Interior-point methods for second-order stationary points of nonlinear semidefinite optimization problems using negative curvature

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Abstract We propose a primal-dual interior-point method (IPM) with convergence to second-order stationary points (SOSPs) of nonlinear semidefinite optimization problems, abbreviated as NSDPs. As far as we know, the current algorithms for NSDPs only ensure convergence to first-order stationary points such as Karush-Kuhn-Tucker points. The proposed method generates a sequence approximating SOSPs while minimizing a primal-dual merit function for NSDPs by using scaled gradient directions and directions of negative curvature. Under some assumptions, the generated sequence accumulates at an SOSP with a worst-case iteration complexity. This result is also obtained for a primal IPM with slight modification. Finally, our numerical experiments show the benefits of using directions of negative curvature in the proposed method.

Keywords Nonlinear semidefinite programming · Primal–dual interior-point method · Negative curvature direction · Second-order stationary points

Mathematics Subject Classification (2000) 90C22 · 90C26 · 90C51

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1 Introduction

We consider the following nonlinear semidefinite optimization problems (NS-DPs), which are possibly nonconvex:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad X(x) \in S^m_+,
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( X : \mathbb{R}^n \to S^m \) are twice continuously differentiable functions and \( \mathbb{R}^n \) and \( S^m \) denote the spaces of \( n \)-dimensional real vectors and \( m \times m \) real symmetric matrices, respectively. Moreover, \( S^m_+ \) denotes the set of positive (semi)definite matrices in \( S^m \). By restricting the range of \( X \) onto the space of diagonal matrices, NS-DPs reduce to the standard inequality-constrained nonlinear optimization problems (NLPs). When all the functions of a problem are affine, the problem is a linear semidefinite optimization problem (LSDP) and has been studied extensively [62].

Though studies of NS-DPs are still fewer than those of LSDPs and NLPs, they are of great importance from a practical point of view. Indeed, NS-DPs arise from various application fields including control [18, 26, 33], statistics [55], finance [31, 32, 37], and structural optimization [6, 29, 34]. Positive semidefinite matrix factorization problems [35, 61] and rank minimization problems [19] are also important applications of NS-DPs. Moreover, NS-DPs have been studied in terms of optimality conditions: For example, the Karush-Kuhn-Tucker (KKT) conditions and the second-order conditions for NS-DPs were studied in detail by Shapiro [57] and Forsgren [20]. Further examples are: the strong second-order conditions by Sun [59], the local duality by Qi [54], sequential optimality conditions by Andreani et al. [2], and the reformulation of optimality conditions via slack variables by Lourenço et al. [39]. We also refer to the book by Bonnans and Shapiro [7]. Supported by those theories, various algorithms have been developed for NS-DPs, including primal interior-point methods (IPMs) [28, 38], primal-dual IPMs [49–51, 65, 67, 68], augmented Lagrangian methods [2, 23], and sequential quadratic semidefinite programming methods [10, 22, 64, 69].

To the best of our knowledge, all the existing algorithms for NS-DPs only ensure global convergence to first-order stationary points such as KKT points \( ^* \). Hence, there exist no algorithms for computing second-order stationary points, SOSPs in short, of NS-DPs. In contrast, many algorithms for computing SOSPs of NLPs have been proposed for several decades, e.g., for unconstrained problems, the negative curvature method by McCormick [41], the trust-region method by Sorensen [58], and the cubic regularized Newton method by Nesterov and Polyak [46]. Even for constrained problems, we can find penalty methods by Auslender [4] and by Facchinei and Lucidi [17], the squared slack variables technique by Mukai and Polak [44], and the negative curvature method with projection by Goldfarb et al. [24].

\( ^* \) Other first-order stationary points are AKKT and TAKKT points [2, Definitions 4, 5].
Recently, motivated by applications in machine learning, a lot of algorithms for NLPs with worst-case iteration complexities for an approximate SOSP are proposed. For example, with respect to unconstrained optimization, we find [11, 13, 14] using the trust-region method and [12] using the negative curvature method. With regard to constrained optimization, there also exist many algorithms depending on constraints. Table 1 summarizes the algorithms for constrained NLPs with convergence to SOSP, where “eq” and “ineq” in the column of constraint stand for equality and inequality, respectively.

One may think of transforming NSDPs to NLPs by means of squared slack variables. Indeed, the semidefinite constraint \( X(x) \in S^n_+ \) can be equivalently transformed into the equality constraint \( X(x) = Y^2 \) with a slack variable \( Y \in S^n \). Hence, NSDPs reduce to the equality-constrained NLPs possessing \((x,Y)\) as variables, and thus existing NLP algorithms with convergence to SOSP are applicable to the transformed NLPs. However, such an approach has drawbacks, e.g., the number of variables increases and, as indicated in [39], there may exist a discrepancy between the set of SOSP of NSDPs and that of SOSP of the transformed NLPs.

| Constraint       | Algorithm                          | Complexity |
|------------------|------------------------------------|------------|
| Nonlinear eq     | Penalty [18] and projection [24]   | No         |
| Linear ineq      | Active-set [21]                    | No         |
| Nonlinear ineq   | Squared slack [44], penalty [4,17]| No         |
|                  | augmented Lagrangian [1,15]        |            |
|                  | **primal-dual IPM** [9,42]         |            |
| Nonlinear eq     | Projection [60] and proximal       | Yes        |
|                  | augmented Lagrangian [63]          |            |
| Linear ineq      | Active-set [40] and trust-region   | Yes        |
| Closed convex set| Second-order Frank-Wolfe [43,47]   | Yes        |
| Nonnegative orthant| **Primal IPM** [52]             | Yes        |
| Nonlinear ineq   | **Primal IPM** [25]               | Yes        |

### 1.1 Contributions

Unlike NLPs, NSDPs have no existing methods with either convergence to SOSP or a worst-case iteration complexity. The main contributions in this paper are summarized in the following two theoretical points.

1) **SOSP of NSDPs** We present the first primal-dual IPM with convergence to SOSP of NSDP (1). In the primal-dual strictly feasible region \( \{(x,Y) | X(x) \in S^n_{++}, Y \in S^n_+ \} \), the primal-dual IPM generates a sequence of approximate SOSP, abbreviated as approx. SOSP, which accumulate at an SOSP of NSDPs under some assumptions. Each approx. SOSP is defined using the primal-dual merit function given in [67]. In order to compute
such approx. SOSP, we utilize scaled gradient directions and directions of negative curvature. The difficulty of theoretical analysis of NSDPs often stems from the so-called sigma term (see Definition 2.4), which reflects the curvature of the positive semidefinite cone. We need to handle it carefully in order to show the convergence to SOSP of NSDPs.

2) a worst-case iteration complexity for approx. SOSP We give a worst-case iteration complexity of the proposed primal-dual IPM for computing an approx. SOSP. Our step size rule makes it possible to keep the generated iterates in the strictly feasible region and obtain a worst-case iteration complexity. Another point to note is that this result is established in the primal-dual framework, whereas previous studies [25, 52] regarding NLPs gave worst-case iteration complexities for primal IPMs. Indeed, we can obtain a primal IPM with convergence to SOSP and a worst-case iteration complexity with slight modification.

1.2 Related works

Primal-dual IPM for NSDP Let us review some existing studies on primal-dual IPM (PDIPM in short) for NSDPs as the most relevant works. Similar to the (path-following) PDIPM for LSDPs (see e.g., [62, Chapter 10]), the fundamental framework of the existing PDIPMs for NSDPs is to approach a KKT triplet of a NSDP by approximately computing perturbed KKT triplets and driving a perturbation parameter to zero. We believe that Yamashita, Yabe, and Harada [67] presented the first PDIPM for NSDPs and showed its global convergence to a KKT triplet of a NSDP using the family of Monteiro-Zhang directions. Its local convergence property was analyzed by Yamashita and Yabe in [66]. Afterward, Kato et al. [30] studied the global convergence of PDIPMs using a different penalty function from [67]. Yamakawa and Yamashita [65] developed a different PDIPM based on the shifted barrier KKT conditions for NSDPs. Okuno [49] analyzed local convergence of a PDIPM using the family of Monteiro-Tsuchiya directions. We re-emphasize that the existing NSDP algorithms including the above PDIPMs have neither convergence guarantees to an SOSP nor iteration complexities even for a KKT point. Though the trust-region based PDIPM, which was presented by Yamashita et al. [68] most recently, is expected to have a convergence guarantee to an SOSP, they only showed convergence to KKT points. Moreover, evaluating an iteration complexity of their PDIPM does not seem easy.

IPM with SOSP and iteration complexities for NLP We review some existing IPMs with convergence to an SOSP of constrained NLPs. The fundamental idea of such IPMs is to approach the set of SOSP by tracking a path-like set formed by SOSP of the problems that are obtained by expressing the inequality constraints of NLPs in the objectives through interior penalty functions

\( ^{02} \text{A scaled gradient direction is intended to be the steepest-descent one premultiplied with a positive definite symmetric matrix.} \)
such as a log-barrier function. To compute SOSP s of such penalized NLP s, Conn et al. [9] used the trust-region method in the framework of PDIPM and Moguerza and Prieto [42] employed the negative curvature method coupled with the modified Newton method. Recently, worst-case iteration complexities of primal IPMs to SOSP s were analyzed by O’Neill and Wright [52] for NLP s with the nonnegative orthant constraint $x \geq 0$ using the Newton-CG method [56], and also by Hinder and Ye [25] for NLP s with general nonlinear inequalities using the trust-region method. We note that there are still no works handling worst-case iteration complexities for PDIPMs. The above IPM papers are shown in bold in the aforementioned Table 1.

1.3 Notation and outline of the paper

Throughout this paper, the following symbols are often used. For $A \in S^m$, $\lambda_i(A)$ denotes its $i$-th largest eigenvalue. In particular, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of $A$, respectively. For $B \in S^m_+$, $B^{1/2}$ denotes the positive semidefinite root of $B$, that is, $B^{1/2} \in S^n$ and $(B^{1/2})^2 = B$. For a matrix $C \in \mathbb{R}^{m \times n}$, the transpose, the Moore-Penrose pseudo-inverse, the rank, and the null space of $C$ are denoted by $C^T$, $C^\dagger$, rank($C$), and Ker($C$), respectively. Also, $\|C\|_F$ denotes the Frobenius norm of $C$, that is, $\|C\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n C_{ij}^2}$, where $C_{ij}$ denotes the $(i, j)$-th entry of $C$. When $C$ is a $m \times m$ real symmetric matrix, $\|C\|_F$ equals to $\sqrt{\sum_{i=1}^m \lambda_i(C)^2}$. In addition, $\|C\|$ denotes the spectral norm, that is, $\|C\| = \sqrt{\lambda_{\max}(C^T C)}$. For a vector $c \in \mathbb{R}^n$, $\|c\|$ denotes the Euclidean norm of $c$ and $c_i$ denote the $i$-th element of the vector. For matrices $Y \in \mathbb{R}^{m \times n}$ and $Z \in \mathbb{R}^{m \times n}$, their inner product is denoted as $\langle Y, Z \rangle := \text{tr}(Y^T Z)$.

We also define $A_i(x) := \partial X/\partial x_i(x)$ for each $i \in \{1, \ldots, n\}$. In addition, we define $A^*(x)$ for $x \in \mathbb{R}^n$ as the following operator from $S^m$ to $\mathbb{R}^n$: $A^*(x)Z := (\langle A_1(x), Z \rangle, \ldots, \langle A_n(x), Z \rangle)^T$, where $Z \in S^m$. Let $d \in \mathbb{R}^n$ and we also define $\Delta X(x; d) := \sum_{i=1}^n A_i(x)d_i$.

Outline. The paper is organized as follows. The rest of this section introduces assumptions of Lipschitz continuities. Section 2 reviews optimality conditions and constraint qualifications of NSDPs. Then, Section 3 describes the proposed IPM and Sections 4 and 5 analyze its convergence, while Section 6 states that, with slight modification, it is possible to make a primal IPM with almost the same convergence properties. Section 7 presents the result of numerical experiments. Section 8 concludes the paper.

1.4 Assumptions on the objective and constraint

In order to establish a worst-case iteration complexity for the proposed IPM, we make assumptions on Lipschitz continuities and boundedness of derivatives.
Let $X_{++}$ denote the strictly feasible region, that is,

$$X_{++} := \{ x \in \mathbb{R}^n : X(x) \in S_{++}^m \}$$

(2)

and $\text{conv}(X_{++})$ denote the convex hull of $X_{++}$. The following Assumption 1 for the objective and constraint on $\text{conv}(X_{++})$ are implicitly supposed throughout the paper. Recall that twice continuously differentiability of the objective and that of the constraint were assumed at the beginning of the paper.

**Assumption 1** The gradient of $f$ is $\tilde{L}_1$-Lipschitz continuous on $\text{conv}(X_{++})$ and the Hessian is $\tilde{L}_2$-Lipschitz continuous on $\text{conv}(X_{++})$, that is, we have

$$\|\nabla f(x) - \nabla f(z)\| \leq \tilde{L}_1 \|x - z\|,$$

(3)

$$\|\nabla^2 f(x) - \nabla^2 f(z)\| \leq \tilde{L}_2 \|x - z\|,$$

(4)

for any $x, z \in \text{conv}(X_{++})$. Moreover, we assume that, for any $x \in \text{conv}(X_{++})$,

$$\sum_{i=1}^{n} \|A_i(x)\|_F \leq L_0,$$

(5)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 X}{\partial x_i \partial x_j}(x) \right\|_F \leq \tilde{L}_1,$$

(6)

where $A_i(\cdot)$ is defined in Section 1.3. For the second-order derivative of $X$, we also assume that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial^2 X}{\partial x_i \partial x_j}(x) - \frac{\partial^2 X}{\partial x_i \partial x_j}(z) \right\|_F \leq \tilde{L}_2 \|x - z\|$$

(7)

for any $x, z \in \text{conv}(X_{++})$. For simplicity, we define $L_1 := \max \{ \tilde{L}_1, \tilde{L}_1 \}$ and $L_2 := \max \{ \tilde{L}_2, \tilde{L}_2 \}$.

The assumptions (3) and (4) on the objective function $f$ are often assumed in deriving iteration complexities [24, 25]. The constants in the remaining assumptions on the constraint function $X$ are ensured to exist if $X$ is affine or $X_{++}$ is bounded.

