ON VARIATIONAL PRINCIPLES FOR METRIC MEAN DIMENSION

RUXI SHI

ABSTRACT. In this note, we show several variational principles for metric mean dimension. First we prove a variational principle in terms of Shapira’s entropy related to finite open covers. Second we establish a variational principle in terms of Katok’s entropy. Finally using these two variational principles we develop a variational principle in terms of Brin-Katok local entropy.

1. INTRODUCTION

The topological entropy is a basic invariant of dynamical systems which was studied for a long time. The interplay between ergodic theory and topological entropy was first investigated by Goodman [Goo71]. Gromov [Gro99] introduced a new topological invariant of dynamical systems called mean topological dimension. Mean topological dimension measures the complexity of dynamical systems of finite entropy. Lindenstrauss and Weiss [LW00] introduced metric mean dimension as an invariant of dynamical systems which majors mean topological dimension. The connection between ergodic theory and metric mean dimension was pioneered by Lindenstrauss and Tsukamoto [LT18]. They established a variational principle which states metric mean dimension as a supremum of certain rate distortion functions over invariant measures of the system. Very recently, Gutman and Śpiwalk [GS20a] showed that it is enough to take supremum over ergodic measures in Lindenstrauss-Tsukamoto variational principle. For further applications of metric mean dimension, we refer to [Tsu18a, Tsu18b, GS19, GS20b] and the references therein.

One of our motivation is Problem 3 in [GS20a], where Gutman and Śpiwalk asked whether metric mean dimension can be expressed in terms of Brin-Katok local entropy. In this note, we give an affirmative answer to this problem and consequently build a variational principle for metric mean dimension in terms of Brin-Katok local entropy. The proof of this variational principle is involved in Section 5. To this end, we show a variational principle in terms of Shapira’s entropy of an open cover. Furthermore, we prove an alternative variational principle for metric mean dimension in terms of Katok’s entropy, which drops $\lim_{\delta \to 0}$ in [VV17]. We remark that Brin-Katok local entropy and Shapira’s entropy are only defined for ergodic measures. Thus the variational
principles established in terms of Brin-Katok local entropy or Shapira’s entropy take the supremum over the set of ergodic measures of the system.

2. Preliminaries

2.1. Topological entropy and variational principle. Let $(\mathcal{X}, d, T)$ be a topological dynamical system, i.e. $(\mathcal{X}, d)$ is a compact metric space and $T : \mathcal{X} \to \mathcal{X}$ is a homeomorphism. Define

$$d_n(x, y) = \max_{0 \leq k \leq n-1} d(T^k x, T^k y),$$

for $n \in \mathbb{N}$. Let $K \subset X$ and $\epsilon > 0$. A subset $E \subset K$ is said to be $(n, \epsilon)$-separated of $K$ if distinct $x, y \in E$ implies $d_n(x, y) > \epsilon$. Denote by $s_n(d, T, K, \epsilon)$ (simply $s_n(K, \epsilon)$ when $d, T$ are fixed) the largest cardinality of any $(n, \epsilon)$-separated subset of $K$. Define

$$S(d, T, K, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(d, T, K, \epsilon).$$

We sometimes write $S(K, \epsilon)$ when $d, T$ are fixed.

A subset $F \subset \mathcal{X}$ is said to be $(n, \epsilon)$-spanning of $K \subset \mathcal{X}$ if for any $x \in K$ there exists $y \in F$ such that $d_n(x, y) \leq \epsilon$. Denote by $r_n(d, T, K, \epsilon)$ (simply $r_n(K, \epsilon)$ when $d, T$ are fixed) the smallest cardinality of any $(n, \epsilon)$-spanning subset in $K$. Define

$$R(d, T, K, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(d, T, K, \epsilon).$$

We sometimes write $R(K, \epsilon)$ when $d, T$ are fixed. It is easy to check that

$$r_n(K, \epsilon) \leq s_n(K, \epsilon) \leq r_n(K, \epsilon)$$

and consequently

$$R(K, \epsilon) \leq S(K, \epsilon) \leq R(K, \epsilon).$$

The topological entropy of $K$ is defined by

$$h_{top}(T, K) = \lim_{\epsilon \to 0} S(d, T, K, \epsilon) = \lim_{\epsilon \to 0} R(d, T, K, \epsilon),$$

which is independent of $d$.

Let $\mu$ be a $T$-invariant measure on $\mathcal{X}$, i.e. $T\mu = \mu$. For a Borel partition $P$ of $\mathcal{X}$, the entropy $H_{\mu}(P)$ of $P$ is defined by

$$H_{\mu}(P) = -\sum_{A \in P} \mu(A) \log \mu(A).$$

For convention, we set $0 \cdot \infty = 0$. Moreover, the (dynamical) entropy of $P$ is defined as

$$h_{\mu}(P) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(\vee_{i=0}^{n-1} T^{-i} P),$$
where $P \lor Q = \{ A \cap B : A \in P, B \in Q \}$. The measure-theoretic entropy $h_\mu(T)$ of $\mu$ is defined by

$$h_\mu(T) = \sup_P h_\mu(P),$$

where the suprema are taken over all Borel partitions $P$ of $\mathcal{X}$. One of the link between topological entropy and measure-theoretic entropy in ergodic theory is the variational principle, which was proved originally by Goodman [Goo71].

