Inverse scattering approach to coupled higher order nonlinear Schrödinger equation and $N$-soliton solutions

Sudipta Nandy *
Institute of Physics, Bhubaneswar
Sachivalaya Marg, Bhubaneswar -751005, India
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Abstract
A generalized inverse scattering method has been applied to the linear problem associated with the coupled higher order nonlinear Schrödinger equation to obtain its $N$-soliton solution. An infinite number of conserved quantities have been obtained by solving a set of coupled Riccati equations. It has been shown that the coupled system admits two different class of solutions, characterised by the number of local maxima of amplitude of the soliton.

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1 Introduction

Optical soliton since it’s discovery occupies a distinguished position in the nonlinear optics research and it is regarded as the next generation carrier in
an all-optical communication system. Because of their remarkable stability solitons are capable of propagating long distance without attenuation [1, 2].

Theoretical study of optical soliton began as early as in seventies with the landmark discovery by Hasegawa and Tappert [3]. They had shown that the nonlinear Schrödinger equation (NLSE) studied by Zakharov and Shbat [4] (but with space and time interchanged) is the appropriate equation for describing the propagation of pico-second optical pulses in optical fibers. Later on Mollenauer et al. [5] successfully demonstrated the transmission of pico-second optical solitons through a monomode fiber. Recently Kodama and Hasegawa [6] proposed a modified NLS model, known as the higher order nonlinear Schrödinger equation (HNLSE), which is suitable for the propagation of femto-second optical pulse. Until now only two integrable HNLS equations [7, 8] which has soliton solutions are known. The dynamics of HNLSE not only takes care of dispersion loss but also takes care of the propagation loss as the optical soliton propagate along the fiber. This is due to the fact that the stimulated Raman scattering effect, which compensates the propagation loss, already exists in the spectrum of the HNLS equation. As a consequence a short pulse can propagate as a soliton for a long distance without distortion. Therefore the importance of the study of HNLS system and to find their soliton solution is unquestionable. The soliton solution may be obtained by several methods like Hirota’s bilinear method [27, 28, 29], Lie group theory [30, 31, 32] and inverse scattering method (IST) [4, 33, 34].
IST is the most elegant and one of the most widely used technique, which eventually proves the complete integrability of the system.

It has been known that a good number of soliton equations in nonlinear optics have multifield generalizations. The coupled nonlinear Schrödinger equation proposed by Manakov [9], is one such example. It describes a system with two interacting optical fields with different states of polarizations. It has an important application in communication systems using optical solitons. Such systems have many other important applications in all optical computations [10, 11, 12, 13], in photorefractive crystals [14, 15]. Recently the existence of multicomponent solitons has been discovered experimentally also [16]. Interestingly the HNLS equation also has it’s multifield generalization and they have been studied in different physical contexts [17, 18, 19, 20]. The coupled HNLS (CHNLS) equations are particularly important for describing the dynamics of ultrashort optical pulse in a system involving two or more interacting optical fields. Recently a CHNLS equation, which incorporates the effect of third order dispersion, kerr nonlinearity and stimulated Raman scattering was proposed by Nakkeeran et. al., [21, 22]. The equation considered in [21, 22] is given by

\[
E_{kz} + \frac{i}{2} \left[ E_{zzz} + (\sum_{j=1}^{n} E_j^* E_j) E_k \right] + \epsilon E_{\sigma \tau \tau \tau} + \epsilon 6 (\sum_{j=1}^{n} E_j^* E_j) E_{\tau \tau} + \epsilon 3 (\sum_{j=1}^{n} (|E_j|^2) \tau E_k) = 0
\]

(1)

