Fourier-Mukai transforms and canonical divisors

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Abstract

Let $X$ be a smooth projective variety. We study a relationship between the derived category of $X$ and that of a canonical divisor. As an application, we will study Fourier-Mukai transforms when $\kappa(X) = \dim X - 1$.

1 Introduction

Let $X$ be a smooth projective variety and $D(X)$ the bounded derived category of coherent sheaves on $X$. Recently $D(X)$ draws much attention from many aspects, especially mirror symmetry, moduli spaces of stable sheaves, and birational geometry. Kontsevich [M.K94] conjectures the existence of equivalence between derived category of $X$ and derived Fukaya category of its mirror. In the physical viewpoint, we can not distinguish the mirror pair by observations or experiments, so this gives a motivation for the new concept of “spaces”. In this respect, the properties which are invariant under Fourier-Mukai transform (i.e. categorical invariant) can be considered as the essential properties of the “spaces”. For example, the Serre functor $S_X = \otimes \omega_X[\dim X]$ is such a categorical invariant.

On the other hand there are many works concerning the derived equivalent varieties. Let $FM(X)$ be a set of isomorphism class of smooth projective varieties which have equivalent derived categories to $X$. In [Muk81], Mukai showed that if $A$ is an abelian variety and $\hat{A}$ is its dual variety, then $\hat{A}$ belongs to $FM(A)$. This fact implies that $D(X)$ does not completely determine $X$. But if we assume that $K_X$ or $-K_X$ is ample, Bondal and Orlov [BD01] showed that $FM(X)$ consists of $X$ itself. When $X$ is a minimal surface, Bridgeland-Maciocia [BM01] described $FM(X)$, and non-minimal case was treated by Kawamata [Kaw02]. In these cases, we can see the following common phenomenon:

“If there exist more information of $K_X$, for example if $\kappa(X, \pm K_X)$ are greater, then $FM(X)$ is smaller.”

The main purpose of this paper is to explain why this phenomenon occurs. The idea is to extract information concerning Serre functors. Here we state the main theorem. Let $Y \in FM(X)$ and $\Phi: D(X) \to D(Y)$ an equivalence of triangulated categories. Let $P \in D(X \times Y)$ be a kernel of $\Phi$. Here the definition of kernel will be given in Definition 2.1. Let $\Psi: D(Y) \to D(X)$ be a quasi-inverse of $\Phi$, and $E \in D(X \times Y)$ be a kernel of $\Psi$. Then we prove that

- $\Phi$ induces an isomorphism of vector spaces, $H^0(X, mK_X) \to H^0(Y, mK_Y)$ for $m \in \mathbb{Z}$. This is also proved in [Cal03]. Let $E \in |mK_X|$ corresponds to $E^\dagger \in |mK_Y|$.

- $\Phi$ induces a bijection between $\pi_0(\bigcap_{i=1}^n E_i)$ and $\pi_0(\bigcap_{i=1}^n E_i^\dagger)$. Here $E_i \in |m_i K_X|$ for $i = 1, \cdots, n$ and $m_i \in \mathbb{Z}$. $n$ and $m_i$ are arbitrary, and $\pi_0$ means connected component. Let $C \in \pi_0(\bigcap_{i=1}^n E_i)$ corresponds to $C^\dagger \in \pi_0(\bigcap_{i=1}^n E_i^\dagger)$.

Then the main theorem is the following:
Theorem 1.1. Assume that \( C \) and \( C^\dagger \) satisfy the following conditions:

- \( C \) and \( C^\dagger \) are complete intersections.
- \( P_L \otimes O_{C \times Y}, E_L \otimes O_{C \times Y} \) are sheaves, up to shift.

Then there exists an equivalence of triangulated categories \( \Phi_C : D(C) \to D(C^\dagger) \) such that the following diagram is 2-commutative:

\[
\begin{array}{ccc}
D(X) & \xrightarrow{L_{C^\ast}} & D(C) & \xrightarrow{i_{C^\ast}} & D(X) \\
\downarrow{\Phi} & & \downarrow{\Phi_C} & & \downarrow{\Phi} \\
D(Y) & \xrightarrow{L_{C^\dagger \ast}} & D(C^\dagger) & \xrightarrow{i_{C^\dagger \ast}} & D(Y).
\end{array}
\]

The assumptions are satisfied if \( |m_iK_X| \) are free, \( E_i \in |m_iK_X| \) are generic members, and \( P \) is a sheaf, up to shift. The above theorem says that “If there are many members in \( |mK_X| \), then we can reduce the problem of describing \( FM(X) \) to lower dimensional case.” As an application, we will study Fourier-Mukai transforms when \( \kappa(X) = \dim X - 1 \). Using this method, we will give a generalization of the theorem of Bondal and Orlov [BD01], and determine \( FM(X) \) when \( \dim X = 3 \) and \( \kappa(X) = 2 \).

In the viewpoint of birational geometry, there are some works concerning derived categories and birational geometry. For example Bridgeland [Bri02] constructed smooth 3-dimensional flops as a moduli space of perverse point sheaves, which are objects in derived category. Surprisingly, his method gives an equivalence of derived categories under flops simultaneously. This result was generalized by Chen [Che02] and Kawamata [Kaw02]. The existence of flops and flips is a very difficult problem in birational geometry, and Bridgeland’s result gives a possibility of treating the problem by moduli theoretic method.

2 Derived categories and Serre functors

Notations and conventions

- Throughout this paper, we assume all the varieties are defined over \( \mathbb{C} \).
- For smooth projective variety \( X \), let \( D(X) := D^b(\text{Coh}(X)) \), i.e. bounded derived category of coherent sheaves on \( X \). The translation functor is written [1], and the symbol \( E[m] \) means the object \( E \) shifted to the left by \( m \) places.
- \( \omega_X \) means canonical bundle, and \( K_X \) means canonical divisor. For a Cartier divisors \( D \), we write the global section of \( \mathcal{O}_X(D) \) as \( H^0(X, D) \), \( |D| \) means linear system, and \( \text{Bs} |D| \) is a base locus as usual.
- For the derived functors, we omit \( R \) or \( L \) if the functors we want to derive are exact.
- For another variety \( Y \), we denote by \( p_i \), projections \( p_1 : X \times Y \to X, p_2 : X \times Y \to Y \).
- For a closed point \( x \in X \), \( \mathcal{O}_x \) means a skyscraper sheaf supported at \( x \).

In this section we recall some definitions and properties concerning derived categories.
Definition 2.1. For an object $P \in D(X \times Y)$, we define a functor $\Phi^P_{X \to Y} : D(X) \to D(Y)$ by

$$\Phi^P_{X \to Y}(E) := Rp_2^*(p_1^* E \otimes P).$$

The object $P$ is called the kernel of $\Phi^P_{X \to Y}$. For a morphism $\mu : P_1 \to P_2$ in $D(X \times Y)$, we also denote by $\Phi^P_{X \to Y} \mu$ the natural transform:

$$\Phi^P_{X \to Y} : \Phi^{P_1}_{X \to Y} \to \Phi^{P_2}_{X \to Y},$$

induced by $\mu$.

The functor of the form $\Phi^P_{X \to Y}$ is called an integral functor. If an integral functor gives an equivalence of categories, then it is called a Fourier-Mukai transform. The following theorem is fundamental in this paper:

Theorem 2.2 (Orlov [Orl97]). Let $\Phi : D(X) \to D(Y)$ gives an equivalence of $\mathbb{C}$-linear triangulated categories. Then there exists an object $P \in D(X \times Y)$ such that $\Phi$ is isomorphic to the functor $\Phi^P_{X \to Y}$. Moreover $P$ is uniquely determined up to isomorphism.

Next we introduce the notion of Fourier-Mukai partners.

Definition 2.3. We define $FM(X)$ as the set of isomorphism classes of smooth projective varieties $Y$, which has an equivalence of $\mathbb{C}$-linear triangulated categories, $\Phi : D(X) \to D(Y)$. If $Y \in FM(X)$, $Y$ is called a Fourier-Mukai partner of $X$.

By Theorem 2.2, if $Y \in FM(X)$, then $D(Y)$ is related to $D(X)$ by a Fourier-Mukai transform. To study the relation between derived categories and canonical divisors, the following Serre functor plays an important role.

Definition 2.4. Let $T$ be a $\mathbb{C}$-linear triangulated category of finite type. An exact equivalence $S : T \to T$ is called a Serre functor if there exists a bifunctorial isomorphism

$$\Hom(E, F) \to \Hom(F, S(E))^*$$

for $E, F \in T$.

As in [BD01] Proposition 1.5, if a Serre functor exists, then it is unique up to canonical isomorphism. If $X$ is a smooth projective variety and $T = D(X)$, then Serre duality implies Serre functor $S_X$ is given by $S_X(E) = E \otimes \omega_X[\dim X]$.

Proposition-Definition 2.5. Let $X, Y, Z$ be varieties, and $p_{ij}$ be projections from $X \times Y \times Z$ onto corresponding factors. Let us take $F \in D(X \times Y)$, $G \in D(Y \times Z)$. We define $G \circ F \in D(X \times Z)$ as

$$G \circ F := Rp_{13*}(p_{12}^* F \otimes p_{23}^* G).$$

Then we have the isomorphism of functors: $\Phi^{G}_{Y \to Z} \circ \Phi^{F}_{X \to Y} \cong \Phi^{G \circ F}_{X \to Z}$, and for a morphism $\mu : F_1 \to F_2$ in $D(X \times Y)$, the isomorphism of natural transforms:

$$\Phi^{G}_{Y \to Z} \circ \Phi^{F}_{X \to Y} \cong \Phi^{G \circ F}_{X \to Z} : \Phi^{G \circ F}_{X \to Z} \to \Phi^{G \circ F}_{X \to Z}.$$

Moreover the operation $\circ$ is associative, i.e. $(H \circ G) \circ F \cong H \circ (G \circ F)$.