**Theorem 2.1.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. Then

$$h(\mathcal{X}, T) = \sup_{\mu \in \mathcal{M}_T(\mathcal{X})} h_\mu(T) = \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h_\mu(T),$$

where $\mathcal{M}_T(\mathcal{X})$ is the collection of $T$-invariant measures and $\mathcal{E}_T(\mathcal{X}) \subset \mathcal{M}_T(\mathcal{X})$ consists of ergodic ones.

### 2.2. Metric mean dimension

Let $(\mathcal{X}, d, T)$ be a topological dynamical system. The **upper metric mean dimension** of the system $(\mathcal{X}, d, T)$ is defined by

$$\text{mdim}_M(\mathcal{X}, d, T) = \limsup_{\epsilon \to 0} \frac{S(d, T, \mathcal{X}, \epsilon)}{\log \frac{1}{\epsilon}},$$

which is also equal to $\limsup_{\epsilon \to 0} \frac{R(d, T, \mathcal{X}, \epsilon)}{\log \frac{1}{\epsilon}}$. Similarly, the **lower metric mean dimension** is defined by

$$\text{mdim}_M(\mathcal{X}, d, T) = \liminf_{\epsilon \to 0} \frac{S(d, T, \mathcal{X}, \epsilon)}{\log \frac{1}{\epsilon}}.$$  
If the upper and lower metric mean dimensions coincide, then we call their common value the metric mean dimension of $(\mathcal{X}, d, T)$ and denote it by $\text{mdim}_M(\mathcal{X}, d, T)$. Unlike the topological entropy, the metric mean dimension depends on the metric $d$.

Lindenstrauss and Tsukamoto [LT19] provided a variational principle for metric mean dimension in terms of certain rate-distortion functions. Velozo and Velozo [VV17] proved an alternative formulation in terms of Katok entropy. Gutman and Spiewak [GS20a] showed another one in terms of Rényi information dimension.

### 2.3. Entropy of an open cover

Let $(\mathcal{X}, d, T)$ be a topological dynamical system. Let $\mathcal{U}$ be a finite open cover of $\mathcal{X}$. The **topological entropy** of $\mathcal{U}$ is defined as

$$h_{top}(\mathcal{U}, T) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{U}^n),$$

where $\mathcal{U}^n = \bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}$ and $\mathcal{N}(\mathcal{U})$ the minimal cardinality of a subcover of $\mathcal{U}$.

The following version of the local variational principle for the entropy of an open cover was first conjectured by Romagnoli [Rom03] and then proved by Glasner and Weiss [GW06].
Theorem 2.2 ([GW06], Theorem 7.11). Let $(X, d, T)$ be a topological dynamical system and let $U$ be a finite open cover of $X$. Then

$$h_{top}(U, T) = \sup_{\mu \in \mathcal{M}_{T}(X)} \inf_{\mathcal{P} \succ U} h_{\mu}(\mathcal{P}, T) = \sup_{\mu \in \mathcal{E}_{T}(X)} \inf_{\mathcal{P} \succ U} h_{\mu}(\mathcal{P}, T)$$

where the infimum is taken over all finite Borel partitions $\mathcal{P}$ of $X$ which refine $U$ (i.e. $A \in \mathcal{P}$ implies that $A \subset B$ for some $B \in U$).

Let $U$ be a finite open cover of $X$. We denote by $\text{diam}(U)$ the diameter of the cover, that is, the maximal diameter of any element of $U$. We denote by $\text{Leb}(U)$ the Lebesgue number of $U$, that is, the largest number $\delta$ with the property that every open ball of radius $\delta$ is contained in an element of $U$. A simple fact which we need is as follow. Indeed it follows by $s_n(X, 3\text{diam}(U)) \leq N(U^n) \leq s_n(X, \text{Leb}(U))$. See [GŚ20a, Lemma 3.5] or [Dow11, Theorem 6.1.8] for details.

Lemma 2.3. Let $(X, d, T)$ be a topological dynamical system. Let $U$ be a finite open cover of $X$. Then

$$S(X, 3\text{diam}(U)) \leq h_{top}(U, T) \leq S(X, \text{Leb}(U)).$$

Let $\mu \in \mathcal{E}_{T}(X)$. Let $U$ be a finite open cover. For $\delta \in (0, 1)$, define $N_{\mu}(U, \delta)$ to be the minimum number of elements of $U$, needed to cover a subset of $X$ whose $\mu$-measure is at least $\delta$. Define

$$h_{top}(U) = \lim_{n \to \infty} \log N_{\mu}(U^n, \delta).$$

The above limit exists and is independent of $\delta$ by Shapira [Sha07, Theorem 4.2]. Moreover, Shapira proved the following theorem.

Theorem 2.4 ([Sha07], Theorem 4.4). Let $(X, d, T)$ be a topological dynamical system and let $U$ be a finite open cover of $X$. Let $\mu \in \mathcal{E}_{T}(X)$. Then

$$h_{top}(U) = \inf_{\mathcal{P} \succ U} h_{\mu}(\mathcal{P}),$$

where $\mathcal{P}$ runs over all partitions refining $U$.

