EXISTENCE AND STABILITY OF STANDING WAVES FOR
NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION WITH
LOGARITHMIC NONLINEARITY

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Abstract. In this paper we consider the nonlinear fractional logarithmic
Schrödinger equation. By using a compactness method, we construct a unique
global solution of the associated Cauchy problem in a suitable functional framework. We also prove the existence of ground states as minimizers of the action
on the Nehari manifold. Finally, we prove that the set of minimizers is a stable
set for the initial value problem, that is, a solution whose initial data is near
the set will remain near it for all time.

1. Introduction

This paper is concerned with the fractional nonlinear Schrödinger equation with
logarithmic nonlinearity

\[ i\partial_t u - (-\Delta)^s u + u \log |u|^2 = 0, \]

where \(0 < s < 1\) and \(u = u(x,t)\) is a complex-valued function of \((x,t) \in \mathbb{R}^N \times \mathbb{R}, N \geq 2\). The fractional Laplacian \((-\Delta)^s\) is defined via Fourier transform as

\[ \mathcal{F} [(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F} u(\xi), \]

where the Fourier transform is given by

\[ \mathcal{F} u(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) e^{-i\xi \cdot x} dx. \]

The fractional Laplacian \((-\Delta)^s\) is a self-adjoint operator on \(L^2(\mathbb{R}^N)\) with quadratic form domain \(H^s(\mathbb{R}^N)\) and operator domain \(H^{2s}(\mathbb{R}^N)\). Moreover, the following spectral properties of \((-\Delta)^s\) are known: \(\sigma_{ess}((-\Delta)^s) = [0, \infty)\) and \(\sigma_p((-\Delta)^s) = \emptyset\) (see, e.g., [26, Example 3.3]). The nonlocal operator \((-\Delta)^s\) can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [2]). Fractional powers of the Laplacian arise in a numerous variety of equations in mathematical physics and related fields; see, e.g., [2, 8, 22] and references therein. Recently, a great attention has been focused on the study of problems involving the fractional Laplacian from a pure mathematical point of view. Concerning the fractional Schrödinger equations, let us mention [15, 10, 19, 21, 20, 18, 16, 23, 24, 28].

The present paper is devoted to the analysis of existence and stability of standing waves of NLS (1.1). If the fractional Laplacian in (1.1) is replaced by a standard Laplacian, this problem is well-known and described in detail in [4, 6, 14, 7, 8]. In this case, one can show that that there exists a unique (up to translations and

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phase shifts) ground state and it is orbitally stable. The classical logarithmic NLS equation was proposed by Bialynicki-Birula and Mycielski [1] in 1976 as a model of nonlinear wave mechanics and has important applications in quantum mechanics, quantum optics, nuclear physics, open quantum systems and Bose-Einstein condensation (see e.g. [25, 29] and the references therein).

The energy functional $E$ associated with problem (1.1) is

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx.$$  \hfill (1.4)$$

Unfortunately, due to the singularity of the logarithm at the origin, the functional fails to be finite as well of class $C^1$ on $H^s(\mathbb{R}^N)$. Due to this loss of smoothness, it is convenient to work in a suitable Banach space endowed with a Luxemburg type norm in order to make functional $E$ well defined and $C^1$ smooth. This space allows to control the singularity of the logarithmic nonlinearity at infinity and at the origin. Indeed, we consider the reflexive Banach space (see Section 2)

$$W^s(\mathbb{R}^N) = \left\{ u \in H^s(\mathbb{R}^N) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^N) \right\},$$  \hfill (1.5)$$

then the energy functional $E$ is well-defined and of class $C^1$ on $W^s(\mathbb{R}^N)$. Moreover, from Lemma 2.4, we have that the operator $u \to (-\Delta)^s u - u \log |u|^2$ is continuous from $W^s(\mathbb{R}^N)$ to $W^{-s}(\mathbb{R}^N)$. Here, $W^{-s}(\mathbb{R}^N)$ is the dual space of $W^s(\mathbb{R}^N)$. Therefore, if $u \in C(\mathbb{R}, W^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, W^{-s}(\mathbb{R}^N))$, then equation (1.1) makes sense in $W^{-s}(\mathbb{R}^N)$.

The next proposition gives a result on the existence of weak solutions to (1.1) in the energy space $W^s(\mathbb{R}^N)$. The proof is contained in Section 3.

**Proposition 1.1.** For any $u_0 \in W^s(\mathbb{R}^N)$, there is a unique solution $u \in C(\mathbb{R}, W^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, W^{-s}(\mathbb{R}^N))$ of (1.1) such that $u(0) = u_0$ and $\sup_{t \in \mathbb{R}} \|u(t)\|_{W^s(\mathbb{R}^N)} < \infty$. Furthermore, the conservation of energy and charge holds; that is,

$$E(u(t)) = E(u_0) \quad \text{and} \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad \text{for all} \ t \in \mathbb{R}.$$  

In this paper we study the existence and stability of standing waves solutions of (1.1) of the form $u(x,t) = e^{i\omega t}\varphi(x)$, where $\omega \in \mathbb{R}$ and $\varphi \in W^s(\mathbb{R}^N)$ is a complex valued function which has to solve the following stationary problem

$$(-\Delta)^s \varphi + \omega \varphi - \varphi \log |\varphi|^2 = 0, \quad x \in \mathbb{R}^N. \hfill (1.6)$$

For $\omega \in \mathbb{R}$, we define the following functionals of class $C^1$ on $W^s(\mathbb{R}^N)$:

$$S_\omega(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx + \frac{\omega + 1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx,$$

$$I_\omega(u) = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx + \omega \int_{\mathbb{R}^N} |u|^2 \, dx - \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx.$$  

Note that (1.6) is equivalent to $S'_\omega(\varphi) = 0$, and $I_\omega(u) = \langle S'_\omega(u), u \rangle$ is the so-called Nehari functional. It was shown in [18] that the problem (1.6) admits a sequence of weak solutions $u_n \in H^s(\mathbb{R}^N)$ with $S_\omega(u_n) \to +\infty$ as $n \to +\infty$.

