COMPARISON THEOREMS IN FINSLER GEOMETRY AND THEIR APPLICATIONS

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Abstract. We prove Hessian comparison theorems, Laplacian comparison theorems and volume comparison theorems of Finsler manifolds under various curvature conditions. As applications, we derive McKean type theorems for the first eigenvalue of Finsler manifolds, as well as generalize a result on fundamental group due to Milnor to Finsler manifolds.

1. Introduction

Finsler geometry, a natural generalization of Riemannian geometry, was initiated by Finsler P. [Fin] in 1918, from considerations of regular problems in the calculus of variations. It developed steadily, with much investigation from the geometric point of view. Chern [Ch1] and many others defined various connections in Finsler manifolds, along the lines of the Levi-Civita connection in Riemannian manifolds; for a comprehensive account, see [BCS].

Recently, there has been a surge of interest in Finsler geometry, especially in its global and analytic aspects. A natural question, that has lately attracted some attention (see e.g., [AL,Ce,Sh2]), is how to generalize the Laplacian from Riemannian manifolds to Finsler manifolds. In Riemannian case, the Laplacian of a function equals to the divergence of gradient of the function, and the spectrum of Laplacian on Riemannian manifolds has been extensively studied. we shall adopt the notion of Laplacian for Finsler manifolds used in [Sh2].

The comparison technique is is widely used in Riemannian geometry. To pursue the global Finsler geometry we would generalize comparison theorems to the Finsler setting. It has been started in [Sh3]. The present paper would continue the investigation on this direction. We derive some Hessian comparison theorems, Laplacian comparison theorems and volume comparison theorems of Finsler manifolds under the various curvature assumptions. Then, we give some applications. We obtain

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some Mckean type theorems for the first eigenvalue of Finsler manifolds, as well as generalize a result on fundamental group due to Milnor to Finsler manifolds.

2. Finsler Geometry

Let \((M, F)\) be a Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\). Let \((x, y) = (x^i, y^i)\) be the local coordinates on \(TM\), and \(\pi : TM\backslash 0 \to M\) the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of \(TM\) rather than \(M\). Some frequently used quantities and relations:

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad \text{(fundamental tensor)}
\]

\[
C_{ijk}(x, y) := \frac{1}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k}, \quad \text{(Cartan tensor)}
\]

\[
(g^{ij}) := (g_{ij})^{-1},
\]

\[
\gamma^k_{ij} := \frac{1}{2} g^{km} \left( \frac{\partial g_{mj}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right),
\]

\[
N^i_j = \gamma^k_{ij} y^k - C^r_{jk} \gamma^k_{rs} y^r y^s.
\]

According to [Ch1], the pulled-back bundle \(\pi^*TM\) admits a unique linear connection, called the Chern connection. Its connection forms are characterized by the structure equation:

- Torsion freeness:
  \[dx^j \wedge \omega^i_j = 0;\]

- Almost \(g\)-compatibility:
  \[dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 2C_{ijk}(dy^k + N^k_i dx^i).\]

It is easy to know that torsion freeness is equivalent to the absence of \(dy^k\) terms in \(\omega^i_j\); namely,

\[\omega^i_j = \Gamma^i_{jk} dx^k,
\]

together with the symmetry

\[\Gamma^i_{jk} = \Gamma^i_{kj}.\]

Let \(V = v^i \partial / \partial x^i\) be a non-vanishing vector field on an open subset \(U \subset M\). One can introduce a Riemannian metric \(g_V\) and a linear connection \(\nabla^V\) on the tangent bundle over \(U\) as following:

\[
g_V(X, Y) := X^i Y^j g_{ij}(x, v), \quad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i};
\]

\[
\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x, v) \frac{\partial}{\partial x^k}.
\]
From the torsion freeness and $g$-compatibility of Chern connection we have

$$
\nabla^V_X Y - \nabla^V_Y X = [X, Y],
$$

(2.1)

$$
X g_V(Y, Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2 C_V(\nabla^V_X V, Y, Z),
$$

(2.2)

here $C_V$ is defined by

$$
C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, v),
$$

and it satisfies

$$
C_V(V, X, Y) = 0.
$$

(2.3)

The Chern curvature $R^V(X, Y)Z$ for vector fields $X, Y, Z$ on $\mathcal{U}$ is defined by

$$
R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X, Y]} Z.
$$

In the Riemannian case this curvature does not depend on $V$ and coincides with the Riemannian curvature tensor. For a flag $(V; \sigma)$ (or $(V; W)$) consisting of a non-zero tangent vector $V \in T_x M$ and a 2-plane $\sigma \subset T_x M$ with $V \in \sigma$ the flag curvature $K(V; \sigma)$ is defined as following:

$$
K(V; \sigma) = K(V; W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_W(W, W) - g_V(V, W)^2}.
$$

Here $W$ is a tangent vector, such that $V, W$ span the 2-plane $\sigma$ and $V \in T_x M$ is extended to a geodesic field, i.e., $\nabla^V_V V = 0$ near $x$. In the Riemannian case the flag curvature is the sectional curvature of the 2-plane $\sigma$ and does not depend on $V$. In the literature there are several connections used in Finsler geometry, but for the definition of the flag curvature it does not make a difference whether one uses the Chern, the Cartan or the Berwald connection. The Ricci curvature of $V$ is defined by

$$
Ric(V) = \sum_i K(V; E_i),
$$

where $E_1, \ldots, E_n$ is the local $g_V$-orthonormal frame over $\mathcal{U}$.

