Sharp gradient estimates for quasilinear elliptic equations with \( p(x) \) growth on nonsmooth domains.

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Abstract

In this paper, we study quasilinear elliptic equations with the nonlinearity modelled after the \( p(x) \)-Laplacian on nonsmooth domains and obtain sharp Calderón-Zygmund type estimates in the variable exponent setting. In a recent work of [10], the estimates obtained were strictly above the natural exponent and hence there was a gap between the natural energy estimates and estimates above \( p(x) \), see (1.3) and (1.4). Here, we bridge this gap to obtain the end point case of the estimates obtained in [10], see (1.5). In order to do this, we have to obtain significantly improved a priori estimates below \( p(x) \), which is the main contribution of this paper. We also improve upon the previous results by obtaining the estimates for a larger class of domains than what was considered in the literature.

Keywords: variable exponent, Calderon-Zygmund theory, end-point estimate.

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1. Introduction

Calderón-Zygmund theory was first developed for the Poisson equation in [14], which related the integrability of the gradient of the solution for the Poisson equation with that of the associated data. This represented the starting point of obtaining a priori estimates in Sobolev spaces for elliptic and parabolic equations.

\textit{All the estimates mentioned in this introduction are quantitative in nature, but to avoid being}
too technical, we only recall the qualitative nature of the bounds. This is sufficient to highlight the
nature of the results that we will prove in this paper.

For the problem
\[
\begin{aligned}
\text{div}(|\nabla u|^{p-2}\nabla u) &= \text{div}(|f|^{p-2}f) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
T. Iwaniec in [30] established the Calderón-Zygmund type estimates, in particular he proved the
following a priori relation
\[|f|^q \in L^q_{loc} \implies |\nabla u|^q \in L^q_{loc} \quad \text{for all } q > p.\]

After this pioneering work, there have been numerous publications which extended these estimates
to other quasilinear elliptic and parabolic equations with the constant \(p\)-growth, see [2, 8, 13, 18,
22, 35, 38] and references therein. In this paper, we are interested in obtaining Calderón-Zygmund
type bounds for the problem
\[
\begin{aligned}
\text{div} A(x, \nabla u) &= \text{div}(|f|^{p(x)-2}f) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.1)

Here \(\Omega\) is a bounded domain of \(\mathbb{R}^n\), \(n \geq 2\), and the quasilinear operator \(A(x, \nabla u)\) is modelled after
well known \(p(x)\)-Laplacian operator having the form \(|\nabla u|^{p(x)-2}\nabla u\). See Section 2 for the precise
assumptions on \(A(\cdot, \cdot), p(\cdot)\) and \(\Omega\). For more on the importance of variable exponent problems, see
[39, 40, 43, 15, 6, 29] and the references therein.

The first estimate for the \(p(x)\)-Laplacian was obtained by Acerbi and Mingione in [1], wherein
they obtained a local Calderón-Zygmund type estimate by proving
\[|f|^{p(\cdot)} \in L^q_{loc} \implies |\nabla u|^{p(\cdot)} \in L^q_{loc} \quad \text{for all } q \in (1, \infty)\]
under the assumption that the variable exponent \(p(\cdot)\) satisfies \(\lim_{r \to 0} r \rho(r) \log \left(\frac{1}{r}\right) = 0\) (see Section
2.1 for the relation between \(\rho\) and \(p(\cdot)\)). This work was subsequently extended in [7] to parabolic systems.

This estimate was further improved upon in [11] for more general equations of the form (1.1), to
obtain global Calderón-Zygmund type estimates, provided the nonlinearity \(A(x, \zeta)\) satisfied a small
BMO (bounded mean oscillation) condition with respect to \(x\) and \(\Omega\) was sufficiently flat in the sense
of Reifenberg (see [12, 26, 31, 37, 41] and the references therein for more about the BMO condition
and Reifenberg flat domains).
On the other hand, there are very few results about $L^{q(\cdot)}$ estimates, i.e., a priori relations of the form

$$|f|^{p(\cdot)} \in L^{q(\cdot)}(\Omega) \implies |\nabla u|^{p(\cdot)} \in L^{q(\cdot)}(\Omega)$$

for a suitably chosen $q(\cdot)$.

The first work in this direction was due to [21] in which they considered the linear Poisson equation to obtain $L^{q(\cdot)}$-estimates. The main approach is based on the boundedness of the associated singular integral operators in the variable exponent Lebesgue spaces $L^{q(\cdot)}$ with the assumption that $q(\cdot)$ is log-Hölder continuous (this is essentially a necessary condition and cannot be avoided). More general extensions are now well known as an application of the theory developed in [16].

Recently in [16], the authors proved a very interesting generalization of the Theory of Extrapolation by Rubia de Francia and Garcia Cuerva in the setting of variable exponent Lebesgue spaces. More specifically, the problem of obtaining $L^{q(\cdot)}$ estimates for constant exponent equations becomes equivalent to obtaining weighted estimates for a fixed exponent $p_0$ with the weight in a suitable Muckenhoupt class $A_{p_0}$ (see [16, Theorem 2.7] and details therein). In the past several decades, there have been a plethora of weighted bounds with weights in Muckenhoupt class obtained for a wide range of constant exponent operators including those considered in [21].

Subsequently, the interesting equations to study now are those where the nonlinear structure depends on the variable exponent (an example being (1.1)). It is easy to obtain energy estimates for solutions $u \in W^{1,p(\cdot)}_0(\Omega)$ on any bounded domain, i.e., the following relation holds:

$$|f|^{p(\cdot)} \in L^1(\Omega) \implies |\nabla u|^{p(\cdot)} \in L^1(\Omega).$$

(1.3)

Recently in [10], the authors obtained the following Calderón-Zygmund type relation provided the nonlinearity satisfies a small BMO condition and the domain is suitably flat in the sense of Reifenberg:

$$|f|^{p(\cdot)} \in L^{p(\cdot)}(\Omega) \implies |\nabla u|^{p(\cdot)} \in L^{q(\cdot)}(\Omega),$$

(1.4)

for any $1 < q^- \leq q(\cdot) \leq q^+ < \infty$ with $q(\cdot)$ being log-Hölder continuous. In particular, they cannot take $q^- = 1$, which would recover (1.3).

The purpose of this paper is twofold: Firstly, we bridge the gap between the estimates (1.3) and (1.4) to obtain the relation

$$|f|^{p(\cdot)} \in L^{p(\cdot)}(\Omega) \implies |\nabla u|^{p(\cdot)} \in L^{q(\cdot)}(\Omega),$$

(1.5)
provided $1 \leq q^- \leq q^+ < \infty$, i.e., we allow $q^- = 1$ and $q(\cdot)$ is log-Hölder continuous (we assume the same structure conditions on the nonlinearity $A$ as in [10]). This represents an end point case of the estimate (1.4).

Secondly, in [10], they considered domains that were sufficiently flat in the sense of Reifenberg. Although this class includes domains that have fractal boundary, it excludes convex domains with sufficiently sharp corners. In this paper, we obtain the end point estimate for quasiconvex domains (see Subsection 2.3 for the details). This class of domains includes both Reifenberg flat domains as considered in [10] and convex domains.

The plan of the paper is as follows: In Section 2, we collect all the assumptions that will be needed on the structure of the nonlinearity $A(\cdot, \cdot)$, the regularity of the boundary of the domain $\Omega$ and the assumptions on the variable exponent $p(\cdot)$. In Section 3, we shall state the main results of this paper. In Section 4, we shall collect all the preliminary results that will be needed in later parts of the paper. Section 5 is devoted to proving the main a priori estimates that will be needed. In Section 6, we shall implement a covering type argument and finally in Section 7, we prove the main theorems.

2. Assumptions on the structures of $A$, $p(\cdot)$ and $\Omega$.

We shall first describe the assumptions imposed on the variable exponent:

2.1. Structure of $p(\cdot)$

**Definition 2.1.** We say that a bounded measurable function $p(\cdot) : \Omega \rightarrow \mathbb{R}$ belongs to the log-Hölder class $\text{Log}^\pm$, if the following conditions are satisfied:

- There exist constants $p^-$ and $p^+$ such that $1 < p^- \leq p(x) \leq p^+ < \infty$ for every $x \in \Omega$.
- $|p(x) - p(y)| \leq \frac{L}{\log |x - y|}$ holds for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and for some $L > 0$.

**Remark 2.2.** We remark that $p(\cdot)$ is log-Hölder continuous in $\Omega$ if and only if there is a nondecreasing continuous function $\rho : [0, \infty) \rightarrow [0, \infty)$ such that

- $\lim_{r \rightarrow 0} \rho(r) = 0$ and $|p(x) - p(y)| \leq \rho(|x - y|)$ for every $x, y \in \Omega$.
- $\rho(r) \log \left(\frac{1}{r}\right) \leq L$ holds for all $0 < r \leq \frac{1}{2}$.

The function $\rho(r)$ is called the modulus of continuity of the variable exponent $p(\cdot)$. 

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2.2. Structure of the nonlinearity $\mathcal{A}(\cdot, \cdot)$

We first assume that $\mathcal{A}(\cdot, \cdot)$ is a Caratheodory function in the sense:

\[ x \mapsto \mathcal{A}(x, \zeta) \text{ is measurable for every } \zeta \in \mathbb{R}^n, \]

\[ \zeta \mapsto \mathcal{A}(x, \zeta) \text{ is continuous for almost every } x \in \Omega. \]

Let $\mu \in [0, 1]$ be given, then there exists two positive constants $\Lambda_0, \Lambda_1$ such that the following holds for almost every $x \in \Omega$ and every $\zeta, \eta \in \mathbb{R}^n$:

\[
(\mu^2 + |\zeta|^2)^{\frac{1}{2}} |D\zeta \mathcal{A}(x, \zeta)| + |\mathcal{A}(x, \zeta)| \leq \Lambda_1 (\mu^2 + |\zeta|^2)^{\frac{p(x)-1}{2}}, \tag{2.1}
\]

\[
(\mu^2 + |\zeta|^2)^{\frac{p(x)-2}{2}} |\zeta|^2 \Lambda_0 \leq \langle D\zeta \mathcal{A}(x, \zeta) \eta, \eta \rangle. \tag{2.2}
\]

We point out that from (2.2), one can derive the following monotonicity bound:

\[
\langle \mathcal{A}(x, \zeta) - \mathcal{A}(x, \eta), \zeta - \eta \rangle \geq \tilde{\Lambda}_0 (\mu^2 + |\zeta|^2 + |\eta|^2)^{\frac{p(x)-2}{2}} |\zeta - \eta|^2, \tag{2.3}
\]

where $\tilde{\Lambda}_0 = \tilde{\Lambda}_0(\Lambda_0, n, p^+, p^-)$. By inserting $\eta = 0$ into (2.3), we also have the following coercivity bound:

\[
\tilde{\Lambda}_2 |\zeta|^{p(x)} \leq \langle \mathcal{A}(x, \zeta), \zeta \rangle + \tilde{\Lambda}_1, \tag{2.4}
\]

where $\tilde{\Lambda}_1 = \tilde{\Lambda}_1(\Lambda_1, \Lambda_0, p^+, p^-, n)$ and $\tilde{\Lambda}_2 = \tilde{\Lambda}_2(\Lambda_1, \Lambda_0, p^+, p^-, n)$.

2.3. Structure of $\Omega$

Let $\gamma > 0$, $\sigma \in (0, 1/4)$ and $S > 0$ be given, then we describe the properties of a $(\gamma, \sigma, S)$-quasiconvex domain $\Omega$ in this subsection:

**Definition 2.3.** $\Omega$ is said to be $(\gamma, \sigma, S)$-quasiconvex if for all $x \in \partial \Omega$ and all $0 < r \leq S$, the following properties hold:

- There exists a ball $B_{\sigma r}(z) \subset \Omega \cap B_r(x)$ where $z$ is relative with respect to $x$ and the given $\sigma \in (0, 1/4)$ is independent of $x$,

- There exist a hyperplane $L(x, r)$ containing $x$, a unit normal vector $\tilde{n}(x, r)$ to $L(x, r)$ and a half space

\[
H(x, r) = \{ y + t\tilde{n}(x, r) : y \in L(x, r), t \geq -\gamma r \}
\]

such that

\[
\Omega \cap B_r(x) \subset H(x, r) \cap B_r(x).
\]
We now state the following two important properties regarding quasiconvex domains. The first lemma says that locally, we can approximate the domain $\Omega$ by convex domains from outside at sufficiently small scales.

**Lemma 2.4** ([32, 33, 34]). Let $\Omega$ be a $(\gamma, \sigma, S)$-quasiconvex domain, then for any $x \in \partial \Omega$ and $r \in (0, S/2]$, there exists a convex domain $F(x, r)$ such that the following holds

\[ B_r(x) \cap \Omega \subset F(x, r) \cap B_r(x) \quad \text{and} \quad D[\partial_w F(x, r), \partial_w \Omega_r(x)] \leq c(\sigma) \gamma r. \]

Here, we have set $D[E, F] := \max \left\{ \sup_{z \in E} d(z, F), \sup_{z \in F} d(z, E) \right\}$ which denotes the Hausdorff distance between two sets $E, F \subset \mathbb{R}^n$, and $\partial_w$ denotes the wiggle part of the boundary, i.e.,

\[ \partial_w \Omega_r(x) := \partial \Omega \cap B_r(x) \quad \text{and} \quad \partial_w F(x, r) := \partial F(x, r) \cap B_r(x). \]

**Remark 2.5.** Following the proof in [32], we see that $F(x, r)$ is constructed to be the set

\[ F(x, r) := \bigcap_{y \in \partial_w \Omega_r(x)} H(y, 2r), \]

where $H(y, 2r)$ is defined in Definition 2.3.

The second property of quasiconvex domains is that locally, it can be approximated by convex domains from the interior at suitably small scales.

**Lemma 2.6** ([32, 33, 34]). Let $\Omega$ be a $(\gamma, \sigma, S)$-quasiconvex domain. For the convex region $F(x, r)$ constructed in Lemma 2.4, there exists another convex region $F^*(x, r)$ such that the following holds:

\[ F^*_r(x) := F^*(x, r) \cap B_r(x) \subset \Omega_r(x) \quad \text{and} \quad D[\partial_w F^*_r(x), \partial_w \Omega_r(x)] \leq \frac{32 \gamma r}{\sigma^3}. \]

**Remark 2.7.** Following the proof in [32], we see that $F^*(x, r)$ is constructed as follows: Using the spherical coordinates centered at $x_0$, it is easy to see that there exists a point $x_0 \in \Omega_r(x)$ such that $B_{\sigma r}(x_0) \subset \Omega_r(x)$. Define

\[ F^*(x, r) := \left\{ (\rho, \theta_1, \ldots, \theta_{n-1}) : \rho \leq \rho' \left( 1 - \frac{16 \gamma}{\sigma^3} \right), (\rho', \theta_1, \ldots, \theta_{n-1}) \in \partial F(x, r) \right\}. \]

A useful property of a $(\gamma, \sigma, S)$-quasiconvex domain is that it satisfies the following measure density estimates:
Lemma 2.8. Since $\Omega$ is $(\gamma, \sigma, S)$-quasiconvex, we have the following estimates: For any $x \in \partial \Omega$ and any $r \in (0, S)$, then there holds

$$|\Omega \cap B_r(x)| = |\Omega_r(x)| \geq |B_{\sigma r}|. \quad (2.5)$$

Analogously, the following bound also holds:

$$|B_r(x) \cap \Omega^c| \geq c(n)[(1 - \gamma)r]^n. \quad (2.6)$$

If we further assume that $\gamma \leq \frac{1}{2}$, then the bound in (2.6) can be made independent of $\gamma$.

2.4. Smallness assumption

In order to prove the main results, we need to assume a smallness condition satisfied by $(p(\cdot), A, \Omega)$.

Definition 2.9. Let $\gamma > 0$, $\sigma > 0$ and $S_0 > 0$ be given, we then say $(p(\cdot), A, \Omega)$ is $(\gamma, \sigma, S_0)$-vanishing if the following three assumptions hold:

(i) Assumption on $p(\cdot)$: The variable exponent $p(\cdot)$ with modulus of continuity $\rho(r)$ as defined in Definition 2.1, is further assumed to satisfy the smallness condition:

$$\sup_{0 < r \leq S_0} \rho(r) \log \left(\frac{1}{r}\right) \leq \gamma. \quad (2.7)$$

(ii) Assumption on $A$: For a bounded open set $U \subset \mathbb{R}^n$, we write

$$\Theta(A, U)(x) := \sup_{\zeta \in \mathbb{R}^n} \left| \frac{A(x, \zeta)}{(\mu^2 + |\zeta|^2)^{\frac{n+1}{2}}} - \left(\frac{A(\cdot, \zeta)}{(\mu^2 + |\zeta|^2)^{\frac{n+1}{2}}} \right)_{U} \right|. \quad (2.8)$$

where we have used the notation $(f)_U := \int_U f(x) \, dx$. Note that if $\mu = 0$, then $\zeta \in \mathbb{R}^n \setminus \{0\}$. Then $A$ satisfies the small BMO condition, i.e., there holds:

$$\sup_{0 < r \leq S_0} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} \Theta(A, B_r(y))(x) \, dx \leq \gamma. \quad (2.9)$$

(iii) Assumption on $\partial \Omega$: We ask that $\Omega$ is a $(\gamma, \sigma, S_0)$-quasiconvex domain in the sense of Definition 2.3.
2.5. Notations

We shall use the following notation throughout the paper:

- In what follows, the function \( \rho(r) \) denotes the modulus of continuity of \( p(x) \) and we denote \( \omega(r) \) for the modulus of continuity of \( q(x) \).

- For the variable exponent \( p(\cdot) \), we shall denote by \( p^{\pm}_{\log} \) to include the constants \( p^+ \), \( p^- \) and those that are part of the log-Hölder continuity structure of \( p(\cdot) \). Analogously, for variable exponents \( q(\cdot) \), \( r(\cdot) \) and \( s(\cdot) \), we shall use \( q^{\pm}_{\log} \), \( r^{\pm}_{\log} \) and \( s^{\pm}_{\log} \) to denote corresponding constants.

- We shall sometimes also write \( q^{\log} \) to denote constants that depend only on the log-Hölder continuity of \( q(\cdot) \) and denote \( q^{\pm} \) to denote the constants \( q^+ \) and \( q^- \).

- Constants with subscripts like radii \( R_1, R_2, \ldots \) and bounding values \( M_1, M_2, \ldots \) will be fixed in subsequent sections once they are chosen.

- We shall use \( \leq \), \( \geq \) and \( \approx \) to suppress writing the constants that could possibly change from line to line as long as they depend on \( n, p^{\pm}_{\log}, q^{\pm}_{\log}, \Lambda_0, \Lambda_1, \sigma, S_0 \) and related quantities.

- We shall sometimes use \( \sim \) to denote variables that occur only within the proof of the concerned result, for example \( \tilde{r}, \tilde{m}, \cdots \).

- Given a variable exponent \( p(\cdot) \), we shall use the following notation:

\[
p_E^- := \inf_{x \in E} p(x) \quad \text{and} \quad p_E^+ := \sup_{x \in E} p(x).
\]

We will drop the set \( E \) and denote \( p^+ := \sup_{x \in \mathbb{R}^n} p(x) \) and \( p^- := \inf_{x \in \mathbb{R}^n} p(x) \).

3. Main theorems

We now state the main results of the paper. The first theorem concerns local estimates around small balls:

**Theorem 3.1.** Assume that \( u \in W^{1,p(\cdot)}_0(\Omega) \) is the weak solution of the problem (1.1) under the structure conditions (2.1) and (2.2). Let \( 0 < \sigma < 1/4 \), \( 0 < S_0 < 1 \), and \( q(\cdot) \) be log-Hölder continuous satisfying \( 1 < q^- \leq q(\cdot) \leq q^+ < \infty \). There exist constants \( \gamma_0 \in (0,1/4) \) and \( \delta_0 \in (0,1/4) \), both depending only on \( \Lambda_0, \Lambda_1, p^{\pm}_{\log}, q^{\pm}_{\log}, n, \sigma \), such that if \( (p(\cdot), A, \Omega) \) is \( (\gamma, \sigma, S_0) \)-vanishing for some
\(\gamma \in (0, \gamma_0)\), then there exists a constant \(C_0 = C_0(\Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm, n, S_0) > 0\) so that for any \(x_0 \in \Omega\), \(\delta \in (0, \delta_0)\) and \(r \in (0, 1/(C_0M_0)]\), we have

\[
\int_{\Omega_r(x_0)} |\nabla u|^{(p(x)-\delta)q(x)} \, dx \leq C \left\{ \int_{\Omega_{4r}(x_0)} |\nabla u|^{p(x)-\delta} \, dx \right\}^{\frac{q_{\Omega_r}(\sigma_0)}{q_{\Omega_{4r}}(\sigma_0)}} + \int_{\Omega_{4r}(x_0)} |f|^{(p(x)-\delta)q(x)} \, dx + 1
\]

for some constant \(C = C(\Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm, n, \sigma) > 0\) and \(M_0\) as defined in (5.14).

