Random Majorana Constellations

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Even the most classical states are still governed by quantum theory. A fantastic array of physical systems can be described by their Majorana constellations of points on the surface of a sphere, where concentrated constellations and highly symmetric distributions correspond to the least and most quantum states, respectively. If these points are chosen randomly, how quantum will the resultant state be, on average? We explore this.

Introduction.— Random matrices sampled from an ensemble with a specific symmetry were introduced by Wigner [1, 2] and Dyson [3–5] to describe spectral properties of quantum many-body systems, such as atomic nuclei [6]. Since then, random matrix theory [7–9] has found applications in fields as diverse as black holes [10–13] and gravity [14], quantum chaos [15], transport in disordered systems [16, 17], spin glasses [18], neural networks [19, 20], and even finance [21]. Quantum information is definitely one of the most recent applications, and a very natural one, too [22–24].

Random quantum states can be seen as arising from the time evolution of arbitrary initial states of quantum analogues of classically chaotic systems [25]. Furthermore, these states emerge when not much is known about a state and one wants to ask about its generic properties, characteristic of a “typical” state [26–28]. Given a random quantum state, one may then ask how quantum is this state: are classical states typical? If one had access to a black box that prepared random states, how much would it be worth and what would its applications be? These questions and more motivate the present study.

When considering the set of pure states living in an N-dimensional complex vector space, the manifold of physical states (i.e., the projective Hilbert space) is the sphere CPN [29]. There is a unique normalized measure that is invariant under all unitary transformations: the associated Haar measure [30]. One can reasonably call this measure the uniform distribution over the unit sphere [31].

The geometrical properties of quantum states are essential to understand where classical intuitions break down and where the next quantum advantage may lie [32–34]. One would expect that the concentration of the measure is a good indicator of quantumness: the most classical states should be described by localized probability distributions and the most quantum states by spread ones [35, 36]. While this notion provides effective analytical tools, the geometry of CPN is not very intuitive and so it is difficult to visualize and to gain insights into the nature of those states.

In this Letter, we propose a novel class of random states based upon a mapping onto a many-qubit system (which, in some cases, might be fictitious). Actually, any system with a finite-dimensional Hilbert space of dimension N = 2S + 1 can be thought of as a spin S [37]. The space manifold CPN admits an appealing representation due to Majorana [38], which maps any state onto 2S points on the unit sphere S2; this is the state’s stellar representation. This picture makes it natural to consider the states associated with random points on the sphere, as opposed to the random states on CPN that induce correlations between the points in the Majorana constellation. We explore here the far-reaching consequences of this intuitive and simple approach.

Majorana constellations.— The Majorana representation can easily be understood by noticing that spin-S states can be obtained as fully symmetrized states of a system of 2S spins 1/2 (or qubits). These systems have many physical applications, ranging from quantum computation to quantum sensing and metrology [39–43]. This idea is also at the heart of the Schwinger map [44, 45], which realizes the set of angular momentum operators in terms of polynomials of bosonic operators.

A pure spin-S state |ψ⟩, which can be expanded as |ψ⟩ = Σm=−S S×m|ψm⟩|S, m⟩ in the standard angular momentum basis, can be rewritten as

$$|ψ⟩ = \frac{1}{N} \prod_{i=1}^{2S} a_{i}^† u_{i} |vac⟩,$$

where ui is a unit direction of spherical coordinates (θi, φi), the rotated bosonic operators are $$a_{i}^† = \cos(\theta/2) a_{i}^† + \sin(\theta/2) e^{iφ} a_{i}$$, with $$a_{i}^†$$ and $$a_{i}$$ creating excitations in two modes (denoted by + and −) from the two-mode vacuum |vac⟩, and N is a normalization factor [46].

The set of 2S (non-necessarily distinct) unit vectors {ui} defines the Majorana constellation of the state.

The states |n⟩ = 1/√(2S)! $$a_{n}^† a_{n}$$ |vac⟩ are pre-
FIG. 1. Multipole distribution \( \sum_{q=-K}^{K} |\varphi_{Kq}|^2 \) for random states with the values of \( S \) indicated in the inset.