The boundedness of the derivatives implies the Lipschitz continuities. Indeed, under Assumption 1, we can show

$$\|X(x) - X(z)\|_F \leq L_0 \|x - z\|,$$

(8)

$$\sum_{i=1}^{n} \|A_i(x) - A_i(z)\|_F \leq L_1 \|x - z\|,$$

(9)

for any $x, z \in \text{conv}(X_{++})$. 
2 Optimality conditions for NSDPs

This section aims to introduce optimality conditions together with constraint qualifications for NSDP (1). First, we define the Fritz-John conditions.

**Definition 2.1 (Fritz-John (FJ) conditions)** Let $L^g(x, \lambda, \Omega) := \lambda f(x) - \langle X(x), \Omega \rangle$ be the generalized Lagrangian of NSDP (1). We say that the FJ conditions for NSDP (1) hold at $\bar{x} \in \mathbb{R}^n$ if there exists some $(\bar{\lambda}, \Omega) \in \mathbb{R} \times \mathbb{S}^m$ such that

\begin{align*}
\nabla_x L^g(\bar{x}, \bar{\lambda}, \Omega) &= 0, \quad \text{(10a)} \\
X(\bar{x})\Omega &= O, \quad \text{(10b)} \\
X(\bar{x}) &\in \mathbb{S}^m_+, \quad \text{(10c)} \\
(\bar{\lambda}, \Omega) &\in \mathbb{R}^+ \times \mathbb{S}^m_+, \quad \text{(10d)} \\
\bar{\lambda} + \|\Omega\|_F &\neq 0, \quad \text{(10e)}
\end{align*}

where $\nabla_x L^g(\bar{x}, \bar{\lambda}, \Omega) = \nabla f(\bar{x}) - A^*(\bar{x})\Omega$. Here $A^*(\cdot)$ is defined in Section 1.3. We call $\bar{x}$ a FJ point for NSDP (1).

The FJ conditions are necessary conditions for the local optimality regardless of any constraint qualifications [7, Proposition 5.87]. Next, we introduce the Mangasarian–Fromovitz constraint qualification (MFCQ) [7, § 5.3.4] to describe the KKT conditions for NSDP (1).

**Definition 2.2 (Mangasarian–Fromovitz constraint qualification)** Let $x \in \mathbb{R}^n$ be a feasible point for NSDP (1). If there exists a vector $h \in \mathbb{R}^n$ such that $X(x) + \Delta X(x; h) \in \mathbb{S}^m_+$, where $\Delta X(x; h)$ is defined in Section 1.3, then we say that the MFCQ holds at $x$.

The KKT conditions for NSDP (1) can be formally defined as follows.

**Definition 2.3 (Karush–Kuhn–Tucker (KKT) conditions)** Define the Lagrangian of NSDP (1) by $L(x, \Lambda) := f(x) - \langle X(x), A \rangle$ for $(x, A) \in \mathbb{R}^n \times \mathbb{S}^m$. We say that the KKT conditions for NSDP (1) hold at $\bar{x}$ if there exists some $\bar{\Lambda} \in \mathbb{S}^m$ satisfying

\begin{align*}
\nabla_x L(\bar{x}, \bar{\Lambda}) &= 0 \text{ (stationarity of the Lagrangian)}, \quad \text{(11a)} \\
X(\bar{x}) &\in \mathbb{S}^m_+ \text{ (primal feasibility)}, \quad \text{(11b)} \\
\bar{\Lambda} &\in \mathbb{S}^m_+ \text{ (dual feasibility)}, \quad \text{(11c)} \\
X(\bar{x})\bar{\Lambda} &= O \text{ (complementarity)}, \quad \text{(11d)}
\end{align*}

where $\nabla_x L(\bar{x}, \bar{\Lambda}) = \nabla f(\bar{x}) - A^*(\bar{x})\bar{\Lambda}$. We call $\bar{x}$ and $\bar{\Lambda}$ a KKT point and a Lagrange multiplier matrix, respectively.

Hereafter, for a KKT point $\bar{x}$, we denote by $A(\bar{x})$ the set of Lagrange multiplier matrices satisfying the KKT conditions. It is well known that, under constraint qualifications such as the MFCQ, $A(\bar{x}) \neq \emptyset$ holds at a local optimum $\bar{x}$ of
NSDP (1), namely, $\bar{x}$ is a KKT point. See [7, Theorem 5.84] for the proof. It is worth mentioning that the MFCQ implies that $\Lambda(\bar{x})$ is nonempty and bounded. Next, we state the second-order necessary conditions and the “weak” second-order necessary conditions.

**Definition 2.4 (second-order necessary conditions)** Let $\bar{x} \in \mathbb{R}^n$ be a KKT point for NSDP (1). We say that $\bar{x}$ satisfies the second-order necessary conditions if

$$\sup_{\Lambda \in \Lambda(\bar{x})} d^\top (\nabla^2_{xx} L(\bar{x}, \Lambda) + H(\bar{x}, \Lambda)) d \geq 0$$

holds for all $d \in C(\bar{x})$, where the critical cone and tangent cone at $x \in \mathbb{R}^n$ are denoted by

$$C(x) := \{ d \in \mathbb{R}^n \mid \Delta X(x; d) \in T_+ S_m(X(x)), \nabla f(x)^\top d = 0 \},$$

$$T_+ S_m(X(x)) := \{ E \in S_m \mid U^\top E U \in S_{++} \},$$

respectively (see [7, §5.3.1] for the derivation). Here, $r := \text{rank}(X(x))$ and $U$ is the $m \times (m-r)$ matrix whose columns form a basis of Ker$(X(x))$. Furthermore, $H(x, \Lambda) \in S^m$ is defined as

$$(H(x, \Lambda))_{ij} := 2 \text{tr}(A_i(x) X(x)^\dagger A_j(x) \Lambda)$$

for each $i, j$. We refer to $d^\top H(x, \Lambda)d$ as a sigma term. Moreover, we call $\bar{x}$ an SOSP of NSDP (1).

Let $\bar{x}$ be a local optimum of NSDP (1). If the MFCQ holds at $\bar{x}$, then $\bar{x}$ is an SOSP [7, Theorem 5.88].

**Definition 2.5 (weak second-order necessary conditions)** Let $\bar{x} \in \mathbb{R}^n$ be a KKT point for NSDP (1). We say that the weak second-order necessary conditions hold at $x$ if inequality (12) holds for all $d \in L(\bar{x})$, where $L(x)$ for $x \in \mathbb{R}^n$ is the linear subspace defined by

$$L(x) := \{ u \in \mathbb{R}^n \mid U^\top (\Delta X(x; u)) U = 0 \}$$

if Ker$(X(x)) \neq \{0\}$, where $U$ is the same matrix as in the definition of $T_+ S_m(X(x))$. In particular, we define $L(x) := \mathbb{R}^n$ if Ker$(X(x)) = \{0\}$. We call $\bar{x}$ a w-SOSP of NSDP (1).

Note that when $X(x)$ has at least one zero-eigenvalue, that is, $x$ is situated on the topological boundary of $X_{++}$, the condition Ker$(X(x)) \neq \{0\}$ holds. Clearly, an SOSP is a w-SOSP by definition, but the converse is not necessarily true. Under the strict complementarity condition, however, these two points are identical as follows.

**Proposition 2.6 ([7, pp. 489-490])** Let $\bar{x}$ be a w-SOSP of NSDP (1). Suppose that the strict complementarity condition holds at $\bar{x}$, i.e., there exists some $\Lambda \in A(\bar{x})$ such that $X(\bar{x}) + \Lambda \in S^m_{++}$. Then, $\bar{x}$ is an SOSP of NSDP (1).
Checking condition (12) is computationally intractable. Even for NLPs, it is known to be in general NP-hard from [45, Theorem 4]. On the other hand, the weak second-order necessary conditions can be checked efficiently by computing a basis of $\mathcal{L}(\bar{x})$ if $\text{Ker}(X(\bar{x})) \neq \{0\}$.

We will show that a sequence generated by the proposed IPM accumulates at w-SOSPs, which are SOSPs under the presence of the strict complementarity condition by Proposition 2.6. As well, in many articles on constrained NLPs, convergence to SOSPs is established through w-SOSP and the strict complementarity condition. For example, see [4, 9, 17, 25].

3 Proposed method

In this section, we present an IPM with convergence to SOSPs of NSDP (1), in which each iteration point $(x, Z) \in \mathbb{R}^n \times \mathbb{S}_m$ satisfies $X(x) \in \mathbb{S}_m^+$ and $Z \in \mathbb{S}_m^+$. The proposed IPM consists of two algorithms:

Decreasing-NC-PDIPM and Fixed-NC-PDIPM.

The former, at each iteration, produces an approximation to an SOSP, which is referred to as an $\varepsilon$-SOSP $(\mu, \nu)$ with parameters $\varepsilon, \mu, \nu > 0$ controlling an approximation degree, for NSDP (1), while the latter plays the role of computing an $\varepsilon$-SOSP $(\mu, \nu)$ by minimizing the primal-dual merit function $\psi_{\mu, \nu} : X_++ \times \mathbb{S}_m^+ \to \mathbb{R}$ defined as

$$\psi_{\mu, \nu}(x, Z) := F_{\mu PB}(x) + \nu F_{\mu BC}(x, Z),$$

where $F_{\mu PB}(x) := f(x) - \mu \log \det X(x)$, $F_{\mu BC}(x, Z) := \langle X(x), Z \rangle - \mu \log \det X(x) - \mu \log \det Z$, and $\mu, \nu > 0$.\footnote{The subscripts “PB” and “BC” represent “primal-barrier” and “barrier complementarity”, respectively in abbreviated form.}

This merit function was first presented in [67]. The partial derivatives of $\psi_{\mu, \nu}$ with respect to $x$ and $Z$ are

$$\nabla_x \psi_{\mu, \nu}(x, Z) = \nabla f(x) - A^*(x)((1 + \nu)\mu X(x)^{-1} - \nu Z),$$

$$\nabla_Z \psi_{\mu, \nu}(x, Z) = \nu(X(x) - \mu Z^{-1}).$$

By differentiating (16) further, we have for each $i, j$,

$$(\nabla^2_{xx} \psi_{\mu, \nu}(x, Z))_{ij} = (\nabla^2_{xx} f(x))_{ij} - \text{tr}\left[(1 + \nu)\mu X(x)^{-1} - \nu Z\right] \frac{\partial^2 X}{\partial x_i \partial x_j}(x) + (1 + \nu)\mu \text{tr}(A_i(x)X(x)^{-1}A_j(x)X(x)^{-1}).$$

Let

$$A := (1 + \nu)\mu X(x)^{-1} - \nu Z.$$}

We obtain $\nabla_x L(x, A) = \nabla_x \psi_{\mu, \nu}(x, Z)$ and

$$(\nabla^2_{xx} \psi_{\mu, \nu}(x, Z))_{ij} = (\nabla^2_{xx} L(x, A))_{ij} + (1 + \nu)\mu \text{tr}(A_i(x)X(x)^{-1}A_j(x)X(x)^{-1}).$$

\footnote{The subscripts “PB” and “BC” represent “primal-barrier” and “barrier complementarity”, respectively in abbreviated form.}
by combining (19) with the derivatives of $\psi_{\mu,\nu}$. The above equations suggest that $\psi_{\mu,\nu}$ serves as a surrogate for the Lagrangian $L$. In Section 4, we will reveal the relationship between $\psi_{\mu,\nu}$ and $L$ more precisely.

Next, we define an $\varepsilon$-SOSP($\mu, \nu$) as follows:

**Definition 3.1** ($\varepsilon$-SOSP($\mu, \nu$)) Given parameters $\varepsilon := (\varepsilon_g, \varepsilon_\mu, \varepsilon_H), \mu, \nu > 0$, we call $(\bar{x}, \bar{Z}) \in X \times S^n_+$ an $\varepsilon$-SOSP($\mu, \nu$) if it holds that

\begin{align}
\|\nabla_x \psi_{\mu,\nu}(\bar{x}, \bar{Z})\| &\leq \varepsilon_g (1 + \mu \|X(\bar{x})^{-1}\|_F + \|\bar{Z}\|_F), \tag{21a} \\
\|\nabla_Z \psi_{\mu,\nu}(\bar{x}, \bar{Z})\|_F &\leq \varepsilon_\mu (1 + \mu \|\bar{Z}^{-1}\|_F), \tag{21b} \\
\lambda_{\min}(\nabla^2_{xx} \psi_{\mu,\nu}(\bar{x}, \bar{Z})) &\geq -\varepsilon_H (1 + \mu \|X(\bar{x})^{-1}\|_F + \|\bar{Z}\|_F)^2. \tag{21c}
\end{align}

In fact, by driving $(\varepsilon_g, \varepsilon_\mu, \varepsilon_H)$ and $(\mu, \nu)$ to zeros, it will be shown that a sequence of $\varepsilon$-SOSP($\mu, \nu$) accumulates at SOSPs of NSDP(1) (cf. Theorem 4.7). Thus, an $\varepsilon$-SOSP($\mu, \nu$) can be regarded as an approximation to an SOP of NSDP(1).

3.1 Decreasing-NC-PDIPM

Decreasing-NC-PDIPM produces a sequence of $\varepsilon$-SOSP($\mu, \nu$) with decreasing $\varepsilon, \mu,$ and $\nu$ to zeros. Each $\varepsilon$-SOSP($\mu, \nu$) is computed by means of Fixed-NC-PDIPM. We denote each iteration by $k$ and define $X_k := X(x^k)$. We parameterize $\varepsilon_g, \varepsilon_\mu, \varepsilon_H$ and $\nu$ by $\mu$, writing $\varepsilon_g(\mu), \varepsilon_\mu(\mu), \varepsilon_H(\mu), \nu(\mu)$, respectively, and assume that, as $k \to \infty$, it holds that

$$
\mu_k \to 0, \varepsilon_g(\mu_k) \to 0, \varepsilon_\mu(\mu_k) \to 0, \varepsilon_H(\mu_k) \to 0, \nu(\mu_k) \to 0.
$$

(22)

The formal description of Decreasing-NC-PDIPM is given in Algorithm 1.