Using a standard covering argument, we can obtain the following global estimates:

**Theorem 3.2.** Let \(M^+ > 1\) be given and let \(r(\cdot)\) be a log-Hölder continuous function satisfying \(1 \leq r^- \leq r(\cdot) \leq r^+ < M^+ < \infty\). Let \(0 < \sigma < 1/4\) and \(0 < S_0 < 1\) be given, then under the assumptions in Theorem 3.1, there is a constant \(\gamma_0 \in (0, 1/4)\) depending only on \(\Lambda_0, \Lambda_1, p_{\log}^\pm, r_{\log}, M^+, n\) and \(\sigma\), such that if \((p(\cdot), A, \Omega)\) is \((\gamma, \sigma, S_0)\)-vanishing for some \(\gamma \in (0, \gamma_0)\), then there exists a constant \(C = C(\Lambda_0, \Lambda_1, p_{\log}^\pm, r_{\log}, M^+, n, \sigma, S_0) > 0\) such that the following global bound holds:

\[
\int_{\Omega} |\nabla u|^{p(\cdot)r(x)} \, dx \leq C \left( \left( \int_{\Omega} |f|^{p(\cdot)r(x)} \, dx \right)^{n(M^+-1)+M^+} + 1 \right).
\]

**Remark 3.3.** In Theorem 3.2, we restrict \(r^+ < M^+\) and we cannot take \(r^+ = M^+\) in general (see Section 7 for details). This is unfortunately an artifact of the variable exponent spaces and the techniques used in this paper. It would be interesting to know if this restriction can be removed!

4. Background material

In this section, we shall collect and in some cases, prove all the necessary details needed in subsequent Sections.

4.1. Sobolev spaces with variable exponents

Let \(\tilde{\Omega}\) be a bounded domain, let \(s(\cdot)\) be an admissible variable exponent as in Section 2.1 and let \(\omega: \tilde{\Omega} \to (0, \infty)\) be any weight function. Given a positive integer \(m\), the variable exponent Lebesgue space \(L_{s(\cdot)}^1(\tilde{\Omega}, \mathbb{R}^m)\) consists of all measurable functions \(f: \tilde{\Omega} \to \mathbb{R}^m\) satisfying

\[
\int_{\tilde{\Omega}} |f(x)|^{s(x)} \omega(x) \, dx < \infty
\]

having the norm

\[
\|f\|_{L_{s(\cdot)}^1(\tilde{\Omega}, \mathbb{R}^m)} := \inf \left\{ \lambda > 0 : \int_{\tilde{\Omega}} \frac{|f(x)|^{s(x)}}{\lambda} \omega(x) \, dx \leq 1 \right\}.
\]
4.2. Muckenhoupt weights in variable exponent spaces

$W^{1,s}_\omega(\tilde{\Omega}, \mathbb{R}^m) := \{ f \in L^s(\tilde{\Omega}, \mathbb{R}^m) : \nabla f \in L^s(\tilde{\Omega}, \mathbb{R}^{mn}) \}$

equipped with the norm

$$\| f \|_{W^{1,s}_\omega(\tilde{\Omega}, \mathbb{R}^m)} := \| f \|_{L^s(\tilde{\Omega}, \mathbb{R}^m)} + \| \nabla f \|_{L^s(\tilde{\Omega}, \mathbb{R}^{mn})}.$$  

We shall denote $W^{1,s}_0(\tilde{\Omega}, \mathbb{R}^m)$ to be the closure of $C_c^\infty(\tilde{\Omega}, \mathbb{R}^m)$ in $W^{1,s}_\omega(\tilde{\Omega}, \mathbb{R}^m)$. For $m = 1$, we write $L^s(\tilde{\Omega})$, $W^{1,s}(\tilde{\Omega})$ and $W^{1,s}_0(\tilde{\Omega})$ for simplicity. In the case $\omega \equiv 1$, the function spaces above become the standard variable exponent spaces as described in [19].

We will sometimes also use the following notation in the case $\omega(\cdot) \equiv 1$:

$$\varrho_{L^s(\tilde{\Omega})}(f) := \int_{\tilde{\Omega}} |f(x)|^{s(x)} \; dx.$$  

We mention the following useful relation between the modular and norm in the variable exponent spaces (see [19, Lemma 3.2.5] for details):

**Lemma 4.1.** For any $f \in L^s(\tilde{\Omega})$, the following holds:

$$\min \left\{ \varrho_{L^s(B)}(f)^{\frac{1}{s(B)}}, \varrho_{L^s(B)}(f)^{\frac{1}{s(B)}} \right\} \leq \| f \|_{L^s(\tilde{\Omega})} \leq \max \left\{ \varrho_{L^s(B)}(f)^{\frac{1}{s(B)}}, \varrho_{L^s(B)}(f)^{\frac{1}{s(B)}} \right\}.$$  

4.2. Muckenhoupt weights in variable exponent spaces

We shall define the class of variable exponent Muckenhoupt weights $A_{s(\cdot)}$ following [20] and use them to prove certain useful bounds on the maximal function.

**Definition 4.2.** We say that a measurable function $w : \mathbb{R}^n \to (0, \infty)$ is an $A_{s(\cdot)}$ weight if

$$[w]_{A_{s(\cdot)}} := \sup_B \frac{1}{|B|^s} \| w \|_{L^s(B)} \left\| \frac{1}{w} \right\|_{L^{s'(B)}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Here we have defined $s_B := \left( \int_B \frac{1}{s(x)} \; dx \right)^{-1}$.

Following [20], we shall collect some of the important properties of $A_{s(\cdot)}$ weights. The first lemma shows that the Muckenhoupt weights form an increasing class (see [20, Lemma 3.1]):

**Lemma 4.3.** Let $s(\cdot), q(\cdot) \in \log^\pm$ with $q(\cdot) \leq s(\cdot)$ pointwise, then there exists a constant $C_{incl} = C_{incl}(s_{log^\pm}, q_{log^\pm})$ such that $[w]_{A_{s(\cdot)}} \leq C_{incl}[w]_{A_{q(\cdot)}}$.  

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The following Lemma lets us compare the behavior of the variable exponent Muckenhoupt weights over balls of different radii (see [20, Lemma 3.3]):

**Lemma 4.4.** Let $s(\cdot) \in \text{Log}^\pm$ and let $w \in A_{s(\cdot)}$ be a given weight, then the following holds for any $x, y \in \mathbb{R}^n$ and $r, R > 0$:

$$w(B_r(x)) \lesssim w(B_R(y)) \left( \frac{r^n}{|x - y|^n + r^n + R^n} \right)^{s^+},$$

where $w(B) := \int_B w(x) \, dx$.

Using the previous Lemma, we can obtain the following fundamental estimates which relate the norm and modular of a characteristic function in the weighted setting, provided the weight is in $A_\infty$. This property will play a crucial role in the next subsection (see [20, Lemma 3.4]):

**Lemma 4.5.** Let $s(\cdot) \in \text{Log}^\pm$ and $w \in A_\infty$ and suppose $B$ is a ball with $\text{diam } B \leq 2$, then for any $x \in B$, there holds

$$\|1\|_{L^{s(\cdot)}(B)} \approx w(B)^{\frac{1}{s}} \approx w(B)^{\frac{1}{s'(-)}} \approx w(B)^{\frac{1}{s'}} \approx w(B)^{\frac{1}{s'}}.$$

For $w \in A_{s(\cdot)}$, we define a *dual weight* by $w'(y) = w(y)^{1-s'(y)}$ where $s'(y) = \frac{s(y)}{s(y) - 1}$. The next lemma proves the variable exponent analogue of the fact that $[w]_{A_s} = [w']^{s-1}_{A_{s'}}$ and so $w \in A_s$ if and only if $w' \in A_{s'}$ (see [20, Proposition 3.8] for the details):

**Proposition 4.6.** Let $s(\cdot) \in \text{Log}^\pm$ and $w \in A_{s(\cdot)}$, then $w' \in A_{s'(\cdot)}$ and

$$|B|^{-s} \|w\|_{L^{s(\cdot)}(B)} \approx \frac{1}{w} \|w'\|_{L^{-s'(\cdot)}(B)} \approx \frac{w(B)}{|B|} \left( \frac{w'(B)}{|B|} \right)^{s-1}.$$}

Let us now recall another basic property which is the variable exponent analogue of the reverse factorization result (See [20, Proposiiton 3.13] for the details):

**Proposition 4.7.** Let $s(\cdot) \in \text{Log}^\pm(\mathbb{R}^n)$ and $w_1, w_2 \in A_1$, then $w_1(x)w_2(x)^{1-s(x)} \in A_{s(\cdot)}$.

### 4.3. Hardy Littlewood maximal function

In this subsection, we shall prove a few important estimates involving the truncated maximal function. We define the truncated maximal function by:

$$\mathcal{M}_{< R} f(x) := \sup_{r < R} \int_{B_r(x)} |f(y)| \, dy.$$
Note that $Mf$ is the standard maximal function of $f$.

Following the ideas in [20, Lemma 5.1], we prove the following important theorem:

**Theorem 4.8.** Let $s(\cdot) \in \Log^\pm_\omega (\mathbb{R}^n)$ and let $\omega \in A_{s(\cdot)}$ be a Muckenhoupt weight. Assume that $M_1, M_2 > 0$ are given constants such that $[\omega]_{A_{s(\cdot)}} \leq M_1$ holds. Then there is $R_1 = R_1(M_1, M_2, s^\pm_{\log}) > 0$ such that for any $2r < R_1$ and for any $f \in L^\omega_s(\mathbb{R}^n)$ satisfying
\[
\int_{B_{2r}} |f(x)|^{s(x)} \omega(x) \, dx + 1 \leq M_2, \tag{4.3}
\]
then the following estimate holds:
\[
\int_{B_r} M_{<r}(|f|)^{s(x)}(x) \omega(x) \, dx \leq C \left( \int_{B_{2r}} |f(x)|^{s(x)} \omega(x) \, dx + 1 \right),
\]
where $C = C(n, M_1, s^\pm_{\log})$.

**Proof.** Since $\omega \in A_{s(\cdot)}$, we have the following bound due to Lemma 4.3:
\[
[\omega]_{A_{s(\cdot)}} \leq C_{\text{incl}}[\omega]_{A_{s(\cdot)}} \leq C_{\text{incl}} M_1. \tag{4.4}
\]
By the self-improvement property of Muckenhoupt weights (see [25, Theorem 9.2.5]), there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(s^+, n, M_1) > 0$ such that
\[
[\omega]_{A_{s(\cdot)}} \leq C \left( \int_{B_r} \omega(x) \, dx \right)^{\frac{1}{\tilde{\varepsilon}}}. \tag{4.5}
\]
We now choose $R_1 = \min \left\{ \tilde{r}_1, \tilde{r}_2, \frac{1}{3} \right\}$, where $\tilde{r}_1$ and $\tilde{r}_2$ satisfy

- With $\tilde{\varepsilon}$ as in (4.5), using Remark 2.2, we see that there exists $\tilde{r}_1(s^+_{\log}) > 0$ such that the following bound holds: $s^+_{B_{2r_1}} - \tilde{\varepsilon} < s^-_{B_{2r_1}}$.
- Let $\tilde{r}_2 < \frac{1}{2M_2}$.

Now fix any radius $r$ satisfying $2r < R_1$, it is then easy to see using the log-Hölder continuity of $s(\cdot)$ such that the following bound holds:
\[
|B_r(y)|^{s^-_{B_r(y)} - s^+_{B_r(y)}} \leq c(s^+_{\log}, n). \tag{4.6}
\]
For any $y \in B_r$, consider the ball $B_{\tau}(y)$ with $\tau < r$ and set $q(x) := \frac{s(x)}{s_{B_{2r}}}$. Then for any $\beta \geq 1$, there holds
\[
\left( \frac{\int_{B_r(y)} |f(x)| \, dx}{q(y)} \right)^{q(y)} \leq \left( \frac{\int_{B_{\tau}(y)} |f(x)|^{q(x)} \, dx}{q(x)} \right)^{\frac{q(x)}{q(y)}} \leq \left( \frac{\int_{B_{\tau}(y)} |f(x)|^{q(x)} \, dx}{q(x)} \right)^{\frac{q(x)}{q(y)}} + \frac{1}{\beta}, \tag{4.7}
\]

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Now set $\beta := \max \left\{ 1, \omega(B_r)^{-\frac{1}{B_{2r}}} \right\}$, then using (4.2), we see that

$$\beta^{q(x)^{-1}} \leq (1 + |x|)^{C(s(x) - s_{B_{2r}}(y))} \leq c(s_{\log, n}^2).$$

(4.8)

Making use of (4.8) into (4.7), we get

$$\left( \int_{B_r(y)} |f(x)| \, dx \right)^{q(y)} \leq |B_r(y)|^{1 - \frac{q(y)}{s_{B_{2r}}(y)}} \varrho_{L^{q(y)}(B_r(y))}(f) \left( \int_{B_r(y)} |f(x)|^{q(x)} \, dx \right)^{\frac{q(y) - s_{B_{2r}}(y)}{s_{B_{2r}}(y)}} + \min \left\{ 1, \omega(B_r)^{-\frac{1}{B_{2r}}} \right\}. \tag{4.9}$$

Let now bound $\varrho_{L^{q(y)}(B_r(y))}(f)^{\frac{q(y)}{s_{B_{2r}}(y)}}$ from above as follows:

$$\varrho_{L^{q(y)}(B_r(y))}(f) \leq \int_{B_r(y)} |f(x)|^{s(x)} \omega(x) \, dx + \int_{B_r(y)} \omega(x)^{-\frac{1}{s_{B_{2r}}(y)}} \, dx$$

$$\leq \int_{B_r(y)} |f(x)|^{s(x)} \omega(x) \, dx + \frac{1}{\omega(B_r)^{s_{B_{2r}}(y)}} \left( \frac{|B_r(y)|^{s_{B_{2r}}(y)}}{\omega(B_r(y))} \right) \frac{1}{s_{B_{2r}}(y)}$$

$$\overset{(a)}{\leq} M_2 + C_{incl} M_1 \left( \frac{|B_r(y)|^{s_{B_{2r}}(y)}}{\omega(B_r(y))} \right)^{s_{B_{2r}}(y)}. \tag{4.10}$$

To obtain (a), we made use of (4.3), (4.5) and (4.8).

Now substituting (4.10) into (4.9), followed by making use of Lemma 4.5, the log-Hölder continuity of $s(\cdot)$ along with the choice of $R_1$ and (4.6), we obtain

$$\left( \int_{B_r(y)} |f(x)| \, dx \right)^{q(y)} \leq \int_{B_r(y)} |f(x)|^{q(x)} \, dx + \min \left\{ 1, \omega(B_r)^{-\frac{1}{s_{B_{2r}}}} \right\}. \tag{4.11}$$

Taking supremum over all $\tau < r$, we get

$$\mathcal{M}_{<r}(|f|^{q(y)}(y) \leq \mathcal{M}_{<r}(|f|^{q(\cdot)}(y) + \min \left\{ 1, \omega(B_r)^{-\frac{1}{s_{B_{2r}}}} \right\}. \tag{4.11}$$

Raising (4.11) to $s_{B_{2r}}$, followed by multiplying with $\omega(y)$ and then integrating over $B_r$, we get

$$\int_{B_r} \mathcal{M}_{<r}(|f|^{s(x)}(x) \omega(x)) \, dx \leq \int_{B_r} \mathcal{M}_{<r}(|f|^{q(\cdot)} s_{B_{2r}}(x) \omega(x)) \, dx +$$

$$+ \omega(B_r) \min \left\{ 1, \omega(B_r)^{-\frac{1}{s_{B_{2r}}}} \right\}$$

$$\overset{(b)}{\leq} \int_{B_r} |f(x)|^{s(x)} \omega(x) \, dx + 1.$$
To obtain (b), we made use of the weighted maximal function bound (see [25, Theorem 9.19]) for exponent $s_{B_2}$. Since we have $\omega \in A^{s_{B_2}}$, this completes the proof of the Theorem.

In the unweighted case i.e., $\omega(x) \equiv 1$, Theorem 4.8 becomes

**Corollary 4.9.** Let $s(\cdot) \in \Log^{\pm}, M_2 > 0$ be given and let $R_2 := \min \left\{ \frac{1}{2M_2}, \frac{1}{\omega_n^{1/n}} \right\}$, where $\omega_n$ is the volume of a unit ball in $\mathbb{R}^n$. Then for any $f \in L^s(\mathbb{R}^n)$ and any radii $2r < R_2$ satisfying

$$\int_{B_{2r}} |f(x)|^s(x) \, dx + 1 \leq M_2,$$

there holds

$$\int_{B_r} M_{<r}(|f|)^s(x) \, dx \leq \int_{B_{2r}} |f(x)|^s(x) \, dx + |B_r|.$$

In the constant exponent case, the results are well known (see for example [25, Chapter 9]):

**Lemma 4.10.** The following bounds hold:

- (Weak 1-1 estimate) For any $f \in L^1(\mathbb{R}^n)$ and for every $\alpha > 0$, there holds

$$|\left\{ x \in \mathbb{R}^n : Mf(x) > \alpha \right\}| \leq \frac{c(n)}{\alpha} \int_{\mathbb{R}^n} |f| \, dx. \quad (4.12)$$

- (Strong s-s estimate) For any $f \in L^s(\mathbb{R}^n)$ with $1 < s < \infty$, then $Mf \in L^s(\mathbb{R}^n)$ and

$$\|Mf\|_{L^s(\mathbb{R}^n)} \leq c(n, s) \|f\|_{L^s(\mathbb{R}^n)}, \quad (4.13)$$

### 4.4. Poincaré type inequalities

We shall recall the modular form of the Poincaré inequality which was proved in [19, Theorem 8.2.4]:

**Lemma 4.11.** Let $B$ be any ball and let $v \in W^{1,s(\cdot)}(B)$, then there exists a constant $C = C(n, s_{\Log}^\pm)$ such that

$$\| v - (v)_B \|_{L^s(\cdot)(B)} \leq C \operatorname{diam}(B) \| \nabla v \|_{L^s(\cdot)(B)}.$$

The Poincaré inequality in Lemma 4.11 is not entirely suitable for our purposes and so we need to prove an integral version of the result. To do this, we shall use the unweighted maximal function bound obtained in Corollary 4.9 to prove the following scaled version of the Poincaré type inequality in the variable exponent spaces. We believe that this result is well known to experts, but we could not find this result in literature and hence we will present the proof.
Lemma 4.12. Let $s(\cdot) \in \text{Log}^\pm$ and $R_3 \geq 1$ be given and define $R_3 := \min \left\{ \frac{1}{2M_3}, \frac{1}{\omega_n^{1/n} \cdot 2} \right\}$. Then for any $\phi \in W^{1,s(\cdot)}(B_{4r})$ with $4r < R_3$ satisfying

$$\int_{B_{4r}} |\nabla \phi(x)|^{s(x)} \, dx + 1 \leq M_3,$$  \hspace{1cm} (4.14)

there exists a constant $C = C(n, s^\pm \text{log})$ such that

$$\int_{B_r} \left( \frac{\phi - (\phi)_{B_r}}{\text{diam}(B_r)} \right)^{s(x)} \, dx \leq C \left( \int_{B_r} |\nabla \phi(x)|^{s(x)} \, dx + |B_r| \right).$$

Since $\text{diam}(B_r) = 2r \leq R_3 < 1$, we also obtain

$$\int_{B_r} |\phi - (\phi)_{B_r}|^{s(x)} \, dx \leq C \left( \int_{B_r} |\nabla \phi(x)|^{s(x)} \, dx + |B_r| \right).$$

Proof. Let $B = B_r$ be any ball of radius $r \leq R_3$ and let $\phi \in W^{1,s(\cdot)}(B_r)$, then using [36, Theorem 1.51], we obtain

$$|\phi(x) - (\phi)_{B}| \leq C(n) \int_B \frac{|\nabla \phi| \chi_B}{|x-y|^{n-1}} \, dy. \hspace{1cm} (4.15)$$

Modifying the proof of [36, Theorem 1.32], we have the following bound (note that $\text{diam}(B) \leq 1$):

$$\int_B \frac{|\nabla \phi| \chi_B}{|x-y|^{n-1}} \, dy \leq C(n)M_{\lesssim 2R}(\nabla \phi \chi_B)(x). \hspace{1cm} (4.16)$$

Combining (4.15) and (4.16), we get

$$\frac{|\phi(x) - (\phi)_{B}|}{\text{diam}(B)} \leq C(n)M_{\lesssim 2R}(\nabla \phi \chi_B)(x). \hspace{1cm} (4.17)$$

We now exponentiate (4.17) by $s(\cdot)$ and then integrate over $B$. To control the maximal function term on the right, we make use of Corollary 4.9 (this is where the restriction on $R_3$ and (4.14) is needed) to get the final conclusion. This completes the proof of the Lemma.

