ABSTRACT

Broadcasting and convergecasting are pivotal services in distributed systems, in particular, in wireless ad-hoc and sensor networks, which are characterized by time-varying communication graphs. We study the question of whether it is possible to disseminate data available locally at some process to all $n$ processes in sparsely connected synchronous dynamic networks with directed links in linear time. Recently, Charron-Bost, Függer and Nowak proved an upper bound of $O(n \log n)$ rounds for the case where every communication graph is an arbitrary directed rooted tree. We present a new formalism, which not only facilitates a concise proof of this result, but also allows us to prove that $O(n)$ data dissemination is possible when the number of leaves of the rooted trees are bounded by a constant. In the special case of rooted chains, only $(n-1)$ rounds are needed. Our approach can also be adapted for undirected networks, where only $(n-1)/2$ rounds in the case of arbitrary chain graphs are needed.

1. INTRODUCTION

We consider a synchronous network of $n$ failure-free nodes with unique ids (uids), which are connected by directed point-to-point links. The nodes execute a deterministic algorithm for disseminating some local data (say, the uids for simplicity), which shall ensure that the uid of at least one node becomes known to all nodes in the system as fast as possible. In a synchronous distributed system, the execution proceeds in the form of lock-step rounds $r = 1, 2, \ldots$, where all processes send and receive round-$r$ messages and simultaneously execute a computing step, which also starts the next round. Communication is unreliable, though: A message adversary $\mathcal{H}$ determines which receiver gets a message from the respective sender in a round: It effectively generates a sequence $G_1, G_2, \ldots$ of directed communication graphs, where $G_r$ contains a directed edge $(p,q)$ if the message from $p$ is received by $q$ in round $r$. We assume that the set of nodes and hence $n$ is fixed, whereas the edges may change over time. Messages may have arbitrary size, i.e., we adhere to the LOCAL model [37].

The particular question asked in this paper is: How many rounds are needed until the uid of some node is known to all $n$ nodes, for a certain message adversary? We will call this quantity the dissemination time, and it is obvious that small dissemination times are beneficial for data distribution applications. This even includes consensus algorithms like [7, 15, 8, 39], since system-wide agreement obviously requires system-wide data dissemination. Moreover, small dissemination times are also interesting for data aggregation, which is a pivotal task in wireless sensor networks [3]. After all, convergecasting is the dual of broadcasting: By reverting the direction of the links and the sequence of communication graphs, a successful broadcast becomes a successful convergecast.

The dissemination time obviously depends heavily upon the message adversary, i.e., the actual sequence

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of communication graphs \(G_1, G_2, \ldots\): If e.g. \(G_1\) contains a star, it is 1, if every graph consists of the same two weakly connected components, it is \(\infty\). We are interested in an upper bound on the dissemination time for at least sparsely connected communication graphs. More specifically, we restrict our attention to the case of an oblivious message adversary \([1, 13]\), where \(G_1, G_2, \ldots\) is an arbitrary sequence of graphs each drawn from a set \(\mathcal{G}\) of candidate graphs with each \(G \in \mathcal{G}\) containing some rooted spanning tree. Note that this is actually the weakest per-graph restriction that guarantees a finite worst-case dissemination time for oblivious message adversaries.

A relatively simple pigeonhole argument (see Lemma 4.1) yields an upper bound of \(O(n^2)\) for the dissemination time in this case.\(^2\) Recently, Charron-Bost, Függer and Nowak improved this to \(O(n \log n)\). We conjecture that this bound can be further tightened to linear time \(O(n)\). Albeit we were not yet able to prove or disprove this, despite considerable efforts, we establish results in this paper that back-up our conjecture.

**Main contributions and paper organization:** After a short discussion of related work in Section 2 and a description of our system model in Section 3, we provide the following results in Section 4:

(i) We introduce the concept of influence and covering sets and apply these techniques in a novel and very concise proof of the known \(O(n \log n)\) upper bound for arbitrary directed rooted trees.

(ii) We show that a dissemination time of \(O(n)\) can be guaranteed for directed rooted trees with a constant number of leaves. In the case of directed rooted paths, i.e., directed rooted trees with only one leaf, the dissemination time is only \((n - 1)\) rounds.

(iii) In Section 5, we adapt our approach to undirected networks and show that only \((n - 1)/2\) rounds are needed for the dissemination time in the case of arbitrary (undirected) chain graphs.

Some outlook in Section 7 concludes our paper.

