Minimal metagravity vs. dark matter and/or dark energy

Yu. F. Pirogov
Theory Division, Institute for High Energy Physics, Protvino,
RU-142281 Moscow Region, Russia

Abstract
The minimal metagravity theory, explicitly violating the general covariance but preserving the unimodular one, is applied to study the evolution of the isotropic homogeneous Universe. The massive scalar graviton, contained in the theory in addition to the massless tensor one, is treated as a source of the dark matter and/or dark energy. The modified Friedmann equation for the scale factor of the Universe is derived. The question whether the minimal metagravity can emulate the LCDM concordance model, valid in General Relativity, is discussed.

1 Introduction
According to the present-day cosmological paradigm, given by General Relativity (GR) and the standard cosmology, the reasonable description of our Universe in toto is achieved in the so-called LCDM concordance model (for a review of cosmology, see, e.g. ref. [1]). In accordance with the model, the Universe is spatially flat, fairly isotropic and homogeneous being filled predominantly with the dark energy, accounted for by the Λ-term, as well as with the cold dark matter in the energy proportion roughly 3 : 1. The energy fraction of the luminous matter is almost negligible. The nature of the dark energy and the dark matter seems to be the main puzzle of the contemporary physics.

Thus, all the sources of the dark substances, including the indirect ones, are to be investigated. With this in mind, we study in the present paper whether the above substances (or the parts of them) can be mimicked by a modification of GR, namely, the minimal metagravity theory proposed earlier [2]. Due to the explicit violation of the general covariance (GC) to the unimodular covariance (UC), such a theory describes the massive scalar graviton in addition to the massless tensor one. The idea is to try and associate the scalar graviton with the dark matter and/or dark energy. In Section 2, the compendium of the minimal metagravity theory is given. In Section 3, the evolution of the isotropic homogeneous Universe in the framework of such a theory is considered, and the modified Friedmann equation

1For a brief exposition of the theory, see, ref. [3].
for the scale factor of the Universe is derived. It is argued then in Resume that the minimal metagravity is not explicitly inconsistent with the LCDM concordance model motivating thus for the further study.

2 Minimal metagravity

To begin with, let us present the short compendium of the metagravity theory. Under the latter, we understand generally the effective field theory of the metric revealing, due to the explicit GC violation, the extra physical degrees of freedom contained in the metric besides those for the massless tensor graviton. In the case of the minimal violation, to be used in what follows, the metagravity preserves the residual UC and describes for this reason only one additional particle, the massive scalar graviton. The generic action of such a minimal metagravity is as follows

\[ S = \int \left( L_g(g_{\mu \nu}) + L_\sigma(g_{\mu \nu}, \sigma) + L_m(\phi_m, g_{\mu \nu}, \sigma) \right) \sqrt{-g} d^4x, \]  

where \( g_{\mu \nu} \) is the dynamical metric, \( \phi_m \) is the generic matter field and

\[ \sigma = \frac{1}{2} \ln \frac{g}{\tilde{g}}. \]  

In the above, \( g = \det g_{\mu \nu} \) and \( \tilde{g} \) is an absolute (nondynamical) scalar density of the same weight as \( g \). Depending on the ratio of the two similar scalar densities, \( \sigma \) is the scalar and thus can serve as the Lagrangian field variable. This field corresponds to the metric compression waves and is to be treated as representing the scalar graviton.

In eq. (1), \( L_g \) is the generally covariant Lagrangian for the tensor graviton being chosen conventionally in the extended Einstein-Hilbert form:

\[ L_g = -\frac{1}{2} m_P^2 R(g_{\mu \nu}) + \Lambda, \]  

where \( R = R^{\mu \nu} R_{\mu \nu} \) is the Ricci scalar, with \( R_{\mu \nu} \) being the Ricci curvature, and \( \Lambda \) is the cosmological constant. Also, \( m_P = (8\pi G_N)^{-1/2} \) is the Planck mass, with \( G_N \) being the Newton’s constant. \( L_\sigma \) is the scalar graviton Lagrangian looking in the lowest order on the derivatives as follows

\[ L_\sigma = \frac{1}{2} f_\sigma \partial \sigma \cdot \partial \sigma - V_\sigma(\sigma). \]  

