Characteristic classes of smooth fibrations

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Abstract

We construct characteristic classes of smooth (Hamiltonian) fibrations as fiber integrals of products of Pontriagin (or Chern) classes of vertical vector bundles over the total space of the universal fibration. We give explicit formulae of these fiber integrals for toric manifolds and get estimates of the dimension of the cohomology groups of classifying spaces.

Keywords: characteristic class; fiber integration; classifying space.

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1 Background

1.1 Universal fibrations and characteristic classes

Let $G$ be a topological group for which the universal principal fibration $G \rightarrow E_G \rightarrow B_G$ is locally trivial. Suppose $G$ acts (smoothly) on a manifold $M$. Consider the associated bundle

$$M \hookrightarrow M_G = E_G \times_G M \rightarrow B_G.$$  

Any fibration $M \hookrightarrow P \rightarrow B$ with structural group $G$ and fiber $M$ is obtained as a pullback of $M_G$ for an appropriate map $f : B \rightarrow BG$

$$
\begin{array}{ccc}
P = f^*M_G & \xrightarrow{f^*} & M_G \\
\pi_B \downarrow & & \pi \\
B & \xrightarrow{f} & BG.
\end{array}
$$

A $\chi$-characteristic class of the $G$-fibration $M \hookrightarrow P \rightarrow B$ is a cohomology class of the form $f^*\chi \in H^*(B)$, where $\chi \in H^*(BG)$.

1.2 Equivariant bundles

A $G$-equivariant bundle $\xi \rightarrow M$ extends to a bundle $\xi_G \rightarrow M_G$ such that the following diagram commutes

$$
\begin{array}{ccc}
\xi & \xrightarrow{i} & \xi_G \\
\downarrow & & \downarrow \text{Id} \\
M & \xrightarrow{i} & M_G \\
\end{array} \xrightarrow{\text{Id}} \begin{array}{c} \text{} \\
\rightarrow \end{array} \xrightarrow{\text{Id}} B_G.
$$
More precisely, $\xi_G = (\xi \times E_G)/G$ and $i^*\xi_G = \xi$, where $i: M \to M_G$ is an inclusion of the fiber. A particularly simple case is when $\xi = TM$ is the tangent bundle of $M$, and the $G$ action is smooth. Then we call $TM_G$ the vertical bundle of the fibration $M_G \to B_G$.

We will need additional structures in these extended bundles, the most important being complex structure. If $G$ acts on a vector bundle preserving a complex structure $J$, then $\xi_G$ admits a complex structure. We can relax somewhat invariance condition of $J$ with respect to $G$ and still obtain a complex structure in $\xi_G$.

**Proposition 1.2.1** Let $\xi \to M$ be a $G$-equivariant real vector bundle. Suppose that there exists a contractible $G$-invariant set $\mathcal{J}$ of complex structures in $\xi$. Then there exists a complex structure $J_G$ in $\xi_G$ whose restriction to each fiber is homotopic to any $J \in \mathcal{J}$. In particular, $i^*c_k(\xi_G, J_G) = c_k(\xi, J)$, for $J \in \mathcal{J}$.

**Proof:** Consider the map $\xi \times E_G \times \mathcal{J} \to M \times E_G \times \mathcal{J}$. This is clearly equivariant with respect to the $G$ action, and thus defines a projection of a bundle

$$\xi'_G = \xi \times E_G \times \mathcal{J}/G \to M \times E_G \times \mathcal{J}/G.$$ 

Since $\mathcal{J}$ is contractible, the base is homotopy equivalent to $M_G$ and $\xi'_G$ is essentially $\xi_G$, that is it is a pullback by the inverse of the homotopy equivalence $E_G \times \mathcal{J}/G \to E_G/G$. The bundle $\xi \times \mathcal{J} \to M \times \mathcal{J}$ admits a tautological complex structure $\tilde{J}$:

$$\tilde{J}_{(m,J)}(v) = J(m)(v),$$

which is $G$-equivariant and thus define a complex structure in $\xi'_G$. Pulling it back we obtain a required complex structure in $\xi_G$.

**QED**

Proposition 1.2.1 is of very general nature and should have many variants for groups acting on vector bundles and preserving geometric structures. The motivating example for the Proposition 1.2.1 comes from symplectic geometry. It is a remark of fundamental importance, due to Gromov, that the set of complex structures $J$ tamed by a symplectic form $\omega$ (i.e. such that $\omega(J\cdot, \cdot)$ is positive definite symmetric form) is contractible. It consists of sections of the bundle over $M$, which is associated to the frame bundle and has a contractible fiber $Sp(2n; \mathbb{R})/U(n)$. 

3
2 Characteristic classes from fiber integrals

Recall that for any fibration $M \hookrightarrow P \to B$, where $M$ is oriented $m$-
dimensional compact manifold, there is a homomorphism of $H^*(B)$-modules

$$\pi_* : H^*(P) \to H^{*-m}(B)$$

called fiber integration \[\text{fiber integration}\] (see [AB], [GS Chapter 10] for detailed description). Although fiber integration behaves badly with respect to the cup product, it has useful properties. The most important to us is its naturality.

Let $M \hookrightarrow P \rightarrow B$ be a fibration with $m$-dimensional fiber and let be

given its pullback

$$f^* \rightarrow \begin{array}{c} P \leftarrow \tilde{f}^* \rightarrow H^*(P) \\ \downarrow \pi' \downarrow \pi \\ B' \rightarrow f \rightarrow B. \end{array}$$

Then the naturality of fiber integration means that the following diagram commutes

$$H^*(f^* P) \leftarrow \tilde{f}^* \rightarrow H^*(P)$$

$$\downarrow \pi'_* \downarrow \pi_*$$

$$H^{*-m}(B') \leftrightarrow f^* \rightarrow H^{*-m}(B).$$

We use fiber integration to construct characteristic classes of $G$-bundles from characteristic classes of $\xi_G$-bundles. This construction has several variants depending on the structure of $\xi_G$ which we describe below. The simple characteristic classes on $M_G$ become fairly involved when pushed down. This of course agrees with the general philosophy of reduction.

