Newtonian perturbations and the Einstein–Yang–Mills-dilaton equations

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Abstract
In this paper, we show that the problem of proving the existence of a countable number of solutions to the static spherically symmetric $SU(2)$ Einstein–Yang–Mills-dilaton (EYMd) equations can be reduced to proving the non-existence of solutions to the linearized Yang–Mills-dilaton equations (lYMd) satisfying certain asymptotic conditions. The reduction from a nonlinear to a linear problem is achieved using a Newtonian perturbation-type argument.

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1. Introduction
Unlike the four-dimensional Yang–Mills (YM) equations which have no static solutions of finite energy [9, 10], the Euclidean $SU(2)$ Yang–Mills-dilaton (YMd) equations were shown numerically to possess a countably infinite sequence of static, globally regular, spherically symmetric solutions [5, 18]. Existence of these solutions was rigorously established using shooting techniques in [15]. The dilaton plays the role of an attractive force which counterbalances the repulsive nature of the Yang–Mills fields and this makes it possible for static solutions to exist on flat space. This is a simpler situation compared to the more well-known Bartnik–McKinnon (BK) static solutions in Einstein–Yang–Mills (EYM) theory [4] where gravity is the counterbalancing force.

In the papers [6, 19] it was found, again numerically, that the YMd solutions persist when the Yang–Mills and dilaton fields are coupled to gravity. The result is a countably infinite sequence of static, globally regular, spherically symmetric solutions to the $SU(2)$ Einstein–Yang–Mills-dilaton (EYMd) equations with the same qualitative behaviour for the Yang–Mills and dilaton fields as when gravity is absent. These solutions limit to the BK solutions as the dilaton coupling constant goes to zero which helps to explain why the Yang–Mills fields for the EYMd solutions have a similar behaviour to the BK solutions where there is no dilaton field.
In this paper, we show that the problem of proving the existence of a countably infinite number of solutions to the static spherically symmetric $SU(2)$ EYMd equations can be reduced to proving the non-existence of solutions to the linearized Yang–Mills-dilaton equations (IYMd) satisfying certain asymptotic conditions. The reduction from a nonlinear to a linear problem is achieved using a Newtonian perturbation-type argument. Unfortunately, we have not been able to exclude the possibility that there exist solutions to the IYMd that satisfy the asymptotic conditions. The main reason for this is that the YMd solutions obtained in [15] about which we linearize are unstable due to the presence of a negative part of the spectrum for the IYMd operator. This means that one cannot expect that a simple integration by parts argument will work to rule out solutions to the linearized equations. Instead we have to directly analyze the linearized equations which is difficult because we do not have much information about the YMd solutions other than that they exist and some asymptotic behaviour. However, we conjecture that there does not exist solutions to the IYMd equations that satisfy the required asymptotic conditions. If this were the case then we would have a full existence proof.

Although the static spherically symmetric solutions to the $SU(2)$-EYMd equations are unstable, they may still be physically relevant as stringy generalizations of the BK solutions which are very similar to sphalerons [11]. Indeed, sphalerons, which are unstable static solutions of the classical equations for the bosonic sector of the electroweak theory, are believed to be responsible for violations of the conservation of baryon numbers at high temperatures [1, 23]. Therefore, it is possible that the static EYMd solutions could play a role in the violation of the conservation of baryon and lepton numbers at high temperatures.

The Newtonian perturbation argument in the form that is employed in this paper was developed by Lottermoser in [20] and subsequently used by Heilig to establish the existence of slowly rotating stars [16]. It was also used by the author to provide an existence proof for the gravitating BPS monopole [22]. The results and the results of this paper show that Newtonian perturbation method is a useful approach to take in investigating the existence problem in general relativity for static or stationary matter models. In addition to establishing existence, the method also provides an analytic deformation from a Newtonian solution to its general relativistic counterpart. The deformation parameter can be interpreted as $1/c^2$ where $c$ is the speed of light. So a Taylor expansion in $1/c^2$ can be considered as a converging post-Newtonian expansion. In this way, the Newtonian perturbation argument can be thought of as the inverse of the Newtonian limit where Newtonian solutions are obtained from general relativistic ones via the limit $1/c^2 \to 0$. An attractive feature of the method is that it produces solutions to the Einstein field equations where the matter fields are uniformly close to their corresponding Newtonian ones. This means that the properties of the Newtonian solution pass directly to the corresponding relativistic solution.

The approach we take to establishing existence of static spherically symmetric solutions is different from previous approaches which rely on the fact that in spherical symmetry the static Einstein equations reduce to ordinary differential equations to which dynamical systems theory can be applied. The papers [7, 24, 25] which contain existence proofs for the BK solutions in EYM theory exemplify the dynamical systems approach to existence. The main advantage of our approach is that it is in principle not restricted to spherical symmetry which is important considering that it is known, numerically at least, that static axially symmetric solutions to the EYMd equations exist [17]. We note that a large amount of work has been done on gravitating gauge fields both numerically and analytically starting with the pioneering work of Bartnik and McKinnon. For a comprehensive review see [26].

This paper is organized as follows: in section 2 we set up the equations in a form suitable to use the Newtonian perturbation method while in section 3 we review the theory of weighted Sobolev spaces. The Banach spaces for our field variables (i.e. the dilaton field, gauge potential
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and metric density) are set up in section 4 and then in section 6 the field equations are shown to be smooth on those spaces. In section 5, we discuss the YMd solutions of [15] and their asymptotic properties. Sections 7–9 contain the Newtonian perturbation argument. In these sections, it is shown that if there are no solutions to the IYMd equations satisfying certain asymptotic conditions then the static spherically symmetric YMd solutions of [15] can be continued smoothly to static spherically symmetric solutions of the full EYMd equations.

2. EYMd equations

For indexing of tensors and related quantities Greek indices, \( \alpha, \beta, \gamma \), etc, will always run from 0 to 4 while Roman indices, \( i, j, k \), etc, will range from 1 to 3. We will use bold letters such as \( \mathbf{x} \) to denote points in \( \mathbb{R}^3 \), i.e. \( \mathbf{x} = (x^0, x^1, x^2, x^3) \).

Let \( g_0 \) denote the Minkowski metric on \( \mathbb{R}^4 \). Fix a global coordinate system \( (x^0, x^1, x^2, x^3) \) so that

\[
\eta_{\alpha\beta} = \text{diag}(-\lambda^{-1}, 1, 1, 1)
\]

(2.1)

where \( \lambda \) is a dimensionless parameter. From the way \( \lambda \) appears in the metric (2.1) it is useful to regard \( \lambda \) as acting like \( 1/c^2 \) in which case the limit \( \lambda \to 0 \) can be thought of as an analogue of the Newtonian limit. Define \( g_{\alpha\beta} \) by \( (g_{\alpha\beta}) := (g_0)_{\alpha\beta}^{-1} \) which gives

\[
\eta_{\alpha\beta} = \text{diag}(-\lambda, 1, 1, 1).
\]

(2.2)

Define the Minkowski metric density \( g_{\alpha\beta} := |g|^{\frac{1}{2}} g_{\alpha\beta} \) where \( |g| := |\det(g_{\alpha\beta})| \).

(2.3)

Assume that \( g_{\alpha\beta} \) is another metric defined on \( \mathbb{R}^4 \). Let \( (g_{\alpha\beta}) := (g_{\alpha\beta})^{-1} \) and introduce the density

\[
g_{\alpha\beta} := |g|^{\frac{1}{2}} g_{\alpha\beta} \quad \text{where} \quad |g| := |\det(g_{\alpha\beta})|.
\]

(2.4)

Following Lottermoser [20], we form the tensor density

\[
\Omega_{\alpha\beta} := \frac{1}{4\lambda^2} (g_{\alpha\beta} - g_{\alpha\beta})
\]

(2.5)

which will be taken as our primary gravitational variable. Observe that the metric \( g_{\alpha\beta} \) can be recovered from \( \Omega_{\alpha\beta} \) by

\[
g_{\alpha\beta} = \frac{1}{\sqrt{|g|}} g_{\alpha\beta}
\]

where \( g_{\alpha\beta} = g_{\alpha\beta} + 4\lambda^2 \Omega_{\alpha\beta} \) and \( |g| := |\det(g_{\alpha\beta})| \).

Letting \( G \) be a fixed constant and \( \lambda^2 G \) be the gravitational coupling constant, the Einstein equations

\[
G_{\alpha\beta} = 8\pi \lambda^2 GT_{\alpha\beta}
\]

(2.6)

can be written in terms of the density (2.5) as [20],

\[
4\pi G|\Omega|T^\alpha\beta = A^\alpha\beta + B^\alpha\beta + C^\alpha\beta + D^\alpha\beta,
\]

(2.7)
where 
\[ \tilde{g}^{\alpha\beta} := \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta}, \] (2.8)
\[ \tilde{g}_{\alpha\beta} := \sqrt{\tilde{g}} \tilde{g}_{\alpha\beta}, \] (2.9)
\[ \tilde{g}^{\alpha\beta} := \sqrt{\tilde{g}} \tilde{g}^{\alpha\beta} + 4\lambda \tilde{g}^{\alpha\beta}, \] (2.10)
\[ \tilde{g}_{\alpha\beta} := \sqrt{\tilde{g}} \tilde{g}_{\alpha\beta}, \] (2.11)
\[ \lambda := \text{det}(\tilde{g}^{\alpha\beta}), \] (2.12)
\[ A^{\alpha\beta} := 2 \left( \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{g}_{\rho\sigma} - \tilde{g}_{\mu\rho} \tilde{g}_{\nu\sigma} \right) \left( \tilde{g}^{\rho\sigma} \tilde{g}^{\mu\nu} - \frac{1}{2} \tilde{g}^{\mu\sigma} \tilde{g}^{\rho\nu} \tilde{g}^{\alpha\beta} \right), \] (2.13)
\[ B^{\alpha\beta} := 4\lambda \tilde{g}_{\alpha\beta} \left( \tilde{g}^{\mu\nu} \tilde{g}^{\rho\sigma} \Gamma^{\alpha\nu}_{\rho\sigma} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{g}^{\rho\sigma} \tilde{g}^{\alpha\beta} \tilde{g}^{\rho\sigma} \right), \] (2.14)
\[ C^{\alpha\beta} := 4\lambda^2 \left( \Gamma^{\alpha\beta}_{\rho\sigma} \Gamma^{\rho\sigma}_{\alpha\beta} - \frac{1}{2} \tilde{g}^{\rho\sigma} \tilde{g}^{\alpha\beta} \tilde{g}^{\rho\sigma} \right), \] (2.15)
\[ D^{\alpha\beta} := g^{\mu\nu} \tilde{g}^{\alpha\beta}_{\mu\nu} + \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} - 2\tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu}, \] (2.16)
and \( T^{\alpha\beta} \) is the stress–energy tensor. Following [16], we choose harmonic coordinates 
\[ \nabla_\alpha \nabla^\alpha \chi^\beta = 0 \] or equivalently 
\[ \Lambda^{\alpha\beta}_{\beta} = 0, \] (2.17)
which allows us to write the full Einstein field equations as 
\[ 4\pi G |\pi| T^{\alpha\beta} = E^{\alpha\beta}, \] (2.18)
where 
\[ E^{\alpha\beta} := \tilde{g}^{\mu\nu} \Lambda^{\alpha\beta}_{\mu\nu} + 4\lambda^2 \left( \Lambda^{\mu\nu} \Lambda^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta} \Lambda^{\mu\nu}_{\mu\nu} - 2\Lambda^{\mu\nu} \Lambda^{\alpha\beta}_{\mu\nu} \right) + A^{\alpha\beta} + B^{\alpha\beta} + C^{\alpha\beta}. \] (2.19)

Equations (2.18) will be called the reduced field equations.

It is important to recognize that for \( \lambda > 0 \) the reduced field equations (2.18) are not equivalent to the Einstein field equations (2.6) or equivalently (2.7). However, it is shown in [16], section 6, that if \( T^{\alpha\beta}_{\beta} = 0 \) and (2.18) can be solved and the stress–energy tensor \( T^{\alpha\beta} \) satisfies certain conditions then the harmonic condition (2.17) will be automatically satisfied. In this case, a solution to (2.18) will actually be a solution to the full Einstein equation (2.6).

We will let \( A = A_\mu dx^\mu \) denote the \( SU(2) \)-gauge potential and \( \psi \) the dilaton field. The \( SU(2) \) Yang–Mills-dilation equations are 
\[ D^\alpha (e^{2\kappa \psi} F_{\alpha\beta}) = 0, \] (2.20)
\[ \nabla^\alpha \nabla_\alpha \psi = -\frac{k \epsilon Y}{\ell_d} e^{2\kappa \psi} \tilde{g}^{\alpha\beta} \tilde{g}^{\alpha\beta} (F_{\alpha\mu}|F_{\beta\nu}) = 0, \] (2.21)
where \( D_\alpha (\cdot) := \nabla_\alpha (\cdot) + [A_\alpha, \cdot] \) is the gauge covariant derivative, \( \epsilon Y \) is the Yang–Mills coupling constant, \( \ell_d, \kappa \) is the dilaton coupling constants, 
\[ F_{\alpha\beta} := A_{\alpha,\beta} - A_{\beta,\alpha} + [A_\alpha, A_\beta] \] (2.22)
is the gauge field strength and \( [\cdot, \cdot] \) is an Ad-invariant, positive definite inner product on \( su(2) \).

