Atomic stability and the quantum mass equivalence

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Abstract

We consider an unexplored aspect of the mass equivalence principle in the quantum realm, its connection with atomic stability. We show that if the gravitational mass were different from the inertial one, a Hydrogen atom placed in a constant gravitational field would become unstable in the long term. In contrast, independently of the relation between the two masses, the atom does not become ionized in an uniformly accelerated frame. This work, in the line of previous analyses studying the properties of quantum systems in gravitational fields, contributes to the extension of that programme to internal variables.

Keywords: Mass equivalence principle; Atomic stability; Quantum systems in gravitational fields
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1 Introduction

The formulation of a quantum theory of the gravitational field remains as one of the main challenges in fundamental physics. Several authors have signaled the need of a deeper understanding of the relation between quantum mechanics and gravitation in order to correctly identify the physical and conceptual roots of the problem. In particular, we must explore at depth the role of the mass equivalence, the equality of inertial and gravitational masses, at the quantum level.

This exploration is part of a more general programme trying to understand the behavior of quantum systems in gravitational fields. In a first development, the influential Colella-Overhauser-Werner (COW) paper showed that terrestrial gravitation acts as an ordinary force at the level of non-relativistic quantum mechanics [1, 2]. Weak gravitational fields can be incorporated into Schrödinger’s equation via the usual prescription for the potential. In addition, the mass
equivalence holds in this experiment. Later, some authors analyzed other aspects of the problem. For instance, in relation with the universality of free fall, the quantum time of flight of a particle in a gravitational field has been computed in [3, 4].

In this work we explore an unnoticed aspect of the problem, the connection existent between atomic stability and the quantum formulation of the mass equivalence principle. Taking advantage of the similarities between our problem and the Stark effect we show that if the inertial and gravitational masses were different a Hydrogen atom placed in a constant gravitational field would become ionized in the long term. In other words, the atom in presence of the field does not have truly stationary solutions [5, 6, 7]. The solutions obtained by the perturbation method, which correctly describe the energy-level structure, really correspond to resonances with a finite lifetime. From a practical point of view these lifetimes are very long and we do not need to care about the effect. However, from a fundamental perspective, the absence of stationary states and their replacement by quasi-stationary ones is relevant. The atomic stability would be lost in the long term if the inertial and gravitational masses were different.

This is not the first time that the behavior of the Hydrogen atom in gravitational fields has been studied. Modifications of the energy-level structure have been evaluated in [8, 9, 10] and even the atom ionization by the action of the field has been described in [11]. However, up to our knowledge, these and other similar analyses have not explored potential connections with the equivalence principle.

In the second part of the paper we consider the Hydrogen atom from the point of view of an uniformly accelerated observer in order to analyze the strong equivalence principle in our problem. We show that the extended Galilean transformation [2, 12], representing the coordinates change to accelerated frames, does not lead to ionization processes even in the case of different inertial and gravitational masses.

Our paper can be viewed as a contribution to the programme initiated by COW, trying to include, in addition to the well understood effects of gravity on the wave properties of matter, some less explored aspects related to the interaction of a gravitational field with the internal variables. Several papers have considered some of these aspects. In [13, 14] the role of internal variables in studies of the equivalence principle has been addressed. How the gravitational interaction can lead to decoherence via the internal degrees of freedom has been studied in [15]. Finally, in relation with the compositeness of quantum systems, in [16] the author analyzed superpositions of stationary states of the Hydrogen atom that can violate the equivalence between active and passive gravitational masses. However, none of these papers has considered the question of the atomic stability.

The main conclusion of the paper, that the mass equivalence is a necessary condition for atomic stability in gravitational fields, shows that that the role
of the principle at the quantum level differs from that at the classical one. We shall consider the implications of this result in the Discussion.

2 The atom in a gravitational field

Let us consider a Hydrogen atom placed in a uniform gravitational field. We restrict our considerations to weak gravitational fields, as the terrestrial one, where according to the COW results we can use the standard form of Schrödinger’s equation. For our system it reads

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_e} \Delta x \psi - \frac{\hbar^2}{2m_p} \Delta y \psi + V \psi + \tilde{m}_e g \cdot x \psi + \tilde{m}_p g \cdot y \psi \]  

(1)

where \( x \) and \( y \) are respectively the electron and proton coordinates, \( m_e \) and \( m_p \) the inertial masses and \( \tilde{m}_e \) and \( \tilde{m}_p \) the gravitational ones. \( V = -e^2/|x - y| \) is the Coulomb potential and \( g \) corresponds to the gravitational field intensity.