Combining Theorem 2.2 and Theorem 2.4, we have an alternative local variational principal of a finite open cover as follows:

Theorem 2.5. Let $(X, d, T)$ be a topological dynamical system and let $U$ be a finite open cover of $X$. Then

$$h_{top}(U, T) = \sup_{\mu \in \mathcal{E}_{T}(X)} h_{top}(U).$$

We remark that $h_{top}(U)$ is only defined for ergodic measure $\mu$. 

\[ \]
2.4. **Brin-Katok local entropy.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. For an invariant measure $\mu \in \mathcal{M}(\mathcal{X})$ and a point $x \in \mathcal{X}$, the *Brin-Katok local entropy* at $x$ is defined by
\[
h_{\mu}^{BK}(x, \epsilon) := \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)),
\]
where $B_n(x, \epsilon) = \{y \in \mathcal{X} : d_n(x, y) < \epsilon\}$. For $\mu$-almost every $x$, the limit $\lim_{\epsilon \to 0} h_{\mu}^{BK}(x, \epsilon)$ exists ([BK83]) and is denoted by $h_{\mu}^{BK}(x)$. If additionally $\mu$ is ergodic, then for $\mu$-a.e. $x$,
- $h_{\mu}^{BK}(x, \epsilon)$ is a constant, denoted by $h_{\mu}^{BK}(\epsilon)$.
- $h_{\mu}^{BK}(x) = h_{\mu}(T)$.

Gutman and Špiewak showed a lower bound for metric mean dimension in terms of Brin-Katok local entropy and asked whether it is also the upper bound. More precisely, they asked [GS20a, Problem 3]:

Let $(\mathcal{X}, d, T)$ be a topological dynamical system. Does the following equality hold?
\[
\text{mdim}_{\mathcal{M}}(\mathcal{X}, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}(\mathcal{X})} h_{\mu}^{BK}(\epsilon).
\]
We show a positive answer to this question in Section 5.

2.5. **Katok’s entropy.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. For $\delta \in (0, 1)$, $n \in \mathbb{N}$ and $\epsilon > 0$, define $N^\delta_n(n, \epsilon)$ to be the smallest number of any $(n, \epsilon)$-dynamical balls (i.e. the balls have radius $\epsilon$ in the metric $d_n$) whose union has $\mu$-measure larger than $\delta$. The *Katok’s entropy* is defined by
\[
h_{\mu}(K, \epsilon, \delta) = \limsup_{n \to \infty} -\frac{1}{n} \log N_{\mu}^\delta(n, \epsilon).
\]
Katok [Kat80] proved that
\[
\lim_{\epsilon \to 0} h_{\mu}(K, \epsilon, \delta) = h_{\mu}(T)
\]
for every $\delta > 0$.

2.6. **Local entropy function.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. For each $\epsilon > 0$ and $x \in X$, we define the *local entropy function*
\[
h_d(x, \epsilon) = \inf \{S(K, \epsilon) : K is a closed neighborhood of x\},
\]
and
\[
\tilde{h}_d(x, \epsilon) = \inf \{R(K, \epsilon) : K is a closed neighborhood of x\},
\]
Clearly, $h_d(x, \epsilon) \geq \tilde{h}_d(x, \epsilon)$. Ye and Zhang showed that [YZ07, Proposition 4.4],
\[
\sup_{x \in \mathcal{X}} \lim_{\epsilon \to 0} h_d(x, \epsilon) = \sup_{x \in \mathcal{X}} \lim_{\epsilon \to 0} \tilde{h}_d(x, \epsilon) = h_{\text{top}}(\mathcal{X}, T).
\]
3. Variational principle I: Shapira’s entropy

Let $F$ be a $(1, \frac{3}{4})$-spanning set of $X$. Obviously, the finite open cover $U := \{B_1(x, \frac{\epsilon}{2}) : x \in F\}$ satisfies that $\text{diam}(U) \leq \epsilon$ and $\text{Leb}(U) \geq \frac{\epsilon}{4}$. Then we have the following lemma. See also [GS20a, Lemma 3.4] for the details.

**Lemma 3.1.** Let $(X, d)$ be a compact metric space. Then for every $\epsilon > 0$ there exists a finite open cover $U$ of $X$ such that $\text{diam}(U) \leq \epsilon$ and $\text{Leb}(U) \geq \frac{\epsilon}{4}$.

Now we show our first variational principle.

**Theorem 3.2.** Let $(X, d, T)$ be a topological dynamical system. Then

$$\text{mdim}_M(X, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_T(X)} \inf_{\text{diam}(U) \leq \epsilon} h^S_{\mu}(U),$$

and

$$\text{mdim}_M(X, d, T) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_T(X)} \inf_{\text{diam}(U) \leq \epsilon} h^S_{\mu}(U),$$

where $U$ runs over all finite open covers.

**Proof.** Let $\epsilon > 0$. By Lemma 3.1, we can find a finite open cover $U_0$ of $X$ with $\text{diam}(U_0) \leq \epsilon$ and $\text{Leb}(U_0) \geq \frac{\epsilon}{4}$. Let $\mu \in \mathcal{E}_T(X)$. By Lemma 2.3 and Theorem 2.5,

$$\sup_{\mu \in \mathcal{E}_T(X)} \inf_{\text{diam}(U) \leq \epsilon} h^S_{\mu}(U) \leq \sup_{\mu \in \mathcal{E}_T(X)} h^S_{\mu}(U_0) = h_{\text{top}}(U_0, T) \leq S(X, \text{Leb}(U_0)) \leq S(X, \frac{\epsilon}{4}).$$

