Extreme-Value Analysis of Standardized Gaussian Increments

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Abstract
Let \( \{X_i, i = 1, 2, \ldots\} \) be i.i.d. standard gaussian variables. Let \( S_n = X_1 + \ldots + X_n \) be the sequence of partial sums and
\[
L_n = \max_{0 \leq i < j \leq n} \frac{S_j - S_i}{\sqrt{j-i}}.
\]
We show that the distribution of \( L_n \), appropriately normalized, converges as \( n \to \infty \) to the Gumbel distribution. In some sense, the random variable \( L_n \), being the maximum of \( n(n+1)/2 \) dependent standard gaussian variables, behaves like the maximum of \( Hn \log n \) independent standard gaussian variables. Here, \( H \in (0, \infty) \) is some constant. We also prove a version of the above result for the Brownian motion.

Keywords: Standardized increments, multiscale statistics, Gumbel distribution, Levy’s continuity modulus, Darling-Erdös theorem, Erdös-Renyi law of large numbers, Pickands’ method of double sums, locally-stationary gaussian fields.
1 Introduction

A basic result in extreme-value theory says that if \( \{X_i, i \in \mathbb{N}\} \) are independent standard normal random variables, then the distribution of \( M_n = \max\{X_1, \ldots, X_n\} \) converges, after appropriate normalization, to the Gumbel law. More precisely, let

\[
a_n = \sqrt{2 \log n} + \frac{-1/2 \log \log n - \log 2\sqrt{\pi}}{\sqrt{2 \log n}}, \quad b_n = \frac{1}{\sqrt{2 \log n}}. \tag{1}
\]

Then, for every \( \tau \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mathbb{P} [M_n \leq a_n + b_n \tau] = \exp(-e^{-\tau}). \tag{2}
\]

It is also well known that the above result remains true for dependent gaussian variables if the dependence is weak enough. We mention only one example, due to Berman (see [25, Chapter 4]). Let \( \{X_i, i \in \mathbb{N}\} \) be a stationary centered gaussian sequence with constant variance 1 such that the covariance function \( r(n) = \text{Cov}(X_1, X_n) \) satisfies \( r(n) = o(1/\log n) \) as \( n \to \infty \). Then (2) holds with the same normalizing constants.

An example of a situation where the dependence cannot be ignored is given by the Darling-Erdős theorem [9].

**Theorem 1.1.** Let \( \{X_i, i \in \mathbb{N}\} \) be i.i.d. standard normal variables. Define \( S_n = X_1 + \ldots + X_n \) and let

\[
M_n = \max_{k \in \{1, \ldots, n\}} \frac{S_k}{\sqrt{k}}.
\]

Then, for every \( \tau \in \mathbb{R} \),

\[
\mathbb{P} [M_n \leq a_n + b_n \tau] \to \exp(-e^{-\tau}),
\]

where

\[
a_n = \sqrt{2 \log \log n} + \frac{1/2 \log \log n - \log 2\sqrt{\pi}}{\sqrt{2 \log n}}, \quad b_n = \frac{1}{\sqrt{2 \log n}}.
\]

The next theorem, together with a strong approximation argument, was used by Darling and Erdős to prove Theorem 1.1.
Theorem 1.2. Let \( \{B(x), x \geq 0\} \) be the standard Brownian motion. For \( n > 1 \) define
\[
M_n = \sup_{x \in [1,n]} \frac{B(x)}{\sqrt{x}}.
\]
Then, for every \( \tau \in \mathbb{R} \),
\[
P [M_n \leq a_n + b_n \tau] \to \exp(-e^{-\tau}),
\]
where the normalizing constants are the same as in the previous theorem.

Theorem 1.2 may be viewed as a distributional convergence version of the law of the iterated logarithm. In somewhat unusual form (see Theorem 14.15 in [34]), the law of the iterated logarithm states that, almost surely,
\[
\lim_{n \to \infty} \frac{1}{\sqrt{2 \log \log n}} \sup_{x \in [1,n]} \frac{B(x)}{\sqrt{x}} = 1.
\]
See [22] for another distributional convergence version of the law of the iterated logarithm.

Of course, the Darling-Erdős theorem is true not only for standard normal variables. A necessary and sufficient condition on the distribution of the i.i.d. variables \( X_i \) for the Darling-Erdős theorem to hold was found by Einmahl [13]. Bertoin [4] proved an analog of the Darling-Erdős theorem for random variables with distributions attracted to stable laws.

The next theorem is the main result of this paper.

Theorem 1.3. Let \( \{X_i, i \in \mathbb{N}\} \) be i.i.d. standard normal random variables. Define \( S_n = X_1 + \ldots + X_n \) and \( S_0 = 0 \). Let
\[
L_n = \max_{0 \leq i < j \leq n} \frac{S_j - S_i}{\sqrt{j - i}}.
\]
Then, for every \( \tau \in \mathbb{R} \),
\[
\lim_{n \to \infty} P [L_n \leq a_n + b_n \tau] = \exp(-e^{-\tau}),
\]
where \( a_n \) and \( b_n \) are given by
\[
a_n = \sqrt{2 \log n} + \frac{1/2 \log \log n + \log H - \log 2 \sqrt{\pi}}{\sqrt{2 \log n}}, \quad b_n = \frac{1}{\sqrt{2 \log n}}
\]
for some constant \( H \in (0, \infty) \).
The constant $H$ is defined as follows. Let \( \{B(t), t \geq 0\} \) be the standard Brownian motion. Let

\[
F(a) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \exp \sup_{t \in [0,T] \cap \mathbb{N}Z} (B(t) - t/2) \right]
\]

and

\[
G(y) = \frac{1}{y^2} F \left( \frac{2}{y} \right)^2. \tag{4}
\]

Then $H = 4 \int_0^\infty G(y) dy$. A more explicit formula for $H$ will be given later in Section 7.

The motivation for studying the distribution of $L_n$ was the fact that $L_n$ as well as related quantities are of interest in statistics [10, 11]. The question about the asymptotic distribution of $L_n$ was studied by Huo [17, 18]. Note, however, that his result does not imply Theorem 1.3. In particular, the normalizing constants in [18] differ from the values given in (3) and are, in fact, random variables.

The next theorem describes the almost sure limiting behavior of $L_n$. It is a consequence of a more general result due to Shao [36], who proved a conjecture of Révész [34, §14.3] (see also [38] for a simplification of Shao’s proof and [24] for a related result).

**Theorem 1.4.** With the notation of Theorem 1.3 we have, almost surely,

\[
\lim_{n \to \infty} \frac{L_n}{\sqrt{2 \log n}} = 1.
\]

The next theorem may be viewed as a distributional convergence version of the Erdős-Rényi law of large numbers in the case of standard normal summands and is a consequence of a more general result of Komlós and Tusnády proved in [23] (see also [31, 39, 40]). We give a short proof of this theorem in Section 5.

\[\text{\footnotesize 1After the second version of this paper was submitted to arXiv, the author became aware that Theorem 1.3 was proved in D. Siegmund, E. S. Venkatraman. Using the generalized likelihood ratio statistic for sequential detection of a change-point. Ann. Statist. 23(1995), 255-271. For a related result see also D. Siegmund, B. Yakir. Tail probabilities for the null distribution of scanning statistics. Bernoulli 6(2000), 191-213.}\]
Theorem 1.5. Let \( \{X_i, i \in \mathbb{N}\} \) be i.i.d. standard normal random variables. Fix some \( c > 0 \) and let \( l_n = [c \log n] \). Define \( S_n = X_1 + \ldots + X_n \) and let

\[
L_{n,c} = \frac{1}{\sqrt{l_n}} \sup_{0 \leq k \leq n - l_n} (S_{k+l_n} - S_k).
\]

Then, for every \( \tau \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left[ L_{n,c} \leq a_n + b_n \tau \right] = \exp(-e^{-\tau}),
\]

where the constants \( a_n \) and \( b_n \) are given by

\[
a_n = \sqrt{2 \log n + \frac{3}{2} \log \log n - \log 2\sqrt{\pi}} - \frac{1}{\sqrt{2 \log n}}, \quad b_n = \frac{1}{\sqrt{2 \log n}}.
\]

We also prove the following continuous counterpart of Theorem 1.3.

Theorem 1.6. Let \( \{B(x), x \geq 0\} \) be the standard Brownian motion. For \( n > 1 \) define

\[
L_n = \sup_{x_1, x_2 \in [0,1]} \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} \text{ s.t. } x_2 - x_1 \geq 1/n.
\]

Then, for every \( \tau \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left[ L_n \leq a_n + b_n \tau \right] = \exp(-e^{-\tau}),
\]

where the constants \( a_n \) and \( b_n \) are given by

\[
a_n = \sqrt{2 \log n + 3/2 \log \log n - \log 2\sqrt{\pi}} - \frac{1}{\sqrt{2 \log n}}, \quad b_n = \frac{1}{\sqrt{2 \log n}}.
\]

Recall that a classical theorem of Lévy on the modulus of continuity of Brownian sample paths (see e.g. [20]) asserts that, almost surely,

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2 \log n}} \sup_{x_1, x_2 \in [0,1]} \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} = 1.
\]

It is not difficult to deduce from this that

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2 \log n}} \sup_{x_1, x_2 \in [0,1]} \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} = 1.
\]
Thus, Theorem 1.6 may be viewed as a distributional convergence version of Lévy’s modulus of continuity.

Since the normalizing constants in Theorems 1.3 and 1.6 are different, it seems to be impossible to deduce Theorem 1.3 from its continuous counterpart Theorem 1.6 by a strong approximation argument as it was done by Darling and Erdős in their proof of Theorem 1.1.

