On Some Properties of Measurable Functions in Abstract Spaces

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Abstract. In this short note we present several infinite dimensional theorems which generalize corresponding facts from the finite dimensional differential inclusions theory.

1. Introduction

In article [2] a concept of a generalized solution to an ODE with nonsmooth right hand side is introduced.

Namely, the following initial value problem

\[ \dot{x} = f(t, x), \quad x(0) = \hat{x} \]

is considered in some domain \( D \subset \mathbb{R}^m, \quad x \in D \). The function \( f \) is defined in \([0, T] \times D\) and measurable with respect to the standard Lebesgue measure. Moreover it is assumed that there exists a function \( u \in L^1(0, T) \) such that

\[ |f(t, x)| \leq u(t) \]

holds for almost all \((t, x) \in [0, T] \times D\).

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Filippov defines a generalized solution $x(t)$ to IVP (1.1) as an absolutely continuous function in $[0, T]$ such that $x(0) = \hat{x}$ and the inclusion

$$\dot{x}(t) \in \bigcap_{r > 0} \bigcap_{N \in \mathcal{N}} \text{conv} f \left( t, B_r(x(t)) \setminus N \right)$$

holds for almost all $t \in [0, T]$.

Here conv stands for the closed convex hull; $B_r(x)$ denotes an open ball of $\mathbb{R}^m$ with the radius $r$ and a center at the point $x$; and we use $\mathcal{N}$ to denote a set of all measure-null subsets of $\mathbb{R}^m$.

Before any differential equations, this definition states up several questions from real analysis. The main ones are:

1) whether the set in the right-hand side of (1.2) is nonvoid for almost all $t$?
2) assume in addition that the function $f$ is continuous in $[0, T] \times D$. Is it true that

$$\bigcap_{r > 0} \bigcap_{N \in \mathcal{N}} \text{conv} f \left( t, B_r(x(t)) \setminus N \right) = \{ f(t, x) \} ?$$

Article [2] answers both questions positively and contains a sketch of the corresponding proofs. Particularly, if the function $f$ is continuous then the definition of a generalized solution turns into the standard definition of the classical solution to an ODE with a continuous right-hand side.

Though this sketch is completely correct we believe that it would be useful to provide a detailed proof; moreover we get rid of some extra hypotheses and generalize Filippov’s assertions to an infinite dimensional case.

Perhaps such a generalization can be of some interest by itself or for the modern studies of the inclusions in infinite dimensional spaces.

## 2. The Main Theorems

Theorems [1, 5] (see below) generalize Filippov’s results which are obtained for the case $X = D, \ Y = \mathbb{R}^m$.

### 2.1. The Case of Measurable Mapping.

Let a set $X$ be equipped with a $\sigma$-algebra and a measure $\mu$.

Let $Y$ stand for a Hausdorff topological space with a countable base. A mapping $f : X \to Y$ is supposed to be measurable with respect to the Borel $\sigma$– algebra in $Y$.

**Theorem 1.** There exists a measurable set $N_0 \subset X$, $\mu(N_0) = 0$ such that

$$\bigcap_{\mu(N) = 0} f(X \setminus N) = f(X \setminus N_0).$$

The intersection is taken over all measurable sets of measure zero; the line denotes the closure.

**Corollary 1.** Assume in addition that $Y$ is a topological vector space then

$$\bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) = \text{conv} f(U \setminus N_0).$$

Indeed, the inclusion

$$\bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) \subset \text{conv} f(U \setminus N_0)$$

does not always follow.
is evident: \( N_0 \) is one of the sets \( N \) from the left side of the formula.

The inverse inclusion is obtained by means of theorem \([1]\) with the help of the formula
\[
\text{conv} \bigcap_{\alpha} U_\alpha \subset \bigcap_{\alpha} \text{conv} U_\alpha.
\]

**Definition 1.** We shall say that \( y \in Y \) is a bad point if it has an open neighbourhood \( U_y \) such that \( \mu(f^{-1}(U_y)) = 0 \). We shall call other points of \( Y \) good.

The notion of a good point has a direct relation to the standard concept of the essential range \([5]\).

Generalizing the definition of the essential range to the infinite dimensional case, we say that the essential range of the function \( f \) coincides with a set of the good points:
\[
\text{ess.im } f := \{ y \in Y \mid y \text{ is good} \}.
\]

A finite dimensional version of the following theorem is well known.

