Oscillation of a Linear Delay Impulsive Differential Equation

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Abstract
The main result of the paper is that the oscillation (non-oscillation)
of the impulsive delay differential equation
\[
\dot{x}(t) + \sum_{k=1}^{m} A_k(t) x[h_k(t)] = 0, \quad t \geq 0,
\]
x(\tau_j) = B_j x(\tau_j - 0), \quad \lim_{\tau_j} = \infty
is equivalent to the oscillation (non-oscillation) of the equation without impulses
\[
\dot{x}(t) = \sum_{k=1}^{m} A_k(t) \prod_{h_k(t) < \tau_j \leq t} B_j^{-1} x[h_k(t)] = 0, \quad t \geq 0.
\]
Explicit oscillation results are presented.

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1 Introduction

Recently results on oscillation of delay differential equations have taken shape of a developed theory presented in monographs [1-4]. At the same time it is an intensively developing field which is an objective of numerous publications.

However, for impulsive differential equations there are only few publications dealing with oscillation problems [1,4,5,6].

The purpose of the present paper is to fill up this gap. The main result is that the oscillation (non-oscillation) of the impulsive delay differential equation is equivalent to the oscillation (non-oscillation) of a certain differential equation without impulses which can be constructed explicitly from an impulsive equation. Thus the oscillation problems (in particular, oscillation and non-oscillation criteria) for an impulsive equation can be reduced to the similar problem for a certain non-impulsive equation.

The method proposed in the present paper for oscillation is new both for impulsive and non-impulsive equations. It is based on the solution representation formula. Recently such formulas are widely used in stability investigations of non-impulsive [7-9] and impulsive equations [5,10-12].

We demonstrate that the existence of a nonoscillating solution is equivalent to the positiveness of the fundamental function. At the same time this is equivalent to the solvability of a certain nonlinear inequality which is similar to "the generalized characteristic equation" from the monograph [2].

The paper is organized as follows. Theorems 1 and 2 are concerned with the equivalence of non-oscillation, positiveness of a fundamental function and solvability of a certain inequality. They lead to explicit non-oscillation results (Theorem 3). Theorem 4 compares non-oscillation conditions for two different impulsive delay differential equations. Theorems 5 and 6 give new oscillation criteria for delay differential equations without impulses. Theorem 7 contains the main result of the paper connecting oscillation of an impulsive and a non-impulsive equation. As a corollary (Theorem 8) we obtain explicit oscillation conditions for an impulsive delay equation.

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2 Preliminaries

We consider a scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x[h_k(t)] = f(t), \quad t \geq 0;$$  \hfill (1) \\
$$x(\tau_j) = B_j x(\tau_j - 0), \quad j = 1, 2, \ldots,$$  \hfill (2)

under the following assumptions

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ are fixed points, $\lim_{j \to \infty} \tau_j = \infty$;

(a2) $A_k, f, k = 1, \ldots, m$ are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $B_j \in \mathbb{R}$, $j = 1, \ldots$, $\mathbb{R}$ is a real axis;

(a3) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$.

Together with (1),(2) we will consider for each $t_0 \geq 0$ an initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x[h_k(t)] = f(t), \quad \text{where } t \geq t_0, \ x(\xi) = \varphi(\xi), \ \xi < t_0, \quad (3)$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad \tau_j > t_0. \quad (4)$$

We assume that for the initial function $\varphi$ the following hypothesis holds

(a4) $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition.** An absolutely continuous on each interval $[\tau_j, \tau_{j+1})$ function $x : [t_0, \infty) \to \mathbb{R}$ is a solution of the impulsive problem (3),(4) if (3) is satisfied for almost all $t \in [0, \infty)$ and the equalities (4) hold.

**Definition.** For each $s \geq 0$ the solution $X(t, s)$ of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x[h_k(t)] = 0, \quad \text{where } t \geq s, \ x(\xi) = 0, \ \xi < s;$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad \tau_j > s, \ x(s) = 1, \quad (5)$$

is a fundamental function of the equation (1),(2).

We assume $X(t, s) = 0$, $0 \leq t < s$.

