Some Metrical Properties of Lattice Graphs of Finite Groups

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Abstract: This paper is concerned with the combinatorial facts of the lattice graphs of $\mathbb{Z}_{p_1^m_1} \times \mathbb{Z}_{p_2^m_2} \times \cdots \times \mathbb{Z}_{p_m^m}$, $\mathbb{Z}_{p_1^m_1} \times \mathbb{Z}_{p_2^m_2}$, and $\mathbb{Z}_{p_1^m_1} \times \mathbb{Z}_{p_2^m_2} \times \cdots \times \mathbb{Z}_{p_m^m}$. We show that the lattice graph of $\mathbb{Z}_{p_1^m_1} \times \mathbb{Z}_{p_2^m_2} \times \cdots \times \mathbb{Z}_{p_m^m}$ is realizable as a convex polytope. We also show that the diameter of the lattice graph of $\mathbb{Z}_{p_1^m_1} \times \mathbb{Z}_{p_2^m_2} \times \cdots \times \mathbb{Z}_{p_m^m}$ is $\sum_{i=1}^{r} m_i$ and its girth is 4.

Keywords: finite group; lattice graph; convex polytope; diameter; girth

1. Introduction

The relation between the structure of a group and the structure of its subgroups constitutes an important domain of research in both group theory and graph theory. The topic has enjoyed a rapid development starting with the first half of the twentieth century.

The main object of this paper is to study the interplay of group-theoretic properties of a group $G$ with graph-theoretic properties of its lattice graph $L(G)$. Every group has a corresponding lattice graph, which can be finite or infinite depending upon the order of the group. This study helps illuminate the structure of the set $H(G)$ of subgroups of $G$.

The lattice graph $L(G)$ of a finite cyclic group $G$ is obtained as follows: Each vertex of $L(G)$ corresponds to an element of $H(G)$, and two vertices corresponding to two elements $H_1, H_2$ of $H(G)$ are connected by an edge if and only if $H_1 \leq H_2$ and that there is no element $K$ of $H(G)$ such that $H_1 \leq K \leq H_2$ (see [1,2]), thus $\leq$ is used when $H_1$ is proper maximal subgroup of $H_2$. The notation $\leq$ is used as subgroup. Degree of a vertex is the number of edges attached to that vertex. A vertex is defined to be even or odd if its degree is even or odd. The degree vector of a graph is the sequence of degrees of its vertices arranged in non-increasing order [3]. The diameter $diam(G)$ of a connected graph $G$ is the maximum distance among vertices of $G$ [4,5]. The girth $g(G)$ of a graph $G$ is the length of a smallest cycle in $G$, and is infinity if $G$ is acyclic [6]. It is a fact that the lattice of subgroups of a given group can rarely be drawn without its edges crossing [1,2]. The crossing number $cr(G)$ of a graph $G$ is the minimum number of crossings of its edges among the drawings of $G$ in the plane. A graph is considered Eulerian if there exists a Eulerian path in which we can start at a vertex, traverse through every edge only once, and return to the same vertex where we started. A connected graph $G$ is Eulerian if each vertex has even degree, and is semi-Eulerian if it has exactly two vertices with odd degrees [7].
A polytope is a finite region of $\mathbb{R}^n$ enclosed by a finite number of hyper-planes. A polytope is called convex if its points form a convex subset of $\mathbb{R}^n$. Combinatorial aspects of the groups can be computed using its lattice graphs [8,9]. The authors discussed some finite simple groups of low rank in [8]. Tarnaucăeanu introduced new arithmetic method of counting the subgroups of a finite Abelian group in [10]. The author of [11] described the finite groups having $|G| - 1$ cyclic subgroups. Saeedi and Farrokhi [12] computed factorization number of some finite groups. Tarnaucăeanu [13] characterized elementary Abelian two-groups. Tarnaucăeanu and Toth discussed cyclicity degree of finite groups in [14]. Tarnaucăeanu [15] discussed finite groups with dismantlable subgroup lattices.

