsilon) < \infty$ such that
\[
\int_{B_R(\varepsilon)(y_\nu)} |u_\nu|^{p+1}dV > \beta - \varepsilon
\]

(iii) Splitting: There exists $\alpha \in (0, \beta)$ with the following properties: For each $\varepsilon > 0$, there exists $\nu_0 \geq 1$ and sets $E_\nu^#, E_\nu^b \subset M$ such that
\[
d(E_\nu^#, E_\nu^b) \to \infty \quad \text{as} \quad \nu \to \infty
\]

and
\[
\left| \int_{E_\nu^#} |u_\nu|^{p+1}dV - \alpha \right| < \varepsilon, \quad \left| \int_{E_\nu^b} |u_\nu|^{p+1}dV - (\beta - \alpha) \right| < \varepsilon
\]

For a statement of the above result in the even more general setting of measured metric spaces, see [CMMT], Appendix A. A couple of lines about the heuristics of the concentration-compactness principle: when we have a sequence of elements in a Banach space with fixed norm, or, in other words, lying on a sphere in the Banach space, we cannot necessarily pick a norm convergent subsequence unless the space itself is finite dimensional. But, we can give an exhaustive list of the behaviour of subsequences, at least in the context of the $L^p$ spaces. That is what the concentration-compactness principle gives. In our case, the only hold we have on the sequence $u_\nu$ is that all of them have same $L^2$-norm. This should make the application of the concentration-compactness argument seem natural. In applications such as ours, the usual line of attack is to rule out vanishing and splitting phenomena, so we are left with concentration phenomenon as the only possibility. From there, we can go to compactness, i.e., convergence of the subsequence, $||u_\nu - u||_{L^2} \to 0$, which has been the goal of the first bullet point.

7.3. Ruling out vanishing and splitting. Following closely the corresponding analysis of [CMMT] and [MT], to rule out vanishing, one has to make the technical assumption
\[
I_\beta < -\frac{(m^2 - \lambda^2)}{2} \beta
\]

It is not clear that we can always have \(7.8\) regardless of the manifold type. We will find more to say on this in the ensuing paragraphs.
Step I: Ruling out vanishing.
Assume vanishing occurs, that is,
\[
\lim_{\nu \to \infty} \sup_{y \in M} \int_{B_R(y)} |u_\nu|^{p+1} dV = 0
\]
We already know that \( u_\nu \) satisfy \( E_{\lambda,X} < I_\beta + 1/\nu \) and that, \( \{u_\nu\} \) is bounded in \( H^1(M) \). By Lemma I.1 on page 231 of [L1], this means,
\[
2 < r < \frac{2n}{n-2} \implies \|u_\nu\|_{L^r(M)} \to 0
\]
Then, by (7.4) and (8.11), we can see that \( u = 0 \), giving
\[
\|u_\nu\|_{L^{p+1}} \to 0
\]
That means,
\[
\|u\|_{H^1}^2 \approx F_{m,\lambda,X}(u) = 2E_{\lambda,X}(u) + \frac{2}{p+1} \int_M |u|^{p+1} dV \] + \((m^2 - \lambda^2)\beta
\]
implies
\[
\|u\|_{H^1}^2 \leq \frac{2}{C^*} I_\beta + \frac{1}{C^*} (m^2 - \lambda^2) \beta
\]
which gives a contradiction. Here \( C^* \) is a constant such that \( C^*\|u\|_{H^1}^2 \leq F_{m,\lambda,X} \).

