Complementary cycles of any length in regular bipartite tournaments

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Abstract

Let $D$ be a $k$-regular bipartite tournament on $n$ vertices. We show that, for every $p$ with $2 \leq p \leq n/2 - 2$, $D$ has a cycle $C$ of length $2p$ such that $D \setminus C$ is hamiltonian unless $D$ is isomorphic to the special digraph $F_{4k}$. This statement was conjectured by Manoussakis, Song and Zhang [K. Zhang, Y. Manoussakis, and Z. Song. Complementary cycles containing a fixed arc in diregular bipartite tournaments. Discrete Mathematics, 133(1-3):325–328,1994]. In the same paper, the conjecture was proved for $p = 2$ and more recently Bai, Li and He gave a proof for $p = 3$ [Y. Bai, H. Li, and W. He. Complementary cycles in regular bipartite tournaments. Discrete Mathematics, 333:14–27, 2014].

Keywords: Cycle factor, complementary cycles, regular bipartite tournaments

1 Introduction

Throughout all the paper, we are dealing with directed graphs or digraphs. Notations not explicitly stated follows [2].

A cycle-factor of a digraph $D$ is a spanning subdigraph of $D$ whose components are vertex-disjoint (directed) cycles. For some positive integer $k$, a $k$-cycle-factor of $D$ is a cycle-factor of $D$ with $k$ vertex-disjoint cycles; it can also be considered as a partition of $D$ into $k$ hamiltonian subdigraphs. In particular, a 1-cycle-factor is a hamiltonian cycle of $D$. The cycles of a 2-cycle-factor are often called complementary cycles.

A tournament is an orientation of a complete graph. A lot of work has been done on cycle-factors in tournament. For instance, the classical result of Camion [5] states that a tournament is strong if and only if it admits an hamiltonian cycle (i.e. a 1-cycle-factor). Reid [12] proved that every 2-connected tournament with at least 6 vertices and not isomorphic to $T_7$ has a 2-cycle-factor, where $T_7$ is the Paley tournament on 7 vertices: it has vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ and is the union of the three directed cycles: the cycle $1, 2, 3, 4, 5, 6, 7$, the cycle $1, 3, 5, 7, 2, 4, 6$ and the

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cycle 1, 5, 2, 6, 3, 7, 4. This was then extended by Chen, Gould and Li [5] who proved that every k-connected tournament with at least 8k vertices contains a k-cycle-factor.

On the other hand, finding cycles of many lengths in different digraphs is a natural problem in Graph Theory [3]. For example, Moon proved in [11] that every vertex of a strong tournament is in a cycle of every length. Concerning cycle-factor with prescribed lengths in tournaments, Song [13], extended the results of Reid [12], and proved that every 2-connected tournament with at least 6 vertices and not isomorphic to $T_7$ has a 2-cycle-factor containing cycles of lengths $p$ and $|V(T)| - p$ for all $p$ such that $3 \leq p \leq |V(T)| - 3$. Li and Shu [9] finally refined the previous result by proving that any strong tournament with at least 6 vertices, a minimum out-degree or a minimum in-degree at least 3, and not isomorphic to $T_7$, has 2-cycle-factor containing cycles of lengths $p$ and $|V(T)| - p$ for all $p$ such that $3 \leq p \leq |V(T)| - 3$. Recently, Kühn, Osthus and Townsend [8] have extended these results by showing that every $O(k^2)$-connected tournament admits a k-cycle-factor with prescribed lengths.

In this paper, we focus on cycle-factors in k-regular bipartite tournaments. A k-regular bipartite tournament is an orientation of a complete bipartite graph $K_{2k,2k}$ where every vertex has out-degree $k$ exactly. The existing results concerning this class of digraphs try to extend what is known about cycle-factors in tournaments. Thus, Zhang and Song [16] proved that any k-regular bipartite tournament with $k \geq 2$ has a 2-cycle-factor. Moreover, Manoussakis, Song and Zhang [15] conjectured the following statement whose proof is the main result of this paper.

**Theorem 1.** For $k \geq 2$ let $D$ be a k-regular bipartite tournament not isomorphic to $F_{4k}$. Then for every $p$ with $2 \leq p \leq k$, $D$ has a 2-cycle-factor containing cycles of length $2p$ and $|V(D)| - 2p$.

The digraph $F_{4k}$ corresponds to the k-regular tournament consisting of four independent sets $K, L, M$ and $N$ each of cardinality $k$ with all possible arcs from $K$ to $L$, from $L$ to $M$, from $M$ to $N$ and from $N$ to $K$. In fact, every cycle of $F_{4k}$ has length $0 \pmod{4}$. Thus, for instance, $F_{4k}$ has no 2-cycle-factor of length 6 and $4k - 6$. Zhang, Manoussakis and Song proved their conjecture when $p = 2$ in their original paper [15]. In 2014, Bai, Li and He proved the conjecture for $p = 3$ [1].

Our proof of Theorem 1 runs by induction on $p$ and so we will use as basis cases the results of Theorem 1 for $p = 2$ [15] and $p = 3$ [1]. To perform induction step, we will need also a weaker form of Theorem 1 given by the following lemma.

**Lemma 1.** For $k \geq 2$ let $D$ be a k-regular bipartite tournament. If $D$ contains a cycle-factor with a cycle $C$ of length $2p$ with $2 \leq p \leq k$, then $D$ contains a $(2p, |V(D)| - 2p)$-cycle-factor $(C', C'')$. Moreover, if $p$ is at least 3 and even and $D[C]$ is not isomorphic to $F_{2p}$, then $D[C']$ is not isomorphic to $F_{2p}$, neither $D[C'']$.

Theorem 1 and Lemma 1 will both need the following result due to Haggkvist and Manoussakis to be proven.

**Theorem 2** (Haggkvist and Manoussakis [7] and Manoussakis [10]). A bipartite tournament containing a cycle-factor either a hamiltonian cycle or a cycle-factor consisting of cycles $C_1, \ldots, C_m$ such that for any $1 \leq i < j \leq m$, there is no arc from $C_j$ to $C_i$.

Section 2 contains introducing tools and definitions we use for the proofs. Lemma 1 and Theorem 1 are proven in Section 3 and 4 respectively. Finally, in Section 5 we give some concluding remarks concerning cycle-factors in bipartite tournaments.
2 Definitions and Notations

Generic definitions Throughout the paper, all digraphs are simple and loopless. Notations not given here are consistent with [3]. The vertex set of a digraph $D$ is denoted by $V(D)$ and its arcs set by $A(D)$. Given a digraph $D$ and a set $X$ of vertices such that $X \subseteq V(D)$, we denote by $D[X]$ the subdigraph with vertex set $X$, and arc set \{uv \in A(D) : u \in X, v \in X\}. If $H$ is a subdigraph of $D$ we abusively write $D[H]$ for $D[V(H)]$. In the following, we say that two digraphs $D_1$ and $D_2$ are isomorphic if there exists a bijection $\varphi : V(D_1) \to V(D_2)$ such that, for every ordered pair $x, y$ of vertices in $D_1$, $xy$ is an arc of $D_1$ if and only if $\varphi(x)\varphi(y)$ is an arc of $D_2$.

The complement digraph of a digraph $D$, denoted by $\overline{D}$, corresponds to the digraph with vertex set $V(D)$ and arc set \{uv : uv \notin A(D)\}.

For any vertices $u$ and $v$ such that $uv$ is an arc of $D$, we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. The out-neighborhood (resp. in-neighborhood) of $u$ in $D$, denoted $N^+_D(u)$ (resp. $N^-_D(u)$), corresponds to the set of vertices which are out-neighbor (resp. in-neighbor) of $u$. The out-degree (resp. in-degree) of a vertex $u$, denoted $d^+_D(u)$ (resp. $d^-_D(u)$), is the size of its out-neighborhood (resp. in-neighborhood). We say that $D$ is regular if, for any vertices $u$ and $v$ of $D$, we have $d^+_u = d^-_u = d^+_v = d^-_v$. If, in addition, we have $d^+_u = d^-_u = k$, we say that $D$ is $k$-regular. For two vertex-disjoint sets $A$ and $B$, if there are all the possible arcs going from $A$ to $B$, then we say that $A$ dominates $B$. The number of arcs from $A$ to $B$ is denoted by $e(A, B)$. We simply write $e(A)$ instead of $e(A, A)$ to denote the number of arcs linking two vertices of $A$.

Similarly, if there is no arc from $u$ to $v$, we say that there is an anti-arc from $u$ to $v$. Moreover, we say that $v$ is an anti-out-neighbor of $u$ and $u$ is an anti-in-neighbor of $v$. The anti-out-neighborhood (resp. anti-in-neighborhood) of $u$ in $D$, denoted $\overline{N}^+_D(u)$ (resp. $\overline{N}^-_D(u)$), corresponds to the set of vertices which are anti-out-neighbor (resp. anti-in-neighbor) of $u$. The anti-out-degree (resp. anti-in-degree) of a vertex $u$, denoted $\overline{d}^+_D(u)$ (resp. $\overline{d}^-_D(u)$), is the size of its anti-out-neighborhood (resp. anti-in-neighborhood). For two vertex-disjoint sets $A$ and $B$ if there are all the possible anti-arcs going from $A$ to $B$ (that is there are no arcs from $A$ to $B$), we say that $A$ anti-dominates $B$.

If there is no ambiguity, we omit the reference to the considered digraph in the previous notations ($N^+(u)$ instead of $N^+_D(u)$, etc...).

Given a digraph $D$ and a set $\{u_1, \ldots, u_t\}$ of $t$ disjoint vertices of $D$, we say that $P = u_1, \ldots, u_t$ is a directed path of length $t-1$ of $D$ if $u_iu_{i+1} \in A(D)$ for $1 \leq i \leq t-1$. The vertices $u_2, \ldots, u_{t-1}$ are called the internal vertices of $P$. In addition, if we also have $u_1u_1 \in A(D)$, then $u_1, \ldots, u_t$ is a directed cycle of length $t$. A cycle of length 2 is also called a digon. In the paper, path and cycle always means directed path and directed cycle, respectively. Symmetrically, given a digraph $D$ and a set of $t$ disjoint vertices $\{u_1, \ldots, u_t\}$ of $D$, we say that $u_1, \ldots, u_t$ is an anti-path if $u_iu_{i+1} \notin A(D)$ for any $1 \leq i \leq t-1$. In addition, if $u_1u_1 \notin A(D)$, then we obtain an anti-cycle. A digraph $D$ is strongly connected (or strong for short) if we have a path from $u$ to $v$ for any vertices $u$ and $v$ of $D$. If $D$ is not strong, a strongly connected component (or strong component for short) of $D$ is a set $X$ of vertices of $D$ such that $D[X]$ is strong and $X$ is maximal by inclusion for that. A strong component $X$ is an initial strong component (resp. a terminal strong component) of $D$ if there is no arc from $V \setminus X$ to $X$ (resp. from $X$ to $V \setminus X$) in $D$. It is well-known that every non-strong digraph contains at least one initial and one terminal strong component. Given a set $X$ of vertices and a cycle $C$, we denote by $C(X)$ the set of the successors of $X$ along $C$. If $X$ is a singleton $\{x\}$, we simply write $C(x)$ instead of $C(\{x\})$.

Finally for a digraph $D$ and integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = |V(D)|$, a $(n_1, \ldots, n_k)$-cycle-factor is a $k$-cycle-factor $(C_1, \ldots, C_k)$ of $D$ such that for each $i = 1, \ldots, k$ the cycle $C_i$ has length $n_i$. The cycle $C_1$ will be called the first cycle of the cycle-factor.
Bipartite tournaments and contracted digraphs A bipartite tournament is an orientation of a complete bipartite graph. Let $D$ be a $k$-regular bipartite tournament with bipartition $(S,T)$. We have $|S| = |T| = 2k$, and for any vertex $u$ of $D$ we have $d^+(u) = d^-(u) = k$. Moreover, the (unoriented) graph on $S \cup T$ containing an edge for every arcs from $S$ to $T$ is a bipartite graph where every vertex has degree $k$. Hence, by Hall’s Theorem \cite{4}, it admits a perfect matching. Let $M$ be a set of arcs of $D$ corresponding to such a perfect matching. For each vertex $u$ of $S$, the vertex $M(u)$ denotes the only vertex of $T$ such that the arc $uM(u)$ is an arc of $M$.

We extend this notation to sets that is, given a subset $X$ of $S$, we define $M(X)$ by $M(X) = \bigcup_{x \in X} M(x)$.

