Quantum Algorithms and Covering Spaces

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Abstract

It’s been recently demonstrated that quantum walks on graphs can solve certain computational problems faster than any classical algorithm. Therefore it is desirable to quantify those purely combinatorial properties of graphs which quantum walks take advantage of and try and separate them from those properties due to the encoding of the problem. In this paper we isolate the combinatorial property responsible (at least in part) for the computational speedups recently observed. We find that continuous-time quantum walks can exploit the covering space property of certain graphs. We formalise the notion of graph covering spaces. Then we demonstrate that a quantum walk on a graph $Y$ which covers a smaller graph $X$ can be equivalent to a quantum walk on the smaller graph $X$. This equivalence occurs only when the walk begins on certain initial states, fibre-constant states, which respect the graph covering space structure. We illustrate these observations with walks on Cayley graphs; we show that walks on fibre-constant initial states for Cayley graphs are equivalent to walks on the induced Schreier graph. We also consider the problem of constructing efficient gate sequences simulating the time evolution of a continuous-time quantum walk. We argue that if $Y \xrightarrow{\pi_N} X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} X_1$ is a tower of graph covering spaces satisfying certain uniformity and growth conditions then there exists an efficient quantum gate sequence simulating the walk. For the case of the walk on the $m$-torus graph $T^m$ on $2^n$ vertices we construct a gate sequence which uses $O(poly(n))$ gates which is independent of the time $t$ the walk is simulated for (and so the sequence can simulate the walk for exponential times). We argue that there exists a wide class of nontrivial

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operators based on quantum walks on graphs which can be measured efficiently using phase estimation. Interestingly, measuring these operators won’t be unitarily equivalent to the quantum fourier transform. Finally, motivated by our results we introduce a new general class of computational problems, HiddenCover, which includes a variant of the general hidden subgroup problem as a subclass. We argue that quantum computers ought to be able to utilise covering space structures to efficiently solve problems from HiddenCover.

1 Introduction

There is a growing belief that quantum computers can solve certain computational problems exponentially faster than any classical computer. Strong evidence for this belief comes in the form of Shor’s algorithm [Sho94] for factorisation, and the graph traversal algorithm of Childs et. al. [CCD+03].

Despite the spectacular success of the known quantum algorithms we believe that there is still only a fairly rudimentary understanding of the properties of quantum mechanics which are useful for computational speedups. We ascribe this poor understanding to at least two causes: (i) we are unsure what sorts of problems might be amenable to quantum-computational speed up; and (ii) even if we firmly believed that an efficient quantum algorithm existed for a problem, it is hard to come up with this putative algorithm because it is difficult (at least for us) to reason within the traditional quantum computing model — the quantum circuit model.

All quantum algorithms are traditionally expressed in the quantum circuit model (see [NC00] and [Pre98] for a detailed description of the quantum circuit model). The quantum circuit model assigns a unit cost to certain elementary quantum gates, such as, for example, CNOT, H, and T (the $\pi/8$ phase gate). This is by no means the only way to express quantum algorithms. Three other computing paradigms are polynomially equivalent to the quantum circuit model: the quantum Turing machine model [BV97], the one-way quantum computer [RB01], and the adiabatic evolution model (which has recently been shown to be polynomially equivalent to the quantum circuit model in [AvDK+04]).

Why might one want to consider computational paradigms other than the quantum circuit model? The reason is that there are conceptual peculiarities with the quantum circuit model which make it hard to work with when designing algorithms the traditional way. Firstly, there is a back-action that quantum superposition induces in gates like the CNOT where there can be a “backwards” flow of quantum information for certain initial states. Secondly, the existence of quantum entanglement seriously confuses the causal structure of quantum circuits because we can think of a Bell pair as sending a qubit backwards in time! However, in favour of the quantum circuit model is the fact that exponentially many degrees of freedom can be summarised succinctly.

Some of the negative features of the quantum circuit model are amply addressed in the quantum walk model of computation. Quantum walks, or the quantum dynamics of a free particle hopping on a graph, are an exciting new
paradigm for quantum computing. (For a review of quantum walks see [Kem03a and references therein.) The attractiveness of quantum walks is that they provide an extremely intuitive way to visualise exponentially many quantum degrees of freedom. In contrast to the quantum circuit model, in a quantum walk there is a clear physical picture of where information is flowing and a definite notion of cause and effect. The price we pay for these intuitive features is that the exponentially ($2^n$ for $n$ qubits) many degrees of freedom of $n$ qubits translate to exponentially vertices in the graph. Additionally, there is no clear way to translate a quantum walk efficiently into the quantum circuit model. Despite these difficulties we believe that quantum walks provide an attractive methodology in the quantum algorithm designer’s toolkit because they appeal to the geometrical intuitions.

The growing number of quantum-walk algorithms might be seen as evidence for the conceptual utility of the quantum walk model. We mention three recent algorithms: (i) the quantum walk search algorithm [SKW03, CG03, and AKR04]; (ii) the graph-traversal algorithm of Childs et. al. [CCD+03]; and (iii) the element distinctness algorithm of Ambainis [Amb03], and relatives [CE03, MSS03, Sze04].

What unites the quantum walk algorithms? (And, more ambitiously, all quantum algorithms?) Let’s concentrate on the results of [CFG02, CCD+03, MR02, Kem03b], which are based on the continuous-time quantum walk. (Note that the quantum walks on the hypercube [MR02, Kem03b] and the graphs in [CFG02] don’t provide the speedups for the solution to any algorithmic problem.) The results in these papers appear to be related phenomena (they all take advantage of “column spaces”), however, this relationship has not yet, to the best of our knowledge, been quantified fully. In this paper we identify and generalise the combinatorial property of these graphs which leads to small hitting times. We believe this is a key ingredient underlying quantum speedups of hitting times. The combinatorial property we isolate is that all of the graphs walked on in these papers are covering spaces for much smaller graphs.

**Quantum computers and covering spaces**

What is a covering space? Suppose that $X$ and $Y$ are arcwise-connected and locally arcwise-connected topological spaces, respectively. Then $(Y, \pi)$ is said to be a covering space of $X$ if $\pi: Y \to X$ is a surjective continuous map with every $x \in X$ having an open neighborhood $U$ such that every connected component of $\pi^{-1}(U)$ is mapped homeomorphically onto $U$ by $\pi$. The preimage of a point in $X$ is called a fibre of $\pi$. An example covering space is shown in figure 1.

