From ALE-instanton Moduli to Super Yang-Mills Theories via Branes

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September 9, 2018

Abstract

A large class of equivalence relations between the moduli spaces of instantons on ALE spaces and the Higgs branches of supersymmetric Yang-Mills theories, are found by means of a certain kind of duality transformation between brane configurations in superstring theories. 4d, N=2 and 5d, N=1 supersymmetric gauge theories with product gauge groups turn out to correspond to the ALE-instanton moduli of type II B and type II A superstring theories, respectively.

PACS: 11.25.Sq, 11.30.Pb
Keywords: ALE-instanton, Super Yang-Mills theories, Branes, Duality transformation

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In a previous paper[1], N. Maru and the present author constructed brane configurations in the space-time of superstring theories, which are equivalent to matrix string theories in the presence of $k$ longitudinal fivebranes. The low energy limits of these brane configurations give $(1+1)$ dimensional $N = (4, 4)$ supersymmetric $U(n)$ gauge theories with an adjoint and a singlet hypermultiplet and $k$ matter hypermultiplets in the fundamental representation of the gauge group.

The latter theory had been studied by Witten[2] and it had been shown that the Higgs branch of the theory (with zero bare mass and FI terms) can be interpreted as the moduli space of $n$-instanton solutions of $U(k)$ gauge theory on $\mathbb{R}^4$, from a point of view of D-flatness of supersymmetric gauge theories and ADHM construction of instantons.

The moduli space of $n$-instanton solutions of $U(k)$ gauge theory on $\mathbb{R}^4$, also admits brane representation. It is a bound state of $n$ Dirichlet(D)-one branes and $k$ D5 branes, in the type II B superstring theory, due to Douglas[3].

Since both Higgs branch of supersymmetric gauge theories and the moduli space of instantons in the gauge theories on $\mathbb{R}^4$, have their own brane representations, it is natural to expect that the equivalence relation between the Higgs branch of $(1+1)$-dimensional supersymmetric gauge theory and the moduli space of the instanton solutions of gauge theory, can be proved more directly at the level of brane configurations.

At first sight, our brane configurations and those of Douglas look quite different since the former model contain D2 branes and D4 branes, while the latter contain D1 and D5 branes. There is, however, simple relation between two types of brane configurations. This is the T-duality transformation in the $x^6$ direction. (See also ref. [4] , for a discussion of this kind of duality).

In our model, the world volume of D2 branes, is $(x^0, x^1, x^6)$ and that of D4 branes, is $(x^0, x^1, x^7, x^8, x^9)$, and $x^6$ direction is compactified on a circle of radius $S$. Consider the small radius limit, $S \to 0$. This limit can be taken by the T-duality transformation of the branes in the $x^6$ direction. Since the world volume of D2 branes contains $x^6$ direction, it loses $x^6$ component upon the T-duality transformation to become $(x^0, x^1)$ which is identified with that of D1 brane. The world volume of D4 branes, on the contrary, does not contain $x^6$ direction and so it acquires $x^6$ direction upon the T-duality transformation to become $(x^0, x^1, x^6, x^7, x^8, x^9)$ which is identified with that of D5 brane.

Now in our brane configuration, Higgs branch of the supersymmetric
gauge theory arises from the sector where D2 branes are connected to the D4 branes, which upon T-duality transformation in $x^6$ direction, is mapped on the sector where D1 branes are bounded to D5 branes, whereby reproducing Douglas’ brane configuration.

A question arises: can one find equivalence relations between the Higgs branch of supersymmetric theories and the moduli space of instantons, in more general examples, by considering them in brane configuration and performing T-duality transformation?

This is the problem which we address in this paper. A natural generalization of the instanton solution of gauge theory on $\mathbb{R}^4$ (flat space) is, the instanton solution of gauge theory on a curved four-dimensional Euclidean space. Among them are the ALE (asymptotically locally Euclidean) spaces, or Gibbons-Hawking gravitational instanton[5]. So, in this paper, we try to find supersymmetric gauge theories, the Higgs branch of which are equivalent to the moduli spaces of instantons of ALE spaces. We will find a large class of equivalence relations between two sets of theories, by means of T-duality transformations between the corresponding brane configurations.

