Extremal projectors of two-parameter Kashiwara algebras

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Abstract

We define a two-parameter Kashiwara algebra $B_{r,s}(g)$ and its extremal projector, and study their basic properties. Applying their properties to the category $O(B_{r,s}(g))$, whose objects are “upper bounded” $B_{r,s}(g)$-modules, we obtain its semi-simplicity and the classification of simple modules.

Keywords: Two-parameter quantum groups; Two-parameter Kashiwara algebras; The skew Hopf pairing; Extremal projectors

1 Introduction

The notion of quantum groups was introduced by Drinfeld and Jimbo, independently, around 1985 in their study of the quantum Yang-Baxter equations and solvable lattice models. Quantum groups $U_q(g)$, depending on a single parameter $q$, are certain families of Hopf algebras that are deformations of universal enveloping algebras of symmetrizable Kac-Moody algebras $g$.

In [K], Kashiwara introduced the so-called Kashiwara algebra $B_q^r(g)$ (he called it the reduced $q$-analogue) and gave the projector $P$ for the Kashiwara algebra of $sl_2$-case in order to define the crystal basis of $U_q^-$. Moreover, let $O(B_q^r(g))$ be the category of $B_q^r(g)$-modules satisfying a finiteness condition, then he affirmed without proof that $O(B_q^r(g))$ is semi-simple and $U_q^-(g)$ is a unique isomorphic classes of simple objects of $O(B_q^r(g))$.

In [N1], Nakashima studied the so-called $q$-Boson Kashiwara algebra $B_q(g)$ and found therein an interesting object $\Gamma$. In [N2], he re-defined the extremal projector $\Gamma$ for $B_q(g)$, clarified its properties and applied it to the representation theory of $q$-Boson Kashiwara algebras. By using the properties of $\Gamma$, he showed that the category $O(B_q(g))$, whose objects are “upper bounded” $B_q(g)$-modules, is semi-simple and classified its simple modules (see also [F]).

From down-up algebras approach, Berkart and Witherspoon recovered Takeuchi’s definition of two-parameter quantum groups of type A and investigated their structures and finite dimensional representation theory in
[BW1, BW2, BW3, BW4]. Since then, a systematic study of the two-parameter quantum groups of other types has been going on. For instance, Bergeron, Gao and Hu developed the corresponding theory of two-parameter quantum groups for type B, C, D in [BGH1, BGH2]. Later on, Hu et al. continued this project (see [HS, BH] for exceptional types $G_2, E_6, E_7, E_8$). In [HP1, HP2], Hu and Pei give a simpler and unified definition for a class of two-parameter quantum groups $U_{r,s}(g)$ associated to a finite-dimensional semi-simple Lie algebra $g$ in terms of the Euler form.

In this paper, we will generalize Nakashima’s results to the two-parameter case. Specifically, we define a two-parameter Kashiwara algebra $B_{r,s}(g)$ and its extremal projector $\Gamma$, and study their basic properties. Applying their properties to the category $\mathcal{O}(B_{r,s}(g))$, whose objects are “upper bounded” $B_{r,s}(g)$-modules, we can give a proof of its semi-simplicity and the classification of simple modules in it.

The organization of this article is as follows. In Sect. 2, we review the definition of the two-parameter quantum group and define the two-parameter Kashiwara algebra $B_{r,s}(g)$ associated to a finite-dimensional semi-simple Lie algebra $g$ and study their properties. In Sect. 3, we introduce the category $\mathcal{O}(B_{r,s}(g))$ of $B_{r,s}(g)$-modules, which we treat in the sequel. In Sect. 4, we review the so-called skew Hopf pairing for the two-parameter quantum group and study its basic properties, especially its non-degeneracy, which plays a crucial role in the next section. In Sect. 5 we define some element $C$ in the tensor product of the two-parameter Kashiwara algebras, which plays a significant role of studying extremal projectors. In the end of this section, we will define the quantum Casimir element $\Omega$. In Sect. 6, we define the extremal projector $\Gamma$ for $B_{r,s}(g)$ and involve its important properties (see Theorem 6.1). In the last section, we apply it to show the semi-simplicity of the category $\mathcal{O}(B_{r,s}(g))$ and classify the simple modules in $\mathcal{O}(B_{r,s}(g))$ (see Theorem 7.1).

2 Two-parameter quantized algebras and two-parameter Kashiwara algebras

Let us start with some notations. For $n > 0$, set

$$(n)_v = 1 + v + v^2 + \cdots + v^{n-1} = \frac{v^n - 1}{v - 1};$$

$$(n)_v^1 = (1)_v(2)_v \cdots (n)_v \quad \text{and} \quad (0)_v^1 = 1;$$

$$\left(\begin{array}{c} n \\ k \end{array}\right)_v = \frac{(n)_v^1}{(k)_v^1(n-k)_v^1} \quad \text{for } n \geq k \geq 0.$$

Let $g$ be a finite-dimensional simple Lie algebra over the field $\mathbb{Q}$ of rational numbers and $A = (a_{ij})_{i,j \in I}$ be the corresponding Cartan matrix of
finite type. Let \( \{d_i|i \in I\} \) be a set of relatively prime positive integers such that \( d_i \alpha_j = d_j \alpha_i \) for \( i, j \in I \). Let \( \Pi = \{\alpha_i|i \in I\} \) be the set of simple roots, \( Q = \oplus_{i \in I} \mathbb{Z} \alpha_i \) a root lattice, \( Q^+ = \oplus_{i \in I} \mathbb{N} \alpha_i \) the semigroup generated by positive roots, and \( \Lambda \) a weight lattice.

Let \( \mathbb{Q}(r, s) \) be the rational function field with two parameters \( r, s \). Let \( \mathbb{K} \) be an algebraically closed field of characteristic \( 0 \) containing \( \mathbb{Q}(r, s) \) as a subfield with assumptions \( r^3 \neq \pm s^3 \) and \( (rs^{-1})\frac{1}{d} \in \mathbb{K} \) with some \( m \in \mathbb{Z}_{>0} \), where \( m \) is the possibly smallest positive integer such that \( m\Lambda \subseteq Q \). We always assume that \( rs^{-1} \) is not a root of unity throughout this article.

Denote \( r_i = r^{d_i}, s_i = s^{d_i} \) for \( i \in I \). As in [HP1], let \( \langle \cdot, \cdot \rangle \) be the Euler bilinear form on \( Q \times Q \) defined by

\[
\langle i, j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} 
    d_i a_{ij} & \text{if } i < j; \\
    d_i & \text{if } i = j; \\
    0 & \text{if } i > j.
\end{cases}
\]

For \( \lambda \in \Lambda \), we linearly extend the bilinear form \( \langle \cdot, \cdot \rangle \) to be defined on \( \Lambda \times \Lambda \) such that \( \langle \lambda, i \rangle = \langle \lambda, \alpha_i \rangle = \frac{1}{m} \sum_{j=1}^{n} a_{j} \langle j, i \rangle \), or \( \langle i, \lambda \rangle = \langle \alpha_i, \lambda \rangle = \frac{1}{m} \sum_{j=1}^{n} a_{j} \langle i, j \rangle \) for \( \lambda = \frac{1}{m} \sum_{j=1}^{n} a_{j} \alpha_j \) with \( a_j \in \mathbb{Z} \).

Now we introduce the symbols \( \{e_i, e''_i, f_i, f'_i, \omega^\pm_i, \omega'^\pm_i (i \in I)\} \). These symbols satisfy the following relations:

\[
\omega_i^\pm \omega_i'^\pm = \omega_i'^\pm \omega_i^\pm = 1, \quad \{\omega_i, \omega_j\} = \{\omega_i', \omega_j'\} = [\omega_i, \omega_j] = 0; \quad (2.1)
\]

\[
\omega_i e_j \omega_i'^{-1} r^{(j,i)} s^{-(j,i)} e_j, \quad \omega_i' e_j \omega_i'^{-1} = r^{-(j,i)} s^{(j,i)} e_j; \quad (2.2)
\]

\[
\omega_i f_j \omega_i'^{-1} = r^{-(j,i)} s^{(j,i)} f_j, \quad \omega_i' f_j \omega_i'^{-1} = r^{(j,i)} s^{-(j,i)} f_j; \quad (2.3)
\]

\[
\omega_i f_j' \omega_i'^{-1} = r^{-(j,i)} s^{(j,i)} f_j', \quad \omega_i' f_j' \omega_i'^{-1} = r^{(j,i)} s^{-(j,i)} f_j'; \quad (2.4)
\]

\[
[e_i, f_j] = \delta_{ij} \frac{\omega_i - \omega_i'}{r^{(j,i)} s^{-(j,i)}}, \quad (2.6)
\]

\[
e''_i f_j = r^{(j,i)} s^{-(j,i)} f_j e''_i + \delta_{ij}; \quad (2.7)
\]

\[
f'_i e_j = r^{(j,i)} s^{-(j,i)} e_j f'_i + \delta_{ij}; \quad (2.8)
\]

\[
\sum_{k=0}^{1-\alpha_{ij}} (-1)^k \binom{1-\alpha_{ij}}{k} r_{s_i}^{-k} c_{ij}^k X_i^k X_j X_i^{1-\alpha_{ij}-k} = 0 \quad (i \neq j) \quad \text{for } X_i = e''_i, f_i; \quad (2.9)
\]

