The Alpha-Beta-Symmetric Divergence and their Positive
Definite Kernels

Mactar Ndaw, Macoumba Ndour, and Papa Ngom
LMA-Laboratoire de Mathématiques Appliquées
Université Cheikh Anta Diop BP 5005 Dakar-Fann Sénégal
e-mail: mactarndaw1@gmail.com, macoumbandour@hotmail.fr, papa.ngom@ucad.edu.sn
September 18, 2018

Abstract

In the field of statistical modeling, the distance or divergence measure is a criterion widely known
and widely used tool for theoretical and applied statistical inference and data processing problems.
In this paper, we deal with the well-known Alpha-Beta-divergences (which we shall refer to as the
AB-divergences) which are a family of cost functions parametrized by two hyperparameters and
their tight connections with the notions of Hilbertian metrics and positive definite (pd) kernels on
probability measures. An attempt is made to describe this dissimilarity measure, which can be sym-
metrized using its two tuning parameters, alpha and beta. We compute the degree of symmetry of the
AB-divergence on the basis of Hilbertian metrics. We investigate the desirable properties that the
proposed approach needs to build a positive definite kernel \( K(x, y) \) corresponding to this symmetric
AB-divergence.

We establish the effectiveness of our approach with experiments conducted on Support Vector Ma-
chine (SVM) and the applicability of this method is described in an algorithm from this symmetric
divergence in image classification.

We perform experiments using the conditionally defined positive \( K \) and the kernel transformed
\( K_t \) and show that these kernels have the same proportion of errors for the Euclidean divergence
and the Hellinger divergence. We also observe large reductions in error for the Itakura-saito divergence
with the K kernel in classifications than classical Kernel methods.

Keywords: Hilbertian metrics, positive definite (pd) kernels, divergence, support vector machine
(SVM).

1 Introduction

Over the last few years, the need for specific design of kernels for a given data structure has been rec-
ognized by the kernel community. Recently, a Hilbert space embedding for probability measures has
been proposed, with applications including dimensionality reduction, independence testing and machine
learning. Therefore the use of specialized metrics and divergences measures in the successful design of
dimensionality reduction techniques has been progressively acquiring much recognition. There are nu-
merous real scenarios and applications for which the parameters of interest belong to non-flat manifolds,
and where the Euclidean geometry results are unsuitable to evaluate the similarities. Indeed, this is
usual case in the comparison of probability density functions. So, for better results the kernel should be
adjusted as accurately as possible to the subjacent structure of the input space. Kernel on probability
measures are very handy for dealing with graph problems, trees, manifolds, acoustic and signal processing
and they became very popular because of their many applications. In probability theory, the distance
between probability measures is used in studying pattern analysis see John Shawe Taylor and Mello
Cristianini (2004). Another application is in giving a bounded probability space \( \mathcal{X} \) and using the kernel
to compare arbitrary sets in that space, by putting e.g the uniform measure on each set. This extremely
useful to compare data of variable length, sequence data in bioinformatics for example, kernel methodes
for predictting protein-protein in interactions (Asa Ben-Hur, William Stafford Noble 2005).

The 20th century witnessed tremendous efforts to exploit new distance/similarity measures for a va-
riety of applications. There are a substantial number of distance/similarity measures encountered in
many different fields such as biology, chemistry, computer science, ecology, information theory, geology, mathematics, physics, statistics, etc. Distance or similarity measures are essential to solve many pattern recognition problems such as classification, clustering, and retrieval problems. Various distance/similarity measures that are applicable to compare two probability density functions. The advantages of discriminative learning algorithms and kernel machines are combined with generative modeling using a novel kernel between distributions. In the probability product kernel, data points in the input space are mapped to distributions over the sample space and a general inner product is then evaluated as the integral of the product of pairs of distributions. Recently, developments in machine learning, including the emergence of support vector machines, have rekindled interest in kernel methods (Vapnik, 1998; Hastie et al., 2001) and take full advantage of well known probabilistic models. These kernel methods have been widely employed to solve machine learning problems such as classification and clustering. Although there are many existing graph kernel methods for comparing patterns represented by undirected graphs, the corresponding methods for directed structures are less developed. In particular for domains such as speech and images kernel functions have been suggested as good ways to combine an underlying generative model in the feature space and discriminant classifiers such as SVM. Of particular concern to mathematicians is that several divergence measures are asymmetric. However, in support vector machine classifier, asymmetric kernel functions are not used so far, although they are frequently used in other kernel classifiers. In this paper, we suggest an alternative procedure by exploiting the symmetric AB-divergence measures, and present an information theoretic kernel method for assessing the similarity between a pair of directed graphs. In particular, we show that our kernel method provides an efficient tool in statistical learning theory, and SVM have demonstrated highly competitive performance in numerous real-world applications, such as medical diagnosis, bioinformatics, face recognition, image processing and text mining, which has established SVM as one of the most popular, state-of-the-art tools for knowledge discovery and data mining. Similar to artificial neural networks, SVM possess the well-known ability of being universal approximators of any multivariate function to any desired degree of accuracy.

The remainder of this paper is organized as follows. In section 2, we set out the basic notations, the definitions and assumptions. We will show the close relationship between Hilbert metrics and pd kernels so that in general, stratements for one category can be easily transferred to another. In section 3, we define a Hilbertian metrics the Alpha-Beta-Symmetric divergence (ABS-divergence) and the property of this divergence are studied and we are given a corresponding positive definite kernels. In section 4, the results of the simulations are presented. Therefore we evaluated the performance of the proposed metrics and kernels in tree classifications. And we proposed and apply a algorithm in experimental dataset to analysis the robustness of the divergence proposed. In the last section we presented the conclusion.