2.1 The Pontriagin classes

Let $G = Diff(M)$ and suppose $\xi$ is a $G$-equivariant vector bundle. For example, $\xi$ might be the tangent bundle. Take $\xi_G \to M_G$ and a monomial in Pontriagin classes $p_I(\xi_G) := p_{i_1}(\xi_G) \cup \cdots \cup p_{i_k}(\xi_G)$. Then integrate it over the fiber of the fibration $M \to M_G \to B_G$, to obtain classes $\pi_*^{\xi_G}(p_I(\xi_G)) \in H^*(B_G)$. This is depicted in the following diagram where $f_{\xi_G} : M_G \to B_{O(n)}$ is a classifying map and the curved arrow is the composition.

\[\text{In case when } M \text{ is not compact } \pi_* \text{ is defined on cohomology with compact support.}\]
2.2 The Pontriagin and the Euler classes

If $G \subset \text{Diff}(M)$ acts on an oriented bundle $\xi$ preserving the orientation then we can also consider the Euler class of $\xi$. Integrating monomials in the Euler and Pontriagin classes along the fiber we get elements in $H^*(B_G)$. Again, if $\xi$ is the tangent bundle of an oriented manifold $M$ then we obtain an information about the cohomology of the classifying space of the group of orientation preserving diffeomorphisms.

For an orientable 2-dimensional manifold $M$ the classes $\pi_*(\text{eu}(TM_G)^k)$ are the, so called, Miller-Morita-Mumford classes and have been extensively studied [Mo].

2.3 The Chern classes

Let $G \subset \text{Diff}(M)$. Suppose now that $\xi$ is an $G$-equivariant vector bundle over $M$ admitting a family of complex structures as in Proposition 2.1. Consider a monomial in the Chern classes $c_I := c_{i_1}(\xi_G) \cup \cdots \cup c_{i_k}(\xi_G)$, and obtain classes $\pi_G^*(c_I) \in H^*(B_G)$. If $\xi$ is a tangent bundle of a symplectic manifold $(M, \omega)$, $G = \text{Symp}(M, \omega)$ and $\mathcal{J}$ is the set of tamed almost complex structures, then we get characteristic classes of symplectic fibrations.

2.4 Hamiltonian actions and coupling class

It is also interesting to consider the group $\text{Ham}(M, \omega) \subset \text{Symp}(M, \omega)$ of Hamiltonian symplectomorphisms. Then an additional characteristic class, arises [GS, Section 9.5],[Po, page 70]:

**Definition 2.4.1** An element $\Omega \in H^2(M_{\text{Ham}(M,\omega)})$ satisfying the following conditions:

1. $\pi_*(\Omega^{n+1}) = 0$
2. $i^*(\Omega) = [\omega]$.

is called the coupling class
Proposition 2.4.2 The coupling class is unique and it is equal to

$$\Omega := \tilde{\omega} - \frac{1}{(n+1)Vol(M,\omega)}\pi^*(\pi_*\tilde{\omega}^{n+1}), \quad (2.1)$$

where $\tilde{\omega} \in H^*(M_{Ham(M,\omega)})$ restricts to the class of symplectic form of the fiber (i.e. $i^*(\tilde{\omega}) = [\omega] \in H^*(M)$) and $Vol(M,\omega) = \frac{1}{n!}\int_M \omega^n$ is the symplectic volume of $(M,\omega)$.

Proof: Since the universal fibration $M_{Ham(M,\omega)} \to BHam(M,\omega)$ is Hamiltonian, then there is a class $\tilde{\omega} \in H^2(M_{Ham(M,\omega)})$ which restricts to the class of symplectic form of the fiber [M[S]. The class given by the above formula satisfies the conditions (1) of Definition 2.4.1.

Since $H^2(M_{Ham(M,\omega)}) \cong H^2(B_{Ham(M,\omega)}) \oplus H^2(M)$ [L[M], then if there were another class $\Omega_1$ satisfying the definition then we would have $\Omega_1 = \Omega + \pi^*\alpha$, for some $\alpha \in H^2(B_{Ham(M,\omega)})$ and

$$\pi_*(\Omega_1^{n+1}) = \pi_*(\Omega + \pi^*\alpha)^{n+1} = \pi_*(\Omega^{n+1} + (n+1)\Omega^n\pi^*\alpha + ...) = (n+1)\alpha \neq 0.$$ 

QED

Let $G = Ham(M,\omega)$ be a the group of Hamiltonian symplectomorphisms. Then integrating monomials in the Chern classes and the coupling class we obtain an information about $H^*(B_{Ham(M,\omega)})$ in the same way as in the previous constructions. It is worth noting that taking the coupling class into account is essential as we shall see in Section 3.1.

Reznikov [Rd] has constructed characteristic classes of Hamiltonian fibration using the Chern-Weil theory. As remarked by McDuff in [M] the classes obtained by Reznikov are fiber integrals of powers of the coupling class.

3 The setup for the computations

We are interested in nonvanishing and independence of classes in $H^*(B_{Diff(M)})$ ($H^*(B_{Symp(M,\omega)})$ or $H^*(B_{Ham(M,\omega)})$). We study this question by pulling these classes back to $H^*(B_H)$ using smooth (symplectic or Hamiltonian) actions of a compact Lie group $H$. Notice that it suffices to restrict to the case of torus actions, as $H^*(B_H) = H^*(B_T)^W \subset H^*(B_T)$, where $T \subset H$ is a maximal torus and $W$ is the Weyl group.
3.1 The detection function

In the sequel the group $G$ will denote the group of Hamiltonian symplectomorphisms or the group of all diffeomorphisms of a symplectic manifold $(M, \omega)$. It is worth of noting that we obtain more information if we take into account several torus actions. This is the reason for the following definition.

**Definition 3.1.1** Let $f_i : T \to G \subset Diff(M), \ i = 1, \ldots, m$ be torus actions. We define the detection function by

$$F : H^*(B_G) \to \bigoplus_{i=1}^m H^*(B_T)$$

$$F(\alpha) := [f_1(\alpha), \ldots, f_m(\alpha)]$$

We compute the image of the detection function on the fiber integrals of characteristic classes which allows to estimate the Betti numbers of $B_{Ham(M,\omega)}$ and $B_{Diff(M)}$.

3.2 Localization formula

Fiber integration for torus actions can be computed using the Atiyah-Bott-Berline-Vergne $[AB, BV]$

$$\pi_*(\alpha) = \sum_P \pi^P_*(\frac{i_P^*\alpha}{E(\nu_P)})$$

for $\alpha \in H^*(M_T)$. Here $P$ is a connected component of the set of fixed points, $i_P : P \hookrightarrow M$ is the inclusion, $\pi^P_* : H^*(P_T) \cong H^*(B_T) \otimes H^*(P) \to H^*(B_T)$ is fiber integration for the trivial action of $T$ on $P$ and $E(\nu_P)$ is the equivariant Euler class of the normal bundle $\nu_P$ to $P$ in $M$.