Now, given a perfect matching $M$ of $D$ made of arcs from $S$ to $T$, we define the contracted digraph according to $M$, denoted $D^M$ and obtained by contracting the arcs of $M$ and only keeping the arcs of $D$ from $T$ to $S$. More formally, the new digraph $D^M$ has vertex set $S$ and arc set \{uv : u \in S, v \in S and M(u)v \in A(D)\}. As the vertex set of $D^M$ is $S$, we also consider vertices of $D^M$ as vertices of $D$. Notice that $D^M$ has 2k vertices and that for every vertex $u$ of $D^M$ we have $N^+_{D^M}(u) = N^+_D(M(u))$ and so, $u$ has out-degree $k$ exactly. Similarly, $u$ has in-neighborhood \{v \in S : M(v) \in N_D(u)\} and so has in-degree $k$ exactly. Notice also that $D^M$ does not contain any parallel arc but may contains cycles on 2 vertices. See Figure 1 which depicts an example of contracted digraph according a matching.

Let $D'$ be a sub-digraph of $D$ and denote by $M'$ the arcs of $M$ with both extremities in $D'$. If $M'$ is also a perfect matching of $D'$, then we abusively denote by $D'^M$ the contracted digraph $D'^M$.

If now $M$ is a perfect matching of $D$ made of arcs from $T$ to $S$, then we can symmetrically define $D^M$ by exchanging $S$ and $T$ in the previous definitions.

Structurally, $u_1,\ldots,u_t$ is a cycle in $D^M$ if and only if $u_1, M(u_1),\ldots,u_t, M(u_t)$ is a cycle of $D$. Thus, to prove Theorem \ref{thm:main} if $D$ is not isomorphic to $F_{2k}$, then for every $p$ with $2 \leq p \leq k - 2$, it suffices to find a $(p,2k - p)$-cycle-factor in $D^M$. Finally, note that the graph $D^M$ contains the same information than $D$ but, most of the time, it will be easier to identify particular structures in the former.

Figure 1: A 2-regular bipartite tournament $D$ and the contracted digraph according to the red matching $M$. In $D^M$, we only keep the vertices of $S$ and the green arcs from $T$ to $S$. Note that the cycle $a,c,b$ in $D^M$ (with bold arcs) corresponds to the cycle $a,M(a),c,M(c),b,M(b)$ in $D$ (also depicted with bold arcs).
3 From cycle-factor to 2-cycle-factor

The aim of this section is to prove Lemma 1 which states that, given a cycle-factor with a cycle of length 2p, we can “merge” the other cycles in order to obtain a (2p, |V(D)| - 2p)-cycle-factor. Moreover, in the case where p is even, we could ask that the new cycle of length 2p is not isomorphic to $F_{2p}$ is the former was not. This condition will be useful in the induction step to prove Theorem 1.

**Lemma 1.** For $k \geq 2$ let $D$ be a $k$-regular bipartite tournament. If $D$ contains a cycle-factor with a cycle $C$ of length $2p$ with $2 \leq p \leq k$, then $D$ contains a $(2p, |V(D)| - 2p)$-cycle-factor $(C', C'')$. Moreover, if $p$ is at least 3 and even and $D[C]$ is not isomorphic to $F_{2p}$, then $D[C']$ is not isomorphic to $F_{2p}$, neither.

**Proof.** As the cases where $p = 2$ and $p = 3$ of Theorem 1 are already proven in [15] and [1], we assume that $p \geq 4$.

Consider a cycle-factor $C'$ of $D$ containing a cycle $C$ of length $2p$, such that $D[C]$ is not isomorphic to $F_{2p}$ if $p$ is even, and such that $C'$ has a minimum total number of cycles. We denote by $C$ the set of cycles of $C'$ different from $C$. Thus, we want to show that $|C| = 1$. By Theorem 2 if $|C| \neq 1$ then we can assume that $C = \{C_1, \ldots, C_{\ell}\}$ with $\ell \geq 2$ and that $C_i$ dominates $C_j$ whenever $i < j$.

Let $(S, T)$ denotes the bipartition of $D$ and for every $i$, we denote by $c_i$ the number of vertices of $V(C_i) \cap S$, that is $C_i$ is of length $2c_i$.

**Claim 1.1.** We have $e(C, C_1) = c_1(2k - c_1)$ and $e(C, C) = c_\ell(2k - c_\ell)$.

**Proof.** A vertex $x_1$ in $C_1$ is an in-neighbor of every vertex in $C_2, \ldots, C_{\ell}$. Hence, we have $\sum_{x \in C_1} d_D^+(x) = e(C_1) + e(C, C_1)$. Thus we get $2kc_1 = c_1^2 + e(C, C_1)$ and the first result holds. The other equality is obtained similarly by reasoning on the out-neighborhood of $C_\ell$.

Now, using Claim 1.1, we have the following.

$$\left(\frac{1}{c_\ell} \sum_{x \in T \cap C_\ell} d_D^+(x) + \frac{1}{c_1} \sum_{x \in T \cap C_1} d_D^+(x)\right) + \left(\frac{1}{c_\ell} \sum_{x \in T \cap C_\ell} d_D^-(x) + \frac{1}{c_1} \sum_{x \in T \cap C_1} d_D^-(x)\right) = \frac{1}{c_\ell} e(C_\ell, C) + \frac{1}{c_1} e(C, C_1) = 4k - (c_1 + c_\ell)$$

Hence, either we have

$$\left(\frac{1}{c_\ell} \sum_{x \in T \cap C_\ell} d_D^+(x) + \frac{1}{c_1} \sum_{x \in T \cap C_1} d_D^+(x)\right) \geq 2k - \frac{(c_1 + c_\ell)}{2}$$

or we have

$$\left(\frac{1}{c_\ell} \sum_{x \in S \cap C_\ell} d_D^+(x) + \frac{1}{c_1} \sum_{x \in S \cap C_1} d_D^+(x)\right) \geq 2k - \frac{(c_1 + c_\ell)}{2}$$

Without loss of generality, we can assume that the former holds (otherwise we exchange in that follows the role of $S$ and $T$).

Denote by $M$ the set of arcs of the digraph induced by the cycle-factor $C, C_1, \ldots, C_{\ell}$ and going from $S$ to $T$ in $D$. It is clear that $M$ forms a perfect matching of $D$ and that $C^M \cup C^M$ is a cycle-factor of $D^M$, where $C^M = \{C^M_1, \ldots, C^M_{\ell}\}$. Moreover, notice that the length of $C^M_i$ is $p$ and for $i$ with $1 \leq i \leq \ell$ the length of $C^M_i$ is $c_i$. By the previous assumption, in $D^M$ we have the following

$$\frac{e(C^M_\ell, C^M)}{c_\ell} + \frac{e(C^M, C^M_1)}{c_1} \geq 2k - \frac{(c_1 + c_\ell)}{2}$$

(1)
Now, we will find suitable vertices in $C^M$ to design the desired 2-cycle-factor. To do so, let $W$ (resp. $R$) be the set of pairs $\{x,y\}$ of distinct vertices of $C^M$ which are “well connected” to $C^M_1$ (resp. from $C^M_1$), that is such that $d_{C^M_1}^+(x) + d_{C^M_1}^+(y) > c_1$ (resp. $d_{C^M_1}^-(x) + d_{C^M_1}^-(y) > c_\ell$). We denote by $w$ (resp. $r$) the cardinal of $W$ (resp. $R$).

**Claim 1.2.** We have $w + r \geq \frac{p(p - 1)}{2}$.

**Proof.** For every pair $\{x,y\}$ of distinct vertices of $V(C^M)$, we have $d_{C^M_1}^+(x) + d_{C^M_1}^+(y) \leq 2c_1$ and, if $\{x,y\}$ is not a pair of $W$, we have more precisely $d_{C^M_1}^+(x) + d_{C^M_1}^+(y) \leq c_1$. Thus, in total,

$$\sum_{\{x,y\} \text{ pair of } V(C^M)} (d_{C^M_1}^+(x) + d_{C^M_1}^+(y)) = \sum_{\{x,y\} \in W} (d_{C^M_1}^+(x) + d_{C^M_1}^+(y)) + \sum_{\{x,y\} \not\in W} (d_{C^M_1}^+(x) + d_{C^M_1}^+(y)) \leq 2wc_1 + \left(\frac{p(p - 1)}{2} - w\right)c_1$$

and

$$\sum_{\{x,y\} \text{ pair of } V(C^M)} (d_{C^M_1}^+(x) + d_{C^M_1}^+(y)) = (p - 1)e(C^M, C^M_1)$$

Thus, we get

$$(p - 1)\frac{e(C^M, C^M_1)}{c_1} \leq w + \frac{p(p - 1)}{2}$$

Similarly, if we do the same reasoning on the arcs from $C^M_1$ to $C^M$ and $R$, we obtain

$$(p - 1)\frac{e(C^M_1, C^M)}{c_\ell} \leq r + \frac{p(p - 1)}{2}$$

Hence, using the inequality [1] we have

$$(p - 1)(2k - \frac{c_1 + c_\ell}{2}) \leq w + r + p(p - 1)$$

Finally, since $C^M \cup C^M_1 \cup C^M_2$ is a subgraph of $D^M$, we have $p + c_1 + c_\ell \leq 2k$ and so $2k - (c_1 + c_\ell)/2 \geq k + p/2$. With the previous inequality we obtain $w + r \geq (p - 1)(k - p/2)$. Finally, using that $k \geq p$, we get the result, that is $w + r \geq p(p - 1)/2$.

Now, for every pair $\{x, x'\}$ of distinct vertices of $C^M$, we color $\{x, x'\}$ in white if it is a pair of $W$, and we color $\{x, x'\}$ in red if $\{y, y'\} \in R$ where $y$ (resp. $y'$) is the out-neighbor of $x$ (resp. $x'$) along $C^M$.

**Claim 1.3.** There exists a pair of vertices colored both in white and red.

**Proof.** If $w + r > p(p - 1)/2$, then we have colored more than $p(p - 1)/2$ pairs of distinct vertices of $C^M$. Thus, at least one pair have been colored both in white and red, yielding the result.

Now, let suppose that $w + r \leq p(p - 1)/2$. By Claim [1, 2] it means that we have $w + r = p(p - 1)/2$ and that all the inequalities leading to the proof of Claim [1, 2] are equalities. In particular, we have $p + c_1 + c_\ell = 2k$ and $p = k$. Notice that, as $c_1$ and $c_\ell$ are at least 2, we have $k \geq 4$. Moreover, [2] is also an equality, we have $d_{C^M_1}^+(x) + d_{C^M_1}^+(y) = 2c_1$ for every pair $\{x, y\}$ of $W$ and $d_{C^M_1}^+(x) + d_{C^M_1}^+(y) = c_1$ for every pair $\{x, y\}$ of vertices of $C^M_2$ which is not in $W$. In particular,
if \( \{x, y\} \in W \) we have exactly \( d_{C^1_i}^+(x) = c_1 \) and \( d_{C^1_i}^+(y) = c_1 \). Similarly, if \( \{x, y\} \in R \), then we have \( d_{C^0_i}^-(x) = c_\ell \) and \( d_{C^0_i}^-(y) = c_\ell \). In particular, we can prove that \( w \neq 0 \). Indeed, if it is not the case, we have \( r = p(p-1)/2 \), that is, every pair of elements of \( C^M \) is a pair of \( R \). Then by the previous remark, every vertex \( x \) of \( C^M \) satisfies \( d_{C^M_i}^- (x) = c_\ell \), and \( C^M \) dominates \( C^M \). So, the out-neighborhood of any vertex \( y \) of \( C^M_i \) would contain the successor of \( y \) along \( C^M_i \) and all the cycle \( C^M \), which is of length \( p = k \), a contradiction to \( d_{C^M_i}^- (y) = k \). Similarly, we have \( r \neq 0 \).

Now, let \( V_W \) (resp. \( V_R \)) be the collection of vertices which belong to at least one pair of \( W \) (resp. \( R \)). Thus, \( V_W \) is not empty and for every vertex \( v \in V_W \), we have \( d_{C^1_i}^+(v) = c_1 \). As every pair \( \{x, y\} \) of distinct vertices of \( C^M \) satisfies \( d_{C^1_i}^+(x) + d_{C^1_i}^+(y) \in \{c_1, 2c_1\} \), it is easy to see that every vertex \( w \notin V_W \) satisfies \( d_{C^0_i}^-(w) = 0 \), and that there is at most one vertex \( a \) which does not belong to \( V_W \). With the same arguments we see that there is at most one vertex \( b \) which does not belong to \( V_R \). So, as \( p = k \geq 4 \) there exists a pair of vertices of \( C^M \) containing neither \( a \) nor the predecessor of \( b \) along \( C^M \). Thus, this pair will colored both in white and red.

