Null octagon from Deift-Zhou steepest descent

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Abstract

A special class of four-point correlation functions in the maximally supersymmetric Yang-Mills theory is given by the square of the Fredholm determinant of a generalized Bessel kernel. In this note, we re-express its logarithmic derivatives in terms of a two-dimensional Riemann-Hilbert problem. We solve the latter in the null limit making use of the Deift-Zhou steepest descent. We reproduce the exact octagonal anomalous dimension in ’t Hooft coupling and provide its novel formulation as a convolution of the non-linear quasiclassical phase with the Fermi distribution in the limit of the infinite chemical potential.
1. **Introduction.** The hexagonalization framework \cite{1, 2, 3} for correlation functions in the maximally supersymmetric Yang-Mills theory is based upon integrable structures \cite{4} of the dual effective two-dimensional theory and provides a truly non-perturbative formalism for their calculation. In generic situations, its use is hampered by one’s inability to perform exact infinite resummations of all excitations propagating on the world-sheet. However, under a judicial choice of the R-symmetry quantum numbers of the operators involved, one can significantly reduce their complexity and furthermore by taking these values to infinity one can suppress the majority of contribution stemming from excitations wrapping compact space-like cycles. This in turn implies that corresponding transitions are saturated by the vacuum state and therefore the sums disappear. It was realized some time ago \cite{5, 6}, that for the particular case of the so-called simplest four-point correlator only two sums remain and moreover their summands factorize such that the original observable falls apart into the product of two simpler objects named octagons. With each of them containing just a single sum, the latter was performed in Refs. \cite{7, 8} where a concise representation was derived in terms of the determinant of a semi-infinite matrix. This result laid out the foundation for the analyses in Ref. \cite{9, 10, 11}, where the octagon was further recast as a Fredholm determinant of an integral operator acting on a semi-infinite line. Its kernel was identified as a convolution of the well-known Bessel kernel with a Fermi-like distribution that depends on the external kinematics and the ’t Hooft coupling $g$.

In Ref. \cite{9}, the null limit of the octagon was addressed. It corresponds to the kinematics where any two nearest-neighbor operators approach a light-like interval such that one of the conformal cross ratios, i.e., $\gamma$, tends to infinity and all other can be set to zero. Using of the method of differential equations, originally developed by Its-Izergin-Korepin-Slavnov for calculations of two-point correlation functions in integrable models, one manages to establish the asymptotic behavior of the octagon as $\gamma \to \infty$ and exactly fix the accompanying coefficients as functions of the coupling,

$$\log \mathfrak{O} = -y^2 \frac{\Gamma(g)}{2\pi^2} + \frac{C(g)}{8} + \ldots .$$  (1)

This Sudakov-like behavior should arise from some kind of a semiclassical limit of the effective two-dimensional path integral. Intuitively this can be observed, making use of the fermionic path integral representation of the octagon devised in Ref. \cite{8, 12} and subsequent Hubbard-Stratonovich transformation performed to get rid of a nonlocality. This picture is obscure, however, within the method employed earlier for the light-cone analysis. Therefore, in the present note an attempt is made to establish a link between the anomalous dimension $\Gamma(g)$ and semiclassics. Indeed, one finds that with a proper method, $\Gamma(g)$ arises as a WKB phase in the Deift-Zhou non-linear steepest descent method applied to Riemann-Hilbert problems (RHP) for Fredholm determinants of integrable operators. While, as a result of this consideration, one merely reproduces the function $\Gamma(g)$ at finite coupling with a different method, there is a hope that its complementary novel representation could shed light on establishing a connection to the same anomalous dimension which arises in a completely different context, i.e., the origin limit of the six-gluon scattering amplitude \cite{13}, which is again governed by a Fredholm determinant akin to the one encountered for the octagon.

2. **Octagon as a Fredholm determinant.** According to the results of Refs. \cite{9, 10, 11}, the octagon is given by the Fredholm determinant

$$\mathfrak{O} = \det (1 - \mathbb{K})$$  (2)
of the integral operator
\[\mathcal{K}f(x') = \int_{-\infty}^{0} dx \chi(x)K(x, x')f(x),\] (3)
with the Bessel kernel \[14, 15\]
\[K(x, x') = \frac{\sqrt{-x'} J_1(\sqrt{-x'})J_0(\sqrt{-x}) - \sqrt{-x} J_1(\sqrt{-x})J_0(\sqrt{-x'})}{2(x - x')},\]
and a cutoff function \(\chi(x)\), which depends, in a generic case, on several space-time and R-symmetry cross ratios and the ‘t Hooft coupling. Compared to the original definition of Refs. \[9, 10, 11\], the semi-infinite interval was reflected through the origin. This is done in order to have a seamless match to the standard conventions adopted for the Bessel model RHP (to be employed below) which is defined on the negative real axis. The focus in this paper will be on the light-like limit when one of the variables, namely \(y\), tends to infinity with the other set to zero. In this case, the cut-off function significantly simplifies and takes on the form of the Fermi distribution
\[\chi(x) = [1 + e^{(\sqrt{-x} - \mu)/T}]^{-1},\] (4)
where the temperature is set by the ‘t Hooft coupling \(T = 2g\) and the chemical potential is proportional to the cross ratio in question \(\mu = 2gy\).

