Quantum phase transitions in the quasi-periodic kicked rotor

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We present a microscopic theory of transport in quasi-periodically driven environments ('kicked rotors'), as realized in recent atom optic experiments. We find that the behavior of these systems depends sensitively on the value of Planck’s constant \( \hbar \): for irrational values of \( \hbar/(4\pi) \) they fall into the universality class of disordered electronic systems and we derive the microscopic theory of the ensuing localization phenomena. In contrast, for rational values the rotor-Anderson insulator acquires an infinite (static) conductivity and turns into a 'super-metal'. Signatures of the corresponding metal/super-metal transition are discussed.

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The quantum kicked rotor (QKR) is one of the most prominent model systems of quantum nonlinear dynamics (quantum chaos) [1]. It comprises a one-dimensional quantum particle subject to periodic boundary conditions and sequential driving in time. In spite of the nominal simplicity of this system — the QKR Hamiltonian is defined in terms of only two dimensionless parameters, kicking strength, \( K \), and time in relation to (effective) Planck’s constant, \( \hbar \) [2] — it displays striking parallels to disordered electronic systems [3, 4], and the ensuing connections to condensed matter physics have been a subject of fundamental research for more than three decades (cf. Ref. [4] and references therein). The general interest in the QKR took a leap forward when the system was experimentally realized in a cold atom setting [5], and localization phenomena otherwise predicted for disordered multi-channel quantum wires were observed. This has been the first observation of strong Anderson localization in a quasi-one dimensional setting (although the full microscopic correspondence to disordered quantum wires was established only recently [4]). The next experimental breakthrough occurred in 2008 when an effectively higher dimensional rotor, the so-called quasi-periodic kicked rotor, was realized in a gas of cold cesium atoms, and an Anderson transition was observed [6]. By now, even signatures of the critical states emerging at the transition point have been seen [7], and it stands to reason that the quasi-periodic QKR makes for an almost ideal environment to study Anderson type critical phenomena.

At the same time, the correspondence between the quasi-periodic QKR and \( (d > 1) \)-dimensional disordered systems is not as well understood as in the \( d = 1 \) case, and this is a gap which we aim to close in this Letter. In the quasi-periodic QKR, \( d \)-dimensional behavior is simulated by modulated driving at \( d \) different frequencies. Below, we will map the low energy physics of this system onto an effective field theory equivalent to the non-linear \( \sigma \)-model of disordered metallic systems [8]. For generic values of the system parameters, this construction establishes the connections to disordered systems, and it explains the observation of Anderson type criticality. However, the rotor is not a genuine metal, and these differences show in anomalies at certain (non-generic) parameter values. Specifically, the one-dimensional QKR displays so-called quantum resonances (cf. Ref. [4] and references therein) at rational values \( \hbar = 4\pi p/q, p, q \in \mathbb{N} \text{ coprime.} \) At resonance, the system behaves like a quasi-one dimensional ring (in the space of rotor-angular momenta) of radius \( q \), and no Anderson localization is observed.

Below we will show that in dimensions \( d > 1 \) the system displays even richer behavior. At \( \hbar = 4\pi p/q \) the system compactifies, e.g., for \( d = 2 \) a nominally two dimensional QKR maps onto the surface of an infinitely long cylinder of finite circumference \( q \) (cf. Fig. 1 top). As far as bulk localization phenomena are concerned, this entails a dimensional reduction to quasi \( d-1 \) dimensions.
(localization along the cylinder axis). However, measurable response functions dominantly couple to its compact dimension, and this has a number of interesting consequences: below we consider a (measurable) observable which, in the light of the above analogies to metallic system, plays a role analogous to an optical conductivity. We find that in the quasi $(d-1)$-dimensional localized regime, the system actually shows ‘super-metallic’ behavior (diverging conductivity). In contrast, for quasi $(d-1)$-dimensional metallic regimes (e.g., the quasi three-dimensional metallic phase realized by driving at four different frequencies), the conductivity remains finite. For $d > 3$, an Anderson transition in quasi $(d-1)$-dimensions may be driven by changing the kicking strength, and this manifests itself in a metal/super-metal transition in observable quantities. A survey of the three different phases realizable in the QKR is shown in table I.

