Semigroups of Operators on Spaces of Fuzzy-Number-Valued Functions with Applications to Fuzzy Differential Equations

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Abstract

In this paper we introduce and study semigroups of operators on spaces of fuzzy-number-valued functions, and various applications to fuzzy differential equations are presented. Starting from the space of fuzzy numbers $\mathbb{R}_F$, many new spaces sharing the same properties are introduced, as for example, with similar notations as in classical functional analysis: $C([a, b]; \mathbb{R}_F)$, $C^p([a, b]; \mathbb{R}_F)$, $L^p([a, b]; \mathbb{R}_F)$, and so on. We derive basic operator theory results on these spaces and new results in the theory of semigroups of linear
operators on fuzzy-number kind spaces. The theory we develop is used to solve “classical” fuzzy systems of differential equations, including, for example, the fuzzy Cauchy problem and the fuzzy wave equation. These tools allow us to obtain explicit solutions to fuzzy initial value problems which bear explicit formulas similar to the crisp case, with some additional fuzzy terms which in the crisp case disappear. The semigroup method displays a clear advantage over other methods available in the literature (i.e., the level set method, the differential inclusions method and other "fuzzification" methods of the real-valued solution) in the sense that the solutions can be easily constructed, and that the method can be applied to a larger class of fuzzy differential equations that can be transformed into an abstract Cauchy problem.

Keywords: fuzzy numbers, fuzzy-number-valued functions, semigroups of operators, fuzzy differential equations.

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1 Introduction

Many physical processes in science are described by models using differential equations. It is a well-known fact that the theory of semigroups of operators on Banach spaces represents a powerful tool for solving many classical differential equations (see e.g. [9 – 10]). Nowadays, differential equations are classified by the different approaches used in order to deal with them: (i) deterministic (ii) random or stochastic and (iii) fuzzy-like. The class (i) is by far the most studied. The second class is usually employed when a deterministic description of the model is not strictly justifiable. However, the first or second descriptions often represent an idealization of real-world situations when imprecision or lack of information in the modelling of the physical process may in fact play a significant role. Indeed, it was this uncertainty or vagueness of the modelling in certain physical processes that suggested the introduction of the so-called fuzzy differential equations [3-4], whose solutions represent fuzzy-number-valued functions.
The goal of this paper is to develop, in Section 3, the theory of the semigroups of operators on spaces of fuzzy-number-valued functions. In Section 4, our goal is to apply this theory to several classes of fuzzy differential equations. The main difficulty of dealing with fuzzy differential equations is the fact that the spaces of fuzzy numbers and fuzzy-number-valued functions are not linear spaces. In particular, they are not groups with respect to addition and the scalar multiplication is not, in general, distributive with respect to usual addition of scalars. However, they are complete metric spaces with their metrics having nice properties. These features allow us to develop a consistent operator theory. To give a better idea of what we are aiming for, let us recall that, as in the classical cases (i) and (ii), many inhomogeneous fuzzy differential equations (iii) can be recasted into the form of an abstract Cauchy problem of the form

\[
\frac{du}{dt}(t) = A[u(t)] \oplus g(t), \quad t \in I, \\
u(t_0) = u_0,
\]

where \( I \) is an interval, \( u : I \to X, \quad u_0 \in X, \quad A : X \to X \) is a linear bounded operator on \( X \), and \( g : I \to X \) is continuous, where \( (X, \oplus, \odot, d) \) is a complete metric space. Our goal is to show that, as in the classical case, the following variation of parameter formula

\[
u(t) = T(t)(u_0) \oplus \int_0^t T(t-s)g(s)ds
\]

always furnishes a solution to the fuzzy Cauchy problem. Furthermore, we also provide the mean for constructing \( T(t)(u_0) \) as follows:

\[
T(t)(u_0) := e^{t \odot A}(u_0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(u_0),
\]

where \( \sum \) is the sum with respect to \( \oplus \). Finally, we apply this formula on some specific examples and show that it always provides the correct answer in the sense that the solutions we construct in the fuzzy case contain additional terms which in the crisp case completely disappear. It is worth mentioning that one can easily formulate the classical Liapunov stability theorem for \( \{T(t)\}_{t \geq 0} \) in this form. While
the above formula provides a constructive mean to obtain solutions, the following important questions still remain open: in the generic case is the solution of the above Cauchy problem unique, and if so, does it depend continuously on the initial data?

2 Preliminaries

In this section we present the main properties of the space of fuzzy numbers and of some other spaces based on it and which have similar properties. Given a set \( X \neq \emptyset \), a fuzzy subset of \( X \) is a mapping \( u : X \to [0, 1] \) and obviously any classical subset of \( X \) can be identified as a fuzzy subset of \( X \) defined by \( \chi_A : X \to [0, 1], \chi_A (x) = 1 \) if \( x \in A \), \( \chi_A (x) = 0 \) if \( x \in X \setminus A \). If \( u : X \to [0, 1] \) is a fuzzy subset of \( X \), then for \( x \in X \), \( u(x) \) is called the membership degree of \( x \) to \( u \) (see e.g. [21]).

**Definition 2.1** (see e.g. [8]). The space of fuzzy numbers denoted by \( R_F \) is defined as the class of fuzzy subsets of the real axis \( R \), i.e., of \( u : R \to [0, 1] \), having the following four properties:

\( i \) \( \forall u \in R_F \), \( u \) is normal, i.e., \( \exists \alpha u \in R \) with \( u(\alpha u) = 1 \);

\( ii \) \( \forall u \in R_F \), \( u \) is a convex fuzzy set, i.e.,

\[ u(t x + (1 - t) y) \geq \min \{ u(x), u(y) \}, \forall t \in [0, 1], x, y \in R; \]

\( iii \) \( \forall u \in R_F \), \( u \) is upper-semi-continuous on \( R \);

\( iv \) \( \{ x \in R; u(x) > 0 \} \) is compact, where \( M \) denotes the closure of \( M \).

**Remarks.** 1) Clearly, \( R \subset R_F \) because any real number \( x_0 \in R \) can be identified with \( \chi_{\{x_0\}} \), which satisfies the properties \( (i) - (iv) \) as in Definition 2.1.

2) For \( 0 < r \leq 1 \) and \( u \in R_F \), let us denote by \( [u]^r = \{ x \in R; u(x) \geq r \} \) and \( [u]^0 = \{ x \in R; u(x) > 0 \} \), the so called level sets of \( u \). Then it is an immediate consequence of \( (i) - (iv) \) that \( [u]^r \) represents a bounded closed (i.e., compact) subinterval of \( R \), denoted by \( [u]^r = [u_-(r), u_+(r)] \), where \( u_-(r) \leq u_+(r) \) for all \( r \in [0, 1] \).

**Definition 2.2** (see e.g. [8]). The addition and the scalar product in \( R_F \) are defined by \( \oplus : R_F \times R_F \to R_F \),

\[ (u \oplus v)(x) = \sup_{y + z = x} \min \{ u(y), v(z) \} \]
and, by $\odot : \mathbb{R} \times \mathbb{R}_F \rightarrow \mathbb{R}_F$, 

$$(\lambda \odot v) (x) = \begin{cases} 
    u (\frac{x}{\lambda}) & \text{if } \lambda \neq 0, \\
    0 & \text{if } \lambda = 0,
\end{cases}$$

where $\tilde{0} : \mathbb{R} \rightarrow [0, 1]$ is $\tilde{0} = \chi_{\{0\}}$.

Also, we can write 

$$[u \oplus v]^r = [u]^r + [v]^r, [\lambda \odot v]^r = \lambda [v]^r,$$

for all $r \in [0, 1]$, where $[u]^r + [v]^r$ means the usual sum of two intervals (as subsets of $\mathbb{R}$) and $\lambda [v]^r$ means the usual product between a real scalar and a subset of $\mathbb{R}$.

If we define $D : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}$ by 

$$D (u, v) = \sup_{r \in [0,1]} \max \{|u_- (r) - v_- (r)|, |u_+ (r) - v_+ (r)|\},$$

where $[u]^r = [u_- (r), u_+ (r)]$, $[v]^r = [v_- (r), v_+ (r)]$, then we have the following.

**Theorem 2.3** (see e.g. [8]). $(\mathbb{R}_F, D)$ is a complete metric space. In addition, $D$ has the following three properties:

(i) $D (u \oplus w, v \oplus w) = D (u, v)$, for all $u, v, w \in \mathbb{R}_F$;
(ii) $D (k \odot u, k \odot v) = |k| D (u, v)$, for all $u, v \in \mathbb{R}_F, k \in \mathbb{R}$;
(iii) $D (u \oplus v, w \oplus e) \leq D (u, w) + D (v, e)$, for all $u, v, w, e \in \mathbb{R}_F$.

Also, the following result is known.

**Theorem 2.4** (see e.g. [1], [8]). (i) $u \oplus v = v \oplus u, u \oplus (v \oplus w) = (u \oplus v) \oplus w$;
(ii) If we denote $\tilde{0} = \chi_{\{0\}}$, then $u \oplus \tilde{0} = \tilde{0} \oplus u = u$, for any $u \in \mathbb{R}_F$;
(iii) With respect to $\tilde{0}$, none of $u \in \mathbb{R}_F \setminus \mathbb{R}$ has an opposite member (regarding $\oplus$) in $\mathbb{R}_F$;
(iv) for any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$ and any $u \in \mathbb{R}_F$, we have 

$$(a + b) \odot u = a \odot u \oplus b \odot u.$$ 

For general $a, b \in \mathbb{R}$, the above property does not hold.

(i) $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$, for all $\lambda \in \mathbb{R}, u, v \in \mathbb{R}_F$;
(ii) $\lambda \odot (\mu \odot u) = (\lambda \mu) \odot u$, for all $\lambda, \mu \in \mathbb{R}, u \in \mathbb{R}_F$;
(vii) If we denote \( \|u\|_F = D\left( u, \bar{0} \right) \), \( u \in \mathbb{R}_F \), then \( \|u\|_F \) has the properties of an usual norm on \( \mathbb{R}_F \), i.e. \( \|u\|_F = 0 \) iff \( u = \bar{0} \), \( \|\lambda \odot u\|_F = |\lambda| \|u\|_F \), \( \|u + v\|_F \leq \|u\|_F + \|v\|_F \), \( \|\|u\|_F - \|v\|_F| \leq D(u, v) \);

(viii) \( D(\alpha \odot u, \beta \odot u) = |\alpha - \beta| D\left( \bar{0}, u \right) \), for all \( \alpha, \beta \geq 0, u \in \mathbb{R}_F \). If \( \alpha, \beta \leq 0 \) then the equality is also valid. If \( \alpha \) and \( \beta \) are of opposite signs, then the equality is not valid.

Remarks. 1) Theorem 2.4 shows that \( (\mathbb{R}_F, \oplus, \odot) \) is not a linear space over \( \mathbb{R} \) and, consequently, \( (\mathbb{R}_F, \|\|_F) \) cannot be a normed space.

2) On \( \mathbb{R}_F \), we can define the substraction \( \ominus \), called the \( H - \) difference (see [12]) as follows: \( u \ominus v \) has sense if there exists \( w \in \mathbb{R}_F \) such that \( u = v \oplus w \). Clearly, \( u \ominus v \) does not exist for all \( u, v \in \mathbb{R}_F \) (for example, \( \bar{0} \ominus v \) does not exists if \( v \neq \bar{0} \)).

In what follows, we define some usual spaces of fuzzy-number-valued functions, which have similar properties to \( (\mathbb{R}_F, D) \).

Denote \( C([a, b]; \mathbb{R}_F) = \{ f : [a, b] \to \mathbb{R}_F ; \ f \text{ is continuous on } [a, b] \} \), endowed with the metric

\[ D^*(f, g) = \sup \{ D(f(x), g(x)) ; x \in [a, b] \}. \]

Since \( (\mathbb{R}_F, D) \) is a complete metric space, by standard technique (see e.g. [20]) we obtain that \( (C([a, b]; \mathbb{R}_F), D^*) \) is also complete metric space. Moreover, if we define

\[ (f \oplus g)(x) = f(x) \oplus g(x), (\lambda \odot f)(x) = \lambda \odot f(x), \]

(for simplicity, the addition and scalar multiplication in \( C([a, b]; \mathbb{R}_F) \) are denoted as in \( \mathbb{R}_F \)), also \( \tilde{0} : [a, b] \to \mathbb{R}_F, \tilde{0}(t) = \bar{0}_{\mathbb{R}_F}, \) for all \( t \in [a, b] \),

\[ \|f\|_F = \sup \left\{ D\left( \tilde{0}, f(x) \right) ; x \in [a, b] \right\}, \]

then we easily obtain the following properties.

