On integration of the Kowalevski gyrostat and the Clebsch problems

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Abstract
For the Kowalevski gyrostat change of variables similar to that of the Kowalevski top is done. We establish one to one correspondence between the Kowalevski gyrostat and the Clebsch system and demonstrate that Kowalevski variables for the gyrostat practically coincide with elliptic coordinates on sphere for the Clebsch case. Equivalence of considered integrable systems allows to construct two Lax matrices for the gyrostat using known rational and elliptic Lax matrices for the Clebsch model. Associated with these matrices solutions of the Clebsch system and, therefore, of the Kowalevski gyrostat problem are discussed. The Kötter solution of the Clebsch system in modern notation is presented in detail.

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1 Introduction

Kowalevski gyrostat is a one parameter integrable extension of the Kowalevski top constructed in 1987 [13], [31]. For the Kowalevski gyrostat the Lax representation with spectral parameter was found in framework of a general group-theoretical approach to integrable systems using the Lie algebras $so(3, 2)$ or $sp(4)$ [24], [1]. For the Kowalevski top the spectral curve generated by this Lax matrix differs from the original Kowalevski curve.

Solutions for the top in terms of the Prym theta-functions were obtained by finite-band integration technique in [4], where it was said that the gyrostat can be integrated in a similar way. We do not know separated variables and separated equations associated with the Lax matrices [4] neither for the top nor for the gyrostat.

Another algebro-geometric approach to study of the Kowalevski top was developed in [10], [1]. In this method detailed analysis of the level surfaces of constant of motion allows to establish a birational isomorphism between the Kowalevski flow and the flows of other integrable systems that are linearizable on abelian varieties of the same type. In [10] the Kowalevski top is related with the Neumann system and in [1] with the Schottky-Manakov top on $so(4)$. The known Lax matrices and separated variables for these systems give rise to the Lax matrices and the separated variables for the Kowalevski top.

The aim of this paper is to extend the Kowalevski treatment of the top to the gyrostat and to construct an isomorphism between the Kowalevski gyrostat and the Clebsch case of the motion of a rigid body in ideal fluid. As a byproduct one gets new rational and elliptic Lax matrices for the Kowalevski gyrostat together with the corresponding integration procedures.

2 The Kowalevski gyrostat

Let two vectors $\mathbf{J}$ and $\mathbf{x}$ are coordinates on the phase space $M$. As a Poisson manifold $M$ is identified with Euclidean algebra $e(3)^*$ with the Lie-Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0,$$

where $\varepsilon_{ijk}$ is the totally skew-symmetric tensor. These brackets have two Casimir functions

$$A = \mathbf{x}^2 \equiv \sum_{k=1}^{3} x_k^2, \quad B = (\mathbf{x} \cdot \mathbf{J}) \equiv \sum_{k=1}^{3} x_k J_k. \quad (2.1)$$

Fixing their values one gets a generic symplectic leaf of $e(3)$

$$\mathcal{O}_{ab} : \quad \{\mathbf{x} \cdot \mathbf{J} : A = a, \ B = b\},$$

which is a four-dimensional symplectic manifold.
The Hamilton function for the original Kowalevski top is given by
\begin{equation}
H_{\text{top}} = \frac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) + cx_1, \quad c \in \mathbb{C}.
\end{equation}
(2.3)
This Hamiltonian and additional integral of motion
\begin{equation}
K_{\text{top}} = \xi_1 \cdot \xi_2,
\end{equation}
(2.4)
are in involution and define a moment map whose fibers are Liouville tori in \(E_{ab}\). Here
\begin{equation}
\xi_1 = z_1^2 - 2c(x_1 + ix_2), \quad \xi_2 = z_2^2 - 2c(x_1 - ix_2)
\end{equation}
(2.5)
and
\begin{equation}
z_1 = J_1 + iJ_2, \quad z_2 = J_1 - iJ_2.
\end{equation}
(2.6)

The Kowalevski gyrostat \cite{13,31} is an integrable extension of the corresponding top defined by the following constants of motion
\begin{equation}
H = H_{\text{top}} - \lambda J_3 = \frac{1}{2}(J_1^2 + J_2^2 + 2J_3^2 - 2\lambda J_3) + cx_1,
\end{equation}
(2.7)
\begin{equation}
K = \xi_1\xi_2 + 4\lambda((J_3 - \lambda)z_1z_2 - (z_1 + z_2)cx_3)
\end{equation}
(2.8)
in involution \(\{H,K\} = 0\). The gyrostat generalization of the Kowalevski top is essential because the corresponding additional terms in the Hamiltonian mimic quantum corrections to the top \cite{13}.

The equations of motion are given by the customary Euler-Poisson equations
\begin{equation}
X : \quad \dot{J} = J \times \frac{\partial H}{\partial J} + x \times \frac{\partial H}{\partial x}, \quad \dot{x} = x \times \frac{\partial H}{\partial J},
\end{equation}
(2.9)
where \(x \times z\) means cross product of two vectors. Equations (2.9) can be rewritten as
\begin{equation}
X \left( \begin{array}{c} J \\ x \end{array} \right) = P_0 dH, \quad P_0 = \left( \begin{array}{cc} J & X \\ -X & 0 \end{array} \right),
\end{equation}
where
\begin{equation}
J = \left( \begin{array}{ccc} 0 & J_3 & -J_2 \\ -J_3 & 0 & J_1 \\ J_2 & -J_1 & 0 \end{array} \right), \quad X = \left( \begin{array}{ccc} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{array} \right).
\end{equation}

We do not show the second commuting flow for brevity.

**Remark 1** \cite{13}: Additional terms in \(K\) are closely related with the third order in momenta integral of the Goryachev-Chaplygin gyrostat
\begin{equation}
G = 2((J_3 - 2\lambda)z_1z_2 + (z_1 + z_2)cx_3)
\end{equation}
and expressed in terms of the Poisson brackets of \(\xi_1, \xi_2\)
\begin{equation}
K = K_{\text{top}} + i\lambda\{\xi_1, \xi_2\} - 16\lambda^2 z_1z_2.
\end{equation}
(2.10)
3 The Kowalevski variables

We will introduce variables $s_1, s_2$ for the Kowalevski gyrostat step by step following original papers [19] and [18] where the separation of variables for the top was constructed.

At first we made a transition from initial variables to new variables $\xi_1, \xi_2$ and organize four constants of motion in the following matrix identity

\[
\begin{pmatrix}
4H & 4cB \\
4cB & 4c^2A - K
\end{pmatrix}
= 4
\begin{pmatrix}
J_3^2 & c^2J_3 \\
c^2J_3 & c^2x_3^2
\end{pmatrix}
- 4\lambda
\begin{pmatrix}
0 & 0 \\
0 & z_1z_2(J_3 - \lambda) - c(z_1 + z_2)x_3
\end{pmatrix}
+ 
\begin{pmatrix}
(z_1 + z_2)^2 & z_1z_2(z_1 + z_2) \\
(z_1z_2(z_1 + z_2) & z_1^2z_2^2
\end{pmatrix}
- \begin{pmatrix}
\xi_1 + \xi_2 & \xi_1z_2 + \xi_2z_1 \\
\xi_1z_2 + \xi_2z_1 & \xi_1z_2^2 + \xi_2z_1^2
\end{pmatrix}
\]

(3.1)

The second step consists of exclusion two variables $x_3$ and $J_3$ using velocities $\dot{z}_i = \{H, z_i\}$

\[
x_3 = \frac{i\dot{z}_1z_2 + z_1\dot{z}_2}{z_1 - z_2}, \quad J_3 = \frac{i(\dot{z}_2 + \dot{z}_1)}{(z_1 - z_2)} + \lambda.
\]

(3.2)

Similarity transform $U^t(\cdot)U$ of the both sides of (3.1) with auxiliary matrix $U$

\[
U = \begin{pmatrix}
z_1 & z_2 \\
-1 & -1
\end{pmatrix}, \quad U^t = \begin{pmatrix}
z_1 & -1 \\
z_2 & -1
\end{pmatrix}
\]

(3.3)

brings us to the following matrix identity for the gyrostat

\[
\begin{pmatrix}
\dot{z}_1^2 & -\dot{z}_1\dot{z}_2 \\
-\dot{z}_1\dot{z}_2 & \dot{z}_2^2
\end{pmatrix}
+ 4i\lambda(z_1 - z_2)
\begin{pmatrix}
\dot{z}_1 & 0 \\
0 & \dot{z}_2
\end{pmatrix}
+ (z_1 - z_2)^2
\begin{pmatrix}
\xi_1 & -2H \\
-2H & \xi_2
\end{pmatrix}
- \begin{pmatrix}
R(z_1, z_1) & R(z_1, z_2) \\
R(z_1, z_2) & R(z_2, z_2)
\end{pmatrix}
= 0.
\]