2 Basic Notation and some results
For a class of Hilbertian metrics, that are metrics which can be isometrically embedded into a Hilbert space. We will also use the following function class to define this subclass of metrics.

2.1 Hilbertian Metrics, Positive Definite Kernels
The positive definite kernel $K(x, y)$ corresponds to an inner product $\langle \phi_x, \phi_y \rangle_H$ in some feature space $H$. The class of conditionally positive definite (cpd) kernel is less well known. Nevertheless this class is of great interest since Schölkopf show in (P. J. Moreno, P. P. Hu, and N Vasconcelos 2003) that all translation invariant kernel methods can also use the larger class of cpd kernels. Therefore we give a short summary of this type of kernels and their connection to Hilbertian metrics.

Definition 2.1. A real valued function $K$ on $\mathcal{X} \times \mathcal{X}$ is positive definite (pd) (resp. conditionally positive definite (cpd) kernel) if and only if $K$ is symmetric and $\sum_{i,j} c_i c_j K(x_i, x_j) \geq 0$, for all $c_i \in \mathbb{R}, i = 1, \ldots, n$, and for all $c_i \in \mathbb{R}, i = 1, \ldots, n$ (resp. for all $c_i \in \mathbb{R}, i = 1, \ldots, n$ with $\sum_i c_i = 0$).

The following theorem describes the class of Hilbertian metrics:

Theorem 2.2. (I. J Schoenberg 1938)
A metric space $(\mathcal{X}, d)$ can be embedded isometrically into a Hilbert space if and only if $-d^2(x, y)$ is conditionally positive definite (cpd).
Lemma 1. (J. P. R. Christensen, C. berg and P. Ressel 1984)
Let $K$ be a kernel defined as $K(x,y) = \hat{k}(x,y) - \hat{k}(x,x_0) - \hat{k}(x_0,y) + \hat{k}(x_0,x_0)$, where $x_0 \in X$. Then $K$ is pd if and only if $\hat{k}$ is cpd.

Similar to pd kernel one can also characterize cpd kernel. Presently one can write all cpd kernel in the form: $K(x,y) = -\frac{1}{2}||\phi_x - \phi_y||_H^2 + f(x) + f(y)$. The cpd kernel corresponding to Hilbertian (semi)-metrics are characterized by $f(x) = 0$ for all $x \in X$, whereas if $K$ is pd it follows that $f(x) = \frac{1}{2}K(x,x) \geq 0$. We also would like to point out that for SVM the class of Hilbertian (semi)-metrics is more important than the class of pd kernels. Namely one can show, (see M. Hein and O. Bousquet 2003), which the solution are characterized by $f$.

Let $P$ and $Q$ be two probability measures on $\mathbb{R}$. Then $d$ on $\mathcal{M}_1^+(X)$ defined as

$$D^2_{\mathcal{M}_1^+}(P,Q) := \int_X d^2_{\mathbb{R}^+}(p(x),q(x))d\mu(x) \tag{1}$$

is a Hilbertian metrics on $\mathcal{M}_1^+(X)$. $D^2_{\mathcal{M}_1^+}(\cdot)$ is independent of the dominating measure $\mu$.

Proof 1. First we show by using the $1/2$-homogeneity of $d_{\mathbb{R}^+}$ is independent of the dominating measure $\mu$. We have

$$\int_X d_{\mathcal{M}_1^+}(\frac{dP}{d\mu}, \frac{dQ}{d\mu})d\mu = \int_X d_{\mathcal{M}_1^+}(dP/d\mu, dQ/d\mu) d\mu = \int_X d_{\mathcal{M}_1^+}(dP/d\nu, dQ/d\nu) d\nu$$

where we use that $d_{\mathbb{R}^+}$ is 1-homogeneous. It is easy that $-D^2_{\mathcal{M}_1^+}(X)$ is conditionally positive definite, simplicity take for every $n \in \mathbb{N}, P_1,\ldots, P_n$ the dominating measure $\sum_{i=1}^n \frac{P_i}{n}$ and use that $-D^2_{\mathbb{R}^+}$ is conditionally positive definite.

Remark 1. It is in principe very easy to build hilbertian metrics on $\mathcal{M}_1^+(X)$ using arbitrary Hilbertian metrics on $\mathbb{R}^+$ and plugging it into the definition 1.

But the key property of the method we propose is the independence of the metric $d$ on $\mathcal{M}_1^+(X)$ of the dominating measure. That is we have generated a metric which is invariant with respect to general coordinate transformations on $X$, therefore we call it a covariant metric.

2.3 $\lambda$-homogeneous Hilbertian Metrics and Positive Definite Kernels on $\mathbb{R}^+$

In this paper we consider the class of Hilbertian metrics on probability measure, therefore the Hilbertian metrics on $\mathbb{R}^+$ is the main element of our approche. This is the event we require that the Hilbertian metrics on $\mathbb{R}^+$ is $\lambda$-homogeneous. The class of $\lambda$-homogeneous Hilbertian metrics on $\mathbb{R}^+$ was characterized by Fuglede:

Definition 2.3. (Topsøe 2003) and B. Fuglede 2004)
A Hilbertian metrics si $\lambda$-homogeneous if and only if $d^2(cp, cq) = c^\lambda d^2(p, q)$ for all $c \in \mathbb{R}^+$. 

