Sharp bounds on the spectrum of triangular Boolean matrices

V. Kaarnioja†

February 11, 2020

Abstract

Let $K_n$ be the set of all nonsingular $n \times n$ lower triangular Boolean $(0, 1)$ matrices.

Hong and Loewy (2004) introduced the numbers

$$c_n = \min \{ \lambda \mid \lambda \text{ is an eigenvalue of } XX^T, \ X \in K_n \}, \ n \in \mathbb{Z}_+.$$ A related family of numbers was considered by Ilmonen, Haukkanen, and Merikoski (2008):

$$C_n = \max \{ \lambda \mid \lambda \text{ is an eigenvalue of } XX^T, \ X \in K_n \}, \ n \in \mathbb{Z}_+.$$ These numbers can be used to bound the singular values of matrices belonging to $K_n$ and they appear, e.g., in the estimation of the spectral radii of GCD and LCM matrices as well as their lattice-theoretic generalizations. In this paper, it is shown that for odd $n$, one has the lower bound

$$c_n \geq \frac{1}{\sqrt{\frac{25}{\varphi^4} \nabla 4n + \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} - \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} - \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} + \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} + \frac{1}{\sqrt{\varphi^2} n (\varphi^2 - 2n)}}},$$

and for even $n$, one has

$$c_n \geq \frac{1}{\sqrt{\frac{25}{\varphi^4} \nabla 4n + \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} - \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} + \frac{2}{\sqrt{\varphi^2} n (\varphi^2 - 2n)} + \frac{1}{\sqrt{\varphi^2} n (\varphi^2 - 2n)}}},$$

where $\varphi$ denotes the golden ratio. These lower bounds improve the estimates derived previously by Mattila (2015) and Altmüller et al. (2016). The sharpness of these lower bounds is assessed numerically and it is conjectured that $c_n \sim 5 \varphi^{-2n}$ as $n \to \infty$. In addition, a new closed form expression is derived for the numbers $C_n$, viz.

$$C_n = \frac{1}{4} \csc \left( \frac{5n}{2} \right) = \frac{4n^2}{\pi^2} + \frac{4n}{\pi^2} + \left( \frac{1}{12} + \frac{1}{\pi^2} \right) + O\left( \frac{1}{n^2} \right), \ n \in \mathbb{Z}_+.$$ 1 Introduction

Let $K_n$ denote the set of all nonsingular $n \times n$ lower triangular Boolean $(0, 1)$ matrices. For example, $K_3$ consists of the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 1School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia (v.kaarnioja@unsw.edu.au). The author gratefully acknowledges the financial support from the Australian Research Council (DP180101356).
and it is easy to see that \( \#K_n = 2^{n(n-1)/2} \) for all \( n \in \mathbb{Z}_+ \).

Hong and Loewy \( \text{[2]} \) introduced the numbers
\[
c_n = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^T, \ X \in K_n\}, \quad n \in \mathbb{Z}_+.
\]
A closely related sequence was introduced by Ilmonen, Haukkanen, and Merikoski \( \text{[3]} \):
\[
C_n = \max\{\lambda \mid \lambda \text{ is an eigenvalue of } XX^T, \ X \in K_n\}, \quad n \in \mathbb{Z}_+.
\]
Let \( \sigma_{\min}(X) \) and \( \sigma_{\max}(X) \) denote the smallest and largest singular values of matrix \( X \), respectively. The numbers \( c_n \) and \( C_n \) are connected to the extremal singular values of matrices belonging to \( K_n \) via
\[
\min_{X \in K_n} \sigma_{\min}(X) = \sqrt{c_n} \quad \text{and} \quad \max_{X \in K_n} \sigma_{\max}(X) = \sqrt{C_n}, \quad n \in \mathbb{Z}_+.
\]

These numbers also appear in the estimation of the spectral radii of certain number-theoretic matrices as the following example illustrates.