(3.4)

Here

\[
R(z_1, z_2) = z_1^2z_2^2 - 2H(z_1^2 + z_2^2) - 4cB(z_1 + z_2) - 4c^2A + K.
\]

(3.5)

The diagonal entries of the identity (3.4) allows to express variables $\xi_{1,2}$ as

\[
\xi_k = \frac{4i\lambda\dot{z}_k}{z_1 - z_2} - \frac{4\dot{z}_k^2 - R(z_k, z_k)}{(z_1 - z_2)^2}, \quad k = 1, 2,
\]

and one gets integrals $H$ and $K$ in terms of biquadratic polynomial $R$ (3.5) and two pairs of
Lagrangian variables \( z_{1,2} \) and \( \dot{z}_{1,2} \)

\[
H = -\frac{4\dot{z}_1\dot{z}_2 + R(z_1, z_2)}{2(z_1 - z_2)^2}, \quad (3.6)
\]

\[
K = -\frac{16\dot{z}_1^2\dot{z}_2^2}{(z_1 - z_2)^2} - 4i\lambda \left( \frac{\dot{z}_1}{\partial z_1} - \frac{\dot{z}_2}{\partial z_2} \right) \frac{R(z_1, z_2)}{(z_1 - z_2)^2} + \frac{(4\dot{z}_1^2 - R(z_1, z_1))(4\dot{z}_2^2 - R(z_2, z_2))}{(z_1 - z_2)^2}. \quad (3.7)
\]

On the level surface of integrals of motion

\[
\Sigma = \{ A = a, B = b, H = h, K = k \} \quad (3.8)
\]

relations (3.6) and (3.7)

\[
\Phi_{1,2}(z_1, z_2, \dot{z}_1, \dot{z}_2, A, B, H, K) \big|_{\Sigma} = 0
\]

can be considered as equations of motion determining two dimensional dynamical system. Unfortunately variables \( z_{1,2} \) do not commute \( \{ z_1, z_2 \} \neq 0 \), so one has to look for more convenient parametrization.

**Remark 2** Associated with the fourth degree polynomials \( R(z_k, z_k) \) (3.5)

\[
R(z_k, z_k) = a_0 z_k^4 + 4a_1 z_k^3 + 6a_2 z_k^2 + 4a_3 z_k + a_4, \quad a_i \in \mathbb{R}
\]

differential equations

\[
\frac{\dot{z}_1}{\sqrt{R(z_1, z_1)}} = \pm \frac{\dot{z}_2}{\sqrt{R(z_2, z_2)}}, \quad (3.9)
\]

originally appeared in the Euler studies of equation of lemniscate and invariance of the corresponding elliptic integrals [6]. In particular Euler proved that equations (3.9) have an algebraic integral

\[
E(z_1, z_2, s) = (z_1 - z_2)^2 s^2 - R(z_1, z_2) s + W = 0, \quad (3.10)
\]

where \( R(z_1, z_2) \) is a mixed biquadratic form similar to (3.5)

\[
R(z_1, z_2) = a_0 z_1^2 z_2^2 + 2a_1 z_1 z_2 (z_1 + z_2) + 3a_2 (z_1^2 + z_2^2) + 2a_3 (z_1 + z_2) + a_4, \quad (3.11)
\]

and

\[
W = \frac{R(z_1, z_2)^2 - R(z_1, z_1)R(z_2, z_2)}{4(z_1 - z_2)^2}.
\]

In algebro-geometric terms [30], Euler studied automorphisms \( (u_1, z_1) \to (u_2, z_2) \) of the algebraic curve of genus one

\[
C : u^2 = R(z, z), \quad (3.12)
\]
which change a sign of the corresponding holomorphic form $dz/u \to \pm dz/u$. Thus every algebraic curve of genus one is isomorphic to a complex torus (cubic elliptic curve), which equivalent to Jacobian of $C$. These automorphisms are parameterized by points of a smooth elliptic curve

$$\Gamma : \eta^2 = P_3(s), \quad P_3(s) = 4s^3 + g_1s^2 + g_2s + g_3,$$

(3.13)

where $g_k$ are functions on initial parameters $a_0, \ldots, a_4$. According to Weil [30], if $O_k = (u_k, z_k)$, $k = 1, 2$, are two points of $C$ and $N_k = (\eta_k, s_k)$, $k = 1, 2$, denote two points of $\Gamma$ related by $O_1 = N_1 + N_2$ and $O_2 = N_1 - N_2$ then

$$\frac{dz_1}{u_1} + \frac{dz_2}{u_2} = \frac{ds_1}{\eta_1}, \quad \frac{dz_1}{u_1} - \frac{dz_2}{u_2} = \frac{ds_2}{\eta_2}.$$

(3.14)

It is infinitesimal version of the Weil interpretation of the Euler results. These results are independent on the choice of affine coordinates $(u, z)$ and $(\eta, s)$ on the curves $C$ and $\Gamma$, respectively.

The third step of Kowalevski in [19] is to apply automorphisms of auxiliary elliptic curve (3.12) given in Remark 2 by introduction her famous variables $s_{1,2}$

$$s_{1,2} = \frac{R(z_1, z_2) \pm \sqrt{R(z_1, z_1)R(z_2, z_2)}}{2(z_1 - z_2)^2},$$

(3.15)

which are transcendental integrals of the corresponding Euler equations (3.9) [6].

The variables $s_{1,2}$ are eigenvalues of an auxiliary spectral problem

$$\begin{pmatrix} R(z_1, z_1) & R(z_1, z_2) \\ R(z_1, z_2) & R(z_2, z_2) \end{pmatrix} \Psi = 2s \sigma_1 (z_1 - z_2)^2 \Psi, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(3.16)

that is naturally extracted from (3.4). Its characteristic polynomial

$$E(s) = (z_1 - z_2)^2 (s - s_1)(s - s_2).$$

(3.17)

coincides with the Euler algebraic integral (3.10). For its analysis see, e.g., Golubev [9].

The matrix of eigenfunctions $\Psi$ of the spectral problem (3.16) reads

$$\Psi = \begin{pmatrix} 1 & 1 \\ \sqrt{R(z_1, z_1)} & \sqrt{R(z_2, z_2)} \\ -\frac{1}{\sqrt{R(z_1, z_1)}} & \frac{1}{\sqrt{R(z_2, z_2)}} \end{pmatrix}.$$

(3.18)

The idea of Kowalevski to pass to new variables $s_{1,2}$ is appeared to be very fruitful in her treatment of the top. For the Kowalevski gyrostat as well as for the top these variables have the following main property:

**Proposition 1** Functions $s_{1,2}$ (3.15) are Poisson commute $\{s_1, s_2\} = 0$. 

6
For $\lambda = 0$ the straightforward proof may be founded in \cite{14,8}. For $\lambda \neq 0$ this unexpected and crucial observation was obtained by direct calculation of the Poisson brackets. It allows to suggest that for the gyrostat variables $s_{1,2}$ provide an essential step to separation of variables and this stimulated us to write down evolutionary equations and the integrals of motion for it in terms of $s_{1,2}$ and their velocities $\dot{s}_{1,2}$.

**Remark 3** In her original paper Kowalevski uses also another preferred coordinate system $(t, w)$ in which curve $\Gamma$ (3.13) has a Weierstrass normal form associated with equation $P_3(w) = 4w^3 + g_2w + g_3$. i.e. with $g_1 = 0$ (see \cite{19}, p. 188). These variables $w_i = s_i - 2H$ do not Poisson commute. The counterpart of (3.4) in $w_{1,2}$ variables gives as its off diagonal entries a nonphysical identity for $w_1, w_2$ and $\dot{w}_1, \dot{w}_2$ instead of (3.24).

