APPROACH TO EQUILIBRIUM OF A BODY COLLIDING SPECULARLY AND DIFFUSELY WITH A SEA OF PARTICLES

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Abstract. We consider a rigid body acted upon by two forces, a constant force and the collective force of interaction with a continuum of particles. We assume that some of the particles that collide with the body reflect elastically (specularly), while others reflect probabilistically with some probability distribution $K$. We find that the rate of approach of the body to equilibrium is $O(t^{-3-p})$ in three dimensions where $p$ can take any value from 0 to 2, depending on $K$.

Contents

1. Introduction 1
1.1. Acknowledgements 3
2. Force on the Body 4
2.1. Force and Flux 4
2.2. Boundary Conditions 6
2.3. Total Force on the Body 9
3. Iteration Scheme 10
3.1. The Iteration Family 10
3.2. Assumptions on $K$ and $f_0$ 12
4. Examples of Collision Kernels 12
5. Main Estimates of the Force 14
5.1. The Right Side 14
5.2. The Left Side 16
5.3. Force Due to Precollisions 17
6. Motion of the Body 18
7. Proof of Existence and Asymptotic Behavior 20
References 21

1. Introduction

The problem that we are considering has a free boundary, the location of the body. The other unknown is the configuration of the particles. The particles may collide with the body elastically or inelastically. Boundary interactions in kinetic theory are very poorly understood, even when the boundaries are fixed. Free boundaries are even more difficult. For this reason we have chosen to consider only the simplest problem of this type, namely, we assume the particles are identical and are rarefied, that is, do not interact among themselves but only with the body. We assume that the whole system, consisting of the body and the particles, starts out rather close to an equilibrium state.

We consider classical particles that are extremely numerous. While one could consider modeling them as a fluid, we instead model them as a continuum like in kinetic (Boltzmann, Vlasov) theory [7] but without any self-interaction. Our focus is on the interaction of the particles with the body at its boundary. In typical physical scenarios this interaction is poorly understood. For instance, the boundary may be so rough that a particle may reflect from it in an essentially random way. There could even be some kind of physical or chemical reaction between the particle and the molecules of the body.

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The present paper treats a problem similar to the series of remarkable papers \[4, 3, 1\] and uses similar methods. In each of these papers the initial velocity \(V_0\) of the body is near its terminal (equilibrium) velocity \(V_\infty\), and the body moves in only one spatial direction (to the right). In \[4\] and \[3\] all the collisions are purely specular. The body’s initial velocity satisfies \(V_0 < V_\infty\) in \[4\], while \(V_0 > V_\infty\) in \[3\]. The latter case is significantly different. In \[1\] the collisions are purely diffusive with the collision kernel \(K(v, u) = C|u_x|e^{-\beta v^2}\) where \(C\) and \(\beta\) are constants. This kernel implies that all the colliding particles are emitted with the same Maxwellian distribution. In the present paper we generalize the boundary behavior to permit a mixture of specular and diffusive reflections. The diffusion part is much more general than in \[4\].

In \[4\] and \[3\] the rate of approach of the velocity to equilibrium is \(O(t^{-d-2})\) in \(d\) spatial dimensions. At first glance it is somewhat surprising that the rate is slower than exponential. This relatively slow rate is due to some particles colliding with the body multiple times over long time periods, which produces a frictional effect on the body that may be called a long tail memory. In \[4\] the rate is slower, namely, \(O(t^{-d-1})\), because the number of collisions is greater due to the diffuse reflections. In the present paper we find various rates of approach depending on the specific law of reflection. We find the rate \(O(t^{-d-\rho})\) where \(\rho\) can take any value from 0 to 2.

In physically realistic situations, many more effects must be included, such as thermal effects, collisions among the particles themselves, or electromagnetic effects. In a plasma the particles are usually modeled kinetically, as for instance the reentry of a space vehicle into the atmosphere. Another way to model particles that interact with a body would be to treat them as a classical fluid. For a general discussion on fluid-structure interaction, see \[5\]. Somewhat related to this paper is the piston problem, where the body is a piston moving back and forth in a finite channel \[6\] and naturally reaching an equilibrium state. However, the piston problem is different primarily because the particles reflect at the ends of the channel and collide an infinite number of times, rather than scattering to infinity. More relevant to this paper are the numerical computations in \[2, 9\], which corroborate the power-law asymptotic behavior for the diffuse boundary conditions of \[1\]. In \[5\] a general convex body, moving horizontally, is considered, and the results are similar to \[4\].

To be specific, here we consider the following problem. The body is a cylinder \(\Omega(t) \subset \mathbb{R}^d\). We write \(x = (x, x_\perp), x_\perp \in \mathbb{R}^{d-1}\). The cylinder is parallel to the \(x\)-axis and the body is constrained to move only in the \(x\) direction with velocity \(V(t)\). There is a constant horizontal force \(E > 0\) acting on the body, as well as the horizontal force \(F(t)\) due to all the colliding particles at time \(t\). Thus

\[
\frac{dX}{dt} = V(t), \quad \frac{dV}{dt} = E - F(t),
\]

In the fictitious situation that none of the particles collide more than once with the body, their collective force on the body is denoted as \(F_0(V)\). (See Lemma \[2, 5\]) Then the equilibrium velocity would be \(V_\infty\), where \(F_0(V_\infty) = E\).

We write the velocity of a particle as \(v = (v_x, v_\perp)\), where \(v_x = v \cdot \hat{1}\) is the horizontal component and \(v_\perp \in \mathbb{R}^{d-1}\). The particle distribution, denoted by \(f(t, x, v)\), satisfies \(\partial_t f + v \cdot \nabla_x f = 0\) in \(\Omega(t)\). We assume the initial velocity \(f(0, x, v) = f_0(v)\) depends only on \(v\) and is even in \(v_\perp\). We also denote the densities before and after a collision with the body by \(f_{\pm}(t, x, v) = \lim_{\epsilon \to 0^+} f(t \pm \epsilon, x \pm \epsilon v, v)\). The assumed law of reflection at the two ends of the cylinder is

\[
f_{+}(t, x, v) = \alpha f_{-}(t, x, 2V(t) - v_x, v_\perp) + (1 - \alpha) \int_{|v - v_x| < V(t)} K(v - iV(t), u - iV(t)) f_{-}(t, x, u) du,
\]

where \(\hat{1}\) is the unit vector in the \(x\)-direction and \(\alpha \in [0, 1]\). The collision kernel \(K\) is assumed to satisfy the conservation of mass condition \(2, 9\). Furthermore, \(K\) and the initial density \(f_0\) satisfy Assumptions A1-A5 in Section 3. Among these conditions are

\[
K(v, u) = k(v_x, u_x)b(u_\perp), \quad c|u_x|^p \leq \int_{v_x \geq 0} v_x^2 k(v_x, u_x) dv_x \leq C|u_x|^p
\]

for some constants \(c, C, p\) and some function \(b(u_\perp)\) where \(0 \leq p \leq 2\). A symmetry assumption implies that the net force on the lateral boundary vanishes (Lemma \[2, 5\]).

**Theorem 1.1.** Given a collision kernel and the initial data \(f_0\) as above. If \(\gamma = V_\infty - V_0\) is sufficiently small and positive, then there exists a solution \((V(t), f(t, x, v))\) of our problem in the following sense.
$V \in C^1(\mathbb{R})$ and $f_\pm \in L^\infty$ for $t \in [0, \infty), x \in \partial \Omega(t), v \in \mathbb{R}^3$, where the force $F(t)$ on the cylinder is given by \(2.3\) and the pair of functions $f_\pm(t, x, v)$ are (almost everywhere) defined explicitly in terms of $V(t)$ and $f_0(x, v)$.

Uniqueness is an open problem, as in \([4,5,6]\).

**Theorem 1.2.** Every solution of the problem (in the sense stated above) satisfies the estimates

$$\gamma e^{-B_\infty t} + \frac{c_p^{p+1} \lambda(2t_0, \infty)(t)}{t^{d+p}} \leq V_\infty - V(t) \leq \gamma e^{-B_0 t} + \frac{C\gamma^{p+1}}{(1 + t)^{d+p}},$$

where $B_0 = \min_{V \in [V_0, V_\infty]} F_0^\prime(V) > 0$ and $B_\infty = \max_{V \in [V_0, V_\infty]} F_0^\prime(V) < \infty$, for some $s$ depending on $\gamma$ and some positive constants $c, C$.

In Section 2 we derive the basic formulas for the total force on a body due to its interaction with the particles. This is done directly from basic principles in a more organized way than in \([1]\). In terms of the boundary conditions the total force is given in Lemma 2.7. We assume that particles are not created or annihilated at the boundary (conservation of mass) (Lemmas 2.3 and 2.4). Under either of two lateral boundary conditions, the force on the lateral side of the body can be ignored (Lemmas 2.5 and 2.6).

Section 3 introduces a family $W$ of possible body motions $W$, in terms of two functions $g(t)$ and $h(t)$ which we determine later. We write the force due to the possible motion $W$ as $F(t) = F_0(t) + R_W(t)$, where $R_W(t)$ is the force due to the collisions occurring before time $t$ (“precollisions”) if the body were to move with velocity $V(\cdot)$. Then $W$ generates a new possible motion $V_W$ by the equation

$$\frac{dV_W}{dt} = E - F_0(t) - R_W(t).$$

The goal is to prove that the mapping $W \rightarrow V_W$ has a fixed point. At the end of the section we list all the assumptions on the collision kernel $K(v, u)$ and the initial state $f_0(v)$ of the particles, stated in as general a form as feasible.

In Section 4 we provide several examples of collision kernels for the two ends of the cylinder. Example 1 is the same gaussian collision law $k(v_x, u_x) = C|u_x| \exp(-\beta v_x^2)$ considered in \([1]\), where the particles comprise a perfect gas in thermal equilibrium. In that paper the exponent $p = 1$ and the authors assume that $V_\infty$ is sufficiently large without specifying how large. We provide an explicit condition \((4.3)\) on the size of $V_\infty$. We also provide an alternative condition \((4.2)\) on the shape of the gaussian that is independent of $V_\infty$.

Example 2 is more interesting. The kernel is $k(v_x, u_x) = C \exp(-v_x^2/|u_x|)$ and the value of $p$ is $\frac{3}{2}$. This means, that for a particle colliding at an incoming velocity $u_x$ close to that of the body $V(t)$, its outgoing velocity $v_x$ upon reflection is given by a narrow gaussian and so is likely to be not very changed. Thus the particles that are almost grazing are deviated only slightly. On the other hand, if $u_x$ is quite far from $V(t)$, that is if the collision is more fierce, the particle’s velocity upon reflection is given by a very wide gaussian and so is likely to take almost any value. It seems that this may be a more realistic scattering scenario than the one in Example 1.

Example 3 proposes a family of kernels that generalize both previous examples, permitting any $p \in [0, 2]$. Example 4 shows that there is no requirement that the kernel is an exponential; all that is needed is some polynomial decay.

