A GEOMETRIC PERSPECTIVE ON $p$-ADIC PROPERTIES OF MOCK MODULAR FORMS

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Abstract. In [BGK12], Bringmann, Guerzhoy and Kane have shown how to correct mock modular forms by a certain linear combination of the Eichler integral of their shadows in order to obtain $p$-adic modular forms in the sense of Serre. In this paper, we give a new proof of their results (for good primes $p$) by employing the geometric theory of harmonic Maass forms developed by the first author [Can14] and the theory of overconvergent modular forms due to Katz and Coleman.

1. Introduction

Over the past decade, there has been a renewed interest in Ramanujan’s mock modular forms and related objects, such as harmonic Maass forms, whose Fourier coefficients seem to encode interesting arithmetic data not elsewhere found in the classical theory of modular forms. In this article, we offer a new perspective on the $p$-adic properties of the Fourier coefficients of mock modular forms, based on the algebro-geometric theory of $p$-adic modular forms of Katz-Coleman ([Kat73], [Col96]). Such $p$-adic properties were originally discovered in [BGK12], [GKO10], but we believe our perspective simplifies some of the arguments and provides a theoretical platform for further exploration.

In order to state our results precisely, let $\tau = u + iv \in \mathfrak{h}$, let $\Gamma_0(N)$ be the congruence subgroup of $\text{SL}_2(\mathbb{Z})$ of matrices that become upper-triangular modulo $N$, and let $\chi$ be a Dirichlet character modulo $N$. Denote by $\mathcal{H}_k(\Gamma_0(N), \chi)$ the vector space of all weight $k$ harmonic Maass forms on $\Gamma_0(N)$ and character $\chi$ (see e.g. [BGK12] §2 for definitions). Any harmonic Maass form $F$ has a decomposition

$$F = F^+ + F^-$$

into a holomorphic part $F^+$ (with poles supported at the cusps) and an anti-holomorphic part $F^-$. The function $F^+: \mathfrak{h} \to \mathbb{C}$ is what is called a mock modular form, since it does not transform like a modular form, but its Fourier coefficients resemble those of a modular form. Harmonic Maass forms map into spaces of classical modular forms via differential operators. In particular, let $M^!(k)(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) be the space of weakly holomorphic modular forms (resp. cusp forms) of weight $k$, level $\Gamma_0(N)$ and character $\chi$. If we let

$$\xi_k := 2iv^k \frac{\partial}{\partial \tau},$$

then $\xi_{2-k}(F) = f \in S_k(\Gamma_0(N), \chi)$ for all $F \in \mathcal{H}_{2-k}(\Gamma_0(N), \overline{\chi})$, and the resulting cusp form is called the shadow of $F$. A fundamental question in the subject is to relate the

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coefficients of a mock modular form $F^+$ to the coefficients of its shadow $f$. In order to
obtain results in this direction, we first have to restrict to normalized newforms $f$ and
then slightly refine the definition of a harmonic Maass form, as follows. Let $K \subseteq \mathbb{C}$ be
a subfield. For $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ a congruence subgroup, we denote by $S_k(\Gamma, K)$ the space of
cusp forms of weight $k$ and level $\Gamma$ whose $q$-expansion coefficients all lie in the field $K$.
Let also $M_k^!(\Gamma, K)$ be the space of weakly holomorphic modular forms with coefficients in $K$.

**Definition 1.1.** Let $f \in S_k(\Gamma_1(N), K)$ be a newform defined over $K$. A harmonic Maass
form $F \in \mathcal{H}_{2-k}(\Gamma_1(N))$ is good for $f$ if

(i) The principal parts of $F$ all lie in $K$.

(ii) $\xi_{2-k}(F) = f/\|f\|^2$, where $\|f\|$ is the Petersson norm of $f$.

Suppose that $f = \sum_{n=1}^{\infty} a_n q^n$ is a (normalized) newform as above, let $F$ be a harmonic
Maass form that is good for $f$, and write $F = F^+ + F^-$ for its holomorphic and anti-
holomorphic parts, with

$$F^+ = \sum_{n > -\infty} c^+(n)q^n.$$ 

Let $E_f = \sum_{n=1}^{\infty} n^{1-k}a_n q^n$ be the Eichler integral of $f$, so that $D^{k-1}(E_f) = f$, where
$D^{k-1}$ is the differential operator on modular forms acting as $(qd/dq)^{k-1}$ on $q$-expansions.
It is shown in [GKO10] (and also in Theorem 1.1 of this paper, by different methods)
that for any $\alpha \in \mathbb{C}$ such that $\alpha - c^+(1) \in K$, the coefficients of

$$\mathcal{F}_\alpha := F^+ - \alpha E_f$$

all lie in $K$, so it makes sense to study their $p$-adic properties. To this end, let $p \nmid N$
be a prime, fix once and for all a choice of complex and $p$-adic embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and
$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and fix a valuation $v_p$ on $\mathbb{C}_p$ extending the $p$-adic valuation of $\mathbb{Q}$. Suppose
the Hecke polynomial $T^2 - \alpha \tau T + \chi(\tau)p^{k-1}$ has roots $\beta, \beta'$ with $v_p(\beta) \leq v_p(\beta')$, and let
$V$ be the operator acting as $q \mapsto q^p$ on $q$-expansions. The $p$-stabilizations of $f$ are the
$p$-adic modular forms

$$f_\beta := f - \beta' V(f), \quad f_{\beta'} := f - \beta V(f),$$

which are easily seen to be eigenvectors for $U$ with eigenvalues $\beta$ and $\beta'$, respectively.
Here, $U$ is defined by $U(\sum_n a_n q^n) = \sum_n a_{pn} q^n$, and our first main result shows that,
for most values of $\alpha$, the $p$-stabilized shadow $f_{\beta'}$ can be recovered $p$-adically from the
corrected mock modular form $\mathcal{F}_\alpha$ by an iterated application of the $U$-operator.

**Theorem 1.2 ([GKO10], Theorem 1.2(i)).** Assume that $v_p(\beta) \neq v_p(\beta')$ and assume
that $v_p(\beta') \neq k - 1$. Then for all but at most one choice of $\alpha$ with $\alpha - c^+(1) \in K$, we have

$$\lim_{w \to +\infty} U^w D^{k-1}(\mathcal{F}_\alpha) = f_\beta.$$

In Section 4, we give a new proof of this result by viewing $f_\beta$ and $f_{\beta'}$ as overconvergent
modular forms, in the sense of [Col96]. Based on ideas of [BDP13], we prove (Theorem
3.3 below) that these two modular forms are $p$-adic representatives of cohomology classes
in the $f$-isotypical component of a certain parabolic cohomology group attached to the modular curve $X_1(N)$. Under the assumptions of Theorem 1.2, the classes $f_{\beta}, f_{\beta'}$ form a basis for this space, and so the modular form $D^{k-1}(F_\alpha)$ (which gives a class in the same space, as shown in [Can14]) can be expressed as a linear combination $f_{\beta}$ and $f_{\beta'}$. Our proof of Theorem 1.2 then follows by analyzing the action of $U$ in cohomology.

