On the slope of hyperelliptic fibrations with positive relative irregularity

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Abstract

Let \( f : S \to B \) be a locally non-trivial relatively minimal fibration of hyperelliptic curves of genus \( g \geq 2 \) with relative irregularity \( q_f \). We show a sharp lower bound on the slope \( \lambda_f \) of \( f \). As a consequence, we prove a conjecture of Barja and Stoppino on the lower bound of \( \lambda_f \) as an increasing function of \( q_f \) in this case, and we also prove a conjecture of Xiao on the ampleness of the direct image of the relative canonical sheaf if \( \lambda_f < 4 \).

1. Introduction

Let \( f : S \to B \) be a fibration (or a family) of curves of genus \( g \geq 2 \), i.e., \( S \) (resp. \( B \)) is a nonsingular complex surface (resp. curve) and the general fiber of \( f \) is a nonsingular complex curve of genus \( g \). If the general fiber of \( f \) is a hyperelliptic curve, then we call \( f \) a hyperelliptic fibration. \( f \) is called relatively minimal, if there is no \((-1)\)-curve contained in fibers of \( f \). Here a curve \( C \) is called a \((-k)\)-curve if it is a smooth rational curve with self-intersection \( C^2 = -k \). Without other statements, we always assume that fibrations in this note are relatively minimal. \( f \) is called smooth if all its fibers are smooth, isotrivial if all its smooth fibers are isomorphic to each other, locally trivial if it is both smooth and isotrivial, and semi-stable if all its singular fibers are semi-stable. Here a singular fiber \( F \) of \( f \) is called semi-stable if it is a reduced nodal curve.

Let \( \omega_S \) (resp. \( K_S \)) be the canonical sheaf (resp. the canonical divisor) of \( S \). Denote by \( \omega_{S/B} = \omega_S \otimes f^* \omega_B^\vee \) (resp. \( K_f = K_{S/B} = K_S - f^* K_B \)) the relative canonical sheaf (resp. the relative canonical divisor) of \( f \). If \( f \) is relatively minimal, \( K_f \) is numerical effective (nef), i.e., \( K_f \cdot C \geq 0 \) for any curve \( C \subseteq S \). Set \( b = g(B) \), \( p_g = h^0(S, \omega_S) \), \( q = h^1(S, \omega_S) \), \( \chi(O_S) = p_g - q + 1 \), and let \( \chi_{\text{top}}(S) \) be the topological Euler characteristic of \( S \). We consider the following relative invariants of \( f \):

\[
\begin{align*}
\chi_f &= \deg f_* \omega_{S/B} = \chi(O_S) - (g - 1)(b - 1), \\
K_f^2 &= \omega_{S/B} \cdot \omega_{S/B} = K_S^2 - 8(g - 1)(b - 1), \\
e_f &= \chi_{\text{top}}(S) - 4(g - 1)(b - 1),
\end{align*}
\]

They satisfy the Noether’s formula:

\[
12\chi_f = K_f^2 + e_f. \tag{1-1}
\]

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If \( f \) is relatively minimal, then these invariants are nonnegative, and \( \chi_f = 0 \) (equivalently, \( K_f^2 = 0 \)) if and only if \( f \) is locally trivial. \( e_f = 0 \) iff \( f \) is smooth.

The relative irregularity \( q_f \) of \( f \) is defined to be

\[
q_f = q - b.
\]

It is clear that \( 0 \leq q_f \leq g \). \( q_f = g \) if and only if \( S \) is birational to \( B \times F \) (cf. \[4\]). And for \( b \geq 1 \), \( q_f = 0 \) if and only if \( f \) is the Albanese map of \( S \).

If \( f \) is not locally trivial, the slope of \( f \) is defined to be

\[
\lambda_f = \frac{K_f^2 \chi_f}{\lambda_f}.
\]

It follows immediately that \( 0 < \lambda_f \leq 12 \). It turns out that the slope of a fibration is sensible to a lot of geometric properties, both of the fibers of \( f \) and of the surface \( S \) itself (cf. \[2\]). We are mainly concerned with a lower bound of the slope. The main known result in this direction is the slope inequality:

\[
\text{If } g \geq 2 \text{ and } f \text{ is not locally trivial, then } \lambda_f \geq \frac{4(g - 1)}{g}. \quad (1-2)
\]

It was first proven by Horikawa and Persson for hyperelliptic fibrations. Xiao gave a proof for general fibrations (cf. \[21\]), and independently, Cornalba and Harris proved it for semi-stable fibrations (cf. \[8\]).

We would like to pay attention to the influence of the relative irregularity \( q_f \) on the slope \( \lambda_f \) of \( f \). It seems that the lower bound of \( \lambda_f \) should be an increasing function of \( q_f \).

The main influence of \( q_f \) is the following Fujita decomposition (cf. \[11\], see also \[14\]):

\[
f_\ast \omega_{S/B} = A \oplus F \oplus \mathcal{O}_B^{\oplus q_f},
\]

with \( A \) ample, \( F \) unitary and \( \dim H^1(B, \Omega^1_B(F)) = 0 \). The first result in this direction is due to Xiao (\[21\]):

\[
\text{If } q_f > 0, \text{ then } \lambda_f \geq 4 \text{ and the equality holds only if } q_f = 1.
\]

In particular, \( q_f = 0 \) if \( \lambda_f < 4 \). He made the following conjecture (\[21\], Conjecture 2):

**Conjecture 1.1** (Xiao). For any locally non-trivial fibration \( f \), \( f_\ast \omega_{S/B} \) has no locally free quotient of degree zero (i.e., \( f_\ast \omega_{S/B} \) is ample) if \( \lambda_f < 4 \).

The above conjecture is confirmed to be true by Bajaj and Zucconi (cf. \[3\]) under the assumption that \( f \) is non-hyperelliptic or \( g \) (or \( b \)) is small.

Related to the influence of \( q_f \) on the lower bound of \( \lambda_f \), Xiao asked the following question (\[22\], Problem 4):

**Problem 1.2** (Xiao). Let \( f : S \to B \) be a fibration of genus \( g \geq 2 \), which is not locally trivial. Find a good relationship between \( \lambda_f \), \( q_f \) and \( g \).

After that, there are lots of important results in this direction, see for instance, \[3\], \[5\], \[9\], \[15\], \[16\] \[18\]. Some explicit lower bounds depending on \( q_f \) are also given in these literature. Recently, Barja and Stoppino made the following conjecture (\[3\], Conjecture 1.1)].

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Conjecture 1.3 (Barja-Stoppino). Let $f : S \to B$ be as in Problem 1.2. If $q_f < g - 1$, then

$$\lambda_f \geq \frac{4(g-1)}{g-q_f}. \quad (1-4)$$

Under some extra conditions, (1-4) is proved to be true in [3, 9, 18]. There are many evidences for this conjecture provided in [3]. We remark that if $q_f = g - 1$, (1-4) is known to be false (cf. [3, 19]).

We are mainly interested in the lower bound of the slope of hyperelliptic fibrations, especially those with positive relative irregularity. Let

$$\lambda_{g,q_f} = \begin{cases} 
8 - \frac{4(g+1)}{(q_f+1)(g-q_f)}, & \text{if } q_f \leq \frac{g-1}{2}; \\
\frac{8(g-1)}{g}, & \text{if } g \text{ is even, and } q_f = \frac{g}{2}; \\
8, & \text{if } g \text{ is odd, and } q_f = \frac{g+1}{2}. 
\end{cases} \quad (1-5)$$

The main result is the following.

Theorem 1.4. Let $f : S \to B$ be a locally non-trivial fibration of hyperelliptic curves of genus $g \geq 2$ with relative irregularity $q_f$. Let $\lambda_{g,q_f}$ be defined in (1-5). Then $q_f \leq \frac{g+1}{2}$, and

$$\lambda_f \geq \lambda_{g,q_f}. \quad (1-6)$$

We will present examples to show that the bound (1-6) is sharp. It is not difficult to show that $\lambda_{g,q_f} \geq \frac{4(g-1)}{g-q_f}$. Therefore, we obtain

Corollary 1.5. For a locally non-trivial hyperelliptic fibration $f$, Conjecture 1.3 is true, and the equality of (1-4) can hold only if $q_f = 0$, $\frac{g-1}{2}$, $\frac{g}{2}$ or $\frac{g+1}{2}$. In particular, $g \leq 3$ if $q_f = 1$ and $\lambda_f = 4$.

According to [13, Theorem 1.6], it can be shown that $\mathcal{F} = 0$ (i.e., there is no non-trivial unitary part) in Fujita’s decomposition (1-3) after a suitable finite étale base change. Hence by our theorem, Conjecture [11] is true when $f$ is hyperelliptic. Combining with the result of Bajaj and Zuccuni (cf. [4]), we prove

Corollary 1.6. Conjecture 1.1 is true.