From the physical point viewpoint, an important role is played by the ground state solution of (1.6). We recall that a function $\varphi \in W^s(\mathbb{R}^N)$ of (1.6) is termed as a ground state if it has some minimal action among all solutions of (1.6). To be
more specific, we consider the minimization problem

\[ d(\omega) = \inf \left\{ S_{\omega}(u) : u \in W^s(\mathbb{R}^N) \setminus \{0\}, I_{\omega}(u) = 0 \right\} \]

\[ = \frac{1}{2} \inf \left\{ \|u\|^2_{L^2(\mathbb{R}^N)} : u \in W^s(\mathbb{R}^N) \setminus \{0\}, I_{\omega}(u) = 0 \right\}, \]

and define the set of ground states by

\[ G_{\omega} = \{ \varphi \in W^s(\mathbb{R}^N) \setminus \{0\} : S_{\omega}(\varphi) = d(\omega), \quad I_{\omega}(\varphi) = 0 \}. \]

The set \( \{ u \in W^s(\mathbb{R}^N) \setminus \{0\}, I_{\omega}(u) = 0 \} \) is called the Nehari manifold. Notice that the above set contains all stationary points of \( S_{\omega} \).

The existence of minimizers for (1.7) will be obtained as a consequence of the stronger statement that any minimizing sequence for (1.7) is, up to translation, precompact in \( W^s(\mathbb{R}^N) \). We will show the following theorem in Section 4.

**Theorem 1.2.** Let \( N \geq 2, \omega \in \mathbb{R} \) and \( 0 < s < 1 \). Let \( \{u_n\} \subseteq W^s(\mathbb{R}^N) \) be a minimizing sequence for \( d(\omega) \). Then there exists a family \( \{y_n\} \subset \mathbb{R}^N \) such that \( \{u_n(\cdot - y_n)\} \) contains a convergent subsequence in \( W^s(\mathbb{R}^N) \). In particular, this implies that \( G_{\omega} \) is not empty set for any \( \omega \in \mathbb{R} \).

We remark that for any \( u \in G_{\omega} \), there exists a Lagrange multiplier \( \Lambda \in \mathbb{R} \) such that \( S'_{\omega}(u) = \Lambda I'_{\omega}(u) \). Thus, we have \( \langle S'_{\omega}(u), u \rangle = \langle I'_{\omega}(u), u \rangle \). The fact that \( \langle S'_{\omega}(u), u \rangle = I_{\omega}(u) = 0 \) and \( \langle I'_{\omega}(u), u \rangle = -2\|u\|^2_{L^2(\mathbb{R}^N)} < 0 \), implies \( \Lambda = 0 \); that is, \( S'_{\omega}(u) = 0 \). Therefore, \( u \) satisfies (1.6).

Now we are ready to state our main result, which is a direct consequence of the result of relative compactness.

**Theorem 1.3.** Let \( N \geq 2, \omega \in \mathbb{R} \) and \( 0 < s < 1 \). Then the set \( G_{\omega} \) is \( W^s(\mathbb{R}^N) \)-stable with respect to NLS (1.1); that is, for arbitrary \( \epsilon > 0 \), there exist \( \delta > 0 \) such that for any \( u_0 \in W^s(\mathbb{R}^N) \), if

\[ \inf_{\psi \in G_{\omega}} \|u_0 - \psi\|_{W^s(\mathbb{R}^N)} < \delta, \]

then the solution \( u(x, t) \) of the Cauchy problem (1.1) with the initial data \( u_0 \) satisfies

\[ \inf_{\psi \in G_{\omega}} \|u(\cdot, t) - \psi\|_{W^s(\mathbb{R}^N)} < \epsilon, \quad \text{for all } t \geq 0. \]

The rest of the paper is organized as follows. In Section 2 we analyse the structure of the energy space \( W^s(\mathbb{R}^N) \). Moreover, we recall several known results, which will be needed later. In Section 3 we give an idea of the proof of Proposition 1.1. In Section 4 we prove, by variational techniques, the existence of a minimizer for \( d(\omega) \). The stability result is proved in Section 5. In the Appendix we list some properties of the Orlicz space associated with \( W^s(\mathbb{R}^N) \).

**Notation.** The space \( L^2(\mathbb{R}^N, \mathbb{C}) \) will be denoted by \( L^2(\mathbb{R}^N) \) and its norm by \( \|\cdot\|_{L^2} \). \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( X' \) and \( X \), where \( X \) is a Banach space and \( X' \) is its dual. Finally, \( 2^*_s := 2N/(N - 2s) \) for \( N \geq 2 \) and \( 0 < s < 1 \). Throughout this paper, the letter \( C \) will denote positive constants whose value may change from line to line.
2. Preliminaries and Functional Setting

For the sake of self-containedness, we first provide some basic properties of the fractional Sobolev spaces $H^s(\mathbb{R}^N)$, which will be needed later. Consider the fractional order Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where $\hat{u} = \mathcal{F}(u)$. The norm is defined by $\|u\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} (|\hat{u}(\xi)|^2 + |\xi|^{2s} |\hat{u}(\xi)|^2) d\xi$. Notice that, by Plancherel’s theorem we have

$$\|u\|_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 d\xi$$

for every $u \in H^s(\mathbb{R}^N)$. Moreover, the space $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for any $q \in [2, 2^*_s]$ and compactly embedded into $L^q_{\text{loc}}(\mathbb{R}^N)$ for any $q \in [2, 2^*_s)$, where $2^*_s = 2N/N - 2s$. See [27] for more details.

Let $0 < s < 1$ and $\Omega$ a smooth bounded domain of $\mathbb{R}^N$, we define $H^s(\Omega)$ as follows

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{2} + s}} \in L^2(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{H^s(\Omega)}^2 = \int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dx dy. \quad (2.1)$$

Also, we denote by $H^s_0(\Omega)$ the Hilbert space defined as the closure of $C_0^\infty(\Omega)$ under the norm $\| \cdot \|_{H^s(\Omega)}^2$ defined in (2.1). The dual space $H^{-s}(\Omega)$ of $H^s_0(\Omega)$ is defined in the standard way. For a general reference on analytical properties of fractional Sobolev spaces, see [27].

The next result is an adaptation of a classical lemma of Lions. For a proof we refer to [5, Lemma 2.8].

**Lemma 2.1.** Let $N \geq 2$ and $2^*_s = 2N/(N - 2s)$. If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and for some $R > 0$ we have

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 dx = 0,$$

then one has $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $2 < r < 2^*_s$.

