Bicyclic graphs with extremal degree resistance distance

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Abstract

Let \( r(u, v) \) be the resistance distance between two vertices \( u, v \) of a simple graph \( G \), which is the effective resistance between the vertices in the corresponding electrical network constructed from \( G \) by replacing each edge of \( G \) with a unit resistor. The degree resistance distance of a simple graph \( G \) is defined as

\[ D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]r(u, v), \]

where \( d(u) \) is the degree of the vertex \( u \). In this paper, the bicyclic graphs with extremal degree resistance distance are strong-minded. We first determine the \( n \)-vertex bicyclic graphs having precisely two cycles with minimum and maximum degree resistance distance. We then completely characterize the bicyclic graphs with extremal degree resistance distance.

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1 Introduction

Topological indices based on the various distances between the vertices of a graph are widely used in theoretical chemistry to establish relations between the structure and the properties of molecules. They provide correlations with physical, chemical and thermodynamic parameters of chemical compounds \[5, 13, 35, 39, 40, 41, 42, 44\].

The graphs considered in this paper are finite, loopless, and containing no multiple edges. Given a graph \( G \), let \( V(G) \) and \( E(G) \) be, respectively, its vertex and edge sets. The ordinary distance \( d(u, v) = d_G(u, v) \) between the vertices \( u \) and \( v \) of the graph \( G \) is the length of the shortest path between \( u \) and \( v \) \[2\]. The Wiener index \[14\] of a connected graph \( G \), denoted by \( W(G) \), is defined as the sum of all distances between unordered pairs of vertices \( u \) and \( v \), i.e.,

\[ W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v). \]
The resistance distance between the vertices \( u \) and \( v \) of the graph \( G \), denoted by \( r(u, v) \), is defined to be the effective resistance between the nodes \( u \) and \( v \) in \( G \). Analogous to Wiener index, the Kirchhoff index is defined as

\[
K_f(G) = \sum_{\{u, v\} \subseteq V(G)} r(u, v)
\]

which has been widely studied \([1, 4, 8, 9, 10, 12, 16, 21, 22, 24, 25, 26, 27, 28, 29, 31, 32, 36, 38, 42, 43, 45]\). For other notations in graph theory, we follow \([2]\) and recent papers \([17, 18, 19, 20, 23, 37]\).

A modified version of the Wiener index is the degree defined as

\[
D(G) = \sum_{\{u, v\} \subseteq V(G)} [d(u) + d(v)]d(u, v),
\]

where \( d(u) = d_G(u) \) is the degree of the vertex \( u \) of the graph \( G \). It was introduced independently by Gutman \([11]\), Dobrynin and Kochetova \([6]\) as a weighted version of the Wiener index.

Analogous to the definition of the degree distance, the degree resistance distance is defined as

\[
D_R(G) = \sum_{\{u, v\} \subseteq V(G)} [d(u) + d(v)]r(u, v),
\]

which was introduced by I. Gutman, L. Feng and G. Yu in \([12]\). Palacios \([30]\) named the same graph invariant “additive degree-Kirchhoff index” and gave tight upper and lower bounds for the degree resistance distance of a connected undirected graph by using Markov chain theory.

Tomescu \([34]\) determined the unicyclic and bicyclic graphs with minimum degree distance \( D(G) \). In \([33]\), the author investigated the properties of connected graphs having minimum degree distance \( D(G) \). Bianchi et al. \([11]\) gave some upper and lower bounds for degree resistance distance \( D_R(G) \) whose expressions do not depend on the resistance distances. Recently, Chen et al. \([4]\) characterized the maximal degree resistance distance of unicyclic graphs. In a natural way, we will consider the bicyclic graphs. A bicyclic graph is a connected graph whose edge number is one more than its vertex number. Obviously a bicyclic graph contains either two or three cycles. Throughout this article, we restrict our consideration on bicyclic graphs with exactly two cycles.

For convenience, let \( \mathcal{B}_n^{p,q} \) be the set of the bicyclic graphs with exactly two cycles \( C_p \) and \( C_q \) as follows: \( C_p = v_1v_2\cdots v_pv_1 \) and \( C_q = u_1u_2\cdots u_qu_1 \) are two cycles such that there is a path \( P = v_1w_1\cdots w_{m-1}u_1 \) joining them. The trees \( T_{v_1}, T_{u_j} \) and \( T_{w_k} \) are rooted at \( v_1, u_j \) and \( w_k \), respectively. We say a tree \( T \) trivial if \( |V(T)| = 1 \), i.e., \( T \) is an singleton vertex, see Fig. 1.
Let $S_{n}^{p,q}$ be the graph obtained from cycles $C_p$ and $C_q$ by attaching $n + 1 - p - q$ pendent edges to the unique common vertex of them. Let $P_{n}^{p,q}$ be the graph consisting of two disjoint cycles $C_p$ and $C_q$ and a path of length $n - p - q + 1$ joining them. If $n - p - q + 1 = 0$, $P_{n}^{p,q}$ coincides with $S_{n}^{p,q}$, see Fig. 2.