Here \( f_\sigma \) is a constant with the dimension of mass and \( V_\sigma \) is the potential producing the mass for the scalar graviton. The dependence of \( \sigma \) on \( \tilde{g} \) explicitly violates the part of GC, namely the local scale covariance, still preserving the residual UC. A priori, one expects \( f_\sigma = \mathcal{O}(m_P) \). Also, one expects that \( V_\sigma \) though being nonzero is suppressed. Finally, \( L_m \) is the matter Lagrangian possessing, generally, only the residual UC.

Varying the action (1), with respect to \( g_{\mu \nu} \), \( \tilde{g} \) being fixed, one arrives at the minimal metagravity equation:

\[ G_{\mu \nu} = \frac{1}{m_P^2} \left( T_{\mu \nu}^{(m)} + T_{\mu \nu}^{(\sigma)} + T_{\mu \nu}^{(\Lambda)} \right), \]  

2
with
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \] (6)

being the gravity tensor. The r.h.s. of eq. (5) is the total energy-momentum of the nontensor-graviton origin, produced by the matter and the scalar gravitons, plus the vacuum energy. \( T^{(m)}_{\mu\nu} \) is the matter energy-momentum tensor including, if required, the real dark matter contribution, too. For the matter as the continuous medium, of interest in cosmology, one has
\[ T^{(m)}_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu - p_m g_{\mu\nu}, \] (7)

with \( \rho_m \) being the energy density, \( p_m \) the pressure and \( u^\mu \) the medium 4-velocity. \( T^{(\sigma)}_{\mu\nu} \) is the scalar graviton contribution:
\[ T^{(\sigma)}_{\mu\nu} = f^2_\sigma (\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} \partial \sigma \cdot \partial \sigma g_{\mu\nu}) + V_\sigma g_{\mu\nu} + \left( f^2_\sigma \nabla \cdot \nabla \sigma + V'_\sigma \right) g_{\mu\nu}, \] (8)

with \( V'_\sigma = \partial V_\sigma / \partial \sigma \), the covariant derivative \( \nabla_\mu \sigma = \partial_\mu \sigma \) and
\[ \nabla \cdot \nabla \sigma = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \sigma). \] (9)

The term \( T^{(\sigma)}_{\mu\nu} \) can be treated as the scalar graviton contribution to the dark matter and/or dark energy. Finally,
\[ T^{(\Lambda)}_{\mu\nu} = -p_\Lambda g_{\mu\nu}, \] (10)

with \( \rho_\Lambda + p_\Lambda = 0 \) and \( \rho_\Lambda = m^2_\Lambda \), is the vacuum contribution to the dark energy. Under \( \Lambda > 0 \), it produces the negative pressure. Due to the Bianchi identity, \( \nabla_\mu G^{\mu\nu} = 0 \), and the property \( \nabla_\lambda g_{\mu\nu} = 0 \), the energy-momentum of the matter and the scalar gravitons is conserved covariantly:
\[ \nabla_\mu (T^{(m)}_{\mu\nu} + T^{(\sigma)}_{\mu\nu}) = 0, \] (11)

whereas the energy-momentum of the matter alone, \( T^{(m)}_{\mu\nu} \), ceases to conserve.

3 Modified Friedmann equation

In the properly chosen observer’s coordinates \( x^\mu = (t, \rho, \theta, \varphi) \) the Friedmann-Robertson-Walker solution for the interval in the isotropic homogeneous Universe is
\[ ds^2 = dt^2 - a^2(t) \left( \frac{1}{1 - \kappa^2 \rho^2} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right), \] (12)

with \( \kappa \) being a constant with the dimension of mass. This interval is form-invariant relative to the shift of the origin of the spatial coordinates, reflecting the isotropy and homogeneity of the Universe.Conventionally, one can rescale the unit of mass so that \( \kappa^2 = k \), with \( k = \pm 1, 0 \). The last three cases correspond, respectively, to the spatially closed, open and flat Universe. In eq. (12), the spatial factor \( 1/(1 - \kappa^2 \rho^2) \) is the geometrical one, given a priori, while the temporal scale factor \( a(t) \) is the dynamical one to be determined by the gravity equations.
Choosing the new radial variable \( r \)

\[
\rho = \frac{r}{1 + \kappa^2 r^2/4}
\]  