Our note is organized as follows. In Section 2, we review some basic properties about a hyperelliptic fibration $f : S \to B$ mainly due to Xiao Gang. By blowing up the isolated fixed points of the hyperelliptic involution, we get a double cover $\pi : \tilde{S} \to \tilde{P}$ of smooth projective surfaces. We then define the local relative invariants $s_i$ for $2 \leq i \leq g + 2$, and show in Theorem 2.7 that the global relative invariants of $f$ can be expressed by those local invariants. In Section 3, we restrict ourselves to the case that the relative irregularity is positive, and prove an inequality (3-1) involving these invariants $s_i$’s. The proof starts from the observation that the double cover $\tilde{\pi} : \tilde{S} \to \tilde{P}$ is fibred. In Section 4, we prove Theorem 1.4 and its corollaries. When $q_f = 0$, (1-6) is nothing new but (1-2). If $q_f > 0$, (1-6) follows from (3-1) and the formulas given in Theorem 2.7. Finally in Section 5, we present examples to show that the bound (1-6) is sharp.
2. Preliminaries

2.1. Double covers

In this subsection, we review some basic properties of double covers (cf. [7 §V.22] and [23 §2]).

A double cover \( \pi : X \to Y \) of a smooth projective surface \( Y \) is determined by a line bundle \( L \) over \( Y \) and a section \( s \in H^0(Y, L^2) \) as

\[
X = \text{Proj} \left( \bigoplus_{i=0}^{+\infty} L^{-i} \right) / \langle s^2 \rangle.
\]

Let \( R \) be the zero divisor of \( s \). Then \( X \) is smooth if and only if \( R \) is smooth. It is well known that

\[
\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus L^{-1},
\]

where the decomposition on the right side is the eigenspace decomposition w.r.t \( \text{Gal}(X/Y) \)-action on \( \pi_* \mathcal{O}_X \).

To obtain a smooth double cover from a double cover \( \pi : X \to Y \) of a smooth projective surface \( Y \), we perform the canonical resolution (cf. [7 §III.7]).

\[
\tilde{X} \xrightarrow{\phi} X_i \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_0} X_0 = X
\]

\[
\tilde{Y} \xrightarrow{\psi} Y_i \xrightarrow{\psi_{i-1}} \cdots \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_1} Y_1 \xrightarrow{\psi_0} Y_0 = Y
\]

where \( \tilde{X} = X_i \) is smooth and \( \psi_i \)'s are successive blowing-ups resolving the singularities of \( R \); \( \pi_i : X_i \to Y_i \) is the double cover determined by \( (R_i, L_i) \) with

\[
R_i = \psi_i^*(R_{i-1}) - 2[m_{i-1}/2] E_i, \quad L_i = \psi_i^*(L_{i-1}) \otimes \mathcal{O}_{Y_i} \left( E_i^{-[m_{i-1}/2]} \right),
\]

where \( E_i \) the exceptional divisor of \( \psi_i \), \( m_{i-1} \) is the multiplicity of the singular point \( y_{i-1} \) in \( R_{i-1} \). \([ \ ] \) stands for the integral part, \( R_0 = R \) and \( L_0 = L \). Let \( \tilde{R} = R_t \), \( \tilde{L} = L_t \). The morphism

\[
\psi \triangleq \psi_1 \circ \cdots \circ \psi_t : \tilde{Y} \to Y
\]

is also called a minimal even resolution of \( R \).

We call a singularity \( y_j \in R_j \subseteq Y_j \) is infinitely closed to \( y_{i-1} \in R_{i-1} \subseteq Y_{i-1} \) \(( j \geq i) \), if

\[
\psi_1 \circ \cdots \circ \psi_j(y_j) = y_{i-1}.
\]

**Definition 2.1.** A singularity \( y_{i-1} \in R_{i-1} \subseteq Y_{i-1} \) above is said to be negligible if \( [m_{i-1}/2] = 1 \), and for any \( y_j \in R_j \subseteq Y_j \) \(( j \geq i) \) infinitely closed to \( y_{i-1} \), we have \( [m_j/2] \leq 1 \). In this case, the blowing-up \( \psi_i : Y_i \to Y_{i-1} \) is called a negligible blowing-up.

It is easy to see that \( \psi \) can be decomposed into \( \tilde{\psi} : \tilde{Y} \to \tilde{Y} \) and \( \psi : \tilde{Y} \to Y \), where \( \tilde{\psi} \) and \( \psi \) are composed of negligible and non-negligible blowing-ups respectively. We call \( \psi \) the minimal even resolution of non-negligible singularities of \( R \).
The invariants of $\tilde{X}$ can be computed as follows.

\[
\begin{align*}
K_{\tilde{X}}^2 &= 2 \left( K_{\tilde{Y}} + \tilde{L} \right)^2 = 2(K_Y + L)^2 - 2 \sum_{i=0}^{t-1} \left( \frac{m_i}{2} - 1 \right) \cdot \left( \frac{m_i}{2} \right), \\
\chi(O_{\tilde{X}}) &= 2 \chi(O_{\tilde{Y}}) + \frac{1}{2} (K_{\tilde{Y}} + \tilde{L}) \cdot \tilde{L} \\
&= 2 \chi(O_Y) + \frac{1}{2} (K_Y + L) \cdot L - \frac{1}{2} \sum_{i=0}^{t-1} \left( \frac{m_i}{2} - 1 \right) \cdot \left( \frac{m_i}{2} \right).
\end{align*}
\]

(2.1)

2.2. Invariants of hyperelliptic fibrations

In this subsection, we review some results about a hyperelliptic fibration $f$, which is mainly due to Xiao (cf. [23; §2] and [25; §5.1]).

Let $f: S \to B$ be a relatively minimal hyperelliptic fibration, i.e., the general fiber of $f$ is a hyperelliptic curve. The relative canonical map of $f$ is generically of degree 2. This map determines an involution $\sigma$ on $S$ whose restriction on a general fiber $F$ of $f$ is the hyperelliptic involution of $F$. $\sigma$ is called the hyperelliptic involution associated to $f$.

Let $\vartheta: \tilde{S} \to S$ be the composition of all the blowing-ups of isolated fixed points of the hyperelliptic involution, and let $\tilde{\sigma}$ be the induced involution on $\tilde{S}$. The quotient space $\tilde{P} = \tilde{S}/\langle \tilde{\sigma} \rangle$ is a smooth surface, and $f$ induces a ruling on $\tilde{P}$:

\[
\tilde{h} : \tilde{P} \to B.
\]

The quotient map $\tilde{\pi}: \tilde{S} \to \tilde{P}$ is a double cover which is determined by the pair $(\tilde{R}, \tilde{L})$, where $\tilde{R}$ is the branch locus of $\tilde{\pi}$ and $\tilde{L}$ is the divisor such that

\[
\tilde{\pi}_* O_{\tilde{S}} \cong O_{\tilde{P}} \oplus \tilde{L}^{-1}.
\]

Lemma 2.2 ([23; 25]). There exists a contraction of rational surfaces $\psi: \tilde{P} \to P$:

\[
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\psi} & P \\
\downarrow{\tilde{h}} & & \downarrow{h} \\
B & \xleftarrow{h} & P
\end{array}
\]

such that $P$ is a geometrical ruled surface (i.e., any fiber of $h$ is $\mathbb{P}^1$), the singularities of $R$ are at most of multiplicity $g+2$, and the self-intersection $R^2$ is the smallest among all such choices, where $(\tilde{R}, \tilde{L})$ is the image of $(\tilde{R}, \tilde{L})$ in $P$.

One sees that $\psi: \tilde{P} \to P$ is a minimal even resolution of $R$, and $\psi$ can be decomposed into $\tilde{\psi}: \tilde{P} \to \tilde{P}$ and $\psi: \tilde{P} \to P$ in the following diagram, where $\tilde{\psi}: \tilde{P} \to P$ is a minimal even resolution of non-negligible singularities of $R$. 

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Let \((\hat{R}, \hat{L})\) be the image of \((\tilde{R}, \tilde{L})\) in \(\hat{P}\). Let \(\hat{\psi}_i : \hat{P}_i \to \hat{P}_{i-1}\) be a blowing-up at \(y_{i-1}\), \(\hat{P}_0 = P\) and \(\hat{P}_1 = \hat{P}\). Let \(\hat{R}_i\) be the image of \(\hat{R}\) in \(\hat{P}_i\). It could happen that there is one or more singular points of \(\hat{R}_i\) over the image \(\hat{E}_i\) of \(\hat{\psi}_i\). We remark that the decomposition of \(\hat{\psi}\) is not unique. If \(y_{i-1}\) is a singular point of \(R_{i-1}\) of odd multiplicity \(2k + 1\) \((k \geq 1)\) and there is only one singular point \(y\) of \(\hat{R}_i\) on the exceptional curve \(\hat{E}_i\) of multiplicity \(2k + 2\), then we always assume that \(\hat{\psi}_{i+1} : \hat{P}_{i+1} \to \hat{P}_i\) is a blowing-up at \(y = y_i\).

**Definition 2.3.** For \(k \geq 1\), a singularity of \(R\) of type \((2k + 1 \to 2k + 1)\) is a pair of \((y_{i-1}, y_i)\) such that \(y_{i-1}\) is a singular point of \(R_{i-1}\) of multiplicity \(2k + 1\), and \(y_i\) is the only one singular point of \(R_i\) on the exceptional curve \(\hat{E}_i\) with the multiplicity equal to \(2k + 2\).

