ORTHOGONAL POWERS AND MÖBIUS CONJECTURE
 FOR SMOOTH TIME CHANGES OF HOROCYCLE FLOWS

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ABSTRACT. We derive, from the work of M. Ratner on joinings of time-changes of horocycle flows and from the result of the authors on its cohomology, the property of orthogonality of powers for non-trivial smooth time-changes of horocycle flows on compact quotients. Such a property is known to imply P. Sarnak’s Möbius orthogonality conjecture, already known for horocycle flows by the work of J. Bourgain, P. Sarnak and T. Ziegler.

1. INTRODUCTION

We set, for brevity, $G = \text{PSL}_2(\mathbb{R})$ and denote by $\Gamma$ a co-compact lattice of $G$. We denote $h_t$ the classical horocycle flow on $\Gamma \backslash G$. Let $\tau \in W^s(\Gamma \backslash G)$ be a strictly positive function of Sobolev order $s > 2$, and let $h^\tau_t$ be the corresponding time change of $h_t$. We recall that the flow $h^\tau_t$ is defined by setting, for any $x \in \Gamma \backslash G$ and $t \in \mathbb{R}$,

$$h^\tau_t(x) := h_{w(x,t)}(x),$$

with $w(x,t)$ the unique function satisfying the identity

$$\int_0^{w(x,t)} \tau(h^u_x) \, du = t$$

for all $(x,t) \in \Gamma \backslash G \times \mathbb{R}$.

Theorem 1. If, for some $0 < p < q$, there exists a non trivial joining of the flows $h^\tau_{pt}$ and $h^\tau_{qt}$, then $\tau$ is cohomologous to a constant.

The goal of this note is to show that the above theorem follows easily from Ratner classification of joinings of times changes of horocycles flows [7] and the characterization of coboundaries given by the authors in [5].

A slightly weaker version of Theorem 1 was first proved by Kanigowski, Lemańczyk and Ulcigrai [6] as a consequence of a general disjointness criterion base on the so-called Ratner property. They proved that if distinct powers of a time-change have a non-trivial joining, then the time change function is cohomologous to a function coming from a harmonic form. They asked us whether the result could be derived directly from Ratner’s work combined with our description of the cohomology of horocycle flows [5]. In this note we answer their question affirmatively by proving a slightly stronger result, which holds for all smooth time-changes. It should be noted that in [6] the proof that the conditions of the disjointness criterion are satisfied by time-changes of horocycle flows is based on the asymptotic of ergodic averages for horocycle flows given by A. Bufetov and the second author in [2], which is a refinement of similar asymptotic results of [5]. These asymptotic results are in turn based on the study of the cohomology of horocycle flows and on renormalization by the geodesic flow.

Theorem 1 implies that the flow $h^\tau_t$ satisfies the AOP property\(^1\) if the function $\tau$ is not cohomologous to a constant. The AOP property was introduced in the paper [4] by El Abdalaoui, Lemańczyk and de la Rue in order to advance in the study of Sarnak’s Möbius orthogonality conjecture([8]).

\(^1\)AOP stands for “Asymptotical orthogonal powers”.

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three authors prove, in the quoted article, the following consequence of the AOP property: let \((X, T)\) be a topological uniquely ergodic dynamical system measurably conjugated to a measurable totally ergodic automorphism \(S\) of a standard probability space \((Y, m)\); then \((X, T)\) satisfies the Möbius orthogonality conjecture: for any continuous function \(f \in C(X)\) of average zero and any \(x \in X\), the Möbius function \(\mu\) satisfies the identity

\[
\lim_{N} \frac{1}{N} \sum_{i=0}^{N-1} f(T^ix)\mu(i) = 0.
\]

Thanks to this work we can conclude, from the above theorem, that the Möbius orthogonality conjecture holds for all non-trivial smooth time-changes of horocycle flows on compact quotients:

**Theorem 2.** Let \(\tau \in W^s(\Gamma \backslash G)\) be a strictly positive function of Sobolev order \(s > 2\), not cohomologous to a constant, and let \(h_\tau^t\) be the corresponding time change of the horocycle flow \(h_t\). Any topological uniquely ergodic dynamical system measurably conjugated to the time-one map \(h_\tau^t\) of the flow \(h_\tau^t\) satisfies the Möbius orthogonality conjecture.