This new proof-template can be applied to similar questions in the theory of mock modular forms. For example, in Section 5 we interpret the exceptional value of $\alpha$ in Theorem 1.2 as giving the precise value for which $F_\alpha$ can be p-adically ‘completed’ to obtain a $p$-adic modular form. This was initially discovered by Bringmann, Guerzhoy and Kane, and we reprove here their results [BGK12] using our $p$-adic analytic/geometric methods. Finally, in Section 6 we discuss the case of when $f$ has CM (also considered [BGK12] and [GKO10]), which requires a different treatment due to the failure of the assumptions in Theorem 1.2.

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2. Harmonic Maass forms: the geometric point of view

We begin by quickly recalling the geometric interpretation of harmonic Maass forms given in [Can14], which will be needed in later sections. For $N > 4$, the moduli functor $\mathcal{M}_1(N)$ of generalized elliptic curves with a point of order $N$ is represented by a smooth and proper scheme over $\mathbb{Z}[1/N]$. Let $E_{\text{gen}} \rightarrow \mathcal{M}_1(N)$ be the universal generalized elliptic curve, and let $\omega$ be its relative dualizing (invertible) sheaf. Let $X := \mathcal{M}_1(N) \times_{\mathbb{Z}[1/N]} \mathbb{Q}$ and $Y := X \setminus C$, where $C$ is the cuspidal subscheme, whose ideal sheaf we denote by $\mathcal{I}_C$. For any subfield $K \subseteq \mathbb{C}$, we denote by $X_K, Y_K$ the base-change to $K$. We have well-known canonical isomorphisms

$$M_k'(\Gamma_1(N), K) \simeq H^0(Y_K, \omega^k), \quad S_k(\Gamma_1(N), K) \simeq H^0(X_K, \omega^k \otimes \mathcal{I}_C),$$

where a modular form $f$ of weight $k$ is identified with the differential $f(dq/q)^k$. Let $\pi : \mathcal{E} \rightarrow Y$ be the universal elliptic curve with $\Gamma_1(N)$-level structure. The relative de Rham cohomology of $\pi : \mathcal{E} \rightarrow Y$ canonically extends to a rank 2 vector bundle $\mathcal{H}^1_{\text{dr}}$ over $X$. For any $r \geq 0$ let

$$\mathcal{H}_r := \text{Sym}^r(\mathcal{H}^1_{\text{dr}}),$$

which is a vector bundle of rank $r+1$ over $X$. The Gauss-Manin connection of $\pi : \mathcal{E} \rightarrow Y$ extends to a connection with logarithmic poles $\nabla : \mathcal{H}^1_{\text{dr}} \rightarrow \mathcal{H}^1_{\text{dr}} \otimes \Omega^1_X(\log C)$ over $X$, and the $r$-th symmetric power of $\nabla$ is a connection with logarithmic poles

$$\nabla_r : \mathcal{H}_r \longrightarrow \mathcal{H}_r \otimes \Omega^1_X(\log C).$$

Define

$$\mathbb{H}^1_{\text{par}}(X, \nabla_r) := \mathbb{H}^1(\mathcal{H}_r \otimes \mathcal{I}_C \xrightarrow{\nabla_r} \mathcal{H}_r \otimes \Omega^1_X),$$

where $\mathbb{H}^*$ denotes hypercohomology. Over $\mathbb{C}$, this group is canonically isomorphic to the classical weight $r$ parabolic cohomology obtained by taking periods of cusp forms. The
formation of this cohomology group is compatible under base-change by a field extension $K \supseteq \mathbb{Q}$ and for all such $K$ and $k \geq 2$ there is a filtration ([Sch85, Thm. 2.7.(i)])

$$0 \rightarrow H^0(X_K, \omega^k \otimes \mathcal{I}_C) \rightarrow H^1_{\text{par}}(X_K, \nabla_{k-2}) \rightarrow H^1(X_K, \omega^{2-k}) \rightarrow 0$$

of $K$-vector spaces, so that $S_k(\Gamma_1(N), K)$ is naturally a subspace of parabolic cohomology. More generally, all parabolic cohomology classes can be represented in terms of classical modular forms. To state this result, recall that for $k \geq 2$ there is an algebraic differential operator of order $k - 1$:

$$D^{k-1}: M^1_{2-k}(\Gamma_1(N), K) \rightarrow M^1_k(\Gamma_1(N), K)$$

which acts as $(q d/dq)^{k-1} = \left(\frac{1}{2\pi i} \frac{\partial}{\partial \tau}\right)^{k-1}$.

Theorem 2.1 ([Can14, Thm. 6]). Let $K \subseteq \mathbb{C}$ be a subfield and let $S^1_k(\Gamma_1(N), K)$ be the subspace of those modular forms in $M^1_k(\Gamma_1(N), K)$ with vanishing constant coefficient in their $q$-expansions at the cusps. Then there is a canonical isomorphism:

$$H^1_{\text{par}}(X_K, \nabla_{k-2}) \cong \frac{S^1_k(\Gamma_1(N), K)}{D^{k-1}M^1_{2-k}(\Gamma_1(N), K)}.$$

Let now $f \in S_k(\Gamma_1(N), K)$ be a newform. Let $H^1_{\text{par}}(X_K, \nabla_{k-2})f$ be the $f$-isotypical component, and let

$$[\phi] \in H^1_{\text{par}}(X_K, \nabla_{k-2})f$$

be a class represented by an element $\phi \in S^1_k(\Gamma_1(N), K)$. By the Shimura isomorphism $H^1_{\text{par}}(X_K, \nabla_{k-2})f \cong \mathbb{C} f \oplus \mathbb{C} \bar{f}$, we may write

$$[\phi] = s_1[f] + s_2[\bar{f}]$$

for some $s_1, s_2 \in \mathbb{C}$. Let now $C^\infty_Y$ (resp. $A^1_Y$) be the sheaf of smooth functions (resp. smooth differential forms) on $Y_C$. The differential $\phi - s_1 f - s_2 \bar{f}$ is smooth over $Y_C$, and it defines a class in

$$H^1(H_{k-2} \otimes C^\infty_Y \nabla_{k-2}, H_{k-2} \otimes A^1_Y) = \frac{H^0(Y_C, H_{k-2} \otimes A^1_Y)}{\nabla_{k-2}H^0(Y_C, H_{k-2} \otimes C^\infty_Y)}.$$ 

