Relativistic quantum mechanics in the einbein field formalism

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Abstract

The system of two relativistic particles with einbein fields is quantized as a constrained system. A method of the introduction of the Newton–Wigner collective coordinate is discussed in presence of different gauge fixing conditions. Some arguments are involved in the favour of Lorentz covariant gauge fixing conditions.

1 Introduction

The motion of the spinless pointlike particle is described by the action

\[ S = \int_{\tau_1}^{\tau_f} Ld\tau, \quad L = -m\sqrt{\dot{x}^2}, \]  

(1.1)

where \( \dot{x}_\mu = \frac{\partial x_\mu}{\partial \tau} \), and \( \tau \) is the parameter specifying the position of the particle along its world line. The action (1.1) is invariant under reparametrization transformation \( \tau \rightarrow f(\tau) \), which makes the theory (1.1) be a constrained theory in the Dirac sense [1], or a gauge theory. It can be more convenient to rewrite the original Lagrange function (1.1) in a different way introducing

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the so-called einbein field $e$

$$L = -\frac{\dot{x}^2}{2e} - \frac{em^2}{2} = -\frac{\mu \dot{x}^2}{2} - \frac{m^2}{2\mu}, \quad \mu = \frac{1}{e}. \tag{1.2}$$

The reparametrization transformations of (1.3) have the form

$$x_\mu(\tau) \rightarrow x_\mu(f(\tau)), \quad f(\tau_i) = \tau_i, \quad f(\tau_f) = \tau_f \tag{1.3}$$

$$\mu(\tau) \rightarrow \mu(f(\tau))/\dot{f}(\tau)$$

There is a lot of reasons to deal with the Lagrange function (1.2) rather than (1.1), starting with the formal ones: the action in the form (1.2) appears naturally in the Feynman-Schwinger representation for the propagator of the relativistic particle [3], and can be most straightforwardly generalized for the case of spinning particle [2, 3]. Besides, the Lagrange function (1.2) is quadratic in velocities, so that the velocity can be expressed explicitly in terms of canonical momentum $p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}$. Another advantage is that while the Lagrange function (1.1) is singular in the limit $m \rightarrow 0$, the Lagrange function (1.2) is not, and it is possible to formulate the Hamiltonian approach for the massless particle.

The theories (1.1) and (1.2) are, of course, equivalent, and it can be easily verified by writing the Euler–Lagrange equation for the field $\mu$:

$$\frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{\mu}} - \frac{\partial L}{\partial \mu} = 0, \tag{1.4}$$

which gives $\mu = m/\sqrt{\dot{x}^2}$. Substituting it into the Lagrange function (1.2), one recovers the original form (1.1). There are two ways to arrange for the Hamiltonian setting of the theory (1.2). One may treat the einbein field as a dynamical variable, and to define the corresponding canonical momentum. Another way makes use of the fact that no time derivatives of $\mu$ enter the Lagrange function; then the standard procedure of canonical description is applied to the Lagrange function (1.2) with the condition $\mu = m/\sqrt{\dot{x}^2}$. If, however, one wishes to perform a transformation of coordinates which involves the field $\mu$, one is left with the first possibility only.

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1In what follows we work with the field $\mu$, referring to it as to einbein field; one should have in mind that the correct definition of einbein is $e = \frac{1}{\mu}$. 

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The simplest nontrivial example of such a situation is the problem of centre–of–mass motion separation in the system of two non–interacting relativistic particles, and the present paper is devoted to the canonical description of this system with einbein fields involved. The paper is organized as follows: in Section 2 we recollect the case of one particle in the einbein field formalism. In Section 3 the Hamilton function and constraints are written out for the case of two particles. Particular cases of the gauge fixing are described in the Subsections 3.1 and 3.2, and the concluding remarks are given in the final Section.