To the best of our knowledge, the extremal degree resistance distances for bicyclic graphs has not been considered so far. In this paper, we firstly characterize $n$-vertex bicyclic graphs with exactly two cycles having minimum and maximum degree resistance distance, and then characterize the bicyclic graphs with extremal degree resistance distance.

## 2 Preliminaries

In this section, we provide some lemmas and graph transformations, that play an important role in the subsequent sections. Let $r(u, v)$ denote the resistance distance between $u$ and $v$ in the graph $G$. Recall that $r(u, v) = r(v, u)$ and $r(u, v) \geq 0$ with equality if and only if $u = v$. 

![Fig. 1. Illustration for a graph in graph $S_{n}^{p,q}$.](image)

![Fig. 2. (a) $S_{n}^{p,q}$ and (b) $P_{n}^{p,q}$.](image)
In what follows, for the sake of simplicity, instead of \( u \in V(G) \) we write \( u \in G \). For a vertex \( v \) in \( G \), we define
\[
K_{f_v}(G) = \sum_{u \in G} r(u, v) \quad \text{and} \quad D_v(G) = \sum_{u \in G} d(u)r(u, v).
\]
The definition of \( D_v(G) \) implies that
\[
D_R(G) = \sum_{v \in G} d(v) \sum_{u \in G} r(u, v).
\]

For a vertex \( v \in G \), \( G - v \) denotes the graph obtained from \( G \) by deleting \( v \) and its incident edges. Let \( \overline{G} \) be the complement of \( G \). Let \( G - e \) (\( G + e \)) denote the graphs obtained from \( G \) by deleting (or adding respectively) the edge \( e \).

Let \( H \) be a subgraph of graph \( G \). For a vertex \( u \in V(H) \), let
\[
r(u|H) = \sum_{v \in V(H)} r(v, u|H), \quad S'(u|H) = \sum_{v \in V(H)} d(v)r(v, u|H).
\]

If \( C_n \) is a cycle with the vertex set \( V(C_n) = \{v_1, \ldots, v_n\} \) and \( n \geq 3 \), by Ohm’s law, we have that, for \( v_i, v_j \in V(C_n) \) with \( i < j \),
\[
r_{C_n}(v_i, v_j) = \frac{(j - i)(n + i - j)}{n}.
\]

**Lemma 2.1 (12).** Let \( x \) be a cut-vertex of a connected graph, and let \( a \) and \( b \) be the vertices occurring in different components which arise upon deletion of \( x \). Then
\[
r(a, b) = r(a, x) + r(x, b).
\]

**Lemma 2.2 (12).** Let \( C_k \) be the cycle of size \( k \), and \( v \in C_k \). Then
\[
K_{f_1}(C_k) = \frac{k^3 - k}{12}, \quad D_R(C_k) = \frac{k^3 - k}{3}, \quad K_{f_v}(C_k) = \frac{k^2 - 1}{6}, \quad D_v(C_k) = \frac{k^2 - 1}{3}.
\]

**Lemma 2.3 (12).** Let \( G_1 \) and \( G_2 \) be connected graphs with disjoint vertex sets, with \( n_1 \) and \( n_2 \) vertices, and with \( m_1 \) and \( m_2 \) edges, respectively. Let \( u_1 \in V(G_1) \), \( u_2 \in V(G_2) \). Constructing the graph \( G \) by identifying the vertices \( u_1 \) and \( u_2 \), and denote the so obtained vertex by \( u \). Then
\[
D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2r(u_1|G_1) + 2m_1r(u_2|G_2) + (n_2 - 1)S'(u_1|G_1) + (n_1 - 1)S'(u_2|G_2).
\]

**Definition 2.1** Let \( G \) be a bicyclic graph with \( V(G) = \{u, v, v_1, \ldots, v_s\} \cup V(C_p) \cup V(C_q) \), for which \( v \) is a vertex of degree \( s + 1 \) such that \( vv_1, vv_2, \ldots, vv_s \) are pendent edges incident with \( v \), and \( u \) is the neighbor of \( v \) distinct from \( v_i \), that is on the cycle \( C_q \). The other cycle \( C_p \) only has one common vertex \( w \) with \( C_q \). We form a graph \( G' = \sigma(G, v) \) by deleting the edges \( vv_1, vv_2, \ldots, vv_s \) and adding new edges \( uv_1, uv_2, \ldots, uv_s \). We say that \( G' \) is a \( \sigma \)-transform of the graph \( G \), see Fig. 3.

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Lemma 2.4 Consider the graphs defined in Definition 2.1 and let \( G' = \sigma(G, v) \) be a \( \sigma \)-transform of the bicyclic graph \( G \), for convenience, let \( r(w, u) = t \). Then \( D_R(G) > D_R(G') \).

