Optical Schrödinger Cat States in One Mode and Two Coupled-Modes Subject to Environments

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Taking the decoherence into account, we investigate nonclassical features of the optical Schrödinger cat states in one mode and two coupled-modes systems with two-photon driving. In the one mode system, the relationship between the Schrödinger cat states and the system parameters is derived. We observe that in the presence of single-photon decay the steady states would be a mixture of Schrödinger cats. The dynamics and steady states of such a cat versus single-photon decay are examined. In the two coupled-modes cases with linear and nonlinear couplings, the dynamics of entanglement and mutual information are examined with two different initial states and single-photon decay. Compared to the linear coupling case, more complicated structure appears in the Wigner function in the nonlinear coupling case. The joint quadrature distributions are also explored. Such nonclassical states can be used not only in exploring the boundary between the classical and the quantum worlds but also in quantum metrology and quantum information processing.

I. INTRODUCTION

Schrödinger cat state has captured plenty of attention since it was first proposed by Schrödinger in the early days of quantum mechanics. Such kinds of macroscopic superposition states have been explored in various systems. For example, it has been studied in an ensemble of two-level atoms coupled with a cavity mode 1 and observed in superconducting quantum interference devices 2-4. In quantum Ising model, topological defects can be put in a non-local superposition—a topological Schrödinger cat state 5. And in qubit-oscillator systems, a scheme based on conditional displacement is proposed to create Schrödinger cat states by coupling the quantum bit and harmonic oscillator 6. Not only two-component Schrödinger cat states but also multicomponent Schrödinger kittens can be generated 7-9. Besides, by designed double-photon processes, a Schrödinger cat state can be confined in a box 10.

In quantum optics, coherent state is a state that is very close to macroscopic state 11. Thus the superposition of two coherent states \(|\beta_1\rangle\) and \(|\beta_2\rangle\) (not overlapping) can be regarded as the Schrödinger cat state, i.e., \(|\phi_{cat}\rangle = p_1|\beta_1\rangle + p_2|\beta_2\rangle\), \(|p_1|^2 + |p_2|^2 = 1\) where \(p_1, p_2, \beta_1\) and \(\beta_2\) are all complex. Usually the superpositions of two coherent states out of 180° in phase are particularly interesting, and the superpositions of more than two coherent states would termed as Schrödinger kittens 12. Since the cat states can be composed of coherent states in quantum optics, several works have been devoted to explore the properties of these states such as squeezing, photon anti-bunching and non-Poisson distribution 13-21. The Schrödinger cat states can be produced in double-photon driven-dissipative system 22, where the dissipation has negative effect on preparing these states 23. Further more, in the presence of single- and double-photon absorption and emission, exact stationary solutions for the diagonal elements of the master equation has been found 24.

In this work, we explore how the Schrödinger cat states emerge in the double-photon pumping and absorbing (decay) processes with single-photon decay in one- and two coupled-modes systems. Such cat states can be regarded as a result of competition between the double-photon generation and destruction 24. Without single-photon decay, the steady state would be a cat state with parity identical to the initial one. When the initial state is a superposition of Fock states with opposite parity, the steady state would be a weighted mixture of even and odd Schrödinger cats, and the weight of each component is determined by the initial condition. Such a relation may be revealed by constructing conserved quantities 25. While intriguing properties emerge in the double-photon driven-dissipative process, single-photon decay is usually inevitable. Further investigation shows that the decay not only diminishes the negative interference fringes in the Wigner functions but also leads to linearly decrease of the photon number in the steady state. These however do not affect the generation of the cat state with small single-photon decay rate, and the double-photon driving can prolong the lifetime of the optical cat state.

By extending the one-mode model to two coupled-modes with linear and nonlinear couplings, the dynamical behaviors of entanglement and mutual information for two types of initial states are examined in the presence of single-photon decay. Single-photon decay not only leads to the vanishing of the negative regions in the Wigner functions. The entanglement and mutual information are

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also suppressed by the single-photon decay. Compared to
the linear coupling case, entanglement and mutual infor-
mation in the nonlinear coupling case are robust against
the single-photon decay. And multicomponent cat states
appear with sub-Planck phase interference structure in
the phase space \cite{27, 28}. Besides being used in explor-
ing the boundary between the classical and the quantum
worlds, the cat states can be used in quantum metrology
\cite{27, 34} and quantum information processing \cite{34–38}.