2. RELATED WORK

Research on broadcasting and gossiping has a long history (see [31] for a survey) and many variations of these problems have been studied: For the classical telephone problem we refer to [23, 40, 80], and for radio broadcasting to [20, 22] and references therein. In [18] the authors consider the rendezvous-communication-model. The widely-used push/pull/push-pull models [21] were studied on several graph classes like the complete graph [27, 48], hypercubes and Erdős-Rényi-graphs [24, 28], random geometric graphs [26], and preferential-attachment graphs [10]. A list-based quasi-randomized algorithm has been studied in [4, 17]. They have in common that they focus on broadcasting a message from a fixed source on a (possibly random but) fixed graph to all nodes.

Radio broadcasting in evolving graphs has been investigated in [14, 33, 2], [19] focused on token-forwarding algorithms, and [30, 11] studied push/pull-algorithms on dynamic graphs. Flooding algorithms have also been studied under several models, including edge-Markovian and related models [5, 13, 12]. In the edge-Markovian model an edge present at time \(t\) stays present with probability \(p\) and disappears with probability \(1 - p\) at time \(t + 1\) and an absent edge appears with probability \(q\) and remains absent with \(1 - q\). Whereas those papers focus on the broadcasting of a single item from a fixed source to all nodes, [31] also consider \(k\)-token dissemination and all-to-all algorithms. All this work above considers undirected communication graphs, however.

3. MODEL

We consider a set of processes \(\Pi = \{1, \ldots, n\}\) with uids, connected by directed point-to-point links. The processes execute a deterministic full-information protocol for distributing a unique local piece of information (for ease of exposition, the uid) to the other processes. The distributed computation proceeds in an infinite number of synchronous lock-step rounds \(r = 1, 2, \ldots\). Each round \(r\) consists of a communication-closed message exchange, specified by the communication graph \(G_r\) determined by an oblivious message adversary \([11]\), followed by a simultaneous local computing step at every process. In a full information protocol, every process sends its complete state in every round: If process \(p\) receives the state of some different process \(q\) (reached at the end of round \(r > 0\) resp. the initial state for \(r = 0\)) in round \(r + 1\), it forwards \(q\)'s state as part of its own state in all following rounds.

An execution of our system is just an infinite sequence of rounds. It can be uniquely described by an initial configuration \(C_0\), which is the vector of the initial states (that includes the uid) to every process, followed by an infinite sequence \(G\) of communication graphs \(G_1, G_2, \ldots\). The configuration reached at the end of round \(r\), after the computation step, is denoted by \(C_r\).

Formally, let \(G\) be a directed or undirected labeled graph on \(n\) vertices and let \(G = (G_r)_{r=1}^{\infty}\) be an infinite sequence of such graphs. Moreover, let \(\sigma_i\) be the finite prefix of \(G\) of length \(i\); we may drop the index if the length is clear from the context. Let \(\text{In}_p(r)\) resp. \(\text{Out}_p(r)\) denote the set of incoming resp. outgoing edges of node \(p\) in \(G_r\).

Thanks to our full information protocol, every node has knowledge \(K_p(r)\) at the end of round \(r\) (with \(K_p(0)\) representing the initial knowledge), which adheres to the following rules:

(1) Initial state: \(K_p(0) = \{p\}\) for all \(p \in \Pi\) (every node knows only its own uid at the beginning).

(2) Updating rule: The knowledge \(K_p(r + 1)\) process

\[^2\]For all the \(O(.)\) terms in this paper, the constants can be computed.
which processes node $K_p(r)$ at the information it gets via all incoming edges in $G_{r+1}$, i.e.,

$$K_p(r + 1) = K_p(r) \cup \bigcup_{q: (q,p) \in \text{In}_p(r+1)} K_q(r).$$

Subsequently, we will use phrases like “$K_p(t)$ at time $t$” for integer times $t \geq 0$, which means $K_p(r)$ at (the end of) round $r = t$ for $t > 0$ and $K_p(0)$ otherwise.

The dissemination time, given a sequence of graphs, is the first time all processes learned the uid from a common process, which is formally defined as follows:

**Definition 1.** Given a class of graphs $\mathcal{G}$ on $n$ nodes and an infinite sequence of graphs $G \in \mathcal{G}^\infty$, i.e., $G_n \in \mathcal{G}$ for $r \geq 1$, the dissemination time in $G$ is defined as

$$B_G = \min \left\{ r : p \in \Pi \,\mid\, K_p(r) \neq \emptyset \right\}.$$

The dissemination time of the class $\mathcal{G}$ is defined as

$$B^{\mathcal{G}} = \max_{G \in \mathcal{G}^\infty} B_G.$$

In this paper, we restrict our attention to classes of graphs where every element is a rooted tree ($\mathcal{T}$), its subclass consisting of directed chains ($\mathcal{C}$), as well as their undirected analogs.

