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Mass scaling of the near-critical 2D Ising model using random currents

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Abstract
We examine the Ising model at its critical temperature with an external magnetic field $h a^{15/8}$ on $a \mathbb{Z}^2$ for $a, h > 0$. A new proof of exponential decay of the truncated two-point correlation functions is presented. It is proven that the mass (inverse correlation length) is of the order of $h^{15/16}$ in the limit $h \to 0$. This was previously proven with CLE-methods in [1]. Our new proof uses instead the random current representation of the Ising model and its backbone exploration. The method further relies on recent couplings to the random cluster model [2] as well as a near-critical RSW-result for the random cluster model [3].

1 Introduction
The square lattice Ising model [4] suggested by Lenz [5] is the archetypal statistical physics model undergoing an order/disorder phase transition. It has been subject of intense study in the past century [6, 7], starting with Periels’ proof of the existence of a phase transition [8] and Onsager’s calculation of the free energy [9]. The rigorous understanding of the critical two-dimensional Ising model has advanced tremendously in the past decade starting with the breakthroughs [10, 11] and with the subsequent works (see, for example, [12]).

One of the questions that remained unsolved until recently is obtaining the speed of the decay of the truncated correlations in the near-critical two-dimensional Ising model. For $a \in (0, 1]$ and $h > 0$ the near critical regime is defined to be the Ising measure on the lattice $a \mathbb{Z}^2$ with the parameter $\beta = \beta_c(\mathbb{Z}^2)$ and external field $a^{15/8} h$. We denote the corresponding correlation functions with $\langle \cdot \rangle_{a,h}$. The following theorem is proved in [1] using the scaling limit of the FK-Ising model which was proved to exist in [13] and its connections to the conformal loop ensemble [14]. See also the review [15].

Theorem 1.1. There exists $B_0, C_0 \in (0, \infty)$ such that for any $a \in (0, 1]$ and $h > 0$ with $ha^{15/8} \leq 1$,

$$0 \leq \langle \sigma_x \sigma_y \rangle_{a,h} - \langle \sigma_x \rangle_{a,h} \langle \sigma_y \rangle_{a,h} \leq C_0 a^{1 \over 2} |x - y|^{-1 \over 4} e^{-B_0 h^{1 \over 8} |x - y|}.$$ 

Accordingly, for $a = 1$ the result on $\mathbb{Z}^2$ is that for any $h \in [0, 1)$,

$$\langle \sigma_y \sigma_x \rangle_{1,h} - \langle \sigma_y \rangle_{1,h} \langle \sigma_x \rangle_{1,h} \leq C_0 |x - y|^{-1 \over 4} e^{-B_0 h^{1 \over 8} |x - y|}.$$ 

In this paper we prove Theorem 1.2 from which we can deduce Theorem 1.1.

Theorem 1.2. For any $h > 0$ and $a \leq 1$ there are functions $C(h) > 0$ and $m(h) > 0$ independent of $a > 0$ such that for any $x, y \in a \mathbb{Z}^2$ it holds that

$$\langle \sigma_x \sigma_y \rangle_{a,h} - \langle \sigma_x \rangle_{a,h} \langle \sigma_y \rangle_{a,h} \leq C(h) a^{1 \over 2} |x - y|^{-1 \over 4} e^{-m(h) |x - y|}.$$ 

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The proof uses first a partial exploration of the backbone of random currents, then a recent coupling between the random current measure with sources and the random cluster model \cite{2}. The proof utilises a new result that extends a result on crossing probabilities for the critical random cluster model to the near critical regime \cite{3}.

Before diving into the details, we briefly show how Theorem 1.1 follows (as explained in \cite{1}) from Theorem 1.2.

**Proof of Theorem 1.1.** Let \( H \) be such that \( a = H \frac{a}{a} \) and \( h = 1 \). Let \( \langle \sigma_0; \sigma_x \rangle = \langle \sigma_0 \sigma_x \rangle_{a,h} - \langle \sigma_0 \rangle_{a,h} \langle \sigma_x \rangle_{a,h} \). Then from Theorem 1.2

\[
\langle \sigma_0; \sigma_x \rangle_{H \frac{a}{a},1} \leq C(1) H \frac{a}{a} |x|^{-\frac{1}{4}} e^{-m(1)|x|}
\]

for \( x \in H \frac{a}{a} \mathbb{Z}^2 \). Using the relation \( \langle \sigma_0; \sigma_x \rangle_{H \frac{a}{a},1} = \langle \sigma_0; \sigma_x \rangle_{1,H} \) whenever \( x' = \frac{x}{H \frac{a}{a}} \) we obtain

\[
\langle \sigma_0; \sigma_{x'} \rangle_{1,H} \leq C(1) |x'|^{-\frac{1}{4}} e^{-mH \frac{a}{a} |x'|}
\]

for \( x' \in \mathbb{Z}^2 \). Rescaling back to \( a \mathbb{Z}^2 \) yields the result. \( \square \)

In \cite{1} a converse inequality is also proved using reflection positivity. A more probabilistic proof of the lower bound was given in \cite{16}. We note that this shows that the correlation length is finite, the mass gap exists, and that the exponent of the correlation length equals \( \frac{3}{17} \). Further, as it is explained in \cite{1} the exponential decay proven in Theorem 1.1 directly translates into the scaling limit. Indeed, as in \cite{1} if \( \Phi^{a,h} \) is the near critical magnetization field given by

\[
\Phi^{a,h} = a \frac{15}{17} \sum_{x \in a \mathbb{Z}^2} \sigma_x \delta_x
\]

with \( \{\sigma_x\}_{x \in a \mathbb{Z}^2} \in \{0,1\}^{a \mathbb{Z}^2} \), it was proven in Theorem 1.4 of \cite{13} that \( \Phi^{a,h} \) converges in law to a continuum (generalized) random field \( \Phi^h \). Let \( C^\infty_0(\mathbb{R}^2) \) denote the set of smooth functions with compact support and let \( \Phi^h(f) \) be \( \Phi^h \) paired against \( f \in C^\infty_0(\mathbb{R}^2) \). Then as in \cite{1} it holds that

**Corollary 1.3.** Let \( f, g \in C^\infty_0(\mathbb{R}^2) \), then there are \( B_0, C_0 \in (0, \infty) \) such that

\[
|\text{Cov}(\Phi^h(f), \Phi^h(g))| \leq C_0 \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} |f(x)||g(y)||x-y|^{-\frac{1}{4}} e^{-B_0h \frac{a}{a} |x-y|} dx dy.
\]

Starting with \cite{17}, there has in the physics community been much interest in the masses of the Ising model \cite{18} including possible connections to the exceptional Lie Algebra \( E_8 \) which has been investigated also experimentally \cite{20,21,22} and numerically \cite{23}. On the mathematical side, exponential decay was first rigorously proven in \cite{24} and in \cite{25} a linear upper bound for the mass was proven. Proving the correct scaling exponent is a further step towards rigorous results in this direction. For further rigorous developments see also \cite{26}.

## 2 Preliminaries

We start by briefly introducing the Ising model and its random cluster and random current representations that we will use to prove the result. Let \( G = (V,E) \) be a finite graph. Then for each spin configuration \( \sigma \in \{\pm 1\}^V \) and \( h \geq 0 \) define the energy

\[
H(\sigma) = - \sum_{xy \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x,
\]

where \( h \) describes the effect of an external magnetic field. For each \( A \subset V \) we let \( \sigma_A = \prod_{x \in A} \sigma_x \) and define the correlation function as

\[
\langle \sigma_A \rangle = \frac{\sum_{\sigma \in \{\pm 1\}^V} \sigma_A \exp(-\beta H(\sigma))}{Z}
\]

2
where \( Z = \sum_{\sigma \in \{-1,1\}^V} \exp(-\beta H(\sigma)) \) is the partition function. In what follows, we will be concerned with the Ising model on the graph \( a\mathbb{Z}^2 \) which is obtained by taking the thermodynamics limit of finite graphs. For discussions about the thermodynamic limit we refer the reader to [27].

In both representations we implement the magnetic field using Griffiths' ghost vertex \( g \). This means that we consider the graph \( G_{\text{ghost}} = (V \cup \{g\}, E \cup E_g) \) where \( E_g = \cup_{v \in V} \{e_{vg}\} \) are additional edges from every original vertex \( v \) to the ghost vertex \( g \) (see for example [27]). We will refer to the edges \( E \) as internal edges and to the edges \( E_g \) as ghost edges.