**Lemma 1** [12] Let (a1)-(a4) hold. Then there exist one and only one solution of the problem (3) with the initial condition $x(t_0) = \alpha_0$ and impulsive conditions

$$x(\tau_j) = B_j x(\tau_j) + \alpha_j$$
that can be presented in the form
\[
x(t) = X(t, t_0)x(t_0) + \int_{t_0}^{t} X(t, s)f(s)ds - \\
- \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)A_k(s)\varphi[h_k(s)]ds + \sum_{\tau_j > t_0} X(t, \tau_j)\alpha_j, \tag{6}
\]
where \(\varphi[h_k(s)] = 0,\) if \(h_k(s) > t_0.\)

### 3 Non-oscillation Criteria for Impulsive Equations

**Definition.** The equation (1),(2) has a non-oscillating solution if there exist \(t_0 > 0,\) \(\varphi(t)\) satisfying (a4) such that for \(f \equiv 0\) the solution of (3),(4) is positive for \(t \geq t_0.\) Otherwise, all solutions of (1),(2) are said to be oscillating.

In sequel we accept that the following hypothesis holds
(a5) delays are bounded: for every \(s > 0\)
\[
\mu_s = \min_k \sup_{t > s} h_k(t) > -\infty
\]
and there exists \(s' \geq s\) such that \(h_k(t) \geq s\) if \(t \geq s'.\)

Denote for any \(s\)
\[
A_k^s(t) = \begin{cases} 
A_k(t), & \text{if } t \geq s, \\
0, & \text{if } t < s,
\end{cases}
\]
\[
h_k^s(t) = \begin{cases} 
h_k(t), & \text{if } t \geq s, \\
s, & \text{if } t < s.
\end{cases}
\tag{7}
\]

The following theorem establishes non-oscillation criteria.

**Theorem 1** Suppose (a1)-(a5) hold, \(A_k(t) \geq 0,\) \(k = 1, \ldots, m,\) and \(B_j > 0,\) \(j = 1, 2, \ldots.\) Then the following hypotheses are equivalent
1) The equation (1),(2) has a non-oscillating solution.
2) There exists \(t_0 \geq 0\) such that \(X(t, s) > 0,\) \(t_0 \leq s < t < \infty.\)
3) For a certain \(t_1 \geq 0\) there exists a non-negative integrable in each interval \([t_1, b]\) solution \(u\) of an inequality
\[
u(t) \geq \sum_{k=1}^{m} A_k^t \exp \left\{ \int_{h_k^t(t)}^{t} u(s)ds \right\} \prod_{h_k^t(t) < \tau_j \leq t} B_j^{-1}, \quad t \geq t_1. \tag{8}
\]
Here and in sequel we assume that a product equals to unit if number of factors is equal to zero.

Proof. The scheme of the proof is 1) \(\Rightarrow\) 3) \(\Rightarrow\) 2) \(\Rightarrow\) 1).

1) \(\Rightarrow\) 3). Let \(x(t)\) be a positive solution of (3),(4) \((f \equiv 0)\). By (a5) for a certain \(t_1 \geq t_0, h_k(t) > t_0, \ t \geq t_1, \ k = 1, \ldots, m.\)

Let us demonstrate that

\[
u(t) = \frac{d}{dt}\ln \left\{ \frac{x(t)}{x(t_1)} \prod_{t_1 < \tau_j \leq t} B_j^{-1} \right\}, \ t \geq t_1.
\]

is a solution of (8). To this end we integrate the latter equality

\[
x(t) = x(t_1) \exp \left\{ -\int_{t_1}^{t} u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j, \ t \geq t_1.
\]

(9)

By setting \(\varphi(t) = x(t)\) for \(t < t_1\) one obtains that \(x(t), \ t \geq t_1\), is a solution of (3),(4), with the initial point \(t = t_1\) and the initial function \(\varphi(t) > 0\). We substitute (9) in (3) \((f \equiv 0)\):

\[
-u(t) \exp \left\{ -\int_{t_1}^{t} u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j + \\
\sum_{k \in N_1} A_k(t) \exp \left\{ -\int_{t_1}^{h_k(t)} u(s) ds \right\} \prod_{t_1 < \tau_j \leq h_k(t)} B_j + \\
\sum_{k \in N_2} A_k(t) \varphi[h_k(t)] = 0, \ t \geq t_1.
\]

(10)

Here \(N_1 = \{k : h_k(t) \geq t_1\}, \ N_2 = \{k : h_k(t) < t_1\}\).