It is well-known that the Bessel kernel belongs to the class of integrable operators in the nomenclature of Ref. \[16\] and therefore the generalized Bessel kernel can be cast into the form
\[\chi(x)K(x, x') = \frac{f(x) \cdot g(x')}{x - x'},\] (5)
with conveniently chosen two two-vectors \(f\) and \(g\) which read
\[f(x) = \frac{\chi(x)}{2\pi i} \begin{pmatrix} J_0(\sqrt{-x}) \\ -i\pi \sqrt{-x} J_1(\sqrt{-x}) \end{pmatrix}, \quad g(x) = \begin{pmatrix} i\pi \sqrt{-x} J_1(\sqrt{-x}) \\ J_0(\sqrt{-x}) \end{pmatrix}.\] (6)
These ensure a regular behavior at coincident points \(f(x) \cdot g(x) = 0\), such that \(\chi(x)K(x, x) = f'(x) \cdot g(x) = -f(x) \cdot g'(x)\). Notice that the entire dependence on the external parameters \(y\) and \(g\) is encoded in the vector \(f\).

The goal is to calculate the Fredholm determinant \(2\). The crucial observation is that its derivative with respect to external parameters (\(\mu\) and \(T\), in the current case) is expressed in terms of the resolvent
\[\mathcal{R} = \frac{\mathcal{K}}{1 - \mathcal{K}},\] (7)
such that \(\partial \log \det(1 - \mathcal{K}) = -\text{tr}(1 + \mathcal{R})\partial\mathcal{K}\). In fact, not only the operator \(\mathcal{K}\) is integrable but also its resolvent \(\mathcal{R}\) as well. A simple calculation demonstrates that its kernel can be written as \[16, 17, 18, 19\]
\[R(x, x') = \frac{F(x) \cdot G(x')}{x - x'},\] (8)
with

$$F(x) = \left[ \frac{1}{1 - K_1} f \right](x), \quad G(x) = \left[ \frac{1}{1 - K_2} g \right](x).$$

(9)

Thus the analysis boils down to the determination of $F$ and $G$.

3. RHP for the Fredholm determinant. It is well known that Fredholm determinants are tau functions associated to RHPs [20] and, in particular, coincide with the Jimbo-Miwa-Ueno conventions [21] for isomonodromic tau functions of monodromy problems in the theory of ordinary differential equations with rational coefficients. Thus, the two-component functions $F$ and $G$ can be determined from a RHP for the two-by-two matrix functions $Y$ and $\tilde{Y}$ [16, 19] (consult also a very comprehensive set of lecture notes [22])

$$F = Y_+ f, \quad G = \tilde{Y}_+ g ,$$

(10)

where here and below $Y_\pm$ are the limiting values of $Y$'s on the real axis, i.e., $Y_\pm(x) \equiv Y(x \pm i 0)$. Introduce the jump matrices such that they possess the components $\{V\}_{ij} = V_{ij}$ and $\{\tilde{V}\}_{ij} = \tilde{V}_{ij}$

$$V_{jk}(x) = \delta_{jk} - 2\pi i f_j(x) g_k(x), \quad \tilde{V}_{jk}(x) = \delta_{jk} + 2\pi i g_j(x) f_k(x),$$

(11)

for $x \in \mathbb{R}_- \equiv (-\infty, 0]$ and $\delta_{jk}$ elsewhere. Observing that $\tilde{V} = (V^{-1})^T$, the matrices $Y$ and $\tilde{Y}$ are related to each other by the very same relation and thus it suffices to determine only one RHP function since

$$\tilde{Y} = (Y^{-1})^T.$$

(12)

The RHP that one has to solve is as follows:

$$Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}_-, \quad Y_+(z) = Y_-(z) V(z), \quad Y(z) \xrightarrow{z \to \infty} 1.$$  

(13)

Having found $Y$, one can immediately calculate the Fredholm determinant in its terms. Namely, making use of the result of Ref. [23], the derivative $\partial$ of the determinant with respect to its parameters $\mu$ or $T$ is

$$\partial \log \det(1 - K) = \omega - \int_{\mathbb{R}_-} dx \partial[f' \cdot g](x) + \frac{1}{2} \int_{\mathbb{R}_-} dx \tr[V'V^{-1}(\partial V)V^{-1}](x),$$

(14)

where the Jimbo-Miwa-Ueno differential reads

$$\omega \equiv \int_{\mathbb{R}_-} \frac{dx}{2\pi i} \tr[Y^{-1}_+ Y'_+(\partial V)V^{-1}](x).$$

(15)

In these formulas the prime on all symbols designates differentiation with respect to the spectral parameter $x$, e.g., $Y'(x) = \partial_x Y(x)$.

