CLASSIFICATION OF STACKED CENTRAL CONFIGURATIONS IN $R^3$

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**Abstract.** We classify the extensions of $n$-body central configurations to $(n + 1)$-body central configurations in $R^3$, in both the collinear case and the non-collinear case. We completely solve the two open questions posed by Hampton (Nonlinearity 18: 2299-2304, 2005). This classification is related with study on co-circular and co-spherical central configurations. We also obtain a general property of co-circular central configurations.

**Key Words:** Newtonian $n$-body problem; stacked central configurations; co-circular configurations; co-spherical configurations; pyramidal central configurations; perverse solutions.

1. Introduction

Central configurations are important in the classical $N$-body problems. They naturally arise in the study of the self-similar solutions, and they are involved in the classification of the topology of integral manifolds [27]. In the collection of important open problems in celestial mechanics compiled by Albouy-Cabral-Santos [2], half of the list is on central configurations. Readers are referred to [1, 2, 3, 20, 26] for introductions, recent advance and open questions.

The $(n + k)$-body central configurations extended from $n$-body central configurations by adding $k$ bodies are called **stacked central configurations**. For instance, the Lagrangian equilateral triangle central configuration is a stacked central configuration. It is also well-known that a **pyramidal central configuration** can be obtained by adding one mass to a co-circular central configuration [11, 13, 25]. Hampton introduced stacked central configurations in 2005 [17]. Many other examples of stacked central configurations were constructed, see [7, 9, 18, 24].
Hampton also raises two questions regarding stacked central configuration \[17\:].

1) In addition to symmetric collinear configurations\[4\] the square or a regular tetrahedron with a mass at its center and the square pyramidal configuration are there any five-body central configurations with a subset forming a four-body central configuration?

2) Are there any five-body non-collinear central configurations all of whose four-body subsets form a central configuration?

There are some works devoted to this two questions. Assuming that a five-body central configuration is co-planar and non-collinear, in 2013, Fernandes-Mello \[12\] and Alvarez Ramírez-Santos-Vidal \[4\] announced independently that such configuration must be a square with equal masses and one mass at the center of the square. Though the paper \[12\] contains several inspirational observations, some argument is problematic. In 2018, Fernandes-Mello \[15\] fix the proof, see Remark \[6\]. However, the two questions remain open.

In this work, we classify the ways by which an \(n\)-body (\(n \geq 2\)) central configuration can extend to an \((n+1)\)-body central configuration in \(R^3\). With this classification, we solve the two open questions completely. We also find one general property of co-circular central configurations. It plays a crucial role in our study of the extensions of co-circular central configurations.

There are two cases. Firstly, the extended \((n+1)\)-body central configuration is collinear. In this case, we show that extensions happen only for \(n = 2\). So the question of collinear extensions has already been answered by Euler \[10\]. Secondly, the extended \((n+1)\)-body central configuration is non-collinear. In this case, it has been proved by Fernandes-Mello \[13\] that it is necessary that the \(n\)-body central configuration lies on a common circle or sphere. Our approach is different from theirs. Our results contain not only the necessary conditions, but sufficient conditions as well. Thus, we can classify all the extensions.

Let \(r\) be the radius of the circle (sphere) and \(r_0 = \left(\frac{m}{\lambda}\right)^{\frac{1}{3}}\), the cubic root of the ratio of total mass \(m\) and the multiplier \(\lambda\) of the \(n\)-body configuration. The co-circular (co-spherical) central configurations can extend if their mass center equals their geometric center. They can also extend if \(r_0 \geq r\) for the co-circular case, and \(r_0 = r\) for the co-spherical case.

Thus, the measurement of \(r\) and \(r_0\) of the co-circular and co-spherical central configurations is important. We obtain a general result for the

\[\text{According to Theorem } \[1\] \text{ there is no five-body collinear central configuration with a subset forming a four-body central configuration.}\]
co-circular ones, with which we prove that \( r_0 > r \) holds for all four, five, and six-body co-circular central configurations. Together with the works of Hampton [16] and Cors-Roberts [8] on the co-circular four-body problem, we find all the extensions of four-body central configurations to five-body central configurations. And we answer Hampton’s two questions completely.

The paper is organized as follows. In Section 2 we state the main results. In Section 3, we prove the main results. In Section 4, we discuss the extensions of two, three, four, and five-body central configurations. In Section 5, we find several examples of co-spherical central configurations whose mass center equals the geometric center.

### 2. Main Results

We are interested in the \( n \)-body central configurations that can extend to \((n+1)\)-body central configurations by adding one mass. If there is no confusion raised, the \( n \) masses are \( m_1, ..., m_n \) and the corresponding configuration is \( \mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_n) \). We denote by \( m_0 \) the added mass and by \( \mathbf{q}_0 \) its position. We denote by \( \overline{\mathbf{q}} = (\mathbf{q}_0, \mathbf{q}_1, ..., \mathbf{q}_n) \) the extended configuration. We will also call the original \( n \)-body configuration \( \mathbf{q} \) the sub configuration. We use \( r_{ij} \) to denote the distance between any two of the \( n+1 \) particles, i.e., \( r_{ij} = |\mathbf{q}_i - \mathbf{q}_j|, 0 \leq i < j \leq n \). We denote by \( m \) the sum of the \( n \) masses, and by \( \overline{m} \) the sum of the \( n+1 \) masses, i.e,

\[
m = \sum_{i=1}^{n} m_i, \quad \overline{m} = \sum_{i=0}^{n} m_i = m_0 + m.
\]

We denote by \( c \) the mass center of the \( n \)-body sub configuration, and \( \overline{c} \) the mass center of the \((n+1)\)-body configuration, i.e,

\[
c = \frac{\sum_{i=1}^{n} m_i \mathbf{q}_i}{m}, \quad \overline{c} = \frac{\sum_{i=0}^{n} m_i \mathbf{q}_i}{m} = \frac{mc + m_0 \mathbf{q}_0}{m + m_0}.
\]

Denote by \( U, I \) and \( \overline{U}, \overline{I} \) the force function and the momentum of inertia of the sub \( n \)-body system and the \((n+1)\)-body system respectively, i.e.,

\[
U = \sum_{1 \leq i < j \leq n} \frac{m_im_j}{r_{ij}}, \quad I = \sum_{1 \leq i < j \leq n} \frac{m_im_j}{m} |\mathbf{q}_i - \mathbf{q}_j|^2 = \sum_{i=1}^{n} m_i |\mathbf{q}_i - c|^2;
\]

\[
\overline{U} = \sum_{0 \leq i < j \leq n} \frac{m_im_j}{r_{ij}} = U + \sum_{i=1}^{n} \frac{m_0m_i}{r_{i0}},
\]

\[
\overline{I} = \sum_{0 \leq i < j \leq n} \frac{m_im_j}{\overline{m}} |\mathbf{q}_i - \mathbf{q}_j|^2 = \sum_{i=0}^{n} m_i |\mathbf{q}_i - \overline{c}|^2.
\]
We assume that both the \((n+1)\)-body configuration and the \(n\)-body sub configuration are central. That is, the configurations \(\bar{q}\) and \(q\) satisfy the following two systems simultaneously,

\[
\nabla U(q) + \lambda/2 \nabla I(q) = 0, \quad \nabla \bar{U}(\bar{q}) + \bar{\lambda}/2 \nabla \bar{I}(\bar{q}) = 0,
\]

where \(\lambda = U/I\) and \(\bar{\lambda} = \bar{U}/\bar{I}\).

