HOMOMORPHISMS TO 3–MANIFOLD GROUPS

DANIEL GROVES, MICHAEL HULL, AND HAO LIANG

ABSTRACT. We prove foundational results about the set of homomorphisms from a finitely generated group to the collection of all fundamental groups of compact 3–manifolds and answer questions of Agol–Liu [2] and Reid–Wang–Zhou [35].

CONTENTS

1. Introduction 1
2. Outline of proofs 4
3. Reduction to \(M_\pi\) 6
4. Generalized geometric decompositions 10
5. The Collapsed Space 16
6. Sequences which are not \(C\)–divergent 23
7. Limits and \(R\)–trees 30
8. JSJ–decompositions and modular automorphisms 34
9. Resolutions and factoring 41
Appendix A. Edge-twisted graphs of groups 48
Appendix B. Relative Linnell and JSJ decompositions 51
References 53

1. INTRODUCTION

There is a well-developed structure theory for compact 3–manifolds based on the Geometrization Theorem of Perelman (conjectured by Thurston). In this paper we develop a structure theory for the collection of all maps between 3–manifolds, from the point of view of their fundamental groups. Let \(\mathcal{M}_\pi\) be the set of all isomorphism classes of fundamental groups of 3–manifolds (see [3] for background). Our main result gives a positive answer to a question of Agol–Liu [2 Question 10.3].

Theorem A. For any finitely generated group \(G\) there is a finitely presented group \(G_0\) and an epimorphism \(\alpha: G_0 \to G\) so that for every \(\Gamma \in \mathcal{M}_\pi\) the map

\[
\alpha^* : \text{Hom}(G, \Gamma) \to \text{Hom}(G_0, \Gamma)
\]

induced by precomposition with \(\alpha\) is a bijection.

Date: July 15, 2024.

The first author is supported in part by NSF grant DMS-1904913. The third author is supported by NSFC (No.11701581 and No.11521101).
Of course, if \( G \) is finitely presented, then one can take \( G_0 = G \), and there is nothing to prove. Since finitely generated 3–manifold groups are finitely presented \[38\], it might appear that Theorem A is not relevant to the study of homomorphisms between 3–manifold groups. On the contrary, Theorem A is an important tool, and in particular implies the following theorem, answering a question of Reid–Wang–Zhou \[35\] Question 3.1.(C2)).

**Theorem B.** Let \( (M_i)_{i \geq 1} \) be a sequence of compact 3–manifolds (possibly with boundary). Every infinite sequence of epimorphisms \( \pi_1(M_1) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(M_n) \twoheadrightarrow \cdots \) contains an isomorphism.

It is worth remarking that Reid, Wang and Zhou ask their question about closed, orientable, aspherical 3–manifolds, whereas the only assumption we need is that they are compact. Theorem B proves the descending chain condition for the following partial order on 3–manifold groups: \( G_1 \geq G_2 \) if there is an epimorphism \( \phi : G_1 \to G_2 \). Per Calegari \[11\], understanding this order is an important question in 3–manifold topology.

Theorem B is proved in Section 2 as a consequence of Theorem A. The proof of Theorem B is by contradiction, and the group \( G \) from Theorem A used in this proof arises as a direct limit of the sequence of maps \( \pi_1(M_i) \twoheadrightarrow \pi_1(M_{i+1}) \).

Partial results about Reid, Wang and Zhou’s question were already known. Reid–Wang–Zhou \[35\] Theorem 3.4] gave a positive answer to this question in case all \( M_i \) are (closed, orientable, aspherical) Seifert 3-manifolds. A key step of their proof is showing that epimorphisms between fundamental groups of aspherical Seifert 3-manifolds with the same \( \pi_1 \) rank are realized by non-zero degree maps, which does not hold in general for epimorphisms between closed aspherical 3-manifolds groups of the same rank (see \[16\]).

Soma gave a positive answer to Reid, Wang and Zhou’s question in case all \( M_i \) are hyperbolic \[44\] Theorem 1]. Since every hyperbolic 3-manifold group embeds into \( \text{PSL}(2,\mathbb{C}) \), Soma’s result can be derived using the classical Hilbert Basis Theorem. To make this approach work in the general case, one would need to not only show that all 3-manifold groups are linear (which is still open for certain graph manifolds), but also to find a single linear group into which every 3-manifold group embeds. Whether this can be done is an open question which seems to be out of reach of current methods.

In order to solve Reid, Wang and Zhou’s question in full generality, a new approach was needed. We avoid questions of linearity and assumptions of positive degree by using Theorem A as a replacement for the Hilbert Basis Theorem. From the point of view of equations over groups, a homomorphism \( h : G \to H \) between groups corresponds to a solution in \( H \) to a system of equations (where the generators of \( G \) are variables, and the relators of \( G \) give equations). In this language, Theorem A is an analogue of the Hilbert Basis Theorem, uniformly for all groups in \( \mathcal{C}^\pi \). In the language of \[17\], Theorem A says that \( \mathcal{C}^\pi \) is an *equationally noetherian* family of groups.

Many other people have studied collections of maps between 3–manifolds or between 3–manifold groups. Examples of previous work include \[36\] \[9\] \[23\] \[33\] \[42\] \[43\] \[48\] \[13\] \[14\] \[7\] \[8\] \[28\] \[2\]. For an overview of the state of the art in 2002 about positive degree maps, see Wang’s ICM address \[47\]. Highlights of work since then have been Agol and Liu’s solution to Simon’s Conjecture (about epimorphisms between knot groups) \[2\], and Liu’s proof that for fixed \( k \geq 1 \), a given closed 3-manifold admits a degree \( k \) map onto only finitely many other closed 3–manifolds \[28\]. However, as far as we are aware, all previous work in this area has made assumptions about the 3–manifolds or the types of maps considered, whereas Theorem A makes no such assumptions.

Since the Borel Conjecture is true in dimension 3 (see for instance \[25\] Section 5), we also obtain:
Corollary C. Let \((M_i)_{i\in \mathbb{N}}\) be a sequence of closed, aspherical 3–manifolds and let \(f_i : M_i \to M_{i+1}\) be \(\pi_1\)-surjective maps. There exists \(N \in \mathbb{N}\) so \(f_i\) is homotopic to a homeomorphism for all \(i \geq N\).

Note that the asphericity assumption in Corollary C cannot be dropped, as can be seen using all \(M_i = S^3\).

We now turn to our approach to proving Theorem A. We fix a finitely generated group \(G\) and consider the collection of all homomorphisms from \(G\) to \(\Gamma\), for all \(\Gamma \in \mathcal{M}\). Our basic approach, inspired by Sela [40, 41] and his work on limit groups, etc., is to consider a sequence of homomorphisms and try to extract limits. In this theory, the goal is usually to find a limiting \(\mathbb{R}\)-tree from a divergent sequence of homomorphisms. At this point, Sela’s shortening argument is used to reduce to the case of non-divergent homomorphisms which are usually constant modulo conjugation.

In order to find limiting \(\mathbb{R}\)-trees, one way is to use actions on \(\delta\)-hyperbolic spaces. For 3–manifold groups, there are the Bass–Serre trees arising from the Kneser–Milnor decomposition or the geometric decomposition. This is the context of previous work by the first two authors [17], who used limiting \(\mathbb{R}\)-trees and a version of Sela’s shortening argument to reduce many questions to the case where a sequence of homomorphisms does not diverge on these trees. This is the starting point of this paper. However, this ‘non-divergent’ case is still highly nontrivial. We do obtain a limiting simplicial action on a tree in this setting, which induces a splitting of the associated limit group \(L\). We call this splitting the geometric decomposition of \(L\). For each vertex group \(V\) of this splitting, there is a corresponding sequence of pieces in the geometric decompositions of the corresponding sequence of 3–manifold groups. For each such piece, we construct a new space called the collapsed space which is a variant of a construction of Farb from [15] (Section 5). This sequence of spaces induces a new limiting action of \(V\) on an \(\mathbb{R}\)-tree (see Section 7), and the collapsed spaces are carefully tuned so that the limiting actions behaves sufficiently well and so we can repeat the shortening argument for this new action (see Section 9).

