STRICT SINGULARITY OF WEIGHTED COMPOSITION OPERATORS ON DERIVATIVE HARDY SPACES

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Abstract. We prove that the weighted composition operator $W_{\phi, \varphi}$ fixes an isomorphic copy of $\ell^p$ if the operator $W_{\phi, \varphi}$ is not compact on the derivative Hardy space $S^p$. In particular, this implies that the strict singularity of the operator $W_{\phi, \varphi}$ coincides with the compactness of it on $S^p$. Moreover, when $p \neq 2$, we characterize the conditions for those weighted composition operators $W_{\phi, \varphi}$ on $S^p$ which fix an isomorphic copy of $\ell^2$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p$ is the space of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} := \left( \sup_{0 < r < 1} \int_{\partial \mathbb{D}} |f(r\xi)|^p dm(\xi) \right)^{1/p} < \infty,$$

where $m$ is the normalized Lebesgue measure on $\partial \mathbb{D}$. From [23, Theorem 9.4], this norm is equal to the following norm:

$$\|f\|_{H^p} = \left( \int_{\partial \mathbb{D}} |f(\xi)|^p dm(\xi) \right)^{1/p},$$

where for any $\xi \in \partial \mathbb{D}$, $f(\xi)$ is the radial limit which exists almost every.

For $p = \infty$, the space $H^\infty$ is defined by

$$H^\infty = \{ f \in H(\mathbb{D}) : \|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty \}.$$

We define the derivative Hardy space $S^p$ by

$$S^p = \{ f \in H(\mathbb{D}) : \|f\|_{S^p} := |f(0)| + \|f'|_{H^p} < \infty \}.$$

For $1 \leq p \leq \infty$, $S^p$ is a Banach algebra and there is an inclusion relation: $S^p \subset H^\infty$ (for the detail structures of $S^p$ spaces, see [6, 7, 12, 13, 16] for references).

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In paper [22], Roan started the investigation of composition operators on the space $S^p$. After his work, MacCluer [18] gave the characterizations of the boundedness and the compactness of the composition operators on the space $S^p$ in terms of Carleson measures. A remarkable result on the boundedness and the compactness of the weighted composition operators on $S^p$ was obtained in [2], in which they are both characterized through the corresponding weighted composition operators on $H^p$. Furthermore, the isometry between $S^p$ was obtained by Novinger and Oberlin in [21], in which they showed that the isometries were closely related to the weighted composition operator.

A bounded operator $T: X \to Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its image. This notion was introduced by Kato [13].

A bounded operator $T: X \to Y$ between Banach spaces is said to fix a copy of the given Banach space $E$ if there is a closed subspace $M \subset X$, linearly isomorphic to $E$, such that the restriction $T|_M$ defines an isomorphism from $M$ onto $T(M)$. The bounded operator $T: X \to Y$ is called $\ell^p$-singular if it does not fix any copy of $\ell^p$.

Laitila, et al [14] recently investigated the strict singularity for the composition operators on $H^p$ spaces. Following their ideas, Mihkkinen [19] studied the strict singularity of $T_g$ on Hardy space $H^p$ and showed that the strict singularity of $T_g$ coincides with its compactness on $H^p$, $1 \leq p < \infty$. Mihkkinen [19] also post an open question which was resolved in [20] by utilizing the generalized Volterra operators. It should be noticed that Hernández, et al [11] investigated the interpolation and extrapolation of strictly singular operators between $L^p$ spaces.

In this paper, we prove that the weighted composition operator $W_{\phi,\varphi}$ fixes an isomorphic copy of $\ell^p$ if the operator $W_{\phi,\varphi}$ is not compact on the derivative Hardy space $S^p$. In particular, this implies that the strict singularity of the operator $W_{\phi,\varphi}$ coincides with the compactness of it on $S^p$. Moreover, when $p \neq 2$, we characterize the conditions for those weighted composition operators $W_{\phi,\varphi}$ on $S^p$ which fix an isomorphic copy of $\ell^2$.

Our main results read as follows:

**Theorem 1.** Let $1 \leq p < \infty$, $\phi \in H(D)$ and $\varphi$ is an analytic self-map of $D$. If the weighted composition operator $W_{\phi,\varphi}: S^p \to S^p$ is bounded but not compact, then $W_{\phi,\varphi}$ fixes an isomorphic copy of $\ell^p$ in $S^p$. In particular, the operator $W_{\phi,\varphi}$ is not strictly singular, that is, strict singularity of bounded operator $W_{\phi,\varphi}$ coincides with its compactness.