In this paper we will show that a continuous-time quantum walk on a graph $Y$ which is a covering space for another graph $X$ (in the natural topology) is isomorphic to a quantum walk on $X$, under certain specific circumstances. This equivalence occurs when the walk starts on fibre-constant quantum states. Additionally, we will argue, and show in some cases, that if there is a tower of
Figure 1: An example of a covering space. In this case the circle $X = S^1$ is covered by another circle $Y$ which has twice the circumference of $X$ under the projection $\pi$. Note that the inverse projection of any point $x \in X$ is a finite set consisting of two points $a$ and $b$ in the covering space $Y$. One can think of this covering space as a subset of the Riemann surface for $f(z) = \sqrt{z}$, which is a covering space for $\mathbb{C}$.

covering spaces

$$Y \xrightarrow{\pi_N} X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} X_1$$

(1)

then there is an efficient gate sequence (i.e. using $O(\text{poly}(n))$ elementary gates) for the quantisation of $Y$. The idea we exploit is to recursively and hierarchically construct the gate sequence from the "elementary gate" $U(X_1)$ and the specification of how $U(X_1)$ lives in $U(X_2)$.

We believe that these results are not specific to continuous-time quantum walks, but rather indicative of a general principle.

We formulate this principle in the following way: consider some collection of mathematical objects like the class of simple graphs, or something with more structure, like the class of finite fields. Let $U(X)$ be a quantisation scheme for this class of objects, by which we mean a way to associate a unitary matrix $U(X)$ acting on a finite-dimensional Hilbert space $\mathcal{H}(X)$ with any object $X$. Suppose, further, that $Y$ is a covering space $\pi: Y \to X$ for another object $X$. If the quantisation scheme $U(X)$ is "sufficiently well-behaved" then $U(Y)$ should be related to $U(X)$ according to a map, the pull-back $\pi^*: U(X) \to U(Y)$. If, further, $Y$ is determined from $X$ in a sufficiently simple way, then it should be possible to construct $U(Y)$ from $U(X)$ according to this specification. Mimicking the recursive and hierarchical construction for gate-sequences for continuous-time quantum walks we expect that if

$$Y \xrightarrow{\pi_N} X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} X_1$$

(2)

is a tower of covering spaces obeying some uniformity and growth conditions then there should be an efficient gate sequence for the quantisation $U(X)$. 

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We retroactively argue that there is an example of where this philosophy has already proven successful. In this example we take as our space of mathematical objects the category $\text{GrpFin}$ of finite groups. We take the quantisation scheme to be the contravariant functor which associates the quantum Fourier transform with every finite group. (For an introduction to category theory see [ML98].) For certain towers of subgroups, i.e. those which are polynomially uniform [MRR03], there is an efficient hierarchical scheme to construct the quantisation $U(Y)$. Here we are taking the “is a subgroup” relation to be the covering space relation in this category. To accord this exactly with the definition of covering space given earlier requires the introduction of an appropriate topology on a finite group. This is something we will avoid, preferring only this qualitative argument — clearly the analogy isn’t perfect.

What sort of problems could these supposed efficient quantisations solve? We will argue that they could be used to solve a problem we call the Hidden Covering Space problem which includes the hidden subgroup problem as a subclass. Roughly speaking, if there is a function $f$ on $Y$ which is periodic on some object $X$ covered by $Y$, then a quantum computer should be able to identify the structure $X$ efficiently.

The general properties of covering spaces are extremely tantalising and suggestive. We hope to convince the reader that the results we have found concerning graph covering spaces and continuous-time quantum walks are indicative of a much larger framework. We believe that many quantum algorithms exhibiting an exponential separation between classical and quantum complexity could exist. We argue that the properties of quantisations of covering spaces may provide many opportunities to design such new algorithms. This is not least because covering spaces are pervasive in mathematics, from number fields and algebraic surfaces to geometric group theory, and, of course, Galois theory.

The outline of this paper is as follows. We begin in §2 by describing the quantisation scheme we study in the remainder of this paper, the continuous-time quantum walk. In §3 we review the theory of graph covering spaces and the heat kernel for graphs. We then apply these results in §4 to demonstrate that quantum walks which begin on fibre-constant states are isomorphic to quantum walks on smaller graphs. We illustrate our results in §5 for the hypercube and Cayley graphs. In §6 we utilise the covering-space properties of the $m$-torus graph on $2^n$ vertices to construct an efficient gate sequence for their quantisation. Motivated by our results we introduce, in §7, a new class of problem, HiddenCover, which quantum computers may be able to solve efficiently. We also solve the hidden cover problem for the $m$-torus graph which, incidentally, provides a (philosophically) different way to solve the abelian hidden subgroup problem.
2 Quantisations of Graphs: the Continuous-Time Quantum Walk

In this section we introduce the continuous-time quantum walk, which is a quantisation scheme for the class of simple graphs. For further details about the graph-theoretic notation and terminology we use in this section and the rest of this paper see Biggs [1993], CDS95, Chm97.

Let us begin by defining the main objects of our study. By a weighted graph $Y$ we mean a vertex set $V = V(Y)$ with an associated weight function $w : V \times V \to \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the real numbers $x \in \mathbb{R}$ such that $x \geq 0$) which satisfies

$$w(u, v) = w(v, u).$$

If $w(u, v) > 0$ then we refer to $\{u, v\}$ as an edge of $Y$, and we say that $u$ and $v$ are adjacent. By a simple graph we mean the special situation where $w(u, v)$ is either 0 or 1 and $w(u, u) = 0$ for all $u \in V$.

We define the degree $d_v$ for a vertex $v$ to be

$$d_v = \sum_{u \in V} w(u, v).$$

A graph is regular if all the degrees are the same.

The Laplacian of a weighted graph $Y$ on $n$ vertices and weight function $w$ is the $n \times n$ matrix $\Delta$ given by

$$\Delta_{u,v} = \begin{cases} d_u - w(v,v) & \text{if } u = v, \\ -w(u,v) & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise,} \end{cases}$$

where $u$ and $v$ are two arbitrary vertices in $V$.

We now consider introduce the (formal) complex vector space $\mathcal{H}(Y)$ spanned by the vectors $|u\rangle$, $u \in V$, which we take to be orthonormal under the natural inner product: $\langle u|v \rangle = \delta_{u,v}$. Of course this vector space is isomorphic to $\mathbb{C}^n$.

We think of the vector space $\mathcal{H}(Y)$ as the space of all complex-valued functions $f$ on the finite set $V(Y)$, where for each vector $|f\rangle = \sum_{u \in V(Y)} f_u |u\rangle \in \mathcal{H}(Y)$ we define the function $f : V(Y) \to \mathbb{C}$ by $f(u) = \langle u|f \rangle$.

We define the adjacency matrix of a weighted graph $Y$ to be the operator

$$A(Y) \triangleq \sum_{u,v \in V(Y)} w(u,v)|u\rangle\langle v|.$$  

The Laplacian, acts in a natural way on $\mathcal{H}(Y)$ as

$$\Delta = \sum_{u \in V(Y)} (d_u - w(u,u))|u\rangle\langle u| - \sum_{u,v \in V(Y)} w(u,v)|u\rangle\langle v|,$$  

where $u$ and $v$ are two arbitrary vertices in $V$. 

where we use the notation $u \sim v$ to mean that $u$ is adjacent to $v$ and $u \neq v$. For a specific vector $|f\rangle = \sum_{u \in V(Y)} f_u |u\rangle$ we have

$$
\Delta |f\rangle = \sum_{u,v \in V(Y) \atop u \sim v} (f(u) - f(v))w(u,v)|u\rangle.
$$

It is worth noting that the Laplacian can be written $\Delta = D(Y) - A(Y)$, where $D(Y) \triangleq \sum_{u,v \in V(Y)} w(u,v)|u\rangle\langle u|$.

