Abstract

The max-product algorithm, which attempts to compute the maximizing assignment of a given objective function, has recently found applications in quadratic minimization and combinatorial optimization. Unfortunately, the max-product algorithm is not guaranteed to converge and, even if it does, is not guaranteed to produce the optimal assignment. In this work, we provide a simple derivation of a new family of message passing algorithms. We first show how to arrive at this general message passing scheme by “splitting” the factors of our graphical model, and then we demonstrate that this construction can be extended beyond integral splitting. We prove that, for any objective function that attains its maximum value over its domain, this new family of message passing algorithms always contains a message passing scheme that guarantees correctness upon convergence to a unique estimate. Finally, we adopt an asynchronous message passing schedule and prove that, under mild assumptions, such a schedule guarantees the convergence of our algorithm.

1. Introduction

Belief propagation was originally formulated by Judea Pearl as a distributed algorithm to perform statistical inference on probability distributions [Pearl (1982)]. His primary observation was that computing marginals is, in general, an expensive operation. However, if the probability distribution can be written as a product of smaller factors that only depend on a small subset of the variables then one could possibly compute the marginals much faster. This “factorization” is captured by a corresponding graphical model. Pearl demonstrated that, when the graphical model is a tree, the belief propagation algorithm is guaranteed to converge to the exact marginals of the input probability distribution. If the algorithm is run on an arbitrary graph that is not a tree, then neither convergence nor correctness are guaranteed.
Max-product, which Pearl dubbed belief revision, is a variant of the belief propagation algorithm where the summations are replaced by maximizations. The goal of the max-product algorithm is to compute the assignment of the variables that maximizes a given objective function. In general, computing such an assignment is an NP-hard problem, but for graphical models possessing a single cycle the algorithm is guaranteed to converge to the maximizing assignment under a few mild assumptions [Weiss (2000)]. Over arbitrary graphical models, the max-product algorithm may fail to converge [Malioutov et al. (2006)] or, worse, may converge to an assignment that is not optimal [Weiss and Freeman (2001)]. Despite these difficulties, max-product and its variants have found empirical success in a variety of application areas including statistical physics, combinatorial optimization [Bayati et al. (2005), Sanghavi and Shah (2005), Sanghavi et al. (2009), Ruozzi and Tatikonda (2008)], computer vision, clustering [Frey and D. (2007), error-correcting codes Berrou et al. (1993)], and the minimization of convex functions [Moallemi and Van Roy (2007); however, rigorously characterizing their behavior outside of a few well-structured instances has proved challenging.

In order to resolve the difficulties presented by the standard max-product algorithm, several alternate message passing schemes have been proposed to compute the maximizing assignment over arbitrary graphical models: MPLP [Globerson and Jaakkola (2007)], tree-reweighted max-product (TRMP) [Wainwright et al. (2005)], and max-sum diffusion (MSD) [Werner (2007)]. Recently, all of these algorithms were shown to be members of a class of “bound minimizing” algorithms for which, under a suitable update schedule, convergence of the algorithms is guaranteed [Meltzer et al. (2009)].

The TRMP algorithm is the max-product analog of the tree-reweighted belief propagation algorithm (TRBP). TRBP, like belief propagation (BP), is an algorithm designed to compute the marginals of a given probability distribution. The key insight that the TRMP algorithm exploits is the observation that the max-product algorithm is correct on trees. The TRMP algorithm begins by choosing a probability distribution over spanning trees of the factor graph and then rewrites the original distribution as an expectation over spanning trees. With this simple rewriting and subsequent derivation of a new message passing scheme, one can show that, for discrete state spaces, TRMP guarantees correctness upon convergence to a unique estimate [Wainwright et al. (2005)]. These results were expanded in subsequent works [Kolmogorov and Wainwright (2005), Kolmogorov (2006), and recently, a serial version of TRMP denoted TRW-S was shown to be provably convergent [Kolmogorov (2006)].

The MPLP algorithm is derived from a special form of the dual linear programming relaxation of the maximization problem. Over discrete state spaces, the algorithm is guaranteed to converge and is correct upon convergence to a unique estimate. Unlike the TRMP algorithm, the MPLP algorithm does not require a choice of parameters. Because choosing the constants for TRMP does require some care, the MPLP algorithm may seem preferable. However, as we will demonstrate by example, the constants provide some flexibility to overcome bad behavior of the algorithm. For example, there are applications over continuous state spaces for which the choice of constants is critical to convergence and correctness. One such example is the quadratic minimization problem. For this application, there exist positive definite matrices for which the TRMP message passing scheme does not converge to the correct minimizing, regardless of the chosen distribution over spanning trees [Ruozzi and Tatikonda (2010)].
We propose a new message passing scheme for the solving the maximization problem based on a simple “splitting” heuristic. Our contributions include:

- A simple and novel derivation of our message passing scheme for general factor graphs.
- A message passing schedule and conditions under which our algorithm converges.
- A simple choice of the parameters such that if each of the beliefs $b_i(x_i)$ has unique argmax then the output of the algorithm is a local optimum of the objective function.
- A simple choice of the parameters such that if each of the beliefs $b_i(x_i)$ has unique argmax then the output of the algorithm is a global optimum of the objective function.
- Conditions under which the algorithm cannot converge to a unique globally optimal estimate.

Unlike MPLP, TRMP, and MSD, the derivation of this algorithm is surprisingly simple, and the update rules closely mirror the standard max-product message updates. Because of its simplicity, we are able to present the algorithm in its most general form: our algorithm is not restricted to binary state spaces or pairwise factor graphs. More importantly, almost all of the intuition for the standard max-product algorithm can be extended with very little effort to our framework.

Like TRMP, our algorithm requires choosing a set of constants. Indeed, TRMP can be seen as a special case of our algorithm. However, unlike TRMP, any choice of non-zero constants will suffice to produce a valid message passing algorithm. In this way, our message passing scheme is more appropriately thought of as a family of message passing algorithms. We will show that, assuming the messages passed by the algorithm are always finite, there is always a simple choice of constants that will guarantee convergence. Further, if we are able to extract a unique estimate from the converged beliefs then this estimate is guaranteed to be the maximizing assignment.

The outline of this paper is as follows: in Section 2 we review the max-product algorithm and other relevant background material, in Section 3 we derive a new passing passing algorithm by splitting factor nodes and prove some basic results, in Section 4 we explore the local and global optimality of the fixed points of our message passing scheme, in Section 5 we provide an alternate message passing schedule under which the algorithm is guaranteed to converge and demonstrate that the algorithm cannot always produce a tight lower bound to the objective function, in Section 6 we show how to strengthen the results of the previous sections for the special case in which the alphabet is binary and the factors are pairwise, and we conclude in Section 7.

2. Preliminaries

Before we proceed to our results, we will briefly review the relevant background material pertaining to message passing algorithms. The focus of this paper will be on solving minimization problems for which we can write the objective function as a sum of functions over fewer variables. These “smaller” functions are called potentials. We note that this is equivalent to the problem of maximizing a product of non-negative potentials, as we can
convert the maximum over a product of potentials into a minimum over a sum by taking negative logs. Although the max-product formulation is more popular in the literature, for notational reasons that will become clear in the sequel, we will use the min-sum formulation.

Let \( f : X^n \to \mathbb{R} \cup \{\infty\} \), where \( X \) is an arbitrary set (e.g. \( \mathbb{R}, \{0,1\}, \mathbb{Z}, \) etc.). Throughout this paper, we will be interested in finding an element \((x_1,\ldots,x_n)\in X^n\) that minimizes \( f \), and as such, we will assume that there is such an element. For an arbitrary function, computing this minimum may be difficult, especially if \( n \) is large. The basic observation of the min-sum algorithm is that, even though the original minimization problem may be difficult, if \( f \) can be written as a sum of functions depending on only a small subset of the variables, then we may be able to minimize the global function by performing a series of minimizations over (presumably easier) sub-problems. To make this concrete, let \( \mathcal{A} \subseteq 2^{\{1,\ldots,n\}} \). We say that \( f \) factorizes over \( \mathcal{A} \) if we can write \( f \) as a sum of real valued potential functions \( \phi_i : X \to \mathbb{R} \cup \{\infty\} \) and \( \psi_\alpha : X^{[\alpha]} \to \mathbb{R} \cup \{\infty\} \) as follows:

\[
f(x) = \sum_i \phi_i(x_i) + \sum_{\alpha \in \mathcal{A}} \psi_\alpha(x_\alpha) \tag{1}\]

Every factorization of \( f \) has a corresponding graphical representation known as a factor graph. The factor graph consists of a node \( i \) for each variable \( x_i \) and a factor node \( \alpha \) for each of the factors \( \psi_\alpha \) with an edge joining the factor node corresponding to \( \alpha \) to the variable node representing \( x_i \) if \( i \in \alpha \). For a concrete example, see Figure 1. The min-sum algorithm is a message passing algorithm on this factor graph. In the algorithm, there are two types of messages: messages passed from variable nodes to factor nodes and messages passed from factor nodes to variable nodes. On the \( t^{th} \) iteration of the algorithm, messages are passed along each edge of the factor graph as follows:

\[
m^t_{i\to\alpha}(x_i) = \kappa + \phi_i(x_i) + \sum_{\beta \in \partial i \setminus \alpha} m^{t-1}_{\beta \to i}(x_i) \tag{2}\]

\[
m^t_{\alpha \to i}(x_i) = \kappa + \min_{x_\alpha \setminus i} \left[ \psi_\alpha(x_\alpha) + \sum_{k \in \alpha \setminus i} m^{t-1}_{k \to \alpha}(x_k) \right] \tag{3}\]

where \( \partial i \) denotes the set of all \( \alpha \in \mathcal{A} \) such that \( i \in \alpha \) (intuitively, this is the set of neighbors of variable node \( x_i \) in the factor graph), \( x_\alpha \) is the vector formed from the entries of \( x \) by selecting only the indices in \( \alpha \), and \( \alpha \setminus i \) is abusive notation for the set-theoretic difference \( \alpha \setminus \{i\} \).
Each message update has an arbitrary normalization factor $\kappa$. Because $\kappa$ is not a function of any of the variables, it only affects the value of the minimum and not where the minimum is located. As such, we are free to choose it however we like for each message and each time step. In practice, these constants are used to avoid numerical issues that may arise during execution of the algorithm. We will think of the messages as a vector of functions indexed by the edge over which the message is passed.

**Definition 1** A vector of messages $m = \{m_{\alpha \rightarrow i}, m_{i \rightarrow \alpha}\}$ is finite if for all $\alpha \in A$, $\forall i \in \alpha$, and $\forall x_i \in X$, $|m_{\alpha \rightarrow i}(x_i)| < \infty$ and $|m_{i \rightarrow \alpha}(x_i)| < \infty$.

Any vector of finite messages is a valid choice for the vector of initial messages $m^0$, but the choice of initial messages can greatly affect the behavior of the algorithm. A typical assumption is that the initial messages are chosen such that $m^0_{\alpha \rightarrow i} \equiv 0$ and $m^0_{i \rightarrow \alpha} \equiv 0$.