Our first result concerns the case that the \((n+1)\)-body central configuration is collinear.

**Theorem 1.** Assume that \(n \geq 2\). Suppose that an \(n\)-body collinear central configuration can extend to an \((n+1)\)-body collinear central configuration by adding one mass, then \(n = 2\).

This reduces study of collinear extensions to study of the well-known three-body collinear central configurations, which has been considered by Euler [10], see Section 4.1.

In what follows, we mainly discuss the non-collinear case. In this case, Fernandes-Mello [13] have showed that the \(n\)-body sub configuration must lie on a common circle or sphere and the added mass is at the geometric center. Their proof employed the Laura-Andoyer equations. Our approach is different from theirs, see Section 3. Our results contain more details, which enables us to provide a complete classification of the non-collinear extensions. We divide our discussion into two cases: \(q_0 = c\) and \(q_0 \neq c\).

**Theorem 2.** Suppose that \(q_0 = c\). Then both the \((n+1)\)-body configuration and the \(n\)-body sub configuration are central if and only if the following two conditions are satisfied

- The \(n\)-body sub configuration is central;
- \(|q_1 - q_0| = |q_2 - q_0| = \cdots = |q_n - q_0|\).

**Theorem 3.** Suppose that \(q_0 \neq c\) and that the \((n+1)\)-body configuration is non-collinear. Then both the \((n+1)\)-body configuration and the \(n\)-body sub configuration are central if and only if the following two conditions are satisfied

- The \(n\)-body sub configuration is central;
- \(\frac{1}{|q_i - q_0|^2} = \frac{\lambda}{m}, \quad i = 1, \ldots, n\).

**Corollary 1.** Suppose that the \((n+1)\)-body non-collinear configuration and the \(n\)-body sub configuration are central, then they are still central if we replace \(m_0\) by an arbitrary mass.

We answer the second question of Hampton [17], see Section 1.
Classification of stacked CC in $\mathbb{R}^3$

**Proposition 1.** In $\mathbb{R}^3$, there are only three types of $(n+1)$-body central configurations all of whose $n$-body subsets form a central configuration, namely, the three-body Eulerian collinear central configurations, the Lagrangian equilateral triangle central configurations and the regular tetrahedron central configurations.

The sub configuration we are looking for lies on a common circle or sphere. These central configurations are called **co-circular central configurations**, in the planar case and **co-spherical central configurations**, in the spatial case. To make it precise, we use the terminology “co-spherical configuration” to indicate that the configuration is not planar. Denote by $r$ the radius of the related circle (sphere). Denote by $r_0$ the cubic root of the ratio of total mass and the multiplier of the sub central configuration, i.e.,

$$r_0 = \left(\frac{m}{\lambda}\right)^{\frac{1}{3}} = \left(\frac{mI}{U}\right)^{\frac{1}{3}} = \left(\frac{\sum_{1 \leq i < j \leq n} m_i m_j r_{ij}^2}{\sum_{1 \leq i < j \leq n} m_i m_j / r_{ij}}\right)^{\frac{1}{3}}.$$

An $(n + 1)$-body spatial central configuration of which $n$ points lie in an affine plane is called a **pyramidal central configuration**.

**Theorem 4.** In $\mathbb{R}^3$, there are only five ways that an $n$-body central configuration can extend to an $(n + 1)$-body non-collinear central configuration.

- **I co-circular to planar:** $n$-body co-circular central configurations whose mass center coincides with the geometric center, extend to $(n + 1)$-body planar central configurations by adding $m_0$ at the geometric center;
- **II co-circular to planar:** $n$-body co-circular central configurations whose mass center does not coincide with the geometric center, but $r = r_0$, extend to $(n + 1)$-body planar central configurations by adding $m_0$ at the geometric center;
- **III co-circular to pyramidal:** $n$-body co-circular central configurations whose mass center may or may not coincide with the geometric center, but $r < r_0$, extend to pyramidal central configurations by adding $m_0$ on the orthogonal axis passing through the center of the circle such that $r_{10} = r_0$;
- **IV co-spherical to spatial:** $n$-body co-spherical central configurations whose mass center coincides with the geometric center, extend to $(n + 1)$-body central configurations by adding $m_0$ at the geometric center;
- **V co-spherical to spatial:** $n$-body co-spherical central configurations whose mass center does not coincide with the geometric center.
center, but \( r = r_0 \), extend to \((n+1)\)-body central configurations by adding \( m_0 \) at the geometric center.

Chenciner [6] asked: Is the regular \( n \)-gon with equal masses the unique co-circular central configuration that the center of mass equals the geometric center? This question is listed as Problem 12 in a collection of open problems in celestial mechanics compiled by Albouy-Cabral-Santos [2]. We may ask one equivalent question: If an \( n \)-body co-circular central configuration can extend to a co-planar central configuration by adding one mass \( m_0 \) at the mass center, does the \( n \)-body central configuration have to be the regular \( n \)-gon with equal masses? Until now, the question has only been answered affirmatively for \( n = 4 \), by Hampton in 2003 [16].

**Corollary 2.** Suppose that an \((n+1)\)-body non-collinear central configuration is obtained from an \( n \)-body co-circular (co-spherical) central configuration by adding one mass \( m_0 \) such that \( r_{i0} = r_0, i = 1, \ldots, n \), i.e., by way II, III, and V of Theorem 4. Let \( \bar{r}_0 = (\bar{m}/\bar{\lambda})^{\frac{1}{3}} \), the cubic root of the ratio of total mass and the multiplier of the extended \((n+1)\)-body central configuration. Then \( \bar{r}_0 \) does not depend on the value of \( m_0 \), and \( \bar{r}_0 = r_0 \).

**Remark 1.** Fernandes-Mello [14] announced that if an \( n \)-body co-circular central configuration can extend to an \((n+1)\)-body central configuration by adding one arbitrary mass at the geometric center, the mass center of the \( n \) bodies must coincide with the geometric center (Lemma 2.3 of [14]). That is to say, the extension way II of Theorem 4 does not exist. This statement is incorrect. The well-known Lagrangian equilateral triangle central configuration is a counterexample, see Remark 4 and Section 4.1. The error in their proof is similar to the one described in Remark 6.

According to Theorem 4, the measurement of \( r \) and \( r_0 \) of the co-circular and co-spherical central configurations is crucial for the classification of stacked central configurations. We obtain a general result for the co-circular case, with which, we could prove that \( r_0 > r \) holds for all four, five, and six-body co-circular central configurations.

