ASYMPTOTIC SPREADING FOR A TIME-PERIODIC PREDATOR-PREY SYSTEM

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Abstract. This paper is concerned with asymptotic spreading for a time-periodic predator-prey system where both species synchronously invade a new habitat. Under two different conditions, we show the bounds of spreading speeds of the predator and the prey, which is proved by the theory of asymptotic spreading of scalar equations, comparison principle and generalized eigenvalue. We show either the predator or the prey has a spreading speed that is determined by the linearized equation at the trivial steady state while the spreading speed of the other also depends on the interspecific nonlinearity. From the viewpoint of population dynamics, our results imply that the predator may play a negative effect on the spreading of the prey while the prey may play a positive role on the spreading of the predator.

1. Introduction. To formulate the invasion process when the initial habitat size of the invader is finite in population dynamics [25, 30], the corresponding asymptotic spreading is an important topic. In particular, asymptotic speed of spreading (for short, spreading speed) is a useful index, which was first defined by Aronson and Weinberger [1] and has been widely studied for monotone semiflows, see Aronson and Weinberger [2], Fang and Zhao [13], Liang et al. [18], Liang and Zhao [19], Lui [24], Weinberger [39, 40], Weinberger et al. [41] and a survey paper by Zhao [46]. For convenience, we first give the following definition.

Definition 1.1. Assume that \( u(x, t) \) is a nonnegative function for \( x \in \mathbb{R}, t > 0 \). Then \( c^* \) is called the spreading speed of \( u(x, t) \) if

(a) \( \lim_{t \to \infty} \sup_{|x| > (c^* + \epsilon)t} u(x, t) = 0 \) for any given \( \epsilon > 0 \),

(b) \( \liminf_{t \to \infty} \inf_{|x| < (c^* - \epsilon)t} u(x, t) > 0 \) for any given \( \epsilon \in (0, c^*) \).

Clearly, the definition of spreading speed states that if an observer were to move to the right or left at a fixed speed greater (less) than \( c^* \), then the local population density \( u(x, t) \) would eventually look like 0 (greater than 0) [41], and it also describes the speed at which the geographic range of the new population expands in population dynamics. In the case of monotone semiflows and cooperative systems,
the works cited above include many important results with applications. When a scalar equation is not monotone due to time delay or discrete, we may refer to [12, 16, 17, 23, 42, 44] and references cited therein. In spatial ecology, the spatial propagation dynamics of predator-prey system has attracted considerable attention [6, 9, 10, 25, 27]. But when the asymptotic spreading in predator-prey systems is involved, the above results do not work, and there are a few results describing the interspecific actions between the prey and the predator. In particular, Lin [20] and Pan [28] studied the asymptotic spreading of the following predator-prey system

\begin{equation}
\begin{aligned}
    &u_t(x,t) = d_1 u_{xx}(x,t) + r_1 u(x,t) [1 - u(x,t) - b_1 v(x,t)], \\
    &v_t(x,t) = d_2 v_{xx}(x,t) + r_2 v(x,t) [1 + b_2 u(x,t) - v(x,t)],
\end{aligned}
\end{equation}

in which \(x \in \mathbb{R}, t > 0\) and all the parameters are positive. They estimated the invasion speeds of \(u, v\) when both \(u, v\) are invaders. Ducrot [8] investigated the spatial propagation for an SIR model which has the same monotonicity as the predator-prey system. Recently, Pan [29] also estimated the asymptotic spreading of a predator-prey system having negative intrinsic growth rate of the predator, which obtained a spreading speed of the predator that equals to the minimal wave speed in Lin [21]. Moreover, Wang [35, 36] and Wang et al. [37] studied the spreading phenomena of (1) with free boundaries and obtained a spreading-vanishing dichotomy.

In this paper, we consider the following time-periodic predator-prey system [22, 38]

\begin{equation}
\begin{aligned}
    &u_t(x,t) = d_1 u_{xx}(x,t) + u(x,t) [r_1(t) - a_1(t) u(x,t) - b_1(t) v(x,t)], \\
    &v_t(x,t) = d_2 v_{xx}(x,t) + v(x,t) [r_2(t) + a_2(t) u(x,t) - b_2(t) v(x,t)],
\end{aligned}
\end{equation}

where \(u(x,t), v(x,t)\) denote the densities for the prey and the predator at time \(t\) and in location \(x\), respectively. Furthermore, \(d_1, d_2 > 0\) are the diffusive rates for the prey and the predator, respectively. The functions \(r_i(t), a_i(t)\) and \(b_i(t), i = 1, 2\) satisfy the following assumptions:

\((A)\) \(r_i(t), a_i(t)\) and \(b_i(t) \in C^\theta(\mathbb{R}, \mathbb{R}), i = 1, 2\) are \(T\)-periodic functions for \(t \in \mathbb{R}\), some \(\theta \in (0, 1)\) and some \(T > 0\). In addition, \(a_1(t) > 0, b_2(t) > 0, a_2(t) \geq 0, b_1(t) \geq 0, t \in [0, T]\) and \(\int_0^T r_i(t) dt > 0, i = 1, 2\).

The purpose of this paper is to investigate the asymptotic spreading that both species synchronously invade a new habitat. So, we shall estimate the spreading speeds of \(u, v\) formulated by the corresponding initial value problem of (2), assuming that the initial values have nonempty compact supports. On the one hand, (2) has a monotone condition similar to that of (1), so we can apply the idea in [20, 28] to study (2). On the other hand, the parameters in (2) depend on \(t\), so there are some significant differences between the study of (1) and (2). For example, we can not obtain the convergence result by dominated converge theorem because of time periodicity. Therefore, we need some techniques different from [20, 28] to show the long time behavior of (2).

In this paper, we obtain the bounds of spreading speed of the predator and the prey in two cases by constructing suitable upper and lower solutions and auxiliary equations, applying the generalized eigenvalue and further combining the theory of asymptotic spreading of scalar equations with comparison principle. To study the long time behavior of solutions, we prove that the solution of (2) converges to the positive \(T\)-periodic solution by the idea in [4, 5]. These results show that either the predator or the prey has a spreading speed that is determined by the linearized
equation
\[ u_t(x, t) = d_1 u_{xx}(x, t) + r_1(t) u(x, t) \quad \text{or} \quad v_t(x, t) = d_2 v_{xx}(x, t) + r_2(t) v(x, t) \]
while the spreading speed of the other species also depends on the interspecific nonlinearity. From the viewpoint of population dynamics, our results imply that the predator may play a negative effect on the spreading of the prey while the prey may play a positive role on the spreading of the predator, which is similar to the phenomena in Fagan and Bishop [11], Owen and Lewis [27].

