A New Family of Weighted One-Parameter Flux Reconstruction Schemes

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Abstract

A novel set of correction functions for flux reconstruction are defined using weighted Sobolev norms that allow an extension to spectral difference schemes. Within this set, corrections that give increased rates of convergence are found theoretically which do not suffer from a loss of explicit temporal stability. Numerical experiments with the Taylor-Green vortex are then used to test, and ultimately confirm, the theoretical findings: that the new correction function can improve numerical accuracy, especially on under-resolved grids.

Keywords: High Order, Flux Reconstruction, Discontinuous Galerkin, Energy Stable, Spectral Difference

1. Introduction

In recent decades discontinuous spectral element methods have emerged as an attractive alternative to classical finite element and finite volume methods for high-order accurate numerical simulations on unstructured grids. Such methods offer the promise of increased accuracy at reduced cost [1, 2].

A popular example of such a method is the Flux Reconstruction (FR) approach of Huynh [3]. Closely related to the lifting collocation penalty

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(LCP) schemes of Gao and Wang [4, 5] and the correction procedure via reconstruction (CPR) method of Haga et al. [6], the FR approach is interesting in the sense that rather than defining one scheme it instead defines an infinite family of schemes. This flexibility enables the FR approach to recover several existing spectral element schemes. These include a nodal form of the discontinuous Galerkin method of Reed and Hill [7] as described in Hesthaven and Warburton [8] and, for a linear flux function, the spectral difference (SD) schemes of Kopriva and Kolias [9] and Sun et al. [10]. The flexibility of the FR framework also enables the construction of new and novel schemes.

In 2011, Vincent et al. [11] discovered a one-parameter family of correction functions, herein referred to as original stable FR (OSFR) schemes, that lead to stable FR schemes for linear advection problems. This work was subsequently extended to linear advection-diffusion problems by Castonguay et al. [12]. More recently Vincent et al. [13] identified a multi-parameter family of linearly stable FR correction functions which are herein referred to as the extended stable FR (ESFR) schemes. In a series of numerical experiments, Vermeire and Vincent [14] observed that several of these schemes are more stable for ILES simulations than NDG.

Further advancements have been made in correction functions with the definition of generalised Sobolev stable FR (GSFR) [15] and generalised Lebesgue stable FR (GLSFR) [16]. These approaches both vastly increase the scope of FR. A pictorial illustration of the current space of FR schemes can be seen in Fig. 1. However, despite this progress several questions around what exactly a correction function is, and more aptly, how one should be chosen, remain. To this end, in this paper we seek to extend the theoretical understanding of FR correction functions. We shall accomplish this through the application of a weighted Sobolev type norm—something which has previously been employed successfully within the summation-by-parts community—in order to expand the definition of SD and DG within FR. The result of this work is a further generalisation of the OSFR schemes.

The remainder of this paper is structured as follows. In section 2 we provide an overview of the FR approach for a one-dimensional advection problem. The one-parameter family of energy stable FR schemes are reviewed in section 3. Our extension of these schemes is presented in section 4 with stability being discussed in section 5. Connections between our new family of schemes and the SD schemes are analysed in section 6. The explicit time-step limits associated with our new family of schemes are reviewed in
section 7. The performance these schemes within the context of the compressible Navier–Stokes equations are assessed in section 8. Finally, in section 9, conclusions are drawn.

2. Flux Reconstruction

Consider using the FR approach to solve the conservative equation:

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

within an arbitrary 1D domain $\Omega$, where $x$ is a spatial coordinate, $t$ is time, $u = u(x, t)$ is a conserved scalar quantity, and $f = f(u)$ is the flux of $u$ in the $x$ direction.

The first stage of the FR approach involves partitioning $\Omega$ into $N$ distinct elements, each denoted $\Omega_n = \{x_n < x < x_{n+1}\}$, such that:

$$\Omega = \bigcup_{n=0}^{N-1} \Omega_n \quad \text{and} \quad \bigcap_{n=0}^{N-1} \Omega_n = \emptyset.$$
The solution $u$ in Eq.(1) is approximated in each $\Omega_n$ by $u^\delta_n = u^\delta_n(x, t)$, which is a polynomial of degree $p$ within $\Omega_n$, and the flux $f$ in Eq.(1) is approximated in each $\Omega_n$ by $f^\delta_n = f^\delta_n(x, t)$, which is a polynomial of degree $p + 1$ within $\Omega_n$. Consequently, a total approximate solution $u^\delta = u^\delta(x, t)$ and a total approximate flux $f^\delta = f^\delta(x, t)$ can be defined within $\Omega$ as

$$ u^\delta = \bigcup_{n=0}^{N-1} u^\delta_n \approx u \quad \text{and} \quad f^\delta = \bigcup_{n=0}^{N-1} f^\delta_n \approx f, $$

(3)

where no level of inter-element continuity in $u^\delta$ is explicitly enforced. However, $f^\delta$ is required to be $C^0$ continuous at element interfaces.

The second stage of the FR approach involves transforming each $\Omega_n$ to a standard element $\hat{\Omega} = \{\zeta | -1 \leq \zeta \leq 1\}$ via the mapping

$$ \zeta = \Theta_n(x) = 2 \left( \frac{x - x_n}{x_{n+1} - x_n} \right) - 1, $$

(4)

which has the inverse

$$ x = \Theta_n^{-1}(\zeta) = \left( \frac{1 - \zeta}{2} \right) x_n + \left( \frac{1 + \zeta}{2} \right) x_{n+1}. $$

(5)

Having performed such a transformation, the evolution of $u^\delta_n$ within any individual $\Omega_n$ (and thus the evolution of $u^\delta$ within $\Omega$) can be determined by solving the following transformed equation within the standard element $\hat{\Omega}$

$$ \frac{\partial \hat{u}^\delta}{\partial t} + \frac{\partial \hat{f}^\delta}{\partial \zeta} = 0, $$

(6)

where

$$ \hat{u}^\delta = \hat{u}^\delta(\zeta, t) = J_n u^\delta_n(\Theta_n^{-1}(\zeta), t) $$

(7)

and

$$ \hat{f}^\delta = \hat{f}^\delta(\zeta, t) = f^\delta_n(\Theta_n^{-1}(\zeta), t), $$

(8)

is a polynomial of degree $p + 1$, and the Jacobian is defined as $J_n = (x_{n+1} - x_n)/2$.

The third stage of the FR approach involves defining the degree $p$ polynomial $\hat{u}^\delta$ in terms of a nodal basis as follows

$$ \hat{u}^\delta = \sum_{i=0}^{p} \hat{u}_i^\delta l_i, $$

(9)
where \( l_i \) are Lagrange polynomials defined as

\[
l_i = \prod_{j=0, j \neq i}^{p} \left( \frac{\zeta - \zeta_j}{\zeta_i - \zeta_j} \right),
\]

\( (10) \)

\( \zeta_i \) (\( i = 0 \) to \( p \)) are \( p + 1 \) distinct solution points within \( \hat{\Omega} \), and \( \hat{u}^\delta_i = \hat{u}^\delta(t) \) (\( i = 0 \) to \( p \)) are values of \( \hat{u}^\delta \) at the solution points \( \zeta_i \).

The fourth stage of the FR approach involves constructing a degree \( p \) polynomial \( \hat{f}^\delta D = \hat{f}^\delta D(\zeta, t) \), defined as the approximate transformed discontinuous flux within \( \hat{\Omega} \). Specifically, \( \hat{f}^\delta D \) is obtained via a collocation projection at the \( p + 1 \) solution points, and can hence be expressed as

\[
\hat{f}^\delta D = \sum_{i=0}^{p} \hat{f}_i^\delta D l_i,
\]

(11)

where the coefficients \( \hat{f}_i^\delta D = \hat{f}_i^\delta D(t) \) are simply values of the transformed flux at each solution point \( \zeta_i \) (evaluated directly from the approximate solution).