Precisely the spin-\( S \) coherent states (CS) [47]. Their constellations consist of only one single point in the antipodal direction \(-n\). It is customary to introduce the CS representation by \( \psi(n) = \langle n|\psi \rangle \): this is a wave function over the sphere \( S_2 \), with probability distribution \( Q_{\psi}(n) \equiv |\langle n|\psi \rangle|^2 \), which is nothing but the Husimi function [48]. Simple algebraic manipulations yield

\[
\psi(n) = \prod_{i=1}^{2S} \left[ \frac{1}{2}(1 - n \cdot u_i) e^{i\Sigma(n,-u_i)} \right],
\]

where \( \Sigma(n,-u_i) \) is the oriented area of the spherical triangle with vertices \((z, n,-u_i)\), with \( z \) the unit vector in the direction of the axis \( Z \). This confirms that the Majorana constellation consists of the zeros of \( \psi(n) \) and, therefore, of the Husimi function.

Because the Majorana representation facilitates a useful geometrical interpretation of quantum states, it has found an increasing number of applications in recent years, with prominent examples being polarimetry and magnetometry [49], Bose-Einstein condensates [50, 51], Berry phases [52–54], and studies of entanglement [55].

Random Majorana constellations.— Since points on the sphere \( S_2 \) correspond uniquely to pure states via the Majorana representation, we have a natural way to generate random states: they correspond to sets of random points on \( S_2 \). Interestingly, the question of distributing points uniformly over a sphere has inspired substantial mathematical research [56–58], as has the question of random polynomials [59–62], in addition to attracting the attention of physicists working in a variety of fields.

We randomize each of the spherical coordinates \((\theta_i, \phi_i)\) independently using the Haar measure \( \sin \theta_i \, d\theta_i \, d\phi_i / 4\pi \) for \( S_2 \). This means that the random operators acting on the state take the form \( U \in SU(2) \otimes 2S \). This is fundamentally tied to the most robust deterministic technique for creating arbitrary bipartite states of light, which uses beam splitters and post selection to sequentially add a photon position on its Poincaré sphere as a new point to a state’s existing Majorana constellation [63, 64]. When the states of the single photons are randomized, the resulting state immediately takes the form of Eq. (1) with random spherical coordinates.

The intrinsic SU(2) symmetry suggests using the germane notion of multipoles [65, 66] to characterize the resulting states. To this end, we expand the density matrix of the system as \( \varrho = \sum_{K=0}^{2S} \sum_{q=-K}^{K} \varrho_{Kq} T_{Kq} \), where the spherical tensor operators read [67]

\[
T_{Kq} = \sqrt{\frac{2K + 1}{2S + 1} \frac{1}{S_m}} \sum_{m_{m'}=S} C_{Sm'_{m'}Sm,0}^S \varrho_{Kq} Y_{Kq}^*(n),
\]

where \( Y_{Kq}^*(n) \) are the spherical harmonics. Since these comprise a complete set of orthonormal functions on \( S_2 \), one can invert this equation to express the multipoles as [39]

\[
\varrho_{Kq} = C_K \int_{S_2} d^2n \, Q_{\varrho}(n) \, Y_{Kq}(n),
\]

with \( C_K = \sqrt{4\pi/(2S + 1)(C_{SS,0}^S)^{-1}} \) and \( d^2n = \sin \theta d\theta d\phi \) the invariant measure on \( S_2 \). The multipoles thus appear as the standard ones in electrostatics, but replacing the charge density by \( Q_{\varrho}(n) \) and distances by directions [69]. They are the \( K \)th directional moments of the state and, therefore, they resolve progressively finer angular features with increasing \( K \).

For each value of \( S \), we average over \( 1.5 \times 10^6 \) samples. In Fig. 1 we show the resulting multipoles plotted in terms of the multipole squared length \( \sum_{q=-K}^{K} |\varrho_{Kq}|^2 \), so that every orientation of the Majorana constellation is represented equally. As we can see, only multipoles with small \( K \) contribute significantly. The maximally contributing multipole \( K_{\text{max}} \) smoothly

FIG. 2. The dots represent the average multipoles for random states with \( S = 60 \). The broken lines delimit the associated variances. The density plot in the back (with the logarithmic scale shown at the right) is the number of trials having the corresponding value of the multipole.