This work is organized as follows: in Sec. II we in-
troduce a model with double-photon driven-dissipative
process for one mode system. Then the dynamics of av-
\begin{equation}
\rho = \frac{\hbar}{2} \left( 1 + \hat{a} \hat{a} \right) \rho + \frac{G_a}{2} \left( \hat{a} \hat{a} \rho \right) + \frac{G_a}{2} \left( \rho \hat{a} \hat{a} \right) - \frac{\eta_a}{2} \left( \hat{a} \hat{a} \rho \hat{a} \hat{a} + \rho \hat{a} \hat{a} \hat{a} \hat{a} \right),
\end{equation}
where $\Delta_a$ is the pump-cavity detuning ($\Delta_a = 0 \$L aefollow-
\begin{equation}
\rho_{ss} = p_+ |\alpha + \rangle \langle \alpha + | + p_- |\alpha - \rangle \langle \alpha - | + p_+ |\alpha - \rangle \langle | - \rangle |\alpha + \rangle,
\end{equation}
where $|\alpha\pm\rangle = N_{\pm}^{-1} (|\alpha\rangle \pm | - \alpha\rangle)$ are the even and odd
Schrödinger cat states, respectively, and $N_{\pm}$ are the
normalization constant. We obtain the critical parameter
$\alpha$ in (3) satisfying,
\begin{equation}
\alpha = \pm \sqrt{\frac{G_a}{\eta_a - U_a}},
\end{equation}
where $|\alpha\rangle$ and $| - \alpha\rangle$ are components for the Schrödinger
cat states. Thus the desired Schrödinger cat states may
be obtained by adjusting the parameters of the system.
Usually large $\alpha$ is desirable since the nonclassical sig-
natures would be more obvious for Schrödinger cat states
with large average photon number.

We will use the Wigner function to characterize non-
classical states in the phase space \cite{39}, which can be
reconstructed by homodyne tomography technique \cite{40}.

The appearance of negative value of Wigner function
is the necessary condition to indicate the appearance of
nonclassical state \cite{11}. It is defined as $W(\alpha) = \frac{\hbar}{2} \left( 1 + \hat{a} \hat{a} \right) \rho + \frac{G_a}{2} \left( \hat{a} \hat{a} \rho \right) + \frac{G_a}{2} \left( \rho \hat{a} \hat{a} \right) - \frac{\eta_a}{2} \left( \hat{a} \hat{a} \rho \hat{a} \hat{a} + \rho \hat{a} \hat{a} \hat{a} \hat{a} \right),$

\section{II. ONE-MODE SYSTEM}

In this case, we consider the situation that the cavity
mode with double-photon driving is coupled to a Marko-

\begin{equation}
\rho = \frac{1}{2} \left[ \rho + \rho_{\gamma} \right] + \frac{1}{2} \left[ \rho - \rho_{\gamma} \right],
\end{equation}
where $\rho_{\gamma}$ is the single-photon dissipation and
$\eta$ is the necessary condition to indicate the appearance
of nonclassical states. It is defined as
\begin{equation}
W(\alpha) = \frac{1}{2} \left[ \rho + \rho_{\gamma} \right] + \frac{1}{2} \left[ \rho - \rho_{\gamma} \right],
\end{equation}
where $\rho_{\gamma}$ is the single-photon dissipation and
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function is reconstructed by experimental data [11]. For the case of its delicate mathematical structure makes it hard to be classicality of a states can be witnessed by P-function, operator. Different from the P-function, Wigner function and \( \hat{\alpha} \) are even and odd Schrödinger cat states and the subscripts denote even and odd Schrödinger cat states. From another point of view, the coherence of the Schrödinger cat state can also be identified by homodyne detection [13, 14]. Such a distribution for a state density \( \rho \) can be written as

