Necessary and sufficient conditions on existence of radial solutions for exterior Dirichlet problem of fully nonlinear elliptic equations

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Abstract: In this paper, we study the exterior Dirichlet problem for the fully nonlinear elliptic equation \( f(\lambda(D^2u)) = 1 \). We obtain the necessary and sufficient conditions of existence of radial solutions with prescribed asymptotic behavior at infinity.

Keywords: fully nonlinear elliptic equation; radial solutions; necessary and sufficient conditions; exterior Dirichlet problem; asymptotic behavior

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1 Introduction

In this paper, we study the exterior Dirichlet problem of the fully nonlinear elliptic equation

\[
\begin{align*}
  f(\lambda(D^2u)) &= 1 \quad \text{in } \mathbb{R}^n \setminus B_1(0), \\
  u &= b \quad \text{on } \partial B_1(0).
\end{align*}
\]  

(1.1) (1.2)

where \( B_1(0) = \{ x \in \mathbb{R}^n : |x| < 1 \} \) is the unit ball in \( \mathbb{R}^n \), \( b \) is a constant, and \( f(\lambda) \) is a given symmetric function of the eigenvalues \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of the Hessian matrix \( D^2u \).

We study \( f \) in an open convex symmetric cone \( \Gamma \subset \mathbb{R}^n \) with vertex at the origin,

\[
\{ \lambda \in \mathbb{R}^n | \lambda_j > 0, 1 \leq j \leq n \} \subset \Gamma \subset \{ \lambda \in \mathbb{R}^n | \sum_{j=1}^n \lambda_j > 0, 1 \leq j \leq n \}.
\]  

(1.3)

Suppose that \( f \in C^\infty(\Gamma) \cap C^0(\Gamma) \) is concave and symmetric in \( \lambda_j \),

\[
f > 0 \text{ in } \Gamma, \ f = 0 \text{ on } \partial \Gamma; \ f_{\lambda_j} > 0 \text{ in } \Gamma \ \forall 1 \leq j \leq n.
\]  

(1.4)

Let \( \sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, k = 1, 2, \ldots, n \), and \( \Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, j = 1, 2, \ldots, k \} \). Then \( (f, \Gamma) = (\sigma_k^\Gamma, \Gamma_k) \) and \( (f, \Gamma) = ((\sigma_{l+k}^\Gamma), \Gamma_k) \), \( 1 \leq l < k \leq n \) are the special cases of \( (f, \Gamma) \). In particular, if \( k = n \), \( (f, \Gamma) = (\sigma_n^\Gamma, \Gamma_k) \) corresponds to the Monge-Ampère operator.

A classical theorem for Monge-Ampère equation states that any classical convex solution of

\[
\det D^2u = 1 \text{ in } \mathbb{R}^n
\]

must be a quadratic polynomial. This theorem was established by Jörgens \[10\] \( (n = 2) \), Calabi \[5\] \( (n \leq 5) \) and Pogorelov \[15\] \( (n \geq 2) \). Later Cheng-Yau \[6\] proved the Jörgens-Calabi-Pogorelov theorem by the simpler and more analytical way along the lines of affine
geometry. This result for viscosity solutions was extended by Caffarelli [3]. Jost-Xin [11] also gave another proof of this theorem. However, Trudinger-Wang [16] proved that if $D$ is an open convex subset in $\mathbb{R}^n$ and $u$ is a convex $C^2$ solution to $\det D^2 u = 1$ in $D$ with $\lim_{x \to \partial D} u(x) = \infty$, then $D = \mathbb{R}^n$.

In 2003, Caffarelli-Li [4] made an extension of the Jörgens-Calabi-Pogorelov theorem to exterior domains. Moreover, Caffarelli-Li [4] also established the existence of solutions with asymptotic behavior at infinity to the exterior Dirichlet problem of Monge-Ampère equations.