2 One free particle

The canonical momenta for the system (1.2) are defined as

\[ p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \pi = \frac{\partial L}{\partial \dot{\mu}} \]  

(2.1)

It follows from equation (2.1) that there is one primary constraint,

\[ \varphi_1 = \pi, \]  

(2.2)

and, in accordance with the general procedure [1], the Hamilton function is

\[ H = H_0 + \lambda \varphi_1, \quad H_0 = -\frac{1}{2\mu}(p^2 - m^2). \]  

(2.3)

The primary constraint (2.2) generates the secondary one as a consequence of the equation of motion,

\[ \varphi_2 = \{ \varphi_1 H \} = \frac{1}{2\mu^2}(p^2 - m^2), \]  

(2.4)

where the Poisson brackets are defined to be

\[ \{ AB \} = \frac{\partial A}{\partial p_\mu} \frac{\partial B}{\partial \dot{x}_\mu} - \frac{\partial A}{\partial \dot{x}_\mu} \frac{\partial B}{\partial p_\mu} + \frac{\partial A}{\partial \pi} \frac{\partial B}{\partial \dot{\mu}} - \frac{\partial A}{\partial \dot{\mu}} \frac{\partial B}{\partial \pi}. \]  

(2.5)

It may be easily checked that no other constraints appear, and one has

\[ \{ \varphi_1 \varphi_2 \} = 0, \]  

(2.6)
i.e. the constraints $\varphi_1$ and $\varphi_2$ are the first class ones. The presence of the first class primary constraint $\varphi_1$ means that there is a gauge freedom associated with the reparametrization symmetry (1.3). As there are no second class constraints, and the Poisson brackets for the variables $x_\nu$ and $p_\mu$ have the canonical form, $\{p_\mu x_\nu\} = g_{\mu\nu}$, one may proceed in the way suggested by Dirac [1]: to postulate the operator valued commutator

$$\hat{[p_\mu, x_\nu]} = -i g_{\mu\nu}, \quad \hat{p}_\mu = -i \frac{\partial}{\partial x_\mu}$$

and obtain, without imposing any gauge fixing conditions, the Klein–Gordon equation

$$(p^2 - m^2)\Psi = 0,$$  \hspace{1cm} (2.8)

which follows from the constraint $\varphi_2$ treated as a weak one, or as the equation for the wave function of the system.

Alternatively, the gauge can be fixed to establish the scale for $\tau$. For example, the time–like gauge can be fixed by imposing the additional primary constraint

$$\varphi_3 = x_0 + \tau,$$  \hspace{1cm} (2.9)

identifying in such a way the evolution parameter $\tau$ with the proper time of the particle.

It proves more convenient [1] to make the canonical transformation of the variables, so that

$$x'_0 = x_0 + \tau$$  \hspace{1cm} (2.10)

and to impose the above constraint in the form

$$\varphi_3 = x'_0$$  \hspace{1cm} (2.11)

clearly getting rid of explicit dependence on time.

One can easily find the corresponding partition function to be

$$F(x, p', \tau) = p'_\mu (x_\mu + \tau g_{\mu 0}),$$  \hspace{1cm} (2.12)

and the modified Hamilton function becomes

$$H' = H + \frac{\partial F}{\partial \tau} = H + p'_0.$$  \hspace{1cm} (2.13)
With constraints $\varphi_3$ and its secondary partner $\varphi_4 = \{\varphi_3 H'\}$ we arrive at the set of four constraints, all of the second class, whereas the physical Hamilton function takes the familiar form

$$H' = \sqrt{p^2 + m^2}$$

(2.14)
on the constraints surface.

3 Hamilton function and constraints in the system of two particles

The Lagrange function for two non-interacting particles,

$$L = \frac{\mu_1 x_1^2}{2} - \frac{m_1^2}{2\mu_1^2} - \frac{\mu_2 x_2^2}{2} - \frac{m_2^2}{2\mu_2^2},$$

(3.1)
is invariant under two independent reparametrization transformations of the kind (1.3). To separate the centre–of–mass motion we introduce the new variables:

$$x_\mu = x_{1\mu} - x_{2\mu}, \quad X_\mu = \frac{\mu_1}{\mu_1 + \mu_2} x_{1\mu} + \frac{\mu_2}{\mu_1 + \mu_2} x_{2\mu},$$

$$\zeta = \frac{\mu_1}{\mu_1 + \mu_2}, \quad M = \mu_1 + \mu_2.$$  

(3.2)

In terms of these new variables the Lagrange function becomes

$$L = -\frac{m_1^2}{2M\zeta} - \frac{m_2^2}{2M(1 - \zeta)} - \frac{1}{2} M \left( X - \zeta x \right)^2 - \frac{1}{2} M \zeta (1 - \zeta) \dot{x}^2$$

(3.3)

The canonical momenta, defined as

$$P_\mu = \frac{\partial L}{\partial \dot{X}_\mu}, \quad p_\mu = \frac{\partial L}{\partial \dot{x}_\mu},$$

$$\Pi = \frac{\partial L}{\partial \dot{M}}, \quad \pi = \frac{\partial L}{\partial \dot{\zeta}},$$