**Remark 1.** In the final section, we prove that the claims of theorem [1] is still true for the case of $p = \infty$.

Denote $E_\varphi = \{\zeta \in \partial D : |\varphi(\zeta)| = 1\}$, then we have

**Theorem 2.** Let $1 \leq p < \infty$, $\phi \in H(D)$ and $\varphi$ is an analytic self-map of $D$. Suppose that $W_{\phi,\varphi}: S^p \to S^p$ is bounded and $m(E_\varphi) = 0$. If $W_{\phi,\varphi}$ is bounded below on an infinite-dimensional subspace $M \subset S^p$, then the restriction $W_{\phi,\varphi}$ on $M$ fixes an isomorphic copy of $\ell^p$ in $M$. In particular, if $p \neq 2$, the operator $W_{\phi,\varphi}$ does not fix any isomorphic copy of $\ell^2$ in $S^p$.

When $m(E_\varphi) > 0$, it holds that
Theorem 3. Let $1 \leq p < \infty$, $\phi \in H(D)$ and $\varphi$ is an analytic self-map of $D$. Suppose that $W_{\phi,\varphi} : S^p \to S^p$ is bounded. If $m(E_\varphi) > 0$ and $\phi \varphi' \neq 0$, then the operator $W_{\phi,\varphi}$ fixes an isomorphic copy of $l^2$ in $S^p$.

Notation: throughout this paper, $C$ will represent a positive constant which may be given different values at different occurrences.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. First, the following Lemma can be deduced from [2, Theorem 2.1] and [9, Theorem 2.2 and Theorem 2.3].

Lemma 1. Let $1 \leq p < \infty$, $\phi \in H(D)$ and $\varphi$ is an analytic self-map of $D$. Then $W_{\phi,\varphi} : S^p \to S^p$ is compact if and only if $\phi \in S^p$ and

$$
\lim_{|a| \to 1^{-}} \int_{\partial D} \frac{1 - |a|^2}{|1 - \overline{a}\varphi(\omega)|^2} |\psi(\omega)\varphi'(\omega)|^p dm(\omega) = 0.
$$

The following lemma is proven in [2, Proposition 3.3(ii)].

Lemma 2. Let $1 \leq p \leq \infty$, $\phi \in H^p(D)$ and $\varphi$ is an analytic self-map of $D$. Then $W_{\phi,\varphi} : S^p \to H^p$ is compact.

We employ the test functions

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^{1/p}}{(1 - \overline{a}\omega)^{2/p}} d\omega, \quad z \in \mathbb{D},$$

where $a \in \mathbb{D}$. They all satisfy $\|f_a\|_{S^p} = 1$ and $f_a$ converges to $0$ uniformly on compact subsets of $\mathbb{D}$, as $|a| \to 1^-$.

Let $L = \{ \xi \in \partial \mathbb{D} : \text{the radial limit } \varphi(\xi) \text{ exists} \}$ and

$$E_\varepsilon := \{ \xi \in L : |1 - \varphi(\xi)| < \varepsilon \}
$$

for any given $\varepsilon > 0$, then $m(\partial \mathbb{D} \setminus L) = 0$. The proof of Theorem 1 relies on the following auxiliary lemma.

Lemma 3. Let $(a_n) \subset \mathbb{D}$ be a sequence such that $0 < |a_1| < |a_2| < \ldots < 1$ and $a_n \to 1$. If the bounded operator $W_{\phi,\varphi} : S^p \to S^p$ is not compact, then we have

$$
\begin{align*}
(1) \lim_{\varepsilon \to 0} \int_{E_\varepsilon} |(W_{\phi,\varphi} f_n)^p| dm = 0 \quad &\text{for every } n \in \mathbb{N}, \\
(2) \lim_{n \to \infty} \int_{\partial \mathbb{D} \setminus E_\varepsilon} |(W_{\phi,\varphi} f_n)^p| dm = 0 \quad &\text{for every } \varepsilon > 0.
\end{align*}
$$

Proof. (1) For each fixed $n$, this follows immediately from the absolute continuity of Lebesgue measure, the boundedness of operator $W_{\phi,\varphi}$ and the fact that $W_{\phi,\varphi}$ is not compact (which implies that $\varphi$ is not identically $1$).