2 Asymptotic Extreme-Value Rate

In this section we are going to introduce the notion of asymptotic extreme-value rate, which will allow us to compare the results of Theorems 1.1, 1.2, 1.3, 1.6 with the classical extreme-value theorem for i.i.d. normal variables stated at the beginning of the paper. Let \( \{\xi_i, i = 1, \ldots, N\} \) and \( \{\eta_i, i = 1, \ldots, N\} \) be two jointly gaussian vectors. We suppose that the variables \( \xi_i \) and \( \eta_i \) are centered and have variance 1. Suppose, moreover, that the variables \( \eta_i \) are independent, whereas \( \xi_i \) are not. Then it is well known that, in some sense, \( \max_{i=1,\ldots,N} \xi_i \) is dominated by \( \max_{i=1,\ldots,N} \eta_i \). One way to make this claim precise is the Slepian Comparison Lemma (see e.g. [25, Corollary 4.2.3]) which states that, for every \( u \),

\[
P\left[ \max_{i=1,\ldots,N} \xi_i > u \right] \leq P\left[ \max_{i=1,\ldots,N} \eta_i > u \right].
\]

Given a dependent vector \( \{\xi_i, i = 1, \ldots, N\} \) of standard normal variables, we would like to determine the number \( f(N) \) of independent standard normal variables \( \{\eta_i, i = 1, \ldots, f(N)\} \) such that behavior of \( \max_{i=1,\ldots,f(N)} \eta_i \) is in some sense close to the behavior of the maximum of the dependent vector \( \xi_i \). By the above, we should have \( f(N) \leq N \). The next definition makes this precise.

**Definition 2.1.** For each \( n \in \mathbb{N} \) let a gaussian field \( \{\xi_n(t), t \in T_n\} \) defined on some parameter space \( T_n \) be given. Suppose that for all \( n \) the field \( \xi_n \) is centered and has constant variance 1. Let \( f : \mathbb{N} \to \mathbb{R} \) be some function. We say that the sequence \( \xi_n \) has asymptotic extreme-value rate \( f \) if, for each \( \tau \in \mathbb{R} \),

\[
\lim_{n \to \infty} P\left[ \sup_{t \in T_n} \xi_n(t) \leq a_{f(n)} + b_{f(n)} \tau \right] = \exp(-e^{-\tau}),
\]

where \( a_n \) and \( b_n \) are constants defined in (1).
Thus, the sequence of gaussian fields $\xi_n$ is said to have asymptotic extreme-value rate $f$ if, for large $n$, the supremum of $\xi_n$ has the same behavior as the supremum of $f(n)$ i.i.d. standard normal variables.

Now we are going to compute the extreme-value rates of gaussian fields defined in Theorems 1.1, 1.2, 1.3, 1.5, 1.6. To this end, we need two simple lemmas. The first one can be proved by a simple calculation. For the second lemma, which is due to Khintchine, see e.g. [25, Theorem 1.2.3].

**Lemma 2.2.** Let the constants $a_n, b_n$ be defined by (1) and let $f(n) = cn(\log n)^b$. Then, as $n \to \infty$,

$$a_{f(n)} = \sqrt{2\log n} + \frac{(-1/2 + b) \log \log n + \log c - \log 2\sqrt{\pi}}{\sqrt{2\log n}} + o\left(\frac{1}{\sqrt{2\log n}}\right),$$

$$b_{f(n)} \sim \frac{1}{\sqrt{2\log n}}.$$

**Lemma 2.3.** Let $M_n$ be a sequence of random variables such that, for some constants $a'_n, b'_n$, the distribution of $(M_n - a'_n)/b'_n$ converges as $n \to \infty$ to some non-degenerate distribution function $G$. Let another constants $a''_n, b''_n$ be given and suppose that

$$\lim_{n \to \infty} b'_n/b''_n = 1, \quad \lim_{n \to \infty} (a'_n - a''_n)/b'_n = 1.$$

Then the distribution of $(M_n - a''_n)/b''_n$ converges to $G$ as well.

Using the above two lemmas, one deduces easily that the gaussian fields considered in Theorems 1.1, 1.2, 1.3, 1.5, 1.6 have asymptotic extreme-value rates given in the following table. The usual notation is used, i.e. $\{X_k, k \in \mathbb{N}\}$ is a sequence of i.i.d. standard normal variables, $S_n = X_1 + \ldots + X_n$ are the
partial sums and \( \{B(x), x \geq 0\} \) is the standard Brownian motion.

| \( T_n \) | \( \xi_n \) | \( f(n) \) |
|---|---|---|
| 1. \( \{1, \ldots, n\} \) | \( \xi_n(k) = X_k \) | \( n \) |
| 2. \( \{1, \ldots, n\} \) | \( \xi_n(k) = \frac{S_k}{\sqrt{k}} \) | \( \log n \log \log n \) |
| 3. \( [1, n] \) | \( \xi_n(x) = \frac{B(x)}{\sqrt{x}} \) | \( \log n \log \log n \) |
| 4. \{\( (i, j) \mid 0 \leq i < j \leq n \}\} | \( \xi_n(i, j) = \frac{S_j - S_i}{\sqrt{j - i}} \) | \( Hn \log n \) |
| 5. \( \left\{ (x_1, x_2) : x_1, x_2 \in [0, 1], x_2 - x_1 \geq 1/n \right\} \) | \( \xi_n(x_1, x_2) = \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} \) | \( n(\log^2 n) \) |
| 6. \( \{0, 1, \ldots, n - [c \log n]\} \) | \( \xi_n(k) = \frac{S_{k+c \log n} - S_k}{\sqrt{c \log n}} \) | \( (4/c)F(4/c)n \) |
| 7. \( \left\{ (x_1, x_2) : x_1, x_2 \in [0, 1], x_2 - x_1 = 1/n \right\} \) | \( \xi_n(x_1, x_2) = \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}} \) | \( n \log n \) |

Note that entry 7 can be easily deduced from Pickands’ results \([27]\) (or see \([25, \text{Chapter 12}]\)).

It is a priori clear that the asymptotic rate of entry 2 in the above table should not be faster than the rate of entry 3. The reason is that the distribution of \( \{S_k/\sqrt{k}, k = 1, \ldots, n\} \) may be identified with the distribution of \( \{B(k)/\sqrt{k}, k = 1, \ldots, n\} \). In fact, as Darling and Erdös showed, the rates in entry 2 and entry 3 are equal. Similarly, there is an embedding of the gaussian vector from the entry 4 into the process from the entry 5, namely one can identify \( \{(S_j - S_i)/\sqrt{j - i}, 0 \leq i < j \leq n\} \) with \( \{(B(j/n) - B(i/n))/\sqrt{(j - i)/n}, 0 \leq i < j \leq n\} \). Thus, it is clear that the rate of entry 4 is not faster than that of entry 5. A somewhat surprising fact is that these rates do not coincide.

The rest of the paper is organized as follows. In Section 3 we recall the definition of locally stationary gaussian fields. The main results of this section are Corollary 3.15 and Corollary 3.18. In Section 4 we prove Theorem 1.6. The main tools are Corollary 3.15 and Berman’s inequality. The proof of Theorem 1.5 is given in Section 5. Finally, Section 6 is devoted to the proof of Theorem 1.3.
3 Locally Stationary Gaussian Fields

Given a centered gaussian field \( \{X(t), t \in \mathbb{R}^d\} \) with constant variance 1 we would like to obtain an exact asymptotics of the so-called high excursion probability of \( X \) over a given compact set \( K \), i.e. a result of the form

\[
P \left[ \sup_{t \in K} X(t) > u \right] \sim C_K u^D e^{-u^2/2}, \quad u \to \infty
\]  

for a number \( D \) depending on the structure of the field and a constant \( C_K \) depending on the set \( K \subset \mathbb{R}^d \) and the structure of the field.

After preliminary results by Cramer, Leadbetter, Volkonski, Rozanov, Ber- man, Slepian and others, this question was studied by Pickands \[27, 28\] (see also \[25, Chapter 12\], \[29\], \[30\]). To state his result, let \( \{X(t), t \in \mathbb{R}\} \) be a stationary centered gaussian process whose covariance function \( r(s) = E[X(0)X(s)] \) satisfies

\[
r(s) = 1 - C|s|^\alpha + o(|s|^\alpha), \quad s \to 0
\]

for some \( \alpha \in (0, 2] \), called the index of the process \( X \), and some \( C > 0 \). Suppose also that \( r(s) = 1 \) holds only for \( s = 0 \). Under these conditions, Pickands proved the asymptotic equality

\[
P \left[ \sup_{t \in [0,l]} X(t) > u \right] \sim lH_\alpha C^{1/\alpha} \frac{1}{\sqrt{2\pi}} u^{2/\alpha - 1} e^{-u^2/2}, \quad u \to \infty,
\]

where \( H_\alpha \in (0, \infty) \) is some constant. Only the values \( H_1 = 1 \) and \( H_2 = 1/\sqrt{\pi} \) are known rigorously. There is a conjecture that \( H_\alpha = 1/\Gamma(1/\alpha) \) (see \[6\]).