\[
\begin{aligned}
\rho(\alpha) &= \frac{1}{\pi} \text{Tr}[\rho \hat{D}_\alpha \hat{D}_\alpha^\dagger], \quad \hat{D}_\alpha = e^{i\alpha \hat{a}^\dagger - \alpha^* \hat{a}}
\end{aligned}
\]

where \( \hat{D}_\alpha \) is the displacement operator and \( \hat{P} = e^{i\pi \hat{a}^\dagger \hat{a}} \) is the photon parity operator. Different from the P-function, Wigner function is a quantum analog of the classical Liouville phase space probability function [11]. Even though the nonclassicality of a states can be witnessd by P-function, its delicate mathematical structure makes it hard to be reconstructed by experimental data [11]. For the case of even and odd Schrödinger cat states \( |\xi\pm\rangle \), the Wigner function is

\[
W(\alpha) = \frac{2}{\pi N_{\alpha}(o)} \left( e^{-2|\alpha - \xi|^2} + e^{-2|\alpha + \xi|^2} + 2e^{-2|\alpha|^2}(-1)^{\mu(o)} \cos[4Im(\alpha^*\xi)] \right)\]

(5)

Here \( N_{\alpha}(o) = (1+(-1)^{\mu(o)} e^{-2|\xi|^2} ) \) are the constants for even and odd Schrödinger cat states and the subscripts \( e \) and \( o \) denote even and odd Schrödinger cat states. From another point of view, the coherence of the Schrödinger cat state can also be identified by the interference fringes in quadrature distributions which can be identified by homodyne detection [13, 14]. Such a distribution for a state density \( \rho \) can be written as

\[
\mathcal{P}(X) = \langle X, \phi | \rho | X, \phi \rangle\]

(6)

where \( |X, \phi\rangle \) is an eigenstate of the quadrature operator

\[
\frac{1}{\sqrt{2}}(\hat{a}^\dagger e^{-i\phi} + \hat{a} e^{i\phi})
\]

Based on the relation

\[
\langle X, \phi | n \rangle = (2^n n! \sqrt{\pi})^{-1/2} H_n(X) e^{-X^2/2} e^{-in\phi},
\]

we gain the quadrature distribution of a state, here \( H_n(X) \) are the Hermite polynomials of order \( n \). Interference patterns in the quadrature distributions is also a signature of nonclassical properties.

State space which is composed of even Fock states does not link to the one composed of odd Fock states by double-photon driving and decay processes. Resulting from the competition between these two double-photon processes, the initial states with deterministic even or odd parity evolve to the one with the same parity. Here they are even and odd Schrödinger cat states respectively. Whereas if the initial state is a superposition of even and odd Fock states, the steady state would be superimposed of even and odd cat states with different weights. Another case is the statistical mixture of even and odd Fock states in which the ratio of the even and odd cat states in the eigenspace of the steady state would be identical to the initial statistical one. This results from the fact that the states in even space and odd space develop separately without single-photon decay or driving process. We display the above arguments by four examples in Fig. [4]

**B. Double-photon driven-dissipative process in the presence of single-photon decay**

One-photon decay leads to mutual exchange of the even and odd Schrödinger cat state since \( \hat{a}|\alpha\pm\rangle = \alpha|\alpha\mp\rangle \) which diminishes the interference fringes in the Wigner functions. The vanishing of the negative regions corresponds to fading away of the interference patterns in

![Figure 1](image1.png)

Figure 1. (a)-(d) show the Wigner functions for the steady states with the initial states, \(|0\rangle\langle 0|, |1\rangle\langle 1|, 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)\) and \(1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)\) respectively. (a) Even Schrödinger cat state; (b) Odd Schrödinger cat state; (c) Superposition of even and odd Schrödinger cat states with different ratios; (d) Statistical mixture of even and odd Schrödinger cat states; The other parameters are \( \Delta = 0, U_0/\eta = 1, G_0/\eta_0 = 10e^{-i\pi/4} \) and \( \gamma_0 = 0 \).