On the other hand, it is clear that for any finite cover $U$ with $\text{diam}(U) \leq \frac{\epsilon}{4}$, the cover $U^n$ refines $U_0^n$ and as a consequence $\mathcal{N}_\mu(U^n, \delta) \geq \mathcal{N}_\mu(U_0^n, \delta)$ for any $0 < \delta < 1$. Thus $\inf_{\text{diam}(U) \leq \frac{\epsilon}{4}} h^S_{\mu}(U) \geq h^S_{\mu}(U_0)$. Then by Lemma 2.3 and Theorem 2.5, we have

$$\sup_{\mu \in \mathcal{E}_T(X)} \inf_{\text{diam}(U) \leq \frac{\epsilon}{4}} h^S_{\mu}(U) \geq \inf_{\mu \in \mathcal{E}_T(X)} h^S_{\mu}(U_0) = h_{\text{top}}(U_0, T) \geq S(X, 3\text{diam}(U_0)) \geq S(X, 3\epsilon).$$

We conclude that

$$S(X, 12\epsilon) \leq \sup_{\mu \in \mathcal{E}_T(X)} \inf_{\text{diam}(U) \leq \epsilon} h^S_{\mu}(U) \leq S(X, \frac{\epsilon}{4})$$

for any $\epsilon > 0$. Therefore we complete the proof by the definition of metric mean dimension. □
4. Variational principle II: Katok’s entropy

Let \((X, d)\) be a compact metric space. For \(\delta \in (0, 1)\), \(n \in \mathbb{N}\) and \(\epsilon > 0\), define \(\tilde{N}_{\mu}^{\delta}(n, \epsilon)\) to be the smallest number of sets with diameter at most \(\epsilon\) in the metric \(d_n\) whose union has \(\mu\)-measure larger than \(\delta\). Recall that \(N_{\mu}^{\delta}(n, \epsilon)\) is the smallest number of any \((n, \epsilon)\)-dynamical balls whose union has \(\mu\)-measure larger than \(\delta\). Clearly,

\[
(4.1) \quad \tilde{N}_{\mu}^{\delta}(n, 2\epsilon) \leq N_{\mu}^{\delta}(n, \epsilon) \leq \tilde{N}_{\mu}^{\delta}(n, \epsilon).
\]

**Lemma 4.1.** Let \((X, d, T)\) be a topological dynamical system. Let \(\mu\) be an ergodic measure. Let \(U\) be a finite open cover of \(X\) with \(\text{diam}(U) \leq \epsilon_1\) and \(\text{Leb}(U) \geq \epsilon_2\). Let \(\delta \in (0, 1)\). Then

\[
\tilde{N}_{\mu}^{\delta}(n, \epsilon_1) \leq N_{\mu}^{\delta}(U^n, \delta) \leq N_{\mu}^{\delta}(n, \epsilon_2).
\]

**Proof.** The inclusion \(\tilde{N}_{\mu}^{\delta}(n, \epsilon_1) \leq N_{\mu}^{\delta}(U^n, \delta)\) is trivial. Let \(F\) be the collection of \((n, \epsilon_2)\)-dynamical balls with \(\sharp F = N_{\mu}^{\delta}(n, \epsilon_2)\) whose union has \(\mu\)-measure larger than \(\delta\). Then for each \(B \in F\), there is \(U_B \in U^n\) such that \(B \subseteq U_B\). Then the union of \(U_B, B \in F\), has \(\mu\)-measure larger than \(\delta\). Thus \(N_{\mu}(U^n, \delta) \leq N_{\mu}^{\delta}(n, \epsilon_2)\). \(\square\)

Our second result on variational principle is as follow.

**Theorem 4.2.** Let \((X, d, T)\) be a topological dynamical system. Then for every \(\delta \in (0, 1)\),

\[
\overline{\text{mdim}}_M(X, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}(X)} h_{\mu}^{K}(\epsilon, \delta),
\]

and

\[
\underline{\text{mdim}}_M(X, d, T) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}(X)} h_{\mu}^{K}(\epsilon, \delta).
\]

**Proof.** By Lemma 3.1, we can find a finite open cover \(U\) of \(X\) with \(\text{diam}(U) \leq \epsilon\) and \(\text{Leb}(U) \geq \frac{\epsilon}{7}\). Fix \(\delta \in (0, 1)\). Let \(\mu \in \mathcal{E}(X)\). Let \(\sigma > 0\). By Lemma 4.1 and (4.1), we have

\[
N_{\mu}^{\delta}(n, \epsilon) \leq N_{\mu}(U^n, \delta) \leq N_{\mu}^{\delta}(n, \frac{\epsilon}{4}).
\]

It follows that

\[
h_{\mu}^{K}(\epsilon, \delta) \leq h_{\mu}^{S}(\epsilon) \leq h_{\mu}^{K}(\frac{\epsilon}{4}, \delta).
\]

Combining this with Lemma 2.3 and Theorem 2.5, we have

\[
\sup_{\mu \in \mathcal{E}(X)} h_{\mu}^{K}(\epsilon, \delta) \leq \sup_{\mu \in \mathcal{E}(X)} h_{\mu}^{S}(\epsilon) = h_{\text{top}}(U, T) \leq S(X, \text{Leb}(U)) \leq S(X, \frac{\epsilon}{4}).
\]
Similarly,
\[
\sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^K_{\mu}(\epsilon, \delta) \geq \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^S_{\mu}(\mathcal{U}) = h_{\text{top}}(\mathcal{U}, T) \\
\geq S(\mathcal{X}, 3\text{diam}(\mathcal{U})) \geq S(\mathcal{X}, 3\epsilon).
\]

We conclude that \( S(\mathcal{X}, 12\epsilon) \leq \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^K_{\mu}(\epsilon, \delta) \leq S(\mathcal{X}, \frac{\epsilon}{4}) \) for any \( \epsilon > 0 \) and \( 0 < \delta < 1 \). Therefore we complete the proof by the definition of metric mean dimension. \[\Box\]

5. Variational principle III: Brin-Katok entropy

In this section, we show the variational principle for metric mean dimension in terms of Brin-Katok local entropy, which also gives a positive answer to Problem 3 in [GŠ20a].

