Holomorphic curves and Toda systems

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Abstract
Geometry of holomorphic curves from point of view of open Toda systems is discussed. Parametrization of curves related this way to non-exceptional simple Lie algebras is given. This gives rise to explicit formulas for minimal surfaces in real, complex and quaternionic projective spaces or complex quadrics. The paper generalizes the well known connection between minimal surfaces in $\mathbb{E}^3$, their Weierstrass representation in terms of holomorphic functions and the general solution to the Liouville equation.

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1 Introduction

Let \( g \) be a simple complex Lie algebra, \( \mathfrak{h} \) its Cartan subalgebra, \( \Delta \subset \mathfrak{h}^* \) the root system and

\[
\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha
\]

the corresponding root space decomposition (see e.g. [1]). We denote by \( \pi \subset \Delta \) a system of simple roots \( \pi = \{\alpha_k\}_{k=1}^m \) \((m = \dim \mathfrak{h})\) and by \( \{h_k\}_{k=1}^m \) the basis of \( \mathfrak{h} \) dual to \( \{\alpha_k\}_{k=1}^m \) with respect to the Killing form of \( \mathfrak{g} \).

The open Toda system related to \( \mathfrak{g} \) [2] is the following equation

\[
2\theta_{,\xi \eta} = \sum_{k=1}^m h_k e^{-2\alpha_k(\theta)},
\]

where

\[
\theta : \mathbb{C}^2 \ni (\xi, \eta) \mapsto \theta(\xi, \eta) \in \mathfrak{h}
\]

is unknown function with values in the Cartan subalgebra.

In this paper we are interested in a particular reduction of the system (2)

\[
\xi = z, \quad \eta = \bar{z}, \quad \theta \in \sum_{k=1}^m \mathbb{R} h_k
\]

obtained from restriction of \( \mathfrak{g} \) to its compact real form.

General solutions to the open Toda systems were found in [3] using the theory of representation of Lie algebras. Recently appeared papers [4, 5, 6, 7] where connection of the open Toda systems to geometry of holomorphic curves was pointed out. This approach is developed in the present Letter. The geometric nature of solutions to the Toda systems, as coming from parametrization of the corresponding holomorphic curves is shown.

Out of such holomorphic curves one can construct minimal surfaces (or more generally, harmonic maps) into complex projective spaces, complex quadrics, Euclidean spheres and quaternionic projective spaces (see [8] and the references given there). The method presented below can be considered as generalization (suggested in [9]) of the well known connection between Weierstrass representation of minimal surfaces in \( \mathbb{E}^3 \) in terms of holomorphic functions, and the simplest Toda system – the Liouville equation.

The next four Sections are devoted to presentation of the approach to the Toda systems related to four classical sequences of simple Lie algebras.
2 Holomorphic curves and $A_n$-Toda systems

Let $\phi : \mathcal{R} \to \mathbb{CP}^n$ be a nondegenerate (i.e. not contained in a proper projective subspace of $\mathbb{CP}^n$) holomorphic curve \cite{1} represented locally by meromorphic vector-function $f : \mathcal{R} \supset \mathcal{O} \to \mathbb{C}^{n+1}$. By $\phi_k : \mathcal{R} \to \mathbb{G}(k, n+1)$ $(k = 1, \ldots, n)$ we denote its associated curves ($\phi_1 = \phi$) represented locally by $k$-plane with matrix homogeneous coordinates

$$M_k = (f, f', \ldots, f^{(k-1)}) \in \mathbb{C}^{n+1}_k.$$

Each of the curves $\phi_k$ induces on the Riemann surface $\mathcal{R}$ the Kähler form $\omega_k$ (we make use the standard Kähler structure on Grassman manifolds) locally expressed as

$$\omega_k|_{\mathcal{O}} = \frac{i}{2} \partial \bar{\partial} \log \det M_k^+ M_k = \frac{i}{2} \partial \bar{\partial} \log ||\Lambda_k||^2.$$

All the resulting Riemannian metrics on $\mathcal{R}$ are conformally equivalent, moreover they are subjected to the so called Plücker formulas

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log ||\Lambda_k(z)||^2 = \frac{||\Lambda_{k-1}(z)||^2||\Lambda_{k+1}(z)||^2}{||\Lambda_k(z)||^4}, \quad k = 1, \ldots, n$$

where $||\Lambda_0||^2 = 1$ and $||\Lambda_{n+1}||^2 = |\det M_{n+1}|^2$.