For horocycle flows on compact quotients the Möbius orthogonality conjecture was proved J. Bourgain, P. Sarnak and T. Ziegler [1]. Hence it also holds for trivial time changes with continuous transfer function, as they are topologically conjugated to the horocycle flow. To the authors best knowledge, it is an open question whether the Möbius orthogonality conjecture holds for all trivial time changes, that is, for all time-changes with measurable or square-integrable transfer function. However, it follows from the authors results in [5] that, within the space of measurably trivial time-changes, the subspace of those with continuous transfer function has finite codimension.

## 2. Setting

In the following we shall have \(G = \text{PSL}_2(\mathbb{R})\), and \(\Gamma, \hat{\Gamma} < G\) co-compact lattices. We denote \(K = \text{PSO}(2) < G\), the usual maximal compact subgroup of \(G\).

The following matrices form a basis Lie algebra \(\mathfrak{sl}_2(\mathbb{R})\) of \(G\):

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

Then \(\Theta = (U - V)/2\) is the generator of \(K\).

The flows \(g_\tau\) and \(h_t\) on quotients \(\Gamma \backslash G\) respectively given by right multiplication by the one-parameter groups \(e^{\tau X}\) and \(e^{tU}\) are, by definition, the (classical) geodesic and horocycle flow on \(\Gamma \backslash G\). They preserve the probability measure \(\mu\) on \(\Gamma \backslash G\) locally defined by a Haar volume form.

Let \(\Delta\) and \(\Box\) be the elements of the enveloping algebra of \(\mathfrak{sl}_2(\mathbb{R})\) defined by \(\Delta = -X^2 - U^2/2 - V^2/2\) and \(\Box = -X^2 - UV/2 - VU/2\). Then \(\Delta\) is positive definite and coincides on \(K\)-invariant function with the Laplace-Beltrami operator for the hyperbolic metric on the Riemann surface \(\Gamma \backslash G/K\).

For any unitary representation of \(G\) on a Hilbert space \(H\) we denote by \(W^s(H)\) the space of Sobolev vectors of order \(s \in \mathbb{R}^+\), i.e. the closed domain of the operator \((1 + \Delta)^{s/2}\). The space \(W^s(H)\) is a Hilbert space for the norm \(\|f\|_s := \|(1 + \Delta)^{s/2}f\|_H\). If \(H\) decomposes as a direct Hilbert sum \(H = \bigoplus_{\alpha \in I} H_\alpha\) of \(G\)-invariant closed subspaces \(H_\alpha\), then the Sobolev space \(W^s(H)\) also splits as a Hilbert direct sum \(W^s(H) = \bigoplus_{\alpha \in I} W^s(H_\alpha)\) of the mutually orthogonal and \(G\)-invariant Sobolev spaces \(W^s(H_\alpha)\). We write \(W^s(\Gamma \backslash G)\) for \(W^s(L^2(\Gamma \backslash G))\). Clearly \(0 \leq s < t\) implies a continuous embedding of \(W^t(H)\) into \(W^s(H)\).
As $G$ acts on $\Gamma \backslash G$ preserving the measure $\mu$, we have a unitary representation of $G$ on $L^2(\Gamma \backslash G)$. It is well known that this representation decomposes as a Hilbert sum of mutually orthogonal irreducible (or primary) sub-representations. We recall that these are parametrized by the spectrum of the Casimir operator $\square$ previously defined\footnote{Functions in $L^2(\Gamma \backslash G)$ which are $K$-invariant are naturally identified with function on the Riemann surface $\Gamma \backslash G/K$. As in \cite{5}, the Casimir operator $\square$ is normalized so to coincide, on these functions, with the Laplacian-Beltrami operator of the Riemannian metric of curvature $-1$ on $\Gamma \backslash G/K$.}. 

In fact, the spectrum of the Casimir $\square$ consists of finitely many values in the interval $(0, 1/4)$, infinitely many countable real values in the interval $(1/4, \infty)$ and the integer values $-n^2 + n$, with $n = 1, 2, \ldots$. (Such subdivision of the spectrum correspond to the classification of irreducible unitary representations of $G$ into complementary, principal and discrete series.)

Given a unitary representation of $G$ on a Hilbert space $H$, the space of distribution of Sobolev order $s \geq 0$ is, by definition, the space $W^{-s}(H)$ dual to $W^s(H)$. A distribution $D \in W^{-s}(H)$ is invariant for the horocycle flow if $D(f \circ h_t) = D(f)$ for all $t \in \mathbb{R}$, and for any $f \in W^s(H)$. The Sobolev order of a distribution $D \in W^{-\infty}(H) := \bigcup_{s \geq 0} W^{-s}(H)$ is the extended real number

$$s_D := \inf \{s \geq 0 \mid D \in W^{-s}(H)\}.$$ 

We recall the following theorem proved in \cite{5}.

