Quantum Fluctuations of Vortex-Lattice State in Ultrafast Rotating Bose Gas

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Quantum fluctuations in an ultrafast rotating Bose gas at zero temperature are investigated. We calculate the condensate density perturbatively to show that no condensate is present in the thermodynamic limit. The excitation from Gaussian fluctuations around the mean field solution causes infrared divergences in loop diagrams, nevertheless, in calculating the atom number density, the correlation functions and the free energy, we find the sum of the divergences in the same loop order vanishes and obtain finite physical quantities. The long-range correlation is explored and the algebraic decay exponent for the single-particle correlation function is obtained. The atom number density distribution is obtained at the one-loop level, which illustrates the quantum fluctuation effects to melt the mean field vortex-lattice. By the non-perturbative Gaussian variational method, we locate the spinodal point of the vortex-lattice state.

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I. INTRODUCTION

The appearance of vortex excitations in response to rotation is a characteristic feature of superfluid[1,2]. Since the discovery of Bose-Einstein condensation (BEC) in atomic gases[4–6], much work has been devoted to the properties of rotating gaseous condensates in traps, and these developments have been reviewed in[7,8]. As the rotation frequency $\Omega$ increases, more and more vortices occur, from a single one to several ones, then they form an Abrikosov lattice, i.e., a triangular array with a surface density $n_s = \frac{\pi a^2}{\hbar \Omega}$ (\(a\) is the mass of the condensed atoms) [9–18]. In the frame co-rotating with harmonic traps, rotation effects are described by a centrifugal force, which effectively reduces the transverse harmonic trapping, and a Coriolis force which has the same mathematical structure as the Lorentz force that an electron experiences in a uniform magnetic field. In the fast rotation regime, when the centrifugal force almost cancels the transverse confinement, the energy levels of the single particle part of the Hamiltonian organize into Landau levels with spacing $2\hbar\Omega$, and the condensates expand radically, leading to a very dilute atom number density, which ensures the mean interaction energy smaller than Landau level spacing, so that the cold atoms are confined in the lowest Landau level (LLL) single particle orbit.

For rotating bosonic atoms in the LLL, the filling fraction, i.e., the ratio of the number of atoms to the number of vortices, is the parameter controlling the nature of the system. At high filling fractions, the condensate is in the mean field quantum-Hall regime and forms an ordered vortex lattice ground state [19,20]. As the filling fraction decreases, there is a zero temperature phase transition from a triangular vortex-lattice to strongly correlated vortex-liquid [21] and the melting point is located by various approaches [22,23,24] to be approximately at the filling fraction $6 \sim 10$. The experiment done by V. Schweikhard etc. [25] created an ordered vortex lattice in the mean-field quantum-Hall regime, and provided evidence that the elastic shear strength of the vortex lattice drops substantially as the BEC enters the mean field quantum-Hall regime.

In the ultrafast rotation limit when the transverse confinement is exactly canceled by the centrifugal force, the condensates expand to be a two dimensional configuration and atoms are frozen in the lowest energy level in the $z$ direction (assuming a strong confinement in the $z$ direction). For such a two dimensional system, J. Sinova etc. [26] find that the solution to the Gross-Pitaevskii (GP) equation is an Abrikosov triangular vortex-lattice and the integral for the fraction of atoms outside the condensate diverges logarithmically with the system size, which implies that no BEC occurs in the thermodynamic limit even at zero temperature.

Fluctuation effects in the transverse plane in an ultrafast rotation limit are important and one has to go beyond mean field treatment [19]. In the present paper, we are devoted to quantum fluctuations in the ultrafast rotation limit and focus on the thermodynamic limit at zero temperature.

Based on analytical calculations in the perturbative framework, we show that the condensate density is zero in the thermodynamic limit, namely, there is no BEC. The single-particle correlation function and the density fluctuation correlation function are shown to fall off as an inverse power of the separation distance in the large distance limit, which indicates an algebraic long-range order and the algebraic decay exponent is obtained. The atom number density distribution is obtained at the one-loop level, which illustrates the quantum fluctuation effects to melt the mean field vortex-lattice.

By loop expansion around the mean field solution, we calculate the free energy density up to two-loop. The mean field solution is an Abrikosov triangular vortex-lattice and the excitation from Gaussian fluctuations has a quadratic dispersion at small wave vectors [27]. We find that the quadratic dispersion causes infrared divergences in the two-loop
diagrams, nevertheless, the sum of the divergences vanishes and the two-loop contribution to the free energy density is finite.

We also study the model by non-perturbative Gaussian variational method. The free energy density calculated by the perturbation theory and that by the Gaussian variational method coincide very well at large filling fractions. The vortex lattice solution exists only when the filling fraction \( \nu \) is greater than a certain value \( \nu_c \), which is found to be about 1.1. The point \( \nu = \nu_c \) is the so-called spinodal point. Between the spinodal point and the melting point is the meta-stable vortex-lattice state.

The paper is organized as follows:

In section II we formulate the model. In section III we explore the long-range correlations. In section IV we calculate the free energy density perturbatively up to two-loop. In section V we use the non-perturbative Gaussian variational method to study the model. In section VI we give a summary and the conclusions.

II. THE MODEL

For a system of \( N \) bosonic atoms in an axisymmetric harmonic trap (with trap frequencies \( \omega_\perp \) and \( \omega_z \)) rotating with angular velocity \( \Omega e_\perp \), the Hamiltonian in the rotating frame is

\[
H = \sum_{i=1}^{N} \left( \frac{\mathbf{p}_i - m\Omega \hat{z} \times \mathbf{r}_i}{2m} \right)^2 + \frac{1}{2}m(\omega_\perp^2 - \Omega^2)(x_i^2 + y_i^2) + \frac{1}{2}m\omega_z^2 z_i^2 + g \sum_{i<j=1}^{N} \delta(\mathbf{r}_i - \mathbf{r}_j),
\]

where \( g = \frac{4 \pi \hbar^2 a_s}{m} \) is the strength of the hard core repulsive interactions, with \( a_s \) the s-wave scattering length. The centrifugal force effectively reduces the radial confinement and the Coriolis force is equivalent to the Lorentz force exerted on a particle with charge \( Q \) by a magnetic field \( \mathbf{B} = \frac{2m\Omega}{Q} \mathbf{e}_z \). In the present paper we are confined to the ultrafast rotation limit by setting \( \Omega = \omega_\perp \), and assume the axial confinement is so strong that atoms are frozen in the lowest harmonic state in the \( z \) direction. Consequently, what we concern is essentially a two-dimensional system of charged bosonic atoms experiencing an effective magnetic field in the \( z \) direction, described by the Hamiltonian

\[
H = \sum_{i=1}^{N} \left( \frac{\mathbf{p}_i - m\Omega \hat{z} \times \mathbf{r}_i}{2m} \right)^2 + g \sum_{i<j=1}^{N} \delta(\mathbf{r}_i - \mathbf{r}_j).
\]