Function $\mathcal{E}(z_1, z_2, s)$ (3.10), (3.17) is a quadratic polynomial with respect to any of its three arguments $z_1, z_2, s$. Its partial derivatives with respect to one of variables are discriminants of the corresponding quadratic equations. Squares of its partial derivatives with respect to one of variables are factorized into functions of the rest two ones

$$
\left( \frac{\partial \mathcal{E}}{\partial s} \right)^2 = R(z_1, z_1)R(z_2, z_2), \quad \left( \frac{\partial \mathcal{E}}{\partial z_k} \right)^2 = R(z_k, z_k)P_3(s), \quad k = 1, 2.
$$

(3.19)

Here polynomial $P_3(s)$ is given by

$$
P_3(s) = 4s^3 - 8H s^2 + 4H^2 s - K s + 4c^2 A s + 4c^2 B
$$

(3.20)

Because of complete differential of $\mathcal{E}(s, z_1, z_2)$ (3.10) is zero

$$
\frac{\partial \mathcal{E}}{\partial s} ds + \frac{\partial \mathcal{E}}{\partial z_1} dz_1 + \frac{\partial \mathcal{E}}{\partial z_2} dz_2 = 0,
$$

one gets relations between the differentials of the variables of both types

$$
\frac{ds_{1,2}}{\sqrt{P_3(s_{1,2})}} = \frac{dz_1}{\sqrt{R(z_1, z_1)}} \pm \frac{dz_2}{\sqrt{R(z_2, z_2)}},
$$

(3.21)

which are the Euler equations (3.14).

In matrix form the relations for velocities look like

$$
\begin{pmatrix}
\dot{s}_1 \\
\dot{s}_2
\end{pmatrix}
\equiv
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}
= \Psi
\begin{pmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{pmatrix},
$$

(3.22)

where we denoted for brevity

$$
\varphi_k \equiv P_3(s_k).
$$

(3.23)

Signs at square roots in (3.21), (3.22) are compatible with definition of $s_1, s_2$ (3.15) and $\Psi$ (3.18).
Using (3.24) – (3.25) we can express integrals of motion $H$ and $K$ (3.6) in terms of cubic polynomial $P_3(s)$, variables $s_{1,2}$ and their velocities $\dot{s}_{1,2}$

$$H = \frac{s_1 - s_2}{2} \left( \frac{\dot{s}_1^2}{\varphi_1} - \frac{\dot{s}_2^2}{\varphi_2} \right) - \frac{s_1 + s_2}{2}, \quad (3.24)$$

$$K = \frac{(2H + s_1 + s_2)\lambda^2 - \lambda\sqrt{-\varphi_1\varphi_2} \left( \frac{\dot{s}_1}{\varphi_1} + \frac{\dot{s}_2}{\varphi_2} \right)}{4} + (s_1 - s_2) \left( \frac{s_2\dot{s}_1^2}{\varphi_1} - \frac{s_1\dot{s}_2^2}{\varphi_2} \right) - s_1s_2 + H^2. \quad (3.25)$$

Here the Hamiltonian $H$ and coefficients of integral $K$ at even powers of gyrostatic parameter $\lambda$ are easy calculated using definitions (3.15) and (3.22) only. For linear in $\lambda$ term in $K = K_2\lambda^2 + K_1\lambda + K_0$ one gets at first

$$K_1 = -4i \frac{\dot{s}_1}{\sqrt{\varphi_1}} \left( \sqrt{R(z_1, z_1)} \frac{\partial}{\partial z_1} - \sqrt{R(z_2, z_2)} \frac{\partial}{\partial z_2} \right) \frac{R(z_1, z_2)}{(z_1 - z_2)^2} \left( \dot{s}_1 \sqrt{\varphi_1} \frac{\partial}{\partial s_1} + \dot{s}_2 \sqrt{\varphi_2} \frac{\partial}{\partial s_2} \right) \frac{R(z_1, z_2)}{(z_1 - z_2)^2}.$$

Due to inverse of (3.21) one converts derivatives $\partial/\partial z_{1,2}$ to $\partial/\partial s_{1,2}$

$$K_1 = -4i \left( \dot{s}_1 \sqrt{\varphi_1} \frac{\partial}{\partial s_1} + \dot{s}_2 \sqrt{\varphi_2} \frac{\partial}{\partial s_2} \right) \frac{R(z_1, z_2)}{(z_1 - z_2)^2}.$$

Minding that from definition (3.15) one gets $s_1 + s_2 = R(z_1, z_2)/(z_1 - z_2)^2$ and including $i$ into square root we obtain finally

$$K_1 = -4 \left( \dot{s}_1 \sqrt{-\frac{\varphi_1}{\varphi_1}} + \dot{s}_2 \sqrt{-\frac{\varphi_2}{\varphi_2}} \right). \quad (3.26)$$

In the section 6 we recover these expressions of $H$ (3.24) and $K$ (3.25) using relation of the Kovalevski gyrostat with the Clebsch system.

Equations (3.24), (3.25) have the form

$$\Phi_{1,2}(s_1, s_2, \dot{s}_1, \dot{s}_2, A, B, H, K)|_{\Sigma} = 0$$

and depend on the commuting variables $s_{1,2}$, their velocities $\dot{s}_{1,2}$ and integrals of motion only. Excluding one of the velocities we obtain two equations of fourth order in $\dot{s}_k$

$$\left( (s_1 - s_2)^2 \dot{s}_k^2 + \lambda\sqrt{-\varphi_1\varphi_2} \dot{s}_k - \beta_k \varphi_k \right)^2 + \lambda^2 \left( \dot{s}_k^2 + \frac{(2H + s_1 + s_2)\varphi_k}{s_1 - s_2} \right) \varphi_k^2 = 0, \quad (3.27)$$

where $k = 1, 2$ and $\beta_k$ is given by

$$\beta_k = (2H + s_1 + s_2)\lambda^2 + s^2 + 2hs + H^2 - \frac{K}{4}. \quad (3.28)$$
At $\lambda = 0$ the equations (3.27) are reduced to the Kowalevski top equations [19, 18]

\[ (-1)^k (s_1 - s_2) \dot{s}_k = \sqrt{P_5(s_k)} , \quad k = 1, 2, \]

(3.29)

which admit integration on $\Sigma$ (3.8) by Jacobi inversion theorem. Here $P_5(s) = P_3(s)P_2(s)$ is a fifth order polynomial, $P_3(s)$ is from (3.20) and

\[ P_2(s) = s^2 + 2Hs + H^2 - \frac{K}{4} \]

(3.30)

is a limiting value of $\beta_k$ (3.28), $P_2(s_k) = \beta_k|_{\lambda=0}$.

To construct separation of variables for the gyrostat one needs to substitute Lagrangian variables $\dot{s}_1, \dot{s}_2$ by momenta $\pi_1, \pi_2$ conjugated to $s_1, s_2$ and to express integrals of motion $H, K$ as functions of Hamiltonian variables. For the Kowalevski top, i.e. at $\lambda = 0$, momenta $\pi_1, \pi_2$ were extracted from evolution equations (3.29) in [8] and [14]. Integrals of motion as functions of $\dot{s}_1, \dot{s}_2, \pi_1, \pi_2$ give rise to separation of variables with separated equations of the form

\[ s_i^2 - 4H_{\text{top}} s_i - \frac{4c^2l^2}{s_i} + \kappa_{\text{top}} = 2c^2(a^2 - \frac{2b^2}{s_i}) \cos (2\sqrt{2s_i} \pi_i) , \quad i = 1, 2, \]

(3.31)

where $\kappa_{\text{top}} = 4H_{\text{top}}^2 - K_{\text{top}} + 2c^2a^2$.

At $\lambda \neq 0$ transition from velocities $\dot{s}_1, \dot{s}_2$ to the corresponding momenta is unknown due to complicated form of equations of motion (3.27) in $s_1, s_2$ variables, thus we cannot claim that $s_1, s_2$ are separation variables. Nevertheless, below we prove that equations (3.27) may be solved in quadratures using relation of the Kowalevski gyrostat with the Clebsch system.

### 4 The Kowalevski gyrostat and the Clebsch system

Let two vectors $l$ and $p$ are coordinates on the phase space $\mathcal{M}$. As a Poisson manifold $\mathcal{M}$ is identified with the algebra $e(3)^*$ equipped with brackets

\[ \{l_i, l_j\} = \varepsilon_{ijk}l_k, \quad \{l_i, p_j\} = \varepsilon_{ijk}p_k, \quad \{p_i, p_j\} = 0. \]

(4.1)

These brackets respect two Casimir elements

\[ \mathcal{A} = (p, p), \quad \mathcal{B} = (p, l). \]

(4.2)

The following integrable case for the Kirchhoff equations on $e(3)$ was found by Clebsch [5]

\[ \mathcal{X} \left( \begin{array}{c} l \\ p \end{array} \right): \quad \dot{l} = p \times Qp, \quad \dot{p} = p \times l. \]

(4.3)

Here $Q$ is a constant symmetric matrix, $\det Q \neq 0$. Equations of motion (4.3) are generated by the brackets (4.1) and the Hamilton function

\[ \mathcal{H} = \frac{1}{2} l^2 + \frac{1}{2} (Qp, p). \]

(4.4)
The second integral of motion reads as
\[ K = (Q l, l) - (Q^\vee p, p), \] (4.5)
where \( Q^\vee \) stands for adjoint matrix, i.e. cofactor matrix. In our case it reads \( Q^\vee = (\det Q) Q^{-1} \).