We will also need the following version of the Poincaré inequality which holds provided the zero set of the function is sufficiently large (In the constant exponent case, this result is well known, see for example [3, Corollary 8.2.7] and references therein). In the variable exponent case, we believe the result is well known to experts (see [23, Lemma 3.3] for a very similar result, in fact our proof follows along the same lines), but we could not find a reference for it. Hence we shall give a proof in Appendix A for the sake of completeness.
Theorem 4.13. Let \( s(\cdot) \in \text{Log}^\pm \) and let \( M_4 \geq 1, \varepsilon \in (0,1) \) be given constants. Define \( R_4 := \min \left\{ \frac{1}{2M_4}, \frac{1}{\omega_n^{1/n}}, \frac{1}{2} \right\} \). For any \( \phi \in W^{1,p(\cdot)}(B_{2r}) \) with \( 2r < R_4 \) satisfying
\[
|\{ \mathcal{N}(\phi) \}| := \left| \{ x \in B_r : \phi(x) = 0 \} \right| > \varepsilon |B_r|
\] (4.18)
and
\[
\int_{B_{2r}} |\nabla \phi(x)|^{s(x)} \, dx + 1 \leq M_4,
\] (4.19)
then there holds
\[
\int_{B_r} \left( \frac{\phi}{\text{diam}(B_r)} \right)^{s(x)} \, dx \leq \int_{B_{2r}} |\nabla \phi(x)|^{s(x)} \, dx + |B_r|.
\]

We now recall the analogue of Hedberg’s lemma with truncated maximal function. The estimate using the non-truncated maximal function is due to Semmes (for a proof, see [28, Section 5]) and the proof of the theorem using truncated maximal function follows along similar lines.

Theorem 4.14. Let \( 1 < q < \infty \) and \( \bar{\Omega} \) be a bounded domain such that \( \bar{\Omega}^c \) satisfies a uniform measure density condition, i.e., there exists \( \varepsilon \in (0,1) \) such that for every \( r > 0 \) and \( x \in \partial \Omega \), there holds \( |\bar{\Omega}^c \cap B_r(x)| \geq \varepsilon |B_r(x)| \). Fix any \( r \geq \text{diam}(\bar{\Omega}) \) and let \( u \in W^{1,q}_0(\Omega) \) be given, then after extending \( u \) to be zero outside \( \bar{\Omega} \), the following estimate holds for a.e \( x, y \in \bar{\Omega} \),
\[
|u(x) - u(y)| \leq C(q, n, \varepsilon) |x - y| \left( \mathcal{M}_{<r}(|\nabla u|^q)\frac{1}{r} (x) + \mathcal{M}_{<r}(|\nabla u|^q)\frac{1}{r} (y) \right).
\]

We need the assumption on the measure density of \( \bar{\Omega}^c \) above to ensure we can apply the constant exponent version of the Poincaré inequality analogous to Theorem 4.13.

4.5. Two important Lemmas

The first Lemma is the well known Gehring’s lemma (see for example [24, Proposition 1.1] for the details):

Lemma 4.15 (Gehring’s Lemma). Let \( 1 < s < \infty \) and \( f, g \in L^s(\bar{\Omega}) \) be nonnegative functions. Suppose that there are constants \( c_g > 0 \) and \( 0 \leq \theta < 1 \) such that the following holds
\[
\int_B f^s \, dx \leq c_g \left( \int_{2B} f \, dx \right)^s + \theta \int_{2B} f^s \, dx + \int_{2B} g^s \, dx
\]
for all balls \( B \) such that \( 8B \subset \bar{\Omega} \). Then for any \( t > 0 \), there are constants \( r = r(n, c_g, \theta) > s \) and \( C = C(n, s, c_g, \theta, r, t) > 0 \) such that the following holds:
\[
\int_B f^r \, dx \leq C \left\{ \left( \int_{2B} f^t \, dx \right)^{\frac{r}{t}} + \int_{2B} g^r \, dx \right\}.
\]
The second lemma is an estimate in $L \log L$-space which can be found in [1] and references therein:

**Lemma 4.16.** Let $\beta > 0$ and $s > 1$, then for any $f \in L^s(\tilde{\Omega})$, we have

$$\int_{\tilde{\Omega}} |f| \left[ \log \left( e + \frac{|f|}{(\|f\|_{\tilde{\Omega}})} \right) \right]^\beta \, dx \leq C(n, s, \beta) \left( \int_{\tilde{\Omega}} |f|^s \, dx \right)^{\frac{\beta}{s}}.$$

4.6. Lipschitz-truncation

In this subsection, we shall give the construction of the Lipschitz function. In this regard, we need the following lemma whose proof can be found in [4, Lemma 3.1].

**Lemma 4.17.** Let $\gamma > 0$, $\sigma \in (0, 1]$ and $S_0 > 0$ be given and suppose that $\tilde{\Omega}$ is a $(\gamma, \sigma, S_0)$-quasiconvex domain. For $s(\cdot) \in \text{Log}^\pm$, let $\varphi \in W_0^{1, s(\cdot)}(\tilde{\Omega})$, and let $\delta \in (0, 1/4)$ be given such that $s_{\tilde{\Omega}}^- - 2\delta > 1$ and let $1 < q < s_{\tilde{\Omega}}^- - 2\delta$ be any exponent. Let $M_5 \geq 1$ be given and define

$$R_5 := \min \left\{ \frac{1}{2M_5}, \frac{1}{\omega_n^{1/n}}, \frac{1}{2}, \frac{S_0}{2} \right\}.$$

After extending $\varphi$ by zero outside $\tilde{\Omega}$, for any $2r \leq R_5$, we define the truncated function $\hat{v}$ by

- If $B_{2r} \subset \tilde{\Omega}$, then we set $\hat{v} = \varphi \phi$ for some cut off function $\phi \in C^\infty(B_{2r})$ with $\chi_{B_r} \leq \phi \leq \chi_{B_{2r}}$.
- If $B_{2r} \cap \partial \tilde{\Omega} \neq \emptyset$, then we set $\hat{v} = \varphi$.

Let $\hat{v}$ satisfy $\int_{\tilde{\Omega}} |\nabla \hat{v}|^{s(x)} \, dx \leq M_5$, and we define the following function:

$$g(x) := \max \left\{ \mathcal{M}_{<4r}(|\nabla \hat{v}|^q)^{1/q}(x), \frac{|\hat{v}(x)|}{d(x, \partial B_{2r})} \right\}.$$

Then the following holds:

- $g(x) \approx \mathcal{M}_{<4r}(|\nabla \hat{v}|^q)^{1/q}(x)$ a.e $x \in \mathbb{R}^n$.
- $\int_{B_{2r}} g(x)^{s(x) - \delta} \, dx \leq \int_{B_{2r}} |\nabla \hat{v}|^{s(x) - \delta} \, dx + |B_r|$.
- $[g^{-\delta}]_{A_{s(\cdot)}} \leq C(s_{\log}^\pm, q, n)[g^{-\delta}]_{A_{s(\cdot)}} \leq C(n, s_{\log}^\pm, q)$.

- The function $g^{-\delta}$ is in the Muckenhoupt class $A_t/q$ for any $t$ such that $t-q > \delta$ with $[g^{-\delta}]_{A_t/q} \leq C = C(n, t, q)$. In particular $g^{-\delta} \in A_{s(\cdot)}$.

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Proof. Since $\Omega$ is $(\gamma, \sigma, S_0)$-quasiconvex, we can use the results from [27, Theorem 2] to get

$$g(x) \approx M_{<4r}(|\nabla \tilde{v}|^{\gamma})^\frac{1}{\gamma}(x).$$

Now we can use Corollary 4.9 to get the estimate

$$\int_{B_{2r}} g(x) s(x) \delta \, dx \leq C(n, s_{\log}, q) \left\{ \int_{B_{2r}} |\nabla \tilde{v}|^{s(x)} \delta \, dx + |B_r| \right\}.$$

Before we prove the last claim, we first show that $g - \delta \in A_{s - \delta}$ as follows:

$$[g^{-\delta}]_{A_{s - \delta}} := \sup_B \left( \int_B g^{-\delta} \, dx \right) \left( \int_B g^{-s - \delta - q} \, dx \right)^{\frac{s - \delta - q}{q}} \leq \left( \inf_{y \in B} M_{<4r}(|\nabla \tilde{v}|^{\gamma})(y) \right)^{\frac{1}{\gamma}} \left( \inf_{y \in B} M_{<4r}(|\nabla \tilde{v}|^{\gamma})(y) \right)^{\frac{1}{\gamma}} \leq C(n, s_{\log}, q).$$

To obtain (a), we made use of [42, Proposition 3.3] concerning $A_1$ weights. Since $s_{\log} \leq \frac{s_{\log}}{q}$, using Lemma 4.3, the following estimate holds:

$$[g^{-\delta}]_{A_{s_{\log}}} \leq C(n, s_{\log}, q) [g^{-\delta}]_{A_{s - \delta}} \leq C(n, s_{\log}, q).$$

Note that all the constants are independent of $\delta$ since we assume $\delta \leq 1/4$, which enables us to remove the dependence on $\delta$. This completes the proof of the Lemma. \qed

We now present an extension lemma which can be found in [44] (see also [4, Lemma 3.2] for the details):\[\text{Lemma 4.18.}\]

Let $\gamma > 0$, $\sigma > 0$, and $S_0 > 0$ be given and let $\overline{\varphi} \in W^{1,s}_{0}(\tilde{\Omega})$ for some $s > 1$. Suppose $\tilde{\Omega}$ is a $(\gamma, \sigma, S_0)$-quasiconvex domain and $\lambda > 0$ is any given constant. Extend $\varphi$ by zero outside $\tilde{\Omega}$ and set

$$F_{\lambda}(\overline{\varphi}, \tilde{\Omega}) := \left\{ x \in \tilde{\Omega} : M_{< \text{diam} \tilde{\Omega}}(|\nabla \overline{\varphi}|^{s})^{\frac{1}{s}}(x) \leq \lambda, |\overline{\varphi}(x)| \leq \lambda d(x, \partial \tilde{\Omega}) \right\}.$$

Then there exists a $c\lambda$-Lipschitz function $\overline{\varphi}_{\lambda}$ defined on $\mathbb{R}^n$ with $c = c(n) > 1$ satisfying the following properties:

- $\overline{\varphi}_{\lambda}(x) = \overline{\varphi}(x)$ and $\nabla \overline{\varphi}_{\lambda}(x) = \nabla \overline{\varphi}(x)$ for a.e. $x \in F_{\lambda}$,
- $\overline{\varphi}_{\lambda}(x) = 0$ for every $x \in \tilde{\Omega}^c$,
- $|\nabla \overline{\varphi}_{\lambda}(x)| \leq c(n) \lambda$ for a.e. $x \in \mathbb{R}^n$.\[18\]
4.7. Some auxiliary results

We will need the following sharp form of Young’s inequality:

**Lemma 4.19.** Let $\tilde{\Omega}$ be any bounded domain and let $s(\cdot) \in \text{Log}^\pm$ and $q(\cdot) > 0$ be any exponent satisfying $q(x) \leq s(x)$ for every $x \in \tilde{\Omega}$. For $f \in L^{s(\cdot)}(\tilde{\Omega})$, Young’s inequality yields

$$
\int_{\tilde{\Omega}} |f(x)|^{q(x)} \, dx \leq \int_{\tilde{\Omega}} |f(x)|^{s(x)} \, dx + |\tilde{\Omega}|.
$$

The next lemma is a reformulation of Calderón-Zygmund decomposition in terms of balls instead of cubes.

**Lemma 4.20 ([32]).** For any $\gamma > 0$, $\sigma \in (0, 1/4)$ and $S_0 > 0$, let $\Omega$ be a $(\gamma, \sigma, S_0)$-quasiconvex domain. Consider the subdomain $\Omega_r(x_0) = \Omega \cap B_r(x_0)$ with $r \in (0, S_0]$ and $x_0 \in \Omega$. Let $C \subset D \subset \Omega_r(x_0)$ be measurable sets and $0 < \varepsilon < 1$ such that

- $|C| < \varepsilon |B_r|$, and
- for all $x \in \Omega$ and $\rho \in (0, 2r]$, if $|C \cap B_\rho(x)| \geq \varepsilon |B_\rho(x)|$, then $B_\rho(x) \cap \Omega_r(x_0) \subset D$.

Then we have the estimate

$$
|C| \leq \varepsilon \left( \frac{10}{\sigma} \right)^n |D|.
$$

We end this section by introducing a well known lemma (see [4, Lemma 4.1] for details).

**Lemma 4.21.** Let $f$ be a measurable function in a bounded open set $\tilde{\Omega} \subset \mathbb{R}^n$ and let $\lambda_0 > 0$ and $N > 1$ be given constants, then for any $0 < q < \infty$ there holds

$$
f \in L^q(\tilde{\Omega}) \iff S := \sum_{k \geq 1} N^{qk} \left| \left\{ x \in \tilde{\Omega} : |f(x)| > N^k \lambda_0 \right\} \right| < \infty
$$

with the estimate

$$
c^{-1} \lambda_0^q S \leq \int_{\tilde{\Omega}} |f|^q \, dx \leq c \lambda_0^q \left( |\tilde{\Omega}| + S \right),
$$

where the constant $c = c(N, q) > 0$.

5. A priori estimates

First let us prove some general a priori estimates.
5.1. General a priori estimates

Let us first recall the following energy estimate:

**Lemma 5.1.** Let $\tilde{\Omega}$ be any bounded domain and suppose that the nonlinearity $A(\cdot, \cdot)$ satisfies (2.1) and (2.2). For any $\tilde{u} \in W^{1,p}(\tilde{\Omega})$ and $\tilde{g} \in L^{p}(\tilde{\Omega}, \mathbb{R}^{n})$, let $\tilde{w} \in W^{1,p}(\tilde{\Omega})$ be the unique solution that solves the following equation

$$
\begin{aligned}
\begin{cases}
\text{div} A(x, \nabla \tilde{w}) = \text{div}(|\tilde{g}|^{p(x)-2}\tilde{g}) & \text{in } \tilde{\Omega}, \\
\tilde{w} \in \tilde{u} + W^{1,p}_{0}(\tilde{\Omega}).
\end{cases}
\end{aligned}
$$

Then the following energy estimate holds:

$$
\int_{\tilde{\Omega}} |\nabla \tilde{w}|^{p(x)} \, dx \leq C \left( \int_{\tilde{\Omega}} |\nabla \tilde{u}|^{p(x)} \, dx + \int_{\tilde{\Omega}} |\tilde{g}|^{p(x)} \, dx + |\tilde{\Omega}| \right),
$$

where the constant $C = C(\Lambda_{0}, \Lambda_{1}, p^{\pm}_{\log}, n)$.

Next we prove a crucial difference estimates below the natural exponent:

**Theorem 5.2** (A priori estimates below the natural exponent). Let $\gamma > 0$, $\sigma \in (0, 1/4)$ and $S_{0} > 0$ be given and suppose that $\hat{\Omega}$ is a $(\gamma, \sigma, S_{0})$-quasiconvex domain. Let $\tilde{u} \in W^{1,p}(\hat{\Omega})$ and $\tilde{g} \in L^{p}(\hat{\Omega}, \mathbb{R}^{n})$ be given. For any $M_{6} \geq 1$, define

$$
R_{6} := \min \left\{ \frac{1}{2M_{6}}, \frac{1}{\omega_{n}^{1/n}}, \frac{1}{2}, \frac{S_{0}}{2} \right\}.
$$

Let $\hat{\Omega}_{r} := \hat{\Omega} \cap B_{r}$ be any region with $2r < R_{6}$ and consider the unique solution $w \in W^{1,p}(\hat{\Omega})$ of the following problem:

$$
\begin{aligned}
\begin{cases}
\text{div} A(x, \nabla \hat{w}) = \text{div}(|\hat{g}|^{p(x)-2}\hat{g}) & \text{in } \hat{\Omega}_{r}, \\
\hat{w} \in \tilde{u} + W^{1,p}_{0}(\hat{\Omega}_{r}).
\end{cases}
\end{aligned}
$$

Suppose that the following bound holds:

$$
\int_{\mathbb{R}^{n}} |\nabla \tilde{u} - \nabla \hat{w}|^{p(x)} \, dx + 1 = \int_{\hat{\Omega}_{r}} |\nabla \tilde{u} - \nabla \hat{w}|^{p(x)} \, dx + 1 \leq M_{6},
$$

then there exists a constant $\delta_{1} = \delta_{1}(\Lambda_{0}, \Lambda_{1}, p^{\pm}_{\log}, n) \in (0, 1/4)$ such that for any $\delta \in (0, \delta_{1})$, there holds

$$
\int_{\hat{\Omega}_{r}} |\nabla \hat{w}|^{p(x)-\delta} \, dx \leq C \left( \int_{\hat{\Omega}_{r}} |\nabla \tilde{u}|^{p(x)-\delta} \, dx + \int_{\hat{\Omega}_{r}} |\tilde{g}|^{p(x)-\delta} \, dx + |\hat{\Omega}_{r}| \right),
$$

where the constant $C = C(\Lambda_{0}, \Lambda_{1}, p^{\pm}_{\log}, n)$.
Proof. Let $2r < R_6$ be fixed, since $\tilde{\Omega}$ is a $(\gamma, \sigma, S_0)$-quasiconvex domain, $\tilde{\Omega}_r$ satisfies the measure density estimate of Lemma 2.8. Let $1 < q < p_{\tilde{\Omega}_r}^- - 2\delta$ be any fixed exponent and let us define

$$g(x) \colon= \max \left\{ M_{< 2r}(\|\nabla \tilde{w} - \nabla \tilde{u}\|), \frac{\|\tilde{w}(x) - \tilde{u}(x)\|}{d(x, \partial \tilde{\Omega}_r)} \right\}.$$ 

Note that we have extended $\tilde{w} - \tilde{u} = 0$ outside $\tilde{\Omega}_r$.

