NEGATIVE EIGENVALUES OF THE RICCI OPERATOR OF SOLVABLE METRIC LIE ALGEBRAS

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ABSTRACT. In this paper we get a necessary and sufficient condition for the Ricci operator of a solvable metric Lie algebra to have at least two negative eigenvalues. In particular, this condition implies that the Ricci operator of every non-unimodular solvable metric Lie algebra or every non-abelian nilpotent metric Lie algebra has this property.

Key word and phrases: left-invariant Riemannian metrics, Lie groups, metric Lie algebras, Ricci operator, eigenvalues of the Ricci operator, Ricci curvature.

2010 Mathematical Subject Classification: 53C30, 17B30.

INTRODUCTION AND THE MAIN RESULTS

Various restrictions on the curvature of a Riemannian manifold allow to obtain some interesting information on its geometric and topological structures. One of the important characteristics of the curvature is the Ricci curvature, that is confirmed by numerous researches of mathematicians and physicists [4]. On the other hand, it should be noted that there are many unsolved problems connected with the Ricci curvature, even in the case of homogeneous Riemannian manifolds (see e.g. the survey [21] for a more detailed information on this subject).

One of this problem is the following: to classify all possible signatures of the Ricci operators of invariant Riemannian metrics on a given homogeneous space. This problem seems to be very hard in general. Now, it is solved only for some very special cases. There are some important results in this direction [3, 4, 8]. For instance, this problem is completely solved for all homogeneous spaces of dimension \( \leq 4 \) (see [14, 15] and references therein). In particular, J. Milnor classifies in [17] all possible signatures of the Ricci operators of left-invariant Riemannian metrics on all Lie groups of dimension \( \leq 3 \), the same result for Lie groups of dimension 4 was obtained by A.G. Kremlyov and Yu.G. Nikonov in [14, 15] (some results in this direction are obtained also in the paper [7] of D. Chen).

For other dimensions we have only partial results. It is necessary to mention the paper [8] of I. Dotti-Miatello, where Ricci signatures of left-invariant Riemannian metrics on two-step solvable unimodular Lie groups are determined, and the paper [13] of A.G. Kremlyov, where the same problem is solved for nilpotent five-dimensional Lie groups.

The project was supported in part by the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (grant NSh-921.2012.1) and by Federal Target Grant “Scientific and educational personnel of innovative Russia” for 2009-2013 (agreement no. 8206, application no. 2012-1.1-12-000-1003-014).
In this paper we restrict our attention to solvable Lie groups with left-invariant Riemannian metrics. It is shown in the papers \cite{11,17} that the scalar curvature of every non-flat left-invariant Riemannian metric on a given solvable Lie group is negative, therefore, the Ricci operator of this metric has at least one negative eigenvalue. The main problem of this paper is to determine, whether the Ricci operator of a given left-invariant metric has at least two negative eigenvalues.

It is convenient to study left-invariant Riemannian metrics on Lie groups in terms of metric Lie algebras (i.e. Lie algebras supplied with inner products) \cite{2,4,21}. Indeed, let $G$ be a Lie group with the Lie algebra $\mathfrak{g}$. Then every inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ uniquely determines a left-invariant Riemannian metric $\rho$ on $G$, and vice versa (see e.g. 7.24 in \cite{4}). As usual, we denote by $[\mathfrak{s},\mathfrak{s}]$ the derived algebra of a Lie algebra $\mathfrak{s}$. For every solvable Lie algebra $\mathfrak{s}$, $[\mathfrak{s},\mathfrak{s}]$ is a nilpotent ideal of $\mathfrak{s}$ and $[\mathfrak{s},\mathfrak{s}] \neq \mathfrak{s}$.

Recall, that an operator $A$ (acting on a given Euclidean space) is called normal, if it commutes with its adjoint $A'$. The main result of this paper is the following

**Theorem 1.** Let $(\mathfrak{s}, Q)$ be a solvable metric Lie algebra, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, $\mathfrak{a}$ be a $Q$-orthogonal complement to $\mathfrak{n}$ in $\mathfrak{s}$. Then one of the following mutually exclusive assertions holds:

1. The ideal $\mathfrak{n}$ is commutative, $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{s}$, and for every $X \in \mathfrak{a}$ the operator $\text{ad}(X)|_{\mathfrak{n}}$ is skew-symmetric with respect to $Q$ (in this case the Ricci operator of $(\mathfrak{s}, Q)$ is zero);

2. The ideal $\mathfrak{n}$ is commutative, $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{s}$, and for every $X \in \mathfrak{a}$ the operator $\text{ad}(X)|_{\mathfrak{n}}$ is trace-free and normal with respect to $Q$, but the subspace

$$\mathfrak{b} = \{X \in \mathfrak{a} \mid \text{ad}(X)|_{\mathfrak{n}} \text{ is skew-symmetric with respect to } Q\}$$

has codimension 1 in $\mathfrak{a}$ (in this case the Ricci operator of $(\mathfrak{s}, Q)$ has only one negative eigenvalue, while all other eigenvalues are zero);

3. The Ricci operator of the metric Lie algebra $(\mathfrak{s}, Q)$ has at least two negative eigenvalues.

**Remark 1.** The structure of the Ricci operator in the items (1) and (2) of the above theorem easily follows from the formula (4). Metric Lie algebras with zero Ricci curvature are flat (i.e. have zero sectional curvature) by the well known result of D.V. Alekseevskii and B.N. Kimmel’fel’d \cite{3}. The case of flat metric Lie algebras has been studied in Theorem 1.5 of the paper \cite{17} by J. Milnor.

The authors of the paper \cite{15} proved (in particular) that the Ricci operator of every non-unimodular solvable metric Lie algebra of dimension $\leq 4$ has at least two negative eigenvalues. Moreover, it has been conjectured in \cite{15}, that the Ricci operator of an arbitrary non-unimodular solvable metric Lie algebra have the same property. This conjecture was confirmed for all non-unimodular solvable metric Lie algebras of dimension $\leq 6$ in \cite{6}, for all completely solvable Lie algebras in \cite{20}, and for all Lie algebras with six-dimensional two-step nilpotent derived algebras in \cite{1}.

Obviously, the cases (1) and (2) of Theorem 1 are impossible for non-unimodular Lie algebras. Hence, Theorem 1 implies immediately a confirmation of the above-mentioned conjecture:
Theorem 2. Let $\mathfrak{s}$ be a non-unimodular solvable Lie algebra. Then for every inner product $Q$ on $\mathfrak{s}$, the Ricci operator of the metric Lie algebra $(\mathfrak{s}, Q)$ has at least two negative eigenvalues.

Using Theorem 1, we can get the following result (see Remark 5 in the last section).

Theorem 3 ([20]). Let $\mathfrak{s}$ be a non-commutative nilpotent Lie algebra. Then for every inner product $Q$ on $\mathfrak{s}$ the Ricci operator of the metric Lie algebra $(\mathfrak{s}, Q)$ has at least two negative eigenvalues.

Note that some partial cases of this theorem were obtained earlier in the paper [13].

From Theorem 1 we easily get also the following two corollaries.

Corollary 1. If the Ricci operator of a solvable metric Lie algebra $(\mathfrak{s}, Q)$ has at least one positive eigenvalue, then it has at least two negative eigenvalues.