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Proof. The proof of $\Phi^T_{Y \to Z} \circ \Phi^T_{X \to Y} \cong \Phi^T_{X \to Z}$ is seen in several references. For example, see [Che02, Proposition 2.3]. The same proof works for natural transforms, formally replacing $F$ by $\mu$. We can check the operation $\circ$ is associative by the same method, but we would like to give the proof for the lack of references. Let $X, Y, Z, W$ be varieties, and take $F \in D(X \times Y)$, $G \in D(Y \times Z)$, and $H \in D(Z \times W)$. We change the index $p_{ij}$ to $p_{XY}$ etc. Let $p_{**}, q_{**}, r_{**},$ and $s_{**}$ be projections, given as in the following diagram:

\[
\begin{array}{c}
X \times Y \times Z \\
\downarrow p_{XY} \downarrow p_{XZ} \downarrow p_{YZ}
\end{array} \quad \begin{array}{c}
X \times Z \times W \\
\downarrow q_{XZ} \downarrow q_{ZW}
\end{array}
\]

\[
\begin{array}{c}
X \times Y \times W \\
\downarrow r_{XY} \downarrow r_{XW}
\end{array} \quad \begin{array}{c}
Y \times Z \times W \\
\downarrow s_{YZ} \downarrow s_{ZW}
\end{array}
\]

Let $\pi_{**}$ or $\pi_{***}$ be projections from $X \times Y \times Z \times W$ onto corresponding factors, for example as in the following diagram:

\[
\begin{array}{c}
X \times Y \times Z \times W \\
\downarrow p_{XY} \downarrow p_{XZ} \downarrow p_{YZ} \downarrow p_{ZW}
\end{array}
\]

Then $H \circ (G \circ F)$ is calculated as

\[
H \circ (G \circ F) \cong Rq_{XW*}(q_{XZ}(G \circ F) \otimes q_{ZW}H)
\]

\[
\cong Rq_{XW*}(q_{XZ}Rp_{XZ*}(p_{XY}^*F \otimes p_{YZ}^*G) \otimes q_{ZW}^*H)
\]

\[
\cong Rq_{XW*}(R\pi_{XZW*}(\pi_{XYZ}^*(p_{XY}^*F \otimes p_{YZ}^*G) \otimes \pi_{XZW}^*q_{ZW}^*H))
\]

\[
\cong R\pi_{XW*}(\pi_{XY}^*F \otimes \pi_{YZ}^*G \otimes \pi_{ZW}^*H).
\]

Here the third isomorphism follows from flat base change, and fourth isomorphism from projection formula. Similarly, $(H \circ G) \circ F$ is calculated as

\[
(H \circ G) \circ F \cong Rr_{XW*}(r_{YW}^*(H \circ G) \otimes r_{XY}^*F)
\]

\[
\cong Rr_{XW*}(r_{YW}^*R\pi_{XW*}(s_{YZ}^*G \otimes s_{ZW}^*H) \otimes r_{XY}^*F)
\]

\[
\cong Rr_{XW*}(R\pi_{XW*}(s_{YZ}^*G \otimes s_{ZW}^*H) \otimes \pi_{XW}^*F)
\]

\[
\cong R\pi_{XW*}(s_{YZ}^*G \otimes s_{ZW}^*H \otimes \pi_{XW}^*F).
\]

Therefore we obtain the isomorphism, $H \circ (G \circ F) \cong (H \circ G) \circ F$. \hfill \qed

Here we give one remark. The category $D(X \times Y)$ is like a category of functors from $D(X)$ to $D(Y)$. In fact an object $F \in D(X \times Y)$ corresponds to a functor $\Phi^T_{X \to Y}$, and a morphism $F \to G$ gives a natural transform $\Phi^T_{X \to Y} \rightarrow \Phi^T_{X \to Y}$. But as remarked in [Cal03], this correspondence is not faithful, i.e. non-trivial morphism $F \to G$ may induce a trivial natural transform. Though natural transform is a categorical concept, it is not useful for our purpose. So sometimes we use
the objects of $D(X \times Y)$ instead of functors, and treat their morphisms as if they are natural transforms.

3 Moduli spaces of stable sheaves

In this section, we introduce the notations of the moduli spaces of stable sheaves, and recall some properties. These are used for the applications of Theorem 1.1. The details are written in the book [DM97]. Let $X$ be a projective scheme and $H$ be a polarization. For a non-zero object $E \in \text{Coh}(X)$, its Hilbert polynomial has the following form:

$$\chi(E \otimes H^\otimes m) = \sum_{i=0}^{\dim(Supp E)} \frac{\alpha_i(E)}{i!} m^i \quad (\alpha_i(E) \in \mathbb{Z}, d = \dim(Supp E)).$$

We define a rank of $E$ and its reduced Hilbert polynomial by

$$\text{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}, \quad p(E, H) := \frac{\chi(E \otimes H^\otimes m)}{\alpha_d(E)}.$$

Now let us introduce the order on $\mathbb{Q}[m]$ as follows: if $p, p' \in \mathbb{Q}[m]$, then $p \leq p'$ if and only if $p(m) \leq p'(m)$ for sufficiently large $m$. We denote $p < p'$ if $p(m) < p'(m)$ for sufficiently large $m$.

**Definition 3.1.** A non-zero object $E \in \text{Coh}(X)$ is said to be $H$-semistable if $E$ is pure, i.e. there exists no subsheaf of dimension lower than $d$, and for all subsheaves $F \subseteq E$, we have $p(F, H) \leq p(E, H)$. $E$ is said to be $H$-stable if $E$ is $H$-semistable and for all subsheaves $F \subseteq E$, we have $p(F, H) < p(E, H)$.

Using the above stability, we can consider the moduli spaces of stable (semistable) sheaves. Also we can consider the relative version of the moduli spaces of such sheaves, under projective morphism $f: X \to S$ and $f$-ample divisor $H$. Let $T$ be a $S$-scheme, and $p_X: X \times_S T \to X$ and $p_T: X \times_S T \to T$ be projections. We define a contravariant functor $\overline{M}^H(X/S): (\text{Sch}/S)^e \to (\text{Sets})$ as follows:

$$\overline{M}^H(X/S)(T) := \left\{ F \in \text{Coh}(X \times_S T), \text{ which are flat over } T, \text{ and for all geometric points } \text{Spec } k(t) \to T, F|_{X \times \text{Spec } k(t)} \right\} / \sim.$$

Here for $E, E' \in \text{Coh}(X \times_S T)$, the equivalence relation $\sim$ is the following:

$$E \sim E' \quad \text{def} \quad E \cong E' \otimes p^*_T \mathcal{L} \quad \text{for some } \mathcal{L} \in \text{Pic}(T).$$

Then there exists a projective scheme

$$\overline{M}^H(X/S) \to S,$$

which corepresents $\overline{M}^H(X/S)$. Let $M^H(X/S) \subseteq \overline{M}^H(X/S)$ be a subset which corresponds to stable sheaves. It is known that $M^H(X/S)$ is an open subscheme of $\overline{M}^H(X/S)$, for example see [DM97].

**Definition 3.2.** Let $M \subset M^H(X/S)$ be an irreducible component. $M$ is called fine if it is projective over $S$ and there exists a universal sheaf on $X \times_S M$.

The following theorem is due to Mukai [Muk87].
Theorem 3.3 (Mukai [Muk87]). For \( x \in M \), we denote by \( E_x \) the corresponding stable sheaf. Then there exists a universal family on \( X \times_S M \) if
\[
g.c.d\{\chi(E_x \otimes N) \mid N \text{ is a vector bundle on } X\} = 1
\]
holds.

We have the following criteria to find the fine moduli scheme:

Lemma 3.4. If \( g.c.d\{\chi(E_x \otimes H^\otimes n) \mid n \in \mathbb{Z}\} = 1 \), then \( M \) is projective over \( S \), i.e. there exists no properly semistable boundary. Hence \( M \) is fine by Theorem 3.3.

Proof. Indeed if there exists some \( x \in \overline{M} \setminus M \), then there exists a subsheaf \( F \subseteq E_x \) such that \( p(F,H) = p(E_x,H) \). If we take \( n_i, \omega_i \in \mathbb{Z} \) such that \( \sum \omega_i \cdot \chi(E_x \otimes H^\otimes n_i) = 1 \), then
\[
\sum \omega_i \cdot \chi(F \otimes H^\otimes n_i) = \alpha_d(F)/\alpha_d(E_x).
\]
Since the left hand side is an integer and \( 0 < \alpha_d(F)/\alpha_d(E_x) < 1 \), we have a contradiction. So by the above theorem \( M \) is fine. \( \square \)

Finally we recall the significant result on the moduli spaces of stable sheaves and derived categories, established by Bridgeland and Maciocia [BM02]. We say that a family of sheaves \( \{U_p\}_{p \in M} \) on \( X \) is complete if the Kodaira-Spencer map
\[
T_pM \to \text{Ext}^1_X(U_p, U_p)
\]
is bijective.

Theorem 3.5 (Bridgeland-Maciocia [BM02]). Let \( X \) be a smooth projective variety of dimension \( n \) and \( \{U_p\}_{p \in M} \) be a complete family of simple sheaves on \( X \) parameterized by an irreducible projective scheme \( M \) of dimension \( n \). Suppose that \( \text{Hom}_X(U_{p_1}, U_{p_2}) = 0 \) for \( p_i \in M, p_1 \neq p_2 \) and the set
\[
\Gamma(\mathcal{U}) := \{(p_1, p_2) \in M \times M \mid \text{Ext}^i_X(U_{p_1}, U_{p_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}
\]
has \( \dim \Gamma(\mathcal{U}) \leq n + 1 \). Suppose also that \( U_p \otimes \omega_X \cong U_p \) for all \( p \in M \). Then \( M \) is a nonsingular projective variety and \( \Phi_{M \to X} : D(M) \to D(X) \) is an equivalence.

