Determination of heat exchange coefficient in heat conductivity problems with asymmetric boundary conditions

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Abstract. This paper depicts an approximate analytical solution of the non-stationary heat conduction problem for an infinite plate under asymmetric boundary conditions of the third kind according to the heat balance integral method. The solution is a simple, engineering-friendly product of exponential and coordinate functions. The coordinate functions are found by the method of undetermined coefficients so that the asymmetric boundary conditions of the third kind meet the main criteria primarily. Satisfactory accuracy of the obtained solution for engineering use is provided by residual orthogonality condition abidance of the differential equation to the coordinate function obtained in this research. This method allows for errors by no more than 1% in the second approximation, in the range $1 < Fo < \infty$ for $Bi = 0$ and $Bi = 0.1$. The heat transfer coefficient was determined a plate surface using the named solution along with data on the temperature change at a fixed plate spot, thus solving the inverse heat conductivity problem.

1. Introduction
Accurate analytical solutions to the boundary coefficient problems of heat conduction for an infinite plate with asymmetric boundary conditions of the third kind have not yet been obtained — only solutions with respect to symmetric boundary conditions have been obtained [1 – 5]. The complexity of using those solutions is that for each Biot number there is a set of eigenvalues obtained from the solution of the Sturm-Liouville boundary value problem. The eigenvalues are ultimately found from the transcendental equation by numerical or graphical methods. It should be mentioned that a boundary-value problem with symmetric boundary conditions of the third kind is only a special case of a wide spectrum of problems with asymmetric boundary conditions. The relevance of obtaining their analytical solutions lies in finding the heat transfer coefficients (that characterize the heat transfer intensity between the plate and the medium) by the experimentally known change in temperature over time at any point of the spatial variable through solving the inverse heat conduction problem. Normally, these coefficients are found from empirically derived non-dimension heat transfer equations which do not take into account numerous factors affecting the value of heat transfer coefficients in each particular case, including their dependence on the temperature field of the body [6 – 10].

2. Mathematical issue
As a specific example, we find a solution of the boundary heat conduction problem for an infinite plate under asymmetric boundary conditions of the third kind
\[
\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}, \quad (t > 0; \quad 0 < x < \delta); \\
T(x,0) = T_0 = \text{const};
\]

(1)

\[
\lambda \frac{\partial T(0,t)}{\partial x} - a_1[T(0,t) - T_{w1}] = 0;
\]

(2)

\[
\lambda \frac{\partial T(\delta,t)}{\partial x} + a_2[T(\delta,t) - T_{w2}] = 0,
\]

(3)

\[
\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t}, \quad (t > 0; \quad 0 < x < \delta);
\]

\[
\lambda \frac{\partial T(0,t)}{\partial x} - a_1[T(0,t) - T_{w1}] = 0;
\]

(4)

\[
\frac{\partial T(\delta,t)}{\partial x} + a_2[T(\delta,t) - T_{w2}] = 0.
\]

\[
T - \text{temperature}; \quad x - \text{coordinate}; \quad t - \text{time}; \quad \lambda, \quad a - \text{coefficients of thermal conductivity and thermal diffusivity}; \quad a_1, \quad a_2 - \text{convective heat transfer coefficients}; \quad T_{w1}, \quad T_{w2} - \text{wall temperatures}; \quad T_0 - \text{initial temperature}; \quad \delta - \text{plate thickness}.
\]

Hereby we describe the following dimensionless variables and parameters:

\[
\Theta = \frac{T - T_{w1}}{T_0 - T_{w1}}; \quad \xi = \frac{x}{\delta}; \quad F_0 = \frac{at}{\delta^2}; \quad B_{i1} = \frac{a_1 \delta}{\lambda}; \quad B_{i2} = \frac{a_2 \delta}{\lambda},
\]

(5)

\[
\Theta, \quad \xi, \quad F_0 \quad \text{are marked as dimensionless temperature, coordinate, time (Fourier number)}; \quad B_{i1}, \quad B_{i2} - \text{Biot numbers}.
\]

