Small oriented cycle double cover of graphs

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Abstract

A small oriented cycle double cover (SOCDC) of a bridgeless graph $G$ on $n$ vertices is a collection of at most $n-1$ directed cycles of the symmetric orientation, $G_s$, of $G$ such that each arc of $G_s$ lies in exactly one of the cycles. It is conjectured that every 2-connected graph except two complete graphs $K_4$ and $K_6$ has an SOCDC. In this paper, we study graphs with SOCDC and obtain some properties of the minimal counterexample to this conjecture.

Keywords: Cycle double cover, Small cycle double cover, Oriented cycle double cover, Small oriented cycle double cover.

1 Introduction

We denote by $G$ a finite undirected graph with vertex set $V$ and edge set $E$ with no loops or multiple edges. The symmetric orientation of $G$, denoted by $G_s$, is an oriented graph obtained from $G$ by replacing each edge of $G$ by a pair of opposite directed arcs. An even graph (odd graph) is a graph such that each vertex is incident to an even (odd) number of edges. A directed even graph is a graph such that for each vertex its out-degree equals to its in-degree. A cycle (a directed cycle) is a minimal non-empty even graph (directed even graph). We denote every directed cycle $C$ and directed path $P$ on $n$ vertices with vertex set $\{v_1, \ldots, v_n\}$ and directed edge set $E(C) = \{v_iv_{i+1}, v_nv_1 : 1 \leq i \leq n-1\}$ and $E(P) = \{v_iv_{i+1} : 1 \leq i \leq n-1\}$ by $C = [v_1, \ldots, v_n]$, and $P = (v_1, \ldots, v_n)$, respectively.

A cycle double cover (CDC) $C$ of a graph $G$ is a collection of cycles in $G$ such that every edge of $G$ belongs to exactly two cycles of $C$. Note that the cycles
are not necessarily distinct. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph has no cut edge which is called a bridgeless graph. Seymour [17] in 1979 conjectured that every bridgeless graph has a CDC. No counterexample to the CDC conjecture is known. It is proved that the minimal counterexample to the CDC conjecture is a bridgeless cubic graph with edge chromatic number equal to 4, which is called a snark.

A small cycle double cover (SCDC) of a graph on $n$ vertices is a CDC with at most $n - 1$ cycles. There exist simple graphs of order $n$ for which any CDC requires at least $n - 1$ cycles (e.g., $K_n$, $n \geq 3$). Furthermore, no simple bridgeless graph of order $n$ is known to require more than $n - 1$ cycles in a CDC. Note that clearly it is false if not restricted to simple graphs. Bondy [3] conjectured that every simple bridgeless graph has an SCDC. For more results on the CDC conjecture see [7, 19].

The CDC conjecture has many stronger forms. In this paper, we consider the oriented version of these conjectures.

An oriented cycle double cover (OCDC) is a CDC in which every cycle can be oriented in such a way that every edge of the graph is covered by two directed cycles in two different directions.

Conjecture 1.1 [8] (Oriented CDC conjecture) Every bridgeless graph has an OCDC.

No counterexample to this conjecture is known. It is clear that the validity of the OCDC conjecture implies the validity of the CDC conjecture. While there is a CDC of the Petersen graph that can not be oriented in such a way that forms an OCDC.

Definition 1.2 A small oriented cycle double cover (SOCDC) of a graph on $n$ vertices is an OCDC with at most $n - 1$ directed cycles.

A perfect path double cover (PPDC) of a graph $G$ is a collection $\mathcal{P}$ of paths in $G$ such that each edge of $G$ belongs to exactly two members of $\mathcal{P}$ and each vertex of $G$ occurs exactly twice as an end of a path in $\mathcal{P}$ [2]. In [11] it is proved that every simple graph has a PPDC.

An oriented perfect path double cover (OPPDC) of a graph $G$ is a collection of directed paths in the symmetric orientation $G_s$ such that each arc of $G_s$ lies in exactly one of the paths and each vertex of $G$ appears just once as a beginning and just once as end of a directed path. Maxová and Nešetřil in [14] showed that two complete graphs $K_3$ and $K_5$ have no OPPDC and in [13], they conjectured every connected graph except $K_3$ and $K_5$ has an OPPDC.