for \( k = 1, 2, \cdots n \). It describes the evolution of a complex vector field \( E \).
with $n$ components, where $n$ is a finite integer. The suffices $z$ and $\tau$ denote the normalised space and time derivatives respectively. $E_k$ and $E^*_k$ respectively represent the amplitude of the $k$th component of a slowly varying field and its complex conjugate. $\epsilon$ is an independent parameter. They have also shown the integrability of (1) through the existence of Lax pair and obtained the one soliton solution (1SS) of (1) through Bačklund transformation. Existence of Lax pair although indicates the integrability of the system but a conclusive sense of integrability is achieved if the system possess an infinite number of conserved charges [26]. It is important to note that the equation (1) is the multifield generalization of the Sasa-Satsuma equation [7]. Interestingly for Sasa-Satsuma (scalar) equation two different 1SS were reported in [7, 23, 24, 25]. One is the conventional sech solution having a single peak and the other is a complex combination of sech function having two peaks. In view of the connection between the Sasa-Satsuma equation and its multifield generalization it is expected that the latter equation should also admit both type of soliton solutions. It should be mentioned that the 1SS with sech envelope function has been obtained in [21, 22]. However, it remained to show that (1) admit a more general class of solutions. The existence of the higher order solitons, which are important to study the soliton interactions also remained unexplored. A possible reason for this may be the difficulty involved in solving the linear problem associated with the system. It is important to note that for a completely integrable system it is necessary that
the system possess $N$-soliton solution.

Our objective in this paper is to obtain the whole hierarchy of the conserved charges in a systematic way to establish integrability of (1) conclusively. We complete the investigation of higher order soliton solutions for (1) by deriving the exact $N$-soliton solutions for the system. We invoke the inverse scattering method for an $(2n+1 \times 2n+1)$ dimensional eigenvalue problem and obtain the $N$-soliton solution by solving the set of $(2n+1)$ Gelfand-Levitan-Marchenko (GLM) equations. Subsequently we discuss some important characteristics of the 1SS.

The paper is organized in the following sequence. In section 2 we use a set of variable transformations to cast the CHNLS equation into a form, suitable for the inverse scattering transform. Subsequently we study the linear problem associated with the modified system. In section 3 we show the existence of infinite number of conserved quantities by solving a set of coupled Riccati equations. The IST scheme and the properties of the scattering data are studied and the generalized GLM equations are derived in the section 4. The exact $N$-soliton solution is obtained in section 5. In Section 6 two class of solutions of the CHNLS equation are discussed. Section 7 is the concluding one.
2 Eigenvalue problem

In order to investigate (1) we use the following change of variables and a Galelian transformation,

\[ q_k(x, t) = E_k(z, \tau) e^{-\frac{i}{6\epsilon}(\tau - \frac{z}{12\epsilon})}, \]

(2)

\[ t = z, \]

(3)

\[ x = \tau - \frac{z}{12\epsilon} \]

(4)

Thus we get a set of complex modified K-dV equations (CMKDV)

\[ q_{kt} + \epsilon q_{kxxx} + 6\epsilon (\sum_{j=1}^{n} q_j^* q_j) q_{kx} + 3\epsilon (\sum_{j=1}^{n} |q_j|^2) x q_k = 0 \]

(5)

We use \( x \) and \( t \) in (5) and in the subsequent expressions to denote the derivatives with respect to \( x \) and \( t \) respectively. The associated spectral problem of (5) may be represented by a pair of linear eigenvalue equations.

\[ \partial_x \Psi = L(x, t, \lambda) \Psi \]

(6)

\[ \partial_t \Psi = M(x, t, \lambda) \Psi \]

(7)

where \( \Psi \) is the \((2n + 1)\) dimensional auxiliary field and \( \lambda \) is the time independent spectral parameter. The pair of matrices \((L, M)\) is called the Lax pair. The explicit form of the Lax pair has been given in [21, 22]. With slight modifications (see eq. 11 in [22]) we express the Lax pair \((L, M)\) by using a pair of matrices \((\Sigma, A)\), where \( \Sigma \) is a c-no. diagonal matrix and the matrix
A(x, t) is a potential function of the eigenvalue problem and consists of the dynamical fields, \( q_k(x, t) \) and \( q^*_k(x, t) \) only,

\[
\Sigma = \sum_{k=1}^{2n} e_{kk} - e_{2n+1, 2n+1}
\]

\[
A(x, t) = \sum_{k=1}^{n} \left( q_k(x, t) e_{2k-1, 2n+1} + q^*_k(x, t) e_{2n+1, 2k-1} - q_k(x, t) e_{2n+1, 2k} \right)
\]

where, \( e_{kj} \) is an \((2n + 1 \times 2n + 1)\) dimensional matrix whose only \((kj)\)th element is unity, the rest elements being zero. By using the properties of \( \Sigma \) and \( A \)

