 SOME LOCAL PROPERTIES OF SUBSOLUTIONS AND
SUPERSOLUTIONS FOR A DOUBLY NONLINEAR NONLOCAL
PARABOLIC \( p \)-LAPLACE EQUATION

Agnid Banerjee, Prashanta Garain and Juha Kinnunen

October 13, 2020

Abstract

We establish a local boundedness estimate for weak subsolutions to a doubly nonlinear parabolic fractional \( p \)-Laplace equation. Our argument relies on energy estimates and a parabolic nonlocal version of De Giorgi’s method. Furthermore, by means of a new algebraic inequality, we show that positive weak supersolutions satisfy a reverse Hölder inequality. Finally, we also prove a logarithmic decay estimate for positive supersolutions.

Key words: Doubly nonlinear parabolic equation, fractional \( p \)-Laplace equation, energy estimates, De Giorgi’s method.

2010 Mathematics Subject Classification: 35K92, 35B45, 35R11.

1 Introduction

This work studies the local behaviour of subsolutions and supersolutions to the doubly nonlinear parabolic nonlocal problem

\[
\partial_t(u^{p-1}) + \mathcal{L}u = 0 \text{ in } \Omega \times (0,T), \quad p > 2,
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded smooth domain, \( T > 0 \) and the operator \( \mathcal{L} \) is defined by

\[
\mathcal{L}u(x,t) = \text{P.V.} \int_{\mathbb{R}^n} |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))K(x,y,t) \, dy,
\]

and where P.V. stands for the principal value. We assume that \( K \) is a symmetric kernel with respect to \( x \) and \( y \) satisfying

\[
\frac{\Lambda^{-1}}{|x-y|^{n+ps}} \leq K(x,y,t) \leq \frac{\Lambda}{|x-y|^{n+ps}},
\]

uniformly in \( t \in (0,T) \) for some \( \Lambda \geq 1 \) and \( s \in (0,1) \). If \( K(x,y,t) = |x-y|^{-(n+ps)} \), then \( \mathcal{L} \) becomes the fractional \( p \)-Laplace operator \( (-\Delta)^p \), which further reduces to the fractional Laplacian \( (-\Delta)^s \) for \( p = 2 \).
The partial differential equation in (1.1) constitutes a nonlocal counterpart of the doubly nonlinear equation,
\[ \partial_t (u^{p-1}) - \text{div}(|\nabla u|^{p-2} \nabla u) = 0. \] (1.3)
We refer the reader to [1] [2] [19] [26] [28] [29] [31] and the references therein. To the best of our knowledge, there is no literature available concerning the corresponding nonlocal equation. This paper is a first step towards a regularity theory where we prove a local boundedness estimate for weak subsolutions to (1.1) when \( p > 2 \). To this end, we establish an energy estimate (Lemma 3.1) and apply De Giorgi’s method to obtain our main result (Theorem 2.14). We also prove a reverse Hölder inequality for strictly positive weak supersolutions (Theorem 2.16) by means of a new algebraic inequality (Lemma 2.9) and a logarithmic decay estimate (Lemma 5.3). In particular, Lemma 2.9 generalizes an inequality due to Felsinger and Kassmann for \( p = 2 \), see Lemma 3.3 in [17]. Finally we note that in the local case as for (1.3), such a reverse Hölder property as well as the logarithmic estimate constitute some of the key ingredients in the proof of weak Harnack inequality, see for instance [26]. To the best of our knowledge, weak Harnack inequality seems to be an open question in the nonlocal case for the doubly nonlinear equation (1.1) and therefore we believe that our results will be important in investigating such question along with further qualitative and quantitative properties of weak solutions to (1.1).

Fractional Laplace equations have been a topic of considerable attention recently. We refer to the survey [14] by Di Nezza, Palatucci and Valdinoci for an elementary introduction to the theory of the fractional Sobolev spaces and fractional Laplace equations. For globally nonnegative solutions of the elliptic fractional Laplace equation \((-\Delta)^s u = 0\), Landkof [30] obtained scale invariant Harnack inequality, which fails for sign changing solutions as shown by Kassman [23]. Indeed, an additional tail term appears in the Harnack estimate. Castro, Kuusi and Palatucci studied local boundedness and Hölder continuity results for the equation \((-\Delta_p)^s u = 0\) with \( p > 1 \) in [13]. They also obtained Harnack inequality with a tail dealing with sign changing solutions in [12]. The nonhomogeneous case \((-\Delta_p)^s u = f\) has been settled for local and global boundedness along with a discussion of eigenvalue problem by Brasco and Parini [8]. Moreover in this case, Brasco, Lindgren and Schikorra established higher and optimal regularity results in [6]. See also [3] [11] and the references therein.

In the parabolic setting, for the fractional heat equation, \( \partial_t u + (-\Delta)^s u = 0 \), weak Harnack inequality has been established by Felsinger and Kassman in [17], see also [10] [24] for related results. Caffarelli, Chan and Vasseur established boundedness and Hölder continuity results in [9] for different type of kernels. For regularity results up to the boundary, see [18]. Bonforte, Sire and Vázquez established optimal existence and uniqueness results in [4], along with a scale invariant Harnack inequality for globally positive solutions. For sign changing solutions, Strömqvist proved Harnack inequality with a tail in [33], see [25] for a different approach.

In the nonlinear framework, we mention the work of Vázquez [35] where global boundedness results for the equation
\[ \partial_t u + (-\Delta_p)^s u = 0 \]
Doubly nonlinear nonlocal $p$-Laplace equation

have been obtained. See also [32]. For such an equation, local boundedness result with a tail term has been investigated by Strömqvist in [34]. More recently, Hölder continuity results has been established for the same equation by Brasco, Lindgren and Strömqvist in [7]. In the doubly nonlinear case, Hynd and Lindgren [21] addressed the question of pointwise behavior of viscosity solutions for the following doubly nonlinear equation

$$|\partial_t u|^{p-2}\partial_t u + (-\Delta)^s u = 0.$$ 

See also [20, 22] for related results in the local case.

This paper is organized as follows: In Section 2, we introduce some basic notations, gather some preliminary results that are relevant to our work and then state our main results. In Section 3-5, we prove our main results. Finally, in Section 6, the appendix, we give a proof of the algebraic inequality in Lemma 2.9 which is applied in the proof of Theorem 2.16.

2 Preliminaries and main results

We first present some facts about fractional Sobolev spaces. For more details we refer the reader to [14].

**Definition 2.1** Let $1 < p < \infty$ and $0 < s < 1$ and assume that $\Omega \subset \mathbb{R}^n$ is an open and connected subset of $\mathbb{R}^n$. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{n+sp}} \in L^p(\Omega \times \Omega) \right\}$$

and endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left( \int_\Omega |u(x)|^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

The fractional Sobolev space with zero boundary values is defined by

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^n) : u = 0 \text{ on } \mathbb{R}^n \setminus \Omega \right\}.$$

Both $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are reflexive Banach spaces, see [14]. The parabolic Sobolev space $L^p(0,T;W^{s,p}(\Omega))$ is the set of measurable functions $u$ on $\Omega \times (0,T)$, $T > 0$, such that

$$\|u\|_{L^p(0,T;W^{s,p}(\Omega))} = \left( \int_0^T \|u(\cdot, t)\|_{W^{s,p}(\Omega)}^p \, dt \right)^{\frac{1}{p}} < \infty.$$ 

The spaces $W_0^{s,p,\text{loc}}(\Omega)$ and $L_0^{s,\text{loc}}(0,T;W_0^{s,p,\text{loc}}(\Omega))$ are defined analogously. Next we discuss Sobolev embedding theorems, see [14]. We write by $C$ to denote a positive constant which may vary from line to line or even in the same line depending on the situation. If $C$ depends on $r_1, r_2, \ldots, r_k$, we write $C = C(r_1, r_2, \ldots, r_k)$.
Theorem 2.2 Let $1 < p < \infty$ and $0 < s < 1$ with $sp < n$ and $\kappa^* = \frac{n}{n-sp}$. For every $u \in W^{s,p}(\mathbb{R}^n)$, we have

$$\|u\|_{L^{s,p}(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy.$$  

If $\Omega$ is a bounded extension domain for $W^{s,p}$ and $u \in W^{s,p}(\Omega)$, then for any $\kappa \in [1,\kappa^*],

$$\|u\|_{L^{s,p}(\Omega)} \leq C(\Omega)\|u\|_{W^{s,p}(\Omega)}.$$  

If $sp = n$, then the above inequalities hold for any $\kappa \in [1, \infty)$. For $sp > n$, the second inequality holds for any $\kappa \in [1, \infty]$.

For the following Sobolev type inequality, we refer to [31, Lemma 2.1]. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}$ denotes the ball in $\mathbb{R}^n$ of radius $r$ and center $x_0$. The barred integral sign denotes the corresponding integral average.

Lemma 2.3 Let $1 < p < \infty$ and $0 < s < 1$. Assume that $u \in W^{s,p}(B_r)$, where $B_r = B_r(x_0)$, and let $\kappa^* = \frac{n}{n-sp}$, if $sp < n$, and $\kappa^* = 2$, if $sp \geq n$. There exists a constant $C = C(n,p,s)$ such that for every $\kappa \in [1,\kappa^*]$, we have

$$\left( \int_{B_r} |u(x)|^{sp} \, dx \right)^{\frac{1}{s}} \leq Cr^{sp-n} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy + C \int_{B_r} |u(x)|^p \, dx.$$  

Next we state the parabolic Sobolev inequality as in [31, Lemma 2.2].

Lemma 2.4 Let $p$, $s$ and $\kappa^*$ be as in Lemma 2.3. Assume that $u \in L^p(t_1,t_2; W^{s,p}(B_r))$. There exists a constant $C = C(n,p,s)$ such that for every $\kappa \in [1,\kappa^*]$, we have

$$\int_{t_1}^{t_2} \int_{B_r} |u(x,t)|^{kp} \, dx \, dt \leq C (r^{sp-n} \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \, dt + \int_{t_1}^{t_2} \int_{B_r} |u(x,t)|^p \, dx \, dt)^{\frac{1}{p}} \left( \sup_{t_1 < t < t_2} \int_{B_r} |u(x,t)|^{p\kappa^*(\kappa-1)} \right)^{\frac{1}{\kappa^*(\kappa-1)}}.$$  

We now state the following weighted Poincaré inequality in fractional Sobolev spaces, see [16, Corollary 6].

Lemma 2.5 Let $1 < p < \infty$, $0 < s_0 \leq s < 1$. Assume that $\phi(x) = \Phi(|x|)$ is a radially decreasing function on $B_1 = B_1(0)$. Then there exists a constant $C = C(p,n,s_0,\phi)$ such that for all $f \in L^p(B_1)$,

$$\int_{B_1} |f(x) - \frac{\int_{B_1} f(y) \phi(y) \, dy}{\int_{B_1} \phi(x) \, dx}|^p \phi(x) \, dx \leq C(1-s) \int_{B_1} \int_{B_1} \frac{|f(x) - f(y)|^p}{|x-y|^{n+ps}} \min\{\phi(x), \phi(y)\} \, dx \, dy,$$

where

$$f^{\phi}_{B_1} = \frac{\int_{B_1} f(y) \phi(y) \, dy}{\int_{B_1} \phi(x) \, dx}.$$
Doubly nonlinear nonlocal $p$-Laplace equation

Using change of variables in Lemma 2.5 we obtain the following weighted Poincaré inequality which will be useful in establishing a logarithmic estimate for weak supersolutions (see Lemma 5.3).

**Lemma 2.6** Let $1 < p < \infty$, $0 < s < 1$ and $\psi(x) = \Psi(|x - x_0|)$ be a radially decreasing function on $B_r = B_r(x_0)$. Then there exists a constant $C = C(n,p,s)$ such that for every $f \in L^p(B_r)$,

$$
\int_{B_r} |f(x) - f_{B_r}^\psi|^p \psi(x) \, dx \leq C r^{ps} \int_{B_r} \int_{B_r} \frac{|f(x) - f(y)|^p}{|x - y|^{n + ps}} \min\{\psi(x), \psi(y)\} \, dx \, dy,
$$

where

$$
f_{B_r}^\psi = \frac{\int_{B_r} f(x) \psi(x) \, dx}{\int_{B_r} \psi(x) \, dx}.
$$

We also need the following real analysis lemmas. For the proof of Lemma 2.7 below, see [15, Lemma 4.1].

**Lemma 2.7** Let $(Y_j)_{j=0}^\infty$ be a sequence of positive real numbers satisfying $Y_{j+1} \leq c_0 b^j Y_j^{1+\beta}$, for some constants $c_0 > 1$, $b > 1$ and $\beta > 0$. If $Y_0 \leq c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}$, then $\lim_{j \to \infty} Y_j = 0$.

The next inequality is as in [13, Lemma 3.1].

**Lemma 2.8** Let $p \geq 1$ and $\epsilon \in (0,1]$. Then for every $a, b \in \mathbb{R}^n$, we have

$$
|a|^p \leq |b|^p + C(p)\epsilon|b|^p + (1 + C(p)\epsilon)\epsilon^{1-p}|a - b|^p,
$$

where $C(p) = (p - 1)\Gamma(\max\{1, p - 2\})$ and $\Gamma$ denotes the gamma function.

The following elementary inequality will play a crucial role in the proof of reverse Hölder inequality for supersolutions as in Theorem 2.16 below. A proof for Lemma 2.9 is given in the appendix. This generalizes an inequality of Felsinger and Kassmann [17] to the $p$-case.

**Lemma 2.9** Let $a, b > 0$, $\tau_1, \tau_2 \geq 0$. Then for any $p > 1$, there exists a constant $C = C(p) > 1$ large enough such that

$$
|b - a|^{p-2}(b - a)(\tau_1^p a^{-\epsilon} - \tau_2^p b^{-\epsilon}) \geq \frac{\zeta(\epsilon)}{C(p)} \left|\tau_2 b^{\frac{p-\epsilon-1}{p}} - \tau_1 a^{\frac{p-\epsilon-1}{p}}\right|^p
$$

$$
- \left(\zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}\right) |\tau_2 - \tau_1|^p (b^{p-\epsilon-1} + a^{p-\epsilon-1}),
$$

where $0 < \epsilon < p - 1$ and $\zeta(\epsilon) = \epsilon \left(\frac{p}{p-\epsilon-1}\right)^p$. If $0 < p - \epsilon - 1 < 1$, we may choose $\zeta(\epsilon) = \frac{\epsilon^p}{p-\epsilon-1}$ in (2.1).