This class is trivial by construction, and so there exists a smooth $H_{k-2}$-valued modular form $F$ such that $\nabla_{k-2}(F) = \phi - s_1 f - s_2 \bar{f}$. The vector bundle $H_{k-2}$ decomposes into line bundles as $H_{k-2} \cong \omega^{2-k} \oplus \omega^{4-k} \oplus \ldots \oplus \omega^{k-2}$, and we let $F := F_{2-k}$ be the component of $F$ of weight $2 - k$. As shown in [Can14 Prop. 4], $F$ is a harmonic Maass form. If we write $F = F^+ + F^-$ for the holomorphic and anti-holomorphic parts of $F$, then

$$D^{k-1}(F^+) = \phi - s_1 f, \quad \frac{2i v^{2-k}}{(-4\pi)^{k-1}} \frac{\partial}{\partial \tau}(F^-) = s_2 \bar{f}.$$ 

To obtain a ‘true’ harmonic Maass form we should insist that $\phi \notin S_k(\Gamma_1(N), K)$, i.e. $s_2 \neq 0$ in Equation (2). Then we may rescale $\phi$ so that $\langle \phi, f \rangle = 1$ (cup-product), which amounts to letting $s_2 = 1/\langle \bar{f}, f \rangle = 1/(-4\pi)^{k-1}\|f\|^2$. With this choice, it is clear from the above that $\xi_{2-k}(F) = f/\|f\|^2$, so that $F$ is good for $f$ in the sense of Definition [11]
3. Overconvergent modular forms

Let \( p \geq 5 \) be a prime and let \( \mathbb{C}_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \). We fix a valuation \( v_p \) on \( \mathbb{C}_p \) such that \( v_p(p) = 1 \) and an absolute value \( \cdot \mid \) on \( \mathbb{C}_p \) which is compatible with \( v_p \). Let \( K_p \subseteq \mathbb{C}_p \) be a complete discretely-valued subfield and let \( R_p \) be its ring of integers. Suppose \( (p, N) = 1 \), and let \( \mathcal{X} := \mathcal{M}_1(N) \times_{\mathbb{Z}[1/N]} R_p \) be the base-change to \( R_p \). Let \( E_{p-1} \in H^0(\mathcal{X} \times_{R_p} K_p, \omega^{p-1}) \) be the global section given by the Eisenstein series of weight \( p - 1 \) and level 1, normalized so that its constant coefficient is 1. As shown in [Col96, §1], for any \( \epsilon \in |R_p| \) there is a unique rigid analytic space \( X_\epsilon \) with the property that

\[
X_{\epsilon}^{\text{cl}} = \{ x \in (\mathcal{X} \times_{R_p} K_p)^{\text{cl}} : |E_{p-1}(x)| \geq \epsilon \},
\]

where by the superscript ‘cl’ we have denoted the set of closed points, and also a unique rigid analytic space \( X_{(\epsilon)} \) with the property that

\[
X_{(\epsilon)}^{\text{cl}} = \{ x \in (\mathcal{X} \times_{R_p} K_p)^{\text{cl}} : |E_{p-1}(x)| > \epsilon \}.
\]

When \( \epsilon = 1 \), the rigid analytic space \( X^{\text{ord}} := X_1 \) is called the ordinary locus of \( X \), since every geometric point of \( X^{\text{ord}} \) reduces mod \( p \) to a point classifying an ordinary elliptic curve. The rigid analytic spaces \( X_{(\epsilon)} \), for \( 0 < \epsilon < 1 \), can be viewed as ‘complements of closed disks’ and are called open neighborhoods of \( X^{\text{ord}} \). They are examples of wide open spaces. For all \( \epsilon \in |R_p| \), we have inclusions \( X_1 \subseteq X_\epsilon \subseteq X_{(\epsilon)} \subseteq X \). The invertible sheaves \( \omega^k \), for \( k \in \mathbb{Z} \), restrict to rigid analytic line bundles over \( X_\epsilon \).

**Definition 3.1.** Let \( \epsilon \in |R_p| \). An overconvergent modular form of weight \( k \in \mathbb{Z} \) is a rigid analytic section \( f \in H^0(X_{(\epsilon)}, \omega^k) \), for \( \epsilon < 1 \).

Note that for \( \epsilon = 1 \) the sections of \( \omega^k \) over \( X^{\text{ord}} \) are Serre’s \( p \)-adic modular forms of integral weight \( k \). Overconvergent modular forms can thus be viewed as \( p \)-adic modular forms which converge not just over \( X^{\text{ord}} \) but on a slightly larger neighborhood of it.

Since \( |E_{p-1}(\epsilon)| = 1 \) at all cusps \( c \in C \), we have that \( C \subseteq X_{(\epsilon)} \) for all \( \epsilon \in |R_p| \). Let

\[
Y^{\text{ord}} := X^{\text{ord}} \setminus C, \quad Y_\epsilon := X_\epsilon \setminus C, \quad Y_{(\epsilon)} := X_{(\epsilon)} \setminus C
\]

be the rigid analytic spaces obtained by removing the cusps.

**Remark 3.2.** For \( \epsilon = 1 \), sections of \( H^0(Y_\epsilon, \omega^k) \) correspond to the \( p \)-adic modular forms of integral weight considered in [BGK12]. As explained in [loc.cit., p. 2394], these can be directly related to the \( p \)-adic modular forms introduced by Serre [Ser73].