We want to use the messages in order to construct an estimate of the min-marginals of $f$. A min-marginal of $f$ is a function of one variable obtained by minimizing the function $f$ over all of the remaining variables. The min-marginal for the variable $x_i$ would be $\min_{x_i': x_i' = x_i} f(x')$ which is a function of $x_i$. Given any vector of messages, $m^t$, we can construct a set of beliefs that are intended to approximate the min-marginals of $f$:

$$b^t_i(x_i) = \kappa + \phi_i(x_i) + \sum_{\alpha \in \partial i} m^t_{\alpha \rightarrow i}(x_i)$$  \hspace{1cm} (4)$$

$$b^t_{\alpha}(x_{\alpha}) = \kappa + \psi_{\alpha}(x_{\alpha}) + \sum_{i \in \alpha} m^t_{i \rightarrow \alpha}(x_i)$$  \hspace{1cm} (5)$$

If $b_i(x_i) = \min_{x_i': x_i' = x_i} f(x')$, then for any $y_i \in \arg\min_{x_i} b_i(x_i)$ there exists a vector $x^*$ such that $x^*_i = y_i$ and $x^*$ minimizes the function $f$. If the $|\arg\min_{x_i} b_i(x_i)| = 1$ for all $i$, then we can take $x^* = y$, but, if the objective function has more than one optimal solution, then we may not be able to construct such an $x^*$ so easily. For this reason, one commonly assumes that the objective function has a unique global minimum. Although this assumption is common, we will not adopt this convention in this work. Unfortunately, because our beliefs are not necessarily the true min-marginals, we can only approximate the optimal assignment by computing an estimate of the argmin:

$$x^*_{i} \in \arg\min_{x_i} b^t_i(x_i)$$  \hspace{1cm} (6)$$

**Definition 2** A vector, $b$, of beliefs admits a unique estimate $x^*$, if $x^*_i \in \arg\min_{x_i} b_i(x_i)$ and the argmin is unique for each $i$.

If the algorithm converges to a collection of beliefs from which we can extract a unique estimate $x^*$, then we hope that the vector $x^*$ is indeed a global minimum of the objective function.

### 2.1 Computation Trees

An important tool in the analysis of the min-sum algorithm is the notion of a computation tree. Intuitively, the computation tree is an unrolled version of the original graph that captures the evolution of the messages passed by the min-sum algorithm needed to compute...
Figure 2: The computation tree at time $t = 4$ rooted at the variable node $x_1$ of the factor graph in Figure 1. The variable nodes have been labeled with their potentials for emphasis.

the belief at time $t$ at a particular node of the factor graph. Computation trees describe the evolution of the beliefs over time, which, in some cases, can help us prove correctness and/or convergence of the message passing updates.

The depth $t$ computation tree rooted at node $i$ contains all of the length $t$ non-backtracking walks in the factor graph starting at node $i$. For any node $v$ in the factor graph, the computation tree at time $t$ rooted at $v$, denoted by $T_v(t)$, is defined recursively as follows: $T_v(0)$ is just the node $v$, the root of the tree. The tree $T_v(t)$ at time $t$ is generated from $T_v(t-1)$ by adding to each leaf of $T_v(t-1)$ a copy of each of its neighbors in $G$ (and the corresponding edge), except for the neighbor that is already present in $T_v(t-1)$. Each node of $T_v(t)$ is a copy of a node in $G$, and the potentials on the nodes in $T_v(t)$, which operate on a subset of the variables in $T_v(t)$, are copies of the potentials of the corresponding nodes in $G$. The construction of a computation tree for the graph in Figure 1 is pictured in Figure 2. Note that each variable node in $T_v(t)$ represents a distinct copy of some variable $x_j$ in the original graph.

Given any initialization of the messages, $T_v(t)$ captures the information available to node $v$ at time $t$. At time $t = 0$, node $v$ has received only the initial messages from its neighbors, so $T_v(1)$ consists only of $v$. At time $t = 1$, $v$ receives the round one messages from all of its neighbors, so $v$’s neighbors are added to the tree. These round one messages depend only on the initial messages, so the tree terminates at this point. By construction, we have the following lemma:

**Lemma 3** The belief at node $v$ produced by the min-sum algorithm at time $t$ corresponds to the exact min-marginal at the root of $T_v(t)$ whose boundary messages are given by the initial messages.

**Proof** See, for example, Tatikonda and Jordan (2002) and Weiss and Freeman (2001).

2.2 Fixed Point Properties

Computation trees provide us with a dynamic view of the min-sum algorithm. After a finite number of time steps, we hope that the beliefs on the computation trees stop changing. In
practice, when the beliefs change by less than some small amount, we say that the algorithm has converged. If the messages of the min-sum algorithm converge then the converged messages must be fixed points of the message update equations.

Ideally, the converged beliefs would be the true min-marginals of the function \( f \). If the beliefs are the exact min-marginals, then the estimate corresponding to our beliefs would indeed be the global minimum. Unfortunately, the algorithm is only known to produce the exact min-marginals on special factor graphs (e.g. when the factor graph is a tree). Instead, we will show that the fixed point beliefs are almost like min-marginals. Like the messages, we will think of the beliefs as a vector of functions indexed by the nodes of the factor graph. Consider the following definitions:

**Definition 4** A vector of beliefs, \( b \), is **admissible** for a function \( f \) if

\[
f(x) = \kappa + \sum_i b_i(x_i) + \sum_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} b_k(x_k) \right]
\]

**Definition 5** A vector of beliefs, \( b \), is **min-consistent** if for all \( \alpha \) and all \( i \in \alpha \):

\[
\min_{x_{\alpha \setminus i}} b_\alpha(x_\alpha) = \kappa + b_i(x_i)
\]

Any vector of beliefs that satisfies these two properties provides a meaningful reparameterization of the original objective function. We can show that any vector of beliefs obtained from a fixed point of the message updates does indeed satisfy these two properties:

**Theorem 6** For any vector of fixed point messages, the corresponding beliefs are admissible and min-consistent.

**Proof** See Wainwright et al. (2004), Proposition 2 and lemmas 7 and 8 below.

For any objective function \( f \) such that for all \( x \), \(|f(x)| < \infty\), there always exists a fixed point of the min-sum message passing updates (see Theorem 2 of Wainwright et al. (2004)). Moreover, the min-sum algorithm is guaranteed to converge to the correct solution on factor graphs that are trees. However, convergence and correctness for arbitrary factor graphs has only been demonstrated for a few special cases Wainwright et al. (2004) Weiss (2000).

3. A General Splitting Heuristic

In this section, we introduce a family of message passing algorithms parameterized by a vector of reals. The intuition for this family of algorithms is simple: given any factorization of the objective function \( f \), we can split any of the factors into several pieces and obtain a new factorization of the objective function \( f \). The standard notation masks the fact that each of the potentials may further factorize into smaller pieces. For example, suppose we are given the objective function \( f(x_1,x_2) = x_1 + x_2 + x_1 x_2 \). There are many different ways that we can factorize \( f \):

\[
f(x_1,x_2) = x_1 + x_2 + x_1 x_2
\]

(7)
Each of these represents a factorization of $f$ into a different number of potentials (the parenthesis indicate a single potential function). All of these can be captured by the standard min-sum algorithm except for the last. Recall that $\mathcal{A}$ was taken to be a subset of $2^{\{1,\ldots,n\}}$. In order to accommodate the factorization given by Equation (10) we will now allow $\mathcal{A}$ to be a multiset over the set $2^{\{1,\ldots,n\}}$. We can then construct the factor graph as before with a distinct factor node for each element of the multiset $\mathcal{A}$. We can use the standard min-sum algorithm in an attempt to compute the minimum of $f$ given this new factorization.

We could, of course, rewrite the objective function in many different ways. However, arbitrarily rewriting the objective function could significantly increase the size of the factor graph, and such rewriting may not make the minimization problem any easier. In this paper, we will focus on one special rewriting of the objective function. Suppose $f$ factorizes over $\mathcal{A}$ as in Equation (1). Let $G$ be the corresponding factor graph. Suppose now that we take one potential $\alpha \in \mathcal{A}$ and split it into $k$ potentials $\alpha_1, \ldots, \alpha_k$ such that for each $j \in \{1\ldots k\}$, $\psi_{\alpha_j} = \frac{\psi_\alpha}{k}$. This allows us to rewrite the objective function, $f$, as

$$f(x) = \sum_i \phi_i(x_i) + \sum_{\beta \in \mathcal{A}} \psi_\beta(x_\beta)$$

$$= \sum_i \phi_i(x_i) + \sum_{\beta \in \mathcal{A}\setminus\alpha} \psi_\beta(x_\beta) + \sum_{j=1}^k \frac{\psi_\alpha(x_\alpha)}{k}$$

$$= \sum_i \phi_i(x_i) + \sum_{\beta \in \mathcal{A}\setminus\alpha} \psi_\beta(x_\beta) + \sum_{j=1}^k \psi_{\alpha_j}(x_\alpha)$$

This rewriting does not change the objective function, but it does produce a new factor graph $F$ (see Figure 3). Now, take some $i \in \alpha$ and consider the messages $m_{i\rightarrow\alpha_j}$ and $m_{\alpha_j\rightarrow i}$ given by the standard min-sum algorithm:

$$m^t_{i\rightarrow\alpha_j}(x_i) = \kappa + \phi_i(x_i) + \sum_{\beta \in \partial F_i \setminus \alpha_j} m^{t-1}_{\beta \rightarrow i}(x_i)$$
with its old potential. For an example of this construction, see Figure 4. This rewriting

\[
m^t_{\alpha_j \rightarrow i}(x_i) = \kappa + \min_{x_\alpha \setminus i} \frac{\psi_\alpha(x_\alpha)}{\kappa} + \sum_{k \in \partial F \setminus \alpha} m^{t-1}_{k \rightarrow \alpha_j}(x_k)
\]

where \(\partial F \setminus i\) denotes the neighbors of \(i\) in \(F\). Notice that there is an automorphism of the graph that maps \(\alpha_l\) to \(\alpha_j\). As the messages passed from any node only depend on the messages received at the previous time step, if the initial messages are the same at both of these nodes, then they must produce identical messages at time 1. More formally, if we initialize the messages identically over each split edge, then, at any time step \(t \geq 0\), \(m^t_{i \rightarrow \alpha_j}(x_i) = m^t_{i \rightarrow \alpha_0}(x_i)\) and \(m^t_{\alpha_j \rightarrow i}(x_i) = m^t_{\alpha_j \rightarrow i}(x_i)\) for any \(l \in \{1, \ldots, k\}\) by symmetry (i.e. there is an automorphism of the graph that maps \(\alpha_l\) to \(\alpha_j\)). Because of this, we can rewrite the message from \(i\) to \(\alpha_j\) as:

\[
m^t_{i \rightarrow \alpha_j}(x_i) = \kappa + \phi_l(x_i) + \sum_{\beta \in \partial F \setminus \alpha_j} m^{t-1}_{\beta \rightarrow i}(x_i)
\]

\[
m^t_{i \rightarrow \alpha_j}(x_i) = \kappa + \phi_l(x_i) + \sum_{\beta \notin j} m^{t-1}_{\alpha_\beta \rightarrow i}(x_i) + \sum_{\beta \in \partial F \setminus \alpha} m^{t-1}_{\beta \rightarrow i}(x_i)
\]

\[
m^t_{i \rightarrow \alpha_j}(x_i) = \kappa + \phi_l(x_i) + (k - 1)m^{t-1}_{\alpha_j \rightarrow i}(x_i) + \sum_{\beta \in \partial F \setminus \alpha} m^{t-1}_{\beta \rightarrow i}(x_i)
\]

Notice that Equation 18 can be viewed as a message passing algorithm on the original factor graph. The primary difference then between Equation 18 and the standard min-sum updates is that the message passed from \(i\) to \(\alpha\) now depends on the message from \(\alpha\) to \(i\).