The join of two simple graphs $G$ and $H$, $G \vee H$, is the graph obtained from the disjoint union of $G$ and $H$ by adding the edges \{$uv : u \in V(G), v \in V(H)$\}. 

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The existence of a PPDC for graphs in general is equivalent to the existence of an SCDC for the bridgeless graph obtained by joining a new vertex to all other vertices [2]. The following theorem denotes a relation between OPPDC and SOCDC.

**Theorem 1.3** [14] Let \( G \) be a connected graph. The graph \( G \) has an OPPDC if and only if \( G \vee K_1 \) has an SOCDC.

In the following theorem a list of some families of graphs that admit an OPPDC is provided. Therefore by Theorem 1.3, the join of graphs satisfying at least one of the conditions in below and \( K_1 \) admit an SOCDC.

**Theorem 1.4** [1, 14] Let \( G \neq K_3 \) be a graph. In each of the following cases, \( G \) has an OPPDC.

(i) \( G \) is a union of two arbitrary trees.

(ii) \( G \) is an odd graph.

(iii) \( G \) has no adjacent vertices of degree greater than two.

(iv) \( G \) is a 2-connected graph of order \( n \) and \( |E(G)| \leq 2n - 1 \).

(v) \( G = L(T) \), for some tree \( T \).

(vi) \( G = L(H) \), where the degree of no adjacent vertices in \( H \) have the same parity.

(vii) \( G \) is a graph with \( \Delta(G) \leq 4 \) and \( \delta(G) \leq 3 \).

(viii) \( G \) is a separable 4-regular graph. (A separable graph is a graph contains cut vertex.)

In what follows we have three sections. Section 2 deals with certain families of graphs with a small oriented cycle double cover. It is conjectured that every 2-connected graph except two complete graphs \( K_4 \) and \( K_6 \) has an SOCDC. In Section 3 we study the properties of the minimal counterexample to this conjecture. Finally in Section 4 some more relations between OPPDC and SOCDC are given.

## 2 The small oriented cycle double cover

The natural question is that which simple bridgeless graphs of order \( n \) have an OCDC with at most \( n - 1 \) cycles (SOCDC)?
Since $K_3$ and $K_5$ have no OPPDC, by Theorem [13], $K_4$ and $K_6$ have no SOCDC. It is known that every $K_{2n-1}, n \geq 4,$ has an OPPDC [1], thus by Theorem [13], every $K_{2n}, n \geq 4,$ has an SOCDC. Moreover, every $K_{2n+1}$ has an SOCDC, since $K_{2n+1}$ has a Hamiltonian cycle decomposition [18].

The following observation shows that if every block of a graph $G$ has an SOCDC, then $G$ has also an SOCDC.

**Observation 2.1** If $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$ which $G_i$ is a graph with an SOCDC, $i = 1, 2$; then $G$ also has an SOCDC.

Moreover, Observation 2.1 directly concludes the following corollaries. A block graph is a graph for which each block is a clique.

**Corollary 2.2** Every block graph with no block of order 2, 4 and 6 has an SOCDC.

Since the line graph of every tree is a block graph, the following result obtained which is an oriented version of existence of SCDC of line graph of trees [12].

**Corollary 2.3** If $T$ is a tree without vertices of degree 2, 4 or 6, then $L(T)$ has an SOCDC.

In the following proposition, we construct some graphs with no SOCDC. In fact, we show that the difference $|C| - (n - 1)$ could be large enough for every OCDC, $C$ of some bridgeless graph of order $n$. 

Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. The collection $C = \{[v_1, v_2, v_4], [v_2, v_1, v_3], [v_3, v_4, v_2], [v_4, v_3, v_1]\}$ is an OCDC of $K_4$. Since $K_4$ has six edges, if $C$ is an arbitrary OCDC of $K_4$, then $|C| \leq (2 \times 6)/3 = 4$. Thus, every OCDC of $K_4$ is of size 4.