Let \((\mathcal{X}, d, T)\) be a topological dynamical system. For a cover \(U\) of \(\mathcal{X}\) and \(\mu \in \mathcal{M}_T(\mathcal{X})\), we define
\[
h_{\mu}(x, U) := \limsup_{n \to \infty} -\frac{1}{n} \log \mu(U^n_x),
\]
where \(U^n_x\) is the union of all cells of the cover \(U^n\) which contain \(x\). If additionally \(\mu\) is ergodic, then \(h^{BK}_{\mu}(x, U)\) is a constant for \(\mu\)-a.e. \(x\), denoted by \(h^{BK}_{\mu}(U)\). Moreover, if \(U\) is a partition, then \(h^{BK}_{\mu}(U) = h_{\mu}(U)\) by Shannon-McMillan-Breiman theorem.

**Lemma 5.1.** Let \((\mathcal{X}, d, T)\) be a topological dynamical system. Let \(U\) be a finite cover of \(\mathcal{X}\). Let \(\epsilon_1, \epsilon_2 > 0\). Suppose that \(\text{diam}(U) \leq \epsilon_1\) and \(\text{Leb}(U) \geq \epsilon_2\). Then
\[
h^{BK}_{\mu}(\epsilon_1) \leq h^{BK}_{\mu}(U) \leq h^{BK}_{\mu}(\epsilon_2),
\]
for any \(\mu \in \mathcal{E}_T(\mathcal{X})\).

**Proof.** It follows by the inclusion \(B_n(x, \epsilon_2) \subset U^n_x \subset B_n(x, \epsilon_1)\) for every \(n \in \mathbb{N}\) and \(x \in \mathcal{X}\). \[\Box\]

Now we present our main result in this section.

**Theorem 5.2.** Let \((\mathcal{X}, d, T)\) be a topological dynamical system. Then
\[
\overline{\text{mdim}}_M(\mathcal{X}, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^{BK}_{\mu}(\epsilon),
\]
and
\[
\underline{\text{mdim}}_M(\mathcal{X}, d, T) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^{BK}_{\mu}(\epsilon).
\]

**Proof.** Fix \(\epsilon > 0\). By Lemma 3.1, we can find a finite open cover \(U\) of \(\mathcal{X}\) with \(\text{diam}(U) \leq \epsilon\) and \(\text{Leb}(U) \geq \frac{\epsilon}{4}\). Since \(\text{diam}(\mathcal{P}) \leq \epsilon\) for any
partition $P > \mathcal{U}$, by Lemma 5.1, we have $h^K_{\mu}(\epsilon) \leq \inf_{P > \mathcal{U}} h(\mathcal{P})$ for any $\mu \in \mathcal{E}_T(\mathcal{X})$. Then by Lemma 2.3 and Theorem 2.2

$$\sup_{\mu \in \mathcal{E}_T(\mathcal{X})} h^K_{\mu}(\epsilon) \leq \sup_{\mu \in \mathcal{E}_T(\mathcal{X})} \inf_{P > \mathcal{U}} h(\mathcal{P}) = h_{\text{top}}(\mathcal{U}, T)$$

$$\leq S(\mathcal{X}, \text{Leb}(\mathcal{U})) \leq S(\mathcal{X}, \epsilon/4).$$

This implies LHS $\geq$ RHS.

It remains to show LHS $\leq$ RHS. Let $\mu \in \mathcal{E}_T(\mathcal{X})$. Let $\sigma > 0$ and let

$$G_{n, \sigma} = \{x \in \mathcal{X} : -\frac{1}{n} \log \mu(B_n(x, \epsilon)) < h^K_{\mu}(\epsilon) + \sigma\}.$$ 

Since $\mu(\bigcup_{N \geq 1} \cap_{n \geq N} G_{n, \sigma}) = 1$ and $\cap_{n \geq N} G_{n, \sigma}$ is increasing as $N$ grows, we have $\lim_{N \to \infty} \mu(\cap_{n \geq N} G_{n, \sigma}) = 1$. Let $\delta \in (0, 1)$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $\mu(G_{n, \sigma}) > \delta$. Pick arbitrarily $n \geq n_0$. Let $H_n$ be a maximal $(n, 2\epsilon)$-separated set of $G_{n, \sigma}$. It follows that $H_n$ is a $(n, 2\epsilon)$-spanning set of $G_{n, \sigma}$. Thus the union of the balls $B_n(x, 3\epsilon), x \in H_n$, are disjoint. It follows that

$$1 \geq \mu(\bigcup_{x \in H_n} B_n(x, \epsilon)) = \sum_{x \in H_n} \mu(B_n(x, \epsilon)) \geq k H_n e^{-n(h^K_{\mu}(\epsilon) + \sigma)},$$

where the last inequality is due to the fact that $H_n \subset G_{n, \sigma}$. Then $H_n \leq e^{n(h^K_{\mu}(\epsilon) + \sigma)}$. Therefore we get

$$N^\delta_{\mu}(n, 3\epsilon) \leq e^{n(h^K_{\mu}(\epsilon) + \sigma)},$$

and consequently

$$h^K_{\mu}(3\epsilon, \delta) \leq h^K_{\mu}(\epsilon) + \sigma.$$ 

Since $\sigma$ is chosen arbitrarily, by Theorem 4.2, this completes the proof.

