Mode Analysis and Duality Symmetry in Different Dimensions

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Abstract
The problem of duality symmetry in free field models is examined in details by performing a mode expansion of these fields which provides a mapping with the purely quantum mechanical example of a harmonic oscillator. By analysing the duality symmetry in the harmonic oscillator, we show that the massless scalar theory in two dimensions display, along with the expected discrete $Z_2$ symmetry, the continuous $SO(2)$ symmetry as well. The same holds for the free Maxwell theory in four dimensions, which is usually regarded to manifest only the $SO(2)$ symmetry. This leads to the new result that, following a proper interpretation, the duality groups in two and four dimensions become identical. Incidentally, duality in quantum mechanics is generally not covered in the literature that considers only $D = 4k$ or $D = 4k + 2$ spacetime dimensions, for integral $k$.

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I. Introduction

The crucial role played by duality symmetry either in field or string theories is becoming increasingly evident [1]. As briefly reviewed below, the conventional interpretation of this symmetry leads to distinct groups in $4k$ or $4k + 2$ dimensions [2, 3]. This shows that there is a basic difference in the treatment of duality symmetry in these dimensions. The motivation of the present work is to show that it is possible to obliterate this difference and reveal duality symmetry in a general framework that encompasses both these dimensions. As a consequence, the duality symmetry groups also become identical, irrespective of the dimensionality. The explicit calculations will be presented in the context of the Maxwell theory in four dimensions and the free massless scalar theory in two dimensions.

Historically, the source free Maxwell’s equations were the first to display the property of duality symmetry which involves a formal $SO(2)$ rotation, apart from a trivial scale factor, in the space of electric and magnetic fields,

$$(E, B) \rightarrow (E', B') = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} (E, B) \quad (1.1a)$$

or, equivalently a $U(1)$ transformation for the combination $(E + iB)$

$$(E + iB) \rightarrow (E' + iB') = e^{-i\theta}(E + iB) \quad (1.1b)$$

Using the language of differential two-forms $F$ and its dual $\tilde{F}$, defined as,

$$F = E_idx^i \wedge dx^j + \frac{1}{2}F_{ij}dx^i \wedge dx^j$$

$$\tilde{F} = -B_idx^i \wedge dx^j + \frac{1}{2}\tilde{F}_{ij}dx^i \wedge dx^j \quad (1.2)$$

with $B_i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ and $E_i = F_{0i} = \frac{1}{2}\epsilon^{ijk}\tilde{F}_{jk}$ being the components of the magnetic and electric fields respectively, (1.1a) is recast as,

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \rightarrow \begin{pmatrix} F' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \quad (1.3)$$

As is well known this is a symmetry of the equations of motion $dF = 0$ and $d\tilde{F} = 0$ only, but not of the action $S = \int d^4xtr(FF - \tilde{F}\tilde{F})$. Incidentally, this analysis is generic for any abelian $N = 2k$-form fields in $D = 4k$ dimensions, for integral $k$.

The corresponding situation in two dimensions (which is generic for $D = 4k + 2$ dimensions, for integral $k$) has also been studied. In the case of the free massless scalar field (which can be regarded as a zero form potential) in two dimensions, the equations of motion are invariant
under $Z_2 \times SO(1, 1)$ transformations, although the action is not. This difference from the four
dimensional example is attributed to the basic identities governing the dual operations,

$$
\tilde{F} = -F; D = 4k
$$
$$
\tilde{F} = F; D = 4k + 2
$$

To elevate the duality at the level of the action, it was naturally imperative to define the
relevant transformations in terms of the basic variables which are the associated potentials
rather than the field tensors. This is possible by rewriting the action in terms of two potentials.
Incidentally, the introduction of a second potential $A'$ is essentially tied to the fact that the
dual field $\tilde{F}$ is closed by the equation of motion, so that one can write $\tilde{F} = dA'$ as an on-
shell relation. It was also shown that the duality groups $G$ preserving the invariance of the
action were the subgroups of those found earlier that preserve the invariance of the equations
of motion. In fact the former was obtained by taking an intersection with $O(2)$, the group of
invariance of the energy-momentum tensor ($T_{\mu\nu} \sim (F_{\mu}F_{\nu} + F_{\nu}F_{\mu})$)(here the unwritten indices
have been summed over). Specifically, these were $2 2$

$$
G = SO(2); D = 4k
$$
$$
G = Z_2; D = (4k + 2)
$$

It is clear, therefore, that a fundamental difference is observed in the study of duality symmetry
in $4k$ and $4k + 2$ dimensions.

To put the above discussion in a proper perspective, it might be useful to mention that the
original study of duality symmetry in the context of the equations of motion can be understood
in an alternative way that does not involve these equations at all. Indeed, it is simple to check
that by only demanding the invariance of the dual operation $F \to \tilde{F}$ under some transformati
(like (1. 3)) yields the $SO(2)$ group for four dimensions. Consider, for instance, the following
transformation,

$$
\begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix} \to \begin{pmatrix}
F' \\
\tilde{F}'
\end{pmatrix} = \begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \begin{pmatrix}
F \\
\tilde{F}
\end{pmatrix}
$$

If we demand that $\tilde{F}'$ is indeed the dual of $F'$, then it follows that $p = s$ and $q = -r$. Hence,
upto a trivial scale factor, the transformation matrix in (1.6) can be identified with the standard
$SO(2)$ matrix (1.3). The same logic holds for two dimensions also where the relevant group
is found to be $Z_2 \times SO(1, 1)$ instead of $SO(2)$. Hence the study of duality symmetry truly
becomes meaningful only with regard to the respective actions.