Section 5 is devoted to our main estimates on how the particles collectively generate a force on the body. Because $E > 0$, there is a difference between the left and the right sides. The bounds on the force employ mainly the first precollisions. The most important conclusion is that $R_W(t) \geq 0$. The estimates are summarized in Lemma \(5.3\).

In Section 6 we apply the estimates of the force $R_W$ to the body’s motion using \((1.2)\). The functions $g(t)$ and $h(t)$ in the definition of $W$ can then be chosen. Finally in Section 7 we deduce the existence theorem, Theorem \(1.1\), by a fixed point argument that iterates the upper bound of $R_W(t)$, thus taking account of all the precollisions. The asymptotic theorem, Theorem \(1.2\), valid for any solution, follows easily.

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2. Force on the Body

2.1. Force and Flux. In this section, we derive the formula of the force on the body assuming conservation of mass. Let \( \Omega(t) \) be the moving body at time \( t \), \( \Omega^c(t) \) its complement (where the particles are located), and \( \mathbf{v} = (v_x, v_\perp) \) the velocity of a particle. We normalize the body’s mass and the particle density to be 1. Then the mass (the total number of particles) is

\[
M(t) = \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} f(t, \mathbf{x}, \mathbf{v}).
\]

Conservation of mass means that the flux across the boundary vanishes; that is,

\[
\int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} \left( \mathbf{v} - \mathbf{V}(t) \right) \cdot \mathbf{n} f(t, \mathbf{x}, \mathbf{v}) = 0,
\]

where \( \partial \Omega(t) \) is the boundary of \( \Omega(t) \), \( \mathbf{n} = \mathbf{n}_x \) is the outward normal on \( \partial \Omega(t) \), and \( \mathbf{V} = (V_x, V_\perp) \) is the velocity of the body. In Section 2.2 we will find boundary conditions so that (2.1) is valid.

**Lemma 2.1.** Assuming conservation of mass (2.1), the horizontal component of the force on the body is given by the formula

\[
F(t) = \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} \left[ \mathbf{v} - \mathbf{V}(t) \right] \cdot \mathbf{n} (v_x - V_x) f(t, \mathbf{x}, \mathbf{v}).
\]

**Proof.** The change of horizontal momentum of the gas and the moving solid together is given by

\[
\frac{d}{dt} \left\{ \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} v_x f(t, \mathbf{x}, \mathbf{v}) + V(t) \right\} = E
\]

Hence \( F(t) \) is

\[
F(t) = \frac{d}{dt} \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} v_x f(t, \mathbf{x}, \mathbf{v})
\]

\[
= \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} (\mathbf{V}(t) \cdot \mathbf{n}) v_x f(t, \mathbf{x}, \mathbf{v}) + \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} v_x \partial_t f(t, \mathbf{x}, \mathbf{v})
\]

\[
= \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} (\mathbf{V}(t) \cdot \mathbf{n}) v_x f(t, \mathbf{x}, \mathbf{v}) - \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} v_x \mathbf{v} \cdot \nabla f(t, \mathbf{x}, \mathbf{v})
\]

\[
= \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} (\mathbf{V}(t) \cdot \mathbf{n}) v_x f(t, \mathbf{x}, \mathbf{v}) + \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} (\mathbf{v} \cdot \mathbf{n}) v_x f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}
\]

\[
= \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} \left[ \mathbf{v} - \mathbf{V}(t) \right] \cdot \mathbf{n} v_x f(t, \mathbf{x}, \mathbf{v}).
\]

Via conservation of mass (2.1), we can write

\[
F(t) = \frac{d}{dt} \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} v_x f(t, \mathbf{x}, \mathbf{v}) - V_x \frac{d}{dt} \int_{\Omega^c(t)} d\mathbf{x} \int_{\mathbb{R}^3} d\mathbf{v} f(t, \mathbf{x}, \mathbf{v})
\]

\[
= \int_{\partial \Omega(t)} dS_x \int_{\mathbb{R}^3} d\mathbf{v} \left[ \mathbf{v} - \mathbf{V}(t) \right] \cdot \mathbf{n} (v_x - V_x) f(t, \mathbf{x}, \mathbf{v}).
\]

\[\square\]

**Lemma 2.2.** If we specialize \( \Omega(t) \) to a cylinder centered at \((X(t), 0, 0)\) with its circular base perpendicular to the \(x\)-axis and moving only horizontally, i.e. \( \mathbf{V} = (V, 0, 0) \), then the horizontal force \( F \) is given by

\[
F(t) = F_{L,R}(t) + F_S(t),
\]

where \( F_{L,R}(t) \) is the contribution from both \( \partial \Omega_L(t) \) and \( \partial \Omega_R(t) \), the left and right ends of the cylinder, and \( F_S(t) \) is the contribution from \( \partial \Omega_S(t) \), the lateral side of the cylinder. Written explicitly in terms of
the incident and reflected particles, they are

\[
F_{L,R}(t) = \int_{\partial \Omega_n(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f_-(t, x, v) \\
+ \int_{\partial \Omega_n(t)} dS_x \int_{v_x > V(t)} dv[v_x - V(t)]^2 f_+(t, x, v) \\
- \int_{\partial \Omega_L(t)} dS_x \int_{v_x > V(t)} dv[v_x - V(t)]^2 f_-(t, x, v) \\
- \int_{\partial \Omega_L(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f_+(t, x, v)
\]

and

\[
F_S(t) = \int_{\partial \Omega_s(t)} dS_x \int_{v_n \leq 0} dv (v \cdot n) (v_x - V) f_-(t, x, v) \\
+ \int_{\partial \Omega_s(t)} dS_x \int_{v_n > 0} dv (v \cdot n) (v_x - V) f_+(t, x, v).
\]

**Proof.** At the right side of the cylinder, \( n = (1, 0, 0) \) and \( V = (V, 0, 0) \). So for the right end, we get from (2.2) the term

\[
\int_{\partial \Omega_n(t)} dS_x \int_{v_n \leq 0} dv [(v - V(t)) \cdot n] (v_x - V) f(t, x, v) \\
= \int_{\partial \Omega_n(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f(t, x, v).
\]

Splitting the above integral into its absorbed (incident) and emitted (scattered) parts as

\[
f(t, x, v) = f_-(t, x, v) \chi (\{v_x \leq V(t)\}) + f_+(t, x, v) \chi (\{v_x > V(t)\}),
\]

we have

\[
\int_{\partial \Omega_n(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f_-(t, x, v)
\]

\[
= \int_{\partial \Omega_n(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f_- (t, x, v) \\
+ \int_{\partial \Omega_n(t)} dS_x \int_{v_x > V(t)} dv[v_x - V(t)]^2 f_+ (t, x, v).
\]

Similarly, at the left end of the cylinder we have

\[
\int_{\partial \Omega_L(t)} dS_x \int_{v_n \leq 0} dv [(v - V(t)) \cdot n] (v_x - V) f(t, x, v)
\]

\[
= - \int_{\partial \Omega_L(t)} dS_x \int_{v_x \leq V(t)} dv[v_x - V(t)]^2 f_- (t, x, v) \\
- \int_{\partial \Omega_L(t)} dS_x \int_{v_x > V(t)} dv[v_x - V(t)]^2 f_+ (t, x, v).
\]

Adding (2.5) and (2.6) gives \( F_{L,R}(t) \).

At the lateral boundary we have

\[
F_S(t) = \int_{\partial \Omega_s(t)} dS_x \int_{v_n \leq 0} dv (v \cdot n) (v_x - V) f(t, x, v) \\
= \int_{\partial \Omega_s(t)} dS_x \int_{v_n \leq 0} dv (v \cdot n) (v_x - V) f(t, x, v).
\]

Here the sign of \( v \cdot n \) indicates the incident and scattered particles, namely,

\[
f(t, x, v) = f_-(t, x, v) \chi (\{v \cdot n \leq 0\}) + f_+(t, x, v) \chi (\{v \cdot n > 0\}),
\]
so that
\[ F_\Sigma(t) = \int_{\partial\Omega_\Sigma(t)} dS_x \int_{V_n \leq 0} dv \cdot n (v_x - V) f_-(t, x, v) + \int_{\partial\Omega_\Sigma(t)} dS_x \int_{V_n \geq 0} dv \cdot n (v_x - V) f_+(t, x, v), \]
which is exactly \((2.4)\).

2.2. Boundary Conditions. In this section, we introduce boundary conditions for the scattering of the particles which satisfy conservation of mass \([2.1]\). There are three boundaries we are considering, namely, \(\partial\Omega_R(t), \partial\Omega_L(t), \partial\Omega_\Sigma(t)\). We will consider first \(\partial\Omega_R(t)\) and \(\partial\Omega_L(t)\), then \(\partial\Omega_\Sigma(t)\).

2.2.1. Boundary Conditions at the Two Ends. On \(\partial\Omega_R(t)\), the right circular base of the cylinder, for \(v_x \geq V(t)\), we assume the boundary condition
\[
 f_+(t, x; v) = \alpha f_-(t, x; 2V(t) - v_x, v_\perp) + (1 - \alpha) \int_{u_x \leq V(t)} K(v - iV(t); u - iV(t)) f_-(t, x; u) du, \quad (2.7)
\]
Similarly, on \(\partial\Omega_L(t)\), the left circular base of the cylinder, for \(v_x \leq V(t)\), we assume the boundary condition
\[
 f_+(t, x; v) = \alpha f_-(t, x; 2V(t) - v_x, v_\perp) + (1 - \alpha) \int_{u_x \geq V(t)} K(v - iV(t); u - iV(t)) f_-(t, x; u) du. \quad (2.8)
\]
We assume \(\alpha \in [0, 1]\) since we are interested in the mixed boundary condition, part specular and part diffusing. We would like to have the same law of reflection on both circular ends of the cylinder so we assume that the kernel \(K(v, u)\) is nonnegative and is even in both \(u_x\) and \(v_x\) separately, namely,
\[
 K(v_x, v_\perp; u_x, u_\perp) = K(-v_x, v_\perp; u_x, u_\perp) = K(v_x, v_\perp; -u_x, u_\perp), \quad \forall u = (u_x, u_\perp), v = (v_x, v_\perp) \in \mathbb{R}^3.
\]