Corollary 2. Let $(\mathfrak{s}, Q)$ be a solvable metric Lie algebra such that a $Q$-orthogonal complement $\mathfrak{a}$ to $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ is not a commutative subalgebra of $\mathfrak{s}$, then the Ricci operator of $(\mathfrak{s}, Q)$ has at least two negative eigenvalues.

We hope that results of this paper will be useful for future research on solvable metric Lie algebras (therefore, on solvable Lie groups with left-invariant Riemannian metrics), in particular, for the study of the Ricci flow on solvable Lie groups (see [9, 16, 22]).

The structure of this paper is the following. In the first section we recall some notations and useful facts and prove also some auxiliary results. The second section is devoted to some convenient formulas for the Ricci operator of solvable metric Lie algebras. In this section, we recall also some important results related to the Ricci curvature. The third section of the paper is devoted to the proof of Theorem 1 for one special (but very involved and important) case. Finally, in the last section we prove Theorem 1 in full generality. It should be noted, that we have used Theorem 3 for this goal, but (for a formal reason) we have proved completely this theorem before its using. Hence, our presentation does not depend on the paper [20].

The author is indebted to Prof. V.N. Berestovskii and Prof. Yu.A. Nikolayevsky for helpful discussions concerning this paper.

1. Notations and auxiliary results

Standard notations and classical results on Lie algebras could be find in [5, 10, 24]. Let $\mathfrak{n}$ be a nilpotent Lie algebra of degree $p$. Consider its lower central series $\{\mathfrak{n}^k\}$, where

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = [\mathfrak{n}, \mathfrak{n}], \quad \ldots, \quad \mathfrak{n}^k = [\mathfrak{n}^{k-1}, \mathfrak{n}] \quad (k \geq 1), \quad \ldots$$

Then $\mathfrak{n}^p = 0$ and $\mathfrak{n}^{p-1} \neq 0$.

Let $\text{Der}(\mathfrak{n})$ and $\text{InnDer}(\mathfrak{n})$ be a space of derivations and a space of inner derivations of the Lie algebra $\mathfrak{n}$ respectively. It is clear that $\text{InnDer}(\mathfrak{n}) \subset \text{Der}(\mathfrak{n})$ and $\text{InnDer}(\mathfrak{n}) \neq \text{Der}(\mathfrak{n})$ since $\mathfrak{n}$ is nilpotent [10].

Lemma 1. For any $A \in \text{Der}(\mathfrak{n})$ we have $A(\mathfrak{n}^k) \subset \mathfrak{n}^k$ for every $k \geq 0$. 
Proof. We prove the lemma by induction. For $k = 0$ we get $A(n^0) = A(n) = n = n^0$. If the lemma is true for all values of $k \leq l$ then (by properties of derivations) we have $A(n^{l+1}) = A([n^l, n]) \subset [A(n^l), n] + [n^l, A(n)] \subset [n^l, n] + [n^l, n] \subset n^{l+1}$. This proves the lemma.

Lemma 2. For any $A \in \text{InnDer}(n)$ we have $A(n^k) \subset n^{k+1}$ for every $k \geq 0$.

Proof. Since $A \in \text{InnDer}(n)$, then there is $X \in n$ such that $A = \text{ad}(X)$. For every $k$ we get $A(n^k) = [X, n^k] \subset [n^k, n] = n^{k+1}$. This proves the lemma.

Lemma 3. Let $n$ be a nilpotent Lie algebra, $b \subset n$ is an abelian subalgebra of codimension 1. Then $b$ is an abelian ideal in $n$.

Proof. One can find the proof of this lemma e.g. in [23], but we give it here for the convenience of the reader. Fix some $Y \in b$. We should prove that $[Y, X] \in b$ for every $X \in n$. If $X \in b$, then $[Y, X] = 0$. Now we suppose that $X \notin b$. Obviously, there are $\alpha \in \mathbb{R}$ and $Z \in b$ such that $\text{ad}(Y)(X) = [Y, X] = \alpha X + Z$. Then we have $\text{ad}^2(Y)(X) = [Y, \alpha X + Z] = \alpha^2 X + \alpha Z, \ldots$, $\text{ad}^k(Y)(X) = \alpha^k X + \alpha^{k-1} Z$, $k \geq 1$. Since $n$ is nilpotent, then $\text{ad}^k(Y)(X) = 0$ for sufficiently large $k$. Therefore, $\alpha = 0$ and $[Y, X] = Z \in b$.

Now, we fix an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $n$ and consider some important properties of the metric Lie algebra $(n, \langle \cdot, \cdot \rangle)$. Let $V_k$ be a $\langle \cdot, \cdot \rangle$-orthogonal complement to $n^k$ in $n^{k-1}$, $k = 1, 2, \ldots, p$. It is clear that $V_p = n^{p-1}$, since $n^p = 0$.

We consider the space $\text{End}(n)$ of linear endomorphisms of the Lie algebra $n$ and its three subspaces $L_1, L_2$ and $L_3$, defined as follows:

$L_1 = \{A \in \text{End}(n) | A(n^k) \subset n^k \text{ for } k = 0, 1, \ldots, p-1\}$,
$L_2 = \{A \in \text{End}(n) | A(V_k) \subset V_k \text{ for } k = 1, \ldots, p\}$,
$L_3 = \{A \in \text{End}(n) | A(n^k) \subset n^{k+1} \text{ for } k = 0, 1, \ldots, p-1\}$.

Obviously, Lemmas 1 and 2 imply the following

Corollary 3. For every nilpotent Lie algebra $n$ the inclusions $\text{Der}(n) \subset L_1$ and $\text{InnDer}(n) \subset L_3$ hold.

In what follows, we denote by $C'$ the adjoint operator to the operator $C \in \text{End}(n)$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on $n$, i.e. $(C(X), Y) = (X, C'(Y))$ for every $X, Y \in n$. If we represent operators from $\text{End}(n)$ by matrices in some $\langle \cdot, \cdot \rangle$-orthonormal basis for $n$, then $C'$ is the transpose of the matrix $C$. We will use also the notation $C$s for a symmetric part of the operator $C$ with respect to $\langle \cdot, \cdot \rangle$, i.e. $C_s = \frac{1}{2}(C + C')$.

Lemma 4. If $A \in \text{Der}(n)$ is such that $A' \in \text{Der}(n)$, then $A, A' \in L_2$.

Proof. Consider any $X \in n^k$ and any $Y \in V_k$, $k = 1, \ldots, p$. By Corollary 3 we get $A, A' \in L_1$. Therefore, $A(n^k) \subset n^k, A'(n^k) \subset n^k, A(n^{k-1}) \subset n^{k-1}$ and $A'(n^{k-1}) \subset n^{k-1}$. Since $V_k$ is the orthogonal complement to $n^k$ in $n^{k-1}$, then we have

$(X, A(Y)) = (A'(X), Y) = 0, \quad (X, A'(Y)) = (A(X), Y) = 0$.

