Wave propagation problem for a micropolar elastic waveguide

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Abstract. A propagation problem for coupled harmonic waves of translational displacements and microrotations along the axis of a long cylindrical waveguide is discussed at present study. Microrotations modeling is carried out within the linear micropolar elasticity frameworks. The mathematical model of the linear (or even nonlinear) micropolar elasticity is also expanded to a field theory model by variational least action integral and the least action principle. The governing coupled vector differential equations of the linear micropolar elasticity are given. The translational displacements and microrotations in the harmonic coupled wave are decomposed into potential and vortex parts. Calibrating equations providing simplification of the equations for the wave potentials are proposed. The coupled differential equations are then reduced to uncoupled ones and finally to the Helmholtz wave equations. The wave equations solutions for the translational and microrotational waves potentials are obtained for a high-frequency range.

Introduction
Modelling of a specific mechanical and multi-physical behaviour of solids (including metamaterials) is an actual problem of modern Continuum Mechanics. The metamaterials exhibit physical properties not usually found in nature. For example, there are materials with negative Poissons ratio (auxetic materials), negative thermal expansion, negative electric permittivity and the magnetic permeability. These paradoxical physical phenomena cannot be described in the frameworks of the conventional Continuum Mechanics. In such a case, microstructure continuum theories are to be usually employed [1–5]. Even now the micropolar continuum models are intensively used in applied mechanics in order to resolve various shortcomings and difficulties arising while employing the classical theory of elasticity. Asymmetric stress and strain tensors are intrinsic to the micropolar elasticity modeling (see discussion in [4] for the subject). The problems of propagating surfaces of discontinuities of displacements and microrotations are previously discussed in a number of studies [8–18]. It is also true for the wave problems formulated for circular micropolar waveguides (see [19]).

The aim of the present paper is to give a method which provides determination of the translational and microrotation waves propagating along an infinite waveguide with circular
cross-section. Microrotation waves in an elastic solid are modelled within the linear micropolar elasticity frameworks by introducing in the governing equations of the Continuum Mechanics additional microrotation degree of freedom. Such a modelling can be regularly carried out in terms of the classical field theory [7], starting from the action integral and the least action variational principle.

The present paper is arranged as follows. In Secs. 1, 2 microrotation waves modelling is realized within the frameworks of linear micropolar elasticity by introducing the axial microrotation vector thus representing additional independent degrees of freedom of the elastic continuum. The coupled vector differential equations of the conventional micropolar elasticity are given. Actually Sec. 2 is to be considered as a script of the equations of the linear micropolar elasticity. Sec. 1 includes a brief discussion of the conventional micropolar elasticity as a field theory based on the least action principle. In Sec. 3 the translational displacements and microrotations vectors in the coupled wave are decomposed into potential and vortex parts. Scalar and vector wave potentials of displacements and microrotations are introduced. The calibrating equations for the vector wave potentials are presented. Coupled vector equations involving the vector wave potentials are obtained. These equations are then uncoupled for harmonic waves of high-frequency range, when the governing differential equations belong to the hyperbolic analytical type. In Sec. 4 the uncoupled equations are reduced to the Helmholtz wave equations. The latter are resolved for the translational and microrotation waves of high-frequency range.

1. Micropolar elasticity as a field theory. The least action principle
The conventional model of micropolar elastic continuum (see [4] for details) is used throughout the paper. Such continua based on the existence of complementary rotational degrees of freedom (represented by the rotation vector \( \phi \)) along with translational degrees of freedom (described by the translational displacement vector \( u \)). The model of micropolar elastic continuum can be expanded as a field theory if the “natural” action density is taken in the following form

\[
\mathcal{L} = \frac{1}{2} \dot{\mathbf{u}} \cdot \rho \cdot \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{\phi}} \cdot \mathbf{\mathcal{J}} \cdot \dot{\mathbf{\phi}} - \psi(e, \gamma),
\]

where \( \rho \) is the mass density tensor, \( \mathbf{\mathcal{J}} \) is the tensor of microinertia, \( \psi \) is the Helmholtz free energy per unit volume, \( \nabla \) is the three-dimensional Hamiltonian differential operator (the nabla symbol), the superimposed dot denotes partial differentiation with respect to time variable. For tensors \( \rho \) and \( \mathbf{\mathcal{J}} \) the symmetry conditions are assumed to be valid.