**Theorem 1.1.** ([4]) Let $\Omega$ be a smooth, bounded, strictly convex domain in $\mathbb{R}^n$, $n \geq 3$ and $\phi \in C^2(\partial \Omega)$. Then for any symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, $\hat{b} \in \mathbb{R}^n$, there exists a constant $c_1 = c_1(n, \Omega, \phi, \hat{b}, A)$ such that for any $\hat{c} > c_1$, there exists a unique solution $u \in C^\infty(\mathbb{R}^n \setminus \Omega) \cap C^0(\mathbb{R}^n \setminus \Omega)$ satisfying

$$
\begin{align*}
\det D^2 u &= 1 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, \\
u &= \phi \text{ on } \partial \Omega, \\
\lim_{|x| \to \infty} |x|^{n-2} \left| u(x) - \left( \frac{1}{2} x^T A x + \hat{b} \cdot x + \hat{c} \right) \right| &= 0.
\end{align*}
$$

For $n = 2$, the existence of solutions to the exterior Dirichlet problem for Monge-Ampère equation $\det D^2 u = 1$ was established by Bao-Li [1] using the Perron method. Ferrer-Martínez-Milán [8, 9] also studied the similar problems by using the complex variable method. We can also refer to Delanoë [7]. Bao-Li-Zhang [2] proved the existence of solutions to the exterior Dirichlet problem for $\det D^2 u = f$ with $f$ being a perturbation of 1 at infinity for $n \geq 2$. Ju-Bao [12] obtained the existence of exterior solutions with $f$ being a perturbation of $f_0(|x|)$ at infinity for $n \geq 3$. For the fully nonlinear elliptic equations (1.1), Li-Bao [13] obtained the existence of solutions of the exterior Dirichlet problem.

The constant $c_1$ in Theorem 1.1 plays an important role in the existence and nonexistence of solutions to the exterior Dirichlet problem. Wang-Bao [17] first studied the constant among the radially symmetric solutions to Hessian equations $\sigma_k(\lambda(D^2 u)) = 1$.

**Theorem 1.2.** ([17]) Let $n \geq 3$, $2 \leq k \leq n$ and $a = (\frac{1}{2\pi})^{1/k}$. There exists a unique radially symmetric solution $u \in C^2(\mathbb{R}^n \setminus \overline{B_1(0)}) \cap C^1(\mathbb{R}^n \setminus B_1(0))$ satisfying

$$
\begin{align*}
\sigma_k(\lambda(D^2 u)) = 1 & \text{ in } \mathbb{R}^n \setminus \overline{B_1(0)}, \\
u &= \hat{a} \text{ on } \partial B_1(0), \\
u &= \frac{a}{2} |x|^2 + \hat{c} + O(|x|^{2-n}) \text{ as } |x| \to \infty,
\end{align*}
$$

if and only if $\hat{c} \geq C^* = \hat{a} - \frac{a}{2} + a \int_1^\infty s((1 - s^{-n})^\frac{1}{k} - 1) ds$. 

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Recently, Li-Lu [14] characterized the existence and nonexistence of solutions in terms of the asymptotic behavior to the exterior Dirichlet problem with the right hand side being 1 or the perturbation of 1 at infinity. In this paper, we obtain the necessary and sufficient conditions of existence of radial solutions to (1.1) and (1.2). We suppose that there exists a constant $c_*$ such that $f(c_*, \ldots, c_*) = 1.$ (1.8)

Let $\Omega$ be a domain in $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is called admissible if at each $x \in \Omega$, $\lambda(D^2u(x)) \in \Gamma$. The condition (1.4) guarantees that (1.1) is elliptic for the admissible functions. Set

$$\Phi = \{u \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B_1}) | u \text{ is an admissible radially symmetric function.}\}$$

The main result is the following.