Lemma 2.3. If the collision kernel \(K(v, u)\) satisfies
\[
 \int_{v_x \geq 0} v_x K(v, u) dv = |u_x|, \quad (2.9)
\]
then across both ends of the cylinder the mass is conserved. This means that
\[
 \int_{\partial\Omega_R(t)} dS_x \int_{\mathbb{R}^3} dv \cdot n (v - V(t)) f(t, x, v) = \int_{\partial\Omega_L(t)} dS_x \int_{\mathbb{R}^3} dv \cdot n (v - V(t)) f(t, x, v) = 0. \quad (2.10)
\]
Proof. Because it is very well known that specular reflection preserves mass, we only consider the diffusing term in \((2.7)\) and \((2.8)\). The proof is the same on the left and the right. Consider the right end. Splitting the integral \((2.10)\) at the right end into its absorbed and emitted parts, we have
\[
 f(t, x, v) = f_-(t, x, v) \chi \{\{v_x \leq V_x(t)\}\} + f_+(t, x, v) \chi \{\{v_x > V_x(t)\}\},
\]
and
\[
 \int_{\partial\Omega_R(t)} dS_x \int_{\mathbb{R}^3} dv \cdot n f(t, x, v) = \int_{\partial\Omega_R(t)} dS_x \int_{v_x \leq V_x(t)} dv \cdot n f_-(t, x, v) + \int_{\partial\Omega_R(t)} dS_x \int_{v_x > V_x(t)} dv \cdot n f_+(t, x, v) = I + II.
\]
Plugging the boundary condition into \(II\), we have
\[
 II = \int_{\partial\Omega_R(t)} dS_x \int_{v_x \geq V_x(t)} dv \cdot n f_-(t, x, u) \int_{u_x \leq V_x(t)} K(v - iV_x(t), u - iV_x(t)) f_-(t, x, u) du
\]
\[
 = \int_{\partial\Omega_R(t)} dS_x \int_{u_x \leq V_x(t)} duf_-(t, x, u) \int_{v_x > V_x(t)} dv \cdot n f_+(t, x, v) \int_{v_x < V_x(t)} K(v - iV_x(t), u - iV_x(t)) f_-(t, x, u) du.
\]
By \((2.9)\) we have
\[
 \int_{v_x > V_x(t)} dv \cdot n f_+(t, x, v) K(v - iV_x(t), u - iV_x(t)) = -(u_x - V_x(t))
\]
Thus the particles that collide with the lateral side must have moved in a straight line from
we can put

Then

Thus we conclude that

\[
\int_{\partial\Omega(t)} dS_x \int_{R^3} dv |(v - V(t))| f_\ast(t, x, v) = 0.
\]

\[\square\]

2.2.2. Boundary Conditions on the Lateral Boundary. As with the ends, we impose a linear combination
of specular and diffusing boundary conditions on \(\partial\Omega_S\). Let \(n_x\) be the outward normal and let \(T_x\)
be the circular tangential direction at \(x \in \partial\Omega_S\). We assume on \(\partial\Omega_S\) the boundary condition

\[
f_\ast(t, x; v) = \alpha_S f_\ast(t, x; u_x, v \cdot T_x, -v \cdot n_x) + (1 - \alpha_S) \int_{u_n \leq 0} K_S (v; u) f_\ast(t, x; u) du (2.11)
\]

for \(v \cdot n_x \leq 0\), where \(K_S \geq 0\) is the lateral collision kernel and \(\alpha_L \in [0, 1]\). We assume

\[K_S (v_x, v_L; u_x, u_L) = K_S (\pm v_x, v_L; \pm u_x, u_L)\]

since we want the same reflection law for the particles coming from the left and the right. (We require
neither the same kernel nor the same \(\alpha\) as at the ends.) Notice that the horizontal speed \(V(t)\) of the
body does not enter the lateral boundary condition. See Subsection 2.2.3 for a different condition.

Because the body moves only horizontally, no particle can collide on the lateral side more than once.
Thus the particles that collide with the lateral side must have moved in a straight line from \(t = 0\). So
we can put \(f_0\) in place of \(f_\ast\) in (2.11), that is,

\[
f_\ast(t, x; v) = \alpha_S f_0(v_x, v \cdot T_x, -v \cdot n_x) + (1 - \alpha_S) \int_{u_n \leq 0} K_S (v; u) f_0(u) du. (2.12)
\]

Lemma 2.4. If the lateral collision kernel \(K_S\) satisfies

\[
\int_{v \cdot n_x \geq 0} (v \cdot n_x) K_S (v; u) dv = |u \cdot n_x| \chi_{u_n \leq 0}, (2.13)
\]

then the mass is conserved across the lateral boundary. That is,

\[
\int_{\partial\Omega_S(t)} dS_x \int_{R^3} dv (v \cdot n_x) f(t, x, v) = 0.
\]

Proof. This is a direct computation. We write

\[
\begin{align*}
\int_{\partial\Omega_S(t)} dS_x \int_{R^3} dv (v \cdot n_x) f(t, x, v) & = \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv (v \cdot n_x) f_\ast(t, x, v) + \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x > 0} dv (v \cdot n_x) f_\ast(t, x, v) \\
& = I + II.
\end{align*}
\]

Then

\[
II = \alpha_S \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x \geq 0} dv (v \cdot n_x) f_\ast(v_x, v \cdot T_x, -v \cdot n_x)
\]

\[+ (1 - \alpha_S) \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x \geq 0} dv (v \cdot n_x) \int_{u_n \leq 0} du K_S (v; u) f_0(u)\]

\[= -\alpha_S \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv (v \cdot n_x) f_\ast(v)\]

\[+ (1 - \alpha_S) \int_{\partial\Omega_S(t)} dS_x \int_{u_n \leq 0} du f_0(u) \left( \int_{v \cdot n_x > 0} dv (v \cdot n_x) K_S (v; u) \right)\]

\[= -\alpha_S \int_{\partial\Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv (v \cdot n_x) f_\ast(v)\]

\[- (1 - \alpha_S) \int_{\partial\Omega_S(t)} dS_x \int_{u_n \leq 0} du (u \cdot n_x) f_0(u) = -I\]

by (2.12). \[\square\]
Lemma 2.5. If the lateral collision kernel $K_S$ satisfies (2.13), then the contribution $F_S(t)$ to the horizontal force from the lateral boundary vanishes.

Proof. Recalling (2.3), we have

$$F_S(t) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \leq 0} dV (v \cdot n) (v_x - V) f(t, x, v).$$

Under (2.12), $f_-$ and $f_+$ are both independent of $V$, so that

$$\frac{\partial F_S(t)}{\partial V} = - \int_{\partial \Omega_S(t)} dS_x \int_{v, x \geq 0} dV (v \cdot n) f(t, x, v) = 0$$

by mass conservation. That is, we can put $V = 0$ when we compute $F_S(t)$. So

$$F_S(t) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \leq 0} dV (v \cdot n) v_x f_0(v)
+ \alpha_S \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \geq 0} dV (v \cdot n) v_x f_0(v_x, v \cdot T_x, -v \cdot n_x)
+ (1 - \alpha_S) \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n_x \geq 0} dV (v \cdot n_x) v_x \int_{u \cdot n_x \leq 0} du K_S (v; u) f_0(u)$$

$$= I + II + III.$$ 

By the assumption that $f_0$ is even in $v_x$, we have

$$\int_{\mathbb{R}} v_x f_0(v) dv_x = 0.$$ 

Thus both $I$ and $II$ are 0. Notice that $K_S (v_x, v_{x \perp}; u_x, u_{x \perp}) = K_S (-v_x, v_{x \perp}; u_x, u_{x \perp})$, we have

$$\int_{\mathbb{R}} v_x K_S (v; u) dv_x = 0$$

so that $III = 0$. □

2.2.3. Alternative Boundary Conditions on the Lateral Boundary. Assume on $\partial \Omega_S$ that

$$f_+(t, x; v) = \int_{u : n_x \leq 0} K_S (v - iV(t); u - iV(t)) f_-(t, x; u) du,$$ \hspace{1cm} (2.14)

where

$$K_S (v_x, v_{x \perp}; u_x, u_{x \perp}) = K_S (\pm v_x, v_{x \perp}; \pm u_x, u_{x \perp}) \geq 0.$$ 

For convenience, we have dropped the specular part since $V(t)$ is unrelated to the specular reflections on $\partial \Omega_S$. A special case of boundary condition (2.14) was studied in [1]. We still have conservation of mass if we assume (2.13) for $K_S$. However, for the alternative boundary condition (2.14) the force does not vanish, as we now show.

Lemma 2.6. Under boundary condition (2.14), the lateral force

$$F_S(t) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \leq 0} dV (v \cdot n_x) (v_x - V(t)) f_0(v)$$ \hspace{1cm} (2.15)

is a nonnegative function depending solely on $V(t)$. It satisfies $\frac{\partial F_S}{\partial V} \geq 0$.

Proof. Again there is no recollision on $\partial \Omega_S$, so we have

$$F_S(t) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \leq 0} dV (v \cdot n_x) (v_x - V) f_0(v)
+ \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \geq 0} dV (v \cdot n_x) (v_x - V) \int_{u \cdot n_x \leq 0} du K_S (v - iV(t); u - iV(t)) f_0(u) du$$

$$= I + II.$$ 

A change of variable of $v_x$ gives

$$II = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n \geq 0} dV (v \cdot n_x) v_x \int_{u \cdot n_x \leq 0} du K_S (v; u - iV(t)) f_0(u) du = 0.$$
because $K_S(v,u)$ is even in $v_x$. Thus $F_S(t) = G(V(t))$ where
\[
G(V) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv \ (v \cdot n_x) (v_x - V) f_0(v).
\]
Now
\[
G(0) = \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv \ (v \cdot n_x) v_x f_0(v) = 0
\]
and since $\int_{\mathbb{R}} v_x f_0(v) dv = 0$ and
\[
G'(V) = - \int_{\partial \Omega_S(t)} dS_x \int_{v \cdot n_x \leq 0} dv \ (v \cdot n_x) f_0(v) \geq 0.
\]
So $G \geq 0$, which means $F_S(t) \geq 0$.

No matter whether we take (2.11) or (2.14) as the boundary condition on $\partial \Omega_S$, $F_S(t)$ is a nonnegative function which depends solely on $V(t)$. So we write $F_S(t)$ as $F_S(V(t))$ from here on.

2.3. Total Force on the Body. We now use the boundary conditions to write the force explicitly and succinctly in terms of $f_-(t,x,v)$.

**Lemma 2.7.** The force is given in terms of $f_-(t,x,v)$ as
\[
F(t) = F_S(V(t)) + \int_{\partial \Omega_R(t) \cup \partial \Omega_L(t)} \ dS_x \int_{v \in \mathbb{R}^3} dv \ 2 v_x (v_x - V_x) f_-(t,x,v),
\]
where the nonnegative function $F_S(V(t))$ either vanishes or is given by (2.15) depending on the choice of boundary condition on $\partial \Omega_i$, and
\[
\ell(w) = (1 + \alpha)u_x^2 + (1 - \alpha) \int_{v_x \geq 0} dv \ v_x^2 K(v,w).
\]

Of course, the last integral could also be taken over $\{v_x \leq 0\}$ due to the evenness of the kernel.