Multiplying (2.20) and (2.21) by \( \sqrt{\lambda |\pi|} \) and \( \lambda |\pi| \), respectively, we find that 
\[ \tilde{g}^{\alpha\beta} (F_{\alpha\beta,\nu} - \Gamma^\mu_{\alpha\nu} F_{\mu\beta} - \Gamma^\mu_{\beta\nu} F_{\mu\alpha} + 2\kappa \psi_{,\nu} F_{\alpha\beta} + 2\kappa \psi_{,\alpha} F_{\nu\beta}) = 0, \] (2.23)
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\[ \bar{g}^{\alpha \beta} \left( \psi_{,\alpha \beta} - \Gamma^\mu_{\alpha \beta} \psi_{,\mu} - \frac{k \ell_y}{\ell_d} e^{2 \kappa} \frac{e^{2 \kappa}}{g^{\mu \nu}} (F_{\alpha \mu} | F_{\beta \nu}) \right) = 0, \]  
(2.24)

where the Christoffel symbols are given by

\[ \Gamma^\mu_{\alpha \beta} = \bar{g}^{\alpha \mu} (2 \bar{g}_{\beta \sigma} \bar{g}^{\sigma \tau} \Lambda_{\tau \sigma} - \bar{g}_{\beta \sigma} \bar{g}^{\sigma \mu}) U_{\sigma \tau, \mu} + 2 \lambda \left( \bar{g}_{\sigma \tau} \delta^\mu_{(\beta} \Lambda_{\sigma \tau \gamma)} - 2 \bar{g}_{\sigma \tau} \Lambda^\mu_{\alpha \gamma} \right). \]  
(2.25)

The stress–energy tensor is given by

\[ T_{\alpha \beta} = \frac{1}{2} \ell_d \left( g^{\alpha \mu} g^{\beta \nu} \psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g^{\alpha \beta} g^{\mu \nu} \psi_{,\mu} \psi_{,\nu} \right) \]  
(2.26)

Using the YMd equations (2.20)–(2.21), it is straightforward to verify that any YMd solution satisfies

\[ T_{\alpha \beta, \beta} = 0 \]  
(2.27)

automatically, irrespective of the metric. Consequently, it will be enough to solve the reduced field equations (2.18) and the YMd equations (2.20)–(2.21) to obtain a solution to the full EYMd field equations.

Let

\[ T^{a \alpha : \beta} := 4 \pi G |d| T^{a \alpha} \]  
(2.28)

so that

\[ T^{a \alpha} = 2 \pi G \ell_d \left( g^{a \mu} g^{b \nu} \psi_{,\mu} \psi_{,\nu} - \frac{1}{2} g^{a \beta} g^{\mu \nu} \psi_{,\mu} \psi_{,\nu} \right) \]  
(2.29)

2.1. Interpretation of solutions for varying \( \lambda \)

Solving equations (2.17), (2.18), (2.20) and (2.21) via the Newtonian perturbation method will produce a one-parameter family of solutions

\[ \{ g_{a \beta}, A^\alpha, \psi \} \quad \lambda \in (0, \Lambda) \]

which will solve the EYMd equations

\[ G_{a \beta} = 8 \pi G \lambda T_{a \beta} \quad G_{\lambda} := \lambda^2 G, \]  
(2.30)

\[ D^a (e^{2 \kappa} F_{a \beta}) = 0, \]  
(2.31)

\[ \nabla^a \nabla_a \psi = \frac{k \ell_y}{\ell_d} e^{2 \kappa} \frac{e^{2 \kappa}}{g^{a \beta}} g^{\alpha \mu} (F_{a \mu} | F_{\beta \nu}) = 0. \]  
(2.32)

The maximum interval \((0, \Lambda)\) for which the one-parameter family of solutions is defined will, in general, depend on the equations and the ‘Newtonian solution’ (i.e. the singular solution at \( \lambda = 0 \)) that is used to start the perturbation argument. We will not discuss methods in this paper to estimate the size of \( \Lambda \) and therefore will have to consider the size of \( \Lambda \) as unknown.

Equation (2.30) shows that the limit \( \lambda \to 0 \) is equivalent to the limit that the gravitational coupling constant \( G_{\lambda} \to 0 \). However, this is not the only interpretation of the limit \( \lambda \to 0 \). Rescaling the fields as

\[ g_{a \beta} := \lambda^{-2} g_{a \beta}, \quad A^\alpha := A^\alpha_{\lambda} \quad \text{and} \quad \psi := \lambda \psi \]

shows that \( \{ g_{a \beta}, A^\alpha, \psi \} \) solve the EYMd equations (2.30)–(2.32) with the following change of coupling constants:

\[ G_{\lambda} \mapsto G, \quad \ell_y \mapsto \ell_y, \quad \ell_d \mapsto \ell_d \quad \text{and} \quad \kappa \mapsto \kappa / \lambda. \]
Thus, the limit $\lambda \to 0$ can also be interpreted as the limit that the dilaton coupling constant $\kappa \to \infty$. We conclude there does not exist a unique interpretation of the limit $\lambda \to 0$ and moreover only certain variables will be defined in the limit $\lambda \to 0$. In our case, the variables that continue to be defined at $\lambda = 0$ are the unscaled dilaton field $\psi$, the gauge potential $A_\alpha$ and the metric density $\mathcal{U}_{\alpha\beta}$. We stress that the metric $g_{\alpha\beta}$ does not exist at $\lambda = 0$.

Since we do not know the size of $\Lambda$, the Newtonian perturbation method does not necessarily produce solutions for all possible values of the coupling constants. Consider $\{G, \ell_d, \kappa, \ell_Y\}$ as a set of coupling constants in some fixed units for which we would like to have a solution to the EYMd equations. If $\Lambda > 1$, then we could choose $\lambda = 1$ and in that case $G_\lambda = G$ and we would have a solution to the EYMd equations for the fixed coupling constants $\{G, \ell_d, \kappa, \ell_Y\}$. On the other hand, if $\Lambda < 1$ then $G_\lambda < G$ and we will have to be satisfied with a solution to the EYMd equation where the gravitational coupling constant is smaller than $G$. As discussed above, we could rescale the metric and the dilaton field to get a solution where $G_\lambda = G$ provided that we change the dilaton coupling constant $\kappa$ to $\kappa/\lambda$. Thus, in general the solutions that the Newtonian perturbation methods produces will have some restriction on the size of at least one of the coupling constants.

3. Weighted Sobolev spaces

Let $V$ denote a finite-dimensional vector space with norm $|\cdot|$.

Definition 3.1. The weighted Lebesgue space $L^p_\delta(\mathbb{R}^n, V)$, $1 \leq p \leq \infty$, with weight $\delta \in \mathbb{R}$ is the set of all measurable maps from $\mathbb{R}^n$ to $V$ in $L^p_{loc}(\mathbb{R}^n, V)$ such that the norm

$$\|u\|_{p, \delta} = \left\{ \begin{array}{ll} \left( \int_{\mathbb{R}^n} |u|^p \sigma^{-k\delta-n} \, dx \right)^{\frac{1}{p}} & \text{if } p < \infty \\ \text{ess sup}_{x \in \mathbb{R}^n} (\sigma^{-\delta}|u|) & \text{if } p = \infty, \end{array} \right.$$ 

is finite. Here $\sigma(x) := \sqrt{|x|^2 + 1}$. If $V = \mathbb{R}$ then we write $L^p_\delta(\mathbb{R}^n)$ instead of $L^p_\delta(\mathbb{R}^n, V)$.

Definition 3.2. The weighted Sobolev space $W^{k,p}_\delta(\mathbb{R}^n, V)$, $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, with weight $\delta \in \mathbb{R}$ is the set

$$W^{k,p}_\delta(\mathbb{R}^n, V) := \left\{ u \in L^p_\delta(\mathbb{R}^n, V) | \partial^I u \in L^p_{\delta-|I|}(\mathbb{R}^n, V) \text{ for all } I : |I| \leq k \right\}$$

with norm

$$\|u\|_{k, p, \delta} := \sum_{|I| \leq k} \|\partial^I u\|_{p, \delta-|I|},$$

where $I = (I_1, I_2, \ldots, I_n)$ is a multi-index and $\partial^I := \partial_1^{I_1} \partial_2^{I_2} \cdots \partial_n^{I_n}$. If $V = \mathbb{R}$ then we will write $W^{k,p}_\delta(\mathbb{R}^n)$ instead of $W^{k,p}_\delta(\mathbb{R}^n, V)$.

From the definition, it is clear that differentiation

$$\partial_j : W^{k,p}_\delta(\mathbb{R}^n, V) \to W^{k-1,p}_{\delta-1}(\mathbb{R}^n, V) \quad (3.1)$$

is a continuous linear map. Also from the definition and Hölder's inequality it is easy to show (see also [2], proposition 1.2 (i)) that if $k_1 \geq k_2$ and $\delta_1 \leq \delta_2$ then

$$W^{k_1,p}_{\delta_1}(\mathbb{R}^n, V) \subset W^{k_2,p}_{\delta_2}(\mathbb{R}^n, V). \quad (3.2)$$

Finally, we note that the set $C_0^\infty(\mathbb{R}^n, V)$ of smooth maps from $\mathbb{R}^n$ to $V$ with compact support is dense in $W^{k,p}_\delta(\mathbb{R}^n, V)$. As above, if $V = \mathbb{R}$ then we write $C_0^\infty(\mathbb{R}^n)$ instead of $C_0^\infty(\mathbb{R}^n, V)$. We will now state some results in weighted Sobolev spaces that will be needed. For proofs see [2, 8].
Lemma 3.3. If there exists a multiplication \( V_1 \times V_2 \to V_3(u, v) \mapsto u \cdot v \) then the corresponding multiplication

\[
W^{k_1, p}_{\delta_1}(\mathbb{R}^n, V_1) \times W^{k_2, p}_{\delta_2}(\mathbb{R}^n, V_2) \to W^{k_3, p}_{\delta_3}(\mathbb{R}^n, V_3) : (u, v) \mapsto u \cdot v
\]

is bilinear and continuous if \( k_1, k_2 \geq k_3, k_3 < k_1 + k_2 - n/p \) and \( \delta_1 + \delta_2 < \delta_3 \).

**Proof.** See lemma 2.5 in [8] for the case \( p = 2 \). For all \( p \) this can be proved easily using theorem 1.2 of [2]. \( \square \)

**Theorem 3.4.** For \( \delta < 0 \) the Laplacian

\[
\Delta := \sum_{j=1}^n \partial^2_j : W^{k, p}_\delta(\mathbb{R}^n, V) \to W^{k-2, p}_\delta(\mathbb{R}^n, V)
\]

is continuous and injective. Moreover, if \( 2 - n < \delta < 0 \) then the Laplacian is an isomorphism.

The inverse is given by

\[
(\Delta^{-1}u)(x) = \frac{1}{(2 - n)\omega_n} \int_{\mathbb{R}^n} \frac{u(y)}{|x - y|^{(n-2)}} \, dy,
\]

where \( \omega_n \) is the area of the unit sphere in \( \mathbb{R}^n \).

**Lemma 3.5.** For \( k_1 > k_2, \delta_1 < \delta_2 \) and \( 1 \leq p < \infty \) the embedding \( W^{k_1, p}_\delta(\mathbb{R}^n, V) \to W^{k_2, p}_\delta(\mathbb{R}^n, V) \) is compact.

As with the Sobolev spaces, we can define a weighted versions of the \( C^{k,\alpha}(\mathbb{R}^n, V) \) spaces. For a map \( u \in C^0(\mathbb{R}^n, V) \) and \( \delta \in \mathbb{R}, \alpha > 0 \), let

\[
\|u\|_{C^{k, \alpha}_\delta} := \|u\|_\delta + \sup_{x \in \mathbb{R}^n} \left( \max_{0 < \sigma < 1} \sup_{y \in B_{|x-y|}(x)} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right).
\]

Using this norm we define the norm \( \|\cdot\|_{C^{k, \alpha}_\delta} \) in the usual way:

\[
\|u\|_{C^{k, \alpha}_\delta} := \sum_{|I| \leq k} \|\partial^I u\|_{C^{\alpha}_\delta}.
\]

So then

\[
C^{k, \alpha}_\delta(\mathbb{R}^n, V) := \{ u \in C^{k}(\mathbb{R}, V) \|u\|_{C^{k, \alpha}_\delta} < \infty \}.
\]

**4. Static spherically symmetric fields**

We assume that all the fields are static and that \( \partial_0 \) is a timelike hypersurface orthogonal Killing vector field for the metric. Therefore, \( \partial_0 \Omega^{0\beta} = 0, \partial_0 A_\alpha = 0, \partial_0 \psi = 0 \) and \( \Omega^{0j} = \Omega^{\beta j} = 0 \).

