RELATING DERIVED EQUIVALENCES FOR SIMPLICES OF HIGHER-DIMENSIONAL FLOPS

W. DONOVAN

Abstract. I study a sequence of singularities in dimension 4 and above, each given by a cone of rank 1 tensors of a certain signature, which have crepant resolutions whose exceptional loci are isomorphic to cartesian powers of the projective line. In each dimension $n$, these resolutions naturally correspond to vertices of an $(n-2)$-simplex, and flops between them correspond to edges of the simplex. I show that each face of the simplex may then be associated to a certain relation between flop functors.

1. Introduction

This note is motivated by a desire to better understand the derived autoequivalence groups of 4-folds and beyond, in particular the contributions coming from birational geometry. I study triangles of birational maps between resolutions of certain singularities, and find that the corresponding flop functors obey a pleasing relation involving spherical twists, extending a well-known story for 3-folds related by Atiyah flops.

1.1. Singularities and resolutions. Consider the $n$-fold singular cone for $n \geq 3$ given by the rank 1 tensors of signature $2^{n-1}$ as follows.

$$Z = \{v_1 \otimes \cdots \otimes v_{n-1} \in V_1 \otimes \cdots \otimes V_{n-1} \} \quad V_i \cong \mathbb{C}^2$$

By a straightforward construction explained later, $Z$ has $n-1$ crepant resolutions. These will be written $X_i$ for $i = 1, \ldots, n-1$. Each is given by replacing the singularity $0 \in Z$ by an exceptional locus $\text{Exc} \cong (\mathbb{P}^1)^{n-2}$ which, for a given $i$, arises as a product of the $\mathbb{P}V_j$ for $j \neq i$.

For $n = 3$, we have the 3-fold Atiyah flop between two resolutions of the cone of singular $2 \times 2$ matrices. In general, assigning each resolution $X_i$ to a vertex of an $(n-2)$-simplex, the edges of the simplex correspond to birational maps $X_i \dashrightarrow X_j$ as illustrated in Figure 1. For $n = 4$ we have a triangle of 4-folds, and for $n = 5$ we have a tetrahedron of 5-folds.

2010 Mathematics Subject Classification. Primary 14F08; Secondary 14J32, 18G80.

Key words and phrases. Calabi–Yau manifolds, crepant resolutions, tensors, derived category, derived equivalence, birational geometry, flops, simplices.

I am supported by the Yau MSC, Tsinghua University, and the Thousand Talents Plan. I am grateful for travel support from EPSRC Programme Grant EP/R034826/1.
1.2. **Equivalences.** The birational maps appearing here are family Atiyah flops, and therefore have associated flop functors, which are derived equivalences, illustrated in Figure 2.

This work grew out of interest in describing ‘derived monodromy’ for Figure 2, namely an autoequivalence given by composition of a 3-cycle of equivalences in the triangle. However, it seems that a question with a neater answer is the following.

*How are routes between two of the vertices of Figure 2 related?*

Theorem A below gives such a relation where we take, without loss of generality, the two vertices $X_1$ and $X_3$. Furthermore it gives a relation associated to each face (2-simplex) in the analogous diagram for $n > 4$.

1.3. **Result.** I prove the following.

**Theorem A** (Theorems 4.3 and 5.9). For $n \geq 4$, write flop functors as follows.

Then there is a natural isomorphism

$$F_3 \cong Tw_2 \circ F_2 \circ F_1$$
where $Tw_2$ is one of the following.

**Case** $n = 4$: a spherical twist around the torsion sheaf $\mathcal{O}_{\text{Exc}}(0, -1)$ on $X_3$, where $\text{Exc} \cong \mathbb{P}V_1 \times \mathbb{P}V_2$, given in Definition 4.2.

**Case** $n > 4$: a family version of this spherical twist, given in Definition 5.6, over base $\mathbb{P}V_4 \times \cdots \times \mathbb{P}V_{n-1}$.

For each $n$, replacing the indices 1, 2, 3 in the above with general $i, j, k$ gives a relation for each face (2-simplex) of the $(n-2)$-simplex.

**Remark.** As a mnemonic for Theorem A, I draw a diagram as follows.

I prove Theorem A by calculating the action of flops and twists on a certain (relative) tilting bundle.

1.4. **Related questions.** As an immediate corollary of Theorem A, we get the following formula for $Tw_2$.

$$F_3 \circ F_1^{-1} \circ F_2^{-1} \cong Tw_2 \quad (*)$$

There are many formulas of the form ‘flop-flop = twist’ (up to taking inverses) in the literature, including [ADM, AL2, Bar, BB, DS1, DW1, DW3, JL, Har, Tod]. I hope (*) gives a hint of how they may be generalized to flop cycles of length 3 and above.

For discussion of the relation of Theorem A to derived monodromy, in particular to the autoequivalence of $D(X_1)$ given by a composition of 3 flop functors, see Remark 4.4.

1.5. **Atiyah flop.** For $n = 3$, we have resolutions $X_1$ and $X_2$ related by an Atiyah flop, and functors as follows, satisfying the relation shown, where $Tw$ is a spherical twist about $\mathcal{O}_{\text{Exc}}(-1)$.

$$D(X_1) \xrightarrow{F_1} D(X_2) \xrightarrow{F_2} \quad \text{id} \cong Tw \circ F_2 \circ F_1$$

The argument for Theorem A is an elaboration of a standard argument for this relation: see Section 4.2 for discussion.

1.6. **Related work.** The example $n = 4$ is studied extensively by Kite [Kit, Sections 5.3 and 7.2]. He gives an action of $\pi_1(\mathcal{M})$ on the derived categories $D(X_i)$.
where $\mathcal{M}$ is a certain Fayet–Iliopoulos parameter space. I expect Theorem A in the case $n = 4$ also follows from his methods.

Kite uses a realization of the $X_i$ as toric GIT quotients, and the technology of ‘fractional magic windows’. This may give an alternative, and perhaps swifter, method to prove Theorem A. However, as it was not needed to obtain the result, I feel the proofs here may be more accessible without it. I also have in mind extensions to more general flops, where a GIT presentation is not available.

Halpern-Leistner and Sam [HLSam] construct similar actions of $\pi_1$ for GIT problems which are ‘quasi-symmetric’. This condition does not hold for the GIT quotients realizing the $X_i$.

**Remark.** The constructions here readily generalize to the case of $V_i \cong \mathbb{C}^d$ for any fixed $d > 2$, and I expect that similar results can be proved, see Remark [2.8]. Kite notes that, for $n = 4$, the above ‘fractional magic windows’ technology may not apply in this new setting [KIT Example 4.37].

### 1.7. Contents

Section 2 describes the resolutions $X_i$ and flop functors between them. Section 3 gives properties of these functors for later use. Sections 4 and 5 prove Theorem A first in dimension 4 and then in higher dimension by a family construction. Section 6 gives further family constructions relating resolutions.

### 1.8. Notation

When I write $X^{(n)}$ and similar notations, the $(n)$ denotes the dimension $n$ of the space, or the relative dimension $n$ of a family. Letters L and R indicate derived functors throughout, but are sometimes dropped in Section 5 for the sake of readability. The bounded derived category of coherent sheaves is denoted by $\mathcal{D}(X)$.

**Acknowledgements.** I am grateful for conversations with Tatsuki Kuwagaki, Xun Lin, Mauricio Romo, Ed Segal, and Weilin Su, and thank the organizers of the conference ‘McKay correspondence, mutation and related topics’ at Kavli IPMU for their work to make the meeting a success during the pandemic. I started calculations for this project while visiting Michael Wemyss at University of Glasgow, and am grateful for his hospitality and support there.

