COMPARISON OF MONGE-AMPÈRE CAPACITIES

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Abstract. Let \((X, \omega)\) be a compact Kähler manifold. We prove that all Monge-Ampère capacities are comparable. Using this we give an alternative direct proof of the integration by parts formula for non-pluripolar products recently proved by M. Xia.

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1. Introduction

Since Yau’s solution to Calabi’s conjecture \cite{29} geometric pluripotential theory has found its important place in the development of differential geometry. An important tool in the theory is the Monge-Ampère capacity introduced by Bedford and Taylor \cite{2}. By analyzing capacities of sublevel sets, Kołodziej \cite{21} has established a fundamental \(L^\infty\)-estimate for complex Monge-Ampère equations. Several capacities have been studied in the literature with interesting applications, see \cite{18, 5, 14, 15, 8, 2, 21, 12} and the references therein. The goal of this note is to quantitatively compare these capacities.

Let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\). Fix a smooth closed real \((1,1)\)-form \(\theta\) such that the De Rham cohomology class \(\{\theta\}\) is big. Given \(\psi \in \)
we define, for a Borel subset $E \subset X$, 

$$\text{Cap}_{\theta, \psi}(E) = \sup \left\{ \int_E \theta^n_u : u \in \text{PSH}(X, \theta), \psi - 1 \leq u \leq \psi \right\}.$$ 

Here $\theta^n_u$ is the non-pluripolar Monge-Ampère measure of $u$, see Section 2.

The fact that these capacities characterize pluripolar sets suggests that they are all comparable. This is the content of our main result:

**Theorem 1.1.** Let $\theta_1, \theta_2$ be smooth closed real $(1, 1)$-forms on $X$ which represent big cohomology classes. Assume that $\psi_1 \in \text{PSH}(X, \theta_1)$ and $\psi_2 \in \text{PSH}(X, \theta_2)$ are such that $\int_X (\theta_1 + dd^c \psi_1)^n > 0$ and $\int_X (\theta + dd^c \psi_2)^n > 0$. Then there exist continuous functions $f, g : [0, +\infty) \to [0, +\infty)$ with $f(0) = g(0) = 0$ such that, for any Borel set $E \subset X$, 

$$\text{Cap}_{\theta_1, \psi_1}(E) \leq f(\text{Cap}_{\theta_2, \psi_2}(E)), \quad \text{Cap}_{\theta_2, \psi_2}(E) \leq g(\text{Cap}_{\theta_1, \psi_1}(E)).$$

A. Trusiani has recently proved in [25] a comparison of Monge-Ampère $\phi$-capacities for model potential $\phi$ using the metric geometry of relative finite energy classes introduced in [24].

Using the comparison of capacities we provide a new proof of the integration by parts formula, a result recently proved in [28]. The proof of [28] relies on a construction of D. Witt-Nyström [27]. Our proof uses a direct approximation method partially inspired by [16].

**Theorem 1.2.** [28] Let $u, v \in L^\infty(X)$ be differences of quasi plurisubharmonic functions. Fix $\phi_j \in \text{PSH}(X, \theta^j), \ j = 2, \ldots, n$ where $\{\theta_j\}$ is big. Then 

$$\int_X u dd^c v \wedge \theta^2_{\phi_2} \wedge \ldots \wedge \theta^n_{\phi_n} = \int_X v dd^c u \wedge \theta^2_{\phi_2} \wedge \ldots \wedge \theta^n_{\phi_n}.$$ 

Here, if $u = \varphi - \psi$ with $\varphi, \psi \in \text{PSH}(X, \eta)$ then, by definition, 

$$dd^c u \wedge \theta^2_{\phi_2} \wedge \ldots \wedge \theta^n_{\phi_n} := \eta_{\varphi} \wedge \theta^2_{\phi_2} \wedge \ldots \wedge \theta^n_{\phi_n} - \eta_{\psi} \wedge \theta^2_{\phi_2} \wedge \ldots \wedge \theta^n_{\phi_n}$$

is a difference of non-pluripolar products, see Section 2.

The integration by parts formula is a key ingredient in the variational approach to solve complex Monge-Ampère equations (see [3], [8]). When the potentials have small unbounded locus, i.e. these are locally bounded outside a closed complete pluripolar set, the above result was proved in [5].

The main idea of our proof of Theorem 1.2 is as follows. We first start with the simple case where $u = \varphi_1 - \varphi_2$, with $\varphi_1, \varphi_2 \in \text{PSH}(X, \omega)$, vanishes in some open neighborhood of the pluripolar set $\{\varphi_1 = -\infty\}$. In this case the result is a simple consequence of the plurifine locality of non-pluripolar products. For the general case we apply the first step with $\varphi_1$ and $\varphi_{2, \lambda} = \max(\varphi_1, \lambda \varphi_2)$ for $\lambda > 1$. We next use the comparison of capacities above to pass to the limit as $\lambda \downarrow 1$. 


Organization of the note. After preparing necessary background materials in Section 2 we systematically compare Monge-Ampère capacities in Section 3, proving Theorem 1.1. A new proof of Theorem 1.2 is given in Section 4.

2. Preliminaries

In this section we recall necessary notions and tools in pluripotential theory. We refer the reader to [5], [3], [9, 8, 7, 10, 11] for more details.

2.1. Quasi plurisubharmonic functions. Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$. Fix a closed smooth real $(1,1)$-form $\theta$. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi plurisubharmonic (qpsh) if locally $u = \rho + \phi$ where $\rho$ is smooth and $\phi$ is plurisubharmonic (psh). If additionally $\theta u := \theta + d\bar{d} u \geq 0$ in the weak sense of currents then $u$ is called $\theta$-psh. We let $\text{PSH}(X, \theta)$ denote the set of all $\theta$-psh functions which are not identically $-\infty$. By elementary properties of psh functions one has that $\text{PSH}(X, \theta) \subset L^1(X)$. Here, if nothing is stated $L^1(X)$ is $L^1(X, \omega^n)$.

The De Rham cohomology class $\{\theta\}$ is big if $\text{PSH}(X, \theta - \varepsilon \omega)$ is non-empty for some $\varepsilon > 0$.

Given $u, v \in \text{PSH}(X, \theta)$ we say that $u$ is more singular than $v$, and denote by $u \preceq v$, if there exists a constant $C$ such that $u \leq v + C$ on $X$. We say that $u$ and $v$ have the same singularity type, and denote by $u \simeq v$, if $u \preceq v$ and $v \preceq u$. There is a natural least singular potential in $\text{PSH}(X, \theta)$ given by $V_{\theta}(x) := \sup \{ u(x) : u \in \text{PSH}(X, \theta), u \leq 0 \text{ on } X \}$.

