The descriptive set theory of the Lebesgue density theorem

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The category algebra.

Work in some perfect Polish space, e.g. $\omega^2$. $\mathcal{B}$ is the collection of all sets with the property of Baire, $\mathcal{M}_{\text{GR}}$ is the ideal of meager sets,

$$\mathcal{B}/\mathcal{M}_{\text{GR}} \cong \mathcal{B}_{\text{OR}}/\mathcal{M}_{\text{GR}} = \mathcal{C}_{\text{AT}}$$

$\mathcal{C}_{\text{AT}}$ is unique up-to isomorphism, i.e. it does not depend on the Polish space. The map

$$\rho : \mathcal{C}_{\text{AT}} \rightarrow \mathcal{RO}$$

is a selector, and $\mathcal{C}_{\text{AT}}$ can be identified with the collection of all regular open sets. $\mathcal{C}_{\text{AT}}$ is a Polish space.
The measure algebra.

\[ \mu \] a continuous probability Borel measure on some perfect Polish space, e.g. the usual Lebesgue measure on \( \omega^2 \). \text{Meas} is the collection of all sets measurable sets, \text{Null} is the ideal of measure-zero sets,

\[ \text{Meas}/\text{Null} \cong \text{Bor}/\text{Null} = \text{Malg} \]

\text{Malg} is unique up-to isomorphism, i.e. it does not depend on \( \mu \). \text{Malg} is a Polish space:

\[ \delta([A], [B]) = \mu(A \Delta B) \]
The Lebesgue density theorem

**Definition**

$x$ has density $r \in [0; 1]$ in $A \subseteq \omega^2$ if

$$D_A(x) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{\mu(A \cap N_x \upharpoonright n)}{\mu(N_x \upharpoonright n)} = r.$$  

**Theorem (Lebesgue)**

Let $A \subseteq \omega^2$ be Lebesgue measurable. Then

$$\Phi(A) = \{ x \in \omega^2 \mid x \text{ has density } 1 \text{ in } A \}$$

is Lebesgue measurable, and $\mu(A \triangle \Phi(A)) = 0$. In other words: $D_A$ agrees with $\chi_A$ almost everywhere.
The Lebesgue density theorem

If $\mu(A \triangle B) = 0$ then $\Phi(A) = \Phi(B)$, so

$$\Phi : \text{MALG} \to \text{MEAS}$$

is a selector. This is the analogue of $\rho : \text{CAT} \to \text{RO}$.

Question

What is the complexity of $\Phi(A)$?
Localization

Definition

The localization of $A$ at $s$ is

$$A_{[s]} = \left\{ x \in \omega^2 \mid s^x \in A \right\}$$

Thus $s^x A_{[s]} = A \cap N_s$.

Trivial observation

$$\mu(A \Delta B) = 0 \iff \forall s \in \omega^2 \left( \mu(A_{[s]}) = \mu(B_{[s]}) \right)$$

Thus a measure class $[A]$ is completely determined by the map $s \mapsto \mu(A_{[s]})$. 
Complexity of $\Phi$

Since

$$x \in \Phi(A) \Leftrightarrow \forall k \exists n \forall m \geq n (\mu(A|\lfloor x|_m]) \geq 1 - 2^{-k-1})$$

then

**Proposition (Folklore)**

For all measurable $A$

$$\Phi(A) \in \Pi^0_3.$$ 

**Question**

Is $\Pi^0_3$ optimal?
The density topology

- $A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B)$,
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,
- $\bigcup_{i \in I} \Phi(A_i) \subseteq \Phi\left(\bigcup_{i \in I} A_i\right)$,
- if $A$ is open, then $A \subseteq \Phi(A)$.

Definition

$$\mathcal{T} = \{A \in \text{MEAS} \mid A \subseteq \Phi(A)\}$$

is the density topology. It is finer than the usual topology.
The density topology

Theorem (Scheinberg 1971, Oxtoby 1971)

\[ A = \Phi(A) \text{ if and only if } A \text{ is open and regular in } \mathcal{T}. \]

\[ \Phi: \text{MALG} \to \text{RO}_\mathcal{T} \]

- **NULL** = **MGR**\(\mathcal{T}\) (Oxtoby, 1971)
- \(\mathcal{T}\) is neither first countable, nor second countable, nor Lindelöf, nor separable.
- \(\mathcal{T}\) is Baire.
Recall that $\Phi(A)$ is always $\Pi^0_3$.

**Theorem**

There is an $A$ such that $\Phi(A)$ is complete $\Pi^0_3$.

Clearly

$$\text{Int}(A) \subseteq \Phi(A) \subseteq \text{Cl}(A).$$

and $A = \Phi(A)$ if $A$ is clopen.

**Question**

Can $\Phi(A)$ be something other than clopen or complete $\Pi^0_3$?