Let $V(K_6) = \{v_1, \ldots, v_6\}$. The collection $C = \{[v_1, v_2, v_3, v_4, v_5, v_6], [v_2, v_6, v_3, v_5, v_4], [v_1, v_5, v_2, v_4, v_3], [v_1, v_4, v_6, v_2, v_5], [v_1, v_6, v_5, v_3, v_2], [v_1, v_3, v_6, v_4]\}$ is an OCDC of $K_6$ of size 6.

**Proposition 2.4** For every integer $r \geq 1$, there exists a bridgeless graph $G$ of order $n$ such that every OCDC of $G$ has $(n - 1) + r$ directed cycles.

**Proof.** Let $P$ be a path of length $r$ with $V(P) = \{v_1, \ldots, v_{r+1}\}$ and $E(P) = \{v_iv_{i+1} : 1 \leq i \leq r\}$. Assume that $G$ is a graph obtained from $P$ by replacing each edge $v_iv_{i+1}$ of $P$ with a clique $K_4$, say $K'_4$, where $V(K'_4) = \{v_i, v'_i, v_{i+1}, v'_{i+1}\}$, $1 \leq i \leq r$. Every OCDC of $G$ is decomposable to $r$ OCDC of $K_4$. Moreover, every OCDC of $K_4$ has four cycles. Therefore, every OCDC of $G$ has $4r$ cycles. Note that $|V(G)| = 3r + 1$, thus every OCDC of $G$ has $(|V(G)| - 1) + r$ cycles.

This fact motivates us to present the following conjecture.
Conjecture 2.5 (SOCDC conjecture) Every simple 2-connected graph except $K_4$ and $K_6$ admits an SOCDC.

The above conjecture has a close relation to the following conjecture.

Conjecture 2.6 [4] (Hajós’ conjecture) If $G$ is a simple, even graph of order $n$, then $G$ can be decomposed into $\lceil (n - 1)/2 \rceil$ cycles.

If the Hajós’ conjecture holds, then every even graph has an SOCDC obtained by taking two copies of the cycles used in its decomposition, in two opposite directions.

An edge of a graph $G$ is said to be contracted if it is deleted and its two ends are identified. A minor of $G$ is a graph obtained from $G$ by deletions of vertices, and deletions and contractions of edges. The graph obtained from $K_6$ by deleting an edge is denoted $K_6^-$. A $K_6^-$-minor free graph is a graph that does not contain $K_6^-$ as a minor.

As the Hajós’ conjecture is true for even graphs with maximum degree four [5], planar graphs [16], projective graphs (a projective graph is a graph $G$ which is embeddable on the projective plane.), and $K_6^-$-minor free graphs [4], these graphs have an SOCDC.

Proposition 2.7 Let $G$ be an even graph. In each of the following cases, $G$ has an SOCDC.

(i) $\Delta(G) = 4$.

(ii) $G$ is planar.

(iii) $G$ is a projective graph.

(iv) $G$ is $K_6^-$-minor free.

Klimmek [9] proved that every even line graph of order $n$ has a cycle decomposition into $\lceil (n - 1)/2 \rceil$ cycles, thus the Hajós’ conjecture holds for such graphs. Since a line graph, $L(G)$, is even if and only if every component of $G$ is either even or odd, the line graph of every even graph and of every odd graph has an SOCDC.

Proposition 2.8 If $G$ is an even or an odd graph, then $L(G)$ has an SOCDC.

The following proposition considers another class of graphs with OCDC which also has SOCDC.

Proposition 2.9 If $G$ has an OCDC, $C$, and the girth of $G$, $g(G)$, is greater than average degree, $\bar{d}(G)$, then $C$ is also an SOCDC of $G$. 5
Proof. Let \( C \) be an OCDC of \( G \). Note that each edge of \( G \) is covered twice by elements of \( C \), therefore,

\[
g(G)|C| \leq \sum_{C \in \mathcal{C}} |E(C)| = 2|E(G)| = \sum_{v \in V(G)} d(v) = |V(G)|d(G).
\]

Since \( g(G) > \bar{d}(G) \), we have \( |C| \leq |V(G)| - 1 \). Hence, \( C \) is an SOCDC of \( G \).

