Quantum multiplex networks described by coupled Bose and Fermi statistics

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Complex networks are very heterogeneous structures in which no simple symmetries have been detected until now. Here we show that, in growing multiplex networks the symmetries of multilayer structures can be exploited by their dynamical rules, forming super-multiplex networks described by coupled Bose-Einstein and Fermi-Dirac quantum statistics. A multiplex network is a multi-layer system formed by $N$ nodes having a copy in each of the $M$ layers, and $M$ layers formed by different networks of interactions between the $N$ nodes. The super-multiplex is formed by layers which are scale-free networks and can display a Bose-Einstein condensation of the links. Moreover here we interpret each layer of the super-multiplex as a pure quantum bipartite state (called also network state), characterizing its entanglement entropy. Interestingly we observe a simple relation between the entanglement entropies of the single network states and the entropy rate of the network. This relation therefore connects the classical non equilibrium growing dynamics of the super-multiplex network with the quantum static characteristics of the pure network states.

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I. INTRODUCTION

Recently, the relation between complex networks, their geometry and evolution [1,2], and more traditional fields of physics such as quantum physics, quantum information [3] and cosmology are starting to attract the interest of scientists. For instance, quantum statistics have been shown to emerge from the evolution of scale-free networks described by the Bose-Einstein statistics and undergoing a Bose-Einstein condensation [4], while the evolution of Cayley trees with a quenched quality of the nodes, are determined by Fermi-Dirac statistics [5,6]. Even ensembles of complex networks have been shown to be related with quantum statistics [7]. Moreover, it has been shown that random geometric networks on a hyperbolic space can show a scale-free network topology [8,9] and that growing networks in hyperbolic space define the so called “network cosmology” [10,11]. From the quantum information perspective, complex networks encode relevant information in their structures, and new quantum entropy measures have been proposed in order to characterize and quantify this information [12,13]. Moreover, quantum communication on complex networks reveals unexpected scenarios [14,15]. Finally complex networks structures can significantly affect quantum critical phenomena defined on them such as the Ising model [16], the Bose-Hubbard model [17], the Jaynes-Cummings model [18], or the random walk giving rise to different definitions of quantum PageRank on networks [20,21]. The long term goal of these works is to find a unifying language able to characterize quantum complex networks and to relate these complex structures to known network structures of major importance in quantum physics and quantum gravity, such as quantum spin networks and chains used in condensed matter physics [22], dynamic triangulations [23], and spin networks for quantum gravity [24,25].

In this paper we characterize the evolution of a multiplex network model with a quenched quality of the nodes called energy, showing that these multiplex networks are characterized by coupled Bose-Einstein and Fermi-Dirac statistics. Moreover we interpret the network in the layer of the multiplex as pure network states and relate the entanglement entropy of these states with the entropy rate of the multiplex network. Multiplex networks are multi-layer network structures that are attracting large interest [26,27]. In fact they are able to describe several types of interactions between the same set of nodes. For example, social networks, where the same people are connected by different means of communications, or different types of social ties, like friendship, collaboration, co-authorship, etc. are better described by multiplex networks. Similarly, if we want to describe diffusion in transportation networks we need to consider the multiplex nature of the underlying transportation networks, where a given location can be linked to other locations by different types of transportation, train, bus, airplane etc. Finally in brain networks the large variety of neuron types and types of interactions between them, will not be fully understood if the multilayer approach will be not adopted [28,29]. These multiplex networks are formed by layers that are usually scale-free and have a number of nodes that increases in time. For these reasons their evolution can be described by growing multiplex networks models [30,31]. Here we consider these recently introduced structures with the goal of exploiting their symmetries in order to build super-multiplex networks described by coupled Fermi-Dirac and Bose-Einstein statistics. Moreover we explore the information content of these structures by looking at each network in a given layer as a quantum network state describing the complex multiplex networks with tools and methods of quantum information.

In particular we consider the case of a duplex (i.e. a multiplex with $M = 2$), but the analysis can be easily extended to a multiplex with a generic value of the number of layers $M$. In these networks we assign to each node an “energy” or quality that affects the multiplex network dynamics. The networks in each of the super-multiplex layers are scale-free and their statistical properties are
The entropy rate [31] of this growing super-multiplex model describes the rate at which the typical number of multiplex networks, that can be realized during the dynamics, grows in time. This quantity is related to the quantum static properties of the networks in the single layers. In fact if we interpret every layer as a pure network state, we observe that the entropy of the entanglement of these states has a very simple relation with the entropy rate of the multiplex networks, describing the classical non-equilibrium dynamics of the multilayer structure.

The paper is structured as follows: In section II we define the super-multiplex model and we provide its mean-field solution. In section III we characterize the super-multiplex structural properties. In section IV we define the entropy rate of the super-multiplex network. In section V we interpret each layer of the super-multiplex as a pure network state and we evaluate its entanglement entropy. In section VI we relate the entanglement entropy of the pure network states with the entropy rate of the super-multiplex. Finally in section VII we give the conclusions.