\( \text{viz,} \)

\[
\Sigma A + A \Sigma = 0, \quad \Sigma^2 = 1
\]

we write the Lax matrices in a simplified form,

\[
L = -i\lambda \Sigma + A \quad M = \epsilon (-A_{xx} + 2i\lambda \Sigma A_x - (AA_x - A_x A) - 2i\lambda \Sigma A^2 + 2A^3 + 4\lambda^2 A - 4i\lambda^3 \Sigma)
\]

The Lax pairs given by (9) is also valid up to an additive constant, since a constant commutes with all other matrices. We assume the constant to be zero for simplicity. The consistency of the Lax equations (6, 7) gives the nonlinear evolution equation for the matrix \( A \),

\[
A_t + \epsilon (A_{xxx} - 3(A_x A^2 + 3A^2 A_x)) = 0
\]
which is nothing but the matrix form of the equation (5). Although the existence of a Lax pair is itself a sign of the integrability of (5) vis. a. vis. (1), it remains to show that the system also possess infinite number of conserved quantities and admit $N$-soliton solutions.

### 3 Conserved quantities

In order to obtain the conserved quantities first we derive the associated Riccati equations. To this aim we write the Lax equation (6) in the component form. For the first $2n$ components of $\Psi$, the equation (6) can be written in the form

\[
\Psi_{2k-1} x = -i\lambda \Psi_{2k-1} + q_k \Psi_{2n+1}
\]

\[
\Psi_{2k} x = -i\lambda \Psi_{2k-1} + q^*_k \Psi_{2n+1}
\]

(11)

for $k = 1, 2, \ldots, n$. But the $(2n+1)$-th component has a different form

\[
\Psi_{2n+1} x = i\lambda \Psi_{2n+1} - \sum_{k=1}^{n} (q^*_k \Psi_{2k-1} + q_k \Psi_{2k})
\]

(12)

Following now a similar procedure as in [24] we write,

\[
\Gamma_k = \frac{\Psi_k}{\Psi_{2n+1}}
\]

(13)
for \( k = 1, 2, \cdots 2n \). By using (11, 12, 13) we may obtain a set of first order differential equations,

\[
\Gamma_{2k-1} + 2i\lambda \Gamma_{2k-1} - \sum_{j=1}^{n} (q_k^* \Gamma_{2k-1} \Gamma_{2j-1} + q_k \Gamma_{2k-1} \Gamma_{2j}) - q_k = 0 \quad (14)
\]

\[
\Gamma_{2k} + 2i\lambda \Gamma_{2k} - \sum_{j=1}^{n} (q_k \Gamma_{2k} \Gamma_{2j-1} + q_k^* \Gamma_{2k} \Gamma_{2j}) - q_k^* = 0 \quad (15)
\]

which are known as the Riccati equations. The solution of the equations are related to the conserved quantities \((\alpha_{2n+1} 2n+1)\) in the following way

\[
\ln \alpha_{2n+1} 2n+1(\lambda) = \ln \Psi_n - i\lambda x \bigg|_{x \to \pm \infty} = - \int_{-\infty}^{\infty} dx \sum_{k=1}^{n} (q_k \Gamma_{2k-1} + q_k^* \Gamma_{2k}) \quad (16)
\]

We will see in the section 4 that \( \alpha_{2n+1} 2n+1 \) is the diagonal element of the scattering matrix and it is time independent.

In order to solve Riccati equations (14) and (15) we assume \( \Gamma_{2k-1} \) and \( \Gamma_{2k} \) in the form

\[
\Gamma_{2k-1} = \sum_{p=0}^{\infty} C_{2k-1}^p(x) \lambda^p \quad (17)
\]

\[
\Gamma_{2k} = \sum_{p=0}^{\infty} C_{2k}^p(x) \lambda^p \quad (18)
\]