Pickands’ result was generalized by Qualls and Watanabe \[32, 33\], who allowed a slightly more general class of covariance functions and considered isotropic fields defined on the \( d \)-dimensional euclidian space; by Bickel and Rosenblatt \[5\], who considered two-dimensional stationary fields; by Al- bin \[1\], who considered non-gaussian stationary processes, as well as by many others. However in this paper, we need an estimate of the form \[5\] for non-stationary gaussian fields. On a heuristical level, Aldous \[2\] applied his method of Poisson clumping heuristic, which is close to Pickands’ method, to many non-stationary fields. In \[19\], Hüsler applied Pickands’ methods to study the high excursion probability for non-stationary centered gaussian processes defined on the real line with covariance function
\[ r(t_1, t_2) = \mathbf{E}[X(t_1)X(t_2)] \] satisfying
\[ r(t, t + s) = 1 - C(t)|s|^\alpha + o(|s|^\alpha), \quad s \to 0 \]
uniformly on compacts in \( t \) for some continuous function \( C(t) > 0 \). Hüsler calls such processes \textit{locally stationary}. It should be noted that not every stationary process is locally stationary. Hüsler proves that, as \( u \to \infty \),
\[
\mathbf{P} \left[ \sup_{t \in [l_1, l_2]} X(t) > u \right] \sim H_\alpha \left( \int_{l_1}^{l_2} C^{1/\alpha}(t) dt \right) \frac{1}{\sqrt{2\pi}} u^{2/\alpha-1} e^{-u^2/2}.
\]
Thus, the function \( C^{1/\alpha}(t) \) may be thought of as a sort of intensity measuring the contribution of the point \( t \) to the high excursion probability.

The notion of locally stationary processes was extended to fields defined on the \( d \)-dimensional euclidian space (or, even more generally, on compact manifolds) by Mikhaleva and Piterbarg in [26] and by Chan and Lai in [7].

First we recall the definition of homogeneous functions.

**Definition 3.1.** A function \( f: \mathbb{R}^d \to \mathbb{R} \) is called homogeneous of order \( \alpha > 0 \) if for each \( s \in \mathbb{R}^d \) and \( \lambda \in \mathbb{R} \)
\[ f(\lambda s) = |\lambda|^\alpha f(s). \]
In particular, homogeneous functions are symmetric, i.e. they satisfy \( f(s) = f(-s) \). Let \( H(\alpha) \) be the set of all continuous homogeneous functions of order \( \alpha \). For \( f \in H(\alpha) \) define \( \|f\| = \sup_{\|t\|_2 = 1} f(t) \). With this norm, \( H(\alpha) \) is a Banach space which can be identified with the space \( C(S^{d-1}) \) of continuous functions on the unit sphere in \( \mathbb{R}^d \).

Let \( H^+(\alpha) \) be the cone of all strictly positive functions in \( H(\alpha) \).

Now we are ready to define locally stationary gaussian fields.

**Definition 3.2 (see [7]).** Let \( \{X(t), t \in D\} \) be a centered gaussian field with constant variance 1 defined on some domain \( D \subset \mathbb{R}^d \). Let \( r(t_1, t_2) = \mathbf{E}[X(t_1)X(t_2)] \) be the covariance function of \( X \) and suppose that it satisfies \( r(t_1, t_2) = 1 \iff t_1 = t_2 \). The field \( X \) is called locally stationary with index \( \alpha \in (0, 2] \) if for each \( t \in D \) a continuous function \( C_t \in H^+(\alpha) \) exists such that the following conditions hold
\[
1. \quad \lim_{\|s\|_2 \to 0} \frac{1 - r(t, t + s)}{C_t(s)} = 1
\]
uniformly on compacts.
2. The map $C \bullet : D \to H^+(\alpha)$, sending $t$ to $C_t$, is continuous.

The collection of homogeneous functions $C_t$ is referred to as the local structure of the field $X$.

The next proposition gives a representation for the local structure of a locally stationary field. Note that it differs from the corresponding representation in [26].

**Proposition 3.3.** Let $\{X(t), t \in D\}$ be a locally stationary gaussian field of index $\alpha$ with local structure $C_t(s)$. Then, for each fixed $t \in D$, the function $C_t(\cdot)$ is negative definite. Moreover, there exists a finite measure $\Gamma_t$ on $\mathbb{S}^{d-1}$ such that the following representation holds

$$C_t(s) = \int_{\mathbb{S}^{d-1}} |(s, x)|^\alpha d\Gamma_t(x).$$

The support of $\Gamma_t$ is not contained in any proper linear subspace of $\mathbb{R}^d$.

**Proof.** Recall (see e.g. [3, p.74]) that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $f(s) = f(-s)$ and $f(0) = 0$ is called negative definite if for each $s_1, \ldots, s_n \in \mathbb{R}^d$ the matrix

$$(f(s_i) + f(s_j) - f(s_i - s_j))_{i,j=1,\ldots,n}$$

is positive definite. For $u > 0$ set $q = q(u) = u^{-2/\alpha}$. Define the gaussian vector $\{Y_i = Y_i(u), i = 1, \ldots, n\}$ by

$$Y_i = u(X(t + qs_i) - u).$$

Consider the joint distribution of $\{Y_i, i = 1, \ldots, n\}$ conditioned on $X(t) = u$. It is (non-centered) gaussian and the well-known formulas for the conditional gaussian distributions show that its covariance matrix is

$$(u^2 r(t + qs_i, t + qs_j) - u^2 r(t, t + qs_i) r(t, t + qs_j))_{i,j=1,\ldots,n}.$$ 

It follows from the definition of local stationarity that, as $u \to \infty$, this converges to

$$(C_t(s_i) + C_t(s_j) - C_t(s_i - s_j))_{i,j=1,\ldots,n}.$$ 

Since the above matrix is positive definite as a limit of positive definite matrices, it follows that the function $C_t(\cdot)$ is negative definite for each $t$. 

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By Schoenberg’s theorem (see e.g. [3, Theorem 2.2]) the function \( \exp(-C_t(\cdot)) \) is positive definite and thus is the characteristic function of some symmetric probability measure \( \mu_t \) on \( \mathbb{R}^d \). Since \( C_t(\cdot) \) is homogeneous of order \( \alpha \), the measure \( \mu_t \) is stable of order \( \alpha \). The remaining part of the proposition follows from the classification of symmetric stable measures on \( \mathbb{R}^d \) (see e.g. [35, Theorem 2.4.3]).

Now we give some examples of locally stationary fields.

**Example 3.4** (see [27]). Let \( \{X(t), t \in \mathbb{R}\} \) be a centered stationary gaussian process with constant variance 1. Suppose that the covariance function \( r(t) = \mathbb{E}\left[X(0)X(t)\right] \) satisfies the Pickands condition

\[
r(s) = 1 - C|s|^{\alpha} + o(|s|^{\alpha}), \quad s \to 0
\]

for some \( C > 0 \) and \( \alpha \in (0,2] \). Then \( X \) is locally stationary of index \( \alpha \). The local structure is given by \( C_t(s) = C \). Examples include, to mention only a few, \( r(t) = \exp(-|t|^{\alpha}) \) (the generalized Ornstein-Uhlenbeck process), \( r(t) = (1 + |t|^{\alpha})^{-\beta} \) for \( \alpha \in (0,2] \) and \( \beta > 0 \) (the generalized Cauchy model, see e.g. [16]), \( r(t) = \max(1 - |t|, 0) \) (the Slepian process). In the latter case, \( \alpha = 1 \).

**Example 3.5** (see [2]). Let \( \{B(t), t \geq 0\} \) be the standard Brownian motion. The *standardized Brownian motion* is the process \( \{X(t), t > 0\} \) defined by

\[
X(t) = B(t)/\sqrt{t}.
\]

The standardized Brownian motion is locally stationary with index \( \alpha = 1 \). The local structure is given by \( C_t(s) = \frac{|s|}{2t} \).

**Proof.** Using that \( \text{Cov}(B(t_1), B(t_2)) = \min(t_1, t_2) \) we obtain, for \( s > 0 \),

\[
r(t, t + s) = \text{Cov}(X(t), X(t + s)) = \frac{t}{\sqrt{t(t + s)}} = 1 - \frac{s}{2t} + O(s^2).
\]

For \( s < 0 \) we obtain

\[
r(t, t + s) = \text{Cov}(X(t), X(t + s)) = \frac{t + s}{\sqrt{t(t + s)}} = 1 + \frac{s}{2t} + O(s^2).
\]

Note also that the \( O \)-term is uniform as long as \( t \) is bounded away from 0. This proves the claim. \( \square \)
Example 3.6 (see [2, 7]). We denote by $\mathbb{H} = \{t = (x, y) \in \mathbb{R}^2 | y > 0\}$ the upper half-plane. Let $\{B(x), x > 0\}$ be the standard Brownian motion. Then the field $\{X(t), t = (x, y) \in \mathbb{H}\}$ of standardized Brownian motion increments is defined by

$$X(t) = \frac{B(x + y) - B(x)}{\sqrt{y}}$$

is locally stationary with index $\alpha = 1$. The local structure is given by

$$C_t(s) = (|s_x| + |s_x + s_y|)/(2y),$$

where $t = (x, y) \in \mathbb{H}$ and $s = (s_x, s_y) \in \mathbb{R}^2$.

Proof. Let $t = (x, y) \in \mathbb{H}$ and $s = (s_x, s_y) \in \mathbb{R}^2$. Suppose first that $s_x > 0$, $s_x + s_y > 0$. Then

$$r(t, t + s) = \text{Cov}(X(t), X(t + s)) = \frac{y - s_x}{\sqrt{y(y + s_y)}} = 1 - \frac{s_x}{y} - \frac{s_y}{2y} + o(s_x, s_y) = 1 - (|s_x| + |s_x + s_y|)/(2y) + o(s_x, s_y).$$

Now suppose that $s_x > 0$, $s_x + s_y < 0$. Then

$$r(t, t + s) = \text{Cov}(X(t), X(t + s)) = \frac{y + s_y}{\sqrt{y(y + s_y)}} = 1 + \frac{s_y}{2y} + o(s_x, s_y) = 1 - (|s_x| + |s_x + s_y|)/(2y) + o(s_x, s_y).$$

The remaining cases can be treated analogously.

