LIST CHROMATIC NUMBERS AND SINGULAR COMPACTNESS

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Abstract. Let $G = (V, E)$ be a graph of size $\mu$ where $\mu > \text{cf}(\mu)$. Assume that $\aleph_0 \leq \lambda < \mu$ and $\eta^\lambda < \mu$ for every $\eta \in \mu$. We prove that if $\text{List}(H) \leq \lambda$ for every $H \leq G$ such that $|H| < \mu$ then $\text{List}(G) \leq \lambda$. In particular, if $\mu$ is a strong limit singular cardinal then singular compactness holds at $\mu$ with respect to the list-chromatic number for every $\lambda \in \mu$. We also use the list-chromatic number in order to prove a statement about the club principle.

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An abelian group $A$ is called almost free if every subgroup $B$ of $A$ of size less than the size of $A$ is free. An interesting problem is whether there is an almost free non-free abelian group or whether almost free implies free. An important parameter here is the cardinality of $A$. In many cases this question is independent of ZFC. This includes the case of a successor of a singular cardinal as proved in [6]. But if the cardinality of $A$ is a singular cardinal $\mu$ then almost free abelian groups of size $\mu$ are free.

This phenomenon has been studied by Shelah in [9] and it is called singular compactness. The range of singular compactness is much wider than free abelian groups, and it is known to exist in many mathematical areas including graph theory. In the general setting we have a family of structures, a property $P$ of these structures and a cardinal $\lambda<\mu$ where $\mu$ is a singular cardinal. We shall say that the family satisfies singular compactness with respect to $P$ iff given any $A$ of size $\mu$ it is true that $P(A) \leq \lambda$ whenever $P(B) \leq \lambda$ for every substructure $B$ of $A$ of size less than $\mu$.

In this paper we deal with singular compactness applied to graph theory. More specifically, we prove a statement about the list chromatic number in infinite graphs. Let $G = (V,E)$ be a graph. A function $f : V \to \text{Ord}$ is a good coloring iff $f(x) \neq f(y)$ whenever $\{x,y\} \in E$. If $\kappa$ is a cardinal then a $\kappa$-assignment for $G$ is a function $F : V \to [\text{Ord}]^\kappa$. The list-chromatic number of $G$, denoted $\text{List}(G)$, is the minimal $\kappa$ such that for every $\kappa$-assignment $F : V^G \to [\text{Ord}]^\kappa$ one can find a good coloring $f : V^G \to \text{Ord}$ so that $f(x) \in F(x)$ whenever $x \in V^G$.

The list chromatic number was defined, independently, by Vizing in [13] and by Erdős, Rubin and Taylor in [2]. Singular compactness with respect to the list chromatic number is the following statement. Suppose that $G = (V,E)$ is a graph where $|V| = \mu$ and $\mu > \text{cf}(\mu)$. If $\lambda < \mu$ and $\text{List}(H) \leq \lambda$ for every $H \leq G$ of size less than $\mu$ then $\text{List}(G) \leq \lambda$ as well. It is known that the list chromatic number fails to satisfy singular compactness without additional assumptions, and a simple example for this failure is given by Komjáth in [4]. Komjáth proved that if $G$ is a bipartite graph with $V^G = A \cup B, |A| = \aleph_0$ and $|B| < 2^{\aleph_0}$ then $\text{List}(G) \leq \aleph_0$. On the other hand, there are many bipartite graphs $G$ such that $V^G = A \cup B, |A| = \aleph_0$ and $|B| = 2^{\aleph_0}$ and $\text{List}(G) > \aleph_0$. Hence if one forces $2^{\aleph_0} = \mu$ where $\mu > \text{cf}(\mu)$ then singular compactness fails with respect to List($G$).

The example of Komjáth generalizes to larger cardinalities, and one concludes that List($G$) does not satisfy singular compactness without further assumptions. However, Usuba proved in [12] that under some cardinal arithmetic assumptions on the singular cardinal $\mu$ and the power-set operation below $\mu$ one can prove singular compactness for List($G$) where $|G| = \mu$. He asked whether the assumption $\eta^\lambda < \mu$ for every $\eta \in \mu$ will be sufficient, and gave some partial results, see [12, Question 3.8]. Our goal is to give a positive answer to this question.
In the last section of the paper we deal with reflection principles which emerge from the list chromatic number (and from another invariant of graphs, the coloring number). We observe that an interesting theorem of Usuba from [12] which assumes diamonds can be proved with the club principle instead of diamond. We conclude from this fact that any ccc forcing notion which forces $2^\omega > \omega_1$ must destroy the club principle at some stationary set.

Our notation is standard. We denote cardinals by $\theta, \kappa, \lambda, \mu, \chi$ and ordinals which are not cardinals by $\alpha, \beta, \gamma, \delta$ and also by $i, j$. If $\lambda = \text{cf}(\lambda) < \kappa$ then $S^\kappa_\lambda = \{\alpha \in \kappa : \text{cf}(\alpha) = \lambda\}$. Graph theoretical notation is usually consistent with [1].
1. Singular compactness

In this section we give sufficient conditions for singular compactness with respect to the list chromatic number in terms of cardinal arithmetic. As a first step we introduce a slight improvement upon [12, Fact 2.4] by proving a similar statement but weakening one of the assumptions. Given a graph $G = (V, E)$ and $x \in V$ let $E^x = \{y \in V : \{x, y\} \in E\}$.

