Vacuum Non Singular Black Hole in Tetrad Theory of Gravitation

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The field equations of a special class of tetrad theory of gravitation have been applied to tetrad space having three unknown functions of radial coordinate. The spherically symmetric vacuum stress-energy momentum tensor with one assumption concerning its specific form generates two non-trivial different exact analytic solutions for these field equations. For large $r$, the exact analytic solutions coincide with the Schwarzschild solution, while for small $r$, they behave in a manner similar to the de Sitter solution and describe a spherically symmetric black hole singularity free everywhere. The solutions obtained give rise to two different tetrad structures, but having the same metric, i.e., a static spherically symmetric nonsingular black hole metric. We then, calculated the energy associated with these two exact analytic solutions using the superpotential method. We find that unless the time-space components of the tetrad go to zero faster than $\frac{1}{\sqrt{r}}$ at infinity, the two solutions give different results. This fact implies that the time-space components of the tetrad must vanish faster than $\frac{1}{\sqrt{r}}$ at infinity.
1. Introduction

In the tetradic theories of gravitation the gravitational field is described by a tetrad field. In the past, several authors have considered these theories to solve some problems of general relativity or to extend it to include other fields. While searching for a unification of gravitation and electromagnetism Einstein considered a tetrad theory \[1\], but soon he rejected it because there was no Schwarzschild solution in his field equation.

Møller has shown that it is impossible to find a satisfactory expression for the energy-momentum complex in the framework of Riemannian geometry \[2\]. In a series of papers, \[2, 3, 4\] Møller was able to obtain a general expression for a satisfactory energy-momentum complex in the absolute parallelism space. The Lagrangian formulation of this tetrad theory of gravitation was first given by Pellegrini and Plebanski \[5\]. In these attempts the admissible Lagrangians were limited by the assumption that the equations determining the metric tensor should coincide with the Einstein equation. Møller \[6\] abandoned this assumption and suggested to look for a wider class of Lagrangians, allowing for possible deviation from the Einstein equation in the case of strong gravitational fields. Sáez \[7\] generalized Møller theory into a scalar tetrad theory of gravitation. Meyer \[8\] showed that Møller theory is a special case of Poincaré gauge theory \[9, 10\].

Hayashi and Nakano \[11\] formulated the tetrad theory of gravitation as the gauge theory of space-time translation group. Hayashi and Shirafuji \[12\] studied the geometrical and observational basis of the tetrad theory, assuming that the Lagrangian be given by a quadratic form of torsion tensor assuming the Lagrangian to be invariant under parity operation involving three unknown parameters to be fixed by experiments, besides a cosmological term. Two of the three parameters were determined by comparing with solar-system experiments, \[12\] while only an upper bound has been estimated for the third one \[12, 13\].

The numerical values of the two parameters found were very small consistent with being equal to zero. If these two parameters are equal to zero exactly, the theory reduces to the one proposed by Hayashi and Nakano \[11\] and Møller, \[6\] which we shall here refer to as the HNM theory for short. This theory differs from general relativity only when the torsion tensor has nonvanishing axial-vector part. It was also shown \[12\] that the Birkhoff theorem can be extended to the HNM theory. Namely, for spherically symmetric case in vacuum, which is not necessarily time independent, the axial-vector part of the torsion tensor should vanish due to the antisymmetric part of the field equation, and therefore, with the help of the Birkhoff theorem \[14\] of general relativity we see that the spacetime metric is the Schwarzschild.