**Theorem 2.5**

(i) \( f \odot g = g \odot f, (f \odot g) \odot h = f \odot (g \odot h) \);

(ii) \( f \odot \tilde{0} = \tilde{0} \odot f, \) for any \( f \in C([a, b]; \mathbb{R}_F) \);

(iii) With respect to \( \tilde{0} \) in \( C([a, b]; \mathbb{R}_F) \), any \( f \in C([a, b]; \mathbb{R}_F) \) with \( f([a, b]) \cap \mathbb{R}_F \neq \emptyset \) has no opposite member (regarding \( \oplus \)) in \( C([a, b]; \mathbb{R}_F) \);
(iv) for all \( \lambda, \mu \in \mathbb{R} \) with \( \lambda, \mu \geq 0 \) or \( \lambda, \mu \leq 0 \) and for any \( f \in C ([a, b]; \mathbb{R}_F) \),
\[
(\lambda + \mu) \odot f = (\lambda \odot f) \oplus (\mu \odot f);
\]

For general \( \lambda, \mu \in \mathbb{R} \), this property does not hold.

(v) \( \lambda \odot (f \oplus g) = \lambda \odot f \oplus \lambda \odot g, \lambda \odot (\mu \odot f) = (\lambda \mu) \odot f \), for any \( f, g \in C ([a, b]; \mathbb{R}_F) \), \( \lambda, \mu \in \mathbb{R} \);

(vi) \( \|f\|_F = 0 \) iff \( f = \tilde{0} \), \( \|\lambda \odot f\|_F = |\lambda| \|f\|_F \), \( \|f \oplus g\|_F \leq \|f\|_F + \|g\|_F \), \( \|f\|_F - \|g\|_F \leq D^* (f, g) \), for any \( f, g \in C ([a, b]; \mathbb{R}_F) \), \( \lambda \in \mathbb{R} \);

(vii) \( D^* (\lambda \odot f, \mu \odot f) = |\lambda - \mu| D^* (\tilde{0}, f) \) for any \( f \in C ([a, b]; \mathbb{R}_F) \), \( \lambda \mu \geq 0 \);

(viii)
\[
D^* (f \oplus h, g \oplus h) = D^* (f, g),
\]
\[
D^* (\lambda \odot f, \lambda \odot g) = |\lambda| D^* (f, g),
\]
\[
D^* (f \oplus g, h \oplus e) \leq D^* (f, h) + D^* (g, e),
\]
for any \( f, g, h, e \in C ([a, b]; \mathbb{R}_F) \), \( \lambda \in \mathbb{R} \).

Remark. It is easy to show that if \( f, g \in C ([a, b]; \mathbb{R}_F) \), then \( F : [a, b] \to \mathbb{R} \), defined by \( F (x) = D (f (x), g (x)) \) is continuous on \([a, b]\).

Now, for \( 1 \leq p < \infty \) and a strongly measurable function \( f \) on \([a, b]\), let us define
\[
L^p ([a, b]; \mathbb{R}_F) = \left\{ f : (L) \int_a^b \left( D \left( \tilde{0}, f (x) \right) \right)^p \, dx < +\infty \right\},
\]
where according to e.g. [7], \( f \) is called strongly measurable if, for each \( x \in [a, b] \), \( f_-(x)(r) \) and \( f_+(x)(r) \) are Lebesgue measurable as functions of \( r \in [0, 1] \) (here again, \( [f (x)]^r = [f_-(x)(r), f_+(x)(r)] \) denotes the \( r \)-level set of \( f (x) \in \mathbb{R}_F \)). The following result shows that \( L^p ([a, b]; \mathbb{R}_F) \) is well defined.

**Theorem 2.6** (i) If \( f : [a, b] \to \mathbb{R}_F \) is strongly measurable then \( F : [a, b] \to \mathbb{R}_+ \) defined by \( F (x) = D \left( \tilde{0}, f (x) \right) \) is Lebesgue measurable on \([a, b]\);

(ii) For any \( f, g \in L^p ([a, b]; \mathbb{R}_F) \), \( F (x) = D (f (x), g (x)) \) is Lebesgue measurable and \( L^p \)-integrable on \([a, b]\). Moreover, if we define
\[
D_p (f, g) = \left\{ (L) \int_a^b \left[ D \left( \tilde{0}, f (x) \right) \right]^p \, dx \right\}^{\frac{1}{p}},
\]
then \((L^p ([a, b] ; R_F), D_p)\) is a complete metric space \((\text{in } L^p ([a, b] ; R_F), f = g \text{ means } f(x) = g(x), \text{ a.e. } x \in [a, b])\) and, in addition, \(D_p\) satisfies the following properties:

\[
D_p (f \oplus h, g \oplus h) = D_p (f, g),
\]

\[
D_p (\lambda \odot f, \lambda \odot g) = |\lambda| D_p (f, g),
\]

\[
D_p (f \oplus g, h \oplus e) \leq D_p (f, h) + D_p (g, e),
\]

for any \(f, g, h, e \in L^p ([a, b] ; R_F), \lambda \in R\).

**Proof.** (i) By definition, we have

\[
D \left( \tilde{0}, f(x) \right) = \sup_{r \in [0,1]} \max \{|f_-(x)(r)|, |f_+(x)(r)|\},
\]

for all \(x \in [a, b]\). Because \(f_-(x)(r)\) and \(f_+(x)(r)\) are left continuous at each \(r \in [0, 1]\) and right continuous at \(r = 0\) (see e.g. [11]), denoting by \(Q\), the set of all rational numbers, we easily obtain

\[
D \left( \tilde{0}, f(x) \right) = \sup_{r \in [0,1] \cap Q} \max \{|f_-(x)(r)|, |f_+(x)(r)|\},
\]

which implies that \(D \left( \tilde{0}, f(x) \right)\) is Lebesgue measurable on \([a, b]\) (as supremum of a countable set of Lebesgue measurable functions).

(ii) We have

\[
F(x) = D (f(x), g(x)) = \sup_{r \in [0,1]} \max \{|f_-(x)(r) - g_-(x)(r)|, |f_+(x)(r) - g_+(x)(r)|\},
\]

and reasoning exactly as for the above point (i), it follows that \(F\) is Lebesgue measurable on \([a, b]\). Moreover, since

\[
D (f(x), g(x)) \leq D \left( f(x), \tilde{0} \right) + D \left( \tilde{0}, g(x) \right),
\]

we immediately obtain

\[
(L) \int_a^b [D (f(x), g(x))]^p \, dx < +\infty.
\]
Then by the properties of $D$, it easily follows that $D_p$ is a metric on $L^p([a,b];\mathbb{R}_F)$ satisfying the properties listed above at $(ii)$.

It remains to prove that $(L^p([a,b];\mathbb{R}_F), D_p)$ is complete. Let $f_n \in L^p([a,b];\mathbb{R}_F)$, $n \in \mathbb{N}$ be a Cauchy sequence with respect to the metric $D_p$. By standard technique (see e.g. [20]), we obtain a subsequence $(f_{n_j})_{j \in \mathbb{N}}$ such that $f_{n_j}(x), j \in \mathbb{N}$, a.e. $x \in [a,b]$ is a Cauchy sequence in the complete metric space $(\mathbb{R}_F, D)$. Therefore, there exists $f(x)$ (a.e. $x \in [a,b]$) such that

$$\lim_{j \to +\infty} D(f_{n_j}(x), f(x)) = 0.$$ 

Then by e.g. [7, Proposition 6.1.4], it follows that $f$ is strongly measurable (on $[a,b]$).

The rest of the proof follows standard technique as in e.g. [20]. This proves the theorem. □

In what follows, we introduce the following definition.

**Definition 2.7** (i) (see e.g. [7]). Let $f : [a, b] \to \mathbb{R}_F$ and $x_0 \in (a, b)$. We say that $f$ is (Hukuhara) differentiable at $x_0$, if there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, there exists $f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and $f'(x_0) \in \mathbb{R}_F$ with the property:

$$\lim_{h \to 0} D \left( \frac{1}{h} \ominus (f(x_0 + h) \ominus f(x_0)), f'(x_0) \right) = 0,$$

and

$$\lim_{h \to 0} D \left( \frac{1}{h} \ominus (f(x_0) \ominus f(x_0 - h)), f'(x_0) \right) = 0.$$

(ii) (see [2]). Let $f : (a, b) \to \mathbb{R}_F$ and $t \in (a, b)$. We say that $f$ is generalized differentiable at $t$, if:

1) There exist $f(t + h) \ominus f(t), f(t) \ominus f(t - h)$, for all $h > 0$ sufficiently small and there exist

$$\lim_{h \searrow 0} \frac{f(t + h) \ominus f(t)}{h} = \lim_{h \searrow 0} \frac{f(t) \ominus f(t - h)}{h} = f'(t) \in \mathbb{R}_F$$

or

2) There exist $f(t) \ominus f(t + h), f(t - h) \ominus f(t)$, for all $h > 0$ sufficiently small and there exist

$$\lim_{h \searrow 0} \frac{f(t) \ominus f(t + h)}{-h} = \lim_{h \searrow 0} \frac{f(t - h) \ominus f(t)}{-h} = f'(t) \in \mathbb{R}_F$$
3) There exist $f(t+h) \ominus f(t), f(t-h) \ominus f(t)$, for all $h > 0$ sufficiently small and there exist
\[
\lim_{h \to 0} \frac{f(t+h) \ominus f(t)}{h} = \lim_{h \to 0} \frac{f(t-h) \ominus f(t)}{-h} = f'(t) \in \mathbb{R}_F
\]
or
4) There exist $f(t) \ominus f(t-h), f(t) \ominus f(t+h)$, for all $h > 0$ sufficiently small and there exist
\[
\lim_{h \to 0} \frac{f(t) \ominus f(t-h)}{h} = \lim_{h \to 0} \frac{f(t) \ominus f(t+h)}{-h} = f'(t) \in \mathbb{R}_F
\]
(Here all the limits are considered in the metric $D$ and $h$ at denominators means $\frac{1}{h} \ominus$)

Obviously the Hukuhara differentiability implies the generalized differentiability but the converse is not valid.

If $f'(x_0)$ exists for all $x_0 \in [a, b]$, by iteration we can define
\[
f''(x_0) = \left( f'(x_0) \right)' , x_0 \in [a, b].
\]

For $p \in \mathbb{N}$, let us denote by
\[
C^p([a,b]; \mathbb{R}_F) = \left\{ f : [a,b] \to \mathbb{R}_F ; \text{there exists } f^{(p)} \text{ continuous on } [a,b] \right\}
\]
(here the derivatives $f'$, $f''$, ..., $f^{(p)}$ are all considered of the same kind as in Definition 2.7) and for $f, g \in C^p([a,b]; \mathbb{R}_F)$, let us define
\[
D^*_p(f,g) = \sum_{i=0}^{p} D^* \left( f^{(i)}, g^{(i)} \right),
\]
where
\[
D^* (f^{(i)}, g^{(i)}) = \sup \left\{ D \left( f^{(i)}(x), g^{(i)}(x) \right) ; x \in [a,b] \right\}.
\]

Then we obtain that $D^*_p(f,g)$ is a metric and by standard technique (as in the case of the real-valued functions), we obtain that $(C^p([a,b]; \mathbb{R}_F), D^*_p)$ is a complete metric space. In addition, $D^*_p$ shares the same properties with $D^*$ (see, Theorem 2.5, (vii) and (viii)).
Other spaces with properties similar to those of \((R_F, D)\) can be constructed as follows. For \(p \geq 1\), let us define

\[
l^p_{R_F} = \left\{ x = (x_n)_n : x_n \in R_F, \forall n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \|x_n\|_{R_F}^p < +\infty \right\},
\]

endowed with the metric

\[
\rho_p(x, y) = \left( \sum_{n=1}^{\infty} [D(x_n, y_n)]^p \right)^{\frac{1}{p}}, \forall x = (x_n)_n, y = (y_n)_n \in l^p_{R_F}.
\]

By the inequality,

\[
D(x_n, y_n) = D(x_n + \tilde{0}, \tilde{0} + y_n) \\
\leq D(x_n, \tilde{0}) + D(\tilde{0}, y_n) \\
= \|x_n\|_F + \|y_n\|_F,
\]

we easily get (by Minkowski’s inequality if \(p > 1\)) \(\rho_p(x, y) < +\infty\). Also, it easily follows that \(\rho_p(x, y)\) is a metric with similar properties to \(D\) (see Theorem 2.3 and Theorem 2.4, (vii), (viii)). Since again \((R_F, D)\) is a complete metric space, by the standard technique, we easily get that \((l^p_{R_F}, \rho_p)\) is also a complete metric space.