A natural question that may arise here concerns the fate of quantum mechanics, which can
be regarded as a field theory in $0 + 1$ dimension ($D = 1$), as this is not covered by the general
equations (1.4,1.5). Likewise, because of dimensional reasons, the form based analysis done
earlier is also inapplicable. The fact that both the free Maxwell theory and the 2-dimensional
massless scalar theory reduce to an assembly of infinite number of harmonic oscillators (HO), as can be seen through a mode expansion, lends further credence to the study of duality symmetry in the quantum mechanical context. One can thus assign a more fundamental status to the corresponding Fourier transformed amplitudes, as these amplitudes, albeit complex, undergo harmonic oscillations. After all, the particle content of such free field theories are identified with the corresponding excitations in various modes. Interpreted in this fashion, there seems to be a mapping of the results for the Maxwell and scalar field theories and the noted difference in the duality groups is clearly not the complete story.

In this paper we explicitly show how the HO manifests a duality symmetry. The corresponding duality group is \(SO(2)\). The method of deriving a duality symmetric lagrangian for the HO is easily generalisable to higher dimensions; in particular this derivation is given for the scalar and Maxwell theories in two and four dimensions respectively.

The results of the HO analysis are next put in a proper perspective with regard to the field theoretical models. As already stated, the free fields can be thought of as an assembly of an infinite number of HOs. Hence the feature of duality symmetry in free field theories should be understandable from the analogous phenomenon for the HO. At this point one is led to an impasse since the duality group for a free scalar theory in two dimensions is known to be \(Z_2\), contrary to the \(SO(2)\) group for the HO. We resolve this apparent paradox by performing a detailed mode analysis in various theories. In this context it becomes desirable to consider the ‘complex’ HO instead of the real one. The former, in contrast to the latter, manifests either the \(Z_2\) or \(SO(2)\) symmetry by a suitable redefinitions of variables. A one to one correspondence between the modes in the free fields and the complex HO is then established. From this correspondence it is shown that the scalar theory manifests both the expected \(Z_2\) as well as the \(SO(2)\) symmetry, depending on the interpretation of the results. Similar conclusions also hold for the Maxwell theory. It may be recalled, however, that the conventional interpretation of duality in the Maxwell theory admits only the \(SO(2)\) symmetry.

The paper is split into five sections. In section II, duality symmetry in the HO both for real and complex variables, is analysed. Sections III and IV describe the corresponding analysis for the scalar and Maxwell theories, including the comparison with the HO formulation. Section V contains the concluding remarks. An appendix is included to derive the explicit form of the duality generator corresponding to the \(k – th\) mode and also to illuminate how the HO itself can be cast in the electromagnetic form.

II. The one-dimensional Harmonic Oscillator
Consider a one dimensional harmonic oscillator given by the lagrangian
\[ L = \frac{1}{2}(q^2 - \omega^2 q^2) \] (2.1)

It is possible to introduce an electromagnetic notation and rewrite (2.1) as
\[ L = \frac{1}{2}(E^2 - B^2) \] (2.2)

with \( E = \dot{q} \) and \( B = \omega q \). With this, the equation of motion
\[ \ddot{q} + \omega^2 q = 0 \] (2.3)

along with the ‘Bianchi identity’ \( \dot{B} = \omega E \) can be expressed compactly as,
\[ \frac{\partial}{\partial t}(E + iB) = i\omega(E + iB) \] (2.4)

which has a manifest \( U(1) \) dual symmetry \( (E + iB) \to e^{i\phi}(E + iB) \), with \( E \) and \( B \) regarded as independent variables. However this is not a symmetry of the Lagrangian, as can be easily seen from (2.2). But our objective is to construct a Lagrangian which is equivalent to the former, and enjoys this duality symmetry. The basic idea [4] is to linearise (2.1) by invoking an additional variable ‘\( p \)’ in an enlarged configuration space as,
\[ L = \frac{1}{2}[\omega(\dot{p}q - q\dot{p}) - \frac{1}{2}\omega^2(q^2 + p^2)] \] (2.5)

Here a symmetrisation of the kinetic term has been performed. As can be easily seen, the equation of motion for \( p \) \( (\omega p = \dot{q}) \), when substituted in (2.5), yields (2.1). It just happens in this case that \( p \) is the momentum conjugate to \( q \). But this need not be true always. In fact we shall regard \( p \) just as an additional variable in an enlarged two dimensional configuration space, as we have mentioned earlier.

By labelling \( q = x_2 \) and \( p = x_1 \), (2.5) takes the form,
\[ L_+ = \frac{1}{2}\omega\epsilon_{\alpha\beta}x_\alpha x_\beta - \frac{1}{2}\omega^2x_\alpha x_\alpha \] (2.6a)

On the other hand, labelling in the reverse order, i.e. \( q = x_1 \) and \( p = x_2 \), one gets,
\[ L_- = -\frac{1}{2}\omega\epsilon_{\alpha\beta}x_\alpha x_\beta - \frac{1}{2}\omega^2x_\alpha x_\alpha \] (2.6b)