Analogously, we can also split the variable nodes. Suppose \(f\) factorizes over \(A\) as in Equation 1. Let \(G\) be the corresponding factor graph. Suppose now that we take one variable \(x_i\) and split it into \(k\) variables \(x_{i_1}, \ldots, x_{i_k}\) such that for each \(l \in \{1 \ldots k\}\), \(\phi_{i_l} = \frac{\phi(x_{i_l})}{k}\). This produces a new factor graph, \(F\). Because \(x_{i_1}, \ldots, x_{i_k}\) are all the same variable, we must add a constraint to ensure that they are indeed the same. Next, we need to modify the potentials to incorporate the constraint and the change of variables. We will construct \(A_F\) such that for each \(\alpha \in A\) with \(i \in \alpha\) there is a \(\beta = (\alpha \setminus i) \cup \{i_1, \ldots, i_k\}\) in \(A_F\). Define \(\psi_\beta(x_\beta) = \psi_\alpha(x_{\alpha \setminus i}, x_{i_1}) - \log \{x_{i_1} = \ldots = x_{i_k}\}\) where \(\{x_{i_1} = \ldots = x_{i_k}\}\) is the 0-1 indicator function for the equality constraint. For each \(\alpha \in A\) with \(i \notin \alpha\) we simply add \(\alpha\) to \(A_F\) with its old potential. For an example of this construction, see Figure 4. This rewriting
produces a new objective function

\[
g(x) = \sum_{j \neq i} \phi_j(x_j) + \sum_{l=1}^{k} \phi_l(x_{i_l}) + \sum_{\alpha \in A_F} \psi_\alpha(x_\alpha) \tag{19}
\]

Minimizing \(g\) is equivalent to minimizing \(f\). Again, we will show that we can collapse the min-sum message passing updates over \(F\) to message passing updates over \(G\) with modified potentials. Take some \(\alpha \in A_F\) containing the new variable \(i_1\) which augments the potential \(\gamma \in A\) and consider the messages \(m_{i_1 \to \beta}\) and \(m_{\alpha \to i_1}\) given by the standard min-sum algorithm:

\[
m^t_{i_1 \to \alpha}(x_{i_1}) = \kappa + \frac{\phi_i(x_{i_1})}{k} + \sum_{\beta \in \partial_{\beta i_1} \setminus \alpha} m^{t-1}_{\beta \to i_1}(x_{i_1}) \tag{20}
\]

\[
m^t_{\alpha \to i_1}(x_{i_1}) = \kappa + \min_{x_\alpha \setminus i_1} \psi_\alpha(x_\alpha) + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{21}
\]

Again, if we initialize the messages identically over each split edge, then, at any time step \(t \geq 0\), \(m^t_{i_1 \to \alpha}(x_i) = m^t_{i_\to \alpha}(x_i)\) and \(m^t_{\alpha \to i_1}(x_i) = m^t_{\alpha \to i_1}(x_i)\) for any \(l \in \{1, ..., k\}\) by symmetry. Using this, we can rewrite the message from \(\alpha\) to \(i_1\) as:

\[
m^t_{\alpha \to i_1}(x_{i_1}) = \kappa + \min_{x_\alpha \setminus i_1} \psi_\alpha(x_\alpha) + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{22}
\]

\[= \kappa + \min_{x_\alpha \setminus i_1} \psi_\gamma(x_\alpha, x_{i_1}) - \log \left\{ x_{i_1} = ... = x_{i_k} \right\} + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{23}
\]

\[= \kappa + \min_{x_\alpha \setminus i_1} \psi_\gamma(x_\alpha, x_{i_1}) - \log \{x_{i_1} = ... = x_{i_k}\} \]

\[+ \sum_{l \neq 1} m^{t-1}_{l_1 \to \alpha}(x_{i_l}) + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{24}
\]

\[= \kappa + \min_{x_\alpha \setminus i_1} \psi_\gamma(x_\alpha, x_{i_1}) + \sum_{l \neq 1} m^{t-1}_{l_1 \to \alpha}(x_{i_l}) + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{25}
\]

\[= \kappa + \min_{x_\alpha \setminus i_1} \psi_\gamma(x_\alpha, x_{i_1}) + (k-1)m^{t-1}_{l_1 \to \alpha}(x_{i_1}) + \sum_{k \in A \setminus i_1} m^{t-1}_{k \to \alpha}(x_k) \tag{26}
\]

By symmetry, we only need to perform one message update to compute \(m^t_{\alpha \to i_l}(x_{i_l})\) for each \(l \in \{1, ..., k\}\). As a result, we can think of these messages as being passed on the original factor graph \(G\).

The combined message updates for each of these splitting operations are presented in Algorithm 1. Observe that if we choose \(c_\alpha = 1\) for each \(\alpha\) and \(c_i = 1\) for each \(i\), then the message updates described in the algorithm are exactly the min-sum message passing updates described in the preliminaries. Rewriting the message updates in this way seems purely cosmetic, but as we will show in the following sections, the choice of the vector \(c\) can influence both the convergence and correctness of the algorithm.

We define the beliefs corresponding to the new message updates as follows:

\[
b^t_i(x_i) = \kappa + \frac{\phi_i(x_i)}{c_i} + \sum_{\alpha \in \partial i} c_\alpha m^t_{\alpha \to i}(x_i) \tag{27}
\]
Algorithm 1: Synchronous Splitting Algorithm

1: Initialize the messages to some finite vector.
2: For iteration $t = 1, 2, \ldots$ update the the messages as follows

$$m_{i \rightarrow \alpha}^t(x_i) = \kappa + \frac{\phi_i(x_i)}{c_i} + (c_\alpha - 1)m_{\alpha \rightarrow i}^{t-1}(x_i) + \sum_{\beta \in \partial \setminus \alpha} c_\beta m_{\beta \rightarrow i}^{t-1}(x_i)$$

$$m_{\alpha \rightarrow i}^t(x_i) = \kappa + \min_{x_{\alpha \setminus i}} \left[ \frac{\psi_\alpha(x_\alpha)}{c_\alpha} + (c_i - 1)m_{i \rightarrow \alpha}^{t-1}(x_i) + \sum_{k \in \alpha \setminus i} c_k m_{k \rightarrow \alpha}^{t-1}(x_k) \right]$$

$$b_\alpha^t(x_\alpha) = \kappa + \frac{\psi_\alpha(x_\alpha)}{c_\alpha} + \sum_{k \in \alpha} c_k \left[ \frac{\phi_k(x_k)}{c_k} + (c_\alpha - 1)m_{\alpha \rightarrow k}^t(x_k) + \sum_{\beta \in \partial k \setminus \alpha} c_\beta m_{\beta \rightarrow k}^t(x_k) \right] \quad (28)$$

Compare these definitions with Equations (4) and (5). Notice that the bracketed expression in Equation (28) is the definition of $m_{k \rightarrow \alpha}$. As we will see in Lemma 7, if we define the beliefs in this way, then any vector of finite messages will produce a vector of admissible beliefs. The beliefs are still approximating the min-marginals of $f$, but each variable node has been split $c_i$ times and each factor node has been split $c_\alpha$ times. Applying Definition 4 to the new factor graph $F$, a vector of beliefs is admissible for our new message passing algorithm if

$$f(x) = \kappa + \sum_i \sum_{x_i} b_i(x_i) + \sum_{\alpha \in A} \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} b_k(x_k) \right] \quad (29)$$

$$= \kappa + \sum_{i \in \{1, \ldots, n\}} c_i b_i(x_i) + \sum_{\alpha \in A} c_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} b_k(x_k) \right] \quad (30)$$

Throughout this discussion, we have assumed that the vector $c$ contained only positive integers. If we allow $c$ to be an arbitrary vector of non-zero reals, then the notion of splitting no longer makes sense. Instead, we will think of the vector $c$ as parameterizing a specific factorization of the function $f$. The definitions of the message updates and the beliefs are equally valid for any choice of non-zero real constants. In what follows, we will explore the properties of this new message passing scheme for a vector $c$ of non-zero reals.

As before, we want the fixed point beliefs produced by our message passing scheme to behave like min-marginals (i.e. they are min-consistent) and they produce a reparameterization of the objective function. Using the definitions above, we have have the following lemmas:

Lemma 7 Let $f$ factorize over $A$. If $c$ is a vector of non-zero reals, then for any vector of finite messages, $m$, the corresponding beliefs are admissible.

Proof Let $m$ be the vector of messages given in the statement of the lemma and $b$ the corresponding vector of beliefs. For any set of messages, we can rewrite the belief $b_\alpha$ as:

$$b_\alpha(x_\alpha) = \kappa + \frac{\psi_\alpha(x_\alpha)}{c_\alpha} + \sum_{k \in \alpha} c_k \left( b_k(x_k) - m_{\alpha \rightarrow k}(x_k) \right) \quad (31)$$
where this equality is obtained by plugging the Equation 27 into Equation 28. Using this observation, we can easily verify that, up to an additive constant, the beliefs satisfy:

$$
\sum_i c_i b_i(x_i) + \sum_{\alpha} c_{\alpha} \left[ b_{\alpha}(x_{\alpha}) - \sum_{k \in \alpha} c_k b_k(x_k) \right]
$$

$$
= \sum_i c_i b_i(x_i) + \sum_{\alpha} c_{\alpha} \left[ \psi_{\alpha}(x_{\alpha}) - \sum_{k \in \alpha} c_k m_{\alpha \rightarrow k}(x_k) \right]
$$

$$
= \sum_i \phi_i(x_i) + \sum_{\alpha \in \partial_i} c_i c_{\alpha} m_{\alpha \rightarrow i}(x_i) + \sum_{\alpha} \left[ \psi_{\alpha}(x_{\alpha}) - \sum_{k \in \alpha} c_k c_{\alpha} m_{\alpha \rightarrow k}(x_k) \right]
$$

$$
= \sum_i \phi_i(x_i) + \sum_{\alpha} \psi_{\alpha}(x_{\alpha})
$$

where the last line follows by observing that \( \sum_i \sum_{\alpha \in \partial_i} c_i c_{\alpha} m_{\alpha \rightarrow i}(x_i) = \sum_{\alpha} \sum_{k \in \alpha} c_k c_{\alpha} m_{\alpha \rightarrow k}(x_k) \). Notice that this subtraction of the messages only makes sense if the messages are finite valued.

The previous lemma guarantees that any vector of finite messages is guaranteed to be admissible, but an analogous lemma is not true for min-consistency. We require a stronger assumption about the vector of messages in order to ensure min-consistency:

**Lemma 8** Let \( m \) be a fixed point of the message updates in Algorithm 4. The corresponding beliefs, \( b \), are min-consistent.

**Proof** Up to an additive constant, we can write,

$$
\min_{x_{\alpha \backslash i}} b_{\alpha}(x_{\alpha}) = \min_{x_{\alpha \backslash i}} \frac{\psi_{\alpha}(x_{\alpha})}{c_{\alpha}} + \sum_{k \in \alpha} c_k \left[ \frac{\phi_k(x_k)}{c_k} + (c_{\alpha} - 1) m_{\alpha \rightarrow k}(x_k) + \sum_{\beta \in \partial k \backslash \alpha} c_{\beta} m_{\beta \rightarrow k}(x_k) \right]
$$

$$
= \min_{x_{\alpha \backslash i}} \frac{\psi_{\alpha}(x_{\alpha})}{c_{\alpha}} + \sum_{k \in \alpha} c_k m_{k \rightarrow \alpha}(x_k)
$$

$$
= \min_{x_{\alpha \backslash i}} \frac{\psi_{\alpha}(x_{\alpha})}{c_{\alpha}} + \sum_{k \in \alpha \backslash i} c_k m_{k \rightarrow \alpha}(x_k) + c_i m_{i \rightarrow \alpha}(x_i)
$$

$$
= m_{\alpha \rightarrow i}(x_i) + m_{i \rightarrow \alpha}(x_i)
$$

$$
= m_{\alpha \rightarrow i}(x_i) + \left[ \frac{\phi_i(x_i)}{c_i} + (c_{\alpha} - 1) m_{\alpha \rightarrow i}(x_i) + \sum_{\beta \in \partial i \backslash \alpha} c_{\beta} m_{\beta \rightarrow i}(x_i) \right]
$$

$$
= \frac{\phi_i(x_i)}{c_i} + \sum_{\beta \in \partial i} c_{\beta} m_{\beta \rightarrow i}(x_i)
$$

$$
= b_i(x_i)
$$

Again, for any objective function \( f \) such that for all \( x \), \(|f(x)| < \infty\), there always exists a fixed point of the min-sum message passing updates (see Theorem 2 of Wainwright et al. (2004)). The proof of this statement can be translated almost exactly for our message passing updates, and we will not reproduce it here.
3.1 Computation Trees

The computation trees produced by the synchronous splitting algorithm are different from their predecessors. Again, the computation tree captures the messages that would need to be passed in order to compute $b_t^i(x_i)$. However, the messages that are passed in the new algorithm are multiplied by a non-zero constant. As a result, the potential at a node $u$ in the computation tree corresponds to some potential in the original graph multiplied by a constant that depends on all of the nodes above $u$ in the computation tree. We summarize the changes as follows:

1. The message passed from $i$ to $\alpha$ may now depend on the message from $\alpha$ to $i$ at the previous time step. As such, we now form the time $t + 1$ computation tree from the time $t$ computation tree by taking any leaf $u$, which is a copy of node $v$ in the factor graph, of the time $t$ computation tree, creating a new node for every $w \in \partial v$, and connecting $u$ to these new nodes. As a result, the new computation tree rooted at node $u$ of depth $t$ contains at least all of the non-backtracking walks of length $t$ in the factor graph starting from $u$ and, at most, all walks of length $t$ in the factor graph starting at $u$.