The asymmetric strain tensor \( e \) and bending–torsion tensor \( \gamma \) are associated with translational displacements \( u \) and microrotations \( \phi \) by the formulae

\[
e = \nabla \otimes u - \phi \cdot \varepsilon,
\]

\[
\gamma = \nabla \otimes \phi.
\]

We employ the notation \( \varepsilon \) for the three-dimensional permutation tensor. The equations (2) are represented in a rectangular co-ordinate system as

\[
e_{ji} = \partial_j u_i - \varepsilon_{jik} \phi_k,
\]

\[
\gamma_{ji} = \partial_j \phi_i.
\]

In general, the variational action integral with “natural” action density (1) is given by following equation (\( d\tau \) is an elementary spatial volume):

\[
\mathcal{I} = \int \mathcal{L}(u, \phi, \dot{u}, \dot{\phi}, \nabla \cdot \phi, \nabla \cdot u) d\tau dt.
\]
The least action principle states that the actual field is realized in such a way that the action of (4) is minimum, i.e. for any admissible variations of the physical fields \( \{u, \phi\} \) and non-variable spatial coordinates and the time variable the following equation is valid

\[
\delta \mathcal{A} = 0. \tag{5}
\]

The differential field equations is simply obtained from the equation (5) as

\[
\partial_\alpha S^\alpha_j - \dot{P}_j = -\frac{\partial \mathcal{L}}{\partial \dot{w}_j} \quad (\alpha = 1, 2, 3; \ j = 1, 2, 3),
\partial_\beta \mathcal{M}^{\beta}_j + A_j - \dot{Q}_j = 0 \quad (\beta = 1, 2, 3; \ j = 1, 2, 3), \tag{6}
\]

wherein the following notations are used

\[
P_j = \frac{\partial \mathcal{L}}{\partial \dot{w}_j}, \quad Q_j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}, \quad S^\alpha_j = -\frac{\partial \mathcal{L}}{\partial (\partial_\alpha w_j)}, \quad \mathcal{M}^{\alpha}_j = -\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi_j)}, \quad A_j = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}. \tag{7}
\]

2. Reminder of the linear micropolar elasticity

For linear micropolar elastic continuum we can expand the Helmholtz free energy \( \psi \) into the Taylor series in the vicinity of the natural state \( e = \gamma = 0 \), while disregarding the terms of higher order than the second ones. In a rectangular coordinate system the following form can be obtained for isotropic, homogeneous and centrosymmetric continua [4]:

\[
\psi = \frac{\mu + \eta}{2} e_{ji}e_{ji} + \frac{\mu - \eta}{2} e_{ji}e_{ij} + \frac{\lambda}{2} e_{kk}e_{nn} + \gamma + \varepsilon \gamma_{ji}\gamma_{ji} + \frac{\gamma - \varepsilon}{2} \gamma_{ji}\gamma_{ij} + \frac{\beta}{2} \gamma_{kk}\gamma_{nn}. \tag{8}
\]

where \( \lambda, \mu, \gamma, \beta, \varepsilon, \eta \) are the constitutive constants of micropolar elastic continuum.

The constitutive equations for the force stress tensor \( \sigma \) and the moment stress tensor \( \mu \) are formulated in virtue of equations (7) as follows

\[
\sigma = (\mu + \eta) e + (\mu - \eta) e^T + \lambda I tr e, \quad \mu = (\gamma + \varepsilon) \gamma + (\gamma - \varepsilon) \gamma^T + \beta I tr \gamma. \tag{9}
\]

In the above equations \( I \) denotes the three-dimensional unit tensor, \( \gamma \) is the strain tensor, \( \kappa \) is the bending-torsion tensor.