2. Preliminaries. We first introduce some notations. In this paper, we use the standard partial ordering in \( \mathbb{R}^2 \). Let
\[ X = \{(u(x), v(x)) | (u(x), v(x)) : \mathbb{R} \to \mathbb{R}^2 \text{ is bounded and uniformly continuous}\}. \]
If \( a, b \in \mathbb{R}^2 \) with \( a \leq b \), then we define
\[ X_{[a, b]} = \{(u(x), v(x)) \in X : a \leq (u(x), v(x)) \leq b, x \in \mathbb{R}\}. \]
For any \( T \)-periodic continuous function \( f(t) \), we denote
\[ f_M = \max_{t \in [0, T]} f(t), \quad f_m = \min_{t \in [0, T]} f(t), \quad \overline{f} = \frac{1}{T} \int_0^T f(t) dt. \]
We now give some properties of the corresponding kinetic systems. Let \( U(t) \) be a positive periodic solution of
\[ u'(t) = u(t) [r_1(t) - a_1(t) u(t)]. \]
By (A), \( U(t) \) is globally asymptotically stable with respect to \( \mathbb{R}^+ \) [7]. Similarly, we can write \( V(t) \) as the unique positive periodic solution of
\[ v'(t) = v(t) [r_2(t) + a_2(t) U(t) - b_2(t) v(t)]. \]
Moreover,
\[ \begin{cases} u'(t) = u(t) [r_1(t) - a_1(t) u(t) - b_1(t) v(t)], \\ v'(t) = v(t) [r_2(t) + a_2(t) u(t) - b_2(t) v(t)] \end{cases} \tag{3} \]
admits a trivial solution \((0, 0)\) and two nonnegative semi-trivial periodic solutions \((U(t), 0)\) and \((0, V(t))\). Furthermore, if
\[ \int_0^T [r_1(t) - b_1(t) V(t)] dt > 0, \tag{4} \]
then system (3) is uniformly persistent and has a positive \( T \)-periodic solution \((u^*(t), v^*(t))\) (see [32, 33]). We say that \((u^*(t), v^*(t))\) is asymptotically stable if
\[ \lim_{t \to \infty} |u(t) - u^*(t)| + |v(t) - v^*(t)| = 0 \]
provided that the initial value of (3) satisfies \( u(0) > 0, v(0) > 0 \). To our knowledge, some sufficient conditions on the asymptotic stability of \((u^*(t), v^*(t))\) have been established. For example, Teng and Chen [34, Theorem 1] implies that \((u^*(t), v^*(t))\) is asymptotically stable if (4) holds and there exist positive constants \( l_1, l_2 \) such that
\[ l_1 a_1(t) \geq 1, \neq l_2 a_2(t), l_2 b_2(t) \geq 1, \neq l_1 b_1(t), t \in [0, T]. \]
There are also some other sufficient conditions on the asymptotic stability and uniqueness of positive periodic solution of (3) and we shall not focus on precise conditions in this paper.
In order to obtain the spreading speeds of predator-prey system described by (2), we consider the following initial value problem

\[
\begin{aligned}
&u_t(x,t) = d_1 u_{xx}(x,t) + u(x,t)[r_1(t) - a_1(t)u(x,t) - b_1(t)v(x,t)], \quad x \in \mathbb{R}, \ t > 0, \\
&v_t(x,t) = d_2 v_{xx}(x,t) + v(x,t)[r_2(t) + a_2(t)u(x,t) - b_2(t)v(x,t)], \quad x \in \mathbb{R}, \ t > 0, \\
&(u(x,0), v(x,0)) = (u(x), v(x)) \in X_{[0,K]}, \ x \in \mathbb{R},
\end{aligned}
\]

in which \(u(x), v(x)\) admit nonempty compact supports and \(K = (U(0), V(0))\).

Obviously, (5) can be analysed by the classical theory of reaction-diffusion systems [14, 43]. Following Fife and Tang [14, Definition 4 and Remark 2], Zhao and Ruan [45], we introduce an admissible pair of irregular upper and lower solutions of (5) as follows, which will be called upper and lower solutions for the sake of convenience.

**Definition 2.1.** Assume that

\[
\begin{aligned}
\bar{\pi}(x,t) &= \min \{ \bar{\pi}_1(x,t), \ldots, \bar{\pi}_n(x,t) \}, \quad \bar{\eta}(x,t) = \min \{ \bar{\eta}_1(x,t), \ldots, \bar{\eta}_n(x,t) \}, \\
\underline{\pi}(x,t) &= \max \{ \underline{\pi}_1(x,t), \ldots, \underline{\pi}_n(x,t) \}, \quad \underline{\eta}(x,t) = \max \{ \underline{\eta}_1(x,t), \ldots, \underline{\eta}_n(x,t) \},
\end{aligned}
\]

where \(\bar{\pi}_i, \underline{\pi}_i, \bar{\eta}_i, \underline{\eta}_i \in C^{2,1}(\mathbb{R} \times [0,T'), \mathbb{R}^+)\), \(i = 1, \ldots, n\) with \(T' > 0\). Further suppose that these continuous functions satisfy

(i) if \(\bar{\pi}(x,t) = \bar{\pi}_i(x,t)\) for some \(i \in \{1, \ldots, n\}\), then

\[
\bar{\pi}_{i,x}(x,t) \geq d_1 \bar{\pi}_{i,xx}(x,t) + \bar{\pi}_{i}(x,t)[r_1(t) - a_1(t)\bar{\pi}_i(x,t) - b_1(t)\bar{\eta}_i(x,t)];
\]

(ii) if \(\bar{\pi}(x,t) = \bar{\eta}_i(x,t)\) for some \(i \in \{1, \ldots, n\}\), then

\[
\bar{\eta}_{i,x}(x,t) \geq d_2 \bar{\eta}_{i,xx}(x,t) + \bar{\pi}_{i}(x,t)[r_2(t) + a_2(t)\bar{\pi}_i(x,t) - b_2(t)\bar{\eta}_i(x,t)];
\]

(iii) if \(\underline{\pi}(x,t) = \underline{\pi}_i(x,t)\) for some \(i \in \{1, \ldots, n\}\), then

\[
\underline{\pi}_{i,x}(x,t) \leq d_1 \underline{\pi}_{i,xx}(x,t) + \underline{\pi}_{i}(x,t)[r_1(t) - a_1(t)\underline{\pi}_i(x,t) - b_1(t)\underline{\eta}_i(x,t)];
\]

(iv) if \(\underline{\pi}(x,t) = \underline{\eta}_i(x,t)\) for some \(i \in \{1, \ldots, n\}\), then

\[
\underline{\eta}_{i,x}(x,t) \leq d_2 \underline{\eta}_{i,xx}(x,t) + \underline{\pi}_{i}(x,t)[r_2(t) + a_2(t)\underline{\pi}_i(x,t) - b_2(t)\underline{\eta}_i(x,t)];
\]

(v) the initial value functions of (5) satisfy

\[
(0,0) \leq (\underline{\pi}(x,0), \underline{\eta}(x,0)) \leq (u(x), v(x)) \leq (\bar{\pi}(x,0), \bar{\eta}(x,0))
\]

for \(x \in \mathbb{R}\).

Then \((\bar{\pi}, \bar{\eta}), (\underline{\pi}, \underline{\eta})\) are said to be a pair of upper and lower solutions of (5), respectively.

**Lemma 2.2.** (5) has a bounded solution \((u(x,t), v(x,t))\) for \(x \in \mathbb{R}, t > 0\) such that

\[
(0,0) \leq (u(x,t), v(x,t)) \leq (U(t), V(t)), \quad x \in \mathbb{R}, t > 0.
\]

Assume that \((\bar{\pi}, \bar{\eta}), (\underline{\pi}, \underline{\eta})\) are a pair of upper and lower solutions of (5) with \(x \in \mathbb{R}, t \in [0,T')\). Then

\[
(u(x,t), v(x,t)) \leq (U(x,t), V(x,t)) \leq (\bar{\pi}(x,t), \bar{\eta}(x,t)), \quad x \in \mathbb{R}, t \in [0,T').
\]

Lemma 2.2 is clear by the classical theory of reaction-diffusion systems (see Smoller [31] and Ye et al. [43]) and the proof is omitted here.

We recall the following time-periodic Fisher-KPP equation

\[
\begin{aligned}
z_t(x,t) &= dz_{xx}(x,t) + z(x,t)[r(t) - a(t)z(x,t)], \quad x \in \mathbb{R}, t > 0, \\
z(x,0) &= z_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]
where \( d > 0, r(t), a(t) \in C^0(\mathbb{R}, \mathbb{R}) \) for some \( \theta \in (0, 1) \) are \( T \)-periodic. Moreover, \( \tau > 0, a(t) > 0, z_0(x) \geq 0 \). Under these conditions, the following lemma on asymptotic spreading is true by Berestycki et al. [3], Liang et al. [18], Nadin [26], Zhao [47, Theorem 3.1.2].