(3.4)
give the Hamilton function

$$H = H_0 + \Lambda \Pi + \lambda \pi,$$
\[ H_0 = \frac{\varepsilon_1^2}{2M\zeta} + \frac{\varepsilon_2^2}{2M(1 - \zeta)} - \frac{P^2}{2M}, \quad (3.5) \]
\[ \varepsilon_1^2 = m_1^2 - p^2, \quad \varepsilon_2^2 = m_2^2 - p^2, \]
and primary constraints
\[ \varphi_1 = \Pi, \quad \varphi_2 = \pi + (P_x). \quad (3.6) \]

Note that, as the transformation (3.2) mixes the space variables \( x_1, x_2 \) and einbein fields \( \mu_1, \mu_2 \), the Lagrange function (3.3) contains explicitly the time derivative of the variable \( \zeta \), and the constrain \( \varphi_2 \) is not as trivial as \( \varphi_1 \). On the other hand, the transformation (3.2) involves the coordinates only; therefore the Poisson brackets are canonical:
\[ \{AB\} = \frac{\partial A}{\partial P_\mu} \frac{\partial B}{\partial X_\mu} - \frac{\partial A}{\partial X_\mu} \frac{\partial B}{\partial P_\mu} + \frac{\partial A}{\partial x_\mu} \frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu} \frac{\partial B}{\partial x_\mu} + \frac{\partial A}{\partial \Pi} \frac{\partial B}{\partial M} - \frac{\partial A}{\partial M} \frac{\partial B}{\partial \Pi} + \frac{\partial A}{\partial \pi} \frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial \pi}. \quad (3.7) \]

Using (3.7), the secondary constraints are calculated as
\[ \varphi_3 = \{\varphi_1 H\} = -\frac{\varepsilon_1^2}{2M^2\zeta} - \frac{\varepsilon_2^2}{2M^2(1 - \zeta)} + \frac{P^2}{2M^2}, \quad (3.8) \]
\[ \varphi_4 = \{\varphi_2 H\} = -\frac{\varepsilon_1^2}{2M^2\zeta^2} + \frac{\varepsilon_2^2}{2M(1 - \zeta)^2} + \frac{(pP)}{M\zeta(1 - \zeta)}. \]

No other constraints arise, and since the constraints \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \) commute with each other they are the first class ones. As there are two independent gauge invariances, we have two primary first class constraints \( \varphi_1 \) and \( \varphi_2 \).

As the set of constraints contains physical and non–physical variables in the mixed way, it is inconvenient to proceed \( a’ la \) Dirac without gauge fixing at all and treating the constraints as weak conditions. In the next section we describe two of many possibilities of gauge fixing.

### 3.1 Time–like gauge

The time–like gauge in both variables, \( x_0 \) and \( X_0 \), can be fixed in the way similar to that used at the end of Section 2. The additional gauge fixing
constraints we are to impose have the form
\[ \varphi_5 = X_0 + \tau, \quad \varphi_6 = x_0 \] (3.1.1)
but the canonical transformation is needed only with respect to \( \varphi_5 \) as \( \varphi_6 \) does not contain time from the very beginning.

Primary constraints (3.1.1) give rise to a couple of secondary ones,
\[ \varphi_7 = \{ \varphi_5 H \} = \frac{P_0}{M} - 1, \quad \varphi_8 = \{ \varphi_6 H \} = \frac{p_0}{M \zeta (1 - \zeta)}, \] (3.1.2)
composing the set of eight second class constraints together with \( \varphi_1, \varphi_2, \varphi_3 \) and \( \varphi_4 \).

The Hamilton function becomes
\[ H = M = P_0 \] (3.1.3)
on the constraints surface.