Mikhail et al. \[15\] derived the superpotential of the energy-momentum complex in the HNM theory and applied it to two spherically symmetric solutions. It was found that in one of the two solutions the gravitational mass does not coincide with the calculated energy. This result was extended to a wider class of solutions with spherical symmetry \[16\]. An explicit expression was given for all the stationary, asymptotically flat solutions with spherical symmetry \[16\], which were then classified according to the asymptotic behavior of the components of \((\lambda_{a0})\) and \((\lambda_{0a})\). It was found that the equality of the gravitational and inertial masses holds only when \((\lambda_{a0})\) and \((\lambda_{0a})\) tend to zero faster than \(1/\sqrt{r}\).
Dymnikova [17] derived a static spherically symmetric nonsingular black hole solution in orthodox general relativity assuming a specific form of the stress-energy momentum tensor. This solution practically coincides with the Schwarzschild solution for large \( r \), for small \( r \) behaves like the de Sitter solution and describe a spherically symmetric black hole singularity free everywhere [17].

The general form of the tetrad, \( \lambda^\mu \), having spherical symmetry was given by Robertson [18]. In the Cartesian form it can be written as

\[
\begin{align*}
\lambda^0 &= iA, \quad \lambda_a^0 = Cx^a, \quad \lambda^\alpha &= iDx^\alpha \\
\lambda_a^\alpha &= \delta_a^\alpha B + Fx^a x^\alpha + \epsilon_{a\alpha\beta} Sx^\beta,
\end{align*}
\]

(1)

where \( A, C, D, B, F, \) and \( S \) are functions of \( t \) and \( r = (x^\alpha x^\alpha)^{1/2} \), and the zeroth vector \( \lambda_0^\mu \) has the factor \( i = \sqrt{-1} \) to preserve Lorentz signature. We consider an asymptotically flat space-time in this paper, and impose the boundary condition that for \( r \to \infty \) the tetrad (1) approaches the tetrad of Minkowski space-time, \( \left( \lambda_i^\mu \right) = \text{diag}(i, \delta_a^\alpha) \). Robertson has shown that:

1- Improper rotation are admitted if and only if \( S = 0 \).

2- The functions \( C \) and \( F \) can be eliminated by a mere coordinate transformations, i.e., by making use of freedom to redefine \( t \) and \( r \), leaving the tetrad (1) having three unknown functions in the Cartesian coordinate, which will be used in section 3 for calculations of HNM field equations but in the spherical polar coordinate.

It is the aim of the present work to find the, asymptotically flat solutions with spherical symmetry which is different from the Schwarzschild solution in the HNM theory. Assuming the same form of the stress-energy momentum tensor as given by Dymnikova [17], we obtain two different exact analytic solutions of the HNM theory. We then, calculate the energy of those solutions using the superpotential of Mikhail et al. [15].

In section 2 we briefly review the tetrad theory of gravitation. In section 3 we first apply the tetrad (1) with three unknown functions of the radial coordinate in spherical polar coordinates to the field equations of HNM theory, then we derived the corresponding partial differential equations. Assuming some conditions on these partial differential equations we obtained two different exact asymptotically flat solutions with spherical symmetry. In section 4 the energy of the gravitating source is calculated by the superpotential method. The final section is devoted to the main results and discussion.

Computer algebra system Maple V Release 4 is used in some calculations.

*In this paper Latin indices (\( i, j, \ldots \)) represent the vector number, and Greek indices (\( \mu, \nu, \ldots \)) represent the vector components. All indices run from 0 to 3. The spatial part of Latin indices are denoted by (\( a, b, \ldots \)), while that of greek indices by (\( \alpha, \beta, \ldots \)). In the present convention, latin indices are never raised. The tetrad \( \lambda_i^\mu \) is related to the parallel vector fields \( b_i^\mu \) of [12] by \( \lambda_0^\mu = i b_0^\mu \) and \( \lambda_a^\mu = b_a^\mu \).*
2. The tetrad theory of gravitation

In this paper we follow Möller’s construction of the tetrad theory of gravitation based on the Weitzenböck space-time. In this theory the field variables are the 16 tetrad components $\lambda_{\mu i}$, from which the metric is derived by

$$g^{\mu\nu} \overset{\text{def.}}{=} \lambda_{\mu i} \lambda_{\nu i}. \quad (2)$$