In the following, let \( \{y_1, z_1\} \) be a pair of vertices of \( V(C^M) \) colored both in white and red and we will denote by \( y_1 \) (resp. \( z_1 \)) the successor of \( y_1 \) (resp. \( z_1 \)) along \( C^M \). Therefore we have \( \{y_1, z_1\} \in W \) and \( \{y_1, z_1\} \in R \). Notice that \( \{y_1, z_1\} \) and \( \{y_1, z_1\} \) are two distinct pairs of vertices but that \( y_1 = z_1 \) or \( z_1 = y_1 \) is possible.

**Claim 1.4.** For every \( i \) with \( 0 \leq i \leq c_1 - 2 \) there exist \( y, z \in C^1_i \) such that \( y, z \in A(D^M) \) and such that the sub-path of \( C^1_i \) from \( y \) to \( z \) contains \( i \) internal vertices exactly. Similarly for every \( i' \) with \( 0 \leq i' \leq c_\ell - 2 \) there exist \( y' \) and \( z' \in C^0_{i'} \) such that \( y', z' \in A(D^M) \) and such that the sub-path of \( C^0_{i'} \) from \( z' \) to \( y' \) contains \( i' \) internal vertices exactly.

**Proof.** Suppose that for every vertex \( y \) of \( N_{C^1_{i'=1}}^+(y_1) \), the vertex \( z \) which is \( i + 1 \) vertices away from \( y \) along \( C^M_i \) does not belong to \( N_{C^1_{i=1}}^+(z_1) \). Thus we have \( |N_{C^1_{i=1}}^+(z_1)| \leq c_1 - |N_{C^1_{i'=1}}^+(y_1)| \), which contradicts \( d_{C^1_i}^+(y_1) + d_{C^1_i}^+(z_1) > c_1 \) as \( \{y_1, z_1\} \) is a pair of \( W \). The proof is similar for the pair \( \{y_1, z_1\} \).

Now, we can construct our 2-cycle-factor from the collection of cycles. To do so, we will build a cycle \( \gamma \) containing \( p \) vertices, such that \( D^M[V(D^M) \setminus V(\gamma)] \) contains a spanning cycle denoted by \( \gamma' \). Let \( s \) (resp. \( s' \)) be the number of vertices in the path \( P \) (resp. \( P' \)) along \( C^M \) from \( y_1 \) to \( z_1 \) (resp. from \( z_1 \) to \( y_1 \)). We have \( s + s' = p \), thus either we have \( s \leq p/2 \) or \( s' \leq p/2 \). We will suppose that the former holds, since an analogous reasoning can be applied for the other case. In the following, we will denote by \( i_0 \) the smallest index \( j \) such that \( s + \sum_{i=1}^{i_0} c_i > p \). Such index exists, since \( s < p \) and \( s + \sum_{i=1}^{i_0} c_i > \sum_{i=1}^{i_0} c_i = 2k - p \). The cycle \( \gamma \) (resp. \( \gamma' \)) will be obtained as the union of the path \( P \) (resp. \( P' \)) and a path \( Q \) (resp. \( Q' \)) well chosen in \( C^1_i \ldots C^\ell_i \). To design \( Q \) and \( Q' \) we consider several cases.

First, assume that we have \( 1 < i_0 < \ell \). According to Claim 1.5 applied with \( i = 0 \) and \( i' = 0 \) there exists a pair of vertices \( \{y, z\} \) of \( C^1_i \) such that \( y \) is the successor of \( z \) along \( C^1_i \) and with \( y_1 z_1 y \in A(D^M) \). Similarly, there is \( \{y', z'\} \) in \( C^0_{i'} \) such \( y' \) is the successor of \( z' \) along \( C^0_{i'} \) and \( z' y_1 y \in A(D^M) \). As \( C^M \) dominates \( C^M_i \) for any \( i < j \), we consider in \( D^M \) the path \( Q \) starting in \( y \), containing every vertices of \( C^M_i \) except \( z \), every vertices of \( C^M_i \), for any \( 2 \leq j \leq i_0 - 1 \) and \( p - s - \sum_{i=1}^{i_0} c_i \) consecutive vertices of \( C^M_i \) and finally ending with the vertex \( y' \). Similarly we construct the path \( Q' \) containing \( z \), the remaining vertices of \( C^M_i \), every vertices of \( C^M_i \), for any \( i_0 < j < \ell \) and every vertices of \( C^M_i \) except \( y' \), that is \( Q' \) ends in \( z' \). As \( y z, y' y, y z \) and \( z' z \) are arcs of \( D^M \gamma = P \cup Q \) and \( \gamma' = P' \cup Q' \) are cycles and they form a

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2-cycle-factor of $D^M$. To conclude this case, it remains to notice that the number of vertices in $\gamma$ is $s + (c_1 - 1) + (\sum_{i=2}^{s-1} c_i) + (p - s - \sum_{i=s}^{s-1} c_i) + 1 = p$.

In the case where $i_0 = 1$, Claim 1.4 applied with $i = p - s - 2$ and $i' = 0$ asserts that there exist $\{y, z\}$ in $C^M_1$ such that there are $p - s - 2$ vertices from $y$ to $z$ along $C^M_1$ with $y_1z, z_1y \in A(D^M)$. As we assume that $p \geq 3$ and we have $s \leq p/2$, we know that $p - s \geq 2$. There also are $\{y', z'\}$ in $C^M_\ell$ such that $y'$ is the successor of $z'$ along $C^M_\ell$ and $z'z, y'y \in A(D^M)$. Thus we construct $Q$ starting from $y$, containing $p - s - 1$ vertices of $C^M_1$ and ending in $y'$. The path $Q'$ starts in $z$, contains the remaining vertices of $C^M_1$, every vertices of $C^M_\ell$, for any $1 < j < \ell$ and every vertices of $C^M_\ell$ except $y'$. That is, $Q'$ ends in $z'$. As previously, we easily check that $\gamma = P \cup Q$ and $\gamma' = P' \cup Q'$ form a 2-cycle-factor of $D^M$ of lengths $p$ and $2k - p$.

The case $i_0 = \ell$ is symmetric to the previous one.

To check the last part of the statement, we have to guarantee that $D[C']$ is not isomorphic to $F_{2p}$, where $C'$ denote the cycle of $D$ corresponding to $\gamma$ (i.e. such that $C^M = \gamma$). To do so, notice that, in all cases, we added $y$ and $y'$ to $P$ in order to close $\gamma$. In $D^M$, as $y \in C^M_1$ and $y' \in C^M_\ell$, we obtain that $y'y'$ is an arc of $D^M$ and $y'y$ is not an arc of $D^M$. Then $C'$ contains four vertices $y, C'(y), y'$, and $C'(y')$ such that $yC(y), C'(y)y', y'C'(y')$ and $y'y'$ are arcs of $D$. However $F_{2p}$ does not contain such a subdigraph. So $D[C']$ is not isomorphic to $F_{2p}$.

4 Proof of Theorem 1

We prove a slightly stronger version of Theorem 1 where we ask for the first cycle of the cycle-factor to be different from $F_{2p}$ if $p$ is even (notice that $F_{2p}$ is not defined for odd $p$). Namely, we prove the following result.

**Theorem 3.** For $k \geq 3$ let $D$ be a $k$-regular bipartite tournament not isomorphic to $F_{4k}$. Then for every $p$ with $3 \leq p \leq k$, $D$ has a 2-cycle-factor $(C_1, C_2)$ where $C_1$ has length $2p$ and if $p$ is even, $C_1$ is not isomorphic to $F_{2p}$.

We prove this statement by induction on $p$. By the result of Bai, Li and He [1] the statement is true for $p = 3$ and the basis case for the induction holds. So for $3 \leq p < k$ we consider $D = (V, A)$.
a $k$-regular bipartite tournament which admits a $(2p, 4k - 2p)$-cycle-factor $(C_1, C_2)$ where $C_1$ is not isomorphic to $F_{2p}$ if $p$ is even. In particular, notice that $D$ is not isomorphic to $F_{2p}$. We want to show that $D$ admits a $(2(p + 1), 4k - 2(p + 1))$-cycle-factor whose first cycle is not isomorphic to $F_{2(p+1)}$ if $p + 1$ is even. In the following we call a good cycle-factor such a cycle-factor.

We denote by $(S, T)$ the bipartition of $D$ and by $(C_1, C_2)$ the $(2p, 4k - 2p)$-cycle-factor of $D$, with $D[C_1]$ not isomorphic to $F_{2p}$. We also denote by $M_u$ the arcs of $C_1 \cup C_2$ going (up) from $S$ to $T$ and by $M_d$ the arcs of $C_1 \cup C_2$ going (down) from $T$ to $S$. It is clear that $M_u$ and $M_d$ are perfect matchings of $D$ and that their union is $C_1 \cup C_2$. For $M$ being either $M_u$ or $M_d$, the digraph $D^M$ admits the 2-cycle-factor $(C_{1}^{M}, C_{2}^{M})$ with $|C_{1}^{M}| = p$ and $|C_{2}^{M}| = 2k - p$. Notice that, for even $p$, having $C_1$ not isomorphic to $F_{2p}$ is equivalent to having $D^M[C_1^M]$ being not isomorphic to the balanced complete bipartite digraph on $p$ vertices.

To form a good cycle-factor from $(C_{1}^{M}, C_{2}^{M})$, we will have a case-by-case study according to the structure of the non-arc in the digraph $D^M$. Prior to this study, we introduce some needed tools.

### 4.1 Switch along an anti-cycle

In this subsection, we first consider a matching $M$ of the $k$-regular bipartite tournament $D$ made from arcs from $S$ to $T$ and we define an operation allowing some local change in $M$.

**Lemma 2.** If $D^M$ contains an anti-cycle $u_1, \ldots, u_t$ with $t \geq 2$, then the set $M'$ of arcs defined in $D$ by $M' = (M \setminus \bigcup_{i=1}^{t-1} u_i M(u_i)) \cup (\bigcup_{i=1}^{t-1} u_{i+1} M(u_i) \cup u_1 M(u_1))$ is a perfect matching of $D$. Moreover, for every $v \notin \{u_1, \ldots, u_t\}$, we have $N_{D^M}^+(v) = N_{D^M}^+(v)$ and, for every $i$ with $2 \leq i \leq t$, we have $N_{D^M}^+(u_i) = N_{D^M}^+(u_{i-1})$ and $N_{D^M}^+(u_1) = N_{D^M}^+(u_t)$.

**Proof.** If $u_1, \ldots, u_t$ is an anti-cycle of $D^M$ it follows by definition that $u_{i+1} M(u_i)$ is an arc of $D$ for every $i$ such that $1 \leq i \leq t - 1$ as well as $u_1 M(u_t)$. Thus, $M'$ is a perfect matching of $D$. For every $i$ with $1 \leq i \leq t - 1$ we have $M'(u_{i+1}) = M(u_i)$ in $D$ (and $M'(u_1) = M(u_t)$), then $N_{D^M}^+(u_{i+1}) = N_{D^M}^+(u_i)$ (and $N_{D^M}^+(u_1) = N_{D^M}^+(u_t)$). The out-neighborhood of the other vertices are unchanged.

The “shifting” operation between matchings $M$ and $M'$ described in the previous lemma is called a switch along the anti-cycle $u_1, \ldots, u_t$. The first easy observation we can make on the new contracted digraph is the following.

**Corollary 3.** If $D^M$ has a cycle-factor and contains an anti-cycle $u_1, \ldots, u_t$ with $t \geq 2$, then the digraph obtained after the switch along the anti-cycle $u_1, \ldots, u_t$ has a cycle-factor.

**Proof.** Let $C$ be the anti-cycle $u_1, \ldots, u_t$ and let $M'$ be the perfect matching obtained after the switch along $C$. Moreover, let $\Delta$ be the subdigraph induced by the arcs of the cycle-factor in $D^M$, and let $\Delta'$ be the subdigraph induced by the switch of $\Delta$ along $C$. By Lemma 2 we make a cyclic permutation on the out-neighborhoods of the vertices of $C$. So for every vertex $x$ in $S$ we have $d_\Delta^+(x) = d_\Delta^+(x) + 1$ and $d_\Delta^-(x) = d_\Delta^-(x) = 1$. Therefore $\Delta'$ is a cycle-factor of $D^{M'}$.