(2) For any given $\varepsilon > 0$, let $\xi \in L \setminus E_\varepsilon$. Then there exists an $N > 0$ such that whenever $n > N$, it holds that

$$
|1 - a_n \varphi(\xi)| = |1 - \varphi(\xi) + \varphi(\xi) - a_n \varphi(\xi)|
\geq |1 - \varphi(\xi)| - |\varphi(\xi) - a_n \varphi(\xi)|
\geq |1 - \varphi(\xi)| - |1 - a_n| > \frac{\varepsilon}{2}.
$$
Now, by definition, we have
\[
\int_{\partial D \setminus E_s} |(W_{\phi, \varphi} f_{a_n})'|^p \, dm \leq C \left( \int_{\partial D \setminus E_s} |\phi' f_{a_n}(\varphi)|^p \, dm + \int_{\partial D \setminus E_s} |\phi \varphi' f_{a_n}'(\varphi)|^p \, dm \right).
\]
Since \( W_{\phi, \varphi} \) is bounded, it follows that \( \phi \in S^p \), that is, \( \phi' \in H^p \). By Lemma 2, \( W_{\phi', \varphi} : S^p \to H^p \) is compact, which implies that
\[
\lim_{n \to \infty} \int_{\partial D \setminus E_s} |\phi' f_{a_n}(\varphi)|^p \, dm \leq \lim_{n \to \infty} \int_{\partial D} |W_{\phi', \varphi} f_{a_n}|^p \, dm = 0.
\]
For the estimate of the second integral, we have
\[
\int_{\partial D \setminus E_s} |\phi \varphi' f_{a_n}'(\varphi)|^p \, dm = \int_{\partial D \setminus E_s} |\phi \varphi'|^p \frac{1 - |a_n|^2}{|1 - \bar{a_n} \varphi|^2} \, dm
\leq \frac{4(1 - |a_n|^2)}{\varepsilon^2} \int_{\partial D} |\phi \varphi'| \, dm,
\]
where \( \int_{\partial D} |\phi \varphi'| \, dm \) is finite due to the boundedness of \( W_{\phi, \varphi} : S^p \to S^p \) and [2] Theorem 2.1 and [3] Theorem 4.

Therefore,
\[
\lim_{n \to \infty} \int_{\partial D \setminus E_s} |\phi \varphi' f_{a_n}'(\varphi)|^p \, dm = 0.
\]
The proof is complete. \( \square \)

Now, we are ready to give a proof of Theorem 1.

**Proof of Theorem 1.** First, we prove that there exists a sequence \( (a_n) \subset \mathbb{D} \) with \( 0 < |a_1| < |a_2| < \ldots < 1 \) and \( a_n \to \omega \in \partial \mathbb{D} \), such that there is a positive constant \( h \) such that
\[
\|W_{\phi, \varphi}(f_{a_n})\|_{H^p} \geq h > 0
\]
holds for all \( n \in \mathbb{N} \).

Since \( W_{\phi, \varphi} : S^p \to S^p \) is not compact, Lemma 1 there exists a sequence \( (a_n) \subset \mathbb{D} \) with \( 0 < |a_1| < |a_2| < \ldots < 1 \) and \( a_n \to \omega \in \partial \mathbb{D} \), such that there is a positive constant \( h \) such that \( \|\phi \varphi' f_{a_n}'(\varphi)\|_{H^p} \geq 2h > 0 \) holds for all \( n \in \mathbb{N} \). Note that
\[
\|W_{\phi, \varphi}(f_{a_n})\|_{H^p} \geq \|\phi \varphi' f_{a_n}'(\varphi)\|_{H^p} - \|\phi' f_{a_n}(\varphi)\|_{H^p}.
\]
By Lemma 2 \( W_{\phi', \varphi} : S^p \to H^p \) is compact, which implies that
\[
\lim_{n \to \infty} \|\phi' f_{a_n}(\varphi)\|_{H^p} = 0.
\]
Hence, there exists a subsequence of \( (a_n) \) (denoted still by \( (a_n) \)) such that the above claim holds. We assume without loss of generality that \( a_n \to 1 \) as \( n \to \infty \) by utilizing a suitable rotation.

