On the Singular Behavior of the Chirality-Odd Twist-3 Parton Distribution $e(x)$

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Abstract

The first moment the chirality-odd twist-3 parton distribution $e(x)$ is related to the pion-nucleon $\sigma$-term which is important for phenomenology. However, the possible existence of a singular contribution proportional to $\delta(x)$ in the distribution prevents from the determination of the $\sigma$-term with $e(x)$ from experiment. There are two approaches to show the existence. The first one is based on an operator identity. The second one is based on a perturbative calculation of a single quark state with finite quark mass. We show that all contributions proportional to $\delta(x)$ in the first approach are in fact cancelled. To the second approach we find that $e(x)$ of a multi-parton state with a massless quark has no contribution with $\delta(x)$. Considering that a proton is essentially a multi-parton state, the effect of the contribution with $\delta(x)$ is expected to be suppressed by light quark masses.

Cross-sections of high energy scattering involving hadrons can be predicted by employing QCD factorization theorem. At the leading power, they can be predicted in the form of convolutions of perturbative coefficient functions with twist-2 parton distributions. With numerous experiments these twist-2 parton distributions are well studied and provide important information about inner structure of hadrons. At the next-to-leading power, twist-3 parton distributions are involved. These distributions contain more information than that of twist-2 parton distributions, but are little known from experiment.

In this letter we study the chirality-odd twist-3 parton distribution $e(x)$. The most interesting quantity related to $e(x)$ is the pion-nucleon $\sigma$-term, determined by the first moment of $e(x)$. This quantity gives important information about explicit chiral symmetry breaking of QCD[1]. It is also phenomenologically interesting for searching physics beyond Standard Model, because the $\sigma$-term is a scalar coupling corresponding to interactions absent in Standard Model. With experimentally determined distribution $e(x)$ one can determine the $\sigma$-term in principle. But it seems impossible because of that $e(x)$ can have a contribution proportional to $\delta(x)$. Such a contribution around $x \sim 0$ cannot be determined by experiment. Therefore, it is not possible to determine $\sigma$-term from $e(x)$ extracted from experimental data.

It is noticed that there are two different approaches in literature showing that $e(x)$ has a contribution proportional to $\delta(x)$. Because of its importance, we examine here the existence of the $\delta(x)$-contribution.

Effects of higher-twist parton distributions are in general suppressed in high energy scattering. Therefore, it is expected that their determination is difficult. However, with progress of experiment, there are already some information about $e(x)$ from experiment. The first extraction is given in [2] from experiment of Semi-Inclusive Deeply Inelastic Scattering(SIDIS) at CLAS[3]. The distribution has been also determined from di-hadron production studied with CLAS experiment[4]. With current high luminosity
facility like JLab\textsuperscript{5} and planned Eic\textsuperscript{6} and EicC\textsuperscript{7}, twist-3 parton distributions can be studied more precisely.

We consider a proton moving fast in the z-direction. We use the light-cone coordinate system, in which a vector $a^\mu$ is expressed as $a^\mu = (a^+, a^-, a_\perp) = ((a^0 + a^3)/\sqrt{2}, (a^0 - a^3)/\sqrt{2}, a^1, a^2)$ and $a_\perp^2 = (a^1)^2 + (a^2)^2$. The transverse metric is given by $g_{\mu\nu} = g_{\mu\nu} - n^\mu n^\nu - n^\nu l^\mu$. In this coordinate system, the momentum of the proton is given by $P^\mu = (P^+, P^-, 0, 0)$. In the case of the unpolarized proton one can defined four chirality-odd parton distributions. We introduce the gauge link:

$$\mathcal{L}_n(x) = P \exp \left\{ - i g_s \int_0^\infty d\lambda n \cdot G(\lambda n + x) \right\}. \quad (1)$$