**Lemma 2.3.** (a) The unique \( T \)-periodic solution \( z^*(t) \) is globally asymptotically stable and attracts all solutions with positive initial value, locally uniformly in \( x \).

(b) Assume that \( z(x,t) \) is a solution of \( (6) \) and \( z_0(x) \) admits a nonempty compact support. Then

\[
\lim_{t \to \infty} \sup_{|x| < (2\sqrt{d} - \varepsilon) t} |z(x,t) - z^*(t)| = 0, \quad \lim_{t \to \infty} \sup_{|x| > (2\sqrt{d} + \varepsilon) t} z(x,t) = 0
\]

for any \( \varepsilon \in \left(0, 2\sqrt{d}\right) \).

**Remark 1.** Assume that \( r > 0, \sigma > 0 \) are fixed and \( z(x,t) \) is a solution of \( (6) \). If \( z(x) > \sigma \) for \( |x - x_0| \leq r \) with some \( x_0 \in \mathbb{R} \), then for any given \( \varepsilon > 0 \), there is a constant \( T_0 = T_0(\varepsilon) \) such that \( z(x_0,t) > z^*(t) - \varepsilon, \ t > T_0 \).

Consider the following initial value problem

\[
\begin{cases}
z_t(x,t) = dzz_x(x,t) + f(x,t,z), & x \in \mathbb{R}, t > 0, \\
z(x,0) = z(x), & x \in \mathbb{R},
\end{cases}
\]

where the function \( f \) satisfies

1. \( f \) is of class \( C^{5,5/2} \) in \( (x,t) \), locally in \( z \) for a given \( \delta \in (0,1) \),
2. \( f \) is locally Lipschitz continuous in \( z \), and of class \( C^1 \) in \( z \in [0, \beta] \) with \( \beta > 0 \) uniformly with respect to \( (x,t) \in \mathbb{R} \times \mathbb{R}^+ \),
3. \( f(x,t,0) = 0 \) and there exists \( \tau > 0 \) such that \( f_z'(x,t,z)|_{z=0} > \tau \) for all \((x,t) \in \mathbb{R} \times \mathbb{R}^+ \),
4. there exists \( K > 0 \) such that \( f(x,t,z) < 0 \) for \( z > K, (x,t) \in \mathbb{R} \times \mathbb{R}^+ \).

Applying generalized eigenvalue, Berestycki et al. [3] proved the following result.

**Lemma 2.4.** Let \( z(x,t) \) be the solution of \( (7) \) with \( z(x) > 0 \). Assume that there exists \( \tilde{r}(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) such that

\[
\lim_{R \to \infty} \left\{ \liminf_{t \to \infty} \left\{ \inf_{|z| \leq R} \left[ 4df_x^2(x \pm \tilde{r}(t), t, z)|_{z=0} - (r'(t))^2 \right] \right\} \right\} > 0.
\]

Then

\[
\lim_{t \to \infty} \inf_{|z| \leq \tilde{r}(t)} z(x,t) > 0.
\]

3. **Main results.** In this section, we shall study system \( (5) \) in two cases. The first case is that \( v \) is stronger than \( u \) in the sense of \( d_0 \bar{r}_2 > d_1 \bar{r}_1 \), and the second one is that \( u \) is stronger than \( v \) in the sense of \( d_1 (\bar{r}_1 - b_1 \bar{V}) > d_2 (\bar{r}_2 + a_2 \bar{U}) \).

Now, we define some constants

\[
c_1 = 2\sqrt{d_1 \bar{r}_1}, \quad c_2 = 2\sqrt{d_2 \bar{r}_2}, \quad c_3 = 2\sqrt{d_1 (\bar{r}_1 - b_1 \bar{V})}, \\
c_4 = 2\sqrt{d_1 (\bar{r}_1 - b_1 \bar{V})}, \quad c_5 = 2\sqrt{d_2 (\bar{r}_2 + a_2 \bar{U})},
\]

where \( U(t), V(t), \bar{V}(t) \) are defined in Section 2.

Assume that \( (4) \) holds. Let \( \tilde{U}(t) \) be the unique positive \( T \)-periodic solution of

\[
u'(t) = u(t)r_1(t) - b_1(t)V(t) - a_1(t)u(t).
\]
3.1. **Case I:** $d_2\overline{r}_2 > d_1\overline{r}_1$. Throughout this subsection, we assume that
\[ d_2\overline{r}_2 > d_1\overline{r}_1, \tag{8} \]
then
\[ c_2 > c_1 > c_3 > c_4. \]
Under the assumption, we shall estimate the spreading speeds of the predator and the prey, which is motivated by Lin [20].

**Theorem 3.1.** Assume that (4) and (8) hold. Then $(u(x, t), v(x, t)), (x, t) \in \mathbb{R} \times (0, \infty)$ defined by (5) satisfies

(i) for any given $\epsilon > 0 \lim_{t \to \infty \mid x > (c_1 + \epsilon)t} u(x, t) = \lim_{t \to \infty \mid x > (c_2 + \epsilon)t} v(x, t) = 0$;

(ii) for any given $\epsilon \in (0, c_2) \liminf_{t \to \infty \mid x < (c_2 - \epsilon)t} [v(x, t) - V(t)] \geq 0$;

(iii) if $c_2 > c_1 + c_3$, then $\lim_{t \to \infty \mid x > (c_1 + \epsilon)t} u(x, t) = 0$ for any given $\epsilon > 0$;

(iv) if $(u^*(t), v^*(t))$ is globally asymptotically stable, then
\[ \lim_{t \to \infty \mid x < (c_4 - \epsilon)t} (|u(x, t) - u^*(t)| + |v(x, t) - v^*(t)|) = 0 \]
for any given $\epsilon \in (0, c_4)$.

**Remark 2.** In view of Lemma 2.3 and Theorem 3.1, we can conclude that

(1) under the assumption (8), the spreading speed of the predator does not change by introducing the prey;

(2) if (8) holds, then the spreading speed of the prey may be slowed by introducing the predator.

In what follows, we shall give several lemmas to prove Theorem 3.1, throughout which the conditions of Theorem 3.1 hold.

**Lemma 3.2.** For any given $\epsilon > 0$, $u(x, t)$ and $v(x, t)$ satisfy
\[ \lim_{t \to \infty \mid x > (c_2 + \epsilon)t} u(x, t) = \lim_{t \to \infty \mid x > (c_2 + \epsilon)t} v(x, t) = 0. \]