II. SUPER-MULTIPLEX NETWORK MODEL

A. The super-multiplex evolution

A multiplex network is a multilayer system formed by $N$ nodes having a copy (or replica) in each of the $M$ layers, and $M$ layers formed by different networks of interactions between the $N$ nodes. We consider for simplicity a multiplex networks formed by $M = 2$ layers and we describe its evolution as a growing multiplex network model in which each node $i$ has quenched quality quantified by the quenched parameter $\epsilon_i \in (0,1)$ called the energy of the node. Moreover, since the preferential attachment rule has been identified to drive the evolution of a large number of complex networks [30] here we assume that the multiplex network dynamics follows a generalized preferential attachment rule inspired to the rule proposed in Ref. [3]. We start at $t = 0$ from a small set of nodes $N_0$ connected in both layers. Each node $i$ of the network has degrees $k_i^{[1]}, k_i^{[2]}$ respectively in layer 1 and layer 2, and energy $\epsilon_i$ drawn from a $g(\epsilon)$ distribution. At each time $t$ we add a node to the multiplex network, each node has two replicas nodes, one each layer and each replica node is attached to $m$ existing nodes in the same layer. In the following we will indicate with $l$ the number of network changes, i.e. links additions or link rewirings, occurring in each layer starting from time $t = 1$. After time $t$ we will have $l = mt$. For each network change in layer 1 we follow the subsequent procedure:

- We extract a number $\sigma_i^{[1]} = 1, 2$. The event $\sigma_i^{[1]} = 1$ occurs with probability $p^{[1]}(1) = \alpha$ while the event $\sigma_i^{[1]} = 2$ occurs with probability $p^{[1]}(2) = 1 - \alpha$.

- If $\sigma_i^{[1]} = 1$ the new node is attached in layer 1 to a node $i_1^{[1]}$ chosen with probability
  \[ \Pi^{[1,1]}(i_1^{[1]} = i) = \frac{e^{-\beta \epsilon_i k_i^{[1]}}}{\sum_j e^{-\beta \epsilon_j k_j^{[1]}}} \]
  i.e. it will be attached preferentially to nodes with low energy and high degree in layer 1, according to a generalized preferential attachment. Instead, if $\sigma_i^{[1]} = 2$ the new node is attached in layer 1 to a node $i_1^{[1]}$ chosen with probability
  \[ \Pi^{[1,2]}(i_1^{[1]} = i) = \frac{e^{-\beta \epsilon_i k_i^{[2]}}}{\sum_j e^{-\beta \epsilon_j k_j^{[2]}}} \]
  i.e. will be attached preferentially to nodes with low energy and high degree in layer 2, according to a generalized preferential attachment.

For each network change in layer 2 we follow the subsequent procedure:

- We extract a number $\sigma_i^{[2]} = 1, 2$. The event $\sigma_i^{[2]} = 1$ occurs with probability $p^{[2]}(1) = 1 - \alpha$ while the event $\sigma_i^{[2]} = 2$ occurs with probability $p^{[2]}(2) = \alpha$.

- If $\sigma_i^{[2]} = 1$ the new node will be attached, in layer 2, to a node $i_2^{[2]}$ chosen with probability
  \[ \Pi^{[2,1]}(i_2^{[2]} = i) = \frac{e^{-\beta \epsilon_i k_i^{[1]}}}{\sum_j e^{-\beta \epsilon_j k_j^{[1]}}} \]
  Instead, if $\sigma_i^{[2]} = 2$ a random link of a node $i_2^{[2]}$ is rewired, i.e. it is detached from node $i_2^{[2]}$ and attached to the new node of the network.

Therefore the network is determined by the sequence of the values \{ $\sigma_i^{[1]}, i_1^{[1]}, \sigma_i^{[2]}, i_2^{[2]}$ \} that fully determines the evolution of the multiplex network given the initial condition.