Even having applied the shortening argument twice, and having a new more restrictive notion of non-divergent, the non-divergent sequences are still far from being constant. Thus, we have yet another layer to our analysis. In this case, we construct a new splitting of our limit group \(L\) dual to a tree that we construct as a quotient of the Cayley graph of \(L\). This splitting has an algebraic property which we call edge-twisted (see Definition 6.5). We prove some technical results about edge-twisted splittings in Appendix A and applying these to our splitting of \(L\) allows us to conclude that each vertex group in the geometric decomposition of \(L\) is itself a limit group coming from a sequence of homomorphisms to geometric 3-manifolds. Finally, we can complete the proof by using the results of the third author [26] to understand these limit groups over geometric 3-manifolds.

This paper is in some sense a culmination of the prior work [17, 26] of the authors. On the other hand, we expect the tools in this paper to be useful for many other questions about maps between 3–manifold groups. In particular, although Theorems A and B are immediate from the Hilbert Basis Theorem in the case of Kleinian groups, we believe that the tools built in this paper will be useful for other questions about maps between and to Kleinian groups, and we intend to pursue this direction in future work.

We now outline the contents of this paper. In Section 2 we outline the proof of Theorem A and use this to derive Theorem B. In Section 2, Theorem A is reduced to four results, proved later in the paper. In Section 3, we prove Theorem 2.8 which allows us to focus on a restricted class of 3–manifold groups. In Section 4 we consider various splittings of limit groups. Later sections are dedicated to the proof of the main technical result, Theorem 2.8.
2. Outline of Proofs

In this section we explain the proof of Theorem A. Specifically, we explain the outline of the proof, and also state four results: Theorem 2.5, Theorem 2.6, Theorem 2.7, and Theorem 2.8 (proved in later sections). Assuming these results, we give a complete proof of Theorem A. We deduce Theorem B from Theorem A.

Our first reduction uses the basic structure of 3–manifolds and 3–manifold groups to restrict the class of 3–manifolds we need to consider. We refer to [3] for background about 3–manifolds, their fundamental groups, etc.

Definition 2.1. Let \( \mathcal{M}_{\mathrm{Gen}} \) be the set of homeomorphism classes of 3–manifolds \( M \) so that:

1. \( M \) is closed, orientable, and irreducible;
2. All geometric pieces are hyperbolic or Seifert-fibered with hyperbolic base orbifold; and
3. The base orbifold of each Seifert-fibered piece is orientable.

Let \( \mathcal{M}^\pi_{\mathrm{Gen}} \) denote the set of isomorphism classes of fundamental groups of elements of \( \mathcal{M}_{\mathrm{Gen}} \).

Throughout this paper, we fix a non-principal ultrafilter \( \omega \) on \( \mathbb{N} \) and use the concepts of \( \omega \)–limits, etc. See [46] for the background and basic results about ultrafilters, ultralimits, etc. Recall that a non-principal ultrafilter \( \omega \) is a finitely additive probability measure \( \omega \colon 2^\mathbb{N} \to \{0,1\} \) such that \( \omega(F) = 0 \) for any finite set \( F \). A statement \( P(i) \) depending on an index \( i \) holds \( \omega \)–almost surely if \( \omega(\{i \mid P(i) \text{ holds}\}) = 1 \).

Definition 2.2. Suppose \( \mathcal{G} \) is a family of groups and \( G \) is a finitely generated group. Let \( \text{Hom}(G, \mathcal{G}) \) denote the set of all homomorphisms from \( G \) to an element of \( \mathcal{G} \). Let \( (\phi_i) \) be a sequence of homomorphisms from \( \text{Hom}(G, \mathcal{G}) \). Associated to \( (\phi_i) \) is the stable kernel \( \text{Ker}^\omega(\phi_i) = \{ g \in G \mid \omega \text{–almost surely } \phi_i(g) = 1 \} \) and a \( \mathcal{G} \)–limit group defined by \( L := G/\text{Ker}^\omega(\phi_i) \). Let \( \phi_\infty \) denote the quotient map \( G \to L \). We refer to the sequence \( (\phi_i) \) as a defining sequence of homomorphisms for the limit group \( L \).

We say that the sequence \( (\phi_i) \) \( \omega \)–factors through the limit if \( \phi_i \) \( \omega \)–almost surely factors through \( \phi_\infty \).

The notion of an equationally noetherian family of groups was introduced by the first and second authors in [17, Definition A]. That the following definition is equivalent to the one from [17] is [17, Theorem 3.6].

Definition 2.3. The family \( \mathcal{G} \) is equationally noetherian if for any finitely generated group \( G \) there is a finitely presented group \( G_0 \) and an epimorphism \( \alpha \colon G_0 \to G \) so that for every \( \Gamma \in \mathcal{G} \) the map

\[
\alpha^* : \text{Hom}(G, \Gamma) \to \text{Hom}(G_0, \Gamma)
\]

induced by precomposition with \( \alpha \) is a bijection.

Note that with this definition, Theorem A is the claim that \( \mathcal{M}^\pi \) is equationally noetherian. In [17] Theorem 3.7 a further characterization of equationally noetherian is given, which is the one we use in the proof of Theorem A.

Theorem 2.4. The family \( \mathcal{G} \) is equationally noetherian if and only if for every finitely generated group \( G \), every sequence from \( \text{Hom}(G, \mathcal{G}) \) \( \omega \)–factors through the limit.

Our first reduction towards the proof of Theorem A is the following, proved in Section 3.
Theorem 2.5. If $\mathcal{M}_\text{Gen}^\pi$ is equationally noetherian then so is $\mathcal{M}^\pi$.

Our proof of Theorem A proceeds by verifying the criterion from Theorem 2.4. One feature a sequence $(\phi_i: G \to \Gamma_i)$ from $\text{Hom}(G, \mathcal{M}_\text{Gen}^\pi)$ might have is being $T$–divergent (see Definition 4.4 – roughly speaking the actions of $\phi_i$ on the trees dual to the geometric decompositions diverge). The main result of [17] implies the following theorem, proved in Section 4.

Theorem 2.6. Suppose that for every finitely generated group $G$, every sequence from $\text{Hom}(G, \mathcal{M}_\text{Gen}^\pi)$ which is not $T$–divergent $\omega$–factors through the limit. Then for every finitely generated group $G$, every sequence from $\text{Hom}(G, \mathcal{M}_\text{Gen}^\pi)$ $\omega$–factors through the limit.

Thus, we fix a non-$T$–divergent sequence $(\phi_i: G \to \Gamma_i)$ defining an $\mathcal{M}_\text{Gen}^\pi$–limit group $L$. The $G$–actions on the trees associated to the $\Gamma_i$ converge to an $L$–action on a (simplicial) tree, inducing the geometric decomposition of $L$ (Definition 4.16). A key set of subgroups are stably parabolic subgroups (Definition 4.5). These are abelian (Lemma 4.6) and edge groups of the geometric decomposition are stably parabolic. The following is proved in Section 4.

Theorem 2.7. Let $L$ be an $\mathcal{M}_\text{Gen}^\pi$–limit group defined by a non-$T$–divergent sequence $(\phi_i)$ and suppose the edge groups of the geometric decomposition of $L$ with respect to $(\phi_i)$ are finitely generated. Then $(\phi_i)$ $\omega$–factors through the limit.

At this point, the following result (together with Theorems 2.5, 2.6 and 2.7) completes the proof of Theorem A.

Theorem 2.8. Let $L$ be an $\mathcal{M}_\text{Gen}^\pi$–limit group defined by a non-$T$–divergent sequence $(\phi_i)$ and suppose the edge groups of the geometric decomposition of $L$ with respect to $(\phi_i)$ are finitely generated. Then $(\phi_i)$ $\omega$–factors through the limit. 