![Figure 2](image2.png)

Figure 2. (a): Quadrature distributions and the corresponding Wigner functions for the steady states \( \rho_{ss} \) without single-photon decay. We have chosen the phase in quadrature operator \( \phi = 0 \). The initial states are \(|0\rangle \) (even) and \(|1\rangle \) (odd) respectively. (b): The quadrature distributions and the corresponding Wigner functions for the steady states when the initial states are \( 1/\sqrt{2}(|0\rangle + |1\rangle) \) for the single-photon decay rates being \( \kappa_0 = 0 \) and \( \kappa_0/\eta_0 = 0.1 \). The asymmetry of the quadrature distribution corresponds to that of Wigner functions. The other parameters are same to those in Fig. [4].
The entropy can be defined as \( \log(\rho) \) reflecting some information of the initial state. The entropy provides more information to dissect the evolution process and it also provides more in-information to the quadrature distributions. With increasing of the single-photon decay strength, the negative stripe in the Wigner functions and interference fringes in the quadrature tends to fade away. And it can be seen that with the initial state \( |\sqrt{2}(0) + |1\rangle \), the steady state become asymmetry in phase space and quadrature distributions.

Compared with the evolution of average photon number \( \langle \hat{n} \rangle \) and parity \( \langle \hat{P} \rangle \), the entropy provides more information to dissect the evolution process and it also reflects some information of the initial state. The entropy can be defined as \( S(t) = -Tr[\rho(t) \log(\rho(t))] \) where \( Tr[\bullet] \) denotes the trace of \( \bullet \). We plot the transient dynamics for \( \langle \hat{n} \rangle \), \( \langle \hat{P} \rangle \) and the entropy \( S(t) \) in Fig. 3 for several single-photon decay strengths with two initial states. It can be seen that while the single-photon decay is weak, the entropy would retrace after it drops from the first peak. This behavior can be interpreted as the competition between the double-photon and single-photon processes, i.e., double-photon process dominates the entropy-decline behavior. But with time going by, as long as there is nonvanishing single-photon decay, the entropy rises and converges to an asymptotic value in long time scale. Such an asymptotic value depends on the initial condition. While the initial state is a vacuum state \( |0\rangle \), the asymptotic value would be close to \( \log(2) \) in the presence of single-photon decay, the decay of a two-component statistical mixture in the eigenspace of the steady state. Whereas if the initial state is \( |0\rangle + |1\rangle \), the asymptotical value would be less than \( \log(2) \). The evolution trajectory for entropy are easier to distinguish than that of \( \langle \hat{P} \rangle \) and \( \langle \hat{n} \rangle \) for different \( \gamma \) which may be employed to reflect the strength of single-photon decay.

Even though the Schrödinger cat states vanish in the steady state of single-photon decay, it can still appear during the evolution. And the life time of a cat state can be prolonged by increasing the double-photon dissipation compared to single-photon decay. The appearance of the Schrödinger cat state in the presence of single-photon decay is shown in Fig. 4. As shown in this figure, when \( \gamma_a \lesssim \eta_a \), the fidelity can reach more than 0.5 during the evolution. The snap plots of the Wigner functions in Fig. 4 indicate the appearance of the cat states. Even though the cat states can be generated during the evolution, single-photon decay leads the fidelity approaching 0.5 in long time scale. Here the states may evolve to the statistical mixture of even and odd Schrödinger cat states with identical weights in the eigenspace. This can be confirmed by the purity defined as \( Tr[\rho^2] \) when \( |0\rangle + |1\rangle \). When there is single-photon decay (\( \gamma_a > 0 \)), the purity converges to 0.5 in long time scale. As mentioned above, the entropy converges to \( \log(2) \) implying that there are mainly two independent components(or cats) with the same weights in the steady state. These signatures suggest that the steady state is a statistical mixture of odd and even Schrödinger cat states.

Then we check the influence of single-photon decay strength to the steady state in detail. The steady state
Figure 5. The influence of the single-photon decay $\gamma_a$ to ‘$a$’ in the steady state. Here ‘$a$’ denotes the ‘size’ of the Schrödinger cat state. The other parameters are same to those in Fig. 1.