4 Correspondences of canonical divisors

In this section we fix two smooth projective varieties \( X \) and \( Y \), such that \( Y \in FM(X) \). The purpose of this section is to establish the relation between the canonical divisors of \( X \) and \( Y \), and state our main theorem. We fix the following notation:

- \( \Phi : D(X) \to D(Y) \) gives an equivalence, and \( \mathcal{P} \in D(X \times Y) \) is a kernel of \( \Phi \).
- \( \Psi : D(Y) \to D(X) \) is a quasi-inverse of \( \Phi \), and \( \mathcal{E} \in D(X \times Y) \) is a kernel of \( \Psi \).
- \( S_X := \otimes \omega_X[\dim X] : D(X) \to D(X) \) is a Serre functor of \( D(X) \).
Since Serre functor is categorical, we have the isomorphism of functors,
\[ \tau : \Phi \circ S_X \sim S_Y \circ \Phi. \]
Note that the kernel of left hand side is \( \mathcal{P} \otimes p_1^* \omega_X[\dim X] \), and right hand side is \( \mathcal{P} \otimes p_2^* \omega_Y[\dim Y] \).
So by Theorem 2.2, we have the isomorphism,
\[ \rho : \mathcal{P} \otimes p_1^* \omega_X[\dim X] \sim \mathcal{P} \otimes p_2^* \omega_Y[\dim Y]. \]
Therefore \( \dim X = \dim Y \), and there exist isomorphism for all \( m \in \mathbb{Z} \),
\[ \rho_m : \mathcal{P} \otimes p_1^* \omega_X \otimes m \sim \mathcal{P} \otimes p_2^* \omega_Y \otimes m. \]
Therefore we can see the following proposition:

**Proposition 4.1.** \( \{ \rho_m \}_{m \in \mathbb{Z}} \) induce the isomorphism of graded \( \mathbb{C} \)-algebras:
\[
\left\{ \rho'_m \right\}: \bigoplus_{m \in \mathbb{Z}} \text{Hom}_{X \times Y}(\mathcal{P}, \mathcal{P} \otimes p_1^* \omega_X \otimes m) \sim \bigoplus_{m \in \mathbb{Z}} \text{Hom}_{X \times Y}(\mathcal{P}, \mathcal{P} \otimes p_2^* \omega_Y \otimes m).
\]

**Proof.** Clear by the above argument.

Next we will compare the vector spaces \( H^0(X, mK_X) \) and \( \text{Hom}_{X \times Y}(\mathcal{P}, \mathcal{P} \otimes p_1^* \omega_X \otimes m) \). Since \( \Phi \) gives an identification of categories, \( \Phi \) must give the bijection between functors \( D(X) \to D(X) \) and functors \( D(X) \to D(Y) \). In this respect, the following lemma is obvious:

**Lemma 4.2.** The following functor,
\[
\mathcal{P} \circ : D(X \times X) \ni a \mapsto \mathcal{P} \circ a \in D(X \times Y)
\]
gives equivalence.

**Proof.** Let \( \Psi \) be a quasi-inverse of \( \Phi \), and \( \mathcal{E} \in D(X \times Y) \) be a kernel of \( \Psi \). Let \( \Delta_X \subset X \times X \) and \( \Delta_Y \subset Y \times Y \) be diagonals. Note that the operations \( \mathcal{O}_{\Delta_X} \circ, \mathcal{O}_{\Delta_Y} \circ \) induce identities. Since \( \mathcal{E} \circ \mathcal{P} \cong \mathcal{O}_{\Delta_X}, \mathcal{P} \circ \mathcal{E} \cong \mathcal{O}_{\Delta_Y} \), the following functor:
\[
\mathcal{E} \circ : D(X \times Y) \ni b \mapsto \mathcal{E} \circ b \in D(X \times X)
\]
gives a quasi-inverse by Proposition-Definition 2.5.

As in the same way, we have equivalence of categories:
\[
\circ \mathcal{P} : D(Y \times Y) \ni a \mapsto a \circ \mathcal{P} \in D(X \times Y).
\]
We have the following lemma:

**Lemma 4.3.** The following diagrams are 2-commutative:
\[
\begin{array}{ccc}
D(X \times X) & \xrightarrow{\mathcal{P} \circ} & D(X \times Y) \\
\Delta \uparrow & & \Delta \uparrow \\
D(X) & & D(Y)
\end{array}
\]
\[
\begin{array}{ccc}
D(Y \times Y) & \xrightarrow{\circ \mathcal{P}} & D(X \times Y) \\
\Delta \uparrow & & \Delta \uparrow \\
D(Y) & & D(Y)
\end{array}
\]

Here \( \Delta \) means diagonal embedding.
Lemma 4.5. \( \sigma \) be the isomorphism of functors, induced by \( \tau \) here

\[ H \]

and the third isomorphism follows from projection formula.

\[ \Phi \]

As the immediate corollary, we have

**Corollary 4.4.** \( \Phi \) induces the isomorphism of graded \( \mathbb{C} \)-algebras:

\[ \{ \phi_m \}_{m \in \mathbb{Z}} : \bigoplus_{m \in \mathbb{Z}} H^0(X, mK_X) \xrightarrow{\sim} \bigoplus_{m \in \mathbb{Z}} H^0(Y, mK_Y). \]

**Proof.** By Lemma 4.3, we have the isomorphism of graded \( \mathbb{C} \)-algebras:

\[
\begin{align*}
\bigoplus_{m \in \mathbb{Z}} \text{Hom}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X^{m}) & \xrightarrow{p_0} \bigoplus_{m \in \mathbb{Z}} \text{Hom}_{X \times Y}(P, P \otimes \mathcal{O}_X^{\otimes m}), \\
\bigoplus_{m \in \mathbb{Z}} \text{Hom}_{Y \times Y}(\Delta_* \mathcal{O}_Y, \Delta_* \omega_Y^{m}) & \xrightarrow{\sigma_p} \bigoplus_{m \in \mathbb{Z}} \text{Hom}_{X \times Y}(P, P \otimes \mathcal{O}_Y^{\otimes m}).
\end{align*}
\]

Since \( H^0(X, mK_X) = \text{Hom}_{X \times X}(\Delta_* \mathcal{O}_X, \Delta_* \omega_X^{m}) \), combining \( \rho'_m \) given in Proposition 4.1, we obtain the corollary.

Now let us interpret the isomorphism \( \phi_m : H^0(X, mK_X) \rightarrow H^0(Y, mK_Y) \) categorically. Take \( \sigma \in H^0(X, mK_X) \) and \( \sigma^\dagger := \phi_m(\sigma) \in H^0(Y, mK_Y) \). Let \( d := \dim X = \dim Y \). Then we can think \( \sigma \) and \( \sigma^\dagger \) as natural transforms,

\[ \sigma : \text{id}_X \rightarrow S^m_X[-md], \quad \sigma^\dagger : \text{id}_Y \rightarrow S^m_Y[-md] \]

Here \( S^m_X[-md] \) is a \( m \)-times composition of the shifted Serre functor, \( S_X[-d] = \otimes \omega_X \). Let

\[ \tau_m : \Phi \circ S^m_X[-md] \xrightarrow{\sim} S^m_Y[-md] \circ \Phi \]

be the isomorphism of functors, induced by \( \tau : \Phi \circ S_X \xrightarrow{\sim} S_Y \circ \Phi \) naturally.

**Lemma 4.5.** \( \sigma^\dagger \) is equal to the following composition:

\[ \text{id}_Y = \Phi \circ \text{id}_X \circ \Phi^{-1} \text{id}_{S^m_X[-dm]} \circ \Phi^{-1} \tau_m \circ \Phi \circ \Phi^{-1} = S^m_Y[-dm]. \]
Lemma 4.6. \( \Phi \) takes \( D_E(X) \) to \( D_{E^\dagger}(Y) \).

**Proof.** Take \( a \in \text{Coh}(X) \cap D_E(X) \). Let \( \sigma^I : \text{id}_X \to S_X^{lmd}[-lmd] = \otimes \omega_X^{\otimes lm} \) be \( l \)-times composition of \( \sigma \). Then

\[
\sigma^I(a) : a \to a \otimes \omega_X^{\otimes lm}
\]

are zero-maps for sufficiently large \( l \). Then by the above categorical interpretation of \( \sigma^I \), we have

\[
(\sigma^I)(\Phi(a)) : \Phi(a) \to \Phi(a) \otimes \omega_Y^{\otimes lm}
\]

are also zero-maps. Since \((\sigma^I)^I\) is a natural transform, locally multiplying the defining equation of \( lE^I \), we have \( \text{Supp} \Phi(a) \subset E^I \). Since \( D_E(X) \) is generated by \( \text{Coh}(X) \cap D_E(X) \), the lemma follows. \( \square \)

For the sake of applications, it is convenient to generalize the above lemma to the intersections of canonical divisors.

**Corollary 4.7.** Take \( E_i \in |m_iK_X| \) and their corresponding divisors \( E_i^\dagger \in |m_iK_Y| \) for \( i = 1, 2, \ldots, n \). There exists a one-to-one correspondence,

\[
\pi_0(\cap_{i=1}^n E_i) \ni C \mapsto C^\dagger \in \pi_0(\cap_{i=1}^n E^\dagger_i),
\]

such that \( \Phi \) takes \( D_C(X) \) to \( D_{C^\dagger}(Y) \). Here \( \pi_0 \) means connected component.

**Proof.** Lemma 4.6 shows that \( \Phi \) takes \( D_{E_i}(X) \) to \( D_{E_i^\dagger}(Y) \). Take a connected component \( C \subset \cap_{i=1}^n E_i \). Since

\[
\text{Hom}_Y(\Phi(\mathcal{O}_{C_{\text{red}}}), \Phi(\mathcal{O}_{C_{\text{red}}})) = \text{Hom}_X(\mathcal{O}_{C_{\text{red}}}, \mathcal{O}_{C_{\text{red}}}) = C,
\]

\( \text{Supp} \Phi(\mathcal{O}_{C_{\text{red}}}) \) is connected. Therefore there exists a unique connected component \( C^\dagger \subset \cap_{i=1}^n E^\dagger_i \) such that \( \text{Supp} \Phi(\mathcal{O}_{C_{\text{red}}}) \subset C^\dagger \). We show \( \Phi \) takes \( D_C(X) \) to \( D_{C^\dagger}(Y) \). It suffices to show \( \Phi \) takes \( \text{Coh}(\mathcal{O}_C) \) to \( D_{C^\dagger}(Y) \). Take a closed point \( x \in C \). Then \( \text{Supp}(\Phi(\mathcal{O}_x)) \) is connected by the same reason. Since there exists a non-trivial morphism \( \mathcal{O}_{C_{\text{red}}} \to \mathcal{O}_x \), we have \( \Phi(\mathcal{O}_x) \in D_{C^\dagger}(Y) \). Let us take a simple \( \mathcal{O}_C \)-module \( \mathcal{F} \). Then since \( \text{Supp}(\Phi(\mathcal{F})) \) is connected and there exists a non-trivial morphism \( \mathcal{F} \to \mathcal{O}_x \) for some closed point \( x \in C \), we have \( \Phi(\mathcal{F}) \in D_{C^\dagger}(Y) \). The lemma follows by taking Harder-Narasimhan filtrations. \( \square \)

Unfortunately the natural functor \( D(C) \to D_C(X) \) does not give an equivalence. (In general, latter has bigger Ext-groups.) However the existence of equivalence between \( D_C(X) \) and \( D_{C^\dagger}(Y) \) leads us to the speculation that \( D(C) \) and \( D(C^\dagger) \) may be equivalent. If \( D(C) \) and \( D(C^\dagger) \) are equivalent, then the relation between \( C \) and \( C^\dagger \) will give us information of the relation between \( X \) and \( Y \). One of the purpose of this paper is to claim this speculation is true, under some technical conditions. We assume the following conditions on \( C, C^\dagger \), and \( \mathcal{P}, \mathcal{E} \in D(X \times Y) \). Recall that \( \mathcal{P}, \mathcal{E} \) are kernels of \( \Phi \) and \( \Phi^{-1} \).
• $C$ and $C^\dagger$ are complete intersections.