**Definition 2.4.** For any singular fiber \(F\) of \(f\) and \(3 \leq i \leq g + 2\), the \(i\)-th singularity index of \(F\) is defined as follows (with respect to the contraction \(\psi\)):

- if \(i\) is odd, \(s_i(F)\) equals the number of \((i \to i)\) type singularities of \(R\) over the image \(f(F)\);
- if \(i\) is even, \(s_i(F)\) equals the number of singularities of multiplicity \(i\) or \(i + 1\) of \(R\) over the image \(f(F)\), not belonging to the second component of \((i - 1 \to i - 1)\) type singularities nor the first component of \((i + 1 \to i + 1)\) type singularities.

We remark that when \(g\) is even, then \(s_{g+2}(F) = 0\); this can be seen from the definition and the assumption that self-intersection \(R^2\) is the smallest among all choices of the contractions in Lemma 2.2. Note also that for \(i \geq 3\), \(s_i(F)\) is non-negative by the definition. As there are finitely many singular fibers contained in \(f\), we also define

\[
s_i \triangleq \sum_{F \text{ is singular}} s_i(F), \quad 3 \leq i \leq g + 2.
\]

Let \(K_{\hat{P}/B} = K_{\hat{P}} - \hat{h}^*K_B\) and \(R' = \hat{R} \setminus \hat{V}\), where \(\hat{V}\) is the union of isolated vertical \((-2)\)-curves in \(\hat{R}\). Here a curve \(C \subset \hat{R}\) is called to be *isolated* in \(\hat{R}\), if there is no other curve \(C' \subset \hat{R}\) such that \(C \cap C' \neq \emptyset\). We define

\[
s_2 \triangleq (K_{\hat{P}/B} + R') \cdot R'.
\]

It is not clear whether \(s_2\) is non-negative or not.

**Lemma 2.5** ([23] [25]). *These singularity indices defined in Definition 2.4 is independent on the choices of \(\psi\) in Lemma 2.2.*
The independence of $s_i$ on the choices of $\psi$ is proved in [23, Lemma 8] for $3 \leq i \leq g + 2$; while the independence of $s_2$ on the choices of $\psi$ is proved in [25, Theorem 5.1.4]. However, one can deduce the later from [23, Lemma 8], which proves that the contraction $\psi : \tilde{P} \to \hat{P}$ is independent on the choices of $\psi$. It follows immediately that $s_2$ is independent on the choices of $\psi$ by the definition of $s_2$.

**Remark 2.6.** In [17, Theorem 1.1], it is proved that if $f$ is semi-stable, then $s_{g+2} = 0$, and $s_2 + 2 \sum_{k=2}^{[g/2]} s_{2k}$ (resp. $s_{2k+1}$ for $1 \leq k \leq [g/2]$) is the number of type 0 (resp. $k$) nodes in fibers of $f$. Here a node $q$ of $F$ is said to be of type 0 (resp. $k$ for $1 \leq k \leq [g/2]$), if the partial normalization of $F$ at $q$ is connected (resp. consists of two connected components of arithmetic genera $k$ and $g - k$).

**Theorem 2.7** ([23, 25]). Let $f : S \to B$ be a fibration of hyperelliptic curves of genus $g \geq 2$, and $s_i$ the singularity indices as above. Then

$$(2g + 1)K_f^2 = (g - 1)(s_2 + (3g + 1)s_{g+2}) + \sum_{k=1}^{[g/2]} a_k s_{2k+1} + \sum_{k=2}^{[g+1/2]} b_k s_{2k},$$

$$(2g + 1)\chi_f = \frac{1}{4}\left(g s_2 + (g^2 - 2g - 1)s_{g+2}\right) + \sum_{k=1}^{[g/2]} k(g - k)s_{2k+1} + \sum_{k=2}^{[g+1/2]} \frac{k(g - k + 1)}{2} s_{2k},$$

$$e_f = s_2 - 2s_{g+2} + \sum_{k=1}^{[g/2]} s_{2k+1} + \sum_{k=2}^{[g+1/2]} 2s_{2k},$$

where $a_k = 12k(g - k) - 2g - 1$ and $b_k = 6k(g - k + 1) - 4g - 2$.

For readers' convenience, we reproduce a proof of Theorem 2.7. To start it, we need the following Lemma.

**Lemma 2.8** ([23, 25]). Let $F$ be a singular fiber of the fibration $f$, and $\tilde{F}$ (resp. $\hat{F}$) the corresponding fiber in $\hat{S}$ (resp. $\hat{P}$). Then those $(-1)$-curves in $\tilde{F}$ are in one-to-one correspondence to isolated $(-2)$-curves of $\hat{R}$, which are also contained in $\hat{F}$. And the number is equal to

$$2s_{g+2}(F) + \sum_{k=1}^{[g/2]} s_{2k+1}(F).$$

**Proof of Theorem 2.7.** Let

$$n = \frac{L^2}{g + 1}. \quad (2.2)$$

Then $K_{P/B} \cdot L = -n$, where $K_{P/B} = K_P - h^*K_B$. As $\psi : \hat{P} \to P$ is a minimal even resolution of non-negligible singularities of $R$, by Definition 2.4, one gets

$$\hat{L}^2 = L^2 - \frac{g^2 + 4g + 5}{2}s_{g+2} - \sum_{k=1}^{[g/2]} (2k^2 + 2k + 1)s_{2k+1} - \sum_{k=2}^{[g+1/2]} k^2 s_{2k},$$

$$K_{\hat{P}/B} \cdot \hat{L} = K_{P/B} \cdot L + (g + 2)s_{g+2} + \sum_{k=1}^{[g/2]} (2k + 1)s_{2k+1} + \sum_{k=2}^{[g+1/2]} ks_{2k}. $$
Hence

\[(K_{\hat{P}/B} + \hat{R}) \cdot \hat{R} = (K_{\hat{P}/B} + 2\hat{L}) \cdot 2\hat{L}\]

\[= 2(2g + 1)n - 2(g^2 + 3g + 3)s_{g+2} - \sum_{k=1}^{\lfloor g/2 \rfloor} 2(4k^2 + 2k + 1)s_{2k+1} - \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} 2k(2k - 1)s_{2k}.\]

On the other hand, by the definition of \(s_2\) and Lemma 2.8, the above number is also equal to

\[s_2 - 4s_{g+2} - 2\sum_{k=1}^{\lfloor g/2 \rfloor} s_{2k+1}.\]

Hence

\[n = \frac{1}{2(2g + 1)}s_2 + \frac{g^2 + 3g + 1}{2g + 1}s_{g+2} + \sum_{k=1}^{\lfloor g/2 \rfloor} \frac{4k^2 + 2k}{2g + 1}s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} \frac{2k^2 - k}{2g + 1}s_{2k}. \quad (2-3)\]

Let \(b = g(B)\) be the genus of \(B\) and \(K_{\hat{P}/B} = K_{\hat{P}} - \hat{h}^*K_B\). Then

\[K_{\hat{P}/B}^2 = -2s_{g+2} - \sum_{k=1}^{\lfloor g/2 \rfloor} 2s_{2k+1} - \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} s_{2k}\]

Note that all singular points of \( \hat{R} \subseteq \hat{P} \) is negligible (cf. Definition 2.1) by construction. Hence by (2-1), we get

\[\chi_f = \chi(\mathcal{O}_{\hat{P}}) - (g - 1)(b - 1) = 2\chi(\mathcal{O}_{\hat{P}}) + \frac{1}{2}(\hat{L}^2 + K_{\hat{P}} \cdot \hat{L}) - (g - 1)(b - 1)\]

\[= \frac{1}{2}(\hat{L}^2 + K_{\hat{P}/B} \cdot \hat{L})\]

\[= \frac{1}{4(2g + 1)}(gs_2 + (g^2 - 2g - 1)s_{g+2}) + \sum_{k=1}^{\lfloor g/2 \rfloor} \frac{k(g - k)}{2g + 1}s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} \frac{k(g - k + 1)}{2(2g + 1)}s_{2k},\]

\[K_f^2 = K_S^2 - 8(g - 1)(b - 1) = 2(\hat{L} + K_{\hat{P}})^2 - 8(g - 1)(b - 1)\]

\[= 2(\hat{L} + K_{\hat{P}/B})^2\]

\[= \frac{g - 1}{2g + 1}s_2 + \frac{3(g^2 - 2g - 1)}{2g + 1}s_{g+2} + \sum_{k=1}^{\lfloor g/2 \rfloor} \left( \frac{a_k}{2g + 1} - 1 \right)s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} \frac{b_k}{2g + 1}s_{2k}.\]

Note that \(\chi_f = \chi_f\), and \(K_f^2 = K_f^2 + 2s_{g+2} + \sum_{k=1}^{\lfloor g/2 \rfloor} s_{2k+1}\) by Lemma 2.8. Combining this with the above equalities and the Noether’s formula (1.1), we prove the theorem. \(\square\)

3. Hyperelliptic fibrations with positive relative regularity

The purpose of the section is to prove the following inequality for a locally non-trivial hyperelliptic fibration with positive relative irregularity.
Proposition 3.1. Let $f : S \to B$ be a hyperelliptic fibration of genus $g$, which is not locally trivial. Let $s_i$ ($2 \leq i \leq g+2$) be the $i$-th singularity index of $f$ defined in Definition 2.4. Assume that the relative irregularity $q_f > 0$. Then

$$s_2 + \sum_{k=1}^{\lfloor g/2 \rfloor} 4k(2k+1)s_{2k+1} + \sum_{k=2}^{\lfloor g/2 \rfloor} 2k(2k-1)s_{2k} \leq \sum_{k=q_f}^{\lfloor (g+1)/2 \rfloor} \frac{(2k+1)(2g+1-2k)}{g+1}s_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \frac{2k(g+1-k)}{g+1}s_{2k} + (g+1)s_{g+2}. \quad (3-1)$$

In order to prove the above proposition, we always assume in the section that $f : S \to B$ is a locally non-trivial hyperelliptic fibration of genus $g$ with positive relative irregularity $q_f$. Let $\tilde{\pi} : \tilde{S} \to \tilde{P}$ be the induced double cover with branched divisor $R \subseteq \tilde{P}$ as in Figure 1. First we recall the following definition.