**Theorem 2.** The set in the left hand side of \((2.1)\) coincides with the essential range of \( f \):
\[
\bigcap_{\mu(N) = 0} f(X \setminus N) = \text{ess.im } f.
\]

**Remark 1.** If \( X \) is a Hausdorff topological space and \( \mu \) is a Borel measure, moreover if
\[
X = Y, \quad f = \text{id}_X
\]
then the set of the good points coincides by definition with the support of the measure \( \mu \).

**Theorem 3.** Let \( Q \subset X \) be a measurable set. We use \( f_Q \) to denote the following restriction
\[
f_Q = f \mid_Q : Q \to Y.
\]
Then there is a measurable set
\[
N_0^Q \subset Q, \quad \mu(N_0^Q) = 0, \quad N_0 \cap Q \subset N_0^Q
\]
such that
\[
\bigcap_{\mu(N) = 0} f_Q(X \setminus N) = f_Q(X \setminus N_0^Q).
\]

**2.2. Some Consequence of Theorem [1]** Zero-measure subsets of measure spaces have an analog in topological spaces. This analog is called a first Baire category set.

Let \( X \) be a topological space. Recall that a set \( A \subset X \) is of the first Baire category if it is contained in a countable union of closed sets and everyone of these sets has empty interior. Other subsets of \( X \) have the second Baire category.

Let \( Y \) denote a Hausdorff topological space with a countable base as above. We do not impose any conditions on a mapping \( f : X \to Y \).

Let \( \mathcal{F} \subset 2^X \) stand for a family of the first Baire category sets.

**Theorem 4.** There exists a set \( E_0 \in \mathcal{F} \) such that
\[
\bigcap_{E \in \mathcal{F}} f(X \setminus E) = f(X \setminus E_0).
\]
This theorem is a direct consequence from theorem\footnote{1} Indeed, to apply theorem\footnote{1} one must point out a $\sigma-$ algebra and a measure in $X$ such that $f$ becomes a measurable function.

On a role of the $\sigma-$algebra we take $2^X$ and introduce a measure

$$\mu : 2^X \rightarrow [0, \infty]$$

such that

$$\mu(B) = \begin{cases} 0, & \text{if } B \in \mathcal{F}; \\ \infty, & \text{otherwise}. \end{cases}$$

### 2.3. The Case of the Continuous at a Point Mapping.

**Theorem 5.** Let $X$ be a topological vector space with the Borel $\sigma$-algebra and a measure $\mu$.

Assume that
1) for any open nonvoid set $S \subset X$ one has $\mu(S) > 0$;
2) $Y$ is a locally convex Hausdorff space;
3) a mapping $f : X \rightarrow Y$ is continuous at a point $\tilde{x} \in X$.

Then it follows that

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \text{conv } f(U \setminus N) = \{f(\tilde{x})\},$$

where $\mathcal{U}$ is a base of the open neighbourhoods at the point $\tilde{x}$.

**Remark 2.** 1) The function $f$ in theorem\footnote{5} needs not be measurable.
2) If in theorem\footnote{5} we will replace condition 2) with "$Y$ is a Hausdorff topological space" then the assertion is replaced with

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} f(U \setminus N) = \{f(\tilde{x})\}.$$  

If in addition $f$ satisfies the conditions of theorem\footnote{7} then this formula implies $f(\tilde{x}) \in \text{ess.im } f$.

### 3. Proofs

#### 3.1. Proof of Theorem \footnote{1}

All the points of $U_y$ are evidently bad provided $y$ is bad. Let $B$ denote the set of bad points. Then we have

$$B = \bigcup_{y \in B} U_y.$$  

(3.1)

Here $U_y$ are the neighbourhoods from definition\footnote{4}

By the second Lindelof theorem\footnote{1} one can extract a countable subcovering

$$\{U_{y_n}\}, \quad \{y_n\}_{n=1}^{\infty} \subset B$$

such that

$$B = \bigcup_{n \in \mathbb{N}} U_{y_n}.$$  

We claim that $N_0 := f^{-1}(B)$ is the set we are looking for.
Indeed,
\[ \mu(N_0) = \mu(f^{-1}(B)) = \mu \left( \bigcup_{n \in \mathbb{N}} f^{-1}(U_{y_n}) \right) \]
\[ = \mu \left( \bigcup_{n \in \mathbb{N}} f^{-1}(U_{y_n}) \right) \leq \sum_{n \in \mathbb{N}} \mu(f^{-1}(U_{y_n})) = 0. \]

The inclusion
\[ \bigcap_{\mu(N)=0} f(X \setminus N) \subset f(X \setminus N_0) \]
is trivial: \( N_0 \) is one of the sets \( N \) from the left side of the formula.

Let us check the backward inclusion "\( \supset \)". Show that \( f(X \setminus N_0) \) does not contain the bad points.