Now we need to introduce some notation. Define

$$F(z) = |z|^2 \log |z|^2$$

for every $z \in \mathbb{C}$, and as in [11], we define the functions $A, B$ on $[0, \infty)$ by

$$A(s) = \begin{cases} -s^2 \log(s^2), & \text{if } 0 \leq s \leq e^{-3}; \\ 3s^2 + 4e^{-3} - e^{-6}, & \text{if } s \geq e^{-3}, \end{cases} \quad B(s) = F(s) + A(s). \quad (2.2)$$
Furthermore, let \( a, b \) be functions defined by
\[
a(z) = \frac{z}{|z|^2} A(\|z\|) \quad \text{and} \quad b(z) = \frac{z}{|z|^2} B(\|z\|) \quad \text{for} \quad z \in \mathbb{C}, \ z \neq 0.
\]
(2.3)

Notice that we have \( b(z) - a(z) = z \log |z|^2 \). It follows that \( A \) is a nonnegative convex and increasing function, and \( A \in C^1 ([0, +\infty)) \cap C^2 ((0, +\infty)). \) The Orlicz space \( L^A(\mathbb{R}^N) \) corresponding to \( A \) is defined by
\[
L^A(\mathbb{R}^N) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : A(\|u\|) \in L^1(\mathbb{R}^N) \},
\]
equipped with the Luxemburg norm
\[
\|u\|_{L^A} = \inf \left\{ k > 0 : \int_{\mathbb{R}^N} A \left( k^{-1}|u(x)| \right) \, dx \leq 1 \right\}.
\]

Here as usual \( L^1_{\text{loc}}(\mathbb{R}^N) \) is the space of all locally Lebesgue integrable functions. It is proved in \[11\] Lemma 2.1 that \( A \) is a Young-function which is \( \Delta_2 \)-regular and \( (L^A(\mathbb{R}^N), \| \cdot \|_{L^A}) \) is a separable reflexive Banach space.

We consider the reflexive Banach space \( W^s(\mathbb{R}^N) = H^s(\mathbb{R}^N) \cap L^A(\mathbb{R}^N) \) equipped with the usual norm \( \|u\|_{W^s(\mathbb{R}^N)} = \|u\|_{H^s(\mathbb{R}^N)} + \|u\|_{L^A} \). The following lemma provides an alternative way of defining the energy space \( W^s(\mathbb{R}^N) \).

**Lemma 2.2.** Let \( 0 < s < 1 \) and \( N \geq 2 \). Then
\[
W^s(\mathbb{R}^N) = \left\{ u \in H^s(\mathbb{R}^N) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^N) \right\}
\]

**Proof.** One easily verifies that for every \( \epsilon > 0 \), there exist \( C_\epsilon > 0 \) such that \( |B(z) - B(z_1)| \leq C_\epsilon (|z|^{1+\epsilon} + |z_1|^{1+\epsilon})|z - z_1| \) for all \( z, z_1 \in \mathbb{C} \). Integrating this inequality on \( \mathbb{R}^N \) with \( \epsilon = (2s - 2)/2 \) and applying Hölder inequality and Sobolev embeddings give
\[
\int_{\mathbb{R}^N} |B(|u|) - B(|v|)| \, dx \leq C \left( \|u\|_{H^s(\mathbb{R}^N)} + \|v\|_{H^s(\mathbb{R}^N)} \right)^\gamma \|u - v\|_{L^2},
\]
with \( \gamma = 2s/2 \). Thus for \( u \in H^s(\mathbb{R}^N) \) we get \( B(|u|) \in L^1(\mathbb{R}^N) \). Lemma 2.2 follows then from the definition of the spaces \( W^s(\mathbb{R}^N) \) and \( L^A(\mathbb{R}^N) \). \( \square \)

The following lemma is a variant of the Brézis-Lieb lemma from \[9\].

**Lemma 2.3.** Let \( \{ u_n \} \) be a bounded sequence in \( W^s(\mathbb{R}^N) \) such that \( u_n \to u \) a.e. in \( \mathbb{R}^N \). Then \( u \in W^s(\mathbb{R}^N) \) and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left\{ |u_n|^2 \log |u_n|^2 - |u_n - u|^2 \log |u_n - u|^2 \right\} \, dx = \int_{\mathbb{R}^N} |u|^2 \log |u|^2 \, dx.
\]

**Proof.** The proof follows along the same lines as \[11\] Lemma 2.3. We omit the details. \( \square \)

It follows from Proposition 1.1.3 in \[12\] that
\[
W^{-s}(\mathbb{R}^N) = H^{-s}(\mathbb{R}^N) + L^{A'}(\mathbb{R}^N),
\]
where the Banach space \( W^{-s}(\mathbb{R}^N) \) is equipped with its usual norm. Here, \( L^{A'}(\mathbb{R}^N) \) is the dual space of \( L^A(\mathbb{R}^N) \) (see \[11\]). It is easy to see that one has the following chain of continuous embedding: \( W^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow W^{-s}(\mathbb{R}^N) \).

Now we will show that the energy functional \( E \) is of class \( C^1 \) on \( W^s(\mathbb{R}^N) \). First we need the following lemma.
Lemma 2.4. The operator \( L : u \to (-\Delta)^s u - u \log |u|^2 \) is continuous from \( W^s(\mathbb{R}^N) \) to \( W^{-s}(\mathbb{R}^N) \). The image under \( L \) of a bounded subset of \( W^s(\mathbb{R}^N) \) is a bounded subset of \( W^{-s}(\mathbb{R}^N) \).

Proof. First, notice that \((-\Delta)^s\) is continuous from \( H^s(\mathbb{R}^N) \) to \( H^{-s}(\mathbb{R}^N) \). Thus, using \( W^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N) \), we obtain that the operator \( u \to (-\Delta)^s u \) is continuous from \( W^s(\mathbb{R}^N) \) to \( W^{-s}(\mathbb{R}^N) \). Secondly, for every \( \epsilon > 0 \), there exist \( C_\epsilon > 0 \) such that \(|b(z) - b(z_1)| \leq C_\epsilon (|z|^\epsilon + |z_1|^\epsilon) |z - z_1|\) for all \( z, z_1 \in \mathbb{C} \). Integrating this inequality on \( \mathbb{R}^N \) with \( \epsilon = (2s - 2)/2 \) and applying Hölder inequality and Sobolev embeddings we obtain

\[
\|b(u) - b(v)\|_{L^2(\mathbb{R}^N)} \leq C \|u - v\|_{H^s(\mathbb{R}^N)} \left( \|u\|_{H^s(\mathbb{R}^N)} + \|u\|_{H^s(\mathbb{R}^N)} \right)^\gamma
\]

where \( \gamma = 2s/(N - 2s) \). Then clearly \( u \to b(u) \) is continuous and bounded from \( H^s(\mathbb{R}^N) \) to \( L^2(\mathbb{R}^N) \), then from \( W^s(\mathbb{R}^N) \) to \( W^{-s}(\mathbb{R}^N) \). Finally, since \( u \to a(u) \) is continuous and bounded from \( L^s(\mathbb{R}^N) \) to \( L^s(\mathbb{R}^N) \) (see \[11\] Lemma 2.6), it follows that the operator \( u \to a(u) - b(u) = -u \log |u|^2 \) is continuous and bounded from \( W^s(\mathbb{R}^N) \) to \( W^{-s}(\mathbb{R}^N) \), and lemma is proved.

