THE MAX-PLUS ALGEBRA OF EXPONENT MATRICES OF TILED ORDERS

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Abstract. An exponent matrix is an $n \times n$ matrix $A = (a_{ij})$ over $\mathbb{N}_0$ satisfying (1) $a_{ii} = 0$ for all $i = 1, \ldots, n$ and (2) $a_{ij} + a_{jk} \geq a_{ik}$ for all pairwise distinct $i, j, k \in \{1, \ldots, n\}$. In the present paper we study the set $\mathcal{E}_n$ of all non-negative $n \times n$ exponent matrices as an algebra with the operations $\oplus$ of component-wise maximum and $\odot$ of component-wise addition. We provide a basis of the algebra $(\mathcal{E}_n, \oplus, \odot)$ and give a row and a column decompositions of a matrix $A \in \mathcal{E}_n$ with respect to this basis. This structure result determines all $n \times n$ tiled orders over a fixed discrete valuation ring. We also study automorphisms of $\mathcal{E}_n$ with respect to each of the operations $\oplus$ and $\odot$ and prove that $\text{Aut}(\mathcal{E}_n, \odot) = \text{Aut}(\mathcal{E}_n, \oplus) = \text{Aut}(\mathcal{E}_n, \odot, \oplus, 0) \cong S_n \times C_2$, $n > 2$.

1. Introduction

Orders over domains is a classical object of study, originated by Dedekind’s ideal theory of maximal orders in algebraic number fields. Apart from their own interest as a “noncommutative arithmetic”, orders have also great importance to the theory of integral representations and to integer representations [25]. Orders of tiled form appeared as structural ingredients in the study of hereditary orders (see [13] or [25]), Bass orders [8] and, more generally, they are used in the context of quasi-Bass orders in [7]. The latter two references witness the essential role of tiled orders in the theory of orders of finite representation type, whereas their importance for the investigation of global dimension stems from Tarsy’s paper [35].

Various aspects of tiled orders have been extensively studied in the literature. These include homological aspects [14, 15, 10, 11, 17, 19, 20, 26, 28], representation theory [27, 32, 33, 34, 39], structure [12, 23, 37, 36, 38], $K$-theory [18, 22] and others. In addition, tiled orders turned out to be useful to prove Krull-Remak-Schmidt-Azumaya type theorems in additive categories [3] and, more recently, a strong connection between cluster categories and Cohen-Macaulay representation theory of some tiled orders was established in [4].

Notice that the term “tiled” for rings was used first time by R. B. Tarsy [35] and independently by D. Eisenbud and J. C. Robson [9]. Since then the term “tiled order” became well established in referring to matrix rings of “tiled form” over a domain, however, orders with tiled structure over non-commutative rings appeared already in [7, 8, 18, 38], and the more general concept of a tiled ring was defined in [9]. Nevertheless, tiled orders
sometimes appear in the literature under other names, such as Schurian orders \[36\] or monomial orders \[37\].

The current paper is concerned with exponent matrices of tiled orders \[16, Chapter 14\] which play a crucial role in characterization of these orders. Exponent matrices are \(n \times n\) matrices over non-negative integers satisfying:

\begin{align*}
\text{(EM1)} & \quad a_{ii} = 0 \text{ for all } i = 1, \ldots, n. \\
\text{(EM2)} & \quad a_{ij} + a_{jk} \geq a_{ik} \text{ for all pairwise distinct } i, j, k \in \{1, \ldots, n\}.
\end{align*}

Of course (EM2) is non-vacuous only starting with \(n = 3\). As the definition suggests, exponent matrices are objects with a strong combinatorial flavour. Throughout the paper, the set of all exponent \(n \times n\) matrices over \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) is denoted by \(E_n\).

The main idea of the present paper is to look at \(E_n\) as an algebra with respect to operations of component-wise maximum, denoted by \(\oplus\) and sometimes called the tropical sum, and component-wise addition, denoted by \(\odot\) and sometimes called the tropical product.

Most of usual axioms of an idempotent semiring hold in the algebra \((E_n, \oplus, \odot, 0)\) where 0 denotes the zero matrix: both of the operations \(\oplus\) and \(\odot\) are associative and commutative, \(\oplus\) is idempotent and \(\odot\) distributes over \(\oplus\). Observe, however, that in our algebra the neutral elements for both of the operations coincide: this is the zero matrix 0.

The equational theory of the algebra \((\mathbb{N}_0, \odot, \oplus, 0)\) was studied in \[1, 2\]. According to J.-E. Pin \[24\] the adjective “tropical”, in relation to a max-plus (or a min-plus) algebra, was coined by Dominique Perrin in honor of the pioneering work of Imre Simon (1943-2009), a mathematician and computer scientist from University of São Paulo, who was first to use min-plus semirings in theoretical computer science. Namely, these semirings are crucial ingredients of I. Simon’s solution of some famous decidability problems on rational languages, treating them from the point of view of Burnside type questions \[29, 30\] (see also \[21\] and \[31\]).

In the current paper we give a basis for the max-plus algebra \((E_n, \odot, \oplus, 0)\) and also study the symmetry of \(E_n\) from various points of view. We now describe this basis. Let \(I \subseteq \{1, 2, \ldots, n\}\) be a proper subset, which means that \(1 \leq |I| \leq n - 1\). We let \(I^c = \{1, 2, \ldots, n\} \setminus I\) be the complement of \(I\). By \(T_I = (t_{ij})\) we denote the matrix given by

\[ t_{ij} = \begin{cases} 1, & i \in I, j \in I^c; \\ 0, & \text{otherwise}. \end{cases} \]

Let \(\mathcal{T}\) be the set of all matrices \(T_I\) where \(I \subseteq \{1, 2, \ldots, n\}\) and \(1 \leq |I| \leq n - 1\). We sometimes call the elements of \(\mathcal{T}\) blocks. It is easy to see that \(\mathcal{T} \subseteq E_n\). We can now state our structure result.

**Theorem 1.1** (Structure Theorem). The matrices \(T_I\), where \(I\) runs through the proper subsets of the set \(\{1, 2, \ldots, n\}\), form a basis of the algebra \((E_n, \odot, \oplus, 0)\). That is, any matrix \(A \in E_n\) can be written in the form

\[ A = B_1 \odot \ldots \odot B_l \oplus \ldots \oplus C_1 \odot \ldots \odot C_m, \]

where all the matrices \(B_1, \ldots, C_m\) are blocks (as usual \(\odot\) is performed prior to \(\oplus\)). Moreover, this basis is the only minimal basis of the algebra \((E_n, \odot, \oplus, 0)\).
Theorem 1.1 is proved in Section 2. Notice that this result gives a way to obtain all tiled orders over a fixed discrete valuation ring from a simply described set of exponent matrices (see [16, pp. 352-353]). In Section 3 we study the automorphisms of the semigroup \((E_n, \circ)\) and prove in Theorem 3.1 that \(\text{Aut}(E_n, \circ) \simeq S_n \times C_2\), if \(n \geq 3\), where \(S_n\) stands for the symmetric group on \(n\) letters and \(C_2\) denotes the cyclic group of order 2. In order to study the automorphisms of \((E_n, \oplus)\) we need some technical preparation which is done in the first part of Section 4, considering strict downsets \(A \Downarrow = \{B \in E_n : B \preceq A\}\) of elements \(A \in E_n\).

In Theorem 4.1 we prove that a matrix \(A \in E_n\) is uniquely determined by its strict downset. This result looks interesting by itself, but it is also used in the proof of Theorem 4.3 which states that \(\text{Aut}(E_n, \oplus) \simeq S_n \times C_2\). It follows that for \(n > 2\) we have

\[
\text{Aut}(E_n, \circ) = \text{Aut}(E_n, \oplus) = \text{Aut}(E_n, \leq) = \text{Aut}(E_n, \circ, \oplus, 0) \simeq S_n \times C_2, \quad (n > 2),
\]

which reflects some harmony between the various structures on \(E_n\). The latter demonstrates some kind of a symmetry which exists in the class of the \(n \times n\)-tiled orders over a fixed discrete valuation ring.