In fact the above formula describes the fiber integration between the localized rings (where the localization is in the ideal generated by the Euler classes of the normal bundles of the connected components of the fixed point set). We apply this to the elements of $H^*(M_T)$ due to the fact that the fiber integration commutes with the localization as depicted in the following diagram, where the right hand side column consists of localized rings:

$$
\begin{array}{ccc}
H^*(M_T) & \longrightarrow & H^*(M_T) \\
\pi_* \downarrow & & \downarrow (\pi_*)_* \\
H^*(B_T) & \longrightarrow & H^*(B_T)
\end{array}
$$
3.3 Symplectic toric varieties, moment maps and Delzant polytopes

Definition 3.3.1 A symplectic toric variety is a symplectic manifold \((M^{2n}, \omega)\) endowed with an effective Hamiltonian action of an \(n\)-dimensional torus \(T\).

Let \(a : G \to \text{Ham}(M, \omega)\) be a Hamiltonian action of a Lie group \(G\) on a symplectic manifold \(M\). Given \(X \in \text{Lie}(G)\) a fundamental vector field \(X\) on \(M\) is Hamiltonian, which means that
\[
\iota_X \omega = -dH_X,
\]
for some function \(H_X : M \to \mathbb{R}\). This function, initially determined up to a constant, is chosen so that the map \(\text{Lie}(G) \to C^\infty(M)\) given by \(X \to H_X\) is a homomorphism of Lie algebras. The Lie algebra structure on functions is given by the Poisson bracket.

Then one defines a moment map \(\Phi : M \to \text{Lie}(G)^*\) by
\[
\Phi(p)(X) = H_X(p).
\]

It is a fundamental result of Atiyah, that for actions of tori the image of a moment map is a convex polytope \(\Delta \subset \mathbb{R}^n\). Moreover \(\Delta\) satisfies the following conditions:

1. There are \(n\) edges meeting at each vertex \(p \in \Delta\) (this means that \(\Delta\) is a simple polytope).

2. Every edge including \(p\) is of the form \(p + tv_i\), where \(v_i \in (\mathbb{Z}^n)^*\).

3. \(v_1, \ldots, v_n\) in (2) can be chosen to be a basis of \((\mathbb{Z}^n)^*\).

Such polytope is called Delzant. Given a Delzant polytope \(\Delta\) there is a symplectic manifold equipped with a Hamiltonian torus action such that the image of the moment map is exactly \(\Delta\) [De], [G, Theorem 1.8]. It is useful to describe \(\Delta\) by a system of inequalities of the form
\[
\langle x, u_i \rangle \geq \lambda_i, \quad i - 1, \ldots, k,
\]
where \(u_i \in \mathbb{Z}^n\). The vectors \(u_i\) can be normalized by requiring them to be primitive. This normalization together with inequalities (3.1) determine \(u_i\)'s uniquely.

Notice that the vectors \(u_i\) may be thought as the inward pointing vectors normal to the faces of the Delzant polytope. The function which associate the vector \(u_i\) to the face \(F_i\) was defined in [D] (page 423) and called the characteristic function.
3.4 Equivariant cohomology of toric manifolds and the face ring

Consider the universal fibration associated to a toric manifold $M \hookrightarrow M_T \xrightarrow{\pi} B_T$. We are interested in the description of the homomorphism $\pi^*: H^*(B_T) \to H^*(M_T)$ induced by the projection in the universal fibration. By a result [DJ], the cohomology ring $H^*(M_T)$ is isomorphic to the face ring of the Delzant polytope $\Delta$. The face ring $St(\Delta)$ of a given $n$-dimensional simple polytope $\Delta$ is defined to be a graded ring generated by the $(n-1)$-dimensional faces $F_1, ..., F_k$ of $\Delta$, all generators being of degree two, subject to the relations $F_i F_j = 0$ iff $F_i \cap F_j = \emptyset$ [S, Chapter II].

Remark 3.4.1 In the sense of the definition given in [S], our face ring is the face ring of the simplicial complex dual to $\Delta$.

Now we can identify the map $\pi^*: H^*(B_T) \to H^*(M_T)$. Since $H^*(B_T) \cong \mathbb{R}[T_1, ..., T_n]$ and $H^*(M_T) \cong St(\Delta)$ are generated by the elements of degree 2, then it suffices to describe $\pi^*$ on the second cohomology. With respect to the bases $\{T_1, ..., T_n\}$ of $H^2(B_T)$ and $\{F_1, ..., F_k\}$ of $H^2(M_T)$, $\pi^*$ is given by the matrix $[u_{ij}]$, where $[u_{i1}, ..., u_{in}] = u_i \in \mathbb{Z}^n$ are the vectors normal to the faces of $\Delta$ [DJ].

Next we describe the Chern classes of the vertical fibration $TM_T$ (see Proposition 1.2.1) in terms of the face ring $St(\Delta)$ [DJ].

**Proposition 3.4.2** The Chern classes $c_i(TM_T) \in St(\Delta) \cong H^*(M_T)$ of the vertical fibration are given by the following formula:

$$c_i(TM_T) = \sum_{1 < j_1 < ... < j_i < k} F_{j_1}...F_{j_i}.$$  

**Proof:** The idea of the proof is to show that the vertical bundle $TM_T$ is stably isomorphic as a complex bundle to the sum of line bundles, whose Chern classes are represented by the faces of the Delzant polytope. This was done by in [DJ] in the real case. The specific choice of orientations of the line bundles provides the isomorphism preserving complex structures.

To be more precise, we associate a line bundle $L_i \to M_T$ to every face $F_i$ of the Delzant polytope in the following way. The face $F_i$ gives a vector $[u_{i1}, ..., u_{in}] \in \mathbb{Z}^n$ as described by the inequalities (3.1). Since $\mathbb{Z}^n \cong \hat{T}^n$ then we obtain a character $\chi_i: T^n \to \mathbb{S}^1$. Now the line bundle $L_i$ is defined as

$$L_i := (M \times E_T \times C)/T,$$

where the action of the torus on $C$ is given by the character $\chi_i$. The choice of the inward pointing normal vectors to the faces of the Delzant polytope

Footnote: Face rings are also sometimes called Stanley-Reisner or Stanley rings.
ensures that the sum of the line bundles $L_i$ is stably isomorphic to $TM_T$ as a complex bundle. \hspace{1cm} \text{QED}

Notice that, since the action is Hamiltonian, then we may ask also about the form of the coupling class in the face ring.

\textbf{Proposition 3.4.3} The coupling class $\Omega \in St(\Delta)$ is given by

$$\Omega = - \sum \lambda_i F_i - \frac{1}{(n+1)Vol(M,\omega)} \pi^*(- \sum \lambda_i F_i)^{n+1},$$

where $\lambda_i \in \mathbb{R}$ are as in (3.1).

