NEGATIVE CORRELATION OF ADJACENT BUSEMANN INCREMENTS

IAN ALEVY AND ARJUN KRISHNAN

ABSTRACT. We consider i.i.d. last-passage percolation on \( \mathbb{Z}^2 \) with weights having distribution \( F \) and time-constant \( g_F \). We provide an explicit condition on the large deviation rate function for independent sums of \( F \) that determines when some adjacent Busemann function increments are negatively correlated. As an example, we prove that Bernoulli(\( p \)) weights for \( p > p^* \), (\( p^* \approx 0.6504 \)) satisfy this condition. We prove this condition by establishing a direct relationship between the negative correlations of adjacent Busemann increments and the dominance of the time-constant \( g_F \) by the function describing the time-constant of last-passage percolation with exponential or geometric weights.

CONTENTS

1. Introduction 1
2. Main Results 4
3. Proofs 7
Appendix A. Conditions implying the existence of Busemann functions 17
Appendix B. Busemann correlations and the KPZ relationship 17
References 19

1. INTRODUCTION

Directed last passage percolation (LPP) is a model for random growth on a directed graph. In this paper we focus on the directed nearest-neighbor lattice \( \mathbb{Z}^2 \), with non-negative i.i.d. vertex weights \( \{\omega_x\}_{x \in \mathbb{Z}^2} \). We say that \( x \prec y \) if \( x_i \leq y_i \) for \( i = 1, 2 \), in which case \( x \) and \( y \) can be connected by an up/right path: this is a sequence of vertices \( \Gamma = \{x = x_0, x_1, \ldots, x_k = y\} \) in which each step is either right or up; i.e., \( x_{i+1} - x_i \in \{e_1, e_2\} \), the canonical unit directions in \( \mathbb{Z}^2 \). The passage-time of \( \Gamma \) is

\[ W(\Gamma) = \sum_{x \in \Gamma} \omega_x. \]

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The last-passage time from \( x \) to \( y \) \((x < y)\) is
\[
G(x, y) = \sup_{\Gamma} W(\Gamma),
\]
where the supremum is over all up/right paths from \( x \) to \( y \). A geodesic is an up/right path between points \( x \) and \( y \) that achieves the supremum in (1). An infinite geodesic is an up/right path that is a geodesic between any two points on it.

If \( \omega_x \) is in \( L^1 \), the classical subadditive ergodic theorem ensures the existence of the so-called time-constant [1, 2], which is the limit
\[
\lim_{n \to \infty} \frac{G(0, [nx])}{n} = g_F(x) \quad \text{a.s and in } L^1
\]
for each \( x \in \mathbb{R}^2 \), where \([y]\) is the only lattice point in \([y, y+1)^2\). The time-constant is a 1-homogeneous, concave function which respects the symmetries of the lattice (e.g. \( g(y, x) = g(x, y) \)). If \( E[w_x^2] < \infty \), then \( g(x) \) is finite for all \( x \in \mathbb{R}^2 \) [2, 1]; see also [3] for a slightly weaker sufficient condition. The limit-shape is the level set \( B = \{ x : g(x) = 1 \} \). For general i.i.d. weights, \( g(x) \) is poorly understood.

Last-passage percolation with exponential or geometrically distributed i.i.d. vertex weights are the only known integrable or solvable cases. Let the weights have mean \( m \) and variance \( \sigma^2 \). Then, the limit shape for both exponential and geometric weights is given by [4]
\[
g_{\text{Exp}}(x, y) = m(x + y) + 2\sigma \sqrt{xy} \quad \forall x, y \in \mathbb{R}^2.
\]
When the weights are exponentials, of course, we must have \( m = \sigma \), but we will use \( g_{\text{Exp}}(x, y) \) to simply mean the function on the right hand side of (2) with parameters \( m \) and \( \sigma \). Martin [5] showed that this shape is asymptotically universal close to the vertical and horizontal axes:
\[
g_F(1, s) = m + 2\sigma \sqrt{s} + o(\sqrt{s}) \quad \text{as } s \to 0.
\]
The random fluctuations of the last-passage time are known to be in the KPZ universality class [6] for growth models, since Johansson [7] proved that
\[
\lim_{N \to \infty} \mathbb{P}\left( \frac{G(0, [Nx]) - Ng_{\text{Exp}}(x)}{c(x)N^\chi} \leq t \right) = F_{\text{GUE}}(t),
\]
where \( \chi = 1/3 \), \( c(x) \) is an explicit function, and \( F_{\text{GUE}}(t) \) is the cdf of the GUE Tracy-Widom distribution [8]. Equation 3 is conjectured to be true for all “nice enough” i.i.d. weights [7]; i.e., the Tracy-Widom distribution is a universal limit. The exponent \( \chi \) is called the fluctuations exponent.

There is another exponent closely associated with \( \chi \) called the geodesic wandering exponent \( \xi \). One way of defining it is as follows [9]: Let \( L_x \) be the straight line between 0 and \( x \), and let \( C(\gamma, x) \subset \mathbb{R}^2 \) be the cylinder with central axis \( L_x \), radius \( N\gamma \) and length \(|x|_1\). Let \( A^N_{\gamma} \) be the event that all geodesics from 0 to \( Nx \) are contained inside the cylinder \( C(\gamma, Nx) \), and let
\[
\xi = \inf\{ \gamma > 0 : \lim_{N \to \infty} \mathbb{P}(A^N_{\gamma}) = 1 \}.
\]
Originally, Johansson [10] proved that \( \xi = 2/3 \) in a related model of two dimensional growth, the Poissonized longest increasing subsequence problem. In the solvable
last-passage percolation models, this was shown by Balázs, Cator, and Seppäläinen [11]. In these solvable models, since $\chi = 1/3$ and $\xi = 2/3$, the first KPZ scaling relationship $\chi = 2\xi - 1$ holds. This scaling relation is again conjectured to be universal [6], in that it is supposed to hold for a large class of growth models with i.i.d. weights, and all dimensions $d \geq 2$. In first-passage percolation, this conjecture has been proven under various unproven hypotheses on the limit shape and the existence of these exponents [12, 13].

In dimension $d = 2$, the exponents conjecturally satisfy the second KPZ relationship $2\chi = \xi$. This second KPZ relationship is related to the fluctuations of the so-called Busemann functions that we define next. These functions were originally used by H. Busemann to study geodesics in metric geometry [14], and they were introduced in first-passage percolation by Newman [15, Theorem 1.1]. They have since been used by various researchers to study geodesics in metric geometry [14], and they were expected to exist in certain directions under unproven differentiability hypotheses on the limit-shape that guarantee the existence of Busemann functions [16, 15, 14].