The flux \( \hat{f}^\delta D \) is termed discontinuous since it is calculated directly from the approximate solution, which is in general discontinuous between elements.

The fifth stage of the FR approach involves evaluating \( \hat{u}^\delta \) at either end of the standard element \( \hat{\Omega} \) (i.e. at \( \zeta = \pm 1 \)). These values, in conjunction with analogous information from adjoining elements, are then used to calculate numerical interface fluxes. The exact methodology for calculating such numerical interface fluxes will depend on the nature of the equations being solved. For example, when solving the Euler equations one may use a Roe type approximate Riemann solver, or any other two-point flux formula that provides for an upwind bias. In what follows the numerical interface fluxes associated with the left and right hand ends of \( \hat{\Omega} \) (and transformed appropriately for use in \( \hat{\Omega} \)) will be denoted \( \hat{f}^\delta I_L \) and \( \hat{f}^\delta I_R \) respectively.

The penultimate stage of the FR approach involves constructing the degree \( p + 1 \) polynomial \( \hat{f}^\delta \), by adding a correction flux \( \hat{f}^\delta C = \hat{f}^\delta C(\zeta, t) \) of degree \( p + 1 \) to \( \hat{f}^\delta D \), such that their sum equals the transformed numerical interface flux at \( \zeta = \pm 1 \), yet in some sense follows \( \hat{f}^\delta D \) within the interior of \( \hat{\Omega} \). In order to define \( \hat{f}^\delta C \) such that it satisfies the above requirements, consider first defining degree \( p + 1 \) correction functions \( h_L = h_L(\zeta) \) and \( h_R = h_R(\zeta) \) which satisfy:

\[
h_L(-1) = 1, \quad h_L(1) = 0,
\]

(12)
and
\[ h_R(-1) = 0, \quad h_R(1) = 1, \quad (13) \]
The exact form of \( h_L \) and \( h_R \) can be varied, determining various stability and accuracy properties; in this sense the FR approach can be considered a family of schemes. In particular, it has been shown that if \( h_L \) and \( h_R \) are the right and left Radau polynomials respectively, then a collocation-based nodal DG scheme is recovered and if \( h_L \) and \( h_R \) are set to zero at a set of \( p \) points within \( \Omega \), located symmetrically about the origin, then SD schemes are recovered for a linear flux. A suitable expression for \( \hat{f}^{\delta C} \) can now be written in terms of \( h_L \) and \( h_R \) as
\[ \hat{f}^{\delta C} = (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R, \quad (14) \]
where \( \hat{f}_L^{\delta D} = \hat{f}_L^{\delta D}(-1, t) \) and \( \hat{f}_R^{\delta D} = \hat{f}_R^{\delta D}(1, t) \). Using this expression, the degree \( p+1 \) approximate transformed total flux \( \hat{f}^{\delta} \) within \( \Omega \) can be constructed from the discontinuous and correction fluxes as follows
\[ \hat{f}^{\delta} = \hat{f}^{\delta D} + \hat{f}^{\delta C} = \hat{f}^{\delta D} + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D})h_L + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D})h_R. \quad (15) \]
The final stage of the FR approach involves evaluating the divergence of \( \hat{f}^{\delta} \) at each solution point \( \zeta_i \) using the expression
\[ \frac{\partial \hat{f}^{\delta}}{\partial \zeta}(\zeta_i) = \sum_{j=0}^{p} \hat{f}_j^{\delta D} \frac{d l_j}{d \zeta}(\zeta_i) + (\hat{f}_L^{\delta I} - \hat{f}_L^{\delta D}) \frac{d h_L}{d \zeta}(\zeta_i) + (\hat{f}_R^{\delta I} - \hat{f}_R^{\delta D}) \frac{d h_R}{d \zeta}(\zeta_i). \quad (16) \]
These values can then be used to advance \( \hat{u}^{\delta} \) in time via a suitable temporal discretisation of the following semi-discrete expression
\[ \frac{d \hat{u}_i^{\delta}}{d t} = -\frac{\partial \hat{f}^{\delta}}{\partial \zeta}(\zeta_i). \quad (17) \]
The nature of a particular FR scheme depends on three factors: the location of the solution points, the interface flux formulation, and the form of the correction functions. In his 2007 paper introducing FR Huynh [3] found that a collocation-based nodal DG scheme (NDG) is recovered if the left and right correction functions are taken to be the right and left Radau polynomials, respectively. He also showed how, at least for a linear flux function, it is possible for FR to recover any SD scheme. Moreover, Huynh also presented several new and novel schemes with various stability and accuracy properties.

Through a tensor-product construction the FR approach can be readily extended to handle quadrilateral and hexahedral elements. Further details can be found in [3, 19].
3. OSFR Schemes

In the preceding section we introduced the FR approach and how correction functions are used to form a piecewise continuous solution. The exact definition of the correction function is the main concern of the present work. Before presenting our extension of the OSFR schemes it is useful to review the key results from the original OSFR paper of Vincent et al. [11]. Specifically, the OSFR corrections functions are obtained by setting:

\[
h_L = \frac{(-1)^p}{2} \left[ L_p - \frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right],
\]

and

\[
h_R = \frac{1}{2} \left[ L_p + \frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right],
\]

where \( L_p = L_p(\zeta) \) is a Legendre polynomial of degree \( p \) and

\[
\eta_p = \frac{c(2p + 1)(a_p p!)^2}{2} \quad \text{and} \quad a_p = \frac{(2p)!}{2^{p}p!^2},
\]

with \( c \) being a free scalar parameter which must lie within the range

\[
\frac{-2}{(2p + 1)(a_p p!)^2} < c < \infty.
\]

Such correction functions satisfy

\[
\int_{-1}^{1} h_L \frac{\partial \hat{u}^\delta}{\partial \zeta} \, d\zeta - c \frac{\partial^p \hat{u}^\delta}{\partial \zeta^p} \frac{d^{p+1}h_L}{d\zeta^{p+1}} = 0,
\]

and

\[
\int_{-1}^{1} h_R \frac{\partial \hat{u}^\delta}{\partial \zeta} \, d\zeta - c \frac{\partial^p \hat{u}^\delta}{\partial \zeta^p} \frac{d^{p+1}h_R}{d\zeta^{p+1}} = 0.
\]

This property ensures that the resulting scheme will be linearly stable in the broken-Sobolev type norm:

\[
\sum_{n=1}^{N} \| \hat{u}_n^\delta \|_{W_2^p}^2 = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} (u_n^\delta)^2 + \frac{c}{2} J_n^2 \left( \frac{\partial^p u_n^\delta}{\partial x^p} \right)^2 \, dx.
\]

\[ \text{(18)} \]

\[ i.e. \text{ the energy is sufficiently constrained to give a bounded functional space of solutions.} \]
4. Weighted Energy Stability

Following on from the definition of OSFR, we may define the weighted Sobolev norm [20] in the reference domain as:

$$\| \hat{u}^\delta \|^2_{W_\omega^2} = \int_{-1}^{1} \left( (\hat{u}^\delta)^2 + \iota \left( \frac{\partial_p \hat{u}^\delta}{\partial \zeta} \right)^2 \right) (1 - \zeta)^\alpha (1 + \zeta)^\beta d\zeta$$  \hspace{1cm} (19)

noting that we have chosen to use $\iota = c/2$ from comparison with section 3.