\textbf{Proof:} As we have observed in Section 2.4, the coupling class is given by the formula (2.1). Thus we have to show that the class $-\sum \lambda_i F_i$ restricts to the class of the symplectic form, that is to say $i^*(-\sum \lambda_i F_i) = [\omega]$. Here $i : M \to M_T$ is an embedding of the fiber. It is known that the class of the symplectic form is equal to $-\sum \lambda_i p_1$, where $p_i \in H^2(M)$ is Poincare dual to the homology classes given by the preimages of the $(n-1)$-dimensional face $F_i$ of the Delzant polytope under the moment map $\mathbb{C}(\text{Appendix 2})$. On the other hand, these classes are exactly equal to $i^*(F_i)$, which completes the proof. \hspace{1cm} \text{QED}

\section{3.5 Specifying the setup to toric varieties}

Let $M$ be a symplectic toric variety of dimension $2n$ and $\Delta$ denotes its Delzant polytope. Recall that the cohomology ring $H^*(M_T)$ is isomorphic to the face ring $St(\Delta)$. In this section, we give an explicit formula for fiber integration

$$\pi_* : H^*(M_T) \cong St(\Delta) \to H^{*-2n}(B_T) \cong \mathbb{R}[T_1,\ldots,T_n].$$

In fact, we give an algorithm which allows to compute fiber integrals of Chern classes (so also Pontriagin) of vertical bundle $TM_T$. The algorithm requires only the data encoded in the Delzant polytope of $M$.

We start with few observations. The first is that the Atiyah-Bott formula become in this case fairly simple, because the action has only isolated fixed points. Thus we have

$$\pi_*(\alpha) = \sum_P \frac{i_P^*\alpha}{E(\nu_P)} ,$$

where $i_P^*\alpha$ and $E(\nu_P)$ are thought as ordinary polynomials.

Next we compute the explicit form of the homomorphism $i_P^* : H^*(M_T) \to H^*(B_T)$ induced by the inclusion of the fixed point $i_P : P \to M$. Notice
that $i^*_P$ is completely defined by its restriction to $H^2(M_T)$, by multiplicativity. We describe it in a matrix form with respect to the following bases: \{F_1, F_2, ..., F_k\} of $H^2(M_T)$ and \{T_1, T_2, ..., T_n\} of $H^2(B_T)$. Recall that $F_i$ denotes the $i$-th face of the Delzant polytope of $M$.

Observe that $i^*_P(F) = 0$ for any face $F$ which does not contain the vertex corresponding to the fixed point $P$. This implies that the column corresponding to $F$ consists of zeros. Moreover, since $i_P$ is a section of the universal bundle then we have that $i^*_P \circ \pi^* = Id$. This two observations completely determine $i^*_P$. Let’s denote the entries of the matrix representing $i^*_P$ by $a_{ij}^P$, where $I_P = \{i_1, ..., i_n\}$ is such that $P = F_{i_1} \cap F_{i_2} \cap ... \cap F_{i_n}$.

**Proposition 3.5.1** With the above notation, the entries of the matrix representing $i^*_P : H^2(M_T) \to H^2(B_T)$ are given by the two following conditions

1. $a_{ij}^P = 0$, if $j \notin I_P$
2. $\sum_j a_{ij}^P \cdot u_{jk} = \delta_i^k$,

where $u_{jk}$ are the entries of the matrix representing $\pi^*$.

QED

Finally, recall that $E(\nu_P) = i^*_P \circ i^*_P(1) = i^*_P(F_{i_1}F_{i_2}...F_{i_n})$ [AB], where $i^*_P : H^*(B_T) \to H^{*+2n}(M_T)$ denotes the push-forward of $i_P$. Thus this observation together with Proposition 3.5.1 make the Atiyah-Bott formula explicitly computable.

4 **The computations for symplectic toric varieties**

4.1 **Rational ruled surfaces**

The rational cohomology ring of the classifying space of symplectomorphism group of a rational ruled surface is known due to [AM]. The aim of this subsection is the “reality test”, that is to check how big part of the cohomology ring of $B_{Ham(M,\omega)}$ is generated by fiber integrals of characteristic classes.

Let $M^k_\lambda$ for $0 \leq k \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$ be a symplectic toric variety, whose Delzant polytope is a quadrilateral with the following vertices:

$$
\begin{align*}
(0, 0) & \quad \text{or} \quad (0, 0) \\
(0, 1) & \quad \text{for $k$ odd,} \quad \lambda > -1 \quad \text{and} \quad (0, 1) \\
(1 + \lambda - \frac{k}{2}, 1) & \quad \text{or} \quad (1 + \lambda - \frac{k}{2}, 1) \\
(0, 2 + \lambda + \frac{k}{2}, 0) & \quad \text{for $k$ even,} \quad \lambda \geq 0 \quad \text{and} \quad (0, 1 + \lambda + \frac{k}{2}, 0)
\end{align*}
$$
These manifolds are symplectic ruled surfaces which means that they are $S^2$-bundles over $S^2$ and the symplectic form restricts to symplectic form on each fiber. Symplectic ruled surfaces (up to rescaling of the symplectic form) are symplectomorphic to either $M^0_\lambda$ for some $\lambda \geq 0$ or $M^1_\lambda$ for some $\lambda > -1 \ [LM1]$. More precisely, we rescale the symplectic forms such that the area of the fiber is equal to 1. Then the Delzant polytopes are rectangular trapezoids of height 1. Manifolds $M^k_\lambda$ and $M^K_\Lambda$ are symplectomorphic if $k \equiv K$ modulo 2 and $\lambda = \Lambda$.

It follows from the above description that we have defined several Hamiltonian torus action on a given symplectic ruled surface provided that $\lambda$ is large enough. For example, manifold $M^2_{1.5}$ (Figure 2) is symplectomorphic to $M^0_{1.5}$ and $M^4_{1.5}$. These are three different actions on one symplectic manifold.

**Proposition 4.1.1** Let $M^k_\lambda$ for $1 + \lambda > \left\lfloor \frac{k}{2} \right\rfloor$ be a symplectic ruled surface and $H^*(B_T) = \mathbb{R}[x, y]$. Then the fiber integrals of the characteristic classes are given by the following formulae.

