FIELD-THEORETICAL TREATMENT
OF NEUTRINO OSCILLATIONS:
THE STRENGTH OF THE CANONICAL OSCILLATION
FORMULA*

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Abstract

We discuss conceptual aspects of neutrino oscillations with the main emphasis on the field-theoretical approach. This approach includes the neutrino source and detector processes and allows to obtain the neutrino transition or survival probabilities as cross sections derived from the Feynman diagram of the combined source – detection process. In this context, the neutrinos which are supposed to oscillate appear as propagators of the neutrino mass eigenfields, connecting the source and detection processes. We consider also the question why the canonical neutrino oscillation formula is so robust against corrections and discuss the nature of the oscillating neutrino state emerging in the field-theoretical approach.

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I. INTRODUCTION

A. History

Neutrino oscillations play a central role in neutrino physics. The idea of neutrino oscillations in analogy with $K^0\bar{K}^0$ oscillations was first discussed by B. Pontecorvo in the late fifties [1]. The concept of 2-neutrino flavour mixing [2] (1962) and the confirmation of the existence of different neutrino flavours $\nu_e$ and $\nu_\mu$ in the same year opened the way to neutrino flavour oscillations [4]. For a general number of neutrinos, neutrino mixing is denoted as

$$\nu_{L\alpha} = \sum_j U_{\alpha j} \nu_{Lj} \quad \text{with} \quad \alpha = e, \mu, \tau, \ldots, \ j = 1, 2, 3, \ldots$$

labelling neutrinos flavours (types) and mass eigenfields, respectively.

The first theory of 2-neutrino oscillations was developed by Gribov and Pontecorvo [5]. The 2-neutrino oscillation probabilities in the form as they are used nowadays were formulated by Bilenky and Pontecorvo in Ref. [6] and the extension to an arbitrary number of neutrinos (with $U$ real) can be found in Refs. [7,8]. The most general form of transition ($\alpha \neq \beta$) and survival ($\alpha = \beta$) probabilities (see, e.g., Refs. [9,10]) reads

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L/E_\nu) = \left| \sum_j U_{\beta j} U_{\alpha j}^* \exp\left(-\frac{m_j^2 L}{2E_\nu}\right) \right|^2,$$

where $U$ denotes the unitary mixing matrix, $L$ the distance between source and detector and $E_\nu$ the neutrino energy. The neutrino masses $m_j$ are associated with the mass eigenfields $\nu_j$. Note that the above formula assumes ultra-relativistic neutrinos. With the ordering $m_1 \leq m_2 \leq \ldots$, Eq. (2) is rewritten as

$$P_{\nu_\alpha \rightarrow \nu_\beta}(L/E_\nu) = \sum_j |U_{\beta j}|^2 |U_{\alpha j}|^2 + 2 \sum_{k>j} \text{Re} \left[ U_{\beta j}^* U_{\alpha j} U_{\beta k} U_{\alpha k}^* \exp\left(-i2\pi L/L_{kj}^{osc}\right) \right],$$

where the oscillation lengths are defined by [8]

$$L_{kj}^{osc} = 4\pi \frac{E_\nu}{\Delta m_{kj}^2}.$$

It was a very important discovery that the oscillation pattern as described by Eqs. (2) and (3) can get significantly modified by the effect of coherent forward scattering in background matter [11]. Otherwise, the canonical oscillation formula (2) or its equivalent form (3) has

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1For a detailed account of the history of neutrino oscillations see the contribution of S.M. Bilenky to this meeting (hep-ph/9908335).
resisted – at least in terrestrial experiments – all attempts to find non-negligible corrections. The strong evidence for neutrino oscillations found in atmospheric neutrino experiments [12] (for reviews on neutrino oscillations see, e.g., Ref. [13]) constitutes a major achievement and progress in neutrino physics. If in addition to the solar neutrino evidence for neutrino oscillations also the LSND evidence will be confirmed by future experiments, then neutrino physics is enriched by a sterile neutrino (or sterile neutrinos), a notion coined by Pontecorvo in Ref. [4]. In this case, in the summation in Eq. (1) there will be sterile neutrino types in addition to the flavours. For a review of the physics of sterile neutrinos see Ref. [14].