With the gauge link one can define the distributions in a gauge invariant way. There are four chirality-odd distributions at twist-3 for an unpolarized proton, which are defined as:

\begin{align*}
M e(x) &= P^+ \int \frac{d\lambda}{4\pi} e^{i\lambda x P^+} \langle P | \bar{\psi}(0) \mathcal{L}_n \bar{\psi}(0) \mathcal{L}_n(n) \psi(n) | P \rangle,
\hat{T}_F(x_1, x_2) &= g_s \int \frac{d\lambda_1 d\lambda_2}{4\pi} e^{-i\lambda_1 x_1 P^+-i\lambda_2 (x_2-x_1) P^+} \langle P | \bar{\psi}(\lambda_1 n) \mathcal{L}_n \bar{\psi}(\lambda_2 n) (i\gamma_{\perp\mu} \gamma^+) G^{\mu\nu}(\lambda_2 n) \mathcal{L}_n \bar{\psi}(\lambda_2 n) \mathcal{L}_n(0) \psi(0) | P \rangle,
E_D(x_1, x_2) &= P^+ \int \frac{d\lambda_1 d\lambda_2}{4\pi} e^{-i\lambda_1 x_1 P^+-i\lambda_2 (x_2-x_1) P^+} \langle P | \bar{\psi}(\lambda_1 n) \mathcal{L}_n \bar{\psi}(\lambda_2 n) (i\gamma_{\perp\mu} \gamma^+) D_\perp^{\mu\nu}(\lambda_2 n) \mathcal{L}_n \bar{\psi}(\lambda_2 n) \mathcal{L}_n(0) \psi(0) | P \rangle,
E_0(x) &= P^+ \int \frac{d\lambda}{4\pi} e^{-i\lambda x P^+} \langle P | \bar{\psi}(\lambda n) \mathcal{L}_n \bar{\psi}(\lambda n) i\gamma_{\perp\mu} \gamma^+ \partial^\mu \mathcal{L}_n(0) \psi(0) | P \rangle, \quad (2)
\end{align*}

with the covariant derivative:

$$D^\mu(x) = \partial^\mu + ig_s G^\mu(x). \quad (3)$$

From symmetries, one can derive:

$$\hat{T}_F(x_1, x_2) = \hat{T}_F(x_2, x_1), \quad E_D(x_1, x_2) = -E_D(x_2, x_1). \quad (4)$$

The defined four parton distributions are not independent. One can derive the relation:

$$E_D(x_1, x_2) = -i\pi \delta(x_1 - x_2) E_0(x_1) + \frac{1}{x_2 - x_1 + i\varepsilon} \hat{T}_F(x_1, x_2). \quad (5)$$

Taking the principal value of the distribution $1/(x_2 - x_1 + i\varepsilon)$, one has:

$$E_D(x_1, x_2) = \hat{T}_F(x_1, x_2) P \frac{1}{x_2 - x_1}, \quad \hat{T}_F(x, x) = E_0(x). \quad (6)$$

Therefore, only one of the last three twist-3 distributions in Eq.\textsuperscript{(2)} is independent. The defined distributions depend on the renormalization scale $\mu$. The dependence has been studied in \cite{8, 9, 10, 11, 12}.

Now we focus on the distribution $e(x)$. It is noted that the factor $M$ in the definition of $e(x)$ is a scale factor to make $e(x)$ dimensionless. It is convenient to take $M$ as the proton mass. In principle it can be any mass quantity at order of $\Lambda_{QCD}$. It is easy to find the first moment:

$$\int_{-1}^1 dx e(x) = \frac{1}{M} \langle P | \bar{\psi}(0) \mathcal{L}_n(0) \psi(0) | P \rangle. \quad (7)$$
By taking $M$ as the mass of proton and summing different flavors of light quarks, the first moment is related to the $\sigma$-term. The sum rule in Eq.(7) gives the possibility to determine the $\sigma$-term by using $e(x)$ extracted from experiment. However, this possibility may not exist. There are evidences that $e(x)$ contains a contribution proportional to $\delta(x)$. If this is the case, then one can never determine the integral in the sum rule from experiment, hence the $\sigma$-term, because the region with $x \sim 0$ can not be accessed in experiment. In this case, the sum rule is violated.