### The random current representation

Let us now introduce the random current representation which is a very effective tool in the study of the Ising model [28, 29, 30, 2, 31, 32, 33, 34]. Further information can be found in [7] and [35]. The central building blocks in the random current representation of the Ising model on a graph \( G_{\text{ghost}} = (V \cup \{g\}, E \cup E_g) \) are the currents \( n \in \mathbb{N}_0^{E \cup E_g} \). For each current \( n \) we can define its sources \( \partial n \) as the \( v \in V \) where \( \sum_{xv \in E \cup E_g} n_{xv} \) is odd. Let further the weight of each current \( n \) be given by

\[
w(n) = \prod_{xy \in E} \frac{\beta n_{xy}^+}{n_{xy}^+ + n_{xy}^-} \prod_{xg \in E_g} \frac{(\beta h)^{n_{xg}}}{n_{xg}^+!}.
\]

A simple identity which connects the random currents to the Ising model is given as (2.4a) in [31]

\[
\langle \sigma_0 \sigma_x \rangle = \frac{\sum_{\partial n = \{0,x\}} w(n)}{\sum_{\partial n = \emptyset} w(n)}.
\]

Further, given a current \( n \) define the traced current \( \hat{\omega} \in \{0,1\}^{E \cup E_g} \) by \( \hat{\omega}(e) = 0 \) if \( n(e) = 0 \) and \( \hat{\omega}(e) = 1 \) if \( n(e) > 0 \). Then \( P^A_G \), the random current measure with sources \( A \subset V \), is the probability measure that satisfies \( P^A_G(n) \propto w(n)1_{\partial n=A} \). If \( A \) and \( B \) are either vertices in or subsets of \( V \cup \{g\} \) we denote the event that they are connected in a configuration \( \omega \in \{0,1\}^{E \cup E_g} \) by \( A \leftrightarrow B \), meaning that one vertex of \( A \) is connected to one vertex of \( B \).

The traced random current measure \( \hat{P}^A_G \) gives each \( \omega \in \{0,1\}^{E \cup E_g} \) the probability

\[
\hat{P}^A_G(\omega) = \sum_{\partial n = A} P^A_G(n)1_{\hat{\omega} = \omega}.
\]

To ease the notation in what follows define \( \hat{P}^{\{0,x\}}_G \) and \( \hat{P}^\emptyset_G \) to be the probability measure which assigns each \( \omega \in \{0,1\}^{E \cup E_g} \) the probability

\[
\hat{P}^{\{0,x\}}_G(\omega) = \frac{1}{Z(\{0,x\})} \sum_{\partial n = \{0,x\}, \partial m = \emptyset} w(n)w(m)1_{\hat{\omega} + \hat{m} = \omega}.
\]

On the square lattice \( a\mathbb{Z}^2 \) in a magnetic field \( a^{15/8} h \) we denote the non traced and traced single current measures by \( P_{a,h}^{A} \) and \( \hat{P}_{a,h}^{A} \) respectively. The main part of what follows proves exponential decay of truncated correlations, but first we obtain the correct front factor \( a^\frac{1}{4} \). We do a similar trick as in [1] where we set the magnetic field \( h \) to 0 in the boxes of radius 1 around 0 and \( x \) and call that magnetic field \( \tilde{h} \).

**Proposition 2.1.** We have

\[
\langle \sigma_0; \sigma_x \rangle_{a,h} \leq \langle \sigma_0; \sigma_x \rangle_{a,\tilde{h}} = \langle \sigma_0 \sigma_x \rangle_{a,\tilde{h}} \cdot \hat{P}^{\{0,x\}}_{a,\tilde{h}}(0 \not\leftrightarrow g) \leq C a^\frac{1}{4} \hat{P}^{\{0,x\}}_{a,\tilde{h}}(0 \not\leftrightarrow g).
\]

**Proof.** The first inequality follows from the GHS inequality [36]. The second step used the switching lemma [36] and [1]. Since the event \( \{0 \not\leftrightarrow g\} \) is decreasing the probability increases when \( \hat{P}^\emptyset_{a,h} \) is removed. Then the last inequality is a standard application of equation (1-arm) below (see also [37]). □
The random cluster model

Each configuration $\omega \in \{0,1\}^{E\cup E_g}$ corresponds to a (spanning) subgraph of $G_{\text{ghost}}$. For each $e \in E \cup E_g$ if $w_e = 1$ we say that $e$ is open and if $w_e = 0$ we say that $e$ is closed. There is a natural partial order $\preceq$ on the configurations where $\omega \preceq \omega'$ if $\omega'$ can be obtained from $\omega$ by opening edges. An event $\mathcal{A}$ is increasing if for any $\omega \in \mathcal{A}$ it holds that $\omega \preceq \omega'$ implies $\omega' \in \mathcal{A}$. Let further $k(\omega)$ be the number of clusters of vertices of the configuration $\omega$.

The random cluster model with free boundary conditions $\phi^0_G$ is a percolation measure on the finite graph $G_{\text{ghost}} = (V \cup \{g\}, E \cup E_g)$ such that for every $\omega \in \{0,1\}^{E\cup E_g}$

$$\phi^0_G(\omega) \propto 2^{\beta(\omega)} \prod_{e \in E \cup E_g} \frac{p_e}{1 - p_e},$$

where $p_e = 1_{\{e \text{ is internal and open}\}} \left(1 - \exp(-2\beta)\right) + 1_{\{e \text{ is ghost edge and open}\}} \left(1 - \exp(-2\beta h)\right) + \frac{1}{2} 1_{\{e \text{ closed}\}}$. In what follows, we will consider the free random cluster model on some finite subsets $\Lambda$ of $a\mathbb{Z}^2$ and we will denote that measure by $\phi^0_\Lambda$ at the same time fixing $\beta = \beta_c = \frac{\log(1+\sqrt{2})}{2}$. Let $\Lambda_k(x)$ denote the box of with side length $k$ around some point $x \in a\mathbb{Z}^2$ and let $\Lambda_k = \Lambda_k(0)$. Notice that $\Lambda_k$ only depends on the distance in $\mathbb{R}^2$ which is not affected when $a$ changes. Further, let $A_{n,m}(x) = \Lambda_m(x)/\Lambda_n(x)$ be the $(n, m)$ annulus around $x$ and $A_{n,m} = A_{n,m}(0)$. The random cluster model has many nice properties that we will use in what follows. Since the boundary conditions are free the random cluster model has stochastic domination in terms of the domain. This means that if $\Lambda_1 \subset \Lambda_2$ then for any increasing event $\mathcal{A}$,

$$\phi^0_{\Lambda_1}(\mathcal{A}) \leq \phi^0_{\Lambda_2}(\mathcal{A}). \quad \text{(MON)}$$

Further, the (FKG)-inequality [7] Theorem 1.6] states that for increasing events $\mathcal{A}, \mathcal{B}$ then

$$\phi^0_\Lambda(\mathcal{A} \cap \mathcal{B}) \geq \phi^0_\Lambda(\mathcal{A})\phi^0_\Lambda(\mathcal{B}). \quad \text{(FKG)}$$

We note that the 1-arm exponent for the random cluster model [37, Lemma 5.4] is given by

$$C_1 a^\frac{1}{4} \leq \phi^0_{\Lambda_1}(0 \leftrightarrow \partial \Lambda_1) \leq C_2 a^\frac{1}{4}. \quad \text{(1-arm)}$$

The following result was proven in [1] and it will also prove useful for us.

**Lemma 2.2.** ([1], Proposition 1) Suppose that configuration of internal edges $\omega$ has clusters $C_1, \ldots, C_n$. Then

$$\phi^0_{\Lambda_1}(C_i \leftrightarrow g|\omega) = \tanh(ha^\frac{n}{4} |C_i|)$$

and the events $\{C_i \leftrightarrow g\}$ given $\omega$ are mutually independent.

Finally, we state a connection between the random currents with sources and the random cluster model.