• $\mathcal{P} \otimes \mathcal{O}_{C \times Y}$ and $\mathcal{E} \otimes \mathcal{O}_{C \times Y}$ are sheaves, up to shift.

These conditions are satisfied, for example, $|m_i K_X|$ are free and $E_i$ are generic members, and $\mathcal{P}$ is a sheaf. Now we can state our main theorem.

**Theorem 4.8.** Under the above conditions, there exists equivalence $\Phi_C : D(C) \to D(C^\dagger)$ such that the following diagram is 2-commutative:

$$
\begin{array}{ccc}
D(X) & \xrightarrow{\mathcal{L}_C^*} & D(C) \\
\Phi & \downarrow & \Phi \\
D(Y) & \xrightarrow{\mathcal{L}_{C^\dagger}^*} & D(C^\dagger)
\end{array}
$$

Here $i_C$, $i_{C^\dagger}$ are inclusions of $C$, $C^\dagger$ into $X$ and $Y$ respectively.

## 5 Proof of Theorem [4.8]

In this section, we give the proof of Theorem 4.8. We use the notation of the previous section. At first, we explain the plan of the proof. We will divide the proof into 4 Steps. In Step 1 and 2, we will show there exists an isomorphism, $\mathcal{P} \otimes \mathcal{O}_{C \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times E^1}$. Using this and the assumptions, we will find a $\mathcal{P}_C \in D(C \times C^\dagger)$, and construct a functor $\Phi_C : D(C) \to D(C^\dagger)$. In Step 4 and 5, we will show $\Phi_C$ gives the desired equivalence.

**Step 1.** There exists an isomorphism $\mathcal{P} \otimes \mathcal{O}_{E_i \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times E^1}$.  

**Proof.** We omit the index $i$, and write $E_i$ as $E$, etc. We have the following exact sequences:

$$
\begin{align*}
0 & \to \mathcal{O}_X \xrightarrow{\sigma} \omega_X^{\otimes m} \to \mathcal{O}_E \otimes \omega_X^{\otimes m} \to 0 \\
0 & \to \mathcal{O}_Y \xrightarrow{\sigma'} \omega_Y^{\otimes m} \to \mathcal{O}_{E^1} \otimes \omega_Y^{\otimes m} \to 0.
\end{align*}
$$

Applying $p_1^*(\mathcal{P}) \otimes \mathcal{P}$ and $p_2^*(\mathcal{P}) \otimes \mathcal{P}$ respectively, we obtain the distinguished triangles:

$$
\begin{align*}
\mathcal{P} \xrightarrow{id \otimes p_1^*} \mathcal{P} \otimes p_1^* \omega_X^{\otimes m} & \to \mathcal{P} \otimes \mathcal{O}_{E \times Y} \otimes p_1^* \omega_X^{\otimes m} \to \mathcal{P}[1] \\
\mathcal{P} \xrightarrow{id \otimes p_2^*} \mathcal{P} \otimes p_2^* \omega_Y^{\otimes m} & \to \mathcal{P} \otimes \mathcal{O}_{X \times E^1} \otimes p_2^* \omega_Y^{\otimes m} \to \mathcal{P}[1].
\end{align*}
$$

On the other hand, byLemma 4.3 and the definition of $\phi_m$ given in Corollary 4.4 we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{id \otimes p_1^*} & \mathcal{P} \otimes p_1^* \omega_X^{\otimes m} \\
\mathcal{P} [\rho_m] & \downarrow & \mathcal{P} \otimes \mathcal{O}_{E \times Y} \otimes p_1^* \omega_X^{\otimes m} \\
\rho_m & \downarrow & \mathcal{P} \otimes \mathcal{O}_{X \times E^1} \otimes p_2^* \omega_Y^{\otimes m}.
\end{array}
$$

Here $\rho_m$ is an isomorphism constructed in the previous section. Therefore there exists a (not necessary unique) isomorphism, $\mathcal{P} \otimes \mathcal{O}_{E \times Y} \otimes p_1^* \omega_X^{\otimes m} \cong \mathcal{P} \otimes \mathcal{O}_{X \times E^1} \otimes p_2^* \omega_Y^{\otimes m}$.

Since $\mathcal{P} \otimes p_1^* \omega_X^{\otimes m} \cong \mathcal{P} \otimes p_2^* \omega_Y^{\otimes m}$, we have an isomorphism, $\mathcal{P} \otimes \mathcal{O}_{E \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times E^1}$.
**Step 2.** There exists an isomorphism,

\[ \mathcal{P} \otimes O_{C \times Y} \cong \mathcal{P} \otimes O_{X \times C^\dagger}. \]

**Proof.** By using the isomorphism of Step 1 \( n \)-times, we can get the isomorphism:

\[ \mathcal{P} \otimes \left( \bigotimes_{1 \leq i \leq n} O_{E_i \times Y} \right) \cong \mathcal{P} \otimes \left( \bigotimes_{1 \leq i \leq n} O_{X \times E_i^\dagger} \right). \]

On the other hand, we have

\[ \bigotimes_{1 \leq i \leq n} O_{E_i \times Y} = \bigoplus_{C \in \pi_0(\bigcap_{i=1}^n E_i)} p_1^* A_C, \quad \bigotimes_{1 \leq i \leq n} O_{X \times E_i^\dagger} = \bigoplus_{C' \in \pi_0(\bigcap_{i=1}^n E_i^\dagger)} p_2^* B_{C'}, \]

for some \( A_C \in D_C(X), B_{C'} \in D_{C'}(Y) \). Therefore we have the following isomorphism:

\[ \bigoplus_{C \in \pi_0(\bigcap_{i=1}^n E_i)} \mathcal{P} \otimes p_1^* A_C \cong \bigoplus_{C' \in \pi_0(\bigcap_{i=1}^n E_i^\dagger)} \mathcal{P} \otimes p_2^* B_{C'}. \]

Now we have the following lemma:

**Lemma 5.1.** \( \mathcal{P} \otimes p_1^* A_C, \mathcal{P} \otimes p_2^* B_{C^\dagger} \) are supported on \( C \times C^\dagger \).

**Proof.** We show \( \mathcal{P} \otimes p_1^* A_C \) is supported on \( C \times C^\dagger \). The rest case follows similarly. We can write,

\[ \mathcal{P} \otimes p_1^* A_C \cong \bigoplus_{C' \in \pi_0(\bigcap_{i=1}^n E_i^\dagger)} \mathcal{R}_{C'}, \]

where \( \mathcal{R}_{C'} \) is supported on \( C \times C' \). Take \( C' \neq C \in \pi_0(\bigcap E_i) \) and assume \( \mathcal{R}_{C'} \) is not zero. Let us take a sufficiently ample line bundle \( \mathcal{L} \) on \( X \). Since \( \Phi(A_C \otimes \mathcal{L}) \in D_{C^\dagger}(Y) \), we have \( R_{p_{2*}}(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L}) = 0 \). On the other hand, if \( \mathcal{L} \) is sufficiently ample and \( H^q(\mathcal{R}_{C'}) \neq 0 \), then \( p_{2*}(H^q(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L}) \neq 0 \) and \( R^p_{p_{2*}}(H^q(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L}) = 0 \) for \( p > 0 \). Since there exists a following spectral sequence:

\[ E_2^{p,q} = R^p p_{2*}(H^q(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L} \Rightarrow R^{p+q} p_{2*}(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L}), \]

we have \( R^q_{p_{2*}}(\mathcal{R}_{C'} \otimes p_1^* \mathcal{L}) \neq 0 \). But this is a contradiction. \( \square \)

By the lemma above we have \( \mathcal{P} \otimes p_1^* A_C \cong \mathcal{P} \otimes p_2^* B_{C^\dagger} \). Since we have assumed \( C \) and \( C^\dagger \) are complete intersections, we have \( A_C = O_C, B_{C^\dagger} = O_{C^\dagger} \) in our case. Combining these, we have the desired isomorphism:

\[ \mathcal{P} \otimes O_{C \times Y} \cong \mathcal{P} \otimes O_{X \times C^\dagger}. \]

\( \square \)
By the assumptions, the object $\mathcal{P} \otimes \mathcal{O}_{C \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times C^\dagger}$ is a sheaf, up to shift. This sheaf is $\mathcal{O}_{C \times Y}$-module and also $\mathcal{O}_{X \times C^\dagger}$-module. Hence this object is a $\mathcal{O}_{C \times C^\dagger}$-module, so there exists an object $\mathcal{P}_C \in D(C \times C^\dagger)$, such that

$$\mathcal{P} \otimes \mathcal{O}_{C \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times C^\dagger} \cong i_{C \times C^\dagger}^* \mathcal{P}_C.$$ 

Let $\Phi_C := \Phi_{C \rightarrow C^\dagger}^C : D(C) \rightarrow D(C^\dagger)$. In what follows, we don’t use the fact these are sheaves up to shift, and show that $\Phi_C$ gives a desired equivalence.