\footnote{This implies that the extension of the cut-off function $\chi$ from $\mathbb{R}_-$ to the entire real axis is $\chi(x) = \{[1 + e^{(\sqrt{-1}x - \mu)/T}]^{-1}, x \leq 0; 0, x > 0\}$.}
4. Dressing procedure. The jump matrices for the RHP (13) are very complicated since they depend on the Bessel functions. Therefore, one has to simplify $V$ first if one hopes to solve the problem. This can be achieved by a standard dressing procedure which allows one to completely eliminate any reference to the ubiquitous Bessel functions. This construction was recently used in the generalized Airy problem in Ref. [24] which is akin to the one studied in this work. Presently, this technique is applied to the current circumstances making necessary adjustments along the way.

There is some freedom in the choice of the dressing function. The RHP for the Bessel model, which is recalled in the Appendix A, is not general enough for the purpose. One can introduce an extra parameter $z_0$ through the shift of the Bessel model RHP contour to the left and thus construct a modified Bessel model solution $\hat{\Psi}$. Employing the latter, one devises the dressed $Y$ via

$$
\Phi_{I,VI} = Y \hat{\Psi}_{I,VI},
$$

$$
\Phi_{II} = Y \hat{\Psi}_{II} \begin{pmatrix} 1 & 0 \\ \frac{1}{1-z} & 1 \end{pmatrix},
$$

$$
\Phi_{III} = Y \hat{\Psi}_{III} \begin{pmatrix} \frac{1}{1-z} & 0 \\ \frac{1}{1-z} & 1 \end{pmatrix},
$$

for the four regions partitioned by the modified Bessel model RHP contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$ as explained in the Appendix A. The additional right-most matrices in Eq. (16) depending on $\chi$ emerge from the factorization of the jump matrix on $C_2$ for the bare product $Y \hat{\Psi}$ in terms of lower-triangular and anti-diagonal matrices. The dressed functions obey the jump relations

$$
\Phi_+ = \Phi_- \begin{pmatrix} 0 & 1 - \chi \\ \frac{1}{1-z} & 0 \end{pmatrix}, \text{ for } z \in C_2
$$

$$
\Phi_+ = \Phi_- \begin{pmatrix} 1 & 0 \\ \frac{1}{1-z} & 1 \end{pmatrix}, \text{ for } z \in C_{1,3}
$$

$$
\Phi_+ = \Phi_- \begin{pmatrix} 1 & 1 - \chi \\ 0 & 1 \end{pmatrix}, \text{ for } z \in C_4
$$

These can be deduced from the jump matrices for $Y$ and $\hat{\Psi}$ together with the following instrumental relation between the two-dimensional vectors $f$ and $g$ and the solutions to the modified Bessel model RHP (69),

$$
2\pi i \chi^{-1}(z)f(z) = \begin{cases} 
    z > z_0 : & \hat{\Psi}_+(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
    z < z_0 : & \hat{\Psi}_+(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} 
\end{cases},
$$

$$
g^T(z) = \begin{cases} 
    z > z_0 : & (0, 1) \hat{\Psi}_+^{-1}(z) \\
    z < z_0 : & (-1, 1) \hat{\Psi}_+^{-1}(z) 
\end{cases}.
$$

The uniqueness of the solution to the RHP (17) follows a standard argument and will not be repeated here, see, e.g., [18].

Making use of the above definitions, the Fredholm determinant can be rewritten in terms of the 21-matrix element of the dressed RHP solution

$$
\partial \log \det(1 - K) = - \int_{R_+} dz \frac{d}{dz} (\hat{\Phi}_+^{-1} \hat{\Phi}_+^\prime)_{21},
$$

4
where

$$
\hat{\Phi}_+ = \begin{cases} 
  z > z_0 : \Phi_1, \\
  z < z_0 : \Phi_2 \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{1-\chi} & 1 \end{array} \right).
\end{cases}
$$