Definition 3.2 (13). A double cover $\pi : X \to Y$ of smooth projective surfaces with branched divisor $R \subseteq Y$ is called fibered if there exist a double cover $\pi' : C \to D$ of smooth projective curves and morphisms $\varpi : X \to C$ and $\epsilon : Y \to D$, such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\varpi \downarrow & & \downarrow \epsilon \\
C & \xrightarrow{\pi'} & D
\end{array}$$

is commutative, $R$ is contained in the fibers of $\epsilon$, and $q(X) - q(Y) = q(C) - q(D)$, where $g(C)$ (resp. $g(D)$) is the genus of $C$ (resp. $D$), $q(X) = h^1(X, \omega_X)$, and $q(Y) = h^1(Y, \omega_Y)$.

The proof of Proposition 3.1 starts from the observation that the double cover $\tilde{\pi} : \tilde{S} \to \tilde{P}$ in Figure 1 is fibred. It is proved in [13] under the extra assumption that $f : S \to B$ is semi-stable. But the proof there does not use this assumption. Hence one gets

Proposition 3.3 ([13] Propositions 4.4 and 4.5). The double cover $\tilde{\pi} : \tilde{S} \to \tilde{P}$ in Figure 1 is fibred, i.e., there exist a double cover $\pi' : B' \to \mathbb{P}^1$ of smooth projective curves and morphisms $\tilde{f}' : \tilde{S} \to B'$ and $\tilde{h}' : \tilde{P} \to \mathbb{P}^1$, such that the diagram

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{\pi}} & \tilde{P} \\
\tilde{f}' \downarrow & & \downarrow \tilde{h}' \\
B' & \xrightarrow{\pi'} & \mathbb{P}^1
\end{array}$$

is commutative, $\tilde{R}$ is contained in the fibers of $\tilde{h}'$ and

$$q_f = q(\tilde{S}) - q(\tilde{P}) = q(B') \leq \frac{g+1}{2}. \quad (3-2)$$

Remark 3.4. Let $f : S \to B$ be as in the above proposition with $b = g(B) \geq 1$. Xiao ([24]) proved a more precise description on $q_f$:

$$\frac{g-1}{d} - 1 \leq q_f \leq \frac{g-1}{d} + 1,$$

where $d \geq 2$ is the degree of the Albanese map $S \to \text{Alb}(S)$. 

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We would like to show that the branched divisor $\tilde{R}$ of $\tilde{\pi}: \tilde{S} \to \tilde{P}$ has a very special form.

Let $\tilde{f}': \tilde{S} \to B'$ be the fibration in Proposition 3.3. Since $b' = g(B') = q f \geq 1$, it follows that any $(-1)$-curve in $\tilde{S}$ is contracted by $\tilde{f}'$. Hence $\tilde{f}'$ factors through $\vartheta: \tilde{S} \to S$. Let $f': S \to B'$ be the induced map. Note that the fibration $\tilde{f}': \tilde{S} \to B'$ in Proposition 3.3 is clearly unique. Hence the hyperelliptic involution $\sigma$ induces an involution $\sigma'$ on $B'$ such that $B'/\langle \sigma' \rangle \cong \mathbb{P}^1$ with the following diagram:

\begin{center}
\begin{tikzpicture}
  \node (S) {$S$};
  \node (B') at (2,0) {$B'$};
  \node (P) at (0,-1.5) {$\mathbb{P}^1 \cong B'/\langle \sigma' \rangle$};
  \node (S') at (-2,1.5) {$\tilde{S}$};
  \node (B) at (-2,-1.5) {$B$};

  \draw[->] (S) to node[above] {$\tilde{f}'$} (B');
  \draw[->] (S) to node[left] {$\tilde{f}$} (B);
  \draw[->] (S) to node[above] {$\vartheta$} (P);
  \draw[->] (B') to node[below] {$\pi'$} (P);
  \draw[->] (B') to node[right] {$\pi_1$} (P);
  \draw[->] (B') to node[below] {$\pi_2$} (B);

  \draw[->] (S') to node[above] {$\tilde{f}$} (B);
  \draw[->] (S') to node[above] {$\vartheta$} (P);
  \draw[->] (B) to node[below] {$\pi'$} (P);

\end{tikzpicture}
\end{center}

Figure 2: Hyperelliptic fibration with positive relative irregularity.

Assume $\pi': B' \to \mathbb{P}^1$ is branched over $\Delta \subseteq \mathbb{P}^1$. Applying Hurwitz formula to the double cover $\pi'$, one sees that $|\Delta| = 2q f + 2$. For any $y \in \Delta$, let $\tilde{\Gamma}'_y = \sum n'_C C$ be the fiber of $\tilde{h}'$ over $y$, and

$$\tilde{\Gamma}'_{y,o} \triangleq \sum_{C \in \tilde{\Gamma}'_y} n'_C C \subseteq \tilde{\Gamma}'_y.$$ 

According to Proposition 3.3, $\tilde{R}$ is contained in the fibers of $\tilde{h}'$. In fact, we can prove an explicit expression of $\tilde{R}$.

**Lemma 3.5.**

$$\tilde{R} = \sum_{y \in \Delta} \tilde{\Gamma}'_{y,o}.$$ 

**Proof.** Let $B' \times_{\mathbb{P}^1} \tilde{P}$ be the fiber-product, and $X \to B' \times_{\mathbb{P}^1} \tilde{P}$ the normalization. By the universal property of the fiber-product (cf. [12, §II-2]), there exists a unique morphism $\gamma': \tilde{S} \to B' \times_{\mathbb{P}^1} \tilde{P}$. Since $\tilde{S}$ is smooth, there also exists a unique morphism $\gamma: \tilde{S} \to X$, such that the following diagram commutes.

\begin{center}
\begin{tikzpicture}
  \node (S) {$\tilde{S}$};
  \node (B) at (-2,-1.5) {$B'$};
  \node (X) at (-2,0) {$X$};
  \node (P) at (2,0) {$\tilde{P}$};

  \draw[->] (S) to node[above] {$\tilde{\pi}$} (P);
  \draw[->] (B) to node[below] {$\pi'$} (P);
  \draw[->] (X) to node[below] {$\pi_2$} (B);
  \draw[->] (X) to node[above] {$\pi_1$} (P);

  \draw[->] (S) to node[above] {$\vartheta$} (X);
  \draw[->] (B) to node[below] {$\tilde{h}'$} (P);
  \draw[->] (B) to node[above] {$f_{o\vartheta}$} (X);

\end{tikzpicture}
\end{center}
Clearly the composition $\pi_1 \circ \pi_2 : X \to \tilde{P}$ is a double cover branched exactly over

$$\sum_{y \in \Delta} \Gamma_{y, o}.$$ 

Therefore, it suffices to prove that $\gamma$ is an isomorphism.

As $\deg \tilde{\pi} = \deg(\pi_1 \circ \pi_2)$, we get $\deg \gamma = 1$, i.e., $\gamma : \tilde{S} \to X$ is a contraction of curves. Note that $\tilde{\pi}$ does not contract any curves. Neither does $\gamma$ because any curve contracted by $\gamma$ must be also contracted by $\tilde{\pi}$. This completes the proof.

The contraction $\psi : \tilde{P} \to P$ is composed of several blowing-ups. We divide those blowing-ups as $\psi = \tilde{\psi} \circ \bar{\psi}$, where $\bar{\psi} : \tilde{P} \to P$ is the largest contraction such that $h'$ factors through $\bar{\psi}$. So we have the following diagram:

$$\begin{array}{ccc}
P^1 & \cong & B'/\langle \sigma' \rangle \\
\hline
\tilde{P} & \xrightarrow{\h' \psi} & \bar{P} \\
\h & \h' \psi & \h \\
B & \xrightarrow{\h} & P
\end{array}$$

Figure 3: Decomposition of $\psi$.

Next we want to show that each blowing-up contained in $\bar{\psi}$ is centered at a singular point of the branched divisor with multiplicity at least $2q_f + 1$.