B. Conceptual questions

It has been indicated in several publications that the standard derivation of Eq. (2) as found, e.g., in Ref. [8] raises a number of conceptual questions (see, e.g., Ref. [15] for a clear exposition). Let us consider some of these questions for a moment:

1. Under which conditions is Eq. (2) valid?

2. Do the neutrino mass eigenstates $|\nu_j\rangle$ have definite momenta or definite energies? Or are both smeared out? Note that in the usual derivation it is assumed that all $|\nu_j\rangle$ have the same momenta $p$ and thus energies $E_j = \sqrt{m_j^2 + p^2}$. One considers their time evolution given by $\exp(-iE_j t)$ and finally makes the replacement $t \to L/c$ ($c$ is the velocity of light). The last step is necessary to make contact with realistic experiments where the distance $L$ between source and detector is known, but no measurement of the time $t$ can be made [16].

3. Why is it allowed to make the replacement $t \to L/c$ when wave functions with definite momentum have infinite extent in space?

4. Why not making the replacement $t \to cL/v_j$ where $v_j = p/E_j$ is the velocity of the neutrino with mass $m_j$?

5. Evidently, the Fock space of neutrinos with definite mass exists. Is this also the case for flavour neutrino states as sometimes assumed in the literature?

One aspect of the limitation of Eq. (2) is treated in terms of a coherence length [17,20]. Some of the above questions are answered in the framework of the wave packet approach [18,19] (see also the review [21] where a list of references can be found), where the neutrino momentum is smeared out, however, the size and form of wave packet is not determined in this approach and remains a subject to reasonable estimates. Let us shortly mention the answers to the above questions offered by the wave packet approach:

Ad 1: One necessary condition for validity of the canonical formula (2) is formulated using the notion of a coherence length. Any coherence length in neutrino oscillations [17] is related to the oscillation length [4] by [22]

$$L_{k,j}^{\text{coh}} = \frac{E_\nu}{\Delta E_\nu} L_{k,j}^{\text{osc}},$$

(5)
where $E_\nu$ is the mean neutrino energy and $\Delta E_\nu$ is the energy spread of the neutrino beam. If $L \gg L_{coh}^{kj}$ holds, the term with $\exp(-i2\pi L/L_{coh}^{kj})$ in Eq. (3) is suppressed.

Ad 2: Obviously, smearing out the neutrino momentum also smears out the neutrino energies.

Ad 3: The wave packet approach deals with the time variable either by putting $t = L/c$ or by averaging over $t$.

Ad 4: Such a replacement in the standard derivation of Eq. (2) is wrong because it leads to $E_j t_j = E_j^2 L/p = pL + m_j^2 L/p$ and consequently misses the factor $1/2$ in the exponents.

Ad 5: In Ref. arguments are given that a Fock space of flavour neutrinos does not exist.

C. The field-theoretical approach

The idea has been put forward to include the neutrino production and detection processes into the consideration of neutrino oscillations [15]. Such an approach can be realized with quantum mechanics – in which case the neutrinos with definite mass are unobserved intermediate states between the source and detection processes [13] – or with quantum field theory where the massive neutrinos are represented by inner lines in a big Feynman diagram depicting the combined source – detection process [25,26]. In the following we will discuss the field-theoretical treatment. The aims and hopes of such an approach are the following:

A. The elimination of the arbitrariness associated with the wave packet approach,

B. the description of neutrino oscillations by means of the particles in neutrino production and detection which are really manipulated in an experiment,

C. a more complete and realistic description in order to find possible limitation of formula (2) in specific experimental situations.

Note that in this approach the question of a Fock space for flavour states has no relevance.

Considering laboratory experiments, there are two typical situations for neutrino oscillation experiments. The first one is decay at rest (DAR) of the neutrino source. Its corresponding Feynman diagram is depicted in Fig. 1. The wave functions of the source and detector particles are localized (peaked) at $\vec{x}_S$ and $\vec{x}_D$, respectively. The other situation is decay in flight (DIF) of the neutrino source as represented in Fig. 2 where it is assumed that a proton hits a target localized at $\vec{x}_T$. The detector particle sits again at $\vec{x}_D$ but the source is not localized. In both situations the distance between source and detection is given by $L = |\vec{x}_D - \vec{x}_S|$. Note that in the Feynman diagrams of Figs. 1 and 2 the neutrinos with definite mass occur as inner lines. In the spirit of our approach, neutrino oscillation probabilities are proportional to the cross sections derived from the amplitudes represented by these diagrams.