The vector field \( \mathcal{X} \) (4.3) is bi-hamiltonian vector field
\[ \mathcal{X} \left( \begin{array}{l} l \\ p \end{array} \right) = \mathcal{P}_0 d\mathcal{H} = \mathcal{P}_1 dK, \] (4.6)
where
\[ \mathcal{P}_0 = \begin{pmatrix} L & P \\ -P & 0 \end{pmatrix}, \quad \mathcal{P}_1 = \frac{1}{2} \begin{pmatrix} Q^{-1} & 0 \\ 0 & I \end{pmatrix} \mathcal{P}_0 \begin{pmatrix} Q^{-1} & 0 \\ 0 & I \end{pmatrix}, \]
and
\[ L = \begin{pmatrix} 0 & l_1 & -l_2 \\ -l_3 & 0 & l_1 \\ l_2 & -l_1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{pmatrix}. \] (4.7)

Here \( I \) stands for \( 3 \times 3 \) unit matrix. Poisson matrices \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \) define two compatible linear brackets on \( \mathcal{M} \) in a standard bi-hamiltonian formulation.

Remark 4 If \( \mathcal{B} = 0 \) the flow (4.3) is equivalent to that of the Neumann system with the Newton equation of motion
\[ \ddot{p} = -Qp + (Qp, p) \dot{p}^2 \] (4.8)
describing the motion of a mass point on the sphere \( p^2 = A \) under influence of the force \( -Qp \).

4.1 Mapping of the Kowalevski gyrostat flow onto the Clebsch flow

The idea of the map Kowalevski top flow (2.9) onto the Neumann flow (4.8) originally appeared in Heine and Horosov [10, 23] to the Kowalevski top and was extended to so(4), so(3, 1) in [15].

Let us introduce the following complex vector-functions
\[ p = \alpha \left( -i \frac{J_1}{J_2}, \frac{J_1^2 + J_2^2 + 1}{2J_2}, i \frac{J_1^2 + J_2^2 - 1}{2J_2} \right), \quad \alpha \in \mathbb{C}, \] (4.9)
and
\[ l_{\text{top}} = \left( -i \frac{cx_3}{J_2}, \frac{2cx_3J_1 - J_3(J_1^2 + J_2^2 - 1)}{2J_2}, i \frac{2cx_3J_1 - J_3(J_1^2 + J_2^2 + 1)}{2J_2} \right), \] (4.10)
such that
\[ A = (p, p) = \alpha^2, \quad B = (p, l_{\text{top}}) = 0. \] (4.11)
We permuted the first and the second entries in original vectors \([10]\) to make gyrostat formulas slightly more symmetric.

In order to describe mapping of the gyrostat flow (2.9) onto the Clebsch flow (4.3) we have to shift vector \(l_{\text{top}}\) by the rule

\[
 l = l_{\text{top}} + \alpha^{-1} \lambda \left( p + i k \times p \right),
\]

(4.12)

where \(k = (1, 0, 0)\) is a unit vector. In compare with (4.11) scalar product of vectors \(l\) and \(p\) for gyrostat becomes differ from zero

\[
 \mathcal{B} = (p, l) = \alpha \lambda.
\]

(4.13)

Adding constraints \(A = a, B = b\) to relations (4.9), (4.12) one gets correspondence \(M \simeq \mathcal{M}\) of the phase manifolds for the Kowalevski gyrostat and the Clebsch system. Initial Poisson structure on \(M\) (2.1) gives rise to the cubic Poisson brackets \(\{\cdot, \cdot\}_3\) on \(M\), for instance

\[
 \{ p_i, p_k \}_3 \equiv \{ p_i(x, J), p_k(x, J) \} = \varepsilon_{ijk} p_k (l_2 + il_3)(p_2 + ip_3).
\]

We shall not use these induced brackets directly and, therefore, remaining brackets are omitted.

On the other hand linear Poisson structure on \(\mathcal{M}\) (4.1) gives rise to the cubic Poisson brackets \(\{\cdot, \cdot\}_3\) on \(\mathcal{M}\). So, \(M\) and \(\mathcal{M}\) are multi-Poisson manifolds for which we constructed the correspondence \(M \simeq \mathcal{M}\) such that their linear brackets map to cubic brackets and vise versa.

**Remark 5** For the Kowalevski top and gyrostat variables \(p, l_{\text{top}}\) and \(p, l\) are coordinates on the different spaces \(\mathcal{M}_{\text{top}}\) and \(\mathcal{M}\) with different brackets (4.1) \(\{\cdot, \cdot\}_{\text{top}}\) and \(\{\cdot, \cdot\}\) forming two sample of \(e(3)\) algebra (4.11). With respect to the top brackets \(\{\cdot, \cdot\}_{\text{top}}\) the gyrostat variables \(p, l\) form the central extension of \(e(3)_{\text{top}}\) which is contracted to \(e(3)\) in the limit \(\lambda \to 0\).

Similar to the top \([10]\) let us introduce symmetric matrix \(Q\) depending on integrals of motion of the Kowalevski gyrostat and the Casimir elements on the initial algebra (2.1)

\[
 Q = \alpha^{-2} \begin{pmatrix}
 -H & -icb & icb \\
 -icb & -\frac{1}{4} + c^2 \kappa & i \left( \frac{1}{4} + c^2 \kappa \right) \\
 icb & i \left( \frac{1}{4} + c^2 \kappa \right) & \frac{1}{4} - c^2 \kappa
\end{pmatrix}, \quad \kappa = a - K/4c^2.
\]

(4.14)

It is easy to prove that this matrix remains constant with respect to dynamics of the Clebsch system on \(\mathcal{M}\) generated linear and cubic Poisson structures.

**Proposition 2** Let us identify \(\mathcal{M}\) with \(M\) by the map \(\{x, J\} \to \{p, l\}\) (4.9), (4.12) such that the Casimir elements are equal to

\[
 A = a, \quad B = b, \quad A = \alpha^2, \quad B = \alpha \lambda,
\]

(4.15)
If the matrix $Q$ is given by (4.14), then
\[ 2\mathcal{H} = -H + \lambda^2, \quad 4\alpha^2 \mathcal{K} = K - 4\lambda^2 H. \] (4.16)

and vector field $X$ (2.9) for the Kowalevski gyrostat on $M$ coincides with vector field $X$ (4.3) for the Clebsch system on $M$

\[ X = P_0 dH = P_0 \mathcal{H} = X. \] (4.17)

The similar equality holds for the second commuting flows of the Kowalevski gyrostat and the Clebsch system.

The proof is straightforward.

According to (4.17) on the space $M$ initial linear Poisson structure $P_0$ and cubic Poisson structure induced by $P_0$ generate the same vector field $X = X$ with respect to a common integral $\mathcal{H} \simeq H$ (4.17). We can embed (4.17) in a standard bi-hamiltonian formulation with two functionally different integrals of motion $\mathcal{H}$ and $K$ (see (4.6)), if we extend Poisson space $M \simeq M$ by additional degree of freedom considering $\lambda$ as an independent dynamical variable. Similar extension was used by Sklyanin [26] when he constructed Lax matrix for the quantum Goryachev-Chaplygin gyrostat.

Thus we arrive at one of the main results of the paper:

**Proposition 3** Solutions of the Clebsch problem give rise to solutions of the Kowalevski gyrostat and vice versa.

We can get solution of the Kowalevski gyrostat problem using either the Kobb-Kharlamova quadratures [20, 12, 11], or the Kötter solution [17] of the Clebsch system in theta functions.

Recall once more that for $\lambda = 0$ solutions of the Neumann system was identified with the Kowalevski solutions of her problem in [10, 15].