As is well-known $V_{\theta}$ is locally bounded in a Zariski open set called the ample locus of $\{\theta\}$. A potential $u \in \text{PSH}(X, \theta)$ has minimal singularity type if it has the same singularity type as $V_{\theta}$. Note that $V_{\theta} \equiv 0$ if and only if $\theta \geq 0$.

Let $\theta^1, ..., \theta^n$ be smooth closed $(1,1)$-forms representing big cohomology classes. Given $u_j \in \text{PSH}(X, \theta^j)$, $j = 1, ..., n$, with minimal singularity type the Monge-Ampère measure $(\theta^1 + d\bar{d} u_1) \wedge \ldots \wedge (\theta^n + d\bar{d} u_n)$ is well defined, by Bedford-Taylor [1, 2], as a positive Borel measure on the intersection of the ample locus of $\{\theta^j\}$ with finite total mass. One extends this measure trivially over $X$, the resulting measure is called the non-pluripolar Monge-Ampère product of $u_1, ..., u_n$. For general $u_j \in \text{PSH}(X, \theta^j)$ one can consider the canonical approximants $u^t_j := \max(u_j, V_{\theta^j} - t)$, $t > 0$, $j = 1, ..., n$. The sequence of measures

$1_{\wedge(u_j \geq V_{\theta^j} - t)}(\theta^1 + d\bar{d} u^t_1) \wedge \ldots \wedge (\theta^n + d\bar{d} u^t_n)$

is increasing in $t$. Its strong limit, denoted by $(\theta^1 + d\bar{d} u_1) \wedge \ldots \wedge (\theta^n + d\bar{d} u_n)$, is a positive Borel measure on $X$. To simplify the notation we also denote the latter by
\( \theta_{u_1} \wedge \ldots \wedge \theta_{u_n} \). When \( u_1 = \ldots = u_n \) and \( \theta^1 = \ldots = \theta^n = \theta \) we obtain the non-pluripolar Monge-Ampère measure of \( u \), denoted by \((\theta + dd^c u)^n\) or simply by \( \theta_u^n \).

We let \( \mathcal{E}(X, \theta) \) denote the set of all \( \theta \)-psh functions \( u \) with full Monge-Ampère mass, i.e. such that \( \int_X (\theta + dd^c u)^n = \int_X (\theta + dd^c V_\theta)^n = \text{Vol}(\theta) \).

### 2.2. Monotonicity of Monge-Ampère mass.

**Lemma 2.1.** Assume that \( u, v \in \text{PSH}(X, \theta) \) have the same singularity type. Then \( \int_X \theta_u^n = \int_X \theta_v^n \).

The above result was first proved by D. Witt-Nyström [27]. A different proof has been recently given in [22] using the monotonicity of the energy functional. We give below a direct proof using a standard approximation process. Another different proof has been recently given in [26] where generalized non-pluripolar products of positive currents are studied.

We also stress that our proof only uses the invariant of the Monge-Ampère mass of bounded \( \omega \)-psh functions. It is thus valid on non-Kähler manifolds \((X, \omega)\) satisfying

\[
\int_X (\omega + dd^c u)^n = \int_X \omega^n, \quad \forall u \in \text{PSH}(X, \omega) \cap C^\infty(X).
\]

As shown in [6] the above condition is equivalent to \( i \ddbar \omega^k = 0 \), for all \( k = 1, \ldots, n-1 \).

**Proof.** **Step 1.** Assume that \( \theta \) is a Kähler form.

We first prove the following claim: if there exists a constant \( C > 0 \) such that \( u = v \) on \( U := \{ \min(u, v) < -C \} \) then \( \int_X \theta_u^n = \int_X \theta_v^n \).

We approximate \( u, v \) by \( u^t := \max(u, -t) \) and \( v^t := \max(v, -t) \). Then, for \( t > C \) we have that \( u^t = v^t \) on the open set \( U \). This results in \( 1_U \theta_u^n = 1_U \theta_v^n \). Observe that \( u^t = u \) on \( \{ u > -t \} \) and \( v^t = v \) on \( \{ v > -t \} \). For \( t > C \) we have \( \{ u \leq -t \} = \{ v \leq -t \} \subset U \). Hence by plurifine locality we have

\[
\theta_u^n = 1_{\{ u > -t \} \} \theta_u^n + 1_{\{ u \leq -t \} \} \theta_u^n = 1_{\{ v > -t \} \} \theta_v^n + 1_{\{ v \leq -t \} \} \theta_v^n,
\]

and

\[
\theta_v^n = 1_{\{ v > -t \} \} \theta_v^n + 1_{\{ v \leq -t \} \} \theta_v^n = 1_{\{ v > -t \} \} \theta_v^n + 1_{\{ v \leq -t \} \} \theta_v^n.
\]

Integrating over \( X \) and noting that \( \int_X \theta_u^n - \int_X \theta_v^n = \text{Vol}(\theta) \), we arrive at

\[
\int_{\{ u > -t \}} \theta_u^n = \int_{\{ v > -t \}} \theta_v^n.
\]

Letting \( t \to +\infty \) we prove the claim.

We now come back to the proof of the lemma in the Kähler case. We can assume that \( v \leq u \leq v + B \), for some positive constant \( B \). For each \( a \in (0, 1) \) we set \( v_a := av \) and \( u_a := \max(u, v_a) \). Choosing \( C > 2Ba(1-a)^{-1} \) we see that \( u_a = v_a \) on \( \{ v_a < -C \} \). It follows from the claim that \( \int_X \theta_{u_a}^n = \int_X \theta_{v_a}^n \). By multilinearity of the
non-pluripolar product we have that \( \int_X \theta^n_{u,v} \to \int_X \theta^n_v \) as \( a \nearrow 1 \). Observe that \( u_a \searrow u \) as \( a \nearrow 1 \) hence, by [8, Theorem 2.3],
\[
\liminf_{a \to 1^-} \int_X \theta^n_{u_a} \geq \int_X \theta^n_v.
\]
We thus have \( \int_X \theta^n_u \leq \int_X \theta^n_v \). Reversing the role of \( u \) and \( v \) we finally have \( \int_X \theta^n_u = \int_X \theta^n_v \), finishing the proof of Step 1.