Claim 3 below is our main application of a switch along an anti-cycle in a contracted digraph. Before stating it, we need the following simple result.

**Lemma 4.** If $D^M$ contains a cycle-factor $\{B_1, \ldots, B_l, B_{l+1}, \ldots, B_{l'}\}$ such that $|B_1| + \cdots + |B_l| = p + 1$ and $D^M[B_1 \cup \cdots \cup B_l]$ is strongly connected and not isomorphic to a balanced complete bipartite digraph, then $D$ admits a good cycle-factor.

**Proof.** For $i = 1, \ldots, l'$ we denote by $\tilde{B}_i$ the cycle of $D$ such that $\tilde{B}^M_i = B_i$. Since $D^M[B_1 \cup \cdots \cup B_l]$ is strongly connected, the digraph $D[\tilde{B}_1 \cup \cdots \cup \tilde{B}_l]$ is also strongly connected and admits a cycle-factor. So, by Theorem 2 it has a Hamiltonian cycle $C$ which is of length $2p + 2$. Moreover, as
Given modulo $t$ in $B$ and the blue arcs form the cycle-factor $H$.

Figure 3: An illustrative case of the proof of Claim 3.1. The dashed arcs form the anti-cycle $H$.

**Claim 3.1.** If $D^M$ contains an anti-cycle $H$ such that $H \setminus V(C^M_1)$ is an anti-path from $x$ to $y$ with $x = C^M_2(y)$ then $D$ admits a good cycle-factor.

**Proof.** To shorten notations, we denote $C^M_1$ by $C$ and $C^2_2$ by $C'$. Let $H = a_1 \ldots a_t$ be an anti-cycle of $D^M$ such that $a_1 \ldots a_s$ is an anti-path of $D^M[C']$, $a_1$ is the successor of $a_s$ along $C'$ and $a_{s+1} \ldots a_t$ is an anti-path of $D^M[C]$. Moreover, for every $i$ with $1 \leq i \leq t$ we denote by $b_i$ the out-neighbor of $a_i$ along $C$ or $C'$. An illustrative case is depicted in Figure 3.

We perform a switch exchange on $H$ to obtain the digraph $D^M$. By Corollary 2, the digraph $D^M$ contains a cycle-factor. More precisely, we pay attention to $B$ the cycle-factor derived from $C \cup C'$. By Lemma 2. the out-neighbor in $B$ of every vertex not in $\{a_1, \ldots, a_t\}$ is its out-neighbor in $C \cup C'$ and the out-neighbor in $B$ of every vertex $a_i$ in $\{a_1, \ldots, a_t\}$ is $b_{i-1}$ (where indices are given modulo $t$). As there is only one arc of $H$ from $C$ to $C'$ and one arc of $H$ from $C'$ to $C$, the only arc of $B$ from $V(C)$ to $V(C')$ or $B$ is $a_{s+1}b_s$ and the only arc of $B$ from $V(C')$ to $V(C)$ is $a_1b_1$. So $B$ contains a subset $B_1$ of cycles covering $V(C) \cup \{a_1\}$ and a subset $B_2$ of cycles covering $V(C') \setminus \{a_1\}$. Thus the cycles of $B_1$ (resp. $B_2$) form a cycle-factor of $D^M[C \cup \{a_1\}]$ (resp. $D^M[C' \setminus \{a_1\}]$) and we denote by $B_1$ (resp. $B_2$) the corresponding cycle-factors of $D$. Moreover, for $i = s + 1, \ldots t$ the arcs $a_iM(a_i)$ belongs to $D$ and as $M(a_i) = M'(a_{i+1})$ they link the cycles of $B_2$ in a strongly connected way. Thus $D[B_2]$ is strongly connected and so by Theorem 2, it has an hamiltonian cycle $B_3$ on $2(p+1)$ vertices. In addition if $p + 1$ is even $D[B_3]$ is not isomorphic to $F_{2(p+1)}$ as it contains $C_1$ as a subdigraph, $C_1$ being a cycle on $2p$ vertices with $p$ odd. Therefore
$B_1 \cup B_3$ forms a cycle-factor of $D$ with a cycle, $D[B_3]$, of length $2(p+1)$ not isomorphic to $F_{2(p+1)}$, and we can conclude with Lemma 3.

The next claim is an easy case where we can insert a vertex of $C_2^M$ into $C_1^M$.

**Claim 3.2.** If $C_2^M$ contains three consecutive vertices $a$, $b$ and $c$ (in this order along $C_2^M$) and $C_1^M$ contains two consecutive vertices $x$ and $y$ (in this order along $C_1^M$) such that $ac$, $xb$ and $by$ are arcs of $D^M$ then $D$ admits a good cycle-factor.

**Proof.** It is clear that using the arcs $ac$, $xb$ and $by$ we can form a 2-cycle-factor of $D^M$, with one cycle of length $2k-(p+1)$ covering $C_2^M \setminus \{b\}$ and the other of length $p+1$ covering $C_1^M \cup \{b\}$. Let us denote by $C$ this latter one. If $p+1$ is even notice that the cycle of $D$ corresponding to $C$ cannot be isomorphic to $F_{2(p+1)}$. Indeed otherwise $C$ would be isomorphic to a complete bipartite digraph but $C$ contains $C$ has a subdigraph which is a cycle on $p$ vertices with $p$ odd, a contradiction.

Now we can prove the following claim, that we will intensively use in the remaining of the proof of Theorem 3.

**Claim 4.1.** Assume that $D^M[C_2^M]$ is not strongly connected and denote by $S_1, \ldots, S_i$ its strongly connected components. If there exists an arc $ab$ of $C_2^M$ such that there is an anti-path in $D^M[C_2^M]$ from $b$ to $a$ and $a \in S_i$, then $D$ admits a good cycle-factor.

**Proof.** In $D^M$, we denote $C_2^M$ by $C'$ and $C_1^M$ by $C$. All the proof stands in $D^M$. First assume that there exists a vertex $c \in C$ such that $ac$ and $cb$ are anti-arcs, then the anti-cycle formed by the anti-path from $b$ to $a$ in $C'$ completed with the anti-arcs $ac$ and $cb$ satisfies the hypothesis of Claim 3.2 and we can conclude.

Hence, we assume that $\overline{N_C^C(a)} \cap \overline{N_C^C(b)} = \emptyset$. In particular, we have $\overline{d_C^C(a)} + \overline{d_C^C(b)} \leq p$. Let us denote by $A$ the set of all the vertices of $C'$ for whom there is a anti-path from $a$ to them, and by $B$ the set $C' \setminus A$. The set $A$ dominates the set $B$, we have $|A| + |B| = 2k - p$ and also $S_j \subseteq A$ and $S_j \subseteq B$ (otherwise $a$ and $b$ would have been in the same connected component of $D^M[C']$). Therefore, we have

$$2k - 2 = d^+(a) + d^-(b) \leq (d_C^+(a) + d_C^-(b)) + d_C^+(a) + d_C^-(b) = p + |A| - 1 + |B| - 1 = 2k - 2$$

and thus we have equalities everywhere. In particular we obtain that $(A, B)$ is a partition of $V(C')$ with $A \setminus \{a\} \subseteq \overline{N^C_C(a)}$ and $B \setminus \{b\} \subseteq \overline{N^C_C(b)}$. As a consequence for every $x_A \in A$ and $x_B \in B$ there exists an anti-path from $x_B$ to $x_A$. Another consequence is that $\overline{d_C^C(a)} + \overline{d_C^C(b)} = p$ and that $C$ admits a partition into $\overline{N_C^C(a)}$ and $\overline{N_C^C(b)}$.

So let $b'$ be the successor of $b$ along $C'$, that is $b' = C'(b)$, and assume first that $b' \in B$. Hence, for every $x \in \overline{N_C^C(a)}$ the arc $xb$ exists in $D^M$ (as we assume that $\overline{N_C^C(a)} \cap \overline{N_C^C(b)} = \emptyset$) and so $C(x)$ is an anti-arc, otherwise we can insert $b$ into $C$ and shortcut the path $abb'$. Thus we have $C(\overline{N_C^C(a)}) \subseteq \overline{N_C^C(b)}$ and in particular, we obtain $\overline{d_C^C(a)} \leq \overline{d_C^C(b)}$. Hence, we have

$$2k - 2 = d^+(a) + d^-(b') = \overline{d_C^+(a)} + |A| - 1 + \overline{d_C^+(b')} + \overline{d_C^-(b')} \\
\leq \overline{d_C^-(b')} + \overline{d_C^-(b')} + |A| - 1 + |B| - 2$$

and then $\overline{d_C^-(b')} + \overline{d_C^-(b')} \geq p + 1$. In particular, there exists $c \in C$ such that $bc$ and $cb'$ are anti-arcs and we can apply Claim 3.1 to the anti-cycle $bb'c$ to obtain a good cycle-factor of $D$. Now, we assume that we have $b' \in A$. And more generally we can assume that $C'$ has no two consecutive vertices lying in $B$. Indeed, otherwise considering a non empty path of $C'[B]$, we can proceed as before with $a = u$, $b = v$ and
We denote by $\mathcal{M}$ the set of all anti-arcs of $D$. Let $u$ be a vertex of $C(A)$ such that $u' \not\in C(u)$ does not belong to $B$. Then let $v$ be a vertex of $C(B)$. As $u' \not\in B$, the arc $vu'$ belongs to $D'$ and then there exists an anti-arc from $u$ to $v$ in $D'$. Moreover, for every vertex $w$ of $T'$, if $C(w)$ does not belongs to $A$ then there

\[ b' = w. \]

Symmetrically, $C'$ has no two consecutive vertices in $A$ and $C'$ alternates between $A$ and $B$. In particular, $p$ is even and we write $C' = (a_1b_1a_2b_2\ldots a_{k-p/2}b_{k-p/2})$ with $A = \{a_1, \ldots, a_{k-p/2}\}$ and $B = \{b_1, \ldots, b_{k-p/2}\}$ (indices in $C'$ will be given modulo $k-p/2$). By the above arguments we also have $A \setminus \{a_i\} \subseteq \overline{N}(a_i)$ and $B \setminus \{b_i\} \subseteq \overline{N}(b_i)$ for every $1 \leq i \leq k-p/2$ implying that $A$ and $B$ are two independent sets of $D^M$.

Notice that, $p$ being even, we have $p \geq 4$ and $k-p/2 > p/2 \geq 2$. So we obtain that $k-p/2 \geq 3$.

Now, for every $i$ with $1 \leq i \leq k-p/2$, we denote by $B_i$ the set $\overline{N}(b_i)$ and by $A_i$ the set $\overline{N}(a_i)$. As $a_ib_i$ is an arc of $C'$, there exists an anti-path from $b_i$ to $a_i$ in $D[C']$ and as $a_i$ and $b_i$ do not belong to the same strongly connected component of $\overline{D^M}[C']$ (because $A$ dominates $B$), we can argue as before with $a_i$ playing the role of $a$ and $b_i$ the one of $b$. In particular, we obtain that $(A_i, B_i)$ is a partition of $V(C)$ for every $i \in \{1, \ldots, k-p/2\}$ with $|A_i| = |B_i| = p/2$. Moreover, assume that there exists $x \in C$ and $i \in \{1, \ldots, k-p/2\}$ such that $a_ix$ and $xb_{i+1}$ are anti-arcs.

In this case, we modify $C'$ into the cycle $C''$ by replacing the subpath $a_{i-1}b_{i-1}a_i a_{i+1}b_{i+1}$ of $C'$ by $a_{i-1}b_{i+1}a_{i+1}b_{i-1}a_i$. Now, $a_ib_{i+1}$ is an arc of $C''$ and there exists an anti-path $P$ in $D^M[C']$ from $b_{i+1}$ to $a_i$. So we can conclude by applying Claim 3.1 to the anti-cycle $P$. It means that $A_i \cap B_{i+1} = \emptyset$ for every $i \in \{1, \ldots, k-p/2\}$. We deduce that all the $A_i$ coincide as well as all the $B_i$ and in particular we have $A$ anti-dominates $A_i$ and $A_i$ dominates $B$ for every $i \in \{1, \ldots, k-p/2\}$.