2. PROOF OF THE STRUCTURE THEOREM

For \(A \in E_n\) and \(k \geq 1\) by \(A \circ^k\) we denote the matrix \(A \circ \ldots \circ A\) where the number of factors \(A\) is \(k\). We also define the partial ordering on \(E_n\) by \(A \leq B\) if and only if \(A \oplus B = B\). This is equivalent to the condition \(a_{ij} \leq b_{ij}\) for all \(1 \leq i, j \leq n\).

We assume that \(A \neq 0\). Fix \(p \in \{1, \ldots, n\}\) such that the \(p\)th row of \(A\) is non-zero. Let \(C\) be the set of numbers in \(\mathbb{N} \cup \{0\}\) which occur in the \(p\)th row of \(A\). We note that \(0 \in C\) since \(a_{pp} = 0\). Let \(r\) be the maximal element of \(C\). Thus \(|C| \geq 2\).

For each \(c \in C\) we define the following sets of indices:

\[
J_c = \{j : a_{pj} = c\}.
\]

We order the elements of \(C\) assuming that

\[
C = \{c_1, \ldots, c_m\} \quad \text{where} \quad c_1 < c_2 < \ldots < c_m.
\]

Observe that \(c_1 = 0\) and \(c_m = r\).

Further, for each \(t \in \{1, \ldots, m - 1\}\) we put

\[
I_t = J_{c_1} \cup \ldots \cup J_{c_t}, \quad \text{and} \quad k_t = c_{t+1} - c_t
\]

and

\[
T(p) = T_{I_1}^{\circ k_1} \circ \ldots \circ T_{I_{m-1}}^{\circ k_{m-1}}.
\]

Let us prove that

\[
T(p) \leq A.
\]

Let \(i, j \in \{1, 2, \ldots, n\}\). We need to show that \(T(p)_{ij} \leq a_{ij}\). The construction of the sets \(J_t\) implies that

\[
J_{c_1} \cup \ldots \cup J_{c_m} = \{1, 2, \ldots, n\}.
\]
Since, in addition, the above union is disjoint, there are unique sets $J_{c_{1}}$ and $J_{c_{2}}$ such that $i \in J_{c_{1}}$ and $j \in J_{c_{2}}$.

For each $t = 1, \ldots, m - 1$ we note that the block $T_{t}$ has 1 precisely at positions with indices $ij$ where $i \in J_{c_{t}} \cup \ldots \cup J_{c_{t+1}}$ and $j \in J_{c_{t+1}} \cup \ldots \cup J_{c_{m}}$. It follows that if $s \geq v$ then $T(p)_{ij} = 0 \leq a_{ij}$. Assume now that $s < v$. Then the matrices $T_{I_{s}}, T_{I_{s+1}}, \ldots, T_{I_{v-1}}$ have 1 at $i, j$ position and all the other matrices $T_{I_{t}}$ have 0 at the same position. It follows that

$$T(p)_{ij} = k_{s} + \ldots + k_{v-1} = c_{v} - c_{s}.$$

It remains to show that $a_{ij} \geq c_{v} - c_{s}$. By condition (R2) in the definition of $E_{n}$ we have the inequality

$$a_{pi} + a_{ij} \geq a_{pj}. \tag{2.5}$$

From $i \in J_{c_{1}}$ and $j \in J_{c_{2}}$ we have that $a_{pi} = c_{s}$ and $a_{pj} = c_{v}$ by (2.1). This and the inequality (2.5) yield that $c_{s} + a_{ij} \geq c_{v}$ so that $a_{ij} \geq c_{v} - c_{s}$, as required.

We now show that the $p$th row of $T(p)$ equals the $p$th row of $A$:

$$T(p)_{pj} = a_{pj} \text{ for all } j \in \{1, \ldots, n\}. \tag{2.6}$$

Indeed, let $j \in \{1, \ldots, n\}$. Assume that $j \in J_{c_{1}}$. Notice that $p \in J_{1} = I_{1} \subset I_{2} \subset \ldots$, as $a_{pp} = c_{1} = 0$. From the construction of the matrix $T(p)$ we have

$$T(p)_{pj} = k_{1} + \ldots + k_{t-1} = c_{t} - c_{0} = c_{t}.$$

But $j \in J_{c_{t}}$ is equivalent to $a_{pj} = c_{t}$, so that (2.6) follows.

From (2.4) and (2.0) we immediately obtain

$$A = \bigoplus \{T(p) : \text{p-th row of A is non-zero}\}, \tag{2.7}$$

which finishes the proof of the fact that the matrices $T_{I}$ form a basis of the algebra $(E_{n}, \odot, \oplus, 0)$.

We are left to prove the claim about minimality. Let $A$ be a non-zero matrix from $E_{n}$ and assume that the index $i$ is such that $a_{ij} \neq 0$ for some $j$. Let $C_{i} = \{j : a_{ij} = 0\}$. Thus $j \notin C_{i}$. We show that $T_{C_{i}} \leq A$. If $C_{i} = \{i\}$, then $T_{\{i\}} \leq A$. Otherwise, let $t \in C_{i}, t \neq i$ and let $k \in C_{i}^{c}$. Since $a_{it} + a_{ik} \geq a_{it} \geq 1$ and since $a_{it} = 0$, it follows that $a_{ik} \geq 1$. This implies $T_{C_{i}} \leq A$, as desired. So for any non-zero $A \in E_{n}$ there is some block matrix, which is less then or equal to $A$. The statement about of the minimality of the basis of block matrices now follows from the fact that any two block matrices are incomparable with respect to $\leq$.

**Remark 2.1.** Assume that all elements of the matrix $A$ are zeros and ones. Then for each $p$ such that the $p$th row of $A$ is non-zero we have that the matrix $T(p)$, as in (2.3), equals $T_{I_{1}}$ (since $m = 2$ and $k_{1} = 1$). It follows that the row decomposition (2.7) of $A$ in this case does not involve the operation $\odot$, and thus $A$ is an $\oplus$-combination of matrices from $T$.

**Remark 2.2.** The construction of the matrix $T(p)$ is carried over as follows. The set $I_{1}$ is the smallest subset of $\{1, \ldots, n\}$ such that the $p$th row (and thus any row) of $T_{I_{1}}$ is less than or equal to the $p$th row of $A$ and $k_{1}$ is the maximal power of $T_{I_{1}}$ such that $T_{I_{1}}^{\otimes k_{1}} \leq A$. Then to construct $T_{I_{2}}$, we find the smallest subset $I_{2}$ of $\{1, \ldots, n\}$ such that the $p$th row of
$T_{I_1}^{\oplus k_1} \odot T_{I_2}$ is less than or equal to the $p$th row of $A$ and we let $k_2$ be the greatest power of $T_{I_2}$ such that the $p$th row of $T_{I_1}^{\oplus k_1} \odot T_{I_2}^{\oplus k_2}$ is less than or equal to the $p$th row of $A$. We construct the subsequent blocks $T_{I_t}$ and their powers $k_t$ similarly.

We emphasize that not only we have proved our theorem but also we have suggested an explicit construction of a decomposition of the form (1.1) which has no more than $n$ summands for every matrix $A \in \mathcal{E}_n$.

We provide an example of the calculation of the matrix $T(p)$.

**Example 2.3.** Let $n = 9$ and let $A \in \mathcal{E}_9$ be a matrix whose first row equals

$$
\begin{bmatrix}
0 & 5 & 0 & 0 & 1 & 3 & 3 & 3 & 5
\end{bmatrix}.
$$

We construct the matrix $T(1)$. Firstly, we have that $C = \{0, 1, 3, 5\}$ is the set of all elements which occur in the given row. Now, we calculate the sets $J_c$ for all $c \in C$:

$$J_0 = \{1, 3, 4\}, \quad J_1 = \{5\}, \quad J_3 = \{6, 7, 8\}, \quad J_5 = \{2, 9\}.$$

Further, for each $t = 1, \ldots, |C| - 1 = 3$ we define the set $I_t$ and the number $k_t$ according to (2.2):

- $I_1 = J_0 = \{1, 3, 4\}$, $I_2 = J_0 \cup J_1 = \{1, 3, 4, 5\}$, $I_3 = J_0 \cup J_1 \cup J_3 = \{1, 3, 4, 5, 6, 7, 8\}$;
- $k_1 = 1 - 0 = 1$, $k_2 = 3 - 1 = 2$, $k_3 = 5 - 3 = 2$.

