UNIQUENESS OF POSITIVE RADIAL SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION IN AN ANNULUS

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ABSTRACT. In this paper, we show the following equation

\[
\begin{aligned}
\Delta u + u^p + \lambda u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

has at most one positive radial solution for a certain range of \( \lambda > 0 \). Here \( p > 1 \) and \( \Omega \) is the annulus \( \{ x \in \mathbb{R}^n : a < |x| < b \} \), \( 0 < a < b \). We also show this solution is radially non-degenerate via the bifurcation methods.

1. Introduction. This paper is concerned with the uniqueness of the positive radial solution of the Dirichlet problem of

\[ \Delta u + f(u) = 0 \]

in the annulus \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) where \( 0 < a < b \).

Problem (1) serves as a model in many different areas of applied mathematics and has been extensively studied by several mathematicians in the last three decades. When \( \Omega \) is a ball or the entire space, it is well known \([5, 6]\) that all positive Dirichlet solutions of (1) must be radially symmetric (up to translation), provided certain conditions on \( f \) hold. Many authors focus on the uniqueness of positive solutions in a finite ball or the entire space \( \mathbb{R}^n \); see \([10, 13, 14, 16, 18, 22]\) and the references therein, while many exact multiplicity results have been obtained, see \([12, 17]\) and the references therein.

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When $\Omega$ is an annulus, the Dirichlet problem is
\[
\begin{align*}
\Delta u + f(u) &= 0 \quad \text{for } a < |x| < b, \\
   u &= 0 \quad \text{for } |x| = a, b,
\end{align*}
\tag{2}
\]
where $0 < a < b$. The positive solution of (2) is not necessarily radially symmetric, and some results of the nonradial solutions have been obtained in [1, 8, 15]. While for some subcritical nonlinearities and for annulus with small interior radius, it has been proved in [9] that all positive solutions of (2) are radial. For $f(u) = u^p \pm u$, the uniqueness of positive radial solutions has been studied by [2, 4, 16, 20, 21].

The aim of this paper is to give some uniqueness results for the following problem
\[
\begin{align*}
\Delta u + u^p + \lambda u &= 0 \quad \text{for } a < |x| < b, \\
   u &= 0 \quad \text{for } |x| = a, b, \\
   u &> 0 \quad \text{for } a < |x| < b
\end{align*}
\tag{3}
\]
where $0 < a < b < \infty$, $n \geq 2$, $p > 1$ and $\lambda > 0$. Ni & Nussbaum [16] have shown the uniqueness results of radial solutions under one of the following conditions:

(i) $1 < p \leq n/(n-2)$, $n \geq 3$.

(ii) the annulus is thin, say, $1 < b/a < c_n$ where $c_n = (n - 1)^{1/(n-2)}$ for $n \geq 3$, and $c_2 = e$ for $n = 2$.

(iii) $\lambda = 0$ (in the special case, there is no restriction on $p$ and the rate of $b/a$).

In fact, in [16], (i)-(iii) have been proved for a more general nonlinearity than the one mentioned here. The condition (ii) has been improved by Korman [12], in which he shows the uniqueness results for $1 < b/a \leq c_n$ where $c_n = (2n - 3)^{1/(n-2)}$ for $n \geq 3$, and $c_2 = e^2$ for $n = 2$. In [21], Yadava shows the uniqueness results for the subcritical and critical case $1 < p \leq (n + 2)/(n - 2)$.

For $0 < \lambda < \lambda_1(a, b)$, the least energy solution (restricted in the radial space) always exists, which is a positive radial solution. Here $\lambda_1(a, b)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in the annulus $\{x \in \mathbb{R}^n : a < |x| < b\}$. For $\lambda = 0$ the radial solution of (3) is unique and radially non-degenerate (see [16]). So, it is also unique for a sufficiently small and positive $\lambda$ by the implicity function theorem. We show that the radial solution of (3) is also unique for a certain range of $\lambda$.

Our uniqueness result is as follows:

**Theorem 1.1.** Let $n \geq 2$, $p > 1$ and $0 < a < b < \infty$, $\lambda > 0$ satisfies one of the following conditions

(i) $1 < p \leq (n + 2)/(n - 2)$ when $n > 2$ and $p > 1$ when $n = 2$.