\[
\sum_{k=0}^{1-\alpha_{ij}} (-1)^k \binom{1-\alpha_{ij}}{k} r_{s_i}^{-k} c_{ij}^k Y_i^{-\alpha_{ij}-k} Y_j Y_i^k = 0 \quad (i \neq j) \quad \text{for } Y_i = e_i, f'_i; \quad (2.10)
\]

where \( c_{ij}^k = r_{s_i}^{-k} \binom{k(k-1)}{2} r^{(j,i)} s^{-(j,i)} \), for \( i \neq j \).
Now we shall define these algebras playing a significant role in this paper.
The algebra $B_{r,s}(\mathfrak{g})$ (resp. $\overline{B}_{r,s}(\mathfrak{g})$) is a unital associative algebra over $\mathbb{K}$ generated by the symbols $\{e_i^\prime, f_i^\prime, \omega_i^{\pm 1}\}_{i \in I}$ (resp. $\{e_i, f_i, \omega_i^{\pm 1}\}_{i \in I}$) with the defining relations (2.1), (2.2), (2.4), (2.6), (2.9) (resp. (2.1), (2.2), (2.4), (2.7) and (2.9)). As in [HP1] and [HP2], the algebra $U_{r,s}(\mathfrak{g})$ is the two-parameter quantum group over $\mathbb{K}$ generated by the symbols $\{e_i, f_i, \omega_i^{\pm 1}\}_{i \in I}$ with the defining relations (2.1), (2.2), (2.4), (2.6), (2.9) and (2.10). We shall call algebras $B_{r,s}(\mathfrak{g})$ and $\overline{B}_{r,s}(\mathfrak{g})$ the two-parameter Kashiwara algebras. Furthermore, we define the following subalgebras:

$$T = \langle \omega_i^{\pm 1}, \omega_i^{\prime \pm 1} | i \in I \rangle = B_{r,s}(\mathfrak{g}) \cap \overline{B}_{r,s}(\mathfrak{g}) \cap U_{r,s}(\mathfrak{g});$$
$$B_{r,s}(\mathfrak{g}) \cap \overline{B}_{r,s}(\mathfrak{g}) = \langle e''_i, f_i (\text{resp. } e_i, f'_i) | i \in I \rangle \subset B_{r,s}(\mathfrak{g}) \cap \overline{B}_{r,s}(\mathfrak{g});$$
$$U_{r,s}^+(\mathfrak{g})(\text{resp. } U_{r,s}^-(\mathfrak{g})) = \langle e_i (\text{resp. } f_i) | i \in I \rangle =: \overline{B}_{r,s}^+(\mathfrak{g})(\text{resp. } B_{r,s}^-);$$
$$U_{r,s}^+(\mathfrak{g})(\text{resp. } U_{r,s}^-(\mathfrak{g})) = \langle e_i (\text{resp. } f_i) | i \in I \rangle =: \overline{B}_{r,s}^+(\mathfrak{g})(\text{resp. } B_{r,s}^-)(\text{resp. } B_{r,s}^-);$$
$$B_{r,s}^+(\mathfrak{g})(\text{resp. } B_{r,s}^-)(\text{resp. } B_{r,s}^-;$$

We will use the abbreviated notations $U, B, \overline{B}, \overline{B'}, \cdots$ for $U_{r,s}(\mathfrak{g}), B_{r,s}(\mathfrak{g}), \overline{B}_{r,s}(\mathfrak{g}), B_{r,s}(\mathfrak{g}), \overline{B}_{r,s}(\mathfrak{g}), \cdots$ if there is no confusion.

For each $\mu = \sum_{i \in I} \mu_i \alpha_i \in Q$, we define elements $\omega_\mu$ and $\omega'_\mu$ by

$$\omega_\mu = \prod_{i \in I} \omega_i^{\mu_i}, \quad \omega'_\mu = \prod_{i \in I} \omega_i^{'\mu_i}.$$  

For $\beta = \sum_{i \in I} m_i \alpha_i \in Q^+$, we set $|\beta| = \sum_i m_i$ and

$$U_{\beta}^+ = \{ x \in U^+ | \omega_\mu x \omega_\mu^{-1} = r^{(\beta, \mu)} s^{-(\mu, \beta)} x, \omega'_\mu x \omega'_\mu^{-1} = r^{-(\mu, \beta)} s^{(\beta, \mu)} x, \forall \mu \in Q \};$$
$$U_{\beta}^- = \{ y \in U^- | \omega_\mu y \omega_\mu^{-1} = r^{-(\beta, \mu)} s^{(\mu, \beta)} y, \omega'_\mu y \omega'_\mu^{-1} = r^{(\mu, \beta)} s^{-(\beta, \mu)} y, \forall \mu \in Q \};$$

and call $|\beta|$ the height of $\beta$ and $U_{\beta}^+$ (resp. $U_{\beta}^-$) a weight space of $U^+$ (resp. $U^-$) with a weight $\beta$ (resp. $-\beta$). We also define $B_{\beta}^+$ and $\overline{B}_{-\beta}$ in a similar manner.

**Proposition 2.1. (a)** We have the following algebra homomorphisms $\Delta : U \rightarrow U \otimes U, \Delta^r : B \rightarrow B \otimes U, \Delta^l : \overline{B} \rightarrow U \otimes \overline{B}, \Delta^b : U \rightarrow \overline{B} \otimes B$ and $\varepsilon : U \rightarrow \mathbb{K}$, which are given by

$$\Delta(\omega_i^{\pm 1}) = \Delta^r(\omega_i^{\pm 1}) = \Delta^l(\omega_i^{\pm 1}) = \Delta^b(\omega_i^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1};$$
$$\Delta(\omega_i^{\prime \pm 1}) = \Delta^r(\omega_i^{\prime \pm 1}) = \Delta^l(\omega_i^{\prime \pm 1}) = \Delta^b(\omega_i^{\prime \pm 1}) = \omega_i^{\prime \pm 1} \otimes \omega_i^{\prime \pm 1};$$
$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes \omega_i + 1 \otimes f_i;$$
$$\Delta^r(e_i^\prime) = (r_i - s_i) \cdot 1 \otimes \omega_i^{\prime -1} e_i + e_i^{\prime \prime} \otimes \omega_i^{\prime -1}, \quad \Delta^r(f_i) = f_i \otimes \omega_i^{\prime} + 1 \otimes f_i;$$

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\( \Delta^{(i)}(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta^{(i)}(f'_i) = (r_i - s_i)\omega_i^{i-1} f_i \otimes 1 + \omega_i^{i-1} \otimes f'_i; \)

\( \Delta^{(b)}(e_i) = \omega_i \otimes \frac{\omega_i^m}{s_i - r_i} + e_i \otimes 1, \quad \Delta^{(b)}(f_i) = 1 \otimes f_i + \frac{\omega_i^{m-1}}{s_i - r_i} \otimes \omega_i^{i}; \)

\( \varepsilon(\omega_i^{\pm 1}) = 1, \quad \varepsilon(e_i) = 0. \)

(b) We have the following anti-isomorphisms \( S : U \rightarrow U \) and \( \varphi : B \rightarrow B, \)

which are given by

\[
S(e_i) = -\omega_i^{-1}e_i, \quad S(f_i) = -f_i\omega_i^{-1}, \quad S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \quad S(\omega_i^{m}) = \omega_i^{m+1}; \quad \varphi(e_i) = -\frac{1}{s_i - r_i}e_i, \quad \varphi(f_i) = -(r_i - s_i)f_i, \quad \varphi(\omega_i^{m}) = \omega_i^{m+1}, \quad \varphi(\omega_i^{m}) = \omega_i^{m-1}. 
\]

**Proof.** By direct calculations, it is not difficult for us to check all the defining relations. Maybe it is the most complicated to check the Serre relations directly. Instead we could first check that the relations \((A1), (A2), \ldots, (A13)\) in \([N2, Appendix A]\) are also true in our case and using them it is relatively easier to check the Serre relations. We will omit the details here. \(\square\)

**Remark 2.1.** As mentioned in \([HP1]\), \( (\Delta, S, \varepsilon), \) which has been given as above, gives a Hopf structure on \( U_{r,s}(g) \).

**Proposition 2.2.** (see \([HP1, Corollary 2.6]\)) \( U_{r,s}(g) \) has the standard triangular decomposition \( U_{r,s}(g) \cong U_{r,s}^- (g) \otimes T \otimes U_{r,s}^+ (g) \).

**Lemma 2.1.** If we set \( f^{(m)}_i = f^m_i / (m)_{r_i s_i}^{-1} \) for \( m \geq 0 \) and \( f^{(m)}_i = 0 \) for \( m < 0 \), then we have the following commutation relations:

\[
e^{m}_i f^{(m)}_j = \begin{cases} \sum_{\nu=0}^{\min(n, m)} (r_i s_i^{-1})(n-\nu)(m-\nu)\binom{n}{\nu} f^{(m-\nu)}_j e^{m-\nu}_i & \text{if } i = j; \\ r^{n(j,i)} s^{-n(i,j)} f^{(m)}_j e^{m}_i & \text{if } i \neq j. \end{cases}
\]

**Proof.** Using (2.7), we get \( e^{m}_i f_j = r^{n(j,i)} s^{-n(i,j)} f_j e^{m}_i + \delta_{ij} (n)_{r_i s_i}^{-1} e^{m-1}_i. \)

By this formula and the following identity:

\[
\binom{m + 1}{n}_{r_i s_i^{-1}} = \binom{m}{n}_{r_i s_i^{-1}} + (r_i s_i^{-1})^{m+1-n} \binom{m}{n-1}_{r_i s_i^{-1}},
\]

we can easily prove this lemma by induction on \( m \). \(\square\)

By Lemma 2.1 and the standard argument, we can obtain the following triangular decomposition of the two-parameter Kashiwara algebras.