This implies $A(V_k) \subset V_k$ and $A'(V_k) \subset V_k$ for all $k = 1, \ldots, p$, i.e. $A, A' \in L_2$. ■
We supply the linear space End\((\mathfrak{n})\) with an inner product \(\langle \cdot, \cdot \rangle\) as follows: \(\langle A, B \rangle = \text{trace}(AB')\), where \(B'\) the adjoint operator to the operator \(B \in \text{End}(\mathfrak{n})\) with respect to \((\cdot, \cdot)\). Using a matrix representation in any \((\cdot, \cdot)\)-orthonormal basis of \(\mathfrak{n}\), we easily get \(\langle A, B \rangle = \text{trace}(AB') = \text{trace}(B'A) = \text{trace}(A'B) = \text{trace}(BA')\). Note also that \(\langle A', B' \rangle = \text{trace}(A'B) = \text{trace}(BA') = \langle A, B \rangle\) for every \(A, B \in \text{End}(\mathfrak{n})\).

**Lemma 5.** If \(A \in L_2\) and \(B \in L_3\), then \(\langle A, B \rangle = 0\).

**Proof.** Since \((A(V_k), V_i) = 0\) is equivalent to \((V_k, A'(V_i)) = 0\), then \(A' \in L_2\). Further, since \(BA'(V_k) \subset B(V_k) \subset B(n^{k-1}) \subset n^k\) and \(BA'(n^{k-1}) = BA'(\bigoplus_{i \geq k} V_i) \subset n^k\), then we get \(BA' \in L_3\). But every operator \(C \in L_3\) is trace-free (it is easy to check by using a basis \(\{e_j\}\) in \(\mathfrak{n}\), such that every \(e_j\) lies in some \(V_i\)). Therefore, \(\langle A, B \rangle = \text{trace}(BA') = 0\). □

**Lemma 6.** For any nilpotent matrix \(L = (l_{ij})\) with real entries the equality

\[
2 \text{trace}(L^s \cdot L^s) = \text{trace}(L \cdot L')
\]

holds, where \(L'\) is the transpose of the matrix \(L\), \(L^s = \frac{1}{2}(L + L')\) is a symmetric part of \(L\).

**Proof.** For any orthogonal matrix \(Q\), the above equality does not change when we replace \(L\) with \(QLQ^{-1}\). Hence, it suffices to consider the case when \(L\) is upper triangular with zeros on the main diagonal. Then we get

\[
4 \text{trace}(L^s \cdot L^s) = \sum_{i,j} (l_{ij} + l_{ji})^2 = 2 \sum_{i,j} l_{ij}^2 + 2 \sum_{i,j} l_{ij}l_{ji} = 2 \sum_{i,j} l_{ij}^2 = 2 \text{trace}(L \cdot L').
\]

This proves the lemma. □

Obviously, \(\text{InnDer}(\mathfrak{n}) \subset \text{Der}(\mathfrak{n}) \subset \text{End}(\mathfrak{n})\). We need the \((\cdot, \cdot)\)-orthogonal projection

\[
P_{\text{inner}} : \text{Der}(\mathfrak{n}) \to \text{InnDer}(\mathfrak{n}),
\]

i. e. \(P_{\text{inner}}(A) \in \text{InnDer}(\mathfrak{n})\) and \(\langle A - P_{\text{inner}}(A), B \rangle = 0\) for every \(A \in \text{Der}(\mathfrak{n})\) and \(B \in \text{InnDer}(\mathfrak{n})\).

**Lemma 7.** Consider any \(A \in \text{Der}(\mathfrak{n})\) and put \(\tilde{A} := P_{\text{inner}}(A)\). Then the equalities

\[
\text{trace}(\tilde{A}A) = 0 \quad \text{and} \quad \text{trace}(\tilde{A}\tilde{A}) = 0
\]

hold, where \(\tilde{A} = A - \tilde{A}\).

**Proof.** Let us consider any \(A \in \text{Der}(\mathfrak{n})\). We define a Lie multiplication \([\cdot, \cdot]_1\) on the direct sum of linear spaces \(\mathfrak{s} := \mathfrak{n} \oplus \mathbb{R}\) as follows:

\[
[(a_1, b_1), (a_2, b_2)]_1 = [(a_1, a_2) + b_1A(a_2) - b_2A(a_1), 0].
\]

Obviously, \((\mathfrak{s}, [\cdot, \cdot]_1)\) is a solvable Lie algebra, and \(\mathfrak{n}\) (we identify each element \(X \in \mathfrak{n}\) with \((X, 0) \in \mathfrak{s}\)) lies in the nilradical \(N(\mathfrak{s})\) of \(\mathfrak{s}\). Consider \(Y \in \mathfrak{n}\) such that \(\text{ad}(Y) = \tilde{A}\). Then \(\text{ad}((Y, 0))|_\mathfrak{n} = \tilde{A}\) and \(\text{ad}((0, 1))|_\mathfrak{n} = A\). We know that the Killing form \(B_\mathfrak{s}\) of any solvable Lie algebra \(\mathfrak{s}\) satisfies the equation \(B_\mathfrak{s}(\mathfrak{s}, N(\mathfrak{s})) = 0\) (see e. g. Remark after Proposition 6 of I.5.5 in [5]). Since \((Y, 0) \in N(\mathfrak{s})\), we get

\[
0 = B_\mathfrak{s}((Y, 0), (Y, 0)) = \text{trace}(\text{ad}((Y, 0)) \cdot \text{ad}((Y, 0))) = \text{trace}(\text{ad}((Y, 0))|_\mathfrak{n} \cdot \text{ad}((Y, 0))|_\mathfrak{n}) = \text{trace}(\tilde{A}\tilde{A})
\]
and
\[ 0 = B_\delta((0, 1), (Y, 0)) = \text{trace}\left(\text{ad}( (0, 1)) \cdot \text{ad}( (Y, 0)) \right) = \text{trace}\left(\text{ad}( (0, 1))|_n \cdot \text{ad}( (Y, 0))|_n\right) = \text{trace}(A\tilde{A}).\]

As a simple corollary, we get also \(\text{trace}(\tilde{A}\tilde{A}) = \text{trace}(\tilde{A}\tilde{A}) - \text{trace}(\tilde{A}\tilde{A}) = 0.\) □

**Proposition 1.** For every \(A \in \text{Der}(n)\) the inequality
\[ \langle A^s, A^s \rangle \geq \frac{1}{2} \langle \tilde{A}, \tilde{A} \rangle \]
holds, where \(\tilde{A} = P_{\text{inner}}(A),\) \(A^s = \frac{1}{2}(A + A').\) Moreover, \(\langle A^s, A^s \rangle = \frac{1}{2}\langle \tilde{A}, \tilde{A} \rangle\) if and only if \(A - \tilde{A}\) is a skew-symmetric derivation of \(n.\)

**Proof.** Put \(\hat{A} := A - \tilde{A}.\) Then \(A = \tilde{A} + \hat{A},\) \(2A^s = \tilde{A} + \hat{A}' + 2\hat{A}^s\) and
\[ 4\langle A^s, A^s \rangle = 2\langle \hat{A}, \tilde{A} \rangle + 4\langle \hat{A}, \hat{A}^s \rangle + 4\langle \hat{A}', \hat{A}^s \rangle + 4\langle \hat{A}', \hat{A}^s \rangle, \]
since \(\langle \hat{A}', \hat{A}' \rangle = \text{trace}(\hat{A}'\hat{A}) = \text{trace}(\tilde{A}\hat{A}') = \langle \tilde{A}, \hat{A} \rangle\) and \(\langle \hat{A}, \hat{A}' \rangle = \text{trace}(\hat{A}\tilde{A}) = 0\) by Lemma 7. Further, since \(\langle \hat{A}', \tilde{A} \rangle = \langle \hat{A}, \hat{A}' \rangle = \text{trace}(\hat{A}\tilde{A}) = 0\) (by Lemma 7) and \(\langle \hat{A}', \hat{A}' \rangle = \langle \tilde{A}, \hat{A} \rangle = 0\) (by definitions of \(\tilde{A}\) and \(\hat{A}\)), then \(\langle \hat{A}, \hat{A}' \rangle = \langle \hat{A}', \hat{A}' \rangle = 0.\) Therefore,
\[ \langle A^s, A^s \rangle = \frac{1}{2}\langle \tilde{A}, \tilde{A} \rangle + \langle \hat{A}^s, \hat{A}^s \rangle \geq \frac{1}{2}\langle \tilde{A}, \tilde{A} \rangle.\]
Clear, that \(\langle \hat{A}^s, \hat{A}^s \rangle = 0\) if and only if the derivation \(\hat{A}\) is skew-symmetric. □

We will need also one well known result on localization of the eigenvalues of a symmetric matrix and one its obvious corollary.