Some **notations** for the co-circular configurations: Edges are line segments connecting two different vertices of a polygon. For a co-circular configuration whose vertices are ordered counterclockwise as \((q_1, \ldots, q_n)\), the edges \( q_i q_j \) are called **exterior sides** if \( |i - j| = 1 \) or \( n - 1 \), and **diagonals** otherwise. An edge and a vertex on that edge are called **incident**.
Theorem 5. Assume that $n \geq 4$. For any $n$-body co-circular central configuration, all the exterior sides are less than $r_0$. At each vertex, there is at least one incident diagonal larger than $r_0$.

Remark 2. For $n = 2, 3$, there is no diagonal. It is easy to see that $r_0 = r_{12}$ for both of the two cases. For $n \geq 4$, there are at least $n/2$ ($n$ even) or $(n + 1)/2$ ($n$ odd) diagonals greater than $r_0$. For $n = 4, 5$, these results have been proved for a larger set of central configurations, namely, the four and five-body planar convex central configurations, by MacMillan-Bartky [22] and Chen-Hsiao [5] respectively. Generally, for large $n$, there would be many diagonals smaller than $r_0$, see the examples in Section 5.1.

Corollary 3. Co-circular central configurations can't lie entirely in a semi-circle.

Remark 3. As suggested by Cors-Roberts [8], this fact also follows nicely from the perpendicular bisector theorem [21].

Proposition 2. For all four, five, and six-body co-circular central configurations, the radius of the circle containing the bodies is smaller than $r_0$.

Remark 4. The two ends of a segment can be placed on a circle with radius equal or greater than half of the segment. Thus, for $n = 2$, we have $r_0/2 \leq r < \infty$. For $n = 3$, we have $r_0 = \sqrt{3}r$.

We find all the extensions of four-body central configurations to five-body central configurations. This answers Hampton’s first question [17], see Section 1.

Theorem 6. There are only three types of five-body central configurations of which a four-body sub configuration is central:

- the square with equal masses and an arbitrary mass $m_0$ at the center;
- any four-body co-circular central configuration and an arbitrary mass $m_0$ on the orthogonal axis passing through the center of the circle. The height of $m_0$ is $h = \sqrt{r_0^2 - r^2}$;
- the regular tetrahedron with equal masses and an arbitrary mass $m_0$ at the center.

For the extension of five and more bodies, we have some partial results.

Proposition 3. • If a five (six)-body co-circular central configuration can extend to a co-planar central configuration by adding
one mass \( m_0 \) at the geometric center, the mass center of the co-
circular central configuration must coincide with the geometric
center.

- Any five and six-body co-circular central configurations can ex-
tend to a pyramidal central configuration by adding an arbitrary
mass \( m_0 \) on the orthogonal axis passing through the center of
the circle. The height of \( m_0 \) is \( h = \sqrt{r_0^2 - r^2} \).

Our classification of the stacked central configurations also enables
us to give a complete characterization of the pyramidal central config-
urations.

**Proposition 4.** Let \( \mathbf{q} = (q_0, q_1, \ldots, q_n) \) be an \((n + 1)\)-body pyramidal
configuration with masses \( m_0, m_1, \ldots, m_n \), where \( q_0 \) is the top vertex
which is off the affine plane containing \( m_1, \ldots, m_n \). Then the pyra-
midal configuration is central if and only if the following conditions are
satisfied

- The sub configuration \( q_1, \ldots, q_n \) is central and co-circular;
- \( r < r_0 \), where \( r, r_0 \) are values associated with the \( n \)-body co-
circular central configuration;
- The top vertex \( q_0 \) is on the orthogonal axis passing through the
center of the circle. The height of \( q_0 \) is \( h = \sqrt{r_0^2 - r^2} \).

The top mass \( m_0 \) is arbitrary.

**Remark 5.** When studying five-body pyramidal configurations, Fayçal
[11] showed that the base must be co-circular and that the top mass is
arbitrary. She also gave a formula for the distance between the top ver-
tex and the base vertices. Ouyang-Xie-Zhang [25] generalized Fayçal’s
result to \( n \)-body pyramidal configurations. Their characterization is
almost the same as ours. They did not compare explicitly the two
values \( r \) and \( r_0 \). Albouy [1] obtained a general proposition on central
configurations in \( \mathbb{R}^N \). Restricted in \( \mathbb{R}^3 \), it immediately implies the base
of any pyramidal central configuration is central and co-circular.

Stacked central configurations are also related with *perverse solutions*
introduced by Chenciner [6]. A solution \( \mathbf{q}(t) = q_1(t), \ldots, q_n(t) \) of the
\( n \)-body problem with masses \( m_1, m_2, \ldots, m_n \) is called a perverse solution
if there exists another system of masses, \( m'_1, m'_2, \ldots, m'_n \), for which \( q(t) \)
is still a solution. Note that any \((n + 1)\)-body non-collinear central
configuration obtained from an \( n \)-body central configuration by adding
one mass at the mass center of the \( n \) bodies, i.e., by way I and IV
of Theorem [4] would provide perverse solutions, namely, the relative
equilibrium and the total collision solution for the planar case, and the
total collision solution for the spatial case, see also Section 5.2.
3. Proofs of the Main Results

We first simplify the central configuration equations (1).

Note that there is a simple but important fact: the three points $c, \bar{c},$ and $q_0$ are collinear. In fact, $\bar{c}$ can also be seen as the mass center of the two material points $c, q_0$ with masses $m, m_0$. Thus, $\bar{c}$ equals $q_0$ if and only if $c$ equals $q_0$. The collinearity of the three points is also revealed in the following equalities

$$ m(\bar{c} - q_0) = m(c - q_0), \quad A - \bar{c} = (A - c) - \frac{m_0}{m}(q_0 - c), $$

where $A$ is an arbitrary point.

We assume that both the $(n+1)$-body configuration and the $n$-body sub configuration are central. The central configuration equations (1) can be written as

$$ \left\{ \begin{aligned}
\sum_{j \neq i,j=1}^n m_i m_j \frac{q_i - q_j}{|q_i - q_j|^3} &= -\lambda m_i (q_i - c), \quad i = 1, \ldots, n, \\
\sum_{j=1}^n m_i m_0 \frac{q_i - q_0}{|q_i - q_0|^3} &= -\hat{\lambda} m_0 (q_0 - c), \\
\sum_{j \neq i,j=1}^n m_i m_j \frac{q_i - q_j}{|q_i - q_j|^3} + \frac{m_i m_0 (q_0 - q_i)}{|q_0 - q_i|^3} &= -\hat{\lambda} m_i (q_i - \bar{c}), \quad i = 1, \ldots, n,
\end{aligned} \right. $$

where $\lambda = U/I$ and $\hat{\lambda} = \bar{U}/\bar{I}$.