3
Theorem 2.4. (B. Fuglede 2004)

A symmetric function \( d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) with \( d(x, y) = 0 \iff x = y \) is a \( \gamma \)-homogeneous, continuous Hilbertian metrics \( d \) on \( \mathbb{R}_+ \) if and only if there exists a (necessarily unique) non-zero bounded measure \( \rho \geq 0 \) on \( \mathbb{R}_+ \) such that \( d^2 \) can be written as

\[
d^2(x, y) = \int_{\mathbb{R}_+} |x^{(\gamma+i\lambda)} - y^{(\gamma+i\lambda)}|^2 d\rho(\lambda)
\]  

(2)

Using lemma [1] we define the corresponding class of pd kernel on \( \mathbb{R}_+ \) by choosing \( x_0 = 0 \). We will see later that this corresponds to choosing the zero-measure as origin of the RKHS (reproducing kernel Hilbert space).

Corollary 1. A symmetric function \( k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) with \( k(x, y) = 0 \iff x = 0 \) is a \( 2\gamma \)-homogeneous continuous pd kernel \( K \) on \( \mathbb{R}_+ \) if and only if there exists a (necessarily unique) non-zero bounded symmetric measure \( \kappa \geq 0 \) on \( \mathbb{R} \) such that \( K \) is given as

\[
K(x, y) = \int_{\mathbb{R}} x^{(\gamma+i\lambda)} y^{(\gamma-i\lambda)} d\kappa(\lambda)
\]

(3)

Proof 2. If we have the form given in (2), then it is obviously \( 2\gamma \)-homogeneous and since \( K(x, y) = x^{2\gamma} \kappa(\mathbb{R}) \) we have \( K(x, y) = 0 \iff x = 0 \).

The other direction follows by first noting that \( K(0, 0) = \langle \phi_0, \phi_0 \rangle = 0 \) and then by applying theorem 2, where \( \kappa \) is the symmetrized version of \( \rho \) around the origin, together with lemma [2] and \( K(x, y) = \langle \phi_x, \phi_y \rangle = \frac{1}{2}(-d^2(x, y) + d^2(x, 0) + d^2(y, 0)) \). \( \square \)

Andrzej Cichocki, Sergio Cruces and Shun-ichi Amari (January 2011) proposed an interesting two-parameter family of metrics, the AB-divergence defined:

\textbf{Definition 2.5.} The function \( d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) defined as:

\[
d_{AB}^{(\alpha, \beta)}(x, y) = \begin{cases} 
-\frac{1}{\alpha \beta}(x^\alpha y^\beta - \frac{\alpha}{(\alpha + \beta)} x^{\alpha + \beta} - \frac{\beta}{(\alpha + \beta)} y^{\alpha + \beta}) & \text{for } \alpha, \beta, \alpha + \beta \neq 0 \\
\frac{1}{\alpha}(x^\alpha \log(\frac{y^\beta}{x^\beta}) - x^\alpha + y^\alpha) & \text{for } \alpha \neq 0, \beta = 0 \\
\frac{1}{\alpha}(\log(\frac{y^\beta}{x^\beta}) + (\frac{y^\beta}{x^\beta})^{-1} - 1) & \text{for } \alpha = -\beta \neq 0 \\
\frac{1}{\beta}(y^\beta \log(\frac{x^\alpha}{y^\alpha}) - y^\beta + x^\alpha) & \text{for } \alpha = 0; \beta \neq 0 \\
\frac{1}{2}(\log x - \log y)^2 & \text{for } \alpha, \beta = 0
\end{cases}
\]

(4)

where \( d_{AB}^{(\alpha, \beta)}(x, y) \) is a divergence on \( \mathbb{R}_+ \)

2.4 A brief recall of Support Vector Machine (SVM)

SVM were developed by Cortes and Vapnik (1995) for binary classification. Their approach may be roughly sketched as follows:

• class separation: basically, we are looking for the optimal separating hyperplane between the two classes by maximizing the margin between the classes closest points, the points lying on the boundaries are called support vectors, and the middle of the margin is our optimal separating hyperplane;

• nonlinearity: when we cannot find a linear separator, data points are projected into an (usually) higher-dimensional space where the data points effectively become linearly separable (this projection is realised via kernel techniques);

• problem solution: the whole task can be formulated as a quadratic optimization problem which can be solved by known techniques etc.

An implicit mapping \( \Phi \) was used by SVM of the input data into a high-dimensional feature space defined by a kernel function, the inner product \( \langle \Phi(x), \Phi(y) \rangle \) between the images of two data points \( x, y \) was returning by the function in the feature space. The kernel function can be represented as

\[
K(x, y) = \langle \Phi(x), \Phi(y) \rangle
\]

(5)
where $\Phi : X \rightarrow H$ is the projection function, this function project $x$ and $y$ into the feature space $H$.

Relationship between the kernel method and SVM: Schölkopf showed that the class of cpd kernel can be used in SVM due to the translation invariant of the maximal margin problem in the RKHS, and the kernel can be used in SVM to the classification, the regression etc if we found a good kernel function. The advantage of kernel method and SVM is that we can found and used a kernel for a problem particular that could be applied directly to data without the need for a feature extraction process. This was used in (M. Hein and O. Bousquet 2003) to show that the properties of the SVM only depend on the Hilbertian metrics. That is all cpd kernel are generated by a Hilbertian metric $d(x, y)$ through $K(x, y) = -d^2(x, y) + g(x) + g(y)$ where $g : X \rightarrow \mathbb{R}$ and the solution of the SVM only depends on the Hilbertian metric $d(x, y)$.