**Proposition 2** (cf. Theorem 4.3.8 in [12]). Let \(\tilde{A}\) be a symmetric \((n \times n)\)-matrix with real entries, \(A\) be a matrix obtained from \(\tilde{A}\) by deleting the last row and the last column. Assume that the eigenvalues \(\lambda_i\) of \(A\) and the eigenvalues \(\tilde{\lambda}_i\) of \(\tilde{A}\) have been arranged in increasing order \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-2} \leq \lambda_{n-1}\) and \(\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_{n-2} \leq \tilde{\lambda}_{n-1} \leq \lambda_n.\) Then the inequality
\[ \tilde{\lambda}_1 \leq \lambda_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_{n-1} \leq \lambda_{n-1} \leq \lambda_n \]
holds.

**Corollary 4.** Let \(A\) be a symmetric \((n \times n)\)-matrix with real entries, \(B\) be a \((m \times m)\)-matrix obtained from \(A\) by deleting of \((n - m)\) rows and \((n - m)\) columns with coincided sets of indexes. If the matrix \(B\) is positive (negative) definite, then the matrix \(A\) has at least \(m\) positive (respectively, negative) eigenvalues.

### 2. The Ricci Operator

Consider a Lie algebra \(\mathfrak{g}\) supplied with an inner product \((\cdot, \cdot)\). We choose some \((\cdot, \cdot)\)-orthonormal basis \(\{X_i\}, 1 \leq i \leq \dim(\mathfrak{g})\), in \(\mathfrak{g}\). Define a vector \(H \in \mathfrak{g}\) by the equality \((H, X) = \text{trace}(\text{ad}(X))\), where \(\text{ad}(X)(Y) = [X, Y], X, Y \in \mathfrak{g}\). Note that
$H = 0$ if and only if the Lie algebra $g$ is unimodular. For the Ricci operator of the metric Lie algebra $(g, (\cdot, \cdot))$ we have the following formula:

$$
\text{Ric} = -\frac{1}{2} \sum_i \text{ad}'(X_i)\text{ad}(X_i) + \frac{1}{4} \sum_i \text{ad}(X_i)\text{ad}'(X_i) - \frac{1}{2} B - \text{ad}^s(H),
$$

(2)

where $B$ is the Killing operator, $\text{ad}'(X_i)$ is the adjoint operator for $\text{ad}(X_i)$ with respect to $(\cdot, \cdot)$, and $\text{ad}^s(H) = \frac{1}{2} (\text{ad}(H) + \text{ad}'(H))$ is a symmetric part of the operator $\text{ad}(H)$ [2].

By $\text{Ric}(X, Y)$ we denote $(\text{Ric} X, Y) = (X, \text{Ric} Y)$, i.e. the value of the Ricci form on the vectors $X, Y$ [2, 4].

Now, we (using some ideas from the paper [18]) get some refinement of the formula (2) for solvable metric Lie algebras. We will use a notation $M'$ for the transpose of a matrix $M$.

Suppose that a solvable Lie algebra $\mathfrak{s}$ is supplied with an inner product $Q$. We are interested in the structures of metric Lie algebras $(\mathfrak{s}, Q)$ and $(\mathfrak{n}, Q|_{\mathfrak{n}})$, where $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is a derived algebra of the Lie algebra $\mathfrak{s}$. Let $\mathfrak{a}$ be the orthogonal complement to $\mathfrak{n}$ in $\mathfrak{s}$ with respect to $Q$. Put $l = \dim(\mathfrak{n})$ and $m = \dim(\mathfrak{a})$.

Let us choose vectors $\{e_i\}$, $1 \leq i \leq l$, that form a $Q$-orthonormal basis in $\mathfrak{n}$. This basis could be completed with a $Q$-orthonormal basis $\{f_1, f_2, ..., f_m\}$ in $\mathfrak{a}$ such that

$$
t := \text{trace}(\text{ad}(f_1)) = t \geq 0, \quad \text{trace}(\text{ad}(f_j)) = 0, \quad 2 \leq j \leq m.
$$

It is easy to see that for a non-unimodular Lie algebra $\mathfrak{s}$ we have $f_1 = \frac{H}{\|H\|}$, where the vector $H \in \mathfrak{s}$ is defined by the equation $Q(H, X) = \text{trace}(\text{ad}(X))$ for all $X \in \mathfrak{s}$. In this case $t = \text{trace}(\text{ad}(f_1)) = \|H\| > 0$. If $\mathfrak{s}$ is unimodular, then we can choose any unit vector from $\mathfrak{a}$ as $f_1$. In this case we get $t = \text{trace}(\text{ad}(f_1)) = 0$.

It is clear that $\text{ad}(f_j)|_{\mathfrak{n}} \in \text{Der}(\mathfrak{n})$, $1 \leq j \leq m$, where $\text{Der}(\mathfrak{n})$ is the Lie algebra of all derivations of $\mathfrak{n}$. We use the basis $\{e_1, ..., e_l, f_1, ..., f_m\}$ in order to represent all operators $\text{ad}(f_j)$ and $\text{ad}(e_i)$ in the matrix form:

$$
\text{ad}(f_j) = \begin{pmatrix} A_j & B_j \\ 0 & 0 \end{pmatrix}, \quad \text{ad}(e_i) = \begin{pmatrix} D_i & C_i \\ 0 & 0 \end{pmatrix},
$$

(3)

for some $(l \times l)$-matrices $A_j, D_i$ and some $(l \times m)$-matrices $B_j, C_i$.

In the basis $\{e_1, ..., e_l, f_1, ..., f_m\}$, the matrix of the Ricci operator of the solvable metric Lie algebra $(\mathfrak{s}, Q)$ has the following form (see the formula (2) or the proof of Theorem 3 in [18]):

$$
\text{Ric} = \begin{pmatrix} R_1 & R_2 \\ R_2' & R_3 \end{pmatrix},
$$

(4)

where

$$
R_1 = \text{Ric}^n + \frac{1}{2} \sum_{j=1}^m [A_j, A_j'] + \frac{1}{4} \sum_{j=1}^m B_j B_j' - t A_1',
$$

$$
R_2 = -\frac{1}{2} \left( \sum_{i=1}^l D_i' C_i + \sum_{j=1}^m A_j' B_j + t B_1 \right),
$$

$$
R_3 = -\frac{1}{2} \sum_{j=1}^m B_j' B_j - L,
$$
where \( \text{Ric}^n \) is the matrix of the Ricci operator of the metric Lie algebra \((n, Q|_n)\) in the basis \( \{e_1, \ldots, e_l\} \). \( L \) is a \((m \times m)\)-matrix with elements \( l_{pq} = \text{trace}(A_p A_q^*) \), \( [A_j, A_j'] = A_j A_j' - A_j' A_j, A_j^* = \frac{1}{2}(A_j^2 + A_j), t = \text{trace}(A^1) = \text{trace}(A_1^*) \geq 0 \).