1. The monomials in the Chern classes and the coupling class:

$$
\pi_*(c_1^i c_2^j \Omega^i) = (x + y)^i (xy)^{j-1} \Omega_1^i + (x - y)^i (-xy)^{j-1} \Omega_2^i + ((k - 1)x - y)^i (-x(kx - y))^{j-1} \Omega_3^i + ((-k - 1)x + y)^i (x(kx - y))^{j-1} \Omega_4^i,
$$

where

$$
\Omega_1 = \frac{1}{3k + 6\lambda} \left( (k^2 + 3k\lambda + 3\lambda^2)x + (k + 3\lambda)y \right)
$$

$$
\Omega_2 = \frac{1}{3k + 6\lambda} \left( (k^2 + 3k\lambda + 3\lambda^2)x - (2k + 3\lambda)y \right)
$$

$$
\Omega_3 = \frac{1}{3k + 6\lambda} \left( (k^2 - 3\lambda^2)x - (2k + 3\lambda)y \right)
$$

$$
\Omega_4 = \frac{1}{3k + 6\lambda} \left( -(2k^2 + 6k\lambda + 3\lambda^2)x + (k + 3\lambda)y \right).
$$
2. The fiber integrals of Pontriagin classes are zero.

3. The fiber integrals of monomials in the Pontriagin and Euler classes:

\[ \pi_* (p^1_1 e^{ij}) = (x^2 + y^2)^i \left( (xy)^{j-1} + (-xy)^{j-1} \right) + \right)
\[(1 + k^2) x^2 - 2kxy + y^2)^i \left( (x(kx - y))^{j-1} + (-x(kx - y))^{j-1} \right). \]

**Proof:** This is a direct computation as described in Section 3.3. QED

**Corollary 4.1.2** Let \((M, \omega)\) be a symplectic ruled surface. Then the following statements hold:

1. If \((M, \omega)\) is different from \(M^0\) and \(M^1_\lambda\) for \(\lambda \in (-1, 0]\), then the rational cohomology ring of the classifying space of the group of symplectomorphisms is generated by fiber integrals of the monomial in the Chern classes and the coupling class.

2. \(H^4(\mathcal{B}Diff(M))\) is nontrivial for \(i = 1, 2, 3\ldots\)

**Proof:** (1) It is known due to Abreu and McDuff [AM], that

\[ H^*(\mathcal{B}Ham(M, \omega)) = \mathbb{Q}[A, X, Y]/\sim, \]
where \(deg(A) = 2, deg(X) = deg(Y) = 4\). The relation is given by a polynomial depending on \(\lambda\) and the diffeomorphism class of \(M\). Using this result it suffices to check that the fiber integration is a surjection on the second and fourth cohomology. This can be done by a direct computation. Indeed, the above formula for the fiber integrals yields that \(\pi_* (c^3_1) \neq 0\) and \(\dim \text{span}\{F(\pi_* (c^3_1)^2), \pi_* (c^3_1 \Omega^2)), \pi_* (c^3_1 \Omega^2))\} = 3\), where \(F\) denotes the detection function (Definition 3.1.1).

(2) The nontrivial elements are given by the fiber integrals of monomials in the Euler and Pontriagin classes. QED

**Remark 4.1.3**

1. The reason that the first part of the theorem does not hold in full generality is that the excluded manifolds don’t admit more than one different actions of torus.

2. Notice that if \(k \neq k'\) then the torus actions associated to \(k\) and \(k'\) are not conjugate in the group of all diffeomorphisms of \(M\). If the actions were conjugate then they would induce equal maps between \(H^*(\mathcal{B}Diff(M))\) and \(H^*(\mathcal{B}T)\). This follows from the fact that any conjugation induces an identity map on the cohomology of the classifying spaces (see [Se]). Hence the fiber integrals of the powers of the Euler class would be equal for different actions. This is a contradiction with the formula in Proposition 4.1.1 (3).

In fact, this can be easily seen directly by looking at the representations on the tangent spaces of the fixed points as was pointed to us by P.Seidel. This remark also applies to subsequent examples.
Our next example consists of the actions on the symplectic manifold diffeomorphic to $\mathbb{CP}^2 \# 2\mathbb{CP}^2$. The cohomology ring of the classifying space of symplectomorphism group is not known in this case. Our computations provide estimates from below for its Betti numbers.

Let $\nu \in \mathbb{R}$ and $k \in \mathbb{Z}$ be such that $\nu > 0$ and $0 \leq k < \left\lceil \frac{\nu}{2} \right\rceil$. Let $M_{\nu,k}$ be a family of symplectic toric varieties whose Delzant polytopes are presented on the Figure 2.

Clearly, $M_{\nu,k}$ are symplectomorphic for fixed $\nu$ and $0 \leq k < \left\lceil \frac{\nu}{2} \right\rceil$. Indeed, $M_{\nu,k}$ are the symplectic blow-ups of ruled surfaces $M_{\nu}^{2k+1}$ of the previous example. Regarding as symplectic manifold they will be denoted by $M_{\nu}$.

According to Proposition 3.4.2 we get that the Chern classes of the vertical bundle correspond to edges and vertices and are of the form

$$c_1 = F_1 + \ldots + F_5 \quad \text{and} \quad c_2 = F_1 F_2 + F_2 F_3 + \ldots + F_5 F_1.$$ 

The direct computation gives the following form of the coupling class

$$\Omega = \frac{1}{2(7 + 4\nu)}((15 - 8k + 12\nu)F_1 + (25 + 8k + 8k^2 + 24\nu + 6\nu^2)F_2 + (31 + 16k + 8k^2 + 24\nu + 6\nu^2)F_3 + (27 + 8k + 12\nu)F_4 + (23 + 12k + 8k^2 + 21\nu + 6\nu^2)F_5).$$
Proposition 4.2.1 Let $M_{x,k}$ be as above and $H^*(B_T) = \mathbb{R}[x, y]$. Then the fiber integrals of the characteristic classes are given by the following formulae.