Using notations (7) the equality (10) can be rewritten in the form

\[
-u(t) \exp \left\{ -\int_{t_1}^{t} u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j + \\
\sum_{k=1}^{m} A_k^{t_1}(t) \exp \left\{ -\int_{t_1}^{h_k^{t_1}(t)} u(s) ds \right\} \prod_{t_1 < \tau_j \leq h_k^{t_1}(t)} B_j + 
\]

5
\[
\sum_{k \in \mathbb{N}_2} A_k(t) \varphi[h_k(t)] = 0, \quad t \geq t_1.
\]

Consequently,
\[
\left( u(t) - \sum_{k=1}^m A^i_k(t) \exp \left\{ \int_{t_1}^t u(s) ds \right\} \prod_{h_k^i(t) < \tau_j \leq t} B_j^{-1} \right) \times
\]
\[
\times \exp \left\{ - \int_{t_1}^t u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j = \sum_{k \in \mathbb{N}_2} A_k(t) \varphi[h_k(t)] \geq 0,
\]
since \(\varphi(t)\) is positive according to our choice of the point \(t_1\), which implies 3).

3) \(\implies\) 2). Consider (3),(4) with the initial function \(\varphi \equiv 0\) and initial value \(x(t_1) = 0\) in a segment \([t_1, b]::
\[
\dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] = f(t), \quad t \in [t_1, b] : \quad x(\xi) = 0, \quad \xi < t_1,
\]
\[
x(t_1) = 0, \quad x(\tau_j) = B_j x(\tau_j - 0), \quad \tau_j > t_1.
\]

Besides, we consider an ordinary impulsive differential equation including the solution \(u(t) \geq 0\) of (8):
\[
\dot{x}(t) + u(t)x(t) = z(t), \quad t \in [t_1, b],
\]
\[
x(\tau_j) = B_j x(\tau_j - 0), \quad x(t_1) = 0.
\]

The solution of (12) can be rewritten in the form [15]
\[
x(t) = \int_{t_1}^t \exp \left\{ - \int_s^t u(\xi) d\xi \right\} \prod_{s < \tau_j \leq t} B_j z(s) ds.
\]

We seek for the solution of (11) of the form (13). By substituting \(x\) and \(\dot{x}\) from (13) and (12) into (11), we obtain
\[
\sum_{k=1}^m A_{k}^{t_1 i}(t) \int_{t_1}^{h_{k}^{t_1}(t)} \exp \left\{ - \int_s^{h_{k}^{t_1}(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h_{k}^{t_1}(t)} B_j z(s) ds = f(t). \quad (14)
\]
The equation (14) is of the type
\[ z - Hz = f, \] (15)
where
\[ (Hz(t) = u(t) \int_{t_1}^{t} \exp \left\{ - \int_{s}^{t} u(\xi)d\xi \right\} \prod_{s<\tau_j \leq t} B_j z(s)ds - \]
\[ \sum_{k=1}^{m} A^i_k(t) \int_{t_1}^{h^i_k(t)} \exp \left\{ - \int_{s}^{h^i_k(t)} u(\xi)d\xi \right\} \prod_{s<\tau_j \leq h^i_k(t)} B_j z(s)ds. \] (16)

It is well known [14] that the integral operator
\[ (Hz)(t) = \int_{t_1}^{b} K(t, s)z(s)ds \]
acting in the space \( L_{[t_1, b]} \) of functions integrable on \([t_1, b]\) is compact if
\[ |K(t, s)| \leq k(t), k \in L_{[t_1, b]} . \] (17)

For the operator \( H \) defined by (16)
\[ |K(t, s)| \leq \sup_{s,t \in [t_1, b]} \prod_{s<\tau_j \leq t} B_j \left( u(t) + \sum_{k=1}^{m} |A^i_k(t)| \right) . \]

Thus the inequality (17) holds and the operator \( H : L_{[t_1, b]} \to L_{[t_1, b]} \) is a compact Volterra integral operator. Therefore [14] its spectral radius is equal to zero. Consequently the equation (15) for any \( f \in L_{[t_1, b]} \) has a single solution
\[ z = (I - H)^{-1} f, \] (18)
where \( I \) is the identity operator.