5. **Deift-Zhou steepest descent.** As one is interested in the leading asymptotics of the Fredholm determinant for $\mu \to \infty$, it is instructive to rescale the complex variable $z$ by $\mu^2$, i.e., $z \to \mu^2 z$, such that the cut-off depends on the ratio $\mu/T$, i.e., $\chi(z) \to \chi(z) = [1 + e(\sqrt{z-1})\mu/T]^{-1}$. However, as one is taking the limit in question, one immediately faces a predicament since the jump matrices for $\Phi$ (see Eqs. (17)) are not decaying to constant matrices. Here is where the nonlinear steepest descent comes to the rescue. The idea of the Deift-Zhou method [26] (see also sect. 4 of [27] for a concise summary) is to perform a set of successive transformations of the original RHP and simplify the problem at each step while determining contributions to the asymptotic solution. The goal is to reduce the problem to jump matrices which exponentially approach the ones with constant matrix elements. The ultimate RHP is a small-norm RHP for a function $R$ with asymptotically unit jumps and normalized behavior at infinity. The analysis is thus divided in the sequence

$$
\Phi \to T \to R,
$$

where in the first step one rotates the “phases” away by constructing a function $g$ which is the analogue of the WKB phase in the linear quasiclassical theory and then in the second step, one finds approximate solutions to the RHP for $T$, which are known as parametrices $P$. The latter asymptotically yield the small-norm RHP for $R = TP^{-1}$.

5. **First transformation.** In the first step of the Deift-Zhou chain, one define the matrix function $T$ by eliminating the exponential growth of the jump matrices as $\mu \to \infty$ by introducing a function $g$ as follows

$$
T = e^{-\mu(V_0/2)\sigma_3} \Phi e^{\mu(g + V_0/2)\sigma_3},
$$

where

$$
V(z) = \frac{1}{\mu} \log(1 - \chi(z)),
$$

and we shifted the origin to $z_0$, such that $V_0 \equiv V(z_0)$. The function $g$ is yet to be determined and is the main focus of a successful RHP analysis. For the transformed $T$, the jump matrices are

$$
T_+ = T_- \begin{pmatrix} 0 & e^{-\mu(g_+ + g_- - V + V_0)} \\ e^{\mu(g_+ + g_- - V + V_0)} & 0 \end{pmatrix}, \quad \text{for } z \in C_2
$$

$$
T_+ = T_- \begin{pmatrix} 1 & 0 \\ e^{\mu(2g_+ - V + V_0)} & 1 \end{pmatrix}, \quad \text{for } z \in C_{1,3},
$$

$$
T_+ = T_- \begin{pmatrix} 1 & e^{-\mu(2g_+ - V + V_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in C_4
$$

A natural way to choose the function $g$ is to impose the condition that $g$ solves the RHP

$$
g_+ + g_- - V + V_0 = 0,
$$

(26)
for $z < z_0$ such that one gets a constant jump matrix for the $C_2$ portion of the contour

$$T_+ = T_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } z \in C_2,$$

(27)

with the asymptotic behavior being

$$T(z) \xrightarrow{z \to \infty} z^{\sigma_3/4} A.$$

(28)

6. Deift-Zhou phase. The phase function is found from the RHP subject to the asymptotic condition

$$g(z) \xrightarrow{z \to \infty} -\sqrt{z} + c + O(1/\sqrt{z})$$

(29)

which is imposed to cancel the untamed behavior of the modified Bessel model RHP solution. It is easier to solve the RHP for $g'$ first and then integrate it back to $g$. Differentiating both sides of (26) with respect to the spectral parameter $z$, one deduces

$$g'_+(z) + g'_-(z) = -\frac{1}{2T} \chi(z),$$

(30)

with the asymptotic behavior at infinity

$$g'(z) \xrightarrow{z \to \infty} -\frac{1}{2\sqrt{z}} + O(z^{-3/2}).$$

(31)

The solution is easily found to be

$$g'(z) = -\frac{1}{2\sqrt{z - z_0}} \left[ 1 + \frac{1}{T} \int_{-\infty}^{z_0} \frac{dz'}{2\pi} \sqrt{\frac{1 - \chi(z')}{z'} \frac{z'}{z' - z}} \right],$$

(32)

while integrating this result, one concludes that

$$g(z) = -\sqrt{z - z_0} + \frac{1}{2T} \int_{-\infty}^{z_0} \frac{dz'}{2\pi i \sqrt{-z'}} \log \frac{\sqrt{z_0 - z'} + i\sqrt{z - z_0}}{\sqrt{z_0 - z'} - i\sqrt{z - z_0}}.$$ 

(33)

In fact, these integrals can be performed exactly for the limiting Fermi distribution as $\mu \to \infty$. Keeping only the leading term, one substitutes $\chi$ by the unit step

$$\chi_\infty(z) = \theta(-1 \leq z \leq 0).$$

(34)

and explicitly obtains

$$g'_\infty(z) = -\frac{1}{2\sqrt{z - z_0}} + \frac{i}{4\pi T} \frac{1}{\sqrt{-z}} \log \frac{\sqrt{z - z_0} - \sqrt{z(z_0 + 1)}}{\sqrt{z - z_0} + \sqrt{z(z_0 + 1)}} - \frac{i}{4\pi T} \frac{1}{\sqrt{z_0 - z}} \log \frac{1 - \sqrt{z_0 + 1}}{1 + \sqrt{z_0 + 1}},$$