Next we will look to rule out the splitting phenomenon. For this, we need a technical lemma:

**Lemma 7.5.** (i) If \( \beta > 0, I_\beta < -\frac{m^2-\lambda^2}{2} \beta, \sigma > 1 \), then
\[
I_{\sigma \beta} < \sigma I_\beta
\]
(ii) If \( 0 < \eta < \beta \) and \( I_\beta < -\frac{m^2-\lambda^2}{2} \beta \), we have
\[
I_\beta < I_{\beta-\eta} + I_\eta
\]

**Proof.** We will save space by skipping these proofs, which are hardly instructive anyway. (i) follows along absolutely similar lines as Proposition 3.1.2 in [CMMT] by defining \( w_\nu = \sigma^{1/2} u_\nu \) and comparing \( E_{\lambda,X}(w_\nu) \) with \( E_{\lambda,X}(u_\nu) \). (ii) follows from (i) exclusively by algebraic manipulation and does not use any other property or form of \( I_\beta \). \( \square \)

Finally, to rule out splitting, we follows the basic line of attack in [CMMT]. We note, however, that certain modifications will be required in our approach. We have

**Proposition 7.6.** Under the setting of Lemma [7.5], if \( \{u_\nu\} \in H^1(M) \) is a minimising sequence, then splitting cannot occur.
Proof. Begin by choosing \( \varepsilon > 0 \) sufficiently small such that
\[
I_\beta < I_\alpha + I_{\beta-\alpha} - C_1 \varepsilon
\] (7.10)
Since \( ||u_\nu||_{H^1} \) and \( ||u_\nu||_{L^{p+1}} \) are uniformly bounded, it follows from (7.6), (7.7), (7.4) and (8.11) that there exists \( \nu_1 \) such that when \( \nu \geq \nu_1 \), we have
\[
\int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dM + \int_{S_\nu} |u_\nu|^{p+1} dM < \varepsilon
\]
where \( S_\nu \) is a set of the form
\[
S_\nu = \{ x \in M : d(x, E_\nu^\#) \leq d_\nu + 2 \} \subset M \setminus (E_\nu^\# \cup E_\nu^b)
\]
for some \( d_\nu > 0 \). Call
\[
\tilde{E}_\nu(r) = \{ x \in M : d(x, E_\nu^\#) \leq r \}
\]
Now define functions \( \chi_\nu^\# \) and \( \chi_\nu^b \) by
\[
\chi_\nu^\#(x) = \begin{cases} 
1, & \text{if } x \in \tilde{E}_\nu(d_\nu) \\
1 - d(x, \tilde{E}_\nu(d_\nu)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 1) \\
0, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 1)
\end{cases}
\]
and
\[
\chi_\nu^b(x) = \begin{cases} 
0, & \text{if } x \in \tilde{E}_\nu(d_\nu + 1) \\
d(x, \tilde{E}_\nu(d_\nu + 1)), & \text{if } x \in \tilde{E}_\nu(d_\nu + 2) \\
1, & \text{if } x \notin \tilde{E}_\nu(d_\nu + 2)
\end{cases}
\]
Just to motivate what we are doing, we want a control on the term \( |\mathcal{E}_{\lambda,X}(u_\nu) - \mathcal{E}_{\lambda,X}(u_\nu^\# + u_\nu^b)| \) i.e., show that
\[
|\mathcal{E}_{\lambda,X}(u_\nu) - \mathcal{E}_{\lambda,X}(u_\nu^\# + u_\nu^b)| = |\mathcal{E}_{\lambda,X}(u_\nu) - [\mathcal{E}_{\lambda,X}(u_\nu^\#) + \mathcal{E}_{\lambda,X}(u_\nu^b)]| \leq \varepsilon
\]
and get a contradiction from the fact that \( |I_\beta - I_\alpha - I_{\beta-\alpha}| > C_1 \varepsilon \) which comes from (7.10). Since we know that
\[
2\mathcal{E}_{\lambda,X}(u_\nu) = F_{m,\lambda,X}(u_\nu) - \frac{2}{p+1}||u_\nu||_{L^{p+1}}^{p+1} - (m^2 - \lambda^2)||u_\nu||_{L^2}^2
\]
we see by triangle inequality that controlling each of the terms
\[
\int_M \left( ||u_\nu||^{p+1} - (||u_\nu^\#||^{p+1} + ||u_\nu||^{p+1}) \right) dM
\]
(7.11)
\[
\int_M \left( ||u_\nu||_{H^1} - (||u_\nu^\#||_{H^1} + ||u_\nu^b||_{H^1}) \right) dM
\]
(7.12) \[|F_{m,\lambda,X}(u_\nu) - (F_{m,\lambda,X}(u_\nu^\#) + F_{m,\lambda,X}(u_\nu^b))|\]
would be sufficient. To that end, we first observe that both \( \chi_\nu^\#(x) \) and \( \chi_\nu^b(x) \) are Lipschitz with Lipschitz constant 1 and the intersection of their supports has measure zero. Also set
\[
u^\# = \chi_\nu^\# u_\nu, \nu^b = \chi_\nu^b u_\nu
\]
Note that
\[ ||u^\#_\nu||^2_{L^2} = \alpha_\nu, \text{ where } |\alpha - \alpha_\nu| < 2\varepsilon \]
and
\[ ||u^b_\nu||^2_{L^2} = \beta_\nu - \alpha_\nu, \text{ where } |(\beta - \alpha) - (\beta_\nu - \alpha_\nu)| < 2\varepsilon \]