Note also that the formula (2) could be simplified for the metric Lie algebra \((n, Q|_n)\) (a nilpotent Lie algebra \( n \) is unimodular and has trivial Killing form). Namely, we get the following formula for the matrix of its Ricci operator in the basis \( \{e_1, \ldots, e_l\} \):

\[
\text{Ric}^n = -\frac{1}{2} \sum_{i=1}^l D_i D_i' + \frac{1}{4} \sum_{i=1}^l D_i D_i'.
\] (5)

We will need the following result.

**Proposition 3** ([17]). Let \( (g, \langle \cdot, \cdot \rangle) \) be a metric Lie algebra, \( X \in g \) is orthogonal to the ideal \([g, g]\). Then the inequality \( \text{Ric}(X, X) \leq 0 \) holds. Moreover, this inequality becomes an equality if and only if the operator \( \text{ad}(X) \) is skew-symmetric with respect to \( \langle \cdot, \cdot \rangle \).

**Remark 2.** Note that this proposition could be easy derived from the formula (4). Indeed, the matrix \( R_3 = -\frac{1}{2} \sum_{j=1}^m B_j B_j' - L \) is negative semi-definite. If \( X = \sum_{j=1}^m \alpha_j f_j \), then \( \text{Ric}(X, X) = \text{Ric}(\sum_{j=1}^m \alpha_j f_j, \sum_{j=1}^m \alpha_j f_j) = \sum_{i,j} \alpha_i \alpha_j \text{Ric}(f_i, f_j) \leq 0 \). The case \( \text{Ric}(X, X) = 0 \) could be easily studied.

We will need also the following remarkable property of nilpotent metric Lie algebras.

**Proposition 4** (Corollary 5 in [19]). Let \( (n, \langle \cdot, \cdot \rangle) \) be a nilpotent metric Lie algebra. Then for every derivation \( A \in \text{Der}(n) \) the inequality

\[
\text{trace}(\text{Ric}^n \cdot [A, A']) = \langle \text{Ric}^n, [A, A'] \rangle \geq 0
\]

holds, where \( \text{Ric}^n \) is the Ricci operator of \( (n, \langle \cdot, \cdot \rangle) \). Moreover, this inequality becomes an equality if and only if \( A' \in \text{Der}(n) \).

3. **One important partial case**

The most difficult case in Theorem 1 is the case, when the derived algebra \( n \) has codimension 1 in the Lie algebra \( s \). We supply the space \( \text{End}(n) \) with the inner product \( \langle A, B \rangle = \text{trace}(AB^*) \), where \( B^* \) means the adjoint of the operator \( A \) with respect to \( Q|_n \).

At first, we refine the formula (4) to this special case. We have \( m = 1 \) and we will use notations \( f \) and \( A \) instead of \( f_1 \) and \( A_1 \) respectively (see (3)). Obviously, \( B_1 \) is trivial because of \( m = 1 \). In the formula (4) for the matrix of the Ricci operator, we get

\[
R_1 = \text{Ric}^n + \frac{1}{2} [A, A'] - tA^*, \quad R_2 = -\frac{1}{2} \sum_{i=1}^l D_i(C_i),
\]

and the matrix \( R_3 \) consists of a unique element \(-r\), where

\[
r = \text{trace}(A^* A^*) = \langle A^*, A^* \rangle.
\] (6)
Remark 3. It is easy to check that the $i$-th entry of the column matrix $R_2$ is equal to
$$-\frac{1}{2} \text{trace}(D_i', A) = -\frac{1}{2} \text{trace}(D_i \cdot A') = -\frac{1}{2} \langle D_i, A \rangle$$
(see [6] for details).

Now, we consider the structural constants $C^k_{ij}$ of the Lie algebra $\mathfrak{n}$ with respect to the basis $\{e_1, \ldots, e_l\}$, i.e. $[e_i, e_j] = \sum_{k=1}^l C^k_{ij} e_k$ for all $i, j, k$. It is clear that the $(j,k)$-th entry of the matrix $D_i$ is equal to $C^j_{ik}$. By the formula (5) we get
$$\text{trace}(\text{Ric}^n) = -\frac{1}{4} \sum_{i=1}^l \text{trace}(D_i D_i') = -\frac{1}{4} \sum_{i,j,k} (C^k_{ij})^2. \tag{7}$$

Further in this section we consider the case $r = \text{trace}(A^s A^s) > 0$, i.e. the operator $A$ is not skew-symmetric. We will prove the following

**Proposition 5.** Let $(\mathfrak{s}, Q)$ be a solvable metric Lie algebra, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, $\mathfrak{a}$ be a $Q$-orthogonal complement to $\mathfrak{n}$ in $\mathfrak{s}$, $\dim(\mathfrak{a}) = 1$, $r = \text{trace}(A^s A^s) > 0$. Then one of the following two exclusive assertions holds:

1. The ideal $\mathfrak{n}$ is commutative and for every nontrivial $X \in \mathfrak{a}$ the operator $\text{ad}(X)|_n$ is trace-free, normal, but not skew-symmetric with respect to $Q$ (in this case the Ricci operator of $(\mathfrak{s}, Q)$ has only one negative eigenvalue, while all other eigenvalues are zero);
2. The Ricci operator of the metric Lie algebra $(\mathfrak{s}, Q)$ has at least two negative eigenvalues.

First, we consider a matrix
$$L = \begin{pmatrix} I & \frac{1}{r} \cdot R_2 \\ 0 & 1 \end{pmatrix}, \tag{8}$$
where $I$ is the identity matrix, and the matrix
$$\overline{\text{Ric}} = L \cdot \text{Ric} \cdot L' = \begin{pmatrix} R_1 + \frac{1}{r} \cdot R_2 R_2' & 0 \\ 0 & -r \end{pmatrix}. \tag{9}$$

By the law of inertia for quadratic forms, the matrices $\overline{\text{Ric}}$ and $\text{Ric}$ have one and the same signature. But the matrix $\overline{\text{Ric}}$ is block-diagonal (with a negative entry in the last block) and we immediately get

**Proposition 6.** The operator $\text{Ric}$ (for $r > 0$) has at least two negative eigenvalues if and only if the matrix
$$R_1 + \frac{1}{r} \cdot R_2 R_2' = \text{Ric}^n + \frac{1}{2} [A, A'] - t A^s + \frac{1}{r} \cdot R_2 R_2' \tag{10}$$
has at least one negative eigenvalue.

Note that a symmetric matrix with negative trace has at least one negative eigenvalues. Therefore, we get
Proposition 7. If $r > 0$ and
\[ \text{trace}(\text{Ric}^n) - t^2 + \frac{1}{r} \text{trace}(R_2 R'_2) < 0, \]
then the operator Ric has at least two negative eigenvalues.