**Step 3. In the diagram of Theorem 4.8 we have the following isomorphisms of functors:**

$$\Phi_C \circ i_C^* \cong L_i^* \Phi_{X \rightarrow C^\dagger}^C,$$

$$i_{C^\dagger}^* \circ \Phi_C \cong \Phi_{X \rightarrow Y}^C (i_{C^\dagger} \times \text{id}_Y)^* \mathcal{P},$$

$$\Phi \circ i_C^* \cong \Phi_{C \rightarrow Y}^C (i_{C^\dagger} \times \text{id}_Y)^* \mathcal{P}.$$

(See the following diagram.)

**Proof.** Let us calculate $\Phi_C \circ Li_C^*$ by using Proposition-Definition 2.5. The rest formulas follow similarly. Let $q_{12} : X \times C \times C^\dagger \rightarrow X \times C$, $q_{23} : X \times C \times C^\dagger \rightarrow C \times C^\dagger$, $q_{13} : X \times C \times C^\dagger \rightarrow X \times C^\dagger$ be projections. Let $\Gamma_C \subset X \times C$ be the graph of the inclusion $i_C$. Let $j$ be the inclusion of $\Gamma_C \times C^\dagger$ into $X \times C \times C^\dagger$. Since $L_i^* = \Phi_{\Gamma_C \rightarrow C}$, we can compute the kernel of $\Phi_C \circ Li_C^*$ as follows:

$$Rq_{13*}(q_{12}^* \mathcal{O}_{\Gamma_C} \otimes q_{23}^* \mathcal{P}_C) \cong Rq_{13*}(\mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C)$$

$$\cong \mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C$$

$$\cong (i_C \times \text{id}_{C^\dagger})_* \mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C$$

$$\cong (i_C \times \text{id}_{C^\dagger})_* (\mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C)$$

$$\cong (i_C \times \text{id}_{C^\dagger})_* \mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C$$

$$\cong (i_C \times \text{id}_{C^\dagger})_* \mathcal{O}_{\Gamma_C \times C^\dagger} \otimes q_{23}^* \mathcal{P}_C$$

Here the third isomorphism follows from $q_{13} \circ j = (i_C \times \text{id}_{C^\dagger}) \circ q_{23} \circ j$ and the last isomorphism follows since $q_{23} \circ j$ is identity. 

**Step 4. $\Phi_C$ gives a desired equivalence.**

By Step 3, to prove the diagram of Theorem 4.8 commutes, we only have to check the followings hold:

$$(i_C \times \text{id}_{C^\dagger})_* \mathcal{P}_C \cong \mathcal{L}(\text{id}_X \times i_C)^* \mathcal{P},$$

$$(\text{id}_{C^\dagger} \times \text{id}_C)_* \mathcal{P}_C \cong \mathcal{L}(\text{id}_X \times \text{id}_C)^* \mathcal{P}.$$ 

There exists a following morphism:

$$\mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{O}_{X \times C^\dagger} \cong i_{C \times C^\dagger}^* \mathcal{P}_C = (\text{id}_X \times i_C)_* (i_C \times \text{id}_{C^\dagger})_* \mathcal{P}_C.$$
Taking its adjoint, we have a morphism $L(id_X \times i_{C^!})^* P \to (i_C \times id_{C^!})_* P_C$. Let us take its distinguished triangle:

$$Q \to L(id_X \times i_{C^!})^* P \to (i_C \times id_{C^!})_* P_C \to Q[1].$$

By applying $(id_X \times i_{C^!})_*$, we get the distinguished triangle,

$$(id_X \times i_{C^!})_* Q \to P \otimes \mathcal{O}_{X \times C^!} \xrightarrow{id_{C \times C^!}} i_{C \times C^!}^* P_C \to (id_X \times i_{C^!})_* Q[1].$$

So, we have $(id_X \times i_{C^!})_* Q = 0$. Therefore $Q = 0$ and the morphism $L(id_X \times i_{C^!})^* P \to (i_C \times id_{C^!})_* P_C$ is an isomorphism. We can prove the isomorphism, $(id_C \times i_{C^!})_* P_C \cong L(id_C \times id_Y)^* P$ similarly.

Finally, we prove $\Phi_C$ gives an equivalence. Let us define $\Psi_C : D(C^!) \to D(C)$ as in the same way of $\Phi_C$, from $\Psi = \Phi^{-1}$. Then the following diagram commutes:

$$
\begin{array}{ccc}
D(X) & \xrightarrow{L_{i_{C^!}}^*} & D(C) \\
\downarrow & & \downarrow \\
D(Y) & \xrightarrow{L_{i_{C^!}}^*} & D(C^!) \\
\downarrow & & \downarrow \\
D(X) & \xrightarrow{L_{i_{C^!}}^*} & D(C^!) \\
\end{array}
$$

Take a closed point $x \in C$. Then by the diagram above, $i_{C^!} \circ \Psi_C \circ \Phi_C \circ \mathcal{O}_x \cong i_{C^!} \circ \mathcal{O}_x$, so $\Psi_C \circ \Phi_C \circ \mathcal{O}_x \cong \mathcal{O}_x$. Then, by [Bri99 Lemma 4.3], kernel of $\Psi_C \circ \Phi_C$ is a sheaf on $C \times C$, therefore it must be a line bundle on its diagonal. Hence $\Psi_C \circ \Phi_C \cong \otimes \mathcal{L}_C$ for some line bundle $\mathcal{L}_C$ on $C$. But, again by the diagram above, we have $\Psi_C \circ \Phi_C \cong \otimes \mathcal{L}_C$ and $\Psi_C \circ \Phi_C \cong \text{id}$. Similarly, $\Phi_C \circ \Psi_C \cong \text{id}$. Therefore $\Phi_C$ is an equivalence and the proof of Theorem 4.8 is completed.

**Remark 5.2.** The conditions of kernels are required to find the object $P_C$ which satisfies

$$P \otimes \mathcal{O}_{C \times Y} \cong P \otimes \mathcal{O}_{X \times C^!} \cong i_{C \times C^!}^* P_C.$$

In fact, if we can find such a $P_C$, then our theorem holds by using $P_C$. In Step 3 and 4, we didn’t use the fact that these are sheaves.

**6 Fourier-Mukai transforms of varieties of $\kappa(X) = \dim X - 1$**

In this section we explain the important situation to which Theorem 4.8 can be applied. Let us consider the situation when $K_X$ (or $-K_X$) is semi-ample, i.e. $|mK_X|$ is free for some $m > 0$ (or $m < 0$). When $K_X$ is semi-ample, we have the following morphism, called Iitaka fibration:

$$\pi_X : X \to Z := \text{Proj} \bigoplus_{m \geq 0} H^0(X, mK_X).$$

Kodaira dimension of its generic fiber is zero. Let $Y \in FM(X)$ and $\Phi : D(X) \to D(Y)$ be an equivalence. Note that $K_Y$ is also semi-ample by Corollary 4.7. By Corollary 4.4, the target of its Iitaka fibration is also $Z$. Let $\pi_Y : Y \to Z$ be the Iitaka fibration. Let us take a general
closed point $p \in \mathbb{Z}$. Let $X_p := \pi^{-1}_X(p)$, and $Y_p := \pi^{-1}_{\mathcal{Y}}(p)$. Assume that kernel of $\Phi$ satisfies the condition as in Theorem 4.8 for example kernel of $\Phi$ is a sheaf. Then Theorem 4.8 states that there exists equivalence $\Phi_p : D(X_p) \to D(Y_p)$ such that the following diagram commutes:

$$
\begin{align*}
D(X) & \xrightarrow{\mathcal{L}_2^p} D(X_p) \xrightarrow{i_p^*} D(X) \\
\Phi & \downarrow \quad \Phi_p \downarrow \quad \Phi \\
D(Y) & \xrightarrow{\mathcal{L}_2^p} D(Y_p) \xrightarrow{j_p^*} D(Y).
\end{align*}
$$

Here $i_p$ and $j_p$ are inclusions, $i_p : X_p \hookrightarrow X$, $j_p : Y_p \hookrightarrow Y$. The conditions of kernels are satisfied if $\kappa(X) = \dim X - 1$. Note that Fourier-Mukai partners of the varieties of $\kappa(X) = \dim X$ are studied in [Kaw02].

**Theorem 6.1.** Let $X$ be a smooth projective variety such that $K_X$ is semi-ample, and $\kappa(X) = \dim X - 1$. Let $Y \in FM(X)$ and $\Phi : D(X) \to D(Y)$ be an equivalence. Then in the above notations, there exists an equivalence $\Phi_p : D(X_p) \to D(Y_p)$ such that the diagram (\textcircled{}) commutes.

**Proof.** Let $\mathcal{P} \in D(X \times Y)$ be a kernel of $\Phi$. It suffices to show $\mathcal{P} \otimes \mathcal{O}_{X_p \times Y}$ is a sheaf, up to shift. Note that

$$
\mathcal{P} \otimes \mathcal{O}_{X_p \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X \times Y},
$$

by Step 2 of Theorem 4.8. By taking the functors whose kernels are left hand side, right hand side respectively, we can obtain the isomorphism of functors:

$$
\Phi(\mathcal{L} \otimes \mathcal{O}_{X_p}) \cong \Phi(\mathcal{L}) \otimes \mathcal{O}_{Y_p}.
$$

Note that the above isomorphism can be also applied to derived categories of quasi-coherent sheaves. Let us consider $\Phi(\mathcal{O}_x)$ for $x \in X_p$. Take a general morphism:

$$
v_x : \text{Spec } \mathbb{C}[t_1, \ldots, t_{d-1}] \longrightarrow X,
$$

which takes a closed point of $\text{Spec } \mathbb{C}[t_1, \ldots, t_{d-1}]$ to $x \in X_p$. Here $d := \dim X$. Let $R_x := v_{x*} \mathbb{C}[t_1, \ldots, t_{d-1}] \in \text{QCoh}(X)$. Then $R_x \otimes \mathcal{O}_{X_p} \cong \mathcal{O}_x$, and

$$
\Phi(\mathcal{O}_x) \cong \Phi(R_x) \otimes \mathcal{O}_{Y_p} \cong j_p* \mathcal{L}_p^* \Phi(R_x).
$$

Since $Y_p$ is one-dimensional, $\mathcal{L}_p^* \Phi(R_x)$ is a direct sum of its cohomologies. Since

$$
\text{Hom}_X(\mathcal{O}_x, \mathcal{O}_x) \cong \text{Hom}_Y(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong \mathbb{C},
$$

we can conclude $\Phi(\mathcal{O}_x)$ is a coherent $\mathcal{O}_{Y_p}$-module, up to shift. We may assume $\Phi(\mathcal{O}_x)$ is a sheaf for general $x \in X_p$. Then for all $x \in X_p$, $\Phi(\mathcal{O}_x)$ is a sheaf. Hence

$$
\mathcal{P} \otimes \mathcal{O}_{X \times Y} \cong \mathcal{P} \otimes \mathcal{O}_{X_p \times Y} \otimes p_{1*}\mathcal{O}_{R_x},
$$

is a sheaf. The above object is calculated by the spectral sequence:

$$
E_2^{p,q} = \mathcal{R}_{\mathcal{P}_p}(H^q(\mathcal{A}), p_{1*}\mathcal{O}_{R_x}) \Rightarrow H^{p+q}(\mathcal{P} \otimes \mathcal{O}_{X \times Y}).
$$

Here $A := \mathcal{P} \otimes \mathcal{O}_{X_p \times Y}$. The above spectral sequence degenerates at $E_2$-terms, since $E_2^{p,q} = 0$ for $p \leq -2$. Therefore if $k \neq 0$, $H^k(A) \otimes p_{1*}\mathcal{O}_{R_x} = 0$ for $x \in X_p$. This implies $H^k(A) = 0$ for $k \neq 0$. 

\[ \square \]
As the immediate application, we generalize the theorem of Bondal and Orlov [BD01].

