AVERAGE REGULARITY OF THE SOLUTION TO AN EQUATION WITH THE RELATIVISTIC-FREE TRANSPORT OPERATOR

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ABSTRACT. Let \( u = u(t, x, p) \) satisfy the transport equation \( \frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial x} = f \), where \( f = f(t, x, p) \) belongs to \( L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \) for \( 1 < p < \infty \) and \( \frac{\partial}{\partial t} + \frac{p}{p_0} \frac{\partial}{\partial x} \) is the relativistic-free transport operator. We show the regularity of \( \int_{\mathbb{R}^3} u(t, x, p)dp \) using the same method as given by Golse, Lions, Perthame and Senti. This average regularity is considered in terms of fractional Sobolev spaces and it is very useful for the study of the existence of the solution to the Cauchy problem on the relativistic Boltzmann equation.

1. INTRODUCTION

We are concerned with the average regularity of the solution to an equation with the relativistic-free transport operator from the relativistic Boltzmann equation. Let us begin with the relativistic Boltzmann equation in the following form:

\[
\frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial x} = Q(u, u),
\]

where \( u = u(t, x, p) \) is a distribution function of a one-particle relativistic gas with the time \( t \in (0, \infty) \), the position \( x \in \mathbb{R}^3 \), and the momentum \( p \in \mathbb{R}^3 \); \( p_0 = (1 + |p|^2)^{1/2} \) denotes the energy of a dimensionless relativistic gas particle with the momentum \( p \); \( \frac{\partial}{\partial t} + \frac{p}{p_0} \frac{\partial}{\partial x} \) is called the relativistic-free transport operator; \( Q(u, u) \) is the relativistic Boltzmann collision operator which can be written as the difference between the gain and loss terms respectively given by Dudyński and Ekiel-Jeżewska \[13\] in the following forms:

\[
Q^+(u, u) = \int_{\mathbb{R}^3 \times S^2} \frac{g s^{1/2}}{p_0 p s_0} \sigma(g, \theta) u(t, x, p') u(t, x, p_s) d\omega dp_s,
\]

\[
Q^-(u, u) = \int_{\mathbb{R}^3 \times S^2} \frac{g s^{1/2}}{p_0 p s_0} \sigma(g, \theta) u(t, x, p) u(t, x, p_s) d\omega dp_s.
\]

It is worth mentioning that the gain and loss terms of the relativistic Boltzmann equation can be expressed in other various forms (see [20]). The other different parts in eqs. (1.2) and (1.3) are explained as follows.

\( p \) and \( p_s \) are dimensionless momenta of two relativistic particles immediately before collision while \( p' \) and \( p'_s \) are dimensionless momenta after collision; \( p_{s0} = (1 + |p_s|^2)^{1/2} \) denotes the dimensionless energy of the colliding relativistic gas particle with the momentum \( p_s \) before collision, and as used below in the same way, \( p'_0 = (1 + |p'|^2)^{1/2} \) and \( p'_{s0} = (1 + |p'_s|^2)^{1/2} \) represent the dimensionless energy of the two relativistic particles after collision. \( s = |p_{s0} + p_{0}|^2 - |p_s + p|^2 \) and \( s^{1/2} \) is the total energy in the center-of-mass frame [13]; \( g = \sqrt{|p_s - p|^2 - |p_{s0} - p_0|^2}/2 \) and \( 2g \) is in fact the value of the relative momentum in the center-of-mass frame [13]; it can be seen that \( s = 4 + 4g^2 \). \( \sigma(g, \theta) \) is the differential scattering cross section of the variable \( g \) and the scattering angle \( \theta \). \( \mathbb{R}^3 \) is a three-dimensional Euclidean space and \( S^2 \) a unit sphere surface with an infinitesimal element \( d\omega = \sin \theta d\theta d\varphi \) for the scattering angle \( \theta \in [0, \pi] \) and the other solid

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angle $\varphi \in [0, 2\pi]$ in the center-of-momentum system, and the scattering angle $\theta$ is defined by
\[
\cos \theta = 1 - \frac{2[(p_0 - p_{0'})\cdot (p_0 - p_{0'}) - (p - p_{0})\cdot (p - p_{0})]}{(4 - s)}.
\]

There is a long history of the study of the relativistic Boltzmann equation as one of the most important in the relativistic kinetic theory. The study of the relativistic kinetic theory began in 1911 when Jüttner [30] derived an equilibrium distribution function of relativistic gases. Lichnerowicz and Marrot [24] were the first to derive the full relativistic Boltzmann equation including the collision operator in 1940. The research of this equation can be roughly classified into four aspects: 1) the derivation of this equation; 2) its relativistic hydrodynamic limit. 3) its Chapman-Enskog approximation and hydrodynamic modes; 4) the existence and uniqueness of the solution to the Cauchy problem on it; For both of 1) and 2), we can see the recent references from Dolan [11], Debbasch & Leeuwen [7] [8], Tsumura & Kumihira [37] and Denicol et al. [9]. For 3), in the early 60’s, many researchers, such as Israel [23], applied the Chapman-Enskog expansion into studying the approximative solution to the relativistic Boltzmann equation. The progress of the last research fields are recently great. In 1967, Bichteler [1] first proved that the relativistic Boltzmann equation admits a unique local solution under the assumptions that the differential scattering cross-section is bounded and that the initial distribution function decays exponentially with energy. In 1988, Dudyński and Ekiel-Jeżewska [13] proved that the Cauchy problem on the linearized relativistic Boltzmann equation has a unique solution in $L^2$ space. Four years later, Dudyński and Ekiel-Jeżewska [14] showed that there exists a DiPerna-Lions renormalized solution [10] to the Cauchy problem on the relativistic Boltzmann equation with large initial data in the case of the relativistic soft interactions. Glassey and Strauss [16] proved in 1993 that a unique global smooth solution to this problem exponentially converges to a relativistic Maxwellian as the time goes to infinity if all initial data are periodic in the space variable and near equilibrium. Then in 1995, Glassey and Strauss extended the above result to the whole space case [17] and found that the solution has the property of polynomial convergence with respect to the time. In 1996, Andréasson [11] showed the regularity of the gain term and the strong $L^1$ convergence to equilibrium for the relativistic Boltzmann equation. Afterward, Jiang gave the global existence of solution to the relativistic Boltzmann equation with hard interactions in the whole space for initial data with finite mass, energy and inertia [26], or in a periodic box for initial data with finite mass and energy [27] [28]. In 2004, Andréasson, Calogero and Illner [2] obtained the property that the solution to the gain-term-only relativistic Boltzmann equation blows up in finite time. In 2006, Glassey [18] showed a unique global solution to the relativistic Boltzmann equation with initial data near vacuum state for certain differential scattering cross section. In 2008, Jiang [29] obtained the global existence of solution to the relativistic Boltzmann equation with hard interactions in the whole space for initial data with finite mass and energy. Recently, Strain gave a Newtonian limit of global solutions to the Cauchy problem on the relativistic Boltzmann equation near vacuum [32], showed the asymptotic stability of the relativistic Boltzmann equation for the soft potentials [33] and made a survey of various equivalent expressions of the relativistic Boltzmann equation [34]. There are also many authors who are contributed to the study of relativistic Boltzmann equation, e.g., Escobedo et al. [15], Ha et al. [21], Hsiao & Yu [22], Swart [35]. Many other relevant results can be found in the references mentioned above.