Remark 4. For non-unimodular Lie algebras, the inequality from the above proposition could be replaced by the inequality
\[ r \cdot \text{trace}(\text{Ric}^n) + \text{trace}(R_2 R'_2) \leq 0, \tag{11} \]
since $t > 0$ for such algebras.

In the remainder of this section we prove Proposition 5. We consider two variants: $R_2 = 0$ and $R_2 \neq 0$.

Claim 1. Proposition 5 is valid for $R_2 = 0$.

Proof. By Proposition 6 (and because of $R_2 = 0$), Ric has at least two negative eigenvalues if and only if the matrix
\[ \text{Ric}^n + \frac{1}{2}[A, A'] - tA^s \tag{12} \]
has at least one negative eigenvalue. The trace of this matrix is (see (7))
\[ -\frac{1}{4} \sum_{i,j,k} (C^k_{ij})^2 - t^2 \]
(since trace([A, A']) = 0 and trace($A^s$) = trace($A$) = $t$). If this trace is not zero, then Ric has at least two negative eigenvalues.

Now, suppose that this trace is zero. Then $t = 0$ and $n$ is abelian (this implies $\text{Ric}^n = 0$ in particular). Further, if $A$ is not normal, then the operator (12), having the form $\frac{1}{2}[A, A']$, is trace-free and non-zero. Hence, it has at least one negative eigenvalue and Ric has at least two negative eigenvalues by Proposition 6. On the other hand, if $A$ is normal, then the operator (12) is zero and Ric has only one negative eigenvalue, while all other its eigenvalues are zero. This proves Proposition 5 for $R_2 = 0$. ■

Now, we consider a much more technically involved claim.

Claim 2. If $R_2 \neq 0$ then the assertion (2) of Proposition 5 is valid.

Proof. Note that the value
\[ \text{trace}(R_2 R'_2) = \frac{1}{4} \sum_{i=1}^{l} (\text{trace}(D_i \cdot A'))^2 \]
(see Remark 3) does not depend on the choice of an orthonormal basis $\{e_i\}$, $1 \leq i \leq l$, in $n$. This assertion has been proved in [20], but we reproduce here a short argument for the convenience of the reader. Consider another orthonormal basis $\{\overline{e}_i\}$, $1 \leq i \leq l$, where \( \overline{e}_i \overline{e}_j = \delta_{ij}, \overline{e}_i \overline{e}_j = 0 \) for $i \neq j$. Let $\overline{R}_2$ be the operator corresponding to $\overline{e}_i$. Then
\[ \text{trace}(\text{Ric}^n) - t^2 + \frac{1}{r} \text{trace}(\overline{R}_2 \overline{R}'_2) < 0, \]
and $\overline{R}_2 \overline{R}'_2$ is positive definite. Therefore, Ric has at least two negative eigenvalues for $\overline{R}_2 \neq 0$. This proves Proposition 5 for $R_2 \neq 0$. ■
in \( n \). Then \( \overline{e}_i = \sum_j q_{ji} e_j \) for all \( i \), where \( (q_{ji}) \) is an orthogonal matrix. Therefore,

\[
\overline{D}_i = \text{ad}(\overline{e}_i) = \sum_j q_{ji} D_j,
\]

\[
\text{trace}(\overline{D}_i A') = \sum_j q_{ji} \text{trace}(D_j A'),
\]

\[
(\text{trace}(\overline{D}_i A'))^2 = \sum_{j,k} q_{ji} q_{ki} \text{trace}(D_j A') \text{trace}(D_k A'),
\]

\[
\sum_i (\text{trace}(\overline{D}_i A'))^2 = \sum_{i,j,k} q_{ji} q_{ki} \text{trace}(D_j A') \text{trace}(D_k A') = \sum_{j,k} \left( \text{trace}(D_j A') \text{trace}(D_k A') \sum_i q_{ji} q_{ki} \right) = \sum_{j,k} \delta_{jk} \text{trace}(D_j A') \text{trace}(D_k A') = \sum_j (\text{trace}(D_j A'))^2.
\]

Hence, we may use some special basis \( \{e_i\} \) (in \( n \)) in order to get more suitable expressions for \( R_2 \) and \( \text{trace}(R_2 R_2') \).

In the linear space \( \text{Der}(n) \subset \text{End}(n) \) of derivations of \( n \), we consider a subspace \( \text{InnDer}(n) \) of inner derivations. We will use the projection \( P_{\text{inner}} : \text{Der}(n) \to \text{InnDer}(n) \) (see (1)). Let us consider \( \tilde{A} = P_{\text{inner}}(A) \).

Let \( I \) be a subspace of codimension 1 in \( n \) such that for any \( X \in I \) the inner derivation \( \text{ad}(X) \) lies in the orthogonal complement to \( \mathbb{R} \cdot \tilde{A} \) with respect to inner product \( \langle \cdot, \cdot \rangle \).

Now we choose a \( Q \)-orthonormal basis \( \{e_i\} \) in \( n \) such that \( e_i \in I \) for \( i \geq 2 \). Hence, we get \( \langle D_i, A \rangle = \text{trace}(D_i A') = 0 \) for \( i \geq 2 \). Recall, that \( i \)-th entry of the column matrix \( R_2 \) is equal to \( -\frac{1}{2} \langle D_i, A \rangle \) (see Remark 3). Since we suppose \( R_2 \neq 0 \), then \( \langle D_1, A \rangle = \text{trace}(D_1 A') \neq 0 \). Therefore,

\[
R_2' = \left(-\frac{1}{2} \langle D_1, A \rangle, 0, 0, \ldots, 0 \right), \tag{13}
\]

\[
R_2 R_2' = \text{diag} \left( \frac{1}{4} \langle D_1, A \rangle^2, 0, 0, \ldots, 0 \right), \tag{14}
\]

\[
4 \text{trace}(R_2 R_2') = (\text{trace}(D_1 \cdot A'))^2 = \langle D_1, A \rangle^2. \tag{15}
\]

Now we will prove the inequality (11) and study possibilities when it becomes an equality.

Recall that \( \text{trace}(\text{Ric}^n) = -\frac{1}{4} \sum_{i,j,k} (C_{ij}^k)^2 \) by (7), On the other hand, by (15) and by the Cauchy–Bunyakovsky–Schwarz inequality we get

\[
4 \text{trace}(R_2 R_2') = \langle D_1, A \rangle^2 = \langle D_1, \tilde{A} \rangle^2 \leq \langle D_1, D_1 \rangle \langle \tilde{A}, \tilde{A} \rangle = \langle \tilde{A}, \tilde{A} \rangle \text{trace}(D_1 \cdot D_1') = \langle \tilde{A}, \tilde{A} \rangle \sum_{j,k} (C_{jk}^j)^2,
\]

where the inequality becomes an equality if and only if \( \tilde{A} = \lambda D_1 \) for some \( \lambda \in \mathbb{R} \) \( \langle D_1 \neq 0 \) by \( R_2 \neq 0 \), \( \langle D_1, \tilde{A} \rangle^2 = \langle D_1, D_1 \rangle \langle \tilde{A}, \tilde{A} \rangle \) iff \( \tilde{A} \) is proportional to \( D_1 \).
Therefore,