**Theorem 6.2.** Let $C$ be an elliptic curve, $Z$ be a smooth projective variety. Assume that $K_Z$ or $-K_Z$ is ample. Then $FM(C \times Z) = \{C \times Z\}$.

**Proof.** We show the theorem when $K_Z$ is ample. The other case follows similarly. Let us take $Y \in FM(C \times Z)$, and let $\Phi: D(C \times Z) \to D(Y)$ be an equivalence. Since $C$ is an elliptic curve, the projection $C \times Z \to Z$ gives Iitaka fibration. Note that $K_Y$ is also semi-ample, and let $\pi: Y \to Z$ be its Iitaka fibration. Take a general closed point $p \in Z$ and fix it. Let $C^\dagger := \pi^{-1}(p)$. Then we can find an object $U \in D(C \times C^\dagger)$ such that $\Phi^U_{C \to C^\dagger}: D(C) \to D(C^\dagger)$ gives equivalence by Theorem 6.1. Note that $C \cong C^\dagger$, since Fourier-Mukai partners of a curve consists of itself. On the other hand, as in Lemma 4.2, the following functor gives equivalence:

$$\circ U: D(C \times Z) \ni a \mapsto a \circ U \in D(C^\dagger \times Z).$$

Let us compose the above equivalence with $\Psi := \Phi^{-1}$. We obtain the equivalence:

$$(\circ U) \circ \Psi: D(Y) \longrightarrow D(C \times Z) \longrightarrow D(C^\dagger \times Z),$$

which takes $O_x$ to $O_{(x,p)}$ for all $x \in C^\dagger$. Therefore we obtain the birational map over $Z$ by Lemma 7.3 below,

$$f: Y \dashrightarrow C^\dagger \times Z.$$

Note that $f$ is defined on the neighborhood of $C^\dagger$. Since $Y$ and $C^\dagger \times Z$ are both minimal models, $f$ is isomorphic in codimension one. We show that $f$ is in fact isomorphism. Let us take an ample divisor $H \subset Y$, and its strict transform $H^\dagger \subset C^\dagger \times Z$. It suffices to show $H^\dagger$ is nef. But this is clear since $H^\dagger$ is effective, and we can deform $H^\dagger$ freely using translations of $C^\dagger$.

---

**7 Fourier-Mukai partners of 3-folds of $\kappa(X) = 2$**

In this section, we study $FM(X)$ when $\dim X = 3$ and $\kappa(X) = 2$. The relative moduli spaces of stable sheaves for three-dimensional Calabi-Yau fibrations are studied in [BM02]. Combining Theorem 4.8 with their results, we can study $FM(X)$ in this case. Before that, we recall some terminology of birational geometry, and give some useful lemmas.

**Definition 7.1.** Let $X$ and $Y$ be projective varieties with only canonical singularities. A birational map $\alpha: X \dashrightarrow Y$ is called crepant, if there exists a smooth projective variety $Z$ and birational morphisms $f: Z \to X$, $g: Z \to Y$, such that $\alpha \circ f = g$, and $f^*K_X = g^*K_Y$. In this case, we say $X$ and $Y$ are $K$-equivalent under $\alpha$.

The following birational transform called “flop” is a special kind of crepant birational map.

**Definition 7.2.** Let $X$ and $Y$ be projective varieties with only canonical singularities. A birational map $\alpha: X \dashrightarrow Y$ is called a flop, if there exist a normal projective variety $W$ and crepant birational morphisms $\phi: X \to W$, $\psi: Y \to W$ which satisfy the following:

- $\phi = \psi \circ \alpha$.
- $\phi$ and $\psi$ are isomorphisms in codimension one.
- Relative Picard numbers of $\phi$, $\psi$ are one.
Let $H$ be a $\phi$-ample divisor on $X$, and $H'$ be its strict transform on $Y$. Then $-H'$ is $\psi$-ample.

In this paper, we will only use flops of smooth 3-folds. In dimension three, crepant birational maps are connected by finite number of flops [Kaw02 Theorem 4.6]. Next we give some useful lemmas.

**Lemma 7.3.** Let $X$ and $Y$ be smooth projective varieties, and $\Phi: D(X) \to D(Y)$ be an equivalence. Assume for some closed point $x \in X$, we have $\dim \text{Supp} \Phi(O_x) = 0$. Then there exists an open neighborhood $U$ of $x$, and $r \in \mathbb{Z}$, such that for $x' \in U$, there exists $f(x') \in Y$ which satisfies $\Phi(O_{x'}) = \mathcal{O}_f(r)$. Moreover $X$ and $Y$ are $K$-equivalent under birational map $f: X \dasharrow Y$.

**Proof.** Since $\Phi$ gives an equivalence, we have $f$ have $\Gamma \subset \text{Supp} \Phi(O_x)$.

Then using the same argument as in [BD01 Proposition 2.2], there exists a point $y \in Y$ and $r \in \mathbb{Z}$ such that $\Phi(O_x) \cong \mathcal{O}_y[r]$. Then as in [BM01 Theorem 2.5], we can find a desired $U$ and a birational map $f$. Let $\mathcal{P} \in D(X \times Y)$ be a kernel of $\Phi$. Since $\mathcal{P} \cong p_1^*\omega_X \cong \mathcal{P} \otimes p_2^*\omega_Y$, as in Section 4, $p_1^*\omega_X$ and $p_2^*\omega_Y$ are numerically equal on $\text{Supp} \mathcal{P}$. By the construction of $f$, we have $\Gamma_f \subset \text{Supp} \mathcal{P}$, where $\Gamma_f$ is a graph of $f$. Therefore $X$ and $Y$ are $K$-equivalent under $f$. 

**Lemma 7.4.** Let $X$ and $Y$ be smooth projective varieties, and $\Phi: D(X) \to D(Y)$ gives equivalence. Then the followings hold:

(i) For a closed point $x \in X$, $\omega_Y$ is numerically zero on $\text{Supp} \Phi(O_x)$.

(ii) If $x \in \text{Bs}|mK_X|$, then $\text{Supp} \Phi(O_x) \subset \text{Bs}|mK_Y|$.

(iii) If $x \notin \text{Bs}|mK_X|$, then $\text{Supp} \Phi(O_x) \cap \text{Bs}|mK_Y| = \emptyset$.

**Proof.** (i) Since $\Phi$ and Serre functor commutes, we have $\Phi(O_x) \otimes \omega_Y \cong \Phi(O_x)$. (i) follows from this.

(ii) This follows from Lemma 4.6 immediately.

(iii) Take $x \notin \text{Bs}|mK_X|$ and assume that there exists $y \in \text{Supp} \Phi(O_x) \cap \text{Bs}|mK_Y|$. Then there exists a non-zero map $\Phi(O_x) \to \mathcal{O}_y[i]$ for some $i$. Therefore there exists a non-zero map $O_x \to \Psi(O_y)[i]$. Since $\Psi(O_y)[i]$ is supported on $\text{Bs}|mK_X|$, this is a contradiction.

Now we state our main theorem of this section.

**Theorem 7.5.** Let $X$ be a smooth projective 3-fold of $\kappa(X) = 2$. Then $Y \in FM(X)$ if and only if one of the following holds:

(1) $X$ and $Y$ are connected by finite number of flops.

(2) There exists a following diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{flops}} & J^H(d) \\
\downarrow \pi' & & \downarrow \pi \\
S & \xleftarrow{\text{flops}} & X
\end{array}
\]

where $\pi: M \to S$ is an elliptic fibration with $\omega_M = 0$, $H \in \text{Pic}(M)$ is a polarization, and $d \in \mathbb{Z}$. $J^H(d) \subset M^H(M/S)$ is an irreducible component which is fine, and contains a point corresponding to line bundles of degree $d$ on smooth fibers of $\pi$.

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"If" direction is already proved in [BM02, Theorem 8.3] and [Bri02], when S is smooth. We can check that the assumption \("S is smooth\) is not required in their proof, hence the "if" direction holds. We prove "only if" direction. Let us take \(Y \in FM(X)\). We use the same notations as in the previous sections. In particular \(\Phi: D(X) \to D(Y)\) gives equivalence, \(P \in D(X \times Y)\) is a kernel of \(\Phi\), and \(\Psi\) is a quasi-inverse of \(\Phi\). Note that, by Lemma 7.3, we may assume \(\dim \text{Supp } \Phi(\mathcal{O}_x) \geq 1\) for all \(x \in X\). In this situation, we will construct a diagram (2), or show (1) holds. We divide the proof into 5 Steps.