**Proposition 2.3.** The multiplication map defines an isomorphism of vector spaces

\[
B_{r,s}^- (g) \otimes T \otimes B_{r,s}^+ (g) \rightarrow B_{r,s} (g), \quad u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3.
\]

We define weight completions of \( L^{(1)} \otimes \cdots \otimes L^{(m)} \), where \( L^{(l)} = B \) or \( U \).

\[
\hat{L}^{(1)} \otimes \cdots \otimes \hat{L}^{(m)} := \lim_{\overline{l}} L^{(1)} \otimes \cdots \otimes L^{(m)} / (L^{(1)} \otimes \cdots \otimes L^{(m)}) L^{+,l}
\]
where $L^{+,l} = \oplus_{|\beta_1|+\cdots+|\beta_m| \geq l} (L^{(1)})_{\beta_1}^+ \otimes \cdots \otimes (L^{(m)})_{\beta_m}^+$. Using the triangular decomposition in Proposition 2.2 and 2.3, it is not difficult to see that the linear maps $\Delta, \Delta^{(r)}, S, \varphi$, the multiplication, etc. are naturally extended for such completions.

3 Category $\mathcal{O}(B)$

Let $\mathcal{O}(B) = \mathcal{O}(B_{r,s}(\mathfrak{g}))$ be the category of left $B$-modules such that

1. Any object $M$ has a weight space decomposition $M = \oplus_{\lambda \in \Lambda} M_{\lambda}$, where $M_{\lambda} = \{u \in M\mid \omega_{\mu} u = r^{(\lambda, \mu)} s^{-1(\mu, \lambda)} u, \omega_{\mu} u = r^{-(\mu, \lambda)} s^{(\lambda, \mu)} u, \forall \mu \in Q\}$ and $\dim M_{\lambda} < \infty$ for any $\lambda \in \Lambda$.

2. For any object $u \in M$ there exists $l > 0$ such that $e_{i_1}^r e_{i_2}^r \cdots e_{i_l}^r u = 0$ for any $i_1, i_2, \ldots, i_l \in I$.

In a single parameter $q$ case, the similar category $\mathcal{O}(B_q(\mathfrak{g}))$ was introduced in [N2], in which he also gave a proof of the semi-simplicity of $\mathcal{O}(B_q(\mathfrak{g}))$. In Section 7, we will give a proof of the semi-simplicity of $\mathcal{O}(B_{r,s}(\mathfrak{g}))$.

For each $\lambda \in \Lambda$, we define the $B$-module $H(\lambda)$ by $H(\lambda) := B/I_\lambda$, where the left ideal $I_\lambda$ is defined as

$$I_\lambda = \sum_{i \in I} B e_i^r + \sum_{\mu \in Q} B (\omega_{\mu} - r^{(\lambda, \mu)} s^{-1(\mu, \lambda)}) + \sum_{\mu' \in Q} B (\omega_{\mu'} - s^{-(\mu', \lambda)} s^{(\lambda, \mu')}).$$

In Section 7, we shall also show that $\{H(\lambda)\mid \lambda \in \Lambda\}$ is a set of representatives of isomorphism classes of simple modules of $\mathcal{O}(B)$.

4 Non-degeneracy of the skew Hopf pairing

**Proposition 4.1.** (see [HP1, Proposition 2.4]) There exists a unique bilinear form

$$\langle \cdot, \cdot \rangle : U^\geq \times U^\leq \longrightarrow \mathbb{K}$$

satisfying the following relations:

$$\langle x, y_1 y_2 \rangle = \langle \Delta(x), y_1 \otimes y_2 \rangle \quad (x \in U^\geq, \ y_1, y_2 \in U^\leq);$$

$$\langle x_1 x_2, y \rangle = \langle x_2 \otimes x_1, \Delta(y) \rangle \quad (x_1, x_2 \in U^\geq, \ y \in U^\leq);$$

$$\langle \omega_{\mu}, \omega_{\mu'} \rangle = r^{(\mu, \mu')} s^{-1(\mu, \mu')} \quad (\mu, \nu \in Q);$$

$$\langle e_i, \omega_{\mu} \rangle = \langle \omega_{\mu}, e_i \rangle = 0 \quad (\forall \ i \in I \text{ and } \mu \in Q);$$

$$\langle e_i, f_j \rangle = \delta_{ij} \frac{1}{s^{-1}} \quad (\forall \ i, j \in I).$$

We will call this bilinear form the skew Hopf pairing of $U$, which enjoys the following properties.
Lemma 4.1. (a) \( \langle S(x), S(y) \rangle = \langle x, y \rangle \) for any \( x \in U^\geq, y \in U^\leq \).

(b) For any \( x \in U^\geq, y \in U^\leq \), we have

\[
    xy = \sum_{x(2), y(2)} \langle x(0), S(y(0)) \rangle \langle x(2), y(2) \rangle x(1)y(1);
\]

\[
    yx = \sum_{x(2), y(2)} \langle x(0), S(y(0)) \rangle \langle x(2), y(2) \rangle x(1)y(1);
\]

where \((\Delta \otimes 1) \circ \Delta(x) = \sum_{x(n)} x(0) \otimes x(1) \otimes x(2) \) and \((\Delta \otimes 1) \circ \Delta(y) = \sum_{y(n)} y(0) \otimes y(1) \otimes y(2)\).

(c) \( \langle x \omega_\nu, y \omega_\mu \rangle = r^{(\mu, \nu)} S^{-(\nu, \mu)} \langle x, y \rangle \) for any \( x \in U^+, y \in U^-, \mu, \nu \in Q \).

(d) \( \langle U_\gamma^+, U^-_\delta \rangle = 0 \) for \( \gamma, \delta \in Q^+ \) with \( \gamma \neq \delta \).

**Proof.** (a) Set \( \langle x, y \rangle' = \langle S(x), S(y) \rangle \). Since \( \langle \cdot, \cdot \rangle' \) satisfies the relations in Proposition 4.1, we have \( \langle x, y \rangle' = \langle x, y \rangle \).

(b) This is clear when \( x \) and \( y \) are the generators of \( U^\geq \) and \( U^\leq \) respectively. The general case is proved by induction using Proposition 4.1.

(c) and (d) could also be easily proved by induction. \( \square \)

The following proposition, which plays a crucial role in the next section, was established for a class of multi-parameter quantum groups (including the two-parameter quantum groups considered in this paper) in [PHR, Proposition 44], and in it they generalized the corresponding one for the two-parameter quantum groups of types \( A, B, C, D \) developed in [BGH2]. Later on, Hu and Pei have given a new and simple proof for the existence of non-degenerate skew Hopf pairing on \( U_{r,s}(\mathfrak{g}) \) (see [HP2, Proposition 38])

**Proposition 4.2.** (see [PHR, Proposition 44] and [HP2, Proposition 38])

For any \( \beta \in Q^+ \), the restriction \( \langle \cdot, \cdot \rangle|_{U_\beta^+ \times U^-_\beta} \) of the pairing \( \langle \cdot, \cdot \rangle \) to \( U_\beta^+ \times U^-_\beta \) is non-degenerate.

### 5 The element \( C \) and the quantum Casimir element \( \Omega \)

In this section, for each \( \beta = \sum_i m_i \alpha_i \in Q^+ \), we will set \( k_\beta := \prod_i \omega_i^{-m_i} \).

Note that \( k_\beta = \omega_\beta^{-1} \). Let \( \{ x_r^\beta \} \) be a basis of \( U_\beta^+ \) and let \( \{ y_r^- \} \) be the dual basis of \( U^-_\beta \) with respect to the skew Hopf pairing owing to Proposition 4.2. We denote the canonical element in \( U_\beta^+ \otimes U^-_\beta \) with respect to the skew Hopf pairing by

\[
    C_\beta := \sum_r x_r^\beta \otimes y^{-\beta}_r.
\]

**Lemma 5.1.** (a) For any \( \beta \in Q^+ \), let \( C_\beta \) be as above and set \( C_\beta' := (1 \otimes S^{-1})(C_\beta) \) and \( C_\beta'' := (\varphi \otimes 1)(C_\beta) \). Then for any \( \beta \in Q^+ \) and \( i \in I \), we have

\[
    [\omega_i^{-1} \otimes e_i'', (1 \otimes k_\beta^{-1} + \alpha_i) \cdot C_{\beta + \alpha_i}] = (1 \otimes k_\beta^{-1})(C_\beta)(\omega_i e_i \otimes (r_i - s_i) \cdot 1); \quad (5.1)
\]
[f_i \otimes \omega_i, C_{\beta+\alpha_i}'' \cdot (1 \otimes k_{\beta+\alpha_i})] = -(1 \otimes f_i)(C_{\beta}''(1 \otimes k_{\beta}). \quad (5.2)

(b) For any \( \beta \in Q^+ \) and \( i \in I \), we have

\[ [1 \otimes e_i, C_{\beta+\alpha_i}] = C_\beta \cdot (e_i \otimes \omega_i) - (e_i \otimes \omega_i) \cdot C_\beta; \quad (5.3) \]
\[ [f_i \otimes 1, C_{\beta+\alpha_i}] = C_\beta \cdot (\omega_i \otimes f_i) - (\omega_i \otimes f_i) \cdot C_\beta. \quad (5.4) \]