The most economic way to exclude from the consideration the non–physical variables, present in the theory because of the second class constraints, is to evaluate these variables from the corresponding constraints, to substitute them into the rest of constraints and into the Hamilton function and to change the Poisson brackets for the physical variables for the so–called Dirac brackets defined as
\[ \{ AB \}^* = \{ AB \} - \sum_a \{ A \varphi_a \} C^{-1}_{ab} \{ \varphi_b B \}, \] (3.1.4)
where \( C^{-1} \) is the inverse matrix with respect to \( C \) constructed from the Poisson brackets of the second class constraints being excluded:
\[ C_{ab} = \{ \varphi_a \varphi_b \}. \] (3.1.5)

The principle of correspondence will read now:
\[ \{ AB \}^* = -i[AB], \] (3.1.6)
where \([AB]\) indicates the quantum commutator.

\[ ^2 \]Hereafter primes at the new variables and the Hamilton function are omitted for simplicity
Choosing as the physical variables the space components of the coordinates and momenta we arrive at the following set of Dirac brackets:

\[
\{ P_i X_k \}^* = \delta_{ik} \\
\{ X_i p_k \}^* = \frac{p_i P_k}{M^2} \\
\{ X_i X_k \}^* = \frac{x_i p_k - x_k p_i}{M^2} \\
\{ X_i x_k \}^* = \frac{x_i P_k}{M^2} + \frac{x_j p_k}{M^2} \left( \frac{1 - \zeta}{\zeta} - \frac{\zeta}{1 - \zeta} \right) \\
\{ p_i x_k \}^* = \delta_{ik} - \frac{P_i P_k}{M^2} - \frac{P_j p_k}{M^2} \left( \frac{1 - \zeta}{\zeta} - \frac{\zeta}{1 - \zeta} \right) \\
\{ P_i P_k \}^* = \{ P_i p_k \}^* = \{ P_i x_k \}^* = \{ p_i p_k \}^* = \{ x_i x_k \}^* = 0
\]  

(3.1.7)

It is convenient to define the spin variable as

\[
S_{ik} = x_i p_k - x_k p_i
\]  

(3.1.8)

Then the following brackets can be obtained from (3.1.7):

\[
\{ X_i X_k \}^* = \frac{S_{ik}}{M^2} \\
\{ X_i S_{kl} \}^* = \frac{1}{M^2} (P_k S_{il} + P_l S_{ki}) \\
\{ S_{ik} S_{mn} \}^* = S_{in} \left( \delta_{km} - \frac{P_k P_m}{M^2} \right) + S_{km} \left( \delta_{in} - \frac{P_i P_m}{M^2} \right) + S_{mi} \left( \delta_{kn} - \frac{P_k P_m}{M^2} \right) + S_{nk} \left( \delta_{im} - \frac{P_i P_m}{M^2} \right)
\]  

(3.1.9)

The brackets (3.1.7) and (3.1.9) are noncanonical, and the Hamilton function (3.1.3) on the constrains surface in terms of \( \vec{p} \) and \( \vec{P} \) is

\[
H = \sqrt{\vec{P}^2 + E_1^2 + E_2^2 + 2\sqrt{E_1^2 E_2^2} + (\vec{p} \vec{P})^2}
\]  

(3.1.10)
where $\mathcal{E}_i = \sqrt{m_i^2 + \vec{p}^2}$, $(i = 1, 2)$.

The set of brackets (3.1.9) is well-known [5, 6, 7]; to bring it into the canonical form the Newton–Wigner variables should be defined:

$$Q_i = X_i - \frac{S_{ik} P_k}{E(M + E)}, \quad E = \sqrt{M^2 - \vec{P}^2}$$

$$J_{ik} = r_i k_k - r_k k_i$$

which commute as

$$\{Q_i Q_k\}^* = 0$$

$$\{Q_i J_{kl}\}^* = \{J_{ik} J_{mn}\}^* = \{J_{ik} J_{mn}\}^* = 0$$

where the proper internal variables are

$$\vec{k} = \vec{p} + \frac{(\vec{p} \vec{P}) \vec{P}}{E(M + E)}$$

$$\vec{r} = \vec{x} + \frac{(\vec{x} \vec{P}) \vec{P}}{E(M + E)} + \frac{(\vec{x} \vec{P}) \vec{k}}{E \omega_1 \omega_2} \left( \omega_1 - \omega_2 - \frac{(\vec{k} \vec{P})}{M} \right)$$

$$M = \sqrt{\vec{P}^2 + (\omega_1 + \omega_2)^2}, \quad \omega_1 = \sqrt{m_1^2 + \vec{k}^2}, \quad \omega_2 = \sqrt{m_2^2 + \vec{k}^2}$$

with the following Dirac brackets:

$$\{k_i r_k\}^* = \delta_{ik}$$

$$\{k_i k_j\}^* = \{r_i r_k\}^* = \{k_i Q_k\}^* = \{r_i Q_k\}^* = 0$$

The physical Hamilton function expressed in terms of the variable $\vec{k}$ from (3.1.13) takes the form:

$$H = \sqrt{\vec{P}^2 + \left( \sqrt{m_1^2 + \vec{k}^2} + \sqrt{m_2^2 + \vec{k}^2} \right)^2};$$
so with the brackets (3.1.12) and (3.1.14) the system is quantized setting
\( \hat{P}_i = -i \frac{\partial}{\partial Q_i} \) and \( \hat{k}_i = -i \frac{\partial}{\partial r_i} \).

