WORLDLINE APPROACH TO QFT
ON MANIFOLDS WITH BOUNDARY

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Abstract: We use the image charge method to compute the trace of the heat kernel for a scalar field on a flat manifold with boundary, representing the trace by means of a worldline path integral and obtain useful non-iterative master formulae for $n$ insertions of the scalar potential. We discuss possible extensions of the method.

1. Worldline formalism on manifolds with boundary

The worldline formalism (see Ref. 1 for a review) is an alternative method to compute effective actions, amplitudes and anomalies in quantum field theory. For example, the one-loop effective action can be written as a “trace log” of a differential operator which be exponentiated using a Schwinger proper time integral and the trace can be written in terms of a quantum mechanical path integral. For the simplest case of a real massless scalar field with self-interaction $U(\phi)$, propagating in a flat boundaryless space, the one-loop effective action formally reads $\Gamma[\phi] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \text{Tr} \ e^{-TH}$ where
\[ H = -\Box + U''(\phi) \] and the partition trace reads
\[
\text{Tr } e^{-TH} = \int_{PBC} \mathcal{D}x \exp \left[ -\int_0^1 \, d\tau \left( \frac{1}{4T} \dot{x}^2 + TV(x(\tau)) \right) \right]
\]
where \( V(x) \equiv U''(\varphi(x)) \) and the path integral is over the space of all closed paths on the unit circle and the Teichmüller parameter \( T \) is the proper length of the circle. The short-time expansion of the operator \( \text{Tr } e^{-TH} \), known as heat kernel expansion, takes the form
\[
\text{Tr } e^{-TH} = \int dDx \ K(T; x, x) = \frac{1}{(4\pi T)^{D/2}} \sum_{n=0}^{\infty} a_n T^n
\]
and the integrated heat kernel coefficients \( a_n \) can be straightforwardly obtained as a short-time expansion of the path integral (1) by Wick contracting the Taylor expansion of the potential.\(^2\)

The worldline path integral approach to QFT on manifolds with boundary has been carried out using Monte Carlo simulations\(^3\) and many interesting results are obtained with this method.\(^4\) However, a serious difficulty one has to face with such a method concerns boundary conditions different than Dirichlet and an alternative method that might help overcoming this difficulty would be quite welcome.

In a manifold with boundary a heat kernel expansion for the trace, cfr. Eq. (2), still holds but the sum involves half-integer powers and the coefficients include boundary contributions as well as bulk contributions.\(^5\) We have developed\(^6\) a method that generalizes analytic worldline techniques to flat manifolds with boundary \( M = \mathbb{R}_+ \times \mathbb{R}^{D-1} \) using the image charge method to map the path integral on a half space to the combination of two path integrals on the whole space
\[
\text{Tr}_M e^{-TH} = \int_M d^Dx \ K(T; x, x) = \int_M d^Dx \ K(T; x, x)
\]
where \( x = (y, \vec{z}) \), \( y \in \mathbb{R}_+ \), \( \vec{z} \in \mathbb{R}^{D-1} \)

\[ x = (y, \vec{z}), \quad y \in \mathbb{R}_+, \quad \vec{z} \in \mathbb{R}^{D-1} \]

and the upper (lower) sign corresponds to Dirichlet (Neumann) boundary conditions (in Ref. 7 an extension of the method to Robin boundary conditions was considered). The above kernels are whole space kernels computed with an evenly extended potential
\[
V(x) \rightarrow \tilde{V}(x) = \theta(y)V(x) + \theta(-y)V(\tilde{x}) = V_+(x) + \epsilon(y)V_-(x) .
\]