When the local holomorphic lift $f$ of the curve $\phi$ is subjected to the normalization condition

$$\det M_{n+1} = \det(f, f', \ldots, f^{(n)}) = 1$$

then one can rewrite \cite{4} the Plücker formulas (7) in one of the standard forms of the open Toda system related to the $\mathfrak{su}(n+1)$ Lie algebra (the compact real form of $A_n \cong \mathfrak{sl}(n+1)$)

$$\varphi^k_{z, \bar{z}} = \exp \left( - \sum_{l=1}^{n} C_{kl} \varphi^l \right), \quad k = 1, \ldots, n$$

where

$$\varphi_k = \log ||\Lambda_k||^2$$

and $C_{kl}$ is the Cartan matrix of the $A_n$ Lie algebra \cite{4}.

**Theorem 1** \cite{4} The Plücker formulas for holomorphic curves in $\mathbb{CP}^n$ form the open Toda system related to the compact real form of $\mathfrak{sl}(n+1)$ Lie algebra.

For our purposes it is more convenient to rewrite the system (7) using functions

$$\theta^k = \log \frac{||\Lambda_k||}{||\Lambda_{k-1}||}, \quad k = 1, \ldots, n+1, \quad \sum_{k=1}^{n+1} \theta^k = 0,$$
then
\[
2\theta_{1,zz}^l = e^{2(\theta^2 - \theta^1)}, \\
2\theta_{l,zz}^l = e^{2(\theta^l+1 - \theta^l)} - e^{2(\theta^l - \theta^l - 1)}, \quad l = 2, \ldots, n \\
2\theta_{n+1,zz}^l = -e^{2(\theta^{n+1} - \theta^l)}.
\] (12)

This matches with the standard representation of the simple roots and Cartan subalgebra of \(A_n\) (we use the scaled Killing form) in terms of the orthonormal basis \(\{e_k\}_{k=1}^{n+1}\) of of \(\mathbb{R}^{n+1}\) (see equation (12))

\[
\pi = \{e_1 - e_2, e_2 - e_3, \ldots, e_n - e_{n+1}\} \cong \{h_1, h_2, \ldots, h_n\}.
\] (13)

Using Gram-Schmidt orthonormalization procedure one can obtain from the natural basis \(f, f', \ldots, f^{(n)}\) along the nondegenerate (normalized) curve the (special) unitary basis \(E_1, E_2, \ldots, E_{n+1}\) along the curve \((\langle E_i|E_j\rangle = \delta_{ij})\). This new basis (called the Frenet basis) satisfies the set of linear equations

\[
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{n+1}
\end{pmatrix}_{zz} = \\
\begin{pmatrix}
\theta_1^1 e^{\theta_2^2 - \theta_1^1} & 0 & \ldots & 0 \\
0 & \theta_2^2 e^{\theta_3^3 - \theta_2^2} & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \theta_n^n e^{\theta_{n+1}^{n+1} - \theta_n^n}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{n+1}
\end{pmatrix}
\] (14)

\[
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{n+1}
\end{pmatrix}_{zz} = \\
\begin{pmatrix}
\theta_1^1 e^{\theta_2^2 - \theta_1^1} & 0 & \ldots & 0 \\
0 & \theta_2^2 e^{\theta_3^3 - \theta_2^2} & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \theta_n^n e^{\theta_{n+1}^{n+1} - \theta_n^n}
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
\vdots \\
E_{n+1}
\end{pmatrix}
\]

and the equations (12) are compatibility conditions of the above linear system.