**Proof.** Recall that $F_{L,R}(t)$ is
\[
F_{L,R}(t) = \int_{\partial \Omega_R(t)} dS_x \int_{v_x \leq V_x(t)} dv \ (v_x - V_x(t))^2 f_-(t,x,v)
+ \int_{\partial \Omega_L(t)} dS_x \int_{v_x \geq V_x(t)} dv \ (v_x - V_x(t))^2 f_+(t,x,v)
- \int_{\partial \Omega_L(t)} dS_x \int_{v_x \geq V_x(t)} dv \ (v_x - V_x(t))^2 f_-(t,x,v)
- \int_{\partial \Omega_R(t)} dS_x \int_{v_x \leq V_x(t)} dv \ (v_x - V_x(t))^2 f_+(t,x,v)
\]
Plugging in the boundary conditions (2.7) and (2.8), it becomes
\[
F_{L,R}(t) = \int_{\partial \Omega_R(t)} dS_x \int_{v_x \leq V_x(t)} dv \ (V(t) - v_x)^2 f_-(t,x,v)
+ \alpha \int_{\partial \Omega_R(t)} dS_x \int_{v_x \geq V_x(t)} dv \ (v_x - V(t))^2 f_+(t,x;2V(t) - v_x,v_{\perp})
+(1 - \alpha) \int_{\partial \Omega_R(t)} dS_x \int_{v_x \geq V_x(t)} dv \ (v_x - V(t))^2 \int_{v_x \leq V(t)} du K(v - iV(t);u - iV(t)) f_-(t,x,u)
- \alpha \int_{\partial \Omega_L(t)} dS_x \int_{v_x \geq V_x(t)} dv \ (v_x - V(t))^2 f_+(t,x;2V(t) - v_x,v_{\perp})
-(1 - \alpha) \int_{\partial \Omega_L(t)} dS_x \int_{v_x \leq V_x(t)} dv \ (v_x - V(t))^2 \int_{v_x \geq V(t)} du K(v - iV(t);u - iV(t)) f_-(t,x,u)
= \int_{\partial \Omega_R(t)} dS_x \int_{v_x \leq V_x(t)} dv \ \ell(v - iV(t)) f_-(t,x,v)
- \int_{\partial \Omega_L(t)} dS_x \int_{v_x \geq V_x(t)} dv \ \ell(v - iV(t)) f_-(t,x,v),
\]
where $\ell(w)$ is defined in (2.16).
2.3.1. *Force without Recollisions.* Putting the initial density \( f_0(v) \) at the place of \( f_-(t,x;v) \) in formula (217), we get the *fictitious force*
\[
F_0(V) = F_S(V) + C \left( \int_{v_x < V} \ell(v - iV)f_0(v)dv - \int_{v_x > V} \ell(v - iV)f_0(v)dv \right),
\]
where \( C \) is the area of the ends of the cylinder. This is the force on the cylinder if all the collisions occurring before time \( t \) were ignored. The basic properties of the fictitious force are stated in the next lemma.

**Lemma 2.8.** Suppose \( f_0(v) \geq 0 \) is even, continuous and \( \not= 0 \). If \( \ell \in C^1 \) and \( \partial_{w_x}\ell(w) < 0 \) for \( w_x \in (-\infty, 0) \), then \( F_0(V) \) is a positive, increasing \( C^1 \) function of \( V \).

**Proof.** First we have
\[
F_0(V) = F_S(V) + C \left( \int_{v_x < V} \ell(v - iV)f_0(v)dv - \int_{v_x > V} \ell(v - iV)f_0(v)dv \right)
\]
\[
\geq C\int_{v_x < -V} (\ell(v - iV) - \ell(v + iV)) f_0(v)dv > 0,
\]
because \( v_x - V \leq v_x + V \) and \( \partial_{w_x}\ell(w) < 0 \) for \( w_x \in (-\infty, 0) \). Using the monotonicity of \( \ell \) again, we deduce the monotonicity of \( F_0(V) \) by
\[
F_0'(V) = -\int_{v \cdot n \leq 0} dv (v \cdot n) f_0(v)
\]
\[
+ C \left( -\int_{v_x \leq V} (\partial_{w_x}\ell)(v - iV)f_0(v)dv - \int_{v_x \leq -V} (\partial_{w_x}\ell)(v + iV)f_0(v)dv \right) > 0.
\]

\[ \square \]

3. Iteration Scheme

**Definition 1.** We define \( \mathcal{W} \) as the family of functions \( W \) that satisfy the following conditions.

(i) \( W : [0, \infty) \to \mathbb{R} \) is Lipschitz and \( W(0) = V_0 \).

(ii) \( W \) is strictly increasing over the interval \( [0, t_0] \) for some \( t_0 \) depending on \( \gamma = V_\infty - V_0 \).

(iii) There exist bounded functions \( h(t) = h(t, \gamma) \) and \( g(t) = g(t, \gamma) \) such that for all \( t, \gamma \in [0, \infty) \) and \( \gamma \in (0, 1) \),
\[
0 < \gamma h(t, \gamma) \leq V_\infty - W(t) < \gamma g(t, \gamma).
\]

We do not assume that \( W(t) \) is increasing in \( [t_0, \infty) \). Specific choices for the functions \( g \) and \( h \) will be made later. For any function \( Y : [0, \infty) \to \mathbb{R} \), we denote its average over time intervals by
\[
\langle Y \rangle_{s,t} = \frac{1}{t-s} \int_s^t Y(\tau) d\tau, \quad \langle Y \rangle_{0,t} = \langle Y \rangle_t.
\]

Thus \( Y \in L^1(\mathbb{R}) \) implies \( \langle Y \rangle_t = O(1/t) \). The family \( \mathcal{W} = \{W\} \) has the following properties.

**Lemma 3.1.** If \( \langle h \rangle_t \geq \gamma g(t) \) for all \( t \geq t_0 \), then

(i) \( W(t) > \langle W \rangle_t, \forall t > 0 \).

(ii) \( \{W\}_t \) is an increasing function. In particular, \( V_0 \leq \langle W \rangle_t \leq W(t) \leq V_\infty \).

(iii) \( \langle W \rangle_{s,t} > \langle W \rangle_t, \forall s \in (0, t) \).

(iv) \( \gamma [\langle h \rangle_t - \gamma g(t)] \chi(t \geq t_0) \leq W(t) - \langle W \rangle_t \leq \gamma g(t) \).
Proof. When \( t \leq t_0 \), (i) follows from the assumption that \( W \) is increasing. When \( t \geq t_0 \), we have

\[
W(t) - \langle W \rangle_t = \frac{1}{t} \int_0^t [(V_\infty - W(\tau)) - (V_\infty - W(t))] \, d\tau
\]

\[
\geq \frac{1}{t} \int_0^t [\gamma h(\tau) - \gamma g(t)] \, d\tau
\]

\[
\geq \gamma \left( \frac{1}{t} \int_0^t h(\tau) \, d\tau - g(t) \right) > 0
\]

by assumption. This proves (i) and part of (iv). Now

\[
\frac{d}{dt} \langle W \rangle_t = \frac{1}{t} (-\langle W \rangle_t + W(t)) > 0
\]

by (i). Thus (ii) is true. Moreover,

\[
\langle W \rangle_{s,t} - \langle W \rangle_t = \frac{1}{t-s} \int_s^t W(\tau) \, d\tau - \frac{1}{t} \int_0^t W(\tau) \, d\tau
\]

\[
= \frac{1}{t-s} \int_0^t W(\tau) \, d\tau - \frac{1}{t} \int_0^s W(\tau) \, d\tau - \frac{1}{t} \int_s^t W(\tau) \, d\tau
\]

\[
= \frac{s}{t-s} \left( \frac{1}{t} \int_0^t W(\tau) \, d\tau - \frac{1}{s} \int_0^s W(\tau) \, d\tau \right) > 0
\]

by (ii). Finally,

\[
W(t) - \langle W \rangle_t = \frac{1}{t} \int_0^t [(V_\infty - W(\tau)) - (V_\infty - W(t))] \, d\tau
\]

\[
\leq \frac{1}{t} \int_0^t [\gamma g(\tau) - \gamma h(t)] \, d\tau \leq \gamma \frac{1}{t} \int_0^t g(\tau) \, d\tau.
\]

The key step in the proof of the theorems will be to prove that \( r^R_W(t) + r^L_W(t) \geq 0 \) where we define

\[
r^R_W(t) = \int_{\partial \Omega_R(t)} dS \int_{u_x \in W(t)} du \, \ell(u - iW(t)) \{ f_-(t, x, u) - f_0(u) \},
\]

\[
r^L_W(t) = \int_{\partial \Omega_L(t)} dS \int_{u_x \geq W(t)} du \, \ell(u - iW(t)) \{ f_-(t, x, u) - f_0(u) \}.
\]

They represent the forces on the right and left of the cylinder due to the precollisions, that is, all the collisions occurring before time \( t \). This will be accomplished via a lower bound of \( r^R_W(t) \) (Lemma [5.1]) and an upper bound of \( r^L_W(t) \) (Lemma [5.2]). Then we will be able to determine \( g \) and \( h \) via the requirement that \( W = \{W\} \) is closed under the map \( W \rightarrow V_W \).

Before beginning the detailed estimates, we consider the meaning of a precollision. In order for a particle to have collisions at two times \( t \) and \( s \) with \( s < t \), it is obviously required that

\[
\int_s^t v(\tau) \, d\tau = \int_s^t W(\tau) \, d\tau.
\]

In order to have no collisions in between \( s \) and \( t \), it is necessary that

\[
(t-s)v_x = \int_s^t W(\tau) \, d\tau, \quad |v_x| \leq \frac{2r}{t-s},
\]

where \( r \) is the radius of the cylinder. Since \( \langle W \rangle_{s,t} \) is a continuous function of \( s \) for any \( t \), the existence of a precollision at some time earlier than \( t \) requires that

\[
v_x \in \left[ \inf_{s \leq t} \langle W \rangle_{s,t}, \sup_{s \leq t} \langle W \rangle_{s,t} \right] = \left[ \langle W \rangle_t, \sup_{s \leq t} \langle W \rangle_{s,t} \right], \quad (3.2)
\]

\[
|v_x| \leq \frac{2r}{t-s}.
\]

We will estimate \( r^R_W(t) \) and \( r^L_W(t) \) by taking only one precollision into account.
3.2. Assumptions on $K$ and $f_0$. We make the following assumptions on the collision kernel $K$ for
the ends of the cylinder and on the initial particle density $f_0$, in addition to the previously stated assumptions
that $K(v, u)$ and $f_0(v)$ are nonnegative, even in $v_x$ and $u_x$, and (2.3) is valid. The first assumption below
implies that at the boundary the momentum is transferred only horizontally.

A1. Let $K$ and $f_0$ have the product form

$$
K(v, u) = k(v_x, u_x) b(v_{\perp}),
$$

with each factor nonnegative and continuous and $f_0$ bounded.

Thus $a_0$ and $k$ are even. Under Assumption A1, $\ell(w)$ actually depends only on $w_x$, that is,

$$
\ell(w) = (1 + \alpha)w_x^2 + (1 - \alpha) \int_{v_x \geq 0} dv_x v_x^2 k(v_x, w_x) = \ell(w_x).
$$

Therefore, at any later time, $f_+ \text{ and } f_-$ must take the product form

$$
f_+(t, x; v) = a_+(t, x; v_x)b(v_{\perp}), \quad f_-(t, x; v) = a_-(t, x; v_x)b(v_{\perp}).
$$

It is then natural to ask whether the analysis is purely one-dimensional. In fact, the dimension does come into play as will be demonstrated in Lemmas 5.1 and 5.2.