Next, let us denote

\[
m_{R_F} = \{ x = (x_n)_n : x_n \in R_F, \forall n \in \mathbb{N} \text{ and } \exists M > 0 \text{ such that } \|x_n\|_F \leq M, \forall n \in \mathbb{N} \},
\]

endowed with the metric

\[
\mu(x, y) = \sup \{ D(x_n, y_n), \forall n \in \mathbb{N} \}.
\]

We easily get that \((m_{R_F}, \mu)\) is a complete metric space and, in addition, \(\mu\) has similar properties to \(D\) (see Theorem 2.3 and Theorem 2.4, (vii), (viii)). Similarly, if we denote

\[
c_{R_F} = \left\{ x = (x_n)_n : x_n \in R_F, \forall n \in \mathbb{N} \text{ and } \exists a \in R_F \text{ such that } D(x_n, a) \xrightarrow{n \to \infty} 0 \right\}
\]

and

\[
c_{R_F} = \left\{ x = (x_n)_n : x_n \in R_F, \forall n \in \mathbb{N}, \text{ such that } D(x_n, \tilde{0}) \xrightarrow{n \to \infty} 0 \right\},
\]

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since \((\mathbb{R}_F, D)\) is complete, by standard technique, it follows that \((c_{\mathbb{R}_F}, \mu)\) and 
\((c_{\mathbb{R}_F}, \mu)\) are complete metric spaces. Note that any finite Cartesian product of the
spaces in this section, endowed with the "box" metric (i.e., \(d = \max_i \rho_i\)) is a complete
metric space and the "Cartesian" metric \(d\) shares the same properties with the
metric \(D\) on \(\mathbb{R}_F\). Clearly, the concept of differentiability in Definition 2.7 remains
valid if \(\mathbb{R}_F\) is replaced by any of the spaces \((X, \oplus, \odot, d)\) introduced above.

3 Operator theory and semigroups of operators

In this section, we consider elements of operator theory on the complete metric
spaces introduced in Section 2. Everywhere in this section one can think of
\((X, \oplus, \odot, d)\) as any of the spaces introduced in the previous section.

**Definition 3.1.** \(A : X \to \mathbb{R}\) is a linear functional if

\[
\begin{align*}
A(x \oplus y) &= A(x) + A(y), \\
A(\lambda \odot x) &= \lambda A(x),
\end{align*}
\]

and, also, \(A : X \to X\) is a linear operator if

\[
\begin{align*}
A(x \oplus y) &= A(x) \oplus A(y), \\
A(\lambda \odot x) &= \lambda \odot A(x),
\end{align*}
\]

for all \(x, y \in X, \lambda \in \mathbb{R}\).

**Remark.** If \(A : X \to \mathbb{R}\) or \(A : X \to X\) is linear and continuous at \(\tilde{0} \in X\), then
the latter does not imply the continuity of \(A\) at each \(x \in X\), because in general, we
cannot write \(x_0 = (x_0 \ominus x) \ominus x\).

However, we can prove the following theorem.

**Theorem 3.2** (i) If \(A : X \to \mathbb{R}\) is linear, then it is continuous at \(\tilde{0} \in X\) if and
only if there exists \(M > 0\) such that

\[|A(x)| \leq M \|x\|_F, \forall x \in X,\]

where \(\|x\|_F = d\left(\tilde{0}, x\right)\).
(ii) If \( A : X \to X \) is linear, then it is continuous at \( \tilde{0} \in X \) if and only if there exists \( M > 0 \) such that

\[
\| A(x) \|_F \leq M \| x \|_F, \forall x \in X.
\]

**Proof.** The proof is similar to that in the case of the usual Banach spaces (see e.g. [20]) by taking into account the properties of \((X, \oplus, \odot, d)\). For example, let us prove (ii). By the continuity of \( A : X \to X \) at \( \tilde{0} \), for \( B(\tilde{0}, 1) = \{ y \in X; d(\tilde{0}, y) < 1 \} \), there exists \( V \in \mathcal{V}(\tilde{0}) \) such that \( A(V) \subset B(\tilde{0}, 1) \). Also, since \( V \in \mathcal{V}(\tilde{0}) \), \( \exists r > 0 \) such that \( B(\tilde{0}, r) = \{ y \in X; d(\tilde{0}, y) < r \} \subset V \). It follows \( A\left( B(\tilde{0}, r) \right) \subset B(\tilde{0}, 1) \). Let \( M = 2 \frac{r}{\alpha} \) and let \( x \in X \) be arbitrary. If \( x = \tilde{0} \), then by additivity, we get \( A(\tilde{0}) = \tilde{0} \), and the inequality is trivial. Now let \( x \neq \tilde{0} \). We get \( \| x \|_F = d(\tilde{0}, x) \neq 0 \) and let us define \( \alpha = \frac{r}{2\| x \|_F} > 0 \). We have \( \| \alpha \odot x \|_F = \alpha \| x \|_F = \frac{r}{2} < r \), i.e. \( \alpha \odot x \in B(\tilde{0}, r) \). It follows \( A(\alpha \odot x) \in B(\tilde{0}, 1) \), i.e., \( \| A(\alpha \odot x) \|_F = d(\tilde{0}, A(\alpha \odot x)) < 1 \), which implies \( \alpha \| A(x) \|_F < 1 \), i.e. \( \| A(x) \|_F < \frac{1}{\alpha} = 2 \frac{r}{\alpha} \| x \|_F = M \| x \|_F \). The converse in (ii) is immediate. \( \square \)

Now, for \( A : X \to \mathbb{R} \) linear and continuous at \( \tilde{0} \), let us denote

\[
\mathcal{M}_A := \{ M > 0; |A(x)| \leq M \| x \|_F, \forall x \in X \},
\]

and for \( A : X \to X \) linear and continuous at \( \tilde{0} \), let us denote

\[
\mathcal{M}_A := \{ M > 0; \| A(x) \|_F \leq M \| x \|_F, \forall x \in X \}.
\]

Furthermore, in both cases, denote \( \| A\|_F = \inf_M \mathcal{M}_A \).

We have the following.

**Theorem 3.3** (i) If \( A : X \to \mathbb{R} \) is linear and continuous at \( \tilde{0} \), then

\[
|A(x)| \leq \| A\|_F \| x \|_F
\]

for all \( x \in X \) and

\[
\| A\|_F = \sup \{ |A(x)|; x \in X, \| x \|_F \leq 1 \}.
\]
(ii) If $A : X \to X$ is linear and continuous at $\tilde{0}$, then

$$\| A(x) \|_F \leq \| A \|_F \| x \|_F$$

(1)

for all $x \in X$ and

$$\| A \|_F = \sup \{ \| A(x) \|_F ; x \in X, \| x \|_F \leq 1 \}.$$  

(2)

**Proof.** The proof follows by standard techniques from functional analysis. For example, let us prove (ii). Let us suppose that (1) does not hold, i.e., $\exists x' \in X$ with $\| A(x') \|_F > \| A \|_F \| x' \|_F$, which means

$$D \left( 0, A(x') \right) > \| A \|_F D \left( 0, x' \right).$$

Denote $\alpha = D \left( 0, x' \right)$. We have $\alpha > 0$ (otherwise $x' = \tilde{0}$, i.e. $A(x') = \tilde{0}$ and $\| A(0) \|_F = 0 > \| A \|_F \| 0 \|_F$, which is impossible). We get $D \left( 0, \frac{1}{\alpha} \circ A(x') \right) > \| A \|_F$, i.e. $D \left( 0, A(x'') \right) > \| A \|_F$ with $x'' = \frac{1}{\alpha} \circ x'$, $\| x'' \|_F = 1$. For $\varepsilon = \| A(x'') \|_F - \| A \|_F > 0$, by $\| A \|_F = \inf M_A$, there exists $M \in M_A$ with $M < \| A \|_F + \varepsilon = \| A(x'') \|_F$, which implies the contradiction

$$\| A(x'') \|_F \leq M \| x'' \|_F = M < \| A(x'') \|_F.$$  

Therefore, (1) must hold.

It remains to prove (2). First, for $x \in X$, $\| x \|_F \leq 1$, by (1) we get $\| A(x) \|_F \leq \| A \|_F$, which implies

$$\sup \{ \| A(x) \|_F ; x \in X, \| x \|_F \leq 1 \} \leq \| A \|_F.$$

If $\| A \|_F = 0$, this inequality becomes equality. So, let us assume $\| A \|_F > 0$. There exists $n_0 \in \mathbb{N}$ such that $\| A \|_F - \frac{1}{n} > 0$, for all $n \geq n_0$. Since $\| A \|_F - \frac{1}{n} \notin M_A$, there exists $x' \in X$, $x' \neq \tilde{0}$ with

$$\| A(x') \|_F > \left( \| A \|_F - \frac{1}{n} \right) \| x' \|_F.$$  

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which can be written as $\|A(x')\|_F > (\|A\|_F - \frac{1}{n})$, for all $n \geq n_0$, with

$$x'' = \frac{1}{\|x'\|_F} \odot x'.$$

Therefore,

$$\sup \{\|A(x')\|_F; x \in X, \|x\|_F \leq 1\} \geq \|A\|_F - \frac{1}{n}.$$ 

Passing to the limit as $n \to \infty$, we get

$$\sup \{\|A(x)\|_F; x \in X, \|x\|_F \leq 1\} \geq \|A\|_F,$$

which proves (2).□

**Corollary 3.4**

(i) If $A : X \to \mathbb{R}$ is additive (i.e., $A(x \oplus y) = A(x) + A(y)$), positive homogeneous (i.e., $A(\lambda \odot y) = \lambda A(x),$ $\forall \lambda \geq 0$) and continuous at $\tilde{0}$, then

$$|A(x)| \leq \|A\|_F \|x\|_F, \forall x \in X.$$ 

(ii) If $A : X \to X$ is additive, positive homogeneous and continuous at $\tilde{0}$, then

$$\|A(x)\|_F \leq \|A\|_F \|x\|_F, \forall x \in X.$$ 

The proof is similar to that of Theorem 3.3, because it uses only the positive homogeneity and additivity (throughout the proof of Theorem 3.2) of $A$.

**Remark.** Examples of operators are the following. Let $X = \mathbb{R}_F$. Define the following operators $A_1, A_4, A_5 : \mathbb{R}_F \to \mathbb{R}$, $A_2, A_3 : \mathbb{R}_F \to \mathbb{R}_F$ by the following expressions:

$$A_1(x) = \int_0^1 [x_-(r) + x_+(r)] \, dr,$$

$$A_2(x) = \left\{ \int_0^1 [x_+(0) - x_+(r)] \, dr \right\} \odot c,$$

$$A_3(x) = \left\{ \int_0^1 [x_-(1) - x_-(r)] \, dr \right\} \odot c, c \in \mathbb{R}_F,$$

$$A_4(x) = \int_0^1 x_-(r) \, dr,$$

$$A_5(x) = \int_0^1 x_+(r) \, dr.$$
where \([x_-(r), x_+(r)] = \{ t \in \mathbb{R}; x(t) \geq r \}\). By e.g. [11], \(x_+(r)\) is bounded non-decreasing on \([0, 1]\), \(x_-(r)\) is bounded non-increasing on \([0, 1]\), both are left continuous on \((0, 1]\) and right continuous at \(r = 0\). It follows that \(x_+(0) - x_+(r) \geq 0, x_-(1) - x_-(r) \geq 0\) and by a simple calculation, \(A_1\) is linear and continuous at each \(x \in \mathbb{R}_F\), and \(A_4, A_5\) are additive, positive homogeneous and continuous at each \(x \in \mathbb{R}_F\). Finally, \(A_2\) and \(A_3\) are linear continuous operators on \(\mathbb{R}_F\). Other examples of linear operators induced by some fuzzy differential equations will be considered in the next section.

Next, let us denote

\[
\mathcal{L}_0^+(X) = \left\{ A : X \to X; A \text{ is additive, positive homogeneous and continuous at } \tilde{0} \right\},
\]

\[
\mathcal{L}_0(X) = \left\{ A : X \to X; A \text{ is linear and continuous at } \tilde{0} \right\},
\]

where \((X, d)\) is any of the spaces described in the beginning of this section. Let us consider the metric \(\Phi : \mathcal{L}_0^+(X) \times \mathcal{L}_0^+(X) \to \mathbb{R}_+\) by

\[
\Phi(A, B) = \sup \{ d(A(\tilde{0}), B(\tilde{0})) ; \|x\|_F \leq 1 \}.
\]

Clearly, we have \(\Phi(A, \tilde{O}) = ||A||_F, A \in \mathcal{L}_0^+(X)\), where \(\tilde{O} : X \to X\) is given by \(\tilde{O}(x) = \tilde{0}, \forall x \in X\).

