Amalgamation of types in pseudo-algebraically closed fields and applications

Zoé Chatzidakis∗(CNRS, UMR 8553 - Ecole Normale Supérieure)

Introduction

Pseudo-algebraically closed fields (henceforth abbreviated by PAC) were introduced by Ax in his famous paper [1] on the theory of finite fields. The elementary theory of arbitrary PAC fields, studied among others by Cherlin-Van den Dries-Macintyre [6], and by Ershov [10], puts in light an interesting dichotomy: definable sets are given, on the one hand by classical algebraic data, and on the other hand by elementary statements concerning the Galois group. Many of the properties of the theory of a PAC field thus reduce to the corresponding properties of its Galois group. For instance, if the subfield of algebraic numbers of the PAC field $F$ is decidable, then Th($F$) will be decidable if and only if the “theory” of its absolute Galois group is decidable. One also knows that the structure of their models is complicated: a result of Duret ([9]) asserts that a PAC field which is not separably closed has the independence property.

Interest for the model theory of PAC fields revived in the mid 90’s, when Hrushovski and Pillay ([14]) were able to use stability theoretic techniques for groups definable in pseudo-finite fields, and more generally in bounded PAC fields (a field is bounded if for each $n > 1$ it has only finitely many algebraic extensions of degree $n$). It was then observed that bounded PAC fields have a simple theory, because they satisfy the independence theorem (1991 result of Hrushovski, only published in 2005, [13]). Other results with a stability-theoretic flavour followed: in [3], the author shows that a PAC field with a simple theory is necessarily bounded; a weak notion of independence is defined, and shown to be implied (in any field) by non-forking. In [4], the study of unbounded PAC fields is continued, with emphasis on the theory of $\omega$-free PAC fields. The author shows that for these fields, forking is the transitive closure of weak independence, and shows versions of the independence theorem for various independence notions, the most difficult one being that $\omega$-free PAC fields of characteristic 0 satisfy the independence theorem with independence being the genuine non-forking. This last result is quite surprising, given that the theories of $\omega$-free PAC fields are not simple. This suggested that more can be done on unbounded PAC fields, and that their study might provide an insight of good behaviours of models of non-simple theories.

∗Most of this work was done while the author was partially supported by MRTN-CT-2004-512234 and by ANR-06-BLAN-0183, while a member of the Equipe de Logique Mathématiques (UMR 7056), in University Paris 7. The work achieved its final form in 2017, while the author was partially supported by ANR-13-BS01-0006.
In this paper we continue the investigation of the behaviour of unbounded PAC fields. Our main result is an amalgamation result for types, similar to the (weak) independence theorem of [4]. This result (Theorem 2.1) isolates the conditions under which amalgamation of types is possible. It has various consequences, notably a weak independence theorem over models for PAC fields $F$ such that $SG(F)$ has a simple theory (Theorem 2.5), and the fact that Frobenius fields satisfy NSOP$_3$ (see 2.9 and 2.10). It also appears as an ingredient in the description of imaginaries in PAC fields of finite degree of imperfection: an imaginary of the PAC field $F$ is equi-definable with a finite collection of pairs $(a, D)$, where $a$ is a tuple of elements of $F$ and $D$ is an imaginary of $SG(F)$ (Theorem 4.2). We show by an example that this result is best possible.

The hope that PAC fields might provide good examples of things happening beyond simplicity was vindicated. Recent results of Chernikov and Ramsey ([7]) show that the weak independence theorem proved in [4] for Frobenius fields implies that the theory of a Frobenius field is NTP$_1$. Thus these fields provide a large family of new examples of structures with an NTP$_1$ theory. This is particularly useful as very few examples of theories with NTP$_1$ were known. The $\omega$-free PAC fields are particularly nice Frobenius fields, in which types and definable sets are well understood. As we show here, imaginaries are equally well understood. So, it can be hoped that a further study of these PAC fields might lead to new insight in NTP$_1$ theories. Clearly, the connections between the neo-stability properties of the Galois group and those of the field also need to be explored further.

The paper is organised as follows. In section 1, after setting up the notation, we recall or prove some technical results on fields and profinite groups. Section 2 contains the main result of this paper, Theorem 2.1 as well as various independence theorems and SOP$_n$ properties for $n \geq 3$. We conclude section 2 with some questions. Section 3 develops the part of the logic of complete systems which is interpretable in fields. In particular, it sets up the formalism which will enable us to deal with definable sets. This is applied in section 4 (Theorem 4.2) to give the description of imaginaries of PAC fields $F$ of finite degree of imperfection.

1 Notation and preliminary results

Recall first that a field $F$ is PAC if every absolutely irreducible variety defined over $F$ has an $F$-rational point. Equivalently, if $F$ is existentially closed in any regular extension. In this section we set up the notation, recall some classical results on PAC fields, and give two additional lemmas. We assume familiarity with elementary results on field extensions, see e.g. Chapter III of [16].

1.1. Notation, conventions. We work in the usual language of rings ($\{+, -, \cdot, 0, 1\}$), sometimes expanded by adding constants for a $p$-basis. The separable closure of a field $K$ is denoted by $K^s$, and its absolute Galois group $Gal(K^s/K)$ by $G(K)$. If $A \subseteq K$, then acl$(A)$ denotes the model-theoretic closure of $A$ in the sense of Th$(K)$. It is known that $K$ is a regular extension of acl$(A)$.

We will often work inside the separable closure of a field $K$. In that case, we will denote by SCF the theory Th$(K^s)$, the notation $tp_{SCF}(\_)$ will refer to the type in the field $K^s$. We use the notation acl$_{K^s}(A)$ to denote the algebraic closure in the sense of Th$(K^s)$, i.e., the smallest subfield of $K^s$ containing $A$ and of which $K^s$ is a regular extension. We will say that two
subsets of \( K \) (or of \( K^* \)) are SCF-independent over some \( E \) if they are independent in the sense of \( \text{Th}(K^*) \).

In addition, unless otherwise specified, all fields will be subfields of some large algebraically closed field \( \Omega \). If \( A, B \) are two subfields, then \( AB \) denotes the composite field.

An extremely useful and fundamental result on PAC fields is the so-called “embedding lemma” of Jarden and Kiehne:

**Theorem 1.2.** (Lemma 20.2.1 in [12]) Let \( E/L \) and \( F/M \) be separable field extensions satisfying: \( E \) is countable and \( F \) is an \( \aleph_1 \)-saturated PAC field; if \( \text{char}(F) = p > 0 \), assume in addition that \( [E : E^p] \leq [F : F^p] \). Assume that there is an isomorphism \( L^* \to M^* \) such that \( \varphi_0(L) = M \), and a commutative diagramme

\[
\begin{array}{ccc}
G(E) & \xrightarrow{\Phi} & G(F) \\
\res & & \res \\
G(L) & \xleftarrow{\Phi_0} & G(M)
\end{array}
\]

where \( \Phi_0 \) is the dual of \( \varphi_0 \), and \( \Phi \) is a (continuous) homomorphism. Then \( \varphi_0 \) extends to an embedding \( \varphi : E^* \to F^* \), with dual \( \Phi \), and such that \( F/\varphi(E) \) is separable.

**Remarks 1.3.** We will use the following essentially immediate consequences of this result.

1. We may replace the countability hypothesis on \( E \) by asking \( F \) to be \( |E|^+ \)-saturated. The proof is identical.

2. We will usually have that the extensions \( E/L \) and \( F/M \) are regular. This means that the restriction maps \( G(E) \to G(L) \) and \( G(F) \to G(M) \) are onto. Note that the conclusion will then be that \( F/\varphi(E) \) is regular.

3. (Notation as above.) Let \( E' \) be a Galois extension of \( E \), and \( \Phi' : G(F) \to Gal(E'/E) \) such that the following diagramme commutes:

\[
\begin{array}{ccc}
Gal(E'/E) & \xleftarrow{\Phi'} & G(F) \\
\res & & \res \\
G(L) & \xleftarrow{\Phi_0} & G(M)
\end{array}
\]

As \( G(F) \) is projective, the map \( \Phi' \) factors through a homomorphism \( \Phi : G(F) \to G(E) \) (see Theorem 11.6.2 in [12]). Applying the embedding lemma therefore gives us an embedding \( \varphi' : E' \to F^* \), with dual \( \Phi' \).

**Complete systems associated to profinite groups**

Cherlin, Van den Dries and Macintyre show in [6] how to associate to any profinite group \( G \) a structure \( SG \) in an \( \omega \)-sorted language \( L_G \), called the *complete system of \( G \)*, which encodes precisely the inverse system of all finite continuous quotients of \( G \). The functor \( G \mapsto SG \) is a contravariant functor, and defines a duality between the category of profinite groups with
continuous epimorphisms and the category of complete systems with embeddings. The functor dual to $S$ is the functor $G$ which to a complete system $S$ associates the inverse limit of the inverse system of finite groups given by $S$. If $F_1$ and $F_2$ are fields and $F_1 \equiv F_2$, then $SG(F_1) \equiv SG(F_2)$. For more details on complete systems and their logic, see [6] or the Appendix of [4]. We will first briefly recall the notation and definitions for arbitrary profinite groups, before going to the setting of Galois groups.

1.4. Definition of the complete system of a profinite group.
Let $G$ be a profinite group, and $L_G$ be the $\omega$-sorted language with sorts indexed by the positive integers, and with non-logical symbols $\{\leq, C, P, 1\}$, where $\leq$ and $C$ are binary relations, $P$ is a ternary relation and 1 is a constant symbol. The complete system associated to $G$ is the $L_G$-structure $S(G)$, with universe the disjoint union $\bigcup_N G/N$ where $N$ ranges over all normal open subgroups of $G$. An element of $G/N$, i.e. a coset $gN$, will be of sort $n$ if and only if $[G : N] \leq n$, and 1 is the only element of sort 1. We have $gN \leq hM \iff N \subseteq M$, $C(gN, hM) \iff gN \subseteq hM$, and $P(g_1N_1, g_2N_2, g_3N_3) \iff N_1 = N_2 = N_3$ and $g_1g_2N_1 = g_3N_1$. The class of complete systems of profinite groups is the class of models of a theory $T_G$. The functor $S$ defines a duality between the category of profinite groups with continuous epimorphisms and the category of models of $T_G$ with embeddings.

1.5. Complete systems of Galois groups, subsystems, double duals.
Let $F$ be a field, $E$ a Galois extension of $F$, and $G = Gal(E/F)$. The universe of $SG$ is the disjoint union of all $Gal(L/F)$ where $L$ is a finite Galois extension of $F$ contained in $E$. The elements of sort $n$ with be the Galois groups of size $\leq n$. The language $L_G$ is interpreted as follows: $Gal(F/F) = 1$; whenever $L_1 \supseteq L_2$, $C \cap Gal(L_1/F) \times Gal(L_2/F)$ is the graph of the restriction maps $Gal(L_1/F) \to Gal(L_2/F)$ and $\leq$ contains $Gal(L_1/F) \times Gal(L_2/F)$, and the ternary relation $P$ encodes the graph of multiplication on each $Gal(L/F)$.

A subset $S$ of $SG$ is a subsystem of $SG$ if it has the following two properties: (i) $\forall \sigma, \tau \in S, \exists \rho \in S (\rho \leq \sigma \land \rho \leq \tau)$; (ii) If $\sigma \in S$ and $\tau \geq \sigma$, then $\tau \in S$. If $A \subset SG$, then $\langle A \rangle$ denotes the subsystem of $SG$ generated by $A$.

One sees easily that if $S$ is a subsystem of $SG$, then $S = SGal(M/F)$, where $M$ is the composite of all Galois extensions $L$ such that $S$ contains $Gal(L/F)$. The inclusion map $S \subset SG$ and the restriction map $Gal(E/F) \to Gal(M/F)$ are dual of each other.

Let $F_1$ and $F_2$ be fields, and $\varphi : F_1^* \to F_2^*$ an embedding such that $\varphi(F_1^*) \cap F_2 = F_1$. We then get a continuous epimorphism $\Phi : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$, defined by $\sigma \mapsto \varphi^{-1}\sigma\varphi$. Applying the functor $S$ to $\Phi$ gives us an embedding $SG(F_1) \to SG(F_2)$, defined as follows: if $L_1$ is a finite Galois extension of $F_1$ and $\sigma \in Gal(L_1/F_1)$, then $S\Phi(\sigma)$ is the unique element of $Gal(F_2\varphi(L_1)/F_2)$ extending the element $\varphi\sigma\varphi^{-1}$ of $Gal(\varphi(L_1)/\varphi(F_1))$. We call the map $S\Phi$ the double dual of $\varphi$.

**Theorem 1.6.** (Cherlin, Van den Dries, Macintyre [6]). Let $F_1$ and $F_2$ be PAC fields, separable over a common subfield $E$. The following conditions are equivalent:

(1) $F_1 \equiv_E F_2$.