A Finsler metric $F$ on $M$ is called reversible if $F(-X) = F(X)$ for all $X \in TM$. In order to consider the non-reversible Finsler metric Rademacher [Ra] introduced the reversibility $\lambda = \lambda(M, F)$ as following

$$
\lambda := \sup_{X \in TM \setminus 0} \frac{F(-X)}{F(X)}.
$$

(2.4)

Clearly $\lambda \in [1, \infty]$ and $\lambda = 1$ if and only if $F$ is reversible.

Let $\gamma(t), 0 \leq t \leq l$ be a geodesic with unit speed velocity field $T$. A vector field $J$ along $\gamma$ is called to be a Jacobi field if it satisfies the following equation

$$
\nabla_T T T J + R^T(J, T) T = 0.
$$
For vector fields $X$ and $Y$ along $\gamma$, the index form $I_\gamma(X, Y)$ is defined by

$$I_\gamma = \int_0^l \left( g_T(\nabla^T X, \nabla^T Y) - g_T(R^T(X, T)T, Y) \right) dt.$$ 

Let

$$ct_c(t) = \begin{cases} \sqrt{c} \cdot \cotan(\sqrt{ct}), & c > 0 \\ \frac{1}{t}, & c = 0 \\ \sqrt{-c} \cdot \coth(-\sqrt{-ct}), & c < 0 \end{cases} \quad (2.5)$$

The following result is fundamental.

**Lemma 2.1** ([BCS], page 254) Let $(M, F)$ be a Finsler manifold and $\gamma(t)$, $0 \leq t \leq l$ be a geodesic with unit speed velocity field $T(t)$. Suppose:

- The flag curvature $K(T; W) \leq c$ for any $W \in T_\gamma(t)M$.
- $J$ is a Jacobi field along $\gamma$ that is $g_T$-orthogonal to $\gamma$.
- $J(0) = 0$.

Then for $0 < t \leq l$ when $c \leq 0$ or $0 < t < \frac{\pi}{\sqrt{c}}$ when $c > 0$,

$$\frac{g_T(\nabla^T J, J)}{g_T(J, J)} \geq ct_c(t).$$

### 3. Laplacian for Finsler Manifolds

In this section we shall introduce the Laplacian for Finsler manifolds adopted in [Sh2]. For this purpose, let us first recall the notion of Legendre transformation.

Given a Finsler manifold $(M, F)$, the dual Finsler metric $F^*$ on $M$ is defined by

$$F^*(\xi_x) := \sup_{Y \in T_x M \setminus 0} \frac{\xi(Y)}{F(Y)}, \forall \xi \in T^* M,$$

and the corresponding fundamental tensor is defined by

$$g^{*kl}(\xi) = \frac{1}{2} \frac{\partial^2 F^2(\xi)}{\partial \xi_k \partial \xi_l}.$$ 

The Legendre transformation $l : TM \to T^* M$ is defined by

$$l(Y) = \begin{cases} g_Y(Y, \cdot), & Y \neq 0 \\ 0, & Y = 0. \end{cases}$$

The following result is well-known (see [BCS, Sh2]).

**Lemma 3.1** For any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_x M \setminus 0$ onto $T^*_x M \setminus 0$, and it is norm-preserving, namely, $F(Y) = F^*(l(Y)), \forall Y \in TM$. Consequently, $g^{ij}(Y) = g^{*ij}(l(Y))$.

Now let $f : M \to \mathbb{R}$ be a smooth function on $M$. The gradient of $f$ is defined by $\nabla f = l^{-1}(df)$. Thus we have

$$df(X) = g_{\nabla f}(\nabla f, X), \quad X \in TM.$$
Let \( U = \{ x \in M : \nabla f |_x \neq 0 \} \). We define the Hessian \( H(f) \) of \( f \) on \( U \) as following:

\[
H(f)(X, Y) := XY(f) - \nabla_X Y f, \quad \forall X, Y \in TM|_U.
\]

From (2.1)-(2.3) we see that \( H(f) \) is symmetric, and it can be rewritten as

\[
H(f)(X, Y) = g_{\nabla f}(\nabla_X \nabla f, Y). \tag{3.2}
\]

It should be noted here that the notion of Hessian here is different from that in [Sh1-2]. In that case \( H(f) \) is in fact defined by

\[
H(f)(X, X) = XX(f) - \nabla_X X f, \quad \forall X \in TM|_U.
\]

In order to define the divergence for vector field, we need the volume form on \( M \). A volume form \( d\mu \) on \( M \) is nothing but a global non-degenerate \( n \)-form on \( M \). A frequently used volume form for \( (M, F) \) is the so-called Busemann-Hausdorff volume form \( dV_F \) which is locally expressed by

\[
dV_F = \sigma_F(x) dx^1 \wedge \cdots \wedge dx^n,
\]

where \( \sigma_F(x) := \frac{\operatorname{vol}(B^n(1))}{\operatorname{vol}\{(y^i) \in R^n : F(x, y^i \frac{\partial}{\partial x^i}) < 1\}} \).

In the following we consider the Finsler manifold \((M, F, d\mu)\) equipped with a volume form \( d\mu \). Let \( X \in TM \). The divergence \( \text{div}(X) \) of \( X \) is defined by

\[
d(X|d\mu) = \text{div}(X)d\mu. \tag{3.3}
\]

In local coordinate system \((x^i)\), express \( d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n \). Then for vector field \( X = X^i \partial/\partial x^i \) on \( M \),

\[
\text{div}(X) = \frac{1}{\sigma} \frac{\partial}{\partial x^i} (\sigma X^i). \tag{3.4}
\]

Applying the Stokes theorem to \( \eta = X|d\mu \) we have

**Lemma 3.2 ([Sh1-2])** Let \((M, F, d\mu)\) be a Finsler n-manifold. Let \( \Omega \) be a compact domain with smooth boundary \( \partial \Omega \) and \( \nu \) denote the outward pointing normal vector. Then for any smooth vector field \( X \) on \( M \),

\[
\int_{\Omega} \text{div}(X)d\mu = \int_{\partial \Omega} g_{\nu}(\nu, X)dA_\mu,
\]

where \( dA_\mu \) is the volume form on \( \partial \Omega \) induced from \( d\mu \).