**Theorem 3.5** \((\mathcal{L}_0^+(X), \Phi)\) is a complete metric space and, in addition, \(\Phi\) has the following properties: if we define

\[
(A \oplus B) (x) = A (x) \oplus B (x)
\]

and

\[
(\lambda \odot A) (x) = \lambda \odot A (x),
\]

then the following hold:

(i) \(\Phi(A \oplus B, C \oplus D) \leq \Phi(A, C) + \Phi(B, D)\),

(ii) \(\Phi(k \odot A, k \odot B) = |k| \Phi(A, B)\),

(iii) \(\Phi(A, B) \leq ||A||_F + ||B||_F\),

(iv) \(\Phi(A \oplus B, C) \leq \Phi(A, C) + \Phi(B, C)\),

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Consequently, \(1\) We have to prove that

\[
\forall \text{ sequence, i.e., }
\]

\[
A
\]

Moreover, if \(x \in X\), we will define \(A : X \to X\) as follows. If \(\|x\|_F \leq 1\), then by the definition of \(A\) we get

\[
d(A_n(x), A_{n+p}(x)) < \varepsilon, \text{ for all } n \geq n_0, p \in \mathbb{N}.
\]

Moreover, if \(\|x\|_F > 1\), denoting \(y = \alpha \circ x\) with \(\alpha = \frac{1}{\|x\|_F} > 0\), we get \(\|y\|_F = 1\),

\[
d(A_n(y), A_{n+p}(y)) = \alpha d(A_n(x), A_{n+p}(x)) < \varepsilon, \text{ i.e.,}
\]

\[
d(A_n(x), A_{n+p}(x)) < \varepsilon \|x\|_F, \text{ for all } n \geq n_0, p \in \mathbb{N}.
\]

Consequently, \((A_n(x))_{n \in \mathbb{N}}\) is a Cauchy sequence in the complete metric space \((X, d)\), i.e., it is convergent. Let us denote \(A(x) = \lim_{n \to \infty} A_n(x)\), i.e.,

\[
\lim_{n \to \infty} d(A_n(x), A(x)) = 0, x \in X.
\]

We have to prove that \(A \in \mathcal{L}_0^+(X)\). First,

\[
0 \leq d(A(x \oplus y), A(x) \oplus A(y)) \leq d(A(x \oplus y), A_n(x \oplus y))
\]

\[
+ d(A_n(x \oplus y), A_n(x) \oplus A_n(y)) + d(A_n(x) \oplus A_n(y), A(x) \oplus A(y))
\]

\[
\leq d(A(x \oplus y), A_n(x \oplus y)) + d(A_n(x), A(x)) + d(A_n(y), A(y)).
\]

Passing to the limit as \(n \to \infty\), we obtain \(0 \leq d(A(x \oplus y), A(x) \oplus A(y)) \leq 0\), i.e.,

\(A(x \oplus y) = A(x) \oplus A(y)\). Let \(\lambda > 0\) and \(x \in X\). Similarly, we get

\[
0 \leq d(A(\lambda \circ x), \lambda \circ A(x)) \leq d(A(\lambda \circ x), A_n(\lambda \circ x))
\]

\[
+ d(A_n(\lambda \circ x), \lambda \circ A_n(x)) + d(\lambda \circ A_n(x), \lambda \circ A(x))
\]

\[
\leq d(A(\lambda \circ x), A_n(\lambda \circ x)) + \lambda d(A_n(x), A(x)),
\]

and passing to the limit as \(n \to \infty\), it follows that \(A(\lambda \circ x) = \lambda \circ A(x)\).
Now, we show that $A$ is continuous at $\tilde{0}$. First, by Corollary 3.4, (ii), we have $\|A_n(x)\|_F \leq \|A_n\|_F \|x\|_F$, $\forall x \in X$, $n \in \mathbb{N}$. We have

$$\|A(x)\|_F = d\left(A(x), \tilde{0}\right) \leq d\left(A(x), A_n(x)\right) + d\left(A_n(x), \tilde{0}\right)$$

But, since $\Phi$ is a metric, we obtain

$$\left|\Phi\left(A_n, \tilde{0}\right) - \Phi\left(A_m, \tilde{0}\right)\right| \leq \Phi\left(A_n, A_m\right),$$

which shows that the sequence of real positive numbers $\left(\Phi\left(A_n, \tilde{0}\right)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence, i.e., it is bounded, implying the existence of $M > 0$ with

$$\|A_n(x)\|_F = \Phi\left(A_n, \tilde{0}\right) \leq M,$$

It follows that

$$\|A(x)\|_F \leq d\left(A(x), A_n(x)\right) + M \|x\|_F, \forall x \in X, n \in \mathbb{N}.$$ 

Passing to the limit as $n \to \infty$, we obtain

$$\|A(x)\|_F \leq M \|x\|_F, \forall x \in X,$$

i.e., according to Theorem 3.2 (ii), $A$ is continuous at $\tilde{0} \in X$. In conclusion, $(\mathcal{L}_0^+(X), \Phi)$ is complete, which proves the theorem. $\square$

**Corollary 3.6** (i) $(\mathcal{L}_0(X), \Phi)$ is a complete metric space.

(ii) If we denote

$$\mathcal{L}^+(X) = \{A \in \mathcal{L}_0^+(X); A \text{ is continuous at each } x \in X\},$$

$$\mathcal{L}(X) = \{A \in \mathcal{L}_0(X); A \text{ is continuous at each } x \in X\},$$

then $(\mathcal{L}^+(X), \Phi)$ and $(\mathcal{L}(X), \Phi)$ are complete metric spaces.

Concerning these spaces of operators, in what follows, we prove the uniform boundedness principle.

**Theorem 3.7** Let $(X, d)$ be any of the spaces listed in the beginning of this section, and let $\mathbf{L}(X)$ be either $\mathcal{L}(X)$ or $\mathcal{L}^+(X)$. If $A_j \in \mathbf{L}(X), j \in J,$ is pointwise
bounded, i.e., for any \( x \in X \), \( \| A_j (x) \|_F = d \left( A_j (x), 0 \right) \leq M_x \), for all \( j \in J \), then there exists \( M > 0 \) such that

\[
\| A_j \|_F \leq M, \forall j \in J,
\]

(i.e., \( (A_j) \) is uniformly bounded).

**Proof.** For any \( n \in \mathbb{N} \), let us denote \( A_n = \{ x \in X ; \| A_j (x) \|_F \leq n, \forall j \in J \} \). It is obvious that \( X = \bigcup_{n \in \mathbb{N}} A_n \). But \( A_n \) are closed sets (because if \( d (x, m, x) \xrightarrow{m \to \infty} 0, x_m \in A_n, \forall m \), then by \( d \left( A_j (x), 0 \right) \leq d (A_j (x), A_j (x_m)) + d \left( A_j (x_m), 0 \right) \leq n + d (A_j (x), A_j (x_m)), \) passing to the limit as \( m \to \infty \) and taking into account the continuity of \( A_j \) at each \( x \), we get \( x \in A_n \)).

Since \( (X, d) \) is a complete metric space, it is of second Baire category, therefore there exists \( m \in \mathbb{N} \), such that \( \text{int} A_m \neq \emptyset \). Let \( x_0 \in \text{int} A_m \) and \( \lambda > 0 \) such that \( B (x_0, \lambda) = \{ x \in X ; d \left( x, 0 \right) < \lambda \} \subset A_m \). For \( x \in X \), denote \( x_1 = x_0 \oplus \frac{\lambda}{2 \| x \|_F} \odot x \).

We have

\[
d (x_1, x_0) = d \left( x_0 \oplus \frac{\lambda}{2 \| x \|_F} \odot x, x_0 \right) = d \left( \frac{\lambda}{2 \| x \|_F} \odot x, 0 \right) = \frac{\lambda}{2} < \lambda,\]

i.e., \( x_1 \in B (x_0, \lambda) \). Then, for all \( x \in X, x \neq 0 \), we have

\[
d \left( A_j (x), 0 \right) = \frac{2 \| x \|_F}{\lambda} d \left( A_j \left( x_0 \oplus \frac{\lambda}{2 \| x \|_F} \odot x \right), A_j (x_0) \right) = \frac{2 \| x \|_F}{\lambda} d (A_j (x_1), A_j (x_0)) \leq \frac{2 \| x \|_F}{\lambda} [\| A_j (x_1) \|_F + \| A_j (x_0) \|_F] \leq \frac{4m}{\lambda} \| x \|_F,
\]

which implies \( \| A_j (x) \|_F \leq \frac{4m}{\lambda} \| x \|_F \) and, therefore, \( \| A_j (x) \|_F \leq M, \forall j \in J \). The theorem is proved. □

**Remark.** Some results in classical functional analysis concerning invertible operators can also be considered but only in a particular case.

Thus, let us recall that \( x \in \mathbb{R}_F \) is called triangular fuzzy number, if we have \( [x] = \{ x_c \}, [x]^0 = [x_l, x_r] \) where \( x_l \leq x_c \leq x_r \) (see e.g. [8]). This implies

\[
[x]^{r} = [x_c - (1 - r) (x_c - x_r), x_c + (1 - r) (x_r - x_c)] , \ r \in [0, 1].
\]
We denote \( x = (x_l, x_c, x_r) \). A triangular number \( x \) is called symmetric if there exists \( \delta \geq 0 \) with \( x = (x_c - \delta, x_c, x_c + \delta) \). Denote by \( R^{TS}_F \) the set of all symmetric triangular fuzzy numbers. Under \( \oplus, \odot \) and passing to limit, \( R^{TS}_F \) is closed, therefore \((R^{TS}_F, D)\) is a complete metric space. Also, for any \( x_1, x_2 \in R^{TS}_F \), there exists \( x_1 \oplus x_2 \in R^{TS}_F \) or \( x_2 \ominus x_1 \in R^{TS}_F \) (see e.g. [7], [17]).

This last property allows us to state the following

**Theorem 3.8** Let \( A \in \mathcal{L}(R^{TS}_F) \). Then \( A^{-1} \in \mathcal{L}(R^{TS}_F) \) if and only if there exists \( \lambda > 0 \) such that

\[
\|A(x)\|_F \geq \lambda \|x\|_F, \forall x \in R^{TS}_F.
\]

In this case, \( \|A^{-1}\|_F \leq \frac{1}{\lambda} \).

**Proof.** First, let us assume \( A^{-1} \in \mathcal{L}(R^{TS}_F) \). Similarly (see the proof of Theorem 3.2., (ii)), there exists \( \alpha > 0 \) such that

\[
\|A^{-1}(y)\|_F \leq \alpha \|y\|_F, \forall y \in A(R^{TS}_F).
\]

For \( x = A^{-1}(y) \), we get \( \|A(x)\|_F \geq \frac{1}{\alpha} \|x\|_F \), i.e., we can take \( \lambda = \frac{1}{\alpha} \).

Conversely, let us suppose that \( \|A(x)\|_F \geq \lambda \|x\|_F, \forall x \in R^{TS}_F \) and take \( x_1, x_2 \in R^{TS}_F \), \( x_1 \neq x_2 \). Then, \( x_1 \ominus x_2 \) or \( x_2 \ominus x_1 \) exists, so let us choose, for example, \( u = x_1 \ominus x_2 \), i.e., \( x_1 = x_2 \oplus u \). We get \( A(x_1) \ominus A(x_2) = A(u) \). Now, if \( A(x_1) = A(x_2) \), we obtain \( A(u) = \tilde{0} \) and by our inequality, it follows \( u = \tilde{0} \), i.e., \( x_1 = x_2 \). Therefore \( A \) is invertible and, moreover, for \( y = A(x) \),

\[
\|A^{-1}(y)\|_F = \|x\|_F \leq \frac{1}{\lambda} \|A(x)\|_F = \frac{1}{\lambda} \|y\|_F,
\]

which implies \( A^{-1} \in \mathcal{L}(R^{TS}_F) \) and \( \|A^{-1}\|_F \leq \frac{1}{\lambda} \). \( \square \)

**Remark.** Unfortunately, if we replace \( R^{TS}_F \) by any from the function spaces \( X = (L^p([a, b]; R_F), D_p), (C^p([a, b]; R_F), D_p^*) \), \( p \in \mathbb{N} \), Theorem 3.8 does not hold because we cannot define, in general, \( x_1 \ominus x_2 \) and \( x_2 \ominus x_1 \) for \( x_1, x_2 \in X \). Effectively, this means that the uniqueness problem in fuzzy differential equations may not be valid in these spaces. At the end of this section, we consider some elements of semigroup theory.
Theorem 3.9 Let \((X, d)\) be any from the spaces described in the beginning of this section, and \((L(X), \Phi)\) be the space \((\mathcal{L}(X), \Phi)\). Let us define \(T(t) = e^{t \odot A}\), \(t \in \mathbb{R}\), by
\[
\lim_{m \to \infty} \Phi \left( T(t), \sum_{p=0}^{m} \frac{t^p}{p!} \odot A^p \right) = 0,
\]
where \(\sum\) is the sum with respect to \(\oplus\), \(A \in L(X)\) and \(A^0 = I, A^p = A^{p-1} \circ A, p = 2, 3, \ldots\). Formally, we write
\[
e^{t \odot A} = \sum_{p=0}^{\infty} \frac{t^p}{p!} \odot A^p.
\]
We have:

(i) \(T(t) \in L(X)\) for all \(t \in \mathbb{R}\);

(ii) \(T(t + s) = T(t)(T(s))\), for all \(t, s \geq 0\), or for all \(t, s \leq 0\);

(iii) There exists
\[
\lim_{h \to 0} \frac{1}{h} \odot [T(h)(x) \ominus x] = A(x),
\]
for all \(x \in X\), where the limit is considered in the metric \(d\);

(iv) \(T(t)\) is continuous as function of \(t \in \mathbb{R}\) and \(T(0) = I\). Also, \(T(t)\) is generalized differentiable with respect to all \(t \in \mathbb{R}\), with the derivative equal to \(A[T(t)]\). More exactly, it is Hukuhara differentiable (in the sense of Definition 2.7, (i), see the remark at the end of Section 2) with respect to \(t \in \mathbb{R}_+\), i.e., for all \(t \geq 0\) we have
\[
\lim_{h \to 0} d\left( \frac{1}{h} \odot (T(t + h)(x) \ominus T(t)(x)), A[T(t)(x)] \right)
= \lim_{h \to 0} d\left( \frac{1}{h} \odot (T(t)(x) \ominus T(t - h)(x)), A[T(t)(x)] \right)
= 0,
\]
and generalized differentiable (in the sense of Definition 2.7,(ii)) with respect to \(t < 0\), i.e., for all \(t < 0\) we have
\[
\lim_{h \to 0} d\left( -\frac{1}{h} \odot (T(t)(x) \ominus T(t + h)(x)), A[T(t)(x)] \right)
= \lim_{h \to 0} d\left( -\frac{1}{h} \odot (T(t - h)(x) \ominus T(t)(x)), A[T(t)(x)] \right)
= 0,
\]
for all \( x \in X \).

(v) If \( u_0 \in X \) and \( g: \mathbb{R} \to X \) is continuous on \( \mathbb{R} \), then

\[
u(t) = T(t)(u_0) \oplus \int_0^t T(t-s)g(s)ds
\]

is generalized differentiable on \( \mathbb{R} \) (more exactly, it is Hukuhara differentiable on \( \mathbb{R}_+\) and generalized differentiable as in the above point (iv) for \( t < 0 \)) and satisfies

\[
u'(t) = A[u(t)] \oplus g(t), \quad t \in \mathbb{R},
\]

\[u(0) = u_0,
\]

where \( u'(t) \) denotes the generalized derivative. Here, the integral for functions defined on a compact interval with values in \( X \) is considered in the Riemann (classical) sense.

**Proof.** (i) Denote \( S_m(t)(x) = \sum_{p=0}^{m} \frac{t^p}{p!} \odot A^p(x) \). By Theorem 3.5 and Corollary 3.6, it suffices to show that \( (S_m(t))_m \) is a Cauchy sequence in the complete metric space \( (L(X), \Phi) \), for all \( t \in \mathbb{R} \). First, since \( A \in L(X) \), it follows \( A^p \in L(X) \) for \( p = 2, 3, ..., \) and \( S_m(t) \in L(X), \forall m = 0, 1, ... \). Then

\[
d(S_n(t)(x), S_{n+p}(t)(x)) = d\left( \sum_{i=0}^{n} \frac{t^i}{i!} \odot A^i(x), \sum_{i=0}^{n+p} \frac{t^i}{i!} \odot A^i(x) \right)
\]

\[
\leq \sum_{i=n+1}^{n+p} d\left( \frac{t^i}{i!} \odot A^i(x) \right) = \sum_{i=n+1}^{n+p} \frac{|t|^i}{i!} d\left( \tilde{0}, A^i(x) \right).
\]

However,

\[
d\left( \tilde{0}, A(A(x)) \right) = \| A^2(x) \|_F \leq \| A \|_F \| A(x) \|_F \leq \| A \|_F^2 \| x \|_F
\]

and, so by induction, one can obtain

\[
\| A^i \|_F \leq \| A \|_F^i, \text{ for all } i = 2, 3, ...
\]

We obtain

\[
d(S_n(t)(x), S_{n+p}(t)(x)) \leq \sum_{i=n+1}^{n+p} \frac{(|t| \| A \|_F \| x \|_F)^i}{i!},
\]

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and passing to supremum with \( \|x\|_F \leq 1 \), we get

\[
\Phi(S_n(t), S_{n+p}(t)) \leq \sum_{i=1}^{n+p} \left( \frac{\|A\|_F}{i} \right)^i,
\]

which immediately implies that \((S_n(t))_n\) is a Cauchy sequence, therefore, its limit \(T(t)\) exists in \(L(X)\). Note that the last inequality actually implies that \((S_n(t))_n\) is a uniformly Cauchy sequence on \([-a, +a]\), for any real number \(a > 0\), which implies that \(\lim_{n \to +\infty} \Phi(S_n(t), T(t)) = 0\) holds uniformly on each compact interval \([-a, a]\), \(a > 0\).

(ii) First let \(t, s \geq 0\). With the notation in the above point (i), a simple calculation shows \(S_m(t)[S_n(s)(x)] = S_{m+n}(t + s)(x)\), for all \(x \in X\). Then

\[
d[T(t)(T(s)(x)), T(t + s)(x)] \leq d[T(t)(T(s)(x)), S_m(t)(T(s)(x))] \\
+ d[S_m(t)(T(s)(x)), S_m(t)(S_n(s)(x))] \\
+ d[S_m(t)(S_n(s)(x)), S_{m+n}(t + s)(x)] \\
+ d[S_{m+n}(t + s)(x), T(t + s)(x)].
\]

Let \(\epsilon > 0\) be arbitrary fixed. There exists \(m_1, n_1 \in \mathbb{N}\), such that for all \(m > m_1\) and \(n > n_1\), we have \(d[S_{m+n}(t + s)(x), T(t + s)(x)] < \epsilon/3\). Because \(S_m(t)(y)\) converges to \(T(t)(y)\), there exists \(m_2 \in \mathbb{N}\) such that \(d[T(t)(T(s)(x)), S_m(t)(T(s)(x))] < \epsilon/3\), \(\forall m > m_2\). Let \(m > \max\{m_1, m_2\} = m_0\) be fixed. Since \(S_m(t)\) is a continuous linear operator and \(S_n(s)(x) \to T(s)(x)\), there exists \(n_2 \in \mathbb{N}\) such that for all \(n > n_2\) we have \(d[S_m(t)(T(s)(x)), S_m(t)(S_n(s)(x))] < \epsilon/3\). Next, choosing \(m > m_0\) and \(n > \max\{n_1, n_2\} = n_0\), we obtain \(d[T(t)(T(s)(x)), T(t + s)(x)] < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\), with arbitrary \(\epsilon > 0\), i.e., \(d[T(t)(T(s)(x)), T(t + s)(x)] = 0\), which proves the assertion.

Now, because \(t, s \geq 0\), a similar calculation shows that \(S_{m+n}(-t - s)(x) = S_m(-t)(S_n(-s)(x))\). Thus, we obtain

\[
T(t + s)(x) = T(t)(T(s)(x)), \forall t, s \leq 0,
\]

which proves the point (ii).
(iii) Since \( S_m(h)(x) = x \oplus \sum_{p=1}^{m} \frac{h^p}{p!} \odot A^p(x) \), we get
\[
\frac{1}{h} \odot [S_m(h)(x) \ominus x] = \sum_{p=1}^{m} \frac{h^{p-1}}{p!} \odot A^p(x).
\]

We need the following auxiliary result in \((X, \oplus, \odot, d)\): if \( d(a_n, a) \to 0 \) and there exist \( a_n \ominus b \), for all \( n \in \mathbb{N} \), then there exists \( a \ominus b \) and \( d(a_n \ominus b, a \ominus b) \to 0 \). Indeed, denote \( c_n = a_n \ominus b, n \in \mathbb{N} \), that is \( a_n = c_n \oplus b \), for all \( n \in \mathbb{N} \). Since
\[
d(c_n, c_m) = d(a_n \ominus b, a_m \ominus b)
\]
\[
= d((a_n \ominus b) \oplus b, (a_m \ominus b) \oplus b)
\]
\[
= d(a_n, a_m),
\]

it follows that \((c_n)_n\) is a Cauchy sequence in the complete metric space \((X, d)\), i.e., it is convergent, so let us denote by \( c \) its limit. We have
\[
d(a, b \oplus c) \le d(a, a_n) + d(c_n \oplus b, c \oplus b) = d(a, a_n) + d(c_n, c) \to 0,
\]
i.e., \( a = b \oplus c \), which implies \( c = a \ominus b \). Therefore, \( d(a_n \ominus b, a \ominus b) = d(c_n, c) \to 0 \).

Since \( S_m(h)(x) \to T(h)(x) \) when \( m \to +\infty \), for all \( h > 0 \) and \( x \in X \), applying the auxiliary result, it follows that for all \( h > 0 \) and \( x \in X \), we have
\[
\lim_{m \to +\infty} d\left( \frac{1}{h} \odot [S_m(h)(x) \ominus x], \frac{1}{h} \odot [T(h)(x) \ominus x] \right) = 0.
\]

Moreover,
\[
d\left( \frac{1}{h} \odot [T(h)(x) \ominus x], A(x) \right)
\]
\[
\le d\left( \frac{1}{h} \odot [T(h)(x) \ominus x], \frac{1}{h} \odot [S_m(h)(x) \ominus x] \right)
\]
\[
+ d\left( \frac{1}{h} \odot [S_m(h)(x) \ominus x], A(x) \right).
\]

Recall that
\[
\frac{1}{h} \odot [S_m(h)(x) \ominus x] = A(x) \oplus \sum_{p=2}^{m} \frac{h^{p-1}}{p!} \odot A^p(x),
\]

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i.e., there exists $\frac{1}{h} \odot [S_m(h)(x) \ominus x] \ominus A(x)$ and
\[
d(\frac{1}{h} \odot [S_m(h)(x) \ominus x] \ominus A(x), \tilde{0}) = d(\frac{1}{h} \odot [S_m(h)(x) \ominus x], A(x))
\]
\[
= d\left(\sum_{p=2}^{m} \frac{h^{p-1}}{p!} \odot A^p(x), \tilde{0}\right)
\]
\[
\leq \sum_{p=2}^{m} \frac{h^{p-1}}{p!} ||A^p(x)||_F
\]
\[
\leq ||x||_F \sum_{p=2}^{m} \frac{h^p}{p!}||A||_F^p
\]
\[
= ||x||_F \frac{e_m(h||A||_F) - 1 - h||A||_F}{h},
\]
where $e_m(h) = 1 + \frac{h}{1!} + ... + \frac{h^m}{m!}$ is the partial sum of order $m$ of the usual exponential $e^h$. Consequently, passing to the limit as $m \to +\infty$, we obtain
\[
d(\frac{1}{h} \odot [T(h)(x) \ominus x], A(x)) \leq ||x||_F \frac{e^h||A||_F - 1 - h||A||_F}{h}, \forall x \in X.
\]
Now passing to the limit as $h \searrow 0$ and taking into account the continuity of the metric $d$ (with respect to its components), it follows
\[
\lim_{h \searrow 0} d(\frac{1}{h} \odot [T(h)(x) \ominus x], A(x)) = 0,
\]
for all $x \in X$, which proves (iii).