Here, \(-q \leq K \leq q\) and \( C_{Sm'_{m'}Sm,0}^S \) is a Clebsch-Gordan coefficient. This expansion is especially friendly because the tensors are orthonormal and transform covariantly under rigid rotations of the Majorana constellation. The expansion coefficients are precisely the state multipoles; viz., \( \varrho_{Kq} = \text{Tr}(\varrho T_{Kq}^j) \). These multipoles determine the angular features of the corresponding Husimi \( \varrho \)-function on \( S_2 \) [68]:

\[
Q_{\varrho}(n) = \sqrt{\frac{4\pi}{2S + 1} \sum_{K=0}^{2S} \sum_{q=-K}^{K} C_{Sm'_{m'}Sm,0}^S \varrho_{Kq} Y_{Kq}^*(n),
\]

where \( Y_{Kq}^*(n) \) are the spherical harmonics. Since these comprise a complete set of orthonormal functions on \( S_2 \), one can invert this equation to express the multipoles as [39]
increases with $S$: a least-squares fitting gives that, for large $S$, $K_{\text{max}} = a \sqrt{S}$, with $a \approx 0.8$.

To gain further insight into this behavior, in Fig. 2 we plot the average multipoles for random states with $S = 60$. The broken lines delimit the corresponding variances, which are clearly not uniform. Since the individual multipole distributions are non-Gaussian, the variances give only partial information. To complete the picture, in the background we display a density plot representing the number of states with a given value of the $K$th multipole. Notice that we use a logarithmic scale in order to better appreciate the behavior of higher $K$s, which have exceedingly small values. We see the emergence of a striking multi-peaked structure. The multipoles with significant yellow areas (i.e., a strong concentration of samples) are those with less variance.

One sensible set of quantities in this scenario is the set of cumulative multipole distributions, defined as $A_M = \sum_{K=1}^{M} \sum_{q=-K}^{K} |\varphi_K q|^2$ [70]. For CS this quantity reaches the value

\[ A_{M,\text{CS}} = \frac{2S}{2S+1} - \frac{[\Gamma(2S+1)]^2}{\Gamma(2S-M)\Gamma(2S+M+2)}, \]

and it has been proven that this value is indeed maximal for every $M \in \{1, \ldots, 2S\}$ [71]. At the opposite extreme we have states whose multipoles vanish up to the highest $M$: they have been dubbed as Kings of Quantumness [72] and they are maximally unpolarized. Therefore, $A_M$ is a good measure of the quantum properties of a state through its “hidden polarization” features [73] that are stored in high-order multipoles.

To appreciate the different behaviors, in Fig. 3 we have plotted the multipole distributions for the most quantum (Kings of Quantumness), the least quantum (CS), and random states for $S = 106$. The differences speak for themselves. A random Majorana constellation is markedly different from a classical state: although the maximal contributions arise from roughly the same multipoles for both states (actually, for CS a quick estimate gives $K_{\text{max}} = \sqrt{S} - 1/2$), random states have much heavier tails, thus hiding their quantum information in higher-order multipoles than CS. This makes them useful for metrological applications, even in the large-$S$ limit, where quantum effects may be expected to vanish. Additionally, such random states are far from the most quantum states, certifying the rarity of the Kings of Quantumness and the effort required for creating them. This behavior is confirmed by the cumulative multipole distribution $A_M$ for the same states, as plotted in Fig. 4.

**Comparison to other random distributions.**— It is worth noting that one could instead consider states with randomized probability amplitudes $\psi_m$. This can be achieved with a random unitary $U \in \text{SU}(2S+1)$ acting on the state, which is often called the circular unitary ensemble (CUE) [4]. Such random unitaries were considered in Refs. [74] and [75], who found the Rényi-Wehrl entropies of such random states to be highly quantum. Moreover, in Ref. [76] it was demonstrated that such random states are useful and robust for metrology. This sort of action may be possible with a physical system such as multплane light conversion elements [77, 78] acting randomly on the orbital angular momentum degrees of freedom of a beam of light (or waveplates and a cross-Kerr element [76]), but it less tied to the geometrical properties of the state.

Actually, we can calculate the average cumulative multipole moments $A_M = \int A_M dU$ for a variety of normalized Haar measures $dU$, through

\[ A_M = \sum_{M} \sum_{K=1}^{M} \sum_{q=-K}^{K} \frac{2K+1}{2S+1} \sum_{m,m'=S} C_{mKq}^{S} C_{mK'q'}^{S} I_{m,m',q}, \]

with $I_{m,m',q} = \int |\psi_{m+n}\psi_{m+n+q}|^2 dU$.