## 2. Resolutions

We construct resolutions of the singularity $Z$ from the introduction, namely

$$Z = \{v_1 \otimes \cdots \otimes v_{n-1} \in V_1 \otimes \cdots \otimes V_{n-1} \} \quad V_i \cong \mathbb{C}^2$$

for $n \geq 3$. To see that $\dim Z = n$, note that $z \in Z - \{0\}$ taken up to scale determines a point of the cartesian power $(\mathbb{P}^1)^{n-1}$. There are various terminologies for $Z$. For instance, we may describe it as the cone of rank 1 tensors of signature $2^{n-1}$, or as the simple hypermatrices of order $n - 1$ in dimension 2.
2.1. Construction. We construct crepant resolutions $X_i$ of $Z$ for $i = 1, \ldots, n - 1$.

**Definition 2.1.** Let $X_1$ be the total space of a rank 2 bundle

$$X_1 : V_1 \otimes O(-1, \ldots, -1) \longrightarrow \mathbb{P}V_2 \times \cdots \times \mathbb{P}V_{n-1}$$

where the line bundle $O(-1, \ldots, -1)$ has degree $-1$ on each factor $\mathbb{P}V_i$. Other spaces $X_i$ are obtained similarly, by applying the cyclic symmetry of the set $\{V_i\}$ to replace $V_1$ by $V_i$.

**Notation 2.2.** Indices will be written in cyclic order. For instance, for $n = 4$ I write the following.

$$X_1 : V_1 \otimes O(-1, -1) \longrightarrow \mathbb{P}V_2 \times \mathbb{P}V_3$$

$$X_2 : V_2 \otimes O(-1, -1) \longrightarrow \mathbb{P}V_3 \times \mathbb{P}V_1$$

$$X_3 : V_3 \otimes O(-1, -1) \longrightarrow \mathbb{P}V_1 \times \mathbb{P}V_2$$

To see the resolution morphism $g_1 : X_1 \to Z$ write $X_1$ as follows, where $L_i$ denotes the tautological subspace bundle on $\mathbb{P}V_i$.

$$X_1 : V_1 \otimes (L_2 \boxtimes \cdots \boxtimes L_{n-1}) \longrightarrow \mathbb{P}V_2 \times \cdots \times \mathbb{P}V_{n-1}$$

Then the inclusions $L_i \subset V_i$ induce the required morphism, which is easily seen to contract the zero section $\text{Exc}_1 \subset X_1$ while being an isomorphism elsewhere. The other $g_i$ are obtained similarly.

**Notation 2.3.** Let $\text{Exc}_i \subset X_i$ be the exceptional locus of the resolution morphism $g_i : X_i \to Z$. I often drop subscripts and write $\text{Exc}$ for readability.

By the following proposition, the $g_i$ are crepant resolutions.

**Proposition 2.4.** Each $X_i$ is Calabi–Yau.

**Proof.** The total space of the bundle $X_1$ is Calabi–Yau because both its determinant, and the canonical bundle of its base $\mathbb{P}V_2 \times \cdots \times \mathbb{P}V_{n-1}$, are isomorphic to $O(-2, \ldots, -2)$. The same holds for the other $X_i$ by the cyclic symmetry. \hfill \Box

I set notation for sheaves and bundles on the $X_i$ for $n = 4$.

**Notation 2.5.** For $X_1$, let $O_{\text{Exc}}(a, b)$ denote a line bundle on $\text{Exc}_1 \cong \mathbb{P}V_2 \times \mathbb{P}V_3$ considered as a torsion sheaf on $X_1$. Let $O(a, b)$ denote a line bundle on $X_1$ given by pullback from the base $\mathbb{P}V_2 \times \mathbb{P}V_3$. Similar notations are used for the other $X_i$.

2.2. Flop functors. If we blow up the zero section $\text{Exc}_i \subset X_i$ for any $i$ we obtain

$$Q : O(-1, \ldots, -1) \longrightarrow \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{n-1}$$

with a blowup map $f_i : Q \to X_i$. Using these $f_i$ to form birational roofs, we have birational maps

$$\phi_{ji} : X_i \dashrightarrow X_j.$$
Definition 2.6. We write functors
\[ F_{ji} = Rf_{j*} \circ Lf_i^*: \mathcal{D}(X_i) \rightarrow \mathcal{D}(X_j). \]

These functors are equivalences. For \( n = 3 \) they are simply the Bondal–Orlov equivalences for the Atiyah flop. For \( n \geq 4 \), the \( \phi_{ji} \) are family Atiyah flops which implies that the \( F_{ji} \) are equivalences. This is explained in Section 3.1 for \( n = 4 \), and a similar argument using Proposition 6.3 suffices for general \( n \).

Remark 2.7. For \( n = 4 \) the birational maps \( \phi_{ji} \) may be drawn as follows. Note that at each \( X_i \) the two birational maps have the same exceptional locus \( \mathbb{P}^1 \times \mathbb{P}^1 \), but that each is a flop of a different ruling.

\[ \begin{array}{ccc}
X_1 & \sim & X_2 \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
X_3 & \swarrow & \searrow
\end{array} \]

Remark 2.8. The constructions in this section extend to the setting where \( V_i \cong \mathbb{C}^d \) for any fixed \( d \geq 2 \). The flops appearing are then families of standard flops of \( \mathbb{P}^{k-1} \). It would be interesting to prove similar results in this case. For comparison, see [ADM] for a ‘flop-flop = twist’ formula for such flops.

2.3. Notation for families. For a given \( n \), each resolution may be constructed as a non-trivial family of the analogous resolution for \( n - 1 \). I explain the \( n = 4 \) case in the following Section 3. The general case, which is analogous but notationally more complex, is deferred to Proposition 6.1. Here I give the notation that will be used in these constructions.

To specify a particular \( n \), the notation \( X_i^{(n)} \) is used. To specify furthermore the vector spaces used in the construction, I write the following.
\[ X_i^{(n)}(V_1, \ldots, V_{n-1}) \]

The construction will often be repeated in a family, replacing the vector spaces \( V_i \) with vector bundles \( \pi_i \) over some base \( B \). The result is denoted as follows.
\[ X_i^{(n)}(\pi_1, \ldots, \pi_{n-1}) \]

Finally, similar notations are used for other constructions, for instance the birational roof \( Q \) from Section 2.2 may be written as \( Q^{(n)}(V_1, \ldots, V_{n-1}) \).

3. Flop calculations

I explain how each resolution for \( n = 4 \) may be constructed as a non-trivial family of the analogous resolutions for \( n = 3 \), and use this to calculate the effect of the flop functors for \( n = 4 \) on certain objects, for use in the proof of Theorem 4.3.
This calculation is routine, but I write it out in full to show how the non-triviality is handled, and anticipating a further family version in Section 5.

3.1. Family construction. Recall that we take the following.

\[ X^{(4)}_1 : V_1 \otimes O(-1, -1) \to \mathbb{P}V_2 \times \mathbb{P}V_3 \]

The proposition below realizes \( X^{(4)}_1 \) as a family of copies of \( X^{(3)}_1 \) over \( \mathbb{P}V_3 \).

**Proposition 3.1.** Using the notation of Section 2.3, we have that

\[ X^{(4)}_1 (V_1, V_2, V_3) \cong X^{(3)}_1 (\pi_1, \pi_2) \]  

(3.A)

where we take bundles

\[ \pi_1 : V_1 \otimes O \to \mathbb{P}V_3 \]
\[ \pi_2 : V_2 \otimes O(-1) \to \mathbb{P}V_3 \]

so that \( \pi_1 \) is the trivial bundle with fibre \( V_1 \).