Compactly, one can write,
\[ L_{\pm}(X) = \pm \frac{1}{2}\omega X^T\epsilon \dot{X} - \frac{1}{2}\omega^2X^TX. \] (2.7)
Here we have used a matrix notation. Thus $X$ stands for the doublet \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) and $\epsilon$ for the $2 \times 2$ matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Note that the cherished aim of obtaining a duality symmetric lagrangian has been fulfilled. The above lagrangians $L_\pm(X)$ are invariant under $SO(2)$ duality rotation $X \to R^+X$ or equivalently, $E \to R^+E$, $B \to R^+B$ in the electromagnetic symbols. Furthermore, the two Lagrangians $L_+$ and $L_-$ are swapped into each other ($L_+ \leftrightarrow L_-$) by an improper $O(2)$ rotation ($X \to R^-X$). Proper (Improper) rotations will be designated by $R^+$ ($R^-$) in the rest of the paper. The generator of the $SO(2)$ duality rotation, in $L_+$ for example, is given by,

\[
G = -\frac{1}{2} \omega X^T X
\]
satisfying $\delta X = \theta \{ X, G \}$, for an infinitesimal $SO(2)$ rotation $\theta$. This can be easily verified from the symplectic bracket following from (2.7),

\[
\{ x_\alpha, x_\beta \} = \frac{1}{\omega} \epsilon_{\alpha\beta}
\]

The lagrangians ($L_\pm$) are refered to as chiral Lagrangians, as the corresponding “angular momentum” (the rotational generator in the internal $x_1x_2$ space)

\[
J_\pm = \pm \frac{1}{\omega} H = \pm \frac{1}{2} \omega X^T X
\]

(2.8)

has positive or negative eigenvalues. Here $H$ is the Hamiltonian common to both $L_\pm$ (2.7). In this sense (2.7) represents the dynamics of chiral oscillators.

It is now an interesting and instructive exercise to show the combination of the left and right chiral oscillators to reproduce the Lagrangian of the usual HO. This is done by following the soldering technique suggested in [5] and developed fully in [6, 7].

Consider therefore $L_+(X)$ and $L_-(Y)$, given as functions of two independent variables $X$ and $Y$, respectively. Under a transformation which preserves the difference $(X - Y)$, i.e.

\[
\delta X = \delta Y = \eta
\]

(2.9)

the Lagrangians $L_\pm(Z)$ transform as,

\[
\delta L_\pm(Z) = \epsilon_{\alpha\beta} \eta_\alpha J_\pm^\beta(Z) = \eta^T \epsilon J_\pm(Z); Z = (X,Y)
\]

(2.10)

where,

\[
J_\pm(Z) = \pm \dot{Z} + \omega \epsilon Z
\]

(2.11)

Introduce a variable $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, that will effect the soldering, transforming as,

\[
\delta B = -\epsilon \eta
\]

(2.12)
Eventually, by a series of iterative steps, it can be shown that the soldered Lagrangian,

\[ L^s = L_+(X) + L_-(Y) - B^T(J^+(X) + J^-(Y)) - B^T B \] (2.13)

is invariant under the complete set of transformations (2.9) and (2.12).

The equation of motion for the auxiliary field \( B \),

\[ B = -\frac{1}{2}(J^+(X) + J^-(Y)) \] (2.14)

which follows from (2.13), when substituted back in the same equation, yields,

\[ L^s = \frac{1}{2}(\dot{Z}^T \dot{Z} - \omega^2 Z^T Z) \] (2.15)

where \( L^s \) is no longer a function of \( X \) and \( Y \) independently, but only of their difference,

\[ Z = \frac{1}{\sqrt{2}}(X - Y) \] (2.16)

The above Lagrangian characterises a bi-dimensional oscillator. It is duality symmetric under the complete \( O(2) \) transformation \( Z \rightarrow R^\pm Z \). It is straightforward to show from (2.14) that the transformation law (2.12) is properly reproduced, thereby serving as a consistency check on the soldering programme.

Let us next consider the example of the 'complex' HO. They occur naturally as the Fourier modes of several free field theories and thus will be useful for the subsequent analysis. Besides, this is an instructive example where distinct variable redefinitions are possible which show a reversal of roles of the duality transformations. Consider, therefore, the following Lagrangian,

\[ L = \frac{1}{2}(\dot{\phi}^* \dot{\phi} - \omega^2 \phi^* \phi) \] (2.17)

Linearising the above Lagrangian, by introducing additional variables \( \pi \) and \( \pi^* \) in an enlarged configuration space, one gets

\[ L = \frac{1}{2} \omega(\pi^* \dot{\phi} + \pi \dot{\phi}^*) - \frac{1}{2} \omega^2 (\pi^* \pi + \phi^* \phi) \] (2.18)

Labelling \( \phi = q_1 \) and \( \pi = q_2 \) and then again in the reverse order i.e. \( \phi = q_2 \) and \( \pi = q_1 \), the following “chiral” forms of the Lagrangian are obtained,

\[ L_\pm(Q) = \pm \frac{1}{2} \omega Q^\dagger \epsilon \dot{Q} - \frac{1}{2} \omega^2 Q^\dagger Q \] (2.19)

with \( Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \).
The occurrence of the $\epsilon$ matrix indicates that, as before, the Lagrangians $L_\pm$ are invariant under the $SO(2)$ transformation $Q \rightarrow R^+ Q$. Similarly, the improper rotations $R^-$ induce a swapping $L_+ \leftrightarrow L_-$. Exactly in analogue with the real HO, the generator of duality rotation is found to be,

$$G = -\frac{1}{2}\omega Q^\dagger Q$$

Now consider the following alternative way of relabelling the $(\phi, \pi)$ variables in (2.18),

$$\phi = q_1; \pi = iq_2 \quad (2.20a)$$

and then as

$$\phi = q_2; \pi = -iq_1 \quad (2.20b)$$

which yields the following structures for the Lagrangians,

$$L_\pm(Q) = \frac{1}{2} (\pm i\omega Q^\dagger \sigma^1 Q - \omega^2 Q^\dagger Q) \quad (2.21)$$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first Pauli matrix. These Lagrangians are invariant under a discrete $Z_2$ transformation $Q \rightarrow \sigma^1 Q$, while the swapping $L_+ \leftrightarrow L_-$ is effected by $Q \rightarrow \epsilon Q$. Clearly therefore, the roles of proper and improper rotations are reversed from the previous case. Compared to the real example, the complex HO has a richer symmetry structure that is essentially tied to the complex nature of the variables, allowing for alternative redefinitions.