For example, if we express our random unitaries by $U \in \text{SU}(2S+1)$, these integrals can be obtained exactly using random matrix theory, as in the Supplemental Material or Refs. [79] and [80]. But there is another intuitive method: all of the nonzero integrals take the form $\int |\psi_{m+n}|^2 |\psi_{n}|^2 dU$ and the distribution for each $\psi_i$ is the same, so we know immediately that $I_{m,m',q} \propto \delta_{q,0} + \delta_{m,m'}$, such that

\[ A_M = \frac{M(M+2)}{(2S+1)(2S+2)}. \]

This means that states with random coefficients $\psi_m$ have $\sum_{q=-K}^{K} |\varphi_K q|^2 \propto 2K + 1$, making them much more quantum
than states with random Majorana constellations and according with other results for this distribution [74, 75]. This is in stark contrast to the notion that both forms of randomness approach each other in the limit of large $S$ in terms of the distance between arbitrary pairs of random states [81], stressing the differences maintained between the forms of randomness for all but $S = 1/2$.

Another possibility of obtaining a random spin-$S$ state is to take a state of $2S$ random qubits and project them onto the symmetric subspace. This is not a unitary operation, instead creating the states from Eq. (1) with the replacement $N \rightarrow \sqrt{(2S)!}$, but it facilitates an analytical calculation whose cumbersome expression we show in the Supplemental Material. Normalizing these averaged multipoles yields a distribution that is remarkably similar to that of CS.

In Fig. 5 we display the conspicuous differences between the multipole distributions for these different randomized states.

Quantifying quantumness.— The conclusion from the discussion thus far is that classical states convey their information in lower multipoles, whereas the opposite occurs for extremal quantum states. However, to assess this behavior in a quantitative way, we need a proper measure of quantumness. Since our analysis has been largely based on multipoles, we will use a recent proposal defined precisely in terms of them [82]; viz.,

$$\mathcal{E}(\varrho) = 1 - \sum_{K=0}^{2S} \sum_{q=-K}^{K} |\varrho_{Kq}|^2 \frac{2K+1}{2K+1}.$$  \hspace{1cm} (9)

Larger values of $\mathcal{E}$ signify more quantum states. This quantity emerges when looking at symmetric superpositions of two-mode states: it turns out that they can be entangled or separable, but this property can change after the Majorana constellation undergoes a rigid rotation. To properly account for this possibility, one can consider their linear entropy of entanglement averaged over all rotated partitions of the two-mode Hilbert space: the final result is precisely (9).

In Fig. 6 we calculate this measure for the different states discussed before for various values of $S$. We see that random Majorana constellations are more quantum than states of random qubits projected onto the symmetric subspace, but less quantum than states with random coefficients in the angular momentum basis. Impressively, the states with random coefficients have $\mathcal{E} = 2S/(2S+2)$ on average, which is very close to the maximum value for a single pure state: $2S/(2S+1)$. The lower continuous line in the figure corresponds to the values of $\mathcal{E}$ for CS, which are the least quantum ones.

Concluding remarks.— We have explored random Majorana constellations that arise as sets of points uniformly distributed on the sphere $S_2$. The concept of state multipoles, intimately linked with the inherent SU(2) symmetry, has served as our main diagnostic tool to capture the amazing properties of these states. Additionally, these multipoles are sensible and experimentally-realizable quantities.

The family of symmetric states contains many metrologically useful states, including GHZ, NOON, and Dicke states, among others. However, all of them are extremely fragile resources. In contradistinction, random Majorana states, from their very same definition, promise to be robust against imperfections such as dephasing and particle losses. Apart from their incontestable geometrical beauty, there surely is plenty of room for the application of these states, whose generation has started to be seriously considered in a variety of physical contexts.

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Supplemental material:
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RANDOM STATES FROM SU(2S + 1)

We would like to compute the following integrals for random state coefficients \( \psi_m = a_m + ib_m \):

\[
I_{m,m',q} = \int \psi_{m+q}^{*}\psi_{m'+q}\psi_{m'}dU.
\]  

(1)

These integrals and more are calculated in Ref. [1], where we can use their Eq. (35) to calculate

\[
\int |\psi_m|^2|\psi_n|^2dU = \frac{\Gamma(2S + 1)}{\Gamma(2S + 1 + 2)}1!!1!, \quad m \neq n,
\]

\[
\int |\psi_m|^4dU = \frac{\Gamma(2S + 1)}{\Gamma(2S + 1 + 2)}2!.
\]

(2)

The asymptotic behavior of such integrals was studied by Weingarten [2].