Let \( W_1 = X_{(p-1/p+1)} \) and \( W_2 = X_{(p-1/p+1)} \), both open neighborhoods of \( X^{\text{ord}} \) with \( W_2 \subseteq W_1 \). Let

\[
U : H^0(W_2, \omega^k) \longrightarrow H^0(W_1, \omega^k) \subseteq H^0(W_2, \omega^k)
\]

\[
V : H^0(W_1, \omega^k) \longrightarrow H^0(W_2, \omega^k)
\]

be the operators defined in the introduction. Let \( f \in S_k(\Gamma_1(N), K) \) be a newform defined over a number field \( K \), and consider \( f \) as an element of \( H^0(W_1, \omega^k) \) by restriction. Then

\[
T_p(f) = U(f) + \chi(p)p^{k-1}V(f) \in H^0(W_2, \omega^k),
\]
where $T_p$ is the $p$-th Hecke operator. In particular, if $f$ is an eigenform of level $\Gamma_0(N)$ and character $\chi$ with $T_p$-eigenvalue equal to $a_p$ then

$$a_p f = U(f) + \chi(p)p^{k-1}V(f) \in H^0(W_2, \omega^k).$$

**Proposition 3.3.** Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ be a newform and let

$$T^2 - a_p T + \chi(p)p^{k-1} = (T - \beta)(T - \beta')$$

be the $p$-th Hecke polynomial of $f$. Then the overconvergent modular forms

$$f_\beta := f - \beta' V(f), \quad f_{\beta'} := f - \beta V(f)$$

in $H^0(W_2, \omega^k)$ are $U$-eigenvectors with eigenvalues $\beta$ and $\beta'$, respectively.

**Proof.** This follows from a straightforward calculation. Indeed, viewing $f$ and $V(f)$ as sections in $H^0(W_2, \omega^k)$, we see that

$$Uf_\beta = Uf - \beta' UV(f) = Uf - \beta' f$$

$$= T_p f - \chi(p)p^{k-1}V(f) - \beta' f$$

$$= (a_p - \beta') f - \chi(p)p^{k-1}V(f)$$

$$= \beta f_\beta,$$

using the relations $a_p = \beta + \beta'$ and $\chi(p)p^{k-1} = \beta\beta'$ for the last equality. The proof for $f_{\beta'}$ is obviously the same. \qed

Let now $W = X(\epsilon)$, with $0 < \epsilon < 1$, be an open neighborhood of $X^{\text{ord}}$, and for any $r \geq 0$ consider the space

$$H^1(W, \nabla_r) := H^1(\mathcal{H}_r | W \to \mathcal{H}_r | W \otimes \Omega^1_W(\log C)).$$

**Theorem 3.4** (See [Col96], §5).

(i) There is a canonical isomorphism

$$H^1(W, \nabla_r) \cong \frac{H^0(W, \omega^{r+2})}{\theta^{r+1}H^0(W, \omega^{-r})}.$$ 

(ii) For any two open neighborhoods $W, W'$ of $X^{\text{ord}}$, there is a canonical isomorphism

$$H^1(W, \nabla_r) = H^1(W', \nabla_r).$$

By restriction, there is an injection

$$H^1_{\text{par}}(X, \nabla_r) \hookrightarrow H^1(W \setminus C, \nabla_r)$$

for any choice of open neighborhood $W$ of $X^{\text{ord}}$. The image of this map can be characterized by $p$-adic residues ([BDP13, Prop. 3.9]). In particular, if $f \in S_k(\Gamma_1(N), K)$ is a newform of weight $k \geq 2$, the cohomology classes

$$\{[f_\beta], [f_{\beta'}]\} \subseteq H^1(W_2 \setminus C, \nabla_{k-2})$$

naturally lie in $H^1_{\text{par}}(X_K, \nabla_{k-2})$, and more precisely they lie in the $f$-isotypical component $H^1_{\text{par}}(X_K, \nabla_{k-2})_f$, which is a two-dimensional $K$-vector space.
Theorem 3.5. Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ be a newform of weight $k \geq 2$, and let $\beta$ and $\beta'$ be the roots of $T^2 - a_p T + \chi(p)p^{k-1}$, ordered so that $v_p(\beta) \leq v_p(\beta')$. Assume that the following two conditions hold:

(i) $\beta \neq \beta'$.
(ii) $v_p(\beta') \neq k - 1$.

Then $\{[f], [V(f)]\}$ is a basis for $H^1_{\text{par}}(X_K, \nabla_{k-2})$.

Proof. Since $H^1_{\text{par}}(X_K, \nabla_{k-2})$ is two-dimensional, it suffices to show that $[f]$ and $[V(f)]$ are linearly independent. By Proposition 3.3 and [Col96, Lem. 6.3], condition (ii) guarantees that $[f_{\beta'}] \neq 0$, and therefore it is an eigenvector of $\text{Ver}$ (see [loc.cit., Thm. 5.4]) acting on parabolic cohomology with eigenvalue $\beta'$. In the same manner, the class $[f_{\beta}]$ is non-trivial, and it is an eigenvector of $\text{Ver}$ with eigenvalue $\beta$. Thus by condition (i), the classes $[f_{\beta}]$ and $[f_{\beta'}]$ are linearly independent, and so must be $[f]$ and $[V(f)]$, since $f_{\beta}$ and $f_{\beta'}$ are linear combinations of $f$ and $V(f)$. $\square$

Remark 3.6. As clear from the proof, condition (ii) in Theorem 3.5 could be weakened to the following:

(ii') either $v_p(\beta') \neq k - 1$ or $[f_{\beta'}] \neq 0$.

By [Col96, Prop. 7.1], condition (ii') fails if $f$ has CM by an imaginary quadratic field in which $p$ splits; conjecturally (see e.g. [Eme]), these are the only cases in which condition (ii') fails, but this is not known in general.

4. Recovering the shadow

Let $f = \sum_{n=1}^{\infty} a(n) q^n$ be a newform satisfying the hypotheses of Theorem 3.5 and let $F$ be a harmonic Maass form which is good for $f$, in the sense of Definition 1.1. The shadow $f$ can be recovered from $F$ by

$$\xi_{2-k}(F) = \frac{f}{\|f\|^2}$$

where $\|f\|$ is the Petersson norm of $f$. By the results in Section 2, the harmonic Maass form $F$ has a holomorphic part $F^+$ with the property that

$$D^{k-1}(F^+) = \phi - s_1 f$$

for some $\phi \in S^1_k(\Gamma_1(N), K)$ and some $s_1 \in \mathbb{C}$.

In [GKO10, Thm. 1.2], Bringmann, Guerzhoy and Kane prove that one of the two $p$-stabilizations of $f$ can be recovered $p$-adically from an iterated application of $U$ to a certain ‘correction’ of $D^{k-1}(F^+)$. In this section, we deduce their result from the $p$-adic techniques developed above. We begin by giving a new proof of [GKO10, Thm. 1.1].