\[
    r \cdot \text{trace}(\text{Ric}^n) + \text{trace}(R_2R'_2) = \\
    -\frac{1}{4} \text{trace}(A^s \cdot A^s) \sum_{i,j,k} (C^k_{ij})^2 + \frac{1}{4} \langle D_1, \tilde{A} \rangle^2 = \\
    -\frac{1}{4} \left( \langle A^s, A^s \rangle \sum_{i,j,k} (C^k_{ij})^2 - \langle D_1, \tilde{A} \rangle^2 \right) \leq \\
    -\frac{1}{4} \left( \langle A^s, A^s \rangle \sum_{i,j,k} (C^k_{ij})^2 - \langle \tilde{A}, \tilde{A} \rangle \langle D_1, D_1 \rangle \right) \leq \\
    -\frac{1}{4} \left( \frac{1}{2} \langle \tilde{A}, \tilde{A} \rangle \sum_{i,j,k} (C^k_{ij})^2 - \langle \tilde{A}, \tilde{A} \rangle \langle D_1, D_1 \rangle \right) = \\
    -\frac{1}{8} \langle \tilde{A}, \tilde{A} \rangle \left( \sum_{i,j,k} (C^k_{ij})^2 - 2 \sum_{j,k} (C^j_{1k})^2 \right) \leq 0,
\]

since \( C^k_{ij} = -C^k_{ji} \), \( \sum_{i,j,k} (C^k_{ij})^2 \geq 2 \sum_{j,k} (C^j_{1k})^2 \) and \( 2 \langle A^s, A^s \rangle \geq \langle \tilde{A}, \tilde{A} \rangle \) by Proposition 1. If \( s \) is non-unimodular (i.e. \( t = \text{trace}(A) \neq 0 \)), then we get that Ric has at least two negative eigenvalue by Proposition 7 and Remark 4.

Now, we suppose that Ric has at most one negative eigenvalue. This implies \( t = \text{trace}(A) = 0 \) and the equality \( r \cdot \text{trace}(\text{Ric}^n) + \text{trace}(R_2R'_2) = 0 \). From the above arguments we get that the latter equality holds if and only if \( 2 \langle A^s, A^s \rangle = \langle \tilde{A}, \tilde{A} \rangle \), \( \tilde{A} = \lambda D_1 \) for some \( \lambda \in \mathbb{R} \) and \( C^k_{ij} = 0 \) for \( 1 \notin \{i, j\} \) simultaneously. Then by Proposition 1 we see that \( A - \tilde{A} \) is a skew-symmetric derivation of \( n \). Since \( A \) is not skew-symmetric, then \( \tilde{A} \neq 0 \) and \( \lambda \neq 0 \). If \( C^k_{ij} = 0 \) for \( 1 \notin \{i, j\} \), then Lin\{\( e_2, e_3, \ldots, e_l \)\} is an abelian subalgebra of codimension 1 in \( n \). By Lemma 3 we get that Lin\{\( e_2, e_3, \ldots, e_l \)\} is an ideal in \( n \). Therefore, \( C^1_{ij} = 0 \) for all \( i, j \).

Let us consider the matrix (10) more closely, put

\[
    \tilde{R} := \text{Ric}^n + \frac{1}{2} [A, A'] + \frac{1}{r} \cdot R_2R'_2.
\]

By Proposition 6 it has no negative eigenvalue. Since \( \text{trace}(\tilde{R}) = \text{trace}(\text{Ric}^n) + 1/r \cdot \text{trace}(R_2R'_2) = 0 \), it means that \( \tilde{R} \) is the zero matrix. Now we will prove, that the latter is impossible (under the above conditions).

Let us suppose the contrary, i.e. \( \tilde{R} = 0 \). At first, consider \( \tilde{R}_{11} \), the \((1, 1)\)-th entry of \( \tilde{R} \). By (5) we get

\[
    \text{Ric}^n = -\frac{1}{2} \sum_{i=1}^{l} D'_i D_i + \frac{1}{4} \sum_{i=1}^{l} D_i D'_i.
\]

Since \( C^1_{ij} = 0 \) for all \( i \) and \( j \), the first column of the matrix \( D_1 \) and the first row of every matrix \( D_i (1 \leq i \leq s) \) are zero. Therefore, \( (D'_i D_1)_{11} = (D_i D'_i)_{11} = 0 \) for all
we get the inequality
\[\frac{1}{2}(D_i, D_i) = \sum_{j=1}^{i} (C_{i1}^j)^2 = \sum_{j=1}^{i} (C_{1i}^j)^2\]
for \(i \geq 2\). Therefore, \((\text{Ric}^n)_{11} = -\frac{1}{2} \sum_{i,j=1}^{i} (C_{i1}^j)^2 = -\frac{1}{2}(D_1, D_1)\).

Since \(\tilde{A} = \lambda D_1\) and \(A - \tilde{A}\) is skew-symmetric, then \(A^s = \tilde{A}^s = \lambda D_1^s\). Since \(D_1\) is a nilpotent operator, then by Lemma 6 we get
\[r = \langle A^s, A^s \rangle = \lambda^2 \langle D_1^s, D_1^s \rangle = \frac{\lambda^2}{2}(D_1, D_1)\).
By (14) we get \((R_2 R_2')_{11} = \frac{1}{4}(D_1, A)^2 = \frac{1}{4}(D_1, \tilde{A})^2 = \frac{1}{4}(D_1, \lambda D_1)^2 = \frac{\lambda^2}{4}(D_1, D_1)^2\).

Hence,
\[\left(\frac{1}{r} R_2 R_2'\right)_{11} = \frac{1}{2}(D_1, D_1)\]
and
\[\tilde{R}_{11} = -\frac{1}{2}(D_1, D_1) + \frac{1}{2}([A, A'])_{11} + \frac{1}{2}(D_1, D_1) = \frac{1}{2}([A, A'])_{11} = 0. \quad (16)\]
Now we multiply the matrix equality
\[\text{Ric}^n + \frac{1}{r} [A, A'] + \frac{1}{r} \cdot R_2 R_2' = 0\]
by the matrix \([A, A']\) from the right and calculate the traces of both sides:
\[\text{trace}(\text{Ric}^n \cdot [A, A']) + \frac{1}{r} \text{trace}([A, A'] \cdot [A, A']) + \frac{1}{r} \text{trace}(R_2 R_2' \cdot [A, A']) = 0.\]
Recall that \(R_2 R_2' = \text{diag}\left(\frac{1}{4}(D_1, A)^2, 0, 0, \ldots, 0\right)\) by (14) and \([A, A']\) is a skew-symmetric \(\lambda\) by (16), hence, \(\text{trace}(R_2 R_2' \cdot [A, A']) = 0\). By Proposition 4 we get the inequality \(\text{trace}(\text{Ric}^n \cdot [A, A']) \geq 0\), that becomes an equality if and only if \(A' \in \text{Der}(n)\). Since \(\text{trace}([A, A'] \cdot [A, A']) = \langle [A, A'], [A, A'] \rangle \geq 0\), then \([A, A'] = 0\), \(\text{trace}(\text{Ric}^n \cdot [A, A']) = 0\), and \(A' \in \text{Der}(n)\).

By Lemma 4 we get \(A \in L_2\) (see the first section for the definitions of \(L_i\)). Since \(D_1 \in \text{InnDer}(n)\), then \(D_1 \in L_3\) by Corollary 3. Finally, by Lemma 5 we get
\[0 = \langle A, D_1 \rangle = \langle \tilde{A}, D_1 \rangle = \lambda \langle D_1, D_1 \rangle.\]
But this is impossible, since \(\lambda \neq 0\) and \(D_1 \neq 0\). Therefore, \(\tilde{R}\) is not a zero matrix.
This contradiction proves the claim. \(\blacksquare\)

Therefore, we have proved Proposition 5.