Let $(\bar{R}, \bar{L}, \bar{\Gamma}_y')$ be the image of $(\tilde{R}, \tilde{L}, \tilde{\Gamma}_y')$ on $\bar{P}$, where $y \in \Delta$. By the construction, any vertical $(−1)$-curve in $\bar{P}$ (here ‘vertical’ means the curve is mapped by $\bar{h}$ to a point in $B$) is mapped surjectively onto $\mathbb{P}^1$ by $\bar{h}'$.

**Lemma 3.6.** Let $\mathcal{E} \subseteq \bar{P}$ be any vertical $(−1)$-curve. Then $\mathcal{E} \cdot \bar{R}$ is even and

$$\mathcal{E} \cdot \bar{R} \geq 2(q_f + 1).$$

**Proof.** Let $\bar{m} = \mathcal{E} \cdot \bar{R}$, $\sigma : \bar{P} \to \mathbb{P}^1$ the contraction of $\mathcal{E}$, $x$ the image of $\mathcal{E}$, and $(\bar{R}_1, \bar{L}_1)$ the image of $(\bar{R}, \bar{L})$ on $\mathbb{P}^1$. Then $x$ is a singularity of $\bar{R}_1$ of multiplicity $\bar{m}$. Let

$$\Gamma_{y, o}' = \sum_{\substack{C \subseteq \Gamma_y \ \\ \ \ \ \ n'_C \text{ is odd}}} \bar{n}'_C C \quad \text{and} \quad \Gamma_{y, r}' = \sum_{\substack{C \subseteq \Gamma_y \ \ \ \ \ \ n'_C = 1}} C; \quad \text{if} \ \Gamma_y = \sum \bar{n}'_C C.$$ 

By Lemma 3.3,

$$\bar{R} = \bar{\psi}(\tilde{R}) = \sum_{y \in \Delta} \Gamma_{y, o}'$$

is contained in fibers of $\bar{h}'$. Hence $\mathcal{E} \not\subseteq \bar{R}$, from which it follows that $\bar{m} = \mathcal{E} \cdot \bar{R}$ is even.

Let

$$\bar{R}_{\text{all}} = \sum_{y \in \Delta} \Gamma_y' \quad \text{and} \quad \bar{R}_r = \sum_{y \in \Delta} \Gamma_{y, r}'.$$
Then $R_t \subseteq R \subseteq R_{\text{all}}$. To complete the proof, it is enough to prove that

$$E \cdot R_t \geq 2(qf + 1).$$

Note that the restricted morphism $\tilde{h}'|_E : E \rightarrow \mathbb{P}^1$ is surjective. For any $p \in E \cap R_{\text{all}}$, let $r_p = l_p(E, R_{\text{all}})$ be the local intersection number. Then by the definition of $R_{\text{all}}$, one has

$$\sum_{p \in E \cap R_{\text{all}}} r_p = E \cdot R_{\text{all}} = \deg(\tilde{h}'|_E) \cdot (2qf + 2).$$

By the definition, $r_p \geq 2$ for any $p \in (E \cap R_{\text{all}}) \setminus (E \cap R_t)$. On the other hand, as $E$ is a $(-1)$-curve, the ramification number of $\tilde{h}'|_E$ is $2\deg(\tilde{h}'|_E) - 2$. So

$$2\deg(\tilde{h}'|_E) - 2 \geq \sum_{p \in E \cap R_{\text{all}}} (r_p - 1) = \sum_{p \in (E \cap R_{\text{all}}) \setminus (E \cap R_t)} (r_p - 1) + \sum_{p \in E \cap R_t} (r_p - 1) \geq \sum_{p \in (E \cap R_{\text{all}}) \setminus (E \cap R_t)} r_p - \frac{\sum_{p \in E \cap R_t} r_p}{2} = \deg(\tilde{h}'|_E) \cdot (qf + 1) - \frac{\sum_{p \in E \cap R_t} r_p}{2}.$$

Therefore

$$E \cdot R_t \geq |\sum_{p \in E \cap R_{\text{all}}} r_p| \geq 2\deg(\tilde{h}'|_E) \cdot (qf + 1) + 4 \geq 2(qf + 1) + 4 = 2(qf + 1).$$

The proof is complete.

We assume that $\tilde{\psi} = \tilde{\psi}_1 \circ \cdots \circ \tilde{\psi}_u$, where $\tilde{\psi}_i : \tilde{P}_i \rightarrow \tilde{P}_{i-1}$ is a blowing-up at $\tilde{x}_{i-1} \subseteq \tilde{P}_{i-1}$ with exception curve $\tilde{E}_i \subseteq \tilde{P}_i$, $\tilde{P}_0 = P$ and $\tilde{P}_u = \bar{P}$. Let $R_i$ be the image of $R$ in $\tilde{P}_i$, and $\tilde{x}_i$ be a singularity of $\tilde{R}_i$ of multiplicity $\tilde{m}_i$.

**Lemma 3.7.** For $1 \leq i \leq u - 1$, we have either $\tilde{m}_i \geq 2(qf + 1)$; or $\tilde{m}_i = 2qf + 1$, $i \leq u - 2$ and $\tilde{x}_{i+1}$ is only one singular point on $\tilde{E}_{i+1} \subseteq \tilde{P}_{i+1}$ of multiplicity $\tilde{m}_{i+1} = 2qf + 2$.

**Proof.** Note that $\tilde{\psi}$ is a part of the even resolution of $R = \bar{R}_0$. So

$$\begin{cases} 
\text{if } \tilde{m}_i \text{ is even, then } \tilde{E}_{i+1} \not\subseteq \tilde{R}_{i+1}, \text{ and so } \tilde{m}_{i+1} \leq \tilde{m}_i; \\
\text{if } \tilde{m}_i \text{ is odd, then } \tilde{E}_{i+1} \subseteq \tilde{R}_{i+1}, \text{ and so } \tilde{m}_{i+1} \leq \tilde{m}_i + 1.
\end{cases} \quad (3-3)$$

By induction, for any singularity $\tilde{x}_{i+j}$, infinitely closed to $\tilde{x}_i$, we have $\tilde{m}_{i+j} \leq \tilde{m}_i$ if $\tilde{m}_i$ is even, and $\tilde{m}_{i+j} \leq \tilde{m}_i + 1$ if $\tilde{m}_i$ is odd. By Lemma 3.6 $\tilde{m}_{i+j} \geq 2(qf + 1)$ for the last infinitely closed singularity $\tilde{x}_{i+j}$ introduced by $\tilde{\psi}$. Thus $\tilde{m}_i \geq 2qf + 1$.

Now we assume $\tilde{m}_i = 2qf + 1$. By the above discussion, $\tilde{m}_{i+1} \geq 2qf + 1$. If $\tilde{m}_{i+1} = 2qf + 2$, then $\tilde{x}_{i+1}$ must be the only one singular point of $\tilde{R}_{i+1}$ on $\tilde{E}_{i+1} \subseteq \tilde{P}_{i+1}$, and we are done. Therefore it is enough to derive a contradiction if $\tilde{m}_{i+1} = 2qf + 1$.

Let $l$ be the smallest number such that $\tilde{m}_{i+l} = 2qf + 1$, where we assume that $\tilde{x}_{i+j}$ is infinitely closed to $\tilde{x}_{i+j-1}$ for $j = 1, \cdots, l$. Such $l$ exists by Lemma 3.6 and (3-3). And
$l \geq 2$ if $\bar{m}_{i+1} = 2q_f + 1$. Note that the exception curve $\hat{E}_{i+j}$ is contained in $\hat{R}_{i+j}$ for $1 \leq j \leq l$, since $\bar{m}_{i+j-1}$ is odd. Because $\bar{m}_{i+l} = \bar{m}_{i+l+1} + 1$, $\bar{x}_{i+l}$ must be the only one singular point of $\hat{R}_{i+l}$ on the exception curve $\hat{E}_{i+l}$.

Let $\hat{E}_{i+l-1} \subseteq \hat{P}_{i+l}$ be the strict transform of $\hat{E}_{i+l-1} \subseteq \hat{P}_{i+l-1}$, $D = \hat{R}_{i+l} - (\hat{E}_{i+l-1} + \hat{E}_{i+l})$, and $D'$ the image of $D$ in $\hat{P}_{i+l+1}$. Then $\bar{x}_{i+l} \in \hat{E}_{i+l-1}$, since $\bar{x}_{i+l}$ is the only one singular point of $\hat{R}_{i+l}$ on the exception curve $\hat{E}_{i+l}$ and $\hat{E}_{i+l-1} \cap \hat{E}_{i+l}$ is a singularity of $\hat{R}_{i+l}$.

Since $\bar{m}_{i+l} = 2(q_f + 1)$, $D$ has multiplicity $\bar{m}_{i+l} - 2 = 2q_f$ at $\bar{x}_{i+l}$. Hence the local intersection

$I_{\bar{x}_{i+l}}(\hat{E}_{i+l-1}, D) \geq 2q_f$, \quad $I_{\bar{x}_{i+l}}(\hat{E}_{i+l}, D) \geq 2q_f$.