By the restriction on $R_6$, we can apply Lemma 4.17 with $M_5 = M_6$ and $\tilde{v} = \tilde{w} - \tilde{u}$ to get

$$g(x) \approx M_{< 2r}(\|\nabla \tilde{w} - \nabla \tilde{u}\|) \tilde{w}(x) \text{ for a.e. } x \in \mathbb{R}^n,$$ 

$$\int_{\tilde{\Omega}_r} g(x)^{p(x) - \delta} \, dx \leq C(p_{\log, n}^+ \tilde{\Omega}) \left( \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x) - \delta} \, dx + |\tilde{\Omega}_r| \right).$$ 

Define the set

$$F_\lambda \colon= \{ x \in \Omega_r : g(x) \leq \lambda \}.$$

We now apply Lemma 4.18 with $s = q$ and $\tilde{v} = \tilde{w} - \tilde{u}$ to get a Lipschitz function $v_\lambda$ which is a valid test function for (5.2), which gives

$$\int_{F_\lambda} (A(x, \nabla \tilde{w}) - A(x, \nabla \tilde{u}) + A(x, \nabla \tilde{u}) - \tilde{g} \tilde{v}) \, dx =$$

$$= -\int_{F_\lambda} \langle A(x, \nabla \tilde{w}), \nabla v_\lambda \rangle \, dx + \int_{F_\lambda} \langle \tilde{g} \tilde{v} \rangle \, dx$$

$$\leq \lambda \int_{F_\lambda} (\mu^2 + |\nabla \tilde{w}|^2)^{\frac{p(x) - 1}{2}} \, dx + \lambda \int_{F_\lambda} |\tilde{g}|^{p(x) - 1} \, dx.$$

Multiplying the above expression by $\lambda^{-1 - \delta}$ and integrating over $(0, \infty)$, we get $I_1 + I_2 + I_3 \leq I_4 + I_5$, where we have set

- $I_1 := \int_0^\infty \lambda^{1 + \frac{\delta}{2}} \int_{F_\lambda} \langle A(x, \nabla \tilde{w}) - A(x, \nabla \tilde{u}), \nabla v_\lambda \rangle \, dx \, d\lambda.$
- $I_2 := \int_0^\infty \lambda^{1 + \frac{\delta}{2}} \int_{F_\lambda} \langle A(x, \nabla \tilde{u}), \nabla v_\lambda \rangle \, dx \, d\lambda.$
- $I_3 := -\int_0^\infty \lambda^{1 + \frac{\delta}{2}} \int_{F_\lambda} \langle \tilde{g} \tilde{v} \rangle \, dx \, d\lambda.$
- $I_4 := \int_0^\infty \lambda^\delta \int_{F_\lambda} |\nabla \tilde{w}|^{p(x) - 1} + \mu^{p(x) - 1} \, dx \, d\lambda.$
- $I_5 := \int_0^\infty \lambda^\delta \int_{F_\lambda} |\tilde{g}|^{p(x) - 1} \, dx \, d\lambda.$

Let us now estimate each of the above terms as follows:
**Estimate for $I_1$:** Applying Fubini’s theorem and (2.3), we get

\[ I_1 = \frac{1}{\delta} \int_{\tilde{\Omega}_r} g(x)^{-\delta} (A(x, \nabla \tilde{w}) - A(x, \nabla \tilde{u}), \nabla \tilde{w} - \nabla \tilde{u}) \, dx \]

\[ \geq \frac{1}{\delta} \int_{\tilde{\Omega}_r} (\mu^2 + |\nabla \tilde{u}|^2 + |\nabla \tilde{w}|^2)^{\frac{p(x)-\frac{\delta}{p(x)} - 1}{p(x)}} |\nabla \tilde{w} - \nabla \tilde{u}|^2 g(x)^{-\delta} \, dx. \]  

(5.5)

In order to estimate (5.5), we need to consider the case if $p(x) \geq 2$ and $p(x) < 2$, hence let us split $\tilde{\Omega}_r = \tilde{\Omega}_1 \cup \tilde{\Omega}_2$, where

\[ \tilde{\Omega}_1 := \{ x \in \tilde{\Omega}_r : p(x) \geq 2 \} \quad \text{and} \quad \tilde{\Omega}_2 := \{ x \in \tilde{\Omega}_r : p(x) < 2 \}. \]

**Case $p(x) \geq 2$:** In this case, we can directly apply Young’s inequality to get

\[ \int_{\tilde{\Omega}_1} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq C(\varepsilon_1) \int_{\tilde{\Omega}_1} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)} g(x)^{-\delta} \, dx + \varepsilon_1 \int_{\tilde{\Omega}_1} g(x)^{p(x)-\delta} \, dx \]

\[ \leq C(\varepsilon_1) \int_{\tilde{\Omega}_1} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)} g(x)^{-\delta} \, dx + \varepsilon_1 \int_{\tilde{\Omega}_1} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx + \varepsilon_1 |\tilde{\Omega}_r|. \]  

(5.6)

Thus combining (5.6) into (5.5), we get

\[ \int_{\tilde{\Omega}_1} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq C(\varepsilon_1) \delta I_1 + \varepsilon_1 \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx + \varepsilon_1 |\tilde{\Omega}_r|. \]  

(5.7)

**Case $p(x) < 2$:** In this case, using Young’s inequality, we obtain

\[ |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \leq C(\varepsilon_1)(\mu^2 + |\nabla \tilde{u}|^2 + |\nabla \tilde{w}|^2)^{\frac{p(x)-2}{p(x)}} |\nabla \tilde{w} - \nabla \tilde{u}|^2 g(x)^{-\delta} + \varepsilon_1(\mu^2 + |\nabla \tilde{u}|^2 + |\nabla \tilde{w}|^2)^{\frac{p(x)-2}{p(x)}} + \varepsilon_1 g(x)^{p(x)-\delta}. \]  

(5.8)

Integrating (5.8) over $\tilde{\Omega}_2$ and using (5.4) and (5.5), we get

\[ \int_{\tilde{\Omega}_2} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq C(\varepsilon_1) \delta I_1 + \varepsilon_1 \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx + \varepsilon_1 |\tilde{\Omega}_r| \]

\[ + C(\varepsilon_1) \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} \, dx. \]  

(5.9)

Thus we can combine (5.7) when $p(x) \geq 2$ and (5.9) when $p(x) < 2$ to get

\[ \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq C(\varepsilon_1) \delta I_1 + \varepsilon_1 \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx + \varepsilon_1 |\tilde{\Omega}_r| \]

\[ + C(\varepsilon_1) \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} \, dx. \]  

(5.10)

**Estimate for $I_2$:** Making use of Fubini’s theorem, Young’s inequality, (2.1) and (5.3), we get

\[ I_2 \lesssim \frac{1}{\delta} \int_{\tilde{\Omega}_r} g(x)^{-\delta} (\mu^2 + |\nabla \tilde{u}|^2)^{\frac{p(x)-1}{p(x)}} |\nabla \tilde{w} - \nabla \tilde{u}| \, dx \]

\[ \lesssim \frac{C(\varepsilon_2)}{\delta} \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} + \frac{C(\varepsilon_2)}{\delta} |\tilde{\Omega}_r| + \varepsilon_2 \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx. \]  

(5.11)
Estimate for $I_3$: Similar to $I_2$, after using Fubini’s theorem, Young’s inequality and (5.3), we get:

$$I_3 \leq \frac{C(\varepsilon_3)}{\delta} \int_{\tilde{\Omega}_r} |\tilde{g}|^{p(x)-\delta} \, dx + \frac{\varepsilon_3}{\delta} \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx. \quad (5.12)$$

Estimate for $I_4$ and $I_5$: Again by using Fubini’s theorem, Young’s inequality and (5.4), we get

$$I_4 + I_5 \leq \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx + \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} \, dx + \int_{\tilde{\Omega}_r} |\tilde{g}|^{p(x)-\delta} \, dx + |\tilde{\Omega}_r|. \quad (5.13)$$

Combining (5.10)–(5.13) followed by using the trivial bound $I_1 \leq |I_2| + |I_4| + I_5$, we obtain

$$\int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq [\varepsilon_1 + C(\varepsilon_1)(\varepsilon_2 + \varepsilon_3 + \delta)] \int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx +$$

$$+ C(\varepsilon_1, \varepsilon_2, \varepsilon_3) \left( \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} \, dx + \int_{\tilde{\Omega}_r} |\tilde{g}|^{p(x)-\delta} \, dx + |\tilde{\Omega}_r| \right).$$

Choosing $\varepsilon_1$ small followed by $\varepsilon_2, \varepsilon_3$ and $\delta_1$, we see that for all $\delta \in (0, \delta_1)$, the following estimate holds:

$$\int_{\tilde{\Omega}_r} |\nabla \tilde{w} - \nabla \tilde{u}|^{p(x)-\delta} \, dx \leq \int_{\tilde{\Omega}_r} |\nabla \tilde{u}|^{p(x)-\delta} \, dx + \int_{\tilde{\Omega}_r} |\tilde{g}|^{p(x)-\delta} \, dx + |\tilde{\Omega}_r|.$$

A simple application of triangle inequality then proves the Theorem.

5.2. Fixing some universal constants

Let us define the following constant:

$$M_0 := \int_{\Omega} \left[|f|^{p(x)} + 1 \right] \, dx + 1. \quad (5.14)$$

Now by applying Lemma 5.1 to (1.1), we see that there is a constant $C = C(p_{\log}^\pm, \Lambda_0, \Lambda_1, n) > 0$ such that the following estimate holds:

$$\int_{\Omega} |\nabla u|^{p(x)} \, dx + 1 \leq C \left( \int_{\Omega} \left[|f|^{p(x)} + 1 \right] \, dx + 1 \right) =: M^u. \quad (5.15)$$

Note that $M^u = M^u(M_0, p_{\log}^\pm, \Lambda_0, \Lambda_1, n)$.

For any $r > 0$, let $\Omega_r := B_r \cap \Omega$ and consider the following first approximation to (1.1):

$$\begin{cases} 
\text{div} \, A(x, \nabla w) = 0 & \text{in } \Omega_r, \\
w \in u + W^{1,p}_{0}(\Omega_r). 
\end{cases} \quad (5.16)$$
Applying Lemma 5.1 to (5.16) and then making use of Lemma 4.19, we obtain a constant $C = C(p^\pm, \Lambda_0, \Lambda_1, n) > 0$ such that the following estimates hold for any $q(x) \leq p(x)$:

$$
\int_{\Omega_r} |\nabla u|^{q(x)} \, dx + 1 \leq C \left( \int_{\Omega} |\nabla u|^{p(x)} \, dx + 1 \right) := M_1^w, \\
\int_{\Omega_r} |\nabla w - \nabla u|^{q(x)} \, dx + 1 \leq C \left( \int_{\Omega} |\nabla u|^{p(x)} \, dx + 1 \right) := M_2^w.
$$

We now define the constant

$$
M^w := M^w(M_0, p^\pm, \Lambda_0, \Lambda_1, n) = \max\{M_1^w, M_2^w\}. \tag{5.17}
$$

5.3. Difference estimate below the natural exponent for the first approximation

In this subsection, we shall prove crucial difference estimates between solutions of (1.1) and (5.16) below the natural exponent.

**Theorem 5.3.** Let $\sigma \in (0, 1/4)$ and $S_0 > 0$ be given and let $\Omega$ be a $(\gamma, \sigma, S_0)$-quasiconvex domain for some $\gamma > 0$ to be chosen later. Suppose that $u \in W_0^{1,p(\cdot)}(\Omega)$ is the unique solution of (1.1).

Under the bounds (5.14) and (5.15), consider the unique solution $w \in W_0^{1,p(\cdot)}(\Omega_r)$ solving (5.16) in $\Omega_r$ for some $r \leq \frac{R_2}{2}$ where

$$
R_7 = \min \left\{ \frac{1}{2M^w}, \frac{1}{2M^w}, \frac{1}{2M_0}, \frac{1}{\omega_n^{1/\theta}}, \frac{1}{2}, \frac{S_0}{2} \right\}.
$$

For any $0 < \varepsilon < 1$, there exist positive constants $\delta_2 = \delta_2(\Lambda_0, \Lambda_1, p^\pm, \omega_0, n, \varepsilon) \in (0, 1)$ and $\gamma_2 = \gamma_2(\Lambda_0, \Lambda_1, p^\pm, \omega_0, n, \varepsilon) \in (0, 1)$ such that for all $\delta \in (0, \delta_2)$ and $\gamma \in (0, \gamma_2)$, we have the following result: Suppose for a given $2r < R_7$, the following bounds hold:

$$
\int_{\Omega_r} |\nabla u|^{p(x)-\delta} \, dx \leq \lambda \quad \text{and} \quad \int_{\Omega_r} |f|^{p(x)-\delta} \, dx \leq \gamma \lambda,
$$

then there exists a constant $C = C(\Lambda_0, \Lambda_1, p^\pm, \omega_0, n)$ such that the following conclusions hold:

$$
\int_{\Omega_r} |\nabla w|^{p(x)-\delta} \, dx \leq C\lambda \quad \text{and} \quad \int_{\Omega_r} |\nabla w - \nabla u|^{p(x)-\delta} \, dx \leq \varepsilon \lambda. \tag{5.18}
$$

**Proof.** Note that the first conclusion in (5.18) follows by applying Theorem 5.2 to (5.16). Hence we shall only need to prove the second conclusion of (5.18).

Since $\Omega$ is a $(\gamma, \sigma, S_0)$-quasiconvex domain, we have that $\Omega_r$ satisfies the measure density estimate of Lemma 2.8. Let $1 < q \leq p_{\Omega_r} - 2\delta$ be any fixed exponent and consider the following function:

$$
g(x) := \max \left\{ M_{<2r}(\nabla w - \nabla u)^{\frac{q}{2}}, \frac{|w(x) - u(x)|}{d(x, \partial\Omega_r)} \right\}.
$$

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Note here that we have extended $w - u = 0$ outside $\Omega_r$. By the restriction $2r \leq R_7$, we can apply Lemma 4.17 with $M_5 = \max\{2M^n, 2M^w\}$ and $\nabla = w - u$ to get

$$g(x) \approx M_{<2r}(|\nabla w - \nabla u|^q)_{\frac{q}{2}}(x) \text{ on } \mathbb{R}^n, \quad \int_{\Omega_r} g(x)^{p(x) - \delta} dx \leq C(p_{\log}^\pm, n) \left( \int_{\Omega_r} |\nabla w - \nabla u|^{p(x) - \delta} dx + |\Omega_r| \right).$$

Define the set $F_\lambda := \{ x \in \Omega_R : g(x) \leq \lambda \}$, we can now apply Lemma 4.18 to get a Lipschitz function $\nu_\lambda$ which is a valid test function for (5.16).

We can now proceed exactly as in the proof of Theorem 5.2 to get for any $\varepsilon \in (0, 1)$,

$$\int_{\Omega_r} |\nabla u - \nabla w|^{p(x)-\delta} \leq C(\Lambda_0, \Lambda_1, p_{\log}^\pm, n) \int_{\Omega_r} |\nabla u|^{p(x)-\delta} dx + \varepsilon |\Omega_r| + \varepsilon \int_{\Omega_r} |\nabla u|^{p(x)-\delta} dx.$$

Now we can choose $\int_{\Omega_R} |\nabla u - \nabla w|^{p(x)-\delta} dx \leq \gamma_2 \lambda$ such that the second estimate in (5.18) holds for all $\gamma \in (0, \gamma_2)$. This completes the proof of the Theorem.

5.4. Boundary higher integrability below the natural exponent

In this subsection, we shall prove a new higher integrability result. To better highlight the result, let us recall the following local versions of (1.1). Let $x_0 \in \overline{\Omega}$ be a fixed point of reference, then in the interior case $\Omega_r(x_0) = B_r(x_0) \subset \Omega$, the equation becomes

$$\text{div } A(x, \nabla u) = \text{div}(|\nabla u|^{p(x)-2} f) \quad \text{in } B_r(x_0),$$

and in the boundary case $B_r(x_0) \not\subset \Omega$ is given by:

$$\begin{cases}
\text{div } A(x, \nabla u) &= \text{div}(|\nabla u|^{p(x)-2} f) \quad \text{in } \Omega_r(x_0), \\
u &= 0 \quad \text{on } \partial_w \Omega_r(x_0). 
\end{cases}$$

**Theorem 5.4.** There exist $R_8 = R_8(p_{\log}^\pm, \Lambda_0, \Lambda_1, n, M_0, S_0) > 0$, $\delta_3 = \delta_3(n, \Lambda_0, \Lambda_1, p_{\log}^\pm)$ and $\sigma_1 = \sigma_1(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)$ such that the following holds for any $2r < R_8$, $\delta \in (0, \delta_3)$ and $\sigma \in (0, \sigma_1)$: if $u \in W_0^{1, p_\Lambda}(\Omega)$ and $f \in L^{p_\Lambda}(\Omega)$ solve (5.19) or (5.20), after extending $u = 0$ and $f = 0$ on $\Omega^c$, there holds

$$\int_{B_r} |\nabla u|^{(p(x)-\delta)(1+\sigma)} dx \leq \left( \int_{B_{2r}} |\nabla u|^{p(x)-\delta} dx \right)^{1+\sigma} + \int_{B_{2r}} |f|^{(p(x)-\delta)(1+\sigma)} dx + 1$$

for all $2r < r$ and $B_{2r} \subset B_r(x_0).$
Proof. We shall only prove the boundary higher integrability. For the interior higher integrability, the only modification we need to do is in the choice of $\tilde{u}$ in (5.21) by taking $\tilde{u} := \phi(u - (u)_{B_{2r}})$.

Let $B_{2r} \subset B_r(x_0)$ be any ball such that $B_{2r} \cap \Omega^c \neq \emptyset$. Let $\phi \in C^\infty(B_{2r})$ be a standard cut off function such that $\chi_{B_{2r}} \geq \phi \geq \chi_{B_r}$ and $|\nabla \phi| \leq \frac{c}{r}$. After extending $u = 0$ on $\Omega^c$, define

$$\tilde{u} := \phi u.$$  (5.21)

For any fixed $q \in (1, p^- - 2\delta)$, we define the following function:

$$g(x) := \max \left\{ M_{< 2r}(|\nabla \tilde{u}|^q + \mu^q)\hat{g}(x), \frac{|\tilde{u}(x)|}{d(x, \partial B_{2r})} \right\}.$$  

For any $q(\cdot) \leq p(\cdot)$, there holds

$$\int_{B_{2r}} |\nabla \tilde{u}|^q dx + 1 \leq \int_{B_{2r}} |\nabla u|^p dx + 1 \leq \int_{B_{2r}} |\nabla u|^p dx + \int_{B_{2r}} \frac{|u|^p(x)}{p} dx + 1 \leq C(n, p(\cdot), M^u) =: M^\tilde{u}.$$  

To obtain (a), we used Lemma 4.19 and to obtain (b), we used Theorem 4.13.