**Proof.** Writing \( \text{Tot}(\mathbb{P}\pi_2) \) for the total space of \( \mathbb{P}\pi_2 \), there is an isomorphism

\[ \text{Tot}(\mathbb{P}\pi_2) \cong \mathbb{P}V_2 \times \mathbb{P}V_3 \]

under which \( O_{\mathbb{P}\pi_2}(-1) \) corresponds to \( O(-1, -1) \), where the second \(-1\) comes from the definition of \( \pi_2 \), giving the claim. \( \square \)

We now extend the argument to birational roofs. We have an isomorphism

\[ X^{(4)}_2 (V_1, V_2, V_3) \cong X^{(3)}_2 (\pi_1, \pi_2) \]  

(3.B)

using that \( \text{Tot}(\mathbb{P}\pi_1) \cong \mathbb{P}V_3 \times \mathbb{P}V_1 \) under which \( O_{\mathbb{P}\pi_1}(-1) \) corresponds to \( O(0, -1) \). Furthermore, we have an isomorphism

\[ Q^{(4)}(V_1, V_2, V_3) \cong Q^{(3)}(\pi_1, \pi_2). \]  

(3.C)

The isomorphisms (3.A), (3.B) and (3.C) intertwine birational roof diagrams of blowup maps \( f_i : Q \to X_i \) as follows.

\[ X^{(4)}_1 (V_1, V_2, V_3) \quad X^{(4)}_2 (V_1, V_2, V_3) \quad X^{(3)}_1 (\pi_1, \pi_2) \quad X^{(3)}_2 (\pi_1, \pi_2) \]

The right-hand diagram is a family of 3-fold Atiyah flops. It follows that the flop functor \( F_{21} \) from Definition 2.6 using the left-hand diagram is an equivalence.

3.2. Flop functors. The above construction lets us calculate the effect of the flop \( X_1 \dashrightarrow X_2 \) for \( n = 4 \) on the derived category. Recall that \( O(a, b) \) denotes a certain line bundle on each of the \( X_i \) by the convention of Notation 2.5.

**Proposition 3.2.** For \( n = 4 \), the flop functor

\[ F : D(X_1) \to D(X_2) \]
acts as follows, where we write projections \( \rho_i : X_i \to \mathbb{P}V_3 \).

(1) For any \( \mathcal{B} \in \mathcal{D}(\mathbb{P}V_3) \), taking \( \mathcal{A}_1, \mathcal{A}_0 \in \mathcal{D}(X_1) \) given by
\[
\mathcal{A}_1 = \rho_1^* \mathcal{B} \otimes \mathcal{O}(-1, 0) \\
\mathcal{A}_0 = \rho_1^* \mathcal{B}
\]
we have the following.
\[
\begin{align*}
F(\mathcal{A}_1) & \cong \rho_2^* \mathcal{B} \otimes \mathcal{O}(1, 1) \\
F(\mathcal{A}_0) & \cong \rho_2^* \mathcal{B}
\end{align*}
\]
(2) There exist canonical inclusions, given at the end of the proof, such that the following diagram commutes, with \( \text{Hom} \) taken in the derived category.

\[
\begin{array}{ccc}
\text{Hom}(\mathcal{A}_1, \mathcal{A}_0) & \xrightarrow{F} & \text{Hom}(F(\mathcal{A}_1), F(\mathcal{A}_0)) \\
& \downarrow & \downarrow \\
& V_2' & \\
\end{array}
\]

**Proof.** The 4-fold \( X_1 \) is isomorphic to the family of 3-folds \( X_1^{(3)}(\pi_1, \pi_2) \) over \( \mathbb{P}V_3 \) by Proposition 3.1. Under this isomorphism \( \mathcal{A}_1, \mathcal{A}_0 \in \mathcal{D}(X_1) \) go to
\[
\rho_1^*(\mathcal{B} \otimes \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{P}V_3}(-1) \quad \text{and} \quad \rho_1^* \mathcal{B} \tag{3.D}
\]
where we reuse \( \rho_1 \) for the projection from \( X_1^{(3)}(\pi_1, \pi_2) \) to \( \mathbb{P}V_3 \). The twist \( \otimes \mathcal{O}(1) \) appearing here is dual to the twist in the definition of \( \pi_2 \). Then the flop
\[
X_1^{(3)}(\pi_1, \pi_2) \dashrightarrow X_2^{(3)}(\pi_1, \pi_2) \tag{3.E}
\]
is a family over \( \mathbb{P}V_3 \) of copies of the 3-fold Atiyah flop \( Y_1 \dashrightarrow Y_2 \) with
\[
\begin{align*}
Y_1 : & \quad W_1 \otimes \mathcal{O}(-1) \to \mathbb{P}W_2 \\
Y_2 : & \quad W_2 \otimes \mathcal{O}(-1) \to \mathbb{P}W_1
\end{align*}
\]
where \( \dim W_i = 2 \). Write \( G : \mathcal{D}(Y_1) \to \mathcal{D}(Y_2) \) for the flop functor.

The following description of the effect of \( G \) is obtained by standard arguments, see for instance [DS2, Proposition 1]. Letting
\[
\mathcal{B}_1 = \mathcal{O}(-1) \quad \mathcal{B}_0 = \mathcal{O}
\]
where \( \mathcal{O}(k) \) denotes a bundle on \( Y_i \) obtained by pullback from \( \mathbb{P}W_i \), we have
\[
G(\mathcal{B}_1) = \mathcal{O}(+1) \quad G(\mathcal{B}_0) = \mathcal{O}.
\]
Furthermore we have a commutative triangle

\[
\begin{array}{ccc}
\text{Hom}(\mathcal{B}_1, \mathcal{B}_0) & \xrightarrow{G} & \text{Hom}(G(\mathcal{B}_1), G(\mathcal{B}_0)) \\
& \downarrow & \downarrow \\
& W_2' & \\
\end{array}
\]
where the inclusions are given by observing the following.

\[
\text{Hom}(B_1, B_0) \cong R\Gamma_{Y_1} \mathcal{O}(+1) \cong R\Gamma_{FV^2_1}(\mathcal{O}(+1) \otimes \text{Sym}^* (W_1^\vee \otimes \mathcal{O}(1))) \\
\text{Hom}(G(B_1), G(B_0)) \cong R\Gamma_{Y_2} \mathcal{O}(-1) \cong R\Gamma_{FV^1_1}(\mathcal{O}(-1) \otimes \text{Sym}^* (W_2^\vee \otimes \mathcal{O}(1)))
\]

These have no higher cohomology by standard vanishing on \(\mathbb{P}^1\), and after taking 0th cohomology, the pieces coming from \(\text{Sym}^0\) and \(\text{Sym}^1\) respectively are both \(W_2^\vee\).

Now repeating the standard arguments in a family, we find that the flop functor for the flop (3.4) applied to the objects (3.2) gives

\[
\rho_2^*(\mathcal{B} \otimes \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(+1) \quad \text{and} \quad \rho_2^* \mathcal{B}
\]

where we reuse \(\rho_2\) for the projection from \(X_2^{(3)}(\pi_1, \pi_2)\) to \(\mathbb{P}V_3\). Under the isomorphism (3.1) it is easily seen that the objects (3.5) go to the required objects on \(X_2\).

For the last part, we also repeat the argument for the Atiyah flop in a family. The inclusions in the statement are given by observing the following.

\[
\text{Hom}(A_1, A_0) \cong R\Gamma_{X_1} \mathcal{O}(1, 0) \\
\cong R\Gamma_{\mathbb{P}V_2 \times \mathbb{P}V_3}(\mathcal{O}(1, 0) \otimes \text{Sym}^* (V_1^\vee \otimes \mathcal{O}(1, 1))) \\
\text{Hom}(F(A_1), F(A_0)) \cong R\Gamma_{X_2} \mathcal{O}(-1, -1) \\
\cong R\Gamma_{\mathbb{P}V_3 \times \mathbb{P}V_1}(\mathcal{O}(-1, -1) \otimes \text{Sym}^* (V_2^\vee \otimes \mathcal{O}(1, 1)))
\]

These have no higher cohomology by standard vanishing on \(\mathbb{P}^1 \times \mathbb{P}^1\), and after taking 0th cohomology, the pieces coming from \(\text{Sym}^0\) and \(\text{Sym}^1\) respectively are both \(V_2^\vee\). \(\Box\)

The following describes the action of the flop functors on certain line bundles, again using the conventions of Notation 2.5.