(c) For any \( \beta \in Q^+ \), we have

\[
\sum_{\gamma, \delta \in Q^+_{\gamma+\delta=\beta}} C_{\gamma} \cdot (\omega_{\delta} \otimes 1)(S \otimes 1)(C_{\delta}) = \begin{cases} 1 & \text{for } \beta = 0, \\ 0 & \text{for } \beta \neq 0; \end{cases} \quad (5.5)
\]

\[
\sum_{\gamma, \delta \in Q^+_{\gamma+\delta=\beta}} (\omega_{\gamma} \otimes 1)(S \otimes 1)(C_{\gamma}) \cdot C_{\gamma} = \begin{cases} 1 & \text{for } \beta = 0, \\ 0 & \text{for } \beta \neq 0. \end{cases} \quad (5.6)
\]

Proof. (a) We first give the proof of (5.1). Since both sides are contained in \( TU^+_{\beta+\alpha_i} \otimes U \), it is sufficient to show that they coincide after applying \( (\cdot, z) \otimes 1 \) for any \( z \in U_{-\beta-\alpha} \), then we obtain

\[
(\langle \cdot, z \rangle \otimes 1)(\text{L.H.S. of (5.1)}) = \sum_r \langle \omega_i^{-1} x^\beta \omega_i^{-1}, z \rangle \otimes e_i'' k_{\beta+\alpha_i}'' S^{-1}(y_r^{-\beta-\alpha_i})
\]

\[
- \sum_r \langle x^\beta \omega_i^{-1}, z \rangle \otimes k_{\beta+\alpha_i}^{-1} S^{-1}(y_r^{-\beta-\alpha_i}) e_i''
\]

(\text{using Lemma 4.1 (c)})

\[
= r^{-1} S^{-1}(z) S^{-1}(z) e_i''
\]

\[
= k_{\beta+\alpha_i}^{-1} (e_i'' S^{-1}(z) - S^{-1}(z) e_i''),
\]

\[
(\langle \cdot, z \rangle \otimes 1)(\text{R.H.S. of (5.1)}) = \sum_r \langle x^\beta \omega_i^{-1} e_i, z \rangle \otimes (r_i - s_i) k_{\beta}^{-1} S^{-1}(y_r^{-\beta}). \quad (5.7)
\]

For \( z \in U_{-\beta-\alpha} \), we can define \( v \in U_{-\beta} \) uniquely by

\[
\Delta(z) = 1 \otimes z + f_i \otimes v \omega_i + \cdots,
\]

where \( \cdots \) implies terms whose left components are elements of \( \bigoplus_{\beta \neq 0, \alpha_i} U_{-\beta} \). By the properties of the skew Hopf pairing, we have

\[
\langle x^\beta \omega_i^{-1} e_i, z \rangle = \langle e_i \otimes x^\beta \omega_i^{-1}, \Delta(z) \rangle
\]

\[
= \langle e_i \otimes x^\beta \omega_i^{-1}, 1 \otimes z + f_i \otimes v \omega_i + \cdots \rangle
\]

\[
= \langle e_i, f_i \rangle \langle x^\beta \omega_i^{-1}, v \omega_i \rangle
\]

\[
= \frac{r_i^{-1} s_i}{s_i - r_i} \langle x^\beta, v \rangle.
\]

Thus, we have

\[
\text{R.H.S. of (5.7)} = -r_i^{-1} s_i k_{\beta}^{-1} S^{-1}(v).
\]
In order to complete the proof of (5.1), it is sufficient to show

$$e_i''S^{-1}(z) - S^{-1}(z)e_i'' = -r_i^{-1}s_i\omega_i^{-1}S^{-1}(v). \quad (5.8)$$

Without loss of generality, we may assume that $z$ is of the form $z = f_1f_2 \cdots f_k \in U_{-\beta - \alpha_i}$ ($\beta + \alpha_i = \alpha_1 + \cdots + \alpha_k$). For $\beta = \sum_j m_j\alpha_j$, we shall show this by induction on $m_i$.

If $m_i = 0$, $z$ is in the form $z = z'f_iz''$, where $z'$ and $z''$ are monomials of $f_i$'s not including $f_i$. By $S^{-1}(f_j) = -\omega_j^{-1}f_j$ and $e_i''(\omega_j^{-1}f_j) = (\omega_j^{-1}f_j)e_i''$ ($i \neq j$) we have

$$e_i''S^{-1}(z') = S^{-1}(z')e_i'', \quad e_i''S^{-1}(z'') = S^{-1}(z'')e_i''.$$ 

Hence, we obtain

$$e_i''S^{-1}(z) = S^{-1}(z'')(e_i''\omega_i^{-1}f_i)S^{-1}(z')$$
$$= S^{-1}(z'')(\omega_i^{-1}f_ie_i'' - r_i^{-1}s_i\omega_i^{-1})S^{-1}(z')$$
$$= S^{-1}(z'')(\omega_i^{-1}f_iS^{-1}(z')e_i'' - r_i^{-1}s_i\omega_i^{-1}S^{-1}(z'')\omega_i^{-1}S^{-1}(z')$$
$$= S^{-1}(z'')S^{-1}(f_iS^{-1}(z')e_i'' - r_i^{-1}s_i\omega_i^{-1}(\alpha_i\beta''\omega_i^{-1}S^{-1}(z'')S^{-1}(z',z'')).$$

where $\beta'' = -\text{wt}(z'')$. Therefore, for $m_i = 0$, we have

L.H.S. of (5.8) = $-r_i^{-1}s_i\omega_i^{-1}(\alpha_i\beta''\omega_i^{-1}S^{-1}(z',z'')$.

In the case $m_i = 0$, we can easily obtain $v = r(\alpha_i\beta'')s^{-\beta''}S^{-1}(z',z'')$ and then

R.H.S. of (5.8) = $-r_i^{-1}s_i\omega_i^{-1}. r(\alpha_i\beta'')s^{-\beta''}S^{-1}(z',z'') = \text{L.H.S. of (5.8)}$.

Thus, the case $m_i = 0$ has been shown.

Suppose that $m_i > 0$. We divide $z = z'z''$ such that $m_i' < m_i$ and $m_i'' < m_i$, where $m_i'$ (resp. $m_i''$) is the number of $f_i$ including in $z'$ (resp. $z''$). Writing

$$\Delta(z') = 1 \otimes z' + f_i \otimes v'\omega_i' + \cdots,$$

$$\Delta(z'') = 1 \otimes z'' + f_i \otimes v''\omega_i' + \cdots,$$

and calculating $\Delta(z',z'')$ directly, we obtain

$$v = z'v'' + r(\alpha_i\beta'')s^{-\beta''}v'z''.$$  

By the hypothesis of the induction, we obtain

$$e_i''S^{-1}(z) = e_i''S^{-1}(z'')S^{-1}(z') = (S^{-1}(z'')e_i'' - r_i^{-1}s_i\omega_i^{-1}S^{-1}(v''))S^{-1}(z')$$
$$= S^{-1}(z'')e_i''S^{-1}(z') - r_i^{-1}s_i\omega_i^{-1}S^{-1}(z'v'')$$
$$= S^{-1}(z'')(S^{-1}(z')e_i'' - r_i^{-1}s_i\omega_i^{-1}S^{-1}(v')) - r_i^{-1}s_i\omega_i^{-1}S^{-1}(z'v'')$$
$$= S^{-1}(z'z'')e_i'' - r_i^{-1}s_i\omega_i^{-1}(S^{-1}(z'v'') + r(\alpha_i\beta'')s^{-\beta''}S^{-1}(v'z''))$$
$$= S^{-1}(z)e_i'' - r_i^{-1}s_i\omega_i^{-1}S^{-1}(v).$$
Now we give the proof of (5.2). Since both sides are contained in \( U \otimes U_{-\beta+\alpha_i} T \), it is sufficient to show that they coincide after applying \( 1 \otimes \langle u, \cdot \rangle \) for any \( u \in U_{\beta+\alpha_i}^+ \), then we obtain

\[
(1 \otimes \langle u, \cdot \rangle)(\text{L.H.S. of (5.2)}) = \sum_r f_r \varphi (x_r^{\beta+\alpha_i}) \otimes \langle u, \omega_r y_r^{\beta-\alpha_i} k_{\beta+\alpha_i} \rangle - \sum_r \varphi (x_r^{\beta+\alpha_i}) f_r \otimes \langle u, y_r^{\beta-\alpha_i} k_{\beta+\alpha_i} \omega'_r \rangle
\]

= \( r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} \sum_r f_r \varphi (\langle u, y_r^{\beta-\alpha_i} k_{\beta} \rangle x_r^{\beta+\alpha_i}) - \sum_r \varphi (\langle u, y_r^{\beta-\alpha_i} k_{\beta} \rangle x_r^{\beta+\alpha_i}) f_r \)

= \( r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} f_r \varphi (u) - \varphi (u) f_r \)

(1 \otimes \langle u, \cdot \rangle)(\text{R.H.S. of (5.2)}) = -\sum_r \varphi (x_r^{\beta}) \otimes \langle u, f_r y_r^{\beta} k_{\beta} \rangle.

