ABELIAN ANALYTIC TORSION AND SYMPLECTIC VOLUME

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ABSTRACT. This article studies the abelian analytic torsion on a closed, oriented, Sasakian three-manifold and identifies this quantity as a specific multiple of the natural unit symplectic volume form on the moduli space of flat abelian connections. This identification effectively computes the analytic torsion explicitly in terms of Seifert data.

CONTENTS

1. Introduction 1
2. Computation of the analytic torsion 7
Appendix A. Review of contact and Sasakian geometry 12
Acknowledgement 19
References 19

1. INTRODUCTION

This article studies the abelian analytic torsion on Sasakian three-manifolds. Recall that the analytic torsion is a topological invariant that was introduced by D.B. Ray and I.M. Singer [1] as an analytic analogue of the combinatorially defined Reidemeister torsion [2]. Note that it is a well known fact that these two torsions agree, as was independently shown by W. Müller, [3], and J. Cheeger, [4], for unimodular representations. More recently an elegant new proof of this equivalence has been given by M. Braverman [5] using the Witten laplacian [6]. We work with the analytic torsion directly in this article, and note that a computation using the purely combinatorial definition would be an interesting complementary approach.

Our main objective in this article is to prove that the (square-root of the) analytic torsion may be identified as a natural symplectic volume form on the moduli space of flat abelian connections for certain Seifert structures on three-manifolds associated with Sasakian structures. This identification is motivated by the work of C. Beasley and E. Witten [8] involving Chern-Simons theory on contact three-manifolds. Recall that A.S. Schwarz [7] has shown that the abelian Chern-Simons partition function is proportional to the analytic torsion and our study is also natural in light of this fact. Our main result also shows that two mathematically a priori different definitions of the abelian Chern-Simons partition function derived from [8] are rigorously equivalent. Our main strategy is to use the the work of M. Rumin and N. Seshadri [9] which naturally connects the analytic torsion with contact structures on three-manifolds.

Before presenting our main results we establish some notation and terminology. Throughout,
$X$ will denote a closed, orientable three-manifold, and $(X, \phi, \xi, \kappa, g)$ will denote $X$ equipped with a Sasakian structure. See [A] for further background on Sasakian and contact geometry.

**Definition 1.** A Seifert manifold is a closed orientable three-manifold that admits a locally free $\mathbb{U}(1)$-action.

**Remark 2.** Note that our definition of a Seifert manifold is not the most general possible. We refer to [10] for the general definition and also for the classification of these manifolds. For us, Seifert manifolds are simply $\mathbb{U}(1)$-bundles over an orbifold $\Sigma$.

It is well known that the topological isomorphism class of a Seifert manifold $X$ is given by the Seifert invariants \([g, n; (\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)]\), $\gcd(\alpha_j, \beta_j) = 1$, where $g$ is the genus of $\Sigma$. Geometrically, the $\mathbb{U}(1)$ action on $X$ is rotations of the fibres over $\Sigma$ and the points in the $\mathbb{U}(1)$ fiber over each orbifold point $p_j$ on $\Sigma$ are fixed by the cyclic subgroup $\mathbb{Z}_{\alpha_j}$ of $\mathbb{U}(1)$. Recall also the following description of the fundamental group of $X$. $\pi_1(X)$ is generated by the following elements \([10]\),

$$a_p, b_p, \quad p = 1, \ldots, g,$$

$$c_j, \quad j = 1, \ldots, M,$$

$$h,$$

which satisfy the relations,

\[
\begin{align*}
[a_p, h] &= [b_p, h] = [c_j, h] = 1, \\
\prod_{p=1}^{g} [a_p, b_p] \prod_{j=1}^{M} c_j &= h^n,
\end{align*}
\]

Geometrically, the generator $h$ is associated to the generic $\mathbb{U}(1)$ fiber over $\Sigma$, the generators $a_p, b_p$ come from the $2g$ non-contractible cycles on $\Sigma$, and the generators $c_j$ come from the small one cycles in $\Sigma$ around each of the orbifold points $p_j$. For convenience we recall that a Sasakian manifold is a normal contact metric manifold, $(X, \phi, \xi, \kappa, g)$ (See [A]), where,

- $\kappa \in \Omega^1(X)$ is a contact form, i.e. $\kappa \wedge d\kappa \neq 0$, $\xi \in \Gamma(TX)$ is the Reeb vector field,
- $\phi \in \text{End}(TX)$, $\phi(Y) := JY$ for $Y \in \Gamma(H)$, $\phi(\xi) = 0$ where $J \in \text{End}(H)$ is an almost complex structure on the contact distribution $H := \ker \kappa \subset TX$, and,
- $g = \kappa \otimes \kappa + d\kappa \circ (\mathbb{I} \otimes \phi)$.

**Remark 3.** Since the analytic torsion is defined with respect to a choice of metric, we naturally work with Sasakian structures in this article. As noted above, the Seifert structures that show up for us are derived from Sasakian structures. More precisely, we use the following: $X$ admits a Sasakian structure $(X, \phi, \xi, \kappa, g) \iff$

- [11] Theorem 7.5.1, (i)] $X$ admits a Seifert structure that is the total space of a principal $\mathbb{U}(1)$ bundle over a Hodge orbifold surface, $\Sigma$. 

Next we establish some notation for abelian Chern-Simons theory. Let $T$ denote a compact, connected abelian Lie group of real dimension $N$, $t$ denote its Lie algebra and $Λ ⊂ t$ the integral lattice. Let $\text{Tors} H^2(X, Λ)$ denote the torsion subgroup of $H^2(X, Λ)$. For $P$ a principal $T$-bundle over $X$, $A_P$ is the affine space of connections on $P$ modeled on the vector space of $\text{T}$-invariant horizontal one-forms on $P$. Let $\langle ·, · \rangle ∈ Ω^2$ be a compact oriented four-manifold such that $∂W = X$, which always exists [12]. Extend $P$ to a $T$-bundle $Q$ over $W$, which is always possible in our case [13]. Given a form $α ∈ Ω^2(P, t)$, let $\tilde{α} ∈ Ω^2(Q, t)$ denote the corresponding extension to $Q$. Thus, for a connection $A ∈ Ω^1(P, t)$, denote the curvature form of the extension $\tilde{A} ∈ Ω^1(Q, t)$ by $\tilde{F}_A ∈ Ω^2(W, t)$.

**Definition 4.** The Chern-Simons action of a $T$-connection $A ∈ A_P$ is defined by,

\[
CS_{X,P}(A) := \frac{1}{4\pi} \int_W \langle \tilde{F}_A ∧ \tilde{F}_A \rangle \mod (2\pi\mathbb{Z}).
\]

We also define the following,

- $m_X := \frac{N}{2} (\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R}))$,
- $A_P$ denotes a flat connection on a principal $U(1)$-bundle $P$ over $X$ with Chern-Simons invariant $CS_{X,P}(A_P)$,
- $[g, n; (α_1, β_1), \ldots, (α_M, β_M)]$, for gcd$(α_j, β_j) = 1$ are the Seifert invariants of a Sasakian manifold $(X, φ, ξ, κ, g)$,
- $η_0 = N \left( \frac{c_1(X)}{6} - 2 \sum_{j=1}^{M} s(α_j, β_j) \right)$ is the adiabatic eta-invariant of the Sasakian manifold $(X, φ, ξ, κ, g)$ introduced in [14],
- $s(α, β) := \frac{1}{πα} \sum_{j=1}^{α-1} \cot \left( \frac{πj}{α} \right) \cot \left( \frac{πjβ}{α} \right) ∈ \mathbb{Q}$ is the classical Rademacher-Dedekind sum,
- The moduli space of flat abelian connections is given by,

\[
\mathcal{M}_X \simeq \prod_{[P] ∈ \text{Tors} H^2(X, Λ)} T^{2g},
\]

and a particular component of $\mathcal{M}_X$ corresponding to a particular bundle class $[P] ∈ \text{Tors} H^2(X, Λ)$ is denoted as,

\[
\mathcal{M}_P \simeq H^1(X, t)/H^1(X, Λ) \simeq T^{2g}.
\]