As mentioned, there are two approaches to show that $e(x)$ contains a contribution proportional to $\delta(x)$. One is given in [13]. To see such a contribution, one starts with the identity for the operator in the definition of $e(x)$:

$$\bar{\psi}(0)L_n^\dagger(0)L_n(\lambda n)\psi(\lambda n) = \bar{\psi}(0)\psi(0) + \int_0^\lambda d\sigma \frac{\partial}{\partial \sigma} \left( \bar{\psi}(0)L_n^\dagger(0)L_n(\sigma n)\psi(\sigma n) \right).$$  

(8)

The integral can be expressed with four terms [14, 15, 16]. The first two terms can be expressed with twist-3 quark-gluon operator and operator of Equation of Motion(EOM) operator respectively. The third term is a total derivative. The last one is proportional to quark mass. The second term gives no contribution by sandwiching the operator into a physical state. The total derivative term will not contribute to $e(x)$. From this identity, one may conclude that $e(x)$ has a contribution proportional to $\delta(x)$, which is given by [13]:

$$Me(x) = \frac{1}{2} \delta(x)\langle P|\bar{\psi}\psi|P\rangle + \cdots,$$  

(9)

where $\cdots$ denote the contribution of the twist-3 quark-gluon operator and that proportional to $m_q$. These contributions are regular. The singular contribution with $\delta(x)$ is given by the first term in Eq.(8). It is clear that this conclusion is correct only if the remaining contributions in Eq.(8) contain no term proportional to $\delta(x)$. But, there are in fact terms with $\delta(x)$ in the remaining contributions. If these terms are cancelled, then there is no contribution proportional to $\delta(x)$.

We need to carefully examine the remaining contribution. With little algebra the derivative of the operator in Eq.(8) can be written:

$$\frac{\partial}{\partial \sigma} \left( \bar{\psi}(0)L_n^\dagger(0)L_n(\sigma n)\psi(\sigma n) \right) = \frac{1}{2} \bar{\psi}(0)L_n^\dagger(0)L_n(\sigma n) \left( \gamma_\perp^\nu \gamma^+ D_\nu + \gamma^+ \gamma \cdot D \right) \psi(\sigma n)$$

$$- \frac{1}{2} \partial^+ \left( \bar{\psi}(0)L_n^\dagger(0) \right) \gamma^- \gamma^+ L_n(\sigma n)\psi(\sigma n)$$

$$+ \frac{1}{2} \partial^+ \left( \bar{\psi}(0)L_n^\dagger(0) \gamma^- \gamma^+ L_n(\sigma n)\psi(\sigma n) \right).$$  

(10)

The derivative in the second line multiplied with $\gamma^-$ can be expressed with EOM as:

$$\partial^+ \left( \bar{\psi}(0)L_n^\dagger(0) \right) \gamma^- = -(D_\nu \bar{\psi}(0))L_n^\dagger(0) \gamma_\perp^\nu + im_q \bar{\psi}(0)L_n^\dagger(0)$$  

(11)

with $m_q$ as the quark mass. Using the identity

$$\int \frac{d\lambda_2dy}{2\pi} dy e^{-i(\lambda_2-\lambda)y} L_n^\dagger(\lambda n)L_n(\lambda_2 n) = 1,$$  

(12)

the matrix element of the derivative term in Eq.(8) becomes:

$$\frac{\partial}{\partial \sigma} \langle P|\bar{\psi}(0)L_n^\dagger(0)L_n(\sigma n)\psi(\sigma n)|P\rangle$$
\[
\frac{1}{2} \int \frac{dyd\lambda}{2\pi} e^{-iy(\lambda_2-\sigma)} \langle P|\bar{\psi}(0)\mathcal{L}_n(0)\gamma^+\gamma^+\mathcal{L}_n(\lambda_2n)\mathcal{L}_n^\dagger(\lambda_2n)\mathcal{L}_n(\sigma_2n)\psi(\sigma_2n)|P\rangle \\
-\frac{1}{2} \int \frac{dyd\lambda}{2\pi} e^{-iy\lambda_2} \langle P|\bar{\psi}(0)\mathcal{L}_n(0)\gamma^+\gamma^+\mathcal{L}_n(\lambda_2n)\mathcal{L}_n^\dagger(\lambda_2n)\mathcal{L}_n(\sigma_2n)\psi(\sigma_2n)|P\rangle \\
-\text{im}_q \langle P|\bar{\psi}(0)\mathcal{L}_n(0)\gamma^+\mathcal{L}_n(\sigma_2n)\psi(\sigma_2n)|P\rangle.
\]