(35)

$$g_\infty(z) = c_1/2 \sqrt{z - z_0} + \frac{\sqrt{-z}}{2\pi T} \log \frac{\sqrt{z - z_0} - \sqrt{z(1 + z_0)}}{\sqrt{z - z_0} + \sqrt{z(1 + z_0)}}.$$
7. Global parametrix. The global parametrix is found from the following RHP: $P_\infty$ is analytic in $\mathbb{C}\setminus(-\infty, z_0)$ and obeys the jump condition

$$P_{\infty,+} = P_{\infty,-} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad \text{for } z \in (-\infty, z_0),$$

with

$$P_\infty(z) \to z^{\sigma_3/4} A.$$  \hspace{1cm} (38)

One can immediately verify that the solution is

$$P_\infty(z) = (z - z_0)^{\sigma_3/4} A.$$  \hspace{1cm} (39)

8. Local parametrix. In the vicinity of $z_0$, the combination entering the exponent of the off-diagonal elements of the jump matrices for $T$ is

$$\left( g - \frac{1}{2} (V - V_0) \right)_\infty \overset{z \to z_0}{\sim} (z - z_0)^{1/2} \left[ c_{1/2} + c_{3/2}(z - z_0)^2 + \ldots \right],$$

with $c_{1/2}$ as determined above in Eq. (37) and

$$c_{3/2} = -\frac{1}{3\pi T z_0 \sqrt{1 + z_0}}.$$  \hspace{1cm} (40)
So it vanishes at least as a square root such that in the $\epsilon$-vicinity of $z_0$, the jump matrices $\hat{M}$ become

$$\hat{M}_{C_1, C_3} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \hat{M}_{C_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{M}_{C_4} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (43)$$

They are the same as in the Airy model reviewed in the Appendix B. Therefore, when one seeks a conformal map from the small disk $U$ of $z_0$

$$z \to m(z), \quad \frac{2}{3} m^{3/2}(z) = g - \frac{1}{7}(V - V_0) \quad (44)$$

to the entire complex plane $\mathbb{C}$ (see Fig. 1), one can reconcile the asymptotic behavior of the nonlinear steepest descent phase with the form of the jump matrices (43) provided one imposes the condition

$$c_{1/2} = 0. \quad (45)$$

Then

$$(g - \frac{1}{7}(V - V_0)) \to z \to z_0 c_{3/2}(z - z_0)^{3/2} + \ldots, \quad (46)$$
as expected for the Airy model RHP (72).

The RHP for the local parametrix $P_{\text{loc}}$ thus reads

$$P_{\text{loc},+} = P_{\text{loc},-} \hat{M}, \quad (47)$$

and $P_{\text{loc}} \to P_{\infty}$ as $\mu \to \infty$ on the boundary $\partial U$. This local parametrix is expressed as

$$T_{\text{loc}}(z) = \left( \frac{z - z_0}{\mu^{2/3} m(z)} \right)^{\sigma_{3/4}} \phi \left( \mu^{2/3} m(z) \right) e^{\mu(g - (V - V_0)/2)\sigma_3}. \quad (48)$$

by means of the solution $\phi(z)$ to the Airy model RHP (73) – (76).

9. Small-norm RHP. Having found the global and local parametrices it is easy to show that

$$R = \begin{cases} T P_{\text{loc}} & \text{for } z \in U, \\ T P_{\infty} & \text{for } z \in \mathbb{C} \setminus U, \end{cases} \quad (49)$$

obeys a small-norm RHP with the asymptotic solution $R \to 1$ as $\mu \to \infty$. Having reached this conclusion, one can invert the set of transformations (22) and determine the asymptotic behavior of $\Phi$ that will feed into the Fredholm determinant (20).

10. Asymptotic behavior of the determinant. With all parametrices in hand, one can finally extract the asymptotic form for the Fredholm determinant of the generalized Bessel kernel (5). In terms of the RHP matrix $T$, the derivative with respect to the chemical potential $\mu$ reads

$$\partial_\mu \log \det(1 - \mathbb{K}) = -\frac{1}{T} \int_{R_-} \frac{dz}{2\pi i} \chi(z) \left( \hat{T}^{-1} \hat{T}' \right)_{21} e^{-\mu(2g_+ - V + V_0)}, \quad (50)$$

where

$$\hat{T}(z) = \begin{cases} z > z_0 : \ T_+(z) \\ z < z_0 : \ T_+(z) \begin{pmatrix} 1 & 0 \\ e^{\mu(2g_+ - V + V_0)} & 1 \end{pmatrix} \end{cases}. \quad (51)$$
The contributions to the integral from the global and local parametrices is found by splitting the real negative axis into three regions $\mathbb{R}_- = (-\infty, z_0 - \epsilon) \cup [z_0 - \epsilon, z_0 + \epsilon] \cup (z_0 + \epsilon, 0]$ and calculating the former in the limit $\mu \to \infty$.