From Lemma 2.4, we have the following consequence:

Proposition 2.5. The operator \( E : W^s(\mathbb{R}^N) \to \mathbb{R} \) is of class \( C^1 \) and for \( u \in W^s(\mathbb{R}^N) \) the Fréchet derivative of \( E \) in \( u \) exists and it is given by

\[
E'(u) = (-\Delta)^s u - u \log |u|^2 - u
\]

Proof. We first show that \( E \) is continuous. Notice that

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} A(|u|) dx - \frac{1}{2} \int_{\mathbb{R}^N} B(|u|) dx.
\]

The first term in the right-hand side of (2.5) is continuous of \( H^s(\mathbb{R}^N) \to \mathbb{R} \), and it follows from Proposition 1.1(i) in Appendix that the second term is continuous of \( L^s(\mathbb{R}^N) \to \mathbb{R} \). Moreover, by (2.4) we get that the third term in the right-hand side of (2.5) is continuous of \( H^s(\mathbb{R}^N) \to \mathbb{R} \). Therefore, \( E \in C(W^s(\mathbb{R}^N), \mathbb{R}) \). Now, direct calculations show that, for \( u, v \in W^s(\mathbb{R}^N), t \in (-1, 1) \) (see \[11\] Proposition 2.7),

\[
\lim_{t \to 0} \frac{E(u + tv) - E(u)}{t} = \langle (-\Delta)^s u - u \log |u|^2 - u, v \rangle.
\]

Thus, \( E \) is Gâteaux differentiable. Then, by Lemma 2.4 we see that \( E \) is Fréchet differentiable and \( E'(u) = (-\Delta)^s u - u \log |u|^2 - u \). \( \square \)

3. The Cauchy Problem

In this section we sketch the proof of the global well-posedness of the Cauchy Problem for (1.1) in the energy space \( W^s(\mathbb{R}^N) \). The proof of Proposition 1.1 is an adaptation of the proof of \[12\] Theorem 9.3.4). So, we will approximate the logarithmic nonlinearity by a smooth nonlinearity, and as a consequence we construct a sequence of global solutions of the regularized Cauchy problem in \( C(\mathbb{R}, H^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^N)) \), then we pass to the limit using standard compactness results, extract a subsequence which converges to the solution of the limiting equation (1.1). Finally, by using special properties of the logarithmic nonlinearity we establish uniqueness of the global solution.
First we regularize the logarithmic nonlinearity near the origin. For \( z \in \mathbb{C} \) and \( m \in \mathbb{N} \), we define the functions \( a_m \) and \( b_m \) by

\[
\begin{align*}
a_m(z) &= \begin{cases} a(z), & \text{if } |z| \geq \frac{1}{m}; \\ m z a\left(\frac{1}{m}\right), & \text{if } |z| \leq \frac{1}{m}; \end{cases} \\
b_m(z) &= \begin{cases} b(z), & \text{if } |z| \leq m; \\ \frac{m}{a(m)}, & \text{if } |z| \geq m, \end{cases}
\end{align*}
\]

where \( a \) and \( b \) were defined in \( 2.3 \). For any fixed \( m \in \mathbb{N} \), we define a family of regularized nonlinearities in the form \( g_m(z) = b_m(z) - a_m(z) \), for every \( z \in \mathbb{C} \).

In order to construct a solution of \( (1.1) \), we solve first, for \( m \in \mathbb{N} \), the regularized Cauchy problem

\[
i \partial_t u_m - (-\Delta)^s u_m + g_m(u_m) = 0. \tag{3.1}
\]

**Proposition 3.1.** Let \( 0 < s < 1 \) and \( N \geq 2 \). For any \( u_0 \in H^s(\mathbb{R}^N) \), there is a unique solution \( u^m \in C(\mathbb{R}, H^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^N)) \) of \( (3.1) \) such that \( u^m(0) = u_0 \). Furthermore, the conservation of energy and charge holds; that is,

\[
\mathcal{E}_m(u^m(t)) = \mathcal{E}_m(u_0) \quad \text{and} \quad \|u^m(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad \text{for all } t \in \mathbb{R},
\]

where

\[
\mathcal{E}_m(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} G_m(u) dx, \quad G_m(z) = \int_0^{|z|} g_m(s) ds.
\]

**Proof.** Our proof is inspired by the results of \( 16 \) Section 4. First, since \( g_m \) is globally Lipschitz continuous \( \mathbb{C} \) \( \rightarrow \mathbb{C} \), one easily verifies that \( \|g_m(u) - g_m(v)\|_{H^{-s}(\mathbb{R}^N)} \leq C(K) \|u - v\|_{H^s(\mathbb{R}^N)} \), provided that \( \|u\|_{H^s(\mathbb{R}^N)} + \|v\|_{H^s(\mathbb{R}^N)} \leq K \). Then, from \( 16 \) Proposition 4.1 we see that there exists a weak solution \( u^m \) of \( (3.1) \) such that

\[
u^m \in L^\infty((-T_{\min}, T_{\max}), H^s(\mathbb{R}^N)) \cap W^{1,\infty}((-T_{\min}, T_{\max}), H^{-s}(\mathbb{R}^N)),
\]

\[
\mathcal{E}_m(u^m(t)) \leq \mathcal{E}_m(u_0) \quad \text{and} \quad \|u^m(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \tag{3.2}
\]

for all \( t \in (-T_{\min}, T_{\max}) \), where \( (-T_{\min}, T_{\max}) \) is the maximal existence time interval of \( u^m \) for initial data \( u_0 \).