(5.10)

For \( u \in U_{\beta+\alpha_i}^+ \) we can define \( w \in U_{\beta}^+ \) uniquely by

\[
\Delta (u) = \omega_{\beta+\alpha_i} \otimes u + e_i \omega_{\beta} \otimes w + \cdots,
\]

where \( \cdots \) implies terms whose right components are elements of \( \bigoplus_{\gamma \neq \beta, \beta+\alpha_i} U_{\gamma}^+ \). By the properties of the skew Hopf pairing, we have

\[
\langle u, f_i y_r^{\beta} k_{\beta} \rangle = \langle \Delta (u), f_i \otimes y_r^{\beta} k_{\beta} \rangle
\]

= \( \langle \omega_{\beta+\alpha_i} \otimes u + e_i \omega_{\beta} \otimes w + \cdots, f_i \otimes y_r^{\beta} k_{\beta} \rangle \)

= \( \langle e_i \omega_{\beta}, f_i \rangle \langle w, y_r^{\beta} k_{\beta} \rangle \)

= \( \frac{1}{s_i - r_i} \langle w, y_r^{\beta} \rangle. \)

Thus, we have

\[
\text{R.H.S. of (5.10)} = (r_i - s_i)^{-1} \varphi (w).
\]

In order to complete the proof of (5.2), it is sufficient to show

\[
r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} f_r \varphi (u) - \varphi (u) f_r = (r_i - s_i)^{-1} \varphi (w).
\]

(5.11)

Without loss of generality, we may assume that \( u \) is of the form \( u = e_{i_1} e_{i_2} \cdots e_{i_k} \in U_{\beta+\alpha_i}^+ \) \( (\beta + \alpha_i = \alpha_{i_1} + \cdots + \alpha_{i_k}) \). For \( \beta = \sum_j m_j \alpha_j \), we shall show this by induction on \( m_i \).

If \( m_i = 0 \), \( u \) is in the form \( u = u' e_i u'' \), where \( u' \) and \( u'' \) are monomials of \( e_j \)'s not including \( e_i \). Then, we obtain

\[
r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} f_r \varphi (u)
\]

= \( r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} f_r \{ \varphi (u'') \{ -(r_i - s_i)^{-1} e_i'' \} \varphi (u') \}
\]

= \( r^{(\alpha,\beta+\alpha_i)} s^{-(\beta+\alpha_i,\alpha_i)} f_r \varphi (u'') \{ -(r_i - s_i)^{-1} e_i'' \} \varphi (u') \)

= \( r^{(\alpha,\beta)} s^{-(\beta,\alpha_i)} (r_i - s_i)^{-1} \varphi (u'') f_i \{ -(r_i - s_i)^{-1} e_i'' \} \varphi (u') \)

= \( \varphi (u'') \varphi (e_i) \varphi (u') f_i + r^{(\alpha,\beta)} s^{-(\beta,\alpha_i)} (r_i - s_i)^{-1} \varphi (u' u'') \),

where \( \beta' = \text{wt}(u') \). Therefore, for \( m_i = 0 \), we have

\[
\text{L.H.S. of (5.11)} = r^{(\alpha,\beta)} s^{-(\beta,\alpha_i)} (r_i - s_i)^{-1} \varphi (u' u'').
\]
In the case $m_i = 0$ we can easily obtain $w = r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} \varphi(u'u'')$ and then

R.H.S. of (5.11) = $r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} (r_i - s_i)^{-1} \varphi(u'u'') = \text{L.H.S. of (5.11)}$.

Thus, the case $m_i = 0$ has been shown.

Suppose that $m_i > 0$. We divide $u = u'u''$ such that $1 \leq m'_i < m_i$ and $1 \leq m''_i < m_i$, where $m'_i$ (resp. $m''_i$) is the number of $e_i$ including in $u'$ (resp. $u''$). Writing

$\Delta(u') = \omega_{\beta'} \otimes u' + e_i \omega_{\beta'-\alpha_i} \otimes w' + \cdots$, 

$\Delta(u'') = \omega_{\beta''} \otimes u'' + e_i \omega_{\beta''-\alpha_i} \otimes w' + \cdots$, 

where $\beta' = \text{wt}(u')$, $\beta'' = \text{wt}(u'')$, and calculating $\Delta(u'u'')$ directly, we obtain

$$w = u'u'' + r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} u'u''.$$ 

By the hypothesis of the induction, we obtain

$$r^{(\alpha_i, \beta'+\alpha_i)} s^{-(\beta'+\alpha_i, \alpha_i)} f_i \varphi(u) = r^{(\alpha_i, \beta'+\alpha_i)} s^{-(\beta'+\alpha_i, \alpha_i)} f_i \varphi(u') \varphi(u'')$$

$$= r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} \{ \varphi(u'') f_i + (r_i - s_i)^{-1} \varphi(w') \} \varphi(u')$$

$$= \varphi(u'' f_i + (r_i - s_i)^{-1} \varphi(w') + (r_i - s_i)^{-1} r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} \varphi(u'w'')$$

$$= \varphi(u'') f_i + (r_i - s_i)^{-1} \varphi(u'' + r^{(\alpha_i, \beta')} s^{-(\beta', \alpha_i)} u'w'')$$

$$= \varphi(u)f_i + (r_i - s_i)^{-1} \varphi(w).$$

(b) Since (5.3) and (5.4) could be proved in exactly the same manner, we only give a proof of (5.3). Since both sides contained in $U^+_{\beta+\alpha_i} \otimes U$, it is sufficient to show that they coincide after applying $\langle \cdot, z \rangle \otimes 1$ for any $z \in U^{-}_{-\beta-\alpha_i}$. Then we have

$$(\langle \cdot, z \rangle \otimes 1)(\text{L.H.S. of (5.3)}) = \langle \cdot, z \rangle \otimes 1(\sum_r x_r^{\beta+\alpha_i} \otimes (e_i y_r^{-\beta-\alpha_i} - y_r^{-\beta-\alpha_i} e_i))$$

$$= \sum_r \langle x_r^{\beta+\alpha_i}, z \rangle (e_i y_r^{-\beta-\alpha_i} - y_r^{-\beta-\alpha_i} e_i)$$

$$= e_i \sum_r \langle x_r^{\beta+\alpha_i}, z \rangle y_r^{-\beta-\alpha_i} - e_i \sum_r \langle x_r^{\beta+\alpha_i}, z \rangle y_r^{-\beta-\alpha_i}$$

$$= e_i z - z e_i.$$

By the definition of $\Delta$, we have

$$\Delta(z) = \sum_{\gamma, \delta \in Q^+} z_{\gamma, \delta} (1 \otimes k^{-1}_{\gamma})$$

with $z_{\gamma, \delta} \in U^{-}_{-\gamma} \otimes U^{-}_{-\delta}$, $z_0^{\beta+\alpha_i} = 1 \otimes z$, $z_{\beta+\alpha_i, 0} = z \otimes 1$, $z_{\alpha_i, \beta} = f_i \otimes u$, $z_{\beta, \alpha_i} = v \otimes f_i$ for some $u, v \in U^{-}_{-\beta}$. Then

$$(\langle \cdot, z \rangle \otimes 1)(\text{R.H.S. of (5.3)})$$
\[ \langle \cdot, z \rangle \otimes 1 \langle \sum_{r} (x_{r}^{\beta} e_{i} \otimes y_{r}^{-\beta} \omega_{i}' - e_{i} x_{r}^{\beta} \otimes \omega_{i} y_{r}^{-\beta}) \rangle = 1 \sum_{r} \langle (x_{r}^{\beta} e_{i}, z) y_{r}^{-\beta} \omega_{i}' - (e_{i} x_{r}^{\beta}, z) \omega_{i} y_{r}^{-\beta} \rangle = (e_{i} \otimes x_{r}^{\beta}, \Delta(z)) y_{r}^{-\beta} \omega_{i}' - (e_{i}, \Delta(z)) \omega_{i} y_{r}^{-\beta} \]

\[ \frac{1}{r_{i} - s_{i}} (\omega_{i} (\sum_{r} (x_{r}^{\beta}, v) y_{r}^{-\beta}) - (\sum_{r} (x_{r}^{\beta}, u) y_{r}^{-\beta}) \omega_{i}' = \frac{1}{r_{i} - s_{i}} (\omega_{i} v - u \omega_{i}'). \]

Therefore it is sufficient to show

\[ e_{i} z - z e_{i} = \frac{1}{r_{i} - s_{i}} (\omega_{i} v - u \omega_{i}). \quad (5.12) \]

Since we have

\[ \Delta_{2}(e_{i}) = e_{i} \otimes 1 \otimes 1 + \omega_{i} \otimes e_{i} \otimes 1 + \omega_{i} \otimes \omega_{i} \otimes e_{i}, \]

\[ \Delta_{2}(z) = \sum_{\gamma_{1}, \gamma_{2}, \gamma_{3} \in Q^{+}} z_{\gamma_{1}, \gamma_{2}, \gamma_{3}} \left( 1 \otimes k_{\gamma_{1}}^{-1} \otimes k_{\gamma_{2}}^{-1} \right), \]

with \( z_{\gamma_{1}, \gamma_{2}, \gamma_{3}} \in \mathbb{U}_{\gamma_{1}}^{-} \otimes \mathbb{U}_{\gamma_{2}}^{-} \otimes \mathbb{U}_{\gamma_{3}}^{-}, \ z_{0, \beta + \alpha, 0} = 1 \otimes z \otimes 1, \ z_{\alpha_{i}, \beta, 0} = f_{i} \otimes u \otimes 1, \ z_{0, \beta, \alpha} = 1 \otimes v \otimes f_{i}, \) we can easily obtain (5.12) using Lemma 4.1 (b).

