On the $L^1$ norm of an exponential sum involving the divisor function

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Abstract. In this paper, we obtain bounds on the $L^1$-norm of the sum $\sum_{n \leq x} \tau(n)e(an)$ where $\tau(n)$ is the divisor function.

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1. Introduction. Let $\tau(n) = \sum_{d|n} 1$ be the divisor function, and

$$S(\alpha) = \sum_{n \leq x} \tau(n)e(n\alpha), \quad e(\alpha) = e^{2\pi i \alpha}.$$ 

In 2001 Brüdern [1] considered the $L^1$ norm of $S(\alpha)$ and claimed to prove

$$\sqrt{x} \ll \int_0^1 |S(\alpha)|d\alpha \ll \sqrt{x}. \quad (1)$$

However there is a mistake in the proof given there which depends on a lemma which is false. In particular, the last two lines of the proof of the lemma in Sect. 2 are incorrect. In this note we prove the following result.

Theorem 1. We have

$$\sqrt{x} \ll \int_0^1 |S(\alpha)|d\alpha \ll \sqrt{x} \log x. \quad (2)$$

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The upper bound here is obtained by following Br"udern’s proof with corrections. The lower bound is based on the method Vaughan introduced to study the $L^1$ norm for exponential sums over primes [4], and also makes use of a more recent result of Pongsriiam and Vaughan [3] on the divisor sum in arithmetic progressions. We do not know whether the upper bound or the lower bound reflects the actual size of the $L^1$ norm here.

2. Proof of the upper bound. Following the initial steps in [1], to prove the upper bound in Theorem 1 it suffices to establish

$$\int_0^1 |T(\alpha)| d\alpha \ll \sqrt{x} \log x \quad (3)$$

where here

$$T(\alpha) = \sum_{u \leq \sqrt{x}} \sum_{u < v \leq x/u} e(\alpha u v).$$

We proceed as in the circle method. Clearly in (3) we can replace the integration range $[0, 1]$ by $[1/Q, 1 + 1/Q]$. By Dirichlet’s theorem for any $\alpha \in [1/Q, 1 + 1/Q]$ we can find a fraction $\frac{a}{q}$, $1 \leq q \leq Q$, $1 \leq a \leq q$, $(a, q) = 1$, with $|\alpha - \frac{a}{q}| \leq 1/(qQ)$. Thus the intervals $[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}]$ cover the interval $[1/Q, 1 + 1/Q]$. Taking

$$2\sqrt{x} \leq Q \ll \sqrt{x}, \quad (4)$$

and

$$M(q, a) = \left[ \frac{a}{q} - \frac{1}{2q\sqrt{x}}, \frac{a}{q} + \frac{1}{2q\sqrt{x}} \right],$$

we obtain

$$\int_0^1 |T(\alpha)| d\alpha \leq \sum_{q \leq Q} \sum_{\substack{a = 1 \\ (a, q) = 1}} \int_{M(q, a)} |T(\alpha)| d\alpha.$$

On each interval $M(q, a)$ we decompose $T(\alpha)$ into

$$T(\alpha) = F_q(\alpha) + G_q(\alpha)$$

where

$$F_q(\alpha) = \sum_{u \leq \sqrt{x}} \sum_{\substack{u < v \leq x/u \\ q | u}} e(u v a)$$

and

$$G_q(\alpha) = \sum_{u \leq \sqrt{x}} \sum_{\substack{u < v \leq x/u \\ q \nmid u}} e(u v a),$$
and have
\[ \int_0^1 |T(\alpha)| d\alpha \leq \sum_{q \leq Q} \sum_{a=1 \atop (a,q)=1}^{q \mathcal{M}(q,a)} \int |F_q(\alpha)| d\alpha + \sum_{q \leq Q} \sum_{a=1 \atop (a,q)=1}^{q \mathcal{M}(q,a)} \int |G_q(\alpha)| d\alpha \]
\[ := I_F + I_G. \]
The upper bound in Theorem 1 follows from the following two lemmas.