- The eta-invariant for the odd signature operator, $L^o$, acting on $Ω^1(X, t) ⊕ Ω^3(X, t)$, is defined by analytic continuation as a limit,

\[
η(L^o) := \lim_{s → 0} \sum_{λ ∈ \text{spec}^+(L^o)} \text{sgn}(λ)|λ|^{-s}.
\]

The eta-invariant is an analytic invariant introduced by Atiyah, Patodi and Singer [15] defined for an elliptic and self-adjoint operator. As in [15, Prop. 4.20], we may
remove some spectral symmetry and the eta-invariant of $L^o$ coincides with the eta-invariant of the operator $\star d$ restricted to $\Omega^1(X, t) \cap \text{Im}(\star d)$. Throughout, we will abuse notation slightly and write,

$$\eta(\star d) = \lim_{s \to 0} \sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

and replace $L^o$ in the notation with $\star d$. We also recall that the expression for the sum,

$$\sum_{\lambda \in \text{spec}^*(\star d)} \text{sgn}(\lambda) |\lambda|^{-s},$$

is defined for large $\text{Re}(s)$ and [15] shows that it has a meromorphic continuation to $\mathbb{C}$ that is analytic at 0. It therefore makes sense to take the limit as $s \to 0$ in (4) and to define the eta-invariant $\eta(\star d)$ as evaluation of this limit.

- $\eta_{\text{grav}}(g)$ denotes the eta-invariant for the operator $\star d$ acting on $\Omega^1(X, \mathbb{R})$, so that,

$$\eta(\star d) = N \cdot \eta_{\text{grav}}(g),$$

where the eta-invariant on the left hand side of (6) is defined on $\Omega^1(X, t)$ and $N = \dim \mathbb{T}$.

- \(^{(7)}\) \(\text{CS}_s(A^g) := \frac{1}{4\pi} \int_X s^* \text{Tr}(A^g \wedge dA^g + \frac{2}{3} A^g \wedge A^g \wedge A^g),\)

is the gravitational Chern-Simons term, where $A^g$ the Levi-Civita connection and $s$ a trivializing section of twice the tangent bundle of $X$. More explicitly, let $H = \text{Spin}(6)$, $Q = TX \oplus TX$ viewed as a principal $\text{Spin}(6)$-bundle over $X$, $g \in \Gamma(S^2(T^*X))$ a Riemannian metric on $X$, $\phi : Q \to \text{SO}(X)$ a principal bundle morphism, and $A^{LC} \in \mathcal{A}_{SO(X)} := \{A \in (\Omega^1(\text{SO}(X)) \otimes \mathfrak{so}(3))^{SO(3)} \mid A(\xi^2) = \xi, \forall \xi \in \mathfrak{so}(3)\}$ the Levi-Civita connection. Then $A^g := \phi^* A^{LC} \in \mathcal{A}_Q := \{A \in (\Omega^1(Q) \otimes \mathfrak{h})^H \mid A(\xi^2) = \xi, \forall \xi \in \mathfrak{h}\}$.

An Atiyah-Patodi-Singer theorem, [16] Prop. 4.19, says that the combination,

$$\eta_{\text{grav}}(g) + \frac{1}{3 \cdot 2\pi} \text{CS}(A^g),$$

is a topological invariant depending only on a two-framing of $X$. Recall that a two-framing is a choice of a homotopy equivalence class $\Pi$ of trivializations of $TX \oplus TX$, twice the tangent bundle of $X$. Note that $\Pi$ is represented by the trivializing section $s : X \to Q$ above. The possible two-framings correspond to $\mathbb{Z}$. The identification with $\mathbb{Z}$ is given by the signature defect defined by,

$$\delta(X, \Pi) = \text{sign}(M) - \frac{1}{6} p_1(2TM, \Pi),$$

where $M$ is a 4-manifold with boundary $X$ and $p_1(2TM, \Pi)$ is the relative Pontrjagin number associated to the framing $\Pi$ of the bundle $TX \oplus TX$. The canonical two-framing $\Pi^c$ corresponds to $\delta(X, \Pi^c) = 0$.

**Remark 5.** Before we present the main quantities of interest in Definitions 6, 7, we note that both definitions implicitly require a choice of base $h^0$ for $H^0(X, \mathbb{R})$ to be well defined. We elaborate on this point in §2.

We are now ready to make the following,
Definition 6. \[17\] Let \( k \in \mathbb{Z} \) and \( X \) a closed, oriented three-manifold. The abelian Chern-Simons partition function, \( Z_T(X, k) \), is the quantity,

\[
Z_T(X, k) = \sum_{P \in \text{Tors} H^2(X, \Lambda)} Z_T(X, P, k),
\]

and,

\[
Z_T(X, P, k) := k^{m_X} e^{ik\text{CS}_{X,P}(A_P)} e^{\frac{\pi i}{4} \left(\frac{\eta_{\text{grav}}(g)}{4} + \frac{1}{12} \text{CS}(A_P^6)\right)} \int_{\mathcal{M}_P} \sqrt{T_X},
\]

where \( m_X = \frac{N}{2} (\dim H^1(X, \mathbb{R}) - 2 \dim H^0(X, \mathbb{R})) \).

Definition 7. \[17\] Let \( k \in \mathbb{Z} \), and let \((X, \phi, \xi, \kappa, g)\) be a closed oriented Sasakian three-manifold with associated principal bundle structure,

\[
\mathbb{U}(1) \rightarrow X \rightarrow \Sigma,
\]

where \( \Sigma \) is an orbifold such that \( X \) has associated Seifert invariants, 

\[
[g, n; (\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)],
\]

for \( \gcd(\alpha_j, \beta_j) = 1 \). Define the symplectic abelian Chern-Simons partition function,

\[
\mathcal{Z}_T(X, k) = \sum_{[P] \in \text{Tors} H^2(X, \Lambda)} \mathcal{Z}_T(X, P, k),
\]

and,

\[
\mathcal{Z}_T(X, P, k) = k^{m_X} e^{ik\text{CS}_{X,P}(A_P)} e^{\frac{\pi i}{4} \left(\frac{\eta_{\text{grav}}(g)}{4} - \frac{1}{12} \text{CS}(A_P^6)\right)} \int_{\mathcal{M}_P} K_X \cdot \omega_P,
\]

where, \( m_X := \frac{N}{2} (\dim H^1(X, \mathbb{R}) - 2 \dim H^0(X, \mathbb{R})) \), \( K_X := \frac{1}{|\text{c}_1(X)|} \frac{1}{\prod_{i=1}^N \alpha_i^{4/N}} \), \( \omega_P := \left(\sum_{j=1}^M \alpha_j \wedge \beta_j\right)^{gN} \).