An alternative route is to explicitly compute the integrals using the normalized Haar measure for SU(2S + 1):

\[
I_{m,m',q} = \frac{1}{\pi^{2S+1}} \int (a_{m+q} + ib_{m+q})(a_m - ib_m)(a_{m'+q} - ib_{m'+q})(a_{m'} + ib_{m'}) \times \exp\left(-a_S^2 - b_S^2 - \cdots - a_{S}^2 - b_{S}^2\right)
\]

\[
\times \frac{1}{(a_S^2 + b_S^2 + \cdots + a_S^2 + b_S^2)^2}da_S db_S db_{S+1} \cdots db_{S+q} db_{S+q+1} \cdots db_{S+q'+1}.
\]

(3)

The integrals vanish unless \( m + q = m \) and \( m' + q = m' \) or \( m + q = m' + q \) and \( m = m' \), because the integrand is otherwise an odd function of some coefficients \( a_m \) and \( b_m \), so we only need consider

\[
I_{m,m',0} = \frac{1}{\pi^{2S+1}} \int (a_{m+q} + b_{m+q})(a_m + b_m)^2 \exp\left(-a_m^2 - b_m^2 - a_{m'}^2 - b_{m'}^2 - r^2\right)
\]

\[
\times \frac{1}{(a_m^2 + b_m^2 + a_{m'}^2 + b_{m'}^2 + r^2)^2}da_m db_m da_{m'} db_{m'} d^4S-r.
\]

\[
I_{m,m,q} = \frac{1}{\pi^{2S+1}} \int (a_{m+q} + b_{m+q})(a_m + b_m)^2 \exp\left(-a_{m+q}^2 - b_{m+q}^2 - a_m^2 - b_m^2 - r^2\right)
\]

\[
\times \frac{1}{(a_{m+q}^2 + b_{m+q}^2 + a_m^2 + b_m^2 + r^2)^2}da_{m+q} db_{m+q} da_m db_m d^4S-r = I_{m,m',0},
\]

\[
I_{m,m,0} = \frac{1}{\pi^{2S+1}} \int r_1^4 \exp(-r_1^2 - r^2) \frac{1}{(r_1^2 + r^2)^2}d^2 r_1 d^4S r,
\]

where we have used the vectors \( r \) to encompass all of the remaining coordinates. Since these integrals all take the form

\[
I = \int |\psi_m|^2|\psi_n|^2dU,
\]

it is tempting to use Isserlis’s theorem to express them in terms of \( \int |\psi_m|^2dU \), which is easier to calculate; however, the coefficients \( \psi_m \) are not themselves normal random variables because they must satisfy normalization constraints. Instead, we can only use the moments-cumulants formula to guarantee that

\[
I_{m,m,0} = 2I_{m,m',0}, \quad m \neq m',
\]

(5)

thus justifying our statement in the main text that

\[
I_{m,m',q} \propto \delta_{q,0} + \delta_{m,m'}.
\]

(6)
We can also directly calculate these volume integrals. The angular dependence can easily be factored out, using

\[ \int d^n r = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty r^{n-1} dr, \]

leading to

\[
I_{m,m',0} = \frac{2^3}{\Gamma(2S - 1)} \int_0^\infty r_1^2 r_2^2 \exp(-r_1^2 - r_2^2) \frac{r_1 r_2 r_3^4 S^{-3} dr_1 dr_2 dr_3}{(r_1^2 + r_2^2 + r_3^2)^2} \\
= \frac{2^3}{\Gamma(2S - 1)} \int_0^{\pi/2} \cos^3 \theta \sin^3 \theta d\theta \int_0^{\pi/2} \cos^2 \phi \sin^{4S-3} \phi d\phi \int_0^\infty x^{4S+1} e^{-x^2} dx \\
= \frac{1}{(2S + 1)(2S + 2)}
\]

and

\[
I_{m,m,0} = \frac{2^2}{\Gamma(2S)} \int_0^\infty r_1^4 S^{-1} \exp(-r_1^2 - r_2^2) \frac{r_1 r_2}{(r_1^2 + r_2^2 + r_3^2)^2} dr_1 dr_2 dr_3 \\
= \frac{2^2}{\Gamma(2S)} \int_0^{\pi/2} \cos^5 \theta \sin^{4S-1} \theta d\theta \int_0^\infty x^{4S+1} e^{-x^2} dx \\
= \frac{2}{(2S + 1)(2S + 2)}.
\]

This yields the result given in Eq. (2).