One can show \([11, 12]\) that all the vectors \(E_k \in \mathbb{C}^{n+1}\) represent harmonic maps (\(\sigma\)-models in physical terminology) from the Riemann surface \(\mathcal{R}\) to the complex projective space \(\mathbb{CP}^n\) equipped with the standard (Fubini – Study) metric.

Open Toda systems related to other simple non-exceptional Lie algebras are various reductions of this general case. It turns out that all the reduced holomorphic curves were studied using the twistor techniques \([13]\).
3 Isotropic curves and $B_m$-Toda systems

Geometry of the $B_m$-Toda systems can be explained with the help of the complex quadric

$$Q_{2m-1} = \{ [z] \in \mathbb{CP}^{2m} : (z|z) = \sum_{k=1}^{2m+1} z_k^2 = 0 \}. \quad (15)$$

Holomorphic curves $\phi : \mathcal{R} \rightarrow Q_{2m-1} \subset \mathbb{CP}^{2m}$ are usually called isotropic curves.

**Definition 1** A nondegenerate holomorphic curve $\phi : \mathcal{R} \rightarrow \mathbb{CP}^{2m}$ is called maximally isotropic if its $m$th associated curve $\phi_m : \mathcal{R} \rightarrow G(m, 2m + 1)$ is made out of isotropic $m$-planes (we write $\phi_m \subset Q_{2m-1}$).

One can observe that $\phi_{m+1} \subset Q_{2m-1}$ cannot hold for nondegenerate curves. When $\phi$ is a maximally isotropic curve then for $k = 1, \ldots, m$ all the associated curves $\phi_k$ are contained in the quadric $Q_{2m-1}$. In terms of local holomorphic lift $f : \mathcal{O} \rightarrow \mathbb{C}^{2m+1}$ this property is equivalent to

$$(f^{(k)}|f^{(k)}) = 0 \quad , \quad k = 0, \ldots, m - 1 . \quad (16)$$

The first $m$ vectors $E_1, \ldots, E_m$ of the Frenet frame are isotropic vectors what implies that together with $E_{m+1}$ and $\bar{E}_1, \ldots, \bar{E}_m$ they form unitary frame along the curve. Moreover, the equations satisfied by $\bar{E}_k$ can be found from those of $E_k$, e.g.

$$\bar{E}_{k,z} = -\theta_k^z E_k - e^{2(\theta_k^z - \theta_k^{k-1})} \bar{E}_{k-1} . \quad (17)$$

If we impose the normalization condition

$$(f^{(m)}|f^{(m)}) = 1 \quad (18)$$

then $E_{m+1} \in \mathbb{RP}^{2m}$ (embedded totally geodesic in $\mathbb{CP}^{2m}$) \cite{12, 14} and the vectors

$$E_1, \ldots, E_m, E_{m+1}, -\bar{E}_m, \ldots, (-1)^k \bar{E}_{m+1-k}, \ldots, (-1)^m \bar{E}_1 \quad (19)$$

form the Frenet frame along the maximally isotropic curve.

From the linear system (14) we obtain that the functions $\theta^k$ defined in (11) are now subjected to the condition

$$\theta^{2m+2-k} = -\theta^k \quad (20)$$

which implies $\theta^{m+1} \equiv 0$ and equations (12) reduce to

$$2\theta_1^{*,z} = e^{2(\theta^2 - \theta^1)} , \quad 2\theta_l^{*,z} = e^{2(\theta^l+1 - \theta^l)} - e^{2(\theta^l - \theta^{l-1})} , \quad l = 2, \ldots, m - 1 , \quad 2\theta_m^{*,z} = e^{-2\theta_m} - e^{2(\theta^m - \theta^{m-1})} . \quad (21)$$
This matches with the standard representation of the simple roots and Cartan subalgebra of $B_m \cong \mathfrak{so}(2m+1)$ in terms of the orthonormal basis of $\mathbb{R}^m$

$$\pi = \{e_1 - e_2, \ldots, e_{m-1} - e_m, e_m\} \cong \{h_1, h_2, \ldots, h_m\} \ .$$

**Theorem 2** [5] The Plücker formulas for maximally isotropic holomorphic curves in $\mathbb{C}P^{2m}$ form the open Toda system related to the compact real form of $\mathfrak{so}(2m+1)$ Lie algebra.