In this paper, we show the regularity of the momentum average of the solution to an equation with the relativistic-free transport operator by use of the same method as given by Golse, Lions, Perthame and Sentis [19]. This regularity is very useful for our further study of the existence and uniqueness of the solution to the Cauchy problem on the relativistic Boltzmann equation. It is presented by the following theorem:

**Theorem 1.1.** Assume that $u$ and $f$ satisfy the following equation:

\[
\frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial x} = f
\]  

(1.4)
where \( \text{supp } u \subset [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R \), \( T \in (0, \infty), \varepsilon_0 \in (0, \frac{T}{2}) \) and \( B_R \) is a ball centered on the origin and with a radius of \( R \). If \( u \) and \( f \) both belong to \( L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \) for \( p \in (1, +\infty) \), we can conclude that \( \tilde{u} = \int_{\mathbb{R}^3} u(t, x, p) dp \), where \( \tilde{u} \in W^{s,p}((0, T) \times \mathbb{R}^3) \) for every \( s \) satisfying \( 0 < s < \inf(1, \frac{1}{4}) \), and what’s more, there exists a constant \( C \) which does not depend on \( u \) and \( f \), such that

\[
\|\tilde{u}\|_{W^{s,p}} \leq C \|u\|_{L^p}^{1-s} \|f\|_{L^p}^s \tag{1.5}
\]

where

\[
\|\tilde{u}\|_{W^{s,p}} = \left( \int_0^T \int_{\mathbb{R}^3} \int_0^T \int_{\mathbb{R}^3} \frac{|u(t_1, x_1) - u(t_2, x_2)|^2}{[(t_1 - t_2)^2 + (x_1 - x_2)^2]^{\frac{3}{2}}} \, dx_1 \, dt_1 \, dx_2 \, dt_2 \right)^{1/p}. \tag{1.6}
\]

The norm of \( \tilde{u} \), given in (1.6), is defined in a fractional Sobolev space which can be seen in the book of Triebel [38]. In order to prove Theorem 1.1, we first have to consider the special case when \( p = 2 \), that is, we investigate the regularity of \( \int_{\mathbb{R}^3} u(t, x, p) dp \) when \( u \) and \( f \) belong to the \( L^2 \) space. This regularity is in fact presented by the following lemma:

**Lemma 1.1** ([25]). Assume that \( u \) and \( f \) satisfy (1.4) and that \( \text{supp } u \subset [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R \), where \( T \in (0, \infty), \varepsilon_0 \in (0, \frac{T}{2}) \) and \( B_R \) is a ball centered on the origin and with a radius of \( R \). If \( u \) and \( f \) belong to \( L^2((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \), we can conclude that \( \tilde{u} = \int_{\mathbb{R}^3} u(t, x, p) dp \), and what’s more, there exists a constant \( C \) which does not depend on \( u \) and \( f \), such that

\[
\|\tilde{u}\|_{H^{\frac{1}{2}}} \leq C \|u\|_{L^2} \|f\|_{L^2} \tag{1.7}
\]

where

\[
\|\tilde{u}\|_{H^{\frac{1}{2}}}((0, T) \times \mathbb{R}^3) = \left( \int_0^T \int_{\mathbb{R}^3} \int_0^T \int_{\mathbb{R}^3} \frac{|u(t_1, x_1) - u(t_2, x_2)|^2}{[(t_1 - t_2)^2 + (x_1 - x_2)^2]^{\frac{3}{2}}} \, dx_1 \, dt_1 \, dx_2 \, dt_2 \right)^{1/2}. \tag{1.8}
\]

Lemma 1.1 was given by Jiang [25] (or see Appendix A) in 1997. His proof is similar to that used by Golse, Lions, Perthame and Sents [19], that is, it is mainly based on the analysis of the integral in the norm \( \|\tilde{u}\|_{H^{\frac{1}{2}}(\mathbb{R}^3 \times \mathbb{R}^3)} \) by use of an estimate of a special integral in a subset in the ball \( B_R = \{ p : |p| < R, p \in \mathbb{R}^3 \} \).