The Lagrangian $L$ is an invariant constructed from $\gamma_{\mu\nu\rho}$ and $g^{\mu\nu}$, where $\gamma_{\mu\nu\rho}$ is the contorsion tensor given by

$$\gamma_{\mu\nu\rho} \overset{\text{def.}}{=} \lambda_{\mu i} \lambda_{\nu i \rho}, \quad (3)$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under parity operation is given by the form

$$\mathcal{L} \overset{\text{def.}}{=} (-g)^{1/2} \left( \alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu} \right), \quad (4)$$

where

$$g \overset{\text{def.}}{=} \det(g_{\mu\nu}), \quad (5)$$

and $\Phi_\mu$ is the basic vector field defined by

$$\Phi_\mu \overset{\text{def.}}{=} \gamma^{\rho}_{\mu \rho}. \quad (6)$$

Here $\alpha_1$, $\alpha_2$, and $\alpha_3$ are constants determined by Möller such that the theory coincides with general relativity in the weak fields:

$$\alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa} (1 - \lambda), \quad (7)$$

where $\kappa$ is the Einstein constant and $\lambda$ is a free dimensionless parameter. The same choice of the parameters was also obtained by Hayashi and Nakano [11].

Möller applied the action principle to the Lagrangian density (4) and obtained the field equation in the form

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (8)$$

$$F_{\mu\nu} = 0, \quad (9)$$

where the Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (10)$$

*Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$. 

Here $H_{\mu\nu}$ and $F_{\mu\nu}$ are given by

$$H_{\mu\nu} \overset{\text{def.}}{=} \lambda \left[ \gamma_{\rho\sigma\mu} \gamma_{\nu}{}^{\rho}{}_{\sigma} + \gamma_{\rho\sigma\mu} \gamma_{\nu}{}^{\rho}{}_{\sigma} + \gamma_{\rho\sigma\nu} \gamma_{\mu}{}^{\rho}{}_{\sigma} + g_{\mu\nu} \left( \gamma_{\rho\sigma\lambda} \gamma_{\lambda}{}^\rho{}_{\sigma} - \frac{1}{2} \gamma_{\rho\sigma\lambda} \gamma_{\rho}{}_{\sigma}{}_{\lambda} \right) \right], \quad (11)$$

and

$$F_{\mu\nu} \overset{\text{def.}}{=} \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\rho} \left( \gamma^{\rho}{}_{\mu\nu} - \gamma^{\rho}{}_{\nu\mu} \right) + \gamma_{\mu\nu} \epsilon^{\rho},_{\rho} \right], \quad (12)$$

and they are symmetric and skew symmetric tensors, respectively.

Møller assumed that the energy-momentum tensor of matter fields is symmetric. In the Hayashi-Nakano theory, however, the energy-momentum tensor of spin-1/2 fundamental particles has nonvanishing antisymmetric part arising from the effects due to intrinsic spin, and the right-hand side of (9) does not vanish when we take into account the possible effects of intrinsic spin. Nevertheless, since in this paper we consider only solutions in which the left-hand side of (9) is identically vanishing, we refer the tetrad theory of gravitation based on the choice of the parameters, (7), as the Hayashi-Nakano-Møller (HNM) theory for short.

It can be shown [12] that the tensors, $H_{\mu\nu}$ and $F_{\mu\nu}$, consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor, $a_{\mu}$, defined by

$$a_{\mu} \overset{\text{def.}}{=} \frac{1}{3} \epsilon^{\mu\rho\sigma} \gamma_{\rho}{}^{\nu}{}_{\sigma}, \quad (13)$$

where $\epsilon^{\mu\rho\sigma}$ is defined by

$$\epsilon^{\mu\rho\sigma} \overset{\text{def.}}{=} (-g)^{1/2} \delta^{\mu\rho\sigma}, \quad (14)$$

with $\delta^{\mu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$. Therefore, both $H_{\mu\nu}$ and $F_{\mu\nu}$ vanish if the $a_{\mu}$ is vanishing. In other words, when the $a_{\mu}$ is found to vanish from the antisymmetric part of the field equations, (9), the symmetric part (8) coincides with the Einstein equation.