A2.

$$
\sup_{v_x \leq \eta} \sup_{v_x} k(v_x, u_x) < \infty.
$$

A3. There is a power $0 \leq p \leq 2$ and there are positive constants $C$ and $c$ such that

$$
e |u_x|^p \leq \int_{v_x \geq 0} v_x^2 k(v_x, u_x) \, dv_x \leq C |u_x|^p
$$

for $u_x \in [-\gamma, 0)$. We also assume that this integral is a $C^1$ function of $u_x$ for $u_x \neq 0$. Note that A3 and

A1 imply that $(1 - \alpha)c|u_x|^p \leq \ell(u_x) \leq (1 + \alpha)u_x^2 + (1 - \alpha)C|u_x|^p$.

A4.

$$
\sup_{v_x \in [-\gamma, 0]} \sup_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_\infty} k(v_x, u_x - \eta)a_0(u_x) \, du_x < \infty.
$$

A5. There exists $\delta > 0$ such that

$$
(1 - \alpha) \inf_{v_x \in [-\gamma, 0]} \inf_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_0} k(v_x, u_x - \eta)a_0(u_x) \, du_x \geq a_0(V_\infty) + \delta.
$$

4. Examples of Collision Kernels

Example 1. The kernel

$$
K(v, u) = C_2 e^{-\beta |v|^2} |u_x|
$$

and the initial density

$$
f_0(v) = C_1 e^{-\beta |v|^2}
$$

were the subject of [1]. They satisfy all the Assumptions A1-A5. The constant $C_2$ is determined so as to
satisfy (2.3). Indeed, it is obvious that they satisfy A1-A4. In order to verify A5, we note that

$$
(1 - \alpha)C_2 \inf_{v_x \in [-\gamma, 0]} \inf_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_0} e^{-\beta u_x^2} |u_x - \eta| e^{-\beta u_x^2} \, du_x = (1 - \alpha)C_2 C^* e^{-\beta \gamma^2},
$$

where

$$
C^* = \inf_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_0} (\eta - u_x) e^{-\beta u_x^2} \, du_x = V_0 \int_{-\infty}^{V_0} e^{-\beta u_x^2} \, du_x + \frac{1}{2\beta} e^{-\beta V_0^2}
$$

depends only on $V_0$ and $\beta$. We may consider $V_\infty$ as fixed and $\gamma$ as small and then $V_0 = V_\infty - \gamma$. From

(4.1) we require $(1 - \alpha)C^* e^{-\beta \gamma^2} > e^{-\beta V_0^2} + \delta$ for some $\delta > 0$ and all sufficiently small $\gamma$. Thus all we require is that

$$
(1 - \alpha)C_2 \left[ V_\infty \int_{-\infty}^{V_\infty} e^{-\beta u_x^2} \, du_x + \frac{1}{2\beta} e^{-\beta V_0^2} \right] > e^{-\beta V_0^2}.
$$
Because of the second term, A5 is true provided
\[ \beta < \frac{1 - \alpha}{2} C_2, \]  
(4.2)
which is a different kind of condition than in [1]. Using instead the first term, we note that
\[ V_\infty \int_{-\infty}^{V_\infty} e^{-\beta u^2} du_x \geq V_\infty \int_{-\infty}^{0} e^{-\beta u^2} dx = V_\infty \sqrt{\frac{\pi}{4\beta}} \]
so that A5 is also satisfied if
\[ V_\infty e^{\beta V^2} > \frac{\sqrt{\pi}}{(1 - \alpha) C_2 \sqrt{\pi}} \]  
(4.3)
In [1] the condition was that \( V_\infty \) be sufficiently large without specifying how large. The inequality (4.3) is a precise condition.

Example 2. Choose
\[ K(v, u) = C_2 e^{-\frac{\beta V^2}{4\pi^2} b(v_\perp)}, \quad f_0(v) = a_0(v_x) b(v_\perp), \]
where once again \( C_2 \) is chosen so that (2.9) is satisfied, \( a_0 \in L^1(\mathbb{R}) \), and \( \int b dv_\perp = 1 \). A2 is easily satisfied, because \( 0 \leq e^{-\frac{\beta V^2}{4\pi^2}} \leq 1 \). Since
\[ C_2 \int_0^{\infty} u_x^2 e^{-\frac{\beta V^2}{4\pi^2}} du_x = C |u_x|^2, \]
A3 is satisfied with \( p = \frac{3}{2} \). We also have
\[ \sup \sup_{v_x} \int_{-\infty}^{V_\infty} e^{-\frac{\beta V^2}{4\pi^2}} a_0(u_x) du_x \leq \int_{\mathbb{R}} a_0(u_x) du_x, \]
which verifies A4. To test A5, we notice that
\[ C_2 (1 - \alpha) \inf_{v_x \in [-\gamma, 0]} \inf_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_0} e^{-\frac{\beta V^2}{4\pi^2}} a_0(u_x) du_x \]
\[ \geq C_2 (1 - \alpha) \inf_{v_x \in [-\gamma, 0]} \inf_{\eta \in [V_0, V_\infty]} \int_{-\infty}^{V_\infty - 1} e^{-\frac{\beta V^2}{4\pi^2}} a_0(u_x) du_x \]
\[ \geq C_2 (1 - \alpha) e^{-\gamma^2} \int_{-\infty}^{V_\infty - 1} a_0(u_x) du_x. \]
Thus if, for small enough \( \gamma \), we have
\[ \int_{-\infty}^{V_\infty - 1} a_0(u_x) du_x > \frac{a_0(V_\infty)}{C_2 (1 - \alpha)}, \]
then A5 is satisfied.

The physical interpretation of such a choice of kernel is the following. Notice that
\[ k(v_x - V(t), u_x - V(t)) = C_2 e^{-\frac{(u_x - V(t))^2}{4\pi^2}}, \]
Thus if \( |u_x - V(t)| \) is big, then there is a wide range of possible emitted velocities. On the other hand, if \( |u_x - V(t)| \) is small, meaning that the incident particle and the body move at almost the same speed, then the same is true for the emitted particles with high probability.

It is then natural to wonder if we can have a family of kernels such that it covers a continuous range of \( p \). This is simply achieved by modifying Example 2.

Example 3. For \( \beta \in [-1, 3] \), consider
\[ K(v, u) = C_2 |u_x|^{\beta} e^{-\frac{\beta V^2}{4\pi^2} b(v_\perp)}, \quad f_0(v) = a_0(v_x) b(v_\perp), \]
where \( C_2 \) is chosen so that (2.9) is satisfied, while \( a_0 \) and \( b \) are as in Example 2. We then have
\[ C_2 |u_x|^{\beta} \int_0^{\infty} u_x^2 e^{-\frac{\beta V^2}{4\pi^2} |u_x|^{\beta - 1}} du_x = C |u_x|^{\frac{\beta - 4}{2}}. \]
Thus \( p \) runs through \([0, 2]\) as \( \beta \) runs through \([-1, 3]\). In particular, if \( \beta = 1 \) we have Example 2. If \( \beta = 0 \) we have Example 3. The same physical interpretation as in Example 2 holds if \( \beta \in [-1, 1] \), meaning that \( p \in (1, 2] \).
It is also natural to inquire whether a gaussian is needed. Actually, it just suffices to have some good decay, as we now illustrate.

Example 4. Let us choose

\[ K(v, u) = C_2 |u_x|^{-N} (v_x)_{t}^{-M}, \quad f_0(v) = (v_x)_{t}^{-P} (v_{\perp})^{-M}, \]

where \( C_2 \) is chosen so that (2.20) is satisfied and \( M > 2, N > 3, P > 2 \). Assumption A1 is true because \( M > 2 \). A2 is obvious. A3 is true because \( N > 3 \). A4 is true because \( P > 2 \). A5 requires

\[ C_2 \int_{-\infty}^{V_0} (V_0 - u_x) |u_x|^{-P} du_x > (\langle V_{\infty} \rangle)^{-P}, \]

which is true for instance if \( V_{\infty} \) is sufficiently large. One can also modify this example to cover a range of \( P \) instead of only \( p = 1 \).

5. Main Estimates of the Force

5.1. The Right Side. In the next lemma we estimate the force on the right side of the cylinder.

Lemma 5.1. Let \( K \) and \( a_q \) satisfy the Assumptions A1-A5. Then for all sufficiently small \( \gamma \) we have the inequalities

\[ \frac{C_\gamma^{p+1} \chi(t \geq t_0)}{t^d-1} \left( (h)_t - g(t) \right)^{p+1} \leq r_W^R(t) \leq C_\gamma^{p+1} (g)^{p+1} (1 + t)^{d-1} + C_\gamma^{p+1} \sup_{\frac{t}{t} \leq \gamma} (g)^{p+1}. \]

We remark that it would seem that the second term in the upper bound of \( r_W^R(t) \) should dominate. Such a statement is actually not true. The first term always acts like \( t^{-(d+p)} \) while the second one is like \( g^{p+1} \). But \( g(t) \) has to act like \( t^{-(d+p)} \) for the sake of the fixed point argument. This will be clarified in the proof of Corollary 6.1.