Theorem 4.1. Let $\alpha \in \mathbb{C}$ be such that $\alpha - c^+(1) \in K$. Then the coefficients of

$$F_\alpha := F^+ - \alpha E_f := \sum_{n \gg -\infty} c^+(n) q^n - \alpha \sum_{n=1}^{\infty} a(n)n^{1-k} q^n$$

are all in $K$.
Proof. Write $\phi = \sum_{n \gg -\infty} d(n)q^n$, with $d(n) \in K$. By (3), we have the formula
\[
eq (d(n) - s_1 a(n)) \binom{n^{k-1}}{n^{k-1}}
\]
where $a(n) := 0$ for $n \leq 0$. The result is thus clear for $n \leq 0$. Now let $n \geq 1$, and write $\alpha = c^+(1) + \gamma$ with $\gamma \in K$, or equivalently, $\alpha = d(1) - s_1 + \gamma$. Using (4), an immediate calculation then reveals that the coefficient of $q^n$ in $F_{\alpha}$ is given by
\[
d(n) - d(1) - \gamma \binom{n^{k-1}}{n^{k-1}}
\]
and the result follows. \qed

Since one can always take $\alpha = c^+(1)$ in Theorem 4.1, the coefficients of $F_{c^+(1)}$ are all in $K$, and so they may be viewed in $\mathbb{C}_p$ via our fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_p$.

The following result is a special case of [GKO10, Thm. 1.2], but the ideas in the proof will allow us to recover their result in its full strength (see Theorem 4.3 below).

**Theorem 4.2.** Assume that $v_p(\beta) < v_p(\beta')$ and that $v_p(\beta') \neq k - 1$. Then
\[
\lim_{w \to +\infty} \frac{U^w D^{k-1}(F_{c^+(1)})}{c^+(1)(p^w)} = f_{\beta},
\]
where we write $D^{k-1}(F_{c^+(1)}) = \sum_{n \gg -\infty} c^+(1)(n)q^n$.

**Proof.** First note that by equation (3) and (4), we have
\[
D^{k-1}(F_{c^+(1)}) = \phi - d(1)f,
\]
which is a weakly holomorphic cusp form of weight $k$ with coefficients in $K$, defining a class in $\mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f$. Now our assumptions clearly imply conditions (i) and (ii) of Theorem 3.5, and so (as shown in the proof of that result) the space $\mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f$ has a basis $\{|f_\beta|, |f_{\beta'}|\}$ of eigenvectors for $U$. In particular, we can write
\[
[D^{k-1}(F_{c^+(1)})] = t_1 |f_\beta| + t_2 |f_{\beta'}|
\]
for some constants $t_1, t_2 \in K$. The differential $D^{k-1}(F_{c^+(1)}) - t_1 f_\beta - t_2 f_{\beta'}$ defines a class in $\mathbb{H}^1(W_2 \setminus C, \nabla_{k-2}) = H^0(W_2 \setminus C, \mathbb{C}^k)/\theta^{k-1}H^0(W_2 \setminus C, \mathbb{C}^{2-k})$. This class is exact, by construction, and thus we may write
\[
D^{k-1}(F_{c^+(1)}) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1}h
\]
for some $h \in H^0(W_2 \setminus C, \mathbb{C}^{2-k})$. Applying $U$ to both sides of the equation gives
\[
UD^{k-1}(F_{c^+(1)}) = t_1 \beta f_\beta + t_2 \beta' f_{\beta'} + U(\theta^{k-1}h)
\]
and more generally, for any power $w \geq 1$, we obtain
\[
U^w D^{k-1}(F_{c^+(1)}) = t_1 \beta^w f_\beta + t_2 \beta^w f_{\beta'} + U^w(\theta^{k-1}h).
\]
Dividing by $\beta^w$ we get
\[
\beta^{-w}U^w D^{k-1}(F_{c^+(1)}) = t_1 f_\beta + t_2 \left(\frac{\beta'}{\beta}\right)^w f_{\beta'} + \beta^{-w}U^w(\theta^{k-1}h).
\]
and taking the limit as \( w \to +\infty \) gives
\[
\lim_{w \to +\infty} \beta^{-w} U^w D^{k-1}(\mathcal{F}_{c^+ (1)}) = t_1 f_{\beta}.
\]
This is because \( v_p (\beta' / \beta) > 0 \) by the hypotheses and the differential \( U^w (\theta^{k-1} h) \) has bounded denominators but its coefficients have arbitrarily high valuation as \( w \to +\infty \).

To determine the value of the constant \( t_1 \), consider the coefficient of \( q^p \) in (5), which is given by
\[
c_{c^+ (1)} (p) = t_1 (a_p - \beta') + t_2 (a_p - \beta) + O(p^{k-1})
= t_1 \beta + t_2 \beta' + O(p^{k-1}).
\]
By applying the multiplicative properties of the Fourier coefficients of newforms we get
\[
c_{c^+ (1)} (p^w) = t_1 \beta^w + t_2 \beta'^w + O(p^{w(k-1)})
\]
and taking the limit we obtain
\[
\lim_{w \to +\infty} \beta^{-w} c_{c^+ (1)} (p^w) = t_1
\]
which gives the result. \( \square \)

Now we modify slightly the argument in Theorem 6.1 to recover [GKO10, Thm. 1.2] in its full strength. This refinement will be key for the results relating mock modular forms to \( p \)-adic modular forms in the next section.

For any \( \alpha \) with \( \alpha - c^+ (1) \in K \), define
\[
\mathcal{F}_\alpha := F^+ - \alpha E_f
\]
and let \( c_\alpha (n) \) denote the \( n \)-th coefficient in the expansion
\[
D^{k-1} (\mathcal{F}_\alpha) = \sum_{n \gg -\infty} c_\alpha (n) q^n.
\]

**Theorem 4.3.** Assume that \( v_p (\beta) \neq v_p (\beta') \) and that \( v_p (\beta') \neq k - 1 \). Then for all but at most one choice of \( \alpha \) with \( \alpha - c^+ (1) \in K \), we have
\[
\lim_{w \to +\infty} \frac{U^w D^{k-1} (\mathcal{F}_\alpha)}{c_\alpha (p^w)} = f_{\beta}.
\]

**Proof.** As in the proof of Theorem 6.1, we can write
\[
[D^{k-1} (\mathcal{F}_{c^+ (1)})] = t_1 [f_{\beta}] + t_2 [f_{\beta'}] \in \mathbb{H}_1 (X_K, \nabla_{k-2}) f
\]
with
\[
t_1 = \lim_{w \to +\infty} \frac{c_{c^+ (1)} (p^w)}{\beta^w}.
\]
Let \( \gamma \in K \) be such that \( \alpha = c^+ (1) + \gamma \), so that \( \mathcal{F}_\alpha = \mathcal{F}_{c^+ (1)} - \gamma E_f \) by definition. Noting that
\[
f = \frac{\beta f_{\beta} - \beta' f_{\beta'}}{\beta - \beta'},
\]
we have
\[
\lim_{w \to +\infty} \frac{U^w D^{k-1} (\mathcal{F}_\alpha)}{c_\alpha (p^w)} = f_{\beta}.
\]