Let us show that \( H \) is a positive operator. The operator \( H \) can be easily rewritten as a sum \( H = H_1 + H_2 \), where
\[
(H_1 z)(t) = \left( u(t) - \sum_{k=1}^{m} A^i_k(t) \exp \left\{ \int_{h^i_k(t)}^{t} u(s)ds \right\} \prod_{h^i_k(t) < \tau_j \leq t} B_j^{-1} \right) \times \\
\times \int_{t_1}^{t} \exp \left\{ - \int_{s}^{t} u(\xi)d\xi \right\} \prod_{s<\tau_j \leq t} B_j z(s)ds,
\]
\[(H_2z)(t) = \sum_{k=1}^{m} A^1_k(t) \int_{h^1_k(t)}^{t} \exp \left\{- \int_{s}^{h^1_k(t)} u(\xi) d\xi \right\} \prod_{s < \tau_j \leq h^1_k(t)} B_j z(s) ds.\]

The inequality (8) implies \(H_1 \geq 0\). So \(H = H_1 + H_2 \geq 0\). Since the spectral radius of \(H\) is equal to zero, then

\[(I - H)^{-1} = I + H + H^2 + \ldots \geq 0.\]

Thus if \(f \geq 0\), then the solution \(z\) of (15) is non-negative: \(z \geq 0\).

The solution of (11) has the form (3), (4), with \(x(t_1) = 0\), \(\varphi \equiv 0\) and

\[f(t) = \exp \left\{- \int_{t_1}^{t} u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j - \sum_{k=1}^{m} A^1_k(t) \exp \left\{- \int_{h^1_k(t)}^{t} u(s) ds \right\} \prod_{h^1_k(t) < \tau_j \leq t} B_j = \exp \left\{- \int_{t_1}^{t} u(s) ds \right\} \prod_{t_1 < \tau_j \leq t} B_j \times \left( u(t) - \sum_{k=1}^{m} A^1_k(t) \exp \left\{ \int_{h^1_k(t)}^{t} u(s) ds \right\} \prod_{h^1_k(t) < \tau_j \leq t} B_j^{-1} \right).\]
Thus (8) implies \( f(t) \geq 0 \). Therefore in view of (6)
\[
x(t) = \int_{t_1}^{t} X(t, s)f(s)ds \geq 0.
\]
Consequently,
\[
X(t, t_1) \geq \exp \left\{ - \int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 < \tau_j \leq t} B_j > 0.
\]

For \( s > t_1 \) the inequality \( X(t, s) > 0 \) can be proven similarly.

2) \( \implies \) 1). Denote \( x(t) = X(t, t_0) \). Then \( x(t) \) is a positive solution of (3),(4) \( (f \equiv 0) \) with the initial function \( \varphi \equiv 0 \). The proof is complete.

Let us consider (1),(2) with coefficients of an arbitrary sign.

Denote \( a^+ = \max\{a, 0\}, a^- = \max\{-a, 0\} \).

**Theorem 2** Suppose (a1)-(a5) hold and \( B_j > 0 \).

Consider three hypotheses:

1) The initial value problem (3),(4) with an initial point \( t_0 > 0 \) \( (f \equiv 0) \) has a positive solution that continuously extend the continuous initial function \( \varphi \).

2) \( X(t, s) > 0, \ t_0 \leq s < t < \infty \).

3) There exists a non-negative integrable on each interval \([t_0, b]\) solution of an inequality
\[
u(t) \geq \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^+ \exp \left\{ \int_{h_k^0(t)}^{t} u(s)ds \right\} \prod_{h_k^0(t) < \tau_j \leq t} B_j^{-1}, \ t \geq t_0.
\]

Then implications 3) \( \implies \) 2), 3) \( \implies \) 1) are valid.

**Proof.** The proof of 3) \( \implies \) 2) coincides with the proof of 3) \( \implies \) 2) in Theorem 1 up to the place where the operator \( H \) is presented as a sum of two terms. Here
\[
H = H_1 + H_2 + H_3,
\]
where
\[
(H_1 z)(t) = \left( u(t) - \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^+ \exp \left\{ \int_{h_k^0(t)}^{t} u(s)ds \right\} \prod_{h_k^0(t) < \tau_j \leq t} B_j^{-1} \right) \times
\]
\[
\times \int_{t_0}^{t} \exp \left\{-\int_{s}^{t} u(\xi) d\xi \right\} \prod_{s \leq \tau_j \leq t} B_j z(s) ds,
\]