One can show [8] that for $k = 1, \ldots, m$ the vectors $E_k$ represent harmonic immersions of the Riemann surface $\mathcal{R}$ into the quadric $Q_{2m-1} \subset \mathbb{C}P^{2m}$ (with the metric structure inherited from the ambient complex projective space) and $E_{m+1}$ represents conformal minimal immersion of $\mathcal{R}$ in $\mathbb{R}P^{2m}$, which lifted to $S^{2m}$ is called pseudoholomorphic immersion [14].

One should say also that all the immersions of Riemann surfaces considered in this paper can have isolated branching points. Such immersions are called branched immersions [15]. The induced metrics on the surface are in fact pseudometrics [8, 10].

The rest of this Section is devoted to parametrization of maximally isotropic curves in terms of meromorphic functions. This provides us with the geometric meaning of the arbitrary functions entering into general solutions of the open Toda systems [5] and gives the geometric interpretation of Liouville-Bäcklund transformations for such systems [16, 17].

It is convenient to use instead of the standard basis $\{e_k\}_{k=1}^{2m+1}$ of $\mathbb{C}^{2m+1}$ the ”maximally isotropic” unitary basis

$$f_{\pm k} = \frac{e_{2k-1} \pm ie_{2k}}{\sqrt{2}} \quad k = 1, \ldots, m \ , \quad f_0 = e_{2m+1} \ .$$

The new coordinates $\{w_k\}_{k=1}^{m}$ can be expressed in terms of the old ones $\{z_i\}_{i=1}^{2m+1}$ as

$$w_{\pm k} = \frac{z_{2k-1} \mp iz_{2k}}{\sqrt{2}} \quad k = 1, \ldots, m \ , \quad w_0 = z_{2m+1}$$

and the form $(|)$ which defines the quadric $Q_{2m-1}$ reads

$$(z|z) = w_0^2 + 2 \sum_{k=1}^{m} w_kw_k = \mathcal{F}(w) \ .$$

When $w_{[1]} : \mathcal{O} \to \mathbb{C}^{2m+1}$ is a local holomorphic lift of the maximally isotropic curve written in terms of the new coordinates then the isotropy condition

$$\mathcal{F}(w_{[1]}) = 0$$
allows to reduce the number of functions necessary to describe the curve. We can represent any isotropic curve in the form

\[ w_1 = u_1 \left( 1, w_2, -\frac{1}{2} F(w_2) \right) , \]

where (we denote by the same letter \( F \) the quadratic form in \( \mathbb{C}^{2m-1} \))

\[ u_1 = w_{[1]-m} , \quad w_{[2]k} = \frac{w_{[1]k}}{u_1} , \quad k = -m + 1, \ldots, m - 1 . \]  

Moreover one can show that

\[ F(w'_1) = u_1^2 F(w'_2) , \]

and the next isotropy condition (see equation (16)) allows to parametrize \( w'_2 \) in the form as above

\[ w'_2 = u_2 \left( 1, w_3, -\frac{1}{2} F(w_3) \right) , \]

and

\[ F(w''_1) = u_1^2 u_2^2 F(w'_3) . \]

This way for maximally isotropic curve one can define \( m \) meromorphic functions \( u_k \) such that

\[ w'_k = u_k \left( 1, w_{[k+1]}, -\frac{1}{2} F(w_{[k+1]}) \right) \in \mathbb{C}^{2m+3-2k} \quad k = 2, \ldots, m . \]