In the following we will work out the amplitude for neutrino oscillations as described by Figs. 1 and 2 in the limit of macroscopic $L$, discuss the consequences and compare this
approach with the wave packet approach. Note that within the field-theoretical approach
the notion *coherent* refers to a summation in the amplitude, whereas *incoherent* means a
summation over squares of amplitudes, i.e., a summation in the cross section \[20\].

**II. ASSUMPTIONS AND THE RESULTING AMPLITUDE**

**A. The basic assumptions**

The further discussion is based on the following assumptions:

I. The wave function \( \phi_D \) of the detector particle does not spread with time, which
amounts to

\[ \phi(\vec{x}, t) = \psi_D(\vec{x} - \vec{x}_D) e^{-iE_{DP}t}, \tag{6} \]

where \( E_{DP} \) is the sharp energy of the detector particle and \( \psi_D(\vec{y}) \) is peaked at \( \vec{y} = \vec{0} \).

II. The detector is sensitive to momenta (energies) and possibly to observables commuting
with momenta (charges, spin).

III. The usual prescription for the calculation of the cross section is valid.

With the amplitude symbolized by Fig. 1 the oscillation probability is obtained by

\[ \langle P_{(\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta})} \rangle \propto \int dP_S \int \frac{d^3p_{D1}}{2E_{D1}} \cdots \frac{d^3p_{Dn_D}}{2E_{Dn_D}} \left| A_{(\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta})} \right|^2. \tag{7} \]

In this equation we have indicated the average over some region \( \mathcal{P} \) in the phase space of the
final particle of the detection process. If no final particle of the neutrino production process
is measured then one has to integrate over the total phase space of these final states. For
DIF symbolized by Fig. 2 there is an additional integration \( \int dP_T \) over the final particles in
the target process. By definition, at the source (detector) a neutrino \( \bar{\nu}_\alpha \) (\( \bar{\nu}_\beta \)) is produced
(detected) if there is a charged lepton \( \alpha^\pm (\beta^\pm) \) among the final states.

In perturbation theory with respect to weak interactions, according to the Feynman diagrams Figs. 1 and 2 one has to perform integrations \( \int d^4x_1, \int d^4x_2 \) and \( \int d^4q \) corresponding
to the Hamiltonian densities for neutrino production, detection and the propagators of the
mass eigenfields. These integrations are non-trivial because \( \psi_D \) and the source (target) wave
functions are not plane waves, but are localized at \( \vec{x}_D \) and \( \vec{x}_S (\vec{x}_T) \), respectively.

**B. Integration over \( d^4x_2 \)**

Let us discuss now in detail the integration \( \int d^4x_2 \) at the interaction vertex of the detector
process. For definiteness we consider \( \bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta \) transitions, which is the typical situation
in experiments with DAR of the neutrino source (see LSND, KARMEN and reactor experiments).
A remark is at order concerning the integration over \( dx_2^0 \). We will perform this
introduction from $-\infty$ to $+\infty$, which corresponds to initial time $t_i = -\infty$ and final time $t_f = +\infty$ of the combined source–detection process. This is certainly only approximately correct for DAR (see Fig. 1), because in this case the time evolution of the source particle should rather be considered starting at a finite time, after the preparation of the state of the source particle. We will later comment on this problem in detail. Looking at the neutrino line in Figs. 1 and 2 which connects the neutrino source with the detector particle, with Assumption II saying that final states are described by plane waves, we read off the following integral:

$$
\int d^4x_2 \exp \left( i \sum_b p'_{Db} \cdot x_2 \right) \int \frac{d^4q}{(2\pi)^4} e^{-iq(\bar{x}_1 - x_2)}
$$

$$
\times \sum_j U_{\alpha j} \frac{q + m_j}{q^2 - m_j^2 + i\epsilon} U^*_{\beta j} \psi_D(\bar{x}_2 - \bar{x}_D) e^{-iE_D p'_{D2}}
$$

$$
= \int dq^0 e^{-iq^0t_1} \delta(q^0 + E_D) \int \frac{d^3q}{(2\pi)^{3/2}} e^{-i\bar{q}(\bar{x}_D - x_1)}
$$

$$
\times \sum_j U_{\alpha j} \frac{q + m_j}{q^2 - m_j^2 + i\epsilon} U^*_{\beta j} \widetilde{\psi}_D(\bar{q} + \bar{p}'_D)
$$