**Theorem 3.** Let $H$ be a Hilbert space on which $\text{PSL}_2(\mathbb{R})$ acts by a unitary irreducible non-trivial representation $\rho$. The subspace $\mathcal{I}(H)$ of $\rho$, invariants distributions in $W^{-\infty}(H)$ has dimension one or two. More precisely:

- If $\rho$ belongs to the principal series $\mathcal{I}(H)$ is spanned by two distributions of Sobolev order $1/2$.
- If $\rho$ belongs to the complementary series\footnote{At most a finite number of such representations occur in $L^2(\Gamma \backslash G)$.} $\mathcal{I}(H)$ is spanned by two distributions of Sobolev order $1/2 - \delta$ and $1/2 + \delta$, with $\delta < 1/2$.
- If $\rho$ belongs to the discrete series $\mathcal{D}_n$ with Casimir value $\square = -n^2 + n$ and $n \in \{1, 2, \ldots\}$, the space $\mathcal{I}(H)$ is a one-dimensional formed by distributions of Sobolev order $n$.

Furthermore, the space $\mathcal{I}(H)$ is invariant under the action of the geodesic flow $g_t$. If $D \in \mathcal{I}(H)$, with Sobolev order $s_D$, then, for some $\lambda_D$ depending on the Casimir value $\square$, we have

$$\langle (g_u)_* D, e^{-\lambda_D u} D \rangle \quad \text{with} \quad \Re \lambda_D = s_D,$$

unless $\square = 1/4$. If $\square = 1/4$ there is a basis of $\mathcal{I}(H)$, for which the matrix of $(g_u)_*$ on $\mathcal{I}(H)$, is given by

$$e^{-u/2} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$ 

We recall that a function $f$ on $\Gamma \backslash G$ is a co-boundary for the horocycle flow $h_t$ with a primitive $g \in W^s(\Gamma \backslash G)$ if, for all $s \in \mathbb{R}$, we have

$$g \circ h_t - g = \int_0^t f \circ h_u \, du.$$ 

Clearly a smooth co-boundary with a smooth primitive is in the kernel of all $h_t$-invariant distributions. The following theorem follows immediately from \cite{5, Theorem 1.3}.

**Theorem 4.** For any $s > 1$, if $f \in W^s(\Gamma \backslash G)$ is a co-boundary for the horocycle flow with a primitive $g \in L^2(\Gamma \backslash G)$, then $D(g) = 0$ for all $h_t$-invariant distributions $D$ of Sobolev order $s_D \leq 1$.

Vanishing of $h_t$-invariant distributions characterizes the space of co-boundaries. More precisely, \cite[Theorems 1.1 and 1.2]{5} implies the following.
Theorem 5. Let $s > 1$. Suppose $f \in W^s(\Gamma \backslash G)$ satisfies $D(f) = 0$ for all $h_t\!-\!$invariant distributions $D$ of Sobolev order $s_D \leq 1$. Then $f$ is a co-boundary with primitive $g \in W^t(\Gamma \backslash G)$ for any $t < \min(1, s - 1)$.

The above theorem needs a bit of explaining, as it is not formulated as such in [5]. By Theorem 3 above, for any $s_1 \in (1, 2)$, the space $\mathcal{I}^{s_1}$ of $h_t\!-\!$invariant distribution of order $\leq s_1$ coincides with $\mathcal{I}^1$. By hypothesis, there exists $s_1 \in (1, 2)$ such that $s_2 \leq s_1$; consequently $g \in W^{s_1}(\Gamma \backslash G)$. Then [5, Theorem 1.2] implies that $g$ is a co-boundary with a primitive in $W^{s_1 - 1 - \epsilon}(\Gamma \backslash G)$, for any $\epsilon > 0$.