(13)

It is noted that the operator in the second and third line is the operator used to define the twist-3 distribution $E_D(x_1, x_2)$ in Eq. (2). The operator in the last line is the one used to define twist-2 parton distribution $f_q$. Finally, we can derive the relation:

\[
M E(x) = \delta(x) \left( \frac{1}{2} \langle P|\bar{\psi}(0)\psi(0)|P\rangle - \frac{1}{4\pi} \int \frac{dx_1dx_2}{x_1x_2} (x_2 - x_1) E_D(x_1, x_2) - m_q \int \frac{dx_1}{x_1} f_q(x_1) \right) \\
+ \frac{1}{4\pi} \int dx_1dx_2 E_D(x_1, x_2) \left( \frac{1}{x_1} \delta(x_1) - \frac{1}{x_2} \delta(x_2) \right) + \frac{m_q}{x} f_q(x).
\]

(14)

Therefore, there are three terms with $\delta(x)$, not only the one given in Eq. (9).

Since there are three terms with $\delta(x)$, it is possible that their sum is zero so that $e(x)$ contains no contribution proportional to $\delta(x)$. In fact, one can show that the sum is zero. For this purpose we can write the quark field as the sum of the $+$ and the $-$ component which are defined as:

\[
\psi^{(+)}(x) = \frac{1}{2} \gamma^+ \gamma^- \psi(x), \quad \psi^{(-)}(x) = \frac{1}{2} \gamma^- \gamma^+ \psi(x).
\]

(15)

With these components the matrix element of $\bar{\psi}\psi$ becomes:

\[
\langle P|\bar{\psi}(0)\psi(0)|P\rangle = \langle P|\bar{\psi}^{(+)}(0)\psi^{(-)}(0)|P\rangle + \langle P|\bar{\psi}^{(-)}(0)\psi^{(+)}(0)|P\rangle.
\]

(16)

The two components are not independent. Using EOM, the $-$ component can be expressed with the $+$ component combined with gauge fields:

\[
\psi^{(-)}(x) = \frac{1}{2} \mathcal{L}_n(x) \int_0^\infty d\lambda \left[ \mathcal{L}_n \gamma^\dagger \left( \gamma_\mu^\dagger D_\mu + \text{im}_q \right) \psi^{(+)} \right] (\lambda n + x).
\]

(17)

Using this expression and the identity in Eq. (12) we can write the matrix element in the form:

\[
\langle P|\bar{\psi}(0)\psi(0)|P\rangle = \frac{1}{2} \int d\lambda d\omega \frac{e^{-i\omega\lambda}}{2\pi} \int \frac{d\lambda_2}{2\pi} dy e^{-i(\lambda_2 - \lambda)y} \\
\left\langle P|\bar{\psi}(0)\mathcal{L}_n^\dagger(\lambda n)\gamma^+\gamma^\dagger D_\mu(\lambda n)\mathcal{L}_n(\lambda n)\mathcal{L}_n^\dagger(\lambda_2n)\mathcal{L}_n(\lambda_2n)\psi(\lambda_2n)|P\rangle \right. \\
+ \frac{1}{2} \text{im}_q \int_0^\infty d\lambda \langle P|\bar{\psi}(0)\mathcal{L}_n(\lambda n)\gamma^+\mathcal{L}_n(\lambda n)\psi(\lambda n)|P\rangle + \text{h.c.}.
\]

(18)

It is noted that the operator in the second- and third line is used to define $E_D$ and $f_q$, respectively. Therefore, the matrix element is related to $E_D$ and $f_q$. The relation is:

\[
\langle P|\bar{\psi}(0)\psi(0)|P\rangle = \frac{1}{2\pi} \int \frac{dx_1dx_2}{x_1x_2} (x_2 - x_1) E_D(x_1, x_2) + 2m_q \int \frac{dx_1}{x_1} f_q(x_1).
\]

(19)

This shows that the sum of three terms with $\delta(x)$ in Eq. (14) is zero. Therefore, the correct relation for $e(x)$ instead of that in Eq. (14) is

\[
M e(x) = \frac{1}{2\pi x} \int dx_2 E_D(x, x_2) + \frac{m_q}{x} f_q(x)
\]

(20)
without $\delta(x)$-terms.