Secondly, we show that there is uniqueness for the problem \( (3.1) \). In fact, let \( I \) be an interval containing \( 0 \) and let \( \Phi, \Psi \in L^\infty(I, H^s(\mathbb{R}^N)) \cap W^{1,\infty}(I, H^{-s}(\mathbb{R}^N)) \) be two solutions of \( (3.1) \). It follows that

\[
\Psi(t) - \Phi(t) = i \int_0^t U(t-s) \left( g_m(\Psi(s)) - g_m(\Phi(s)) \right) ds \quad \text{for all } t \in I,
\]

where \( U(t) = e^{-it(-\Delta)^s} \). Since \( g_m \) is Lipschitz continuous \( L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \), there exists a constant \( C > 0 \) such that

\[
\|\Psi(t) - \Phi(t)\|_{L^2}^2 \leq C \int_0^t \|\Psi(s) - \Phi(s)\|_{L^2}^2 ds,
\]

and therefore the uniqueness follows by Gronwall’s Lemma. Furthermore, since

\[
\left| \int_{\mathbb{R}^N} G_m(u) dx \right| \leq C \|u\|_{L^2}^2,
\]

from \( (3.2) \) we get \( \|u^m(t)\|_{H^s(\mathbb{R}^N)}^2 \leq C \|u_0\|_{L^2}^2 + 4\mathcal{E}_m(u_0) \) for all \( t \in (-T_{\min}, T_{\max}) \). The continuity argument implies that all solutions of \( (3.1) \) are global and uniformly bounded in \( H^s(\mathbb{R}^N) \).

Finally, we prove that the weak solution \( u^m \) of \( (3.1) \) satisfies the conservation of energy. Indeed, fix \( t_0 \in \mathbb{R} \). Let \( \varphi_0 = u^m(t_0) \) and let \( w \) be the solution of \( (3.1) \) with \( w(0) = \varphi_0 \). By uniqueness, we see that \( w(\cdot - t_0) = u^m(\cdot) \) on \( \mathbb{R} \). From \( (3.2) \), we deduce in particular that

\[
\mathcal{E}_m(u_0) \leq \mathcal{E}_m(\varphi_0) = \mathcal{E}_m(u^m(t_0))
\]
Therefore, we have that both \( \|u^m(t)\|_{L^2}^2 \) and \( \mathcal{E}_m(u^m(t)) \) are constant on \( \mathbb{R} \). The inclusion \( u^m \in C(\mathbb{R}, H_s^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^N)) \) follows from conservation laws. This completes the proof of Proposition 3.1. \( \square \)

For the proof of Proposition 3.1 we will use the following lemma.

**Lemma 3.2.** Let \( 0 < s < 1 \) and \( N \geq 2 \). Given \( k \in \mathbb{N} \), set \( \Omega_k = \{ x \in \mathbb{R}^N : |x| < k \} \). Let \( \{u^m\}_{m \in \mathbb{N}} \) be a bounded sequence in \( L^\infty(\mathbb{R}, H^s(\mathbb{R}^N)) \). If \( \{u^m_{|\Omega_k}\}_{m \in \mathbb{N}} \) is a bounded sequence of \( W^{1,\infty}(\mathbb{R}, H^{-s}(\Omega_k)) \) for \( k \in \mathbb{N} \), then there exists a subsequence, which we still denote by \( \{u^m\}_{m \in \mathbb{N}} \), and there exist \( u \in L^\infty(\mathbb{R}, H^s(\mathbb{R}^N)) \), such that the following properties hold:

(i) \( u^m_{|\Omega_k} \in W^{1,\infty}(\mathbb{R}, H^{-s}(\Omega_k)) \) for every \( k \in \mathbb{N} \).

(ii) \( u^m(t) \to u(t) \) in \( H^s(\mathbb{R}^N) \) as \( m \to \infty \) for every \( t \in \mathbb{R} \).

(iii) For every \( t \in \mathbb{R} \) there exists a subsequence \( m_j \) such that \( u^{m_j}(x,t) \to u(x,t) \) as \( j \to \infty \), for a.e. \( x \in \mathbb{R}^N \).

(iv) \( u^m(x,t) \to u(x,t) \) as \( m \to \infty \), for a.e. \( (x,t) \in \mathbb{R}^N \times \mathbb{R} \).

**Proof.** The proof follows a similar argument as in Lemma 9.3.6 of [12] and we do not repeat here. \( \square \)

**Proof of Proposition 3.1.** Our proof follows the ideas of Cazenave [12, Theorem 9.3.4]. Applying Proposition 3.1, we see that for every \( m \in \mathbb{N} \) there exists a unique global solution \( u^m \in C(\mathbb{R}, H^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, H^{-s}(\mathbb{R}^N)) \) of (3.1), which satisfies

\[
\mathcal{E}_m(u^m(t)) = \mathcal{E}_m(u_0) \quad \text{and} \quad \|u^m(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad \text{for all} \ t \in \mathbb{R},
\]

where

\[
\mathcal{E}_m(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \Phi_m(|u|) dx - \frac{1}{2} \int_{\mathbb{R}^N} \Psi_m(|u|) dx,
\]

and the functions \( \Phi_m \) and \( \Psi_m \) defined by

\[
\Phi_m(z) = \frac{1}{2} \int_0^{|z|} a_m(s) ds \quad \text{and} \quad \Psi_m(z) = \frac{1}{2} \int_0^{|z|} b_m(s) ds.
\]

It follows from (3.3) that \( u^m \) is bounded in \( L^\infty(\mathbb{R}, L^2(\mathbb{R}^N)) \). Moreover, we have that the sequence of approximating solutions \( u^m \) is bounded in the space \( L^\infty(\mathbb{R}, H^s(\mathbb{R}^N)) \). It also follows from the NLS equation (3.1) that the sequence \( \partial_t u^m_{|\Omega_k} \) is bounded in the space \( L^\infty(\mathbb{R}, H^{-s}(\Omega_k)) \), where \( \Omega_k = \{ x \in \mathbb{R}^N : |x| < k \} \). Such statements can be proved along the same lines as the Step 2 of Theorem 9.3.4 in [12]. Therefore, we have that \( \{u^m\}_{m \in \mathbb{N}} \) satisfies the assumptions of Lemma 3.2. Let \( u \) be the limit of \( u^m \).