**Example 1.1** (cf. \( \text{[3]} \)). Let \((P, \preceq, \wedge, 0)\) be a locally finite meet semilattice, where \( \preceq \) is a partial ordering on the set \( P \), \( \wedge \) denotes the meet (or greatest lower bound) of two elements in \( P \), and \( 0 \in P \) is the least element such that \( 0 \preceq x \) for all \( x \in P \). Let \( S = \{x_1, \ldots, x_n\} \subset P \) be a lower closed set such that \( x_i \preceq x_j \) only if \( i \leq j \). Let \( f : P \to \mathbb{R} \) be a function and define the \( n \times n \) meet matrix \( A \) elementwise by setting \( A_{i,j} = f(x_i \wedge x_j) \) for \( i,j \in \{1, \ldots, n\} \). Define the function
\[
J_{P,f}(x) = \sum_{0 \preceq z \preceq x} f(z)\mu(z, x) \quad \text{for all } x \in P,
\]
where \( \mu \) denotes the Möbius function of \( P \). If \( J_{P,f}(x) > 0 \) for all \( x \in S \), then
\[
\lambda_{\min}(A) \geq c_n \min_{x \in S} J_{P,f}(x) \quad \text{and} \quad \lambda_{\max}(A) \leq C_n \max_{x \in S} J_{P,f}(x).
\]

For example, in the case of the divisor lattice \((\mathbb{Z}_+, |, \gcd)\) and the identity function \( f(x) = x \) for \( x \in \mathbb{Z}_+ \), the function \( J_{P,f} \) is precisely Euler’s totient function. See \( \text{[3]} \) for a rigorous statement of this result and see \( \text{[2]} \) for the special case of greatest common divisor matrices.

Mattila \( \text{[4]} \) derived the following lower bounds for \( c_n \):
\[
c_n \geq \begin{cases} 
16 & \text{for even } n, \\
36 & \text{for odd } n.
\end{cases}
\]

The lower bounds \( (1.1) \) and \( (1.2) \) were subsequently improved in \( \text{[1]} \):
\[
c_n \geq \begin{cases} 
48 & \text{for even } n, \\
\frac{48}{n^2/2 + 48} & \text{for odd } n.
\end{cases}
\]

where \((F_n)_{n=1}^{\infty} \) denotes the Fibonacci sequence. However, a straightforward numerical investigation shows that the bounds \( (1.1)-(1.3) \) are not sharp. It is the goal of this article to remedy this situation by developing a new sharp lower bound for the numbers \( c_n \). In addition, a new characterization for the numbers \( C_n \) is also derived in this paper.

This paper is structured as follows. Section \( \text{[2]} \) begins with the development of a new sharp lower bound on Hong and Loewy’s numbers \( c_n \). In Subsection \( \text{[2.1]} \) it is shown that this lower bound can be expressed in a much simplified form, which is the main contribution of this paper. The sharpness of this lower bound is assessed by numerical experiments in Subsection \( \text{[2.2]} \) A novel characterization for the closely related sequence of Ilmonen–Haukkanen–Merikoski numbers \( C_n \) is proved in Section \( \text{[3]} \). Finally, some conclusions and thoughts about future work are given at the end of the paper.
2 Hong and Loewy’s numbers \( c_n \)

Altıntaş et al. [1] proved the following characterization

\[
c_n = \lambda_{\min}(Z_n^{-1}) \quad \text{for all } n \in \mathbb{Z}_+,
\]

(2.1)

where \( Z_n \) is the symmetric \( n \times n \) matrix defined elementwise by

\[
(Z_n)_{i,j} = \begin{cases} 
1 + \sum_{k=i+1}^{n} F_{k-i}^2 & \text{if } i = j, \\
(-1)^{i-j}(F_{i-j} + \sum_{k=j+1}^{n} F_{k-i} F_{k-j}) & \text{if } i < j, \\
(-1)^{i-j}(F_{i-j} + \sum_{k=i+1}^{n} F_{k-i} F_{k-j}) & \text{if } i > j,
\end{cases}
\]

for \( i, j \in \{1, \ldots, n\} \) and the sequence of Fibonacci numbers is defined by the recurrence relation \( F_0 = 0, F_1 = 1, \) and \( F_k = F_{k-1} + F_{k-2} \) for \( k \geq 2. \)

The following technical result will serve as the basis for the analysis in Subsection 2.1.