Yes!
**Wadge degrees**

**Definition**

A is Wadge reducible to $B$

$$A \leq_{W} B$$

just in case $A = f^{-1}(B)$ for some continuous $f: \mathbb{ω}^2 \rightarrow \mathbb{ω}^2$.

$A \equiv_{W} B$ iff $A \leq_{W} B \land B \leq_{W} A$.

The equivalence classes $[A]_{W}$ are called Wadge degrees.

For $d \subseteq \Pi^0_3$ a Wadge degree, let

$$\mathcal{W}_d = \{ [A] \mid \Phi(A) \in d \}$$
The sets $\mathcal{W}_d$ are non-empty, in fact are dense in the topological space $\mathcal{M}_{\text{ALG}}$:

$$\forall \varepsilon \forall A \forall d \subseteq \Pi^0_3 \exists C \in \Pi^0_1 \exists U \in \Sigma^0_1 \\left( \Phi(C) = \Phi(U) \in d \land \mu(A \triangle C) < \varepsilon \right).$$
Density
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The motivation
Complete Boolean algebras
Lebesgue’s theorem
The density topology

Results
\[ \Pi^0_3 \] -completeness
Wadge degrees
Dualistic sets
Comeagerness
Forcing
\[ \hat{\Phi} \] is Borel

Would you like to see some proofs?
\[ \Pi^0_1 \] -completeness
Inside \[ \Delta^0_3 \]

Dualistic sets

Recall that \( D_A(x) = 0, 1 \) for \textit{almost} all \( x \).

Definition

A set \( A \) is dualistic (or Manichæan) if \( D_A(x) = 0, 1 \) for all \( x \).
\( M \) is the Boolean algebra of all dualistic sets.

Clearly being dualistic depends on the equivalence class of \( A \), so

\[
A \in M \iff \Phi(A) \in M.
\]

Fact

\( A = \Phi(A) \) is dualistic iff \( A \) is \( \mathcal{T} \)-clopen, i.e.,

\[
M \cap \text{ran}(\Phi) = \Delta^0_1-\mathcal{T}
\]
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Inside $\Delta^0_n$

Dualistic sets

Proposition

$$\forall A \in \text{Meas} \ (A \in \mathcal{M} \Rightarrow \Phi(A) \in \Delta^0_2).$$

We can refine the Metric Approximation Theorem for $\Delta^0_2$
degrees:

$$\forall \varepsilon > 0 \forall A \forall d \subseteq \Delta^0_2 \exists C \in \Pi^0_1 \exists U \in \Sigma^0_1$$

$$(\Phi(C) = \Phi(U) \in \mathcal{W}_d \cap \mathcal{M} \land \mu(A \triangle C) < \varepsilon)$$
Theorem

Let $d = \Pi^0_3 \setminus \Delta^0_3$ be the degree of the complete $\Pi^0_3$ sets. Then $\mathcal{W}_d$ is comeager in $\text{MALG}$. 
Another comeager setC’m on, we all knew that...

Given any measurable $A$ there are $F \subseteq A \subseteq G$ with $F \in \Sigma^0_2$ and $G \in \Pi^0_2$ such that $\mu(A) = \mu(F) = \mu(G)$.

**Theorem**

$$\{ [A] \mid [A] \cap \Delta^0_2 = \emptyset \} \text{ is comeager in } \text{MALG}.$$
Dense sets in boolean algebras

By the Metric Approximation Theorem, the $\mathcal{W}_d$ are *topologically* dense in $\mathsf{MALG}$. But $\mathsf{MALG}$ is a Boolean algebra (i.e. a forcing notion) so there is a competing notion of *density*.

**Theorem**

Let $d = \Pi^0_3 \setminus \Delta^0_3$ be the degree of the complete $\Pi^0_3$ sets. If $\emptyset \neq A = \Phi(A)$ has empty interior, then $A \in d$. Therefore $\mathcal{W}_d$ contains a dense open set.
Recall that \( \Phi \) induces a map \( \hat{\Phi} : \text{MALG} \to \Pi_3^0 \),

\[ \hat{\Phi}(\lbrack A \rbrack) = \Phi(A). \]

Fix some standard coding \( \pi : \omega^2 \to \Pi_3^0 \).

**Proposition**

\( \hat{\Phi} \) is Borel, i.e. there is a Borel \( \mathcal{F} : \text{MALG} \to \omega^2 \) such that

\[ \hat{\Phi}(\lbrack A \rbrack) = \pi(\mathcal{F}(\lbrack A \rbrack)). \]
Sketch of the proof for $\Pi^0_3$ completeness

- $T$ a pruned tree such that $[T]$ has positive measure and empty interior. Thus $- [T] = \bigcup_n N_{t_n}$.
- $n < m \Rightarrow \text{lh}(t_n) < \text{lh}(t_m)$ and $\exists \infty n (\text{lh}(t_n) + 1 < \text{lh}(t_{n+1}))$.
- For all $t \in T$ there is a shortest $s \supset t$ such that $s \notin T$. $s$ is the target of $t$.
- Let $\tau(t) = \text{lh} (\text{target of } t) - \text{lh}(t), \tau : T \to \omega \setminus \{0\}$.
- For $x \in [T],$
  \[ x \in \Phi([T]) \iff \lim_{n \to \infty} \tau(x \restriction n) = \infty. \]
Sketch of the proof for $\Pi^0_3$ completeness, ctd.