For $v, k > 0$, the auxiliary function defined by

$$
\xi((v - k)_+) = \int_k^{v^{p-1}} (\eta^{p-1} - k)_+ \, d\eta = (p - 1) \int_k^v (\eta - k)_+ \eta^{p-2} \, d\eta,
$$

would be very useful to deduce the energy estimate below. Indeed, from [21 Lemma 2.2], we have the following result.
Lemma 2.10 There exists a constant $\lambda = \lambda(p) > 0$ such that for all $v, k > 0$, we have
\[
\frac{1}{\lambda}(v + k)^{p-2}(v - k)^2_+ \leq \xi((v - k)_+) \leq \lambda(v + k)^{p-2}(v - k)^2_+.
\]
For more applications of such functions in the doubly nonlinear context, we refer to \[2, 19, 29\].

For $t_0 \in (r^{sp}, T - r^{sp})$, we consider the space-time cylinders
\[
U^-(r) = U^-(x_0, t_0, r) = B_r(x_0) \times (t_0 - r^{sp}, t_0)
\]
and
\[
U^+(r) = U^+(x_0, t_0, r) = B_r(x_0) \times (t_0, t_0 + r^{sp}).
\]
We denote the positive and negative parts of $u$ by
\[
u_+(x, t) = \max\{u(x, t), 0\} \quad \text{and} \quad \nu_-(x, t) = \max\{-u(x, t), 0\},
\]
respectively. For any $a, b \in \mathbb{R}$, we have $|a_+ - b_+| \leq |a - b|$ which implies $u_+ \in W^{s,p}(\Omega)$ when $u \in W^{s,p}(\Omega)$. Analogously, we have $u_- \in W^{s,p}(\Omega)$. Throughout the paper, we denote by
\[
A(u(x, y, t)) = |u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t)) \quad \text{and} \quad d\mu = K(x, y, t) \, dx \, dy.
\]
It is well known that a tail term appears in nonlocal problems. If $u$ is a measurable function in $\mathbb{R}^n \times (0, T)$ and $x_0 \in \mathbb{R}^n$, $r > 0$, $0 < t_1 < t_2 < T$, the parabolic tail of $u$ with respect to $x_0$, $r$, $t_1$ and $t_2$ is defined by
\[
\text{Tail}_{\infty}(u; x_0, r, t_1, t_2) = (r^{sp} \sup_{t_1 < t < t_2} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(x, t)|^{p-1}}{|x - x_0|^{n+sp}} dx)^{\frac{1}{p-1}}. \tag{2.2}
\]

Next we define the notion of weak sub and supersolution.

Definition 2.11 A function $u \in L^\infty(0, T; L^\infty(\mathbb{R}^n))$, with $u > 0$ in $\mathbb{R}^n \times (0, T)$, is a weak sub-solution (or supersolution) of the equation (1.1) if $u \in C_{\text{loc}}(0, T; L^p_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{s,p}_{\text{loc}}(\Omega))$ and for every $\Omega' \times [t_1, t_2] \subseteq \Omega \times (0, T)$, and nonnegative test function $\phi \in W^{1,p}_{\text{loc}}(0, T; L^p(\Omega')) \cap L^p_{\text{loc}}(0, T; W^{s,p}_{0,\text{loc}}(\Omega'))$, one has
\[
\int_{\Omega'} u(x, t_2)^{p-1} \phi(x, t_2) \, dx - \int_{\Omega'} u(x, t_1)^{p-1} \phi(x, t_1) \, dx - \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{p-1} \partial_t \phi(x, t) \, dx \, dt
\]
\[
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x, y, t))(\phi(x, t) - \phi(y, t)) \, d\mu \, dt \leq 0 \quad (\text{or} \geq 0)
\]
respectively.

Remark 2.12 The assumption $u \in L^\infty(0, T; L^\infty(\mathbb{R}^n))$ ensures that the last term in the right hand side of (2.3) and the term Tail_{\infty} defined in (2.2) are finite.

Remark 2.13 The test functions in the energy estimates would depend on the solution $u$ itself and the use of the term $u_\ell$ can be justified by using the mollification in time defined for $f \in L^1(\Omega \times I)$ by
\[
f_{h}(x, t) = \frac{1}{h} \int_{0}^{t} e^{\frac{s-x}{h}} f(x, s) \, ds.
\]
For more details of $f_h$, we refer to \[2, 27\].
Statement of the main results.

We now state our main results. Our first main result is following local boundedness estimate for subsolutions.

**Theorem 2.14** Let \( p > 2, \) \( x_0 \in \mathbb{R}^n, \) \( r > 0 \) and \( t_0 \in (r^{sp}, T). \) Assume that \( u \) is a weak subsolution of (1.1) with \( u > 0 \) in \( \mathbb{R}^n \times (t_0 - r^{ps}, t_0). \) Then there exists a positive constant \( C = C(n, p, s, \Lambda) \) such that for any \( \delta \in (0, 1), \) we have

\[
\sup_{(x,t) \in U - (\frac{r}{2})} u(x, t) \leq C \delta^{\frac{(p-1)\kappa}{sp\kappa}} \left( \int_{U - (r)} u(x, t)^p \, dx \, dt \right)^{\frac{1}{p}} + \delta \text{Tail}_\infty(u; x_0, \frac{r}{2}, t_0 - r^{sp}, t_0),
\]

where \( \kappa = \frac{n+sp}{n}, \) if \( sp < n, \) and \( \kappa = \frac{3}{2}, \) if \( sp \geq n. \)

**Remark 2.15** One should note that in the case when \( \Omega = \mathbb{R}^n, \) for the validity of Theorem 2.14 we only require \( u \in L^p((0, T); W^{s,p}((\mathbb{R}^n)) \cap L^\infty((0, T); L^p(\mathbb{R}^n)) \) which ensures that the Tail\( _\infty \) is finite and thus one can avoid the qualitative boundedness assumption on \( u \) unlike that in the case of bounded domain.

Our second main result constitutes the following reverse Hölder inequality for positive supersolutions.

**Theorem 2.16** Let \( p > 2. \) Suppose that \( u \) is a weak supersolution of (1.1) with \( u \geq \rho > 0 \) in \( \mathbb{R}^n \times (t_0, t_0 + r^{ps}), \) where \( t_0 \in (0, T - r^{ps}). \) Then for any \( \theta \in (\frac{1}{2}, 1) \) there exists positive constants \( \mu = \mu(\kappa, p) \) and \( C = C(n, p, q, s, \Lambda) \geq 1 \) such that

\[
\left( \int_{U^+(x_0, t_0, \theta r)} u(x, t)^q \, dx \, dt \right)^{\frac{1}{q}} \leq \left( \frac{C}{(1 - \theta)^\mu} \int_{U^+(x_0, t_0, r)} u(x, t)^q \, dx \, dt \right)^{\frac{1}{q}}, \tag{2.4}
\]

for all \( 0 < \bar{q} < q < q_0 \) where \( q_0 = \kappa(p-1), \) provided \( \kappa = \frac{n+sp}{n}, \) if \( sp < n \) and \( \kappa = \frac{3}{2}, \) if \( sp \geq n. \)

**Remark 2.17** We would like to emphasize that the constant \( C \) in the reverse Hölder inequality (2.4) above is independent of \( \bar{q} \) as \( \bar{q} \to 0 \) and this is precisely where the algebraic lemma 2.9 plays a crucial role. It is well known that such a stable behaviour of the constant \( C \) is needed in order to establish the Harnack inequality for local equations using the approach of Bombieri as in [3] (see also [20] for an adaptation of such an idea in the case of (1.3)). We therefore believe that such a reverse Hölder inequality will have similar future applications in the nonlocal case.

3  Energy estimate

To prove Theorem 2.14 we need the following Caccioppoli type estimate for subsolutions.
Lemma 3.1 Let $p > 2$, $x_0 \in \mathbb{R}^n$, $0 < \tau_1 < \tau_2$ and $l > 0$ with $(\tau_1 - l, \tau_2) \subset (0, T)$. Assume that $u$ is a weak sub-solution of (1.1) with $u > 0$ in $\mathbb{R}^n \times (\tau_1 - l, \tau_2)$. Let $k \in \mathbb{N}$ and denote $w(x, t) = (u - k)_+(x, t)$. Then there exists a positive constant $C = C(n, p, s, \Lambda)$ such that

$$
\int_{\tau_1 - l}^{\tau_2} \int_{B_r} \int_{B_r} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt + C \sup_{\tau_1 < t < \tau_2} \int_{B_r} w(x, t)^p \psi(x)^p \, dx
$$

$$
\leq \int_{\tau_1 - l}^{\tau_2} \int_{B_r} \int_{B_r} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt + C \sup_{\tau_1 < t < \tau_2} \int_{B_r} \xi(w(x, t)\psi(x)^p) \, dx
$$

$$
\leq C \left( \int_{\tau_1 - l}^{\tau_2} \int_{B_r} \int_{B_r} \max\{w(x, t), w(y, t)\}^p |\psi(x) - \psi(y)|^p \eta(t)^p \, d\mu \, dt
$$

$$
+ \left( \sup_{x \in \text{supp} \psi} \int_{\tau_1 - l}^{\tau_2} \int_{B_r} \frac{w(y, t)^{p-1}}{|x - y|^{n+ps}} \, dy \right) \int_{\tau_1 - l}^{\tau_2} \int_{B_r} w(x, t)\psi(x)^p \eta(t)^p \, dx \, dt
$$

$$
+ \int_{\tau_1 - l}^{\tau_2} \int_{B_r} \xi(w(x)\psi(x)^p) \partial_t \eta(t)^p \, dx \, dt \right),
$$

for all nonnegative $\psi \in C^\infty_0(B_r)$ and nonnegative $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) = 0$ for $t \leq \tau_1 - l$ and $\eta(t) = 1$ for $t \geq \tau_1$.

**Proof.** Since $p > 2$, we observe that the first inequality i.e.

$$
\int_{\tau_1 - l}^{\tau_2} \int_{B_r} \int_{B_r} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt + C \sup_{\tau_1 < t < \tau_2} \int_{B_r} w(x, t)^p \psi(x)^p \, dx
$$

$$
\leq \int_{\tau_1 - l}^{\tau_2} \int_{B_r} \int_{B_r} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt + C \sup_{\tau_1 < t < \tau_2} \int_{B_r} \xi(w(x, t)\psi(x)^p) \, dx
$$

follows directly from Lemma 2.10. Therefore, it is enough to prove the second inequality.

Let $t_1 = \tau_1 - l$ and $t_2 = \tau_2$ and for fixed $t_1 < l_1 < l_2 < t_2$ and $\epsilon > 0$ small enough, following [2] we define the function $\zeta_\epsilon \in W^{1,\infty}((t_1, t_2), [0, 1])$ by

$$
\zeta_\epsilon(t) := \begin{cases} 
0 & \text{for } t_1 \leq t \leq l_1 - \epsilon, \\
1 + \frac{t - l_1}{\epsilon} & \text{for } l_1 - \epsilon < t \leq l_1, \\
1 & \text{for } l_1 < t \leq l_2, \\
1 - \frac{t - l_2}{\epsilon} & \text{for } l_2 < t \leq l_2 + \epsilon, \\
0 & \text{for } l_2 + \epsilon < t \leq t_2,
\end{cases}
$$

and we choose

$$
\phi(x, t) = w(x, t)\psi(x)^p \zeta_\epsilon(t)^p
$$

as a test function in (2.3). We denote

$$
v_h^{p-1} = (u^{p-1})_h \quad \text{and} \quad \mathcal{V}(u(x, y, t)) = \mathcal{A}(u(x, y, t))K(x, y, t).
$$

Then following [2] [27], we observe that the sub-solution $u$ of (1.1) satisfies the following mollified inequality

$$
I_{h, \epsilon} + J_{h, \epsilon} \leq 0,
$$

(3.1)
Indeed, we can write
\[
I_{h,\epsilon} = \int_{t_1}^{t_2} \int_{B_r} \partial_t v_h^{p-1} \phi(x, t) \, dx \, dt = \int_{t_1}^{t_2} \int_{B_r} \partial_t v_h^{p-1} w(x, t) \psi(x)^p \zeta(t) \eta(t)^p \, dx \, dt,
\]
and
\[
J_{h,\epsilon} = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{V}(u(x, y, t)))_h (\phi(x, t) - \phi(y, t)) \, dx \, dy \, dt
= \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{V}(u(x, y, t)))_h (w(x, t) \psi(x)^p - w(y, t) \psi(y)^p) \zeta(t) \eta(t)^p \, dx \, dy \, dt.
\]

**Estimate of** $I_{h,\epsilon}$: Proceeding similarly as in the proof of [2, Proposition 3.1], we have
\[
\lim_{\epsilon \to 0} \lim_{h \to 0} I_{h,\epsilon} \geq \int_{B_r} \xi(w(x, l_2)) \psi(x)^p \eta(l_2) \, dx - \int_{B_r} \xi(w(x, l_1)) \psi(x)^p \eta(l_1) \, dx
- \int_{t_1}^{t_2} \int_{B_r} \xi(w)(x, t) \psi(x)^p \partial_t \eta(t)^p \, dx \, dt.
\]  
(3.2)