Since \( \sum_{S=S}^{S=S} \langle C_{Sm,Kq}^{Sm+q} \rangle \) vanishes for \( q = 0 \), we only need to further consider the \( I_{m,m,q} \) terms to calculate the averaged multipoles. We then verify the final result of Eq. (8).

\[
\mathcal{A}_{M}^{(S)} = \frac{1}{(2S + 1)(2S + 2)} \sum_{K=1}^{M} \sum_{q=-K}^{K} \frac{2K + 1}{2S + 1} \sum_{m=S}^{S} \left( C_{Sm,Kq}^{Sm+q} \right)^2 = \frac{1}{(2S + 1)(2S + 2)} \sum_{K=1}^{M} (2K + 1) = \frac{M(M + 2)}{(2S + 1)(2S + 2)}. \tag{10}
\]

**SYMmetric PROJECTIONS OF 2S RANDOM QUBITS**

Random Majorana constellations are intimately related to the symmetrized tensor products of 2S random qubits, albeit with profound differences. To start, we can relate the coefficients of a state given in the angular momentum basis to the coordinates of the Majorana constellation through

\[ \psi_k = \sqrt{\frac{(S + k)! (S - k)!}{N}} \sum_{\text{perm } i \neq j} \prod_{i \neq j} \cos \left( \frac{\theta_i}{2} \right) \prod_{i = j} \sin \left( \frac{\theta_i}{2} \right) e^{i \phi_i}, \tag{11} \]

where the sum is over all symmetric permutations \( j \) of the integers from 1 to 2S. Here, \( N \) is equal to the permanent of the matrix \( M \) with elements \[ M_{ij} = \cos \left( \frac{\theta_i}{2} \right) \cos \left( \frac{\theta_j}{2} \right) + \sin \left( \frac{\theta_i}{2} \right) \sin \left( \frac{\theta_j}{2} \right) e^{i(\phi_i - \phi_j)}. \tag{12} \]

Next, we consider the state of 2S qubits projected onto the symmetric subspace. The translation between the two pictures is given by

\[ |S, m\rangle = \frac{1}{\sqrt{\binom{2S}{S+m}}} \sum_{\text{perm}} |+\rangle^{\otimes S+m} \otimes |-\rangle^{\otimes S-m}, \tag{13} \]

facilitating the projection operator \( P_S = \sum_{m=S}^{S} |S, m\rangle \langle S, m| \) onto the symmetric subspace. A symmetric state can always be composed from a symmetrized SU(2)^S⊗2S rotation operator, which we remark is no longer a unitary operator,

\[ U_{PS} = P_S U^{(1)} \otimes \cdots \otimes U^{(2S)} P_S \tag{14} \]
acting on the fiducial state $|+\rangle^{\otimes 2S}$, where
\begin{equation}
U^{(i)} |+\rangle_i = u^{(i)}_0 |+\rangle_i + u^{(i)}_1 |−\rangle_i = \cos \frac{\theta_i}{2} |+\rangle_i + \sin \frac{\theta_i}{2} e^{i\phi_i} |−\rangle_i .
\end{equation}

In fact, this process immediately yields
\begin{equation}
\langle S, k | U_{1S} \rangle |+\rangle^{\otimes 2S} = \left( \frac{2S}{S+k} \right)^{1/2} \prod_{i,j=0}^{2S} \cos \left( \frac{\theta_i}{2} \right) \prod_{i=1}^{i<j} \sin \left( \frac{\theta_i}{2} \right) e^{i\phi_i},
\end{equation}
which is equivalent to $\psi_k$ found in Eq. S(11) up to replacing the normalization factor by $\mathcal{N} \rightarrow \sqrt{(2S)!}$. This crucial difference of the constellation-dependent normalization factor significantly changes the properties of random Majorana constellations versus symmeterized projections of random qubits. In fact, it is the absence of the normalization constant in the latter case that makes the present integrals analytically tractable, which reinforces the need to present numerical studies of random Majorana constellations.