(iv) We distinguish three cases: $t > 0$, $t = 0$ or $t < 0$. We have
\[
d(T(t)(x), T(t+h)(x)) \leq d(T(t)(x), S_n(t)(x))
\]
\[
+ d(S_n(t)(x), S_n(t+h)(x))
\]
\[
+ d(S_n(t+h)(x), T(t+h)(x)),
\]
where
\[
S_n(t)(x) = \sum_{p=0}^{n} \frac{t^p}{p!} \odot A^p(x).
\]
Passing to the limit as $n \to \infty$ in the above inequality, and using (viii) of Theorem 2.4, we obtain
\[
d(T(t)(x), T(t+h)(x)) \leq \sum_{p=1}^{\infty} \frac{|(t+h)^p - t^p|}{p!} d(\tilde{0}, A^p(x)).
\]
While for \( t = 0 \), the above inequality is trivial, when \( t > 0 \) or \( t < 0 \) we choose a sufficiently small \( \delta > 0 \) such that for all \( |h| < \delta \), we have either \( t \pm h > 0 \) or \( t \pm h < 0 \), respectively. Passing to supremum with \( \|x\|_\mathcal{F} \leq 1 \), we get
\[
\Phi (T(t), T(t+h)) \leq \sum_{p=1}^{\infty} \frac{|(t+h)^p - t^p|}{p!} \| |A| \|_\mathcal{F}^p .
\]

But for \( t \neq 0 \), there exists \( \sigma \) with \( |\sigma| < 2|t| \), such that for \( |h| \) sufficiently small (e.g. for \( |h| < |t| \)), we have
\[
|(t+h)^p - t^p| = p |\sigma|^{p-1} |h| \leq p|2t|^{p-1} |h| ,
\]
which implies
\[
\Phi (T(t), T(t+h)) \leq |h| \| |A| \|_\mathcal{F} \sum_{p=1}^{\infty} \frac{1}{(p-1)!} (2|t| \| |A| \|_\mathcal{F})^{p-1} .
\]

Passing to the limit as \( h \to 0 \), it follows that \( \lim_{h \to 0^+} \Phi (T(t+h), T(t)) = 0 \), i.e., \( T(t) \) is continuous in \( t \in \mathbb{R} \).

Now, let \( t \geq 0 \) and \( h > 0 \). Replacing (in the proof of relation (iii)) \( x \) by \( T(t)(x) \), we get
\[
\lim_{h \to 0^+} d\left( \frac{1}{h} \odot [T(h)(T(t)(x)) \ominus T(t)(x)], A(T(t)(x)) \right) = 0,
\]
and since \( T(h)(T(t)(x)) = T(t+h)(x) \), passing to supremum with \( \|x\|_\mathcal{F} \leq 1 \), it follows that
\[
\lim_{h \to 0^+} \Phi \left( \frac{1}{h} \odot [T(t + h) \ominus T(t) \ominus T(t)], A[T(t)] \right) = 0.
\]
Similarly, denoting \( t = t - h + h \), for \( 0 < h < t \), replacing in the proof of relation (iii) \( x \) by \( T(t - h)(x) \) we obtain
\[
\begin{align*}
&\lim_{h \to 0} d\left( \frac{1}{h} \odot [T(t)(x) \ominus T(t - h)(x)], A[T(t - h)(x)] \right) \\
&\leq \|T(t - h)(x)\|_\mathcal{F} \frac{e^{h\| |A| \|_\mathcal{F} - 1 - h\| |A| \|_\mathcal{F}}}{h}.
\end{align*}
\]
Passing to the limit as \( h \searrow 0 \), from the continuity of \( A, T(t) \) and reasoning as above, we arrive at
\[
\lim_{h \searrow 0} d\left( \frac{1}{h} \odot [T(t)(x) \ominus T(t - h)(x)], A(T(t)(x)) \right) = 0,
\]
and passing to supremum with $||x||_F \leq 1$, it follows that
\[
\lim_{h \to 0^+} \Phi \left( \frac{1}{h} \diamond [T(t) \ominus T(t-h)], A[T(t)] \right) = 0.
\]
Now, let $t < 0$ and $h > 0$. Since by the above, (i) we have $S_m(k)(x) \to T(k)(x)$, for all $k < 0$ and $x \in X$, repeating word for word the proof in (iii), we immediately obtain
\[
\lim_{k \to 0} d(\frac{1}{k} \diamond [T(k)(x) \ominus x], A(x)) = 0.
\]
Replacing $k$ by $-h$ and $x$ by $T(t)(x)$, by using (ii), we get
\[
\lim_{h \searrow 0} d(-\frac{1}{h} \diamond [T(t-h)(x) \ominus T(t)(x)], A[T(t)(x)]) = 0.
\]
On the other hand, replacing $x$ by $T(t-k)(x)$ (where for $k$ sufficiently close to zero, we have $t - k < 0$) and then $k$ by $-h$, we arrive to
\[
\lim_{h \searrow 0} d(-\frac{1}{h} \diamond [T(t)(x) \ominus T(t+h)(x)], A[T(t)(x)]) = 0,
\]
which proves (iv).

(v) Let $t \geq 0$. We have
\[
u'(t) = [T(t)(u_0)]' \oplus \left( \int_0^t T(t-s)g(s)ds \right)_{t},
\]
where according to the point (iv), we have $[T(t)(u_0)]' = A[T(t)(u_0)]$. Denoting
\[
F(t) = \int_0^t T(t-s)(g(s))ds,
\]
we will show that $F'(t)$ exists and we shall find it. To this end, let $h > 0$. Denoting $F(t) = \int_0^t T(t-s)(g(s))ds$, we have $t - s \geq 0$ and, using (ii), we get
\[
F(t+h) = \int_0^{t+h} T(t-s+h)(g(s))ds = T(h)[\int_0^{t+h} T(t-s)(g(s))ds]
\]
\[
= T(h)[F(t) \oplus \int_t^{t+h} T(t-s)(g(s))ds]
\]
\[
= T(h)[F(t)] \oplus T(h)[\int_t^{t+h} T(t-s)(g(s))ds].
\]
Using (iii), there exists $T(h)[F(t)] \ominus F(t)$, which implies

$$T(h)[F(t)] = [T(h)(F(t)) \ominus F(t)] \ominus F(t).$$

Replacing above and multiplying with $\frac{1}{h} \odot$, we obtain

$$\frac{1}{h} \odot [F(t + h) \ominus F(t)] = \frac{1}{h} \odot [T(h)(F(t)) \ominus F(t)] \oplus T(h)[\frac{1}{h} \odot \int_t^{t+h} T(t - s)(g(s))ds].$$

Passing to the limit as $h \searrow 0$, by (iii) it follows

$$\lim_{h \searrow 0} \frac{1}{h} \odot [F(t + h) \ominus F(t)] = A[F(t)] \oplus \lim_{h \searrow 0} T(h)[\frac{1}{h} \odot \int_t^{t+h} T(t - s)(g(s))ds].$$

Because for $h \searrow 0$, we have $T(h) \rightarrow T(0) = I$, it remains to show that

$$\lim_{h \searrow 0} \frac{1}{h} \odot \int_t^{t+h} T(t - s)(g(s))ds = g(t).$$

Indeed, since $g(t) = T(0)(g(t)) = \frac{1}{h} \odot \int_t^{t+h} T(0)(g(t))ds$, we obtain

$$d\left(\frac{1}{h} \odot \int_t^{t+h} T(t - s)(g(s))ds, \frac{1}{h} \odot \int_t^{t+h} T(0)(g(t))ds\right) \leq \frac{1}{h} \int_t^{t+h} H_t(s)ds,$$

where $H_t(s) = d[T(t - s)(g(s)), T(0)(g(s))]$ is continuous on $[t, t + h]$ as function of $s$, since $T(.)$ and $g$ are continuous. Consequently,

$$\frac{1}{h} \int_t^{t+h} H_t(s)ds \rightarrow H_t(t) = 0, \text{ as } h \searrow 0.$$

Therefore,

$$\lim_{h \searrow 0} \frac{1}{h} \odot [F(t + h) \ominus F(t)] = A[F(t)] \oplus g(t).$$

On the other hand, for $0 < h \leq t$, we have $t - h \geq 0$ and $F(t) = F(u + h)$, with $u = t - h \geq 0$. Reasoning as above, we obtain

$$\frac{1}{h} \odot [F(u + h) \ominus F(u)] = \frac{1}{h} \odot [F(u + h) \ominus F(u)]$$

$$= \frac{1}{h} \odot [T(h)[F(u)] \ominus F(u)] \oplus T(h)[\frac{1}{h} \odot \int_u^{u+h} T(u - s)(g(s))ds].$$
Because $F(t)$ is continuous (from the continuity of $T(.)$), clearly, $F(u) \to F(t)$ as $h \searrow 0$, which easily implies

$$
\lim_{h \searrow 0} \frac{1}{h} \odot [T(h)(F(u)) \ominus F(u)] = A[F(t)].
$$

Then, as in the case of the difference $\frac{1}{h} \odot [F(t + h) \ominus F(t)]$, we obtain

$$
\lim_{h \searrow 0} T(h) \left[ \frac{1}{h} \odot \int_{u}^{u+h} T(u - s)(g(s))ds \right] = g(t).
$$

In conclusion, for $t \geq 0$ we have

$$
\left( \int_{0}^{t} T(t - s)(g(s))ds \right)' = A[F(t)] \oplus g(t),
$$

which implies

$$
u'(t) = A[T(t)(u_0)] \oplus A[F(t)] \oplus g(t) = A[u(t)] \oplus g(t).$$

Now, let $t < 0$ and $h > 0$. As in the proof of the above point (iv), we use the relation

$$
\lim_{k \searrow 0} d\left( \frac{1}{k} \odot [T(k)(x) \ominus x], A(x) \right) = 0,
$$

and repeating the above reasonings, we arrive at $u'(t) = A[u(t)] \oplus g(t)$. As before, the derivative $u'(t)$ is considered in the sense of Definition 2.7, (ii), 2). This proves (v) and consequently the theorem. □

Theorem 3.9 allows us to call $(e^{t\otimes A})_{t \geq 0}$ the one-parameter fuzzy-semigroup generated by $A \in L(X)$.

**Remark.** Let $A, B : \mathbb{R}_F \to \mathbb{R}_F$ be defined by

$$
A(x) = \left[ x_- (1) - \int_{0}^{1} x_- (r)dr \right] \odot c,
$$

$$
B(x) = \left[ x_+ (1) - \int_{0}^{1} x_+ (r)dr \right] \odot c,
$$

where $[x]^r = [x_-(r), x_+(r)]$ and $c \in \mathbb{R}_F$ is a constant chosen such that

$$
\mu = c_- (1) - \int_{0}^{1} c_- (r)dr > 0.
$$
By the Remark after Corollary 3.4, we have \( A, B \in L(\mathbb{R}_F) \). Then, by a simple calculation, we obtain that the fuzzy-semigroups generated by \( A \) and \( B \) are

\[
e^{t \odot A}(x) = x \oplus \left( \frac{x_-(1) - \frac{1}{\mu} \int_0^1 x_-(r) \, dr}{e^{\mu} - 1} \right) (e^{\mu} - 1) \odot c, \quad t \geq 0
\]

and

\[
e^{t \odot B}(x) = x \oplus \left( \frac{x_+(0) - \frac{1}{\mu} \int_0^1 x_+(r) \, dr}{e^{\mu} - 1} \right) (e^{\mu} - 1) \odot c, \quad t \geq 0,
\]

respectively. For example, by mathematical induction it easily follows

\[
A^n(x) = \mu^{n-1} \left( x_-(1) - \int_0^1 x_-(r) \, dr \right) \odot c
\]

and, using the Taylor series formula

\[
e^{t \odot A}(x) = x \oplus \frac{t}{1!} \odot A(x) \oplus \frac{t^2}{2!} \odot A^2(x) \oplus \frac{t^3}{3!} \odot A^3(x) \oplus \ldots \oplus \frac{t^n}{n!} \odot A^n(x) \oplus \ldots,
\]

we easily get the formula for \( e^{t \odot A}(x) \).

Another important result of this section is the following

**Theorem 3.10** For \((X, d)\) as in the statement of Theorem 3.9 and \( A \in L(X) \), let us define, formally,

\[
T(t) = \cosh[t \odot A]
\]

in the sense that

\[
\lim_{m \to +\infty} \Phi(T(t), \sum_{p=0}^{m} \frac{t^{2p}}{(2p)!} \odot A^p) = 0,
\]

(formally, we write \( T(t) = \cosh[t \odot A] = \sum_{p=0}^{+\infty} \frac{t^{2p}}{(2p)!} \odot A^p \)). We have:

(i) \( T(t) \in L(X) \), for all \( t \in \mathbb{R} \);

(ii) \( T(t) \) is continuous as function of \( t \in \mathbb{R} \), \( T(0) = I \) and if there exists \( M > 0 \) such that \( ||A||^p_F \leq M, \forall p = 0, 1, \ldots \), then \( T(t) \) is twice generalized differentiable on \( \mathbb{R} \), \( T'(0) = \tilde{0}_X \) and \( T''(t) = A[T(t)] \). More exactly, for each \( t \geq 0 \) we have

\[
\lim_{h \searrow 0} \Phi \left( \frac{1}{h} \odot [T'(t + h) \odot T'(t)], A[T(t)] \right) = 0,
\]

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\[
\lim_{h \to 0} \Phi \left( \frac{1}{h} \circ [T'(t) \ominus T'(t-h)], A[T(t)] \right) = 0,
\]
where \( T'(t) \) is given by
\[
\lim_{h \to 0} \Phi \left( \frac{1}{h} \circ [T(t+h) \ominus T(t)], T'(t) \right) = 0,
\]
\[
\lim_{h \to 0} \Phi \left( \frac{1}{h} \circ [T(t) \ominus T(t-h)], T'(t) \right) = 0,
\]
i.e.,
\[
T'(t)(x) = \sinh(t \circ A) = \sum_{p=1}^{+\infty} \frac{t^{2p-1}}{(2p-1)!} \circ A^p(x),
\]
and for \( t < 0 \), we have
\[
\lim_{h \to 0} \Phi \left( -\frac{1}{h} \circ [T'(t) \ominus T'(t+h)], A[T(t)] \right) = 0,
\]
\[
\lim_{h \to 0} \Phi \left( -\frac{1}{h} \circ [T'(t-h) \ominus T'(t)], A[T(t)] \right) = 0,
\]
where \( T'(t) \) is given by
\[
\lim_{h \to 0} \Phi \left( -\frac{1}{h} \circ [T(t) \ominus T(t+h)], T'(t) \right) = 0,
\]
\[
\lim_{h \to 0} \Phi \left( -\frac{1}{h} \circ [T(t-h) \ominus T(t)], T'(t) \right) = 0.
\]

**Proof.** (i) Let us denote \( C_m(t)(x) = \sum_{p=0}^{m} \frac{(t+h)^{2p}}{(2p)!} \circ A^p(x). \) Similarly, (see the proof of Theorem 3.9,(i)) we obtain that \( C_m(t)(x), m \in \mathbb{N} \) is a Cauchy sequence and therefore its limit \( T(t) = \cosh(t \circ A) \) exists in \( L(X) \).