For \( y \in T_xM\setminus 0 \), define

\[
\tau(y) := \log \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma} \tag{3.5}
\]

\( \tau \) is called the distorsion of \((M, F, d\mu)\). To measure the rate of the distorsion along geodesics, we define

\[
S(y) := \frac{d}{dt} \tau(\gamma(t))|_{t=0}, \tag{3.6}
\]
where \( \gamma(t) \) is the geodesic with \( \dot{\gamma}(0) = y \). \( S \) is called the S-curvature [Sh2], and it is an important non-Riemannian curvature for Finsler manifold. In local coordinates it can be expressed by [Sh2]

\[
S(y) = N_i^j(x, y) - \frac{y^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x). \tag{3.7}
\]

Now we are ready to introduce the Laplacian \( \Delta f \) of \( f \) as \( \Delta f = \text{div}(\nabla f) = \text{div}(\text{div}^{-1}(df)) \).

By Lemma 3.1 and (3.4) we have the following local express for \( \Delta f \).

\[
\Delta f = \frac{1}{\sigma(x)} \frac{\partial}{\partial x^i} \left( \sigma(x) g^{ij}(df) \frac{\partial f}{\partial x^j} \right) \tag{3.8}
\]

For later use we need the following invariant express for \( \Delta f \).

**Lemma 3.3** Let \((M, F, d\mu)\) be a Finsler \( n \)-manifold, and \( f : M \to \mathbb{R} \) the smooth function on \( M \). Then on \( U = \{ x \in M : \nabla f \neq 0 \} \) we have

\[
\Delta f = \sum_a H(f)(e_a, e_a) - S(\nabla f) := \text{tr}_{\nabla} H(f) - S(\nabla f),
\]

where \( e_1, \ldots, e_n \) is the local \( g_{\nabla f} \)-orthonormal frame on \( U \).

**Proof.** Write

\[
e_a = u_a^i \frac{\partial}{\partial x^i}, \quad \frac{\partial}{\partial x^i} = v_a^i e_a, \tag{3.9}
\]

then

\[
v_a^i u_b^i = \delta_a^b, \quad u_a^i v_b^i = \delta_a^b, \quad g^{ij} = \sum_a u_a^i u_a^j. \tag{3.10}
\]

Substituting (3.9) and (3.10) into (3.8) we get

\[
\Delta f = \sum_a e_a e_a(f) + (\nabla f)(\log \sigma) + \sum_b e_a(v_b^a) e_a(f). \tag{3.11}
\]

From (3.7) one has

\[
S(\nabla f) = N_i^j(\nabla f) - (\nabla f)(\log \sigma). \tag{3.12}
\]

Let \( \{\omega_a^b\} \) be the Chern connection form with respect to \( \{e_a\} \), then it is easy to deduce that (see [BCS], page 42)

\[
\omega_a^b = (du_b^i) v_a^i + u_b^j \omega_a^j v_a^i, \tag{3.13}
\]

\[
\omega_a^b + \omega_b^a = -2C_{abc} v_c^i (dy^i + N_i^j dx^j). \tag{3.14}
\]

Thus from (3.13) and (3.14) we have

\[
\sum_a \nabla_{e_a} f e_a = \sum_a \omega_a^b(e_a) e_b
\]
Noting that $\Gamma_{jk}^i (\nabla f)^k = N_j^i (\nabla f)$ ([BCS], page43), we deduce from (2.3), (3.9), (3.10) and (3.15) that

$$
\sum_a \nabla_{e_a} e_a(f) = - \sum_b \left( e_a(u_b^i)v_a^i e_b(f) + \Gamma_{ji}^i u_j^i u^k_b \frac{\partial f}{\partial x^k} \right) = - \sum_b e_a(u_b^i)v_a^i e_b(f) - N_j^i (\nabla f).
$$

Combining (3.1), (3.11), (3.12) and (3.16) we obtain the desired result.

### 4. The Hessian Comparison Theorem

In this section let us study the Hessian comparison theorem for distance function. For this purpose, let us first compute the Hessian of distance function.

Let $(M, F, d\mu)$ be a Finsler $n$-manifold, and $r = d_F(p, \cdot)$ is the distance function on $M$ from a fixed point $p \in M$. It is well-known that $r$ is smooth on $M \setminus \{p\}$ away from the cut points of $p$. Now we assume $\gamma$ is a unit-speed geodesic without a conjugate point up to distance $r$ from $p$. It is known that $F(\nabla r) = 1$ (see [Sh2], page 38), which together with the first variation of arc length implies that $\nabla r = T := \dot{\gamma}$. For any vector $X \in T_{\gamma(r)} M$, there exists a unique Jacobi field $J$ such that $J(0) = 0$, $J(r) = X$. We have, by (2.1)-(2.3) and (3.2),

$$
H(r)(X, X) = g_T(\nabla_X^T T, X) = g_T(\nabla_j^T T, J) \bigg|_{\gamma(0)} = \int_0^r \frac{d}{dt} g_T(\nabla_j^T T, J) dt
$$

$$
= \int_0^r \left( g_T(\nabla_T^T \nabla_j^T T, J) + g_T(\nabla_j^T T, \nabla_T^T J) \right) dt
$$

$$
= \int_0^r \left( g_T(R^T(T, J)T, J) + g_T(\nabla_T^T J, \nabla_T^T J) \right) dt
$$

$$
= I_\gamma(J, J) = I_\gamma(J^\perp, J^\perp) = g_T(\nabla_T^T J^\perp, J^\perp),
$$

where $J^\perp = J - g_T(T, J)T$. Now we can prove the following Hessian comparison theorem.