(2) (i) $F_1$ and $F_2$ have the same degree of imperfection,

\footnote{Warning: in earlier papers by the author they are called substructures.}
(ii) There is \( \varphi \in G(E) \) such that \( \varphi(F_1 \cap E^*) = F_2 \cap E^* \), and the double dual \( S\Phi : S\mathcal{G}(F_1 \cap E^*) \to S\mathcal{G}(F_2 \cap E^*) \) of \( \varphi \), is a partial elementary \( \mathcal{L}_G \)-map \( S\mathcal{G}(F_1) \to S\mathcal{G}(F_2) \).

(In particular, \( S\mathcal{G}(F_1) \equiv S\mathcal{G}(F_2) \)).

From this result, one easily deduces a description of types:

**Theorem 1.7.** (Cherlin, Van den Dries, Macintyre [6]). Let \( F \) be a PAC field, separable over some subfield \( E \). Let \( a \) and \( b \) be tuples of elements of \( F \), and \( A = \operatorname{acl}_{K^s}(E, a) \cap F \), \( B = \operatorname{acl}_{K^s}(E, b) \cap F \). The following conditions are equivalent:

1. \( \operatorname{tp}(a/E) = \operatorname{tp}(b/E) \).
2. There is an \( E \)-isomorphism \( \varphi : A^* \to B^* \), with \( \varphi(a) = b \), \( \varphi(A) = B \), such that the double dual \( S\Phi : S\mathcal{G}(A) \to S\mathcal{G}(B) \) is a partial elementary \( \mathcal{L}_G \)-map of \( S\mathcal{G}(F) \).

1.8. Important facts and remarks. If \( F \) is a PAC field and \( A \subset F \), then \( \operatorname{acl}(A) = \operatorname{acl}_{F^s}(A) \cap F \) (see 4.5 in [5]). Let \( E \subset A, B \) be subfields of \( F \), and assume that \( A \) and \( B \) are SCF-independent over \( E \). If \( \text{char}(F) = p > 0 \) and \( [F : F^p] < \infty \) assume moreover that \( E \) contains a \( p \)-basis of \( K \). Then \( \operatorname{acl}_{F^s}(AB) = (\operatorname{acl}_{F^s}(A)\operatorname{acl}_{F^s}(B))^\ast \). Hence we also have \( \operatorname{acl}(AB) = (\operatorname{acl}(A)\operatorname{acl}(B))^\ast \cap F \).

**Frobenius and \( \omega \)-free PAC fields**

**Definition 1.9.** (1) A profinite group \( G \) has the embedding property if for any finite groups \( A, B \), whenever \( f : G \to A \) and \( g : B \to A \), \( f' : G \to B \) are (continuous) epimorphisms, then there exists an epimorphism \( h : G \to B \) such that \( f = g \circ h \). This property translates into a property of \( S\mathcal{G} \) which is axiomatisable in the language \( \mathcal{L}_G \). See section 24.3 of [12] for more details and properties of these groups.

(2) A Frobenius field is a PAC field whose absolute Galois group \( \mathcal{G}(F) \) has the embedding property.

(3) Recall that a PAC field \( F \) is \( \omega \)-free if whenever \( F_0 \prec F \) is countable, then \( \mathcal{G}(F_0) \simeq \hat{\tilde{F}}_\omega \), the free profinite group on \( \aleph_0 \) generators. In particular, all finite groups occur as finite quotients of \( \mathcal{G}(F) \), and \( F \) is Frobenius. Being Frobenius is an elementary property of a field \( F \). When dualized, and if \( S\mathcal{G}(F) \) is countable, it says that any \( \mathcal{L}_G \)-isomorphism between two finite subsystems of \( S\mathcal{G}(F) \) extends to an automorphism of \( S\mathcal{G}(F) \). In particular this implies the following:

**Theorem 1.10.** ([6]) If \( F \) is a Frobenius field, then any \( \mathcal{L}_G \)-isomorphism between two subsystems of \( S\mathcal{G}(F) \) is elementary.

Thus, in Theorems 1.6 and 1.7 the conditions stating that the partial maps \( S\Phi \) are elementary can be removed.
1.11. \(\omega\)-sorted logic behaves very much like ordinary one-sorted logic, provided one works sort by sort. Our \(\omega\)-sorted structure \(S\) can be viewed as the countable union of structures \(S_n, n \geq 1\), where each \(S_n\) has universe the elements of sort \(\leq n\), and is a structure in the language with \(n\) sorts, relational symbols \(P, C\) and \(\leq\), constant symbol 1. The theory of \(S\) is then naturally the limit of the theories of the \(S_n\)’s. For instance, let \(G\) be a profinite group with the embedding property, \(SG\) its complete system. Then the above characterisation of countable models of \(\text{Th}(SG)\) translates into: \(\text{Th}(SG)\) is \(\aleph_0\)-categorical (see [6]). Notions such as stability or \(\omega\)-stability easily generalise: one just counts types in each sort. Notions which are local immediately generalise, as they only involve finitely many sorts. For instance, the usual definition of forking of a formula over a set; and therefore forking of a type: a type will fork over a set if it contains a formula which forks over that set. Hence one can define the property of a theory of being simple. We will use the fact that the result of Kim and Pillay characterizing simple theories via the properties of the forking relation go through.

Before proving our two technical lemmas, we first recall some results from [4]:

**Lemma 1.12.** (Lemma 2.1 in [4]) Let \(A, B\) be fields (contained in \(\Omega\)), and assume that the field composite \(AB\) is a regular extension of \(A\) and of \(B\). If \(E = A \cap B\), then \(E^* = A^* \cap B^*\).

**Lemma 1.13.** Let \(A, B, C, E\) be separably closed fields (\(\subset \Omega\)), with \(A, B, C\) separable extensions of \(E\), and \(AB\) a separable extension of \(A\) and of \(B\), free from \(C\) over \(E\), and with \(A \cap B = E\). Then

\[
\begin{align*}
(i) \quad & (AB)^* \cap (AC)^*(BC)^* = AB; \quad (b) \quad (AB)^*C \cap (AC)^*(BC)^* = ABC \\
(ii) \quad & (AC)^* \cap (AB)^*(BC)^* = AC; \quad (b) \quad (AC)^*B \cap (AB)^*(BC)^* = ABC.
\end{align*}
\]

**Proof.** Items (4) and (2) of Lemma 2.5 in [4] give (i)(a)(b) and (ii)(a). By (3) of that same lemma, we have \((AC)^*(AB)^* \cap (BC)^*(AB)^* = C(AB)^*\), which implies \((AC)^*B \cap (AB)^*(BC)^* \subseteq C(AB)^* \cap (AC)^*B \subseteq ABC\) by (i)(b), and gives us (ii)(b).

**Lemma 1.14.** Let \(A, B, C\) (contained in \(\Omega\)) be regular extensions of a field \(E\), and assume that \(AB\) is a regular extension of \(A\) and of \(B\), that \(A \cap B = E\), and that \(AB\) is free from \(C\) over \(E\). Consider the map

\[
\rho : \text{Gal}((AB)^*(AC)^*(BC)^*/ABC) \rightarrow \mathcal{G}(AB) \times \mathcal{G}(AC) \times \mathcal{G}(BC)
\]

defined by

\[
\sigma \mapsto (\sigma|_{(AB)^*}, \sigma|_{(AC)^*}, \sigma|_{(BC)^*}).
\]

Then the image of \(\rho\) is the subgroup of \(\mathcal{G}(AB) \times \mathcal{G}(AC) \times \mathcal{G}(BC)\) consisting of the triples \((\sigma_1, \sigma_2, \sigma_3)\) such that

\[
\sigma_1|_{A^*} = \sigma_2|_{A^*}, \quad \sigma_1|_{B^*} = \sigma_3|_{B^*}, \quad \sigma_2|_{C^*} = \sigma_3|_{C^*}.
\]

**Proof.** The compatibility conditions are clearly necessary, it remains to show that they are sufficient. Let \((\sigma_1, \sigma_2, \sigma_3) \in \mathcal{G}(AB) \times \mathcal{G}(AC) \times \mathcal{G}(BC)\) satisfy the required conditions. We will first show that there is some \(\sigma \in \text{Gal}(A^*B^*C^*/ABC)\) which agrees with \(\sigma_1\) on \(A^*B^*\), with \(\sigma_2\) on \(A^*C^*\) and with \(\sigma_3\) on \(B^*C^*\). First note that \(\sigma_1, \sigma_2, \sigma_3\) all agree on \(E^*\).
As $C$ is free from $AB$ over $E$, and is a regular extension of $E$, we know that $C^*$ is linearly disjoint from $(AB)^*$ over $E^*$, and therefore from $A^*B^*C^*$ over $CE^*$. Hence there is $\sigma \in \mathcal{G}al(A^*B^*C^*/ABC)$ which agrees with $\sigma_1$ on $A^*B^*$ and with $\sigma_2$ on $C^*$; by hypothesis, $\sigma$ therefore agrees with $\sigma_2$ on $A^*C^*$, and with $\sigma_3$ on $B^*$ and on $C^*$, i.e., on $B^*C^*$.

By Lemma 1.12, $A^* \cap B^* = E^*$ and we may apply Lemma 1.13 to obtain:

- the Galois extensions $(AB)^*C^*$, $(AC)^*B^*$ and $A^*(BC)^*$ are linearly disjoint over $A^*B^*C^*$ (use (i)(b) and (ii)(b)),
- $(AB)^*$ and $(AC)^*(BC)^*$ are linearly disjoint over $A^*B^*$ (by (i)(a)), so that
  \[ \mathcal{G}al((AB)^*C^*/A^*B^*C^*) \simeq \mathcal{G}(A^*B^*), \]
- $(AC)^*$ and $(AB)^*(BC)^*$ are linearly disjoint over $A^*C^*$ (by (ii)(a)), so that
  \[ \mathcal{G}al((AC)^*B^*/A^*B^*C^*) \simeq \mathcal{G}(A^*C^*), \]
- $(BC)^*$ and $(AB)^*(AC)^*$ are linearly disjoint over $B^*C^*$ (by (ii)(a)), so that
  \[ \mathcal{G}al((BC)^*A^*/A^*B^*C^*) \simeq \mathcal{G}(B^*C^*). \]

This gives in particular that
\[ \mathcal{G}al((AB)^*(AC)^*(BC)^*/A^*B^*C^*) \simeq \mathcal{G}(A^*B^*) \times \mathcal{G}(A^*C^*) \times \mathcal{G}(B^*C^*). \]

Hence, the automorphism $\sigma \in \mathcal{G}al(A^*B^*C^*/ABC)$ can be lifted uniquely to an element $\sigma$ of $\mathcal{G}al((AB)^*(AC)^*(BC)^*/ABC)$ which agrees with $\sigma_1$ on $(AB)^*$, with $\sigma_2$ on $(AC)^*$ and with $\sigma_3$ on $(BC)^*$.

**Lemma 1.15.** Let $E \subset A$ and $E_1$ be algebraically closed subsets of a PAC field $F$, and assume that $\varphi_0 : E^* \to E_1^*$ is an isomorphism and restricts to an elementary map $E \to E_1$. If the characteristic is $p > 0$ and $[F : F^p] < \infty$, then assume that $E$ contains a $p$-basis of $F$. Let $S\Psi : S\mathcal{G}(A) \to S \subset S\mathcal{G}(F)$ be a partial $L_G$-elementary isomorphism extending the double dual $S\Phi_0 : S\mathcal{G}(E) \to S\mathcal{G}(E_1)$ of $\varphi_0$. Then in some elementary extension $F^*$ of $F$, there is $B$, which is SCF-independent from $F$ over $E_1$, and an isomorphism $\varphi : A^* \to B^*$ sending $A$ to $B$, extending $\varphi_0$ and with double dual $S\Psi$. Moreover, $(B, E_1)$ realises tp($A, E$).

**Proof.** Let $M$ be the Galois extension of $F$ with Galois group over $F$ corresponding to $S$, i.e., the restriction map $\text{res} : \mathcal{G}(F) \to \mathcal{G}(M/F)$ is dual to $S \subset S\mathcal{G}(F)$. (Without the requirement that $B$ be SCF-independent from $F$ over $E_1$, we could just apply Theorem 1.12.)