To complete the analysis, the soldering of the complex ‘chiral’ oscillators (2.19) is done to reproduce the complex HO. Consider a transformation,

$$W \rightarrow W' = W + \eta \quad (2.22)$$

Under this, (2.19) transforms as,

$$\delta L_\pm(W) = \eta^\dagger \epsilon J^\pm - J_\pm^\dagger \epsilon \eta; W = (Q, R) \quad (2.23)$$

where,

$$J^\pm(W) = \pm \frac{1}{2}\omega W + \frac{1}{2}\omega^2 \epsilon W \quad (2.24)$$

is the counterpart of (2.11).

Introduce a (column) matrix-valued variable $B$ transforming as,

$$\delta B = -\epsilon \eta \quad (2.25)$$

Then the first iterated Lagrangian $L^{(1)}_\pm$, defined as,

$$L^{(1)}_\pm = L_\pm - B^\dagger J^\pm - J_\pm^\dagger B \quad (2.26)$$

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can be shown to transform as,
\[ \delta L^{(1)} = \mp \frac{1}{2} \omega (B^{\dagger} \dot{\eta} + \dot{\eta}^{\dagger} B) - \frac{1}{2} \omega^2 (B^{\dagger} \epsilon \eta - \eta^{\dagger} \epsilon B) \] (2.27)

Assuming that \( B \) does not depend on \( W \), one finds,
\[ \delta L^{(1)}(Q) + \delta L^{(1)}(R) = -\omega^2 (B^{\dagger} \epsilon \eta - \eta^{\dagger} \epsilon B) \] (2.28)

Just as in the previous section, the soldered Lagrangian \( L^s \) is defined as,
\[ L^s = L^{(1)}(Q) + L^{(1)}(R) - \omega^2 B^{\dagger} B \] (2.29)

Using (2.25) and (2.28), one can easily show that \( L^s \) is invariant
\[ \delta L^s = 0 \] (2.30)

under the above transformations (2.22) and (2.25). Eliminating the auxiliary variables \( B \) and \( B^{\dagger} \) from (2.29), by using the corresponding equations of motion, one obtains
\[ L^s = \frac{1}{2} (\dot{S}^{\dagger} \dot{S} - \omega^2 S^{\dagger} S) \] (2.31)

where,
\[ S = \frac{1}{\sqrt{2}} (Q - R) \] (2.32)

is a ‘gauge invariant’ combination of variables. Thus starting with chiral forms of Lagrangians \( L_+(Q) \) and \( L_-(R) \), given as functions of \( Q \) and \( R \) respectively, we have constructed a soldered Lagrangian \( L^s \), which is a function of the difference \( S \) (2.32) only. Thus the bi-dimensional complex HO is manifestly invariant under the simultaneous transformation, \( \delta Q = \delta R = \eta \). Incidentally, the same conclusions are obtained if one starts from (2.21) instead of (2.19).

Using these concepts, the duality symmetry in the context of free field theories is better understood, as evolved in the subsequent sections.

III. Massless scalar fields in 1 + 1 dimensions

The HO is quite ubiquitous in field theoretical models. This is because a large number of free field models can be thought of as an assembly of infinite number of free HOs, each designated by the mode vector \( k \). In this section, we shall carry out the mode analysis of the massless scalar fields in (1+1) dimension and study the duality symmetry through these modes,
simultaneously revealing the close connection with the HO analysis carried out in the previous section.

The Lagrangian of the model is given by,

\[ L = \frac{1}{2} \int dx (\dot{\phi}^2 - \phi^2) \]  \hspace{1cm} (3.1)

Putting the system in a box of length \( L \), one can make the Fourier decomposition of the real scalar field \( \phi(x) \) as,

\[ \phi(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \phi_k(t) \]  \hspace{1cm} (3.2)

Here \( k \) represents the space component of a 2-vector \( k^\mu \), satisfying \( k^\mu k_\mu = \omega_k^2 - k^2 = 0 \) and \( \phi_k^* = \phi_{-k} \). Substituting (3.2) in (3.1), one gets

\[ L = \sum_k L_k \]  \hspace{1cm} (3.3a)

with

\[ L_k = \frac{1}{2} (\dot{\phi}_k^* \dot{\phi}_k - \omega_k^2 \phi_k^* \phi_k) \]  \hspace{1cm} (3.3b)

representing a “complex” HO for the \( k \)-th mode (see (2.17)), as \( \phi_k \) is a complex number in general. Thus one can proceed just as in the preceding section to linearise the Lagrangian and then relabel the variables in the appropriate manner to obtain the following duality invariant forms of the Lagrangian,

\[ L_k^\pm(Q_k) = \pm \frac{1}{2} \omega_k Q_k^* \dot{Q}_k - \frac{1}{2} \omega_k^2 Q_k^* Q_k \]  \hspace{1cm} (3.4a)

and

\[ L_k^\pm(R_k) = \frac{1}{2} (\pm i \omega_k Q_k^* \sigma^1 Q_k - \omega_k^2 Q_k^* Q_k) \]  \hspace{1cm} (3.4b)

with \( Q_k = \begin{pmatrix} q_{1k} \\ q_{2k} \end{pmatrix} \). Note that these expressions are just (2.19) and (2.21), but with only an additional subscript \( k \)-the mode index. It is clear that while the duality group is \( SO(2) \) for (3.4a), it is \( Z_2 \) for (3.4b). Recall that, expressed in terms of the original scalar fields, only the latter is manifested \[ 3, 4, 5 \].