Lemma 2.1. It holds for all \( n \in \mathbb{Z}_+ \) that

\[
c_n \geq \frac{1}{\sqrt{1 + \sum_{i=2}^{n}(1 + F_i F_{i-1})^2 + 2 \sum_{i=2}^{n} \sum_{j=2}^{i} (F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j})^2}}.
\]

(2.2)

Proof. Let \( n \in \mathbb{Z}_+ \). By the characterization (2.1), it holds that

\[
c_n = \lambda_{\max}(Z_n)^{-1} = \|Z_n\|_2^{-1} \geq \|Z_n\|_F^{-1},
\]

(2.2)

where \( \| \cdot \|_2 \) denotes the spectral norm, \( \| \cdot \|_F \) is the Frobenius norm, and the final inequality is due to \( \| \cdot \|_2 \leq \| \cdot \|_F \). To prove the claim, it is sufficient to compute the value of the Frobenius norm appearing in (2.2).

Making use of the block structure of the matrices \( Z_n \), it is possible to write

\[
Z_1 = (1) \quad \text{and} \quad Z_n = \begin{pmatrix} a_n & b_n^T \\ b_n & Z_{n-1} \end{pmatrix} \quad \text{for } n \geq 1,
\]

where \( a_n = (Z_n)_{1,1} = 1 + F_n F_{n-1} \) and the \((n-1)\)-vector \( b_n = [(Z_n)_{2,1}, \ldots, (Z_n)_{n,1}]^T \in \mathbb{R}^{n-1} \)\) clearly satisfies

\[
\|b_n\|^2 = \sum_{j=2}^{n} \left(F_{j-1} + \sum_{k=j+1}^{n} F_{k-1} F_{k-j}\right)^2.
\]

Hence \( \|Z_n\|_F^2 = a_n^2 + 2\|b_n\|^2 + \|Z_{n-1}\|_F^2 \), which yields the recurrence relation

\[
\|Z_1\|_F^2 = 1,
\]

\[
\|Z_n\|_F^2 = \|Z_{n-1}\|_F^2 + (1 + F_n F_{n-1})^2 + 2 \sum_{j=2}^{n} \left(F_{j-1} + \sum_{k=j+1}^{n} F_{k-1} F_{k-j}\right)^2, \quad n \geq 2.
\]

This recurrence can be used to produce the expression

\[
\|Z_n\|_F^2 = 1 + \sum_{i=2}^{n} \left(1 + F_i F_{i-1}\right)^2 + 2 \sum_{i=2}^{n} \sum_{j=2}^{i} \left(F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j}\right)^2, \quad n \in \mathbb{Z}_+,
\]

which, together with the inequality (2.2), proves the assertion. \(\square\)

Lemma 2.1 gives a computable, albeit rather unwieldy, lower bound on the numbers \( c_n \). However, it is shown in the following section that this lower bound can be recast into a much simpler closed form expression.
2.1 Simplifying the lower bound on $c_n$

In this section, a closed form expression for the term inside the square root in Lemma 2.1 is derived. To this end, recall that the sequence of Lucas numbers can be defined by the recursion $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 2$. Moreover, it is convenient to extend both the Fibonacci numbers and the Lucas numbers to negative indices using the formulæ

$$ F_k = (-1)^{k+1}F_{-k} \quad \text{and} \quad L_k = (-1)^k L_{-k} \quad \text{for all } k \in \mathbb{Z}_. $$

Notice in particular that both the Fibonacci–Binet formula and the Lucas–Binet formula hold for all indices regardless of sign, i.e.,

$$ F_k = \frac{\phi^k - (-\phi)^{-k}}{\sqrt{5}} \quad \text{and} \quad L_k = \phi^k + (-\phi)^{-k} \quad \text{for all } k \in \mathbb{Z}, \quad (2.3) $$

where $\phi$ denotes the golden ratio.

The main result of this paper is given by the following theorem. It is a simplified version of the lower bound given in Lemma 2.1.

**Theorem 2.2.** It holds for all $n \in \mathbb{Z}_+$ that

$$ c_n \geq \frac{1}{\sqrt{\frac{1}{25} \phi^{-4n} + \frac{3+(-1)^n}{25} \phi^{-2n} - \frac{2}{5\sqrt{5}} n \phi^{-2n} + \frac{1+(-1)^n}{50} + \frac{2+(-1)^n}{25} \phi^{-2n} + \frac{2}{5\sqrt{5}} n \phi^{-2n} + \frac{1}{25} \phi^{-4n}}}.$$