3 Main results

Generally the AB-metrics is not symmetric. We extend and improve this family an two-parameter symmetric. The metrics we propose is very interesting since it is a symmetric and smoothed variant from AB-metrics. This allows us to recover all proprety in Hilbertian metrics on $M^1_+(X)$ from the family of two-parameter.

Theorem 3.1. The function $d : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined as :

$$d^{\alpha,\beta}(x, y) = \begin{cases} 
\frac{1}{\alpha^2}(x^\alpha - y^\alpha)(x^\beta - y^\beta) & \text{for } \alpha, \beta \neq 0 \\
\frac{1}{\alpha}(x^\alpha - y^\alpha) \log \left( \frac{x^\alpha}{y^\alpha} \right) & \text{for } \alpha \neq 0, \beta = 0 \\
\frac{1}{\beta}(x^\beta - y^\beta) \log \left( \frac{x^\beta}{y^\beta} \right) & \text{for } \alpha = 0, \beta \neq 0 \\
\frac{1}{\beta}\left( (x^\alpha - y^\alpha) \log \left( \frac{x^\alpha}{y^\alpha} \right) + \left( \frac{x^\beta}{y^\beta} \right)^{-1} + \left( \frac{x^\beta}{y^\beta} \right)^{-1} - 2 \right) & \text{for } \alpha = -\beta \neq 0 \\
\frac{1}{2}(\log x - \log y) & \text{for } \alpha = \beta = 0 
\end{cases}$$

is a $\gamma$-homogeneous Hilbertian metrics on $\mathbb{R}^+$, note that $d^{(\alpha,\beta)}$ is symmetric.

Proof 3. The proof for the symmetry is trivial because this function is symmetric by construction and $d^{(\alpha,\beta)}(cx, cy) = c^{\alpha+\beta}d^{(\alpha,\beta)}(x, y)$ where $\gamma = \alpha + \beta$ then $d^{(\alpha,\beta)}$ is $\gamma$-homogeneous. Second for simplicity note that $K(x, y) = -d^2(x, y)$, where $d^2$ is a Hilbertian metrics. The all conditions for theorem of Schoenberg satisfied we have $-d^2$ cpd. □

We can now apply the principle to construct Hilbertian metrics on $M^1_+(X)$, of building Hilbertian metrics on $M^1_+(X)$ and use the family of $\gamma$-homogeneous Hilbertian metrics $d^{(\alpha,\beta)}_{ABS}$ on $\mathbb{R}^+$.

Definition 3.2. We proposed two ways to build the symmetry divergence:

**Thype-1 :**

$$D^{(\alpha,\beta)}_{ABS} = \frac{1}{2}[D^{(\alpha,\beta)}_{AB}(P, Q) + D^{(\alpha,\beta)}_{AB}(Q, P)]$$

**Type-2:**

$$D^{(\alpha,\beta)}_{ABS}(P, Q) = \frac{1}{2}[D^{(\alpha,\beta)}_{AB}(P, \frac{P + Q}{2}) + D^{(\alpha,\beta)}_{AB}(Q, \frac{P + Q}{2})]$$

For the construction ABS-divergence we used the definition above and we apply the proposition (1) from the section(2). We obtain the symmetric ABS-divergence (Type-1) defined as:

Definition 3.3. : Let $P$ and $Q$ two probability measures on $X$ (probability space) and $\mu$ an dominating
measure of P, Q and $d_{ABS}^{(α,β)}$ a γ-homogeneous Hilbertian metrics on $\mathbb{R}_+$, then $D_{ABS}^{(α,β)}$ defined as:

$$
D_{ABS}^{(α,β)}(P,Q) = \begin{cases}
\frac{1}{α} \int_X (p^α - q^α) (p^β - q^β) d\mu(x) & \text{for } α ≠ 0, β ≠ 0 \\
\frac{1}{α} \int_X (p^α - q^α) \log \left( \frac{p^α}{q^α} \right) d\mu(x) & \text{for } α ≠ 0, β = 0 \\
\frac{1}{α} \int_X ((p^α - q^α) \log \left( \frac{p^α}{q^α} \right) + \frac{q^α}{p^α} + \frac{p^α}{q^α} - 2) d\mu(x) & \text{for } α = -β ≠ 0 \\
\frac{1}{α} \int_X (p^β - q^β) \log \left( \frac{p^β}{q^β} \right) d\mu(x) & \text{for } α = 0, β ≠ 0 \\
\frac{1}{α} \int_X (\log x - \log y) d\mu(x) & \text{for } α = β = 0
\end{cases}
$$

(9)

is a γ-homogeneous Hilbertian metrics on $\mathcal{M}_1^+(X)$.

The ABS-divergence has the following basic properties:

Properties 1. :

1. Convexity: $D_{ABS}^{(α,β)}(P,Q)$ is convex with respect to both P and Q.
2. Strict Positvity: $D_{ABS}^{(α,β)}(P,Q) ≥ 0$ and $D_{ABS}^{(α,β)}(P,Q) = 0$ if and only if $P = Q$.
3. Continuity: The ABS-divergences is continuous function of real variant $(α, β)$ in the whole range including singularities.
4. Symmetric: $D_{ABS}^{(α,β)}(P,Q) = D_{ABS}^{(α,β)}(Q,P)$
5. γ-homogeneous: $D_{ABS}^{(α,β)}(cP,cQ) = c^{α+β}D_{ABS}^{(α,β)}(P,Q)$

We used the instrument of building Hilbertian metrics on $\mathcal{M}_1^+(X)$ and use the family of $(α + β)$-homogeneous Hilbertian metrics $d_{ABS}^{(α,β)}$ on $\mathbb{R}_+$. This yield as special case the following measures on $\mathcal{M}_1^+(X)$.