**Theorem 4.1** Let $(M, F, d\mu)$ be a Finsler $n$-manifold, $r = d_F(p, \cdot)$, the distance function from a fixed point $p$. Suppose that the flag curvature of $M$ satisfies $K(V; W) \leq c$ (resp. $K(V; W) \geq c$) for any $V, W \in TM$. Then for any vector $X$ on $M$ the following inequality holds whenever $r$ is smooth:

$$
H(r)(X, X) \geq (\text{resp.} \leq) \text{ct}_c(r) \left( g_{\nabla r}(X, X) - g_{\nabla r}(\nabla r, X)^2 \right).
$$
Proof. First we note that
\[ g_T(J^\perp, J^\perp)_{|_{\gamma(r)}} = g_T(X, X) - g_T(T, X)^2, \]
by (4.1) and Lemma 2.1 we conclude that in the case \( K(V; W) \leq c \) one has
\[ H(r)(X, X) \geq c \gamma(r)(g_{\varpi_r}(X, X) - g_{\varpi_r}(\nabla r, X)^2). \]
Now we consider the case \( K(V; W) \geq c \). For given \( X \), by parallel transformation along \( \gamma \) we obtain a vector field \( X(t) \) along \( \gamma \). We define a vector field \( W(t) \) along \( \gamma \) by
\[ W(t) = \frac{s_c(t)}{s_c(r)} X(t), \]
where
\[ s_c(t) = \begin{cases} \sin(\sqrt{ct}), & c > 0 \\ t, & c = 0 \\ \sinh(\sqrt{-ct}), & c < 0 \end{cases}. \]
It is clear that \( W(0) = J(0) = 0, W(r) = J(r) \), and consequently, \( W^\perp(0) = J^\perp(0) = 0, W^\perp(r) = J^\perp(r) \). Thus from (4.1) and the basic index lemma (see [BCS], page 182) we have
\[ H(r)(X, X) = I_{\gamma}(J^\perp, J^\perp) \leq I_{\gamma}(W^\perp, W^\perp) \]
\[ = \frac{g_T(X^\perp, X^\perp)}{s_c(r)^2} \int_0^r \{s_c'(t)^2 - K(T(t); W(t))s_c(t)^2\} dt \]
\[ \leq \frac{g_T(X^\perp, X^\perp)}{s_c(r)^2} \int_0^r \{s_c'(t)^2 - cs_c(t)^2\} dt = ct_c(r)g_T(X^\perp, X^\perp), \]
so we are done.

5. The Laplacian Comparison Theorems

In this section we shall derive some Laplacian comparison theorems for distance function. First of all, by Lemma 3.3 and Theorem 4.1 we have

**Theorem 5.1** Let \((M, F, d\mu)\) be a Finsler \(n\)-manifold, \( r = d_F(p, \cdot) \), the distance function from a fixed point \( p \). Suppose that the flag curvature of \( M \) satisfies \( K(V; W) \leq c \) for any \( V, W \in TM \). Then the following holds whenever \( r \) is smooth.
\[ \Delta r \geq (n - 1)ct_c(r) - \|S\|, \]
where \( \|S\| \) is the pointwise norm function of S-curvature which is defined by
\[ \|S\|_x = \sup_{X \in T_xM \setminus 0} \frac{S(X)}{F(X)}. \]
When \( M \) has nonpositive flag curvature we have the following Laplacian comparison theorem in terms of Ricci curvature.
Theorem 5.2 Let \((M, F, d\mu)\) be a Finsler \(n\)-manifold with nonpositive flag curvature. If the Ricci curvature of \(M\) satisfies \(Ric_M \leq c < 0\), then the following holds whenever \(r\) is smooth.

\[ \triangle r \geq ct_c(r) - \|S\|. \]

**Proof.** We need only to prove \(\text{tr}_{\nabla r} H(r) \geq ct_c(r)\). Suppose that \(r\) is smooth at \(q \in M\), then \(r\) is also smooth near \(q\). Let \(S_p(r(q))\) be the forward geodesic sphere of radius \(r(q)\) centered at \(p\). We choose the local \(g_T\)-orthonormal frame \(E_1, \ldots, E_{n-1}\) of \(S_p(r(q))\) near \(q\), here \(T = \nabla r\). By parallel transformation along geodesic rays we get local vector fields \(E_1, \ldots, E_{n-1}, E_n = T\) of \(M\). Then for any \(1 \leq i, j \leq n - 1\), we have by (2.1)-(2.3) and (3.2),

\[
\frac{d}{dr} (H(r)(E_i, E_j)) = \frac{d}{dr} g_T (\nabla^T_{E_i} T, E_j) = g_T (\nabla^T_T \nabla^T_{E_i} T, E_j)
\]

\[
= g_T (R^T(T, E_i) T, E_j) + g_T (\nabla^T_{[T, E_i]} T, E_j)
\]

\[
= -g_T (R^T(E_i, T) T, E_j) - g_T (\nabla^T_{E_i T} T, E_j)
\]

\[
= -g_T (R^T(E_i, T) T, E_j) - \sum_k g_T (\nabla^T_{E_k T} T, E_j) g_T (\nabla^T_{E_k} T, E_j),
\]

and consequently,

\[
\frac{d}{dr} \text{tr}_{\nabla r} H(r) = -Ric(\nabla r) - \sum_{i,j} (H(r)(E_i, E_j))^2. \tag{5.1}
\]