The main tool on the proof of Theorem 1 is the classification of the joinings of time changes of horocycle flows proved by Marina Ratner. Ratner’s theorem states

Theorem 6 (Ratner, [7]). Let $\tau_1, \tau_2 \in C^1(\Gamma \backslash G)$ be strictly positive functions and suppose that they have the same average with respect to the Haar measure. If there exists a non-trivial joining of the reparametrized flows $h_{\tau_1}$ and $h_{\tau_2}$ then there exists a finite index subgroup $\Gamma \subset \Gamma$ and finite $G\!-\!$equivariant covers

$$p_1 : \hat{\Gamma} \backslash G \to \Gamma \backslash G \quad \text{and} \quad p_2 : \hat{\Gamma} \backslash G \to \Gamma \backslash G$$

such that the function

$$\tau_1 \circ p_1 - \tau_2 \circ p_2$$

is cohomologous to 0 for the flow $h_t$ on $\hat{\Gamma} \backslash G$. Under the assumption that $\tau_1, \tau_2 \in W^s(\Gamma \backslash G)$ with $s > 2$ the primitive of the function $\tau_1 \circ p_1 - \tau_2 \circ p_2$ belongs to $W^t(\hat{\Gamma} \backslash G)$, for all $t < 1$.

We remark that, by the Sobolev embedding theorem, the theorem holds under the assumption that $\tau_1, \tau_2 \in W^s(\Gamma \backslash G)$ for any $s > 5/2$. As the original theorem yields only a measurable primitive, the last part of the statement requires the cocycle rigidity lemma proved below.

Lemma 1. A function $f \in W^s(\Gamma \backslash G)$ with $s > 2$ is a co-boundary for the horocycle flow $h_t$ with measurable primitive if and only if it is a co-boundary with primitive in $W^t(\Gamma \backslash G)$, for all $t < 1$.

Proof. Since the converse implication is immediate, let us assume that $f$ is a co-boundary with measurable primitive and derive that in fact the primitive belongs to $W^t(\hat{\Gamma} \backslash G)$, for all $t < 1$. By Luzin’s theorem, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that the following holds. For any $t > 0$ there exists a measurable set $B_{c,t} \subset \Gamma \backslash G$ volume $\text{vol}(B_{c,t}) \geq 1 - \epsilon$ such that

$$\left| \int_0^t f(h_u x) du \right| \leq C_\epsilon, \quad \text{for all } x \in B_{c,t}.$$  

If the function $f$ is not in the kernel of all horocycle invariant distributions supported on irreducible representations of the principal and complementary series, it follows from the results of the authors [5, Theorem 5] that the $L^2$ norm of ergodic integrals $f$ diverge (polynomially) and every weak limit of the random variables

$$\frac{\int_0^t f(h_u x) du}{\| \int_0^t f(h_u x) du \|_{L^2(\hat{\Gamma} \backslash G)}},$$

in the sense of probability distributions, is a (compactly supported) distribution on the real line not supported at the origin. The argument is given in detail in the proof of [5, Corollary 5.6] which established that the Central Limit Theorem does not hold for horocycle flows. Refined theorems on limit distributions for horocycle flows were proved in [2] but under the slightly stronger (technical) assumption that $f \in W^s(\Gamma \backslash G)$ with $s > 11/2$. It follows that in this case property (1) implies that $f$ is in the kernel of all invariant distributions supported on irreducible representations of the principal and complementary series.

It follows then from the results of [5] (in particular, from the formulas for invariant distributions of sections 3.1 and 3.2, and from Theorem 5 above) that $f$ is cohomologous to a function given by
a harmonic form on the unit tangent bundle, with a primitive in $W^t(\Gamma \backslash G)$ for all $t < 1$. Our argument is then reduced to the proof that a non-zero harmonic form cannot be a co-boundary with measurable primitive. Indeed, this statement follows from results of D. Dolgopyat and O. Sarig \cite{3} on the so-called windings of the horocycle flow, for instance, from \cite[Theorems 3.2 and 5.1]{3} or from \cite[Lemma 5.10]{3}.

\hfill \Box

\section{Proof of Theorem 1}

Let $\tau \in W^s(\Gamma \backslash G)$ be a strictly positive function of Sobolev order $s > 2$, and let $h^*_i$ be the corresponding time change of $h_i$. Suppose $0 < p < q$.

The flows $h^{*p}_{\mu}$ and $h^{*q}_{\mu}$ can be rewritten as the flows $h^{*p'}_{\mu}$ and $h^{*q'}_{\mu}$ with $p'' = \tau/p$ and $q'' = \tau/q$.

Now setting $\tau_1 = \tau \circ g_{\sigma_1}$ with $\sigma_1 = (\log p)/2$ and $\tau_2 = \tau \circ g_{\sigma_2}$ with $\sigma_2 = (\log q)/2$ we have

$$g_{\sigma_1} \circ h^{*p}_{\mu} = g_{\sigma_1} \circ h^{*p'}_{\mu} = h^{*p'}_{\mu} \circ g_{\sigma_1} \quad \text{and} \quad g_{\sigma_2} \circ h^{*q}_{\mu} = g_{\sigma_2} \circ h^{*q'}_{\mu} = h^{*q'}_{\mu} \circ g_{\sigma_2}.$$ 

Let $\rho$ be a joining of the flows $h^{*p}_{\mu}$ and $h^{*q}_{\mu}$. Then the measure $\rho_{p,q} = (g_{\sigma_1} \times g_{\sigma_2})_*\rho$ is a joining of the flows $h^{*p}_{\mu}$ and $h^{*q}_{\mu}$.