We can now proceed with the soldering of these two Lagrangians \( L_{k^+}(Q) \) and \( L_{k^-}(R) \), for two independent variables \( Q \) and \( R \), as we have done in the previous section to finally get

\[ L^*_k = \frac{1}{2} (\dot{S}_k^* \dot{S}_k - \omega_k^2 S_k^* S_k) \]  \hspace{1cm} (3.5a)
where,

\[ S_k = \frac{1}{\sqrt{2}}(Q_k - R_k) \tag{3.5b} \]

is the ‘gauge invariant’ combination of variables \( Q_k \) and \( R_k \). Note that the above result follows irrespective of whether one starts from (3.4a) or (3.4b). The soldered Lagrangian, which is just the expression for the \( k \)-th mode, is thus manifestly invariant under the simultaneous transformation, \( \delta Q_k = \delta R_k = \eta_k \). At this stage we can sum over all the modes to get the complete soldered Lagrangian \( L^s \) as,

\[ L^s = \sum_k L^s_k = \frac{1}{2} \sum_k (\dot{S}_k^\dagger \dot{S}_k - \omega_k^2 S_k^\dagger S_k) \tag{3.6} \]

Using the inverse Fourier transform, this can be easily shown to yield

\[ L^s = \frac{1}{2} \int dx \partial_\mu S^\dagger(x) \partial^\mu S(x) \tag{3.7} \]

where

\[ S(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} S_k(t) \tag{3.8} \]

is a doublet of real scalar fields. This is again given in terms of the difference,

\[ S(x) = \frac{1}{\sqrt{2}}(Q(x) - R(x)) \tag{3.9} \]

where \( Q(x) \) and \( R(x) \) are obtained from \( Q_k \) and \( R_k \) using expressions similar to (3.8).

On the other hand, as shown in [4], the original model (3.1) can be reexpressed, after a suitable redefinition of variables, in a linearised form as,

\[ \mathcal{L}_\pm(\Phi) = \frac{1}{2} (\pm \dot{\Phi}^T \sigma^1 \Phi' - \Phi'^T \Phi') \tag{3.10} \]

where \( \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \). The matrix swapping \( \mathcal{L}_+ \leftrightarrow \mathcal{L}_- \) is \( \epsilon \). Again as shown in [4], the soldering of \( \mathcal{L}_+(Q) \) and \( \mathcal{L}_-(R) \), where \( Q \) and \( R \) denote the independent fields corresponding to the positive and negative components of the Lagrangian given in (3.10), yields,

\[ \mathcal{L}^s = \frac{1}{2} \partial_\mu S^\dagger \partial^\mu S \tag{3.11} \]

where \( S \) is identical to (3.9). Note that this is precisely the Lagrangian density appearing in (3.7). This shows that writing the original model in the chiral form and then soldering, as in [4], yields the same result as the one obtained by first making a Fourier decomposition (3.2)
and then expressing this Lagrangian (a “complex HO”) in a linearised chiral oscillator form $L_{k\pm}$ (3.4), next soldering to get $L^s_k$ (3.5), followed by a final summation over all the modes to get (3.7).

It may be recalled that (3.10) is the conventional form of the duality symmetric action in two dimensions \cite{2,3,4}. Nevertheless, expressed in terms of its modes, the massless scalar theory (3.3b) gets mapped to the complex HO, thereby manifesting either the $Z_2$ or the $SO(2)$ symmetry depending on the variable redefinitions. To establish compatibility with (3.10) where only the $Z_2$ symmetry is revealed, recall that (3.10) was obtained \cite{4} by rewriting (3.1) in its linearised version,

$$\mathcal{L} = \frac{1}{2}(P\dot{\phi} - \dot{P}\phi - P^2 - \phi^2) \quad (3.12)$$

where $P$ is an additional variable in an extended configuration space. In order to get the form $\mathcal{L}_+(3.10)$ for example, one has to make the following relabelling

$$\phi = \phi_1$$
$$P = \phi'_2 \quad (3.13a,b)$$

Incidentally, the existence of the second scalar field $\phi_2(x)$ is understood in the following manner. Since $\phi(x) = \phi_1(x)$ can be regarded as a zero-form potential, the field one-form

$$F = d\phi_1 = (\dot{\phi}_1 dt + \phi'_1 dx)$$

has the dual

$$\tilde{F} = - (\dot{\phi}_1 dx + \phi'_1 dt)$$

which is closed on-shell, so that in the absence of any nontrivial topology it must be exact. In other words there exists another function $\phi_2(x)$ satisfying $\tilde{F} = -d\phi_2$. As can be easily seen, here $P = \dot{\phi}_1 = \phi'_2$.

To get the form $\mathcal{L}_-$, the relabelling has to be done in the reverse order i.e. ($\phi = \phi_2$; $P = \phi'_1$). In the rest of this section, we shall only consider $\mathcal{L}_+$ for convenience.