The kinetic part of the Hamiltonian has equally spaced Landau levels (with spacing \( 2\hbar \Omega \)) and the interaction part is a small perturbation, \( ng \ll 2\hbar \Omega \) (\( n \) is the mean number density of the atoms). Without thermal fluctuations at zero temperature, all atoms are confined in the LLL. In the LLL subspace, the kinetic part of the Hamiltonian is quenched and represented by \( \hbar \Omega \), hence we have the grand Hamiltonian in second quantized form (with the unit \( \hbar = 1 \))

\[
\hat{H} - \mu \hat{N} = \int d^2 r \left[ (\Omega - \mu) \Psi^\dagger(r) \Psi(r) + \frac{g}{2} \Psi^\dagger(r) \Psi^\dagger(r) \Psi(r) \Psi(r) \right],
\]

and the grand-canonical partition function in the functional formalism

\[
Z(\beta, \mu) = \int \mathcal{D}[\Psi^*, \Psi] e^{-S[\Psi^*, \Psi]},
\]

where the action \( S[\Psi^*, \Psi] \) takes the form

\[
\int_0^\beta d\tau \int d^2 r \left[ \Psi^*(r, \tau)(\partial_\tau - \mu + \Omega)\Psi(r, \tau) + \frac{1}{2}g|\Psi(r, \tau)|^4 \right],
\]

in which \( \beta = \frac{1}{k_B T} \) and \( \mu \) is the chemical potential. By variable rescaling: \( \frac{1}{mg\Omega}(\mu - \Omega) = a_\mu, \tau = \frac{1}{mg\Omega} \tau', \beta = \frac{1}{mg\Omega} \beta', r = \frac{1}{\sqrt{2mg\Omega}} \Psi = \sqrt{2mg\Omega} \Psi' \), the partition function simplifies

\[
Z'(\beta', a_\mu) = \int \mathcal{D}[\Psi'^* \Psi'] \exp - \int_0^{\beta'} d\tau' \int d^2 r' \left[ \Psi'^*(r', \tau')(\partial_\tau' - a_\mu)\Psi'(r', \tau') + |\Psi'(r', \tau')|^4 \right].
\]

With all primes omitted, we have
\[
Z(\beta, a_\mu) = \int \mathcal{D}[\Psi^* \Psi] \exp - \int_0^\beta d\tau \int d^2r \left[ \Psi^*(r, \tau) (\partial_\tau - a_\mu) \Psi(r, \tau) + |\Psi(r, \tau)|^4 \right].
\] (7)

Note that the kinetic part is absorbed in a shift of the effective chemical potential \(a_\mu\), and only the interaction part is relevant. Hereafter we will work with the rescaled variables.

Using Landau gauge \(A = (-By, 0)\) for the effective magnetic field \(B = \frac{2m\Omega}{e} \mathbf{e}_z\), we have the LLL magnetic Bloch representation \[29, 30]\n
\[
\varphi_k(r) = 3^\frac{1}{2} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{2\pi}{\sqrt{3}} \left( \frac{y}{d} - \frac{\sqrt{3}}{2} n - \frac{\sqrt{3}}{4\pi} k_x d \right)^2 + i \left( \frac{\pi}{2} (n^2 - n) + 2\pi n \frac{x}{d} + \frac{\sqrt{3}}{2} nk_y d + k_x x \right) \right].
\] (8)

\(\varphi(r)\), i.e. \(\varphi_k(r)\) with \(k = 0\), is a superposition of the lowest Landau levels of the kinetic part of the Hamiltonian and corresponds to an Abrikosov triangular vortex-lattice with lattice spacing \(d\). In the rescaling length unit, \(d\) equals \(\sqrt{\frac{4\pi}{\sqrt{3}}}\) and the primitive vectors of the vortex-lattice are \(d_1 = \sqrt{\frac{4\pi}{\sqrt{3}}} (1, 0)\), \(d_2 = \sqrt{\frac{4\pi}{\sqrt{3}}} (\frac{1}{2}, \frac{\sqrt{3}}{2})\). \(\varphi_k(r)\) describes the transverse oscillations of the vortex-lattice, and the primitive vectors of the reciprocal lattice are \(\hat{d}_1 = \sqrt{\frac{4\pi}{\sqrt{3}}} (\frac{1}{2}, -\frac{1}{2})\), \(\hat{d}_2 = \sqrt{\frac{4\pi}{\sqrt{3}}} (0, 1)\). The vortex cores, i.e. the zero point of the function \(\varphi_k(r)\) with \(k = k_1 \hat{d}_1 + k_2 \hat{d}_2\), are uniformly distributed at sites \((n_1 - k_2)d_1 + (n_2 + \frac{1}{2} + k_1)d_2\), where \(n_1\) and \(n_2\) are integers. Note that both the lattice cell and the Brillouin zone have an area of \(2\pi\), and the vortex number density is \(n_v = \frac{1}{2\pi}\) in the rescaling units.

## III. THE LONG-RANGE CORRELATIONS

As is known \[26\], for the ultrafast rotating two dimensional Bose gas, there is no BEC in the thermodynamic limit even at zero temperature. In spite of that, we will calculate \(U(1)\) invariant quantities, like the atom number density, the free energy and the correlation functions, in the perturbative framework, and show that the infrared divergences are canceled, similar to the method used in two dimensional non-linear \(\sigma\) model \[31, 32\]. In two dimensional \(O(N)\) non-linear \(\sigma\) model, David in \[32\] proved that, using the “wrong” spontaneously broken symmetry phase, any \(O(N)\) invariant observable has an infrared finite weak coupling perturbative expansion. In this paper, though we will only show some \(U(1)\) invariant quantities are also free of infrared divergences at most to two loops in the perturbative framework, we believe that it is true to all orders similar to the two dimensional \(O(N)\) non-linear \(\sigma\) model. This method was extensively used to study the vortex lattice in type II superconductors, for example, in Ref. \[33, 34\].

In this section, We calculate the condensate density perturbatively and show it is zero in the thermodynamic limit. By calculating the single-particle correlation function and the density fluctuation correlation function, we obtain the algebraic decay exponent. The atom number local density is also calculated, which shows that at large filling fractions, the number density retains the vortex-lattice configuration, while at small filling fractions, quantum fluctuations tend to smooth away the vortex-lattice.