For $z \in (-\infty, z_0)$, one finds
\[
(\hat{P}_0^{-1} \hat{P}_0')_{21} = \mu (2g_0' - V') e^{\mu (2g_0 - V + V_0)} - \frac{i}{2(z - z_0)} e^{\mu (2g_0 - V + V_0)} \cos (\mu (2g_0 - V + V_0)) .
\]

The first term is dominant as $\mu \to \infty$ and its contribution to Eq. (50) evaluates to
\[
\int_{-\infty}^{z_0} \frac{dz}{2\pi i} (2g_0' - V') = \frac{\sqrt{1 + z_0}}{\pi} + \frac{1}{\pi^2 T} \left[ \sqrt{1 + z_0} \log \frac{1 + \sqrt{1 + z_0}}{\sqrt{-z_0}} - \log \sqrt{-z_0} \right] .
\]

For $z \in [z_0 - \epsilon, z_0 + \epsilon]$, one immediately finds using Eq. (48),
\[
(\hat{P}_{\text{loc}}^{-1} \hat{P}_{\text{loc}}')_{21} = \mu^{2/3} m'(z) \left( \phi_{+}^{-1} \phi'_{+} \right)_{21} (\mu^{2/3} m(z)) e^{\mu (2g_0 - V + V_0)} ,
\]
where the matrix element evaluates to
\[
(\phi_{+}^{-1} \phi'_{+})_{21} (z) = iK_{\text{Ai}}(z, z) ,
\]
and is expressed via the Airy kernel at coincident points
\[
K_{\text{Ai}}(z, z') = \frac{\text{Ai}(z) \text{Ai}'(z') - \text{Ai}(z') \text{Ai}'(z)}{z - z'} .
\]

Cumulatively,
\[
(\hat{P}_{\text{loc}}^{-1} \hat{P}_{\text{loc}}')_{21} = i\mu^{2/3} m'(z) K_{\text{Ai}} (s^{2/3} m(z), s^{2/3} m(z)) e^{\mu (2g_0 - V + V_0)} ,
\]
and yields a subleading effect compared to the contribution of the global parametrix to the determinant.

Finally, for $z \in (z_0 + \epsilon, 0]$, making use of the definition [51], one deduces that
\[
(\hat{T}^{-1} \hat{T}')_{21} \simeq (\hat{P}_0^{-1} \hat{P}_0')_{21} = \frac{i}{4(z - z_0)} + \ldots .
\]
Taking into account the behavior of the exponential factor in the integrand of Eq. (50) as $\mu \to \infty$, the evaluation of the integral results again in a power-suppressed effect compared to the $(-\infty, z_0)$-region. Thus, it does not, similarly to the local parametrix, affect the leading scaling behavior of the Fredholm determinant and, therefore, can be ignored.

As was just determined, the leading asymptotics arises only from the global parametrix in the region $(-\infty, z_0)$ and is given by the convolution of the (derivative of the) Deift-Zhou phase and the Fermi distribution. The result is a function of the parameter $z_0$. The latter is in turn related to the temperature $T$ by solving Eq. (45), which stems from the sewing condition between the global and local parametrices at the disk boundary $\partial U$. Employing Eq. (37), it can be inverted and one finds
\[
z_0 = -\frac{1}{\cosh^2(\pi T)} .
\]
Its substitution to Eq. (53) yields the \( \mu \) derivative of the Fredholm determinant
\[
\partial_\mu \log \det (1 - K) \xrightarrow{\mu \to \infty} -\frac{\mu}{T} \int_{-1}^{\infty} \frac{dz}{2\pi i} (2g' - V') = -\frac{\mu}{(\pi T)^2} \log \cosh(\pi T) \ .
\] (60)

Recalling that \( T = 2g \) and \( \mu = 2gy \) and integrating both sides of the above equation with respect to \( y \), one uncovers the leading \( y^2 \)-behavior in Eq. (1).