1. Monomials in the Chern classes and the coupling class:

$$
\pi_*(c_1^i c_2^j \Omega^l) = (x + y)^i (xy)^j \Omega^l + \pi^*(-y(x + y))^{j-1} \Omega^l + \pi^*(-x(x + y))^{j-1} \Omega^l + (2kx - y)^i (-x((2k + 1)x - y))^{j-1} \Omega^l + (-2k + 2x - y)^i (-x(-(2k + 1)x + y))^{j-1} \Omega^l,
$$

where

$$
\Omega_1 = \frac{1}{3(7 + 4\nu)} ((25 + 8k + 8k^2 + 24\nu + 6\nu^2)x + (15 - 8k + 12\nu)y)
\Omega_2 = \frac{1}{3(7 + 4\nu)} ((25 + 8k + 8k^2 + 24\nu + 6\nu^2)x + (-6 - 8k)y)
\Omega_3 = \frac{1}{3(7 + 4\nu)} ((4 + 8k + 8k^2 + 12\nu + 6\nu^2)x + (-27 - 8k - 12\nu)y)
\Omega_4 = \frac{1}{3(7 + 4\nu)} ((4 + 50k + 8k^2 - 9\nu + 24k\nu - 6\nu^2)x + (-27 - 8k - 12\nu)y)
\Omega_5 = \frac{1}{3(7 + 4\nu)} ((-38 - 34k + 8k^2 - 33\nu - 24k\nu - 6\nu^2)x + (15 - 8k + 12\nu)y).
$$

2. Monomials in the Pontriagin classes:

$$
\pi_*(p_1^i p_2^j) = \frac{1}{xy} [(x^2 y^2)^j (x^2 + y^2)^i] - \frac{1}{xy(x + y)} [y (x^2 (x + y)^2)^j (2x^2 + 2xy + y^2)^i + x (y^2 (x + y)^2)^j (x^2 + 2xy + 2y^2)^i],
$$

3. Powers of the Euler class:

$$
\pi_*(eu^i) = (-x((2k + 1)x - y)^i - (x((2k + 1)x - y))^{i-1} + (xy)^i - (x(x + y))^i + (-x(x + y))^i + (-y(x + y))^i - (-y(y + 2k))^i.
$$

4. Monomials in the Euler and Pontriagin classes:

$$
\pi_*(p_1^i eu^j) = (xy)^{j-1} (x^2 + y^2)^i + (-x(x + y))^{j-1} (2x^2 + 2xy + y^2)^i + (-y(x + y))^{j-1} (x^2 + 2y(x + y))^i + (x(x + 2kx - y))^{j-1}(-2x(x + 2kx - y) + (-2(1 + k)x + y)^2)^i + (-x(x + 2kx - y))^{j-1}((2 + 4k + 4k^2)x^2 + y^2 - 2x(x + 2ky))^i.
$$
Corollary 4.2.2 Let \((M, \omega) = M_\nu\) be as above. The following statements are true:

1. If \(\nu > 2\), then \(\dim H^2(B_{\text{Ham}}(M, \omega)) \geq 4\),
2. If \(0 < \nu \leq 2\), then \(\dim H^2(B_{\text{Ham}}(M, \omega)) \geq 2\),
3. \(\dim H^4(B_{\text{Diff}}(M)) \geq 2\). Moreover \(H^{4k}(B_{\text{Diff}}(M))\) is nontrivial for \(k = 1, 2, \ldots\).

Proof: The first two statements follow from the direct application of the detection function. The assumption on \(\nu\) in the first one ensures that there are at least two different actions, which contributes to the detection function. Notice that the first estimate is the best we can get. Indeed, we integrate the following classes: \(\Omega^3, \Omega^2 c_1, \Omega c_1^2, c_2^3, c_2 \Omega, c_2 c_1\). \(\pi_* (\Omega^3) = 0\) from the definition and \(\pi_*(c_2 c_1) = 0\) according to the \(T\)-strict multiplicativity of the Todd genus.

To get the last statement we integrate \(p_1^2\) and \(p_1 e u\). This fiber integrals are linearly independent in \(H^4(B_{\text{Diff}}(M))\). Also \(\pi_*(p_1 e u) \neq 0\) which implies that \(H^{4k}(B_{\text{Diff}}(M))\) is nontrivial for \(k = 1, 2, \ldots\).

QED

4.3 Dimension 6: projectivizations of complex bundles

Let \(L_k \to \mathbb{CP}^1\) be a line bundle with \(c_1(L_k) = k\). Let \(1 \leq \mu \in \mathbb{R}\) be greater than \(k\) and \(l\). Consider a symplectic toric variety \(M_{\mu, k, l} := \mathbb{P}(L_k \oplus L_l \oplus L_0)\), which is the projectivization of complex bundle and whose Delzant polytope looks as in the following figure.

![Figure 3](image-url)
The Delzant polytope for $M_{\mu,k,l}$

**Lemma 4.3.1** Two symplectic toric varieties $M_{\mu,k,l}$ and $M_{\mu',k',l'}$ are symplectomorphic provided that

1. $k + l \equiv k' + l' \pmod{3}$ and
2. $3\mu - (k + l) = 3\mu' - (k' + l')$.

**Proof:** First observe that any of the above manifolds is the total space of a symplectic bundle $\mathbb{CP}^2 \to M \to S^2$, whose structure group is $PU(3)$. Since $\pi_1(PU(3)) \cong \mathbb{Z}_3$, then it is clear that $M_{\mu,k,l}$ and $M_{\mu',k',l'}$ are isomorphic (as symplectic bundles) if $k + l \equiv k' + l' \pmod{3}$. Moreover, the symplectic structures on these manifolds are of the form $\Omega + K\pi^*\omega_{S^2}$, where $\Omega$ denotes the coupling form and $\pi : M \to S^2$ is the projection. One can see easily that $K = \mu - \frac{1}{3}(k + l)$ for $M_{\mu,k,l}$.

Let $M_0$ and $M_1$ be two such isomorphic bundles and $H : S^1 \times [0,1] \to PU(3)$ be a chosen homotopy between loops defining $M_0$ and $M_1$. By the usual Thurston argument, there exist a number $K \in \mathbb{R}$ such that $\omega_t := \Omega_t + K\pi_t^*\omega_{S^2}$ is a symplectic form on each $M_t$. Notice that $\omega_t$ is an isotopy of symplectic form which does not change the cohomology class, hence it follows from Moser’s argument that $(M_0,\omega_0)$ and $(M_1,\omega_1)$ are symplectomorphic.

**QED**

**Remark 4.3.2** A detailed discussion of the coupling parameter $K$ as well as exact computations for certain bundles was done by Polterovich in [Po1]. Thus the symplectomorphism type depends only on $\mu$ and on $\lambda := (k+l)3\mathbb{Z} \in \mathbb{Z}/3\mathbb{Z}$. We denote it by $M_{\mu,\lambda}$. The explicit formulae of fiber integrals are quite complicated, so we do not present them. As in the previous examples, they allow to estimate the dimension of cohomology groups.

**Proposition 4.3.3** Let $(M,\omega) = M_{\mu,\lambda}$ be as above. Then

1. $\dim H^2(B_{Ham(M,\omega)}) \geq 1$,
2. $\dim H^4(B_{Ham(M,\omega)}) \geq 8$, provided that $(M,\omega)$ admits at least two different Hamiltonian actions of $T^2$.
3. $H^{4k}(B_{Diff(M)})$ is nontrivial for $k = 1, 2, \ldots$.