At this stage, we can Fourier analyse the field $\phi_\alpha(x)$ ($\alpha = 1, 2$) and $P(x)$ as,

$$\phi_\alpha(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \phi_{\alpha k}(t) \quad (3.14)$$
$$P(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} k\pi_k(t) \quad (3.15)$$

We can easily see that (3.13b) implies, in terms of momentum space variables,

$$\pi_k = i\phi_{2k} \quad (3.16)$$
Again, reality of $\phi_\alpha(x)$ implies,

\begin{align}
\phi_{\alpha k}^* &= \phi_{\alpha(-k)} \\
\pi_k^* &= -\pi_{-k}
\end{align}

One can then proceed, as for the model (3.1), to obtain two equivalent forms for $L_k$, starting from (3.12). Using the Fourier decomposition of both $\phi$ and $P$ fields, we get,

\[ L_k = \frac{1}{2} [k(\pi_k \dot{\phi}_k^* + \pi_k^* \dot{\phi}_k) - \omega_k^2 (\phi_k^* \phi_k + \pi_k^* \pi_k)] \] (3.18)

Note that in the Fourier decomposition of the field $P(x)$ in (3.15), we had intentionally incorporated an additional factor of $k$ in front of $\pi_k$, so that the form of (3.18) looks exactly like that of ‘complex’HO (2.18). Also note that the relation $\omega_k^2 = k^2$ was used crucially in these expressions, indicating that the above structure for the Lagrangian is strictly valid for massless fields.

Mimicking the steps of the complex HO, it is simple to show that the above Lagrangian displays either the $Z_2$ or $SO(2)$ symmetry, based on a suitable relabelling of fields.

**IV. Maxwell field in 4D**

In this section, we shall carry out a similar analysis for the free Maxwell field. But because of the inherent gauge invariance of the model, we shall not start with a Fourier analysis right at the beginning. Rather the Gauss constraint of the model will be imposed strongly to isolate the physical degrees of freedom. Mode analysis then reveals the HO structure just as in the scalar case with the difference that to each mode $k$ there are two orthogonal transverse oscillators. Following the HO example, this model is then linearised and written in ‘chiral’ forms. We then carry out the soldering of the ‘chiral’ forms of the Lagrangian followed by a summation over all the modes, to get hold of the final soldered Lagrangian. This part is just the same as we did for the scalar field. To that end, consider

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2) \] (4.1)

where $E_i = (\partial_0 A_i - \partial_i A_0)$ and $B^k = \epsilon^{ijk} \partial_j A_j$ are the electric and magnetic fields. At this stage, the Gauss constraint ($\nabla \cdot E = 0$) can be solved for $A_0$ yielding,

\[ A_0 = \frac{\partial_0}{\nabla^2} (\nabla \cdot A) \] (4.2)
Time preservation of (4.2) is guaranteed by the equations of motion. Hence it is possible to eliminate $A_0$ from the Lagrangian by using (4.2) to get 
\[
\mathcal{L} = \frac{1}{2}[(\dot{A}^T)^2 - (\nabla \times A^T)^2]
\] (4.3)

where the longitudinal component $A^L$ drops out automatically and only the physical (transverse) component $A^T$ survives. In terms of the gauge field $A$, this is given by,
\[
A^T_i = P_{ij} A_j = (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) A_j
\] (4.4)

with $P_{ij}$ being the projection operator satisfying $P^2 = P$. Therefore the Fourier decomposition has to be carried out keeping this in mind. Hereafter, we shall omit the superscript (T) from $A^T$ and write simply $A$.

So finally carrying out a Fourier decomposition,
\[
A(x) = \frac{1}{\sqrt{V}} \sum_k e^{ik \cdot x} A_k(t)
\] (4.5)

Note that $A_k(t)$ can be written as,
\[
A_k(t) = \sum_{\lambda=1}^{2} A_{k\lambda}(t) \epsilon_{\lambda}(k)
\] (4.6)

where $\epsilon_{\lambda}(k)$ are the polarisation vectors orthogonal to $k$ ($k \cdot \epsilon_{\lambda}(k) = 0$). This orthogonality projects out the transverse component of the vector potential.

As expected, the Lagrangian $L(= \int d^3x \mathcal{L})$ can then be written as,
\[
L = \sum_k L_k
\] (4.7)

with
\[
L_k = \frac{1}{2}(\dot{A}_k^* \cdot \dot{A}_k - \omega_k^2 A_k^* \cdot A_k)
\] (4.8)

Comparing with (3.3b), we can see that $A_k$ are now 3-vectors in $C^3$ in contrast to $\phi_k$, which are just complex scalars.

Thus one can proceed just as in the scalar theory to linearise (4.8) by invoking additional vector-valued variables $\Pi_k$ and its complex conjugates in an enlarged configuration space, to write
\[
L_k = \frac{1}{2}\omega_k(\Pi_k^* \cdot \dot{A}_k + \Pi_k \cdot \dot{A}_k^*) - \frac{1}{2}\omega_k^2 (\Pi_k^* \cdot \Pi_k + A_k^* \cdot A_k)
\] (4.9)
associated with each mode $k$. Parametrising $q_{1k} = A_k$ and $q_{2k} = \Pi_k$ and then in the reverse order i.e. $q_{2k} = A_k$ and $q_{1k} = \Pi_k$, one gets the following “chiral” forms of the Lagrangian,

$$L_{k\pm}(Q_k) = \pm \frac{1}{2} \omega_k Q_k^\dagger \epsilon \cdot Q_k - \frac{1}{2} \omega_k^2 Q_k^\dagger Q_k$$  \hspace{1cm} (4.10)$$

where $Q_k$ is the doublet $\left( \begin{array}{c} q_{1k} \\ q_{2k} \end{array} \right)$. The above lagrangian is invariant, mode by mode, under the usual $SO(2)$ transformation $Q_k \rightarrow \epsilon Q_k$. Similarly, under the transformation $Q_k \rightarrow \sigma^1 Q_k$ the lagrangians $L_{k+}$ and $L_{k-}$ are swapped into one another.