Note that $D$ is nothing but the strict transform of $D'$, and $(\hat{\psi}_{i+l-1})^* (\hat{E}_{i+l-1}) = \hat{E}_{i+l-1} + \hat{E}_{i+l}$. Thus

$I_{\bar{x}_{i+l-1}}(\hat{E}_{i+l-1}, D') = I_{\bar{x}_{i+l}}(\hat{\psi}_{i+l-1}^* (\hat{E}_{i+l-1}), D) = I_{\bar{x}_{i+l}}(\hat{E}_{i+l-1} + \hat{E}_{i+l}, D) \geq 4q_f$.

Note that $\hat{\psi}_{i+l-2}(D') = \hat{R}_{i+l-2}$, and the multiplicity $\bar{m}_{i+l-2}$ of $\hat{R}_{i+l-2}$ at $\bar{x}_{i+l-2}$ equals to the intersection number $\hat{E}_{i+l-1} \cdot D'$. Hence

$2q_f + 1 = \bar{m}_{i+l-2} = \hat{E}_{i+l-1} \cdot D' \geq I_{\bar{x}_{i+l-1}}(\hat{E}_{i+l-1}, D') \geq 4q_f$,

which is a contradiction, since $q_f \geq 1$. So we finish the proof.

According to Lemma 3.7, it follows that $\hat{\psi}$ is composed of a sequence blowing-ups of singularities of type $(2k+1 \rightarrow 2k+1)$ with $k \geq q_f$, or singularities with multiplicity at least $2(q_f + 1)$. Let $\bar{s}_{2k-1}$ be the number of singularities of $R$ of type $(2k+1 \rightarrow 2k+1)$ introduced by $\hat{\psi}$, and $\bar{s}_g$ be the number of singularities of $R$ of multiplicity $2k$ or $2k + 1$ introduced by $\hat{\psi}$ not belonging to the second component of $(2k-1 \rightarrow 2k-1)$ type singularities nor the first component of $(2k + 1 \rightarrow 2k + 1)$ type singularities. Then

$\bar{s}_{2k+1} = 0$, \quad $\forall 1 \leq k \leq q_f - 1$; \quad $\bar{s}_{2k} = 0$, \quad $\forall 2 \leq k \leq q_f$. \hspace{1cm} (3-4)

Let $\bar{s}_i \neq s_i - \bar{s}_i$. Then $\bar{s}_i, s_i \geq 0$, and one has by (3-1) that

$s_{2k+1} = \bar{s}_{2k+1}$, \quad $\forall 1 \leq k \leq q_f - 1$; \quad $s_{2k} = \bar{s}_{2k}$, \quad $\forall 2 \leq k \leq q_f$. \hspace{1cm} (3-5)

**Proof of Proposition 3.1.** According to Lemma 3.3 and the decomposition of $\psi$ in Figure 3, we see that $\overline{R}$ is contained in fibers of $\hat{h}'$, hence it is semi-negative definite. By the definition of $\hat{s}_i$'s, there are at least $\sum_{k=q_f}^{[g/2]} \bar{s}_{2k+1} + 2\bar{s}_{g+2}$ isolated $(-2)$-curves contained in $\overline{P}$. Thus

$$\overline{R}^2 \leq -2 \left( \sum_{k=q_f}^{[g/2]} \bar{s}_{2k+1} + 2\bar{s}_{g+2} \right).$$
On the other hand, by the definition,

\[
\overline{R}^2 = R^2 - \sum_{k=\lfloor q/2 \rfloor}^{\lfloor (g+1)/2 \rfloor} 4(2k^2 + 2k + 1)\hat{s}_{2k+1} - \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} 4k^2\hat{s}_{2k} - 2(g^2 + 3g + 1)\hat{s}_{g+2}.
\]

As \( R^2 = 4L^2 = 4(g+1)n \) by (2-2), we get

\[
(g+1)n \leq \sum_{k=\lfloor q/2 \rfloor}^{\lfloor (g+1)/2 \rfloor} \left(2k^2 + 2k + \frac{1}{2}\right)\hat{s}_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} k^2\hat{s}_{2k} + \frac{(g+1)(g+3)}{2}\hat{s}_{g+2}.
\]

Hence

\[
s^2 + \sum_{k=1}^{\lfloor q/2 \rfloor} 4k(2k+1)s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} 2k(2k-1)s_{2k} \\
\leq \hat{s}^2 + \sum_{k=1}^{\lfloor q/2 \rfloor} 4k(2k+1)s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} 2k(2k-1)s_{2k} + 2(g^2 + 3g + 1)\hat{s}_{g+2} \quad \text{by (3-3)},
\]

\[
\leq \sum_{k=q_f}^{\lfloor (g+1)/2 \rfloor} \frac{(2k+1)(2g+2-2k)}{g+1}\hat{s}_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \frac{2k(g+1-k)}{g+1}\hat{s}_{2k} + (g+1)\hat{s}_{g+2} \quad \text{by (2-3) and (3-6)},
\]

\[
\leq \sum_{k=q_f}^{\lfloor q/2 \rfloor} \frac{(2k+1)(2g+2-2k)}{g+1}s_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \frac{2k(g+1-k)}{g+1}s_{2k} + (g+1)s_{g+2}.
\]

The proof is complete.

4. Proof of Theorem 1.4 and its corollaries

The section aims to prove our main results Theorem 1.4 and its corollaries. They follow from (3-1) and the formulas given by Theorem 2.7.

Proof of Theorem 1.4  According to (3-2), it is known that \( q_f \leq \frac{g+1}{2} \). Recall the definition of \( \lambda_{q, q_f} \) in (1-5). If \( q_f = 0 \), then \( \lambda_{g, 0} = \frac{4(g-1)}{g} \), and so (1-6) holds by (1-2). Thus we assume \( q_f \geq 1 \) in the following.

First we prove that

\[
K^2 \geq \lambda_{q, q_f} \cdot \chi_f + \alpha s_{g+2} + \sum_{k=1}^{q_f - 1} \alpha_{k} s_{2k+1} + \frac{q_f}{2} \beta_{k} s_{2k} + \sum_{k=q_f}^{\lfloor q/2 \rfloor} \gamma_{k} s_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \delta_{k} s_{2k}, \quad (4-1)
\]
where

\[
\begin{align*}
\alpha &= \frac{(g-1)(8 - \lambda_{g,q_f})}{4}, \\
\alpha_k &= k^2\lambda_{g,q_f} - (2k - 1)^2, \quad \forall 1 \leq k \leq q_f - 1, \\
\beta_k &= k\lambda_{g,q_f} - 4(k - 1), \quad \forall 2 \leq k \leq q_f, \\
\gamma_k &= \frac{8(4k(g-k) - 1) - (4k(g-k) + g) \cdot \lambda_{g,q_f}}{4(g+1)}, \quad \forall q_f \leq k \leq \lfloor g/2 \rfloor, \\
\delta_k &= \frac{k(g+1 - k)(8 - \lambda_{g,q_f}) - 4(g+1)}{2(g+1)}, \quad \forall q_f + 1 \leq k \leq \lfloor (g+1)/2 \rfloor.
\end{align*}
\]

Indeed, it is clear that \(\lambda_{g,q_f} \geq \frac{4(g-1)}{g}\). Hence by Theorem \(2.7\) and \((3.1)\), one obtains

\[
(2g + 1) \left( K_f^2 - \lambda_{g,q_f} \cdot \chi_f \right)
= \left( g - 1 - \frac{g}{4} \cdot \lambda_{g,q_f} \right) s_2 + \left( 3g^2 - 2g - 1 \right) - \left( g^2 - 2g - 1 \right) \cdot \frac{\lambda_{g,q_f}}{4} \right) s_{g+2} + \sum_{k=1}^{\lfloor g/2 \rfloor} (a_k - k(g-k) \cdot \lambda_{g,q_f}) s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor - 1} \left( b_k - \frac{1}{2}k(g+1-k) \cdot \lambda_{g,q_f} \right) s_{2k}
\geq \left( g - 1 - \frac{g}{4} \cdot \lambda_{g,q_f} \right) \cdot \Lambda_h + \left( 3g^2 - 2g - 1 \right) - \left( g^2 - 2g - 1 \right) \cdot \frac{\lambda_{g,q_f}}{4} \right) s_{g+2} + \sum_{k=1}^{\lfloor g/2 \rfloor} (a_k - k(g-k) \cdot \lambda_{g,q_f}) s_{2k+1} + \sum_{k=2}^{\lfloor (g+1)/2 \rfloor} \left( b_k - \frac{1}{2}k(g+1-k) \cdot \lambda_{g,q_f} \right) s_{2k}
= (2g + 1) \left( \alpha s_{g+2} + \sum_{k=1}^{q_f-1} \alpha_k s_{2k+1} + \sum_{k=2}^{q_f} \beta_k s_{2k+1} + \sum_{k=q_f}^{\lfloor g/2 \rfloor} \gamma_k s_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \delta_k s_{2k} \right),
\]

where

\[
\Lambda_h = \sum_{k=q_f}^{\lfloor g/2 \rfloor} \frac{(2k+1)(2g+1 - 2k)}{g+1} s_{2k+1} + \sum_{k=q_f+1}^{\lfloor (g+1)/2 \rfloor} \frac{2k(g+1-k)}{g+1} s_{2k} + (g+1)s_{g+2} - \sum_{k=1}^{q_f-1} 4k(2k+1)s_{2k+1} - \sum_{k=2}^{q_f} 2k(2k-1)s_{2k}.
\]

Therefore, \((4.1)\) follows.