(c) Since (5.5) and (5.6) could be proved in exactly the same manner, we only give a proof of (5.5). The case \( \beta = 0 \) is trivial. Assume that \( \beta \in Q^{+} \setminus \{0\} \). Since the left-hand side is contained in \( U_{\beta}^{+} \otimes U \), it is sufficient to show

\[ \langle \cdot, z \rangle \otimes 1 \langle \text{L.H.S. of (5.5)} \rangle = 0 \text{ for any } z \in U_{\beta}^{-}. \]

Let

\[ \Delta(z) = \sum_{\delta, \gamma \in Q^{+}} z_{\delta, \gamma}^{\delta, \gamma} (1 \otimes k_{\delta}^{-1}) \quad (z_{\delta, \gamma}^{\delta, \gamma} \in U_{\delta}^{-} \otimes U_{\gamma}^{-}) \]

\[ z_{\delta, \gamma}^{\delta, \gamma} = \sum_{m} z_{\delta, m}^{\delta, \gamma} \otimes z_{\gamma, m}^{\delta, \gamma} \quad (z_{\delta, m}^{\delta, \gamma} \in U_{-\delta}^{-}, \ z_{\gamma, m}^{\delta, \gamma} \in U_{-\gamma}^{-}) \]

Then we have

\[ \langle \cdot, z \rangle \otimes 1 \langle \text{L.H.S. of (5.5)} \rangle = \langle \cdot, z \rangle \otimes 1 \langle \sum_{\gamma_{1}, \gamma_{2}, \gamma_{3}} x_{\gamma_{1}}^{\gamma_{1}} \omega_{\delta} S(x_{\gamma_{2}}^{\delta}) \otimes y_{\gamma_{2}}^{-\gamma} y_{\gamma_{3}}^{-\delta} \rangle = \sum_{\gamma_{1}, \gamma_{2}, \gamma_{3}} \langle x_{\gamma_{1}}^{\gamma_{1}} \omega_{\delta} S(x_{\gamma_{2}}^{\delta}), z \rangle y_{\gamma_{2}}^{-\gamma} y_{\gamma_{3}}^{-\delta} = \sum_{\gamma_{1}, \gamma_{2}, \gamma_{3}} \langle \omega_{\delta} S(x_{\gamma_{2}}^{\delta}) \otimes x_{\gamma_{1}}^{\gamma_{1}}, \Delta(z) \rangle y_{\gamma_{2}}^{-\gamma} y_{\gamma_{3}}^{-\delta} = \sum_{\gamma_{1}, \gamma_{2}, \gamma_{3}} \langle \omega_{\delta} S(x_{\gamma_{2}}^{\delta}), z_{\delta, m}^{\delta, \gamma} \rangle x_{\gamma_{1}}^{\gamma_{1}} z_{\gamma, m}^{\delta, \gamma} k_{\delta}^{-1} \rangle y_{\gamma_{2}}^{-\gamma} y_{\gamma_{3}}^{-\delta} \]
\[
\begin{align*}
\text{Proof.} & \quad (\text{commute with each other by (5)} & \text{then the element } \\
\text{Proposition 5.1.} & \quad (\text{in } B) \\
\text{Here note that } & \quad = (S^{-1} \circ m \circ (1 \otimes S) \circ \Delta)(z) \\
\text{if we set} & \quad = \varepsilon(z) \\
\text{then the element } C & \quad = 0.
\end{align*}
\]

If we set
\[
C := \sum_{\beta \in Q^+} (1 \otimes k^{-1}_\beta)(1 \otimes S^{-1})(C_\beta) \in U^+ \hat{\otimes} U^- = U^+ \hat{\otimes} B^-,
\] (5.13)

then the element \( C \) satisfies the following relations.

**Proposition 5.1.** (a) For any \( i \in I \), we have
\[
\begin{align*}
(\omega_i^{-1} \otimes e''_i)C & = C(\omega_i^{-1} \otimes e''_i) + (r_i - s_i)\omega_i^{-1} \otimes e_i \otimes 1, & (5.14) \\
(f_i \otimes \omega'_i + 1 \otimes f_i)\{(\varphi \otimes S)(C)\} & = \{((\varphi \otimes S)(C))(f_i \otimes \omega'_i). & (5.15)
\end{align*}
\]

Here note that (5.14) is the equation in \( U_{r,s}(g) \hat{\otimes} B_{r,s}(g) \) and (5.15) is the equation in \( B_{r,s}(g) \hat{\otimes} B_{r,s}(g) \).

(b) The element \( C \) is invertible and the inverse is given as
\[
C^{-1} = \sum_{\beta \in Q^+} r^{-(\beta, \beta)} s^{(\beta, \beta)} (\omega_\beta \otimes k^{-1}_\beta)(S^{-1} \otimes S^{-1})(C_\beta). \tag{5.16}
\]

**Proof.** (a) If \( \beta \in Q^+ \) does not include \( \alpha_i \), since \( e''_i \) and \( S^{-1}(z) \ (z \in U^-_{\beta}) \) commute with each other by (5.9), we have
\[
(\omega_i^{-1} \otimes e''_i)(1 \otimes k^{-1}_\beta)(C'_\beta) = (1 \otimes k^{-1}_\beta)(C'_\beta)(\omega_i^{-1} \otimes e''_i).
\]

Thus, we have
\[
\begin{align*}
(\omega_i^{-1} \otimes e''_i)C & = C(\omega_i^{-1} \otimes e''_i) \\
& = \sum_{\gamma \in Q^+} \{(\omega_i^{-1} \otimes e''_i)(1 \otimes k^{-1}_\gamma)(C'_\gamma) - (1 \otimes k^{-1}_\gamma)(C'_\gamma)(\omega_i^{-1} \otimes e''_i)\} \\
& = \sum_{\beta \in Q^+} \{(\omega_i^{-1} \otimes e''_i)(1 \otimes k^{-1}_{\beta+\alpha_i})(C'_{\beta+\alpha_i}) - (1 \otimes k^{-1}_{\beta+\alpha_i})(C'_{\beta+\alpha_i})(\omega_i^{-1} \otimes e''_i)\} \\
& = \sum_{\beta \in Q^+} [\omega_i^{-1} \otimes e''_i, (1 \otimes k^{-1}_{\beta+\alpha_i})(C'_{\beta+\alpha_i})] \\
& = \sum_{\beta \in Q^+} (1 \otimes k^{-1}_\beta)(C'_\beta)((r_i - s_i)\omega_i^{-1}e_i \otimes 1) \quad \text{by (5.1)}
\end{align*}
\]

\[\text{13}\]
Then we obtain (5.14).

If $\beta \in Q^+$ does not include $\alpha_i$, using (2.4) and (2.7), it is not difficult to show

$$ (f_i \otimes \omega'_i)(C''_{\beta})(1 \otimes k_\beta) = (C''_{\beta})(1 \otimes k_\beta)(f_i \otimes \omega'_i). $$

Thus we have

$$ (f_i \otimes \omega'_i)(\varphi \otimes S(C)) - (\varphi \otimes S(C))(f_i \otimes \omega'_i) $$

$$ = \sum_{\gamma \in Q^+} \left\{ (f_i \otimes \omega'_i)(\varphi \otimes 1(C_\gamma))(1 \otimes k_\gamma) - (\varphi \otimes 1(C_\gamma))(1 \otimes k_\gamma)(f_i \otimes \omega'_i) \right\} $$

$$ = \sum_{\beta \in Q^+} \left\{ (f_i \otimes \omega'_i)(C''_{\beta+\alpha_i})(1 \otimes k_{\beta+\alpha_i}) - (C''_{\beta+\alpha_i})(1 \otimes k_{\beta+\alpha_i})(f_i \otimes \omega'_i) \right\} $$

$$ = \sum_{\beta \in Q^+} \left[ (f_i \otimes \omega'_i, C''_{\beta+\alpha_i} \cdot (1 \otimes k_{\beta+\alpha_i}) \right] $$

$$ = -\sum_{\beta \in Q^+} (1 \otimes f_i)(\varphi \otimes 1(C_\beta))(1 \otimes k_\beta) \quad \text{by (5.2)} $$

$$ = -(1 \otimes f_i)\{(\varphi \otimes S(C)\}. $$

Then we obtain (5.15).

(b) Set $\hat{C} := \sum_{\beta \in Q^+} (rs^{-1})^{(\beta, \beta)}(1 \otimes k_\beta) (S \otimes 1)(C_\beta)$. Then we claim that $\hat{C}$ is invertible and $\hat{C}^{-1} := \sum_{\beta \in Q^+} (rs^{-1})^{(\beta, \beta)}(\omega^{-1}_\beta \otimes k_\beta)(C_\beta) \in \hat{U} \otimes \hat{U}$.