**Step 2.** We treat the general case, \( \{\theta\} \) is merely big. Fix \( s > 0 \) so large that \( \theta + s\omega \) is Kähler. For \( t > s \) we have, by the first step,
\[
\int_X (\theta + t\omega + dd^c u)^n = \int_X (\theta + t\omega + dd^c v)^n.
\]
The multilinearity of the non-pluripolar product then gives, for all \( t > s \),
\[
\sum_{k=0}^n \binom{n}{k} \int_X \theta_u^k \wedge \omega^{n-k} t^{n-k} = \sum_{k=0}^n \binom{n}{k} \int_X \theta_v^k \wedge \omega^{n-k} t^{n-k}.
\]
We thus obtain an equality between two polynomials and identifying the coefficients we infer the desired equality. \( \square \)

2.3. Quasi-psh envelopes and model potentials. Let \( f = f_1 - f_2 \) be a difference of two quasi-psh functions. We let \( P_\theta(f) \) denote the largest \( \theta \)-psh function on \( X \) lying below \( f \):
\[
P_\theta(f)(x) := (\sup \{ u(x) : u \in \text{PSH}(X, \theta), u \leq f \text{ on } X \})^*.
\]
Here, the inequality \( u \leq f \) is understood as \( u + f_2 \leq f_1 \) on \( X \). A potential \( \phi \in \text{PSH}(X, \theta) \) is called a model potential if \( \int_X (\theta + dd^c \phi)^n > 0 \) and \( P_\theta[\phi] = \phi \), where \( P_\theta[\phi] \) is the envelope of singularity type of \( \phi \), introduced by J. Ross and D. Witt-Nyström [23]:
\[
P_\theta[\phi] := \left( \lim_{t \to +\infty} P_\theta(\min(\phi + t, 0)) \right)^*.
\]
Given a model potential \( \phi \) we let \( \mathcal{E}(X, \theta, \phi) \) denote the set of \( u \in \text{PSH}(X, \theta) \) more singular than \( \phi \) such that \( \int_X (\theta + dd^c u)^n = \int_X (\theta + dd^c \phi)^n \).

**Lemma 2.2.** If \( u \in \mathcal{E}(X, \omega) \) then \( P_\theta(u) \in \mathcal{E}(X, \theta) \).

**Proof.** Since \( u \in \mathcal{E}(X, A\omega) \) for any \( A \geq 1 \), we can assume that \( \omega \geq \theta \).

We first claim that, for all \( b \geq 1 \), \( P_\theta(bu + (1 - b)V_\theta) \in \text{PSH}(X, \theta) \). Indeed, set \( u_j := \max(u, -j) \), \( v_j := P_\theta(bu_j + (1 - b)V_\theta) \), and
\[
D := \{ v_j = bu_j + (1 - b)V_\theta \}, \quad \varphi_j := b^{-1}v_j + (1 - b^{-1})V_\theta.
\]
Since \( \varphi_j \leq u_j \) with equality on \( D \), using [11, Lemma 4.5] we have
\[
1_D(\theta + dd^c \varphi_j)^n \leq 1_D(\omega + dd^c \varphi_j)^n \leq 1_D(\omega + dd^c u_j)^n.
\]
We choose \( t > 0 \) so large that \( b^n \int_{\{u \leq -b^{-1}t\}} (\omega + dd^cu_j)^n < \text{Vol}(\theta) \). For \( j > b^{-1}t \), by plurifine locality, we have
\[
b^n \int_{\{u \leq -b^{-1}t\}} (\omega + dd^c u_j)^n = b^n \int_X (\omega + dd^c u_j)^n - b^n \int_{\{u > -b^{-1}t\}} (\omega + dd^c u_j)^n = b^n \int_X (\omega + dd^c u)^n - b^n \int_{\{u > -b^{-1}t\}} (\omega + dd^c u)^n \]
\[
= b^n \int_{\{u \leq -b^{-1}t\}} (\omega + dd^c u)^n < \text{Vol}(\theta) .
\]

By [11, Lemma 4.4] we have that \((\theta + dd^c v_j)^n\) is supported on \(D\), hence
\[
\int_{\{u \leq -t\}} (\theta + dd^c v_j)^n = \int_{\{u \leq -t\} \cap D} (\theta + dd^c v_j)^n \leq b^n \int_{\{u \leq -t\} \cap D} (\theta + dd^c \varphi_j)^n \leq b^n \int_{\{u \leq -b^{-1}t\}} (\omega + dd^c u_j)^n < \text{Vol}(\theta) .
\]
It thus follows that \(\sup_X v_j > -t\), hence \(v_j \downarrow v \in \text{PSH}(X, \theta)\), proving the claim.

Observe also that \(P_\theta(u) \geq b^{-1}P_\theta(bu + (1-b)V_\theta) + (1-b^{-1})V_\theta\). It thus follows from [27] and multilinearity of the non-pluripolar product that
\[
\int_X (\theta + dd^c P_\theta(u))^n \geq (1-b^{-1})^n \text{Vol}(\theta) \to \text{Vol}(\theta)
\]
as \(b \to +\infty\). This proves that \(P_\theta(u) \in \mathcal{E}(X, \theta)\). \(\square\)

2.4. Monge-Ampère capacities. Fix a \(\theta\)-psh function \(\psi \leq 0\). We define, for each Borel set \(E \subset X\),
\[
\text{Cap}_{\theta, \psi}(E) := \sup \left\{ \int_E \theta_u^n : u \in \text{PSH}(X, \theta), \psi - 1 \leq u \leq \psi \right\}.
\]
Given a Borel subset \(E\), the global \(\phi\)-extremal function is defined by
\[
V_{E, \theta, \phi}(x) := \sup\{v(x) : v \in \text{PSH}(X, \theta), v \leq \phi, v \leq \phi \text{ on } E\}, x \in X .
\]

It was shown in [8, 10], when \(\phi\) is a model potential and \(E\) is non pluripolar, that \(V_{E, \theta, \phi}^*\) is a \(\theta\)-psh function having the same singularity type as \(\phi\). Moreover \(V_{E, \theta, \phi}^* = \phi\) on \(E\) modulo a pluripolar set. We set \(M_{E, \theta, \phi} := \sup_X V_{E, \theta, \phi}^*\). In case when \(\phi = V_\theta\) we will simplify the notation by setting \(V_E := V_{E, \theta, \phi}\) and \(M_{E, \theta} := M_{E, \theta, \phi}\).

Lemma 2.3. Let \(\phi \in \text{PSH}(X, \theta)\) be such that \(\int_X \theta_\phi^n > 0\). If \(E \subset X\) is a Borel set and \(P \subset X\) is a pluripolar set then \(V_{E \cup P, \theta, \phi}^* = V_{E, \theta, \phi}^*\).
Proof. It follows from the definition that $V_{E,\theta,\phi} \geq V_{E \cup P,\theta,\phi}$ since $E \subset E \cup P$. Let now $u \in \text{PSH}(X, \theta)$ be a candidate defining $V_{E,\theta,\phi}$. We claim that there exists $v \in \text{PSH}(X, \theta)$ such that $v \leq \phi$ and $P \subset \{v = -\infty\}$. Indeed, it follows from [18, 19] that there exists $v_0 \in \mathcal{E}(X, \omega)$ such that $P \subset \{v_0 = -\infty\}$. By Lemma 2.2 we have $P_\theta(v_0) \in \mathcal{E}(X, \theta)$, hence [11, Lemma 5.1] ensures that $v := P_\theta(\min(\phi, v_0)) \in \text{PSH}(X, \theta)$. Note also that $P \subset \{v = -\infty\}$ and $v \leq \phi$. This proves the claim.