**Theorem 1.3.** Let $n \geq 3$ and (1.8) hold. There exists a unique function $u \in \Phi$ satisfying (1.1), (1.2) and

$$u(x) = \frac{c_*}{2}|x|^2 + c + O(|x|^{2-n})$$

if and only if $c \in [c_0, +\infty)$ where $c_0 = \mu(W^{-1}_s(\gamma_0)) + b - c_*/2$, $0 \leq \gamma_0 < c_*$,

$$\mu(\alpha) = \int_1^{+\infty} s(W(s, \alpha) - c_*)ds$$

and $W_s(\alpha) = W(s, \alpha)$ satisfies (2.11) and (2.12).

## 2 Proof of Theorem 1.3

**Lemma 2.1.** ([17]) Let $\lambda = (\beta, \gamma, \ldots, \gamma) \in \Gamma$. Then $\gamma > 0$.

**Lemma 2.2.** Let $\lambda = (\beta, \gamma, \ldots, \gamma) \in \mathbb{R}^n$, $n \geq 2$ and $f(\lambda) = 1$. Then $\lambda \in \Gamma$ if and only if there exists a constant $0 \leq \gamma_0 < c_*$ such that

$$\gamma_0 < \gamma < +\infty.$$  

**Proof.** Consider the equation

$$f(\beta, \gamma, \ldots, \gamma) = 1 \text{ for } (\beta, \gamma, \ldots, \gamma) \in \Gamma.$$  

Similar to [13], from (1.4) and (1.8), we know that

$$f(c_*, \gamma, \ldots, \gamma) > 1, \text{ if } \gamma > c_*.$$

By (1.3), for $(\beta_0, \gamma, \ldots, \gamma) \in \partial\Gamma$,

$$f(\beta_0, \gamma, \ldots, \gamma) = 0.$$  

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Hence from the intermediate value theorem and (1.4), for each $\gamma > c^*$, there exists a unique $g(\gamma)$ between $\beta_0$ and $c^*$ such that $(g(\gamma), \gamma, \ldots, \gamma) \in \Gamma$, and
\[
f(g(\gamma), \gamma, \ldots, \gamma) = 1, \quad \forall \gamma > c^*.
\] (2.3)

Then we can define the continuous and differentiable function $g$ such that
\[
\beta = g(\gamma), \quad \gamma \geq c^*
\]
and
\[
g(c^*) = c^*.
\]
Moreover, differentiating (2.3) with respect to $\gamma$, we get
\[
f_{\lambda_1}(g(\gamma), \gamma, \ldots, \gamma)g'(\gamma) + \sum_{j=2}^{n} f_{\lambda_j}(g(\gamma), \gamma, \ldots, \gamma) = 0, \quad \forall \gamma \geq c^*.
\] (2.4)

So by (1.4),
\[
\frac{d\beta}{d\gamma} = g'(\gamma) = -\frac{\sum_{j=2}^{n} f_{\lambda_j}(g(\gamma), \gamma, \ldots, \gamma)}{f_{\lambda_1}(g(\gamma), \gamma, \ldots, \gamma)} < 0.
\] (2.5)

In particular, by the symmetry of $f$, we can deduce that
\[
g'(c^*) = 1 - n.
\] (2.6)

Let
\[
F(\beta, \gamma, \ldots, \gamma) = -\frac{\sum_{j=2}^{n} f_{\lambda_j}(\beta, \gamma, \ldots, \gamma)}{f_{\lambda_1}(\beta, \gamma, \ldots, \gamma)}.
\]

Then
\[
\frac{d\beta}{d\gamma} = F(\beta, \gamma, \ldots, \gamma).
\]