Choose any extension $\varphi$ of $\varphi_0$ to $A^*$ such that $\varphi(A^*)$ and $F^*$ are linearly disjoint over $E_1^*$, and let $B = \varphi(A)$. Then the double dual $S\Phi$ of $\varphi$ extends $S\Phi_0$, and the dual $\Phi$ of $\varphi$ defines an isomorphism $\mathcal{G}(B) \to \mathcal{G}(A)$ which induces the dual $\Phi_0$ of $\varphi_0$. $\Phi_0 : \mathcal{G}(E_1) \to \mathcal{G}(E)$. The dual $\Psi$ of $S\Psi$ defines an isomorphism $\Psi : \mathcal{G}(M/F) \to \mathcal{G}(A)$, which also induces $\Phi_0$ on $\mathcal{G}(E)$. Consider the profinite group
\[ H = \{ (\Phi^{-1}(\sigma), \Psi^{-1}(\sigma)) \mid \sigma \in \mathcal{G}(A) \}. \]
Then $H$ is the graph of $\Psi^{-1}\Phi : \mathcal{G}(B) \to \mathcal{G}(M/F)$, and can be identified with a closed subgroup of $\mathcal{G}(B^*M/BF) \simeq \mathcal{G}(B) \times_{\mathcal{G}(E_1)} \mathcal{G}(M/F)$. Let $L$ be the subfield of $B^*M$ fixed by the elements of $H$. Since $H$ projects onto $\mathcal{G}(B)$ and onto $G(S) = \mathcal{G}(M/F)$, it follows that $L$ is a regular extension of $B$ and of $F$, and that $\mathcal{G}(B^*M/L) = H$ canonically identifies with $\mathcal{G}(B)$ and with $\mathcal{G}(M/F) = G(S)$ via the restriction maps. It follows that the restriction $\mathcal{G}(B^*F^*/L) \to \mathcal{G}(F)$ is an isomorphism. By Theorem 1.2 there is an elementary extension $F^*$ of $F$ containing $B$, and such that $F^* \cap B^*F^* = L$. Then the map $\varphi : A^* \to B^*$ is our desired map: by construction, inside $SG(F^*)$, we have $SG(B) = S$, and the double dual of $\varphi$ coincides with $S\Psi$, which is an elementary map. This proves the first assertion, and 1.7 gives the moreover part.

**Remark 1.16.** Let $E,E_1,A,\varphi_0$ be as above. Let $L$ be a Galois extension of $A$, and assume that we have a partial elementary $\mathcal{L}_G$-map $S\Psi^\prime : S\mathcal{G}(L/A) \to S^\prime \subset S\mathcal{G}(F)$ which extends the restriction of $S\Phi_0$ to $S\mathcal{G}(L \cap E^*/E)$. Then there is a subsystem $S$ of $S\mathcal{G}(E)$, such that the map $S\Psi^\prime$ extends to an elementary $\mathcal{L}_G$-map $S\mathcal{G}(A) \to S$. Thus in the above lemma, we may replace $S\mathcal{G}(A)$ by a subsystem.

## 2 Amalgamation of types

**Theorem 2.1.** Let $F$ be a PAC field, and let $E,A,B,C_1,C_2$ be algebraically closed subsets of $F$, with $E$ contained in $A,B,C_1,C_2$. Assume that $A \cap B = E$, that $A$ and $C_1$, and $B$ and $C_2$, are SCF-independent over $E$, and that if the degree of imperfection of $F$ is finite, then $E$ contains a $p$-basis of $F$. Moreover, assume that there is an $E^*$-isomorphism $\varphi : C_1^* \to C_2^*$ such that $\varphi(C_1) = C_2$, and that there is $S_0 \subset S\mathcal{G}(F)$, and elementary (in $S\mathcal{G}(F)$) isomorphisms

$$
S\Psi_1 : \langle S\mathcal{G}(C_1), S\mathcal{G}(A) \rangle \to \langle S_0, S\mathcal{G}(A) \rangle
$$

$$
S\Psi_2 : \langle S\mathcal{G}(C_2), S\mathcal{G}(B) \rangle \to \langle S_0, S\mathcal{G}(B) \rangle
$$

such that

(i) $S\Psi_1$ is the identity on $S\mathcal{G}(A)$, $S\Psi_2$ is the identity on $S\mathcal{G}(B)$, $S\Psi_1(S\mathcal{G}(C_1)) = S_0$ and

(ii) If $S\Phi : S\mathcal{G}(C_1) \to S\mathcal{G}(C_2)$ is the morphism double dual to $\varphi$, then

$$
S\Psi_2 S\Phi = S\Psi_1|_{S\mathcal{G}(C_1)}.
$$

Then, in some elementary extension $F^*$ of $F$, there is $C$ which is SCF-independent from $(A,B)$ over $E$, realises $\text{tp}(C_1/A) \cup \text{tp}(C_2/B)$, and with $S\mathcal{G}(C) = S_0$ (The variables for $\text{tp}(C_1/A)$ and $\text{tp}(C_2/B)$ are identified via $\varphi$.)

**Proof.** We may assume that $F$ is sufficiently saturated; then $S\mathcal{G}(F)$ will also be sufficiently saturated. We work inside $\Omega$. Choose $C$ realising $\text{tp}_{\text{SCF}}(C_1/E)$, and SCF-independent from $F$ over $E$. Let $\varphi_1 : C^* \to C_1^*$ and $\varphi_2 : C^* \to C_2^*$ be $E^*$-isomorphisms such that

$$
\varphi_1(C) = C_1, \quad \varphi_2(C) = C_2, \quad (\text{whence } \varphi_2(C) = C_2).
$$
As \( A \) is linearly disjoint from \( C \) and from \( C_1 \) over \( E \), we have that \( A^s \) is linearly disjoint from \( C^s \) and from \( C_1^s \) over \( E^s \), and we may therefore extend \( \varphi_1 \) to an \( A^s \)-isomorphism

\[
\varphi'_1 : (AC)^s \to (AC_1)^s.
\]

Similarly, we extend \( \varphi_2 \) to a \( B^s \)-isomorphism

\[
\varphi'_2 : (BC)^s \to (BC_2)^s.
\]

Let \( D_1 = \varphi'^{-1}_1(\text{acl}(AC_1)) \), and \( D_2 = \varphi'^{-1}_2(\text{acl}(BC_2)) \). Then \( D_1 \subseteq (AC)^s \), \( D_2 \subseteq (BC)^s \).

Because \( S\Psi_1 \) and \( S\Psi_2 \) are elementary and \( S\varpi(F) \) is sufficiently saturated, there are subsystems \( S_1 \) and \( S_2 \) of \( S\varpi(F) \), and elementary isomorphisms

\[
S\Psi'_1 : S\varpi(\text{acl}(AC_1)) \to S_1, \quad S\Psi'_2 : S\varpi(\text{acl}(BC_2)) \to S_2
\]

extending \( S\Psi_1 \) and \( S\Psi_2 \) respectively.

For \( i = 0, 1, 2 \) we let \( L_i \) be the Galois extension of \( F \) such that the restriction map \( \varpi(F) \to \text{Gal}(L_i/F) \) is dual to \( S_i \subset S\varpi(F) \). Let \( S\Psi'_i \) be the double dual of \( \varphi'_i \) for \( i = 1, 2 \), and define

\[
S\Theta_i = S\Psi'_i S\varphi'_i : S\varpi(D_i) \to S_i,
\]

and let

\[
\Theta_i : \text{Gal}(L_i/F) \to \varpi(D_i)
\]

be the homeomorphism dual to \( S\Theta_i \) (i.e., \( S\Theta_i \) sends isomorphically Galois groups of finite subextensions of \( L_i \) over \( F \) to finite Galois groups of finite extensions of \( D_i \), and passing to the limit gives an isomorphism \( \Theta_i \)).

We will show that there is a continuous morphism (not necessarily onto)

\[
\Theta : \text{Gal}((AB)^s L_1 L_2/F) \to \text{Gal}((AB)^s(AC)^s(BC)^s/\text{ABC})
\]

which induces \( \Theta_i \) on \( \text{Gal}(L_i/F) \) for \( i = 1, 2 \), the identity on \( \text{Gal}(\text{acl}(AB)) \), and whose image \( U \) projects onto \( \varpi(D_1) \), \( \varpi(D_2) \) and \( \varpi(\text{acl}(AB)) \) (via the restriction maps).

Since \( A = \text{acl}(A) \) and \( B = \text{acl}(B) \), we know that \( F \) is a regular extension of \( A \) and of \( B \); hence \( AB \) is a regular extension of \( A \) and of \( B \). By Lemma 1.13, we may identify \( \text{Gal}((AB)^s(AC)^s(BC)^s/\text{ABC}) \) with the set of triples \( (\sigma_1, \sigma_2, \sigma_3) \in \varpi(AB) \times \varpi(AC) \times \varpi(BC) \) satisfying

\[
\sigma_1|_{A^s} = \sigma_2|_{A^s}, \quad \sigma_1|_{B^s} = \sigma_3|_{B^s}, \quad \sigma_2|_{C^s} = \sigma_3|_{C^s}.
\]

Define

\[
\Theta(\sigma) = (\sigma|_{(AB)^s}, \Theta_1(\sigma|_{L_1}), \Theta_2(\sigma|_{L_2})) = \text{def} (\sigma_1, \sigma_2, \sigma_3).
\]

We need to show that \( \Theta(\sigma) \in \text{Gal}((AB)^s(AC)^s(BC)^s/\text{ABC}) \). Because \( \varphi'_1 \) is an \( A^s \)-isomorphism, \( S\varphi'_1 \) is the identity on \( S\varpi(A) \), and because \( S\varphi'_1 \) extends \( S\Psi_1 \), so is \( S\Psi'_1 \). Hence \( S\Theta_1 \) is the identity on \( S\varpi(A) \). This shows that \( \sigma_1|_{A^s} = \sigma_2|_{A^s} \). Similarly, \( \sigma_1|_{B^s} = \sigma_3|_{B^s} \). We still need to show that \( \sigma_2|_{C^s} = \sigma_3|_{C^s} \). By duality, it is enough to show that \( S\Theta_1 \) and \( S\Theta_2 \) agree on \( S\varpi(C) \).

We know that \( \varphi(\varphi_1 = \varphi_2 \), and that \( \varphi_i \) extends \( \varphi_i \). Hence

\[
S\Phi S\varphi'_1|_{S\varpi(C)} = S\Phi'_2|_{S\varpi(C)}.
\]
We also have that 
\[ S\Psi_1|_{S\mathcal{G}(C_1)} = S\Psi_2 S\Phi. \]

Hence
\[
S\Theta_1|_{S(G(C))} = S\Psi_1|_{S\mathcal{G}(C_1)} S\Psi'_1|_{S\mathcal{G}(C)} = (S\Psi_2 S\Phi)(S\Phi^{-1} S\Phi'_2|_{S\mathcal{G}(C)}) = S\Theta_2|_{S\mathcal{G}(C)},
\]
so that \( \Theta \) takes its values in \( \mathcal{G}(\langle AB \rangle^* (AC)^*(BC)^*/ABC) \). Moreover, observe that since \( \varphi'_1(D_1) = acl(AC_1) \), and by definition of \( \Theta_1 \), we get that \( \Theta_1 \) defines a homeomorphism between \( \mathcal{G}(L_1/F) \) and \( \mathcal{G}(D_1) \). Similarly, \( \Theta_2 \) defines a homeomorphism between \( \mathcal{G}(L_2/F) \) and \( \mathcal{G}(D_2) \).

As \( \mathcal{G}(\langle AB \rangle^* L_1 L_2/F) \) projects onto \( \mathcal{G}(acl(AB)) \), onto \( \mathcal{G}(L_1/F) \) and onto \( \mathcal{G}(L_2/F) \) (via the restriction maps), we get that \( U = \Theta(\mathcal{G}(\langle AB \rangle^* L_1 L_2/F)) \) projects onto \( \mathcal{G}(acl(AB)) \), onto \( \mathcal{G}(D_1) \) and onto \( \mathcal{G}(D_2) \). Let \( D \) be the subfield of \( \langle AB \rangle^* (AC)^*(BC)^* \) fixed by \( U \). Then \( D \) is a regular extension of \( acl(AB) \), and of \( D_1 \) and \( D_2 \). By Theorem 1.2 there is an elementary extension \( F^* \) of \( F \), such that \( F^* \cap \langle AB \rangle^* (AC)^*(BC)^* = D \). Note that \( D_1 = acl(AC) \) and \( D_2 = acl(BC) \).

To finish the proof, we need to show that \( C \) realises \( tp(C_1/A) \cup tp(C_2/B) \), and that \( S\mathcal{G}(C) = S_0 \). Consider \( \varphi'_1 : (AC)^* \rightarrow (AC_1)^* \). Then \( \varphi'_1(C) = C_1 \) and \( \varphi'_1(acl(AC)) = acl(AC_1) \) (because \( F^* \) is a regular extension of \( D_1 \)). We therefore only need to show that the double-dual \( S\Psi'_1 \) is elementary in the structure \( S\mathcal{G}(F^*) \). By definition, \( S\Psi'_1 = S\Psi'^{-1} S\Theta_1 \). By definition of \( D \), the Galois extensions \( L_1 F^* \) and \( D_1 F^* \) are equal, and \( S\Theta_1 \) is the identity on \( \mathcal{G}(L_1 F^*/F^*) = \mathcal{G}(D_1 F^*/F^*) \). Also, \( S\Psi'_1 \) is an elementary isomorphism of \( S\mathcal{G}(F) \), hence also of \( S\mathcal{G}(F^*) \). This shows that \( S\Psi'_1 \) is elementary, and therefore that \( tp(C/A) = tp(C_1/A) \). Similarly one shows that \( tp(C/B) = tp(C_2/B) \). From the definition of \( F \) one also deduces that \( F^* L_0 = F^* C^* \), which finishes the proof.