Let $U^c := \{x: x_1 + x_2 = 1, x_1, x_2 > 0\}$, the set of directions in $\mathbb{R}^2_{\geq 0}$ relevant to last-passage percolation. Suppose $\{x_n \in \mathbb{Z}^2\}$ is such that $|x_n| \to \infty$ but $x_n/|x_n| \to x \in U^c$. A Busemann function in direction $x$ is defined by the limit

$$B^x(a, b) = \lim_{n \to \infty} G(a, x_n) - G(b, x_n)$$

if it exists. Let $D = \{x \in U^c: g'(x) \text{ exists}\}$. For any $x \in D$, the limit in (4) is expected to exist. This result is known in first-passage percolation [16]; however, for last-passage percolation, it is only known conditional on unproven but mild differentiability hypotheses on the limit-shape [17] (see Appendix A). It is also expected that there is a unique Busemann function associated with each tangent line or gradient of the limit shape, and thus we will index Busemann functions using $G = \{\nabla g(x): g(x) \text{ is differentiable at } x \in U^c\}$. However, we will not require uniqueness in this paper.

A heuristic argument that was communicated to us by Newman, Alexander and others in a conference held at the American Institute for Mathematics in 2016, connects the covariance of Busemann increments to the second KPZ relationship. Consider a down/right lattice path along the antidiagonal defined as follows. Let $x_i = e_i$ for odd $i$, $x_i = -e_2$ for even $i$, $v_0 = 0$, and $v_k = \sum_{i=1}^{k} x_i$ be the $k$th point on the down/right path. For some $a \in G$, suppose

$$\begin{align*}
\text{Cov}(B^a(0, e_1), B^a(v_k, v_{k+1})) &\leq 0 \quad \forall k \geq 1, \\
\text{Cov}(B^a(e_2, 0), B^a(v_k, v_{k+1})) &\leq 0 \quad \forall k \geq 0.
\end{align*}$$

Then, the argument shows that $2\chi \leq \xi$ (see Appendix B for details).

In this paper, for last-passage percolation with i.i.d. vertex weights having distribution $F$, mean $m$ and variance $\sigma^2$, the main theorem (Theorem 2.1) provides an easily verifiable sufficient condition that determines whether some adjacent Busemann increments in (6) with $k = 0$ are negatively correlated for all $a \in G$. To demonstrate the use of this criterion, we show that i.i.d. Bernoulli weights with parameter $p > p^* (p^* \approx 0.5004)$ have negatively correlated adjacent Busemann increments. The criterion is explicit: it only involves the large deviation rate function for i.i.d. sums of $F$. Theorem 2.3 is proved by establishing that the negative covariance of
adjacent Busemann increments for all \( a \in \mathcal{G} \) is equivalent to \( g_F(x) \leq g_{\text{Exp}}(x) \) for all \( x \in \mathbb{R}^2_{\geq 0} \), where \( g_{\text{Exp}} \) is the function in (2) with parameters \( m \) and \( \sigma \). Our criterion provides a sufficient condition for this last inequality involving time-constants to hold, and thus proves negative correlation.

**Question 1** (Newman, Alexander and others). Fix a down/right path and some Busemann function \( B^u \). Are any two distinct Busemann increments of the form \( B^u(y, y + e_1) \) or \( B^u(z, z - e_2) \) on the down/right path negatively correlated? In particular, for the particular down/right path that goes along the main anti-diagonal, do (5) and (6) hold?

**Question 2.** Simulations (see Figures 1 and 2) indicate that for many distributions \( F \) (Uniform\([0, 1]\), Bernoulli\((p)\)), \( g_F(x) \leq g_{\text{Exp}}(x) \) for all \( x \in \mathbb{R}^2_{\geq 0} \). Is this true for all distributions, or is there a counterexample? Are there examples of distributions where \( g_F(x) \geq g_{\text{Exp}}(x) \) for all \( x \in \mathbb{R}^2_{\geq 0} \), in which case Lemma 2.2 shows that adjacent Busemann increments are positively correlated?

2. Main Results

**Assumption 1.** Associated with each \( u \in \mathcal{G} \), there is a stationary Busemann function \( B^u \) satisfying

1. Correct Expectation (see [18, Theorem 2.3]):
   \[
   \mathbb{E}[B^u(0, e_1), B^u(0, e_2)] = u
   \]
   (7)
2. Additivity (see [18, Definition 3.1(c)]):
   \[
   B^u(x, y) + B^u(x, z) = B^u(x, y + z) \text{ for all } x, y, z \in \mathbb{Z}^2
   \]
   (8)
3. Recovery/Dynamic Programming. (see [18, eq. 2.14])
   \[
   \omega_x = \min_{i=1,2} (B^u(x, x + e_i)).
   \]
   (9)

Assumption 1 holds under the mild differentiability hypotheses of [18, Theorem 2.3] (see Appendix A). The assumption is generally expected to be true without any hypotheses on the time-constant by adapting the methods of Ahlberg and Hoffman [16] for first-passage percolation. Assumption 1 implicitly applies to all the results stated below.

Suppose the weights \( w_e \) have distribution function \( F \) and moment generating function \( M(t) = \mathbb{E}[e^{w_e t}] \). We assume that for some small interval around the origin \(-\delta < t < \delta, M(t) < \infty\). The corresponding large deviation rate function is the Legendre transform:

\[
I(a) = \sup_t \{at - \log M(t)\}. \tag{10}
\]

**Theorem 2.1.** Suppose \( F \) has mean \( m \), variance \( \sigma^2 > 0 \), and moment generating function \( M(t) \) which is finite in some nontrivial interval around the origin. Suppose

\[
\log(4) \frac{s}{s + 1} < I \left( \frac{g_{\text{Exp}}(1, s)}{1 + s} \right) \quad \forall s \in (0, 1), \tag{11}
\]

where \( g_{\text{Exp}}(1, s) \) is the function in (2) with parameters \( m \) and \( \sigma \). Then,

\[
\text{Cov}(B^u(e_2, 0), B^u(0, e_1)) \leq 0 \quad \forall u \in \mathcal{G}.
\]
Theorem 2.1 follows from Lemma 2.2 and Theorem 2.3 below.

**Lemma 2.2.** Under the conditions of Theorem 2.1, let $g_F$ be the corresponding time-constant for i.i.d. last-passage percolation. Then,

$$g_F(x) \leq \text{Exp}(x) \quad \forall x \in \mathbb{R}_{\geq 0}$$

$$\quad \iff \quad \text{Cov}(B^u(e_2,0), B^u(0,e_1)) \leq 0 \quad \forall u \in \mathcal{G}.$$ \hfill (12)

The equivalence in (12) holds with both inequalities reversed, thus providing an analogous equivalence for positive correlation.

The second ingredient needed for proving Theorem 2.1 is the following theorem, which gives a sufficient condition on the rate function for i.i.d. sums of $F$ that determines when $g_F(x) \leq K$ for some $K > m$.

**Theorem 2.3.** Under the conditions of Theorem 2.1, let $g_F$ be the time-constant for i.i.d. last-passage percolation with weight distribution $F$, and let $I$ be the large deviation rate function for i.i.d. sums of $F$. Let $s \in (0, \infty)$ and $K > m$ be such that

$$\log(4) \frac{s}{s+1} < I \left( \frac{K}{1+s} \right).$$ \hfill (13)

Then, we have $g_F(1,s) \leq K$. 