**Remark 3.2** The proposed algorithm gradually decreases the weight parameter $\nu$ in $\psi_{\mu,\nu}$ to zero, whereas it is fixed in many PDIPMs \cite{30,49,65,67}. This device is required for technical reasons concerning the proofs. For example, see Proof of (10d) in Theorem 4.2 and (36) in Theorem 4.7.

**Algorithm 1:** Decreasing Negative Curvature Primal-Dual Interior Point Method (Decreasing-NC-PDIPM)

1. **Input:** $x^1, Z_1, \varepsilon_g(\mu), \varepsilon_\mu(\mu), \varepsilon_H(\mu), \nu(\mu), \mu_1, L_0, L_1, L_2$
2. **/ /** $(x^1, Z_1) \in X_+^n \times S^n_+$
3. $k = 1$
4. **while** not converge **do**
5. Set $\mu_{k+1} > 0$ so that $\mu_{k+1} < \mu_k$
6. $(\mu, \nu, \varepsilon_g, \varepsilon_\mu, \varepsilon_H) \leftarrow (\mu_{k+1}, \nu(\mu_{k+1}), \varepsilon_g(\mu_{k+1}), \varepsilon_\mu(\mu_{k+1}), \varepsilon_H(\mu_{k+1}))$
7. $(x^{k+1}, Z_{k+1}) \leftarrow$ output of Fixed-NC-PDIPM($\mu, \nu, \varepsilon_g, \varepsilon_\mu, \varepsilon_H)$ started from $(x^k, Z_k)$
8. $k \leftarrow k + 1$
9. **end while**
3.2 Fixed-NC-PDIPM

We describe Fixed-NC-PDIPM formally in Algorithm 2, where we define $X_\ell := X(x_\ell)$ for each $\ell$. Given the parameters $(\varepsilon, \mu, \nu)$ and an initial point from Decreasing-NC-PDIPM, Fixed-NC-PDIPM computes an $\varepsilon$-SOSP($\mu, \nu$).

Each of the Updates in Fixed-NC-PDIPM employs a specific direction:

- $\frac{\kappa_{\min}}{\kappa_{\max}} \mathcal{H}_\ell(\nabla_Z \psi_{\mu,\nu})$ in Update 1,
- $\frac{h_{\min}}{h_{\max}} \mathcal{H}_\ell \nabla_x \psi_{\mu,\nu}$ in Update 2,
- an eigenvector corresponding to $\lambda_{\min}(\nabla^2_{xx} \psi_{\mu,\nu})$ in Update 3,

where $\kappa_{\min}, \kappa_{\max}, h_{\min}, h_{\max} > 0$ are prefixed constants, $\mathcal{H}_\ell$ is a symmetric linear operator from $\mathbb{S}^m$ to $\mathbb{S}^m$, and $\mathcal{H}_\ell$ is an $n \times n$ positive definite symmetric matrix satisfying

$$0 < \kappa_{\min} \leq \langle D, \mathcal{H}_\ell(D) \rangle \leq \kappa_{\max}, \forall D \in \mathbb{S}^m \text{ with } ||D||_F = 1,$$

$$0 < h_{\min} I_n \preceq \mathcal{H}_\ell \preceq h_{\max} I_n. \quad (23)$$

In order to obtain a faster convergence, it is desirable to set the inverse of the Hessian or their approximation as $\mathcal{H}_\ell$ and $\mathcal{H}_\ell$. When $\kappa_{\min} = \kappa_{\max} = h_{\min} = h_{\max} = 1$, $\mathcal{H}_\ell = I_m$ and $\mathcal{H}_\ell(\cdot)$ is the identity mapping, the directions in Updates 1 and 2 reduce to the usual partial derivatives.

The three directions are devised so as to ensure conditions (21a), (21b), and (21c), respectively. If any of those conditions is violated, proceeding along the corresponding direction with the step size $\alpha_\ell$

- keeps the strict feasibility (Proposition 5.1),
- decreases $\psi_{\mu,\nu}$ by more than a certain constant (Lemma 5.3).

The step size $\alpha_\ell$ is determined according to the Lipschitz constants $L_0, L_1$, and $L_2$. Without knowing these constants a priori, backtracking line search enables us to make Updates 1, 2, and 3 work. In numerical experiments, we implement this technique. Actually, the same convergence guarantees hold even if the order of Procedures 1, 2, and 3 is changed in Fixed-NC-PDIPM. However, taking into consideration the computation cost of eigenvalue decomposition, it would be favorable to prioritize the use of the gradient directions over the minimum eigenvalue direction.

4 Convergence to SOPS

In this section, supposing that Decreasing-NC-PDIPM produces an infinite sequence of $\varepsilon$-SOSP($\mu, \nu$), we show that the sequence accumulates at SOPS in the following order. First, Section 4.1 asserts the convergence to FJ points under certain reasonable assumptions. Second, in Section 4.2, the additional assumption of the MFCQ yields the convergence to KKT points. Finally, in Section 4.3, we prove that these KKT points are equivalent to w-SOPS.
Algorithm 2: Fixed Negative Curvature Primal-Dual Interior Point Method with $\mu$, $\nu$, $\varepsilon_\mu$, $\varepsilon_\nu$, $\varepsilon_H$ (Fixed-NC-PDIPM ($\mu, \nu, \varepsilon_\mu, \varepsilon_\nu, \varepsilon_H$))

1. **Input:** $x^1, Z_1, \mu, \nu, \varepsilon_\mu, \varepsilon_\nu, \varepsilon_H, L_0, L_1, h_{\text{min}}, h_{\text{max}}, \kappa_{\text{min}}, \kappa_{\text{max}}$
2. // $(x^1, Z_1) \in X_{++} \times S^n_{++}$
3. for $\ell = 1, \ldots$ do
4. if $\|\nabla_Z \psi_{\mu,\nu}(x^\ell, Z^\ell)\|_F > \varepsilon_\mu (1 + \mu \|Z^{-1}_\ell\|_F)$ // Procedure 1
5. then
6. Set $Z^{\ell+1}$ by Update 1
7. $x^{\ell+1} \leftarrow x^\ell$
8. else if $\|\nabla_x \psi_{\mu,\nu}(x^\ell, Z^\ell)\| > \varepsilon_\nu \left(1 + \mu \|X^{-1}_\ell\|_F + \|Z^\ell\|_F\right)$ // Procedure 2
9. then
10. Set $x^{\ell+1}$ by Update 2
11. $Z^{\ell+1} \leftarrow Z^\ell$
12. else if $\lambda_{\text{min}}(\nabla^2_{xx} \psi_{\mu,\nu}(x^\ell, Z^\ell)) < -\varepsilon_H \left(1 + \mu \|X^{-1}_\ell\|_F + \|Z^\ell\|_F\right)^2$ // Procedure 3
13. then
14. Set $x^{\ell+1}$ by Update 3
15. $Z^{\ell+1} \leftarrow Z^\ell$
16. else
17. Output $(x^\ell, Z^\ell)$
18. end if
19. end for

Update 1: Use of the scaled partial derivative w.r.t. $Z$ at $(x^\ell, Z^\ell)$

1. **Input:** $x^\ell, Z^\ell, H^\ell(\cdot), \kappa_{\text{min}}, \kappa_{\text{max}}$
2. **Output:** $Z^{\ell+1}$
3. $d^\ell \leftarrow -\frac{\lambda_{\text{min}}(\nabla_Z \psi_{\mu,\nu})}{\kappa_{\text{min}}} H^\ell(\nabla_Z \psi_{\mu,\nu})$
4. $l_Z(Z^\ell) \leftarrow 2\mu \|Z^{-1}_\ell\|_F^2$ // “Local” Lipschitz constant of $\nabla_Z \psi_{\mu,\nu}$
5. $\alpha_\ell \leftarrow \min\left(\frac{1}{l_Z(Z^\ell)}, \frac{\lambda_{\text{min}}(Z^\ell)}{2\|d^\ell\|_F}\right)$ // Step size
6. $Z^{\ell+1} \leftarrow Z^\ell + \alpha_\ell d^\ell$

Update 2: Use of the scaled partial derivative w.r.t. $x$ at $(x^\ell, Z^\ell)$

1. **Input:** $x^\ell, Z^\ell, H^\ell, L_0, L_1, h_{\text{min}}, h_{\text{max}}$
2. **Output:** $x^{\ell+1}$
3. $d^\ell \leftarrow -\frac{\lambda_{\text{min}}(H^\ell)}{h_{\text{min}}} H^\ell(\nabla_x \psi_{\mu,\nu})$
4. $l_x(x^\ell, Z^\ell) \leftarrow L_1 + \nu_\ell \|Z^\ell\|_F + 2(1 + \nu)\mu L_0 \|X^{-1}_\ell\|_F + (1 + \nu)\mu L_1 \|X^{-1}_\ell\|_F$ // “Local” Lipschitz constant of $\nabla_x \psi_{\mu,\nu}$
5. $\alpha_\ell \leftarrow \min\left(\frac{\lambda_{\text{min}}(X^\ell)}{2l_x(X^\ell, Z^\ell)}, \frac{1}{l_x(x^\ell, Z^\ell)}\right)$ // Step size
6. $x^{\ell+1} \leftarrow x^\ell + \alpha_\ell d^\ell$
Update 3: Use of the negative curvature w.r.t. $x$ at $(x^{\ell}, Z^{\ell})$

1. **Input:** $x^{\ell}, Z^{\ell}, L_0, L_1, L_2$
2. **Output:** $x^{\ell+1}$
3. $d_{x^{\ell}} \leftarrow$ The normalized eigenvector corresponding to $\lambda_{\min}(\nabla^2_{xx} \psi_{\mu, \nu}(x^{\ell}, Z^{\ell}))$
4. $l_{xx}(x^{\ell}, Z^{\ell}) \leftarrow L_2 + \nu L_2 \|Z^{\ell}\|_F + (1 + \nu)\mu \left(L_2 \|X^{\ell-1}\|_F^2 + 4L_1 L_0 \|X^{\ell-1}\|_F^2 + 6L_0 \|X^{\ell-1}\|_F^3 \right)$
5. // “Local” Lipschitz constant of $\nabla^2_{xx} \psi_{\mu, \nu}$
6. if $d_{x^{\ell}}^\top x^{\ell} \nabla_{xx} \psi_{\mu, \nu}(x^{\ell}, Z^{\ell}) > 0$ then
7. $d_{x^{\ell}} \leftarrow -d_{x^{\ell}}$
8. end if
9. // Reverse the direction in order to ensure $d_{x^{\ell}}$ is a descent direction
10. $\alpha^{\ell} \leftarrow \min \left(\frac{-2\lambda_{\min}(\nabla^2_{xx} \psi_{\mu, \nu}(x^{\ell}, Z^{\ell}))}{l_{xx}(x^{\ell}, Z^{\ell})}, \frac{\lambda_{\min}(X^{\ell})}{2L_0 \|d_{x^{\ell}}\|} \right)$ // Step size
11. $x^{\ell+1} \leftarrow x^{\ell} + \alpha^{\ell} d_{x^{\ell}}$

Since each iterate of Decreasing-NC-PDIPM is an $\varepsilon_k$-SOSP($\mu_k, \nu_k$), the following equations hold:

$$X_k \in S^+_m,$$  \hspace{1cm} (25a)
$$Z_k \in S^+_m,$$  \hspace{1cm} (25b)
$$\|\nabla_x \psi_{\mu, \nu}(x^k, Z_k)\|_F \leq \varepsilon_g(\mu_k) \left(1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F\right),$$  \hspace{1cm} (25c)
$$\|\nabla Z \psi_{\mu, \nu}(x^k, Z_k)\|_F \leq \varepsilon_{\mu}(\mu_k) \left(1 + \mu_k \|Z_k^{-1}\|_F\right),$$  \hspace{1cm} (25d)
$$\lambda_{\min}(\nabla^2_{xx} \psi_{\mu, \nu}(x^k, Z_k)) \geq -\varepsilon_H(\mu_k) \left(1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F\right)^2.$$  \hspace{1cm} (25e)

**Remark 4.1** The subsequent analysis on convergence to SOSPs focuses only on the behavior of $\varepsilon_k$-SOSP($\mu_k, \nu_k$) as $k$ tends to $\infty$. The analysis never requires Assumption 1, which is the assumption of the Lipschitz constants of the functions.

4.1 Convergence to FJ points

In this section, we will prove that any accumulation points of Decreasing-NC-PDIPM are FJ points under the following assumption.

**Assumption 2**

(i) The sequence $\{x^k\}$ is bounded.

(ii) The sequence $\{Z_k - \mu_k X_k^{-1}\}$ is bounded.

The first assumption is often found in many articles for NLPs or NSDPs, for example, [42, 67, 68]. In contrast, the second one is actually closely related to (25d), and may be peculiar to the analysis in this paper. On the central-path-like set formed by 0-SOSP($\mu, \nu$), $Z = \mu X^{-1}$ holds from (25d). In view of this
fact, the assumption is interpreted as that the sequence is not too distant from the path-like set.

**Theorem 4.2 (convergence to FJ points)** Suppose that Assumption 2 holds. Then, \( \{(x^k, \lambda_k, \Omega_k)\} \) with

\[
\lambda_k := \frac{1}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F}, \quad \Omega_k := \frac{A_k}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F}
\]

is bounded and any accumulation points satisfy the FJ conditions, where \( A_k = (1 + \nu(\mu_k))\mu_k X_k^{-1} - \nu(\mu_k)Z_k \) as defined in (19).

**Proof** Note that from (22) and Assumption 2 (ii)

\[
\lim_{k \to \infty} \nu(\mu_k) \|\mu_k X_k^{-1} - Z_k\|_F = 0 \tag{26}
\]

holds. Since \( \{x^k\} \) is bounded from Assumption 2 (i) and \( X \) is continuous, \( \{X_k\} \) is bounded. Next, we prove the boundedness of \( \{(\Omega_k, \lambda_k)\} \). From the definition of \( \Omega_k \), we have

\[
\|\Omega_k\|_F = \frac{\|\mu_k X_k^{-1} + \nu(\mu_k)(\mu_k X_k^{-1} - Z_k)\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \leq \frac{\mu_k \|X_k^{-1}\|_F + \|Z_k\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \leq 1 + \nu(\mu_k) \|\mu_k X_k^{-1} - Z_k\|_F.
\]

By taking \( \lim \sup \) of \( \|\Omega_k\|_F \) and the rightmost hand and using (26), we see that \( \lim \sup_{k \to \infty} \|\Omega_k\|_F \leq 1 \), implying the boundedness of \( \{\Omega_k\} \). Since \( 0 < \lambda_k = \frac{1}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \leq 1 \), \( \{\lambda_k\} \) is bounded. Consequently, we obtain the boundedness of \( \{(x^k, \lambda_k, \Omega_k, X_k)\} \).