The linear vector partial differential equations determining the coupled translational and microrotation waves in a micropolar elastic continuum read (see [4]):

\[
\begin{cases}
(\lambda + 2\mu) \nabla \nabla \cdot u - (\mu + \eta) \nabla \times (\nabla \times u) + 2\eta \nabla \times \phi = \rho \ddot{u}, \\
(\beta + 2\gamma) \nabla \cdot \phi - (\gamma + \varepsilon) \nabla \times (\nabla \times \phi) + 2\eta \nabla \times u - 4\eta \phi = \ddot{\phi}.
\end{cases} \tag{10}
\]

In view of equations (9) and (2) after chain of transformations the following equations for the traction vector \( t \) and the moment vector \( m \) can be obtained:

\[
t = \lambda (\nabla \cdot u) n + 2\mu (n \cdot \nabla) u + (\mu - \eta)(n \times \nabla) \times u + 2\eta n \times \phi, \\
m = \beta (\nabla \cdot \phi) n + 2\gamma (n \cdot \nabla) \phi + (\gamma - \varepsilon)(n \times \nabla) \times \phi, \tag{11}
\]

where \( n \) is the unit normal vector determining the spatial orientation of the plane surface element.

The latter equations are required in order to formulate boundary conditions on the free from tractions and moments sidewall of the waveguide:

\[
t = 0, \quad m = 0. \tag{12}
\]
3. Wave potentials of translational displacements and microrotations

The vector partial differential equations (10) are coupled and can be uncoupled following a scheme described in this Section.

At the beginning we introduce wave vector potentials corresponding to the translational displacements and microrotations. The vector fields \( \mathbf{u} \) and \( \phi \) are decomposed into potential and vortex parts according to the Helmholtz representations:

\[
\mathbf{u} = \nabla \Phi + \nabla \times \Psi, \\
\phi = \nabla \Sigma + \nabla \times \mathbf{H},
\]

where \( \Phi, \Sigma \) are the scalar wave potentials, and \( \Psi, \mathbf{H} \) are the vector wave potentials of translational displacements and microrotations respectively. Discussion on the completeness and relations between such potentials and stress functions in micropolar elasticity see in [20].

Second, in the present study we shall impose the usual calibration conditions for the wave potentials \( \Psi, \mathbf{H} \). Those read the two scalar equations

\[
\nabla \cdot \Psi = 0, \\
\nabla \cdot \mathbf{H} = 0.
\]

Then by substituting equations (13) in (10) and taking into consideration the calibrating equations (14) it is seen that the system (10) is satisfied if the scalar and the vector wave potentials are to be solutions of the uncoupled scalar equations

\[
\Delta \Phi - \frac{1}{c^2} \frac{\ddot{\Phi}}{\parallel} = 0, \\
\Delta \Sigma - \frac{1}{\mu c^2} \frac{\ddot{\Sigma}}{\parallel} - \frac{\Omega^2}{\mu c^2} \Sigma = 0,
\]

and the following system of coupled vector partial differential equations:

\[
\left\{ \begin{array}{l}
\Delta \Psi - \frac{1}{\nu c^2 \parallel} \frac{\ddot{\Psi}}{\parallel} + 2d_\perp \nabla \times \mathbf{H} = 0, \\
\Delta \mathbf{H} - \frac{1}{\mu c^2 \parallel} \frac{\ddot{\mathbf{H}}}{\parallel} - \frac{\Omega^2}{\mu c^2 \parallel} \mathbf{H} + \frac{\Omega^2}{2\mu c^2 \perp} \nabla \times \Psi = 0,
\end{array} \right.
\]

wherein we have employed the notations for given below set of constitutive constants:

\[
\Omega^2 = \frac{4\eta}{3}, \quad c^2_\parallel = \frac{\lambda + 2\mu}{\rho}, \quad \mu c^2_\parallel = \frac{2\gamma}{3}, \quad \mu c^2_\perp = \frac{\mu + \eta}{3}, \\
\nu^2 = \frac{\mu + \eta}{\rho}, \quad c^2_\perp = \frac{\eta}{\rho}, \quad d^2_\perp = \frac{\nu^2}{\nu^2}.
\]