Thus, the criterion on energy stability in time for a finite polynomial solution is that:

$$\frac{d}{dt} \| \hat{u}^\delta \|^2_{W_\omega^2} = \frac{d}{dt} \int_{-1}^{1} \left( (\hat{u}^\delta)^2 + \iota \left( \frac{\partial_p \hat{u}^\delta}{\partial \zeta} \right)^2 \right) w_{\alpha,\beta}(\zeta) d\zeta \leq 0$$ \hspace{1cm} (20)

where

$$w_{\alpha,\beta}(\zeta) = (1 - \zeta)^\alpha (1 + \zeta)^\beta$$ \hspace{1cm} (21)

and for brevity we will define the average as:

$$\frac{1}{2} \int_{-1}^{1} w_{\alpha,\beta}(\zeta) d\zeta = \overline{w}_{\alpha,\beta}$$ \hspace{1cm} (22)

Let us now consider applying FR to a linear advection problem. Without loss of generality we shall assume a unit convection velocity such that $f(u) = u$. It follows that:

$$\frac{d\hat{u}^\delta}{dt} = - \frac{\partial \hat{u}^\delta}{\partial \zeta} - (\hat{u}^\delta_L - \hat{u}^\delta_R) \frac{d h_L}{d\zeta} - (\hat{u}^\delta_R - \hat{u}^\delta_L) \frac{d h_R}{d\zeta}$$ \hspace{1cm} (23)

Previously, for the proof of OSFR, Eq.(23) would be multiplied by $\hat{u}^\delta$ and integrated over the reference domain. However, as we want to use a weight function, we will shall defer this integration step, as this simplifies the use of the product rule. Multiplying Eq.(23) by $\hat{u}^\delta$ we obtain:

$$\frac{d\hat{u}^\delta}{dt} \hat{u}^\delta = - \hat{u}^\delta \frac{\partial \hat{u}^\delta}{\partial \zeta} - (\hat{u}^\delta_L - \hat{u}^\delta_R) \frac{d h_L}{d\zeta} - (\hat{u}^\delta_R - \hat{u}^\delta_L) \frac{d h_R}{d\zeta}$$ \hspace{1cm} (24)

for which the product rule can be used to get:

$$\frac{1}{2} \frac{d(\hat{u}^\delta)^2}{dt} = - \frac{1}{2} \frac{\partial (\hat{u}^\delta)^2}{\partial \zeta} - (\hat{u}^\delta_L - \hat{u}^\delta_R) \left( \frac{\partial h_L \hat{u}^\delta}{\partial \zeta} - h_L \frac{d \hat{u}^\delta}{d\zeta} \right) - (\hat{u}^\delta_R - \hat{u}^\delta_L) \left( \frac{\partial h_R \hat{u}^\delta}{\partial \zeta} - h_R \frac{d \hat{u}^\delta}{d\zeta} \right)$$ \hspace{1cm} (25)
This step is important as it allows the formation of the conserved variable at the interface as well ensuring that only derivatives of $\hat{u}^{\delta}$ are present. We may now proceed to multiply by the weight function and integrate over the reference domain as:

$$\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} (\hat{u}^{\delta})^2 w_{\alpha,\beta} d\zeta = - \frac{1}{2} \int_{-1}^{1} \frac{\partial (\hat{u}^{\delta})^2}{\partial \zeta} w_{\alpha,\beta} d\zeta$$

$$- (\hat{u}^{\delta I}_L - \hat{u}^{\delta I}_L) \int_{-1}^{1} \left( \frac{\partial h_L \hat{u}^{\delta}}{\partial \zeta} - h_L \frac{d\hat{u}^{\delta}}{d\zeta} \right) w_{\alpha,\beta} d\zeta$$

$$- (\hat{u}^{\delta I}_R - \hat{u}^{\delta I}_R) \int_{-1}^{1} \left( \frac{\partial h_R \hat{u}^{\delta}}{\partial \zeta} - h_R \frac{d\hat{u}^{\delta}}{d\zeta} \right) w_{\alpha,\beta} d\zeta$$

(26)

Now proceeding to form the second component of the weighted Sobolev norm we first take the $p^{th}$ spatial derivative of Eq.(23):

$$\frac{d}{dt} \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} = - \frac{\partial^{p+1} \hat{u}^{\delta}}{\partial \zeta^{p+1}} - (\hat{u}^{\delta I}_L - \hat{u}^{\delta I}_L) \frac{d^{p+1} h_L}{d\zeta^{p+1}} - (\hat{u}^{\delta I}_R - \hat{u}^{\delta I}_R) \frac{d^{p+1} h_R}{d\zeta^{p+1}}$$

(27)

Given $\hat{u}^{\delta}$ is a $p^{th}$ order polynomial, Eq.(27) may be multiplied by the $p^{th}$ derivative of $\hat{u}^{\delta}$ and integrated over the reference domain with the weighting function to give:

$$\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} \left( \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \right)^2 w_{\alpha,\beta} d\zeta = - 2(\hat{u}^{\delta I}_L - \hat{u}^{\delta I}_L) \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_L}{d\zeta^{p+1}} w_{\alpha,\beta}$$

$$- 2(\hat{u}^{\delta I}_R - \hat{u}^{\delta I}_R) \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_R}{d\zeta^{p+1}} w_{\alpha,\beta}$$

(28)

This simplification can be made due to the respective orders of $\hat{u}^{\delta}$, $h_L$, and $h_R$. To now form the complete weighted Sobolev norm, we take Eq.(26) and
add \( \iota \) times Eq.(28) as:

\[
\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} \left( (\hat{u}^{\delta})^2 + \iota \left( \frac{\partial \hat{u}^{\delta}}{\partial \zeta} \right)^2 \right) w_{\alpha,\beta} d\zeta = -\frac{1}{2} \int_{-1}^{1} \frac{\partial (\hat{u}^{\delta})^2}{\partial \zeta} w_{\alpha,\beta} d\zeta
\]

\[-(\hat{u}^{\delta}_L - \hat{u}^{\delta}_r) \int_{-1}^{1} \left( \frac{\partial h_L \hat{u}^{\delta}}{\partial \zeta} - h_L \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta
\]

\[-(\hat{u}^{\delta}_L - \hat{u}^{\delta}_R) \int_{-1}^{1} \left( \frac{\partial h_R \hat{u}^{\delta}}{\partial \zeta} - h_R \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta
\]

\[-2(\hat{u}^{\delta}_L - \hat{u}^{\delta}_L) \int_{-1}^{1} \left( \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_L}{d \zeta^{p+1}} w_{\alpha,\beta} \right) d\zeta
\]

\[-2(\hat{u}^{\delta}_R - \hat{u}^{\delta}_R) \int_{-1}^{1} \left( \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_R}{d \zeta^{p+1}} w_{\alpha,\beta} \right) d\zeta
\]

Hence, by analogy to Vincent et al. [17] and section 3, if the following conditions are imposed on the correction function:

\[
\int_{-1}^{1} \left( h_L \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta \quad \text{and} \quad \int_{-1}^{1} \left( h_R \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta = 0
\]

\[
\int_{-1}^{1} \left( \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_L}{d \zeta^{p+1}} w_{\alpha,\beta} \right) d\zeta
\]

\[
\int_{-1}^{1} \left( \frac{\partial^p \hat{u}^{\delta}}{\partial \zeta^p} \frac{d^{p+1} h_R}{d \zeta^{p+1}} w_{\alpha,\beta} \right) d\zeta
\]

then

\[
\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} \left( (\hat{u}^{\delta})^2 + \iota \left( \frac{\partial \hat{u}^{\delta}}{\partial \zeta} \right)^2 \right) w_{\alpha,\beta} d\zeta = -\frac{1}{2} \int_{-1}^{1} \frac{\partial (\hat{u}^{\delta})^2}{\partial \zeta} w_{\alpha,\beta} d\zeta
\]

\[-(\hat{u}^{\delta}_L - \hat{u}^{\delta}_L) \int_{-1}^{1} \left( \frac{\partial h_L \hat{u}^{\delta}}{\partial \zeta} - h_L \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta
\]

\[-(\hat{u}^{\delta}_R - \hat{u}^{\delta}_R) \int_{-1}^{1} \left( \frac{\partial h_R \hat{u}^{\delta}}{\partial \zeta} - h_R \frac{d \hat{u}^{\delta}}{d \zeta} \right) w_{\alpha,\beta} d\zeta
\]

In order to find correction functions that can meet the conditions of Eq.(30 & 31) we need to define the projection of \( \hat{u}^{\delta} \) and \( h_L \) into the Jacobi polynomial basis:

\[
\hat{u}^{\delta} = \sum_{i=0}^{p} \tilde{u}_i J_i^{(\alpha,\beta)} \quad \text{and} \quad h_L = \sum_{i=0}^{p+1} \tilde{h}_{L,i} J_i^{(\alpha,\beta)} \quad \text{and} \quad h_R = \sum_{i=0}^{p+1} \tilde{h}_{R,i} J_i^{(\alpha,\beta)}
\]
Before proceeding we will lay out some results for Jacobi polynomials that will be used throughout, firstly the orthogonality condition:

\[
\int_{-1}^{1} J_m^{(\alpha,\beta)} J_n^{(\alpha,\beta)} w_{\alpha,\beta} d\zeta = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{2n+\alpha+\beta+1 \Gamma(n+\alpha+\beta+1) \delta_{mn}}
\]

where \(\Gamma(x)\) is the gamma function and \(\delta_{mn}\) is the Kronecker delta function.