**Proof.** Let $c = c_2 + \frac{5}{2}$. We define constants
\[ \gamma_{11} = \frac{c - \sqrt{c^2 - 4d_i\overline{r}_i}}{2d_i}, \quad \gamma_{12} = \frac{c + \sqrt{c^2 - 4d_i\overline{r}_i}}{2d_i}, \quad i = 1, 2, \]
then $0 < \gamma_{11} < \gamma_{12}, 0 < \gamma_{21} < \gamma_{22}$. Select a constant $\eta > 1$ such that
\[ \eta \in \left(1, \min_{1 \leq i, j \leq 2} \left\{ \frac{\gamma_{12}}{\gamma_{11}}, \frac{\gamma_{21} + \gamma_{11}}{\gamma_{11}} \right\} \right), \tag{9} \]
We further define the following continuous functions
\[ \varphi_1(t) = e^{\gamma_{11}[d_1\gamma_{11} - c\gamma_{11} + r_1(s)]ds}, \quad \varphi_2(t) = e^{\gamma_{11}[d_2\gamma_{11}^2 - c\gamma_{21} + r_2(s)]ds}, \quad t \geq 0, \tag{10} \]
and
\[ \bar{u}(x, t) = \min \left\{ e^{\gamma_{11}(x + ct + t_1)}\varphi_1(t), e^{\gamma_{11}(-x + ct + t_1)}\varphi_1(t), U(t) \right\}, \]
\[ \bar{v}(x, t) = \min \left\{ e^{\gamma_{21}(x + ct + t_1)} + qe^{\gamma_{21}(x + ct + t_1)}\varphi_2(t), \right. \]
\[ \left. e^{\gamma_{21}(-x + ct + t_1)} + qe^{\gamma_{21}(-x + ct + t_1)}\varphi_2(t), V(t) \right\}, \]
\[ u(x, t) = 0, \quad v(x, t) = 0, \]
where \( t_1 > 0 \) is sufficiently large such that \( \bar{u}(x,0) \geq u(x), \bar{v}(x,0) \geq v(x), q = \max(q_1, q_2) \) defined by

\[
q_1 = \frac{2(a_2 \varphi_1)_M}{-\left[d_2(\eta \gamma_{21})^2 - c \eta \gamma_{21} + \nu_2\right]} + \frac{V_M}{(\varphi_2)_m},
\]
\[
q_2 = \left(\frac{2(a_2 \varphi_1)_M}{-\left[d_2(\eta \gamma_{21})^2 - c \eta \gamma_{21} + \nu_2\right]}\right) \frac{\gamma_{21}^{21}}{(\varphi_2)_m} + \frac{V_m}{(\varphi_2)_m}.
\]

Since

\[
\lim_{t \to \infty} \sup_{|x| > (c_2 + t)} \bar{u}(x,t) = \lim_{t \to \infty} \sup_{|x| > (c_2 + t)} \bar{v}(x,t) = 0,
\]

then our conclusion holds by Lemma 2.2 if \((\bar{u}, \bar{v}), (\underline{u}, \underline{v})\) are a pair of upper and lower solutions of system (5). Now, we verify the inequalities in Definition 2.1, and the inequalities on \( \bar{u}(x,t), \underline{v}(x,t) \) are clear.

(1) (i) If \( \varpi(x,t) = U(t) \), then

\[
d_1 \varpi_{xx} + \varpi[r_1(t) - a_1(t)\varpi - b_1(t)\underline{v}] - \varpi_t \leq U(t)[r_1(t) - a_1(t)U(t)] - \frac{dU(t)}{dt} = 0.
\]

(ii) If \( \varpi(x,t) = e^{\gamma_{11}(x + ct + t_1)} \varphi_1(t) \), then

\[
d_1 \varpi_{xx} + \varpi[r_1(t) - a_1(t)\varpi - b_1(t)\underline{v}] - \varpi_t \leq d_1 \varpi_{xx} + r_1(t)\varpi - \varpi_t = d_1 \gamma_{11}^2 \varpi + r_1(t)\varpi - [d_1 \gamma_{11}^2 + r_1(t)]\varpi = 0.
\]

(iii) Similarly, the inequality still holds if \( \varpi(x,t) = e^{\gamma_{11}(-x + ct + t_1)} \varphi_1(t) \).

(2) (i) If \( \varpi(x,t) = V(t) \), then \( \varpi(x,t) \leq U(t) \) such that

\[
d_2 \varpi_{xx} + \varpi[r_2(t) + a_2(t)\varpi - b_2(t)\bar{v}] - \varpi_t \leq V(t)[r_2(t) + a_2(t)U(t) - b_2(t)V(t)] - \frac{dV(t)}{dt} = 0.
\]

(ii) If \( \varpi(x,t) = [e^{\gamma_{21}(x + ct + t_1)} + q e^{\gamma_{21}(x + ct + t_1) + \phi_2(t)}] \varphi_2(t) \leq V(t) \), then

\[
\varpi(x,t) \leq e^{\gamma_{11}(x + ct + t_1)} \varphi_1(t), \quad q e^{\gamma_{21}(x + ct + t_1)} \varphi_2(t) \leq V(t)
\]

imply that \( x + ct + t_1 < 0 \) by \( q \geq V_M / \varphi_2.m \) such that

\[
d_2 \varpi_{xx} + \varpi[r_2(t) + a_2(t)\varpi - b_2(t)\bar{v}] - \varpi_t \leq d_2 \varpi_{xx} + \varpi[r_2(t) + a_2(t)\varpi] - \varpi_t \\
\leq (d_2(\eta \gamma_{21})^2 - c \eta \gamma_{21} + \nu_2) q \varphi_2(t) e^{\gamma_{21}(x + ct + t_1)} \\
+ a_2(t) \varphi_1(t) \varphi_2(t) \left(e^{(\gamma_{11} + \gamma_{21})(x + ct + t_1)} + q e^{(\gamma_{11} + \gamma_{21})(x + ct + t_1)} \right) \\
\leq 0
\]

provided that

\[
(d_2(\eta \gamma_{21})^2 - c \eta \gamma_{21} + \nu_2) q e^{\gamma_{21}(x + ct + t_1)} + 2a_2(t) \varphi_1(t) e^{(\gamma_{11} + \gamma_{21})(x + ct + t_1)} \leq 0 \tag{11}
\]

and

\[
(d_2(\eta \gamma_{21})^2 - c \eta \gamma_{21} + \nu_2) + 2a_2(t) \varphi_1(t) e^{\gamma_{11}(x + ct + t_1)} \leq 0. \tag{12}
\]

For (11), \( e^{\gamma_{21}(x + ct + t_1)} \geq e^{(\gamma_{11} + \gamma_{21})(x + ct + t_1)} \) for \( x + ct + t_1 < 0 \) by (9). Therefore, \( q \geq q_1 \) implies (11).
For (12), \( q e^{\gamma_2 t(x+ct+t_1)} \varphi_2(t) \leq V(t) \) such that \( (x+ct+t_1) \leq \frac{1}{q \gamma_2} \ln \left( \frac{\sqrt{m}}{\sqrt{q \gamma_2} M} \right) \).

Therefore, (12) is true since \( q \geq q_2 \).

(iii) Similarly, we may verify the inequality if

\[
\tau(x, t) = \left[ e^{\gamma_2 t(-x+ct+t_1)} + q e^{\gamma_2 t(-x+ct+t_1)} \right] \varphi_2(t).
\]

This completes the proof. \( \square \)

**Lemma 3.3.** For any given \( \epsilon > 0 \), \( u(x, t) \) satisfies \( \lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon) t} u(x, t) = 0. \)

**Proof.** Define continuous functions

\[
u(x, t) = v(x, t) = 0, \quad \tau(x, t) = V(t), \quad \tau(x, t) = \min \{ U(t), \ V \left( e^{\frac{\epsilon}{\gamma_2}(\pm x + ct) + t^\prime} \varphi_1(t) \right) \},
\]

where \( t^\prime > 0 \) is large enough such that \( u(x) \leq \tau(x, 0), \ \varphi_1(t) \) is defined by (10).

Similar to the proof of Lemma 3.2, we can easily verify that \( (\tau, \nu) \), \( (u, v) \) are a pair of upper and lower solutions of system (5).