**Theorem 2.3.** ([2], Theorem 3.2] Let $\{X(e)\}_{e \in E}$ be independent Bernoulli percolation with parameter $\left(1 - \exp(-\beta_e)\right)$ with $\beta_e = \beta$ for $e \in E$ and $\beta_e = \beta h$ for $e \in E_g$. Then define for each $e \in E \cup E_g$ the configuration

$$\omega(e) = \max\{\tilde{n}(e), X(e)\}.$$

where $\tilde{n}$ has the law of $\tilde{\beta}^{\{x\leftrightarrow y\}}$ the traced random current with sources $\partial n = \{x, y\}$. Then $\omega$ has the law of $\tilde{\beta}^{\{x \leftrightarrow y\}}$ which is the random cluster measure conditioned on $\{x \leftrightarrow y\}$. Hence, if $\mathcal{A}$ is a decreasing event then

$$\tilde{\beta}^{\{x\leftrightarrow y\}}(\mathcal{A}) \geq \phi^0_G(\mathcal{A} | x \leftrightarrow y).$$

A key part in our result is the backbone exploration which we turn to next.
for all \( k \) remaining edges such that (at least one)

\[ i \]

\[ \text{as} \]

\[ u_i \]

for our purpose in this paper we include it.

there is always at least one such \( i \)

\[ n \]

is an algorithmic way of step-by-step constructing such a path until it hits some set of vertices \( A \supset \{g, y\} \)

To do that, we define the sets of (explored) edges \( \emptyset = S_0 \subset S_1 \ldots \) inductively. For each \( i \geq 0 \) the set \( S_i \) is defined in such a way that \( n \) restricted to \( S_i \) has sources \( \{x\} \cap \{u_i\} \) for some vertex \( u_i \). We will say that the backbones path up to step \( i \) is \( x = u_0, u_1, \ldots u_i \).

If \( u_i \notin A \) we continue as follows. If \( i = 0 \) we consider the five edges incident to \( u_0 = x \).

Order them as \( e_0, e_1, \ldots, e_4 \) with \( e_0 = e_{xg} \) and the other edges in arbitrary order. Since \( x \) is a source of \( n \) there is at least one \( i \) such that \( n(e_i) \) is odd. Let \( k \) be the least such \( i \) and let \( S_i = \{e_0, \ldots, e_k\} \). Then \( u_1 \) is such that \( e_k = e_{u_0u_1} \).

In words, the backbone explored the edges \( \{e_0, \ldots, e_k\} \) and walked to the vertex \( u_1 \).

For \( i \geq 1 \) we call the edge \( e_{u_1} \), the incoming edge to the vertex \( u_1 \).

We can define an order on the remaining edges such that \( (e_0, e_1, e_2, e_3) = (e_{u_0g}, e_{u_0g}, e_{u_0g}, e_{u_0g}) \) where \( e_{u_0g}, e_{u_0g}, e_{u_0g} \) denotes the edges that are right, left and straight with respect to the incoming edge.

The backbone path is the path of explored vertices \( x = u_0, u_1, \ldots \) and the explored backbone in step \( i \) is \( S_i \).

This sequence \( \{S_i\} \) stabilizes after a finite number of steps and we call the terminating set the backbone starting from \( x \) explored up to \( A \) and denote it by \( \gamma_{x,A}(n) \).

There is a path from \( x \) to \( A \) along the vertices \( x = u_0, u_1, \ldots, u_{\text{end}} \) with \( u_{\text{end}} \in A \) such that every edge \( e \) in the path obeys that \( e \in \gamma_{x,A}(n) \) and has \( n(e) \) odd. We call this path \( \gamma_{x,A}(n) \).

The vertex \( u_{\text{end}} \) we call \( u_{\text{end}}(n) \). If \( u_{\text{end}} \) is the ghost \( g \) we say that the backbone hits the ghost.

In what follows, we will work with events of the \( Q = \{F = \gamma_{0,1}(n)\} \) where \( \Gamma \supset \{g, x\} \) is a set of vertices, and \( F \) is a set of edges. Notice that by construction we can tell whether the explored backbone is \( F \) only by looking at the edges in \( F \) which means that 1_{\Gamma}(n) = 1_{\Gamma}(n_F) \) where \( n_F \) is the current restriction to the set \( F \).

The partial backbone exploration is useful because of the following Markov property.

**Theorem 2.4.** Let \( \Gamma \supset \{g, x\} \) be a set of vertices, \( F \) be a set of edges, \( Q = \{F = \gamma_{0,1}(n)\} \) and on the event \( Q \) let \( \bar{x} = \gamma_{0,1}^\text{end}(n) \) be the unique vertex in \( \Gamma \) connected to \( x \) in \( \gamma_{0,1}(n) \). Let \( \mathcal{A} \) be an event such that

\[ 1_{\mathcal{A}}(n_\lambda)1_{\mathcal{Q}}(n_\lambda) = 1_{\mathcal{A}}(n_\lambda/F)1_{\mathcal{Q}}(n_F). \]

Then whenever \( \mathbb{P}_\Lambda^{(0,x)}(Q) > 0 \) it holds that

\[ \mathbb{P}_\Lambda^{(0,x)}(\mathcal{A} \mid Q) = \mathbb{P}_\Lambda^{(\bar{x},x)}(\mathcal{A}). \]

\[ \text{[1]} \]

In the constructions in the literature the set corresponding to \( A \) usually does not necessarily contain the ghost, but for our purpose in this paper we include it.

\[ \text{[2]} \]

Note that if \( h = 0 \) then \( \gamma_{x,(g,y)}(n) \) explores some edges in and around the path \( \gamma_{x,(g,y)}(n) \) from \( x \) to \( y \) where \( n(e) \) is odd for all traversed edges.
Proof. That $n \in Q$ means that the explored backbone of $n$ up to $\Gamma$ is $F$. Thus, $F$ is the terminating set of the sequence $\{S_i\}_{i \in \mathbb{N}}$. Thus, on the event $Q$ the current $n$ restricted to $F$ must have sources $\partial n_F = \{0\} \triangle \{w_{\text{end}}\} = \{\tilde{x}\}$. Since $n = n_{\Lambda \setminus F} + n_F$ it holds that

$$\{0, x\} = \partial n = \partial n_{\Lambda \setminus F} \triangle \partial n_F = \partial n_{\Lambda \setminus F} \triangle \{0, \tilde{x}\}.$$  

So for $n \in Q$ then $\partial n_{\Lambda \setminus F} = \{\tilde{x}, x\}$. The map $n \mapsto (n_F, n_{\Lambda \setminus F})$ is a bijection from $\{n \in Q \mid \partial n = \{0, x\}\}$ to $\{(n_F, n_{\Lambda \setminus F}) \mid n_F + n_{\Lambda \setminus F} \in Q, \partial n_F = \{0, \tilde{x}\}, \partial n_{\Lambda \setminus F} = \{\tilde{x}, x\}\}$ with inverse $(n_F, n_{\Lambda \setminus F}) \mapsto n_F + n_{\Lambda \setminus F}$. Thus, for any function $f : N_0^{E_{\tilde{F}_x}} \to \mathbb{R}$ it holds that

$$\sum_{\partial n = \{0, x\}} f(n) 1_Q(n) = \sum_{\partial n_F = \{0, \tilde{x}\}} f(n) 1_Q(n).$$

Since $w(n_F + n_{\Lambda \setminus F}) = w(n_F) \cdot w(n_{\Lambda \setminus F})$ and the fact that $Q$ only depends on edges in $F$, the double sum below factorizes and

$$P^{\{0, x\}}_{\Lambda, x}(A \mid Q) = \frac{\sum_{\partial n = \{0, x\}} w(n) 1_A(n) 1_Q(n)}{\sum_{\partial n = \{0, x\}} w(n) 1_Q(n)} = \frac{\sum_{\partial n_F = \{0, \tilde{x}\}} w(n_F) 1_Q(n_F) \sum_{\partial n_{\Lambda \setminus F} = \{\tilde{x}, x\}} w(n_{\Lambda \setminus F}) 1_A(n_{\Lambda \setminus F})}{\sum_{\partial n_F = \{0, \tilde{x}\}} w(n_F) 1_Q(n_F) \sum_{\partial n_{\Lambda \setminus F} = \{\tilde{x}, x\}} w(n_{\Lambda \setminus F})} = P^{\{\tilde{x}, x\}}_{\Lambda, x}(A).$$

3 Main result

We now prove Theorem 1.2 given a result that we then prove later.