**Lemma 2** (Brüdern). We have
\[ I_F \ll \sqrt{x}. \]

**Lemma 3.** We have
\[ I_G \ll \sqrt{x} \log x. \]

In what follows we always assume \((a,q)=1\), and define the new variable \(\beta\) by
\[ \alpha = \frac{a}{q} + \beta. \] (5)

**Proof of Lemma 2.** The proof follows from the estimate
\[ F_q(\alpha) \ll \begin{cases} \min \left( x, \frac{1}{|\beta|} \right) \frac{\log 2\sqrt{x}}{q} & \text{if } q \leq \sqrt{x}, \ |\beta| \leq \frac{1}{2q\sqrt{x}}; \\ 0 & \text{if } q > \sqrt{x}; \end{cases} \] (6)
since this implies
\[ I_F \ll \sum_{q \leq \sqrt{x}} q \int_0^{1/(2q\sqrt{x})} \min \left( x, \frac{1}{|\beta|} \right) \log \frac{2\sqrt{x}}{q} \ d\beta \]
\[ \ll \sum_{q \leq \sqrt{x}} \log \frac{2\sqrt{x}}{q} \left( \int_0^{1/(2x)} x \ d\beta + \int_{1/(2x)}^{1/(2q\sqrt{x})} \frac{1}{\beta} \ d\beta \right) \]
\[ \ll \sum_{q \leq \sqrt{x}} \left( \log \frac{2\sqrt{x}}{q} \right)^2 \ll \sqrt{x}. \]

To prove (6), we first note that the conditions \(q|u\) and \(u \leq \sqrt{x}\) force \(F_q(\alpha) = 0\) when \(q > \sqrt{x}\). Next, when \(q \leq \sqrt{x}\), we write \(u = jq\) and have
\[ F_q(\alpha) = \sum_{j \leq \frac{\sqrt{x}}{q}} \sum_{jq \leq v \leq \frac{\sqrt{x}}{jq}} e(jqv\beta). \]

Making use of the estimate
\[ \sum_{N_1 < n \leq N_2} e(n\alpha) \ll \min \left( N_2 - N_1, \frac{1}{\|\alpha\|} \right), \] (7)
we have
\[ F_q(\alpha) \ll \sum_{j \leq \frac{\sqrt{x}}{q}} \min \left( \frac{x}{jq}, \frac{1}{\|jq\beta\|} \right). \]
In this sum \( jq \leq \sqrt{x} \) so that \( jq|\beta| \leq |\beta|\sqrt{x} \), and hence the condition \( |\beta| \leq \frac{1}{2q\sqrt{x}} \) implies \( jq|\beta| \leq \frac{1}{2q} \leq \frac{1}{2} \). Hence \( \|jq\beta\| = jq|\beta| \), and we have
\[
F_q(\alpha) \ll \sum_{j \leq \sqrt{x}} \frac{1}{jq} \min \left( x, \frac{1}{|\beta|} \right) \ll \min \left( x, \frac{1}{|\beta|} \right) \frac{\log \frac{2\sqrt{x}}{q}}{q}.
\]

\[\Box\]

Proof of Lemma 3. The proof follows from the estimate,
\[
G_q(\alpha) \ll (\sqrt{x} + q) \log q, \quad \text{for } \alpha = \frac{a}{q} + \beta, \quad |\beta| \leq \frac{1}{2q\sqrt{x}},
\]

since this implies
\[
I_G \ll \sum_{q \leq Q} \frac{1/(2q\sqrt{x})}{q} \int_0^{1/(2q\sqrt{x})} (\sqrt{x} + q) \log q \, d\beta
\]

\[
\ll \frac{1}{\sqrt{x}} Q(\sqrt{x} + Q) \log Q \ll \sqrt{x} \log x
\]

by (4). To prove (8), we apply (7) to the sum over \( v \) in \( G_q(\alpha) \) and obtain
\[
G_q(\alpha) \ll \sum_{\substack{u \leq \sqrt{x} \\text{mod } q, u \nmid u}} \min \left( x, \frac{1}{\|u\alpha\|} \right).
\]

Recalling \( \|x\| = \| - x\| \) and the triangle inequality \( \|x + y\| \leq \|x\| + \|y\| \), and using the conditions \( 1 \leq u \leq \sqrt{x}, q \nmid u, |\beta| \leq \frac{1}{2q\sqrt{x}} \), we have
\[
\|u\alpha\| \geq \left\| \frac{au}{q} \right\| - u|\beta| \geq \left\| \frac{au}{q} \right\| - \frac{1}{2q} \geq \frac{1}{2} \left\| \frac{au}{q} \right\|,
\]

and therefore
\[
G_q(\alpha) \ll \sum_{\substack{u \leq \sqrt{x} \\text{mod } q, u \nmid u}} \frac{1}{\left\| \frac{au}{q} \right\|}.
\]