### A. The atom number density

In a usual fashion \[35, 38\], we separate the field as the condensate part and the fluctuation part

\[
\Psi(r, \tau) = \sqrt{n_c} \varphi(r) + \psi(r, \tau),
\] (9)

where \(n_c\), the condensate number density, is a real number minimizing the free energy and the fluctuation part \(\psi(r, \tau)\) can be expanded as

\[
\psi(r, \tau) = \frac{1}{\sqrt{A_{BZ}}} \sum_{k \in BZ} \sum_m \psi_{km} \varphi_k(r) e^{-\frac{i}{2} \theta_k e^{-i\omega_m \tau}}.
\] (10)

In powers of \(\psi^*_m\) and \(\psi_{km}\), we divide the action \(S\) into four parts
\[ S_0 = \beta A \left( -a_\mu n_c + \beta_A n_c^2 \right), \]

\[ S_2 = \sum_{p} \left[ -i\omega_m - a_\mu + 4n_c\beta_k \right] \psi_{km}^* \psi_{km} + n_c |\gamma_k| \left( \psi_{km}^* \psi_{-k-m}^* + \psi_{km} \psi_{-k-m} \right), \]

\[ S_3 = 2\sqrt{n_c} \frac{1}{\sqrt{A\beta}} \sum_{p_{1,2,3,4}} \left( \psi_{p_1}^* \psi_{p_2}^* \psi_{p_3} P_{p_1,p_2,p_3,0} + \text{c.c.} \right), \]

\[ S_4 = \frac{1}{\sqrt{A\beta}} \sum_{p_{1,2,3,4}} \psi_{p_1}^* \psi_{p_2}^* \psi_{p_3} P_{p_1,p_2,p_3,p_4}, \]

where \( p = (k, \omega_m) \), \( \psi_p = \psi_{km} \) and \( \sum_p = \sum_{k \in BZ} \sum_m \). The quadratic part can be diagonalized as

\[ S_2 = \sum_{p} \left( -i\omega_m + \epsilon(k) \right) a_{km}^* a_{km}, \]

by the following Bogoliubov transformation:

\[ a_{km} = u_k \psi_{km} + v_k \psi_{-k-m}^*, \]

\[ a_{km}^* = u_k \psi_{km}^* + v_k \psi_{-k-m}, \]

and inversely,

\[ \psi_{km} = u_k a_{km} - v_k a_{-k-m}^*, \]

\[ \psi_{km}^* = u_k a_{km}^* - v_k a_{-k-m}, \]

where

\[ u_k = \left\{ \frac{1}{2} \left( \frac{\epsilon_0(k)}{\epsilon(k)} + 1 \right) \right\}^{\frac{1}{2}}, \]

\[ v_k = \left\{ \frac{1}{2} \left( \frac{\epsilon_0(k)}{\epsilon(k)} - 1 \right) \right\}^{\frac{1}{2}}, \]

and

\[ \epsilon_0(k) = (-a_\mu + 4n_c\beta_k), \]

\[ \epsilon(k) = \sqrt{(-a_\mu + 4n_c\beta_k)^2 - 4n_c^2|\gamma_k|^2}. \]

Note that the free energy density \( F \) depends on two parameters, \( a_\mu \) and \( n_c \), nevertheless, \( n_c \) is related to \( a_\mu \) by the constraint \( \frac{\partial F(n_c,a_\mu)}{\partial n_c} = 0 \) and hence \( n_c \) is renormalized order by order. In the zero-loop order,

\[ F_c = -a_\mu n_c + \beta_A n_c^2, \]

and the condensate density equals

\[ n_c^{(0)} = \frac{a_\mu}{2\beta_A}, \]

which is also the total number density in the zero-loop order, denoted as \( n_0 \). Now we want to calculate the total number density and the condensate density to one-loop. The total number density \( n \) is given by

\[ \frac{1}{A} \int d^2 r \left\langle \Psi^*(r) \Psi(r) \right\rangle = n_c + \frac{1}{A\beta} \sum_p \left\langle \psi_{km}^* \psi_{km} \right\rangle \]
where \( n_c \) is the condensate density and \( \frac{1}{\beta A} \sum_p \langle \psi_{km}^* \psi_{km} \rangle \) is the density of the atoms outside the condensate. First we need to know the one-loop correction to the condensate density, denoted as \( n_c^{(1)} \). Up to one-loop order, the free energy density takes the form
\[
\mathcal{F}_{0+1}(a_\mu, n_c) = -a_\mu n_c + \beta A n_c^2 + \frac{1}{4\pi} \int_{BZ} \frac{d^2 k}{2\pi} \sqrt{(-a_\mu + 4n_c\beta k)^2 - 4n_c^2|\gamma_k|^2}.
\]  
(22)

Minimizing \( \mathcal{F}_{0+1} \) with respect to \( n_c \) leads to
\[
n_c = \frac{a_\mu}{2\beta A} - \frac{1}{2\pi \beta A} \int_{BZ} \frac{d^2 k}{2\pi} \frac{\beta_k (-a_\mu + 4n_c\beta k) - n_c|\gamma_k|^2}{\sqrt{(-a_\mu + 4n_c\beta k)^2 - 4n_c^2|\gamma_k|^2}}.
\]
(23)

from which we see that the one-loop correction to \( n_c \) equals
\[
n_c^{(1)} = -\frac{1}{4\pi \beta A} \int_{BZ} \frac{d^2 k}{2\pi} \frac{2\beta_k (2\beta_k - \beta_A) - |\gamma_k|^2}{\sqrt{(2\beta_k - \beta_A)^2 - |\gamma_k|^2}}.
\]
(24)

The density of the atoms outside the condensate is given by
\[
\frac{1}{A\beta} \sum_p \langle \psi_{km}^* \psi_{km} \rangle_{1\text{-loop}} = \frac{1}{4\pi} \int_{BZ} \frac{d^2 k}{2\pi} E_0(k),
\]
(25)

where \( E_0(k) \) and \( E(k) \) are defined as
\[
E_0(k) = 2\beta_k - \beta_A,
\]
(26)
\[
E(k) = \sqrt{(2\beta_k - \beta_A)^2 - |\gamma_k|^2}.
\]
(27)

The total number density, \( n_{0+1} \), is equal to \( n_0^{(0)} + n_1^{(1)} + \frac{1}{\beta A} \sum_p \langle \psi_{km}^* \psi_{km} \rangle_{1\text{-loop}} \). The filling fraction, \( n/n_c \), is given by \( 2\pi n_{0+1} \) at the one-loop level.