It can be proved that an OCDC for planar graphs can be obtained from their planar embedding and some planar graph has also SOCDC.

**Proposition 2.10** Every bridgeless planar graph \( G \) with \( |E(G)| < 2|V(G)| - 2 \), has an SOCDC.

**Proof.** Let \( G \) be a bridgeless planar graph. Since we can orient the edges of each face of \( G \) in such a way that the collection of the boundary of its faces, \( \mathcal{F} \), is an OCDC. By Euler’s formula, \( |\mathcal{F}| = 2 + |E(G)| - |V(G)| \). Since \( |E(G)| < 2|V(G)| - 2 \), we conclude \( |\mathcal{F}| < |V(G)| \). Hence, \( G \) has an SOCDC.

Since in every simple triangle-free planar graph \( G \) with at least three vertices, \( |E(G)| \leq 2|V(G)| - 4 \), we obtain the following corollary.

**Corollary 2.11** Let \( G \) be a bridgeless planar graph. If \( G \) is triangle-free, then \( G \) admits an SOCDC.

The following proposition presents an SOCDC for the well-known non-planar triangle-free graphs.

**Proposition 2.12** Every \( K_{n,m} \), \( n, m \geq 2 \), has an SOCDC.

**Proof.** Assume that \( V(K_{n,m}) = \{v_1, \ldots, v_n; w_1, \ldots, w_m\} \), \( n \leq m \). Let

\[
C_i = [v_1, w_i, v_2, w_{i+1}, v_3, w_{i+2}, \ldots, v_{n-1}, w_{i+n-2}, v_n, w_{i+n-1}],
\]

be a directed cycle, where subscripts are reduced modulo \( m \). It is easy to check that \( \mathcal{C} = \{C_i : 1 \leq i \leq m\} \) is an SOCDC of \( K_{n,m} \), \( n, m \geq 2 \).

Let \( G \) be a simple graph and \((D, f)\) be an ordered pair where \( D \) is an orientation of \( E(G) \) and \( f \) is a weight on \( E(G) \) to \( \mathbb{Z} \). For each \( v \in V(G) \), denote

\[
f^+(v) = \sum f(e) \quad \text{and} \quad f^-(v) = \sum f(e),
\]

where the summation is taken over all directed edges of \( G \) (under the orientation \( D \)) with tails and heads, respectively, at the vertex \( v \). An integer flow of \( G \) is an ordered pair \((D, f)\) such that for every vertex \( v \in V(G) \), \( f^+(v) = f^-(v) \). A nowhere-zero \( k \)-flow of \( G \) is an integer flow \((D, f)\) such that \( 0 < |f(e)| < k \), for every edge \( e \in E(G) \) and is denoted by \( k\)-NZF \([19]\).
Theorem 2.13 [19] Every cubic graph \( G \) admits a 4-NZF if and only if \( \chi'(G) = 3 \).

Theorem 2.14 [19] A graph \( G \) admits a 4-NZF if and only if \( G \) has an OCDC consists of four directed even subgraphs.

The following theorem concludes from Theorems 2.13 and 2.14.

Theorem 2.15 Every cubic graph with edge chromatic number 3 admits an OCDC.

Theorem 2.16 [10] If \( C \) is a CDC of a cubic graph \( G \) of order \( n \), then \( |C| \leq n/2+2 \).

Since from every OCDC of a graph a CDC for the graph is obtained, we have the following corollary.

Corollary 2.17 Every OCDC, \( C \), of a cubic graph of order \( n \geq 6 \), is an SOCDC.

The following corollary concludes directly from Theorem 2.13 and Corollary 2.17.

Corollary 2.18 Every cubic graph with edge chromatic number 3, \( G \neq K_4 \), has an SOCDC.

3 The minimal counterexample to the SOCDC conjecture

If the CDC conjecture is false, then it must have a minimal counterexample. In this section, we study the properties of the minimal counterexample to the SOCDC conjecture.

Observation 3.1 If \( G \) is a graph with an SOCDC and \( G' \) is the graph obtained from \( G \) by dividing one edge of \( G \), then \( G' \) also admits an SOCDC.