From Lemma 4.17, we see that $[g^{-\delta}]_{A_p(1)} \leq C_0(n, p_{\log}^\pm, q)$ and hence we can apply Theorem 4.8 with $M_1 = C_0(n, p_{\log}^\pm, q)$ and $M_2 = M^\tilde{u}$ to obtain an $R_1 = R_1(M_1, M_2, p_{\log}^\pm)$ such that for any $2r \leq R_1$ and for any $h \in L_{g^{\delta}}(B_{2r})$ with $\int_{B_{2r}} |h|^{\frac{p(\cdot)}{p}} g^{-\delta} dx + 1 \leq M^\tilde{u}$, there holds:

$$\int_{B_{2r}} M_{< r}(|h|)^\frac{p(\cdot)}{p} g^{-\delta} dx \leq \int_{B_{2r}} |h|^{\frac{p(\cdot)}{p}} g^{-\delta} dx + 1.$$  (5.22)

Further choose $\tilde{R}_1$ such that

$$\rho(2\tilde{R}_1) \leq \sqrt{\frac{n + 1}{n}} - 1 < 1.$$  (5.23)

Now choose $R_8 \leq \min \left\{ \frac{R_1}{2}, \frac{\tilde{R}_1}{2}, \frac{1}{2M^u}, \frac{1}{\omega_n^{1/n}} \right\}$. With this choice of $R_8$, we have from Lemma 4.17 that

$$g(x) \approx M_{< 2f}(|\nabla \tilde{u}|^q + \mu^q)\hat{g}(x)$$ on $\mathbb{R}^n$,  (5.24)

$$\int_{B_{2r}} g(x)^{p(x)-\delta} dx \leq C(p_{\log}^\pm, n) \left( \int_{B_{2r}} |\nabla \tilde{u}|^{p(x)-\delta} dx + |B_{2r}| \right).$$  (5.25)

We can now apply Lemma 4.18 to get a Lipschitz test function $v_\lambda \in W^{1, \infty}_0(\Omega_{2r})$. Using this in (5.20) and then multiplying by $\lambda^{-1-\delta}$ and integrating over $(0, \infty)$ with respect to $\lambda$, for $F_\lambda := \{ x \in \Omega_{2r} : g(x) \leq \lambda \}$, we get

$$I_1 - I_2 = I_3 - I_4,$$  

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where

\[ I_1 := \int_0^\infty \lambda^{-1-\delta} \int_{F_\lambda} \langle A(x, \nabla u), \nabla \lambda \rangle \, d\lambda, \quad I_2 := \int_0^\infty \lambda^{-1-\delta} \int_{F_\lambda} \langle |f|^p(x)^{-2} f, \nabla \lambda \rangle \, d\lambda. \]

\[ I_3 := \int_0^\infty \lambda^{-1-\delta} \int_{F_\lambda} \langle |f|^p(x)^{-2} f, \nabla \lambda \rangle \, d\lambda. \]

\[ I_4 := \int_0^\infty \lambda^{-1-\delta} \int_{F_\lambda} \langle A(x, \nabla u), \nabla \lambda \rangle \, d\lambda. \]

Let us now estimate each of the terms as follows:

**Estimate for** \( I_1 \): Applying Fubini’s theorem, we get

\[ I_1 = \frac{1}{\delta} \int_{B_{2r}} g^{-\delta} \langle A(x, \nabla u), \nabla \hat{u} \rangle \, dx = \frac{1}{\delta} \left( \int_{B_r} + \int_{D_1} + \int_{D_2} \right) g^{-\delta} \langle A(x, \nabla u), \nabla \hat{u} \rangle \, dx, \]

where we have set

\[ D_1 := \left\{ x \in B_{2r} \setminus B_r : M_{< \epsilon} (|\nabla \hat{u}| + \mu)^{\frac{1}{q}} (x) \leq \delta M_{< \epsilon} (|\nabla u|^q + \mu^q) \chi_{B_{2r}} (x) \right\}, \quad (5.26) \]

\[ D_2 := B_{2r} \setminus (D_1 \cup B_r). \quad (5.27) \]

**Estimate on** \( B_r \): Since \( \phi \equiv 1 \) on \( B_r \), we see that

\[ \int_{B_r} g^{-\delta} \langle A(x, \nabla u), \nabla \hat{u} \rangle \, dx \geq \int_{B_r} g^{-\delta} (\mu^2 + |\nabla u|^2)^{\frac{p(x)-2}{2}} |\nabla u|^2 \, dx \]

\[ \geq \left\{ \begin{array}{ll} \int_{B_r} g^{-\delta} |\nabla u|^p(x) \, dx & \text{if } \mu = 0, \\ \int_{B_r} g^{-\delta} |\nabla u|^p(x) \, dx - \frac{|B_r|}{\mu^q} & \text{if } \mu \neq 0. \end{array} \right. \quad (5.28) \]

We only prove this theorem for the case \( \mu \neq 0 \) since the case \( \mu = 0 \) can be obtained in the same way. We now apply (5.22) to obtain the estimate:

\[ |B_r| + \int_{B_r} g^{-\delta} |\nabla u|^p(x) \, dx \geq \int_{B_r} g^{-\delta} M_{< \epsilon} (|\nabla u|^q \chi_{B_r}) \frac{p(x)}{q} \, dx. \]

By definition of the maximal function, we have

\[ g(x) \leq C_1 M_{< \epsilon} (|\nabla u|^q \chi_{B_r})^{1/q} (x) + C_2 \left( \int_{B_{2r}} |\nabla u|^q \, dx \right)^{1/q} + C_3 \mu \quad \text{for all } x \in B_{r/2}. \]

Define the set

\[ G := \left\{ x \in B_{r/2} : C_1 M_{< \epsilon} (|\nabla u|^q \chi_{B_r})^{1/q} (x) \geq C_2 \left( \int_{B_{2r}} |\nabla u|^q \, dx \right)^{1/q} + C_3 \mu \right\}. \]
Thus for $x \in G$, we see that $g(x) \leq 2C_1 \mathcal{M}_{<2F}((|\nabla u|^q \chi_{B'_r})^{1/q}(x)$. Using this, we get

$$
\int_{B_{2r}} g^{-\delta} \mathcal{M}_{<2F}((|\nabla u|^q \chi_{B'_r})^{\frac{p(x)}{q}} dx \geq \int_{G} g^{-\delta} \mathcal{M}_{<2F}((|\nabla u|^q \chi_{B_r})^{\frac{p(x)}{q}} dx
\geq \int_{B_{2r}} |\nabla u|^{p(x)-\delta} dx - \int_{B_{r}\setminus G} \mathcal{M}_{<2F}((|\nabla u|^q \chi_{B_r})^{\frac{p(x)}{q}} dx
\geq \int_{B_{2r}} |\nabla u|^{p(x)-\delta} dx - \int_{B_{2r}} \left(\int_{B_{2r}} |\nabla u|^q dx\right)^{\frac{p(x)-\delta}{q}} dy - \mu |B_r|.
$$

We shall now estimate $\int_{B_{2r}} \left(\int_{B_{2r}} |\nabla u|^q dx\right)^{\frac{p(x)-\delta}{q}} dy$ as follows: fix any $y \in B_{2r}$ and set $t := \frac{p_{B_{2r}} - \delta}{q} > 1$, then we have

$$
\left(\int_{B_{2r}} |\nabla u|^q dx\right)^{\frac{p(x)-\delta}{q}} \leq \left(\int_{B_{2r}} \left(|\nabla u| + 1\right)^{\frac{p(x)-\delta}{q}} dx\right)^{\frac{p(x)-\delta}{q}}
\leq \left(\int_{B_{2r}} \left(|\nabla u| + 1\right)^{\frac{p(x)-\delta}{q}} dx\right)^{\frac{p(x)-p_{B_{2r}}}{q}}
\leq \left(\int_{B_{2r}} |\nabla u|^{\frac{p(x)-\delta}{q}} dx\right)^{t} \left(\frac{1}{|B_{2r}|} M^u\right)^{\frac{p(x)-p_{B_{2r}}}{q}}
\leq \left(\int_{B_{2r}} |\nabla u|^{\frac{p(x)-\delta}{q}} dx\right)^{t} + 1.
\tag{5.30}
$$

Thus using (5.30) into (5.29), we get

$$
\int_{B_{r}} g^{-\delta} \langle A(x, \nabla u), \nabla \bar{u} \rangle dx \geq \int_{B_{2r}} |\nabla u|^{p(x)-\delta} dx - |B_{r}| \left(\int_{B_{2r}} |\nabla u|^{\frac{p(x)-\delta}{q}} dx\right)^{t} - \frac{|B_{r}|}{\mu^\delta}.
\tag{5.31}
$$

**Estimate on $D_1$**: Recall that $\mu \in (0, 1)$, after using (2.1), we get

$$
\int_{D_1} g^{-\delta} \langle A(x, \nabla u), \nabla \bar{u} \rangle dx \leq \int_{D_1} \mathcal{M}_{<2F}((|\nabla \bar{u}|^q + \mu^q)^{\frac{1-\delta}{q}} \mathcal{M}_{<2F}((|\nabla u|^q + \mu^q) \chi_{B_{2r}})^{\frac{p(x)-1}{q}} dx
\leq \delta^{1-\delta} \int_{D_1} \mathcal{M}_{<2F}((|\nabla \bar{u}|^q + \mu^q) \chi_{B_{2r}})^{\frac{p(x)-1}{q}} dx
\leq \delta^{1-\delta} \left(\int_{B_{2r}} |\nabla u|^{p(x)-\delta} dx + |B_{2r}|\right).
\tag{5.32}
$$

**Estimate on $D_2$**: Using the expansion $\nabla \bar{u} = \phi \nabla u + u \nabla \phi$, we see that

$$
\int_{D_2} g^{-\delta} \langle A(x, \nabla u), \nabla \phi \rangle dx \geq 0.
$$
Hence we can ignore this term and bound the other term from above as follows:

\[
\int_{D_2} g^{-\delta} \langle A(x, \nabla u), \nabla \phi \rangle u \ dx \lesssim \int_{D_2} g^{-\delta} (\mu^{p(x)-1} + |\nabla u|^{p(x)-1}) \left| \frac{u}{2^F} \right| \ dx
\]

\[\lesssim \int_{D_2} M_{<2F}(|\nabla \tilde{u}|^q + \mu^q)^{-\delta} M_{<2F}((|\nabla \tilde{u}|^q + \mu^q)^{\chi_{B_{2F}}}) \left| \frac{u}{2^F} \right| \ dx\]

\[\lesssim \delta^{-\delta} \int_{D_2} M_{<2F}((|\nabla \tilde{u}|^q + \mu^q)^{\chi_{B_{2F}}}) \left| \frac{u}{2^F} \right| \ dx.\]

We can now apply Young’s inequality along with Corollary 4.9 to get

\[
\int_{D_2} g^{-\delta} \langle A(x, \nabla u), \nabla \phi \rangle u \ dx \lesssim \varepsilon_1 \int_{B_{2^F}} |\nabla u|^{p(x)-\delta} \ dx +
\]

\[+ \varepsilon_1 |B_{2^F}| \left\{ \int_{B_{2^F}} \left| \frac{u}{2^F} \right|^{p^+_{\partial B_{2^F}} - \delta} \ dx + 1 \right\}.\]

To control the term \( \int_{B_{2^F}} \left| \frac{u}{2^F} \right|^{p^+_{\partial B_{2^F}} - \delta} \ dx \), we apply the Sobolev-Poincaré inequality with \( s := \sqrt{\frac{n+1}{n}} \) (see [36] or [4, Theorem 3.3] and references therein for the details) to obtain

\[
\int_{B_{2^F}} \left| \frac{u}{2^F} \right|^{p^+_{\partial B_{2^F}} - \delta} \ dx \lesssim \left( \int_{B_{2^F}} |\nabla u|^{p^+_{\partial B_{2^F}} - \delta} \ dx \right)^{\frac{s(p^+_{\partial B_{2^F}} - \delta)}{p^+_{\partial B_{2^F}} - \delta}}.\]

(5.34)

This is possible since (5.23) implies \( \frac{p^+_{\partial B_{2^F}} - \delta}{s} \leq s \), hence \( \frac{p^+_{\partial B_{2^F}} - \delta}{s} \geq \frac{n(p^+_{\partial B_{2^F}} - \delta)}{n+1} \geq \frac{n(p^+_{\partial B_{2^F}} - \delta)}{n+(p^+_{\partial B_{2^F}} - \delta)} \), which implies \( \left( \frac{p^+_{\partial B_{2^F}} - \delta}{s} \right)^s = p^+_{B_{2^F}} - \delta \).

Thus we can further bound (5.34) using the log-Hölder continuity of \( p(\cdot) \) to obtain:

\[
\left( \int_{B_{2^F}} |\nabla u|^{p^+_{\partial B_{2^F}} - \delta} \ dx \right)^{\frac{s(p^+_{\partial B_{2^F}} - \delta)}{p^+_{\partial B_{2^F}} - \delta}} \lesssim \left( \int_{B_{2^F}} |\nabla u|^{p(x)-\delta} + 1 \ dx \right)^{\frac{s(p(x)-\delta)}{p(x)-\delta}} \left( \int_{B_{2^F}} |\nabla u|^{\frac{p(x)-\delta}{s}} + 1 \ dx \right)^s
\lesssim \left( \frac{1}{|B_{2^F}|} M^n \right)^{\frac{s(p(x)-\delta)}{p(x)-\delta}} \left( \int_{B_{2^F}} |\nabla u|^{\frac{p(x)-\delta}{s}} + 1 \ dx \right)^s
\lesssim \left( \int_{B_{2^F}} |\nabla u|^{\frac{p(x)-\delta}{s}} \ dx \right) + 1.\]

(5.35)

Thus combining (5.35) and (5.34) into (5.33), we get

\[
\int_{D_2} g^{-\delta} \langle A(x, \nabla u), \nabla \phi \rangle u \ dx \lesssim \varepsilon_1 \int_{B_{2^F}} |\nabla u|^{p(x)-\delta} \ dx + C(\varepsilon_1)|B_{2^F}| \left\{ \left( \int_{B_{2^F}} |\nabla u|^{\frac{p(x)-\delta}{s}} \ dx \right)^s + 1 \right\}.\]

(5.36)
**Estimate for $I_2$:** Proceeding as in the proof of Theorem 5.2, after applying Fubini’s theorem, (5.25) and Theorem 4.13, we get

$$I_2 \leq \frac{C(\varepsilon_2)}{\delta} \int_{B_{2r}} |f|^{p(x)-\delta} \, dx + \frac{\varepsilon_2}{\delta} \int_{B_{2r}} |\nabla u|^{p(x)-\delta} \, dx + \frac{\varepsilon_2}{\delta} |B_{2r}|. \quad (5.37)$$

**Estimate for $I_3$ and $I_4$:** Using the bound $|\nabla v_\lambda| \leq C\lambda$, followed by applying Fubini’s theorem, (5.25) and Theorem 4.13, we get

$$I_3 \leq \int_{B_{2r}} |\nabla u|^{p(x)-\delta} \, dx + \int_{B_{2r}} |f|^{p(x)-\delta} \, dx + |B_{2r}|, \quad (5.38)$$

$$I_4 \leq \int_{B_{2r}} |\nabla u|^{p(x)-\delta} \, dx + |B_{2r}|. \quad (5.39)$$

Combining (5.31), (5.32) and (5.36)–(5.39), for $\kappa := \min\{t, s\}$ where $t = \frac{p_{B_{2r}} - \delta}{q}$ as used to obtain (5.31) and $s = \sqrt{\frac{n+1}{n}}$ as used to obtain (5.34), we get

$$\int_{B_{\frac{r}{2}}} |\nabla u|^{p(x)-\delta} \, dx \leq C(\varepsilon_1)|B_{2r}| \left( \int_{B_{2r}} |\nabla u|^{\frac{p(x)\kappa}{\kappa - \delta}} \, dx \right)^{\kappa} + (\delta^{1-\delta} + \varepsilon_1 + \varepsilon_2 + \delta) \int_{B_{2r}} |\nabla u|^{p(x)-\delta} \, dx$$

$$+ (C(\varepsilon_2) + \delta) \int_{B_{2r}} |f|^{p(x)-\delta} \, dx + (1 + \delta^{1-\delta} + \delta + C(\varepsilon_1))|B_{2r}| + \frac{|B_{2r}|}{\mu^\delta}.$$

Choosing $\varepsilon_1, \varepsilon_2$ small followed by $\delta \in (0, \delta_3)$ with $\delta_3$ small, for some $\theta \in \left(0, \frac{1}{2}\right)$, we get

$$\int_{B_{\frac{r}{2}}} |\nabla u|^{p(x)-\delta} \, dx \leq \left( \int_{B_{2r}} |\nabla u|^{\frac{p(x)\kappa}{\kappa - \delta}} \, dx \right)^{\kappa} + \theta \int_{B_{2r}} |\nabla u|^{p(x)-\delta} \, dx + \int_{B_{2r}} |f|^{p(x)-\delta} \, dx + \frac{1}{\mu^\delta} + 1.$$

In the case $\mu = 0$, we can eliminate the term $\frac{1}{\mu^\delta}$ above because of (5.28). We can now apply Lemma 4.15 to obtain the desired higher integrability. □

### 5.5. Estimates for the homogeneous problem

In this section, we obtain useful higher integrability results for the homogeneous problem that was proved in [10, Lemma 3.5].

**Theorem 5.5 ([10]).** Let $\Omega$ be a $(\gamma, \sigma, S_0)$-quasiconvex domain for some $\gamma > 0$, $\sigma > 0$ and $S_0 > 0$. Let $u \in W^{1,p(\cdot)}_0(\Omega)$ solve (1.1) and $w \in W^{1,p(\cdot)}(\Omega_r)$ solve (5.16) with $r \leq \frac{R_0}{2}$ where

$$R_0 := \min \left\{ \frac{1}{2M^n}, \frac{1}{2M^w}, \frac{1}{2M_0}, \frac{1}{\omega_n^{1/n}}, S_0, 1 \right\} \quad \text{and} \quad \rho(R_0) \leq \frac{1}{2n} < 1.$$
Recall that $\rho$ is the log-Hölder modulus of continuity function for $p(\cdot)$.

Then there exists $\sigma_2 = \sigma_2(\Lambda_0, \Lambda_1, p_{\log}^+, n) \in (0, 4(p^--1)]$ such that for any $\tilde{\sigma} \in (0, \sigma_2]$, $t > 0$ and any $0 < 2\tilde{r} \leq r$ with $B_{2\tilde{r}}(x_0) \subset B_r$, there holds:

$$\int_{B_{\tilde{r}}(x_0)} |\nabla w|^{p(x)(1+\tilde{\sigma})} \, dx \leq C \left( \int_{B_{2\tilde{r}}(x_0)} |\nabla w|^{p(x)t} \, dx \right)^{\frac{1}{1+\tilde{\sigma}}} + 1,$$

where the constant $C = C(\Lambda_0, \Lambda_1, p_{\log}^+, n, t)$.

As a consequence, we obtain the following important corollary.

**Corollary 5.6.** Under the assumptions of Theorem 5.5, we have the ameliorated estimate for any $2\tilde{r} < r$ such that $B_{2\tilde{r}}(x_0) \subset B_r$ with $2\tilde{r} \leq R_9$:

$$\int_{B_{\tilde{r}}(x_0)} |\nabla w|^{p(x)} \, dx \leq C \left( \int_{B_{2\tilde{r}}(x_0)} (|\nabla w| + 1) \, dx \right)^{p_{\tilde{r}}} + 1,$$

for some $C = C(p_{\log}^+, \Lambda_0, \Lambda_1, n)$.

**Proof.** From Hölder’s inequality and Theorem 5.5, setting $t = \frac{1}{p^+}$, we get

$$\int_{B_{\tilde{r}}(x_0)} |\nabla w|^{p(x)} \, dx \leq \left( \int_{B_{2\tilde{r}}(x_0)} (|\nabla w| + 1) \, dx \right)^{p_{\tilde{r}}} + 1,$$

which proves the Corollary. \qed

We can further ameliorate the estimates in Theorem 5.5 and Corollary 5.6 as follows:

**Lemma 5.7.** Under the assumptions of Theorem 5.5, there exists $R_{10} = R_{10}(p_{\log}^+, \Lambda_0, \Lambda_1, n, M_0) \in (0, R_9/2)$ such that for any solution $w \in W^{1,p(\cdot)}(\Omega_{4r})$ with $4r < R_{10}$ solving

$$\begin{cases}
\text{div} A(x, \nabla w) = 0 & \text{in } \Omega_{4r}, \\
w \in u + W_0^{1,p(\cdot)}(\Omega_{4r}),
\end{cases}$$

the following estimates hold:

$$\int_{\Omega_3r} |\nabla w|^{p_{\tilde{r}}(1+\frac{\tilde{\sigma}}{4})} \, dx \leq C \int_{\Omega_{4r}} |\nabla w|^{p(x)} \, dx + 1,$$

$$\int_{\Omega_3r} |\nabla w|^{p_{\tilde{r}}(1+\frac{\tilde{\sigma}}{4})} \, dx \leq C \left( \int_{\Omega_{4r}} |\nabla w|^{p(x)} \, dx \right)^{\frac{1}{1+\tilde{\sigma}}} + 1,$$

$$\int_{\Omega_3r} |\nabla w|^{p_{\tilde{r}}(1+\frac{\tilde{\sigma}}{4})} \, dx \leq C \left( \int_{\Omega_{4r}} |\nabla w|^{p(x)} \, dx \right)^{\frac{1}{1+\tilde{\sigma}}} + 1,$$

where the constants $C = C(\Lambda_0, \Lambda_1, p_{\log}^+, n)$.  