(ii) The proof of continuity of \( T(t) = \cosh(t \circ A), t \in \mathbb{R} \), i.e., \( \lim_{h \to 0} \Phi(T(t), T(t+h)) = 0, \forall t \in \mathbb{R} \), is similar to the proof for \( T(t) = e^{t \circ A}, \) see the proof of Theorem 3.9,(iii). Now let \( t \geq 0 \) and \( h > 0 \). We have
\[
C_m(t+h)(x) = \sum_{p=0}^{m} \frac{(t+h)^{2p}}{(2p)!} \circ A^p(x)
\]
\[
= C_m(t)(x) \odot h \odot [t \odot A(x) \odot \frac{t^3}{3!} \odot A^2(x)
\]
\[
\odot ... \odot \frac{t^{2m-1}}{(2m-1)!} \odot A^m(x)] \odot E_m(t, h, x),
\]

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where
\[ E_m(t, h, x) = \frac{h^2}{2!} \odot A(x) \oplus \left[ \frac{(2k)}{4!} t^{2k-1} + \frac{(2k)}{3!} t^{2k} + \frac{h^2}{4!} \right] \odot A^2(x) \oplus \ldots \]
\[ \oplus \left[ \frac{(2k)}{4!} t^{2k-2} + \frac{(2k)}{(2k)!} t^{2k-3} + \ldots + \frac{h^{2k}}{(2k)!} \right] \odot A^k(x) \oplus \ldots \]
\[ \oplus \left[ \frac{(2m)}{4!} t^{2m-2} + \frac{(2m)}{(2m)!} t^{2m-3} + \ldots + \frac{h^{2m}}{(2m)!} \right] \odot A^m(x). \]

Let us denote
\[ P_m(t)(x) := t \odot A(x) \oplus \frac{t^3}{3!} \odot A^2(x) \oplus \ldots \oplus \frac{t^{2m-1}}{1!(2m-1)!} \odot A^m(x). \]

As in the proof of Theorem 3.9, we can show that \((P_m(t)(x))_m\) is a Cauchy sequence, therefore it is convergent and let \(\sinh(t \odot A)(x)\) be its limit in \((X, \oplus, \odot, d)\). Moreover, note that for any \(T > 0\), \((C_m(t)(x))_m\) and \((P_m(t)(x))_m\) are uniformly Cauchy sequences on \([0, T]\). We get
\[ \frac{1}{h} \odot [C_m(t + h)(x) \odot C_m(t)(x)] = P_m(t)(x) \oplus \frac{1}{h} \odot E_m(t, h, x). \]

On the other hand we have
\[ d(\frac{1}{h} \odot E_m(t, h, x), \tilde{0}) \leq \frac{1}{h} M[ch_m(t + h) - ch_m(t) - sh_m(t)], \]
where \(ch_m(t) = 1 + \frac{t^2}{2!} + \ldots + \frac{t^{2m}}{(2m)!}\) and \(sh_m(t) = t + \frac{t^3}{3!} + \ldots + \frac{t^{2m}}{(2m-1)!}\). Passing to the limit as \(h \searrow 0\), by the L’Hôpital’s rule, we obtain
\[ \lim_{h \searrow 0} \frac{1}{h} [ch_m(t + h) - ch_m(t) - sh_m(t)] = 0, \]
that is
\[ \lim_{h \searrow 0} \frac{1}{h} \odot [C_m(t + h)(x) \odot C_m(t)(x)] = P_m(t)(x). \]

Writing \(C_m(t)(x) = C_m(t - h + h)(x)\), for \(0 < h < t\), by a similar reasoning, we get
\[ \lim_{h \searrow 0} \frac{1}{h} \odot [C_m(t)(x) \odot C_m(t - h)(x)] = P_m(t)(x). \]

Consequently, there exists the usual (Hukuhara) derivative in Definition 2.7, (i), \(C'_m(t)(x) = P_m(t)(x)\), for all \(x \in X\). Thus, we have obtained the following: for any \(x \in X\),
\[ \lim_{m \to +\infty} C_m(t)(x) = \cosh(t \odot A)(x) \]
and
\[ \lim_{m \to +\infty} C'_m(t)(x) = sinh(t \odot A)(x), \]
uniformly with respect to \( t \in [0, T] \), for each \( T > 0 \). Now, we show that the sequences
\[ F_m(h) = \frac{1}{h} \odot [C_m(t + h)(x) \odot C_m(t)(x)], \quad m \in \mathbb{N} \]
and
\[ G_m(h) = \frac{1}{h} \odot [C_m(t)(x) \odot C_m(t - h)(x)], \quad m \in \mathbb{N} \]
are uniformly Cauchy sequences with respect to \( h \in (0, T] \), for any \( T > 0 \). Let us prove the claim for \( F_m(h) \) (the case of \( G_m(h) \) is similar). It is enough to show that \( H_m(h) = \frac{1}{h} \odot E_m(t, h, x) \), \( m \in \mathbb{N} \), is a uniformly Cauchy sequence for \( h \in (0, T] \), \( T > 0 \) arbitrary (here \( t > 0 \) and \( x \in X \) are fixed). We obtain (by using the mean value theorem)
\[
d(\frac{1}{h} \odot E_m(t, h, x), \tilde{0}) \leq \frac{1}{h}M[sh_m(\xi) \odot hsh_m(t)] \\
= M[sh_m(\xi) \odot sh_m(t)] \\
= M|\xi - t|ch_m(\eta) \leq Mch_m(\eta),
\]
where \( \xi \in (t, t + h), \eta \in (t, \xi) \).

When \( h \in (0, T] \), we have \( \xi \in (t, t + T] \) and \( \eta \in (t, t + T] \) and \( ch_m(\eta) \leq \)
\( ch_m(t + T) \leq cosh(t + T) \), which implies
\[
d(\frac{1}{h} \odot E_m(t, h, x), \tilde{0}) \leq M\cosh(t + T), \forall m \in \mathbb{N},
\]
and
\[ H_m(h) = \frac{1}{h} \odot E_m(t, h, x) \to \tilde{0}, \]
uniformly with respect to \( h \in (0, T] \) and, therefore, \( (H_m(h))_m \) are uniformly Cauchy for \( h \in (0, T] \).

From the uniform convergence of the sequences \( (C_m(t))_m \), \( (C'_m(t))_m \) with respect to \( t \in (0, T] \) and of \( (H_m(h))_m \) with respect to \( h \in (0, T] \), by standard reasoning
(similar to that for the uniform convergence of derivative of the usual sequences of functions), we obtain

\[ T'(t)(x) = [\cosh(t \odot A)]'(x) = \sinh(t \odot A)(x). \]

Repeating the procedure for \( T'(t)(x) \), we arrive at \( T''(t) = A[\cosh(t \odot A)] = A[T(t)]. \)

The case \( t < 0 \) is analogous. The proof of the theorem is finished. \( \square \)

4 Applications to fuzzy differential equations

In this section, we apply the main results of the previous sections to solve fuzzy differential equations. As it was pointed out in the Introduction, imprecision due to uncertainty or vagueness suggests of considering fuzzy differential equations (i.e., whose solutions represents fuzzy-number-valued functions), rather than random differential equations. The simplest model is the following fuzzy Cauchy problem

\[
\frac{du}{dt}(t) = A[u(t)], \quad t \in I, \\
u(t_0) = u_0,
\]

where \( I \) is an interval, \( u : I \to \mathbb{R}_F, u_0 \in \mathbb{R}_F \) and \( A : C(I; \mathbb{R}_F) \to C(I; \mathbb{R}_F) \).

The study of solutions of fuzzy differential equations has been considered by e.g. [3]–[7], [12]–[16], [18]. The main tools exploited are the so-called level set and the differential inclusion methods. These methods are based on the idea that to each fuzzy number \( x \in \mathbb{R}_F \) one can attach the family of closed bounded real intervals

\[ [x]^r = \{ t \in \mathbb{R} : x(t) \geq r \} = [x_-(r), x_+(r)], \quad r \in [0, 1]. \]

However, none of these approaches use the powerful theory of semigroups of operators to solve fuzzy differential equations, simply because the theory was not developed until the present paper.

In what follows, we apply the theory of fuzzy-semigroups on some examples. First, let us consider the general fuzzy Cauchy problem (3). Here \( \frac{du}{dt} \) means the derivative in the sense of the Definition 2.7, where \( u(t) \in X, t \geq 0, A \in L(X) \) and
\((X, d)\) can be chosen any of the following spaces: \((\mathbb{R}_x, D)\), \((L^p_{\mathbb{R}_x}, \rho_p)\), \((m_{\mathbb{R}_x}, \mu)\), \((c_{\mathbb{R}_x}, \mu)\), \((\tilde{c}_{\mathbb{R}_x}, \mu)\), \((L^p([a,b]; \mathbb{R}_x), D_p)\), for \(1 \leq p < \infty\), \((C([a,b]; \mathbb{R}_x), D^*)\), \((C^p([a,b]; \mathbb{R}_x), D^*_p)\), \(p \in \mathbb{N}\), or any finite Cartesian product of them endowed with the box metric. According to Theorem 3.9,

\[
u(t) = T(t)(u_0) = e^{t \otimes A}(u_0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(u_0),
\]

where \(\sum\) is the sum with respect to \(\oplus\), formally furnishes a solution to the fuzzy Cauchy problem (3). We can first apply Theorem 3.9 to the following fuzzy partial differential equation with initial conditions

\[
\begin{aligned}
&\frac{du}{dt}(t, x) = v(t, x) \\
&\frac{dw}{dt}(t, x) = u(t, x), t \geq 0, x \in [a,b], \\
&u(0, x) = u_0(x), \\
&v(0, x) = v_0(x), x \in [a,b],
\end{aligned}
\]

where \(\frac{du}{dt}\) means the derivative of \(u\) with respect to \(t\), see Definition 2.7, (i), and \(u(t, \cdot), v(t, \cdot) \in C([a,b]; \mathbb{R}_x)\), for all \(t \geq 0\).

Let us now set

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, w = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.
\]

Then, problem (4) can be written as

\[
\begin{aligned}
&\frac{dw}{dt}(t) = \tilde{A}[w(t)], t \geq 0, \\
w(0) = w_0,
\end{aligned}
\]

where

\[
\tilde{A} \in [C([a,b]; \mathbb{R}_x)]^2 \to [C([a,b]; \mathbb{R}_x)]^2,
\]

is defined by

\[
\tilde{A}(w) = A \otimes w = \begin{pmatrix} v \\ u \end{pmatrix}.
\]
Clearly, \( A \) is a linear operator and continuous at each \( w \). Then, as in e.g. [9], we easily get that
\[
T(t) = e^{t\odot \tilde{A}} = \left( \begin{array}{cc} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{array} \right),
\]
where \( \sin h(t) = \frac{e^{t} - e^{-t}}{2} \), \( \cos h(t) = \frac{e^{t} + e^{-t}}{2} \), \( t \geq 0 \). Consequently, a solution of (4) will be given by \( w(t) = T(t)(w_0) \), i.e.,
\[
\begin{align*}
    u(x,t) &= \cosh(t) \odot u_0(x) \oplus \sinh(t) \odot v_0(x), \\
v(x,t) &= \sinh(t) \odot u_0(x) \oplus \cosh(t) \odot v_0(x),
\end{align*}
\]
such that \((u, v)\) is a solution of the fuzzy system (4).