Corollary 3.2 Let \( G \) be the minimal counterexample to the SOCDC conjecture, then the minimum degree of \( G \) is at least 3.

Theorem 3.3 The minimal counterexample to the SOCDC conjecture is 3-connected.

Proof. Let \( G \), the minimal counterexample to the SOCDC conjecture be a 2-connected graph of order \( n \) with vertex cut \( \{v_1, v_2\} \) and \( G = G_1 \cup G_2 \), where \( V(G_1) \cap V(G_2) = \{v_1, v_2\} \) and \( |V(G_i)| = n_i \), \( i = 1, 2 \). Assume that \( G_i \cup \{v_1 v_2\} \) has an SOCDC, \( C_i, \ i = 1, 2 \). Let \( C_j, j = 1, 2 \), be the two directed cycles in \( C_i, \ i = 1, 2 \), which include the directed edge \( v_j v_{j+1} \), where subscripts are reduced modulo 2. In each of the following cases, we show that \( G \) admits an SOCDC, which is a contradiction.
(I) If $v_1v_2 \in E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2 \} \setminus \{C_1^1, C_2^2\}.$$  

The collection $\mathcal{C}$ is an OCDC of $G$, where

$$|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 1 \leq (n_1 - 1) + (n_2 - 1) - 1$$

$$\leq (n_1 + n_2) - 3$$

$$\leq (n + 2) - 3 = n - 1.$$  

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $C_2$, then let $C_1 = [v_1, v_2, v_4], C_2 = [v_1, v_4, v_3, v_2], C_3 = C_2^2 \cup (v_1, v_3, v_4, v_2) \setminus \{v_1v_2\}$, and $C_4 = C_2^2 \cup (v_2, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = C_2 \cup \{C_1, C_2, C_3, C_4\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of $G$.

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $C_2$, then let $C_1 = [v_1, v_2, v_4, v_6, v_3, v_5], C_2 = [v_1, v_3, v_6, v_2], C_3 = [v_1, v_4, v_2, v_5, v_6], C_4 = [v_1, v_5, v_2, v_3, v_4], C_5 = C_2^2 \cup (v_1, v_6, v_5, v_4, v_3, v_2) \setminus \{v_1v_2\}$, and $C_6 = C_2^2 \cup (v_2, v_6, v_4, v_5, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = C_2 \cup \{C_1, C_2, C_3, C_4, C_5, C_6\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of $G$.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ or $G_1 \cup \{v_1v_2\} = K_4$ and $G_2 \cup \{v_1v_2\} = K_6$ or $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6$, then by Theorem 1 in [1], $G \setminus v_1$ admits an OPPDC, thus by Theorem 1.3 $G$ has an SOCDC.

(II) If $v_1v_2 \notin E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2, C_1^2 \Delta C_2^1 \} \setminus \{C_1^1, C_2^2, C_1^2, C_2^1\}.$$  

The collection $\mathcal{C}$ is an OCDC of $G$, where $|\mathcal{C}| \leq n - 2$.

Furthermore, if $G_1 \cup \{v_1v_2\} = K_4$ or $K_6, v_1v_2 \notin E(G)$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, by the similar argument in above using the given SOCDC for $K_4$ and $K_6$ of size 4 and 6, an SOCDC for $G$ is obtained.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ with $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $V(G_2) = \{v_1, v_2, v_5, v_6\}$, then

$$\mathcal{C} = \{v_1, v_4, v_3, v_2, v_5, v_6\}, [v_1, v_5, v_2, v_3], [v_1, v_3, v_4, v_2, v_6, v_5], [v_1, v_6, v_2, v_4]\}$$

is an SOCDC of $G$.  

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If \( G_1 \cup \{v_1v_2\} = K_4 \) with \( V(G_1) = \{v_1, v_2, v_3\} \) and \( G_2 \cup \{v_1v_2\} = K_6 \) \( V(G_2) = \{v_1, v_2, v_5, v_6, v_7, v_8\} \), then
\[
\mathcal{C} = \{[v_1, v_6, v_5, v_7, v_8, v_2, v_3], [v_1, v_4, v_4, v_2, v_8], [v_1, v_7, v_6, v_8, v_5, v_2, v_4], [v_1, v_5, v_8, v_7, v_2, v_6], [v_1, v_8, v_6, v_2, v_7, v_5], [v_1, v_4, v_3, v_2, v_5, v_6, v_7]\}\]
is an SOCDC of \( G \).