(ii) $p > (n + 2)/(n - 2)$, $n > 2$ and $\lambda \in (0, \lambda_1) \cup [\lambda^*, \infty)$ where

$$
\lambda_1 = \frac{(\beta - 2)L}{\beta b^2}, \quad \lambda^* = \frac{(\beta - 2)L}{\beta a^2}.
$$

Here $\beta$ and $L$ are defined in (5) with $\nu = n - 1$. Then the Dirichlet problem (3) has at most one radial solution. Moreover, this unique radial solution is non-degenerate in the space of the radially symmetric function.

In [21], they focus on the non-degeneracy of the solution, while we mainly establish the uniqueness result first and then the non-degeneracy. We mainly focus on the super-critical case and our methods can be also applied for the subcritical and critical case, provided another proof of the result by Yadava [21].
For the radially symmetric solution, we will consider a more general equation

\[
\begin{cases}
  u'' + \frac{\nu}{r} u' + u^p + V(r)u = 0, & \text{for } a < r < b, \\
u u = 0, & \text{for } r = a, b, \\
u u > 0, & \text{for } a < r < b,
\end{cases}
\]  

(4)

where \( \nu \geq 0, p \in (1, \infty) \), and \( 0 < a < b < \infty \) are constant. We assume that \( V \) satisfies the following conditions:

1. \( V \in C^1[a, b] \).
2. \( G(r) = r^\beta V(r) - Lr^{\beta - 2} \) satisfies one of the three conditions:
   (i) \( G(r) \) is nonnegative in \( (a, b) \);
   (ii) \( G'(r) \) does not change signs in \( (a, b) \);
   (iii) \( G'(a) > 0 \) and \( G'(r) \) change signs exactly once in \( (a, b) \).

Here

\[
\alpha = \frac{2\nu}{p + 3}, \quad \beta = (p - 1)\alpha, \quad L = \alpha(\nu - 1 - \alpha).
\]

(5)

**Theorem 1.2.** Assume that \( \nu \geq 0, p \in (1, \infty) \), \( 0 < a < b < \infty \) and \( V \) satisfies (V1) and (V2). Then the problem (4) has at most one solution. Moreover, this unique solution is non-degenerate.

In the special case, \( \nu = n - 1 \) and \( V(r) \equiv \lambda \) is a positive constant, we have

\[
G(r) = \lambda r^\beta - Lr^{\beta - 2}, \quad G'(r) = r^{\beta - 3}(\lambda r^2 - L(\beta - 2))
\]

and

\[
\beta - 2 = \frac{2(n - 2)p - 2(n + 2)}{p + 3}, \quad L = \frac{2(n - 1)}{(p + 3)^2}[(n - 2)p + (n - 4)]
\]

If \( n = 2 \), then \( 0 < \beta < 2 \), \( L < 0 \), and \( G(r) > 0 \).

If \( n \geq 3 \), \( 1 < p \leq (n + 2)/(n - 2) \), then \( 0 < \beta \leq 2 \), \( L > 0 \), and \( G \) is increasing in \( (0, \infty) \).

If \( n \geq 3 \), \( p > (n + 2)/(n - 2) \), then \( \beta > 2 \), \( L > 0 \), and \( G \) is decreasing-increasing in \( (0, \infty) \) with the unique minimum point \( r = \bar{r} = \sqrt{(\beta - 2)L/(\lambda^* \beta)} \). If \( \bar{r} \notin (a, b) \), i.e., \( \lambda \in (0, \lambda_*) \cup [\lambda^*, \infty) \), then \( G \) is monotone in \( [a, b] \), so condition (ii) is satisfied.

**Remark 1.** Under the assumption of Theorem 1.1, if we set \( V(r) \equiv \lambda \) and \( \nu = n - 1 \), then \( V \) satisfies (V1),(V2). Thus, Theorem 1.1 follows directly from Theorem 1.2.