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Proof. We make the following choice for $R_{10}$:

$$p_{14, r}^+ - p_{14, r}^- \leq \rho(8r) \leq \rho(2R_{10}) \leq \frac{\sigma_2}{4} \quad \text{and} \quad R_{10} \leq \frac{R_9}{2}.$$  

Then for any $x \in \Omega_{4r}$, we see that

$$p_{14, r}^+ \leq p(x) \left(1 + \frac{p_{14, r}^+ - p_{14, r}^-}{p^-}\right) \leq p(x)(1 + \rho(8r)). \quad (5.43)$$

and from the restriction of $\sigma_2 \leq 4(p^- - 1)$ (see Theorem 5.5), we get

$$p_{14, r}^+ \left(1 + \frac{\sigma_2}{4}\right) \leq (p(x) + p_{14, r}^+ - p_{14, r}^-) \left(1 + \frac{\sigma_2}{4}\right) \leq p(x) \left(1 + \frac{\sigma_2}{4}\right) + (p_{14, r}^+ - p_{14, r}^-)p^- \leq p(x) \left(1 + \frac{\sigma_2}{4} + \rho(8r)\right). \quad (5.44)$$

Since $R_{10} \leq \frac{1}{4Mw}$, making use of the log-Hölder continuity of $p(\cdot)$, we get

$$\left(\int_{B_{4r}} |\nabla w|^p(x) \, dx\right)^{\rho(8r)} \leq \left(\frac{1}{|B_{4r}|} M^w\right)^{\rho(8r)} \leq C(p_{\log, n}). \quad (5.45)$$

First estimate in (5.42): Using Lemma 2.8, we get

$$\int_{\Omega_{4r}} |\nabla w|^{p_{14, r}^+} \, dx \lesssim \int_{B_{4r}} |\nabla w|^{p_{14, r}^+} \, dx \lesssim \int_{B_{4r}} |\nabla w|^{p(x)(1+\rho(8r))} \, dx + 1 \quad (5.46)$$

Second estimate in (5.42): Again making use of Lemma 2.8, we get

$$\int_{\Omega_{4r}} |\nabla w|^{p_{14, r}^+ \left(1 + \frac{\sigma_2}{4}\right)} \, dx \lesssim \int_{B_{4r}} |\nabla w|^{p(x)(1+\frac{\sigma_2}{4} + \rho(8r))} \, dx + 1 \quad (5.47)$$

Third estimate in (5.42): Using Hölder’s inequality, we obtain the following chain of estimate:

$$\int_{\Omega_{4r}} |\nabla w|^{p_{14, r}^+ - \delta(1 + \frac{\sigma_2}{4})} \, dx \lesssim \left(\int_{\Omega_{4r}} |\nabla w|^{p(x)} \, dx\right)^{1 + \frac{\sigma_2}{4}} + 1 \quad (5.48)$$
To obtain the last inequality above, we observed that \( \frac{p_{\Omega R}^+ - \delta}{p_{\Omega R}^+} \leq 1 \) holds.

This proves the lemma. \( \square \)

5.6. Approximation by constant exponent operator

**Remark 5.8.** In order to simplify the exposition, we only consider the following two situations in this section:

- **In the interior case,** we have \( \Omega_R = B_R \).
- **In the boundary case,** we only consider regions of the form \( \Omega_R(x_0) = B_R(x_0) \cap \Omega \) with \( x_0 \in \partial \Omega \).

This simplifies the exposition for this section.

Let us consider the problem (5.2) in \( \Omega_{8r} \) with \( 8r < R_0 \) where \( R_0 \) is defined in Definition 5.9, i.e., we have the following PDE:

\[
\begin{cases}
\text{div } A(x, \nabla w) = 0 \quad \text{in } \Omega_{8r}, \\
w \in u + W_0^{1,p(\cdot)}(\Omega_{8r}).
\end{cases}
\] (5.49)

Let us define \( B = B(x, \zeta) : \Omega_{8r} \times \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
B(x, \zeta) := A(x, \zeta)(\mu^2 + |\zeta|^2)^{\frac{p_{\Omega_{8r}}^+ - p(x)}{2}}.
\] (5.50)

then the following holds for every \( \eta, \zeta \in \mathbb{R}^n \) and all \( x \in \Omega_{8r} \) (see [11] for the details):

\[
(\mu^2 + |\zeta|^2)^{\frac{1}{2}} |D_\zeta B(x, \zeta)| + |B(x, \zeta)| \leq 3 \Lambda_1 (\mu^2 + |\zeta|^2)^{\frac{p_{\Omega_{8r}}^+ - 1}{2}},
\] (5.51)

\[
\langle D_\zeta B(x, \zeta) \eta, \eta \rangle \geq \frac{\Lambda_0}{2} (\mu^2 + |\zeta|^2)^{\frac{p_{\Omega_{8r}}^+ - 2}{2}} |\eta|^2.
\] (5.52)

Now define \( \overline{B} = \overline{B}(\zeta) : \mathbb{R}^n \to \mathbb{R}^n \) which denotes the integral average of \( B(\cdot, \zeta) \) on \( B_{\Omega_{8r}}^+ \), such that

\[
\overline{B}(\zeta) := \int_{B_{\Omega_{8r}}^+} B(x, \zeta) \, dx.
\] (5.53)

It is easy to see that \( \overline{B} \) also satisfies (5.51) and (5.52) with \( B(x, \zeta) \) replaced by \( \overline{B}(\zeta) \). Moreover a
direct calculation now gives

$$\sup_{\zeta \in \mathbb{R}^n} \frac{|B(x, \zeta) - \overline{B}(\zeta)|}{(\mu^2 + |\zeta|^2)^{\frac{p^*}{2} - 1}} = \sup_{\zeta \in \mathbb{R}^n} \left| \frac{B(x, \zeta)}{(\mu^2 + |\zeta|^2)^{\frac{p^*}{2} - 1}} - \frac{B(x)}{(\mu^2 + |\zeta|^2)^{\frac{p^*}{2} - 1}} \right| \int_{B^+_r} \frac{A(x, \zeta)}{(\mu^2 + |\zeta|^2)^{\frac{p^*}{2} - 1}} \, dx$$

$$= \Theta(A, B^+_r)(x),$$

where $\Theta(A, B^+_r)(x)$ is defined in (2.8).

From the choice of $R_0$ (see Definition 5.9), we see that Lemma 5.7 is applicable and thus $w \in W^{1, p^+}_{\Omega_{3r}}(\Omega_{3r})$. We now consider the following problem:

$$\begin{cases} 
\text{div} B(\nabla v) = 0 & \text{in } \Omega_{3r}, \\
v \in w + W^{1, p^+}_{\Omega_{3r}}(\Omega_{3r}).
\end{cases}$$

Given $w$ as in (5.49) and $v$ as in (5.55), we see that there is a constant $C = C(p^+_{\log}, \Lambda_0, \Lambda_1, n)$ such that the following energy estimates hold trivially:

$$\int_{\Omega_{3r}} |\nabla v|^{p^+_{\Omega_{3r}}} \, dx \leq C \left( \int_{\Omega_{3r}} |\nabla w|^{p^+_{\Omega_{3r}}} + 1 \, dx \right),$$

$$\int_{\Omega_{3r}} |\nabla v - \nabla w|^{p^+_{\Omega_{3r}}} \, dx \leq C \left( \int_{\Omega_{3r}} |\nabla w|^{p^+_{\Omega_{3r}}} + 1 \, dx \right).$$

Using (5.17), we see that for any $q \leq p^+_{\Omega_{3r}}$, there holds

$$\int_{\Omega_{3r}} |\nabla v|^q \, dx \leq C(M^w + 1) =: M^v_1,$$

$$\int_{\Omega_{3r}} |\nabla v - \nabla w|^q \, dx \leq C(M^w + 1) =: M^v_2.$$

Let us now define

$$M^v := \max\{M^v_1, M^v_2\}. \quad (5.56)$$

5.7. Fixing a few more constants

The first constant that we define concerns the higher integrability exponent:

$$\sigma_0 = \sigma_0(n, p^+_{\log}, \Lambda_0, \Lambda_1) := \min\{\sigma_1, \sigma_2\}, \quad (5.57)$$

where $\sigma_1$ is from Theorem 5.4 and $\sigma_2$ is from Theorem 5.5.
Definition 5.9. Let us now define $R_0 = R_0(p^\pm_\log, \Lambda_1, \Lambda_1, n, M_0)$, where $M_0$ is defined in (5.14) to satisfy all the following properties:

- $R_0 \leq \min \left\{ \frac{1}{4M_0} \cdot \frac{1}{4M^e} \cdot \frac{1}{4M^w} \cdot \frac{1}{4} \cdot \frac{S_0}{2} \cdot \frac{R_8}{2} \cdot \frac{R_{10}}{2} \right\}$, where $M_0$ is from (5.14), $M^u$ is from (5.15), $M^w$ is from (5.17), $R_8$ is from Theorem 5.4, $R_{10}$ is from Lemma 5.7, and $S_0 > 0$ is a universal constant.

- $\rho(2R_0) \leq \min \left\{ \sqrt{\frac{n+1}{n}} - 1, \frac{\Lambda_0}{2\Lambda_1}, \frac{\sigma_0}{4}, \frac{1}{2n}, \frac{1}{4} \right\}$ where $\rho$ is the modulus of continuity of $p(\cdot)$.

- $\omega(2R_0) \leq \min \left\{ \frac{q^-\sigma_0}{8}, \frac{q^-\tilde{\sigma}}{2}, \frac{(\varphi^-)^2}{4q^+}, \frac{\sigma_0}{4}, \frac{1}{4} \right\}$ where $\omega$ is the modulus of continuity of $q(\cdot)$ and the exponent $\tilde{\sigma}$ is chosen to be $\tilde{\sigma} := \min \left\{ \frac{q^- - 1}{2}, 1 \right\}$.

Definition 5.10. For any $\varepsilon \in (0, 1)$, let us now set $\delta_0 = \delta_0(n, p^\pm_\log, q_\log^\pm, \Lambda_0, \Lambda_1, \varepsilon)$ such that the following holds:

- $\delta_0 = \min \left\{ \delta_1, \delta_2, \delta_3 \right\}$ where $\delta_1$, $\delta_2$ and $\delta_3$ are from Theorem 5.2, Theorem 5.3 and Theorem 5.4, respectively.

- $\delta_0 \leq \delta_4$ where $\delta_4$ is defined in Theorem 5.13.

Definition 5.11. For any $\varepsilon \in (0, 1)$, let us now fix the exponent $\gamma_0 = \gamma_0(n, p^\pm_\log, q_\log^\pm, \Lambda_0, \Lambda_1, \varepsilon)$ such that the following holds:

- $\gamma_0 = \min \left\{ \gamma_2, \gamma_3, \gamma_4 \right\}$ where $\gamma_2$, $\gamma_3$ and $\gamma_4$ are from Theorem 5.3, Theorem 5.13 and Theorem 5.15, respectively.

Remark 5.12. We point out that the above constants $\delta_0$ and $\gamma_0$ are independent of $q_\log^\pm$ in Section 5. In Section 6, however, $\delta_0$ and $\gamma_0$ are dependent on $q_\log^\pm$, see Lemma 6.2.

5.8. Estimates satisfied by the constant exponent approximation in (5.55)

Theorem 5.13. For any $\sigma \in (0, 1/4)$ and $S_0 > 0$, let $\Omega$ be a $(\gamma, \sigma, S_0)$-quasiconvex domain for a $\gamma \in (0, \gamma_3)$ with $\gamma_3$ to be chosen and let $r$ be such that $8r < R_0$ with $R_0$ as defined in Definition 5.9. Let $u \in W^{1,p(\cdot)}_0(\Omega)$, $w \in W^{1,p(\cdot)}(\Omega_{8r})$ and $v \in W^{1,p^+_{\log}(\Omega_{8r})}$ solve (1.1), (5.49) and (5.55), respectively and fix any $\lambda > 1$. Since $u = 0$ on $\partial\Omega$, we extend $u = 0$ on $\Omega^c$ followed by $w = u$.
are satisfied, we see that, we get such that for any \( \delta \in (0, \delta_4) \) and \( \gamma \in (0, \gamma_3) \), the following holds:

\[
\int_{\Omega_{8r}} |\nabla u|^{p(x) - \delta} \, dx \leq \lambda \quad \text{and} \quad \int_{\Omega_{8r}} |f|^{p(x) - \delta} \, dx \leq \gamma \lambda, 
\]

(5.58)

then there exists a constant \( C = C(\Lambda_0, \Lambda_1, p_{\log}^\pm, n, \sigma) \) such that the following conclusions hold:

\[
\int_{\Omega_{3r}} |\nabla v|^{p_{\Omega_{8r}}^\pm - \delta} \, dx + \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq C \lambda,
\]

\[
\int_{\Omega_{3r}} |\nabla w - \nabla v|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq \varepsilon \lambda.
\]

**Proof.** Let us first prove \( \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq C \lambda \): Applying Hölder’s inequality and Jensen’s inequality along with (5.42) and Corollary 5.6, we get

\[
\int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq \left( \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm} \, dx \right)^{\frac{p_{\Omega_{8r}}^\pm - \delta}{p_{\Omega_{8r}}^\pm}} \leq \left( \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm} \, dx + 1 \right)^{\frac{p_{\Omega_{8r}}^\pm - \delta}{p_{\Omega_{8r}}^\pm}} \leq \left( \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm} \, dx + 1 \right)^{\frac{\rho_{\Omega_{8r}}^\pm - \delta}{\rho_{\Omega_{8r}}^\pm}}.
\]

(5.59)

From the restriction \( 8r \leq R_0 \), we see that the following bound holds:

\[
\left( \int_{B_{8r}} |\nabla w|^{p(x) - \delta} + 1 \, dx \right)^{\frac{p_{\Omega_{8r}}^\pm - \delta}{p_{\Omega_{8r}}^\pm}} \leq \left( \frac{1}{|B_{8r}|} M^w \right)^{C \rho(8r)} \leq C.
\]

(5.60)

Since all the assumptions of Theorem 5.3 are satisfied, we see that \( \int_{\Omega_{8r}} |\nabla w|^{p(x) - \delta} \, dx \leq C \lambda \) holds for all \( \delta \in (0, \delta_2) \) with \( \delta_2 \) as in Theorem 5.3. Using this and (5.60) into (5.59), we get the estimate:

\[
\int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq C \lambda.
\]

(5.61)

Let us now prove \( \int_{\Omega_{3r}} |\nabla v|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq C \lambda \): Since \( B(\cdot) \) is a constant exponent operator, we can apply Theorem 5.2 such that for any \( \delta \in (0, \delta_1) \) with \( \delta_1 \) as in Theorem 5.2, the following estimate holds:

\[
\int_{\Omega_{3r}} |\nabla v|^{p_{\Omega_{8r}}^\pm - \delta} \, dx \leq C(n, p_{\log}^\pm, \Lambda_0, \Lambda_1) \left( \int_{\Omega_{3r}} |\nabla w|^{p_{\Omega_{8r}}^\pm - \delta} \, dx + 1 \right).
\]

(5.62)
Alternatively, we can also apply [4, Corollary 3.5] to obtain (5.62) for suitably small $\delta$. Using (5.61) into (5.62) gives the desired estimate for any $\delta \leq \min\{\delta_1, \delta_2\}$ where $\delta_1$ is coming from applying Theorem 5.2 to (5.55) and $\delta_2$ is obtained from Theorem 5.3.

Let us now prove $\int_{\Omega_3r} |\nabla w - \nabla v|^{\tilde{p}_{isr}, -\delta} \, dx \leq \varepsilon \lambda$: Since $w - v \in W_0^{1, p_{isr}}(\Omega_{3r})$, we shall define

$$g(x) := \max \left\{ M_{< 6r}(|\nabla w - \nabla v|^q)^{1/q}(x), \frac{|w(x) - v(x)|}{d(x, \partial \Omega_{3r})} \right\}.$$

Using Lemma 4.17 and Lemma 4.18, we get a Lipschitz function denoted by $v_\lambda$ which is a valid test function for (5.55). Using this as a test function, we get

$$\int_{\Omega_{3r}} \langle B(\nabla w) - B(\nabla v), \nabla v \rangle \, dx = \int_{\Omega_{3r}} \langle B(\nabla w) - B(x, \nabla w), \nabla v_\lambda \rangle \, dx +$$

$$+ \int_{\Omega_{3r}} \langle B(x, \nabla w) - A(x, \nabla w), \nabla v_\lambda \rangle \, dx. \tag{5.63}$$

Define the set $F_\lambda := \{ x \in \Omega_{3r} : g(x) \leq \lambda \}$, now multiplying (5.63) by $\lambda^{-1-\delta}$ and integrating over $(0, \infty)$, we obtain

$$I_1 = I_2 + I_3 + I_4 + I_5 + I_6, \tag{5.64}$$

where

- $I_1 := \int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(\nabla w) - B(\nabla v), \nabla w - \nabla v \rangle \, dx.$
- $I_2 := \int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(\nabla w) - B(x, \nabla w), \nabla w - \nabla v \rangle \, dx.$
- $I_3 := \int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(x, \nabla w) - A(x, \nabla w), \nabla w - \nabla v \rangle \, dx.$
- $I_4 := -\int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(\nabla v) - B(\nabla v_\lambda), \nabla v_\lambda \rangle \, dx.$
- $I_5 := \int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(x, \nabla w) - B(x, \nabla w), \nabla v_\lambda \rangle \, dx.$
- $I_6 := \int_0^{\infty} \lambda^{-1-\delta} \int_{F_\lambda} \langle B(x, \nabla w) - A(x, \nabla w), \nabla v_\lambda \rangle \, dx.$

Estimate for $I_1$: Applying Fubini’s theorem and Young’s inequality and following the calculation of (5.10), we get

$$\int_{\Omega_{3r}} |\nabla w - \nabla v|^{\tilde{p}_{isr}, -\delta} \, dx \leq C(\varepsilon_1)\delta I_1 + \varepsilon_1 \int_{\Omega_{3r}} |\nabla w|^{\tilde{p}_{isr}, -\delta} \, dx + \varepsilon_1 |\Omega_{3r}|. \tag{5.65}$$
Estimate for $I_2$: Using Fubini's theorem and Young’s inequality along with (5.54), we get

$$|I_2| \leq \frac{1}{\delta} \int_{\Omega_{3r}} \Theta(A, B_{3r}^+) (\mu^2 + |\nabla w|^2) \frac{r_{1 \alpha_r}^{-\delta}}{\delta} \frac{1}{\delta} |\nabla w - \nabla v|^{\alpha_r} \, dx$$

$$\leq \frac{\varepsilon_2}{\delta} \int_{\Omega_{3r}} |\nabla w - \nabla v|^{p_{1 \alpha_r} - \delta} \, dx + \frac{c(\varepsilon_2)}{\delta} \int_{\Omega_{3r}} \Theta(A, B_{3r}^+) r_{1 \alpha_r}^{-\delta} (1 + |\nabla w|)^{p_{1 \alpha_r} - \delta} \, dx.$$

Let us now estimate $I_{2,2}$ as follows:

$$I_{2,2} \leq \left( \int_{\Omega_{3r}} \Theta(A, B_{3r}^+) \frac{r_{1 \alpha_r}^{-\delta} (1+\sigma_0/4)}{\sigma_0/4} \, dx \right)^{\sigma_0/4} \left( \int_{\Omega_{3r}} |\nabla w|^{p_{1 \alpha_r} - \delta} (1+\sigma_0/4) \, dx + 1 \right)^{\lambda_0/4}. \tag{5.66}$$

Using Theorem 5.4 and (5.59), we also get

$$\left( \int_{\Omega_{3r}} |\nabla w|^{p_{1 \alpha_r} - \delta} (1+\sigma_0/4) \, dx + 1 \right)^{\lambda_0/4} \leq \left( \int_{\Omega_{3r}} |\nabla w|^{p(x)} \, dx \right)^{\sigma_0/4} + 1 \leq \lambda. \tag{5.67}$$

Using Lemma 2.8, we have the bound

$$|\Omega_{3r} \setminus B_{3r}^+| \leq C(n) \left( \frac{96 \gamma r}{\sigma^4} \right)^n. \tag{5.69}$$

As a consequence, after applying Young’s inequality and using the bound $\Theta(A, B_{3r}^+) \leq 2\Lambda_1$, we get

$$\int_{\Omega_{3r}} \Theta(A, B_{3r}^+) \frac{r_{1 \alpha_r}^{-\delta} (1+\sigma_0/4)}{\sigma_0/4} \, dx \leq \frac{B_{3r}}{|\Omega_{3r}|} \left( 2\Lambda_1 \right)^{\sigma_0/4} \int_{B_{3r}^+} \Theta(A, B_{3r}^+) \, dx +$$

$$+ \left( 2\Lambda_1 \right)^{\sigma_0/4} \left( \frac{r_{1 \alpha_r}^{-\delta} (1+\sigma_0/4)}{\sigma_0/4} \right) \frac{|\Omega_{3r} \setminus B_{3r}^+|}{|\Omega_{3r}|}. \tag{5.68}$$

Using Lemma 2.8 along with (2.9) and (5.69), we get

$$\int_{\Omega_{3r}} \Theta(A, B_{3r}^+) \frac{r_{1 \alpha_r}^{-\delta} (1+\sigma_0/4)}{\sigma_0/4} \, dx \leq C_0 (\gamma + \gamma^n) \leq C_0 \gamma, \tag{5.70}$$

where $C_0 = C_0(\Lambda_0, \Lambda_1, \sigma, \mu_{\log}, n)$. Here we have used $\gamma < 1$ to bound $\gamma^n < \gamma$.