## 5 Lax representations

Equations of motion for the Clebsch system may be expressed in a Lax form
\[ \frac{d}{dt} \mathcal{L}(y) = [\mathcal{L}(y), \mathcal{A}(y)], \] (5.1)

which automatically exhibits constants of motion as eigenvalues of $\mathcal{L}$ and leads to the linearization of the flow on the Jacobi or Prym varieties of the algebraic curve $\det (\mathcal{L}(y) - \mu I) = 0$. Here $y$ is an auxiliary variable (spectral parameter).

There are few different Lax matrices for the Clebsch system associated with two different integration procedures. These matrices depend on rational and elliptic matrix-functions of the spectral parameter.
The rational $3 \times 3$ Lax matrix for the Clebsch system was found by Perelomov \cite{22}

\[ \mathcal{L}_r(y) = Q + Ly - Ny^2, \quad \mathcal{A}_r(y) = y^{-1}Q + L, \quad (5.2) \]

where $Q$ is symmetric matrix \cite{11,22}, $L$ is given by \cite{11} and $N = p \otimes p$, $N_{ij} = p_ip_j$. The corresponding spectral curve is equal to

\[ \tau_1(y,\mu) = B^2y^4 + \left( A\mu^2 + (2H - A\text{tr} \, Q)\mu - K \right)y^2 - \det (Q - \mu I) = 0. \quad (5.3) \]

We do not know separated variables for the Clebsch system associated with this curve. If $B = 0$ the separated variables are well-known \cite{21} and may be obtained by various integration schemes, for instance, by intersection of two algebraic curves related with Lax matrix $\mathcal{L}_r(y)$ \cite{15}. The regular way is provided by the Sklyanin method \cite{27} that is appeared to be nontrivial in the considered case:

**Proposition 4** In the Neumann case at $B = 0$ the separated variables $u_{1,2}$ are poles of the corresponding Baker-Axiezer function $\Psi$, such that

\[ \mathcal{L}_r(y)\Psi = \mu \Psi, \quad \text{and} \quad (\alpha, \Psi) = 1, \]

with dynamical normalization $\alpha = p \equiv (p_1, p_2, p_3)$. Canonical variables $u_k$ and their momenta $p_{u_k}$ lie on the algebraic curve (5.3) that gives rise to separated equations.

The proof consists of direct comparison of the known separated variables \cite{21} with poles of the Baker-Axiezer functions. Integration procedure of the generic Clebsch system in these variables will be considered in the next section.

Another Lax matrices for the Clebsch system is related with Kötter’s approach \cite{17}. Let $Q = \text{diag}(a_1, a_2, a_3)$ is a diagonal matrix. Introduce two vectors $t(\mu)$ and $s(\mu)$

\[ t = W(\mu) l + W^\vee(\mu) p, \quad s = W(\mu) p, \quad W = (\mu I - Q)^{1/2}, \quad (5.4) \]

where $W(\mu) = \text{diag}(w_1, w_2, w_3)$ is diagonal and $W^\vee$ is its adjoint matrix. In a special uniformisation of the spectral parameter $\mu$ diagonal entries $w_k = \sqrt{\mu - a_k}$ can be considered as basic elliptic functions (see \cite{17,22,28}).

The equations of motion

\[ \dot{t}(\mu) = s(\mu) \times t(\mu), \quad (5.5) \]

may be rewritten in the Lax form (see \cite{2}) using matrix

\[ \mathcal{L}_e(\mu) = \sum_{k=1}^3 t_k(\mu)\sigma_k \equiv \sum_{k=1}^3 \left( w_k l_k + \frac{w_1w_2w_3}{w_k} p_k \right) \sigma_k, \quad \mathcal{A}_e(\mu) = \sum_{k=1}^3 s_k(\mu)\sigma_k, \quad (5.6) \]

where $\sigma_k$ are the Pauli matrices.
The corresponding spectral curve reads
\[
\tau_2(w, \mu) = w^2 - t^2(\mu) = w^2 - \left( A\mu^2 + (2\mathcal{H} - A \text{ tr } Q)\mu - K \right) - 2B\sqrt{-\det(Q - \mu I)} = 0. \tag{5.7}
\]
The linearization of the flow associated with the curve (5.7) and expressions of initial variables \(l\) and \(p\) in theta-functions were done by Kötter [17] using new Lagrangian variables \(z_1, z_2, \dot{z}_1, \dot{z}_2\) which satisfy to nice evolutionary equations and may be considered as candidates for separation variables. These variables will be studied in the next section.

According to [3] let us consider one parametric transformation \(f_\mu : so(4) \to e(3)\)
\[
f_\mu : \quad p_i = w_i(S_i - T_i), \quad l_i = \frac{w_1 w_2 w_3}{w_i}(S_i + T_i). \tag{5.8}
\]
where \(S_i, T_i\) are coordinates on \(so(4) = so(3) \oplus so(3)\) with the Lie-Poisson brackets
\[
\{S_i, S_j\} = \varepsilon_{ijk} S_k, \quad \{S_i, T_j\} = 0, \quad \{T_i, T_j\} = \varepsilon_{ijk} T_k. \tag{5.9}
\]

**Remark 6** The inverse transformation \(f_\mu^{-1}\) reads as
\[
S_i = \frac{w_i}{2w_1 w_2 w_3} l_i + \frac{1}{2w_i} p_i, \quad T_i = \frac{w_i}{2w_1 w_2 w_3} l_i - \frac{1}{2w_i} p_i, \tag{5.10}
\]
where \(w_k\) depends on the parameter \(\mu\). The map \(f_\mu^{-1}\) is easy generalized to two parametric mapping \(f_{\mu,\nu}^{-1}\) if we substitute \(w_k(\mu) = \sqrt{\mu - a_k}\) and \(w_k(\nu) = \sqrt{\nu - a_k}\) in the definition \(S_k\) and \(T_k\) respectively.

In Section 7 we show that Kötter used namely this mapping to integrate the Clebsch system.

The mapping \(f_\mu\) is a twisted Poisson map, which identifies two bi-Hamiltonian manifolds \(e(3)\) and \(so(4)\) such that
\[
\begin{array}{ccc}
e(3) & so(4) & e(3) & so(4) \\
\{\ldots\} & \{\ldots\} & \{\ldots\} \leftrightarrow \{\ldots\} & \text{instead of} \\

\{\ldots\}^* & \{\ldots\}^* & \{\ldots\}^* \leftrightarrow \{\ldots\}^*
\end{array}
\]
for the usual Poisson map.

Here second compatible brackets \(\{\ldots\}^*\) on \(e(3)\) are equal to
\[
\{l_i, l_j\}^* = \varepsilon_{ijk} w_k^2 l_k, \quad \{l_i, p_j\}^* = \varepsilon_{ijk} w_j p_k, \quad \{p_i, p_j\}^* = \varepsilon_{ijk} l_k. \tag{5.11}
\]

The polynomial \(t^2(\nu)\) is a Casimir function of the corresponding Poisson pencil \(\{\ldots\} = \{\ldots\} - \nu \{\ldots\}^*\), where \(\nu \neq \mu\). For brevity the second linear brackets \(\{\ldots\}^*\) on \(so(4)\) will be omitted because they are completely determined by mapping \(f_\mu\) (5.8). The similar twisted Poisson map related to Steklov integrable cases on \(e(3)\) and \(so(4)\) is discussed in [29].
The map $f_\mu$ identifies the Clebsch system on $e(3)$ with the Schottky–Manakov system on $so(4)$ and, according to [3], identify the corresponding $2 \times 2$ elliptic Lax matrices. At the same time the Schottky–Manakov system on $so(4)$ has another $4 \times 4$ Lax matrix with linear dependence on spectral parameter. It allows us to construct two-parametric family of Lax matrices for the Clebsch system

$$
\tilde{L}(\nu, \mu) = \begin{pmatrix}
\nu W^\vee + W^\vee L W^{-1} & W^\vee p \\
-(W^\vee p)^T & 0
\end{pmatrix},
$$

$$
\tilde{A}(\nu, \mu) = \begin{pmatrix}
0 & W p \\
-(W p)^T & \nu \det W
\end{pmatrix},
$$

(5.12)

where $W = (\mu I - Q)^{1/2}$ and $L$ is given by (4.7). This family of Lax matrices leads to a spectral surface, rather than a spectral curve. The nature of this surface is discussed in [1].

At $\mu = 0$ the Lax matrix (5.12) becomes rational with the following spectral curve

$$
\tau_3(y, \nu) = y^4 - \text{tr} Q^\vee \nu y^3 + (\det Q \text{tr} Q \nu^2 - K) y^2
$$

$$
- \det Q \left( \nu^2 \det Q - 2H + \text{tr} Q A \right) \nu y - \det Q \left( A \det Q \nu^2 + B^2 \right) = 0.
$$

(5.13)

The separated variables for the Clebsch system associated with this curve are unknown. The corresponding solutions in theta-functions were obtained by algebro-geometric methods in [32].