The proof of our theorem is based on the ideas developed in [14], [11] and [4]. In section 2, we proceed to prove that the solution is unique by a contradiction argument, assuming there is more than one solution. In doing so, we first characterize two possible positive solutions by the number of crossing points and then we use an energy analysis to get a contradiction. In section 3, we prove the non-degeneracy of the positive solution by a contradiction argument, assuming the positive solution \( \varphi \) is degenerate. We construct a perturbed equation, with \( \varphi \) as a trivial solution, and use the bifurcation methods to get a contradiction.

2. The uniqueness of positive radial solutions. In this section, we will prove the uniqueness part of Theorem 1.2, that is, (4) has at most one solution.

Before proving the uniqueness result, we point out the existence of solutions of (4). It is clear that a necessary condition for the existence is \( V \) satisfies \( \lambda_1(V) > 0 \) where \( \lambda_1 = \lambda_1(V) \) is the first eigenvalue of

\[
\begin{cases}
  \phi'' + \frac{\nu}{r} \phi' + V(r)\phi + \lambda \phi = 0, & \text{for } a < r < b, \\
\phi = 0, & \text{for } r = a, b
\end{cases}
\]

(6)
On the other hand, the solution of (4) is a critical point of the functional
\[ J(u) = \int_a^b \left\{ \frac{1}{2} u'^2 - V(r)u^2 - \frac{1}{p+1} u^{p+1} \right\} r'' dr, \]  
for \( u \in H_0^1(a,b) \). If \( V \) satisfies \( \lambda_1(V) > 0 \), then the functional \( J \) has the mountain pass structure and there exists a mountain pass solution [19], which is a solution of (4).

**Lemma 2.1.** (1) Any two distinct solutions \( u_1, u_2 \) of (4) must intersect at least once in \((a,b)\).

(2) Suppose that (4) has two distinct solutions \( u_1, u_2 \) such that
\[ 0 < u_1(r) < u_2(r) \]  
for \( r \in (a,\sigma) \),
\[ u_1(r) > u_2(r) > 0 \]  
for \( r \in (\sigma,b) \) for some \( \sigma \in (a,b) \). Then
\[ \frac{d}{dr} \left( \frac{u_1(r)}{u_2(r)} \right) > 0 \]  
for \( r \in (a,b) \).

**Proof.** Denote
\[ \xi(r) = r'' u_2(r)^2 \frac{d}{dr} \left( \frac{u_1(r)}{u_2(r)} \right) = r''(u_1 u_2 - u_1 u_2'). \]
Since \( u_1, u_2 \) are solutions of (4), we have
\[ \xi'(r) = r'' u_1 u_2 (u_2^{p-1} - u_1^{p-1}). \]
It immediately follows that the interior zero of \( \xi' \) is just the intersection points of \( u_1(r) \) and \( u_2(r) \). Observing that \( \xi(a) = \xi(b) = 0 \), we know \( \xi' = 0 \) has at least one interior zero, that is, \( u_1(r) \) and \( u_2(r) \) intersect at least once in \((a,b)\).

If (8) holds, then the interior zero (which is \( \sigma \)) of \( \xi' \) is unique, \( \xi'(r) > 0, r \in (a,\sigma) \) and \( \xi'(r) < 0, r \in (\sigma,b) \). Combining this with the fact \( \xi(a) = \xi(b) = 0 \), we get \( \xi(r) > 0 \) for \( r \in (a,b) \). This completes the proof.

If the solution of problem (4) is not unique, then we want to find two solutions \( u_1, u_2 \) which intersect exactly once, i.e., \( u_1, u_2 \) satisfies (8). In order to do this, we will use the solution of the initial value problem
\[ \begin{cases} u'' + \frac{r}{r} u' + |u|^{p-1} u + V(r)u = 0, \\ u(a) = 0, u'(a) = \gamma. \end{cases} \]  
Let \( u = u(r;\gamma) \) be the solution of (10) and let \( R = R(\gamma) \) be the first zero of \( u(\cdot;\gamma) \), i.e.,
\[ R(\gamma) = \sup \{ R' > a : u(r;\gamma) > 0, r \in (a,R') \}. \]

**Lemma 2.2.** Assume that \( V \in C[a,b] \). Then we have that
(i) \( \lim_{\gamma \to \infty} R(\gamma) = a. \)
(ii) \( \lim_{\gamma \to \infty} \max_{0 < r < R(\gamma)} u(r;\gamma) = \infty. \)
(iii) \( u(r;\gamma) \) has exactly one critical point in \([0,R(\gamma)]\) for all sufficiently large \( \gamma \).