We define the heat equation and Schrödinger equation for $Y$ to be the differential equations

$$
\frac{\partial}{\partial \tau} |\psi(\tau)\rangle = -\Delta |\psi(\tau)\rangle,
$$

and

$$
i \frac{\partial}{\partial t} |\psi(t)\rangle = \Delta |\psi(t)\rangle,
$$

respectively. Note that the Schrödinger equation is equivalent to the heat equation with the replacement $\tau = it$.

The Laplacian $\Delta$ for a graph on $n$ vertices is a symmetric matrix and so we can write its spectral decomposition

$$
\Delta = \sum_{j=0}^{n-1} \lambda_j |\lambda_j\rangle\langle \lambda_j|.
$$

We often refer to an eigenstate of $\Delta$ as a harmonic eigenfunction.

The spectral decomposition of the Laplacian allows us to define the heat kernel and propagator

$$
H(\tau) = \sum_{j=0}^{n-1} e^{-\lambda_j \tau} |\lambda_j\rangle\langle \lambda_j|,
$$

and

$$
U(t) = \sum_{j=0}^{n-1} e^{-i \lambda_j t} |\lambda_j\rangle\langle \lambda_j|,
$$

respectively. Note that $U(t) = H(it)$. Using the the heat kernel and propagator we can solve the heat and Schrödinger equations $\partial \psi(\tau) / \partial \tau = -\Delta \psi(\tau)$ and $i \partial \psi(t) / \partial t = \Delta \psi(t)$ with initial state $|\psi(0)\rangle$ by defining $|\psi(\tau)\rangle = H(\tau)|\psi(0)\rangle$ and $|\psi(t)\rangle = U(t)|\psi(0)\rangle$:

$$
\frac{\partial}{\partial \tau} |\psi(\tau)\rangle = \frac{\partial H(\tau)}{\partial \tau} |\psi(0)\rangle = -\Delta H(\tau)|\psi(0)\rangle.
$$

The result for the propagator follows by substituting $\tau = it$.

**Definition 2.1.** A continuous-time quantum walk on a weighted graph $Y$ is the propagator $U[Y](t)$ associated with the Laplacian $\Delta[Y]$ for $Y$. 

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3 Graph Covering Spaces and the Heat Kernel

In this section we review the theory of graph covering spaces and the heat kernel for weighted graphs. For an introduction to graph covering spaces see [Big93] and [GM80]. For further results in this area see [CY99] and [ST96, ST01, Ter99, ST00, Ter02]. We follow [CY99] closely for most of this section. The principal result of this section is that the spectrum of a graph $X$ covered by another graph $Y$ is contained in the spectrum of $Y$. The eigenvectors of $X$ also induce eigenvectors in $Y$.

We begin with some definitions.

**Definition 3.1.** Suppose we have two graphs $X$ and $Y$ with weight functions $w_X(x,y)$ and $w_Y(u,v)$, respectively. We say $Y$ is a covering space (alternatively, $X$ is covered by $Y$) if there is a set map $\pi: V(Y) \to V(X)$ satisfying the following two properties:

1. There is a $\mu \in \mathbb{R}^+$ (the positive real numbers including 0), called the index of $\pi$, such that for $u, v \in V(X)$ the equality holds:
   \[
   \sum_{x \in \pi^{-1}(u)} w_Y(x,y) = \mu \sqrt{||\pi^{-1}(u)||} w_X(u,v). \tag{15}
   \]

2. For $x, y \in V(Y)$ with $\pi(x) = \pi(y)$ and $v \in V(X)$, we have
   \[
   \sum_{z \in \pi^{-1}(v)} w_Y(z,x) = \sum_{z' \in \pi^{-1}(v)} w_Y(z',y). \tag{16}
   \]

We now translate Definition 3.1 into a more useful statement about the adjacency matrices of $Y$ and $X$. In doing so we make contact with the theory of equitable partitions [Big93].

We begin by considering an arbitrary projection operator $\pi: V(Y) \to V(X)$ and define the corresponding pull-back operator $P: \mathcal{H}(Y) \to \mathcal{H}(X)$,

\[
P = \sum_{u \in V(X)} \frac{1}{\sqrt{||\pi^{-1}(u)||}} |u\rangle \langle x|. \tag{17}
\]

(It is easy to establish the identity $(P^\dagger P)^2 = P^\dagger P$ via multiplication, which shows that $P^\dagger P$ is a projector. Similarly, one can show that $PP^\dagger = I_{\mathcal{H}(X)}$. This implies that $P^\dagger$ is an isometric imbedding $P^\dagger: \mathcal{H}(X) \to \mathcal{H}(Y)$.) What are the conditions that $\pi$ must satisfy in order that $\pi$ be a legal covering map? We answer this question in two steps, by translating parts (i) and (ii) of Definition 3.1 into conditions on $P$.

\footnote{Note that our definition of (unramified) graph covering spaces accords with the standard definition given in the introduction when we take for the topology of the graphs $X$ and $Y$ the weak topology. See [Hat02] and [ST01, ST00] for further discussion and details.}
Using the operator $P$ we define the quotient matrix $A_\pi(X) \triangleq PA(Y)P^\dagger$, which we want to identify with the adjacency matrix of $X$. Expanding this expression and utilising Definition 3.1 we obtain

\[
A_\pi(X) = \sum_{u \in V(X)} \sum_{v \in V(X)} \sum_{l \in \pi^{-1}(u)} \sum_{m \in \pi^{-1}(v)} \frac{w_Y(x, y)}{\sqrt{|\pi^{-1}(u)||\pi^{-1}(v)|}} |u\rangle \langle l| |x\rangle \langle y|m\rangle |v\rangle
\]

\[
= \sum_{u \in V(X)} \sum_{v \in V(X)} \sum_{l \in \pi^{-1}(u)} \sum_{m \in \pi^{-1}(v)} \frac{w_Y(l, m)}{\sqrt{|\pi^{-1}(u)||\pi^{-1}(v)|}} |u\rangle \langle v|
\]

\[
= \sum_{u \in V(X)} \sum_{v \in V(X)} \mu w_X(u, v) |u\rangle \langle v| = \mu A(X).
\]

This expression shows us that the adjacency matrix $w_Y(u, v)$ for $Y$ satisfies the condition (i) of Definition 3.1 to be a covering space for the graph $X$ with adjacency matrix $w_X(u, v)$. Note that part (i) of Definition 3.1 is really saying that $Y$ is a covering space for any graph which has as its adjacency matrix a scalar multiple of $PA(Y)P^\dagger$.