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorem 1 in full generality. Consider a subspace
\[\tilde{a} = \{X \in a | \text{ad}(X) \text{ is skew-symmetric in } (s, Q)\} \}
There are three mutually exclusive cases:
\[1) \ \dim(a) - \dim(\tilde{a}) \geq 2, \quad 2) \ \dim(a) - \dim(\tilde{a}) = 1, \quad 3) \ \dim(a) = \dim(\tilde{a}).\]
Case 1). Choose a subspace \( b \subset \mathfrak{a} \) such that \( \mathfrak{a} = \tilde{\mathfrak{a}} \oplus b \). Then for every \( X \in b \) the operator \( \text{ad}(X) \) is not skew-symmetric and by Proposition 3 we get \( \text{Ric}(X, X) < 0 \). Since \( \dim(b) \geq 2 \), then the operator \( \text{Ric} \) has at least two negative eigenvalues (see Corollary 4).

Case 2). We can choose a basis \( \{ f_1, f_2, \ldots, f_m \} \) such that \( f_i \in \tilde{\mathfrak{a}} \) for \( i \geq 2 \). Since the operator \( \text{ad}(f_i) \), \( 2 \leq i \leq m \), is skew-symmetric, then the matrix \( A_i \) are skew-symmetric and \( B_i = 0 \) for \( i \geq 2 \) (see (3)). Then \( B_1 = 0 \) also. By (4) we get

\[
\text{Ric} = \begin{pmatrix} R_1 & R_2 & \vdots & R_3 \\ R_2 & R_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ R_3 & \cdots & \cdots & R_1 \end{pmatrix},
\]

where \( R_1 = \text{Ric}^n + \frac{1}{2}[A_1, A'_1] - tA_1^a \),

\[
R_2 = -\frac{1}{2} \left( \sum_{i=1}^{l} D_i C_i \right), \quad R_3 = -L,
\]

where \( \text{Ric}^n \) is the matrix of the Ricci operator of the metric Lie algebra \( (\mathfrak{n}, Q|_{\mathfrak{n}}) \) in the basis \( \{ e_1, \ldots, e_l \} \), \( L \) is a \( (m \times m) \)-matrix with elements \( l_{pq} = \text{trace}(A^a_p A^a_q) \). Since \( A_i \) are skew-symmetric for \( i \geq 2 \), then \( L = \text{diag}(\text{trace}(A^a_1 \cdot A^a_1), 0, 0, \ldots, 0) \).

Now, let us consider \( \tilde{\mathfrak{s}} = \text{Lin}(\mathfrak{n}, f_1) \subset \mathfrak{s} \). It is clear that \( \tilde{\mathfrak{s}} \) is closed under the Lie multiplication \([\cdot,\cdot]\), i.e. \( \tilde{\mathfrak{s}} \) is a subalgebra of the Lie algebra \( \mathfrak{s} \). Supply it with the inner product \( Q|_{\tilde{\mathfrak{s}}} \). Let \( \text{Ric} \) be a matrix of the Ricci operator of metric Lie algebra \( (\tilde{\mathfrak{s}}, Q|_{\tilde{\mathfrak{s}}}) \) in the basis \( \{ e_1, e_2, \ldots, e_l, f_1 \} \). Using the formula (4) one more time, we see that \( \text{Ric} \) is submatrix of Ric corresponding to rows and columns with the numbers 1, 2, \ldots, \( l, l + 1 \). By Proposition 5, we have two possibilities for the metric Lie algebra \( (\tilde{\mathfrak{s}}, Q|_{\tilde{\mathfrak{s}}}) \):

2a) the ideal \( \mathfrak{n} \) is commutative and the operator \( \text{ad}(f_1)|_{\mathfrak{n}} \) is trace-free, normal, but not skew-symmetric with respect to \( Q|_{\mathfrak{n}} \);

2b) the Ricci operator \( \text{\tilde{Ric}} \) of the metric Lie algebra \( (\tilde{\mathfrak{s}}, Q|_{\tilde{\mathfrak{s}}}) \) has at least two negative eigenvalues.

If 2a) holds, then for the metric Lie algebra \( (\mathfrak{s}, Q) \) the possibility (2) of Theorem 1 holds.

If 2b) holds, then the matrix \( \text{Ric} \) has at least two negative eigenvalues, since this property has its submatrix \( \text{\tilde{Ric}} \) (see Corollary 4). Hence, we prove Theorem 1 in case 2).

Before studying the case 3), we prove the following

**Lemma 8.** If \( \mathfrak{s} \) is a non-abelian nilpotent Lie algebra, then the cases 2a) and 3) are impossible for the metric Lie algebra \( (\mathfrak{s}, Q) \).

**Proof.** At first, prove that the case 2a) is impossible. Suppose the contrary. Then (in the above notations) the operator \( \text{ad}(f_1)|_{\mathfrak{n}} \) is trace-free and normal, but not skew-symmetric. On the other hand, it is nilpotent, but the only nilpotent normal operator is the zero operator. We get a contradiction, since \( \text{ad}(f_1)|_{\mathfrak{n}} \) is not skew-symmetric.

Now, we prove that the case 3) is impossible. Suppose the contrary. Every operator \( \text{ad}(f_i) \) are both skew-symmetric (hence, normal) and nilpotent. Therefore, \( \text{ad}(f_i) \) is the zero operator for all \( i = 1, 2, \ldots, l \), and \( \mathfrak{a} \) lies in the center of the Lie algebra \( \mathfrak{s} \).
Hence,
\[ n = [s, s] = [n \oplus a, n \oplus a] = [n, n]. \]
But this is impossible, since \( n \) is nilpotent and \( n \neq 0. \]

**Remark 5.** Lemma 8 implies that for a non-abelian nilpotent metric Lie algebra \((s, Q)\), only the cases 1) and 2b) are possible. In both these cases the Ricci operator of \((s, Q)\) has at least two negative eigenvalue. This proves Theorem 3.

Finally, we consider the case 3). Since all operators \( \text{ad}(f_i), 1 \leq i \leq m, \) are skew-symmetric, then (for all \( i \)) the matrix \( A_i \) is skew-symmetric, \( B_i = 0 \) and \( A_i^2 = 0 \) (see (3)). By (4) we get
\[
\text{Ric} = \begin{pmatrix}
R_1 & R_2 \\
R_2 & R_3
\end{pmatrix},
\]
and \( \text{Ric}^n \) is the matrix of the Ricci operator of the metric Lie algebra \((n, Q|_n)\) in the basis \( \{e_1, \ldots, e_l\} \).

We have two possibilities: 3a) \( n \) is abelian; 3b) \( n \) is non-abelian.

If 3a) holds, then for the metric Lie algebra \((s, Q)\) the possibility (3) of Theorem 1 holds.

If 3b) holds, then by Corollary 4 the matrix \( \text{Ric} \) has at least two negative eigenvalues, since this property has its submatrix \( \text{Ric}^n \) (the latter is a statement of Theorem 3, that we have proved in Remark 5). Hence, we have proved Theorem 1 in full generality.

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