**Proof.** Set
\[ u(r;\gamma) = \frac{\gamma}{l} v(s;\gamma), \quad s = l(r-a), \]
where \( l = \gamma^{(p-1)/(p+1)} \). Then \( v(s) = v(s; \gamma) \) satisfies the following equation

\[
\begin{cases}
  v_{ss} + \frac{\nu}{a+l+s} v_s + |v|^{p-1} v + \frac{1}{s} V(a + s/l)v = 0, \\
v(0) = 0, v_a(0) = 1,
\end{cases}
\]

which is a perturbed equation of

\[
\begin{cases}
  \tilde{v}_{ss} + |\tilde{v}|^{p-1} \tilde{v} = 0, \\
  \tilde{v}(0) = 0, \tilde{v}_s(0) = 1.
\end{cases}
\]  

The solution \( \tilde{v} \) of (12) and its derivative \( \tilde{v}_s \) is bounded in the interval \([0, \infty)\), by the fact that \((\tilde{v}^2 - 1)/2 + \tilde{v}^{p+1}/(p + 1) = 0\).

We claim that the solution \( \tilde{v} \) of (12) has a positive root. Otherwise, \( \tilde{v} \) is positive and strictly concave in the interval \((0, \infty)\). Then there are two possibilities. (i) \( \tilde{v} \) is always nondecreasing. In this case, for some fixed \( s_1 > 0 \), we have \( \tilde{v}(s) \leq \tilde{v}(s_1) > 0 \), \( \tilde{v}''(s) = -\tilde{v}''(s_1) < -\tilde{v}''(s_1) < 0 \) for \( s > s_1 \). It follows that for every \( s \in (s_1, \infty) \)

\[0 < \tilde{v}(s) \leq \tilde{v}(s_1) + \tilde{v}_s(s_1)(s - s_1) - \frac{1}{2} \tilde{v}''(s_1)(s - s_1)^2.\]

This yields a contradiction. (ii) \( \tilde{v}_s(s_1) < 0 \) for some \( s_1 > 0 \). Noting that \( \tilde{v} \) is concave in \((0, \infty)\), we deduce that \( \tilde{v}_s(s) < \tilde{v}_s(s_1) < 0 \), and

\[0 < \tilde{v}(s) \leq \tilde{v}(s_1) + \tilde{v}_s(s_1)(s - s_1)\]

for every \( s \in (s_1, \infty) \). This also yields a contradiction. Finally, we know \( \tilde{v} \) has a positive root, and the first positive root is denoted by \( s_0 \). Since (11) is a regular perturbation of (12), we can see that

\[v(s; \gamma) \to \tilde{v}(s) \text{ in } C^2_{\text{loc}}[0, \infty) \text{ as } \gamma \to \infty.\]  

Furthermore, we conclude that as \( \gamma \to \infty \), the first positive root \( s_\gamma \) of \( v(s; \gamma) \) is close to \( s_0 \), the first zero \( R(\gamma) \) of \( u(\cdot; \gamma) \) exists and satisfies

\[\lim_{\gamma \to \infty} \gamma^{\frac{p-1}{2p}} (R(\gamma) - a) = \lim_{\gamma \to \infty} s_\gamma = s_0.\]

This finishes (i).

Let \( \hat{s} \) be the unique critical point of \( \tilde{v} \) in \([0, s_0]\). It is obvious that \( \tilde{v}_s > 0 \) in \([0, \hat{s}]\), \( \tilde{v}_s < 0 \) in \((\hat{s}, s_0]\), and \( \tilde{v}_s(\hat{s}) < 0 \). When \( \gamma \) is sufficiently large, we see from (13) that \( v(\cdot; \gamma) \) has exactly one critical point \( \hat{s}(\gamma) \) in \([0, s_0]\). The conclusion (iii) immediately follows by the fact that \( u'(r; \gamma) = \gamma v_s(s; \gamma) \). The conclusion (ii) follows by the fact that

\[\lim_{\gamma \to \infty} \gamma^{-\frac{2}{p+2}} \max_{r \in [0, R(\gamma)]} u(r; \gamma) = \lim_{\gamma \to \infty} \max_{s \in [0, s_\gamma]} v(s; \gamma) = \lim_{\gamma \to \infty} v(\hat{s}_\gamma; \gamma) = \tilde{v}(\hat{s}) > 0.\]

This completes the proof. \(\square\)

For a set \( S \), we use \#(S) to denote the number of elements in \( S \).