2. The messages are weighted by the elements of $c$. This changes the potentials at the nodes in the computation tree. For example, suppose the computation tree was rooted at variable node $i$ and that $b_t^i$ depends on the message from $\alpha$ to $i$. Because $m_{\alpha \rightarrow i}$ is multiplied by $c_{\alpha}$ in $b_t^i$, every potential along this branch of the computation tree is multiplied by $c_{\alpha}$. To make this concrete, we can associate a weight to every edge of the computation tree that corresponds to the constant that multiplies the message passed across that edge. To compute the new potential at a variable node $i$ in the computation tree, we now need to multiply the corresponding potential $\phi_i$ by each of the weights corresponding to the edges that appear along the path from $i$ to the root of the computation tree. An analogous process can be used to compute the potentials at each of the factor nodes. The computation tree produced by the splitting algorithm at time $t = 2$ for the factor graph in Figure 1 is pictured in Figure 5. Compare this with computation tree produced by the standard min-sum algorithm in Figure 2.

If we make these adjustments and all of the weights are positive, then the belief, $b_t^i(x_i)$, at node $i$ at time $t$ is given by the min-marginal at the root of $T_i(t)$. If some of the weights are negative, then $b_t^i(x_i)$ is computed by maximizing over each variable in $T_i(t)$ whose self-potential has a negative weight and minimizing over each variable whose self-potential has a non-negative weight. In this way, the beliefs correspond to marginals at the root of these computation trees.

4. Optimality of Fixed Points

Empirically, the standard min-sum algorithm need not converge and, even if it does, the estimate produced at convergence need not actually minimize the objective function. Up until this point, we have not placed any restriction on the vector $c$ except that all of its entries are non-zero. Still, we know from the TRMP case that certain choices of the parameters are better than others: some ensure that the estimate obtained at a fixed point is correct.
Figure 5: Construction of the computation tree rooted at node $x_1$ at time $t = 2$ produced by Algorithm 1 for the factor graph in Figure 1. The message passing tree with edge weights corresponding to the constant that multiplies the message passed across the edge (left) is converted into the new computation tree (right). Notice that the potentials in the new computation tree are now weighted by elements of the parameter vector $c$.

From the previous section, we know that the fixed point beliefs produced by Algorithm 1 are admissible and min-consistent. From these fixed point beliefs, we construct a fixed point estimate $x^*$ such that $x^*_i \in \arg\min b_i$. If the objective function had a unique global minimum and the fixed point beliefs were the true min-marginals, then $x^*$ would indeed be the global minimum. Now, suppose that the $b_i$ are not the true min-marginals. What can we say about the optimality of any vector $x^*$ such that $x^*_i \in \arg\min b_i$? What can we say if there is a unique vector $x^*$ with this property? Our primary tool for answering these questions will be the following lemma:

**Lemma 9** Let $b$ be a vector of min-consistent beliefs. If there exists a unique estimate $x^*$ that minimizes $b_i(x_i)$ for each $i$, then $x^*$ also minimizes $b_\alpha(x_\alpha)$ and, for any $i \in \alpha$, $x^*$ minimizes $b_\alpha(x_\alpha) - b_i(x_i)$.

**Proof** Because the beliefs are min-consistent for any $i \in \alpha$, we have:

$$\min_{x_\alpha \setminus i} b_\alpha(x_\alpha) = \kappa + b_i(x_i)$$

From this, we can conclude that there is a some $x_\alpha$ that minimizes $b_\alpha$ with $x_i = x^*_i$. Further, because the minimum is unique for each $b_i$, $x^*_\alpha$ must minimize $b_\alpha$. Now fix a vector $x$ and consider

$$b_\alpha(x^*_\alpha) - b_i(x^*_i) = \min_{x_\alpha \setminus i} b_\alpha(x^*_i, x_{\alpha \setminus i}) - b_i(x^*_i)$$

$$= \min_{x_\alpha \setminus i} b_\alpha(x_i, x_{\alpha \setminus i}) - b_i(x_i)$$

$$\leq b_\alpha(x_\alpha) - b_i(x_i)$$

This lemma will be a crucial building block of many of the theorems in this paper, and many variants of this lemma have been proven in the literature (e.g. Lemma 4 in Wainwright et al. (2004) and Theorem 1 in Weiss et al. (2007)).
Using this lemma and the observation of Lemma 7 that \( f \) can be written as a sum of the beliefs, we can convert questions about the optimality of the vector \( x \) into questions about the choice of parameters. We will show how to choose the \( c_i \) and \( c_\alpha \) such that we will be guaranteed some form of optimality for a collection of admissible and min-consistent beliefs.

4.1 Local Optimality

A function \( f \) has a local optimum at the point \( x \in \mathcal{X}^n \) if there is some neighborhood of \( x \) such that \( f \) does not increase in that neighborhood. The definition of neighborhood is metric dependent, and in the interest of keeping our results applicable to a wide variety of spaces, we will choose the metric to be the Hamming distance. For any two vectors \( x, y \in \mathcal{X}^n \), the Hamming distance is the number of entries in which the two vectors differ. For the purposes of this paper, we will restrict our definition of local optimality to vectors within Hamming distance one:

**Definition 10** \( x \in \mathcal{X}^n \) is a **local minimum** of the objective function, \( f \), if for every vector \( y \) that has at most one entry different from \( x \), \( f(x) \leq f(y) \).

Our notion does not necessarily coincide with other notions of local optimality from the literature Wainwright et al. (2004). If the standard min-sum algorithm converges to unique estimate, then \( x^* \) is locally optimal in the following sense: \( x^* \) is a global minimum of the reparameterization when it is restricted to factor-induced subgraphs of the factor graph that contain exactly one cycle Wainwright et al. (2004). However, \( x^* \) is not necessarily a global optimum of the objective function. Suppose the factor graph consists of only pairwise factors. In this case, the collection of nodes formed by taking some variable node \( i \) and every node in its two-hop neighborhood must be a tree, \( T \). The restriction of the reparameterization to this tree is given by:

\[
R(x_T) = \sum_{j \in T} b_j(x_j) + \sum_{\alpha \in T} [b_\alpha(x_\alpha) - \sum_{k \in \alpha} b_k(x_k)]
\]  

(32)

\( R \) contains every part of the reparameterization that depends on the variable \( x_i \), and \( x^* \) minimizes \( R \). As a result, we observe that if we change only the value of \( x_i^* \), then we cannot decrease the value of \( R \) and, consequently, we cannot decrease the objective function. In this case, the local optimality condition in Wainwright et al. (2004) does imply local optimality in our sense. However, if the factorization is not pairwise, then the two-hop neighborhood of any node is not necessarily cycle free (see Figure 5). Consequently, the notion of optimality from Wainwright et al. (2004) need not correspond to Definition 10 for graphs where the factorization is not pairwise.

We will show that there exist choices of the parameters for which any fixed point estimate extracted from a vector of admissible and min-consistent beliefs that simultaneously minimizes all of the beliefs is guaranteed to be locally optimal with respect to the Hamming distance. In order to prove such a result, we first need to relate the minima of the fixed point beliefs to the minima of the objective function. By Lemma 7 the objective function \( f \) can be written as a sum of the beliefs. Let \( b \) be a vector of admissible beliefs for the
Figure 6: A factor graph for which the two-hop neighborhood of every node is not a tree.

Define $-j = \{1, \ldots, n\} \setminus \{j\}$. For a fixed $x_{-j}$, we can lower bound the optimum value of the objective function as follows:

$$\min_{x_j} f(x_j, x_{-j}) = \min_{x_j} \left[ \kappa + \sum_i c_i b_i(x_i) + \sum_{\alpha} c_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \right]$$

$$= g_j(x_{-j}) + \min_{x_j} \left[ c_j b_j(x_j) + \sum_{\alpha \in \partial j} c_\alpha \left[ b_\alpha(x_\alpha) - c_j b_j(x_j) \right] \right]$$

$$= g_j(x_{-j}) + \min_{x_j} \left[ (1 - \sum_{\alpha \in \partial j} c_\alpha) c_j b_j(x_j) + \sum_{\alpha \in \partial j} c_\alpha b_\alpha(x_\alpha) \right]$$

$$\geq g_j(x_{-j}) + \min_{x_j} [(1 - \sum_{\alpha \in \partial j} c_\alpha) c_j b_j(x_j)] + \min_{x_j} \left[ \sum_{\alpha \in \partial j} c_\alpha b_\alpha(x_\alpha) \right]$$

where $g_j(x_{-j})$ is the part of the reparameterization that does not depend on $x_j$. The last inequality is tight whenever there is a value of $x_j$ that simultaneously minimizes each component of the sum. If the coefficients of $b_j$ and each of the $b_\alpha$ in Equation 36 were non-negative, then we could rewrite this bound as

$$\min_{x_j} f(x) \geq g_j(x_{S(j)}) + (1 - \sum_{\alpha \in \partial j} c_\alpha) c_j \min_{x_j} [b_j(x_j)] + \sum_{\alpha \in \partial j} c_\alpha \min_{x_j} [b_\alpha(x_\alpha)]$$

which depends on the minima of each of the beliefs. Recall from Lemma 9 that any unique estimate must simultaneously minimize $b_i$, $b_\alpha$, and, for $i \in \alpha$, $b_\alpha - b_i$. So, in general, we want to know if we can write

$$(1 - \sum_{\alpha \in \partial j} c_\alpha) c_j b_j(x_j) + \sum_{\alpha \in \partial j} c_\alpha b_\alpha(x_\alpha)$$

$$= d_{jj} b_j(x_j) + \sum_{\alpha \in \partial j} d_{aa} b_\alpha(x_\alpha) + \sum_{\alpha \in \partial j} d_{ja} [b_\alpha(x_\alpha) - b_j(x_j)]$$

for some vector of non-negative constants $d$. This motivates the following definition:

**Definition 11** A function, $h$, can be written as a conical combination of the beliefs if there exists a vector of non-negative reals, $d$, such that

$$h(x) = \kappa + \sum_{i, \alpha : i \in \alpha} d_{i\alpha} [b_\alpha(x_\alpha) - b_i(x_i)] + \sum_{\alpha} d_{aa} b_\alpha(x_\alpha) + \sum_i d_i b_i(x_i)$$
The set of all conical combinations of a collection of vectors in $\mathbb{R}^n$ forms a cone in $\mathbb{R}^n$ in the same way that a convex combination of vectors in $\mathbb{R}^n$ forms a convex set in $\mathbb{R}^n$. The above definition is very similar to the definition of “provably convex” in Weiss et al. (2007). There, an entropy approximation is provably convex if it can be written as a conical combination of the entropy functions corresponding to each of the factors. In contrast, our approach follows from a reparameterization of the objective function.