Now, we have
\[
\int_M \left( |u_\nu|^{p+1} - (|u^\#_\nu|^{p+1} + |u_\nu|^{p+1}) \right) dM \leq \int_{S_\nu} |u_\nu|^{p+1} dM \lesssim \varepsilon
\]
(7.13)
\[
\int_M \left( |u_\nu|^2 - (|u^\#_\nu|^2 + |u_\nu|^2) \right) dM \leq \int_{S_\nu} |u_\nu|^2 dM \lesssim \varepsilon
\]
(7.14)

Using \( \nabla u^\#_\nu = \chi^\#_\nu \nabla u_\nu + (\nabla \chi^\#_\nu) u_\nu \), the corresponding identity for \( \nabla u^b_\nu \) and the fact that both \( \chi^\#_\nu(x) \) and \( \chi^b_\nu(x) \) have Lipschitz constant 1 we see that
\[
\int_M \left( |\nabla u_\nu|^2 - (|\nabla u^\#_\nu|^2 + |\nabla u_\nu|^2) \right) dM \leq \int_{S_\nu} |\nabla u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dMd \lesssim \varepsilon
\]
(7.13) and (7.14) together give (7.11). Now we are left with (7.12).

We start by observing that it suffices to control
\[
|(X^2 u_\nu, u_\nu) - (X^2 u^\#_\nu, u^\#_\nu) - (X^2 u^b_\nu, u^b_\nu)|
\]
or, equivalently,
\[
\left| \int_M \langle |Xu_\nu|^2 - |Xu^\#_\nu|^2 - |Xu^b_\nu|^2 \rangle dM \right|
\]

Now, as before, \( Xu^\#_\nu = \chi^\#_\nu u_\nu + X(\chi^\#_\nu) u_\nu \), so
\[
\left| \int_M (|Xu_\nu|^2 - |Xu^\#_\nu|^2 - |Xu^b_\nu|^2) dM \right| \leq \int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |Xu_\nu|^2 dM
\]
(7.15)
\[
\lesssim \int_{S_\nu} |u_\nu|^2 dM + \int_{S_\nu} |\nabla u_\nu|^2 dM \lesssim \varepsilon
\]
(7.16)

the last observation coming from the fact that \( X \) is bounded. Lastly, we can also control
\[
|(iXu_\nu, u_\nu) - (iXu^\#_\nu, u^\#_\nu) - (iXu^b_\nu, u^b_\nu)|
\]
by using the Cauchy-Schwarz inequality. That completes the proof. \( \square \)

Now that we have ruled out the alternatives, we can say that the minimising sequence \( u_\nu \) will concentrate. Recall that this means
Corollary 7.7. Under the setting of Lemma (7.5), there is a sequence of points \( y_\nu \in M \) such that for all \( \varepsilon > 0 \), there exists \( R(\varepsilon) < \infty \) (quite independent of \( \nu \)) such that

\[
\int_{M \setminus B_{R(\varepsilon)}(y_\nu)} |u_\nu|^2 dM < \varepsilon
\]

However, unfortunately, to go from concentration to compactness, one needs certain homogeneous space like properties of \( M \). [CMMT] calls spaces like these weakly homogeneous spaces, defined as follows:

Definition 7.8. There exists a group \( G \) of isometries of \( M \) and a number \( D > 0 \) such that for every \( x, y \in M \), there exists \( g \in G \) such that \( d(x, g(y)) \leq D \).