Now we show that the limiting function \( u \in L^\infty(\mathbb{R}, H^s(\mathbb{R}^N)) \) is a weak solution of the logarithmic NLS equation (1.1). To do so, we first write a weak formulation of the NLS equation (3.1). Indeed, for any test function \( \psi \in C_0^\infty(\mathbb{R}^N) \) and \( \phi \in C_0^\infty(\mathbb{R}) \), we have

\[
\int_{\mathbb{R}} \left< i u^m \psi, \phi' - u^m (-\Delta)^s \phi \right> dt + \int_{\mathbb{R}} \int_{\mathbb{R}^N} g_m(u^m) \psi \phi dx dt = 0. \quad (3.4)
\]

Furthermore, since \( g_m(z) \to z \log |z|^2 \) pointwise in \( z \in \mathbb{C} \) as \( m \to +\infty \), we apply the properties (ii)-(iv) of Lemma 3.2 to the integral formulation (3.4) and obtain
the following integral equation (see proof of Step 3 of [12 Theorem 9.3.4])
\[
\int_{\mathbb{R}} \left[ -\langle i u, \psi \rangle \phi(t) - \langle u, (-\Delta)^s \psi \rangle \phi(t) \right] dt + \int_{\mathbb{R}} \int_{\mathbb{R}^N} u \log |u|^2 \psi \phi \, dx \, dt = 0. \tag{3.5}
\]
In addition, \( u(0) = u_0 \) by property (ii) of Lemma 3.2. Moreover, it is easy to see that \( u \in L^\infty(\mathbb{R}, L^4(\mathbb{R}^N)) \). Therefore, by integral equation (3.5), \( u \in L^\infty(\mathbb{R}, W^s(\mathbb{R}^N)) \) is a weak solution of the logarithmic NLS equation (1.1). In particular, from Lemma 2.4 we deduce that \( u \in W^{1,\infty}(\mathbb{R}, W^{-s}(\mathbb{R}^N)) \). Now we show uniqueness of the solution in the class \( L^\infty(\mathbb{R}, W^s(\mathbb{R}^N)) \cap W^{1,\infty}(\mathbb{R}, W^{-s}(\mathbb{R}^N)) \). Indeed, let \( u \) and \( v \) be two solutions of (1.1) in that class. On taking the difference of the two equations and taking the weak solution in the class \( L \), we see that
\[
\langle u_t - v_t, u - v \rangle = -\Im \int_{\mathbb{R}^N} \left( u \log |u|^2 - v \log |v|^2 \right) (\nabla - \nabla) dx.
\]
Thus, from [12 Lemma 9.3.5] we obtain
\[
\|u(t) - v(t)\|_{L^2}^2 \leq 8 \int_0^t \|u(s) - v(s)\|_{L^2}^2 \, ds.
\]
Therefore, the uniqueness of a solution follows by Gronwall’s Lemma. Finally, the conservation of charge and energy, and the continuity of the solution \( u \in C(\mathbb{R}, W^s(\mathbb{R}^N)) \cap C^1(\mathbb{R}, W^{-s}(\mathbb{R}^N)) \) in time \( t \) follow from the arguments identical to the case of the classical logarithmic NLS equation (see proof of Step 4 of [12 Theorem 9.3.4]). This finishes the proposition. \( \Box \)

4. Variational Analysis

The aim of this section is to prove Theorem 1.2. First we recall the fractional logarithmic Sobolev inequality. For a proof we refer to [17].

**Lemma 4.1.** Let \( f \) be any function in \( H^s(\mathbb{R}^N) \) and \( \alpha > 0 \). Then
\[
\int_{\mathbb{R}^N} |f(x)|^2 \log \left( \frac{|f(x)|^2}{\|f\|_{L^2}^2} \right) dx + \left( N + \frac{N}{s} \log \alpha + \log \frac{s \Gamma(N)}{\Gamma(N/s)} \right) \|f\|_{L^2}^2 \leq \frac{\alpha^2}{\pi^2} \|(-\Delta)^s f\|_{L^2}^2.
\]
(4.1)

**Lemma 4.2.** Let \( \omega \in \mathbb{R} \). Then, the quantity \( d(\omega) \) is positive and satisfies
\[
d(\omega) \geq \frac{1}{2} \left( \frac{s \Gamma(N)}{\Gamma(N/s)} \right)^{N} \pi^2 e^{\omega+N}.
\]
(4.2)

**Proof.** Let \( u \in W^s(\mathbb{R}^N) \setminus \{0\} \) be such that \( I_\omega(u) = 0 \). Using the fractional logarithmic Sobolev inequality with \( \alpha = \pi^2 \), we see that
\[
\left( \omega + N(1 + \log(\sqrt{\pi})) + \log \frac{s \Gamma(N)}{\Gamma(N/s)} \right) \|u\|_{L^2}^2 \leq \left( \log \|u\|_{L^2}^2 \right) \|u\|_{L^2}^2.
\]
Thus, by the definition of \( d(\omega) \) given in (1.7), we get (4.2). \( \Box \)

**Lemma 4.3.** Let \( \omega \in \mathbb{R} \) and \( 0 < s < 1 \). If \( \{u_n\} \) is a minimizing sequence of problem (1.7), then there is a subsequence \( \{u_{n_j}\} \) and a sequence \( \{y_n\} \subset \mathbb{R}^N \) such that
\[
v_n(x) := u_{n_j}(x + y_n)
\]
converges weakly in \( W^s(\mathbb{R}^N) \) to a function \( \varphi \neq 0 \). Moreover, \( \{v_n\} \) converges to \( \varphi \) a.e and in \( L^q_{loc}(\mathbb{R}^N) \) for every \( q \in [2, 2s_*) \).
Proof. Let \( \{ u_n \} \subseteq W^s(\mathbb{R}^N) \) be a minimizing sequence for \( d(\omega) \), then the sequence \( \{ u_n \} \) is bounded in \( W^s(\mathbb{R}^N) \). Indeed, it is clear that the sequence \( \| u_n \|_{L^2}^2 \) is bounded. Moreover, using the fractional logarithmic Sobolev inequality and recalling that \( I_\omega(u_n) = 0 \), we obtain
\[
\left( 1 - \frac{\alpha^2}{\pi^2} \right) \| (-\Delta)^{\frac{s}{2}} u_n \|_{L^2}^2 \leq \text{Log} \left[ \left( \frac{e^{-(\omega+N)} \Gamma (N/2s)}{s \alpha^s \Gamma (N/2)} \right) \| u_n \|_{L^2}^2 \right] \| u_n \|_{L^2}^2.
\]
Taking \( \alpha > 0 \) sufficiently small, we see that \( \| (-\Delta)^{\frac{s}{2}} u_n \|_{L^2}^2 \) is bounded, so the sequence \( \{ u_n \} \) is bounded in \( H^s(\mathbb{R}^N) \). Then, using \( I_\omega(u_n) = 0 \) again, and (2.4) we obtain
\[
\int_{\mathbb{R}^N} A(|u_n|) \, dx \leq \int_{\mathbb{R}^N} B(|u_n|) \, dx + |\omega| \| u_n \|_{L^2}^2 \leq C,
\]
which implies, by (6.1) in the Appendix, that the sequence \( \{ u_n \} \) is bounded in \( W^s(\mathbb{R}^N) \). Furthermore, since \( W^s(\mathbb{R}^N) \) is a reflexive Banach space, there is \( v \in W^s(\mathbb{R}^N) \) such that, up to a subsequence, \( u_n \rightharpoonup v \) weakly in \( W^s(\mathbb{R}^N) \).