**Proof.** The proof is based on simplifying the term inside the square root in Lemma 2.1.

The claim is clearly true with equality for $n = 1$. In the following analysis, let $n \geq i \geq j \geq 2$ be integers. Using the formulæ (2.3), it is straightforward to check that

$$ F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} = \frac{1}{5} \left( L_{2i-j} + \frac{5}{2} F_{j-1} - \frac{1}{2} (-1)^{i-j} L_{j-1} \right), $$

where the sum is taken to be 0 if the index set is empty. In consequence, it follows that

$$ \left( F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2 = \frac{1}{25} L_{2i-j}^2 + \frac{1}{5} F_{j-1} L_{2i-j} - \frac{1}{25} (-1)^{i-j} L_{j-1} L_{2i-j} + \frac{1}{4} F_{j-1}^2 - \frac{1}{10} (-1)^{i-j} F_{j-1} L_{j-1} + \frac{1}{100} L_{j-1}^2. \quad (2.4) $$

Using the summation formulæ

$$ \sum_{j=2}^{i} L_{2i-j}^2 = L_{2i-2} L_{2i-1} - L_i L_{i-1} \quad \sum_{j=2}^{i} F_{j-1}^2 = F_{i-1} F_i $$

$$ \sum_{j=2}^{i} F_{j-1} L_{2i-j} = (i-1) F_{2i-1} + \frac{L_{2i-2}}{5} + \frac{2}{5} (-1)^i \sum_{j=2}^{i} (-1)^j F_{j-1} L_{j-1} = (-1)^i \frac{F_{i+1}^2 - F_{i-2}^2}{4} $$

$$ \sum_{j=2}^{i} (-1)^j L_{j-1} L_{2i-j} = 2 + \frac{1+(-1)^i}{2} L_{2i-1} - L_{2i-2} \quad \sum_{j=2}^{i} L_{j-1}^2 = L_{i-1} L_i - 2 $$

\[\text{Page 4}\]
in conjunction with (2.4) yields that
\[
\sum_{j=2}^{i} \left( F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2 = \frac{1}{25} L_{2i-2} L_{2i-1} - \frac{3}{100} L_i L_{i-1} + \frac{1}{5} F_{2i-1} - \frac{1}{5} F_{2i-1}
\]
\[
+ \frac{1}{25} L_{2i-2} + \frac{1}{25} (-1)^i L_{2i-2} - \frac{1}{50} L_{2i-1} - \frac{1}{50} (-1)^i L_{2i-1}
\]
\[
+ \frac{1}{4} F_i F_{i-1} + \frac{1}{40} F_{i-2} - \frac{1}{40} F_{i+1} - \frac{1}{50}.
\]
Applying the summation formulae
\[
\sum_{i=2}^{n} L_{2i-2} L_{2i-1} = n - 3 + F_{4n-1}
\]
\[
\sum_{i=2}^{n} F_{2i-1} = n F_{2n+1} - (n + 1) F_{2n-1}
\]
\[
\sum_{i=2}^{n} L_{2i-2} = L_{2n-1} - 1
\]
\[
\sum_{i=2}^{n} (-1)^i L_{2i-2} = (-1)^n F_{n-1} L_n
\]
leads to the identity
\[
\sum_{i=2}^{n} \sum_{j=2}^{i} \left( F_{j-1} + \sum_{k=j+1}^{i} F_{k-1} F_{k-j} \right)^2
\]
\[
= \frac{3}{20} + \frac{(-1)^n}{8} + \frac{3}{200} (-1)^n + \frac{n}{50} + \frac{1}{40} F_{n-2} F_{n-1} + \frac{1}{4} F_n - \frac{1}{5} F_{2n}
\]
\[
- \frac{1}{40} F_{n+1} F_{n+2} - \frac{1}{5} F_{2n-1} - \frac{1}{5} n F_{2n-1} + \frac{1}{5} n F_{2n+1} + \frac{1}{25} F_{4n-1}
\]
\[
+ \frac{1}{25} (-1)^n F_{n-1} L_n - \frac{1}{20} L_{2n} - \frac{1}{50} (-1)^n F_{n-1} L_{n+1} + \frac{1}{25} L_{2n-1}.
\]
Meanwhile, it holds that
\[
1 + \sum_{i=2}^{n} (1 + F_i F_{i-1})^2 = n + 2 \sum_{i=2}^{n} F_i F_{i-1} + \sum_{i=2}^{n} F_i^2 F_{i-1}^2
\]
\[
= 2 F_n^2 + (-1)^n - 1 + \frac{24}{25} n + \frac{1}{25} F_{4n} + \frac{2}{25} (-1)^n F_n L_n,
\]
since \(\sum_{i=2}^{n} F_i F_{i-1} = F_n^2 + \frac{(-1)^n - 1}{2}\) and \(\sum_{i=2}^{n} F_i^2 F_{i-1}^2 = -\frac{n}{25} + \frac{1}{25} F_{4n} + \frac{2}{25} (-1)^n F_n L_n.\)
Putting the previous formulae together results in the equation