$$
D_{ABS}^{α,β}(P,Q) = \int_X ϕ_{α,β}(p(x)q(x)) d\mu(x)
$$

| Divergence | Function $ϕ_{α,β}(p(x)q(x))$ | Name |
|------------|-------------------------------|------|
| $D_{ABS}^{(1,1)}(P,Q)$ | $(p(x) - q(x))^2$ | Euclidian |
| $D_{ABS}^{(1/2,1)}(P,Q)$ | $2(\sqrt{p(x)} - \sqrt{q(x)})(p(x) - q(x))$ | $V_1$-Hellinger |
| $D_{ABS}^{(1/2,-1)}(P,Q)$ | $2(\sqrt{p(x)} - \sqrt{q(x)})(p(x) - q(x)) / (p(x)q(x))$ | $V_2$-Hellinger |
| $D_{ABS}^{(1/2,1/2)}(P,Q)$ | $4(\sqrt{p(x)} - \sqrt{q(x)})^2$ | Helliger |
| $D_{ABS}^{(1,0)}(P,Q)$ | $(p(x) - q(x)) \log \left( \frac{p(x)}{q(x)} \right)$ | Jeffrey |

Table (a): $D_{ABS}^{α,β}(P,Q)$ divergence

$D_{ABS}^{(1,1)}$ corresponds to the square of euclidian metric, $D_{ABS}^{(1/2,1/2)}$ corresponds to the Hellinger metric which is well known in the statistics community, $D_{ABS}^{(1,0)}$ correspond to the Jeffreys metric, $D_{ABS}^{(1/2,1)}$ $V_1$-Hellinger and $D_{ABS}^{(1/2,-1)}$ $V_2$-Hellinger is a variant of Hellinger metrics.

For completeness we also give the corresponding pd kernels on $\mathcal{M}_1^+(X)$, where we take in lemma the zero measure as $x_0 \in M_1^+(X)$. This choice seems strange at first since we are dealing with probability
measures. But in fact the whole framework presented in this paper can easily be extended to all finite, positive measure on $\mathcal{X}$. For this, set zero measure is a natural choice of the origin.

$$\mathcal{K}_{(1,1)}(P, Q) = \int_{\mathcal{X}} p(x)q(x)d\mu(x)$$

$$\mathcal{K}_{(1/2,1)}(P, Q) = \int_{\mathcal{X}} (q(x)\sqrt{p(x)} + p(x)\sqrt{q(x)})d\mu(x)$$

$$\mathcal{K}_{(1/2,1/2)}(P, Q) = 4 \int_{\mathcal{X}} \sqrt{p(x)q(x)}d\mu(x)$$

$$\mathcal{K}_{(1,0)}(P, Q) = \frac{1}{2} \left( \int_{\mathcal{X}} (q(x) - p(x))\log\left(\frac{p(x)}{q(x)}\right) - p(x) - q(x) \right)d\mu(x)$$

Using two-parameters is so difficult in practice that it's the reason why we will proposed a one-parameter family to improve the ABS-divergence.

**Proposition 2.** The function $d : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined as:

$$d_t^2(x, y) = \begin{cases} \frac{1}{2t}(x - y)^2 & \text{for } t \neq 0 \\ \frac{1}{2}(\log x - \log y) & \text{for } t = 0 \end{cases}$$

(10)

is a $2t$-homogeneous Hilbertian metric on $\mathbb{R}_+$ if $t \neq 0$.

**Proof 4.** Note that $d_t^2$ is symmetric by construction and it’s easy verified the property of Hilbertian metric. Therfore we show that: $d_t^2(cx, cy) = 1 - \frac{1}{t}((cx)^t - (cy)^t)^2 = \frac{e^{2t}}{2t}((x^t - y^t)^2 = e^{2t}d_t^2(x, y). □$

We used the proposition 1 of building Hilbertian metrics on $\mathcal{M}_1^+(\mathcal{X})$ and use the family of $2t$-homogeneous Hilbertian metrics $d_t^2$ on $\mathbb{R}_+$. Therefor we obtain as special case the following measures on $\mathcal{M}_1^+(\mathcal{X})$.

$$D_t^2(P, Q) = \int_{\mathcal{X}} \varphi_t(p(x)q(x))d\mu(x)$$

| Divergence $D_t^2(P, Q)$ | function $\varphi_t(p(x)q(x))$ | Name |
|---------------------------|---------------------------------|------|
| $D_1^2(P, Q)$            | $\frac{1}{2}(p(x) - q(x))^2$    | Euclidian |
| $D_{1,2}^2(P, Q)$        | $\frac{1}{2}\left(\frac{1}{p(x)} - \frac{1}{q(x)}\right)^2$ | S-Euclidian |
| $D_{1/2}^2(P, Q)$        | $2(\sqrt{p(x)} - \sqrt{q(x)})^2$ | Hellinger |
| $D_1^2(P, Q)$            | $2\left(\frac{1}{\sqrt{p(x)}} - \frac{1}{\sqrt{q(x)}}\right)^2$ | S-Itakura Saito |

Table (b): Divergence using $2t$-homogeneous Hilbertian metric

$D_t^2$ correspond to the square of euclidian metric, $D_{1,2}^2$ is an other version of euclidian metric, $D_{1/2}^2$ corresponds two Hellinger metric. The Hellinger metric is well known in the statistics community. $D_{1/2}^2$ is a symmetrized Itakura-Saito distance (called also the COSH distance) modified.