Secondly, it will be useful to differentiate a Jacobi polynomial and express the result as a series of Jacobi polynomials in the same basis. From Doha [21] we find:

\[
\frac{d_m J_n^{(\alpha,\beta)}}{d\zeta^m} = 2^{-m}(n+\alpha+\beta+1) \sum_{i=0}^{n-m} D_{n-m,i}(\alpha+m,\beta+m,\alpha,\beta) J_i^{(\alpha,\beta)}
\]

where

\[
D_{j,i}(\gamma,\delta,\alpha,\beta) = \binom{j+\gamma+\delta+1}{j-i} \Gamma(2i+\alpha+\beta+1) \times 3F2\left(\begin{array}{ccc}
  i-j, & j+i+\gamma+\delta+1, & i+\alpha+1 \\
  i+\gamma+1, & 2i+\alpha+\beta+2, & 2 \end{array}\right)
\]

We define here that that \((x)_i\) is the rising Pochhammer function and \(3F2(\ldots; z)\) is the 3-2 generalised hypergeometric function [22, 23]. In the interest of brevity in later sections we will also define the constant:

\[
b_p^{(\alpha,\beta)} = \frac{\partial_p J_p^{(\alpha,\beta)}}{\partial\zeta_p} = 2^{-p}(p+\alpha+\beta+1)_p
\]

With these definitions, we may now substitute Eq.(33) into Eq.(30) to get:

\[
\int_{-1}^{1} \left( \sum_{i=0}^{p+1} \sum_{j=0}^{p} \bar{h}_{L,i} \bar{u}_j J_i^{(\alpha,\beta)} \frac{dJ_j^{(\alpha,\beta)}}{d\zeta} \right) w_{\alpha,\beta} d\zeta - \iota \bar{u}_p \bar{h}_{L,p+1} b_p^{(\alpha,\beta)} b_{p+1}^{(\alpha,\beta)} q_0^{(\alpha,\beta)} = 0
\]

To solve this integral condition we may constrain \(\bar{h}_L\), without loss of generality, to only have terms in \(p-1\), \(p\) and, \(p+1\). We then observe that the only contribution to the equality in Eq.(38) is from the \(p-1\)th term of \(\bar{h}_L\). Therefore, if we can find a closed form of \(D_{p-1,p-1}(\alpha+1,\beta+1,\alpha,\beta)\) we can...
relate $\tilde{h}_{L,p-1}$ to $\tilde{h}_{L,p+1}$ through $\iota$. Hence, substituting the values into Eq.(36) we find that the hypergeometric function component becomes $\binom{3}{2}F_2(0,\ldots;1)$ and, from the definition of the rising Pochhammer function, this must have a value of unity, hence we may write:

$$D_{p-1,p-1}(\alpha + 1, \beta + 1, \alpha, \beta) = \frac{(2p + \alpha + \beta - 1)(2p + \alpha + \beta)}{2(p + \alpha + \beta)}$$

which leads to:

$$\iota = \frac{\tilde{h}_{L,p-1}}{\tilde{h}_{L,p+1}} \left( \frac{(p + \alpha + \beta + 1)(p + \alpha + \beta + 2)p_{p-1}^{(\alpha,\beta)}}{2(p + \alpha + \beta)p_{p-1}^{(\alpha,\beta)}b_p^{(\alpha,\beta)}h_p^{(\alpha,\beta)}q_0^{(\alpha,\beta)}} \right)$$

Using the fact that $h_L(-1) = 1$ and $h_L(1) = 0$ we find:

$$(-1)^p = \tilde{h}_{L,p} \left( \frac{\beta + 1}{p!} \right) \left[ \frac{(p + \alpha)}{p + \beta} \frac{up(p + 1) + (p + \beta)(p + \beta + 1)A_p}{up(p + 1) + (p + \alpha)(p + \alpha + 1)A_p} + 1 \right]_{\kappa_p^{(\alpha,\beta)}}$$

$$\tilde{h}_{L,p+1} = -\frac{A_pJ_p^{(\alpha,\beta)}(1)h_{L,p}}{\iota J_{p-1}^{(\alpha,\beta)}(1) + A_pJ_{p+1}^{(\alpha,\beta)}(1)} = -\frac{A_p(p + 1)(p + \alpha)\tilde{h}_{L,p}}{up(p + 1) + (p + \alpha + 1)(p + \alpha)A_p}$$

Putting this all together we obtain:

$$h_L = \frac{(-1)^p}{\kappa_p^{(\alpha,\beta)}} \left( J_p^{(\alpha,\beta)} - \iota \frac{(p + 1)(p + \alpha)J_{p-1}^{(\alpha,\beta)} + (p + 1)(p + \alpha)A_pJ_{p+1}^{(\alpha,\beta)}}{up(p + 1) + (p + \alpha + 1)(p + \alpha)A_p} \right)$$

$$h_R = \frac{1}{\kappa_p^{(\alpha,\beta)}} \left( J_p^{(\alpha,\beta)} + \iota \frac{(p + 1)(p + \alpha)J_{p-1}^{(\alpha,\beta)} + (p + 1)(p + \alpha)A_pJ_{p+1}^{(\alpha,\beta)}}{up(p + 1) + (p + \alpha + 1)(p + \alpha)A_p} \right)$$

5. Limits on Stability

In order for the correction function to be stable Eq.(20) must be satisfied. Moreover, the norm described by Eq.(19) must indeed be a norm; i.e. it must be positive, definite, homogeneous, and obey the triangle inequality. Of particular concern to us is the positivity condition for the remaining conditions
follow immediately due to the linear nature of differentiation. Hence, it is required that:

$$0 < \| \hat{u}^\delta \|^2_{W_2^*,W} = \int_{-1}^{1} \left( (\hat{u}^\delta)^2 + \iota \left( \frac{\partial^p \hat{u}^\delta}{\partial \zeta^p} \right)^2 \right) (1 - \zeta)^\alpha (1 + \zeta)^\beta \, d\zeta < \infty \quad (45)$$

for $\hat{u}^\delta \neq 0$. Therefore, following the method of [17] we can substitute Eq.(33) into Eq.(45) as:

$$0 < \sum_{i=0}^{p} \tilde{u}_i^2 q_i^{(\alpha,\beta)} + \iota \left( b_p^{(\alpha,\beta)} \right)^2 q_0^{(\alpha,\beta)} < \infty \quad (46)$$

Then grouping terms of the same order:

$$0 < \sum_{i=0}^{p-1} \tilde{u}_i^2 q_i^{(\alpha,\beta)} + \left( q_p^{(\alpha,\beta)} + \iota \left( b_p^{(\alpha,\beta)} \right)^2 q_0^{(\alpha,\beta)} \right) \tilde{u}_p^2 < \infty \quad (47)$$