Numerical results in Fig. 5 indicate that the modulus of $\alpha$ in the Schrödinger cat states decreases linearly as $\gamma_a$ increases. But the influence to the modulus of $\alpha$ is more obvious than that to the angle of $\alpha$. This means larger $\gamma_a$ leads to more degree of reduction of the average photon number in the steady state since $\langle \pm |\hat{n}| \pm \alpha \rangle = |\alpha|^2$.

III. TWO COUPLED-MODES SYSTEM

Based on the one-mode system discussed above, we now consider the two coupled-modes. Two cases with different kinds of couplings are considered in following, namely the case with linear coupling and the one with nonlinear coupling. These two cases can be expressed as

$$\begin{align*}
\text{linear} & : g_l(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \\
\text{nonlinear} & : g_{nl}(\hat{a}^3 + \hat{a}^\dagger \hat{b}^2),
\end{align*}$$

where $g_{l(nl)}$ denotes the linear (nonlinear) coupling strength between the two modes. In linear coupling case, we consider two identical systems except for the phase of double-photon driving terms which influence the distributions of the Schrödinger cat states in phase space (as [4] shows). The nonlinear case is similar to the quantum description for second harmonic generation which has been considered in [4]. We investigate the Schrödinger cat states in the coupled modes and correlation between the modes characterized by entanglement and mutual information. Both entanglement and mutual information reflect the correlation between the coupled systems merely through different aspects. They play important roles in quantum information theory [43, 44]. The sketches for the two coupled modes are shown in Fig. 6.

A. Influence of single-photon decay in the case with linear coupling

First, we consider linear couplings between the two modes as shown in Fig. 6. In this case, the evolution for the total system is described by

$$\begin{align*}
\partial_t \rho & = -i[\hat{H}_a + \hat{H}_b, \rho] - ig_l[\hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b}], \\
& + (L^{(1)}_a + L^{(2)}_a + L^{(1)}_b + L^{(2)}_b)\rho.
\end{align*}$$

Here $\rho$ is the density matrix for the total system. $\hat{H}_a$ and $\hat{H}_b$ are Hamiltonians for the two coupled modes. In this case, not only coherence behavior but also correlation between modes are worth examining. For the one mode system, the nonclassical character can be described by negative value of the Wigner functions. For the two coupled-modes, however, the entanglement and mutual information may be interesting to be examined. Since the appearance of negative eigenvalues for the partial transposed density matrix of a compound system is the necessary condition for nonseparability [43, 44], the absolute value for the summation of the negative eigenvalues of the partial transposed density matrix can be used as the entanglement indicator, i.e., the $|E^T| = \sum_{\lambda_i<0} |\lambda_i|$; where $\lambda_i$ denotes the negative eigenvalues of the partial.
transposed density matrix. Besides, quantum mutual information, or von Neumann mutual information provide us with a way to characterize the information of a variable ‘A’ in one system by exploring its partner ‘B’ in another system [43,44]. It can be regarded as the quantum version of Shannon mutual information defined as 
\[ I = S_A + S_B - S_{AB} \]
where \( S_{A(B)} \) is the entropy for \( A(B) \) mode and \( S_{AB} \) is that for the compound bipartite system. For comparison, we calculate and plot the dynamics for the mutual information \( I \) along with the entanglement indicator \( |E^T| \) in Fig. 6 for three examples. In the presence of single-photon decay in one of the coupled modes, the larger coupling strength leads to sharp changes in \( |E^T| \). This can be interpreted as fast exchange of correlations between the two modes caused by larger couplings. This corresponds to single-photon decay caused destruction of nonclassical characters and the sharp destruction of the entanglement. The stronger the coupling is, the sharper the change of the mutual information. Similarly, strong couplings lead to depression of the entanglement at long time scale. Sum up, single-photon decay has negative effect on the entanglement since \( |E^T| \) decreases sharply when the single-photon decay is nonvanishing. But the mutual information behaves robustly against this decay in the case of linear coupling.