Note that the third part of system (3) can be written as

$$ \lambda m_i (q_i - c) + \frac{m_0 m_i}{r_{i0}^3} (q_i - q_0) = \hat{\lambda} m_i (q_i - \bar{c}), \quad i = 1, \ldots, n. $$

Summing up all the $n$ equations gives the second part of system (3).

Furthermore, by (2), the $n$ equations are equivalent to

$$ (\lambda - \hat{\lambda}) (q_i - c) = \left( \frac{m_0}{r_{i0}^3} - \frac{\hat{\lambda} m_0}{m} \right) (q_0 - c), \quad i = 1, \ldots, n. $$

Therefore, system (1) is equivalent to the following system,

$$ \left\{ \begin{aligned}
\sum_{j \neq i,j=1}^n m_i m_j \frac{q_i - q_j}{|q_i - q_j|^3} &= -\lambda m_i (q_i - c), \quad i = 1, \ldots, n, \\
(\lambda - \hat{\lambda}) (q_i - c) &= \left( \frac{m_0}{r_{i0}^3} - \frac{\hat{\lambda} m_0}{m} \right) (q_0 - c), \quad i = 1, \ldots, n,
\end{aligned} \right. $$

where $\lambda = U/I$ and $\hat{\lambda} = \bar{U}/\bar{I}$.

Proof of Theorem 1. Note that equations (5) can be written as

$$ (\lambda - \hat{\lambda}) (q_i - q_0) + \frac{m_0 (q_i - q_0)}{r_{i0}^3} + (\lambda - \frac{\hat{\lambda}}{m} m) (q_0 - c) = 0, \quad i = 1, \ldots, n. $$
Assume that all the particles are on the \( x \)-axis, and use \( x_i \) to denote the position of \( m_i \). The equations become

\[
(7) \quad (\lambda - \overline{\lambda})(x_i - x_0) + \frac{m_0(x_i - x_0)}{|x_i - x_0|^3} + (\lambda - \frac{\overline{\lambda}}{m})(x_0 - c) = 0, \quad i = 1, \ldots, n.
\]

Let \( y_i = x_i - x_0 \), \( \alpha = (\lambda - \frac{\overline{\lambda}}{m})(x_0 - c) \), and \( \beta = \lambda - \overline{\lambda} \). The above equation implies that each of the \( n \) distinct real nonzero values \( \{y_1, \ldots, y_n\} \) satisfies a common algebraic equation,

\[
\beta y + \frac{m_0 y}{|y|^3} + \alpha = 0.
\]

Assume that there are \( k \) particles on the left side of \( m_0 \) and \( n - k \) particles on the right, \( 0 \leq k \leq n \). Then the cubic equation

\[
(8) \quad -m_0 z^3 + \alpha z + \beta = 0,
\]

has at least \( k \) distinct negative roots, and the cubic equation

\[
(9) \quad m_0 z^3 + \alpha z + \beta = 0
\]

has at least \( n - k \) distinct positive roots.

Assume that \( z_1, z_2, z_3 \) are the roots of equation (8) and \( z_4, z_5, z_6 \) are the roots of equation (9). We are going to finish the proof by showing that the sum of the number of negative roots of equation (8) and the number of positive roots of equation (9) is not greater than 2. This is obviously true if \( \alpha = 0 \) or \( \beta = 0 \). So we assume that \( \alpha \neq 0 \) and \( \beta \neq 0 \).

Recall that a generic cubic equation \( az^3 + bz^2 + cz + d = 0 \), \( a \neq 0 \) has only one real root and two conjugate imaginary roots if and only if

\[
\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 < 0.
\]

For our two cubic equations, we have

\[
\Delta_1 = 4m_0\alpha^3 - 27m_0^2\beta^2, \quad \Delta_2 = -4m_0\alpha^3 - 27m_0^2\beta^2.
\]

Obviously, at least one of \( \Delta_1 \) and \( \Delta_2 \) is negative. Without loss of generality, we assume that \( \Delta_1 < 0 \), then only one of \( z_1, z_2, z_3 \) is real, say, \( z_1 \), and the other two are conjugate imaginary numbers.

By Vieta’s formulas, \( z_4 + z_5 + z_6 = 0 \), which implies that at most two of the roots of equation (9) are positive. Thus, we assume that \( z_1 < 0 \). By Vieta’s formulas, \( z_4 z_5 z_6 = -z_1 z_2 z_3 > 0, z_4 + z_5 + z_6 = 0 \), which implies that equation (9) has only one positive root. In words, the sum of the number of negative roots of equation (8) and the number of positive roots of equation (9) is not greater than 2. This completes the proof.
Proof of Theorem 2 The proof of the necessary conditions: First, the \( n \)-body sub configuration must be central. In this case, we have \( \mathbf{q}_0 = c = \bar{c} \), so \( \mathbf{q}_i \neq c, i = 1, \ldots, n \). Then the second part of system (6) holds and the proof is completed. □

The proof of the sufficient conditions: The first part of system (6) evidently holds. Since \( \mathbf{q}_0 = c = \bar{c} \) and \( r_{10} = \ldots = r_{n0} \), we have

\[
\bar{I} = \sum_{i=0}^{n} m_i |\mathbf{q}_i - c|^2 = \sum_{i=1}^{n} m_i |\mathbf{q}_i - c|^2 = I = m r_{10}^2,
\]

\[
\bar{\lambda} = \frac{U + \sum_{i=1}^{n} m_i m_0}{I} = \lambda + m_0 \frac{m}{I_{r_{10}}} = \lambda + m_0 \frac{1}{r_{10}^3},
\]

which implies that \( \frac{1}{|\mathbf{q}_i - \mathbf{q}_0|^3} = \frac{\bar{\lambda} - \lambda}{m_0} \) for \( i = 1, \ldots, n \). Thus, the second part of system (6) holds and the proof is completed.

Proof of Theorem 3 The proof of the necessary conditions: First, the \( n \)-body sub configuration must be central. There exists some body not on the line \( \overline{\mathbf{q}_i} \) since the configuration \( \mathbf{q} \) is non-collinear. Suppose that \( \mathbf{q}_k \notin \overline{\mathbf{q}_0} \), then the \( k \)-th equation of the second part of system (6) holds only if

\[
\frac{1}{|\mathbf{q}_k - \mathbf{q}_0|^3} = \frac{\bar{\lambda} - \lambda}{m_0} = \frac{\bar{\lambda}}{m}.
\]

Note that \( |\mathbf{q}_k - \mathbf{q}_0| = |\mathbf{q}_j - \mathbf{q}_0| \) also holds if \( \mathbf{q}_k \notin \overline{\mathbf{q}_0}, \mathbf{q}_j \in \overline{\mathbf{q}_0} \). By (4), we have

\[
\lambda(\mathbf{q}_i - c) + \frac{m_0(\mathbf{q}_i - \mathbf{q}_0)}{r_{i0}^3} = \bar{\lambda}(\mathbf{q}_i - \bar{c}), i = k, j.
\]