The set

$$P = \{ z \in \omega \times \omega 2 \mid \forall m \forall \infty n \ z(n, m) = 0 \}.$$  

is complete $\Pi^0_3$.

Given $a : n \times n \rightarrow 2$ construct a node $\varphi(a) \in T$ so that

$$a \subset b \Rightarrow \varphi(a) \subset \varphi(b),$$

and

$$\omega \times \omega 2 \rightarrow [T], \quad z \mapsto \bigcup_n \varphi(z \upharpoonright n \times n)$$

witnesses $P \leq_w \Phi([T])$.  

/
Sketch of the proof for $\Pi^0_3$ completeness, ctd.

Let $a: (n + 1) \times (n + 1) \to 2$. (Say $n = 4$)

Case 1:

|   | $a_{0,4}$ | $a_{1,4}$ | $a_{2,4}$ | $a_{3,4}$ | 0   |
|---|-----------|-----------|-----------|-----------|-----|
|   | $a_{0,3}$ | $a_{1,3}$ | $a_{2,3}$ | $a_{3,3}$ | 0   |
|   | $a_{0,2}$ | $a_{1,2}$ | $a_{2,2}$ | $a_{3,2}$ | 0   |
|   | $a_{0,1}$ | $a_{1,1}$ | $a_{2,1}$ | $a_{3,1}$ | 0   |
|   | $a_{0,0}$ | $a_{1,0}$ | $a_{2,0}$ | $a_{3,0}$ | 0   |

Then pick $t \supset \varphi(a \upharpoonright n \times n)$ such that

$$\tau(t) \geq \max \{n + 1, \tau(\varphi(a \upharpoonright n \times n))\}.$$
Sketch of the proof for $\Pi^0_3$ completeness, ctd.

Let $a: (n + 1) \times (n + 1) \rightarrow 2$. (Say $n = 4$)

Case 2:

|   | $a_{0,4}$ | $a_{1,4}$ | $a_{2,4}$ | $a_{3,4}$ | $a_{4,4}$ |
|---|-----------|-----------|-----------|-----------|-----------|
| $a_{0,3}$ | $a_{1,3}$ | $a_{2,3}$ | $a_{3,3}$ | $a_{4,3}$ |
| $a_{0,2}$ | $a_{1,2}$ | $a_{2,2}$ | $a_{3,2}$ | $a_{4,2}$ |
| $a_{0,1}$ | $a_{1,1}$ | $a_{2,1}$ | $a_{3,1}$ | $0$      |
| $a_{0,0}$ | $a_{1,0}$ | $a_{2,0}$ | $a_{3,0}$ | $0$      |

Then pick $t \supset \varphi(a \upharpoonright n \times n)$ such that

$$\tau(t) = 3.$$
A set $A$ (or degree) is self dual if $A \equivW \neg A$. Otherwise it is non-self-dual.

- Self-dual and non-self-dual pairs alternate.
- At all limit levels there is a non-self-dual pair.
Given $f: \omega \to \omega \setminus \{0\}$ and sets $A_0, A_1, \ldots$ consider the set

$$\text{Rake}^-(f; (A_n)_n)$$

$0(\infty)$

$0(3) 1(f(3))$

$0(4) 1(f(4))$

$01(f(1))$

$1(f(0))$

$A_0$

$A_1$

$A_2$

$A_3$

$A_4$
How to construct larger degrees.

If $\exists \infty n \left( f(n) \geq 2 \right)$ and the $A_n$'s are $\mathcal{T}$-regular, i.e. $\Phi(A_n) = A_n$ then so is $\text{Rake}^-(f; (A_n)_n)$. Moreover

- if $A = A_0 = A_1 = \ldots$ are self-dual, then $\text{Rake}^-(f; (A_n)_n)$ is non-self-dual and immediately above $A$,
- if $A_0 <_W A_1 <_W A_2 <_W \ldots$ then $\text{Rake}^-(f; (A_n)_n)$ is non-self-dual and immediately above the $A_n$'s.