In the present article, we are interested in the lattices of finite groups. We demonstrate that the lattice graph of $\mathbb{Z}_{p_1 \times p_2 \times \cdots \times p_m}$ can be viewed as a convex polytope, and that the diameter of the lattice graph of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times \cdots \times p_r^{m_r}}$ is $\sum_{i=1}^{r} m_i$ and its girth is 4. We also compute many other properties of these lattices. We are interested developing some combinatorial invariants of the groups coming from their lattice graphs. The main motivation comes from the fact that finite graphs are relatively easier to handle than the finite groups. One is interested in capturing facts of the group while studying the graph associated to it. This study ultimately relates some parameters of the graphs with some parameters of the groups. Several authors computed some combinatorial aspects of finite graphs such as diameter, girth and radius but, for the lattice graph, similar questions are still open and need to be addressed. This article can be considered as a step forward in this direction. We, naturally, pose problems about the exact values of these parameters for more general classes of groups such as $S_n$, $A_n$, and sporadic groups.

2. The Results

In this section, we give the combinatorial results about the lattice graphs of $\mathbb{Z}_{p_1 \times p_2 \times \cdots \times p_m}$, $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$, and $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3}$. In the end, we give the general results about $L(\mathbb{Z}_n)$.

2.1. $L(\mathbb{Z}_{p_1 \times p_2 \times \cdots \times p_m})$

**Theorem 1.** The lattice graph of $\mathbb{Z}_n$, $n = p_1 \times p_2 \times \cdots \times p_m$ is realizable as a convex polytope with $2^m$ vertices.

**Proof.** We prove it using all possible maximal chains of subgroups of $\mathbb{Z}_n$. It is clear that $\mathbb{Z}_n$ is finite cyclic group, thus, for each possible divisor of $n$, there exist a unique subgroup of this group, up to isomorphism. This leads us to the set of all subgroups of $\mathbb{Z}_n$; $\mathcal{V}(G) = \{< 1 >, < p_1 >, < p_2 >, < p_3 >, \ldots, < p_{m-1} >, < p_m >, < p_1 \times p_2 >, < p_1 \times p_3 >, \ldots, < p_1 \times p_m >, < p_1 \times p_2 \times p_3 >, \ldots, < p_1 \times p_2 \times p_m >, \ldots, < p_1 \times p_2 \times p_3 \times \ldots \times p_m >\}$. Now, the maximal chains: The subgroup $H_0 = < 0 >$ is contained immediately in $m$ subgroups of $\mathbb{Z}_n$: $H_1 = < p_1 \times p_2 \times \cdots \times p_{m-1} >, H_{12} = < p_1 \times p_2 \times \cdots \times p_{m-2} \times p_m >, H_{13} = < p_2 \times p_3 \times \cdots \times p_{m-1} \times p_m >$, as shown in Figure 1.

![Figure 1. First step graph.](image)

Each $H_1$ is contained in $m - 1$ subgroups of $\mathbb{Z}_n$: For instance, $H_1 = < p_1 \times p_2 \times \cdots \times p_{m-1} >$ is contained in $H_2 = < p_1 \times p_2 \times \cdots \times p_{m-2} >$, $H_{22} = < p_1 \times p_2 \times \cdots \times p_{m-3} \times p_{m-1} >, \ldots, H_{2m-1} = < p_2 \times p_3 \times \cdots \times p_{m-1} >$ (see Figure 2).
Similarly, every $H_2$ is contained in $m - 2$ subgroups of $\mathbb{Z}_n$: $H_{2_1} = < p_1 \times p_2 \times \ldots \times p_{m-2} >$ is contained in $H_{3_r}$ subgroups of $\mathbb{Z}_n$ (see Figure 3).

The process will continue until we receive $< p_1 >, < p_2 > \ldots < p_m >$ at the second last stage. Now, each one of these is contained in $< 1 > = \mathbb{Z}_n$ and the process is finished (see Figure 4).