**Estimate of** $J_{h,\epsilon}$: First we claim that $\lim_{h \to 0} J_{h,\epsilon} = J_\epsilon$, where
\[
J_\epsilon = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{V}(u(x, y, t)) (w(x, t) \psi(x)^p - w(y, t) \psi(y)^p) \zeta(t) \eta(t)^p \, dx \, dy \, dt.
\]
Indeed, we can write
\[
J_{h,\epsilon} - J_\epsilon = L_{h,\epsilon} + N_{h,\epsilon},
\]  
(3.3)

where
\[
L_{h,\epsilon} = \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} ((\mathcal{V}(u(x, y, t)))_h - \mathcal{V}(u(x, y, t))) (w(x, t) \psi(x)^p - w(y, t) \psi(y)^p) \zeta(t) \eta(t)^p \, dx \, dy \, dt,
\]
and
\[
N_{h,\epsilon} = 2 \int_{t_1}^{t_2} \int_{B_r} \int_{\mathbb{R}^n \setminus B_r} ((\mathcal{V}(u(x, y, t)))_h - \mathcal{V}(u(x, y, t))) w(x, t) \psi(x)^p \zeta(t) \eta(t)^p \, dx \, dy \, dt.
\]

**Estimate of** $L_{h,\epsilon}$: We can rewrite $L_{h,\epsilon}$ as
\[
L_{h,\epsilon} = \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} ((\mathcal{V}(u(x, y, t)))_h - \mathcal{V}(u(x, y, t))) \frac{(w(x, t) \psi(x)^p - w(y, t) \psi(y)^p) \zeta(t) \eta(t)^p}{|x - y| \frac{1}{p} \frac{2}{p + \frac{1}{p}}} \, dx \, dy \, dt,
\]
and using Hölder’s inequality with exponents $p' = \frac{p}{p-1}$ and $p$, we obtain
\[
L_{h,\epsilon} \leq \left( \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \left| ((\mathcal{V}(u(x, y, t)))_h - \mathcal{V}(u(x, y, t))) \frac{|x - y| \frac{2}{p + \frac{1}{p}}}{p'} \right| \, dx \, dy \, dt \right)^{\frac{1}{p'}}
\cdot \left( \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \frac{|w(x, t) \psi(x)^p - w(y, t) \psi(y)^p| \zeta(t) \eta(t)^p}{|x - y|^{n+\frac{p}{p}}} \, dx \, dy \, dt \right)^{\frac{1}{p}}.
\]  
(3.4)
Now using the property \[^{[12]}\], we observe that,
\[
|x - y|^{\frac{n+ps}{p}} |V(u(x, y, t))| \leq \Lambda \frac{|u(x, t) - u(y, t)|^{p-1}}{|x - y|^{\frac{n+ps}{p}}} \in L^{p'}((t_1, t_2) \times B_r \times B_r),
\]
From \[^{[27]}\] Lemma 2.2, we have
\[
((V(u(x, y, t))))_h - V(u(x, y, t)) \to 0 \text{ in } L^{p'}((t_1, t_2) \times B_r \times B_r),
\]
and therefore from \(^{[33]}\), it follows that \(\lim_{h \to 0} L_{h, \epsilon} = 0\).

**Estimate of \(N_{h, \epsilon}\):** We note that given the pointwise convergence of mollified functions together with the fact that \(u \in L^{\infty}((t_1, t_2); L^{\infty}(\mathbb{R}^n))\), we can therefore apply the Lebesgue dominated convergence theorem to conclude that \(\lim_{h \to 0} N_{h, \epsilon} = 0\).

**Estimate of \(J_\epsilon\):** We can rewrite \(J_\epsilon = J^1_\epsilon + J^2_\epsilon\), where
\[
J^1_\epsilon = \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} A(u(x, y, t))(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\zeta_\epsilon(t)\eta(t)^p d\mu dt,
\]
and
\[
J^2_\epsilon = 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} A(u(x, y, t))w(x, t)\psi(x)^p\zeta_\epsilon(t)\eta(t)^p d\mu dt.
\]

**Estimate of \(J^1_\epsilon\):** To estimate the integral \(J^1_\epsilon\), we mainly adapt an idea from the proof of \[^{[13]}\] Theorem 1.4\]. By symmetry we may assume \(u(x, t) \geq u(y, t)\). In this case, for every fixed \(t\), we observe that
\[
|u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t))(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\eta(t)^p
= (u(x, t) - u(y, t))^{p-1}(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\eta(t)^p
\]
\[=
\begin{cases}
(u(x, t) - u(y, t))^{p-1}(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\eta(t)^p, & \text{if } u(x, t), u(y, t) > k, \\
(u(x, t) - u(y, t))^{p-1}w(x, t)\psi(x)^p\eta(t)^p, & \text{if } u(x, t) > k, u(y, t) \leq k, \\
0, & \text{otherwise}.
\end{cases}
\]
Thus
\[
|u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t))(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\eta(t)^p
\geq |w(x, t) - w(y, t)|^{p-1}(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\eta(t)^p.
\]
This implies,
\[
J^1_\epsilon \geq \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} (w(x, t) - w(y, t))^{p-1}(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)\zeta_\epsilon(t)\eta(t)^p d\mu dt.
\]
Let us now consider the case when \(w(x, t) > w(y, t)\) and \(\psi(x) \leq \psi(y)\). By Lemma \[^{[28]}\] we obtain
\[
\psi(x)^p \geq (1 - C(p)\epsilon)\psi(y)^p - (1 + C(p)\epsilon)^{1-p} |\psi(x) - \psi(y)|^p
\] (3.5)
Doubly nonlinear nonlocal p-Laplace equation

for any $\epsilon \in (0, 1]$ where $C(p) = (p - 1)\Gamma(\max\{1, p - 2\})$. Now by letting
\[
\epsilon = \frac{1}{\max\{1, 2C(p)\}} \frac{w(x, t) - w(y, t)}{w(x, t)} \in (0, 1],
\]
we deduce from above that the following inequality holds for some positive constant $C = C(p)$,
\[
(w(x, t) - w(y, t))^{p-1}w(x, t)\psi(x)^p \geq (w(x, t) - w(y, t))^{p-1}w(x, t)\max\{\psi(x), \psi(y)\}^p
- \frac{1}{2}(w(x, t) - w(y, t))^p \max\{\psi(x), \psi(y)\}^p - C \max\{w(x, t), w(y, t)\}^p|\psi(x) - \psi(y)|^p.
\]

Note that over here, we used that under the assumption $\psi(x) \leq \psi(y)$, we have $\max\{\psi(x), \psi(y)\} = \psi(y)$. In the other cases $w(x, t) \geq w(y, t), \psi(x) \geq \psi(y)$ or $w(x, t) = w(y, t)$, the above estimate is clear. Therefore, when $w(x, t) \geq w(y, t)$, we have
\[
(w(x, t) - w(y, t))^{p-1}(w(x, t)\psi(x)^p - w(y, t)\psi(y)^p)
\geq (w(x, t) - w(y, t))^{p-1}(w(x, t)\max\{\psi(x), \psi(y)\}^p - w(y, t)\psi(y)^p)
- \frac{1}{2}(w(x, t) - w(y, t))^p \max\{\psi(x), \psi(y)\}^p - C \max\{w(x, t), w(y, t)\}^p|\psi(x) - \psi(y)|^p
\]

(3.6)

If $w(x, t) < w(y, t)$, we may interchange the roles of $x$ and $y$ above to obtain (3.6). We then observe that
\[
|w(x, t)\psi(x) - w(y, t)\psi(y)|^p \leq 2^{p-1}|w(x, t) - w(y, t)|^p \max\{\psi(x), \psi(y)\}^p
+ 2^{p-1} \max\{w(x, t), w(y, t)\}^p|\psi(x) - \psi(y)|^p.
\]

(3.7)

Now (3.6) and (3.7) gives
\[
J_t^1 \geq c \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \zeta_\epsilon(t)\eta(t)^p \, d\mu \, dt
- C \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \max\{w(x, t), w(y, t)\}^p|\psi(x) - \psi(y)|^p \zeta_\epsilon(t)\eta(t)^p \, d\mu \, dt,
\]

(3.8)

for some positive constants $c = c(p), C = C(p)$.

**Estimate of $J_t^2$:** To estimate $J_t^2$, we observe that
\[
|u(x, t) - u(y, t)|^{p-2}(u(x, t) - u(y, t))w(x, t) \geq -(u(y, t) - u(x, t))^{p-1}w(x, t)
\geq -(u(y, t) - k)^{p-1}w(x, t)
\geq -w(y, t)^{p-1}w(x, t).
\]

As a consequence, we obtain,
\[
J_t^2 \geq - \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} K(x, y, t)w(y, t)^{p-1}w(x, t)\psi(x)^p \zeta_\epsilon(t)\eta(t)^p \, dx \, dy \, dt
\geq -\lambda \left( \sup_{t_1 < t < t_2, x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_r} \frac{w(y, t)^{p-1}}{|x - y|^{n + ps}} \, dy \right) \int_{t_1}^{t_2} \int_{B_r} w(x, t)\psi(x)^p \zeta_\epsilon(t)\eta(t)^p \, dx \, dt.
\]

(3.9)
Therefore from (3.8) and (3.9), we obtain for some positive constants $c = c(p)$ and $C = C(p)$,

$$\lim_{\epsilon \to 0} J_{h, \epsilon} = \lim_{\epsilon \to 0} J_{\epsilon} = \lim_{\epsilon \to 0} (J_{1, \epsilon}^1 + J_{2, \epsilon}^2)$$

$$\geq c \int_{t_1}^{t_2} \int_{B_r} \int_{B_{2r}} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt$$

$$- C \int_{t_1}^{t_2} \int_{B_r} \int_{B_{2r}} \max\{|w(x, t), w(y, t)|^p|\psi(x) - \psi(y)|^p \eta(t)^p \, d\mu \, dt$$

$$- \Lambda \left( \sup_{t_1 < t < t_2, \, x \in \supp \psi} \int_{\mathbb{R}^n \setminus B_r} \frac{w(y, t)^{p-1}}{|x - y|^{n+ps}} \, dy \right) \int_{t_1}^{t_2} \int_{B_r} w(x, t)\psi(x)^p \eta(t)^p \, dx \, dt.$$

Now employing the estimates (3.2) and (3.10) into (3.1) and then first letting $l_1 \to t_1$ and then by $l_2 \to t_2$, we get

$$\int_{t_1}^{t_2} \int_{B_r} \int_{B_{2r}} |w(x, t)\psi(x) - w(y, t)\psi(y)|^p \eta(t)^p \, d\mu \, dt$$

$$\leq C \left( \int_{t_1}^{t_2} \int_{B_r} \int_{B_{2r}} \max\{|w(x, t), w(y, t)|^p|\psi(x) - \psi(y)|^p \eta(t)^p \, d\mu \, dt$$

$$+ \left( \sup_{t_1 < t < t_2} \int_{\mathbb{R}^n \setminus B_r} \frac{w(y, t)^{p-1}}{|x - y|^{n+ps}} \, dy \right) \int_{t_1}^{t_2} \int_{B_r} w(x, t)\psi(x)^p \eta(t)^p \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \int_{B_r} \xi(w)\psi(x)^p \partial_t \eta(t)^p \, dx \, dt \right) .$$

Again using (3.2) and (3.11) and then first letting $l_1 \to t_1$ and then by choosing $l_2 \in (\tau_1, \tau_2)$ such that

$$\int_{B_r} \xi(w)(l_2)\psi(x)^p \, dx \geq \frac{1}{2} \sup_{\tau_1 < t < \tau_2} \int_{B_r} \xi(w)(x, l_2)\psi(x)^p \, dx,$$

we observe that

$$\sup_{\tau_1 < t < \tau_2} \int_{B_r} \xi(w)(x, t)\psi(x)^p \, dx$$

$$\leq C \left( \int_{t_1}^{t_2} \int_{B_r} \int_{B_{2r}} \max\{|w(x, t), w(y, t)|^p|\psi(x) - \psi(y)|^p \eta(t)^p \, d\mu \, dt$$

$$+ \left( \sup_{t_1 < t < t_2} \int_{\mathbb{R}^n \setminus B_r} \frac{w(y, t)^{p-1}}{|x - y|^{n+ps}} \, dy \right) \int_{t_1}^{t_2} \int_{B_r} w(x, t)\psi(x)^p \eta(t)^p \, dx \, dt$$

$$+ \int_{t_1}^{t_2} \int_{B_r} \xi(w)\psi(x)^p \partial_t \eta(t)^p \, dx \, dt \right) .$$

Now from (3.11) and (3.12), we get the required estimate.