| properties in quasi $(d-1)$-dimensions | non-generic Planck’s constant | generic Planck’s constant |
|----------------------------------------|-------------------------------|---------------------------|
| localized                              | super-metallic, infinite conductivity | insulating, zero conductivity |
| metallic                               | metallic, $(d \geq 4)$ | metallic, $(d \geq 3)$ |
| finite conductivity                    | finite conductivity           |

TABLE I. Phases realizable in the quasi-periodic QKR

In dimensionless units [2], the time-dependent Hamiltonian of the system is defined as $\hat{H}(t) = \frac{1}{2}\hbar^2 \hat{n}^2 + KV(t) \sum_{m} d(t - m)$. Here, $\theta$ is the rotor’s angular variable, $\hat{n} = -i \partial_\theta$ the angular momentum operator, and $V(t) \equiv V(\cos \theta, \cos(\theta_1 + \omega_1 t), ..., \cos(\theta_{d-1} + \omega_{d-1} t))$, the kicking potential. The $(d-1)$ frequencies, $\omega_1, ..., \omega_{d-1}$, are incommensurate to the kicking frequency $2\pi$ and among themselves, and $\theta_1, ..., \theta_{d-1}$ are $d-1$ arbitrary phases. We assume $V$ to be symmetric with respect to its $d$ arguments, and of unit characteristic variation. Later we will see that the detailed form of $V$ determines the diffusion constant of the system, but is largely inessential otherwise. We assume $\hbar / 4\pi = p/q$ where the limit of an irrational value may be taken by sending $p,q \to \infty$.

As in the experimental applications, we consider the evolution of a wave function initially uniform in $\theta$. The localization properties of such states are probed by the correlation function

$$E(t) = \frac{1}{2} \sum_{n} |\langle n|\hat{U}^\dagger(0)|\hat{U}^\dagger(m)|0\rangle|^2 n^2$$

(1)

where $\hat{U}(m) \equiv \exp(i\hbar \hat{V}(m))$ is the Floquet operator and the overline stands for the average over the parameters $\theta_i$. The mapping to an effectively higher dimensional system [2] is achieved by interpreting $|\theta\rangle \equiv |\theta, \theta_1, ..., \theta_{d-1}\rangle$ as a $d$-dimensional coordinate vector, comprising a ‘real’ angular coordinate, and a generalization of the parameters $\theta_i$ to ‘virtual’ coordinates. Similarly, we introduce a $d$-dimensional angular momentum state, $|N\rangle = |n, n_1, ..., n_{d-1}\rangle$ where $n_i = -i \partial_{\theta_i}$ is conjugate to $\theta_i$, with eigenvalues $n_i \in \mathbb{Z}$. Defining the operator $\hat{\Phi}(m) = \exp(-im\sum_\theta \omega_i \hat{n}_i)$, it is then not difficult to verify that the ‘gauge transformed’ Floquet operator $\hat{U}(m) \equiv \hat{\Phi}(m + 1)\hat{U}(m)\hat{\Phi}^{-1}(m)$ becomes time-independent, viz. $\hat{U} = T(N)\hat{W}(\theta), T(N) \equiv e^{i\sum \omega_i \hat{n}_i}$. $\hat{W}(\theta) \equiv \exp[i\hbar K V(\cos \theta, \cos \theta_1, ..., \cos \theta_{d-1})]$, and that $E(t) = \frac{1}{2} \sum_{N} |\langle N|\hat{U}(0)|\rangle|^2 n^2$. We have, thus, traded the time dependence of the original problem for an effective expansion to a multi-dimensional Hilbert space spanned by the states $|N\rangle$.