In $e(x)$ the transverse momentum of the parton is integrated over. One can define a transverse-momentum-dependent parton distribution $e(x, k_\perp)$ by undoing the integration. The defined distribution has a similar relation like that of $e(x)$ in Eq. (14) shown in [17], where there are three terms with $\delta(x)$ corresponding to those in Eq. (14). One can use EOM as doing in the above to show that the sum of the three terms with $\delta(x)$ is zero. Therefore, there is no term with $\delta(x)$ explicitly.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a): A diagram for one-loop correction to $e(x)$ of a single quark state. (b) and (c): Tree-level diagrams for $e(x)$ of a multi-parton state. The black dots in all diagrams represent the insertion of the quark field in the definition of $e(x)$.}
\end{figure}

Based on our results presented in the above, one can not find any contribution proportional to $\delta(x)$ in $e(x)$ or $e(x, k_\perp)$ at operator level from the identity of operators. However, there is another approach to show that a $\delta(x)$-term can exist. In this approach [18], one calculates the distribution of $e(x)$ of a quark state with the momentum $p$ perturbatively. Because $e(x)$ is a chirality-odd distribution, the quark must have a nonzero mass to obtain nonzero result. At tree-level, the result is:

$$M e(x) = m_q \delta(1-x) + O(\alpha_s).$$

In the light-cone gauge $n \cdot G = 0$, the one-loop contribution consists of two parts: One is the correction of external legs of the tree-level diagram, another one is given by the diagram Fig.1a. The contribution from Fig.1a has a $\delta(x)$-term. The diagram is an uncut diagram. With a cut diagram one can miss the contribution with $\delta(x)$ [19].

To understand this term, we examine the contribution from Fig.1a. It is:

$$M e(x)\bigg|_{1a} = g_s^2 C_F \frac{1}{2} \sum_s \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2} p^+ \delta(k^+ - x p^+) \left\{ \text{tr} \left[ \frac{\gamma_\alpha}{k^2 - m_q^2 + i\epsilon} (k^2 - m_q^2 + i\epsilon) \right] \right\} u(p,s) \bar{u}(p,s) \left( g^{\alpha\beta} - \frac{n^\alpha q^\beta}{n \cdot q} - \frac{n^\beta q^\alpha}{n \cdot q} \right) \frac{-i}{q^2 + i\epsilon}, \quad q = p - k,$$

where $k$ is the momentum carried by the quark propagator connecting the black dot. The gluon line carries the momentum $q = p - k$. For $e(x)$ $\Gamma$ is an unit matrix. It is noted that we treat Fig.1a as a contribution of forward scattering matrix element. We will only keep the leading order of $m_q$. The collinear divergence associated with the limit $m_q \to 0$ will be regularized with dimensional regularization. Working out the trace and the contraction of Lorentz indices, we have:

$$M e(x)\bigg|_{1a} = 2m_q g_s^2 C_F p^+ \int \frac{d^4 k}{(2\pi)^4} i \delta(k^+ - x p^+) \left( \frac{-2p^+}{k^+ - p^+ (k^2 - m_q^2 + i\epsilon)(q^2 + i\epsilon)} - i \right) \frac{-i}{(k^2 - m_q^2 + i\epsilon)^2} + O(m_q^2) \right).$$

\begin{equation} \tag{23} \end{equation}
It is noted that the second term in \((\cdots)\) is proportional to the integral studied in detail in [20]:

\[
\int \frac{dk^-}{2\pi} \frac{-i}{(k^2 - m_q^2 + i\epsilon)^2} = \frac{1}{2} \delta(k^+) \frac{1}{k^2 + m_q^2}.
\]  

(24)

with the fixed \(k^+ = xp^+\). This integral is zero with \(x \neq 0\), but becomes singular at \(x = 0\). It is proportional to \(\delta(x)\). This gives \(e(x)\) a contribution proportional to \(\delta(x)\). The calculation of one-loop correction is straightforward. We obtain:

\[
Me(x) = m_q \delta(1 - x) + m_q \frac{\alpha_s C_F}{2\pi} \left( -\frac{2}{\epsilon_c} + \ln \frac{\mu^2 e^\gamma}{4\pi \mu_c^2} \right) \left[ \frac{1}{2} \delta(1 - x) + \delta(x) + \frac{2}{(1 - x)_+} \right] + O(\alpha_s^2 m_q) + O(\alpha_s m_q^3).
\]  

(25)

where the pole in \(\epsilon_c = 4 - d\) represents collinear singularities. This results indicates that \(e(x)\) contains a contribution proportional to \(\delta(x)\) from a perturbative calculation with a single quark state. It is also noted in [18] that without the contribution the sum rule in Eq.(7) can not be satisfied.