Lemma 1. Let $G = (V, E)$ be a graph and suppose that $\lambda \leq \tau$ are infinite cardinals. Assume that there are $W_0, W_1 \subseteq V$ such that $|W_0| = \tau, |W_1| \geq \tau^\lambda, W_0 \cap W_1 = \emptyset$ and $|E^x \cap W_0| \geq \lambda$ for every $x \in W_1$. Then $\text{List}(G) > \lambda$.

Proof. Since $H \leq G$ implies $\text{List}(H) \leq \text{List}(G)$ it is sufficient to find some $H \leq G$ for which $\text{List}(H) > \lambda$. Hence we assume that $|E^x \cap W_0| = \lambda$ for every $x \in W_1$ by eliminating some edges from $E$ if needed.

Fix a disjoint family $\{A_y : y \in W_0\} \subseteq [\tau]^\lambda$. The cardinality of the product $\prod_{y \in W_0} A_y$ is $\tau^\lambda$. Let $H$ be the graph induced by $W_0 \cup W_1$. We shall define a $\lambda$-assignment for $H$ so that no function $f : V^H \to \text{Ord}$, amenable to this assignment, will be good. This will show that $\text{List}(H) > \lambda$ and hence $\text{List}(G) > \lambda$ as desired.

Enumerate the elements of $\prod_{y \in W_0} A_y$ by $(g_z : z \in W_1)$. If $|W_1| > \tau^\lambda$ then either take a subset of $W_1$ of size $\tau^\lambda$ or use repetitions. For every $y \in W_0$ let $F(y) = A_y$ and for every $z \in W_1$ let $F(z) = \{g_z(y) : y \in W_0, \{y, z\} \in E\}$. Observe that $F$ is a $\lambda$-assignment since $|E^x \cap W_0| = \lambda$ for every $x \in W_1$ and since $A_y \cap A_z = \emptyset$ whenever $y, z \in W_0$ and $y \neq z$.

Suppose that $f$ is a coloring of $H$ for which $f(x) \in F(x)$ whenever $x \in W_0 \cup W_1 = V^H$. Notice that $f \upharpoonright W_0 \in \prod_{y \in W_0} A_y$ and choose $z \in W_1$ such that $f \upharpoonright W_0 = g_z$. By definition, $F(z) = \{g_z(y) : y \in E^z \cap W_0\} = \{f(y) : y \in E^z \cap W_0\}$. But $f(z) \in F(z)$ so $f(z) = f(y)$ for some $y \in E^z \cap W_0$. This means that $f$ is not a good coloring, so we are done. \qed

The above lemma supplies a condition under which $\text{List}(G)$ is large, namely larger than $\lambda$. The next lemma goes in the opposite direction and bounds $\text{List}(G)$ by $\lambda$. This lemma is a quotation from [12]. Recall that if $A$ is a set and $\delta$ is an ordinal then a $\delta$-filtration of $A$ is a $\subseteq$-increasing and continuous sequence $(A_\alpha : \alpha \in \delta)$ such that $|A_\alpha| < |A|$ for every $\alpha \in \delta$ and $A = \bigcup_{\alpha \in \delta} A_\alpha$. If $G = (V, E)$ and $A_\alpha \subseteq V$ then $\text{List}(A_\alpha)$ is the list-chromatic number of the subgraph of $G$ induced by $A_\alpha$. The following is [12, Lemma 3.5].

Lemma 2. Let $G = (V, E)$ be a graph and let $\lambda$ be an infinite cardinal. Let $(A_\alpha : \alpha \in \delta)$ be a filtration of $V$ such that:

1. $\text{List}(A_\alpha) \leq \lambda$ for every $\alpha \in \delta$.
2. $|E^x \cap A_\alpha| < \lambda$ for every $\alpha \in \delta$ and every $x \in V - A_\alpha$. Then $\text{List}(G) \leq \lambda$.

We can prove now our main result in a similar way to that of [12, Proposition 3.6], but using Lemma 1 we can relax the assumptions needed for
singular compactness. Notice that if our singular cardinal is strong limit then singular compactness holds at \( \mu \) for the list chromatic number with no additional parameters. That is, for every \( \lambda < \mu \) if \( \text{List}(H) \leq \lambda \) whenever \( H \leq G \) is of size less than \( \mu \) then \( \text{List}(G) \leq \lambda \) as well.

**Theorem 3.** Assume that:

(a) \( \mu > \text{cf}(\mu) = \theta \) and \( \aleph_0 \leq \lambda < \mu \).
(b) \( \eta^\lambda < \mu \) for every \( \eta < \mu \).
(c) \( G = (V, E) \) is a graph of size \( \mu \).

If \( \text{List}(H) \leq \lambda \) for every \( H \leq G \) of size less than \( \mu \) then \( \text{List}(G) \leq \lambda \). Hence if \( \mu \) is strong limit then the statement holds for every \( \lambda < \mu \).

**Proof.** Let \( (\mu_i : i \in \theta) \) be an increasing continuous sequence of cardinals such that \( \theta, \lambda < \mu_0 \) and \( \mu = \bigcup_{i \in \theta} \mu_i \). For every \( i \in \theta \) let \( \nu_i = \mu_i^\lambda \), so \( \nu_i < \mu \) for every \( i \in \theta \) by virtue of (b).

The following observation is central to the proof. If \( U \subseteq V \) and \( |U| = \mu_i \) then \( |Y_U| < \nu_i \) where \( Y_U = \{ x \in V : |E_x \cap U| \geq \lambda \} \). Indeed, if \( U \) forms a counterexample then one can apply Lemma 1 to the subgraph of \( G \) induced by \( U \cup Y_U \) and conclude that \( \text{List}(U \cup Y_U) > \lambda \) contrary to the assumption that \( \text{List}(H) \leq \lambda \) whenever \( H \leq G \) and \( |H| < \mu \).