On the other hand, let \( 2 < q < 2^*_s \) and \( 0 < \delta < 1 \). Notice that \( \int_{\mathbb{R}^N} |u_n|^q \, dx = \| (-\Delta)^{\frac{s}{2}} u_n \|_{L^q}^2 + \omega \| u_n \|_{L^q}^2 \geq -M(\omega) \), for sufficiently large \( n \), where \( M(\omega) \) is a positive constant depending only on \( \omega \). Arguing as in the proof of Lemma 3.3 in [11], it is easy to see that
\[
2d(\omega) \leq (\delta^{(q-2)} - \left( (q-2)\text{Log}\delta^2 \right)^{-1}) \int_{\mathbb{R}^N} |u_n|^q \, dx - (\text{Log}\delta^2)^{-1} M(\omega).
\]
We now choose \( \delta \) such that \( (\text{Log}\delta^2)^{-1} M(\omega) = d(\omega) ((q-2)/(q+2)) \). Easy computations permit us to obtain
\[
\int_{\mathbb{R}^N} |u_n|^q \, dx \left( e^{M(\omega) (q+2)/2d(\omega)} + \frac{d(\omega)}{M(\omega) (q+2)} \right) \geq 2d(\omega) \left( \frac{q+6}{q+2} \right) \geq d(\omega).
\]
Combining (4.3) and Lemma 2.1 implies
\[
\sup_{y \in \mathbb{R}^N} \int_{B_{\epsilon}(y)} |u_n|^2 \, dx \geq \epsilon > 0,
\]
In this case we can choose \( \{ y_n \} \subseteq \mathbb{R}^N \) such that
\[
\int_{B_{\epsilon'}(0)} |u_n(\cdot + y_n)|^2 \, dx \geq \epsilon',
\]
where \( 0 < \epsilon' < \epsilon \), and hence, due to the compactness of the embedding \( H^s_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N) \), we deduce that the translated sequence \( v_n := u_n(\cdot + y_n) \) has a weak limit \( \varphi \) in \( H^s(\mathbb{R}^N) \) that is not identically zero. Also, it follows that \( \{ v_n \} \) converges to \( \varphi \) strongly in \( L^q_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in [2, 2^*_s) \). Therefore, we infer that, after a translation if necessary, \( \{ u_n \} \) converges weakly in \( W^s(\mathbb{R}^N) \) and a.e. to a function \( \varphi \neq 0 \). Hence the result is established. \qed

**Proof of Theorem 1.2** The proof follows basically the same idea as the proof of [1] Proposition 1.3 and Lemma 3.1 (see also [1]). Let \( \{ u_n \} \subseteq W^s(\mathbb{R}^N) \) be a minimizing sequence for \( d(\omega) \). From Lemma 4.3 there exist \( \varphi \in W^s(\mathbb{R}^N) \setminus \{ 0 \} \) such that, \( v_n := u_{j_n}(\cdot + y_n) \rightharpoonup \varphi \) weakly in \( W^s(\mathbb{R}^N) \) and \( \{ v_n \} \) converges to \( \varphi \) a.e. and in \( L^q_{\text{loc}}(\mathbb{R}^N) \) for every \( q \in [2, 2^*_s) \).

Now we prove that \( \varphi \in \mathcal{G}_\omega \), that is, \( I_\omega(\varphi) = 0 \) and \( S_\omega(\varphi) = d(\omega) \). First, assume by contradiction that \( I_\omega(\varphi) < 0 \). By elementary computations, we can see that
there is $0 < \lambda < 1$ such that $I_\omega(\lambda \varphi) = 0$. Then, from the definition of $d(\omega)$ and the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm, we have
\[
d(\omega) \leq \frac{1}{2} ||\lambda \varphi||^2_{L^2} < \frac{1}{2} ||\varphi||^2_{L^2} \leq \frac{1}{2} \liminf_{n \to \infty} ||v_n||^2_{L^2} = d(\omega),
\]
which is impossible. On the other hand, assume that $I_\omega(\varphi) > 0$. Since the embedding $W^s(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$ is continuous, we see that $v_n \rightharpoonup \varphi$ weakly in $H^s(\mathbb{R}^N)$. Thus, by (4.5) and applying the same argument as above, we see that
\[
\|(-\Delta)\tilde{z} v_n\|^2_{L^2} - \|(-\Delta)\tilde{z} \varphi\|^2_{L^2} \to 0, \quad (4.4)
\]
and
\[
\|v_n\|^2_{L^2} - \|v_n - \varphi\|^2_{L^2} \to 0, \quad (4.5)
\]
as $n \to \infty$. Combining (4.4), (4.5) and Lemma 2.3 leads to
\[
\lim_{n \to \infty} I_\omega(v_n - \varphi) = \lim_{n \to \infty} I_\omega(v_n) - I_\omega(\varphi) = -I_\omega(\varphi),
\]
which combined with $I_\omega(\varphi) > 0$ give us that $I_\omega(v_n - \varphi) < 0$ for sufficiently large $n$. Thus, by (4.5) and applying the same argument as above, we see that
\[
d(\omega) \leq \frac{1}{2} \lim_{n \to \infty} ||v_n - \varphi||^2_{L^2} = d(\omega) - \frac{1}{2} ||\varphi||^2_{L^2},
\]
which is a contradiction because $||\varphi||^2_{L^2} > 0$. Then, we deduce that $I_\omega(\varphi) = 0$. In addition, by the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm, we have
\[
d(\omega) \leq \frac{1}{2} ||\varphi||^2_{L^2} \leq \frac{1}{2} \liminf_{n \to \infty} ||v_n||^2_{L^2} = d(\omega), \quad (4.6)
\]
which implies, by the definition of $d(\omega)$, that $\varphi \in G_\omega$.