**Proposition 3.3.** For \(n = 4\), considering the flop functors

\[
\begin{array}{ccc}
D(X_1) & \xrightarrow{F_{21}} & D(X_2) \\
\downarrow & & \downarrow \\
F_{31} & \xrightarrow{F_{31}} & D(X_3)
\end{array}
\]

between

\[
X_1 : \ V_1 \otimes \mathcal{O}(-1, -1) \rightarrow \mathbb{P}V_2 \times \mathbb{P}V_3 \\
X_2 : \ V_2 \otimes \mathcal{O}(-1, -1) \rightarrow \mathbb{P}V_3 \times \mathbb{P}V_1 \\
X_3 : \ V_3 \otimes \mathcal{O}(-1, -1) \rightarrow \mathbb{P}V_1 \times \mathbb{P}V_2
\]
we have the following.

\[ F_{21} : \quad O(0, b) \mapsto O(b, 0) \]
\[ \quad O(-1, b) \mapsto O(b + 1, 1) \]

\[ F_{31} : \quad O(a, 0) \mapsto O(0, a) \]
\[ \quad O(a, -1) \mapsto O(1, a + 1) \]

Remark 3.4. From this proposition we may deduce the action of all \( F_{ji} \) by cyclic symmetry. Because of our conventions, the statements for \( F_{32} \) and \( F_{13} \) read the same as for \( F_{21} \), and so on.

Proof. The claim for \( F_{21} \) is from Proposition 3.2 using isomorphisms as follows, and the claim for \( F_{31} \) is obtained by symmetry.

\[ O(0, b) \cong \rho_1^*O(b) \quad \text{and} \quad O(-1, b) \cong \rho_1^*O(b) \otimes O(-1, 0) \quad \text{on } X_1 \]
\[ O(b, 0) \cong \rho_2^*O(b) \quad \text{and} \quad O(b + 1, 1) \cong \rho_2^*O(b) \otimes O(1, 1) \quad \text{on } X_2 \]

Remark 3.5. Propositions 3.2 and 3.3 may also be obtained directly by adapting an argument of Kawamata [Kaw, Proposition 3.1], after calculating the canonical bundles of the roofs \( Q \).

The following straightforward observation about the flop functors \( F_{ji} \) will be used in the proof of Theorem 4.3.

**Proposition 3.6.** Writing restriction functors

\[ \text{res}_i : \mathcal{D}(X_i) \to \mathcal{D}(X_i - \text{Exc}_i) \]

we have intertwinements

\[ \text{res}_j \circ F_{ji} \cong g_{ji*} \circ \text{res}_i \]

where we write \( g_{ji} : X_i - \text{Exc}_i \to X_j - \text{Exc}_j \) for the isomorphism induced by the birational map \( \phi_{ji} \).

Proof. Each \( \text{res}_i \) is by definition a pullback along an open immersion, so this follows using flat base change. \qed

4. Dimension 4

Here I prove Theorem A for dimension 4. Though the argument is standard, I write it in detail, anticipating a family version in Section 5. After the proof, some discussion of the method is given in Section 4.2.

4.1. **Proof.** Recall that \( \text{Exc} \cong \mathbb{P}V_1 \times \mathbb{P}V_2 \) is the exceptional locus of \( X_3 \).

**Proposition 4.1.** The object \( \mathcal{E} = \mathcal{O}_{\text{Exc}}(0, -1) \) is spherical in \( \mathcal{D}(X_3) \).
Proof. Noting that $X_3$ is Calabi–Yau by Proposition 2.4, we require
\[ \text{Hom}(\mathcal{E}, \mathcal{E}) \cong H^*(S^4) \]
which follows by a standard spectral sequence calculation, using that for the normal bundle $\mathcal{N}$ of Exc we have
\[ \mathcal{N} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)^{\oplus 2} \]
\[ \Lambda^2 \mathcal{N} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -2). \]
See for instance [Huy, Examples 8.10(v)] for a similar calculation. \qed

Definition 4.2. Take the spherical twist autoequivalence on $X_3$
\[ \text{Tw}_2 = \text{Tw}(\mathcal{O}_{\text{Exc}}(0, -1)) \]
where the subscript 2 is used because pullback of $\mathcal{O}_{\mathbb{P}^2}(-1)$ gives $\mathcal{O}_{\text{Exc}}(0, -1)$. The autoequivalence $\text{Tw}(\mathcal{E})$ is defined so that there is a triangle of Fourier–Mukai functors
\[ \text{Tw}(\mathcal{E}) = \text{Cone}(\text{RHom}_{X_3}(\mathcal{E}, -)^L \otimes \mathcal{E} \to \text{id}), \]
see [ST] or [Huy, Section 8.1].

Similarly, we have an autoequivalence on $X_2$
\[ \text{Tw}_3 = \text{Tw}(\mathcal{O}_{\text{Exc}}(-1, 0)) \]
where the subscript 3 is used because pullback of $\mathcal{O}_{\mathbb{P}^3}(-1)$ gives $\mathcal{O}_{\text{Exc}}(-1, 0)$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (X1) at (0,0) {$\text{D}(X_1)$};
\node (X2) at (2,0) {$\text{D}(X_2)$};
\node (X3) at (2,-2) {$\text{D}(X_3)$};
\node (F21) at (1,-1) {$\text{F}_{21}$};
\node (F23) at (1,-1.5) {$\text{F}_{23}$};
\node (F31) at (1,-2.5) {$\text{F}_{31}$};
\node (F32) at (1,-2) {$\text{F}_{32}$};
\node (Tw2) at (3,0) {$\text{Tw}_2$};
\node (Tw3) at (3,-2) {$\text{Tw}_3$};

\draw[->] (X1) to (X2);
\draw[->] (X1) to (X3);
\draw[->] (X2) to (X3);
\draw[->] (F21) to (F23);
\draw[->] (F21) to (F31);
\draw[->] (F21) to (F32);
\draw[->] (F23) to (F31);
\draw[->] (F23) to (F32);
\draw[->] (F31) to (F32);
\draw[->, dashed] (Tw2) to (X1);
\draw[->, dashed] (Tw3) to (X2);
\draw[->, dashed] (Tw3) to (X3);
\end{tikzpicture}
\caption{Relations between functors from Theorem 4.3}
\end{figure}

Theorem 4.3 (Theorem A). For $n = 4$ there are natural isomorphisms
\[ F_{21} \cong \text{Tw}_3 \circ F_{23} \circ F_{31} \]
\[ F_{31} \cong \text{Tw}_2 \circ F_{32} \circ F_{21} \]
of functors from $\text{D}(X_1)$ to $\text{D}(X_2)$ and $\text{D}(X_3)$ respectively, illustrated in Figure 3.