### 3. Spherically symmetric nonsingular black hole solutions

The tetrad space having three unknown functions of radial coordinate with spherical symmetry in spherical polar coordinates, can be written as

$$\left( \lambda^\mu \right) = \begin{pmatrix}
  iA & iDr & 0 & 0 \\
  0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & -\frac{B \sin \phi}{r \sin \theta} \\
  0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\
  0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0
\end{pmatrix}, \quad (15)$$

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where \( i = \sqrt{-1} \).

Applying (15) to the field equations (8), (9) we note that the two tensors \( H_{\mu \nu} \) and \( F_{\mu \nu} \) are vanishing identically regardless of the values of the functions \( A, B \) and \( D \). Thus Møller field equations reduce for the tetrad (15) to Einstein’s equations. Then the field equations (8) and (9) take the form

\[
\kappa T_{00} = \frac{1}{r A^2 B^4} \left[ \left\{ \left( 3D^2 + 8B^2 \right) D - 2 \left( 2BB'' + B'D' \right) B \right\} r^3 B^2 D - \right.
\]

\[
\left. \left\{ 2 \left( DB'' + B'D' \right) B - 5DB'^2 \right\} r^5 D^3 \right. - \left( 2BB'' - 3D^2 - 3B'^2 \right) rB^4 + \right.
\]

\[
2 \left( BD' - 4DB' \right) r^4 BD^3 + \left( BD' - 6DB' \right) r^2 B^3 D - 4B^5 B', \right.
\]

\[
\kappa T_{01} = \frac{D}{AB^4} \left[ \left\{ 2 \left( DB'' + B'D' \right) B - 5DB'^2 \right\} r^3 D + \left( 2BB'' - 3D^2 - 3B'^2 \right) rB^2 - \right.
\]

\[
2 \left( BD' - 4DB' \right) r^2 BD + 4B^3 B', \right.
\]

\[
\kappa T_{11} = \frac{1}{r AB^4} \left[ \left\{ \left( 3D^2 + B^2 \right) A + 2BA'B' \right\} rB^2 + \left( 2 \left( DB'' + B'D' \right) B - 5DB'^2 \right) r^3 AD + \right.
\]

\[
2 \left( BD' - 4DB' \right) r^2 ABD - 2AB^3 B' - 2B^4 A', \right.
\]

\[
\kappa T_{22} = \frac{r}{A^2 B^4} \left[ \left\{ \left( DA'' + 3A'D' \right) B - 3DA'B' \right\} ABD + \left\{ \left( 2BB'' + 5B'D' \right) BD - \right. \right.
\]

\[
\left. \left. \left( DD'' + D'^2 \right) B^2 - 5D^2 B'^2 \right\} A^2 - 2B^2 D^2 A'^2 \right\} r^3 + \right.
\]

\[
\left. \left\{ \left( B'^2 - 3D^2 \right) A^2 - AB^2 A'' - B'' BA^2 + 2B^2 A'^2 \right\} rB^2 - \right.
\]

\[
2 \left\{ \left( 3BD' - 4DB' \right) A - 2BDA' \right\} r^2 ABD + A^2 B^3 B' + AB^4 A', \right.
\]

\[
T_{33} = \sin \theta^2 T_{22}, \tag{16}
\]

where \( A' = \frac{dA}{dr}, B' = \frac{dB}{dr} \) and \( D' = \frac{dD}{dr} \).