According to the notations proposed in (5), problem (1) – (4) will be

\[
\frac{\partial \Theta(\xi,F_0)}{\partial F_0} = \frac{\partial^2 \Theta(\xi,F_0)}{\partial \xi^2}, \quad (F_0 > 0; \quad 0 < \xi < 1);
\]

(6)

\[
\Theta(\xi,0) = 1;
\]

(7)

\[
\frac{\partial \Theta(0,F_0)}{\partial \xi} - B_{i1} \Theta(0,F_0) = 0;
\]

(8)

\[
\frac{\partial \Theta(1,F_0)}{\partial \xi} + B_{i1} \left[\Theta(1,F_0) + D\right] = 0,
\]

(9)

where \( D = (T_{w1} - T_{w2}) / (T_0 - T_{w1}) \).

3. Analytical solution method

The solution of the problem (6) – (9) can be written as

\[
\Theta(\xi,F_0) = \psi(\xi) + q(F_0) \varphi(\xi),
\]

(10)

where \( q(F_0); \psi(\xi) \) is marked as the function that has been determined so that the solution (10) to satisfy the inhomogeneous boundary condition (9); \( \varphi(\xi) \) – coordinate function that is determined so that solution (10) satisfies the homogeneous boundary conditions (8), (9), i.e. at \( D = 0 \).

The function \( \psi(\xi) \) is found in the form \( \psi(\xi) = F_1 \xi^2 \), where \( F_1 \) – constant that is determined from the boundary condition (9). It’s formula is \( F_1 = -B_{i1} D / (2 + B_{i2}) \), therefore, we get

\[
\psi(\xi) = -B_{i2} D \xi^2 / (2 + B_{i2}).
\]

(11)

The coordinate function \( \varphi(\xi) \) is found in the form

\[
\varphi(\xi) = F_2 + F_3 \xi - \xi^2,
\]

(12)

where \( F_2, \quad F_3 \) – constants determined from the homogeneous boundary conditions (8), (9) (at \( D = 0 \)).

Formulas for them will be: \( F_2 = (2 + B_{i2}) / D_1 \); \( F_3 = B_{i1} F_{i2} \), where \( D_1 = B_{i1} + B_{i2} + B_{i1} B_{i2} \).

Taking into account the determined values \( F_2, \quad F_3 \), relation (12) will be
\[ \varphi (\xi) = \frac{(2 + B_i \xi)(1 + B_i \xi)}{D_1} - \xi^2. \]  

(13)

Substituting (11), (13) into (10), we can find

\[ \Theta (\xi, F_0) = -B_i \xi D_2 \xi^2 / (2 + B_i \xi) + q(F_0) \left( \frac{(2 + B_i \xi)(1 + B_i \xi)}{D_1} - \xi^2 \right). \]  

(14)

The relation (14) satisfies the boundary conditions (8), (9). In the first approximation, it is required to satisfy (6), averaged equations, i.e., the heat balance integral [11 – 15]:

\[ \int_0^1 \partial \Theta (\xi, F_0) d\xi = \int_0^1 \partial^2 \Theta (\xi, F_0) d\xi. \]  

(15)

Substituting (14) into (15), after determining the integrals with accordance to the unknown function \( q(F_0) \), we obtain the ordinary differential equation

\[ \frac{1}{6} \frac{D_1}{D} \frac{d q(F_0)}{d F_0} + 2 \left( \frac{D_3}{2 + B_i \xi} \right) = 0, \]  

(16)

where \( D_2 = 4 \left( 3 + B_i + B_i \right) + B_i B_i \).

Integrating equation (16), we find

\[ q(F_0) = -\frac{B_i D}{2 + B_i \xi} + C \exp \left\{ -\frac{12 D \xi F_0}{D_2} \right\}, \]  

(17)

where \( C \) – constant of integration.

Substituting (17) into (14), we get

\[ \Theta (\xi, F_0) = \left( C \exp (-D_2 F_0) - D_3 \right) \left( D_3 - \xi^2 \right) - D_2 \xi^2. \]  

(18)

where \( D_1 = \frac{B_i D}{2 + B_i \xi} \); \( D_3 = \frac{12 D}{D_2} \); \( D_3 = \frac{(2 + B_i \xi)(1 + B_i \xi)}{D_1} \).