Bearing in mind the above observation we choose an array \( (M_i^\alpha : i \in \theta, \alpha \in \lambda^+) \) of elementary submodels of \( \mathcal{H}(\chi) \) where \( \chi \) is a sufficiently large regular cardinal, such that the following requirements are met:

(\( \alpha \)) \( |M_i^\alpha| = \nu_i \) and \( \nu_i + 1 \subseteq M_i^\alpha \) for every \( i \in \theta, \alpha \in \lambda^+ \).

(\( \beta \)) \( (M_i^\alpha : i \in \theta) \) is \( \subseteq \)-increasing and continuous for every \( \alpha \in \lambda^+ \).

(\( \gamma \)) If \( \alpha \in \lambda^+ \) is a limit ordinal then \( M_i^\alpha = \bigcup_{\beta \in \alpha} M_i^\beta \) for every \( i \in \theta \).

(\( \delta \)) \( (M_i^\alpha : i \in \theta) \in M_0^{\alpha+1} \) for every \( \alpha \in \lambda^+ \).

(\( \varepsilon \)) \( \mu, G \in M_i^\alpha \) for every \( i \in \theta, \alpha \in \lambda^+ \).

We may assume, without loss of generality, that \( V = \mu \) so \( G = (\mu, E) \). For every \( i \in \theta \) let \( M_i = \bigcup_{\alpha \in \lambda^+} M_i^\alpha \) and let \( A_i = M_i \cap \mu \). Notice that \( M_i^\alpha \subseteq M_i \) for every \( i \in \theta, \alpha \in \lambda^+ \) and \( \mu \subseteq \bigcup_{i \in \theta} M_i \). Also, \( |A_i| = \nu_i < \mu \) and hence \( (A_i : i \in \theta) \) is a filtration of \( V \). By the assumptions of the theorem it is true that \( \text{List}(A_i) \leq \lambda \) for every \( i \in \theta \), so our filtration satisfies the first assumption of Lemma 2.

We claim that for every \( i \in \theta \) and every \( x \in V - A_i \) it is true that \( |E_x \cap A_i| < \lambda \). If we prove this claim then from Lemma 2 we will infer that \( \text{List}(G) \leq \lambda \) and the proof of the theorem will be accomplished. Assume, therefore, that the claim fails and fix \( i \in \theta \) and \( x \in \mu - A_i \) such that \( |E_x \cap A_i| \geq \lambda \). Since \( A_i = M_i \cap \mu = \bigcup_{\alpha \in \lambda^+} M_i^\alpha \cap \mu \) one can choose \( \alpha \in \lambda^+ \) for which \( |E_x \cap M_i^\alpha| \geq \lambda \).

Define \( Y = \{ y \in \mu : |E_y \cap M_i^\alpha| \geq \lambda \} \). The set \( Y \) is definable in \( M_i \) since \( M_i^\alpha \in M_i \) as well as the other parameters, so \( Y \in M_i \). By the observation at the beginning of the proof we have \( |Y| < \nu_i \) and hence \( Y \subseteq M_i \). But \( x \) satisfies \( |E_x \cap M_i^\alpha| \geq \lambda \) and hence \( x \in Y \subseteq M_i \). This means that \( x \in M_i \cap \mu = A_i \), which is impossible as \( x \in \mu - A_i \), so we are done. \( \square \)
As mentioned in the introduction, the invariant of list chromatic numbers does not satisfy singular compactness without further assumptions. One may wonder whether $\eta^\lambda < \mu$ for every $\eta < \mu$ is the optimal assumption for singular compactness. There is no much room for improving this assumption since $2^\omega = \mu$ already implies the failure of singular compactness with respect to the list chromatic number at $\mu$, and $\omega$ can be replaced by every strongly inaccessible cardinal $\kappa$.

Pondering the possibility that $2^\lambda < \mu$ is insufficient for singular compactness at $\mu$ (with $\lambda$ as a parameter) we conclude from the above theorem that in such a case there must be $\eta < \mu$ so that $\eta^\lambda \geq \mu$. The following claim shows that the first such cardinal $\eta$ has some interesting properties.

**Claim 4.** Suppose that $2^\lambda < \mu, \mu$ is a singular cardinal and singular compactness for the list chromatic number fails at $\mu$ with respect to $\lambda$. Then there is a singular cardinal $\eta < \mu$ which satisfies $\text{cf}(\eta) \leq \lambda, \eta^\lambda \geq \mu$ and $\theta < \eta \Rightarrow \theta^\lambda < \eta$.

**Proof.** Let $\eta$ be the first cardinal below $\mu$ for which $\eta^\lambda \geq \mu$. Such a cardinal exists by virtue of Theorem 3 and the assumptions of the claim. If $\theta < \eta$ and $\theta^\lambda \geq \eta$ then $\theta^\lambda = (\theta^\lambda)^\lambda \geq \eta^\lambda \geq \mu$, and this is impossible since $\eta$ is the first cardinal which satisfies $\eta^\lambda \geq \mu$, hence $\theta < \eta \Rightarrow \theta^\lambda < \eta$. Thus necessarily $\text{cf}(\eta) \leq \lambda$. Indeed, if $\text{cf}(\eta) > \lambda$ then $\eta^\lambda = \bigcup_{\theta < \eta} \theta^\lambda = \eta < \mu$. The proof of the claim is accomplished. 