Following (2.3), we obtain

$$T(1) = T_{\{1,3,4\}} \odot T_{\{1,3,4,5\}}^{\oplus 2} \odot T_{\{1,3,4,5,6,7,8\}}^{\oplus 2}.\]$$

We call the decomposition (2.7) the *row decomposition* of the matrix $A$. We now introduce the notion of the column decomposition of the matrix $A$. Let

$$A^t = \bigoplus \{T(p) : \text{pth column of } A \text{ is non-zero}\},$$

be the row decomposition of the transpose $A^t$ of the matrix $A$.

Since clearly the operation $\oplus$ commutes with taking the transpose, transposing the latter equality we obtain

$$A = \bigoplus \{T(p)^t : \text{pth column of } A \text{ is non-zero}\}.$$

Note that for a block $T_I$ its transpose $T_I^t$ is the block $T_{I^c}$, where by $I^c$ we denote the complement $\{1, \ldots, n\} \setminus I$. Since the operation $\odot$ also commutes with taking the transpose, we can readily calculate the transpose of each summand $T(p)$. If

$$T(p) = T_{I_1}^{\oplus k_1} \odot \ldots \odot T_{I_{m-1}}^{\oplus k_{m-1}},$$

then we put

$$S(p) = T(p)^t = T_{I_1}^{\oplus k_1} \odot \ldots \odot T_{I_{m-1}}^{\oplus k_{m-1}}.$$

We call the decomposition

(2.8) 
$$A = \bigoplus \{S(p) : \text{pth column of } A \text{ is non-zero}\}. $$
the column decomposition of \( A \).

The technique we have developed so far may be effectively used to verify if a given \( n \times n \) matrix over \( \mathbb{N} \cup \{0\} \) belongs to \( \mathcal{E}_n \). Firstly, for any such a matrix we can calculate the matrices \( T(p) \) and \( S(p) \) using our constructions.

**Proposition 2.4.** Let \( A \) be an \( n \times n \) matrix over \( \mathbb{N} \cup \{0\} \). The following statements are equivalent:

1. \( A \in \mathcal{E}_n \).
2. \( T(p) \leq A \) for every non-zero row of \( A \).
3. \( S(p) \leq A \) for every non-zero column of \( A \).

**Proof.** The implication (1) \( \Rightarrow \) (2) was shown in the proof of Theorem 1.1. For the converse implication, we observe that the corresponding part of the proof of Theorem 1.1 shows that also \( T(p) \leq A \) implies that that all the inequalities (R2) of the form \( a_{pq} + a_{ij} \geq a_{pj} \) hold. The equivalence (1) \( \Leftrightarrow \) (3) follows from (1) \( \Leftrightarrow \) (2) and the observation that \( A \in \mathcal{E}_n \) implies that \( A^t \in \mathcal{E}_n \). The remaining equivalence now also follows. \( \square \)

We now provide an example of the calculation of the row decomposition and the column decomposition of a matrix \( A \in \mathcal{E}_n \).

**Example 2.5.** Let \( A = \begin{pmatrix} 0 & 2 & 5 & 5 \\ 4 & 0 & 3 & 3 \\ 6 & 2 & 0 & 2 \\ 4 & 4 & 2 & 0 \end{pmatrix} \). The row decomposition of \( A \) is

\[
A = T(1) \oplus T(2) \oplus T(3) \oplus T(4) = (T_{\{1\}}^{\odot 2} \oplus T_{\{1,2\}}^{\odot 3}) \oplus (T_{\{2\}}^{\odot 3} \oplus T_{\{2,3,4\}}^{\odot 4}) \oplus (T_{\{3\}}^{\odot 2} \oplus T_{\{2,3,4\}}^{\odot 2}) \oplus (T_{\{4\}}^{\odot 2} \oplus T_{\{3,4\}}^{\odot 2}).
\]

The column decomposition of \( A \) is

\[
A = S(1) \oplus S(2) \oplus S(3) \oplus S(4) = (T_{\{2,3,4\}}^{\odot 4} \oplus T_{\{3\}}^{\odot 2}) \oplus (T_{\{1,3,4\}}^{\odot 2} \oplus T_{\{4\}}^{\odot 2}) \oplus (T_{\{1,2,4\}}^{\odot 2} \oplus T_{\{1,2\}}^{\odot 2}) \oplus (T_{\{1,2,3\}}^{\odot 2} \oplus T_{\{1,2\}} \oplus T_{\{1\}}^{\odot 2}).
\]

From \( T(1), T(2), T(3), T(4) \leq A \) we see that we indeed have \( A \in \mathcal{E}_4 \).

3. Automorphisms of \((\mathcal{E}_n, \odot)\)

In this section, we study automorphisms of the semigroup \((\mathcal{E}_n, \odot) = (\mathcal{E}_n, +)\). We denote by \( e_{ij} \) the matrix whose entry at position \( i, j \) equals 1 and all the other entries are 0’s.

We begin by observing that if \( A \in \mathcal{E}_n \) then \( A^t \in \mathcal{E}_n \), too. We thus have an action of the two-element group \( C_2 = \{ e, a \} \) on \((\mathcal{E}_n, \odot, \oplus, 0)\) where \( e \) is the identity map, and \( a \) acts by \( a \cdot A = A^t \). Furthermore, let \( \sigma \in \mathcal{S}_n \) and \( A = (a_{ij}) \in \mathcal{E}_n \). We put \( \sigma \cdot A = (a_{\sigma(i)\sigma(j)}) \). We observe that \( \sigma \cdot A \in \mathcal{E}_n \) and that we have an action of \( \mathcal{S}_n \) on \((\mathcal{E}_n, \oplus, \odot, 0)\). It is clear that this action commutes with the action of \( C_2 \), and we obtain an action by automorphisms of the group \( C_2 \times \mathcal{S}_n \) on \((\mathcal{E}_n, \oplus, \odot, 0)\). This action is faithful if \( n > 2 \), since \( \sigma \cdot T_{\{i\}} = T_{\{\sigma(i)\}} \neq T_{\{i\}} \)
\( T_{[i]} = T^t_{[i]}, \) for any \( \sigma \in S_n \) and \( i = 1, \ldots, n. \) As to the case \( n = 2, \) the action of the unique nontrivial permutation in \( S_2 \) coincides with the transpose.

In the case \( n = 2 \) we easily have that \( \text{Aut}(E_2) = C_2. \) Indeed, observe that any nonnegative \( 2 \times 2 \)-matrix whose diagonal entries are 0’s is exponent and the unique minimal generating set of \( E_2 = (E_2, \odot) \) is \( \{e_{12}, e_{21}\}. \) Then any automorphism of \( E_2 \) preserves \( \{e_{12}, e_{21}\}, \) and consequently, a non-trivial automorphism maps \( e_{12} \mapsto e_{21} \) and \( e_{21} \mapsto e_{12}. \)

For \( n > 2 \) we prove that any automorphism of \( (E_n, \odot) \) belongs to \( C_2 \times S_n. \)

**Theorem 3.1.** Let \( \varphi \) be an automorphism of \( (E_n, \odot) \) where \( n > 2. \) Then \( \varphi \in C_2 \times S_n. \)

So assume for the rest of the section that \( n > 2 \) and let \( \varphi: A \mapsto \varphi(A) \) be an automorphism of \( (E_n, \odot). \) Let us introduce some notation. We put

\[
\mathcal{L} = \{T_{[i]} : 1 \leq i \leq n\}, \quad \mathcal{C} = \{T_{[i]} : 1 \leq i \leq n\}.
\]

For each ordered pair \( (i, j) \) where \( i \neq j \) and \( i, j \in \{1, \ldots, n\} \) we put

\[
A_{ij} = T_{[i]} \oplus T_{[j]}^r, \quad L_{ij} = T_{[i,j]} \oplus T_{[j]}, \quad C_{ij} = T_{[i,j]}^r \oplus T_{[i]}^r.
\]

Let, further, \( \mathcal{A}, \mathcal{L}, \) and \( \mathcal{C} \) be the sets consisting of all matrices \( A_{ij}, L_{ij} \) and \( C_{ij}, \) respectively.