**Remark 3.1.** Following a commonly used definition, we could also define a dissemination time $B_G(p) = \min \left\{ r : p \in \Pi \,\mid\, K_p(r) = \{p\} \right\}$ for node $p$ in the first place, i.e., the time when $p$ becomes known to all nodes in a graph sequence $G$. The dissemination time is then $B_G = \min_{p \in \Pi} B_G(p)$. Whereas this alternative expression has been used rarely for dynamic graphs, it has been employed for static graphs; an analogon of (2) has also been studied in [22]. On the other hand, in the classic telephone problems and its variants, one is interested in $\max_{p \in \Pi} B_G(p)$.

**Example 3.2.** To illustrate the definitions above we give a short example (see Figure 7).

The dissemination times are $B_G = B_G(3) = B_G(5) = 3$.

## 4. ROUTED TREES

In this section, we will present our main results on rooted trees $\mathcal{T}$. We use influence sets (see [54] Lemma 3.2. (b))) for this purpose, which are dual to knowledge sets: While the knowledge set $K_p(r)$ describes which processes node $p$ has already heard of at the end of round $r$, the influence set $S_p(r)$ describes which processes have already heard of $p$:

**Definition 2.** The influence set $S_p(r)$ of process $p$ at time $r$ is the set of processes that know about $p$ at time $r$, i.e., $S_p(r) = \{q \in \Pi : p \in K_q(r)\}$.

### 4.1 Influence Sets and Coverings

Obviously, there are always $n$ influence sets, and each node can be element of multiple of these. In the following lemma, we collect some elementary properties of influence sets.

**Lemma 4.1.** For all $p \in \Pi$ and $r \geq 0$, we have:

(i) Initial state: $S_p(0) = \{p\}$.

(ii) Updating rule: $S_p(r + 1) = S_p(r) \cup \bigcup_{q \in S_p(r)} \{q' : (q,q') \in \text{Out}_q(r+1)\}$.

(iii) Given $G$, the dissemination time $B_G = \min \left\{ r : \max_{p \in \Pi} |S_p(r)| = n \right\}$.

(iv) $S_p(r) \subseteq S_p(r + 1)$ for all $p$ and all $r$.

(v) If $p$ is the root in $G_{r+1} \in \mathcal{G}$, $p \in S_q(r)$, and $|S_q(r)| < n$, then $|S_q(r + 1)| > |S_q(r)|$.

**Proof.** Properties (i) – (iv) are an immediate consequence of the definition. Let us prove (v): Define $X := \Pi \setminus S_q(r)$ as the nonempty set of nodes which don’t know from $q$. Since $p$ is the root, for all $v \in \Pi$ (and in particular for all $v \in X$) there exists a path from $p$ to $v$ in $G_r$. Thus there must be an edge from $S_q(r)$ to $X$ and hence $S_q(r)$ grows at least by 1.

From Property (v) of this lemma, we obtain directly the trivial $\mathcal{O}(n^2)$-bound on the dissemination time for rooted trees $\mathcal{T}$: By the pigeonhole principle, after $n(n - 2) + 1$ rounds, one node was at least $n - 1$ times the root and hence its influence set has size $n$.

Whereas influence sets will turn out to be sufficient for establishing our results on chain graphs, we need the extended concept of coverings for dealing with general rooted trees.

**Definition 3.** For $r \geq t$ let

$$C_{I(r)}(r) = \{S_p(r) \mid p \in I(t)\}$$

be a class of influence sets, for some given index set $I(t) \subseteq \Pi$. It is called covering if $\bigcup S_p(r) \in C_{I(r)}(r) = \Pi$. The influence sets that make up a covering are called covering sets. The size of a covering is the number of covering sets it consists of, i.e., the size of its index set $I(t)$.

(A sacle of simplicity we will drop the index sometimes in the following if $r = t$ and the index set is clear from the context.)

A sequence of coverings $(C_{I(r)}(r))_{r \geq 0}$ with the additional property $I(r + 1) \subseteq I(r)$ we denote by $\mathcal{C}$.