The main motivation for this work is the conjectural equivalence of the rigorous topological invariants \( Z_T(X, k) \) and \( \mathcal{Z}_T(X, k) \). Note that this conjecture arises simply due to the fact that the rigorous definitions of \( Z_T(X, k) \) and \( \mathcal{Z}_T(X, k) \) are derived from the same heuristic Chern-Simons partition function in physics. We note that part of this conjectural equivalence is motivated by \[18\] which argues that \( \sqrt{T_X} \) is proportional to a specific scalar multiple of the natural unit symplectic volume form \( \omega \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R}) \) by using the group structure on the moduli space \( \mathcal{M}_X \),

\[
\sqrt{T_X} = C \cdot \left(\frac{1}{\sqrt{|\text{Tors} H^2(X, \Lambda)|}} \cdot \omega\right),
\]

where \( 0 \neq C \in \mathbb{R} \). Note that \[18\] works with the case where \( X \) is endowed with a regular Sasakian structure, which corresponds to a principle \( \mathbb{U}(1) \) bundle over a surface without orbifold points. This article studies the more general case of a three-manifold \( X \) that admits a Sasakian structure. One of the main results of this article is Theorem 8. We are able to calculate the square-root of \( T_X \) explicitly as a specific scalar multiple of a natural symplectic volume form on the moduli space \( \mathcal{M}_X \) using a theorem of M. Rumin and N. Seshadri [9, Theorem 5.4]. We obtain the following,
Theorem 8. Let \((X, \phi, \xi, \kappa, g)\) be a closed Sasakian three-manifold. Then,
\[
\sqrt{T_X} = (2\pi)^{-Ng} \left| c_1(X) \cdot \prod_i \alpha_i \right|^{N/2} \sqrt{\delta_{\text{dR}}(\nu^1)},
\]
where \(\delta_{\text{dR}}(\nu^1) : \bigwedge^{\max} H^1(X, t) \to \mathbb{R}^+\) is the volume form associated to the base given by \(\delta_{\text{dR}}(\nu^1)\). Alternatively,
\[
\sqrt{T_X} = \frac{1}{c_1(X) \cdot \prod_i \alpha_i} \cdot \omega,
\]
where,
\[
\omega := \frac{\Omega g^N}{(gN)!(2\pi)^{2gN}},
\]
and,
\[
\Omega := \sum_{1 \leq i \leq gN} d\theta_i \wedge d\bar{\theta}_i.
\]

We also recall an explicit description of the moduli space of flat abelian connections \(\mathcal{M}_X\) on a closed, Sasakian three-manifold.

Theorem 9. [19, Theorem 8.1], [20] Given a closed oriented Sasakian three-manifold \(X\) (so that \(c_1(X) \neq 0\)) then,
\[
\mathcal{M}_X \simeq \mathbb{T}^{2g} \times \text{Tors}(H^2(X, \Lambda)) \simeq \text{Hom}(\pi_1(X), \mathbb{T}),
\]
where, \(|\text{Tors} H^2(X, \Lambda)| = |c_1(X) \cdot \prod_{j=1}^M \alpha_j|^N\).

We note that Theorem 8 combined with Theorem 9 leads to an explicit computation of the symplectic volume of the moduli space. Thus, we have the following,

Corollary 10. Given a closed oriented Sasakian three-manifold \((X, \phi, \xi, \kappa, g)\) with associated Seifert data,
\[
[g, n; (\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)],
\]
for \(\gcd(\alpha_j, \beta_j) = 1\), the symplectic volume of the moduli space \(\mathcal{M}_X\) with respect to the symplectic volume form \(\sqrt{T_X} \in \Omega^{2gN}(\mathcal{M}_X, \mathbb{R})\) is given by,
\[
\int_{\mathcal{M}_X} \sqrt{T_X} = \frac{1}{|\text{Tors} H^2(X, \Lambda)|} = \left| c_1(X) \cdot \prod_j \alpha_j \right|^{N/2},
\]
where \(c_1(X) = n + \sum_{j=1}^M \frac{\beta_j}{\alpha_j}\) is the first orbifold Chern number of \((X, \phi, \xi, \kappa, g)\) and \(N = \dim \mathbb{T}\).

As a consequence of Theorem 8 we obtain the following verification of the above conjecture,

Corollary 11. Let \(k \in \mathbb{Z}\), and let \((X, \phi, \xi, \kappa, g)\) be a closed oriented Sasakian three-manifold. Then the magnitudes of \(Z_T(X, k)\) and \(\mathbb{Z}_T(X, k)\) agree identically,
\[
|Z_T(X, k)| = |\mathbb{Z}_T(X, k)|,
\]
and,
\[
|Z_T(X, k)| = k^{m_X} \cdot \frac{\left| \sum_{[P] \in \text{Tors} H^2(X, \Lambda)} e^{ik \langle CS_X, p(A_P) \rangle} \right|}{\sqrt{|\text{Tors} H^2(X, \Lambda)|}},
\]
where \( m_X := \frac{N}{2}(\dim H^1(X; \mathbb{R}) - 2 \dim H^0(X; \mathbb{R})) \), and \( \text{Tors} H^2(X, \Lambda) \) is the torsion part of \( H^2(X, \Lambda) \) with values in the integral lattice \( \Lambda \).

We note that it may be possible to compute the quantity,

\[
\sum_{[P] \in \text{Tors} H^2(X, \Lambda)} e^{ik_{CS, P}(A_P)} ,
\]

explicitly in terms of Seifert data using the work of P. Kirk and E. Klassen \[21\]. Lastly, we note that it is an open problem to compute the abelian Chern-Simons partition function \( Z_T(X, k) \) explicitly on an arbitrary closed three-manifold \( X \).

2. Computation of the analytic torsion

In this section we compute the square root of the analytic torsion \( \sqrt{T_X} \) as a symplectic volume form on the moduli space of flat abelian connections,

\[
\mathcal{M}_X \simeq \bigsqcup_{[P] \in \text{Tors} H^2(X, \Lambda)} T^{2g} ,
\]

in the case that \( X \) admits a Sasakian structure, \((\xi, \kappa, \phi, g)\).

**Remark 12.** We will denote a single component of the moduli space corresponding to a bundle class \([P]\) as \( \mathcal{M}_P \). Also, we note that \( \sqrt{T_X} \) is more naturally viewed as an element of a determinant line over \( \mathcal{M}_X \) and in order to compare \( \sqrt{T_X} \) to a symplectic volume form we must choose a base for the zeroth cohomology of \( X \). See Remarks \[13\] and \[14\] for further elaboration on this point. We indicate here that such a choice is assumed implicitly in what follows.

Let,

\[
\Omega := \sum_{1 \leq i \leq g, 1 \leq j \leq N} d\theta_{i,j} \wedge d\bar{\theta}_{i,j} ,
\]

be the standard symplectic form on \( \mathcal{M}_P \simeq \mathbb{T}^{2g} \), where \( N = \dim T \) is the dimension of the Lie group for gauge transformations. The natural symplectic volume form that we consider is defined as,

\[
\omega := \frac{\Omega^{gN}}{(gN)!(2\pi)^{2gN}} .
\]

As noted above, we will assume \( T = U(1) \) in this section and set \( N = 1 \).

**Remark 13.** The natural quantity that shows up in the symplectic abelian Chern-Simons path integral is \( \omega \) multiplied by \( 1/|\text{Vol}(I)| \), where

\[
I := \{ g \in G | A_P \cdot g = A_P \} \simeq U(1) < G ,
\]

is the isotropy subgroup of the gauge group of a given abelian connection \( A_P \in \mathcal{A}_P \). The volume of the isotropy group, \( \text{Vol}(I) \), requires a choice of measure on \( I \simeq U(1) \), which boils down to a choice of base \( h^0 \) for \( H^0(X, \mathbb{R}) \). We recall some of the details presently.
In our study of abelian Chern-Simons theory [17], the natural invariant metric on the group $G$ is defined in terms of the Hodge star $\star$ for the given Sasakian metric $g$ on $X$,

$$G_G(\theta, \phi) := \int_X \langle \theta \land \star \phi \rangle,$$

where $\theta, \phi \in \text{Lie } G \simeq \Omega^0(X, \mathbb{R})$. Observe that $G_G$ restricted to constant functions $\theta, \phi \in \mathbb{R} \subset \text{Lie } G$ is given as follows,

$$G_G(\theta, \phi) = \left( \int_X \star 1 \right) \cdot \langle \theta, \phi \rangle.$$