It is worth mentioning that Lemma 1.1 can be regarded as a special case of the well established result given by Rein [31] in 2004. Rein’s result includes the fact that if \( u \) has compact supports in \( L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R) \) and satisfies

\[
\frac{\partial u}{\partial t} + p \frac{\partial u}{\partial x} = g_0 + \text{div}_p g_1 \tag{1.9}
\]

for \( g_0 \) and \( g_1 \) in \( L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R) \), then the momentum average \( \tilde{u} \) belongs to \( H^{1/4}(\mathbb{R} \times \mathbb{R}^3) \). When \( g_1 \) is equal to zero, (1.9) is reduced to be (1.4). Jiang’s result in Lemma 1.1 shows that in this case \( \tilde{u} \) belongs to \( H^{1/2}(\mathbb{R} \times \mathbb{R}^3) \). Since \( u \in L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R) \), it can be further found that \( \tilde{u} \) belongs to \( L^2(\mathbb{R} \times \mathbb{R}^3) \) as well. Hence, by use of Proposition 1.32 in [3], it can be known that \( \tilde{u} \) belongs to \( H^{1/4}(\mathbb{R} \times \mathbb{R}^3) \) in Lemma 1.1 too.

Theorem 1.1 is a generalization of Lemma 1.1 into the \( L^p \) space for \( p \in (1, +\infty) \) and it can be proved by the analysis of the operator \( \tilde{g} : (u, f) \rightarrow \tilde{u} \), with the help of the complex interpolation method (see [5 [36]). The detailed proof of Theorem 1.1 will be shown in the next section.

2. **Proof for Theorem 1.1**

In order to prove Theorem 1.1 we first observe some properties of the solutions to Eqn. (1.4). Notice that \( u \) and \( f \) satisfy Eqn. (1.4), and that \( \text{supp } u \subset [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R \). It follows that \( u \) is in fact a unique solution to the Cauchy problem on Eqn. (1.4) with its initial datum zero.
when $f \in L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $p \in (1, +\infty)$. Denote $u^s(t, x, p) \triangleq u(t, x + \frac{p}{p_0}t, p)$ and $f^s(t, x, p) \triangleq f(t, x + \frac{p}{p_0}t, p)$. Then Eqn. (1.4) can be rewritten as

$$\frac{\partial u^s}{\partial t} + \frac{p}{p_0} \frac{\partial u^s}{\partial x} = f^s. \quad (2.1)$$

(2.1) is equivalent to the following equation:

$$\frac{du^s}{dt} = f^s. \quad (2.2)$$

Hence

$$u^s(t, x, p) = \int_0^t f^s(s, x, p)ds, \quad (2.3)$$

that is,

$$u(t, x, p) = \int_0^t f(s, x + \frac{p}{p_0}(s-t), p)ds. \quad (2.4)$$

Let $h(t, x, p) \triangleq u(t, x, p) + f(t, x, p)$. Then Eqn. (1.4) can be rechanged as

$$u + \frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial x} = h, \quad (2.5)$$

thus $u(t, x, p)$ can be expressed as

$$u(t, x, p) = \int_0^t e^{s-t}h(s, x + \frac{p}{p_0}(s-t), p)ds, \quad (2.6)$$

and it is still a unique solution to the Cauchy problem on Eqn. (2.5) with its initial datum zero when $f \in L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $p \in (1, +\infty)$. Notice that $h(t, x, p)$ has compact support since $\text{supp}u \subseteq [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R$ and that $h$ belong to $L^q((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ if $u$ and $f$ both belong to $L^q((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $q \in [1, +\infty]$. Then we can define a linear operator $\mathfrak{F}$ from $L^q_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $L^q((0, T) \times \mathbb{R}^3)$ as follows:

$$\mathfrak{F}(h) = \tilde{u}, \quad (2.7)$$

for every $q \in [1, +\infty]$, where $L^q_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ is defined by

$$L^q_R = \{ h \in L^q((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) : h \text{ has compact support in } (0, T) \times \mathbb{R}^3 \times B_R \} \quad (2.8)$$

and $\tilde{u}(t, x) = \int_{\mathbb{R}^3} u(t, x, p)dp$ for any $u(t, x, p)$ in (2.6).

We below prove that $\mathfrak{F}$ is bounded from $L^q_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $L^q((0, T) \times \mathbb{R}^3)$ for every $q \in [1, +\infty]$. By Hölder’s inequality, we obtain the following estimate:

$$\|\tilde{u}\|_{L^q((0, T) \times \mathbb{R}^3)} = \left( \int_0^T \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} u(t, x, p)dp \right|^q dx dt \right)^{\frac{1}{q}} \leq \left( \int_0^T \int_{\mathbb{R}^3} \left( \int_{B_R} |u(t, x, p)|^q dp \right)^{\frac{1}{q}} dx dt \right)^{\frac{1}{q}} \leq \left( \frac{4}{3} \pi R^3 \right)^{\frac{1}{q}} \left( \int_0^T \int_{B_R} |u(t, x, p)|^q dp dx dt \right)^{\frac{1}{q}}$$

$$= \left( \frac{4}{3} \pi R^3 \right)^{\frac{1}{q}} \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)|^q dp dx dt \right)^{\frac{1}{q}} \leq \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)|^q dp dx dt \left( \frac{4}{3} \pi R^3 \right)^{\frac{1}{q}} \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)|^q dp dx dt \right)^{\frac{1}{q}}$$

$$= \left( \frac{4}{3} \pi R^3 \right)^{\frac{1}{q}} \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)|^q dp dx dt \right)^{\frac{1}{q}} \quad (2.9)$$
for \( q \in (1, +\infty) \). Because of (2.6), using Hölder’s inequality again, we have

\[
\left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)|^q dp dx dt \right)^{\frac{1}{q}}
\]

\[
= \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} e^{s-t} h(s, x + \frac{p}{p_0}(s-t), p) ds dp dx dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \left( \int_0^t e^{\frac{s-t}{q}} ds \right)^{1-1/q} \left( \int_0^t |h(s, x + \frac{p}{p_0}(s-t), p)|^q ds \right)^{1/q} dp dx dt \right)^{\frac{1}{q}}
\]

\[
\leq C_1 \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t |h(s, x + \frac{p}{p_0}(s-t), p)|^q ds dp dx dt \right)^{\frac{1}{q}},
\]  