Combining (5.70) and (5.68) into (5.67), we get $I_{2,2} \leq \gamma^{\sigma_0/4} \lambda$ which is then substituted into (5.66) to get

$$|I_2| \leq \frac{\varepsilon_2}{\delta} \int_{\Omega_{3r}} |\nabla w - \nabla v|^{p_{1 \alpha_r} - \delta} \, dx + \frac{c(\varepsilon_2)}{\delta} |\Omega_{3r}| \gamma^{\sigma_0/4} \lambda. \tag{5.71}$$

Recall the definition of $\sigma_0$ from (5.57).
Estimate for $I_3$: Applying Fubini’s theorem followed by Young’s inequality, we get

$$|I_3| \leq \frac{\varepsilon_3}{\delta} \int_{\Omega_{3r}} |\nabla w - \nabla \eta|^p |\nabla \eta - \delta| dx + \frac{C(\varepsilon_3)}{\delta} \int_{\Omega_{3r}} |\mathcal{B}(x, \nabla w) - \mathcal{A}(x, \nabla w)|^{\frac{p}{p-1}} dx. \quad (5.72)$$

We shall now proceed with estimating the second term in $(5.72)$ as follows: Denote $\Omega_{3r} = \{ x \in \Omega_{3r} : \mu^2 + |\nabla w|^2 > 0 \}$, then using $(5.50)$, we see that

$$|\mathcal{B}(x, \nabla w) - \mathcal{A}(x, \nabla w)| = |\mathcal{A}(x, \nabla w)| \left| 1 - (\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}} \right|.$$  

For each $x \in \Omega_{3r}$, in view of the mean value theorem applied to $(\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}}$, there exists $\xi_x \in [0, 1]$ such that we get

$$(\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}} - 1 = \frac{p}{p-\beta} (\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}-1} \xi_x \log(\mu^2 + |\nabla w|^2). \quad (5.73)$$

This implies

$$|\mathcal{A}(x, \nabla w)| \left| 1 - (\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}} \right| \leq \frac{\Lambda_1 \rho(8r)}{2} (\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}-1_{\xi_x + p(x) - 1}} \log(\mu^2 + |\nabla w|^2).$$

Let us now define the sets

$$\overline{\Omega}_{3r} := \{ x \in \Omega_{3r} : |\nabla w(x)| \leq 1 \} \quad \text{and} \quad \Omega_{3r}^2 := \{ x \in \Omega_{3r} : |\nabla w(x)| > 1 \}. \quad (5.74)$$

Recall that $\mu \leq 1$ and hence using the inequality $t^\gamma |\log t| \leq \max \left\{ \frac{1}{e}, 2^\beta \log 2 \right\}$ which holds for all $t \in (0, 2]$ and any $\beta > 0$, we get for $x \in \overline{\Omega}_{3r}$

$$|\mathcal{A}(x, \nabla w)| \left| 1 - (\mu^2 + |\nabla w|^2)^{\frac{p}{p-\beta}} \right| \leq \frac{\Lambda_1 \rho(8r)}{2} \max \left\{ \frac{1}{e^\frac{p}{e}}, 2^{\frac{2p+1}{2p-1}} \log 2 \right\}. \quad (5.75)$$

To obtain the above estimate, with $\beta(x) := \frac{t_x(p_{1\beta r} - p(x)) + p(x) - 1}{2}$, there holds

$$\frac{p^- - 1}{2} \leq \beta(x) \leq \frac{p_{1\beta r} - 1}{2} \leq \frac{p^+ - 1}{2}.$$ 

Hence using $(5.74)$ and combining $(5.75)$ into $(5.73)$, we get

$$|\mathcal{B}(x, \nabla w) - \mathcal{A}(x, \nabla w)| \leq \chi_{\Omega_{3r}} \frac{\Lambda_1 \rho(8r)}{2} \max \left\{ \frac{1}{e^\frac{p}{e}}, 2^{\frac{2p+1}{2p-1}} \log 2 \right\} + \chi_{\Omega_{3r}} \rho(8r)|\nabla w|^{|\frac{p}{p-\beta} - \beta(x)|_{1_x + p(x) - 1}} \log(\mu^2 + |\nabla w|^2). \quad (5.76)$$
Combining (5.76) and (5.72), we get

\[
\int_{\Omega_{3r}} |B(x, \nabla w) - A(x, \nabla w)|^{\frac{p_{13r}^+}{p_{13r}^+ - \delta}} dx \leq \rho(8r)^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} |\Omega_{3r}| (1 + J),
\]  

(5.77)

where \( J := \int_{\Omega_{3r}} |\nabla w|^{p_{13r}^+ - \delta} \log(e + |\nabla w|) |\Omega_{3r}^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}}. \) Using the inequality \( \log(e + ab) \leq \log(e + a) + \log(e + b) \) for \( a, b > 0, \) we get

\[
J \leq \int_{\Omega_{3r}} |\nabla w|^{p_{13r}^+ - \delta} \left[ \log \left( e + \frac{|\nabla w|^{p_{13r}^+ - \delta}}{|\nabla w|^{p_{13r}^+ - \delta}_{\Omega_{3r}}} \right) \right]^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} dx + \int_{\Omega_{3r}} |\nabla w|^{p_{13r}^+ - \delta} \left[ \log \left( e + \frac{|\nabla w|^{p_{13r}^+ - \delta}}{|\nabla w|^{p_{13r}^+ - \delta}_{\Omega_{3r}}} \right) \right]^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} dx

=: J_1 + J_2.

Estimate for \( J_1: \) We now apply Lemma 4.16 with \( \beta = \frac{p_{13r}^+ - \delta}{p_{13r}^+} s = \frac{p_{13r}^+ - \delta}{p_{13r}^+ - \delta} \) and \( f = |\nabla w|^{p_{13r}^+ - \delta} \) to get

\[
J_1 \leq \left( \int_{\Omega_{3r}} |\nabla w|^{p_{13r}^+ - \delta} dx \right)^{\frac{p_{13r}^+ - \delta}{p_{13r}^+}} \lambda. \tag{5.59}
\]

Estimate for \( J_2: \) We see that

\[
\log \left( e + \left( |\nabla w|^{p_{13r}^+ - \delta} \right)_{\Omega_{3r}} \right) \leq \log \left( e + C \left( |\nabla w|^{p(x) - \delta} + 1 \right)_{\Omega_{3r}} \right) \leq \log \left( \frac{1}{|\Omega_{3r}|} \right) + \log(e|\Omega_{3r}| + M^w).
\]

Since we have \( R_0 \leq \frac{1}{2M^w}, \) we get

\[
\log \left( e + \left( |\nabla w|^{p_{13r}^+ - \delta} \right)_{\Omega_{3r}} \right) \leq \log \left( \frac{1}{|\Omega_{3r}|} \right) + \log(M^w) + 1 \leq \log \left( \frac{1}{8r} \right).
\]

Thus we can estimate \( J_2 \) by:

\[
J_2 \leq \int_{\Omega_{3r}} |\nabla w|^{p_{13r}^+ - \delta} \left[ \log \left( \frac{1}{8r} \right) \right]^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} dx \leq \left( \log \left( \frac{1}{8r} \right) \right)^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} \lambda. \tag{5.79}
\]

Now combining (5.79) and (5.78) into (5.77), we get

\[
\int_{\Omega_{3r}} |B(x, \nabla w) - A(x, \nabla w)|^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} dx \leq |\Omega_{3r}| \left( \rho(8r) \log \left( \frac{1}{8r} \right) \right)^{\frac{p_{13r}^+ - \delta}{p_{13r}^+ - 1}} \lambda. \tag{5.80}
\]

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Hence using (5.80) into (5.72) and from (2.7), we get

\[ |I_3| \leq \frac{\varepsilon_3}{\delta} \int_{\Omega_4} |\nabla w - \nabla v|^{p_{\Omega_4}^+-\delta} \, dx + \frac{C(\varepsilon_3)}{\delta} |\Omega_4| \gamma^{p_{\Omega_4}^+-\delta} \lambda. \]  

(5.81)

**Estimate for I₄:** Applying Fubini's theorem and Young's inequality, (5.62), (5.1) and the standard maximal function bound, we get

\[ I_4 \leq \int_{\Omega_4} |\nabla w - \nabla v|^{p_{\Omega_4}^+-\delta} \, dx + |\Omega_4|. \]  

(5.82)

**Estimate for I₅:** Again making use of Fubini's theorem and Young's inequality, we get

\[ I_5 \leq \int_{\Omega_5} g(x)^{p_{\Omega_5}^+-\delta} \, dx + \int_{\Omega_4} |B(\nabla w) - B(x, \nabla w)|^{p_{\Omega_4}^+-\delta} \, dx. \]  

(5.83)

Note that the last term in (5.83) is exactly the same as the last term in (5.66) and thus we can directly use (5.71) to obtain the estimate

\[ I_5 \leq \int_{\Omega_5} |\nabla w - \nabla v|^{p_{\Omega_5}^+-\delta} \, dx + |\Omega_4| \gamma^{p_{\Omega_4}^+-\delta} \lambda. \]  

(5.84)

**Estimate for I₆:** Exactly as the estimate for I₅, we analogously get (see (5.72) for the details)

\[ I_6 \leq \int_{\Omega_6} |\nabla w - \nabla v|^{p_{\Omega_6}^+-\delta} \, dx + |\Omega_4| \gamma^{p_{\Omega_4}^+-\delta} \lambda. \]  

(5.85)

We now combine (5.71), (5.81), (5.82), (5.84) and (5.85) into (5.65) and use (5.64) to get

\[
\int_{\Omega_4} |\nabla w - \nabla v|^{p_{\Omega_4}^+-\delta} \, dx \leq C(\varepsilon_1)(\varepsilon_2 + \varepsilon_3 + \delta) \int_{\Omega_4} |\nabla w - \nabla v|^{p_{\Omega_4}^+-\delta} \, dx + C(\varepsilon_1, \varepsilon_2, \varepsilon_3) |\Omega_4| \left( \gamma^{\frac{\sigma_0}{\varepsilon_{\Omega_4}}} + \gamma^{\frac{\varepsilon_{\Omega_4}}{\varepsilon_{\Omega_4}}} \right) \lambda \\
+ \varepsilon_1 \int_{\Omega_4} |\nabla w|^{p_{\Omega_4}^+-\delta} \, dx + (\varepsilon_1 + \delta) |\Omega_4|. 
\]

Choose $\varepsilon_1$ small followed by $\varepsilon_2$ and $\varepsilon_3$, then finally $\delta$ and $\gamma$ small and lastly making use of (5.59), the proof follows.

\[ \square \]

**5.9. $L^\infty$ gradient estimates up to the boundary**

We need $L^\infty$ bound for the gradient in both the interior and boundary case for the constant coefficient equation (5.55). Note that the operator $\overline{B}$ defined in (5.53) is independent of $x$. Hence in the interior case, we get
Theorem 5.14. Let $\Omega_{3r} = B_{3r}$ and (5.58) hold. Suppose that $v \in W^{1,p}_{0\Omega_{3r}}(B_{3r})$ with $B_{3r} \subset \Omega$ solves
\[
\begin{aligned}
\begin{cases}
\text{div}(\nabla v) = 0 \\
v \in w + W^{1,p}_{0\Omega_{3r}}(B_{3r})
\end{cases}
\end{aligned}
\tag{5.86}
\]
then the following holds:
\[
\|\nabla v\|_{L^\infty(B_r)} \leq \left( \int_{B_{3r}} |\nabla v|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq C\lambda}.
\]

Proof. We only need to show $\left( \int_{B_{3r}} |\nabla v|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq C\lambda}$ as the first bound is from [17]. Thus from the energy estimate in Lemma 5.1 applied to (5.86), we get
\[
\left( \int_{B_{3r}} |\nabla v|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq (\int_{B_{3r}} |\nabla w|^{p_{0\Omega_{3r}}} \, dx + 1)^{\frac{1}{p_{0\Omega_{3r}}}}.
\]

Similar to how (5.59) was obtained, we see that
\[
\left( \int_{B_{3r}} |\nabla w|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq \int_{B_{3r}} |\nabla w|^{p_{0\Omega_{3r}}} \, dx \leq 1.
\]

Using the calculations from (5.86), we can conclude the following bound holds:
\[
\left( \int_{B_{3r}} |\nabla v|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq \int_{B_{3r}} |\nabla w|^{p_{0\Omega_{3r}}} \, dx + 1 \leq C\lambda.
\]

\]

In the boundary case, we need to make use of the fact that our domain $\Omega$ is $(\gamma, \sigma, S_0)$-quasiconvex. Let us consider the following problem with $x_0 \in \partial\Omega$:
\[
\begin{aligned}
\begin{cases}
\text{div}(\nabla \overline{V}) = 0 \\
\overline{V} = 0
\end{cases}
\end{aligned}
\tag{5.87}
\]
Here $F^*_r$ is defined in Lemma 2.6 and we have extended $\overline{V} = 0$ on $B_{2r}(x_0) \setminus F^*_r(x_0)$.

Theorem 5.15 ([9]). For any $\varepsilon \in (0,1)$, there exists $\gamma_4 = \gamma_4(\Lambda_0, \Lambda_1, p_{\log}, n, \varepsilon)$ such that if (5.58) holds and $\Omega$ is a $(\gamma_4, \sigma, S_0)$-quasiconvex domain for any $\gamma \in (0, \gamma_4)$ and $\sigma \in (0, 1/4)$, then the unique weak solution of (5.87) satisfies the estimate:
\[
\|\nabla \overline{V}\|_{L^\infty(\partial F^*_r(x_0))} \leq C \left( \int_{F^*_r(x_0)} |\nabla \overline{V}|^{p_{0\Omega_{3r}}} \, dx + 1 \right)^{\frac{1}{p_{0\Omega_{3r}}} \leq C\lambda},
\]

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where the constant $C = C(\Lambda_0, \Lambda_1, p_{\text{log}}^+, n, \sigma)$.

Furthermore, suppose that $v \in W^{1, p_{\text{log}}^+}(\Omega_{3r})$ solves (5.55) with the bound $\int_{\Omega_{3r}} |\nabla v|^{p_{\text{log}}^+ - \delta} \leq C \lambda$, then there holds for any $\delta \in (0, 1/4)$:

$$
\int_{B_{2r}(x_0)} |\nabla v|^{p_{\text{log}}^+ - \delta} \, dx \leq C \lambda \quad \text{and} \quad \int_{B_{2r}(x_0)} |\nabla v - \nabla w|^{p_{\text{log}}^+ - \delta} \, dx \leq \varepsilon \lambda. \quad (5.89)
$$

**Proof.** Similar to how we obtained the estimate in Theorem 5.14, we can analogously obtain

$$
\left( \int_{F_{\gamma}(x_0)} |\nabla v|^{p_{\text{log}}^+} \, dx + 1 \right)^{\frac{1}{p_{\text{log}}^+}} \leq C \lambda.
$$

The first bound in (5.88) is from [9, Lemma 4.2] and applying Hölder’s inequality to [9, Corollary 4.4] proves (5.89) (see also [5, Corollary 2.7] for an expanded version of the calculation). \qed

### 6. Covering arguments

Let $\gamma \in (0, \gamma_0)$, $\delta \in (0, \delta_0)$, $\sigma \in (0, 1/4)$ and $S_0 > 0$ be given, where $\gamma_0$ and $\delta_0$ are from Definition 5.11 and Definition 5.10, respectively. Assume that $(p(\cdot), \mathcal{A}, \Omega)$ is $(\gamma, \sigma, S_0)$-vanishing in the sense of Definition 2.9. Let $r \leq R_0/4$, where $R_0$ is from Definition 5.9 and let $\Omega_{kr} = \Omega_{kr}(x_0)$ for $x_0 \in \Omega$ and $k \in \mathbb{N}$, we then set

$$
\mathcal{M}^*(\nabla u)(x) := \mathcal{M} \left( |\nabla u|^{(p(\cdot) - \delta) \frac{\lambda}{\delta} \frac{1+\sigma}{\mathcal{A}_{12r}} \chi_{\Omega_{12r}}}(x) \right),
$$

$$
\mathcal{M}^*_1(\nabla u)(x) := \mathcal{M} \left( \left[ |\nabla u|^{(p(\cdot) - \delta) \frac{\lambda}{\delta} \frac{1+\sigma}{\mathcal{A}_{12r}} \chi_{\Omega_{12r}}}(x) \right]^{\frac{1+\sigma}{1}} \right),
$$

where $\sigma$ is given in Definition 5.9, $\mathcal{M}$ is the standard maximal operator and $\chi$ is the standard characteristic function. For a fixed $\varepsilon \in (0, 1)$ and $N > 1$, we further define

$$
\lambda_0 := \frac{1}{\varepsilon} \left\{ \int_{\Omega_{r}} |\nabla u|^{p(x) - \delta} \, dx + \left( \int_{\Omega_{r}} |f|^{p(x) - \delta} \chi_{\Omega_{12r}} \, dx \right)^{\frac{1+\sigma}{1}} + 1 \right\}, \quad (6.1)
$$

and for $k \in \mathbb{N} \cup \{0\}$, the upper-level sets

$$
C_k := \left\{ x \in \Omega_r : \mathcal{M}^*(\nabla u)(x) > N^{k+1} \lambda_0 \right\},
$$

$$
D_k := \left\{ x \in \Omega_r : \mathcal{M}^*(\nabla u)(x) > N^k \lambda_0 \right\} \cup \left\{ x \in \Omega_r : \mathcal{M}^*_1(\nabla u)(x) > \gamma N^k \lambda_0 \right\}.
$$

Note that $\varepsilon$ and $N$ are to be chosen later depending only on $\Lambda_0, \Lambda_1, p_{\text{log}}^+, q_{\text{log}}^+, n, \sigma$.

We now verify two assumptions in Lemma 4.20.
Lemma 6.1. There exists a constant $N_1 = N_1(\Lambda_0, \Lambda_1, P_{\log}^\pm, q_{\log}^\pm, n, \sigma) > 1$ such that for any fixed $N \geq N_1$ and $k \in \mathbb{N} \cup \{0\}$, there holds

$$|C_k| < \frac{\varepsilon}{(1000)^n}|B_r|.$$  

Proof. The proof is similar to that for [10, Lemma 4.1]. 

Lemma 6.2. There exist $N_2 = N_2(\Lambda_0, \Lambda_1, P_{\log}^\pm, q_{\log}^\pm, n, \sigma) > 1$, $\gamma_0 = \gamma_0(\Lambda_0, \Lambda_1, P_{\log}^\pm, q_{\log}^\pm, n, \sigma, \varepsilon) \in (0, 1/4)$ and $\delta_0 = \delta_0(\Lambda_0, \Lambda_1, P_{\log}^\pm, q_{\log}^\pm, n, \sigma, \varepsilon) \in (0, 1/4)$ such that for any fixed $N \geq N_2$, $k \in \mathbb{N} \cup \{0\}$, $y_0 \in C_k$, and $r_0 \leq \frac{\sigma r}{1000}$ if

$$\frac{|C_k \cap B_{r_0}(y_0)|}{|B_{r_0}(y_0)|} \geq \varepsilon,$$

then $B_{r_0}(y_0) \cap \Omega_r \subset D_k$.

Proof. The proof is similar to that for [10, Lemma 4.2]. For the sake of convenience, we outline the proof.

Step 1. For simplicity, we write $\lambda_k = N^k \lambda_0 > 1$, where $N \geq N_2 > 1$. If not, suppose that $B_{r_0}(y_0) \cap \Omega_r \not\subset D_k$. Then there is a point $y_1 \in B_{r_0}(y_0) \cap \Omega_r$ such that $y_1 \not\in D_k$, that is,

$$\int_{B_r(y_1)} |\nabla u|^{(p(x) - \delta) \frac{\alpha(x)}{q_{\log}^\pm}} \chi_{\Omega_{2r}} \, dx \leq \lambda_k,$$

and

$$\left(\int_{B_r(y_1)} |\nabla u|^{(p(x) - \delta) \frac{\alpha(x)}{q_{\log}^\pm}} \chi_{\Omega_{2r}} \, dx\right)^{\frac{1}{1 + \phi}} \leq \gamma \lambda_k$$

for all $r > 0$.