Subtracting the two equations, we obtain

\[
\lambda(\mathbf{q}_k - \mathbf{q}_j) + \frac{m_0(\mathbf{q}_k - \mathbf{q}_0)}{r_{k0}^3} - \frac{m_0(\mathbf{q}_j - \mathbf{q}_0)}{r_{j0}^3} = \bar{\lambda}(\mathbf{q}_k - \mathbf{q}_j),
\]

\[
(\mathbf{q}_k - \mathbf{q}_0) - (\mathbf{q}_j - \mathbf{q}_0) = \frac{\bar{\lambda} - \lambda}{m_0} (\mathbf{q}_k - \mathbf{q}_j) = \frac{(\mathbf{q}_k - \mathbf{q}_0) - (\mathbf{q}_j - \mathbf{q}_0)}{r_{k0}^3},
\]

which implies that \( |\mathbf{q}_k - \mathbf{q}_0| = |\mathbf{q}_j - \mathbf{q}_0| \). Thus, we obtain

\[
\frac{1}{|\mathbf{q}_i - \mathbf{q}_0|^3} = \frac{\bar{\lambda} - \lambda}{m_0} = \frac{\bar{\lambda}}{m}, i = 1, \ldots, n.
\]

The equality \( \frac{\bar{\lambda} - \lambda}{m_0} = \frac{\bar{\lambda}}{m} \) implies that \( \frac{\lambda}{m} = \frac{\bar{\lambda}}{m} \), so we have \( \frac{1}{|\mathbf{q}_i - \mathbf{q}_0|^3} = \frac{\lambda}{m} \) for \( i = 1, \ldots, n \).
The proof of the sufficient conditions: The first part of system (6) obviously holds. With the condition \( \frac{1}{|q_i - q_0|} = \frac{\lambda}{m}, i = 1, \ldots, n \), we obtain
\[
\bar{\lambda} = \bar{U} / (m \bar{I}) = U + \sum_{i=1}^{n} \frac{m_0 m_i}{r_{10}} = \frac{U + \sum_{i=1}^{n} m_0 m_i}{m \bar{I} + \sum_{i=1}^{n} m_0 m_i r_{10}^2}
\]
\[
= \frac{\lambda I + m_0 \lambda r_{10}^2}{m I + m_0 m_r^2} = \frac{\lambda}{m}.
\]

The equality \( \frac{\lambda}{m} = \frac{\bar{\lambda}}{m} \) implies that \( \frac{\lambda}{m} = \frac{\bar{\lambda} - \lambda}{m_0} = \frac{\bar{\lambda}}{m} \), so we have
\[
\frac{1}{|q_i - q_0|^3} = \frac{\bar{\lambda} - \lambda}{m_0} = \frac{\bar{\lambda}}{m}, \quad i = 1, \ldots, n.
\]
Thus, the second part of system (6) holds and the proof is completed.

Proof of Proposition 1. In the collinear case, Theorem 1 implies that it is possible if and only if \( n = 2 \). For the non-collinear case, Theorem 2 and Theorem 3 implies that \( r_{01} = \ldots = r_{0n} = r_{12} = \ldots = r_{1n} = r_{n-1,n} \), which happens only if the \( n + 1 \) bodies form a regular polytope. In \( \mathbb{R}^3 \), that is, the equilateral triangle and the regular tetrahedron. On the other hand, it is well-known that these configurations with arbitrary masses are central. This completes the proof.

Proof of Theorem 4. It is clear from Theorem 2 and Theorem 3.

Proof of Corollary 2. It is clear from the proof of Theorem 3.

Proof of Theorem 5. Assume that \( q_i = (\cos \theta_i, \sin \theta_i) \) and \( 0 = \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi \). The configuration is central if and only if
\[
\sum_{k \neq j} m_k \left( \frac{1}{r_{kj}} - \frac{\lambda}{m} \right) (q_k - q_j) = \sum_{k \neq j} m_k S_{kj} (q_k - q_j) = 0,
\]
for \( j = 1, \ldots, n \), where \( S_{kj} = \frac{1}{r_{kj}} - \frac{1}{r_{i0}} \).

We first show that the two exterior sides incident with \( q_1 \) are smaller than \( r_0 \) by contradiction. Note that the sequence \( \{r_{12}, r_{13}, \ldots, r_{1n}\} \) is either monotonic or at first increasing then decreasing.

Case I: \( r_{12} \geq r_0, \ r_{n1} \geq r_0 \). Then we have
\[
r_{k1} > \min \{r_{12}, r_{n1}\} \geq r_0, \quad \frac{1}{r_{k1}^3} - \frac{1}{r_{i0}^3} < 0, \quad k = 3, \ldots, n - 1.
\]
That is, \( S_{k1} \leq 0 \) for \( k = 2, \ldots, n \). Denote by \( l \) the line perpendicular with the tangent of the circle at \( q_1 \) (the dashed line), see Figure 1,
left, and by $P_l u$ the orthogonal projection of vector $u$ along the line $l$. Then it is easy to see that

$$P_l \left( \sum_{k \neq 1} m_k S_{k1}(q_k - q_1) \right) = \sum_{k \neq 1} m_k S_{k1} P_l(q_k - q_1) \neq 0.$$ 

Therefore, the first equation of system (10), $0 = \sum_{k \neq 1} m_k S_{k1}(q_k - q_1)$, can not hold. This is a contradiction.

**Figure 1.** Co-circular configuration with $r_{12} \geq r_0$, $r_{n1} \geq r_0$, left, and $r_{12} \geq r_0$, $r_{n1} < r_0$, right.

Case II: Only one of the incident exterior side is smaller than $r_0$, say, $r_{12} \geq r_0$, $r_{n1} < r_0$. Suppose that $\theta_2 < \theta_k < \ldots < \theta_L < \theta_{L+1} < \ldots < \theta_n < 2\pi$ and that $r_{1L} \geq r_0$, $r_{1,L+1} < r_0$.

That is, $S_{k1} \leq 0$ for $k = 2, \ldots, L$, and $S_{k1} > 0$ for $k = L + 1, \ldots, n$. Connect $q_1$ and one point between $q_L$ and $q_{L+1}$ on the circle by the dashed line, and denote by $l$ the line perpendicular with the dashed line, see Figure 1, right. Then it is easy to see that

$$P_l \left( \sum_{k=2}^{L} m_k S_{k1}(q_k - q_1) + \sum_{k=L+1}^{n} -m_k S_{k1}(q_1 - q_k) \right) \neq 0.$$ 

Therefore, the first equation of system (10), $0 = \sum_{k \neq 1} m_k S_{k1}(q_k - q_1)$, can not hold. This is a contradiction.

We conclude that the two exterior sides incident with $q_1$ are smaller than $r_0$, i.e., $S_{12} > 0$, $S_{1n} > 0$. If the values $S_{12}, S_{13}, \ldots, S_{1n}$ are all positive, the equation $0 = \sum_{k \neq 1} m_k S_{k1}(q_k - q_1)$ can not hold neither, see Figure 1, left. Thus, there is at least a negative one that must
correspond to a diagonal. Hence, there is at least one diagonal incident with \( q_1 \) that is larger than \( r_0 \).