(2.10)

where \( C_1 = \left( \frac{q-1}{q} \right)^{\frac{q-1}{q}} (1 - e^{-T}) \). Let \( y = x + \frac{p}{p_0}(s-t) \). By Fubini’s theorem, we can first calculate the last integral in (2.10) with respect to \( x \), thus, by transforming \( x \) into \( y \), giving

\[
\left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t |h(s, x + \frac{p}{p_0}(s-t), p)|^q ds dp dx dt \right)^{\frac{1}{q}}
\]

\[
= \left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t |h(s, y, p)|^q ds dp dy dt \right)^{\frac{1}{q}}.
\]  

(2.11)

It follows that

\[
\left( \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t |h(s, y, p)|^q ds dp dy dt \right)^{\frac{1}{q}}
\]

\[
\leq (T)^{1/q} \left( \int_{\mathbb{R}^3} \int_{B_R} \int_0^T |h(s, y, p)|^q ds dp dy \right)^{\frac{1}{q}}
\]

\[
= (T)^{1/q} \|h\|_{L^q}.
\]  

(2.12)

By (2.9), (2.10), (2.11) and (2.12), we conclude that

\[
\|\tilde{u}\|_{L^q} \leq C_2 \|h\|_{L^q},
\]  

(2.13)

where \( C_2 = \left( \frac{4\pi R^4}{q} \right)^{1-\frac{1}{q}} T^{1/q} \left( \frac{q-1}{q} \right)^{\frac{q-1}{q}} \). Similarly, we can consider two special cases: one is for \( p = 1 \) and another is for \( p = +\infty \). In the \( p = 1 \) case, for any \( h \in L^1_{L^1}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \), we have

\[
\|\tilde{u}\|_{L^1_{L^1}(0, T) \times \mathbb{R}^3}) = \int_0^T \int_{\mathbb{R}^3} |\tilde{u}(t, x)| dx dt
\]

\[
= \int_0^T \int_{\mathbb{R}^3} \int_{B_R} u(t, x, p) dp dx dt
\]

\[
\leq \int_0^T \int_{\mathbb{R}^3} \int_{B_R} |u(t, x, p)| dp dx dt
\]

\[
= \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t e^{s-t} h(s, x + \frac{p}{p_0}(s-t), p) ds dp dx dt
\]

\[
\leq \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t e^{s-t} h(s, x + \frac{p}{p_0}(s-t), p) ds dp dx dt
\]

\[
\leq \int_0^T \int_{\mathbb{R}^3} \int_{B_R} \int_0^t |h(s, x + \frac{p}{p_0}(s-t), p)| ds dp dx dt.
\]  

(2.14)
By Fubini’s theorem, transforming $x$ into $y = x + \frac{P}{p_0}(s - t)$, we can get

$$
\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |h(s, x + \frac{P}{p_0}(s - t), p)| dsdpdxdt
$$

$$
= \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |h(s, y, p)| dsdpdydt
$$

$$
\leq T \int_{\mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |h(s, y, p)| dsdpdy
$$

$$
\leq C_3\|h\|_{L^1},
$$

(2.15)

where $C_3 = T$. (2.14) and (2.15) yield

$$
\|\tilde{u}\|_{L^1} \leq C_3\|h\|_{L^1}.
$$

(2.16)

In the $p = +\infty$ case, for $h \in L^\infty_{\mathbb{R}}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, we can also get

$$
\|\tilde{u}\|_{L^\infty((0, T) \times \mathbb{R}^3)} = \sup_{(t, x) \in (0, T) \times \mathbb{R}^3} |\tilde{u}(t, x)|
$$

$$
= \sup_{(t, x) \in (0, T) \times \mathbb{R}^3} \left| \int_{\mathbb{B}_R} u(t, x, p)dp \right|
$$

$$
= \sup_{(t, x) \in (0, T) \times \mathbb{R}^3} \left| \int_{\mathbb{B}_R} \int_0^t e^{s-t} h(s, x + \frac{P}{p_0}(s - t), p) dsdp \right|
$$

$$
\leq \sup_{(t, x) \in (0, T) \times \mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |e^{s-t} h(s, x + \frac{P}{p_0}(s - t), p)| dsdp
$$

(2.17)

By Fubini’s theorem, transforming $x$ into $y = x + \frac{P}{p_0}(s - t)$, we have

$$
\sup_{(t, x) \in (0, T) \times \mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |h(s, x + \frac{P}{p_0}(s - t), p)| dsdp
$$

$$
= \sup_{(t, y) \in (0, T) \times \mathbb{R}^3} \int_{\mathbb{B}_R} \int_0^t |h(s, y, p)| dsdp
$$

$$
\leq \sup_{(t, y) \in (0, T) \times \mathbb{R}^3} \left( \frac{4}{3} \pi R^3 T \right) \sup_{(s, p) \in (0, t) \times \mathbb{B}_R} |h(s, y, p)|
$$

$$
\leq \left( \frac{4}{3} \pi R^3 T \right) \sup_{(s, y, p) \in (0, T) \times \mathbb{R}^3 \times \mathbb{B}_R} |h(s, y, p)|
$$

$$
\leq C_4\|h\|_{L^\infty},
$$

(2.18)

where $C_4 = \frac{4}{3} \pi R^3 T$. By (2.17) and (2.18), we get

$$
\|\tilde{u}\|_{L^\infty} \leq C_4\|h\|_{L^\infty}.
$$

(2.19)

Hence $\Phi$ is bounded from $L^q_{\mathbb{R}}((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ to $L^q((0, T) \times \mathbb{R}^3)$ for every $1 \leq q \leq \infty$. 