If we add this technical restriction, we can proceed as follows. We will map the sequence \( y_\nu \) into a compact region so that any subsequence which concentrates will have compact Sobolev embeddings by Rellich’s theorem. This concludes the (admittedly long-drawn) discussion about the first bullet point of page 23. How does this produce an energy minimiser? The fact that \( u_\nu \) converges to \( u \) in the weak* topology of \( H^1_r(M) \) implies that

\[
\| \nabla u \|_{L^2} \leq \liminf_{\nu \to \infty} \| \nabla u_\nu \|_{L^2}
\]

which means

\[
\mathcal{E}_{\lambda,X}(u) \leq \liminf_{\nu \to \infty} \mathcal{E}_{\lambda,X}(u_\nu) = I_\beta
\]

Finally we have an energy minimiser.

7.4. Comments on the technical assumption (7.8). We would like to make a few comments regarding when the technical assumption (7.8) holds. To make the problem simpler for now, we look at the corresponding problem for the ground waves (see [CMMT]):

What kind of manifolds \( M \) possess the following property that given any \( \beta > 0 \), one can find \( u_\beta \in H^1_r(M) \) such that

\[
\mathcal{E}(u_\beta) = \frac{1}{2} \| \nabla u_\beta \|_{L^2}^2 - \frac{1}{p+1} \int_M |u_\beta|^{p+1} dM < 0
\]

We assert

Lemma 7.9. Choosing \( u_\beta \) as above is possible on Euclidean spaces.

Proof. Here we use similar scaling techniques as used in Section 4. To start with, pick \( u \in H^1_r(\mathbb{R}^n) \) such that \( \|u\|_{L^2}^2 = \beta \).

Define

\[
u_{\lambda,\alpha}(x_1, ..., x_n) = \lambda u(\lambda^\alpha x_1, ..., \lambda^\alpha x_n)\]

Now, on calculation,

\[
\|u_{\lambda,\alpha}\|_{L^p}^p = \lambda^{p-\alpha n} \|u\|_{L^p}^p
\]
realized by the usual change of variables. This means, in particular,

$$||u^{\lambda,\alpha}||_{L^2}^2 = \lambda^{2-\alpha n}||u||_{L^2}^2$$

We will want all the $u^{\lambda,\alpha}$ to have the same $L^2$ norm, so we have to select $\alpha = 2/n$. Also,

$$||\nabla u^{\lambda,\alpha}||_{L^2}^2 = \lambda^{2+\alpha n}||\nabla u||_{L^2}^2 = \lambda^{2/n}||\nabla u||_{L^2}^2$$

by similar techniques. Now we have,

$$||u^{\lambda,\alpha}||_{L^{p+1}}^2 = \lambda^{p-1}||u||_{L^{p+1}}^2 = \lambda^{p-1-4/n}||u||_{L^{p+1}}^2$$

(7.19)

Since we know that $p \in (1, 1 + 4/n)$, we label $p - 1 - 4/n = r < 0$. Then from (7.19) we see that as $\lambda \to 0$, $u^{\lambda,\alpha}$ has the same $L^2$ norm, but the value of

$$\frac{||u^{\lambda,\alpha}||_{L^{p+1}}}{||\nabla u^{\lambda,\alpha}||_{L^2}^2}$$

becomes very large. At some point, it will go above $\frac{p+1}{2}$, whence we will have some $u$ for which $E(u) < 0$.