It is clear that there are $m$ possibilities of subgroups of $\mathbb{Z}_n$ containing trivial group, $m - 1$ possibilities of each of $H_{1_1}$ to be contained in other subgroups of $\mathbb{Z}_n$, $m - 2$ possibilities for each of $H_{2_2}$ to be contained in next subgroups and so on. Thus, by the product rule, the number of all possibilities of maximal chains is $(m)(m - 1)(m - 2) \ldots 3.2.1 = m!$. Now, we put all these chains of subgroup in the plane such that each subgroup is identified to itself occurring in all these series. Thus, the same subgroups that appear in more than one series, appear only as a single vertex of the lattice graph in the plane. This lattice graph starts at the identity and finishes at $\mathbb{Z}_n$ because these subgroups appear in all series. This may have crossings. If we imagine this gluing in $\mathbb{R}^3$ avoiding all crossings in higher dimension, we get a convex polytope with $2^m$ vertices, one vertex corresponding to each element of $H(\mathbb{Z}_n)$. For instance, the cases for $m = 3, 4, 5$ are shown in Figures 5–7, respectively.
Corollary 1.
(a) The number of maximal subgroups of $\mathbb{Z}_n$ is $m$.
(b) The length of each maximal series is $m + 1$.
(c) The diameter of $L(\mathbb{Z}_n)$ is $m$.
(d) $L(\mathbb{Z}_n)$ is $m$-regular.

Proof.
(a) It is clear that $\mathbb{Z}_n$ is cyclic, thus, for each divisor, we have a unique subgroup. Prime divisor $p_i$ of $n$ yields maximum quotient, thus these numbers correspond to the maximal subgroups of $\mathbb{Z}_n$, which are $\langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle, \ldots, \langle p_{m-1} \rangle, \langle p_m \rangle$.
(b) This can be shown by simply counting the number of vertices of $L(\mathbb{Z}_n)$ along a path from $<0>$.
to $<1>$; see, for instance, a typical series: $0 < p_1 \times p_2 \times \ldots \times p_{m-1} \leq \ldots \leq p_1 \times p_2 \leq \ldots < 1 = \mathbb{Z}_n$.

(c) $\text{diam}(L(Z_n))$ is one less than the length of a maximal series, which is the length of each path from $<0>$ to $<1>$.

(d) At the first stage, the degree of the vertex $H_0$ is $m$ because it is adjacent to $m$ vertices $H_i$, $i = 1, 2, \ldots, m$ in the second stage, as shown in the construction of $L(Z_n)$.

The degree of each vertex at the second stage is $m$ as each vertex $H_i$ is adjacent to $m - 1$ vertices lying higher to it and to one vertex lying below it. For instance, $H_1 = < p_1 \times p_2 \times \ldots \times p_{m-1} >$ is attached with $H_2 = < p_1 \times p_2 \times \ldots \times p_{m-2} >$, $H_3 = < p_1 \times p_2 \times \ldots \times p_{m-3} \times p_{m-1} >$, $\ldots$, $H_{2m-1} = < p_2 \times p_3 \times \ldots \times p_{m-1} >$. Similarly, $\text{deg}(H_2)$ is $m$. Continuing this process, we receive that the vertices corresponding to $< p_1 >, < p_2 >, \ldots, < p_m >$ are adjacent to the vertex corresponding to $<1>$, giving the degree of the last vertex $m$ too. Thus, each vertex of $L(Z_n)$ has degree $m$, and the proof is finished.

Remark 1. It should be remarked that the lattice graph of $\mathbb{Z}_n$ is a small-world network in which nearly all nodes are not neighbors of one another, but the neighbors of any given node are likely to be neighbors of each other and most nodes can be reached from every other node by a small number of steps. Thus, it is a connected graph [16].