With the above discussion, the origin of the \(\delta(x)\)-term is clear. If we can use massless quark propagators instead of massive quark propagators with \(k\) in Fig.1a, then the term does not appear. But, this is inconsistent with a single quark state with a nonzero mass. If we take a massless quark-state, \(e(x)\) is zero because of helicity conservation of QCD. But, in the reality a single quark does not exist. We only observe hadrons. A hadron consists of quarks, antiquarks and gluons, i.e, it is a multi-parton state. If we calculate \(e(x)\) of a multi-parton state, in which quarks are massless, then the contribution proportional to \(\delta(x)\) is absent. The sum rule in Eq.(7) should be satisfied without such a contribution. We examine this in the below.

We introduce the multi-parton state as a superposition of a single quark state and a quark-gluon state:

\[
|n[\lambda]\rangle = |q(p, \lambda)[\lambda]\rangle + c_1 |q(p_1, \lambda_q)g(p_2, \lambda_g)[\lambda]\rangle,
\]  

(26)

with \(p_1 + p_2 = p\). The state has the helicity \(\lambda\). In the first term the quark helicity is given by \(\lambda_q = \lambda\). For the \(qg\)-state, the total helicity is the sum \(\lambda_q + \lambda_g = \lambda\). It is noted that the helicity of the single quark is always opposite to that of the quark in the \(qg\)-state, i.e., \(\lambda_q = -\lambda\). The \(qg\)-state is in the fundamental representation. The quark \(q\) in the multi-parton state is massless. \(c_1\) is a coefficient with the dimension as a mass. All partons in the state move in \(z\)-direction with \(p_1^x = x_0 p^+\) and \(p_2^x = (1 - x_0) p^+ = \bar{x}_0 p^+\). Such a multi-parton state has been used to study factorization problems of single transverse-spin asymmetry in [21,22] and evolutions of chirality-odd operators in [12]. If we calculate the distribution \(e(x)\) of the state, only the interference of the single quark state with the \(qg\)-state gives nonzero contribution because of helicity conservation of QCD. At tree-level, the contributions are from Fig.1b and 1c. We have:

\[
Me^{(0)}(x) = -c_1 g_s C_F \sqrt{2x_0} \left[ \delta(1 - x) - \frac{1}{x_0} \delta(x - x_0) \right].
\]  

(27)

At tree-level, the matrix element is

\[
\langle n | \bar{\psi} \psi | n \rangle^{(0)} = c_1 g_s C_F \sqrt{2x_0} \frac{\bar{x}_0}{x_0}.
\]  

(28)

At this order the sum rule is satisfied.
At one-loop level, there are corrections from diagrams by adding extra one-gluon exchange in Fig.1b and Fig.1c. The calculation is straightforward. We skip the detail of the calculation and give the result:

\[
Me^{(1)}(x) = -c_1 g_s \sqrt{2} x_0 - \frac{\alpha_s C_F}{2\pi} \left\{ \frac{-2}{\epsilon_c} + \ln \frac{\mu^2 e^\gamma}{4\pi\mu^2_c} \right\} \left\{ \frac{1}{2N_c} \theta(-x) \theta(x_0 + x) \frac{x_0 + x}{x_0(x_0 - x)} + \theta(x) \left( C_F \left( \frac{3}{2} \delta(1 - x) - \frac{1}{x_0} \delta(x_0 - x) \right) + \frac{x}{(1 - x)_+} - \theta(x_0 - x) - \theta(x_0 - x) \frac{x_0(x_0 - x)_+}{(x_0 - x)_+} \right) \right\} + \frac{1}{2N_c} \left( \frac{2}{x_0} \theta(x_0 - x) + \frac{\ln x_0}{x_0} \left( \delta(1 - x) - \delta(x_0 - x) \right) + \frac{x - x_0}{(1 - x)_+} \left( \frac{1}{x_0} \theta(x - x_0) - \frac{1}{x_0} \theta(x_0 - x) \right) + \theta(x_0 - x) \frac{x_0 + x}{x_0(x_0 - x)_+} \right) \right\},
\]