Then by Lemma 3 and induction method, we are able to choose a decreasing positive sequence \( (\varepsilon_n) \) such that \( E_{\varepsilon_1} = \partial \mathbb{D} \) and \( \lim_{n \to \infty} \varepsilon_n = 0 \), and a subsequence...
(\(b_n\)) \(\subset\) (\(a_n\)) such that the following three conditions hold:

1. \(\left( \int_{\mathbb{E}_n} |(W_{\phi,\varphi}f_b_k)'|^p \, dm \right)^{1/p} < 4^{-n}\delta_h, \quad k = 1, \ldots, n - 1;\)
2. \(\left( \int_{\partial D \setminus \mathbb{E}_n} |(W_{\phi,\varphi}f_b_n)'|^p \, dm \right)^{1/p} < 4^{-n}\delta_h;\)
3. \(\left( \int_{\mathbb{E}_n} |(W_{\phi,\varphi}f_b_n)'|^p \, dm \right)^{1/p} > \frac{h}{2}\)

for every \(n \in \mathbb{N}\), where \(\delta > 0\) is a small constant whose value will be determined later.

Now we are ready to prove that there is a \(C > 0\) such that the inequality \(\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \|_{S^p} \geq C \| (c_j) \|_{\ell^p}\) holds. By the triangle inequality in \(L^p\), we have

\[
\left\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \right\|_{S^p}^p \geq \left\| \sum_{j=1}^{\infty} (c_j W_{\phi,\varphi}(f_{b_j}))' \right\|_{L^p}^p
\]

\[
= \sum_{n=1}^{\infty} \int_{E_n \setminus E_{n+1}} \left| \sum_{j=1}^{\infty} (c_j W_{\phi,\varphi}(f_{b_j}))' \right|^p \, dm
\]

\[
\geq \sum_{n=1}^{\infty} |c_n| \left( \int_{E_n \setminus E_{n+1}} |(W_{\phi,\varphi}(f_{b_n}))'|^p \, dm \right)^{\frac{1}{p}}
\]

\[
- \sum_{j \neq n} |c_j| \left( \int_{E_n \setminus E_{n+1}} |(W_{\phi,\varphi}(f_{b_j}))'|^p \, dm \right)^{\frac{1}{p}}
\]

Observe that for every \(n \in \mathbb{N}\), we have

\[
\left( \int_{E_n \setminus E_{n+1}} |(W_{\phi,\varphi}(f_{b_n}))'|^p \, dm \right)^{\frac{1}{p}}
\]

\[
= \left( \int_{E_n} |(W_{\phi,\varphi}(f_{b_n}))'|^p \, dm - \int_{E_{n+1} \setminus E_n} |(W_{\phi,\varphi}(f_{b_n}))'|^p \, dm \right)^{1/p}
\]

\[
\geq \left( \left( \frac{h}{2} \right)^p - (4^{-n-1}\delta_h)^p \right)^{1/p}
\]

\[
\geq \frac{h}{2} - 4^{-n-1}\delta_h
\]

according to conditions (1) and (3) above, where the last estimate holds for \(1 \leq p < \infty\).

Moreover, by condition (1) and (2), it holds that

\[
\left( \int_{E_n \setminus E_{n+1}} |(W_{\phi,\varphi}(f_{b_j}))'|^p \, dm \right)^{\frac{1}{p}} < 2^{-n-1}\delta_h \quad \text{for} \ j \neq n.
\]
Consequently, we obtain that
\[
\left\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \right\|_{S^p} \geq \left( \sum_{n=1}^{\infty} \left( |c_n| \left( \frac{h}{2} - 2^{-n-1} \delta h \right) - 2^{-n} \delta h \|c_j\|_{\ell^p} \right)^p \right)^{1/p} \\
\geq \left( \sum_{n=1}^{\infty} \left( |c_n| \left( \frac{h}{2} - 2^{-n+1} \delta h \|c_j\|_{\ell^p} \right)^p \right) \right)^{1/p} \\
\geq \frac{h}{2} \|c_j\|_{\ell^p} - \delta h \|c_j\|_{\ell^p} \left( \sum_{n=1}^{\infty} 2^{-n-1} \right)^{1/p} \\
\geq h \left( \frac{1}{2} - \delta \left( 1 - 2^{-p} \right) \right) \|c_j\|_{\ell^p} \geq C \|c_j\|_{\ell^p},
\]
where the last inequality holds when we choose \( \delta \) small enough.