Clearly, a trivial example of a covering is the set of all influence sets. To exclude such trivial cases, we introduce a subclass of coverings that contain no redundant sets.

**Definition 4.** A strict covering of a covering $S_{C_{I(r)}(r)} = \{S\} \forall S \in SC(r)$ is not a covering. A unique node is a node that is element of only one covering set.
The following lemma states some useful properties of coverings:

**Lemma 4.2.** In the case of rooted trees $\Xi$, every covering satisfies the following properties:

(i) A covering is a strict covering iff each covering set contains a unique node.

(ii) In a strict covering every covering set, except possibly one, loses at least one of its unique nodes. The only covering set that may not lose one of its unique nodes is the one of the root of $G_r$.

(iii) Let $C_{I(r)(r)}$ be a strict covering at time $r$. Assume that at time $r + t$, for some $t > 0$, there exists a covering set $S$ in $X := C_{I(r)(r + t)}$ containing no unique node. Then, $X' := X \setminus \{S\}$ is still a covering and $X'$ has at most $|S|$ more unique nodes than $X$. By repeating this argument, one can reduce $X$ to a new strict covering $SC_{I(r+t)}(r + t)$ with strictly smaller index set $I(r + t) \subset I(r)$.

(iv) Let $l$ be the number of leaves in $G_r$. In a strict covering $SC_{I(r-1)}(r - 1)$, at most $l$ influence sets do not grow in round $r$.

(v) If a strict covering consists of only one set, then dissemination has been completed.

(vi) At time $t = 2$, there is always a covering consisting of covering sets of size at least 2.

(vii) For each covering $C_{I(r)}(r)$ there exists a $p \in I(r)$ with $|S_p(r + 1)| > |S_p(r)|$.

**Proof.** The properties (i) and (v) are obvious.

(ii) Let $q$ be a unique node. Then $q$ is a) the root, or b) has a predecessor which is only in the same influence as $q$ (and thus unique too), or c) has a predecessor which is in another influence set too. In case c) $q$ is not unique anymore. In case b) we repeat the argument with the predecessors of $q$ until we stop in case c) or a). If we stop in case c), then one of the predecessor of $q$ is not unique anymore. If we stop in case a) (and not in c)!) then $q$ and all its predecessors remain unique. Thus the only influence set which may not lose one of its unique nodes is the set of the root.

(iii) Since $S$ contains no unique nodes all these nodes must be contained in other sets too, and thus we can remove $S$ from $X$ still having a covering. Repeating this procedure leads to a covering $X'$ containing no unique nodes. Hence it is a strict covering.

(iv) Note that an influence $S$ which does not grow has the property that for all $p \in S$ also all successors of $p$ must be contained in $S$. Moreover, each set in $SC(r - 1)$ contains a unique node. Now let $P_i$ denote the unique path from the root to the leaf $l_i$ ($1 \leq i \leq \ell$). We will show that each path $P_i$ contains unique nodes from at most one non-growing influence set. Let $p$ be a unique node of a non-growing influence set $S$. Then $S$ contain all successors of $p$ and those nodes can not be unique nodes from another non-growing set. On the other hand, if there is a unique node $p'$ from another non-growing set $S'$ on the path from the root to $p$ then $S'$ would contain all successors from $p'$, hence also $p$, and $p$ would not be a unique node. Consequently there can be at most $\ell$ non-growing covering sets.

(vi) The only influence sets of size 1 after round 1 are the sets $S_{l_i}(1), \ldots, S_{l_i}(1)$ where the $l_i$ are the leaves in the tree $G_1$. But node $l_i$ is surely contained in the influence set of $l_i$th predecessor. So take any covering that does not contains the influence sets of leaves but the influence sets of predecessors of leaves.

(vii) Since $C(r)$ is a covering, one set must contain the root. By Lemma 4.1 (v), this set grows.

### 4.2 Bounds on dissemination time

Equipped with the properties from Lemma 4.1 we can now give a novel, concise proof of the $O(n \log n)$-bound established in [9, 10].

**Fact 4.3.** ([9, Lemma 4] and [10, Lemma 1].) For the class $\Xi$ of rooted trees, dissemination is completed within $B^\Xi = O(n \log n)$ rounds.

**Proof.** Let $C_{I(r)}(r)$ be a strict covering of size $x + 1$, and $z$ be the number of unique nodes in $C(r)$. Lemma 4.2 (ii) ensures that, after $t := \lceil \frac{x}{2} \rceil$ rounds, there exists an influence set in $C_{I(r)}(r + t)$ with no unique nodes, which can be removed. The new strict covering $SC_{I(r+t)}(r+t)$ resulting from this procedure is of size at most $x$.