We can directly calculate the multipoles from the symmeterized projection states because $|+\rangle^{\otimes 2S}$ and $T_{Kq}$ are already symmetric:
\begin{equation}
\mathcal{Q}_{Kq} = \langle + |^{\otimes 2S} U_{1S} U_{1S}^\dagger T_{Kq} U_{1S} |+\rangle^{\otimes 2S} = \langle + |^{\otimes 2S} U^{(1)}_1 \otimes \cdots \otimes U^{(2S)}_1 T_{Kq} U^{(1)}_1 \otimes \cdots \otimes U^{(2S)}_1 |+\rangle^{\otimes 2S} \nonumber 
\end{equation}
\begin{equation}
= \sum_{i,j=0}^{2S} u^{(1)}_{i0} \cdots u^{(2S)}_{j0} |i\rangle^T_{K_q} |j\rangle u^{(1)*}_{j0} \cdots u^{(2S)*}_{i0},
\end{equation}
where the sums run over all $2^{2S}$ bit strings of length $2S$ and $|i\rangle = |i_1\rangle \otimes \cdots \otimes |i_{2S}\rangle_{2S}$, with the convenient translation $(|0\rangle, |1\rangle) \rightarrow (|+\rangle, |−\rangle)$.

Averaging over the squares of the multipoles is straightforward because, for all $m = 1, \ldots, 2S$,
\begin{equation}
\int |\mathcal{Q}_{Kq}|^2 dU = \sum_{i,j,k,l=0}^{2S} \langle i| T^\dagger_{K_q} |j\rangle \langle j| T_{K_q} |k\rangle \prod_{m=1}^{2S} \delta_{i_m, j_m} \delta_{k_m, l_m} + \delta_{i_m, k_m} \delta_{j_m, l_m} / 6.
\end{equation}

This lets us express the average squared multipole as
\begin{equation}
\int |\mathcal{Q}_{Kq}|^2 dU = \sum_{i,j,k,l=0}^{2S} \langle i| T^\dagger_{K_q} |j\rangle \langle j| T_{K_q} |k\rangle \prod_{m=1}^{2S} \delta_{i_m, j_m} \delta_{k_m, l_m} + \delta_{i_m, k_m} \delta_{j_m, l_m} / 6.
\end{equation}

Then, noting that each state of the form $|i\rangle$ is not symmeterized and writing $|i\rangle = \sum_{m=1}^{2S} i_m$, we can calculate expressions like
\begin{equation}
\langle i| T_{K_q} |k\rangle = \sqrt{\frac{2K+1}{2S+1}} \left[ \begin{array}{c} (2S) \\ |i\rangle \\ |k\rangle \end{array} \right]^{1/2} C_{S,S-|i\rangle,|k\rangle,|i\rangle,|k\rangle}^{S,S,|i\rangle,|k\rangle},
\end{equation}
which set the constraints $|i\rangle = |k\rangle - q$. This leads to the closed-form expression
\begin{equation}
\int |\mathcal{Q}_{Kq}|^2 dU = \frac{2K+1}{2S+1} \sum_{i,j,k,l=0}^{2S} \delta_{|i\rangle,|k\rangle+q} \delta_{|i\rangle,|j\rangle+q} \left( \begin{array}{c} (2S) \\ |i\rangle \\ |j\rangle \end{array} \right)^{1/2} C_{S,S-|i\rangle,|k\rangle,|q\rangle,|i\rangle,|j\rangle,|k\rangle,|q\rangle}^{S,S,|i\rangle,|k\rangle,|q\rangle} \prod_{m=1}^{2S} \delta_{i_m, j_m} \delta_{k_m, l_m} + \delta_{i_m, k_m} \delta_{j_m, l_m} / 6.
\end{equation}

As it stands, Eq. (21) can be used to exactly compute all of the averaged multipoles. However, the number of terms in the sum grows exponentially with $S$, making the expression rather cumbersome, so we can invoke a number of counting arguments to simplify the expressions. The terms contributing to the sum must be of the form
\begin{equation}
(i_m, j_m, k_m, l_m) \in \{(0,0,0,0), (0,0,1,1), (1,1,0,0), (1,1,1,1), (0,1,0,1), (1,1,1,0)\}
\end{equation}
and we must partition the sums by counting how many such terms are present, because we find an extra factor of 2 for every index $m$ with $i_m = j_m = k_m = l_m$. Denoting these numbers of terms by $n_1, \ldots, n_{6}$, respectively, we have the following constraints:
\begin{align}
n_1 + n_2 + n_3 + n_4 + n_5 + n_6 &= 2S \\
n_3 + n_4 + n_6 &= |i| \\
n_3 + n_4 + n_5 &= |j| = |i| - q \\
n_2 + n_4 + n_6 &= |k| \\
n_2 + n_4 + n_5 &= |l| = |k| - q.
\end{align}
These imply that $n_6 = n_5 + q$ and $n_3 = n_2 + |i| - |k|$. For a given pair of $|i|$ and $|k|$, we are left with two constraints on the remaining four numbers:

\[
\begin{align*}
n_1 + 2n_2 + n_4 + 2n_5 &= 2S + |k| - |i| - q \\
n_2 + n_4 + n_5 &= |k| - q.
\end{align*}
\]  