Obviously, \( n_c^{(1)} \) and \( \frac{1}{\beta A} \sum_p \langle \psi_{km}^* \psi_{km} \rangle_{1\text{-loop}} \) both contain infrared divergences, but in the sum the divergences are canceled and \( n_1^{(1)} + \frac{1}{\beta A} \sum_p \langle \psi_{km}^* \psi_{km} \rangle_{1\text{-loop}} \), which is the one-loop correction to the total number density, equals
\[
n_1 = -\frac{1}{4\pi \beta A} \int_{BZ} \frac{d^2 k}{2\pi} E(k) \approx -0.023.
\]
(28)

The condensate density, \( n_c = n_c^{(0)} + n_c^{(1)} \), is infrared divergent, \( n_c^{(0)} + n_c^{(1)} = c(1 - \alpha \ln L) \), where \( c \) and \( \alpha \) are positive constants and \( L \) is the system size. If we can calculate \( n_c \) at all loops, \( n_c \approx c(1 - \alpha \ln L + \frac{1}{2}(\alpha \ln L)^2 + \cdots) \approx c \exp(-\alpha \ln L) = cL^{-\alpha} \). When we take \( L \to \infty \), \( n_c \to 0 \). Therefore, in the thermodynamic limit, there will be no condensate, as is also shown by J. Sinova etc. [26]. For a finite system, there is a finite infrared cutoff \( \sim \frac{1}{L} \), and the condensate density, \( n_c = cL^{-\alpha} \), will be finite.

Up to one-loop, the local density \( n(r) \) is given by
\[
\langle \Psi^\dagger(r)\Psi(r) \rangle = (n_0^{(0)} + n_1^{(1)})\varphi^*(r)\varphi(r) + \frac{1}{4\pi} \int_{BZ} \frac{d^2 k}{2\pi} \frac{E_0(k)}{E(k)} \varphi_k^*(r)\varphi_k(r),
\]
(29)

where \( n_0^{(0)} \varphi^*(r)\varphi(r) \) is the mean field local density, denoted as \( n_0(r) \). Obviously, in Eq. (29) \( n_1^{(1)} \) and the last term both contain infrared divergences, nevertheless, by arranging them properly we see the divergences are canceled and the one-loop correction to the total local density is obtained as
\[
n_1(r) = n_1^{(1)}\varphi^*(r)\varphi(r) + \frac{1}{4\pi} \int_{BZ} \frac{d^2 k}{2\pi} \frac{E_0(k)}{E(k)} \varphi_k^*(r)\varphi_k(r)
\]
(30)
\[
= n_1\varphi^*(r)\varphi(r) + \frac{1}{4\pi} \int_{BZ} \frac{d^2 k}{2\pi} \frac{E_0(k)}{E(k)} (\varphi_k^*(r)\varphi_k(r) - \varphi^*(r)\varphi(r)),
\]
in which the second term is free of divergences.
Note that the mean field local density, $n_0(r) = n_c(0)\varphi^*(r)\varphi(r)$, forms a vortex-lattice, and the one-loop correction, $n_1(r)$, includes quantum fluctuations. In order to explore quantum fluctuation effects on the atom number density configuration, we plot the mean field density distribution, $\frac{n_0(r)}{n_0} = \varphi^*(r)\varphi(r)$, and the total density distribution, $\frac{n_0(r)+n_1(r)}{n_0+n_1}$, at different filling fractions. From Fig. 1, one finds that quantum fluctuations tend to smooth the vortex-lattice. At large filling fractions, the number density still retains the vortex-lattice configuration, while at small filling fractions, the vortex-lattice is smoothed away.

B. The single-particle correlation function

Up to one-loop, the single-particle correlation function $\langle \Psi^\dagger(r_1)\Psi(r_2) \rangle$ is equal to

$$\langle \Psi^\dagger(r_1)\Psi(r_2) \rangle = \left(n_c^{(0)} + n_c^{(1)}\right)\varphi^*(r_1)\varphi(r_2) + \langle \psi^\dagger(r_1)\psi(r_2) \rangle$$

$$= \left(n_c^{(0)} + \lambda n_c^{(1)}\right)\varphi^*(r_1)\varphi(r_2) + \lambda \langle \psi^\dagger(r_1)\psi(r_2) \rangle ,$$

where $\lambda$ is used to keep trace of the loop order and should be set to 1 in the end. Similar to Eq. [29], the above equation can be arranged as

$$(n_c^{(0)} + \lambda n_c^{(1)} + \lambda \frac{1}{4\pi} \int_{BZ} \frac{d^2k}{2\pi} \frac{E_0(k)}{E(k)} \varphi^*(r_1)\varphi(r_2) + \lambda f(r_1, r_2)$$

$$= \left(n_c^{(0)} + \lambda n_1\right)\varphi^*(r_1)\varphi(r_2) + \lambda f(r_1, r_2) ,$$

where $f(r_1, r_2)$ is given by $\frac{1}{4\pi} \int_{BZ} \frac{d^2k}{2\pi} \frac{E_0(k)}{E(k)} [\varphi^*_k(r_1)\varphi_k(r_2) - \varphi^*(r_1)\varphi(r_2)]$ and free of divergences.

Now we define the relative coordinate $r \equiv r_1 - r_2$, and the center coordinate $R \equiv \frac{r_1+r_2}{2}$. We take $R = 0$ for simplicity and analyze the large $|r|$ limit. Based on numerical calculations, we find

$$\lim_{|r| \to \infty} f(r_1, r_2) \simeq -0.18 \ln |r|\varphi^*(r_1)\varphi(r_2)$$

Therefore

$$\lim_{|r| \to \infty} \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \simeq \left(n_c^{(0)} + \lambda n_1\right)\varphi^*(r_1)\varphi(r_2) - 0.18\lambda \ln |r|\varphi^*(r_1)\varphi(r_2)$$

$$= \left(n_c^{(0)} + \lambda n_1\right)\varphi^*(r_1)\varphi(r_2) (1 - \alpha \ln |r|)$$

where $\alpha = \frac{0.18\lambda}{n_c + \lambda n_1}$, and to one loop order $\alpha \approx \frac{0.18\lambda}{n_c} = \frac{0.18}{n_c}$. Of course this result is not physical if we only include one-loop correction, as $\lim_{r \to \infty} \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \to -\infty$. We shall include all order in perturbation theory to get the physical result. We argue, in all order, similar to the calculation in Ref. [34].
\[ \lim_{r \to \infty} \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \propto \varphi^*(r_1)\varphi(r_2) \left( 1 - \alpha' \ln |r| + \frac{1}{2}(\alpha' \ln |r|)^2 + \ldots \right) \]

\[ = \varphi^*(r_1)\varphi(r_2) |r|^{-\alpha'} \]

To one loop, \( \alpha' = \alpha = \frac{0.18}{\hbar c} \). similar calculations can be done in usual BKT (Berezinsky, Kosterlitz and Thouless) phase transition systems, and the algebraic decay exponent can be obtained correctly \[39, 40\].

