Abstract
The scattering of photons off photons at the one-loop level is investigated. We give a short review of the weak field limit, as given by the first order term in the series expansion of the Heisenberg–Euler Lagrangian. The dispersion relation for a photon in a radiation gas is presented. Based on this, a wave kinetic equation and a set of fluid equations is formulated. These equations describe the interaction between a partially coherent electromagnetic pulse and an intense radiation gas. The implications of the results are discussed.

1. Introduction
Photon–photon scattering is a non-classical effect arising in quantum electrodynamics (QED) due to virtual electron–positron pairs in vacuum. Effectively, the interaction between photons and these virtual pairs will result in what is known as elastic photon–photon collisions. Formulated as an effective field theory, using the Heisenberg–Euler Lagrangian [1, 2], this results in nonlinear corrections to Maxwell’s vacuum equations, which to lowest order in the fine structure constant are cubic in the electromagnetic (EM) field. These correction takes the same form as in nonlinear optics, where the material properties of, e.g., optical fibers, gives rise to cubic nonlinearities in Maxwell’s equations, so called Kerr nonlinearities. Since the effective self-interaction term in proportion to the fine structure constant squared, this means that the field strengths need to reach appreciable values until these effects becomes important [3]. Higher order corrections can easily be incorporated, as will be done here, by taking into account higher vertex and loop order diagrams. Possible detection techniques [4, 5], as well as physical implications, such as the formation of light bullets [6–10], of photon–photon scattering have attracted a historical and current, of the research in this area, see Refs. [4, 5] and references therein. The concept of self-trapping of photons due to vacuum nonlinearities has also been discussed in the context of the one-dimensional nonlinear Schrödinger equations [13]. The non-trivial propagation of photons in strong background electromagnetic fields, due to effects of nonlinear electrodynamics, has been considered in a number of papers (e.g., Ref. [11]). The main focus in these papers has been on the photon splitting and birefringence of vacuum, something which has attracted attention when it comes to the extreme magnetic fields outside magnetars [14, 15]. It has even been suggested that the nontrivial refractive index due to photon–photon scattering could induce a lensing effect in the neighborhood of a magnetar [16].

Here we will investigate the propagation of incoherent high frequency photons on a radiation background, both of arbitrary intensities. It will be shown that the nontrivial dispersion relation for a single photon on a given background gives rise to novel collective nonlinear effects. Furthermore, the general dispersion relation for a perturbation in the system will be derived, and applications will be considered.

2. Basic Relations
The effective field theory of one-loop photon–photon scattering in constant background EM fields $[E, B]$ can be described by the Lagrangian density $L = L_0 + L_1$, where $L_0 = -F$ is the classical free field Lagrangian, and [2]

$$L = -\frac{1}{8\pi^2} \int_0^\infty ds s^3 e^{-s}\left[\epsilon_{a ba} \cot(\epsilon_{a ba}) \cot(\epsilon_{a ba}) \right]$$

$$+ \frac{1}{4} \left| \epsilon_{a ba} \right|^2 - \frac{1}{4} \left| \epsilon_{a ba} \right|^2 - 1,$$

where $m$ is the electron mass, $e = |e|$ is the electron charge, $a = (|E|^2 + |G|^2)^{1/2} + |F|^{1/2}$, $b = (|E|^2 + |G|^2)^{1/2} - |F|^{1/2}$, $F \equiv F_{\mu \nu} F^{\mu \nu}/4 = (|E|^2 - |B|^2)/2$, $G \equiv F_{\mu \nu} F^{\mu \nu}/4 = -E \cdot B$, $T_{\mu \nu} = \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}/2$. Thus, $F = (a^2 - b^2)/2$ and $|G| = ab$.

Following Ref. [17], the above one-loop Lagrangian yields the dispersion relation for a test photon in a background EM field of arbitrary field strength. This can be formulated according to [18]

$$\beta = E \times B / n,$$

$$\Lambda = E \times B / n.$$  

Furthermore, the “effective action charge” $\zeta$ is given by [18]

$$\zeta = \frac{1}{2} \left[ \frac{\epsilon_{a ba}}{\epsilon_{a ba}} + \frac{\epsilon_{a ba}}{\epsilon_{a ba}} \right] L.$$  

From Eqs. (2) and (4) we see that as the background EM fields vanish, so does the nonlinear effects, i.e., $n \mapsto 1$, as expected.