By using Lemma 1.1, we below show that $\tilde{\mathcal{F}}$ is bounded from $L^2_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $H^{1/2}_R((0, T) \times \mathbb{R}^3)$. Setting $\alpha = 1$ in (A.23) and using Plancherel’s identity, we have
\[
\|\tilde{u}\|^2_{H^{1/2}_R(\mathbb{R} \times \mathbb{R}^3)} = \int_{\mathbb{R} \times \mathbb{R}^3} |\mathbf{z}^2 + \tau^2|^{1/2} \int_{\mathbb{R}^3} \tilde{u}(\tau, \mathbf{z}, \mathbf{p}) d\mathbf{p} d\tau d\tau \\
\leq 2C_R(\int_{\mathbb{R} \times \mathbb{R}^3} |\tilde{u}|^2 d\mathbf{p} d\tau d\tau + \int_{\mathbb{R} \times \mathbb{R}^3} |\mathbf{z}|^2 \mathbf{p} d\mathbf{p} d\tau d\tau) \\
= 2C_R(\int_{\mathbb{R} \times \mathbb{R}^3} |u|^2 d\mathbf{p} d\tau d\tau + \int_{\mathbb{R} \times \mathbb{R}^3} |\partial_x u| d\mathbf{p} d\tau d\tau) \\
= 2C_R(\int_{\mathbb{R} \times \mathbb{R}^3} |u|^2 d\mathbf{p} d\tau d\tau + \int_{\mathbb{R} \times \mathbb{R}^3} |f|^2 d\mathbf{p} d\tau d\tau) \\
\leq 6C_R(\int_{\mathbb{R} \times \mathbb{R}^3} |u|^2 d\mathbf{p} d\tau d\tau + \int_{\mathbb{R} \times \mathbb{R}^3} |h|^2 d\mathbf{p} d\tau d\tau),
\]
where $C_R$ is the same as in Lemma 1.1. Because $u(t, \mathbf{x}, \mathbf{p}) = \int_{0}^{t} e^{s-t} h(s, \mathbf{x} + \frac{\mathbf{p}}{p_0} (s-t), \mathbf{p}) ds$, by using (2.10), (2.11) and (2.12) with $q = 2$, (2.20) yields
\[
\|\tilde{u}\|_{H^{1/2}_R} \leq C_5 \|h\|_{L^2}.
\]
By using complex interpolation method ([5] [36], we can know that $\tilde{\mathcal{F}}$ is bounded from $L^2_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $H^{1/2}_R((0, T) \times \mathbb{R}^3)$. Hence the operator $\tilde{\mathcal{F}}$ is bounded from $L^2_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $H^{1/2}_R((0, T) \times \mathbb{R}^3)$.

By using complex interpolation method ([5] [36], we can know that $\tilde{\mathcal{F}}$ is bounded from $(L^q_R, L^2_R)_[\theta](\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $(L^q, H^{1/2}_R)_[\theta](\mathbb{R} \times \mathbb{R}^3)$. Here, for $1 < q < +\infty$,
\[
(L^q_R, L^2_R)_[\theta] = L^p_R \quad \text{for} \quad 1 - \frac{\theta}{q} = \frac{1}{2}, \quad 0 \leq \theta \leq 1.
\]
and
\[
(L^q, H^{1/2}_R)_[\theta] = W^{s,p} \quad \text{for} \quad 1 - \frac{\theta}{q} = \frac{1}{2}, \quad s = \frac{\theta}{2}, \quad 0 \leq \theta \leq 1.
\]
In particular, for $q = 1$,
\[
1 - \frac{\theta}{2} = \frac{1}{2}, \quad \frac{\theta}{2} = 1 - \frac{1}{p}.
\]
and for $q = +\infty$,
\[
1 - \frac{\theta}{2} = \frac{1}{p}, \quad \frac{\theta}{2} = \frac{1}{p}.
\]
Hence it is necessary to set the following condition in Theorem 1.1
\[
0 < s < \inf(1 - \frac{1}{p}, \frac{1}{p}).
\]
According to the above analysis, we can conclude that $\tilde{\mathcal{F}}$ is a bounded linear operator from $L^p_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ into $W^{s,p}((0, T) \times \mathbb{R}^3)$ for any $1 < p < \infty$ and $0 < s < \inf(1 - \frac{1}{p}, \frac{1}{p})$. It follows that $\tilde{u} \in W^{s,p}((0, T) \times \mathbb{R}^3)$ if $u(t, x, p)$ and $f(t, x, p)$ both belong to $L^p_R((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for any $1 < p < \infty$. Since $\text{supp} u \subset [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R$, we can extend the domain of $u$ from $(0, T) \times \mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. Moreover, using Minkowski’s inequality, we have
\[
\|\tilde{u}\|_{W^{s,p}((0, T) \times \mathbb{R}^3)} = \left( \int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| \frac{u(t_1, x_1) - u(t_2, x_2)}{t_1 - t_2} \right|^2 dt_1 dx_1 dt_2 dx_2 \right)^{1/p} \\
\leq C_6 \|u\|_{L^p} + \|f\|_{L^p},
\]
where \( C_6 = \max(\frac{4}{3}\pi R^3, T, C_4, C_5) \). Let \( u_\lambda(t, \mathbf{x}, \mathbf{p}) = u(\lambda t, \lambda \mathbf{x}, \mathbf{p}) \) with \( \lambda > 0 \). Then we have the following equalities:

\[
\|u_{\lambda}\|_{W^{s,p}(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_{\lambda}(t_1, \mathbf{x}_1) - u_{\lambda}(t_2, \mathbf{x}_2)|^2}{[(t_1 - t_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2]^{\frac{4-s}{2}}} \, d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \right)^{1/p}
\]

\[
= \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(\lambda t_1, \lambda \mathbf{x}_1) - u(\lambda t_2, \lambda \mathbf{x}_2)|^2}{[(t_1 - t_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2]^{\frac{4-s}{2}}} \, d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \right)^{1/p}
\]

\[
= \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(t_1, \mathbf{x}_1) - u(t_2, \mathbf{x}_2)|^2}{[(t_1 - t_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2]^{\frac{4-s}{2}}} \, d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \right)^{1/p},
\]

\[
||u_{\lambda}\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u(\lambda t, \lambda \mathbf{x}, \mathbf{p})|^p \, d\mathbf{p} \, d\mathbf{x} \right)^{\frac{1}{p}} \tag{2.28}
\]

\[
\leq \lambda^{\frac{4}{p}} \|u\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)},
\]

\[
\left|\frac{\partial u_{\lambda}}{\partial t} + \frac{p}{p_0} \frac{\partial u_{\lambda}}{\partial \mathbf{x}}\right|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\partial u_{\lambda}}{\partial t} + \frac{p}{p_0} \frac{\partial u_{\lambda}}{\partial \mathbf{x}} \right|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} \tag{2.29}
\]

\[
= \lambda^{1-\frac{2}{p}} \left| \frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial \mathbf{x}} \right|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} \tag{2.30}
\]