There are some interesting upshots which follow from this claim. Call $\mu$ a **weird** cardinal iff $\mu$ is singular and there is some $\lambda < \mu$ such that $2^\lambda < \mu$ and singular compactness for list chromatic numbers fails at $\mu$ with respect to $\lambda$. If $\lambda$ is specified then we shall say that $\mu$ is $\lambda$-weird. Let us show that in some sense the collection of weirdos is small. Fix a cardinal $\kappa$ of uncountable cofinality. Suppose that $\mu < \kappa$ is $\lambda$-weird for some $\lambda < \mu$. From Claim 4 we see that there is some $\eta < \mu$ so that $\eta^\lambda \geq \mu$. If $\eta^\lambda < \kappa$ for every $\mu < \kappa$ then $\kappa$ will be called a **sane** cardinal.

**Corollary 5.** If $\kappa$ is sane then the set of weird cardinals below $\kappa$ is not stationary in $\kappa$.

**Proof.** As a first step, we fix an infinite cardinal $\lambda$ and show that the set of $\lambda$-weird cardinals below $\kappa$ is not stationary. Call this set $S$, and assume towards contradiction that $S$ is stationary in $\kappa$. Applying Claim 4 to each element of $S$, let $\eta_\mu < \mu$ be the pertinent cardinal, and since $\kappa$ is sane we see that $\eta_\mu^\lambda < \kappa$ for every $\mu \in S$.

The function $h(\mu) = \eta_\mu$ is regressive on $S$, hence there is a stationary subset $T \subseteq S$ and a fixed $\eta$ such that $\mu \in T \Rightarrow \eta_\mu = \eta$. Since $\kappa$ is sane we see that $\eta^\lambda < \kappa$. On the other hand, since $T$ is unbounded in $\kappa$ one can pick some $\mu \in T$ so that $\eta^\lambda < \mu$ and this is impossible since $\eta^\lambda = \eta_\mu^\lambda \geq \mu$.

For the second step, let $S_0$ be the set of weird cardinals below $\kappa$. For every $\mu \in S_0$ there exists some $\lambda_\mu < \mu$ such that $\mu$ is $\lambda_\mu$-weird. If $S_0$ is
stationary then there are a fixed $\lambda$ and a stationary subset $S_1$ of $S_0$ so that $\mu \in S_1 \Rightarrow \lambda_\mu = \lambda$. Apply the first step to $S_1$ in order to obtain the desired contradiction. □

If $\kappa$ is a strong limit cardinal then the above corollary follows simply from the fact that the set of strong limit singular cardinals below $\kappa$ is a club, and every weird cardinal is not strong limit. However, the above corollary applies to cases in which $\kappa$ is not strong limit, and it shows that the set of weirdos is topologically small, in the sense that it is not stationary. This leads us to the possibility of an unbounded subset of weirdos below a sane cardinal.

In the following theorem we employ a reflection argument with respect to a supercompact cardinal. A similar argument works at many local properties, e.g. the fact that if there is a huge cardinal above a supercompact cardinal then there are unboundedly many huge cardinals below the supercompact, see [7].

**Theorem 6.** Let $\kappa$ be a supercompact cardinal. If $\kappa < \lambda < \mu$ and $\mu$ is $\lambda$-weird then the set of weird cardinals below $\kappa$ is unbounded in $\kappa$.

**Proof.** Let $\mu$ be $\lambda$-weird and notice that necessarily $\lambda > \kappa$. Recall that $\mu$ is a singular cardinal, $2^\lambda < \mu$ and singular compactness fails at $\mu$ and $\lambda$ with respect to the list chromatic number. Let $G$ be a graph of size $\mu$ which witnesses this fact in $V$. Namely, if $H \subseteq G$ and $|H| < \mu$ then $\text{List}(H) \leq \lambda$ despite the fact that $\text{List}(G) > \lambda$.

Choose a sufficiently large $\delta > \mu$ (the cardinal $\delta = (2^\mu)^+$ is also ondo) and an elementary embedding $j : V \rightarrow M$ so that $\kappa = \text{crit}(j), j(\kappa) > \lambda$ and $\delta M \subseteq M$. By the closure properties of $M$ we see that $G \in M$ and it witnesses the fact that $\mu$ is $\lambda$-weird. But $\mu < j(\kappa)$ and hence $M$ knows that there exists a $\lambda$-weird cardinal below $j(\kappa)$. Apply elementarity and conclude that there exists a weird cardinal $\theta$ below $\kappa$ in $V$. Moreover, if $\gamma \in \kappa$ then $j(\gamma) = \gamma$ since $\kappa = \text{crit}(j)$. Hence for every $\gamma \in \kappa$ we know, in $M$, that there is a weird cardinal $\mu < j(\kappa)$ such that $\mu > j(\gamma)$. Therefore, in $V$ there is a weird cardinal $\theta < \kappa$ such that $\theta > \gamma$, and since $\gamma \in \kappa$ was arbitrary we are done. □

As we have seen already, the set of weirdos below $\kappa$ will be unbounded (under the assumption of the theorem) but it will never be stationary. We do not know if the existence of a weird cardinal, especially above a supercompact cardinal, is consistent at all. Actually, one may wonder whether singular compactness for the list chromatic number can fail above a supercompact cardinal. The answer is positive, since the counterexample of [4] with $\omega$ works at every strongly inaccessible cardinal $\lambda$ instead of $\omega$. For completeness, we unfold the details.