Let an arbitrary accumulation point of \( \{(x^k, X_k, \lambda_k, \Omega_k)\} \) be \( (\bar{x}, \bar{X}, \bar{\lambda}, \bar{\Omega}) \). To prove the convergence to FJ points, let us check conditions (10a)-(10e) one by one.

**Proof of (10a)** We have, from condition (25c),

\[
\|\nabla_x \psi_{\mu, \nu}(x^k, Z_k)\| \leq \varepsilon_{4}(\mu_k) (1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F).
\]

By substituting (16) into this inequality and dividing both the sides with \( 1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F \), we obtain

\[
\|\lambda_k \nabla f(x^k) - A^*(x^k) \Omega_k\| \leq \varepsilon_{4}(\mu_k) \quad \xrightarrow{k \to \infty} 0
\]

and thus (10a) is confirmed at \( (\bar{x}, \bar{X}, \bar{\Omega}) \).

**Proof of (10b)** From (22) and Assumption 2 (ii), we have

\[
\|A_k X_k\|_F = \|\mu_k I_m + \nu(\mu_k) (\mu_k I_m - Z_k X_k)\|_F \leq \|\mu_k I_m\|_F + \nu(\mu_k) \|X_k\|_F \|\mu_k X_k^{-1} - Z_k\|_F \quad \xrightarrow{k \to \infty} 0.
\]
Therefore we have
\[ \|X_k \Omega_k\|_F = \frac{\|X_k A_k\|_F}{1 + \mu_k \|X_k^{-1}\|_F} \leq \|X_k A_k\|_F \xrightarrow{k \to \infty} 0 \]
and thus (10b) holds at \((X, \Omega)\).

Proof of (10c) From condition (25a), any accumulation points of \(\{X_k\}\) satisfy (10c).

Proof of (10d) Since \(\lambda_k \geq 0\) at each \(k\), its accumulation point is also nonnegative. We next show that any accumulation point of \(\{\Omega_k\}\) is positive semidefinite. First, it follows that
\[ \nu(\nu_k) \lambda_{\min}(\mu_k X_k^{-1} - Z_k) \leq \nu(\nu_k) \|\mu_k X_k^{-1} - Z_k\|_F \]
and thus the left-hand side converges to zero as \(k \to \infty\) from (26). Since \(\lambda_{\min}(A) + \lambda_{\min}(B) \leq \lambda_{\min}(A + B)\) for \(A, B \in S^+_m\) (see [27, Theorem 4.3.1]), we have, from the above inequality,
\[ \liminf_{k \to \infty} \lambda_{\min}(\Omega_k) \geq \liminf_{k \to \infty} \frac{\lambda_{\min}(\mu_k X_k^{-1})}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} + \liminf_{k \to \infty} \left( \nu(\mu_k) \frac{\lambda_{\min}(\mu_k X_k^{-1} - Z_k)}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \right) \geq 0. \]
Therefore (10d) is confirmed at \(\Omega\).

Proof of (10e) \(\lambda_k + \|\Omega_k\|_F = \frac{1 + \|\mu_k X_k^{-1} + \nu(\mu_k)(\mu_k X_k^{-1} - Z_k)\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \geq \frac{1 + \|\mu_k X_k^{-1}\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} - \frac{\nu(\mu_k)\|\mu_k X_k^{-1} - Z_k\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \)
To obtain a contradiction, assume that there exists a subsequence \(\{\lambda_k + \|\Omega_k\|_F\}_{k \in S}^\infty\) converging to zero. Taking the limit of the above inequality and using (26) again yield
\[ 0 = \lim_{k \to \infty} \left( \lambda_k + \|\Omega_k\|_F \right) \geq \lim_{k \to \infty} \frac{1 + \|\mu_k X_k^{-1}\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F}, \]
which entails
\[ \lim_{k \to \infty} \frac{1 + \|\mu_k X_k^{-1}\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} = 0. \]
We consider the two cases where \( \{Z_k\}_{k \in \mathcal{S}} \) is bounded and unbounded and then derive contradictions in both two cases. If \( \{Z_k\}_{k \in \mathcal{S}} \) is bounded, by Assumption 2 (ii), \( \{\mu_k X_k^{-1}\}_{k \in \mathcal{S}} \) is also bounded. Then, the left-hand side in (27) must be positive because it is bounded from below by \( 1/\limsup_{k \to \infty} (1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F) \), but this contradicts (27). Next, consider the case where \( \{Z_k\}_{k \in \mathcal{S}} \) is unbounded. Notice that it holds that

\[
\frac{1 + \|\mu_k X_k^{-1}\|_F}{1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F} \geq 1 + \frac{\|\mu_k X_k^{-1} - Z_k\|_F - \|Z_k\|_F}{1 + \|\mu_k X_k^{-1} - Z_k\|_F + 2\|Z_k\|_F} =: v_k,
\]

where the last inequality follows by the fact \( \|A + B\|_F^2 \geq (\|A\|_F - \|B\|_F)^2 \) for any \( A, B \in \mathbb{S}^m \) and the substitution \( (A, B) = (\mu_k X_k^{-1} - Z_k, Z_k) \). Assumption 2 (ii) and the unboundedness of \( \{\|Z_k\|_F\}_{k \in \mathcal{S}} \) yield \( \limsup_{k \to \infty} v_k \geq 1/2 \). Meanwhile, (27), (28), and \( v_k \geq 0 \) lead to \( \lim_{k \to \infty} v_k = 0 \), giving a contradiction again. So, (10e) holds at \( (\bar{x}, \bar{\lambda}) \).

The proof is complete. \( \square \)

4.2 Convergence to KKT points

In this section, we prove that any accumulation points of a sequence generated by Decreasing-NC-PDIPM are KKT points by assuming the MFCQ in addition to Assumption 2.

Assumption 3 (MFCQ) The MFCQ holds at any accumulation point of \( \{x^k\} \).

Theorem 4.3 (convergence to KKT points) Suppose that Assumptions 2 and 3 hold. Then \( \{(x_k, A_k)\} \) is bounded and any accumulation points of \( \{(x_k, A_k)\} \) satisfy the KKT conditions. In addition, \( \{Z_k\} \) and \( \{\mu_k X_k^{-1}\} \) are bounded.

Proof Let \( \{\lambda_k\} \) and \( \{\Omega_k\} \) be the same ones as defined in Theorem 4.2. From Theorem 4.2, \( \{(x^k, \lambda_k, \Omega_k)\} \) is bounded and furthermore, by letting \( (\bar{x}, \bar{\lambda}, \bar{\Omega}) \) be its arbitrary accumulation point, \( (\bar{x}, \bar{\lambda}, \bar{\Omega}) \) satisfies the FJ conditions. Under the MFCQ, \( \bar{x} \) is a KKT point and \( \bar{\lambda} \neq \emptyset \). This fact together with the boundedness of \( \{\Omega_k\} \) yields the boundedness of \( \{A_k\} = \lambda_k^{-1} \Omega_k \). Therefore, any accumulation point of \( \{(x^k, A_k)\} \) satisfies the KKT conditions. We thus obtain the former assertion.

Since an arbitrary accumulation point of \( \{\lambda_k\} \) is nonzero, we see that \( \{Z_k\} \) and \( \{\mu_k X_k^{-1}\} \) are bounded by recalling the definition of \( \lambda_k \). The proof is complete. \( \square \)

Assumption 4 The sequence \( \left\{ \frac{\sigma_k(\mu_k)}{\mu_k} \right\} \) converges to zero.

Theorem 4.4 Suppose that Assumptions 3 and 4 hold. Then we have \( \|\mu_k X_k^{-1} - Z_k\|_F \to 0 \), \( \|A_k - Z_k\|_F \to 0 \).
Proof From (25d) and (17), we have
\[ \|X_k - \mu_k Z_k^{-1}\|_F \leq \frac{\varepsilon_\mu(\mu_k)}{\nu(\mu_k)} (1 + \mu_k \|Z_k^{-1}\|_F), \]
and thus get
\[
\|\mu_k X_k^{-1} - Z_k\|_F = \|X_k^{-1} (\mu_k Z_k^{-1} - X_k) Z_k\|_F \\
\leq \|X_k^{-1}\|_F \|\mu_k Z_k^{-1} - X_k\|_F \|Z_k\|_F \\
\leq \frac{\varepsilon_\mu(\mu_k)}{\nu(\mu_k)} \|\mu_k X_k^{-1}\|_F \|Z_k\|_F (1 + \mu_k \|Z_k^{-1}\|_F) \xrightarrow{k \to \infty} 0,
\]
where the last convergence follows from Assumption 4 and the boundedness of \( \{\mu_k X_k^{-1}\} \) and \( \{Z_k\} \) from Theorem 4.3. We also have \( \|A_k - Z_k\|_F = (1 + \nu(\mu_k)) \|\mu_k X_k^{-1} - Z_k\|_F \xrightarrow{k \to \infty} 0 \). The proof is complete. \qed

4.3 Convergence to SOSPs

Before presenting the theorem regarding the convergence to w-SOSPs, we define some terminology. Let \( \mathcal{D}_\alpha \) and \( \mathcal{D}_\alpha^{\pm(+)} \) be the set of \( p \times p \) diagonal matrices and the set of \( p \times p \) positive (semi) definite diagonal matrices, respectively. We also define \( \mathcal{O}^{p \times q} := \{U \in \mathbb{R}^{p \times q} \mid U^T U = I_q\} \). If \( p = q \), we abbreviate \( \mathcal{O}^{p \times p} \) as \( \mathcal{O}^p \). Under Assumption 2 (i), \( \{X_k\} \) and \( \{x^k\} \) are bounded and thus have accumulation points, say \( X_* \) and \( x^* \), respectively. There exists a subsequence \( K \) such that
\[
\lim_{k \in K \to \infty} x^k = x^*, \quad \lim_{k \in K \to \infty} X_k = X_*.
\]
In what follows, we introduce some specific sequences for the subsequence \( K \).

For each \( k \in K \), an eigenvalue decomposition of \( X_k \in \mathcal{S}^m_{++} \) is expressed as
\[
X_k = U_k D_k U_k^\top, \quad U_k \in \mathcal{O}^m, \quad D_k \in \mathcal{D}^m_{++}.
\]
Since \( \{U_k\} \) and \( \{D_k\} \) are bounded, they have accumulation points, say \( U_* \in \mathcal{O}^m \) and \( D_* \in \mathcal{D}^m_+ \). We have \( X_* = U_* D_* U_*^\top \). Let
\[
r_* := \text{rank}(X_*)
\]
and \( \mathcal{D}^p \) be the unique submatrix of \( D_* \) such that it is in \( \mathcal{D}^p_{++} \). Extracting appropriate \( r_* \) columns of \( U_* \), written as \( U_*^p \in \mathcal{O}^{m \times r_*} \), we can rewrite \( X_* \) as \( X_* = U_*^p D_*^p (U_*^p)^\top \). Moreover, let \( U_*^N \in \mathcal{O}^{m \times (m-r_*)} \) be the remaining columns of \( U_* \) after excluding \( U_*^p \) from \( U_* \). Without loss of generality, we assume
\[
D_* = \begin{bmatrix} D_*^p & O \\ O & O \end{bmatrix}, \quad U_* = \begin{bmatrix} U_*^p & U_*^N \end{bmatrix}, \quad \lim_{k \in K \to \infty} (U_k, D_k) = (U_*, D_*).
\]
Notice that $U_k^N$ spans $\text{Ker}(X)$, while $U_k^P$ does its orthogonal complement space. In relation to the above decomposition of $X$, there exist some matrices $D_k^P \in D_{+ \times +}, U_k^P \in O^{m \times r}$, $D_k^N \in D_{- \times -}$ and $U_k^N \in O^{m \times (m-r)}$ such that

$$X_k = U_k^P D_k^P (U_k^P)^\top + U_k^N D_k^N (U_k^N)^\top$$

and

$$D_k = \begin{bmatrix} D_k^P & 0 \\ 0 & D_k^N \end{bmatrix}, \quad U_k = [U_k^P, U_k^N].$$

Notice that

$$\lim_{k \to \infty} (D_k^P, D_k^N, U_k^P, U_k^N) = (D_*, D_*, U_*, U_*),$$

by comparing the above equations and (29). Furthermore, we define $X_k^P \equiv U_k^P D_k^P (U_k^P)^\top$ and $X_k^N \equiv U_k^N D_k^N (U_k^N)^\top$, leading to

$$X_k^P = U_k^P D_k^P (U_k^P)^\top,$$

$$X_k^N = U_k^N D_k^N (U_k^N)^\top,$$

$$X_k^{-1} = (X_k^P)^\dagger + (X_k^N)^\dagger.$$ (32)

If $\text{Ker}(X) = \{0\}$, i.e., $x^* \in X_{++}$, the sequences $\{U_k^N\}_{k \in K}, \{D_k^N\}_{k \in K}, \{X_k^N\}_{k \in K}$, and the points $U_*^N$ and $D_*^N$ are not well-defined. Even in that case, the above argument makes sense just by neglecting the terms including them.

### 4.3.1 Theorem for convergence to SOSP

We require an additional assumption on the subsequence $K$. To begin with, for each $k \in K$, let $G_k \in \mathbb{R}^{(m-r+1) \times n}$ be the matrix whose each row is expressed as

$$[(U_k^N)^\top A_1(x^k) (U_k^N)_q, \ldots, (U_k^N)^\top A_n(x^k) (U_k^N)_q] \quad (1 \leq p \leq q \leq m-r),$$

where $A_i(\cdot) (i = 1, \ldots, n)$ are defined in Section 1.3 and moreover, let

$$L_k := \text{Ker}(G_k) = \left\{ u \in \mathbb{R}^n \bigg| (U_k^N)^\top \Delta X(x^k; u) U_k^N = 0 \right\}.$$ (33)

Notice that $\{G_k\}_{k \in K}$ has a limit, denoted by $G_*$, as $\lim_{k \to \infty} (x^k, U_k^N) = (x^*, U_*^N)$. It follows that $L(x^*) = \text{Ker}(G_*)$ from the definitions of $L(x^*)$ and $G_*$. If $\text{Ker}(X) = \{0\}$, the above symbols are not well-defined and $L(x^*) = \mathbb{R}^n$. In this case, we define $L_k := \mathbb{R}^n$ for each $k \in K$.