Proof. I prove the second statement, as the first follows by the same argument. Using the results of Section 3.2 we evaluate the functors in the proposed isomorphism on the following bundle $\mathcal{T}$ on $X_1$. Note that $\mathcal{T}$ is tilting, by standard methods using the Beilinson tilting bundles $\mathcal{O} \oplus \mathcal{O}(-1)$ on $\mathbb{P}^V_2$ and $\mathbb{P}^V_3$.
\[ \mathcal{T} = \mathcal{O} \oplus \mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, -1) \]
By applying functors to the summands using Proposition 3.3, we find that
\[ F_{21}(\mathcal{T}) \cong \mathcal{O} \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,1) \]
\[ F_{31}(\mathcal{T}) \cong \mathcal{O} \oplus \mathcal{O}(0,-1) \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,0) \]
and furthermore by Remark 3.4 that
\[ F_{32}F_{21}(\mathcal{T}) \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,0), \]
where we let
\[ \mathcal{F} = F_{32}(\mathcal{O}(1,1)) \cong F_{32}F_{21}(\mathcal{O}(-1,0)). \] (4.A)
This \( \mathcal{F} \) may be described as follows.
\[
\mathcal{F} \cong F_{32} \text{Cone}(\land^2 V_3^\vee \otimes \mathcal{O}(-1,1) \xrightarrow{\psi_2} V_3^\vee \otimes \mathcal{O}(0,1)) \\
\cong \text{Cone}(\land^2 V_3^\vee \otimes \mathcal{O}(2,1) \xrightarrow{\psi_3} V_3^\vee \otimes \mathcal{O}(1,0))
\]
The first line uses the pullback of the Euler short exact sequence from \( \mathbb{P}V_3 \cong \mathbb{P}^1 \) to \( X_2 \). The second line follows by Proposition 3.3, where we let \( \psi_3 = F_{32}(\psi_2) \).

To describe \( \psi_3 \) we apply Proposition 3.2(2) with \( B = \mathcal{O}(1) \) on \( \mathbb{P}V_1 \) and cycling the indices as in Remark 3.4. As in this proposition, \( \text{Hom}(\mathcal{O}(-1,1), \mathcal{O}(0,1)) \) on \( X_2 \) has a canonical summand \( V_3^\vee \). By construction, \( \psi_2 \) is induced by this summand. If follows from the proposition that \( \psi_3 \) is induced by the canonical summand \( V_3^\vee \) of \( \text{Hom}(\mathcal{O}(2,1), \mathcal{O}(1,0)) \) on \( X_3 \). Using the Koszul resolution of the exceptional locus \( \text{Exc} \) on \( X_3 \), we therefore find the following.
\[
\mathcal{F} \cong \text{Cone}(\mathcal{O}(0,-1) \xrightarrow{\text{res}} \mathcal{O}_{\text{Exc}}(0,-1))[\![-1] \\
\cong \text{Tw}_2^{-1}(\mathcal{O}(0,-1)) \\
\cong \text{Tw}_2^{-1}F_{31}(\mathcal{O}(-1,0)) \] (4.B)
For the second isomorphism we use that
\[
\text{Tw}_2^{-1} = \text{Tw}(\mathcal{E})^{-1} \cong \text{Cone}(\text{id} \rightarrow \text{RHom}_{X_3}(-, \mathcal{E})^\vee \otimes \mathcal{E})[\![-1] \] (4.C)
with \( \mathcal{E} = \mathcal{O}_{\text{Exc}}(0,-1) \), where the morphism is an adjunction unit and
\[
\text{RHom}_{X_3}(\mathcal{O}(0,-1), \mathcal{O}_{\text{Exc}}(0,-1))^\vee \cong \Gamma_{\text{Exc}}(\mathcal{O}_{\text{Exc}})^\vee = \mathbb{C} \] (4.D)
byprojectivity.

I now argue that we have the following.
\[
\text{Tw}_2^{-1}F_{31}(\mathcal{T}) \cong F_{32}F_{21}(\mathcal{T}) \] (4.E)
We first take a splitting \( \mathcal{T} = \mathcal{U} \oplus \mathcal{F} \) with \( \mathcal{U} \) given below, and show that (4.E) holds with \( \mathcal{T} \) replaced by \( \mathcal{U} \).
\[
\mathcal{U} = \mathcal{O} \oplus \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,-1)
\]
By the above argument \( F_{31}(\mathcal{U}) \cong F_{32}F_{21}(\mathcal{U}) \) with
\[
F_{31}(\mathcal{U}) \cong \mathcal{O} \oplus \mathcal{O}(1,1) \oplus \mathcal{O}(1,0). \]
This is in the kernel of
\[ \text{RHom}_{X_3}(-, O_{\text{Exc}}(0, -1)), \]
and therefore is unchanged up to isomorphism by applying \( \text{Tw}_{-1}^2 \). We deduce that (4.E) holds with \( T \) replaced by \( U \). Combining with the definition (4.A) and description (4.B) of \( F \), we conclude (4.E).

Finally, we prove the claim by considering the ‘difference’ of the two sides of the claimed isomorphism, namely proving the following natural isomorphism of functors on \( D(X_1) \).

\[ \Psi = F_{-1}^{21} \circ F_{-1}^{32} \circ \text{Tw}_{-1}^2 \circ F_{31} \cong \text{id} \quad (4.F) \]

By (4.E), \( \Psi(T) \cong T \), so \( \Psi \) induces a composition as follows.

\[ \text{End}(T) \xrightarrow{\sim} \text{End}(\Psi(T)) \cong \text{End}(T) \quad (4.G) \]

We will show this is the identity, and thence that (4.F) holds by the tilting equivalence.

Recall the restriction functors \( \text{res}_i : D(X_i) \to D(X_i - \text{Exc}) \) of Proposition 3.6. We have

\[ \text{res}_1 \circ \Psi \cong \text{res}_1 \]

by combining the intertwinements of Proposition 3.6 with

\[ \text{res}_2 \circ \text{Tw}_{-1}^2 \cong \text{res}_2 \quad (4.H) \]

which follows from definition of the twist, in particular that the spherical object is supported on \( \text{Exc} \). It follows immediately that (4.G) intertwines \( \Psi \) with the identity on \( \text{End}(\text{res}_1 T) \). Noting that \( \text{res}_1 T \) is just the bundle \( T|_{X_1 - \text{Exc}} \), that \( \text{Exc} \) is codimension 2, and \( X_1 \) is smooth thence normal, we deduce that (4.G) is the identity. Using that \( T \) is tilting so that there is an equivalence \( D(\text{End}(T)) \cong D(X_1) \), we find that (4.F) holds, and this completes the proof. \( \square \)

**Remark 4.4.** I briefly explain how Theorem 4.3 above relates to calculating the derived monodromy around the triangle formed by the \( D(X_i) \), namely, to determining the composition \( F_{13} \circ F_{32} \circ F_{21} \). Using the theorem we have the following.

\[ F_{13} \circ F_{32} \circ F_{21} \cong F_{13} \circ \text{Tw}_{-1}^2 \circ F_{31} \]

\[ \cong (F_{13} \circ F_{31}) \circ (F_{31}^{-1} \circ \text{Tw}_{-1}^2 \circ F_{31}) \]

The two brackets may then be calculated by standard techniques. The first may be expressed as a product of twists of spherical objects by, for instance, flop-flop formulas for toric variation of GIT in [HLShi]. The second may be expressed as a twist by a spherical object using the following.

\[ \Phi \circ \text{Tw}(\mathcal{E}) \circ \Phi^{-1} \cong \text{Tw}(\Phi \mathcal{E}) \quad \iff \quad \Phi \circ \text{Tw}^{-1}(\mathcal{E}) \circ \Phi^{-1} \cong \text{Tw}^{-1}(\Phi \mathcal{E}) \]

It would be interesting to carry out this calculation, for this example and more generally.
4.2. Discussion. For some intuition for the above Theorem 4.3, I include Figure 4 showing the action of the flop functors on line bundles on the $X_i$. The arrows in Figure 4 indicate source and target for the given functor, up to isomorphism. The dotted arrows indicate the same thing, but where the target $\mathcal{O}(a, b)$ should be replaced with

$$\text{Tw}^{-1}_{\mathcal{O}_{\text{Exc}}(a, b)} \mathcal{O}(a, b) \cong \text{Cone}(\mathcal{O}(a, b) \to \mathcal{O}_{\text{Exc}}(a, b))[-1] \cong \mathcal{I}_{\text{Exc}}(a, b)$$

using a similar argument to the proof of Theorem 4.3. Note also that each flop functor takes $\mathcal{O}$ to $\mathcal{O}$. Therefore in the diagrams each flop “cycles” the bundles by $2\pi/3$, up to spherical twists.