Before starting the proof, we go through some notation that we will use throughout the main section. Let $n$ be such that $x \in A_{9n, 9(n+1)}$. In what follows, we will consider the case $|x| \geq 36$ which means that
n \geq 4$. In the proof of Theorem $1.2$ we tie it together with the case $|x| < 36$. We also let $\Lambda$ be any box which contains $\Lambda_{0(n+1)}$. Everything we prove will be independent of this $\Lambda$. Later we let $\Lambda \ni \mathbb{Z}^2$.

We will explore the backbone partially in steps up to the annuli $A_{0, 9(i+1)}$. Suppose that in this exploration the backbone does not hit the ghost $g$, which we can assume in our application. Then define $x_i = \gamma_{0, \Lambda_{0,i}}^n(n) \in A_{0, 9(i+1)}$. Thus, $x_i$ is random variable corresponding to the first vertex the backbone hits in the $i$-th annulus of the form $A_{0, 9(i+1)}$ see also Figure $3$. Further, to ease notation we let $\gamma_i = \gamma_{0, \Lambda_{0,i}}(n)$. Thus, $\gamma_i$ is the path $u_0, \ldots, u_{\text{end}} = x_i = \gamma_{0, \Lambda_{0,i}}^n(n)$ explored until the backbone hits the $i$-th annulus. Let $\Omega_\gamma$, be the event that the backbone explored is $\gamma_i$.

A technical detail is that to account for removing the magnetic field in the box of size $1$ around $x$ we explore the backbone partially also from $x$ until it leaves the box of size $2$. Denote the explored backbone $\gamma_{x, A_2(x)}(n)$ by $\gamma_x$, name the first point hit outside $A_2(x)$ by $\tilde{x} = \gamma_{x, A_2(x)}(n)$ and let the event that the explored backbone is $\gamma_x$ be denoted $\Omega_{\gamma_x}$.

The set $\gamma_i$ contains the path $\gamma_i$ from $0$ to $x_i$. Thus, $\gamma_i$ will intersect $\partial \Lambda_1(x_i)$ one or more times. Since $x_i$ is the first time the annulus $A_{0, 9(i+1)}$ is intersected the set $\gamma_i \cap \partial \Lambda_1(x_i)$ is contained within one half of $\partial \Lambda_1(x_i)$. Let $d$, $d'$ denote the points in $\gamma_i \cap \partial \Lambda_1(x_i)$ that are most clockwise and anticlockwise with respect to some way of walking around $\partial \Lambda_1(x_i)$, see also Figure $2$.

More formally, we consider an order $\preceq$ of the points in $\partial \Lambda_1(x_i)$ and then define $d$, $d'$ to be the minimal and maximal element of $\gamma_i \cap \partial \Lambda_1(x_i)$ with respect to this ordering. Let us define the order in the case where $x_i$ is in the right side of the annulus (which is the case $x_1, x_2, x_3$ on Figure $5$ generalising to the other cases is straightforward. To do that, we split $\partial \Lambda_1(x_i)$ and define the order with respect to the segments and arrows as shown on Figure $3$. Now, define $\Omega_i = \{v \in \partial \Lambda_1(x_i) \mid d' \preceq v \preceq d\}$ and let $\Omega_i = \partial \Lambda_1(x_i) \setminus \Omega_i$. Let $\Sigma_i$ be the graph obtained by removing $\gamma_i \cup \gamma_x$ from $\Lambda$. Then we can define the domain $D_i$ to be the connected component of $x_i$ in the graph induced by the vertices of $\Sigma_i$ without $\tilde{\Omega}_i$. See also Figure $2$. In the following claim we show how our order of exploration with respect to the incoming edge implies that $D_i$ is a bounded domain.

**Claim 3.1.** The set $D_i \subseteq \Lambda_{0(i+1)}$ and it only depends on the current $\mathbf{n}\Lambda_{0i}$.

**Proof.** Since the vertex $d$ is explored by the backbone it is either on the backbone path $\gamma_i$ in which case we let $v = d$. Otherwise, there is an edge $e_{dc}$ from $d$ to a vertex on the backbone path that we call $v$. Similarly, we can define a vertex $v'$ taking $d'$ as the starting point. Let $P_i$ be the subpath of $\gamma_i$ which goes either from $v$ to $v'$ or from $v'$ to $v$ and extended by the edges $\{e_{dc}\}$ and/or $\{e'_{dc}\}$ if $d$, $d'$ are not on the backbone path. By construction the path $P_i$ is edge self-avoiding, but we do not know that $P_i$ is vertex self-avoiding and hence non self-intersecting. However, due to the way we explore the edges of the backbone with respect to the incoming edge if there is a vertex which is hit by the backbone path twice (i.e. $u_i = u_j$ for some $i \neq j$) then the backbone path must turn $90^\circ$ twice at that vertex. This means that we can deform the path slightly to be non-intersecting (see Figure $4$).

Since $P_i$ is a path between $d$ and $d'$ and further $\tilde{\Omega}_i$ is also a path between $d$ and $d'$ along $\Lambda_1(x_i)$ that by definition of $d$ and $d'$, $P_i$ and $\tilde{\Omega}_i$ do not intersect. Thus, if we glue them together then $P_i \cup \tilde{\Omega}_i$ is a closed non-intersecting path, which therefore encloses a domain $Q_i$. Now, assume for contradiction that $\delta : x_i \to Q_i$ is a path. Since we have removed all the vertices in $\tilde{\Omega}_i$ including $d$ and $d'$ it is impossible for
Suppose that for all \( i \), \( \{n_1, n_2\} \in \mathcal{R}_i^* \) then \( n_2 \in \mathcal{R}_i^* \). Thus, \( \mathcal{R}_i^* \) only depends on the traced and not on the full current. Define the corresponding connection event either for the traced current or for the random cluster measure by

\[
\mathcal{R}_i^* = \{ \partial \Lambda_2(x_i) \setminus (D_i \cup \gamma_i) \neq \emptyset \partial \Lambda_4(x_i) \} \subset \{0,1\} A_{2,4}(x_i) \setminus (D_i \cup \gamma_i).
\]

Notice that \( \mathcal{R}_i^* = \{ \hat{n} \mid n \in \mathcal{R}_i^* \} \). Hence, it holds that

\[
\mathbb{P}_{\Lambda \setminus (\gamma_i \cup \gamma_{\bar{i}}), \hat{\lambda}}^{\{x, \hat{x}\}} (\mathcal{R}_i^*) = \mathbb{P}_{\Lambda \setminus (\gamma_i \cup \gamma_{\bar{i}}), \hat{\lambda}} (\mathcal{R}_i^*) \geq c
\]

uniformly in any (x-dependent) \( \Lambda \) sufficiently large. Then

\[
\langle \sigma_0; \sigma_{\bar{x}} \rangle_{a, h} \leq C a^{\frac{1}{2}} \exp (-M(h) |x|)
\]

for \( |x| \geq 36 \) and where \( M(h) \) does not depend on \( a \).

**Proof.** Let \( H \) be the event that the backbone explored from 0 hits \( x \) (i.e. does not hit the ghost). Notice that \( \{\hat{x} = g\} \cap Q_{\gamma_i} \cap H = \emptyset \) so when we condition \( H \) on all possible events \( Q_{\gamma_i} \), we can omit those where \( \hat{x} = g \). In other words, if \( \hat{x} = g \) then the backbone explored from 0 would necessarily hit the ghost since after the partial backbone exploration then \( g \) and 0 would be the only two vertices with odd degree.

Further, \( 1_H (n_{\Lambda}) \mathbb{P}_{A, \hat{\lambda}}^{\{0, x\}} (Q_{\gamma_i}) = 1_H (n_{\Lambda \setminus \gamma_i}) \mathbb{P}_{A, \hat{\lambda}}^{\{0, \hat{x}\}} (Q_{\gamma_i}) \) so by the backbone exploration Theorem 2.4

\[
\mathbb{P}_{A, \hat{\lambda}}^{\{0, x\}} (H) = \sum_{\gamma_i} \mathbb{P}_{A, \hat{\lambda}}^{\{0, x\}} (H \mid Q_{\gamma_i}) \mathbb{P}_{A, \hat{\lambda}}^{\{0, x\}} (Q_{\gamma_i}) = \sum_{\gamma_i} \mathbb{P}_{A \setminus \gamma_i, \hat{\lambda}} (H) \mathbb{P}_{A, \hat{\lambda}}^{\{0, x\}} (Q_{\gamma_i})
\]
where we from now on assume that $\bar{x} \neq g$ which means that $\bar{x} \in \partial \Lambda_2(x)$.