To conclude, consider $s, t \in \{1, k-p/2\}$ such that $b_{si}a_i$ is an anti-arc (such an anti-arc exists since there exists an anti-path from $b$ to $A$). If there exists an anti-arc from $x_s \in A_s$ to $x_t \in B_t$, then we conclude with Claim 3.1 applied to the anti-cycle $x_s x_t b_s b_t a_t$. Otherwise it means that $A_t$ dominates $B_t$ and so as $A_t$ dominates $B$, we have $A_t$ anti-dominates $A$. But then we obtain $\{b_s\} \cup A_t \cup A \setminus \{a_t\} \subseteq \overline{N}(a_t)$ and $\overline{d^{-2}}(a_t) \geq k$, a contradiction. \hfill \Box

4.2 Properties of the 2-cycle-factor $(C_1, C_2)$.

Now, we define four different properties of the 2-cycle-factor $(C_1, C_2)$ of $D$. For this, we first need the next claim.

Claim 4.2. Let $C$ be $C_1$ or $C_2$. We have:

- either $D^{M_a}[C^{M_a}]$ or $D^{M_b}[C^{M_b}]$ has exactly one terminal strong component
- or $|C|$ is congruent to 0 modulo 4 and $D[C]$ is isomorphic to $F_{|C|}$.

Similarly, we have:

- either $D^{M_a}[C^{M_a}]$ or $D^{M_b}[C^{M_b}]$ has exactly one initial strong component
- or $|C|$ is congruent to 0 modulo 4 and $D[C]$ is isomorphic to $F_{|C|}$.

Proof. We prove the statement for terminal strong components. The proof for initial ones is similar. We denote by $D'$ the digraph $D[C]$ with bipartition $(S', T')$ where $S' = S \setminus V(C)$ and $T' = T \setminus V(C)$. So, assume that $D^{M_a}$ has at least two terminal strong components and denote by $A$ and $B$ two such components. In $D^{M_a}$, it means that $A$ dominates $D^{M_a} \setminus A$ and that $B$ dominates $D^{M_a} \setminus B$. In $D'$, by considering that $A$ and $B$ are subsets of $S'$, we then have $C(A)$, the successors of the vertices of $A$ along $C$, dominates $S' \setminus A$ and $C(B)$ dominates $S' \setminus B$. Notice that $A$ and $B$ are two disjoint sets of $S'$ and that $C(A)$ and $C(B)$ are two disjoint sets of $T'$ with $|C(A)| = |A|$ and $|C(B)| = |B|$.

Now, assume that there exists a vertex $u$ of $C(A)$ such that $u' = C(u)$ does not belong to $B$. Then let $v$ be a vertex of $C(B)$. As $u' \not\in B$, the arc $vu'$ belongs to $D'$ and then there exists an anti-arc from $u$ to $v$ in $D^{M_a}$. Moreover, for every vertex $w$ of $T'$, if $C(w)$ does not belongs to $A$ then there
Proof. Let \( C \) be the arc \((u, v)\) in \( D' \). In particular, we have \(|C'(A)| \leq |B| = |C(B)| \leq |A| = |C(A)|\) and so \(|A| = |B| = |C(A)| = |C(B)|\). Moreover, as \( C \) contains all the vertices of \( D' \), \((A, B)\) is a partition of \( S' \) and \((C(A), C(B))\) is a partition of \( T' \) with \( C(A) \) dominates \( B \) and \( C(B) \) dominates \( A \). In particular, \( |C| \) is congruent to 0 modulo 4. As all the arcs of \( C \) from \( T' \) to \( S' \) go from \( C(A) \) to \( B \) or from \( C(B) \) to \( A \) and that \( C(A) \) dominates \( B \) and \( C(B) \) dominates \( A \) in \( D' \), the sets \( C(A) \) and \( C(B) \) respectively induce an independent set in \( D'[M] \), that is a complete digraph in \( D'[M] \). As they form a partition of the vertex set of \( D[M] \), it has exactly one initial strong component except if there is no arc between \( C(A) \) and \( C(B) \). In this later case, it means that all the arcs from \( A \) to \( C(A) \) and from \( B \) to \( C(B) \) are contained in \( D' \). But, then \( D'[C] \) is isomorphic to \( F[C]\). \( \Box \)

In the case where \( D[M][C_2^{M^*}] \) has exactly one initial strong component, we say that the 2-cycle-factor \((C_1, C_2)\) has property \( Q_{up} \). And in the case where \( D[M][C_1^{*}] \) has exactly one initial strong component, we say that \((C_1, C_2)\) has property \( Q_{down} \). By Claim 4.2, we know that either \((C_1, C_2)\) has property \( Q_{up} \) or \( Q_{down} \) or that \(|C| \) is congruent to 0 modulo 4 and that \( D[C_2] \) is isomorphic to \( F[C_2]\).

Now, let us define another pair of properties for the 2-cycle-factor \((C_1, C_2)\). As \( D \) is a bipartite tournament every vertex of \( C_2 \) has an in-neighbor or an out-neighbor in \( C_1 \). Moreover, as \( D \) is \( k \)-regular and \(|C_1| = 2p < 2k\) there exists at least one arc from \( C_1 \) to \( C_2 \) and one arc from \( C_2 \) to \( C_1 \). So, it is easy to check that there exists an arc \((u, v)\) of \( C \) such that \( N_{C_1}(u) \neq \emptyset \) and that \( N_{C_2}(v) \neq \emptyset \). If we have \( u \in S \) and \( v \in T \) we say that \((C_1, C_2)\) has property \( P_{down} \). In this case, it means that the arc \((u, v)\) is an arc of \( M_\uparrow \), and in \( D[M] \) the vertex \( u \) has an in-neighbor and an out-neighbor in \( C_1^{M^*} \). Otherwise, that is when we have \( v \in S \) and \( u \in T \), we say that \((C_1, C_2)\) has property \( P_{up} \). In this case, let \( u' \) be the predecessor of \( u \) along \( C_2 \), that is \( u = C_2(u') \). So, in \( D[M] \), \( u'v \) is an arc of \( C_2^{M^*} \) and \( v \) has an in-neighbor in \( C_1^{M^*} \).

Claim 4.3. If \((C_1, C_2)\) does not satisfies \( P_{down} \), then in \( D[M] \) every vertex of \( C_2^{M^*} \) anti-dominates \( C_1^{M^*} \) or is anti-dominated by \( C_1^{M^*} \).

Proof. Let \( x \) be a vertex of \( C_2^{M^*} \). Since \( D \) does not satisfies \( P_{down} \), then in \( D \), either \( C_2(x) \) is dominated by \( C_1 \cap S \) or \( x \) dominates \( C_1 \cap T \). In the first case, it means that there is no arc from \( x \) to \( C_1^{M^*} \) in \( D[M] \), while in the latter, it means that there is no arc from \( C_1^{M^*} \) to \( x \). \( \Box \)

Notice that if \((C_1, C_2)\) satisfies property \( Q_{up} \), then exchanging the role of \( S \) and \( T \) in \( D \) (and then of \( M_\uparrow \) and \( M_\downarrow \)) leads to \((C_1, C_2)\) satisfies property \( Q_{down} \), and conversely. We also have the similar property with \( P_{up} \) and \( P_{down} \).

Thus, without loss of generality, we assume that \((C_1, C_2)\) has the property \( P_{up} \). Then, we study, in this order, the three different cases: either \((C_1, C_2)\) has property \( Q_{up} \), or \( D[C_2] \) is isomorphic to \( F[C_2] \), or \((C_1, C_2)\) satisfies property \( Q_{down} \).

4.3 Case A: \((C_1, C_2)\) has property \( Q_{up} \)

So, we know that \( D[M][C_2^{M^*}] \) has exactly one initial strong component. If it is not strong itself, there exists an arc of \( C_2^{M^*} \) entering into its unique initial strong component and we can directly conclude with Claim 4.1. Then, we assume that \( D[M][C_2^{M^*}] \) is strongly connected. We consider several cases.
There exist two vertices \( a \in A \) and \( b \in B \) such that there is an anti-path \( P \) from \( b \) to \( a \) in \( D^{M_s}[C] \) (depicted in blue) and, there are two vertices \( a' \) and \( b' \) in \( C' \) such that \( aa' \) and \( bb' \) are anti-arcs, and \( b' \) is the successor of \( a' \) along \( C' \). We can use the red anti-path from \( a' \) to \( b' \) and \( P \) to form an anti-cycle.

**Case 1:** \( D^{M_s}[C] \) is strongly connected too. As \((C_1, C_2)\) has the property \( P_{uu'} \), there exist \( u \) and \( v \) in \( D^{M_s} \) such that \( v \) is the successor of \( u \) along \( C_2^{M_s} \). \( u \) has an anti-out-neighbor \( u' \) in \( C_1^{M_s} \) and \( v \) has an anti-in-neighbor \( v' \) in \( C_1^{M_s} \). As both \( D^{M_s}[C] \) and \( D^{M_s}[C_2^{M_s}] \) are strongly connected, there exists an anti-path from \( v \) to \( u \) in \( D^{M_s}[C] \) and an anti-path from \( u' \) to \( v' \) in \( D^{M_s}[C_1^{M_s}] \). So, we can form an anti-cycle in \( D^{M_s} \) which satisfies the hypothesis of Claim 3.1 and we conclude that \( D \) contains a good cycle-factor.

**Case 2:** \( D^{M_s}[C] \) is not strongly connected. In what follows, to shorten the notation, we denote \( C_1^{M_s} \) by \( C \) and \( C_2^{M_s} \) by \( C' \). So, as \( D^{M_s}[C] \) is not strong, there exists a partition \((A, B)\) of \( V(C) \) such that there is no anti-arcs from \( A \) to \( B \), thus we have \( A \) dominating \( B \) in \( D^{M_s} \).

**Case 2.1:** There exist two vertices \( a \in A \) and \( b \in B \) such that there is an anti-path \( P \) from \( b \) to \( a \) in \( D^{M_s}[C] \). So, in \( D^{M_s} \) we have

\[
\overline{d^+(a)} + \overline{d^-(b)} = 2k - 2
\]

Since \( \overline{d^+(a)} \leq |A| - 1 + \overline{d^-(a)} \) and \( \overline{d^-(b)} \leq |B| - 1 + \overline{d^+(b)} \) and \( |A| + |B| = p \), we get

\[
\overline{d^+(a)} + \overline{d^-(b)} \geq 2k - p
\]

If \( \overline{d^+(a)} + \overline{d^-(b)} > 2k - p = |C'| \) then there exist two vertices \( a' \) and \( b' \) in \( C' \) such that \( aa' \) and \( bb' \) are anti-arcs, and \( b' \) is the successor of \( a' \) along \( C' \). As \( D^{M_s}[C'] \) is strongly connected, there exists an anti-path \( Q \) from \( a' \) to \( b' \) in \( D^{M_s}[C'] \). So the concatenation of the paths \( P, Q \) and the arcs \( a' \) and \( bb' \) forms an anti-cycle of \( D^{M_s} \) satisfying the conditions of Claim 3.1 and we can conclude that \( D \) contains a good cycle-factor. See Figure [1] which depicted this subcase.

So, we assume that \( \overline{d^+(a)} + \overline{d^-(b)} \leq 2k - p \). It implies that \( \overline{d^+(a)} = |A| - 1 \) and \( \overline{d^-(b)} = |B| - 1 \) and that \( a \) anti-dominates \( A \) and \( B \) anti-dominates \( b \). We deduce that for every \( a' \in A \) and \( b' \in B \) there exists an anti-path from \( b' \) to \( a' \) and applying the same arguments to \( a' \) and \( b' \) we can assume that \( A \) and \( B \) are both independent sets of \( D^{M_s} \). As \( C \) has then to alternate between \( A \) and \( B \), we have \( |A| = |B| = p/2 \), and in particular \( p \) is even.

Now, let \( B' \) be the set of vertices in \( V(C') \) which have an anti-out-neighbor in \( B \). More formally, we define \( B' = \{ c' \in V(C') : \exists b \in B \text{ such that } c'b \text{ is an anti-arc} \} \). For any \( b \in B \) we have \( \overline{d^+(b)} = k - p/2 \) implying that \( |B'| \geq k - p/2 \). Similarly, we define \( A' = \{ c' \in C' : \exists a \in A, ac' \text{ is an anti-arc} \} \). We have the analogous result \( |A'| \geq k - p/2 \).
Assume first that there is an arc $b'z'$ of $C'$ with $b' \in B'$ and $z'$ having an anti-in-neighbor $z$ in $C$. Then $b'$ has an anti-out-neighbor $b$ in $C$ and there exists an anti-path $P$ from $b$ to $z$ in $D^{M_z}[C]$. Moreover, as $D^{M_z}[C']$ is strongly connected, there exists also an anti-path $Q$ from $z'$ to $b'$ in $D^{M_z}[C]$ and we can conclude with Claim 3.1 applied on the anti-cycle $P \cup Q$.