Since \( \Omega^{0\beta} \) is symmetric, i.e. \( \Omega^{0\beta} = \Omega^{\beta 0} \), we define the following subspace of the \( 4 \times 4 \) matrices:

\[
\mathcal{S} := \{ X = (X^{\alpha\beta}) \in M_{4 \times 4} | X^{\alpha\beta} = X^{\beta\alpha} \text{ and } X^{0j} = X^{\beta j} = 0 \}.
\]

Then letting \( \Omega = (\Omega^{\alpha\beta}), \Omega \) takes values in \( \mathcal{S} \). We will also assume that \( A_0 = 0 \). Therefore, if we write the gauge potential \( A_i \) as a 3-tuple \( A = (A_1, A_2, A_3) \) then the gauge potential \( A \) takes values in the space \( \text{su}(2)^3 \) which carries a norm \( |A|^2 := \sum_{i=1}^3 |A_i|^2 \). Therefore, \( W^{k, p}_\delta(\mathbb{R}^3, \mathcal{S}) \), \( W^{k, p}_\delta(\mathbb{R}^3, \text{su}(2)^3) \) and \( W^{k, p}_\delta(\mathbb{R}^3, \text{su}(2)^3) \) are appropriate functions spaces for the static metric densities, dilaton fields and gauge potentials, respectively.

In addition to being static, we will also assume that our fields are spherically symmetric. To define what we mean by spherical symmetry we first need to specify an action of
$SO(3)$ on spacetime $\mathbb{R}^4$. We want $SO(3)$ to act on the hypersurfaces orthogonal to the timelike killing vector field $\partial_t$. So using the matrix representation of $SO(3)$ given by $SO(3) = \{ a \in M_{3 \times 3} \vert a^T a = a^{-1} \text{ and } \det(a) = 1 \}$ we define a $SO(3)$ action on spacetime by $\Phi : SO(3) \times \mathbb{R}^4 \to \mathbb{R}^4 : (a, (x^0, x)) \to \Phi_a(x^0, x) := (x^0, ax)$ where we are treating $x$ as a column vector and $ax$ denotes matrix multiplication. We then get the induced action on functions via pullbacks. Therefore, $SO(3)$ acts on the dilaton field $\psi(x)$ as follows: $\Phi_a(\psi)(x) := \psi(ax)$. Lifting the $SO(3)$ action on spacetime to the tensor bundle, we get the following action on the metric densities: $\Phi_a(\Omega)(x) := \bar{a}\Omega(ax)\bar{a}^T$ where $\bar{a} := \text{diag}(1, a)$. Let $\widetilde{C}_0^{\infty}(\mathbb{R}^3)$ denote the set of smooth $SO(3)$-invariant functions with compact support, i.e. $\widetilde{C}_0^{\infty}(\mathbb{R}^3) := \{ \psi \in C_0^{\infty}(\mathbb{R}^3) \vert \psi = \Phi_a\psi \text{ for all } a \in SO(3) \}$. In other words, $\widetilde{C}_0^{\infty}(\mathbb{R}^3)$ is the set of radial functions on $\mathbb{R}^3$. Similarly, define

$$
\widetilde{C}_0^{\infty}(\mathbb{R}^3, S) := \{ \Omega \in C_0^{\infty}(\mathbb{R}^3, S) \vert \Omega = \Phi_a\Omega \text{ for all } a \in SO(3) \}.
$$

In addition to being spherically symmetric, we will assume that our gauge potential is purely magnetic. Choosing an appropriate gauge, the gauge potential can then be written as [3]

$$
A_j(x) := u(x)\epsilon_j^i k x^i \tau_j
$$

where $u(x) = u(|x|)$ and

$$
\tau_1 = \frac{1}{2i} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \tau_2 = \frac{1}{2i} \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \tau_3 = \frac{1}{2i} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
$$

is a basis for $\mathfrak{su}(2)$. This form of the gauge potential is known as the Witten ansatz.

We then define the set of smooth static spherically symmetric purely magnetic gauge potentials with compact support by

$$
\mathcal{A}_0^{\infty} := \{ A : \mathbb{R}^3 \to \mathfrak{su}(2)^3 \vert A_j(x) = u(x)\epsilon_j^i k x^i \tau_j \text{ for some } u \in \widetilde{C}_0^{\infty}(\mathbb{R}^3) \}.
$$

Note that every $A \in \mathcal{A}_0^{\infty}$ satisfies

$$
\text{div } A := \sum_{j=1}^3 \partial_j A_j = 0. \quad (4.1)
$$

So then the spherically symmetric Sobolev spaces we consider are

$$
D^{k,p}_\delta := \mathcal{C}_0^{\infty}(\mathbb{R}^3) \subset W^{k,p}_\delta(\mathbb{R}^3), \quad (4.2)
$$

$$
U^{k,p}_\delta := \mathcal{C}_0^{\infty}(\mathbb{R}^3, S) \subset W^{k,p}_\delta(\mathbb{R}^3, S), \quad (4.3)
$$

and

$$
\mathcal{A}^{k,p}_\delta := \mathcal{A}_0^{\infty} \subset W^{k,p}_\delta(\mathbb{R}^3, \mathfrak{su}(2)^3). \quad (4.4)
$$

Because of (4.1) we have

$$
\text{div } A = 0 \quad \text{for all } A \in \mathcal{A}^{k,p}_\delta \quad (4.5)
$$

by the density of $\mathcal{A}_0^{\infty}$ in $\mathcal{A}^{k,p}_\delta$ and the continuity of differentiation (see (3.1)). Therefore, each $A \in \mathcal{A}^{k,p}_\delta$ automatically satisfies the Coulomb gauge condition.

We now analyse the Laplacian $\Delta = \sum_{j=1}^3 \partial_j^2$ on the spherically symmetric spaces (4.2)–(4.4).
Proposition 4.1. For $-1 < \delta < 0$ the Laplacian $\Delta : \mathcal{D}^{k,p}_\delta \to \mathcal{D}^{k,p}_{\delta-2}$ is an isomorphism.

Proof. Straightforward calculation shows that $\Delta(\overline{C}^\infty_0(\mathbb{R}^3)) \subset \overline{C}^\infty_0(\mathbb{R}^3)$. Using formula (3.3), it is not difficult to verify that if $\psi \in \overline{C}^\infty_0(\mathbb{R}^3)$ then $\Phi_\Delta(\Delta^{-1}\psi) = \Delta^{-1}\psi$ for all $a \in SO(3)$. The proposition then follows from these two results and theorem 3.4. \qed

The next proposition is proved in the same fashion.

Proposition 4.2. For $-1 < \delta < 0$ the Laplacian $\Delta : \mathcal{U}^{k,p}_\delta \to \mathcal{U}^{k-2,p}_{\delta-2}$ is an isomorphism.

We will often use the following notation:

$$r := |x| \quad \text{and} \quad (\cdot)' = \frac{d(\cdot)}{dr}.$$  

The next proposition is interesting because it shows that on the space of the spherically symmetric gauge potentials, the Laplacian is invertible for a larger range of weights $\delta$ than one would expect from theorem 3.4. The reason for this is that the Laplacian (see equation (4.6)) acting on the space of spherically symmetric gauge potentials is essentially equivalent to the Laplacian acting on the space of spherically symmetric functions on $\mathbb{R}^5$.

We note that this observation has also been used in [14] to construct global solutions of the Yang–Mills equations on Minkowski spacetime.

Proposition 4.3. For $-2 < \delta < 1$, $\delta \neq -1, 0$, the Laplacian $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ is an isomorphism.

Proof. By definition of $\mathcal{A}^\infty_0$, if $A \in \mathcal{A}^\infty_0$ then $A_i = u(r)\epsilon^i_jk^x\tau_j$ for some $u \in \overline{C}^\infty_0(\mathbb{R}^3)$. So

$$\Delta A_i = \left(\frac{4}{r^2}\frac{d}{dr}\frac{d}{dr}(r)\right)\epsilon^i_jk^x\tau_j$$  

and hence $\Delta(\mathcal{A}^\infty_0) \subset \mathcal{A}^\infty_0$. Therefore, $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ is continuous by the density of $\mathcal{A}^\infty_0$ and the continuity of $\Delta : \mathcal{U}^{2,p}(\mathbb{R}^3, su(2)^3) \to \mathcal{U}^{0,p}(\mathbb{R}^3, su(2)^3)$.

Suppose $A \in \mathcal{A}^{2,p}_{\delta}$ satisfies $\Delta A = 0$. Then by elliptic regularity $A \in C^\infty$ and hence $A_i = u(r)\epsilon^i_jk^x\tau_j$ for some smooth function $u(r)$ on $[0, \infty)$ that satisfies (i) $u(r) = u_0 + O(r^2)$ as $r \to 0$ for some constant $u_0$ and (ii) the differential equation

$$u''(r) + \frac{4}{r}u'(r) = 0.$$  

However, the general solution to this equation is $u(r) = c_1 + c_2r^3$ for some constants $c_1, c_2$. This shows that $u(r) = u_0$ as $u(r)$ is bounded near $r = 0$. So $A_i(x) = u_0\epsilon^i_jk^x\tau_j$. Any positive definite invariant function of $a(x)$ is given by $(AB) = -2\rho Tr(AB)$ for some $\alpha > 0$. A short calculation then shows that $|\epsilon^i_jk^x\tau_j|^2 = 2\alpha r^2$. Using this we find that $|A(x)| = \sqrt{2\alpha|u_0|r}$. This shows that $A \in \mathcal{A}^{2,p}_{\delta}$ for $\delta < 1$ only if $u_0 = 0$. This establishes that $\ker \Delta|_{\mathcal{A}^{2,p}_{\delta}} = 0$ for $\delta < 1$.

It follows from theorem 1.10 in [2] that $\Delta : \mathcal{U}^{2,p}(\mathbb{R}^3, su(2)^3) \to \mathcal{U}^{0,p}(\mathbb{R}^3, su(2)^3)$ has closed range for $-2 < \delta < 1$ and $\delta \neq -1, 0$. This implies that $\Delta : \mathcal{A}^{2,p}_{\delta} \to \mathcal{A}^{0,p}_{\delta-2}$ also has closed range for the same values of $\delta$. With respect to the pairing $(A, B) = \int (AB) f^1x$ the Laplacian has a formal adjoint $\Delta^* = \Delta$. Since $\mathcal{U}^{0,p}(\mathbb{R}^3, su(2)^3)^* = \mathcal{U}^{0,p-1}(\mathbb{R}^3, su(2)^3)$ where $p' = p/(p - 1)$, it follows from propositions 1.6 and 1.14 of [2] that $\ker \Delta^* \subset \mathcal{U}^{2,p-1}(\mathbb{R}^3, su(2)^3)$. Therefore, the arguments in the previous paragraph show that

$$\dim \ker \Delta|_{\mathcal{A}^{2,p}_{\delta}} = 0$$  

for $-2 < \delta < 1$. Hence, $\Delta$ is an isomorphism for $\delta \neq 0, -1$ and $-2 < \delta < 1$. \qed
5. Yang–Mills-dilaton solutions

To employ the Newtonian perturbation method, we need static solutions to the Euclidean YMd equations

$$\Delta \alpha - \frac{\kappa \ell_Y}{\ell_d} e^{2u} \delta^{ij} \delta^{kl} \left( F^W_{ik} F^W_{jl} \right) = 0,$$

$$\delta^{ij} \left( \partial_i F^W_{ij} + 2\kappa F^W_{ij} \partial_i \alpha + \left[ W_i, F^W_{ij} \right] \right) = 0.$$  

(5.1)

Assuming that $\alpha$ is a function of $r$ only and

$$W_i(r) := \frac{w(r) - 1}{r^2} e^{i/4} \kappa^4 \tau_j,$$

(5.3)

the YMd equations (5.1)–(5.2) become

$$w'' = -2\kappa \alpha' w' + \left( w^2 - 1 \right) w,$$

(5.4)

$$\left( r^2 \alpha' \right)' = 4\kappa \ell_Y e^{2u} \left( w^2 + \frac{(w^2 - 1)^2}{2r^2} \right).$$

(5.5)

It is easy to check that

$$\bar{w}(r) := w \left( \sqrt{8\kappa} \sqrt{\frac{\ell_Y}{\ell_d}} r \right)$$

and

$$\bar{\alpha}(r) := 2\kappa \alpha \left( \sqrt{8\kappa} \sqrt{\frac{\ell_Y}{\ell_d}} r \right)$$

satisfy (5.4)–(5.5) with $\kappa = 1/2$ and $4\kappa \ell_Y / \ell_d = 1$. Therefore, we can, without any loss of generality, consider the equations

$$w'' = -\alpha' w' + \frac{(w^2 - 1)w}{r^2},$$

(5.6)

$$\left( r^2 \alpha' \right)' = e^u \left( w^2 + \frac{(w^2 - 1)^2}{2r^2} \right).$$

(5.7)

We note that these equations have a scaling symmetry. To be precise, if $(w(r), \alpha(r))$ is a solution then

$$w\beta(r) := w \left( e^{\beta/2} r \right)$$

will also solve the equations for any $\beta \in \mathbb{R}$.

The next theorem provides the existence of an infinite number of solutions to (5.6)–(5.7).