**Proposition 7.** Let $\kappa$ be a supercompact cardinal. Let $\lambda$ be a strongly inaccessible cardinal greater than $\kappa$. One can force the existence of a singular cardinal $\mu > \lambda$ such that singular compactness fails at $\lambda$ and $\mu$ with respect to the list chromatic number, while preserving the supercompactness of $\kappa$. 

Proof. Based on \cite{5} we may assume that $\kappa$ is Laver-indestructible. Let $\mathbb{P}$ be a $\kappa$-directed closed forcing notion which forces $2^\lambda = \mu$ for some singular cardinal $\mu > \lambda$. Notice that $\kappa$ remains supercompact in the generic extension. Let us show that $\mu$ and $\lambda$ satisfy the statement of the proposition.

Let $G$ be the complete bipartite graph over $V = A \cup B$, where $A = \{a_\alpha : \alpha \in \lambda\}$ and $B = \{b_\beta : \beta \in \mu\}$. Fix a family of disjoint sets of ordinals $\{S_\alpha : \alpha \in \lambda\}$ so that $|S_\alpha| = \lambda$ for every $\alpha \in \lambda$. We define an assignment $F$ which exemplifies the fact that $\text{List}(G) > \lambda$. Firstly, let $F(a_\alpha) = S_\alpha$ for each $\alpha \in \lambda$. Secondly, fix a bijection $h$ between $B$ and $\prod_{\alpha \in A} S_\alpha$ which maps $b_\beta$ to $g_\beta$, and for each $\beta \in \mu$ let $F(b_\beta) = \{g_\beta(a_\alpha) : \alpha \in \lambda\}$.

Suppose that $f$ is a good coloring of $G$ and $f(x) \in F(x)$ for every $x \in V$. In particular, if $g = f \upharpoonright A$ then $g \in \prod_{\alpha \in A} S_\alpha$. Let $\beta \in \mu$ be such that $h(b_\beta) = g$. It follows that $f(b_\beta) = F(b_\beta) = \{g_\beta(a_\alpha) : \alpha \in \lambda\}$ and hence $f(b_\beta) = g(a_\alpha)$ for some $\alpha \in \lambda$. But $a_\alpha E b_\beta$ and $f(b_\beta) = g(a_\alpha) = f(a_\alpha)$ as $g = f \upharpoonright A$, so $f$ is not good after all.

Suppose now that $H \leq G$ and $|H| < \mu$. Let $V_H = A_H \cup B_H$ and without loss of generality $A_H = A$. Suppose that $|B_H| = \theta < \mu = 2^\lambda$, let $A_H = \{a_\alpha : \alpha \in \lambda\}$ and let $B_H = \{b_\beta : \beta \in \theta\}$. Given $F : A_H \cup B_H \to [\text{Ord}]^\lambda$, we wish to find an $F$-amenable good coloring.

For every $\alpha \in \lambda$, the set of functions $\alpha \to \omega$ is of size less than $\lambda$, so one can choose for every $\varphi \in \alpha \to \omega$ an element $c_\varphi \in F(a_\alpha)$ such that $\varphi, \psi \in \alpha \to \omega, \varphi \neq \psi = c_\varphi \neq c_\psi$. We do this for every $\alpha \in \lambda$. Now for each $h \in \lambda^2$ we define the set $T_h = \{c_{h \upharpoonright \alpha} : \alpha \in \lambda\}$. Notice that $|T_h| = \lambda$ whenever $h \in \lambda^2$. Likewise, if $h \neq h'$ then $|T_h \cap T_{h'}| < \lambda$ since $h \upharpoonright \alpha \neq h' \upharpoonright \alpha$ over an end-segment of $\alpha \in \lambda$.

Therefore, if $\beta \in \theta$ then $F(b_\beta) \subseteq T_h$ is possible for at most one $h \in \lambda^2$. Since $\theta < \mu = 2^\lambda$, one can choose an element $h \in \lambda^2$ such that $-(F(b_\beta) \subseteq T_h)$ for every $\beta \in \theta$. Fix such a function $h$ and define a good coloring $f$ by the following procedure. If $\alpha \in \lambda$ then $f(a_\alpha) = c_{h \upharpoonright \alpha}$ which belongs to $F(a_\alpha)$ by definition. If $\beta \in \theta$ then $f(b_\beta)$ is any element from $F(b_\beta) - T_h$. One can verify that $f$ is a good coloring so we are done.

One may wonder whether the failure of singular compactness with respect to the list-chromatic number is limited to one cardinality. Namely, if $\mu > \text{cf}(\mu), |G| = \mu, \lambda < \mu$ and $\text{List}(H) \leq \lambda$ for every small subgraph $H$ of $G$, is it true that $\text{List}(G) \leq \lambda^+$ (or a similar bound which depends on $\lambda$)? It turns out that $\text{List}(G)$ can be arbitrarily large from the $\aleph$-scale point of view. For instance, if $2^\omega = 2^{\omega_1} = \mu$ and $\mu > \text{cf}(\mu)$ then Komjath’ example of the complete bipartite graph $G$ of size $\omega \times 2^\omega$ satisfies $\text{List}(H) \leq \aleph_0$ for every $H \leq G$ of size less than $\mu$, but $\text{List}(G) > R_1$. This can be generalized as in Proposition \cite{7} It seems, however, that the cofinality of $\mu$ is restricted in such examples.