For completeness we also give the corresponding pd kernels on $\mathcal{M}_1^+(\mathcal{X})$, where we take in lemma 1 the zero measure as $x_0 \in \mathcal{M}_1^+(\mathcal{X})$. This choice seems strange at firts since we are dealing with probability measures. But in fact the whole framework presented in this paper can easily be extended to all finite, positive measures on $\mathcal{X}$. For this set zero measure is a natural choice the origin.

$$\mathcal{K}_{\frac{1}{2}}(P, Q) = \frac{1}{2} \int_{\mathcal{X}} \frac{x^ty^t}{t^2}d\mu$$

$$\mathcal{K}_{\frac{1}{2}}(P, Q) = \frac{1}{2} \int_{\mathcal{X}} p(x)q(x)d\mu(x)$$

7
\[ K_{-1}(P, Q) = \frac{1}{2} \int_X \frac{1}{p(x)q(x)} d\mu(x) \]

\[ K_{\frac{1}{2}}(P, Q) = 2 \int_X \sqrt{p(x)q(x)} d\mu(x) \]

\[ K_{\frac{1}{2}}(P, Q) = 2 \int_X \frac{1}{\sqrt{p(x)q(x)}} d\mu(x) \]

4 Numerical studies

To show the interest of our study the examples of applications have been proposed. For the SVM we made studies on the classification of the genes and on the sex of the cats knowing the weight of the heart and the body. As regards classification of images, we use our divergences to separate them into two classes.

4.1 Application in SVM

The performance of metrics and kernels has been compared in classification using some data sets. All data sets were split into a training (80\%) and a test (20\%) set. For the problem we use SVM method. For all experiments we use the one-parameter family \( d_t^2 \) of hilbertian metric, the corresponding kernel cpd is \( K = -D_t^2 \) with varying penalty constants \( C \) in the SVM, and we use the transformed kernel (gaussian transformation):

\[ K_t(P, Q) = e^{-D_t^2(P, Q)/2\sigma^2} \]

The test error was evaluated by the best parameters \( C \) and \( \sigma \). The best constant penalty \( C \) and \( \sigma \) was found by cross-validation.

We evaluated the performance of the proposed metrics and kernels in three classification tasks. Firstly we generated a artificial data and we consider the test error for kernels proposed. Secondly as a real-world application, let us test the ability of SVM to predict the class of a tumour from gene expression data. We use a publicly available datasets of gene expression data for 128 different individuals with acute lymphoblastic leukemia (ALL). Here we focus on predicting the type of the disease (B-cell or T-cell). Therefore we test a SVM classifier for cancer diagnosis from gene expression data, and we test the ability of a SVM to predict the class of the disease from gene expression. Finally we apply the data from support functions and datasets for Venables and Ripley’s MASS. We use the anatomical data from domestic cats, the heart and body weights of samples of male and female cats used for digitalis experiments. The cats were all adult, over 2 kg body weight. We presented the classification error according to the sex of the cats.

The tables shows the test errors for the kernels corresponding to \( t = \{-1; -1/2; 1/2; 1\} \) from the ABS-divergence resp. and their gaussian transformation. The first line shows the kernels directly (dir) and the second line the gaussian transformation (tran).

| Divergence | Euclidian error C \( \sigma \) | Hellinger error C \( \sigma \) | Itakura-Saito error C \( \sigma \) | S-Euclidian error C \( \sigma \) |
|------------|-------------------------------|-----------------------------|-------------------------------|-------------------------------|
| Data (dir) | 0.0001 10 -                   | 0.005 10 -                  | 0.26 100 -                    | 0.13 10 -                     |
| Artificial (tran) | 0.0001 10 1.5 | 0.005 10 0.5 | 0.125 100 1.5 | 0.17 100 1.5 |
| Table 1: Test error using data artificial |

Table 1 shows that the errors committed using the conditionally defined positive \( K \) and the kernel transformed \( K_t \) are in the same proportion for the Euclidean divergence and the Hellinger divergence with \( \sigma = 1.5 \) and the constant \( C = 10 \). While for the Itakura-saito divergence the errors committed in the classifications with the \( K_t \) kernel are smaller than that of the \( K \) kernel with \( \sigma = 1.5 \) and \( C = 100 \). So the classification with the transformed core is better than the one used directly. For the S-Euclidean divergence by varying the constant \( C, C = 10 \) for the kernel \( K \) and \( C = 100 \) for the kernel transforms \( K_t \), we find that the transformed nuclei offers a better classification result. Thus in all cases we notice that the transformed nuclei give the best results. For the sake of using the transformed nuclei for artificial data.

| Divergence | Euclidian error C \( \sigma \) | Hellinger error C \( \sigma \) | Itakura-Saito error C \( \sigma \) | S-Euclidian error C \( \sigma \) |
|------------|-------------------------------|-----------------------------|-------------------------------|-------------------------------|
| Data”ALL” (dir) | 0.679 10 -                   | 0.414 10 -                  | 0.257 10 -                    | 0.461 10 -                     |
| gene (tran) | 0.0234 10 0.5 | 0.156 100 1.5 | 0.0468 100 1.5 | 0.164 100 1.5 |
| Table 2: Test error using data “ALL” gene |
With the data gene (ALL), we find that the kernel transformed offers the best results for the classification. However with the use of kernels $K$ it is the kernel constructed with the divergence of Itakura-Saito that gives the best classification results, followed by Hellinger, S-Euclidean and the Euclidean divergence. Whereas if we use kernel transformed, it is those built with Euclidean divergence that give the best results, followed by that of Itakura-Saito, Hellinger and S-Euclidian.