In order to evaluate the structure of the atom we must separate the equation into its center of mass and internal parts. The CM and relative coordinates are given by \( R = (m_e x + m_p y)/M \) and \( r = x - y \), with \( M = m_e + m_p \). The Schrödinger equation becomes

\[ i\hbar \frac{\partial \psi_{CM}}{\partial t} = -\frac{\hbar^2}{2M} \Delta R \psi_{CM} + \tilde{M} g \cdot R \psi_{CM} \]  

(3)

and

\[ i\hbar \frac{\partial \psi_{rel}}{\partial t} = -\frac{\hbar^2}{2\mu} \Delta r \psi_{rel} + V \psi_{rel} - \frac{\tilde{m}_p m_e - \tilde{m}_e m_p}{\tilde{M}} g \cdot r \psi_{rel} \]  

(4)

with \( \mu = m_e m_p/M \) the reduced mass and \( \tilde{M} = \tilde{m}_e + \tilde{m}_p \).

The equation can be separated by introducing the coordinates change \( \psi(R, r, t) = \psi_{CM}(R, t) \psi_{rel}(r, t) \). We obtain the two expressions

(3)

and

(4)

When the mass equivalence holds, \( m_e = \tilde{m}_e \) and \( m_p = \tilde{m}_p \), we have \( \tilde{M} = M \) and \( \tilde{m}_p m_e = \tilde{m}_e m_p \) and we recover the usual equation for both the internal and CM variables. In contrast, when the masses are different there are non trivial effects. On the one hand, the CM fall of the atom in the field depends on \( \tilde{M} \), which differs from \( M \). On the other hand, and more important, the external field also affects to the internal dynamics.

These internal changes manifest in two different ways: modifying the energy-level structure and preventing the existence of truly stationary states. Although the first aspect is not fundamental for our purposes, we shall briefly discuss it in the next section by the sake of completeness.
3 Energy-level structure

The energy levels are obtained from the time-independent Schrödinger equation

\[ \hat{H}_\text{rel} \psi_{\text{rel}} = -\frac{\hbar^2}{2\mu} \Delta r \psi_{\text{rel}} + V \psi_{\text{rel}} - \mathbf{Mg} \cdot \mathbf{r} \psi_{\text{rel}} = E \psi_{\text{rel}} \]

(5)

where, by the matter of simplicity, we have introduced the notation \( MM = \bar{m}_p m_e - \bar{m}_e m_p \).

To solve this equation we note its resemblance to that describing the Stark effect. In both cases we have an atom placed in an external constant field. As in the Stark effect we resort to parabolic coordinates, \( \xi, \eta \) and \( \phi \), given by \[ 17 \]:

\[ x_1 = \sqrt{\xi \eta} \cos \phi, \quad x_2 = \sqrt{\xi \eta} \sin \phi \quad \text{and} \quad x_3 = (\xi - \eta)/2, \]

where \( x_3 \) is taken as the direction of the external field. As the external field is very small when compared to the Coulomb interaction even in strong gravitational fields, we can invoke a perturbative treatment where we can decompose the Hamiltonian in the form \( \hat{H} = \hat{H}_0 + \hat{H}_G \) with \( \hat{H}_0 = -\frac{\hbar^2}{2\mu} \Delta r + V \) and \( \hat{H}_G = -\mathbf{Mg} \cdot \mathbf{r} \).

The quantum numbers in parabolic coordinates for the unperturbed problem are denoted as \( n_1, n_2 \) (parabolic quantum numbers) and \( m \) (magnetic quantum number). They are related to the principal quantum number by the relation \( n = n_1 + n_2 + |m| + 1 \) \[ 17 \]. When the perturbation is taken into account at first order the energy becomes

\[ E(n,k) = E_0(n) - \frac{3Mgh}{2\mu\alpha c nk} \]

(6)

with \( E_0 \) the unperturbed energy, \( \alpha = e^2/(4\pi\varepsilon_0 \hbar c) \) the fine structure constant, \( \varepsilon_0 \) the free space permittivity, \( c \) the light speed and \( k = n_1 - n_2 \) that takes the values \( k = 0, \pm 1, \cdots, \pm (n - 1) \). Every level with \( n > 1 \) splits in \( 2n - 1 \) equally separated levels. This separation is proportional to \( nMg \) \[ 17 \].