One will wonder in usual BKT phase transition systems, the phase transition is continuous. In the vortex lattice phase, from Eq. (35), the correlation decays algebraically, and the rotational symmetry is broken as the factor \( \varphi^*(r_1)\varphi(r_2) \) in Eq. (35) is not rotationally invariant. In the vortex liquid phase, the correlation decays exponentially, and the rotational symmetry is unbroken. Therefore, the phase transition is the spontaneous breaking of the rotational symmetry. As in usual solid to liquid phase transition, the phase transition is a first order melting transition.

C. The density fluctuation correlation function

In the following, we will calculate the density fluctuation correlation to one-loop,

\[ \langle \delta n(r_1)\delta n(r_2) \rangle = \langle [\hat{n}(r_1) - \langle \hat{n}(r_1) \rangle] [\hat{n}(r_2) - \langle \hat{n}(r_2) \rangle] \rangle = \langle \hat{n}(r_1)\hat{n}(r_2) \rangle - \langle \hat{n}(r_1) \rangle \langle \hat{n}(r_2) \rangle. \]

As shown in subsection IIIA there is no condensate and thus the system retains the \( U(1) \) symmetry, therefore, only the \( U(1) \) gauge invariant terms in the contractions remain

\[ \langle \hat{n}(r_1)\hat{n}(r_2) \rangle = \langle \hat{n}(r_1) \rangle \langle \hat{n}(r_2) \rangle \]

\[ = \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \langle \Psi^\dagger(r_2)\Psi(r_2) \rangle + \langle \Psi^\dagger(r_1)\Psi(r_1)\Psi^\dagger(r_2)\Psi(r_2) \rangle. \]

The term \( \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \) is investigated in the last subsection. In the large \(|r_1 - r_2|\) limit, the first term in Eq. (37) falls off as an inverse power of the separation distance \(|r_1 - r_2|\),

\[ \lim_{|r_1 - r_2| \to \infty} \langle \Psi^\dagger(r_1)\Psi(r_2) \rangle \sim |r_1 - r_2|^{-2\alpha}. \]

The second term in Eq. (37) is the two-body connected Green’s function and it is hard to calculate non-perturbatively. But we speculate that this term will not alter the asymptotic behavior of \( \langle \delta n(r_1)\delta n(r_2) \rangle \) in the large \(|r_1 - r_2|\) limit, and hence the density fluctuation correlation function also decays algebraically with the distance in the large distance limit. The result will indicate, near the Brag peak, the structure function \( S(Q + k) \) where \( Q \) belongs to the reciprocal lattice and \( k \) is small will have a scaling \( \lim_{k \to 0} S(Q + k) \propto |k|^{-2\alpha} \).

Similar calculation can be done for vortex lattice in type II superconductors and the density fluctuation correlation will be shown to have similar behavior. For the vortex lattice near the Brag peak, the structure function \( S(Q + k) \) will have a scaling \( \lim_{k \to 0} S(Q + k) \propto |k|^{-\eta} \) (details will be published elsewhere).

IV. THE FREE ENERGY DENSITY CALCULATION BY LOOP EXPANSION

In this section, we shall calculate the free energy density by loop expansion up to two-loop and show the cancelation of infrared divergences.

A. Mean-field contribution

In the saddle point approximation,

\[ \frac{\delta S[\Psi^*, \Psi]}{\delta \Psi^*} = 0, \quad \frac{\delta S[\Psi^*, \Psi]}{\delta \Psi} = 0, \]

(39)
with the LLL constraint, one obtains \[3, 41\]

\[
\Psi_0(r) = \sqrt{\frac{a_\mu}{2\beta_A}} \varphi(r),
\]

(40)

where \(\varphi(r) = \varphi_{k=0}(r), \beta_A = \frac{1}{i} \int_{\text{cell}} d^2 r |\varphi(r)|^4\). Obviously, the saddle point approximation is equivalent to the mean field GP equation. We have the mean field contribution to the free energy density

\[
\mathcal{F}_0(a_\mu) = -\frac{a_\mu^2}{4\beta_A},
\]

(41)

and the mean field contribution to the number density

\[
n_0(a_\mu) = -\partial \mathcal{F}_c(a_\mu) / \partial a_\mu = a_\mu / 2\beta_A.
\]

(42)

**B. One-loop correction**

Following the loop expansion procedure presented in \[42\], we set

\[
\Psi(r, \tau) = \sqrt{\frac{a_\mu}{2\beta_A}} \varphi(r) + \psi(r, \tau),
\]

(43)

in which \(\sqrt{\frac{a_\mu}{2\beta_A}} \varphi(r)\) is the mean field part and \(\psi(r, \tau)\) is the higher order corrections, and then expand \(S[\Psi^*, \Psi]\) in powers of \(\psi^*(r, \tau)\) and \(\psi(r, \tau)\). In the magnetic Bloch representation, \(\psi(r, \tau)\) can be expanded as

\[
\psi(r, \tau) = \frac{1}{\sqrt{A\beta}} \sum_{k \in BZ} \sum_m \psi_{km} \varphi_k(r) e^{-i\theta_k} e^{-i\omega_m \tau}
\]

(44)

where \(A\) is the area of the sample, \(\omega_m = \frac{2\pi m}{\beta}\) is the bosonic Matsubara frequency, and \(\theta_k\) is defined in Eq. \[46\]. In powers of \(\psi_{km}^*\) and \(\psi_{km}\), we divide the action into four parts

\[
S_0 = A\beta \left( -\frac{a_\mu^2}{4\beta_A} \right),
\]

\[
S_2 = \sum_p \left[ -i \omega_m + \frac{a_\mu}{\beta_A} (2\beta_k - \beta_A) \right] \psi_{km}^* \psi_{km} + \frac{1}{2} \frac{a_\mu}{\beta_A} \gamma_k \left[ \psi_{km}^* \psi_{-k-m}^* + \psi_{km} \psi_{-k-m} \right],
\]

\[
S_3 = \sqrt{\frac{2a_\mu}{\beta A\beta}} \sum_{p_1, p_2, p_3} \left[ \psi_{p_1}^* \psi_{p_2}^* \psi_{p_3} P_{p_1, p_2, p_3, 0} + c.c. \right],
\]

\[
S_4 = \frac{1}{A\beta} \sum_{p_1, p_2, p_3, p_4} \psi_{p_1}^* \psi_{p_2}^* \psi_{p_3} \psi_{p_4} P_{p_1, p_2, p_3, p_4},
\]

