Generator Matrix Elements For $G_2 \supset SU(3)$ II:

Generic Representations

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Abstract

Basis states and generator matrix elements are given for the generic representation \((a, b)\) of \(G_2\) in an \(SU(3)\) basis.
1 Introduction

In a recent paper by Farell et al (1994), to be referred to as I, generator matrix elements of \( G_2 \) in an \( SU(3) \) basis are given for degenerate representation (one Dynkin label zero) of \( G_2 \). In this paper the analysis is extended to generic representations.

In the degenerate case \((a, 0)\) and \((0, b)\) the subgroup provides a complete set of internal labels, so the states are automatically orthogonal and can be normalized straightforwardly; in the generic case \((a, b)\) there is one missing label and it is convenient to use non–orthogonal unnormalized states.

Section 2 deals with the basis states and their non-orthonormality is discussed. In Section 3 the generator matrix elements are derived. Section 4 contains some concluding remarks. The earlier paper I contains comments on physical applications of our results and references to previous work on the subject; special reference should be made to the early work of Sviridov, Smirnov and Tolstoy (1975).

2 Basis States

We will use “character states”, for which the integrity basis is provided by the \( G_2 \) character generator (see I). It is best to start with the \( G_2 \supset SU(3) \) branching rules generating function (Gaskell et al 1978)

\[
G(A, B; P, Q) = [(1 - AP)(1 - AQ)(1 - BP)(1 - BQ)]^{-1} \times
\]

\[
\]
\[
(1 - APQ)^{-1} + B (1 - B)^{-1}.
\]  \hspace{1cm} (2.1)

We are using Dynkin’s labelling of the \(G_2\) fundamental representations: \((1, 0)\) is the 14–plet and \((0, 1)\) the septet. In I and in Gaskell et al (1978) that numbering is reversed.

The power series expansion of (2.1) gives the \(G_2 \supset SU(3)\) branching rules: the coefficient of \(A^a B^b P^p Q^q\) is the multiplicity of the \(SU(3)\) representation \((p, q)\) in the \(G_2\) representation \((a, b)\). But equation (2.1) does more that count multiplicities. We interpret \(AP \sim \lambda\), \(AQ \sim \nu^*\), \(APQ \sim \alpha\), \(BP \sim \eta\), \(BQ \sim \zeta^*\), \(B \sim \theta\) as the highest states of the \(SU(3)\) representations contained in the fundamental \(G_2\) representations (see Fig. 1 of I); the highest state of any \(SU(3)\) representation in any \(G_2\) representation is then given by the appropriate product of powers of them.

We distinguish two types of state, called \(\alpha\)–states and \(\theta\)–states according to whether the highest state of the \(SU(3)\) representation to which they belong contains \(\alpha\) or \(\theta\) (by (2.1) it cannot contain both). For \(\alpha\)–states we adopt the exponent of \(\zeta^*\) as the missing label; for \(\theta\)–states the missing label is the exponent of \(\nu^*\). We call the missing label \(i\) in both cases. Then a highest \(\alpha\)–state is

\[
\begin{vmatrix}
 p & q & i \\
 p & p & (p + 2q)/3
\end{vmatrix}_\alpha = \eta^{b-i} \zeta^{*i} \lambda^{a-q+i} \nu^{s+a+b-p-i} \alpha^{p+q-a-b},
\]  \hspace{1cm} (2.2)

and a highest \(\theta\)–state is

\[
\begin{vmatrix}
 p & q & i \\
 p & p & (p + 2q)/3
\end{vmatrix}_\theta = \eta^{p-a+i} \zeta^{*q-i} \theta^{a+b-p-q} \lambda^{a-i} \nu^{*i}.
\]  \hspace{1cm} (2.3)
We remark that a state for which \( p + q = a + b \) may be called, and labelled as, an \( \alpha \)-state or a \( \theta \)-state (\( \alpha \) and \( \theta \) both absent); its \( i \) label as an \( \alpha \)-state is \( q \) less its \( i \) label as a \( \theta \)-state. We suppress the \( G_2 \) representation labels \( (a, b) \). The internal \( SU(3) \) labels are respectively \( t, m, y \) with the isospin labels doubled to avoid half–odd values. The ranges of the labels \( p, q, i \) are such that the exponents in (2.2) and (2.3) take all non–negative integer values.