| Divergence | Euclidian error C σ | Hellinger error C σ | Itakura-Saito error C σ | S-Euclidian error C σ |
|------------|---------------------|---------------------|-------------------------|----------------------|
| Data “MASS” (dir) | 0.236 1 - | 0.326 1 - | 0.340 1 - | 0.326 10 - |
| cats (tran) | 0.194 1 0.5 | 0.326 10 0.5 | 0.368 1 0.5 | 0.181 1 0.5 |

Table 3: Test error using data cats “MASS”

With the data cats (MASS), if we use kernel $K$ we have almost the same results for all measures of divergence. However with the modified kernel the classification obtained with the S-Euclidean divergence and the Euclidean divergence offer the best results. It should also be noted that for the divergence of Itakura-Saito it is the kernel $K$ that gives the best results. In conclusion we retain that the choice of the best method depends on data and constants.

4.2 Applications in imagess classifications

In this work we apply the above proposed algorithm and also implement with the metrics proposed in the section above, to color image segmentation. The divergence we proposed gives a good classification with different threshold ($k$). In our experiments we used the data from the image (a) and apply our algorithm with different threshold. From these figures, we can see the results experimented by our algorithm. These results have been obtained with our divergences following some ($k$) values.

---

**Algorithm**

$X \in M_{l,c}(\mathbb{R})$ matrix origin

$X' \in M_{l,c}(\mathbb{R})$ matrix sortie

$X' \leftarrow$ matrix null order $l \times c$

for $i$ rang(1, $l - 1$):

$P_1 \leftarrow X_{i,j-1}$

$P_2 \leftarrow X_{i-1,j}$

If $\text{norm}(P_1, P_2) < k$

$P_0 \leftarrow (225, 225, 225)$

else:

$P_0 \leftarrow (0, 0, 0)$

$X' \leftarrow P_0$
4.2.1 Facial image segmentation

Fig 1: Results facial image segmentation

In this part of the work we will make a brief presentation of the classification of an image. A presentation of the transformation with the corresponding divergence and the optimal threshold. The color image segmentation results is that: (a) corresponds to the original image, (b) image $D_{ABS}^{(1,0)}$-divergence with $k=2.4$, (c) image $D_{ABS}^{(1,1)}$-divergence with $k=1354$, (d) image $D_{ABS}^{(1/2,-1)}$-divergence and $k=2.5$, (e) image $D_{ABS}^{(1/2,-1)}$-divergence with $k=3.4$, (f) image $D_{ABS}^{(1,0)}$-divergence with $k=3.5$, (g) image $D_{ABS}^{(1/2,1/2)}$-divergence with $k=4.5$, (h) image $D_{ABS}^{(1,1)}$-divergence with $k=1350$, (c) image $D_{ABS}^{(1,1)}$-divergence with $k=1354$, (i) image $D_{ABS}^{(1/2,1/2)}$-divergence with $k=5.5$, (j) image $D_{ABS}^{(1/2,1)}$-divergence with $k=50$, (k) image $D_{ABS}^{(1/2,1)}$-divergence with $k=80$.

From these figures (Fig 1), we can observe the experimental results of our algorithm on a facial image. Using on divergences in the proposed algorithm, we can observe the separation into two classes of our image according to the metric used with an adequate threshold. We can see some differences between the segmentation results images (b-k). For example, the images (b) and (f) given by the $D_{ABS}^{(1,0)}$ divergence showed a bleary delimitation of the original image (a); for the images (c,e,h,l) we used (resp.) $D_{ABS}^{(1,1)}$, $D_{ABS}^{(1/2,-1)}$, $D_{ABS}^{(1,1)}$, and $D_{ABS}^{(1/2,1/2)}$ divergences they correctly delineates the contour of the image (a); for the images (d,g,j) the $D_{ABS}^{(1/2,-1)}$, $D_{ABS}^{(1/2,1/2)}$, and $D^{(1/2,1)}$ divergences results are relatively homogeneous and do a good work.
4.2.2 Fruit image segmentation

![Fruit image segmentation](image)

The images in Figure 2 correspond to a classification using the proposed algorithm. The classification uses our divergences with a certain decision threshold $k$. Image 1 corresponds to the original images. The image 2 corresponds to the classification using the divergence $D_{ABS}^{(1,1)}$ with a threshold $k = 0.1$, the image 3 is that of this divergence with the threshold $k = 0.5$. The images 4 and 5 correspond to the classification the divergence $D_{ABS}^{(\frac{1}{2}, \frac{1}{2})}$ with the thresholds $k = 0.1$ and $k = 0.5$ respectively. The classification made with the divergence $D_{ABS}^{(\frac{1}{2}, 1)}$ with the threshold $k = 0.1$ and $k = 0.5$ respectively corresponding to the 6 and 7. The images 8 and 9 are obtained with the divergence $D_{ABS}^{(\frac{1}{2}, -1)}$ with the thresholds $k = 0.1$ and $k = 0.5$. The images 10 and 11 correspond to the classification with the divergence $D_{ABS}^{(1,0)}$ with $k = 0.1$ and $k = 0.5$ respectively.

From these figures, we can observe the experimental results of our algorithm on a fruit image. Using on divergences in the proposed algorithm, we can observe the separation into two classes of our image according to the metrics used with an adequate threshold. For $k=0.1$ figures (2, 4, 6, 8, 10) present the better representation image than $k=0.5$ images (3, 5, 7, 9, 11). The fruits images representations depend two thing $k$ and the divergences. But in this case we have a few differences between our divergences.