Proof of Theorem A. Suppose that $L$ is an $\mathcal{M}_\text{Gen}^\pi$–limit group defined by a non-$T$–divergent sequence. Edge groups of the geometric decomposition of $L$ are finitely generated by Theorem 2.8, so by Theorem 2.7 the defining sequence for $L$ $\omega$–factors through the limit. Theorem 2.6 implies that all sequences of homomorphisms to $\mathcal{M}_\text{Gen}^\pi$ $\omega$–factor through the limit, so by Theorem 2.4 $\mathcal{M}_\text{Gen}^\pi$ is equationally noetherian. Therefore, by Theorem 2.5, $\mathcal{M}^\pi$ is equationally noetherian, as required. \[\square\]

2.1. Proof of Theorem B. We now deduce Theorem B from Theorem A. Suppose that $(\Gamma_i)_{i=1}^\infty$ is a sequence from $\mathcal{M}^\pi$, and that for each $i \geq 1$ there is a surjection $\tau_i: \Gamma_i \to \Gamma_{i+1}$. Define $(\rho_i)$ from $\text{Hom}(\Gamma_1, \mathcal{M}^\pi)$ by $\rho_i = \tau_i \circ \cdots \circ \tau_1: \Gamma_1 \to \Gamma_{i+1}$. By Theorem A, $\mathcal{M}^\pi$ is equationally noetherian, so by Theorem 2.4, $(\rho_i)$ $\omega$–factors through the limit map $\rho_\omega$.

Now, $\text{Ker}(\rho_{i+1}) \subseteq \text{Ker}(\rho_i)$, so

$$\text{Ker}(\rho_\omega) = \bigcup_{i=1}^\infty \text{Ker}(\rho_i).$$
Definition 3.3. Let $GL$ subgroup of spaces $X$. For the precise definition. In all of our applications of ultra-limits of metric spaces, the spaces with fixed basepoints noetherian, then $G$.

Lemma 3.2. [17, Lemma 3.20]

Lemma 3.1. Equationally noetherian families.

3. Reduction to $\mathcal{M}_\text{gen}^2$

The purpose of this section is to record some basic facts about (limit groups over the family of) 3-manifold groups, and in particular to prove Theorem 2.5.

3.1. Equationally noetherian families. In this subsection, we record some facts about equationally noetherian families not particular to 3-manifold groups. The first follows from [41 Theorem B1] and [17 Lemma 3.9].

Lemma 3.1. For any field $K$ and $n \geq 1$, if $\mathcal{G}$ is a family of groups so each $\Gamma \in \mathcal{G}$ is a subgroup of $GL(n,K)$, then $\mathcal{G}$ is equationally noetherian.

Lemma 3.2. [17] Lemma 3.20] Suppose $(\phi_i)$ is a sequence from $\text{Hom}(G;\mathcal{G})$ and the associated limit group is finitely presented relative to subgroups $\{P_1, ..., P_n\}$. Suppose for each $P_j$, there is $P_j \leq G$ so that $\phi_\omega(P_j) = P_j$ and that $\phi_i|_{P_j}$ $\omega$–almost surely factors through $\phi_\omega|_{P_j}$. Then $(\phi_i)$ $\omega$–factors through the limit.

Theorem 1 of [5] states that if $H$ is a finite index subgroup of $G$ and $H$ is equationally noetherian, then $G$ is equationally noetherian. We generalize this to a family of groups.

Definition 3.3. Let $\mathcal{G}$ be a family of groups and let $n \geq 1$. Then $\mathcal{G}^\leq n$ is the family of groups containing a subgroup from $\mathcal{G}$ of index at most $n$. 

By Theorem A for $\omega$–almost all $j$ the homomorphism $\rho_j$ factors through $\rho_\omega$. Fix such a $j$. Then $\text{Ker}(\rho_\omega) \subseteq \text{Ker}(\rho_j)$, and since $\text{Ker}(\rho_j) \subseteq \text{Ker}(\rho_\omega)$, the two are equal. Thus, the ascending sequence of kernels stabilizes after $\rho_j$, and for all $k \geq j$ the map $\tau_i$ is an isomorphism, completing the proof.

2.2. Notation and conventions. Throughout this paper a $\delta$–hyperbolic space is a geodesic metric space in which all geodesic triangles are $\delta$–thin in the sense of [10] Definition III.H.1.16. We remark that if a geodesic metric space has $\nu$–slim triangles (in the sense of [10] Definition III.H.1.1) then it has $4\nu$–thin triangles. If $X$ is a geodesic metric space and $x,y \in X$ then we denote any geodesic between $x$ and $y$ by $[x,y]$.

Let $G$ be a group, and $\mathcal{A}$ and $\mathcal{H}$ families of subgroups closed under conjugation. An $(\mathcal{A},\mathcal{H})$–splitting of $G$ is an identification of $G$ with the fundamental group of a graph of groups in which all edge groups are in $\mathcal{A}$ and all groups in $\mathcal{H}$ are conjugate into vertex groups. When we consider a group as being the fundamental group of a graph of groups, we consider it to come with a specific choice of vertex groups and edge groups, rather than just conjugacy classes arising from an action on a tree. We also consider the underlying graph to have a fixed spanning tree, which, together with the choice of vertex and edge groups, induces a specific Bass-Serre presentation for the fundamental group of the graph of groups. This allows us to identify the fundamental group of any connected sub-(graph of groups) with a subgroup of the fundamental group of the original graph of groups. A graph of groups is minimal if there are no proper invariant subtrees of the Bass–Serre tree.

Given a sequence of real numbers $(\omega_i)$, we denote the $\omega$–limit of the sequence by $\lim^\omega a_i$. See [45] or [17] Section 4.1 for precise definitions. If $X_i$ is a sequences of metric spaces with fixed basepoints $o_i \in X_i$, then we denote the ultra-limit of the metric spaces $X_i$ with basepoints $o_i$ by $\lim^\omega(X_i,o_i)$. This ultra-limit is another metric space, see [17] Section 4.1 for the precise definition. In all of our applications of ultra-limits of metric spaces, the spaces $X_i$ are $\delta_i$–hyperbolic with $\lim^\omega \delta_i = 0$. In these cases, the corresponding ultra-limit is an $\mathbb{R}$–trees by [17] Lemma 4.1. 

3. Reduction to $\mathcal{M}_\text{gen}^2$

The purpose of this section is to record some basic facts about (limit groups over the family of) 3-manifold groups, and in particular to prove Theorem 2.5.

3.1. Equationally noetherian families. In this subsection, we record some facts about equationally noetherian families not particular to 3-manifold groups. The first follows from [41 Theorem B1] and [17 Lemma 3.9].

Lemma 3.1. For any field $K$ and $n \geq 1$, if $\mathcal{G}$ is a family of groups so each $\Gamma \in \mathcal{G}$ is a subgroup of $GL(n,K)$, then $\mathcal{G}$ is equationally noetherian.

Lemma 3.2. [17] Lemma 3.20] Suppose $(\phi_i)$ is a sequence from $\text{Hom}(G;\mathcal{G})$ and the associated limit group is finitely presented relative to subgroups $\{P_1, ..., P_n\}$. Suppose for each $P_j$, there is $P_j \leq G$ so that $\phi_\omega(P_j) = P_j$ and that $\phi_i|_{P_j}$ $\omega$–almost surely factors through $\phi_\omega|_{P_j}$. Then $(\phi_i)$ $\omega$–factors through the limit.

Theorem 1 of [5] states that if $H$ is a finite index subgroup of $G$ and $H$ is equationally noetherian, then $G$ is equationally noetherian. We generalize this to a family of groups.

Definition 3.3. Let $\mathcal{G}$ be a family of groups and let $n \geq 1$. Then $\mathcal{G}^\leq n$ is the family of groups containing a subgroup from $\mathcal{G}$ of index at most $n$. 

By Theorem A for $\omega$–almost all $j$ the homomorphism $\rho_j$ factors through $\rho_\omega$. Fix such a $j$. Then $\text{Ker}(\rho_\omega) \subseteq \text{Ker}(\rho_j)$, and since $\text{Ker}(\rho_j) \subseteq \text{Ker}(\rho_\omega)$, the two are equal. Thus, the ascending sequence of kernels stabilizes after $\rho_j$, and for all $k \geq j$ the map $\tau_i$ is an isomorphism, completing the proof.
Theorem 2.4. Suppose that $\mathcal{G}$ is an equationally noetherian family of groups. For any $n \geq 1$, the family $\mathcal{G}^\leq n$ is equationally noetherian.