**Remark/Corollary 2.2.** If \( F \) is Frobenius, then the condition on the \( S\Psi_i \) can be relaxed to their being \( \mathcal{L}_G \)-isomorphisms. The existence of \( S_0 \) is still required, for trivial reasons: one could have \( S\mathcal{G}(C_1) \subset S\mathcal{G}(A) \) and \( S\mathcal{G}(C_2) \not\subset S\mathcal{G}(A) \).

**Definition 2.3.** Let \( F \) be a PAC field, let \( a, b, c \) be subsets of \( F \), \( C = acl(c), A = acl(c,a) \) and \( B = acl(c,b) \). We say that \( a \) and \( b \) are weakly independent over \( c \) if \( A \) and \( B \) are SCF-independent over \( C \) and \( tp(S\mathcal{G}(A)/S\mathcal{G}(B)) \) does not fork over \( S\mathcal{G}(C) \).

**Remark.** Note that this notion will only be symmetric if \( Th(S\mathcal{G}(F)) \) is simple.

**Theorem 2.4.** Let \( F \) be a PAC field, let \( E = acl(E) \subset F \), let \( a, b, c_1, c_2 \) be tuples of elements of \( F \), and let \( A = acl(Ea), B = acl( Eb) \) and \( C_i = acl(Ec_i) \) for \( i = 1, 2 \). Assume that \( E \) contains a \( p \)-basis of \( F \) if the degree of imperfection of \( F \) is finite, and moreover that

(i) \( A \cap B = E \).

(ii) \( A \) and \( C_1 \) are weakly independent over \( E \), \( B \) and \( C_2 \) are weakly independent over \( E \).

(iii) \( tp(c_1/E) = tp(c_2/E) \).
Then there is $c$ realizing $tp(c_1/A) \cup tp(c_2/B)$, weakly independent from $(a,b)$ over $E$.

**Proof.** Clear by Theorem 2.1

**Theorem 2.5.** Let $F$ be a PAC field, and assume that $Th(S\mathcal{G}(F))$ is simple. Then $Th(F)$ satisfies the weak independence theorem over submodels, i.e., in the notation of 2.4, if $E \prec F$, $a$ and $b$ are weakly independent over $E$, and $a,b,c_1,c_2$ satisfy the hypotheses (ii) – (iii) of 2.4 then there is $c$ realizing $tp(c_1/Ea) \cup tp(c_2/Eb)$, weakly independent from $(a,b)$ over $E$.

**Proof.** Apply Theorem 2.4: (i) follows from the weak independence of $a$ and $b$ over $E$, and (iv) because $S\mathcal{G}(F)$ satisfies the independence theorem over models, by a result of Kim-Pillay [15].

**Theorem 2.6.** Let $F$ be a Frobenius field, $E \subset F$. Let $a,b,c_1,c_2$ be tuples in $F$, and $A = acl(Ea), B = acl(Eb), C_1 = acl(Ec_1)$ and $C_2 = acl(EC_2)$. Assume that

(i) $A \cap B = E$.

(ii) $a$ and $c_1$ are SCF-independent over $E$, and $b$ and $c_2$ are SCF-independent over $E$.

(iii) $tp(c_1/E) = tp(c_2/E)$.

(iv) $acl(S\mathcal{G}(A)) \cap acl(S\mathcal{G}(C_1)) = S\mathcal{G}(E), acl(S\mathcal{G}(B)) \cap acl(S\mathcal{G}(C_2)) = S\mathcal{G}(E)$.

Then there is $c$ realizing $tp(c_1/Ea) \cup tp(c_2/Eb)$ and which is weakly independent from $(b,c)$ over $E$.

**Proof.** By Proposition 4.1 in [2], two subsets of $S\mathcal{G}(F)$ are independent over the intersection of their algebraic closure. Apply Theorem 2.5

**Remark 2.7.** Condition (iv) is a little bit awkward. (When $F$ is Frobenius), it is equivalent to $S\mathcal{G}(A) \cap S\mathcal{G}(C_1) = S\mathcal{G}(B) \cap S\mathcal{G}(C_2) = S\mathcal{G}(E)$ in the following two cases

- if $F$ is $\omega$-free, or more generally, is $c$-Frobenius (see (6.6) in [4] for a definition),
- or if $E \prec F$.

**Proof.** The first assertion follows from the fact that if $F$ is a $c$-Frobenius field, then any subsystem of $S\mathcal{G}(F)$ is algebraically closed. (In that particular case, the result already appears in Theorem 6.4 of [4].)

To show the second assertion, observe first that $S = S\mathcal{G}(E)$ is algebraically closed: this is because $E \prec F$. Proposition 4.1 of [2] tells us that if $S,S_1,S_2$ are substructures of $S\mathcal{G}(F)$ with $S_1 \cap S_2 = S$ and $S$ algebraically closed, then $S_1 \downarrow_S S_2$, and therefore $acl(S_1) \cap acl(S_2) = acl(S) = S$. 

11
Definition 2.8. Let $n$ be an integer $> 2$. A theory $T$ has NSOP$_n$ if for every formula $\varphi(x, y)$ (with $x, y$ of the same length), if $M$ is a model of $T$, and $a_i$, $i \in \omega$, is an infinite sequence of elements of $M$ such that $M \models \varphi(a_i, a_j)$ whenever $i < j$, then there are $b_1, \ldots, b_n$ in $M$ such that $M \models \varphi(b_i, b_{i+1})$ for $i = 1, \ldots, n - 1$, and $M \models \varphi(b_n, b_1)$. An easy application of compactness gives the following equivalent formulation: for all $m$ and $E \subseteq M$, if $p(x, y)$ is a $2m$-type over such that $\bigwedge_{i \in \mathbb{N}} p(x_i, x_{i+1})$ is consistent, then $\bigwedge_{i \leq n-1} p(x_i, x_{i+1}) \land p(x_n, x_1)$ is also consistent.

Theorem 2.9. Let $F$ be a PAC field, and assume that Th$(SG(F))$ has NSOP$_n$ for some $n > 2$. Then Th$(F)$ satisfies NSOP$_n$.

Proof. If char$(F) = p > 0$ and $[F : F^p] < \infty$, we will assume that the language contains constant symbols for elements of a $p$-basis. Assume that we have an infinite sequence $a_i$, $i \in \omega$, and a formula $\varphi(x, y)$ such that $\varphi(a_i, a_j)$ holds whenever $i < j$. We may assume that the sequence $a_i$ is indiscernible, and of length $\aleph_1$. By stability of the theory of separably closed fields of a given degree of imperfection, there is some $\alpha < \aleph_1$ such that for $\beta \geq \alpha$, $tp_{SCF}(a_\beta/a_\gamma, \gamma < \beta)$ does not fork over $E = acl(a_\gamma | \gamma < \alpha)$. Then the sequence $a_\beta, \beta \geq \alpha$, is an infinite sequence of indiscernibles over $E$.

So, we have reduced to the case where: we have an infinite sequence $a_i$, $i \in \omega$, of tuples which are indiscernible and SCF-independent over $E = acl(E)$, and which satisfy $\varphi(a_i, a_j)$ whenever $i < j$; moreover $E$ contains a $p$-basis of $F$ if char$(F) = p > 0$ and $[F : F^p] < \infty$. Let $A_i = acl(E, a_i)$ for $i \in \omega$, and $K_{i,j} = acl(A_i, A_j)$. Then for $i < j$, all tuples $(SG(K_{i,j}), SG(A_i), SG(A_j))$ realize the same $L_G(SG(E))$-type as $(SG(K_{i,2}), SG(A_i), SG(A_2))$. Since Th$(SG(F))$ satisfies NSOP$_n$, there is a sufficiently saturated extension $F^*$ of $F$, and $S_1, \ldots, S_n \subseteq SG(F^*)$ such that $(S_1, S_{i+1})$ and $(S_n, S_1)$ realize $tp_L(SG(A_1), SG(A_2))$ for $1 \leq i < n$. There are also $S_{i,i+1}$, $1 \leq i < n$ and $S_{n,1}$ such that the tuples $(S_{i,j}, S_{i,1})$ realize $tp_L(SG(K_{i,j}), SG(A_i), SG(A_j)/SG(E))$ for $(i, j) \in I := \{ (1,2), (2,3), \ldots, (n-1,n), (n,1) \}$. Note that for $1 \leq i < n$, the tuples $(S_{i,i+1}, S_i, S_j)$ also realize $tp_L(SG(K_{i,j}), SG(A_j), SG(A_i)/SG(E))$. We fix $L_G(SG(E))$-elementary isomorphisms $S\Phi_1 : SG(K_{i,i+1}) \to S_{i,i+1}$, which send $SG(A_i)$ to $S_i$ and $SG(A_{i+1})$ to $S_{i+1}$ for $i < n$, and an $L_G(SG(E))$-elementary isomorphism $S\Phi_n : SG(K_{0,1}) \to S_{n,1}$, sending $SG(A_0)$ to $S_n$ and $SG(A_1)$ to $S_1$.

We now apply repeatedly Lemma 1.13 (and the remark following it): in some saturated extension $F^*$ of $F$, we find some $B_1$ realizing $tp(A_1/E)$, $E^*$-isomorphism $\varphi_0 : A_1^* \to B_1^*$ such that $\varphi_0(A_1) = B_1$, and the double dual of $\varphi_0$ coincides with the restriction of $S\Phi_1$ to $SG(A_1)$. Again, using Lemma 1.13 applied to the extension $K_{1,2}$ of $A_1$, the isomorphisms $\varphi_0$ and $S\Phi_1$, we now find some $B_2$ in $F^*$ realizing $\varphi_0(tp(A_2/A_1))$ and SCF-independent from $B_1$ over $E$, an $E^*$-isomorphism $\varphi_1 : (A_1A_2)^* \to (B_1B_2)^*$ extending $\varphi_0$, whose double-dual coincides with $S\Phi_1$.

Induction step: At stage $i \leq n$, we have found $B_1, \ldots, B_i \subseteq F^*$ which are SCF-independent over $E$, and for each $1 \leq j < i$, $E^*$-isomorphisms $\varphi_j(A_jA_{j+1})^* \to (B_jB_{j+1})^*$, with $\varphi_j$ and $\varphi_{j+1}$ agreeing on $A_j$, such that $\varphi_j(K_{j,j+1}) = (B_jB_{j+1})^* \cap F^*$, and the double dual of $\varphi_j$ coincides with $S\Phi_j$. We now again apply Lemma 1.13 to the isomorphism $\varphi_i|_{A_i} : A_i^* \to B_i^*$, to the isomorphism $S\Phi_{i,i+1} : SG(K_{i,i+1}) \to S_{i,i+1}$, and find some $B_{i+1}$ SCF-independent from $B_1 \cdots B_i$, an isomorphism $\varphi_{i+1} : (A_iA_{i+1})^* \to (B_iB_{i+1})^*$ which extends $\varphi_i|_{A_i}$ and sends $K_{i,i+1}$ to $(B_iB_{i+1})^* \cap F^*$, with double dual $SG(B_i)$. Observe that via $S\Phi_{n,1}, SG(B_i) = SG(B_{i+1})$. By Theorem 2.1 there is $B_i$ realizing $tp(B_{n+1}/B_2, \ldots, B_n) \cup tp(B_1/B_2)$. Then $(B_1, B_2, \ldots, B_n)$ is our desired tuple.
Theorem 2.10. Let \( F \) be a Frobenius field. Then \( \text{Th}(F) \) satisfies \( \text{NSOP}_3 \).

Proof. We know that \( \text{Th}(S\mathcal{G}(F)) \) is \( \omega \)-stable by Theorem 2.4 in [2], and therefore satisfies \( \text{NSOP}_3 \). The result follows from Theorem 2.9.

Theorem 2.11. Let \( F \) be a PAC field, let \( E, A, B \) be algebraically closed subsets of \( F \), with \( E \) containing a \( p \)-basis of \( F \) if the degree of imperfection of \( F \) is finite, and assume that \( A \) and \( B \) are weakly independent over \( E \). Then, if \( B_i, i \in I \), is an indiscernible sequence of realisations of \( \text{tp}(B/E) \), which is SCF-independent over \( E \), and if \( p_i \) denotes the image of \( \text{tp}(A/B) \) by an \( E \)-automorphism of \( F \) sending \( B \) to \( B_i \), the type \( \bigcup_{i \in I} p_i \) is consistent, and has a realisation which is weakly independent from \( \bigcup_{i \in I} B_i \) over \( E \).