States other than the highest ones given by Equations (2.2, 2.3) are obtained by applying the \( SU(3) \) lowering generators \( E_{21}, E_{32}, E_{31} \). In differential form, suitable for acting on the states (2.2, 2.3) each is given by the sum of the two expressions for it in Eq. (I 2.6) and Eq. (I 2.16).

The basis states of the \( G_2 \) representation \( (a, b) \) are polynomials of degree \( a \) in the \((1,0)\) states and degree \( b \) in the \((0,1)\) states. Thus only stretched representations (representation labels additive) in the direct product of \( a \) copies of \((1,0)\) and \( b \) copies of \((0,1)\) are to be retained. It is known (I) that the elementary unstretched, and therefore unwanted, representations are all of degree 2.

The \( G_2 \) character generator is needed in dealing with the unwanted states. Interpreted as describing the integrity basis for general basis states, it tells us that certain pairs of fundamental representation states are incompatible, \( i.e. \), never appear multiplied. The incompatible pairs of a particular weight are equal in number to the
unwanted states of that weight. Equating each unwanted state to zero allows us to solve for each incompatible pair in terms of pairs that are compatible. When an incompatible pair appears we eliminate it by means of these incompatibility equations.

We have used a version of the character generator in which all fundamental basis states appear in the denominator factors, as opposed to the version of Gaskell and Sharp (1981) in which only exterior states appear in denominators. Since the incompatibility rules characterize the character generator completely we content ourselves with presenting the rules in tabular form. Incompatibilities between (1, 0) and (0, 1) states are shown in Table 1. Those between pairs of (1, 0) states are shown in Figure 2 of I (ν, ν*, δ are compatible with all (1, 0) states). The only incompatible pair of (0, 1) states is ξξ*. We should mention that δ is the m = 0 state of an SU(2) triplet and κ is an SU(2) scalar.

We have determined all the unwanted states and, equating then to zero, found the equations by which incompatible pairs are to be eliminated. We give only the replacements that are actually needed in Section 3.

(1, 0) × (0, 1) states

\[ \alpha \zeta = (2/3)^{1/2} \kappa \eta - (4/3)^{1/2} \nu^* \xi^* - 6^{-1/2} \lambda \theta , \]
\[ \gamma \xi = 2^{-1/2} \delta \eta + 6^{-1/2} \kappa \eta - (4/3)^{1/2} \nu^* \xi^* , \]
\( \beta \xi^* = -\alpha \eta^* - (2/3)^{1/2} \kappa \zeta^* - 6^{-1/2} \nu^* \theta, \)

\( \alpha \theta = (2/3)^{1/2} \lambda \zeta^* + (2/3)^{1/2} \nu^* \eta, \)

\( \alpha \xi^* = -\gamma \zeta^* - 3^{-1/2} \lambda \eta, \)

\( \lambda^* \zeta^* = \nu^* \eta^* - 2^{-1/2} \mu \theta, \)

\( \mu^* \zeta^* = 2^{-1/2} \lambda \theta + \nu^* \xi^*, \)

\( \lambda \zeta = \nu \eta - 2^{-1/2} \mu^* \theta, \)

\( \mu^* \xi = -\lambda^* \eta - \nu^* \zeta, \)

\( \mu \xi^* = -\lambda \eta^* - \nu \zeta^*, \)

\( \kappa \theta = \nu^* \zeta + \nu \zeta^*, \quad (2.4) \)

\((1,0)^2 \) states

\( \beta \mu^* = -6^{-1/2} \kappa \nu^* - 2^{-1/2} \delta \nu^* + 3^{-1/2} \mu \lambda, \)

\( \gamma \mu = 1/2 \alpha \nu + 2^{-1/2} \lambda \delta + (12)^{-1/2} \mu^* \nu^*, \)

\( \alpha \lambda^* = -6^{-1/2} \nu^* \kappa + 2^{-1/2} \nu^* \delta - 3^{-1/2} \lambda \mu, \)

\( \beta \gamma = 6^{-1/2} \alpha \kappa + 2^{-1/2} \alpha \delta + 1/3 \lambda \nu^*, \)

\( \lambda \kappa = (3/2)^{1/2} \alpha \nu - 2^{-1/2} \mu^* \nu^*. \)

\( \alpha \mu = \beta \lambda - 3^{-1/2} \nu^* \theta^2, \)

\( \alpha \mu^* = 3^{-1/2} \lambda^2 - \gamma \nu^*, \)

\( \mu \mu^* = -\nu \nu^* - \lambda \lambda^*, \quad (2.5) \)
\[(0, 1)^2 \text{ states}
\]
\[
\xi \xi^* = -\eta \eta^* - \zeta \zeta^* - 1/2 \theta^2.
\] (2.6)

It should be remarked that although our states correspond one–to–one to all states of all \(G_2\) representations they still contain admixtures of unwanted states belonging to lower representations. That does not matter for the purpose of computing generator matrix elements, to which we are about ready to turn.