Inspecting Figure 4 we see, for instance, that $F_{32} \circ F_{21}$ and $F_{31}$ give the same results on all summands of the tilting bundle $\mathcal{T}$ except for $\mathcal{O}(-1, 0)$. The spherical twist $\text{Tw}_2$ in Theorem 4.3 accounts precisely for the disparity.

![Diagram](image_url)

(A) Effect of $F_{21}, F_{32}, F_{13}$.

(B) Effect of $F_{31}, F_{12}, F_{23}$.

Figure 4. Flop functors for $n = 4$, with summands of $\mathcal{T}$ highlighted.

Remark 4.5. There is a similar diagram, albeit simpler and more well-known, for the case $n = 3$, given in Figure 5. Here we have $X_1$ and $X_2$ related by an Atiyah flop, and functors as follows.

$$D(X_1) \xrightarrow{F_{21}} D(X_2) \xleftarrow{F_{12}}$$

Then Figure 5 shows the action of the flop functors. The dotted arrow indicates that the target $\mathcal{O}(a)$ should be replaced with

$$\text{Tw}^{-1}_{\mathcal{O}_{\text{Exc}}(a)} \mathcal{O}(a) \cong \text{Cone}(\mathcal{O}(a) \to \mathcal{O}_{\text{Exc}}(a))[-1] \cong \mathcal{I}_{\text{Exc}}(a)$$

by standard arguments. This can be used to prove the relation $\text{id} \cong \text{Tw} \circ F_2 \circ F_1$ from Section 1.5 for the Atiyah flop, by a simple analogue of the argument of Theorem 4.3. Namely, we check the relation on the tilting bundle $\mathcal{O}(-1) \oplus \mathcal{O}$, and then deduce that it holds in general.
5. Higher dimension

Having proved the result for \( n = 4 \) in Section 4, I explain how to extend to \( n > 4 \) by a family construction.

5.1. Family construction. I realize \( X_1^{(n)} \) as a family of \( X_1^{(4)} \) over base

\[ B = \mathbb{P}V_4 \times \cdots \times \mathbb{P}V_{n-1}. \]

**Proposition 5.1.** For \( n > 4 \) we have for \( i = 1, 2, 3 \)

\[ X_i^{(n)}(V_1, \ldots, V_{n-1}) \cong X_i^{(4)}(\pi_1, \pi_2, \pi_3) \]

where we take the following bundles.

\[ \begin{align*}
\pi_1 : & \quad V_1 \otimes \mathcal{O} \longrightarrow B \\
\pi_2 : & \quad V_2 \otimes \mathcal{O} \longrightarrow B \\
\pi_3 : & \quad V_3 \otimes \mathcal{O}(-1, \ldots, -1) \longrightarrow B
\end{align*} \]

**Proof.** Recalling that we have

\[ X_1^{(n)} : \quad V_1 \otimes \mathcal{O}(-1, -1, -1, \ldots, -1) \longrightarrow \mathbb{P}V_2 \times \mathbb{P}V_3 \times \mathbb{P}V_4 \times \cdots \times \mathbb{P}V_{n-1} \]

the result follows by the methods of Section 3.1. \( \square \)

Proposition 5.1 also holds trivially for \( n = 4 \), if we take \( B \) to be a point.

**Remark 5.2.** The isomorphisms of Proposition 5.1 express each \( n \)-fold as a family of \( 4 \)-folds, similarly to how the isomorphisms (3.A) and (3.B) expressed \( 4 \)-folds as a family of \( 3 \)-folds.

5.2. Family spherical twist. I construct a family spherical twist on \( X_3 \) for \( n \geq 4 \) generalizing the twist \( \text{Tw}_2 \) for \( n = 4 \) from Definition 4.2. Via the isomorphism of Proposition 5.1 the exceptional locus on \( X_3 \) is

\[ \text{Exc} = \mathbb{P}_{\pi_1} \times \mathbb{P}_{\pi_2}, \]

a bundle over \( B \) with fibre \( \mathbb{P}V_1 \times \mathbb{P}V_2 \). Write

\[ \mathcal{O}_{\text{Exc}}(a, b) = \mathcal{O}_{\mathbb{P}\pi_1}(c) \boxtimes_{\mathbb{P}} \mathcal{O}_{\mathbb{P}\pi_2}(d) \]

and consider this as a torsion sheaf on \( X_3 \) via the inclusion. This is a relative analog of Notation 2.5. Continuing the analogy, write \( \mathcal{O}(a, b) \) for the pullback of \( \mathcal{O}_{\text{Exc}}(a, b) \) to \( X_3 \), and use similar notation for the other \( X_i \).
I define the following functor, which I will show to be spherical. Here and in the next subsection, the letters L and R on derived functors are often dropped for the sake of readability.

**Definition 5.3.** Take the functor

\[ S = \mathcal{E} \otimes \tau^*(-) : \mathcal{D}(B) \to \mathcal{D}(X_3) \]

where \( \mathcal{E} = \mathcal{O}_{\text{Exc}}(0, -1) \) on \( X_3 \) and \( \tau : X_3 \to B \) denotes the projection morphism.

**Proposition 5.4.** We have adjoints to \( S \) as follows.

\[ L = \tau_! \text{Hom}(\mathcal{E}, -) \cong \tau_* \text{Hom}(-, \mathcal{E})^\vee \]
\[ R = \tau_* \text{Hom}(\mathcal{E}, -) \]

*Proof.* First note that \( \mathcal{E}^\vee \otimes - \), where we take derived dual, is a two-sided adjoint to \( \mathcal{E} \otimes - \). We then use that \( \tau_! \dashv \tau_* \dashv \tau^* \). The isomorphism follows after noting that \( \tau_! \cong \tau_*(\mathcal{E})^\vee \). \( \square \)

**Proposition 5.5.** \( S \) is a spherical functor.

*Proof.* Using the framework of Anno–Logvinenko [AL1], \( S \) yields a cotwist endofunctor \( C \) of \( \mathcal{D}(B) \), satisfying the following.

\[ C \cong \text{Cone(id} \to RS)[1] \]

It suffices that this is an equivalence, along with a certain Calabi–Yau condition, namely that a canonical natural transformation \( R \to CL[1] \) is an isomorphism.

Now we have that

\[ RS \cong \tau_* \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \tau^*(-)) \]
\[ \cong \tau_*(\text{Hom}(\mathcal{E}, \mathcal{E}) \otimes \tau^*(-)) \]
\[ \cong \tau_* \text{Hom}(\mathcal{E}, \mathcal{E}) \otimes - \]

where all functors are derived. Using a family version of the argument of Proposition 4.1, we may then calculate \( \tau_* \text{Hom}(\mathcal{E}, \mathcal{E}) \) and obtain an isomorphism

\[ RS \cong \text{id} \oplus (- \otimes \omega_B)[-4] \]

where \( \omega_B \cong \mathcal{O}(-2, \ldots, -2) \). The \( \omega_B \) in this calculation arises from the determinant of the relative normal bundle of \( \text{Exc} \) over \( B \), namely

\[ \wedge^2 \mathcal{N}_{\text{rel}} \cong \mathcal{O}_{\text{Exc}}(-2, -2) \otimes \sigma^* \omega_B \]

where we take projection \( \sigma : \text{Exc} \to B \). It follows that the cotwist

\[ C \cong (- \otimes \omega_B)[-5] \]

which is an autoequivalence.