This is because \( v_p (\beta' / \beta) > 0 \) by the hypotheses and the differential \( U^w (\theta^{k-1} h) \) has bounded denominators but its coefficients have arbitrarily high valuation as \( w \to +\infty \).
and substituting into the expression corresponding to (6) for $F_{\alpha}$ in place of $F_{c^+(1)}$, we obtain

$$[D^{k-1}(F_{\alpha})] = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) [f_\beta] + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) [f_{\beta'}],$$

and hence we have the equality

(7) \[ D^{k-1}(F_{\alpha}) = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) f_\beta + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) f_{\beta'} + \theta^{k-1} h \]

as sections in $H^0(W_2 \setminus C, \omega^k)$, for some $h \in H^0(W_2 \setminus C, \omega^{2-k})$. Applying $U^w$ to both sides of this equation and letting $w \to +\infty$ as in the proof of Theorem 6.1, we deduce that

(8) \[ \lim_{w \to +\infty} U^w D^{k-1}(F_{\alpha}) = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) f_\beta. \]

On the other hand, arguing again as in Theorem 6.1 we find that the $p^w$-th coefficient of $D^{k-1}(F_{\alpha})$ is given by

$$c_\alpha(p^w) = \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) \beta^w + \left(t_2 + \gamma \frac{\beta'}{\beta - \beta'}\right) \beta'^w + O(p^{w(k-1)}),$$

and hence

(9) \[ \left(t_1 - \gamma \frac{\beta}{\beta - \beta'}\right) = \lim_{w \to +\infty} \frac{c_\alpha(p^w)}{\beta^w}. \]

Therefore, except in the case where

(10) \[ \gamma = \frac{t_1(\beta - \beta')}{\beta} = (\beta - \beta') \lim_{w \to +\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}, \]

combining (8) and (9) we recover $f_\beta$ from $F_{\alpha}$ as in the statement of the theorem. \hfill \Box

5. Mock modular forms as $p$-adic modular forms

We now let $\alpha$ range over the larger set of values

$$c^+(1) + \mathbb{C}_p := \{c^+(1) + \gamma : \gamma \in \mathbb{C}_p\},$$

and interpret the exceptional value of $\alpha$ in Theorem 4.3 as the only one for which the ‘corrected’ mock modular form

$$F_{\alpha} = F^+ - \alpha E_f$$

gives rise to a $p$-adic modular form upon $p$-stabilization. Recall that we let $\beta$ and $\beta'$ be the roots of the $p$-th Hecke polynomial of $f$, ordered so that $v_p(\beta) \leq v_p(\beta')$.

Definition 5.1. For any $\alpha \in c^+(1) + \mathbb{C}_p$, define

$$F^*_\alpha := F_\alpha - p^{1-k}\beta' F_\alpha|V$$

and write

$$D^{k-1}(F^*_\alpha) = \sum_{n \gg -\infty} c^*_\alpha(n) q^n.$$

Our first result shows that, similarly as in Theorem 4.3 for $F_\alpha$, the $p$-stabilization $f_\beta$ of the shadow of $F^+$ can be recovered $p$-adically from $F^*_\alpha$. 
Theorem 5.2. Assume that \( v_p(\beta) \neq v_p(\beta') \) and that \( v_p(\beta') \neq k - 1 \). Then for all but at most one choice of \( \alpha \in c^+(1) + \C_p \), we have

\[
\lim_{w \to +\infty} \frac{U_w D^{k-1}(F^*_\alpha)}{c^*_\alpha(p^w)} = f_\beta.
\]

Proof. The proof is quite similar to the proof of Theorem 4.3. Writing \( \alpha = c^+(1) + \gamma \) with \( \gamma \in \C_p \), an immediate calculation reveals that

\[
(D^{k-1}(F^*_\alpha)) = D^{k-1}(F_{c^+(1)})(1 - \beta'V) - \gamma f_\beta.
\]

As in the proof of Theorem 4.2, we write

\[
[D^{k-1}(F_{c^+(1)})] = t_1[f_\beta] + t_2[f_\beta] \in H^1_{\text{par}}(X_K, \nabla_{k-2}),
\]

with \( t_1 = \lim_{w \to +\infty} \beta^{-w} c_{c^+(1)} \). Applying the operator \( 1 - \beta' V \) to this last equality, and noting that \( V = U^{-1} \) on cohomology, we obtain

\[
[D^{k-1}(F_{c^+(1)})(1 - \beta'V)] = t_1 \frac{(\beta - \beta')}{\beta}[f_\beta],
\]

and hence by (11):

\[
[D^{k-1}(F^*_\alpha)] = \left( \frac{t_1(\beta - \beta')}{\beta} - \gamma \right)[f_\beta].
\]

Arguing again as in the proof of Theorem 6.1, we obtain the equalities

\[
\lim_{w \to +\infty} \frac{U_w D^{k-1}(F^*_\alpha)}{\beta^w} = \left( \frac{t_1(\beta - \beta')}{\beta} - \gamma \right) f_\beta
\]

and

\[
\frac{t_1(\beta - \beta')}{\beta} - \gamma = \lim_{w \to +\infty} \frac{c^*_\alpha(p^w)}{\beta^w}.
\]

Therefore, except in the case where

\[
\gamma = \frac{t_1(\beta - \beta')}{\beta} = (\beta - \beta') \lim_{w \to +\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}},
\]

the combination of (8) and (9) recovers \( f_\beta \) from \( F^*_\alpha \) as in the statement of the theorem. \( \square \)

Considering the exceptional value of \( \alpha \) arising in the proof of Theorem 5.2, we recover the result of [BGK12, Thm. 1.1].