To explore the effect of single-photon decay on the coherence, we display the Wigner functions for the coupled modes at scaled time \( \eta_t = 2 \) in Fig. 8. It can be seen that while there is single-photon decay in one of the coupled modes, the strong linear couplings lead to severe destruction to the negative parts in the Wigner functions. When the single-photon decay exists in both modes, the negative values in Wigner functions disappear at long time scales.

### B. Influence of single-photon decay in the case with nonlinear coupling

In the case with nonlinear couplings, we consider a toy model in which the \( a \)-mode acts as the double-photon driving for another mode (\( b \)-mode, i.e. the sub-converted mode) with double-photon decay. The sketch of such a model is shown in Fig. 6. The dynamics is governed by the equation,

\[
\partial_t \rho = -i[H_a + H_b, \rho] - ig_{nl}[^{a}1^{b}2^{2} + ^{a}1^{b}2^{2} , \rho] \\
+ (\mathcal{L}_{a}^{(1)} + \mathcal{L}_{a}^{(2)} + \mathcal{L}_{b}^{(1)} + \mathcal{L}_{b}^{(2)}) \rho, \\
\tilde{H}_b = -\Delta b^{\dagger}b^{\dagger} + \frac{U_b}{2} b^{\dagger}b^{\dagger}.
\]

This can be seen as the cascaded double-photon down conversion scheme. In this case, the entanglement \( |E^T| \), mutual information \( I \), and nonclassical signatures behave differently with respect to those in the linear coupling case. We study the influence of single-photon decay in both modes to \( |E^T| \), \( I \), and coherence by three examples, and the results are shown in Fig. 8 and Fig. 10. When single-photon decay exists in \( a \)-mode, with the increasing of nonlinear coupling strength \( g_{nl} \), \( |E^T| \) would decrease but \( I \) is enhanced. When single-photon decay is added in \( b \)-mode (\( \gamma_b \neq 0 \)), the \( |E^T| \) decreases obviously but the mutual information is robust against this decay.

Figure 8. \( (a_1), (a_2) \) and \( (a_3) \): The Wigner functions for \( a \)-mode in the case with linear couplings at time \( \eta_t = 2 \). \( (b_1), (b_2) \) and \( (b_3) \): The Wigner functions for \( b \)-mode corresponding to those in \( (a_1), (a_2) \) and \( (a_3) \) respectively. The other parameters in both modes are the same as those in Fig. 7.
Figure 9. Transient dynamics for $|E_T|$ and mutual information $I$ in the case with nonlinear couplings. We set $\gamma_a/\gamma_a=0.5$ in this case. The other parameters are the same as those in Fig. 1. The initial state is $|0\rangle$ for both modes.

In Fig. 10 intriguing nonclassical signature emerges in the $b$-mode when there is only double-photon decay in this mode. To reveal the intriguing patterns in both $a$ and $b$-mode, we calculate their components and show the results in Fig. 11. It can be seen that the state of $a$-mode is a superposition of even and odd Schrödinger cat states. Whereas $b$-mode is superimposed by two four-component cat-like states. The scars in the Wigner functions indicate that qubits could be encoded by these states. E.g., the qubits here can be encoded as $|\alpha_i\rangle_i \rightarrow |0\rangle_i$ and $|-\alpha_i\rangle_i \rightarrow |1\rangle_i$, where $i$ denotes the mode index and $|\pm \alpha_i\rangle$ are the coherent states. The Schrödinger cat states might find potential applications in quantum metrology and may be valuable in efficient storage and communication for quantum information processing. This is due to the sensitivity of the states to the structure change in phase space, for instance, rotation and shift.

The intriguing patterns in $b$-mode may be influenced by $a$-mode though coupling. In the following, we will check the Wigner functions of $a$-mode and $b$-mode at time $\eta_a t = 2$ by varying a set of parameters in $a$-mode in Fig. 12. We find that the double-photon driving or photon self-interaction in $a$-mode is necessary for the emergence of intriguing checkerboard patterns in $b$-mode. In turn, the pattern in $b$-mode depends on the parameters of $a$-mode. This may provide us a method to estimate system parameters by examining the coupled two modes.