I briefly explain how the Calabi–Yau condition mentioned above is obtained from the Calabi–Yau property of the target space \( X_3 \). Note that for the fibration
\( \tau: X_3 \to B \) we have \( \omega_\tau \cong \tau^* \omega_B \) because \( X_3 \) is Calabi–Yau by Proposition 2.4. Then recall
\[
\tau_! = \tau_* (\tau^* \omega_B) = \tau_* (\tau^* \omega_B) \cong \tau_* (\tau^* \omega_B)[4]
\]
where we use the projection formula. The condition then follows using Proposition 5.4. \( \square \)

**Definition 5.6.** Take the spherical twist autoequivalence on \( X_3 \)

\[ \text{Tw}_2 = \text{Tw}(S) \]

where the subscript 2 is used because under the isomorphism \( \text{Exc} \cong \mathbb{P}V_1 \times \mathbb{P}V_2 \times B \) the bundle \( \mathcal{O}_{\text{Exc}}(0, -1) \) on \( \text{Exc} \) appearing in the definition of \( S \) is the pullback of \( \mathcal{O}_{\mathbb{P}V_2}(-1) \). The autoequivalence \( \text{Tw}(S) \) is defined so that there is a triangle of Fourier–Mukai functors

\[ \text{Tw}(S) \cong \text{Cone}(SR \to \text{id}) \]

as explained in [AL1].

Similarly, we have an autoequivalence \( \text{Tw}_3 \) on \( X_2 \), by repeating the construction of this subsection using instead \( \mathcal{E} = \mathcal{O}_{\text{Exc}}(-1, 0) \) on \( X_2 \) and projection \( \tau: X_2 \to B \).

**Remark 5.7.** Definition 5.6 reduces to Definition 4.2 when \( n = 4 \), using Proposition 5.4.

**Remark 5.8.** The above twists can be formulated as EZ-twists [Hor], see for instance [Huy, Definition 8.43]. Indeed, taking projection \( \sigma: \text{Exc} \to B \) and inclusion \( i: \text{Exc} \to X_3 \) we get that

\[ S \cong i_*(\mathcal{E} \otimes \sigma^*(-)) \]

using \( \tau \circ i = \sigma \) and the projection formula.

5.3. **Proof.** The following uses a family version of the argument of Theorem 4.3 to generalize the result there to dimension \( n > 4 \).

**Theorem 5.9 (Theorem A).** For \( n \geq 4 \) there are natural isomorphisms

\[ F_{21} \cong \text{Tw}_3 \circ F_{23} \circ F_{31} \]
\[ F_{31} \cong \text{Tw}_2 \circ F_{32} \circ F_{21} \]

of functors from \( D(X_1) \) to \( D(X_2) \) and \( D(X_3) \) respectively, where the twists \( \text{Tw} \) are defined in Definition 5.6 and following it.

**Proof.** As before, we prove the second statement, with the first following by the same argument. We replace the spaces \( X_i \) for \( i = 1, 2, 3 \) in the proof of Theorem 4.3 with

\[ X_i^{(4)}(\pi_1, \pi_2, \pi_3), \]

writing projections \( \tau_i: X_i^{(4)} \to B \), or simply \( \tau \). The proof proceeds by repeating the argument of Theorem 4.3 in a family over \( B \). I explain the key modifications needed for this relative context.
The tilting bundle $\mathcal{T}$ of Theorem 4.3, namely

$$\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(-1, 0) \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, -1)$$

makes sense in the relative context using the notation of Section 5.2. It is now a relative tilting bundle over $B$, as follows. Taking $\mathcal{R} = \tau_* \mathcal{E}nd(\mathcal{T})$, a sheaf of algebras on $B$, we have

$$\mathcal{D}(\mathcal{R}) \cong \mathcal{D}(X_1)$$

where we take right $\mathcal{R}$-modules, with mutually inverse equivalences as follows.

$$\mathcal{R} \tau_* \mathcal{H}om(\mathcal{T}, -) \xrightarrow{\tau^{-1}(-) \otimes_{\tau^{-1} \mathcal{R}} \mathcal{T}}$$

The description of the flop functors is similar to Propositions 3.2 and 3.3. We replace the $\mathcal{A}_i$ as written there with $\mathcal{A}_i(\mathcal{C})$ for any $\mathcal{C} \in \mathcal{D}(B)$, where we put

$$\mathcal{A}(\mathcal{C}) = \tau^{-1}(\mathcal{C}) \otimes_{\tau^{-1} \mathcal{O}_B} \mathcal{A}.$$ 

In place of the commuting diagram in Proposition 3.2(2) it suffices to prove the result with the following diagram.

$$\xymatrix{ \tau_* \mathcal{H}om(\mathcal{A}_1, \mathcal{A}_0) \ar[rr]^F \ar[rd]_{V_2^\vee \otimes \mathcal{O}} & & \tau_* \mathcal{H}om(F(\mathcal{A}_1), F(\mathcal{A}_0)) \ar[ld] \ar[dd] \ar@{.>}[др] \ar@{.>}[l] \\
& V_3^\vee \otimes \mathcal{O}(1, \ldots, 1) & \mathcal{F} }$$

Given this, the calculation of the action of the flop functors on $\mathcal{T}$ at the beginning of the proof of Theorem 4.3 proceeds in the relative context. Note that when Proposition 3.2(2) is applied to describe $\psi_3$, it takes the following form, with the twist $\mathcal{O}(1, \ldots, 1)$ being dual to the twist in the definition of $\pi_3$.

$$\xymatrix{ \tau_* \mathcal{H}om(\mathcal{A}_1, \mathcal{A}_0) \ar[rr]^F \ar[rd]_{V_3^\vee \otimes \mathcal{O}(1, \ldots, 1)} & & \tau_* \mathcal{H}om(F(\mathcal{A}_1), F(\mathcal{A}_0)) \ar[ld] \ar[dd] \ar@{.>}[др] \ar@{.>}[l] \\
& \mathcal{F} & }$$

For equation (4.B) in the relative context we have

$$\mathcal{F} = F_{32} F_{21} (\mathcal{O}(-1, 0))$$

$$\cong \text{Cone}(\mathcal{O}(0, -1) \xrightarrow{\text{res}} \mathcal{O}_{\text{Exc}}(0, -1))[-1]$$

$$\cong \text{Tw}_2^{-1} (\mathcal{O}(0, -1))$$

$$\cong \text{Tw}_2^{-1} F_{31} (\mathcal{O}(-1, 0))$$

$$\cong \text{Cone}(\text{id} \rightarrow SL)[-1]$$

(5.A)
and (4.D) is replaced by
\[
L(O(0, -1)) \cong \tau_* \mathcal{H}om(O(0, -1), \mathcal{E})^\vee \\
\cong \tau_* \mathcal{H}om(O(0, -1), O_{Exc}(0, -1))^\vee \\
\cong \sigma_*(O_{Exc})^\vee \\
\cong O_B.
\]
Here \( \sigma : \text{Exc} \to B \) denotes the projection morphism, and we obtain the last line using that \( \sigma \) is a bundle with fibre \( \mathbb{P}V_1 \times \mathbb{P}V_2 \). Finally, we use \( S(O_B) \cong \mathcal{E} = O_{Exc}(0, -1) \) to get (5.A).

The analogs of the intertwinements of Proposition 3.6 and (4.H) follow by similar arguments, using that all the Fourier–Mukai functors are relative to \( B \), that is their kernels are pushed forward from the fibre product over \( B \).