(45)

where \(p \equiv (k, \omega_m)\), \(\psi_p \equiv \psi_{km}\), \(\sum_p \equiv \sum_{k \in BZ} \sum_m\), and

\[
\beta_k = \frac{1}{2\pi} \int_{\text{cell}} d^2 r \varphi_k^*(r) \varphi^*(r) \varphi_k(r) \varphi(r),
\]

\[
\gamma_k = \frac{1}{2\pi} \int_{\text{cell}} d^2 r \varphi_k^*(r) \varphi^*(r) \varphi_k(r) \varphi_{-k}(r),
\]

\[
e^{i\theta_k} = \frac{\gamma_k}{|\gamma_k|},
\]

(46)

\[
P_{p_1, p_2, p_3, p_4} = \delta_{m_1 + m_2, m_3 + m_4} \int_{\text{cell}} d^2 r \frac{2}{2\pi} \varphi_k^*(r) \varphi_{k_1}^*(r) \varphi_{k_2}(r) \varphi_{k_3}(r) \varphi_{k_4}(r) e^{i\theta_k_{k_1} + \theta_k_{k_2} - \theta_k_{k_3} - \theta_k_{k_4}}.
\]

In the one-loop approximation, we keep only the quadratic part of the action and diagonalize it as
\[
S_2 = \sum_p (-i\omega_m + \epsilon(k)) a_{km}^* a_{km},
\]
where
\[
\epsilon(k) = \frac{a_{\mu}}{\beta_A} \sqrt{(2\beta_k - \beta_A)^2 - |\gamma_k|^2},
\]
by the following Bogoliubov transformation:
\[
a_{km} = u_k \psi_{km} + v_k \psi_{-km}^*, \\
a_{km}^* = u_k \psi_{km}^* + v_k \psi_{-km},
\]
and inversely,
\[
\psi_{km} = u_k a_{km} - v_k a_{-km}^*, \\
\psi_{km}^* = u_k a_{km}^* - v_k a_{-km},
\]
where \(u_k = \sqrt{\frac{1}{2} \frac{E_0(k)}{E_0(k)} + 1}\) and \(v_k = \sqrt{\frac{1}{2} \frac{E_0(k)}{E_0(k)} - 1}\). \(E_0(k)\) and \(E(k)\) are defined in Eqs. (26) and (27). By Taylor expanding \(\beta_k\) and \(|\gamma_k|\), we find the excitation \(\epsilon(k)\) has a quadratic dispersion at small wave vectors, i.e. \(\lim_{k \to 0} \epsilon(k) \sim k^2\), which is consistent with previous results [26, 33, 11]. The one-loop contribution to the free energy density, \(F_1(a_{\mu})\), takes the form
\[
-\frac{1}{A} \frac{1}{\beta} \ln \int D[a^* a] \exp - \sum_p (-i\omega_m + \epsilon(k)) a_{km}^* a_{km} = \frac{1}{A} \sum_{k \in BZ} \left[ \frac{1}{2} \epsilon(k) + \frac{1}{\beta} \ln (1 - e^{-\beta\epsilon(k)}) \right].
\]
By setting the area \(A\) to infinity and the temperature \(T\) to zero, we obtain
\[
F_1(a_{\mu}) = \frac{a_{\mu}}{4\pi\beta_A} \langle E(k) \rangle_k,
\]
where \(\langle \cdots \rangle_k \equiv \int \frac{d^2k}{2\pi}\), means average over the Brillouin zone. The one-loop correction to the number density, \(n_1(a_{\mu})\), is equal to
\[
-\frac{\partial F_1(a_{\mu})}{\partial a_{\mu}} = -\frac{1}{4\pi\beta_A} \langle E(k) \rangle_k,
\]
which is consistent with the result obtained in the last section as in Eq. (25).

C. Two-loop correction

By the Bogoliubov transformation shown in Eqs. (51) and (52), we switch to the field \(a_{km}^*, a_{km}\), and write the cubic and quartic part as
\[
S_3 = \sum_{p_1,p_2,p_3} \frac{2a_{\mu}}{\beta_A} \frac{1}{\sqrt{A\beta}} \left[ a_{p_1} a_{p_2} a_{p_3}^* (\Lambda_{p_1p_2p_3} - \Lambda'_{p_1p_2p_3}) + a_{p_1} a_{p_2} a_{p_3} (\Pi_{p_1p_2p_3} - \Pi'_{p_1p_2p_3}) \right] + c.c.
\]
and
\[
S_4 = \left\{ \frac{1}{A\beta} \sum_{p_1,p_2,p_3,p_4} \left[ a_{p_1} a_{p_2} a_{p_3}^* P_{p_1p_2p_3p_4} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} - 2a_{p_1} a_{p_2} a_{p_3}^* P_{p_1p_2p_3p_4} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} + P_{p_2p_3p_4p_1} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} \right] + c.c. \right\} + \frac{1}{A\beta} \sum_{p_1,p_2,p_3,p_4} \left[ a_{p_1} a_{p_2} a_{p_3}^* P_{p_1p_2p_3p_4} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} \right] + c.c.
\]
and
\[
+ P_{p_1p_2p_3p_4} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} + P_{p_3p_4p_1p_2} u_{k_1}^* u_{k_2} u_{k_3} u_{k_4} \right],
\]
where $a_p \equiv a_{km}$ and

$$\Pi_{p_1_p_2_p_3} = \frac{1}{3} (P_{p_1_p_2_p_3} u_{k_2} u_{k_3} u_{k_1} + P_{p_0-p_1_p_2_p_3} u_{k_1} u_{k_3} u_{k_2} + P_{p_0-p_3_p_2_p_1} u_{k_1} u_{k_2} u_{k_3}),$$

$$\Pi'_{p_1_p_2_p_3} = \frac{1}{3} (P_{p_3_p_2-p_1} u_{k_2} u_{k_3} u_{k_1} + P_{p_1-p_3-p_2} u_{k_1} u_{k_2} u_{k_3} + P_{p_1-p_3-p_2} u_{k_1} u_{k_2} u_{k_3}),$$

and

$$\Lambda_{p_1_p_2_p_3} = P_{p_0-p_1-p_2} u_{k_1} u_{k_2} u_{k_3} + P_{p_0-p_2-p_1} u_{k_1} u_{k_2} u_{k_3} + P_{p_0-p_3-p_2} u_{k_1} u_{k_2} u_{k_3},$$

$$\Lambda'_{p_1_p_2_p_3} = P_{p_2-p_3-p_0} u_{k_1} u_{k_2} u_{k_3} + P_{p_4-p_3-p_0} u_{k_1} u_{k_2} u_{k_3} + P_{p_1-p_3-p_0} u_{k_1} u_{k_2} u_{k_3}.$$  (58)