Proof. Let $G$ be finitely generated and let $(φ_i : G → \Gamma_i)$ be a sequence with $Γ_i ∈ \mathcal{G}^\leq n$. By Theorem 2.4 it suffices to show that $ω$–almost surely $\text{Ker}(φ_ω) ⊆ \text{Ker}(φ)$. So, let $Γ'_i ∈ \mathcal{G}$ be so that $|Γ'_i : Γ_i| ≤ n$. Let $H$ be the intersection of all subgroups of $G$ of index at most $n$. Then $φ_i(H) ⊆ Γ'_i$, so $φ_i|_H ∈ \text{Hom}(H, \mathcal{G})$. Since $\mathcal{G}$ is equationally noetherian by Theorem 2.4, $ω$–almost surely $\text{Ker}(φ_ω|_H) ⊆ \text{Ker}(φ_i|_H)$.

Let $g ∈ G$. If $φ_ω(g) / φ_ω(H)$, then for any $h ∈ H$, $gh / \text{Ker}(φ_ω)$, suppose now that $φ_ω(g) ∈ φ_ω(H)$. Fix $k ∈ H$ such that $φ_ω(g) = φ_ω(k)$. Now, $ω$–almost surely $gk^{-1} ∈ \text{Ker}(φ)$ and $\text{Ker}(φ_ω|_H) ⊆ \text{Ker}(φ_i|_H)$. For any such $i$ and for any $h ∈ H$, if $gh ∈ \text{Ker}(φ_i|_H)$ then $kh ∈ \text{Ker}(φ_ω) ⊆ \text{Ker}(φ)$. Hence $gh = gk^{-1}kh ∈ \text{Ker}(φ)$. Putting together these two cases, we see that for any $g ∈ G$ $ω$–almost surely $\text{Ker}(φ_ω)∩ gH ⊆ \text{Ker}(φ_i)$. Repeating this argument over a finite of coset representatives of $H$ in $G$ proves that $ω$–almost surely $\text{Ker}(φ_ω) ⊆ \text{Ker}(φ_i)$. □

Remark 3.5. The bound on the index in Proposition 3.4 is essential. Indeed, it is straightforward to see that if one considers the family $\mathcal{G}^\text{fin}$ of groups which have a finite index subgroup contained in $\mathcal{G}$ then $\mathcal{G}$ may be equationally noetherian while $\mathcal{G}^\text{fin}$ is not. This is true, for example, in case $\mathcal{G}$ is the family consisting of the trivial group, since then $\mathcal{G}^\text{fin}$ is the family of finite groups, which is not equationally noetherian [17, Example 3.15].

Definition 3.6. If $\mathcal{G}$ is a family of groups, then let $\mathcal{G}^*$ be the family of all free products of elements of $\mathcal{G}$.

Theorem 3.7. ([17] Cor. C) If $\mathcal{G}$ is equationally noetherian then so is $\mathcal{G}^*$.

3.2. Reduction to closed, orientable, irreducible 3–manifolds. Recall that $\mathcal{M}^x$ is the set of all fundamental groups of all 3–manifolds. Recall that we are trying to prove Theorem 2.5 so we are interested in homomorphisms from a finitely generated group $G$ to elements of $\mathcal{M}^x$. Of course, all such homomorphisms have finitely generated image, and since $\mathcal{M}^x$ is closed under passing to subgroups there is no harm in restricting to the finitely generated elements of $\mathcal{M}^x$. See [17] Lemma 3.11.

By Scott’s Theorem [38, Theorem 2.1], every finitely generated 3–manifold group is in fact finitely presented, and moreover is the fundamental group of a compact 3–manifold (see also [38] or [37]). Doubling along the boundary embeds any compact 3–manifold with boundary into a closed 3–manifold as a retract. Since it is a retract, the fundamental group of the manifold with boundary injects into the fundamental group of the closed 3–manifold. Therefore, on the level of groups, we may assume that any homomorphism from $G$ to an element of $\mathcal{M}^x$ is in fact a homomorphism to the fundamental group of a closed 3–manifold. In summary, we have the following, where $\mathcal{M}_c$ is the set of fundamental groups of closed 3–manifolds.

Theorem 3.8. If $\mathcal{M}_c$ is equationally noetherian then so is $\mathcal{M}^x$.

We now move on to consider how to prove $\mathcal{M}_c$ is equationally noetherian, by considering the structure of closed 3–manifolds. We first make another reduction to the closed, orientable, irreducible case, and then we consider the geometric decomposition of these 3–manifolds.

Here are some pertinent classes of 3–manifolds.

Definition 3.9. Let $\mathcal{M}^x_{\text{cl}}$ be set of the fundamental groups of closed, orientable 3–manifolds. Let $\mathcal{M}^x_{\text{irr}}$ be the set of fundamental groups of closed, orientable and irreducible 3–manifolds.
Let $\mathcal{M}_{\text{Geo}}^\pi$ be the set of fundamental groups of compact geometric 3–manifolds with all boundary components tori.

**Lemma 3.10.** If $\mathcal{M}_{\text{Ori}}^\pi$ is equationally noetherian then so is $\mathcal{M}_{\text{Geo}}^\pi$.

**Proof.** By Theorem 3.8, it suffices to consider only closed manifolds. For each closed 3–manifold $M$, $\pi_1(M)$ has an index at most 2 subgroup belonging to $\mathcal{M}_{\text{Ori}}^\pi$, so the result follows from Proposition 3.4. \[ \square \]

Each closed, orientable 3–manifold $M$ has a unique Milnor–Kneser decomposition which induces a free product decomposition of $\pi_1(M)$:

$$\pi_1(M) \cong \pi_1(M_1) \ast \ldots \ast \pi_1(M_k) \ast F_n$$

where each $\pi_1(M_i) \in \mathcal{M}_{\text{Ori}}^\pi$ (and $F_n$ is a free group of rank $n$). The following is thus an immediate consequence of Theorem 3.7

**Corollary 3.11.** If $\mathcal{M}_{\text{Ori}}^\pi$ is equationally noetherian then so is $\mathcal{M}_{\text{Geo}}^\pi$.

### 3.3. The geometric decomposition

If $M$ is a closed, orientable, irreducible 3–manifold then the Geometrization Theorem (see [31, 32, 33, 12, 24, 29]) gives a (possibly trivial) decomposition of $M$ along incompressible tori into geometric pieces. This induces a splitting of $\pi_1(M)$ which we call the geometric decomposition. We refer to [3, 39], or [45] for the definition of a geometric 3–manifold, and to [39] for details about each of the 8 Thurston geometries.

Assume $M$ is a closed, orientable, irreducible 3–manifold and also that $M$ is not a geometric 3–manifold (in particular, $M$ is not a torus bundle over a circle). Then each piece of the geometric decomposition of $M$ is either hyperbolic, Seifert-fibered with hyperbolic base orbifold, or a twisted $I$–bundle over a Klein bottle. Replacing $M$ by a double cover as in [51, Lemma 2.4] we may assume there are no twisted $I$–bundles over a Klein bottle. By a similar argument, there is a double cover so all the base orbifolds of Seifert-fibered pieces with hyperbolic base orbifold are orientable. Thus:

**Lemma 3.12.** Let $M$ be a closed, orientable, irreducible, non-geometric 3–manifold. Then $M$ has a cover of degree at most 4 which lies in $\mathcal{M}_{\text{Gen}}^\pi$.

An important property of elements of $\mathcal{M}_{\text{Gen}}^\pi$ is the following result, which follows immediately from the proof of [51, Lemma 2.4]. Recall a group action on a tree $T$ is $k$–acylindrical if the stabilizer of any segment in $T$ of length at least $k + 1$ is trivial.

**Lemma 3.13.** Suppose $\Gamma \in \mathcal{M}_{\text{Gen}}^\pi$ and that $T$ is the Bass–Serre tree dual to the geometric decomposition of $\Gamma$. The $\Gamma$–action on $T$ is $2$–acylindrical.

**Definition 3.14.** Suppose that $\Gamma \in \mathcal{M}_{\text{Gen}}^\pi$ and that $T$ is the Bass–Serre tree dual to the geometric decomposition of $\Gamma$. A vertex group of the associated graph of groups decomposition of $\Gamma$ is of hyperbolic type if the associated sub-manifold in the geometric decomposition is hyperbolic, and of SFH-type if the associated sub-manifold is of Seifert-fibered type (with orientable hyperbolic base orbifold).