For each $\lambda \in (0, 1)$ the function $u_\lambda := \lambda v + (1 - \lambda)u$ is $\theta$-psh and satisfies $u_\lambda \leq \phi$, $u_\lambda \leq \phi$ on $E \cup P$. We thus have $u_\lambda \leq V_{E \cup P,\theta,\phi}^\ast$. Letting $\lambda \rightarrow 0^+$ we obtain $u \leq V_{E \cup P,\theta,\phi}^\ast$, modulo a pluripolar set on $X$. This finally gives $V_{E,\theta,\phi}^\ast \leq V_{E \cup P,\theta,\phi}^\ast$. \hfill $\square$

Proposition 2.4. If $\psi \in \text{PSH}(X, \theta)$ satisfies $\int_X (\theta + dd^c \psi)^n > 0$ then $\text{Cap}_{\theta,\psi}$ characterizes pluripolar sets: for all Borel sets $E$ we have

\[ \text{Cap}_{\theta,\psi}(E) = 0 \iff E \text{ is pluripolar}. \]

The proof below is quasi identical to that of [8, Lemma 4.3].

Proof. If $E$ is pluripolar then by definition $\text{Cap}_{\theta,\psi}(E) = 0$. Conversely, assume that $E$ is not pluripolar. Then there exists a compact set $K \subset E$ such that $K$ is not pluripolar.

Let $V_{K,\theta}$ be the global extremal $\theta$-psh function of $K$. Then $V_{K,\theta}^\ast \in \text{PSH}(X, \theta)$ has minimal singularity type. For $t > 0$ we set

\[ u_t := P_\theta(\min(\psi + t, V_{K}^\ast)). \]

It is well known that $\theta^n_{V_{K,\theta}^\ast}$ is supported on $K$. By [8, Lemma 3.7]

\[ 0 < \int_X \theta^n_{\psi} = \int_X \theta^n_{u_t} \leq \int_{\{u_t = \psi + t\}} \theta^n_{\psi} + \int_K \theta^n_{u_t}. \]

The first term on the right-hand side converges to 0 as $t \rightarrow +\infty$. Thus for $t > 1$ large enough we have $\int_K \theta^n_{u_t} > 0$, hence $\text{Cap}_{\theta,\psi}(K) > 0$. \hfill $\square$

A sequence of functions $u_j$ converges in capacity to $u$ if, for any $\varepsilon > 0$,

\[ \lim_{j \rightarrow +\infty} \text{Cap}_{\omega}(\{x \in X : |u_j(x) - u(x)| > \varepsilon\}) = 0. \]

We will also need the following convergence result whose proof is quasi identical to the proof of [9, Corollary 2.9]:

Theorem 2.5. Assume that $\mu_j$ is a sequence of positive Borel measures converging weakly to $\mu$. Assume that there exists a continuous function $f : [0, +\infty) \rightarrow [0, +\infty)$ with $f(0) = 0$ such that, for any Borel set $E$,

\[ \mu_j(E) + \mu(E) \leq f(\text{Cap}_{\omega}(E)). \]
Let \( u_j \) be a sequence of uniformly bounded quasi-continuous function which converges in capacity to \( u \). Then \( u_j \mu_j \Rightarrow u \mu \) in the sense of measures on \( X \).

**Proof.** Fixing \( \varepsilon > 0 \) there exist \( v, v_j \) continuous functions on \( X \) such that
\[
\text{Cap}_\omega(\{x \in X : u_j(x) \neq v_j(x) \text{ or } u(x) \neq v(x)\}) < \varepsilon.
\]
Let \( A > 0 \) be a constant such that \(|u_j| + |v_j| + |u| + |v| \leq A \) on \( X \). Fix \( \delta > 0 \). For \( j > N \) large enough we have, by the assumption that \( u_j \) converges in capacity to \( u \), that
\[
\text{Cap}_\omega(\{x \in X : |u_j(x) - u(x)| > \delta\}) < \varepsilon.
\]
Fixing a continuous function \( \chi \), it follows from the above that
\[
\left| \int_X (\chi u_j \mu_j - \chi u d\mu) \right| \leq \int_X |\chi (u_j - u)| \mu_j + \int_X \chi u (\mu_j - \mu)
\]
\[
\leq \delta \int_X |\chi| \mu_j + C \sup_X |\chi| \text{O}(|\varepsilon|) + \int_X \chi (u - v)(\mu_j - \mu) + \int_X \chi v (\mu_j - \mu)
\]
\[
\leq \delta \int_X |\chi| \mu_j + 2C \sup_X |\chi| \text{O}(|\varepsilon|) + \int_X \chi v (\mu_j - \mu).
\]
Since \( v \) is continuous on \( X \) the last term converges to 0 as \( j \to +\infty \). This completes the proof. \( \square \)

**3. Comparison of Monge-Ampère capacities**

In this section we establish a comparison between Monge-Ampère capacities. We first prove a version of the Chern-Levine-Nirenberg inequality.

**Lemma 3.1.** Assume that \( u, v \in \text{PSH}(X, \omega) \) have the same singularity type, \( v \leq u \leq v + B \), and \( \psi \) is a bounded \( \omega \)-psh function. Then
\[
\int_X \psi \omega^n_u \geq \int_X \psi \omega^n_v - nB \int_X \omega^n.
\]
**Proof.** We can assume that \( u \geq v \). We first prove the lemma under the assumption that \( u = v \) on the open set \( U := \{v < -C\} \), for some positive constant \( C \).

We approximate \( u \) and \( v \) by \( u^t := \max(u, -t) \) and \( v^t := \max(v, -t) \). For \( t > 0 \) we apply the integration by parts formula for bounded \( \omega \)-psh functions, which is a consequence of Stokes theorem, to get
\[
\int_X \psi(\omega^n_u - \omega^n_v) = \int_X (u^t - v^t) dd^c \psi \wedge S^t,
\]
where $S_t := \sum_{k=0}^{n-1} \omega^{k}_{u_t} \wedge \omega^{n-k}_{v_t}$. Since $u^t \geq v^t$ we can continue the above estimate and obtain
\[
\int_X \psi (\omega^n_{u_t} - \omega^n_{v_t}) = \int_{\Omega} (u^t - v^t) (\omega^t \wedge S^t - \omega^t \wedge S^t) \geq -Bn \int_X \omega^n.
\]
For $t > B + C$ we have that $u^t = v^t$ on the open set $U$ which contains $\{u \leq -t\} = \{v \leq -t\}$. It thus follows that $1_U \omega^n_{u_t} = 1_U \omega^n_{v_t}$. Thus, for $t > B + C$ we have
\[
\int \chi_{\{v > -t\}} \psi (\omega^n_{u_a} - \omega^n_{v_a}) = \int_X \psi (\omega^n_{u_t} - \omega^n_{v_t}) \geq -Bn \int_X \omega^n.
\]
Letting $t \to +\infty$ we prove the claim.