**QED**
5 Restrictions coming from multiplicativity of genera

When computing fiber integrals one observes many more linear dependencies between them than we a priori expected. This is due to multiplicity properties of certain genera. Recall that a genus is a ring homomorphism

\[ K : \Omega \otimes \mathbb{Q} \to R, \]

where \( \Omega \) is a cobordism ring and \( R \) is an integral domain over \( \mathbb{Q} \). Since we are working in the symplectic category then in the sequel \( \Omega \) will denote the complex cobordism ring. Every genus is defined for a stable complex manifold \( M \) by a multiplicative sequence \( K := \{ K_r \} \), \( K_r \in R[x_1, ..., x_r] \), as follows. Given a complex vector bundle \( E \) of rank \( n \)

\[ K(E) := K(c_1(E), ..., c_n(E)) := 1 + K_1(c_1(E)) + K_2(c_1(E), c_2(E)) + ... \]

and

\[ K(M) := \langle K(TM); [M] \rangle \in R. \]

5.1 G-strict multiplicativity

Let \( G \) be a group. Multiplicative sequence \( K \) is said to be \( G \)-strictly multiplicative or \( G \)-sm for short if

\[ \pi^*(K(TM_G)) \in H^0(B_G) \]

for every manifold \( M \) on which \( G \) acts. Here \( \pi : M_G \to B_G \) is the universal fibration associated to the action and \( TM_G \to M_G \) is the complex vector bundle tangent to fibers. In other words, \( G \)-strict multiplicativity says that any fiber integral \( \pi^*(K_r(c_1(TM_G), ..., c_r(TM_G))) = 0 \) for \( r \neq \dim M \). Hence for \( r > \dim M \) we obtain relations in \( H^*(B_G) \) between fiber integrals of characteristic classes.

The name multiplicativity is justified by the following fact.

**Proposition 5.1.1** Let \( M \to P \to B \) be a \( G \)-bundle over a compact stably complex base. Then \( P \) admits a stable complex structure. If \( K \) is a \( G \)-strict multiplicative sequence then

\[ K(P) = K(M)K(B). \]

**Proof:** Since \( G \)-strict multiplicativity is a property of the universal fibration then it holds for every \( G \)-fibration. Hence we have that

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where $Vert \to P$ is the bundle tangent to the fibers of $\pi: P \to B$. The statement follows from the following computation.

\[
\begin{align*}
K(P) &= \langle K(TP); [P] \rangle = \langle K(Vert \oplus \pi^*TB); [P] \rangle \\
&= \langle K(Vert)K(\pi^*TB); [P] \rangle \\
&= \langle \pi_*[K(Vert) \cup \pi^*K(TB)]; [B] \rangle \\
&= \langle \pi_*K(Vert) \cup K(TB); [B] \rangle \\
&= \langle K(TM); [M] \rangle K(TB); [B] \rangle = K(M)K(B)
\end{align*}
\]

QED

Notice that if a multiplicative sequence $K$ is $S^1$-sm then it is also $G$-sm for any compact Lie group $G$ [Och]. Indeed, $H^*(BG)$ is a subalgebra in $H^*(BT)$ where $T \subset G$ is a maximal torus. For any polynomial $p \in \mathbb{R}[t_1, ..., t_n] = H^*(BT) = \mathbb{R}[t]$ of degree $k$ (i.e. $p \in H^{2k}(BT)$) there exists a homomorphism $f: S^1 \to T$ such that $H^{2k}(BT)(p) \neq 0$. To see this, notice that any such map is determined by $n$ integers, say $[k_1, ..., k_n]$ and $H^{2k}(BT)(p) = p(k_1, ..., k_n)t^k$. Thus if $K$ were not $G$-sm for some manifold $M$ then it would be not $S^1$-sm.

5.2 $G$-strict multiplicativity of $\chi_y$-genus

Recall that $\chi_y: \Omega \otimes \mathbb{Q} \to \mathbb{Q}[y]$ is a genus whose value on a Kähler manifold is given by

\[
\chi_y(M) := \sum_{p,q}(-1)^q h^{p,q}y^p,
\]

where $h^{p,q}$ are the Hodge numbers [HBJ].

**Theorem 5.2.1** $\chi_y$-genus is $G$-strictly multiplicative for every compact Lie group $G$.

**Proof:** According to the observation in the previous subsection, it is sufficient to prove strict multiplicativity for $S^1$. This is equivalent to the fact that $\chi_y$-genus is multiplicative for any fibration $M \to P \to \mathbb{CP}^k$, that is

\[
\chi_y(P) = \chi_y(M)\chi_y(\mathbb{CP}^k).
\]

We compute $\chi_y(P)$ with the use of the localization formula for circle action. The formula due to Kosniowski and Lusztig allows to compute the genus in terms of the genera of the fixed points:
\[
\chi_y(P) = \sum_{i=1}^{m} (-y)^{s_i} \chi_y(F_i).
\]

Here \( F_i \) is the component of the fixed point set and \( s_i := \dim V_{>0} \) the dimension of the subspace of the tangent space to any point of \( F_i \) on which the circle acts positively \((z \cdot v = z^k v, \text{ where } k > 0)\).

We start with defining a circle action on \( P \). Let \( \alpha : S^1 \to Aut(M) \) be a given action on \( M \) preserving the stable complex structure. Moreover let \( \beta : S^1 \to Aut(\mathbb{C}P^k) \) be an action given by \( \beta(t)[z_0 : \ldots : z_k] = [t^{i_0} z_0 : \ldots : t^{i_k} z_k] \) with isolated fixed points. We use the same notation for the action lifted to \( S^{2k+1} \). The bundle \( P \) is associated to the principal bundle \( S^{2k+1} \to \mathbb{C}P^k \), i.e.

\[
P := S^{2k+1} \times_{S^1} M,
\]

where \((z, x) \simeq (z_1, x_1)\) iff \( z_1 = sz \) and \( x_1 = \alpha(s^{-1})(x) \), for \( s \in S^1 \).

We define an action \( \phi : S^1 \to Aut(P) \) by

\[
\phi(t)[z, x] = [\beta(t)(z), \alpha(t)(x)].
\]

Clearly, this action is well defined since \( S^1 \) is commutative. Further, fixed points lie in the fibers over the fixed points of \( \beta \). When restricted to the fixed fiber, the fixed points of \( \phi \) are those of \( \alpha \). More precisely, let \( F_j \) denote a path component of the fixed point set of \( \alpha \). Denote by \( F_{ij} \) the \( j \)-th component of the fixed point det of \( \phi \) lying in the fiber over the \( i \)-th fixed point of \( \beta \). Clearly, \( F_{ij} \cong F_j \).