By Ratner's Theorem 6, if $\rho_{p,q}$ is not the product joining, there exist a finite index subgroup $\Gamma_0$ of $\Gamma$ and two coverings $p_i : \Gamma_0 \backslash G \to \Gamma \backslash G$ commuting with the right action of $G$, such that the functions $\tau_1 \circ p_i$ are cohomologous for the horocycle flow $h_i$ on $\Gamma_0 \backslash G$.

In conclusion, the functions $\tau \circ g_{\sigma_1} \circ p_1$ and $\tau \circ g_{\sigma_2} \circ p_2$, are cohomologous for the horocycle flow on $\Gamma_0 \backslash G$. Equivalently, $\tau \circ p_1$ and $\tau \circ p_2 \circ g_{\sigma_2 - \sigma_1}$, are cohomologous functions for the horocycle flow on $\Gamma_0 \backslash G$. Since $p < q$ we have $\sigma := \sigma_2 - \sigma_1 > 0$.

Denote by $T^s(\Gamma \backslash G)$ and $T^s(\Gamma_0 \backslash G)$ the space of $h_i$-invariant distributions of order $\leq s$ on the respective spaces. Our goal is to show that

\begin{equation}
D(\tau) = 0
\end{equation}

for all $h_i$-invariant distributions $D \in T^1(\Gamma \backslash G)$.

What we know, by theorem 4, is that

\begin{equation}
D(\tau \circ p_1 - \tau \circ p_2 \circ g_{\sigma_2 - \sigma_1}) = 0
\end{equation}

for all $D \in T^1(\Gamma_0 \backslash G)$, as the function $\tau \circ p_1 - \tau \circ p_2 \circ g_{\sigma_2 - \sigma_1}$ is a co-boundary with a primitive in $W^t(\Gamma_0 \backslash G)$ for any $t < \min(1, s - 1)$.

Remark that Sobolev norms on $\Gamma \backslash G$ are defined by the local right $G$ action on $\Gamma \backslash G$. The pullback operators

$$p_i^*: W^s(\Gamma \backslash G) \to W^s(\Gamma_0 \backslash G), \quad p_i^*(f) = f \circ p_i, \quad f \in W^s(\Gamma \backslash G),$$

intertwine the $G$ actions on these spaces; hence they preserve the norms in any Sobolev space, i.e. they are isometric embeddings of $W^s(\Gamma_0 \backslash G)$ onto $G$-invariant subspaces $V_i \subset W^s(\Gamma_0 \backslash G)$. Regarding $p_i^*$ as a $G$-equivariant isomorphisms $W^s(\Gamma \backslash G) \to V_i$ we have dual $G$-equivariant isometric isomorphims $(p_i)_*: V_i^* \to W^{-s}(\Gamma \backslash G)$, which is the restriction to $V_i^*$ of the surjective map $(p_i)_*.$

Consider the the orthogonal $G$-invariant decomposition

$$W^s(\Gamma \backslash G) = V_i \oplus V_i^\perp,$$

and the associated orthogonal projections $\pi_i, \pi_i^\perp$ and inclusions $j_i, j_i^\perp$.

Define a $G$-equivariant linear isometric immersion

$$J_i : W^{-s}(\Gamma \backslash G) \to W^{-s}(\Gamma_0 \backslash G)$$

by

$$J_i(D)(f) = \begin{cases} D(g), & \text{if } f \in V_i, \text{ with } f = p_i^*g \quad (g \in W^{-s}(\Gamma \backslash G)); \\ 0 & \text{if } f \in V_i^\perp. \end{cases}$$

Clearly $(p_i)_* \circ J_i(D)(g) = D(g)$ for all $g \in W^{-s}(\Gamma \backslash G)$, i.e. $(p_i)_* \circ J_i(D) = D$. 


Set \( P_1 = (p_2)_* \circ J_1, \quad P_2 = (p_1)_* \circ J_2. \)

By definition both these operators are weak contractions since they are compositions of a contraction and of an isometry.