Since $\partial F/\partial \beta$ is continuous, then by the extension theorem of ODE, $\beta = g(\gamma), \gamma > c^*$ can be extended to the left of $c^*$. And because $g'(\gamma) < 0$ and due to Lemma 2.1 $\gamma > 0$, then $\beta = g(\gamma)$ can be extended to $\gamma_0, 0 \leq \gamma_0 < c^*$ such that $\lim_{\gamma \to \gamma_0^+} g(\gamma) = +\infty$. Then the maximum existence interval of $\beta = g(\gamma)$ is $(\gamma_0, +\infty), 0 \leq \gamma_0 < c^*$, that is,
\[
\beta = g(\gamma), \quad \gamma \in (\gamma_0, +\infty).
\] (2.7)

On the other hand, differentiating (2.4) with respect to $\gamma$, we have that
\[
g''(\gamma) = -\frac{\Lambda^T \left( \frac{\partial^2 f}{\partial \gamma \partial \lambda_1} \right) \Lambda}{f_{\lambda_1}},
\]
where $\Lambda^T = (g'(\gamma), 1, \ldots, 1)$. Since $f$ is concave and $f_{\lambda_1} > 0$, then
\[
g''(\gamma) > 0.
\] (2.8)
Then
\[ g'(\gamma) > g'(c_*) = 1 - n \text{ for } \gamma > c_* \] (2.9)
Since \( \lim_{\gamma \to c_*^+} g(\gamma) = +\infty \), then we declare that
\[ g(\gamma) > (1 - n)\gamma, \quad \gamma_0 < \gamma < +\infty. \] (2.10)
On the contrary, since \( g'(\gamma) < 0 \) and \( g(c_*) = c_* \), then there exists some \( \tilde{\gamma} \in (c_*, +\infty) \) such that \( g(\tilde{\gamma}) = (1 - n)\tilde{\gamma} \) and \( g(\gamma) < (1 - n)\gamma \) for \( \gamma > \tilde{\gamma} \). So
\[
g'(\tilde{\gamma}) = \lim_{\gamma \to \tilde{\gamma}^+} \frac{g(\gamma) - g(\tilde{\gamma})}{\gamma - \tilde{\gamma}} \leq \lim_{\gamma \to \tilde{\gamma}^+} \frac{(1 - n)\gamma - (1 - n)\tilde{\gamma}}{\gamma - \tilde{\gamma}} = 1 - \gamma
\]
which contradicts with (2.9). Hence (2.10) holds and then \( g(\gamma) + (n-1)\gamma > 0 \). Moreover, if \( \gamma_0 < \gamma < +\infty \), then \( f(g(\gamma), \gamma, \ldots, \gamma) = 1 \) and \( g(\gamma) \) may be positive. So \( (g(\gamma), \gamma, \ldots, \gamma) \in \Gamma \).

The Lemma is proved. \(\square\)

**Lemma 2.3.** Let \( \alpha > 0 \) and \( g \) be the same function as (2.7). Then the problem
\[
d\frac{W}{dr} = g(W) - \frac{W}{r}, \quad r > 1, \quad W(1) = \alpha \] (2.11)
has a unique solution \( W = W(r, \alpha) \) and
\[
\lim_{r \to \infty} W(r, \alpha) = c_* . \] (2.13)

**Proof.** If \( W > c_* \), then by (2.5), we have \( g(W) < g(c_*) = c_* \). Thus \( g(W) - W < 0 \), that is, \( dW/dr < 0 \).

If \( W < c_* \), then by (2.5), we have \( g(W) > g(c_*) = c_* \). Thus \( g(W) - W > 0 \), that is, \( dW/dr > 0 \).

Let \( G(W, r) = \frac{g(W) - W}{r} \), then \( \partial G/\partial W \) is continuous. In addition, we know that \( W = c_* \) is a special solution of (2.11). Hence by the existence and uniqueness theorem of solutions to the ODE equation, we know that (2.11) and (2.12) has a unique solution \( W = W(r, \alpha) \). Then by the extension theorem of solutions, we know that (2.13) holds.