Since our state are non–orthonormal it is preferable to define the matrix element \((i \mid G \mid j)\) of a generator \(G\) between two states \(|i\rangle\) and \(|j\rangle\) as the coefficient of \(|i\rangle\) when \(G\) acts on \(|j\rangle\), rather than as the overlap \(\langle i \mid G \mid j \rangle\). Matrices so defined can be multiplied in the usual way. The Wigner–Eckart theorem holds for them. Matrices for operators can be diagonalized and their eigenvalues and eigenstates found by the usual standard techniques.

### 3 The Generator Matrix Elements

We now calculate generator matrix elements with respect to \(\alpha\)–states (highest state given by (2.2)) and \(\theta\)–states (highest state given by (2.3)). The six significant generators are the components of two \(SU(3)\) tensors \(G^{(10)}\) and \(G^{(01)}\) which transform by the indicated \(SU(3)\) representations.
According to the SU(3) Wigner–Eckart theorem the matrix elements of $G^{(10)}$ are given in terms of its reduced matrix elements (double bars) by

$$
\langle \begin{array}{c} p_2 q_2 i' \\ t_2 \ m_2 \ y_2 \end{array} | G_{i,m,y}^{(10)} | \begin{array}{c} p_1 q_1 i \\ t_1 \ m_1 \ y_1 \end{array} \rangle = \langle \begin{array}{c} p_2 q_2 i' \\ t_2 \ m_2 \ y_2 \end{array} | G_{i,m,y}^{(10)} | \begin{array}{c} p_1 q_1 i \\ t_1 \ m_1 \ y_1 \end{array} \rangle
$$

$$
\times \langle \begin{array}{c} p_1 q_1 \\ t_1 \ m_1 \ y_1 \end{array} | \begin{array}{cc} 1 & 0 \\ t & m y \end{array} | \begin{array}{c} p_2 q_2 \\ t_2 \ m_2 \ y_2 \end{array} \rangle \times [(p_2 + 1)(q_2 + 1)(p_2 + q_2 + 2)/2]^{-1/2}.
$$

(3.1)

The second factor on the right is an SU(3) Clebsh–Gordon coefficient. A similar formula exists for the matrix elements of $G^{(01)}$, we will see below that the reduced matrix elements of $G^{(01)}$ may be expressed in terms of those of $G^{(10)}$.

We may write

$$
G_{0,0,-\frac{2}{3}}^{(10)} \left| \begin{array}{c} p \ q \ i \\ p \ p \ (p + 2q)/3 \end{array} \right\rangle = \sum_{i'} A_{i',i} \left| \begin{array}{c} p + 1 \ q \ i' \\ p \ p \ (p + 2q - 2)/3 \end{array} \right\rangle + \sum_{i'} B_{i',i} \left| \begin{array}{c} p - 1 \ q \ i' \\ p \ p \ (p + 2q - 2)/3 \end{array} \right\rangle + \sum_{i'} C_{i',i} \left| \begin{array}{c} p - 1 \ q \ i' \\ p \ p \ (p + 2q - 2)/3 \end{array} \right\rangle
$$

(3.2)

where $A_{i',i}$, $B_{i',i}$, $C_{i',i}$ are matrix elements of $G_{0,0,-\frac{2}{3}}^{(10)}$ to be determined. We have suppressed a subscript $\alpha$ or $\theta$ on the states in (3.2) and, correspondingly, a superscript $\alpha$ or $\theta$ on the coefficients $A_{i',i}$, $B_{i',i}$, $C_{i',i}$.

We remark that an ambiguous state ($p + q = a + b$) is transformed by $G^{(10)}$ into an $\alpha-$state, an ambiguous state or a $\theta-$state. An $\alpha-$state always goes to an $\alpha-$state except when $p + q = a + b + 1$ when it can also go to an ambiguous state; and a
\( \theta \)-state always goes to a \( \theta \)-state except when \( p + q = a + b - 1 \) when it can also go to an ambiguous state.