**Step 1. Application of Theorem 4.8**

At first, we apply Theorem 4.8, and give the preparation for the proof. Since \(\dim X = \dim Y = 3\), we can run minimal model programs, and obtain birational minimal models \(X_{\text{min}}\) and \(Y_{\text{min}}\):

\[
\begin{array}{ccc}
X & \xrightarrow{\text{MMP}} & X_{\text{min}} \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
Z & \xleftarrow{\text{MMP}} & Y_{\text{min}}
\end{array}
\]

Here \(\pi_X, \pi_Y\) are Iitaka fibrations. Note that \(\dim Z = 2\), and generic fibers of \(\pi_X, \pi_Y\) are elliptic curves. Then for sufficiently large \(m\), we obtain isomorphisms,

\[
X \setminus \text{Bs}|mK_X| \xrightarrow{\cong} X_{\text{min}} \setminus C_X, \quad Y \setminus \text{Bs}|mK_Y| \xrightarrow{\cong} Y_{\text{min}} \setminus C_Y,
\]

for some closed subsets \(C_X \subset X_{\text{min}}, C_Y \subset Y_{\text{min}}\) with \(\dim C_X \leq 1, \dim C_Y \leq 1\). Let us take general members \(E_i \in |mK_X|\), for \(i = 1, 2\). By Corollary 4.4 we have the isomorphism of linear systems:

\[
|mK_X| \xrightarrow{\sim} |mK_Y|.
\]

Let \(E_i^\dagger \in |mK_Y|\) corresponds to \(E_i\). Also note that we have the isomorphisms:

\[
H^0(X, mK_X) \cong H^0(X_{\text{min}}, mK_{X_{\text{min}}}), \quad H^0(X, mK_Y) \cong H^0(Y_{\text{min}}, mK_{Y_{\text{min}}}'),
\]

for sufficiently divisible \(m\). Let

\[
E_i^\dagger \in |mK_{X_{\text{min}}}|, \quad E_i^\dagger \in |mK_{Y_{\text{min}}}|,
\]

correspond to \(E_i, E_i^\dagger\) under the above isomorphisms respectively. Then if we choose \(E_i\) sufficiently general, then we have,

\[
E_1^\dagger \cap E_2^\dagger \cap C_X = \emptyset.
\]

Therefore we have the following decompositions:

\[
E_1 \cap E_2 = (E_1^\dagger \cap E_2^\dagger) \bigoplus \text{Bs}|mK_X|, \quad E_1^\dagger \cap E_2^\dagger = (E_1^\dagger \cap E_2^\dagger) \bigoplus \text{Bs}|mK_Y|.
\]

Now let us take \(C \in \pi_0(E_1^\dagger \cap E_2^\dagger)\). We can consider \(C\) as a curve on \(X\). Using Corollary 4.7 and Lemma 7.3, we can find \(C^\dagger \in \pi_0(E_1^\dagger \cap E_2^\dagger)\) such that \(\Phi\) takes \(D_C(X)\) to \(D_{C^\dagger}(Y)\). Now using the same argument as in Theorem 6.1, we can see \(P \otimes \mathcal{O}_{C \times Y}\) is a sheaf, up to shift. Then we can apply Theorem 4.8 so there exists an equivalence \(\Phi_C: D(C) \to D(C^\dagger)\), such that the diagram of Theorem 4.8 commutes.

**Step 2. Construction of M.**
In this step, we will construct a desired $M$. We will construct $M$ as a moduli space of stable sheaves on $Y$. Let us take $x \in C$, and consider $\Phi_C(\mathcal{O}_x) \in D(C^\dagger)$. As in Theorem 4.8, we may assume $\Phi_C(\mathcal{O}_x)$ is a simple sheaf on $C^\dagger$. Since $C^\dagger$ is an elliptic curve, $\Phi_C(\mathcal{O}_x)$ is a stable sheaf on $C^\dagger$. Let $\text{rk} \Phi_C(\mathcal{O}_x) = a$ and $\deg \Phi_C(\mathcal{O}_x) = b$. By the commutative diagram of Theorem 4.8, $\Phi(\mathcal{O}_x)$ is a stable sheaf on $Y$ supported on $C^\dagger$, with respect to any polarization. Then take a polarization $H' \in \text{Pic}(Y)$, and consider moduli space of stable sheaves $M^{H'}(Y/\text{Spec} \mathbb{C})$. Let

$$M \subset M^{H'}(Y/\text{Spec} \mathbb{C})$$

be an irreducible component, which contains a point corresponding to $\Phi(\mathcal{O}_x) \in \text{Coh}(Y)$. Note that there exists a birational map:

$$f_1: X \dashrightarrow M,$$

which takes a general point $x \in X$ to a point of $M$, corresponding to a stable sheaf $\Phi(\mathcal{O}_x)$. We show $M$ is a fine moduli scheme, or (1) holds. For $E, F \in D(X)$, we define $\chi(E, F)$ as follows:

$$\chi(E, F) := \sum (-1)^i \dim \text{Ext}^i_X(E, F).$$

Since $\chi(\Phi(\mathcal{O}_X), \Phi(\mathcal{O}_x)) = \chi(\mathcal{O}_X, \mathcal{O}_x) = 1$, Riemann-Roch implies that

$$b \cdot \text{ch}_0 \Phi(\mathcal{O}_X)^* + a(c_1(\Phi(\mathcal{O}_X)^*) \cdot C^\dagger) = 1.$$

Here $\Phi(\mathcal{O}_X)^*$ means derived dual of $\Phi(\mathcal{O}_X)$. We divide into 2-cases:

**Case 1.** $b = 0$

If $b = 0$, then $a = c_1(\Phi(\mathcal{O}_X)^*) \cdot C^\dagger = 1$. Therefore there exists an effective divisor $E$ on $Y$ such that $E \cdot C^\dagger = 1$. There exists a birational map:

$$f_2: Y \dashrightarrow M,$$

such that $f_2$ takes general point $y \in Y$ to a point corresponding to $\mathcal{O}_{C_y}(E \cap C_y - y)$, a degree zero line bundle on $C_y$. Here $C_y$ is a compact fiber of the Iitaka fibration $Y \dashrightarrow Z$, which contains $y$. Composing these we obtain a birational map,

$$f := f_2^{-1} \circ f_1: X \dashrightarrow M \dashrightarrow Y,$$

which satisfies $f(x) \in \text{Supp} \Phi(\mathcal{O}_x)$ for general $x \in X$. Therefore $\Gamma_f \subset \text{Supp} \mathcal{P}$, where $\Gamma_f$ is a graph of $f$. Since $p_1^* K_X \equiv p_2^* K_Y$ on $\text{Supp} \mathcal{P}$, it is also true on $\Gamma_f$. Therefore $X$ and $Y$ are $K$-equivalent under birational map $f$.

**Case 2.** $b \neq 0$

Let us replace $H'$ to $\det \Phi(\mathcal{O}_X)^* \pm lbH'$ for $l \gg 0$. Then we may assume $\text{g.c.d}(a(H' \cdot C^\dagger), b) = 1$. Then

$$\text{g.c.d}\{\chi(\Phi(\mathcal{O}_x) \otimes H'^{\otimes m}) \mid m \in \mathbb{Z}\} = \text{g.c.d}\{ma(H' \cdot C^\dagger) + b \mid m \in \mathbb{Z}\} = 1.$$

By Lemma 3.4, this implies that $M$ is a fine moduli scheme.

**Step 3.** $M$ is smooth and the universal sheaf $U \in \text{Coh}(Y \times M)$ gives an equivalence

$$\Phi_M := \Phi_M^{H'}: D(M) \longrightarrow D(Y).$$
Proof. For \( p \in M \), let \( \mathcal{U}_p \in \text{Coh}(Y) \) be the corresponding stable sheaf. We check that the conditions of Theorem 3.15 are satisfied. First we show \( \mathcal{U}_p \otimes \omega_Y \cong \mathcal{U}_p \). Let

\[
\begin{array}{c}
\tilde{X} \\
\downarrow g \\
X \leftarrow \leftarrow M
\end{array}
\]

be an elimination of indeterminacy. Consider morphisms,

\[
g \times \text{id}: \tilde{X} \times Y \rightarrow X \times Y, \quad h \times \text{id}: \tilde{X} \times Y \rightarrow M \times Y,
\]

and objects,

\[
L(g \times \text{id})^* \mathcal{P} \in D(\tilde{X} \times Y), \quad (h \times \text{id})^* \mathcal{U} \in \text{Coh}(\tilde{X} \times Y).
\]

Take \( x \in \tilde{X} \) and let \( i_{x \times Y}: x \times Y \rightarrow X \times Y \) be an inclusion. Then

\[
L_{i_{x \times Y}}^* \circ L(g \times \text{id})^* \mathcal{P} = L_i^* \mathcal{P} = \Phi(\mathcal{O}_{g(x)})
\]

\[
L_{i_{x \times Y}}^* \circ (h \times \text{id})^* \mathcal{U} = \mathcal{U}_{h(x)}.
\]

Take open subsets \( X^0 \subset X \), \( Y^0 \subset Y \), \( Z^0 \subset Z \) such that the rational maps \( X \rightarrow Z \), \( Y \rightarrow Z \) are defined on \( X^0 \), \( Y^0 \), and \( X^0 \rightarrow Z^0 \), \( Y^0 \rightarrow Z^0 \) are smooth projective. From here, we will shrink \( Z^0 \) if necessary. Since \( f_1 \) is defined on \( X^0 \), we can think \( X^0 \) as an open subset of \( \tilde{X} \). So if \( x \in X^0 \subset \tilde{X} \), then \( \Phi(\mathcal{O}_{g(x)}) = \mathcal{U}_{h(x)} \). This implies

\[
\text{Supp}(h \times \text{id})^* \mathcal{U} \cap (X^0 \times Y) = \text{Supp}(g \times \text{id})^* \mathcal{P} \cap (X^0 \times Y).
\]

Therefore by Lemma 7.6 below, we have

\[
\text{Supp}(h \times \text{id})^* \mathcal{U} \subset \text{Supp}(g \times \text{id})^* \mathcal{P} \subset \tilde{X} \times Y.
\]

Therefore for all \( x \in \tilde{X} \), we have

\[
\text{Supp}(h \times \text{id})^* \mathcal{U} \cap (x \times Y) \subset \text{Supp}(g \times \text{id})^* \mathcal{P} \cap (x \times Y).
\]

So \( \text{Supp} \mathcal{U}_{h(x)} \subset \text{Supp} \Phi(\mathcal{O}_{g(x)}) \) follows. Since \( \omega_Y \) is numerically zero on \( \text{Supp} \Phi(\mathcal{O}_{g(x)}) \), this is also true on \( \text{Supp} \mathcal{U}_{h(x)} \), hence on \( \text{Supp} \mathcal{U}_p \) for all \( p \in M \). Therefore \( \mathcal{U}_p \otimes \omega_Y \) is also \( H' \)-stable, and its reduced Hilbert polynomial is equal to \( \mathcal{U}_p \), i.e.

\[
p(\mathcal{U}_p, H') = p(\mathcal{U}_p \otimes \omega_Y, H').
\]

On the other hand, there exists a non-trivial map \( \mathcal{U}_p \rightarrow \mathcal{U}_p \otimes \omega_Y \) by semi-continuity. So \( \mathcal{U}_p \cong \mathcal{U}_p \otimes \omega_Y \) for all \( p \in M \).