We may therefore write $\sqrt{G_G} = \left( \int_X \star 1 \right)^{1/2}$. Now we choose the measure $\sqrt{G_G} d\sigma$ on $I \simeq \text{U}(1)$ such that $d\sigma = d\theta/2\pi$ setting $\int_{\text{U}(1)} d\sigma = 1$. Let $\mathcal{H}^0(X, \mathbb{R})$ denote the harmonic 0-forms on $X$. Note that by definition of the de Rham map $\delta_{\text{dR}}^0 : \mathcal{H}^0(X, \mathbb{R}) \to \mathcal{H}^0(X, \mathbb{R})$, this choice of measure may be viewed as a choice of base $h^0$ for $\mathcal{H}^0(X, \mathbb{R}) \cong \text{Lie } \text{U}(1)$ such that $\delta_{\text{dR}}^0(2\pi) = h^0$. We have,

$$\text{Vol}(I) := \int_{\text{U}(1)} \sqrt{G_G} d\sigma,$$

$$= \sqrt{G_G}, \text{ since } \int_{\text{U}(1)} d\sigma = 1,$$

$$= \left[ \int_X \star 1 \right]^{1/2}.$$

Since the Hodge star $\star$ is defined in terms of the given Sasakian metric, we have,

$$\text{Vol}(I) = \left[ \int_X \kappa \land d\kappa \right]^{1/2} = |c_1(X)|^{1/2},$$

where $c_1(X) = n + \sum \frac{\beta_j}{\alpha_j} > 0$ is the orbifold first Chern number of $(X, \phi, \xi, \kappa, g)$ as a Seifert manifold with Seifert invariants,

$$[g, n; (\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)],$$

where $\gcd(\alpha_j, \beta_j) = 1$.

[18] shows,

$$\sqrt{T_X} = C \cdot \frac{1}{|\text{Vol}(I)|^{\omega}},$$

for some non-zero constant $C \in \mathbb{R}^*$ when $X$ admits a regular Sasakian structure. In the general case of a Sasakian structure, we find that $C = 1$, provided that $|\text{Vol}(I)|$ is replaced by $|c_1(X)| \cdot \prod \alpha_i|^{1/2}$, where

$$[g, n; (\alpha_1, \beta_1), \ldots, (\alpha_M, \beta_M)], \text{ } \gcd(\alpha_j, \beta_j) = 1,$$

are the Seifert invariants corresponding to the Sasakian structure given. Of course, when $X$ admits a regular Sasakian structure we have that $|\text{Vol}(I)| = |c_1(X)|^{1/2}$, since there are no exceptional fibers in this case. Our main result follows from [9, Theorem 5.4], where the analytic torsion is computed on a closed Sasakian three-manifold twisted by a unitary
representation $\rho : \pi_1(X) \to U(r)$. A proof of our main result then follows by substituting some known special values of the Riemann-zeta function.

We start by recalling the definition of the analytic torsion $T_X$. We will restrict ourselves to the $U(1)$ structure group case for simplicity. Before we make our definition we recall some standard notation. Let $(M, g)$ be a closed oriented Riemannian manifold of dimension $m$ and let $\rho : \pi_1(M) \to U(1)$ be a representation of the fundamental group of $M$. Recall that $\rho$ corresponds to a flat principal $U(1)$ bundle $P$ over $M$ equipped with a flat connection $A_\rho \in \mathcal{A}_P$. Given a representation $\chi : U(1) \to \text{Aut} \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we obtain an associated line bundle $\mathcal{E}_\chi := P \times_\chi \mathbb{F}$ in the usual way.

Remark 14. We will only need to assume $\chi$ to be the standard representation on $\mathbb{C}$ as in [1] and [9]. We see no reason to restrict to the standard representation in general, however. We also speculate on extending this theory to any field $\mathbb{F}$ of arbitrary characteristic. We will not pursue this here. We will therefore state our definition of the torsion making this choice explicit. The choice of standard representation seems to be common in the literature and for us it natural to loosen this restriction. Also, it is important to note that the torsion $T_X$ that shows up most naturally in abelian Chern-Simons theory is defined in terms of the adjoint representation $\chi = \text{Ad} : U(1) \to \text{Aut}(u(1)) \simeq \mathbb{R}^\times$. This will not concern us because we obtain our desired results for abelian Chern-Simons theory by restricting to the trivial representation $\rho_0 : \pi_1(M) \to U(1)$ and working in the standard representation over $\mathbb{C}$.

Let,

$$d_{A_\rho}^\chi : \Omega^q(M, \mathcal{E}_\chi) \to \Omega^{q+1}(M, \mathcal{E}_\chi),$$

denote the covariant derivative associated to $A_\rho$ and let,

$$\Delta^\chi_q(\rho) := (d_{A_\rho}^\chi)^*d_{A_\rho}^\chi + d_{A_\rho}^\chi (d_{A_\rho}^\chi)^* : \Omega^q(M, \mathcal{E}_\chi) \to \Omega^q(M, \mathcal{E}_\chi),$$

denote the corresponding Laplacian. We make the following,

Definition 15. [1] Let $M$ be a closed oriented Riemannian manifold of dimension $m$ and let $\rho : \pi_1(M) \to U(1)$ be a representation of the fundamental group of $M$ and let $\chi : U(1) \to \text{Aut} \mathbb{F}$ be a representation of $U(1)$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Let $\Delta^\chi_q(\rho) : \Omega^q(M, \mathcal{E}_\chi) \to \Omega^q(M, \mathcal{E}_\chi)$ denote the Laplacian in the representation $\chi$. For each $0 \leq q \leq m$, let $h^q$ be a preferred base for the cohomology group $H^q(M, \rho)$. Let $\mathcal{H}^q(M, \rho)$ denote the harmonic $q$-forms and let $\nu^q$ be an orthonormal base for $\mathcal{H}^q(M, \rho)$. Let $\delta^q_{\text{dR}} : \mathcal{H}^q(M, \rho) \to H^q(M, \rho)$ denote the de Rham map, sending $\nu^q$ into a base $\delta^q_{\text{dR}}(\nu^q)$ for $H^q(M, \rho)$. Let $[\delta^q_{\text{dR}}(\nu^q)/h^q]$ denote the determinant of the change of base map to the preferred base. The analytic torsion is defined as,

$$T^\chi_{M}(\rho)\{h^q\} := \exp \left( \sum_{0 \leq q \leq m} (-1)^q \left[ \frac{1}{2} q \zeta_q'(0) + \log \left| [\delta^q_{\text{dR}}(\nu^q)/h^q] \right| \right] \right),$$

where $\zeta_q(s)$ is the zeta-function for $\Delta^\chi_q(\rho)$ defined for $\text{Re}(s) \gg 0$ by,

$$\zeta_q(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr}(e^{t\Delta^q} - \Pi_q) dt,$$
analytically continued to $\mathbb{C}$ as usual, and $\Pi_q : \Omega^q(M, \rho) \to \mathcal{H}^q(M, \rho)$ is orthogonal projection. It is also useful to define the scalar quantity,

$$T^{sX}_{M}(\rho) := \exp \left( \frac{1}{2} \sum_{q=0}^{m} (-1)^{q} q \zeta_{q}(0) \right),$$

which we also call the torsion.

Observe that the torsion $T^{X}_{M}(\rho)$ is naturally viewed as an element of the determinant line,

$$| \det H^{*}(M, d_{A_{\rho}}) | := \bigotimes_{j=0}^{3} | \det H^{j}(M, d_{A_{\rho}})^{(-1)^{j}} |.$$

Before we quote the main result that we need we note that [9] uses the terminology “CR-Seifert” manifold, whereas we use the terminology “Sasakian” manifold. As shown in Prop. 53 the two structures are equivalent. We also note that [9] defines and studies a new type of analytic torsion on contact manifolds called the contact analytic torsion, denoted by $T^{C}_{X}$. It is shown in [9], however, that the analytic torsion is equivalent to the contact analytic torsion on Sasakian manifolds. Note that our definition of $T^{X}_{X}$ is the inverse of the definition used in [9].