The internal momentum \( \vec{k} \) can be expressed in terms of the single particle variables \( \vec{p}_1 \) and \( \vec{p}_2 \) with the result familiar from textbooks (see e.g. [8]):

\[
\vec{k} = \vec{p}_1 - \frac{\vec{P} \omega_1 (\vec{p}_1)}{E} + \frac{(\vec{P} \vec{p}_1) \vec{P}}{E (E + M)}, \quad \vec{P} = \vec{p}_1 + \vec{p}_2 \tag{3.1.16}
\]

If one deals with 3–dimensional momenta \( \vec{p}_1 \) and \( \vec{p}_2 \), then the substitution (3.1.16) for the relative momentum \( \vec{k} \) is the only way to separate the centre–of–mass motion and to obtain the Hamilton function in the form (3.1.15).

### 3.2 \((P_x) = 0\) gauge

In this subsection we present another, in our opinion more elegant way of gauge fixing. Instead of (3.1.1) we impose the constraint

\[
\varphi_5 = (P_x), \tag{3.2.1}
\]

which is manifestly covariant. The secondary constraint generated by primary constraint (3.2.1) is

\[
\varphi_6 = \frac{(pP)}{M \zeta (1 - \zeta)} \tag{3.2.2}
\]

and is also a covariant one.

With this gauge fixing constraints \( \varphi_5 \) and \( \varphi_6 \) the subset \( \{ \varphi_2, \varphi_4, \varphi_5, \varphi_6 \} \) is a second class one, while \( \varphi_1 \) and \( \varphi_3 \) remain commuting with anything else, and are still the first class ones. It offers the possibility to define the preliminary Dirac brackets as

\[
\{AB\}' = \{AB\} - \sum_{a,b} \{A\varphi_a\} C^{-1}_{ab} \{\varphi_b B\}, \tag{3.2.3}
\]

where

\[
C_{ab} = \{ \varphi_a \varphi_b \},
\]

and indices \( a \) and \( b \) take the value 2, 4, 5, 6 only. As now constraints \( \varphi_4 \) and \( \varphi_6 \) are of the second class, then at the constraints surface one has

\[
\zeta = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}, \tag{3.2.4}
\]
the condition that holds strongly. Similarly, the condition \( \pi = 0 \), which follow from the second class constraints \( \varphi_2 \) and \( \varphi_5 \), is also a strong one, and we eliminate these variables.

Defining the spin variable \( S_{\mu\nu} = x_{\mu}p_{\nu} - x_{\nu}p_{\mu} \) we have the following set of preliminary Dirac brackets (3.2.3):

\[
\{ P_\mu X_\nu \}' = g_{\mu\nu} \\
\{ X_\mu X_\nu \}' = \frac{S_{\mu\nu}}{P^2} \\
\{ P_\mu P_\nu \}' = 0 \\
\{ X_\mu S_{\alpha\beta} \}' = \frac{1}{P^2} (P_\beta S_{\alpha\mu} + P_\alpha S_{\mu\beta}) \\
\{ S_{\mu\nu} S_{\alpha\beta} \}' = S_{\mu\beta} \left( g_{\nu\alpha} - \frac{P_\nu P_\alpha}{P^2} \right) + S_{\alpha\mu} \left( g_{\nu\beta} - \frac{P_\nu P_\beta}{P^2} \right) \\
\quad + S_{\beta\nu} \left( g_{\mu\alpha} - \frac{P_\mu P_\alpha}{P^2} \right) + S_{\nu\alpha} \left( g_{\mu\beta} - \frac{P_\mu P_\beta}{P^2} \right)
\]

and

\[
\{ X_\mu p_\nu \}' = \frac{p_\mu P_\nu}{P^2} \\
\{ X_\mu x_\nu \}' = \frac{x_\mu P_\nu}{P^2} \\
\{ p_\mu x_\nu \}' = g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \\
\{ P_\mu p_\nu \}' = \{ P_\mu x_\nu \}' = \{ p_\mu p_\nu \}' = \{ x_\mu x_\nu \}' = 0
\]

The set of Dirac brackets (3.2.5) is typical for the systems obeying the condition \( S_{\mu\nu} P_\nu = 0 \) [5, 7], the latter being conveniently provided by constraint (3.2.3) and its conjugated partner (3.2.2). These brackets are noncanonical but covariant, and there exists a rather smart way to make them canonical [9].