Apply \( E_{12} E_{23} \) to both sides of (3.2). The result is

\[
G^{(10)}_{1,1,1} \begin{bmatrix} p & q & i \\ p & p & (p + 2q)/3 \end{bmatrix} = \sqrt{\frac{(p + 1)(p + q + 2)}{p + 2}} \sum_{i'} \begin{bmatrix} p + 1 & q & i' \\ p & p & (p + 2q - 2)/3 \end{bmatrix} A_{i',i}. \tag{3.3}
\]

The states in the second and third sums on the right-hand side of (3.2) are annihilated and we can read off the allowed values of \( i' \) as well as the matrix element \( A_{i',i} \). We find, for \( \alpha \)-states, \( i' \) can be only \( i \) and

\[
A^{\alpha}_{i',i} = (-i - p + a + b) \sqrt{\frac{3(2 + p)}{(1 + p)(2 + p + q)}}. \tag{3.4}
\]

For \( \theta \)-states \( i' \) can be \( i \) or \( i - 1 \) and

\[
A^{\theta}_{i',i} = (i - p - q + a + b) \sqrt{\frac{2(2 + p)}{(1 + p)(2 + p + q)}}, \tag{3.5}
\]

\[
A^{\theta}_{i'-1,i} = i \sqrt{\frac{2(2 + p)}{(1 + p)(2 + p + q)}}. \tag{3.6}
\]

Next apply \( E_{23} \) to both sides of (3.2), after transferring the first sum to the left.

The result is

\[
G^{(10)}_{1,-1,1} \begin{bmatrix} p & q & i \\ p & p & (p + 2q)/3 \end{bmatrix} = - \sqrt{\frac{p + q + 2}{(p + 1)(p + 2)}} \sum_{i'} A_{i',i} E_{21} \begin{bmatrix} p + 1 & q & i' \\ p + 1 & p + 1 & (p + 2q + 1)/3 \end{bmatrix} E_{23} \begin{bmatrix} p - 1 & q + 1 & i' \\ p - 1 & p - 1 & (p + 2q - 2)/3 \end{bmatrix}. \tag{3.7}
\]
We can read the values of $i'$ and the matrix elements $B_{\alpha,i'}$. For $\alpha$–states $i'$ can be $i$ or $i + 1$, and

$$B_{i,i}^\alpha = \frac{(i - q + a)(2 - i + p + a + b)}{(1 + p) \sqrt{1 + q}}, \quad (3.8)$$

$$B_{i+1,i}^\alpha = \frac{(i - b)(1 - 2i - p + 2a + 2b)}{(1 + p) \sqrt{1 + q}}. \quad (3.9)$$

For $\theta$–states $i'$ can also be only $i$ or $i + 1$, and

$$B_{i,i}^\theta = \frac{(1 + i + p)(-i - p + a)}{(1 + p) \sqrt{1 + q}}, \quad (3.10)$$

$$B_{i+1,i}^\theta = \frac{(-i + a)(2 + i + p - q + a + b)}{(1 + p) \sqrt{1 + q}}. \quad (3.11)$$

Finally transfer the first two sums to the left side of (3.2). Only the third sum remains on the right and we can read the allowed values of $i'$ and the matrix elements $C_{\alpha,i'}$:

For $\alpha$–states $i'$ can be $i, i - 1$ or $i + 1$ and

$$C_{i,i}^\alpha = \frac{1}{\sqrt{3}(1 + p)(1 + q)(2 + p + q)} \left( -4i - 4i^2 + 4i^3 - 6p - 7ip - 2i^2p + 3i^3p - 8p^2 - 4ip^2 + i^2p^2 - 2p^3 - ip^3 - 6q - 3iq - 4i^2q + i^3q - 13pq - 7ipq - 2i^2pq - 8p^2q - 3ip^2q - p^3q - 5q^2 - iq^2 - i^2q^2 - 7pq^2 - 2ipq^2 - 2p^2q^2 - q^3 - pq^3 + 6b + 2ib - 8i^2b + 6pb - 6i^2pb - ip^2b + 2gb + 6iqb - 2i^2qb + 4pqb + 3ipqb + p^2qb + 2iq^2b + pq^2b + 2b^2 + 4ib^2 + 2pb^2 + 3ipb^2 - 2qb^2 + iq^2b - pqb^2 - q^2b^2 + 6a - 3ia - 4i^2a + 3pa - 3ipa - 3i^2pa - 4p^2a - p^3a + 2iqa - i^2qa - 2pqa + ipqa - 2p^2qa - q^2a + i^2a - pq^2a + 7ba + 3iba + 11 \right)
\[ 7pba + 2ipba - qba + iqba - q^2ba + b^2a + pb^2a + 5a^2 - \]
\[ ia^2 + 5pa^2 - ipa^2 + qa^2 + pqa^2 + 2ba^2 + 2pba^2 + a^3 + pa^3, \quad (3.12) \]