Proof. Let \( H \) be a subgraph \( G'[\{u, v, v_1, \ldots, v_s\} \cup V(C_q)] \) and \( T = G'[\{u, v, v_1, \ldots, v_s\}] \) (\( H' \) be a subgraph \( G''[\{u, v, v_1, \ldots, v_s\} \cup V(C_q)] \) and \( T' = G''[\{u, v, v_1, \ldots, v_s\}] \), respectively). Note that \( H \) and \( C_p \) (\( H' \) and \( C_p \), respectively) share a common vertex \( w \). By Lemmas 2.2 and 2.3, we get

\[
D_R(G) = D_R(C_p) + D_R(H) + 2(s + 1 + q)r(w|C_p) + 2pr(w|H) + (q + s)S'(w|C_p) + (p - 1)S'(w|H),
\]
\[
D_R(H) = D_R(C_q) + D_R(T) + 2(s + 1)r(u|C_q) + 2qr(u|T) + (s + 1)S'(u|C_q) + (q - 1)S'(u|T),
\]
\[
D_R(G') = D_R(C_p) + D_R(H') + 2(s + 1 + q)r(w|C_p) + 2pr(w|H') + (q + s)S'(w|C_p) + (p - 1)S'(w|H'),
\]
\[
D_R(H') = D_R(C_q) + D_R(T') + 2(s + 1)r(u|C_q) + 2qr(u|T') + (s + 1)S'(u|C_q) + (q - 1)S'(u|T'),
\]
\[
D_R(G) - D_R(G') = 2qr(u|T) + (q - 1)S'(u|T) + 2pr(w|H) + (p - 1)S'(w|H) - 2qr(u|T') - (q - 1)S'(u|T') - 2pr(w|H') - (p - 1)S'(w|H')
\]
\[
= 2q(2s + 1) + (q - 1)(s + 1 + 2s) + 2p\left[\frac{q^2 - 1}{6} + t + 1 + (t + 2)s\right] + (p - 1)\left[\frac{q^2 - 1}{3} + (t + 1)(s + 1) + (t + 2)s\right] - 2q(s + 1) - (q - 1)(s + 1)
\]
\[
- 2p\left[\frac{q^2 - 1}{6} + (t + 1)(s + 1)\right] - (p - 1)\left[\frac{q^2 - 1}{3} + (t + 1)(s + 1)\right]
\]
\[
= 2ps + (p - 1)(t + 2)s + 2s(2q - 1) > 0.
\]

Lemma 2.5 Let \( G_0 \) be a bicyclic graph with \( V(G) = \{v_1, \ldots, v_s\} \cup V(C_p) \cup V(C_q) \), for which \( u \) is a vertex of degree \( s \) in the cycle \( C_q \) of the bicyclic graph \( G_0 \), and \( uv_1, uv_2, \ldots, uv_s \) are pendent edges incident with \( u \). The other cycle \( C_p \) only has one common vertex \( w \) with \( C_q \). Let graph \( G_1 \) delete the edges \( uv_1, u v_2, \ldots, u v_s \), and add new edges \( w v_1, w v_2, \ldots, w v_s \). For convenience, let \( r(w, u) = t \). Then \( D_R(G_0) > D_R(G_1) \).
Proof. Let $H_0$ be a subgraph $G'[\{v_1, \ldots, v_s\} \cup V(C_q)]$ and $S = G'[\{u, v_1, \ldots, v_s\}]$ ($H_1$ be a subgraph $G_1[\{v_1, \ldots, v_s\} \cup V(C_q)]$ and $S_1 = G'[\{w, v, v_1, \ldots, v_s\}]$, respectively). By Lemmas 2.2 and 2.3, we get

$$D_R(G_0) = D_R(C_p) + D_R(H_0) + 2(s + 1) r(w|C_p) + 2pr(w|H_0) + (q + s - 1) S'(w|C_p) + (p - 1) S'(w|H_0),$$

$$D_R(H_0) = D_R(C_q) + D_R(S) + 2sr(w|C_q) + 2qr(u|S) + sS'(u|C_q) + (q - 1) S'(u|S),$$

$$D_R(G_1) = D_R(C_p) + D_R(H_1) + 2(s + 1) r(w|C_p) + 2pr(w|H_1) + (q + s - 1) S'(w|C_p) + (p - 1) S'(w|H_1),$$

$$D_R(H_1) = D_R(C_q) + D_R(S_1) + 2sr(w|C_q) + 2qr(w|S_1) + sS'(w|C_q) + (q - 1) S'(w|S_1),$$

$$D_R(G_0) - D_R(G_1) = 2pr(w|H_0) + (p - 1) S'(w|H_0) - 2pr(w|H_1) - (p - 1) S'(w|H_1)$$

$$= 2p \left[ \frac{q^2 - 1}{6} + (t + 1)s \right] + (p - 1) \left[ \frac{q^2 - 1}{3} + (t + 1)s \right]$$

$$- 2p \left[ \frac{q^2 - 1}{6} + s \right] - (p - 1) \left[ \frac{q^2 - 1}{3} + s \right]$$

$$= (3p - 1)ts > 0.$$  

Definition 2.2 Let $G$ be a bicyclic graph $G$ with $V(G) = \{u, v, v_1, \ldots, v_s\} \cup V(C_p) \cup V(C_q)$, for which $v$ is a vertex of degree $s + 1$ such that $v v_1, v v_2, \ldots, v v_s$ are pendent edges incident with $v$, and $u$ is the neighbor of $v$ distinct from $v_i$ that is on the cycle $C_q$. The other cycle $C_p$ only has one common vertex $w$ with $C_q$. We form a graph $G'' = \pi(G, v)$ by deleting the edges $v v_1, v v_2, \ldots, v v_s$ and connecting $v_i$ and all the isolated vertices into a path $v v_1 \ldots v_s$. We say that $G''$ is a $\pi$-transform of the graph $G$, see Fig. 4.