Now we are going to find some special solutions to the partial differential equations (16), assuming that the stress-energy momentum tensor has the form [17]

\[
T_{0}^0 = T_{1}^1, \quad T_{2}^2 = T_{3}^3. \tag{17}
\]

A first non-trivial solution can be obtained by taking \( D(r) = 0 \), and solving for \( A(r) \) and \( B(r) \), then we obtain
\[ A = \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}}, \]
\[ B = \sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}, \] (18)

where \( R \) is a new radial coordinate defined by \( R = r/B \) and
\[ r_1^3 = r_gr_0^2, \]
\[ r_g = 2m, \]
\[ r_0^2 = \frac{3}{8\pi\epsilon_0}. \] (19)

The form of the energy-momentum tensor is
\[ T_0^0 = T_1^1 = \epsilon_0 e^{-R^3/r_1^3}, \]
\[ T_2^2 = T_3^3 = \epsilon_0 e^{-R^3/r_1^3} \left(1 - \frac{3R^3}{2r_1^3}\right), \] (20)

and the tetrad (15) takes the form
\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \sin \theta \cos \phi \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}} & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\
  0 & \sin \theta \sin \phi \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}} & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
  0 & \cos \theta \frac{1}{\sqrt{1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)}} & -\frac{\sin \theta}{R} & 0 \\
\end{pmatrix}
\] (21)

with the associated Riemannian metric
\[ ds^2 = -\eta_1 dt^2 + \frac{dR^2}{\eta_1} + R^2 d\Omega^2, \] (22)

where
\[ \eta_1 = \left[1 - \frac{2m}{R} \left(1 - e^{-R^3/r_1^3}\right)\right], \] (23)

and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), which is a static spherically symmetric nonsingular black hole solution [17].
A second non-trivial solution can be obtained by taking $A(r) = 1, B(r) = 1, D(r) \neq 0$ and solving for $D(r)$. In this case the resulting field equations of (16) can be integrated directly to give

$$ D(r) = \sqrt{\frac{2m}{r^3}} \left(1 - e^{-r^3/r_1^3}\right). \tag{24} $$

Substituting for the value of $D(r)$ as given by (24) into (15), we get

$$ \left(\lambda^\mu_i\right) = \begin{pmatrix} i & i\sqrt{\frac{2m}{r}} \left(1 - e^{-r^3/r_1^3}\right) & 0 & 0 \\ 0 & \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\ 0 & \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\ 0 & \cos \theta & -\frac{\sin \theta}{r} & 0 \end{pmatrix}, \quad \tag{25} $$

with the associated metric

$$ ds^2 = -\left[1 - \frac{2m}{r} \left(1 - e^{-r^3/r_1^3}\right)\right] dt^2 - 2\sqrt{\frac{2m}{r}} \left(1 - e^{-r^3/r_1^3}\right) dr dt + dr^2 + r^2 d\Omega^2, \quad \tag{26} $$

it is to be noted that $m$ in the metric (22) and (26) is a constant of integration that will play the role of the mass producing the field in the calculations of the energy. Also the form of the stress-energy momentum tensor for this solution is the same as given by (20).

Using the coordinate transformation

$$ dT = dt + \frac{Dr}{1 - D^2 r^2} dr, \quad \tag{27} $$

we can eliminate the cross term of (26) to obtain

$$ ds^2 = -\eta_2 dT^2 + \frac{dr^2}{\eta_2} + r^2 d\Omega^2, \quad \tag{28} $$

where $\eta_2$ is defined by (23) and $R = r$.

Thus we have two exact solutions of HNM field equations, each of which leads to the same metric, a static spherically symmetric nonsingular black hole in the spherical polar coordinate.

The solutions (18) and (24) are the exact solutions of the HNM field equations. They practically coincide with the Schwarzschild solution for $r >> r_1$ and behave like the de Sitter solution, for $r << r_1$.