3. Derivation of the Governing Equations
Suppose that a plane wave pulse travels through an approximately isotropic and homogeneous medium with refractive index $n$. The relations $E \cdot B = 0$ and $|B| = |E| n$ then holds. We note that if...
the background medium is the plane wave field itself, the only physical solution to Eq. (2) is \( n = 1 \), i.e., a plane wave cannot self-interact. If the background medium is a radiation gas, which we consider as an ensemble \( \{ E_g, B_g \} \) of plane waves, the interaction contribution to the dispersion relation (2) will be nonzero. We first note that, since we are interested in the case of arbitrary intense fields, we have to take the effect on each photon in both the pulse and the gas into account. The self-interaction within the gas will also be nonzero, i.e., each photon in the gas will experience the refractive index \( n \) due to the gas and possible partially coherent plane wave pulses. Thus, taking the ensemble average over the radiation gas background in Eq. (2), we find the relation [17]

\[
\frac{2}{\sqrt{1 - 2\lambda_p E_g}} + \frac{2}{\lambda_p E_g} \left( 1 - 1/2 \frac{\nabla \cdot E_g}{E_g} \right) \tag{5a}
\]

for the refractive index of a radiation gas, as experienced by a plane wave pulse. The relation is valid for a weakly anisotropic and inhomogeneous radiation gas, since we have neglected the contribution from the averaged radiation gas momentum density. Moreover, we have used \( \langle \hat{\mathbf{k}}_p \cdot E_g \rangle = \langle \mathbf{B}_g \rangle = 0 \), i.e., \( n/\sqrt{E_g} \) as \( E = \langle |E_g|^2 \rangle \), and \( \mathbf{B}_g \) denotes the unit wavevector of the pulse. In other words, the radiation gas is assumed to be close to equilibrium. The relation (5a) looks deceptively simple, but one has to keep in mind that \( \lambda_p = \lambda_p(n_g, E) \) is determined through Eq. (4). Thus, Eq. (5a) constitutes an implicit relation for the refractive index.

In order to determine \( \lambda_p \), we have to evaluate the derivatives of the Lagrangian (1). For a radiation gas background, where each photon experiences the radiation gas through its refractive index, the invariants satisfy \( u = \langle \sigma_r - 1 \rangle \sqrt{E_g} \) and \( b = 0 \). Neglecting terms proportional to \( b \) in the denominator of Eq. (4), we obtain the approximate expression

\[
\lambda_p = \frac{\sqrt{E_g}}{\sqrt{E_{\text{crit}}}} \frac{F(u, E_{\text{crit}})}{F(u, E_{\text{crit}})} \tag{5b}
\]

for the effective action charge. Here \( F(u, E_{\text{crit}}) = 4\pi e/(\mu_0 c)^2 \) and \( E_{\text{crit}} = m_e^2/e - 10^{18} \text{eV}/\text{cm} \) is the Schwinger field [2].

In Fig. 1 we have plotted the dimensionless number \( \lambda_p E_{\text{crit}} \) as a function of the dimensionless variable \( u/E_{\text{crit}} \). However, since \( u \) is a function of \( n \), Eq. (5a) still has to be solved for numerically in order to obtain the refractive index as a function of the radiation gas energy density \( E \).

The case of a photon in random motion in the background field of a plane wave pulse, represented by the slowly varying wave amplitude \( E_p \), can be treated in a similar way. Taking the ensemble average of Eq. (2) with respect to a random ensemble of test photons yields

\[
\frac{2}{\sqrt{1 - 2\lambda_p |E_p|^2}} + \frac{2}{\lambda_p |E_p|^2} \left( 1 - 1/2 \frac{\nabla \cdot |E_p|^2}{E_g} \right) \tag{6a}
\]

Note that this can easily be generalized to incorporate the self-interaction within the gas (which is non-zero in general). We only need to make the replacement \( |E_p|^2 \rightarrow E_{\text{gas}} \), where \( E_{\text{gas}} = E + |E_p|^2 \). Equation (6a) has to be supplemented by an expression for the effective action charge, obtained from Eq. (4) according to

\[
\lambda_p = \frac{2}{\sqrt{n_{\text{gas}}} F(u_p, E_{\text{gas}})}, \tag{6b}
\]

where \( u_p = (n_p - 1)|E_p|^2 \). By the replacement \( |E_p|^2 \rightarrow E_{\text{gas}} \) we also include the gas self-interactions.

In the weak field limit, \( n_{\text{gas}} \) is close to unity, and the approximation \( n_{\text{gas}} \approx 1 \) may therefore be used in the evaluation of \( \lambda_p \). We then obtain \( \lambda_p = \lambda_p(n_p, E) \). Keeping the first order terms in the energy densities of the pulse and the gas, Eqs. (5a) and (6a) yields

\[
n_p = 1 + 1/2 \frac{E_p}{E}, \tag{7a}
\]

and

\[
n_p = 1 + 1/2 |E_p|^2, \tag{7b}
\]

respectively. These results coincide with the ones obtained in Ref. [6].

3.1. Collective interactions

The relation between the energy density \( E \) and the refractive index \( n(E) \), as given by Eq. (5) or in the weak field limit by Eq. (7a), can be used to derive a wave kinetic equation, determining the collective dynamics of a partially coherent pulse of high frequency photons [20].

\[
\frac{\partial I}{\partial t} + \nabla I + \frac{1}{n_{\text{gas}}(E) \sqrt{E_{\text{gas}}}} \frac{\partial}{\partial E} \frac{\partial I}{\partial E} = 0 \tag{8a}
\]

obtained using Hamilton’s ray equations for the individual photons. Here, the specific intensity \( I(\tau, r, \mathbf{k}) \) is normalised such that

\[
\left( |E_p|^2 \right) = \int 1 d^3 k \tag{8b}
\]

is the energy density of the photons, and the angular brackets is the ensemble average over the partially coherent photons in the pulse.