Later, it will be convenient to have another representation of the field of standardized Brownian motion increments, which differs from (6) by a simple coordinate change.

Example 3.7. Let $D = \{(x_1, x_2) | x_2 > x_1\}$. Define a field $\{Y(t), t = (x_1, x_2) \in D\}$ by

$$Y(x_1, x_2) = \frac{B(x_2) - B(x_1)}{\sqrt{x_2 - x_1}}.$$ 

Then the field $Y$ is locally stationary with $\alpha = 1$. The local structure is given by

$$C_t(s_1, s_2) = \frac{|s_1| + |s_2|}{2(x_2 - x_1)},$$

where $t = (x_1, x_2) \in D$ and $(s_1, s_2) \in \mathbb{R}^2$. 

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Example 3.8 (see [2]). Let \( \{B(t), t \in [0, 1]\} \) be the Brownian bridge. Recall that the covariance function of \( B \) is given by 
\[
\text{Cov}(B(t_1), B(t_2)) = \min(t_1, t_2) - t_1 t_2.
\]
Then the standardized Brownian bridge \( \{X(t), t \in (0, 1)\} \) defined by
\[
X(t) = B(t)/\sqrt{t(1-t)}
\]
is locally stationary with index \( \alpha = 1 \) and local structure \( C_t(s) = \frac{|s|}{2t(1-t)} \).

The next example is a multidimensional generalization of Example 3.6.

Example 3.9 (see [2]). Let \( \{\xi(A), A \in \mathcal{B}\} \) be a white noise on \((\mathbb{R}^d, \mathcal{B}, \text{Leb})\). This means that we are given a centered gaussian process \( \xi \) indexed by the collection \( \mathcal{B} \) of all Borel subsets of \( \mathbb{R}^d \) such that
\[
\text{Cov}(\xi(A_1), \xi(A_2)) = \text{Leb}(A_1 \cap A_2) \quad \text{for each } A_1, A_2 \in \mathcal{B},
\]
where \( \text{Leb} \) denotes the Lebesgue measure. A set of the form
\[
[x_1, y_1] \times \ldots \times [x_d, y_d], \quad x_i < y_i, \ i = 1, \ldots, d
\]
is called rectangle. Let
\[
\mathcal{R} = \{(x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d} \mid x_i < y_i, \ i = 1, \ldots, d\}
\]
be the collection of all rectangles. Define a process \( \{X(R), R \in \mathcal{R}\} \) indexed by rectangles by
\[
X(R) = \xi(R)/\sqrt{\text{Leb}(R)}.
\]
Then \( X \) is locally stationary on \( \mathcal{R} \) of index \( \alpha = 1 \). The local structure is given by
\[
C_t(s) = \sum_{i=1}^d (|s_{ix}| + |s_{ix} + s_{iy}|)/(2y_i),
\]
where
\[
t = (x_1, y_1, \ldots, x_d, y_d) \in \mathcal{R}, \quad s = (s_{1x}, s_{1y}, \ldots, s_{dx}, s_{dy}) \in \mathbb{R}^{2d}.
\]

Example 3.10. The Brownian motion with multidimensional time, introduced by Lévy, is a centered gaussian process \( \{B(t), t \in \mathbb{R}^d\} \) with the covariance function
\[
\text{Cov}(B(t), B(s)) = \frac{1}{2} (\|t\|_2 + \|s\|_2 - \|t - s\|_2),
\]

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where $\|t\|_2$ denotes the euclidian norm of $t$. Then the process $\{X(t), t \in \mathbb{R}^d \setminus \{0\}\}$ defined by

$$X(t) = B(t) / \sqrt{\|t\|_2}$$

is locally stationary with index $\alpha = 1$. The local structure is given by

$$C_t(s) = \frac{\|s\|_2^2}{2\|t\|_2}, \quad t \in \mathbb{R}^d \setminus \{0\}, s \in \mathbb{R}^d.$$

To state the main theorems of this section, we need the following two definitions.

**Definition 3.11.** Let $\{X(t), t \in D\}$ be a gaussian field defined on some domain $D \subset \mathbb{R}^d$. Suppose that $X$ is locally stationary with index $\alpha$ and local structure $C_t(s)$. For each $t \in D$, let $\{Y_t(s), s \in \mathbb{R}^d\}$ be a gaussian field defined by

$$E[Y_t(s)] = -C_t(s) \quad (7)$$

and

$$\text{Cov}(Y_t(s_1), Y_t(s_2)) = C_t(s_1) + C_t(s_2) - C_t(s_1 - s_2). \quad (8)$$

Then $Y_t$ is called the tangent field of $X$ at the point $t$ conditioned on $X(t) = \infty$.

The existence of $Y_t$ is guaranteed by Proposition 3.3. Moreover, the field $\tilde{Y}_t(s) = Y_t(s) + C_t(s)$ is $\alpha$-self-similar and has stationary increments. That is, for every $\lambda \in \mathbb{R}$, the field $\tilde{Y}_t(\lambda s)$ has the same finite-dimensional distributions as $|\lambda|^\alpha \tilde{Y}_t(s)$, and, for every $s_0 \in \mathbb{R}^d$, the finite-dimensional distributions of the fields $\tilde{Y}_t(s_0 + s) - \tilde{Y}_t(s_0)$ and $\tilde{Y}_t(s)$ coincide. The next proposition, which will not be used in the sequel, may serve as a justification for the use of the term tangent field.

**Proposition 3.12.** Assume that the assumptions of the previous definition are satisfied. Let $q = q(u) = u^{-2/\alpha}$. For $t \in D$ and $u \in \mathbb{R}$, define a gaussian field $\{Y^u_t(s), s \in \mathbb{R}^d\}$ as the field $u(X(t + sq) - X(t))$ conditioned on $X(t) = u$. Then, for each fixed $t \in D$, the finite-dimensional distributions of $Y^u_t(s)$ converge, as $u \to \infty$, to the finite-dimensional distributions of $Y_t(s)$ from the previous definition.

**Definition 3.13.** With the above notation,

$$H(t) = \lim_{T \to \infty} \frac{1}{Td} E \left[ \exp \left( \sup_{s \in [0,T]^d} Y_t(s) \right) \right]. \quad (9)$$
is called the high excursion intensity of the field $X$.

It was proved in [7] that $H(t) \in (0, \infty)$ exists and is continuous in $t$. Alternatively, $H(t)$ can be defined by

$$H(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^\infty \mathbb{P} \left[ \sup_{s \in [0,T]^d} Y_t(s) > w \right] e^w dw. \quad (10)$$

The next theorem, proved in [7], describes the asymptotic behavior of the high excursion probability of a locally stationary gaussian field.

**Theorem 3.14** (see [26, 7]). Let $\{X(t), t \in D\}$ be a gaussian field defined on some domain $D \subset \mathbb{R}^d$. Suppose that $X$ is locally stationary of index $\alpha$ with local structure $C_t(s)$. Let $K \subset D$ be a compact set with positive Jordan measure. Then, as $u \to \infty$,

$$\mathbb{P} \left[ \sup_{t \in K} X(t) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left( \int_K H(t) dt \right) u^{2d-1} e^{-u^2/2},$$

where the function $H(t) : D \to (0, \infty)$ is the high excursion intensity of $X$ defined in (9).

We are interested in the following special case of the above theorem.

**Corollary 3.15** (see [2, 7]). Let $\{X(t), t \in \mathbb{H}\}$ be the field of standardized Brownian motion increments defined in Example 3.6. Let $K \subset \mathbb{H}$ be a compact set with positive Jordan measure. Then, as $u \to \infty$,

$$\mathbb{P} \left[ \sup_{t \in K} X(t) > u \right] \sim \frac{1}{4\sqrt{2\pi}} \int_K \frac{dx dy}{y^2} u^3 e^{-u^2/2}.$$

We also need the following theorem, which describes the asymptotic behavior of the high excursion probability over a finite grid with mesh size going to 0.

**Theorem 3.16.** Suppose that the conditions of Theorem 3.14 are satisfied. Let $u \to +\infty$ and $q \to +0$ in such a way that $qu^{2/\alpha} \to a$ for some constant $a > 0$. Then, as $u \to \infty$,

$$\mathbb{P} \left[ \sup_{t \in K \cap q\mathbb{Z}^d} X(t) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left( \int_K H_a(t) dt \right) u^{2d-1} e^{-u^2/2},$$

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where
\[ H_a(t) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \exp \left( \sup_{s \in [0,T] \cap \alpha Z} Y_t(s) \right) \right]. \]

Furthermore, \( \lim_{a \downarrow 0} H_a(t) = H(t) \), where \( H(t) \) is the high excursion intensity of \( X \).

We omit the proof of Theorem 3.16 since it is an adaptation of the proof of Lemma 12.2.4 from [25] to locally stationary fields.

**Corollary 3.17.** Let \( \{X(t), t \in \mathbb{R}\} \) be the Slepian process defined in Example 3.4. Let \( u \to +\infty \) and \( q \to +0 \) in such a way that \( qu^2 \to a \) for some constant \( a > 0 \). Then
\[
\mathbb{P} \left[ \sup_{t \in \{0,1\} \cap \alpha Z} X(t) > u \right] \sim 2F(2a) \frac{1}{\sqrt{2\pi}} ue^{-u^2/2},
\]
where
\[
F(a) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \exp \sup_{s \in [0,T] \cap \alpha Z} (B(s) - s/2) \right].
\]

Here, \( \{B(s), s \geq 0\} \) is the standard Brownian motion. Further, \( \lim_{a \downarrow 0} F(a) = 1/2 \).