Here \( \left\| \frac{au}{q} \right\| = \frac{b}{q} \) for some integer \( 1 \leq b \leq \frac{q}{2} \), and since the integers \( \{au : 1 \leq u \leq \frac{q}{2}\} \) are distinct modulo \( q \) since \( (a, q) = 1 \), we see
\[
\sum_{\substack{1 \leq u \leq \frac{q}{2} \\text{mod } q, u \nmid u}} \frac{1}{\left\| \frac{au}{q} \right\|} = \sum_{\substack{1 \leq b \leq \frac{q}{2}}} \frac{q}{b} \ll q \log q.
\]

If \( q > \sqrt{x} \), then
\[
\sum_{\substack{u \leq \sqrt{x} \\text{mod } q, u \nmid u}} \left\| \frac{au}{q} \right\| \leq 2 \sum_{\substack{1 \leq u \leq \frac{q}{2} \\text{mod } q, u \nmid u}} \left\| \frac{au}{q} \right\| \ll q \log q,
\]
while if \( q \leq \sqrt{x} \), then the sum bounding \( G_q(\alpha) \) can be split into \( \ll \frac{\sqrt{x}}{q} \) sums of this type and

\[
G_q(\alpha) \ll \frac{\sqrt{x}}{q} (q \log q) \ll \sqrt{x} \log q.
\]

\( \Box \)

3. Proof of the lower bound. Following Br"udern, consider the intervals \( |\alpha - \frac{a}{q}| \leq 1/(4x) \) for \( 1 \leq a \leq q \leq Q \), where we take \( \frac{1}{2} \sqrt{x} \leq Q \leq \sqrt{x} \). These intervals are pairwise disjoint because for two distinct fractions \( |a/q - a'/q'| \geq 1/(qq') \geq 1/x \). (We will see later why these intervals have been chosen shorter than required to be disjoint.) Hence, using (5)

\[
\int_0^1 |S(\alpha)| \, d\alpha = \int_{1/Q}^{1+1/Q} |S(\alpha)| \, d\alpha \geq \sum_{q \leq \frac{1}{2} \sqrt{x}} \sum_{\frac{1}{4} \sqrt{x} - 1/(4x)}^{1/(4x)} \sum_{a=1}^q |S\left(\frac{a}{q} + \beta\right)| \, d\beta.
\]

Next we follow Vaughan’s method [4] and apply the triangle inequality to obtain the lower bound

\[
\int_0^1 |S(\alpha)| \, d\alpha \geq \sum_{q \leq \frac{1}{2} \sqrt{x}} \sum_{\frac{1}{4} \sqrt{x} - 1/(4x)}^{1/(4x)} \left| \sum_{a=1}^q S\left(\frac{a}{q} + \beta\right) \right| \, d\beta.
\]

Letting

\[
U_q(x; \beta) := \sum_{a=1}^q S\left(\frac{a}{q} + \beta\right) = \sum_{n \leq x} \tau(n) c_q(n) e(n\beta), \quad (9)
\]

where

\[
c_q(n) = \sum_{\frac{a}{q} = 1}^q \sum_{(a,q) = 1} e\left(\frac{an}{q}\right)
\]

is the Ramanujan sum, our lower bound may now be written as

\[
\int_0^1 |S(\alpha)| \, d\alpha \geq \sum_{q \leq \frac{1}{2} \sqrt{x}} \sum_{\frac{1}{4} \sqrt{x} - 1/(4x)}^{1/(4x)} |U_q(x; \beta)| \, d\beta. \quad (10)
\]

To complete the proof of the lower bound we need the following lemma, which we prove at the end of this section.

**Lemma 4.** For \( q \geq 1 \) we have

\[
U_q(x; 0) = \frac{\varphi(q)}{q} x (\log(x/q^2) + 2\gamma - 1) + O\left(q \tau(q)(x^{\frac{1}{4}} + q^{\frac{1}{2}}) x^\epsilon\right), \quad (11)
\]

where \( \gamma \) is Euler’s constant.
Proof of the lower bound in Theorem 1. For any exponential sum $T(x; \beta) = \sum_{n \leq x} a_n e(n\beta)$ we have by partial summation or direct verification

$$T(x; \beta) = e(\beta x)T(x; 0) - 2\pi i \beta \int_1^x e(\beta y)T(y; 0) \, dy.$$ 