C. Schrödinger cat states in the two-mode system with two different initial states

We calculate and show the dynamics of entanglement $|E_T|$ and mutual information $I$ in the two modes system for two different initial states in Fig. 13. Compared to those in Fig. 7 and Fig. 9, we find that with both linear and nonlinear couplings, $|E_T|$ and $I$ have been suppressed. However, with nonlinear coupling, the entanglement has been suppressed more obviously compared to those in Fig. 9, while the mutual information is robust in this case.

Next we explore the change of coherence for two different initial states. Compared to Fig. 8 and Fig. 10 we examine the Wigner functions at time $\eta_a t = 2$ in Fig. 14 with the same parameters as in Fig. 13. The negative region in the Wigner functions approximately disappears in $a$-mode and small area with negative value of Wigner function left in $b$-mode in the linear coupling case. The influence on $b$-mode distribution is similar to that in the one-mode case as shown in Fig. 2, the distribution becomes asymmetric in phase space. The distribution for $a$-mode has also become asymmetric. With nonlinear couplings, compared to those in Fig. (b) and (bb), the interference fringes in Fig. (a2) and (b2) also changes obviously in $b$-mode more than that in $a$-mode. This may result from that the nonlinear coupling terms act like single-photon decay for $a$-mode. Compared to the symmetric pattern in the Wigner function (see Fig. 10), the asymmetric signatures in Fig. (b2) too, resemble that in the one-mode case.
Figure 11. (a), (a1) and (a2) : The Wigner functions for \( a \)-mode and its two major components in its eigenspace at time \( \eta_a t = 3 \) with parameters as the same as in Fig. 10(b). (b), (b1) and (b2) are those for \( b \)-mode with the same parameters as that in Fig. 10(bb). The decimals on the panels are the ratios for the components in the eigenspaces. The summation of major ratios are more than 0.99.

D. The joint quadrature distribution in the two coupled-modes cases

In the two coupled-modes case, the joint quadrature distribution defined by \( \langle X_a, X_b | \rho | X_a, X_b \rangle \) provides with the other aspects to witness the nonclassical characters of the cat states. Here \( X_a \) and \( X_b \) are the quadrature operators of \( a \)-mode and \( b \)-mode in (6) with \( \phi = 0 \) and \( \rho \) being the total density matrix of the coupled system. We show the Wigner functions and the corresponding joint quadrature distributions at two different times in the linear and nonlinear coupling cases in Fig.15 and Fig.16 respectively. While there is negative regions in the Wigner functions, interference fringes also appear in the joint quadrature distributions. When the Wigner function is symmetric (asymmetric), the joint quadrature distribution is also symmetric (asymmetric). Considering the interference fringes in the quadrature distributions in one-mode case, we can see that measurement of the quadrature \( X_a \) may project the system to a Schrödinger cat state. The joint quadrature distribution has been used to explore Einstein-Podolsky-Rosen paradox, see Refs. [50–54]. The purity trends to 0.25 which hints that \( a \)- and \( b \)-mode approaches to a statistical mix-

Figure 12. In (a)-(f) we set: \( \gamma_a = (0.5,0.5,0.5,0.5,0.5,0), \) \( \eta_a = (1,1,0,1,0,1,1), \) \( U_a = (1,1,1,1,0,1), \) \( U_b = 1, \Delta_b = 0, \eta_b = 1, \) \( \gamma_b = 0, g_{nl} = 1. \) Here we choose \( |G_a|/10 \) as the energy units since both \( \eta_a = 0 \) and \( U_a = 0 \) situations are examined in these examples. The insets are the Wigner functions for the corresponding \( a \)-modes with the same color map to \( b \)-modes.