As $\tilde{u}_i^2$ is always positive, the limit on the value of $\iota$ is:

$$-\iota_{\text{crit}} = - \frac{q_p^{(\alpha,\beta)}}{\left( b_p^{(\alpha,\beta)} \right)^2 q_0^{(\alpha,\beta)}} \leq \iota < \infty \quad (48)$$

The value of $\iota_{\text{crit}}$ can be evaluated in a closed form that shows it is always positive, and hence quasi-DG (qDG) schemes ($\iota = 0$) will always have a positive norm, leading us expect a large region of stability.

$$\iota_{\text{crit}} = \frac{q_p^{(\alpha,\beta)}}{\left( b_p^{(\alpha,\beta)} \right)^2 q_0^{(\alpha,\beta)}} = \left( \frac{\alpha + \beta + 1}{2p + \alpha + \beta + 1} \right) \left( \frac{(\alpha + 1)_p (\beta + 1)_p}{(\alpha + \beta + 1)_p} \right) \times \left( \frac{2^p}{(p + \alpha + \beta + 1)_p} \right)^2 \quad (49)$$

To show this is compatible with previous work, let us suggest $(\alpha, \beta) = (0, 0)$. Remembering that $(1)_p = p!$, we get:

$$\frac{q_p^{(0,0)}}{\left( b_p^{(0,0)} \right)^2 q_0^{(0,0)}} = \frac{(p!)^3}{(2p + 1) \left( \frac{2^p}{(2p)!} \right)^2} \quad (50)$$

Which is identical to the result of section 3 where $\iota = c/2$. 
6. Spectral Difference Schemes

It has been shown [3, 18, 17] that FR is able to recover SD schemes for equation sets with linear and homogeneous flux functions. The implication being that the treatment of aliasing, introduced by non-linear or heterogeneous flux functions, is different in FR to a native SD scheme. To obtain an SD scheme within FR the roots of the left and right corrections functions must be symmetric about \( \zeta = 0 \). The simplest method of achieving this is by prescribing the interior zeros to be a set of quadrature points. In the first proof of energy stability of SD schemes the canonical Gauss–Legendre quadrature [18] was utilised, but other quadratures have been considered, for example Lui et al. [24] saw the potential for a Gauss–Legendre–Lobatto quadrature. As the choice of quadrature is arbitrary, it is proposed that this be extended to the full set of Gauss–Jacobi quadratures, where the \( p \)th Gauss–Jacobi points are the roots of the \( p \)th order Jacobi polynomial [25]. This gives rise to the correction functions:

\[
\begin{align*}
  h_{L,SD} &= \frac{1 - \zeta}{2} \frac{J_p^{(\alpha,\beta)}(\zeta)}{J_p^{(\alpha,\beta)}(-1)} \quad \text{and} \quad h_{R,SD} = \frac{1 + \zeta}{2} \frac{J_p^{(\alpha,\beta)}(\zeta)}{J_p^{(\alpha,\beta)}(1)} \quad (51)
\end{align*}
\]

where the correction functions are normalised for the edge value, as Jacobi polynomials do not guarantee unit magnitude at \( \pm 1 \). The recurrence relation for Jacobi polynomials [26] leads us to the following form:

\[
\begin{align*}
  h_{L,SD} &= \left( - \frac{(p + \alpha)(p + \beta)}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta)} J_p^{(\alpha,\beta)} - \frac{1}{2} \left( 1 + \frac{\alpha^2 - \beta^2}{(2p + \alpha + \beta + 2)(2p + \alpha + \beta)} \right) J_p^{(\alpha,\beta)} \right. \\
  &\quad \left. + \frac{1}{2} \frac{p + 1 + \alpha + \beta}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta + 2)} J_p^{(\alpha,\beta+1)} \right) \frac{(-1)^p \Gamma(p + 1) \Gamma(\beta + 1)}{\Gamma(p + \beta + 1)} \\
  &= \left( - \frac{(p + \alpha)(p + \beta)}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta)} J_p^{(\alpha,\beta)} - \frac{1}{2} \left( 1 + \frac{\alpha^2 - \beta^2}{(2p + \alpha + \beta + 2)(2p + \alpha + \beta)} \right) J_p^{(\alpha,\beta)} \right. \\
  &\quad \left. + \frac{1}{2} \frac{p + 1 + \alpha + \beta}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta + 2)} J_p^{(\alpha,\beta+1)} \right) \frac{(-1)^p \Gamma(p + 1) \Gamma(\beta + 1)}{\Gamma(p + \beta + 1)} \quad (52)
\end{align*}
\]
and

\[
\begin{align*}
  h_{R,SD} = & \left( \frac{(p + \alpha)(p + \beta)}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta)} J^{(\alpha,\beta)}_{p-1} \right. \\
  & + \left. \frac{1}{2} \left( 1 - \frac{\alpha^2 - \beta^2}{(2p + \alpha + \beta + 2)(2p + \alpha + \beta)} \right) J^\alpha_p \right) \\
  & + \frac{(p + 1)(p + 1 + \alpha + \beta)}{(2p + \alpha + \beta + 1)(2p + \alpha + \beta + 2)} J^{(\alpha,\beta)}_{p+1} \frac{\Gamma(p + 1)\Gamma(\alpha + 1)}{\Gamma(p + \alpha + 1)} \tag{53}
\end{align*}
\]

The advantage of this method of extending the set of SD correction functions is that we may use the work of the previous section in order to prove the theoretical energy stability. We therefore assert:

**Lemma 6.1. SD Energy Stability.** A given SD correction function with \( \alpha, \beta \in (-1, \infty) \) will always give the norm \( \| \hat{u}^\delta \|_{W^2} \) to be positive.

**Proof.** Using Eq.(40) to produce an expression of \( \iota \) for SD schemes, which, after normalisation by \( \iota_{\text{crit}} \), gives:

\[
\frac{\iota_{SD}}{\iota_{\text{crit}}} = \frac{p}{p + 1} \tag{54}
\]

As \( \iota_{\text{crit}} \) is always positive, so must \( \iota_{SD} \) always be positive. \( \Box \)

Lastly, we will note that this definition of SD schemes may be extended to include all quadratures in \([-1, 1]\) by defining the correction function as:

\[
h_L = \left( 1 - \zeta \right) J^\alpha_{\langle m,\beta \rangle}(\zeta) J^{(\alpha,\beta)}_{m}\left( -1 \right) \text{ for } |m|_1 = p \tag{55}
\]

where \( m = (m_1, m_2, \ldots) \) is a multiindex and \( (\alpha, \beta) \) may vary with \( m_i \). This is included for completeness, however the focus will put on the Jacobi SD function of Eq.(51).

To briefly touch on some of the characteristic of the Jacobi SD correction function, consider the Gauss–Jacobi quadratures that make up the interior zeros of the correction function. Figure 2 aims to demonstrate how the quadrature is effected by the variation of \( \alpha \) and \( \beta \), primarily that for \( \alpha = \beta \) the quadrature is symmetric. Furthermore, in the limit as \( \alpha, \beta \to -1 \) the
\(\alpha = -1, \beta = 10\)

\[\begin{array}{ccccc}
-1 & -0.5 & 0 & 0.5 & 1 \\
\end{array}\]

\(\alpha = \beta = 10\)

\[\begin{array}{ccccc}
-1 & -0.5 & 0 & 0.5 & 1 \\
\end{array}\]

\(\alpha = \beta = 0\)

\[\begin{array}{ccccc}
-1 & -0.5 & 0 & 0.5 & 1 \\
\end{array}\]

\(\alpha = \beta = -1\)

\[\begin{array}{ccccc}
-1 & -0.5 & 0 & 0.5 & 1 \\
\end{array}\]

\(\alpha = 10, \beta = -1\)

\[\begin{array}{ccccc}
-1 & -0.5 & 0 & 0.5 & 1 \\
\end{array}\]

Figure 2: Selected Gauss-Jacobi quadratures for \(n = 4\).