Proof. To establish upper and lower bounds of \( r_W^R(t) \), we need upper and lower bounds of \( f_+(t, x; v) \). Recall the boundary condition (2.7) on the right of the cylinder

\[ f_+(t, x; v) = \alpha f_-(t, x; 2W(t) - v_x, v_\perp) + (1 - \alpha) \int_{u_x \in W(t)} K(v - iW(t); u - iW(t)) f_-(t, x; u) du. \]

In light of condition (5.2), we denote the precollision characteristic function by

\[ \chi_0(t, u) = \chi \left\{ u : \forall s \in (0, t), \text{ either } u_x \neq (W)_{s, t} \text{ or } |u_x| \geq 2 \frac{r}{t - s} \right\}, \]

\[ \chi_1(t, u) = \chi \left\{ u : \exists s \in (0, t) \text{ s.t. } u_x = (W)_{s, t} \text{ and } |u_x| \geq 2 \frac{r}{t - s} \right\}. \]

In the case the precollisions occurred at a sequence of earlier times \( t_j \to t \), it would follow that \( v_x = W(t) \), so there would be no contribution to the force since \( \ell(0) = 0 \). Thus we can assume that there is a first precollision, that is, a collision that occurs at an earlier time closest to \( t \). In that case let \( \tau \) be the time and \( \xi \) be the position of that first precollision. Of course, \( \tau \) and \( \xi \) depend on \( t, x, u \). We can then write

\[ f_-(t, x; u) = f_+(\tau, \xi; u) \chi_1(t, u) + f_0(u) \chi_0(t, u). \] (5.1)

Plugging (5.1) into the boundary condition, we have

\[ f_+(t, x; v) = \alpha f_-(t, x; 2W(t) - v_x, v_\perp) + (1 - \alpha) \int_{u_x \leq W(t)} K(v - iW(t); u - iW(t)) f_-(t, x; u) du \]

\[ = \alpha \left\{ f_+(\tau, \xi; 2W(t) - v_x, v_\perp) \chi_1(t, 2W(t) - v_x, v_\perp) + f_0(2W(t) - v_x, v_\perp) \chi_0(t, 2W(t) - v_x, v_\perp) \right\} \]

\[ + (1 - \alpha) \int_{u_x \leq W(t)} K(v - iW(t); u - iW(t)) \left\{ f_+(	au, \xi; u) \chi_1(t, u) + f_0(u) \chi_0(t, u) \right\} du. \]

Since the momentum is only transferred horizontally, we can rewrite this formula as

\[ a_+(t, x; v_x) b(v_\perp) \]

\[ = \alpha \left\{ a_+(\tau, \xi; 2W(t) - v_x) b(v_\perp) \chi_1(t, 2W(t) - v_x, v_\perp) + a_0(2W(t) - v_x) b(v_\perp) \chi_0(t, 2W(t) - v_x, v_\perp) \right\} \]

\[ + (1 - \alpha) b(v_\perp) \int_{u_x \leq W(t)} b(v_x - W(t), u_x - W(t)) \left\{ a_+(\tau, \xi; u) b(u_\perp) \chi_1(t, u) + a_0(u_x) b(u_\perp) \chi_0(t, u) \right\} du. \]
We do not divide by $b(v_{\perp})$ on both sides because it could possibly vanish. Now

$$
a_+(t, x; v_{\perp})b(v_{\perp}) \leq \alpha \{ a_+(\tau, \xi; 2W(t) - v_{\perp})b(v_{\perp}) + a_0(2W(t) - v_{\perp})b(v_{\perp}) \}
$$

$$+ b(v_{\perp}) \left\{ \sup_{\tau, u_{\perp} \in \partial \Omega(\tau)} a_+(\tau, \xi; u_{\perp}) \right\} \int_{(W_\tau)^c} k(v_{\perp} - W(t), u_{\perp} - W(t))du_{\perp} \times
$$

$$+ b(v_{\perp}) \int_{\mathbb{R}^n} k(v_{\perp} - W(t), u_{\perp} - W(t))a_0(u_{\perp})du_{\perp}
$$

by (5.2). Since $W(t) - (W_\tau) \leq \gamma = V_\infty - V_0$, we have

$$a_+(t, x; v_{\perp})b(v_{\perp}) \leq \alpha \{ a_+(\tau, \xi; 2W(t) - v_{\perp})b(v_{\perp}) + a_0(2W(t) - v_{\perp})b(v_{\perp}) \}
$$

$$+ b(v_{\perp})C_\gamma \left\{ \sup_{\tau, u_{\perp} \in \partial \Omega(\tau)} a_+(\tau, \xi; u_{\perp}) \right\} + b(v_{\perp})C
$$

by A2 and A4. Hence, taking the supremum over all times $t$, positions $x \in \partial \Omega(t)$ and velocities $v_{\perp} \in \mathbb{R}$, we have

$$b(v_{\perp}) \left\{ \sup_{\tau, u_{\perp} \in \partial \Omega(\tau)} a_+(\tau, \xi; u_{\perp}) \right\} \leq \frac{Cb(v_{\perp})}{1 - \frac{\alpha}{1 - C\gamma}} \leq Cb(v_{\perp}), \quad \text{for } \gamma < \frac{1 - \alpha}{C}.
$$

(5.2)

which is an upper bound for $f_+(t, x; v)$.

In order to get a lower bound of $f_+(t, x; v)$, we use Assumption A5 to deduce, for $V_0 \leq v_{\perp} \leq V_\infty$, that

$$f_+(t, x; v) = a_+(t, x; v_{\perp})b(v_{\perp})
$$

$$\geq (1 - \alpha)b(v_{\perp}) \int_{u_{\perp} \in W(t)} k(v_{\perp} - W(t), u_{\perp} - W(t))a_0(u_{\perp})b(u_{\perp})\chi_0(t, u)du
$$

$$\geq (1 - \alpha)b(v_{\perp}) \int_{\mathbb{R}^n} k(v_{\perp} - W(t), u_{\perp} - W(t))a_0(u_{\perp})du_{\perp}
$$

$$\geq (a_0(V_\infty) + \delta)b(v_{\perp})
$$

by A5.

We are now ready to establish upper and lower bounds of $r_+(t)$. We begin with the crucial lower bound because it is the main reason why $r_+^R + r_+^L \geq 0$. Using the lower bound of $f_+(\tau, \xi, \mathbf{u})$, we get

$$r_+^R(t) = \int_{\partial \Omega(t)} ds_x \int_{u_{\perp} \leq W(t)} \mathbf{u} \cdot (u_{\perp} - W(t)) \{ f_-(t, x, \mathbf{u}) - f_0(\mathbf{u}) \}
$$

$$= \int_{\partial \Omega(t)} ds_x \int_{u_{\perp} \leq W(t)} \mathbf{u} \cdot (u_{\perp} - W(t))[f_+(\tau, \xi, \mathbf{u})\chi_1(t, \mathbf{u}) + f_0(\mathbf{u})\chi_0(t, \mathbf{u}) - f_0(\mathbf{u})]
$$

$$= \int_{\partial \Omega(t)} ds_x \int_{|u_{\perp}| \leq \frac{1}{2} \mathbf{W}_1} du_{\perp} \int_{(W_\tau)^c} k(u_{\perp} - W(t))f_+(\tau, \xi, \mathbf{u}) - f_0(\mathbf{u})
$$

$$\geq \int_{\partial \Omega(t)} ds_x \int_{|u_{\perp}| \leq \frac{1}{2} \mathbf{W}_1} du_{\perp} b(u_{\perp}) \int_{(W_\tau)^c} du \cdot (u_{\perp} - W(t)) (a_0(V_\infty) + \delta - a_0(u_{\perp}))
$$

by (5.3). For small enough $\gamma$, we have by continuity of $a_0$ that

$$a_0(V_\infty) + \delta - a_0(u_{\perp}) \geq \frac{\delta}{2} > 0.
$$

We then deduce via Assumption A3 and Lemma 3.1(iv) that

$$r_+^R(t) \geq C_\gamma \frac{\delta}{2} \int_{|u_{\perp}| \leq \frac{1}{2} \mathbf{W}_1} b(u_{\perp})du_{\perp} \int_{(W_\tau)^c} du \cdot (u_{\perp} - W(t)) \geq C_\gamma \frac{\delta}{2} \frac{(W(t) - (W_\tau)^c)^p}{(1 + t)^{p-1}} \geq 0.$$
Thus
\[ r_W^R (t) \geq \frac{C \delta}{(1 + t)^{d-1}} \gamma^{P+1} ((h)_t - g(t))^{P+1} \]
for \( t \geq t_0 \) and small enough \( \gamma \). This is the desired lower bound of \( r_W^R \).

We now determine an upper bound for \( r_W^R \). Using the upper bound \( \delta \) of \( f_+ (\tau, \xi, u) \) and Lemma 3.1 (iv), we have
\[
 r_W^R (t) = \int_{\Omega(t)} dS_x \int_{u_x \in W(t)} du_x \ell(u_x - W(t)) \{ f_+ (\tau, \xi, u) - f_0(u) \}
\]
\[
= \int_{\Omega(t)} dS_x \int_{W(t)} du_x \int_{[u_x \leq \frac{\gamma}{d} \tau]} du_x \ell(u_x - W(t))(f_+ (\tau, \xi, u) - f_0(u))
\]
\[
\leq C \int_{W(t)} du_x \int_{[u_x \leq \frac{\gamma}{d} \tau]} du_x \ell(u_x - W(t)) b(u_\perp).
\]
We split the integral according to whether \( \tau < t/2 \) or \( \tau \geq t/2 \). Thus
\[
r_W^R (t) \leq \frac{C (W(t) - (W(t))^{P+1})}{(1 + t)^{d-1}} + C \int_{W(t)} du_x \int_{[u_x \leq \frac{\gamma}{d} \tau \geq \frac{t}{2}]} du_x \ell(u_x - W(t)) b(u_\perp)
\]
\[
\leq \frac{C (\gamma \int_{\tau}^t g(\tau) d\tau)^{P+1}}{(1 + t)^{d-1}} + C \int_{W(t)} du_x \int_{[u_x \leq \frac{\gamma}{d} \tau \geq \frac{t}{2}]} du_x \ell(u_x - W(t)) b(u_\perp)
\]
by Assumption A3. For the second term in this estimate, by the precollision condition \( \delta \) we notice that
\[
u_x = W(t) - \frac{1}{1 - \tau} \int_{\tau}^t (W(t) - W(s)) ds \geq W(t) - \frac{1}{1 - \tau} \int_{\tau}^t (V_\infty - W(s)) ds \geq W(t) - \gamma (g)_{\tau,t}.
\]
By Assumption A3 again, this inequality allows us to estimate the second term as
\[
\int_{W(t) - (g)_{\tau,t}}^W du_x \int_{[u_x \leq \frac{\gamma}{d} \tau \geq \frac{t}{2}]} du_x \ell(u_x - W(t)) b(u_\perp)
\]
\[
\leq C \sup_{\frac{t}{2} \leq \tau \leq t} \int_{- (g)_{\tau,t}}^0 du_x |u_x|^p C \gamma^{P+1} \sup_{\frac{t}{2} \leq \tau \leq t} \langle g \rangle^{P+1}_{\tau,t}.
\]
Hence
\[
r_W^R (t) \leq \frac{C \gamma^{P+1} \langle g \rangle^{P+1}_{\tau,t}}{(1 + t)^{d-1}} + C \gamma^{P+1} \sup_{\frac{t}{2} \leq \tau \leq t} \langle g \rangle^{P+1}_{\tau,t}.
\]
\[
\square
\]
5.2. The Left Side. We now proceed to bound the force \( |r_W^L| \) on the left side of the cylinder.

Lemma 5.2. Under the same assumptions as in Lemma 5.1 for \( \gamma \) small enough, we have
\[
|r_W^L (t)| \leq C \gamma^{P+1} \chi \{ t \geq t_0 \} \left( \frac{g^{P+1}(t)}{t^{d-1}} + \sup_{\frac{t}{2} \leq \tau \leq t} \langle g \rangle^{P+1}_{\tau,t} \right).
\]
This estimate is different from the upper bound of \( r_W^R(t) \) because it is the second term that is the dominant one.

Proof. We first notice that \( r_W^L (t) = 0 \) for all \( t \leq t_0 \) because \( W \) is increasing. Indeed, suppose that on the left there is a precollision at time \( \tau \) and a later collision at time \( t \leq t_0 \). If the velocity of the particle in the time period \( (\tau, t) \) is \( u \), then \( u_x \leq W(\tau) \) and \( u_x \geq W(t) > W(\tau) \) which is a contradiction.