4 Proof of Theorem 2.14

Let us first assume that $sp < n$ and for $j \in \mathbb{N}$, we denote by

$$r_0 = r, \quad r_j = \frac{1 + 2^{-j}}{2} r, \quad s_j = \frac{r_j + r_{j+1}}{2},$$
Moreover, we choose $\bar{\kappa} > 0$ to be chosen later, we let

$$k_j = (1 - 2^{-j})\bar{\kappa}, \quad \bar{k}_j = \frac{k_{j+1} + k_j}{2}, \quad w_j = (u - k_j)_+ \text{ and } \bar{w}_j = (u - \bar{k}_j)_+.$$  

We observe that

$$\bar{k}_j > k_j, \quad w_j \geq \bar{w}_j \quad \text{and} \quad w_j^p \geq (2^{-j-2\bar{\kappa}})^{p-1}\bar{w}_j = (k_j - \bar{k}_j)^{p-1}\bar{w}_j.$$  

Moreover, we choose $\psi_j \in C_0^\infty(B_j)$, $\eta_j \in C^\infty(\Gamma_j)$ such that $0 \leq \psi_j \leq 1$ in $B_j$, $\psi_j \equiv 1$ on $B_{j+1}$, $|\nabla \psi_j| \leq \frac{2^{j+3}}{p}$ in $B_j$ and $0 \leq \eta_j \leq 1$ in $\Gamma_j$, and $\eta_j \equiv 1$ on $\Gamma_{j+1}$ with $\eta_j = 0$ on $\Gamma_j \setminus \Gamma_j$ and $|\partial_n \eta_j| \leq \frac{2^{j+3}}{p^n}$ in $\Gamma_j$. By Lemma 2.3 with $\kappa = \frac{n+p}{n}$ and $\kappa^* = \frac{n}{n-sp}$, we have

\begin{equation}
\begin{aligned}
&\int_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^{p^*} dx \, dt \\
&\leq C_{j+1} \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+sp}} dx \, dy \, dt \cdot \left( \sup_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^p dx \right)^{\frac{p}{p^*}} \\
&\quad + C \int_{\Gamma_{j+1}} \int_{B_{j+1}} \bar{w}_j^p dx \, dt \cdot \left( \sup_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^p dx \right)^{\frac{p}{p^*}} \\
&= C_{j+1} \int_{\Gamma_{j+1}} \int_{B_{j+1}} \left( \frac{I_2}{|B_{j+1}|} \right)^{\frac{p}{p^*}} dx \, dt + C \int_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^p dx \, dt \cdot \left( \frac{I_2}{|B_{j+1}|} \right)^{\frac{p}{p^*}},
\end{aligned}
\end{equation}

where

$$I_1 = \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+sp}} dx \, dy \, dt \quad \text{and} \quad I_2 = \sup_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^p dx.$$

Let $U_j = B_j \times \Gamma_j$ and $\bar{U}_j = \bar{B}_j \times \bar{\Gamma}_j$. Since $r_{j+1} < r_j$, $s_j < r_j$, we have $\bar{B}_j \subset B_j$, $\bar{\Gamma}_j \subset \Gamma_j$, $B_{j+1} \subset B_j$ and $\Gamma_{j+1} \subset \Gamma_j$. To estimate $I_1$ and $I_2$ we apply Lemma 3.1 with $r = r_j, \tau_1 = t_0, \tau_1 = t_0 - r_{j+1}^p, l = s_j - r_{j+1}^p$ and $\phi_j(x,t) = \psi_j(x)\eta_j(t)$ with $\eta_j(t) = 0$ if $t \leq \tau_1 - l$ and $\eta_j(t) = 1$ if $t \geq \tau_1$. Observing that $B_{j+1} \subset \bar{B}_j$ and $\Gamma_{j+1} \subset \bar{\Gamma}_j$, using Lemma 3.1 for some positive constant $C = C(n,p,s,\Lambda)$ we get

\begin{equation}
\begin{aligned}
I_1 + C I_2 &\leq \int_{\Gamma_j} \int_{B_j} \int_{B_j} |\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p \psi_j(x)^p \eta_j(y)^p \, d\mu \, dt \\
&\quad + C \sup_{\Gamma_{j+1}} \int_{B_j} \bar{w}_j(x,t)^p \psi_j(x)^p dx \\
&\leq C(J_1 + J_2 + J_3),
\end{aligned}
\end{equation}

where

\begin{align*}
J_1 &= \int_{\Gamma_j} \int_{B_j} \int_{B_j} \max\{\bar{w}_j(x,t)^p, \bar{w}_j(y,t)^p\} |\psi_j(x) - \psi_j(y)|^p \eta_j(t)^p \, d\mu \, dt, \\
J_2 &= \sup_{t \in \Gamma_j, x \in \text{supp } \psi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{\bar{w}_j(y,t)^{p-1}}{|x-y|^{n+sp}} dy \int_{B_j} \bar{w}_j(x,t) \psi_j(x)^p \eta_j(t)^p dx,
\end{align*}
where $J_j = \int_{\Gamma_j} \int_{B_j} \xi(\bar{w}_j)(x,t)\psi_j(x)^p \partial_t \eta_j(t)^p \, dx \, dt$.

Now we estimate each $J_i$, $i = 1, 2, 3$ separately.

**Estimate of $J_1$:** Using $r_j < r$ and $\bar{w}_j \leq w_j$, we have

$$J_1 = \int_{\Gamma_j} \int_{B_j} \max\{\bar{w}_j(x,t)^p, \bar{w}_j(y,t)^p\} |\psi_j(x) - \psi_j(y)|^p \eta_j(t)^p \, d\mu \, dt$$

$$\leq C(n, p, s, \Lambda) \left( \sup_{x \in B_j} \int_{B_j} \frac{|\psi_j(x) - \psi_j(y)|^p}{|x-y|^{n+sp}} \, dy \right) \int_{\Gamma_j} \int_{B_j} \bar{w}_j(x,t)^p \, dx \, dt \tag{4.3}$$

$$\leq C(n, p, s, \Lambda) \frac{2^j(n+sp+p)}{r^{sp}} \int_{\Gamma_j} \int_{B_j} w_j(x,t)^p \, dx \, dt.$$

**Estimate of $J_2$:** Without loss of generality, we may assume $x_0 = 0$. Using the fact that $\bar{w}_j \leq w_0$, under the assumptions on $\psi_j$, we have for $x \in \text{supp} \psi_j$, and $y \in \mathbb{R}^n \setminus B_j$,

$$\frac{1}{|x-y|} = \frac{1}{|y|} \left( \frac{|x-(x-y)|}{|x-y|} \right) \leq \frac{1}{|y|} \left( 1 + 2^{j+3} \right) \leq \frac{2^{j+4}}{|y|}.$$

This implies

$$J_2 = \sup_{t \in \Gamma_j, x \in \text{supp} \psi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{\bar{w}_j(y,t)^{p-1}}{|x-y|^{n+sp}} \, dy \int_{\Gamma_j} \int_{B_j} \bar{w}_j(x,t)\psi_j(x)^p \eta_j(t)^p \, dx \, dt$$

$$\leq C \frac{2^j(n+sp+p)}{r^{sp} \kappa^{p-1}} \text{Tail}^{p-1}(w_0; x_0, \frac{r}{2}, t_0 - r^{sp}, t_0) \int_{\Gamma_j} \int_{B_j} w_j(x,t)^p \, dx \, dt \tag{4.4}$$

$$\leq C \frac{2^j(n+sp+p)}{\delta^{p-1} r^{sp}} \int_{\Gamma_j} \int_{B_j} w_j(x,t)^p \, dx \, dt,$$

where we also used the fact $\bar{w}_j \leq \left( \frac{2^{j+2}}{\kappa} \right)^{p-1} w_j^p$ and also that $\kappa$ would be chosen finally such that $\kappa \geq \delta \text{Tail}^{\infty}(w_0; x_0, \frac{r}{2}, t_0 - r^{sp}, t_0)$.

**Estimate of $J_3$:** To estimate $J_3$, we first note that by Lemma 2.10 and the fact that $p > 2$ we have,

$$J_3 = \int_{\Gamma_j} \int_{B_j} \xi(\bar{w}_j)(x,t)\psi_j(x)^p \partial_t \eta_j(t)^p \, dx \, dt$$

$$\leq C(p) \int_{\Gamma_j} \int_{B_j} (\bar{w}_j(x,t) + \bar{k}_j)^{p-2} \bar{w}_j(x,t)^2 \psi_j(x)^p \partial_t \eta_j(t)^p \, dx \, dt \tag{4.5}$$

$$= J_4 + J_5,$$

where

$$J_4 = \int_{(\Gamma_j \times B_j) \cap \{0 < u - \bar{k}_j < \bar{k}_j\}} (\bar{w}_j(x,t) + \bar{k}_j)^{p-2} \bar{w}_j(x,t)^2 \psi_j(x)^p \partial_t \eta_j(t)^p \, dx \, dt,$$
and
\[ J_5 = \int_{(\Gamma_j \times B_j) \cap \{ \tilde{\omega}_j \geq \bar{k}_j \}} (\tilde{w}_j(x,t) + \bar{k}_j)^{p-2} \tilde{w}_j(x,t)^2 \psi_j(x)^p |\partial_t \eta_j(t)^p| \, dx \, dt \]
\begin{align*}
\leq 2^{p-2} \int_{(\Gamma_j \times B_j) \cap \{ \tilde{\omega}_j \geq \bar{k}_j \}} \tilde{w}_j(x,t)^p \psi_j(x)^p |\partial_t \eta_j(t)^p| \, dx \, dt \\
\leq C(p,s) \frac{2j^{(n+sp+p)}}{r_j^{ps}} \int_{\Gamma_j \setminus B_j} w_j(x,t)^p \, dx \, dt,
\end{align*}
where to deduce the estimate (4.6) we have again used the fact that \( p > 2 \).

**Estimate of \( J_4 \):** Now we estimate \( J_4 \) by adapting some ideas from [19]. Indeed, we denote by \( A_j = (\Gamma_j \times B_j) \cap \{ 0 < u - \bar{k}_j < \bar{k}_j \} \) and using binomial expansion we observe that,
\begin{align*}
J_4 &= \int_{A_j} (\tilde{w}_j(x,t) + \bar{k}_j)^{p-2} \tilde{w}_j(x,t)^2 \psi_j(x)^p |\partial_t \eta_j(t)^p| \, dx \, dt \\
&= \sum_{d=0}^{\infty} \int_{A_j} \left( p - 2 \right) \frac{\tilde{w}_j(x,t)^{p-2}}{d} \left( \frac{\tilde{w}_j(x,t)}{\bar{k}_j} \right)^d |\partial_t \eta_j(t)^p| \, dx \, dt \\
&= J_4^1 + J_4^2,
\end{align*}
where
\begin{align*}
J_4^1 &= \sum_{d=0}^{[p-2]} \int_{A_j} \left( p - 2 \right) \frac{\tilde{w}_j(x,t)^{p-2}}{d} \left( \frac{\tilde{w}_j(x,t)}{\bar{k}_j} \right)^d |\partial_t \eta_j(t)^p| \, dx \, dt,
\end{align*}
and
\begin{align*}
J_4^2 &= \sum_{d=[p-2]+1}^{\infty} \int_{A_j} \left( p - 2 \right) \frac{\tilde{w}_j(x,t)^{p-2}}{d} \left( \frac{\tilde{w}_j(x,t)}{\bar{k}_j} \right)^d |\partial_t \eta_j(t)^p| \, dx \, dt.
\end{align*}

**Estimate of \( J_4^1 \):** Let us estimate \( J_4^1 \) as follows. Using Hölder’s inequality, we obtain
\[ J_4^1 \leq \sum_{d=0}^{[p-2]} \left| \left( p - 2 \right) \frac{\tilde{w}_j(x,t)^{p-2}}{d} \left( \frac{\tilde{w}_j(x,t)}{\bar{k}_j} \right)^d \right| |A_j|^1 \, |A_j|^{\frac{d+2}{p}}. \]
Now, since \( u \geq \bar{k}_j \) in \( A_j \), we observe that,
\[ \int_{A_j} w_j^p \, dx \, dt \geq (\bar{k}_j - \bar{k}_j)^p |A_j| = (\frac{\bar{k}}{2j+1})^p |A_j|. \]
Therefore, we obtain
\[ |A_j| \leq \left( \frac{2j+2}{k} \right)^p \int_{A_j} w_j^p \, dx \, dt. \] (4.8)
Now using (4.8) together with the fact \( \tilde{w}_j \leq w_j, \bar{k}_j < \bar{k}, r_j < r \) and also by using the bounds on \( |\partial_t \eta_j| \), we get
\begin{align*}
J_4^1 &\leq C(p) \sum_{d=0}^{[p-2]} \left| \left( p - 2 \right) \frac{\tilde{w}_j(x,t)^{p-2}}{d} \left( \frac{\tilde{w}_j(x,t)}{\bar{k}_j} \right)^d \right| |A_j|^1 \, |A_j|^{\frac{d+2}{p}} \left( \int_{A_j} w_j^p \, dx \, dt \right)^{1-\frac{d+2}{p}} \\
&\leq C(p) \frac{2j^{p(s+1)}}{r_j^{sp}} \int_{A_j} w_j^p \, dx \, dt.
\end{align*} (4.9)
**Estimate of $J_4^2$:** Now since $\tilde{w}_j < \tilde{k}_j$, therefore for all $d \geq |p-2|+1$, we have that $\tilde{w}_j^{d-|p-2|-1} \leq \tilde{k}_j^{d-|p-2|-1}$. Thus $\tilde{k}_j^{d-|p-2|+2} \leq \tilde{k}_j^{d-|p-2|+3}$ and consequently we obtain

$$J_4^2 \leq \sum_{d=|p-2|+1}^{\infty} \left| \left( \begin{array}{c} p-2 \\ d \end{array} \right) \right| \int_{A_j} \tilde{k}_j^{p-3-|p-2|} |\tilde{w}_j^{p-2} + 3| \partial_t \eta_j^p \, dx \, dt.$$ 

Finally by using $\tilde{k}_j^{p-3-|p-2|} < \tilde{w}_j^{p-3-|p-2|}$, we have

$$J_4^2 \leq \sum_{d=|p-2|+1}^{\infty} \left| \left( \begin{array}{c} p-2 \\ d \end{array} \right) \right| \int_{A_j} \tilde{w}_j^p |\partial_t \eta_j^p| \, dx \, dt$$

where we have also used the fact that the series $\sum_{d=0}^{\infty} |\left( \begin{array}{c} p-2 \\ d \end{array} \right)|$ is convergent. Therefore, using (4.9) and (4.10) into (4.7), we obtain

$$J_4 \leq C(p) \frac{2^{j_p(s+1)}}{r_j^p} \int_{B_j} \int_{B_j} \tilde{w}_j^p \, dx \, dt.$$

(4.11)

Now using the estimates (4.10) and (4.11) in (4.5) we conclude

$$J_3 \leq C(p, s) \frac{2^{j(n+p+s+p)}}{r_j^p} \int_{B_j} \int_{B_j} \tilde{w}_j^p \, dx \, dt.$$

(4.12)

Then using $\tilde{w}_j^{pn} \geq (2^{-j-2} \tilde{k}_j^p(k-1)) w_{j+1}^p$ in (4.11), we get

$$I = (2^{-j-2} \tilde{k}_j^p(k-1)) \int_{B_{j+1}} \int_{B_{j+1}} w_{j+1}^p \, dx \, dt$$

$$\leq \frac{C r_j^{p-n}}{|B_{j+1}|} I_1 \left( \frac{I_2}{|B_{j+1}|} \right)^{\frac{p}{n}} + C \int_{B_{j+1}} \int_{B_{j+1}} w_{j+1}^p \, dx \, dt \cdot \left( \frac{I_2}{|B_{j+1}|} \right)^{\frac{p}{n}}.$$ 

(4.13)