If \( G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6 \) with \( V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( V(G_2) = \{v_1, v_2, v_7, v_8, v_9, v_{10}\} \), then
\[
\mathcal{C} = \{[v_1, v_6, v_4, v_5, v_3, v_2, v_7, v_9, v_8, v_{10}], [v_1, v_3, v_5, v_4, v_6, v_2, v_{10}, v_8, v_9, v_7], [v_1, v_4, v_3, v_6, v_5, v_2, v_9, v_{10}, v_7, v_8], [v_1, v_5, v_6, v_3, v_4, v_2, v_8, v_7, v_{10}, v_9], [v_1, v_8, v_2, v_4],
[v_1, v_10, v_2, v_6], [v_1, v_9, v_2, v_5], [v_1, v_7, v_2, v_3]\}\]
is an SOCDC of \( G \).

\[\blacksquare\]

**Corollary 3.4** The minimal counterexample to the SOCDC conjecture is 3-edge-connected.

An edge cut \( F \), is called trivial if one of the component in \( G \setminus F \) be an isolated vertex.

**Theorem 3.5** The minimal counterexample to the SOCDC conjecture has no non-trivial edge cut of size 3.

**Proof.** Let \( G \) be the minimal counterexample to the SOCDC conjecture. We know that \( G \) is 2-connected and 3-edge-connected. Assume that \( G \) has a non-trivial edge cut of size 3. We consider the following cases.

(1) \( G = G_1 \cup G_2 \cup \{u_1v_1, u_2v_2, u_3v_3\} \), where \( G_1 \cap G_2 = \emptyset \), the vertices \( u_i \) are distinct vertices of \( G_1 \), and the vertices \( v_i \) are distinct vertices of \( G_2 \), \( i = 1, 2, 3 \).

Denote by \( H_i \) the graph obtained by contracting the subgraph \( G_{i+1} \) to a single vertex \( w_i \), \( i = 1, 2 \), where subscripts are reduced modulo 2. Since \( \deg(w_i) = 3 \), \( H_i \neq K_6 \), \( i = 1, 2 \). By the minimality of \( G \), \( H_i \) has an SOCDC or \( H_i = K_4 \). Therefore, \( H_i \) has an OCDC, \( C_i \), \( i = 1, 2 \). Let \( C_i^j \), \( j = 1, 2, 3 \), be the three directed cycles in \( C_i \) which include \( w_i \), \( i = 1, 2 \), where without loss of generality, we assume that \( C_i^1 \) includes directed path \( (w_{j-1}, w_1, u_{j+1}) \), and \( C_i^2 \) includes directed path \( (v_{j+1}, w_2, v_{j-1}) \), where subscripts are reduced modulo 3, \( j = 1, 2, 3 \). Define \( C_i^j = P_i^j \setminus w_i \), \( i = 1, 2 \), \( j = 1, 2, 3 \). Define \( C_i^j = P_i^j \cup P_i^k \cup \{u_{j-1}v_{j-1}, v_{j+1}u_{j+1}\} \), \( C' = \{C_i^j \ : \ j = 1, 2, 3\} \), and \( C'' = \{C_i^j \ : \ i = 1, 2, j = 1, 2, 3\} \). Thus, \( \mathcal{C} = C_1 \cup C_2 \cup C' \cup C'' \) is an OCDC of \( G \), where \( |\mathcal{C}| = |C_1| + |C_2| - 3 \). Note that every OCDC of \( K_4 \) has 4 cycles, therefore, in both cases \( |\mathcal{C}| \leq |V(G)| - 1 \), which is a contradiction.
(II) \( G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_3\} \), where \( G_1 \cap G_2 = \emptyset \), the vertices \( u_1 \) and \( u_2 \) are distinct vertices of \( G_1 \), and the vertices \( v_i \) are distinct vertices of \( G_2 \), \( i = 1, 2, 3 \).