In order to determine the integration constant \( C \), the integral of the weighted residual of the initial condition has to be calculated:

\[ \int_0^1 (\Theta (\xi, 0) - 1) d\xi = 0. \]  

(19)

Substituting (18) into (19), we can find

\[ \int_0^1 [(C - D_3)(D_3 - \xi^2) - D_2 \xi^2 - 1] d\xi = 0. \]  

(20)

By determining the integrals in (20), in accordance with the integration constant, we get an algebraic linear equation, resulting from which we can find

\[ C = 3 \left[ \frac{B_i D (2 + B_i \xi) + 2 D_1}{D_2} \right] / D_2. \]  

(22)

Having found the integration constant, the solution of the problem (6) – (9) in a first approximation will be

\[ \Theta (\xi, F_0) = \left[ 3 \frac{B_i D (2 + B_i \xi) + 2 D_1}{D_2} \right] \exp (-D_2 F_0) - D_3 \left( D_3 - \xi^2 \right) - D_2 \xi^2. \]  

(21)

If \( B_i = 0; \ B_i = 10000; \ D = 0 \) then the problem (6) – (9) is reduced to a problem with symmetric boundary conditions of the first kind

\[ \frac{\partial \Theta (0, F_0)}{\partial \xi} = 0; \ \Theta (1, F_0) = 0. \]  

(22)

The solution here will be
\[ \Theta (\xi, F_o) = 1.5 \exp (-3 F_o)(1 - \xi^2). \] (23)

The calculation results by formula (23) compared to the accurate solution [4] are shown in Figure 1. From their analysis it follows that in the range \( 0.15 \leq F_o < \infty \) the highest expected error is 6%.

Figure 2 shows the calculation results by formula (21) at \( B_i = 0; B_i = 0.1; \xi = 1; T_0 = 100^\circ C; T_{s_1} = 0^\circ C; T_{s_2} = 0^\circ C \) compared to the accurate solution [4]. From their analysis it follows that in the range \( 0.15 \leq F_o < \infty \) the error is to be within 8%.

In order to increase the solution accuracy, we require relation (14) to satisfy the heat balance integral:

\[ \int_0^1 \frac{d\Theta (\xi, F_o)}{dF_o} \varphi (\xi) d\xi = \int_0^1 \frac{\partial \Theta (\xi, F_o)}{\partial \xi^2} \varphi (\xi) d\xi. \] (24)

In contrast to relation (15), the relation (24) requires the residual orthogonality of equation (6) to the coordinate function \( \varphi (\xi) \) (this has to be considered way of obtaining a solution as the second approximation).

Substituting (14) into (24), with respect to an unknown function \( q (F_o) \), we obtain an ordinary differential equation

\[ D_s \frac{dq (F_o)}{dF_o} + D_s q (F_o) + D_s = 0, \] (25)

where \( D_s = (120 + 80 (B_i + B_i) + B_i B_i, (42 + 7 (B_i + B_i) + B_i B_i) + 16 (B_i^2 + 16 B_i^2)) / (30 D_i^3) \);

\[ D_s = \left[ 24 + 4 B_i, (5 + B_i) + 8 B_i, B_i, B_i, (6 + B_i) \right] / \left[ 3 D_i, (2 + B_i) \right]; \]

\[ D_s = D \left[ 4 B_i, (3 + B_i) + B_i, B_i, (4 + B_i) \right] / \left[ 3 D_i, (2 + B_i) \right]. \]

Integrating equation (25), we can find

\[ q (F_o) = -D_s, C_1, \exp \left( -\frac{D_s}{D_{10}} F_o \right), \] (26)

where \( C_1 \) — constant of integration; \( D_{10} = 10 D_i (B_i B_i + 4 (B_i + B_i) + 12); D_{10} = B_i B_i, (B_i B_i + + 7 (B_i + B_i) + 42) + 16 (B_i^2 + B_i^2) + 80 (B_i + B_i) + 120. \)

Substituting (26) into (14), we obtain

\[ \Theta (\xi, F_o) = (C_1, \exp \left( -\frac{D_s}{D_{10}} F_o \right) - D_s) \left( D_s - \xi^2 \right) - D_s \xi^2. \] (27)