**Remark 1.** The statement of Lemma 2.2 suggests the use of a convex ordering inequality in a manner similar to Berg and Kesten [19]. Given two distributions $F_1$ and $F_2$, we say $F_2$ is more variable than $F_1$ and write $F_1 \ll F_2$ if

$$\int \phi dF_1 \leq \int \phi dF_2$$ \hfill (14)

for all convex, non-decreasing integrable functions $\phi$. If $F_1 \ll F_2$, since the last-passage time is a convex non-decreasing function of the edge-weights, the associated time-constants satisfy $g_1(x) \leq g_2(x) \forall x \in \mathbb{R}_{\geq 0}$. For this to apply in our case, we would need to find a distribution $F$ with the same mean and variance as an exponential distribution $\text{Exp}(\lambda)$, such that $F \ll \text{Exp}(\lambda)$. Unfortunately, the following proposition shows this to be impossible.

**Proposition 2.4.** Let $X$ and $Y$ be random variables with different distributions $F$ and $G$. If $E[X] = E[Y]$ and $E[X^2] = E[Y^2]$ then $G \ll F$ cannot hold.

Next, we utilize the criterion in Theorem 2.1 to demonstrate a distribution for which adjacent Busemann increments are negatively correlated.

**Proposition 2.5.** Consider last-passage percolation with i.i.d. Bernoulli($p$) weights. If $p > p^*$, where $p^* \approx 0.6504$ (see Prop. 3.6 for the definition of $p^*$), then the criterion in Theorem 2.1 is satisfied.

Numerical computations show that the condition in Theorem 2.1 holds for Bernoulli($p$) weights for all parameters $p \geq 1/2$, and we believe that Prop. 2.5 can be easily extended to this setting (see Fig. 5). It follows trivially that shifted and scaled Bernoulli random variables also have negatively correlated Busemann function increments.
Figure 1. Simulated boundaries of limit shapes \( \{ x : g_F(x) = 1 \} \) for \( F = \text{Bernoulli}(p) \) with various values of \( p \). The limit shapes are in red, and are compared with the level set \( \{ x : g_{\text{Exp}}(x) = 1 \} \) in dashed blue, where \( g_{\text{Exp}}(x) \) is the function in (2) with parameters \( m = p \) and \( \sigma = \sqrt{p(1-p)} \). The limit shapes were approximated by plotting the scaled, occupied set of lattice sites at the largest time possible on a 5000 \( \times \) 5000 grid (see the shape-theorem in [5, Theorem 5.1]).

Figure 2. Simulated boundaries of limit shapes \( \{ x : g_F(x) = 1 \} \) for additional distributions. See Figure 1 for details. It appears as though \( g_F \) is not dominated by \( g_{\text{Exp}} \) for the Lognormal distribution, but it is too close to tell for certain. It is worth noting that the Lognormal distribution does not have a finite moment generating function. The Exponential distribution is included for comparison.

Corollary 2.6. Any random variable \( Y \) with \( P(Y = a) = 1 - p \) and \( P(Y = b) = p \) for \( p^* < p < 1 \) as in Prop. 3.6, also satisfies the criterion in Theorem 2.1.

Remark 2. Since our coarse graining method is far from optimal, the criterion fails for exponentially distributed weights. In the exponential case, the covariance of any two distinct Busemann increments on a down/right path is 0; in fact, they are independent.
3. Proofs

3.1. Covariance and the time-constant: Proof of Lemma 2.2. Consider a slice of the time-constant
\[
\gamma(s) = \begin{cases} 
g(1, s) & \text{if } 0 \leq s < \infty, \\
-\infty & \text{if } s < 0.
\end{cases}
\]
In last-passage percolation the time-constant is concave and \(-\gamma(s)\) is a convex function. Consider the Legendre transform (with a change of coordinates) of \(-\gamma(s)\) given by
\[
f(a) = \sup_{s>0} (-sa + \gamma(s)).
\]
From the trivial bound \(\gamma(s) > m(1 + s)\) for \(s > 0\), it follows that \(f(a)\) is only finite for \(a > m\). From Legendre duality, we have
\[
\gamma(s) = \inf_{a>m} (sa + f(a)). \tag{15}
\]
Since \(g(\lambda x) = \lambda g(x)\), we must have \(\partial_\lambda g(\lambda x) = \nabla g(\lambda x) \cdot x\), and hence \(g(x) = \nabla g(\lambda x) \cdot x\) for all \(\lambda > 0\). In terms of \(\gamma(s)\), this translates to \(\gamma(s) = \nabla g(1, s) \cdot e_1 + s\gamma'(s)\). Therefore,
\[
\nabla g(1, s) = (f(a), a),
\]
where \(a = \gamma'(s)\). The set of derivatives of \(\gamma(s)\) are in one-to-one correspondence with gradients of the time-constant. Thus, we will index Busemann functions by a single variable \(a \in \mathbb{R}\), which stands for the gradient. When we write \(a \in \mathcal{G}\) below, we mean \((f(a), a) \in \mathcal{G}\) where \(\mathcal{G}\) is the set of gradients of the time-constant.

Lemma 3.1. Let \(F\) have mean \(m\) and variance \(\sigma^2\). The covariance of adjacent Busemann increments is negative; i.e., \(\text{Cov}(B^a(0, e_1), B^a(e_2, 0)) \leq 0\) for all \(a \in \mathcal{G}\), if and only if
\[
f(a) \leq m + \frac{\sigma^2}{a - m} \quad \forall a \in \mathcal{G}. \tag{16}
\]

Proof. Fix \(a \in \mathcal{G}\), and from Assumption 1, there exists a Busemann function \(B^a\) satisfying
\[
\mathbb{E}[B^a(0, e_1), B^a(0, e_2)] = (f(a), a).
\]
From the additivity property (8), we have
\[
B^a(x, x + e_2) + B^a(x + e_2, x + e_1) = B^a(x, x + e_1).
\]
Inserting this into the recovery property (9), setting \(x = 0\), and rearranging, we get
\[
B^a(0, e_1) = \omega_0 + B^a(e_2, e_1)^+, \tag{17}
\]
\[
B^a(0, e_2) = \omega_0 + B^a(e_2, e_1)^-. \tag{18}
\]
where \(f^\pm = \max(\pm f, 0)\). It follows from (17) and (18) that
\[
\mathbb{E}[(B^a(e_2, e_1)^+, B^a(e_2, e_1)^-)] = (f(a) - m, a - m).
\]
Thus, the covariance of $B^a(0, e_1)$ and $B^a(e_2, 0)$ can be written as
\[
\text{Cov}(B^a(0, e_1), B^a(e_2, 0)) = -\text{Cov}(\omega_0, \omega_0) - \text{Cov}(B^a(e_2, e_1)^+, B^a(e_2, e_1)^-) \\
= -\sigma^2 + E[B^a(e_2, e_1)^+]E[B^a(e_2, e_1)^-] \\
= -\sigma^2 + (f(a) - m)(a - m),
\]

using bi-linearity of covariance and the fact that $B^a(e_2, e_1)$ is independent of the weight $\omega_0$ [18, Theorem 3.3]. Equations (17), (18) and (19) are due to T. Seppalainen [20], who noted that zero-correlation of all adjacent Busemann increments (eq. (19) is identically 0) implies that the limit-shape must be given by (2). Equation (19) shows that the covariance is negative if and only if
\[
f(a) \leq m + \frac{\sigma^2}{a - m}.
\]