Plugging the estimates (4.3), (4.4) and (4.12) into (4.2), we have

$$I_1, I_2 \leq C(n, p, s, \Lambda) \frac{2^{j(n+p+s+p)}}{r_j^p} \int_{B_j} \int_{B_j} w_j^p \, dx \, dt.$$

(4.14)

Then using (4.14) in (4.13), we get

$$I \leq C(n, p, s, \Lambda) \left( \frac{2^{j(n+p+s+p)}}{r_j^p} \int_{B_j} \int_{B_j} w_j^p \, dx \, dt \right)^{\frac{n+p}{n}}.$$

We now let

$$A_j = \left( \int_{U_j} w_j^p \, dx \, dt \right)^{\frac{1}{p}}.$$
Then we have
\[
2^{-j} \bar{k}^{-p(\kappa-1)} A_{j+1}^p \leq C(n, p, s, \Lambda) \left( \frac{2^{j(n+ps+p)}}{\delta^{p-1}} A_j^p \right)^{\frac{n+ps}{n}}.
\]
Then for some positive constant \( C = C(n, p, s, \Lambda) \) we have
\[
\frac{A_{j+1}}{k} \leq C \frac{2^{j(n+ps+p)}}{k^{\kappa-1}} \left( \frac{2^{j(n+ps+p)}}{\delta^{p-1}} A_j^p \right)^{\frac{n+ps}{np}} = C \left( \frac{2^{j(n+ps+p)(\kappa)}}{\delta^{p-1}} \right)^{\frac{n+ps}{np}} \left( \frac{A_j}{k} \right)^{\kappa},
\]
with \( \kappa = \frac{n+ps}{n} \). Noting that \( w_0 = u \), we now let
\[
\bar{k} = \delta T \text{ail}_\infty(u; x_0, \frac{r}{2}, t_0 - r^{ps}, t_0) + C \frac{n}{p} 2 r^{2 \frac{2}{p-1}} \left( \kappa - \frac{n+ps+p}{p} \right) \frac{\delta - \frac{(p-1)n}{ps}}{\delta^{p-1}} \left( \int_{U_r(u)} w^p \, dx \, dt \right)^{\frac{1}{p}},
\]
such that for
\[
\beta = \frac{ps}{n}, \quad c_0 = \frac{C}{\delta^{p-1}} > 1, \quad b = 2^{\kappa-1+(n+ps+p)(\frac{\kappa}{p})} > 1 \quad \text{and} \quad Y_j = \frac{A_j}{k},
\]
the hypothesis of Lemma 2.7 is satisfied and consequently we have that
\[
\sup_{U_r(u)} u \leq \bar{k},
\]
which proves Theorem 2.14. In the case when \( sp \geq n \), the proof follows by similar arguments.

5 Some qualitative and quantitative properties of supersolutions

In this section, we prove some qualitative and quantitative properties of supersolutions which are strictly bounded away from zero. Throughout this section, by a global supersolution \( u \) in \( \mathbb{R}^n \times (0, T) \), we refer to a bounded positive function \( u \) which satisfies the hypothesis of Definition 2.11 in \( \Omega \times (0, T) \) where \( \Omega \) is any bounded domain in \( \mathbb{R}^n \).

The following lemma is the nonlocal analogue of Lemma 3.1 in [26] which states that the inverse of a supersolution is a subsolution.

**Lemma 5.1** Let \( p > 2 \) and \( u \geq \rho > 0 \) in \( \mathbb{R}^n \times (0, T) \) be a supersolution of (1.1), then \( u^{-1} \) is a subsolution of (1.1).

**Proof.** Let \( v = u^{-1} \) and \( \psi \in W^{1,p}_0(0, T; L^p(\Omega')) \cap L^p_0(0, T; W_0^{s,p}(\Omega')) \) be nonnegative. Since \( u \) is a weak supersolution of (1.1), by formally choosing \( \phi(x, t) = u(x, t)^{2(1-p)} \psi(x, t) \) as a test function in (2.3) which can be justified by mollifying in time as in the proof of Lemma 3.1, we obtain for every \( [t_1, t_2] \subseteq (0, T) \),
\[
0 \leq I_1 + I_2,
\]
(5.1)
Here we thus obtain from above, and with

\[
I_1 = \int_{\Omega'} u(x, t_2)^{p-1} \phi(x, t_2) \, dx - \int_{\Omega'} u(x, t_1)^{p-1} \phi(x, t_1) \, dx - \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{p-1} \partial_t \phi(x, t) \, dx \, dt,
\]

\[
= \int_{\Omega'} u(x, t_2)^{1-p} \psi(x, t_2) \, dx - \int_{\Omega'} u(x, t_1)^{1-p} \psi(x, t_1) \, dx - I_3,
\]

with

\[
I_3 = \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{p-1} \left( u(x, t)^2 (1-p) \partial_t \psi(x, t) - 2(p-1) \psi(x, t) u(x, t)^{1-2p} \partial_t u(x, t) \right) \, dx \, dt
\]

\[
= \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{1-p} \partial_t \psi(x, t) \, dx \, dt - 2(p-1) \int_{t_1}^{t_2} \int_{\Omega'} \psi(x, t) u(x, t)^{-p} \partial_t u(x, t) \, dx \, dt
\]

\[
= \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{1-p} \partial_t \psi(x, t) \, dx \, dt - 2 \int_{t_1}^{t_2} \int_{\Omega'} u(x, t)^{1-p} \partial_t \psi(x, t) \, dx \, dt + 2I_4,
\]

and

\[
I_4 = \int_{\Omega'} u(x, t_2)^{1-p} \psi(x, t_2) \, dx - \int_{\Omega'} u(x, t_1)^{1-p} \psi(x, t_1) \, dx.
\]

We thus obtain from above,

\[
I_1 = -\left( \int_{\Omega'} v(x, t_2)^{p-1} \psi(x, t_2) \, dx - \int_{\Omega'} v(x, t_1)^{p-1} \psi(x, t_1) \, dx - \int_{t_1}^{t_2} \int_{\Omega'} v^{p-1} \partial_t \psi \, dx \, dt \right).
\]

Here

\[
I_2 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(u(x, y, t))(\phi(x, t) - \phi(y, t)) \, d\mu \, dt
\]

\[
= \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x, t) - u(y, t)|^{p-2} (u(x, t) - u(y, t)) \cdot (u(x, t)^2 (1-p) \psi(x, t) - u(y, t)^2 (1-p) \psi(y, t)) \, d\mu \, dt
\]

\[
= -\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(v(x, y, t)) \left( \left( \frac{v(x, t)}{v(y, t)} \right)^{-p-1} \psi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{-p-1} \psi(y, t) \right) \, d\mu \, dt.
\]

Now we estimate $I_2$. Let us first consider the case when $v(x, t) \geq v(y, t)$. In this case, we have

\[
A(v(x, y, t)) \left( \left( \frac{v(x, t)}{v(y, t)} \right)^{-p-1} \psi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{-p-1} \psi(y, t) \right) \geq A(v(x, y, t)) (\psi(x, t) - \psi(y, t)).
\]

Likewise when $v(x, t) < v(y, t)$, we have

\[
A(v(x, y, t)) \left( \left( \frac{v(x, t)}{v(y, t)} \right)^{-p-1} \psi(x, t) - \left( \frac{v(y, t)}{v(x, t)} \right)^{-p-1} \psi(y, t) \right)
\]

\[
= |v(y, t) - v(x, t)|^{p-2} (v(y, t) - v(x, t)) \left( \left( \frac{v(y, t)}{v(x, t)} \right)^{-p-1} \psi(y, t) - \left( \frac{v(x, t)}{v(y, t)} \right)^{-p-1} \psi(x, t) \right)
\]

\[
\geq A(v(y, x, t)) (\psi(y, t) - \psi(x, t)).
\]
Therefore in either case we obtain

\[ I_2 \leq -\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{A}(v(x,y,t))(\psi(x,t) - \psi(y,t)) \, d\mu \, dt. \]

By inserting the above estimates for \( I_1 \) and \( I_2 \) into (5.1), we get

\[ \int_{\Omega'} v(x,t_2) (\psi(x,t_2))^p \psi(x,t_2) \, dx - \int_{\Omega'} v(x,t_1) (\psi(x,t_1))^p \psi(x,t_1) \, dx - \int_{t_1}^{t_2} \int_{\Omega'} v(x,t) (\psi(x,t) - \psi(y,t)) \, d\mu \, dt \]

\[ + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{A}(v(x,y,t))(\psi(x,t) - \psi(y,t)) \, d\mu \, dt \leq 0. \]

Hence \( v = u^{-1} \) is a subsolution of (1.1).

Now we prove an energy estimate for strictly positive supersolutions of (1.1) which is the key ingredient needed to deduce reverse Hölder inequality for strictly positive supersolutions.

**Lemma 5.2** Let \( p > 2 \), \( x_0 \in \mathbb{R}^n \), \( r > 0 \) and \( \alpha \in (0, p-1) \). Suppose that \( u \) is a weak supersolution of (1.1) with \( u \geq 0 \) in \( \mathbb{R}^n \times (\tau_1, \tau_2 + l) \), \( (\tau_1, \tau_2 + l) \subset (0, T) \). Then there exists positive constants \( C = C(n, p, s, \Lambda) \) and \( c = c(p) \) large enough such that

\[
\frac{p - 1}{\alpha} \sup_{\tau_1 < t < \tau_2} \int_{B_r} \psi(x)^p u(x,t)^\alpha \, dx + \frac{\zeta(\epsilon)}{c(p)} \int_{\tau_1}^{\tau_2+l} \int_{B_r} |\psi(x)u(x,t)\psi(x)u(x,t)^\alpha - \psi(y)u(y,t)\psi(y)u(y,t)^\alpha|^p \eta(t) \, d\mu \, dt \\
\leq \left( \zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}} \right) \int_{\tau_1}^{\tau_2+l} \int_{B_r} |\psi(x) - \psi(y)|^p \eta(t) \, d\mu \, dt \\
+ C(\Lambda) \sup_{x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_r} dy \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p \eta(t) \, dx \, dt \\
+ \frac{p - 1}{\alpha} \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p |\partial_t \eta(t)| \, dx \, dt,
\]

where \( \epsilon = p - \alpha - 1 \) and \( \zeta(\epsilon) = \frac{\epsilon^p}{\alpha^p} \), if \( \alpha \geq 1 \) and \( \zeta(\epsilon) = \frac{\epsilon^p}{\alpha} \) if \( \alpha \in (0, 1) \). Moreover \( \psi \in C_0^\infty(B_r) \) is taken to be nonnegative and \( \eta \in C^\infty(\mathbb{R}) \) is also non-negative such that \( \eta(t) = 1 \) if \( \tau_1 \leq t \leq \tau_2 \) and \( \eta(t) = 0 \) if \( t \geq \tau_2 + l \).

**Proof.** Let \( t_1 \in (\tau_1, \tau_2) \) and \( t_2 = \tau_2 + l \). We consider \( \eta \in C^\infty(t_1, t_2) \) such that \( \eta(t_2) = 0 \) and \( \eta(t) = 1 \) for all \( t \leq t_1 \). Let \( \epsilon \in (0, p-1) \) and \( \alpha = p - \epsilon - 1 \). Then since \( u \) is a strictly positive weak supersolution of (1.1), choosing \( \phi(x,t) = u(x,t)^{-\epsilon} \psi(x)^p \eta(t) \) as a test function in (2.3) (which is again justified by mollifying in time), we obtain

\[ 0 \leq I_1 + I_2 + 2I_3, \]

where

\[ I_1 = \int_{t_1}^{t_2} \int_{B_r} \frac{\partial}{\partial t} (u^{p-1}) \phi(x,t) \, dx \, dt, \]

\[ I_2 = \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \mathcal{A}(u(x,y,t))(u(x,t)^{-\epsilon} \psi(x)^p - u(y,t)^{-\epsilon} \psi(y)^p) \eta(t) \, dx \, dy \, dt, \]

and

\[ I_3 = \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \frac{\partial}{\partial t} (u^{p-1}) \phi(x,t) \eta(t) \, dx \, dy \, dt. \]
and
\[ I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} A(u(x, y, t))u(x, t)^{-\epsilon}\psi(x, t)^p\eta(t) \, d\mu \, dt. \]

We observe that for any \( x \in B_r \) and \( y \in \mathbb{R}^n \setminus B_r \), we have that the integrand in \( I_3 \) is non-negative precisely in the set where \( u(x, t) \geq u(y, t) \). In view of this, we observe that \( I_3 \) can be estimated from above in the following way,
\[
I_3 \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} u(x, t)^{p-\epsilon}\psi(x)^p\eta(t) \, d\mu \, dt \leq C(\Lambda) \sup_{x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_r} \frac{dy}{|x-y|^{n+ps}} \int_{t_1}^{t_2} \int_{B_r} u(x, t)^{p-\epsilon}\psi(x)^p\eta(t) \, dx \, dt.
\]

Then we note that \( I_2 \) can be estimated using Lemma 2.9 as follows,
\[
I_2 \leq \frac{\zeta(\epsilon)}{C(p)} \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} |\psi(x)u(x, t)|^p - |\psi(y)u(y, t)|^p \eta(t) \, d\mu \, dt + \left( \zeta(\epsilon) + 1 + \frac{1}{e^{p-1}} \right) \int_{t_1}^{t_2} \int_{B_r} |\psi(x) - \psi(y)|^p(u(x, t)^{p-\epsilon} + u(y, t)^{p-\epsilon})\eta(t) \, d\mu \, dt.
\]

For \( I_1 \) we have
\[
I_1 = -\frac{p-1}{p-\epsilon-1} \int_{B_r} u^{p-\epsilon-1}(x, t_1)\psi(x)^p \, dx - \frac{p-1}{p-\epsilon-1} \int_{t_1}^{t_2} \int_{B_r} u(x, t)^{p-\epsilon-1}\psi(x)^p\partial_t\eta(t) \, dx \, dt.
\]