Now, if $\frac{||u||_{L^{p+1}}}{||\nabla u||_{L^2}}$ increases without bound, so does $\frac{||u||_{L^{p+1}}}{||\nabla u||_{L^2}^2}$. Now

$$\frac{||u||_{L^{p+1}}^2}{||\nabla u||_{L^2}^2 + \beta} = \frac{||u||_{L^{p+1}}^2}{||\nabla u||_{L^2}^2 + ||u||_{L^2}^2} = \frac{||u||_{L^{p+1}}^2}{||u||_{H^1}^2}$$

Now if $\frac{||u||_{L^{p+1}}}{||u||_{H^1}}$ increases without bound, at some point it will go above $\frac{C(p+1)}{2}$, where $C$ is a constant such that $F_{m,\lambda,X}(u) \leq C||u||_{H^1}^2$. Then onwards,

$$C||u||_{H^1}^2 - \frac{2}{p+1}||u||_{L^{p+1}}^2 < 0 \implies F_{m,\lambda,X}(u) - \frac{2}{p+1}||u||_{L^{p+1}}^2 < 0$$

$$\implies 2E_{\lambda,X}(u) + (m^2 - \lambda^2)\beta < 0$$

So we have been able to say something about the technical assumption (7.8). Of course, this technique per se would not work in spaces where one cannot scale.
8. Small perturbations of $X$ and corresponding minimisers

Previously, we have established existence of $F_{m,\lambda,X}$ minimisers on non-compact manifolds of type $M = N \times [0, \infty)$ which have bounded geometry and metrics of the type $g = dr^2 + \phi(r)g_N$, with smooth $\phi$. Similarly, we can establish the existence of constrained $F_{\lambda,X}$ minimisers.

Now, we raise the following perturbation question: if we perturb the Killing field $X$ slightly, can we prove that the corresponding constrained minimisers also vary slightly? It certainly seems believable on a compact manifold, but the question is more involved on a non-compact setting. We study this question for the $F_{\lambda,X}$ minimisers in connection with the NLS equation.

**Proposition 8.1.** Small perturbations of the Killing field in the sup norm will also lead to small variations in the constrained minimisers in the sup norm. More formally, call $X_n = X + \varepsilon_n X''$, where $X, X''$ are bounded Killing fields on $M$, and $\varepsilon_n \to 0$ is a decreasing sequence of positive real numbers. Let $u_n$ be constrained minimisers of $F_{\lambda,X_n}(u)$ subject to (2.3). Then, there exists a subsequence of $u_n$, still called $u_n$, such that

$$\sup_{x \in M} |u_m - u_n| \to 0, \text{ as } m, n \to \infty$$

**Proof.** So, let us suppose, we change the Killing field $X$ to $X' = X + \varepsilon X''$, where $X''$ is such that $(X'', X'') \leq C$ (wanting that $X'$ be a Killing field is definitely some kind of restriction on the manifold; for the generic manifold, slight perturbations of a Killing field might not at all give another Killing field). We first want to see how much $F_{\lambda,X'}(u)$ varies. Now

$$F_{\lambda,X'}(u) = (-\Delta u + \lambda u - iX'u, u)$$

$$= (-\Delta u - iXu - i\varepsilon X''u + \lambda u, u)$$

$$= F_{\lambda,X}(u) - (i\varepsilon X''u, u) \to F_{\lambda,X}(u) \text{ as } \varepsilon \to 0$$

Also,

$$|F_{\lambda,X'}(u) - F_{\lambda,X}(u)| = |(i\varepsilon X''u, u)| \leq ||\varepsilon X''u|||u||$$

$$= ||\varepsilon X''.\nabla u|||u|| \leq ||\varepsilon X''|||\nabla u|||u||$$

$$\leq C||u||^2_{H^1}$$

the last step using the bound on $X''$ and the Cauchy-Schwarz inequality.