### 3.4. Reduction to $\mathcal{M}_{\text{Gen}}^\pi$

Consider the case $\mathcal{M}_{\text{SFH}}^\pi$ of fundamental groups of Seifert–fibered manifolds with orientable hyperbolic base orbifold. Let $\mathcal{M}_{\text{Orb}}^\pi$ be the set of fundamental groups of orientable hyperbolic 2–orbifolds of finite-type. Each $\Gamma \in \mathcal{M}_{\text{SFH}}^\pi$ admits a short exact sequence which is a central extension:

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \Gamma_B \rightarrow 1,$$
for some \( \Gamma_B \in \mathcal{M}_\text{ Orb} \) (the fundamental group of the base orbifold). From this, the following result follows quickly.

**Lemma 3.15.** Let \( L \) be an \( \mathcal{M}_\text{SFH}^\pi \)-limit group. Then there is a central extension

\[ 1 \to A \to L \to B \to 1, \]

where \( A \) is abelian and \( B \) is an \( \mathcal{M}_\text{ Orb}^\pi \)-limit group.

**Definition 3.16.** A group \( G \) is \( \mathcal{A} \)-slender if every abelian subgroup of \( G \) is finitely generated.

Our analysis of \( \mathcal{M}_\text{SFH}^\pi \)-limit groups is based on combining Lemma 3.15 with the following theorem of the third author.

**Theorem 3.17.** [26, Theorem 1.2] Let \( B \) be an \( \mathcal{M}_\text{ Orb}^\pi \)-limit group. Then \( B \) is finitely presented and \( \mathcal{A} \)-slender.

**Corollary 3.18.** Let \( L \) be an \( \mathcal{M}_\text{SFH}^\pi \)-limit group. Then \( L \) is finitely presented and \( \mathcal{A} \)-slender.

**Proof.** Consider the central extension from Lemma 3.15. Since \( B \) is finitely presented by Theorem 3.17, \( A \) is finitely generated, for example by [18, Lemma 12.1]. It is now easy to see \( L \) is finitely presented. The fact that \( A \) is finitely generated and abelian subgroups of \( B \) are finitely generated implies that all abelian subgroups of \( L \) are finitely generated. \( \square \)

**Lemma 3.19.** Let \( M \) be a Seifert fibered 3–manifold with orientable hyperbolic base orbifold \( O \). Let \( K \) be the kernel of the natural quotient map \( \pi_1(M) \to \pi_1(O) \). Let \( g, h \in \pi_1(M) \). If \( [g, h] \in K \), then \( [g, h] = 1 \in \pi_1(M) \).

**Proof.** All abelian subgroups of \( \pi_1(O) \) are cyclic, so the images of \( g \) and \( h \) in \( \pi_1(O) \) generate a cyclic subgroup. A central extension of a cyclic group is abelian. \( \square \)

The following result follows quickly from the fact that finite subgroups of elements of \( \mathcal{M}_\text{ Orb}^\pi \) are cyclic.

**Lemma 3.20.** Let \( B \) be an \( \mathcal{M}_\text{ Orb}^\pi \)-limit group. Any finite subgroup of \( B \) is cyclic.

The following result is now immediate from Lemmas 3.19 and 3.20.

**Corollary 3.21.** Suppose that \( L \) is an \( \mathcal{M}_\text{SFH}^\pi \)-limit group, and let \( 1 \to A \to L \to B \to 1 \) be as in Lemma 3.15. Suppose that \( E \leq L \) has finite image in \( B \). Then \( E \) is abelian.

**Lemma 3.22.** Let \( \Gamma \in \mathcal{M}_\text{Gen}^\pi \) and let \( \Gamma_e \) be an edge group of the geometric decomposition of \( \Gamma \), with adjacent vertex groups \( \Gamma_v, \Gamma_w \). The pre-images in \( \Gamma_e \) of the centers \( Z(\Gamma_v) \) and \( Z(\Gamma_w) \) intersect trivially.

**Proof.** The only way that \( Z(\Gamma_v) \) and \( Z(\Gamma_w) \) can both be nontrivial is if the corresponding pieces are both Seifert–fibered. The centers of adjacent Seifert-fibered pieces intersect trivially, since otherwise the connecting torus is not part of the characteristic sub-manifold. \( \square \)

**Theorem 3.23.** \( \mathcal{M}_\text{Geo}^\pi \) is an equationally noetherian family of groups.

**Proof.** It is a straightforward observation that any finite union of equationally noetherian families is equationally noetherian [17, Lemma 3.8]. Hence it suffices to deal with each geometry individually.
Suppose first that $M$ supports one of the following geometries: $S^3$, $S^2 \times \mathbb{R}$, $E^3$, NIL or SOL. Then (as, for example, explained in [18] §7 – 11) $\pi_1(M)$ has a subgroup of index at most 240 which is either cyclic (finite or infinite), $\mathbb{Z}$, or some semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Each of these groups embeds in $\text{SL}(3, \mathbb{Z})$. Therefore, the union of the set of fundamental groups of manifolds supporting these five geometries is equationally noetherian by Proposition 3.4 and Lemma 3.1.

The fundamental group of any hyperbolic 3–manifold is (isomorphic to) a subgroup of some semi-direct product $G \rtimes \mathbb{Z}$, with $G$ either cyclic (finite or infinite), $\mathbb{Z}$, or SOL. Then (as, for example, explained in [18, §11]) it suffices to consider $\mathcal{M}_{\text{SFH}}^\pi$–limit groups. If $L$ is an $\mathcal{M}_{\text{SFH}}^\pi$–limit group then $L$ is finitely presented by Lemma 3.11. It follows that the sequence of maps defining $L$ $\omega$–factors through the limit (see Lemma 3.2). Thus the family $\mathcal{M}_{\text{SFH}}^\pi$ is equationally noetherian by Theorem 2.4.

Summarizing the above discussion and the previous results in the section, we have the following result from Section 2.

**Theorem 2.5.** If $\mathcal{M}_{\text{Gen}}^\pi$ is equationally noetherian then so is $\mathcal{M}^\pi$.

**Proof.** Suppose that $\mathcal{M}_{\text{Gen}}^\pi$ is equationally noetherian. By Lemmas 3.10 and 3.11, in order to prove that $\mathcal{M}^\pi$ is equationally noetherian it is enough to prove that $\mathcal{M}_{\text{Gen}}^\pi$ is. By Theorem 3.23, $\mathcal{M}_{\text{Gen}}^\pi$ is equationally noetherian, so it is enough to consider closed, orientable, irreducible, non-geometric 3–manifolds. By Lemma 3.12 any such 3–manifold has a degree at most 4 cover which lies in $\mathcal{M}_{\text{Gen}}$. Therefore, under the assumption that $\mathcal{M}_{\text{Gen}}$ is equationally noetherian, we see that $\mathcal{M}^\pi$ is equationally noetherian, as required. □

## 4. Generalized geometric decompositions

In order to prove Theorem 1A it remains to prove Theorems 2.6, 2.7 and 2.8. In this section, we provide the setting and some basic definitions, and prove Theorems 2.6 and 2.7.

We fix the following setup. Let $G = \langle S \rangle$ be a finitely generated group, and let $(\phi_i : G \to \Gamma_i)$ be a sequence from $\text{Hom}(G, \mathcal{M}_{\text{Gen}}^\pi)$. Let $L = G/\text{Ker}^\omega(\phi_i)$ be the $\mathcal{M}_{\text{Gen}}^\pi$–limit group associated to $(\phi_i)$, with associated limit map $\phi_\omega : G \to L$ (see Definition 2.2).

**Definition 4.1.** Suppose that $G, (\phi_i)$ and $L$ are as above, and that $g \in L$. An $\omega$–approximation to $g$ is a sequence $(g_i)$ so that each $g_i \in \Gamma_i$ and there is an element $\bar{g} \in G$ so that $\phi_\omega(\bar{g}) = g$ and for each $i$ we have $\phi_i(\bar{g}) = g_i$.

Similarly, if $F \subset L$ is a finite ordered set then an $\omega$–approximation to $F$ is a sequence of (ordered) tuples $(F_i)$ from $\Gamma_i$ so that for some lift $\bar{F}$ of $F$ to $G$ (so $\phi_\omega$ restricts to an ordered bijection from $\bar{F}$ to $F$) we have $\phi_i(\bar{F}) = F_i$ (again, as an ordered bijection).

We often apply this definition to a finite set $F$ without explicitly mentioning the ordering. The ordering in these cases can be chosen arbitrarily, the point is that for each $g \in F$ and each $i$, there is a fixed $g_i \in F_i$ such that $(g_i)$ is an $\omega$–approximation to $g$.