As both maps \( P_i \) intertwine the \( G \)-action, they map into themselves the spaces \( \mathcal{I}(\mu) \) of \( h_i \)-invariant distributions of a given Casimir parameter \( \mu \), and also the generalised eigenspaces of \( (g_\sigma)_* \) on \( \mathcal{I}(\mu) \) (Jordan blocks for \( (g_\sigma)_* \) appear only when \( \mu = 1/4 \)).

Let \( \mathcal{I}_\alpha \subset \mathcal{I}(\mu) \) be the eigenspace defined by

\[ (g_\sigma)_*, D = e^{\alpha \sigma} D. \]

Recall that we always have \( \Re(\alpha) < 0 \). Suppose \( D \in \mathcal{I}_\alpha \) is an eigenvector of \( P_1 \) of eigenvalue \( \lambda \) with \( |\lambda| \leq 1 \). Then, for all \( \sigma \in \mathbb{R} \),

\[
D(\tau) = J_1(D)(\tau \circ p_1) = J_1(D)(\tau \circ p_2 \circ g_\sigma) \\
= e^{\alpha \sigma} J_1(D)(\tau \circ p_2) = e^{\alpha \sigma} P_1(D)(\tau) = e^{\alpha \sigma} \lambda \tau.
\]

This implies that \( D(\tau) = 0 \), unless \( e^{\alpha \sigma} = 1 \). In particular, since \( |\lambda| \leq 1 \) and \( \sigma \Re(\alpha) < 0 \), this implies \( D(\tau) = 0 \).

Let \( \mathcal{E} \subset \mathcal{I}_\alpha \) be a \( P_1 \)-invariant subspace of maximal dimension such that \( E(\tau) = 0 \) for all \( E \in \mathcal{E} \). We claim that \( \mathcal{E} = \mathcal{I}_\alpha \). Otherwise \( P_1 \) has an eigenvector \( D + \mathcal{E} \) in the quotient space \( \mathcal{I}_\alpha / \mathcal{E} \), of eigenvalue \( \lambda \) with \( |\lambda| \leq 1 \). Then, for some \( E \in \mathcal{E} \),

\[
D(\tau) = J_1(D)(\tau \circ p_1) = J_1(D)(\tau \circ p_2 \circ g_\sigma) \\
= e^{\alpha \sigma} J_1(D)(\tau \circ p_2) = e^{\alpha \sigma} P_1(D)(\tau) = e^{\alpha \sigma}(\lambda D + E)(\tau) = e^{\alpha \sigma} \lambda \tau.
\]

From this follows, as above that \( D(\tau) = 0 \), proving the claim.

If \( \mu = 1/4 \) we have proved that, for all \( D \) belonging to the eigenspace \( \mathcal{I}_{-1/2} \subset \mathcal{I}(1/4) \), we have \( D(\tau) = 0 \). Since \( \mathcal{I}_{-1/2} \) is \( P_1 \) invariant, we can pass to the quotient space \( \mathcal{I}(1/4)/\mathcal{I}_{-1/2} \). The action of \( g_\sigma \) on this subspace is diagonalizable (with eigenvalue \( e^{-\sigma/2} \)).

Let \( D + \mathcal{I}_{-1/2} \subset \mathcal{I}(1/4)/\mathcal{I}_{-1/2} \) be an eigenvector of \( P_1 \) of eigenvalue \( \lambda \) with \( |\lambda| \leq 1 \). Then, for some \( E, E' \in \mathcal{I}_{-1/2} \)

\[
D(\tau) = J_1(D)(\tau \circ p_1) = J_1(D)(\tau \circ g_\sigma \circ p_2) \\
= P_1(D)(\tau \circ g_\sigma) = \lambda D(\tau \circ g_\sigma) + E(\tau \circ g_\sigma) \\
= [e^{-\sigma/2} \lambda D + E'](\tau) + e^{-\sigma/2} E(\tau) = e^{\lambda \tau},
\]

This proves that \( D(\tau) = 0 \) for all \( D \in \mathcal{I}(1/4) \) which are eigenvectors of \( P_1 \) on \( \mathcal{I}(1/4)/\mathcal{I}_{-1/2} \). We can prove, in a similar way as above, that the maximal subspace of \( \mathcal{I}(1/4)/\mathcal{I}_{-1/2} \) annihilating \( \tau \) coincides with \( \mathcal{I}(1/4)/\mathcal{I}_{-1/2} \), and conclude that \( D(\tau) = 0 \) for all \( D \in \mathcal{I}(1/4) \).

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