Replacing \( u \) with \( u_{\lambda} \) in (2.27) and using (2.28), (2.29) and (2.30), we can obtain

\[
\lambda^{\frac{s-4}{p}} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(t_1, \mathbf{x}_1) - u(t_2, \mathbf{x}_2)|^2}{[(t_1 - t_2)^2 + (\mathbf{x}_1 - \mathbf{x}_2)^2]^{\frac{4-s}{2}}} \, d\mathbf{x}_1 \, dt_1 \, d\mathbf{x}_2 \, dt_2 \right)^{1/p} \leq C_6(\lambda^{\frac{4}{p}} \|u\|_{L^p} + \lambda^{1-\frac{4}{p}} \left| \frac{\partial u}{\partial t} + \frac{p}{p_0} \frac{\partial u}{\partial \mathbf{x}} \right|_{L^p}) \tag{2.31}
\]

Setting \( \lambda = \|u\|_{L^p}/\|f\|_{L^p} \), we have

\[
||\tilde{u}||_{W^{s,p}(0,T \times \mathbb{R}^3)} \leq C_6 \|u\|_{L^p}^{\frac{4}{p}} \|f\|_{L^p} \tag{2.32}
\]

This completes the proof of Theorem 1.1.

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In order to prove Lemma 1.1 we need two estimates related to a subset in the ball $B_R$. First, we have the following lemma:

**Lemma A.1** (24). For any given $\varepsilon > 0$, if $e' \in \mathbb{R}$, $e \in \mathbb{R}^3$, and $|e|^2 + e'^2 = 1$, then for all $R > 0$, there exists a constant $C_R$ which only depends on $R$, such that

$$\text{mes}(E_R) \leq C_R \varepsilon,$$

where $E_R = \{ p \in B_R : -\varepsilon - e' \leq \frac{p \cdot e}{\sqrt{|p|^2 + 1}} \leq \varepsilon - e' \}$.

**Proof.** Since we know that

$$\text{mes}(E_R) \leq \text{mes}(B_R) = 4 \pi R^3 \frac{\varepsilon}{2} \leq \frac{8}{3} \pi R^3 \varepsilon$$

for $\varepsilon \geq 1/2$, in order to show (A.1), it suffices to consider the case when $0 < \varepsilon < 1/2$.

For any $\varepsilon \in (0, 1/2)$, since $p \in B_R$, we can show that $|\frac{p \cdot e}{\sqrt{|p|^2 + 1}} + e'| > \varepsilon$ if we have the following inequality:

$$|e'| - R|e| > \varepsilon.$$  

Because $|e|^2 + e'^2 = 1$, (A.3) is equal to $\sqrt{1 - |e|^2} - R|e| > \varepsilon$ which implies that

$$|e| < -R\varepsilon + \sqrt{R^2 + 1 - \varepsilon}.$$  

Therefore we have

$$|e| \geq -R\varepsilon + \sqrt{R^2 + 1 - \varepsilon}$$

when $|\frac{p \cdot e}{\sqrt{|p|^2 + 1}} + e'| \leq \varepsilon$. We introduce the following rotation transform:

$$\phi(e_1, e_2, e_3)^T = (|e|, 0, 0),$$

thus getting

$$\text{mes}(E_R) = \text{mes}\{ p \in B_R : -\varepsilon - e' \leq \frac{p \cdot e}{\sqrt{|p|^2 + 1}} \leq \varepsilon - e' \}$$

$$= \text{mes}\{ \phi p \in B_R : -\varepsilon - e' \leq \frac{(\phi p) \cdot (\phi e)}{\sqrt{|\phi p|^2 + 1}} \leq \varepsilon - e' \}$$

$$= \text{mes}\{ \phi p \in B_R : -\varepsilon - e' \leq \frac{(\phi p) \cdot |e|}{\sqrt{|\phi p|^2 + 1}} \leq \varepsilon - e' \}$$

$$\leq 4R^2 \text{mes}\{ \phi p \in B_R \text{ and } -\varepsilon - e' \leq \frac{(\phi p)_{|e|}}{\sqrt{|\phi p|^2 + 1}} \leq \varepsilon - e' \}$$

$$\leq 4R^2 \text{mes}\{ \phi p \in B_R \text{ and } -\varepsilon - e' \leq \frac{(\phi p)_{|e|}}{|e| \sqrt{1 + R^2}} \leq \varepsilon - e' \sqrt{1 + R^2} \}$$

$$\leq 8R^2 \frac{\varepsilon}{|e|}$$

$$\leq 8R^2 \frac{\varepsilon}{\sqrt{1 + R^2}} \sqrt{\frac{R^2 + 1 - \varepsilon - R\varepsilon}{R^2 + 1 - \varepsilon - R\varepsilon}}$$

$$\leq 16R(1 + R^2)^{3/2} \varepsilon.$$  

Put $C_R = \max(\frac{8}{3} \pi R^3, 16R(1 + R^2)^{3/2})$. Then, by (A.2) and (A.7), (A.1) holds. This completes the proof of Lemma A.1. □
Then we have the following estimate of an integral:

Lemma A.2 (25). For any given \( \varepsilon > 0, e' \in \mathbb{R}, e \in \mathbb{R}^3 \), and \( |e|^2 + e'^2 = 1 \), then for all \( R > 0 \), there exists a constant \( C_R \) which only depends on \( R \), such that

\[
\int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \left| \frac{p \cdot e}{|p|^2 + 1} + e' \right|^2 dp \leq \frac{2C_R}{\varepsilon}, \quad (A.8)
\]

where \( C_R \) is equal to that given in Lemma A.1.