**Theorem 16.** [9, Theorem 4.2] Let $(X, \phi, \xi, \kappa, g)$ be a closed Sasakian (CR-Seifert) three-manifold, $\rho : \pi_{1}(X) \to U(N)$ a unitary representation, and $\chi_{0} : U(N) \to \text{Aut}(\mathbb{C}^{N})$ the standard representation. Let $T^{X}_{X}$ and $T^{C}_{X}$ denote the analytic torsion and the contact analytic torsion, respectively, in the standard representation; e.g. $T^{X}_{X} := T^{X}_{X}$. Then the analytic torsion $T^{X}_{X}$ and the contact analytic torsion $T^{C}_{X}$ satisfy:

$$T^{X}_{X}(\rho) = T^{C}_{X}(\rho).$$

Also, the scalar part of the torsions agree, $T^{sX}_{M}(\rho) = T^{Cs}_{X}(\rho)$.

The main result that we need is given as follows.

**Theorem 17.** [9, Theorem 5.4] Let $(X, \phi, \xi, \kappa, g)$ be a closed Sasakian three-manifold. Split $E_{\chi}$ into irreducibles $E_{\chi}^{0}$. The torsion function spectrally decomposes as,

$$K(s) = \sum_{E_{\chi}^{0}} K_{\theta}(s),$$

such that,

- On $E_{\chi}^{\theta}$ with $\theta \in (0, 1)$, i.e. $\chi \circ \rho(h) = e^{2\pi i \theta} \neq 1$, we have,

$$K_{\theta}(s) = \dim(E_{\chi}^{\theta})(\zeta(2s, \theta) + \zeta(2s, 1 - \theta)) + \sum_{i,j} \frac{1}{\alpha_{i}^{2s}} (\zeta(2s, \theta_{i,j}) + \zeta(2s, 1 - \theta_{i,j})).$$

- Let $E_{\chi}^{0,i} = \ker(1 - \chi \circ \rho(c_{i}))$. Then we have,

$$K_{0}(s) = K(X, \rho)(2\zeta(2s) + 1) + 2\zeta(2s) \sum_{i} \dim(E_{\chi}^{0,i})(\alpha_{i}^{-2s} - 1) + \sum_{\{i,j : \theta_{i,j} \neq 0\}} \frac{1}{\alpha_{i}^{2s}} (\zeta(2s, \theta_{i,j}) + \zeta(2s, 1 - \theta_{i,j})).$$
Remark 18. We note that the proof of this theorem relies heavily on the assumption that $X$ admits a Sasakian structure and follows by application of the Riemann-Roch-Kawasaki formula \cite{22}, \cite{23}. It is precisely the rigidity of the Sasakian structure that allows \cite{9} to compare the spectra of the ordinary and contact Laplacians.

The case of interest for us is the trivial representation $\rho_0 : \pi_1(X) \to U(1)$. Since this is already scalar we have,

\begin{equation}
K(s) = K_0(s),
\end{equation}

where, by Theorem 17, we have,

\begin{equation}
K_0(s) = K(X, \rho)(2\zeta(2s) + 1) + 2\zeta(2s) \sum_i (\alpha_i^{-2s} - 1).
\end{equation}

Now we use the identification of the analytic torsion and the contact analytic torsion given in Theorem 16 to write $T^s_X(\rho_0) = \exp(-K'_0(0)/2)$. We compute $K'_0(0)$ using Theorem 17.

Using the special values of the Riemann-zeta function, $\zeta(0) = -1/2$ and $\zeta'(0) = -\ln(2\pi)/2$ \cite{28}, and $K(X, \rho) = 2 \dim H^0(X, t) - \dim H^1(X, t)$ \cite{9} Eq. 42, we obtain,

\begin{equation}
-K'_0(0)/2 = (2 - 2g) \ln(2\pi) - \sum_i \ln(\alpha_i).
\end{equation}

Thus,

\begin{equation}
T^s_X(\rho_0) = \frac{(2\pi)^{2-2g}}{\prod_i \alpha_i}.
\end{equation}

It is easy to see that $T^{s,\text{Ad}}_X(\rho) = T^{s,\chi}_X(\rho_0)$ when $\rho_0 \equiv 1$ is the trivial representation, $\chi$ is the standard representation, and $\rho : \pi_1(X) \to U(1)$ is arbitrary. This follows because the spectra of the corresponding Laplacians are identical. That is, for the standard representation $\chi$, the Laplacian at the trivial representation $\rho_0$ is given by,

\begin{equation}
\Delta^\chi_j(\rho_0) := d^*d + dd^* : \Omega^j(X, \mathbb{C}) \to \Omega^j(X, \mathbb{C}),
\end{equation}

where $d^\chi_{A,\rho_0} = d$ is just the ordinary de Rham derivative. Also, for the adjoint representation,

\begin{equation}
\Delta^\text{Ad}_j(\rho) := d^*d + dd^* : \Omega^j(X, \mathbb{R}) \to \Omega^j(X, \mathbb{R}),
\end{equation}

since $d^{\text{Ad}}_{A,\rho} = d$ for any representation $\rho$. Clearly, these operators have identical spectra.

By Poincaré duality $H^3(X, d)^{-1}$ is canonically isomorphic to $H^0(X, d)$, and $H^1(X, d)^{-1}$ is canonically isomorphic to $H^2(X, d)$. Thus,

\begin{equation}
T^{\text{Ad}}_X(\rho) \in | \det H^0(X, d_{A,\rho}) |^{\otimes 2} \bigotimes | \det H^1(X, d_{A,\rho})^{-1} |^{\otimes 2},
\end{equation}

and we naturally view,

\begin{equation}
\sqrt{T^{\text{Ad}}_X(\rho)} \in | \det H^0(X, d_{A,\rho}) | \bigotimes | \det H^1(X, d_{A,\rho})^{-1} |.
\end{equation}

Note that since the adjoint representation is trivial on $\mathbb{R}$, we suppress the dependence on $\rho$ and write,

\begin{equation}
\sqrt{T^{\text{Ad}}_X} = \sqrt{T^{\text{Ad}}_X(\rho)} \in | \det H^0(X, \mathbb{R}) | \bigotimes | \det H^1(X, \mathbb{R})^{-1} |.
Remark 19. Observe that if $\nu^0$ is an orthonormal base for $\mathcal{H}^0(X, \mathbb{R}) = \mathbb{R}$, then it may be identified as a scalar $\nu^0 \in \mathbb{R}$ such that,

\[
1 = ||\nu^0||^2,
\]

\[
= \int_X \nu^0 \wedge \ast \nu^0,
\]

\[
= |\nu^0|^2 \int_X \kappa \wedge d\kappa,
\]

\[
= |\nu^0|^2 \cdot c_1(X).
\]

Thus, $|\nu^0| = 1/|c_1(X)|^{1/2}$. In order to view the analytic torsion as a volume form on $\mathcal{M}_X$, we must choose a base $h^0$ for $H^0(X, \mathbb{R})$ and compute

\[
\left[ \delta_0 \mathrm{dR}(\nu^0)/h^0 \right] \in |\det H^1(X, \mathbb{R})|^{-1}.
\]

Theorem 20. Let $(X, \phi, \xi, \kappa, g)$ be a closed Sasakian three-manifold. Then,

\[
\sqrt{T_X} := \sqrt{T_X^{\text{Ad}}|_{h^0}} \in |\det H^1(X, \mathbb{R})|^{-1}.
\]

Overall we have the following,

\[
\sqrt{T_X} = \frac{(2\pi)^{-N_g}}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \left| \bigwedge \delta_{\text{dR}}^1(\nu^1) \right|^*,
\]

where $\left| \bigwedge \delta_{\text{dR}}^1(\nu^1) \right|^* : \bigwedge^\text{max} H^1(X, t) \to \mathbb{R}^+$ is the volume form associated to the base given by $\delta_{\text{dR}}^1(\nu^1)$. Alternatively,

\[
\sqrt{T_X} = \frac{1}{|c_1(X) \cdot \prod_i \alpha_i|^{N/2}} \cdot \omega,
\]

where,

\[
\omega := \frac{\Omega g^N}{(gN)! (2\pi)^{2gN}},
\]

and,

\[
\Omega := \sum_{1 \leq i \leq gN} d\theta_i \wedge d\bar{\theta}_i.
\]

Remark 21. Note that the generalization to the case of an arbitrary torus $\mathbb{T}$ that occurs in Prop. 20 is straightforward. We also point out that the extra factor of $(2\pi)^{gN}$ that occurs in $\omega$ in Prop. 20 above is due to the corresponding factor of $\sqrt{2\pi}$ in the norm of each orthonormal basis element for the first cohomology.