By symmetry, the statement made for the edges incident with \( q_1 \) also holds for the edges incident with any other vertex. Therefore, we have proved that all the exterior sides are less than \( r_0 \) and that there is at least one incident diagonal larger than \( r_0 \) at each vertex. \( \square \)

**Proof of Corollary 3.** If a co-circular configuration lies entirely in a semi-circle, then there is one exterior side longer than all the diagonals. By Theorem 5, it is not central. \( \square \)

**Proof of Corollary 2.** We only prove for the six-body case. The other cases are similar. Order the six masses sequentially on the circle as in Figure 2. First note that the center of the circle, \( O \), must be in the convex hull of the six masses since the masses are not in a semi-circle. Assume that \( r \geq r_0 \). Then Theorem 5 implies that

\[
r_{12}, r_{23}, r_{34}, r_{45}, r_{56}, r_{61} < r.
\]

This implies that each of the six angles \( \angle q_1 O q_2, \angle q_2 O q_3, \angle q_3 O q_4, \angle q_4 O q_5, \angle q_5 O q_6, \angle q_6 O q_1 \), is strictly less than \( \pi/3 \). It is a contradiction since the sum has to be \( 2\pi \).

\( \square \)

We postpone the proof of Theorem 6 and Proposition 3 to Section 4.

**Proof of Proposition 4.** The proof of the necessary conditions: By Albouy [1], Ouyang-Xie-Zhang [25], the sub configuration \( q \) must be central. Thus, both the pyramidal configuration and the sub configuration

![Figure 2. An example of a co-circular configuration. The center of the circumscribing circle is marked with \( O \).](image)
are central and \(q_0 \neq c\). Then the other conditions follows easily from Theorem 3.

The proof of the sufficient conditions: By Theorem 3 if these conditions are satisfied, the pyramidal configuration \(\vec{q}\) must be central. \(\Box\)

4. THE EXTENSIONS OF TWO, THREE, FOUR, AND MORE BODY CENTRAL CONFIGURATIONS

In this section, we discuss the extensions of \(n\)-body central configurations to \((n+1)\)-body central configurations for small \(n\). If \(n \leq 4\), we understand thoroughly the extensions. If \(n \geq 5\), we can only get some partial results.

4.1. Two bodies to three. There is only one two-body central configuration, namely, a segment with two arbitrary masses at the ends. It is obviously co-circular and the circumscribed circle is not unique. The radius is in the range \([\frac{m_0}{2}, \infty)\).

- I co-circular to planar: It is easy to see that the mass center coincides the geometric center if and only if the two masses are equal. In this case, we could extend it by adding an arbitrary mass \(m_0\) at the center, which is a three-body collinear central configuration.
- II co-circular to planar: Since \(r_0 = r_{12}\), the range of radius of the circumscribed circle is \([\frac{m_0}{2}, \infty)\). It is easy to see that we can extend it by adding one arbitrary mass \(m_0\) on the orthogonal bisector of \(q_1q_2\) such that \(r_{01} = r_{02} = r = r_0 = r_{12}\). The three masses are all arbitrary and the triangle is equilateral. In other words, we have provided another proof of the well-known fact that the equilateral triangle with three arbitrary masses is a central configuration [19].
- III co-circular to pyramidal: Not exist.
- IV and V: Not exist.

The two-body central configurations can also extend to other three-body collinear central configurations. Assume that the central configuration is on the \(x\)-axis, with positions \(x_1, x_2, x_1 < x_2\). For any given mass \(m_0\), it is easy to show that there is a unique position \(x_0\) in each of the three intervals, \((-\infty, x_1), (x_1, x_2)\), and \((x_2, \infty)\), such that the configuration \((x_1, x_2, x_3)\) is central, which is the well-known three-body collinear Eulerian central configurations [10].

For \(n \geq 3\), by Theorem 1 any \(n\)-body central configuration can not extend to an \((n+1)\)-body collinear central configuration. And that an
$n$-body collinear configuration can not extend to an $(n + 1)$-body non-collinear central configuration by the perpendicular bisector theorem [21]. Therefore, we only need to discuss extension of the $n$-body non-collinear central configurations in the following cases.

4.2. **Three bodies to four.** This has been considered by Hampton [17]. In the three-body case, the only non-collinear central configuration is the equilateral triangle with three arbitrary masses, which is co-circular. It is easy to see that $r_{0} = r_{12} = r_{13} = r_{23} = \sqrt{3}r$.

- **I** co-circular to planar: It is easy to see that the mass center coincides the geometric center if and only if the three masses are equal. In this case, we could extend it by adding an arbitrary mass $m_0$ at the center.

- **II** co-circular to planar: Not exist, since $r_0 > r$.

- **III** co-circular to pyramidal: As mentioned above, $r_0 = \sqrt{3}r$ holds for any equilateral triangle central configuration. Thus, any equilateral triangle central configuration can extend to a pyramidal central configuration by adding one arbitrary mass $m_0$ such that $r_{10} = r_{20} = r_{30} = r_0 = r_{12}$. In other words, we have provided another proof of the well-known fact that the regular tetrahedron with four arbitrary masses is a central configuration.

- **IV** and **V**: Not exist.

4.3. **Four bodies to five.** In the four-body case, the only spatial central configuration is the regular tetrahedron with arbitrary masses, which is co-spherical. On the other hand, the co-circular central configurations are very rich, and it has been studied thoroughly by Cors-Roberts [8].

- **I** co-circular to planar: It has been proved by Hampton [16] that there is only one four-body co-circular central configuration with mass center at the geometric center, namely, the square with equal masses. In this case, we could extend it by adding an arbitrary mass $m_0$ at the center.

- **II** co-circular to planar: Not exist, since that $r_0 > r$ for any four-body co-circular central configuration by Corollary 2.

- **III** co-circular to pyramidal: Any four-body co-circular central configuration [8] could extend to a five-body pyramidal central configuration.

- **IV** co-spherical to spatial: It is easy to see that the mass center of the regular tetrahedron central configuration coincides the geometric center if and only if the four masses are equal. In
this case, we could extend it by adding an arbitrary mass $m_0$

at the center.

- V co-spherical to spatial: Not exist, since that $r_0 = r_{12} = \frac{2\sqrt{6}}{3}r$
for any regular tetrahedron central configuration.

This discussion proves Theorem 6.

**Remark 6.** Fernandes-Mello in 2013 [12] also announced that a four-body co-circular central configuration can be extended to a five-body co-planar central configuration if and only if the configuration is a square with equal masses. However, their original proof is incorrect. On page 302 of [12], where the authors claim that the equation $r^3 = \bar{m}/\bar{\lambda}$, (with our notations), leads to a quadratic polynomial in $m_0$. But from the proof of Theorem 3, we see that if $r^3 = m/\lambda$, in which no $m_0$ is involved, then $r^3 = \bar{m}/\bar{\lambda}$ is just an identity for any $m_0$. Chen-Hsiao pointed out this error in 2018 [5]. After a preliminary vision of this paper was completed, we were informed that Fernandes-Mello have corrected their proof in 2018 [15].