We come back to the proof of the lemma. By approximating $\psi$ from above by smooth $\omega$-psh functions, see [13], [4], we can assume that $\psi$ is smooth. We fix $a \in (0, 1)$ and set $v_a := av, u_a := \max(u, v_a)$. Then for some constant $C > 0$ large enough we have that $u_a = v_a$ on $\{v_a < -C\}$. We can thus apply the first step to get
\[
\int_X \psi \omega^n_{u_a} \geq \int_X \psi \omega^n_{v_a} - nB \int_X \omega^n.
\]
Letting $a \to 1$ and using [8] we obtain
\[
\int \psi \omega^n_u \geq \int_X \psi \omega^n_v - B'.
\]

Lemma 3.2. Let $\phi \in$ PSH$(X, \omega)$ be such that $P_{\omega} [\phi] = \phi$ and $\int_X \omega^n_{\phi} > 0$. Then for any Borel set $E \subset X$ we have
\[
\text{Cap}_{\omega, \phi}(E) \leq \frac{C}{M_{E, \omega, \phi}},
\]
where $C > 0$ is a uniform constant independent of $\phi$.

Note that the above estimate holds for a big class $\{\theta\}$ as well but to prove this we need to invoke the integration by parts formula in Section 4.

Proof. Fix $C_0$ a positive constant such that for all $v \in$ PSH$(X, \omega)$ with $\sup_X v = 0$ we have $\int_X |v| \omega^n \leq C_0$. The existence of $C_0$ follows from [18, Proposition 2.7].

We can assume that $0 < M_{E, \omega, \phi} < +\infty$. Let $u \in$ PSH$(X, \omega)$ be such that $\phi - 1 \leq u \leq \phi$. Observe that the function $V^*_E, \omega, \phi - M_{E, \omega, \phi} is \omega$-psh satisfying $\sup_X (V^*_E, \omega, \phi - M_{E, \omega, \phi}) = 0$. As recalled above we thus have
\[
\int_X |V^*_E, \omega, \phi - M_{E, \omega, \phi} - \phi| \omega^n \leq 2C_0.
\]
We also have that $|V_{E,\omega,\phi}^* - M_{E,\omega,\phi} - \phi| = M_{E,\omega,\phi}$ on $E$ modulo a pluripolar set. By Lemma 3.1 we have that, for all negative $v \in \text{PSH}(X, \omega)$,
\[
\int_X |v| \omega^n_u - \omega^n_\phi \leq n \int_X \omega^n.
\]
By [8, Theorem 3.8] we also have that $\omega^n_\phi \leq \omega^n$. We thus have, for all $v \in \text{PSH}(X, \omega)$ normalized by $\sup_X v = 0$,
\[
\int_X |v| \omega^n_u \leq n \int_X \omega^n + C_0.
\]
It thus follows from Lemma 3.1 and the triangle inequality that
\[
\int_E \omega^n_u \leq \frac{1}{M_{E,\omega,\phi}} \int_X |V_{E,\omega,\phi}^* - M_{E,\omega,\phi} - \phi| \omega^n_u \leq \frac{1}{M_{E,\omega,\phi}} \left( 2n \int_X \omega^n + 4C_0 \right).
\]
Taking the supremum over all candidates $u$ we obtain the desired inequality. □

Lemma 3.3. Fix $\varphi, \psi \in \text{PSH}(X, \theta)$ such that $\psi \leq \varphi$ and $\int_X \theta^n_\varphi = \int_X \theta^n_\psi$. Then there exists a continuous function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that, for all Borel sets $E$,
\[
\text{Cap}_{\theta,\psi}(E) \leq g \left( \text{Cap}_{\theta,\varphi}(E) \right).
\]
Our proof uses an idea in [17].

Proof. We can assume that $\varphi \leq 0$. Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be an increasing function such that $\chi(-\infty) = -\infty$ and
\[
A := \int_X |\chi(\psi - 1 - \varphi)| \theta^n_\psi < +\infty.
\]
We claim that if $v \in \text{PSH}(X, \theta)$ with $\varphi - t \leq v \leq \varphi$ then for any Borel set $E$ we have
\[
\int_E \theta^n_v \leq \max(t, 1)^n \text{Cap}_{\theta,\varphi}(E).
\]
Indeed, for $t \geq 1$, the function $v_t := t^{-1}v + (1 - t^{-1})\varphi$ is $\theta$-psh and $\varphi - 1 \leq v \leq \varphi$. We thus have
\[
t^{-n} \int_E \theta^n_v \leq \int_E \theta^n_{v_t} \leq \text{Cap}_{\theta,\varphi}(E),
\]
yielding the claim. Let $u$ be a $\theta$-psh function such that $\psi - 1 \leq u \leq \psi$. Fix $t > 0$ and set $u_t := \max(u, \varphi - 2t)$, $E_t := E \cap \{u > \varphi - 2t\}$, $F_t := E \cap \{u \leq \varphi - 2t\}$. By plurifine locality and the claim we have that
\[
\int_{E_t} \theta^n_u = \int_{E_t} \theta^n_{u_t} \leq (2t)^n \text{Cap}_{\theta,\varphi}(E_t) \leq (2t)^n \text{Cap}_{\theta,\varphi}(E).
\]
On the other hand, using the inclusions
\[ F_t \subset \left\{ \psi - 1 \leq \frac{u + \varphi}{2} - t \right\} \subset \{ \psi - 1 \leq \varphi - t \} \]
and the comparison principle [8, Corollary 3.6] we infer
\[ \int_{F_t} \theta^n_u \leq 2^n \int_{\{\psi \leq \varphi - t + 1\}} \theta^n_{\psi} \leq \frac{2^n}{|\chi(-t)|} \int_X |\chi(\psi - 1 - \varphi)| \theta^n_{\psi}. \]
Taking the supremum over all candidates \( u \) we obtain
\[ \text{Cap}_{\theta,\psi}(E) \leq (2t)^n \text{Cap}_{\theta,\phi}(E) + \frac{2^n A}{|\chi(-t)|}. \]
Taking \( t := (\text{Cap}_{\theta,\varphi}(E))^{-1/2n} > 1 \), we get \( \text{Cap}_{\theta,\psi}(E) \leq g \left( \text{Cap}_{\theta,\varphi}(E) \right) \), where \( g \) is defined on \([0, +\infty)\) by
\[ g(s) := 2^n s^{1/2} + \frac{2^n A}{|\chi(-s^{-1/2n})|}. \]

\textbf{Lemma 3.4.} Assume that \( \phi \in \text{PSH}(X, \omega), \int_X \omega^n_\phi > 0 \) and \( P_\omega[\phi] = \phi \). Then there exists a constant \( A > 0 \) such that for any Borel set \( E \) we have
\[ A^{-1} \text{Cap}_\omega(E)^n \leq \text{Cap}_{\omega,\phi}(E) \leq A \left( \text{Cap}_\omega(E) \right)^{1/n}. \]

The proof uses an idea in [9].

