Bifurcation, mode coupling and noise in a nonlinear multimode superconducting microwave resonator

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The addition of nonlinearity to an harmonic resonator provides a route to complex dynamical behaviour of resonant modes, including coupling between them. We present a superconducting device that makes use of the nonlinearity of Josephson junctions to introduce a controlled, tunable, nonlinear inductance to a thin film coplanar waveguide resonator. Considering the device as a potential quantum optical component in the microwave regime, we create a sensitive bifurcation amplifier and then demonstrate spectroscopy of other resonant modes via the intermode coupling. We find that the sensitivity of the device approaches within a factor two quantitative agreement with a quantum model by Dykman, but is limited by a noise that has its source(s) on-chip.

Superconducting circuits that involve the quantisation of charge or flux have been the subject of great attention in recent years as potential building blocks for quantum computers. In this context nonlinear effects have become a strong focus because of unprecedentedly strong coupling parameters, tunability and engineering flexibility. In the weak nonlinear regime the effects are classified into two types: Self-Kerr effects, which give rise to nonlinear phase shifts in a single mode, have been used to generate squeezed states for parametric amplification and to perform high-fidelity measurements of quantum bit while cross-Kerr effects, which dispersively couple two distinct modes, have been discussed in the context of QND measurements and intermode coupling. Nonlinearities have also been exploited in other systems.

In this paper, we take a slightly different viewpoint for these systems by considering them as controllable (quantum) optical devices in the microwave range. The development of “on-chip superconducting microwave optical components” (SMOCs) such as phase shifters, mixers, splitters and detectors, especially in the single and few photon regime, will be essential for any future superconducting QIP technology. However, despite strong recent advances, non-linear quantum dynamics is relatively poorly understood and exploited and we study here a relatively simple generic SMOC device. Superconducting coplanar resonators, that may be thought of as analogous to optical Fabry-Perot resonators in terms of mode structure, exhibit weak intrinsic Kerr nonlinearities. However, the strength of these nonlinearities may be greatly enhanced and controlled by inserting nonlinear, non-dissipative lumped elements based on Josephson junctions. For example, inserting a SQUID in the central conductor couples the SQUID itself to any harmonic mode that has a non-zero current at the SQUID’s location. It also adds an additional inductance whose value depends on the total current through the junctions which may arise from both externally applied flux and the presence of photons in any coupled harmonic modes. This dependence induces nonlinearities in the coupled modes as well as couplings between these modes. The SQUID also enables flux-tunability of the frequencies of the coupled harmonic modes, together with enhancement and tunability of the Kerr nonlinearities.

We explore Kerr effects in a capacitively coupled, multimode superconducting coplanar resonator which incor-
porates a seven-element SQUID array located at the mid-
point in the central conductor of the resonator[23]. The
resonator exhibits electromagnetic standing-wave modes
with a current node at the location of the capacitors.
With the addition of the SQUID array at the cavity mid-
point, antinodes of current occur at the location of the
SQUIDs for odd-numbered modes. The even-numbered
harmonics, in contrast, possess nodes of electrical cur-
rent at the location of the SQUIDs and do not exhibit
enhanced nonlinearities. We first use the self-Kerr ef-
fect to operate an odd-numbered mode as a cavity bi-
furcation amplifier[23]. We then use the sensitivity of this
mechanism in conjunction with the cross-Kerr coupling
to demonstrate spectroscopy of other cavity modes. We
observe that our device is affected by a source of para-
meterically coupled noise at low frequency; elimination of
such effects is a grand challenge in this field[23,24] and so
we investigate some of its properties. We then compare
our results to the calculation by Dykman et al.[25] and show
that our device operates only a factor two worse than
this theory predicts. More details and measurements
on this sample can be found in Palacios-Laloy et al.[22].