Cors-Roberts [8] showed that the four-body co-circular central configurations form a two-dimensional surface. Thus, the five-body pyramidal central configurations also form a two-dimensional surface. The property of the five-body pyramidal central configurations are really rich. We state some properties about them. They are straightforward corollaries of the results in [8].

**Proposition 5.** Not all choices of five positive masses lead to a five-body pyramidal central configuration.

**Proposition 6.** For a five-body pyramidal central configuration, let $m_1, m_2, m_3, m_4$ be the four masses of the co-circular base. If just two of the four masses are equal, then the base configuration is symmetric, either a kite or an isosceles trapezoid. If any three of the four masses are equal, then the base configuration is a square and all four masses are necessarily equal.

4.4. **Five and more bodies.** In the five and more body case, both the co-circular and co-spherical central configurations are rich, but much less research has been done in this direction. We only state some results known to us.

- I co-circular to planar: Obviously, the regular $n$-gon ($n \geq 5$) with equal masses are examples. We could extend them by adding an arbitrary mass $m_0$ at the center. However, until now, we do not know that whether there exist other examples or not, since the question of Chenciner remains unsolved for $n \geq 5$, see the comment after Theorem 4.
• II co-circular to planar: Not exist for five and six-body case, since that $r_0 > r$ in these cases by Corollary \textsuperscript{2}. For more bodies, we have not found any such example yet.

• III co-circular to pyramidal: Any five and six-body co-circular central configuration could extend to a six and seven-body pyramidal central configuration. For more bodies, we have no general results, see Subsection \textsuperscript{5.1}

• IV co-spherical to spatial: For the five to ten-body cases, We have some examples of co-spherical central configurations whose mass center equals the geometric center, see Section \textsuperscript{5.2}. In those cases, we could extend it by adding an arbitrary mass $m_0$ at the center.

• V co-spherical to spatial: For the five and more body case, We do not know that whether there exist co-spherical central configurations with $r = r_0$ or not.

This discussion proves Proposition \textsuperscript{3}

5. Regular polygons and some examples of co-spherical central configurations

In this section, we discuss the regular polygonal central configurations and construct some co-spherical central configurations. Some of them have mass center at the sphere center. Thus, they can extend by adding one mass at the center.

5.1. Regular polygons. Consider the regular $n$-gon with equal masses. Obviously, they can extend to planar central configurations by adding one mass at the center. Whether they can extend to pyramidal central configurations depends on the measurement of $r_0$ and $r$. The following result was first showed by Ouyang-Xie-Zhang \textsuperscript{25}.

**Proposition 7.** The regular $n$-gon with equal masses can extend to pyramidal central configurations if and only if $n \leq 472$.

**Proof.** Assume that $m_k = 1$, the radius of the circle is 1, and that the positions are $q_k = e^{\sqrt{-1}\theta_k}$, $\theta_k = \frac{2k\pi}{n}$, $k = 1, ..., n$. Then we have

$$
 r_0^3 = \frac{\sum_{j,k=1}^{n} m_j m_k r_{jk}^2}{\sum_{j,k=1, j \neq k}^{n} m_j m_k / r_{jk}} = \frac{\sum_{k=1}^{n-1} |1 - \rho_k|^2}{\sum_{k=1}^{n-1} |1 - \rho_k|} = \frac{n}{A(n)},
$$

where $A(n) = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{|1 - \rho_k|} = \frac{1}{4} \sum_{k=1}^{n-1} \csc \left( \frac{k\pi}{n} \right)$. 

It has been found by Moeckel-Simó [23] that \( \frac{n}{A(n)} \) is decreasing, and that the asymptotic expansion of \( \frac{A(n)}{n} \) for \( n \) large is
\[
\frac{A(n)}{n} \sim \frac{1}{2\pi} \left( \gamma + \log \frac{2n}{\pi} \right) + \sum_{k \geq 1} \frac{(-1)^k (2^{2k-1} - 1) B_{2k}^2 \pi^{2k-1}}{(2k)(2k)! n^{2k}},
\]
where \( \gamma \) and \( B_{2k} \) stand for the Euler-Mascheroni constant and the Bernoulli numbers respectively.

Computation by Matlab shows
\[
\frac{A(472)}{472} \approx 1.001, \quad \text{while} \quad \frac{A(473)}{473} \approx 0.9998.
\]
So \( r_0 = \left( \frac{n}{A(n)} \right)^{\frac{1}{3}} > r \) if and only if \( n \leq 472 \). This completes the proof. \( \square \)

5.2. **Co-spherical central configurations.** There are much less research on co-spherical central configurations, compared with the co-circular ones. Corbera-Llibre-Pérez [7] constructed three families of central configurations, each consisting of a regular polyhedron and its dual. In each family, there is a co-spherical one, and the mass center equals its geometric center.

We construct some co-spherical central configurations related with some co-circular ones. Let us introduce some **notations** that will be used only in this subsection. For co-circular central configurations, we denote by \( r, r_0 \) the radius of the circumscribing circle and the cubic root of the ratio of total mass and the multiplier respectively. These planar configurations will extend to co-spherical central configurations. For the co-spherical ones, we denote by \( R, R_0 \) the radius of the circumscribing sphere and the cubic root of the ratio of total mass and the multiplier respectively. In this subsection, the mass at the top vertex of an \( (n+1) \)-body pyramidal configuration is denoted by \( m_{n+1} \) and the position by \( q_{n+1} \).

We want to construct co-spherical central configurations that can extend by way IV and V of Theorem 4. That is, we want the mass center to be at the sphere center, or, \( R_0 = R \).

5.2.1. **Pyramidal central configurations.** Recall that an \( (n+1) \)-body pyramidal central configuration is obtained by adding one arbitrary mass \( m_{n+1} \) to a co-circular central configuration with the property \( r_0 > r \). The top vertex \( q_{n+1} \) is on the orthogonal axis passing through the center of the circle, and the height is \( h = \sqrt{r_0^2 - r^2} \). Obviously, pyramidal configurations are co-spherical.
Proposition 8. Let \( q \) be an \((n + 1)\)-body pyramidal central configuration. Assume that the mass center of the co-circular base is at the center of the circumscribing circle. We can choose \( m_{n+1} \) such that the mass center of the pyramidal central configuration coincides with the circumscribing sphere center if and only if \( r_0 > \sqrt{2}r \).

\[ \text{Figure 3. Slice of a pyramidal central configuration.} \]

The centre of the circumscribing sphere is marked with \( O \).

\[ \text{Proof.} \] Obviously, the sphere center \( O \) is between the base and \( m_{n+1} \) if and only if \( h = \sqrt{r_0^2 - r^2} > r \), see Figure 3, left. That is, \( r_0 > \sqrt{2}r \). The mass center is always between the base and \( m_{n+1} \). Thus, to make the mass center equals the sphere center, we must have \( r_0 > \sqrt{2}r \).