![Fig. 4. π-transform of a vertex v.](image-url)

Lemma 2.6 Consider the graphs defined in Definition 2.1 and let $G'' = \pi(G, v)$ be a $\pi$-transform of the bicyclic graph $G$, for convenience, let $r(w, u) = t$. Then $D_R(G) < D_R(G'')$. 


Proof. Let $H$ be a subgraph $G\{u, v, v_1, \ldots, v_s\} \cup V(C_q)$ and $T = G\{u, v, v_1, \ldots, v_s\}$ ($H''$ be a subgraph $G''\{u, v, v_1, \ldots, v_s\} \cup V(C_q)$ and $P = G''\{u, v, v_1, \ldots, v_s\}$, respectively). By Lemmas 2.2 and 2.3, we get

$$D_R(G) = D_R(C_p) + D_R(H) + 2(s + 1 + q)r(w|C_p) + 2pr(w|H) + (q + s)S'(w|C_p) + (p - 1)S'(w|H),$$

$$D_R(H) = D_R(C_p) + D_R(T) + 2(s + 1)r(w|C_p) + 2q(r(u|T) + (q + s)S'(w|C_p) + (q - 1)S'(w|T),$$

$$D_R(G'') = D_R(C_p) + D_R(H'') + 2(s + 1 + q)r(w|C_p) + 2pr(w|H'') + (q + s)S'(w|C_p) + (q - 1)S'(w|H''),$$

$$D_R(H'') = D_R(C_p) + D_R(P) + 2(s + 1)r(w|C_p) + 2q(r(u|P) + (s + 1)S'(w|C_p) + (q - 1)S'(w|P),$$

$$D_R(G) - D_R(G'') = D_R(T) + 2q(r(u|T) + (q - 1)S'(w|T) + 2pr(w|H) + (p - 1)S'(w|H)$$

$$- D_R(P) - 2q(r(u|P) - (q - 1)S'(w|P) - 2pr(w|H'') - (p - 1)S'(w|H'') = (3s^2 - s) + 2q(2s + 1) + (q - 1)(s + 1 + 2s) + 2p\left[\frac{q^2 - 1}{3} + t + 1 + (t + 2)s\right]$$

$$+ (p - 1)\left[\frac{q^2 - 1}{3} + (t + 1)(s + 1) + (t + 2)s\right] - \frac{2}{3}s^3 + s^2 + \frac{1}{3}$$

$$- 2q\left[\frac{1 + s + 1)(s + 1)}{2} - (q - 1)\left[(s + 1)s + s + 1\right]\right.$$

$$- 2p\left[\frac{q^2 - 1}{6} + t(s + 1) + \frac{(1 + s + 1)(s + 1)}{2}\right]$$

$$- (p - 1)\left[\frac{q^2 - 1}{3} + (t + 1 + t)s + t + s + 1\right]$$

$$= (s - s^2)(2q + 2p - 2) - \frac{2}{3}s(s - 1)(s - 2) < 0.$$

Lemma 2.7 Let $G_0$ be a bicyclic graph with the vertex set $V(C_p) \cup V(C_q) \cup V(P_{s+1})$, in which $V(C_p) \cap V(P_{s+1}) = \{v\}$ and $V(C_q) \cap V(P_{s+1}) = \{w\}$. For $wa \in E(P_{s+1})$ and $u \in V(C_q)$, let $G_1 = (G - \{aw\}) \cup \{ua\}$. For convenience, let $r(u, w) = t$. Then $D_R(G_0) > D_R(G_1)$.

Proof. By Lemmas 2.2 and 2.3, we get

$$D_R(G_0) = D_R(H_0) + 2(p + s - 1)r(a|H_0) + 2(q + 1)r(a|H) + (p + s - 2)S'(a|H_0) + qS'(a|H),$$

$$D_R(G_1) = D_R(H_1) + 2(p + s - 1)r(w|H_1) + 2(q + 1)r(w|H) + (p + s - 2)S'(w|H_1) + qS'(w|H),$$

$$D_R(G_0) - D_R(G_1) = 2(p + s - 1)r(a|H_0) + (p + s - 2)S'(a|H_0)$$

$$- 2(p + s - 1)r(w|H_1) - (p + s - 2)S'(w|H_1)$$

$$= 2(p + s - 1)(1 + \frac{q^2 - 1}{6} + q) + (p + s - 2)\left[\frac{q^2 - 1}{3} + 3 + 2q\right]$$

$$- 2(p + s - 1)\left[\frac{q^2 - 1}{6} + t + 1\right] - (p + s - 2)\left[\frac{q^2 - 1}{3} + 2t + 1\right]$$

$$= 2(p + s - 1)(q - t) + (p + s - 2)(2q - 2t + 2) > 0.$$
3 Main results

In this section, we will characterize $n$-vertex bicyclic graphs with exactly two cycles having minimum and maximum degree resistance distances.