The effective Floquet operator $\hat{U}$ possesses two fundamental symmetries: time reversal symmetry [10] $T: t \to -t, \hat{\Theta} \to -\hat{\Theta}$, and invariance under the translation $\hat{n} \to \hat{n} + q$. Exploiting the latter, it is straightforward to verify that the variable $E(t)$ affords the representation

$$E(t) = \frac{q}{2} \int_0^{2\pi/q} \frac{d\phi}{2\pi} \partial^2_{\phi_\perp} \text{tr} \left( \hat{U}^\dagger_\perp \delta_{N_0}^{\perp} \hat{U}^\dagger_\perp \right) |_{\phi_\perp = \phi}$$

(2)

where ‘$\text{tr}$’ is a trace over all states $|N\rangle$ whose real coordinate $n \in \{0, ..., q\}$ is restricted to a compact ‘unit cell’, and we have defined the ‘Bloch-Floquet’ operator $\hat{U}_0 \equiv T(N)\hat{W}(\Theta + \phi)$, where $\Theta = \phi \equiv \{\phi, \theta_1, ..., \theta_{d-1}\}$. As in the quantum mechanics of periodic structures, the summation over a Bloch phase, $\phi$, enables us to compactify $n$-space to a ring of circumference $q$ with periodic boundary conditions. The shift of the angular variable $\hat{\theta} \to \hat{\theta} + \phi$ shows that $\phi$ couples to the system as an Aharonov-Bohm flux, cf. Fig. [1] top.

Eqs. [1] and [2] are different (yet equivalent) ways of probing the time dependent spreading of angular momentum states. Anticipating a competition of classical diffusion and quantum localization, we expect three qualitatively distinct cases (cf. Fig. [1] bottom): if the localization length, $\xi$ is infinitely large, unbound diffusive spreading $n^2 \sim Dt$ characterized by a diffusion coefficient, $D$, leads to a linear increase $E(t) \sim Dt$ (metallic regime). In contrast, for $\xi < q$ we expect saturation, $E(t) \sim \xi^2 / D \sim \text{const. (localized regime)}$. (A finite size correction $\sim \xi^2$ exponentially small in $e^{-\eta/\xi}$ will be ignored.) Finally, in cases where $\xi > q$, the system behaves similar to a finite quantum system of characteristic quasilevel spacing $\sim 1/(q\xi^d)$. For large times, $t \gg q\xi^d$, individual states of this system can be resolved and a formal decomposition of $\hat{U}$ in quasi-energy states shows that $E(t) \sim \xi^2$ (super-metallic regime).

To quantitatively describe these regimes, we define the resolvent operators $\hat{G}^\perp_\phi(\omega) \equiv (1 - (e^{i\phi \hat{U}^\dagger_\perp} + 1)^{-1})^{-1}$, where $\omega_\perp \equiv \omega_0 \pm \frac{1}{2} (\omega + 0)$. The Fourier transform $E(\omega) = \int_0^\infty dt e^{i\omega t} E(t)$, then assumes the form $E(\omega) = \int_0^{2\pi/q} \frac{d\phi}{2\pi} \partial^2_{\phi_\perp} \text{tr} \left( \hat{G}^\perp_{\phi_\perp}(\omega_\perp) \delta_{N_0} \hat{G}^\perp_{\phi_\perp}(\omega_\perp) \right) |_{\phi_\perp = \phi}, \quad \text{and} \quad (\langle \omega \rangle_\perp = \int_0^{2\pi/q} \frac{d\phi}{2\pi}$.
Apart from an overall factor \( \omega \), this resembles the two-particle response function employed to compute the optical conduction properties of electronic systems.

To make further progress, we describe the correlation function \( Y \) in terms of a low energy effective field theory. The technical details of this mapping [11] are nearly identical to those of our earlier treatment of the one-dimensional rotor [1], and we here restrict ourselves to a brief sketch of the principal steps. We start from a representation of the correlation function \( Y \) in terms of Gaussian integral over superfields [8]:

\[
Y(\phi_+, \phi_-, \omega) = \int d\lambda \int D(\psi, \psi) \left< e^{-\bar{\psi}G^{-1}\psi} \right>_{\omega_0} X[\psi, \psi].
\]

Here, the superfield \( \psi = \{\psi_{N,\lambda,\alpha}\} \) where \( \alpha = b, f \) distinguishes between commuting and anti-commuting components, and \( \lambda = \pm \), retarded and advanced components. The pre-exponential term is given by \( X[\bar{\psi}, \psi] = \psi_{N,+b} \bar{\psi}_{N,+b} - \psi_{N,-b} \bar{\psi}_{N,-b} \) and \( G^{-1} = \text{diag}(G_{\phi+}(\omega_+), G_{\phi-}(\omega_-)) \) is a matrix block-diagonal in advanced/retarded space. To make progress with this expression, we apply the color-flavor transformation [22], an integral transform that trades the integral over \( \psi \) and \( \omega_0 \) for the integration over an auxiliary field, \( Z \):

\[
K(\phi, \omega) = \int D(\bar{Z}, Z) \left< \ldots \right> \exp(-S[Z, \bar{Z}]),
\]