The two-loop contribution to the free energy density, $\mathcal{F}_2(a_\mu)$, takes the form

$$\mathcal{F}_2(a_\mu) = \frac{1}{\beta A} \ln \left[ \frac{\int D[a^*, a] \exp - (S_2 + S_3 + S_4)}{\int D[a^*, a] \exp - S_2} \right]_{2\text{-loop}} = \frac{1}{\beta A} \left( (S_4) - \frac{1}{2} (S_3 S_3) \right),$$  (60)

where $(\cdots)$ denotes the sum of all the connected Feynman diagrams with $G_p = \frac{1}{i\omega_m + \epsilon(k)}$ as a propagator. The two-loop Feynman diagrams are depicted in Fig. 2. The contribution from the diagram “$\odot$” equals

$$\frac{1}{\beta^2 A^2} \sum_{p_1, p_2} \left[ 4P_{p_1-p_2-p_2} u_{k_1}^2 v_{k_2}^2 + 4P_{p_1-p_2-p_1} v_{k_1} u_{k_2} v_{k_2}^2 + 2P_{p_1-p_2-p_1} \left( u_{k_2}^2 + v_{k_2}^2 \right) \right] G_{p_1} G_{p_2} G_{p_3} = \frac{1}{4\pi^2 \beta A} \left( (\gamma_{k} | u_{k} u_{k}^* \rangle \langle k |)^2 + \frac{1}{8\pi^2} (\beta_{k_1-k_2} (u_{k_1}^2 + v_{k_1}^2) (u_{k_2}^2 + v_{k_2}^2)) \right)_{k_1,k_2};$$  (61)

and the contribution from the diagram “$\odot \odot$” equals

$$\frac{12a_\mu}{\beta A^2} \sum_{p_1, p_2, p_3} \left| \Pi_{p_1_p_2_p_3} - \Pi'_{p_1_p_2_p_3} \right|^2 G_{p_1} G_{p_2} G_{p_3} = -\frac{3}{\pi^2} < \left| \Pi_{k_1,k_2(-k_1-k_2)} - \Pi_{k_1,k_2(-k_1-k_2)} \right|^2 \frac{1}{E(k_1) + E(k_2) + E((k_1 + k_2))} >_{k_1,k_2};$$  (63)

and the contribution from the diagram “$\odot \odot \odot$” equals

$$\frac{8a_\mu}{\beta A^2} \sum_{p_1, p_2, p_3} \left( \Lambda_{p_1_p_2_p_1} - \Lambda'_{p_1_p_2_p_1} \right) \left( \Lambda_{p_3_p_2_p_3} - \Lambda'_{p_3_p_2_p_3} \right) G_{p_1} G_{p_2} G_{p_3} = \frac{1}{4\pi^2 \beta A} \left( [\beta_{k} (u_{k}^2 + v_{k}^2) - (\gamma_{k} | u_{k} u_{k}^* \rangle \langle k |)] \right)^2.$$  (64)

We have set the temperature $T$ to zero and the area $A$ to infinity in the end. The notation $(\cdots)_{k_1,k_2} \equiv \int_{BZ} \frac{d^2k_1}{2\pi} \int_{BZ} \frac{d^2k_2}{2\pi}$, means average over the Brillouin zone, and $(k_1 + k_2)$ represents the reduced wave vector in the Brillouin zone. Obviously, $\frac{\partial}{\partial a_\mu} \mathcal{F}_2(a_\mu) = 0$, and hence the two-loop correction to the atom number density is zero, $n_2(a_\mu) = 0$. By Taylor expansion, one can confirm that each of the three contributions in Eqs. (62), (63) and (64) has infrared divergences. However, all the divergences are exactly canceled if they are summed up. By numerical integration, we find that

$$\mathcal{F}_2(a_\mu) = 0$$  (65)

It is amazing that it precisely amounts to zero, and the reason is still under investigations.
D. The filling fraction

Up to two-loop, the free energy density has been obtained as
\[
F_{0+1+2}(a_{\mu}) = -\frac{a_{\mu}^2}{4\beta_A} + \frac{a_{\mu}}{4\pi\beta_A} \langle E(k) \rangle_k, \tag{66}
\]
and the atom number density equals
\[
n_{0+1+2}(a_{\mu}) = \frac{a_{\mu}}{2\beta_A} - \frac{1}{4\pi\beta_A} \langle E(k) \rangle_k. \tag{67}
\]
The filling fraction, \(n_{0+1+2}/n_c\), is given by
\[
\nu = \frac{\pi}{\beta_A} n_{\mu} - \frac{1}{2\beta_A} \langle E(k) \rangle_k. \tag{68}
\]
Using the filling fraction \(\nu\) as a parameter, we have the free energy density
\[
F_{0+1+2}(\nu) = -\frac{\beta_A^2}{4\pi^2} \nu^2 + \frac{1}{16\pi^2\beta_A} (\langle E(k) \rangle_k)^2, \tag{69}
\]
and numerically it is \(F_{0+1+2}(\nu) = -0.029\nu^2 + 0.00062\). To quantify the significance of quantum fluctuations, we calculate the ratio \(|F_{1+2}/F_0|\), given by \(0.107\nu + 0.0536\), from which one can infer that at very large filling fractions, the quantum fluctuation effect is negligible and the mean field theory is adequate; at very small filling fractions, the quantum correction is even greater than the mean field contribution and the mean field vortex-lattice will be unstable due to drastic fluctuations. The quantum melting of the vortex-lattice has been intensively studied \[25–27\]. In the next section, we will locate the spinodal point of the vortex-lattice state, i.e. the terminal point of the meta-stable vortex lattice.