To this end we, following the method of [9], define the tetrade of vectors
associated with the four–vector $P_\mu$ as

$$e_{0\mu} = \frac{P_\mu}{\sqrt{P^2}}, \quad e_{i\mu}e_{j\mu} = -\delta_{ij}, \quad e_{0\mu}e_{j\mu} = 0,$$  \hspace{1cm} (3.2.7)

and introducing the Cristoffel symbols, which define the transport of the tetrade,

$$\Gamma_{ij\alpha} = e_{i\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu}, \quad \Gamma_{0j\alpha} = e_{0\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu}. \hspace{1cm} (3.2.8)$$

Note that here the indeces $i, j$ are the tetrade, not the three–vector ones.

It can be shown that the quantity

$$Z_\mu = X_\mu + \frac{1}{2} S_{ij} \Gamma_{ij\mu},$$  \hspace{1cm} (3.2.9)

where $S_{ij} = e_{i\mu}e_{j\nu} S_{\mu\nu}$, is a properly commuting variable:

$$\{Z_\mu Z_\nu\}' = 0$$  \hspace{1cm} (3.2.10)

The proper internal variables are defined as the tetrade components of

$$x_\mu, \quad p_\mu,$$  \hspace{1cm} (3.2.11)

and commute in the canonical way:

$$\{x_i x_j\}' = \{p_i p_j\}' = \{Z_\mu x_i\}' = \{Z_\mu p_i\}' = \{Z_\mu S_{ik}\}' = 0$$  \hspace{1cm} (3.2.12)

The details of the tetrade formalism which allow to prove the formulae (3.2.10) and (3.2.12) are given in the Appendix.

We are still left with two first class constraints, $\varphi_1$ and $\varphi_3$, but now, having the Dirac brackets for the variables $P_\mu, Z_\mu, x_i$ and $p_i$ in the canonical form, we can treat the trajectory constraint which follows from the constraint $\varphi_3$ with account of (3.2.4)

$$P^2 - M^2 = 0, \quad M^2 = \left( \sqrt{m_1^2 + p_i^2} + \sqrt{m_2^2 + p_i^2} \right)^2,$$  \hspace{1cm} (3.2.13)
as a weak condition, and apply the Dirac quantization procedure arriving at
the Klein–Gordon equation for the wave function:
\[
\left( \hat{P}^2 - M^2(\hat{p}_i) \right) \Psi = 0,
\]
where \( \hat{P}_\mu = -i \frac{\partial}{\partial Z_\mu} \), \( \hat{p}_i = -i \frac{\partial}{\partial x_i} \).

As the internal gauge was fixed by means of covariant condition (3.2.1),
no surprise that the generator of Lorentz transformations
\[
M_{\mu\nu} = X_{\mu} P_{\nu} - X_{\nu} P_{\mu} + S_{\mu\nu} =
Z_{\mu} P_{\nu} - Z_{\nu} P_{\mu} - \frac{1}{2} S_{ij}(\Gamma_{ij\mu} P_{\nu} - \Gamma_{ij\nu} P_{\mu}) + S_{\mu\nu} \tag{3.2.15}
\]
commutes in the proper way
\[
\{ M_{\mu\nu} M_{\alpha\beta} \}' = g_{\nu\beta} M_{\mu\alpha} - g_{\nu\alpha} M_{\mu\beta} + g_{\mu\beta} M_{\alpha\nu} - g_{\mu\alpha} M_{\beta\nu} \tag{3.2.16}
\]
so that the system of the preliminary Dirac brackets is Poincaré–covariant
at the classical as well as quantum level, while the coordinate \( Z_\mu \) is not a
four–vector, as it follows from (3.2.15).