**Proof.** Actually, this was proved already in [27]. According to Example 3.4, the Slepian process is locally stationary, the tangent process being \( Y_t(s) = B(2s) - s \). It remains to use Theorem 3.16. See [25] Chapter 12 for the proof that \( \lim_{a \downarrow 0} F(a) = 1/2 \).

**Corollary 3.18.** Let \( \{X(t), t \in \mathbb{H}\} \) be the field of standardized Brownian motion increments defined in Example 3.6. Let \( K \subset \mathbb{H} \) be a compact set with positive Jordan measure. Let \( u \to +\infty \) and \( q \to +0 \) in such a way that \( qu^2 \to a \) for some constant \( a > 0 \). Then
\[
\mathbb{P} \left[ \sup_{t \in K \cap \alpha Z^2} X(t) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left( \int_K G(y) dxdy \right) u^3 e^{-u^2/2},
\]
where
\[
G(y) = \frac{1}{y^2} F \left( \frac{a}{y} \right)^2
\]
and the function \( F \) is defined by (11). Furthermore, we have \( G(y) \sim 1/(4y^2) \) as \( y \to +\infty \) and, for fixed \( y \), \( \lim_{a \to 0} G(y) = 1/(4y^2) \).
**Proof.** It is more convenient to use the notation of Example 3.7 rather than that of Example 3.6. Let \( \{B_1(s), s \in \mathbb{R}\} \) and \( \{B_2(s), s \in \mathbb{R}\} \) be two independent standard Brownian motions and let \( W_1(s) = B_1(s) - s/2 \), \( W_2(s) = B_2(s) - s/2 \). The tangent process of \( X \) is given, in the notation of Example 3.7, by

\[
Y_{(x_1,x_2)}(s_1, s_2) = W_1 \left( \frac{s_1}{x_2 - x_1} \right) + W_2 \left( \frac{s_2}{x_2 - x_1} \right).
\]

Now we use Theorem 3.16. A simple change of variables shows that the high excursion intensity is given by

\[
H_a(t) = \frac{1}{(x_2 - x_1)^2} \lim_{T \to \infty} \frac{1}{T^2} \mathbb{E} \left[ \exp \sup_{(s_1, s_2) \in [0,T]^2} \frac{a}{x_2 - x_1} Z^2 (W_1(s_1) + W_2(s_2)) \right].
\]

Since the processes \( W_1, W_2 \) are independent, this is equal to

\[
\frac{1}{(x_2 - x_1)^2} \lim_{T \to \infty} \frac{1}{T^2} \left( \mathbb{E} \left[ \exp \sup_{s_1 \in [0,T]} a \frac{Z}{x_2 - x_1} W_1(s_1) \right] \right)^2,
\]

which is, by definition, \( \frac{1}{(x_2 - x_1)^2} F \left( \frac{a}{x_2 - x_1} \right)^2 \). The lemma follows by switching to the notation of Example 3.6.

\[ \square \]

### 4 Standardized Brownian Motion Increments

In this section we prove Theorem 1.6. Let us describe briefly the method of the proof and fix the notation.

Let

\[
\mathbb{H} = \{t = (x, y) \in \mathbb{R}^2 \mid y > 0\}
\]

denote the open upper half-plane. A point \( t = (x, y) \in \mathbb{H} \) will be often identified with the interval \([x, x + y] \subset \mathbb{R}\). There is a natural action of the group of affine transformations of the real line on \( \mathbb{H} \) defined as follows. If \( g : x \mapsto ax + b \), where \( a > 0, b \in \mathbb{R} \), is an affine transformation of \( \mathbb{R} \), then the action of \( g \) on \( \mathbb{H} \) is given by

\[
g(t) = (ax + b, ay), \quad t = (x, y) \in \mathbb{H}.
\]
Let \( \{B(x), x \geq 0\} \) be the standard Brownian motion. Recall that the random field \( \{X(t), t = (x, y) \in \mathbb{H}\} \) of \textit{standardized Brownian motion increments} was defined in Example 3.6 by

\[
X(t) = \frac{B(x + y) - B(x)}{y^{1/2}}.
\]

(13)

Note that the field \( X \) is centered gaussian. For each \( t \in \mathbb{H} \) the distribution of \( X(t) \) is standard normal.

The following invariance property of the field \( X \) will be useful

**Proposition 4.1.** Let \( g \) be an affine transformation of \( \mathbb{R} \). Then, for each \( t_1, \ldots, t_n \in \mathbb{H} \), the joint distribution of \( X(g(t_1)), \ldots, X(g(t_n)) \) coincides with the joint distribution of \( X(t_1), \ldots, X(t_n) \).

The proof follows from the scaling property of the Brownian motion. The above proposition allows us to state Theorem 1.6 in the following, equivalent form.

**Theorem 4.2.** For \( n > 1 \) let \( H(n) \) be the triangle

\[
\{(x, y) \in \mathbb{H} \mid x \in [0, n], y \in [1, n-x]\}
\]

Define the random field \( X \) by (13). Then, for each \( \tau \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mathbb{P}\left[ \sup_{t \in H(n)} X(t) \leq a_n + b_n \tau \right] = \exp(-\tau),
\]

where \( a_n, b_n \) are constants defined by (1).

The rest of the section is devoted to the proof of Theorem 4.2.

Let \( \tau \in \mathbb{R} \) be fixed. Let \( u_n = a_n + b_n \tau \) with \( a_n, b_n \) defined by (1). Note that \( u_n \sim \sqrt{2\log n} \) as \( n \to \infty \).

**Remark 4.3.** We have, as \( n \to \infty \),

\[
\frac{1}{4\sqrt{2\pi}} u_n^3 e^{-u_n^2/2} \sim e^{-\tau}/n.
\]

For \( l > 1 \) define \( H(n, l) = \{(x, y) \in H(n) \mid y \in [1, l]\} \).
Lemma 4.4. The following holds for the high excursion probability over the triangle $H(n) \backslash H(n,l)$.

$$\lim_{l \to \infty} \limsup_{n \to \infty} P \left[ \sup_{t \in H(n) \backslash H(n,l)} X(t) > u_n \right] = 0.$$ 

**Proof.** Divide $\mathbb{H}$ into rectangles

$$R_{k,l} = [2^{l+1}k, 2^{l+1}(k+1)] \times [2^l, 2^{l+1}], \quad k, l \in \mathbb{Z}.$$ 

Note that all rectangles can be obtained from $R_{0,1}$ by the action of the one-dimensional affine group on $\mathbb{H}$. Thus, by the affine invariance of $X$ (Proposition 4.1), the probability $P \left[ \sup_{t \in R_{k,l}} X(t) > u_n \right]$ is independent of $k, l$ and, by Corollary 3.15 and Remark 4.3,

$$P \left[ \sup_{t \in R_{k,l}} X(t) > u_n \right] \sim \frac{e^{-\tau}}{n} \int_{R_{k,l}} \frac{dx dy}{y^2} = \frac{e^{-\tau}}{n}, \quad n \to \infty.$$ 

It is easy to see that $H(n) \backslash H(n,l)$ is covered by at most $\lceil 2n/l \rceil$ rectangles of the form $R_{k,l}$. Thus

$$\limsup_{n \to \infty} P \left[ \sup_{t \in H(n) \backslash H(n,l)} X(t) > u_n \right] \leq \frac{2e^{-\tau}}{l}.$$ 

The statement of the lemma follows. \hfill \Box

Lemma 4.5. We have

$$\lim_{n \to \infty} P \left[ \sup_{t \in H(n,l)} X(t) \leq u_n \right] = \exp \left( -e^{-\tau}(l - 1)/l \right).$$ 

**Proof.** Let

$$H^*(n, l) = [0, n - 1] \times [1, l], \quad H_s(n, l) = [0, n - l] \times [1, l].$$ 

Then $H_s(n, l) \subset H(n, l) \subset H^*(n, l)$. So we have to prove that

$$\lim_{n \to \infty} P \left[ \sup_{t \in H^*(n,l)} X(t) \leq u_n \right] = \exp \left( -e^{-\tau}(l - 1)/l \right). \quad (14)$$
The same statement with $H_*(n, l)$ instead of $H^*(n, l)$ can be proved analogously and the lemma follows.

For $i = 0, \ldots, n - 2$ define $R_i = [i, i + 1] \times [1, l]$. Then, by Corollary 3.15 and Remark 4.3

$$
P \left[ \sup_{t \in R_i} X(t) > u_n \right] \sim \frac{e^{-\tau}}{n} \int_{R_i} \frac{dx dy}{y^2} = \frac{e^{-\tau}}{n} (l - 1)/l, \quad n \to \infty. \quad (15)$$

Note, that by the affine invariance, the above probability is independent of $i$. If the events $\sup_{t \in R_i} X(t) > u_n$ were independent, we could finish the proof by applying the Poisson limit theorem. However, some additional work is required to overcome the dependence.

Fix $\varepsilon, a > 0$. Define $q_n = a/[2 \log n]$ and

$$R_i(\varepsilon) = [i + \varepsilon, i + 1 - \varepsilon] \times [1, l], \quad R_i(\varepsilon, a) = R_i(\varepsilon) \cap q_n \mathbb{Z}^2.$$

Note that $R_i(\varepsilon, a)$ is a finite set depending on $n$. Let

$$H^*(n, l, \varepsilon, a) = \bigcup_{i=0}^{n-2} R_i(\varepsilon, a).$$

Lemma 4.6. Let

$$\Delta_1(\varepsilon, a) = \lim_{n \to \infty} n P \left[ \max_{t \in R_0(\varepsilon, a)} X(t) > u_n \right] - e^{-\tau}(l - 1)/l.$$

Then $\lim_{a \downarrow 0} \lim_{\varepsilon \downarrow 0} \Delta_1(a, \varepsilon) = 0$.