On the other hand, if \( r_0 > \sqrt{2}r \), it is easy to find \( m_{n+1} \) such that the two centers equal. \( \square \)

Proposition 9. Let \( q \) be an \((n + 1)\)-body pyramidal central configuration extended from an \( n \)-body co-circular central configuration. Then \( R_0 = R \) if and only if \( r_0 = \frac{2}{\sqrt{3}}r \).

\[ \text{Proof.} \] By Corollary [2], we see that \( R_0 = r_0 \). Note that \( r_0 = R \) if and only if that \( \alpha = 60^\circ \), or, \( \sin \alpha = \frac{r}{r_0} = \frac{\sqrt{3}}{2} \), see Figure 3, right. \( \square \)

Examples: Consider the central configurations of regular \( n \)-gon with equal masses. Suppose that the \( m_1 = 1 \), the radius of the circle is 1, and that the positions are \( q_k = e^{\sqrt{-7\theta_k}}, \ \theta_k = \frac{2k\pi}{n}, \ k = 1, ..., n \). Recall that \( \frac{r_0}{r} = \left( \frac{n}{A(n)} \right)^{\frac{1}{3}} \), and it is decreasing with respect to \( n \).

Computation by Matlab shows

\[
\left( \frac{8}{A(8)} \right)^{\frac{1}{3}} > \sqrt{2}, \left( \frac{9}{A(9)} \right)^{\frac{1}{3}} < \sqrt{2}, \left( \frac{52}{A(52)} \right)^{\frac{1}{3}} > \frac{2}{\sqrt{3}}, \left( \frac{53}{A(53)} \right)^{\frac{1}{3}} < \frac{2}{\sqrt{3}}.
\]
We can draw two conclusions about the \((n + 1)\)-body pyramidal central configurations extended from the regular \(n\)-gon central configurations \((n \leq 472)\), see Figure 4 left.

1. \(R_0 \neq R\);
2. Only for \(n = 3, 4, 5, 6, 7, 8\), we can choose a proper top mass to make that the mass center of the pyramidal central configuration coincides with the sphere center. They can extend to \((n+2)\)-body central configuration by adding one arbitrary mass at the center. As commented after Proposition [4], the total collision solutions associated with them are perverse solutions of the \((n+2)\)-body problem [6]. This was noticed first by Ouyang-Xie-Zhang [25].

5.2.2. **Bi-Pyramidal central configurations.** By bi-pyramidal configurations, we mean configurations of \(n + 2\) bodies of which \(n\) bodies are co-planar and the other two being off the plane and in opposite directions. The regular \(n\)-gon with equal masses also generates \((n+2)\)-body bi-pyramidal co-spherical central configurations. Similar construction has been considered by Zhang-Zhou [28].

Place the \(n\)-gon with equal masses on the equator, and two equal masses at the north and south pole, see Figure 4 right. Assume that the masses are \(m_1 = 1, ..., m_n = 1, m_{n+1} = m_{n+2} = a\), and the positions are \(q_k = (\cos \theta_k, \sin \theta_k, 0), \quad \theta_k = \frac{2k\pi}{n}, k = 1, ..., n, \quad q_{n+1} = (0, 0, 1), \quad q_{n+2} = (0, 0, -1).\)

**Figure 4.** Examples of co-spherical central configuration generated from the equilateral triangle configuration.
The mass center is at the origin. The symmetry reduces the central configuration equations to the following system,
\[- \lambda q_1 = \sum_{i=2}^{n} \frac{q_i - q_1}{r_{k1}^3} + a \frac{q_{n+1} - q_1}{r_{n+1,1}^3} + a \frac{q_{n+2} - q_1}{r_{n+1,1}^3} = -(A(n) + \frac{a}{\sqrt{2}})q_1,\]
\[- \lambda q_{n+1} = \sum_{i=1}^{n} \frac{q_k - q_{n+1}}{r_{n+1,k}^3} + a \frac{q_{n+2} - q_{n+1}}{r_{n+1,n+2}^3} = -(\frac{n}{2\sqrt{2}} + \frac{a}{4})q_{n+1}.\]

Here we use the fact that \(r_{n+1,1} = \ldots = r_{n+1,n} = \sqrt{2}, r_{n+1,n+2} = 2\) and that the sub configuration on the equator is central,
\[\sum_{i=2}^{n} \frac{q_k - q_1}{r_{k1}^3} = -\frac{1}{r_0^3} (\sum_{i=1}^{n} m_i) q_1 = -\frac{A(n)}{n} n q_1.\]

The system holds for positive \(a = m_{n+1}\) if and only if \(\frac{n}{A(n)} > 2\sqrt{2}\). We have showed that this happens if and only if \(3 \leq n \leq 8\).

**Proposition 10.** The bi-pyramidal \((n + 2)\)-body configurations constructed above are central with positive masses if and only if \(3 \leq n \leq 8\).

For all of them, the mass center equals the sphere center. Thus, they extend to \((n + 3)\)-body configurations by adding one arbitrary mass at the center. Direct computation shows \(R_0 > R = 1\) for all of them.

6. Conclusions

We have classified the extensions of \(n\)-body central configurations to \((n + 1)\)-body central configurations in \(R^3\). For the collinear case, the extensions happen only if \(n = 2\), so it is well understood. For the non-collinear case, the \(n\)-body central configurations must be co-circular or co-spherical. The co-circular (co-spherical) central configurations can extend if the mass center equals the geometric center, or \(r_0 \geq r\) \((r_0 = r\) for the co-spherical case). We also obtain a property on the value of \(r_0\) for co-circular central configurations. This enables us to prove the inequality \(r_0 > r\) for all four, five and six-body co-circular central configurations. We solve the two questions of Hampton completely. It might be worth noting that most of our proof remains valid for more general potentials and higher dimensional spaces.

There exist many research works on co-circular central configurations. We hope that this work may spark similar interest to the co-spherical ones. The value \(r_0 = (\frac{m}{\lambda})^{\frac{1}{3}}\) has showed its importance in the study of four and five-body planar convex central configurations [4, 22]. Our work reveals its another role in the study of central configurations. Many questions arise for the value \(r_0\). For example, except from the
trivial case $n = 2$, do there exist co-circular or co-spherical central configurations with the property $r_0 = r$? Can one obtain some general property of $r_0$ for the co-spherical central configurations? We hope to explore some of these questions in future work.

7. ACKNOWLEDGMENTS

Xiang Yu is supported by NSFC(No.11701464) and the Fundamental Research Funds for the Central Universities (No. JBK1805001). Shuqiang Zhu is supported by NSFC(No.11801537, No.11721101) and the Fundamental Research Funds for the Central Universities (No.WK0 010450010).

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