(8)

with the definitions

$$
\bar{p}'_D = \sum_{b=1}^{n_D} \bar{p}'_{Db} \quad \text{and} \quad E_D = \sum_{b=1}^{n_D} E'_{Db} - E_{DP}.
$$

(9)

The Fourier transform of $\psi$ is denoted by $\widetilde{\psi}$. Note that $\bar{x}_1$ is identical with the source position $\bar{x}_S$ for DIF. One can show for DAR with a wave function of the neutrino source analogous to the detector wave function (6) that integration over $\bar{x}_1$ causes the replacement of $\bar{x}_1$ by $\bar{x}_S$ in the expression (8) \[25\].

C. The integration over $d^3q$ and the asymptotic limit $L \to \infty$

The next step in treating the expression (8) is obviously the integration over $d^3q$. Actually, we need the result of this integration only in the asymptotic limit of macroscopic distance between source and detector $L = |\bar{x}_D - \bar{x}_S|$. This limit is handled by the following theorem \[25\].

Theorem: Suppose we have an integral of the form

$$
J(\bar{L}) = \int d^3q \Phi(\bar{q}) \frac{e^{-i\bar{q}\bar{L}}}{A - q^2 + i\epsilon},
$$

(10)

where $\Phi$ is three times continuously differentiable and $\Phi, \nabla_j \Phi, \nabla_j \nabla_k \Phi$ all decrease at least like $1/\bar{q}^2$ for $L \equiv |\bar{L}| \to \infty$. Furthermore, $A$ is a constant, i.e., independent of $\bar{q}$ and $\bar{L}$. Then in the asymptotic limit $L \to \infty$ the integral (10) is given by
\[ J(\vec{L}) = -\frac{2\pi^2}{L} \Phi \left( -\sqrt{A \frac{\vec{L}}{L}} \right) e^{i\sqrt{A} L} + \mathcal{O} \left( L^{-3/2} \right) \quad \text{for} \quad A > 0, \quad (11) \]
\[ J(\vec{L}) = \mathcal{O} \left( L^{-2} \right) \quad \text{for} \quad A < 0. \quad (12) \]

Applying this theorem to the integration \[ \int d^3 q e^{-i\vec{q} \cdot \vec{L}} \ldots \] in the expression (8), for each \( j \) we identify \( A \) with \((q^0)^2 - m_j^2\) and \( \vec{L} \) with \( \vec{x}_D - \vec{x}_S \). Apart from a factor, the result of the integration and the asymptotic limit \( L \to \infty \), which picks out the term proportional to \( L^{-1} \), is described by the following operations on the right-hand side of Eq. (8) for each \( j \):

1. The denominator in the neutrino propagator is removed.
2. In the three places in expression (8) where \( \vec{q} \) occurs, i.e., in the exponential, in the numerator of the neutrino propagator and in \( \tilde{\psi}_D \), the replacement \( \vec{q} \to -q_j \vec{\ell} \) with \( q_j = \sqrt{E_D^2 - m_j^2} \) and \( \vec{\ell} = (\vec{x}_D - \vec{x}_S)/L \)

has to be made.

This amounts to having neutrino \( j \) on mass shell with four-momentum \[ k_j = \left( E_D, q_j \vec{\ell} \right). \quad (14) \]

Note that with \( k_j \) the numerator of the neutrino propagator is trivially rewritten as
\[ -k_j^0 + m_j = -\sum_{\pm} v_j(k_j, s) \bar{v}(k_j, s), \quad (15) \]

which is the projection operator on the negative-energy solutions of the free Dirac equation. Of course, due to the left-handedness of the weak charged currents the term with negative helicity is negligible in Eq. (15).

This consideration conforms to our physical intuition: if we have a production process for a particle with which one performs a scattering experiment at a distance macroscopically separated in space from the production process, then the amplitude of the combined production – scattering (detection) process is the product of the production and scattering amplitudes. This is the physical content of the above theorem. In the case of neutrino oscillations the particles with which the scattering is performed are the massive neutrinos.