In order to establish the matrix version of part (ii) of Definition 3.1 we need to make use of the following result of Godsil and McKay [GM80], which is the fundamental connection between $P$ and graph covering spaces. We include a proof for completeness.

**Lemma 3.2 (Godsil and McKay [GM80]).** The graph $Y$ is a covering space for $X$ if and only if $PA(Y) = A(X)P$.

**Proof.** For a vertex $v \in V(Y)$, $1 \leq j \leq |V(X)|$ define $h_{vj}$ to be the sum of the weights of the edges connecting vertices in $\pi^{-1}(u_j)$ with $v$, where $u_j \in V(X)$. That is, $h_{vj} = \sum_{x \in \pi^{-1}(u_j)} w_1(v, x)$. For $1 \leq j \leq |V(X)|$, $v \in V(Y)$, we have

\[
\langle u_j | PA(Y) | v \rangle = \frac{1}{\sqrt{|\pi^{-1}(u_j)|}} \sum_{x \in \pi^{-1}(u_j)} w_1(x, v) = \frac{h_{vj}}{\sqrt{|\pi^{-1}(u_j)|}}
\]

(19)

and

\[
\langle u_j | A(X)P | v \rangle = \frac{w_2(u_j, u_k)}{\sqrt{|\pi^{-1}(u_k)|}}.
\]

(20)

where $v \in \pi^{-1}(u_k)$.

By comparing (19) and (20) we note that if $PA(Y) = A(X)P$, then $h_{vj}$ equals $\sqrt{|\pi^{-1}(u_j)|} w_2(u_j, u_k)$ for all $v \in \pi^{-1}(u_j)$ and so $\pi$ satisfies part (ii) of definition 3.1.

For the converse, suppose $\pi$ satisfies the conditions of Definition 3.1. We see that

\[
PA(Y) = PA(Y)P^\dagger P = A(X)P.
\]

(21)

\[\square\]
It is well known that the spectrum of a graph $Y$ which is a covering space for $X$ is determined, in part, by the spectrum of $X$. Let’s make this a bit more precise. Suppose we have a harmonic eigenfunction of $|f⟩$ of $X$ with eigenvalue $\lambda$. We can lift or pull back this eigenfunction to a harmonic eigenfunction $|g⟩$ of $Y$, by defining for each vertex $l \in Y$, $(|l⟩|g⟩) = (|\pi(l)⟩|f⟩)/\sqrt{|\pi^{-1}(l)⟩}$. Equivalently, $|g⟩ = PA|f⟩$.

We use the notation $|\pi^{-1}(u)⟩ = 1/\sqrt{|\pi^{-1}(u)⟩} \sum_{l \in \pi^{-1}(u)} |l⟩$ and write

$$|g⟩ = \sum_{l \in V(X)} f(l)|\pi^{-1}(l)⟩.$$  

(22)

It follows from Definition 3.1 that $|g⟩$ is a harmonic eigenfunction for $Y$.

**Lemma 3.3.** Let $Y$ be a covering space for $X$ with projection map $\pi$. Then for any eigenvector $|f⟩$ and scalar $\lambda$, $A(X)|f⟩ = \lambda|f⟩$ if and only if $A(Y)P|f⟩ = \lambda P|f⟩$.

**Proof.** If $A(X)|f⟩ = \lambda|f⟩$, then $PA(X)|f⟩ = \lambda P|f⟩$ and so $A(Y)P|f⟩ = \lambda P|f⟩$, by lemma 3.2. If $A(Y)P|f⟩ = \lambda P|f⟩$, then $PA(Y)P|f⟩ = \lambda P|f⟩$ and so $A(X)|f⟩ = \lambda|f⟩$.

We define the Laplacian $\triangle(X)$ for the covered graph $X$ via $\triangle(X) ≜ PD(Y)PA(Y)P − P$. Let’s make this a bit more precise. Suppose we have a harmonic eigenfunction of $|f⟩$ of $X$ with eigenvalue $\lambda$. We can lift or pull back this eigenfunction to a harmonic eigenfunction $|g⟩$ of $Y$, by defining for each vertex $l \in Y$, $(|l⟩|g⟩) = (|\pi(l)⟩|f⟩)/\sqrt{|\pi^{-1}(l)⟩}$. Equivalently, $|g⟩ = PA|f⟩$.

The main result for this section is the following lemma relating the heat kernels and propagators of a graph $X$ and an arbitrary covering space $Y$.

**Lemma 3.4.** Suppose $Y$ is a covering space for $X$. Let $H[Y](\tau)$ and $H[X](\tau)$ and $U[Y](t)$ and $U[X](t)$ denote the heat kernels and propagators for $Y$ and $X$, respectively. Then we have

$$H[X](\tau) = PH[Y](\tau)P, \quad \text{and} \quad U[X](t) = PU[Y](t)P.$$  

(23)

**Proof.** We establish the lemma for the heat kernel. The result for the propagator follows from substituting $\tau = it$.

Note first that $A(X)^r = PA(Y)^r P$, which follows from a simple induction using lemma 3.2.

Expanding the heat kernel in an absolutely convergent power series in $\tau$ gives:

$$H[X](\tau) = \sum_{j=0}^{\infty} \frac{(-\tau PA(Y)P)^j}{j!}$$

$$= \sum_{j=0}^{\infty} P \frac{(-\tau A(Y))^j}{j!} P^t = PH[Y](\tau)P.$$  

(24)
Because we are living in finite Hilbert spaces, where everything is well-behaved, we don’t have to worry about the convergence of series. For this reason, knowing the heat kernel \( H(\tau) \) is equivalent to knowing the propagator \( U(t) \). We take advantage of this fact by only stating all our results about the propagator in terms of the heat kernel in imaginary time.

4 Induced Quantum Walks — the Quotient Walk

In this section we show that if a graph \( Y \) is a covering space for another graph \( X \) with projection \( \pi : V(Y) \to V(X) \) then a quantum walk on \( Y \) which begins on a state constant on fibres of \( \pi \) is isomorphic to a quantum walk on \( X \). This result provides an insight into the hitting-time speedups observed by Kempe [Kem03b] and Childs et. al. [CFG02, CCD+03].

**Definition 4.1.** Let \( Y \) be a covering space for \( X \) with projection \( \pi : V(Y) \to V(X) \). We say that a quantum state \(|\psi\rangle \in H(Y)\) is fibre-constant for \( \pi \) if 
\[
|\psi\rangle = \sum_{u \in V(X)} c_u |\pi^{-1}(u)\rangle,
\]
where we are using the notation
\[
|\pi^{-1}(u)\rangle = P^\dagger|u\rangle = \sqrt{1/|\pi^{-1}(u)|} \sum_{x \in \pi^{-1}(u)} |x\rangle
\]
of the previous section. We write \( |\psi\rangle = P^\dagger|\phi\rangle \), where \(|\phi\rangle = \sum_{u \in V(X)} c_u |u\rangle\).