Proof. We may assume that \( I = \mathbb{N} \). Using induction and 2.4 one shows that for every \( n \), there is \( A' \) realising \( \bigcup_{i \leq n} p_i \), weakly independent from \( \bigcup_{i \leq n} B_i \).

Remark 2.12. The fact that the \( B_i \)'s form an indiscernible sequence over \( E \) is completely unnecessary. We included this hypothesis so as to make it look more like the usual criterion for forking. What we really prove, is that if the \( B_i \)'s are SCF-independent over \( E \), and for all \( i \in I \), \( p_i \) is an extension of \( \text{tp}(A/E) \), having a realisation which is weakly independent from \( B_i \) over \( E \), then \( \bigcup_{i \in I} p_i \) has a realisation which is weakly independent from \( \bigcup_{i \in I} B_i \) over \( E \).

Questions 2.13. We conclude this section with several questions. Throughout, \( F \) is a PAC field. I believe that the answer to most questions is positive.

1. (Strengthening of Theorem 2.9) If \( \text{Th}(S\mathcal{G}(F)) \) does not have the strict order property, then neither does \( \text{Th}(F) \).
2. Assume that \( \text{Th}(S\mathcal{G}(F)) \) satisfies \( n \)-amalgamation. Then so does \( \text{Th}(F) \) for the weak independence relation.
3. If \( F \) is \( \omega \)-free, then \( \text{Th}(F) \) satisfies \( n \)-amalgamation for the strong independence relation. (Recall that \( A \) and \( B \) are strongly independent over \( E \) if they are SCF-independent over \( E \) and \( \text{acl}(AB) = \text{acl}(A)\text{acl}(B) \).
4. Characterize \( p \)-forking, and whether it is equivalent to \( p \)-forking at the level of the Galois groups.
5. Is the theory of \( \omega \)-free PAC fields rosy? Superrosy?
6. . . .

3 More on the logic of complete systems, and codes of their formulas

3.1. Notation. Let \( G \) be a profinite group, \( S\mathcal{G} \) its associated system. It is convenient to consider the equivalence relation \( \sim \) associated to \( \leq \): if \( \alpha, \beta \in S\mathcal{G} \) we define \( \alpha \sim \beta \) if and only if \( \alpha \leq \beta \) and \( \beta \leq \alpha \). We denote by \( [\alpha] \) the \( \sim \)-equivalence class of \( \alpha \): it comes with a group law
with graph given by $[\alpha]^3 \cap P$, and for $\beta \geq \alpha$, with a group epimorphism, denoted $\pi_{\alpha \beta} : [\alpha] \to [\beta]$, with graph given by $C \cap [\alpha] \times [\beta]$. The set of $\sim$-equivalence classes, equipped with the induced partial ordering $\leq$, is a modular lattice with sup ($\vee$) and inf ($\wedge$). For $\alpha, \beta \in SG$, we let $\alpha \vee \beta$ denote the identity element of $[\alpha] \vee [\beta]$, and $\alpha \wedge \beta$ the identity element of $[\alpha] \wedge [\beta]$. In other words: if $N_1$ is the kernel of the natural epimorphism $G \to [\alpha]$ and $N_2$ the kernel of the natural epimorphism $G \to [\beta]$, then $\alpha \vee \beta$ is the identity element of $[\alpha] \vee [\beta] = G/N_1 \cap N_2$, and $\alpha \wedge \beta$ the identity element of $[\alpha] \wedge [\beta] = G/N_1 \cap N_2$.

If $A$ is a substructure of $SG$ and $\alpha \in SG$, we denote by $\alpha \vee A$ the smallest (for $\leq$) element of $\{\alpha \vee \beta \mid \beta \in A\}$.

3.2. Action of $G$. If $G$ is a profinite group, then $G$ acts on itself by conjugation. This induces an action of $G$ on $S(G)$, which respects the $L_G$-structure of $S(G)$. The action of an element $g$ on a given $\sim$-equivalence class $G/N$ is then given by conjugation by $gN$, and so does not depend on the choice of the coset representative for $gN$. This also defines an action of $G$ on all cartesian powers $S(G)^m$.

Let $\sigma$ be a tuple of elements of $S(G)$, and $\theta(\xi, \zeta)$ an $L_G$-formula. If $N$ is an open subgroup of $G$ such that (the coset) $N$ is $\leq$ than all the elements of $\sigma$, then conjugation by the elements of $N$ leaves the elements of the tuple $\sigma$ fixed, so that the set defined by the formula $\theta(\xi, \sigma)$ will be invariant under conjugation by $N$. Hence, the set defined by $\theta(\xi, \sigma)$ will be invariant under conjugation by $G$ if and only if, for all $\tau \in G/N$, the sets defined by $\theta(\xi, \tau^{-1}\sigma \tau)$ and by $\theta(\xi, \sigma)$ coincide. Observe that in any case, the formulas $\bigvee_{\tau \in G/N} \theta(\xi, \tau^{-1}\sigma \tau)$ and $\bigwedge_{\tau \in G/N} \theta(\xi, \tau^{-1}\sigma \tau)$ define sets which are invariant under the action of $G$.

One can also define an action of $G^n$ on $S(G)^{m_1} \times \cdots \times S(G)^{m_n}$ in the natural manner.

3.3. Codes. Let $L$ be a Galois extension of $K$ of degree $n$, and let $\sigma_1, \ldots, \sigma_m$ elements of $Gal(L/K)$. We say that a tuple of elements of $K$ is a code for $(L, \sigma_1, \ldots, \sigma_m)$ if it is of the form $(a, b_1, \ldots, b_m)$, where, if $a = (a_1, \ldots, a_n)$, the polynomial $p(T) = T^n + \sum_{i=1}^n a_{n-i}T^i$ is the minimal monic polynomial over $K$ of some generator $\alpha$ of $L$ over $K$, and if $b_i = (b_{i1}, \ldots, b_{im})$, then $\sigma_i(\alpha) = \sum_{j=1}^n b_{ij}\alpha^{j-1}$. Note that a tuple coding $(L, \sigma_1, \ldots, \sigma_m)$ will also code $(L, \tau^{-1}\sigma_1\tau, \ldots, \tau^{-1}\sigma_m\tau)$ for any $\tau \in Gal(L/K)$. By abuse of language, we will say that $(L, \sigma, \alpha, p(T))$ is the data associated to the code $(a, b_1, \ldots, b_m)$ $(\sigma = (\sigma_1, \ldots, \sigma_m))$. We will also say that $(a, b_1, \ldots, b_m)$ codes $(L, \sigma_1, \ldots, \sigma_m, \alpha)$.

Note also that the above remark shows that any orbit in $Gal(L/K)^n$ (under the action of $Gal(L/K)$) is in fact an imaginary of $K$.

3.4. Definable subsets of $SG(K)$. Let $K$ be a field. While we know (6) that the elementary equivalence of two fields implies the elementary equivalence of the complete systems associated to their absolute Galois groups, the proof of this result does not show that subsets of $SG(K)$ are “definable over $K$”. This is easy to see: let $S \subset SG(K)^m$ be definable over $Gal(L/K)$, and let $\tau \in G(K)$: then $\tau$ leaves $K$ fixed, but its double dual sends $S$ to $\tau S\tau^{-1}$. One however has the following result:

Proposition 3.5. (Cherlin-van den Dries-Macintyre (6)). Given an $L_G$-formula $\theta(\xi)$, which in particular says that the elements of the tuple $\xi$ live in the same $\sim$-equivalence class, there is a formula of the language of fields $\theta^*(x)$ such that for any field $K$, and code $a$ for an $(L, \sigma)$ of the appropriate sort,
\[ K \models \theta^*(a) \iff SG(K) \models \theta(\sigma). \]

The proof is easy: if \( tp(b) = tp(a) \), there is an automorphism of \( K \) which sends \( a \) to \( b \). This automorphism extends to an automorphism of \( K' \), with double dual sending \( \sigma \) to a tuple \( \sigma' \) coded by \( b \). Then \( tp(\sigma) = tp(\sigma') \) (in \( SG(K) \)). Hence, if \( \sigma \) satisfies \( \theta(\xi) \), there is some formula \( \theta_a(x) \in tp(a) \) which “implies” \( \theta \), i.e., such that if \( b \) satisfies \( \theta_a \), then any tuple \( \sigma' \) coded by \( b \) will satisfy \( \theta \). By compactness we get the formula \( \theta^*(x) \).

The difficulties occur when one deals with an arbitrary \( \mathcal{L}_G \)-formula \( \theta(\xi) \), and in general one cannot hope for a similar result. The problem is that if \( \sigma = (\sigma_1, \ldots, \sigma_n) \) and \( a_i \) is a code for \((L_i, \sigma_i)\), then \( a_i \) only defines \( \sigma_i \) up to conjugation by the elements of \( Gal(L_i/K) \). Thus already a formula of the form \( \xi_i = \xi_j \) poses problem: one cannot expect to have a formula \( \theta(x_i, x_j) \) which expresses this property of all elements coded by \( x_i \) and \( x_j \). This problem can be addressed by adapting the definition of codes, however is quite unpleasant to formulate in the general case. Here, we will deal with a particular case.

**Definition 3.6.**

1. Let \( L \) be a finite Galois extension of \( K \), \( \sigma \) a tuple of elements in \( Gal(L/K) \), and \( \alpha, \beta \in L \) such that \( L = K(\alpha) \). We say that \( a \) codes \((L, \sigma, \alpha, \beta)\) if it is of the form \((b, c)\) where \( b \) is a code for \((L, \sigma, \alpha)\) (see 3.9), and \( c \) gives the coordinates of \( \beta \) with respect to the basis \( \{1, \alpha, \ldots, \alpha^{[L:K]-1}\} \) of \( L \) over \( K \).

2. Let \( L_1 \) and \( L_2 \) be finite Galois extensions of a field \( K \), \( \sigma_1 \) and \( \sigma_2 \) tuples of elements in \( Gal(L_1/K) \), \( Gal(L_2/K) \) respectively, and \( L_0 = L_1 \cap L_2 \). Let \( \alpha_0, \alpha_1, \alpha_2 \) be such that \( L_i = K(\alpha_i), i = 0, 1, 2 \). We say that \((a_1, a_2)\) is a 2-code for \((L_1, L_2, \sigma_1, \sigma_2, \alpha_2, \alpha_1, \alpha_0)\) if \( a_1 \) is a code for \((L_1, \sigma_1, \alpha_1, \alpha_0)\) and \( a_2 \) is a code for \((L_2, \sigma_2, \alpha_2, \alpha_0)\).

3. We say that \( a \) is a 2-code for \((L_1, L_2, \sigma_1, \sigma_2)\) if it is a 2-code for \((L_1, L_2, \sigma_1, \sigma_2, \alpha_0, \alpha_1, \alpha_2)\) for some \( \alpha_0, \alpha_1, \alpha_2 \).

4. An \( \mathcal{L}_G \)-formula \( \theta(\xi) \) is codable if it implies that the elements of the tuple \( \xi \) are \~{}-equivalent.

5. An \( \mathcal{L}_G \)-formula \( \theta(\xi_1, \xi_2) \) is 2-codable if it implies that the elements of the tuple \( \xi_i \) are \~{}-equivalent, for \( i = 1, 2 \).

**Remark 3.7.** One checks easily that being a 2-code of some \((L_1, L_2, \sigma_1, \sigma_2)\) (with \([L_1L_2 : K] \leq n \) for a fixed \( n \)), is an elementary property of a tuple (again, one uses that a tuple having the same type as a 2-code, is a 2-code; see also 3.9). One also notes that if \((a_1, a_2)\) is a 2-code for \((L_1, L_2, \sigma_1, \sigma_2)\), then \((a_1, a_2)\) codes exactly the tuples \((L_1, L_2, \rho_{L_1}, \sigma_1, \rho_{L_1}^{-1}, \rho_{L_2}, \sigma_2, \rho_{L_2}^{-1})\) where \( \rho \in Gal(L_1L_2/K) \). Indeed, assume that \((a_1, a_2)\) codes \((L_1, L_2, \sigma_1, \sigma_2, \alpha_0, \alpha_1, \alpha_2)\) and \((L_1, L_2, \tau_1, \beta_0, \beta_1, \beta_2)\). Then \( \beta_1 = \rho_1(\alpha_1) \) for some \( \rho_1 \in Gal(L_1/K) \), and \( \beta_0 = \rho_1(\alpha_0) \). Similarly, \( \beta_2 = \rho_2(\alpha_2) \) for some \( \rho_2 \in Gal(L_2/K) \), and \( \beta_0 = \rho_2(\alpha_0) \). Then \( \beta_1 = \rho_2(\alpha_1) \). This implies that \( \rho_1 \) and \( \rho_2 \) agree on \( L_0 = L_1 \cap L_2 \), and that they can be extended to a common \( \rho \in Gal(L_1L_2/K) \).