\[ C^\alpha_{i+1,i} = \frac{(i-b)(-i-p+a+b)}{\sqrt{3}(1+p)(1+q)(2+p+q)} \sqrt{3}(1+p)(1+q)(2+p+q) \]
\[ pq + 3b + 2pb + qb + 3a + 2pa + qa), \quad (3.13) \]

\[ C^\alpha_{i-1,i} = \frac{i}{\sqrt{3}(1+q)(2+p+q)} \]

For \( \theta \)-states \( i' \) takes the values \( i \) or \( i-1 \) and

\[ C^\theta_{i,i} = \frac{(i-q)(1-i+q+a)(4+i+p+q+a+b)}{\sqrt{2}(1+q)(2+p+q)}, \quad (3.15) \]

\[ C^\theta_{i-1,i} = \frac{i(1+i+p)(-i-p+a)}{\sqrt{2}(1+q)(2+p+q)}. \quad (3.16) \]

When one of the states in the matrix element is ambiguous its label \( \alpha \) or \( \theta \) must be the same as for the other state. If both are ambiguous they should both be given the same label \( \alpha \) or \( \theta \).

The matrix elements \( A^{(\alpha,\theta)}_{i',i} \), \( B^{(\alpha,\theta)}_{i',i} \), \( C^{(\alpha,\theta)}_{i',i} \) are all we need to obtain the corresponding reduced matrix elements. Also we can get the needed matrix elements of \( G^{(01)} \) by following the steps used above for \( G^{(10)} \). If we use lowest states (and near lowest) of the \( SU(3) \) IR \( (q,p) \) instead of highest states (and near highest) of \( (p,q) \), and use hermitian conjugates of all the generators, then apart from some changes in phase, the steps are identical in every detail, with starred and unstarred variables interchanged.
For $\alpha$—states the results are

$$\langle p+1 \ q \ i \ | \ G^{(10)} \ | \ p \ q \ i \ \rangle = - \langle q \ p+1 \ i \ | \ G^{(01)} \ | \ q \ p \ i \ \rangle$$

$$= (-i - p + a + b) \sqrt{\frac{3 (2 + p) (1 + q) (3 + p + q)}{2}}, \quad (3.17)$$

$$\langle p-1 \ q+1 \ i \ | \ G^{(10)} \ | \ p \ q \ i \ \rangle = \langle q+1 \ p-1 \ i \ | \ G^{(01)} \ | \ q \ p \ i \ \rangle$$

$$= (i - q + a) (2 - i + p + a + b) \sqrt{\frac{(2 + q) (2 + p + q)}{2 (1 + p)}}, \quad (3.18)$$

$$\langle p-1 \ q+1 \ i+1 \ | \ G^{(10)} \ | \ p \ q \ i \ \rangle = \langle q+1 \ p-1 \ i+1 \ | \ G^{(01)} \ | \ q \ p \ i \ \rangle$$

$$= (i - b) (1 - 2i - p + 2a + 2b) \sqrt{\frac{(2 + q) (2 + p + q)}{2 (1 + p)}}, \quad (3.19)$$

$$\langle p \ q-1 \ i \ | \ G^{(10)} \ | \ p \ q \ i \ \rangle = - \langle q-1 \ p \ i \ | \ G^{(01)} \ | \ q \ p \ i \ \rangle$$