B. Mean-field solution of the super-multiplex model

When studying growing networks with preferential attachment, in general large attention is given to the degree sequence of the network. In order to predict the degree distribution of these models, mean-field approaches have been extensively studied, finding that in general they give a very good prediction of the structural properties of
the network \[37\]. In this paper we analyse the super-multiplex model with the mean-field theory, leaving to subsequent works the analysis with the master equation approach. In the mean-field approach, one assumes that the degree of each node has no fluctuations, and therefore identifies the degrees \(k_i^{[1]}, k_i^{[2]}\) at time \(t\) with their average over the multiplex network realization. Moreover this approximation is also called the continuous approximation because it is assumed that both the degrees of the nodes and the time are continuous variables. Therefore the mean-field equations for the super-multiplex model read
\[
\frac{dk_i^{[1]}}{dt} = me^{-\beta_i} \left[ \frac{\alpha}{\sum_j e^{-\beta_j} k_j^{[1]}} k_i^{[1]} + \frac{(1-\alpha)}{\sum_j e^{-\beta_j} k_j^{[2]}} k_i^{[2]} \right],
\]
\[
\frac{dk_i^{[2]}}{dt} = me^{-\beta_i} \left[ \frac{(1-\alpha)}{\sum_j e^{-\beta_j} k_j^{[1]}} k_i^{[1]} - \frac{\alpha}{\sum_j e^{-\beta_j} k_j^{[2]}} k_i^{[2]} \right].
\]
Using an approach similar to the one used in the Bianconi-Barabási model \[3\], we will assume self consistently that
\[
\lim_{t\to\infty} \frac{\sum_j e^{-\beta_j} k_j^{[1]}}{mt} = c_1,
\]
\[
\lim_{t\to\infty} \frac{\sum_j e^{-\beta_j} k_j^{[2]}}{mt} = c_2,
\]
where \(c_1\) and \(c_2\) are constants independent of the network realization. Therefore, asymptotically in time we have
\[
\sum_j e^{-\beta_j} k_j^{[1]} \approx mtc_1,
\]
\[
\sum_j e^{-\beta_j} k_j^{[2]} \approx mtc_2.
\]
If we define the vector of the degree of each node as
\[
k_i = \begin{pmatrix} k_i^{[1]}(t) \\ k_i^{[2]}(t) \end{pmatrix},
\]
and we substitute for \(t \gg 1\) the asymptotic expression for the normalization sums Eq. \[7\] we obtain that the mean-field Eqs. \[5\] can be written as
\[
\frac{dk_i}{dt} = e^{-\beta_i} \frac{1}{t} A k_i,
\]
where the matrix \(A\) is defined as
\[
A = \begin{pmatrix} \alpha/c_1 & (1-\alpha)/c_2 \\ (1-\alpha)/c_1 & -\alpha/c_2 \end{pmatrix}.
\]
The solution of Eq. \[8\] is given in terms of the eigenvalue and the eigenvector of the matrix \(A\). These eigenvalue are respectively positive and negative for every value of the parameter \(\alpha\) of the model. We will indicate the eigenvalues of \(A\) as \(\lambda_+\) and \(\lambda_-\) in correspondence of their sign. These eigenvalues are given by
\[
\lambda_+ = \frac{1}{2} \left[ \frac{\alpha}{c_1} - \frac{\alpha}{c_2} + \sqrt{\Delta} \right],
\]
\[
\lambda_- = \frac{1}{2} \left[ \frac{\alpha}{c_1} - \frac{\alpha}{c_2} - \sqrt{\Delta} \right],
\]
with
\[
\Delta = \left( \frac{\alpha}{c_1} - \frac{\alpha}{c_2} \right)^2 + 4\alpha^2 + (1-\alpha)^2.
\]
We have therefore that the constants \(c_1\) and \(c_2\) can be expressed as a function of \(\lambda_+\) and \(\lambda_-\) as
\[
c_1 = \frac{(1-\alpha)^2 + \alpha^2}{2\alpha} \left[ \frac{1}{\lambda_+} + \frac{1}{\lambda_-} \right] + \sqrt{\Delta},
\]
\[
c_2 = \frac{(1-\alpha)^2 + \alpha^2}{2\alpha} \left[ -\frac{1}{\lambda_+} + \frac{1}{\lambda_-} \right] + \sqrt{\Delta},
\]
with
\[
\Lambda = \left( \frac{1}{\lambda_+} + \frac{1}{\lambda_-} \right)^2 - \frac{4\alpha^2 + 1}{[(1-\alpha)^2 + \alpha^2] \lambda_+ \lambda_-}.
\]
Moreover, we indicate by \(u^+ = (u_1^+, u_2^+)\) and \(u^- = (u_1^-, u_2^-)\) the eigenvectors corresponding respectively to the eigenvalues \(\lambda_+\) and \(\lambda_-\). The components of these eigenvectors are given by
\[
u_1^+ = \frac{c_1}{2(1-\alpha)} \left( \frac{\alpha}{c_2} + \frac{\alpha}{c_1} + \sqrt{\Delta} \right),
\]
\[
u_2^+ = 1,
\]
\[
u_1^- = \frac{c_1}{2(1-\alpha)} \left( \frac{\alpha}{c_2} + \frac{\alpha}{c_1} - \sqrt{\Delta} \right),
\]
\[
u_2^- = 1.
\]
Therefore, solving the Eqs. \[8\] the degrees of node \(i\) in the two layers can be calculated in the mean-field approximation to be
\[
k_i(t) = d^+ u^+ \left( \frac{t}{t_i} \right)^{-\beta_i \lambda_+} + d^- u^- \left( \frac{t}{t_i} \right)^{-\beta_i \lambda_-},
\]
where \(t_i\) is the time at which the node \(i\) is arrived in the network and where \(d^-\) and \(d^+\) are constants determined by the initial condition \(k_i(t_i) = m1\) where \(1\) is the column vector of components \((1, 1)\). Starting form Eq. \[14\], the initial condition can be also written as
\[
k_i(t_i) = m1 = Ud,
\]
where \(U\) is the matrix with column vectors given by the eigenvectors \(u^+\) and \(u^-\) i.e.
\[
U = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix},
\]
and the column vector \( \mathbf{d} \) has components \( \mathbf{d} = (d^+, d^-) \). This equation can be solved finding that the constants \( d^+ \) and \( d^- \) are given by

\[
\mathbf{d} = m \mathbf{U}^{-1} \mathbf{1} = \frac{m}{u_1^+ u_2^+ - u_1^- u_2^-} \left( \begin{array}{c} u_1^- - u_2^- \\ u_1^+ - u_2^+ \end{array} \right).
\]