Remark 3. The proof also shows that $\text{Cov}(B^a(0, e_1), B^a(e_2, 0)) \geq 0$ for all values of $a \in G$ if and only if
\[
f(a) \geq m + \frac{\sigma^2}{a - m}.
\]

Proof of Lemma 2.2. First suppose the Busemann increments are negatively correlated. Combining (15) and Lemma 3.1, we find
\[
g_F(1, s) = \gamma(s) = \inf_{m < a < \infty} (f(a) + sa) \\
\leq \inf_{m < a < \infty} (m + \frac{\sigma^2}{a - m} + sa) \\
= m(1 + s) + 2\sigma\sqrt{s} = g_{\text{Exp}}(1, s).
\]

By the 1-homogeneity, continuity and symmetry of $g_F$ and $g_{\text{Exp}}$, it follows that $g_F(x) \leq g_{\text{Exp}}(x) \forall x \in \mathbb{R}_{\geq 0}^2$ (see Prop. 3.5).

Next suppose $g_F(1, s) \leq g_{\text{Exp}}(1, s)$ for all $s \in [0, \infty)$. Then, for $m < a < \infty$, we have
\[
f(a) = \sup_{s > 0} (\gamma(s) - sa) \\
\leq \sup_{s > 0} (m(1 + s) + 2\sigma\sqrt{s} - sa) \\
= m + \frac{\sigma^2}{a - m}.
\]

3.2. Coarse graining argument: Proof of Theorem 2.3. Recall that $I$ is the large deviation rate function for i.i.d. sums of $F$. In this section, we prove Theorem 2.3, which states that if for any $s \in (0, \infty)$ and $K > m$, we have
\[
\frac{\log(4)s}{1 + s} < I \left( \frac{K}{1 + s} \right),
\]
then $g_F(s) \leq K$. 

\[\square\]
Let $s \in (0, \infty)$. Since $\log(4)(s/(1 + s)) < I(K/(1 + s))$, choose $r \in \mathbb{Q}_{>0}$ such that
\[
\frac{\log(4)(s + r)}{(1 + s)(1 + r)} < I \left( \frac{K}{1 + s} \right).
\] (22)
We define an event on which the weights in $[0, N] \times [0, Ns]$ are not too large:
\[
\text{GOOD}_N = \bigcup_{1 \leq i \leq N} \{ \omega : \omega_{ij} \leq b_N \}. \tag{23}
\]
Since $F$ has moment generating function $M(t)$ which is finite for $t \in (-\delta, \delta)$ where $\delta > 0$, applying a union bound to (23) gives
\[
P(\text{GOOD}_C^N) \leq cN^2 se^{-\lambda b_N} \tag{24}
\]
for constants $\lambda > 0$ and $c > 0$ that only depend on $F$.

Let $\text{PATH}_N$ be the set of all up/right paths from $(0, 0)$ to $N(1, s)$. Consider the event
\[
A_N = \bigcup_{\Gamma \in \text{PATH}_N} \{ \omega \in \Omega : G(\Gamma) > NK \}. \tag{25}
\]
If $\lim_{N \to \infty} P(A_N) = 0$, then we have
\[
g_F(s) \leq K. \tag{26}
\]
To show that $A_N$ has vanishing probability as $N \to \infty$, we use a coarse graining argument to reduce the number of allowed paths in $A_N$ (entropy reduction), and then use a union bound. This strategy is inspired by Damron and Wang [21].

Let $M$ be a positive integer (to be fixed after the proof of Lemma 3.2) such that $Mr$ is an integer as well. For $k \in \mathbb{Z}_{\geq 0}$ define the anti-diagonal lines $L_k = \{(x, y) \in \mathbb{R}^2 : y = r(kM - x)\}$, and $\mathcal{L} = \bigcup_{k \in \mathbb{Z}_{>0}} L_k$. Let $L \leq M$ be any positive real number such that $Lr$ is an integer, and let $C_g$ be the coarse grid consisting of points in $\mathcal{L} \cap \mathbb{Z}^2$ a diagonal distance $L\sqrt{1 + r^2}$ apart; i.e.,
\[
C_g = \{ p \in \mathcal{L} \cap \mathbb{Z}^2 : |p - (kM, 0)|_1 = qL(1 + r) \text{ for } k, q \in \mathbb{Z}_{\geq 0} \}.
\]
Here we use $|x|_p$ to denote the $\ell^p$ norm.

Define the free zone $F_g$ with
\[
F_g = \bigcup_{k \geq 1} F_g^k := \bigcup_{k \geq 1} \{(x, y) \in \mathbb{Z}^2 : r(kM - x) \leq y \leq r(kM - x) + Lr\},
\]
and define two extra sets of anti-diagonal lines that flank each $L_k$:
\[
L_k^\pm := \{(x, y) \in \mathbb{R}^2_{\geq 0} : y = r(kM - x) \pm Lr\} \quad k \in \mathbb{Z}_{\geq 0}
\]
$F_g^k$ is the set of lattice points between $L_k$ and $L_k^+$.

We form another set of paths $\text{PATH}'_N$ by considering up/right paths from $(0, 0)$ to $(N, Ns)$ that
\begin{itemize}
  \item[(1)] only intersect lines in $\mathcal{L}$ at coarse grid points $C_g$ except for the final point $(N, Ns)$, and
  \item[(2)] are up/right everywhere except for the free zones, where they are allowed to move in all 4 directions and have $\ell^1$ length at most $2L$ in each $F_g^k$.
\end{itemize}
Figure 3. Coarse grid in red with parameters $r = 1/2, M = 8, L = 2$. $\Gamma \in \text{PATH}_N$ is shown in green. The modifications in blue result in a path $\Gamma' \in \text{PATH}'_N$: $\Gamma'$ follows the green path $\Gamma$ until it encounters a blue modification and takes that instead until it rejoins the green path.

Figure 4. A zoomed-in view of the construction of $\Gamma'$ between the lines $\mathcal{L}_k^-$ and $\mathcal{L}_k^+$.

For each $\omega \in A_N$, by definition, there exists a path $\Gamma \in \text{PATH}_N$ with $G(\Gamma) > NK$. This path can be modified between the lines $\mathcal{L}_k^-$ and $\mathcal{L}_k^+$ to obtain a path $\Gamma' \in \text{PATH}'_N$. This construction is illustrated in Figures 3 and 4: $\Gamma$ is modified so that $\Gamma'$ passes through a coarse grid point, and then takes a detour in the free-zone to rejoin $\Gamma$. We describe the construction precisely next.