Appendix A. Review of contact and Sasakian geometry

In this section we briefly review some definitions and results in contact geometry and also review the definition and some properties of Sasakian three-manifolds. We primarily follow [11]. Recall the following,
Definition 22. A \((2n+1)\)-dimensional manifold \(M\) is a contact manifold if there exists a one-form \(\kappa \in \Omega^1(M, \mathbb{R})\), called a contact one-form, on \(M\), such that,

\[\kappa \wedge (d\kappa)^n \neq 0,\]

everywhere on \(M\). A contact structure on \(M\) is an equivalence class of such one-forms, where \(\kappa' \sim \kappa \iff \exists 0 \neq f \in C^\infty(M)\) such that \(\kappa' = f\kappa\). The subbundle \(\ker(\kappa) =: H \subset TM\) will be called the contact subbundle of \(M\).

Remark 23. Note that the condition \(\kappa \wedge (d\kappa)^n \neq 0\) is equivalent to \(d\kappa\) being non-degenerate on \(H\). There are several different perspectives and more general approaches to defining a contact structure on an odd dimensional manifold \(M\) that we will not pursue here \([29], [30]\). Let the line bundle \(\mathcal{L}_H\) be defined as the annihilator bundle of the contact subbundle \(\ker(\kappa) =: H \subset TM\), i.e. \(\mathcal{L}_H := H^0\). We only note that a more general definition involves allowing the bundle \(\mathcal{L}_M \subset T^*M\) over \(M\) to be non-trivial. In Def. 22 we have assumed that \(\mathcal{L}_H\) is trivial and \(\kappa \in \Gamma(\mathcal{L}_H)\) represents a choice of trivializing section. In order to distinguish the two cases, we will refer to the case where \(\mathcal{L}_H\) is trivial as a strict (or co-orientable) contact structure, and the case where \(\mathcal{L}_H\) is non-trivial as a non-strict contact structure. The two-fold cover of a non-strict contact manifold is strict, and in particular, every simply connected contact manifold is strict.

Example 24. One of the most important examples of a contact manifold is \(\mathbb{R}^{2n+1}\) with contact form given by \(\kappa = dt - \sum_i y_idx_i\). The contact subbundle \(H\) is spanned by \(\{\partial/\partial x_i + y_i\partial/\partial y_i\}\). This clearly defines a contact structure on \(\mathbb{R}^{2n+1}\) according to Def. 22. For example \(d\kappa = dx_i \wedge dy_i\) is easily seen to be non-degenerate on \(H\). This is called the standard contact structure on \(\mathbb{R}^{2n+1}\). Note by the contact version of Darboux’s theorem that every contact manifold is locally contactomorphic to \(\mathbb{R}^{2n+1}\) with the standard contact structure \([29]\). Recall that a map \(\Psi : (M, \kappa) \to (M', \kappa')\) between contact manifolds is called a contactomorphism if it is a diffeomorphism that preserves the contact form, \(\Psi^*\kappa' = \kappa\). If there exists an contactomorphism \(\Psi : (M, \kappa) \to (M', \kappa')\), then \((M, \kappa) \simeq (M', \kappa')\) are said to be contactomorphic.

Example 25. Let \(M = S^{2n+1}\), the unit \((2n+1)\)-sphere. Let \(\alpha := \sum_{i=0}^n (x_idy_i - y_idx_i) \in \Omega^1(\mathbb{R}^{2n+2}, \mathbb{R})\), where \(\mathbb{R}^{2n+2}\) is given the standard Cartesian coordinates \((x_0, \ldots, x_n, y_0, \ldots, y_n)\). Define,

\[\kappa := \alpha|_{S^{2n+1}}.\]

It is straightforward to see that \(\kappa \wedge (d\kappa)^n \neq 0\) everywhere on \(S^{2n+1}\). This defines the standard contact structure on \(S^{2n+1}\).

There is a very interesting generalization of Example 25 above due to Gray \([31]\).

Proposition 26. \([11]\) Prop. 6.1.17] Let \(M\) be an immersed hypersurface in \(\mathbb{R}^{2n+2}\) such that no tangent space of \(M\) contains the origin of \(\mathbb{R}^{2n+2}\). Then \(M\) admits a contact structure.

It is interesting to note that the following provides an example of a contact manifold for which the results of this article do not apply.

Example 27. Let \(M = T^3 := \mathbb{R}^3/\mathbb{Z}^3\). Let \((x, y, z)\) denote the standard Cartesian coordinates in \(\mathbb{R}^3\) and let

\[\kappa := \sin(y)dx + \cos(y)dz.\]

Then \(\kappa \wedge d\kappa = -dx \wedge dy \wedge dz\), and the contact subbundle is spanned by \(\{\partial/\partial y, \cos(y)\partial/\partial x - \sin(y)\partial/\partial z\}\).
Recall that every oriented surface admits a symplectic structure. The analogue of this fairly trivial fact in the three-manifold case is the reason why studying contact structures on three-manifolds is so natural. We have the following,

**Theorem 28.** [32] Every orientable three-manifold admits a contact structure.

We recall the following standard fact.

**Lemma 29.** On a contact manifold \((M, \kappa)\) there is a unique vector field \(\xi \in \Gamma(TM)\) called the Reeb vector field, satisfying the two conditions,

\[
i_\xi \kappa = 1, \quad i_\xi d\kappa = 0.
\]

We note that the Reeb vector field \(\xi\) depends strongly on the choice of contact form \(\kappa\) and one can obtain very different Reeb vector fields for different choices of contact forms within a contact structure.

**Example 30.** Consider \((S^{2n+1}, \kappa)\) with the standard contact structure as in Example 25. Let \(H_i := x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\). It is straightforward to see that the Reeb vector field of \(\kappa\) is given by,

\[
\xi = \sum_i H_i.
\]

Let \(w = (w_0, \ldots, w_n) \in \mathbb{R}^{n+1}\) be some positive vector so that \(w_i > 0 \forall \ 0 \leq i \leq n\). Let,

\[
f_w(x) := \frac{1}{\sum_{i=0}^{n} w_i(x_i^2 + y_i^2)}, \text{ for } x \in S^{2n+1}.
\]

Define a deformed contact form,

\[
\kappa_w := f_w \kappa.
\]

It is easy to see that,

\[
\xi_w = \sum_i w_i H_i,
\]

is the corresponding Reeb field for \(\kappa_w\). Clearly, the Reeb field changes drastically depending on the choice of vector \(w\). If the components of \(w\) are rational numbers, for example, the orbits of the Reeb field turn out to be all circles. If we choose one of the components to be irrational, however, we may obtain Reeb orbits that do not close. Yet, since \(\kappa\) and \(\kappa_w := f_w \kappa\) differ by a non-zero function \(f_w\), these different choices amount to the same underlying contact structure.

The Reeb vector field is sometimes called the characteristic vector field and the one-dimensional foliation \(\mathcal{F}_\xi\) uniquely determined by \(\xi\) is called the characteristic foliation of \((M, \kappa)\).