Denote by \( H_i \) the graph obtained by contracting the subgraph \( G_{i+1} \) to a single vertex \( w_i \), \( i = 1, 2 \), and removing the multiple edge in \( H_1 \), where subscripts are reduced modulo 2. Since \( \deg(w_i) = 2 \) or 3, \( H_1 \neq K_4 \) and \( H_i \neq K_6 \), \( i = 1, 2 \). By the minimality of \( G \), \( H_i \) has an SOCDC or \( H_2 = K_4 \). Therefore, \( H_i \) has an OCDC, \( C_i \), \( i = 1, 2 \). Let \( C_1^1 \) and \( C_1^2 \) be two directed cycles in \( C_1 \) which include \( w_1 \), where without loss of generality, we assume that \( C_1^1 \) includes directed path \( (u_j, w_1, u_{j+1}) \), where subscripts are reduced modulo 2, \( j = 1, 2 \), and \( C_2^k \), \( k = 1, 2, 3 \), be the three directed cycles in \( C_2 \) which include \( w_2 \), where without loss of generality, we assume that \( C_2^k \) includes directed path \( (v_k, w_2, v_{k-1}) \), where subscripts are reduced modulo 3, \( k = 1, 2, 3 \). Let \( P_1^j = C_1^j \setminus w_1, j = 1, 2 \), and \( P_2^k = C_2^k \setminus w_2, k = 1, 2, 3 \). Define \( C^1 = P_1^1 \cup P_2^3 \cup \{u_1v_2, v_3u_2\}, C^2 = P_2^1 \cup P_2^1 \cup \{u_2v_3, v_1u_1\}, \) and \( C^3 = P_2^2 \cup \{u_1v_1, v_2u_1\} \). Let \( C' = \{C^1, C^2, C^3\} \), and \( C'' = \{C_1^1, C_1^2, C_2^1, C_2^2\} \). Thus, \( C = C_1 \cup C_2 \cup C' \setminus C'' \) is an OCDC of \( G \), where \( |C| = |C_1| + |C_2| - 2 \). Note that every OCDC of \( K_4 \) has 4 cycles, therefore, in both cases \( |C| \leq |V(G)| - 1 \), which is a contradiction.

(III) \( G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_2\} \), where \( G_1 \cap G_2 = \emptyset \), the vertices \( u_1 \) and \( u_2 \) are distinct vertices of \( G_1 \), and the vertices \( v_1 \) and \( v_2 \) are distinct vertices of \( G_2 \).

Denote by \( H_i \) the graph obtained by contracting the subgraph \( G_{i+1} \) to a single vertex \( w_i \), \( i = 1, 2 \), and removing the multiple edges, where subscripts are reduced modulo 2. Since \( \deg(w_i) = 2 \), \( H_i \neq K_4 \) or \( K_6 \), \( i = 1, 2 \). By the minimality of \( G \), \( H_i \) has an SOCDC, \( C_i \), \( i = 1, 2 \).

Let \( C_1^j, j = 1, 2 \), be the two directed cycles in \( C_1 \) which include \( w_i \), \( i = 1, 2 \), where without loss of generality, we assume that \( C_1^1 \) includes directed path \( (u_j, w_1, u_{j+1}) \), and \( C_1^2 \) includes directed path \( (v_j, w_2, v_{j+1}) \), where subscripts are reduced modulo 2, \( j = 1, 2 \). Let \( P_1^j = C_1^j \setminus w_1, i = 1, 2 \), \( j = 1, 2 \). Define \( C^1 = P_1^1 \cup P_2^2 \cup \{u_1v_2, v_2u_2\}, C^2 = P_2^1 \cup \{u_2v_3, v_1u_1\}, \) and \( C^3 = P_2^2 \cup \{u_1v_1, v_2u_1\} \). Let \( C' = \{C^1, C^2, C^3\} \), and \( C'' = \{C_1^1, C_1^2, C_2^1, C_2^2\} \). Thus, \( C = C_1 \cup C_2 \cup C' \setminus C'' \) is an OCDC of \( G \), where \( |C| = |C_1| + |C_2| - 1 \). Therefore, \( |C| \leq |V(G)| - 1 \), which is a contradiction.