**Lemma 2.3.** Assume that \( V \in C[a, b] \). Let \( u_1 \) be a solution of (4). Then there exists a number \( \bar{\gamma} \in (0, \infty) \) such that for every \( \gamma \geq \bar{\gamma} \), \( R(\gamma) < b \) and

\[\# \{ r \in (a, R(\gamma)) : u(r; \gamma) = u_1(r) \} = 1.\]

**Proof.** Let \( c_1 \in (a, b) \) be the first critical point of \( u_1 \) and \( M_1 = u_1(c_1) \). From Lemma 2.2, there exists a large constant \( \gamma_1 \) such that for every \( \gamma \geq \gamma_1 \), \( R(\gamma) < c_1 \) and the function \( u = u(r; \gamma) \) has a unique critical point (denoted by \( c = c(\gamma) \)) in \((a, R(\gamma))\).
We choose $\bar{\gamma} > \gamma_1$ large such that
\[
\frac{\bar{\gamma}^\nu}{c_1^\nu} - (M_1^p + \Lambda M_1)\frac{c_1^\nu + 1 - \bar{\gamma}^\nu + 1}{(\nu + 1)c_1^\nu} > \max_{r \in [a, b]} u_1'(r) > 0
\] (14)
where $\Lambda = \max_{r \in [a, b]} |V(r)|$. Let $\gamma \geq \bar{\gamma}$ be fixed and let $\bar{r}$ be as follows
\[
\bar{r} = \sup\{s \in (a, c(\gamma)) : u'(r; \gamma) > 0, u(r; \gamma) < M_1, r \in (a, s)\}.
\]
From $u$-equation, we have for $r \in (0, \bar{r})$
\[
(r'^\nu u')' = -r'^\nu (u^p + V(r)u) > -r'^\nu (M_1^p + \Lambda M_1)
\]
and integrating over $[a, r]$ we get
\[
u \gamma
\]
From the fact that the right-hand side of (15) is decreasing in $r$ and (14), we obtain that $u'(r; \gamma) > u_1'(r) > 0$, $r \in (a, \bar{r})$ and $u(\bar{r}) = M_1$. In particular, we have $u(r; \gamma) > u_1(r)$ for $r \in (a, \bar{r})$. In the interval $[\bar{r}, c(\gamma)]$, we have $u(r; \gamma) > M_1 > u_1(r)$.
In the interval $(c(\gamma), R(\gamma)]$, we have $u'(r; \gamma) < 0 < u_1'(r)$. Combining this with the fact that $u(R(\gamma); \gamma) = 0 < u_1(R(\gamma))$, we get the function $r \mapsto u(r; \gamma) - u_1(r)$ has a unique zero in $(a, R(\gamma))$. This completes the proof.

\begin{lemma}
Suppose that (4) has two distinct solutions $u_1, u_2$ such that $0 < \gamma_1 < \gamma_2$ where $\gamma_i = u_i'(a)$, $i = 1, 2$. Then (4) has a solution $u_3$ such that $u_3''(a) \geq u_2''(a)$ and the number of intersections of $u_3$ and $u_1$ is one,
\[
\#\{r \in (a, b) : u_3(r) = u_1(r)\} = 1.
\]
\end{lemma}