Now, consider a decreasing sequence $\varepsilon_n \to 0$ as $n \to 0$ and consider the perturbations $X_n = X + \varepsilon_n X''$ of $X$. Suppose for each $n$, $u_n \in H^1(M)$ is a minimiser of $F_{\lambda,X_n}$ subject to $||u||_{L^{p+1}}^{p+1} = \beta$ (constant). We will start by arguing that $u_n$ has a convergent subsequence in the $L^{p+1}$-norm (let us point out that getting a result about a subsequence is about as good as it gets. For a particular value of $n$, the constrained minimisation problem might have several solutions, some of which must be “incomparable” norm-wise with
some of the constrained minimisers for another particular $n$.
First, we argue that $||u_n||_{H^1}^2$ is uniformly bounded. We need to see that
(8.1) $I_{\beta,n}$ uniformly bounded $\iff F_{\lambda,X_n}(u_n)$ uniformly bounded
which finally implies
(8.2) $\implies ||u_n||_{H^1}^2$ uniformly bounded
where $I_{\beta,n} = \inf\{F_{\lambda,X_n}(u) : u \in H^1(M), ||u||_{L^{p+1}}^{p+1} = \beta\}$.
We observe that $F_{\lambda,X_n}(u) \geq 0$ for all $n, u$. So, fixing an integer $k$, we have
(8.3) $|I_{\beta,k+q} - I_{\beta,k}| \leq |F_{\lambda,X_{k+q}}(u_k) - F_{\lambda,X_k}(u_k)|$
(8.4) $\leq \varepsilon_k||u_k||_{H^1}^2$
for all positive integers $q$. That means,
(8.5) $I_{\beta,k+q} \leq I_{\beta,k} + C\varepsilon_k||u_k||_{H^1}^2$
which gives that $I_{\beta,n}$ is uniformly bounded, which means that $F_{\lambda,X_n}(u_n)$ is uniformly bounded.
Now we know, $||u||^2_{H^1(M)} \leq C(n)F_{\lambda,X_n}(u)$ for all $u \in H^1(M)$. That gives, for $m > k$, $k$ being fixed,
$$||u||^2_{H^1(M)} \leq C^k((-\Delta - iX_k + \lambda)u, u)$$
$$= C^k((-\Delta - iX_m + \lambda)u, u) + C^k((iX_m - iX_k)u, u)$$
$$\leq C^kF_{\lambda,X_m}(u) + C^k(|\varepsilon_m - \varepsilon_k|(X''u, u))$$
$$\leq C^kF_{\lambda,X_m}(u) + CC^k\varepsilon_k||u||_{H^1(M)}^2$$
which finally implies
(8.6) $||u||^2_{H^1(M)} \leq \frac{C^kF_{\lambda,X_m}(u)}{1 - CC^k\varepsilon_k}$
That means, in particular
(8.7) $||u_n||^2_{H^1(M)} \leq \frac{C^kF_{\lambda,X_m}(u_n)}{1 - CC^k\varepsilon_k}$
which means that finally we have, $\{||u_n||_{H^1}^2\}$ is uniformly bounded.
Since we have $||u_n||_{H^1}^2 \leq K$ uniformly, we can say that $u_n$ converges in the weak* topology of $H^1(M)$. Since we are working on manifolds of bounded geometry, we also have, when $2 < s < \infty$, and relatively compact $U$,
(8.8) $\int_{M\setminus U}|u_n|^sdM \leq ||u_n||^s_{L^\infty(M\setminus U)}\int_{M\setminus U}|u_n|^2dM$
(8.9) $\leq ||u_n||^s_{L^\infty(M\setminus U)}||u_n||_{H^1(M)}^2$
Also, the Rellich theorem gives us
$$H^1(M) \hookrightarrow L^s(\Omega) \text{ compactly }, \forall s \in \left[2, \frac{2n}{n - 2}\right)$$
given \( \Omega \subset M \) relatively compact. This, together with (8.8), gives
\[
\begin{align*}
  u_n &\in H^1_r(M) \Rightarrow u_n \in L^s(M) \quad \forall \ s \in \left[2, \frac{2n}{n-2}\right) \\
  &\Rightarrow u_n \in L^{p+1}(M) \quad \forall \ p \in \left[1, \frac{n+2}{n-2}\right)
\end{align*}
\]
Passing to a subsequence if necessary and without changing the notation, \( u_n \to u \in H^1_r(M) \) weakly implies, by Rellich compactness,
\[
\text{(8.10)} \quad u_n \to u \text{ in } L^{p+1}(U) - \text{norm for all relatively compact } U.
\]