The above partial suppression of the degeneracy provides a potential method to test the mass equivalence in the quantum realm. The observation of the above splitting in a gravitational field would be an unequivocal demonstration of the quantum inequality of both masses. In the absence of an observable splitting, the method would provide bounds on the possible values of the mass differences. However, a simple calculation shows that the separation between sublevels is much smaller than its linewidth.

4 Atomic stability

An interesting property of the Stark effect is that in presence of an external electric field the Hydrogen atom can become unstable. This property has been presented in the literature in two different forms. From a physical perspective the electron can tunnel the potential barrier ionizing the atom \[ 6 \]. From a more
mathematical point of view we say that the internal atomic Hamiltonian has no eigenvalues and the atomic states are no longer stationary ones \[5, 6, 7\]. Both approaches are complementary and lead to the same final result. The ionization of the atom is the physical consequence of the absence of stationary states.

A similar property holds for constant gravitational fields. By similitude with the Stark effect we have that the electron can tunnel through the potential barrier ionizing the atom. In other words, the time-independent Schrödinger’s equation (5) does not have solutions, that is, the operator \(H_0 + H_G\) has no eigenvalues \[5, 7\]. This contrasts with the behaviour of the system in absence of the external field, when the equation \(H_0\psi = E\psi\) has well-known solutions. More technically, the spectrum of \(H\) is continuous while that of \(H_0\) is discrete. The solutions to the problem are not truly bound but correspond to resonances. The perturbative solutions described in the previous section are good approximations to the resonances of the system. We can describe the resonance as a perturbation of a true bound state. The resonances decay after a very long but finite time to a state of the continuum.

For all practical purposes the time-decay of the resonances in our problem is very long, specially for the lower energy states. We can evaluate the corrections to the energy levels, the transition probabilities, ... in the usual way. However, this property has important consequences from a more fundamental perspective. We must resort to a quasi-stationary picture, where bound states are only a practical approximation. The atom is no longer stable. After a very long but finite time the atom will lose the electron.

We can estimate the order of magnitude of this time by invoking the lifetime of the Hydrogen atom in the ground state due to ionization by the Stark effect \[6\]. Replacing the electric field by the gravitational one we obtain

\[
\tau = \frac{Mgh^2}{4m^2c^2\alpha^3} \exp\left(\frac{m^2c^3\alpha^3}{Mgh}\right)
\]

In statistical terms this time is very long, but with a low probability we can observe events of this type for much shorter times.

We conclude that a violation of the mass equivalence would lead to an unstable behavior of atomic matter in gravitational fields. Events of this type would only occur with a very low probability, but they would exist.

5 Accelerated frames

In this section we move from gravitational fields to accelerated observers. It is well-known that the extended Galilean transformation of coordinates associated with uniform acceleration \[2, 12\], introduces a potential into Schrödinger’s equation that can be identified with a constant gravitational field. At least in this formal sense, the strong version of the equivalence principle (local equivalence of gravitation and acceleration) is preserved in the quantum realm.
First of all, we derive the Schrödinger equation in the accelerated frame. The coordinates change for this transformation is named the extended Galilean transformation [2, 12]. In the two-particle case it can be written as

\[ x = x' + Z(t), \quad y = y' + Z(t), \quad t = t' \]  

(8)

where \( x' \) and \( y' \) are the electron and proton coordinates in the new frame and \( Z(t) \) is any function of time describing a linear displacement. \( Z(t) \) can also be defined as the solution of the equation \( \frac{d^2 Z}{dt^2} = a \) with \( a \) the acceleration of the moving coordinate system. We can consider any arbitrary translational acceleration, but here we are only interested into constant ones. Note that every point in the frame is accelerating at the same rate, so we have a rigid coordinate system.

As in the one-particle case we express the accelerated wave function in the form \( \psi'(x', y', t') = \exp(i\Phi(x', y', t'))\varphi(x', y', t') \). With the choice \( \hbar \Phi(x', y', t') = -m_e \dot{Z} \cdot x' - m_p \dot{Z} \cdot y'(M/2)\dot{Z}^2t' \) the Schrödinger equation in the accelerated frame reads

\[ i\hbar \frac{\partial \varphi}{\partial t'} = -\frac{\hbar^2}{2m_e}\Delta_{x'}\varphi - \frac{\hbar^2}{2m_p}\Delta_{y'}\varphi + V\varphi - m_e \dot{Z} \cdot x'\varphi - m_p \dot{Z} \cdot y'\varphi \]  

(9)

Note that \( V = -e^2/|x' - y'| \), the Coulomb potential, is invariant under the extended transformation.