\begin{proof}
To prove the conclusion, we argue by contradiction. Suppose that
\[
\#\{r \in (a, b) : u(r; \gamma_2) = u_1(r)\} \geq 2.
\]
We denote by $\sigma_1(\gamma)$, $\sigma_2(\gamma)$ the first and second intersection points of $u(r; \gamma)$ and $u(r; \gamma_1)$. Clearly, $\sigma_1(\gamma)$, $\sigma_2(\gamma)$ exist and belong to $(a, b)$ at least for $\gamma$ close to $\gamma_2$.
By the uniqueness of solutions of the initial value problem at $r = \sigma_i(\gamma)$, we know
\[
u \gamma
\]
Hence by implicit function theorem $\sigma_1(\gamma)$ vary continuously as $\gamma$ moves.
We claim that there exists $\gamma_3 \in (\gamma_2, \infty)$ such that
\[
\lim_{\gamma \uparrow \gamma_3} \sigma_2(\gamma) = b.
\]
Let $[\gamma_2, \gamma^\ast)$ be the maximal interval in which both $\sigma_1(\gamma)$, $\sigma_2(\gamma)$ exist and belong to $(a, b)$, that is,
\[
\gamma^\ast = \sup\{\gamma' > \gamma_2 : \sigma_2(\gamma) \in (a, b)\text{ for } \gamma \in [\gamma_2, \gamma')\}.
\]
We show that for $\gamma \in [\gamma_2, \gamma^\ast)$,
\[
u \gamma
\]
Indeed, if (16) is false for some $\gamma \in [\gamma_2, \gamma^\ast)$. Since (16) holds $\gamma = \gamma_2$, we know there exists a minimal $\gamma_0 \in (\gamma_2, \gamma^\ast)$ such that
\[
u \gamma
\]
It immediately follows that
\[
u \gamma

By the uniqueness of solutions of the initial value problem at \( r = r_0 \), we know \( u(r; \gamma_0) \equiv 0 \). This contradicts \( u'(r; \gamma_0) = \gamma_0 > 0 \).

By Lemma 2.3 we know \( \gamma^* \) is finite, and hence (16) holds also for \( \gamma = \gamma^* \), and \( \lim_{\gamma \to \infty} \sigma_2(\gamma) = b \). Setting \( \gamma_3 = \gamma^* \) and \( u_3 = u(r; \gamma_3) \), we get \( u_3(b) = 0 \), \( u_3 \) is a solution of (4) and \( \#\{r \in (a, b) : u_3(r) = u_1(r)\} = 1 \). \( \square \)

Now we set
\[
w(r) = r^\alpha u(r), \quad \alpha = \frac{2\nu}{p+3}.
\]
Then \( w \) satisfies
\[
r^\beta w'' + \left( \frac{\beta}{2} r^{\beta-1} - L r^{\beta-2} \right) w = 0
\]
where \( \beta \) and \( L \) are constants given in (5). Denote
\[
G(r) = V(r)r^\beta - L r^{\beta-2},
\]
and
\[
E(r) = E(r, w) = \frac{1}{2} r^\beta w^2 + \frac{w^{p+1}}{p+1} + \frac{1}{2} G(r) w^2.
\]
(17)
Then
\[
E' = \frac{1}{2} G'(r) w^2.
\]

**Lemma 2.5.** Assume that \( V \) satisfies (V1)-(V2). Then \( E(r) > 0 \) for \( r \in [a, b] \).

**Proof.** (i) If \( G(r) \) is nonnegative in \((a, b)\), then we have \( E > 0 \) in \([a, b]\) since \( w \) and \( w' \) can not vanish simultaneously.

(ii) If \( G'(r) \) does not change signs in \((a, b)\), then \( E \) is monotone and hence \( E > 0 \) in \([a, b]\) since \( E(a) > 0 \) and \( E(b) > 0 \).

(iii) If \( G'(a) > 0 \) and \( G'(r) \) change signs exactly once in \((a, b)\), then \( E \) is increasing and decreasing and hence \( E > 0 \) in \([a, b]\) since \( E(a) > 0 \) and \( E(b) > 0 \). \( \square \)

Now we are ready to prove the uniqueness part of Theorem 1.2.

**Theorem 2.6.** Under the assumption (V1) and (V2), we know (4) admits at most one solution.

**Proof.** Suppose that (4) has two distinct solutions, then by Lemma 2.4, Lemma 2.1, we know (4) has two distinct solutions \( u_1, u_2 \) satisfying (8) and (9).