Now we can assume that every arc $b'z'$ of $C'$ with $b' \in B'$ satisfies $\overline{N}(z') \cap V(C) = \emptyset$. Symmetrically, we can assume that every arc $z'a'$ of $C'$ with $a' \in A'$ satisfies $\overline{N}^{-1}(z') \cap V(C) = \emptyset$.

But now, we will obtain the contradiction that $(C_1, C_2)$ cannot satisfy property $P_{\text{up}}$. Indeed, in particular, we have $A' \cap C'(B') = \emptyset$ and as $|A'| \geq k-p/2$ and $|C'(B')| = |B'| \geq k-p/2$ we obtain that $(A', C'(B'))$ is a partition of $V(C')$. As $(C_1, C_2)$ satisfies property $P_{\text{up}}$, the cycle $C'$ should contain two vertices $v$ and $u$ such that $v$ is the successor of $u$ along $C'$, and $u$ has an anti-out-neighbor in $C$ and $v$ has an anti-in-neighbor in $C$. By the previous arguments, we must have $u \notin B'$ and $v \notin A'$, a contradiction to the fact that $(A', C'(B'))$ is a partition of $V(C')$.

**Case 2.2:** For every $a \in A$ and $b \in B$, there are no anti-paths from $b$ to $a$. Thus, the set $A$ dominates the set $B$ and $B$ dominates $A$. Suppose without loss of generality that $|B| \leq |A|$ and let $b$ be a vertex in $B$. If $\overline{d}_{C'}^{-}(b) + \overline{d}_{C'}^{+}(b) > 2k - p$, then we can find two vertices $v$ and $u$ in $C'$ such that $ab$ and $bv$ are anti-arcs, and $v$ is the successor of $u$ along $C'$. In that case we conclude with Claim 3.1 considering the anti-cycle $P \cup b$ where $P$ is an anti-path from $v$ to $u$ in $D^{M_z}[C']$ (which exists as $D^{M_z}[C']$ is strongly connected). Therefore, we can assume that for every $b \in B$, $\overline{d}_{C'}^{+}(b) + \overline{d}_{C'}^{-}(b) \leq 2k - p$. Thus, we have

$$\overline{d}_{B}^{+}(b) + \overline{d}_{B}^{-}(b) = (\overline{d}_{C'}^{+}(b) + \overline{d}_{C'}^{-}(b)) - (\overline{d}_{C'}^{+}(b) + \overline{d}_{C'}^{-}(b)) \geq (2k - 2) - (2k - p) \geq p - 2$$

Finally, since $\overline{d}_{B}^{+}(b) + \overline{d}_{B}^{-}(b) \leq 2|B| - 2 \leq |A| + |B| - 2 = p - 2$, we have equalities everywhere in the previous computation. In particular, we have $|A| = |B| = p/2$ and $p$ is even. Moreover, for every $b \in B$ we have $\overline{d}_{B}^{+}(b) = \overline{d}_{B}^{-}(b) = p/2 - 1$, implying that $B$ is an independent set. Symmetrically for $A$, as $|A| = |B|$ either we can conclude as previously with Claim 3.1 or $A$ is also an independent set. In this latter case, $C$ would induce a complete bipartite graph in $D^{M_z}$ and $D[C_1]$ would be isomorphic to $F_{2k-p}$, a contradiction to our induction hypothesis.

To conclude this section, notice that if $D^{M_z}[C_{2-M_z}]$ has exactly one terminal strong component, then we can conclude similarly. Indeed, we have seen that if $D^{M_z}[C_{2-M_z}]$ is strongly connected, then $D$ admits a good cycle-factor, and if $D^{M_z}[C_{2-M_z}]$ is not strong, then there exists an arc of $C_{2-M_z}$ leaving its unique terminal component and we can directly conclude with Claim 4.1.

**4.4 Case B:** $D[C_2]$ is isomorphic to $F_{2k-p}$.

As $D[C_1]$ is not isomorphic to $F_p$, by Claim 4.2 we can assume without loss of generality that $D^{M_z}[C_{M_z}]$ has exactly one strong initial component. As usual, we denote $C_{1-M_z}$ by $C$ and $C_{2-M_z}$ by $C'$. As $D[C_2]$ is isomorphic to $F_{2k-p}$, $p$ is even and the digraph $D^{M_z}[C']$ is the complete bipartite digraph $K_{(k-p/2,k-p/2)}$. We denote by $(A, B)$ its bipartition.

**Claim 4.4.** If there exists an arc $ab$ of $C$ such that there is an anti-path in $C$ from $b$ to $a$, an anti-arc from $a$ to $A$ and an anti-arc from $A$ to $b$, then $D$ admits a good cycle-factor.

**Proof.** First denote by $a'$ the end of an anti-arc from $a$ to $A$ and by $b'$ the beginning of an anti-arc from $A$ to $b$. We can assume that there exists a vertex $c$ of $B$ such that $b'ca'$ is a subpath of $C'$. Indeed, as $D^{M_z}[C']$ is isomorphic to a complete bipartite digraph, there exists a hamiltonian cycle
Theorem 2 in $D$ of $A$ on $k$ cycles of $C$ containing all the vertices of $b$. We denote by $D'$ the new contracted matching in $D$. By Lemma 2 and Corollary 3 it is easy to see that we obtain a new cycle-factor $C$ of $D^{M_u}$ containing the 3-cycle $C_3 = a'bca$, a cycle $C_s$ containing all the vertices of $C' \setminus \{a', c\}$ and other cycles included into $C$. Notice that $D^{M_u}[C_s]$ is a bipartite complete digraph on $2k - p - 2$ vertices and we can replace it with two cycles : $C_p'$ on $p - 2$ vertices and $C_2'$ on $2k - 2p$ vertices (with $2k - 2p \geq 2$). Now, as $C_3$ contains a vertex of $A$ and $C_2'$ a vertex of $B$, the union of these two cycles is strongly connected in $D^{M_u}$. So using Theorem 2 in $D^{M_u}$ there exists a cycle of length $p + 1$ spanning $C_p' \cup C_3$. Using this cycle and the cycles of $(C \setminus \{C_3, C_s\}) \cup C_2'$, by Lemma 1 we form a 2-cycle-factor of $D^{M_u}$ with one being of length $p + 1$. In particular, as $p + 1$ is odd, this cycle-factor of $D^{M_u}$ corresponds to a good cycle-factor of $D$.

As $A$ and $B$ have a symmetric role, Claim 4.3 still holds by replacing $A$ by $B$ in its statement.

**Case 1:** $D^{M_u}[C]$ is strongly connected In this case, let $v$ be a vertex of $C$ and $u$ its predecessor along $C$. As $|C| = p < k$, the vertex $u$ is the beginning of an anti-arc which ends in $C'$, that is in $A$ or $B$. So, if we cannot conclude with Claim 4.3 it means that $v$ is dominated by $A$ or $B$. As this is true for every vertex $v$, by a direct counting argument, there exist a set $X_A$ of $p/2$ vertices of $C$ which are dominated by $A$ and a set $X_B$ of $p/2$ vertices which are dominated by $B$. Moreover, no vertex $x$ of $A$ dominates a vertex of $X_B$ (otherwise we would have $d_{M_u}(x) > k$) and no vertex of $B$ dominates a vertex of $X_A$. So, for every vertex $v$ of $C$ we have $d_C^{-}(v) = k - (k - p/2) = p/2$.

Using the same argument between a vertex $v$ and its successor along $C$ we obtain that every vertex $v$ of $C$ satisfies $d_C^{-}(v) = p/2$. It means that in $D$, the bipartite tournament $D[C_1]$ contains $2p$ vertices, is not isomorphic to $F_2p$, and satisfies $d^+(u) = d^-(u) = p$ for each of its vertex $u$.

Now, assume first that $p \geq 5$. By induction, provided that $p \geq 5$, the bipartite tournament $D[C_1]$ has at least 10 vertices and admits a 2-cycle-factor $(C_{ind}, C'_{ind})$ with $C_{ind}$ being of length 6. In $D^{M_u}$, we have a 2-cycle-factor $(F_{ind}, F'_{ind})$ with $F_{ind}$ being of length 3. Let $xy$ be an arc of $F_{ind}$. As $d_{C'}(x) \geq k - p > 0$ and $d_{C'}(y) \geq k - p > 0$, there exist $x'$ and $y'$ in $C'$ such that $xx'$ and $yy'$ are arcs of $D^{M_u}$. Now, $D^{M_u}[C']$ being a complete bipartite digraph on $2k - p$ vertices, it admits a 2-cycle-factor $(C_s, C'_s)$ such that $C_s$ contains $x'$ and $y'$ and is of length $p - 2$. So, using Lemma 4 as $D^{M_u}[C_s \cup F_{ind}]$ is strongly connected, $D$ admits a good cycle-factor.

To conclude, assume that $p = 4$. As every vertex $x$ of $C$ satisfies $d_C^+(x) = d_C^-(x) = 2$ and $D^{M_u}[C]$ is strongly connected, it is easy to see that the anti-arcs of $C$ form a cycle of length 4. If
Claim 4.5. For all vertices of \( Y \) from \( x \), otherwise, let \( B \) from \( Y \) denote \( C \). Besides, if \((x',x'',y',y'')\) such that the in- and out-neighborhood of \( a \) and \( a' \) in \( Y \) are exactly \( B \) and \( Y \) in \( C' \) are exactly \( B \). Moreover, as every pair of vertices are linked by at least one arc in \( D_{M^a}[C] \), we can assume that \( aa' \) and \( bb' \) are arcs of \( D_{M^a} \). Then, as \( k > p \), we have \( 2k - p \geq 6 \) and we can select three vertices \( a_1, a_2, a_3 \) in \( A \) and three vertices \( b_1, b_2, b_3 \) in \( B \). Then, we form a good cycle-factor by applying Lemma 1 to the cycles \( aa'a_1b_2b_3 \) and \( bb'b_1a_2a_3 \) of length \( p + 1 = 5 \) and a cycle covering \( C' \setminus \{a_1, a_2, a_3, b_1, b_2, b_3\} \).

Case 2: \( D_{M^a}[C] \) is not strongly connected. However, as \((C_1,C_2)\) satisfies \( P_{up} \), we know that \( D_{M^a}[C] \) contains only one strong initial component, denoted by \( Y_1 \). Let \( Y_2 = V(C) \setminus Y_1 \). In particular, all the arcs from \( Y_2 \) to \( Y_1 \) exist in \( D_{M^a} \). Moreover, there exists an arc \( xy \) of \( C \) with \( x \in Y_2 \) and \( y \in Y_1 \). As \( Y_1 \) is the only initial component of \( D_{M^a}[C] \), there exists an anti-path from \( y \) to \( x \) in \( D_{M^a}[C] \). As \( |C| = p < k \), there exists a vertex \( x' \) in \( C' \) such that \( xx' \) is an anti-arc of \( D_{M^a} \). Without loss of generality, we can assume that \( x' \in A \). By Claim 4.4, if \( y \) has an anti-in-neighbor in \( A \), then \( D \) admits a good cycle-factor. So we can assume that \( A \) dominates \( y \). Similarly, \( y \) has an anti-in-neighbor in \( C' \) which must be in \( B \) and if we cannot conclude with Claim 4.4, it means that \( x \) dominates \( B \). As \( x \) dominates also \( Y_1 \) and \( y \) is dominated by \( Y_2 \), we have \( |Y_1| = |Y_2| = p/2 \) and the out-neighborhood of \( x \) is exactly \( B \cup S_1 \) and the in-neighborhood of \( y \) is exactly \( A \cup Y_2 \). Now, assume that \( z \) is the successor of \( y \) along \( C \) is in \( Y_1 \). Notice that \( zy \) is an anti-arc of \( D_{M^a} \). As \( Y_2 \) and \( y \) dominate \( z \), it has at least one anti-in-neighbor in \( A \) and at least one anti-in-neighbor in \( B \) (otherwise, we would have \( d^-(z) \geq k - p/2 + p/2 + 1 = k + 1 \)). As \( y \) has an anti-out-neighbor in \( C' \), wherever it is, in \( A \) or \( B \), we can conclude with Claim 4.4. Otherwise, it means that \( z \) is in \( Y_2 \). Symmetrically, the predecessor of \( x \) along \( C \) is in \( Y_1 \). Repeating the argument, we conclude that \( C \) alternates between \( Y_1 \) and \( Y_2 \). Indeed \( C \) cannot induce a path of positive length in \( Y_1 \) for instance: the first vertex of such a path would be the end of an arc from \( Y_2 \) to \( Y_1 \) and the second vertex of the path would lie in \( Y_1 \) then. Also the conclusions we had for \( x \) and \( y \) respectively hold for all vertices of \( Y_1 \) and \( Y_2 \). So, we have \( Y_1 \) is dominated by \( A \) and \( Y_2 \) and is an independent set of \( D_{M^a} \), and \( Y_2 \) dominates \( B \) and \( Y_1 \) and is also an independent set of \( D_{M^a} \). We deduce that the in-neighborhood of \( B \) is exactly \( A \cup Y_2 \) and then the out-neighborhood of \( Y_1 \) is \( A \cup Y_2 \) also. In particular, \( Y_1 \) dominates \( Y_2 \) and there is no anti-path from \( Y_1 \) to \( Y_2 \), providing a contradiction, as \( Y_1 \) is the only initial strong component of \( D_{M^a}[C] \).