\( 'str' \) is the supertrace [8], and we have temporarily suppressed the pre-exponential terms for notational simplicity. Here, \( Z = \{Z_{N,\alpha}^{b}\} \) is a bi-local supermatrix field, subject to the constraints \( \bar{Z}_{b}Z_{b} = Z_{b}^{\dagger}Z_{b} = -Z_{t}^{\dagger}Z_{t} \) and \( |Z_{b}Z_{b}^{\dagger}| < 1 \). The anti-commuting blocks \( Z_{\alpha,\alpha'} \) and \( \bar{Z}_{\alpha,\alpha'} \), \( \alpha \neq \alpha' \) are independent. Physically (cf. Ref. [1] for a more extensive discussion), the field \( Z_{N,\alpha}^{b} \sim \psi_{N,\alpha}^{b} \) describes the pair propagation of a retarded and an advanced single particle amplitude at a slight difference in frequency, \( \omega \), and Aharonov-Bohm flux \( \varphi \equiv \phi_+ - \phi_- \). The structure of the action \( S[Z, \bar{Z}] \) shows that at these values field configurations \( \bar{U}_{\phi+}^{\dagger}ZU_{\phi+} \sim Z \) near-stationary under the adjoint action of the Bloch-Floquet operator dominantly contribute to the field integral. The identification of these ‘slow modes’ is facilitated by passing to a Wigner representation, \( Z_{N_{1},\alpha}^{b} \rightarrow Z_{N,\alpha} \), where \( N = (N_{1} + N_{2})/2 \), and \( \Phi \) is dynamically conjugate to \( N \). Due to the fast relaxation of the dynamics in the space of angular variables, \( \Phi \), the modes of lowest action \( Z_{N,\alpha} \) depend only on \( N \). These angular zero modes then produce the low energy representation

\[
Y(\varphi, \omega) = -\int d\lambda \int dQ e^{-\bar{S}[Q]} \text{str}(Q N P) \text{str}(Q \bar{P}),
\]

\[
S[Q] = -\frac{1}{8} \int d\omega \text{str}(D(\partial_\varphi Q)^2 + 2i\omega Q^3_{\text{AR}}).
\]  

Here, \( P = E_{\text{AR}}^{11} \otimes E_{\text{BF}}^{11} \) and \( \bar{P} = E_{\text{AR}}^{21} \otimes E_{\text{BF}}^{11} \), where the \( 2 \times 2 \)-matrices \( E_{\text{AR}}^{ij} \) act in the space of \( \lambda/\alpha \)-indices, carry a unity at position \((i, j)\), and zero elsewhere. The matrix field \( Q \) is given by

\[
Q = \begin{pmatrix} 1 & Z \\ \bar{Z} & 1 \end{pmatrix}^{-1},
\]

where \( \sigma_{\text{AR}}^3 = E_{\text{AR}}^{11} - E_{\text{AR}}^{22} \). The fluctuations of these fields are governed by an action \( S[Q] \) identical to the action of the ‘diffusive’ nonlinear \( \sigma \)-model of disordered metals (subject to an Aharonov-Bohm flux). The rotor’s diffusion constant is given by \( D = \frac{C}{\omega^2} \), where \( C = (\sin \theta \theta_{|x=0} \phi)^2 \) is a constant of \( O(1) \) whose detailed value depends on the kicking potential. (The perturbative integration over non-zero mode configurations \( Z_{N,\Phi} \) generates corrections to \( D = O(1/\K) \) which we do not discuss here.) Finally, \( \partial_{\varphi} \equiv (\partial_{n} + i\varphi[\sigma_{\text{AR}}^3], \partial_{n,1}, \ldots, \partial_{n,3}) \), is a covariant derivative accounting for the coupling to an AB flux in the compact \( n \)-direction.