Putting all of the above ideas together, we have the following theorem:

**Theorem 12** Let $b$ be a vector of admissible and min-consistent beliefs for the function $f$ with a corresponding weighting vector of non-zero real numbers, $c$, such that for all $i$, $c_i b_i(x_i) + \sum_{\alpha \in \partial i} c_\alpha \left[ b_\alpha(x_\alpha) - c_i b_i(x_i) \right]$ can be written as a conical combination of the beliefs. If the beliefs admit a unique estimate, $x^*$, then $x^*$ is a local minimum (with respect to the Hamming distance) of the objective function.

**Proof** By Lemma 7, the beliefs reparameterize the objective function. Choose a $j \in \{1, ..., n\}$. By assumption, the portion of the objective function that depends on the variable $x_j$ can be written as a conical combination of the beliefs. By admissibility, we can write

$$ f(x^*) = \kappa + \sum_i c_i b_i(x_i^*) + \sum_\alpha c_\alpha \left[ b_\alpha(x_\alpha^*) - \sum_{k \in \alpha} c_k b_k(x_k^*) \right] $$

$$ = g_j(x_j^*) + c_j b_j(x_j^*) + \sum_{\alpha \in \partial j} c_\alpha \left[ b_\alpha(x_\alpha^*) - c_j b_j(x_j^*) \right] $$

$$ = g_j(x_j^*) + d_j \sum_{\alpha \in \partial j} d_{\alpha j} b_{\alpha j}(x_j, x_{\alpha \backslash j}) + \sum_{\alpha \in \partial j} d_{\alpha j} \left[ b_\alpha(x_j, x_\alpha^{* \backslash j}) - b_j(x_j) \right] $$

for any $x_j \in X$ where the inequality follows from Lemma 7. We can repeat this proof for each $j \in \{1, ..., n\}$. $\blacksquare$

Theorem 12 tells us that, under suitable choices of the parameters, no vector $x$ within Hamming distance one of $x^*$ can decrease the objective function. For a differentiable function $f$, we can infer that the gradient of $f$ at the point $x^*$ must be zero. Further, by the second derivative test and the observation that the function can only increase in value along the coordinate axes, $x^*$ is either a local minimum or a saddle point of $f$. For a convex differentiable function $f$, this condition is equivalent to global optimality:

**Corollary 13** Let $b$ be a vector of admissible and min-consistent beliefs for a differentiable convex function $f$ with a corresponding weighting vector of non-zero real numbers, $c$, such that $\forall i$, $c_i b_i(x_i) + \sum_{\alpha \in \partial i} c_\alpha \left[ b_\alpha(x_\alpha) - c_i b_i(x_i) \right]$ can be written as a conical combination of the beliefs. If the beliefs admit a unique estimate, $x^*$, then $x^*$ is a global minimum of the objective function.

**Corollary 14** The standard min-sum algorithm with $c_i = 1$ for all $i$ and $c_\alpha = 1$ for all $\alpha$ always satisfies the conditions of Theorem 12.
4.2 Global Optimality

We now extend the approach of the previous section to show that there are choices of the vector $c$ that guarantee the global optimality of any unique estimate produced from admissible and min-consistent beliefs. As before, suppose $b$ is a vector of admissible beliefs for the function $f$. If $f$ can be written as a conical combination of the beliefs, then we can lower bound the optimal value of the objective function as follows:

$$\min_x f(x) = \min_x \left[ \kappa + \sum_i c_i b_i(x_i) + \sum_{\alpha} c_\alpha \left( b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right) \right]$$

(40)

$$= \min_x \left[ \kappa + \sum_{i, \alpha : i \in \alpha} d_{ia} (b_\alpha(x_\alpha) - b_i(x_i)) + \sum_{\alpha} d_{\alpha \alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} b_i(x_i) \right]$$

(41)

$$\geq \kappa + \sum_{i, \alpha : i \in \alpha} d_{ia} \min_{x_\alpha} (b_\alpha(x_\alpha) - b_i(x_i)) + \sum_{\alpha} d_{\alpha \alpha} \min_{x_\alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} \min_{x_i} b_i(x_i)$$

(42)

This analysis provides us with our first global optimality result. We note that the following theorem also appears as Theorem 1 in Meltzer et al. (2009), and Theorem 2 in Wainwright et al. (2005) provides a similar proof for the TRMP algorithm.

Theorem 15 Let $b$ be a vector of admissible and min-consistent beliefs for the function $f$ with a corresponding weighting vector of non-zero real numbers, $c$, such that $f$ can be written as a conical combination of the beliefs. If the beliefs admit a unique estimate, $x^*$, then $x^*$ minimizes the objective function.

Proof Choose $x \in X^n$. Using Definition 11 and Lemma 9, we can write

$$f(x) = \kappa + \sum_{i, \alpha : i \in \alpha} d_{ia} (b_\alpha(x^*_\alpha) - b_i(x^*_i)) + \sum_{\alpha} d_{\alpha \alpha} b_\alpha(x^*_\alpha) + \sum_i d_{ii} b_i(x^*_i)$$

(43)

$$\leq \kappa + \sum_{i, \alpha : i \in \alpha} d_{ia} (b_\alpha(x_\alpha) - b_i(x_i)) + \sum_{\alpha} d_{\alpha \alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} b_i(x_i)$$

(44)

$$= f(x)$$

(45)

Theorem 15 also provides us with a simple proof that the standard min-sum algorithm is correct on a tree:

Corollary 16 Suppose the factor graph is a tree. If the admissible and min-consistent beliefs produced by the standard min-sum algorithm admit a unique estimate, $x^*$, then $x^*$ is the global minimum of the objective function.

Proof Let $b$ be the vector of min-consistent and admissible beliefs obtained from running the standard min-sum algorithm. Choose a node $r \in G$ and consider the factor graph as a tree rooted at a variable node, $r$. Let $p(\alpha)$ denote the parent of factor node $\alpha \in G$. We can
now write,

\[
\begin{align*}
  f(x) &= \sum_i b_i(x_i) + \sum_{\alpha \in A} \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} b_k(x_k) \right] \\
  &= b_r(x_r) + \sum_{\alpha \in A} \left[ b_\alpha(x_\alpha) - b_{p(\alpha)}(x_{p(\alpha)}) \right]
\end{align*}
\]

Hence, we can conclude that \( f \) can be written as a conical combination of the beliefs and apply Theorem [15].

Given Theorem [15], starting with the vector \( d \) seems slightly more natural than the starting with the vector \( c \). Consider any non-negative real vector \( d \), we now show that we can find a vector \( c \) such that \( f \) has a conical decomposition in terms of \( d \) provided \( d \) satisfies a mild condition.

Choose the vector \( c \) as follows:

\[
\begin{align*}
  c_\alpha &= d_{\alpha\alpha} + \sum_{i \in \alpha} d_{i\alpha} \\
  c_i &= \frac{d_{ii} - \sum_{\alpha \in \partial_i} d_{i\alpha}}{1 - \sum_{\alpha \in \partial_i} c_\alpha}
\end{align*}
\]

These equations are valid whenever \( 1 - \sum_{\alpha \in \partial_i} c_\alpha \neq 0 \). Note that any valid reparameterization must have \( c_i \neq 0 \) and \( c_\alpha \neq 0 \) for all \( \alpha \) and \( i \). Hence, \( d_{\alpha\alpha} + \sum_{i \in \alpha} d_{i\alpha} \neq 0 \) and \( d_{ii} - \sum_{\alpha \in \partial_i} d_{i\alpha} \neq 0 \). Now, for this choice of \( c \), we have:

\[
\begin{align*}
  f(x) &= \kappa + \sum_i c_i b_i(x_i) + \sum_\alpha c_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \\
  &= \kappa + \sum_i c_i (1 - \sum_{\alpha \in \partial_i} c_\alpha) b_i(x_i) + \sum_\alpha c_\alpha b_\alpha(x_\alpha) \\
  &= \kappa + \sum_i (d_{ii} - \sum_{\alpha \in \partial_i} d_{i\alpha}) b_i(x_i) + \sum_\alpha (d_{\alpha\alpha} + \sum_{i \in \alpha} d_{i\alpha}) b_\alpha(x_\alpha) \\
  &= \kappa + \sum_i d_{ii} b_i(x_i) + \sum_\alpha d_{\alpha\alpha} b_\alpha(x_\alpha) + \sum_{i,\alpha: i \in \alpha} d_{i\alpha} (b_\alpha(x_\alpha) - b_i(x_i))
\end{align*}
\]

In the case that \( 1 - \sum_{\alpha \in \partial_i} c_\alpha = 0 \), \( c_i \) can be chosen to be any non-zero real. Again, any valid reparameterization must have \( c_i \neq 0 \) and \( c_\alpha \neq 0 \) for all \( \alpha \) and \( i \). Hence, \( d_{\alpha\alpha} + \sum_{i \in \alpha} d_{i\alpha} \neq 0 \), but, unlike the previous case, we must have \( d_{ii} - \sum_{\alpha \in \partial_i} d_{i\alpha} = 0 \). The remainder of the proof then follows exactly as above.

We now address the following question: given a factorization of the objective function \( f \), is it always possible to choose the vector \( c \) in order to guarantee that any unique estimate produced from min-consistent and admissible beliefs minimizes the objective function? The answer to this question is yes, and we will provide a simple condition on the vector \( c \) that will ensure this. Again, suppose \( b \) is a vector of admissible beliefs for the function \( f \). We can lower bound \( f \) as

\[
\begin{align*}
  \min_x f(x) &= \min_x \left[ \kappa + \sum_i c_i b_i(x_i) + \sum_\alpha c_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \right]
\end{align*}
\]
\[ \begin{align*}
= \min_x \left[ \kappa + \sum_i (1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i b_i(x_i) + \sum_{\alpha} c_{\alpha} b_{\alpha}(x_{\alpha}) \right] \\
\geq \kappa + \sum_i \min_x [(1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i b_i(x_i)] + \sum_{\alpha} \min_{x_{\alpha}} [c_{\alpha} b_{\alpha}(x_{\alpha})]
\end{align*} \] (53)

Observe that if \((1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i \geq 0\) for all \(i\) and \(c_{\alpha} \geq 0\) for all \(\alpha\), then we can further rewrite the bound as:

\[ \min_x f(x) \geq \kappa + \sum_i (1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i \min_{x_i} [b_i(x_i)] + \sum_{\alpha} c_{\alpha} \min_{x_{\alpha}} [b_{\alpha}(x_{\alpha})] \] (55)

This analysis yields the following theorem:

**Theorem 17** Let \(b\) be a vector of admissible and min-consistent beliefs for the function \(f\) with a corresponding weighting vector of non-zero real numbers, \(c\), such that

1. For all \(i\), \((1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i \geq 0\)
2. For all \(\alpha\), \(c_{\alpha} > 0\)

If the beliefs admit a unique estimate, \(x^*\), then \(x^*\) minimizes the objective function.

**Proof** By Lemma 7, up to a constant, we can write

\[ f(x) = \sum_i c_i b_i(x_i) + \sum_{\alpha} c_{\alpha} \left[ b_{\alpha}(x_{\alpha}) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \]

\[ = \sum_i \left[ (1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i b_i(x_i) \right] + \sum_{\alpha} c_{\alpha} b_{\alpha}(x_{\alpha}) \]

Now, by assumption, \((1 - \sum_{\alpha \in \partial i} c_{\alpha}) c_i\) and \(c_{\alpha}\) are non-negative real numbers. Therefore, by Lemma 8 we can conclude that the assignment \(x^*\) simultaneously minimizes each of the beliefs and hence minimizes the function \(f\).