Proof. Note that $\lim_{n \to \infty} q_n u_n^2 = a$. We have, by Corollary 3.18 and Remark 4.3

$$P \left[ \sup_{t \in R_0(\varepsilon, a)} X(t) > u_n \right] \sim \left( \int_{R_0(\varepsilon)} 4G(y)dx dy \right) e^{-\tau}/n, \quad n \to \infty.$$

Here, the function $G$ is defined by (12). Thus

$$\Delta_1(\varepsilon, a) = e^{-\tau} \left( \int_{R_0(\varepsilon)} 4G(y)dx dy - (l - 1)/l \right).$$

Letting $\varepsilon$ to 0, we obtain

$$\lim_{\varepsilon \downarrow 0} \Delta_1(\varepsilon, a) = e^{-\tau} \left( \int_{R_0} 4G(y)dx dy - (l - 1)/l \right).$$

To finish the proof note that $\lim_{a \to 0} G(y) = 1/(4y^2)$ by Corollary 3.18 \qed
Lemma 4.7. We have
\[
\limsup_{n \to \infty} \left( P \left[ \sup_{t \in H^*(n, l, \varepsilon, a)} X(t) \leq u_n \right] - P \left[ \sup_{t \in H^*(n, l)} X(t) \leq u_n \right] \right) \leq \Delta_1(\varepsilon, a),
\]
where \( \Delta_1(\varepsilon, a) \) was defined in the previous lemma.

Proof. We have, evidently,
\[
P \left[ \sup_{t \in H^*(n, l, \varepsilon, a)} X(t) \leq u_n \right] - P \left[ \sup_{t \in H^*(n, l)} X(t) \leq u_n \right] =
P \left[ \sup_{t \in H^*(n, l \setminus H^*(n, l, \varepsilon, a))} X(t) > u_n \right] \bigwedge \sup_{t \in H^*(n, l, \varepsilon, a)} X(t) \leq u_n.
\]
The last probability is not greater than
\[
\sum_{i=0}^{n-2} P \left[ \sup_{t \in R_i \setminus R_i(\varepsilon, a)} X(t) \leq u_n \right] =
\left( n - 1 \right) P \left[ \sup_{t \in R_0} X(t) > u_n \right] - (n - 1) P \left[ \sup_{t \in R_0(\varepsilon, a)} X(t) > u_n \right].
\]
To finish the proof it remains to use \([15]\) for the first and Lemma 4.6 for the second term.

Let \( \{Y(t), t \in H^*(n, l, \varepsilon, a)\} \) be standard normal variables with the following covariance matrix:
\[
\begin{align*}
\mathbf{E}[Y(t_1)Y(t_2)] &= \mathbf{E}[X(t_1)X(t_2)] \quad \text{if } \exists i : t_1, t_2 \in R_i(\varepsilon, a), \\
\mathbf{E}[Y(t_1)Y(t_2)] &= 0 \quad \text{otherwise}.
\end{align*}
\]
Thus, we remove the dependence between \( X(t_1) \) and \( X(t_2) \) if \( t_1 \) and \( t_2 \) are in different \( R_i \)'s.
The next lemma is known as Berman’s Inequality, see e.g. \([25]\) Theorem 4.2.1.
Lemma 4.8. Suppose \( \xi_1, \ldots, \xi_N \) are standard normal variables with covariance matrix \( \Lambda^1 = (\Lambda_{ij}^1) \), and \( \eta_1, \ldots, \eta_N \) similarly with covariance matrix \( \Lambda^2 = (\Lambda_{ij}^2) \), and let \( \rho_{ij} = \max(|\Lambda_{ij}^1|, |\Lambda_{ij}^2|) \). Then

\[
P \left[ \max_{1 \leq i \leq N} \xi_i \leq u \right] - P \left[ \max_{1 \leq i \leq N} \eta_i \leq u \right] \leq \frac{1}{2\pi} \sum_{1 \leq i < j \leq N} |\Lambda_{ij}^1 - \Lambda_{ij}^2|(1 - \rho_{ij}^2)^{-1/2} \exp \left( -\frac{u^2}{1 + \rho_{ij}} \right).
\]

The next lemma shows that the high excursion behavior of the gaussian vector \( X(t) \) coincides with that of \( Y(t) \).

Lemma 4.9. We have, for fixed \( \varepsilon \) and \( a \),

\[
\lim_{n \to \infty} \left( P \left[ \sup_{t \in H^*(n, l, \varepsilon, a)} X(t) \leq u_n \right] - P \left[ \sup_{t \in H^*(n, l, \varepsilon, a)} Y(t) \leq u_n \right] \right) = 0
\]
which is smaller than
\[
K \exp\left(-u_n^2/(1 + \delta)\right) \sum_{t_1, t_2 \in H^*(n, l, \varepsilon, a)} \Lambda_{t_1 t_2}^X,
\]

for some constant \(K\) depending on \(\varepsilon\) but not on \(n\).

Recall that \(R_i(\varepsilon, a) = q_n \mathbb{Z}^2 \cap R_i(\varepsilon)\). It follows that the number of elements of \(R_i(\varepsilon, a)\) is less than \(O(\log^2 n)\), where the constant in the \(O\)-term depends only on \(a\) and \(l\).

It is easy to see that \(X(t_1)\) and \(X(t_2)\) are independent provided that \(t_1 \in R_{i_1}\) and \(t_2 \in R_{i_2}\) with \(|i_1 - i_2| > l + 1\). Consequently, the number of pairs \((t_1, t_2)\) such that \(X(t_1)\) and \(X(t_2)\) are dependent is less than \(O(n \log^4 n)\). Thus
\[
P\left[\max_{t \in H(n, \varepsilon, a)} X(t) \leq u_n\right] - P\left[\max_{t \in H(n, \varepsilon, a)} Y(t) \leq u_n\right] \leq K'n(\log^4 n) e^{-u_n^2/(1+\delta)}.
\]

where \(K'\) depends on \(\varepsilon\) and \(a\), but not on \(n\). Recall that \(u_n \sim \sqrt{2 \log n}\). The statement of the lemma follows.

**Lemma 4.10.** Let
\[
\Delta_2(\varepsilon, a) = \limsup_{n \to \infty} \left| P\left[\max_{t \in H^*(n, l, \varepsilon, a)} Y(t) \leq u_n\right] - \exp(-e^{-\tau(l-1)/l}) \right|.
\]

Then \(\lim_{a \downarrow 0} \lim_{\varepsilon \downarrow 0} \Delta_2(\varepsilon, a) = 0\).

**Proof.** Since \(Y(t_1)\) and \(Y(t_2)\) are independent if \(t_1\) and \(t_2\) are in different \(R_i\)’s, we have
\[
P\left[\max_{t \in H^*(n, l, \varepsilon, a)} Y(t) \leq u_n\right] = \left(1 - P\left[\max_{t \in R_0(\varepsilon, a)} Y(t) > u_n\right]\right)^{n-1} = \left(1 - P\left[\max_{t \in R_0(\varepsilon, a)} X(t) > u_n\right]\right)^{n-1}.
\]

Using this and Lemma 4.6, we obtain
\[
\lim_{n \to \infty} P\left[\max_{t \in H^*(n, l, \varepsilon, a)} Y(t) \leq u_n\right] = \exp(-e^{-\tau(l-1)/l + \Delta_1(\varepsilon, a)})
\]

where \(\lim_{a \downarrow 0} \lim_{\varepsilon \downarrow 0} \Delta_1(\varepsilon, a) = 0\). This proves Lemma 4.10. 

\[24\]
Now we are able to finish the proof of Lemma 4.5. Recall that we have to prove (14). Using Lemmas 4.9 and 4.10, we obtain

$$\limsup_{n \to \infty} \left| P \left[ \max_{t \in H^*(n,l,\varepsilon,a)} X(t) \leq u_n \right] - \exp(-e^{-\tau}(l - 1)/l) \right| = \Delta_2(\varepsilon, a).$$

Now use Lemma 4.7 to obtain

$$\limsup_{n \to \infty} \left| P \left[ \max_{t \in H^*(n,l)} X(t) \leq u_n \right] - \exp(-e^{-\tau}(l - 1)/l) \right| \leq \Delta_1(\varepsilon, a) + \Delta_2(\varepsilon, a).$$

To finish the proof let $\varepsilon, a \downarrow 0$. \hfill \Box

**Proof of Theorem 4.2.** It follows from $H(n,l) \subset H(n)$ that

$$\limsup_{n \to \infty} P \left[ \sup_{t \in H(n)} X(t) \leq u_n \right] \leq \limsup_{n \to \infty} P \left[ \sup_{t \in H^*(n,l)} X(t) \leq u_n \right],$$

which is equal to $\exp(-e^{-\tau}(l - 1)/l)$ by Lemma 4.5. Letting $l \to \infty$ we obtain

$$\limsup_{n \to \infty} P \left[ \sup_{t \in H(n)} X(t) \leq u_n \right] \leq \exp(-e^{-\tau}).$$

On the other hand, we have

$$P \left[ \sup_{t \in H(n)} X(t) \leq u_n \right] \geq P \left[ \sup_{t \in H(n,l)} X(t) \leq u_n \right] - P \left[ \sup_{t \in H(n) \setminus H(n,l)} X(t) > u_n \right].$$

Letting $n \to \infty$, $l \to \infty$ and using Lemma 4.5 for the first and Lemma 4.4 for the second term, we obtain

$$\liminf_{n \to \infty} P \left[ \sup_{t \in H(n)} X(t) \leq u_n \right] \geq \exp(-e^{-\tau}),$$

which finishes the proof of Theorem 4.2. \hfill \Box
5 Distributional Convergence in the Erdős-Rényi Law