\[
(H_2 z)(t) = \sum_{k=1}^{m} \left( A_{k}^{t_0}(t) \right)^{+} \int_{h_{k}^{t_0}(t)}^{t} \exp \left\{-\int_{s}^{t} u(\xi) d\xi \right\} \prod_{s \leq \tau_j \leq h_{k}^{t_0}(t)} B_j z(s) ds,
\]

\[
(H_3 z)(t) = \sum_{k=1}^{m} \left( A_{k}^{t_0}(t) \right)^{-} \int_{t_0}^{h_{k}^{t_0}(t)} \exp \left\{-\int_{s}^{h_{k}^{t_0}(t)} u(\xi) d\xi \right\} \prod_{s \leq \tau_j \leq h_{k}^{t_0}(t)} B_j z(s) ds.
\]

Again, like in Theorem 1, \(H_1 \geq 0\), \(H_2 \geq 0\), \(H_3 \geq 0\), which implies \(H = H_1 + H_2 + H_3 \geq 0\). The end of the proof completely repeats the corresponding one of Theorem 1.

3) \(\Rightarrow\) 1). Let us consider the problem (3),(4). Let \(\mu_{t_0}\) be chosen as in the hypothesis (a5). We extend to the interval \([\mu_{t_0}, t_0)\) the coefficients \(A_k(t)\) by zero and the delays \(h_k(t)\) such that \(h_k(t) \leq t\). Let \(u(t)\) be a non-negative function satisfying (20). We extend it by zero to \([\mu_{t_0}, t_0)\). Then \(u(t)\) is a solution of (20), where \(t_0\) is changed by \(\mu_{t_0}\).

Consider a corresponding extension of (3),(4) to the interval \([\mu_{t_0}, \infty)\). As proven above, 3) \(\Rightarrow\) 2), therefore \(X(t, s) > 0\) for \(\mu_{t_0} \leq s < t < \infty\).

Assuming

\[
\varphi(t) = X(t, \mu_{t_0}) \text{ for } \mu_{t_0} \leq t < t_0 \quad \text{and} \quad x(t) = X(t, \mu_{t_0}) \text{ for } t \geq t_0,
\]

we obtain that \(x(t)\) is a positive solution of (3),(4) \((f \equiv 0)\), with an initial point \(t_0\), that continuously extends the continuous initial function \(\varphi\). This completes the proof of the theorem.

Now we proceed to explicit non-oscillation results.

Denote

\[
h^{t_0}_k(t) = \min \limits_{k} h^{t_0}_k(t),
\]

where \(h^{t_0}_k(t)\) is defined by (7).

**Theorem 3** Suppose (a1)-(a5) hold, \(B_j > 0\) and at least one of the following three hypotheses hold:

1) \(A_k(t) \leq 0\), \(t \geq t_0\).

2) \(\text{vrai sup}\ \sum_{t \geq t_0} \int_{h^{t_0}_k(t)}^{t} \left( A_{k}^{t_0}(s) \right)^{+} \prod_{h^{t_0}_k(s) \leq \tau_j \leq s} B_j^{-1} z(s) ds \leq 1/e. \quad (21)\)

10
3) \[ \sum_{k=1}^{m} \int_{h_0^k(t)}^{t} \left( A_{k}^0(s) \right)^{+} ds \leq 1/e \left( 1 + \sum_{h_0^k(t)<\tau_j \leq t, \ B_j<1} \ln B_j \right), \quad t \geq t_0. \]  

Then the fundamental matrix \( X(t, s) \) is positive for \( t_0 \leq s < t < \infty \) and there exists a positive solution of (3),(4) \( (f \equiv 0) \) continuously extending a continuous initial function \( \varphi \).

**Proof.** Obviously 1) is a special case of 2). Let us prove the theorem assuming (21) holds. To this end we will demonstrate that a function

\[ u(t) = e^{m \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^{+} \prod_{h_0^k(t)<\tau_j \leq t} B_j^{-1}} \]

is a non-negative solution of the inequality (20). By substituting \( u \) in (20) one obtains

\[ e^{m \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^{+} \prod_{h_0^k(t)<\tau_j \leq t} B_j^{-1}} \geq \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^{+} \times \]

\[ \times \exp \left\{ e^{\int_{h_0^k(t)}^{t} \sum_{i=1}^{m} \left( A_{i}^0(s) \right)^{+} \prod_{h_0^i(s)<\tau_j \leq s} B_j^{-1} ds} \right\} \prod_{h_0^k(t)<\tau_j \leq t} B_j^{-1}. \]