This result is quite general; for any choice of \(c\) such that \(c_{\alpha} > 0\) for all \(\alpha \in \mathcal{A}\), there exists a choice of \(c_i\) for each \(i\) such that the conditions of the above theorem are satisfied. The following corollary is an immediate consequence of this observation and Theorem 17:

**Corollary 18** Given any function \(f(x) = \sum_i \phi_i(x_i) + \sum_{\alpha} \psi_{\alpha}(x_{\alpha})\), there exists a choice of a non-zero parameter vector \(c\) such that for any vector of admissible and min-consistent beliefs for the function \(f\), if the beliefs admit a unique estimate, \(x^*\), then this estimate minimizes \(f\).

Up until this point, we have been assuming that the estimate produced at the fixed point was unique when, in fact, all of the previous theorems are equally valid for any vector that simultaneously minimizes all of the beliefs. However, finding such a vector may be difficult outside of special cases.
Lastly, we note that there are choices of the parameters for which we are guaranteed local optimality but not global optimality. The difference between Theorem 12 and Theorem 15 is that the former only requires that the part of the reparameterization depending on a single variable can be written as a conical combination of the beliefs, whereas the latter requires the entire reparameterization to be a conical combination of the beliefs. The standard min-sum algorithm always guarantees local optimality, and there are applications for which the algorithm is known to produce local optima that are not globally optimal (Weiss and Freeman, 2001).

4.3 Relation to TRMP

The TRMP algorithm is a special case of our algorithm. We consider the algorithm on a pairwise factor graph \( G \) with corresponding objective function \( f \). Let \( T \) be the set of all spanning trees on \( G \), and let \( \mu \) be a probability distribution over \( T \). We define \( c_i = 1 \) for all \( i \) and \( c_{ij} = Pr_{\mu}[(i,j) \in T] \) corresponding to the edge appearance probabilities. Let \( b \) be a vector of admissible and min-consistent beliefs for \( f \). We can write the objective function \( f \) as

\[
f(x) = \sum_{i \in G} b_i(x_i) + \sum_{(i,j) \in G} c_{ij} [b_{ij}(x_i, x_j) - b_i(x_i) - b_j(x_j)] \tag{56}
\]

\[
= \sum_{T \in T} \mu(T) \left[ \sum_{i \in T} b_i(x_i) + \sum_{(i,j) \in T} [b_{ij}(x_i, x_j) - b_i(x_i) - b_j(x_j)] \right] \tag{57}
\]

The remainder of this argument is now a generalized version of Corollary 16. For each \( T \in T \), designate a variable node \( r_T \in T \) as the root of \( T \). Let \( p_T(\alpha) \) denote the parent of factor node \( \alpha \in T \). We can now write,

\[
f(x) = \sum_{T \in T} \mu(T) \left[ \sum_{i \in T} b_i(x_i) + \sum_{(i,j) \in T} [b_{ij}(x_i, x_j) - b_i(x_i) - b_j(x_j)] \right] \tag{58}
\]

\[
= \sum_{T \in T} \mu(T) \left[ b_{r_T}(x_{r_T}) + \sum_{i \in T \cap \partial \neq r_T} [b_{p_T(i)}(x_i, x_{p_T(i)}) - b_{p_T(i)}(x_{p_T(i)})] \right] \tag{59}
\]

Because \( \mu(T) \geq 0 \) for all \( T \in T \), we can conclude that \( f \) can be written as a conical combination of the beliefs. By Theorem 15, convergence of the TRMP algorithm to a unique estimate implies correctness. A similar argument can be made if \( \mu \) is a distribution over all subgraphs of \( G \) containing at most one cycle.

Computing the vector \( c \) for the TRMP algorithm requires finding a distribution on spanning trees. For arbitrary graphs, computing such a distribution is nontrivial especially if we want to preserve the distributed nature of the algorithm; we would need to compute enough spanning trees so that every edge is contained in at least one spanning tree. If we had simply chosen the \( c_{ij} > 0 \) such that for all \( i, \sum_{k \in \partial i} c_{ki} \leq 1 \), we would have been guaranteed global optimality by Theorem 17 without the additional work of choosing a distribution over spanning trees.
4.4 Interpreting the Beliefs

We conclude this section with an exploration of the following question: what is the relationship between min-consistent and admissible beliefs, \( b \), and the true min-marginals? Let \( f_i(x_i) \equiv \min_{x_{-i}} f(x) \). For the standard min-sum algorithm, when the factor graph is a tree and the initial messages are all chosen to be the zero message, we know that \( b_i(x_i) = f_i(x_i) + \kappa \). This is a direct consequence of the observation that the beliefs are the true min-marginals on the computation trees.

Now, let \( b \) be a vector of beliefs that is admissible and min-consistent for the objective function \( f \). If the parameter vector, \( c \), satisfies the conditions of Theorem 12, then we can lower bound the min-marginal \( f_j \) as:

\[
\min_{x_{-j}} f(x) = \min_{x_{-j}} \left[ \kappa + \sum_i c_i b_i(x_i) + \sum_{\alpha} c_{\alpha} \left[ b_{\alpha}(x_{\alpha}) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \right] 
\geq \min_{x_{-j}} g_j(x_{-j}) + \min_{x_{-j}} \left[ c_j b_j(x_j) + \sum_{\alpha \in \partial j} c_{\alpha} \left[ b_{\alpha}(x_{\alpha}) - c_j b_j(x_j) \right] \right] 
= \kappa' + \min_{x_{-j}} \left[ d_{jj} b_j(x_j) + \sum_{\alpha \in \partial j} d_{\alpha j} \left[ b_{\alpha}(x_{x_{\alpha}}) - b_j(x_j) \right] + \sum_{\alpha \in \partial j} d_{\alpha \alpha} b_{\alpha}(x_{\alpha}) \right] 
\geq \kappa' + d_{jj} b_j(x_j) + \sum_{\alpha \in \partial j} d_{\alpha \alpha} b_{\alpha}(x_{\alpha}) 
= \kappa'' + (d_{jj} + \sum_{\alpha \in \partial j} d_{\alpha \alpha}) b_j(x_j) 
\]

In other words, \( f_i \) is lower bounded by \( b_i \) after an appropriate rescaling and shifting. A similar argument can be made if \( f \) can be written as a conical combination of the beliefs. This argument results in the following theorem:

**Theorem 19** Let \( b \) be a vector of admissible and min-consistent beliefs for the function \( f \). If the parameter vector \( c \) satisfies the conditions for either global optimality as in Theorem 13 or local optimality as in Theorem 12, then for all \( i \) there exists \( d_i > 0 \) and a constant \( \kappa_i \) such that \( f_i(x_i) \geq \kappa_i + d_i b_i(x_i) \).

Any collection of admissible and min-consistent beliefs, after an appropriate scaling and shifting, lower bound the true min-marginals. There are two special cases of this theorem worth noting:

1. If \( c \) is the all ones vector, corresponding to the standard min-sum algorithm, then \( d_i = 1 \) for all \( i \). Further, if the factor graph is a tree then the inequality is actually an equality.

2. If \( c_\alpha > 0 \) for all \( \alpha \), \( c_i = 1 \) for all \( i \), and \( \sum_{\alpha \in \partial i} c_\alpha \leq 1 \) for all \( i \), then \( d_i = 1 \) for all \( i \).

In both of these cases, the scale factor is one, and as a result, up to a constant, any collection of admissible and min-consistent beliefs lower bound the true min-marginals.
5. Convergence

Throughout the previous section, we assumed that we somehow obtained beliefs that were both admissible and min-consistent. We know that, for bounded objective functions, beliefs satisfying these two properties are produced by Algorithm 1 if it converges, but thus far, we have avoided the issue of convergence. In order to apply the the results of Section 4, we will need that the beliefs are converging to a collection of admissible and min-consistent beliefs. Traditionally, the min-sum algorithm is said to converge if the beliefs at two consecutive time steps do no change by more than some $\epsilon > 0$. There are two potential algorithmic behaviors that we would like to avoid:

1. The beliefs do not converge to a vector of admissible and min-consistent beliefs.
2. The vector of messages at time $t$ is not finite as a result of performing the message updates.

If the vector of messages is not finite, we may not be able to construct a corresponding vector of admissible beliefs as in Lemma 7. Unfortunately, there are distributions for which it is possible that the min-sum algorithm will generate infinite messages at some time step (e.g. the quadratic minimization problem Ruozzi and Tatikonda 2010). However, case 2 cannot arise for any distribution $f$ such that $\phi_{\alpha}$ and $\psi_{\alpha}$ are bounded real-valued functions for each $\alpha \in \mathcal{A}$ and each $i$.

In this section, we will focus on case 1. The standard approach to force convergence of the messages and beliefs is to apply a damping factor to the message updates. This tends to work well empirically, but choosing the correct damping factor remains somewhat of an art form. In what follows, we will pursue a different approach: modifying the message passing schedule to ensure convergence.

5.1 Asynchronous Message Passing

Consider the local and global optimality results of Section 4. Each of these results relied on a particular lower bound on the objective function (or at least part of the objective function) being tight. Similar in spirit to the recent work of Kolmogorov (2006) and Meltzer et al. (2009), we will present a message passing schedule that is guaranteed to improve a specific lower bound on the objective function at each time step. We will say that this algorithm has converged if the lower bound cannot be improved by subsequent iteration.

Empirically, Algorithm 1 does not always converge, but the synchronous message passing schedule of Algorithm 1 is only one such schedule for the message updates. Consider the alternative message passing schedule in Algorithm 2. This asynchronous message passing schedule fixes an ordering on the variables and for each $i$, in order, updates all of the messages from each $\alpha \in \partial i$ to $i$ as if $i$ were the root of the subtree containing only $\alpha$ and its neighbors. We will show that, for certain choices of the parameter vector $c$, this message passing schedule improves a specific lower bound at each iteration.

By using the asynchronous schedule in Algorithm 2 we seem to lose the distributed nature of the parallel message updates. Fortunately, for some asynchronous schedules, we can actually parallelize the updating process by performing concurrent updates as long as the simultaneous updates do not form a cycle (e.g. we could randomly select a subset
of the message updates that do not interfere). We also note that updating over larger subtrees may be advantageous. Other algorithms, such as those in Kolmogorov (2006) and Sontag and Jaakkola (2009), perform updates over specific trees.

Algorithm 2 Asynchronous Splitting Algorithm

1: Initialize the messages uniformly to zero.
2: Choose some ordering of the variables, and perform the following update for each variable $i$
3: for each edge $(j, \beta)$ do
4: For all $i \in \beta \setminus j$ update the message from $i$ to $\beta$

$$m_{i \rightarrow \beta}(x_i) = \kappa + \frac{\phi_i(x_i)}{c_i} + (c_\beta - 1)m_{\beta \rightarrow i}(x_i) + \sum_{\alpha \in \partial i \setminus \beta} c_\alpha m_{\alpha \rightarrow i}(x_i)$$

5: Update the message from $\beta$ to $j$

$$m_{\beta \rightarrow j}(x_j) = \kappa + \min_{x_\beta, j} \left[ \frac{\psi_\beta(x_\beta)}{c_\beta} + (c_j - 1)m_{j \rightarrow \beta}(x_j) + \sum_{k \in \beta \setminus j} c_k m_{k \rightarrow \beta}(x_k) \right]$$

6: end for

We know that the ordinary min-sum algorithm is exact on trees. More specifically, given any choice of boundary messages, if we pass messages from the leaves to the root and back down on any subtree, then the beliefs over that subtree are guaranteed to be correct for that choice of boundary messages. In this case, we are guaranteed to produce a vector of beliefs that is min-consistent over that tree. For arbitrary choices of $c$, we are not guaranteed such a property on trees. Instead, we will show that, under a restriction on the choice of $c$, the message updates in Algorithm 2, which are performed sequentially over one variable node of the graph at a time, ensure that the beliefs satisfy a weak notion of consistency over the factors with respect to that variable. Our primary tool in this section will be a lemma similar to Lemma 9:

**Lemma 20** Let $c$ be a vector such that $c_i = 1$ for all $i$. Suppose we perform the update for the edge $(j, \beta)$ as in Algorithm 2. If the vector of messages is finite after the update, then $b_\beta$ is min-consistent with respect to $b_j$.