(24)

These lead to the overall sums

\[
\int |qKq|^2 dU = \frac{2K + 1}{2S + 1} \sum_{i, k = 0}^{2S} \left[ \left( \frac{2S}{i - q} \right) \left( \frac{2S}{k - q} \right) \left( \frac{2S}{k} \right) \right]^{-1/2} C_{S,S-|i|-2M} C_{S,S-|j|-2M} C_{S,S-|k|-2M} C_{S,S-|l|-2M} \frac{1}{2} \left( \frac{2S}{i} \right) \\
\times \sum_{n_2, n_5} \left( \frac{2S - |i| - n_2 - n_5, n_2, n_5}{2S - |i| - n_2 - n_5, n_2, n_5 + |i| - |k|}, |k| - q - n_2 - n_5, n_5 + q \right).
\]  

(25)

These can be explicitly calculated by summing the $O(S^4)$ terms for each $K$ and $q$. While all pure spin-$S$ states satisfy $A_{2S}^{(S)} = 1 - A_{0}^{(S)} = 2S/(2S + 1)$, the projection of a 2S-qubit state will in general have $A_{2S}^{(S)} < 2S/(2S + 1)$.

An alternate counting argument yields an even simpler result, reducing the number of sums by one, as follows. First, there are $\binom{2S}{|i|}$ choices for the locations of the 0s and 1s in $i$. Then, there are $\binom{2S-|i|-|k|-2M}{|i|-2M}$ choices for which of the locations of the 1s of $i$ match those of $i$ and $\binom{2S-|i|-|k|-2M}{|i|-2M}$ choices for which the locations of the 1s of $i$ do not match. Both of $j$ and $k$ must have 1s at the $M$ matching locations, leaving $\binom{|i|+|j|+|k|}{|i|} \binom{|i|+|j|+|k|-2M}{|i|-2M}$ choices for the matching locations. The remaining 1s of $j$ and $k$ must have 1s at the $M$ matching locations, leaving $\binom{|i|+|j|+|k|-2M}{|i|-2M}$ choices for the remaining 1s of $j$ and $k$ among the as-yet unpaired 1s of $i$ and $l$ (note: $|i| + |j| = |j| + |k|$). The extra factors of 2 arise $M$ times from the terms with $i_m = j_m = k_m = l_m = 1$ and $2S - |i| - |j| + M$ times when $i_m = j_m = k_m = l_m = 0$.

These considerations lead to the overall sums

\[
\prod_{m=1}^{2S} \delta_{i_m, j_m} \delta_{k_m l_m} + \delta_{i_m k_m} \delta_{j_m l_m} = \delta_{|i|+|j|, |k|+|k|} \sum_{M} \binom{2S-|i|-|k|-2M}{|i|-2M} \binom{|i|+|j|+|k|-2M}{|i|-2M} |M|^{4M},
\]  

where the constraints on the allowed values of $M$ are directly enforced by the binomial coefficients. Finally,

\[
\int |qKq|^2 dU = \frac{2K + 1}{2S + 1} \sum_{n, n' = -S}^{S} \left[ \left( \frac{2S}{S + m + q} \right) \left( \frac{2S}{S + m' + q} \right) \right]^{-1/2} C_{S,m+q} C_{S,m+q} C_{S,m+q} C_{S,m+q} \frac{2m+2q}{6S} \left( \frac{2S}{S + m} \right) \\
\times \sum_{M} \binom{S - m - q - M}{S - m - q - M} \binom{S + m + q}{S + m + q} \binom{2S - m - m' - q - 2M}{S - m - q - M}^{4M}
\]  

(27)

These averaged multipoles can now be computed by summing only $O(S^3)$ terms.

To facilitate a comparison of the relative strengths of the multipoles of different orders, we artificially normalize all of the multipoles by an $S$-dependent constant such that the strengths of the multipoles versus $K$ can be compared to the same relationship for CS and states with random Majorana constellations.

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