This inequality can be deduced from the following one

\[ e^{m \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^{+} \prod_{h_0^k(t)<\tau_j \leq t} B_j^{-1}} \geq \]

\[ \exp \left\{ e^{\int_{h_0^k(t)}^{t} \sum_{k=1}^{m} \left( A_{k}^0(s) \right)^{+} \prod_{h_0^k(s)<\tau_j \leq s} B_j^{-1} ds} \right\} \times \]

\[ \times \sum_{k=1}^{m} \left( A_{k}^0(t) \right)^{+} \prod_{h_0^k(t)<\tau_j \leq t} B_j^{-1}. \]

After dividing this inequality by its left-hand side and logarithmizing it we obtain

\[ \sum_{k=1}^{m} \int_{h_0^k(t)}^{t} \left( A_{k}^0(s) \right)^{+} \prod_{h_0^k(s)<\tau_j \leq s} B_j^{-1} z(s) ds \leq 1/e, \]
which obviously results from (21).

Let 3) hold. We will prove that
\[ u(t) = e^{\sum_{k=1}^{m} (A_{k}^{0}(t))^{+}} \]
is a solution of the inequality (20) which after substituting takes form
\[
e^{\sum_{k=1}^{m} (A_{k}^{0}(t))^{+}} \geq \sum_{k=1}^{m} (A_{k}^{0}(t))^{+} \exp \left\{ e \int_{t}^{\tau_{j}} \sum_{i=1}^{m} (A_{i}^{0}(s))^{+} ds \right\} \prod_{k=1}^{m} B_{k}^{-1}.
\]
This inequality can be deduced from
\[
e^{\sum_{k=1}^{m} (A_{k}^{0}(t))^{+}} \geq \sum_{k=1}^{m} (A_{k}^{0}(t))^{+} \exp \left\{ e \int_{t}^{\tau_{j}} \sum_{k=1}^{m} (A_{k}^{0}(s))^{+} ds \right\} \prod_{k=1}^{m} B_{k}^{-1},
\]
where the product contains only factors for which \( B_{j} < 1 \). The latter inequality after dividing by the left-hand side and logarithmizing coincides with (22). This completes the proof of the theorem.

Let us compare oscillation properties of (1),(2) and an impulsive equation
\[
\dot{x}(t) + \sum_{k=1}^{m} \tilde{A}_{k}(t)x(\tilde{h}_{k}(t)) = f(t), \; t \in [0, \infty),
\]
x(\tau_{j}) = \tilde{B}_{j}x(\tau_{j} - 0).
\]
\[
(23)
\]

**Theorem 4** Let the hypotheses (a1)-(a5) hold for the equations (1),(2) and (23), \( \tilde{A}_{k}(t) \geq 0, B_{j} > 0 \). Suppose that any (therefore, all) of the hypotheses 1)-3) of Theorem 1 holds for (1),(2).

Then if \( A_{k}(t) \geq \tilde{A}_{k}(t), \; B_{j} \leq \tilde{B}_{j} \) and at least one of the hypotheses
1) \( h_{k}(t) \leq \tilde{h}_{k}(t), \; \tilde{B}_{j} \leq 1, \; j = 1, 2, \ldots; \)
2) \( h_{k}(t) = \tilde{h}_{k}(t), \)

holds then for the equation (23) the assertions 1)-3) of Theorem 1 are valid.
Proof. By the hypothesis of the theorem there exists a non-negative function \( u(t) \) satisfying (7). Besides, for any non-negative function \( u \) under the hypotheses of the theorem the inequality

\[
\sum_{k=1}^{m} A_k(t) \exp \left\{ \int_{h_k(t)}^{t} u(s)ds \right\} \prod_{h_k(t)<\tau_j\leq t} B_j^{-1} \geq \sum_{k=1}^{m} \tilde{A}_k(t) \exp \left\{ \int_{\tilde{h}_k(t)}^{t} u(s)ds \right\} \prod_{\tilde{h}_k(t)<\tau_j\leq t} \tilde{B}_j^{-1}
\]

holds. Consequently if \( u \) is a solution of the inequality (7) then \( u \) is a solution of this inequality, where \( A_k, h_k, B_j \) are changed by \( \tilde{A}_k, \tilde{h}_k, \tilde{B}_j \). Then by Theorem 1 the other assertions of this theorem also hold.