The brane configuration which gives the moduli space of $U(k)$ gauge instanton, with instanton number $n$, on the ALE space with the metric asymptotic to $\mathbb{R}^4/Z_m$ is discussed by Douglas and Moore[6]. In the case of type II B superstring, it is a bound state of $n$D3 branes with $k$D7 branes which are placed at the singularity of $\mathbb{C}^2/Z_m$ orbifold. Let the world volume of D-branes be;

- $nm$ D3 branes: $(x^0, x^1, x^2, x^3)$
- $k$ D7 branes: $(x^0, x^1, x^2, x^3, x^6, x^7, x^8, x^9)$

Then the four-dimensional Euclidean space in question is parametrized by $(x^6, x^7, x^8, x^9)$, so $\mathbb{C}^2$ is parametrized by two complex numbers;

$$
\begin{align*}
  z^1 &= x^6 + ix^7 \\
  z^2 &= x^8 + ix^9
\end{align*}
$$

(1)

and the $Z_m$ action on these numbers is;

$$
\begin{align*}
  z^1 &\rightarrow e^{2\pi i/m} z^1 \\
  z^2 &\rightarrow e^{-2\pi i/m} z^2
\end{align*}
$$

(2)

The $A_{m-1}$ singularity of $\mathbb{C}^2/Z_m$ is at the point $(z^1, z^2) = (0, 0)$, which is $x^6 = x^7 = x^8 = x^9 = 0$. The resolved ALE space has a metric of Gibbons-
Hawking gravitational instanton[5], [6].

\[ ds^2 = V^{-1} \left( dt + \vec{A} d\vec{x} \right)^2 + V (d\vec{x})^2 \]  

\[ V = \sum_{j=1}^{m} \frac{1}{|\vec{x} - \vec{x}_j|} \]

\[-\vec{\nabla} V = \vec{\nabla} \times \vec{A} \]

\[ x^6 = t, \quad \vec{x} = (x^7, x^8, x^9) \]

For our purpose, the \( A_{m-1} \) singularity of \( \mathbb{C}^2/\mathbb{Z}_m \) can be conveniently represented by \( m \) copies of NS five branes with the world volume:

- \( m \) NS5 branes: \((x^0, x^1, x^2, x^3, x^4, x^5)\)
  - since these branes have definite \((x^6, x^7, x^8, x^9)\) value corresponding to their position coordinates.

Now under the T-duality transformation in \( x^6 \) direction, branes are transformed as follows:

- \( nm \) D3 branes \((x^0, x^1, x^2, x^3)\)
  - \( \rightarrow n \) D4 branes \((x^0, x^1, x^2, x^3, x^6)\)
  - \( k \) D7 branes \((x^0, x^1, x^2, x^3, x^6, x^7, x^8, x^9)\)
  - \( \rightarrow mk \) D6 branes \((x^0, x^1, x^2, x^3, x^7, x^8, x^9)\)

- since D3 branes acquire \( x^6 \) component, and D7 branes lose \( x^6 \) component in the world volume. On the other hand, \( m \) NS5 branes with the world volume \((x^0, x^1, x^2, x^3, x^4, x^5)\) are not changed under the duality transformation.

The brane configuration after the duality transformation is depicted in fig. 1. From this configuration, one can read off the effective supersymmetric gauge theory, to which it reduces at low energies. The world volumes of D4 branes are \((x^0, x^1, x^2, x^3, x^6)\), but they have only finite length in \( x^6 \) direction, so macroscopically, the D4-brane effective field theory is on the four-dimensional space-time \((x^0, x^1, x^2, x^3)\). In the course of T-duality transformation, the original type II B string is transformed into type II A string, which has \( N=2 \) supersymmetry in ten dimensions. This is equivalent to \( N=8 \) supersymmetry in four dimensions. In the presence of NS5 branes, D4 branes and D6 branes, one quarter of the original supersymmetry is preserved, so it gives \( N=2 \) supersymmetry in four dimensions. The presence of \( n \) D4 branes gives \( U(n) \) gauge group, but they are suspended between \( m \) NS5 branes located at equal spacing \( 2\pi/m \) in the circle of radius \( S \) in \( x^6 \) direction. Therefore, the gauge group becomes a product of \( U(n) \) groups, which is \( U(n)^m \).
Type II B theory is transformed into type II A theory and gives this brane configuration of $n$ D4, $mk$ D6 and $m$ NS5 branes. This figure is drawn for $n = m = k = 3$. The effective field theory is a 4-dimensional supersymmetric Yang-Mills theory.