Substituting (17,18 ) in (14) we get

\[
C_{0}^{2k-1} = 0, \quad C_{1}^{2k-1} = \frac{q_k}{2i} \quad (19)
\]

\[
2iC_{p+2}^{2k-1} = -(C_{p+1}^{2k-1})_x + \sum_{l=1}^{n} \sum_{m=0}^{p+1} (C_{p-m+1}^{2k-1} C_{m}^{2l-1} q_l^* + C_{p-m+1}^{2k-1} C_{m}^{2l} q_l) \quad (20)
\]
similarly substituting (17,18) in (15) we get

\[ C_{0}^{2k} = 0, \quad C_{1}^{2k} = \frac{q_{k}^{*}}{2l} \]  

\[ 2iC_{p+2}^{2k} = -(C_{p+1}^{2k})_{x} + \sum_{l=1}^{n} \sum_{m=0}^{p} (C_{p-m+1}^{2l-1} q_{l}^{*} + C_{p-m+1}^{2l} C_{m}^{ql} q_{l}) \]  

Now the infinite number of Hamiltonians (conserved quantities) may explicitly be determined in terms of the \( q_{k}, q_{k}^{*} \) and their derivatives by expanding \( \alpha_{2n+1 \ 2n+1} \) in the form

\[ \ln(\alpha_{2n+1 \ 2n+1}) = n \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(2i)^{2l+1}} H_{l} \lambda^{-1} \]  

The first few conserved quantities are given by

\[ H_{1} = \frac{1}{n} \int dx \sum_{k=1}^{n} q_{k}^{*} q_{k} \]  

\[ H_{3} = \frac{1}{n} \int dx (2 \sum_{k=1}^{n} |q_{k}|^{2})^{2} - \sum_{k=1}^{n} q_{k}^{*} x q_{k} x \]  

\[ H_{5} = \frac{1}{n} \int dx [\sum_{k=1}^{n} q_{k}^{*} xx q_{k} xx + 2(\sum_{k=1}^{n} |q_{k}|^{2})^{3} - (\sum_{k=1}^{n} (|q_{k}|^{2})_{x})^{2}] 
- \sum_{k=1}^{n} |q_{k}|^{2} \sum_{k=1}^{n} q_{k}^{*} x q_{k} x - \sum_{k=1}^{n} q_{k}^{*} q_{k} x \sum_{l=1}^{n} q_{l}^{*} x q_{l} x \]  

Notice that the hamiltonians with odd indices only survive. The even indexed hamiltonians become trivial. It is easy to see using the equation of motion that \( H_{1}, H_{3}, H_{5} \) are indeed the constants of motion. If we choose the field \( q \), a scalar field then we are able to show that \( H_{1}, H_{3}, H_{5} \) reduce to the conserved quantities for the scalar Sasa-Satsuma equation [24].
4 Gelfand-Levitan-Marchenko equations

We now generalise the inverse scattering method for the \((2n + 1 \times 2n + 1)\) dimensional Lax operators (9). The generalization however, is a nontrivial one and crucially depends on the scattering data matrix. We have broadly followed the treatment of Manakov, developed in the context of \(3 \times 3\) Lax operators [9].

In order to formulate the scattering problem we assume that the family of Jost functions \(\Phi^{(k=1,2,...2n+1)}\) and \(\Psi^{(k=1,2,...2n+1)}\) of (6) satisfy the following boundary conditions for real values of \(\lambda\),

\[
\Phi^{(k)} \bigg|_{x \to -\infty} \to e_k e^{-i\lambda x} \quad (27)
\]

for \(k = 1, 2 \ldots 2n\), but the \((2n+1)\)-th component satisfies a different boundary condition

\[
\Phi^{(2n+1)} \bigg|_{x \to -\infty} \to e_{2n+1} e^{i\lambda x} \quad (28)
\]

Similarly, other set of Jost functions satisfy the boundary conditions,

\[
\Psi^{(k)} \bigg|_{x \to \infty} \to e_k e^{-i\lambda x} \quad (29)
\]

for \(k = 1, 2 \ldots 2n\) and the \((2n + 1)\)-th component satisfies,

\[
\Psi^{(2n+1)} \bigg|_{x \to \infty} \to e_{2n+1} e^{i\lambda x}. \quad (30)
\]

In the equations (27-30) \(e_k\)’s are the basis vectors in an \((2n + 1)\)-dimensional vector space. Note that the set of jost functions (27-30) also satisfy the orthogonality condition, That is,
for $k, j = 1, 2, \ldots 2n + 1$. Since vectors $\Psi^{(k)}$ form a complete set of solutions of (6) hence,
\[ \Phi^{(k)}(x, \lambda) = \sum_{j=1}^{2n+1} \alpha_{kj}(\lambda) \Psi^{(j)}(x, \lambda) \]  
(32)
where $\alpha_{kj}(\lambda)$ is the $(kj)^{th}$ element of scattering data matrix ($\det[\alpha_{kj}]=1$).