Also, using (8.9), using (8.10) with very large \( U \)'s and the fact that \( u_n, u \) vanish at infinity (this is Lemma 7.1), we have
\[
\text{(8.11)} \quad u_n \to u \text{ in } L^{p+1}(M \setminus U) - \text{norm}
\]
meaning, finally that
\[
\text{(8.12)} \quad ||u||_{p+1}^{p+1} = \beta
\]
which gives \( ||u_{k+q} - u_k||_{L^{p+1}} \) is small beyond some integer \( k \), for all positive integers \( q \). Now, since these \( u_n \)'s give constrained minimisers, they are actually solutions to the auxiliary NLS’s, which are elliptic for all \( X_n \). Since \( u_{k+q} - u_k \in H^1_r(M) \), they vanish at infinity, which means they have small \( L^\infty \) norm outside a finite ball of big enough radius (clearly, the radius of this ball might actually depend on the chosen value of \( k \)).

Inside a relatively compact ball, we can apply the Moser estimates to bound \( ||u_{k+q} - u_k||_{L^\infty} \) by the \( L^2 \)-norm bounds of \( (u_{k+q} - u_k) \). But we are through, as having a control on \( L^{p+1} \)-norm of \( (u_{k+q} - u_k) \) will give a control on the \( L^2 \) norm on \( (u_{k+q} - u_k) \) inside a relatively compact ball by Sobolev embedding. So, ultimately, we have argued that with small perturbations of the Killing fields in the sup norm, we get small variations in the constrained minimisers in the sup norm.

9. Two nonlinearities and their interactions

In this section, we comment on standing vortex solutions to nonlinear Schrödinger equations with two power-type nonlinearities:
\[
\text{(9.1)} \quad i\partial_t v + \Delta v = -|v|^{p-1}v - |v|^{q-1}v
\]
where, without any loss of generality, \( q > p \). We will just be content with pointing out that an entirely analogous study can be carried out for travelling waves along absolutely similar lines.

Writing \( v(t, x) = e^{i\lambda t}w(x) \) we get, as usual, the following auxiliary equation:
\[
\text{(9.2)} \quad -\Delta u + \lambda u = |u|^{p-1}u + |u|^{q-1}u
\]
Now, let $F_\lambda(u) = ||\nabla u||^2 + \lambda ||u||^2 - \frac{2}{p+1} \int_M |u|^{p+1} dM$. We will first establish that minimising $F_\lambda(u)$ subject to $||u||^{q+1}_{L^{q+1}} = \beta$ (constant) will give a solution to the auxiliary equation. To that end, we find on calculation,

$$\frac{d}{d\tau} \bigg|_{\tau=0} F_\lambda(u + \tau v) = 2 \text{Re}(-\Delta u, v) + 2\lambda \text{Re}(u, v) - 2\int_M \text{Re}|u|^{p-1} \langle u, v \rangle dM$$

$$= 2(-\Delta u + \lambda u - |u|^{p-1} u, v)$$

Since $u, v$ take values in an inner product space $V$, here $\langle u, v \rangle = \text{Re} \int_M \langle u, v \rangle dM$, where $\langle ., . \rangle$ represents the inner product in $V$.

The above calculation proves that minimising $F_\lambda(u)$ subject to $||u||^{q+1}_{L^{q+1}} = \beta$ does indeed give a solution to the auxiliary equation. So now we need to argue the existence of a constrained minimiser.