\[ \square \]

**Example 5.3.** Let $\mathcal{X} = [0, 1]^\mathbb{Z}$ be the infinite product of the unit interval. Let $\sigma : \mathcal{X} \to \mathcal{X}$ be the (left) shift defined by $(x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$. Define a distance $d$ on $\mathcal{X}$ by

$$d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n|.$$ 

It is known that $\text{mdim}_M([0, 1]^\mathbb{Z}, d, \sigma) = 1$ (see for instance [LT19, Example 1.1]). Let $\mathcal{L}$ be the Lebesgue measure on $[0, 1]$ and $\mu = \mathcal{L}^\otimes \mathbb{Z}$. We will calculate $h^K_{\mu}(\epsilon)$ for $\epsilon > 0$.

Let $\epsilon > 0$ and $x \in [0, 1]^\mathbb{Z}$. Set $\ell = \lceil \log_2 \frac{\epsilon}{4} \rceil$. Then $\sum_{|n| > \ell} 2^{-|n|} \leq \epsilon/2$.

Let

$$I_n(x, \epsilon) := \{y \in [0, 1]^\mathbb{Z} : y_k \in x_k + [-\frac{\epsilon}{6}, \frac{\epsilon}{6}], \forall \ell \leq k \leq n + \ell\},$$
and
\[ J_n(x, \epsilon) := \{ y \in [0, 1]^Z : y_k \in x_k + [-\epsilon, \epsilon], \forall 0 \leq k \leq n \}. \]

It is easy to see that
\[ I_n(x, \epsilon) \subset B_n(x, \epsilon) \subset J_n(x, \epsilon). \]

Since \( \mu(I_n(x, \epsilon)) \geq \left( \frac{\epsilon}{n+1} \right)^n \) and \( \mu(J_n(x, \epsilon)) \leq (4\epsilon)^n \), we obtain that
\[ \log \frac{1}{4\epsilon} \leq h_{BK}^\mu(\epsilon) \leq \log \frac{3}{\epsilon}. \]

Therefore
\[ \lim_{\epsilon \to 0} \frac{h_{BK}^\mu(\epsilon)}{\log \frac{1}{\epsilon}} = 1 = \text{mdim}_M([0, 1]^Z, d, \sigma). \]

6. Discussion on Katok’s entropy and Brin-Katok local entropy

Let \((X, d, T)\) be a topological dynamical system. For an invariant measure \( \mu \in \mathcal{M}_T(X) \) and a point \( x \in X \), analogous to Brin-Katok entropy, we define
\[ h^{-\mu}(x, \epsilon) := \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)). \]

For \( \mu \)-almost every \( x \), the limit \( \lim_{\epsilon \to 0} h^{-\mu}(x, \epsilon) \) exists and is denoted by \( h^{-\mu}(x) \). If additionally \( \mu \) is ergodic, then \( h^{-\mu}(x, \epsilon) \) is a constant for \( \mu \)-a.e. \( x \), denoted by \( h^{-\mu}(\epsilon) \) and as a consequence \( h^{-\mu}(x) = h_{\mu}(T) \).

For a cover \( \mathcal{U} \) of \( X \) and \( \mu \in \mathcal{M}_T(X) \), we define
\[ h^{-\mu}(x, \mathcal{U}) := \lim_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{U}_n^x). \]

If additionally \( \mu \) is ergodic, then \( h^{-\mu}(x, \mathcal{U}) \) is a constant for \( \mu \)-a.e. \( x \), denoted by \( h^{-\mu}(\mathcal{U}) \). Same as Lemma 5.1, we have the following lemma.

Lemma 6.1. Let \((X, d, T)\) be a topological dynamical system. Let \( \mathcal{U} \) be a finite open cover of \( X \). Let \( \epsilon_1, \epsilon_2 > 0 \). Suppose that \( \text{diam}(\mathcal{U}) \leq \epsilon_1 \) and \( \text{Leb}(\mathcal{U}) \geq \epsilon_2 \). Then
\[ h^{-\mu}(\epsilon_1) \leq h^{-\mu}(\mathcal{U}) \leq h^{-\mu}(\epsilon_2), \]

for any \( \mu \in \mathcal{E}_T(X) \).

In the proof of Theorem 5.2, we see that \( h^K_\mu \) is bounded from above by \( h^{BK}_\mu \). We show in the following proposition that \( h^K_\mu \) is bounded from below by \( h^{-\mu} \).

Proposition 6.2. Let \((X, d, T)\) be a topological dynamical system. Then for every \( \delta \in (0, 1) \) and \( \epsilon > 0 \),
\[ h^K_\mu \left( \frac{\epsilon}{4}, \delta \right) \geq h^{-\mu}(\epsilon). \]
Proof. By Lemma 3.1, we can find a finite open cover $\mathcal{U}$ of $\mathcal{X}$ with $\text{diam}(\mathcal{U}) \leq \epsilon$ and $\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{7}$. Fix $\delta \in (0, 1)$. Let $\mu \in E_T(\mathcal{X})$. Let $\sigma > 0$. Let

$$F_{n, \sigma} = \{x \in \mathcal{X} : -\frac{1}{n} \log \mu(U^n_x) > h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma\}.$$ 