Using (31) and (32), $\alpha_{kj}$ is expressed in the form
\[ \alpha_{kj}(\lambda) = \Psi^{(j)^\dagger}(x, \lambda) \Phi^{(k)}(x, \lambda) \]  
(33)
It is interesting to see that using the unitary property of $[\alpha_{kj}]$ we can write $\alpha_{kj}^*$ as the cofactor of the elements of the matrix $[\alpha_{kj}]$, that is,
\[ \alpha_{2n+1 k}^* = (-1)^{2n+1+k} \det[\tilde{\alpha}_{2n+1 k}] \]  
(34)
where $[\tilde{\alpha}_{2n+1 k}]$ is an $(2n \times 2n)$ dimensional matrix, constructed from the matrix $[\alpha_{kj}]_{2n+1 \times 2n+1}$ with $(2n+1)$-th row and $k$-th column being omitted.

Now by using (32) and (34) we obtain the following useful relations among the jost functions,
\[ \frac{1}{\alpha_{2n+1 2n+1}(\lambda)} \sum_{j=1}^{2n} (\text{Adj}[\tilde{\alpha}_{2n+1 2n+1}])_{kj} \Phi^{(j)} e^{i\lambda x} = \Psi^{k} e^{i\lambda x} - \frac{\alpha_{2n+1 k}^*(\lambda)}{\alpha_{2n+1 2n+1}(\lambda)} \Psi^{2n+1} e^{i\lambda x} \]  
(35)
for $k = 1, 2, \cdots 2n$. The $(2n+1)$-th jost function satisfy another relation,
\[ \frac{1}{\alpha_{2n+1 2n+1}} \Phi^{(2n+1)} e^{-i\lambda x} = \Psi^{(2n+1)} e^{-i\lambda x} + \frac{1}{\alpha_{2n+1 2n+1}} \sum_{j=1}^{2n} \alpha_{2n+1 j} \Psi^{j} e^{-i\lambda x} \]  
(36)
Notice that in deriving (35, 36) we have used the following property of scattering matrix,

$$\alpha^*_{2n+1} k \delta_{ij} = \sum_{l=1}^{2n} [\tilde{\alpha}_{2n+1} k]_{ij} (\text{Adj} [\tilde{\alpha}_{2n+1} k])_{lj}$$  \hspace{1cm} (37)

In order to obtain the complete analytic behaviour of the jost functions \textit{vis a vis} scattering data the domain of \(\lambda\) is extended to complex plane. It can be shown that the functions \(\Phi^{(k)} e^{i\lambda x}\) for \(k = 1, 2, \cdots 2n\) and \(\Psi^{(2n+1)} e^{-i\lambda x}\) are analytically continued into the upper half-plane \((Im \lambda \geq 0)\) whereas \(\Psi^{*(k)} e^{i\lambda x}\) for \(k = 1, 2, \cdots 2n\) and \(\Phi^{*(2n+1)} e^{-i\lambda x}\) are analytically continued into the lower half-plane. Consequently the scattering element, \(\alpha^*_{2n+1} 2n+1 (\lambda)\) and all elements of the matrix \([\tilde{\alpha}_{2n+1} 2n+1 (\lambda)]\) are analytic in the upper half-plane and \(\alpha_{2n+1} 2n+1 (\lambda)\) and all elements of the matrix \([\tilde{\alpha}^*_{2n+1} 2n+1 (\lambda)]\) are analytic in the lower half-plane. It is important to note that the bound states of the eigenvalue equation (6) correspond to zeros of \(\alpha_{2n+1} 2n+1 (\lambda)\) in the lower half-plane. We assume that the bound states are located at \(\lambda^*_j (Im \lambda \geq 0)\) for \(j = 1, 2, \cdots N\), where the jost function \(\Phi^{(2n+1)}\) becomes,

$$\Phi^{(2n+1)}(x, \lambda^*_j) = \sum_{m=1}^{2n} C^{(j)}_{2n+1 m} \Psi^{(m)}(x, \lambda^*_j)$$  \hspace{1cm} (38)

In (38), \(C^{(j)}_{2n+1 m}\) represent the value of the scattering parameter \(\alpha_{2n+1 m}\) at the position of the \(j^{th}\) pole.