We define $\Gamma'$ using an inductive construction over the lines $\mathcal{L}_k$. Let $\Gamma' = \Gamma$ until $\Gamma$ reaches $\mathcal{L}_1^-$. Fix $k \geq 1$ and assume $\Gamma$ and $\Gamma'$ have been defined up to the line
\( \mathcal{L}_k^-, \) and \( \Gamma \cap \mathcal{L}_k^- = \Gamma' \cap \mathcal{L}_k^- \). Let \( \Gamma_k, \Gamma_k^\pm \) be the points at which \( \Gamma \) intersects \( \mathcal{L}_k \) and \( \mathcal{L}_k^\pm \) respectively. If \( \Gamma_k \in C_g \), let \( \Gamma' = \Gamma \) from \( \mathcal{L}_k^- \) to \( \mathcal{L}_{k+1}^- \). If not, we construct \( \Gamma' \) between \( \mathcal{L}_k^- \) and \( \mathcal{L}_k^+ \) as follows. Let \( c_1, c_2 \) be points in \( C_g \cap \mathcal{L}_k \) that are on either side of \( \Gamma_k \), where \( c_1 \) has smaller \( x \)-coordinate.

Let \( c_3^- = c_2 - e_1 L r \). Consider the triangle formed by \( c_1, c_2 \) and \( c_3^- \), and let \( \Gamma^- \) be the point at which \( \Gamma \) intersects the sides \( [c_3^-, c_2] \) or \( [c_3^-, c_1] \). Let us consider the case where \( \Gamma \) intersects \( [c_3^-, c_2] \); the case where it intersects \( [c_3^-, c_1] \) can be handled analogously. Let \( \Gamma' \) follow \( \Gamma \) from \( \Gamma_k^- \) to \( \Gamma^- \) and then go horizontally in the \( e_1 \) direction to \( c_2 \). Similarly, we consider the triangle formed by the points \( c_3^+ := c_1 + e_1 L r, c_1 \) and \( c_2 \). Let \( \Gamma^+ \) be the point where \( \Gamma \) exits the triangle formed by \( c_1, c_2 \), and \( c_3^+ \). From \( c_2, \Gamma' \) takes the shortest path along the sides \( (c_1, c_3^+) \) and \( (c_2, c_3^+) \) until it meets \( \Gamma^+ \). Thereafter, \( \Gamma' \) follows \( \Gamma \) until it reaches \( \mathcal{L}_{k+1}^- \), and this completes the induction step. Finally, if \((N, N s)\) falls between an \( \mathcal{L}_k^- \) and \( \mathcal{L}_k^+ \), we just have \( \Gamma' \) follow \( \Gamma \).

The next lemma shows that replacing \( \Gamma \) by \( \Gamma' \) does not change the passage time substantially. Proposition 3.4 shows that \( \Gamma \) crosses a total of \( N \frac{(r + s)}{M r} \) lines \( \mathcal{L}_r \) in \( \mathcal{L} \). The modified path \( \Gamma' \) could incur a detour of length at most \( \frac{2L}{M r} \) between each \( \mathcal{L}_k^+ \) and \( \mathcal{L}_k^- \), and so \( \Gamma' \) has length at most
\[
|\Gamma'| \leq N(1 + s) + 2L \frac{N(r + s)}{M r}.
\]

Since \( \Gamma' \) does not coincide with \( \Gamma \) only between \( \mathcal{L}_k^- \) and \( \mathcal{L}_k^+ \), on the event \( \text{GOOD}_N \), we have
\[
G(\Gamma') \geq G(\Gamma) - 2L \frac{N(r + s)}{M r} (b_N - a).
\]

Inspired by (28), we define the event \( \tilde{A}_N \) where
\[
\tilde{A}_N = \left\{ \omega \in \text{PATH}_N \mid G(\Gamma') > N \left( K - 2L \frac{(r + s)}{M r} (b_N - a) \right) \right\}.
\]

Lemma 3.2. We have \( A_N \cap \text{GOOD}_N \subseteq \tilde{A}_N \cap \text{GOOD}_N \) and thus \( P(A_N) \leq P(\tilde{A}_N) + csN^2 e^{-\lambda b_N} \) for positive constants \( c, \lambda \) from (24).

Proof. For \( \omega \in A_N \cap \text{GOOD}_N \), consider any \( \Gamma \in \text{PATH}_N \). By the construction described above, there is a corresponding path \( \Gamma' \in \text{PATH}'_N \). By (28), it follows that \( \omega \in \tilde{A}_N \cap \text{GOOD}_N \). Thus,
\[
P(A_N) = P(A_N \cap \text{GOOD}_N) + P(A_N \cap \text{GOOD}'_N)
\leq P(\tilde{A}_N \cap \text{GOOD}_N) + P(\text{GOOD}'_N)
= P(\tilde{A}_N) + csN^2 e^{-\lambda b_N}.
\]

Let \( \alpha, \beta, \gamma > 0 \) with \( \alpha + \gamma < \beta \), let \( b_N = N^\gamma \), and let \( L \) and \( M \) be the smallest integers larger than \( \lfloor N^\alpha \rfloor \) and \( \lfloor N^\beta \rfloor \) respectively, such that \( L r \) and \( M r \) are integers. From (27), \( n = |\Gamma'| \) satisfies the bound
\[
N(1 + s) \leq n \leq N(1 + s) + O(N^{1+\alpha-\beta}).
\]
Using a union bound over all paths in $\text{PATH}'_N$, gives

$$P(\tilde{A}_N)$$

$$\leq |\text{PATH}'_N| \sum_{N(1+s) \leq n \leq N(1+s)+O(N^{1-\delta})} P\left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq \frac{N}{n} \left( K - 2L(b_N-a) \frac{(s+r)}{M} \right) \right),$$

(31)

$$\leq |\text{PATH}'_N| \sup_{N(1+s) \leq n} P\left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq \frac{K}{1+s} - O(N^{\alpha+\gamma-\beta}) \right),$$

(32)

where $X_i$ are iid with distribution $F$, and $\delta = 1 + \alpha - \beta < 1$. We estimate the term $|\text{PATH}'_N|$ next.

**Proposition 3.3.** For fixed $s > 0$, $r \in \mathbb{Q}_{>0}$, and $L$ and $M$ as described above,

$$\frac{1}{N} \log |\text{PATH}'_N| = \frac{s+r}{1+r} \log(4) + O(N^{-(\beta-\alpha)}).$$

**Proof.** Fix a path $\Gamma' \in \text{PATH}'_N$, and suppose $\Gamma'$ intersects the line $L^+_k$ at a point $x$. $\Gamma'$ lies inside a triangle $\triangle_{M'}$ with coordinates $x, x + M'e_1$ and $x + M' e_2$, where $M' = M - L$, until it intersects the line $L_{k+1}$, which is the hypotenuse of $\triangle_{M'}$. $\Gamma'$ must exit the triangle at a coarse-grained point in $C_g \cap L_{k+1}$. For the purpose of getting an upper bound on the number of up/right paths in $\triangle_{M'}$ that exit at a coarse-grained point, we may translate $\triangle_{M'}$ to the origin $(x=0)$ and replace $M'$ with $M$. Let $T_M$ be the number of such paths in the triangle $\triangle_M$.