What is the time evolution of a fibre-constant state \(|\psi\rangle\) for covering map \( \pi \)? The following simple lemma shows us that it is isomorphic to a walk on \( X \).

**Lemma 4.2.** Suppose \( Y \) is a covering space for \( X \) with covering map \( \pi \). Let 
\[
|\psi\rangle = P^\dagger|\phi\rangle
\]
be a fibre-constant state. Then the time evolution of \(|\psi(t)\rangle\) on \( Y \) is isomorphic to the time evolution of \(|\phi(t)\rangle\) on \( X \).

**Proof.** Let \( H[Y](\tau) \) be the heat kernel for \( Y \). The time evolution of \(|\psi(\tau)\rangle\) on \( Y \) is given by
\[
|\psi(\tau)\rangle = H[Y](\tau)P^\dagger|\phi\rangle
= P^\dagger H[X](\tau)|\phi\rangle \quad \text{by lemma 3.4}
\]
This equation makes it clear that the time-evolution of \(|\psi(\tau)\rangle\) is determined from that of \(|\phi\rangle\) on \( X \). 

5 Examples: the Hypercube \( Q^n \) and Cayley and Schreier Graphs

We now illustrate this result for the hypercube and the Cayley and Schreier graphs of a finite group.
**Definition 5.1.** Let $G$ be a group, and let $S \subset G$ be a set of group elements such that: (1) The identity element $e \not\in S$; (2) If $x \in S$, then $x^{-1} \in S$. The Cayley graph $X(G,S)$ associated with $G$ and $S$ is then defined as the simple graph having one vertex associated with each group element and directed edges $(g,h)$ whenever $gh^{-1} \in S$. (It is easy to check that with our definition of the generating set this graph is well-defined.)

**Example 5.2.** We think of the hypercube graph $Q_n$ of dimension $n$ as a Cayley graph of the the abelian group $(\mathbb{Z}/2\mathbb{Z})^n$, the $n$-fold direct product of $\mathbb{Z}/2\mathbb{Z}$. Specifically, we write $Q_n = X((\mathbb{Z}/2\mathbb{Z})^n, \{e_1,e_2,\ldots,e_n\})$, where $e_j = (0,\ldots,1,\ldots,0)$ is the unit vector with a one in the $j$th entry. Note that $|V(Q_n)| = 2^n$. It is easily verified that $Q_n$ is a covering for the weighted path $P_n^Q$ of size $|V(P_n^Q)| = n + 1$ with adjacency matrix

$$A(P_n^Q) = \begin{bmatrix} 0 & \sqrt{n} & 0 & \cdots & 0 \\ \sqrt{n} & 0 & \sqrt{2(n-1)} & \cdots & 0 \\ 0 & \sqrt{2(n-1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \sqrt{n} \\ 0 & 0 & \cdots & \sqrt{n} & 0 \end{bmatrix}. \quad (28)$$

The projection map $\pi : V(Q_n) \to V(P_n^Q)$ maps the collection of vertices of $V(Q_n)$ of hamming weight $j$ to the single vertex $v_j \in V(P_n^Q)$ at position $j$.

**Remark 5.3.** Because $Q_n$ is vertex-transitive (i.e. the automorphism group $\text{Aut}(Q_n)$ acts transitively on $V(Q_n)$) a quantum walk on $Q_n$ beginning on the state concentrated on any single vertex is equivalent to a walk on the weighted path $P_n^Q$ beginning at the first vertex. (Note that all Cayley graphs $Y$ are vertex-transitive, where the automorphism group $\text{Aut}(Y)$ acts transitively on the vertex set.) The adjacency operator $(28)$ of the weighted path is exactly the angular momentum operator $J_x$ in the spin-$\frac{2}{\sqrt{2}}$ irrep. of $SU(2)$. One can envisage the quantum state propagating from the first vertex to the last vertex in $P_n^Q$ as a localised state at $z = +\frac{\pi}{2}$ on north pole of the Bloch sphere propagating across the sphere (and hence becoming completely delocalised around the equator in the process) to the south pole $z = -\frac{\pi}{2}$.

Our second example concerns the Cayley graph for an arbitrary finite abelian or nonabelian group $G$. To introduce it we need a definition.

**Definition 5.4.** Given a subgroup $H$ of a finite group $G$ and a generating subset $S$ (recall $s \in S$ implies $s^{-1} \in S$ and $e \not\in S$) of $G$, which we call the edge set, the Schreier graph $X = X(G/H,S)$ has as vertices the cosets $gH$, $g \in G$. Two vertices $gH$ and $s^{-1}gH$ are joined by an edge for all $s \in S$.

**Remark 5.5.** Note that unless $H$ is a normal subgroup $H \trianglelefteq G$ then the Schreier graph $X(G/H,S)$ will contain loops. In the case that $H$ is normal then $X(G/H,S)$ is exactly the Cayley graph of the group $G/H$ with generating set $S$. Note also that our definition of graph covering spaces includes graphs with loops (i.e. when $w(u,u) > 0$) so that the following results are well-defined.
Lemma 5.6. The adjacency matrix $A(X)$ for a Cayley graph $X(G,S)$ can be written in the following way

$$A(X) = \sum_{s \in S} \sum_{x \in G} |x\rangle\langle s^{-1}x|.$$  \hfill (29)

Similarly, the adjacency matrix $A(X(G/H,S))$ for the Schreier graph $X(G/H,S)$ is given by

$$A(X(G/H,S)) = \sum_{s \in S} \sum_{gH \in H} |gH\rangle\langle s^{-1}gH|.$$  \hfill (30)

We aim to show that if $Y$ is a Cayley graph $Y = X(G,S)$ for a finite group $G$ with generating set $S$ then it is a covering space for the Schreier graphs $X(G/H,S)$ for all $H \leq G$. In order to show this we define the projection map

$$\pi: Y \rightarrow X(G/H,S)$$

by

$$\pi(g) = gH.$$  \hfill (31)

In order to establish our claim we need to show that $PA(Y)$ and $A(X(G/H,S))P$ are equal and apply lemma 3.2.

$$A(X(G/H,S))P = \frac{1}{|H|} \sum_{s \in S} \sum_{gH \in G/H} \sum_{kH \in G/H} \delta_{s^{-1}gH,kH} |gH\rangle\langle k'|$$

$$= \frac{1}{|H|} \sum_{s \in S} \sum_{kH \in G/H} \sum_{k' \in kH} |kH\rangle\langle s^{-1}k'|$$

$$= \frac{1}{|H|} \sum_{s \in S} \sum_{kH \in G/H} \sum_{k' \in kH} |kH\rangle\langle s^{-1}k'|$$

$$= PA(Y).$$  \hfill (32)

The transition from the second-last line to the last follows by a change of variable $kH \rightarrow skH$ and $k' \rightarrow sk'$, and the definition of $P$ and $A(Y)$. This discussion, together with lemma 4.2, constitutes the following corollary.