The time dependence of the scattering data may be easily obtained from the asymptotic limit of (7), which gives the following time dependence of the
scattering data.

\[
\begin{align*}
\alpha_{2n+1} k(t) &= \alpha_{2n+1} k(0) e^{-8i\lambda^3 t} \\
\alpha_{2n+1} 2n+1(t) &= \alpha_{2n+1} 2n+1(0), \\
C^{(j)}_{2n+1} k(t) &= C^{(j)}_{2n+1} k(0) e^{-8i\lambda^3 t} \\
C^{(j)}_{2n+1} 2n+1(t) &= C^{(j)}_{2n+1} 2n+1(0)
\end{align*}
\]

(39) (40) (41) (42)

In order to derive the GLM equation we consider an integral representation of the Jost functions

\[
\Psi^{(j)}(x, \lambda) = e^{j e^{-i\lambda x}} + \int_x^\infty dy K^{(j)}(x, y) e^{-i\lambda y}
\]

(43) with \( j = 1, 2, \ldots, 2n \), while the \((2n + 1)\)th Jost function is considered as

\[
\Psi^{(2n+1)}(x, \lambda) = e^{2n+1 e^{i\lambda x}} + \int_x^\infty dy K^{(2n+1)}(x, y) e^{i\lambda y}.
\]

(44) where, the kernels \( K^{(j)} \) and \( K^{(2n+1)} \) are \((2n + 1)\) dimensional column vectors, which may be written explicitly in the component form as

\[
K^{(j)}(\tau, y) = \sum_{m=1}^{2n} K^{(j)}_m(\tau, y) e_m
\]

(45) \( K^{(2n+1)}(x, y) = \sum_{m=1}^{2n+1} K^{(n+1)}_m(x, y) e_m
\]

(46) Multiplying (36) with \( \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda y} d\lambda, (y > x) \) and using (43,44,38) together with the analytic properties of the associated scattering data we obtain the desired GLM equation for the kernel \( k^{(2n+1)} \)

\[
K^{(2n+1)}(x, y) + \sum_{p=1}^{2n} e_p F_p(x + y) + \sum_{p=1}^{2n} \int_x^\infty dsk^{(p)}(x, s) F_p(s + y) = 0
\]

(47)
where,
\[ F_p(x + y) = i \sum_{j=1}^{N} \frac{C_{2n+1}^{(j)}(\alpha_{2n+1} p t e^{-i\lambda_j^* (x+y)})}{\alpha_{2n+1} (\lambda_j^*)} + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \frac{\alpha_{2n+1} p(\lambda)}{2n+1} e^{-i\lambda (x+y)} \] (48)

The ' over \( \alpha_{2n+1} \) denotes derivative with respect to \( \lambda \).

The other integral equations for kernel \( K^p \) is obtained from (36) in a way similar to that (47) and the resultant equations are

\[ K^{(p)}(x, y) + e_{2n+1} F^*_i(x + y) + \int_{x}^{\infty} ds K^{(2n+1)}(x, s) F^*_i(s + y) = 0 \] (49)

for \( p = 1, 2, \cdots 2n \) In deriving (49) we have used the identity,

\[ C_{2n+1}^* m = \alpha_{2n+1} m(\lambda_j) = \sum_{i=1}^{2n} [\tilde{\alpha}_{2n+1} m(\lambda_j)] k_i(Adj[\tilde{\alpha}_{2n+1} m(\lambda_j)])_{ij} \] (50)

The set of equations (47,49) may be called generalized GLM equations. Substituting (46,49) in (47) we get the GLM equation for the \( p \)-th component of the kernel \( k^{(2n+1)} \), which is given by,

\[ K_p^{(2n+1)}(x, z) + F_p(x + z) + \sum_{m=1}^{n} \left( \int_{x}^{\infty} ds K_p^{(2n+1)}(x, s) \right) \]
\[ + \int_{x}^{\infty} dy F_{2m-1}(y + z) F^*_{2m-1}(y + s) \int_{x}^{\infty} dy F^*_{2m}(y + z) F_{2m}(y + s) \] = 0 \] (51)