2.2. $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$

The construction of $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is given by using all maximal chains from the set $H(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}) = \{<1>, <p_1>, <p_1^2>, \ldots, <p_1^{m_1-1}>, <p_1^{m_1}>, <p_2>, <p_2^2>, \ldots, <p_2^{m_2-1}>, <p_2^{m_2}>, \ldots, <p_1 \times p_2>, \ldots, <p_1^{m_1} \times p_2^{m_2}>, \ldots, <p_1 \times p_2^{m_2}>, \ldots, <p_1^{m_1} \times p_2^{m_2}>, \ldots\}$. The subgroup $<0> = <p_1^{m_1} \times p_2^{m_2}>$ is contained immediately in two subgroups $<p_1^{m_1} \times p_2^{m_2-1}>$ and $<p_1^{m_1-1} \times p_2^{m_2}>$ which are further contained in $<p_1^{m_1-1} \times p_2^{m_2-1}>$. The subgroup $<p_1^{m_1-1} \times p_2^{m_2-1}>$ is contained immediately in two subgroups $<p_1^{m_1-2} \times p_2^{m_2-1}>$ and $<p_1^{m_1-3} \times p_2^{m_2-1}>$, which is contained in next two subgroups, $<p_1^{m_1-2} \times p_2^{m_2-1}>$ and $<p_1^{m_1-3} \times p_2^{m_2-1}>$. The process will continue until we receive a subgroup $<p_1 \times p_2> = \mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$ that is contained in next two subgroups, $<p_1 \times p_2^{m_2}>$ and $<p_1^{m_1} \times p_2^{m_2-1}>$. Both are further contained in $<p_2^{m_2-1}>$, which itself is contained in $<p_2^{m_2-2}>$. This process is continued until we obtain a subgroup $<p_2>$, which is continued in $<1> = \mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$, and thus we receive a series of subgroups of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$; for instance, $<0> \leq <p_1^{m_1-2} \times p_2^{m_2} > \leq <p_1^{m_1-2} \times p_2^{m_2-1} > \leq \ldots \leq <p_2^{m_2-1} > \leq <p_2^{m_2-2} > \leq \ldots \leq <p_2 > \leq <1>$. We obtain all other series of subgroups similarly. Now, gluing all maximal series, we obtain the lattice graph of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$, as shown in Figure 8.

From the construction, we reach at the results:

Proposition 1.

(a) $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$ is planar.

(b) Length of maximal series of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$ is $(m_1 + m_2 + 1)$.

(c) The diameter of lattice graph of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}}$ is $m_1 + m_2$.

Proof.

(a) This is obvious from the construction of $L(\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2}})$.

(b) Again, this is obvious; see a typical maximal series: $0 \leq <p_1^{m_1-1} \times p_2^{m_2} > \leq <p_1^{m_1-2} \times p_2^{m_2} > \leq \ldots \leq <p_2^{m_2-1} > \leq <p_2^{m_2-2} > \leq \ldots \leq <p_2 > \leq <1>$.

(c) In this case, the diameter is exactly one less than the number of elements in a maximal series. □
2.3. On the Lattice Graph of $\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}}$.

The set of subgroups of $\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \mathbb{Z}_{p_3}$ is $H(\mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \mathbb{Z}_{p_3}) = \{ \langle p_1^{m_1} \times p_2^{m_2} \times p_3 \rangle, \langle p_1^{m_1-1} \times p_2^{m_2} \times p_3 \rangle, \ldots, \langle p_2 \times p_3 \rangle \} \cup \{ \langle p_1 \times p_2^{m_2} \rangle, \langle p_1 \times p_2^{m_2-1} \times p_3 \rangle, \ldots, \langle p_1 \times p_2 \rangle, \langle p_1 \times p_2^{m_2-1} \times p_3 \rangle, \ldots, \langle p_1 \rangle \} \cup \{ \langle p_1^{m_1} \times p_2 \rangle, \ldots, \langle p_2^2 \rangle, \langle 1 \rangle \}$. The subgroup $\langle 0 \rangle = \langle p_1^{m_1} \times p_2^{m_2} \times p_3 \rangle$ is contained immediately in three subgroups $\langle p_1^{m_1} \times p_2^{m_2-1} \times p_3 \rangle, \langle p_1^{m_1-1} \times p_2^{m_2} \times p_3 \rangle$ and $\langle p_1^{m_1} \times p_2^{m_2-1} \times p_3 \rangle$. The first two of these are contained in $\langle p_1^{m_1-1} \times p_2^{m_2-1} \times p_3 \rangle$. The subgroup $\langle p_1^{m_1-1} \times p_2^{m_2} \times p_3 \rangle$ is contained in $\langle p_1^{m_1-2} \times p_2^{m_2} \times p_3 \rangle$ which is further contained in two subgroups $\langle p_1^{m_1-2} \times p_2^{m_2-1} \times p_3 \rangle$ and $\langle p_1^{m_1-3} \times p_2^{m_2} \times p_3 \rangle$. The subgroup $\langle p_1^{m_1-3} \times p_2^{m_2} \times p_3 \rangle$ is contained in other two subgroups of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3}$. The process will continue till we receive a subgroup $\langle p_1 \times p_2^{m_2} \times p_3 \rangle$. It is contained in three subgroups $\langle p_1 \times p_2^{m_2} \rangle, \langle p_1 \times p_2^{m_2-1} \times p_3 \rangle$ and $\langle p_2^{m_2} \times p_3 \rangle$. The subgroup $\langle p_2^{m_2} \times p_3 \rangle$ is contained in $\langle p_2^{m_2} \rangle$. It is contained in $\langle p_2^{m_2-1} \rangle$. Continuing this process till we obtain $\langle p_2 \rangle$ which is contained in $\langle 1 \rangle = \mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3}$. And we will receive series of subgroups of $\mathbb{Z}_{p_1^{m_1} \times p_2^{m_2} \times p_3}$, for instance, $\langle 0 \rangle \subseteq \langle p_1^{m_1-1} \times p_2^{m_2} \times p_3 \rangle \subseteq \langle p_1^{m_1-2} \times p_2^{m_2} \times p_3 \rangle \subseteq \ldots \subseteq \langle p_2 \rangle$. Similarly, all the remaining subgroups will form other series of subgroups. At the end, gluing the vertices occurring in all maximal series, we obtain the required graph, as shown in Figure 9.
Proposition 2. The crossing number of $L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}})$ is $2m_1m_2$.