We say that \( A \in E_n \) is \( \odot \)-irreducible, if \( A \) can not be decomposed as \( A = B \odot C = B + C \) where \( B, C \neq A. \)

**Lemma 3.2.** The matrices \( A_{ij}, L_{ij} \) and \( C_{ij} \) are \( \odot \)-irreducible for all ordered pairs \( (i, j) \) where \( i \neq j \) and \( i, j \in \{1, \ldots, n\}. \)

**Proof.** We prove the claim for \( i = 1 \) and \( j = 2, \) the general case follows applying some \( \sigma \in S_n \) satisfying \( \sigma(1) = i \) and \( \sigma(2) = j. \) Assume that \( A_{12} = B \odot C \) where \( B, C \in E_n, B, C \neq A. \) By Theorem 1.1 both \( B \) and \( C \) can be decomposed as \( (\odot, \oplus) \)-expressions in blocks. Since, clearly, all the blocks \( T \) in this decomposition satisfy \( T \leq B, C \leq A_{12}, \) it follows that we must have \( T = T_{[1]} \) or \( T = T_{[2]} \). Furthermore, both \( T_{[1]} \) and \( T_{[2]} \) must appear in the decompositions of \( B \) and \( C \) (at least once in the two decompositions). Also, if \( T_{[1]} \) does not appear in one of the decompositions, say of \( B, \) then it must appear in the other decomposition, as otherwise \( A_{12} = B \odot C \) can not hold. But then, applying distributivity, it follows that \( T_{[1]} \oplus T_{[2]} \) is a summand of a decomposition of \( A_{12} \) which is impossible because the 1,2-entry of \( T_{[1]} \odot T_{[2]} \) equals 2, while the 1,2-entry of \( A_{12} \) is 1.

We now analyze the sums of entries of the matrices \( T_I \in \mathcal{T}. \) We have that the some of entries of \( T_I \) equals the number of positions with 1. The latter number equals \( |I|(n - |I|). \) It is easy to see that \( 1 \cdot (n - 1) \leq 2(n - 2) \leq \ldots \leq \lceil n/2 \rceil (n - \lfloor n/2 \rfloor), \) moreover the number of non-zero entries of \( T_I \) and \( T_I^r \) are the same. In particular, we have:

**Lemma 3.3.** The set of non-zero matrices in \( E_n \) with minimal sum of entries is \( \mathcal{L} \cup \mathcal{C}. \)

For \( A \in E_n \) let \( \#(A) \) denote the sum of all entries of \( A. \) We set \( U \in E_n \) to be the matrix whose diagonal entries are 0’s, and all other entries are 1’s, that is,

\[
U = \sum_{i=1}^{n} T_{[i]} = \sum_{i=1}^{n} T_{[i]}^r.
\]
This matrix will play an important role in our considerations. Furthermore, let \( \mathcal{G} \subseteq \mathcal{E}_n \) denote the set of \( \circ \)-irreducible matrices. Clearly, \( \mathcal{T} \subseteq \mathcal{G} \). Also, by Lemma 3.2, \( \mathcal{A}, \mathcal{L}, \mathcal{C} \subseteq \mathcal{G} \).

Let \( \varphi \) be an automorphism of \((\mathcal{E}_n, \circ)\). Clearly \( \varphi(\mathcal{G}) = \mathcal{G} \). The following important step is in observing that \( \varphi \) fixes \( U \) and that \( #(A) \) is invariant under the action of \( \varphi \) for any \( A \in \mathcal{G} \).

**Lemma 3.4.**

1. \( \sum \{ A : A \in \mathcal{G} \} = tU \) for some \( t \in \mathbb{N} \).
2. \( \varphi(U) = U \).
3. \( #(\varphi(A)) = #(A) \).
4. \( \varphi(\mathcal{L} \cup \mathcal{C}) = \mathcal{L} \cup \mathcal{C} \).
5. \( \varphi(\mathcal{L}) = \mathcal{L} \) or \( \varphi(\mathcal{C}) = \mathcal{C} \) (and thus, respectively, \( \varphi(\mathcal{C}) = \mathcal{C} \) or \( \varphi(\mathcal{C}) = \mathcal{L} \)).

**Proof.** (1) We begin by observing that \( \mathcal{G} \) is invariant under the action of \( \mathcal{S}_n \). It follows that \( \mathcal{G} \) is a union of several \( \mathcal{S}_n \)-orbits. If \( A = (a_{ij}) \in \mathcal{G} \) then the entry at the position \( pq \) with \( p \neq q \) of the sum of all elements of the orbit of \( A \) equals \( \sum_{a \in \mathcal{S}_n} a_{(p-1)(q-1)} \). It follows that this entry equals \( (n - 2)! \#(A) \). Thus the sum of all the matrices in the orbit of \( A \) equals \( n! 2)! \#(A)U \), and (1) follows.

(2) follows form (1) as \( \varphi(\mathcal{G}) = \mathcal{G} \).

(3) For each \( i \geq 1 \) let \( M_i = \{ A \in \mathcal{G} : #(A) = i \} \). Let \( i_0 \) be the minimal \( i \) for which \( M_i \neq \emptyset \) (it is easy to see that \( i_0 = n - 1 \)). Let \( A \in M_i \). As above, the sum of all matrices of the orbit of \( A \) under the action of \( \mathcal{S}_n \) equals \( (n - 2)! i_0 U \), and, because the latter matrix is fixed by \( \varphi \), we have the sum of all matrices of the orbit of \( A \) is fixed by \( \varphi \), too: \( \sum_{\sigma \in \mathcal{S}_n} \varphi(\sigma(A)) = \sum_{\sigma \in \mathcal{S}_n} \varphi(\sigma(A)) \). On both sides of this equality we have a sum of \( n! \) matrices. The sum in the right-hand side is such that the sum of entries of each its member is minimal possible, \( i_0 \). It follows that the same must be true about the sum in the left-hand side, which, too, has \( n! \) summands. We thus have that \( \#(\varphi(\sigma(A))) = i_0 \) for all \( \sigma \in \mathcal{S}_n \). This implies that \( \varphi(M_{i_0}) = M_{i_0} \).

Let \( i_i \) be the minimal \( i > i_0 \) such that \( M_i \neq \emptyset \). Repeating the argument above and using the fact that \( \varphi(M_{i_0}) = M_{i_0} \) we obtain that \( \varphi(M_{i_i}) = M_{i_i} \). Applying induction, it follows that \( \varphi(M_i) = M_i \) for all \( i \geq 1 \).

(4) As was mentioned in the proof of (3) above, \( i_0 = n - 1 \) and we have \( M_{i_0} = \mathcal{L} \cup \mathcal{C} \). Thus \( \varphi(\mathcal{L} \cup \mathcal{C}) = \mathcal{L} \cup \mathcal{C} \), as desired.

(5) By the above we have that, for each \( i = 1, \ldots, n \), \( \varphi(T_{(i)}) \) equals either some \( T_{(j)} \), or some \( T_{(j)} \). Applying \( \varphi \) to the equality \( \sum_{i=1}^{n} T_{(i)} = U \), we obtain

\[
T_{(i_1)} + \ldots + T_{(i_p)} + T_{(i_{p+1})} + \ldots + T_{(i_n)} = U,
\]

But, unless \( p = 0 \) or \( p = n \), the sum in the left-hand side has at least one entry which is greater than one which is a contradiction. The statement follows.

We now turn to the behavior of the image of the set \( \mathcal{A} \cup \mathcal{L} \cup \mathcal{C} \) with respect to \( \varphi \). We first observe that \( #(A) = 2n - 3 \) for \( A \in \mathcal{A} \cup \mathcal{L} \cup \mathcal{C} \). Note also that for any such \( A \) all its entries are 0's and 1's: indeed, assuming that \( a_{ij} \geq 2 \) we would have \( a_{ik} + a_{kj} \geq a_{ij} \geq 2 \) for all \( k = 1, \ldots, n \), yielding \( #(A) \geq 2n \), a contradiction.
Lemma 3.5. \( \varphi(\mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}}) = \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \).

Proof. We divide the proof into several cases.