5 N-soliton Solution

To obtain a closed form solution of the GLM equation (51) we assume \( (\alpha_{2n+1} p(\lambda) = 0) \). This is justified because our primary interest is to obtain the soliton solutions, which is obtained for the reflectionless potential. We
also note that, substituting (44) in (6) we get a relation between the Kernels of the integral equations (51) and the 'potential' of the eigenvalue equation (6),

\[ q_{2i-1}(x) = -2K_{2i-1}^{(2n+1)}(x, x) \]  
\[ q_{2i}^*(x) = -2K_{2i}^{(2n+1)}(x, x) \]

for \( i = 1, 2, \ldots n \). To obtain the general \( N \)-soliton solution diagonal element of the scattering matrix, \( \alpha_{2n+1 \ 2n+1}(\lambda) \) is considered to have \( N \)-pairs of zeros located symmetrically about the imaginary axes in the lower-half plane. That is,

\[ \alpha_{nn}(\lambda_j^*) = \prod_{j=1}^{N} \frac{(\lambda - \lambda_j^*)(\lambda + \lambda_j)}{(\lambda - \lambda_j)(\lambda + \lambda_j^*)} \]  

Note that unlike the CNLSE, where a zero corresponds to a single soliton, in the CSSE a pair of zeros corresponds to a single soliton. Finally we solve the set of GLM equations, by assuming that kernel of the integral equations are of the form,

\[ K_p^{(2n+1)}(x, z) = \sum_{j=1}^{N} R_{pj}(x, t)e^{-\lambda_j^*x} + S_{pj}(x, t)e^{\lambda_jx} \]  

and we obtain the \( N \)-soliton solution of (5) for the \( k^{th} \) component of the field,

\[ q_k(z, \tau) = -2 \sum_{j=1}^{2N} (BC^{-1})_{kj} e^{-i\lambda_j^*x} \]

\( B \) and \( C \) in (56) are respectively \( 2n \times 2N \) and \( 2N \times 2N \) matrices whose elements \( b_{kj}, c_{kj} \) are given by,

\[ b_{kj} = \begin{cases} 
 p_{kj}e^{-i\lambda_j^*x} & 1 \leq j \leq N \\
 0 & N + 1 \leq j \leq 2N 
\end{cases} \]
and

\[
c_{lm} = \left\{ \begin{array}{l}
\sum_{j=1}^{2n} p_{km} p_{kj} j+N e^{-i(2\lambda_{j+N}^*+\lambda_{j}^*)x} \\
\sum_{k=1}^{N} p_{km} e^{-i(2\lambda_{j+N}^*+\lambda_{j}^*)x} \end{array} \right. - \delta_{lm} \quad \forall \quad 1 \leq m \leq N, \\
\sum_{j=1}^{2n} p_{km} p_{kj} j+N e^{-i(2\lambda_{j+N}^*+\lambda_{j}^*)x} \\
\sum_{k=1}^{N} p_{km} e^{-i(2\lambda_{j+N}^*+\lambda_{j}^*)x} \end{array} \right. - \delta_{lm} \quad \forall \quad N + 1 \leq m \leq 2N,
\]

where,

\[
p_{kj} = i \frac{C_{2n+1}^{(j)} k(t)}{\alpha'_{2n+1} 2n+1(\lambda_{j}^*)}
\]

with the constraints

\[
\lambda_{j+N} = -\lambda_{j}^* \\
p_{j+N} = p_{j}^*
\]

for \(j = 1, \ldots, N\). The \(N\)-soliton solution for the equation (1) is obtained from (57) by using the inverse variable transformations (2,3,4).