Moreover the normalization condition (18) implies

\[ u_1^2 u_2^2 \cdots u_m^2 v^2 = 1 \quad (v = w'_{m+1} \in \mathbb{C}) . \]

One can reverse this process, and starting from \( m \) meromorphic local functions \( u_k \) (and some constants of integration, which should play more important role in the general case of complex Lie algebras) and using equations (27) (32) (33) reconstruct locally the normalized maximally isotropic curve, and then the corresponding solution to the Toda system (21). Details and examples can be found in [19].

The parametrization presented above can be considered also as a method of construction of maximally isotropic curves in \( \mathbb{C}P^{2m} \) from such curves in \( \mathbb{C}P^{2m-2} \) (see also [18] where \( \sigma \)-models with finite action were constructed in a similar way). On the level of Toda systems this process corresponds to the Liouville–Bäcklund transformation [16] between open \( B_m \)-system and \( B_{m-1} \) system plus Laplace equation.
4 Horizontal curves and $C_m$-Toda systems

Geometry of $C_m$-Toda systems can be described in terms of the celebrated Penrose fibration $\mathbb{CP}^{2m-1} \xrightarrow{\mathbb{C}^1} \mathbb{HP}^{m-1}$ (here we consider the right quaternionic projective space) obtained from the standard identification of the quaternionic space $\mathbb{H}$ with $\mathbb{C}^2$.

\[ \mathbb{H} \ni q = a + jb \mapsto (a, b) \in \mathbb{C}^2 \quad . \tag{34} \]

In terms of homogeneous coordinates

\[ \pi[(z_1, z_2)]_\mathbb{C} = [z_1 + jz_2]_\mathbb{H} \quad , \quad z_1, z_2 \in \mathbb{C}^m . \tag{35} \]

The fiber over $\pi[(z_1, z_2)]_\mathbb{C}$ is the projective line in $\mathbb{CP}^{2m-1}$ through $[(z_1, z_2)]_\mathbb{C}$ and $[(-\bar{z}_2, \bar{z}_1)]_\mathbb{C}$ (note that $(z_1 + jz_2)j = -\bar{z}_2 + j\bar{z}_1$).

The complement to the fibre orthogonal with respect to the standard Fubini-Study metric in $\mathbb{CP}^{2m-1}$ defines the horizontal distribution $\mathcal{H}$. The vector $v \in T_{[z]}\mathbb{CP}^{2m-1}$ is horizontal if

\[ \langle (z_1, z_2)|j|(v_1, v_2) \rangle = \langle (\bar{z}_2, \bar{z}_1)|(v_1, v_2) \rangle = (z_1|v_2) - (z_2|v_1) = 0 \quad . \tag{36} \]

**Definition 2** A non-degenerate holomorphic curve $\phi : \mathcal{R} \to \mathbb{CP}^{2m-1}$ is called horizontal if it is tangent to the distribution $\mathcal{H}$.

In terms of a local holomorphic lift $f = (f_1, f_2) : \mathcal{O} \to \mathbb{C}^m$ this property is equivalent to

\[ \langle f_j|f'_l \rangle = (f_1|f'_2) - (f_2|f'_1) = 0 \quad . \tag{37} \]

One can generalize the notion of horizontality to the associated curves.

**Definition 3** A nondegenerate holomorphic curve $\phi : \mathcal{R} \to \mathbb{CP}^{2m-1}$ is called superhorizontal if its $(m - 1)$th associated curve $\phi_{m-1} : \mathcal{R} \to \mathbb{G}(m-1, 2m)$ is horizontal.