For a discussion of the equivalence of (4.10) with the standard form [1, 2, 3, 4] of duality invariant electromagnetic action, we refer the reader to the appendix. The explicit form of the duality generator is also derived there. Alternatively, parametrising,

$$\Phi_{1k} = A_k$$

$$\Phi_{2k} = -i \Pi_k$$  \hspace{1cm} (4.11)$$

and then in the reverse order, the lagrangian (4.9) is expressed in the chiral form as,

$$L_{k\pm} = \frac{1}{2} (\pm i \omega_k \dot{\Phi}_k^\dagger(t) \sigma^1 \Phi_k(t) - \omega_k^2 \Phi_k^\dagger(t) \Phi_k(t))$$  \hspace{1cm} (4.12)$$

which reveals the $Z_2$ invariance, instead of the usual $SO(2)$. The analogy with the ‘complex’ HO is therefore complete. Not surprisingly therefore, a similar equation (3.18) had also occurred earlier in the case of scalar field. The only additional feature in this case is that $\Phi_k = \left( \begin{array}{c} \Phi_{1k} \\ \Phi_{2k} \end{array} \right)$ is now a doublet of vector fields.

It is quite straightforward to solder the two ‘chiral’ forms of the Lagrangians $L_{k+}(Q_k)$ and $L_{k-}(R_k)$ in the lines of the scalar case to get,

$$L^s_k = \frac{1}{2} (S_k^\dagger \cdot \dot{S}_k - \omega_k^2 S_k^\dagger \cdot S_k)$$  \hspace{1cm} (4.13a)$$

where,

$$S_k = \frac{1}{\sqrt{2}} (Q_k - R_k)$$  \hspace{1cm} (4.13b)$$

is a doublet of vectors. Contrast this with (3.5b), where $S_k$ stands for a doublet of scalars.

Now to obtain the final soldered Lagrangian, we have to sum over all the modes ($L^s = \sum_k L^s_k$), as we did for the scalar case (3.6). This yields,

$$L^s = -\frac{1}{4} \int d^3x G^{\alpha}_{\mu
u} G_{\alpha\mu\nu}$$  \hspace{1cm} (4.14a)$$

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where
\[ G^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu \] (4.14b)
is a doublet of abelian field strengths ($\alpha = 1, 2$). This is the same result, which was obtained in [4]. We have thus been able to provide an alternative derivation by starting from the basic HO example.

Finally, introducing a doublet of divergence free vector field,
\[ S(x) = \frac{1}{\sqrt{V}} \sum_k e^{i k \cdot x} S_k(t) \] (4.15)
one can also cast $L^s$ in the pattern of scalar fields (3.7) as,
\[ L^s = \frac{1}{2} \int d^3 x \partial_\mu S^\dagger(x) \partial^\mu S(x) \] (4.16)
Again, the only difference with (3.7) is that $S(x)$ appearing there is a doublet of scalar fields, in contrast to the case here, where $S(x)$ is a doublet of vector fields.

The reason that the soldered Lagrangian for electrodynamics can be cast in the form of scalars is rooted to the fact that, at the level of modes, both of them represent an infinite number of decoupled HOs. The only additional feature of electrodynamics is that to each mode $k$, there exists two orthogonal HOs associated to two polarisation states.

V. Conclusions

This paper showed that duality symmetry in certain free field theories had their origin in a similar symmetry in a quantum mechanical example-the ‘complex’ harmonic oscillator (HO). While a clear distinction is made in the literature concerning duality in $D = 4k$ and $D = (4k+2)$ dimensions, nothing specific is mentioned regarding the quantum mechanical case, which can be regarded as a field theory in $D = 1$ dimension. Our analysis, on the other hand, clearly revealed that the study of duality symmetries in the HO case is fundamental to properly understand the corresponding phenomenon in the field theoretic case, at least for the models considered in this paper. Indeed by performing an explicit mode analysis, the free scalar and Maxwell theories were mapped to the complex HO. The one to one correspondence between duality symmetry in the HO and the field theories was easily established.

An algebraic consistency check was also provided for the mode analysis. This was done by taking recourse to the soldering mechanism that was earlier advocated by one of us [4, 7]. It was shown that the soldering of duality symmetric lagrangians ($\mathcal{L}_+$ and $\mathcal{L}_-$) before the
mode decomposition yields identical results by, alternatively, first doing a mode analysis of the individual lagrangians, then soldering the various modes and finally summing over all the modes.

To understand the new feature in this paper it is necessary to recall the development of duality symmetry. Originally, by considering the transformations on the electric and magnetic fields, an invariance of the equations of motion was found although the Lagrangean flipped its sign. In fact, as shown here, this duality is obtained directly from the algebraic transformation theory and need not consider any equations of motion. Obviously, therefore, it was necessary to look at the invariance of the action. Moreover, since the electric and magnetic fields are derived quantities from the potential, it was reasonable to study duality symmetry through these potentials. Simultaneously this brought out a new feature, namely, the invariance of the action itself. Nevertheless, a distinction between twice odd and twice even dimensions prevailed since the duality groups in the two cases differed. By pushing this development to its logical conclusion of considering the potential not as a basic field, but as a quantity derived from its Fourier modes, and then investigating duality symmetry through these modes, we obtained new results. The invariance of the action was now demonstrated for both the duality groups $Z_2$ and $SO(2)$, irrespective of the dimensionality of space time. The explicit computations were done for the scalar theory in $D = 2$ dimensions and the Maxwell theory in $D = 4$ dimensions. Indeed, by mapping these models to the HO, it became clear that these have a common origin. By suitable field redefinitions it was possible to discuss the role of either $Z_2$ or $SO(2)$ as a duality group in both the models. The germ of this feature was obviously contained in the HO, which displayed both the symmetries depending on the change of variables. This may be compared with the general algebraic arguments [3] regarding $Z_2$ and $SO(2)$ as the duality groups for $D = (4k + 2)$ and $D = 4k$ dimensions, respectively. Nevertheless, it should be clarified that there is no clash between the general algebraic arguments and our findings. The duality groups suggested by these arguments are obtained from the intersection of two groups which, in turn, follow from the transformation properties of the N-form potentials. The crucial point is that these forms are real whereas the modes with which we work are, in general, complex. Hence such form based analysis does not cover our formalism. Yet another way to look at this issue is to realise that the duality symmetry in field theories were mapped to the corresponding properties in the H.O. The latter, being defined in one dimension, obviously has no form related interpretation. Clearly, therefore, our explicit calculations provided fresh insights that cannot be otherwise gained from purely general reasoning. We feel that this analysis of duality symmetry through a mode expansion can be pursued for other examples.