Step 2. We divide the proof into two cases: the interior ($B_{9r_0}(y_1) \subset \Omega$) and boundary ($B_{9r_0}(y_1) \not\subset \Omega$) case. We only prove the boundary case since the interior one can be proved in a similar way.

Step 3. Let $B_{9r_0}(y_1) \not\subset \Omega$. We can find a boundary point $\tilde{y}_1 \in \partial \Omega \cap B_{9r_0}(y_1)$. Then it satisfies that

$$\Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{20r_0}(\tilde{y}_1) \quad \text{and} \quad \Omega_{\frac{4\sigma r_0}{\sigma}}(\tilde{y}_1) \subset \Omega_{\frac{8\sigma r_0}{\sigma}}(y_0) \subset \Omega_{2r}. \quad (6.3)$$

Then we have

$$\int_{\Omega_{\frac{4\sigma r_0}{\sigma}}(\tilde{y}_1)} |\nabla u|^{p(x) - \delta} \, dx \leq c_1 \lambda_k \quad \text{and} \quad \int_{\Omega_{\frac{4\sigma r_0}{\sigma}}(\tilde{y}_1)} |f|^{p(x) - \delta} \, dx \leq c_1 \delta \lambda_k$$
for some constant \( c_1 = c_1(\Lambda_0, \Lambda_1, p^\pm_{\log}, q^\pm_{\log}, n, \sigma) > 0 \).

**Step 4.** Applying Theorem 5.3, Theorem 5.13, and Theorem 5.15, there exist the constants \( \gamma_0 \) and \( \delta_0 \), both depending only on \( \Lambda_0, \Lambda_1, p^\pm_{\log}, q^\pm_{\log}, n, \sigma, \eta \), such that

\[
\int_{\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1)} |\nabla u - \nabla \tilde{V}|^{(p(x)-\delta)\frac{q(x)}{q_{14r}}} \ dx \leq c_1 \eta \lambda_k \frac{q_{14r}}{q_{14r}(\tilde{y}_1)},
\]

\[
\int_{\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1)} |\nabla w - \nabla \tilde{V}|^{\frac{p_{14r}}{\delta}} \ dx \leq c_1 \eta \lambda_k \frac{q_{14r}}{q_{14r}(\tilde{y}_1)},
\]

\[
\|\nabla \tilde{V}\|_{L^\infty(\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1))} \leq c_1 \lambda_k \frac{q_{14r}}{q_{14r}(\tilde{y}_1)}.
\]

**Step 5.** We next obtain

\[
\int_{\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1)} |\nabla u - \nabla \tilde{V}|^{(p(x)-\delta)\frac{q(x)}{q_{14r}}} \ dx \leq c_2 \eta \lambda_k \eta r_k,
\]

and

\[
\left\| \nabla \tilde{V}\right\|_{L^\infty(\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1))} \leq c_2 \lambda_k
\]

for some constant \( c_2 = c_2(\Lambda_0, \Lambda_1, p^\pm_{\log}, q^\pm_{\log}, n, \sigma) > 0 \).

**Step 6.** We now have

\[
C_k \cap \Omega_{\tau_0}(y_0) \subset \left\{ x \in \Omega_{\tau_0}(y_0) : \mathcal{M} \left( |\nabla u - \nabla \tilde{V}|^{(p(x)-\delta)\frac{q(x)}{q_{14r}}} \chi_{\Omega_{\tilde{\tau}_{\text{r}}}(\tilde{y}_1)} \right)(x) > \lambda_k \right\}
\]

provided that \( N \geq N_2 \geq \max \left\{ 2 \frac{\eta^+}{\eta^-}, (1 + c_2), 3^n \right\} \).

**Step 7.** We finally conclude, using (4.12), (6.5), (6.3), and (6.4), that

\[
|C_k \cap B_{r_0}(y_0)| \leq \frac{1}{\lambda_k} \int_{\Omega_{r_0}(y_0)} |\nabla u - \nabla \tilde{V}|^{(p(x)-\delta)\frac{q(x)}{q_{14r}}} \ dx \leq c_2 \eta \lambda_k r_0 \ |B_{r_0}(y_0)| < \varepsilon |B_{r_0}(y_0)|,
\]

by taking \( \eta \) sufficiently small. As a consequence \( \gamma_0 \) and \( \delta_0 \) are also determined, which is a contradiction to (6.2).

Applying Lemma 4.20, we finally obtain the following power decay estimate:
Corollary 6.3. Let \( N = \max\{N_1, N_2\} > 1 \), where \( N_1 \) and \( N_2 \) are given in Lemma 6.1 and Lemma 6.2, respectively. Then there exist \( \gamma_0 \in (0, 1/4) \) and \( \delta_0 \in (0, 1/4) \), both depending only on \( \Lambda_0, \Lambda_1, \rho_{\log}^+, \eta_{\log}^+ \), \( n, \sigma, \varepsilon \), such that
\[
|C_k| \leq \varepsilon \left(\frac{10}{\sigma}\right)^n |D_k| \quad \text{for } k \in \mathbb{N} \cup \{0\}.
\]
Moreover, by iteration, we obtain
\[
|\{x \in \Omega : \mathcal{M}^*(\nabla u)(x) > N^k \lambda_0\}|
\leq \varepsilon_1^n \left|\{x \in \Omega : \mathcal{M}^*(\nabla u)(x) > \lambda_0\}\right| + \sum_{i=1}^{k} \varepsilon_1^n \left|\{x \in \Omega : \mathcal{M}^*_{i+\sigma}(f)(x) > \gamma N^{k-i} \lambda_0\}\right|,
\]
where \( \varepsilon_1 := \varepsilon \left(\frac{10}{\sigma}\right)^n \).

7. Calderón-Zygmund type estimates

We are ready to prove our main theorems. In this section, we omit \( x_0 \) in the \( \Omega_r(x_0) \).

Proof of Theorem 3.1. We first recall Section 5.7. Note that \( \max \{M^u, M^v, M^\nu\} \lesssim M_0 \). Fix any \( x_0 \in \Omega \) and \( r \in \left(0, \frac{1}{C_0M_0}\right] \) with
\[
\frac{1}{C_0(n, \Lambda_0, \Lambda_1, \rho_{\log}^+, \eta_{\log}^+, \sigma, \varepsilon)} := \min \left\{ \frac{1}{4}, \frac{S_0}{2}, \frac{R_8}{2}, \rho^{-1}(d_1), \frac{\omega^{-1}(d_2)}{2}, \frac{\omega^{-1}(d_1)}{2} \right\},
\]
where
\[
d_1 := \min \left\{ \sqrt{\frac{n+1}{n} - 1}, \frac{\Lambda_0}{2\Lambda_1}, \frac{\sigma_0}{4}, \frac{1}{2n} \right\}, \quad d_2 := \min \left\{ \frac{q\sigma_0}{8}, \frac{q^{-\sigma_0}}{2}, \frac{(q^{-2})^2}{4q^2}, \frac{\sigma_0}{2}, \frac{1}{4} \right\},
\]
\[
\rho^{-1}(t) := \sup\{r \in (0, 1) : \rho(r) \leq t\}, \quad \omega^{-1}(t) := \sup\{r \in (0, 1) : \omega(r) \leq t\},
\]
for \( t > 0 \). Note that \( \rho^{-1} \) and \( \omega^{-1} \) is well defined by the definition of \( \rho \) and \( \omega \), respectively. Thus we can apply all results in Section 6.

Let \( \delta_0 \) be such that Lemma 6.2 and Corollary 6.3 holds and let \( \delta \in (0, \delta_0) \) be given. Set
\[
S := \sum_{k=1}^{\infty} N^{kq_0\varepsilon} \left|\left\{x \in \Omega_r : \mathcal{M}^*(\nabla u)(x) > N^k \lambda_0\right\}\right|.
\]
By Corollary 6.3, Lemma 4.21, Fubini’s theorem and Lemma 4.10, we deduce
\[
S \leq \sum_{k=1}^{\infty} \left(N^k q_1 \varepsilon_1\right)^k \left\{2|\Omega_r| + \frac{C}{(\gamma \lambda_0)^{\delta_{\varepsilon_1}} \int_{\Omega_2} \left|f\right|^{(p(x)-\delta)q(x)} + 1} \right\} dx.
\]
Now we select \( \varepsilon = \varepsilon(\Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm, n, \sigma) > 0 \) such that \( Nq^+ \varepsilon \left(\frac{10}{\sigma}\right)^n = Nq^+ \varepsilon_1 = \frac{1}{2} \) and a corresponding \( \gamma_0 \) and \( \delta_0 \), both depending only on \( \Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm, n, \sigma \). Then we see

\[
S \leq 2|\Omega_r| + C_{(\Lambda_0)}^{\bar{\Omega}_{4\varepsilon}^r} \int_{\Omega_{2r}} \left[ |f|^{(p(x)-\delta)q(x)} + 1 \right] \, dx. \tag{7.2}
\]

According to Lemma 4.21, (7.2), (6.1), and Hölder’s inequality, we obtain

\[
\int_{\Omega_r} |\nabla u|^{(p(x)-\delta)q(x)} \, dx \leq \int_{\Omega_r} M^* (\nabla u)^{\bar{q}_{\tilde{\Omega}_{4\varepsilon}}^r} \, dx \leq C \lambda_0^{\bar{\Omega}_{4\varepsilon}^r} \left( 1 + \frac{S}{|\Omega_r|} \right)
\]

\[
\leq C \left\{ \lambda_0^{\bar{\Omega}_{4\varepsilon}^r} + \int_{\Omega_{2r}} \left[ |f|^{(p(x)-\delta)q(x)} + 1 \right] \, dx \right\}
\]

\[
\leq C \left( \int_{\Omega_{4\varepsilon}} |\nabla u|^{(p(x)-\delta)q(x)} \, dx \right)^{\bar{q}_{\tilde{\Omega}_{4\varepsilon}}^r} + C \left( \int_{\Omega_{4\varepsilon}} |f|^{(p(x)-\delta)(1+\tilde{\sigma})} \, dx \right)^{\frac{\bar{q}_{\tilde{\Omega}_{4\varepsilon}}^r}{1+\tilde{\sigma}}}
\]

\[
+ C \int_{\Omega_{4\varepsilon}} |f|^{(p(x)-\delta)q(x)} \, dx + C
\]

\[
\leq C \left\{ \left( \int_{\Omega_{4\varepsilon}} |\nabla u|^{(p(x)-\delta)q(x)} \, dx \right)^{\bar{q}_{\tilde{\Omega}_{4\varepsilon}}^r} + \int_{\Omega_{4\varepsilon}} |f|^{(p(x)-\delta)q(x)} \, dx + 1 \right\},
\]

for some constant \( C = C(\Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm, n, \sigma) > 0 \). Here the last inequality above have used, in Definition 5.9, the fact that \( 1 + \tilde{\sigma} \leq q^- \leq \bar{q}_{\tilde{\Omega}_{4\varepsilon}}^r \).

We extend the local estimate (7.3) up to the boundary.

**Proof of Theorem 3.2.** We first choose \( r = \frac{1}{C_0 M_0} \), where \( C_0 \) and \( M_0 \) are given in (7.1). From the standard covering argument, we can find finitely many disjoint open balls \( \{ B_r(y_k) \}_{k=1}^m \), \( y_k \in \Omega \), such that \( \bar{\Omega} \subset \bigcup_{k=1}^m B_r(y_k) \). Note that for an integrable function \( f \) there exists a constant \( C = C(n) > 0 \) so that

\[
\sum_{k=1}^m \int_{\Omega_{4\varepsilon}(y_k)} f \, dx \leq C \int_{\Omega} f \, dx.
\]

Let \( \delta_0 \) be as in Theorem 3.1 and let \( \delta \in (0, \delta_0) \).

Then it follows from (7.3) that

\[
\int_{\Omega} |\nabla u|^{(p(x)-\delta)q(x)} \, dx \leq \sum_{k=1}^m \int_{\Omega_{4\varepsilon}(y_k)} |\nabla u|^{(p(x)-\delta)q(x)} \, dx
\]

\[
\lesssim \sum_{k=1}^m \left\{ r^n \left( \int_{\Omega_{4\varepsilon}(y_k)} |\nabla u|^{(p(x)-\delta)} \, dx \right)^{\bar{q}_{\tilde{\Omega}_{4\varepsilon}(y_k)}^r} + \int_{\Omega_{4\varepsilon}(y_k)} \left[ |f|^{(p(x)-\delta)q(x)} + 1 \right] \, dx \right\}
\]

\[
\lesssim r^{n(1-q^-)} \left( \int_{\Omega} \left[ |\nabla u|^{p(x)-\delta} + 1 \right] \, dx \right)^{q^+} + \int_{\Omega} \left[ |f|^{(p(x)-\delta)q(x)} + 1 \right] \, dx.
\]
From (5.15) and the definition of \( r \), we obtain

\[
\int_\Omega |\nabla u|^{p(x)q(x)} \, dx \lesssim \left( \int_\Omega |f|^{p(x)} \, dx \right)^{n(q^+ - 1) + q^+} + \int_\Omega |f|^{(p(x) - \delta)q(x)} \, dx + |\Omega| + 1. \tag{7.5}
\]

Let \( M^+ \) and \( M^- \) be any two constants such that additionally we have \( 1 < M^- \leq q^- \leq q(\cdot) \leq q^+ \leq M^+ < \infty \). Following the proof of Theorem 3.1, we see that \( \delta_0 \) can be chosen to depend on \( M^+ \) instead of \( q(\cdot) \). This, in particular, implies that we can choose \( \delta_0 \) independent of \( M^- \).

Let us now define \( r(x) := \frac{p(x) - \delta}{p(x)}q(x) \) for \( \delta \in (0, \delta_0) \) (it is important to note that we cannot take \( \delta = 0 \)), then we trivially have

\[
\begin{align*}
  r^- &\geq \left( \frac{p(x) - \delta}{p(x)} \right)^- M^- \quad \text{and} \quad r^+ \leq \left( \frac{p(x) - \delta}{p(x)} \right)^+ M^+.
\end{align*}
\]

Note that \( r(\cdot) \) is clearly log-Hölder continuous with the log-Hölder constants equivalent to the ones satisfied by \( q(\cdot) \).

Since all the estimates above are independent of \( M^- \) and \( \delta_0 \) is independent of \( M^- \), we can choose \( M^- \) small such that \( \left( \frac{p(x) - \delta_0}{p(x)} \right)^- M^- \leq 1 \). This in particular allows \( r^- = 1 \).

For this choice of the exponent \( r(\cdot) \), we conclude from (7.5) that

\[
\int_\Omega |\nabla u|^{p(x)r(x)} \, dx \leq C \left\{ \left( \int_\Omega |f|^{p(x)r(x)} \, dx \right)^{n(q^+ - 1) + q^+} + 1 \right\}
\leq C \left\{ \left( \int_\Omega |f|^{p(x)r(x)} \, dx \right)^{n(M^+ - 1) + M^+} + 1 \right\}
\]

for some constant \( C = C(A_0, A_1, p_{\log}^\pm, r_{\log}, M^+, n, \sigma, \Omega, S_0) > 0 \), which completes the proof. \( \square \)

**Appendix A. Proof of Theorem 4.13**

**Proof.** Let \( B = B_r \) be a ball of radius \( 2r < R_4 \leq 1 \) and \( \phi \in W^{1, s(\cdot)}(2B) \) be a function satisfying all the hypothesis. We can now apply Lemma 4.12 to obtain the estimate

\[
\int_B |\phi - (\phi)_B|^{s(x)} \, dx \leq \frac{n\pm}{n-\pm} \frac{1}{|B|} \int_B |\nabla \phi|^{s(x)} \, dx + 1.
\]

Using the trivial bound \( |N(\phi)| \leq |B| \), we get

\[
\int_B |\phi - (\phi)_B|^{s(x)} \, dx \leq \frac{n\pm}{n-\pm} \frac{1}{|N(\phi)|} \int_B |\nabla \phi|^{s(x)} \, dx + 1. \tag{A.1}
\]
The following bound holds:

\[ \int_B |\phi|^s(x) \, dx \leq \int_B |\phi - (\phi)_B|^s(x) \, dx + \int_B |(\phi)_B|^s(x) \, dx. \]  
(A.2)

The first term in the right of (A.2) can be controlled using (A.1). We shall now estimate the second term on the right on (A.2) as follows:

Set \( u(x) := |\phi(x) - (\phi)_B| \) and consider the cut-off function \( \eta \in C^\infty_c(2B) \) such that \( \eta \equiv 1 \) on \( \overline{B} \) and \( |\nabla \eta| \leq \frac{c}{\text{diam}(B)} \). Define \( \varpi(x) := \frac{u(x)\eta(x)}{|(\phi)_B|} \), then we see that \( \text{spt}(\varpi) \subset 2B \). Thus from the definition of \( N(\phi) \), we see that

\[ \varpi(x)^s(x) \geq \chi_{N(\phi)}(x), \quad \text{and} \quad \varpi(x)^s(x) = 1 \quad \text{for} \quad x \in N(\phi). \]

**Claim:** The following bound holds:

\[ \int_{2B} |\nabla (v\eta)|^s(x) \, dx \leq \int_{2B} |\nabla \phi|^s(x) + |B|. \]  
(A.3)

To prove this, we estimate as follows:

\[ \int_{2B} |\nabla (v\eta)|^s(x) \, dx \leq \int_{2B} \left( \frac{|\phi - (\phi)_B|}{\text{diam}(B)} \right)^s(x) \, dx + \int_{2B} |\nabla \phi|^s(x) \, dx + \int_{2B} |\nabla \phi|^s(x) \, dx + \text{diam}(B)^{-\frac{s+1}{2s}} |B| \max\{|(\phi)_B - (\phi)_B|^s 2B, \, |(\phi)_2B - (\phi)_B|^s 2B\}. \]

Observe that

\[ |(\phi)_2B - (\phi)_B| \leq \int_{2B} |(\phi)(x) - (\phi)_2B| \, dx \leq \text{diam}(B) \int_{2B} |\nabla \phi| \, dx \]

\[ \leq \text{diam}(B) \left( \int_{2B} |\nabla \phi|^s(x) \, dx \right)^{\frac{1}{s}}. \]

Using the log-Hölder continuity of \( s(\cdot) \) along with the restriction \( r \leq \frac{1}{M_4} \) with \( M_4 \) as in (4.19), we get

\[ \text{diam}(B)^{-\frac{s+1}{2s}} |B| \max\{|(\phi)_2B - (\phi)_B|^s 2B, \, |(\phi)_2B - (\phi)_B|^s 2B\} \]

\[ \leq \max \left\{ \text{diam}(B)^{s_2B - s_2B} \int_{2B} |\nabla \phi|^s 2B \, dx, |B|^{1 - \frac{s_2B}{s_2B} \left( \int_{2B} |\nabla \phi|^s 2B \, dx \right)^{\frac{s_2B}{s_2B}}} \right\} \]

\[ \leq \max \left\{ \text{diam}(B)^{s_2B - s_2B}, |B|^{1 - \frac{s_2B}{s_2B}} \right\} \int_{2B} |\nabla \phi|^s 2B \, dx \leq \int_{2B} |\nabla \phi|^s(x) \, dx + |B|. \]
This proves the Claim of estimate (A.3).

We can thus make use of Lemma 2.8 along with [19, Theorem 8.2.4] and (4.1) to get

$$|N(\phi)| \leq \int_{2B} \varpi(x)^{s(x)} \, dx \leq \max \left\{ \left( \int_{2B} |\nabla \varpi(x)|^{s(x)} \, dx \right)^{\frac{1}{s^*_{2B}}}, \left( \int_{2B} |\nabla \varpi(x)|^{s(x)} \, dx \right)^{\frac{1}{s^*_{2B}}} \right\}.$$  

Depending on the cases if $\int_{2B} |\nabla \varpi(x)|^{s(x)} \, dx \geq 1$ or $\int_{2B} |\nabla \varpi(x)|^{s(x)} \, dx < 1$ or $\int_{B} \phi(x) \, dx \geq 1$ or $\int_{B} \phi(x) \, dx \leq 1$, we can use the log-Hölder continuity of $s(\cdot)$ and (A.3) to obtain

$$\int_{B} |(\phi)|_{B}^{s(x)} \, dx \leq \frac{1}{|N(\phi)|} \int_{2B} |\nabla \phi(x)|^{s(x)} + 1 \, dx.$$  

This completes the proof of the Theorem. \qed

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