The reflection property of the potential also allows to extend the overall Riemannian integral to the whole space, so that the above two contributions

\[ V(x) \rightarrow \tilde{V}(x) = \theta(y)V(x) + \theta(-y)V(\tilde{x}) = V_+(x) + \epsilon(y)V_-(x) .
\]
can be written as
\[
K_{\text{dir}}(T) \equiv \int_M dDx K(T; x, x) = \frac{1}{2} \int dDx K(T; x, x)
\]
\[
= \frac{1}{2} \int_{PBC} Dx \exp \left[ - \int_0^1 d\tau \left( \frac{1}{4T} \dot{x}^2 + T \tilde{V}(x(\tau)) \right) \right] \quad (5)
\]
\[
K_{\partial M}(T) \equiv \int_M dDx K(T; \tilde{x}, x) = \frac{1}{2} \int dDx K(T; \tilde{x}, x)
\]
\[
= \frac{1}{2} \int_{(A)PBC} Dx \exp \left[ - \int_0^1 d\tau \left( \frac{1}{4T} \dot{x}^2 + T \tilde{V}(x(\tau)) \right) \right] \quad (6)
\]
and will be referred to as the “direct contribution” and the “indirect contribution” respectively. Above, the suffix \((A)_{PBC}\) indicates that the coordinate \(y(\tau)\) satisfies (anti)-periodic boundary conditions, whereas coordinates \(\tilde{z}(\tau)\) satisfy periodic boundary conditions. The potential \(\tilde{V}\) includes a distribution \(\delta(y)\) and the naive application of the Wick theorem results nontrivial. However, we demonstrated that (i) upon Fourier representing the sign function \(\delta(y) = \int dp \pi p \sin(py) = \int ev i\pi p e^{ipy}\) and (ii) upon carefully separating out bulk contributions from boundary contributions, one can safely use it: two new coefficients for the half-space, \(a_4\) and \(a_{9/2}\), were computed.

### 1.1. Indirect contribution to the heat kernel trace

For this contribution the coordinate \(y(\tau)\) is antiperiodic and therefore its kinetic action has no zero mode. Hence, \(\int_{ABC} Dy e^{-\frac{1}{4T} \int_0^1 d\tau y^2} = \frac{1}{2}\) and we can safely Taylor expand the potential about the boundary \((0, \tilde{z})\)

\[
\tilde{V}(y(\tau), \tilde{z}+\tilde{z}(\tau)) = e^{\tilde{z}(\tau)} \tilde{\partial} \left[ e^{y(\tau)} \partial_y V_+(0, \tilde{z}) + \int_{ev} \frac{dp}{2\pi} e^{y(\tau)D^0(p)} V_-(0, \tilde{z}) \right] \quad (7)
\]

with \(D^0(p) = \partial_y + ip\). Inserting the latter into (6) one obtains

\[
K_{\partial M}(T) = \frac{1}{4(4\pi T)^{\frac{D-1}{2}}} \sum_{n=0}^{\infty} \frac{(-T)^n}{n!} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n \int_{\partial M} d^{D-1}z \times \exp \left[ -\frac{T}{2} \sum_{i,j=1}^{n} \left( G_P(\tau_i, \tau_j) \partial_i \cdot \partial_j + G_A(\tau_i, \tau_j) D^0_i(p) D^0_j(p) \right) \right] \
\times \prod_{k=1}^{n} \left[ V_+^{(k)}(0, \tilde{z}) + \int_{ev} \frac{dp_k}{2\pi} V_-^{(k)}(0, \tilde{z}) \right] \quad (8)
\]

and the suffix \(i\) on the derivative means that it acts on the term of potential labelled accordingly. The ABC propagator appearing above is
given by \( G_A(\tau, \sigma) = |\tau - \sigma| - \frac{1}{2} \), whereas the expression for the PBC propagator depends on the prescription one adopts for factoring out the zero mode \( \tilde{z} \). For example, in the "String Inspired" method we have \( G_P(\tau, \sigma) = |\tau - \sigma| - (\tau - \sigma)^2 \), and using (worldline) DBC we instead have \( G_P(\tau, \sigma) = |\tau - \sigma| + \frac{1}{2}(1 - 2\tau)(1 - 2\sigma) - \frac{1}{2} \). The two methods yield different unintegrated heat kernel expansions and their difference resides on total derivative terms. However, since here these terms are boundary total derivatives their integrals vanish and the integrated expression (8) is scheme-independent. Scheme-independence will be slightly more subtle for the direct contribution that we describe next.