Starting at $r = 0$ and $x + 1 = n$ and using $z \leq n$, we can bound the dissemination time by $B^\Xi \leq \sum_{i=1}^{n} \left\lceil \frac{n}{2^i} \right\rceil = O(n \log n)$.
If we restrict ourselves to rooted trees with a fixed number of leaves, it is possible to prove that data dissemination can be completed even in linear time.

**Theorem 4.4.** For the class of rooted trees $\mathcal{S}_{k-1}$ with exactly $k-1$ leaves, data dissemination is $B^{\mathcal{S}_{k-1}} \leq k \cdot (n-3) + 2$ rounds.

**Proof.** Let $\mathcal{C}$ be a sequence of strict coverings with $I(0) \subseteq I(1)$ such that at time $r = 2$ all influence sets are of size at least 2 (see Lemma 4.2). Let $S_{p_1}(r), \ldots, S_{p_k}(r)$ be the $k$ smallest influence sets in $\mathcal{C}$ at time $r$ and let $s_{p_i}(r)$ denote the size of $S_{p_i}(r)$. Due to Lemma 4.2 (iv) in every round at least one of them grows by at least one, hence

$$\sum_{i=1}^{k} s_{p_i}(r) \geq 2k + (r - 1).$$

Thus, if $2k + (r - 1) = k(n - 1) + 1$ one set must contain $n$ elements and dissemination is done. Solving this equation for $r$ yields $r = (k(n - 3) + 2)$. $\square$

In particular, in the special case of a directed chain graph (i.e., $k = 2$), one can do even better: The following theorem shows that the dissemination time is only $n - 1$ rounds in this case. Since in the constant chain graph, it takes the root $n - 1$ rounds to disseminate its value, this bound is tight.

**Theorem 4.5.** Let $\mathcal{C}$ be the class of directed chains. At the end of round $r$, there exists a collection $S(r) = \{S_{p_1}(r), \ldots, S_{p_{n-r}}(r)\}$ of $n - r$ influence sets with

1. $|S_{p_i}| \geq r + 1$ for $1 \leq i \leq n - r$, and
2. for all $0 \leq r \leq n - 1$,

$$\bigcup_{i \in I} S_{p_i}(r) \geq r + 1 + |I| - 1 \quad \forall I \subseteq [1, n-r]. \quad (3)$$

Thus, $B^{\mathcal{C}} \leq n - 1$ rounds.

**Proof.** We do it by induction on $r$. If $r = 0$, then $S_{p_0}(0) = \{p\}$ and thus $|S_{p_0}(0)| = 1$ for all $p$. Obviously, $|U_{i \in I} S_{p_0}(0)| = |I|$. Assume that the induction hypothesis holds for $r$. We will show the the stated assertion also holds for $(r + 1)$.

We take successively a set $S_{p_i}(r + 1)$ (where $S_{p_i}(r)$ was contained in $S(r)$) and add it to $S(r+1)$ iff the following two conditions hold:

(i) $|S_{p_i}(r+1)| \geq r + 2$ and
(ii) inequality (3) holds for all collection of sets of $S(r+1)$ and $S_{p_i}(r+1)$.

Note that condition (i) holds for all but at most one set from $S(r)$: A set of size $m$ does not grow iff it captures the last $m$ elements of the chain. Since due to inequality (3) all sets of size $(r + 1)$ are pairwise different only one of these sets can be completely at the end of the chain, hence at most one set does not grow. If such a set exists we denote it by $S_{p_k}$.

Assume now that we have already added successfully $(L-1)$ sets to $S(r+1)$ and that by adding $S_{p_L}(r+1)$ condition (ii) is violated, i.e., there exists $k \leq (L-1)$ sets $S_{p_1}(r+1), \ldots, S_{p_k}(r+1) \in S(r+1)$ such that

$$\bigcup_{i=1}^{k} S_{p_i}(r+1) = r + 1 + k$$

and

$$\bigcup_{i=1}^{k} S_{p_i}(r+1) \cup S_{p_L}(r+1) = r + 1 + k.$$

This means that $S_{p_L}(r+1) \subseteq \bigcup_{i=1}^{k} S_{p_i}(r+1)$.