We indicate that the situation of $\kappa < \lambda < \eta < \mu$ with $\kappa$ being supercompact, $2^\lambda < \eta$ and $\eta^{\aleph_0} \geq \mu$ is forceable. Begin with a Laver-indestructible supercompact cardinal $\kappa$, choose $\lambda < \eta < \mu$ so that $\kappa < \lambda$ and add many
Cohen subsets to $\eta$ of size $\lambda$ (the supercompactness will be preserved). As in the proof of Theorem 6, unboundedly many triples $\lambda' < \eta' < \mu'$ below $\kappa$ will satisfy $2^{\lambda'} < \eta'$ and $\eta'^{\lambda'} \geq \mu'$. We do not know, however, if this gives weird cardinals.

Question 8. Is it consistent that there exists a weird cardinal?
2. Tiltan and Reflection

Suppose that $\kappa = \text{cf}(\kappa) > \aleph_0$ and $S \subseteq \kappa$ is stationary. The club principle $\clubsuit_S$ is a prediction principle introduced by Ostaszewski in [8]. It says that there exists a sequence $(T_\alpha : \alpha \in S)$ where each $T_\alpha$ is an unbounded subset of $\alpha$, and for every unbounded $A \subseteq \kappa$ the set $S_A = \{\alpha \in S : T_\alpha \subseteq A\}$ is stationary. Henceforth we call the club principle tiltan, to avoid double meaning and overuse of the term club.

It is clear that $\lozenge S$ implies $\clubsuit S$ and this is true, in particular, where $S = \omega_1$. For the converse, $\clubsuit_{\omega_1} \land 2^{\omega_1} = \omega_1$ implies $\lozenge_{\omega_1}$. However, one can force tiltan at $\omega_1$ along with $2^{\omega_1} > \omega_1$, in which case diamond fails at $\omega_1$. Hence tiltan is strictly weaker than diamond.

The methods for obtaining tiltan with a large value of $2^{\omega_1}$ divide into two groups. One approach is to use collapses. Shelah proved in [10] that if one begins with diamond at every stationary subset of $\omega_1$ and $\omega_2$, increases $2^{\omega_1}$ and collapses with Lévy($\aleph_0, \aleph_1$) then tiltan holds at every stationary subset of $\omega_1$ in the generic extension. Of course, $2^{\omega_1} > \omega_1$ and hence diamond fails despite the fact that tiltan holds. Todorčević, [11], employs a similar idea. He iterates proper forcing notions with countable support where the length of the iteration is more than $\omega_2$.

Another approach is taken in [3], and here no collapse is involved. The forcing there is not $\aleph_1$-cc but a special argument proves that $\aleph_1$ is preserved. Clearly, if one collapses $\aleph_1$ as in the first approach then the forcing is not $\aleph_1$-cc. One may wonder, therefore, if this is always the case. That is, whether a ccc forcing notion which increases $2^{\omega_1}$ necessarily destroys tiltan at some stationary set. The main result of this section is a positive statement in this direction, provided that there exists a supercompact cardinal in $V$.

Let us consider another phenomenon related to compactness with respect to list-chromatic numbers. This is a reflection principle discussed in [12] and denoted by $\text{RP}(\text{List}, \lambda)$. It says that for every graph $G$ such that $|G| \leq \lambda$, if $\text{List}(G) > \aleph_0$ then there exists $H \leq G$ such that $|H| = \aleph_1$ and $\text{List}(H) > \aleph_0$.

Another invariant of graphs is called the coloring number. Given a graph $G$, the coloring number $\text{Col}(G)$ is the minimal $\kappa$ such that there exists a well-ordering $<_* \text{ of } V_G$ which satisfies $|\{u \in E^v : u <_* v\}| < \kappa$ for every $v \in V_G$. Notice that $\text{List}(G) \leq \text{Col}(G)$ for every graph $G$. The corresponding reflection principle is defined in the same way. Thus $\text{RP}(\text{Col}, \lambda)$ is the statement that for every graph $G$ with $|G| \leq \lambda$, if $\text{Col}(G) > \aleph_0$ then there exists $H \leq G$ such that $|H| = \aleph_1$ and $\text{Col}(H) > \aleph_0$. By writing $\text{RP}(\text{Col})$ or $\text{RP}(\text{List})$ we assert that these principles hold at every $\lambda$. By [12, Fact 2.12] the principle $\text{RP}(\text{Col})$ is preserved in ccc forcing extensions.

The following is a modification of a theorem from [12] in which $\lozenge_S$ is replaced by $\clubsuit_S$. The proof is essentially the same proof, but there are some points in which one has to change the argument slightly. We spell-out the details for the sake of completeness.
Theorem 9. Assume that \( \lambda = \text{cf}(\lambda) < \kappa = \text{cf}(\kappa) \). Suppose that \( \clubsuit_S \) holds at every stationary \( S \subseteq S^\lambda_\kappa \). Let \( G \) be a graph with \( V_G = \kappa \). Suppose that \( \text{Col}(G) > \lambda \) and \( \text{Col}(H) \leq \lambda \) for every \( H \leq G \) of size less than \( \kappa \). Then \( \text{List}(G) > \lambda \).

Proof. Let \( S = \{ \alpha \in S^\lambda_\kappa : \exists \beta \geq \alpha, |\alpha \cap E^\beta| \geq \lambda \} \). By [12] Fact 2.7 one knows that \( S \) is a stationary subset of \( \kappa \). For every \( \alpha \in S \) choose \( \beta(\alpha) \geq \alpha \) such that \( |\alpha \cap E^{\beta(\alpha)}| \geq \lambda \). Without loss of generality, if \( \alpha_0, \alpha_1 \in S \) and \( \alpha_0 < \alpha_1 \) then \( \beta(\alpha_0) < \beta(\alpha_1) \).