In $\triangle_M$, suppose $\Gamma'$ exits at a point $(x, y) \in C_g$ on the hypotenuse. Since paths must take a total of $x + y = x + r(M - x)$ steps from the origin to the hypotenuse, the number of possible paths is $\binom{x + r(M - x)}{x}$. This binomial coefficient is maximized when $x + r(M - x) = 2x$, so there are at most $\binom{2M}{Mr} \frac{1+r}{1+\frac{Mr}{L}}$ paths to $(x, y)$. Since the coarse grid points are a $\ell^2$ distance $L\sqrt{1+r^2}$ apart on the hypotenuse of $\triangle_M$,

$$T_M \leq \frac{M}{L} \binom{2M}{Mr} \frac{1+r}{1+\frac{Mr}{L}}.$$

Once $\Gamma'$ has intersected $L_{k+1}$, in the free zone $F^k_g$, $\Gamma'$ has length at most $2L$ so there are at most $4^{2L}$ such paths. This is a very crude bound, of course, but it makes no difference asymptotically at $N \to \infty$. Thus, we have accounted for all paths between $L^+_k$ to $L^+_k$. The number of paths from the origin to $L^+_k$ produces an identical bound.

A coarse-grained path crosses at most $\frac{N(1+s)}{M^r}$ lines $L_k$, which gives the estimate

$$|\text{PATH}'_N| \leq (T_M 4^{2L}) \frac{N(1+s)}{M^r} \leq \left( \frac{M}{L} \frac{2M}{Mr} 4^{2L} \right)^{\frac{N(1+s)}{M^r}}.$$ 

Define the binary entropy function $H(p)$ by

$$H(p) = -p \log(p) - (1-p) \log(1-p).$$

(33)
Using Stirling’s formula to estimate the binomial coefficients gives
\[
\log |\text{PATH}_N'| = \frac{N(s + r)}{Mr} \left( \log \left( \frac{M}{L} \right) + \frac{2Mr}{1+r}H(1/2) + O(1) + 2L \log(4) \right)
\]
\[
= \frac{N(s + r)}{1 + r} \log(4) + O(N^{1-\beta + \alpha}).
\]
Since \(\alpha < \beta < 1\) the error terms are all of order less than \(N\). \(\square\)

Now we have all the tools needed to prove Theorem 2.3.

**Proof of Theorem 2.3.** From (31),
\[
P(\tilde{A}_N) \leq |\text{PATH}_N| \max_{N(1+s) \leq n} P \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq \frac{K}{1+s} - o(1) \right),
\]
where the \(o(1)\) term goes to 0 as \(N \to \infty\).

The standard large deviations estimate (Cramér’s theorem) \([22]\) applied to the i.i.d. sum \(n^{-1} \sum_{i=1}^{n} X_i\) gives
\[
\lim_{n \to \infty} \frac{1}{n} \log P \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq x \right) = -I(x),
\]
where \(I(x)\) is the large deviations rate function (10). Inserting this into (34) and using the estimate for \(|\text{PATH}_N'|\) from Prop. 3.3, we find that as \(N \to \infty\),
\[
\frac{1}{N(1+s)} \log P(A_N) \leq \frac{\log(4)(s + r)}{(1+s)(1+r)} - I \left( \frac{K}{1+s} + o(1) \right) + o(1).
\]

The continuity of \(I\) and the choice of \(r\) in (22) imply that the right hand side of (35) converges to zero as \(N \to \infty\) and thus \(g_F(s) \leq K\). \(\square\)

**Figure 5.** \(\phi(s)\) for different values of \(p\). Notice that the criterion in Theorem 2.1 for Bernoulli weights (see Prop. 2.5) is satisfied for \(p = 1/2\) and \(3/4\) since \(\phi(s) < 0\) for \(0 < s < s^*\). The criterion is not satisfied for \(p = 1/4\), and more generally (not shown here) for \(p < 1/2\). Despite not satisfying the criterion, simulations (Fig. 1) show that \(g_F \leq g_{\text{Exp}}\) when \(F = \text{Bernoulli}(p)\) for \(p < 1/2\).

We used the following elementary count of the numbers lines in \(L\) crossed by \(\Gamma\) and \(\Gamma'\) in the proof of Theorem 2.3.

**Proposition 3.4.** A path from \((0,0)\) to \(N(1,s)\) intersects a total of \(N(r + s)/(Mr) + O(1)\) lines in \(L\).
Proof. An up/right path from $(0,0)$ to $N(1,s)$ must cross every diagonal line in $L$ once. Consider the rectangle $[0,N] \times [0,Ns]$. There are $\lfloor N/M \rfloor$ lines in $L$ that intersect the eastern boundary of the box, and $\lfloor (Ns)/(rM) \rfloor$ lines that intersect the northern. Therefore the path crosses a total of $\lfloor N/M \rfloor + \lfloor (Ns)/(rM) \rfloor = \frac{N(r+s)}{rM} + O(1)$ diagonal lines.

Next, we prepare to prove Theorem 2.1 by making a simplification that allows us reduce to the case $0 < s < 1$.

**Proposition 3.5.** Let $g_i, i = 1, 2$ be continuous, 1-homogeneous functions on $\mathbb{R}^2_{\geq 0}$ satisfying $g_i(x,y) = g_i(y,x) \ \forall x,y \in \mathbb{R}_{\geq 0}$. If $g_1(1,s) \leq g_2(1,s)$ for all $s \in (0,1)$ then $g_1(x) \leq g_2(x)$ for all $x \in \mathbb{R}^2_{\geq 0}$.

**Proof.** Suppose $s > 1$. Since $1/s < 1$ we have

$$g_1(1,1/s) \leq g_2(1,1/s).$$

Using $g_i(x,y) = g_i(y,x)$, we get

$$g_1(1/s,1) \leq g_2(1/s,1).$$

Using 1-homogeneity, $g_i(\lambda x, \lambda y) = \lambda g_i(x,y)$ for $\lambda \geq 0$, and choosing $\lambda = s$ we conclude

$$g_1(1,s) \leq g_2(1,s).$$

Thus, for arbitrary $x,y \in \mathbb{R}_{>0}$, we have

$$g_1(x,y) = xg_1(1,y/x) \leq xg_2(1,y/x) = g_2(x,y),$$

and by continuity, the result extends to $\mathbb{R}^2_{\geq 0}$. \hfill \Box

**Proof of Theorem 2.1.** Since $\log(4)s/(1 + s) < I(g_{\text{Exp}}(1,s)/(1 + s))$ for all $s \in (0,1)$, Theorem 2.3 implies $g_F(1,s) \leq g_{\text{Exp}}(1,s) \ \forall s \in (0,1)$. Since $g_F$ and $g_{\text{Exp}}$ are continuous [5, Theorem 2.4], 1-homogeneous functions satisfying $g_i(x,y) = g_i(y,x) \ \forall x,y \in \mathbb{R}^2_{\geq 0}, \ i \in \{F, \text{Exp}\}$, Prop. 3.5 completes the proof. \hfill \Box

### 3.3. Failure of convex ordering.

This section contains the short proof of Prop. 2.4, that shows that two distinct random variables cannot be comparable in the convex ordering if they have equal first and second moments.