(29)

As expected, the one-loop correction have no contribution proportional to \( \delta(x) \). From \( e^{(1)}(x) \) we have its first moment:

\[
\int dx Me^{(1)}(x) = \frac{3}{4} c_1 g_s \alpha_s C_F^2 \pi \left\{ \frac{-2}{\epsilon_c} + \ln \frac{\mu^2 e^\gamma}{4\pi\mu^2_c} \right\} \sqrt{2} x_0 - \frac{\alpha_s C_F}{2\pi} \theta(-x) \theta(x_0 + x) \frac{x_0 + x}{x_0(x_0 - x)}
\]

(30)

Calculating the one-loop correction of the matrix element directly, one finds that it is exactly the above result. Therefore, the sum rule in Eq.(17) is satisfied at one-loop level with our multi-parton state, and \( e(x) \) does not have a contribution proportional to \( \delta(x) \). Our result for \( e(x) \) also satisfied the sum rule with the second moment. The second moment is zero in our case, because the quark is massless.

Since a proton contains more than one parton, it is in general a superposition of multi-parton states. The distribution of \( e(x) \) of a proton receives contributions not only from a single quark state, but also contributions from interference between different states like a single quark state and multi-parton state. The contribution from a single quark state will be proportional to quark mass from the argument of perturbative theory, while the contribution from interference will survive in the massless limit, as shown from our study here. The contribution from interference to \( Me(x) \) is expected at the order \( 1/R \) with \( R \) as the size of proton from argument of dimension. This enables us to decompose \( e(x) \) into two parts:

\[
Me(x) = \frac{1}{R} e_I(x) + m_q e_s(x),
\]

(31)

where \( e_I \) denotes the interference contribution, and \( e_s \) the single quark contribution. Therefore, relative to the interference contribution the single quark contribution is suppressed by \( m_q/m_N \) with \( m_N \) as nucleon mass. Combining the result in [18] and ours, the possible \( \delta(x) \)-term only exists in \( e_s \). Therefore, its effect is suppressed by \( m_q R \sim m_q/m_N \). If we neglect quark mass, it is expected that \( e(x) \) will not contain a contribution proportional to \( \delta(x) \). It should be kept in mind that the conclusion made here is based on arguments from perturbative QCD.

We notice from [18] that the contribution with \( \delta(x) \) exists not only in \( e(x) \) but also in \( h_L(x) \), a twist-3 distribution of longitudinally polarized proton. Our results about \( e(x) \) do not apply for the case of \( h_L \). It seems that the existence of such a contribution in higher-twist parton distributions is quite general. The origin of the contribution can be zero-modes of partons and their long-range order inside hadrons, as discussed in [23]. Although the contribution can not be studied with experiment, it can be studied with large momentum effective theory [24] through Lattice QCD simulations as argued in [23].

To summarize: We have shown that at operator level one can not find a contribution proportional to \( e(x) \). It is true that \( e(x) \) of a single quark has such a contribution. However, in \( e(x) \) of a multi-parton state containing massless quarks, there is no such a contribution, as shown through our one-loop
calculation. Since a hadron is a superposition of multi-parton states, we can decompose $e(x)$ of a proton as the sum of the contribution from a single quark and that from interference of different states. Based on arguments from perturbative QCD, the single quark contribution can have a contribution with $\delta(x)$ but proportional to the quark mass, and the interference contribution is nonzero with massless quarks and has no contribution proportional to $\delta(x)$. Therefore, it is expected that the effect of the contribution with $\delta(x)$ is suppressed by the quark mass. If we neglect the masses of light quarks in proton, the sum rule of $e(x)$ related to the pion-nucleon $\sigma$ term is not violated. In this case, one can still use the sum rule to determine the $\sigma$-term.

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