Theorem 5.3. Assume that \( v_p(\beta) \neq v_p(\beta') \) and that \( v_p(\beta') \neq k - 1 \). Then among all values of \( \alpha \in c^+(1) + \C_p \), the value

\[
\alpha = c^+(1) + (\beta - \beta') \lim_{w \to +\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}
\]

is the unique one such that \( F^*_\alpha \) is a \( p \)-adic modular form of weight \( 2 - k \).
Proof. Write \( \alpha = c^+(1) + \gamma \) with \( \gamma \in \mathbb{C}_p \). Since \([f_\beta] \neq 0 \) (see the proof of Theorem 5.3), we deduce from (12) and (15) that the class of \( D^{k-1}(F^*_\alpha) \) in \( H^1_{\text{par}}(X_K, \nabla_{k-2}) \) vanishes only for the value of \( \alpha \) given in the statement. Now, since the natural restriction map

\[
H^1_{\text{par}}(X_K, \nabla_k) \to H^1(W_2 \setminus C, \nabla_{k-2}) = \frac{H^0(W_2 \setminus C, \omega^k)}{\theta^{k-1}H^0(W_2 \setminus C, \omega^{2-k})}
\]

is injective, the above value of \( \alpha \) is also the unique one such that the class of \( D^{k-1}(F^*_\alpha) \) becomes trivial in \( H^1(W_2 \setminus C, \nabla_{k-2}) \), and hence such that \( F^*_\alpha \in H^0(W_2 \setminus C, \omega^{2-k}) \).

Next we consider a second modification of \( F_\alpha \).

**Definition 5.4.** For any \( \delta \in \mathbb{C}_p \), define

\[
F_{\alpha,\delta} := F_\alpha - \delta(E_f - \beta E_f|\nabla).
\]

Our next result explores the values of \( \alpha \) and \( \delta \) for which \( F_{\alpha,\delta} \) is a \( p \)-adic modular form, recovering the content of [BGK12, Thm 1.2(2)].

**Theorem 5.5.** Assume that \( v_p(\beta) \neq v_p(\beta') \) and that \( v_p(\beta') \neq k - 1 \). Then there exists a unique pair of values \((\alpha, \delta)\) for which \( F_{\alpha,\delta} \) is a \( p \)-adic modular form. In fact, \( \alpha \) is as in Theorem 5.3 and

\[
\delta = \lim_{w \to +\infty} a_{F_\alpha}(p^w)p^{w(1)} / \beta^{nw}.
\]

Here, we write \( F_\alpha = \sum_{n \gg -\infty} a_{F_\alpha}(n)q^n \).

**Proof.** With the same notations as in the proof of Theorem 4.3, we can write the equality

\[
[D^{k-1}(F_{\alpha,\delta})] = \left( t_1 - \gamma \frac{\beta'}{\beta - \beta'} \right) [f_\beta] + \left( t_2 + \gamma \frac{\beta'}{\beta - \beta'} - \delta \right) [f_\beta']
\]

in \( H^1_{\text{par}}(X_K, \nabla_{k-2}) \). Since we may check the triviality of these classes upon restriction to \( W_2 \setminus C \), it follows that \( F_{\alpha,\delta} \) is a \( p \)-adic modular form of weight \( 2 - k \) if and only if the class \([D^{k-1}(F_{\alpha,\delta})]\) vanishes. As in the proof of Theorem 5.3, the classes \([f_\beta], [f_\beta']\) form a basis for \( H^1_{\text{par}}(X_K, \nabla_{k-2}) \), and hence \( F_{\alpha,\delta} \) is a \( p \)-adic modular form if and only if the coefficients in the right-hand side of (16) both vanish. In particular (second coefficient), this shows that the value of \( \gamma \) is given by (10), and therefore the necessary value of \( \alpha = c^+(1) + \gamma \) is the same as in Theorem 5.3. To determine the value of \( \delta \), we first rewrite Equation (17) for the above value of \( \alpha \) (so that the first summand in the right-hand side of that equation vanishes):

\[
D^{k-1}(F_\alpha) = \left( t_2 + \gamma \frac{\beta'}{\beta - \beta'} \right) f_\beta' + \theta^{k-1}h_c.
\]

Equating the \( p^w \)-th coefficients in this equality, we obtain

\[
c_\alpha(p^w) = \left( t_2 + \gamma \frac{\beta'}{\beta - \beta'} \right) \beta^w + O(p^{w(1)})
\]

and hence dividing by \( \beta^w \) and letting \( w \to +\infty \) we deduce

\[
\lim_{w \to +\infty} \frac{c_\alpha(p^w)}{\beta^w} = \left( t_2 + \gamma \frac{\beta'}{\beta - \beta'} \right).
\]
(Note that the assumption \( v_p(\beta') < k - 1 \) is being used here.) Finally, substituting (16) into (16), we see that the necessary value for \( \delta \) is given by

\[
\delta = \lim_{w \to +\infty} \frac{c_\alpha(p^w)}{\beta^w} = \lim_{w \to +\infty} \frac{\alpha_{F_\alpha}(p^w)p^{w(k-1)}}{\beta^w},
\]

as was to be shown.

6. The CM case

In this section we treat the case in which \( f \) has CM. This case is of special interest, since then one can choose a good harmonic Mass form \( F \) for \( f \) as in Section 2 with \( F^+ \) having algebraic coefficients. Conjecturally, this characterize the CM property of \( f \) (see [GKO10, p.6170]). Thus assume that \( f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_1(N), K) \) has CM by an imaginary quadratic field \( M \) of discriminant prime to \( p \), and let \( F = F^+ + F^- \) be a good harmonic Maass form attached to \( F \). We also assume (upon enlarging \( K \), if necessary) that \( K \) contains a primitive \( m \)-th root of unity, where \( m = N \cdot \text{disc}(M) \).

Then by [BOR08 Thm. 1.3], \( F^+ \) has coefficients in \( K \), and so \( D^{k-1}(F^+) \) defines a class in \( \mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f \).

We first treat the case in which \( p \) is inert in \( M \). In this case \( a_p = \beta + \beta' = 0 \), and so by the proof of Theorem 5.3 the space \( \mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f \) admits a basis given by the classes \([f_\beta]\) and \([f_{\beta'}]\).

**Lemma 6.1.** Assume that \( p \) is inert in \( M \), and write \( [D^{k-1}(F^+)] = t_1[f_\beta] + t_2[f_{\beta'}] \). Then

\[
\lim_{w \to +\infty} \frac{\alpha_{D^{k-1}(F^+)}(p^{2w+1})}{\beta_{2w+1}} = t_1 - t_2.
\]

**Proof.** The proof will be obtained by arguments similar to the proof of Theorem 4.2, but some adjustments are necessary due to the fact the condition \( v_p(\beta) \neq v_p(\beta') \) is not satisfied in this case. Instead, we shall exploit the extra symmetry \( \beta' = -\beta \).