In the present study only coupled high-frequency waves of translational displacements and microrotations are considered. That means we assume \( \omega > \Omega \). In such a case the following differential equations for the wave potentials can be obtained

\[
\left\{ \begin{array}{l}
(\Delta + \alpha^2_\parallel) \Phi = 0, \\
(\Delta + \beta^2_\parallel) \Sigma = 0; \\
(\Delta + \alpha^2_\perp) \Psi + 2d^2_\perp \nabla \times \mathbf{H} = 0, \\
(\Delta + \beta^2_\perp) \mathbf{H} + \frac{\Omega^2}{2\mu c^2_\perp} \nabla \times \Psi = 0.
\end{array} \right.
\]
In the above equations we have introduced the constants defined according to
\[
\alpha^2_\parallel = \frac{\omega^2}{c^2_\parallel}, \quad \beta^2_\parallel = \frac{\omega^2 - \Omega^2}{\mu c^2_\parallel}, \quad \alpha^2_\perp = \frac{\omega^2}{\epsilon^2_\perp}, \quad \beta^2_\perp = \frac{\omega^2 - \Omega^2}{\mu c^2_\perp}.
\] (19)

The two vector equations of system (18) related to the vortex potentials \( \Psi \) and \( H \) are still coupled. Uncoupled equations involved them can be obtained however this requires higher order differential operators.

At last, for the vortex wave potentials \( \Psi \) and \( H \) the two separate equations are derived in the following identical forms
\[
(\Delta + K^2_1)(\Delta + K^2_2)\Psi = 0, \quad (\Delta + K^2_1_2)(\Delta + K^2_2)H = 0,
\] (20)

wherein both the constants \( K_1, K_2 \) are real for the harmonic wave of the high-frequency range. They are defined by
\[
K^2_{1,2} = -\Delta_{1,2},
\]
\[
\Delta_{1,2} = \frac{-(\alpha^2_\perp + \beta^2_\perp + \sigma^2_\perp) \pm \sqrt{(-\alpha^2_\perp - \beta^2_\perp + \sigma^2_\perp)^2 + 4\beta^2_\perp \sigma^2_\perp}}{2},
\] (21)
\[
\sigma^2_\perp = \frac{d^2 \Omega^2}{\mu c^2_\perp}.
\]

4. Displacement and microrotation fields in propagating wave

We proceed the discussion to determination of the displacement and microrotation vectors in the coupled propagating wave. The coupled translational and microrotation wave fields in a cylindrical waveguide are determined in the cylindrical coordinates \( r, \phi, z \) by the separation of variables method (see details in [2]). This method provides investigation of waves of an arbitrary azimuthal number \( n \) propagating along the infinite waveguide.

Before all it is necessary to underline that the differential operators
\[
(\Delta + K^2_1)
\]
and
\[
(\Delta + K^2_2)
\]
commute. This fact enables us to expand the vector wave potential \( \Psi \) into the sum
\[
\Psi = \Psi_{(1)} + \Psi_{(2)}
\]
with the summands satisfying the standard Helmholtz wave equations
\[
(\Delta + K^2_1)\Psi_{(1)} = 0, \quad (\Delta + K^2_2)\Psi_{(2)} = 0,
\]
and it is also true for the vector wave potential \( H \), i.e.
\[
H = H_{(1)} + H_{(2)},
\]
where
\[
(\Delta + K^2_1)H_{(1)} = 0, \quad (\Delta + K^2_2)H_{(2)} = 0.
\]
For the scalar wave potentials $\Phi, \Sigma$ of the coupled wave of displacements and microrotations the following representations are obtained:

$$
\Phi = C_1 I_n(p_1 r) \begin{cases}
\cos n\varphi \\
- \sin n\varphi
\end{cases} e^{\pm ikz},
$$

$$
\sum = -C_2 I_n(p_2 r) \begin{cases}
\sin n\varphi \\
\cos n\varphi
\end{cases} e^{\pm ikz},
$$

wherein the time harmonic exponent multipliers have been omitted; $k$ denotes the wavenumber of the propagating wave; $C_1$ and $C_2$ are arbitrary constants; $I_n(\cdot)$ is the standard Bessel function of the first kind of an imaginary argument;

$$
p_1^2 = k^2 - \alpha_0^2, \quad p_2^2 = k^2 - \beta_0^2.
$$

The vortex wave potentials of translational displacements and microrotations are obtainable in a more complicated forms given by the following formulae for their physical components in the cylindrical coordinate net:

$$
\Psi_{\varphi} = \begin{cases}
[C_3 I_{n-1}(q_1 r) + C_4 I_{n+1}(q_1 r) + C_5 I_{n-1}(q_2 r) + C_6 I_{n+1}(q_2 r)] \begin{cases}
\sin n\varphi \\
\cos n\varphi
\end{cases} e^{\pm ikz},
\end{cases}
$$

$$
H_{\varphi} = \begin{cases}
[L_3 I_{n-1}(q_1 r) + L_4 I_{n+1}(q_1 r) + L_5 I_{n-1}(q_2 r) + L_6 I_{n+1}(q_2 r)] \begin{cases}
- \cos n\varphi \\
\sin n\varphi
\end{cases} e^{\pm ikz},
\end{cases}
$$

wherein the time harmonic exponent multipliers have been omitted; $C_3' - C_5'$, $C_3'' - C_5''$, $L_3' - L_5'$ and $L_3'' - L_5''$ are arbitrary constants and

$$
q_1^2 = k^2 - K_1^2, \quad q_2^2 = k^2 - K_2^2.
$$

The coordinate representations (23) and (24) of the vortex wave potentials are derived from a formal solution

$$
\Gamma_r = [C_1 I_{n-1}(q_1 r) + C_2 I_{n+1}(q_1 r)] \begin{cases}
\sin n\varphi \\
\cos n\varphi
\end{cases} e^{\pm ikz},
$$

$$
\Gamma_{\varphi} = [C_1 I_{n-1}(q_2 r) - C_2 I_{n+1}(q_2 r)] \begin{cases}
\cos n\varphi \\
- \sin n\varphi
\end{cases} e^{\pm ikz},
$$

$$
\Gamma_z = C_3 I_n(q_3 r) \begin{cases}
\sin n\varphi \\
\cos n\varphi
\end{cases} e^{\pm ikz},
$$

of the vector Helmholtz wave equation

$$
(\Delta + k_0^2) \Gamma = 0,
$$

(26)
which can be obtained by separating variables in the cylindrical coordinates $r, \phi, z$.

We should also note that the constant $q_*$ coming into the solution (25) of the Helmholtz wave equation (26) is defined according to

$$q_*^2 = k^2 - k_*^2.$$ 

In view of equations (22), (23) and (24) the physical components of displacement and microrotation vectors in the propagating coupled wave can be obtained. We give these components for any azimuthal number $n$ and omitting as previously the time harmonic exponent multipliers:

$$u_r = \left[ C_1 \left( p_1 I_{n+1}(p_1 r) + \frac{n}{r} I_n(p_1 r) \right) + \frac{n}{r} \left( C'_5 I_n(q_1 r) + C''_5 I_n(q_2 r) \right) + \right. $$

$$+ \left. \left( \mp ik \right) \left( C'_5 I_{n-1}(q_1 r) - C'_4 I_{n+1}(q_1 r) + C''_3 I_{n-1}(q_2 r) - C''_4 I_{n+1}(q_2 r) \right) \right] \times$$