\[ 1 + \sum_{i=2}^{n}(1 + F_iF_{i-1})^2 + 2\sum_{i=2}^{n}\sum_{j=2}^{i}\left(F_{j-1} + \sum_{k=j+1}^{i}F_{k-1}F_{k-j}\right)^2 \]

\[ = \frac{1}{25}F_{4n} + \frac{2}{25}F_{4n-1} + n\frac{2}{5}nF_{2n-1} + \frac{2}{5}nF_{2n+1} \]

\[ + \frac{7}{10} + \frac{2}{25}(-1)^nF_{n-1}L_n + \frac{2}{25}(-1)^nF_nL_n - \frac{1}{25}(-1)^nF_{n-1}L_{n+1} \]

\[ + \frac{32}{25}(-1)^n + \frac{1}{20}F_{n-2}F_{n-1} + \frac{5}{2}F_n^2 - \frac{2}{5}F_{2n} - \frac{1}{25}F_{n+1}F_{n+2} - \frac{2}{5}F_{2n-1} - \frac{1}{10}L_{2n} + \frac{2}{25}L_{2n-1}. \]

(2.5)

At this juncture, one can proceed as follows.

- To simplify row (2.5), use the identity
  \[ \frac{1}{25}F_{4n} + \frac{2}{25}F_{4n-1} = \frac{1}{25}L_{4n}. \]
  
- To simplify row (2.6), apply
  \[ \frac{2}{5}nF_{2n+1} = \frac{2}{5}n(F_{2n} + F_{2n-1}). \]

- To cope with row (2.7), use the identity
  \[ -\frac{7}{10} + \frac{2}{25}(-1)^nF_{n-1}L_n + \frac{2}{25}(-1)^nF_nL_n - \frac{1}{25}(-1)^nF_{n-1}L_{n+1} = (-1)^n\frac{L_{2n}}{25} \]

- Finally, to handle row (2.8), utilize the identity
  \[ \frac{32}{25}(-1)^n + \frac{1}{20}F_{n-2}F_{n-1} + \frac{5}{2}F_n^2 - \frac{2}{5}F_{2n} - \frac{1}{25}F_{n+1}F_{n+2} - \frac{2}{5}F_{2n-1} - \frac{1}{10}L_{2n} + \frac{2}{25}L_{2n-1} \]
  \[ = \frac{3}{25}L_{2n} + \frac{13(-1)^n}{50}. \]

It is straightforward to verify the validity of each of these formulae. Altogether, the above formulae yield

\[ 1 + \sum_{i=2}^{n}(1 + F_iF_{i-1})^2 + 2\sum_{i=2}^{n}\sum_{j=2}^{i}\left(F_{j-1} + \sum_{k=j+1}^{i}F_{k-1}F_{k-j}\right)^2 \]

\[ = \frac{1}{25}L_{4n} + \frac{3}{25}L_{2n} + \frac{2}{5}nF_{2n} + \frac{13(-1)^n}{50} + n. \]

The claim follows by expanding the Fibonacci and Lucas numbers in terms of the golden ratio using (2.3).

It is evident that Theorem 2.2 can be recast in the following way.

**Corollary 2.3.** For $n$ odd, it holds that

\[ c_n \geq \frac{1}{\sqrt{\frac{1}{25}\phi^{-4n} + \frac{2}{25}\phi^{-2n} - \frac{2}{5\sqrt{5}}n\phi^{-2n} - \frac{23}{25} + n + \frac{2}{25}\phi^{2n} + \frac{2}{5\sqrt{5}}n\phi^{2n} + \frac{1}{25}\phi^{4n}}}, \]

and for $n$ even, one has

\[ c_n \geq \frac{1}{\sqrt{\frac{1}{25}\phi^{-4n} + \frac{4}{25}\phi^{-2n} - \frac{2}{5\sqrt{5}}n\phi^{-2n} - \frac{2}{5} + n + \frac{4}{25}\phi^{2n} + \frac{2}{5\sqrt{5}}n\phi^{2n} + \frac{1}{25}\phi^{4n}}}. \]
Table 1: Tabulated values of the constant $c_n$ and the lower bound of Theorem 2.2 for $n \in \{1, \ldots, 10\}$, which suggest that the lower bound becomes sharper as $n$ increases.