The properties of the minimal counterexample to the OPPDC conjecture is studied in [1, 13], and it is shown that this counterexample is a 2-connected and 3-edge-connected graph with minimum degree at least 4. Regarding to the relation between existence of an OPPDC for a graph and an SOCDC (Theorem 1.3), the following relation between the order and the number of edges of these two minimal counterexamples can be obtained.

Assume that \( G_p \) and \( G_c \) are the minimal counterexamples to the OPPDC conjecture and the SOCDC conjecture, respectively. Let \( G'_p \) be a graph obtained from \( G_p \) by joining a new vertex to all vertex of \( G_p \). By Theorem 1.3, \( G'_p \) has no SOCDC. Note that \( G_p \) is connected, so \( G'_p \) is 2-connected. Thus, \( G'_p \) is a counterexample to
the SOCDC conjecture. Therefore by the minimality of $G_c$, 

$$|V(G_c)| + |E(G_c)| \leq |V(G'_p)| + |E(G'_p)| = 2|V(G_p)| + |E(G_p)| + 1.$$ 

Since $\delta(G_c) \geq 3$, 

$$|V(G_c)| \leq \frac{2}{3}(2|V(G_p)| + |E(G_p)| + 1).$$

4 SOCDC and the Cartesian product

In [15] infinite classes of graphs with an SCDC are obtained using the Cartesian product of graphs. In this section, we proved the similar results in the oriented version.

The Cartesian product of two graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u, v)$ and $(x, y)$ are adjacent if and only if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

**Theorem 4.1** If $G$ has an OPPDC, then $G \Box P_2$ has an SOCDC with $|V(G)|$ directed cycles. Furthermore, if $G$ has an SOCDC, then $G \Box P_n$, $n \geq 3$, has an SOCDC with at most $|V(G \Box P_n)| - 1$ directed cycles.

**Proof.** The proof is similar to the proof of Theorem 1 in [15].

Now we need a theorem about the existence of OPPDC for the Cartesian product of graphs.

**Theorem 4.2** [1] If $G$ and $H$ have an OPPDC, then $G \Box H$ also has an OPPDC.

The following corollary follows directly from Theorems 4.1 and 4.2.

**Corollary 4.3** If $G$ has an OPPDC, then for all $l \geq 2$, $G^l \Box P_2$ has an SOCDC, where $G^l = G \Box \cdots \Box G$.

**Corollary 4.4** Every hypercube graph $Q_n$, $n \geq 2$, has an SOCDC.

With the similar argument as above, the following theorems, which are the oriented version of some results in [15] for SCDC, can be proved.

**Theorem 4.5** If $G$ has an OPPDC and an SOCDC, then for any tree, $T$, $G \Box T$ has an SOCDC.
Theorem 4.6 If $G$ has an OPPDC, then for all $k \geq 2$, $G \square C_{2k}$ has an SOCDC. Furthermore, if $G$ has an SOCDC, then $G \square C_{2k-1}$ has an SOCDC.

The following corollary concludes directly from Theorems 4.2 and 4.6.

Corollary 4.7 If $G$ has an OPPDC, then for all $k, l \geq 2$, $G^l \square C_{2k}$ has an SOCDC, where $G^l = \overbrace{G \square \cdots \square G}^{l \text{ times}}$.

Theorem 4.8 If $G$ has an SOCDC, then for $n \geq 2|V(G)| + 1$, $G \square C_n$ has an SOCDC.

Proof. Let $\mathcal{F}$ be the union of the corresponding SOCDC’s of copies of $G$ and the corresponding SOCDC’s of copies of $C_n$. Since every OCDC of $C_n$ has two directed cycles, and $n \geq 2|V(G)| + 1$, we have $|\mathcal{F}| \leq n(|V(G)| - 1) + 2|V(G)| \leq n(|V(G)| - 1) + (n - 1) \leq n|V(G)| - 1 = |V(G \square C_n)| - 1$. Therefore, $\mathcal{F}$ is an SOCDC of $G \square C_n$. $\blacksquare$

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