**Proposition 3.8.** Let \( \theta(\xi_1, \xi_2) \) be a 2-codable \( \mathcal{L}_G \)-formula. There is a formula \( \theta^*(x_1, x_2) \) of the language of fields, such that in any field \( K \), if \((a_1, a_2)\) is a 2-code for \((L_1, L_2, \sigma_1, \sigma_2)\), then

\[ K \models \theta^*(a_1, a_2) \iff SG(K) \models \theta(\sigma_1, \sigma_2). \]
**Proof.** Reason as in the proof of Proposition 3.4.

### 3.9. Some remarks about codes

We saw earlier that being a code, or a 2-code, is an elementary property. If one wishes to show this result constructively, one needs to work a little.

To express that \((a_1, a_2)\) is a code for \((L, \sigma)\) is fairly easy. One says first of all that the monic polynomial \(p(T)\) whose coordinates are given by \(a_i\) is irreducible over \(K\) and separable. Then, if \(\alpha\) is a root of \(p(T)\), one can interpret in \(K\) the pair of fields \((K(\alpha), K)\), by identifying \(K(\alpha)\) with \(K \oplus K\alpha \oplus \cdots \oplus K\alpha^{n-1}\) where \(n\) is the degree of \(p(T)\). One then says that \(K(\alpha)\) contains all \(n\) roots of \(p(T)\), and that the tuple \(a_2\) consists of coordinates of some of these roots (indeed, one can code an element \(\tau\) of \(\text{Gal}(K(\alpha)/K)\) by specifying the coordinates of \(\tau(\alpha)\)).

We now want to express the fact that \(((a_1, a_2, a_3), (b_1, b_2, b_3))\) is a 2-code. That \((a_1, a_2, a_3)\) is a code for some \((L, \sigma, \alpha, \gamma)\) is expressible, follows from the previous paragraph, and similarly that \((b_1, b_2, b_3)\) is a code for some \((M, \tau, \beta, \delta)\). As before, one can interpret in \(K\), using the parameters \((a_1, a_2, a_3)\) the structure \((L, K, \sigma, \alpha, \gamma)\), and similarly, using the parameters \((b_1, b_2, b_3)\), the structure \((M, K, \tau, \beta, \delta)\). To express that \(((a_1, a_2, a_3), (b_1, b_2, b_3))\) is a 2-code, we need to express the following:

- that \(\gamma\) and \(\delta\) are conjugates over \(K\).
- that \(L \cap M = K(\gamma) = K(\delta)\).

The first item is easy: in the structure \((K(\alpha), K, \alpha, \gamma)\), one can define the coefficients of the minimal (monic) polynomial \(r(T)\) of \(\gamma\) over \(K\), and similarly one can define in \((M, K, \beta, \delta)\) the coefficients of the minimal polynomial of \(\delta\) over \(K\). It then suffices to say that these two minimal polynomials are the same, and that \(K(\gamma)\) contains all roots of \(r(T)\).

For the second item, observe that in fact the triples \((L, K(\gamma), K, \alpha, \gamma)\) and \((M, K(\delta), K, \beta, \delta)\) are interpretable from \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\). In \((L, K(\gamma), K, \alpha, \gamma)\), the coefficients of the minimal polynomial \(q(\gamma, T)\) of \(\alpha\) over \(K(\gamma)\) are definable. It therefore suffices to say that \(q(\delta, T)\) is irreducible over \(M\), and this is expressible in the structure \((M, K, \beta, \delta)\). The first item gives us that \(L \cap M \supseteq K(\gamma)\). The irreducibility of \(q(T, \delta)\) over \(M\) implies that \([L : K(\delta)] = [LM : M]\), so that \(L \cap M = K(\delta)\).

All this is done uniformly in the length of the parameters involved, and so gives the first-order expressibility of \(\langle x, y \rangle\) is a 2-code. Note however that the partition of the variables of the formula needs to be fixed, i.e., one needs to know \([L : K] = n, |\sigma| = i, [M : K] = m\) and \(|\tau| = j\): if \(x = (x_1, \ldots, x_n)\) then \(a_1\) will correspond to \((x_1, \ldots, x_n), a_2\) to \((x_{n+1}, \ldots, x_{n+i-1})\), and \(a_3\) to \((x_{n+i+1}, \ldots, x_{n+i+2})\), and similarly for the elements of the tuple \(y\).

### 3.10. An easy observation

Let \(\rho_1, \ldots, \rho_n\) enumerate an \(\sim\)-equivalence class of \(S(G)\). Then the elements of the subsystem of \(S(G)\) generated by \(\rho_i\) are in the definable closure of \(\rho_1, \ldots, \rho_n\). Indeed, each \(\tau \in \langle \rho_i \rangle\) is \(\geq \rho_1\); consider the set \(I(\tau)\) of indices \(j\) such that \(C(\rho_j, \tau)\) holds. Because the \(\rho_i\) enumerate the \(\sim\)-class of \(\rho_1\), the element \(\tau\) is uniquely defined by the formula

\[
\bigwedge_{j \in I(\tau)} C(\rho_j, \xi) \land \bigwedge_{j \notin I(\tau)} \neg C(\rho_j, \xi).
\]

**Notation 3.11.** Let \(S_1, S_2\) be subsets of \(S(G)\). We denote by \(tp^2(S_2/S_1)\) the set of all formulas of the form \(\theta(\xi_1, \xi_2) \in tp(S_2/S_1)\), where the \(L_G\)-formula \(\theta(\xi_1, \xi_2)\) is 2-codable.

Let \(K\) be a field, and \(A, B\) subfields of \(K\) such that \(F\) is a regular extension of \(A\) and of \(B\). We denote by \(tp^*(SG(A)/SG(B))\) the set of formulas \(\theta^*(X, B)\), where \(\theta(\Xi_1, \Xi_2)\) is a 2-codable formula, and \(\theta(\Xi_1, SG(B)) \in tp^2(SG(A)/SG(B))\).

16
Lemma 3.12. Let $S_1$ and $S_2$ be subsystems of $S(G)$, and assume that $S_3$ satisfies $\text{tp}^2(S_2/S_1)$. Then $\text{tp}(S_3/S_1) = \text{tp}(S_2/S_1)$.

Proof. By compactness, we may assume that $S_1$, $S_2$ and $S_3$ are finite. For $i = 1, 2$, let $\sigma_i$ be an enumeration of the greatest $\sim$-equivalence class of $S_i$, and let $\sigma_3$ be the subtuple of $S_3$ corresponding to $\sigma_2 \subset S_2$. By assumption, $\text{tp}(\sigma_3/\sigma_1) = \text{tp}(\sigma_2/\sigma_1)$, and by Observation 3.10 we get the result.

Remarks 3.13. (1) A finite disjunction of 2-codable formulas is not necessarily 2-codable, but a result analogous to 3.8 holds nevertheless:

Let $\theta_i(\xi_1, \xi_2)$ be 2-codable formulas, $i = 1, \ldots, n$. Then for every field $K$, and codes $(a_i, b_i)$ for $(L_i, M_i, \sigma_i, \tau_i)$, we have

$$K \models \bigvee_{i=1}^n \theta_i^*(a_i, b_i) \iff S\mathcal{G}(K) \models \bigvee_{i=1}^n \theta_i(\sigma_i, \tau_i).$$

(2) This result becomes false if one replaces the disjunctions by conjunctions. However, note that a Boolean combination of 2-codable formulas in the same variables is 2-codable.

Lemma 3.14. Let $S_1$ and $S_2$ be subsystems of some $S(G)$, and let $\Xi_1, \Xi_2$ enumerate the variables of $\text{qftp}(S_1)$ and $\text{qftp}(S_2)$ (the quantifier-free types). Let $\theta(\xi_1, \xi_2)$ be a Boolean combination of 2-codable formulas, $\xi_i \subset \Xi_i$. Then there is a 2-codable formula $\theta'(\xi_1, \xi_2)$, $(\xi_i \subset \Xi_i)$ such that

$$\text{qftp}(\langle \xi_1 \rangle) \cup \text{qftp}(\langle \xi_2 \rangle) \vdash \theta(\xi_1, \xi_2) \leftrightarrow \theta'(\xi_1, \xi_2).$$

Proof. Say that $\theta(\xi_1, \xi_2)$ is a Boolean combination of the 2-codable formulas $\theta_i(\xi_{i1}, \xi_{i2})$. Let $\xi_1 \subset \Xi_1$ enumerate a $\sim$-equivalence class such that $\text{qftp}(S_1) \vdash \bigwedge_i (\xi_1 \leq \xi_{i1})$, and let $\xi_2$ be defined similarly for $\xi_{i2}$. By Observation 3.10 there are 2-codable formulas $\theta_i'(\xi_1, \xi_2)$ such that

$$\text{qftp}(S_1) \cup \text{qftp}(S_2) \vdash \theta_i(\xi_{i1}, \xi_{i2}) \leftrightarrow \theta_i'(\xi_1, \xi_2).$$

Any Boolean combination of the $\theta_i'(\xi_1, \xi_2)$ is 2-codable, and this gives the result (get rid of the extra variables of $\text{qftp}(S_i)$).

Definition 3.15. Let $\theta(\xi)$ be a codable formula of $\mathcal{L}_G$. We define $\theta^*(x)$ to be the formula of the language of fields which satisfies the following condition, in any field $K$:

For any tuple $a$ in $K$, $K \models \theta^*(a)$ if and only if $a$ is a code for some $(L, \sigma)$, and $S\mathcal{G}(K) \models \theta(\sigma)$.

Similarly, if $\theta(\xi_1, \xi_2)$ is a 2-codable formula, we let $\theta^*(x, y)$ be the formula of the language of fields which satisfies the following, for every field $K$:

For any tuple $(a, b)$ in $K$, $K \models \theta^*(a, b)$ if and only if $(a, b)$ is a 2-code for some $(L, M, \sigma, \tau)$ and $S\mathcal{G}(K) \models \theta(\sigma, \tau)$.

The formulas $\theta^*(x)$ and $\theta^*(x, y)$ exist, by the discussion above and by 3.4, 3.8. Note that $(\neg\theta)^* \neq \neg(\theta^*)$. 

17
3.16. Definition of $tp^*$. Let $K$ be a field, and $A, B$ subfields of $K$ such that $K \cap A^* = A, K \cap B^* = B$. We denote by $tp^*(SG(B)/SG(A))$ the set of formulas $\theta^*(X_1, A)$, where $\theta(\Xi_1, \Xi_2) \in tp^2(SG(B)/SG(A))$, and $\theta^*(X_1, X_2)$ is a formula of the language of fields satisfying the conclusion of $3.8$ for the formula $\theta(\Xi_1, \Xi_2)$, and in addition expressing that $(X_1, X_2)$ is a $2$-code.

Here we need a word of explanation about the variables. The elements of $X_1$ correspond to an enumeration of $B$, and similarly for the variables of $X_2$. That $C$ satisfies $tp^*(SG(B)/SG(A))$ will mean that we have fixed an enumeration of $C$ corresponding to the elements of $X_1$. From the definition of the formulas $\theta^*(X_1, X_2)$ we obtain the following:

Remarks 3.17. Assume that $C$ satisfies $tp^*(SG(B)/SG(A))$.

1. There is an elementary $SG(A)$-isomorphism $f : SG(C) \to SG(B)$, which respects the coding, i.e., if $a \subset A, b \subset B$ are such that $(b, a)$ is a $2$-code for some $(L_1, L_2, \sigma, \tau)$, and if $c \subset C$ is the subtuple of $C$ corresponding to $b \subset B$, then $(c, a)$ is a $2$-code for $(L_3, L_2, F(\sigma), \tau)$, where $L_3$ is defined by $Gal(L_1/B) = f(Gal(L_3/C))$.

2. If the correspondence between the elements of $B$ and the elements of $C$ (given by $X$) defines a field isomorphism, then $f$ is in fact induced by some extension of this isomorphism to an $A^*$-isomorphism with domain $B^*$.

4 Imaginaries of PAC fields

In this section, we will show how the type amalgamation result gives information about imaginaries. In the later part of this chapter, we will fix a large PAC field $F$, of characteristic $p$. If $p > 0$, then we assume that its degree of imperfection is finite, and add to the language of rings constant symbols for elements of a $p$-basis. All our subfields of $F$ will contain these distinguished elements. This has two consequences which we will use:

1. The theory of separably closed fields in this expanded language, together with axioms saying that the new constants form a $p$-basis, is complete and eliminates imaginaries (S).

2. If $A$ and $B$ are subfields of $F$ closed under the $\lambda$-functions of $F$, then $AB$ is also closed under the $\lambda$-functions of $F$. This implies (4.5 in [5]) that $acl(AB) = F \cap (AB)^*$.

Before starting with the description of imaginaries, we will take a closer look at subsets of $SG(F)$ which are definable in $F$.

Definition 4.1. A basic imaginary of $F$ is a pair $(a, D)$, where $a$ is a tuple of elements of $F$, and $D$ is (the code of) a definable subset $D$ of $SG(F)^m$ for some $m$, which is stable by conjugation under elements of $G(F)$.