| $n$ | $c_n$       | $\| Z_n \|_F^{-1}$   |
|-----|-------------|----------------------|
| 1   | 1.000000000 | 1.000000000          |
| 2   | 0.381966011 | 0.377964473          |
| 3   | 0.198062264 | 0.196116135          |
| 4   | 0.087003112 | 0.086710997          |
| 5   | 0.037068335 | 0.037037037          |
| 6   | 0.014827585 | 0.014824986          |
| 7   | 0.005816999 | 0.005816805          |
| 8   | 0.002245345 | 0.002245332          |
| 9   | 0.000862203 | 0.000862202          |
| 10  | 0.000330004 | 0.000330004          |

2.2 Numerical experiments

The sharpness of the lower bound presented in Theorem 2.2 is assessed by numerical experiments. The characterization (2.1) provides an easy way of computing the numerical value of $c_n$ for $n \in \mathbb{Z}_+$. The value of the lower bound corresponding to $c_n$ is denoted by $\| Z_n \|_F^{-1}$.

In Table 1 the values of both $c_n$ and the lower bound of Theorem 2.2 have been tabulated for $n \in \{1, \ldots, 10\}$. The results suggest that the lower bound becomes sharper with increasing $n$. This observation is further backed by the results illustrated in Figure 1 where the absolute errors, relative errors as well as the number of common significant digits between $c_n$ and the lower bound in Theorem 2.2 have been tabulated for $n \in \{2, \ldots, 100\}$. All numerical experiments were carried out by using 150 digit precision computations in Mathematica 11.2.

3 The Ilmonen–Haukkanen–Merikoski numbers $C_n$

To conclude this paper, the following new characterization is proved for the Ilmonen–Haukkanen–Merikoski numbers $C_n$. 

Figure 1: Left and middle images: both the absolute error and the relative error between $c_n$ and the lower bound of Theorem 2.2 decay at an exponential rate. Right image: the number of common significant digits between $c_n$ and the lower bound of Theorem 2.2 is displayed for increasing $n$. 

All numerical experiments were carried out by using 150 digit precision computations in Mathematica 11.2.
Lemma 3.1. It holds for all $n \in \mathbb{Z}_+$ that

$$C_n = \frac{1}{4} \csc^2 \left( \frac{\pi}{4n+2} \right) = \frac{4n^2}{\pi^2} + \frac{4n}{\pi^2} + \left( \frac{1}{12} + \frac{1}{\pi^2} \right) + O\left( \frac{1}{n^2} \right).$$

Proof. The second identity is a consequence of the Laurent expansion of the first expression developed at infinity. It is therefore enough to focus on proving the first identity.

It is easy to check that the claim holds for $n = 1$. Let $n \geq 2$. By [3], it is known that

$$C_n = \lambda_{\text{max}}(W_n),$$

where $W_n$ is the $n \times n$ matrix defined elementwise by

$$(W_n)_{i,j} = \min\{i,j\}, \quad i,j \in \{1,\ldots,n\}.$$

It is easy to see that its matrix inverse $B_n = W_n^{-1}$ is given by

$$B_n = \begin{pmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & 2 & -1 \\
& & \ddots & \ddots & \ddots \\
& & & -1 & 2 & -1 \\
& & & & -1 & 1
\end{pmatrix}$$

and hence $C_n = \lambda_{\text{min}}(B_n)^{-1}$. In other words, it is sufficient to find the reciprocal of the minimal eigenvalue of $B_n$. Noting that $B_n$ is a special case of a second order finite difference matrix subject to mixed Dirichlet–Neumann boundary conditions (see also the Remark following this proof), it is well known that the eigenvalues of $B_n$ are roots of certain Chebyshev polynomials and, as such, the roots have closed form solutions. A brief derivation is presented in the following for completeness.