7. Convergence and CFL Study

In the previous sections we introduced a new class of energy stable correction functions and demonstrated how these functions can be used to recover a range of new and novel schemes. In this section we shall seek to characterise the numerical properties of these schemes. Our focus will be on two specific subsets: qDG schemes which correspond to choices of \((\alpha, \beta)\) for which \(i = 0\); and the SD schemes of section 6.

The analytical means by which we will assess the behaviour of different corrections is via the rate of convergence of the solution with mesh refinement and the explicit temporal integration (CFL) limit. Rate of convergence has previously been studied for DG via FR [27], but we will extend this analysis by varying the correction function.

If we introduce the semi-discretised form of the linear advection equation for plane waves with wavenumber \(k\) and unit wave speed

\[
\frac{\partial u^\delta_j}{\partial t} = -Q(k)u^\delta_j
\] (56)
The exact details of the definition of $Q$ are not included as this is secondary to the aim of the paper, however details can be found in [3, 11, 27, 28]. Then defining the diagonalisation of $Q$ as:

$$Q = ikW\Lambda W^{-1}$$

(57)

By diagonalising the semi-discrete operator in this manner, where $W$ is the eigenvector matrix and $\Lambda = \text{diag}(\lambda_0 \ldots \lambda_p)$, we may form the initial interpolation of the solution as:

$$u^\delta_j(t = 0) = \exp(ikx_j)Wv_0 = \exp(ik(J_j(\zeta + 1) + x_j))$$

(58)

To calculate the rate of convergence, we need to monitor the semi-discrete error of the solution for different grid spacings. The derivation of the analytical semi-discrete and fully-discrete error can be found in [27, 28] and as the derivation is secondary to the aim, here we will jump to the end result for the semi-discrete error.

$$\epsilon^\delta_j(t, J) = u^\delta_j(t) - u_j(t) = \exp(ik(x_j - t))\sum_{n=0}^{p} \left( \exp(ikt(\lambda_n + 1)) - 1 \right)v_{0,n}w_n$$

(59)

Here $w_n$ is the $n$\textsuperscript{th} column vector of $W$. If we define the $l_2$ norm of the error as $\|\epsilon(t, J)\|_2 = E(t, J)$ then the grid convergence rate is:

$$r_h(t) = \frac{\log(E(t, J_1)) - \log(E(t, J_2))}{\log J_1 - \log J_2}$$

(60)

where $J_1$ and $J_2$ are the Jacobians of the two grids over which the convergence rate is to be calculated.

The second method used to understand these new correction functions is analysis of the explicit temporal integration stability. The importance of this is that, as is often the case in numerical methods, there is a trade off between spatial accuracy and temporal stability. We wish to know if a compromise can be found or if a scheme exists that breaks this trade off, being both accurate and temporally stable. To investigate the maximum stable time step we will use von Neumann’s stability conditions [29] on the fully discretised form:

$$u^{\delta, n+1}_j = R(Q(k))u^{\delta, n}_j$$

(61)

where $R$ is the update matrix that advances the solution from the $n$\textsuperscript{th} time level to the $n + 1$\textsuperscript{th} level. The definition of the update matrix varies with
explicit integration method. As an example consider, forward Euler $R = I - \tau Q$, where $\tau$ is the explicit time step and $I$ is the identity matrix. Therefore, von Neumann's theorem states that for stability the spectral radius must be less than unity $\rho(R) \leq 1$ or, in other words, from Banach's fixed point theorem [30], the update matrix must cause a contraction.

7.1. Quasi-DG

As has already been described these are the set of schemes where, for a given $\alpha$ & $\beta$, the correction parameter is set as $\iota = 0$. In the case of $\alpha = \beta = 0$ this is known to be equivalent to Nodal DG. Therefore, when the space of correction functions is extended using the Jacobi weighting function, the set of schemes may be considered to be quasi-DG (qDG) for the different bases. The property that sets DG apart from other correction functions is that it achieves super-convergence [3, 27, 31, 32, 19], i.e. for sufficiently smooth and well resolved specific cases, the rate of error convergence can be of order $2p+1$, whereas other FR schemes would normally be expected to obtain less.

Because of this feature we will initially look at the rate of convergence of the qDG, with a focus on the case when $\alpha = \beta$, i.e. symmetric weight functions. There are two rates of convergence that can be considered, the initial rate as $t \to 0$ and the long time rate as $t \to \infty$, with the initial rate being dictated by the secondary modes and the long time rate by the primary mode. Due to the secondary modes having very short half-lives at a well-resolved wavenumbers [28], we consider the most important convergence rate to be $r_h(t) \to \infty$, and this is shown in Figure 3. It is evident then that for both the central difference and upwinded interface case improvement may be made to the rate of convergence over standard DG schemes. It may be possible, then, to improve the practical performance of FR using a qDG scheme.

Second, we go on to study the stability of the qDG once the temporal integration is discretised. For this we will make use of low storage RK schemes [33] as they can be simply added to FR schemes practically and do not prohibitively increase memory usage, a major concern with the current cost of graphics RAM.

Figure 4 then shows that for both RK33 and RK44, as $\alpha$ or $\beta$ is decreased the CFL limit is reduced. Furthermore, as $\beta \to -1$ temporal stability is lost, and in order to understand this it can be useful to consider the shape of the correction function as $\beta \to -1$ which is shown in Figure 5. From this it is evident that the maximum gradient increases, with significant changes
Figure 3: Variation of the rate of error convergence with grid for quasi-DG correction functions, ($\iota = 0$) when, $p = 4$, $\alpha = \beta$, $J_2/J_1 = 0.5$, $k = 3\pi/4$ and, $t/T = 1000$. The dotted line is for $\alpha = \beta = 0$.

Figure 4: CFL limit of various temporal integration methods for $\iota = 0$ with upwinded interfaces. The dashed line is for $\alpha = \beta$ and the dotted lines are $\alpha = 0$ and $\beta = 0$. 
occurring across the whole domain. This is the opposite of what Huynh [3] proposed as good characteristics for temporal stability, where most of gradient was lumped at one end.

7.2. Spectral Difference

Spectral Difference via FR is often considered in some sense as being a canonical FR correction function [3, 17, 34, 14]. This is with good reason as its dispersion and dissipation characteristics are quite favourable [11]. Yet the Jacobi generalisation allows us to extend the definition of the SD correction functions, allowing us to investigate whether yet more favourable characteristics can be achieved. To restrict the space of possible functions we will use a detail outlined in section 6, that in the case of $\alpha \neq \beta$ the correction functions are not symmetric about $\zeta = 0$. This will bias convection due to directional variation in the phase and group velocities, which for most practical calculations would be unacceptable. Because of this our study into the rate of convergence will focus on the case of $\alpha = \beta$.

The long time rate of convergence is then presented in Figure 6 for both upwinded and central differenced interfaces. In both cases, it is clear that improvements may be made. This improvement is only minor in the case of upwinded interfaces, however when the interface is centrally differenced, a three order increase in the rate of convergence can be seen for $\alpha = \beta \approx 2 \times 10^{-2}$. This is similar in size to the increase seen for qDG with central differencing.

After the peak rate of convergence, there is a sharp drop off in the rate of convergence towards unity order. A similar drop off in Figure 6a is also seen
but closer to the peak value. When practically implemented, an approximate Riemann solver is used across the interface, which is likely to give a mix of upwinding and central flux - so the rate of convergence will vary somewhere between the Figure 3a & 3b. Because of this, to avoid hitting the penalty of the sharp decrease in order as the degree of upwinding changes, it may be more robust to use an $\alpha = \beta$ value closer to $5 \times 10^{-3}$ (the upwind optimal) rather than the central optimal value.