4.5 Case C: \((C_1,C_2)\) satisfies \( Q_{down} \)

As we assume that we are not in Case A, the digraph \( D_{M^a}[C]_{K^2_{M^a}} \) has at least two initial strong components, and at least two terminal strong components (as noticed at the end of Case A).

Besides, if \((C_1,C_2)\) satisfies \( P_{down} \), then by exchanging the role of \( S \) and \( T \), we are in the symmetrical case of Case A. Then, we can assume that \((C_1,C_2)\) does not satisfy \( P_{down} \). Once again, we denote \( C_{M^a} \) by \( C \) and \( C_{M^b} \) by \( C' \). So, by Claim 4.3, for every vertex \( x \) of \( C' \) either there is no arc from \( x \) to \( C \) or there is no arc from \( C \) to \( x \).

Before concluding the proof of Theorem 3 we need the two following claims.

Claim 4.5. If \( D_{M^a}[C] \) contains a vertex \( x \) such that there is no arc between \( x \) and \( C \), then \( D \) admits a good cycle-factor.

Proof. Otherwise, let \( x \) be such a vertex and call \( x,y,z,t \) the subpath of \( C' \) of length 3 starting from \( x \). We know that either there is no arc from \( C \) to \( t \) or no arc from \( t \) to \( C \). Assume that the
latter holds, the other case could be treated symmetrically. If there also exists an anti-arc from \( C \) to \( t \), then we can find three vertices \( a, b \) and \( c \) in \( C \) such that \( a, b, c \) is a subpath of \( C \) of length 2 and that \( \{a, x\} \) and \( \{c, t\} \) are independent sets of \( D^{M_a} \). So, we will exchange some small paths between \( C \) and \( C' \).

First, notice that \( p > 3 \). Indeed, if we denote by \( A_C \) (resp. \( B_C \)) the set of vertices of \( C' \) which anti-dominate \( C \) (resp. are anti-dominating by \( C \)), we have \( V(C') = A_C \cup B_C, x \in A_C \cap B_C \) and then \( |A_C| + |B_C| \geq |A_C \cup B_C| + 1 = |C'| + 1 = 2k - p + 1 \geq 2k - 2 \) if \( p \leq 3 \). So, as \( |A_C| \leq k - 1 \) and \( |B_C| \leq k - 1 \), we have \( |A_C| = |B_C| = k - 1 \) and \( A_C \cap B_C = \{x\} \). Moreover, we have \( t \in A_C \setminus \{x\} \) and so \( \overline{N}(c) \) contains \( B_C \cup \{t\} \) of size \( k \), a contradiction.

Now, to perform the path exchange, let us depict the situation in \( D \): \( C_1 \) contains the path \( ac_1(\{a\})bc_1(\{b\})c_1(\{c\}) \) and \( C_2 \) contains the path \( C_2(x)yC_2(y)zC_2(z)tC_2(t) \). Moreover, in \( D \), \( x \) dominates \( V(C_1) \cap T \), \( C_2(x) \) is dominated by \( V(C_1) \cap S \), there is an arc from \( x \) to \( C_2(t) \) and an arc from \( t \) to \( C_1 \). So, we replace in \( C_1 \) the path \( aC_1(\{a\})bc_1(\{b\})c_1(\{c\}) \) by \( aC_2(x)yC_2(y)zC_2(z)tC_1(c) \) to obtain the cycle \( \tilde{C}_1 \) and we replace in \( C_2 \) the path \( xyC_2(x)yC_2(z)C_2(t) \) by \( xC_1(a)bc_1(b)c_2(t) \) to obtain the cycle \( \tilde{C}_2 \). The cycles \( \tilde{C}_1, \tilde{C}_2 \) form a \((2(p+1), 4k-2(p+1))-\)cycle-factor of \( D \). Moreover, \( \tilde{C}_1 \) is not isomorphic to \( \tilde{C}_2 \). Indeed, as \( p > 3 \), the cycle \( \tilde{C}_1 \) contains the predecessor \( u \) of \( a \) along \( C_1 \) (which is not \( C_1(c) \) then). Let call \( d \) the vertex of \( C_1 \) with \( C_1(d) = u \). As \( C_2(x) \) is dominated by \( V(C_1) \cap S \) in \( D \), there is an arc from \( d \) to \( C_2(x) \). Thus, \( \tilde{C}_1 \) is not isomorphic to \( F_{2(p+1)} \), as it contains the path \( dC_1(d)aC_2(x) \) and the arc \( dC_2(x) \) while \( F_{2(p+1)} \) does not contain such a sub-structure.

So, we can assume that \( C \) dominates \( t \). Call by \( Y \) the strong component of \( \overline{D^{M_a}[C']} \) containing \( t \). There exists \( Y_{term} \) a terminal strong component of \( \overline{D^{M_a}[C']} \) distinct from \( Y \). Let \( Y' \) be the union of the strong components of \( \overline{D^{M_a}[C']} \) different from \( Y_{term} \) and \( Y \). As \( Y_{term} \) is a terminal strong component of \( \overline{D^{M_a}[C']} \), for any vertex \( u \) of \( Y_{term} \), we have \( k - 1 \leq d_{\overline{C'}}(u) \leq |Y_{term}| - 1 \) and then \( |Y_{term}| \geq k - p \) (noticed that a symmetrical reasoning holds for initial strong components also). Moreover, as \( C \) and \( Y_{term} \) dominate \( t \) we must have exactly \( |Y_{term}| = k - p \) and the in-neighborhood of \( t \) is exactly \( C \cup Y_{term} \). In particular, \( z \) belongs to \( Y_{term} \) and \( Y \) is the other terminal strong component of \( \overline{D^{M_a}[C']} \). Moreover, as \( Y' \cup Y \) has size \( k \) and is dominated by \( Y_{term} \), then \( Y' \cup Y \) is exactly the out-neighborhood of each vertex of \( Y_{term} \). Thus there is no arc from \( Y_{term} \) to \( C \). We look at two cases to conclude the proof.

First assume that \( C \) dominates \( z \). As \( z \) is dominated by \( Y \) (recall that \( Y \) is a terminal strong component of \( \overline{D^{M_a}[C']} \)) and \( Y \) has size at least \( k - p \), then we have that \( Y \cup C \) is exactly the in-neighborhood of \( z \) and that \( |Y'| = k - p \). Finally, \( Y' \) is non empty (of size \( 2k - 2p \)) and there is no arc from \( Y' \) to \( \{z, t\} \). To conclude, let \( w \) be an arc of \( C' \) with \( u \in Y_{term} \cup Y \) and \( v \in Y' \) (such an arc exists as \( Y' \neq \emptyset \)). By the previous arguments there exists an anti-path from \( v \) to \( u \) in \( D^{M_a}[C'] \) and by Claim 4.1, we conclude that \( D \) admits a good cycle-factor.

Now, if there is an anti-arc from \( C \) to \( z \), then we will conclude as previously using a small path exchange between \( C \) and \( C' \). Indeed, if such an anti-arc exists between a vertex \( b' \) of \( C \) and \( z \), call \( a' \) the predecessor of \( b' \) of \( C \) and \( z \). So, \( \{a', x\} \) and \( \{b', z\} \) are independent sets of \( D^{M_a} \). Then, in \( D \) we exchange the paths \( a'C(a')b'C(b') \) of \( C_1 \) with the path \( a'C(x)yC'(y)zC'(z) \) to obtain the cycle \( \tilde{C}_1 \). Similarly, we exchange the path \( xC(x)yC'(y)zC'(z) \) of \( C_2 \) with the path \( xC(a')b'C(z) \) to obtain the cycle \( \tilde{C}_2 \). Then, \( (\tilde{C}_1, \tilde{C}_2) \) forms a good cycle-factor of \( D \), as in particular, denoting by \( c' \) the predecessor of \( a' \) along \( C \), \( \tilde{C}_1 \) contains in \( D \) the path \( c'C(c')a'C(x) \) and the arc \( c'C(z) \) and so \( \tilde{C}_1 \) is not isomorphic to \( F_{2(p+1)} \) which does not contain such a sub-structure.

The last claim will show that every arc of \( C' \) is contained in a digon.

**Claim 4.6.** If \( D^{M_a}[C'] \) contains an arc \( xy \) such \( yx \) is not an arc of \( D^{M_a} \), then \( D \) admits a good cycle-factor.
Proof. Assume that \( xy \) is an arc of \( C' \) such that \( yx \) is an anti-arc of \( D^{M_u}[C'] \). If \( x \) and \( y \) are not in the same strong component of \( D^{M_u}[C'] \), then we conclude with Claim 4.1. Otherwise, we assume that \( x \) and \( y \) lie in the same strong component \( Y \) of \( D^{M_u}[C'] \). As \( D^{M_u}[C'] \) has at least two initial and two terminal strong components, there exist an initial strong component \( Y_{\text{init}} \) and a terminal strong component \( Y_{\text{term}} \) of \( D^{M_u}[C'] \) which are different from \( Y \) (with possibly \( Y_{\text{init}} \) = \( Y_{\text{term}} \)). As previously noticed, we have \( |Y_{\text{init}}| \geq k - p \) and \( |Y_{\text{term}}| \geq k - p \). And as \( Y_{\text{init}} \) and \( Y_{\text{term}} \) are different from \( Y \) then, \( Y \) dominates \( Y_{\text{init}} \) and is dominated by \( Y_{\text{term}} \). In particular, as \( x \) and \( y \) lie in \( Y \) and \( xy \) is an arc of \( C' \), \( x \) has at least an anti-out-neighbor \( x' \) in \( C' \) and \( y \) has at least an anti-in-neighbor \( y' \) in \( C \) (otherwise, considering \( Y_{\text{init}} \) or \( Y_{\text{term}} \), we would have \( d^{+}_{D^{M_u}}(x) \geq k + 1 \) or \( d^{-}_{D^{M_u}}(y) \geq k + 1 \)).

If \( x' y \) or \( y' x \) is not an arc of \( D^{M_u} \) then we conclude with Claim 3.1 using an anti-path from \( y \) to \( x \) in \( Y \). So we assume that \( x' y \) and \( y' x \) are arcs of \( D^{M_u} \). Similarly, if we have \( d^{+}_{C}(x) + d^{+}_{C}(y) > p \) then there exists a vertex \( u \in C \) such that \( xu \) and \( uy \) are anti-arcs of \( D^{M_u} \) and we conclude with Claim 3.1. Thus we assume that we have \( d^{+}_{C}(x) + d^{+}_{C}(y) \leq p \) and then that \( d^{-}_{C}(x) + d^{-}_{C}(y) \geq 2k - p \). If there is no vertex \( z \in V(C') \) such that \( yz \) is an anti-cycle, then we can partition \( V(C') \setminus \{x,y\} \) into two sets \( X \) and \( Y \) such that \( x \) anti-dominates \( X \) and \( y \) anti-dominates \( Y \). Call \( Y_X \) the set of strongly connected components of \( D^{M_u}[C'] \) which are included into \( X \). Notice that \( Y_X \) is not empty as it contains all the terminal strong component of \( D^{M_u}[C'] \), but does not contain any initial strong component of \( D^{M_u}[C'] \). Now, consider any arc \( uv \) of \( C' \) going from a component \( Y_1 \) of \( Y_X \) to a component \( Y_2 \) not belonging to \( Y_X \). As \( Y_2 \) contains a vertex of \( \{x,y\} \cup Y \), there exists an anti-path from \( v \) to \( x \). And as \( Y_1 \) belongs to \( Y_X \), there is an anti-arc from \( x \) to \( u \). So using Claim 4.1 we can conclude that \( D \) admits a good cycle-factor.

