Wolfenstein Parametrization Reexamined

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Abstract

The Wolfenstein parametrization of the $3 \times 3$ Kobayashi-Maskawa (KM) matrix $V$ is modified by keeping its unitarity up to the accuracy of $O(\lambda^6)$. This modification can self-consistently lead to the off-diagonal asymmetry of $V$: $|V_{ij}|^2 - |V_{ji}|^2 = Z \sum_k \epsilon_{ijk}$ with $Z \approx A^2 \lambda^6 (1 - 2\rho)$, which is comparable in magnitude with the Jarlskog parameter of $CP$ violation $J \approx A^2 \lambda^6 \eta$. We constrain the ranges of $J$ and $Z$ by using the current experimental data, and point out that the possibility of a symmetric KM matrix has almost been ruled out.

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Within the standard electroweak model, the $3 \times 3$ Kobayashi-Maskawa (KM) matrix $V$ offers a natural description of quark mixing and $CP$ violation [1]. The unitarity of $V$ leads to a rephasing invariant measure of $CP$ violation (the Jarlskog parameter $J$ [2])

$$\text{Im} \left( V_{il} V_{jm} V_{im}^{*} V_{jl}^{*} \right) = J \sum_{k,n} \epsilon_{ijk} \epsilon_{lmn}$$

and an off-diagonal asymmetry of $V$ (denoted by $Z$)

$$|V_{ij}|^2 - |V_{ji}|^2 = Z \sum_{k} \epsilon_{ijk},$$

where $i, j, k, l, m, n = 1, 2, 3$. Confronting these two relations with the existing and forthcoming experimental data may provide a stringent test of the standard model.

It proves convenient in practice to use a parametrization form of the KM matrix [1,3,4]. Among the proposed parametrizations, the Wolfenstein form [3]

$$V_{W} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 \left[ \rho - i\eta \left( 1 - \frac{1}{2} \lambda^2 \right) \right] \\
-\lambda & 1 - \frac{1}{2} \lambda^2 - iA^2 \lambda^4 \eta & A \lambda^2 \left( 1 + i \lambda^2 \eta \right) \\
A \lambda^3 (1 - \rho - i\eta) & -A \lambda^2 & 1
\end{pmatrix}$$

is very popular for phenomenological applications. However, there are two minor drawbacks associated with $V_{W}$: (a) its unitarity is only kept up to the accuracy of $O(\lambda^4)$; and (b) it cannot self-consistently describe the off-diagonal asymmetry $Z$. These are of course unsatisfactory when we apply $V_{W}$ to more precise experimental data of quark mixing and $CP$ violation. Noticing drawback (a), Kobayashi [5] has recently presented an exactly unitary parametrization of the KM matrix $V$ in terms of the Wolfenstein parameters. The exactness of this parametrization, accompanied with a complicated form, reduces its phenomenological practicability on the other hand.

Following the same approach as that of Kobayashi and keeping unitarity up to the accuracy of $O(\lambda^6)$, here we present a modified version of the Wolfenstein parametrization as follows:

$$V'_{W} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 - \frac{1}{8} \lambda^4 & \lambda & A \lambda^3 (\rho - i\eta) \\
-\lambda \left[ 1 + \frac{1}{2} A^2 \lambda^4 (2\rho - 1) + iA^2 \lambda^4 \eta \right] & 1 - \frac{1}{2} \lambda^2 - \frac{1}{8} \left( 4A^2 + 1 \right) \lambda^4 & A \lambda^2 \\
A \lambda^3 (1 - \rho - i\eta) & -A \lambda^2 \left[ 1 + \frac{1}{2} \lambda^2 (2\rho - 1) + i\lambda^2 \eta \right] & 1 - \frac{1}{2} A^2 \lambda^4
\end{pmatrix}.$$  

One can observe a few different features of $V'_{W}$ from $V_{W}$:

1. The matrix elements $|V_{12}|$ and $|V_{21}$ given by $V'_{W}$ are not symmetric.
(2) \(|V_{22}|\) is smaller than \(|V_{11}|\) and the difference is of the order \(A^2\lambda^4/2\). This is in agreement with the prediction from a variety of quark mass Ansätze [6].

(3) The relation between \(V_{ub}\) and \(V_{cb}\), described by \(\lambda, \rho,\) and \(\eta\), becomes simpler in \(V'_W\). Thus it should be more convenient to confront the ratio \(V_{ub}/V_{cb}\) with the data of \(B\)-meson physics and \(CP\) violation [7].

(4) Both the normalization conditions and orthogonality relations of \(V'_W\) can be given to the degree of accuracy \(O(\lambda^6)\).

With the help of \(V'_W\), we are now able to carry out a self-consistent calculation of \(Z\) (as well as \(J\)) and obtain
\[
Z \approx A^2\lambda^6(1 - 2\rho) , \quad J \approx A^2\lambda^6\eta .
\]
Clearly \(Z\) is of the same order as \(J\). Note that \(Z\) is approximately independent of \(\eta\), a parameter necessary for \(CP\) violation. In the case of \(\rho = 1/2\), \(Z \approx 0\) holds, which implies a symmetric quark mixing matrix \(V\) up to the accuracy of \(O(\lambda^6)\). The experimental value of \(\rho\) lies in the range \(-0.6 \leq \rho \leq 0.5\), but it is most likely around zero [8]. Thus the possibility of a symmetric KM matrix \(V\) has almost been ruled out [9]. In Fig. 1 we plot the ranges of \(J\) and \(Z\) allowed by the current data on \(A, \lambda, \rho,\) and \(\eta\). The linear relation between \(J\) and \(Z\), given by
\[
Z \approx \frac{1 - 2\rho}{\eta} J ,
\]
can be clearly observed from Fig. 1. We expect that Eq. (6) could serve as a good test of unitarity of the KM matrix in the near future.

In summary, a modified form of the Wolfenstein parametrization has been presented, in which unitarity is kept up to the accuracy of \(O(\lambda^6)\). With this new parametrization we have carried out a self-consistent calculation of the off-diagonal asymmetry of \(V\) and found that it is comparable in magnitude with the rephasing invariant measure of \(CP\) violation. The constraints on these two parameters are given by using the current experimental data. We conclude that the possibility of a symmetric KM matrix has almost been ruled out.

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Figure 1: The ranges of $J$ and $Z$ allowed by the current experimental data on $A, \lambda, \rho$, and $\eta$. 