The Lemma is proved. \(\square\)

**Remark 2.1.** For the proof of (2.13), we can also refer to Lemma 2.2 in [13].
Lemma 2.4. Let $u \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B}_1)$ be a radial solution of (1.1) and (1.2) and

$$\alpha = u'(1).$$

Then $\lambda(D^2u) \in \Gamma$ if and only if

$$\sup_{r \geq 1} W_r^{-1}(\gamma_0) \leq \alpha < +\infty,$$

where $W_r(\alpha) = W(r, \alpha)$ satisfies (2.11) and (2.12).

Proof. Let $u(x) = u(|x|) = u(r) \in C^1(\mathbb{R}^n \setminus B_1) \cap C^2(\mathbb{R}^n \setminus \overline{B}_1)$ be a radial solution of (1.1) and (1.2), then the eigenvalues of the Hessian matrix $D^2u$ are

$$\lambda_1 = u'', \lambda_2 = \cdots = \lambda_n = \frac{u'}{r}.$$

So

$$f\left(\frac{u''}{r}, \frac{u'}{r}, \ldots, \frac{u'}{r}\right) = 1.$$

By Lemma 2.2 we have that

$$\gamma_0 < \frac{u'}{r} < +\infty.$$

Let $W(r) = u'(r)/r$, then

$$\gamma_0 < W(r) < +\infty,$$

and

$$u''(r) = rW'(r) + W(r).$$

On the other hand, by (2.16) and (2.7), we know that

$$u''(r) = g\left(\frac{u'}{r}\right) = g(W(r)), \quad \gamma_0 < W(r) < +\infty.$$

So $W(r) = u'(r)/r$ satisfies (2.11) and (2.12). In the following, we denote $W(r) = W(r, \alpha) = W_r(\alpha)$.

Differentiating (2.11) and (2.12) with respect to $\alpha$, we know that $V = \partial W(r, \alpha)/\partial \alpha$ satisfies

$$\begin{cases}
\frac{\partial V}{\partial r} = \left(\frac{g'(W(r, \alpha)) - 1}{r}\right)V, & r > 1, \\
V(1) = 1.
\end{cases}$$

Then

$$\frac{\partial W(r, \alpha)}{\partial \alpha} = \exp \int_1^r \frac{g'(W(t, \alpha)) - 1}{t} dt.$$

And then $W(r, \alpha)$ is strictly increasing in $\alpha$. Next we prove that

$$W(r, \alpha) \to +\infty, \quad \text{as} \quad \alpha \to +\infty.$$
Indeed, if $\alpha \to +\infty$, that is, $W(1) \to +\infty$, as the proof of Lemma 2.3, we can know that $W(r, \alpha) > c_*$. Hence by (2.8) and (2.6), we obtain that

$$g'(W(r, \alpha)) = g'(c_*) = 1 - n.$$ 

Then by (2.19), we get that

$$\frac{\partial W(r, \alpha)}{\partial \alpha} > r^{-n}.$$ 

And thus $W(r, \alpha) > c_*r^{-n} + W(r, 0)$. So (2.20) holds.

Since $W(r, \alpha)$ is strictly increasing in $\alpha$, then $W^{-1}(\alpha)$ exists, and from (2.17), we know that (2.13) holds.

On the other hand, if (2.14) holds, then (2.17) holds, by Lemma 2.2, we know that $\lambda(D^2u) \in \Gamma$. \hfill \Box

**Proof of Theorem 1.3** Due to the Proposition 2.1 in [13],

$$u(x) = \frac{c_*}2 |x|^2 + \mu(\alpha) + b - \frac{c_*}2 + O(|x|^{2-n}),$$

where $\mu(\alpha)$ is the same as (1.10). Moreover, by the Proposition 2.1 in [13], $\mu(\alpha)$ is strictly increasing in $\alpha$ and

$$\lim_{\alpha \to +\infty} \mu(\alpha) = +\infty.$$ 

Then by Lemmas 2.2 and 2.4 we know that Theorem 1.3 is true. \hfill \Box

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