**Proof.** We argue by contradiction. Suppose $G \ll F$; then, by Theorem 2 in [23], this is equivalent to the fact that there exists a coupling of $X$ and $Y$ such that $E[X|Y] \leq Y$ a.s. Since $E[X] = E[Y]$, the tower property of conditional expectation implies $E[X|Y] = Y$ a.s. Applying the conditional Jensen’s inequality, we have

$$E[Y^2] = E[E[X|Y]^2] \leq E[E[X^2|Y]] = E[X^2].$$

However, by assumption, $E[X^2] = E[Y^2]$ and thus the inequality above must be an equality. Equality in the conditional Jensen’s inequality holds if and only if $X = f(Y)$ a.s. Then, $E[X|Y] = Y$ a.s. implies that $X = Y$ a.s. \hfill \Box
3.4. Bernoulli weights. In this section we prove Proposition 2.5: We demonstrate that Bernoulli weights with $p \geq p^*$ ($p^* \approx 0.6504$) satisfy the criterion in Theorem 2.1, and thus have negatively correlated adjacent Busemann increments. The criterion requires

$$\log(4) \frac{s}{1 + s} - I \left( m + 2\sigma \frac{\sqrt{s}}{1 + s} \right) < 0 \quad \forall s \in (0, 1),$$

where $m = p$ and $\sigma = \sqrt{p(1 - p)}$. For convenience, we define

$$u_s = \frac{2\sigma \sqrt{s}}{1 + s}.$$ 

Proposition 3.6. Let

$$\phi(s) = \log(4) \frac{s}{1 + s} - I(p + u_s).$$

Let $0 < s^*(p) < 1$ be the unique solution of $p + u_s = 1$ for $1 > p > 1/2$ (see (39)), and let $p^*$ be the unique solution of $\log(4) = \frac{1 - p + u_s}{1 + u_s}$ in $(1/2, 1)$. If $X \sim Ber(p)$ with $p^* < p < 1$ then

$$\phi(s) = \begin{cases} 
  < 0 & \text{if } 0 < s < s^*(p) \\
  0 & \text{if } s = 0 \\
  -\infty & \text{if } s \geq s^* 
\end{cases}.$$

Proof. The rate function $I(x)$ for Bernoulli weights is

$$I(x) = \begin{cases} 
  x \log \left( \frac{x}{p} \right) + (1 - x) \log \left( \frac{1 - x}{1 - p} \right) & 0 < x < 1 \\
  -\infty & \text{otherwise}
\end{cases}.$$ 

So in our case, we only have to consider $s \in (0, 1)$ such that $0 < p + u_s < 1$. For ease of computation we write $\phi(s)$ in terms of the entropy function defined in (33).

We have

$$\phi(s) = \log(4) \frac{s}{1 + s} - (p + u_s) \log \left( \frac{p + u_s}{p} \right) - (1 - (p + u_s)) \log \left( \frac{1 - (p + u_s)}{1 - p} \right)$$

$$= \log(4) \frac{s}{1 + s} + H(p + u_s) + u_s \log \left( \frac{p}{1 - p} \right) - H(p).$$

Note that $\phi(0) = -H(p) + H(p) = 0$. We show $\phi'(s) < 0$ for $0 < s < s^*$. Using $H'(x) = \log \left( \frac{1 + x}{x} \right)$, we get

$$\phi'(s) = \frac{\log(4)}{(1 + s)^2} + u_s' \left( \log \left( \frac{p}{1 - p} \right) + \log \left( \frac{1 - (p + u_s)}{p + u_s} \right) \right)$$

$$= u_s' \left( \frac{\log(4) \sqrt{s}}{\sigma(1 - s)} - \log \left( \frac{1 + u_s}{p} \right) + \log \left( \frac{1 - u_s}{1 - p} \right) \right).$$

Now we focus on the term inside the parentheses in (37),

$$\psi(s) := \frac{\log(4) \sqrt{s}}{\sigma(1 - s)} - \log \left( \frac{1 + u_s}{p} \right) + \log \left( \frac{1 - u_s}{1 - p} \right),$$

and so
and bound this from above. Note that the condition for the rate function to be finite is $p + u_s < 1$ which implies

$$\frac{u_s}{p} < \frac{1-p}{p} < 1,$$

since $1/2 < p < 1$. We now use the following two elementary inequalities for the logarithm

$$\log(1+x) > \frac{x}{2}, \quad \log(1-x) < -x \quad \forall \ 0 < x < 1.$$

Inserting these inequalities gives

$$\psi(s) \leq \log(4) \frac{\sqrt{s}}{\sigma(1-s)} - \frac{u_s}{2p} - \frac{u_s}{1-p} = \frac{\log(4) \sqrt{s}}{\sigma(1-s)} - \frac{(1+p) \sqrt{s}}{\sigma(1+s)}.$$

Plugging this back into (37) gives

$$\phi'(s) \leq \frac{1}{(1+s)^3} \left((1+s) \log(4) - (1+p)(1-s)\right).$$

Integrating this inequality from 0 to $s$, we get

$$\phi(s) \leq \frac{s((1+s) \log(4) - (1+p)(1-s))}{(s+1)^2} \quad \text{(38)}$$

using the fact that $\phi(0) = 0$. Thus, we find $\phi(s) < 0$ if $\log 4 < \frac{1+p}{1+s}$.

Solving $p + g_s = 1$ for $s$ in terms of $p$ gives

$$s^*(p) = \frac{1 - 3p + 2\sqrt{p(2p-1)}}{p-1}. \quad \text{(39)}$$

Since $0 < s < s^*(p)$, it is enough to show $\log 4 < \frac{1+p}{1+s^*(p)}$. Since $s^*(p)$ is a strictly decreasing function for $p > 1/2$ (see (39)), if $\log(4) < \frac{1+p}{1+s^*(p)}$ for some $p^*$, then $\phi(s) < 0 \quad \forall s \in (0, s^*(p))$ for any $p > p^*$. Let $p^*$ be the solution of $\log(4) = \frac{1+p}{1+s^*(p)}$. Solving this numerically for $p^*$, we find that $p^* \approx 0.6504$.

Finally we verify that $s^*(p)$ is a decreasing function of $p$ for $p > 1/2$ by computing its derivative:

$$\frac{ds^*}{dp} = \frac{-3p + 2\sqrt{p(2p-1)} + 1}{(p-1)^2 \sqrt{p(2p-1)}},$$

$$\leq \frac{1 - 3p + 3p - 1}{(p-1)^2 \sqrt{p(2p-1)}} = 0, \quad \text{(40)}$$

where we have applied the AM-GM inequality since $p > 1/2$.