**Proposition 9.1.** On a manifold $M$ we can minimise $F_\lambda(u)$ subject to the constraint $||u||^{q+1}_{L^{q+1}} = \beta$ (constant), thus giving a solution to (9.2). On a compact manifold, $p \in (1, \frac{n+2}{n-2})$, $q < p$ and on a noncompact manifold, $p, q \in (1, \frac{n+2}{n-2})$.

**Proof.** First we establish the existence of a constrained infimum. We have

$$F_\lambda(u) = ||\nabla u||^2 + \lambda ||u||^2 - \frac{2}{p+1} \int_M |u|^{p+1} dM$$

$$= F_\lambda(u) - \frac{2}{p+1} \int_M |u|^{p+1} dM$$

Now, if we are on a compact manifold $M$, we can write

$$F_\lambda(u) \approx ||u||^2_{H^1} = F_\lambda(u) + \frac{2}{p+1} \int_M |u|^{p+1} dM$$

$$\leq F_\lambda(u) + C ||u||_{L^2}^{(p+1)(1-\gamma)} ||u||_{H^1}^{\gamma(p+1)}$$

$$\leq F_\lambda(u) + C ||u||_{L^{q+1}}^{(p+1)(1-\gamma)} ||u||_{H^1}^{\gamma(p+1)}$$

In the second step, we have used the Gagliardo-Nirenberg inequality and the last step uses the inclusion $L^{q+1} \subset L^2$, where $q \geq 1$. Since $\gamma(p+1) < 2$, the last equation implies two things:

- There exists a constrained infimum, let us call it $I_\beta$.
- If $u_k$ represents a sequence of functions in $H^1_\pi(M)$ such that $||u||^{q+1}_{L^{q+1}} = \beta$ and $F_\lambda(u) \rightarrow I_\beta$, then $||u_k||^{2}_{H^1}$ remain bounded.

Again, if we are on a non-compact manifold, let us consider the case $2 < p+1 < q+1$. Then, we have by the $L^p$ interpolation identity

$$||u||^{p+1}_{L^{p+1}} \leq ||u||^{1-\theta}_{L^2} ||u||^{\theta}_{L^{q+1}}$$

where $\theta$ satisfies

$$\frac{1}{p+1} = \frac{1-\theta}{2} + \frac{\theta}{q+1}$$
which gives upon calculation,
\[ \theta = \frac{(p - 1)(q + 1)}{(q - 1)(p + 1)} \]

Now we have
\[
F_\lambda(u) = F_\lambda(u) + \frac{2}{p + 1} \int_M |u|^{p+1} dM \\
\leq F_\lambda(u) + \frac{2}{p + 1} ||u||_{L^2}^{(p+1)(1-\theta)} ||u||_{L^{p+1}}^{(p+1)\theta} \\
= F_\lambda(u) + \frac{2}{p + 1} ||u||_{L^{q-1}}^{2(q-p)} ||u||_{L^{q+1}}^{(p-1)(q+1)} \\
\leq F_\lambda(u) + \frac{2}{p + 1} K ||u||_{H^1}^{2(q-p)}
\]

On calculation we see that \( p > 1 \implies \frac{2(q-p)}{q-1} < 2 \). Once again, we have
- There exists a constrained infimum, let us call it \( I_\beta \).
- If \( u_k \) represents a sequence of functions in \( H^1_\pi(M) \) such that \( ||u||_{L^{q+1}} = \beta \) and \( F_\lambda(u) \rightarrow I_\beta \), then \( ||u_k||_{H^1}^2 \) remain bounded.

So then, in the compact case, we use the chain of arguments in Proposition (2.2) to see that \( u_k \rightarrow u \) weak* in \( H^1(M) \) and \( u \) provides a constrained minimiser giving a solution to the auxiliary equation. Similarly, for the non-compact case, we follow the line of argument in Proposition (7.3) to establish the existence of constrained minimisers.

\[ \square \]

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