**Theorem 5.1** (theorem 1, [15]). There exists a sequence $n = 1, 2, 3, \ldots$, of analytic solutions $(w_n(r), \alpha_n(r))$ to the YMd equations (5.6)–(5.7) defined on $(0, \infty)$ such that $w_n$ has precisely $n$ zeros and $\lim_{r \to \infty} w_n(r) = (-1)^n$.

**Remark 5.2.** It is also established in [15] that the solutions $(w_n(r), \alpha_n(r))$ from theorem 5.1 satisfy the following:

1. $\lim_{r \to \infty} \alpha_n(r) = \text{const}$,
2. $|w| \leq 1$, $w'_n \in o(r^{-1})$ and $\alpha'_n = O(r^{-2})$ as $r \to \infty$.
3. $w_n(r)$ and $\alpha_n(r)$ are analytic in a neighbourhood of $r = 0$ and

$$w_n(r) = 1 - \beta_n r^2 + O(r^4) \quad \text{as} \quad r \to 0$$

for a constant $\beta_n > 0$. 

4. \( w_n' \) is either strictly positive or negative for large enough \( r \).

By using the scaling transformation (5.8), we can assume \( \lim_{r \to \infty} \alpha_n(r) = 0 \).

These are the solutions that we will use to start our perturbation argument. However, for these solutions to be useful for our purposes we will need more information about their large \( r \) behaviour. The required information is contained in the next proposition.

**Proposition 5.3.** Suppose \((w(r), \alpha(r))\) is a solution to the flat YM\(d\) equations (5.6)–(5.7) defined on \((0, \infty)\) that satisfies \( |w(r)| < 1 \) for all \( r \in (0, \infty) \), \( \lim_{r \to 0} w(r) = 0 \), \( \lim_{r \to \infty} w(r) = 1 \) or \( \lim_{r \to \infty} w(r) = -1 \), \( \lim_{r \to \infty} \alpha(r) = 0 \), \( w' = o(1/r) \) and \( \alpha' = O(r^{-2}) \) as \( r \to \infty \). Furthermore, suppose that there exist an \( R > 0 \) such that \( w'(r), w(r) > 0 \) or \( w'(r), w(r) < 0 \) for all \( r \geq R \). Then for any \( \epsilon \in (0, 1) \): \( w'' = O(r^{-2\epsilon-2}), w' = O(r^{-2\epsilon}) \), \( w - 1 \) or \( w + 1 = O(r^{-2\epsilon+1}) \), \( \alpha'' = O(r^{-2}) \) and \( \alpha = O(r^{-1}) \) as \( r \to \infty \).

**Proof.** Define

\[
 u := 1 - w^2 
\]

and

\[
 Z_{\pm} := 1 - \frac{w^2}{r^2} \pm \frac{w'}{r^{1/2}}. 
\]

Note that

\[ 0 < u(r) \leq 1 \quad \forall r \in (0, \infty) \]

as \( |w| < 1 \) on \((0, \infty)\). Also note that \( Z \) can be written as

\[
 Z_{\pm} = \frac{u}{r} \pm \frac{w'}{r^{1/2}}. 
\]

**Lemma 5.4.** If \( w'(r), w(r) > 0(w'(r), w(r) < 0) \) for \( r \geq R \) and there exist an \( R^* \geq R \) such that \( Z_{\pm}(r) < 0(Z_{\pm}(r) < 0) \) for all \( r \geq R^* \) then \( w' = O(r^{-2}) \) as \( r \to \infty \).

**Proof.** We only prove the case where \( w'(r) \) and \( w(r) \) are both positive for sufficiently large \( r \). The other case can be handled using similar arguments. Since \( w(r) > 0 \) for \( r \geq R \), \( Z_{\pm}(r) < 0 \) implies that

\[
 \frac{1}{r} \leq -\frac{w'}{u} \quad \forall r \geq R^* 
\]

as \( u > 0 \). Integrating this expression between \( R^* \) and \( r \) yields

\[
 \ln \left( \frac{r}{R^*} \right) < \ln \left( \frac{u(R^*)}{u(r)} \right),
\]

or equivalently

\[
 u(r) < \frac{C}{r} \quad \forall r \geq R^* 
\]

where \( C = u(R^*)\sqrt{R^*} \). Note that (5.6) can be written as \((e^w w')' = -r^{-2}wu'\). Then for \( r \geq R^* \), integration yields

\[
 e^{\alpha(r)} w'(r) = \int_r^{\infty} \frac{wu}{\rho^2} \, d\rho \quad \text{(since } \lim_{r \to \infty} e^{\alpha(r)} w'(r) = 0\text{)} \]

\[
 \leq \int_r^{\infty} \frac{C}{\rho^3} \, d\rho \quad \text{(by (5.9) and } |w| \leq 1\text{)} \]

\[
 = \frac{2C}{3} \frac{1}{r^2}. 
\]

The result then follows since \( w'(r) > 0 \) for \( r \geq R \) and \( \lim_{r \to \infty} \alpha(r) = 0 \).  \( \square \)
Lemma 5.5. If \( w'(r), w(r) > 0 (w'(r), w(r) < 0) \) for \( r \geq R \) and there exists an \( R^* \geq R \) such that \( Z_+(r) > 0 (Z_-(r) > 0) \) for all \( r \geq R^* \) then for any \( \varepsilon \in (0, 1) \) \( w' = O(r^{-2\varepsilon}) \).

Proof. Again, we only prove the case where \( w'(r) \) and \( w(r) \) are both positive for sufficiently large \( r \), with the other cases following from similar arguments. Since \( \lim_{r \to \infty} w(r) = 1 \) there exists an \( \hat{R} \geq R^* \) such that \( w(r) > 0 \) for all \( r \geq \hat{R} \). Therefore, \( Z_+(r) > 0 \) for \( r \geq R^* \) implies that
\[
\frac{w(1 - w^2)}{r^2} - \frac{2w^2w'}{r} + \frac{ww'}{r^{3/2}} > 0 \quad \forall \ r \geq \hat{R}
\]
as \( w' > 0 \) for all \( r \geq R^* \). It then follows from (5.6) that
\[
-w'' - \alpha w' > \frac{2w^2w'}{r} - \frac{ww'}{2r^{3/2}} \quad \forall \ r \geq \hat{R}.
\]
Fix \( \varepsilon \in (0, 1) \). As \( \lim_{r \to \infty} w(r) = 1 \), there exists an \( R_\varepsilon \geq \hat{R} \) such that \( w(r) \geq \sqrt{\varepsilon} \) for all \( r \geq R_\varepsilon \). Thus,
\[
-w'' - \alpha w' > \frac{2\varepsilon}{r}w' - \frac{w'}{2r^{3/2}} \quad \forall \ r \geq R_\varepsilon.
\]
Note that in deriving this inequality we have also used \( |w| \leq 1 \). Dividing (5.10) by \( w' \) yields
\[
\frac{-w''}{w'} > -\alpha' + \frac{2\varepsilon}{r} - \frac{1}{2r^{3/2}} \quad \forall \ r \geq R_\varepsilon.
\]
Integrating gives
\[
\ln \left( \frac{w'(R_\varepsilon)}{w'(r)} \right) > \alpha(r) - \alpha(R_\varepsilon) + \ln \left( \frac{r}{R_\varepsilon} \right) \geq -\frac{1}{\sqrt{\varepsilon}} \quad \forall \ r \geq R_\varepsilon,
\]
and hence
\[
w'(r) < \left( \frac{w'(R_\varepsilon) e^{\alpha(r)} e^{2R_\varepsilon^{-1/2}}}{e^{\alpha(R_\varepsilon)}} \right) \frac{1}{r^{2\varepsilon}} \quad \forall \ r \geq R_\varepsilon.
\]
The proof then follows as \( \lim_{r \to \infty} \alpha(r) = 0 \) and \( w'(r) > 0 \) for all \( r \geq R_\varepsilon \).

Lemma 5.6.
\[
w' = O(r^{-2\varepsilon}) \quad \text{for any} \quad \varepsilon \in (0, 1).
\]

Proof. We need to consider two cases, namely \( w'(r), w(r) > 0 \) and \( w'(r), w(r) < 0 \) for \( r \geq R \). We will prove the lemma assuming that \( w'(r), w(r) > 0 \) for \( r \geq R \) with the other case following from similar arguments. We may assume that there exists a sequence \( \{r_n\}_{n=1}^\infty \) such that \( R \leq r_1 < r_2 < r_3 < \cdots, \lim_{n \to \infty} r_n = \infty \) and \( Z_+(r_n) = 0 \), where \( n = 1, 2, 3, \ldots, \) because otherwise we are done by lemmas 5.4 and 5.5. From (5.6), it is easy to verify that \( u = 1 - w^2 \) satisfies
\[
u'' = \frac{2w^2}{r^2}u - 2|u'|^2 + 2wu'w'.
\]
Define
\[
f(r) := 2|u'|^2 + \frac{wu}{r^2} + \left( \frac{3}{2r^{3/2}} + \frac{\alpha^2}{r^{2/3}} - 2w\alpha' \right) w'.
\]
Since \( \alpha' = O(r^{-2}) \), \( |w| \leq 1 \) and \( w'(r) > 0 \) for all \( r \geq R \), there exists an \( \hat{R} \geq R \) such that
\[
f(r) > 0 \quad \forall \ r \geq \hat{R}.
\]
Choose \( m \in \mathbb{N} \) large enough so that
\[ r_m \geq \bar{R}. \] (5.14)

By definition of the \( r_m \) we have
\[ Z_+(r_m) = \frac{u(r_m)}{r_m} + u'(r_m) + \frac{w'(r_m)}{r_m} = 0. \] (5.15)

Consider the following initial value problem:
\[ v'' = \frac{2}{r^2} v + \frac{3w'}{2r^{3/2}} + \frac{\alpha'w'}{r^{1/2}} + \frac{wu}{r^2}, \] (5.16)
\[ v(r_m) = u(r_m), \quad v'(r_m) = u'(r_m). \] (5.17)

From (5.11) and (5.16) we see that
\[ (v - u)'' = \frac{2}{r^2} (v - w^2w) + f(r). \] (5.18)

Then \(|w| \leq 1, (5.12), (5.13), (5.17)\) and (5.18) imply that \((v - u)''(r_m) > 0\). Therefore, there exists a \( \sigma > 0 \) such that \( v'(r) > u'(r) \) for \( r_m \leq r < r_m + \sigma \) and hence \( v(r) > u(r) \) for \( r_m \leq r < r_m + \sigma \). Let \( r_* \) be the first \( r \) greater than \( r_m \) for which \( v'(r) = u'(r) \). Using \( v(r_*) \geq u(r_*), \) \(|w| \leq 1, (5.12), (5.13)\) and (5.18), we see that \((u - v)''(r_*) > 0\) which contradicts \( v'(r_*) = u'(r_*) \). Therefore, \( v'(r) > u'(r) \) for all \( r \geq r_m \) which implies that
\[ 1 - w^2(r) < v(r) \quad \forall r \geq r_m. \] (5.19)

The general solution to (5.16) is
\[ v = \frac{C_1}{r} + C_2 r^2 - \frac{1}{r} \int_{r_m}^{r} \rho^{1/2} w' \rho \, d\rho, \] (5.20)
where \( C_1 \) and \( C_2 \) are arbitrary constants. So then
\[ C_2 = \frac{1}{3r} \left( \frac{v}{r} + v' + \frac{w'}{r^{1/2}} \right) \]
and hence
\[ C_2 = \frac{1}{3r_m} \left( \frac{u(r_m)}{r_m} + u'(r_m) + \frac{w'(r_m)}{r_m} \right) \quad \text{by (5.17)} \]
\[ = 0 \quad \text{by (5.15)}. \]

Therefore,
\[ 1 - w^2(r) \leq \frac{C_1}{r} - \frac{1}{r} \int_{0}^{r} \rho^{1/2} w' \rho \, d\rho \quad \forall r \geq r_m. \] (5.21)

As \( w' = o(r^{-1}) \) it is easy to see that there exists a \( C > 0 \) such that
\[ 1 - w^2(r) \leq \frac{C}{r^{1/2}} \quad \forall r \geq r_m. \]

Using the same arguments as in lemma 5.4 it follows from this inequality that \( w' = O(r^{-3/2}) \). Using this back in (5.21) we see that
\[ 1 - w^2(r) \leq \frac{C}{r^{1/2}} \quad \forall r \geq r_m \]
for any \( \epsilon \in (0, 1) \). Again using the arguments from lemma 5.4 we find that \( w' = O(r^{-2\epsilon}) \) for any \( \epsilon \in (0, 1) \). \( \square \)
Since \( \lim_{r \to \infty} w(r) = 1 \), we have \( 1 - w(r) = \int_r^\infty w'(\rho) \, d\rho \) and hence
\[
|1 - w(r)| \leq \int_r^\infty |w'(\rho)| \, d\rho. \tag{5.22}
\]
But \( w' = O(r^{-2\epsilon}) \) by lemma 5.6, and hence \( 1 - w(r) = O(r^{-2\epsilon + 1}) \) by (5.22). Writing (5.6) as
\[
w'' = -\alpha' w' + (w - 1)(w + 1)w \tag{5.23}
\]
we see that \( w'' = O(r^{-2\epsilon - 1}) \) since \( |w| \leq 1, w' = O(r^{-2\epsilon}), 1 - w(r) = O(r^{-2\epsilon + 1}) \) and \( \alpha' = O(r^{-\epsilon}). \) Using (5.7), \( \lim_{r \to \infty} \alpha(r) = \infty \) and similar arguments, it is straightforward to show that \( \alpha = O(r^{-1}) \) and \( \alpha'' = O(r^{-2}). \)

We can now use the previous proposition to show that the gauge potential and its corresponding field arising from the solutions in theorem 5.1 lie in certain weighted spaces.