Corollary 5.7. A quantum walk on a Cayley graph $X(G,S)$ which begins on the fibre-constant state $|\phi_H\rangle = \sqrt{|H|/|G|} \sum_{gH \in G/H} c_{gH} |\pi^{-1}(gH)\rangle$ is isomorphic to a quantum walk on induced the Schreier graph $X(G/H,S)$ for the subgroup $H$ which begins on the induced state $|\phi'\rangle = \sqrt{|H|/|G|} \sum_{gH \in G/H} c_{gH} |gH\rangle$.

6 Efficient Gate Sequences for Continuous-Time Quantum Walks

In this section we explain how to construct efficient gate sequences simulating continuous-time quantum walks on certain classes of graphs $Y$. The principle
A feature intuitively exploited in these gate sequences is that the graph $Y$ covers a tower $Y \xrightarrow{\pi_{N-1}} X_N \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} X_1$ of graphs $X_j$. An interesting feature of these gate sequences is that their length doesn’t depend on the time $t$ the walk is simulated for.

Consider the cycle $C_{2^n}$ on $2^n$ vertices, which is the Cayley graph $C_{2^n} = X(\mathbb{Z}/2^n\mathbb{Z}, \{\pm 1\})$ of the cyclic group. We begin by presenting a quantum circuit which simulates the quantum walk on the cycle $C_{2^n}$ which uses only $O(\text{poly}(n))$ gates. A special feature of this gate sequence is that it can simulate the walk for any given time $t$, including exponential times $O(t) = 2^{\text{poly}(n)}$.

Recall [Lub95] that the eigenvalues and eigenstates of the adjacency matrix $A(C_m)$ are given in terms of sums of the characters $\mu^j = e^{i \frac{2\pi j}{m}}$, $j = 0, \ldots, m - 1$, of $\mathbb{Z}/m\mathbb{Z}$. Specifically,

$$A(C_m) = \sum_{j=0}^{m-1} (\mu^j + \mu^{-j}) |W(j)\rangle\langle W(j)|,$$  \hspace{1cm} (33)

where

$$|W(j)\rangle = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \mu^j |k\rangle.$$  \hspace{1cm} (34)

Note that the vectors $|W(j)\rangle$ are precisely those given by applying the quantum fourier transformation to $|j\rangle$:

$$|W(j)\rangle = \text{QFT}|j\rangle = \frac{1}{\sqrt{N}} \sum_{k,l=0}^{m-1} \mu^{kl} |k\rangle\langle l|.$$  \hspace{1cm} (35)

Our objective is to simulate, with a quantum circuit, the evolution

$$U[C_{2^n}](t) = \sum_{j=0}^{m-1} e^{-i 2 \cos \left( \frac{2\pi j}{m} t \right)} |W(j)\rangle\langle W(j)|,$$  \hspace{1cm} (36)

where $m = 2^n$. We note that this unitary can be written as

$$U[C_{2^n}](t) = \text{QFT}\Phi(t)\text{QFT}^\dagger,$$  \hspace{1cm} (37)

where

$$\Phi(t) = \sum_{j=0}^{m-1} e^{-i 2 \cos \left( \frac{2\pi j}{m} t \right)} |j\rangle\langle j|.$$  \hspace{1cm} (38)

The phase operation $\Phi(t)$ can be performed efficiently using, for instance, the phase kickback trick [CEMM98]. The phase kickback trick shows that if the computation

$$|j\rangle \mapsto |j\rangle|f(j)\rangle,$$  \hspace{1cm} (39)

where $|j\rangle$ is a computational basis state of $n$ qubits and $f(j)$ is an $n$ bit approximation to $(2 \cos \left( \frac{2\pi j}{m} t \right)) \bmod 2\pi$ (calculable efficiently classically), is implementable efficiently, then the phase changing operation

$$|j\rangle \mapsto e^{-i 2 \cos \left( \frac{2\pi j}{m} t \right)} |j\rangle,$$  \hspace{1cm} (40)
is implementable using $O(\text{poly}(n))$ quantum gates \cite{CEMM98}. (It is easy to obtain arbitrary accuracy by appending more qubits to the register for $|f(x)|$.)

Hence, because there is an efficient (i.e., using $O(\text{poly}(n))$ elementary gates) quantum circuit for the quantum fourier transform \cite{NC00}, there is an efficient gate sequence approximating the propagator $U[C_m]|(t)$. Moreover, the number of gates required to simulate the propagator does not depend on the time $t$.

We now note some features of our gate sequence for $U[C_m](t)$. Firstly, we point out that the circuits for $U[C_m](t)$ can be easily generalised to simulate quantum walks on a many other families of graphs. This is because any circulant matrix is implementable using $O(\text{poly}(n))$ quantum gates \cite{NC00}. This idea is explored further in \cite{Osb04}.

We believe that the existence of covering spaces structures in graphs may be utilised more generally to supply efficient quantum circuits for quantum walks on graphs. We sketch one result which is indicative of this idea.

We restrict our attention to regular graphs. Let $Y$ be a regular graph on $2^n$ vertices which is a covering space for a tower of graphs $X \xrightarrow{\pi_N} Y \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} X_1$ with the property that $|V(X_{j+1})| \approx \sqrt{|V(X_j)|}$. In this case $N$ must be $O(\text{poly}(\log(|V(Y)|)))$.

We assume that $m = |\pi_{N}^{-1}(u)|$ is the same for all $u \in V(X_N)$. Consider the transition matrix $A(u,v)$ for an edge $(u,v) \in E(X_N)$, which is the $2m \times 2m$ matrix whose $(x,y)$ entry, where $x,y \in \pi_{N}^{-1}(u) \cup \pi_{N}^{-1}(v)$, is given by $A(Y)_{x,y}$.

\footnote{Given a finite field $\mathbb{F}_q$ with $q$ elements, the Paley graph $P(\mathbb{F}_q)$ is the graph with vertex set $V = \mathbb{F}_q$ where two vertices are joined when their difference is a square in the field. This is an undirected graph when $q$ is congruent $1(\text{mod} 4)$ \cite{Chm97}. Note that Paley graphs have edge density approximately $\frac{1}{2}$, and so are not efficiently simulatable using the recipe of \cite{ATSO03}.}
Suppose we can label the vertices in $\pi_N^{-1}(u)$ so that

$$A(u,v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_m,$$

or

$$A(u,v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \frac{1}{\sqrt{m}} J_m,$$

(42)

where $I_m$ is the $m \times m$ identity matrix and $J_m$ is the $m \times m$ all 1’s matrix. We call such a transition matrix trivial, and we say that the edge $(u,v)$ has trivial pull-back.