For each $1 \leq i \leq n$ let $\hat{G}_i$ be the event that the backbone hits the annulus $A_{\gamma_i, \theta(i+1)}$ before hitting the ghost. Given a current configuration in $\hat{G}_i$ we know that the vertices $x_j$ exist for $1 \leq j \leq i$. Further, if we define $\mathcal{G}_0 = \{ n | \partial n = \{0, x\} \}$ then $\hat{G}_{i+1} \subset \mathcal{G}_i$ for each $0 \leq i \leq n-1$ as well as $\{ 0 \neq g \} \subset \mathcal{G}_n$.

Since $P_{W}^{\Lambda_{\gamma_i}}(H) \leq P_{W}^{\Lambda_{\gamma_i}, \hat{h}}(\mathcal{G}_i)$ and bounding $P_{W}^{\Lambda_{\gamma_i}, \hat{h}}(H)$ uniformly in $\gamma_i$ bounds $P_{W}^{\Lambda, \hat{h}}(H)$ through $\mathcal{G}_0$. It means that to bound $P_{W}^{\Lambda, \hat{h}}(0 \neq g)$ it suffices to show exponential decay of $P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_n)$ uniformly in $\gamma_i$. Notice that given $\mathcal{G}_{\gamma_{i-1}}$ then $\hat{G}_i$ depends only on edges in $\Lambda(\gamma_{i-1} \cup \gamma_x)$ so

$$1_{\mathcal{G}_i}(n_{\Lambda_{\gamma_{i}}}^{\gamma_i})1_{\mathcal{Q}_{\gamma_{i-1}}}^{\gamma_i} = 1_{\mathcal{G}_i}(n_{\Lambda_{\gamma_{i-1}} \cup \gamma_x})1_{\mathcal{Q}_{\gamma_{i-1}}}^{\gamma_i} (n_{\Lambda_{\gamma_{i-1}} \cup \gamma_x})$$

and by the backbone exploration Theorem 2.4

$$P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_n) = \left[ \prod_{i=1}^{n} P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_i) | G_{i-1} \right] P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_0) = \prod_{i=1}^{n} \sum_{\mathcal{Q}_{\gamma_{i-1}} \in \mathcal{G}_{i-1}} P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_i) P_{W}^{\Lambda, \hat{h}}(\mathcal{Q}_{\gamma_{i-1}} | G_{i-1})$$

$$\leq \prod_{i=1}^{n} \sum_{\mathcal{Q}_{\gamma_{i-1}} \in \mathcal{G}_{i-1}} P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_i) P_{W}^{\Lambda, \hat{h}}(\mathcal{Q}_{\gamma_{i-1}} | G_{i-1}) \leq (1 - c)^{n-2}$$

where in the last step we used $P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_i) \leq P_{W}^{\Lambda, \hat{h}}(\mathcal{G}_i) P_{W}^{\Lambda, \hat{h}}(\mathcal{Q}_{\gamma_{i-1}} | G_{i-1}) \leq 1 - c$. Now, by the remarks in the beginning of the proof it follows from Proposition 2.1 that

$$\langle \sigma_0 ; \sigma_x \rangle_{\Lambda, \hat{h}} \leq C a \hat{h} P_{W}^{\Lambda, \hat{h}}(0 \neq g) \leq C a \hat{h}(1 - c)^{n-2} = C a \hat{h} \exp(-M(h) |x|).$$

The inequality passes to the infinite volume limit since the constants are independent of $\Lambda$.
Proof of Theorem 4.2. By \( \phi \), the monotone coupling from Theorem 2.3 along with the fact that \( \mathcal{R}_i^* \) is a decreasing event and Proposition 3.3 we get that

\[
p_{\{x, \tilde{x}\}}^{\Sigma_i}(\mathcal{R}_i^*, 0, a, \tilde{h}) = p_{\{x, \tilde{x}\}}^{\Sigma_i}(\mathcal{R}_i^*) \geq \phi^{0, a}_{\Lambda(\gamma_i \cup \gamma_0), \tilde{h}}(\mathcal{R}_i^* \mid x_i \leftrightarrow \tilde{x}) \geq c.
\]

Thus, we can apply Proposition 3.2. In Proposition 3.3 we only have the result for sufficiently small \( h \), but this suffices by the GHS-inequality \( 36 \). To account for the constraint \( |x| \geq 36 \) in Proposition 3.2 and get the correct front factor notice that from the GHS inequality and Proposition 5.5 in \( 37 \) some \( B > 0 \) it holds for all \( x \in a \mathbb{Z}^2 \) that

\[
\langle \sigma_0; \Sigma_x \rangle_{a, \tilde{h}} \leq \langle \sigma_0; \Sigma_x \rangle_{a, 0} \leq B \left( \frac{a}{|x|} \right)^{\frac{1}{4}}.
\]

Using that for \( |x| \geq K(h) \) it holds that

\[
\langle \sigma_0; \Sigma_x \rangle_{a, \tilde{h}} \leq C a^\frac{1}{4} \exp \left( -\frac{M(h)}{2} \frac{|x|}{2} \right) \exp \left( -\frac{M(h)}{2} \frac{|x|}{2} \right) \leq C \left( \frac{a}{|x|} \right)^{\frac{1}{4}} \exp \left( -\frac{M(h)}{2} \frac{|x|}{2} \right).
\]

By putting \( C(h) = \max\{B, C\} \exp \left( \frac{M(h)}{2} K(h) \right) \) and \( m(h) = \frac{M(h)}{2} \) our main result Theorem 1.2 follows.

\[\square\]

4 Proof of Proposition 3.3

Recall that \( \Sigma_i = \Lambda(\gamma_i \cup \gamma_x) \). To ease the notation here and in what follows we define \( \phi^{0, a}_{\Sigma_i, \tilde{h}} = \phi^{0, a}_{\Lambda(\gamma_i \cup \gamma_0), \tilde{h}} \). Further, for a set \( \Gamma \) let \( x \leftrightarrow y \) denote the event that \( x \) and \( y \) are connected in \( \Gamma \cup \{g\} \) and similarly by \( x \overset{T}{\leftrightarrow} y \) that \( x \) and \( y \) are connected in \( \Gamma \) not using the ghost. Define the domain \( T_i \) to be all points in \( D_i \) as well as all points in \( \Lambda_i(x_i) \) that can be reached from \( x_i \) without using edges in \( \gamma_i \) or \( \partial\Lambda_2(x_i) \). Further, define \( \{x_i \overset{T}{\leftrightarrow} g\} \) to be the event that \( x_i \) is connected within the domain \( T_i \) to some vertex \( v \) where the edge from \( v \) to \( g \) is open. Define \( \overset{T}{D} \) and \( \overset{T}{T} \) similarly to \( D_i \) and \( T_i \) with \( \overset{T}{\tilde{x}} \) instead of \( x_i \). Define also \( A_i = \partial\Lambda_2(x_i) \cap T_i^* \) and \( \overset{T}{A} = \partial\Lambda_2(\overset{T}{\tilde{x}}) \cap \overset{T}{T}^c \).

Proposition 4.1. Suppose for some \( 1 < i < n \) that \( \phi_{\Sigma_i}^{0, a} \left( x_i \overset{T}{\leftrightarrow} g \mid x_i \leftrightarrow A_i \right) \geq c \) and \( \phi_{\Sigma_i} \left( \overset{T}{\tilde{x}} \overset{T}{\leftrightarrow} g \mid \overset{T}{\tilde{x}} \leftrightarrow \overset{T}{A} \right) \geq c \). Then for all \( h \leq h_0 \) it holds that \( \phi_{\Sigma_i}^{0, a} \left( \mathcal{R}_i^* \mid x_i \leftrightarrow \overset{T}{\tilde{x}} \right) \geq c. \)

We first state the recent result that the random cluster model still has the RSW property at scales up to the correlation length. Here we need the wired boundary condition which is introduced in for example 7.