Proof. Notice that the integral on the right of (A.8) can be rewritten as

\[
\begin{align*}
\int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \left| \frac{p \cdot e}{|p|^2 + 1} + e' \right|^2 dp \\
= 2 \int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \int_0^{+\infty} \frac{1}{t^2} dt \: dp \\
= 2 \int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \int_\varepsilon^{+\infty} t^{-3} \chi(|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon, t) (t) dt \: dp \\
= 2 \int_\varepsilon^{+\infty} t^{-3} \int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \chi(t, p) dp dt,
\end{align*}
\]

where \( \chi(|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon, t) \) and \( \chi(|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon, t) \) are characteristic functions. It follows from Lemma A.1 that

\[
\int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \chi(|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon, t) (p) dp \\
\leq \text{mes}(p \in B_R : |p \cdot e|/|p|^2 + 1 + e'| > \varepsilon, t) \leq C_R t. \quad (A.10)
\]

Putting (A.10) into (A.9), we can get

\[
\begin{align*}
\int B_R (|p \cdot e|/|p|^2 + 1 + e'| > \varepsilon) \left| \frac{p \cdot e}{|p|^2 + 1} + e' \right|^2 dp \\
\leq 2C_R \int_\varepsilon^{+\infty} t^{-2} dt \leq 2C_R/\varepsilon.
\end{align*}
\]

\( \square \)

Proof of Lemma A.1. We first consider \( \| \tilde{u} \|_{H^\frac{1}{2}(0,T \times \mathbb{R}^3)} \). Since \( \text{supp} u \subset [\varepsilon_0, T - \varepsilon_0] \times \mathbb{R}^3 \times B_R \), we extend the domain of \( u = u(t, x, p) \) from \( (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \) into \( \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \) such that \( u = 0 \) outside \( (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \) and then define \( \tilde{u}(\tau, z, p) \) as the Fourier transformation of \( u(t, x, p) \) with respect to \( t \) and \( x \), that is,

\[
\tilde{u}(\tau, z, p) = \left( \frac{1}{\sqrt{2\pi}} \right)^4 \int \int_{\mathbb{R} \times \mathbb{R}^3} u(t, x, p) e^{-i\tau t - i x \cdot z} \: dx \: d\tau.
\]

Because \( u(t, x, p) \) and \( \frac{\partial}{\partial t} + \frac{p \cdot x}{|p|^2} \frac{\partial}{\partial z} (t, x, p) \) both belong to \( L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \), using Plancherel’s identity, we know that \( \tilde{u} \) and \( (\tau + \frac{p \cdot z}{|1+p|^2}) \tilde{u} \) both belong to \( L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \) too. By the definition of the norm of \( \tilde{u} \) in \( H^\frac{1}{2}(\mathbb{R} \times \mathbb{R}^3) \), we know that

\[
\| \tilde{u} \|_{H^\frac{1}{2}(\mathbb{R} \times \mathbb{R}^3)} = \left( \int_{\mathbb{R} \times \mathbb{R}^2} \left[ |z^2 + \tau^2|^{1/2} \right] \int_{\mathbb{R}^3} \tilde{u}(\tau, z, p) dp \: dz \: d\tau \right)^{1/2}. \quad (A.13)
\]
It is worth mentioning that \( \text{(1.8)} \) and \( \text{(A.13)} \) are equivalent norms (see proposition 1.37 in [3]). Then we estimate the integral in \( \text{(A.13)} \). For every \( \alpha > 0 \), we have

\[
\| \hat{u} \|_{H^2_\tau(R \times R^3)}^2 = \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{R^3} \hat{u}(\tau, z, p) d\tau d\tau \\
= \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} \hat{u}(\tau, z, p) d\tau d\tau \\
+ \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 > \alpha} \hat{u}(\tau, z, p) d\tau d\tau \\
\triangleq I_1 + I_2, \tag{A.14}
\]

where,

\[
I_1 = \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} \hat{u}(\tau, z, p) d\tau d\tau, \tag{A.15}
\]

\[
I_2 = \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 > \alpha} \hat{u}(\tau, z, p) d\tau d\tau. \tag{A.16}
\]

We below estimate \( I_1 \) and \( I_2 \) respectively. For \( I_1 \), using the Cauchy-Schwarz inequality, we have

\[
I_1 = \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} \hat{u}(\tau, z, p) d\tau d\tau \\
\leq \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \left( \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} |\hat{u}(\tau, z, p)|^2 d\tau \right) \left( \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} |p|^2 d\tau \right)^{1/2}.
\]

From Lemma \( \text{[A.1]} \) we obtain

\[
|z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 \leq \alpha} d\tau \leq C_R \alpha. \tag{A.18}
\]

It follows from \( \text{(A.17)} \) and \( \text{(A.18)} \) that

\[
I_1 \leq C_R \alpha \int_{R \times R^3} |\hat{u}|^2 d\tau d\tau. \tag{A.19}
\]