Following Ref. [6], the dynamics of the low frequency radiation gas, giving rise to the energy density \( E \), can be formulated in terms of a coupled set of fluid equations. Assuming a particle frame, and an equation of state \( P = E/3 \), where \( P \) is the radiation gas pressure, the fluid hierarchy is closed at the second moment to yield

\[
\frac{\partial E}{\partial t} + \nabla \left( \frac{\partial}{\partial E} \frac{\partial I}{\partial E} \right) = E \frac{\partial}{\partial E} \frac{\partial I}{\partial E} \frac{\partial I}{\partial E} \tag{9a}
\]

where \( E(\tau, r) \) is weakly varying, and

\[
\frac{\partial I}{\partial t} + \frac{1}{3} \nabla E = E \frac{\partial}{\partial E} \frac{\partial I}{\partial E} \frac{\partial I}{\partial E} \tag{9b}
\]
where $|H|$ is small compared to $E$, and $E_{\text{lin}} = E + (|E|p)^2_i$. The equations (8) together with (9) give the collective interaction between partially coherent high frequency photons and a radiation gas.

4. Stability Analysis

We now turn our attention to the stability of perturbations of the system (8) and (9). Starting with

$$I = I_0(t, r, k), \quad E = E_{\text{lin}} + E_i(t, r),$$

and

$$\Pi = \Pi_i(t, r),$$

where $I_0 \gg |I|$, $E_{\text{lin}} \gg |E_i|$, $E_i \gg ||H|$, we derive the general dispersion relation for the linear perturbation. Inserting the form (10) into Eqs. (8) and (9), we obtain

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial r} \left( I E_i + \frac{|E|}{n} I' (|E|p)^2_i \right) = 0,$$

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial r} \left( E_i E_i' + \frac{|E|}{n} E_i' (|E|p)^2_i \right) = 0,$$

(11a)

and

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial r} \left( |E| (E_i E_i') + \frac{|E|}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial r} \left( |E| (E_i E_i') + \frac{|E|}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

(11b)

where $n = n_0$, $|E| = n_0 (|E| E_i')$, $E_0 = dE_0/dE_0$, $n_0 = dE_0/dE_0$, and

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial r} \left( E_0 (E_i E_i') + \frac{E}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

(11c)

Inserting the form (10) into Eqs. (8) and (9), we obtain

$$\delta \Phi = \frac{\partial}{\partial r} \left( \delta \Phi (|E|p)^2_i \right),$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial r} \left( E_0 (E_i E_i') + \frac{E}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

(12)

for a perturbation of the system (8) and (9). Here $\alpha = E_0 (E_i E_i')/n_0$. The case of a one-dimensional beam in the $z$-direction, $E_i = |E| (E_i E_i')$, and Eq. (13) becomes

$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial r} \left( E_i E_i' + \frac{E}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

(13a)

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial r} \left( E_0 (E_i E_i') + \frac{E}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

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(13b)

and

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial r} \left( E_0 (E_i E_i') + \frac{E}{n} (E_i E_i') (|E|p)^2_i \right) = 0,$$

(13c)

We can see that the characteristics between longitudinal ($K_z = 0$) and transverse ($K_z \neq 0$) perturbations are very different. The former is a stable perturbation, while the latter gives an instability growth rate $\gamma = -Q$ according to

$$\gamma^2 = \left\{ \frac{1}{2} \left[ \frac{k}{\rho_0} (1 + \alpha) + \rho_0 \rho_0' \right] \right\},$$

(14)

Furthermore, if the beam has a spread in $k$-space, as in the case of a Gaussian or Lorentzian distribution, there will be damping due to the poles of the integrand in Eq. (13) when the dimension of the physical system exceeds one.

5. Discussion and Conclusions

Currently, laser intensities can reach $10^{22}$ W/cm$^2$ and in the near future will go beyond this value, possibly reaching $10^{24}$ W/cm$^2$ [21]. The intensity limit of ordinary laser techniques are well below the Schwinger critical field strength [21], but laser-plasma systems hold the promise of surpassing these limits, coming closer to the Schwinger intensity $10^{29}$ W/cm$^2$ [22, 23], at which the vacuum becomes fully nonlinear. When laser intensities reach the above field strength, the plasma particles will achieve highly relativistic quiver velocities. Thus, the relativistic ponderomotive force will reach appreciable values, expelling the plasma particles [24, 25]. Due to this expulsion, plasma channels will form and provide conditions for elastic photon-photon scattering [26, 27]. Thus, the next generation laser-plasma systems will provide conditions at which Eqs. (8) and (9) will be applicable.

In this paper, we have presented the equations describing the collective interaction between a pulse of incoherent photons and a radiation gas close to equilibrium, at arbitrary intensities. The derivation of these equations was based on the one-loop Lagrangian of Schwinger, and the resulting refractive index of the background electromagnetic fields. The equations where linearized around a constant background, and the general dispersion relation for the perturbation spectrum was derived. It was shown that transverse perturbations are unstable, and the corresponding instability growth rate was found. Applications of the results to the next generation laser-plasma systems were discussed.

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