Notice that

\[
\lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon) t} \tau(x, t) = \lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon) t} u(x, t) = 0.
\]

Then Lemma 2.2 implies the desirable limit. This completes the proof. \( \square \)

**Lemma 3.4.** For any given \( \epsilon \in (0, c_2) \), we have \( \liminf_{t \to \infty} \inf_{|x| < (c_2 - \epsilon) t} [w(x, t) - V(t)] \geq 0. \)

**Proof.** Clearly, \( v(x, t) \) is an upper solution of the initial value problem

\[
\begin{align*}
w_1(x, t) &= d_2 x w_x(x, t) + w(x, t) [r_2(t) - b_2(t) w(x, t)], \\
w(0, t) &= v(0) \in [0, V(0)]
\end{align*}
\]

By the comparison principle and Lemma 2.3, we obtain that

\[
\liminf_{t \to \infty} \inf_{|x| < (c_2 - \epsilon) t} [w(x, t) - V(t)] \geq \liminf_{t \to \infty} \inf_{|x| < (c_2 - \epsilon) t} [w(x, t) - V(t)] = 0.
\]

This completes the proof. \( \square \)

**Lemma 3.5.** If \( c_2 > c_1 + c_3 \) holds, then \( \lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon) t} u(x, t) = 0 \) for any given \( \epsilon > 0 \).

**Proof.** It suffices to consider the case \( c_2 - \epsilon > c_1 + c_4 \). Let \( \delta > 0 \) be small enough such that

\[
(c_3 + \epsilon/4)^2 = 4d_1 \left[ r_1 - \frac{b_1}{2} (V - \delta) \right].
\]

Define a positive constant

\[
M := \frac{2d_1}{c_1 - (c_3 + \epsilon/4)} \int_0^T \left( \left| b_1(s)V(s) - \frac{b_1}{2} \right| + \left| b_1(s) - \frac{b_1}{2} \right| \right) ds + 1,
\]

then the proof of Lemma 3.4 implies that for any given \( \delta \in (0, 1) \), there exists a \( T_1 > 0 \) such that

(a) the solution \( w(x, t) \) of (13) satisfies

\[
\inf_{|x| < (c_2 - \epsilon) t} [w(x, t) - (V(t) - \delta)] \geq 0, \quad t > T_1,
\]

(b) \( (c_2 - c_1 - c_3 - \frac{3\epsilon}{4}) T_1 > M. \)
We define constants
\[ \gamma_3 = \frac{c_1}{2d_1}, \quad \gamma_4 = \frac{c_3 + \epsilon/4}{2d_1} \]
and continuous functions
\[ \varphi_3(t) = e^{\int_0^t \left( r_1(s) - b_1(s) - \frac{\epsilon}{4} \right) ds}, \]
\[ u(x, t) = 0, \quad v(x, t) = w(x, t), \quad \varpi(x, t) = V(t), \]
\[ \varpi(x, t) = \min \left\{ U(t), e^{\gamma_4(x + c_1) t + 2\varphi_1(t)}, e^{\gamma_4(x + (c_3 + \frac{\epsilon}{4}) t) + 2\varphi_3(t)} \right\}, \]
where \( w(x, t) \) satisfies (13), \( \varphi_1(t) \) is given by (10) and \( t_2 > 0 \) is large such that \( \varpi(x, T_1) \geq u(x, T_1) \) for all \( x \in \mathbb{R} \), here \( t_2 \) is admissible by Lemmas 3.2-3.3.

Notice that
\[ \lim_{t \to \infty} \sup_{|x| > (c_3 + \epsilon/4) t} \varpi(x, t) = 0. \]
If \( (\varpi, \varpi) \), \( (u, v) \) are a pair of upper and lower solutions of system (5), then our conclusion by Lemma 2.2.

Next, we shall show that \( (\varpi, \varpi) \) and \( (u, v) \) are a pair of upper and lower solutions of (5) for \( x \in \mathbb{R}, t > T_1 \). In view of the proof of Lemmas 3.2-3.3, \( u, v, \varpi \) satisfy Definition 2.1, and it suffices to verify the inequality of \( \varpi \).

1. If \( \varpi(x, t) = U(t) \), then the proof is the same as that of Lemma 3.2.
2. If \( \varpi(x, t) = e^{\gamma_4(x + c_1) t + \varphi_1(t)} \), then the verification is similar to that in Lemma 3.2.
3. If \( \varpi(x, t) = e^{\gamma_4(x + (c_3 + \epsilon/4) t) + \varphi_3(t)} \), then
\[ e^{\gamma_4(x + (c_3 + \epsilon/4) t) + \varphi_3(t)} \leq e^{\gamma_4(x + c_1) t + \varphi_1(t)} = e^{\gamma_4(x + c_1) t + t_2 e^{\int_0^t r_1(s) - \varpi ds}} \]
yields
\[ -x \leq \left( c_1 + c_3 + \frac{\epsilon}{4} \right) t + \frac{2d_1}{c_1 - (c_3 + \epsilon/4)} \int_0^t \left[ b_1(s)(V(s) - \delta) - b_1(V(s) - \delta) \right] ds. \]
Since \( b_1(t), V(t) \) are continuous and \( T \)-periodic, then
\[ \left| \frac{2d_1}{c_1 - (c_3 + \epsilon/4)} \int_0^t \left[ b_1(s)(V(s) - \delta) - b_1(V(s) - \delta) \right] ds \right| < M. \]
Furthermore, it follows from the item (b) that
\[ -x < (c_2 - \epsilon/2) t \]
for any fixed \( \epsilon > 0 \) small enough and \( t > T_1 \), which indicates that \( v(x, t) \) satisfies the item (a) for \( -x < (c_2 - \epsilon/2) t, t > T_1 \).

Due to the positivity of \( v(x, t) \) and the item (a), we obtain that for any \( t > T_1 \),
\[ d_1 \varpi_{xx} + \varpi \left( r_1(t) - a_1(t) \varpi - b_1(t) v \right) - \varpi_t \]
\[ \leq d_1 \varpi_{xx} + \varpi \left( r_1(t) - b_1(t) \left( V(t) - \delta \right) \right) - \varpi_t \]
\[ \leq d_1 \gamma_2^2 \varpi + \varpi \left( r_1(t) - b_1(s)(V(s) - \delta) \right) - \varpi \left[ \gamma_4 (c_3 + \epsilon/4) + r_1(t) - b_1(s)(V(s) - \delta) - \left( \varpi - b_1(V(s) - \delta) \right) \right] \]
\[ = 0. \]
If \( \varpi(x, t) = \min \left\{ e^{\gamma_4(-x + c_1) t + \varphi_1(t)}, e^{\gamma_4(-x + (c_3 + \epsilon/4) t) + \varphi_3(t)}, U(t) \right\} \), then the proof is similar to what we have done. This completes the proof. \qed
Lemma 3.6. For any given $\varepsilon \in (0, c_4)$, we have $\liminf_{t \to \infty} \inf_{|x| < (c_4 - \varepsilon)t} [u(x, t) - U(t)] \geq 0$.

Proof. Recalling the upper and lower solutions of (5) in Lemma 3.2, $u(x, t)$ satisfies

$$\begin{cases}
u_t(x, t) \geq d_1u_{xx}(x, t) + u(x, t)[r_1(t) - b_1(t)V(t) - a_1(t)u(x, t)], \\
u(x, 0) = u(x) \in [0, U(0)].
\end{cases}$$

By the comparison principle and Lemma 2.3, we have

$$\liminf_{t \to \infty} \inf_{|x| < (c_4 - \varepsilon)t} [u(x, t) - U(t)] \geq 0.$$

This completes the proof.

To prove the convergence of $(u(x, t), v(x, t))$, we first prove the following result by the idea in [4, 5].