As is clear from (20), the spherically symmetric stress-energy momentum tensor is really anisotropic. The difference between the principle pressures

$$ T^k_k = -p_k, \quad \tag{29} $$
correspond to the well known anisotropic character of evolution of the space-time inside a black hole undergoing a spherically symmetric gravitational collapse [19]. For $r \ll r_1$ isotropization occurs and the stress-energy momentum tensor takes the isotropic form

$$T_{\alpha\beta} = \epsilon g_{\alpha\beta}. \quad (30)$$

When $r \to 0$ the energy density tends to $\epsilon_0$. For $r >> r_1$ all the components of the stress-energy momentum tensor tend to zero very rapidly.

The important result obtained in this section is that we have been able to derive two different solutions for HNM theory; the Riemannian metric associated with these two solutions are identical, namely spherically symmetric nonsingular black hole. Since HNM theory is a pure gravitational theory, the above two solutions have to be equivalent in the sense that they describe the same physical situation. In what follows we examine the equivalence of these solutions by calculating the energy associated with each of them, using the superpotential derived for Møller’s theory by Mikhail et al. [15].

4. The Energy Associated with each Solution

The superpotential of the HNM theory is given by Mikhail et al. [15] as

$$U_{\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma\tau}^{\nu\mu\lambda\tau} \left[ \Phi_{\rho}^{\sigma} g_{\gamma\tau\chi} - \lambda g_{\gamma\tau\chi} - (1 - 2\lambda) g_{\gamma\tau\chi}^{\sigma\rho\chi} \right], \quad (31)$$

where $P_{\chi\rho\sigma\tau}^{\nu\mu\lambda\tau}$ is

$$P_{\chi\rho\sigma\tau}^{\nu\mu\lambda\tau} \overset{\text{def}}{=} \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\lambda} + \delta_{\rho}^{\nu} g_{\sigma\chi}^{\nu\lambda} - \delta_{\sigma}^{\nu} g_{\chi\rho}^{\nu\lambda} \quad (32)$$

with $g_{\rho\sigma}^{\nu\lambda}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\lambda} \overset{\text{def}}{=} \delta_{\rho}^{\nu} \delta_{\sigma}^{\lambda} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\lambda}. \quad (33)$$

The energy is expressed by the surface integral [20]

$$E = \lim_{r \to \infty} \int_{r=\text{constant}} U_{0\alpha}^{0\alpha} n_{\alpha} dS, \quad (34)$$

where $n_{\alpha}$ is the unit 3-vector normal to the surface element $dS$.

Now we are in a position to calculate the energy associated with the two solutions (18) and (24) using the superpotential (31). As is clear from (34), the only components which contributes to the energy is $U_{0\alpha}^{0\alpha}$. Thus substituting from the first solution (18) into (31) we obtain the following non-vanishing value

$$U_{0\alpha}^{0\alpha} = \frac{2X^{\alpha} m}{\kappa r^2} \left( 1 - e^{-r^3/r^{1/3}} \right). \quad (35)$$
Substituting from (35) into (34) we get

\[ E(r) = m \left( 1 - e^{-r^3/r^3} \right). \]  

(36)

This is a satisfactory result and should be expected.

Now let us turn our attention to the second solution (24). Calculating the necessary components of the superpotential, we get

\[ \mathcal{U}_0^{0\alpha} = \frac{4X^\alpha}{kr^2} \left( 1 - e^{-r^3/r^3} \right). \]  

(37)

Substituting from (37) into (34) we get

\[ E(r) = 2m \left( 1 - e^{-r^3/r^3} \right). \]  

(38)

That is twice (36): it is clear from (36) and (38) that if \( r \to 0 \), \( E(r) \to 0 \) and as \( r \to \infty \), \( E(r) \to m \) for (36) and \( E(r) \to 2m \) for (38). Note that \( E(r) > 0 \) for all \( r: 0 \leq r < \infty \).

5. Main results and Discussion

In this paper we have studied the nonsingular black hole spherically symmetric solution in the HNM tetrad theory of gravity. The axial vector part of the torsion, \( a^\mu \) for these solutions is identically vanishing.