Proof. The proof is clear from the Table 1:

| $m_1, m_2$ | 1, 1 | 2, 1 | 3, 1 | ... | $m_1, 2$ | $m_1, 3$ | ... | $m_1, m_2$ |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $cr(L)$   | 2   | 4   | 6   | ... | $2m_1$ | $4m_1$ | $6m_1$ | $2(m_1)(m_2)$ |

The following proposition relates the number of crossings and number of subgroups of $Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}$.

Proposition 3. $cr(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}})) = |V(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}))| - 2(m_1 + m_2 + 1)$.

Proof. We proceed as follows: $|V(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}))| = 2(m_1 + 1)(m_2 + 1) = 2(m_1m_2 + m_1 + m_2 + 1) = 2m_1m_2 + 2(m_1 + m_2 + 1)$. However, as in Proposition 2.4, $(cr(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}))) = 2(m_1)(m_2)$ we have, $|V(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}))| = cr(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}})) + 2(m_1 + m_2 + 1)$, which leads to the given result. □

Similarly, the following corollary provides a link between number of crossings and number of subgroups of $Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}$.

Corollary 2. $cr(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}})) = \frac{|V(L(Z_{p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}}))|}{2} - 2.$
The set of subgroups of $G$ is $H(G) = \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \ldots \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \ldots \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \}$. This set consists of $(m_1 + 1)(m_2 + 1)(m_3 + 1) \ldots (m_{r-1} + 1)(m_r + 1)$ elements; this number is actually the number of divisors of the order of $G$. Now, take elements of $H(G)$ and form a (maximal) series from $< 0 >$ to $< 1 >$. A typical series of such kind is:

\[
\begin{align*}
&< 0 > = < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \\
&\subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}-1} > \\
&\subseteq \ldots \subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \\
&\subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}-1} > \\
&\subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}-2} > \\
&\subseteq \ldots \subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}-2} > \\
&\subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}-1} > \\
&\subseteq < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >
\end{align*}
\]

Finally, the information about the diameter and girth is enclosed in the most general result:

**Theorem 2 (The Main Theorem).** If $G$ is the group $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} \times \ldots \times \mathbb{Z}_{m_r}$, then:

(a) The length of the maximal series of $G$ is $1 + r \sum_{i=1}^{r} m_i$.

(b) The diameter of $L(G)$ is $\sum_{i=1}^{r} m_i$.