**Theorem 3.1** Let $G \in B_{n}^{p,q}$ and $G \neq S_{n}^{p,q}$. Then $D_{R}(G) > D_{R}(S_{n}^{p,q})$.

**Proof.** Suppose that a bicyclic graph $G_{0}$ has minimal degree resistance distance among graphs in $B_{n}^{p,q}$. For $G_{0}$, we prove the following claims.

**Claim 1.** In Fig. 1, $T_{v_{i}}$, $T_{u_{j}}$ and $T_{w_{k}}$ are all stars with their centers at $v_{i}$, $u_{j}$ and $w_{k}$ for each $i$, $j$ and $k$.

Without loss of generality, suppose that tree $T_{v_{i}}$ is not a star. Let $G_{1}$ be constructed from $G_{0}$ by deleting all the edges of $T_{v_{i}}$ and connecting all the isolated vertices to $v_{i}$; that is, $T_{v_{i}}$ is a star in $G_{1}$ with its center at $v_{i}$ and denote it by $S_{v_{i}}$. By Lemma 2.4, $D_{R}(G_{0}) > D_{R}(G_{1})$, which contradicts the choice of $G_{0}$. Hence Claim 1 holds. ■

**Claim 2.** The length of $P$ is 0.

Suppose to the contrary that the length of $P$ is $k (k \geq 1)$. Assume that $v_{1} = w_{0}, u_{1} = w_{k}$. Let $e = w_{i}w_{i+1}$ be an edge of $P$. Let $G_{2}$ be the graph obtained from $G_{0}$ by first contracting $e$ and then attaching a pendent edge $w_{i}a$ to $w_{i}$. Assume that $G_{01}$ and $G_{02}$ are two components of $G_{0} - e$ and $G_{21}$ and $G_{22}$ are copies of $G_{01}$ and $G_{02}$ in $G_{2}$, respectively. See Fig. 5.

![Fig. 5. Graphs $G_{0}$ and $G_{2}$](image)

In the following, we prove $D_{R}(G_{2}) < D_{R}(G_{0})$.

**Proof.** By Lemma 2.3, we get

\[
D_{R}(G_{0}) = D_{R}(G_{01}) + D_{R}(G_{02} + w_{i}w_{i+1}) + 2(m_{02} + 1)r(w_{i}|G_{01}) + 2m_{02}r(w_{i}|G_{02} + w_{i}w_{i+1})
+ n_{02}S'(w_{i}|G_{01}) + (n_{01} - 1)S'(w_{i}|G_{02} + w_{i}w_{i+1}),
\]

\[
D_{R}(G_{2}) = D_{R}(G_{21}) + D_{R}(G_{22} + w_{i}a) + 2(m_{22} + 1)r(w_{i}|G_{21}) + 2m_{22}r(w_{i}|G_{22} + w_{i}a)
+ n_{22}S'(w_{i}|G_{21}) + (n_{21} - 1)S'(w_{i}|G_{22} + w_{i}a),
\]


Claim 3. In Fig. 1, if \( p + q \leq n \), then only \( T_{v_1}(T_{v_1} = T_{u_1}) \) is nontrivial.

Without loss of generality, suppose to the contrary that tree \( T_{u_i}(i \neq 1) \) is nontrivial. By Lemma 2.5, \( D_R(G_0) > D_R(G_1) \), which contradicts the choice of \( G_0 \). Hence Claim 3 holds.

Claims 1-3 yield Theorem 3.1.

![Graphs G0 and G2](image)

**Theorem 3.2.** Let \( G \in \mathscr{B}^{p,q}_n \) and \( G \neq P^{p,q}_n \). Then \( D_R(G) < D_R(P^{p,q}_n) \).

**Proof.** Suppose that a bicyclic graph \( G_0 \) has maximal degree resistance distance among graphs in \( \mathscr{B}^{p,q}_n \). For \( G_0 \), we prove the following Claims.

**Claim 1.** In Fig. 1, \( T_{v_i}, T_{u_j} \) and \( T_{w_k} \) are all paths with their end vertices \( v_i, u_j \) and \( w_k \) for each \( i, j \) and \( k \).

Without loss of generality, suppose that tree \( T_{v_i} \) is not a path. Let \( G_1 \) be the graph constructed from \( G_0 \) by deleting all the edges of \( T_{v_i} \) and connecting \( v_i \) and all the isolated vertices into a path; that is, \( T_{v_i} \) is a path with end vertex \( v_i \) in \( G_1 \) and denote it by \( P_{v_i} \). By Lemma 2.6, \( D_R(G_1) > D_R(G_0) \), which contradicts the choice of \( G_0 \). Hence Claim 1 holds.

**Claim 2.** Assume that \( T_{w_0} = T_{v_1} \) and \( T_{w_{m}} = T_{u_1} \), then \( T_{w_i} \) is trivial \( (0 \leq i \leq m) \).