$$\times \left\{ \begin{array}{c}
\cos n\phi \\
- \sin n\phi
\end{array} \right\} e^{\pm ikz},$$

$$u_\phi = \left[ -\frac{n}{r} C_1 I_n(p_1 r) - C'_5 \left( q_1 I_{n+1}(q_1 r) + \frac{n}{r} I_n(q_1 r) \right) + \right. $$

$$+ \left. \left( \pm ik \right) \left( C'_5 I_{n-1}(q_1 r) + C'_4 I_{n+1}(q_1 r) + C''_3 I_{n-1}(q_2 r) + C''_4 I_{n+1}(q_2 r) \right) - \right] \times$$

$$\times \left\{ \begin{array}{c}
\sin n\phi \\
\cos n\phi
\end{array} \right\} e^{\pm ikz},$$

$$\phi_r = \left[ \frac{n}{r} C_2 I_n(p_2 r) - L'_5 \left( q_1 I_{n+1}(q_1 r) + \frac{n}{r} I_n(q_1 r) \right) + \right. $$

$$+ \left. \left( \mp ik \right) \left( L'_5 I_{n-1}(q_1 r) - L'_4 I_{n+1}(q_1 r) + L''_3 I_{n-1}(q_2 r) - L''_4 I_{n+1}(q_2 r) \right) \right] \times$$

$$\times \left\{ \begin{array}{c}
\sin n\phi \\
\cos n\phi
\end{array} \right\} e^{\pm ikz},$$

$$\phi_\phi = \left[ -\frac{n}{r} C_2 I_n(p_2 r) - L'_5 \left( q_1 I_{n+1}(q_1 r) + \frac{n}{r} I_n(q_1 r) \right) + \right. $$

$$+ \left. \left( \pm ik \right) \left( L'_5 I_{n-1}(q_1 r) + L'_4 I_{n+1}(q_1 r) + L''_3 I_{n-1}(q_2 r) + L''_4 I_{n+1}(q_2 r) \right) - \right] \times$$

$$\times \left\{ \begin{array}{c}
- \cos n\phi \\
\sin n\phi
\end{array} \right\} e^{\pm ikz},$$

$$\phi_z = \left[ C_2 p_2 I_n(q_2 r) + \left( L'_3 - L'_4 \right) q_1 I_n(q_1 r) + \right. $$

$$+ \left. \left( L''_3 - L''_4 \right) q_2 I_n(q_2 r) \right] \times$$

$$\times \left\{ \begin{array}{c}
\sin n\phi \\
\cos n\phi
\end{array} \right\} e^{\pm ikz}.$$
are to be expanded in virtue of formulae (11), representing the traction and moment vectors via the displacement and microrotation vectors, and the Helmholtz decompositions (13).

Note that the calibration equations (14) actually may not enter the script of the requisite equations related to the problem under consideration. In general they might be replaced by

$$\nabla \cdot \Psi = F, \quad \nabla \cdot H = G$$

with arbitrary functions $F$ and $G$ of spatial coordinates and the time variable. This immediately leads to the conclusion that the equations (33) in fact may not be taken into consideration.

Conclusion
• The constitutive and governing equations of the linear micropolar elasticity have been furnished as a tool for mathematical modeling of microrations wave.
• The mathematical model of the linear (or even nonlinear) micropolar elasticity has been expanded to a field theory model by variational least action integral the least action principle.
• Propagation problem for coupled harmonic waves of translational displacements and microrotations along the axis of a long circular waveguide has been resolved.
• The translational displacements and microrotations in the coupled wave have been decomposed into potential and vortex parts and the coupled differential equations of motion has been reduced to uncoupled ones.
• The wave problem solution has been obtained for a coupled wave of the high-frequency range providing the hyperbolicity of the requisite equations.
• An excluded case of the coupled wave of the low-frequency range has been shown need a further analysis be carried out.

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