Now by the precollision condition, we have
\[
u_x = W_{\tau,t} = \frac{1}{1 - \tau} \int_{\tau}^t W(s) ds \leq \frac{1}{1 - \tau} \int_{\tau}^t (V_\infty - \gamma h(s)) ds \leq V_\infty.
\]
Recalling the boundary condition \( \delta \) on the left side of the cylinder, we have
\[
f_+ (t, x; v) = \alpha f_-(t, x, 2W(t) - v_x, v_\perp) + (1 - \alpha) \int_{u_x \geq W(t)} K(v - iW(t), u - iW(t)) f_-(t, x; u) du.
\]
Again, plugging in the precollision condition (5.5.1), namely \( f_-(t, x; u) = f_+(\tau, \xi; u) \chi_1(t, u) + f_0(u) \chi_0(t, u) \), we then have

\[
f_+(t, x; v) = \alpha \{ f_+(\tau, \xi; 2W(t) - v_x, v_\perp) \chi_1(t, 2W(t) - v_x, v_\perp) + f_0(2W(t) - v_x, v_\perp) \chi_0(t, 2W(t) - v_x, v_\perp) \} \\
+ (1 - \alpha) \int_{u_x \geq W(t)} K(v - iW(t), u - iW(t)) \{ f_+(\tau, \xi; u) \chi_1(t, u) + f_0(u) \chi_0(t, u) \} \, du.
\]

Together with \( u_x \leq V_\infty \), we have

\[
f_+(t, x; v) = a_+(t, x; v_x) b(v_\perp) = \alpha \{ a_+(\tau, \xi; 2W(t) - v_x) \chi_1(t, 2W(t) - v_x, v_\perp) + a_0(2W(t) - v_x) \chi_0(t, 2W(t) - v_x, v_\perp) \} b(v_\perp) \\
+ (1 - \alpha) b(v_\perp) \int_{u_x \geq W(t)} k(v_x - W(t), u_x - W(t)) \{ a_+(\tau, \xi; u_\perp) \chi_1(t, u) + a_0(u_x) \chi_0(t, u) \} du \\
\leq \alpha \{ a_+(\tau, \xi; 2W(t) - v_x) \chi_1(t, 2W(t) - v_x, v_\perp) + a_0(2W(t) - v_x) \chi_0(t, 2W(t) - v_x, v_\perp) \} b(v_\perp) \\
+ (1 - \alpha) b(v_\perp) \int_{W(t)}^{V_\infty} k(v_x - W(t), u_x - W(t)) a_+(\tau, \xi; u_x) du_x \\
\leq \alpha \{ a_+(\tau, \xi; 2W(t) - v_x) \chi_1(t, 2W(t) - v_x, v_\perp) + a_0(2W(t) - v_x) \chi_0(t, 2W(t) - v_x, v_\perp) \} b(v_\perp) \\
+ C b(v_\perp) \left\{ \sup_{\tau, u_x, \xi \in \Theta(t)} a_+(\tau, \xi; u_x) \right\} + C
\]

for \( v_x \in [V_0, V_\infty] \), where in the last line we used Assumptions A2 and A5. Hence, taking suprema as in the earlier estimate (5.5.2), we have

\[
b(v_\perp) \left\{ \sup_{\tau, u_x, \xi \in \Theta(t)} a_+(\tau, \xi; u_x) \right\} \leq b(v_\perp) \{ \alpha + C \gamma \} \left\{ \sup_{\tau, u_x, \xi \in \Theta(t)} a_+(\tau, \xi; u_x) \right\} + C b(v_\perp) = C b(v_\perp).
\]

Since \( \alpha < 1 \) and \( \gamma \) is small, we deduce that

\[
b(v_\perp) \left\{ \sup_{\tau, u_x, \xi \in \Theta(t)} a_+(\tau, \xi; u_x) \right\} \leq C b(v_\perp).
\]

Using this upper bound of \( f_+(\tau, \xi, u) \), we get

\[
|r^L_W(t)| = \left| \int_{\Theta(t)} dS \int_{u_x \geq W(t)} \left| \mathcal{L}(u_x - W(t)) \{ f_-(t, x; u) - f_0(u) \} \right| \, du \right| \\
\leq \int_{\Theta(t)} dS \int_{u_x \geq W(t)} \left| \mathcal{L}(u_x - W(t)) \right| \left| f_-(t, x; u) - f_0(u) \right| \, du \\
\leq C \int_{u_x \leq \frac{V_\infty}{2}} du_x \int_{W(t)}^{V_\infty} du_x \left| \mathcal{L}(u_x - W(t)) b(u_\perp) \right|
\]

As before, we split the integral at \( \tau = t/2 \). So, as in the proof of the previous lemma, we obtain

\[
|r^L_W(t)| \leq \frac{C (V_\infty - W(t))^{p+1}}{(1 + t)^{d-1}} \chi\{ t \geq t_0 \} + C \sup_{\frac{t}{2} \leq \tau \leq t} \left( \frac{\gamma}{\tau} \int_{\tau}^{t} g(s) ds \right)^{p+1} \\
\leq C \gamma^{p+1} g^{p+1}(t) \left( \frac{V_\infty - W(t)}{(1 + t)^{d-1}} \chi\{ t \geq t_0 \} + C \gamma^{p+1} \sup_{\frac{t}{2} \leq \tau \leq t} \left( g(\tau) \right)^{p+1} \right)
\]

for small \( \gamma \), by Assumption A3.

\[\Box\]

5.3. Force Due to Precollisions.

Lemma 5.3. Define

\[ R_W(t) = r^R_W(t) + r^L_W(t). \]

Assume that \( g \) is non-increasing and that there is a power \( M > \frac{d+1}{p+1} \) and a constant \( G \) such that

\[ g(t) \leq G (1 + t)^{-M}. \]

(5.4)
Then
\[ R_W(t) \leq C_\gamma^{p+1} \frac{G^{p+1}}{(1+t)^{d+p}}. \] (5.5)

Furthermore, for \( t \geq \max((2G/H)^{1/(M-1)} A^q (2^M G/H)^{q(p+1)}) \), we have
\[ R_W(t) \geq C_\gamma^{p+1} \frac{\chi(t \geq t_0)}{t^{d+p}} H^{p+1}, \] (5.6)

where \( H = \int_0^1 h(s)ds \) and \( q = \{M(p+1) - (p+d)\}^{-1} \). Here and below, the constant \( C \) may change from line to line but is always independent of \( t, \gamma, H, G, g(0) \).

Proof. By the monotonicity, \( (g)_t/t^2 \leq g(t/2) \). So by Lemmas 5.1 and 5.2, we have
\[ R_W(t) \leq C_\gamma^{p+1} \left\{ \frac{1}{(1+t)^{d-1}} (g(t))^{p+1} + g^{p+1}(t/2) \right\}. \]

Thus by (5.4) we have
\[ R_W(t) \leq C_\gamma^{p+1} \frac{G^{p+1}}{(1+t)^{d+p}}. \]

which is the desired upper bound. Next, by Lemmas 5.1 and 5.2 we have the lower bound
\[ R_W(t) \geq \chi(t \geq t_0) C_\gamma^{p+1} \left\{ \frac{(h)_t - g(t))^{p+1}}{t^{d-1}} - g^{p+1}(t/2) \right\}, \]
\[ \geq \chi(t \geq t_0) C_\gamma^{p+1} \left\{ \frac{H}{t} - \frac{G t^{p+1}}{t^{d-1}} \right\} \] \[ \geq \chi(t \geq t_0) C_\gamma^{p+1} \left\{ \frac{H^{p+1}}{t^{d+p}} - G^{p+1} \left( \frac{2}{t} \right)^{M(p+1)} \right\}, \]
provided \( t > (2G/H)^{1/(M-1)} \). Therefore
\[ R_W(t) \geq C_\gamma^{p+1} \frac{H^{p+1}}{4t^{d+p}} \]
provided also that \( t > 4^q (2^M G/H)^{q(p+1)} \).

6. Motion of the Body

Combining Lemmas 5.1, 5.2 and 5.3, we can now determine upper and lower bounds of \( V_\infty - V_W(t) \).

Lemma 6.1. Define the quotient
\[ Q(t) = \frac{F_0(V_\infty) - F_0(W(t))}{V_\infty - W(t)}, \]
the two positive constants
\[ B_0 = \min_{V \in [V_0, V_\infty]} F_0'(V), \quad B_\infty = \max_{V \in [V_0, V_\infty]} F_0'(V) \]
and the cutoff time
\[ t_0 = \frac{1}{2B_\infty} \log \frac{B_0}{\gamma^p}. \]

Assuming (5.3) and that \( \gamma \) is small enough, we then have the following conclusions.

(i): As a function of \( t \), \( V_W(t) \) is differentiable with bounded derivatives and it is increasing over the interval \([0, t_0] \).

(ii): For \( t \geq 0 \), we have the upper bound
\[ V_\infty - V_W(t) \leq \gamma e^{-B_0 t} + C_\gamma^{p+1} G^{p+1} (1 + t)^{-d-p}. \]

(iii): For \( t \geq 0 \), we have the lower bound
\[ V_\infty - V_W(t) \geq \gamma e^{-B_\infty t} + C H^{p+1} t^{p+1} (1 + t)^{-d-p} \chi_{(2t_0, \infty)}(t). \]
Applying the upper bound on $R$ thus:

$$V_\infty - V_W(t) = \gamma e^{-f_0}Q(t) + \int_0^t e^{-f_0}Q(s) ds$$

since $V_\infty - V_W(0) = V_\infty - V_0 = \gamma$. By Lemma 2.8, since $V_0 \leq V_\infty - \gamma g \leq W(t) \leq V_\infty - \gamma h \leq V_\infty$. By Lemma 2.8,

$$0 < B_0 = \min_{V \in [V_0, V_\infty]} F_0(V) \leq Q(t) = \frac{1}{V_\infty - W(t)} \int_{W(t)}^{V_\infty} F_0'(s) ds \leq \max_{V \in [V_0, V_\infty]} F_0'(V) = B_\infty < \infty.$$

Thus (6.2) together with the positivity of $R$ implies the lower bound

$$V_\infty - V_W(t) \geq \gamma e^{-f_0}Q \geq \gamma e^{-B_\infty t}.$$

Now (6.1) gives us the upper bound

$$\frac{d}{dt}(V_\infty - V_W(t)) \leq - \min [Q(t)] \cdot \min [V_\infty - V_W(t)] + R_W(t) \leq - B_0 \gamma e^{-B_\infty t} + R_W(t).$$

Applying the upper bound on $R_W(t)$, we then turn this estimate into

$$\frac{d}{dt}(V_\infty - V_W(t)) \leq - B_0 \gamma e^{-B_\infty t} + C \gamma^{p+1} < 0$$

for $0 \leq t \leq t_0$, provided \((B_0/\gamma^p)e^{-B_\infty t_0} > C\). We choose $t_0$ as above so that $t_0 \to +\infty$ as $\gamma \to 0$, and

\((B_0/\gamma^p)e^{-B_\infty t_0} = e^{+B_\infty t_0} > > 0)$$.