This implies that for $k = 1, \ldots, m - 1$ all the curves $\phi_k$ are also horizontal. In terms of a local holomorphic lift it is equivalent to (compare with (37))

\[ \langle f^{(k)}_j|f^{(k+1)}_l \rangle = (f^{(k)}_1|f^{(k+1)}_2) - (f^{(k)}_2|f^{(k+1)}_1) = 0 \quad , \quad k = 0, \ldots, m - 2 \quad . \tag{38} \]

The superhorizontal curve defines a special type of complex frame (called the symplectic frame). Let us take the first $m$ vectors $E_1, \ldots, E_m$ of the Frenet frame. The superhorizontality of the curve implies that together with the vectors $E_1j, \ldots, E_mj$ they form unitary frame along the curve. Moreover, the equations satisfied by $E_{kj}$ can be found from those of $E_l$, e.g.

\[ (E_{kj})_z = -\theta^k z E_{kj} - e^{\theta_k - \theta^{k-1}} E_{k-1, j} \quad . \tag{39} \]
One can show that the vector $E_{m+1}$ of the natural unitary Frenet frame is proportional to $E_m j$. The phase factor can be fixed by the following normalization of the local holomorphic lift

$$\langle f^{(m-1)} j | f^{(m)} \rangle = 1$$

which implies $E_{m+1} = E_m j$.

Putting all the facts together we obtain that the vectors

$$E_1, \ldots, E_m, E_m j, \ldots, (-1)^k E_{m-k} j, \ldots, (-1)^{m-1} E_1 j$$

form the Frenet frame along the superhorizontal curve with the corresponding functions

$$\theta^1, \ldots, \theta^m, -\theta^m, \ldots, -\theta^1$$

and equations (41) reduce to

\begin{align*}
2\theta^1_{z \bar{z}} &= e^{2(\theta^2 - \theta^1)} \\
2\theta^l_{z \bar{z}} &= e^{2(\theta^{l+1} - \theta^l)} - e^{2(\theta^l - \theta^{l-1})}, \quad l = 2, \ldots, m-1 \\
2\theta^m_{z \bar{z}} &= e^{-4\theta^m} - e^{2(\theta^m - \theta^{m-1})}.
\end{align*}

This matches with the standard representation of the simple roots and Cartan subalgebra of $C_m \cong \mathfrak{sp}(m)$ in terms of the orthonormal basis of $\mathbb{R}^m$

$$\pi = \{e_1 - e_2, \ldots, e_{m-1} - e_m, 2e_m \} \cong \{h_1, h_2, \ldots, h_m \}.$$ 

**Theorem 3** The Plücker formulas for superhorizontal holomorphic curves in $\mathbb{C}P^{2m-1}$ form the open Toda system related to the compact real form of $\mathfrak{sp}(m)$ Lie algebra.

One can show [8] that the vectors of the symplectic frame represent harmonic branched immersions of $\mathcal{R}$ into $\mathbb{HP}^{m-1}$.

The general solution to the system (43) can be found from parametrization of the superhorizontal curves subjected to the normalization condition (40). When $f_{[1]} : \mathcal{O} \to \mathbb{C}^{2m}$ is a local holomorphic lift of such a curve then the horizontality condition

$$\langle f_{[1]} j | f'_{[1]} \rangle = 0$$

allows to represent it (we arrange coordinates of a point $z \in \mathbb{C}^{2m}$ in pairs) as

$$f_{[1]} = u_1 \left(1, -\int \langle f_{[2]} j | f'_{[2]} \rangle dz, f_{[2]} \right),$$

where

$$u_1 = f_{[1]1}, \quad f_{[2]k} = \frac{f_{[1]k+2}}{u_1} \quad k = 1, \ldots, 2m-2.$$
The next condition

$$\langle f'[1]|f'|1]\rangle = u^2_1 \langle f'[2]|f'|2]\rangle = 0 \quad (48)$$

allows to represent $f'[2]$ similarly.