Having fixed the constant \( d^+ \) and \( d^- \), we can rewrite Eq. (14) as

\[
k_i = \mathbf{B} v_i
\]

with \( v_i \) indicating the column vector

\[
v_i = \left( \frac{1}{t_i} e^{-\beta \epsilon_i \lambda_+} \right) \left( \frac{1}{t_i} e^{-\beta \epsilon_i \lambda_-} \right),
\]

and the matrix \( \mathbf{B} \) given by

\[
\mathbf{B} = \begin{pmatrix} d^+ u_1^+ & d^- u_1^- \\ d^+ u_2^+ & d^- u_2^- \end{pmatrix}.
\]

Therefore we have also that

\[
v_i = \mathbf{B}^{-1} k_i
\]

with

\[
\mathbf{B}^{-1} = \frac{1}{m} \left( \begin{array}{cc} u_2^- - u_1^- & -u_2^- \\ -u_2^+ & u_1^+ - u_2^+ \end{array} \right)
\]

Since \( \lambda_+ \) and \( \lambda_- \) have respectively positive and negative sign, Eq. (15) defines the two linear combination of the degrees \( k^{[1]} \) and \( k^{[2]} \) that respectively increases and decreases as a power-law of time. Therefore we have found the solution of the model, once the constants \( c_1 \) and \( c_2 \) are given. In order to find the correct values of the constants \( c_1 \) and \( c_2 \) given by Eqs. (16), we need to close our self-consistent argument. Since we have assumed that the constants \( c_1 \) and \( c_2 \) are independent on the network realization, determined in the mean-field approximation by the quenched disorder of the assignment of the energies to the nodes, the constants \( c_1 \) and \( c_2 \) can be evaluated performing the following limits:

\[
\lim_{t \to \infty} \frac{\left( \sum_j e^{-\beta \epsilon_j k_j^{[1]}} \right)}{mt} = c_1, \\
\lim_{t \to \infty} \frac{\left( \sum_j e^{-\beta \epsilon_j k_j^{[2]}} \right)}{mt} = c_2.
\]

Where in Eqs. (16) the average is performed over the distribution of the energies of the nodes. Therefore, by multiplying each equation of Eq. (16) by \( e^{-\beta \epsilon_i} \), integrating over the continuous time \( t_i \), and averaging over the \( g(\epsilon) \) distribution we get that the self-consistent equations determining the constants \( c_1 \) and \( c_2 \), or equivalently \( \lambda_+ \) and \( \lambda_- \), are given by

\[
mc = BJ
\]

where the column vector \( \mathbf{J} = (J_+, J_-) \) has component given by

\[
J_+ = \int g(\epsilon) \frac{1/\lambda_+}{e^{\beta \epsilon / \lambda_+} - 1}, \\
J_- = \int g(\epsilon) \frac{1/\lambda_-}{e^{\beta \epsilon / \lambda_-} - 1}.
\]

Inverting the Eqs. (20) we can express \( \mathbf{J} \) as

\[
\mathbf{J} = m \mathbf{B}^{-1} \mathbf{c} = \left( \frac{c_2 u_2^- - c_2 u_2^+}{u_1^- - u_1^+}, \frac{c_2 u_2^+ - c_1 u_2^+}{u_1^- - u_1^+} \right).
\]

By defining the two constants \( \mu_B \) and \( \mu_F \), as in the following,

\[
\lambda_+ = e^{\beta \mu_B}, \\
-\lambda_- = e^{\beta \mu_F}
\]

and multiplying \( J_+ \) by \( \lambda_+ \) and \( J_- \) by \( -\lambda_- \) we get the following self-consistent equation, fixing the “chemical potentials” \( \mu_B \) and \( \mu_F \),

\[
I_B = \int g(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu_B)} - 1} = G_B, \\
I_F = \int g(\epsilon) \frac{1}{e^{\beta(\epsilon - \mu_F)} + 1} = G_F,
\]

with \( G_B \) and \( G_F \) independent on the energy distribution and only function of the inverse temperature \( \beta \) and the two “chemical potentials” \( \mu_B \) and \( \mu_F \). In fact we have,

\[
\mathbf{G} = \begin{pmatrix} G_B \\ G_F \end{pmatrix} = \left( \frac{c_2 u_2^- - c_2 u_2^+}{c_1 u_2^- - c_1 u_2^+}, \frac{c_2 u_2^+ - c_1 u_2^+}{u_1^- - u_1^+} \right).
\]