\textit{Proof.} By inner regularity of the capacity we can assume that \( E \) is compact. By Lemma 3.2 and [8, Lemma 4.9] we have
\[ \text{Cap}_\omega(E)^n \leq CM_{E,\omega}^{-n} \leq CM_{E,\omega,\phi}^{-n} \leq C' \text{Cap}_\omega,\phi(E), \]
proving the left-hand side inequality. We next prove the right-hand side inequality. By [11, Lemma 4.3] there exists a constant \( b > 1 \) such that \( P_\omega(\lambda \phi) \in \text{PSH}(X, \omega) \). Set
\[ v := (1 - b^{-1})V_{E,\omega}^* + b^{-1} P_\omega(b\phi). \]
Recall that \( V_{E,\omega} = V_{E,\omega,0} \) is the global extremal function of \( E \) which takes values 0 on \( E \) modulo a pluripolar set. As \( V_{E,\omega}^* \) is bounded we have that \( v \in \text{PSH}(X, \omega), v \leq \phi \), and \( v \leq \phi \) on \( E \) modulo a pluripolar set. By Lemma 2.3 we thus have \( v \leq V_{E,\omega,\phi}^* \). Set \( C_0 := -\sup_X P_\omega(b\phi) \geq 0 \) and \( G := \{ P_\omega(2\phi) \geq -C_0 - 1 \} \). Note that \( G \) has positive Lebesgue measure, hence \( G \) is non pluripolar. In particular \( M_{G,\omega} < +\infty \). We have
\[ \sup_X V_{E,\omega,\phi}^* \geq \sup_X v \geq \sup_X (1 - b^{-1}) V_{E,\omega}^* - C_1. \]
On the other hand we have that $u := V_{E,\omega} - \sup_G V_{E,\omega}^*$ is $\omega$-psh and $u \leq 0$ on $G$. It thus follows that $u \leq M_{G,\omega} < +\infty$, hence $\sup_G V_{E,\omega}^* \geq V_{E,\omega}^* - M_{G,\omega}$. Taking the supremum over $X$ we get $\sup_G V_{E,\omega}^* \geq V_{E,\omega}^* - M_{E,\omega}$. Therefore

$$M_{E,\omega,\phi} \geq (1 - b^{-1})M_{E,\omega} - C_2.$$ 

It follows from [18, Proposition 7.1] that $\text{Cap}_{\omega}(E) \geq C_3 M_{E,\omega}^{-n}$, for some uniform constant $C_3 > 0$. Set $a = C_3^{-1}(2b(b - 1)^{-1}C_2)^{-n}$. If $\text{Cap}_{\omega}(E) \leq a$ then

$$(1 - b^{-1})M_{E,\omega} \geq (1 - b^{-1})(aC_3)^{-1/n} = 2C_2.$$ 

From the above we thus have $M_{E,\omega,\phi} \geq C_5 M_{E,\omega}$. Then by Lemma 3.2 and [18, Proposition 7.1] we have

$$\text{Cap}_{\omega,\phi}(E) \leq C_6 \text{Cap}_{\omega}(E)^{1/n}.$$ 

Observe that $\text{Cap}_{\omega,\phi}(E) \leq \int_X \omega^n$. Let $C_7$ be a positive constant such that $C_7 \geq C_6$ and $C_7 a^{1/n} \geq \int_X \omega^n$. We then have

$$\text{Cap}_{\omega,\phi}(E) \leq C_7 \text{Cap}_{\omega}(E)^{1/n}. \qed$$

The main result of this note is a direct consequence of the following:

**Theorem 3.5.** Assume that $\psi \in \text{PSH}(X, \theta)$ and $\int_X \theta^n \psi_\omega > 0$. Then there exist continuous functions $f, g : [0, +\infty) \to [0, +\infty)$ with $f(0) = g(0) = 0$ such that, for any Borel set $E$,

$$\text{Cap}_{\theta,\psi}(E) \leq f \left( \text{Cap}_{\omega}(E) \right) \quad \text{and} \quad \text{Cap}_{\omega}(E) \leq g \left( \text{Cap}_{\theta,\psi}(E) \right).$$

**Proof.** By inner regularity of the capacities we can assume that $E$ is compact. By scaling we can assume that $\theta \leq \omega$. Set $\phi := P_\omega[\psi]$. It follows from Lemma 3.3 that

$$\text{Cap}_{\omega,\psi} \leq f \left( \text{Cap}_{\omega,\phi} \right),$$

for some continuous function $f$ with $f(0) = 0$, while Lemma 3.4 gives

$$\text{Cap}_{\omega,\phi} \leq A \text{Cap}_{\omega}^{1/n},$$

for some positive constant $A$. Combining these two inequalities we obtain the first inequality of the theorem. We next prove the second one. By [18, Proposition 7.1] and [8, Lemma 4.9] we have

$$\text{Cap}_{\omega}(E) \leq C M_{E,\omega}^{-1} \leq C M_{E,\theta,\phi}^{-1} \leq C' \text{Cap}_{\theta,\phi}(E)^{1/n}.$$ 

Since $\int_X (\theta + dd^c \psi)^n > 0$, by [22, Corollary 3.20] $P_\theta(2\psi - \phi) \in \mathcal{E}(X, \theta, \phi)$. Setting $u := P_\theta(2\psi - \phi) + \phi \leq \psi$, by Lemma 3.3 we have $\text{Cap}_{\theta, u} \leq g(\text{Cap}_{\theta, \psi})$, for some continuous
function \( g \) with \( g(0) = 0 \). The proof is finished if we can show that \( \text{Cap}_{\theta,\phi} \leq 2^n \text{Cap}_{\omega,u} \). Take \( v \in \text{PSH}(X, \theta) \) such that \( \phi - 1 \leq v \leq \phi \). Then
\[
u - 1 \leq h := \frac{v + P_0(2\psi - \phi)}{2} \leq \nu,
\]
and hence
\[
\int_E \theta^n_v \leq 2^n \int_E \theta^n_h \leq 2^n \text{Cap}_{\theta,\omega}(E).
\]
Taking the supremum over all \( v \) we obtain \( \text{Cap}_{\theta,\phi} \leq 2^n \text{Cap}_{\theta,u} \). \( \square \)

4. Integration by parts

The integration by parts formula was recently studied in [28] using Witt-Nyström’s construction. In this section we give an alternative direct proof which also applies to the setting of complex \( m \)-Hessian equations considered in [22]. We first start with the following key lemma.

