A recent parameterisation scheme developed for the quark mixing matrix is shown to be easily applicable to the lepton mixing matrix as well.

1 The parameterisation

Recently, guided by experiments, an explicit form was suggested [1] for the quark mixing matrix to emphasise its Cabibbo substructure. In this parameterisation, the smallest angle, denoted by $\theta_3$ is of order $\lambda^2$, where $\lambda$ is the parameter introduced by Wolfenstein in his empirical scheme [2]. An essential feature is that one may easily choose any arbitrary angle of the experimentally studied unitarity triangle to be one of the parameters of the matrix. Evidently, one could choose any angle of any of the other triangles as well.

The purpose of this note is to show how, within this framework, the currently known lepton mixing matrix is reproduced.

The idea in Ref. [1] was to write the mixing matrix in the exact form

$$V = V_0 + s_3 V_1 + (1 - c_3) V_2$$

where $s_3 = \sin \theta_3$, $c_3 = \cos \theta_3$ and the matrices $V_j$, $j = 0 - 2$, are given by

$$V_0 = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ b_1^* & b_2^* & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ <B| \end{pmatrix} |A>$$

$$V_2 = \begin{pmatrix} |A><B| & 0 \\ 0 & 0 \end{pmatrix}$$

Here $U$ is a two-by-two unitary matrix and

$$|A> = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad |B> = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
and \((|A><B|)_{ij} \equiv a_ib_j^*\). Furthermore, vector \(A\) is normalised to one and the two vectors \(A\) and \(B\) are related to one another, \(|B> = -U|A>\). Thus we have

\[
<A|A> = <B|B> = 1, \quad |A> = -U|B>, \quad |B> = -U^\dagger|A>
\]  

(6)

It is a trivial task to check that \(V\) is unitary, as was described in Ref.\[1\]. For the case of quarks, the obvious choice for \(U\) is to take it to be a rotation matrix, which was denoted by \(R_2(\Phi)\) in Ref.\[1\]. Thereby the Cabibbo substructure is manifestly exhibited. Finally, the CP violation parameter \(J\) defined by

\[
Im(V_{\alpha j}V_{\beta k}^*V^*_{\gamma l}) = J \sum_{\gamma,l} \epsilon_{\alpha\beta\gamma} \epsilon_{jkl}
\]

(7)

is in this parameterisation given by

\[
J = s_3^2 c_3 \sin\Phi \cos\Phi Im(a_1^*a_2) = s_3^2 c_3 \sin\Phi \cos\Phi Im(b_1^*b_2)
\]

(8)

We now turn to the case of leptons.

2 Lepton mixing matrix

The lepton mixing matrix is a product of two matrices. One of these contains the so called Majorana phases and the other has the same form as the quark mixing matrix. For simplicity, in the following we shall refer to the latter matrix as the lepton mixing matrix. The values of the matrix elements of this matrix may be found in the article by Kayser in the Review of Particle Physics \[3\]. Kayser quotes

\[
V \approx \begin{bmatrix}
    c & s & s_{13} e^{-i\delta} \\
    -s/\sqrt{2} & c/\sqrt{2} & 1/\sqrt{2} \\
    s/\sqrt{2} & -c/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}
\]

(9)

To obtain this form in the above parameterisation we first note that Eq.(1) yields

\[
V_{33} = c_3, \quad V_{13} = a_1 s_3, \quad V_{23} = a_2 s_3, \quad V_{31} = b_1^* s_3, \quad V_{32} = b_2^* s_3
\]

(10)

Therefore we put \(c_3 = \frac{1}{\sqrt{2}}\) and choose

\[
U = R_2(\theta) = \begin{pmatrix}
    c & s \\
    -s & c
\end{pmatrix}
\]

(11)

where \(c\) and \(s\) are the parameters appearing in Eq.(9). Next we put

\[
|A> = \left(\hat{s}e^{-i\delta} \atop \hat{c}\right) \sim \left(\hat{\theta}e^{-i\delta} \atop 1\right) + O(\hat{\theta}^2)
\]

(12)

From Eq.(6) follows that

\[
|B> \sim \left(s \atop -c\right) - \hat{\theta}e^{-i\delta} \left(c \atop s\right) + O(\hat{\theta}^2)
\]

(13)
The two-by-two matrix in the upper left corner of Eq.(11) is in the present case given by

\[
\begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix} = R_2(\theta) + (1 - c_3)|A > < B|
\]

(14)

A simple computation yields

\[
\begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix} = \begin{pmatrix}
c & s \\
-sc_3 & cc_3
\end{pmatrix} + (1 - c_3)\hat{\theta} \begin{pmatrix}
se^{-i\delta} & -cc^{-i\delta} \\
-cc^{i\delta} & -se^{i\delta}
\end{pmatrix} + O(\hat{\theta}^2)
\]

(15)

Inserting these results into the matrix \(V\) we find

\[
V \simeq \begin{pmatrix}
c & s & \hat{\theta}e^{-i\delta}/\sqrt{2} \\
-sc_3 & cc_3 & 1/\sqrt{2} \\
s/\sqrt{2} & c/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\]

(16)

where we have substituted \(c_3 = 1/\sqrt{2}\), and following Ref.[3] we have systematically neglected terms of order \(\hat{\theta}\) as compared to the leading terms. Identifying \(\hat{\theta}/\sqrt{2}\) with \(s_{13}\) we see that the two matrices in Eqs.(9) and (16) have exactly the same entries.

Finally, remembering that \(\Phi\) has been replaced by \(\theta\) and \(c_3 = 1/\sqrt{2}\), from Eqs.(8) and (12) we obtain

\[
J \simeq \frac{sc}{2\sqrt{2}} \hat{\theta}sin\delta = \frac{sc}{2}s_{13}sin\delta
\]

(17)

This quantity is not necessarily small and leaves us with the hope that CP violation in neutrino oscillations may be observable.

References

[1] C. Jarlskog, hep-ph/0503199

[2] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945

[3] B. Kayser in Review of Particle Physics, S. Eidelman et al., Phys. Lett B592 (2004) 1