In the case of neutrino mixing (11) several different massive neutrinos are produced and the amplitude is the sum over the products of production and detection of the neutrino mass eigenstates with mass \( m_j \). In this way interference effects arise if certain conditions, which will be explored in the following, are fulfilled. In summary, after having performed the integrations over \( d^4 x_2 \) and \( d^4 q \), in the asymptotic limit \( L \to \infty \) only the neutrinos on mass shell contribute to the amplitude in Eq. (7) which can be written as \[ 25,26 \]
\[ \mathcal{A}_{\nu_\alpha \to \nu_\beta}^\infty = \sum_j A_j^{\nu_\alpha} A_j^{D \nu_\beta} U_{aj} U_{bj}^* e^{iq_j L}, \quad (16) \]
where $A_j^S$ and $A_j^D$ denote the amplitudes for production and detection of a neutrino with mass $m_j$. Note that $E_D$ is the energy on the neutrino line in Figs. 1 and 2. It is independent of $m_j$ and determined by the energies of the final states of the detection process. This is an immediate consequence of the assumptions in the previous section, in particular, of Assumption I. Furthermore, due to the above-mentioned integrations and the asymptotic limit we obtain

$$A_j^D \propto \tilde{\psi}_D(-q_j \ell + \vec{p}_D').$$

(17)

In the case of DAR an analogous function $\tilde{\psi}_S$ appears ($\psi_S$ is localized at $\vec{x}_S$), such that

$$A_j^S \propto \tilde{\psi}_S(q_j \ell + \vec{p}_S') \text{ with } \vec{p}_S' = \sum_{b=1}^{n_S} \vec{p}_{Sb}'. \tag{18}$$

The amplitude (17) looks very similar to the amplitude appearing in the standard oscillation formula (2), and the question when the standard formula arises amounts to a discussion of the conditions under which the amplitudes $A_j^S$ and $A_j^D$ are independent of $j$. This will be done in the next section.

In Ref. [26] it has been shown that the finite lifetime of the neutrino source particle can be incorporated with the help of the Weisskopf–Wigner approximation, thus performing perturbation theory starting at a final initial time $t_i = 0$. Consequently, in the $d^4x_2$ integration, the time variable $x_2^0$ is integrated from 0 to $\infty$ and, therefore, the $dq_0$ integrand is not simply given by $\delta(q_0 + E_D)$ (see Eq. (8)). Nevertheless, it has been demonstrated in Ref. [26] that in the limit of macroscopic $L$ one recovers this delta function and all our manipulations at the detector vertex and the inner neutrino line discussed in this section are valid also for explicit finite lifetime of the neutrino source. However, there are effects stemming from the source vertex. These will be discussed in the next section without derivations. For details we refer the reader to Refs. [26,27].

### III. RESULTS

The preceding discussion based on the assumptions stated in Section II A leads us to the following conclusions:

(i) Since in the asymptotic limit $L \to \infty$ the neutrinos on the inner line in the Feynman diagrams Fig. 1 and 2 are on mass shell we are allowed to make the interpretation that we have neutrino mass eigenstates $|\nu_j\rangle$ characterized by the energy $E_\nu \equiv E_D \forall j$ and momenta $q_j = \sqrt{E_D^2 - m_j^2}$. Thus they have all the same energy determined by the detection process, but the momenta are different [26]. The same conclusion has also been reached in [16,23], however, for other reasons.

(ii) The summation over $E_D$ is incoherent, i.e., it occurs in the cross section (see Eq. (7)).
not in the amplitude (16). In this sense there are no neutrino wave packets\(^2\) in experiments conforming to our assumptions \(^2\).

(iii) It has been shown in Ref. \([22]\) that the relation (3) holds irrespective of the coherent or incoherent nature of the energy spread. However, with the two points just made it follows that any coherence length originates in an incoherent energy spread and reflects the inability to measure the energies of the final states of the detector process more precisely than \(\Delta E_\nu\) \([26]\). Note that in \([22]\) it has already been pointed out that a coherent or incoherent neutrino energy spread cannot be distinguished in neutrino oscillation experiments. Since the neutrino energy can in principle be determined with arbitrary precision the coherence length can theoretically be increased solely by detector manipulations \([22,29]\).