**Lemma 4.1.** Let \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in \text{PSH}(X, \theta) \) be such that \( \varphi_1 \simeq \varphi_2 \) and \( \psi_1 \simeq \psi_2 \). Then
\[
\int_X (\varphi_1 - \varphi_2) (\theta^n_{\psi_1} - \theta^n_{\psi_2}) = \int_X (\psi_1 - \psi_2)(S_1 - S_2),
\]
where \( S_j := \sum_{k=0}^{n-1} \theta_{\varphi_j} \wedge \theta_{\psi_1}^{k} \wedge \theta_{\psi_2}^{n-k-1}, \ j = 1, 2 \).

**Proof.** It follows from [8, Theorem 2.4] that \( \int_X (\theta^n_{\psi_1} - \theta^n_{\psi_2}) = \int_X (S_1 - S_2) = 0 \). By adding a constant we can assume that \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) are negative.

**Step 1.** We assume that \( \theta \) is Kähler and \( \psi_1, \psi_2, \varphi_1, \varphi_2 \) are \( \lambda \theta \)-psh for some \( \lambda \in (0,1) \).

**Step 1.1.** We also assume that there exists \( C > 0 \) such that \( \psi_1 = \psi_2 \) on the open set \( U := \{ \min(\psi_1, \psi_2) < -C \} \) and \( \varphi_1 = \varphi_2 \) on the open set \( V := \{ \min(\varphi_1, \varphi_2) < -C \} \).

For a function \( u \) we consider its canonical approximant \( u^t := \max(u, -t), \ t > 0 \). It follows from Stokes theorem that
\[
\int_X (\varphi_1^t - \varphi_2^t) (\theta^n_{\psi_1^t} - \theta^n_{\psi_2^t}) = \int_X (\psi_1^t - \psi_2^t)(S_1^t - S_2^t),
\]
where \( S_j^t := \sum_{k=0}^{n-1} \theta_{\varphi_j}^t \wedge \theta_{\psi_1}^k \wedge \theta_{\psi_2}^{n-k-1}, \ j = 1, 2 \). Fix \( t > C \). Since \( \psi_1^t = \psi_2^t \) in the open set \( U \) and \( \{ \psi_1 \leq -t \} = \{ \psi_2 \leq -t \} \subset U \) it follows that \( 1_{\{\psi_1 \leq -t\}} \theta^n_{\psi_1} = 1_{\{\psi_1 \leq -t\}} \theta^n_{\psi_2} \). Moreover, by plurifine locality of the non-pluripolar product we have
\[
\int_X (\varphi_1^t - \varphi_2^t) (\theta^n_{\psi_1^t} - \theta^n_{\psi_2^t}) = \int_{\{\psi_1 > -t\}} (\varphi_1^t - \varphi_2^t) (\theta^n_{\psi_1^t} - \theta^n_{\psi_2^t})
\]
\[
= \int_{\{\psi_1 > -t\}} (\varphi_1^t - \varphi_2^t) (\theta^n_{\psi_1} - \theta^n_{\psi_2}).
\]
Letting $t \to +\infty$ we obtain
\[
\lim_{t \to +\infty} \int_X (\varphi^t_1 - \varphi^t_2) \left( \theta^n_{\psi_1^t} - \theta^n_{\psi_2^t} \right) = \int_X (\varphi_1 - \varphi_2) \left( \theta^n_{\psi_1} - \theta^n_{\psi_2} \right).
\]
Using the fact that $\varphi^t_1 = \varphi^t_2$ on $\{\varphi_1 \leq -t\} = \{\varphi_2 \leq -t\}$ which is contained in the open set $V$ we have that
\[
1_{\{\varphi_1 \leq -t\}} S^t_1 = 1_{\{\varphi_1 \leq -t\}} S^t_2.
\]
We thus have
\[
\int_X (\psi^t_1 - \psi^t_2)(S^t_1 - S^t_2) = \int_{\{\psi_1 > -t\} \cap \{\psi_2 > -t\}} (\psi_1 - \psi_2)(S_1 - S_2).
\]
Letting $t \to +\infty$ we finish Step 1.1.

**Step 1.2.** We remove the assumptions made in Step 1.1.

It follows from \[8, Theorem 2.4\] that $\int_X (\theta^n_{\psi_1} - \theta^n_{\psi_2}) = \int_X (S_1 - S_2) = 0$. Thus adding a constant we can assume that $\varphi_1 \leq \varphi_2$ and $\psi_1 \leq \psi_2$. Let $B > 0$ be a constant such that $\varphi_2 \leq \varphi_1 + B$ ; $\psi_2 \leq \psi_1 + B$.