Since \(M\) has nonpositive flag curvature, it is easy to see from Theorem 4.1 that the eigenvalues of \(H(r)\) are nonnegative, which implies that

\[
\sum_{i,j} (H(r)(E_i, E_j))^2 \leq (\text{tr}_{\nabla r} H(r))^2,
\]

and (5.1) can be rewritten as

\[
\frac{d}{dr} \text{tr}_{\nabla r} H(r) \geq -c - (\text{tr}_{\nabla r} H(r))^2. \tag{5.2}
\]

Note that in this case \(c < 0\), and \(ct_c(r) = \sqrt{-c} \cdot \text{cotanh}(\sqrt{-c}r)\), from (5.2) we have

\[
\frac{d}{dr} (\text{tr}_{\nabla r} H(r) - ct_c(r)) \geq - (\text{tr}_{\nabla r} H(r))^2 + ct_c(r)^2. \tag{5.3}
\]

Putting

\[
A = \text{tr}_{\nabla r} H(r) - ct_c(r), \quad B = \text{tr}_{\nabla r} H(r) + ct_c(r),
\]

then (5.3) becomes

\[
\frac{dA}{dr} + AB \geq 0. \tag{5.4}
\]

We have again by the nonpositivity of flag curvature and Theorem 4.1 that

\[
\text{tr}_{\nabla r} H(r) \geq \frac{n-1}{r},
\]
which implies that there exist small $\varepsilon > 0$ so that
\[
A(r) \geq \frac{n-1}{r} - ct_c(r) \geq 0, \quad \forall r \in (0, \varepsilon].
\tag{5.5}
\]

On the other hand, from (5.4) we have
\[
\frac{d}{dr} \left( A(r) \exp \left( \int_{\varepsilon}^{r} B(\tau)d\tau \right) \right) \geq 0,
\]
which yields
\[
A(r) \exp \left( \int_{\varepsilon}^{r} B(\tau)d\tau \right) \geq A(\varepsilon) \geq 0,
\]
so we are done.

**Remark** In the Riemannian case, $S = 0$, and Theorem 5.2 was obtained by [Ding] (see also [Xin1]).

For the case where the curvature is bounded from below, we have the following comparison theorem.

**Theorem 5.3** Let $(M, F, d\mu)$ be a Finsler $n$-manifold with Ricci curvature satisfying $\text{Ric}_M \geq (n-1)c$. Then the following holds whenever $r$ is smooth.

\[
\triangle r \leq (n-1)ct_c(r) + \|S\|.
\tag{5.6}
\]

**Proof.** Let $r = d_F(p, \cdot)$ is smooth at $q \in M$, and $\gamma : [0, r(q)] \to M$ be the unit-speed geodesic from $p$ to $q$, and $T = \dot{\gamma}$. Let $e_1, \cdots, e_{n-1}, e_n = T$ be the $g_T$-orthonormal basis of $T_qM$. By parallel transformation along $\gamma$ we obtain the parallel vector fields $E_1(t), \cdots, E_n(t)$ along $\gamma$. For $1 \leq i \leq n-1$, let $J_i$ be the unique Jacobi field along $\gamma$ such that $J_i(0) = 0$, $J_i(r(q)) = e_i$, and $W_i(t) = \frac{s_c(t)}{s_c(r(q))} E_i(t)$, where $s_c(t)$ is defined by (4.2). Clearly, we have $W_i(0) = J_i(0) = 0, W_i(r(q)) = J_i(r(q))$. Thus from (4.1) and the basic index lemma (see [BCS], page 182) we have

\[
\text{tr}_{\nabla r} (H(r)) |_q = \sum_{i=1}^{n} H(r)(e_i, e_i) = \sum_{i=1}^{n-1} I_{\gamma}(J_i, J_i) \leq \sum_{i=1}^{n-1} I_{\gamma}(W_i, W_i)
= \frac{1}{s_c(r(q))^2} \int_{0}^{r(q)} \left\{ (n-1)s_c'(t)^2 - \text{Ric}(T(t))s_c(t)^2 \right\} dt
\leq \frac{1}{s_c(r(q))^2} \int_{0}^{r(q)} \left\{ (n-1)s_c'(t)^2 - (n-1)c s_c(t)^2 \right\} dt = (n-1)ct_c(r(q)),
\]
which together with Lemma 3.3 yields (5.6).
6. Volume Comparison Theorems

In this section we shall use the Laplacian comparison theorems to derive some volume comparison theorems for Finsler manifolds.

Let \((M, F, d\mu)\) be a Finsler \(n\)-manifold. Fix \(p \in M\), let \(I_p = \{v \in T_pM : F(v) = 1\}\) be the indicatrix at \(p\). For \(v \in I_p\), the cut-value \(c(v)\) is defined by
\[
c(v) := \sup\{t > 0 : d_F(p, \exp_p(tv)) = t\}.
\]
Then, we can define the tangential cut locus \(C(p)\) of \(p\) by \(C(p) := \{c(v)v : c(v) < \infty, v \in I_p\}\), the cut locus \(C(p)\) of \(p\) by \(C(p) = \exp_p C(p)\), and the injectivity radius \(i_p\) at \(p\) by \(i_p = \inf\{c(v) : v \in I_p\}\), respectively. It is known that \(C(p)\) has zero Hausdorff measure in \(M\). Also, we set \(D_p = \{tv : 0 \leq t < c(v), v \in I_p\}\) and \(D_p = \exp_p D_p\). It is known that \(D_p\) is the largest domain, starlike with respect to the origin of \(T_pM\), for which \(\exp_p\) restricted to that domain is a diffeomorphism, and \(D_p = M \setminus C(p)\).