branes give matter hypermutiplets, first $k$ of which are in the $(n, 1, 1, \cdots, 1)$ representation of $U(n)^m$ next $k$ of which are in the $(1, n, 1, \cdots, 1)$, and so on.

The last $k$ hypermultiplets are in the $(1, \cdots, 1 n)$ representation of $U(n)^m$.

There arise also $m$ hypermultiplets in the bi-fundamental representations of $U(n)^m$, which are $(n, \bar{n}, 1, \cdots, 1), (1, n, \bar{n}, 1, \cdots, 1), \cdots, (\bar{n}, 1, \cdots, 1, n)$. Now we proceed to the instanton moduli on ALE in the type II A superstring. In type II A superstring, the brane configuration of Douglas and Moore is, a bound state of $n$ D4 branes with $k$ D8 branes, as opposed to $n$ D3 branes with $k$ D7 branes in the type II B case. Under the T-duality transformation in $x^6$ direction, branes are transformed as follows:

- $nm$ D4 branes $(x^0, x^1, x^2, x^3, x^4) \rightarrow n$ D5 branes $(x^0, x^1, x^2, x^3, x^4, x^6)$
- $k$ D8 branes $(x^0, x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9) \rightarrow mk$ D7 branes $(x^0, x^1, x^2, x^3, x^4, x^7, x^8, x^9)$

On the other hand $m$ NS 5branes with the world volume $(x^0, x^1, x^2, x^3, x^4, x^5)$
are not changed under the duality transformation. The brane configuration after the duality transformation is depicted in fig. 2. The crucial difference of this type IIA case (IIB after the duality transformation) from the above type II B case (IIA after the duality transformation) is that the dimension of the space-time of the low-energy effective field theory is 5, instead of 4, since there are D5 branes with world volume \( (x^0, x^1, x^2, x^3, x^4, x^6) \), which looks macroscopically like the five-dimensional space-time \( (x^0, x^1, x^2, x^3, x^4) \). Therefore, we have five-dimensional N=1 supersymmetric gauge theory as an effective theory. The gauge group and hypermultiplets and their representations are the same as in the case of type II B (II A after the duality transformation).

In conclusion, we have found a large class of equivalence relations between the moduli spaces of instantons on ALE spaces and the Higgs branches of supersymmetric Yang-Mills theories, by means of T-duality transformation in \( x^6 \) direction between brane configurations in type II A and type II B superstring theories.

The equivalence relations we have found are summarized as follows;

The moduli spaces of \( U(k) \) gauge instanton with instanton number \( n \) on the ALE space with the metric asymptotic to \( \mathbb{R}^4/Z_m \) in type IIB (type II
A) superstring is, isomorphic to the Higgs branch of 4-dimensional $N=2$ (5-dimensional $N=1$) supersymmetric $U(n)^m$ Yang-Mills theory with $mk$ hypermultiplets in $(n, 1, \ldots, 1)$, $(1, n, 1, \ldots, 1)$, $\ldots$, $(1, 1, \ldots, n)$ representations of $U(n)^m$ and $m$ hypermultiplets in the bi-fundamental representations of $U(n)^m$, $(n, \bar{n}, 1, \ldots, 1)$, $(1, n, \bar{n}, 1, \ldots, 1)$, $\ldots$, $(\bar{n}, 1, \ldots, 1, n)$.

Acknowledgements

The author would like to thank S. Kitakado, H. Ikemori and N. Maru for useful discussions.
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