Now using (5.3), (5.4), (5.5) into (5.2) and letting \( t_1 \to \tau_1 \), we obtain
\[
\frac{\zeta(\epsilon)}{C(p)} \int_{\tau_1}^{\tau_2+1} \int_{B_r} \int_{B_r} |\psi(x)u(x, t)|^p - |\psi(y)u(y, t)|^p \eta(t) \, d\mu \, dt \leq \left( \zeta(\epsilon) + 1 + \frac{1}{e^{p-1}} \right) \int_{\tau_1}^{\tau_2+1} \int_{B_r} \int_{B_r} |\psi(x) - \psi(y)|^p(u(x, t)^{p-\epsilon} + u(y, t)^{p-\epsilon})\eta(t) \, d\mu \, dt + C(\Lambda) \sup_{x \in \text{supp} \psi} \int_{\mathbb{R}^n \setminus B_r} \frac{dy}{|x-y|^{n+ps}} \int_{\tau_1}^{\tau_2+1} \int_{B_r} u(x, t)^{p-\epsilon}\psi(x)^p\eta(t) \, dx \, dt
\]
\[
+ \frac{p-1}{\alpha} \int_{\tau_1}^{\tau_2+1} \int_{B_r} u(x, t)^{p-\epsilon}\psi(x)^p|\partial_t\eta(t)| \, dx \, dt.
\]

We then choose \( t_1 \) such that
\[
\int_{B_r} u(x, t_1)^{p-\epsilon-1}\psi(x)^p \, dx \geq \frac{1}{2} \sup_{\tau_1 < t < \tau_2} \int_{B_r} u(x, t)^{p-\epsilon-1}\psi(x)^p \, dx.
\]

Again using (5.3), (5.4), (5.5) and (5.7), we get
\[
\frac{p - 1}{\alpha} \sup_{\tau_1 < \tau < \tau_2} \int_{B_r} \psi(x)^p u(x,t)^\alpha \, dx \\
\leq \left( \zeta(\epsilon) + 1 + \frac{1}{e^{p-1}} \right) \int_{\tau_1}^{\tau_2 + l} \int_{B_r} |\psi(x) - \psi(y)|^p (u(x,t)^\alpha + u(y,t)^\alpha) \eta(t) \, d\mu \, dt \\
+ C(\Lambda) \sup_{x \in \text{supp} \psi} \int_{\mathbb{R}^n} \frac{dy}{|x - y|^{n+sp}} \int_{\tau_1}^{\tau_2 + l} \int_{B_r} u(x,t)^\alpha \psi(x)^p \eta(t) \, dx \, dt \\
+ \frac{p - 1}{\alpha} \int_{\tau_1}^{\tau_2 + l} \int_{B_r} u(x,t)^\alpha \psi(x)^p |\partial_t \eta(t)| \, dx \, dt.
\]  

Therefore from (5.6) and (5.5) we get the required estimate.

Following the energy estimate, we now proceed with the proof of the reverse Hölder inequality for strictly positive supersolutions as in Theorem 2.16.

**Proof of Theorem 2.16** We will prove the Theorem when \( sp < n \) with \( \kappa = \frac{n+ps}{n} \). Similar argument holds in case when \( sp \geq n \). Let us denote by

\[
r_0 = r, \quad r_j = \left( 1 - (1-\theta) \frac{1 - 2^{-j}}{1 - 2^{-m}} \right) r, \quad \delta_j = 2^{-j} r, \quad j = 1, 2, \ldots, m
\]

and \( U_j = B_j \times \Gamma_j = B_{r_j}(x_0) \times (t_0, t_0 + r_j^{ps}) \). We shall fix \( m \) later. Now we choose nonnegative test functions \( \psi_j \in C_c^\infty(B_j) \) such that \( 0 \leq \psi_j \leq 1 \) in \( B_j \), \( \psi_j \equiv 1 \) in \( B_{j+1} \), \( |\nabla \psi_j| \leq \frac{2^{j+3}}{(1-\theta)^{p}} \) and \( \text{dist}(\text{supp} \psi_j, \mathbb{R}^n \setminus B_j) \geq \frac{\delta_j(1-\theta)}{2} \). Moreover we choose \( \eta_j \in C^\infty(\Gamma_j) \) such that \( 0 \leq \eta_j \leq 1 \) in \( \Gamma_j \), \( \eta_j \equiv 1 \) in \( \Gamma_{j+1} \), and \( |\partial_t \eta_j| \leq \frac{2^{ps(j+3)}}{(1-\theta)^{ps}} \). If \( \alpha = p - \epsilon - 1 \) where \( \epsilon \in (0, p - 1) \). Then \( \alpha \in (0, p - 1) \). Denote by \( v = u^\frac{p}{n} \). Let \( r = r_j, \tau_1 = t_0, \tau_2 = t_0 + r_j^{ps}, l = r_j^{ps} - r_{j+1}^{ps} \). By the Sobolev embedding theorem (Lemma 2.4), we obtain

\[
\int_{\Gamma_{j+1}} \int_{B_{j+1}} |v|^{ps} \, dx \, dt \leq C \left( \frac{r_j^{ps-n} I_1 + \int_{\Gamma_{j+1}} \int_{B_{j+1}} |v|^p \, dx \, dt}{|B_{j+1}|} \right)^{\frac{ps}{n}},
\]

where

\[
I_1 = \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|v(x,t) - v(y,t)|^p}{|x - y|^{n+sp}} \, dx \, dy \, dt,
\]

and

\[
I_2 = \sup_{\tau_1 \leq t \leq \tau_2} \int_{B_{j+1}} |v|^p \, dx.
\]

Using the fact that \( \psi_j \equiv 1 \) on \( B_{j+1} \) and also that \( \eta_j \equiv 1 \) on \( \Gamma_{j+1} \), we obtain using Lemma 5.2 that the following holds,

\[
I_1, I_2 \leq C(J_1 + J_2 + J_3),
\]

for some positive constant \( C \) which is independent of \( \alpha \) as long as \( \alpha \) is away from \( p - 1 \), where

\[
J_1 = \int_{\Gamma_j} \int_{B_j} \int_{B_j} (v(x,t)^p + v(y,t)^p) \frac{|\psi_j(x) - \psi_j(y)|^p}{|x - y|^{n+sp}} \eta_j(t) \, dx \, dy \, dt
\]

\[
\leq C \left( \frac{2^{j(n+sp+p)}}{(1-\theta)^{ps}} \right) \int_{\Gamma_j} \int_{B_j} v(x,t)^p \, dx \, dt,
\]
Doubly nonlinear nonlocal \( p \)-Laplace equation

since \( r_j < r \),

\[
J_2 = \sup_{x \in \text{supp} \psi_j} \int_{\mathbb{R}^n \setminus B_j} \frac{dy}{|x - y|^{n + sp + p}} \int_{\Gamma_j} \int_{B_j} v(x, t)^p \psi_j(x)^p \eta_j(t) \, dx \, dt \\
\leq C \frac{2^{j(n + sp + p)}}{r_j^{ps}} \int_{\Gamma_j} \int_{B_j} v(x, t)^p \, dx \, dt,
\]

and

\[
J_3 = \int_{\Gamma_j} \int_{B_j} \psi_j(x)^p v(x, t)^p |\partial_t \eta_j(t)| \, dx \, dt \leq C \frac{2^{j(n + ps + p)}}{(1 - \theta)r_j^{ps}} \int_{\Gamma_j} \int_{B_j} v(x, t)^p \, dx \, dt,
\]

again since \( r_j < r \).

Therefore using (5.11), (5.12) and (5.13) into (5.10) we obtain

\[
I_1, I_2 \leq C \frac{2^{j(n + ps + p)}}{(1 - \theta)r_j^{ps}} \int_{\Gamma_j} \int_{B_j} v(x, t)^p \, dx \, dt,
\]

for some positive constant \( C \) independent of \( \alpha \) as long as \( \alpha \) is away from \( p - 1 \), but may depend on \( n, p, s, \Lambda \).

Using the estimate (5.14) and the fact that \( r_{j+1} < r_j < 2r_{j+1} \) for every \( j \), we obtain from (5.9), since \( v = u^{\frac{p}{q}} \) that

\[
\left( \int_{U_{j+1}} |u|^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}} \leq C \left( \frac{2^{j(n + ps + p)}}{(1 - \theta)^p} \int_{U_j} |u|^\alpha \, dx \, dt \right)^{\frac{1}{\alpha}},
\]

for some positive constant \( C \) independent of \( \alpha \) (given that our choice of \( \alpha \) will be away from \( p - 1 \)) but may depend on \( n, p, s, \Lambda \). Now we use Moser iteration technique into (5.15). Since \( \alpha \in (0, p - 1) \), we have \( \kappa \alpha \in (0, \kappa(p - 1)) \). This gives the upper bound \( q_0 = \kappa(p - 1) \) for \( q \).

Let us fix \( q, \bar{q} \) such that \( 0 < \bar{q} < q \) and \( m \) such that \( \bar{q} \kappa^{-1} \leq q \leq \bar{q} \kappa^m \). Let \( t_0 \) be such that \( t_0 \leq \bar{q} \) and \( q = \kappa^m t_0 \). Denote by \( t_j = \kappa^j t_0 \) for \( j = 0, 1, \ldots, m \). Then observing that \( r_m = \theta r \) and \( r_0 = r \), we get \( U_m = U^+(\theta r) \) and \( U_0 = U^+(r) \). Hence

\[
\left( \int_{U^+(\theta r)} u^q \, dx \, dt \right)^{\frac{1}{q}} = \left( \int_{U_m} u^q \, dx \, dt \right)^{\frac{1}{q}} \leq \left( \frac{C_{\text{prod}}(m)}{(1 - \theta)^m} \int_{U_{m-1}} u^{t_{m-1}} \, dx \, dt \right)^{\frac{1}{t_{m-1}}},
\]

where

\[
C_{\text{prod}}(m) = C \prod_{j=0}^{m-1} \left( \frac{2^{n + ps + p}(j + 1)}{p} \right)^{p \kappa^{-j}},
\]

and

\[
m^* = p \sum_{j=0}^{m-1} \kappa^{-j} = \frac{p \kappa}{\kappa - 1} (1 - \kappa^{-m}).
\]
It can be easily seen that $C_{\text{prod}}(m)$ is a positive constant uniformly bounded on $m$, where $C$ is independent of $\bar{q}$ but depends on $q$ due to the singularity of the constants involved in the energy inequality in Lemma 5.2 at $\epsilon = 0$. Finally using Hölder’s inequality, we obtain
\[
\left( \int_{U^+(\theta r)} u^q \, dx \, dt \right)^{\frac{1}{q}} \leq \left( \frac{C}{(1-\theta)m^r} \right)^{\frac{1}{q}} \left( \int_{U^+(r)} u^q \, dx \, dt \right)^{\frac{1}{q}}.
\]

Now, since $\bar{q} \kappa^{m-1} \leq t_0 \kappa^m$, we have $t_0 \geq \frac{\bar{q}}{\kappa}$. As a consequence the required estimate follows with $\mu = \frac{p c^2}{n-1}$.

In closing, we prove the following logarithmic estimate for strictly positive supersolutions which constitutes the nonlocal analogue of Lemma 6.1 in [26] and also constitutes one of the key ingredients in the proof of weak harnack in the local case.

**Lemma 5.3** Let $p > 2$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $t_0 \in (r^{ps}, T - r^{ps})$. Suppose that $u$ is a weak supersolution to (1.1) with $u \geq \rho > 0$ in $\mathbb{R}^n \times (t_0 - r^{ps}, t_0 + r^{ps})$. Then there exists a positive constant $C = C(n, p, s, \Lambda)$ such that
\[
\left| \{(x, t) \in U^+(x_0, t_0, r) : \log u(x, t) < -\lambda - b \} \right| \leq \frac{C r^{n+ps}}{\lambda^{p-1}} \quad (5.16)
\]
and
\[
\left| \{(x, t) \in U^-(x_0, t_0, r) : \log u(x, t) > \lambda - b \} \right| \leq \frac{C r^{n+ps}}{\lambda^{p-1}} \quad (5.17)
\]
where
\[
b = b(u(\cdot, t_0)) = - \frac{\int_{B_{\frac{3r}{2}}(x_0)} \log u(x, t_0) \psi(x)^p \, dx}{\int_{B_{\frac{3r}{2}}(x_0)} \psi(x)^p \, dx}.
\]

**Proof.** Following Lemma 6.1 in [26], we only prove (5.16) because the proof of (5.17) is analogous. Without loss of generality, we may assume $x_0 = 0$. Let $\psi \in C_0^\infty(B_{\frac{3r}{2}})$ be a nonnegative radially decreasing function such that $0 \leq \psi \leq 1$ in $B_{\frac{3r}{2}}$, $\psi \equiv 1$ in $B_r$, $|\nabla \psi| \leq \frac{C}{r}$ in $B_{\frac{3r}{2}}$. Since $u$ is a strictly positive supersolution of (1.1), choosing $\phi(x, t) = \psi(x)^p u(x, t)^{1-p}$ as a test function in (2.2), we get
\[
I_1 + I_2 + 2I_3 \geq 0, \quad (5.18)
\]
where for any $t_0 - r^{ps} \leq t_1 < t_2 \leq t_0 + r^{ps}$, we have
\[
I_1 = \int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \frac{\partial}{\partial t} (u(x, t)^{p-1}) \phi(x, t) \, dx \, dt = (p - 1) \int_{B_{\frac{3r}{2}}} \log u(x, t) \psi(x)^p \, dx \bigg|_{t=t_1}^{t_2}, \quad (5.19)
\]
\[
I_2 = \int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} A(u(x, y, t)) (\phi(x, t) - \phi(y, t)) \, d\mu \, dt,
\]
and
\[
I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} A(u(x, y, t)) \phi(x, t) \, d\mu \, dt.
\]
Following the arguments in the proof of [13] Lemma 1.3, we obtain for some positive constant $C = C(p)$,

$$I_2 = \int_{t_1}^{t_2} \int_{B_{3r}^+} \int_{B_{3r}^+} \mathcal{A}(u(x,y,t))(\phi(x,t) - \phi(y,t)) \, d\mu \, dt$$

$$\leq -\frac{1}{C} \int_{t_1}^{t_2} \int_{B_{3r}^+} \int_{B_{3r}^+} K(x,y,t) \log u(x,t) - \log u(y,t)|\psi(y)^p \, dx \, dy \, dt$$

$$+ C \int_{t_1}^{t_2} \int_{B_{3r}^+} \int_{B_{3r}^+} K(x,y,t)|\psi(x) - \psi(y)|^p \, dx \, dy \, dt$$