The following lemmas are immediate.

**Lemma 4.2.** Suppose that $g \in L$ and that $(g_i)$ and $(g'_i)$ are $\omega$–approximations to $g$. Then $\omega$–almost surely $g_i = g'_i$. Similarly, if $F \subset L$ is finite and $(F_i)$ are $\omega$–approximations to $F$ then $\omega$–almost surely $F_i = F'_i$. 
Lemma 4.3. Suppose \( g^1, \ldots, g^t \in L \) and for each \( 1 \leq j \leq s \), \((g^j_i)\) is an \( \omega \)-approximation to \( g^j \). For any word \( w(x_1, \ldots, x_t) \in \{ \pm \}^t \) we have \( w(g^1_i, \ldots, g^t_i) = 1 \) in \( L \) if and only if for \( \omega \)-almost all \( i \) \( w(g^1_i, \ldots, g^t_i) = 1 \) in \( L \).

We are only concerned with properties of \( \omega \)-approximations that hold \( \omega \)-almost surely for terms in the sequence, and so it is always irrelevant which \( \omega \)-approximation we choose for an element (or finite subset) of \( L \). We use this observation frequently without mention.

Let \( S \) be a finite generating set for \( G \) and \( S_i = \phi_i(S) \). Let \( T_i \) be the tree dual to the geometric decomposition of \( \Gamma \).

\[
\| \phi_i \|_{T_i} = \inf_{t \in T_i} \max_{s \in S_i} d_{T_i}(t, s, t).
\]

Definition 4.4. The sequence \((\phi_i)\) is \( \mathcal{T} \)-divergent if \( \lim_{i} \| \phi_i \|_{T_i} = \infty \).

We are now ready to prove the following theorem from Section 2.

Theorem 2.6. Suppose that for every finitely generated group \( G \), every sequence from \( \text{Hom}(G, \mathcal{M}^\infty_{\text{Gen}}) \) which is not \( \mathcal{T} \)-divergent \( \omega \)-factors through the limit. Then for every finitely generated group \( G \) every sequence from \( \text{Hom}(G, \mathcal{M}^\infty_{\text{Gen}}) \) \( \omega \)-factors through the limit.

\[
\text{Proof.} \quad \text{The action of any } \Gamma \in \mathcal{M}^\infty_{\text{Gen}} \text{ on its geometric tree is } 2-\text{acylindrical (see Lemma 3.13). In particular, } \mathcal{M}^\infty_{\text{Gen}} \text{ is (in the terminology of [17]) a uniformly acylindrical family of groups (see, for example, [17] p.7142). Moreover, any sequence from } \text{Hom}(G, \mathcal{M}^\infty_{\text{Gen}}) \text{ which is not } \mathcal{T} \text{-divergent is } \text{“non-divergent” in the sense of [17]. Thus, Theorem 2.6 is an immediate consequence of [17] Theorem B}. \]

It now remains to prove Theorems 2.7 and 2.8. Thus, we henceforth assume that the defining sequence \((\phi_i)\) is not \( \mathcal{T} \)-divergent.

Definition 4.5. Let \( g \in L \) and let \((g_i)\) be an \( \omega \)-approximation to \( g \). Then \( g \) is stably parabolic with respect to \((\phi_i)\) if \( \omega \)-almost surely \( g_i \) fixes an edge in \( T_i \). A subgroup \( H \leq L \) is stably parabolic with respect to \((\phi_i)\) if for any finite \( F \subset H \) and any \( \omega \)-approximation \((F_i)\) of \( F \) \( \omega \)-almost surely there is an edge in \( T_i \) fixed by each \( f \in F_i \). Given \( V \leq L \), the set of stably parabolic subgroups of \( V \) with respect to \((\phi_i)\) is \( \mathcal{M}_{V,(\phi_i)} \), or just \( \mathcal{M}_V \) when \((\phi_i)\) is implied.

Since each subgroup of \( \Gamma \) which is the stabilizer of an edge in \( T_i \) is isomorphic to \( \mathbb{Z}^2 \), the following result is immediate from Lemma 4.3.

Lemma 4.6. Any stably parabolic subgroup of \( L \) is abelian.

Definition 4.7. A subgroup \( H \) of \( L \) is \( \omega \)-geometric if for any finite subset \( F \subset H \) and any \( \omega \)-approximation \((F_i)\) of \( F \) \( \omega \)-almost surely there is a vertex \( v_i \) of \( T_i \) so that \( F_i \) fixes \( v_i \). Note that \( v_i \) is unique when \( H \) is non-abelian. When \( H \) is abelian and \( F_i \) fixes multiple vertices of \( T_i \) we choose one and denote it by \( v_i \).

Terminology 4.8. Let \( H \) be an \( \omega \)-geometric subgroup of \( L \) and \( v_i \) be as in Definition 4.7. Then either

1. \( \omega \)-almost surely \( v_i \) is of hyperbolic type; or
2. \( \omega \)-almost surely \( v_i \) is of SFH-type.

In the first case \( H \) is hyperbolic-type and in the second \( H \) is SFH-type.
Definition 4.9. Let \( H \) be an \( \omega \)-geometric subgroup of \( L \), with associated sequence \( v_i \) of vertices of \( T_i \), and let \( \Gamma_{i,v_i} \) be the associated vertex group of \( \Gamma_i \). The stable center of \( H \), denoted \( Z^\omega(H) \), is the set of \( g \in H \) so that for any \( \omega \)-approximation \((g_i)\) of \( g \), \( \omega \)-almost surely \( g_i \in Z(\Gamma_{i,v_i}) \).

Observe that \( Z^\omega(H) \) is a subgroup and that if \( H \) is a hyperbolic-type \( \omega \)-geometric subgroup of \( L \) then \( Z^\omega(H) = \{ 1 \} \). Observe also that all elements of the stable center are stably parabolic.

As we see in Definition 4.16 below, \( L \) admits a splitting arising from a limiting action on the trees \( T_i \). This geometric decomposition is the natural splitting associated to \( L \), but for technical reasons in Section 9 we need a more general class of splittings of \( L \).

Definition 4.10. A minimal abelian splitting \( G \) of \( L \) is a generalized geometric decomposition (or GGD) with respect to \((\phi_i)\) if

1. Each vertex group of \( G \) is \( \omega \)-geometric.
2. For any adjacent vertices \( v \) and \( w \) of \( G \) with sequences \((v_i)\) and \((w_i)\) as in the definition of \( \omega \)-geometric (Definition 4.7), \( \omega \)-almost surely \( v_i \) and \( w_i \) are adjacent in \( T_i \).
3. \( G \) is an \((\mathcal{H}_L,\mathcal{H}_L(\phi))\)-splitting. That is to say, the edge groups are stably parabolic, and all stably parabolic subgroups are elliptic.
4. For each SFH-type vertex group \( G_v \) of \( G \), \( Z^\omega(G_v) \) is contained in each edge group of \( G \) adjacent to \( G_v \).

For the rest of this subsection, fix the above notation, and let \( G \) be a GGD of \( L \) with respect to \((\phi_i)\). The sequences \((v_i)\) and \((w_i)\) in Definition 4.10 are implicitly fixed.

Lemma 4.11. If \( G_v \) is an SFH-type vertex group of \( G \) then \( Z^\omega(G_v) \) is central in \( G_v \).

Proof. Let \( g \in Z^\omega(G_v) \) and \( h \in G_v \). If \((g_i)\) and \((h_i)\) are \( \omega \)-approximations to \( g \) and \( h \), respectively then \( \omega \)-almost surely there is an SFH-type vertex group \( \Gamma_{i,v_i} \) of \( \Gamma_i \) so that \( g_i \in Z(\Gamma_{i,v_i}) \) and \( h_i \in \Gamma_{i,v_i} \). Therefore \( \omega \)-almost surely \([g_i,h_i] = 1 \) in \( \Gamma_i \). By Lemma 4.8 \([g,h] = 1 \) in \( L \).

Let \( V \) be a vertex group of \( G \). By Lemma 4.11 \( Z^\omega(V) \leq V \), and we define \( \overline{V} = V/Z^\omega(V) \). Let \( \pi : V \rightarrow \overline{V} \) be the quotient map.