Thus we thus that there exists a vertex \( z \in V(C') \) such that \( yxz \) is an anti-cycle of \( D^{M_u} \). By Claim 4.5 we can assume that there exists an arc \( zz' \) from \( z \) to \( C \). We will perform a switch exchange along the anti-cycle \( yzz' \) and show that the 2-cycle-factor that we obtain will satisfy \( P_{\text{down}} \) and \( Q_{\text{down}} \).

Denote by \( M'_u \) the perfect matching of \( D \) obtain from \( M_u \) by switch exchange along \( yz \) (that is \( M'_u = (M_u \setminus \{zM_u(x), yM_u(y), zM_u(z)\}) \cup \{zM_u(y), yM_u(z), zM_u(x)\} \)). So, when performing the switch exchange along \( yz \), by Lemma 2 we obtain \( N^+_{D^{M'_u}}(x) = N^+_{D^{M_u}}(y), N^+_{D^{M'_u}}(y) = N^+_{D^{M_u}}(z) \) and \( N^+_{D^{M'_u}}(z) = N^+_{D^{M_u}}(x) \). In \( D^{M_{u'}} \), call by \( P_1 \) the sub-path of \( C' \) going from the successor of \( y \) (along \( C' \)) to the predecessor of \( z \) and by \( P_2 \) the sub-path of \( C' \) going from the successor of \( z \) to the predecessor of \( x \). Then, after the switch exchange, the cycle \( C' \) becomes in \( D^{M_{u'}} \) the cycle \( C'' = xP_1zP_2yP_1 \). We denote by \( C'_2 \) its corresponding cycle in \( D \). Notice that the strong components of \( D^{M_u}[C'] \) are the same than the ones of \( D^{M_{u'}}[C'] \). Indeed, as the anti-cycle \( yzz' \) becomes the anti-cycle \( yxz \) in \( D^{M_{u'}} \), the permutation of the anti-out-neighborhoods of \( x \), \( y \) and \( z \) does not affect the strong components of \( D^{M_{u'}}[C'] \) and their relationships. So, \( (C_1, C'_2) \) still satisfies \( Q_{\text{down}} \).

Finally, remind that in \( D^{M_{u'}} \), the vertex \( z \) has an out-neighbor \( z' \) in \( C \) and \( y \) an in-neighbor \( y' \) in \( C \). As we have \( N^+_{D^{M_{u'}}}(y) = N^+_{D^{M_{u'}}}(z) \), the arcs \( y'y \) and \( yz' \) belong to \( D^{M_{u'}} \). Thus, \( (C_1, C'_2) \) now satisfies \( P_{\text{down}} \) and we can conclude with the symmetrical case of Case A.

Finally we can assume that every arc of \( C' \) is in a digon. We write \( C' = u_1, \ldots, u_l \) with \( l = 2k - p \). The indices of vertices of \( C' \) will be given modulo \( l \). Then we consider two cases:

**Case 1:** \( p \) is odd. Then \( l = 2k - p \) is also odd. As \( D^{M_u}[C'] \) is not strongly connected, there exists \( i \in \{1, \ldots, l\} \) such that \( u_{i}u_{i-2} \) is an arc of \( D^{M_u} \). Without loss of generality, we can assume that \( u_{i}u_{i-2} \) is an arc of \( D^{M_u} \). We consider then the set \( X = \{u_1, u_5, u_7, \ldots, u_{l-p+1}\} \), that is all the vertices \( u_i \) with odd \( i \) between 1 and \( l - p + 1 \), except \( u_3 \). Notice that \( X \) has size \((l-p+1-1)/2-1 = k-p \). If there is no arc between \( X \) and \( u_3 \), as \( u_3 \) has no arc to \( C \) or no arc from \( C \), we would have \( d^{-}_{D^{M_u}}(u_3) \geq k - 1 \) or \( d^{-}_{D^{M_u}}(u_3) \geq k \), a contradiction. So, there exists an arc between
$u_3$ and some $u_i \in X$. If $i = 1$, then we consider the cycle-factor $C'$ on $2k - p - 1$ vertices containing $C$ and the cycles with vertex sets $\{u_1, u_2, u_3\}, \{u_4, u_5, \ldots, u_{i-2}, u_{i-1}\}$ and the cycle-factor $C$ on $p + 1$ vertices containing the cycles with vertex set $\{u_{i-p}, u_{i-p+1}\}, \{u_{i-p+2}, u_{i-p+3}\}, \ldots, u_{i-1}, u_i$. Notice that $D^{M_c}[u_{i-p}, \ldots, u_i]$ is strongly connected and is not a complete bipartite graph, as it contains the cycle $u_{i-2}, u_{i-1}, u_i$. So, by Lemma 3, the digraph $D$ admits a good cycle-factor.

Now, if $i = u_{i-p+1}$, then we consider the cycle-factor $C$ on $p + 1$ vertices containing the cycles with vertex sets $\{u_{i-p+2}, u_{i-p+3}\}, \ldots, \{u_{i-1}, u_i\}$, and the cycle-factor $C'$ on $2k - p - 1$ vertices containing $C$ and only the cycle with vertex set $\{u_3, u_4, \ldots, u_{i-p}, u_{i-p+1}\}$. We conclude as previously. Finally, if $i \in \{5, 7, \ldots, l-p-1\}$, then we choose $C$ to be the cycle-factor on $p + 1$ vertices containing the cycles with vertex set $\{u_{i-p}, u_{i-p+1}\}, \{u_{i-p+2}, u_{i-p+3}\}, \ldots, \{u_{i-1}, u_i\}$ and $C'$ the one containing the cycles with vertex set $\{u_1, u_2\}, \{u_3, u_4, \ldots, u_i\}, \{u_{i+1}, u_{i+2}\}, \ldots, \{u_{i-p-2}, u_{i-p-1}\}$. Once again, we conclude as previously.

Case 2: $p$ is even. If there exists an arc between two vertices at distance 2 along $C'$, then we will proceed almost as in the case where $p$ is odd. The difference here, is that $C$ will contain cycles all of length 2 except one of length 3. Indeed, assume for instance that $u_{i-1}u_2$ is an arc of $D^{M_c}$. We consider once again the set $X = \{u_1, u_5, u_7, \ldots, u_{i-p+1}\}$ and show, as previously, that there exists an arc between a vertex $u_i$ of $X$ and $u_3$. If $i = 1$, then we consider the cycle-factor $C'$ on $2k - p - 1$ vertices containing $C$ and the cycles with vertex sets $\{u_1, u_2, u_3\}, \{u_4, u_5, \ldots, u_{i-2}, u_{i-1}\}$ and the cycle-factor $C$ on $p + 1$ vertices containing the cycles with vertex set $\{u_{i-p}, u_{i-p+1}\}, \{u_{i-p+2}, u_{i-p+3}\}, \ldots, \{u_{i-4}, u_{i-3}\}, \{u_{i-2}, u_{i-1}, u_i\}$. Once again, we conclude with Lemma 3 that $D$ admits a good cycle-factor. The cases where $i \in \{5, 7, \ldots, l-p-1\}$ are similar to the corresponding cases where $p$ is odd.

Finally, assume that there is no arc between two vertices at distance 2 along $C'$. Then, we denote by $A$ the vertices $u_i$ with odd indices and by $B$ the vertices $u_i$ with even indices. The sets $A$ and $B$ form two strong connected components of $D^{M_c}[C]$ and so their are both initial and terminal strong components of $D^{M_c}[C]$. In particular, $D^{M_c}$ contains all the arcs from $A$ to $B$ and all the arcs from $B$ to $A$. The vertex $u_1$ must have a neighbor in $A$. Indeed, otherwise, as there is no arc from $u_1$ to $C$ or from $C$ to $u_1$, we will conclude that $d^{D^{M_c}}(u_1) \geq k$ or $d^{D^{M_c}}(u_1) \geq k$, which is not possible. So, assume that $u_1$ is adjacent to a vertex $u_i$ of $A$. Then, instead $C'$ we consider the cycle $C'' = u_1, u_2, u_3, u_4, \ldots, u_{i-1}, u_{i+1}, \ldots, u_i$. As there exist all the possible arcs between $A$ and $B$, then all the arcs of $C''$ are contained in a digon and there exists an arc between two vertices at distance 2 along $C''$ (the arc $u_1u_i$). Thus, we are in the previous case and we conclude that $D^{M_c}$ admits a good anti-cycle.

5 Concluding remarks

We finish this paper with some conjectures about the problem of cycle-factor in bipartite or multipartite tournaments. First of all, we have to mention the two related conjectures appearing in the original paper of Zhang and al. [15]. The first one adds a new hypothesis imposing an arc in the 2-cycle-factor.

Conjecture 5 (Zhang and al. [15]). Let $D$ be a $k$-regular bipartite tournament, with $k$ an integer greater than 2. Let $uv$ be any specified arc of $D$. If $D$ is isomorphic neither to $F_{4k}$ nor to some other specified families of digraphs, then for every even $p$ with $4 \leq p \leq |V(D)| - 4$, $D$ has a cycle $C$ of length $p$ such that $D \setminus C$ is hamiltonian and such that $C$ goes through the arc $uv$.

The second conjecture, conversely, imposes that the cycles contain specific vertices.
Conjecture 6 (Zhang and al. [15]). Let \( D \) be a \( k \)-regular bipartite tournament, with \( k \) an integer greater than 2. Let \( u \) and \( v \) be two specified vertices of \( D \). If \( D \) is isomorphic neither to \( F_{2k} \) nor to some other specified families of digraphs, then for every even \( p \) with \( 4 \leq p \leq |V(D)| - 4, D \) has a cycle \( C \) of length \( p \) such that \( D \setminus C \) is hamiltonian and such that \( C \) contains \( u \) and \( D \setminus C \) contains \( v \).

Throughout the proof of Theorem 1 we intensively used the regularity of the bipartite tournament. It seems that we cannot get rid of this condition as we can easily find an infinite family of bipartite tournament with \( |d^+(u) - d^-(u)| \leq 1 \) for every vertex \( u \) and \( |d^+(u) - d^+(v)| \leq 1 \) for every pair of vertices \( \{u, v\} \), which does not contain any cycle-factor. For instance, for any \( k \geq 1 \) consider the bipartite tournament, inspired by \( F_{2k} \), consisting of four independent sets \( K, L, M \) and \( N \) with \( |K| = |N| = k \) and \( |L| = |M| = k + 1 \) with all possible arcs from \( K \) to \( L \), from \( L \) to \( M \), from \( M \) to \( N \) and from \( N \) to \( K \).

Let \( D \) be a \( c \)-partite tournament, and denote by \( I_1, \ldots, I_c \) its independent sets. We say that \( D \) is \( k \)-fully regular if, for any distinct \( i \) and \( j \) with \( 1 \leq i, j \leq c, D[I_i \cup I_j] \) is a \( k \)-regular bipartite tournament. In particular, all the sets \( I_i \) have size \( 2k \).

In [14], Yeo proved that if \( c \geq 5 \) then, in every \( c \)-partite regular tournament \( D \), every vertex is contained in a cycle of length \( l \) for \( l = 3, \ldots, |V(D)| \). He also conjectured the following.

Conjecture 7 (Yeo [14]). Every regular \( c \)-partite tournaments \( D \), with \( c \geq 5 \), contains a \( (p, |V(D)| - p) \)-cycle-factor for all \( p \in \{3, \ldots, |V(D)| - 3\} \).

An extension of our results and a weaker form of Yeo’s Conjecture could be the following.

Conjecture 8. Let \( D \) be a \( k \)-fully regular \( c \)-partite tournament with \( c \geq 5 \). Then for every even \( p \) with \( 4 \leq p \leq |V(D)| - 4, D \) has a \( (p, |V(D)| - p) \)-cycle-factor.

We can see that if \( c \) is even and there is at least one pair \( \{I_i, I_j\} \) such that \( D[I_i, I_j] \) is not isomorphic to \( F_{2k} \), then our result implies the Conjecture 8 by properly partitioning the sets \( I_i \) into two parts and applying Theorem 1 on the bipartite lying between the two parts. However, the case where \( c \) is odd seems more complicated to handle.

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