From the self-consistent Eq. (23) the two constants \( \mu_B \) and \( \mu_F \) can be interpreted as “chemical potentials” of coupled Bose and Fermi gases and fully determine the evolution of the super-network, as long as the equations can be satisfied. Only the left hand side of the Eqs. (23) depends on the energy distribution \( g(\epsilon) \) while the right hand side does not depend on it. Moreover the quantities \( G_B \) and \( G_F \) depend on both the chemical potentials \( \mu_B \) and \( \mu_F \) and can be explicitly expressed as

\[
G_B = e^{\beta \mu_B} \frac{(2 - \alpha)e^{-\beta \mu_B} + \alpha e^{-\beta \mu_F} + (1 - 2\alpha)\sqrt{\lambda}}{\alpha e^{-\beta \mu_B} + (2 - \alpha)e^{-\beta \mu_F} + (1 - 2\alpha)\sqrt{\lambda}} \times \frac{(1 - \alpha)^2 + \alpha^2}{2\alpha} \left[ e^{-\beta \mu_B} + e^{-\beta \mu_F} + \sqrt{\lambda} \right], \\
G_B = e^{\beta \mu_F} \frac{-\alpha e^{-\beta \mu_B} + (2 - \alpha)e^{-\beta \mu_F} + (1 - 2\alpha)\sqrt{\lambda}}{-(2 - \alpha)e^{-\beta \mu_B} + \alpha e^{-\beta \mu_F} + (1 - 2\alpha)\sqrt{\lambda}} \times \frac{(1 - \alpha)^2 + \alpha^2}{2\alpha} \left[ e^{-\beta \mu_B} + e^{-\beta \mu_F} + \sqrt{\lambda} \right],
\]
where $\Lambda$ is given by
\[
\Lambda = \left( e^{-\beta \mu_B} - e^{-\beta \mu_F} \right)^2 + \frac{4\alpha^2}{(1-\alpha)^2 + \alpha^2} e^{-\beta(\mu_B + \mu_F)}. \tag{24}
\]

The self-consistent Eqs. (20) that fix the chemical potential $\mu_B$ and $\mu_F$ fully determine the mean-field solution of this model. In the super-multiplex network, nevertheless there can be two phenomena that implies a breakdown of this solution. On one side we can observe a condensation of the links in correspondence of the regime of high values of $\beta$ where the Eqs. (20) do not have a solution. This phenomenon will be discussed more in depth in the next section. On the other side, it is possible to observe in the model stochastic effects that are not captured by the mean-field solution.

III. STRUCTURAL PROPERTIES OF THE SUPER-MULTIPLEX NETWORK

The mean-field solution of the model well capture the main characteristics of the super-multiplex as long as $\alpha$ is not too large. In fact we found very good agreement of the mean-field theory with the simulation results as long as $\alpha$ is lower than 0.5. For higher values of $\alpha$ in the second layer the rewiring process has a higher rate of the process of addition of new links and therefore non-trivial stochastic effects set in that are not captured by the mean-field solution. For this reason, here we focus of the regime $\alpha < 0.5$ where we find very good agreement between the theory and the simulations results. We will describe the structural properties of the super-multiplex networks, covering the degree distribution of the networks in the two layers, different types of correlations typical of multiplex networks, and we will describe the phenomenology related to the super-multiplex condensation transition in which one node acquires a finite fraction of all the links in both layers.

A. Degree distribution

The degree distribution in the network is scale-free in both layers, as predicted by the mean-field solution. In fact if we consider the dynamical Eq. (14) for the degree $k_i^{[1]}$ and the degree $k_i^{[2]}$ and we take only the leading term in the limit $t \gg 1$ we found
\[
k_i^{[1]} \simeq d^+ u_1^+ \left( \frac{t}{t_i} \right)^{-\beta(\epsilon - \mu_B)},
\]
\[
k_i^{[2]} \simeq d^+ u_2^+ \left( \frac{t}{t_i} \right)^{-\beta(\epsilon - \mu_B)}. \tag{25}
\]

Therefore, using the same mean-field arguments that are used to show that growing complex networks with preferential attachment are scale-free, we can approximate the degree distributions $P^{[1]}(k)$ and $P^{[2]}(k)$ in the two layers as
\[
P^{[1]}(k) \propto \int d\epsilon g(\epsilon) e^{\beta(\epsilon - \mu_B)} \left( \frac{d^+ u_1^+}{k} \right) \frac{1}{k},
\]
\[
P^{[2]}(k) \propto \int d\epsilon g(\epsilon) e^{\beta(\epsilon - \mu_B)} \left( \frac{d^+ u_2^+}{k} \right) \frac{1}{k}. \tag{26}
\]

These expression reveals that the degree distributions in the two layers can be seen as a convolution of power-law networks with exponents $\gamma(\epsilon) = e^{\beta(\epsilon - \mu_B)}+1 > 2$. In Figure 1 we show the degree distribution of the two layers for $g(\epsilon) = (\theta + 1) \epsilon^\theta$ and $\theta = 0.5$ for different values of $\beta = 0., 1, 5$. The mean-field theory valid as long as the super-multiplex is not condensed, is in very good agreement with the simulation results.