Let \(B_p(R)\) be the forward geodesic ball of \(M\) with radius \(R\) centered at \(p\). The volume of \(B_p(R)\) with respect to \(d\mu\) is defined by
\[
\text{vol}(B_p(R)) = \int_{B_p(R)} d\mu.
\]
In order to compute the volume, we need the polar coordinates on \(D_p\). Let \(\theta^\alpha, \alpha = 1, \ldots, n - 1\) be the local coordinates that are intrinsic to \(I_p\). For any \(q \in D_p\), the polar coordinates of \(q\) is defined by \((r, \theta) = (r(q), \theta^1(q), \ldots, \theta^{n-1}(q))\), where \(r(q) = F(v), \theta^\alpha(q) = \theta^\alpha(\frac{v}{F(v)}),\) and \(v = \exp_p^{-1}(q)\). Then by the Gauss lemma (see [BCS], page 140), the unit radial coordinate vector \(\frac{\partial}{\partial r}\) is \(g_{\alpha\beta}\)-orthogonal to coordinate vectors \(\frac{\partial}{\partial \theta^\alpha}\) for \(\alpha = 1, \ldots, n - 1\). Therefore, writing \(d\mu = \sigma(r, \theta) dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} := \sigma(r, \theta) dr \wedge d\theta\), we have, from (3.8),
\[
\Delta r = \frac{\partial}{\partial r} \log \sigma.
\]
For \(r > 0\), let \(D_p(r) \subset I_p\) be defined by
\[
D_p(r) = \{v \in I_p : rv \in D_p\}.
\]
It is easy to know that \(D_p(r_1) \subset D_p(r_2)\) for \(r_1 > r_2\) and \(D_p(r) = I_p\) for \(r < i_p\). Since \(C(p)\) has zero Hausdorff measure in \(M\), we have
\[
\text{vol}(B_p(R)) = \int_{B_p(R)} d\mu = \int_{B_p(R) \cap D_p} d\mu = \int_{\exp_p^{-1}(B_p(R)) \cap D_p} \exp_p^*(d\mu) = \int_0^r dr \int_{D_p(r)} \sigma(r, \theta) d\theta.
\]
(6.2)
For real numbers \(c, \Lambda\) and positive integer \(n\), let
\[
V_{c,\Lambda,n}(r) = \text{vol}(S^{n-1}(1)) \int_0^r e^{\Lambda t} S_c(t)^{n-1} dt.
\]
(6.3)
We have

**Theorem 6.1** Let \((M, F, d\mu)\) be a complete Finsler \(n\)-manifold which satisfies \(K(V; W) \leq c\) and \(\|S\| \leq \Lambda\). Then the function

\[
\frac{\text{vol}(B_p(r))}{V_{c,-\Lambda,n}(r)}
\]

is monotone increasing for \(0 < r \leq i_p\), where \(i_p\) is the injectivity radius of \(p\). In particular, for \(d\mu = dV_F\), the Busemann-Hausdorff volume form, one has

\[
\text{vol}(B_p(r)) \geq V_{c,-\Lambda,n}(r), \quad r \leq i_p.
\]

(6.4)

**Proof.** By (6.1), Theorem 5.1 and the assumptions of the theorem, we have

\[
\frac{\partial}{\partial r} \log \sigma \geq (n - 1)ct_c(r) - \Lambda = \frac{d}{dr} \log \left( e^{-\Lambda r} s_c(r)^{n-1} \right),
\]

namely, the function

\[
\frac{\sigma(r, \theta)}{e^{-\Lambda r} s_c(r)^{n-1}}
\]

is monotone increasing in \(r\) for any \(\theta\). Let

\[
\sigma_p(r) = \int_{D_p(r)} \sigma(r, \theta) d\theta, \quad \sigma_{c,-\Lambda,n}(r) = \text{vol}(S^{n-1}(1)) e^{-\Lambda r} s_c(r)^{n-1}.
\]

Then from (6.2) and (6.3) we have

\[
\text{vol}(B_p(r)) = \int_0^r \sigma_p(t) dt, \quad V_{c,-\Lambda,n}(r) = \int_0^r \sigma_{c,-\Lambda,n}(t) dt.
\]

Noting that \(D_p(r) = I_p\) for \(r < i_p\), \(\frac{\sigma_p(r)}{\sigma_{c,-\Lambda,n}(r)}\) is also monotone increasing for \(r \leq i_p\). Thus by the standard argument [Ch2], the function

\[
\frac{\int_0^r \sigma_p(t) dt}{\int_0^r \sigma_{c,-\Lambda,n}(t) dt} = \frac{\text{vol}(B_p(r))}{V_{c,-\Lambda,n}(r)}
\]

is still monotone increasing for \(r \leq i_p\). From [Sh3] we see that for \(d\mu = dV_F\),

\[
\lim_{r \to 0} \frac{\text{vol}(B_p(r))}{V_{c,-\Lambda,n}(r)} = 1,
\]

thus we have (6.4).

The following theorem can be shown similarly by use of Theorem 5.2.

**Theorem 6.2** Let \((M, F, d\mu)\) be a complete and simply connected Finsler \(n\)-manifold with nonpositive flag curvature. If the Ricci curvature of \(M\) satisfies \(\text{Ric}_M \leq c < 0\) and \(\|S\| \leq \Lambda\), then the function

\[
\frac{\text{vol}(B_p(r))}{V_{c,-\Lambda,2}(r)}
\]
is monotone increasing. In particular, for \( d\mu = dV_F \),
\[
\text{vol}(B_p(r)) \leq \frac{\text{vol}(B^n(1))}{\text{vol}(B^2(1))} V_{c,-\Lambda,2}(r).
\]

The following theorem was first obtained in [Sh3], and here we provide another proof by use of Laplacian comparison theorem.