Corollary 1. Suppose the hypotheses (a1)-(a5) hold for (1),(2) and \( B_j > 0 \). Besides, let \( 0 \leq A_k(t) \leq A_k, \quad t - h_k(t) \leq h_k, \quad B_j \leq 1 \).

If there exists a non-oscillating solution of the equation with constant coefficients and delays

\[
\dot{x}(t) + \sum_{k=1}^{m} A_k x(t - h_k) = f(t), \quad t \in [0, \infty),
\]

\[
x(\tau_j) = B_j x(\tau_j - 0),
\]

then there exists a non-oscillating solution of the equation (1),(2).

Corollary 2. Let (a1)-(a5) hold and \( A_k(t) \geq 0 \). If there exists a non-oscillation solution of the equation (1) without impulses and \( B_j \geq 1 \), then there exists a non-oscillating solution of the impulsive equation (1),(2).

4 Oscillation Properties of Impulsive and Non-impulsive Equations

Consider a non-impulsive differential equation

\[
\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x[h_k(t)] = f(t), \quad t \geq 0.
\]

Denote by \( x(t, s) \) the fundamental function of the equation (24). After substituting \( B_j \equiv 1 \) Theorems 1 and 2 immediately yield the following results.
Theorem 5 Suppose (a2)-(a5) hold for (24) and $a_k(t) \geq 0, k = 1, 2, \ldots$. Then the following hypotheses are equivalent:

1) The equation (24) has a non-oscillating solution ($f \equiv 0$).
2) There exists $t_0 \geq 0$ such that $x(t, s) > 0$ for $t_0 \leq s < t < \infty$.
3) For a certain $t_1 \geq 0$ there exists a non-negative integrable on each interval $[t_1, b]$ solution $u$ of the inequality

$$u(t) \geq \sum_{k=1}^{m} a_k^{t_1}(t) \exp \left\{ \int_{h_k^{t_1}(t)}^{t} u(\xi)d\xi \right\}, \ t \geq t_1. \tag{25}$$

Theorem 6 Suppose (a2)-(a5) hold for (24). Consider three hypotheses:

1) The initial value problem for (24) ($f \equiv 0$) with an initial point $t_0 \geq 0$ has a positive solution that is a continuous expansion of a continuous initial function $\varphi$;
2) $x(t, s) > 0, \ t_0 \leq s < t < \infty$;
3) There exists a non-negative integrable on each interval $[t_0, b]$ solution $u$ of the inequality

$$u(t) \geq \sum_{k=1}^{m} \left( a_k^{t_0}(t) \right)^+ \exp \left\{ \int_{h_k^{t_0}(t)}^{t} u(s)ds \right\}, \ t \geq t_0$$.

Then implications $3) \Rightarrow 2), 3 \Rightarrow 1)$ are valid.

Corollary of Theorem 3 for the equation (24) coincides with the known non-oscillation result for equations without impulses [1,2,4].

In this paper we present a fundamental result that enables to reduce the oscillation problem for (1),(2) to the oscillation problem for an equation without impulses. To this end consider an auxiliary equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t) \prod_{h_k^j(t) < 0 \leq t} B_j^{-1}x[h_k(t)] = 0, \ t \in [0, \infty), \tag{26}$$

where $h_k^0(t) = \begin{cases} h_k(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$

Denote by $Y(t, s)$ a fundamental function of the equation (26).
Theorem 7 Suppose (a1)-(a5) hold, $A_k \geq 0$, $B_j > 0$.

Then
1) There exists $t_0 > 0$, such that $X(t,s) > 0$, $t_0 \leq s < t < \infty \iff$ there exists $t_1 > 0$, such that $Y(t,s) > 0$, $t_1 \leq s < t < \infty$.
2) All solutions of (1),(2) ($f \equiv 0$) are oscillating $\iff$ all solutions of (26) are oscillating.
3) There exists a non-oscillating solution of (1),(2) ($f \equiv 0$) $\iff$ there exists a non-oscillating solution of (26).