Lemma 4.2. (3, Lemma 8.5) For any sufficiently large \( C > 0 \), there is an \( \varepsilon > 0 \) such that if \( n \geq 1 \) and \( H \geq 0 \) are such that \( H n^{2} \phi_{\Lambda_n, t}^{0, a} = 0 \leftrightarrow \partial\Lambda_n \leq \varepsilon \) then

\[
\phi_{T_n, a}^{1, n} = \phi_{T_n, a}^{1, n} \left( \Lambda_n \left( A_n \psi \right) \partial\Lambda_{2n} \right) \leq C \phi_{T_n, a}^{1, n} \left( \Lambda_n \left( A_n \psi \right) \partial\Lambda_{2n} \right).
\]

Translating the lemma into our setting yields the following lemma.

Lemma 4.3. There exists a \( h_0 > 0 \) such that for \( a \leq 1 \) and \( h \leq h_0 \) it holds that

\[
\phi^{0}_{\Sigma_i, \tilde{h}}(\mathcal{R}_i^*) \geq c.
\]

where \( c > 0 \) is (as always) independent of \( 0 < a \leq 1, \Sigma_i \) and \( 0 \leq h < h_0 \). Further, for any event \( E \) depending only on edges in \( D_i \) it holds that

\[
\phi^{0}_{\Sigma_i}(\mathcal{R}_i^* \mid E) \geq c.
\]
Proof. If \( n = \frac{2}{a} \) and \( H = a^{\frac{n}{2}} h \) then using equation (1-arm)

\[
H^n \psi_{A_n,h=0}^{0,a=1}(0 \leftrightarrow \partial \Lambda_n) \leq C(an)^{\frac{n}{2}} h = C2^{\frac{n}{2}} h \leq \varepsilon
\]

which can be satisfied by choosing \( h \) sufficiently small (independent of \( a \)). Therefore

\[
\phi_{A_{2\times 4 \times h}}(A_2 \not\leftrightarrow \partial \Lambda_4) = \phi_{A_{2h} \times \frac{1}{2}}(A_2 \not\leftrightarrow \partial A_4) \geq C \phi_{A_{2h} \times \frac{1}{2}}(A_2 \not\leftrightarrow \partial A_4) = C \phi_{A_{4 \times h}}(A_2 \not\leftrightarrow \partial \Lambda_4).
\]

Since \( \tilde{R}_i^* \) is decreasing it follows by (MON) and the RSW for usual rectangles [37] that

\[
\phi_{ \Sigma_{\tilde{h}}}^{0,a}(\tilde{R}_i^*) \geq \phi_{ \Sigma_{\tilde{h}}}^{0,a}(\tilde{R}_i^*) \geq \phi_{ \Sigma_{\tilde{h}}}^{0,a}(A_2(x_i) \not\leftrightarrow \partial \Lambda_4(x_i)) \geq \phi_{A_{h}}(A_2(x_i) \not\leftrightarrow \partial \Lambda_4(x_i)) \geq \phi_{A_{h}}(A_2 \not\leftrightarrow \partial \Lambda_4) \geq c.
\]

Using comparison between boundary conditions, and that because of the argument in Claim 3.1 removing all explored edges of the backbone acts as a free boundary condition it holds that

\[
\phi_{ \Sigma_{\tilde{h}}}^{0}(\tilde{R}_i^* \mid E) \geq \phi_{A_{1 \times h}}^{0,\text{on}}(A_2 \not\leftrightarrow \partial \Lambda_4) \geq \phi_{A_{2 \times h}}^{0,a}(A_2 \not\leftrightarrow \partial \Lambda_4) \geq c.
\]

\[Q.E.D. \]

Next, we need the general result that we can do mixing also with a magnetic field at scales up to the correlation length and that we, up to constants, can decorrelate events in \( T_i \) from events in \( (T_i \cup A_4(x_i)) \). Define \( J_i = (T_i \cup A_4(x_i)) \) and \( J = (T \cup A_4(\tilde{x})) \). Define also \( \hat{R}_i \) similarly to \( \tilde{R}_i \) as an event in the vicinity of \( \tilde{x} \) instead of \( x_i \).

Lemma 4.4. (Mixing) Let \( E_1, E_2 \) be increasing events that only depend on edges in the boxes \( T_i \cup A_4(x_i) \) and \( T \cup A_4(\tilde{x}) \) respectively. Then for \( 1 \leq i \leq n - 2 \) it holds that

\[
\phi_{\Sigma_{\tilde{x}}}^{0}(E_1 \cap E_2) = \phi_{J_{i \tilde{x}}}^{0,a}(E_1 \cap J_{i \tilde{x}}(E_2)
\]

Similarly, if \( l \leq 1 \), \( z_1, z_2 \) are such that \( \Lambda_4(z_1) \cup \Lambda_4(z_2) = \emptyset \) and \( E_1, E_2 \) are increasing events that only depend on edges in the boxes \( \Lambda_4(z_1), \Lambda_4(z_2) \) respectively. Then

\[
\phi_{\Sigma_{\tilde{x}}}^{0}(E_1 \cap E_2) = \phi_{\Lambda_4(z_1) \cup \Lambda_4(z_2)}^{0,\Lambda_4(z_1) \cup \Lambda_4(z_2)}(E_1 \cap E_2)
\]

Proof. We prove the first statement first. Define \( E = E_1 \cap E_2 \). It follows from Lemma 4.3 that

\[
c \phi_{\Sigma_{\tilde{x}}}^{0}(E) \leq \phi_{\Sigma_{\tilde{x}}}^{0}(E \mid \hat{R}_i^*)
\]

and similarly we can condition on \( \hat{R}_i^* \). Using that closed dual paths inside the annulus give rise to monotonicity properties as free boundary conditions (which is for example proven in Lemma 11 in [1]) we obtain that

\[
\phi_{\Sigma_{\tilde{x}}}^{0}(E) \leq \phi_{\Sigma_{\tilde{x}}}^{0}(E_1, E_2 \mid \hat{R}_i^*, \hat{R}_i^*) \leq \phi_{J_{i \tilde{x}}}^{0}(E_1, E_2) = \phi_{J_{i \tilde{x}}}^{0}(E_1 \cap J_{i \tilde{x}}(E_2)
\]

Since the reverse inequality is (FKG) and (MON) the first result follows. The second assertion follows mutatis mutandis using that the estimates from the proof of Lemma 4.3 by which Lemma 4.2 also work on smaller scales and using the event \( \{ \partial \Lambda_4(z_i) \not\leftrightarrow \partial \Lambda_4(z_i) \} \) for \( i = 1, 2 \) instead of \( \tilde{R}_i \) and \( R_i^* \). \( \Box \)

With the lemmas proven we continue to the proof of the Proposition 4.1.

Proof of Proposition 4.7. First, notice that

\[
\phi_{\Sigma_{\tilde{x}}}^{0}(\tilde{R}_i^* \mid x \rightarrow \tilde{x}) \geq \phi_{\Sigma_{\tilde{x}}}^{0}(\tilde{R}_i^* \mid x \rightarrow \tilde{x} \rightarrow g, \tilde{x} \rightarrow \tilde{g}) \phi_{\Sigma_{\tilde{x}}}^{0}(x \rightarrow \tilde{x} \rightarrow g, \tilde{x} \rightarrow \tilde{g} \mid x \rightarrow \tilde{x})
\]

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From the mixing argument in Lemma 4.3 it follows that \( \phi_{\Sigma_i}(x_i, T, g, x, \bar{z}, \bar{g}) \geq c \). Thus, we just need to prove that \( \phi_{\Sigma_i}(x_i, \Sigma^+, z, \bar{z}, g) \geq c \). Notice that
\[
\{x_i, \Sigma^+, \bar{z}\} \subset \left(\{x_i, T_i, g\} \cup \{x_i \leftrightarrow A_i\}\right) \cap \left(\{\bar{z}, \bar{g}\} \cup \{\bar{x} \leftrightarrow \bar{A}\}\right).
\]

Now, by first a union bound, then (Mixing) and the assumption and finally (MON) and (FKG)
\[
\phi_{\Sigma_i}(x_i, \Sigma^+, \bar{z}) \leq \phi_{\Sigma_i}(x_i, T_i, g, \bar{z}, g) + \phi_{\Sigma_i}(x_i, T_i, g, \bar{x}, \bar{g}) + \phi_{\Sigma_i}(x_i \leftrightarrow A_i, \bar{z}, g) + \phi_{\Sigma_i}(x_i \leftrightarrow A_i, \bar{g}, \bar{z}) \\
\leq 4C\phi_{\Sigma_i}^{a,a}(x_i, T_i, g) \phi_{\Sigma_i}^{a,a}(\bar{x}, g) \leq c\phi_{\Sigma_i}(x_i, T_i, g, \bar{x}, \bar{g}).
\]
\[
\square
\]

We now turn to the main technical part for proving Proposition 3.3. From now on, we will assume, for notational reasons, without loss of generality that \( x_i \) is on the right side of the inner boundary of the annulus \( A_{9i,9(i+1)} \), i.e. that \( x_i = (\frac{9i}{2}, y) \) for some \( y \) such that \( -\frac{9}{2} \leq y \leq \frac{9}{2} \). Then define \( L = [1,2] \times [0,\frac{9}{4}] + x_i \) and \( \bar{L} \) similarly to be a rectangle in the vicinity of \( \bar{x} \).