Technically, Eq. (3) represents the main result of the present paper. We have described the low energy physics of the quasi-periodic QKR in terms of the nonlinear \( \sigma \)-model of disordered metals [13]. The construction is parametrically controlled by the parameters \( h/\K, \omega \ll 1 \) and corrections to the effective action are small in these parameters. In the following, we discuss a number of physical predictions deriving from the representation (3).

**Metallic regimes.** In dimensions \( d \geq 3 \) (\( q = \infty \)) or \( d \geq 4 \) (\( q \) finite) the system supports an Anderson (metal/insulator) transition. In the metallic phase, \( \K/h \gg 1 \), fluctuations are weak and the action may be expanded to quadratic order in the generators \( Z \). Doing the Gaussian integral over \( Z \), one then obtains

\[
Y(\varphi, \omega) = \frac{1}{D_{\omega} \varphi^2 - i\omega},
\]

where \( D_{\omega} \sim D \) is the diffusion constant weakly renormalized by non-linear and frequency dependent corrections to the quadratic theory. Substituting this result into the expression for \( E(t) \), we obtain diffusive growth \( E(t) \sim Dt \), corresponding to a finite optical conductivity.

**Localized regimes.** For \( q = \infty \), the system is in a localized phase in low dimensions, \( d < 3 \), or below the Anderson transition at \( K/h = O(1) \) in \( d \geq 3 \). In these regimes, the diffusion constant is undergoing strong renormalization, \( D_{\omega} \rightarrow i\omega \). This in turn leads to saturation \( E(t) \sim t^{3/2} \) const., and vanishing (static) conductivity. For \( d \geq 3 \), the condition of scale invariance of the critical conductivity leads to the predictions \( D_{\omega} \sim (-i\omega)^{d/2} \) and \( E(t) \sim t^{d/2} \) at the critical point. This asymptotic agrees with the experimental observation [17] on the scaling of \( E(t) \) in \( d = 3 \).
Super-metallic regimes. For finite $q < \xi$ and $d \leq 3$ the system is localized in the virtual directions and delocalized along the real angular momentum direction (cf. Fig. 1 c)). Resonant transmission through the discrete levels of the ensuing system of effectively finite size then leads to ‘super-metallic’ growth $E(t) \approx Ct^2$ at large time scales, and a corresponding diverging (static) conductivity. The absence of rigorous descriptions of strong localization notwithstanding (for the exceptional case of $d = 2$ see below), we may apply phenomenological reasoning to estimate both the coefficient, $C$, and the crossover time, $t_c$, to super-metallic scaling: at short times, the uncertainty in quasilevel resolution, $\sim t^{-1}$, is larger than the characteristic quasilevel spacing $\Delta t \equiv 1/(q L_t^{d-1})$ of a fictitious system of size $q \times L_t^{d-1}$, where $L_t \equiv (D t^{-1})^{1/2}$ is the characteristic extension of a diffusive process of duration $t$ in the virtual directions, and $D_t^{-1}$ is a shorthand for the diffusion coefficient renormalized down to frequency scales $\omega \sim t^{-1}$. The level mixing then leads to diffusive growth $E(t) \sim Dt$. The borderline condition $t^{-1} \approx \Delta t$ marks the crossover to long-time dynamics, $t > t_c$, governed by localization effects. In this regime, individual levels are no longer mixed by quantum uncertainty. The coherent propagation through individual states then leads to $E(t) \approx Ct^2$, where the coefficient $C$ is fixed by the matching condition $D_t t_c \approx Ct_c^2$, i.e. $C = D t_c / t_c$.

The quantitative determination of both $\xi$ and $t_c$ involves localization phenomena in slab geometries and is more difficult. Generally speaking, the above condition $q < \xi$ on the transverse extension refers to the bulk $d$-dimensional localization length. In contrast, the scale $t_c$ is determined by the localization length characterizing the quasi $(d - 1)$-dimensional low energy regimes. The application of scaling arguments [8] leads to $t_c \approx q D$ and $\xi \approx \exp(2qD)$ in dimensions $d = 2$ and $d = 3$, respectively. In dimensions $d > 3$, the situation is more complicated in that the system supports an Anderson transition in the underlying $(d - 1)$-dimensional system. While the finite transverse extension of the system makes it difficult to determine the transition point, it is clear as a matter of principle that lowering the bare value of $D \sim (K/h)^2$ will trigger a localization transition, which will manifest itself as a metal/super-metal transition in the long time scaling of the observable $E$.