Again, when using these correction functions practically, a method of discrete time integration will be used. Figure 7 displays the CFL limit for $p = 4$ for two low storage RK explicit temporal integration schemes. It is evident that, although previously we showed that $\iota_{SD}$ would lead to the norm always being positive. This, together with the result for qDG, indicates that the critical value of $\iota$ is a necessary but not sufficient condition for stability and that additional contributions from Eq.(32) will further restrict the stability. This is not unexpected, as when $\alpha, \beta < 0$ it is clear that the weight function is ill-defined at the end-points; something which is not reflected in the value of $\iota_{crit}$. However, $\iota_{crit}$ is not without merit and Figures 4 & 7 clearly show that there is a well defined stable region of correction functions which contain the optimal points found in Figures 3 & 6. In the case of optimal SD, the CFL limit is approximately the same as for the original SD scheme ($\alpha = \beta = 0$).
Figure 7: CFL limit of various temporal integration methods for Jacobi SD with updiffed interfaces. The dashed line is for $\alpha = \beta$ and the dotted lines are $\alpha = 0$ and $\beta = 0$.

and the performance will be further explored in the next section.

For completeness, we include in Figure 8 the dispersion and dissipation relationships for the correction functions that give optimal rates of convergence. These relations can be extracted from the diagonal matrix $\Lambda$ and for the case of central differenced interfaces the dissipation is zero. The dispersion relations for central differenced interfaces are also split between two modes, one active at low frequencies and the other active at high frequencies. Because of this we have included both modes. Comparison made between these relations and their OSFR counterparts found previously by Vincent et al. [11] show that the difference is only slight. This should give us confidence that the correction functions found will at least provide a reasonable answer practically.
Figure 8: Dispersion ($\Re(\hat{\omega})$) and dissipation ($\Im(\hat{\omega})$) relations of the SD and qDG $p = 4$ correction functions found to give improved rates of convergence.
8. Numerical Experiments

We shall now consider applying our new correction functions to the compressible Navier–Stokes equations. This will inform us if practical improvements have been made and if theoretical expectations align with numerical experiments. The case under consideration is the Taylor–Green Vortex (TGV) [35], simulated with implicit LES (ILES). The implication of this is that no explicit sub-grid-scale model and no filtering will be used. It was previously demonstrated by Vermeire and Vincent [14] that FR applied as ILES can be beneficial when flow physics is under-resolved. Therefore, we will also investigate the performance of the proposed FR schemes as cell Reynolds numbers is varied for the TGV. Performance will be evaluated by two metrics, kinetic energy dissipation and enstrophy based dissipation rate, respectively defined as:

\[
\begin{align*}
\epsilon_1 &= -\frac{1}{2\rho_0 U_0^2 |\Omega|} \frac{d}{dt} \int_{\Omega} \rho \mathbf{u} \cdot \mathbf{u} \, d\mathbf{x} \\
\epsilon_2 &= \frac{\mu}{(\rho_0 U_0)^2 |\Omega|} \int_{\Omega} \rho \mathbf{\omega} \cdot \mathbf{\omega} \, d\mathbf{x}
\end{align*}
\]

where \(|\Omega|\) is the volume of the domain \(\Omega\), \(U_0\) is a characteristic velocity, and \(\rho_0\) is the stagnation density. For this case we will use the initial conditions for \([\rho, u, v, w, p]^T\) as is set out in DeBonis [36] with the non-dimensional constants set as:

\[
R_e = 1600 = \frac{\rho_0 U_0 L}{\mu}, \quad P_r = 0.71 = \frac{\mu \gamma R}{\kappa (\gamma - 1)}, \quad M_a = 0.08 = \frac{U_0}{\sqrt{\gamma R T_0}}
\]

where we will set the characteristic velocity to be \(U_0 = 1\); the characteristic length scale to be \(L = 1\); and the stagnation density to be \(\rho_0 = 1\). With such a low Mach number, this case may be considered incompressible, hence we should see that \(\epsilon_1 = \epsilon_2\) and any deviation will highlight under-resolution. This has previously been used to assess the impact of polynomial order on the resolution of this case in many investigations.

The domain of interest is \(\Omega = [-\pi, \pi]^3\), which is uniformly subdivided to produce a regular cuboid mesh of elements. Inside of each element the solution and flux points are taken to be a tensor-product construction of Gauss–Legendre quadrature points. Inviscid fluxes are calculated using a Rusanov type approximate Riemann solver [37] with Davis [38] wave speeds.
(a) Kinetic energy dissipation  
(b) Enstrophy based dissipation

Figure 9: Comparison of the Jacobi SD correction function ($\alpha = \beta = 2.1 \times 10^{-2}$) with the SD correction function recovered by OSFR for two under-resolved grids.
We will begin by evaluating the performance of the SD correction function found in section 7. Again, we note here that SD via FR and native SD are only identical for linear homogeneous equation sets. In Figure 9, we compare two grids that are marginally underresolved. For the TGV small scales are generated at times approximately in the range $3 < t < 7$. Moderate levels of over-dissipation around these times are indicative of dispersion due to under-resolution. Looking at $\epsilon_1$ and comparing against the reference line we observe similar performance for both SD schemes. However, looking at $\epsilon_2$ it is clear that on both grids the Jacobi SD correction function has an improved dissipation rated compared with the SD scheme recovered by OSFR. This suggests that the extra flexibility provided by our Jacobi correction functions may enable us to obtain more physical sub grid-scale dissipation.

In order to explore this further, we will investigate how the error in the dissipation varies with time over a range of Jacobi SD correction functions at different grid resolution levels. For this investigation we will use the $l_2$ error in $\epsilon_1$ and $\epsilon_2$ which we will define as $\| \epsilon_1 \|_2 = \| \epsilon_1 - \epsilon_{1,\text{ref}} \|_2$ and similarly for $\epsilon_2$, where $\epsilon_{1,\text{ref}}$ is the dissipation rate of the reference DNS solution [36, 41].

The resulting errors for several grid resolutions over the range of correction

![Graph showing kinetic energy dissipation error and enstrophy based dissipation error](image)

(a) Kinetic energy dissipation error  (b) Enstrophy based dissipation error

- $R_{e,\text{cell}} = 133$  - $R_{e,\text{cell}} = 100$  - $R_{e,\text{cell}} = 80$  - $R_{e,\text{cell}} = 67$

Figure 10: Time averaged error in dissipation rates for the Taylor–Green Vortex, $p = 4$, with symmetric Jacobi SD correction functions.

Viscous fluxes are handled using the BR1 approach of Bassi and Rebay [39, 40].
functions \(-1 < \alpha = \beta \leq 0.5\) are shown in Figure 10.

Before analysing the results further we will remark on the temporal integration method. RK44 [33] explicit temporal integration was used with a time step at \(R_{e,\text{cell}} = 100\) of \(\Delta t = 10^{-3}\). This explicit time step was then linearly scaled as the grids were refined or coarsened, such that the acoustic CFL number remains approximately constant. A time step of \(\Delta t = 10^{-3}\) has previously been used for high resolution DNS simulations and is understood to be more than sufficient to resolve the temporal dynamics of the TGV [36]. As such we shall opt to discard any correction function which is found to be temporally unstable with this time step—for such functions are unlikely to be of practical utility. Finally, we note that in the case of \(R_{e,\text{cell}} = 133\), some of the schemes with particularly large errors were found to blow up at some time \(t > 10\).

Looking at Figure 10a it is apparent that the integrated \(\epsilon_1\) error is less sensitive to variations in the correction function. As a result this may be useful in giving more general information about grid resolution and order and will be discussed later. However, far higher sensitivity in the integrated \(\epsilon_2\) error is observed. There is clearly an optimal region around \(0 < \alpha = \beta < 0.2\), which is similar to the region that was found theoretically to give optimal convergence for linear advection.