(5.20)

$$\leq -\frac{1}{C} \int_{t_1}^{t_2} \int_{B_{3r}^+} \int_{B_{3r}^+} K(x,y,t) \log u(x,t) - \log u(y,t)|\psi(y)^p \, dx \, dy \, dt$$

$$+ C(t_2 - t_1)r^{n-sp},$$

where the last inequality is obtained using the properties of $\psi$. Again following the proof of [13] Lemma 1.3, we get that

$$I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_{3r}} \int_{B_{3r}^+} \mathcal{A}(u(x,y,t))\phi(x,t) \, d\mu \, dt \leq C(t_2 - t_1)r^{n-sp}. \quad (5.21)$$

Therefore using the estimates (5.19), (5.20) and (5.21) into (5.18), we obtain

$$\frac{1}{C} \int_{t_1}^{t_2} \int_{B_{3r}^+} \int_{B_{3r}^+} K(x,y,t) \log u(x,t) - \log u(y,t)|\psi(y)^p \, dx \, dy \, dt$$

$$- (p - 1) \int_{B_{3r}^+} \log u(x,t)|\psi(x)^p \, dx \bigg|_{t=t_1}^{t_2} \leq C(t_2 - t_1)r^{n-sp}. \quad (5.22)$$

Let $v(x,t) = -\log u(x,t)$ and

$$V(t) = \frac{\int_{B_{3r}^+} v(x,t)\psi(x)^p \, dx}{\int_{B_{3r}^+} \psi(x)^p \, dx}.$$ 

Since $0 \leq \psi \leq 1$ in $B_{3r}$ and $\psi \equiv 1$ in $B_r$, therefore we have that $\int_{B_{3r}^+} \psi(x)^p \, dx \approx r^n$. Hence dividing by $\int_{B_{3r}^+} \psi(x)^p \, dx$ on both sides of (5.22), we obtain using the weighted Poincaré inequality in Lemma 2.6 that the following holds,

$$V(t_2) - V(t_1) + \frac{r^{-sp}}{c(p - 1)} \int_{t_1}^{t_2} \int_{B_r} \phi(x,t) - V(t)^p \, dx \, dt \leq \frac{cr^{-sp}}{p - 1}(t_2 - t_1).$$

Let $A_1 = C(p - 1)$, $A_2 = \frac{C}{p - 1}$,

$$\bar{w}(x,t) = v(x,t) - A_2r^{-sp}(t - t_1) \quad \text{and} \quad \bar{W}(t) = V(t) - A_2r^{-sp}(t - t_1).$$

Therefore $v(x,t) - V(t) = \bar{w}(x,t) - \bar{W}(t)$. Hence we get

$$\bar{W}(t_2) - \bar{W}(t_1) + \frac{1}{A_1r^{n+ps}} \int_{t_1}^{t_2} \int_{B_r} |\bar{w}(x,t) - \bar{W}(t)|^p \, dx \, dt \leq 0. \quad (5.23)$$
Therefore $\tilde{W}(t)$ is a monotone decreasing function in $t_0 - r^{ps} \leq t_1 < t_2 \leq t_0 + r^{ps}$. Hence $\tilde{W}(t)$ is differentiable almost everywhere with respect to $t$. Dividing by $t_2 - t_1$ on both sides of (5.23), we obtain after letting $t_2 \to t_1$,

$$\tilde{W}'(t) + \frac{1}{A_1 r^{n+ps}} \int_{B_r} |\tilde{w}(x,t) - \tilde{W}(t)|^p \, dx \leq 0. \quad (5.24)$$

Let $t_1 = t_0$, then $\tilde{W}(t_0) = V(t_0)$ and we denote by $b(u(\cdot, t_0)) = \tilde{W}(t_0)$. Let

$$\Omega_t^+(\lambda) = \{ x \in B_r : \tilde{w}(x,t) > b + \lambda \}.$$ 

Then for any $x \in \Omega_t^+(\lambda)$ and $t \geq t_0$, since $\tilde{W}(t) \leq \tilde{W}(t_0) = b$, we have

$$\tilde{w}(t,x) - \tilde{W}(t) \geq b + \lambda - \tilde{W}(t) \geq b + \lambda - \tilde{W}(t_0) = \lambda > 0.$$ 

Hence from (5.24), we have

$$\tilde{W}'(t) + \frac{|\Omega_t^+(\lambda)|}{A_1 r^{n+ps}} (b + \lambda - \tilde{W}(t))^p \leq 0.$$ 

Therefore, we have

$$|\Omega_t^+(\lambda)| \leq \frac{A_1 r^{n+sp}}{p-1} \partial_t (b + \lambda - \tilde{W}(t))^{1-p}.$$ 

Integrating over $t_0$ to $t_0 + r^{sp}$, we obtain

$$|\{(x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \tilde{w}(x,t) > b + \lambda\}| \leq \frac{A_1 r^{n+sp}}{p-1} \int_{t_0}^{t_0 + r^{sp}} \partial_t (b + \lambda - \tilde{W}(t))^{1-p} \, dt,$$

which gives

$$|\{(x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \log u(x,t) + A_2 r^{-sp}(t - t_0) < -\lambda - b\}| \leq \frac{A_1}{p-1} \frac{r^{n+sp}}{\lambda^{p-1}}. \quad (5.25)$$

Finally we note that,

$$|\{(x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \log u(x,t) < -\lambda - b\}| \leq A + B, \quad (5.26)$$

where

$$A = \{ (x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \log u(x,t) + A_2 r^{-sp}(t - t_0) < -\frac{1}{2} - b \} \leq \frac{C r^{n+sp}}{\lambda^{p-1}},$$

which follows from (5.25) and

$$B = \{ (x,t) \in B_r \times (t_0, t_0 + r^{sp}) : A_2 r^{-sp}(t - t_0) > \frac{1}{2} \} \leq \left( 1 - \frac{\lambda}{2A_2} \right) r^{n+sp}.$$ 

If $\frac{\lambda}{2A_2} < 1$, then

$$B \leq \left( 1 - \frac{\lambda}{2A_2} \right) r^{n+sp} < r^{n+sp} < \left( \frac{2A_2}{\lambda} \right)^{p-1} r^{n+sp}.$$
If $\frac{\lambda}{2A^2} \geq 1$, then $B = 0$. Hence in either case we have

$$B \leq \frac{C_{r}^{n+ps}}{\lambda^{p-1}}.$$ 

Inserting the above estimates of $A$ and $B$ into (5.26), we obtain

$$\left\{|(x, t) \in B_r \times (t_0, t_0 + r^{sp}) : \log u(x, t) < -\lambda - b\right\} \leq \frac{C_{r}^{n+ps}}{\lambda^{p-1}},$$

for some positive constant $C = C(n, p, s, \Lambda)$, which proves (5.16). The proof of (5.17) is analogous.

6 Appendix

In this section, we prove Lemma [2.9] To this end, we establish the following auxiliary lemmas. Throughout this section, we assume $p > 1$.

**Lemma 6.1** Let $f, g \in C^1([a, b])$. Then

$$\left|\frac{f(b) - f(a)}{b - a} + \left|\frac{g(b) - g(a)}{b - a}\right|^p \leq \max_{t \in [a, b]} \left|f'(t) + |g'(t)|^p\right|,\right.$$ 

**Proof.** Suppose the result does not hold, then by contradiction, we get

$$\left|\frac{f(b) - f(a)}{b - a} + \left|\frac{g(b) - g(a)}{b - a}\right|^p > f'(t) + |g'(t)|^p,\right.$$ 

for all $t \in [a, b]$. Integrating over $a$ to $b$, we obtain

$$\left|\frac{g(b) - g(a)}{b - a}\right|^p > \frac{1}{b - a} \int_a^b |g'(t)|^p \, dt,$$

which contradicts Jensen’s inequality.

**Lemma 6.2** Let $a, b > 0, 0 < \epsilon < p - 1$. Then we have

$$|b - a|^{p-2}(b - a)(a^{-\epsilon} - b^{-\epsilon}) \geq \zeta(\epsilon)\left|b^{\frac{p-1}{p}} - a^{\frac{p-1}{p}}\right|^p,$$

where $\zeta(\epsilon) = \frac{p^\epsilon}{(p-\epsilon-1)^p}$. Moreover, if $0 < p - \epsilon - 1 < 1$, then we may choose $\zeta(\epsilon) = \frac{p^\epsilon}{p-\epsilon-1}$.

**Proof.** Let $0 < \epsilon < p - 1$ and $\zeta(\epsilon) = \frac{p^\epsilon}{(p-\epsilon-1)^p}$. Let $f(t) = t^{\frac{1}{p-\epsilon}}$ and $g(t) = t^{\frac{p-1}{p}}$. By Lemma 6.1 we have

$$\frac{1}{\zeta(\epsilon)} \left|b^{\frac{p-1}{p}} - a^{\frac{p-1}{p}}\right|^p \leq 0.$$

If $b \geq a$, multiplying by $(b - a)^p$, we obtain

$$(b - a)^{p-1}(a^{-\epsilon} - b^{-\epsilon}) \geq \zeta(\epsilon)\left|b^{\frac{p-1}{p}} - a^{\frac{p-1}{p}}\right|^p.$$

If $b < a$, interchanging $a$ and $b$, the Lemma follows. If $0 < (p - \epsilon - 1)^p < p - \epsilon - 1$, we have $\zeta(\epsilon) \geq \frac{p^\epsilon}{p-\epsilon-1}$ and (6.1) implies

$$(b - a)^{p-1}(a^{-\epsilon} - b^{-\epsilon}) \geq \frac{p^\epsilon}{p - \epsilon - 1}\left|b^{\frac{p-1}{p}} - a^{\frac{p-1}{p}}\right|^p.$$

Hence the claim follows with $\zeta(\epsilon) = \frac{p^\epsilon}{p-\epsilon-1}$ when $0 < p - \epsilon - 1 < 1$. 
6.1 Proof of Lemma 2.9

We denote the left-hand and right-hand sides of (2.1) by L.H.S and R.H.S, respectively. Let 
\[ \zeta_1(\epsilon) = \frac{\epsilon}{\epsilon(p)} \] and \[ \zeta_2(\epsilon) = \zeta(\epsilon) + 1 + \frac{1}{\epsilon^p}. \] Then \( \zeta_1(\epsilon) - \zeta_2(\epsilon) < -1 \) since \( C(p) > 1 \) (to be finally chosen appropriately).

Case 1. If \( \tau_1 = \tau_2 = 0 \), then (2.1) holds trivially.

Case 2. If \( \tau_1 > 0 \) and \( \tau_2 = 0 \). In this case, we note that if \( b > a \), then

\[
\text{L.H.S} = |b - a|^{p-2} (b - a) (\tau_1^p a^{-\epsilon} - \tau_2^p b^{-\epsilon}) = (b - a)^{p-1} \tau_1^p a^{-\epsilon}
\]

and

\[
\text{R.H.S} = \zeta_1(\epsilon) \tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon) \tau_1^p (b^{p-\epsilon-1} + a^{p-\epsilon-1})
= (\zeta_1(\epsilon) - \zeta_2(\epsilon)) \tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon) \tau_1^p b^{p-\epsilon-1}.
\]

Now L.H.S is positive and since \( \zeta_1(\epsilon) - \zeta_2(\epsilon) < 0 \) and \( \zeta_2(\epsilon) > 0 \), the R.H.S is negative. Therefore we have \( \text{L.H.S} \geq \text{R.H.S} \). On the other hand if \( b \leq a \), then

\[
\text{L.H.S} = -(a - b)^{p-1} \tau_1^p a^{-\epsilon} \geq -\tau_1^p a^{p-\epsilon-1},
\]

and since \( \zeta_1(\epsilon) - \zeta_2(\epsilon) < -1 \) and \( \zeta_2(\epsilon) > 0 \), we have

\[
\text{R.H.S} = (\zeta_1(\epsilon) - \zeta_2(\epsilon)) \tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon) \tau_1^p b^{p-\epsilon-1} < -\tau_1^p a^{p-\epsilon-1} \leq \text{L.H.S}.
\]

Case 3. If \( \tau_1 = 0 \) and \( \tau_2 > 0 \). Then we have

\[
\text{L.H.S} = -|b - a|^{p-2} (b - a) \tau_2^p b^{-\epsilon},
\]

and

\[
\text{R.H.S} = (\zeta_1(\epsilon) - \zeta_2(\epsilon)) \tau_2^p b^{p-\epsilon-1} - \zeta_2(\epsilon) \tau_2^p a^{p-\epsilon-1}.
\]

If \( b > a \), then

\[
\text{L.H.S} = -(b - a)^{p-1} \tau_2^p b^{-\epsilon} \geq -\tau_2^p b^{p-\epsilon-1},
\]

and since \( \zeta_1(\epsilon) - \zeta_2(\epsilon) < -1 \) and \( \zeta_2(\epsilon) > 0 \), we have

\[
\text{R.H.S} = (\zeta_1(\epsilon) - \zeta_2(\epsilon)) \tau_2^p b^{p-\epsilon-1} - \zeta_2(\epsilon) \tau_2^p a^{p-\epsilon-1} < -\tau_2^p b^{p-\epsilon-1} \leq \text{L.H.S}.
\]

If \( b \leq a \), then the L.H.S is nonnegative and the R.H.S is negative. Therefore we have \( \text{L.H.S} \geq \text{R.H.S} \).