**Assumption 5** If $\text{Ker}(X) \neq \{0\}$, then $\text{rank}(G_k) = \text{rank}(G_*)$ for any sufficiently large $k \in K$. 

Associated with this assumption, we give the following proposition that will play an important role to prove the theorem on convergence to SOSPs. An analogous proposition is found in [3, Lemma 3.1], and the following one can be proved in a manner quite similar to this lemma, so we omit the proof here.

**Proposition 4.5** Assume that Assumption 5 holds. For arbitrarily chosen \( \bar{u} \in \mathcal{L}(x^*) \) with \( \mathcal{L} \) defined in (14), there exists a sequence \( \{u^k\}_{k \in K} \) converging to \( \bar{u} \) such that \( u^k \in L_k \) for each \( k \in K \).

Before moving onto the theorem on SOSPs, we make a remark about a sufficient condition for Assumption 5.

**Remark 4.6** A sufficient condition for Assumption 5 is the nondegeneracy condition (NC) [62, p. 86] at \( x^* \), a constraint qualification of NSDP (1). Indeed, suppose \( \text{Ker}(X(x^*)) \neq \{0\} \). Since \( \lim_{k \to \infty} G_k = G_* \) and the NC means that \( G_* \) is of full row rank, we obtain \( \text{rank}(G_k) = \text{rank}(G_*) \) for all \( k \in K \) large enough, meaning Assumption 5.

Now, we are ready to present the theorem on convergence to SOSPs. The key ingredient of the proof is that we show that, for any sequence \( \{u^k\}_{k \in K} \) with \( u^k \in L_k \),

\[
\liminf_{k \to \infty} u^k \top \nabla_{xx}^2 L(x^k, A_k) u^k + u^k \top \tilde{H}(x^k, A_k) u^k \geq 0,
\]

where \( \tilde{H}(x^k, A_k)_{ij} := 2 \text{tr}(A_i(x^k)X^{-1}_k A_j(x^k)A_k) \) (33)

for each \( i, j \). In fact, the second term \( u^k \top \tilde{H}(x^k, A_k) u^k \) converges to the sigma-term at an accumulation point of \( \{x^k\} \), which leads us to the weak second-order necessary conditions.

**Theorem 4.7 (convergence to w-SOSPs)** Suppose that Assumptions 2, 3, and 5 hold. Then any accumulation points of \( \{(x^k, A_k)\} \) are w-SOSPs of NSDP (1), which are identical to SOSPs when the strict complementarity condition holds there.

**Proof** The sequence \( \{x^k\} \) is bounded due to Assumption 2 (i), and has an accumulation point, say \( x^* \), for which the subsequence \( K \) is constructed as at the beginning of this subsection. Since \( \{A_k\}_{k \in K} \) is bounded by Theorem 4.3, we have an accumulation point \( A_* \). Without loss of generality, taking a subsequence further if necessary, we assume \( \lim_{k \to \infty} A_k = A_* \). We then have

\[
(H(x^*, A_*))_{ij} = 2 \text{tr}(A_i(x^*)X^{-1}_i A_j(x^*)A_*).
\]

where \( \tilde{H}(x^k, A_k) \) and \( H \) are defined in (33) and (13), respectively.

To begin with, we assume \( \text{Ker}(X(x^*)) \neq \{0\} \). From (20), we have

\[
(\nabla_{xx}^2 L(x^k, A_k))_{ij} = \left( \nabla_{xx}^2 \psi_{\mu, \nu}(x^k, Z_k) \right)_{ij} - (1 + \nu(\mu_k)) \mu_k \text{tr}(A_i(x^k)X^{-1}_k A_j(x^k)X^{-1}_k).
\]
Choose \( \bar{u} \in \mathcal{L}(x^*) \) arbitrarily. From Proposition 4.5, there exists a sequence \( \{u^k\} \) such that \( \lim_{k \to \infty} u^k = \bar{u} \) and \( u^k \in \mathcal{L}_k \) for each \( k \in \mathcal{K} \). Note that

\[
\begin{align*}
    u^k \nabla_x^2 L(x^k; A_k) u^k + u^k \nabla H(x^k; A_k) u^k \\
    = u^k \nabla_x^2 \psi_{\mu, \nu}(x^k, Z_{\mu}) u^k + 2 \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) A_k \right) \\
    \quad - (1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^N)^\dagger \Delta X(x^k; u^k) (X_k^N)^\dagger \right) \\
    \quad - (1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) (X_k^P)^\dagger \right) \\
    \quad - 2(1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) (X_k^N)^\dagger \right),
\end{align*}
\]

where the equality follows from definition (19) of \( A_k \) and (32). Since \( u^k \in \mathcal{L}_k \), we have \( (U_k^N)^\dagger \Delta X(x^k; u^k) U_k^N = 0 \) which together with (31) implies (B) = 0. Substituting definition (19) of \( A_k \) into (A) and using (32) again imply

\[
(A) = 2(1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) (X_k^P)^\dagger \right) \\
    + 2(1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) (X_k^N)^\dagger \right) \\
    - 2\nu(\mu_k) \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) Z_k \right),
\]

which together with (35) and (B) = 0 yields

\[
u^k \nabla_x^2 L(x^k; A_k) u^k + u^k \nabla H(x^k; A_k) u^k = L_k + M_k + N_k,
\]

where, for each \( k \),

\[
L_k := u^k \nabla_x^2 \psi_{\mu, \nu}(x^k, Z_{\mu}) u^k,
\]
\[
M_k := (1 + \nu(\mu_k)) \mu_k \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) (X_k^N)^\dagger \right),
\]
\[
N_k := -2\nu(\mu_k) \text{tr} \left( \Delta X(x^k; u^k) (X_k^P)^\dagger \Delta X(x^k; u^k) Z_k \right).
\]

Hereafter, we aim to prove

\[
\begin{align*}
    (F.1) \lim_{k \to \infty} L_k \geq 0, \quad (F.2) \lim_{k \to \infty} M_k \geq 0, \quad (F.3) \lim_{k \to \infty} N_k = 0,
\end{align*}
\]

which together with (34) actually entails \( \bar{u}^T \nabla_x^2 L(x^*; A_*) \bar{u} + \bar{u}^T H(x^*; A_*) \bar{u} \geq 0 \). Since \( \bar{u} \) is an arbitrary point in \( \mathcal{L}(x^*) \) and the KKT conditions hold at \((x^*, A_*)\) from Theorem 4.3, \((x^*, A_*)\) is a w-SOSP, meaning that the proof is complete. Let us move on to the proofs of (F.1), (F.2), and (F.3).

*Proof of (F.1)* From condition (25e), we have \( L_k \geq -\varepsilon_H(\mu_k)(1 + \mu_k \|X_k^{-1}\|_F + \|Z_k\|_F)^2 \|u^k\|^2 \), which together with (22) and the boundedness of \( \{u^k\}, \{Z_k\}, \) and \( \{\mu_k X_k^{-1}\} \) gives (F.1).
Proof of (F.2) Since $(D_k^P)^{-1} = \left((D_k^P)^{-1}\right)^2$, we obtain

$$M_k = (1 + \nu(\mu_k))\mu \left\|\left((D_k^P)^{-1/2}(U_k^P)\right)\Delta X(x^k; u^k)\left((U_k^P)(D_k^P)^{-1/2}\right)\right\|_F^2 \geq 0$$

and therefore (F.2) holds.

Proof of (F.3) In the case of $N_k$, we have

$$|N_k| \leq 2\nu(\mu_k)\left\|\Delta X(x^k; u^k)(X_k^P)^\dagger\right\|_F\left\|\Delta X(x^k; u^k)Z_k\right\|_F. \quad (36)$$

From Assumption 2 (i), the continuity of $\mathcal{A}_i$ for all $i \in \{1, \ldots, n\}$, and the boundedness of $\{u^k\}$ and $\{Z_k\}$, we have the boundedness of $\{\Delta X(x^k; u^k)\}$ and $\{\Delta X(x^k; u^k)Z_k\}$. Recall $\lim_{k\to\infty} D_k^P = D^P \in \mathcal{D}_+^*$ from equation (30). Hence, $\{(D_k^P)^{-1}\}$ is bounded, and thus $\{(X_k^P)^\dagger\}$ is also bounded. These facts ensure that $\{\Delta X(x^k; u^k)(X_k^P)^\dagger\}$ is bounded. Therefore by driving $k \in \mathcal{K} \to \infty$, $\nu(\mu_k) \to 0$ holds and hence we obtain (F.3).

We next consider the case where $\ker(X^*) = \{0\}$, i.e., $X(x^*) \in S^m_{++}$. Then, we have $\mathcal{L}_k = L(x^*) = \mathbb{R}^n$ for any $k \in \mathcal{K}$ and also have $\mathcal{A}_k = O$ by the complementarity condition. Moreover, note that $H(x^*, \mathcal{A}_k) = O$ holds. Then, by almost the same argument as above ignoring the terms including $X_k^N$ and $U_k^N$, we can reach the desired conclusion again.

The proof is complete. \hfill \Box

5 Worst-case iteration complexity

In this section, we analyze the worst-case iteration complexity of Fixed-NC-IPM. Throughout this section, we suppose that $\mathcal{X}_{++} \neq \emptyset$ and the initial point $(x^0, Z_1) \in \mathcal{X}_{++} \times S^m_{++}$ can be found.

The step size rules in Updates 1, 2, and 3 yield

$$\|x^\ell - x^{\ell + 1}\| \leq \frac{\lambda_{\min}(X_\ell)}{2L_0}, \quad \|Z_\ell - Z_{\ell + 1}\|_F \leq \frac{\lambda_{\min}(Z_\ell)}{2} \quad (37)$$

for each $\ell \geq 1$. Relevant to these inequalities, we introduce the following sets for each $\ell \geq 1$:

$$\mathcal{N}^X_\ell := \left\{ x \in \mathbb{R}^n \left| \|x - x^\ell\| \leq \frac{\lambda_{\min}(X_\ell)}{2L_0} \right. \right\},$$

$$\mathcal{N}^Z_\ell := \left\{ Z \in S^m \left| \|Z - Z_\ell\|_F \leq \frac{\lambda_{\min}(Z_\ell)}{2} \right. \right\}.$$ 

It is clear that $x^{\ell + 1} \in \mathcal{N}^X_\ell$ and $Z_{\ell + 1} \in \mathcal{N}^Z_\ell$ from (37). Hereafter, we often write $X := X(x)$ for brevity.

We next show that $(x^\ell, Z_\ell) \in \mathcal{X}_{++} \times S^m_{++}$ holds for each $\ell \geq 1$. 
Proposition 5.1 Suppose that Assumption 1 holds. For each $\ell \geq 1$, the following properties hold.

1. For any $x \in N_X$, it holds that $0 < \frac{1}{2} \lambda_i(X) \leq \lambda_i(X) \leq \frac{3}{2} \lambda_i(X)$ for each $i \in \{1, \ldots, m\}$. Therefore $X \in S_{++}^m$. Besides, $\frac{1}{2} \|X\|_F \leq \|X\|_F \leq \frac{3}{2} \|X\|_F$ and $\|X^{-1}\|_F \leq 2 \|X^{-1}\|_F$ hold.

2. For any $Z \in N^\ell_Z$, $\|Z^{-1}\|_F \leq 2 \|Z^{-1}\|_F$ and $Z \in S_{++}^m$ holds.

In particular, $\emptyset \neq N_X \times N^\ell_Z \subseteq X_{++} \times S_{++}^m$ for each $\ell$, and thus $\{(x^\ell, Z^\ell)\} \subseteq X_{++} \times S_{++}^m$.

Proof Proposition 5.1 (2) can be verified in a similar manner, and therefore we only show Proposition 5.1 (1). From the Wielandt-Hoffman theorem [27, Corollary 6.3.8] and (8), we have, for each $i \in \{1, \ldots, m\}$, $|\lambda_i(X) - \lambda_i(X)| \leq \|X - X\|_F \leq L_0 \|x^\ell - x\|$, which together with the first inequality in (37) implies

$$|\lambda_i(X^\ell) - \lambda_i(X)| \leq \frac{\lambda_{\min}(X^\ell)}{2} \leq \frac{\lambda_i(X)}{2},$$

which leads to $\frac{\lambda_i(X^\ell)}{2} \leq \lambda_i(X) \leq \frac{\lambda_i(X)}{2}$. We hence obtain $\frac{1}{2} \|X^\ell\|_F \leq \|X\|_F \leq \frac{3}{2} \|X^\ell\|_F$ and $\|X^{-1}\|_F = \sqrt{\sum_{i=1}^n \left(\frac{1}{\lambda_i(X)}\right)^2} \leq \sqrt{\sum_{i=1}^n \left(\frac{2}{\lambda_i(X)}\right)^2} = 2 \|X^{-1}\|_F$.

Finally, we consider the last assertion. Since $(x^1, Z^1) \in X_{++} \times S_{++}^m$, it follows that $N_X \times N^\ell_Z \neq \emptyset$. By Proposition 5.1 (2) and Proposition 5.1 (1) with $\ell = 1$, Inductively, $\emptyset \neq N_X \times N^\ell_Z \subseteq X_{++} \times S_{++}^m$ and $(x^{\ell+1}, Z^{\ell+1}) \in X_{++} \times S_{++}^m$ for each $\ell \geq 2$. Hence, we obtain the desired conclusion.

The proof is complete.

Later on, we will prove that if one of the approximate stationary conditions (21) is violated, there exists a direction that provides a sufficient decrease of the primal-dual merit function $\psi_{\mu,\nu}$. In order to prove this, we show the following “local” Lipschitz continuities of the derivatives and the Hessian of $\psi_{\mu,\nu}$. We provide the proof of this lemma in Appendix.

Lemma 5.2 (local Lipschitz continuities) Suppose that Assumption 1 holds. For each $\ell \geq 1$, we have the following properties, where $l_Z$, $l_x$, and $l_{xx}$ are defined in Update 1, Update 2, and Update 3, respectively.