Upon restriction to \( W_2 \subset C \), we can write

\[
D^{k-1}(F^+) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1} h
\]

for some \( h \in H^0(W_2 \subset C, \omega^{2-k}) \). Taking \( p^{2w+1} \)-st coefficients in this identity, we immediately obtain

\[
a_{D^{k-1}(F^+)}(p^{2w+1}) = t_1 \beta^{2w+1} + t_2 \beta'^{2w+1} + O(p^{(2w+1)(k-1)})
\]

\[
= (t_1 - t_2) \beta^{2w+1} + O(p^{(2w+1)(k-1)}),
\]

and hence dividing by \( \beta^{2w+1} \) and letting \( w \to +\infty \) the result follows.

**Definition 6.2.** For any \( \alpha \in \mathbb{C}_p \), define

\[
\tilde{F}_\alpha := F^+ - \alpha E_{f|V}.
\]

Armed with Lemma 6.1 in Corollary 6.4 below we will determine the values of \( \alpha \) for which \( \tilde{F}_\alpha \) is a \( p \)-adic modular form, thus recovering [BGK12 Thm. 1.3]. This will be an immediate consequence of the following result.
Theorem 6.3. Assume that \( p \nmid N \) is inert in \( M \), and for any \( \alpha \in \mathbb{C}_p \) define
\[
G_{\alpha} := F^+ - \alpha(E_f - \beta E_f|V).
\]
Then there exists a unique value of \( \alpha \) such that \( G_{\alpha} \) is a \( p \)-adic modular form of weight \( 2 - k \), and it is given by
\[
\alpha = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}}.
\]

Proof. We will deduce this result by first determining the values of \( \alpha \) and \( \delta \) for which the form \( \mathcal{F}_{\alpha,\delta} \) of Definition 5.4 is a \( p \)-adic modular form. Note that this case is not covered by Theorem 5.5 since the proof of that result relies crucially on the assumption that \( v_p(\beta) \leq v_p(\beta') \). Instead, we will exploit again the fact that \( \mathcal{F}_{\alpha,\delta} \) is a \( p \)-adic modular form of weight \( 2 - k \) if and only if both coefficients in the right-hand side of Equation (19) vanish; in particular, we need to have
\[
[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = (t_1 - \frac{\alpha}{2})[f_\beta] + (t_2 - \frac{\alpha}{2} - \delta)[f_\beta'].
\]
By Theorem 6.3, the classes \([f]\) and \([V(f)]\) form a basis for \( \mathbb{H}^1_{par}(X_K, \nabla_{k-2})_f \), and rewriting (19) in terms of them we arrive at
\[
[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = (t_1 + t_2 - \alpha - \delta)[f] + \beta(t_1 + t_2 - \alpha - \delta)[V(f)].
\]
Now, \( \mathcal{F}_{\alpha,\delta} \) is a \( p \)-adic modular form of weight \( 2 - k \) if and only if both coefficients in the right-hand side of Equation (20) vanish; in particular, we need to have
\[
\alpha + \delta = t_1 - t_2 = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}},
\]
where we used Lemma 6.1 for the second equality. The necessary vanishing of (20) also forces the vanishing of \( t_2 \) and hence from (19) we deduce that \( \delta = -\frac{\alpha}{2} \), or equivalently, \( \alpha + \delta = \frac{\alpha}{2} \). Finally, noting that
\[
\mathcal{F}_{\alpha,\delta} = F^+ - \frac{\alpha}{2}(E_f - \beta E_f|V) = G_{\frac{\alpha}{2}},
\]
we conclude from (21) that \( G_{\alpha} \) is a \( p \)-adic modular form if and only if \( \alpha \) is given by the \( p \)-adic limit in the statement. \( \square \)

Corollary 6.4. Assume that \( p \nmid N \) is inert in \( M \). Then there exists a unique value of \( \alpha \) such that \( \tilde{\mathcal{F}}_{\alpha} \) is a \( p \)-adic modular form of weight \( 2 - k \), and it is given by
\[
\alpha = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w}}.
\]

Proof. Comparing the definitions of \( \tilde{\mathcal{F}}_{\alpha} \) and \( G_{\alpha} \), we see that
\[
G_{\alpha} = \tilde{\mathcal{F}}_{\alpha} - \bar{\alpha} E_f,
\]
with \( \alpha = \bar{\alpha} \beta \). Since \( E_f \) is easily seen to be a \( p \)-adic modular form under our hypotheses (see [BGK12], Prop. 4.2), which remains true in our case \( p \nmid N \), the result follows from Theorem 6.3. \( \square \)
We conclude this section by dealing with the case in which \( f \) has CM by an imaginary quadratic field \( M \) in which \( p \) splits, characterizing the values of \( \alpha \in \mathbb{C}_p \) for which \( \mathcal{F}_\alpha^* \) is a \( p \)-adic modular form. As noted in Remark 3.6, the class \([f_\beta']\) vanishes in this case, and so the proofs of Theorem 5.2 and Theorem 5.3 break down. However, based on the observation that (using the algebraicity of \( c^+(1) \) to set \( \alpha = \gamma \))

\[
\mathcal{F}_\alpha^* = (F^+ - \alpha E_f)(1 - p^{1-k}V) = \mathcal{F}_0^* - \alpha E_{f_\beta},
\]

we can easily prove the following result (cf. [BGK12, Thm. 1.2]).

**Theorem 6.5.** Assume that \( p \nmid N \) splits in \( K \). Then among all values of \( \alpha \in \mathbb{C}_p \), the value \( \alpha = 0 \) is the unique one for which \( \mathcal{F}_\alpha^* \) is a \( p \)-adic modular form of weight \( 2 - k \).

**Proof.** As we have already argued in preceding proofs, \( \mathcal{F}_\alpha^* \) is a \( p \)-adic modular form of weight \( 2 - k \) if and only if the class \([D^{k-1}(\mathcal{F}_\alpha^*)]\) vanishes, and from (22) we see that

\[
[D^{k-1}(\mathcal{F}_\alpha^*)] = 0 \iff \alpha[f_\beta] = [D^{k-1}(\mathcal{F}_0^*)].
\]

In particular, this shows that \( \mathcal{F}_\alpha^* \) is a \( p \)-adic modular form of weight \( 2 - k \) for \( \alpha = 0 \), and so \([D^{k-1}(\mathcal{F}_0^*)]\) = 0. On the other hand, since \([f_\beta] \neq 0 \) (see the proof of Theorem 3.5), equivalence (23) shows that \([D^{k-1}(\mathcal{F}_\alpha^*)]\) \neq 0 for \( \alpha \neq 0 \), yielding the result. \( \square \)

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