The argument concludes by showing the following.
\[
\Psi = F_{21}^{-1} \circ F_{32}^{-1} \circ T_{\mathbb{P}^1}^{-1} \circ F_{31} \cong \text{id}
\]

For this, we study the endomorphism induced by \( \Psi \) of the sheaf of algebras \( \mathcal{R} = \tau_* \mathcal{E}nd(\mathcal{T}) \), similarly to the end of the proof of Theorem 4.3.

\[\square\]

Remark 5.10. It would be interesting to give a global analog of Theorem \( \Delta \) taking, for instance, a collection of quasiprojective \( Y_i \) having a diagram of birational maps as for the \( X_i \), and further having formal completions isomorphic to the formal completions of the \( X_i \) along \( \text{Exc}_i \). A first step could be to extend existing methods for proving relations between derived equivalences on 3-folds to this setting, for instance [DW1, Section 7.6] which follows [Tod].

6. Family constructions

I conclude with some straightforward constructions which realize each \( n \)-fold resolution as a family of \( k \)-fold resolutions for some \( k < n \). I first explain the statement for \( k = n - 1 \). This coincides with Proposition 3.1 for \( n = 4 \) and a case of Proposition 5.1 for \( n = 5 \).

Proposition 6.1. For \( n \geq 4 \) we have for \( i = 1, \ldots, n - 2 \)
\[
X_i^{(n)}(V_1, \ldots, V_{n-1}) \cong X_i^{(n-1)}(\pi_1, \ldots, \pi_{n-2})
\]
where we take the following bundles.
\[
\begin{align*}
\pi_1 &: V_1 \longrightarrow \mathbb{P}V_{n-1} \\
\vdots \\
\pi_{n-3} &: V_{n-3} \longrightarrow \mathbb{P}V_{n-1} \\
\pi_{n-2} &: V_{n-2} \otimes O(-1) \longrightarrow \mathbb{P}V_{n-1}
\end{align*}
\]
Proof. We may write
\[ X_1^{(n)} : V_1 \otimes \mathcal{O}(-1, \ldots, -1, -1, -1) \rightarrow (\mathbb{P}V_2 \times \cdots \times \mathbb{P}V_{n-3}) \times \mathbb{P}V_{n-2} \times \mathbb{P}V_{n-1} \]
so that the claim again follows by the methods of Section 3.1. Similar arguments suffice for the other \( i \).

\[ \square \]

Remark 6.2. Note for completeness that the \( n = 3 \) case of the above Proposition 6.1 is true too, when suitably interpreted. For this, we take \( X_1^{(2)}(V_1) \) as simply \( V_1 \), because \( Z^{(2)}(V_1) = V_1 \) so no resolution is needed here. By extension we take \( X_1^{(2)}(\pi_1) \) as simply \( \pi_1 \). We may then put
\[ \pi_1 : V_1 \otimes \mathcal{O}(-1) \rightarrow \mathbb{P}V_2 \]
so that the statement follows by definition of \( X_1^{(3)}(V_1, V_2) \).

By iterating the above Proposition 6.1 in families, we may realize \( X_1^{(n)} \) as a family of \( X_1^{(k)} \). The following is a direct construction to show this fact, which coincides with the above when \( k = n - 1 \).

**Proposition 6.3.** For \( n > k \geq 3 \) we have for \( i = 1, \ldots, k - 1 \)
\[ X_1^{(n)}(V_1, \ldots, V_{n-1}) \cong X_1^{(k)}(\pi_1, \ldots, \pi_{k-1}) \]
where we take the bundle
\[ \pi_{k-1} : V_{k-1} \otimes \mathcal{O}(-1, \ldots, -1) \rightarrow \mathbb{P}V_k \times \cdots \times \mathbb{P}V_{n-1} \]
and \( \pi_1, \ldots, \pi_{k-2} \) bundles with constant fibre \( V_1, \ldots, V_{k-2} \) over the same base.

Proof. Note first for consistency that the dimension of the base is \( n - k \). Observe that, taking \( k \geq 4 \), we may write the following.
\[ X_1^{(n)} : V_1 \otimes \mathcal{O}(-1, \ldots, -1, -1, -1, -1, \ldots, -1) \]
\[ \rightarrow (\mathbb{P}V_2 \times \cdots \times \mathbb{P}V_{k-2}) \times \mathbb{P}V_{k-1} \times (\mathbb{P}V_k \times \cdots \times \mathbb{P}V_{n-1}) \]
We deduce the claim for \( i = 1 \) for \( k \geq 4 \). A similar argument gives the claim for other \( i \), and also \( k = 3 \). \[ \square \]

References

[ADM] N. Addington, W. Donovan, and C. Meachan, Mukai flops and \( \mathbb{P} \)-twists, *J. Reine Angew. Math. (Crelle)* 748 (2019) 227–240, arXiv:1507.02595

[AL1] R. Anno and T. Logvinenko, Spherical DG functors, *J. Eur. Math. Soc* 19 (9) (2017) 2577–2656.

[AL2] R. Anno and T. Logvinenko, \( \mathbb{P}^n \)-functors, arXiv:1905.05740

[Bar] F. Barbacovi, Spherical functors and the flop-flop autoequivalence, arXiv:2007.14415

[BB] A. Bodzenta and A. Bondal, Flops and spherical functors, arXiv:1511.00665

[DS1] W. Donovan and E. Segal, Window shifts, flop equivalences and Grassmannian twists, *Compos. Math.* 150 (6) (2014), 942–978.

[DS2] W. Donovan and E. Segal, Mixed braid group actions from deformations of surface singularities, *Comm. Math. Phys.* 335 (1) (2014) 497–543.
[DW1] W. Donovan and M. Wemyss, Noncommutative deformations and flops, *Duke Math. J.* 165 (8) (2016) 1397–1474.

[DW3] ———, Twists and braids for general 3-fold flops, *J. Eur. Math. Soc.* 21 (6) (2019) 1641–1701.

[HLSam] D. Halpern-Leistner and S. Sam, Combinatorial constructions of derived equivalences, *Adv. Math.* 303 (2016) 1264–1299.

[HLShi] D. Halpern-Leistner and I. Shipman, Autoequivalences of derived categories via geometric invariant theory, *Adv. Math.* 303 (2016) 1264–1299.

[Har] W. Hara, On derived equivalence for Abuaf flop: mutation of non-commutative crepant resolutions and spherical twists, [arXiv:1706.04417](https://arxiv.org/abs/1706.04417).

[Hor] R. P. Horja, Derived category automorphisms from mirror symmetry, *Duke Math. J.* 127 (2005) 1–34.

[Huy] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Oxford University Press, 2006.

[JL] Q. Jiang, N. C. Leung, Derived category of projectivization and flops, [arXiv:math/1811.12525](https://arxiv.org/abs/math/1811.12525).

[Kaw] Y. Kawamata, Derived equivalence for stratified Mukai flop on $\mathbb{G}(2,4)$, [arXiv:math/0503101](https://arxiv.org/abs/math/0503101).

[Kit] A. F. Kite, *Fundamental Group Actions on Derived Categories*, thesis, King’s College London, 2018, available at [http://www.homepages.ucl.ac.uk/~ucaheps](http://www.homepages.ucl.ac.uk/~ucaheps).

[ST] P. Seidel and R. P. Thomas, Braid group actions on derived categories of sheaves, *Duke Math. J.* 108 (2001) 37–108.

[Tod] Y. Toda, On a certain generalization of spherical twists, *Bulletin de la Société Mathématique de France* 135 (1) (2007) 119–134.

W. DONOVAN, YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, HAIDIAN DISTRICT, BEIJING 100084, CHINA.

*Email address:* donovan@mail.tsinghua.edu.cn