Case 1. Assume first that \( n > 6 \). Let \( A \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \). We have \( \#(\varphi(A)) = 2n - 3 \) by part (3) of Lemma 3.4. As all the entries of \( \varphi(A) \) are 0's and 1's, Remark 2.1 implies that \( \varphi(A) \) can be expressed as a \( \oplus \)-combination of blocks. Any matrix \( B \) involved in such a combination combination must satisfy \( \#(B) \leq 2n - 3 \). Thus we must have \( B = T_I \) or \( B = T_{I'} \) with \( |I| = 1 \) or \( |I| = 2 \) (because if \( |I| \geq 3 \) we would have \( \#(B) \geq 3(n - 3) = 3n - 9 > 2n - 3 \) as \( n > 6 \)). Note that \( \#(B) = n - 1 \) if \( |I| = 1 \) and \( \#(B) = 2n - 4 \) if \( |I| = 2 \). It easily follows that for pairwise distinct \( B_1, B_2, B_3 \) equal to \( T_I \) or \( T_{I'} \) with \( |I| = 1 \) or \( |I| = 2 \) we have \( \#(B_1 \oplus B_2 \oplus B_3) > 2n - 3 \), so we are left to consider only the case where \( \varphi(A) = B_1 \oplus B_2 \) with \( B_1, B_2 \) of the form \( T_I \) or \( T_{I'} \) where \( |I| = 1 \) or \( |I| = 2 \). This and \( \#(\varphi(A)) = 2n - 3 \) yield that \( \varphi(A) \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \), as desired.

Case 2. Assume that \( n = 6 \). In this case \( 2n - 3 = 9 \). We first prove that \( \#(A) = 9 \) implies that \( A \in T_I \) where \( |I| = 3 \), or \( A \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \). As all entries of \( A \) are zeros and ones, we have that \( A \) is expressed as a \( \oplus \)-combination of matrices \( T_I \). If such a combination contains \( B = T_I \) with \( |I| = 3 \) then \( \#(B) = 9 \) and we must have \( A = B \). Assume that such a combination contains only matrices of the form \( T_I \) or \( T_{I'} \) where \( |I| = 1 \) or \( |I| = 2 \). Then \( \#(B) = 5 \) or \( \#(B) = 8 \). A similar analysis as in the previous case leads to \( A \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \).

Let \( T_3 = \{ \epsilon \} \cup \{ T_I : |I| = 3 \} \). From the previous paragraph and part (3) of Lemma 3.4 it follows that \( \varphi(\mathcal{A} \cup \mathcal{L} \cup \mathcal{C}) = \mathcal{T}_3 \cup \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \). The needed equality \( \varphi(\mathcal{A} \cup \mathcal{L} \cup \mathcal{C}) = \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \) will follow if we prove that \( \varphi(T_3) = T_3 \). Let \( A \in T_3 \). We assume that \( A = T_{(p,q,r)} \). Consider the equality

\[ T_{(p,q,r)} + T_{(p,q)} + T_{(q,r)} + T_{(r,p)} = T_{(p,q,r)} + T_{(p,q)} + T_{(q,r)} + T_{(r,p)}. \]

We apply \( \varphi \) to both sides of this equality. In view of part (5) of Lemma 3.4 we obtain either

\[ \varphi(T_{(p,q,r)}) + T_{(i_1,r)} + T_{(i_2,p)} + T_{(i_3,q)} = \varphi(T_{(p,q,r)}) + T_{(j_1,q)} + T_{(j_2,r)} + T_{(j_3,p)}, \]

or

\[ \varphi(T_{(p,q,r)}) + T_{(i_1,p)} + T_{(i_2,q)} + T_{(i_3,r)} = \varphi(T_{(p,q,r)}) + T_{(j_1,r)} + T_{(j_2,q)} + T_{(j_3,p)}, \]

where \( i_1, i_2, i_3 \), as well as \( j_1, j_2, j_3 \) are pairwise distinct. Without loss of generality, we assume that we obtain the former equality. The matrix \( T_{(j_1)} + T_{(j_2)} + T_{(j_3)} \) has in the rows \( j_1, j_2, j_3 \) all elements, but the diagonal ones, equal to 1. It follows that the matrix in the left-hand side, \( \varphi(T_{(p,q,r)}) + T_{(i_1,p)} + T_{(i_2,q)} + T_{(i_3,r)} \), must be greater than or equal to this matrix. It easily follows that \( \varphi(T_{(p,q,r)}) \geq T_{(j_1,j_2,j_3)} \) and thus, as \( \#(\varphi(T_{(p,q,r)})) = 9 \), we get the equality \( \varphi(T_{(p,q,r)}) = T_{(j_1,j_2,j_3)} \). It follows that \( \varphi(T_3) = T_3 \), as desired.

Case 3. Assume that \( n = 5 \). Then \( 2n - 3 = 7 \). If \( |I| = 1 \), \( \#(T_{I'}) = \#(T_{I'}) = 4 \), if \( |I| = 2 \), \( \#(T_{I'}) = \#(T_{I'}) = 6 \). Considering \( \oplus \)-combinations of such matrices, similarly as in Case 1 above, yields that if \( A \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \) then \( \varphi(A) \in \mathcal{A} \cup \tilde{\mathcal{L}} \cup \tilde{\mathcal{C}} \), too.

The cases where \( n = 3 \) and \( n = 4 \) can be treated similarly and are left to the reader. \( \square \)

Lemma 3.6.
(1) There is $\sigma \in S_n$ such that we have either $\varphi(T_{(i)}) = T_{\{\sigma(i)\}}$ and $\varphi(T_{(i)^c}) = T_{\{\sigma(i)\}^c}$, or $\varphi(T_{(i)}) = T_{\{\sigma(i)\}}$ and $\varphi(T_{(i)^c}) = T_{\{\sigma(i)\}^c}$.

(2) $\varphi(\mathcal{A}) = \mathcal{A}$.

(3) If $i \neq j$ then $\varphi(A_{ij}) = \varphi(T_{(i)}) \oplus \varphi(T_{(j)^c})$.

Proof. (1) From part (5) of Lemma [3.1] we know that $\varphi(\mathcal{L}) = \mathcal{L}$ or $\varphi(\mathcal{L}) = \mathcal{C}$ (and thus, respectively, $\varphi(\mathcal{C}) = \mathcal{C}$ or $\varphi(\mathcal{C}) = \mathcal{L}$). We assume that $\varphi(\mathcal{L}) = \mathcal{L}$ and $\varphi(\mathcal{C}) = \mathcal{C}$, the other case being treated similarly. As $\varphi$ is a bijection, there are $\sigma, \tau \in S_n$ such that $\varphi(T_{(i)}) = T_{\{\sigma(i)\}}$ and $\varphi(T_{(i)^c}) = T_{\{\tau(i)\}^c}$ for all $i = 1, \ldots, n$. Hence, we need only to prove that $\tau = \sigma$.

Observe that for each $i \in \{1, \ldots, n\}$ the following equality holds:

$$
(3.1) \quad \sum_{k \neq i} A_{ik} + T_{(i)^c} = (n - 2)T_{(i)} + U.
$$

Applying $\varphi$ to both sides of this equality, we get

$$
(3.2) \quad \sum_{k \neq i} \varphi(A_{ik}) + T_{(\tau(i))^c} = (n - 2)T_{(\sigma(i))} + U,
$$

or, equivalently,

$$
(3.3) \quad \sum_{k \neq i} \varphi(A_{ik}) = (n - 2)T_{(\sigma(i))} + U - T_{(\tau(i))^c},
$$

Assume, from the converse that $\tau(i) \neq \sigma(i)$. Denote the matrix in the right-hand side of (3.3) by $B = (b_{ij})$ and the matrix in the left-hand side by $C = (c_{ij})$. Let $k \neq \tau(i), \sigma(i)$. Then $b_{\sigma(i)k} + b_{\sigma(i)j} = n$. On the other hand, as $\varphi(A_{ik}) \in \mathcal{A} \cup \mathcal{L} \cup \mathcal{C}$, for any $p \neq q$ we have that $(\varphi(A_{ik}))_{pq} + (\varphi(A_{ik}))_{qp} \leq 1$. It follows that $c_{pq} \leq n - 1$ for all $p \neq q$. Thus the equality $B = C$ can not hold. The obtained contradiction shows that $\tau(i) = \sigma(i)$.

(2) Observe that in (3.3) (we already know that $\tau = \sigma$) the matrix in the right-hand side does not have any zero column. Thus neither does the matrix in the left-hand side. It follows that $\varphi(A_{ik}) \in \mathcal{A} \cup \mathcal{L} \cup \mathcal{C}$. Switching rows and columns, we can write the ‘transpose’ of the equality (3.1), then apply $\varphi$ to it and get the ‘transpose’ of (3.3). We similarly conclude that $\varphi(A_{ik}) \in \mathcal{A} \cup \mathcal{L}$. Therefore, $\varphi(A_{ik}) \in \mathcal{A}$, as desired.