Another example of a system of fuzzy differential equation is the following
\[
\begin{cases}
    \frac{du}{dt}(t, x) = u(t, x) \oplus v(t, x) \\
    \frac{dv}{dt}(t, x) = (-1) \odot u(t, x) \oplus (-1) \odot v(t, x), t \geq 0, x \in [a, b], \\
    u(0, x) = u_0(x), \\
    v(0, x) = v_0(x), x \in [a, b],
\end{cases}
\]
where \( u(t, \cdot), v(t, \cdot) \in C([a, b]; \mathbb{R}_x) \), for all \( t \geq 0 \) and the derivatives are meant in the sense of Definition 2.7, (i). Let us set
\[
A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad w = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.
\]
Then, problem (5) can be written as
\[
\begin{cases}
    \frac{dw}{dt}(t) = \tilde{A}[w(t)], t \geq 0, \\
    w(0) = w_0,
\end{cases}
\]
where
\[
\tilde{A} \in [C([a, b]; \mathbb{R}_x)]^2 \to [C([a, b]; \mathbb{R}_x)]^2
\]
is defined by
\[
\tilde{A}[w] = \begin{pmatrix} u \oplus v \\ (-1) \odot [u \oplus v] \end{pmatrix}.
\]
Once more $A$ is a linear operator and continuous at each $w$. We will calculate $e^{t \circ \tilde{A}}$.

In this case, we observe that

$$
\tilde{A}^2(w) = \left( u \oplus v \oplus (-1) \odot [u \oplus v] \right),
$$

$$
\tilde{A}^n(w) = 2^{n-2} \odot \left( u \oplus v \oplus (-1) \odot [u \oplus v] \right), \quad n \geq 2,
$$

since denoting $E(u, v) = (u \oplus v) \odot (-1) \odot (u \oplus v)$, we easily get $(-1) \odot E(u, v) = E(u, v)$. Thus,

$$
T(t)(w) = e^{t \circ \tilde{A}}(w) = w \oplus \frac{t}{1!} \odot \tilde{A}(w) \oplus \frac{t^2}{2!} \odot \tilde{A}^2(w) \oplus ... \oplus \frac{t^n}{n!} \odot \tilde{A}^n(w) \oplus ...
$$

$$
= \left( \begin{array}{c} u \\ v \end{array} \right) \oplus \left( t \odot (u \oplus v) \right) \oplus \left( (-t) \odot (u \oplus v) \right) \oplus ... \oplus \left( \frac{t^n}{n!} \odot \tilde{A}^n(w) \right).
$$

Let us denote

$$
h(t) := \sum_{k=2}^{\infty} \frac{t^k}{k!} 2^{k-2}.
$$

It easy to check that

$$
\frac{1}{4} e^{2t} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{t^k}{k!} (e^{2t})^{(k)} = \frac{1}{4} \sum_{k=0}^{\infty} \frac{t^k}{k!} 2^k = h(t) + \frac{1}{4} \left( \frac{2t}{1!} + 1 \right),
$$

i.e., $h(t) = \frac{1}{4} (e^{2t} - 2t - 1)$.

Henceforth, we have

$$
T(t)[u_0] = \left( \begin{array}{c} u_0 \\ v_0 \end{array} \right) \oplus \left( t \odot (u_0 \oplus v_0) \right) \oplus \left( (-t) \odot (u_0 \oplus v_0) \right) \oplus \left( E(u_0, v_0) \right) \oplus h(t).
$$

Thus, an explicit solution of (5) is given by

$$
\begin{align*}
&u(t, x) = u_0(x) \odot (t) \odot (u_0(x) \oplus v_0(x)) \oplus E(u_0, v_0) \odot h(t), \\
v(t, x) = v_0(x) \odot (-t) \odot (u_0(x) \oplus v_0(x)) \oplus E(u_0, v_0) \odot h(t).
\end{align*}
$$

**Remark.** We don’t know if the solution of the general fuzzy Cauchy problem (3) is unique. In the classical case, i.e., when the differential equations and their
corresponding solutions are in the setting of Banach space valued functions, it is known that the uniqueness phenomenon holds. For example, in this case, we know that the unique solution of (5) is given by

\[ u(t, x) = u_0(x) + t[u_0(x) + v_0(x)], \quad v(t, x) = v_0(x) - t[u_0(x) + v_0(x)] \]

(note that it is obvious that \( E(u_0, v_0) = 0 \)). However, if \( u_0(x), v_0(x) \in R_F \setminus R \), then

\[ u(t, x) = u_0(x) \oplus t \odot [u_0(x) \oplus v_0(x)], \quad v(t, x) = v_0(x) \oplus (-t) \odot [u_0(x) \oplus v_0(x)] \]

is actually not a solution of (5), because

\[ \frac{\partial u}{\partial t} = u_0(x) \oplus v_0(x) \]

is a quantity which is essentially different from

\[ u(t, x) \oplus v(t, x) = u_0(x) \oplus v_0(x) \oplus t \odot E(u_0, v_0). \]

In what follows we apply Theorem 3.10 on two more examples. Let us consider the following system of fuzzy partial differential equation with initial conditions

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, x) = u(t, x) \oplus v(t, x) \\
\frac{\partial^2 v}{\partial t^2}(t, x) = (-1) \odot u(t, x) \oplus (-1) \odot v(t, x), t \geq 0, x \in [a, b], \\
u(0, x) = u_0(x), \\
v(0, x) = v_0(x), \frac{\partial u}{\partial t}(0, x) = \frac{\partial v}{\partial t}(0, x) = \tilde{0}, x \in [a, b],
\end{cases}
\]

(6)

where \( u(t, \cdot), v(t, \cdot) \in C([a, b]; R_F) \), for all \( t \geq 0 \) and where the second derivatives \( \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2} \) are meant in the sense of Definition 2.7, (i). As in the previous example, let us set

\[ A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, w = \begin{pmatrix} u \\ v \end{pmatrix}, \tilde{0} = \begin{pmatrix} \tilde{0} \\ \tilde{0} \end{pmatrix} \quad \text{and} \quad w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \]

Then, system (6) can be written as

\[
\begin{cases}
\frac{\partial^2 w(t)}{\partial t^2} (t) = \tilde{A}[w(t)], t \geq 0, \\
w(0) = w_0, \frac{\partial w}{\partial t}(0) = \tilde{0},
\end{cases}
\]

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where

\[ \tilde{A} \in [C([a, b]; \mathbb{R}_F)]^2 \to [C([a, b]; \mathbb{R}_F)]^2 \]

is defined by

\[ \tilde{A}[w] = \begin{pmatrix} u \oplus v \\ (-1) \odot [u \oplus v] \end{pmatrix}. \]

According to Theorem 3.10, a solution is given by the formula

\[ T(t)(w_0) = \cosh(t \odot \tilde{A})(w_0). \]

We will calculate \( \cosh(t \odot \tilde{A}) \). Similarly to the previous example, we have

\[ \tilde{A}^2(w) = \begin{pmatrix} u \oplus v \oplus (-1) \odot [u \oplus v] \\ u \oplus v \oplus (-1) \odot [u \oplus v] \end{pmatrix}, \]

\[ \tilde{A}^n(w) = 2^{n-2} \odot \begin{pmatrix} u \oplus v \oplus (-1) \odot [u \oplus v] \\ u \oplus v \oplus (-1) \odot [u \oplus v] \end{pmatrix}, \quad n \geq 2, \]

so that

\[ T(t)(w) = \cosh(t \odot \tilde{A})(w) = w \oplus \frac{t^2}{2!} \odot \tilde{A}(w) \oplus \frac{t^4}{4!} \odot \tilde{A}^2(w) \]

\[ \oplus ... \oplus \frac{t^{2n}}{(2n)!} \odot \tilde{A}^n(w) \oplus ... \]

\[ = \begin{pmatrix} u \\ v \end{pmatrix} \oplus \left( \begin{pmatrix} \frac{t^2}{2!} \odot (u \oplus v) \\ -\frac{t^2}{2!} \odot (u \oplus v) \end{pmatrix} \right) \]

\[ \oplus \left( u \oplus v \oplus (-1) \odot [u \oplus v] \right) \odot \left( \frac{t^4}{4!} + \frac{t^6}{6!} + ... + \frac{t^{2n}}{(2n)!} 2^{n-2} + ... \right). \]

Let us set

\[ h(t) := \sum_{k=2}^{\infty} \frac{t^{2k}}{(2k)!} 2^{k-2} \]

and

\[ E(u, v) := u \oplus v \oplus (-1) \odot [u \oplus v]. \]

It easy to check that

\[ h(t) = \frac{1}{4} \frac{t^2}{\sqrt{2}} \sum_{k=2}^{\infty} \frac{(\sqrt{2})^{2k}}{(2k)!} = \frac{1}{4} \frac{\cosh(t\sqrt{2}) - t^2 - 1}{2}. \]
Finally, we then have
\[ T(t)[w_0] = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \oplus \left( \begin{pmatrix} \frac{t^2}{2} \odot (u_0 \oplus v_0) \\ -\frac{t^2}{2} \odot (u_0 \oplus v_0) \end{pmatrix} \right) \oplus \left( \begin{pmatrix} E(u_0, v_0) \\ E(u_0, v_0) \end{pmatrix} \right) \odot h(t), \]
such that an explicit solution of (6) is given by
\[
\begin{aligned}
u(t, x) &= u_0(x) + \frac{t^2}{2} \odot [u_0(x) \oplus v_0(x)], \\
v(t, x) &= v_0(x) - \frac{t^2}{2} \odot [u_0(x) \oplus v_0(x)].
\end{aligned}
\]

Remark. As in the previous example we don’t know if this solution is unique, but in the case when \( u_0(x) \) and \( v_0(x) \) are real-valued functions, we obtain the unique solution (since \( E(u_0, v_0) = 0 \))
\[
\begin{aligned}
u(t, x) &= u_0(x) + t^2 \odot [u_0(x) \oplus v_0(x)], \\
v(t, x) &= v_0(x) - t^2 \odot [u_0(x) \oplus v_0(x)].
\end{aligned}
\]
Note that if \( u_0(x), v_0(x) \in \mathbb{R}_F \setminus \mathbb{R} \), then
\[
\begin{aligned}
u(t, x) &= u_0(x) + \frac{t^2}{2} \odot [u_0(x) \oplus v_0(x)], \\
v(t, x) &= v_0(x) + \frac{t^2}{2} \odot [u_0(x) \oplus v_0(x)]
\end{aligned}
\]
is actually not a solution of (6).

As our final example, let us consider the initial value problem for the fuzzy wave equation
\[
\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t, x) &= \frac{\partial^2 u}{\partial x^2}(t, x), \\
u(0, x) &= u_1(x), \\
\frac{\partial u}{\partial t}(0, x) &= u_2(x), x \in [a, b],
\end{aligned}
\]
where \( \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2} \) are Hukuhara-kind derivatives and \( u_1 : \mathbb{R} \to X \) is supposed to be Hukuhara differentiable of any order (in the sense of Definition 2.7, (i)), such that there exists \( M > 0 \) satisfying
\[
d(\tilde{0}, u^{(2p)}_1(x)) \leq M, \forall x \in X, p = 0, 1, \ldots.
\]
Then \( A(u) = \frac{\partial^2 u}{\partial x^2} \) and
\[
cosh(t \odot A)(u_1(x)) = \sum_{p=0}^{+\infty} \frac{t^{2p}}{(2p)!} \odot u^{(2p)}_1(x)
\]
is a convergent series in \((X, d)\), with \(\sum_{p=0}^{m} \frac{t^{2p}}{(2p)!} \odot u_1^{(2p)}(x)\) being a uniform approximation (with respect to \(t\) on compact subintervals) of \(\cosh(t \odot A)(u_1(x))\). As a consequence,

\[ u(t, x) = \cosh(t \odot A) \oplus t \odot u_2(x) \]

furnishes a solution to the above fuzzy wave equation. Furthermore,

\[ u_m(t, x) = \sum_{p=0}^{m} \frac{t^{2p}}{(2p)!} \odot u_1^{(2p)}(x) \oplus t \odot u_2(x), \quad m \in \mathbb{N}, \]

represents a sequence of uniform approximation (with respect to \(t\) on compact subintervals) for the exact solution \(u(t, x)\).

**Remark.** In view of Theorems 3.9 and 3.10 we can also take into the above examples the differentiability in the generalized sense as well (see Definition 2.7, (ii)).

We can conclude that, as in the classical theory of differential equations, the theory of semigroups of operators developed in this paper may become a powerful tool to study solutions of fuzzy partial differential equations.

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