In this section we sketch a proof of Theorem 1.5. Let \( \{X(t), t \in \mathbb{R}\} \) be the Slepian process, i.e. the stationary gaussian process defined by \( X(t) = \int_t^{t+1} dW \), where \( dW \) is the white noise on \( \mathbb{R} \). Equivalently, \( X \) can be defined as a stationary gaussian process with the covariance function given by

\[
\text{Cov}(X(0), X(t)) = \begin{cases} 
1 - |t|, & \text{if } |t| \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

Let \( c \) be a positive constant and define \( l_n = \lceil c \log n \rceil \). Let \( q_n = 1/l_n \). Finally, fix \( \tau \in \mathbb{R} \) and let

\[
u_n = \sqrt{2 \log n + \frac{-1/2 \log \log n + \log(2F(4/c)/(c\sqrt{\pi})) + \tau}{\sqrt{2 \log n}}},
\]

where the function \( F \) is defined by (11). It is easy to see that the random variables \( \{X(kq_n), k = 0, \ldots, n - l_n\} \) have the same joint law as \( \{(S_{k+l_n} - S_k)/\sqrt{l_n}, k = 0, \ldots, n - l_n\} \). It follows from Corollary 3.17 with \( a = \lim_{n \to \infty} q_n u_n^2 = 2/c \) that

\[
P \left[ \max_{k=0,\ldots,l_n-1} X(kq_n) > u_n \right] \sim \frac{1}{\sqrt{2\pi}} u_n e^{-u_n^2/2} 2F(4/c) \sim \frac{l_n}{n} e^{-\tau}.
\]

Now we would like to apply the Poisson limit theorem to the events

\[
\max_{k=ml_n,\ldots,(m+1)l_n-1} X(kq_n) > u_n, \quad m = 0, \ldots, n/l_n - 1.
\]

To prove the approximate independence of the above events, one can use Berman Inequality as it was done in Lemma 4.9. We omit the details. Thus, by the Poisson limit theorem,

\[
\lim_{n \to \infty} P \left[ \max_{k=0,\ldots,n-l_n} X(kq_n) > u_n \right] = \lim_{n \to \infty} \left( 1 - \frac{1}{l_n} e^{-\tau} \right)^{n/l_n-1} = \exp(-e^{-\tau}).
\]

This proves Theorem 1.5.
6 Standardized Increments of the Gaussian Random Walk

In this section we prove Theorem 1.3. First we introduce some notation. Let $\tau \in \mathbb{R}$ be fixed. Define
$$u_n = \sqrt{2 \log n + \frac{1}{2} \log \log n - \log (2^{-1/\pi} + \tau)}$$
and let $q_n = 1/\log n$. Note that $\lim_{n \to \infty} q_n u_n^2 = 2$.

**Remark 6.1.** We have, as $n \to \infty$,
$$\frac{1}{\sqrt{2\pi}} u_n^3 e^{-u_n^2/2} \sim \frac{\log n}{n} e^{-\tau}.$$

Let $\{B(x), x \geq 0\}$ be the standard Brownian motion. Recall that $\mathbb{H}$ denotes the upper half-plane and that the random field of standardized Brownian increments $\{X(t), t = (x, y) \in \mathbb{H}\}$ was defined in Example 3.6 by
$$X(x, y) = \frac{B(x + y) - B(x)}{\sqrt{y}}.$$

Let
$$T(n) = \{(xq_n, yq_n) | x = 0, \ldots, n; y = 1, \ldots, n - x\}.$$
Then it is easy to see that the random vector $\{(S_j - S_i)/\sqrt{j - i}, 0 \leq i < j \leq n\}$ has the same distribution as $\{X(t), t \in T(n)\}$. Thus, our aim is to prove that
$$\lim_{n \to \infty} \mathbb{P}\left[ \sup_{t \in T(n)} X(t) \leq u_n \right] = \exp \left( -e^{-\tau} \int_0^\infty G(y)dy \right). \quad (16)$$
Here, $G$ is defined by (14) or, equivalently, by (12) with $a = 2$. First, we prove that the integral $\int_0^\infty G(y)dy$ is finite.

**Lemma 6.2.** $\int_0^\infty G(y)dy$ is finite.

**Proof.** Since $G(y) \sim 1/(4y^2)$ as $y \to \infty$ by Corollary 3.18, we have only to prove that $\int_0^1 G(y)dy$ is finite.
Fix some $0 < l < 1$. Let $K = [0, 1] \times [l, 1]$. Again using Corollary 3.18 and Remark 6.1 we obtain, as $n \to \infty$,

\[ P \left[ \sup_{t \in T(n) \cap K} X(t) > u_n \right] \sim \left( \int_{l}^{1} G(y)dy \right) \frac{\log n}{n} e^{-\tau}. \]

On the other hand, since $T(n) \cap K$ consists of at most $\log^2 n$ points, we have, evidently,

\[ P \left[ \sup_{t \in T(n) \cap K} X(t) > u_n \right] \leq (\log^2 n)(1 - \Phi(u_n)), \]

where $\Phi$ is the standard normal distribution function. Using that $1 - \Phi(u) \sim \frac{1}{\sqrt{2\pi}} u e^{-u^2/2}$ as $u \to \infty$, as well as Remark 6.1 we obtain that the right-hand side is asymptotically equivalent to

\[ (\log^2 n) \frac{1}{\sqrt{2\pi}} \frac{1}{u_n} e^{-u_n^2/2} \sim \frac{1}{4} \log n \frac{1}{n} e^{-\tau}. \]

It follows that $\int_{l}^{1} G(y)dy \leq 1/4$ for all $l > 0$, which proves the lemma. \hfill \Box

For $0 \leq l_1 < l_2 \leq \infty$ define

\[ T(n, l_1, l_2) = T(n) \cap \{(x, y) \in \mathbb{H} \mid y \in (l_1, l_2)\}. \]

**Lemma 6.3.** We have

\[ \lim_{l_1 \to 0} \lim_{n \to \infty} \sup_{u_n} P \left[ \max_{t \in T(n, 0, l_1)} X(t) > u_n \right] = 0. \]

**Proof.** The number of elements in the finite set $T(n, 0, l_1)$ does not exceed $l_1 n \log n$. We have, as $n \to \infty$,

\[ P \left[ \max_{t \in T(n, 0, l_1)} X(t) > u_n \right] \leq l_1 n (\log n)(1 - \Phi(u_n)) \sim l_1 n (\log n) \frac{1}{\sqrt{2\pi}} \frac{1}{u_n} e^{-u_n^2/2}. \]

Using Remark 6.1 we obtain

\[ \lim_{n \to \infty} \sup_{u_n} P \left[ \max_{t \in T(n, 0, l_1)} X(t) > u_n \right] \leq \frac{1}{4} e^{-\tau} l_1. \]

This finishes the proof. \hfill \Box
Lemma 6.4. We have
\[
\lim_{l_2 \to +\infty} \lim_{n \to \infty} P \left[ \max_{t \in T(n,l_2,+,\infty)} X(t) > u_n \right] = 0.
\]

Proof. The proof is analogous to the proof of Lemma 4.4 and is therefore omitted.

Lemma 6.5. We have
\[
\lim_{n \to \infty} P \left[ \sup_{t \in T(n,l_1,l_2)} X(t) \leq u_n \right] = \exp \left( -e^{-\tau} \int_{l_1}^{l_2} G(y)dy \right).
\]

Proof. Let
\[
T^*(n,l_1,l_2) = q_n \mathbb{Z}^2 \cap ([0, [nq_n - l_1]] \times [l_1, l_2]),
\]
\[
T_*(n,l_1,l_2) = q_n \mathbb{Z}^2 \cap ([0, [nq_n - l_2]] \times [l_1, l_2]).
\]
Then \( T_*(n,l_1,l_2) \subset T(n,l_1,l_2) \subset T^*(n,l_1,l_2) \). Thus, to prove Lemma 6.5 we have to show that
\[
\lim_{n \to \infty} P \left[ \sup_{t \in T^*(n,l_1,l_2)} X(t) \leq u_n \right] = \exp \left( -e^{-\tau} \int_{l_1}^{l_2} G(y)dy \right),
\]
since the proof of the corresponding statement with \( T_*(n,l_1,l_2) \) instead of \( T^*(n,l_1,l_2) \) is analogous.

For \( i = 0, \ldots, \lceil nq_n - l_1 \rceil - 1 \) define
\[
R_i = q_n \mathbb{Z}^2 \cap ([i, i + 1] \times [l_1, l_2]).
\]
Recall that \( \lim_{n \to \infty} q_n u_n^2 = 2 \). Then, by Corollary 3.18 with \( a = 2 \) and Remark 6.1
\[
P \left[ \sup_{t \in R_i} X(t) > u_n \right] \sim e^{-\tau \log n / n} \int_{l_1}^{l_2} G(y)dy, \quad n \to \infty. \tag{18}
\]
By the affine invariance (Proposition 4.1), the above probability is independent of \( i \). As in the previous section, the difficulty is the dependence of the events "\( \sup_{t \in R_i} X(t) > u_n \)". If the events were independent, we were done by the Poisson limit theorem. Fix \( \varepsilon > 0 \). Define
\[
R_i(\varepsilon) = q_n \mathbb{Z}^2 \cap ([i + \varepsilon, i + 1 - \varepsilon] \times [l_1, l_2]).
\]
\[ T^*(n, l_1, l_2, \varepsilon) = \bigcup_{i=0}^{\lceil nq_n - l_1 \rceil - 1} R_i(\varepsilon). \]

Note that the finite set \( R_i(\varepsilon) \) depends on \( n \).