B. Multilayer degree correlations

In the multiplex networks one relevant correlation is between the degrees of the replica nodes. In particular in a duplex it is interesting to investigate if a hub in a network is also typically a hub in the other network or if it is typically a low degree node. In the super-multiplex network model, we observe that for $\alpha < 0.5$, these correlations are positive. In fact in the mean-field solution, approximating the degrees $k_i^{[1]}$ and $k_i^{[2]}$ for $t \rightarrow$
expectation given by Eq. (27). This quantity characterizes the degree correlation in the multiplex network and is defined as

$$\langle k_2[k_1] \rangle = \sum_{k_2} k_2^2 P(k_2|k_1),$$

where $P(k_2|k_1)$ is the conditional distribution of having a node of degree $k_2$ in layer 2 given that it has degree $k_1$ in layer 1. Since in the mean-field approximation the degrees of a node are deterministic variable, the mean-field expectation for $\langle k_2[k_1] \rangle$ is given by Eq. 27. In Figure 2 we display $\langle k_2[k_1] \rangle$ showing that this is an increasing function of $k_1$ indicating that the degrees of the same node in the two layer of the super-multiplex are positively correlated as long as $\alpha < 0.5$. Moreover the conditional average $\langle k_2[k_1] \rangle$ is well approximated by the mean-field expectation given by Eq. 27.

C. Bose-Einstein condensation in the super-multiplex network

The Bianconi-Barabási model [3] describing a growing scale-free networks that can mapped to a Bose-Einstein gas, displays the Bose-Einstein condensation in complex networks. This condensation transition is a structural phase transition occurring in the network when the mapped Bose gas is in the the Bose-Einstein condensation phase. Below this phase transition, in the network, one node grabs a finite fraction of the links and non trivial non-equilibrium process determine the network evolution.

A similar phenomena occurs also in the super-multiplex model, where the condensation occurs simultaneously on the two replicas of the same node. In this model the condensation phase transition occurs at $\beta = \beta_c$ for which $\mu_B = 0$. Therefore the equations determining the condensation phase transition are

$$\begin{align*}
I_B &= \int \frac{1}{e^{\beta g(\epsilon) - 1}} = G_B, \\
I_F &= \int \frac{1}{e^{\beta g(\epsilon) - 1}} = G_F, \\
\mu_B &= 0.
\end{align*}$$

Below this phase transition a single node grabs a finite fraction of all the links in both layers. This condensation can possibly occur at low enough temperatures $T = 1/\beta$ only if the integral $I_B(\mu_B = 0)$ converges. Therefore as in the classical Bose-Einstein condensation a necessary condition for this condensation to occur is that $g(\epsilon) \to 0$ as $\epsilon \to 0$.

Since the condensation occurs in the same node in the two layers, we observe that below the condensation transition the two layer develop another type of correlation. In fact we observe that the total overlap of the links in the two layers becomes significant below the condensation transition. The total overlap $O^{[1,2]}$ of the links [3] is defined as

$$O^{[1,2]} = \sum_{i<j} a_{ij}^{[1]} a_{ij}^{[2]},$$

where $a_{ij}^{[1]}$ and $a_{ij}^{[2]}$ are the matrix elements of the adjacency matrix of layer 1 and layer 2 respectively. In Figure 3 we plot the fraction of the links linked to the most connected node in layer 1 and in layer 2. The absence of finite size effects below the condensation phase transition shows that in the super-multiplex there is one node that grabs a finite fraction of all the links. Moreover, in Figure 3 we plot also the total overlap $O^{[1,2]}$ of the links, showing that below the condensation transition the total overlap becomes significant.

IV. ENTROPY RATE OF THE SUPER-MULTIPLEX NETWORK

Given the initial condition, the super-multiplex evolution up to time $t$ is fully determined by the sequence
of symbols $X = \{\sigma_1^{[1]}, i_1^{[1]}, \sigma_2^{[2]}, i_2^{[2]}\}_{l=1,2,...,mt}$. Therefore, similarly to what happens for growing network models \cite{31}, it is possible to define an entropy rate of the growing super-multiplex network. This entropy rate $H(X)$ can be used for example if aim at compressing the network, as we could aim at extending the Shannon’s noiseless coding theorem \cite{2} to the sequence $X = \{\sigma_1^{[1]}, i_1^{[1]}, \sigma_2^{[2]}, i_2^{[2]}\}_{l=1,2,...,mt}$ encoding the full network evolution. The entropy rate of the super-multiplex network model when in the super-multiplex we have already observed $l-1$ network changes, is given by

$$H(X) = - \sum_{x_l} P(x_l|x_{\ell=1,...,l-1}) \times \log P(x_l|x_{\ell=1,...,l-1}),$$