Lemma 4.5. For each \( 1 < i < n \) it holds that
\[
\phi_{\Sigma_i}^0(x_i \leftrightarrow L \mid x_i \leftrightarrow A_i) \geq c \quad \text{as well as} \quad \phi_{\Sigma_i}^0(\bar{x} \leftrightarrow L \mid \bar{x} \leftrightarrow \bar{A}) \geq c.
\]
Let \( C(x_i) \) be number of points in \( \Lambda_2(x_i) \) connected to \( x_i \) without using edges to the ghost and \( N_{M,i} = \{C(x_i) \geq M\} \) for each \( M \in \mathbb{N} \).

Lemma 4.6. Let \( k > 0 \). Then,
\[
\phi_{\Sigma_i}^0\left(\frac{N_{k\alpha}}{k\alpha^{\frac{3}{2}}} \mid x_i \leftrightarrow L, x_i \leftrightarrow A_i\right) \geq c.
\]

Then let us start proving the lemmas. To do that we need to show that crossings of topological rectangles exist with constant probability. This is done in [89, Theorem 1.1] if the discrete extremal length \( l_{\Omega}[(ab),(cd)] \) is bounded. From (88, (3.7)) (see also 89) we have the following characterisation of the discrete extremal length
\[
l_{\Omega}((ab),(cd)) = \sup_{g : E(T) \to \{0,1\}} \left(\frac{\inf_{\gamma : (ab) \leftrightarrow (cd)} \sum_{e \in \gamma} g_e}{\sum_{\bar{e}} g_{\bar{e}}}\right)^2
\]
where the supremum is over all non-negative, not identically zero functions on the edges. Using this representation we obtain the following lemma (which is equivalent to Rayleigh’s monotonicity law).

Lemma 4.7. Let \( T \) be a topological rectangles with marked points (abcd). Let \( e, f \) be points on (ad) and \( \gamma \) a path from \( e \) to \( f \) inside \( T \). Let \( \bar{T} \) be the points in \( T \) reachable from \( b \) in \( T \setminus \gamma \). Note \( \bar{T} \) is a new topological rectangle with marked points (abcd). Then,
\[
l_T((ad),(bc)) \geq l_T((ad),(bc)) \quad \text{as well as} \quad l_T((ad),(bc)) \geq l_T((ab),(cd))
\]
\[
\text{Proof.} \quad \text{We prove the first inequality, the second follows similarly. Since the graph } \bar{T} \text{ is finite the supremum and infimum are attained and we get some maximizing function } \bar{g} \text{ for } \bar{T}. \text{ Now, define the function } g \text{ by extending } \bar{g} \text{ with } g(e) = 0 \text{ whenever } e \notin T \setminus \bar{T}. \text{ Then,}
\]
\[
l_T((ad) \leftrightarrow (bc)) = \left(\frac{\inf_{\gamma : (ad) \leftrightarrow (bc)} \bar{T} \sum_{e \in \gamma} g_e}{\sum_{\bar{e}} g_{\bar{e}}}\right)^2 \leq \left(\frac{\inf_{\gamma : (ad) \leftrightarrow (bc)} \sum_{e \in \gamma} g_e}{\sum_{\bar{e}} g_{\bar{e}}}\right)^2 \leq l_T((ad) \leftrightarrow (bc)).
\]
The second equality follows since any path \( \gamma : (ad) \leftrightarrow (bc) \) in \( T \) has a subpath \( \bar{\gamma} : (ad) \leftrightarrow (bc) \) in \( \bar{T} \). The inequality follows since the function \( g \) is just one element in the supremum defining the discrete extremal length.
\[
\square
\]
Using Lemma \ref{lem:open_rectangles}, we can now prove Lemma \ref{lem:open_path}.

**Proof of Lemma \ref{lem:open_path}**. Define the explored vertices $V$ of the backbone to be all vertices with at least one incident explored edge. Then define $U$ to be the set of vertices in $V(\Lambda) \setminus (V \cup D_i)$ with at least one edge to $V$. Since $\gamma_i \cap \Lambda_{2, R}(x_i) = \emptyset$ there exists at least one $\leftrightarrow$-path (i.e. a path that can also jump diagonally) $P_i^*$ in $U$ from $d$ to a vertex in $\partial \Lambda_2(x_i)$. From such a $\leftrightarrow$-path $P_i^*$ we can construct a usual path $P_i$ just going around the plaquette every time $P_i^*$ jumps diagonally. Let the first vertex that $P_i$ hits in $\Lambda_2(x_i)^c$ be $d_1$ and denote henceforth the path $P_i$ by $(dd_1)$.

Define $d_i''$ similarly following the outside of the backbone from $d'$.

Now, let $A_1(x_i) \cap \Lambda_i$ denote the right half of the box $A_1(x_i)$. Define $a_i = x_i + (1, -1)$, $b_i = x_i + (2, -1)$, $a_i' = x_i' + (1, 1)$ and $b_i' = x_i + (2, 1)$ see Figure \ref{fig:open_path}. Define $T_{i, 1} = \Lambda_2(x_i) \setminus D_i$. Then let $S_i \in \{0, 1\}^{\mathcal{E}(T_{i, 1})}$ be the event defined by

$$S_i = \biggl\{ (a_i b_i) \xrightarrow{\phi_i} (dd_1) \biggr\} \bigcap \biggl\{ (a_i' b_i') \xrightarrow{\phi_i} (d'd_1') \biggr\} \bigcap \biggl\{ (a_i a_i') \xrightarrow{\phi_i} (b_i b_i') \biggr\}.$$

I.e. $S_i$ ensures that any path from $x_i$ to $\Lambda_{2, i+1}^c$ will intersect a cluster of open edges that in particular hits $L$. We claim that

**Claim 4.8.** $\phi_{\Sigma_i}^0 (S_i) \geq c$.

**Proof.** We prove that each of the three events defining $S_i$ has a positive probability. That $\phi_{\Sigma_i}^0 ((a_i a_i') \xrightarrow{\phi_i} (b_i b_i')) \geq c$ follows from RSW for usual rectangles \cite{37}. Thus, by symmetry it suffices to prove $\phi_{\Sigma_i}^0 ((a_i b_i) \xrightarrow{\phi_i} (dd_1)) \geq c$.

Notice that the path $(dd_1)$ does not leave the left half of the box $\Lambda_2(x_i)$ since the backbone is only in the left half. If we consider a new topological rectangle $T_{i, 2}$ to be $\Lambda_2(x_i) \setminus$ union the top-right quarter of $A_{1, 2}(x_i)$ with the four marked points $a_i, b_i, d, d_i$ and where we use the part of $\partial A_1(x_i)$ from $x_i + (0, 1)$ until $d$ as the boundary twice as shown on Figure \ref{fig:open_path}. Then the path $(dd_1)$ has the form of $\gamma$ in Lemma \ref{lem:open_rectangles}, so we conclude that

$$l_{T_{i, 1}} ((a_i b_i), (dd_1)) \leq l_{T_{i, 2}} ((a_i b_i), (dd_1)).$$

Define $c_i = x_i + (0, -1)$ and $d_i = x_i + (0, -2)$. Then $c_i, d_i$ are on the segment $(dd_1)$ and thus

$$l_{T_{i, 2}} ((a_i b_i), (dd_1)) \leq l_{T_{i, 2}} ((a_i b_i), (c_i d_i)) \leq l_{T_{i, 1}} ((a_i b_i), (c_i d_i)) \leq c$$

where we in the last step considered a new topological rectangle $T_{i, 3}$ where we used the part of $A_{1, 2}(x_i)$ enclosed by $(a_i b_i)$ and $(c_i d_i)$ as shown on Figure \ref{fig:open_path} which has bounded discrete extremal length. Therefore $l_{T_{i, 1}} ((a_i b_i), (dd_1)) \leq c$ which means by \cite{38}, Theorem 1.1 if that

$$\phi_{\Sigma_i}^0 ((a_i b_i) \xrightarrow{\phi_i} (dd_1)) \geq c.$$

That $\phi_{\Sigma_i}^0 (S_i) \geq c$ then follows from (FKG).