**Proposition 5.7.** Suppose \((w(r), \alpha(r))\) is one of the solutions from theorem 5.1. If \( W_\alpha \) is given by (5.3) for \( \alpha = 1, 2, 3 \), \( W_0 = 0 \) and \( F_{\alpha\beta}^{W} = \partial_\alpha W_\beta - \partial_\beta W_\alpha + [W_\alpha, W_\beta] \) (5.24) then \( W_\alpha \in \mathcal{A}_2^{p, \delta} \) for any \( \delta > -1, 1 < p < \infty \) and \( F_{\alpha\beta}^{W} \in W_2^{p, \delta}(\mathbb{R}^3, su(2)) \) for any \( \delta > -2, 1 < p < \infty \).

**Proof.** A short calculation shows that non-zero components of \( F_{\alpha\beta}^{W} \) are
\[
F_{ij}^{W} = \varepsilon_{ijk} \left[ \frac{w'(r)}{r} \left( \frac{x^k x^l}{r^2} \right) + \frac{w^2 - 1}{r^4} x^k x^l \right] \tau_j. \tag{5.25}
\]
The proof then follows directly from theorem 5.1, proposition 5.3, formulae (5.3), (5.25) and the definition of the spaces \( \mathcal{A}_2^{p, \delta}, W_2^{p, \delta}(\mathbb{R}^3, su(2)) \).

**Remark 5.8.** For the remainder of this report we will always assume that \((w(r), \alpha(r))\) is one of the solutions to the Euclidean Yang–Mills-dilaton equations (5.6)–(5.7) from theorem (5.1).

### 5.1. Solutions of the linearized Yang–Mills equations

As will be seen later in section 8, the main obstacle to having a complete proof of the existence of EYMd solutions is that we do not yet have a complete understanding of the solutions to the lYMd equations
\[
v'' + \phi' w' + \alpha' v' - \frac{(3w^2 - 1)}{r^2} v = 0, \tag{5.26}
\]
\[
(r^2 \phi')' - e^\alpha \left( w^2 + \frac{(w^2 - 1)^2}{2r^2} \right) \phi - 2e^\alpha \left( w' v' + \frac{(w^2 - 1)}{r^2} wv \right) = 0, \tag{5.27}
\]
that satisfy the boundary conditions
\[
v(r) = O(r^2) \quad \text{and} \quad \phi(r) = O(1) \quad \text{as} \quad r \to 0. \tag{5.28}
\]
Using the fact that \( w(r) \) and \( \alpha(r) \) are analytic in a neighbourhood and that \( w(r) = 1 - \beta r^2 + O(r^4) \) and \( \alpha(r) = O(1) \) as \( r \to 0 \) (see remark 5.2), it can be shown using theorem 5.0.6 of [21] that there are exactly two \( C^2 \) linearly independent solutions \((v_1(r), \phi_1(r))\)
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and \((v_2(r), \phi_2(r))\) to (5.26)–(5.27) which satisfy the boundary conditions (5.28). The solutions are uniquely determined by their expansions near \(r = 0:\)

\[
v_1(r) = -\beta r^2 + O(r^4) \quad \phi_1(r) = -1 + O(r^2)
\]

and

\[
v_2(r) = -r^2 + O(r^4) \quad \phi_2(r) = O(r^2)
\]
as \(r \to 0\). It also follows from theorem 5.0.6 of [21] that the solutions are analytic in a neighbourhood of \(r = 0\). This coupled with the fact that \((w(r), \alpha(r))\) are analytic for \(r > 0\) implies that the solutions \((v_1(r), \phi_1(r))\) and \((v_2(r), \phi_2(r))\) are also analytic for \(r > 0\).

The following lemma shows that we can exactly determine the solution \((v_1(r), \phi_1(r))\).

**Lemma 5.9.**

\[
\phi_1(r) = \frac{r}{2} \alpha'(r) - 1 \quad v_1(r) = \frac{r}{2} w'(r)
\]

**Proof.** From (5.8) we see that \(w_\beta(r) = w(e^{\beta/2}r)\) and \(\alpha_\beta = \alpha (e^{\beta/2}r) - \beta\) defines a one-parameter family of solutions passing through the solution \((w(r), \alpha(r))\). Therefore,

\[
v(r) := \frac{d}{d\beta} \bigg|_{\beta=0} w_\beta(r) = \frac{r}{2} w'(r) \quad \text{and} \quad \phi(r) := \frac{d}{d\beta} \bigg|_{\beta=0} \alpha_\beta(r) = \frac{r}{2} \alpha'(r) - 1
\]

must satisfy the linearized equations (5.26)–(5.27). The fact that this solution satisfies (5.29) follows from the expansions \(w(r) = 1 - \beta r^2 + O(r^4)\) and \(\alpha(r) = O(1)\). \(\square\)

The fall of conditions for \(w(r)\) and \(\alpha(r)\) as \(r \to \infty\) imply that

\[
\lim_{r \to \infty} (v_1(r), \phi_1(r)) = (0, -1).
\]

At present, we do not have an understanding of the asymptotic behaviour of \(r \to \infty\) for the solution \((v_2(r), \phi_2(r))\). This is the main obstacle in our having a complete existence proof. However, we conjecture that

\[
\lim_{r \to \infty} |v_2(r)| + |\phi_2(r)| = \infty.
\]

If this were not true, then there would exist a bounded solution to the IYMd equations. It would then be natural to expect that there exists a one-parameter family of bounded solutions to the YMd equations which when differentiated gives rise to the bounded IYMd solution. As shown above, this is how the solution \((v_1(r), \phi_1(r))\) arises. We note, however, that numerical evidence does not support the existence of one-parameter families of bounded solutions that pass through the YMd solutions from theorem 5.1 other than the family that arises via scaling (5.8). These solutions appear to be unique up to scaling.

The difficulty in proving (5.33) is that even though (5.26)–(5.27) are linear equations, we do not have very much information about the coefficients because they depend on the functions \(w(r)\) and \(\phi(r)\) of which we know very little. This makes it difficult to determine the behaviour of the solution \((v_2(r), \phi_2(r))\). However, we will show in the following sections how to prove existence of solutions to the EYMd equations under the assumption that (5.33) is true.

**6. Differentiability of the field equations**

In this section, we establish that the reduced field equations and the YMd equations define differentiable maps. In fact they define analytic maps. Before we proceed we first introduce some definitions.
Let $\mathcal{L}_k(B_1, B_2)$ denote the Banach space of $k$-linear and continuous maps from the Banach space $B_1$ into $B_2$ with norm

$$
\|T\|_{\mathcal{L}_k(B_1, B_2)} := \sup\{\|T(x_1, \ldots, x_k)\|_{B_2} | \sup\{\|x_1\|_{B_1}, \ldots, \|x_k\|_{B_1}\} \leq 1\}.
$$

**Definition 6.1.** Let $X_1$ and $X_2$ be Banach spaces and let $U$ be an open subset of $X_1$. A map $T : U \to V_2$ is said to be analytic if for each $x \in U$ there exists an $R > 0$ and a sequence $T_k \in \mathcal{L}_k(X_1, X_2)$ of $k$-linear symmetric maps such that

$$
\sum_{k=0}^{\infty} R^k \|T_k\|_{\mathcal{L}_k(X_1, X_2)} < \infty
$$

and

$$
T(y) = \sum_{k=0}^{\infty} T_k(y - x, \ldots, y - x) \quad \text{for all } y \text{ with } \|y - x\|_{X_1} < R.
$$

We use $C^{\omega}(U, X_2)$ to denote the set of all the analytic maps from $U$ to $X_2$.

An open ball in a Banach space $X$ will be denoted by

$$
B_X(x; R) := \{y \in X | \|x - y\|_X < R\}.
$$

We then have the following useful proposition:

**Proposition 6.2.** Suppose $u \in C^{\omega}(B_{R^2}(0; R), \mathbb{R})$ satisfies $u(0) = 0$. Furthermore, suppose $X$ is a commutative Banach algebra where $C$ is any constant such that $\|xy\|_X \leq C \|x\|_X \|y\|_X$ for all $x, y \in X$. Then the map

$$
\hat{u} : X^n \to X : (x_1, \ldots, x_n) \mapsto \sum_{|I| = 1}^{\infty} \frac{1}{|I|!} (\partial^I u(0)) x_1^{I_1} \cdots x_n^{I_n}
$$

is of class $C^{\omega}(B_X(0; \rho), X)$ for $\rho = R/C$.

Note that

$$
(\hat{g}^{\alpha\beta}_\omega) = \begin{pmatrix}
-\lambda & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

so that

$$
(\hat{g}^{\alpha\beta}_\omega)|_{\lambda = 0} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

As in [16], we define for any weakly differentiable map $u$

$$
u^{\alpha\beta}_\omega := (\hat{g}^{\alpha\beta}_\omega)|_{\lambda = 0} u, \beta = \begin{cases}
\partial_\alpha u & \text{for } \alpha \neq 0 \\
0 & \text{for } \alpha = 0.
\end{cases}
$$

We now collect some results from [16] concerning the analyticity of various quantities involving the density $\hat{u}$. 

Proposition 6.3 (proposition 3.10, [16]). Suppose $p > 3/2$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that the following maps are of class $C^\alpha$:

\[
(\Lambda, \Lambda) \times B_{W^2_0}(\mathbb{R}^3, S) \to W^{2, p}_0(\mathbb{R}^3, S) : (\lambda, \Omega) \mapsto (\tilde{g}^{\alpha\beta} - \tilde{g}^{\alpha\beta})
\]

and

\[
(\Lambda, \Lambda) \times B_{W^{2, p}_0}(\mathbb{R}^3, S) : (\lambda, \Omega) \mapsto |D|^{\alpha/2} - 1
\]

Proposition 6.4 (proposition 6.2, [16]). Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a $\Lambda > 0$ such that the Christoffel symbols

\[
\Gamma^\alpha_{\beta\gamma} : (\Lambda, \Lambda) \times B_{W^2_0}(\mathbb{R}^3, S) : (\lambda, \Omega) \mapsto |D|^{\alpha/2} - 1
\]

are of class $C^\alpha$ for all $\alpha, \beta, \gamma = 0, 1, 2, 3$. Moreover, the following expansions are valid:

\[
|D| - 1 = -4\lambda \Omega^{00} + O(\lambda^2), \quad \sqrt{D} - 1 = -2\lambda \Omega^{00} + O(\lambda^2),
\]

\[
\frac{1}{\sqrt{D}} - 1 = 2\lambda \Omega^{00} + O(\lambda^2), \quad (\tilde{g}^{\alpha\beta} - \tilde{g}_{\alpha\beta}) = -4\lambda \Omega^{00} + O(\lambda^2).
\]

Proposition 6.5. Suppose $p > 3$ and $-1 < \delta < 0$. Then for any $R > 0$ there exists a $\Lambda$ such that the map

\[
(E - \Delta) : (\Lambda, \Lambda) \times B_{W^2_0}(\mathbb{R}^3, S) : (\lambda, \Omega) \mapsto (E^\alpha\beta - \Delta \Omega^{\alpha\beta})
\]

is of class $C^\alpha$ where $E^\alpha\beta$ is defined by (2.19). Moreover,

\[
D_2(E - \Delta)(0, \Omega) \cdot \delta \Omega = (\delta \Omega^{00,\alpha} \Omega^{00,\beta} + \delta \Omega^{\alpha\beta} \Omega^{00,\alpha} - \tilde{g}^{\alpha\beta} |_{\lambda=0} \delta \Omega^{00,\alpha} \Omega^{00,\beta})
\]

Proof. The proof of this proposition is contained in the proof of proposition 4.2 in [16].

Let

\[
\gamma^2 := \tilde{g}^{\alpha\beta} \left( \psi_{,\alpha\beta} - \Gamma^\mu_{\alpha\beta} \psi_{,\mu} - \frac{\kappa e^{-\psi}}{\ell_d} \sqrt{|D|} \tilde{g}^{\alpha\nu} (F_{\alpha\mu} F_{\beta\nu}) \right), \quad (6.3)
\]

and

\[
\gamma^1 := \tilde{g}^{\alpha\beta} (F_{\alpha\beta} - \Gamma^\mu_{\alpha\beta} F_{\mu} - \Gamma^\mu_{\alpha\beta} F_{\alpha\mu} + 2\kappa \psi_{,\alpha} F_{\alpha\beta} + \{A_{\alpha}, F_{\alpha\beta}\}). \quad (6.4)
\]

The YMd equations are then $\gamma^1 = 0$ and $\gamma^2 = 0$. Note that the Yang–Mills equation $\gamma^1 = 0$ appears to be missing a component. However, due to our assumption that the fields are static and spherically symmetric it follows that $F_{\alpha\beta} = 0$ and $\Gamma^\mu_{\alpha\beta} = 0$, and hence that the $\beta = 0$ component of the Yang–Mills equation (2.23) is automatically satisfied.