Our device consists of a 200 nm sputter-deposited Nb
film fabricated through optical lithography into a me-
andering half-wavelength coplanar waveguide resonator
with end capacitors of design value $C = 7 \text{pF}$ (Figure 1).
The fundamental frequency is $\simeq 1.77 \text{GHz}$. An array of
$N = 7$ Al/AI0.5%/Al SQUIDs was formed with standard e-
beam lithography and double-angle shadow evaporation.
The areal dispersion of the SQUID array is 4%, accord-
ing to SEM imaging. The purpose of using a short array is
to be able to vary the critical current and hence the
magnitude of the nonlinearities at the design stage, while
maintaining a fixed overall inductance of the array. The
sample is magnetically and electrically shielded and at-
tached to the mixing chamber of a dilution refrigerator.
The measurement circuit is shown in Fig. 1. Carefully
controlled microwave pulses are transmitted through the
sample using lines designed to avoid thermal and am-
plifier noise from reaching the sample. The pulses are
homodyne detected and measured using an oscilloscope
or analogue-to-digital converters. We use a pulse repeti-
tion rate (5 kHz) that is sufficiently slow that the mode
relaxes to its ground state between pulses. For CW mea-
surements an Anritsu Vector Network Analyzer is used.
The measurement bandwidth is limited by the circu-
tors to approximately 4 – 8 GHz, covering the 2nd, 3rd
and 4th modes. The third mode is used as our bifurca-
tion amplifier the 2nd and 4th modes are uncoupled and
can be ignored.

We write the Hamiltonian of the system, consisting of
the 3rd mode and some other odd $n$th mode to which it
couples, as $H = H_n + H_3 + H_{3,n}$, with:

$$H_n = h\nu_n \left( a_n^\dagger a_n + 1/2 \right),$$  \hspace{1cm} (1)$$

$$H_3 = h\nu_3 \left( a_3^\dagger a_3 + 1/2 \right) + hK_3 \left( a_2^\dagger a_2 \right)^2,$$ \hspace{1cm} (2)$$

$$H_{3,n} = h\lambda_{3,n} a_n^\dagger a_3 a_n a_n^\dagger.$$ \hspace{1cm} (3)$$

$a_n(a_n^\dagger)$ represents the annihilation (creation) operator of
the $n$th mode of resonant frequency $\nu_n$. $K_n$ is the self-
Kerr parameter of the $n$th mode and $\lambda_{3,n}$ is the cross-
Kerr coefficient between the $n$th and $m$th modes. Here
we assume that all self-Kerr terms except for $K_3$ may
be neglected, as these modes are either operated at low
power far from bifurcation or are empty. $K_3$, the self-
Kerr coefficient of the 3rd harmonic, is related to the
critical number of photons $N_3^c$ required for bifurca-
tion via $N_3^c = 2\gamma_3/\sqrt{3K_3}$, with $\gamma_3$ being the linewidth
of the mode. $\lambda_{3,n}$, the cross-Kerr parameter, defines the
frequency shift of the 3rd mode per photon present in the
$n$th coupled mode.

The Kerr coefficients are related to the circuit param-
eters, and to each other, by

$$K_3 = \lambda_{3,3} \frac{\nu_3}{\nu_n} = \frac{\beta^2 h\nu_3}{N E_j},$$ \hspace{1cm} (4)$$

where $E_j$ is the Josephson energy of a single SQUID of
inductance $L_j$ and $N$ is the number of SQUIDs in the
array. $\beta = L_{array}/L_{tot}$ is the ratio between the total in-
ductance of the SQUID array ($L_{array} = N L_j$) and the
total inductance of the resonator ($L_{tot} = L_{wg} + L_{array}$,
where $L_{wg}$ is the inductance of the waveguide). Both self-
and cross-Kerr coupling parameters in this device are
flux-tunable, though not independently. This arises
from the fact that $E_j$, $\nu_3$, and $\nu_n$ can all be tuned by
varying the magnetic flux through the SQUIDs. Remark-
ably, Eq. (4) has an upper limit given by $\approx 2n\frac{2\epsilon}{\pi R_k} \approx 0.01$
where $\epsilon_0$ is the characteristic impedance of the wave-
guide and $R_k$ is the resistance quantum.