One can proceed further, and fix the gauge in the centre–of–mass variables also. For example, the time–like gauge is fixed as described in Section 2.
Straightforward but rather irksome calculations lead to the following expres-
sion for the Newton–Wigner coordinate expressed already not through the
tetrad components of spin but through convenient 3–dimensional ones:
\[
Z_i = X_i + \frac{S_{ik} P_k}{H(E + H)}, \quad H = \sqrt{E^2 + \vec{P}^2} \tag{3.2.17}
\]
where \( E \) is the energy of the system in the centre–of–mass frame. The form
(3.2.17) is the correct of the Newton–Wigner variable for a system gauged
by the condition \( S_{\mu\nu} P_\nu = 0 \) \cite{5, 7}. Note that this Newton–Wigner coordinate
certainly coincides with that defined in \( \text{(3.1.13)} \), but to see it explicitly one
is to express them both in terms of the same variables, \( \vec{x}_1, \vec{x}_2, \vec{p}_1 \) and \( \vec{p}_2 \) for
example (see e.g. \cite{7}).
4 Discussion

A pragmatically reader may ask the question: why to bother with powerful machinery of Dirac brackets to obtain the result familiar from the first’s year textbooks? Apart from general consideration of inner beauty, there might be more practical reasons in using the formalism of einbein fields.

Indeed, in relativistic quantum mechanics the problem of gauge fixing is closely connected to the problem of eliminating of the relative time variable whatever it means. We have demonstrated that the einbein formalism provides the natural environment for solving this problem. It can be done in more standard way by defining the three–dimensional Newton–Wigner variables (3.1.1), as well as in more sophisticated way of introducing the covariant analogue of these variables (3.2.9). The latter procedure allows to retain the Poincaré–invariance of the theory even after gauge fixing, and one cannot overestimate this feature. The method suggested can be generalized for the case of interacting particles, in particular when the interaction depends on velocities.

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Appendix

For reference purpose we collect here the formulae from the paper [9] relevant for the definition of the variable $Z_\mu$ and calculation of the Dirac brackets. It follows from equation (3.2.8) that the tetrade vectors $e_{i\mu}$ and $e_{0\mu}$ satisfy the relations

$$\frac{\partial e_{i\mu}}{\partial P_\alpha} = \Gamma_{ij\alpha} e_{j\mu} - \Gamma_{i0\alpha} e_{0\mu},$$

$$\frac{\partial e_{0\mu}}{\partial P_\alpha} = \Gamma_{0i\alpha} e_{i\mu}.\quad (A.1)$$

Treating these relations as a system of equations for $e_{i\mu}$ and $e_{0\mu}$, one obtains:

$$\frac{\partial \Gamma_{0i\alpha}}{\partial P_\beta} - \frac{\partial \Gamma_{0i\beta}}{\partial P_\alpha} + \Gamma_{ij\beta} \Gamma_{0i\alpha} - \Gamma_{ij\alpha} \Gamma_{0i\beta} = 0$$

$$\frac{\partial \Gamma_{jm\beta}}{\partial P_\alpha} - \frac{\partial \Gamma_{jma}}{\partial P_\beta} - \Gamma_{ij\alpha} \Gamma_{jm\beta} + \Gamma_{ij\beta} \Gamma_{jma} + \Gamma_{i0\alpha} \Gamma_{0m\beta} - \Gamma_{ij\beta} \Gamma_{0ma} = 0\quad (A.2)$$

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as the conditions under which the system (A.1) has nontrivial solutions. The tetrade vectors depend only on $P_\mu$, and therefore

$$\{X_\mu e_{i\nu}\}' = -\frac{\partial e_{i\nu}}{\partial P_\mu}$$

$$\{X_\mu e_{0\nu}\}' = -\frac{\partial e_{0\nu}}{\partial P_\mu}$$

(A.3)

$$\{X_\mu \Gamma_{ij\nu}\}' = -\frac{\partial \Gamma_{ij\nu}}{\partial P_\mu}$$

With the help of commutators (3.2.5) the relations (A.1) – (A.3) give formulae (3.2.10) and (3.2.12).

The explicit choice of the tetrade is not unique as it is clearly seen from its definition. In the given reference frame the tetrade may be chosen, for example, as

$$e_{0\mu} = \frac{P_\mu}{\sqrt{P^2}}, \quad e_{i0} = \frac{P_i}{\sqrt{P^2}}, \quad e_{ik} = \delta_{ik} + \frac{P_i P_k}{\sqrt{P^2}(P_0 + \sqrt{P^2})}$$

(A.4)

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