Now we claim that $v_n \to \varphi$ strongly in $W^s(\mathbb{R}^N)$. Indeed, by (4.5), we infer that $v_n \to \varphi$ in $L^2(\mathbb{R}^N)$. Moreover, since the sequence $\{v_n\}$ is bounded in $H^s(\mathbb{R}^N)$, from (2.4) we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} B(|v_n(x)|) \, dx = \int_{\mathbb{R}^N} B(|\varphi(x)|) \, dx,
\]
which combined with $I_\omega(v_n) = I_\omega(\varphi) = 0$ for any $n \in \mathbb{N}$, gives
\[
\lim_{n \to \infty} \left[\|(-\Delta)\tilde{z} v_n\|^2_{L^2} + \int_{\mathbb{R}^N} A(|v_n(x)|) \, dx\right] = \|(-\Delta)\tilde{z} \varphi\|^2_{L^2} + \int_{\mathbb{R}^N} A(|\varphi(x)|) \, dx. \quad (4.7)
\]
Moreover, by (14), the weak lower semicontinuity of the $L^2(\mathbb{R}^N)$-norm and Fatou lemma, we deduce (see e.g. [13, Lemma 2.4.4])
\[
\lim_{n \to \infty} \|(-\Delta)\tilde{z} v_n\|^2_{L^2} = \|(-\Delta)\tilde{z} \varphi\|^2_{L^2} \quad (4.8)
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} A(|v_n(x)|) \, dx = \int_{\mathbb{R}^N} A(|\varphi(x)|) \, dx. \quad (4.9)
\]
Since $v_n \rightharpoonup \varphi$ weakly in $H^s(\mathbb{R}^N)$, it follows from (4.8) that $v_n \to \varphi$ in $H^s(\mathbb{R}^N)$. Finally, by Proposition 6.1 ii) in Appendix and (4.9) we have $v_n \to \varphi$ in $L^A(\mathbb{R}^N)$. Thus, by definition of the $W^s(\mathbb{R}^N)$-norm, we infer that $v_n \to \varphi$ in $W^s(\mathbb{R}^N)$ which concludes the proof. \qed
5. Stability of the Standing Waves

Proof of Theorem 1.3 We argue by contradiction. Suppose that $G_\omega$ is not $W^s(\mathbb{R}^N)$-stable. Then there exist $\varepsilon > 0$, a sequence $(u_{n,0})_{n \in \mathbb{N}}$ in $W^s(\mathbb{R}^N)$ such that

$$\inf_{\psi \in G_\omega} \|u_{n,0} - \psi\|_{W^s(\mathbb{R}^N)} < \frac{1}{n},$$

(5.1)

and a sequence $(t_n)_{n \in \mathbb{N}}$ such that

$$\inf_{\psi \in G_\omega} \|u_{n}(t_n) - \psi\|_{W^s(\mathbb{R}^N)} \geq \varepsilon,$$

(5.2)

where $u_n$ denotes the solution of the Cauchy problem (1.1) with initial data $u_{n,0}$. Set $v_n(x) = u_n(x, t_n)$. By (5.1) and conservation laws, we obtain

$$\|v_n\|_{L^2} = \|u_{n}(t_n)\|_{L^2}^2 = \|u_{n,0}\|_{L^2}^2 \to 2d(\omega)$$

(5.3)

$$S_\omega(v_n) = S_\omega(u_n(t_n)) = S_\omega(u_{n,0}) \to d(\omega),$$

(5.4)

as $n \to \infty$. Moreover, by combining (5.3) and (5.4) lead us to $I_\omega(v_n) \to 0$ as $n \to \infty$. Next, define the sequence $f_n(x) = \rho_n v_n(x)$ with

$$\rho_n = \exp\left(\frac{I_\omega(v_n)}{2\|v_n\|_{L^2}^2}\right),$$

where $\exp(x)$ represent the exponential function. It is clear that $\lim_{n \to \infty} \rho_n = 1$ and $I_\omega(f_n) = 0$ for any $n \in \mathbb{N}$. Furthermore, since the sequence $(v_n)$ is bounded in $W^s(\mathbb{R}^N)$, we get $\|v_n - f_n\|_{W^s(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Then, by (5.1), we have that $(f_n)$ is a minimizing sequence for $d(\omega)$. Thus, by Theorem 1.2 up to a subsequence, there exist $(y_n) \subset \mathbb{R}^N$ and a function $\varphi \in G_\omega$ such that

$$\|f_n(y_n) - \varphi\|_{W^s(\mathbb{R}^N)} \to 0 \quad \text{as} \quad n \to +\infty.$$  

(5.5)

Since $\varphi(y_n) \in G_\omega$, remembering that $v_n = u_n(t_n)$ and substituting this in (5.5), we get

$$\inf_{\psi \in G_\omega} \|u_n(t_n) - \psi\|_{W^s(\mathbb{R}^N)} \leq \|v_n - f_n\|_{W^s(\mathbb{R}^N)} + \inf_{\psi \in G_\omega} \|f_n - \psi\|_{W^s(\mathbb{R}^N)} \to 0$$

as $n \to +\infty$, which is a contradiction with (5.2). This finishes the proof. \qed

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6. Appendix

We list some properties of the Orlicz space $L^A(\mathbb{R}^N)$ that we have used above. For a proof of such statements we refer to [11] Lemma 2.1.

Proposition 6.1. Let $(u_n)$ be a sequence in $L^A(\mathbb{R}^N)$, the following facts hold:

i) If $u_n \to u$ in $L^A(\mathbb{R}^N)$, then $A(|u_n|) \to A(|u|)$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

ii) Let $u \in L^A(\mathbb{R}^N)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$ and if

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} A(|u_n(x)|) \, dx = \int_{\mathbb{R}^N} A(|u(x)|) \, dx,$$

then $u_n \to u$ in $L^A(\mathbb{R}^N)$.
then \( u_m \to u \) in \( L^A(\mathbb{R}^N) \) as \( n \to \infty \).

iii) For any \( u \in L^A(\mathbb{R}^N) \), we have

\[
\min \left\{ \|u\|_{L^A}, \|u\|_{L^2}^2 \right\} \leq \int_{\mathbb{R}^N} A(|u(x)|) \, dx \leq \max \left\{ \|u\|_{L^A}, \|u\|_{L^2}^2 \right\}.
\] (6.1)

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