Similarly, applying the Cauchy-Schwarz inequality again, we have

\[
I_2 = \int_{R \times R^3} |z^2 + \tau^2|^{1/2} \int_{p \in \mathbb{R}^3, z^2 + \tau^2 > \alpha} \hat{u}(\tau, z, p) d\tau d\tau \\
\leq \int_{R \times R^3} \left( \int_{p \in \mathbb{R}^3, z^2 + \tau^2 > \alpha} |z^2 + \tau^2|^{1/2} \left( \frac{p \cdot z}{\sqrt{1 + p^2}} + \frac{\tau}{\sqrt{z^2 + \tau^2}} \right) \right) \left( \int_{p \in \mathbb{R}^3, z^2 + \tau^2 > \alpha} \frac{|p \cdot z}{\sqrt{1 + p^2}} + \frac{\tau}{\sqrt{z^2 + \tau^2}} |\hat{u}|^2 d\tau d\tau \right). \tag{A.20}
\]

From Lemma \( \text{[A.2]} \) we obtain

\[
\int_{R \times R^3} \int_{\frac{p \cdot z}{\sqrt{1 + p^2}} + \frac{\tau}{\sqrt{z^2 + \tau^2}} > \alpha} |z^2 + \tau^2|^{1/2} d\tau d\tau d\tau \\
= \int_{R \times R^3} \frac{1}{\sqrt{z^2 + \tau^2}} \left( \int_{\frac{p \cdot z}{\sqrt{1 + p^2}} + \frac{\tau}{\sqrt{z^2 + \tau^2}} > \alpha} \frac{1}{\sqrt{z^2 + \tau^2}} d\tau d\tau \right) \\
\leq \frac{1}{\sqrt{z^2 + \tau^2}} 2C_R \frac{\sqrt{z^2 + \tau^2}}{\alpha} = \frac{2C_R}{\alpha}. \tag{A.21}
\]
By (A.20) and (A.21), it follows that
\[ I_2 \leq \frac{2C_R}{\alpha} \int_{\mathbb{R}^3} \frac{\mathbf{p} \cdot \mathbf{z}}{\sqrt{1 + \mathbf{p}^2}} + \tau |\hat{u}|^2 d\mathbf{p} d\tau. \] (A.22)
Hence, by (A.14), (A.19) and (A.22), we know that
\[ \|\tilde{u}\|^2_{H^1_{\alpha}(\mathbb{R} \times \mathbb{R}^3)} = \int_{\mathbb{R} \times \mathbb{R}^3} |\mathbf{z}|^2 + \tau^2 |\hat{u}(\tau, \mathbf{z}, \mathbf{p})|^2 d\mathbf{z} d\tau \]
\[ \leq C_R \int_{\mathbb{R} \times \mathbb{R}^3} |\tilde{u}|^2 d\mathbf{p} d\tau + \frac{2C_R}{\alpha} \int_{\mathbb{R} \times \mathbb{R}^3} \frac{\mathbf{p} \cdot \mathbf{z}}{\sqrt{1 + \mathbf{p}^2}} + \tau |\hat{u}|^2 d\mathbf{p} d\tau \]
\[ \leq 2C_R (\alpha \int_{\mathbb{R}^3} |\tilde{u}|^2 d\mathbf{p} d\tau + \frac{1}{\alpha} \int_{\mathbb{R} \times \mathbb{R}^3} |\frac{\mathbf{p} \cdot \mathbf{z}}{\sqrt{1 + \mathbf{p}^2}} + \tau |\hat{u}|^2 d\mathbf{p} d\tau)^{1/2}. \] (A.23)
In (A.23), we set
\[ \alpha = \left( \int_{\mathbb{R} \times \mathbb{R}^3} |\tilde{u}|^2 d\mathbf{p} d\tau \right)^{1/2} \left( \int_{\mathbb{R} \times \mathbb{R}^3} \frac{1}{\alpha} \int_{\mathbb{R} \times \math{R}^3} |\frac{\mathbf{p} \cdot \mathbf{z}}{\sqrt{1 + \mathbf{p}^2}} + \tau |\hat{u}|^2 d\mathbf{p} d\tau \right)^{1/2}, \] (A.24)
thus getting
\[ \|\tilde{u}\|^2_{H^1_{\alpha}(\mathbb{R} \times \mathbb{R}^3)} = \int_{\mathbb{R} \times \mathbb{R}^3} |\mathbf{z}|^2 + \tau^2 |\hat{u}(\tau, \mathbf{z}, \mathbf{p})|^2 d\mathbf{z} d\tau \]
\[ \leq 2C_R \left( \int_{\mathbb{R} \times \mathbb{R}^3} |\tilde{u}|^2 d\mathbf{p} d\tau \right)^{1/2} \left( \int_{\mathbb{R} \times \mathbb{R}^3} \frac{1}{\alpha} \int_{\mathbb{R} \times \mathbb{R}^3} |\frac{\mathbf{p} \cdot \mathbf{z}}{\sqrt{1 + \mathbf{p}^2}} + \tau |\hat{u}|^2 d\mathbf{p} d\tau \right)^{1/2} \]
\[ = 2C_R \|u\|_{L^2} \left\| \frac{\partial u}{\partial t} + \frac{\mathbf{p}}{p_0} \frac{\partial u}{\partial x}(t, \mathbf{x}, \mathbf{p}) \right\|_{L^2}. \] (A.25)
Therefore \( \tilde{u} = \int_{\mathbb{R}^3} u(t, \mathbf{x}, \mathbf{p}) d\mathbf{p} \) belongs to \( H^1_{\alpha}(\mathbb{R} \times \mathbb{R}^3) \). Since (1.3) and (A.13) are equivalent, we know from (A.25) that
\[ \|\tilde{u}\|_{H^1_{\alpha}((0,T) \times \mathbb{R}^3)} = \int_{\mathbb{R} \times \mathbb{R}^3} \int_{\mathbb{R} \times \mathbb{R}^3} \left| \frac{u(t_1, \mathbf{x}_1) - u(t_2, \mathbf{x}_2)}{t_1 - t_2} \right|^2 dt_1 d\mathbf{x}_1 dt_2 d\mathbf{x}_2 \]
\[ \leq \sqrt{2C_R} \|u\|_{L^2}^{1/2} \|f\|_{L^2}^{1/2}. \] (A.26)
This completes our proof of Lemma 1.1 □

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