Lemma 3.7. Assume that $(u^*(t), v^*(t))$ is asymptotically stable. Let $\sigma > 0$ be a fixed constant. For any $\varepsilon > 0$, there exist $r = r(\varepsilon, \sigma)$ and $T_0 = T_0(\varepsilon, \sigma)$ such that if $(u(x), v(x)) \geq (\sigma, \sigma), |x - x_0| < r$ for any given $x_0 \in \mathbb{R}$, then $(|u(x_0, t) - u^*(t)|, |v(x_0, t) - v^*(t)|) \leq (\varepsilon, \varepsilon)$ with $t \in [T_0, T_0 + T]$.

Proof. We write

$$f_1(t, u, v) := u[r_1(t) - a_1(t)u - b_1(t)v], \quad f_2(t, u, v) := v[r_2(t) + a_2(t)u - b_2(t)v].$$

Without loss of generality, we consider the case $x_0 = 0$ in the following proof. The asymptotic stability implies that $(u^*(t), v^*(t))$ satisfies

$$\limsup_{t \to \infty} x \in \mathbb{R} (|w_1(x, t) - u^*(t)| + |w_2(x, t) - v^*(t)|) = 0, \quad (14)$$

where $w_i(x, t)$ is the solution of

$$\begin{cases}
w_{i,t}(x, t) = d_1w_{i,xx}(x, t) + f_i(t, w_1, w_2), x \in \mathbb{R}, t > 0, \\
w_i(x, 0) = w_i(x), x \in \mathbb{R},
\end{cases}$$

with

1. $\omega_i(x) \geq \sigma, x \in \mathbb{R}, i = 1, 2$,
2. if $|x| \leq r$ with $r > 0$ clarified later, then $(w_1(x), w_2(x)) = (u(x), v(x))$.

For any $\varepsilon > 0$, (14) implies that there exists $T_0 > 0$ such that

$$\sup_{x \in \mathbb{R}} (|w_1(x, t) - u^*(t)| + |w_2(x, t) - v^*(t)|) < \varepsilon/2, t \geq T_0. \quad (15)$$

Consider (5) for $t \in [0, T_0 + T]$, it suffices to show that there exists a large number $r = r(\varepsilon, \sigma) > 0$ such that

$$\sup_{t \in [T_0, T_0 + T]} (|u(0, t) - w_1(0, t)| + |v(0, t) - w_2(0, t)|) < \varepsilon/2 \quad (16)$$

provided that $(u(x), v(x)) \geq (\sigma, \sigma)$ for $|x| < r$.

Let

$$\varphi_1(x, t) := u(x, t) - w_1(x, t), \varphi_2(x, t) := v(x, t) - w_2(x, t),$$

then

$$\begin{cases}
\varphi_{i,t}(x, t) = d_1\varphi_{i,xx}(x, t) + h_i(x, t), x \in \mathbb{R}, t > 0, \\
\varphi_1(x, 0) = \varphi_2(x) = u(x) - w_1(x), x \in \mathbb{R}, \\
\varphi_2(x, 0) = \varphi_2(x) = v(x) - w_2(x), x \in \mathbb{R},
\end{cases} \quad (17)$$
where \( h_i(x,t) := f_i(t,u,v) - f_i(t,w_1,w_2) \) and \( \varphi_i(x,0) = 0 \) for \( |x| < r, i = 1, 2 \). Due to the Lipschitz continuity of \( h_i(x,t) \), there exists a constant \( L > 0 \) such that
\[
-L(|\varphi_1| + |\varphi_2|) \leq h_i(x,t) \leq L(|\varphi_1| + |\varphi_2|) := H(\varphi_1,\varphi_2)
\]
for \( i = 1, 2, t > 0, x \in \mathbb{R} \).

Because of the uniform boundedness of \( u, v \) and \( w_1, \varphi_i(x,t) \) is bounded and well defined for all \( t > 0, x \in \mathbb{R}, i = 1, 2 \). To estimate \( (\varphi_1, \varphi_2) \), we introduce a pair of generalized upper and lower solutions \((\varphi_1, \varphi_2)\) and \((\bar{\varphi}_1, \bar{\varphi}_2)\) for (17), which satisfy

\[
\begin{align*}
\varphi_i(x,t) &\geq d_i \varphi_i,xx(x,t) + H(\varphi_1, \varphi_2), x \in \mathbb{R}, t > 0, \\
\varphi_i(x,0) &\geq |u(x) - w_1(x)| + |v(x) - w_2(x)|, x \in \mathbb{R}
\end{align*}
\]

and

\[
\begin{align*}
\bar{\varphi}_i(x,t) &\leq d_i \bar{\varphi}_i,xx(x,t) - H(\varphi_1, \varphi_2), x \in \mathbb{R}, t > 0, \\
\bar{\varphi}_i(x,0) &\leq -|u(x) - w_1(x)| - |v(x) - w_2(x)|, x \in \mathbb{R}
\end{align*}
\]

Let \( c > 0 \) and \( \lambda > 0 \) be constants such that \( c\lambda - d_i\lambda^2 \geq 2L \) for all \( i = 1, 2 \). We define continuous functions

\[
\begin{align*}
\varphi_i(x,t) &:= \min \left\{ K e^{2Lt}, K e^{\lambda(\pm x - 2r + ct)} \right\}, i \in \{1, 2\}, \\
\bar{\varphi}_i(x,t) &:= \max \left\{ -K e^{2Lt}, -K e^{\lambda(\pm x - 2r + ct)} \right\}, i \in \{1, 2\}
\end{align*}
\]

for \( t \geq 0, x \in \mathbb{R}, K > 0 \) (that is independent of \( r \)) such that
\[
|u(x) - w_1(x)| + |v(x) - w_2(x)| \leq \min \left\{ K, K e^{\lambda(\pm x - 2r)} \right\}, x \in \mathbb{R}.
\]

By directly calculating, we can prove that these continuous functions \((\varphi_1, \varphi_2)\), \((\bar{\varphi}_1, \bar{\varphi}_2)\) satisfy (18)-(19) and are a pair of generalized upper and lower solutions for (17). Due to the classical theory of reaction-diffusion system (see [31, 43]), we obtain
\[
\varphi_i(x,t) \leq \varphi_i(x,t) \leq \varphi_i(x,t), t > 0, x \in \mathbb{R}, i = 1, 2.
\]

Let \( r > 0 \) be sufficiently large such that
\[
K e^{\lambda(-r+\epsilon(T_0+T))} < \varepsilon/4.
\]

Then we obtain
\[
\sup_{t \in [T_0, T_0 + T]} (|u(0,t) - w_1(0,t)| + |v(0,t) - w_2(0,t)|) < \varepsilon/2.
\]

Therefore, with (15), we prove that \((|u(0,t) - u^*(t)|, |v(0,t) - v^*(t)|) \leq (\varepsilon, \varepsilon)\) with \((u(x), v(x)) \geq (\sigma, \sigma)\) for \( |x| < r \) and \( t \in [T_0, T_0 + T] \). This completes the proof. \( \square \)

**Lemma 3.8.** Assume that (4) and (8) hold. If \((u^*(t), v^*(t))\) is asymptotically stable, then
\[
\lim_{t \to \infty, |x| < (c_4 - \epsilon)t} (|u(x,t) - u^*(t)| + |v(x,t) - v^*(t)|) = 0
\]
for any \( \epsilon \in (0, c_4) \).