**Proof** Let $m$ be the vector of messages before the update and let $m^+$ be the vector of messages after the update. The proof of this lemma is similar to that of Lemma 8. Observe that for each $i \in \beta \setminus j$,

$$m^+_{i \rightarrow \beta}(x_i) = \kappa + \phi_i(x_i) + (c_\beta - 1)m_{\beta \rightarrow i}(x_i) + \sum_{\alpha \in \partial i \setminus \beta} c_\alpha m_{\alpha \rightarrow i}(x_i)$$

$$= \kappa + \phi_i(x_i) + (c_\beta - 1)m^+_{\beta \rightarrow i}(x_i) + \sum_{\alpha \in \partial i \setminus \beta} c_\alpha m^+_{\alpha \rightarrow i}(x_i)$$
Similarly,
\[ m_{\beta \rightarrow j}^+(x_j) = \kappa + \min_{x_{\beta \setminus j}} \left[ \frac{\psi_\beta(x_\beta)}{c_\beta} + \sum_{k \in \beta \setminus j} m_{k \rightarrow \beta}^+(x_k) \right] \]

Up to an additive constant, we have:
\[
\min_{x_{\beta \setminus j}} b_{\beta}^+(x_\beta) = \min_{x_{\beta \setminus j}} \frac{\psi_\beta(x_\beta)}{c_\beta} + \sum_{k \in \beta \setminus j} m_{k \rightarrow \beta}^+(x_k) + \left[ \phi_j(x_j) + \sum_{\alpha \in \partial_j \setminus \beta} c_\alpha m_{\alpha \rightarrow j}^+(x_j) \right] + (c_\beta - 1)m_{\beta \rightarrow j}^+(x_j)
\]
\[ = m_{\beta \rightarrow j}^+(x_j) + \left[ \phi_j(x_j) + \sum_{\alpha \in \partial_j \setminus \beta} c_\alpha m_{\alpha \rightarrow j}^+(x_j) + (c_\beta - 1)m_{\beta \rightarrow j}^+(x_j) \right]
\[ = b_j^+(x_j) \]

Observe that, after performing all of the message updates for a node \( j \) as in Algorithm 2, \( b_\beta \) is min-consistent with respect to \( b_j \) for every \( \beta \in \mathcal{A} \) containing \( j \). The most important conclusion we can draw from this is that there is an \( x_j^* \) that simultaneously minimizes \( b_j \), \( \min_{x_{\beta \setminus j}} b_\beta \), and \( \min_{x_{\beta \setminus j}} b_\beta - b_j \).

5.2 Global Convergence

We will show that, under certain conditions, the vector of beliefs generated from the messages at any point in Algorithm 2 are converging to a vector of admissible and min-consistent beliefs. We note that this theorem does not show convergence in the normal sense (i.e. the messages may not be converging to a fixed point of the message passing equations), but as the theorems from the previous section only require that the beliefs be admissible and min-consistent, this will suffice.

As in the previous section, suppose \( m \) is a vector of finite messages with corresponding beliefs \( b \). If \( f \) can be written as a conical combination of the beliefs, then we can lower bound the optimal value of the objective function as follows:
\[
\min_x f(x) = \min_x \left[ \kappa + \sum_{i, \alpha : i \in \alpha} d_{i\alpha}(b_\alpha(x_\alpha) - b_i(x_i)) + \sum_{\alpha} d_{\alpha\alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} b_i(x_i) \right] \quad (66)
\]
\[
\geq \kappa + \sum_{i, \alpha : i \in \alpha} d_{i\alpha} \min_{x_\alpha} (b_\alpha(x_\alpha) - b_i(x_i)) + \sum_{\alpha} d_{\alpha\alpha} \min_{x_\alpha} b_\alpha(x_\alpha) + \sum_i d_{ii} \min_{x_i} b_i(x_i) \quad (67)
\]

This lower bound is a function of the message vector, \( m \). Further, if the beliefs satisfy the conditions of Theorem 17, then we can write this lower bound as:
\[
\min_x f(x) \geq \kappa + \sum_i c_i (1 - \sum_{\alpha \in \partial_i} c_\alpha) \min_{x_i} b_i(x_i) + \sum_{\alpha} c_\alpha \min_{x_\alpha} b_\alpha(x_\alpha) \quad (68)
\]
Define \( LB(m) \) to be the lower bound in Equation 68. Using Lemma 20, we can show that, for certain choices of the parameter vector, each variable update as in Algorithm 2 can only increase this lower bound:

**Theorem 21** Suppose \( c_i = 1 \) for all \( i \), \( c_\alpha > 0 \) for all \( \alpha \), and \( \sum_{\alpha \in \partial i} c_\alpha \leq 1 \) for all \( i \). If the vector of messages is finite at each step of the asynchronous splitting algorithm, then the algorithm converges to a collection of beliefs such that \( LB(m) \) cannot be improved by further iteration of the asynchronous algorithm. Further, if upon convergence of the lower bound the beliefs admit a unique estimate \( x^* \), then \( x^* \) is the global minimum of the objective function.

**Proof** We will show that the message updates performed for variable node \( j \) cannot decrease the lower bound. Let \( LB_j(m) \) denote the terms in \( LB \) that involve the variable \( x_j \). We can upper bound \( LB_j \) as follows:

\[
LB_j(m) \leq \min_{x_j}(1 - \sum_{\beta \in \partial_j} c_\beta b_j(x_j) + \sum_{\beta \in \partial_j} c_\beta \min_{x_\beta \mid j} b_\beta(x_\beta)) \\
= \min_{x_j}[(1 - \sum_{\beta \in \partial_j} c_\beta b_j(x_j) + \sum_{\beta \in \partial_j} c_\beta \min_{x_\beta \mid j} \psi_\beta(x_\beta)} c_\beta + \sum_{k \in \beta} (b_k(x_k) - m_{\beta \rightarrow k}(x_k))]) \\
= \min_{x_j}[\phi_j(x_j) + \sum_{\beta \in \partial_j} c_\beta \min_{x_\beta \mid j} \psi_\beta(x_\beta)} c_\beta + \sum_{k \in \beta \mid j} (b_k(x_k) - m_{\beta \rightarrow k}(x_k))]
\]

Notice that this upper bound does not depend on the choice of the messages from \( \beta \) to \( j \) for any \( \beta \in \partial j \). As a result, any choice of these messages for which the inequality is tight must maximize \( LB_j \). Observe that the upper bound is tight iff there exists an \( x_j \) that simultaneously minimizes \( b_j \) and \( \min_{x_\beta \mid j} b_\beta \) for each \( \beta \in \partial j \). By Lemma 20 this is indeed the case after performing the updates in Algorithm 2 for the variable node \( j \). As this is the only part of the lower bound affected by the update, we have that \( LB \) cannot decrease. Let \( m \) be the vector of messages before the update for variable \( j \) and \( m^+ \) the vector after the update. Define \( LB_{-j} \) to be the sum of the terms of the lower bound that do not involve the variable \( x_j \). By definition and the above we have:

\[
LB(m) = LB_{-j}(m) + LB_j(m) \\
\leq LB_{-j}(m) + \min_{x_j}[(1 - \sum_{\beta \in \partial j} c_\beta b_j(x_j) + \sum_{\beta \in \partial j} c_\beta \min_{x_\beta \mid j} b_\beta(x_\beta)) \\
= LB_{-j}(m^+) + \min_{x_j}[(1 - \sum_{\beta \in \partial_j} c_\beta b_j^+(x_j) + \sum_{\beta \in \partial j} c_\beta \min_{x_\beta \mid j} b_\beta^+(x_\beta)] \\
= LB(m^+)
\]

\( LB(m) \) is bounded from above by \( \min_x f(x) \). From this we can conclude that the value of the lower bound converges.

Finally, the lower bound has converged if no single variable update can improve the bound. By the arguments above, this must mean that there exists an \( x_j \) that simultaneously minimizes \( b_j \) and \( \min_{x_\beta \mid j} b_\beta \) for each \( \beta \in \partial j \). These beliefs may or may not be
min-consistent. Now, if there exists a unique minimizer \( x^* \), then \( x^* \) must simultaneously minimize \( b_j \) and \( \min_{x \in A_j} b_j \) for each \( j \in \partial \). From this we can conclude that \( x^* \) simultaneously minimizes all of the beliefs and therefore, using the argument from Theorem 17, must minimize the objective function.

We note that, even with this restricted choice of the parameter vector, Algorithm 2 is not strictly a member of the family of bound minimizing algorithms discussed in Meltzer et al. (2009). The disparity occurs because the definition of a bound minimizing algorithm as presented therein would require \( b_{\alpha} \) to be min-consistent with respect to \( x_j \) for all \( j \in \alpha \) after the update is performed over the edge \((i, \alpha)\). Instead, Algorithm 2 only guarantees that \( b_{\alpha} \) is min-consistent with respect to \( x_i \) after the update.

Although the restriction on the parameter vector in Theorem 21 seems strong, we observe that it captures the TRMP algorithm, and for any objective function \( f \), we can choose the parameters such that the theorem is sufficient to guarantee convergence and global optimality:

**Corollary 22** Define \( d = \max_i |\partial_i| \). If \( c \) is chosen as in Theorem 21 and the vector of messages is finite at each step of the asynchronous splitting algorithm, then the lower bound converges. Further, if the converged beliefs admit a unique estimate, then \( x^* \) is a global minimum of \( f \).

We note that Theorem 21 is similar to a special case of Theorem 1 in Hazan and Shashua (2009). For an appropriate setting of the parameters in their algorithm, the message updates for the special case in Theorem 21 are identical to those in Algorithm 2 and the two message passing algorithms differ only in the order of the updates. This difference turns out to be significant, and the proofs of convergence for the two algorithms are entirely different.

### 5.3 Local Convergence

Even if the objective function cannot be written as a conical combination of the beliefs, we can still use lower bounds on the objective function to explain the behavior of the asynchronous message passing algorithm. Suppose \( m \) is a vector of finite messages with corresponding beliefs \( b \). If \( c \) is a vector of non-zero reals, such that for all \( i \), \( c_i b_i(x_i) + \sum_{\alpha \in \partial_i} c_\alpha [b_\alpha(x_\alpha) - c_i b_i(x_i)] \) can be written as a conical combination of the beliefs, then we can lower bound the optimal value of the objective function as follows:

\[
\min_x f(x_j, x_{-j}) = \min_x \left[ \kappa + \sum_i c_i b_i(x_i) + \sum_{\alpha} c_\alpha \left[ b_\alpha(x_\alpha) - \sum_{k \in \alpha} c_k b_k(x_k) \right] \right] \tag{69}
\]

\[
= \min_{x_{-j}} \left[ g_j(x_{-j}) + \min_{x_j} \left[ c_j b_j(x_j) + \sum_{\alpha \in \partial_j} c_\alpha \left[ b_\alpha(x_\alpha) - c_j b_j(x_j) \right] \right] \right] \tag{70}
\]

\[
\geq \min_{x_{-j}} g_j(x_{-j}) + \sum_{j \in \partial_j} d_{jj} \min_{x_j} b_j(x_j) + \sum_{\alpha \in \partial_j} d_{\alpha j} \min_{x_\alpha} b_\alpha(x_\alpha) - b_j(x_j)
\]

\[
+ \sum_{\alpha \in \partial_j} d_{\alpha \alpha} \min_{x_\alpha} b_\alpha(x_\alpha) \tag{71}
\]

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Applying Lemma 20 to this lower bound yields the following theorem:

**Theorem 23** Let $c$ be a vector such that $c_i = 1$ for all $i$ and $c_\alpha > 0$ for all $\alpha$. If the vector of messages is finite at each step of the asynchronous splitting algorithm, then the message update for the variable node $j$ in Algorithm 2 cannot decrease the lower bound in Equation 74.