**Theorem 6.3** [Sh3]

Let \((M, F, d\mu)\) be a complete Finsler \(n\)-manifold. Suppose that
\[
\text{Ric}_M \geq (n-1)c, \quad \|S\| \leq \Lambda.
\]

Then the function
\[
\frac{\text{vol}(B_p(r))}{V_{c,\Lambda,n}(r)}
\]

is monotone decreasing in \( r \). In particular, for \( d\mu = dV_F \),
\[
\text{vol}(B_p(r)) \leq V_{c,\Lambda,n}(r).
\]

**Proof.** By (6.1), Theorem 5.3 and the assumptions of the theorem we have
\[
\frac{\partial}{\partial r} \log \sigma \leq (n-1)ct_{c} + \Lambda = \frac{d}{dr} \log \left(e^{\Lambda R_{c}(r)^{n-1}}\right),
\]

thus the function
\[
\frac{\sigma(r, \theta)}{e^{\Lambda R_{c}(r)^{n-1}}}
\]

is monotone decreasing. Noting that \( D_p(R) \subset D_p(r) \) for \( R > r > 0 \), we have for \( R > r > 0 \),
\[
\frac{\sigma_p(r)}{\sigma_{c,\Lambda,n}(r)} = \frac{1}{\text{vol}(S^{n-1}(1))} \int_{D_p(r)} \frac{\sigma(r, \theta)}{e^{\Lambda R_{c}(r)^{n-1}}} d\theta \geq \frac{1}{\text{vol}(S^{n-1}(1))} \int_{D_p(R)} \frac{\sigma(r, \theta)}{e^{\Lambda R_{c}(r)^{n-1}}} d\theta = \frac{\sigma_p(R)}{\sigma_{c,\Lambda,n}(R)},
\]

namely, \( \frac{\sigma_p(r)}{\sigma_{c,\Lambda,n}(r)} \) is also monotone decreasing. Now the theorem can be verified easily.

### 7. The First Eigenvalue

In this section we shall study the first eigenvalue for Finsler manifolds and proof some Mckean type theorems. We need some lemmas.

**Lemma 7.1** Let \((M, F)\) be a Finsler manifold with finite reversibility \( \lambda \), then \( |\xi(X)| \leq \lambda F^*(\xi)F(X) \) for any \( X \in TM, \xi \in T^*M \).

**Proof.** By the definition of \( F^* \) we actually have \( \xi(X) \leq F^*(\xi)F(X) \). On the other hand, from the definition of reversibility one has \(-\xi(X) = \xi(-X) \leq F^*(\xi)F(-X) \leq \lambda F^*(\xi)F(X)\), so the lemma follows.

Now let \((M, F, d\mu)\) be a Finsler \(n\)-manifold, \( \Omega \subset M \) a domain with compact
closure and nonempty boundary \( \partial \Omega \). The first eigenvalue \( \lambda_1(\Omega) \) of \( \Omega \) is defined by (see [Sh2], page 176)

\[
\lambda_1(\Omega) = \inf_{f \in L^2_{1,0}(\Omega) \setminus \{0\}} \left\{ \frac{\int_{\Omega} (F^*(df))^2 d\mu}{\int_{\Omega} f^2 d\mu} \right\},
\]

where \( L^2_{1,0}(\Omega) \) is the completion of \( C_0^\infty \) with respect to the norm

\[
\| \varphi \|^2 = \int_{\Omega} \varphi^2 d\mu + \int_{\Omega} (F^*(df))^2 d\mu.
\]

If \( \Omega_1 \subset \Omega_2 \) are bounded domains, then \( \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \geq 0 \). Thus, if \( \Omega_1 \subset \Omega_2 \subset \cdots \subset M \) be bounded domains so that \( \bigcup \Omega_i = M \), then the following limit

\[
\lambda_1(M) = \lim_{i \to \infty} \lambda_1(\Omega_i) \geq 0
\]

exists, and it is independent of the choice of \( \{ \Omega_i \} \). We have the following lemma which is crucial in this section.

**Lemma 7.2** Let \((M, F, d\mu)\) be a Finsler manifold with finite reversibility \( \lambda \), \( \Omega \subset M \) a domain with compact closure and nonempty boundary, and \( X \) a vector field on \( \Omega \) so that \( \|X\|_{\infty} = \sup_{\Omega} F(X) < \infty \) and \( \inf_{\Omega} \text{div}(X) > 0 \). Then

\[
\lambda_1(\Omega) \geq \frac{\left[ \inf_{\Omega} \text{div}X \right]^2}{2 \lambda \|X\|_{\infty}},
\]

(7.1)

**Proof.** Let \( f \in C_0^\infty \), then vector field \( f^2 X \) has compact support in \( \Omega \). Now computing the divergence of \( f^2 X \) we have by Lemma 7.1,

\[
\text{div}(f^2 X) = 2fX(f) + f^2 \text{div}X
\]

\[
\geq -2\lambda |f| \cdot \sup_{\Omega} F(X) \cdot F^*(df) + \inf_{\Omega} \text{div}X \cdot f^2.
\]

(7.2)

Using the inequality

\[
-2 |f| \cdot F^*(df) \geq -\varepsilon f^2 - \frac{1}{\varepsilon} (F^*(df))^2
\]

for all \( \varepsilon > 0 \), we have from (7.2) that

\[
\text{div}(f^2 X) \geq \lambda \cdot \sup_{\Omega} F(X) \cdot \left( -\varepsilon f^2 - \frac{1}{\varepsilon} (F^*(df))^2 \right) + \inf_{\Omega} \text{div}X \cdot f^2.
\]

(7.3)