**Definition 31.** An almost contact structure on a differentiable manifold \(M\) is a triple \((\xi, \kappa, \phi)\), where \(\phi : TM \to TM\) is a tensor field of type \((1,1)\), \(\xi\) is a vector field, and \(\kappa \in \Omega^1(M, \mathbb{R})\) is a one-form which satisfy,

\[
\kappa(\xi) = 1, \quad \phi^2 = -\mathbb{I} + \xi \otimes \kappa,
\]

where \(\mathbb{I}\) is the identity endomorphism of \(TM\). A smooth manifold with such a structure is called an almost contact manifold. An almost contact structure is said to be normal if,

\[
[\phi, \phi] + 2d\kappa \otimes \xi = 0,
\]
where,
\[
[\phi, \phi](Y_1, Y_2) := \phi^2[Y_1, Y_2] - [\phi Y_1, \phi Y_2] - \phi[\phi Y_1, Y_2] - \phi[Y_1, \phi Y_2],
\]
is the Nijenhuis torsion of \(\phi\).

**Definition 32.** Let \(M\) be an almost contact manifold. A Riemannian metric \(g\) on \(M\) is said to be compatible with the almost contact structure if for any vector fields \(Y_1, Y_2 \in \Gamma(TM)\), we have
\[
g(\phi Y_1, \phi Y_2) = g(Y_1, Y_2) - \kappa(Y_1)\kappa(Y_2).
\]
An almost contact structure with a compatible metric is called an almost contact metric structure.

We have the following,

**Proposition 33.** [11] Every almost contact manifold admits a compatible metric.

We will need the following,

**Definition 34.** Let \((M, \kappa)\) be a contact manifold with contact distribution \(H\). Then an almost contact structure \((\xi, \kappa', \phi)\) is said to be compatible with the contact structure if \(\kappa = \kappa'\), \(\xi\) is the Reeb vector field, and the endomorphism \(\phi\) satisfies,
\[
d\kappa(\phi Y_1, \phi Y_2) = d\kappa(Y_1, Y_2), \quad \text{for all } Y_1, Y_2 \in \Gamma(TM),
\]
and,
\[
d\kappa(\phi Y_0, Y_0) > 0, \quad \text{for all } Y_0 \in \Gamma(H).
\]
Denote by \(\mathcal{AC}(\kappa)\) the set of compatible almost contact structures on \((M, \kappa)\).

**Proposition 35.** [11] Prop. 6.4.3] Let \((M, \kappa)\) be a contact manifold. The set of associated Riemannian metrics are in one-to-one correspondence with the set of compatible almost contact structures, \(\mathcal{AC}(\kappa)\), on \((M, \kappa)\).

Finally, the following is the basic definition that we need for this article.

**Definition 36.** A contact manifold \((M, \kappa)\) with a compatible almost contact metric structure \((\xi, \kappa, \phi, g)\) such that,
\[
g(Y_1, \phi Y_2) = d\kappa(Y_1, Y_2), \quad \text{for all } Y_1, Y_2 \in \Gamma(TM),
\]
is called a contact metric structure, and \((M, \xi, \kappa, \phi, g)\) is called a contact metric manifold.

**Definition 37.** A K-contact manifold is a manifold \(M\) with a contact metric structure \((\phi, \xi, \kappa, g)\) such that the Reeb field \(\xi\) is Killing for the associated metric \(g\), \(\mathcal{L}_{\xi} g = 0\).

**Definition 38.** The characteristic foliation \(\mathcal{F}_\xi\) of a contact manifold \((M, \kappa)\) is said to be quasi-regular if there is a positive integer \(j\) such that each point has a foliated coordinate chart \((U, x)\) such that each leaf of \(\mathcal{F}_\xi\) passes through \(U\) at most \(j\) times. If \(j = 1\) then the foliation is said to be regular.

Definitions 37 and 38 together define a quasi-regular K-contact manifold, \((M, \phi, \xi, \kappa, g)\). The following result provides several different perspectives on K-contact structures.

**Proposition 39.** [11] Prop. 6.4.8] On a contact metric manifold \((M, \phi, \xi, \kappa, g)\), the following conditions are equivalent:

(i) The characteristic foliation \(\mathcal{F}_\xi\) is a Riemannian foliation.

(ii) \(g\) is bundle-like.
(iii) The Reeb flow is an isometry.
(iv) The Reeb flow is a CR-transformation.
(v) The contact metric structure \((\phi, \xi, \kappa, g)\) is \(K\)-contact.

Recall the following,

**Definition 40.** A normal contact metric manifold \((M, \xi, \kappa, \phi, g)\) is called a Sasakian manifold.

**Remark 41.** Since every Sasakian three-manifold, \((X, \phi, \xi, \kappa, g)\), is \(K\)-contact \([29, \text{Corollary 6.3}]\) and every \(K\)-contact manifold admits a quasi-regular \(K\)-contact structure \([11, \text{Theorem 7.1.10}]\), then every Sasakian three-manifold admits a quasi-regular \(K\)-contact structure, \((X, \phi, \xi, \kappa, g)\). We implicitly take a Sasakian three-manifold \((X, \phi, \xi, \kappa, g)\) to be quasi-regular.

The following theorem is important for this article as it allows us to conclude a particularly nice form for the metric structures on Sasakian three-manifolds.

**Theorem 42.** \([11, \text{Theorem 6.3.6}]\) If \((X, \phi, \xi, \kappa, g)\) is a quasi-regular Sasakian three-manifold, then the metric takes the form,

\[
g = \kappa \otimes \kappa + \pi^* h,
\]

where \(h\) is a metric on the base, \(\Sigma\), of \(X\).

Sasakian geometry may also be viewed as an odd dimensional analogue of Kähler geometry. Let \(C(X) := \mathbb{R}^+ \times X\) be the cone on \(X\) with coordinate \(r\) on the \(\mathbb{R}^+\) factor and metric \(g_C := dr^2 + r^2 g\). One may define a Sasakian structure \((\kappa, \Phi, \xi, g)\) by requiring the associated structure on the metric cone \((g_C, d(r^2 \kappa), J_C)\) to be Kähler. Note that trivial \(U(1)\)-bundles over a surface \(\Sigma_g\), \(X = U(1) \times \Sigma_g\), admit no Sasakian structure \([33]\) and our results do not apply in this case.

**Example 43.** All three-dimensional lens spaces \(L(p, q)\) and the three-sphere \(S^3\) admit quasi-regular Sasakian structures. Identify \(S^3 = \{z = (x, y) \in \mathbb{C}^2 : ||z||^2 = ||x||^2 + ||y||^2 = 1\}\), and let,

\[
\kappa_0 = \sum_{j=0}^{1} (y_j dx_j - x_j dy_j) \big|_{S^3},
\]

\[
\xi_0 = \sum_{j=0}^{1} \left( y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right) \big|_{S^3},
\]

\[
\Phi_0 \big|_{\ker \kappa_0} = \sum_{j=0}^{1} \left( \frac{\partial}{\partial y_j} \otimes dx_j - \frac{\partial}{\partial x_j} \otimes dy_j \right) \big|_{\ker \kappa_0}, \quad \Phi_0(\xi_0) = 0,
\]

\[
g_0 = \kappa_0 \otimes \kappa_0 + d\kappa_0 \circ (I \otimes \Phi_0).
\]

Then \((\Phi_0, \xi_0, \kappa_0, g_0)\) defines the standard Sasakian structure on \(S^3\). This construction yields the Hopf fibration \(U(1) \hookrightarrow S^3 \rightarrow S^2\). One may obtain a Sasakian structure on the Lens space \(L(p, q)\) by taking the quotient of the standard Sasakian \(S^3\) by the usual Lens space \(\mathbb{Z}_p\) action.