If not, without loss of generality, suppose that there is nontrivial \( T_{w_j} \). By Claim 1, we know that \( T_{w_j} \) is a path with \( w_j \) as its end vertex and assume that \( u \) is the other end vertex. Let \( G_2 = G_0 - w_jw_{j+1} + uw_{j+1} \) (if \( j = m \), \( G_2 = G_0 - w_{j-1}w_j + uw_{j-1} \)). Assume that \( G_{01} \) and \( G_{02} \) are two components of \( G_0 - w_jw_{j+1} \) and \( G_{21} \) and \( G_{22} \) are two components of \( G_2 - uw_{j+1} \). See Fig. 4.

In the following, we prove \( D_R(G_2) > D_R(G_0) \).
Proof. By Lemma 2.3, we get

\[ D_R(G_0) = D_R(G_{01} + w_jw_{j+1}) + D_R(G_{02}) + 2m_{02}^r(w_{j+1}|G_{01} + w_jw_{j+1}) + 2(m_{01} + 1)r(w_{j+1}|G_{02}) + (n_{02} - 1)S'(w_{j+1}|G_{01} + w_jw_{j+1}) + n_{01}S'(w_{j+1}|G_{02}), \]

\[ D_R(G_2) = D_R(G_{21} + uw_{j+1}) + D_R(G_{22}) + 2m_{22}^r(w_{j+1}|G_{21} + uw_{j+1}) + 2(m_{21} + 1)r(w_{j+1}|G_{22}) + (n_{22} - 1)S'(w_{j+1}|G_{21} + uw_{j+1}) + n_{21}S'(w_{j+1}|G_{22}), \]

\[ D_R(G_{01} + w_jw_{j+1}) = D_R(G_0) + D_R(w_jw_{j+1}) + 2r(w_j|G_0) + 2m_{01}r(w_j|w_jw_{j+1}) + S'(w_j|G_0) + (n_{01} - 1)S'(w_jw_{j+1}), \]

\[ D_R(G_{21} + uw_{j+1}) = D_R(G_{21}) + D_R(uw_{j+1}) + 2r(u|G_{21}) + 2m_{21}r(u|uw_{j+1}) + S'(u|G_{21}) + (n_{21} - 1)S'(u|uw_{j+1}), \]

\[ D_R(G_0) - D_R(G_2) = 2r(w_j|G_0) + S'(w_j|G_0) + 2m_{02}^r(w_{j+1}|G_{01} + w_jw_{j+1}) + (n_{02} - 1)S'(w_{j+1}|G_{01} + w_jw_{j+1}) - 2r(u|G_{21}) - S'(u|G_{21}) - 2m_{22}^r(w_{j+1}|G_{21} + uw_{j+1}) - (n_{22} - 1)S'(w_{j+1}|G_{21} + uw_{j+1}) < 2m_{02}^r\left[r(w_{j+1}|G_0) + 1\right] + (n_{02} - 1)\left[S'(w_{j+1}|G_0) + 3\right] - 2m_{22}^r\left[r(w_{j+1}|G_{21}) + 1\right] - (n_{22} - 1)\left[S'(w_{j+1}|G_{21}) + 2\right] = (n - 1)\left[S'(w_{j+1}|G_0) - S'(w_{j+1}|G_{21}) + 1\right] < 0. \]

We obtain \( D_R(G_2) > D_R(G_0) \). This contradicts the hypothesis. Hence Claim 2 holds.

Claim 3. In Fig. 1, if \( p + q \leq n \), then \( T_{v_i} \text{ and } T_{u_j} \) are trivial for each \( i \) and \( j \).

Without loss of generality, suppose to the contrary that \( T_{v_i}(i \neq 1) \) is nontrivial. By Lemma 2.7, \( D_R(G_0) < D_R(G_1) \), which contradicts the choice of \( G_0 \). Hence Claim 3 holds.

Claims 1-3 yield Theorem 3.2.

We now compute \( D_R(S_n^{p,q}) \) and \( D_R(P_n^{p,q}) \).

For \( S_n^{p,q} \), see Fig. 2(a).

\[ D_R(S_n^{p,q}) = D_R(C_q) + D_R(H) + 2(p + s)r(v_1|C_q) + 2qr(v_1|H) + (p + s - 1)S'(v_1|C_q) + (q - 1)S'(v_1|H), \]

\[ D_R(H) = D_R(C_p) + D_R(S) + 2sr(v_1|C_p) + 2pr(v_1|S) + sS'(v_1|C_p) + (p - 1)S'(v_1|S), \]

\[ r(v_1|H) = r(v_1|C_p) + r(v_1|S), \]

\[ S'(v_1|H) = S'(v_1|C_p) + S'(v_1|S), \]

\[ D_R(S_n^{p,q}) = \frac{q^3 - q}{3} + \frac{p^3 - p}{3} + 3(s + 1)^2 - 7(s + 1) + 4 + 2s^2 - \frac{1}{6} + 2ps + (p - 1)s + 2(p + s)\frac{q^2 - 1}{6} + 2qs + (p + s - 1)\frac{p^2 - 1}{3} + (q - 1)s + (q - 1), \]

\[ = \frac{1}{3}(q^3 + 3q + 3q^2 + 2q + 2q^2p + 2sp^2 + 2spq + 9sq + 9sq - q^2 - p^2 - 3q - 3p + 9s^2 - 13s + 2). \]
And $s = n + 1 - p - q$, so we get

$$D_R(S_n^{p,q}) = \frac{1}{3}[-p^3 - q^3 + (2n + 1)(p^2 + q^2) + (1 - 9n)(p + q) + 9n^2 + 5n - 2]. \quad (1)$$