To determine the constant of integration \( C_1 \), we find the residual of the initial condition (7) and require it to be performed orthogonally to the coordinate function \( \varphi (\xi) \)

\[ \int_0^1 (\Theta (\xi, 0) - 1) \varphi (\xi) d\xi = 0. \] (28)

Substituting (27) into (28), after determining the integrals with respect to the constant of integration, we have an algebraic linear equation

\[ (D_s (C_1 - D_s) - 1) \left( \frac{1}{2} F_s + F_s \right) + \frac{1}{3} (D_s (D_s - C_s) + 1 - C_s F_s) + C_s \left( \frac{1}{5} - \frac{1}{4} F_s \right) = 0 \]

from the solution of which we find

\[ C_1 = \frac{10 D_s}{D_{10}} \] (29)

where \( D_{10} = (D_s D_s + 1)(3 F_s + 6 F_s - 2); D_{12} = 30 D_s (F_s + 2 F_s) - 20 (D_s + F_s) - 15 F_s + 12. \)

After determination of the constant of integration \( C_1 \), the solution of problem (6) – (9) in the second approximation is as follows
\[
\Theta (\xi, F_0) = -D_1 \xi^2 + \left(10 \frac{D_{12}}{D_{10}} \exp \left(-\frac{D_{10}}{D_{10} F_0}\right) - D_1\right) (D_1 - \xi^2).
\] (30)

If \( B_{i_1} = 0; \ B_{i_2} = 10000; \ D = 0, \) then formula (30) will be
\[
\Theta (\xi, F_0) = 1.25 \exp (-2.5 F_0) \left(1 - \xi^2\right).
\] (31)

4. Results and discussions

The calculation results by formula (31) compared to the accurate solution [4] are shown in Figure 3. From their analysis it follows that in the second approximation the solution of problem (6) – (9) has more accurate results. In particular, in the range \( 0.15 \leq F_0 < \infty \) the discrepancy relative to the accurate solution is within 3%.

The calculation results by formula (30) for \( B_{i_1} = 0; \ B_{i_2} = 0.1; \ \xi = 1; \ T_0 = 100^\circ C; \ T_{i_1} = 0^\circ C; \ T_{i_2} = 0^\circ C \) in comparison with the accurate solution [4] are shown in Figure 2. From their analysis it follows that in the range \( 1.0 \leq F_0 < \infty \) the error regarding to the accurate analytical solution [4] does not exceed 1%.

The solution obtained makes it possible to understand the physics of the processes, since it explicitly contains all the main parameters of the problem, which enables us to evaluate the influence of each of them on the value of the initial function. This property of the obtained solution is very useful for their employment in technological process control tasks, as well as in solving inverse heat conduction problems aimed at identifying any of problem parameters.

5. Inverse problem solution

Formula (30) by an experimentally derived temperature value that varies over time at any point of the plate through solving the inverse heat conduction problem allows for identifying one of the unknown heat transfer coefficients (\( \alpha_1 \) or \( \alpha_2 \)). It is assumed that the temperature change at a point \( \xi = 1.0 \) in the range of the Fourier number \( F_{o_1} \leq F_0 \leq F_{o_4} \).

![Figure 1. Temperature distribution (first approximation).](image-url)
Figure 2. Temperature distribution (first and second approximations). $B_{1_1} = 0; \quad B_{1_2} = 0.1; \quad D = 0; \quad \xi = 1$. • – accurate solution [4]; —— by formula (30); △ – by formula (21).

Figure 3. Temperature distribution (second approximation). $B_{1_1} = 0; \quad B_{1_2} = 10000; \quad D = 0; \quad • – accurate solution [4]; —— by formula (31).