Since we have

$$ \hat{C}^{-1} \cdot \hat{C} $$

$$ = \left\{ \sum_{\gamma \in Q^+} \left( rs^{-1} \right)^{(\gamma, \gamma)}(\omega^{-1}_\gamma \otimes k_\gamma)(C_\gamma) \right\} \left\{ \sum_{\delta \in Q^+} \left( rs^{-1} \right)^{(\delta, \delta)}(1 \otimes k_\delta)(S \otimes 1)(C_\delta) \right\} $$

$$ = \sum_{\gamma, \delta \in Q^+} \left( rs^{-1} \right)^{(\gamma, \gamma)}(1 \otimes k_\gamma)(S \otimes 1)(C_\gamma) \cdot (\omega^{-1}_\delta \otimes k_\delta)(S \otimes 1)(C_\delta) $$

$$ = \sum_{\beta \in Q^+} \left\{ \sum_{\gamma, \delta \in Q^+} \left( rs^{-1} \right)^{(\gamma, \gamma)}(1 \otimes k_\gamma)(S \otimes 1)(C_\gamma) \cdot (\omega^{-1}_\delta \otimes k_\delta) \right\} $$

$$ \times (\omega^{-1}_\beta \otimes k_\beta) $$

(using $\omega_\beta S(x_\delta) \omega^{-1}_\beta = r^{(\delta, \beta)} S^{-1}(\delta, \beta) S(x_\delta), \ k^{-1}_\beta y_\delta k_\beta = r^{(\delta, \beta)} S^{-1}(\delta, \beta) y_\delta$)

$$ = \sum_{\beta \in Q^+} \left\{ \sum_{\gamma, \delta \in Q^+} \left( rs^{-1} \right)^{(\gamma, \gamma)}(1 \otimes k_\gamma)(S \otimes 1)(C_\gamma) \cdot (\omega^{-1}_\delta \otimes k_\delta) \right\} $$

$$ \times (\omega^{-1}_\beta \otimes k_\beta) $$

$$ = \sum_{\beta \in Q^+} \left( rs^{-1} \right)^{(\beta, \beta)} \left\{ \sum_{\gamma, \delta \in Q^+} \left( \omega^{-1}_\delta \otimes k_\delta \right) \right\} $$

(by (5.5))

$$ = 1. $$
In exactly the same manner we can show that $\tilde{C} \cdot \tilde{C}^{-1} = 1$.

Note that
\[
(S^{-1} \otimes S^{-1})(\tilde{C}) = \sum (rs^{-1})^{(\beta,\beta)}(1 \otimes S^{-1})\{(1 \otimes k_{\beta})(C_{\beta})\} \\
= \sum (rs^{-1})^{(\beta,\beta)}\{(1 \otimes S^{-1})(C_{\beta})\}(1 \otimes k_{\beta}^{-1}) \\
= \sum (1 \otimes k_{\beta}^{-1})(1 \otimes S^{-1})(C_{\beta}) \\
= C.
\]

Thus, we obtain
\[
C^{-1} = (S^{-1} \otimes S^{-1})(\tilde{C}^{-1}) \\
= \sum (rs^{-1})^{(\beta,\beta)}\{(S^{-1} \otimes S^{-1})(C_{\beta})\}(\omega_{\beta} \otimes k_{\beta}^{-1}) \\
= \sum (r^{-1}s)^{(\beta,\beta)}(\omega_{\beta} \otimes k_{\beta}^{-1})(S^{-1} \otimes S^{-1})(C_{\beta}).
\]

Now we have completed the proof of Proposition 5.1. \hfill \Box

In the end, we will define the quantum Casimir element for the two-parameter quantum group $U_{r,s}(g)$, which has been introduced in [BGH2] and [PHR].

For each $\beta \in Q^+$, we can define
\[
\Omega_{\beta} = (m \circ (S \otimes 1) \circ \sigma)(C_{\beta}), \\
\Omega = \sum_{\beta \in Q^+} \Omega_{\beta} \in \tilde{U}.
\]

Let $\Psi$ be the $\mathbb{K}$-algebra automorphism of $U$ given by
\[
\Psi(\omega_i^{\pm 1}) = \omega_i^{\pm 1}, \quad \Psi(\omega_i'^{\pm 1}) = \omega_i'^{\pm 1}, \quad \Psi(e_i) = \omega_i^{-1}\omega_i'e_i, \quad \Psi(f_i) = f_i\omega_i\omega_i'^{-1} \forall i \in I.
\]

We will call $\Omega$ the quantum Casimir element for the two-parameter quantum group $U$ by the following proposition.

**Proposition 5.2.** For any $u \in U$, we have $\Psi(u)\Omega = \Omega u$.

**Proof.** Since the assertion is clear for $u = \omega_i^{\pm 1}$ or $\omega_i'^{\pm 1}$, we have only to show
\[
\omega_i^{-1}\omega_i'e_i\Omega = \Omega e_i, \quad f_i\omega_i\omega_i'^{-1}\Omega = \Omega f_i.
\]

They follow easily by applying $m \circ (S \otimes 1) \circ \sigma$ to both sides of (5.3) and (5.4) respectively. \hfill \Box

**Remark 5.1.** (1) By the explicit form of $C^{-1}$ in (5.16), we find that $C^{-1} \in U_{r,s}(g) \hat{\otimes} U_{r,s}(g) = U_{r,s}(g) \hat{\otimes} B_{r,s}(g)$.

(2) There are two misprints in [N2]. The first one is that (4.1) in [N2] should be
\[
\langle xq^h, yq^{h'} \rangle = q^{-(h|h')}(x, y), \quad \text{for} \ x \in U^+, \ y \in U^-, \ h, h' \in P^*.
\]

The second one is that (4.5) in [N2] should be
\[
(f_i \otimes t_i^{-1} + 1 \otimes f_i)(\varphi \otimes S(C)) = (\varphi \otimes S(C))(f_i \otimes t_i^{-1}).
\]
6 Extremal Projectors

Let $\mathcal{C}$ be as in Section 5. We define the extremal projector of $B_{r,s}(\mathfrak{g})$ by

$$
\Gamma := m \circ \sigma \circ (\varphi \otimes 1)(\mathcal{C}) = \sum_{\beta \in Q^*, r} k_{\beta}^{-1} S^{-1}(y_{r}^{-\beta} \varphi(x_{r}^{\beta})) \tag{6.1}
$$

where $m : a \otimes b \mapsto ab$ is the multiplication and $\sigma : a \otimes b \mapsto b \otimes a$ is the permutation.

Here note that $\Gamma$ is a well-defined element in $\widehat{B}_{r,s}(\mathfrak{g})$.

Example 6.1. In $\mathfrak{sl}_2$-case, it is not difficult to get the following explicit form of $\Gamma$ (here $f^{(n)}$ is defined as in Lemma 2.1).

$$
\Gamma = \sum_{n \geq 0} (-1)^{n} (r s^{-1})^{m(n-1)} f^{(n)} e_{m}.
$$

Theorem 6.1. The extremal projector $\Gamma$ enjoys the following properties:

(a) $e''_i \Gamma = 0$, \quad $\Gamma f_i = 0 \quad (\forall i \in I)$.

(b) $\Gamma^{2} = \Gamma$.

(c) There exists $a_k \in B_{r,s}^{+}(\mathfrak{g}) = U_{r,s}^{-}(\mathfrak{g})$, $b_k \in B_{r,s}^{-}(\mathfrak{g})$ such that

$$
\sum_{k} a_k \Gamma b_k = 1.
$$

(d) $\Gamma$ is a well-defined element in $\widehat{B}_{r,s}(\mathfrak{g})$.

Proof. (a) At first we shall show $e''_i \Gamma = 0$. Here let us write $\mathcal{C} = \sum_k c_k \otimes d_k$, where $c_k \in U_{r,s}^{+}(\mathfrak{g})$ and $d_k \in B_{r,s}^{-}(\mathfrak{g})$. Thus, we have

$$
\Gamma = \sum_k d_k \varphi(c_k).
$$

Equation (5.1) can be written as follows:

$$
\sum_k \omega^{-1}_i c_k \otimes e''_i d_k = \sum_k \{c_k \omega^{-1}_i \otimes d_k e''_i + (r_i - s_i)c_k \omega^{-1}_i e_i \otimes d_k\}. \tag{6.2}
$$

Applying $m \circ \sigma \circ (\varphi \otimes 1)$ to both sides of (6.2), we get

$$
\sum_k e''_i d_k \varphi(c_k) \omega_i = \sum_k d_k e''_i \omega_i \varphi(c_k) - \sum_k d_k e''_i \omega_i \varphi(c_k) = 0,
$$

and then $e''_i \Gamma \omega_i = 0$, which implies the desired result since $\omega_i$ is invertible.

Let $\mathcal{C} = \sum_k c_k \otimes d_k$ be as above, then equation (5.2) can be written as follows:

$$
\sum_k f_i \varphi(c_k) \otimes \omega'_i S(d_k) + \sum_k \varphi(c_k) \otimes f_i S(d_k) = \sum_k \varphi(c_k) f_i \otimes S(d_k) \omega'_i. \tag{6.3}
$$

Applying $m \circ \sigma \circ (1 \otimes S^{-1})$ to both sides of (6.3), we get

$$
\sum_k \omega^{-1}_i d_k \varphi(c_k) f_i = \sum_k d_k \omega^{-1}_i f_i \varphi(c_k) + \sum_k d_k (-\omega^{-1}_i f_i) \varphi(c_k) = 0,
$$

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and then \( \omega^{-1}_i \Gamma f_i = 0 \), which implies the desired result since \( \omega^{-1}_i \) is invertible.

(b) Owing to the explicit form of \( \Gamma \) in (6.1), we find that

\[
1 - \Gamma \in \sum_i B_{r,s}(\mathfrak{g})e''_i.
\]

Therefore (b) is an immediate consequence of (a).