Integrating (7.3) on \( \Omega \) and using Lemma 3.2 we have

\[
0 = \int_{\Omega} \text{div}(f^2 X) d\mu
\]

\[
\geq \lambda \cdot \|X\|_{\infty} \int_{\Omega} \left( -\varepsilon f^2 - \frac{1}{\varepsilon} (F^*(df))^2 \right) d\mu + \inf_{\Omega} \text{div}X \cdot \int_{\Omega} f^2 d\mu.
\]
Therefore,
\[
\int_{\Omega} \left( F^*(df) \right)^2 \, d\mu \geq \frac{\varepsilon}{\lambda \|X\|_\infty} \left( \inf_{\Omega} \text{div} X - \lambda \cdot \|X\|_\infty \cdot \varepsilon \right) \int_{\Omega} f^2 \, d\mu.
\]
Choosing \( \varepsilon = \inf_{\Omega} \text{div} X/(2\lambda \cdot \|X\|_\infty) \) we have
\[
\int_{\Omega} \left( F^*(df) \right)^2 \, d\mu \geq \left[ \inf_{\Omega} \text{div} X / (2\lambda \|X\|_\infty) \right]^2 \int_{\Omega} f^2 \, d\mu. \tag{7.4}
\]
Since (7.4) holds for any \( f \in C^\infty_0(\Omega) \), we have (7.1).

Now we are in the position to prove the first main result of this section.

**Theorem 7.3** Let \((M, F, d\mu)\) be a Finsler \(n\)-manifold with finite reversibility \(\lambda\) and flag curvature \(K(V; W) \leq c\) for any \(V, W \in TM\). Let \(B_p(R)\) be the forward geodesic ball of \(M\) with radius \(R\) centered at \(p\), and \(R < i_p\), where \(i_p\) denotes the injectivity radius about \(p\). Suppose that
\[
(n-1)c_t c(R) - \sup_{B_p(R)} \|S\| > 0,
\]
then
\[
\lambda_1(B_p(R)) \geq \left[ \frac{(n-1)c_t c(R) - \sup_{B_p(R)} \|S\|}{2\lambda} \right]^2. \tag{7.5}
\]

**Proof.** For \(R > \varepsilon > 0\), let \(\Omega_{\varepsilon} = B_p(R) \setminus B_p(\varepsilon)\). Then \(r = d_f(p, \cdot)\) is smooth on \(\Omega_{\varepsilon}\), and thus \(X = \nabla r\) is a smooth vector field on \(\Omega_{\varepsilon}\). Noting that \(F(X) = F(\nabla r) = 1\) and \(\text{div} X = \triangle r\), we deduce from Theorem 5.1 and Lemma 7.2 that
\[
\lambda_1(\Omega_{\varepsilon}) \geq \left[ \frac{(n-1)c_t c(R) - \sup_{B_p(R)} \|S\|}{2\lambda} \right]^2.
\]
Letting \(\varepsilon \to 0\) we get (7.5).

By Theorem 7.3 we have the following result which is the Finsler version of Mckean’s theorem.

**Theorem 7.4** Let \((M, F, d\mu)\) be a complete noncompact and simply connected Finsler \(n\)-manifold with finite reversibility \(\lambda\) and flag curvature \(K(V; W) \leq -a^2\) \((a > 0)\). If \(\sup_M \|S\| < (n-1)a\), then
\[
\lambda_1(M) \geq \frac{((n-1)a - \sup_M \|S\|)^2}{4a^2}.
\]

The following result can be verified by use of Theorem 5.2 which is another Finsler version of Mckean’s theorem in term of Ricci curvature.

**Theorem 7.5** Let \((M, F, d\mu)\) be a complete noncompact and simply connected Finsler \(n\)-manifold with finite reversibility \(\lambda\) and nonpositive flag curvature. If \(\text{Ric}_M \leq -a^2\) \((a > 0)\) and \(\sup_M \|S\| < a\), then
\[
\lambda_1(M) \geq \frac{(a - \sup_M \|S\|)^2}{4a^2}.
\]
8. ON CURVATURE AND FUNDAMENTAL GROUP

In 1968 Milnor [Mi] studied the curvature and fundamental group of Riemannian manifolds and proved that the fundamental group of a compact Riemannian manifold with strictly negative sectional curvature has at least exponential growth. The key in the proof is that the fundamental group can be identified with the deck transformation group of the universal covering space, and any geodesic ball in universal covering space can be covered by the union of a number of translations of the fundamental domain. Combining with the estimate of the volume growth Milnor was able to obtain his result. His result was generalized in [Y] and [Xin2]. Milnor’s idea can be generalized to the Finsler setting. As the first step, we have, from Theorems 6.1 and 6.2,

Lemma 8.1 Let $(M, F, d\mu)$ be a simply connected and complete Finsler $n$-manifold with $\|S\| \leq \Lambda$. Suppose that one of the following two conditions holds:
(i) the flag curvature of $M$ satisfies $K(V; W) \leq -a^2$ with $a > \Lambda/(n-1)$;
(ii) $M$ has nonpositive flag curvature and $\text{Ric}_M \leq -a^2$ with $a > \Lambda$.
Then the volume of the forward geodesic ball of $M$ grows at least exponentially.

For the universal covering space of a Finsler manifold, we can endow the covering space with a pulled-back Finsler metric so that the covering map is a local isometry. With Lemma 8.1 at hand, we can prove the following theorem by the almost same argument as in [Mi].

Theorem 8.2 Let $(M, F, d\mu)$ be a compact Finsler $n$-manifold with $\|S\| \leq \Lambda$. Suppose that one of the following two conditions holds:
(i) the flag curvature of $M$ satisfies $K(V; W) \leq -a^2$ with $a > \Lambda/(n-1)$;
(ii) $M$ has nonpositive flag curvature and $\text{Ric}_M \leq -a^2$ with $a > \Lambda$.
Then the fundamental group of $M$ grows at least exponentially.

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