At $\nu = 0$ the Lax matrix (5.12) becomes elliptic matrix, which spectral curve coincides with spectral curve (5.3) of the rational Lax matrix up to transformation $\tilde{y} = B y$.

5.1 Algebraic curves associated with the Kowalevski gyrostat

From relation of the Kowalevski gyrostat with the Clebsch system established in the section [4] one gets three Lax matrices $L_1(y), L_e(\mu)$ and $L(\mu)$ for the Kowalevski gyrostat.

The spectral curve of the rational Lax matrix $L_1(y)$ looks like

$$
\tau_1(y, \mu) = \lambda^2 y^4 + \left( \mu^2 + (\mu + H)\lambda^2 - \frac{K}{4} \right) y^2 - \frac{P_3(\mu)}{4},
$$

(5.14)

where $P_3(\mu)$ is given by (3.30). It is a biquadratic function of $y$ with coefficients being quadratic and cubic polynomials of $\mu$. At $\lambda = 0$ it is a famous Kowalevski curve and the corresponding variables $u_{1,2}$ give rise to the Kowalevski separated variables $s_{1,2}$ (see discussion in the next section).

The $2 \times 2$ elliptic matrix $L_e(\mu)$ for the Kowalevski gyrostat has another spectral curve

$$
\tau_2(w, \mu) = w^2 - \left( \mu^2 + (\mu + H)\lambda^2 - \frac{K}{4} \right) - \lambda \sqrt{P_3(\mu)}.
$$
Associated with this curve expressions of initial variables \( x \) and \( J \) in theta-functions may be obtained using Kötter formulae and results of the section (4).

At \( \lambda = 0 \) the two-dimensional family of \( 4 \times 4 \) Lax matrix \( \tilde{L}(\nu, \mu) \) was obtained by Adler and van Moerbeke [1] using directly correspondence of the Kowalevski top flow and the Schottky-Manakov flow. For the Kowalevski gyrostat third algebraic curve (5.13) is given by

\[
\tau_3(y, \nu) = y^4 - \left( A c^2 - \frac{K}{4} \right) \nu y^3 + \left( \nu^2 \left( A c^2 - \frac{K}{4} \right) H^2 - (c B \nu - \lambda)(c B \nu + \lambda) H - \frac{K}{4} \right) y^2
\]

\[
- \left( \left( A c^2 - \frac{K}{4} \right) H - c^2 B^2 \right) \left( \nu^2 \left( A c^2 - \frac{K}{4} \right) H - (c B \nu - \lambda)(c B \nu + \lambda) \right) (y \nu - 1)
\]

Associated with this curve solutions of the gyrostat may be obtained using theta-functions expressions for the Clebsch variables from [32] and change of variables from the section 3.

The fourth algebraic curve associated with gyrostat is due to rational Lax matrices of [4]. It is given by

\[
\tau_4(y, \mu) = y^4 - y^2 \left( 2(\mu^2 - H) - \lambda^2 + \frac{c^2 A}{\mu^2} \right) + (\mu^2 - H)^2 - \frac{K}{4} + c^2 A - \frac{c^2 B^2}{\mu^2}.
\]

6 Integration of the Clebsch system in elliptic coordinates

Minkowski [20] identified the Clebsch system with the Jacobi problem of geodesic motion on ellipsoid for which elliptic coordinates \( u_{1,2} \) were introduced by Jacobi. In 1895 Kobb started the integration procedure in the Euler angles and passed to variables \( \xi = \tan(\theta/2), \nu = \tan(\phi/2) \), which are equivalent to variables \( u_{1,2} \) [12]. In 1959 Kharlamova [11] used directly elliptic coordinates \( u_{1,2} \) for integration of the second flow of the Clebsch system associated with \( K \).

In order to explain the method proposed in [11, 12] we reproduce some simple formulae. Using equations of motion (4.3) and the Casimir elements (4.2) we express angular momenta \( l \) via Lagrangian variables \( p, \dot{p} \)

\[
l = \frac{1}{A} (B p + \dot{p} \times p).
\]

Then we introduce variables \( u_{1,2} \) as roots of the following function

\[
e(\mu) = (\mu - u_1)(\mu - u_2) = \mu^2 + \left( \frac{Q p, p}{A} - \text{tr} Q \right) \mu + \left( \frac{Q^\dagger p, p}{A} \right).
\]

Substituting \( u_{1,2} \) and their velocities \( \dot{u}_{1,2} \) into (4.4) and (4.5) one gets the Hamilton function

\[
\mathcal{H} = \mathcal{T} + \mathcal{V}
\]

\[
\mathcal{T} = \frac{u_1 - u_2}{2} \left( \dot{\varphi}_1^2 - \dot{\varphi}_2^2 \right) + \frac{B^2}{2A} \quad \mathcal{V} = \frac{1}{2} (\text{tr} Q - u_1 - u_2) A,
\]

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and the second integral of motion
\[
\mathcal{K} = (u_1 - u_2) \left( \frac{u_2 \dot{u}_1^2}{\varphi_1} - \frac{u_1 u_2^2}{\varphi_2} \right) - \frac{B}{\sqrt{A}} \left( u_1 \sqrt{-\frac{\varphi_2}{\varphi_1}} + u_2 \sqrt{-\frac{\varphi_1}{\varphi_2}} \right) + \frac{B^2}{A} \left( \text{tr} Q - u_1 - u_2 \right) - u_1 u_2 A.
\]
in terms of variables \( u_{1,2} \), their velocities \( \dot{u}_{1,2} \) and the cubic polynomial
\[
\varphi_k = 4 \det (Q - u_k I).
\]
Below this polynomial will be identified with the cubic polynomial \( \varphi_k \) for the Kowalevski gyrostat for which we used the same notation.

Excluding one of the velocities from these equations we obtain two equations of fourth degree in each of velocities depending on both variables \( u_1 \) and \( u_2 \)
\[
\left( A(u_1 - u_2)^2 \dot{u}_k^2 + B \sqrt{-A \varphi_1 \varphi_2} \dot{u}_k + \beta_k \varphi_k \right)^2 + B^2 \left( A \dot{u}_k^2 + \left( \frac{A^2(u_1 + u_2 - \text{tr} Q) + 2AH - B^2}{u_1 - u_2} \right) \varphi_k \right) \varphi_k^2 = 0.
\]
Here \( \beta_k \) is a cubic polynomial also depending on \( u_1 \) and \( u_2 \)
\[
\beta_k = B^2(u_1 + u_2 + u_k - \text{tr} Q) + A \left( A \dot{u}_k^2 + (2H - A \text{tr} Q) u_k - \mathcal{K} \right).
\]

Momenta conjugated to \( u_{1,2} \) are introduced by \( p_{u_{1,2}} = \frac{\partial T}{\partial \dot{u}_{1,2}} \), where \( T \) is kinetic energy (6.2) and the Casimir operator \( \mathcal{B} \) depends on velocities \( \dot{u}_k \).

Equations (6.3) were solved in quadratures in [11, 12]. We have to underline only that variables \( \{u_{1,2}, p_{u_{1,2}}\} \) are not the separated variables in the standard meaning.