We will split the gauge potential $A$ and the dilaton field $\psi$ as follows:

\[
A(x) = W(r) + Y(x) = W_\alpha(r) \, dx^\alpha + Y_\alpha(x) \, dx^\alpha \quad (6.5)
\]
where \( W_a(r) = \delta_a^c r^{-2}(w(r) - 1)\epsilon_i^c x^i \tau_j \) and \( \alpha(r) \) are to be considered as fixed. Recall that we are assuming that \((w(r), \alpha(r))\) is one of the solutions to the Euclidean Yang–Mills-dilaton equations from theorem 5.1. Under the splitting (6.5), the gauge potential decomposes as

\[
F_{\alpha\beta} = F^W_{\alpha\beta} + F^\gamma_{\alpha\beta} + [Y_\alpha, W_\beta] + [W_\alpha, Y_\beta]
\]

where \( F^W \) is defined by (5.24) and

\[
F^\gamma_{\alpha\beta} := \partial_\alpha Y_\beta - \partial_\beta Y_\alpha + [Y_\alpha, Y_\beta].
\]

Note that only \( F_{\alpha\beta} \) and \( F^W_{\alpha\beta} \) define field strengths. The quantity \( F^\gamma_{\alpha\beta} \) does not define a field strength as \( Y_\alpha \) does not transform as a gauge potential under gauge transformations.

**Proposition 6.6.** Suppose \(-2 > \delta > -1\) and \( p > 3 \). Then for any \( R > 0 \) and \( \alpha, \beta, \gamma = 0, 1, 2, 3 \) the following maps are \( C^\omega \)

\[
B_{W_{\gamma}^{1},(\mathbb{R}^3,\mathfrak{su}(2))}(0; R) \to W_{4-1}^{1,p}(\mathbb{R}^3, \mathfrak{su}(2)) : (Y_j) \mapsto F_{\alpha\beta},
\]

and

\[
B_{W_{\gamma}^{1},(\mathbb{R}^3,\mathfrak{su}(2))}(0; R) \to W_{4-2}^{0,p}(\mathbb{R}^3, \mathfrak{su}(2)) : (Y_j) \mapsto [A_\alpha, F_{\beta\gamma}].
\]

where \( A_\alpha \) and \( F_{\alpha\beta} \) are given by the formulae (6.5) and (6.7), respectively.

**Proof.** The proof is a direct consequence of lemma 3.3 and proposition 5.7. \( \square \)

**Proposition 6.7.** Suppose \( p > 3 \) and \(-1 < \delta_1 < 0\) and \(-2 < \delta_2 < -1\). Then for any \( R > 0 \) there exists a \( \Lambda > 0 \) such that the map

\[
\Upsilon : (-\Lambda, \Lambda) \times B_{W_{\gamma}^{1},(\mathbb{R}^3,\Sigma)}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3,\mathfrak{su}(2))}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3)}(0; R) \to W_{4-1}^{1,p}(\mathbb{R}^3, \mathfrak{su}(2)) : (\lambda, \Upsilon, Y, \xi) \mapsto (\Upsilon^1, \Upsilon^2)
\]

is of class \( C^\omega \).

**Proof.** Follows easily from lemma 3.3 and propositions 6.2, 6.3, 6.4 and 6.6. \( \square \)

**Proposition 6.8.** Suppose \( p > 3 \) and \(-1 < \delta_1 < 0\) and \(-2 < \delta_2 < -1\). Then for any \( R > 0 \) there exists a \( \Lambda > 0 \) and \( \epsilon > 0 \) such that the maps

\[
T : (-\Lambda, \Lambda) \times B_{W_{\gamma}^{1},(\mathbb{R}^3,\Sigma)}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3,\mathfrak{su}(2))}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3)}(0; R) \to W_{4-1}^{1,p}(\mathbb{R}^3, \mathfrak{su}(2)) : (\lambda, \Upsilon, Y, \xi) \mapsto (T^\alpha)^\beta
\]

and

\[
T : (-\Lambda, \Lambda) \times B_{W_{\gamma}^{2},(\mathbb{R}^3,\Sigma)}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3,\mathfrak{su}(2))}(0; R) \times B_{W_{\gamma}^{2},(\mathbb{R}^3)}(0; R) \to W_{4-1}^{1,p}(\mathbb{R}^3, \mathfrak{su}(2)) : (\lambda, \Upsilon, Y, \xi) \mapsto (T^\alpha)^\beta
\]

are of class \( C^\omega \).

**Proof.** Follows easily from lemma 3.3 and propositions 6.2, 6.3 and 6.6. \( \square \)

We now prove spherically symmetric versions of propositions 6.5, 6.7 and 6.8.
Proposition 6.9. Suppose \( p > 3 \) and \(-1 < \delta < 0\). Then for any \( R > 0 \) there exists a \( \Lambda \) such that the map

\[
(E - \Delta) : (-\Lambda, \Lambda) \times B_{t_\lambda^2}(0; R) \rightarrow \mathcal{U} = (\lambda, U) \mapsto (E^{\alpha\beta} - \Delta^{\alpha\beta})
\]

is of class \( C^0 \). Moreover,

\[
D_2(E - \Delta)(\lambda, U) - \delta \mathcal{U} = (\delta U^{00,\alpha}U^{00,\beta} + \delta U^{00,\alpha}U^{00,\beta} - \delta^{\alpha\beta}|_{V=0}U^{00,\alpha}U^{00,\beta}).
\]

Proof. Given \( R \), let \( \Lambda \) be determined as in proposition 6.5. By straightforward calculation it can be shown that if \( U \in \mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S}) \) and \( B_{t_\lambda^2}(0; R) \) then \((E - \Delta)(U) \in \mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S})\). Consequently, \((E - \Delta)(\mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S})) \subset \mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S})\). Therefore, \((E - \Delta)(B_{t_\lambda^2}(0; R)) \subset \mathcal{U}^{00,\alpha} \) by the density of \( \mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S}) \) in \( \mathcal{U}^{00,\alpha} \) for \( \eta \in \mathbb{R} \) and continuity of the map \((E - \Delta)\) by proposition 6.5. The proposition now follows from proposition 6.5. \( \square \)

Proposition 6.10. Suppose \( p > 3 \) and \(-1 < \delta_1 < 0 \) and \(-2 < \delta_2 < -1\). Then for any \( R > 0 \) there exists \( \Lambda > 0 \) such that the map

\[
\Upsilon : (-\Lambda, \Lambda) \times B_{t_\lambda^2}(0; R) \times B_{A_{t_\lambda^2}}(0; R) \times B_{r_\lambda^2}(0; R)
\]

\[
\rightarrow \mathcal{A}^{00,\alpha} \times \mathcal{D}^{00,\alpha} : (\lambda, U, Y, \xi) \mapsto (\Upsilon^1, \Upsilon^2)
\]

is of class \( C^0 \).

Proof. As in the proof of proposition 6.9, straightforward calculation shows that if \( U \in \mathcal{C}_0^{0}(\mathbb{R}^3, \mathcal{S}) \) and \( B_{t_\lambda^2}(0; R), \mathcal{Y} \in \mathcal{A}^{00,\alpha} \cap B_{r_\lambda^2}(0; R) \) and \( \xi \in \mathcal{C}_0^{0}(\mathbb{R}^3) \) then \( \Upsilon(Y, \alpha) \in \mathcal{C}_0^{0}(\mathbb{R}^3) \) \( \cap \mathcal{C}^2 \times \mathcal{A}^{00,\alpha} \cap \mathcal{C}^2 \). We then argue in the same manner as proposition 6.9. \( \square \)

Proposition 6.11. Suppose \( p > 3 \) and \(-1 < \delta_1 < 0 \) and \(-2 < \delta_2 < -1\). Then for any \( R > 0 \) there exists \( \Lambda > 0 \) and \( \epsilon > 0 \) such that the maps

\[
T : (-\Lambda, \Lambda) \times B_{t_\lambda^2}(0; R) \times B_{A_{t_\lambda^2}}(0; R) \times B_{r_\lambda^2}(0; R)
\]

\[
\rightarrow \mathcal{U}^{1,\rho} \times (\lambda, U, Y, \xi) \mapsto (T^{\alpha\beta})
\]

and

\[
\tilde{T} : (-\Lambda, \Lambda) \times B_{t_\lambda^2}(0; R) \times B_{A_{t_\lambda^2}}(0; R) \times B_{r_\lambda^2}(0; R)
\]

\[
\rightarrow \mathcal{U}^{1,\rho} \times (\lambda, U, Y, \xi) \mapsto (T^{\alpha\beta})
\]

are of class \( C^0 \).

Proof. See the proofs of propositions 6.9 and 6.10. \( \square \)

From (3.2) and propositions 4.2, 6.9 and 6.11 we get the following:

Proposition 6.12. Suppose \(-1 < \delta_1 < 0, -2 < \delta_2 < -1 \) and \( p > 3 \). Then for any \( R > 0 \) there exists \( \Lambda > 0 \) such that

\[
\Xi : (-\Lambda, \Lambda) \times B_{t_\lambda^2}(0; R) \times B_{A_{t_\lambda^2}}(0; R) \times B_{r_\lambda^2}(0; R)
\]

\[
\rightarrow \mathcal{U}^{0,\rho} : (\lambda, U, Y, \xi) \mapsto (\Delta^{\alpha\beta} - \Delta^{-1}(T^{\alpha\beta} - (E^{\alpha\beta} - \Delta^{\alpha\beta})))
\]

is of class \( C^0 \).

From the definition of \( \Xi \), it is clear that the reduced field equations (2.17) are equivalent to \( \Xi = 0 \).
7. Solving the reduced field equations

We now employ the same method as in [16] to find solutions to the reduced field equations. Namely, we first solve the reduced equations for \( \lambda = 0 \), and then use an implicit function argument to show that there exist a solution for \( \lambda \) small enough.

7.1. \( \lambda = 0 \)

Assume \(-1 < \delta_1 < 0\), \(-2 < \delta_2 < -1\), \(p > 3\) and for fixed \( R > 0 \) let \( \Lambda > 0 \) be as in proposition 6.12. From the expansions in proposition 6.3 and (2.10) we see that

\[
E^{\alpha\beta}_{|\lambda=0} = \Delta U^{\alpha\beta} + \begin{cases} -\hat{\Omega}^{00,\alpha} \hat{\Omega}^{\alpha\beta,0} + \frac{1}{2} \delta^{\alpha\beta} |\text{grad} \hat{\Omega}^{00}|^2 & \text{if } \alpha \neq 0, \beta \neq 0 \\ 0 & \text{otherwise,} \end{cases}
\]

and

\[
T^{\alpha\beta}_{|\lambda=0} = 2 \pi G \xi_d ((\bar{\delta}^{\mu\nu} \bar{\delta}^{\rho\sigma})_{|\lambda=0} \psi_{\mu\rho} \psi_{\nu\sigma} - \frac{1}{2} (\bar{\delta}^{\mu\nu} \bar{\delta}^{\rho\sigma})_{|\lambda=0} \psi_{\rho\sigma} \psi_{\mu\nu})
+ 4 \pi G \xi_\gamma e^{2\psi} ((\bar{\delta}^{\mu\nu} \bar{\delta}^{\rho\sigma})_{|\lambda=0} (F_{\mu\rho} | F_{\nu\sigma}) - \frac{1}{4} (\bar{\delta}^{\mu\nu} \bar{\delta}^{\rho\sigma} \bar{\delta}^{\rho\sigma})_{|\lambda=0} (F_{\mu\rho} | F_{\nu\sigma})).
\]

Therefore,

\[
\hat{T}^{\alpha\beta}_{|\lambda=0} = 0, \quad \hat{T}^{\alpha\gamma}_{|\lambda=0} = 0 \quad \text{and} \quad \hat{T}^{ij}_{|\lambda=0} = 4 \pi G \hat{T}^{Nij},
\]

where

\[
\hat{T}^{ij} := \frac{1}{\Lambda} \epsilon_d \left( \delta^{ik} \delta^j \psi_{k\rho} \psi_{j\rho} - \frac{1}{2} \delta^{ij} \delta^{kl} \psi_{k\rho} \psi_{l\rho} \right)
+ \epsilon_\gamma e^{2\psi} \left( \delta^{ik} \delta^{lm} \delta^{jn} (F_{km} | F_{ln}) - \frac{1}{4} \delta^{ij} \delta^{mn} (F_{km} | F_{ln}) \right)
\]

is the stress–energy tensor for the Euclidean YMd equations on \( \mathbb{R}^3 \). So then

\[
\exists (0, U, Y, \xi) = 0 \iff \begin{cases} \Delta \hat{\Omega}^{00} = 0 \\ \Delta \hat{\Omega}^{0j} = 0 \\ \Delta \hat{\Omega}^{ij} = \hat{\Omega}^{00,ij} - \frac{1}{2} \delta^{ij} |\text{grad} \hat{\Omega}^{00}|^2 + 4 \pi G \hat{T}^{Nij} \end{cases}
\]

The first equation \( \Delta \hat{\Omega}^{00} = 0 \) can be interpreted as the Newtonian gravitational equation for the gravitational potential \( \hat{\Omega}^{00} \) [20]. The vanishing of the mass density \( (T^{00})_{|\lambda=0} = 0 \) decouples the Newtonian potential from the YMd fields in the limit \( \lambda \to 0 \). For other matter fields such as perfect fluids, this decoupling does not occur as \( T^{00} \neq 0 \) [16, 20].