Next, we note that \([9]\) study “CR-Seifert” manifolds. Next we will show that a CR-Seifert structure naturally induces a quasi-regular Sasakian structure and furthermore that the two structures are equivalent. First we review some CR geometry. We begin with the following,
Definition 44. [34] Def. 1.1 and 1.2] An almost CR structure on a manifold $X$, with $\dim X = m$, is a subbundle $T_{(1,0)} = T_{(1,0)}(X) \subset T^C X$ of complex rank $n$ of the complexified tangent bundle such that,

$$T_{(1,0)}(X) \cap T_{(0,1)}(X) = 0,$$

where $T_{(0,1)}(X) := \overline{T_{(1,0)}(X)}$ the complex conjugate. An almost CR structure is called a CR structure if,

$$[T_{(1,0)}(X), T_{(1,0)}(X)] \subset T_{(1,0)}(X),$$

so that $T_{(1,0)}(X)$ is an integrable subbundle of $T^C X$. The integers $n$ and $l = m - 2n$ are called the CR dimension and CR codimension of the almost CR structure and $(n, l)$ denotes its type. The pair $(X, T_{(1,0)})$ is called an (almost) CR manifold of type $(n, l)$.

We are mainly interested in almost CR structures of type $(1, 1)$ in this article.

Definition 45. Let $(X, T_{(1,0)})$ be an (almost) CR manifold of type $(n, k)$. The maximal complex, or Levi distribution is the real rank $2n$ subbundle defined as,

$$L(X) = \mathcal{R}(T_{(1,0)} \oplus T_{(1,0)}).$$

$L(X)$ carries the complex structure $J_L : L(X) \to L(X)$ defined by,

$$J_L(Y + \overline{Y}) = i(Y + \overline{Y}),$$

for any $Y \in T_{(1,0)}$.

As noted in Remark 23 above, given a contact manifold $(X, \kappa)$ with contact distribution $H$, a contact form is naturally viewed as a section of the annihilator bundle $H^0$. Generalizing this to CR manifolds of type $(n, 1)$, which are also called CR manifolds of hypersurface type, we let $H^0$ denote the annihilator bundle of the Levi distribution $H = L(X)$. It is easy to see that $H^0$ is a subbundle of $T^* X$ that is isomorphic to $TX/H$. Assume $X$ is orientable. Then since $H$ is oriented by the complex structure $J_L$, it follows that $H$ is orientable. Any orientable real line bundle over a connected manifold is trivial, so there exist globally defined nowhere vanishing sections $\theta \in \Gamma(H)$.

Definition 46. [34] Def. 1.6] Let $(X, T_{(0,1)})$ be an oriented CR manifold of type $(n, 1)$ with $H = L(X)$. Then any choice of $\theta \in \Gamma(H)$ is referred to as a pseudo-Hermitian structure on $X$. Given a pseudo-Hermitian structure $\theta$ on $X$ the Levi form $L_\theta$ is defined by

$$L_\theta(Z, \overline{W}) = -i d\theta(Z, \overline{W}),$$

for any $Z, W \in T_{(1,0)}$.

We now make the following,

Definition 47. [34] Def. 1.7] Let $(X, T_{(0,1)})$ be an oriented CR manifold of type $(n, 1)$ with $H = L(X)$. We say that $(X, T_{(0,1)})$ is nondegenerate if the Levi form $L_\theta$ is non-degenerate for some (and hence any) choice of pseudo-Hermitian structure $\theta$ on $X$. If $L_\theta$ is positive definite (i.e. $L_\theta(Z, \overline{Z}) > 0$, $\forall 0 \neq Z \in T_{(0,1)}$) for some $\theta$, then $(X, T_{(0,1)})$ is said to be strictly pseudoconvex. Of course, this does not apply to all choices of $\theta$ since $L_\theta$ positive definite implies that $L_{-\theta}$ is negative definite.

Next we observe that one may obtain a natural contact metric structure $(X, \xi, \kappa, \phi, g)$ from a CR structure $(X, T_{(0,1)})$ of type $(n, 1)$ with pseudo-Hermitian structure $\kappa$. First we need the following,
Proposition 48. [34] Prop. 1.2] Given \((X, T_{(0,1)})\) a type \((n, 1)\) CR manifold with pseudo-Hermitian structure \(\kappa\), there exists a unique globally defined nowhere zero tangent vector field \(\xi\) on \(X\) such that,

\[\iota_\xi \kappa = 1, \quad \iota_\xi d\kappa = 0,\]

and \(\xi\) is transverse to the Levi distribution \(H = L(X)\).

We also have,

Proposition 49. [34] Prop. 1.4] Given \((X, T_{(0,1)})\) a type \((n, 1)\) CR manifold with pseudo-Hermitian structure \(\kappa\), Levi distribution \(H = L(X)\) and \(\xi\) as in Prop. 48 above, then,

\[TX \cong H \oplus \mathbb{R}\xi.\]

By setting \(\phi(Y) = J_L Y\) for all \(Y \in H\), and \(\phi \xi = 0\), one can show that \((X, \kappa, \xi, \phi)\) defines an almost contact manifold. If the Levi form \(L_\kappa\) is non-degenerate then \((X, \kappa)\) is a contact manifold. Let,

\[g(Y_1, Y_2) = d\kappa(Y_1, J_L Y_2).\]

Then \(g(J_L Y_1, J_L Y_2) = g(Y_1, Y_2)\) since the Nijenhuis tensor of \(J_L\) vanishes when \(X\) is a CR manifold. We may now extend \(g\) to all of \(TX\) by using the splitting \(TX \cong H \oplus \mathbb{R}\xi\) and defining \(g(Y, \xi) = 0\) and \(g(\xi, \xi) = 1\). The resulting form \(g\) is called the Webster metric of \((X, \kappa)\). If the pseudo-Hermitian structure \(\kappa\) on \(X\) is strictly pseudoconvex, then \(g\) defines a Riemannian metric and \((X, \xi, \kappa, \phi, g)\) defines a contact metric manifold. Consider the following,

Example 50. Let \((X, \kappa, \xi, \phi)\) be an almost contact manifold with distribution \(H = \ker \kappa\). The restriction of \(\phi\) to \(H\) determines the decomposition,

\[H^C = H_{(1,0)} \oplus H_{(0,1)} \subset T^C X,\]

where \(H_{(1,0)}\) and \(H_{(0,1)}\) are the \(+i\) and \(-i\) eigenbundles of \(J := \phi|_H\), respectively. Taking \(T_{(1,0)} X = H_{(1,0)}\) determines an almost CR structure on \(X\). This construction clearly also applies to the case where \((X, \kappa)\) is a contact manifold and \((X, \kappa, \xi, \phi)\) is choice of compatible almost complex structure. Whether or not this construction yields a CR structure (i.e. integrable distribution) has been determined by S. Tanno in [35].

As observed in the above Example 50 a given contact metric manifold \((X, \xi, \kappa, \phi, g)\) does not always induce a (strongly pseudoconvex) CR structure. The case of dimension three is special, however, and we have the following,

Proposition 51. [29] Corollary 6.4] A three-dimensional contact metric manifold is a strongly pseudoconvex CR-manifold.

We will see that that a “CR-Seifert” manifold naturally corresponds to a contact metric structure that is quasi-regular. We make the following,

Definition 52. A CR-Seifert manifold is a three-dimensional compact manifold endowed with both a strictly pseudoconvex CR structure \((H, \phi)\) and a Seifert structure that are compatible in the sense that the circle action \(\psi : \mathbb{U}(1) \to \text{Diff}(X)\) preserves the CR structure and is generated by a Reeb field \(\xi\).

Thus, given a CR-Seifert manifold as in Def. 52 we obtain a natural contact metric structure associated by the above construction. It remains to show that this contact metric structure is indeed quasi-regular Sasakian. By Prop. 39 we see that this contact metric
structure must be $K$-contact since the circle action of the CR-Seifert structure acts by CR-transformations by definition and therefore this structure is Sasakian by [29, Corollary 6.5]. By the results of Thomas [36] we also know that the locally free circle action associated to the Seifert structure on $X$ is equivalent to the Reeb foliation being quasi-regular. Conversely, given a quasi-regular Sasakian structure, it can similarly be shown that one may obtain a CR-Seifert structure such that this correspondence is one-to-one. We have the following,

**Proposition 53.** There is a natural one-to-one correspondence between the CR-Seifert structures and the quasi-regular Sasakian structures on a closed orientable three-manifold $X$.

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