Appendix
One may have observed that the entire discussion of duality symmetry for the Maxwell theory did not invoke the familiar form of the Lagrangian

\[ L_\pm = \frac{1}{2} (\pm \epsilon_{\alpha\beta} E_\alpha B_\beta - B_\alpha B_\beta) \]  

(A1a)

where,

\[ E_{i\alpha} = \partial_0 A_{i\alpha} - \partial_i A_{0\alpha} \]

and

\[ B_{i\alpha} = \epsilon^{ijk} \partial_j A_{k\alpha} \]  

(A1b)

represent the electric and magnetic fields in the internal space. This is further simplified to

\[ L_\pm = \frac{1}{2} (\pm \epsilon_{\alpha\beta} \dot{A}_\alpha^a B_\beta^a - B_\alpha^a B_\beta^a) \]  

(A2)

since the \( A_0 \) piece merely contributes a boundary term. The above form of the duality invariant Lagrangian was obtained in various ways [9, 2, 4] and has also been the starting point of several recent investigations [10, 11, 12, 13]. It is therefore desirable to establish some sort of connection of our analysis with this structure. Note that henceforth we only consider the positive ‘chiral’ component of \( (A2) \) here.

Performing a mode analysis of \( (A2) \), we obtain, for the Lagrangian \( L(= \int d^3x L) \):

\[ L = \sum_k L_k \]  

(A3a)

where,

\[ L_k = \frac{1}{2} (\epsilon_{\alpha\beta} \dot{A}_{\alpha k}^a B_{\beta k}^a - B_{\alpha k}^* B_{\alpha k}) \]  

(A3b)

Using \( (A1b) \) and the fact that only the transverse components of the fields are relevant, one finds the following relations,

\[ B_{1k} = \frac{i}{\omega_k} \mathbf{k} \times \mathbf{A}_{1k} \]

\[ B_{2k} = \omega_k \Pi_k \]  

(A4)

\[ A_{2k} = \frac{i}{\omega_k^2} \mathbf{k} \times \Pi_k = \frac{i}{\omega_k^2} \mathbf{k} \times B_{2k} \]

Inserting these in \( (A3) \) yields,

\[ L_k = \frac{1}{2} \omega_k (\Pi_k^* \dot{\mathbf{A}}_{1k} + \Pi_k \dot{\mathbf{A}}_{1k}^* - \frac{1}{2} \omega_k^2 (\Pi_k^* \Pi_k + \mathbf{A}_{1k}^* \mathbf{A}_{1k}) \]  

(A5)

which reproduces \( (4.9) \). This shows the equivalence of the duality invariant Maxwell action derived here with the conventional form.
It is well known [3, 4] that the generator of the duality rotation is given by the Chern-Simons structure,

$$G = -\frac{1}{2} \int d^3 x A_\alpha . B_\alpha$$ \hspace{1cm} (A6)

Using the identification (A4), this reduces to,

$$G_k = \frac{i}{2} k \cdot \left( q^*_k \times q_k \right)$$ \hspace{1cm} (A7)

where $q_{1k} = A_k$ and $q_{2k} = \Pi_k$. This gives the explicit form of the generator corresponding to the $k$th mode (4.10). It bears a striking resemblance with the expression obtained earlier for the complex H.O.

We now show how a real HO can also be cast in the electromagnetic form. Noting that a one dimensional HO can be regarded as a linear oscillator undergoing its motion in a line passing through the origin and pointed in an arbitrary direction in $\mathbb{R}^3$, it is possible to write (2.1) as,

$$L = \frac{1}{2} (\dot{Q}_1^2 - \omega^2 Q_1^2) = \frac{1}{2} (\dot{Q}_1^T \dot{Q}_1 - \omega^2 Q_1^T Q_1)$$ \hspace{1cm} (A8)

where $Q_1 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \in \mathbb{R}^3$ is a triplet of real numbers. Note that we have an enlarged 3-dimensional configuration space, although the original HO undergoes its motion in 1-dimension. In analogy with (A4), one can define,

$$B_1 = k \times Q_1$$ \hspace{1cm} (A9)

where $k$ is an arbitrary fixed vector in $\mathbb{R}^3$, which is orthogonal to $Q_1 (k . Q_1 = 0)$ and the magnitude is $\omega = |k|$. Similarly, define,

$$B_2 = \dot{Q}_1$$ \hspace{1cm} (A10)

and

$$Q_2 = \frac{1}{\omega^2} k \times B_2$$ \hspace{1cm} (A11)

Note that $\dot{Q}_1$ and $Q_1$ are parallel. Since we have assumed that $\dot{k} = 0$, we can easily see that the vectors $(Q_1, Q_2, k)$ form an orthogonal triplet. Then (A8) can be written as,

$$L = \frac{1}{2} (\epsilon_{\alpha\beta} \dot{Q}_\alpha . B_\beta - B_\alpha . B_\alpha)$$ \hspace{1cm} (A12)

which maps with the usual duality invariant form (A2) of the Maxwell theory.
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