To prove \((4.3)\), it suffices to prove that those coefficients \(\alpha, \alpha_k, \beta_k, \delta_k\) and \(\gamma_k\) in \((4.1)\) are all non-negative. It is clear that

\[
\alpha \geq 0; \quad \alpha_k > 0, \quad \forall 1 \leq k \leq q_f - 1; \quad \text{and} \quad \beta_k > 0, \quad \forall 2 \leq k \leq q_f.
\]

\[(4.2)\]
If \( q_f \leq \frac{g-1}{2} \), then

\[
\gamma_k = \frac{4k(g-k) + g}{(q_f + 1)(g - q_f)} - 2 \geq \frac{4q_f(g - q_f) + g}{(q_f + 1)(g - q_f)} - 2 = \frac{2(q_f - 1)(g - q_f) + g}{(q_f + 1)(g - q_f)} > 0, \quad \forall \ q_f \leq k \leq \lfloor g/2 \rfloor.
\]

\[
\delta_k = 2 \left( \frac{k(g - 1 - k)}{(q_f + 1)(g - q_f)} - 1 \right) \geq 2 \left( \frac{(q_f + 1)(g + 1 - (q_f + 1))}{(q_f + 1)(g - q_f)} - 1 \right) = 0, \quad \forall \ q_f + 1 \leq k \leq \lfloor (g + 1)/2 \rfloor.
\]

Hence (1-6) follows from (4-1) by noting that \( s_i \geq 0 \) for any \( i \geq 3 \).

If \( g \) is even and \( q_f = \frac{g}{2} \), by virtue of (4-1) and (4-2), it suffices to show \( \gamma_{g/2} \geq 0 \). By the definition,

\[
\gamma_{g/2} = \frac{1}{4(g + 1)} \cdot \left( 8 \cdot \left( \frac{4 \cdot \frac{g}{2} \cdot (g - \frac{g}{2})}{2} - 1 \right) - \left( \frac{4 \cdot \frac{g}{2} \cdot (g - \frac{g}{2})}{2} + g \right) \cdot \frac{8(g - 1)}{g} \right) = 0.
\]

Hence, (1-6) also holds in this case.

Finally, if \( g \) is odd and \( q_f = \frac{g + 1}{2} \), then by (4-1) and (4-2), we have already showed that \( K_f \geq \lambda_{g, q_f} \cdot \chi_f \). So (1-6) holds in this case too. This completes the proof. \( \square \)

**Remark 4.1.** When \( q_f = \frac{g + 1}{2} \), \( \frac{g}{2} \) or \( \frac{g - 1}{2} \), (1-6) was already obtained by Xiao in [24].

**Proof of Corollary 1.5** If \( g \) is even and \( q_f = \frac{g}{2} \), or \( g \) is odd and \( q_f = \frac{g + 1}{2} \), then by the definition,

\[
\lambda_{g, q_f} = \frac{4(g - 1)}{g - q_f}.
\]

If \( q_f \leq \frac{2g - 1}{2} \), i.e., \( g - 2q_f - 1 \geq 0 \), then

\[
\lambda_{g, q_f} - \frac{4(g - 1)}{g - q_f} = \frac{4q_f(g - 2q_f - 1)}{(q_f + 1)(g - q_f)} \geq 0;
\]

and ‘\( = \)’ holds only if \( q_f = 0 \) or \( \frac{2g - 1}{2} \). Therefore, our corollary is a consequence of (1-6). \( \square \)

**Proof of Corollary 1.6** Let \( f : S \to B \) be a fibration of genus \( g \geq 2 \), which is not locally trivial and \( \lambda_f < 4 \). We need to prove that \( f_*\omega_{S/B} \) has no locally free quotient of degree zero. By [3, Theorem 1], we may assume that \( f \) is a hyperelliptic fibration. Recall that we have the following decomposition (cf. [11], see also [14]):

\[
f_*\omega_{S/B} = A \oplus F_1 \oplus \cdots \oplus F_k \oplus O_B^{\omega_{q_f}}, \quad (4-3)
\]

with \( A \) ample, \( F_i \) irreducible unitary and \( \dim H^1(B, \Omega^1_{B}(F_i)) = 0 \) for \( 1 \leq i \leq k \). Since \( \lambda_f < 4 \), we may assume that \( q_f = 0 \) by Corollary 1.5. Clearly \( A \) has no non-trivial locally free quotient of degree zero. Hence it suffices to prove that \( F_i = 0 \) in the above decomposition (4-3).

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Assume that $F_i \neq 0$ for some $i$. By construction, $F_i$ corresponds to a unitary representation of the fundamental group
\[ \rho_i : \pi_1(B) \rightarrow U(r_i), \] where $r_i = \text{rank } F_i$.

If the image of $\rho_i$ is finite, then after a suitable finite étale base change, $F_i$ becomes trivial, which implies that $q_f > 0$ after such a finite étale base change. However, it is a contradiction by Corollary 1.5, since the slope does not change under any finite étale base change. Hence we may assume that $\rho_i$ has infinite image.

By a stable reduction theorem (cf. [1, 10]), there exists a base change $\phi : \bigtriangledown B \rightarrow B$ of finite degree, possibly ramified, such that the pull-back fibration $\tilde{f} : \tilde{S} \rightarrow \bigtriangledown B$ is semi-stable.

Here the pull-back fibration $\tilde{f} : \tilde{S} \rightarrow \bigtriangledown B$ is constructed as follows. Let $S_1$ be the resolution of singularities of $S \times _B \bigtriangledown B$. Then $\tilde{f} : \tilde{S} \rightarrow \bigtriangledown B$ is just the relatively minimal model of $S_1$.

It is known that (cf. [20, P331]) there is an inclusion $\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B} \subseteq \phi^* f_* \omega_{S/B}$. Because
\[ \text{rank } \tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B} = \text{rank } \phi^* f_* \omega_{S/B} = g, \]
the quotient $Q := (\phi^* f_* \omega_{S/B})/ (\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B})$ is torsion. By projection, we get a morphism:
\[ \text{pr}_i : \tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B} \rightarrow \phi^* F_i. \]

The quotient $Q_i := (\phi^* F_i)/ \text{pr}_i (\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B})$ is also torsion. Note that $\deg \left( \text{pr}_i (\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B}) \right) \geq 0$, since it is a quotient of $\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B}$. Note also that $\deg Q_i \geq 0$, and
\[ 0 = \deg \phi^* F_i = \deg Q_i + \deg \left( \text{pr}_i (\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B}) \right). \]

We obtain that $Q_i$ is zero, and $\text{pr}_i$ is surjective.

By the construction, $\phi^* F_i$ comes from the following unitary representation
\[ \pi_1(B) \xrightarrow{\phi_*} \pi_1(B) \rightarrow U(r_i), \]
\[ \phi_* \pi_1(B) \xrightarrow{\rho_i} U(r_i). \]

It follows that $\rho_i$ has infinite image, since $\rho_i$ has infinite image and $\phi_* \left( \pi_1(B) \right)$ has finite index in $\pi_1(B)$.

On the other hand, because $\tilde{f}$ is a semi-stable hyperelliptic fibration, by [18, Theorem 1.6], after a suitable finite étale base change, the Fujita decomposition of $\tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B}$ is as follows,
\[ \tilde{f}_* \omega_{\tilde{S}/\bigtriangledown B} = \tilde{A} \oplus C_B^{\oplus d_f}, \] with $\tilde{A}$ ample.

---
By the same argument above, replacing $\tilde{B}$ by a finite étale cover, there will be still a surjective morphism $\text{pr}_i : \tilde{f}_* \omega_{\tilde{S}/\tilde{B}} \to F_i$, and $\phi^* F_i$ is unitary corresponding to a representation $\tilde{\rho}_i : \pi_1(\tilde{B}) \to U(r_i)$ with infinite image. Since $\tilde{A}$ is ample, it maps to zero by $\text{pr}_i$. Therefore we have a surjective morphism:

$$\text{pr}_i : O_{\tilde{B}}^{\oplus q_j} \to F_i.$$

This implies that the representation $\tilde{\rho}_i$ corresponds to a quotient representation of the trivial representation of $\pi_1(\tilde{B})$ corresponding to $O_{\tilde{B}}^{\oplus q_j}$. In particular, the representation $\tilde{\rho}_i$ is trivial. It is a contradiction, since the representation $\tilde{\rho}_i$ has infinite image by construction.

This completes the proof.

5. Examples

In the section, we construct examples to show that the bound (1-6) is sharp.