Two different exact analytic solutions of the HNM field equations are obtained for the case of spherical symmetry. The two solutions give rise to the same Riemannian metric (spherically symmetric nonsingular black hole metric). The exact solutions (18) and (24) represent a black hole which contain the de Sitter world instead of a singularity. The stress-energy momentum tensor (20) responsible for geometry describes a smooth transition from the standard state at infinity to isotropic state at \( r \to 0 \) through anisotropic state in intermediate region. This agrees with the Poisson-Israel prediction concerning ”non-inflationary material at the interface” [21].

It was shown by Møller [3] that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have applied the superpotential method given by Mikhail et al.[15] to calculate the energy of the central gravitating body. It is shown that the two solutions give two different values of the energy content. The following suggestions may be considered to get out of this inconsistency:

(a) The energy-momentum complex suggested by Møller [3] is not quite adequate; though it has the most satisfactory properties.

(b) Many authors believe that a tetrad theory should describe more than a pure gravitation
field. In fact, Møller himself [3] considered this possibility in his earlier trials to modify general relativity. In these theories, the most successful candidates for the description of the other physical phenomenon are the skew-symmetric tensors of the tetrad space, e.g., $\Phi_{\mu,\nu} - \Phi_{\nu,\mu}$. The most striking remark here is that: All the skew-symmetric tensors vanish for the first solution; but not all of them do so for the second one. Some authors; e.g; [22, 23], believe that these tensors are related to the presence of an electromagnetic field. Others; e.g.; [23] believe that these tensors are closely connected to the spin phenomenon. There are a lot of difficulties to claim that HNM theory deserves such a wider interpretation. This needs a lot of investigations before arriving at a concrete conclusion.

(c) Other possibility is that HNM theory is in need to be generalized rather than to be reinterpreted. There are already some generalizations of HNM theory. Møller himself considered this possibility at the end of his paper [3]; by including terms in the Lagrangian other than the simple quadratic terms. Sáez [7] has generalized HNM theory in a very elegant and natural way into a scalar tetradic theories of gravitation. In these theories the question is: Do the field equations fix the tetradic geometry in the case of spherical symmetry? This question was discussed in length by Sáez [23]. The results of the present paper can be considered as a first step to get a satisfactory answer to this question. Mayer [8] has shown that Møller’s theory is a special case of the poincaré gauge theory constructed by Hehl et al. [10]. Thus poincaré gauge theory can be considered as another satisfactory generalization of Møller’s theory.

(d) Mikhail et al. [15] calculated the energy of two spherically symmetric solutions and found that the energy in one of the two solutions does not coincide with the gravitational mass. Shirafuji et al. [16] extended the calculations to all the stationary asymptotically flat solutions with spherical symmetry, dividing them into two classes, the one in which the components $(\lambda_0^a)$ and $(\lambda_0^a)$ of the parallel vector fields $(\lambda_a^\mu)$ tend to zero faster than $1/\sqrt{r}$ for large $r$ and the other, in which those components go to zero as $1/\sqrt{r}$. It was found that the equality of the energy and the gravitational mass holds only in the first class. It is of interest to note that the two tetrad structures (21) and (25) have those properties, i.e., the first tetrad structure (21) the components $(\lambda_0^a)$ and $(\lambda_0^a)$ go to zero faster than $1/\sqrt{r}$ for large $r$ and so its energy is the same as that obtained before [27]. As for the second tetrad structure (25) the components $(\lambda_0^a)$ and $(\lambda_0^a)$ go to zero as $1/\sqrt{r}$. So its energy content is different from the energy content of the first solution (18) and from that given by Yang [26] and Radinschi [27] by factor 2.

It is of interest to note that we have obtained two exact solutions for the partial differential equations (16) under some special constraint. The general solution for the partial differential equation in the case when the energy-momentum tensor does not vanish is not yet obtained. This will be studied in future work.

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