V. GAUSSIAN VARIATIONAL CALCULATION

In this section we are going to study the model by the Gaussian variational method. As in section III, we set
\[
\Psi(r, \tau) = \sqrt{n_c} \phi(r) + \psi(r, \tau), \tag{70}
\]
where \(n_c\) is a real number given by minimizing the free energy and \(\psi(r, \tau)\) is expanded as
\[
\psi(r, \tau) = \frac{1}{\sqrt{A\beta}} \sum_{k \in BZ} \sum_m O_{km} + iA_{km} \frac{n_c \varphi_k(r)}{\sqrt{2}} e^{-\frac{iE_k}{\beta} e^{-i\omega_m \tau}}, \tag{71}
\]
where \(O_{km}^* = O_{-k-m}, A_{km}^* = A_{-k-m}\). In powers of \(O_{km}\) and \(A_{km}\), we divide the action \(S[\Psi^* \Psi]\) into four parts \(S_c, S_2, S_3, S_4\).
\[
S_c = A\beta \left(-a_\mu n_c + \beta_A n_c^2\right), \quad S_2 = \frac{1}{2} \sum_p \left(O_{-p} A_p\right) \left(\frac{E_k^O}{-\omega_m} \frac{\omega_m}{E_k^A} \right) \left(\frac{O_p}{A_p}\right), \tag{72}
\]
where \(O_p \equiv O_{km}, A_p \equiv A_{km}\), and
\[
E_k^O = -a_\mu + 4n_c\beta_k + 2n_c|\gamma_k|, \quad E_k^A = -a_\mu + 4n_c\beta_k - 2n_c|\gamma_k|. \tag{73}
\]
Following the standard Gaussian variational procedure \[15 \[17\], we first define the Gaussian variational kernel
\[
G[\varepsilon^O, \varepsilon^A] = \frac{1}{2} \sum_p \left(O_{-p} A_p\right) \left(\frac{\varepsilon_k^O}{-\omega_m} \frac{\omega_m}{\varepsilon_k^A} \right) \left(\frac{O_p}{A_p}\right), \tag{74}
\]
FIG. 3: The free energy density $F$ and the filling fraction $\nu$ are shown. The blue square symbol represents the data obtained by Gaussian variational method, and the red circle symbol represents the result of the perturbation theory up to one-loop.

where $\varepsilon_k^O$ and $\varepsilon_k^A$ are real variational parameters. The grand-canonical partition function can be written as

$$Z = \int D[OA] e^{-G} e^{-(S_0+S_2+S_3+S_4)}$$

$= e^{-S_0} \int D[OA] e^{-G} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (S_2 - G + S_3 + S_4)^n,$

and the free energy density, $F[n_c, G]$, is given by

$$\frac{1}{\beta A} \left[ S_c - \ln \int D[OA] e^{-G} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle (S_2 - G + S_3 + S_4)^n \rangle_G \right],$$

where $\langle \rangle_G$ denotes the sum of all the connected Feynman diagrams with $G$ as a propagator. Truncated to the first order, the free energy density $F[n_c, G]$ takes the form

$$\frac{1}{\beta A} \left[ S_c - \ln \int D[OA] e^{-G} + \langle S_2 - G + S_3 + S_4 \rangle_G \right].$$

Minimizing $F[n_c, G]$ with respect to $\varepsilon_k^O$, $\varepsilon_k^A$ and $n_c$ leads to the following coupled equations

$$\varepsilon_k^O = E_k^O + \frac{1}{2\pi} \left( \beta_{k-k'} \left( \sqrt{\frac{\varepsilon_k^O}{\varepsilon_{k'}^O}} + \sqrt{\frac{\varepsilon_k^O}{\varepsilon_{k'}^O}} \right) \right) + \frac{\gamma_k}{4\pi\beta A} \left( \sqrt{\frac{\varepsilon_k^A}{\varepsilon_{k'}^O}} - \sqrt{\frac{\varepsilon_k^O}{\varepsilon_{k'}^O}} \right)_{k'},$$

$$\varepsilon_k^A = E_k^A + \frac{1}{2\pi} \left( \beta_{k-k'} \left( \sqrt{\frac{\varepsilon_k^A}{\varepsilon_{k'}^O}} + \sqrt{\frac{\varepsilon_k^A}{\varepsilon_{k'}^O}} \right) \right) - \frac{\gamma_k}{4\pi\beta A} \left( \sqrt{\frac{\varepsilon_k^A}{\varepsilon_{k'}^O}} - \sqrt{\frac{\varepsilon_k^O}{\varepsilon_{k'}^O}} \right)_{k'},$$

$$n_c = \frac{\alpha}{2\beta A} - \frac{1}{8\pi\beta A} \left( \sqrt{\frac{\varepsilon_k^O}{\varepsilon_k^O}} (2\beta_k - |\gamma_k|) \right)_{k'} - \frac{1}{8\pi\beta A} \left( \sqrt{\frac{\varepsilon_k^O}{\varepsilon_k^O}} (2\beta_k + |\gamma_k|) \right)_{k}.$$
which can be solved numerically. Note that the effective chemical potential $a_\mu$ is the only physical parameter. Beginning with $a_\mu = 10$ and slowly lowering it, we find that these equations cease admitting a solution when $a_\mu < 0.47$, which indicates that the system can no longer support a vortex-lattice solution and hence it is the spinodal point. The atom number density can be obtained as the partial derivative of the free energy density with respect to the effective chemical potential and then the filling fraction is easily calculated. At the spinodal point, the filling fraction equals 1.1, smaller than that at the quantum melting point \cite{25–27}. In Fig. 3 the free energy density and the filling fraction obtained by the Gaussian variational method are shown, together with those obtained by the perturbation theory and expressed in Eqs. (66) and (68). To take a closer look at the very small differences of the results obtained by the two approaches, the relative difference, that is the ratio of the result of the Gaussian variational method minus that of the perturbation theory to the result of the perturbation theory, is plotted in Fig. 4. From the two figures, we see that the differences are very small and the higher the filling fraction, the smaller the relative difference. It reveals the fact that in the vortex-lattice state the perturbation theory and the Gaussian variational method coincide very well, and as the filling fraction gets higher, the two methods get closer and finally both are identical to the mean field theory in the large filling fraction limit.

### VI. SUMMARY

We calculate the condensate density perturbatively to show that no condensate is present in the thermodynamic limit. By calculating the single-particle correlation function and the density fluctuation correlation function, we obtain the algebraic decay exponent. We calculate the free energy density to two-loop and show the cancelation of the two-loop infrared divergences. The atom number density distribution to one-loop is obtained, which illustrates the quantum fluctuation effects to smooth away the mean field vortex-lattice. By the non-perturbative Gaussian variational method, we locate the spinodal point of the vortex-lattice, where the filling fraction, $\nu_s$, is numerically obtained to be about 1.1, lower than the quantum melting point obtained by various approaches \cite{25–27}. Between the spinodal point and the melting point is the meta-stable vortex-lattice state. From Fig. 4 we find that at the one-loop level, near the melting point where the filling fraction equals 6 ∼ 10, the atom number density still retains the lattice configuration, while near the spinodal point where the filling fraction is about 1.1, the lattice is almost smoothed away.

In order to determine the melting point accurately, we shall obtain the free energy density of the vortex liquid phase in the future. The study of the vortex liquid will be our focus in the future studies.
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