**Remark 7** After a suitable rotation
\[
\tilde{p} = Vp, \quad \tilde{l} = Vl, \quad Q \to \tilde{Q} = VQV^{-1} = \text{diag}(a_1, a_2, a_3)
\]
which diagonalize the matrix \( Q \), coordinates \( u_{1,2} \) coincide with elliptic coordinates on sphere \( p^2 = A \) defined by
\[
e(\mu) = (\mu - u_1)(\mu - u_2) = \det(Q - \mu I) \left( \frac{\tilde{p}_1^2}{\mu - a_1} + \frac{\tilde{p}_2^2}{\mu - a_2} + \frac{\tilde{p}_3^2}{\mu - a_3} \right).
\]

**Remark 8** If \( \mathcal{B} = 0 \) there are considerably simple equations
\[
\left( A(u_1 - u_2)^2 \dot{u}_k^2 + 4\varphi_k \beta_k \right)^2 = 0 \quad k = 1, 2.
\]
In this case both integrals are quadratic polynomials in momenta \( p_{u_{1,2}} = \pm \frac{(u_1 - u_2)\dot{u}_k}{2 \varphi_k} \) and the corresponding Neumann system belongs to the Stäckel family of integrable systems. Moreover, variables \( \{u_k, p_k\} \) are separated variables which lie on the spectral curve (5.3) of the rational Lax matrix.
6.1 The Kowalevski gyrostat in $s$ variables

Without loss of generality we put $\alpha = |p| = 1$. Inserting $p$ (4.9) and $l$ (4.12) into the generating function $e(\mu)$ (6.1) of $u$-variables one gets
\[ e(\mu) = \frac{1}{(z_1 - z_2)^2} E(s)\big|_{s=-\mu-H}, \]
where $E(s)$ (3.17) is the generating functions of the $s$-variables. Combining this fact with the Proposition 2 we have
\[ u_k = -s_k - H, \quad \dot{u}_k = -\dot{s}_k. \]
Here $\dot{s}_k = \{H, s_k\}_1$ and $\dot{u}_k = \{H, u_k\}_2$ and $\{\}_1,2$ means the Poisson brackets (2.1) on $M$ and the Poisson brackets (4.1) on $M$, respectively.

Inserting constant of motion (4.16), (4.15) and variables (6.7) into the equations (6.2) we arrive at the same expressions of integrals $H$ and $K$ (3.24)-(3.25) in terms of $s$-variables and polynomials $\varphi_{1,2}$. However, in contrast with the Clebsch system expression for Hamiltonian $H$ (3.24) contains both integrals $H$ and $K$ via polynomial $\varphi(s)$. Expressions of $H$ and $K$ in $s_{1,2}$ and $\dot{s}_{1,2}$ variables may be obtained by solving equations (3.27). Unfortunately, we cannot use very complicated resulting formulae to calculate associated with $s$ variables momenta as in the Clebsch case.

7 The Kötter solution of the Clebsch system

In 1888 Minkowski proved that the Clebsch flow is isomorphic to geodesic flow on the ellipsoid [20]. Then in 1891 Schottky found that the Clebsch flow is isomorphic to integrable motion of four-dimensional rigid body, which may be integrated in a special case [25] associated with the Neumann system. In 1892 Kötter [17] joined these results together and integrated the Clebsch system completely.

Very brief description of Kötter approach was presented in [7]. Below we reproduce the essence of Kötter derivation in modern notations. Instead of two vectors $t(\mu)$ and $s(\mu)$ (5.4) defining pair of the Lax matrices $L_\mu(\mu)$ and $A_\mu(\mu)$ (5.6) we take two vectors $t(\mu)$ and $t(\nu)$ defining two samples of the first Lax matrix $L_\mu$ and pass to their linear combinations
\[ \xi = a(\mu, \nu) t(\mu) + b(\mu, \nu) t(\nu) = W_+(\mu, \nu)l + \tilde{W}_+(\mu, \nu)p, \]
\[ \eta = a(\mu, \nu) t(\mu) - b(\mu, \nu) t(\nu) = W_-(\mu, \nu)l + \tilde{W}_-(\mu, \nu)p, \]
(7.1)
depending on two auxiliary functions $a, b$ having zero time derivatives $\dot{a} = \dot{b} = 0$. Here we denoted for brevity
\[ W_\pm(\mu, \nu) = a(\mu, \nu)W(\mu) \pm b(\mu, \nu)W(\nu), \quad \tilde{W}_\pm(\mu, \nu) = a(\mu, \nu)W(\mu) \vee \pm b(\mu, \nu)W \vee(\nu). \]

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Using inverse relations

\[ l = -Z^{-1}(\tilde{W}_- \xi - \tilde{W}_+ \eta), \quad p = Z^{-1}(W_- \xi - W_+ \eta), \]  

(7.2)

where

\[ Z = 2a(\mu, \nu)b(\mu, \nu)\left(W(\mu)W^\nu(\nu) - W(\nu)W^\nu(\mu)\right), \]

and the Kirchhoff equations on \( e(3) \) (4.3) one gets the following evolutionary equations

\[ \dot{\eta} = (C_{11} \eta + C_{12} \xi) \times \eta + (C_{21} \eta + C_{22} \xi) \times \xi, \]

\[ \dot{\xi} = (C_{11} \eta + C_{12} \xi) \times \xi + (C_{21} \eta + C_{22} \xi) \times \eta, \]

(7.3)

where

\[ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} Z^{-1} [W(\mu) + W(\nu)] & 0 \\ 0 & Z^{-1} [W(\mu) - W(\nu)] \end{pmatrix} \begin{pmatrix} -W_+ & W_- \\ -W_- & W_+ \end{pmatrix}, \]

By definition we have \( C_{11}C_{22} - C_{12}C_{21} = 0 \).

Let us postulate the following brackets between \( \xi \) and \( \eta \)

\[ \{ \eta_i, \eta_j \} = \varepsilon_{ijk}(\eta_k + \gamma \xi_k), \quad \{ \eta_i, \xi_j \} = \varepsilon_{ijk}(\xi_k + \gamma \eta_k), \quad \{ \xi_i, \xi_j \} = \varepsilon_{ijk}(\xi_k + \gamma \xi_k), \]

(7.4)

which coincide with the Lie-Poisson brackets on \( so(4) \) (5.9) after the following change of variables

\[ S_i = \frac{\eta_i - \xi_i}{2(1 - \gamma)}, \quad T_i = \frac{\eta_i + \xi_i}{2(1 + \gamma)}, \]

here \( \gamma \) is arbitrary.

The equations of motion (7.3) are Hamiltonian equations with respect to the brackets (7.4) and the following quadratic Hamilton function

\[ H = (\tilde{C}_{11} \eta, \eta) + (\tilde{C}_{12} \xi, \eta) - (\tilde{C}_{22} \xi, \xi), \]

where

\[ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{12} & \tilde{C}_{22} \end{pmatrix}. \]

So, in fact Kötter constructed two parametric transformation \( f_{\mu \nu} : so(4) \to e(3) \) defined by relations (7.1) that identifies the Clebsch system on \( e(3) \) with the Schottky–Manakov system on \( so(4) \), it means that after mapping \( f_{\mu \nu} \) the corresponding Kirchhoff equations on \( e(3) \) (4.3) and on \( so(4) \) (7.3) coincide.

If \( a = b \) then \( \gamma = 0 \) and the Kötter mapping \( f_{\mu \nu} \) (7.1) is equivalent to the twisted Poisson map \( f_\mu \) (5.8) (see Remark (6)). In generic case map \( f_{\mu \nu} \) (7.1) is the twisted Poisson map if \( a(\mu, \nu) \) and \( b(\nu, \mu) \) are numerical functions. In order to prove it we have to check that
transformation $f_{\mu\nu}$ (7.1) gives rise to the second Poisson brackets $\{.,.\}^*$ on $e(3)$ and so(4), which are compatible with initial ones.

According to [25, 17] (see also Remarks (4) and (8)) equations (7.3) may be integrated in elliptic coordinates if one of the Casimir elements on so(4) is equal to zero

$$(\xi, \eta) = 0, \quad \xi^2 + \eta^2 = \text{const}. \tag{7.5}$$

However in our case we have

$$(\xi, \eta) = a^2(\mu, \nu)t^2(\mu) - b^2(\mu, \nu)t^2(\nu), \quad \xi^2 + \eta^2 = 2a^2(\mu, \nu)t^2(\mu) + 2b^2(\mu, \nu)t^2(\nu). \tag{7.5}$$

Using the two parametric Kötter mapping $f_{\mu\nu}$ (7.1) the necessary relation $(\xi, \eta) = 0$ may be achieved if one puts $\mu = s_i$ and $\nu = s_k$, where $s_{i,k}$ are any of the roots of the equation

$$t^2(s_j) \equiv \det L_e(s_j) = 0, \quad j = 1, 2, 3, 4. \tag{7.7}$$

Of course, these substitutions destroy the Poissonity of the mapping $f_{\mu\nu}$. As a sequence, substituting $\mu, \nu = s_1, s_2$ and $\mu, \nu = s_3, s_4$ conversely into (7.1) we can construct two different pair of orthogonal vectors

$$\xi = \xi|_{\mu = s_1, \nu = s_2}, \quad \eta = \eta|_{\mu = s_1, \nu = s_2}, \quad \text{and} \quad \tilde{\xi} = \tilde{\xi}|_{\mu = s_3, \nu = s_4}, \quad \tilde{\eta} = \tilde{\eta}|_{\mu = s_3, \nu = s_4}$$

that satisfy dynamical equations (7.3) and algebraic relations

$$\xi^2 + \eta^2 = 0, \quad (\xi, \eta) = 0, \quad \text{and} \quad \tilde{\xi}^2 + \tilde{\eta}^2 = 0, \quad (\tilde{\xi}, \tilde{\eta}) = 0. \tag{7.6}$$