This way any superhorizontal curve defines $m$ meromorphic functions $u_k$ such that for $k = 2, \ldots, m-1$

$$f'[k] = u_k \left( 1, - \int \langle f[k+1]|f'[k+1]\rangle dz, f[k+1] \right) \in \mathbb{C}^{2m+2-2k}, \quad (49)$$

and

$$f'[m] = u_m(1, v) \in \mathbb{C}^2. \quad (50)$$

The normalization condition (40) implies

$$u^2_1 u^2_2 \cdots u^2_m v' = 1. \quad (51)$$

One can reverse this process and from $m$ meromorphic local functions $u_k$ (and some constants of integration) reconstruct the normalized local holomorphic lift of the superhorizontal curve, and then the corresponding solution to the $C_m$-Toda system (43).

## 5 Isotropic curves and $D_m$-Toda systems

Geometric interpretation of the open $D_m \cong so(2m)$ Toda systems is similar to that of $B_m \cong so(2m+1)$. We start from nondegenerate holomorphic curve $\phi : \mathcal{R} \to \mathbb{C}P^{2m-1}$ subjected to the maximal isotropy condition

$$\phi_{m-1} \subset \mathcal{Q}_{2m-2}. \quad (52)$$

In terms of local holomorphic lift $f : \mathcal{O} \to \mathbb{C}^{2m}$ this condition is equivalent to

$$\langle f^{(k)}|f^{(k)}\rangle = 0, \quad k = 0, \ldots, m-2. \quad (53)$$

Let us take the first $m-1$ vectors $E_1, \ldots, E_{m-1}$ of the Frenet frame along the curve. These vectors are isotropic, what implies that they are orthogonal to their conjugate $\bar{E}_1, \ldots, \bar{E}_{m-1}$. The next vector $\bar{E}_m$ of the Frenet frame (we add the tilde for convenience) is orthogonal to the space $V$ spanned by $E_1, \ldots, E_{m-1}, \bar{E}_1, \ldots, \bar{E}_{m-1}$ and can be decomposed into sum of a pair of complementary isotropic vectors $E_m, \bar{E}_m \in V$ (such a pair is given up to a phase factor). When $\bar{\theta}^m$ is the ”old” function corresponding to $\bar{E}_m$ then we can write

$$e^{\bar{\theta}^m} \bar{E}_m = aE_m + b\bar{E}_m. \quad (54)$$

We fix the phase factor by the condition

$$a = e^{\bar{\theta}^m} \in \mathbb{R}_+ \quad (55)$$
defining this way new local function $\theta^m$. The normalization condition
\[(f^{(m-1)}|f^{(m-1)}) = 2\] (56)
implies $b = a^{-1}$, and this way
\[e^{\tilde{\theta}^m} \tilde{E}_m = e^{\theta^m} E_m + e^{-\theta^m} \tilde{E}_m, \quad e^{2\tilde{\theta}^m} = 2 \cosh 2\theta^m .\] (57)

The new vectors are subjected to the equations
\[E_{m,z} = \theta^m E_m - e^{-\theta^m} \theta^{m-1} \tilde{E}_{m-1},\]
\[E_{m,\bar{z}} = -\theta^m E_m - e^{\theta^m} \theta^{m-1} \bar{E}_{m-1},\]
\[\bar{E}_{m,z} = -\theta^m \bar{E}_m - e^{-\theta^m} \theta^{m-1} E_{m-1},\]
\[\bar{E}_{m,\bar{z}} = \theta^m \bar{E}_m - e^{\theta^m} \theta^{m-1} \bar{E}_{m-1} .\] (58)