Let $A_n = B_n + e_ne_n^T$, where $e_n = [0,0,\ldots,0,1]^T \in \mathbb{R}^n$. Let $p_n(\lambda) = \det(A_n - \lambda I_n)$ and $q_n(\lambda) = \det(B_n - \lambda I_n)$ be the characteristic polynomials of $A_n$ and $B_n$, respectively. It is easy to check that

$$p_n(\lambda) = U_n \left( 1 - \frac{\lambda}{2} \right),$$

where $U_n$ denotes the Chebyshev polynomial of the second kind. Developing the Laplace cofactor expansion of $\det(B_n - \lambda I_n)$ across the final column and using the properties of Chebyshev polynomials of the second kind yield that

$$q_n(\lambda) = (1 - \lambda)p_{n-1}(\lambda) - p_{n-2}(\lambda)$$

$$= 2 \left( 1 - \frac{\lambda}{2} \right) U_{n-1} \left( 1 - \frac{\lambda}{2} \right) - U_{n-2} \left( 1 - \frac{\lambda}{2} \right) - U_n \left( 1 - \frac{\lambda}{2} \right)$$

$$= U_n \left( 1 - \frac{\lambda}{2} \right) - U_{n-1} \left( 1 - \frac{\lambda}{2} \right)$$

$$= \frac{\sin \left( (n+1) \arccos \left( 1 - \frac{\lambda}{2} \right) \right) - \sin \left( n \arccos \left( 1 - \frac{\lambda}{2} \right) \right)}{\sqrt{1 - (1 - \frac{\lambda}{2})^2}}.$$

The previous expression can be used to solve the roots of $q_n$ by elementary means, i.e.,

$$\lambda_j = 4 \cos^2 \left( \frac{j\pi}{2n+1} \right), \quad j \in \{1,\ldots,n\}.$$
Since the smallest root of $q_n$ is $\lambda_n$ for all $n \in \mathbb{Z}_+$, it follows that

$$C_n = \frac{1}{\lambda_{\min}(B_n)} = \frac{1}{4} \sec^2 \left( \frac{n\pi}{2n+1} \right) = \frac{1}{4} \csc^2 \left( \frac{\pi}{4n+2} \right),$$

completing the proof.

Remark. The matrix $B_n$ is (up to a scalar multiple) precisely the finite difference matrix corresponding to the Dirichlet–Neumann problem

$$-u''(x) = f(x) \quad \text{for } x \in (a, b), \quad u(a) = 0, \quad u'(b) = 0.$$  

The properties of finite difference matrices for this problem are very well known in the literature; see, e.g., [5] for a comprehensive treatment of the topic.

It was shown in [3] that the numbers $C_n$ can be bounded by

$$C_n \leq \sqrt{(2n-1) + 4(2n-3) + 9(2n-5) + \cdots + 3(n-1)^2 + n^2}, \quad n \in \mathbb{Z}_+,$$

but the closed form solution stated in Lemma 3.1 appears to have eluded the authors of the aforementioned paper.

Conclusions

The numerical experiments presented in this paper suggest that the lower bound obtained for the numbers $c_n$ is extremely sharp as $n$ tends to infinity. The numerical evidence leads the author to further conjecture that $c_n \sim 5\phi^{-2n}$ as $n \to \infty$, based on the dominating term that appears in Theorem 2.2. Proving this asymptotic result appears to require developing mathematical techniques which are beyond the scope of this paper, posing an interesting challenge for researchers working in this area.

References

[1] E. Altınısk, A. Keskin, M. Yıldız, and M. Demirbüken. On a conjecture of Ilmonen, Haukkanen and Merikoski concerning the smallest eigenvalues of certain GCD related matrices. *Linear Algebra Appl.*, 493:1–13, 2016.

[2] S. Hong and R. Loewy. Asymptotic behavior of eigenvalues of greatest common divisor matrices. *Glasgow Math. J.*, 46(3):551–569, 2004.

[3] P. Ilmonen, P. Haukkanen, and J. K. Merikoski. On eigenvalues of meet and join matrices associated with incidence functions. *Linear Algebra Appl.*, 429:859–874, 2008.

[4] M. Mattila. On the eigenvalues of combined meet and join matrices. *Linear Algebra Appl.*, 466:1–20, 2015.

[5] A. R. Mitchell and D. F. Griffiths. *The Finite Difference Method in Partial Differential Equations*. John Wiley & Sons, 1980.