**Lemma 6.6.** We have
\[
0 \leq \limsup_{n \to \infty} \left( \mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2, \varepsilon)} X(t) \leq u_n \right] - \mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2)} X(t) \leq u_n \right] \right) < c_1 \varepsilon.
\]
for some constant \( c_1 \) depending only on \( l_1, l_2 \).

**Proof.** Proceeding as in Lemma 4.7, we obtain
\[
\mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2, \varepsilon)} X(t) \leq u_n \right] - \mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2)} X(t) \leq u_n \right] \leq \left( \lceil nq_n - l_1 \rceil - 1 \right) \left( \mathbb{P} \left[ \max_{t \in R_0(\varepsilon)} X(t) > u_n \right] - \mathbb{P} \left[ \max_{t \in R_0(\varepsilon)} X(t) > u_n \right] \right).
\]
By Corollary 3.18 with \( a = 2 \) and Remark 6.1
\[
\mathbb{P} \left[ \max_{t \in R_0(\varepsilon)} X(t) > u_n \right] \sim e^{-\frac{\log n}{n}} (1 - 2\varepsilon) \int_{l_1}^{l_2} G(y) dy, \quad n \to \infty. \tag{19}
\]
Using this together with (18), we obtain the statement of the lemma. \( \square \)

Let \( \{Y(t), t \in T^*(n, l_1, l_2, \varepsilon)\} \) be a gaussian vector with the following covariance structure
\[
\mathbb{E}[Y(t_1)Y(t_2)] = \mathbb{E}[X(t_1)X(t_2)] \quad \text{if}\ \exists i : t_1, t_2 \in R_i(\varepsilon),
\]
\[
\mathbb{E}[Y(t_1)Y(t_2)] = 0 \quad \text{otherwise}.
\]
Thus, we remove the dependence between \( X(t_1) \) and \( X(t_2) \) if \( t_1 \) and \( t_2 \) are in different \( R_i(\varepsilon) \)'s.

**Lemma 6.7.** We have
\[
\lim_{n \to \infty} \left( \mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2, \varepsilon)} X(t) \leq u_n \right] - \mathbb{P} \left[ \max_{t \in T^*(n, l_1, l_2, \varepsilon)} Y(t) \leq u_n \right] \right) = 0.
\]

**Proof.** The proof, which we omit, uses Berman’s inequality and is analogous to the proof of Lemma 4.9. \( \square \)
Lemma 6.8. We have
\[
\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \mathbb{P} \left[ \max_{t \in T^*(n,l_1,l_2,\epsilon)} Y(t) \leq u_n \right] = \exp \left( -e^{-\tau} \int_{l_1}^{l_2} G(y) dy \right)
\]

**Proof.** Since \(Y(t_1)\) and \(Y(t_2)\) are independent provided that \(t_1\) and \(t_2\) are in different \(R_i(\epsilon)\)'s, we have
\[
\mathbb{P} \left[ \max_{t \in T^*(n,l_1,l_2,\epsilon)} Y(t) \leq u_n \right] = \left( 1 - \mathbb{P} \left[ \max_{t \in R_0(\epsilon)} Y(t) > u_n \right] \right)^{[nq_n-l_1]-1} = \left( 1 - \mathbb{P} \left[ \max_{t \in R_0(\epsilon)} X(t) > u_n \right] \right)^{[nq_n-l_1]-1}.
\]

Recall that \([nq_n-l_1]-1 \sim n/\log n, n \to \infty\). Using (19), we obtain
\[
\lim_{n \to \infty} \mathbb{P} \left[ \max_{t \in T^*(n,l_1,l_2,\epsilon)} Y(t) \leq u_n \right] = \exp \left( -e^{-\tau} (1 - 2\epsilon) \int_{l_1}^{l_2} G(y) dy \right)
\]
and the lemma follows by letting \(\epsilon \downarrow 0\).

Now we can finish the proof of Lemma 6.5. We have to show (17). But it follows easily from Lemmas 6.6, 6.7 and 6.8.

Proof of Theorem 1.3. Recall that we have to prove (16). The evident inequality
\[
\mathbb{P} \left[ \max_{t \in T(n)} X(t) \leq u_n \right] \leq \mathbb{P} \left[ \max_{t \in T(n,l_1,l_2)} X(t) \leq u_n \right]
\]
together with Lemma 6.5 imply that
\[
\limsup_{n \to \infty} \mathbb{P} \left[ \max_{t \in T(n)} X(t) \leq u_n \right] \leq \exp \left( -e^{-\tau} \int_{0}^{\infty} G(y) dy \right).
\]

Now, using Lemmas 6.5, 6.3 and 6.4 and the inequality
\[
\mathbb{P} \left[ \max_{t \in T(n)} X(t) \leq u_n \right] \geq \mathbb{P} \left[ \max_{t \in T(n,l_1,l_2)} X(t) \leq u_n \right] - \mathbb{P} \left[ \max_{t \in T(n,l_1)} X(t) > u_n \right] - \mathbb{P} \left[ \max_{t \in T(n,l_2,\infty)} X(t) > u_n \right]
\]
we obtain, by letting \(l_1 \to 0\) and \(l_2 \to \infty\),
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \max_{t \in T(n)} X(t) \leq u_n \right] \geq \exp \left( -e^{-\tau} \int_{0}^{\infty} G(y) dy \right).
\]
This finishes the proof of Theorem 1.3.
7 An Explicit Formula for the Constant $H$

Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of i.i.d. standard gaussian variables. Let $S_n = \sum_{i=1}^n \xi_i, S_0 = 0$ be the gaussian random walk and recall that the maximum of standardized gaussian random walk increments was defined by

$$L_n = \max_{0 \leq i < j \leq n} \frac{S_j - S_i}{\sqrt{j - i}}.$$ 

It was shown in Theorem $1.3$ that the extreme-value rate as $n \to \infty$ of $L_n$ is $H n \log n$. Here, $H > 0$ is a constant which was defined as follows. Let $\{B(t), t \geq 0\}$ be the standard Brownian motion. Let

$$F(a) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \exp \sup_{t \in [0,T] \cap \mathbb{N}} (B(t) - t/2) \right]$$

and

$$G(y) = \frac{1}{y^2} F \left( \frac{2}{y} \right)^2.$$ 

Then $H = 4 \int_0^\infty G(y) dy$. This formulae do not allow to calculate the constant $H$ numerically. Our goal is to obtain a different representation of $H$ which makes numerical calculations possible.

**Theorem 7.1.** Let $\Phi$ be the standard normal distribution function. We have

$$H = \int_0^\infty \exp \left\{ -4 \sum_{k=1}^\infty \frac{1}{k} \Phi(-\sqrt{k/(2y)}) \right\} dy.$$ 

A numerical calculation shows that $H \approx 0.21$. The rest of the section is devoted to the proof of the above theorem.

Fix some $a > 0$. Let $\{X_i, i = 1, \ldots\}$ be i.i.d. gaussian random variables with $\mathbb{E}X_i = -a/2, \text{Var}X_i = a$. Define the negatively drifted gaussian random walk $Z_n = \sum_{i=1}^n X_i, Z_0 = a$. Note that $Z_n$ drifts to $-\infty$ a.s. The behavior of one-dimensional random walks is well-studied, see [15, Chapters XII and XVIII], [37] as well as [21] for the drifted gaussian case.

Let $p_\infty(a) = \mathbb{P}[Z_n < 0 \forall n \in \mathbb{N}]$ be the probability that $Z_n$ never enters the upper half-line. By Spitzers Identity

$$p_\infty(a) = \exp \left\{ -\sum_{k=1}^\infty \frac{1}{k} \mathbb{P}(Z_k > 0) \right\}.$$ 

Theorem [7.1] is then easily seen to follow from
Theorem 7.2. We have $F(a) = p^2_\infty(a)/a$.

Proof. Let $M_n = \max_{i=0,...,n} Z_i$. It is easy to see from the definition of $F$ that

$$F(a) = \frac{1}{a} \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[e^{M_n}].$$

Thus, we concentrate on the calculation of the above limit. Let

$$g(w) = \sum_{n=0}^{\infty} w^n \mathbb{E}[e^{M_n}].$$

By [37], equation (1) on page 207, we have

$$g(w) = (1 - w)^{-1} \exp \left\{ - \sum_{k=1}^{\infty} \frac{w^k}{k} \mathbb{E}[(1 - e^{Z_k})1_{Z_k > 0}] \right\}.$$

Now, recalling that $Z_k \sim \mathcal{N}(-ak/2, ak)$,

$$\mathbb{E}[e^{Z_k}1_{Z_k > 0}] = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{ak}/2}^{\infty} e^{\sqrt{ak}x - ak/2} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{ak}/2}^{\infty} e^{-(x-\sqrt{ak})^2/2} dx = 1 - \mathbb{P}[Z_k > 0].$$

Thus, we obtain

$$g(w) = (1 - w)^{-1} \exp \left\{ - \sum_{k=1}^{\infty} \frac{w^k}{k} (2\mathbb{P}[Z_k > 0] - 1) \right\} = (1 - w)^{-2} \exp \left\{ -2 \sum_{k=1}^{\infty} \frac{w^k}{k} \mathbb{P}[Z_k > 0] \right\}$$

and, consequently,

$$g(w) \sim \frac{p^2_\infty(a)}{(1 - w)^2} \quad \text{as } w \uparrow 1.$$

By a well-known Tauberian Theorem (see e.g. [15] Theorem 5 on p. 447) it follows that

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[e^{M_n}] = p^2_\infty(a).$$

This finishes the proof.

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