**Proof.** Since \( c_2 > c_4 \), for any given \( \epsilon \in (0, c_4) \), Lemmas 3.4 and 3.6 imply that
\[
\begin{align*}
\liminf_{t \to \infty} \liminf_{|x| < (c_4 - \epsilon)t} |u(x,t) - U(t)| &\geq 0, \\
\liminf_{t \to \infty} \liminf_{|x| < (c_4 - \epsilon)t} |v(x,t) - V(t)| &\geq 0.
\end{align*}
\]
Define $\sigma_1 = \min\{U_m/2, V_m/2\} > 0$. It is evident that there exists a constant $T_2 > 0$ such that $(u(x, t), v(x, t)) \geq (\sigma_1, \sigma_1)$ for any $t \geq T_2$ and $|x| < (c_4 - \epsilon/2)t$. According to Lemma 3.7, for any $\epsilon_1 > 0$, there exist $T_3 > 0$ and $r_1 > 0$ such that
\[
(\|u(x, t) - u^*(t)\|, \|v(x, t) - v^*(t)\|) \leq (\epsilon_1/2, \epsilon_1/2)
\] (20)
for all $t \in [t_3 + T_3, t_3 + T_3 + T]$ and $|x| < (c_4 - \epsilon/2)t_3 - r_1$. Furthermore, for any given $\epsilon$ and fixed positive constant $r_1$, there exists a constant $T_4 > 0$ such that
\[
(c_4 - \epsilon)t < (c_4 - \epsilon/2)t_3 - r_1
\]
with $t_3 > T_3$ and $t \in [t_3, t_3 + T]$. Therefore, for any $t_3 \geq \max\{T_2 + T_3, T_4\}$, $u(x, t)$ and $v(x, t)$ satisfy (20) with $t \in [t_3, t_3 + T]$ and $|x| < (c_4 - \epsilon)t$. Selecting $t > \max\{T_2 + T_3, T_4\}$, we have
\[
\|u(x, t) - u^*(t)\| + \|v(x, t) - v^*(t)\| \leq \epsilon_1, \ |x| < (c_4 - \epsilon)t.
\]
Since $\epsilon_1$ is arbitrary, the desirable result holds. This completes the proof. \qed

3.2. Case II: $d_1(\bar{r}_1 - \bar{b}_1V) > d_2(\bar{r}_2 + \bar{a}_2U)$. Assume that
\[
d_1(\bar{r}_1 - \bar{b}_1V) > d_2(\bar{r}_2 + \bar{a}_2U).
\] (21)
We shall confirm that the spreading speed of the prey is $2\sqrt{d_1\bar{r}_1}$ and give the lower bounds of the spreading speed of the predator. Here we refer to Pan [28]. Let $\bar{V}(t)$ be the unique positive $T$-periodic solution of
\[
v'(t) = v(t)[r_2(t) + a_2(t)U(t) - b_2(t)v(t)].
\]
Then we define two positive constants
\[
c_6 := 2\sqrt{d_2(r_2 + a_2U)m}, \quad c_7 := 2\sqrt{d_2(\bar{r}_2 + \bar{a}_2\bar{U})}.
\]
Furthermore, by (21), we have
\[
c_1 > c_4 > c_5 > c_7 > c_2.
\]

**Theorem 3.9.** Assume that (4) and (21) hold. Then $(u(x, t), v(x, t)), (x, t) \in \mathbb{R} \times (0, \infty)$ defined by (5) satisfies
(i) for any given $\epsilon \in (0, c_1)$,
\[
\lim_{t \to \infty} \inf_{|x| < (c_1 - \epsilon)t} \|u(x, t) - U(t)\| \geq 0, \quad \lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon)t} u(x, t) = 0;
\]
(ii) if $(r_2 + a_2U)m > \bar{r}_2$, then $c_6 > c_2$ and
\[
\lim_{t \to \infty} \inf_{|x| < (c_6 - \epsilon)t} \|v(x, t) - \bar{V}(t)\| \geq 0
\]
for any given $\epsilon \in (0, c_6)$;
(iii) if $(u^*(t), v^*(t))$ is globally asymptotically stable and (ii) holds, then
\[
\lim_{t \to \infty} \inf_{|x| < (c_6 - \epsilon)t} (|u(x, t) - u^*(t)| + |v(x, t) - v^*(t)|) = 0
\]
for any given $\epsilon \in (0, c_6)$.

**Remark 3.** In view of Lemma 2.3 and Theorem 3.9, we can conclude that
(1) from the item (i), the spreading speed of the prey is exactly $c_1 = 2\sqrt{d_1\bar{r}_1}$, which illustrates that the interspecific action does not change the spreading speed of the prey when (21) holds,
(2) the item (ii) gives a lower bounds $c_6$ of the spreading speed for the predator if the intrinsic growth rate $r_2(t)$ has a small amplitude, which implies that the prey may accelerate the asymptotic spreading of the predator.

Subsequently, we shall prove several lemmas to verify our main results.

**Lemma 3.10.** Assume that (4) and (21) hold. Then
\[
\lim_{t \to \infty} \sup_{|x| > (c_3 + \epsilon) t} v(x, t) = 0
\]
for any given $\epsilon > 0$.

**Proof.** Due to the proof of Lemma 3.2, $u(x, t) \leq U(t)$ implies that $v(x, t)$ satisfies
\[
\begin{cases}
v_t(x, t) \leq d_2 v_{xx}(x, t) + v(x, t)[r_2(t) + a_2(t)U(t) - b_2(t)v(x, t)], \\
v(x, 0) = v(x) \in [0, V(0)].
\end{cases}
\]
By Lemma 2.3 and the comparison principle, we have
\[
\lim_{t \to \infty} \sup_{|x| > (c_3 + \epsilon) t} v(x, t) = 0.
\]
This completes the proof. \(\square\)

**Lemma 3.11.** Assume that (4) and (21) hold. For any given $\epsilon \in (0, c_1)$, we have
\[
\liminf_{t \to \infty} \inf_{|x| < (c_1 - \epsilon) t} u(x, t) > 0.
\]

**Proof.** For any given $\epsilon \in (0, c_1)$, let $\epsilon' \in (0, 1)$ be a constant such that
\[
(c_1 - \epsilon)^2 < 4d_1 \left( \frac{a_1}{b_1} \right).
\]
According to Lemmas 3.6 and 3.10, $c_5 < c_4$ implies that there exists $\hat{T}_1 > 0$ such that
\[
\begin{align*}
& (i) \quad \sup_{2|x| < (c_4 + c_5) t} \{v(x, t)/u(x, t)\} < 2K_2/U(t), \quad t > \hat{T}_1, \\
& (ii) \quad \sup_{2|x| \geq (2c_5 + c_4) t} v(x, t) < \epsilon', \quad t > \hat{T}_1,
\end{align*}
\]
where $K_2 = \frac{2(r_2)_{M} + (a_2)_{M}}{(b_2)_{M}}$ with $K_1 = \frac{2(r_1)_{M}}{(a_1)_{M}}$.
Then $u(x, t)$ satisfies
\[
u_t(x, t) \geq d_1 u_{xx}(x, t) + u \left[ r_1(t) - \epsilon'b_1(t) - \frac{a_1(t)U(t) + 2K_2b_1(t)}{U(t)} u(x, t) \right]
\]
for all $t > \hat{T}_1, x \in \mathbb{R}$. Then Lemma 2.3 indicates the desirable limit. This completes the proof. \(\square\)

**Lemma 3.12.** Assume that (4) and (21) hold. Then
\[
\liminf_{t \to \infty} \inf_{|x| < (c_1 - \epsilon) t} [u(x, t) - U(t)] \geq 0, \quad \limsup_{t \to \infty} u(x, t) = 0
\]
for any given $\epsilon \in (0, c_1)$.