The invertibility of the Laplacian (theorem 3.4) then implies that

\[
\hat{\Omega}^{00} = 0, \quad \hat{\Omega}^{0j} = 0 \quad \text{and} \quad \hat{\Omega}^{ij} = 4 \pi G^2 \Delta^{-1} \hat{T}^{Nij}
\]

solve \( \exists (0, U, Y, \xi) = 0 \) for any \( Y \in B_{\delta_2,0}(0; R) \) and \( \xi \in B_{\delta_2,0}(0; R) \).

7.2. \( \lambda \neq 0 \)

Proposition 7.1. Suppose \(-1 < \delta_1 < 0\), \(-2 < \delta_2 < -1\) and \(p > 3\). Then there exists a \( \Lambda > 0, \epsilon > 0 \) and a \( C^\infty \) map

\[
\hat{U} : (-\Lambda, \Lambda) \times B_{\delta_2,0}(0; \epsilon) \times B_{\delta_2,0}(0; \epsilon) \to \hat{U}_{\delta_2}^{\lambda,p} : (\lambda, Y, \xi) \to \hat{U}(\lambda, Y, \xi) = (\hat{U}\alpha(\lambda, Y, \xi))
\]

such that \( \exists (\lambda, \hat{U}(\lambda, Y, \xi), Y, \xi) = 0 \) for all \( (\lambda, Y, \xi) \in (-\Lambda, \Lambda) \times B_{\delta_2,0}(0; \epsilon) \times B_{\delta_2,0}(0; \epsilon) \). Moreover, \( \hat{U} \) satisfies \( \hat{U}^{00}(0, 0, 0) = 0 \) and \( \hat{U}^{ij}(0, 0, 0) = 0 \) and \( \hat{D}_2 \hat{U}^{00}(0, 0, 0) = 0 \) and \( \hat{D}_2 \hat{U}^{ij}(0, 0, 0) = 0 \).
Newtonian perturbations and the Einstein–Yang–Mills-dilaton equations

**Proof.** Fix $R > 0$ and let $\Lambda > 0$ be chosen so that the maps $\Xi$, $E - \Delta$ and $\mathcal{T}$ are of class $C^\nu$ which we can do by propositions 6.9, 6.11 and 6.12. Then we can solve $\Xi(0, \Omega, 0, 0) = 0$ by (7.2). Let $\Omega_0$ denote the solution. Note that $\Omega_0^\mu = 0$ by (7.2). So $D_2(E - \Delta)(\lambda, \Omega_0) = 0$ by proposition 6.9. From the expansions in proposition 6.3 and formula (2.10) it follows that $D_2 T(0, \Omega_0, 0, 0) = 0$. Therefore, from the definition of $\Xi$ it is clear that

$$D_2 \Xi(0, \Omega_0, 0, 0) = \mathbb{I}_{1\#}, \tag{7.3}$$

and hence by the implicit function theorem there exists a $\tilde{\Lambda} > 0$, $\epsilon > 0$ and a $C^\infty$ map

$$\hat{\Upsilon} : (\tilde{\Lambda}, \Lambda) \times B_{A_{t^2}}(0; \epsilon) \times B_{D_{\Xi t^2}}(0; \epsilon) \to \hat{U}_{\tilde{\Lambda}_1}^{0, p} : (\lambda, Y, \xi) \to \hat{\Upsilon}(\lambda, Y, \xi) = (\hat{\Upsilon}^0(\lambda, Y, \xi))$$

such that

$$\Xi(\lambda, \hat{\Upsilon}(\lambda, Y, \xi), Y, \xi) = 0 \tag{7.4}$$

for all $(\lambda, Y, \xi) \in (\tilde{\Lambda}, \Lambda) \times B_{A_{t^2}}(0; \epsilon) \times B_{D_{\Xi t^2}}(0; \epsilon)$. Differentiating (7.4) with respect to $Y$ and using (7.3) we find

$$D_2 \hat{\Upsilon}^0(0, 0, 0) = D_2 \Xi^0(0, \Omega_0, 0, 0). \tag{7.5}$$

But

$$D_2 \Xi^0(\lambda, \Omega_0, Y, \xi) \cdot \delta Y = \left( -4\pi G \Delta^{-1} \left\{ \frac{\ell Y}{\sqrt{|g|}} e^{-2\psi} (\tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu} \tilde{g}^{\gamma\tau} [\delta F_{\mu\nu\sigma}] F_{\gamma\tau}) + \langle F_{\sigma \tau} [\delta F_{\tau\gamma}] \tilde{g}^{\gamma\mu} \tilde{g}^{\sigma\nu} \tilde{g}^{\alpha\beta} \rangle \right\} \right)$$

where

$$\delta F_{\alpha\beta} = \partial_\alpha Y_\beta - \partial_\beta Y_\alpha + [\delta Y_\alpha, Y_\beta] + [Y_\alpha, \delta Y_\beta] + [\delta Y_\alpha, W_\beta] + [W_\alpha, \delta Y_\beta] \tag{7.6}$$

and $F_{\alpha\beta}$ is given by formula (6.7). Setting $\lambda = 0$ we get, by (2.10), (6.2) and the expansions of proposition 6.3, that $D_2 \Xi^0(0, \Omega_0, 0, 0) = 0$. Therefore, $D_2 \hat{\Upsilon}^0(0, 0, 0) = 0$ by (7.5). Similar calculations show that $D_2 \hat{\Upsilon}^0(0, 0, 0) = 0$. \hfill $\square$

**8. Solving the YMd equations**

Suppose $-1 < \delta_1 < 0, -2 < \delta_2 < -1, \rho > 3$ and let $\Lambda$, $\epsilon$ and $\hat{\Upsilon}$ be as in proposition 7.1. Then by the results of propositions 6.10 and 7.1 the map

$$\hat{\Gamma} : (-\Lambda, \Lambda) \times B_{A_{t^2}}(0; \epsilon) \times B_{D_{\Xi t^2}}(0; \epsilon) \to D_{b_{t-2}}^{0, p} \times A_{b_{t-2}}^{0, p} \tag{8.1}$$

defined by

$$\hat{\Gamma}(\lambda, Y, \xi) := \Upsilon(\lambda, \hat{\Upsilon}(\lambda, Y, \xi), Y, \xi) \tag{8.2}$$

is $C^\infty$. Define

$$\hat{\Gamma}^a_{b \tau}(\lambda, Y, \xi) := \Gamma^a_{b \tau}(\lambda, \hat{\Upsilon}(\lambda, Y, \xi)).$$

Then (2.25), (2.10), (6.2), the expansions of proposition 6.3 and proposition 7.1 show that

$$\hat{\Gamma}^a_{b \tau}(0, 0, 0) = 0, \quad D_2 \hat{\Gamma}^a_{b \tau}(0, 0, 0) = 0 \quad \text{and} \quad D_4 \hat{\Gamma}^a_{b \tau}(0, 0, 0) = 0.$$

Using this result along with (2.10), (4.5), (6.2) and the expansions of proposition 6.3, we find after straightforward calculation that

$$\hat{\Gamma}^2(0, 0, 0) = \Delta \alpha - \frac{k \ell Y}{\ell_d} e^{2a \tilde{g}^{ij} \tilde{g}^{kl}} \langle F^W_{ik} | F^W_{jl} \rangle \tag{8.3}$$
\[
\hat{T}^1(0, 0, 0) = (\delta^{ik} (\partial_k F_{ij}^W + 2\kappa F_{ij}^W \partial_k \alpha + [W_k, F_{ij}^W]))
\]
(8.4)

and
\[
D_2\hat{T}^1(0, 0, 0) \cdot \delta Y = \left( \Delta \delta Y_j + \delta^{ik} ([\delta Y_i, \partial_k W_j] + [W_i, \partial_k \delta Y_j] + 2\kappa \mathcal{F}(\delta Y)_{ij} \partial_k \alpha + [\delta Y_k, F_{ij}^W] + [W_k, \mathcal{F}(\delta Y)_{ij}]) \right).
\]
(8.5)

\[
D_2\hat{T}^1(0, 0, 0) \cdot \delta \xi = (2\kappa \delta^{ik} F_{ij}^W \partial_k \delta \xi).
\]
(8.6)

\[
D_2\hat{T}^2(0, 0, 0) \cdot \delta Y = -2\frac{\kappa \beta_Y}{\ell_d} e^{2\kappa \alpha} \delta^{ij} \delta^{kl} (F_{ik}^W | \mathcal{F}(\delta Y)_{jl}),
\]
(8.7)

\[
D_2\hat{T}^2(0, 0, 0) \cdot \delta \xi = \Delta \delta \xi + 2\kappa \frac{\kappa \beta_Y}{\ell_d} \delta \xi e^{2\kappa \alpha} \delta^{ij} \delta^{kl} (F_{ik}^W | F_{jl}^W),
\]
(8.8)

where
\[
\mathcal{F}(\delta Y)_{ij} := \partial_i \delta Y_j - \partial_j \delta Y_i + [\delta Y_i, W_j] + [W_i, \delta Y_j].
\]
(8.9)

Observe that
\[
\hat{T}^1(0, 0, 0) = 0 \quad \text{and} \quad \hat{T}^2(0, 0, 0) = 0
\]
(8.10)
since \((W, \alpha)\) satisfies (5.1)–(5.2).

We can collect (8.5)–(8.8) into a single matrix expression
\[
K \begin{pmatrix} \delta Y \\ \delta \xi \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} \delta Y \\ \delta \xi \end{pmatrix} + K \begin{pmatrix} \delta Y \\ \delta \xi \end{pmatrix}
\]
(8.11)

where
\[
K := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix},
\]

and
\[
K_{11} := (\delta^{ik} ([\delta Y_i, \partial_k W_j] + [W_i, \partial_k \delta Y_j] + 2\kappa \mathcal{F}(\delta Y)_{ij} \partial_k \alpha + [\delta Y_k, F_{ij}^W] + [W_k, \mathcal{F}(\delta Y)_{ij}]),
\]
(8.12)

\[
K_{22} := -2\kappa \frac{\kappa \beta_Y}{\ell_d} e^{2\kappa \alpha} \delta^{ij} \delta^{kl} (F_{ik}^W | F_{jl}^W).
\]
(8.15)

In order to use the implicit function theorem we need
\[
K : A^{1, p}_{b_2} \times D^{2, p}_{b_1} \rightarrow A^{0, p}_{b_2} \times D^{0, p}_{b_1}
\]
to be an isomorphism. As the next result shows, it will be enough to establish that ker \(K = \{0\} \).

**Proposition 8.1.** ker \(K = \{0\}\) if and only if \(K\) is an isomorphism.

**Proof.** Since \(-2 < \delta_2 < -1\) and \(-1 < \delta_1 < 0\), there exist an \(\epsilon > 0\) such that \(K \left( A^{1, p}_{b_2} \times D^{2, p}_{b_1} \right) \subset A^{1, -2+\epsilon}_{b_2} \times D^{2, -1+\epsilon}_{b_1}\) by lemma 3.3. But the embedding \(A^{1, p}_{b_2} \times D^{2, p}_{b_1} \rightarrow A^{0, p}_{b_2} \times D^{0, p}_{b_1}\) is compact by lemma 3.5 and hence \(K : A^{1, p}_{b_2} \times D^{2, p}_{b_1} \rightarrow A^{0, p}_{b_2} \times D^{0, p}_{b_1}\) is compact. As \(\Delta \oplus \Delta : A^{1, p}_{b_2} \times D^{2, p}_{b_1} \rightarrow A^{0, p}_{b_2} \times D^{0, p}_{b_1}\) is an isomorphism by propositions 4.1
and 4.3 it follows by compactness of $K$ that index $(\Delta \oplus \Delta + K) = 0$ and the proof is complete.

The difficulty in proving that $\ker K = \{0\}$ lies with the fact that the spectrum of $K$ contains both a (strictly) negative and positive component. The negative part of the spectrum accounts for the well-known instability of the Yang–Mills-dilaton solutions. It also means that one cannot expect that $\ker K = \{0\}$ can be proved by a integration by parts argument.

**Proposition 8.2.** If (5.33) is valid, then
\[
K : \mathcal{A}_{K}^{p} \times D_{K}^{p} \to \mathcal{A}_{K}^{p} \times D_{K}^{p}
\]
is an isomorphism.