Note that the rake $\text{Rake}^-(f; (A_n)_n)$ has no pole, i.e., $0^{(\infty)}$ does not belong to this set. In order to construct the dual degrees we need another kind of rake, a pole and densely packed tines.
How to construct larger degrees.

\[ \text{Rake}^+ (f; (A_n)_n) \]
If \( \lim_{n} f(n) = \infty \) then and the \( A_n \)'s are \( \mathcal{T} \)-regular, i.e. \( \Phi(A_n) = A_n \) then so is \( \text{Rake}^+(f; (A_n)_n) \). Moreover

\[
\text{Rake}^+(f; (A_n)_n) \equiv^w \neg \text{Rake}^-(f; (A_n)_n).
\]

If \( A \) and \( B \) are \( \mathcal{T} \)-regular then so is

\[
A \oplus B = 0^\infty A \cup 1^\infty B.
\]

Arguing this way, we can climb up to \( \Delta_2^0 \).
Jumping $\omega_1$ levels.

Wadge defined two operations $A^\natural$ and $A^\flat$ on subsets of the Baire space

$$A^\natural = \left\{ s_0^+ \circ 0 \circ s_1^+ \circ 0 \circ \ldots \circ s_n^+ \circ 0 \circ x^+ \mid n \in \omega, s_i \in <\omega, x \in A \right\}$$

$$A^\flat = A^\natural \cup \{ x \in \omega \mid \exists \infty n (x(n) = 0) \}$$

where $s^+$ and $x^+$ are the sequences obtained from $s$ and $x$ by adding a 1 to all entries.

The idea is that $A^\natural$ is the union of $\omega$ many layers of the form

$$A^+ = \{ x^+ \mid x \in A \}$$
Jumping $\omega_1$ levels.

**Theorem (Wadge)**

If $A$ is self-dual, then $A^\h^b$ and $A^b$ form a non-self-dual pair and

$$\|A^\h^b\|_W = \|A^b\|_W = \|A\|_W \cdot \omega_1.$$  

The operations $A^\h^b$ and $A^b$ together with the (analogs of) the Rake operations, are sufficient to construct sets of rank $< \omega_1^{\omega_1}$, i.e. the $\Delta^0_3$ sets.
Jumping $\omega_1$ levels.

An analogue of $A^+$. 

- $s \upharpoonright i = \bar{s} \upharpoonright ii$, for $s \in <\omega 2$.
- $\bar{x} = \bigcup_n x \upharpoonright n$, for $x \in \omega 2$.
- Replace $A$ with $\{ \bar{x} \mid x \in A \}$, but...
- Does not work, since $\{ \bar{x} \mid x \in \omega 2 \}$ is of measure 0!
- The cure: enlarge $\{ \bar{x} \mid x \in A \}$ like $\text{Rake}^-$ was enlarged to $\text{Rake}^+$. The resulting set is called $\text{Plus}(A)$.
- In fact we construct $\text{Plus}(A; r)$ (with $r \in (0; 1)$) so that $\mu \left( \text{Plus}(A; r)_{[\bar{s}]} \right) \geq r$ for all $s$.
- If $A$ is $\mathcal{T}$-regular (i.e., $A = \Phi(A)$), then so is $\text{Plus}(A; r)$. 


Jumping $\omega_1$ levels.

Construct $\text{Nat}(A)$ and $\text{Flat}(A)$: they are the analogs of $A^\natural$ and $A^\flat$, and have rank $\|A\|_W \cdot \omega_1$.

Using the operations $\text{Nat}(A)$, $\text{Flat}(A)$, $\text{Rake}^- A$, $\text{Rake}^+ A$, and $\oplus$ it is possible to construct a closed sets $C$ such that $\Phi(C)$ is of any given Wadge degree in $\Delta^0_3$. 
Fix $0 < r < 1$. $\text{Nat}(A)$ is composed of $\omega$-many layers:

- $\text{Plus}(A; r)$
- $\text{Plus}(A; r)$
- $\text{Plus}(A; r)$

If $x$ settles inside a layer, then $x = s^{\uparrow}y$ and the density of $x$ in $\text{Nat}(A)$ will be ‘similar’ to the density of $y$ in $A$.

Every time we climb to a higher level, the density drops momentarily to $\leq 1/2$. So if $x$ climbs infinitely many layers, then $x$ will not have density 1 in $\text{Nat}(A)$. 
Fix $0 < r_0 < r_1 < r_2 < \cdots \to 1$.

Flat$(A)$ is the set is composed of $\omega$-many layers

\[
\vdots
\]

\[
\text{Plus}(A; r_2)
\]

\[
\text{Plus}(A; r_1)
\]

\[
\text{Plus}(A; r_0)
\]

- If $x$ settles inside a layer, then $x = s^\omega y$ and the density of $x$ in Flat$(A)$ will be ‘similar’ to the density of $y$ in $A$.

- In the layer $n$, the density will always be $\geq r_n$. So if $x$ climbs infinitely many layers, then $x$ will have density 1 in Flat$(A)$.
Density

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