(iv) From Eq. (17) it follows that with \(\Delta m^2_{kj} \equiv m_k^2 - m_j^2\) \((k > j)\) the condition

\[
q_j - q_k \simeq \frac{\Delta m^2_{kj}}{2E_D} \lesssim \sigma_D \quad \text{or} \quad \sigma_{xD} \lesssim \frac{1}{4\pi} L^\text{osc}_{kj} \quad \text{(DAR + DIF)} \tag{19}
\]

is necessary for having neutrino oscillations with \(\Delta m^2_{kj}\) (see Eq. (3)), where \(\sigma_D\) and \(\sigma_{xD}\) are the widths of the wave function of the detector particle in momentum and coordinate space, respectively. Otherwise, the standard formula (2) has to be corrected by suppressing the oscillatory term containing \(\Delta m^2_{kj}\). The second part of equation (19) is obtained by rewriting the first part using the oscillation length \([4,18,15,25]\). In realistic experiments condition (19) holds because \(\sigma_{xD}\) is a microscopic quantity, whereas, dropping from now on the indices \(j, k\), the oscillation length

\[
L^\text{osc} \simeq 2.48 \text{ m} \left( \frac{1 \text{ eV}^2}{\Delta m^2} \right) \left( \frac{E_\nu}{1 \text{ MeV}} \right) \tag{20}
\]

is a macroscopic quantity.

(v) In the case of DAR a condition analogous to (19)

\[
\sigma_{xS} \lesssim \frac{1}{4\pi} L^\text{osc} \quad \text{(DAR)} \tag{21}
\]

exists for the width of the neutrino source wave function. Note that the conditions (19) and (21) are independent of \(L\), i.e., Eqs. (19) and (21) do not define a coherence length. Such conditions were first discussed in the wave packet approach by Kayser \([18]\).

(vi) Now we want to consider effects of the finite lifetime \(\tau_S = 1/\Gamma\) of the neutrino source. The decay width \(\Gamma\) introduces an energy spread \(\Delta E_\nu \sim \Gamma\). However, the corresponding

\(^2\)The same conclusion but for a different reason was presented in Ref. \([28]\).
coherence length (5) is much too long to be of interest in realistic experiments if we take the decay widths of muons or charged pions. If the neutrino source particle can be conceived as a free, unbound particle its wave functions spreads during its lifetime. It has been shown [26] that, in order not to wash out neutrino oscillations, the condition

\[ \frac{\Delta m^2 \sigma_S}{m_S E_D} \lesssim \Gamma \]  \hspace{1cm} (22)

has to be fulfilled, where \( \sigma_S \) is the spread of \( \tilde{\psi}_S \). Defining a velocity spread of the source particle at rest by \( \sigma_S/m_S \), where \( m_S \) is its mass, we can rewrite Eq. (22) as

\[ \Delta v_S \tau_S \lesssim \frac{1}{4\pi} L_{osc}. \]  \hspace{1cm} (23)

We can interpret this relation by saying that the spreading of the source wave function during the lifetime of the source particle should be less than the oscillation length in order not to destroy the oscillation pattern.

Some comments concerning these results are at order. The points (i) – (iv) refer only to the detector. In particular, (i) – (iii) mean that the nature of the neutrino energy spread is determined by the properties of the detector. This is an immediate consequence of the Assumptions I, II, III in Section II A. Making a simple numerical estimate of Eq. (22) for the LSND [30] and KARMEN [31] experiments, where the source particles are stopped muons, we have \( \sigma_S \lesssim 0.01 \text{ MeV} \) [32], \( \Delta m^2 \sim 1 \text{ eV}^2 \) and \( E_\nu \sim 30 \text{ MeV} \). Thus we get \( \Delta m^2 \sigma_S/m_\mu E_\nu \lesssim 3 \times 10^{-18} \text{ MeV} \ll \Gamma_\mu \approx 3 \times 10^{-16} \text{ MeV} \). Therefore, the coherence condition (22) is very well fulfilled (see Ref. [27] for a detailed derivation and discussion of this condition). The quantum-mechanical conditions (19) and (21) have classical analogues: the uncertainty in the position of the detector (source) particle has to be less than the oscillation length, otherwise oscillations are washed out by these macroscopic, classical averaging processes [8]. Consequently, if these classical conditions, which involve macroscopic lengths, are fulfilled, then the conditions (19) and (21) involving the microscopic spread of wave functions hold to a much better degree. In summary, in the framework discussed here all corrections to Eq. (2) are negligible.