Now we set
\[
F(r) = E(r, w_2) - \left( \frac{w_2}{w_1} \right)^2 E(r, w_1).
\]
Then
\[
\frac{d}{dr} F(r) = -E(r, w_1) \frac{d}{dr} \left( \frac{w_2}{w_1} \right)^2 = -E(r, w_1) \frac{d}{dr} \left( \frac{u_2}{u_1} \right)^2 > 0
\]
for \( r \in (a, b) \) by (9) and Lemma 2.5.

On the other hand, we have that
\[
F(a) = \lim_{r \to a} F(r) = 0, \quad F(b) = \lim_{r \to b} F(r) = 0.
\]
This is a contradiction. This completes the proof. \( \square \)

**Remark 2.** The condition (V2) is used just to guarantee the positivity of \( E(r, w) \) in Lemma 2.5.
3. The non-degeneracy of positive radial solutions. In this section we prove the non-degeneracy of the unique solution by a contradiction argument based on the bifurcation theory of a perturbed equation. We say the solution \( u \) of (4) is non-degenerate if the linearized problem

\[
\begin{aligned}
&\frac{w''}{r} + \frac{\nu}{r} w' + [p r^{p-1} + V(r)] w = 0, \\
&w(a) = w(b) = 0,
\end{aligned} \tag{18}
\]

does not admit any non-trivial solution.

Let us denote by \( \varphi = \varphi(r) \) the unique positive solution of (4). For any \( \delta \in \mathbb{R} \), we consider a perturbed equation related to (4)

\[
\begin{aligned}
&u'' + \frac{\nu}{r} u' + (1 + \delta) u^p + (V(r) - \delta r^{p-1}) u = 0, \\
&u(a) = u(b) = 0.
\end{aligned} \tag{19}
\]

Clearly, \( u = \varphi \) is always a solution of (19).

We first prove a uniform bounded result by the blow up methods.

**Lemma 3.1.** Let \( \Lambda > 0 \) be any fixed constant. Then there exists a constant \( K \) depending only on \( \Lambda \) such that if \( u \) is a solution of (4) with \( |V(r)| < \Lambda \) then \( \|u\|_{C^0[a,b]} < K \).

**Proof.** The proof is based on the blow up methods [7]. We argue by contradiction and suppose that there exists a sequence of solutions \( u_k \) (related to \( V = V_k \) with \( |V_k| < \Lambda \)) such that \( M_k = u_k(r_k) = \|u_k\|_{C^0[a,b]} \to \infty \) as \( k \to \infty \). Set

\[
v_k(s) = \frac{1}{M_k} u_k(M_k^{-(p-1)/2} s + r_k), s \in I_k
\]

where \( I_k = (\hat{a}_k, \hat{b}_k) \) is the interval \( (M_k^{(p-1)/2}(a - r_k), M_k^{(p-1)/2}(b - r_k)) \). Then \( v_k \) satisfies

\[
(v_k)_s s + \frac{\nu M_k^{-(p-1)/2}}{M_k^{-(p-1)/2} s + r_k} (v_k)_s + v_k^p + M_k^{1-p} V(M_k^{-(p-1)/2} s + r_k)v_k = 0.
\]

Note that \( \hat{b}_k - \hat{a}_k = M_k^{(p-1)/2}(b - a) \to \infty \). Without loss of generality, we assume that \( \hat{b}_k \to \infty \). Note that for any \( R > 0 \), \( \|v_k\|_{C^2[0,R]} \) is bounded as \( k \to \infty \). By passing subsequence if necessary, we may assume that \( v_k \) converges in \( C^{1+1/2}_{loc}[0, \infty) \) to a function \( v \), and \( v \) is a weak (then classical) solution to

\[
v_{ss} + (v^p) = 0 \quad \text{in} \quad [0, \infty)
\]

with \( 0 \leq v \leq 1 \), \( v(0) = 1 \). Here we use the fact \( M_k^{-(p-1)/2} s + r_k > a > 0 \). Since \( v \neq 1 \), we know there is a \( s_1 \) such that \( v_s(s_1) < 0 \), and hence \( v_s(s) \leq v_s(s_1) < 0 \) for \( s > s_1 \), this leads \( v \) to be unbounded from below, a contradiction. \( \square \)

The corresponding “energy” to (19) is denoted by \( E^\delta = E^\delta(r, u) \) as follows

\[
E^\delta(r, u) = \frac{1}{2} r^\beta [(r^\alpha u)^2 + (1 + \delta)(r^\alpha u)^{p+1}] + \frac{1}{2} \left( \frac{(r^\beta V(r) - \delta r^{p-2} - \delta r^{p-1}(r)) \langle r^\alpha u \rangle^2}{p+1} \right).
\]

(20)

Now we turn to prove the uniqueness of positive solutions of (19).