Here, for us, a definable set is an imaginary element, i.e., we identify definable sets with their codes. Thus, if $L$ is a finite Galois extension of $F$, then $Gal(L/F)$ is an imaginary, and so is $L$.

It is clear from the discussion in section 3 that basic imaginaries are indeed imaginaries of the field $F$. The requirement that the set $D$ be stable under conjugation is necessary, as elements of $G(F)$ will fix elements of $F^{eq}$.
Theorem 4.2. Let $F$ be a PAC field, of finite degree of imperfection if the characteristic is positive, and expand the language by adding constants for elements of a $p$-basis if necessary (call $\mathcal{L}$ the language). Let $e \in F^{eq}$. Then $e$ is equi-definable with a finite set of basic imaginaries.

Proof. The proof is fairly long, and proceeds with a series of steps. We will assume that $F$ is sufficiently saturated. Let $E = \text{acl}^{eq}(e) \cap F$, $E_0 = \text{dcl}^{eq}(e) \cap F$. Then $E$ is a Galois extension of $E_0$, and every element of $Gal(E/E_0)$ lifts to an automorphism of $F^{eq}$ fixing $e$.

If $e \in \text{acl}^{eq}(E)$, we are done: $e$ is coded by a tuple of elements of $E_0$. We will therefore assume that this is not the case, and fix a 0-definable map $f$, and a tuple $a$ such that $f(a) = e$. We let $A = \text{acl}(E, a)$, and consider the set $P$ of realisations of $tp(A/E)$. We also write $f(A) = e$. The first step is by now a routine argument.

We will consider the fundamental order on types (in the sense of Th($F^s$)), denoted by $\leq_{fo}$, and $\sim_{fo}$ will denote the associated equivalence relation. We refer to chapter 13 of [17] for the definition and properties of the fundamental order. Recall that if $p$ and $q$ are stationary types, then $p \sim_{fo} q$ iff they have a common non-forking extension. In our setting, we have that if $D = \text{acl}(D)$ and $d$ is a tuple in $F$, then $tp_{SCF}(d/D)$ is stationary. And of course, any type in the sense of Th($F^s$) over a separably closed field is stationary.

**Step 1.** There is $B \in P$, with $f(B) = e$, and which is SCF-independent from $EA$ over $E$.

By Lemma 1.4 of [11], there is $B$ realising $tp(A/\text{acl}^{eq}(E, e))$ such that $\text{acl}^{eq}(E, B) \cap \text{acl}^{eq}(E, A) = \text{acl}^{eq}(E, e)$, whence

$$f(B) = e \text{ and } \text{acl}(E, A) \cap \text{acl}(E, B) \cap F = E. \quad (*)$$

Observe that because $A$ is a regular extension of $E$, $tp_{SCF}(A/E)$ is stationary, and so is every non-forking extension. Consider the set $\mathcal{B}$ of realisations of $tp(A/\text{acl}^{eq}(E, e))$ which satisfy $(*)$, and choose $B \in \mathcal{B}$ such that $tp_{SCF}(B/(EA)^s)$ is maximal for the fundamental order in the set $\{tp_{SCF}(B'/EA)^s) \mid B' \in \mathcal{B} \}$. Let $C$ realise $tp(B/\text{acl}(EA))$, SCF-independent from $B$ over $EA$. (Note that $e \in \text{acl}(EA)$, and so $C$ realises $tp(A/\text{acl}^{eq}(E, e))$.) Then

$$tp_{SCF}(C/(EAB)^s) \sim_{fo} tp_{SCF}(B/(EA)^s), \text{ and } f(C) = e.$$  

Since $tp_{SCF}(C/(EAB)^s) \leq_{fo} tp_{SCF}(C/(EB)^s)$, we obtain

$$tp_{SCF}(B/(EA)^s) \leq_{fo} tp_{SCF}(C/(EB)^s).$$

Moreover,

$$\text{acl}(EB) \cap \text{acl}(EA) \cap F \subseteq \text{acl}(EAB) \cap \text{acl}(EC) \cap F = \text{acl}(EA) \cap \text{acl}(EC) \cap F = E,$$

so that the pair $(B, C)$ satisfies $(*)$. Since $B$ and $C$ both realise $tp(A/\text{acl}^{eq}(E, e))$, the maximality of $tp_{SCF}(B/(EA)^s)$ among the extensions of $tp_{SCF}(A/\text{acl}^{eq}(E, e))$ satisfying $(*)$ and the inequality $tp_{SCF}(B/(EA)^s) \leq_{fo} tp_{SCF}(C/(EB)^s)$ imply that

$$tp_{SCF}(B/(EA)^s) \sim_{fo} tp_{SCF}(C/(EB)^s).$$

Hence

$$tp_{SCF}(C/(EAB)^s) \sim_{fo} tp_{SCF}(B/(EA)^s) \sim_{fo} tp_{SCF}(C/(EB)^s),$$

19
and $tp_{SCF}(C/(EAB)^*)$ does not fork over $EB$. By elimination of imaginaries in SCF (recall that the degree of imperfection is finite, and that $E$ contains a $p$-basis), we obtain that $tp_{SCF}(C/EAB)$ does not fork over $acl_{SCF}(EA) \cap acl_{SCF}(EB) = E^*$, and therefore does not fork over $E$. Since $tp(C/(EA)^*) = tp(B/(EA)^*)$, we have $tp_{SCF}(B/EA)$ does not fork over $E$, which proves the result.

**Step 2.** Let $B \in P$, SCF-independent from $a$ over $E$, and with $f(B) = e$. Assume that $C \in P$ is SCF-independent from $B$ over $E$, and that there is an $E^*$-isomorphism $\varphi : B^* \to C^*$, whose double dual $S\Phi : SG(B) \to SG(C)$ is an $L_G(SG(A))$-elementary map. Then $f(C) = e$.

We will apply the results of Theorem 2.11 to show that $tp(B/A) \cup tp(C/B)$ is consistent (the identification between the variables of $tp(B/A)$ and of $tp(C/B)$ being given by the fact that these types extend $tp(A/E)$). In the notation of this theorem, we let $\varphi = \varphi$, $S_0 = SG(C)$, $S\Psi_2$ be the identity of $SG(C), SG(B))$, and $S\Psi_1$ be the partial (elementary) isomorphism on $\langle SG(B), SG(A) \rangle$ extending $S\Phi$ and the identity of $SG(A)$.

If $C$ realises $tp(B/A) \cup tp(C/B)$, then $f(C') = f(B) = e$, and this implies that $f(C) = e$.

**Step 3.** Let $B, C \in P$ with $f(B) = e$. Assume that there is an $E^*$-isomorphism $\varphi : B^* \to C^*$, whose double dual $S\Phi : SG(B) \to SG(C)$ is an $L_G(SG(A))$-elementary map. Then $f(C) = e$.

By Step 1, there is $B' \in P$, with $f(B') = e$, and which is SCF-independent from $A$ over $E$. As $tp(B'/A)$ has a realisation which is SCF-independent from $C$ over $A$, we may assume that $B'$ is SCF-independent from $AC$ over $E$. Apply Step 2 to $A, B', C$.

**Step 4.** There is a 2-codable formula $\theta(\Xi, \Upsilon)$ such that, if $C \in P$, then $f(C) = e$ if and only if $SG(C)$ satisfies $\theta(\Xi, SG(A))$.

Here we need a word about the variables. If $B \in P$, then by definition we have an $E$-isomorphism $\varphi : B \to A$ (which is elementary). This isomorphism extends to an $E^*$-isomorphism $B^* \to A^*$, whose double dual is an $SG(E)$-isomorphism $SG(B) \to SG(A)$.

For each $B \in P$ such that $f(B) = e$, we know by Step 3 and by Lemma 3.12 that

$$tp(B/E) \cup tp^2(SG(B)/SG(A)) \vdash f(X) = e.$$ 

Hence there is $\theta_B(\Xi, \Upsilon)$ such that $\theta_B(\Xi, SG(A)) \in tp^2(SG(B)/SG(A))$ and

$$tp(B/E) \cup \theta_B^*(X, A) \vdash f(X) = e.$$ 

By compactness, a finite disjunction of the $\theta_B^*(X, A)$ is equivalent to $f(X) = e$ modulo $tp(B/E)$. By Lemma 3.11 we may replace this disjunction by $\theta^*(X, A)$ for some 2-codable formula $\theta(\Xi, \Upsilon)$.

**Step 5.** Let $B, C \in P$. Then $C$ satisfies $\theta^*(X, B)$ if and only if $f(C) = f(B)$.

Indeed, there is an $E$-automorphism of $F$ which sends $A$ to $B$. Then the elements of $P$ satisfying $\theta^*(X, B)$ are precisely those satisfying $f(X) = f(B)$.

**Step 6.** There is a set $D$, definable over $SG(A)$, such that an $E$-automorphism $\sigma$ of $F$ fixes $e$ if and only some (any) extension $\tilde{\sigma}$ of $\sigma$ to $F^*$ leaves $D$ invariant.

We know by Step 4 that

$$tp(A/E) \vdash \theta^*(X, A) \iff f(X) = e.$$ 

We also know that the set $D$ of realisations of $\theta(\Xi, SG(A))$ is stable under conjugation by elements of $SG(F)$. Let $\sigma \in Aut(F/E)$ fix $e$, and $\tilde{\sigma}$ an extension of $\sigma$ to $F^*$. Then $\sigma$ induces an
automorphism of the set $P$, which leaves invariant the set of realisations of $\theta^*(X, A)$. Hence, it leaves invariant the set $D$ of realisations of $\theta(\Xi, SG(A))$. I.e., $D \in \text{dcl}^{eq}(E, e)$.

Conversely, let $\sigma \in \text{Aut}(F/E)$, and $\tilde{\sigma}$ an extension of $\sigma$ to $F^s$ which leaves $D$ invariant. Then $\tilde{\sigma}$ leaves invariant the set $P$, as well as any $\mathcal{L}_{SG(E)}$-definable subset of $D$ which is stable by conjugation. Hence it leaves invariant the set of realisations of $\theta(\Xi, SG(A))$ which are a subtuple of some realisation of $tp^2(SG(A)/SG(E))$. By Lemma 1.13 there is some $B \in P$ such that $SG(B)$ satisfies $\theta(\Xi, SG(A))$. I.e., $B$ satisfies $\theta^*(X, A)$ and $f(B) = e$. So $e \in \text{dcl}^{eq}(E, D)$.

**Step 7.** The result. It follows that the imaginary $D$ and $e$ are equi-definable over $E$. Hence there is a finite tuple $a$ of elements of $E$ such that they are equi-definable over $a$. Then $e$ is equi-definable with the set of conjugates of $(a, D)$ over $e$.

### 4.3. Remark

This result is not totally satisfactory: it would have been better to obtain a single basic imaginary. We will show by an example below that it is not always possible. One can however observe that $e$ can be squeezed between two basic imaginaries: namely $e \in \text{dcl}^{eq}(a, D)$, and if $b$ codes the set of conjugates of $a$ over $e$, and $D'$ the set of conjugates of $D$ over $e$, then $e$ is algebraic over $(b, D')$.

It turns out that the interaction between $F$ and $SG(F)$ at the level of algebraic closure is very weak:

**Proposition 4.4.** Let $F$ be a PAC field. If $F$ is of characteristic $p > 0$, we assume that its degree of imperfection is finite and that we have constant symbols for elements of a $p$-basis of $F$. Let $e = \{(a_1, D_1), \ldots, (a_n, D_n)\} \in F^{eq}$, where each $(a_i, D_i)$ is a basic imaginary.

1. $\text{acl}^{eq}(e) \cap F = \text{acl}_{SG}(a_1, \ldots, a_n) \cap F$ (\textit{= acl}(a_1, \ldots, a_n) in the sense of the theory of $F$).
2. $\text{acl}^{eq}(e) \cap SG(F) = \text{acl}^{eq}(SG(\text{acl}(a_1, \ldots, a_n), D_1, \ldots, D_n)) \cap SG(F)$. Here, by $D_1, \ldots, D_n$, we mean codes for the sets defined by $D_1, \ldots, D_n$.