(3) Observe the $\sigma(i)$th row of the matrix in the right-hand side of (3.3) (we already know that $\tau = \sigma$) has all non-diagonal entries equal to $n - 1$. As this is achieved as a sum of $n - 1$ matrices of the form $A_{ik}$, we conclude that $\varphi(A_{ik}) \in \{A_{(i\sigma)}: t \neq \sigma(i)\}$. Similarly, switching rows and columns, we get $\varphi(A_{ki}) \in \{A_{(\sigma(i))}: t \neq \sigma(i)\}$. Therefore, $\varphi(A_{ik}) = A_{\sigma(i)\sigma(k)}$, and the desired equality follows.

We now conclude the proof of Theorem [3.1]. We denote by $M_n^+$ the additive semigroup of all non-negative integer $n \times n$-matrices with zero diagonal. For any $\varphi \in \text{Aut}(\mathcal{E}_n, \circ)$ we define an endomorphism $\tilde{\varphi}$ of $M_n^+$ by $\tilde{\varphi}(e_{ij}) = \varphi(T_{(i)}) + \varphi(T_{(j)^c}) - \varphi(A_{ij})$, $i \neq j$. Assume that $\varphi(\mathcal{L}) = \mathcal{L}$. It follows that there is $\sigma \in S_n$ such that

$$
\tilde{\varphi}(e_{ij}) = \varphi(T_{(i)}) + \varphi(T_{(j)^c}) - \varphi(A_{ij}) = T_{(\sigma(i))} + T_{(\sigma(j))^c} - A_{\sigma(i)\sigma(j)} = c_{\sigma(i)\sigma(j)}.
$$
Thus, \( \hat{\varphi} \) is an automorphism of \( M_n^+ \) and for \( A = (a_{ij}) \) we have \( \hat{\varphi}(A) = (a_{\sigma(i)\sigma(j)}) \). It also follows that the restrictions of \( \varphi \) and \( \hat{\varphi} \) to \( L \cup C \cup A \) coincide. Moreover, as any \( A \in E_n \) can be written as \( A = \sum_{i,j: i \neq j} \alpha_{ij} e_{ij} = \sum_{i,j: i \neq j} \alpha_{ij}(T_{ij} + T_{ij}^e - A_{ij}) \), the restriction of \( \hat{\varphi} \) to \( E_n \) coincides with \( \varphi \). The case where \( \varphi(L) = C \) is similar, as matrix transposing commutes with the action of \( S_n \).

4. Automorphisms of \((E_n, \oplus)\)

4.1. Strict downsets of elements of \( E_n \). A strict downset of \( A \in E_n \), denoted by \( A^\diamond \) is the set of all \( B \in E_n \) satisfying \( B \leq A \). Clearly, a strict downset of an element does not allow to reconstruct this element, as, for example, all minimal elements of \( E_n \), which are the elements of the set \( T \), have the same strict downset, consisting of the zero matrix. In this section we prove the following.

**Theorem 4.1.** Let \( A, B \in E_n \) and \( A \notin T \) and assume that \( A^\diamond = B^\diamond \). Then \( A = B \).

Hence, a non-minimal element of \( E_n \) is uniquely determined by its strict downset. This result looks interesting on itself, but we also use it later on for studying automorphisms of \((E_n, \oplus)\). The remainder of this subsection will be devoted to the proof of Theorem 4.1.

So assume that \( A, B \in E_n \) are such that \( A \notin T \) and that \( A^\diamond = B^\diamond \). We write \( A = (a_{ij}) \) and \( B = (b_{ij}) \). Let \( \text{Max}(A^\diamond) \) be the set of maximal elements of \( A^\diamond \). Clearly, \( A^\diamond = B^\diamond \) if and only if \( \text{Max}(A^\diamond) = \text{Max}(B^\diamond) \).

Since the \( \oplus \) operation on \( E_n \) coincides with the join with respect to the natural partial order, we have

\[
\oplus\{C: C \in \text{Max}(A^\diamond)\} \leq A
\]

and thus we have that either \( \oplus\{C: C \in \text{Max}(A^\diamond)\} = A \) or \( \oplus\{C: C \in \text{Max}(A^\diamond)\} \in \text{Max}(A^\diamond) \). In the latter case we have that \( |\text{Max}(A^\diamond)| = 1 \).

In the former case we have \( A = \oplus\{C: C \in \text{Max}(A^\diamond)\} = \oplus\{C: C \in \text{Max}(B^\diamond)\} = B \). So, to prove Theorem 4.1 we can suppose that \( |\text{Max}(A^\diamond)| = |\text{Max}(B^\diamond)| = 1 \).

Consider the column decomposition of \( A \). As \( |\text{Max}(A^\diamond)| = 1 \), no \( \oplus \)-sum of several matrices which are strictly less than \( A \) is equal to \( A \). It follows that at least one \( \oplus \)-summand in the column decomposition of \( A \) equals \( A \). We thus have \( S(p) = A \) for some \( p \) (and a similar statement is true for the row decomposition, but we do not need it here).

We will say that \( A \in T^\circ m \) if \( A \) can be decomposed as a product of precisely \( m \) \( \circ \)-factors. If \( A = S(p) = T_{i_1}^{\circ k_1} \circ \cdots \circ T_{i_l}^{\circ k_l} \) with \( k_1 + \cdots + k_l = m \) then \( A \in T^\circ m \) and the maximal element in the \( p \)th column of \( A \) is \( m \)(which is also the maximal element of \( A \)).

**Lemma 4.2.** Let \( A, B \in E_n \) and \( A, B \notin T \) and assume that \( A^\diamond = B^\diamond \). Assume also that \( |\text{Max}(A^\diamond)| = |\text{Max}(B^\diamond)| = 1 \).

1. If \( a_{ij} > b_{ij} \) then \( a_{ij} = b_{ij} + 1 \).
2. \( a_{ij} > b_{ij} \) implies that \( a_{ij} = m \) and \( b_{ij} = m - 1 \).
3. \( a_{ij} = 0 \) if and only if \( b_{ij} = 0 \).
Proof. (1) Assume \( a_{ij} \geq b_{ij} + 2 \). Assume that \( A = S(p) = T_{I_1}^{c_{k_1}} \odot \cdots \odot T_{I_l}^{c_{k_l}} \), where \( I_1 \supset \cdots \supset I_l \) (the inclusions are strict) and \( k_1 + \cdots + k_l \geq 2 \) as \( A \) is not minimal. Let \( I_t \) be such that \( i \in I_t \), and consider the matrix obtained from \( S(p) \) by removing one factor \( I_t \). The obtained matrix belongs to \( E_{ij} \), and its \( ij \)-entry equals \( a_{ij} - 1 \). Hence, the obtained matrix is strictly less than \( A \) and is not less than \( B \) which contradicts the assumption that \( A^\psi = B^\psi \). It follows that \( a_{ij} = b_{ij} + 1 \), as required.

(2) Assume \( a_{ij} > b_{ij} \) but that \( a_{ij} \neq m \). This implies that there is some \( t \) such that \( i \notin I_t \). Consider the matrix obtained from \( A = S(p) \) by removing the factor \( T_{I_t}^{c_{k_t}} \). This matrix is strictly less than \( A \) but its \( ij \)-entry equals \( a_{ij} > b_{ij} \). Thus \( A - T_{I_t}^{c_{k_t}} \) belongs to \( A^\psi \setminus B^\psi \), which contradicts our assumption.

(3) Assume \( b_{ij} = 0 \) and \( a_{ij} \neq 0 \). Then \( a_{ij} > b_{ij} \) which, by (2) above, means \( a_{ij} = 1 = m \).

This is a contradiction with \( m > 1 \).