Two new terms appear with respect to the inertial equation. When the equivalence of inertial and gravitational masses holds, we can identify them with the potential describing the interaction between the particles and an uniform gravitational field. The direction and intensity of the field is given by \(-\dot{Z}\), that is, by the acceleration of the second frame. This result is many times presented as a quantum (non-relativistic) version of the strong equivalence principle: at the level of Schrödinger’s equation an uniformly accelerated frame is indistinguishable from a constant gravitational field.

Let us analyze what happens when the mass equivalence does not hold. The relation \( m_e \dot{Z} \cdot x' + m_p \dot{Z} \cdot y' = M\dot{Z} \cdot R' = (M/M)\bar{M}\dot{Z} \cdot R' \) shows that at variance with a real gravitational field the acceleration only affects the CM coordinates. The CM variables behave as those of a composed particle in a constant gravitational field but with an effective total gravitational mass \((M/M)\bar{M}\) instead of \(\bar{M}\). On the other hand, the internal dynamics is not modified by the extended Galilean transformation and, consequently, the energy-level structure does not change with respect to that in an inertial frame. In particular, there are stationary states and no ionization event can take place, even if the mass equivalence would be violated. Atomic matter would be stable in an uniformly accelerated frame even in these circumstances. We conclude that if inertial and gravitational masses were not equal the internal dynamics in accelerated frames and gravitational fields would differ.
6 Discussion

We have analyzed the behavior of the Hydrogen atom in constant gravitational fields and uniformly accelerated frames under the assumption of different inertial and gravitational masses. The discussion has focused on the internal properties of the system, a subject scarcely explored in the context of the equivalence principle.

Our main conclusion is that the atom can become unstable in these conditions. The validity of the mass equivalence is a necessary condition for the stability of atomic systems in gravitational fields. Classically, the mass equivalence is related to the universality of free fall. At the quantum level it plays another role, it guarantees the atomic stability in a gravitational field.

The situation for accelerated observers is completely different. The violation of the mass equivalence would only affect to the CM variables. The internal dynamics, in particular the atomic stability, would remain unaltered with respect to that in an inertial frame. The mass equivalence does not play any role in the atomic stability for accelerated observers. This result shows that there are scenarios where gravitational fields and accelerated frames are not equivalent, a property not in the spirit of the equivalence principle. Note, however, that although the mass equivalence is irrelevant for the stability, its validity is a necessary condition for the strong equivalence to hold in the quantum realm (in the specific sense related to the Galilean transformations) because as discussed in the previous section the fictitious gravitational potential appearing in the equations after the Galilean transformation depends on $M$ instead of $\bar{M}$.

Our analysis also highlights the fact that a complete understanding of the mass equivalence in the quantum realm demands the consideration of the internal degrees of freedom. This agrees with other studies showing that the internal variables must be taken into account in order to get a full understanding of the behavior of quantum systems in gravitational fields [13, 14, 15].

The capacity of the gravitational potential (when the mass equivalence is violated) to modify the atomic spectra and the stability conditions does not depend on its intensity. Even a tiny potential is enough to transform a discrete spectra into a continuous one [6, 7]. This is a qualitative, rather than quantitative, characteristic of the effect.

We have restricted our considerations to the COW- or weak gravitational field-regime, where non-relativistic quantum mechanics describes the behavior of the atom. Even with this limitation our approach can be relevant to study the interplay between gravitation and quantum theory. Note, at this respect, that it has been proposed to use the non-relativistic COW experiment as a low-energy window to look into the structure of space-time [18]. The non-relativistic approach notoriously simplifies the mathematical treatment of the problem. In order to complete the analysis of the problem we must consider relativistic quantum mechanics to see if the same results hold in the new regime. We can consider the relativistic corrections to the Hydrogen atom or resort directly
to the Dirac equation. In any of the two cases, these considerations would enlarge too much the paper and must be analyzed separately. As signaled in the Introduction several authors have studied the Hydrogen atom in a gravitational field, both in the relativistic [8, 9, 10] and non-relativistic [11] regimes (although none of them in connection with the equivalence principle). The relativistic considerations indicate that one must analyze in detail the decomposition into CM and internal variables [9, 10].

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