For $P_n^{p,q}$, see Fig. 2(b).

$$D_R(P_n^{p,q}) = D_R(C_p) + D_R(H) + 2(q + m)r(v_1|C_p) + 2pr(v_1|H) + (q + m - 1)S'(v_1|C_p) + (p - 1)S'(v_1|H),$$

$$D_R(H) = D_R(C_q) + D_R(P_{m+1}) + 2qr(u_1|P_{m+1}) + 2mr(u_1|C_q) + (q - 1)S'(u_1|P_{m+1}) + sS'(u_1|C_q),$$

$$r(v_1|H) = r(v_1|C_p) + r(v_1|P_{m+1}),$$

$$S'(v_1|H) = S'(v_1|C_q) + S'(v_1|P_{m+1}),$$

$$D_R(P_n^{p,q}) = \frac{q^3 - q}{3} + \frac{p^3}{3} + \frac{p^3 - p}{3} + \frac{2}{3}m^3 + m^2 + \frac{1}{3}m + 2q\frac{(1 + m)m}{2} + 2m\frac{q^2 - 1}{6} + (q - 1)m^2 + \frac{q^2 - 1}{3}m$$

$$+ 2(q + m)\frac{p^2 - 1}{6} + 2p\frac{(1 + m)m}{2} + m(q - 1) + \frac{q^2 - 1}{6}$$

$$+ (q + m - 1)\frac{p^2 - 1}{3} + (p - 1)[(1 + m - 1)(m - 1) + 3m + 2(m - 1) + \frac{q^2 - 1}{3}]$$

$$= \frac{1}{3}[p^3 + q^3 + (2q + 2m - 1)p^2 + (2p + 2m - 1)q^2 + (6m^2 - 3m - 3)(p + q)$$

$$+ 12mpq + 2m^3 - 3m^2 - 3m + 2].$$

And $m = n + 1 - p - q$, so we get

$$D_R(P_n^{p,q}) = \frac{1}{3}[3p^3 + 3q^3 + (4n + 5)(p^2 + q^2) + (3n + 3)(p + q) + 2n^3 + 3n^2 - 3n - 2]. \quad (2)$$



### 4 Bicyclic graphs with extremal degree resistance distance

By Theorems 3.1 and 3.2, $n$-vertex bicyclic graphs in $P_n^{p,q}$ with minimal and maximal degree resistance distance must belong to the classes of $S_n^{p,q}$ and $P_n^{p,q}$, respectively. In what follows, we will determine those which has the extremal degree resistance distance among graphs in $P_n^{p,q}$.

**Theorem 4.1** Among all $n$-vertex bicyclic graphs, the graph $S_n^{3,3}$ has the minimum degree resistance distance.

$$D_R(S_n^{3,3}) = 3n^2 - \frac{13}{3}n - \frac{32}{3}, \quad n \geq 5.$$ 

**Proof.** Let $u_1, u_2, w$ be three successive vertices lying on the cycle $C_p$ of the bicyclic graph $G_1$. The other cycle $C_q$ only has one common vertex $w$ with $C_p$. And $wv_1, wv_2, \ldots, wv_s$ are pendent edges incident with $w$. Let the graph $G_2$ is obtained by deleting the edges $u_1u_2$ and adding the edge $wv_2$. Then $D_R(G_2) < D_R(G_1)$.

$$D_R(G_1) = D_R(C_p) + D_R(H_1) + 2pr(w|H_1) + 2(s + q)r(w|C_p) + (p - 1)S'(w|H_1) + (q + s - 1)S'(w|C_p),$$

$$D_R(G_2) = D_R(C_p) + D_R(H_2) + 2pr(w|H_2) + 2(s + q)r(w|C_p) + (p - 1)S'(w|H_2) + (q + s - 1)S'(w|C_p) - pr(w|H_1) - pr(w|H_2).$$