We now complete the proof of Theorem 4.1. Let \( A, B \in \mathcal{E}_n \), \( A \neq B \) and \( A, B \notin \mathcal{T} \) and assume that \( A^\psi = B^\psi \). (Note that if \( A \in \mathcal{T} \) and \( B \notin \mathcal{T} \) the equality \( A^\psi = B^\psi \) can not hold.) We can suppose that \( |\text{Max}(A^\psi)| = |\text{Max}(B^\psi)| = 1 \). Let \( A = S(j) = T_{J_1}^{c_{r_1}} \circ \cdots \circ T_{J_s}^{c_{r_s}} \) with \( k_1 + \cdots + k_l = m \) and let \( B = S(q) = T_{J_1}^{\circ_{r_1}} \circ \cdots \circ T_{J_s}^{\circ_{r_s}} \) with \( r_1 + \cdots + r_s = l \) be the column decompositions of \( A \) and \( B \). As \( B \nsubseteq A \) there is an index \( ij \) such that \( a_{ij} > b_{ij} \). By Lemma 4.2 we have \( a_{ij} = m \) and \( b_{ij} = m - 1 \). This equality and \( J_1 \supset \cdots \supset J_s \) mean that \( i \in J_1, \ldots, J_{s-1} \) and \( i \notin J_s \) (also \( r_s = 1 \)). Let \( p \in J_s \). This and \( i \in J_s^c \) implies that \( b_{pi} \neq 0 \). On the other hand, as \( a_{ij} = m \), it follows that \( i \in I_1, \ldots, I_t \), so \( i \notin I_t^c \) for all \( t = 1, \ldots, l \). It follows that \( a_{ti} = 0 \) for all \( t = 1, \ldots, n \). In particular \( a_{pi} = 0 \). This, together with \( b_{pi} \neq 0 \), contradicts part (3) of Lemma 4.2. This finishes the proof.

4.2. Automorphisms of \((\mathcal{E}_n, \oplus)\). Throughout this section, if not stated otherwise, we assume that \( n \geq 3 \). In this subsection we prove that the automorphisms of the semigroup \((\mathcal{E}_n, \oplus)\) are the same as those of the semigroup \((\mathcal{E}_n, \odot)\) (cf. Theorem 3.1).

Theorem 4.3. Let \( \varphi \) be an automorphism of \((\mathcal{E}_n, \oplus)\) where \( n > 2 \). Then \( \varphi \in C_2 \times S_n \).

For \( A, B \in \mathcal{E}_n \) we have \( A \leq B \) if and only if \( A \oplus B = B \). Therefore, for \( \varphi \in \text{Aut}(\mathcal{E}_n, \oplus) \) we have that \( A \leq B \) if and only if \( \varphi(A) \leq \varphi(B) \). In other words, an automorphism of \((\mathcal{E}_n, \oplus)\) is an order-automorphism of \((\mathcal{E}_n, \leq)\). And conversely, since \( A \oplus B \) is the join of \( A \) and \( B \) with respect to \( \leq \), it follows that any order-automorphism of \((\mathcal{E}_n, \leq)\) is an automorphism of \((\mathcal{E}_n, \oplus)\). This observation will be important in what follows and will be used without further mention.

Again the case \( n = 2 \) is very easy. Indeed, since \( \varphi \in \text{Aut}(\mathcal{E}_2) \) preserves the partial order, it preserves the set of minimal matrices \( \{e_{12}, e_{21}\} \), and consequently, \( \varphi \) is either the identity map, or the transposition of matrices.

Throughout this section, if not stated otherwise, we assume that \( n \geq 3 \).

Lemma 4.4. Let \( \varphi, \psi \in \text{Aut}(\mathcal{E}_n, \oplus) \) are such that \( \varphi(T) = \psi(T) \) for all \( T \in \mathcal{T} \). Then \( \varphi = \psi \).
Proof. For $A \in \mathcal{E}_n$ let $h(A)$ be the biggest integer $k$ such that there exist $A_1, \ldots, A_k \in \mathcal{E}_n$ such that $A_1 \leq A_2 \leq \cdots \leq A_k = A$. Applying $\varphi$, it follows that $h(\varphi(A)) \geq h(A)$. As $A = \varphi^{-1} \varphi(A)$, we similarly obtain the opposite inequality, whence $h(\varphi(A)) = h(A)$.

We prove the statement of the lemma by induction on $h(A)$. Notice that $h(A) = 1$ if and only if $A \in \mathcal{T}$ and the equality $\varphi(A) = \psi(A)$ for $A \in \mathcal{T}$ holds by the assumption of the lemma. We assume that $\varphi(A) = \psi(A)$ for any $A$ with $h(A) \leq t$, where $t \geq 1$, and prove that $\varphi(A) = \psi(A)$ for any $A$ with $h(A) = t + 1$. Assume that $\varphi(A) \neq \psi(A)$. Since $h(\varphi(A)) = h(\psi(A)) = t + 1$, it follows from Theorem 4.1 that $\varphi(A)^\diamond \neq \psi(A)^\diamond$. Without loss of generality, there is $C \in \mathcal{E}_n$ satisfying $C \leq (\varphi(A))$ but $C \not\leq \psi(A)$. Hence, $\varphi^{-1}(C) \leq A$ and $\varphi^{-1}(C) \not\leq \varphi^{-1}(C)$. Observe that $h(\varphi^{-1}(C)) \leq t$ and thus, by the inductive assumption, we have $C = \varphi\varphi^{-1}(C) = \psi\varphi^{-1}(C)$. Hence, $\psi^{-1}(C) = \varphi^{-1}(C)$ whence $\psi^{-1}(C) \leq A$, and applying $\psi$, we get $C \leq \psi(A)$, which contradicts our assumption on $C$ that $C \not\leq \psi(A)$. Therefore, we have proved that $\varphi(A) = \psi(A)$.

Let $I_1, I_2 \subseteq \{1, \ldots, n\}$, $|I_1|, |I_2| \in \{1, \ldots, n - 1\}$. If the inequality

\[(4.1) \quad T_J \leq T_{I_1} \oplus T_{I_2}\]

holds for some $J \subseteq \{1, \ldots, n\}$ with $|J| \in \{1, \ldots, n - 1\}$ we say that $T_J$ is a solution of (4.1). We say that a solution $T_J$ of (4.1) is proper if $J \neq I_1$ and $J \neq I_2$.

Lemma 4.5. Let $I_1, I_2 \subseteq \{1, \ldots, n\}$ and $|I_1|, |I_2| \in \{1, \ldots, n - 1\}$. Then

1. If $|I_1 \cap I_2| \in \{1, \ldots, n - 1\}$ then $T_{I_1 \cap I_2}$ is a solution of (4.1).
2. If $|I_1 \cup I_2| \in \{1, \ldots, n - 1\}$ then $T_{I_1 \cup I_2}$ is a solution of (4.1).
3. If $T_J$ is a proper solution of (4.1) then either $J = I_1 \cap I_2$ or $J = I_1 \cup I_2$.

Proof. (1) Let $T_{I_1 \cap I_2} = (a_{ij})$, $T_{I_1} = (b_{ij})$ and $T_{I_2} = (c_{ij})$. (1) We need to show that if $a_{ij} = 1$ then $b_{ij} + c_{ij} = 1$. We have $a_{ij} = 1$ if and only if $i \in I_1 \cap I_2$ and $j \in (I_1 \cap I_2)^c = I_1^c \cup I_2^c$. If $i \in I_1$ we have $b_{ij} = 1$, if $i \in I_2$ we have $c_{ij} = 1$, as needed.

(2) is similar to (1).

(3) Let $T_J = (a_{ij})$, $T_{I_1} = (b_{ij})$ and $T_{I_2} = (c_{ij})$. We first show that $I_1 \cap I_2 \subseteq J$. If $I_1 \cap I_2 = \emptyset$, we are done. Otherwise, if $I_1 \cap I_2 \subseteq J$, take some $j \in (I_1 \cap I_2) \cap J^c$. Then take some $i \in J$. We have $a_{ij} = 1$, but $b_{ij} = c_{ij} = 0$, which contradicts $T_J \leq T_{I_1} \oplus T_{I_2}$.

Assume that $J \cap (I_1 \setminus I_2) \neq \emptyset$ and show that $J \supseteq I_1$. If the latter does not hold, we take some $i \in J \cap (I_1 \setminus I_2)$ and $j \in I_1 \setminus J$. We have $a_{ij} = 1$, $b_{ij} = c_{ij} = 0$, which again contradicts $T_J \leq T_{I_1} \oplus T_{I_2}$.

It follows by symmetry that if $J \cap (I_2 \setminus I_1) \neq \emptyset$ then $J \supseteq I_2$.