Suppose the edges of $A(X_N)$ have trivial pull-back for all but $O(\text{poly}(\log(|V(Y)|)))$ edges. In the case we can write the adjacency matrix for $A$ as

$$A(Y) = A(X_N) \otimes I_m + D,$$

or

$$A(Y) = A(X_N) \otimes \frac{1}{\sqrt{m}} J_m + D,$$

(43)

where $D$ is a symmetric $|V(Y)| \times |V(Y)|$ matrix whose $(x,y)$ entry, $x,y \in \pi_N^{-1}(u) \cup \pi_N^{-1}(v)$, is nonzero only when there is an edge $(u,v) \in E(X_N)$ whose transition matrix is nontrivial. When the edges of $A(X_N)$ have trivial pull-back for all but $O(\text{poly}(\log(|V(Y)|)))$ edges, $D$ is sparse, and has $O(\text{poly}(\log(|V(Y)|)))$ nonzero entries in each row (in the language of [ATS03], $D$ is row sparse).

We want to find a quantum circuit which simulates $U[Y](t) = e^{-iA(Y)t}$ (note that because $Y$ is regular the action of the adjacency matrix and laplacian are equivalent). To do this we apply the Trotter formula (see [NC00] for details and further discussion)

$$\lim_{n \to \infty} (e^{-iAt/n} e^{-iBt/n})^n = e^{-i(A+B)t}.$$  

(44)

By taking $O(n) = \text{poly}(\|A\|, \|B\|, \log(|V(Y)|))$ we gain a good approximation to the time evolution of $A + B$ for time $O(t) = \text{poly}(n)$ [NC00].

Applying the Trotter formula to (43) we find that

$$U[Y](t) \approx ((e^{-iA(X_N) \otimes I_{m t/n}}) e^{-i D t/n})^n,$$

or

$$U[Y](t) \approx ((e^{-iA(X_N) \otimes J_m t/n}) e^{-i D t/n})^n.$$  

(45)

Because $D$ is row sparse, the simulation algorithm of [ATS03] can be applied to simulate $e^{-i D t/n}$ efficiently. (In order to guarantee the applicability of the Trotter formula we assume that $\|A(Y)\|$ grows polylogarithmically with $|V(Y)|$.)

We recursively reapply this construction to $A(X_N)$ (assuming, at each step, that the pull-back of the edges $X_j$ is trivial for all but a small number of edges) until we have expressed all instances of $A(Y)$ with $A(X_1)$. This construction furnishes an efficient (i.e. using $O(\text{poly}(\log(|V(Y)|)))$ elementary quantum gates) quantum circuit which simulates the propagator $U[Y](t)$ accurately for times $t$ which are $O(\text{poly}(\log(|V(Y)|)))$.

The calculations in this section should be seen as representative of a general theory analogous to that initiated for quantum fourier transforms in [MRR03]. That is, given a tower of covering spaces $Y \xrightarrow{\pi_N} X_N \xrightarrow{\pi_{N-1}} X_{N-1} \xrightarrow{\pi_{N-2}} \cdots \xrightarrow{\pi_1} X_1$, what are the conditions the $X_j$ must satisfy in order to give rise to an efficient gate sequence? We believe that such a theory is worth developing because it will potentially provide a class of quantum circuits whose behaviour
may be interesting from an algorithmic point of view. Interestingly, except for Cayley graphs, such quantum circuits will be unrelated to discrete Fourier transforms.

7 The Hidden Cover Problem

The discussion in the previous section indicates that continuous-time quantum walks on certain graphs admit efficient gate decompositions whose size doesn’t depend on the length of time the walk is simulated for. This feature can be exploited to give polynomial time (in \(\log(|V(Y)|)\)) quantum algorithms which can measure the Hamiltonian \(A(Y)\). As a simple corollary of this we obtain an alternative observable for the hidden subgroup problem which is not immediately equivalent to that employed by Shor’s algorithm (and related quantum algorithms). Motivated by these new (efficiently implementable) observables we propose another generalisation of the hidden subgroup problem: the hidden covering space problem \(\text{HiddenCover}\).

The (coherent sampling) hidden subgroup problem for a group \(G\) (see, for example, [NC00] and references therein, for a discussion of the hidden subgroup problem) consists of a black box which outputs at random quantum states \(|\psi_{gH}\rangle\) which are equal superpositions of elements of cosets of a hidden subgroup \(H \leq G\). (The Hilbert space \(\mathcal{H}\) in this case is taken to be the group algebra \(\mathbb{C}[G]\).) The problem is to determine \(H\) using as few of the coset states \(|\psi_{gH}\rangle\) and as little quantum computational time as possible.

The solution of the specific case \(G = \mathbb{Z}/pq\mathbb{Z}\), where \(p\) and \(q\) are prime and \(H = \mathbb{Z}/p\mathbb{Z}\) or \(\mathbb{Z}/q\mathbb{Z}\), is well-known — this is Shor’s factoring algorithm.

In the following we refer to a quantum state \(|\phi\rangle\) in \(\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]\), \(n = pq\), as a constant-coset state on cosets of a subgroup \(\mathbb{Z}/q\mathbb{Z} \leq \mathbb{Z}/pq\mathbb{Z}\) if it can be written

\[
|\phi\rangle = \sum_{j=0}^{p-1} c_j |\alpha_j\rangle,
\]

where \(|\alpha_j\rangle = 1/\sqrt{q} \sum_{l=0}^{p-1} |j + lp\rangle\), \(j = 1, \ldots, p - 1\).

The expansion of a constant coset state \(|\phi\rangle\) in the basis \(|W(j)\rangle\) can be found via (this is essentially the discrete Poisson summation formula [Ter99]):

\[
\langle W(k)|\alpha_j\rangle = \frac{1}{q\sqrt{p}} \sum_{l=0}^{q-1} e^{-\frac{2\pi i (j + lp)k}{pq}}
\]

\[
= e^{-\frac{2\pi i jk}{pq}} \frac{1}{q\sqrt{p}} \sum_{l=0}^{q} e^{-\frac{2\pi i lk}{q}}
\]

\[
= e^{-\frac{2\pi i jk}{pq}} \frac{1}{\sqrt{p}} \delta_{k,\lambda q}, \quad \lambda = 0, \ldots, p - 1.
\]

The expansion coefficients \(\langle W(k)|\alpha_j\rangle\) are nonzero only when \(k\) is a multiple of \(q\).
In the abelian hidden subgroup problem we have a black box which outputs at random the constant-coset states $|\alpha_j\rangle$. The standard solution of the hidden subgroup problem proceeds by applying the quantum fourier transform to the $|\alpha_j\rangle$ and then measuring in the computational basis. This yields an approximation $\tilde{r}/q$ to the number $r/q$, for random integer $r$. After enough samples the identity of the hidden subgroup can be inferred by applying the continued-fractions algorithm to $r/q$.