Since $S'(w|H_1) < S'(w|H_2)$, we have $D_R(G_2) < D_R(G_1)$. Therefore, $S_n^{3,3}$ has the minimum degree resistance distance among all $n$-vertex bicyclic graphs.
\[D_R(G_2) = D_R(C'_p) + D_R(H_1) + 2pr(w|H_1) + 2(s + q)r(w|C'_p) + (p - 1)S'(w|H_1) + (q + s - 1)S'(w|C'_p),\]
\[D_R(G_2) - D_R(G_1) = D_R(C'_p) + 2(s + q)r(w|C'_p) + (q + s - 1)S'(w|C'_p) - D_R(C_p) - 2(s + q)r(w|C_p) - (q + s - 1)S'(w|C_p) = D_R(C_{p-1}) + D_R(u_1w) + 2(p - 1)r(w|u_1w) + 2r(w|C_{p-1}) + (p - 1)S'(w|u_1w) + S'(w|C_{p-1}) - D_R(C_p) + 2(s + q)r(w|C'_p) - 2(s + q)r(w|C_p) + (s + q - 1)S'(w|C'_p) - (s + q - 1)S'(w|C_p) = (p - 1)^3 - (p - 1) + 2 + 2p - 2 + 2\left(\frac{(p - 1)^2 - 1}{6}\right) + p - 1 + \frac{(p - 1)^2 - 1}{3} - \frac{p^3 - p}{3} + 2(s + q)\left[\frac{(p - 1)^2 - 1}{6} + 1 - \frac{p^2 - 1}{6}\right] + (s + q - 1)\left[\frac{(p - 1)^2 - 1}{3} + 1 - \frac{p^2 - 1}{3}\right] = -\frac{1}{3}p^2 - \frac{8}{3}p - 1 - \frac{4}{3} + \frac{2}{3}p + \frac{s}{3}(11 - 4p) + \frac{q}{3}(11 - 4p) = -\frac{1}{3}p^2 - 2p - \frac{7}{3} + \frac{s}{3}(11 - 4p) + \frac{q}{3}(11 - 4p) < 0.
\]
Combining the above discussions, Eq. (1) and \(p = q = 3\), we can get
\[D_R(S^{3,3}_n) = 3n^2 - \frac{13}{3}n - \frac{32}{3}, \quad n \geq 5.
\]

**Theorem 4.2** Among all \(n\)-vertex bicyclic graphs, the graph \(P^{3,3}_n\) has the maximal degree resistance distance.

\[D_R(P^{3,3}_n) = \frac{2}{3}n^3 + n^2 - 19n + \frac{88}{3}, \quad n \geq 5.
\]

**Proof.** Let \(u_1, w, u_2\) be three successive vertices lying on the cycle \(C_p\) of the bicyclic graph \(G_3\). The cycles \(C_p\) and \(C_q\) are linked with two end vertexes \(v\) and \(w\) of \(P_{s+1}\). Let the graph \(G_4\) is obtained by deleting the edge \(wu_2\) and adding the edge \(u_1u_2\). Then \(D_R(G_4) > D_R(G_3)\).

\[D_R(G_3) = D_R(C_p) + D_R(H) + 2(q + s)r(w|C_p) + 2pr(w|H) + (q + s - 1)S'(w|C_p) + (p - 1)S'(w|H),\]
\[D_R(G_4) = D_R(C'_p) + D_R(H) + 2(q + s)r(w|C'_p) + 2pr(w|H) + (q + s - 1)S'(w|C'_p) + (p - 1)S'(w|H),\]
\[D_R(C'_p) = D_R(C_{p-1}) + D_R(u_1C'_p) + 2(p - 1)r(u_1|C'_p) + 2(p - 1)r(u_1|wu_1) + S'(u_1|C'_p) + (p - 2)S'(u_1|wu_1) = \frac{(p - 1)^3 - (p - 1)}{3} + 2 + 2\left(\frac{(p - 1)^2 - 1}{6}\right) + 2(p - 1) + \frac{(p - 1)^2 - 1}{3} + (p - 2) = \frac{p^3}{3} + \frac{p^2}{3} + \frac{8p}{3} + \frac{7}{3},\]
\[r(w|C'_p) = 1 + \frac{(p - 1)^2 - 1}{3} + (p - 1) = \frac{p^2}{6} + \frac{2p}{3},\]
\[S'(w|C'_p) = 3 + \frac{(p - 1)^2 - 1}{3} + 2(p - 1) = \frac{p^2 - 2p}{3} + 2p + 1,
\]

\[12\]
\[ D_R(G_4) - D_R(G_3) = D_R(C_p') + 2(q + s)r(w|C_p') + (q + s - 1)S'(w|C_p') \]
\[ - D_R(C_p) - 2(q + s)r(w|C_p) - (q + s - 1)S'(w|C_p) \]
\[ = \frac{p^3}{3} + \frac{p^2}{3} + \frac{8p}{3} - \frac{7}{3} + 2(q + s)(\frac{p^2}{6} + \frac{2p}{3}) \]
\[ + (q + s - 1)(\frac{p^2 - 2p}{3} + p + 1) - \frac{p^3 - p}{3} \]
\[ - 2(q + s)\frac{p^2 - 1}{6} - (q + s - 1)\frac{p^2 - 1}{3} \]
\[ = \frac{p^3}{3} + 3p + 2(q + s)(\frac{2p}{3} + \frac{1}{6}) + (q + s - 1)(\frac{4p}{3} + \frac{4}{3}) - \frac{7}{3} > 0. \]

Combining above discussion, and by Eq. (2) and \( p = q = 3 \). So we can get
\[ D_R(P_n^{3,3}) = \frac{2}{3}n^3 + n^2 - 19n + \frac{88}{3}, \quad n \geq 5. \]

5 Concluding remarks

In this paper, we completely characterized the bicyclic graphs with exactly two cycles having extremal degree resistance distances. Although our research is restricted to a small family of graphs, it motivates one to further extend this to the larger classes of graphs. For example, can one determine the maximum or minimum degree resistance distance for tricyclic graphs or any general graphs? We plan to investigate this question for larger classes of graphs for future research in this area.

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