In view of the phenomenological nature of these arguments, it is reassuring that for $d = 2$ a much more sophisticated approach can be formulated. In this case, the theory is defined on an infinitely long cylinder, $N \in (n, n_1) \in [0, q) \times \mathbb{R}$. Fluctuations of the field $Q$ inhomogeneous in $n$-direction are penalized by an action cost (cf. Eq. [3]) of at least $\sim D/q^2$, which we identify as the inverse of the diffusion time in $n$-direction. For frequency scales smaller than $q$, fluctuations can be neglected, and we are left with the quasi one-dimensional field $Q_N = Q_{n_1}$, with effective action $S[Q] = -\frac{2}{3} \int dn_1 \text{str}(D(\partial_n, Q)^2 + Dq^2(\sigma^3_{\text{SR}})^2 + 2\omega Qq^3_{\text{SR}})$.

The correlation function $Y(\varphi, \omega)$ of this theory can be computed by adaption [11] of Efetov’s transfer matrix technique [8]. The qualitative features mentioned above then follow from scaling properties of the corresponding solutions. For a quantitative discussion including pre-factors, we refer to Ref. [11].

Summarizing, we have introduced a microscopic theory of quantum phase transitions in the quasi-periodic QKR. For irrational values of Planck’s constant, the system is described by a $d$-dimensional nonlinear $\sigma$-model which entails a near perfect analogy to the physics of $d$-dimensional disordered metals. However, for rational values, its effective topology changes, and a dimensional reduction to a quasi $(d - 1)$-dimensional system takes place. We discussed the ensuing consequences, including the existence of a metal/super-metal quantum phase transition in $d \geq 4$. It stands to reason, that the interplay of localization and these reduction phenomena should be observable in experiment.

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[1] B. Chirikov and D. Shepelyansky, Scholarpedia 3, 3550 (2008); S. Fishman, ibid. 5, 9816 (2010).
[2] Throughout we will work in dimensionless units where (kicking strength and Planck’s constant) are scaled by (the inverse of) the kicking period. The moment of inertia of particle is set to unity.
[3] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982); A. Altland, ibid. 71, 69 (1993); A. Altland and M. R. Zirnbauer, ibid. 77, 4536 (1996); C. Tian, A. Kamenev and A. Larkin, ibid. 93, 124101 (2004).
[4] C. Tian and A. Altland, New J. Phys. 12, 043043 (2010).
[5] F. L. Moore, J. C. Robinson, C. F. Bharucha, Bala Sundaram, and M. G. Raizen, Phys. Rev. Lett. 75, 4598 (1995).
[6] J. Chabé, G. Lemarié, B. Gréaud, D. Delande, P. Szriftgiser, and J. C. Garreau, Phys. Rev. Lett. 101, 255702 (2008).
[7] G. Lemarié, H. Lignier, D. Delande, P. Szriftgiser, and J. C. Garreau, Phys. Rev. Lett. 105, 090601 (2010).
[8] B. E. Efetov, Supersymmetry in disorder and chaos (Cambridge, UK, 1997).
[9] G. Casati, I. Guarneri, and D. L. Shepelyansky, Phys. Rev. Lett. 62, 345 (1989).
[10] R. Blümel and U. Smilansky, Phys. Rev. Lett. 69, 217 (1992).
[11] C. Tian and A. Altland, in preparation.
[12] M. R. Zirnbauer, J. Phys. A 29, 7113 (1996).
[13] More precisely, Eq. [3] represents a $\sigma$-model of unitary symmetry [8], as relevant for systems of broken T-invariance. In the present context, T-model of unitary symmetry [8], as relevant for systems of broken T-invariance. In the present context, T-breaking is so weak that we should actually employ
the T-invariant form of the QKR nonlinear $\sigma$-model (cf. Ref. [3]). However, this extension is not of great consequence and will be discussed elsewhere [11].