By induction hypothesis,

$$\bigcup_{i=1}^{k} S_{p_i}(r) \cup S_{p_L}(r) \geq r + 1 + k.$$

Thus the set $\bigcup_{i=1}^{k} S_{p_i}(r) \cup S_{p_L}(r)$ did not grow in round $(r+1)$, or equivalently, it captures the last $(r + 1 + k)$ elements of the chain in round $(r+1)$. Firstly, in case of the existence of $S_{p_k}$, also this set contains exactly the last $(r + 1)$ elements, which gives

$$\bigcup_{i=1}^{k} S_{p_i}(r) \cup S_{p_L}(r) \cup S_{p_k}(r) = r + 1 + k$$

which is a contradiction to the induction hypothesis. Hence – if $S_{p_k}$ exists – we can add all influence sets of $S(r)$ but $S_{p_k}$ to $S(r+1)$ and the assertion of this theorem holds in this case. On the other hand we are allowed to delete one set from our 'good' sets, so we do not add $S_{p_L}(r+1)$ to $S(r+1)$. The remaining question is: Can there be an index $L' > L$ such that again condition (ii) is violated? So assume that there is such an index and $k_1 < k$ sets $S_{p_1}, \ldots, S_{p_{k_1}}$ with

$$\bigcup_{i=1}^{k_1} S_{p_i}(r+1) \cup S_{p_L}(r+1) = r + 1 + k_1.$$

Again, due to induction hypothesis, the set $\bigcup_{i=1}^{k_2} S_{p_i}(r) \cup S_{p_L}(r)$ captures the last $(r + 1 + k_1)$ elements of the chain in round $(r+1)$. If $k_1 \geq k$ then we take the $k_1$ influence sets from here together with $S_{p_L}(r)$ and $S_{p_{L'}}(r)$ and obtain

$$\bigcup_{i=1}^{k_1} S_{p_i}(r) \cup S_{p_L}(r) \cup S_{p_L}(r) = r + 1 + k_1$$

which is a contradiction. If $k_1 < k$ we take the $k$ influence sets from above together with $S_{p_L}(r)$ and $S_{p_{L'}}(r)$ again yielding a contradiction. So the theorem is proven. $\square$

But not only trees with a few number of leaves admit linear-time data-dissemination (see Theorem 4.4), also trees with only a few inner nodes do:
Theorem 4.6. For trees with \( \ell \) leaves we have \( B_{T}^{\ell} \leq (n - \ell)(n - 1) + 2 - \max(n, 2(n - \ell)) \). In particular, in trees with only \( k \) inner nodes (i.e., \( (n - k) \) leaves), data-dissemination is linear. In fact, \( B_{T}^{n-k} \leq k(n - 1) + 2 - \max(n, 2k) \).

Proof. After the first round we can find a covering of size \( (n - \ell) \) sets (by taking all influence sets except those of the leaves), all of size at least 2. By Lemma 4.2 (vii), in all following rounds at least of them must grow.

Remark 4.7. Note that in case of the star graph (i.e., \( n - 1 \) leaves and 1 inner node) this theorem indeed gives dissemination time of 1.

Turning back to general rooted trees \( T \), the following theorem presents a lower bound on the dissemination time. It reveals that, in the worst-case, it takes more time than in the case of chain graphs.

Theorem 4.8. For the class of rooted trees \( T \), \( B_{T} \geq \lceil \frac{3n-1}{2} \rceil - 2 \) rounds.

Proof. We will construct a specific sequence \( G \) of graphs with \( B_{G} = \lceil \frac{3n-1}{2} \rceil - 2 \). This sequence consists of three different graphs \( G^{(1)}, G^{(2)}, G^{(3)} \) where each graph will be applied for multiple rounds.

The first graph, \( G^{(1)} \), is the simple chain rooted in the process 1 and edges \( i \rightarrow i + 1 \). The tree \( G^{(2)} \) is rooted in \( n \) and contains edges \( (n \rightarrow 1), (n \rightarrow n - 1) \). Furthermore \( (i \rightarrow i + 1) \) for \( 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \), and \( (i \rightarrow i - 1) \) for \( n - 1 \geq i \geq \lceil \frac{n}{2} \rceil + 2 \).

Finally, \( G^{(3)} \) is rooted in \( \lceil \frac{n}{2} \rceil \) with edges \( i \rightarrow i + 1 \) for \( \lceil \frac{n}{2} \rceil \leq i \leq n - 1 \) and edges \( i \rightarrow i + 1 \) for \( 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \). Furthermore there is an edge \( n \rightarrow 1 \).