We now supply an alternative procedure to the standard quantum fourier transform which uses a quantum walk on the Cayley graph $X(\mathbb{Z}/pq\mathbb{Z}, \{\pm1\})$. We don’t claim that this is any different to the standard quantum fourier transform algorithm for the HSP on $\mathbb{Z}/pq\mathbb{Z}$. The point is that the generalisation of this procedure to other graphs will not be equivalent to the quantum fourier transform method. At the moment, however, we can only perform the algorithm for cyclic groups.

The eigenvalues of the hamiltonian (i.e. the Laplacian $\triangle$) for the quantum walk on the cycle $X(\mathbb{Z}/pq\mathbb{Z}, \{\pm1\})$ are given by

$$\lambda_j = \cos\left(\frac{2\pi j}{pq}\right).$$

Consequently, if the hamiltonian $\triangle[X]$ is measured exactly on a constant-coset state $|\phi\rangle$ then, by the discussion surrounding (47), the only eigenvalues that can be measured are those of the form $\lambda_{jq} = \cos\left(\frac{2\pi j}{p}\right)$, for some random $0 \leq j \leq q - 1$.

We can effectively measure $\triangle[X]$ using $U[X](t)$ — which, as discussed in §6, can be implemented efficiently, using the discretisation of von Neumann’s prescription for measuring a hermitian operator given by CDF02. Implementing this measurement yields an approximation $\tilde{\lambda}_j$ to an eigenvalue of $\triangle[X]$ for random $j$. Because $\cos(x)$ is continuous, the function

$$\tilde{j}/q = \cos^{-1}(\tilde{\lambda}_j)$$

is a good approximation to the ratio $j/q$, for random $0 \leq j \leq q - 1$. Applying the continued fractions algorithm yields $q$.

The previous result is suggestive of generalisations in the following way.

Imagine we have a graph $Y$ which is a covering space $\pi : Y \rightarrow X$ for $X$. Imagine, further, we have a black box which outputs at random fibre-constant states $|\psi_\pi\rangle$. Recall that any fibre-constant state can be written in terms of the pull-backs of the eigenstates of $X$:

$$|\psi_\pi\rangle = \sum_{j=0}^{V(X)-1} c_j P_j |E_j(X)\rangle,$$

where $|E_j(X)\rangle$ are the eigenstates of $X$.

This means that if the hamiltonian $\triangle[Y]$ is measured on $|\psi_\pi\rangle$ then it can only report eigenvalues of $X$, rather than eigenvalues from the full possible spectrum.
of \(Y\). If the spectrum of \(\lambda(X)\) can be distinguished from the spectra \(\lambda(Z_l)\) of the other graphs \(Z_l\) that \(Y\) is a covering space for then this measurement can identify the hidden graph \(X\). As we showed previously, in the case where \(Y = X(Z/pq\mathbb{Z}, \{\pm 1\})\), and \(X = X(Z/q\mathbb{Z}, \{\pm 1\})\), then this is equivalent to solving the abelian hidden subgroup problem.

Thus, generalising boldly, we believe that quantum computers ought to be able to efficiently solve the following problem.

**The hidden covering space problem HiddenCover**

**Input:**
1. A class \(\mathcal{C}\) of mathematical objects.
2. A quantisation scheme which associates a unitary matrix \(U(Y)\), with each \(Y \in \mathcal{C}\), acting on an associated Hilbert space \(\mathcal{H}(Y)\).
3. An object \(Y\) from \(\mathcal{C}\) with the promise that \(Y\) is a covering space \(\pi_l: Y \to X_l\) only for a (known) set of objects \(X_l, l = 0, \ldots, n\).
4. A black box which randomly emits fibre-constant states \(|\psi_{\pi_l}\rangle\) for some (unknown) projection \(\pi_l\), for some \(l = 0, \ldots, n\).

**Task:** Determine the projection \(\pi_l\), and hence identify the base space \(X_l\).

We are willing to conjecture that HiddenCover is efficiently solvable on a quantum computer for certain classes of simple graphs using continuous-time quantum walks.

A natural question arises: what happens when we apply the algorithms sketched above to the Cayley graph for the dihedral group \(D_n = \langle s, t \mid s^n = t^2 = e \rangle\)? In this case, we are after a hidden transposition \(\langle e, s^l t \rangle\). Unfortunately, it can be verified that the Schreier graphs for the \(n\) hidden transpositions are isospectral, so that, with one fibre-constant state \(|\psi_{\pi}\rangle\) it is impossible to tell which transposition \(\langle e, s^l t \rangle\) is hidden.

### 8 Conclusions and Future Directions

In this paper we have explored two related ideas, both of which are connected with covering space structures. The first is that a quantum evolution for an object \(Y\) which is a covering space for another object \(X\) can be equivalent to an evolution on the smaller object \(X\). The second is that covering space structures can be exploited to give efficient gate sequences for their quantisations. The first idea can be exploited to give hitting-time speedups for such evolutions as quantum walks. Additionally, the first idea leads to the notion that hidden covering spaces can be identified spectrally. The second leads to the notion that quantum computers can do this efficiently.

We shall conclude with a list of future directions.
1. For the dihedral group consider walking on a graph $Y$ which is not a simple cartesian product of $\log |D_n|$ copies of the Cayley graph of $D_n$ (this is what happens when you measure the propagator $\log |D_n|$ times). Possible graphs to try might be certain graph products of the Cayley graph $X(D_n, S)$ for the dihedral group with generating set $S$.

2. What about other mathematical objects? There are some promising candidates, such as algebraic number fields, smooth manifolds, and knots which have natural structures amenable to quantisation.

3. What sorts of computational problems are expressible as variants of HiddenCover? Are there any interesting computational problems?

Finally, a word on discrete-time quantum walks. Discrete-time quantum walks represent another quantisation scheme for simple graphs where the topology of the graph is encoded in the unitary quantisation. It is natural to ask how the ideas of this paper extend to the discrete-time quantum walks? It is interesting to remark that a coined quantum walk on a graph $X$ is in fact a discrete-time quantum walk on a certain directed graph $Y$ (namely $Y$ is the line digraph of $X$ — see, for example, [GKWS98] for an application of line digraph in the design of algorithms) which is homomorphic to $X$. It can be observed that $Y$ covers $X$ (the spectrum of $Y$ is the spectrum of $X$ plus a zero eigenvalues with the appropriate multiplicity [Ros01]). Now, the construction mentioned above is only one possible quantisation of graphs. Are there other natural quantisations? Perhaps one can pick out a canonical quantisation by demanding that it respect covering space structures?

Given a matrix $M$ (over any field), a directed graph $X$ is said to be the graph of $M$ if the $uv$-th entry of $M$ is nonzero if and only if there is a directed arc $(u, v)$ for every pair of vertices $u, v$. A characterisation of graphs of unitary matrices is still missing (see, for example, the open problems section of [ZKSS03]). In general, what is the relation between graphs of unitary matrices and covering spaces? The study of this problem may be useful in understanding the combinatorial properties of certain combinatorial designs like weighing matrices.

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