The following diagram describes the new unitary and isotropic frame (the arrow denotes differentiation with respect to $z$)

\[
\left( \begin{array}{c} \theta^1 \\ E_1 \end{array} \right) \rightarrow \ldots \rightarrow \left( \begin{array}{c} \theta^{m-1} \\ E_{m-1} \end{array} \right) \rightarrow \left( \begin{array}{c} \theta^m \\ E_m \\ -\theta^m \\ -E_{m-1} \end{array} \right) \rightarrow \ldots \left( \begin{array}{c} \theta^{m-k} \\ (-1)^k \bar{E}_{m-k} \end{array} \right) \rightarrow \left( \begin{array}{c} \theta^1 \\ (-1)^{m-1} E_1 \end{array} \right)
\]

and the new equations read
\[2\theta^1_{,z\bar{z}} = e^{2(\theta^2 - \theta^1)}\]
\[2\theta^l_{,z\bar{z}} = e^{2(\theta^{l+1} - \theta^l)} - e^{2(\theta^l - \theta^{l-1})}, \quad l = 2, \ldots, m - 2\]
\[2\theta^{m-1}_{,z\bar{z}} = e^{2(\theta^m - \theta^{m-1})} + e^{-2(\theta^m + \theta^{m-1})} - e^{2(\theta^{m-1} - \theta^{m-2})},\]
\[2\theta^m_{,z\bar{z}} = -e^{2(\theta^m - \theta^{m-1})} + e^{-2(\theta^m + \theta^{m-1})}.\] (59)

This matches with the standard representation of the simple roots and Cartan subalgebra of $D_m \cong \mathfrak{so}(2m)$ in terms of the orthonormal basis of $\mathbb{R}^m$
\[\pi = \{ e_1 - e_2, \ldots, e_{m-1} - e_m, e_{m-1} + e_m \} \cong \{ h_1, h_2, \ldots, h_m \} .\] (60)

**Theorem 4** The Plücker formulas (modified by conditions (57)) for maximally isotropic holomorphic curves in $\mathbb{C}P^{2m-1}$ form the open Toda system related to the compact real form of $\mathfrak{so}(2m)$ Lie algebra.

Parametrization of the maximally isotropic curves in $\mathbb{C}P^{2m-1}$ is similar to that in $\mathbb{C}P^{2m}$. The only difference (except for the fact that we use now the fully isotropic basis in $\mathbb{C}^{2m}$ dropping the vector $f_0$, see (23)) is at the last step of the construction. To show this it is enough to consider a simple case, say $m = 3$. 11
The isotropic curve $\phi : \mathcal{R} \to \mathcal{Q}_4 \subset \mathbb{CP}^5$ can be represented locally in isotropic coordinates as

$$w_{[1]} = u_1 \left(1, w_{[2]}, -\frac{1}{2} \mathcal{F}(w_{[2]})\right) \in \mathbb{C}^6,$$  \hspace{1cm} (61)

where $w_{[2]} : \mathcal{O} \to \mathbb{C}^4$. The next isotropy condition implies that also

$$w'_{[2]} = u_2 \left(1, w_{[3]}, -\frac{1}{2} \mathcal{F}(w_{[3]})\right),$$  \hspace{1cm} (62)

and the normalization condition (56) implies

$$u_1^2 u_2^2 u_3^2 v = 1 \quad , \quad (w'_{[3]} = u_3(1, v) \in \mathbb{C}^2).$$  \hspace{1cm} (63)

The details are left to the Reader.

6 Conclusion

Geometry of special holomorphic curves in relation to the open Toda systems for non-exceptional simple Lie algebras was described. The legs of the Frenet frame along the curves give rise to (pseudoholomorphic) harmonic maps into complex projective spaces, quadrics, Euclidean spheres and quaternionic projective spaces.

Parametrization of the curves under consideration, which generalizes the Weierstrass representation for minimal surfaces in $\mathbb{E}^3$ in terms of meromorphic functions was also found. This provides with the geometric origin of the general solution to the open Toda systems and their Liouville–Bäcklund transformations.

Also the known reductions of the $A_n$ systems to $B_m$ and $C_m$ systems were explained on the geometric level. Moreover, embedding of the $D_m$ Toda system into $A_{2m-1}$ system was given.

In the forthcoming paper [21] it will be shown how periodic Toda systems (which are genuine soliton systems) can be also described in relation to harmonic maps of Riemann surfaces.

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