1. **local Lipschitz continuity of $X^{-1}$:** For any $x \in N_X^\ell$,

$$\|X^{-1}_\ell - X^{-1}\|_F \leq 2L_0 \|X^{-1}\|_F \|x^\ell - x\|.$$

2. **local Lipschitz continuity of $\nabla_x \psi_{\mu,\nu}$:** For any $x \in N_X^\ell$,

$$\|\nabla_x \psi_{\mu,\nu}(x^\ell, Z^\ell) - \nabla_x \psi_{\mu,\nu}(x, Z^\ell)\| \leq l_x(x^\ell, Z^\ell) \|x^\ell - x\|.$$

3. **local Lipschitz continuity of $\nabla_Z \psi_{\mu,\nu}$:** For any $Z \in N^\ell_Z$,

$$\|\nabla_Z \psi_{\mu,\nu}(x^\ell, Z^\ell) - \nabla_Z \psi_{\mu,\nu}(x^\ell, Z)\| \leq l_Z(Z^\ell) \|Z^\ell - Z\|_F.$$
(4) local Lipschitz continuity of $\nabla_{xx}^{2} \psi_{\mu,\nu}$: For any $x \in \mathcal{N}_{Z}$,
\[ \| \nabla_{xx}^{2} \psi_{\mu,\nu}(x', Z_{t}) - \nabla_{xx}^{2} \psi_{\mu,\nu}(x, Z_{t}) \| \leq \ell_{xx}(x', Z_{t}) \| x' - x \|. \]
In order to derive a worst-case iteration complexity, we show the following descent lemmas for Updates 1, 2, and 3 by using Lemma 5.2.

Lemma 5.3 (descent lemmas) Suppose that Assumption 1 holds. Then we have the following descent lemmas.

(1) descent lemma for Update 1: Suppose that the approximated complementarity condition (21b) is violated, that is, $\| \nabla_{Z} \psi_{\mu,\nu}(x', Z_{t}) \|_{F} > \varepsilon_{g}(1 + \mu\|Z_{t}^{-1}\|_{F})$. Then, Update 1 decreases $\psi_{\mu,\nu}$ at least by $\sigma_{2} \doteq \min \{ \mu_{\nu} \varepsilon_{h_{\min}}^{2} \circ \frac{e_{H}^{2} h_{\min}^{2}}{4 L_{1} h_{\max}^{2}}, \frac{e_{H}^{2} h_{\min}^{2}}{8 L_{1} h_{\max}^{2}} \}$. 

(2) descent lemma for Update 2: Suppose that the approximated stationarity condition (21a) is violated, that is, $\| \nabla_{x} \psi_{\mu,\nu}(x', Z_{t}) \| > \varepsilon_{g}(1 + \mu\|X_{t}^{-1}\|_{F} + \|Z_{t}\|_{F})$. Then, Update 2 decreases $\psi_{\mu,\nu}$ at least by
\[ \sigma_{1} \doteq \min \{ \mu_{\nu} \varepsilon_{h_{\min}}^{2} \circ \frac{e_{H}^{2} h_{\min}^{2}}{4 L_{1} h_{\max}^{2}}, \frac{e_{H}^{2} h_{\min}^{2}}{8 L_{1} h_{\max}^{2}} \} \].

(3) descent lemma for Update 3: Suppose that the approximated second-order condition (21c) is violated, that is, $\lambda_{\min}(\nabla_{xx}^{2} \psi_{\mu,\nu}(x', Z_{t})) < -\varepsilon_{H}(1 + \mu\|X_{t}^{-1}\|_{F} + \|Z_{t}\|_{F})^{2}$. Then, Update 3 decreases $\psi_{\mu,\nu}$ at least by
\[ \sigma_{3} \doteq \min \left\{ \frac{2 \varepsilon_{H}^{2}}{5 L_{1}^{2}}, \frac{2 \varepsilon_{H}^{2}}{3 L_{2}^{2}}, \frac{2 \varepsilon_{H}^{2}}{15 L_{3}^{2}}, \frac{2 \varepsilon_{H}^{2}}{(1 + \nu) L_{1}^{2} L_{2}^{2}}, \frac{2 \varepsilon_{H}^{2}}{40 (1 + \nu)^{2} L_{1}^{2} L_{2}^{2}}, \frac{2 \varepsilon_{H}^{2}}{1350 (1 + \nu)^{2} L_{1}^{2} L_{2}^{2}} \right\}. \]

Proof of Lemma 5.3 (1) The first-order descent lemma [5, Lemma 5.7] together with Lemma 5.2 (3) yields, for any $Z \in \mathcal{N}_{Z}$,
\[ \psi_{\mu,\nu}(x', Z) \leq \psi_{\mu,\nu}(x', Z_{t}) + \alpha_{t} \nabla_{Z} \psi_{\mu,\nu}(x', Z_{t})^{\top} d_{Z_{t}} + \frac{\alpha_{t}^{2} l_{Z}(Z_{t})}{2} \| d_{Z_{t}} \|^{2}, \]
which, together with $d_{Z_{t}} = -\mu_{\nu} \lambda_{\min} \nabla_{Z} \psi_{\mu,\nu}(Z_{t})$ and $Z = Z_{t+1}$, implies
\[ \psi_{\mu,\nu}(x', Z_{t+1}) \]
\[ \leq \psi_{\mu,\nu}(x', Z_{t+1}) - \frac{\alpha_{t} l_{Z}(Z_{t})}{2 \lambda_{\min}} \| \nabla_{Z} \psi_{\mu,\nu}(x', Z_{t}) \|_{F}^{2} \]
\[ + \frac{\alpha_{t}^{2} l_{Z}(Z_{t})}{2 \lambda_{\min}} \| \nabla_{Z} \psi_{\mu,\nu}(x', Z_{t}) \|^{2} \]
\[ \leq \psi_{\mu,\nu}(x', Z_{t+1}) - \frac{\alpha_{t} l_{Z}(Z_{t})}{2 \lambda_{\min}} \| \nabla_{Z} \psi_{\mu,\nu}(x', Z_{t}) \|_{F}^{2} \left( 1 - \frac{l_{Z}(Z_{t})}{2} \right), \]
where the last inequality follows from (23) and the symmetry of $H_t(\cdot)$. Since $\alpha_t \leq \frac{1}{2\xi(x_t)}$ by the step size rule, we have $\psi_{\mu,\nu}(x^t, Z_{t+1}) \leq \psi_{\mu,\nu}(x^t, Z_t)$.

Therefore we see that $\psi_{\mu,\nu}$ decreases at least by $\xi_t := \frac{\alpha_t\kappa_{\min}^2}{\kappa_{\max}^2} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F^2$. In what follows, we evaluate a lower bound of $\xi_t$ for each choice of $\alpha_t = \frac{\lambda_{\min}(Z_t)}{4\|dz_t\|_F}$ and $\frac{1}{\xi(x_t)} \left( \frac{1}{2\mu\|Z_t\|_F^2} \right)$. The choice of $\alpha_t$ satisfies $\lambda_{\min}(Z_t) \|dZ_t\|_F^{\frac{1}{2}} \|F\|_{\psi_{\mu,\nu}}(x^t, Z_t) \|_F \leq \frac{\kappa_{\min}}{\kappa_{\max}} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F$, where we used $\| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F \geq \epsilon_\mu (1 + \mu \|Z_t^{-1}\|_F)$ in the first inequality.

With respect to the choice of $\alpha_t = \frac{1}{\xi(x_t)}$, we have

$$\xi_t = \frac{\lambda_{\min}(Z_t)\kappa_{\min}^2}{4\|dz_t\|_F \kappa_{\max}^2} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F^2 \geq \frac{\mu \lambda_{\min}(Z_t)}{4\kappa_{\max}} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F^2 \geq \frac{\kappa_{\min}^2}{4\mu \kappa_{\max}^2} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F^2.$$

These lead to the desired conclusion and the proof is complete.

Proof of Lemma 5.3 (2) The first-order descent lemma with Lemma 5.2 (2) yields, for any $x \in \mathcal{N}_t$, $\psi_{\mu,\nu}(x, Z_t) \leq \psi_{\mu,\nu}(x^{t+1}, Z_t) + \alpha_t \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \cdot d_x + \frac{\alpha_t^2 \|d_x\|^2}{2}$. From $d_x = -\frac{h_{\min}}{h_{\max}} H_t \nabla_x \psi_{\mu,\nu}(x^{t+1}, Z_t)$ and $x = x^{t+1}$, we obtain

$$\psi_{\mu,\nu}(x^{t+1}, Z_t) \leq \psi_{\mu,\nu}(x^{t+1}, Z_t) - \alpha_t \frac{h_{\min}}{h_{\max}} \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \cdot H_t \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \cdot H_t \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \leq \psi_{\mu,\nu}(x^{t+1}, Z_t) - \alpha_t \frac{h_{\min}^2}{h_{\max}^2} \| \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \|^2 \left( 1 - \frac{\|d_x(x^t, Z_t)\|}{2} \right),$$

where the last inequality follows from (24). Since $\alpha_t \leq \frac{1}{\xi(x_t)}$ by the step size rule, the above inequality implies $\psi_{\mu,\nu}(x^{t+1}, Z_t) \leq \psi_{\mu,\nu}(x^t, Z_t) - \alpha_t \frac{h_{\min}^2}{h_{\max}^2} \| \nabla_x \psi_{\mu,\nu}(x^t, Z_t) \|^2$. Now, we evaluate a lower bound of the function decrease $\delta_t := \frac{\alpha_t\kappa_{\min}^2}{\kappa_{\max}^2} \| \nabla_z \psi_{\mu,\nu}(x^t, Z_t) \|_F^2$ for each choice of $\alpha_t = \frac{\lambda_{\min}(X_t)}{2\xi(x_t)} \|d_x\|_F$ and $\frac{1}{\xi(x_t)}$. 




The choice of \( \alpha_\ell = \frac{\lambda_{\min}(X_\ell)}{L_0 \|d_\ell\|} \), together with \( \|d_\ell\| = \frac{\lambda_{\min}}{h_{\max}} \|H x_\ell \psi_{\mu,\nu}(x^\ell, Z_\ell)\| \)

\[
\delta_\ell = \frac{\lambda_{\min}^2(X_\ell)}{2h_{\max}^2} \left( \frac{\lambda_{\min}(X_\ell)}{2L_0 \|d_\ell\|} \right) \|H x_\ell \psi_{\mu,\nu}(x^\ell, Z_\ell)\|^2 \\
\geq \frac{\lambda_{\min}(X_\ell)}{2L_0} \left( 1 + \mu \|X_\ell^{-1}\|_F + \|Z_\ell\|_F \right) \\
\geq \frac{\lambda_{\min}(X_\ell)}{2 L_0 h_{\max}} \|X_\ell^{-1}\|_F \geq \frac{\lambda_{\min}(X_\ell)}{4 L_0 h_{\max}}.
\]

where the last inequality is due to \( \lambda_{\min}(X_\ell)^{-1} \leq \|X_\ell^{-1}\|_F \). With respect to the choice of \( \alpha_\ell = \frac{1}{l_\ell(x^\ell, Z_\ell)} \), we have

\[
\delta_\ell > \frac{\lambda_{\min}^2(X_\ell)}{2h_{\max}^2} \left( \frac{1 + \mu \|X_\ell^{-1}\|_F + \|Z_\ell\|_F}{\|X_\ell^{-1}\|_F + \|Z_\ell\|_F} \right)^2 \\
\geq \frac{\lambda_{\min}(X_\ell)^2}{2h_{\max}^2} \left( 1 + \mu \|X_\ell^{-1}\|_F + \|Z_\ell\|_F \right) \\
= \frac{\lambda_{\min}(X_\ell)^2}{2h_{\max}^2} \left( \frac{1}{4 \|Z_\ell\|_F} \cdot \frac{1}{8(1 + \nu) \mu L_0^2 \|X_\ell^{-1}\|_F^2} \cdot \frac{1}{4(1 + \nu) \mu L_1 \|X_\ell^{-1}\|_F} \right) \\
\geq \frac{\lambda_{\min}^2(X_\ell)}{2h_{\max}^2} \cdot \frac{\lambda_{\min}(X_\ell)^2}{2 \|Z_\ell\|_F} \cdot \frac{\mu \|X_\ell^{-1}\|_F^2}{4(1 + \nu) \mu L_1 \|X_\ell^{-1}\|_F} \\
\geq \frac{\lambda_{\min}(X_\ell)^2}{2h_{\max}^2} \cdot \frac{\lambda_{\min}(X_\ell)^2}{2 \|Z_\ell\|_F} \cdot \frac{\mu \|X_\ell^{-1}\|_F^2}{4(1 + \nu) \mu L_1 \|X_\ell^{-1}\|_F} \\
\geq \min \left\{ \frac{\lambda_{\min}(X_\ell)^2}{2h_{\max}^2}, \frac{\lambda_{\min}(X_\ell)^2}{2 \|Z_\ell\|_F}, \frac{\mu \|X_\ell^{-1}\|_F^2}{4(1 + \nu) \mu L_1 \|X_\ell^{-1}\|_F} \right\},
\]

where the fact that \( \frac{1}{a+\frac{1}{b+c+d}} \geq \frac{1}{4} \min \left\{ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d} \right\} \) for \( a, b, c, d > 0 \) derives the second inequality. These lead to the conclusion and the proof is complete.