Now, to finish the proof of the lemma note that since $\{x_i \leftrightarrow A_i\} \cap S_i \subset \{x_i \leftrightarrow L\}$ then by (FKG)

$$\phi_{\Sigma_i}^0 (x_i \leftrightarrow L | x_i \leftrightarrow A_i) \geq \phi_{\Sigma_i}^0 (S_i | x_i \leftrightarrow A_i) \geq \phi_{\Sigma_i}^0 (S_i) \geq c.$$

We end by proving Lemma \ref{lem:open_path}.

\end{proof}
Proof of Lemma 4.6. We use some ideas from Lemma 3.1 in [40]. Consider a square \( B = \Lambda_{3/2, 1/2} + x_i \) corresponding with the previous lemma such that \( L \) passes through \( B \). Define the event \( S \) to be the \( S_i \) where there is also a crossing of each of the four (overlapping) rectangles that make up the annulus \( A_{1/2, 1} \) around \( B \) as shown on Figure 7. By RSW for usual rectangles and (FKG) we know that \( \phi_0^{\Sigma_i} (S) \geq c \). By the definition of \( S_i \) from Lemma 4.5 \( \{ x_i \leftrightarrow L \} \cap S \) and so by (FKG) and Lemma 4.5 we get that
\[
\phi_0^{\Sigma_i} (x_i \leftrightarrow B \mid x_i \leftrightarrow L, x_i \leftrightarrow A_i, S) \geq \phi_0^{\Sigma_i} (x_i \leftrightarrow B \mid x_i \leftrightarrow L, x_i \leftrightarrow A_i, S) \geq c.
\]
Now, since \( \phi_0^{\Sigma_i} (x_i \leftrightarrow B \mid x_i \leftrightarrow A_i, S) \geq c \), \( \phi_0^{\Sigma_i} (S) \geq c \) and then we do a dyadic summation for each \( B \) partitioning \( B \) into annuli \( A_{2^{-k}, 2^{-k-1}} (z) \) for \( k \) such that
\[
E(N_B \mid x_i \leftrightarrow B, x_i \leftrightarrow A_i, S) = \sum_{z \in B} \phi_0^{\Sigma_i} (x_i \leftrightarrow z \mid x_i \leftrightarrow B, x_i \leftrightarrow A_i, S) \geq \sum_{z \in B} \phi_0^{\Sigma_i} (z \leftrightarrow \partial A_{2^{-k}} (z) \mid x_i \leftrightarrow B, x_i \leftrightarrow A_i, S) \geq \sum_{z \in B} \phi_0^{\Sigma_i} (z \leftrightarrow \partial A_{2^{-k}} (z)) \geq \frac{1}{100a^2} Ca^4 = ca^{-1/2}.
\]

Let us then consider the second moment. First we use that \( \phi_0^{\Sigma_i} (x_i \leftrightarrow B \mid x_i \leftrightarrow A_i) \geq c \), \( \phi_0^{\Sigma_i} (S) \geq c \) and then we do a dyadic summation for each \( z \) partitioning \( B \) into annuli \( A_{2^{-k-1}, 2^{-k}} (z) \) for \( k \) such that
Proof of Proposition 3.3.\[−m ≤ k ≤ 0\] where \(m \in \mathbb{N}\) is chosen such that \(A_{2k−1,2k}(x) = ∅\) for \(k < m\).

\[
\mathbb{E}(N_B^2 | x_i \leftrightarrow B, x_i \overset{T}{\leftrightarrow} A_i, S) = \sum_{z,y \in B} \phi_{\Sigma_i}^0(x_i \leftrightarrow z, x_i \leftrightarrow y | x_i \leftrightarrow B, x_i \overset{T}{\leftrightarrow} A_i, S)
\]

\[
≤ \frac{c}{\phi_{\Sigma_i}^0(x_i \overset{T}{\leftrightarrow} A_i)} \sum_{z,y \in B} \phi_{\Sigma_i}^0(x_i \leftrightarrow z, x_i \leftrightarrow y | x_i \overset{T}{\leftrightarrow} A_i)
\]

\[
≤ \frac{c}{\phi_{\Sigma_i}^0(x_i \overset{T}{\leftrightarrow} A_i)} \sum_{z \in B} \sum_{y \in A_{2k−1,2k}(z)} \phi_{\Sigma_i}^0(z \leftrightarrow \partial A_{2k−1,2k}(z), y \leftrightarrow \partial A_{2k−1,2k}(y), x_i \leftrightarrow \partial A_{2k−1,2k}(x_i))
\]

\[
≤ \frac{c \phi_{\Sigma_i}^0(x_i \overset{T}{\leftrightarrow} A_i)}{\phi_{\Sigma_i}^0(x_i \overset{T}{\leftrightarrow} A_i)} \sum_{z \in B} \sum_{y \in A_{2k−1,2k}(z)} \phi_{\Sigma_i}^0(z \leftrightarrow \partial A_{2k−1,2k}(z), y \leftrightarrow \partial A_{2k−1,2k}(y))
\]

\[
≤ \sum_{z \in B} c_1 \left(\frac{2^k}{a}\right)^2 \left(\frac{a}{2^k−2}\right)^\frac{1}{2} c_2 a^{−2} a^{−\frac{1}{2}} \sum_{k=m}^0 2^{2k−\frac{k}{2}} = c_3 a^{−\frac{11}{2}}
\]

where we also used (1-arm) several times and that (Mixing) holds at all scales smaller than our fixed macroscopic scale. The conclusion follows from the Paley-Zygmund inequality

\[
\phi_{\Sigma_i}^0\left(N_{\frac{15}{2},15} | x_i \leftrightarrow B, x_i \overset{T}{\leftrightarrow} A_i, S\right) ≥ c a^{\frac{15}{2} − \frac{11}{2}} = c.
\]

Finally, let us prove Proposition 3.3.

**Proof of Proposition 3.3.** By Proposition 4.1 it suffices to prove \(\phi_{\Sigma_i}^0\left(x_i \overset{T}{\leftrightarrow} g | x_i \leftrightarrow A_i\right) ≥ c\) and the similar inequality for \(\tilde{T}\) and \(\tilde{x}\) which will follow in the same way. First, notice that by Lemma 4.5 and Lemma 4.6

\[
\phi_{\Sigma_i}^0\left(N_{\frac{15}{2},15} | x_i \leftrightarrow A_i\right) ≥ \phi_{\Sigma_i}^0\left(N_{\frac{15}{2},15} | x_i \leftrightarrow L, x_i \leftrightarrow A_i\right) φ_{\Sigma_i}^0(x_i \leftrightarrow L | x_i \leftrightarrow A_i) ≥ c \cdot c
\]

Now, we can make the following observation using Lemma 2.2 to find

\[
\phi_{\Sigma_i}^0(x_i \overset{T}{\leftrightarrow} g | x_i \leftrightarrow A_i) ≥ \phi_{\Sigma_i}^0\left(N_{\frac{15}{2},15} | x_i \overset{T}{\leftrightarrow} g, x_i \overset{T}{\leftrightarrow} A_i\right)
\]

\[
= \phi_{\Sigma_i}^0\left(x_i \overset{T}{\leftrightarrow} g | N_{\frac{15}{2},15} | x_i \leftrightarrow A_i\right) φ_{\Sigma_i}^0\left(N_{\frac{15}{2},15} | x_i \leftrightarrow A_i\right)
\]

\[
≥ \tanh\left(k a^{−\frac{11}{2}} ha^{\frac{k}{2}}\right) c ≥ c(k, h).
\]
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