**IV. WHAT REMAINS TO BE DISCUSSED**

It remains to point out what is still missing or what was not discussed in the field-theoretical approach. In the case of DIF, conditions concerning the target process (replacing conditions concerning the source process for DAR) have to be worked out. For an attempt in this direction see Ref. [33]. Note that we have not been able to build in the interaction of the final state particles in the source with the environment. This is a potential source of a finite coherence length [14,22] according to the following consideration: neutrino emission is interrupted at the time when one of the final particles in the source process interacts with the environment. This leads to a finite length \( \ell_\nu \) of the neutrino wave train and thus to an energy spread \( \Delta E_\nu \propto 1/\ell_\nu \), resulting in a finite coherence length (4). Such a coherence
length might play a role for solar neutrinos with $\Delta m^2 \sim 10^{-5}$ eV$^2$ (MSW effect \[11\]), in which case the mass-squared difference and $L = 150 \times 10^6$ km could be large enough that the neutrino wave packets corresponding to the mass eigenstates are separated when they arrive on earth \[14\] \[22\]. However, the different averaging processes involved in the calculation of the solar neutrino flux lead in any case effectively to the consideration of the neutrino state arriving on earth as an incoherent mixture of neutrino mass eigenstates (see, e.g., Ref. \[34\]). A further problem is the inclusion of the MSW effect in the field-theoretical approach. This problem was studied in Ref. \[35\]. In the theorem discussing the asymptotic limit $L \to \infty$ (see Section \[11C\]) we have assumed that $L$ pertaining to a neutrino oscillation experiment is so large that in the specific situation $L$ is in the asymptotic region of the integral \[8\]. It was found in Ref. \[36\] that under certain conditions there is an intermediate range for $L$ which results in an oscillation formula different from \[2\]. There is also the possibility that in some cases (e.g., the KARMEN experiment) it is not realistic to use the conventional procedure to calculate the cross section \[14\] by taking the asymptotic limit of the final time to infinity.

V. CONCLUDING REMARKS

The field-theoretical treatment of neutrino oscillations includes the neutrino production and detection processes and allows thus a description in terms of quantities which are really manipulated in an experiment. This treatment has demonstrated the robustness of the canonical oscillation formula \[2\] against corrections in realistic experimental situations. The reason is that in the field-theoretical approach, for the validity of Eq. \[2\], only the conditions \[14\], \[21\] and \[22\], which do not involve $L$, have to be fulfilled. The first two conditions simply state that the macroscopic oscillation length \[20\] has to be much larger than the microscopic spatial extension of the wave functions of the neutrino detection and source particles. Condition \[22\] refers to unstable neutrino sources and seems to be well fulfilled in view of \[30\] $\Delta m^2 \lesssim 2 \text{ eV}^2$ for all realistic sources. Conditions involving $L$ (oscillation lengths) stemming from interactions of final state particles of the source process with surrounding matter have not yet been included in the field-theoretical approach. However, we do not expect that their inclusion drastically changes the heuristic estimates of coherence lengths.

A theoretical aspect of the approach presented here is that the properties of the oscillating neutrino state is largely determined by the detector. With the reasonable assumptions introduced in Section \[11A\] we arrive at the conclusion that no neutrino wave packets are needed for the description of neutrino oscillations. In some sense, neutrino wave packets are replaced by the wave functions of the neutrino detector and source (target) particles. Using neutrino flavour states

$$|\nu_\alpha; t, x\rangle = \sum_j U_{\alpha j}^* |\nu_j\rangle e^{-i(E_\nu t - q_j x)} \quad \text{with} \quad q_j = \sqrt{E_\nu^2 - m_j^2} \quad (24)$$

and $\langle \nu_j | \nu_k \rangle = \delta_{jk}$ in the discussion of neutrino oscillations and performing neutrino energy summations in the probabilities \[2\] but not in the states \[24\] comes close to the spirit of the field-theoretical approach.
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FIG. 1. Feynman diagram for decay at rest (DAR) of the neutrino source particle. The source (S) and detector (D) processes are symbolized by the circles. The labels $\vec{x}_S$ and $\vec{x}_D$ represent the coordinates where the wave functions of the source and detector particles are peaked, respectively. We have also indicated the $n_S$ ($n_D$) momenta of the final particles originating from the source (detector) process and the neutrino propagator of the neutrino field with mass $m_j$.

FIG. 2. Feynman diagram for decay in flight (DIF) of the neutrino source particle which is produced by a proton with momentum $\vec{P}$ hitting a target (T) particle localized at $\vec{x}_T$. In addition to the final momenta of DAR there are $n_T$ final momenta originating from the target process.