**Lemma 3.2.** Assume that \( V \) satisfies (V1) and (V2). Then there is a small constant \( \delta_0 \) such that for \( \delta \in (-\delta_0, \delta_0) \), problem (19) has a unique positive solution, which is \( u = \varphi \).
Proof. Let $S_\delta$ denote the set of positive solutions of (19). By Lemma 3.1, we know that $\cup_{\delta \in [-1, 1]} S_\delta$ is a bounded set in $C^0[a, b]$ and then in $C^2[a, b]$, and it is a pre-compact of $C^1[a, b]$. Since $S_0$ contains a single element $\varphi$, we have

$$\lim_{\delta \to 0} \sup_{u \in S_\delta} \|u - \varphi\|_{C^1[a, b]} = 0.$$  

(21)

From Lemma 2.5, $E^0(r, \varphi) > 0$ in $[a, b]$. Then there exist a positive constant $\delta_0$ such that $E^\delta(r, u) > 0$, $r \in [a, b]$ for $u \in S_\delta$ and for $\delta \in [-\delta_0, \delta_0]$. Therefore, by Theorem 2.6, the positive solution of (19) is unique for $\delta \in [-\delta_0, \delta_0]$.

Theorem 3.3. The solution of (4) is non-degenerate.

Proof. In order to derive the non-degeneracy, we argue indirectly. Suppose that this unique solution $\varphi$ of (4) is degenerate. Noting that $u = \varphi$ is always a solution of (19), we can use the local bifurcation theorem (See [3]) to obtain another solution which will be a contradiction. Denote by $F : C^2_0[a, b] \times \mathbb{R} \to C[a, b]$ the map as follows

$$F(u, \delta) = u'' + \frac{\nu}{r} u' + (1 + \delta)|u|^{p-1}u + (V(r) - \delta \varphi^{p-1})u,$$

where $\delta$ is the bifurcation parameter, $C^2_0[a, b] = \{u \in C^2[a, b] : u(a) = u(b) = 0\}$. Note that $F$ is $C^2$. It is easy to calculate the Frechlet derivatives at $(u, \delta) = (\varphi, 0)$,

$$L_0 = F_u(\varphi, 0) = \partial_{rr} + \frac{\nu}{r} \partial_r + (p\varphi^{p-1} + V),$$

and $F_{u\delta}(\varphi, 0) = (p - 1)\varphi^{p-1}$.

By the degeneracy of $\varphi$, we know 0 is a simple eigenvalue of $L_0$. It is clear that the kernel of $L_0$ is spanned by some function $\phi$ ($\phi \neq 0$) and the range space of $L$ is characterized by

$$R(L_0) = \{ h \in C[a, b] : \int_a^b \phi(r) h(r) r^\nu dr = 0 \}.$$

Since $\int_a^b \varphi^{p-1}(r) \phi^2(r) r^\nu dr > 0$, we know $F_{u\delta}(\varphi, 0)[\phi] \not\in R(L_0)$. From the simple eigenvalue bifurcation theorem, we know that near $(\varphi, 0)$ all the zeros of $F = 0$ lie either on the trivial solution curve $(\varphi, \delta)$ or lie on the continuous solution curve $\Gamma$ which is given by

$$\delta = \delta(s) = o(1), \quad u = u(s) = \varphi + s\phi + o(s) \text{ for } |s| \ll 1.$$

It immediately follows that for some $s$ (with $|s|$ small), $F$ has a non-trivial zero, which is another positive solution of (19). This contradicts the uniqueness proved in Lemma 3.2. This completes the proof.

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