Let \( (r_\alpha, t_\alpha : \alpha \in S) \) be a sequence of functions where both \( r_\alpha \) and \( t_\alpha \) are partial functions from \( \alpha \) into \( \alpha \) and if \( f, g : \kappa \to \kappa \) then \( r_\alpha \subseteq f \land t_\alpha \subseteq g \) for stationarily many \( \alpha \in S \). We may assume that the domain of each \( r_\alpha \) and each \( t_\alpha \) is unbounded in \( \alpha \).

Let \( (A_\beta : \beta \in \kappa) \) be a sequence of pairwise disjoint sets where \( A_\beta \in [\kappa]^\lambda \) for every \( \beta \in \kappa \). Recall that \( |\alpha \cap E^{\beta(\alpha)}| \geq \lambda \) for every \( \alpha \in S \), and fix a set \( x_\alpha \subseteq \alpha \cap E^{\beta(\alpha)} \) of size \( \lambda \) for each \( \alpha \in S \). Based on the proof of [12] Proposition 3.1] we define two \( \lambda \)-assignments \( F_0, F_1 \) and we show that (at least) one of them witnesses \( \text{List}(G) > \lambda \).

Suppose that \( \beta \in \kappa \). The substantial part of the definition lies on the cases in which \( \beta = \beta(\alpha) \) for some \( \alpha \in S \), so if \( \beta \neq \beta(\alpha) \) for every \( \alpha \in S \) then simply let \( F_0(\beta) = F_1(\beta) = A_\beta \). If \( \beta = \beta(\alpha) \) for some \( \alpha \in S \) then \( \alpha \) is unique, and we distinguish three cases:

(a) If \( |r_\alpha''x_\alpha| = \lambda \) then \( F_0(\beta) = r_\alpha''x_\alpha \) and \( F_1(\beta) = A_\beta \).

(b) If \( |r_\alpha''x_\alpha| < \lambda \) and \( |t_\alpha''x_\alpha| = \lambda \) then \( F_0(\beta) = A_\beta \) and \( F_1(\beta) = t_\alpha''x_\alpha \).

(c) If both \( |r_\alpha''x_\alpha| < \lambda \) and \( |t_\alpha''x_\alpha| < \lambda \) then \( F_0(\beta) = F_1(\beta) = A_\beta \).

Assume that \( f : \kappa \to \kappa \) is \( F_0 \)-amenable and \( g : \kappa \to \kappa \) is \( F_1 \)-amenable. Our goal is to show that (at least) one of them is not good. Suppose, therefore, that \( f \) is good. Choose \( \alpha \in S \) such that \( r_\alpha \subseteq f \land t_\alpha \subseteq g \). In particular, \( r_\alpha \upharpoonright x_\alpha \subseteq f \upharpoonright x_\alpha \) and \( t_\alpha \upharpoonright x_\alpha \subseteq g \upharpoonright x_\alpha \) since \( x_\alpha \subseteq \alpha \).

Let \( \beta = \beta(\alpha) \). We consider the above three possibilities. If (a) holds, namely \( |r_\alpha''x_\alpha| = \lambda \) then \( F_0(\beta) = r_\alpha''x_\alpha = f''x_\alpha \). Since \( f \) is \( F_0 \)-amenable we see that \( f(\beta) \in f''x_\alpha \). Choose \( \eta \in x_\alpha \) such that \( f(\beta) = f(\eta) \) and notice that \( \eta \neq \beta \) since \( \eta \in x_\alpha \subseteq \alpha \) and \( \beta = \beta(\alpha) \geq \alpha \). However, \( x_\alpha \subseteq E^\beta \) and hence \( (\eta, \beta) \in E \) yet \( f(\eta) = f(\beta) \). This is impossible since \( f \) is good.

We conclude, therefore, that (a) fails, that is to say \( |r_\alpha''x_\alpha| < \lambda \). Recall that we fixed an ordinal \( \alpha \in S \) such that \( r_\alpha \subseteq f \) and \( t_\alpha \subseteq g \). We may assume, without loss of generality, that both \( r_\alpha \) and \( t_\alpha \) are defined on every \( \gamma \in x_\alpha \). Indeed, define \( x = \bigcup_{\alpha \in S} x_\alpha \) and consider all the ordinals \( \alpha \) for which \( r_\alpha \subseteq f \upharpoonright x \) and \( t_\alpha \subseteq g \upharpoonright x \). By Fodor’s lemma there must be some \( \alpha \in S \) and \( y_\alpha \in [x_\alpha]^\lambda \) for which both \( r_\alpha \) and \( t_\alpha \) are defined on every \( \gamma \in y_\alpha \). Since all we need in the following argument is just a set of size \( \lambda \) we may assume that \( y_\alpha = x_\alpha \).

Let \( A = \{ \gamma \in x_\alpha : F_0(\gamma) = A_\gamma \} \). We claim that \( |A| < \lambda \). To see this, notice that \( f''A \) is one-to-one since \( f(\gamma) \in A_\gamma = F_0(\gamma) \) for each \( \gamma \in A \) and \( A_\gamma \cap A_\delta = \emptyset \) whenever \( \gamma \neq \delta \). Hence if \( |A| = \lambda \) then \( |f''A| = \lambda \) as well. But
then $|r_{\alpha}''A| = |f''A| = \lambda$ since $A \subseteq x_{\alpha}$ and then $|r_{\alpha}''x_{\alpha}| = \lambda$ contradicting the fact that $|r_{\alpha}''x_{\alpha}| < \lambda$.