Figure 13. (a1) and (b1) show the entanglement \( |E^{T}_{\text{linear}}| \) and mutual information \( I_{\text{linear}} \) respectively in the case with linear couplings when \( \gamma_b/\eta_a = 0 \) and \( g_{nl}/\eta_a = 0.5. \) (a2) and (b2) are those in the case with nonlinear couplings when \( \gamma_b/\eta_a = 0 \) and \( g_{nl}/\eta_a = 0.5. \) The initial state for \( b \)-mode are taken to be \( |\psi\rangle = |0\rangle + |1\rangle \) \( \rangle \) while it is the vacuum state for \( a \)-mode. The other parameters are the same as those in Fig. 8 (a2), (b2) and Fig. 10 (b1), (bb) for the case with linear couplings and Fig. 10 (b), (bb) for the case with nonlinear couplings respectively.
Figure 14. (a₁) and (b₁) show the Wigner functions for \(a\)-mode and \(b\)-mode in the linear coupling case at time \(\eta_a t = 2\). (a₂) and (b₂) are the Wigner functions in the nonlinear coupling case. The initial states and parameters in (a₁), (a₂), (b₁) and (b₂) are the same as those in Fig. 13. In the linear coupling case, the two peaks in the Wigner function in \(a\)-mode are asymmetric which are more obvious than that in the nonlinear coupling case.

Figure 15. (a) and (b) are the evolution of the entanglement indicator \(|E^T|\) and the purity with linear couplings for different initial states (solid curve: \(|\phi_b\rangle = |0\rangle\); dashed curve: \(|\phi_b\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\). (c₁) and (c₂) are the Wigner functions for \(b\)-mode and the corresponding joint quadrature distributions at time \(\eta_a t = 0.4\) when \(|\phi_b\rangle = |0\rangle\). (d₁) and (d₂) are the Wigner functions for \(b\)-mode and the corresponding joint quadrature distributions at time \(\eta_a t = 2\). The insets are those with initial states of \(b\)-mode: \(|\phi_b\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\) as shown in (a) and (b) while \(|\phi_a\rangle = |0\rangle\). We have set \(\gamma_a/\eta_a = \gamma_b/\eta_a = 0.1\) and \(g_l/\eta_a = 1\) and the other parameters are the same as those in Fig. 7.

IV. SUMMARY

In summary, we have explored the Schrödinger cat states in two-photon driven dissipative systems of one mode and two cases of coupled-modes. In one mode case, the steady state is a superposition of coherent states, the weight of each component in the state is determined by the system parameters. Compared to average photon number and parity, entropy can characterize the dynamics more precisely. Single-photon decay leads to vanishing of the interference fringes in the quadrature distributions which corresponds to the disappearance of negative regions in the Wigner functions. But the Schrödinger cat states can still emerge in the presence of weak single-photon decay. The single-photon decay slightly decreases the modulus of these coherent states in the Schrödinger cats when the system reaches the steady state. Whereas the double-photon dissipation can prolong the lifetime of the Schrödinger cats.

In the two-mode systems with both linear and nonlinear couplings, single-photon decay suppresses the entanglement at long time scales. And this deleterious effect of the single-photon decay on the entanglement is severer in the linear coupling case than that in the nonlinear coupling case. However the coupling benefits the mutual information, and intriguing nonclassical features appear in the low frequency mode in the nonlinear coupling case. But they are quite fragile against the single-photon decay. Such cat states not only provide us with candidates to explore the boundary between quantum and classical world, but also can be applied in precision measurement and quantum information processing.

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Figure 16. (a) and (b) are the evolution of $|E^\gamma_t|$ and the purity in the case with nonlinear coupling for different initial states (solid curve: $|\phi_0\rangle = |0\rangle$; dashed curve: $|\phi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$). $(c_1)$ and $(c_2)$ are the Wigner functions for $a$-mode and the corresponding joint quadrature distributions at time $\eta=0.4$ when $|\phi_0\rangle = |0\rangle$. $(d_1)$ and $(d_2)$ are the Wigner functions for $b$-mode and the corresponding joint quadrature distributions at time $\eta=2$. The insets are those with initial states of $|\phi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ as shown in (a) and (b). We have set $\gamma_0/\eta_0=\gamma_0/\eta_0=0.1$ and $g_{a_1}/g_{a_1}=0.5$ and the other parameters are the same as those in Fig. 9.

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