(c) $g(L(G)) = \begin{cases} \infty & \text{when } r = 1 \\ 4 & \text{when } r \neq 1 \end{cases}$

**Proof.**

(a) The set of subgroups of $G$ is $H(G) = \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \ldots \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \ldots \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \} \cup \ldots \cup \{ < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} >, < p_{m_1}^{m_1} \times p_{m_2}^{m_2} \times p_{m_3}^{m_3} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_{r}^{m_{r}} > \}$. This set consists of $(m_1 + 1)(m_2 + 1)(m_3 + 1) \ldots (m_{r-1} + 1)(m_r + 1)$ elements; this number is actually the number of divisors of the order of $G$. Now, take elements of $H(G)$ and form a (maximal) series from $< 0 >$ to $< 1 >$. A typical series of such kind is:
Theorem 3.

(a) $L(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n})$ is Eulerian if $m$ is even and is non-Eulerian if $m(> 1)$ is odd.

(b) $L(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2})$ is non-Eulerian if $m_1 \neq 1, 2$ and $m_2 \neq 1$.

(c) $L(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \mathbb{Z}_{p_3})$ is non-Eulerian.
Proof. The proof is given in the following Table 2:

Table 2. Degree vectors and graph types.

| Lattice Graphs | $m$ | Degree Vector | Graph Type          |
|----------------|-----|---------------|---------------------|
| $L(Z_n)$, $n = p_1 \times p_2 \times \ldots \times p_{m-1} \times p_m$ | 1   | $V_1 = [1 \ 1]$ | Semi-Eulerian       |
|                | 2   | $V_2 = [2 \ 2 \ 2 \ 2]$ | Eulerian            |
|                | 3   | $V_3 = [3 \ 3 \ldots 8 \times 3]$ | Non-eulerian        |
|                | 4   | $V_4 = [4 \ 4 \ldots 16 \times 4]$ | Eulerian            |
|                | \vdots | \vdots | \vdots |
|                | $m$ | $V_m = [m \ m \ldots 2^m \times m]$ | Eulerian if $m = \text{even}$ Non-Eulerian if $m = \text{odd}$ |
| $L(Z_{p_1^{m_1}} \times p_2^{m_2})$ | 1, 1 | $V'_1 = [2 \ 2 \ 2 \ 2]$ | Eulerian            |
|                | 2, 1 | $V'_2 = [3 \ 3 \ 2 \ 2 \ 2 \ 2]$ | Semi-Eulerian       |
|                | 3, 1 | $V'_3 = [3 \ 3 \ 3 \ 3 \ 2 \ 2 \ 2 \ 2]$ | Non-Eulerian        |
|                | 4, 1 | $V'_4 = [3 \ 3 \ 3 \ 3 \ 3 \ 2 \ 2 \ 2 \ 2]$ | Non-Eulerian        |
|                | \vdots | \vdots | \vdots |
| $L(Z_{p_1^{m_1}} \times p_2^{m_2} \times p_3^{m_3})$ | 1, 1 | $V''_1 = [3 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3]$ | Non-Eulerian        |
|                | 2, 1 | $V''_2 = [4 \ 4 \ 4 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3 \ 3]$ | Non-Eulerian        |
|                | \vdots | \vdots | \vdots |

3. Conclusions

In this article, we discuss some metrical aspects of the lattice graphs of some families of finite groups. We obtain the diameter and girth as well as many other aspects of these lattice graphs. Along with many other results, the following are the main contributions of this article.

Theorem 4 (The Main Theorem). If $G$ is the group $\mathbb{Z}_{p_1^{m_1}} \times p_2^{m_2} \times \ldots \times p_{r-1}^{m_{r-1}} \times p_r^{m_r}$, then:

(a) The length of the maximal series of $G$ is $1 + \sum_{i=1}^{r} m_i$.

(b) The diameter of $L(G)$ is $\sum_{i=1}^{r} m_i$.

(c) $g(L(G)) = \begin{cases} \infty & \text{when } r = 1 \\ 4 & \text{when } r \neq 1 \end{cases}$
and

**Theorem 5.**

(a) $L(Z_{p_1} \times \cdots \times Z_{p_m})$ is Eulerian if $m$ is even and is non-Eulerian if $m(>1)$ is odd.

(b) $L(Z_{p_1^{m_1}} \times Z_{p_2^{m_2}})$ is non-Eulerian if $m_1 \neq 1, 2$ and $m_2 \neq 1$.

(c) $L(Z_{p_1^{m_1}} \times Z_{p_2^{m_2}} \times Z_{p_3})$ is non-Eulerian.

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