(13)

one gets

\[
ds^2 = dt^2 - \frac{a^2(t)}{(1 + \kappa^2 r^2/4)^2} dx^2,
\]

(14)

with \( x^\mu = (x^0 = t, \{x^m\} = x) \), \( m = 1, 2, 3 \) and \( r^2 = x^2 \). In other terms, the metric looks like

\[
g_{00} = 1, \quad g_{m0} = 0,
\]

\[
g_{mn} = -\frac{a^2(t)}{(1 + \kappa^2 r^2/4)^2} \delta_{mn},
\]

(15)

with \( \sqrt{-g} = a^3/(1 + \kappa^2 r^2/4)^3 \). These coordinates will be understood in what follows.

From eq. (8), one gets

\[
T^{(\sigma)}_{00} = f^2 \sigma \partial_0 \sigma,\]

(16)

where \( \dot{\sigma} = d\sigma/dt \) and

\[
\sigma = \ln \frac{a^3(t)}{\sqrt{-g}(1 + \kappa^2 r^2/4)^3}.
\]

(17)

Hence, generally, \( T^{(\sigma)}_{m0} \neq 0 \). On the other hand due to the isotropy, there should fulfil \( G_{m0} = 0 \) and \( T^{(\sigma)}_{m0} = 0 \). This requires \( T^{(\sigma)}_{00} = 0 \), too, and thus \( \partial_0 \sigma = 0 \). To achieve this the spatial parts of \( g \) and \( \tilde{g} \) should coincide, so that

\[
\sqrt{-\tilde{g}} = \frac{\tilde{a}^3(t)}{(1 + \kappa^2 r^2/4)^3},
\]

(18)

with \( \tilde{a}(t) \) being a free temporal factor. Altogether one has

\[
\sigma(t) = 3 \ln \frac{a(t)}{\tilde{a}(t)}.
\]

(19)

The minimal metagravity equation (5) results now in the two following equations

\[
G^0_0 = 3 \left( \frac{\dot{a}}{a} \right)^2 + \kappa^2 \frac{a^2}{a^2} = \frac{1}{m_P^2} \rho,
\]

\[
G^m_n = \left( 2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \kappa^2 \frac{a^2}{a^2} \right) \delta^m_n = -\frac{1}{m_P^2} p \delta^m_n,
\]

(20)

with \( \ddot{a} = d^2 a/dt^2 \), and \( \rho \) and \( p \) being the total energy density and pressure, respectively:

\[
\rho = \rho_m + \rho_\sigma + \rho_\Lambda,
\]

\[
p = p_m + p_\sigma + p_\Lambda
\]

(21)

(remind that \( p_\Lambda = -\rho_\Lambda \)). The continuous medium is taken to be nonrelativistic: \( a^0 = 1, \ a^m = 0 \). The energy density and pressure for the scalar gravitons are formally defined as for the continuous medium:

\[
T^{0}_{00} = f^2 \sigma \left( \frac{1}{2} \dot{\sigma}^2 + \frac{3}{a} \ddot{a} \dot{\sigma} + \dot{\sigma} \right) + (V_\sigma + V'_\sigma) \equiv \rho_\sigma,
\]

\[
T^{m}_{n0} = \left[ f^2 \sigma \left( -\frac{1}{2} \dot{\sigma}^2 + 3 \frac{\dot{a}}{a} \dot{\sigma} + \ddot{\sigma} \right) + (V_\sigma + V'_\sigma) \right] \delta^m_n \equiv -p_\sigma \delta^m_n,
\]

(22)
with the effective “equation of state”

\[ \rho_{\sigma} + p_{\sigma} = f_{\sigma}^2 \dot{\sigma}^2. \]  