$$= \frac{1}{\sqrt{6 (1 + p) (1 + q) (2 + p + q)}} (-4i - 4i^2 + 4i^3 - 6p - 7ip - 2i^2p + 3i^3p - 8p^2 - 4ip^2 + i^2p^2 - 2p^3 - ip^3 - 6q - 3iq - 4i^2q + i^3q - 13pq - 7ipq - 2i^2pq - 8p^2q - 3ip^2q - p^3q - 5q^2 - iq^2 - i^2q^2 - 7pq^2 - 2ipq^2 - 2p^2q^2 - q^3 - pq^3 + 6b + 2ib - 8i^2b + 6pb - 6i^2pb - ip^2b + 2qb + 6iqb - 2i^2qb + 4pqb + 3ipqb + p^2qb + 2iq^2b + pq^2b + 2b^2 + 4ib^2 + 2p^2b + 3ipl^2 - 2q^2b - iq^2b - pq^3 - q^2b^2 + 6a - 3ia - 4i^2a + 3pa - 3ipa - 3i^2pa - 4p^2a - p^3a + 2iqa - i^2qa - 2pqa + ipqa - 2p^2qa - q^2a + iq^2a - pq^2a + 7ba + 3iba + 7pba + 2ipba - qba + iqba - q^2ba + b^2a + pb^2a + 5a^2 - ia^2 + 5pa^2 - ipa^2 + qa^2 + pqa^2 + 2ba^2 + 2pba^2 + a^3 + pa^3), \quad (3.20)$$
\[
\langle p q - 1 i + 1 \| G^{(10)} \| p q i \rangle = - \langle q - 1 p i + 1 \| G^{(01)} \| q p i \rangle
\]
\[
= \frac{(i - b) (-i - p + a + b)}{\sqrt{6} (1 + p) (1 + q) (2 + p + q)} (1 - 3i - p - 2ip - p^2 - iq - pq + 3b + 2pb + qb + 3a + 2pa + qa),
\] (3.21)
\[
\langle p q - 1 i - 1 \| G^{(10)} \| p q i \rangle = - \langle q - 1 p i - 1 \| G^{(01)} \| q p i \rangle
\]
\[
= i (-4 + i - p - 2q - a - b) (1 + i + a) \sqrt{\frac{1 + p}{6 (1 + q) (2 + p + q)}},
\] (3.22)

For \(\theta\)-states we find for the reduced matrix elements
\[
\langle p + 1 q i \| G^{(10)} \| p q i \rangle = - \langle q p + 1 i \| G^{(01)} \| q p i \rangle
\]
\[
= (i - p - q + a + b) \sqrt{3 (2 + p) (1 + q) (3 + p + q)},
\] (3.23)
\[
\langle p + 1 q i - 1 \| G^{(10)} \| p q i \rangle = - \langle q p + 1 i - 1 \| G^{(01)} \| q p i \rangle
\]
\[
= i \sqrt{\frac{3 (2 + p) (1 + q) (3 + p + q)}{2}},
\] (3.24)
\[
\langle p - 1 q + 1 i \| G^{(10)} \| p q i \rangle = \langle q + 1 p - 1 i \| G^{(01)} \| q p i \rangle
\]
\[
= (1 + i + p) (-i - p + a) \sqrt{\frac{(2 + q) (2 + p + q)}{2 (1 + p)}},
\] (3.25)
\[
\langle p - 1 q + 1 i + 1 \| G^{(10)} \| p q i \rangle = \langle q + 1 p - 1 i + 1 \| G^{(01)} \| q p i \rangle
\]
\[
= (-i + a) (2 + i + p - q + a + b) \sqrt{\frac{(2 + q) (2 + p + q)}{2 (1 + p)}},
\] (3.26)
\[
\langle p q - 1 i \| G^{(10)} \| p q i \rangle = - \langle q - 1 p i \| G^{(01)} \| q p i \rangle
\]
\[
= (i - q) (1 - i + q + a) (4 + i + p + q + a + b) \sqrt{\frac{1 + p}{4 (1 + q) (2 + p + q)}},
\] (3.27)
\[ \langle p \ q - 1 \ i - 1 \ | \ G^{(10)} \ | \ p \ q \ i \rangle = - \langle q - 1 \ p \ i - 1 \ | \ G^{(01)} \ | \ q \ p \ i \rangle = i (1 + i + p) (-i - p + a) \sqrt{\frac{1 + p}{4(1 + q)(2 + p + q)}}. \] (3.28)

4 Concluding Remarks

We have used non-orthonormal generic states; that is convenient when there is a missing label. Our results can be compared with those of I by setting one of the \( G_2 \) representation \( a \) or \( b \) labels equal to zero, omitting the “missing” label \( i \), and replacing the normalization constants \( N_{p,q} \) of I by unity.
Table 1: Incompatibility table. Each incompatible pair of $(1, 0) \times (0, 1)$ states is marked with a cross. The $(0, 1)$ states $\eta, \eta^*$ and the $(1, 0)$ states $\nu, \nu^*$ are compatible with all.
5 References

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