Repeating this series of tests for qDG, we can see from comparison of Figure 11 & 10 that for a range of \(\alpha\) values the error of qDG is less than that on SD schemes. In some cases, \(\alpha \approx 0.25\), the error in the enstrophy is very low and moderately invariant with \(R_{e,\text{cell}}\). Referencing section 7, there is no appreciable degradation in CFL limit at this point although it is near to the limit of theoretical stability. Thus the Jacobi SD schemes trade a small amount of accuracy for an appreciable gain in the CFL limit.

A key point to note about both the Jacobi SD and qDG tests is that in both cases correction functions were found that performed better when considering \(\epsilon_2\) than in \(\epsilon_1\), this seems to indicate that the resolved turbulence and the implicit sub-grid model—which comes from the correction function—lead to physical vortical motions. However, the larger \(\epsilon_1\) means that there is still non-physical dissipation, but this must be in the larger bulk movement of the fluid, and hence not displayed in the gradients. As was mentioned previously, the variation \(\epsilon_1\) is smaller than \(\epsilon_2\) when considering changes in correction function. This indicates that \(\epsilon_1\) is controlled more by other factors such as the polynomial order, or as this investigation shows, \(R_{e,\text{cell}}\).
Figure 11: Time averaged error in dissipation rates for the Taylor–Green Vortex, $p = 4$, with qDG correction functions for $\alpha = \beta$.

9. Conclusions

A new set of FR correction functions, defined in terms of Jacobi polynomials and stable with regards to a weighted Sobolev type norm, have been presented. Using these functions it is possible to recover, at least for a linear flux function, a range of spectral difference schemes. Theoretical convergence studies and von Neumann analysis were then used to show that these new schemes can be used to increase accuracy without impacting temporal stability relative to previous FR schemes. Moreover, the newly derived schemes also enable the definition of quasi-DG schemes which correspond to DG schemes with a different polynomial basis. The same convergence study showed that these have the potential to increase the order of accuracy of DG, and in the case of $p = 4$ with central interfaces, an increase in the rate of convergence of two orders was found. Numerical experiments were then performed to validate some of the findings, which showed that the optimal correction function for the full Navier–Stokes equation lay in the same region predicted by theoretical linear investigations. A summary of how the present work is positioned relative to the literature is shown in Figure 12.
Figure 12: Euler diagram to show the interconnection of the spaces of FR correction functions: Nodal DG (NDG) [3]; Original Stable FR (OSFR) [17]; Extended range Stable FR (ESFR) [13]; Generalised Sobolev stable FR (GSFR) [15]; Generalised Lebesgue Stable FR (GLSFR) [16] and, the Generalised Jacobi stable FR (GJFR) of the present work. Some specific examples of specific schemes are given, notably Huynh’s Lumped Chebyshev-Lobatto (LCL) [3] scheme and the original Legendre spectral difference (LSD) scheme [9, 3, 18]

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A. Nomenclature

**Roman**

| Symbol | Description |
|--------|-------------|
| \( a_p \) | \((2p)!/(2^p p!)^2\) |
| \( A_p \) | constant relating the ratio of \( h_{L,p-1} \) over \( h_{L,p+1} \) and \( \iota \) |
| \( b_{p}^{(a,b)} \) | \( p^{th} \) derivative of \( p^{th} \) Jacobi polynomial |
| \( c \) | OSFR correction function variable (\( \iota = c/2 \)) |
| \( D_{i,j}(\gamma, \delta, \alpha, \beta) \) | Doha’s Jacobi differentiation constant |
| \( e_j(t, J) \) | analytical error at solution points for time \( t \) with element Jacobian \( J \) |
| \( e_1(t) \) | error in \( \epsilon_1 \) at time \( t \) |
| \( e_2(t) \) | error in \( \epsilon_2 \) at time \( t \) |
| \( h_L \) & \( h_R \) | left and right correction functions |
| \( J_n \) | \( n^{th} \) Jacobian of the mapping \( x \rightarrow \zeta \) |
| \( J_1^{(a,b)} \) | \( i^{th} \) order Jacobi polynomial of the first kind |
| \( k \) | wavenumber |
| \( L_n \) | \( n^{th} \) order Legendre polynomial |
| \( M_a \) | Mach number |
| \( P_r \) | Prandtl number |
| \( q_{n}^{(a,b)} \) | value of the \( n^{th} \) Jacobi orthogonality integral |
| \( Q \) | FR operator matrix for linear advection equation |
| \( R_e \) | Reynolds number |
| \( r_{h}(t) \) | rate of convergence with cell width at time \( t \) |
| \( w_{\alpha,\beta}(x) \) | Jacobi weight function \((1-x)^{\alpha}(1+x)^{\beta}\) |
| \( w_n \) | \( n^{th} \) column of \( W \) |
| \( W \) | eigenevector matrix of \( Q \) |
| \( W_{2}^{c} \) | Sobolev space in 2 norm, modified by factor \( c \) |
| \( W_{2}^{\iota, w} \) | Sobolev space in 2 norm, modified by factor \( \iota \) and weighted by \( w_{\alpha,\beta} \) |

**Greek**

| Symbol | Description |
|--------|-------------|
| \( \alpha \) | first Jacobi polynomial control parameter |
| \( \beta \) | second Jacobi polynomial control parameter |
| \( \gamma \) | ratio of specific heat capacities |
| \( \Gamma(x) \) | Gamma function evaluated at \( x \) |
| \( \epsilon_1 \) | global kinetic energy dissipation rate |
| \( \epsilon_2 \) | global enstrophy based dissipation rate |
| \( \zeta \) | 1D spatial variable in reference domain |
| \( \eta_p \) | OSFR derived parameter |
| \( \Theta_n \) | linear transformation of \( n^{th} \) element from \( x \) to \( \zeta \). \( \Theta_n : \Omega_n \rightarrow \hat{\Omega} \) |
\( \tau \) correction function parameter
\( \tau_{\text{crit}} \) critical correction function parameter
\( \kappa \) thermal conductivity
\( \kappa_{p}^{(\alpha,\beta)} \) derived parameter for Jacobi correction functions
\( \Lambda \) diagonal eigenvalue matrix of \( Q \)
\( \mu \) dynamic viscosity
\( \omega \) vorticity
\( \Omega \) spatial domain
\( \Omega_{n} \) \( n^{\text{th}} \) partition of the domain \( \Omega \)

**Subscript**
- \( \bullet_{L} \) variable at left of cell
- \( \bullet_{R} \) variable at right of cell

**Superscript**
- \( \delta^{C} \) correction to the discontinuous function
- \( \delta^{D} \) uncorrected discontinuous value
- \( \delta^{I} \) common value at interface
- \( \delta^{T} \) vector or matrix transpose
- \( \delta \) discontinuous value
- \( \hat{\bullet} \) variable transformed to reference domain
- \( \bar{\bullet} \) function averaged of the reference domain
- \( \tilde{\bullet} \) function transformed to polynomial space

**Symbols or Operators**
- \( (x)_{i} \) rising Pochhammer function of \( x \) with \( i \) steps
- \( _{n}F_{m}(N,M;z) \) the \( n \)-\( m \) generalised hypergeometric function at \( z \) [22, 23]

**Abbreviations**
- CPR Correction Procedure via Reconstruction
- DG Discontinuous Galerkin
- FR Flux Reconstruction
- GJFR Generalised Jacobi stable Flux Reconstruction
- GLSFR Generalised Lebesgue Stable Flux Reconstruction
- GSFR Generalised Sobolev stable Flux Reconstruction
- ILES Implicit Large Eddy Simulation
- LCL Lumped Chebyshev-Lobatto
| Abbreviation | Description                      |
|--------------|----------------------------------|
| LCP          | Lifting Collocation Penalty      |
| LES          | Large Eddy Simulation            |
| LSD          | Legendre Spectral Difference     |
| NDG          | Nodal Discontinuous Galerkin     |
| OSFR         | Original Stable Flux Reconstruction |
| qDG          | quasi Discontinuous Galerkin     |
| SD           | Spectral Difference              |
| TGV          | Taylor-Green Vortex              |