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**A. (Modified) Bessel model RHP.** The solution to the Bessel model RHP is analytic everywhere in the complex plane except on the contour \( C = C_1 \cup C_2 \cup C_3 \) shown in the panel (a) of Fig. 2 where the former develops jumps
\[
\Psi_+(z) = \Psi_-(z) M \ , \quad \text{for} \quad z \in C \, ,
\] (61)
with corresponding jump matrices
\[
M_{C_1,C_3} = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \, , \quad M_{C_2} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \, ,
\] (62)
and possesses a uniform asymptotic behavior
\[
\Psi(z) \xrightarrow{z \to \infty} (\pi \sqrt{z})^{-\sigma_3/2} A e^{\sqrt{\pi} \sigma_3} \, ,
\] (63)
where \( A \equiv (1 + i\sigma_1)/\sqrt{2} \) and the Pauli matrices \( \sigma_1 \) and \( \sigma_3 \). The solution can be written concisely as
\[
\Psi(z) = \left( \begin{array}{cc} I_0(\sqrt{z}) & \frac{2}{\pi} K_0(\sqrt{z}) \\ i\pi \sqrt{z} I_1(\sqrt{z}) & \sqrt{z} K_1(\sqrt{z}) \end{array} \right) H(0, z) \, ,
\] (64)
\[
H_I(0, z) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \, , \quad H_{II}(0, z) = \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) \, , \quad H_{III}(0, z) = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \, ,
\] (65)

in terms of the modified Bessel functions of the first and second kind with

What one needs in the dressing procedure used in the body of the paper is the solution to the modified Bessel model RHP. Namely, shifting the imaginary portion of the contour to the left by a real negative parameter \( z_0 \), one ends up with a contour shown in the middle panel (b) of Fig. 2. The RHP then reads: \( \hat{\Psi} \) is analytic in \( \mathbb{C} \setminus C \) with \( C = C_1 \cup C_2 \cup C_3 \cup C_4 \) with the jump conditions
\[
\hat{\Psi}_+(z) = \hat{\Psi}_-(z) \hat{M} \, ,
\] (67)
and the jump matrices being
\[ \tilde{M}_{C_1, C_3} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \tilde{M}_{C_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{M}_{C_4} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \] (68)

\( \tilde{\Psi} \) has the same asymptotic behavior as \( \Psi \) in Eq. (63). Then the solution to this modified Bessel model RHP is obtained from the previous one by the substitution \( H(0, z) \to H(z_0, z) \), i.e., [28],
\[ \tilde{\Psi}(z) = \begin{pmatrix} I_0(\sqrt{z}) \\ \frac{\pi}{i\sqrt{2}}K_0(\sqrt{z}) \\ \sqrt{z}I_1(\sqrt{z}) \end{pmatrix} H(z_0, z), \] (69)
and the same assignment of \( H \) in the three regions as above (of course, \( \Pi = \hat{\Pi} \), \( \III = \hat{\III} \) and \( \I = \hat{\I} \cup \hat{\IV} \)).

B. Airy model RHP. The solution to the Airy model RHP is analytic everywhere in the complex plane except on the contour \( C = C_1 \cup C_2 \cup C_3 \cup C_4 \) shown in the panel (c) of Fig. 2, where it develops jumps
\[ \phi_+ = \phi \tilde{M} \] (70)
with matrices
\[ \tilde{M}_{C_1, C_3} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \tilde{M}_{C_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{M}_{C_4} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \] (71)
and the asymptotics
\[ \phi(z) \to z^{-3/4} A e^{-\frac{\sigma_3}{2} z^{3/2}}. \] (72)

The model was introduced in Ref. [29] and the solution is
\[ \phi_I(z) = \sqrt{2\pi} \begin{pmatrix} \Ai(z) & -\omega^2 \Ai(\omega^2 z) \\ -i\Ai'(z) & i\omega \Ai'(\omega^2 z) \end{pmatrix}, \] (73)
\[ \phi_{II}(z) = \phi_I(z) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \] (74)
\[ \phi_{III}(z) = \phi_{IV}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \] (75)
\[ \phi_{IV}(z) = \sqrt{2\pi} \begin{pmatrix} \Ai(z) & \omega \Ai(\omega z) \\ -i\Ai'(z) & -i\omega^2 \Ai'(\omega z) \end{pmatrix}, \] (76)
where \( \omega = e^{2\pi i/3} \).
References

[1] B. Basso, S. Komatsu and P. Vieira, “Structure Constants and Integrable Bootstrap in Planar $\mathcal{N}=4$ SYM Theory,” arXiv:1505.06745 [hep-th].

[2] T. Fleury and S. Komatsu, “Hexagonalization of Correlation Functions,” JHEP 01 (2017) 130 arXiv:1611.05577 [hep-th].

[3] B. Eden and A. Sfondrini, “Tessellating cushions: four-point functions in $\mathcal{N}=4$ SYM,” JHEP 10 (2017) 098 arXiv:1611.05436 [hep-th].

[4] N. Beisert, et al. “Review of AdS/CFT Integrability: An Overview,” Lett. Math. Phys. 99 (2012) 3 arXiv:1012.3982 [hep-th].

[5] F. Coronado, “Perturbative four-point functions in planar $\mathcal{N}=4$ SYM from hexagonalization,” JHEP 01 (2019) 056 arXiv:1811.00467 [hep-th].

[6] F. Coronado, “Bootstrapping the Simplest Correlator in Planar $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory to All Loops,” Phys. Rev. Lett. 124 (2020) 171601 arXiv:1811.03282 [hep-th].