Example 5.1. Hyperelliptic fibration $f$ of genus $g$ with relative irregularity $q_f$ satisfying $g + 1 = m(q_f + 1)$ for some $m \geq 2$, and

$$\lambda_f = \frac{K^2_f}{\chi_f} = \lambda_{g,q_f}, \quad \text{where } \lambda_{g,q_f} \text{ is defined in } (1-5).$$

Let $P = P_{\mathbb{P}^1}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(e))$ be the rational ruled surface with invariant $e \geq 1$. Let $h : P \to B = \mathbb{P}^1$

be the ruling, $\Gamma \subseteq P$ a general fiber of $h$, and $C_0 \subseteq P$ the unique section with self-intersection $C_0^2 = -e$. According to [12, § V-2], the divisor $mC_0 + b\Gamma$ is very ample if and only if $b > me$. Let $mC_0 + b\Gamma$ be a very ample divisor. Then by Bertini’s theorem (cf. [12, § II-8]), a general member $D \in |mC_0 + b\Gamma|$ is smooth and any two general members $D_1, D_2 \in |mC_0 + b\Gamma|$ intersect with each other transversely. Let $D, D'$ be two general members in $|mC_0 + b\Gamma|$, and $\Lambda$ the pencil generated by $D$ and $D'$. Then $\Lambda$ defines a rational map $\varphi_\Lambda : P \dashrightarrow \mathbb{P}^1$. By blowing up the base points of $\Lambda$, we get a fibration $\tilde{h} : \tilde{P} \to \mathbb{P}^1$.

$$\tilde{P} \xrightarrow{\psi} P \xrightarrow{\tilde{h}} \mathbb{P}^1$$

where $\psi : \tilde{P} \to P$ is composed of blowing-ups centered at the base points of $\Lambda$. Let $\tilde{\Gamma}'$ be a general fiber of $\tilde{h}'$, $K_{\tilde{P}}$ the canonical divisor of $\tilde{P}$. Then

$$K^2_{\tilde{P}} = 8 - x, \quad K_{\tilde{P}} \cdot \tilde{\Gamma}' = \frac{(m - 1)x}{m} - 2m, \quad \left(\tilde{\Gamma}'\right)^2 = 0,$$

(5-1)

where $x = (mC_0 + b\Gamma)^2$ is the number of blowing-ups contained in $\psi$. Let $\Delta \subseteq \mathbb{P}^1$ be a set of $2(q_f + 1)$ general points, and $R = (\tilde{h}')^*(\Delta)$ the corresponding fibers of $\tilde{h}'$. Let $\pi' : B' \to \mathbb{P}^1$ be the double cover ramified over $\Delta$, and $\tilde{S}$ be the normalization of the fiber-product $\tilde{P} \times_{\mathbb{P}^1} B'$.
Note that if $\Delta$ is general on $\mathbb{P}^1$, then $\tilde{R}$ is both reduced and smooth. Hence $\tilde{S}$ is also smooth. Note that $\tilde{R} \equiv (2q_f + 2)\tilde{\Gamma}'$, where $\tilde{\Gamma}'$ is a general fiber of $\tilde{h}'$. So by (2-1) and (5-1), one gets
\[ K^2_{\tilde{S}} = 2 \left( K_{\tilde{P}} + \frac{1}{2} \tilde{R} \right)^2 = \left( \frac{4(m-1)(q_f + 1)}{m} - 2 \right) x - (8m(q_f + 1) - 16), \]
\[ \chi(\mathcal{O}_{\tilde{S}}) = 2\chi(\mathcal{O}_{\tilde{P}}) + \frac{1}{2} \left( K_{\tilde{P}} + \frac{1}{2} \tilde{R} \right) \cdot \frac{\tilde{R}}{2} = \frac{(m-1)(q_f + 1)}{2m} x - (m(q_f + 1) - 2). \]

The ruling $h : P \to B \cong \mathbb{P}^1$ induces a fibration $\tilde{h} : \tilde{P} \to B$ and hence a fibration $f : \tilde{S} \to B$.

It is easy to show that the induced map $\tilde{h}|_{\tilde{R}} : \tilde{R} \to B$ is of degree $m \cdot 2(q_f + 1) = 2(g + 1)$, hence $f$ is a hyperelliptic fibration of genus $g$. By construction, the relative irregularity of $f$ is just $q_f = g(B')$, and
\[ K^2_f = K^2_{\tilde{S}} - 8(g-1)(g(B)-1) = \left( \frac{4(m-1)(q_f + 1)}{m} - 2 \right) x, \]
\[ \chi_f = \chi(\mathcal{O}_{\tilde{S}}) - (g-1)(g(B)-1) = \frac{(m-1)(q_f + 1)}{2m} x. \]

Actually, $f$ is relatively minimal. To see this, let $R$ be the image of $\tilde{R}$ in $P$. Then the singular points of $R$ are all of multiplicity $2(q_f + 1)$. Hence $s_{2k+1} = 0$ for all $k \geq 1$, which implies that $\tilde{S}$ is relatively minimal by Lemma [2.8]. Hence $f$ is a relatively minimal locally non-trivial hyperelliptic fibration of genus $g$ with relative irregularity $q_f = \frac{g+1}{2} - 1 \leq \frac{g-3}{2}$, and
\[ \lambda_f = \frac{K^2_f}{\chi_f} = 8 - \frac{4m}{(m-1)(q_f + 1)} = 8 - \frac{4(g+1)}{(g-q_f)(q_f + 1)} = \lambda_{g,q_f}. \]

Remarks 5.2. (i). Let $g = 3$ and $m = 2$ in the above example. Then we get a hyperelliptic fibration $f$ of genus 3 with relative irregularity $q_f = 1$ and slope $\lambda_f = 4$.

(ii). According to Bertini’s theorem (cf. [12] §II-8), for a general member $D \in |mC_0 + b_0 \Gamma|$, the projection of $h|_D : D \to B$ has at most simply ramified points. Hence if $\Sigma'$ is general enough, then the fibration $f$ obtained in the above example is semi-stable.

Example 5.3. Hyperelliptic fibration $f$ of genus $g$ with
\[
\begin{align*}
q_f &= \frac{g}{2}, & \lambda_f &= \frac{K^2_f}{\chi_f} = \frac{8(g-1)}{g}, & \text{if } g \text{ is even;} \\
q_f &= \frac{g+1}{2}, & \lambda_f &= \frac{K^2_f}{\chi_f} = 8, & \text{if } g \text{ is odd.}
\end{align*}
\]
Let $F$ be the hyperelliptic curve of genus $g$ defined by $u^2 = v^{2g+2} - 1$, and $\tau_1$ be an involution of $F$ defined by $\tau_1(u, v) = (-u, -v)$. Then $\tau_1$ has exactly two fixed points if $g$ is even, and $\tau_1$ has no fixed point if $g$ is odd. Let $\phi : B \to B$ be a double cover between two projective curves of genus $\tilde{b} = g(\tilde{B})$ and $b = g(B)$ respectively, $\Sigma \subseteq B$ the branched divisor, $\Sigma = \phi^{-1}(\Sigma)$, and $\tau_2$ the induced involution of $B$ such that $B = \tilde{B}/\langle \tau_2 \rangle$. Let $\tau = (\tau_1, \tau_2)$ be an involution of $X = F \times \tilde{B}$ defined by $\tau(p, q) = (\tau_1(p), \tau_2(q))$, where $p \in F$ and $q \in \tilde{B}$. Then $X/\langle \tau \rangle$ has a natural fibration of genus $g$ over $B$. Let $f : S \to B$ be the relatively minimal smooth model of $X/\langle \tau \rangle$ as follows.

Assume $\Sigma \neq \emptyset$. Then we see that $f$ is a non-trivial hyperelliptic fibration. If $g$ is even, then $\tau$ has exactly two fixed points over each fiber in $\tilde{f}^*(\Sigma)$, and so $X/\langle \tau \rangle$ has only rational singularities of type $A_1$ (cf. [7, §III-3]); and if $g$ is odd, then $\tau$ is fixed-point-free, and hence $S = X/\langle \tau \rangle$ is already smooth and relatively minimal. Let $|\Sigma|$ be the number of points in $\Sigma$. Then one can compute that

$$K^2_f = 2(g - 1) \cdot |\Sigma|,$$

$$\chi_f = \begin{cases} 
g \cdot |\Sigma|, & \text{if } g \text{ is even;} 
g - 1 \cdot |\Sigma|, & \text{if } g \text{ is odd.} 
\end{cases}$$

Therefore, we obtain hyperelliptic fibrations with required slopes. To compute the relative irregularity, we consider another projection of $X$, i.e., $\tilde{h} : X \to F$. It induces a fibration $h' : X/\langle \tau \rangle \to B' = F/\langle \tau_1 \rangle$, and hence also a fibration $h : S \to B'$ with

$$g(B') = \frac{g}{2}, \quad \text{if } g \text{ is even,} \quad \text{and} \quad g(B') = \frac{g + 1}{2}, \quad \text{if } g \text{ is odd.}$$

In particular, $q_f \geq g(B')$. Combining this with (3-2), we see that $q_f = g(B')$ as required.

**Remark 5.4.** Taking $g = 2$ in the above example, we get a hyperelliptic fibration of genus 2 with relative irregularity $q_f = 1$ and slope $\lambda_f = 4$.

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