**Proof of Lemma 5.3 (3)** Notice that Update 3 is performed when

\[
\lambda_{\min}(\nabla^2_{xx} \psi_{\mu,\nu}(x^\ell, Z_\ell)) < 0.
\]

Since \( d_\ell \) is a normalized eigenvector corresponding to \( \lambda_{\min}(\nabla^2_{xx} \psi_{\mu,\nu}(x^\ell, Z_\ell)) \) such that \( d_\ell^T \nabla^2_{xx} \psi_{\mu,\nu}(x^\ell, Z_\ell) \leq 0 \), the second-order descent lemma [46, Lemma 1] together with Lemma 5.2 (4) yields, for any \( x \in X_X \),

\[
\psi_{\mu,\nu}(x, Z_\ell) \leq \psi_{\mu,\nu}(x^\ell, Z_\ell) + \alpha_\ell^2 \left( \frac{\lambda_{\min}(\nabla^2_{xx} \psi_{\mu,\nu}(x^\ell, Z_\ell))}{2} + \frac{l_{xx}(x^\ell, Z_\ell) \alpha_\ell}{6} \right).
\]

(38)
Note that $0 < \alpha_\ell \leq \frac{2\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell))}{2}$ and $l_{xx}(x^\ell, Z_\ell) > 0$ derive
\[
\lambda_{\min}\left(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell)\right) + \frac{l_{xx}(x^\ell, Z_\ell)\alpha_\ell}{6} \leq \frac{\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell))}{6} < 0. \tag{39}
\]
By (38) and (39), $\psi_{\mu,\nu}$ decreases at least by $\eta_\ell := -\alpha_\ell \lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell)) > 0$.
We evaluate a lower bound of $\eta_\ell$ for each choice of $\alpha_\ell = -\frac{2\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell))}{l_{xx}(x^\ell, Z_\ell)}$ and $\frac{\lambda_{\min}(X_\ell)}{2L_0\|d_{x^\ell}\|}$. Recall the definition of $l_{xx}(x^\ell, Z_\ell)$ together with
\[
\lambda_{\min}\left(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell)\right) < -\varepsilon_H (1 + \mu\|X^{-1}_\ell\|_F + \|Z_\ell\|_F)^2.
\]
The choice of $\alpha_\ell = -\frac{2\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell))}{l_{xx}(x^\ell, Z_\ell)}$ implies
\[
\eta_\ell = \frac{-2(\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell)))^3}{3l_{xx}(x^\ell, Z_\ell)^2} \geq \frac{2\alpha_\ell^3}{3} \left(1 + \mu\|X^{-1}_\ell\|_F + \|Z_\ell\|_F\right)^6 \left(L_2 + \nu L_2\|Z_\ell\|_F + (1 + \nu)\left(L_2\|X^{-1}_\ell\|_F + 4L_1L_0\|X^{-1}_\ell\|_F^2 + 6L_3^3\|X^{-1}_\ell\|_F^3\right)\right)^{-2}
\]
\[
\geq \frac{2\alpha_\ell^3}{75} \min \left\{ \frac{1}{L_2^2}, \frac{1}{\nu^2 L_2^2}, \frac{1}{(1 + \nu)^2 L_2^2}, \frac{1}{16\mu^2(1 + \nu)^2 L_2^2\|X^{-1}_\ell\|_F^4}, \frac{1}{36\mu^2(1 + \nu)^2 L_2^2\|X^{-1}_\ell\|_F^6} \right\}
\]
where the fact that $\frac{1}{\frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}} \geq \frac{1}{5} \min \left\{ \frac{1}{\mu}, \frac{1}{\nu}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e} \right\}$ for $a, b, c, d, e > 0$ derives the second inequality. The choice of $\alpha_\ell = \frac{\lambda_{\min}(X_\ell)}{2L_0\|d_{x^\ell}\|}$ together with $\|d_{x^\ell}\| = 1$ implies
\[
\eta_\ell = \frac{-\left(\lambda_{\min}(X_\ell)\right)^2}{2L_0} \frac{\lambda_{\min}(\nabla^2_{xx}\psi_{\mu,\nu}(x^\ell, Z_\ell))}{6} \geq \frac{(\lambda_{\min}(X_\ell))^2}{2L_0} \frac{2\varepsilon_H (1 + \mu\|X^{-1}_\ell\|_F + \|Z_\ell\|_F)^2}{6} \geq \frac{(\lambda_{\min}(X_\ell))^2}{2L_0} \frac{2\varepsilon_H ^2 (\mu^2\|X^{-1}_\ell\|_F^2)}{24L_0^2}
\]
where the last inequality follows from $\|X^{-1}_\ell\|_F \geq \lambda_{\min}(X_\ell)^{-1}$. These lead to the conclusion and the proof is complete. \qed
To establish the convergence theorem for Fixed-NC-IPM, let us make one more assumption.

**Assumption 6** For each $\mu, \nu > 0$, $\psi_{\mu,\nu}$ is bounded below on $\mathcal{X}_+ \times \mathbb{S}_+^m$. That is, with some $\psi^*_\mu,\nu$, we have $\psi_{\mu,\nu}(x, Z) \geq \psi^*_\mu,\nu$ for all $(x, Z) \in \mathcal{X}_+ \times \mathbb{S}_+^m$.

**Theorem 5.4** (iteration complexity for approximate SOSPs) Suppose that Assumptions 1 and 6 hold. Then Fixed-NC-IPM terminates at $\varepsilon$-SOSP($\mu, \nu$) within $O\left(\frac{\psi_{\mu,\nu}(x_1, Z_1) - \psi^*_\mu,\nu}{\min \{\sigma_1, \sigma_2, \sigma_3\}}\right)$ iterations.

**Proof** By Lemma 5.3, we see that (21a)-(21c) are satisfied after $\psi_{\mu,\nu}(x_1, Z_1) - \psi^*_\mu,\nu \min\{\sigma_1, \sigma_2, \sigma_3\}$ iterations, where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are defined in Lemma 5.3. With easy calculation, we find that $\psi_{\mu,\nu}(x_1, Z_1) - \psi^*_\mu,\nu = O\left(\frac{\psi_{\mu,\nu}(x_1, Z_1) - \psi^*_\mu,\nu}{\min \{\sigma_1, \sigma_2, \sigma_3\}}\right)$. The proof is complete.

**Remark 5.5** In practice, more iterations are often required for minimizing $\psi_{\mu,\nu}$ as $\mu$ gets smaller. We can observe this fact from $\frac{1}{\min \{\sigma_1, \sigma_2, \sigma_3\}} \geq O(\mu^{-4})$ when $\varepsilon_\gamma, \varepsilon_\mu, \varepsilon_H,$ and $\nu$ are controlled carefully.

### 6 Extension to primal IPM

We can derive a primal IPM from the proposed primal-dual IPM with the following modification. In Algorithm 1, we fix $\nu(\mu_k) \equiv 0$ for every $k$, and thus $\psi_{\mu,0}$ is the merit function under this setting. In Algorithm 2, we set $Z_\ell := \mu X_{\ell-1}^{-1}$ at each iteration so that we have $\nabla Z\psi_{\mu,0}(x^\ell, Z_\ell) = O$ for every $\ell$ (see (17)) and remove Procedure 1. Algorithm 4 describes the resulting algorithm.

With these changes, it is possible to establish the convergence of the primal IPM to SOSPs with an iteration complexity in almost the same argument as the primal-dual IPM. Specifically, Fixed-NC-PIPM (Algorithm 4) converges to an $\varepsilon$-SOSP($\mu, 0$) within $O\left(\frac{\psi_{\mu,0}(x_1, Z_1) - \psi^*_\mu,0}{\min \{\sigma_1^*, \sigma_2^*, \sigma_3^*\}}\right)$ iterations under Assumption 1 and Assumption 6 with $\nu = 0$. Since $\nabla Z\psi_{\mu,0}(x^\ell, Z_\ell) = O$ for every $\ell$, the term $\varepsilon^2_\mu$ is no longer necessary in Theorem 5.4. As well, the convergence of $\varepsilon$-SOSP($\mu, 0$) to w-SOSP can be proved under the same assumptions as Theorem 4.7, but without Assumption 2 (ii) and Assumption 4. Indeed, $Z_\ell = \mu X_{\ell-1}^{-1}$ unconditionally yields Assumption 2 (ii) together with the assertion in Theorem 4.4. As a result, Assumption 4 becomes redundant because it is required by the primal-dual IPM only for the sake of establishing Theorem 4.4.

### 7 Numerical Experiments

We conduct numerical experiments to examine the efficiency of directions of negative curvature. For the sake of practical implementation, we integrate
Algorithm 4: Fixed Negative Curvature Primal Interior Point Method with $\mu, \nu, \epsilon_g, \epsilon_H$ (Fixed-NC-PIPM ($\mu, \nu, \epsilon_g, \epsilon_H$))

1. **Input:** $x^1, Z^1 = \mu X(x^1)^{-1}, \mu, \nu, \epsilon_g, \epsilon_H, L_0, L_1, h_{\min}, h_{\max}, \kappa_{\min}, \kappa_{\max}$
2. **//** $X(x^1), Z^1 \in S^m_+$
3. **for** $\ell = 1, \ldots$ **do**
4.   **if** $\|\nabla_x \psi_{\mu,0}(x^\ell, Z^\ell)\| > \epsilon_g (1 + \mu \|X^{-1}\|_p + \|Z\|_p)^2$ **// Procedure 2**
5.     **then**
6.       Set $x^{\ell+1}$ by Update 2
7.     **Z**_{\ell+1} \leftarrow \mu X^{-1}_{\ell+1}
8.   **else if** $\lambda_{\min}(\nabla^2_x \psi_{\mu,0}(x^\ell, Z^\ell)) < -\epsilon_H (1 + \mu \|X^{-1}\|_p + \|Z\|_p)^2$ **// Procedure 3**
9.     **then**
10.       Set $x^{\ell+1}$ by Update 3
11.     **Z**_{\ell+1} \leftarrow \mu X^{-1}_{\ell+1}
12. **else**
13.       **Output** $(x^\ell, Z^\ell)$
14. **end if**
15. **end for**

line-search procedures: We use the upper bound of the step sizes $\alpha_\ell$ given in Proposition 5.1 as initial values and backtrack until $\psi_{\mu,\nu}$ decreases by

- $\alpha_{\ell h_{\min}}^2 \|\nabla_x \psi_{\mu,0}(x^\ell, Z^\ell)\|^2$ for the scaled gradient w.r.t. $x$,
- $\alpha_{2h_{\max}} \|\nabla Z \psi_{\mu,0}(x^\ell, Z^\ell)\|^2$ for the scaled gradient w.r.t. $Z$,
- $\alpha_{2} \lambda_{\min}(\nabla^2_x \psi_{\mu,0}(x^\ell, Z^\ell))$ for the negative curvature.

All experiments are implemented in Python with JAX [8] on a computer with AMD Ryzen 7 2700X and Ubuntu 18.04.

7.1 Problem setting: shifted positive semidefinite factorization

We consider positive semidefinite factorization (PSF) problems [61], which are generalization of nonnegative matrix factorization (NMF) problems [36, 53]. As well as NMF problems, PSF problems are nonconvex. Let $V$ be an $m \times n$ nonnegative matrix and $q < \min \{m, n\}$. We solve the following PSF problem shifted by adding $r I_q$ to the constraints:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^m \sum_{j=1}^n (V_{ij} - \langle A_i, B_j \rangle)^2 \\
\text{subject to} & \quad A_i + r I_q \in S^q_+, \quad i = 1, \ldots, m, \\
& \quad B_j + r I_q \in S^q_+, \quad j = 1, \ldots, n,
\end{align*}
\]

(40)

where $r$ is a positive constant and $\{A_i\}, \{B_j\}$ are matrices to be determined. The origin, an interior point, is a strict saddle point of problem (40) unless $V = O$. 
To produce the matrix $V \in \mathbb{R}^{m \times n}$, we repeat the following procedures in order until the nonnegativity of $V$ is ensured.

1. Generate $A^*_i \in \mathcal{S}^q$ and $B^*_j \in \mathcal{S}^q$ by multiplying a matrix with elements randomly generated from uniform distribution over $[0, 1)$ and its transpose.
2. Randomly set 20% of eigenvalues of $A^*_i$ and $B^*_j$ as $-r$ so that the solution of problem (40) is on the boundary.
3. Generate $V$ by $V_{ij} := \langle A^*_i, B^*_j \rangle$ for each $i,j \in \{1, \ldots, m\} \times \{1, \ldots, n\}$.

7.2 Details of the algorithms

In order to examine the effects of directions of negative curvature, we also implement Decreasing-NC-PDIPM without negative curvature, namely, Procedure 3 in Fixed-NC-PDIPM. We set $L_0 = 1$ and also set an initial value of $\mu$ to $0.3$. With respect to scaling of gradient directions, we set $H_\ell = I_n$ and $H_\ell(\cdot)$ to be an identity mapping. The update rules for the parameters are as follows:

$$
\mu_{k+1} = \min(0.8\mu_k, 10\mu_k^{1.5}), \nu(\mu) = \mu^{0.1}, \varepsilon_\mu(\mu) = \mu,
$$

$$
\varepsilon_H(\mu) = \mu, \varepsilon_H(\mu) = \mu^{1.2}.
$$

Let $x \in \mathbb{R}^{(q+1)(n+m)}$ be the vector obtained by combining the vectorization of $A_i (i = 1, \ldots, n)$ and $B_j (j = 1, \ldots, m)$. We choose an initial point $x^1 \in \mathbb{R}^{(q+1)(n+m)}$ randomly from a uniform distribution over $[0, 10^{-6})$ in order to place $x^1$ in the neighborhood of the strict saddle point so that there will be the use of directions of negative curvature. From the above generation of $x^1$, $x^1$ is not guaranteed to be feasible. Therefore we generate $x^1$ until we find feasible $x^1$.

With regard to $Z$ in Algorithm 1, we selected $Z_1 = \mu_1 X_1^{-1}$ as an initial value, where $X : \mathbb{R}^{(q+1)(n+m)} \rightarrow \mathcal{S}^{(n+m)}$ is a function generated by combining constraints of problem (40) into a single positive semidefinite constraint, that is, $X(A_1, \ldots, A_m, B_1, \ldots, B_n) := (A_1 + rI_q) \oplus \cdots \oplus (A_m + rI_q) \oplus (B_1 + rI_q) \cdots \oplus (B_n + rI_q)$, where $\oplus$ denotes the direct sum of matrices.

7.3 Results

We set $(m, n, q, r) = (5, 5, 4, 0.3)$ in problem (40) and see the behavior of both algorithms in their first 300 iterations. Table 2 summarizes the results of the experiments. “N/A” in the second column means not applicable, because the algorithm of the corresponding row is not equipped with negative curvature.

All of the results indicate that the algorithm with negative curvature obtains lower function value. Figure 1 shows the history of the function value in the experiment with seed 3, in which we see that the directions of negative curvature depicted by $\times$ decreases function value drastically. Even though the
algorithm with negative curvature takes more time to reach 300 iterations, less function value is also confirmed if we compare the two algorithms with respect to function value vs. time.