Taking $T(x; \beta) = U_q(x; \beta)$ we thus obtain from (10) and the triangle inequality

$$\int_0^1 |S(\alpha)| \, d\alpha \geq \sum_{q \leq \frac{1}{2} \sqrt{x}} \int_{-1/(4x)}^{1/(4x)} \left( |U_q(x; 0)| - 2\pi |\beta| \int_1^x |U_q(y; 0)| \, dy \right) d\beta. \quad (12)$$

By Lemma 4, with $q \leq \frac{1}{2} \sqrt{x},$

$$\int_1^x |U_q(y; 0)| \, dy \leq \frac{\varphi(q)}{q} \left( \int_1^x y \log(y/q^2) + (2\gamma - 1)y \, dy \right) + O(xq\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon)$$

$$\leq \frac{\varphi(q)}{q} \left( \int_1^{q^2} y \log(y^2/q) \, dy + \int_{q^2}^x y \log(y/q^2) \, dy + \frac{1}{2} x^2 (2\gamma - 1) \right)$$

$$+ O(xq\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon)$$

$$= \frac{x}{2} \left( \frac{\varphi(q)}{q} \left( x(\log(x/q^2) + 2\gamma - 1) - \frac{x}{2} + \frac{q^4}{x} \right) + O(q\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon) \right)$$

$$\leq \frac{x}{2} U_q(x, 0) + O(q\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon).$$

Using $|\beta| \leq 1/(4x),$ we have

$$|U_q(x; 0)| - 2\pi \beta \int_1^x |U_q(y; 0)| \, dy \geq \left( 1 - \frac{\pi}{4} \right) |U_q(x; 0)| - O(q\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon).$$

We conclude, returning to (12) and making use of Lemma 4 again,

$$\int_0^1 |S(\alpha)| \, d\alpha \geq \frac{4 - \pi}{8x} \sum_{q \leq \frac{1}{2} \sqrt{x}} \left( |U_q(x; 0)| - O(q\tau(q)(x^{\frac{1}{2}} + q^{\frac{1}{2}})x^\epsilon) \right)$$

$$\geq \frac{4 - \pi}{8} \sum_{q \leq \frac{1}{2} \sqrt{x}} \frac{\varphi(q)}{q} (\log(x/q^2) + 2\gamma - 1) - O(x^{\frac{1}{2} + \epsilon}).$$

That the above sum is $\gg \sqrt{x}$ then follows from partial summation and the well-known fact (see for example [2, Theorem 330]) that

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} n^2 + O(n \log n).$$
Proof of Lemma 4. Pongsriiam and Vaughan [3] recently proved the following very useful result on the divisor function in arithmetic progressions. For integer $a$ and $d \geq 1$ and real $x \geq 1$, we have

$$\sum_{n \leq x \atop n \equiv a (\text{mod } d)} \tau(n) = \frac{x}{d} \sum_{r|d} c_r(a) \left( \log \frac{x}{r^2} + 2\gamma - 1 \right) + O((x^{1/2} + d^{1/2})x^\epsilon),$$

where $\gamma$ is Euler's constant and $c_r(a)$ is the Ramanujan sum. We need the special case when $a = 0$ which along with the situation $(a, d) > 1$ is explicitly allowed in this formula. Hence we have

$$\sum_{n \leq x \atop d|n} \tau(n) = \frac{x}{d} f_x(d) + O((x^{1/2} + d^{1/2})x^\epsilon) \quad (13)$$

where

$$f_x(d) = \sum_{r|d} g_x(r), \quad g_x(r) = \frac{\varphi(r)}{r} (\log(x/r^2) + 2\gamma - 1). \quad (14)$$

Making use of

$$c_q(n) = \sum_{d|n} d \mu \left( \frac{q}{d} \right),$$

and (13) we have

$$U_q(x; 0) = \sum_{n \leq x} \tau(n)c_q(n)$$

$$= \sum_{d|q} \left( \frac{q}{d} \right) \sum_{n \leq x \atop d|n} \tau(n)$$

$$= x \sum_{d|q} \mu \left( \frac{q}{d} \right) f_x(d) + O(q\tau(q)(x^{1/2} + d^{1/2})x^\epsilon).$$

We evaluate the sum above using Dirichlet convolution and the identity $1 * \mu = \delta$ where $\delta(n)$ is the identity for Dirichlet convolution defined to be 1 if $n = 1$ and zero otherwise. Hence

$$\sum_{d|q} \left( \frac{q}{d} \right) f_x(d) = (f_x * \mu)(q) = ((g_x * 1) * \mu)(q) = (g_x * \delta)(q) = g_x(q),$$

and Lemma 4 is proved. \qed

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