The execution (fig. 2) is constructed in the following way:

- \( G_{r} = G^{(1)} \) for \( 1 \leq r \leq \lceil \frac{n-1}{2} \rceil \),
- \( G_{r} = G^{(2)} \) for \( \lceil \frac{n-1}{2} \rceil + 1 \leq r \leq n - 2 \), and
- \( G_{r} = G^{(3)} \) for \( n - 1 \leq r \leq \lceil \frac{3n-1}{2} \rceil - 2 \).

In this sequence, the first time an influence set has size \( n \) is the last round. Hence \( \lceil \frac{3n-1}{2} \rceil - 2 \) is a lower bound for broadcasting in directed trees.

\[
\text{for } 1 \leq r \leq \left\lfloor \frac{n-1}{2} \right\rfloor : S_{i}(r) = \{i, \ldots, \min(r + i, n)\} \\
\text{for } \left\lfloor \frac{n-1}{2} \right\rfloor < r \leq n - 2 : S_{i}(r) = \{i, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor + i\} \text{ for } i \leq \frac{n}{2}, \\
S_{i}(r) = \{\max\left(\frac{n}{2}, 1, i - (r - \left\lfloor \frac{n-1}{2} \right\rfloor)\right), \ldots, n, 1, \ldots, r - \left\lfloor \frac{n-1}{2} \right\rfloor\} \text{ for } i > \frac{n}{2}, \\
\text{for } n - 2 < r \leq \left\lfloor \frac{3n-1}{2} \right\rfloor - 2 : S_{i}(r) = \{\max\left(\frac{n}{2}, 1, i + 2 + \left\lfloor \frac{n-1}{2} \right\rfloor\right), \ldots, n, 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor - 2\} \text{ for } i \leq \left\lfloor \frac{3n+1}{2} \right\rfloor - 2 - r, \\
S_{i}(r) = \{i, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor + i - (n - 2)\} \text{ for } \frac{n}{2} \geq i > \left\lfloor \frac{3n+1}{2} \right\rfloor - 2 - r \\
\text{for all } 0 \leq r \leq (n - 1)/2 \\
\left| \bigcup_{i \in I} S_{i}(r) \right| \geq 2r + 1 + |I| - 1 \quad \forall I \subseteq [1, n-2r]. \quad (4)
\]

Thus, \( B_{G_{n}} \leq \lceil (n - 1)/2 \rceil \) rounds.

Note that this bound is also tight, as the constant chain graph reveals.

Proof. The proof runs along the same lines as the proof for the analogous result for rooted chains and uses induction on \( r \). For \( r = 0 \) it is obvious true.

Now let’s do the induction step: Again, we take successively a set \( S_{i}(r + 1) \) (where \( S_{i}(r) \) was contained in \( S(r) \)) and add it to \( S(r + 1) \) iff the following two conditions hold:

- \( |S_{i}(r + 1)| \geq 2r + 3 \) and

5. UNDIRECTED TREES

In undirected graphs, dropping the direction of the edges speeds-up data dissemination. In the case where \( \mathcal{G} \) is the class of undirected and connected graphs, after \( n - 1 \) rounds, even all-to-all dissemination is completed (see [31][Theorem 3.1]). But what can be said about the dissemination time? The following theorem shows that dissemination is twice as fast as all-to-all dissemination in undirected chains, and roughly at least \( 3/2 \) as fast as in directed rooted trees.

Theorem 5.1. Let \( \mathcal{C}_{n} \) be the class of undirected chains. At the end of round \( r \), there exists a set \( S(r) \) of \( n - 2r \) influence sets \( S_{p_{1}}(r), \ldots, S_{p_{n-2r}}(r) \) with

- \( |S_{p_{i}}| \geq 2r + 1 \) for \( 1 \leq i \leq n - 2r \), and
Again, the set \( \bigcup_i S_i \) that again condition (ii) is violated, there would be also to \( S \) site end) we are allowed to delete one set from our sets \( S \). Hence – if two sets \( S \) which is a contradiction to the induction hypothesis. 