Since $\mu(\bigcup_{n \geq 1} \cap_{n \geq N} F_{n, \sigma}) = 1$ and $\cap_{n \geq N} F_{n, \sigma}$ is increasing as $N$ grows, we have $\lim_{N \to \infty} \mu(\cap_{n \geq N} F_{n, \sigma}) = 1$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $\mu(F_{n, \sigma}) > 1 - \frac{\delta}{2}$. Since $\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}$, we see that a $(n, \frac{\epsilon}{7})$-dynamical ball containing a point $x \in F_{n, \sigma}$ is entirely contained in $U^n_x$, so its measure is at most $e^{-n(h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma)}$. For $n > n_0$, note that the $\mu$-measure of the intersection between the complement of $F_{n, \sigma}$ and any union of $(n, \epsilon)$-dynamical balls in $\mathcal{X}$ whose measure larger than $\delta$ is smaller or equal to $\delta/2$. Thus

$$N^{\delta}(n, \frac{\epsilon}{4}) \geq \frac{\delta}{2} e^{nh_{\mathcal{U}}(\mathcal{U}) - n \sigma}, \forall n > n_0.$$ 

It follows that $h^K_{\mu} (\frac{\epsilon}{4}, \delta) \geq h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma$. Since $\sigma$ is arbitrary, by Lemma 6.1 we get

$$h^K_{\mu} (\frac{\epsilon}{4}, \delta) \geq h^\mu_{\mathcal{U}}(\mathcal{U}) \geq h^\mu_{\mathcal{U}}(\epsilon).$$

This completes the proof. \qed

**Open problem:** Does the variational principle for metric mean dimension hold in terms of $h^\mu_{\mathcal{U}}$?

### 7. On local entropy function

In this section, we show that the metric mean dimension is related to the local entropy function. Tsukamoto [Tsu18b, Lemma 2.5] showed a formula of metric mean dimension in terms of local quantity. We develop an alternative formula in terms of local entropy function.

**Theorem 7.1.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. Then

$$\underline{\text{mdim}}_M(\mathcal{X}, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in \mathcal{X}} h_d(x, \epsilon).$$

**Proof.** Since $h_d(x, \epsilon) \leq S(\mathcal{X}, \epsilon)$ for all $x \in \mathcal{X}$, it is obvious that LHS $\geq$ RHS. The other inequality follows from Lemma 7.2 and $h_d(x, \epsilon) \geq \tilde{h}_d(x, \epsilon)$. \qed

By same argument, we also have that

$$\overline{\text{mdim}}_M(\mathcal{X}, d, T) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in \mathcal{X}} h_d(x, \epsilon).$$

**Lemma 7.2.** Let $K$ be a closed subset of $X$. Then $\sup_{x \in K} \tilde{h}_d(x, \epsilon) \geq R(K, \epsilon)$. 

---

**ON VARIATIONAL PRINCIPLES FOR METRIC MEAN DIMENSION**

---

**Proof.** Using Lemma 3.1, we can find a finite open cover $\mathcal{U}$ of $\mathcal{X}$ with $\text{diam}(\mathcal{U}) \leq \epsilon$ and $\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}$. Fix $\delta \in (0, 1)$. Let $\mu \in E_T(\mathcal{X})$. Let $\sigma > 0$. Let

$$F_{n, \sigma} = \{x \in \mathcal{X} : -\frac{1}{n} \log \mu(U^n_x) > h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma\}.$$ 

Since $\mu(\bigcup_{n \geq 1} \cap_{n \geq N} F_{n, \sigma}) = 1$ and $\cap_{n \geq N} F_{n, \sigma}$ is increasing as $N$ grows, we have $\lim_{N \to \infty} \mu(\cap_{n \geq N} F_{n, \sigma}) = 1$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $\mu(F_{n, \sigma}) > 1 - \frac{\delta}{2}$. Since $\text{Leb}(\mathcal{U}) \geq \frac{\epsilon}{4}$, we see that a $(n, \frac{\epsilon}{7})$-dynamical ball containing a point $x \in F_{n, \sigma}$ is entirely contained in $U^n_x$, so its measure is at most $e^{-n(h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma)}$. For $n > n_0$, note that the $\mu$-measure of the intersection between the complement of $F_{n, \sigma}$ and any union of $(n, \epsilon)$-dynamical balls in $\mathcal{X}$ whose measure larger than $\delta$ is smaller or equal to $\delta/2$. Thus

$$N^{\delta}(n, \frac{\epsilon}{4}) \geq \frac{\delta}{2} e^{nh_{\mathcal{U}}(\mathcal{U}) - n \sigma}, \forall n > n_0.$$ 

It follows that $h^K_{\mu} (\frac{\epsilon}{4}, \delta) \geq h^\mu_{\mathcal{U}}(\mathcal{U}) - \sigma$. Since $\sigma$ is arbitrary, by Lemma 6.1 we get

$$h^K_{\mu} (\frac{\epsilon}{4}, \delta) \geq h^\mu_{\mathcal{U}}(\mathcal{U}) \geq h^\mu_{\mathcal{U}}(\epsilon).$$

This completes the proof. \qed

**Open problem:** Does the variational principle for metric mean dimension hold in terms of $h^\mu_{\mathcal{U}}$?

### 7. On local entropy function

In this section, we show that the metric mean dimension is related to the local entropy function. Tsukamoto [Tsu18b, Lemma 2.5] showed a formula of metric mean dimension in terms of local quantity. We develop an alternative formula in terms of local entropy function.