where $x_l = \{\sigma_1^{[1]}, i_1^{[1]}, \sigma_2^{[2]}, i_2^{[2]}\}$. We note here that the entropy rate of the super-multiplex, as the entropy rate of growing networks has a very characteristic feature, i.e. contains a leading term of order of $\log(t)$. Therefore it does not converge in the thermodynamic limit $t \to \infty$. This is due to the fact that the attachment probability in these networks are non-local. This occurs also in the growth of other networks \cite{31} such as random trees, where each new node is attached to a random node of the network with probability $1/t$, where $t$ is the number of nodes in the network. In fact it is easy to see that also for this basic, non-local model we have

$$H(X) = \log(t).$$

**V. THE SUPER-MULTIPEX NETWORK AS A SET OF BIPARTITE NETWORK STATES**

Here we perform a mapping between the super-multiplex network described previously and two bipartite quantum states. In particular we will consider the states $|A^{[1]}\rangle, |A^{[2]}\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$ with $\mathcal{H}_a \simeq \mathcal{H}_b \simeq \mathbb{C}^N$, given by

$$|A^{[1]}\rangle = \frac{1}{\|A^{[1]}\|} \sum_{i,j} e^{-\beta \epsilon_{ij}/2} a_{ij}^{[1]} |i\rangle |j\rangle,$$

$$|A^{[2]}\rangle = \frac{1}{\|A^{[2]}\|} \sum_{i,j} e^{-\beta \epsilon_{ij}/2} a_{ij}^{[2]} |i\rangle |j\rangle$$

where $a_{ij}^{[1]}$ is the adjacency matrix of the first layer of the super-multiplex network, $a_{ij}^{[2]}$ is the adjacency matrix of the second layer of the super-multiplex network. Moreover, $|A\rangle = \sqrt{\operatorname{Tr} A^\dagger A}$ denotes the Frobenius norm of the matrix $A$ and the matrix elements $(i,j)$ of the matrices $A^{[1]}$ and $A^{[2]}$ are given respectively by

$$A_{ij}^{[1]} = e^{-\beta \epsilon_{ij}/2} a_{ij}^{[1]}$$

and

$$A_{ij}^{[2]} = e^{-\beta \epsilon_{ij}/2} a_{ij}^{[2]}.$$  

Using the terminology of \cite{13} we will refer to $|A^{[1]}\rangle$ and $|A^{[2]}\rangle$ as pure network states.

**VI. THE ENTANGLEMENT ENTROPY OF THE SUPER-MULTIPEX NETWORK**

Here we want to consider the entanglement entropy of the pure network states $|A^{[1]}\rangle$ and $|A^{[2]}\rangle$, therefore we defined the reduced density matrices $\rho^{[1]}, \rho^{[2]}$ given by

$$\rho^{[1]} = \operatorname{Tr}_b |A^{[1]}\rangle \langle A^{[1]}|,$$

$$\rho^{[2]} = \operatorname{Tr}_b |A^{[2]}\rangle \langle A^{[2]}|$$

and we calculate the entanglement entropy $S^{[1]}$ and $S^{[2]}$ given by

$$S^{[1]} = - \operatorname{Tr}_a \rho^{[1]} \log \rho^{[1]},$$

$$S^{[2]} = - \operatorname{Tr}_a \rho^{[2]} \log \rho^{[2]}.$$

Using the explicit expression for the reduced density matrices in terms of the degree of the nodes in the different layers $\rho^{[1]}$ and $\rho^{[2]}$, i.e.

$$\rho^{[1]} = \sum_i \sum_{k_{ij}^{[1]}} e^{-\beta \epsilon_{ij}/2} k_{ij}^{[1]} |i\rangle \langle i|,$$

$$\rho^{[2]} = \sum_i \sum_{k_{ij}^{[2]}} e^{-\beta \epsilon_{ij}/2} k_{ij}^{[2]} |i\rangle \langle i|$$

FIG. 3: The fraction of links $k_{\max}^{[1]}/(mN)$ and $k_{\max}^{[2]}/(mN)$ linked to the most connected node in layer 1 and in layer 2 are shown together with the total overlap of the links $O_{1,2}$ versus $T = 1/\beta$. The data is shown for networks of $N = 10^3, 10^4, 10^5$ nodes with energy distribution $g(\epsilon) = (1 + \theta) e^\theta$ and $\theta = 0.5$, $\alpha = 0.3$. The curves are averaged over 100 multiplex network realizations for $N = 10^3$ and $N = 10^4$, and over 30 multiplex network realization for $N = 10^5$.\[1000]}
we found that the entropies $S^{[1]}$ and $S^{[2]}$ are given by

$$S^{[1]} = -\sum_{i} \frac{e^{-\beta \epsilon_{i} k_{i}^{[1]}}}{\sum_{j} e^{-\beta \epsilon_{j} k_{j}^{[1]}}} \log \left( \frac{e^{-\beta \epsilon_{i} k_{i}^{[1]}}}{\sum_{j} e^{-\beta \epsilon_{j} k_{j}^{[1]}}} \right),$$

$$S^{[2]} = -\sum_{i} \frac{e^{-\beta \epsilon_{i} k_{i}^{[2]}}}{\sum_{j} e^{-\beta \epsilon_{j} k_{j}^{[2]}}} \log \left( \frac{e^{-\beta \epsilon_{i} k_{i}^{[2]}}}{\sum_{j} e^{-\beta \epsilon_{j} k_{j}^{[2]}}} \right).$$