Now if $\gamma \in x_{\alpha} - A$ (and there are $\lambda$-many such ordinals) then $F_0(\gamma) \neq A_\gamma$. By definition, $F_1(\gamma) = A_\gamma$ for each such ordinal $\gamma$, and this will happen at $\lambda$-many $\gamma$s. Since $g$ is $F_1$-amenable, if $\gamma, \delta \in x_{\alpha} - A$ and $\gamma \neq \delta$ then $g(\gamma) \neq g(\delta)$.

Hence $|r_{\alpha}''x_{\alpha}| = |g''x_{\alpha}| = \lambda$ and one concludes that necessarily (b) holds.

Let us show that (b) implies that $g$ is not good, and this will accomplish the proof. Recall that $\beta = \beta(\alpha)$ and (b) asserts that $F_1(\beta) = t_{\alpha}''x_{\alpha}$. Pick $\eta \in \alpha$ such that $g(\beta) = t_{\alpha}(\eta)$. Since $t_{\alpha}''x_{\alpha} \subseteq g''x_{\alpha}$ we see that $g(\beta) = g(\eta)$. But $\eta \neq \beta$ since $\eta \in x_{\alpha} \subseteq \alpha \leq \beta(\alpha) = \beta$, so $(\eta, \beta) \in E$ and hence $g$ is not good as required.

From the above theorem one can derive an interesting conclusion with regard to the club principle. We are interested in the properties of forcing notions which increase the size of the continuum but preserve tiltan at various stationary sets.

**Theorem 10.** Let $\kappa$ be supercompact and let $G \subseteq \text{Lévy}(\omega_1, < \kappa)$ be $V$-generic. Let $\mathbb{P}$ be a forcing notion in $V[G]$ and let $H \subseteq \mathbb{P}$ be $V[G]$-generic. If the continuum hypothesis fails in $V[G][H]$ then for some stationary $S \subseteq \omega_1$ one has $V[G][H] \models \neg \clubsuit_S$.

**Proof.** We commence with the following general statement. Assume that $\lambda = \text{cf}(\lambda) < \kappa$. Assume further that $\clubsuit_S$ holds whenever $S \subseteq S_\lambda^\theta$ for every $\theta \in \text{Reg} \cap (\lambda, \kappa)$. Then for every graph $G$ of size less than $\kappa$ it is true that $\text{Col}(G) > \lambda$ if and only if $\text{List}(G) > \lambda$. Of course, one direction is immediate since the inequality $\text{List}(G) \leq \text{Col}(G)$ is always true. For the opposite direction suppose that $\text{Col}(G) > \lambda$. Let $H \leq G$ be such that $\text{Col}(H) > \lambda$ and for every $I \leq H$ if $|I| < |H|$ then $\text{Col}(I) \leq \lambda$. Shelah proved in [9] that the coloring number satisfies singular compactness. Hence if $H$ is of minimal cardinality then necessarily $|H|$ is a regular cardinal which falls within the scope of Theorem 9. Applying Theorem 9 to the graph $H$ one concludes that $\text{List}(H) > \lambda$ and hence $\text{List}(G) > \lambda$.

In particular, if $\clubsuit_S$ holds at every stationary $S \subseteq \omega_1$ and $G$ is a graph of size $\aleph_1$ such that $\text{Col}(G) = \aleph_1$ then $\text{List}(G) = \aleph_1$. Therefore, if $\clubsuit_S$ holds at every stationary $S \subseteq \omega_1$ and $\text{RP}(\text{Col})$ is true then $\text{RP}(\text{List})$ is true as well. Indeed, let $G$ be any graph of size $\kappa > \aleph_1$ such that $\text{List}(G) > \aleph_0$. We know that $\text{Col}(G) \geq \text{List}(G) > \aleph_0$ so from $\text{RP}(\text{Col})$ we choose $H \leq G$, $|H| = \aleph_1$ such that $\text{Col}(H) > \aleph_0$. From the above argument we conclude that $\text{List}(H) > \aleph_0$, that is to say $\text{RP}(\text{List})$.

From [12] we know that if $\kappa$ is supercompact and $G$ is generic for the collapse of the predecessors of $\kappa$ then $\text{RP}(\text{Col})$ holds in $V[G]$. Notice further that $\Diamond_S$ holds at every stationary $S \subseteq \omega_1$ and, in particular, $2^{\omega_1} = \omega_1$. Since $\mathbb{P}$ is ccc we know that $\text{RP}(\text{Col})$ holds in $V[G][H]$.

Assume towards a contradiction that $\clubsuit_S$ holds in $V[G][H]$ whenever $S \subseteq \omega_1$ is stationary. By the above considerations $\text{RP}(\text{List})$ is true, and in particular $\text{RP}(\text{List}, 2^{\omega_1})$ holds in $V[G][H]$. But $\text{RP}(\text{List}, 2^{\omega_1})$ implies $2^{\omega_1} = \omega_1$.
as proved in [12, Lemma 4.1]. We are assuming that \( P \) forces \( 2^{\omega} > \omega_1 \), thus we arrived at a contradiction.

We believe that the limitation on forcing notions which increase the continuum and preserve tiltan as expressed in the above theorem does not depend on the existence of large cardinals in the ground model. Thus we phrase the following:

**Conjecture 11.** Suppose that \( V = L \). Let \( P \) be a ccc forcing notion and let \( G \subseteq P \) be generic. If \( 2^{\omega} > \omega_1 \) in \( V[G] \) then \( \clubsuit_S \) fails in \( V[G] \) for some stationary set \( S \subseteq \omega_1 \).
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