**Proof.** Suppose $(\delta Y, \delta \xi) \in \mathcal{A}^{p} \times D^{p}$ is a solution to
\[
K \begin{pmatrix} \delta Y \\ \delta \xi \end{pmatrix} = 0.
\]
(8.16)

We observe that $K$ is uniformly elliptic and has $C^\infty$ coefficients since $W$ and $\alpha$ are $C^\infty$ by (5.1). Therefore by elliptic regularity, see [13] theorem 9.19 or [12] theorem 3.6, $\delta Y \in C^\infty \cap \mathcal{A}^{p}$ and $\delta \xi \in C^\infty \cap D^{p}$. Letting
\[
\phi = \delta \xi \quad \text{and} \quad \delta Y_i = \frac{v(r)}{r^2} \epsilon_i \epsilon \tau_j,
\]
shows that $(v(r), \phi(r))$ satisfy equations (5.26)–(5.27). Also since $\delta Y$ and $\delta \xi$ are smooth it follows that $v(r)$ and $\phi(r)$ satisfy the boundary condition (5.28). From our discussion in section 5.1, we know that there must exist constant $c_i$, where $i = 1, 2, 3, 4$, such that $v(r) = c_1 v_1 (r) + c_2 v_2 (r)$ and $\phi(r) = c_3 \phi_1 (r) + c_4 \phi_2 (r)$. Assuming that $v_2 (r)$ satisfies (5.33), it then follows from (5.32) that $\delta \xi \notin D^{p}$ and $\delta Y \notin \mathcal{A}^{p}$ and hence $\ker K = \{0\}$. 

We are now ready to solve the YMd equations.

**Proposition 8.3.** Suppose $-1 < \delta_1 < 0$, $-2 < \delta_2 < -1$, $p > 3$ and let $\Lambda$ and $\epsilon$ be as in (8.1). If $(w(r), \alpha(r))$ is one of the solutions from theorem (5.1) of the Euclidean YMd equations (5.1)–(5.2) and (5.33) holds then there exists $\hat{\Lambda} \in (0, \Lambda)$ and two $C^\infty$ maps
\[
\hat{Y} : (-\hat{\Lambda}, \hat{\Lambda}) \to B_{K^{-2}}(0; \epsilon) \quad \text{and} \quad \hat{\xi} : (-\hat{\Lambda}, \hat{\Lambda}) \to B_{D^{-2}}(0; \epsilon)
\]
such that $\hat{Y}(0) = 0$, $\hat{\xi}(0) = 0$ and $\hat{Y}(\hat{\lambda}), \hat{\xi}(\hat{\lambda}), \hat{\xi}(\lambda)) = 0$ for all $\lambda \in (-\hat{\Lambda}, \hat{\Lambda})$.

**Proof.** Because $K : \mathcal{A}^{p} \times D^{p} \to \mathcal{A}^{p} \times D^{p}$ is an isomorphism by propositions 8.2 we can apply the implicit functions theorem to get the desired result.

9. **Solving the EYMd field equations**

By propositions 7.1 and 8.3 we can solve the reduced field equations (2.18) and the YMd equations (2.23)–(2.24). Using the following result of Heilig [16], we will see that this solution will actually be a solution to the full EYMd equations.

**Proposition 9.1** (proposition 6.1, [16]). Suppose $-1 < \delta < 0$, $p > 3$ and $\Lambda > 0$. Furthermore, suppose
\[
T : [0, \Lambda] \to W_{\delta, p}^{1, 1}(\mathbb{R}^3, S^1) \cap C^{1}(\mathbb{R}^3, S^1) : \lambda \mapsto (T_{\lambda}^{\delta p})
\]
and
\[ \Omega : [0, \Lambda] \to W^{2,p}_\theta(\mathbb{R}^3, S^3) : \lambda \mapsto (\Omega^{\alpha\beta}_\lambda) \]
are two continuous maps such that for every \( \lambda \in [0, \Lambda] \): \( (\lambda, \Omega^{\alpha\beta}_\lambda, T^{\alpha\beta}_\lambda) \) is a solution to the reduced field equations (2.17), \( \nabla_\beta T^{\alpha\beta}_\lambda = 0 \) and \( \partial_\gamma T^{\alpha\beta}_\lambda \in B_{W^{1,p}_{\theta,\gamma}}(\mathbb{R}^3)(0, R) \) for some \( R > 0 \) independent of \( \lambda \) and \( \alpha, \beta, \gamma \). Then there exists a constant \( \hat{\Lambda} \in (0, \Lambda) \) such that \( \partial_\gamma \Omega^{\alpha\beta}_\lambda = 0 \) for all \( \lambda \in [0, \hat{\Lambda}] \).

We are now ready to show that to each one of the Euclidean YMd solutions \((u_\mu(r), \alpha_\mu(r))\), where \( n = 1, 2, 3, \ldots \), from theorem 5.1 for which (5.32) holds, there exists a solution to the full EYMd equations.

**Theorem 9.2.** Suppose \(-1 < \delta_1 < 0, -2 < \delta_2 < 1, p > 3 \) and let \( (u_\mu(r), \phi_\mu(r)) \) be one of the solutions to the Euclidean Yang–Mills-dilaton equations from theorem 5.1. If condition (5.33) holds for the solution \((u_\mu(r), \alpha_\mu(r))\) then there exist a \( \Lambda > 0 \) and \( C^\infty \) maps \( \Omega : [-\Lambda, \Lambda] \to U^{2,p}_{\theta^2} : \lambda \mapsto (\Omega^{\alpha\beta}_\lambda), Y : [-\Lambda, \Lambda] \to A^{2,p}_{\theta^2} : \lambda \mapsto (Y^a_\lambda) \) and \( \xi : [-\Lambda, \Lambda] \to D^{2,p}_{\theta^2} : \lambda \mapsto \xi^\lambda, \) such that \( (Y^0, \xi^0) = (0, 0) \) and for any \( \lambda \in (0, \Lambda] \) \( (\Omega^{\alpha\beta}_\lambda, A^\lambda = W^\mu + Y^\lambda, \psi^\lambda = \alpha_\mu + \xi^\lambda) \) is a \( C^2 \) solution to the EYMd equations (2.7), (2.23) and (2.24). Moreover, the solution is static, spherically symmetric and asymptotically flat.

**Proof.** Let \((u_\mu, \alpha_\mu)\) be one of the solutions to the Euclidean YMd equations from theorem 5.1 and let \( W_\theta = \delta_2 \theta^{-2}(u_\mu(r) - 1)\epsilon_1 \lambda^4 \tau_j \). If we assume that (5.33) holds for the solution \((u_\mu(r), \alpha_\mu(r))\) then by propositions 7.1 and 8.3 there exists a \( \Lambda > 0 \) and \( C^\infty \) maps \( \Omega : [-\Lambda, \Lambda] \to U^{2,p}_{\theta^2} : \lambda \mapsto (\Omega^{\alpha\beta}_\lambda), Y : [-\Lambda, \Lambda] \to A^{2,p}_{\theta^2} : \lambda \mapsto (Y^a_\lambda) \) and \( \xi : [-\Lambda, \Lambda] \to D^{2,p}_{\theta^2} : \lambda \mapsto \xi^\lambda, \) such that \( (Y^0, \xi^0) = (0, 0), \)
\[ \varepsilon(\lambda, Y(\lambda), (\xi(\lambda)) = 0, \quad Y(\lambda) = \Omega(\lambda) = \xi(\lambda) = 0, \]
for all \( \lambda \in (-\Lambda, \Lambda) \). To reduce notation, we will often write \( \Omega, Y \) and \( \xi \) instead of \( \Omega^\lambda, Y^\lambda \) and \( \xi^\lambda \).

**Lemma 9.3.** There exists a \( \Lambda^* \in (0, \Lambda] \) such that \( A^\lambda = W^\mu + Y^\lambda, \psi^\lambda = \alpha_\mu + \xi^\lambda \in C^2 \) for all \( \lambda \in (-\Lambda^*, \Lambda^*) \).

**Proof.** Let \( B_R \subset \mathbb{R}^3 \) be an open ball of radius \( R \) centred at the origin. Then \( \psi, \psi_\mu, \Omega^{\alpha\beta}_\mu, A_\mu, A_{\alpha\beta} \in W^{1,p}(B_R) \), where recall that \( A = W^\mu + Y \) and \( \psi = \alpha + \xi \). As \( W^{1,p}(B_R) \) is a Banach algebra, we have
\[ f := \Gamma^{\mu}_{\alpha\beta}\psi_\mu \Omega^{\alpha\beta}_\lambda = \frac{\kappa_F}{\ell_d} \hat{g}_\lambda^{\alpha\beta}(F_{\alpha\mu}F_{\beta\mu} - 2\kappa \psi_\lambda F_{\alpha\beta} - [A_\lambda, F_{\alpha\beta}]) \in W^{1,p}(B_R), \]
\[ h = (h_j) := (\hat{g}_\lambda^{ji}(\Gamma^j_{\alpha\mu}F_{\mu\beta} + \Gamma^j_{\beta\mu}F_{\alpha\mu} - 2\kappa \psi_\lambda F_{\alpha\beta} - [A_\lambda, F_{\alpha\beta}])) \in W^{1,p}(B_R, \mathbb{R}^3), \]
\[ \hat{g}_\lambda^{ij} = \delta^i_j + 4\lambda^2 \delta_\lambda^i \in W^{1,p}(B_R), \]
\[ Q^{ij} := (Q^{ij}_j) := (\delta^i_j + 4\lambda^2 \delta_\lambda^i \delta^j_\lambda - 4\lambda^2 \delta^{ij}_\lambda \delta^j_\lambda) \in W^{1,p}(B_R, \mathbb{M}_{3\times3}), \]
and hence \( f, h_j, \hat{g}_\lambda^{ij}, Q^{ij}_j \in C^{0,1-3p}(B_R) \) by the Sobolev embedding theorem. Note that YMd equations \( \Upsilon(\lambda), \Upsilon(\lambda), (\xi(\lambda)) = 0 \) can be written as
\[ \hat{g}_\lambda^{ij} \partial_\lambda \hat{g}_\lambda^{ij} = f \quad \text{and} \quad Q^{ij}_j \partial_\lambda Q^{ij}_j = h_j. \]
By the weighted Sobolev inequality, \( [2, \text{theorem 1.2(v)}] \), the embedding \( W^{1,p}_{\theta_1}(\mathbb{R}^3, S^3) \to C^{0,1-3p}_{\theta_1}(\mathbb{R}^3, S^3) \) is continuous and hence the map \( (-\Lambda, \Lambda) \to C^{0,1-3p}_{\theta_1}(\mathbb{R}^3, S^3) : \lambda \mapsto U(\lambda) \) is continuous. Therefore, there exists a \( \Lambda^* \in (0, \Lambda) \) such that the operators \( \hat{g}_\lambda^{ij} \partial_\lambda \hat{g}_\lambda^{ij} \) and \( Q^{ij}_j \partial_\lambda Q^{ij}_j \)
are uniformly elliptic with coefficients in $C^{0,1-3/p}(\mathbb{R}^3)$ for all $\lambda \in [-\Lambda^*, \Lambda^*]$. By elliptic regularity, $A^I_j = W^I_j + Y^I_j$ and $\psi^J = \alpha + \psi^J$ are in $C^2(B_R)$ for all $\lambda \in [-\Lambda^*, \Lambda^*]$. As $\Lambda^*$ is independent of $R$ the result follows.

It follows immediately from equation (2.27), proposition 6.8 and the above lemma that the hypotheses of proposition 9.1 are satisfied. Therefore, we conclude that there exist a constant $\tilde{\Lambda} \in (0, \Lambda^*)$ such that

$$\partial_\lambda U^\mu_\lambda = 0 \quad (9.1)$$

for all $\lambda \in [0, \tilde{\Lambda}]$. This implies that the full EYMd equations are equivalent to $\Xi = 0$ and $\Upsilon = 0$ and hence $(\lambda, U^\mu_\nu, A^k = W^k + Y^k, \psi^\lambda = \alpha + \psi^\lambda)$ satisfy the EYMd equations for all $\lambda \in (0, \tilde{\Lambda})$.

Using (9.1), the reduced field equations $\Xi = 0$ can be written as

$$\overline{g}^{ij} \partial^2_{ij} U^\mu_\lambda = H^{\mu\nu}$$

where $H^{\mu\nu} = -A^{\mu\nu} - B^{\mu\nu} - C^{\mu\nu} + 4\pi G \bar{\partial} |T^\mu_\nu|$. We can then argue as in lemma 9.3 to conclude that $U^\mu_\lambda \in C^2$. □

10. Conclusion

In this paper, we have shown how to reduce the existence problem for static spherically symmetric solutions to the $SU(2)$ EYMd equations to that of proving the non-existence of solutions to the IYMd equations by using a Newtonian perturbation argument. We conjectured that solutions to the IYMd equations satisfy (5.33) and showed that if this is true then there exists a countably infinite number of static spherically symmetric solutions.

Numerically, it has been found that the EYMd equations also admit static axially symmetric solutions [17]. There is nothing in principle from generalizing the results of this paper to the non-spherically symmetric case. To make progress in the non-spherically symmetric case a PDE proof that the IYMd equations have only the trivial solution would be needed. However, as discussed in this paper, even in the spherically symmetric case, this is a difficult problem and would represent a significant advance. The other main problem would be to try and prove that static axially symmetric solutions to the YMd equations exist. Even though this problem would be much simpler than proving the existence of solutions to the full EYMd equations it still represents an extremely difficult problem.

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