Secondly we show the set

\[
\Gamma(\mathcal{U}) := \{(p_1, p_2) \in M \times M \mid \text{Ext}_Y^i(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\}
\]

has \( \dim \Gamma(\mathcal{U}) \leq 4 \). It suffices to show if \( (p_1, p_2) \in \Gamma(\mathcal{U}) \setminus \Delta_M \), where \( \Delta_M \) is a diagonal, then \( p_i \in M \setminus f_1(X^0) \). Assume that \( p_1 \in f_1(X^0) \). Since \( \text{Ext}_Y^i(\mathcal{U}_{p_1}, \mathcal{U}_{p_2}) \neq 0 \), we have \( \text{Supp} \mathcal{U}_{p_1} \cap \text{Supp} \mathcal{U}_{p_2} \neq \emptyset \). Take an irreducible component \( l \subset \text{Supp} \mathcal{U}_{p_2} \) such that \( \text{Supp} \mathcal{U}_{p_1} \cap l \neq \emptyset \). Since we have assumed \( p_1 \in f_1(X^0) \), we have

\[
\text{Supp} \mathcal{U}_{p_1} \cap \text{Bs} |mK_Y| = \emptyset.
\]
So it follows that \( l \) is not contained in \( \text{Bs} |mK_Y| \). Furthermore \( K_Y \cdot l = 0 \), since \( K_Y \) is numerically zero on \( \text{Supp} \mathcal{U}_p \). Therefore \( l \backslash \text{Bs} |mK_Y| = \emptyset \) and \( l \) is contained in the fiber of the Itkawa fibration, \( Y \backslash \text{Bs} |mK_Y| \to Z \). This implies \( l = \text{Supp} \mathcal{U}_{p_1} \) and therefore \( \text{Supp} \mathcal{U}_{p_2} = \text{Supp} \mathcal{U}_{p_1} \), since \( \text{Supp} \mathcal{U}_{p_2} \) is connected. Therefore \( \mathcal{U}_{p_2} \) is a stable sheaf on \( \text{Supp} \mathcal{U}_{p_1} \), so \( p_2 \in f_1(X^0) \). Let \( q_i \in X^0 \) be points such that \( p_i = f_1(q_i) \). Then \( \text{Ext}^i(X_{p_1}, \mathcal{U}_{p_2}) = \text{Ext}^i(X_{q_1}, \mathcal{O}_{q_2}) \neq 0 \) implies \( q_1 = q_2 \) and \( p_1 = p_2 \). But this contradicts to \( (p_1, p_2) \notin \Delta_M \).

In the above proof, we used the following lemma:

**Lemma 7.6.** \( \text{Supp}(h \times \text{id}) \mathcal{U} \) is irreducible.

**Proof.** Let \( \tilde{f} : \tilde{X} \times Y \to \tilde{X} \) be a projection. Note that a general fiber of the restriction of \( \tilde{f} \) to \( \text{Supp}(h \times \text{id}) \mathcal{U} \) is an elliptic curve. Therefore if \( \text{Supp}(h \times \text{id}) \mathcal{U} \) is not irreducible, then there exists \( p \in \text{Ass}((h \times \text{id}) \mathcal{U}) \) such that \( \dim \mathcal{O}_{\tilde{X}, \tilde{f}(p)} \geq 1 \). Take a non-zero element of the maximal ideal \( t \in m_{\tilde{f}(p)} \subset \mathcal{O}_{\tilde{X}, \tilde{f}(p)} \). Then \( \mathcal{O}_{\tilde{X}, \tilde{f}(p)} \to \mathcal{O}_{\tilde{X}, \tilde{f}(p)} \) is injective. Since \( (h \times \text{id}) \mathcal{U} \) is flat over \( \tilde{X} \), we have an injection, \( ((h \times \text{id}) \mathcal{U})_p \to ((h \times \text{id}) \mathcal{U})_p \), and \( \tilde{f}^*t \in m_p \mathcal{O}_{\tilde{X} \times Y, p} \). But this contradicts to \( p \in \text{Ass}((h \times \text{id}) \mathcal{U}) \).

**Step 4.** \( X \) and \( M \) are connected by finite number of flops, and \( M \) has an elliptic fibration \( \pi : M \to S \) with \( \omega_M \equiv_\pi 0 \).

**Proof.** Consider the following composition:

\[
\Psi \circ \Phi_M : D(M) \longrightarrow D(Y) \longrightarrow D(X).
\]

This is an equivalence and for general points \( p \in M \), we have

\[
\dim \text{Supp} \Psi \circ \Phi_M(\mathcal{O}_p) = 0.
\]

Therefore \( X \) and \( M \) are connected by finite number of flops. Since \( \text{Supp} \mathcal{U} \subset Y \times M \) is irreducible and all the fibers of the projection \( \text{Supp} \mathcal{U} \to M \) are one-dimensional, this is a well-defined family of proper algebraic cycles in the sense of \cite{L.K90}. Therefore there exists a morphism \( M \to \text{Chow}(Y) \) which takes \( p \in M \) to an algebraic cycle whose support is equal to \( \text{Supp} \mathcal{U}_p \). Let

\[
M \overset{\pi}{\longrightarrow} S \to \text{Chow}(Y)
\]

be a stein factorization. We show that \( \omega_M \equiv_\pi 0 \). Let us take \( p, p' \in M \) such that \( \pi(p) = \pi'(p) \). Then by the definition of \( \pi \), it follows that

\[
\text{Supp} \Phi_M(\mathcal{O}_p) = \text{Supp} \Phi_M(\mathcal{O}_{p'}).
\]

Let us take \( q \in \text{Supp} \Phi_M(\mathcal{O}_p) \). Then \( p' \in \text{Supp}(\Phi_M)^{-1}(\mathcal{O}_q) \). Therefore \( \pi^{-1}\pi(p) \subset \text{Supp}(\Phi_M)^{-1}(\mathcal{O}_q) \). This implies \( \omega_M \equiv_\pi 0 \).

**Step 5.** There exists a polarization \( H \subset M \), \( d \in \mathbb{Z} \), such that \( J^H(d) \) is fine, and smooth. Moreover \( Y \) and \( J^H(d) \) are connected by finite number of flops.
Proof. We continue the same argument. Let us take a general closed point \( y \in Y \). The object

\[(\Phi_M)^{-1}(O_y) \in D(M)\]

is a stable sheaf on a general fiber of \( \pi \). Let its rank and degree be \( c \) and \( d \) respectively. Let \( H \in \text{Pic}(M) \) be a polarization, and take an irreducible component \( M^+ \subset M^H(M/S) \) which contains a point corresponding to \((\Phi_M)^{-1}(O_y)\). Similarly, take an irreducible component \( J^H(d) \subset M^H(M/S) \) which contains a point corresponding to line bundles of degree \( d \) on smooth fibers of \( \pi \). By the same argument as before, we can choose \( H \) such that \( \pi'' : M^+ \to S, \pi' : J^H(d) \to S \), are fine moduli schemes (or \( X \) and \( Y \) are connected by finite number of flops if \( d = 0 \)). By [BM02] Theorem 8.3, \( M^+ \) and \( J^H(d) \) are smooth, \( \omega_{M^+} \equiv_{x^0} 0, \omega_{J^H(d)} \equiv_{x^0} 0 \), and the universal sheaf \( \Phi \) gives an equivalence

\[\Phi_{M^+} := \Phi_{M^+ \to M} : D(M^+) \sim \to D(M).\]

Since the composition

\[\Phi_M \circ \Phi_{M^+} : D(M^+) \sim \to D(M) \sim \to D(Y)\]

takes general points to general points, \( Y \) and \( M^+ \) are connected by finite number of flops. By [Ati57] Theorem 6, there exists a following birational map over \( S \):

\[M^+ \ni E \mapsto \lambda \circ E \in J^H(d).\]

Since they are both minimal over \( S \), \( M^+ \) and \( J^H(d) \) are connected by finite number of flops. Now we have obtained the diagram (2).

If \( X \) is minimal we have a better result. By the abundance theorem in dimension three, \( K_X \) is semi-ample. Let \( \pi_X : X \to Z \) be its Iitaka fibration. We define \( \lambda_X > 0 \) as follows:

\[\lambda_X := \text{g.c.d}\{c_1(E) \cdot f_X | E \in D(X)\},\]

where \( f_X \) is a cohomology class of a general fiber of \( \pi_X \). For a polarization \( H \) on \( X \), let \( J^H(b) \subset M^H(X/Z) \) be as in the Theorem 7.5. The proof of the following theorem is almost the same as in the previous theorem and is left to the reader.

**Theorem 7.7.** Let \( X \) be a smooth minimal 3-fold with \( \kappa(X) = 2 \). Then \( Y \in FM(X) \) if and only if there exists some \( b \in \mathbb{Z} \) which is co-prime to \( \lambda_X \), and there exists a polarization \( H \) on \( X \), for which \( J^H(b) \) is a fine moduli scheme, such that \( Y \) and \( J^H(b) \) are connected by finite number of flops.

Because \( J^H(b + \lambda_X) \cong J^H(b) \), birational classes of \( FM(X) \) are finite in the above case. By [Kaw97], the number of 3-dimensional minimal model in a fixed birational class is finite. So we obtain the next corollary.

**Corollary 7.8.** Let \( X \) be a smooth minimal 3-fold with \( \kappa(X) = 2 \). Then \( \sharp FM(X) < \infty \).

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