We need to figure out the dimension \( s_{ij} \) of the subspace of the tangent space to any fixed point \( x \in F_{ij} \) on which the circle acts positively. Notice that the infinitesimal action in the direction normal to the fiber is the same as infinitesimal action induced by \( \beta \) on the tangent space to the fixed point in \( \mathbb{C}P^k \). Thus \( s_{ij} = s_i + s_j \), where \( s_i \) (\( s_j \) respectively) denote the appropriate dimension with respect to the action \( \beta \) (\( \alpha \) respectively).

Now, plugging the above observations into the localization formula we get the statement as follows.
\[ \chi_y(P) = \sum_{i=1}^{k+1} \sum_{j} (-y)^{s_{ij}} \chi_y(F_{ij}) \]
\[ = \sum_{i=1}^{k+1} \sum_{j} (-y)^{s_{i}+s_{j}} \chi_y(F_{ij}) \]
\[ = \sum_{j} \left( \sum_{i=1}^{k+1} (-y)^{s_{i}} (-y)^{s_{j}} \right) \chi_y(F_j) \]
\[ = \left( \sum_{j} (-y)^{s_{j}} \chi_y(F_j) \right) \left( \sum_{i=1}^{k+1} (-y)^{s_{i}} \right) \]
\[ = \chi_y(M) \chi_y(\mathbb{C}P^k). \]

QED

**Corollary 5.2.2** The Todd genus and the signature is \(G\)-sm for any compact Lie group.

**Proof:** The Todd genus is equal to \(\chi_y\)-genus for \(y = 0\) and the signature for \(y = 1\).

QED

It would be very interesting to know to what extent strict multiplicativity is true. For example, the signature is \(Ham(M, \omega)\)-sm, since the group \(Ham(M, \omega)\) of Hamiltonian symplectomorphisms is connected. These considerations motivate for the following

**Question:** Is the Todd (or \(\chi_y\)) genus \(Ham(M, \omega)\)-strictly multiplicative for any compact symplectic manifold?

**Example 5.2.3** (The Todd genus is not \(Symp\)–strictly multiplicative.)
Let \(\Sigma_h \to M \to \Sigma_g\) be a symplectic surface bundle over surface with nonzero signature, \(\sigma(M) \neq 0\) [3].

\[ 3\sigma(M) = \langle p_1(TM), [M] \rangle = \langle c_1(TM)^2 - 2c_2(TM), [M] \rangle \]
\[ = \langle c_1(Vert \oplus \pi^*T\Sigma_g)^2, [M] \rangle - 2 \langle c_2(Vert \oplus \pi^*T\Sigma_g), [M] \rangle \]
\[ = \langle c_1(Vert)^2 + 2c_1(Vert)\pi^*c_1(T\Sigma_g), [M] \rangle - 2 \langle c_1(Vert)\pi^*c_1(T\Sigma_g), [M] \rangle \]
\[ = \langle c_1(Vert)^2, [M] \rangle \]
\[ = \langle \pi_*(c_1(Vert)^2), [\Sigma_g] \rangle \neq 0 \]

Since the Todd polynomial \(T_2(c_1, c_2) = \frac{1}{12}(c_1^2 + c_2)\), then the Todd genus is not strictly multiplicative in this case. In this example signature measures the defect of strict multiplicativity.
References

[A] M.F.Atiyah, *The signature of fibre-bundles*, Global Analysis, papers in honor of K. Kodaira. University of Tokyo Press and Princeton University Press (1969), 73-84.

[AB] M.F.Atiyah, R.Bott *The moment map and the equivariant cohomology*, Topology 23 No. 1, 1-28.

[AM] M.Abreu, D.McDuff, *The topology of the groups of symplectomorphisms of ruled surfaces*, J.Amer.Math.Soc. 13 (2000), no.4,971-1009.

[BV] N.Berline, M.Vergne, *Classes caracteristiques quivariantes. Formule de localisation en cohomologie quivariante*, C. R. Acad. Sci. Paris Sr. I Math. 295 (1982), no. 9, 539–541.

[BT] R.Bott, L.W.Tu *Differential forms in algebraic topology*, Springer-Verlag, 1982.

[DJ] M.Davis, T.Januszkiewicz *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. Journal 62, No.2(1991),417-451.

[De] T.Delzant, *Hamiltoniens priodiques et images convexes de l'application moment*, Bull. Soc. Math. France 116 (1988), no. 3, 315–339.

[G] V. Guillemin *Moment Maps and Combinatorial invariants of Hamiltonian T^n-spaces*, Birkhauser 1994.

[GS] V. Guillemin, S. Sternberg, *Supersymmetry and Equivariant deRham Theory*, Springer, 1999.

[HBJ] F. Hirzebruch, T. Berger, R. Jung, *Manifolds and Modular Forms*, Aspect of Mathematics E 20, Vieweg-Verlag, 1992.

[LM1] F.Lalonde, D.McDuff *J-holomorphic spheres and the classification of rational and ruled symplectic 4-manifolds* in Contact and Symplectic Geometry, ed.C.Thomas, CUP (1996).

[LM2] F.Lalonde, D.McDuff *Symplectic structures on fibre bundles*, [math.SG/0010273](http://arxiv.org/abs/math.SG/0010273).

[MS1] D.McDuff and D.Salamon, *Introduction to Symplectic Topology*, 2nd edition, OUP, Oxford (1998).

[M] D.McDuff, *Lectures on Groups of Symplectomorphisms*, [math.SG/0201032](http://arxiv.org/abs/math.SG/0201032).
[Mo] S.Morita, *Structure of the mapping class groups of surfaces: a survey and a prospect*, Geometry and Topology Monographs Volume 2: Proceedings of the Kirbyfest, 349-406.

[Och] S. Ochanine, *Genres Eliptiques Equivariants*, LNM 1326, 1986, 107-122.

[Po] L.Polterovich *The Geometry of the Group of Symplectic Diffeomorphisms*, Birkhauser 2000.

[Po1] L.Polterovich *Gromov’s K-area and symplectic rigidity*, GAFA 6 (1996), 726-739.

[Re] A.Reznikov *Characteristic classes in symplectic topology*, Selecta Mathematica, New Series 3 (1997) 601-642.

[Se] G.Segal *Classifying spaces and spectral sequences*, Inst. Hautes tudes Sci. Publ. Math. No. 34, 1968 105–112.

[S] R.P.Stanley *Combinatorics and Commutative Algebra*, second edition, Birkhauser 1996.

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