Using relations (7.2) it is easy to prove that vectors $\xi, \eta$ and $\tilde{\xi}, \tilde{\eta}$ are linearly dependent

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = U \begin{pmatrix} \tilde{W}_+ - W_+ \\ \tilde{W}_- - W_- \end{pmatrix}_{\mu = s_3, \nu = s_4} \begin{pmatrix} W_- - W_+ \\ \tilde{W}_- - \tilde{W}_+ \end{pmatrix}_{\mu = s_1, \nu = s_2} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{7.7}$$

where

$$U = Z(s_3, s_4)Z^{-1}(s_1, s_2) \begin{pmatrix} W_- \tilde{W}_+ - W_+ \tilde{W}_- \end{pmatrix}_{\mu = s_1, \nu = s_2}^{-1}(s_3, s_4);$$

Inserting (7.7) into (7.6) we obtain four algebraic equations, which relate six dynamical variables $\xi_i$ and $\eta_i$. These equations have non-trivial solutions $\xi_i = g_i(\eta_1, \eta_2, \eta_3)$ for the special choice of the functions $a(\mu, \nu)$ and $b(\mu, \nu)$ (7.1) only.

For instance, Kötter uses the following functions

$$a(\mu, \nu) = \frac{1}{\sqrt{\psi'(\mu)}}, \quad b(\mu, \nu) = \frac{i}{\sqrt{\psi'(\nu)}},$$

where

$$\psi(\mu) = (\mu - s_1)(\mu - s_2)(\mu - s_3)(\mu - s_4), \quad \psi'(\mu) = \frac{d\psi(\mu)}{d\mu}.$$
In this case relation
\[ W_\pm(s_1, s_2) W_\pm^{-1}(s_3, s_4) = \tilde{W}_\pm(s_1, s_2) \tilde{W}_\pm^{-1}(s_3, s_4) \]
allows us to reduce equations (7.7) to more simple form \( \tilde{\xi} = D\xi \) and \( \tilde{\eta} = D^{-1}\eta \), such that four algebraic equations (7.6) give rise to three independent equations only
\[
\sum_{j=1}^3 (\xi_j^2 + \eta_j^2) = 0, \quad \sum_{j=1}^3 \xi_j \eta_j = 0, \quad \sum_{j=1}^3 \left( d_j^2 \xi_j^2 + \eta_j^2 \frac{d_j}{d_j^2} \right) = 0 . \quad (7.8)
\]
Here \( d_j \) are diagonal elements of the matrix \( D = W_+(s_3, s_4) W_+^{-1}(s_1, s_2) \)
\[
d_j = \frac{\tilde{w}_j(s_4) + i\tilde{w}_j(s_3)}{\tilde{w}_j(s_1) + i\tilde{w}_j(s_2)}, \quad \tilde{w}_j(\mu) = \frac{w_j(\mu)}{\sqrt{\mu}} .
\]
Matrix \( D \) obeys the property \( D^*D = -1 \), where \( D^* = W_-(s_3, s_4) W_-^{-1}(s_1, s_2) \) is its Hermitian conjugate.

Introduce elliptic coordinates \( z_{1,2} \) as the roots of equation
\[
A (z - z_1)(z - z_2) = \frac{\eta_1^2}{z - d_1^2} + \frac{\eta_2^2}{z - d_2^2} + \frac{\eta_3^2}{z - d_3^2} = 0 , \quad (7.9)
\]
where \( \chi(z) = (z - d_1^2)(z - d_2^2)(z - d_3^2) \) and \( A = \eta_1^2 + \eta_2^2 + \eta_3^2 \). From equations (7.8) and (7.9) one gets
\[
\eta_j = \sqrt{A (z_1 - d_j^2)(z_2 - d_j^2) \chi'(d_j^2)}, \quad \xi_j = \eta_j \frac{\sqrt{z_1 z_2}}{d_1 d_2 d_3 (z_1 - z_2)} \frac{\sqrt{P_5(z_1)} - \sqrt{P_5(z_2)}}{z_2 (z_2 - d_j^2)} ,
\]
where \( P_5(z) \) is a fifth order polynomial
\[
P_5(z) = (z - d_1^2)(z - d_2^2)(z - d_3^2) \chi(z) = z (z - d_1^2 d_2^2 d_3^2)(z - d_1^2)(z - d_2^2)(z - d_3^2) . \quad (7.10)
\]
By definition (7.9) velocities of the coordinates \( z_{1,2} \) are given by
\[
(-1)^j (z_1 - z_2) \dot{z}_j = \frac{2\chi(z_j)}{A} \sum_{k=1}^{3} \frac{\eta_k \dot{\eta}_k}{z_j - d_k^2} , \quad j = 1, 2 .
\]
Excluding \( \dot{\eta}_j \) from equations of motion (7.3) one gets
\[
(-1)^j (z_1 - z_2) \dot{z}_j = (dw_1 z_j + dw_2) \sqrt{P_5(z_j)} , \quad j = 1, 2 , \quad (7.11)
\]
where \( dw_{1,2} \) are values of the constants of motion, which we reproduce in the Kötter form
\[
dw_1 = 2 \sum_{k=1}^{3} \eta_k \dot{\eta}_k \left( \frac{\sqrt{P_5(z_1)}}{z_1 (z_1 - d_k^2)(z_1 - d_1^2 d_2^2 d_3^2)} - \frac{\sqrt{P_5(z_2)}}{z_2 (z_2 - d_k^2)(z_2 - d_1^2 d_2^2 d_3^2)} \right) , \quad (7.12)
\]
\[
dw_2 = 2 \sum_{k=1}^{3} \eta_k \dot{\eta}_k \left( \frac{\sqrt{P_5(z_1)}}{(z_1 - d_k^2)(z_1 - d_1^2 d_2^2 d_3^2)} - \frac{\sqrt{P_5(z_2)}}{(z_2 - d_k^2)(z_2 - d_1^2 d_2^2 d_3^2)} \right) .
\]
Applying the Abel-Jacobi inversion theorem to the equations (7.11), Kötter then expressed initial variables \( l \) and \( p \) in theta-functions.

On the next step we have to rewrite coefficients of \( P_5(z) \) (7.10) and integrals \( dw_{1,2} \) (7.12) as functions of the initial Clebsch integrals and have to prove that the Kötter variables \( z_{1,2} \) (7.9) are in involution with respect to initial Poisson brackets for the Clebsch system. It is not an easy task since \( \{ \eta_i, \eta_j \} \neq 0 \) and \( \{ d_k, \eta_j \} \neq 0 \). We suppose that or \( z_{1,2} \) commute or their commutativity may be restored by using another functions \( a(\mu, \nu) \) and \( b(\mu, \nu) \) in (7.1) because the Kötter solution in theta-functions was reproduced in framework of the finite-band integration technique [2]. Recall that in [19] Kowalevski used non-commutative variables too (see Remark 3).

On the other hand, the matrix \( L_e(\mu) \) (5.6) belongs to the family of the Lax matrices for elliptic or XYZ Gaudin magnet. The separated variables for the generic Gaudin model were constructed in [28] in classical and quantum mechanics. So, we can construct the separated variables for the Clebsch system and, therefore, for the Kowalevski gyrostat using the Sklyanin method, which is free from the difficulties of the Kötter approach.

8 Conclusion

Our treatment shows that including gyrostatic term into the Hamiltonian of the Kowalevski top we arrive at the essentially more complicate dynamic system. Recall once more that quantum corrections to the Kowalevski top looks similar to the gyrostatic term.

The main result of this paper is that we establish one to one correspondence between the Kowalevski gyrostat and the Clebsch system. It allows us to construct solution of the gyrostat problem using various known solutions of the Clebsch model. The separation of variables for the Clebsch system that is unknown now becomes a matter of a primary importance.

The similar correspondence may be obtained for the Kowalevski gyrostat on \( so(4) \) algebra, which is equivalent to the generalized Kowalevski gyrostat on \( e(3) \) [16]. For the \( so(4) \) top initial vector \( L_{so(4)} \) was obtained in [15]. In the gyrostat case we have to substitute it by the rule (4.12).

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