**Theorem 7.1.** Let $(\mathcal{X}, d, T)$ be a topological dynamical system. Then

$$\underline{\text{mdim}}_M(\mathcal{X}, d, T) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in \mathcal{X}} h_d(x, \epsilon).$$

**Proof.** Since $h_d(x, \epsilon) \leq S(\mathcal{X}, \epsilon)$ for all $x \in \mathcal{X}$, it is obvious that LHS $\geq$ RHS. The other inequality follows from Lemma 7.2 and $h_d(x, \epsilon) \geq \tilde{h}_d(x, \epsilon)$. \qed

By same argument, we also have that

$$\overline{\text{mdim}}_M(\mathcal{X}, d, T) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in \mathcal{X}} h_d(x, \epsilon).$$

**Lemma 7.2.** Let $K$ be a closed subset of $X$. Then $\sup_{x \in K} \tilde{h}_d(x, \epsilon) \geq R(K, \epsilon)$.
Proof. Let \( \{B^1_1, B^1_2, \ldots, B^1_n\} \) be a cover of \( K \) consisting of closed balls with diameter at most 1. Then there exists \( j_1 \) such that
\[
R(K, \epsilon) = R(B^1_{j_1} \cap K, \epsilon).
\]
Cover \( B^1_{j_1} \cap K \) by closed balls \( B^2_1, B^2_2, \ldots, B^2_n \) with diameter at most \( \frac{1}{2} \). Then there exists \( j_2 \) such that
\[
R(K, \epsilon) = R(B^1_{j_1} \cap K, \epsilon).
\]
By induction, for every \( k \geq 2 \), there exists a closed ball \( B^k_{j_k} \) with diameter at most \( \frac{1}{k} \) such that
\[
R(K, \epsilon) = R(B^k_{j_k} \cap K, \epsilon).
\]
Let \( \bar{x} = \cap_{k \in \mathbb{N}} B^k_{j_k} \) (which is equal to \( \cap_{k \in \mathbb{N}} (B^k_{j_k} \cap K) \) by above construction). For any closed neighborhood \( K' \) of \( \bar{x} \), we can find sufficiently large \( k \in \mathbb{N} \) such that \( B^k_{j_k} \cap K \subset K' \), which implies that
\[
R(K', \epsilon) \geq R(B^k_{j_k} \cap K) = R(K, \epsilon),
\]
that is, \( h_d(\bar{x}, \epsilon) \geq R(K, \epsilon) \). This completes the proof.

\[\Box\]

Acknowledgement

We thank Adam Śpiewak for pointing out a mistake on an earlier draft. We are grateful to Yonatan Gutman for valuable remarks.

References

[BK83] Michael Brin and Anatole Katok. On local entropy. In Geometric dynamics, pages 30–38. Springer, 1983.

[Dow11] Tomasz Downarowicz. Entropy in dynamical systems, volume 18. Cambridge University Press, 2011.

[Goo71] Tim NT Goodman. Relating topological entropy and measure entropy. Bulletin of the London Mathematical Society, 3(2):176–180, 1971.

[Gro99] Misha Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. Math. Phys. Anal. Geom., 2(4):323–415, 1999.

[GŚ19] Yonatan Gutman and Adam Śpiewak. New uniform bounds for almost lossless analog compression. In 2019 IEEE International Symposium on Information Theory (ISIT), pages 1702–1706. IEEE, 2019.

[GŚ20a] Yonatan Gutman and Adam Śpiewak. Around the variational principle for metric mean dimension. arXiv:2010.14772, 2020.

[GŚ20b] Yonatan Gutman and Adam Śpiewak. Metric mean dimension and analog compression. IEEE Transactions on Information Theory, 2020.

[GW06] E. Glasner and B. Weiss. On the interplay between measurable and topological dynamics. In Handbook of dynamical systems. Vol. 1B, pages 597–648. Elsevier B. V., Amsterdam, 2006.

[Kat80] Anatole Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Publications Mathématiques de l’Institut des Hautes Études Scientifiques, 51(1):137–173, 1980.
[LT18] Elon Lindenstrauss and Masaki Tsukamoto. From rate distortion theory to metric mean dimension: variational principle. *IEEE Transactions on Information Theory*, 64(5):3590–3609, 2018.

[LT19] Elon Lindenstrauss and Masaki Tsukamoto. Double variational principle for mean dimension. *Geometric and Functional Analysis*, pages 1–62, 2019.

[LW00] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000.

[Rom03] Pierre-Paul Romagnoli. A local variational principle for the topological entropy. *Ergodic Theory and Dynamical Systems*, 23(5):1601, 2003.

[Sha07] Uri Shapira. Measure theoretical entropy of covers. *Israel Journal of Mathematics*, 158(1):225–247, 2007.

[Tsu18a] Masaki Tsukamoto. Large dynamics of yang–mills theory: mean dimension formula. *Journal d’Analyse Mathématique*, 134(2):455–499, 2018.

[Tsu18b] Masaki Tsukamoto. Mean dimension of the dynamical system of brody curves. *Inventiones mathematicae*, 211(3):935–968, 2018.

[VV17] Anibal Velozo and Renato Velozo. Rate distortion theory, metric mean dimension and measure theoretic entropy. *arXiv:1707.05762*, 2017.

[YZ07] Xiangsdong Ye and Guohua Zhang. Entropy points and applications. *Transactions of the American Mathematical Society*, 359(12):6167–6186, 2007.

Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland

Email address: rshi@impan.pl