To extract some basic parameters of the system, we used
continuous-wave measurements to acquire the
forward-scattering parameter, $S_{21}$, for the 2nd, 3rd
and 4th modes at different values of magnetic flux. Figure 2
shows the resonant frequencies as a function of reduced
magnetic flux. As expected, the flux modulations of the
uncoupled modes $\nu_2$ and $\nu_4$ are negligible, while $\nu_3$
shows a modulation of $\sim 30\%$. We obtain the following
physical parameters at $\Phi/\Phi_0 = 0$: $L_{array} \approx (0.34 \pm 0.02)\mu\text{H}$,
the average critical current of a SQUID $I_c \approx 6.72 \mu\text{A}$,
$\beta = (2.54 \pm 0.02)\%$, $\nu_3 \approx 5.32 \text{GHz}$ and $K_3 \approx 940 \text{Hz}$.
Hence $K_3/\nu_3 \approx 2 \times 10^{-7}$, which, given the linewidth
of the mode in the linear regime $\gamma_3 \approx 212 \text{kHz}$, estab-
lishes that $N_3^c$, the critical number of photons required for bi-
furcation, is $\sim 260$, which is in reasonable agreement with
the estimated microwave power at the sample.

When a cavity mode, in our case the 3rd, is probed with
a number of photons greater than the critical value, the
resonator response switches (stochastically and hystere-
tically) from the low- to the high-amplitude metastable
state (see Fig. 1F). The probability of the system switch-
ing $(P_S)$ depends weakly on the quality factor of the
mode and on the self-Kerr coefficient $K_3$ while it strongly
depends on the power and frequency of the probing
pulses, and on the resonant frequency $\nu_3$ of the mode in
the linear regime. Figure 3 shows 50 switching probability curves as a function of the driving frequency \( \nu_d \).

The 10%-90% width of the S-shaped averaged probability curve (white line) is \( \Delta S = (4.5 \pm 0.5) \text{kHz} \) or \( \Delta S/\nu_3 \approx 0.9 \text{ppm} \). This value is in reasonable agreement with the theoretical value calculated from Dykman’s model (\( \Delta S/\nu_3 \approx 0.5 \text{ppm} \)) given by

\[
\Delta S = \frac{3^{2/3}}{4} \left( \frac{\beta^2}{N} \right)^{2/3} \left( \frac{k_B T_{\text{eff}}}{E_J} \right)^{2/3} \left( \frac{\delta \nu}{\nu_3} \right)^{1/3},
\]

where \( \delta \nu = \nu_d - \nu_3 \) is the frequency detuning of the microwave driving and \( T_{\text{eff}} = \frac{h \nu_3}{2k_B} \coth \frac{h \nu_3}{2k_B T} \) is the effective temperature. The effective temperature \( T_{\text{eff}} \) is the physical temperature if \( T > T_{co} \) or \( h \nu_3/2k_B \) for \( T < T_{co} \), where \( T_{co} = h \nu_3/4k_B \approx 70 \text{mK} \) is the crossover temperature from the classical regime to the quantum regime. For comparison, the linewidth of the same mode in the linear regime is \( \gamma_3 = 212 \text{kHz} \) or \( \gamma_3/\nu_3 = 40 \text{ppm} \).

\( \Delta S \) represents the sensitivity of the device - i.e. it expresses the ability to resolve a frequency shift of \( \Delta S \) with a high level of confidence within a single shot measurement. This bifurcation technique, known as cavity bifurcation amplification, is a sensitive threshold measurement which may now be used to investigate other aspects of the resonator.

To demonstrate the potential utility of this SMOC, we performed spectroscopy of nearby coupled modes. We first tuned the bifurcation amplifier to a switching probability of 10% and then injected a low-power continuous-wave signal and swept its frequency in the vicinity of the coupled mode to be detected. As shown in Fig. 4, we were able to detect the change in photon occupation of the 1st, 5th, 7th, and 9th modes via a variation in the switching probability of the 3rd mode, thus demonstrating the cross-Kerr coupling between the modes. The resonant frequencies of the coupled modes are obtained via a Lorentzian fit providing a value for \( \beta \) at zero magnetic flux of \( \beta = (2.55 \pm 0.1)\% \) that is compatible with the previously extracted value. In this way, the cross-Kerr effect enables spectroscopy of modes that lie outside our measurement bandwidth and thus are undetectable via the usual transmission detection. Based on the estimate \( \Delta S/\nu_3 \approx 10 \), our device is sensitive to \( \approx 10 \) photons in the 1st mode. The 9th mode is split for reasons that we do not understand; possibly the mode is bifurcating.