Lemma 4.12. Let \( V \) be an SFH-type vertex group of \( G \). If \( H \) is an abelian subgroup of \( \overline{V} \) then \( \pi^{-1}(H) \) is an abelian subgroup of \( V \).

Proof. Let \( \overline{g}, \overline{h} \in H \), let \( g \in \pi^{-1}(\overline{g}) \) and \( h \in \pi^{-1}(\overline{h}) \), and let \((g_i)\) and \((h_i)\) be \( \omega \)-approximations to \( g \) and \( h \), respectively. Since \([\overline{g},\overline{h}] = 1 \), we have \([g,h] \in Z^\omega(V) \), and so \( \omega \)-almost surely \([g_i,h_i] \in Z(\Gamma_{v_i}) \). Since \( Z(\Gamma_{v_i}) \) is the kernel of \( p_i : \Gamma_{v_i} \twoheadrightarrow \pi_1(O_i) \), where \( O_i \) is the base orbifold of the Seifert-fibered manifold corresponding to \( \Gamma_{v_i} \), by Lemma 3.19 we have \([g_i,h_i] = 1 \) \( \omega \)-almost surely. By Lemma 4.3 \([g,h] = 1 \), as required.

Let \( V, \overline{V} \) be as above, let \( \mathcal{E} \) be the set of edge groups of \( G \) adjacent to \( V \), and \( \overline{\mathcal{E}} \) the image of \( \mathcal{E} \) in \( \overline{V} \). Since \( L \) is finitely generated, \( V \) is finitely generated relative to \( \mathcal{E} \). Hence \( \overline{V} \) is finitely generated relative to \( \overline{\mathcal{E}} \). Therefore, by Corollary 3.3 \( \overline{V} \) admits a graph of groups decomposition \( \mathcal{L} \) rel \( \mathcal{E} \) so that all edge groups of \( \mathcal{L} \) have cardinality at most 4, and no vertex group of \( \mathcal{L} \) splits rel \( \mathcal{E} \) over a subgroup of size at most 4. We call \( \mathcal{L} \) the relative 4–Linnell decomposition of \( \overline{V} \) rel \( \overline{\mathcal{E}} \). Notice that by Corollary 3.21 the preimages in \( V \) of edges in \( \mathcal{L} \) are abelian.
Definition 4.13. Let $V$ be an SFH-type vertex group of $G$. Let $\overline{V} = V/Z^\omega(V)$. Let $\mathbb{L}$ be the relative 4–Linell decomposition of $\overline{V}$ relative to (images of) edge groups of $G$ adjacent to $V$, and $\mathbb{L}'$ the induced abelian splitting of $V$ relative to adjacent edge groups. The Linell refinement of $G$, denoted $G'$, is a refinement of $G$ obtained by replacing each SFH-type vertex group $V$ by $\mathbb{L}$.

Remark 4.14. Suppose that $V$ is a vertex group of $G$. The edge groups of $G$ adjacent to $V$ are elliptic the relative 4–Linell decomposition $\mathbb{L}$ of $V$ by definition. However, such an edge group might not be conjugate into a unique vertex group of $\mathbb{L}$. Therefore, there is a refinement $G'$ of $G$ as described in Definition 4.13 but it might not be unique. None of our constructions or results in this paper depend on which refinement is chosen, so we henceforth ignore this ambiguity.

Observe that the hyperbolic type vertex groups of $G$ and $G'$ are the same.

Definition 4.15. Suppose $V$ is a vertex group of $G'$. A good relative generating set for $V$ is a finite set $A$ so that

1. $A$ together with the adjacent edge groups generates $V$; and
2. For each edge group $E \supseteq Z^\omega(V)$ adjacent to $V$, there exists $a \in A \cap E \setminus Z^\omega(V)$.

The existence of good relative generating sets is clear.

4.1. The geometric decomposition. Let $(\phi_i : G \to \Gamma_i)$ be a sequence from $\text{Hom}(G, \mathcal{M}_{\text{Gen}}^\omega)$ which is not $\mathcal{F}$–divergent and let $T_i$ be the minimal $\phi_i(G)$–invariant subtree of the tree dual to the geometric splitting of $\Gamma_i$. Let $o_i$ be a point in $T_i$ so $\max_{s \leq S} d_{T_i}(o_i, \phi_i(s)o_i) = \|\phi_i\|_{\mathcal{T}}$.

Let $T_m = \lim^\omega(T_i, o_i)$. Since each $T_i$ is a tree in which edges have length one, $T_m$ is a simplicial tree. Moreover, the actions of $G$ on $T_i$ induce a 2–acylindrical action of $L$ on $T_m$; see [17 Proposition 6.1].

Definition 4.16. Let $G, S, (\phi_i), T_i, L$ and $T_m$ be as above. The geometric tree of $L$, denoted $T_{\text{geom}}$, is the minimal $L$–invariant subtree of $T_m$.

The geometric decomposition of $L$ is the splitting dual to $T_{\text{geom}}$.

It is straightforward to check that the geometric decomposition of $L$ is in fact a GGD in the sense of Definition 4.10 which we record in the following lemma.

Lemma 4.17. Let $L$ be an $\mathcal{M}_{\text{Gen}}^\omega$–limit group defined by a sequence $(\phi_i : G \to \Gamma_i)$ (so $L = G/\text{Ker}^\omega(\phi_i)$). If $(\phi_i)$ is not $\mathcal{F}$–divergent then the geometric splitting of $L$ is a GGD with respect to $(\phi_i)$.

We now prove the following result from Section 2.

Theorem 2.7. Let $L$ be an $\mathcal{M}_{\text{Gen}}^\omega$–limit group defined by a non–$\mathcal{F}$–divergent sequence $(\phi_i)$ and suppose the edge groups of the geometric decomposition of $L$ with respect to $(\phi_i)$ are finitely generated. Then $(\phi_i)$ $\omega$–factors through the limit.

Proof. Let $G$ be the geometric decomposition of $L$. The edge groups of $G$ are finitely generated, so the vertex groups are also and hence each vertex group of $G$ is a $\mathcal{M}_{\text{Geo}}^\omega$–limit group. By Theorem 5.23 the defining sequence for each vertex group $\omega$–factors through the limit. The edge groups of $G$ are finitely generated and abelian, so $L$ is finitely presented relative to the vertex groups of $G$, and by Lemma 4.2 the defining sequence for $L$ $\omega$–factors through the limit, as required. □

Given Theorem 2.7, our goal is to prove that the edge groups of $G$ are finitely generated. One way to achieve this is to show that the following question has a positive answer.
Question 1. Suppose the edge groups in an acylindrical splitting of a finitely generated group are abelian. Moreover, all vertex groups of this splitting have the property that their finitely generated subgroups are $\mathcal{A}$-slender. Are the edge groups of such a splitting always finitely generated?

However it is not clear to us how to answer the above question. Hence we obtain more information about the geometric decomposition of $L$ before we show that its edge groups are finitely generated.

4.2. Assumptions. It remains to prove Theorem 2.8 and the remainder of the paper is dedicated to its proof. Let $G_0$ be a finitely presented group and $\pi : G_0 \to G$ be a surjective map. Then the sequence of homomorphisms $\{\phi_i \circ \pi\}$ defines the same limit group $L$ as $\{\phi_i\}$. Since Theorem 2.8 is about properties of $L$, we can assume $G$ to be finitely presented in its proof without loss of generality. For the remainder of the paper we make the following assumptions, which are key for the proof of Theorem 2.8.

Standing Assumption 4.18. Let $G$ be a finitely presented group with finite generating set $S$. Let $(\phi_i : G \to \Gamma_i = \pi_1(M_i))$ be a sequence from $\text{Hom}(G, \mathcal{M}_{\text{Gen}}^\mathcal{F})$ and let $T_i$ be the minimal $\phi_i(G)$-invariant subtree of the geometric tree associated to $M_i$. Suppose $(\phi_i)$ is not $\mathcal{F}$-divergent. Let $L = G/\text{Ker}(\phi_i)$ be the $\mathcal{M}_{\text{Gen}}^\mathcal{F}$-limit group, and $\phi_{\infty} : G \to L$ the limit map. Let $\mathcal{G}$ be a GGD of $L$ wrt $(\phi_i)$ and $T_\mathcal{G}$ the Bass–Serre tree. Let $\mathcal{G}^{\omega}$ be the Linnell refinement of $\mathcal{G}$ (Definition 4.13). Fix a vertex group $V$ of $\mathcal{G}$ associated to the vertex $v$, and a good relative generating set $A$ of $V$ (Definition 4.15). Let $(A_i)$ be an $\omega$-approximation to $A$. Let $H_V$ be the set of stably parabolic subgroups of $V$.