(36)

Moreover, in the asymptotic limit $t \gg 1$ we can evaluate the entropy using the asymptotic relations given by Eqs. 7. We found that

$$E^{[1]} - TS^{[1]} + T \log(mt_{1}) = F^{[1]}$$

$$E^{[2]} - TS^{[2]} + T \log(mt_{2}) = F^{[2]}$$

(37)

where

$$E^{[\nu]} = \langle \epsilon \rangle_{\nu}$$

$$F^{[\nu]} = -T \langle \log k^{[\nu]} \rangle_{\nu}$$

(38)

with $\nu = 1, 2$ and

$$\langle f \rangle_{\nu} = \sum_{i} \frac{e^{-\beta \epsilon_{i} k_{i}^{[\nu]}}}{\sum_{j} e^{-\beta \epsilon_{j} k_{j}^{[\nu]}}} f_{i}$$

(39)

Finally by using the mean-field solution of the model we can evaluate the energies $E^{[1]}, E^{[2]}$, and the free energies $F^{[1]}, F^{[2]}$, as long as the super-multiplex is not in the condensed phase and $\alpha < 0.5$. If we define $E_B$ and $E_F$ respectively as the average energies calculated over the Bose and Fermi distributions with chemical potentials $\mu_B$ and $\mu_F$, i.e.

$$E_B = \int d\epsilon \frac{\epsilon}{e^{\beta (\epsilon - \mu_B)} - 1}$$

$$E_F = \int d\epsilon \frac{\epsilon}{e^{\beta (\epsilon - \mu_F)} + 1}$$

(40)

we have, in the mean-field approximation,

$$\left( \begin{array}{c} E^{[1]} \\ E^{[2]} \end{array} \right) = V \left( \begin{array}{c} E_B \\ E_F \end{array} \right).$$

with

$$V = \left( \begin{array}{cc} \frac{mc_1}{mc_2} d^+ u^+_1 \lambda^+ & -\frac{mc_1}{mc_2} d^- u^-_1 \lambda^- \\ \frac{1}{mc_2} d^+ u^+_2 \lambda^+ & -\frac{1}{mc_2} d^- u^-_2 \lambda^- \end{array} \right).$$

Similarly also the “free energies” $F^{[1]}$ and $F^{[2]}$ can be estimated using the mean-field solution of the model.

VII. RELATION BETWEEN THE ENTANGLEMENT ENTROPIES OF THE SUPER-MULTIPLEX AND ITS ENTROPY RATE

We note here a surprising result. In fact the entanglement entropies $S^{[1]}$ and $S^{[2]}$ have a immediate classical meaning because they can be linked to the entropy rate of the network evolution. In fact, by calculating explicitly the entropy rate of the super-multiplex, defined in Eq. 30, we get

$$H(X) = S^{[1]} + S^{[2]} + 2h(\alpha)$$

(41)

where $h(\alpha)$ is given by

$$h(\alpha) = -\alpha \log(\alpha) - (1 - \alpha) \log(1 - \alpha).$$

(42)

Therefore, the entanglement entropies of super-multiplex network interpreted as two quantum bipartite network states, are related to the entropy rate of the super-multiplex, which is described by a classical non-equilibrium process. The relation between the entropy rate $H(X)$ and the entanglement entropies $S^{[1]}$ and $S^{[2]}$ remains valid for every value of $\alpha \in [0, 1]$ and also below the condensation phase transition. Nevertheless, the scaling of the entanglement entropies with the system size changes below the condensation phase transition.

VIII. CONCLUSIONS

In conclusion here we have proposed to investigate the properties of the super-multiplex network model. This is a growing multiplex network model that can be fully characterized by coupled quantum Bose-Einstein and Fermi-Dirac statistics. The super-multiplex is a duplex, i.e., is formed by $M = 2$ layers. The dynamic rules exploit the symmetries of this multilayer structure as it is reflected by the self-consistent equations determining their dynamics that couples the “chemical potentials” $\mu_B$ and $\mu_F$ of the underlying Bose-Einstein and Fermi-Dirac statistics. The resulting super-multiplex network is formed by two scale-free layers and can undergo a Bose-Einstein condensation of the links. Moreover if the two networks in the two layers of the super-multiplex are interpreted as pure network states, and interesting relation has been shown to exists between the entanglement entropies of these states, characterizing these bipartite quantum network states and the entropy rate of the classical super-multiplex network. In conclusion, in this work the complexity of the super-multiplex networks, and its underlying symmetries have been shown to be connected to quantum statistics and quantum information. Our hope that this work will open new perspective for the emerging field of quantum complex networks.
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