We note that this theorem does not necessarily guarantee the convergence of the asynchronous message passing algorithm as the bound being improved is different for each variable. Hence, stronger assumptions are necessary in order to ensure convergence when the parameter vector only satisfies the conditions for local optimality.

### 5.4 Tightness of the Lower Bounds

Although Theorem 21 guarantees convergence, the lower bound in Equation 68 evaluated at the converged beliefs may not actually be tight. If the bound is tight, then any minimum of the objective function must simultaneously minimize each of the beliefs. In this section, we will provide a condition under which the lower bound cannot be tight. The intuition behind this condition is that the min-sum algorithm, in attempting to solve the minimization problem on one factor graph, is actually attempting to solve the minimization problem over an entire family of equivalent (in some sense) factor graphs. To make this more precise, we introduce the notion of a graph cover:

**Definition 24** A graph $H$ covers a graph $G$ if there exists a graph homomorphism $\phi : H \rightarrow G$ such that $h : H \rightarrow G$ is an isomorphism on $\partial v$ for all vertices $v \in H$. If $h(v) = u$, then we say that $v \in H$ is a copy of $u \in G$. Further, $H$ is a $k$-cover of $G$ if every vertex of $G$ has exactly $k$ copies in $H$.

If $H$ covers the factor graph $G$, then $H$ has the same local properties as $G$. To any cover $H$ we can associate a collection of potentials in which the potential at node $i \in H$ is equal to the potential at node $h(i) \in G$. The min-sum algorithm is incapable of distinguishing the two factor graphs $H$ and $G$ given that the initial messages to and from each node in $H$ are identical to the nodes they cover in $G$. Observe that for every node $v \in G$ the messages received and sent by this node at time $t$ are exactly the same as the messages sent and received at time $t$ by any copy of $v$ in $H$. As a result, if we use the min-sum algorithm to deduce an assignment for $v$, the algorithm run on the graph $H$ must deduce the same assignment for each copy of $v$. We summarize this equivalence with the following lemma:

**Lemma 25** Let $b^G$ be a vector of beliefs that is admissible and min-consistent for a function $f^G$ with corresponding factor graph $G$ and a non-zero parameter vector $c$. If $H$ is a graph cover of $G$ via $h : H \rightarrow G$ with corresponding objective function $f^H$, then the vector of beliefs $b^H$ such that

- For all $i \in H$, $b^H_i = b^G_{h(i)}$
- For all $\alpha \in H$, $b^H_\alpha = b^G_{h(\alpha)}$

is admissible and min-consistent for $f^H$. 

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In essence, the $b^H$ beliefs are simple copies of the beliefs on $G$. We could create an estimate $x^H$ from an estimate $x^G$ by duplicating the components in the same way as in the statement of the lemma. A minimum of the objective function $f^H$ need not be a copy of some minimum of the objective function $f^G$. Even worse, minima of $f^G$ need not correspond to minima of $f^H$. This idea is the basis for the theory of pseudocodewords in the LDPC community [Vontobel and Koetter (2005)]. In this community, solutions on covers that are not copies of solutions of the original problem are referred to as pseudocodewords.

Theorem 15 guarantees us correctness upon convergence to a unique estimate under an appropriate choice of parameters. The same correctness argument can be made for any cover:

**Theorem 26** Let $b^G$ be a vector of beliefs that is admissible and min-consistent for a function $f^G$ with corresponding factor graph $G$ and a non-zero parameter vector $c$ such that $f$ can be written as a conical combination of the beliefs. Suppose $x^G$ simultaneously minimizes each of the beliefs $b^G$. If $h : H \rightarrow G$ is a covering homomorphism, then $x^H$ defined such that $x^H_i = x^G_{h(i)}$ is a global minimum of $f^H$.

**Proof** Let $H$ be a cover of $G$. By Lemma 25, we know that we can construct a vector of beliefs $b^H$ that is admissible and min-consistent for $f^H$ by copying beliefs from $b^G$. If $x^G$ minimizes each of the beliefs in $b^G$, then by construction, $x^H$ minimizes all of the beliefs in $b^H$. We can then apply the proof of Theorem 15 to conclude that $x^H$ is a global minimum of $f^H$. $lacksquare$

From this theorem, we can conclude that if the objective function can be written as a conical combination of the beliefs, then unless there is an assignment that uniquely minimizes $f^G$ and $f^H$ for any cover $H$ of $G$, neither Algorithm 1 nor Algorithm 2 can converge to a unique estimate.

6. Pairwise Binary Factorizations

For special cases, we can strengthen and extend the results of the previous sections. One such special case is when the state space is binary (e.g. $X = \{0, 1\}$) and the objective function can be written as a sum of self-potentials and factor potentials that are a function of at most two of the variables. Many important problems can be captured by this restriction: the Ising model, the maximum weight independent set problem (MWIS), minimizing a quadratic function over a binary state space, etc.

6.1 Partial Solutions

In the previous sections, we examined the situation in which all of the $b_i$ had a unique argmin. As we discussed in the preliminaries, if this is not the case, then we may not always be able to extend these partial solutions into a global minimum. Suppose $b$ is a vector of admissible beliefs for the function $f$. If $f$ can be written as a conical combination of these beliefs, then, under certain conditions, we can demonstrate that the partial solution constructed from each $i$ such that $b_i$ has a unique argmin can be extended to a global
minimum. This result is identical to the results of Kolmogorov and Wainwright (2005) and Weiss et al. (2007), but we will present it here for completeness:

**Theorem 27** For pairwise factorizations, if $b_j \equiv \text{constant}$ for each variable $j$ such that there is a factor containing $j$ and some fixed variable $i$, then the partial assignment $x^*_F$ given by the fixed variables can be extended to a global minimum of the objective function.

**Proof** See the corollary to Theorem 2 in Weiss et al. (2007) or Theorem 2 in Kolmogorov and Wainwright (2005).

6.2 Tightness of the Lower Bound

Perhaps the most interesting property of pairwise binary factorizations is that the lower bound, $LB$, is maximized for any collection of admissible and min-consistent beliefs. This was demonstrated in Kolmogorov and Wainwright (2005), but we will extend their result and provide an alternate proof. Specifically, we will produce a 2-cover and an assignment that minimizes the objective function on that 2-cover. Hence, we will have that the $LB$ is tight for the objective function on the 2-cover. By our arguments in Section 5.4, we know that no vector of min-consistent and admissible beliefs can do better than this.

**Theorem 28** Let $b$ be a vector of admissible and min-consistent beliefs for the objective function $f_G$ with factor graph $G$ obtained from a collection of messages $m$. If the graphical model is pairwise binary and the objective function can be written as a conical combination of the beliefs, then the beliefs maximize the lower bound, $LB(m)$.

**Proof** Without loss of generality we can assume that $X = \{0, 1\}$. Because the factors are pairwise, we can draw $G$ as a graph with a node for each variable and an edge joining two nodes if there is a factor $\alpha = \{i,j\} \in A$. We will construct a 2-cover, $H$, of the factor graph $G$ and an assignment $x^*_H$ such that $x^*_H$ minimizes $f_H$. We will index the copies of variable $i \in G$ in the factor graph $H$ as $i_1$ and $i_2$. First, we will construct the assignment. If $\arg \min_{x_i} b_i(x_i)$ is unique, then set $x^*_i = x^*_{i_1} = \arg \min_{x_i} b_i(x_i)$. Otherwise, set $x^*_i = 0$ and $x^*_i = 1$. Now, we will construct a 2-cover, $H$, such that $x^*$ minimizes each of the beliefs. We will do this factor by factor. Consider the factor $\alpha = \{i,j\} \in A$. There are several possibilities:

1. $b_i$ and $b_j$ have unique argmins. In this case, $b_\alpha$ is minimized at $b_\alpha(x^*_{i_1}, x^*_{j_1})$. So, we can add the edges $(i_1, j_1)$ and $(i_2, j_2)$ to $H$. The corresponding beliefs $b_{i_1,j_1}$ and $b_{i_2,j_2}$ are minimized at $x^*$.

2. $b_i$ has a unique argmin and $b_j$ is minimized at both 0 and 1 (or vice versa). In this case, we have $x^*_{i_1} = x^*_{i_2}, x^*_{j_1} = 0,$ and $x^*_{j_2} = 1$. By min-consistency, we can conclude that $b_\alpha$ is minimized at $(x^*_{i_1}, 0)$ and $(x^*_{i_1}, 1)$. Therefore, we can add the edges $(i_1, j_1)$ and $(i_2, j_2)$ to $H$.

3. $b_i$ and $b_j$ are minimized at both 0 and 1. In this case, we have $x^*_{i_1} = 0, x^*_{i_2} = 1, x^*_{j_1} = 0,$ and $x^*_{j_2} = 1$. By min-consistency, there is an assignment that minimizes
Consider the special case in which \( c_i = 1 \) for all \( i \). For a pairwise factorization, if none of the variable nodes in the factor graph are split, then the message passing equations in Algorithm 1 simplify to the following message update equation:

\[
m^t_{i \to j}(x_j) = \min_{x_i} \phi_i(x_i) + \frac{\psi_{ij}(x_i, x_j)}{c_{ij}} + (c_{ij} - 1)m^{t-1}_{j \to i}(x_j) + \sum_{k \in \partial i \setminus j} c_{ki}m^{t-1}_{k \to i}(x_i) \quad (72)
\]

Similarly, the beliefs can be simplified to:

\[
b^t_i(x_i) = \phi_i(x_i) + \sum_{k \in \partial i} c_{ki}m^t_{k \to i}(x_i) \quad (73)
\]

\[
b^t_{ij}(x_i, x_j) = \frac{\psi_{ij}(x_i, x_j)}{c_{ij}} + \phi_i(x_i) + \phi_j(x_j) + (c_{ij} - 1)m^{t-1}_{j \to i}(x_i) + \sum_{k \in \partial i \setminus j} c_{ki}m^{t-1}_{k \to i}(x_i)
\]

\[+ (c_{ij} - 1)m^{t-1}_{i \to j}(x_j) + \sum_{k \in \partial j \setminus i} c_{kj}m^{t-1}_{k \to j}(x_j) \quad (74)
\]

Now, suppose that \( f \) admits a pairwise binary factorization and that \( c \) is a positive vector such that \( \sum_{k \in \partial i} c_{ik} < 1 \) for all \( i \). As a consequence of Theorem 28, the asynchronous splitting algorithm provides us with a convergent algorithm that is guaranteed to produce a solution on a 2-cover of the original problem. Moreover, as a consequence of the proof, we can construct such a 2-cover and the optimal assignment on this 2-cover in polynomial time given any vector of min-consistent and admissible beliefs. Hence, Theorem 28 and Theorem 21, taken together, provide a nearly complete characterization of the behavior of the min-sum algorithm for this special case. Only the rate of convergence remains unresolved.

7. Discussion

The splitting technique described in this paper has a wide variety of applications. The proposed algorithm closely mirrors the traditional min-sum algorithm, but it can be proven to be both convergent and correct under appropriate message passing schedules. Most important is the simplicity of the derivation: a naive rewriting of the factorization. Other rewritings may produce even more families of message passing algorithms whose convergence and correctness properties may be even better than the splitting algorithm considered in this paper.

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