Assume now that we have already added successfully \((2r + 1) \) sets to \( S \). Thus the set \( \bigcup_i S_i(r + 1) \cup S_{PL}(r + 1) \) such that

\[
\left| \bigcup_{\ell=1}^k S_{p_{\ell}}(r + 1) \cup S_{PL}(r + 1) \right| \leq 2r + k + 2.
\]

By induction hypothesis,

\[
\left| \bigcup_{\ell=1}^k S_{p_{\ell}}(r) \cup S_{PL}(r) \right| \geq 2r + 1 + k.
\]

Thus the set \( \bigcup_{\ell=1}^k S_{p_{\ell}}(r) \cup S_{PL}(r) \) grew only by 1 in round \((r + 1)\), or equivalently, it captures the last \( (2r + 1 + k) \) elements of one end of the chain in round \((r + 1)\). 

Firstly, in case of the existence of \( S_{p_i}^* \) at the same end, also this set contains exactly the last \( (2r + 1) \) elements, which gives

\[
\left| \bigcup_{\ell=1}^k S_{p_{\ell}}(r) \cup S_{PL}(r) \cup S_{p_i}^*(r) \right| = 2r + 1 + k
\]

which is a contradiction to the induction hypothesis. Hence – if two sets \( S_{p_i}^* \) exists – we can add all influence sets of \( S(r) \) but \( S_{p_i}^* \) to \( S(r + 1) \) and the assertion of this theorem holds in this case.

Secondly, if exactly one set \( S_{p_i} \) exists (at the opposite end) we are allowed to delete one set from our sets fulfilling condition (ii), so we do not add \( S_{PL}(r + 1) \) to \( S(r + 1) \). If there would be an index \( L' > L \) such that again condition (ii) is violated, there would be also \( k_1 < L' \) sets \( S_{p_{i_1}}(r + 1), \ldots, S_{p_{i_{k_1}}}(r + 1) \) with

\[
\left| \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r + 1) \cup S_{PL}(r + 1) \right| = 2r + 3 + k_1.
\]

Again, the set \( \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r) \cup S_{PL}(r) \) captures the last \((2r + 1 + k_1) \) elements of the same end of the chain in round \((r + 1)\). If \( k_1 \geq k \) then we take the \( k_1 \) influence sets from here together with \( S_{PL}(r) \) and \( S_{PL}(r) \) and obtain

\[
\bigg| \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r) \cup S_{PL}(r) \cup S_{PL}(r) \bigg| = 2r + 1 + k_1
\]

which is a contradiction. If \( k_1 < k \) we take the \( k \) influence sets from above together with \( S_{PL}(r) \) and \( S_{PL}(r) \) again yielding a contradiction.

Thirdly, In case of absence of sets \( S_{p_i} \) we are allowed to delete two sets from \( S(r) \). Since by the argument above at one end at most one set violates condition (ii) we are done. So the theorem is proven. 

\[
\bigg| \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r) \cup S_{PL}(r) \cup S_{PL}(r) \bigg| = 2r + 1 + k_1
\]

which is a contradiction. If \( k_1 < k \) we take the \( k \) influence sets from above together with \( S_{PL}(r) \) and \( S_{PL}(r) \) again yielding a contradiction.

Thirdly, In case of absence of sets \( S_{p_i} \) we are allowed to delete two sets from \( S(r) \). Since by the argument above at one end at most one set violates condition (ii) we are done. So the theorem is proven. 

\[
\bigg| \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r) \cup S_{PL}(r) \cup S_{PL}(r) \bigg| = 2r + 1 + k_1
\]

which is a contradiction. If \( k_1 < k \) we take the \( k \) influence sets from above together with \( S_{PL}(r) \) and \( S_{PL}(r) \) again yielding a contradiction.

\[
\bigg| \bigcup_{\ell=1}^{k_1} S_{p_{i_\ell}}(r) \cup S_{PL}(r) \cup S_{PL}(r) \bigg| = 2r + 1 + k_1
\]

which is a contradiction. If \( k_1 < k \) we take the \( k \) influence sets from above together with \( S_{PL}(r) \) and \( S_{PL}(r) \) again yielding a contradiction.

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7. OUTLOOK

We presented a number of lower and upper bounds for the dissemination time in dynamic networks under oblivious message adversaries, where the set of admissible graphs is restricted to contain trees or fixed sub-structures of trees. For rooted directed trees, the best upper bound is \( O(n \log(n)) \) and the best lower bound is \( \Omega(n) \). Hence, it is still an open question whether the dissemination time is indeed linear or not. Besides our interest in finally closing this question, we are wondering whether the worst case dissemination time is somehow connected to the maximum path length, diameter or the maximum node degree of the individual communication graphs or their dynamic transitive closure in the graph sequence. Moreover, it remains to be seen whether our chain upper bound also holds for undirected trees.
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