On the other hand, we are to prove the converse inequality:
\[
\left\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \right\|_{S^p} \leq C \|c_j\|_{\ell^p}.
\]

By definition,
\[
\left\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \right\|_{S^p} = \left\| \sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0)) \right\| + \left\| \sum_{j=1}^{\infty} (c_j W_{\phi,\varphi}(f_{b_j}))' \right\|_{H^p}.
\]

First, we note that a straightforward variant of the above procedure also gives
\[
\left\| \sum_{j=1}^{\infty} (c_j W_{\phi,\varphi}(f_{b_j}))' \right\|_{H^p} \leq C \|c_j\|_{\ell^p}.
\]

Next, when \( p = 1 \), since \( \lim_{j \to \infty} f_{b_j}(\varphi(0)) = 0 \), it is trivial that
\[
\left\| \sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0)) \right\| \leq C \|c_j\|_{\ell^1}.
\]

When \( 1 < p < \infty \), we can choose a subsequence of \((b_j)\) (still denoted by \((b_j)\)) such that \( \{(1 - |b_j|^2)^{1/p}\}_{j=1}^{\infty} \in \ell^q \), where \( 1/p + 1/q = 1 \). Then by Hölder’s inequality,
\[
\left\| \sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0)) \right\| \leq C \|c_j\|_{\ell^p}.
\]

Accordingly, the desired inequality follows.

By choosing \( \phi = 1 \) and \( \varphi = z \), we obtain that
\[
C \|c_j\|_{\ell^p} \leq \left\| \sum_{j=1}^{\infty} c_j f_{b_j} \right\|_{S^p} \leq C \|c_j\|_{\ell^p}.
\]

Thus, we have
\[
\left\| \sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j}) \right\|_{S^p} \geq C \sum_{j=1}^{\infty} \left\| c_j f_{b_j} \right\|_{S^p}
\]

The proof is complete. \( \square \)
3. Proof of Theorem 2

In this section, we give the proof of Theorem 2.

**Proof of Theorem 2.** Since $M$ is the infinite-dimensional subspace of $S^p$ and polynomials are dense in $S^p$ (see [15]), there exists a sequence $(f_n)$ of unit vectors in $M$ such that $f_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since $W_{\phi,\varphi}$ is bounded below on $M \subset S^p$, there exists $h > 0$ such that

$$\|W_{\phi,\varphi}f_n\|_{S^p} > h,$$

for all $n \in \mathbb{N}$. For $k \geq 1$, denote $E_k := \{ \xi \in \partial \mathbb{D} : |\varphi(\xi)| \geq 1 - 1/k \}$. Since by assumption, $\lim_{k \to \infty} m(E_k) = m(E_{\varphi}) = 0$, it holds that

$$\lim_{k \to \infty} \int_{E_k} |(W_{\phi,\varphi}f_n)'|^p dm = 0$$

for every $n \in \mathbb{N}$. Moreover, since $f_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\lim_{n \to \infty} \int_{\partial \mathbb{D} \setminus E_k} |(W_{\phi,\varphi}f_n)'|^p dm = 0$$

for every $k \in \mathbb{N}$.

The remainder of the proof is an argument that goes exactly as the proof of Theorem 1, so we omit it. Thus, the proof is complete. □

4. Proof of Theorem 3

In this last section, we give a proof for Theorem 3.

**Proof of Theorem 3.** We define the subspace $S^p_0$ of $S^p$ by

$$S^p_0 := \{ f \in S^p : f(0) = 0 \}.$$

Then the integral operator $T_z : f \mapsto \int_0^z f(\zeta) d\zeta$ is an isometric isomorphism from $H^p$ onto $S^p_0$. Then the weighted composition operator $W_{\phi,\varphi}$ on $S^p_0$ is unitary similar to the operator

$$T := W_{\phi,\varphi} \circ T_z + W_{\phi',\varphi} \circ T_z$$

on $H^p$.