(23)

Note that \( \rho_{\sigma} \) and \( p_{\sigma} \) are coordinate dependent, in distinction with the scalars \( \rho_m \) and \( p_m \). In the above, one has

\[ \dot{\sigma} = 3 \left( \frac{\dot{a}}{a} - \tilde{H} \right), \]
\[ \ddot{\sigma} = 3 \left( \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \dot{\tilde{H}} \right), \]  

(24)

with \( \tilde{H} \equiv \dot{a}/a \), and use is made of the relation

\[ \nabla \cdot \nabla \sigma = \ddot{\sigma} + 3 \frac{\dot{\sigma}}{a}. \]  

(25)

With account for equations (22) and (24), the two lines of eq. (20) substitute the similar equations valid in GR. The first line of eq. (20), the modified Friedmann equation, determines the scale factor \( a(t) \), with the second line giving the consistency condition. Introducing the “critical” energy density

\[ \rho_c = 3m_P^2 \left( \frac{\dot{a}}{a} \right)^2 \]  

(26)

one can bring the modified Friedmann equation to the form

\[ \Omega \equiv \frac{\rho}{\rho_c} = 1 + \frac{\kappa^2 \dot{a}^2}{a^2}. \]  

(27)

Further, differentiating the first line of eq. (20) and combining the result with the second line, one can substitute the latter by the continuity condition:

\[ \dot{\rho} + 3\frac{\dot{a}}{a} (\rho + p) = 0, \]  

(28)

with \( \rho_{\sigma} + p_{\sigma} = f_{\sigma}^2 \dot{\sigma}^2 \) and \( \rho_{\Lambda} + p_{\Lambda} = 0 \).

To really solve these equations one should specify the free functions entering the theory. For the continuous medium, there are conventionally two extreme cases: the dust

\[ \rho_m = \frac{\rho_0}{a^3}, \quad p_m = 0 \]  

(29)

and the radiation

\[ \rho_m = 3p_m = \frac{\rho_0}{a^4}. \]  

(30)

For the scalar graviton, the good starting point would conceivably be the assumption \( \ddot{a} = \text{Const} \) and thus \( \ddot{H} = 0 \). As for the potential \( V_{\sigma} \), little can be said about it a priori, and probably it should be looked for by the trial-and-error method. With these caveats, the equations above are ready for use in working out the cosmological scenarios in the framework of the minimal metagravity.
4 Resume

The above system of the modified Friedmann equation plus the continuity condition is much more intricate compared with the similar system valid in GR and urge to a special investigation. Nevertheless, to show that the given approach is not completely unrealistic suppose that there indeed exists such a solution which mimics the observationally consistent GR solution for the spatially flat Universe ($\kappa^2 = 0$). The latter solution corresponds to the LCDM concordance model and requires the existence of the cold dark matter ($p_{\text{dm}} = 0$) in the energy fraction $\rho_{\text{dm}}/\rho_c$ roughly $1/4$. For this to be reconciled in the minimal metagravity, one should get effectively for the scalar graviton $p_\sigma \simeq 0$ and thus, according to eq. (23), $\rho_\sigma \simeq f_\sigma^2 \dot{\sigma}^2$. Now, putting $p_\sigma/\rho_c \simeq 1/4$, as observations imply, and assuming $\dot{H} = 0$ one gets $f_\sigma \simeq m_p/2\sqrt{3}$ as the necessary condition. The relation $f_\sigma = \mathcal{O}(m_p)$ appears to be quite natural. Similarly, one can look for the minimal metagravity solutions which mimic the (part of) dark energy together with the cold dark matter. Whether such an approach can really be made consistent with observations remains yet to be seen.

References

[1] M. Trodden and S. M. Carroll, astro-ph/0401547.
[2] Yu. F. Pirogov, Phys. At. Nucl. 69, 1338 (2006) [Yad. Fiz. 69, 1374 (2006)]; gr-qc/0505031.
[3] Yu. F. Pirogov, gr-qc/0609103