5 Conclusion

A general method to build Hilbertian metrics on probability measures from Hilbertian on $\mathbb{R}_+$ was presented. Using results from Cichocki and Amari from the Alpha-Beta-divergence, then we generalized this framework by incorporating the symmetry property. We propose a new variant of Alpha-Beta-Symmetric divergence metrics (ABS-divergence) and kernels associated. Our main contributions consist of, first is to construct a new family of metrics, ABS-divergence and kernels, and second is to be integrated into SVM and algorithm classification. Our results, which are based on a choice of ABS divergence parameters leading to symmetric kernel $K$, are very efficient compared to classical $K$-based methods.

6 Acknowledgments

This research was supported, in part, by grants from NLAGA project, "Non linear Analysis, Geometry and Applications Project". (Supported by the University Cheikh Anta Diop UCAD).
References

[1] Andrzej Cichocki, Sergio Cruces, and Shun-ichi Amari, Log-Determinant Resivited: Alpha-Beta and Gamma log-det divergences, Entropy, vol. 17, pp. 2988-3034, 2015.

[2] Andrzej Cichocki, Sergio Cruces, and Shun-ichi Amari Generalized Alpha-Beta Divergences and Their Application to Robust Nonnegative Matrix Factorization. Entropy, vol. 13, pp. 134-170, 2011.

[3] Asa Ben-Hur and Willian Stafford Noble, Kernel Methods for Predicting protein-protein-interactions, vol. 21, pp. i38-i46, 2005.

[4] B. Frénay and M. Verleysen, Parameter-free kernel extreme learning for non-linear support vector regression, Neurocomputing, vol. 74, pp. 2526-2531, 2011.

[5] B. Fuglede, Spirals in Hilbert space, With an application in information theory, Expositiones Mathematicae, vol. 23, pp. 23-45, 2005.

[6] B. Scolkeropf and A. Smola, Learning with kernels, MIT Press, Cambridge, MA, 2002. Bharath K. Sriperumbudur and al, Hilbert space embeddings and metrics on probability measures, Journal of Machine Learning Research, vol. 11, pp. 1517-1561, 2010.

[7] B. Fuglede, Spirals in Hilbert space. With an application in information theory , TO appear in Expositiones Mathematicae (2004).

[8] Christopher M. Bishop, Pattern Recognition and Machine Learning, Springer 2006.

[9] D. B. Thiyam and al, Optimization of alpha-beta log-det divergences and their application in the spatial filtering of two class motor imagery movements, Entropy, vol. 19, pp. 89, 2017.

[10] D. Olszewski and B. Ster, Asymmetric clustering using the alpha-beta divergence, Pattern recognition, vol. 47, pp. 2014-2041, 2014.

[11] F. Topsoe, Jensen-shannon divergence and norm-based measures of discrimination and variation, Preprint (2003).

[12] Ho Chung Leon Law, Dougal J. Sutherland, Dino Sejdinovic and Seth Flanman, Bayesian Approches to distribution regression, Proceedings of the 21st International Conference on Artificial Intelligence and Statistics (AISTATS), Lamzorote, Spain. PMLR, vol. 84, 2018.

[13] I.J. Schoenberg, Metric space and positive definite function, Trans. Amer. Math. Soc, vol.44, pp.522-536, 1938.

[14] F. Amara, M. Fezari and H. Bouboura, An improved GMM-SVM system based on distance metric for voice pathology detection, Applied Mathematics and Information Sciences An International Journal, vol. 10, N. 3, pp. 1061-1070, 2016.

[15] I.W. Sumarjaya, A survey of kernel-type estimators for copula and their applications, Journal of physics, conf.series 893, 012027, 2017.

[16] J. Lafferty and G. Lebanon, Diffusion kernels on statistical manifolds, Journal of machine learning Research, vol. 3175, pp. 129-163, 2005.

[17] J. P. R. Christensen C. Berg and P. Ressel, Harmonic Analysis on Semigroups, Springer, New York , vol. 29, pp. 438, 1984.

[18] John Shawe-Taylor and Mello Cristianini, Kernel Methods for Pattern Analysis 2004. Krikamol Muandet and al, Kernel mean embedding of distribution, A review and Beyond, Foundations and trends in Mchine Learning, vol. 10, N. 1-2, pp. 1-141, 2017.

[19] M. Hein, T. N. Lal, and O. Bousquet, Hilbertian metrics on probability measures and thier application in SMVs, Acceptide at DAGM Springer , vol. 3175 , pp. 270-277, 2004.

[20] M. Hien, O. Bousquet, and B. Scholkopf, Maximal margin classification for metric spaces, Journal of Computer and System Sciences, vol. 71, pp. 333-359, 2005.
[21] O. Chappelle, P. Haffiner, and V. Vapnik, *SVMs for histogram-based image classification*, IEEE Transaction on Neural Networks, vol. 10, pp. 1055-1064, 1999.

[22] P. J. Moreno, P. P. Hu, and Vasconcelos, *A Kullback-Leibler divergence based kernel for SVM classification in multimedia application*, NIPS, vol. 16, 2003.

[23] T. Jebara and R. Kondor, *Bhattacharyya and expected likelihood kernels*, In 16th Annual Conference on Learning Theory (COLT), pp. 57-71, 2003.

[24] T. J. Abrahamsen, L. K. Hansen and O. Winther, *Kernel Methods for Machine Learning*, Technical University of Denmark (DTU), PHD, N. 299, 2013.

[25] V. Vapnik, *Statistical Learning Theory*, Wiley, New York 1998.