**Proof.** By Lemma 3.3, we obtain
\[
\lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon) t} u(x, t) = 0.
\]
According to Lemma 3.11, there exist $\sigma_2 > 0$ and $\hat{T}_2 > 0$ such that
\[
u(x, t_1) > \sigma_2 \quad \text{for all} \quad t_1 \geq \hat{T}_2, \quad |x| \leq (c_1 - \epsilon/2)t_1.
\]
Remark 1 and the comparison principle imply that for any \( \epsilon_2 \in (0, U_m) \), if \( |x| \leq (c_1 - \epsilon/2)t_1 - r_2 \) with some positive constant \( r_2 > 0 \), then there is \( \hat{T}_3 > 0 \) such that
\[
U(t) - \epsilon_2, \quad t \in [t_1 + \hat{T}_3, t_1 + \hat{T}_3 + T].
\]
Furthermore, for any given \( \epsilon > 0 \), we can find a constant \( \hat{T}_4 > 0 \) such that
\[
(c_1 - \epsilon)t \leq (c_1 - \epsilon/2)t_1 - r_2
\]
for \( t_1 \geq \hat{T}_4 \) and \( t \in [t_1, t_1 + T] \). Take \( t_1 > \max \{ \hat{T}_2 + \hat{T}_3, \hat{T}_4 \} \) and \( t \in [t_1, t_1 + T] \), then
\[
u(x, t) \geq U(t) - \epsilon_2, \quad |x| < (c_1 - \epsilon)t.
\]
Therefore, we have \( u(x, t) \geq U(t) - \epsilon_2 \) for any \( t > \max \{ \hat{T}_2 + \hat{T}_3, \hat{T}_4 \} \) and \( |x| < (c_1 - \epsilon)t \), which implies that
\[
\inf_{|x| < (c_1 - \epsilon)t} \min \{ u(x, t) - U(t) + \epsilon_2 \} \geq 0, \quad t > \max \{ \hat{T}_2 + \hat{T}_3, \hat{T}_4 \}.
\]
Since \( \epsilon_2 \) is arbitrary, the desirable result holds. This completes the proof.

\begin{lemma}
Assume that (4) and (21) hold. If \( (r_2 + a_2U)_m > \overline{V}_2 \), then
\[
\liminf_{t \to \infty} \inf_{|x| < (c_6 - \epsilon)t} \min \{ v(x, t) - \overline{V}(t) \} \geq 0
\]
for any given \( \epsilon \in (0, c_6) \).
\end{lemma}

\begin{proof}
Let \( \delta > 0 \) be a constant such that
\[
(c_6 - \epsilon/4)^2 = 4d_2 \left[ r_2 + a_2(U - \delta) \right]_m.
\]
In view of Lemma 3.12, there exists \( \hat{T}_5 := kT > 0 \) with some \( k \in \mathbb{N}^+ \) such that
\[
u(x, t) \geq U(t) - \delta, \quad |x| < (c_1 - \epsilon/2)t, \quad t > \hat{T}_5.
\]
For any \( t > \hat{T}_5 \), we define a continuous function \( h(x, t) \) satisfying
\begin{enumerate}
\item \( h(x, t) = U(t) - \delta, |x| < (c_1 - \epsilon)t \),
\item \( h(x, t) = 0, |x| > (c_1 - \epsilon)t + \epsilon \),
\item \( 0 \leq h(x, t) \leq \overline{U}(t) - \delta, 0 < |x| - (c_1 - \epsilon)t \leq \epsilon \),
\item \( h(x, t) \) is smooth enough for \( x \in \mathbb{R}, t \geq \hat{T}_5 \).
\end{enumerate}
From the above results, \( v(x, t) \) satisfies
\[
\begin{align*}
&\begin{cases}
v_t(x, t) \geq d_2v_{xx}(x, t) + v(x, t)[r_2(t) + a_2(t)h(x, t) - b_2(t)v(x, t)], x \in \mathbb{R}, t > \hat{T}_5, \\
v(x, \hat{T}_5) \in [0, \overline{V}(\hat{T}_5)].
\end{cases}
\end{align*}
\]
Let \( w(x, t) \) be defined by
\[
\begin{align*}
&\begin{cases}
w_t(x, t) = d_2w_{xx} + w(x, t)[r_2(t) + a_2(t)h(x, t) - b_2(t)w(x, t)], x \in \mathbb{R}, t > \hat{T}_5, \\
w(x, \hat{T}_5) = v(x, \hat{T}_5).
\end{cases}
\end{align*}
\]
We denote \( g(x, t) := r_2(t) + a_2(t)h(x, t) \), then
\[
g(x, t) \geq r_2(t) + a_2(t)(U(t) - \delta) \geq \left[ r_2 + a_2(U(t) - \delta) \right]_m, \quad |x| < (c_1 - \epsilon/2)t, \quad t > \hat{T}_5.
\]
Let \( r(t) = (c_6 - \epsilon/2)t \), then for every \( R > 0 \),
\[
\liminf_{t \to \infty} \left\{ \inf_{|x| < R} \left[ 4d_2g(x \pm r(t), t) - (r'(t))^2 \right] \right\} > 0,
\]
and the above limit is independent on $R$. Therefore,

$$\lim_{R \to \infty} \left\{ \lim_{t \to \infty} \inf_{|x| < R} \left[ 4d_2 g(x \pm r(t), t) - (r'(t))^2 \right] \right\} > 0.$$ 

Due to Lemma 2.4 and the comparison principle, we obtain

$$\lim_{t \to \infty} \inf_{|x| < (c_6 - \epsilon/2)t} v(x, t + T_5) \geq \lim_{t \to \infty} \inf_{|x| < (c_6 - \epsilon/2)t} w(x, t + T_5) > 0,$$

which implies that there exists a constant $\sigma_3 > 0$ such that

$$v(x, t_2) > \sigma_3, \quad |x| < (c_6 - \epsilon/2)t_2, \quad t_2 > T_5.$$

Remark 1 and the comparison principle imply that for any $\epsilon_3 \in (0, \tilde{V}_m)$, if $|x| \leq (c_6 - \epsilon/2)t_2 - r_3$ with some constant $r_3 > 0$, then there is $\tilde{T}_6 > 0$ such that

$$v(x, t) \geq \tilde{V}(t) - \epsilon_3, \quad t \in [t_2 + T_6, t_2 + T_6 + T].$$

Furthermore, for any given $\epsilon > 0$, we can find a constant $\tilde{T}_7 > 0$ such that

$$(c_6 - \epsilon)t \leq (c_6 - \epsilon/2)t_2 - r_3$$

for $t_2 \geq \tilde{T}_7$ and $t \in [t_2, t_2 + T]$. Take $t_2 > \max \{ \tilde{T}_5 + \tilde{T}_6, \tilde{T}_7 \}$ and $t \in [t_2, t_2 + T]$, then

$$v(x, t) \geq \tilde{V}(t) - \epsilon_3, \quad |x| < (c_6 - \epsilon)t.$$

Therefore, we have $v(x, t) \geq \tilde{V}(t) - \epsilon_3$ for any $t > \max \{ \tilde{T}_5 + \tilde{T}_6, \tilde{T}_7 \}$ and $|x| < (c_6 - \epsilon)t$, which implies that

$$\inf_{|x| < (c_6 - \epsilon)t} \left[ v(x, t) - \tilde{V}(t) + \epsilon_3 \right] \geq 0, \quad t > \max \{ \tilde{T}_5 + \tilde{T}_6, \tilde{T}_7 \}.$$ 

By the arbitrariness of $\epsilon_3$, we obtain the desirable result. This completes the proof. \qed

The proof of (iii) in Theorem 3.9 is similar to that of Lemma 3.8, so we omit it here. From what we have done, the proof of Theorem 3.9 is complete.

**Remark 4.** We have proved that $c_6$ is a lower bounds of spreading speed of the predator if $r_3(t)$ has a small amplitude. Inspired by the result in autonomous sense [28, Theorem 4.1], we conjecture that the lower bounds of spreading speed of the predator is not less than $c_7$.

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