For each $\varepsilon \in (0, \frac{1}{\lambda} - 1)$ we define
\[
\psi_{2, \varepsilon} := \max(\psi_1, (1 + \varepsilon)\psi_2) ; \quad \varphi_{2, \varepsilon} := \max(\varphi_1, (1 + \varepsilon)\varphi_2).
\]
Observe that $\psi_1 \leq \psi_{2, \varepsilon} \leq \psi_1 + B$ and $\varphi_1 \leq \varphi_{2, \varepsilon} \leq \varphi_1 + B$. These are $\omega$-psh functions satisfying the assumptions in Step 1.1 with $C = B + B\varepsilon^{-1}$. Indeed, if $\varphi_1(x) < -C$ then
\[
(1 + \varepsilon)\varphi_{2, \varepsilon}(x) = \varphi_2(x) + \varepsilon\varphi_2(x) \leq \varphi_1(x) + B + \varepsilon(B - C) \leq \varphi_1(x).
\]
We can thus apply Step 1.1 to $\psi_1$, $\psi_{2, \varepsilon}$, $\varphi_{1, \varepsilon}$, $\varphi_{2, \varepsilon}$ to obtain
\[
\int_X (\varphi_1 - \varphi_{2, \varepsilon}) \left( \theta^n_{\psi_1} - \theta^n_{\psi_{2, \varepsilon}} \right) = \int_X (\psi_1 - \psi_{2, \varepsilon})(S_{1, \varepsilon} - S_{2, \varepsilon}),
\]
where $S_{1, \varepsilon} := \sum_{k=0}^{n-1} \theta_{\varphi_1} \wedge \theta_{\psi_1}^{n-k-1}$ and $S_{2, \varepsilon} := \sum_{k=0}^{n-1} \theta_{\varphi_{2, \varepsilon}} \wedge \theta_{\psi_{2, \varepsilon}}^{n-k-1}$. By Theorem 3.5 there exists a continuous function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$ such that for every Borel set $E$,
\[
\text{Cap}_{\theta, \psi}(E) \leq f(\text{Cap}_\theta(E)),
\]
where $\psi := \frac{\varphi_1 + \varphi_2 + \psi_1 + \psi_2}{5} - B$ is a $\theta$-psh function with $\int_X \theta^n_{\psi} > 0$. Using
\[
\psi \leq \frac{\varphi_1 + \psi_1 + \varphi_{2, \varepsilon} + \psi_{2, \varepsilon}}{5} \leq \psi + B,
\]
and $S_{j, \varepsilon} \leq C(5\theta + dd^c(\varphi_1 + \varphi_{2, \varepsilon} + \psi_{2, \varepsilon} + \psi_1))^n$ we obtain, for any Borel set $E$, that
\[
\int_E S_{j, \varepsilon} \leq C' f(\text{Cap}_\theta(E)), \quad \forall \varepsilon \in (0, 1), j = 1, 2.
\]
For each \(j \in \{1, 2\}\) we also have that \(S_{j, \varepsilon} \to S_j, \theta^n_{\psi_{2, \varepsilon}} \to \theta^n_{\psi_2}\) as \(\varepsilon \to 0\) in the weak sense of measures (see [8, Theorem 2.3]). These measures are uniformly dominated by \(\text{Cap}_g\). Note also that \(\varphi_1 - \varphi_{2, \varepsilon}, \varphi_1 - \varphi_2, \psi_{2, \varepsilon} - \psi_1, \psi_2 - \psi_1\) are uniformly bounded, quasi-continuous. Moreover, \(\psi_{2, \varepsilon} - \psi_1 \to \psi_2 - \psi_1\), and \(\varphi_1 - \varphi_{2, \varepsilon} \to \varphi_1 - \varphi_2\) in capacity as \(\varepsilon \to 0\). It thus follows from Theorem 2.5 that

\[
\lim_{\varepsilon \to 0} \int_X (\varphi_1 - \varphi_{2, \varepsilon}) \left( \theta^n_{\psi_1} - \theta^n_{\psi_{2, \varepsilon}} \right) = \int_X (\varphi_1 - \varphi_2) \left( \theta^n_{\psi_1} - \theta^n_{\psi_2} \right)
\]

and

\[
\lim_{\varepsilon \to 0} \int_X (\psi_1 - \psi_{2, \varepsilon}) (S_{1, \varepsilon} - S_{2, \varepsilon}) = \int_X (\psi_1 - \psi_2) (S_1 - S_2),
\]

finishing the proof of Step 1.2.

**Step 2.** We merely assume that \(\{\theta\}\) is big. We can assume that \(\theta + \omega\) is a Kähler form. For \(s > 2\) we apply the first step for \(\theta_s := \theta + s\omega\), which is also Kähler, to get

\[
\int_X u ((\theta_s + dd^c\psi_1)^n - (\theta_s + dd^c\psi_2)^n) = \int_X v T_s,
\]

where \(u = \varphi_1 - \varphi_2, v = \psi_1 - \psi_2\) and

\[
T_s = \sum_{k=0}^{n-1} (\theta_s + dd^c\varphi_1) \wedge (\theta_s + dd^c\psi_1)^k \wedge (\theta_s + dd^c\psi_2)^{n-k-1}
- \sum_{k=0}^{n-1} (\theta_s + dd^c\varphi_2) \wedge (\theta_s + dd^c\psi_1)^k \wedge (\theta_s + dd^c\psi_2)^{n-k-1}.
\]

We thus obtain an equality between two polynomials in \(s\). Identifying the coefficients we arrive at the conclusion. \(\square\)

**Proof of Theorem 1.2.** We first assume that \(\theta\) is Kähler, \(u = \varphi_1 - \varphi_2\) and \(v = \psi_1 - \psi_2\) where \(\psi_1, \psi_2, \varphi_1, \varphi_2\) are \(\theta\)-psh. Fix \(\phi \in \text{PSH}(X, \theta)\) and for each \(s \in [0, 1]\), \(j = 1, 2\), we set \(\psi_{j, s} := s\psi_j + (1-s)\phi\). Note that \(\psi_{1, s} \simeq \psi_{2, s}\). It follows from Lemma 4.1 that for any \(s \in [0, 1]\),

\[
\int_X u \left( \theta^n_{s\psi_1 + (1-s)\phi} - \theta^n_{s\psi_2 + (1-s)\phi} \right) = \int_X sv T_s,
\]

where \(T_s := \sum_{k=0}^{n-1} \theta_{\varphi_1} \wedge \theta^k_{\psi_{1, s}} \wedge \theta^{n-k-1}_{\psi_{2, s}} - \sum_{k=0}^{n-1} \theta_{\varphi_2} \wedge \theta^k_{\psi_{1, s}} \wedge \theta^{n-k-1}_{\psi_{2, s}}\). We thus have an identity between two polynomials in \(s\). Taking the first derivative in \(s = 0\) we obtain

\[
\int_X u dd^c v \wedge \theta^{n-1}_\phi = \int_X v dd^c u \wedge \theta^{n-1}_\phi.
\]
For the general case we can write \( u = \phi_1 - \phi_2 \) and \( v = \psi_1 - \psi_2 \) where \( \psi_1, \psi_2, \phi_1, \phi_2 \) are \( A\omega \)-psh, for some \( A > 0 \) large enough. We apply the first step with \( \theta \) replaced by \( \theta + t\omega \), for \( t > A \) to get

\[
\int_X u \, dd^c v \wedge (t\omega + \theta \phi)^{n-1} = \int_X v \, dd^c u \wedge (t\omega + \theta \phi)^{n-1}.
\]

Identifying the coefficients of these two polynomials in \( t \) we obtain

\[
\int_X u \, dd^c v \wedge \theta^{n-1} \phi = \int_X v \, dd^c u \wedge \theta^{n-1} \phi.
\]

We now consider \( \theta = s_2 \phi^2 + \ldots + s_n \phi^n \), \( \phi := s_2 \phi^2 + \ldots + s_n \phi^n \) with \( s_2, \ldots, s_n \in [0,1] \) and \( \sum s_j = 1 \). We obtain an identity between two polynomials in \( (s_2, \ldots, s_n) \) and identifying the coefficients we arrive at the result.

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