(c) By the remark (1) in the last section, we can write

\[
C^{-1} = \sum_k b'_k \otimes a_k \in U^+(\mathfrak{g}) \otimes B^-_{r,s}(\mathfrak{g}).
\]

Then, we have

\[
\sum_{j,k} b'_k c_j \otimes a_k d_j = 1 \otimes 1. \tag{6.4}
\]

Applying \( m \circ \sigma \circ (\varphi \otimes 1) \) to both sides of (6.4), we obtain

\[
\sum_{j,k} a_k d_j \varphi(c_j) \varphi(b'_k) = \sum_k a_k \Gamma \varphi(b'_k) = 1.
\]

Here setting \( b_k := \varphi(b'_k) \), we get (c).

(d) It is easy to see (d) by the explicit forms of the antipode \( S \), the anti-isomorphism \( \varphi \) and \( \Gamma \) in (6.1).

7 Representation theory of \( \mathcal{O}(B) \)

As an application of the extremal projector \( \Gamma \), we shall show the following theorem.

Theorem 7.1. (a) The category \( \mathcal{O}(B) \) is a semi-simple category.

(b) The module \( H(\lambda) \) is a simple object of \( \mathcal{O}(B) \) and for any simple object \( M \) in \( \mathcal{O}(B) \) there exists some \( \lambda \in \Lambda \) such that \( M \cong H(\lambda) \). Furthermore, \( H(\lambda) \) is a rank one free \( B^-_{r,s}(\mathfrak{g}) \)-module.

In order to show this theorem, we need to prepare several things.

For an object \( M \) in \( \mathcal{O}(B) \), set

\[
K(M) := \{ v \in M | e''_i v = 0 \text{ for any } i \in I \}.
\]

Lemma 7.1. For an object \( M \) in \( \mathcal{O}(B) \), we have

\[
\Gamma \cdot M = K(M). \tag{7.1}
\]

Proof. By Theorem 6.1 (a), we have \( e''_i \Gamma = 0 \) for any \( i \in I \). Thus, it is trivial to see that \( \Gamma \cdot M \subseteq K(M) \). Owing to the explicit form of \( \Gamma \) in (6.1), we find that

\[
1 - \Gamma \in \sum_i B_{r,s}(\mathfrak{g}) e''_i.
\]

Therefore, for any \( v \in K(M) \) we get \( (1 - \Gamma)v = 0 \), which implies that \( \Gamma \cdot M \supseteq K(M) \).
Lemma 7.2. For an object $M$ in $\mathcal{O}(B)$, we have

$$M = B_{-r,s}(g) \cdot (K(M)). \quad (7.2)$$

Proof. By Theorem 6.1 (c), we have $1 = \sum_k a_k \Gamma b_k$ ($a_k \in B_{-r,s}(g)$, $b_k \in B_{r,s}^+(g)$). For any $u \in M$,

$$u = \sum_k a_k (\Gamma b_k u).$$

By Lemma 7.1, we have $\Gamma b_k u \in K(M)$. Then we obtain the desired result. \qed

Proposition 7.1. For an object $M$ in $\mathcal{O}(B)$, we have

$$M = K(M) \oplus (\sum_i \text{Im}(f_i)). \quad (7.3)$$

Proof. By (7.2), we get

$$M = K(M) + (\sum_i \text{Im}(f_i)).$$

Thus, it is sufficient to show

$$K(M) \cap (\sum_i \text{Im}(f_i)) = \{0\}. \quad (7.4)$$

Let $u$ be a vector in $K(M) \cap (\sum_i \text{Im}(f_i))$. Since $u \in \sum_i \text{Im}(f_i)$, there exist $\{u_i \in M\}_{i \in I}$ such that $u = \sum_{i \in I} f_i u_i$. Since $u \in K(M)$, by the argument in the proof of Lemma 7.1, we have $\Gamma u = u$. It follows from Theorem 6.1 (a) that

$$u = \Gamma u = \sum_{i \in I} (\Gamma f_i) u_i = 0,$$

which implies (7.4). \qed

Lemma 7.3. If $u, v \in M$ ($M$ is an object in $\mathcal{O}(B)$) satisfies $v = \Gamma u$, then there exists $P \in B_{r,s}(g)$ such that $v = Pu$.

Proof. By the definition of the category $\mathcal{O}(B)$, there exists $l > 0$ such that $\varphi(\sigma_r^\beta)u = 0$ for any $r$ and $\beta$ with $|\beta| > l$. Thus, by the explicit form of $\Gamma$ in (6.1), we can write

$$v = \Gamma u = (\sum_{|\beta| \leq l} k_{\beta}^{-1} S^{-1}(y_r^{-\beta}) \varphi(x_r^\beta)) u,$$

which implies the desired result. \qed

Now we start the proof of Theorem 7.1.
Proof. (a) Let $L \subset M$ be objects in the category $O(B)$. We shall show that there exists a submodule $N \subset M$ such that $M = L \oplus N$.

Since $K(M)$ (resp. $K(L)$) is invariant under the action of any $\omega_{\mu}$ and $\omega'_{\mu}$ ($\mu \in Q$), we have the following weight space decomposition:

$$K(M) = \bigoplus_{\lambda \in \Lambda} K(M)_{\lambda} \quad \text{(resp. } K(L) = \bigoplus_{\lambda \in \Lambda} K(L)_{\lambda}).$$

There exist subspaces $N_{\lambda} \subset K(M)_{\lambda}$ such that $K(M)_{\lambda} = K(L)_{\lambda} \oplus N_{\lambda}$, which is a decomposition of a vector space. Here set $N := \bigoplus_{\lambda} N_{\lambda}$. We have

$$K(M) = K(L) \oplus N.$$ 

Let us show $M = L \oplus B_{r,s}(g) \cdot N$. (7.5)

Since $M = B_{r,s}(g)(K(M)) = B_{r,s}(g)(K(L) \oplus N)$, we get $M = L + B_{r,s}(g) \cdot N$. Thus, it is sufficient to show

$$L \cap B_{r,s}(g) \cdot N = \{0\}. \quad (7.6)$$

For $v \in L \cap B_{r,s}(g) \cdot N$, we have by Theorem 6.1 (c)

$$v = \sum_{k} a_k (\Gamma b_k v).$$

It follows from $v \in L$ that $\Gamma b_k v \in K(L)$ for all $k$, and from $v \in B_{r,s}(g) \cdot N$ that $\Gamma b_k v \in \Gamma(B_{r,s}(g) \cdot N) = N$ for all $k$. These imply

$$\Gamma b_k v \in K(L) \cap N = \{0\} \quad \text{for all } k.$$ 

Hence we get $v = 0$ and then (7.6).

Next, let us show (b). As an immediate consequence of Proposition 2.3 we can see that $H(\lambda)$ is a rank one free $B_{r,s}(g)$-module.

Let $\pi_\lambda : B_{r,s}(g) \to H(\lambda)$ be the canonical projection and set $u_\lambda := \pi_\lambda(1)$. Here we have

$$H(\lambda) = B_{r,s}(g) \cdot u_\lambda = \mathbb{K} u_\lambda + \sum_{i} \text{Im}(f_i).$$

It follows from this, Proposition 7.1, and $\mathbb{K} u_\lambda \subset K(H(\lambda))$ that $H(\lambda) = \mathbb{K} u_\lambda + \sum_{i} \text{Im}(f_i)$ and then

$$\Gamma \cdot H(\lambda) = K(H(\lambda)) = \mathbb{K} u_\lambda. \quad (7.7)$$

In order to show the irreducibility of $H(\lambda)$, it is sufficient to see that for any arbitrary $u \ (\neq 0)$, $v \in H(\lambda)$, there exists $P \in B_{r,s}(g)$ such that $v = Pu$. Set $v = Qu_\lambda$ ($Q \in B_{r,s}(g)$). By Theorem 6.1 (c), we have

$$u = \sum_{k} a_k (\Gamma b_k u) \neq 0.$$
Then, for some $k$ we have $\Gamma b_k u \neq 0$, which implies that $c \Gamma b_k u = u_\lambda$ for some non-zero scalar $c$. Therefore, by Lemma 7.3, there exists some $R \in B_{r,s}(\mathfrak{g})$ such that $u_\lambda = Ru$ and then we have

$$v = Qu_\lambda = QRu.$$ 

Thus, $H(\lambda)$ is a simple module in $\mathcal{O}(B)$.

Suppose that $L$ is a simple module in $\mathcal{O}(B)$. First, let us show

$$\dim(K(L)) = 1. \quad (7.8)$$

For $x, y (\neq 0) \in K(L)$, there exists $P \in B_{r,s}(\mathfrak{g})$ such that $y = Px$. Since $x \in K(L)$, we can take $P \in B_{r,s}^-(\mathfrak{g})$. Because $y \in K(L)$ and $K(L) \cap (\sum_i \text{Im}(f_i)) = \{0\}$, we find that $P$ must be a scalar, say $c$. Thus, we have $y = cx$, which implies (7.8).

Let $u_0$ be a basis vector in $K(L)$. The space $K(L)$ is invariant by the action of any $\omega_\mu$ and $\omega'_\mu$ ($\mu \in Q$), then we get $u_0 \in L_\lambda$ for some $\lambda \in \Lambda$. Therefore, since $H(\lambda)$ is a rank one free $B_{r,s}^-(\mathfrak{g})$-module, the map

$$\phi_\lambda : H(\lambda) \to L \quad Pu_\lambda \mapsto Pu_0, \quad (P \in B_{r,s}^-(\mathfrak{g})), $$

is a well-defined nontrivial homomorphism of $B_{r,s}(\mathfrak{g})$-modules. Thus, by Schur’s lemma, we obtain $H(\lambda) \cong L$. \hfill \Box

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