Table 2: With negative curvature vs. without negative curvature.

| Seed | # negative curvature | Function value  | Time  |
|------|----------------------|-----------------|-------|
| 1    | N/A                  | 1.40 × 10⁻³     | 1.90 s|
| 2    | N/A                  | 6.43 × 10⁻⁴     | 1.80 s|
| 3    | N/A                  | 5.95 × 10⁻³     | 3.80 s|
| 4    | N/A                  | 6.75 × 10⁻⁴     | 1.94 s|
| 5    | N/A                  | 1.69 × 10⁻¹     | 1.80 s|
| 6    | N/A                  | 1.29 × 10⁻¹     | 1.92 s|

Fig. 1: Function value vs. iterations (seed 3).

8 Discussion and conclusion

We have proposed a primal-dual IPM with convergence to SOSP of NSDPs using directions of negative curvature. Utilizing "local" Lipschitz continuities of derivatives of \( \psi_{\mu,\nu} \) makes it possible to establish a worst-case iteration complexity of our primal-dual IPM for computing approx. SOSP of NSDPs. Difficulty of our analysis comes from the sigma-term that appears in the second-order necessary conditions (13) for NSDPs. Our numerical experiments show that the use of negative curvature is beneficial in nonconvex settings.
We believe that our primal-dual IPM can be extended to the equality constrained NSDPs and it is possible to use trust-region methods in place of negative curvature methods in inner iterations.

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A Proofs of Lemma 5.2

A.1 Proof of Lemma 5.2 (1)

Proof Notice that
\[\|X^{-1} - X^{-1}\|_F = \|X^{-1}(X - X)\|_F \leq \|X^{-1}\|_F \|X - X\|_F.\]

The last equation in the proof of Proposition 5.1 (1) along with (8) yields
\[\|X^{-1} - X^{-1}\|_F \leq 2L_0 \|X^{-1}\|_F \|x - x\|_F.\]

The proof is complete. □

A.2 Proof of Lemma 5.2 (2)

Proof Using (16), we obtain
\[
\|\nabla x_{\nu,\mu}(x', Z_\ell) - \nabla x_{\nu,\mu}(x, Z_\ell)\| = \underbrace{\|\nabla f(x') - \nabla f(x)\|}_{(A)} + \nu\|A^*(x')Z_\ell - A^*(x)Z_\ell\| + (1 + \nu)\mu\|A^*(x')X^{-1} - A^*(x)X^{-1}\|.\]

In what follows, we evaluate upper bounds of (A), (B), and (C).

Evaluation of (A) : From (3), we have
\[(A) \leq L_1 \|x' - x\|.\] (A-1)

Evaluation of (B) :
\[
(B) \leq \nu \sum_{i=1}^n \|\text{tr}\left(Z_\ell \left(A_i(x') - A_i(x)\right)\right)\|
\leq \nu \sum_{i=1}^n \|Z_\ell\|_F \|A_i(x') - A_i(x)\|_F \leq \nu L_1 \|Z_\ell\|_F \|x' - x\|,\] (A-2)

where the third inequality follows from the (9).

Evaluation of (C) : Let \((D) := (1 + \nu)\mu\|A^*(x') - A^*(x)\|X^{-1}\|X^{-1}\|_F\) and \((E) := (1 + \nu)\mu\|A^*(x')X^{-1} - A^*(x)X^{-1}\|.\) As \((C) \leq (D) + (E),\) we will evaluate upper bounds of \((D)\) and \((E).\) With regard to \((D),\) we obtain from (9) that
\[
(D) \leq (1 + \nu)\mu\|X^{-1}\|_F \sum_{i=1}^n \|A^*(x') - A^*(x)\|_F
\leq (1 + \nu)\mu L_1 \|X^{-1}\|_F \|x' - x\|.\]
Similarly, it follows from (5), that
\[
(E) \leq (1 + \nu)\mu \sum_{i=1}^{n} \left| \text{tr} \left( A_i \left( x^\ell \right) \left( X_{\ell}^{-1} - X^{-1} \right) \right) \right|
\]
\[
\leq (1 + \nu)\mu \sum_{i=1}^{n} \| A_i(x) \|_F \left \| X_{\ell}^{-1} - X^{-1} \right \|_F
\]
\[
\leq 2(1 + \nu)\mu L_0 \sum_{i=1}^{n} \| A_i(x) \|_F \left \| X_{\ell}^{-1} \right \|^2 \| x^\ell - x \| \quad \text{(by Lemma 5.2 (1))}
\]
\[
\leq 2(1 + \nu)\mu L_0^2 \left \| X_{\ell}^{-1} \right \|^2 \| x^\ell - x \|. \quad \text{(by (6))}
\]
Therefore
\[
(C) \leq \left( (1 + \nu)\mu L_1 \left \| X_{\ell}^{-1} \right \|_F + 2(1 + \nu)\mu L_0^2 \left \| X_{\ell}^{-1} \right \|^2 \| x^\ell - x \| \right)
\]
\[
\text{(A-3)}
\]
Lastly, from (A-1)-(A-3), we obtain
\[
\left \| \nabla_{x} \psi_{\mu,\nu}(x^\ell, Z_\ell) - \nabla_{x} \psi_{\mu,\nu}(x, Z_\ell) \right \|
\]
\[
\leq (L_1 + \nu L_1) \| Z_\ell \|_F + 2(1 + \nu)\mu L_0^2 \left \| X_{\ell}^{-1} \right \|^2 + (1 + \nu)\mu L_1 \left \| X_{\ell}^{-1} \right \|_F \| x^\ell - x \|
\]
The proof is complete. \(\square\)

A.3 Proof of Lemma 5.2 (3)

\textbf{Proof} Note that
\[
\left \| \nabla_{Z} \psi_{\mu,\nu}(x^\ell, Z_\ell) - \nabla_{Z} \psi_{\mu,\nu}(x^\ell, Z) \right \|_F \leq \mu \nu \left \| Z^{-1} - Z_\ell^{-1} \right \|_F
\]
\[
\leq \mu \nu \left \| Z^{-1}\|_F \| Z_\ell - Z \| F^{-1} \| \right \|_F.
\]
Since \( \| Z^{-1} \|_F \leq 2 \| Z^{-1} \|_F \), from Proposition 5.1 (2), we have
\[
\left \| \nabla_{Z} \psi_{\mu,\nu}(x^\ell, Z_\ell) - \nabla_{Z} \psi_{\mu,\nu}(x^\ell, Z) \right \|_F \leq 2 \mu \nu \left \| Z^{-1} \right \|_F \| Z_\ell - Z \|_F.
\]
The proof is complete. \(\square\)

A.4 Proof of Lemma 5.2 (4)

\textbf{Proof} We have \((\nabla_{x}^2 \psi_{\mu,\nu}(x^\ell, Z_\ell))_{ij} - (\nabla_{x}^2 \psi_{\mu,\nu}(x, Z_\ell))_{ij} \equiv (A_{ij}) + (B_{ij}) + (C_{ij})\), where
\[
(A_{ij}) = \left( \nabla^2 f(x^\ell) \right)_{ij} - \left( \nabla^2 f(x) \right)_{ij},
\]
\[
(B_{ij}) = (1 + \nu)\mu \left( \text{tr} \left( A_i(x^\ell)X_{\ell}^{-1}A_j(x^\ell)X_{\ell}^{-1} \right) - \text{tr} \left( A_i(x)X^{-1}A_j(x)X^{-1} \right) \right),
\]
\[
(C_{ij}) = \text{tr} \left( (1 + \nu)\mu X_{\ell}^{-1} - \nu Z_\ell \right) \frac{\partial^2 X}{\partial x_i \partial x_j} (x)
\]
\[
- \text{tr} \left( (1 + \nu)\mu X_{\ell}^{-1} - \nu Z_\ell \right) \frac{\partial^2 X}{\partial x_i \partial x_j} (x^\ell).\]
In what follows, we evaluate \((A_{ij}), (B_{ij}),\) and \((C_{ij}).\)

**Evaluation of \((A_{ij}).\)** From (4), we have

\[
\|\nabla^2 f(x') - \nabla^2 f(x)\| \leq L_2\|x' - x\|. \tag{A-4}
\]

**Evaluation of \((B_{ij}).\)** \((B_{ij})\) can be written as

\[
(B_{ij}) = (1 + \nu)\mu \left( \text{tr} \left( A_i(x)X^{-1}A_j(x)X^{-1} - \text{tr} \left( A_i(x)X^{-1}A_j(x)X^{-1} \right) \right) \right)
\]

\[
+ (1 + \nu)\mu \left( \text{tr} \left( A_i(x)X^{-1}A_j(x)X^{-1} - \text{tr} \left( A_i(x)X^{-1}A_j(x)X^{-1} \right) \right) \right).
\]

Here, \((D_{ij})\) can be bounded as

\[
|\langle D_{ij} \rangle| \leq (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}
\]

\[
\leq (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}
\]

\[
+ (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}.
\]

In the above, \((F_{ij})\) can be bounded as

\[
|\langle F_{ij} \rangle| \leq (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}
\]

\[
+ (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}
\]

\[
\leq (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}
\]

\[
+ (1 + \nu)\mu \|A_i(x)X^{-1}A_j(x)X^{-1} - A_i(x)X^{-1}A_j(x)\|_{F}.
\]

Taking the sum of \((F_{ij})\) over \(i, j\) together with (5) and (9) yields

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle F_{ij} \rangle| \leq 2(1 + \nu)\mu LoL_1 \|X^{-1}\|_F \|x' - x\|. \tag{A-5}
\]

For \((G_{ij}),\) we obtain, from Lemma 5.2 (1),

\[
|\langle G_{ij} \rangle| \leq (1 + \nu)\mu \|X^{-1}\|_F \|A_i(x)\|_F \|A_j(x)\|_F \|X^{-1} - X^{-1}\|_F
\]

\[
\leq 2(1 + \nu)\mu Lo \|X^{-1}\|_F^3 \|A_i(x)\|_F \|A_j(x)\|_F \|x' - x\|.
\]

which together with (5) yields

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle G_{ij} \rangle| \leq 2(1 + \nu)\mu Lo \|X^{-1}\|_F^3 \|x' - x\|. \tag{A-6}
\]

By combining inequalities (A-5) and (A-6), the sum of \((D_{ij})\) is bounded from above by

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle D_{ij} \rangle| \leq 2 \left( (1 + \nu)\mu Lo \|X^{-1}\|_F^3 + (1 + \nu)\mu LoL_1 \|X^{-1}\|_F \|x' - x\| \right). \tag{A-7}
\]
With regard to \((E_{ij})\), we have
\[
\| (E_{ij}) \| \leq (1 + \nu) \mu \| A_i(x) \|_F \| A_j(x) \|_F \| X^{-1} \|_F \| X^{-1} - X \|_F.
\]
By taking the sum over \(i, j\) of \(\| (E_{ij}) \|\), we have
\[
\sum_{i=1}^n \sum_{j=1}^n \| (E_{ij}) \| \leq 4(1 + \nu) \mu L_3 \| X^{-1} \|_F^3 \| x^f - x \|.
\]  \(\text{(A-8)}\)

Combining inequalities \((A-7)\) and \((A-8)\) yields
\[
\sum_{i=1}^n \sum_{j=1}^n \| (B_{ij}) \| \leq \left( 6(1 + \nu) \mu L_3 \| X^{-1} \|_F^3 + 2(1 + \nu) \mu L_0 L_3 \| X^{-1} \|_F^3 \right) \| x^f - x \|.
\]  \(\text{(A-9)}\)

**Evaluation of \((C_{ij})\):** Note that \((C_{ij})\) can be written as
\[
(C_{ij}) = \nu \left( \text{tr} \left( Z_i \left( \frac{\partial^2 X}{\partial x_i \partial x_j} (x) - \frac{\partial^2 X}{\partial x_i \partial x_j} (x') \right) \right) 
+ (1 + \nu) \mu \text{tr} \left( X^{-1} \left( \frac{\partial^2 X}{\partial x_i \partial x_j} (x) - \frac{\partial^2 X}{\partial x_i \partial x_j} (x') \right) \right) \right).
\]  \(\text{(K}_{ij}\text{)}\)

By the Cauchy-Schwarz inequality, we have
\[
\sum_{i=1}^n \sum_{j=1}^n \| (K_{ij}) \| \leq \nu \sum_{i=1}^n \sum_{j=1}^n \| Z_i \|_F \left\| \frac{\partial^2 X}{\partial x_i \partial x_j} (x) - \frac{\partial^2 X}{\partial x_i \partial x_j} (x') \right\|_F \| x^f - x \|. \quad \text{(A-10)}
\]

With regard to the sum of \(\| (K_{ij}) \|\),
\[
\sum_{i=1}^n \sum_{j=1}^n \| (K_{ij}) \| \leq \nu \sum_{i=1}^n \sum_{j=1}^n \| Z_i \|_F \left\| \frac{\partial^2 X}{\partial x_i \partial x_j} (x) - \frac{\partial^2 X}{\partial x_i \partial x_j} (x') \right\|_F \| x^f - x \| + (1 + \nu) \mu L_2 \| X^{-1} \|_F \| x^f - x \|. \quad \text{(A-11)}
\]
where the inequality follows from \((6)\), \((7)\), and Lemma 5.2 \((1)\). Moreover, combining \((A-10)\) and \((A-11)\) yields
\[
\sum_{i=1}^n \sum_{j=1}^n \| (C_{ij}) \| \leq \nu L_2 \| Z_i \|_F + 2(1 + \nu) \mu L_0 L_1 \| X^{-1} \|_F + (1 + \nu) \mu L_0 \| X^{-1} \|_F \| x^f - x \|. \quad \text{(A-12)}
\]
Lastly, from (A-4), (A-9), and (A-12), we have

\[
\left\| \nabla_x^2 \psi_{\mu,\nu}(x', z_\ell) - \nabla_x^2 \psi_{\mu,\nu}(x, z_\ell) \right\|
\leq \left( L_2 + \nu L_2 \|Z_\ell\|_F 
\right)
\left( + (1 + \nu)\mu \left( L_2 \|X^{-1}_\ell\|_F + 4L_1L_0 \|X^{-1}_\ell\|_F^2 + 6L_0^3 \|X^{-1}_\ell\|_F^3 \right) \right)
\cdot \|x' - x\|
\]

The proof is complete. \qed