(ii) By (6.2), we have the upper bound

$$V_\infty - V_W(t) \leq \gamma e^{-B_0 t} + \int_0^t e^{-B_0(t-s)} R_W(s) ds.$$

We split the integral into two parts. On the one hand, the integral from $t/2$ to $t$ is bounded above by

$$\int_{t/2}^t e^{-B_0(t-s)} R_W(s) ds \leq \int_{t/2}^t e^{-B_0(t-s)} C \gamma^{p+1} \frac{G^{p+1}}{(1+s)^{d+p}} ds \leq C \gamma^{p+1} \frac{G^{p+1}}{(1+t)^{d+p}}.$$

On the other hand, the integral from 0 to $t/2$ is bounded more simply by

$$\int_0^{t/2} e^{-B_0(t-s)} C \gamma^{p+1} ds \leq \frac{C \gamma^{p+1}}{B_0} e^{-B_0 t/2}.$$

Thus

$$V_\infty - V_W(t) \leq \gamma e^{-B_0 t} + C \gamma^{p+1} \frac{G^{p+1}}{(1+t)^{d+p}}.$$

(iii) On the other hand, by (6.2) we have the lower bound

$$V_\infty - V_W(t) \geq \gamma e^{-B_\infty t} + \int_0^t e^{-B_\infty(t-s)} R_W(s) ds \geq \gamma e^{-B_\infty t} + C \gamma^{p+1} \int_{t_0}^t e^{-B_\infty(t-s)} s^{-d-p} ds$$

$$\geq \gamma e^{-B_\infty t} + C \gamma^{p+1} \frac{1 - e^{-B_\infty(t-t_0)}}{B_\infty} t^{-d-p}$$

by (5.6). Now for $t \geq 2t_0$, we have $1 - e^{-B_\infty(t-t_0)} \geq 1 - e^{-B_\infty t_0} \geq \frac{1}{2}$ for large $t_0$ (small $\gamma$). Thus we have the desired lower bound

$$V_\infty - V_W(t) \geq \gamma e^{-B_\infty t} + C \gamma^{p+1} \frac{1 - e^{-B_\infty(t-t_0)}}{B_\infty} t^{-d-p} \chi_{[2t_0, \infty)}(t)$$

with a different constant $C$. \qed

By (3.1) we summarize the requirements on $g$ and $h$ as follows.
Corollary 6.1. $V_W(t) \in W$ provided the following conditions are satisfied.

\begin{align}
\gamma e^{-B_0 t} + C \gamma^{p+1} \frac{G^{p+1}}{(1 + t)^{d+p}} &< \gamma g(t), \\
\gamma e^{-B_\infty t} + CH^{p+1} t^{d-p} &> \gamma h(t), \\
2t_0 &\geq \max\{(2G/H)^{1/(M-1)}, 4^p (2MG/H)^{q(p+1)}\}, \\
g(t) &\leq G(1 + t)^{-M} \text{ with } M > \frac{p + d}{d + p}.
\end{align}

Corollary 6.2. One can choose constants $A_+$ and $A_-$ so that the pair

\begin{align}
g(t) &= e^{-B_0 t} + \frac{\gamma^p A_+}{(1 + t)^{d+p}} \\
h(t) &= e^{-B_\infty t} + \frac{\gamma^p A_-}{t^{d+p}} x(2t_0, \infty)(t)
\end{align}

satisfies the conditions of Corollary 6.1.

Proof of Corollary 6.2. Notice for (6.8) we can take $G = 1 + \gamma A_+$. For (6.9) we require

\[ C_0^p \frac{G^{p+1}}{(1 + t)^{d+p}} < \frac{\gamma^p A_+}{(1 + t)^{d+p}}, \]

which is true provided $A_+ > 2C$ and $\gamma$ is sufficiently small. Notice that $H = \int_0^1 h(s)ds = \int_0^1 \exp(-B_\infty s)ds = (1 - \exp(-B_\infty))/B_\infty > 0$. For (6.6) we require

\[ CH^{p+1} \frac{t^{d+p}}{t^{d+p}} x(2t_0, \infty)(t) > \frac{\gamma^p A_-}{t^{d+p}} x(2t_0, \infty)(t), \]

which merely requires $A_- < CH^{p+1}$. Finally, (6.7) is true for small $\gamma$ because $t_0 \to \infty$ as $\gamma \to 0$. \qed

7. Proof of Existence and Asymptotic Behavior

Proof of Theorem 7.1. As in Definition 1, $W$ is defined as the set of all Lipschitz functions $W(t)$ on the half line $0 \leq t < \infty$, non-decreasing in $[0, t_0]$, such that

\[ W(0) = V_0, \quad \lim_{t \to \infty} W(t) = V_\infty, \quad 0 < \gamma h(t) \leq V_\infty - W(t) < \gamma g(t). \]

We define the “ball” of radius $L$ in $W$ as

\[ \mathcal{K} = \{W \in W \mid \text{esssup}_{0 \leq t < \infty}(|W(t)| + |\dot{W}(t)|) \leq L\} \]

for any positive constant $L$. Define $C_0([0, \infty))$ to be the space of continuous bounded functions on $[0, \infty)$. Of course, $\mathcal{K}$ is a compact and convex subset of $C_0([0, \infty))$.

Given $W \in \mathcal{K}$, recall that $V_W$ is defined as the solution of the differential equation

\[ \dot{V}_W = E - F_0(V_W) - R_W(t), \quad V_W(0) = V_0. \]

Keeping in mind that $|R_W(t)| \leq C\gamma^{p+1}$ according to Lemma 5.3 we choose

\[ L = \max\{V_\infty, E - F_0(V_\infty) + C\gamma^{p+1}\}. \]

We then consider the mapping $A : W \to V_W$. By Corollary 6.1, $A$ maps $\mathcal{K}$ into $\mathcal{K}$. By Lemma 7.1 below, $A$ is continuous in the topology of $C_0([0, \infty))$, that is, with respect to uniform convergence. By the Schauder fixed point theorem, $A$ has a fixed point in $\mathcal{K}$, which is our desired solution. Hence we have concluded the proof of Theorem 7.1. \qed

Lemma 7.1. The mapping $A$ is continuous in the topology of $C_0([0, \infty))$.

Proof. Let $W_j \to W$ in $C_0([0, \infty))$ where each $W_j \in \mathcal{K}$. By (7.1) it suffices to prove that $R_W(t) \to R_W(t)$ uniformly in $[0, \infty)$. Fix any $T > 0$. For any $j$ and any $N$, we define $B_{W}^N = \{(x, v_x) : \exists \text{ trajectory passing through } (T, x, v_x) \text{ which has collided at least } N + 1 \text{ times in } [0, T]\}$. Its complement is $A_{W}^N = \{(x, v_x) : \text{ no trajectory passing through } (T, x, v_x) \text{ has collided more than } N \text{ times in } [0, T]\}$. We can then write $R_{W}(t)$ as a sum of contributions from $A_{W}^N$ and $B_{W}^N$, namely,

\[ R_{W}(t) = R_{W} \left( t; A_{W}^N \right) + R_{W} \left( t; B_{W}^N \right). \]
Taking account of only the first precollision of each particle, we proved in Lemmas 5.1 and 5.2 that
\[ 0 \leq R_W(t) \leq \frac{C_{\gamma}^{p+1}}{(1 + t)^{p+d}} \quad (7.2) \]
with \( C \) independent of both \( t \in \mathbb{R} \) and \( W \in \mathcal{W} \). Iterating the same argument \( N \) times, we have
\[ \sup_{0 \leq t < \infty} |R_W(t; B_W^N)| \leq (C_{\gamma}^{p+1})^N. \quad (7.3) \]
Thus it is natural to choose \( \gamma < C^{-\frac{1}{p+d}} \).

Now we may write
\[
\sup_{0 \leq t < \infty} |R_{W,j}(t) - R_W(t)| \leq \sup_{T \leq t < \infty} \left| R_{W,j}(t) - R_W(t) \right| + \sup_{0 \leq t < T} \left| R_W(t; B_W^N) \right| + \sup_{0 \leq t < T} \left| R_{W,j}(t; B_{W,j}^N) \right|
\]
\[
+ \sup_{0 \leq t < T} \left| R_{W,j}(t; A_{W,j}^N) - R_W(t; A_W^N) \right|
\]
\[ = I + II + III + IV. \]

Let \( \varepsilon > 0 \). By estimate (7.2), we may choose \( T = T_\varepsilon \) so large that \( |I| < \varepsilon/3 \). By estimate (7.3), we may then choose \( N = N_\varepsilon \) so large that
\[ |II| + |III| \leq 2 (C_{\gamma}^{p+1})^{N+1} < \varepsilon/3. \]

In IV, there are no more than \( N \) collisions. Therefore we can express both terms in IV as iterates of \( N \) integrals by repeated use of the collision boundary condition. The resulting finite number of iterated integrals contain \( W_j \) in a finite number of places. Therefore they converge as \( j \to \infty \) to the same expression with \( W_j \) replaced by \( W \). Thus we can choose \( j \) so large that \( |IV| < \varepsilon/3 \).

Therefore \( R_{W,j}(t) \to R_W(t) \) in \( C_b([0, \infty)) \). Hence \( A \) is continuous in the topology of \( C_b([0, \infty)) \). \( \Box \)

**Proof of Theorem 1.3** Let \( (V(t), f(t, x, v)) \) be a solution of the problem in the sense of Theorem 1.1. Then \( V \) is a fixed point of \( A \) so that \( V \in \mathcal{W} \) and Corollary 6.1 is valid for it. Letting \( g(t) \) and \( h(t) \) be given by Corollary 6.2 we then have
\[ \gamma h(0) = \gamma = V_\infty - V(0) < \gamma g(0), \]
so that \( V_\infty - V(t) < \gamma g(t) \) for small enough \( t \). Furthermore,
\[ \frac{dV}{dt} \bigg|_{t=0} = F_0(V_\infty) - F(V_0) - F_0(V_\infty) - \gamma < \gamma \max_{V \in [V_0, V_\infty]} F'_0(V) = \gamma B_\infty, \]
so that \( V_\infty - V(t) > \gamma e^{-B_\infty t} = \gamma h(t) \) at least for small enough \( t > 0 \). Let
\[ T = \inf\{s \mid \gamma h(t) < V_\infty - V(t) < \gamma g(t), \forall 0 < t < s\} \leq \infty. \]
In the interval \( (0, T) \) the inequalities (5.3) and (6.4) are satisfied. If \( T \) were finite, we would have
\[ V_\infty - V(T) = \gamma h(T) \quad \text{or} \quad V_\infty - V(T) = \gamma g(T), \]
contradicting Corollary 6.1. Hence \( T = \infty \). This proves Theorem 1.2. \( \Box \)

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