6 One soliton solution

The one soliton solution for (5) is obtained by choosing \(N = 1\) in (56). It implies from (54), that \(\alpha_{2n+1} 2n+1(\lambda_{j})\) has a pair of zeros located symmetrically about the imaginary line in the lower half-plane. We assign them as \((-\lambda_{1}, \lambda_{1}^*)\), where \(\lambda_{1} = (-\xi + i\eta)/2\) with \(\xi, \eta > 0\). By reinstating the transformations (2,3,4) in (56) we obtain the \(k\)-th component of one soliton solution for (1), which is given as,
\[ E_k(z, \tau) = \frac{2\eta \ p_{k1} \ e^{iB(e^A + c \ e^{-A})}}{\sqrt{\sum_{j=1}^{2n} p_{j1}(|c|^{-1}e^{2A} + |c|e^{-2A} + 2|c|)}} \]  

(62)

where

\begin{align*}
A &= \eta \tau - \eta \epsilon(\eta^2 - 3\xi^2 + \frac{1}{12\epsilon^2}) \zeta - \gamma \\
B &= \xi \tau + \xi \epsilon(\xi^2 - 3\eta^2 - \frac{1}{12\epsilon^2} - \frac{1}{108\epsilon^3\xi}) \zeta + \delta \\
c &= 1 - i\frac{\eta}{\xi} \\
\ e^{\gamma+i\delta} &= \frac{\sum_{j=1}^{2n} |p_{j1}|}{\eta c^*} \tag{66}
\end{align*}

Notice that each component of the soliton solution (62) is defined completely by a set of four parameters \(viz, \eta, \xi, p_{k1}, \) and \(\sum_{j=1}^{2n} p_{j1}\). It represents an envelope wave moving with a group velocity, \(\epsilon(\eta^2 - 3\xi^2 + \frac{1}{12\epsilon^2})\) undergoing internal oscillation. Interestingly the group velocity depends on both real part \((\eta)\) and imaginary part \((\xi)\) of \(\lambda_1\). If we specialize to the scalar limit of the CHNLS equation the solution (62) reduces to the one soliton solution of the Sasa-Satsuma equation (see Eqs 42. in [7]). In order to investigate the shape of the pulse we take the derivative of \(|E_k|^2\) with respect to \(\tau\). This gives the following conditions for maxima of \(|E_k|^2\),

\[ e^{2A} = |c|^2 - 2 \pm \sqrt{(|c|^2 - 2)^2 - |c|^2} = 0 \]  

(67)
It is clear from (67) that for $|c|$, such that $1 \leq |c| \leq 2$ the solution has a single peak. Interestingly however, for $|c| > 2$, there are two values (real) for $e^{2A}$, that is (62) has two maxima. This corresponds to two peaks. As $|c|$ increases further the two peaks gradually shifts apart from each other. Finally at $|c| \to \infty$, that is, when the two zeros of $\alpha_{2n+1} 2n+1$ merge on the imaginary line in the lower half-plane the solution (62) then reduces to,

$$E_k(z, \tau) = \frac{\lambda p_{k1}}{2n} \text{sech}(\eta \tau - \epsilon \eta^3 z + \frac{1}{12 \epsilon} z) e^{i(\eta^2 - \frac{1}{108\epsilon^2} \eta^3 z)}$$ \hspace{1cm} (68)

It is important to note that the phase factor arises in (68) is purely from the variable transformations (2,3, 4). The solution (68) represents an wave moving with a group velocity, $(\epsilon \eta^2 + \frac{1}{12 \epsilon})$. Unlike the earlier case (62), the group velocity depends only on the real part $\eta$ of $\lambda_1$. This is the solution reported in [21, 22]. It is important to note that the shape of the solitons remains invariant with respect to space and time for all values of $|c|$. The two class of solutions obtained for 1SS may be extended straightforwardly for $N$- soliton solutions.

7 Conclusion

In this paper we have studied the CHNLS equation by applying a generalized inverse scattering method developed to solve the $(2n + 1 \times 2n + 1)$ dimensional linear problem associated with (5) vis a vis (1). We have shown
the integrability of the system by showing the existence of infinite number of conserved quantities. The $N$-soliton solutions for the system have been obtained by solving a set of generalized GLM equation. We have shown two different class of solutions by considering the zero’s of the diagonal element of the scattering data on the imaginary line and a pair of zero’s lying symmetrically about the imaginary line in the lower-half plane. By a suitably defined parameter we have shown how the double-peak soliton reduces to the single peak soliton. The results, we have obtained predicts that CHNLS equation allows dispersionless propagation of the ultrashort optical soliton in the shape of single hump pulse or double hump pulse. This may have interesting consequences in the propagation of optical solitons through nonlinear fiber.

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