By Lemma 2 and the expression of the operator $T$, we see that $W_{\phi,\varphi}$ is bounded on $S^p$ if and only if $\phi \in S^p$ and $W_{\phi,\varphi}$ is bounded on $S^p_0$. Now, for the operator $W_{\phi,\varphi}$ on $H^p$, we can deduce from the proof of [17, Theorem 2] that there exists a sequence of integers $(n_k)$ satisfying $\inf_k (n_{k+1}/n_k) > 1$ and a positive constant $C$ such that

$$\left\| \sum_k c_k W_{\phi,\varphi}(e_{n_k}) \right\|_{H^p} \geq C \| (c_k) \|_{l^2},$$

where $e_{n_k} := z^{n_k}$ is the unit vector in $H^p$. Since the operator $W_{\phi,\varphi} \circ T_z : H^p \to H^p$ is compact (it is equivalent to the compactness of $W_{\phi,\varphi} : S^p \to H^p$, which is claimed by Lemma 2), then for any $\varepsilon > 0$, there exists a subsequence of $(n_k)$ (still denoted as $(n_k)$) such that

$$\left\| \sum_k c_k W_{\phi',\varphi} \circ T_z(e_{n_k}) \right\|_{H^p} \leq \varepsilon \| (c_k) \|_{l^2}.$$

Thus,

$$\left\| \sum_k c_k T(e_{n_k}) \right\|_{H^p} \geq C \| (c_k) \|_{l^2},$$
which implies that the weighted composition operator $W_{\phi,\varphi}$ on $S_0^p$ is bounded below:

$$\| \sum_k c_k W_{\phi,\varphi}(g_{n_k}) \|_{S^p} \geq C \| (c_k) \|_{\ell^2},$$

where $g_{n_k} := T_z(e_{n_k})$ is the unit vector in $S_0^p$.

Since Paley’s theorem (see [10]) implies that the closed linear span $M := \{ e_{n_k} : k \geq 1 \}$ in $H^p$ is isomorphic to $\ell^2$, which implies that the closed linear span $T_z(M) = \{ g_{n_k} : k \geq 1 \}$ in $S_0^p$ is isomorphic to $\ell^2$. Hence, $W_{\phi,\varphi}$ fixes an isomorphic copy of $\ell^2$ in $S_0^p$. Accordingly, it follows that $W_{\phi,\varphi}$ fixes an isomorphic copy of $\ell^2$ in $S^p$ since $S_0^p \subset S^p$ and $\| W_{\phi,\varphi} \|_{S^p} \geq \| W_{\phi,\varphi} \|_{S_0^p}$, which is the desired result. \hfill \Box

5. The strict singularity of $W_{\phi,\varphi}$ on $S^\infty$

Here we show that the claims of theorem [1] is still true for the case of $p = \infty$. We have known that the weighted composition operator $W_{\phi,\varphi}$ on $S_0^\infty$ is unitary similar to the operator

$$T := W_{\phi',\varphi'} \circ T_z + W_{\phi,\varphi'} \circ W_{\phi',\varphi'} \circ T_z$$

on $H^\infty$. It follows from [1] that any weakly compact weighted composition operator on $H^\infty$ is compact. Since by Lemma 2 the operator $W_{\phi,\varphi'}$ is compact on $H^\infty$, it holds that $T$ is weakly compact on $H^\infty$ is and only if $T$ is compact on $H^\infty$. Moreover, Bourgain [1] established that a bounded linear operator on $H^\infty$ is weakly compact if and only if it does not fix any copy of $\ell^\infty$. Thus, $T$ is compact on $H^\infty$ if and only if it does not fix any copy of $\ell^\infty$. Therefore, the weighted composition operator $W_{\phi,\varphi}$ on $S_0^\infty$ is compact if and only if it does not fix any copy of $\ell^\infty$.

By Lemma 2, Theorem 2.1] and the expression of the operator $T$, we see that $W_{\phi,\varphi}$ is bounded on $S^\infty$ if and only if $\phi \in S^\infty$ and $W_{\phi,\varphi}$ is bounded on $S_0^\infty$. Moreover, under the assumption for the boundedness of $W_{\phi,\varphi}$ on $S^\infty$, $W_{\phi,\varphi}$ is bounded on $S^\infty$ if and only if $W_{\phi,\varphi}$ is bounded on $S_0^\infty$.

Therefore, the weighted composition operator $W_{\phi,\varphi}$ on $S^\infty$ is compact if and only if it does not fix any copy of $\ell^\infty$. In particular, the noncompact operator $W_{\phi,\varphi}$ on $S^\infty$ is not strictly singular, that is, strict singularity of bounded operator $W_{\phi,\varphi}$ on $S^\infty$ coincides with its compactness.

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