Case 4. Let both \( \tau_1, \tau_2 > 0 \). By symmetry, we may assume that \( b \geq a \). Let \( t = \frac{b}{a} \geq 1 \), \( s = \frac{a}{\tau_1} > 0 \) and \( \lambda = s^pt^{-\epsilon} \). It can be easily seen that the inequality (2.1) is equivalent to the following inequality

\[
\zeta_1(\epsilon)|st|^{\frac{p^2-1}{p}} - |t|^p \leq (t - 1)^{p-1}(1 - \lambda) + \zeta_2(\epsilon)|s - 1|^p(t^{p-\epsilon-1} + 1).
\]

(6.2)
We first estimate the following term.

\[
|s \cdot t^{\frac{p-\epsilon-1}{p}} - 1|^p = |s \cdot t^{\frac{p-\epsilon-1}{p}} - t^{\frac{p-\epsilon-1}{p}} + t^{\frac{p-\epsilon-1}{p}} - 1|^p \\
= |(s-1)t^{\frac{p-\epsilon-1}{p}} + (t^{\frac{p-\epsilon-1}{p}} - 1)|^p \\
\leq 2^{p-1}|s-1|t^{p\rho_{p-\epsilon}-1} + 2^{p-1}|t^{\frac{p-\epsilon-1}{p}} - 1|^p \\
= A + B,
\]

where

\[
A = 2^{p-1}|s-1|t^{p\rho_{p-\epsilon}-1} \quad \text{and} \quad B = 2^{p-1}|t^{\frac{p-\epsilon-1}{p}} - 1|^p.
\]

By Lemma 6.3, we have

\[
B \leq \frac{2^{p-1}(t-1)^{p-1}(1-\epsilon)}{\zeta(\epsilon)}.
\]

As a consequence, we obtain

\[
|s \cdot t^{\frac{p-\epsilon-1}{p}} - 1|^p \leq 2^{p-1}|s-1|t^{p\rho_{p-\epsilon}-1} + \frac{2^{p-1}(t-1)^{p-1}(1-\epsilon)}{\zeta(\epsilon)}.
\]

We observe that

\[
1 - t^{-\epsilon} = 1 - \lambda + \lambda - t^{-\epsilon} = 1 - \lambda + (s^p - 1)t^{-\epsilon} = 1 - \lambda + |s-1|^p t^{-\epsilon} + (s^p - 1 - |s-1|^p)t^{-\epsilon}.
\]

Therefore, we get

\[
|s \cdot t^{\frac{p-\epsilon-1}{p}} - 1|^p \leq 2^{p-1}\left(1 + \frac{1}{\zeta(\epsilon)}\right)|s-1|t^{p\rho_{p-\epsilon}-1} + \frac{2^{p-1}(t-1)^{p-1}(1-\lambda)}{\zeta(\epsilon)}
\]

\[+ \frac{2^{p-1}}{\zeta(\epsilon)}(t-1)^{p-1}(s^p - 1 - |s-1|^p)t^{-\epsilon}.
\]

Next we estimate the term \( T = \frac{2^{p-1}}{\zeta(\epsilon)}(t-1)^{p-1}(s^p - 1 - |s-1|^p)t^{-\epsilon} \) for different values of \( t \) and \( s \).

Case (a). If \( t > 1 \) and \( s \geq 2 \). Then using the fact that \( s \geq 2 \), it can be easily seen that there exists constant \( C(p) \) large enough such that \( s^p - 1 - (s-1)^p \leq C(p)(s-1)^p \). Therefore we get

\[
T \leq \frac{C(p)}{\zeta(\epsilon)}|s-1|^p t^{p\rho_{p-\epsilon}-1}.
\]

(6.4)

By inserting (6.3) into (6.4), we get

\[
|s \cdot t^{\frac{p-\epsilon-1}{p}} - 1|^p \leq C(p)\left(1 + \frac{1}{\zeta(\epsilon)}\right)|s-1|^p t^{p\rho_{p-\epsilon}-1} + \frac{C(p)}{\zeta(\epsilon)}(t-1)^{p-1}(1-\lambda).
\]

(6.5)

Case (b). If \( t = 1 \) or \( 0 < s \leq 1 \). Then \( T \leq 0 \). Hence we get the estimate in (6.5).

Case (c). If \( t > 1 \), \( s \in (1, 2) \). Let \( r \geq p \) be the nearest integer to \( p \). Again it follows that there exists a positive constant \( C(p) \) large enough such that \( s^p - 1 - |s-1|^p \leq C(p)|s-1| \).

We have further subcases.
Case (i). If
\[ t - 1 < \frac{r2^{r-1}}{\epsilon} t(s - 1). \]
Note that we can choose \( C(p) \) large enough such that \( r2^{r-1} \leq C(p) \). Hence we have
\[ T \leq \frac{C(p)}{\epsilon^{p-1} \zeta(\epsilon)} t^{p-\epsilon-1} |s - 1|^p. \] (6.6)

By inserting (6.6) into (6.3), we get
\[ |st^{\frac{p-\epsilon}{p}} - 1|^p \leq C(p) \left( 1 + \frac{1}{\zeta(\epsilon)} (1 + \frac{1}{\epsilon^{p-1}}) \right) |s - 1|^p t^{p-\epsilon-1} + \frac{C(p)}{\zeta(\epsilon)} (t - 1)^{p-1} (1 - \lambda). \] (6.7)

Case (ii). If
\[ t - 1 \geq \frac{r2^{r-1}}{\epsilon} t(s - 1). \]
Since \( r \) is an integer, we observe that
\[ s^r + s - 2 = (s - 1)(s^{r-1} + s^{r-2} + \cdots + s + 2). \]

By the mean value theorem there exists \( \eta \in (1, t) \) such that \( t' - 1 = \epsilon t^\eta - 1(t - 1) \) and so \( \epsilon = \frac{t' - 1}{t^\eta - 1(t - 1)} \). Now, we have
\[ \frac{s^r + s - 2}{t} = \frac{s - 1}{t\eta - 1(t - 1)} \]
\[ \leq \frac{r2^{r-1}}{\epsilon} \]
\[ \leq \frac{t' - 1}{t}. \]
which gives \( t^\eta - 1(s^r + s - 2) \leq t' - 1 \).

Now, the fact \( \epsilon > 0 \) and \( 1 < \eta < t \) gives \( t^\eta - 1 > \eta' > 1 \). Therefore since \( r \geq p \) and \( s > 1 \), we get \( s^p + s - 2 < s^r + s - 2 < t^\eta - 1(s^r + s - 2) \leq t' - 1 \). Hence we have \( s - 1 \leq t' - s^p = t^\epsilon (1 - \lambda) \). Thus
\[ T \leq \frac{C(p)}{\zeta(\epsilon)} (t - 1)^{p-1} (1 - \lambda). \] (6.8)

Using (6.8) into (6.3) we get
\[ |st^{\frac{p-\epsilon}{p}} - 1|^p \leq C(p) \left( 1 + \frac{1}{\zeta(\epsilon)} \right) |s - 1|^p t^{p-\epsilon-1} + \frac{C(p)}{\zeta(\epsilon)} (t - 1)^{p-1} (1 - \lambda). \] (6.9)

Finally from the estimates (6.3), (6.7) and (6.9), we obtain
\[ |st^{\frac{p-\epsilon}{p}} - 1|^p \leq C(p) \left( 1 + \frac{1}{\zeta(\epsilon)} (1 + \frac{1}{\epsilon^{p-1}}) \right) |s - 1|^p (p^{p-\epsilon-1} + 1) \]
\[ + \frac{C(p)}{\zeta(\epsilon)} (t - 1)^{p-1} (1 - \lambda). \] (6.10)

Multiplying \( \frac{\zeta(\epsilon)}{C(p)} \) on both sides of (6.10), we obtain
\[ \frac{\zeta(\epsilon)}{C(p)} |st^{\frac{p-\epsilon}{p}} - 1|^p \leq \left( \zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}} \right) |s - 1|^p (p^{p-\epsilon-1} + 1) + (t - 1)^{p-1} (1 - \lambda), \]
which corresponds to the inequality (6.2). The lemma thus follows.
Acknowledgements

A.B. is supported in part by SERB Matrix grant MTR/2018/000267 and by Department of Atomic Energy, Government of India, under project no. 12-R & D-TFR-5.01-0520. P.G. and J.K. are supported by the Academy of Finland.

References

[1] Verena Bögelein, Frank Duzaar, Juha Kinnunen, and Christoph Scheven. Higher integrability for doubly nonlinear parabolic systems. *arXiv e-prints*, page arXiv:1810.06039, Oct 2018.

[2] Verena Bögelein, Frank Duzaar, and Naian Liao. On the Hölder regularity of signed solutions to a doubly nonlinear equation. *arXiv e-prints*, page arXiv:2003.04158, March 2020.

[3] E. Bombieri and E. Giusti. Harnack's inequality for elliptic differential equations on minimal surfaces. *Invent. Math.*, 15:24–46, 1972.

[4] Matteo Bonforte, Yannick Sire, and Juan Luis Vázquez. Optimal existence and uniqueness theory for the fractional heat equation. *Nonlinear Anal.*, 153:142–168, 2017.

[5] Lorenzo Brasco and Erik Lindgren. Higher Sobolev regularity for the fractional $p$-Laplace equation in the superquadratic case. *Adv. Math.*, 304:300–354, 2017.

[6] Lorenzo Brasco, Erik Lindgren, and Armin Schikorra. Higher Hölder regularity for the fractional $p$-Laplacian in the superquadratic case. *Adv. Math.*, 338:782–846, 2018.

[7] Lorenzo Brasco, Erik Lindgren, and Martin Strömqvist. Continuity of solutions to a nonlinear fractional diffusion equation. *arXiv e-prints*, page arXiv:1907.00910, Jul 2019.

[8] Lorenzo Brasco and Enea Parini. The second eigenvalue of the fractional $p$-Laplacian. *Adv. Calc. Var.*, 9(4):323–355, 2016.

[9] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.

[10] Jamil Chaker and Moritz Kassmann. Nonlocal operators with singular anisotropic kernels. *Comm. Partial Differential Equations*, 45(1):1–31, 2020.

[11] Matteo Cozzi. Fractional De Giorgi classes and applications to nonlocal regularity theory. In *Contemporary research in elliptic PDEs and related topics*, volume 33 of *Springer INdAM Ser.*, pages 277–299. Springer, Cham, 2019.

[12] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Nonlocal Harnack inequalities. *J. Funct. Anal.*, 267(6):1807–1836, 2014.
[13] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci. Local behavior of fractional $p$-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(5):1279–1299, 2016.

[14] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[15] Emmanuele DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993.

[16] Bartłomiej Dyda and Moritz Kassmann. On weighted Poincaré inequalities. *Ann. Acad. Sci. Fenn. Math.*, 38(2):721–726, 2013.

[17] Matthieu Felsinger and Moritz Kassmann. Local regularity for parabolic nonlocal operators. *Comm. Partial Differential Equations*, 38(9):1539–1573, 2013.

[18] Xavier Fernández-Real and Xavier Ros-Oton. Boundary regularity for the fractional heat equation. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 110(1):49–64, 2016.

[19] Ugo Gianazza and Vincenzo Vespri. A Harnack inequality for solutions of doubly nonlinear parabolic equations. *J. Appl. Funct. Anal.*, 1(3):271–284, 2006.

[20] Ryan Hynd and Erik Lindgren. A doubly nonlinear evolution for the optimal Poincaré inequality. *Calc. Var. Partial Differential Equations*, 55(4):Art. 100, 22, 2016.

[21] Ryan Hynd and Erik Lindgren. Hölder estimates and large time behavior for a nonlocal doubly nonlinear evolution. *Anal. PDE*, 9(6):1447–1482, 2016.

[22] Ryan Hynd and Erik Lindgren. Lipschitz regularity for a homogeneous doubly nonlinear PDE. *SIAM J. Math. Anal.*, 51(4):3606–3624, 2019.

[23] Moritz Kassmann. A new formulation of Harnack’s inequality for nonlocal operators. *C. R. Math. Acad. Sci. Paris*, 349(11-12):637–640, 2011.

[24] Moritz Kassmann and Russell W. Schwab. Regularity results for nonlocal parabolic equations. *Riv. Math. Univ. Parma (N.S.)*, 5(1):183–212, 2014.

[25] Yong-Cheol Kim. Nonlocal Harnack inequalities for nonlocal heat equations. *J. Differential Equations*, 267(11):6691–6757, 2019.

[26] Juha Kinnunen and Tuomo Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. *Math. Ann.*, 337(3):705–728, 2007.

[27] Juha Kinnunen and Peter Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Ann. Mat. Pura Appl. (4)*, 185(3):411–435, 2006.
[28] Tuomo Kuusi, Rojbin Laleoglu, Juhana Siljander, and José Miguel Urbano. Hölder continuity for Trudinger’s equation in measure spaces. *Calc. Var. Partial Differential Equations*, 45(1-2):193–229, 2012.

[29] Tuomo Kuusi, Juhana Siljander, and José Miguel Urbano. Local Hölder continuity for doubly nonlinear parabolic equations. *Indiana Univ. Math. J.*, 61(1):399–430, 2012.

[30] N. S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[31] Naian Liao. Remarks on parabolic De Giorgi classes. *arXiv e-prints*, page arXiv:2004.14324, April 2020.

[32] José M. Mazón, Julio D. Rossi, and Julián Toledo. Fractional $p$-Laplacian evolution equations. *J. Math. Pures Appl. (9)*, 105(6):810–844, 2016.

[33] Martin Strömqvist. Harnack’s inequality for parabolic nonlocal equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(6):1709–1745, 2019.

[34] Martin Strömqvist. Local boundedness of solutions to non-local parabolic equations modeled on the fractional $p$-Laplacian. *J. Differential Equations*, 266(12):7948–7979, 2019.

[35] Juan Luis Vázquez. The Dirichlet problem for the fractional $p$-Laplacian evolution equation. *J. Differential Equations*, 260(7):6038–6056, 2016.

Agnid Banerjee  
Tata Institute of Fundamental Research  
Centre For Applicable Mathematics  
Bangalore-560065, India  
Email: agnidban@gmail.com

Prashanta Garain  
Department of Mathematics  
P.O. Box 11100  
FI-00076 Aalto University, Finland  
Email: pgarain92@gmail.com

Juha Kinnunen  
Department of Mathematics  
P.O. Box 11100  
FI-00076 Aalto University, Finland  
Email: juha.k.kinnunen@aalto.fi