[7] I. Kostov, V. B. Petkova and D. Serban, “Determinant Formula for the Octagon Form Factor in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory,” Phys. Rev. Lett. 122 (2019) 231601 arXiv:1903.05038 [hep-th].

[8] I. Kostov, V. B. Petkova and D. Serban, “The Octagon as a Determinant,” JHEP 11 (2019) 178 arXiv:1905.11467 [hep-th].

[9] A. V. Belitsky and G. P. Korchemsky, “Exact null octagon,” JHEP 05 (2020) 070 arXiv:1907.13131 [hep-th].

[10] A. V. Belitsky and G. P. Korchemsky, “Octagon at finite coupling,” JHEP 07 (2020) 219 arXiv:2003.01121 [hep-th].

[11] A. V. Belitsky and G. P. Korchemsky, “Crossing bridges with strong Szego limit theorem,” arXiv:2006.01831 [hep-th].

[12] I. Kostov and V. B. Petkova, “Octagon with finite bridge: free fermions and determinant formulas,” (to appear).

[13] B. Basso, L. J. Dixon and G. Papathanasiou, “Origin of the Six-Gluon Amplitude in Planar $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory,” Phys. Rev. Lett. 124 (2020) 161603 arXiv:2001.05460 [hep-th].

[14] P. J. Forrester, “The spectrum edge of random matrix ensembles,” Nucl. Phys. B402 (1993) 709.

[15] C. A. Tracy and H. Widom, “Level spacing distributions and the Bessel kernel,” Commun. Math. Phys. 161 (1994) 289 hep-th/9304063.

[16] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, “Differential equations for quantum correlation functions,” Int. J. Mod. Phys. B4 (1990) 1003.
[17] J. Harnad and A. R. Its, “Integrable Fredholm operators and dual isomonodromic deformations,” Commun. Math. Phys. 226 (2002) 497 [arXiv:solv-int/9706002 [nlin.SI]].

[18] P.A. Deift, A.R. Its and X. Zhou, “A Riemann-Hilbert Approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and also in the Theory of Integrable Statistical Mechanics,” Ann. Math. 146 (1997) 149.

[19] P.A. Deift, “Integrable Operators,” Amer. Math. Soc. Transl. 189 (1999) 69.

[20] B. Malgrange, “Sur les déformations isomonodromiques. I. Singularités régulières,” In Mathematics and Physics (Birkhäuser, 1983) 401 (Progr. Math. 37);
M. Bertola, “The dependence on the monodromy data of the isomonodromic tau function,” Comm. Math. Phys., 294 (2010) 539 arXiv:0902.4716 [nlin.SI]; “CORRIGENDUM: The dependence on the monodromy data of the isomonodromic tau function,” arXiv:1601.04790 [math-ph].

[21] M. Jimbo, T. Miwa and K. Ueno, “Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and tau-function,” Physica D 2 (1981) 306.

[22] P.A. Deift, “Riemann-Hilbert Problems,” Course Notes from Fall 2009, https://www.ams.org/open-math-notes/omn-view-listing?listingId=110694.

[23] M. Bertola and M. Cafasso, “The transition between the gap probabilities from the Pearcey to the Airy process; a Riemann-Hilbert approach,” Int. Math. Res. Notices 2012 (2012) 1519 arXiv:1005.4083 [math-ph].

[24] M. Cafasso and T. Claeys, “A Riemann-Hilbert approach to the lower tail of the KPZ equation,” arXiv:1910.02493 [math-ph].

[25] A.B.J. Kuijlaars, K.T.-R. McLaughlin, W. Van Assche and M. Vanlessen, “The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on [−1,1],” Adv. Math. 188 (2004) 337 arXiv:math/0111252 [math.CA].

[26] P.A. Deift and X. Zhou, “Asymptotics for the PainlevéII equation,” Comm. Pure Appl. Math. 48 (1995) 277; “A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation,” Ann. of Math. 137 (1993) 295;
P.A. Deift, S. Venakides and X. Zhou, “New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems,” Internat. Math. Res. Notices 1997 (1997) 286.

[27] P.A. Deift, T. Kriecherbauer, K. T.R. McLaughlin, S. Venakides and X. Zhou, “Strong asymptotics of orthogonal polynomials with respect to exponential weights,” Comm. Pure and Applied Math. 52 (1999) 1491.

[28] C. Charlier and A. Doeraene, “The generating function for the Bessel point process and a system of coupled Painlevé V equations,” in Random Matrices. Theory and Applications (2019) arXiv:1709.07365 [math-ph].
[29] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides and X. Zhou, “Strong asymptotics of orthogonal polynomials with respect to exponential weights,” Comm. Pure Appl. Math. 52 (1999) 1491.