1.2. Direct contribution to the heat kernel trace

Here all the coordinates have periodic boundary conditions and we Taylor expand the potential insertions about the zero modes \((y, \tilde{z})\) and get

\[
K^{dir}(T) = \frac{1}{2(4\pi T)^D} \sum_{n=0}^{\infty} \frac{(-T)^n}{n!} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n \int_{-\infty}^{\infty} dy \int_{\partial M} d^{D-1}z 
\times \exp \left[ -\frac{T}{2} \sum_{i,j=1}^n G_P(\tau_i, \tau_j) \partial_i \partial_j \right] \prod_{k=1}^n \left[ V_+^{(k)}(x) + (\epsilon(y)V_-(x))^{(k)} \right] 
\]

(9)

where the notation is meant to convey that derivatives may act on \( \epsilon(y) \) as well as on \( V_\pm \). When derivatives act on \( \epsilon(y) \), \( \delta \) functions or derivatives thereof are generated giving rise to boundary terms. Contributions where no derivatives act on \( \epsilon(y) \) are bulk terms. Namely

\[
K^{dir}_M(T) = \frac{1}{2(4\pi T)^D} \sum_{n=0}^{\infty} \frac{(-T)^n}{n!} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n \int_{-\infty}^{\infty} dy \int_{\partial M} d^{D-1}z 
\times \exp \left[ -\frac{T}{2} \sum_{i,j=1}^n G_{D,S}(\tau_i, \tau_j) \partial_i \partial_j \right] \prod_{k=1}^n \left[ V_+^{(k)}(x) + (\epsilon(y)V_-(x))^{(k)} \right] . 
\]

(10)

Subtracting (10) from (9) yields the boundary terms associated to the direct contribution, namely

\[
K^{dir}_{\partial M}(T) = \frac{1}{2(4\pi T)^D} \sum_{n=0}^{\infty} \frac{(-T)^n}{n!} \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n \int_{-\infty}^{\infty} dy \int_{\partial M} d^{D-1}z 
\times \int_0^1 dw \frac{\partial}{\partial w} \exp \left[ -\frac{T}{2} \sum_{i,j=1}^n G_P(\tau_i, \tau_j) D_i(wp) \cdot D_j(wp) \right] 
\times \prod_{k=1}^n \left[ V_+^{(k)}(x) + \int_{ev} \frac{dp_k}{i\pi p_k} e^{ip_k y} V_-^{(k)}(x) \right] 
\]

(11)
where $D(wp) = (D^0(wp), \bar{\partial})$ and the total derivative on $w$ takes care of the aforementioned subtraction. The evaluation of (11) is done as follows:

(i) Taylor expand potentials about the boundary: it is safe as by construction all terms in (11) are boundary terms;
(ii) integrate over $y$: it yields a $\delta$ function involving various $p$’s and $\partial$’s;
(iii) $w$ derivative cancels one (spurious) pole;
(iv) integrate over all $p$’s, then over $w$ and finally over $\tau_i$.

Expressions (10) and (11) separately are scheme-dependent but the scheme-dependent terms cancel out in the sum (9). In fact as mentioned above scheme-dependence of the bulk part is encoded in a set of total derivative terms that upon integration yield boundary terms.

2. Outlook

We discussed a path integral method to compute the heat kernel trace for a self-interacting scalar field on a flat manifold with boundary. A natural generalization is the inclusion of interaction with external fields. This is clearly feasible by evenly extending to the whole space the coupling $A_M(y, z)x^M$ or $h_{MN}(y, z)x^Mx^N$ for the spin-one and spin-two case. Another possible generalization involves the inclusion of particles with spin in the loop, e.g. by representing the effective action in terms of a spinning particle action on the circle.

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