We now consider in more details the sensitivity of our device. We have already noted that Dykman’s calculation predicts a width \( \Delta S \) due to quantum fluctuations that is below our experimental value. The probability curves repetitively acquired under nominally iden-
tical conditions (Fig. 3) show that the switching probability $P_S$ is affected by fluctuations that significantly exceed the expected statistical fluctuations. A noise source, whose low frequency components appear in these measurements, may also be present at higher frequencies where it may increase $\Delta S$ and be the source of the discrepancy between our experimental values and the theory of Dykman. Such a noise source would be parametrically coupled to our bifurcation amplifier, either through the frequency of the cavity or through the amplitude and frequency of the biasing pulses. Estimates of the extrinsic sources of noises, such as temperature fluctuations and power (frequency) fluctuations of the biasing pulses, suggest that they cannot account for the observed fluctuations in $P_S$. Fluctuations in the microwave pulse power at the sample holder, measured with the cryostat warm, were found to contribute $\lesssim 10\%$ to the observed $\Delta S$. Hence, in the following, we focus our attention on noise sources intrinsic to the device.

The upper panel in Fig. 5 shows the dependence of the width $\Delta S$ on applied magnetic flux. We first note that $\Delta S$ displays a minimum of $\Delta S_0 \simeq 4.5 $ kHz. We achieve a reasonable fit by assuming that $\Delta S_0$ is an additive constant at all flux values and that the additional increase of $\Delta S$ away from zero flux is caused by a quasi-static flux noise. The RMS amplitude of the flux noise, integrated over the bandwidth of the experiment, can be extracted using the measured dependence of $\nu_1$ on magnetic flux. This leads to an RMS noise amplitude of $\simeq 5 \mu \Phi_0$, which is comparable to commercial SQUID amplifiers and probably indicates the quality of our magnetic shielding. The second-order contribution to $\Delta S$ from this flux noise at zero flux is $\simeq 1$ Hz, which is an insignificant contribution to the observed $\Delta S_0 = 4.5$ kHz. Hence, $\Delta S$ at zero flux is not due to flux noise. The center panel in Fig. 5 shows measurements of $\Delta S$ as a function of the refrigerator temperature over the range $8 - 400 $ mK at zero magnetic flux. The expected dependence of $\Delta S$ with temperature is given by Eq. (5), with $T_{e,ff} = \frac{h \nu_0}{2k} \coth \frac{h \nu_0}{2kT}$ (shown as the dotted line in the central panel of Fig. 3). We observe that the values for $\Delta S$ are generally within a factor two of Dykman’s predictions and follow the weak temperature dependence of the theory down to the lowest temperature. The lower panel in Fig. 5 shows $\Delta S$ as a function of the microwave power in units of the critical power for bifurcation. $\Delta S$ is found to be almost constant over one order of magnitude above the critical power and is again within a factor two of the predictions of Dykman’s model.

In summary, we have created a device to assess the usefulness of on-chip superconducting microwave optical components, or “SMOCs”. By inserting a controllable, nonlinear element into a superconducting microwave resonator, we are able to create a SMOC that simultaneously exploits the self- and cross-Kerr effects. We have shown that high-sensitivity spectroscopy of nearby modes may be performed through measurements of the probability of switching of another mode via non-linear coupling. Analysis of the device sensitivity suggests that there is a source of noise that manifests as fluctuations in the cavity resonant frequency, it has a low frequency component and it arises on-chip. The noise may be related to the ubiquitous noise seen in similar SQUID and qubit systems; we have excluded microscopic flux noise and the remaining sources include two-level fluctuations in the environment and critical current noise in the tunnel barriers. However, we note that the intrinsic sensitivity of the device is close to the limits given by the theory of Dykman.