We finally show that $J \subseteq I_1 \cup I_2$. If $I_1 \cup I_2 = \{1, \ldots, n\}$, then we are done. Otherwise, assume that $J \setminus (I_1 \cup I_2) \neq \emptyset$ and take some $i \in J \setminus (I_1 \cup I_2) = J \cap I_1^c \cap I_2^c$ and any $j \in J^c$. We have $a_{ij} = 1$ but $b_{ij} = c_{ij} = 0$, a contradiction with $T_J \leq T_{I_1} \oplus T_{I_2}$.\[\square\]

Lemma 4.5 tells us that for any given $I_1, I_2 \subseteq \{1, \ldots, n\}$ with $|I_1|, |I_2| \in \{1, \ldots, n - 1\}$, (4.1) has at most two proper solutions.

Lemma 4.6.

1. We have that $|I_1| \in \{1, n - 1\}$ if and only if for any $I_2 \subseteq \{1, \ldots, n\}$ with $|I_2| \in \{1, \ldots, n - 1\}$ the inequality (4.1) has at most one proper solution.
Let $\varphi \in \text{Aut}(\mathcal{E}_n, \oplus)$. Then $\varphi(\mathcal{L} \cup \mathcal{C}) = \mathcal{L} \cup \mathcal{C}$.

Let $\varphi \in \text{Aut}(\mathcal{E}_n, \oplus)$. Then either $\varphi(\mathcal{L}) = \mathcal{L}$ or $\varphi(\mathcal{C}) = \mathcal{C}$ (and then, respectively, $\varphi(\mathcal{L}) = \mathcal{L}$ or $\varphi(\mathcal{C}) = \mathcal{C}$).

Proof. (1) Assume that $|I_1| = 1$. Then we have that either $I_1 \cap I_2 = \emptyset$, or, otherwise, $I_1 \cap I_2 = I_1$. Thus $T_{I_1 \cap I_2}$ can not be a proper solution of (4.1). Assume that $|I_1| = n - 1$. Then we have that either $I_1 \cup I_2 = \{1, \ldots, n\}$ or, otherwise, $I_2 \subseteq I_1$. Thus $T_{I_1 \cup I_2}$ can not be a proper solution of (4.1).

Assume now that for any $I_2 \subseteq \{1, \ldots, n\}$ with $|I_2| \in \{1, \ldots, n - 1\}$ the inequality (4.1) has at most one proper solution and let us prove that $|I_1| \in \{1, n - 1\}$. Assume that $|I_1| \notin \{1, n - 1\}$. Let $x,y,s,t \in \{1, \ldots, n\}$ be such that $x \neq y$, $s \neq t$, $x,y \in I_1$ and $s,t \in I_2$. Let $I_2 = (I_1 \setminus \{x\}) \cup \{s\}$. In this case we have $T_{I_1 \cup I_2}$ and $T_{I_1 \cap I_2}$ are two proper solutions of (4.1).

(2) follows from (1) because $\mathcal{L} = \{T_I: |I| = 1\}$ and $\mathcal{C} = \{T_I: |I| = n - 1\}$ and an automorphism preserves the property about proper solutions given in (1).

(3) As $\varphi(T) = T$, we have $\varphi(U) = \varphi(\oplus_{T \in T} T) = \oplus_{T \in T} T = U$. Now, from $\oplus_{i=1}^n \varphi(T_{\{i\}}) = U$, we have $\oplus_{i=1}^n \varphi(T_{\{i\}}) = U$. As $\#(U) = n(n - 1)$ and $\#(A) = n - 1$ for any $A \in \mathcal{L} \cup \mathcal{C}$, no two of the matrices $\varphi(T_{\{i\}})$ above can have overlapping occurrences of 1. It follows that the set of all $\varphi(T_{\{i\}})$, $1 \leq i \leq n$, is either $\mathcal{L}$ or $\mathcal{C}$.

Let $\varphi \in \text{Aut}(\mathcal{E}_n, \oplus)$. We assume that $\varphi(\mathcal{L}) = \mathcal{L}$. By the lemma above we have that there are $\sigma, \tau \in S_n$ such that $\varphi(T_{\{i\}}) = T_{\{\sigma(i)\}}$ and $\varphi(T_{\{i\}^c}) = T_{\{\tau(i)\}^c}$ for all $i = 1, \ldots, n$.

We now proceed with the proof of Theorem 4.3. Let $I \subseteq \{1, \ldots, n\}$ be such that $|I| \in \{1, \ldots, n - 1\}$. For $S \in \mathcal{T}$ consider the following sets of conditions:

\begin{align}
(4.2) & \quad S \leq \oplus_{i \in I} T_{\{i\}} \text{ and } S \not\leq \oplus_{i \in I \setminus \{j\}} T_{\{i\}} \text{ for any } j \in I; \\
(4.3) & \quad S \leq \oplus_{i \in I^c} T_{\{i\}^c} \text{ and } S \not\leq \oplus_{i \in I \setminus \{j\}} T_{\{i\}^c} \text{ for any } j \in I^c.
\end{align}

It is straightforward to verify that there is precisely one $S \in \mathcal{T}$ which satisfies (4.2): this is $S = T_I$ and also there is precisely one $S \in \mathcal{T}$ which satisfies (4.3): this is again $S = T_I$. Applying $\varphi$ to sets of conditions (4.2) and (4.3), we obtain

\begin{align}
(4.4) & \quad \varphi(S) \leq \oplus_{i \in I} T_{\{\sigma(i)\}} \text{ and } S \not\leq \oplus_{i \in I \setminus \{j\}} T_{\{\sigma(i)\}} \text{ for any } j \in I; \\
(4.5) & \quad \varphi(S) \leq \oplus_{i \in I^c} T_{\{\tau(i)\}^c} \text{ and } S \not\leq \oplus_{i \in I \setminus \{j\}} T_{\{\tau(i)\}^c} \text{ for any } j \in I^c.
\end{align}

By uniqueness of solution of (4.4) and (4.5) we obtain that $\varphi(S) = T_{\sigma(I)} = T_{\tau(I)}$. It follows in particular that $\sigma(I) = \tau(I)$ for any $I \subseteq \{1, \ldots, n\}$ with $|I| \in \{1, \ldots, n - 1\}$. If we take $|I| = 1$ this implies that $\sigma = \tau$.

It follows that if $\varphi \in \text{Aut}(\mathcal{E}_n, \oplus)$ and $\varphi(\mathcal{L}) = \mathcal{L}$, there is $\sigma \in S_n$ such that for all $A \in \mathcal{T}$ we have $\varphi(T) = \sigma \cdot T$. Now, it follows from Lemma 4.4 that $\varphi(A) = \sigma \cdot A$ for all $A \in \mathcal{E}_n$. This completes the proof for the case where $\varphi(\mathcal{L}) = \mathcal{L}$. The case where $\varphi(\mathcal{L}) = \mathcal{C}$ is considered similarly.

Corollary 4.7. $\text{Aut}(\mathcal{E}_n, \circ) = \text{Aut}(\mathcal{E}_n, \oplus) = \text{Aut}(\mathcal{E}_n, \leq) = \text{Aut}(\mathcal{E}_n, \circ, \oplus, 0) \cong S_n \times C_2$, if $n \geq 3$. 

Notice, that each \( n \times n \)-tiled order over a discrete valuation ring \( \mathcal{O} \) is conjugate by a matrix from \( GL_n(\mathcal{O}) \) to a tiled order with non-negative exponent matrix (see, for example, [14][15]). The set \( \text{Tiled}(n, \mathcal{O}) \) of all \( n \times n \) tiles orders over a fixed \( \mathcal{O} \) is a partially ordered set with respect to the set-theoretic inclusion \( \subseteq \), which is anti-isomorphic to \( (\mathcal{E}_n, \leq) \). In addition, there is an anti-isomorphism between \( (\text{Tiled}(n, \mathcal{O}), \cap) \) and \( (\mathcal{E}_n, \oplus) \). Consequently,

\[
\text{Aut}(\text{Tiled}(n, \mathcal{O}), \subseteq) \cong \text{Aut}(\mathcal{E}_n, \leq) \cong \text{Aut}(\text{Tiled}(n, \mathcal{O}), \cap) \cong \text{Aut}(\mathcal{E}_n, \oplus).
\]

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THE MAX-PLUS ALGEBRA OF EXPONENT MATRICES OF TILED ORDERS

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