**Proof of Corollary 2.6.** Let $Y = (b-a)X + a$ where $X$ is Bernoulli$(p)$. Using (10), it is easy to see that the rate functions $I_X$ and $I_Y$ of $X$ and $Y$ satisfy

$$I_Y(s) = I_X\left(\frac{s-a}{b-a}\right).$$
Let $m_X, m_Y, \sigma_X, \sigma_Y$ be the means and variances of $X$ and $Y$. Since $m_Y = m_X(b-a) + a$ and $\sigma_Y = (b-a)\sigma_X$, 

$$I_Y \left( m_Y + \frac{2\sigma_Y \sqrt{s}}{1 + s} \right) = I_X \left( m_X + \frac{2\sigma_X \sqrt{s}}{1 + s} \right),$$

and Proposition 3.6 applies.

\[\square\]

Appendix A. Conditions implying the existence of Busemann functions

For any $x \in U^\circ$, let $H_x = \{ \lambda x : \lambda > 0 \}$ be the line beginning at the origin that passes through $x$. Since the time-constant $g$ is 1-homogeneous, $g'(x)$, if it exists, is constant along any $H_x$. Let $L_x$ be a tangent line of the limit shape $B$ that intersects $H_x$; then there exist $x_L, x_R \in (0, 1)$ that are the smallest and largest numbers, not necessarily distinct, such that the lines $H_{(x_L, 1-x_L)}$ and $H_{(x_R, 1-x_R)}$ intersect $L_x \cap B$. In last-passage percolation, the limit in (4) is known to exist and produce Busemann functions satisfying Assumption 1 when $g$ is differentiable at both endpoints $(x_L, 1-x_L)$ and $(x_R, 1-x_R)$ [17]. In first-passage percolation, similar results have been proved in [24]. More recently, in first-passage percolation, Ahlberg and Hoffman [16] removed the differentiability requirements at the points $x_L, 1-x_L$ and $x_R, 1-x_R$, and showed that there is a unique Busemann function associated with each tangent line of the limit shape. It is expected that their techniques can be extended to prove a similar result for last-passage percolation.

Since the time-constant of last-passage percolation is concave, it is not differentiable on at most a countable number of points in $U^\circ$. However, the only available result about differentiability is in first-passage percolation, where the weights have support bounded away from 0, and the minimum element in the support has probability larger than the critical probability for percolation. Here, we know that the boundary of the limit shape is a straight line between two angles $\theta_1 < \theta_2$ that are symmetric about $(1/2, 1/2)$ in the positive quadrant [25, 26] —the so-called percolation cone— and that the limit shape is differentiable at the end points $\{\theta_1, \theta_2\}$ [27].

Appendix B. Busemann correlations and the KPZ relationship

Suppose

$$B^2(0, Nv) = \nabla g(x) \cdot Nv + \Theta(N^{1/2}),$$

for some $v$ that is not parallel to $x$, and $\Theta(x)$ means that the quantity is bounded above and below by a constant times $x$. In the case of exponential or geometric weights, this is known to be true since Busemann increments are i.i.d. exponentials or geometrics respectively on any lattice path that only goes down or right, and the CLT implies their diffusive behavior [11]. Assuming (41), the following heuristic argument due to Newman, Alexander and others shows that $2\chi = \xi$ in $d = 2$.

From Johansson’s theorem (3), it follows that it is not unreasonable to expect that in general, for any $x \in U^\circ$,

$$G(0, Nx) = Ng(x) + \Theta(N^\chi).$$

(42)
Under the differentiability hypothesis described in Appendix A, the Busemann function associated with $x$ is known to exist. Then, 
\[ B^x(a, b) = \lim_{M \to \infty} G(a, Mx) - G(b, Mx), \] (43)
and moreover, 
\[ \mathbb{E}[B^x(x, y)] = \nabla g(x) \cdot (y - x). \] (44)
The dynamic-programming or recovery property (9) can be used to recover geodesics from Busemann functions. Let 
\[ \alpha(x) = \arg\min_{y \in \{e_1, e_2\}} B(x, x + y) \] (45)
be the arrow at $x$. In case of a tie in (45), we may always assume that $\alpha(x) = e_1$. Given any $x \in \mathbb{Z}^2$, we can form an up/right lattice path as follows: let $X_0 = x$, and 
\[ X_n = X_{n-1} + \alpha(X_{n-1}) \]
for $n \geq 1$. It can be shown these paths formed by following arrows always produce geodesics, and these are called Busemann geodesics [18, eq. (2.14)].

The Licea-Newman argument [28, 24] shows that Busemann geodesics from any two points coalesce almost surely [18, Theorem 4.5]. Since the geodesics from 0 to $N^x$ and $N^\xi v$ to $N^x$ fluctuate on the $N^\xi$ scale, it is expected that there is a random tight constant $a_N$ such that the geodesics from 0 and $N^\xi v$ would have merged after $a_N N^x$ steps. Indeed, this is known to be true in the exponential and geometric cases [29]. Then, from (42),
\[ B^x(0, N^\xi v) = G(0, a_N N^x) - G(0, a_N N^x), \]
for some large enough $N$. Inserting (42) and (41) into the above, we get
\[ \nabla g(x) \cdot N^\xi v + \Theta(N^{\xi/2}) = Ng(a_N x) - Ng(a_N x - N^{\xi-1} v) + \Theta(N^\chi) \]
\[ \approx \nabla g(a_N x) \cdot N^\xi v + \Theta(N^\chi). \]
By the homogeneity of $g(u)$, $\nabla g(a_N x) = \nabla g(x)$, and thus $\xi = 2\chi$.
An initial step towards proving (41) is to show
\[ \var{B^x(0, N^v)} \leq O(N). \] (46)
If the covariance inequalities in (5) and (6) hold, we have
\[ \var{B^x(0, N^v)} = \var{\sum_{i=0}^{N-1} B^x(v_i, v_{i+1})} \]
\[ = \sum_{i=0}^{N-1} \var{B^x(v_i, v_{i+1})} + 2 \sum_{i<j} \cov{B^x(v_i, v_{i+1}), B^x(v_j, v_{j+1})} \]
\[ \leq \var{B^x(0, v)} N, \] (47)
where the first equality follows from additivity and the last uses the stationarity of Busemann functions. If only (46) is available, the heuristic gives $\xi \geq 2\chi$.

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Ian Alevy, University of Rochester, Mathematics Department, Hylan 1008, Rochester, NY 14627, USA.

Email address: ian.alevy@rochester.edu
URL: https://people.math.rochester.edu/faculty/ialevy/

Arjun Krishnan, University of Rochester, Mathematics Department, Hylan 817, Rochester, NY 14627, USA.

Email address: arjunkc@gmail.com
URL: https://people.math.rochester.edu/faculty/akrish11/