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1. A. Walraff, D. I Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature, 431, 162-167 (2004).
2. I. Chiorescu, P. Bertet, K. Semba, Y. Nakamura, C. J. P. M. Harmans, and J. E. Mooij, Nature, 431, 159-162 (2004).
3. A. Blais, R.S. Huang, A. Wallraff, S.M. Girvin and R.J. Schoelkopf, Phys. Rev. A, 69, 062320 (2004)
4. M. A. Castellano-Beltran, K. D. Irwin, G. C. Hilton, L. R. Vale, and K. W. Lehnert, Nature Phys. 4, 929 (2008).
5M. A. Castellano-Beltran, and K. W. Lehnert, Appl. Phys. Lett. 91, 083509 (2007).
6R. Vijay, D. H. Slichter, and I. Siddiqi, Phys. Rev. Lett. 106, 110502 (2011).
7M. Metcalfe, E. Boaknin, V. Manucharyan, R. Vijay, I. Siddiqi, C. Rigetti, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret, Phys. Rev. B 76, 174516 (2007).
8F. Mallet, F. R. Ong, A. Palacios-Laloy, F. Nguyen, P. Bertet, D. Vion, and D. Esteve, Nature Phys. 5, 791 (2009).
9F. R. Ong, M. Boissonneault, F. Mallet, A. Palacios-Laloy, A. Dewes, A. C. Doherty, A. Blais, P. Bertet, D. Vion, and D. Esteve, Phys. Rev. Lett. 106, 167002 (2011).
10E. Buks and B. Yurke, Phys. Rev. A 73, 032815 (2006).
11G. Kumar and D. P. DiVicenzo, Phys. Rev. B 82, 014512 (2010).
12O. Suchi, B. Abdo, E. SegoV, O. Shtempluck, M. P. Bencow, and E. Buks, Phys. Rev. B 81, 174525 (2010).
13F. R. Ong, M. Boissonneault, F. Mallet, A. C. Doherty, A. Blais, D. Vion, D. Esteve, and P. Bertet, Phys. Rev. Lett. B 110, 047001 (2013).
14G. Kirchmair, B. Vlastakis, Z. Leghtas, S. E. Nigg, H. Paik, E. Ginossar, M. Mirrahimi, L. Frunzio, S. M. Girvin, and R. J. Schoelkopf, Nature 495, 205 (2013).
15T. J. Kippenberg, S. M. Spillane, and K. J. Vahala, Phys. Rev. Lett. 93, 8 (2004).
16H. J. R. Weistra, M. Poot, H. S. J. van der Zant, and W. J. Venstra, Phys. Rev. Lett. 105, 117205 (2010).
17H. Schmidt and A. Imamoglu, Opt. Lett. 21, 1936 (1996).
18L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature 397, 594 (1999).
19H. Kang and Y. Zhu, Phys. Rev. Lett. 91, 093601 (2003).
20M. Mucke, E. Figuroa, J. Bochmann, C. Hahn, K. Murr, S. Ritter, C. J. Villa-Boas, and G. Rempe, Nature 465, 755 (2010).
21F. Boaknin, V. Manucharyan, S. Fissette, M. Metcalfe, L. Frunzio, R. Vijay, I. Siddiqi, A. Wallraff, R. J. Schoelkopf, and M. H. Devoret, arXiv:cond-mat/0702445.
22A. Palacios-Laloy, F. Nguyen, F. Mallet, P. Bertet, D. Vion, and D. Esteve, J. Low Temp. Phys. 151, 1034 (2008).
23D. J. Van Harlingen, T. L. Robertson, B. L. T. Plourde, P. A. Reichardt, T. A. Crane, and J. Clarke, Phys. Rev. B 70, 064517 (2004).
24R. C. Bialczak, R. McDermott, M. Ansmann, M. Hofheinz, N. N. Katz, E. Lucero, M. Neeley, A. D. O’Connell, H. Wang, A. N. Cleland, and J. M. Martinis, Phys. Rev. Lett. 99, 187006 (2007).
25S. M. Anton, C. Muller, J. S. Birenbaum, S. R. O’Kelley, A. D. Fefferman, D. S. Golubev, G. C. Hilton, H.-M. Cho, K. D. Irwin, F. C. Wellstood, G. Shon, A. Shnirman, and J. Clarke, Phys. Rev. B 85, 224505 (2012).
26M. I. Dykman and V. N. Smelyanski, Zh. Eksp. Teor. Fiz. 94, 61-74 (1988).
27M. I. Dykman and M. A. Krivoglaz, Zh. Eksp. Teor. Fiz. 77, 60-73 (1979).