The temperature curve is shown in Figure 4. Let us approximate this curve by the following function

$$\Theta (1, \alpha_1) = \sum_{k=0}^{3} b_k \alpha_1^k,$$

where $b_k$, $(k = 0, 3)$ are unknown coefficients. To determine them, experimental data in figure 4 is used.
Points 1, 2, 3, 4 of the curve shown in Figure 4 correspond to the following time values: 

- $F_0 = 0.05; \xi = 0.05$
- $F_0 = 0.1; \xi = 0.1$
- $F_0 = 0.15; \xi = 0.15$
- $F_0 = 0.2; \xi = 0.2$

According to the experiment, for $\xi = 1.0$ the following temperatures correspond to these time values:

1. $1.0; 0.05 \Theta = 0.7941$
2. $1.0; 0.1 \Theta = 0.6283$
3. $1.0; 0.15 \Theta = 0.5312$
4. $1.0; 0.2 \Theta = 0.4617$

When writing relation (32) for each of the points on the curve (Figure 4) and taking into account the known values of times and temperatures at these points with respect to the unknowns $b_k$, we have a system of algebraic linear equations

\[
\begin{align*}
&b_0 + 0.05b_1 + 0.0025b_2 + 0.000125b_3 = 0.7941; \\
&b_0 + 0.1b_1 + 0.01b_2 + 0.001b_3 = 0.6283; \\
&b_0 + 0.15b_1 + 0.0225b_2 + 0.003375b_3 = 0.5312; \\
&b_0 + 0.2b_1 + 0.04b_2 + 0.008b_3 = 0.4617.
\end{align*}
\]

(33)

Solving the system of equations (33), we can find $b_0 = 1.0697; b_1 = -6.884; b_2 = 30.18; b_3 = -54.8$.

Relation (32), taking into account the found values of the coefficients $b_k$, will be

$$\Theta(1.0; Fo) = 1.0697 - 6.884Fo + 30.18Fo^2 - 54.8Fo^3.$$  (34)

The calculation results by formula (34) presented in Figure 4 allow one to conclude that they practically coincide with experimental data. Note that a cubic parabola of the form (32) does not allow temperature jumps between the points at which it was approximated.

Suppose that from the solution of the inverse problem it is necessary to identify (restore) the dimensionless heat transfer coefficient $Bi$ for the following initial data

$$T_0 = 100^\circ C; \quad T_{w1} = 0^\circ C; \quad T_{w2} = 0^\circ C; \quad Bi = 0.$$

Substituting (34) into the left-hand side of solution (30) and determining the integral of the obtained relation within $F_0 \leq Fo \leq F_{0,4}$, we obtain

$$\int_{F_{0,1}}^{F_{0,3}} \left(1.0697 - 6.884Fo + 30.18Fo^2 - 54.8Fo^3\right) dFo =$$
By defining the integrals in (35) at $\xi = 1$, with respect to the dimensionless heat transfer coefficient, the following transcendental equation is resulting

$$
\exp \left[ -\frac{0.25 B_i \lambda (3 + B_i \lambda)}{15 + 10 B_i \lambda + 2 B_i \lambda^2} \right] - \exp \left[ -\frac{B_i \lambda (3 + B_i \lambda)}{15 + 10 B_i \lambda + 2 B_i \lambda^2} \right] = 0.088768 B_i \lambda.
$$

From the solution of equation (36) we can find: $B_i = 1.4181$.

With plate thickness $\delta = 0.03 \text{ m}$ and $\lambda = 37 W / (m K)$, the heat transfer coefficient will be

$$
\alpha_{\lambda} = \frac{B_i \lambda}{\delta} = 1749 \frac{W}{m^2 K}.
$$

It is critical of importance to mention that on the basis of solution (30) and using the experimental data on the temperature change over time at a fixed point on the plate, one can find not only $B_i$, but also any other constant included in this solution ($\lambda$, $B_i$, $\delta$, $T_{w1}$, $T_{w2}$).

6. Conclusions

An approximate analytical solution for the non-stationary heat conduction problem under inhomogeneous and asymmetric boundary conditions of the third kind has been obtained using the orthogonal methods of weighted residuals. This solution was obtained due to the developed in the research method for constructing systems of coordinate functions that accurately satisfy the boundary conditions of the boundary value problem.

The heat transfer coefficient at the boundary between the wall and medium using experimental data on the temperature change over time at one of the boundary points of the region is determined by solving the inverse heat conduction problem. From the inverse problem solution, any other parameters of the boundary-value problem can also be found, such as thermal conductivity coefficients, thermal diffusivity coefficients, medium temperatures and plate thickness.

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8. References

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