Indeed,
\[ f(X \setminus N_0) \cap B = f((X \setminus N_0) \cap f^{-1}(B)) = f((X \setminus N_0) \cap N_0) = f(\emptyset) = \emptyset. \]

Here we employ a set theory formula [3]:
\[ f(A) \cap B = f \left( A \cap f^{-1}(B) \right). \]

It follows that \( f(X \setminus N_0) \subset Y \setminus B \). By formula (3.1) the set \( B \) is open, thus \( Y \setminus B \) is closed. We consequently obtain
\[ f(X \setminus N_0) \subset Y \setminus B. \] (3.2)

This means that \( f(X \setminus N_0) \) consists of the good points only.

Observe that if \( y \in Y \setminus B \) is a good point then for any measure-null set \( N \subset X \) we have
\[ y \in f(X \setminus N). \]

Indeed, take a set \( N \), \( \mu(N) = 0 \). Let \( y \) be a good point. This means that any open neighbourhood \( V_y \) of the point \( y \) is such that
\[ \mu(f^{-1}(V_y)) > 0. \]

Thus there exists a point \( x' \in f^{-1}(V_y) \setminus N \). This implies
\[ f(x') \in V_y \cap f(X \setminus N). \]

Consequently for any open neighbourhood \( V_y \) the set \( V_y \cap f(X \setminus N) \) is nonvoid. Therefore we have \( y \in f(X \setminus N) \).

Since \( N \) is an arbitrary measure-null set and \( y \) is an arbitrary good point we conclude that the set
\[ \bigcap_{\mu(N)=0} f(X \setminus N) \] (3.3)
contains all the good points and thus
\[ f(X \setminus N_0) \subset \bigcap_{\mu(N)=0} f(X \setminus N). \]

The theorem is proved.

3.2. Proof of Theorem [2]. We have proved that set (3.3) contains all the good points. Due to formula (3.2) we see that set (3.3) coincides with the set of the good points. This gives (2.2).

The theorem is proved.
3.3. Proof of Theorem 3. Let \( B_Q \) stand for a set of bad points of the mapping \( f_Q \). It is clear that \( B \subset B_Q \). Thus one has
\[
N_0 \cap Q \subset f^{-1}(B_Q) \cap Q = f_Q^{-1}(B_Q) = N_0^Q.
\]
The theorem is proved.

3.4. Proof of Theorem 5. First let us check that for any \( U \in \mathcal{U} \) and for any measurable \( N \), \( \mu(N) = 0 \) it follows that \( \tilde{x} \in \overline{U \setminus N} \).
(3.4)
Indeed, assume the converse: \( \tilde{x} \notin \overline{U \setminus N} \). Since \( U \setminus N \) is a closed set there exists an open set \( \tilde{U} \in \mathcal{U} \) such that
\[
\tilde{U} \cap (U \setminus N) = \emptyset,
\]
all the more so
\[
\tilde{U} \cap (U \setminus N) = \emptyset.
\]
(3.5)
On the other hand we have
\[
\tilde{U} \cap (U \setminus N) = (\tilde{U} \cap U) \setminus (\tilde{U} \cap N).
\]
Since \( \tilde{x} \in \tilde{U} \cap U \) the set \( \tilde{U} \cap U \) is open, nonvoid thus \( \mu(\tilde{U} \cap U) > 0 \); and \( \mu(\tilde{U} \cap N) = 0 \).
Therefore
\[
\mu(\tilde{U} \cap (U \setminus N)) > 0.
\]
This contradicts to (3.5).

Inclusion (3.4) is proved.

Let us prove the inclusion
\[
\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) \supset \{ f(\tilde{x}) \}.
\]
(3.6)
From the properties of continuous functions [3] inclusion (3.4) implies \( f(\tilde{x}) \in \overline{f(U \setminus N)} \). Now the following trivial observation
\[
\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) \supset \bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \overline{f(U \setminus N)}
\]
implies inclusion (3.6).

Let us check the inclusion
\[
\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) \subset \{ f(\tilde{x}) \}.
\]
(3.7)
The point \( f(\tilde{x}) \) has a base of convex closed neighbourhoods \( \mathcal{V} \).

Thus for any \( V \in \mathcal{V} \) there exists \( U \in \mathcal{U} \) such that \( f(U) \subset V \); therefore
\[
\text{conv} f(U \setminus N) \subset V.
\]
We consequently have
\[
\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N) = 0} \text{conv} f(U \setminus N) \subset V.
\]
Since \( V \) is an arbitrary element of the base \( \mathcal{V} \) the inclusion (3.7) is proved.

The theorem is proved.
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