**Proof.** (1) Using Lemma 3.10 we may assume that all the $D_i$’s are definable over $Gal(L/F)$, for some finite Galois extension $L$ of $F$. Let $A = \text{acl}(a_1, \ldots, a_n)$, $b$ any finite tuple of elements of $F \setminus A$ such that $A(b)$ contains a code of the extension $L$ and of the elements of $Gal(L/F)$. Let $B = \text{acl}(A, b)$, and choose an $A^s$-automorphism $\varphi$ of $\Omega$ such that $\varphi(B) = C$ is linearly disjoint from $F$ over $A$. Let $\Phi : G(B) \rightarrow G(C)$ be the dual of $\varphi^{-1}$, and consider the subgroup $H = \{ \langle \sigma, \Phi(\sigma|_B) \rangle \mid \sigma \in G(F) \}$. Let $M$ be the subfield of $C^{F^s}$ fixed by $H$. Then $M$ is a regular extension of $C$ and of $F$, and the restriction map $Gal(C^{F^s}/M) \rightarrow G(F)$ is an isomorphism. By Theorem 1.2 $F$ has an elementary extension $F^*$ containing $M$, regular over $M$. Then $tp_{F^s}(C/A) = tp_{F^s}(B/A) = tp_F(B/A)$. By definition of $H$, the tuple of $Gal(L/F)$ coded by $\varphi(b)$ is the same (up to conjugation) as the tuple coded by $b$. Moreover, as $b$ was any finite tuple of $F \setminus A$, and $\varphi(b) \neq b$, it follows that $b \notin \text{acl}^{eq}(e)$.

(2) Let $S_0 = \text{acl}^{eq}(SG(A), D_1, \ldots, D_n)) \cap SG(F)$. Going to some sufficiently saturated extension $F^s$ of $F$ and using again Lemma 1.4 of [11], we find $S'$ realising $tp(SG(F)/S_0)$ and such that $S' \cap SG(F) = S_0$. (Note that both $S'$ and $SG(F)$ are algebraically closed). We fix some $\mathcal{L}_{G}$-elementary map $S\Psi : S' \rightarrow SG(F)$ which is the identity on $S_0$. By Lemma 1.13 there is $F_1$ realising $tp(F/A)$, and an $A^s$-isomorphism $\psi : F_1^s \rightarrow F^s$, with double dual $S\Psi$. Since $\psi$ is the identity on $A^s$, and $S\Psi$ is the identity on $S_0$, it follows that $\psi(e) = e$. This shows (2).
4.5. Imaginaries in complete systems of Frobenius fields. Recall that a Frobenius field is a PAC field $F$, whose absolute Galois group $G(F)$ has the embedding property, see [1,9].

The properties of $SG(F)$ we will use are the following (see [2], sections 2 and 4):

- $\text{Th}(SG(F))$ is $\omega$-stable.
- (Description of the types) Let $\beta, \gamma$ be tuples of elements of the equivalence class $[\beta], [\gamma]$ respectively, let $S$ be a substructure of $SG(F)$, and let $\delta = \beta \cup S$. Then $tp(\beta/S) = tp(\gamma/S)$ if and only if $\gamma \cup S = \delta$, and there is an isomorphism $f : [\beta] \to [\gamma]$ such that $f(\beta) = \gamma$, and $\pi_{\beta,\delta} = \pi_{\gamma,\delta} f$ (i.e., $f$ induces the identity on $[\delta]$).
- If $S$ is a substructure of $SG(F)$, then the quantifier-free type of $S$ implies its type.
- Let $A$ be a substructure of $SG(F)$, and $\alpha, \beta \in SG(F)$, with $\alpha = \beta \cup A$. Then $\beta \in acl(A)$ if and only if $\beta \in acl(\alpha)$.

One cannot expect the theory of a complete system $SG$ to eliminate imaginaries, simply because most finite groups do not eliminate imaginaries: consider for instance $\mathbb{Z}/5\mathbb{Z}$. Then its subset $\{1,2\} \subset SG$ cannot be coded by any finite tuple of elements of $\mathbb{Z}/5\mathbb{Z}$. However, in case of profinite groups with the embedding property, one obtains the next best thing: weak elimination of imaginaries.

**Theorem 4.6.** Let $G$ be a profinite group with the embedding property, $SG$ its associated system. Then $\text{Th}(SG)$ weakly eliminate imaginaries. Furthermore, any imaginary is equi-definable with an imaginary of the form $([\alpha], \varepsilon)$, where $\alpha \in SG$, and $\varepsilon$ is an imaginary of $[\alpha]$.

**Proof.** Let $D$ be a definable subset of $SG^m$, defined over some algebraically closed substructure $A$. By observation [3,10] we may assume that if the $m$-tuple $\beta$ is in $D$, then all its elements are $\sim$-equivalent. Since $\text{Th}(SG)$ is $\omega$-stable, $D$ contains only finite many types of maximal Morley rank, say $p_i = tp(\beta_i/A), i = 1, \ldots, r$. Then, each $p_i$ is definable over $[\alpha_i]$, where $\alpha_i = \beta_i \cup A$: if $B$ is a substructure of $SG$ containing $A$, then the unique non-forking extension of $p_i$ to $B$ is given by $p_i|_{[\alpha_i]} \cup \{\neg(\xi \geq \gamma) \mid \gamma \in B, \gamma < \alpha_i\}$. I.e., $[\alpha_i]$ is the (algebraic closure of a) canonical base for $p_i$. This shows weak elimination of imaginaries. The last assertion follows immediately from our first reduction.

**Theorem 4.7.** Let $F$ be a Frobenius field, of finite invariant if the characteristic is positive, and in that case assume that the language contains symbols of constants for the $p$-basis of $F$. Then every imaginary of $F$ is equidefinable with a finite set of basic imaginaries $(a,D)$, where furthermore $D$ is an imaginary of some group $[\alpha]$.

**Proof.** This is clear from the discussion above and Theorem [1,2].

4.8. An example. Let $a, b, c, d$ be elements which are algebraically independent over $\mathbb{Q}$, and $\zeta$ a primitive 3-rd root of 1. We let $E_0 = \mathbb{Q}(a,b)^{alg}$, $E_1 = E_0(c, d, \sqrt[d+1]{d+a}(c+b)^2, \sqrt[d+1]{d+b}(c+a)^2)$, and let $F$ be an $\omega$-free PAC field which is a regular extension of $E_1$. Then any automorphism of $E_0$, or of $E_1$, is elementary in the sense of $\text{Th}(F)$.
Let $\alpha_1^2 = c + a$, $\beta_1^2 = c + b$, and $\alpha_2^3 = d + a$, $\beta_2^3 = d + b$. Let $L = F(\alpha_1, \beta_1)$, and $\sigma \in \text{Gal}(L/F)$ be defined by $\sigma(\alpha_1) = \zeta \alpha_1$ and $\sigma(\beta_1) = \zeta^2 \beta_1$. Then $F(\alpha_1) = F(\beta_1)$ and $F(\beta_1) = F(\alpha_2)$, $\sigma(\alpha_2) = \zeta^2 \alpha_2$, $\sigma(\beta_2) = \zeta \beta_2$. Consider the basic imaginaries $e_1 := (a, (L, \sigma))$ and $e_2 = (b, (L, \sigma^2))$. Note that because $\text{Gal}(L/F)$ is abelian, we do not have to worry about conjugation. Consider the imaginary $e := \{e_1, e_2\}$. So, we have $e \in \text{dcl}^a(a, b, (L, \sigma))$, and letting $f_1 = (ab, a + b)$, $f_2 = (L, \{\sigma, \sigma^2\})$, we have $f_1, f_2 \in \text{dcl}^a(e)$.

We will show that $e$ is not equidefinable with any basic imaginary $(E, \varepsilon)$. Assume by way of contradiction that $\text{dcl}^a(e) = \text{dcl}^a(E, \varepsilon)$.

Claim. $\text{dcl}^a(e) \cap F = Q(f_1)$.

We know by Proposition 4.4 that $\text{dcl}^a(e) \cap E \subset \text{acl}(a, b) = E_0$. Let $\rho_1$ be any automorphism of $E_1$ which fixes $c, d$ and $\zeta$, and exchanges $a$ and $b$. Then $\rho_1(e_1) = e_2$: Indeed, $\rho_1$ extends to an automorphism $\rho_1'$ of $L_1 := E_1(\alpha_1, \beta_1)$, which sends $(\alpha_1, \beta_1)$ to $(\zeta' \alpha_1, \zeta^2 \beta_1)$ for some $i, j$; one then computes that $\rho_1' \sigma \rho_1'^{-1} = \sigma^2$. So, $\rho_1(e) = e$. This being true for any $\rho \in \text{Gal}(E_1/Q(f_1, \zeta, c, d))$, we get that $E \subset \text{dcl}(f_1, \zeta)$. Consider now any automorphism $\rho_2$ of $E_1$ which fixes $a, b$, exchanges $c$ and $d$, and sends $\zeta$ to $\zeta^2$. Then again one computes that $\rho_2(e) = e$. As $\rho_2$ moves $\zeta$ and fixes $f_1$, we obtain that $E = Q(f_1)$.

Now, there is also an automorphism $\rho$ of $E_1$, which exchanges $a$ and $b$, and $c$ and $d$, and is the identity on $\zeta$. One computes that it induces the identity on $\text{Gal}(L/F)$, and therefore $\rho(e) \neq e$. By Proposition 4.4 using the fact that $G(E_0) = 1$, we know that $\varepsilon \in \text{acl}^a(\text{Gal}(L/F)) = [\text{Gal}(L/F)]$. This shows that $e \notin \text{dcl}^a(E, \varepsilon)$.

4.9. A non-example. Our first attempt was in fact wrong. We started with $a, b, c, d, \zeta, \alpha_1, \beta_1$ as above, but set $F \cap Q(a, b, c, d)^{alg} = Q(\zeta, a, b, c, d, \sqrt[3]{d + a}(c + b)^2, \sqrt[3]{d + b}(c + a)^2)$. Then, as before, we let $L = F(\alpha_1, \beta_1, \sigma) as above, and $e = \{(a, (L, \sigma)), (b, (L, \sigma^2))\}$. Consider now the imaginaries $\gamma_1 = \text{Gal}(F(\sqrt{a})/F)$ and $\gamma_2 = \text{Gal}(F(\sqrt{b})/F)$. Then $\gamma_1 \in \text{dcl}^a(a)$, $\gamma_2 \in \text{dcl}^a(b)$. So, $e$ is coded by the basic imaginary $(a + b, ab, \delta)$ where $\delta$ is a code for the imaginary $\{(\gamma_1, (L, \sigma)), (\gamma_2, (L, \sigma^2))\}$.

References

[1] J. Ax, The elementary theory of finite fields, Annals of Math. 88 (1968), 239 – 271.
[2] Z. Chatzidakis, Model theory of profinite groups having IP, Illinois J. of Math. 42 No 1 (1998), 70 – 96.
[3] Z. Chatzidakis, Simplicity and Independence for Pseudo-algebraically closed fields, in: Models and Computability, S.B. Cooper, J.K. Truss Ed., London Math. Soc. Lect. Notes Series 259, Cambridge University Press, Cambridge 1999, 41 – 61.
[4] Z. Chatzidakis, Properties of forking in $\omega$-free pseudo-algebraically closed fields, J. of Symb. Logic 67 Nr 3 (2002), 957 – 996.
[5] Z. Chatzidakis, A. Pillay, Generic structures and simple theories, Ann. Pure Applied Logic 95 (1998), 71 – 92.
[6] G. Cherlin, L. van den Dries, A. Macintyre, Decidability and Undecidability Theorems for PAC-Fields, Bull. of the AMS 4 (1981), 101-104.
[7] A. Chernikov, N. Ramsey, On model-theoretic tree properties, J. Math. Log. 16 No. 2 (2016), 1650009. DOI: 10.1142/S0219061316500094.

[8] F. Delon, Idéaux et types sur les corps séparément clos, Supplément au Bull. de la S.M.F, Mémoire 33, Tome 116, 1988.

[9] J. -L. Duret, Les corps faiblement algébriquement clos non séparément clos ont la propriété d'indépendance, in: Model theory of Algebra and Arithmetic, Pacholski et al. ed., Springer Lecture Notes 834 (1980), 135 –157.

[10] Yu. L. Ershov, Undecidability of regularly closed fields (Russian) Algebra i Logika 20 (1981), no. 4, 389 – 394, 484.

[11] D.M. Evans, E. Hrushovski, On the automorphism group of finite covers, APAL 62 (1993), 83 – 112.

[12] M. Fried, M. Jarden, Field Arithmetic, Ergebnisse 11, 3rd edition, Springer Berlin-Heidelberg 2008.

[13] E. Hrushovski, Pseudo-finite fields and related structures, in: Model Theory and Applications, Bélaïr et al. ed., Quaderni di Matematica Vol. 11, Aracne, Rome 2005, 151 –212.

[14] E. Hrushovski, A. Pillay, Groups definable in local fields and pseudo-finite fields, Israel J. of Math. 85 (1994), 203 –262.

[15] B. Kim, A. Pillay, Simple theories, APAL 88 Nr 2-3 (1997), 149 – 164.

[16] S. Lang, Introduction to algebraic geometry, Addison-Wesley Pub. Co., Menlo Park 1973.

[17] B. Poizat, A Course in Model Theory, Universitext, Springer-Verlag New York Berlin Heidelberg 2000.

Current address:
Département de Mathématiques et Applications
Ecole Normale Supérieure
45 rue d’Ulm
75230 Paris Cedex 05
France
e-mail: zchatzid@dma.ens.fr