Proof. 1). Let $X(t,s) > 0$, $t_0 \leq s < t < \infty$. Then by Theorem 1 there exists a solution of the inequality (7) for $t \geq t_1$. This inequality coincides with (25) under

$$a_k(t) = A_k(t) \prod_{h_k(t) < \tau_j \leq t} B_j^{-1}.$$ 

Therefore by Theorem 5 $Y(t,s) > 0$, $t_1 \leq s < t < \infty$. The converse can be proven similarly.

2). Suppose all solutions of (1),(2) ($f \equiv 0$) are oscillating and (26) has a positive solution, beginning with a certain $t_0$. Then by Theorem 5 $Y(t,s) > 0$ for $t_1 \leq s < t < \infty$. Then, as proven in 1), $X(t,s) > 0$ for $t_2 \leq s < t < \infty$. Consequently, by Theorem 1 the equation (1),(2) has a non-oscillating solution, which contradicts to the hypothesis. The converse is proven similarly.

Besides, 2) implies 3), which completes the proof.

By applying Theorem 7 and known oscillation (non-oscillation) results on equations without impulses, one obtains oscillation results for impulsive equations. As an example we present the following statement.

Denote $h(t) = \min_k h_k(t)$, $\bar{h}(t) = \max_k h_k(t)$.

Theorem 8 Let (a1)-(a5) hold for (1),(2), $A_k(t) \geq 0$ and $B_j > 0$. Then if at least one of the following inequalities holds

1) $\lim_{t \to \infty} \inf_{h(t)} \int_{h(t)}^{t} \sum_{k=1}^{m} A_k(s) \prod_{h_k(s) \leq \tau_j \leq s} B_j^{-1} ds > 1/e$,

2) $\lim_{t \to \infty} \sup_{\bar{h}(t)} \int_{\bar{h}(t)}^{t} \sum_{k=1}^{m} A_k(s) \prod_{h_k(s) \leq \tau_j \leq s} B_j^{-1} ds > 1$,

then all the solutions of (1),(2) are oscillating.
This statement is obtained by applying Theorem 7 and oscillation results for equations without impulses from the monographs [1,2,4].

References

[1] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, Inc, New York and Basel, 1987.

[2] I. Gyory, G. Ladas, Oscillation Theory of Delay Differential Equations, Clarendon Press, Oxford, 1991,

[3] D. D. Bainov, D. P. Mishev, Oscillation Theory for Neutral Differential Equation with Delay, Adam Hilger, Bristol, Philadelphia and New York, 1991.

[4] K. Gopalsamy, Stability and Oscillation in Delay Differential Equation of Popular Dynamics, Kluwer Academic Publishers, Dordrecht, Boston, London, 1992.

[5] K. Gopalsamy and B. G. Zhang, On delay differential equations with impulses, J. Math. Anal. Appl. 139 (1989), 110-122.

[6] D. D. Bainov, Y. Domshlak, P. S. Simeonov, On oscillation properties of first order delay differential equations with impulses (submitted).

[7] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, New York, 1990.

[8] C. Corduneanu, Integral representation of solutions of linear Volterra functional differential equations, Libertas Mathematics 9 (1989), 133-146.

[9] L. Berezansky, The positiveness of the Cauchy function and the stability of linear differential equations with after-effect, Differential Equations 22 (1990), 1092-1100.

[10] A. Anokhin, L. Berezansky and E. Braverman, Stability of linear delay impulsive differential equations (to appear in Dynamic Systems & Applications).
[11] **L. Berezansky and E. Braverman**, Preservation of the Exponential Stability under Perturbations of Linear Delay Impulsive Differential Equations, *Zeitschrift fur Analysis und ihre Anwendungen* **14**(1995), 157-174.

[12] **A. Anokhin, L. Berezansky, and E. Braverman**, Exponential stability of linear delay impulsive differential equations (to appear in *J. Math. Anal. Appl.*).

[13] **L. Berezansky and E. Braverman**, Impulsive stabilization of linear delay impulsive differential equations (in preparation).

[14] **M. A. Krasnoselskiĭ, P. P. Zabreiko, E.I. Pustylnik, and P. E. Sobolevskii**, Integrable Operators in the Spaces of Summable Functions. Noordhoff, Leyden, 1976.

[15] **V. Lakshmikantham, D.D. Bainov and P.S. Simeonov**, "Theory of Impulsive Differential Equations", World Scientific, Singapore, 1989.