4.3. Outline of the proof of Theorem 2.8. As explained above, in order to prove Theorem 2.8, it remains to prove Theorem 2.8. In this subsection, we provide an outline of the proof of Theorem 2.8.

First, recall the statement:

Theorem 2.8. Let $L$ be an $\mathcal{M}_{\text{Gen}}^\mathcal{F}$-limit group defined by a non-$\mathcal{F}$-divergent sequence $(\phi_i)$. The edge groups of the geometric decomposition of $L$ with respect to $(\phi_i)$ are finitely generated.

Recall from Definition 4.10 and the fact that the geometric decomposition is a GGD that the edge groups are stably parabolic. We prove that all stably parabolic subgroups are finitely generated. Stably parabolic subgroups were defined in Definition 4.5 and in Lemma 4.6 we observed that stably parabolic subgroups are abelian. Therefore, in order to prove that stably parabolic subgroups are finitely generated, it suffices to embed them in a finitely generated abelian group. This is achieved via the combination of the following two results. Recall that in Standing Assumption 4.18 we fixed a defining sequence $(\phi_i : G \to \Gamma_i)$ of the $\mathcal{M}_{\text{Gen}}^\mathcal{F}$-limit group $L$ and a GGD $\mathcal{G}$ of $L$ with respect to $(\phi_i)$. In Definition 6.1 below we define what it means for $(\phi_i)$ to be $\mathcal{C}$-divergent with respect to $\mathcal{G}$. However, for the logical structure of the proof of Theorem 2.8 the definition is not important, and we just need the dichotomy that a defining sequence either is or is not $\mathcal{C}$-divergent with respect to a given GGD. Theorem 4.19 is proved in Section 6 modulo the technical result Theorem 6.7 proved in Appendix A. Theorem 4.20 is proved in Section 9 building on the work in Sections 7 and 8.

Theorem 4.19. Let $L$ be an $\mathcal{M}_{\text{Gen}}^\mathcal{F}$-limit group defined by a non-$\mathcal{F}$-divergent sequence $(\phi_i)$. If there is a GGD $\mathcal{G}$ of $L$ with respect to which $(\phi_i)$ is not $\mathcal{C}$-divergent then all stably parabolic subgroups of $L$ are finitely generated.
Theorem 4.20. Suppose that $L$ is an $\mathcal{M}_{\text{Gen}}^3$–limit group whose defining sequence of homomorphisms is not $\mathcal{T}$–divergent. There exists $k \geq 0$ and a sequence of epimorphisms:

$$L = S_0 \xrightarrow{\eta_1} S_1 \xrightarrow{\eta_2} S_2 \xrightarrow{\eta_3} \cdots \xrightarrow{\eta_k} S_k \xrightarrow{\eta_{k+1}} S_{k+1},$$

so that

1. Each $S_i$ is an $\mathcal{M}_{\text{Gen}}^3$–limit group;
2. For each $1 \leq i \leq k$, the map $\eta_i : S_{i-1} \rightarrow S_i$ is a proper quotient map and $\eta_{k+1}$ is an isomorphism;
3. For each $1 \leq i \leq k+1$, $\eta_i$ injectively maps the stably parabolic subgroups of $S_{i-1}$ into stably parabolic subgroups of $S_i$; and
4. There is a GGD for $S_{k+1}$ with respect to which $S_{k+1}$ is not $\mathcal{C}$–divergent.

Given Theorems 4.19 and 4.20, Theorem 2.8 is easily proved.

Proof of Theorem 2.8. Recall that we have an $\mathcal{M}_{\text{Gen}}^3$–limit group $L$ defined by a non-$\mathcal{T}$–divergent sequence $(\phi_i)$, and we have to prove that the edge groups of the geometric decomposition of $L$ with respect to $(\phi_i)$ are finitely generated. We apply Theorem 4.20 to $L$ and $(\phi_i)$. By Condition (3) of Theorem 4.20 and induction, any stably parabolic subgroup $H$ of $L$ embeds into a stably parabolic subgroup of the group $S_{k+1}$ given by Theorem 4.20. Moreover, by Theorem 4.19, the stably parabolic subgroups of $S_{k+1}$ are finitely generated. The stably parabolic subgroups of $S_k$ are abelian by Lemma 4.6, so the stably parabolic subgroups of $L$ are subgroups of finitely generated abelian groups, and hence are finitely generated. The geometric decomposition of $L$ is a GGD by Lemma 4.17, and so the edge groups of the geometric decomposition of $L$ are stably parabolic by this lemma and Definition 4.10.\(\blacksquare\)

So, it finally remains to prove Theorem 4.19 and Theorem 4.20. We briefly comment on the tools we use to prove these two results.

In Section 5, we define a new space, the collapsed space. This space arises from $\mathbb{H}^2$ or $\mathbb{H}^3$ by collapsing an invariant collection of tubes around geodesics to lines, and an invariant collection of balls and horoballs to points. This space is defined because we do not know how to make the shortening argument work for the actions of different geometric pieces of 3–manifold groups on $\mathbb{H}^2$ and $\mathbb{H}^3$. Therefore, we construct the collapsed space carefully in order to make limiting $\mathbb{R}$–trees and the shortening argument work in “nearly” the usual way. Section 7 is devoted to analysing limiting $\mathbb{R}$–trees that arise from degenerations of actions on the collapsed space. Section 8 is devoted to proving that the appropriate JSJ-decomposition works for $\mathcal{M}_{\text{Gen}}^3$–limit groups. Section 9 is devoted to the shortening argument, and the proof of Theorem 4.20. Though there is substantial work to do in all of these sections, these will be relatively familiar to the experts. In fact, this is the benefit of the collapsed space – it is designed to make these arguments run as “standard” a way as possible.

The downside to the use of the collapsed space is that it is possible for interesting sequences of homomorphisms to not lead to a divergent sequence of actions on the collapsed space(s). Section 6 is devoted to dealing with this issue. We define what it means for a sequence of homomorphisms to be $\mathcal{C}$–divergent (Definition 6.1), which is when a sequence of homomorphisms leads to actions on the collapsed space which diverge. The goal of Section 6 is to prove Theorem 4.19. This is achieved by finding yet another limiting tree, the cut point tree (see Subsection 6.2), and introducing and analyzing a new kind of graph of groups, which we call edge-twisted (see Definition 6.5). These definitions are carefully...
tuned to the situation of \( \mathcal{M}^\Gamma_{\text{Gen}} \)-limits groups and the associated actions on the collapsed spaces, and designed to prove Theorem 4.19.

5. The Collapsed Space

Suppose \( \Delta \in \mathcal{M}^\Gamma_{\text{Gen}} \), and consider the geometric decomposition of \( \Delta \). There are two kinds of vertex groups: (i) Hyperbolic vertex groups which act on \( \mathbb{H}^3 \); and (ii) Seifert-fibered vertex groups with hyperbolic base orbifold which (through projection to the base orbifold and a choice of hyperbolic structure on this base orbifold) act on \( \mathbb{H}^2 \). Each of \( \mathbb{H}^3 \) and \( \mathbb{H}^2 \) are \( \delta \)-hyperbolic. However, these actions do not have properties that are well suited for the arguments in this paper.

In this section we build the collapsed space, a variation on Farb’s construction of the “Electric” space from [15]. The collapsed space is geodesic and \( \delta \)-hyperbolic for a uniform \( \delta \) (see Proposition 5.8 and Theorem 5.9). The key advantage of the collapsed space over \( \mathbb{H}^2 \) and \( \mathbb{H}^3 \) is given by Lemma 5.17. As well as these results, we prove a collection of basic structural results which are used in later sections.

5.1. The collapsed space for Kleinian and Fuchsian groups. Let \( \Gamma \) be a finitely generated Fuchsian or torsion-free Kleinian group. In case \( \Gamma \) is Kleinian, let \( \mathbb{H}^* = \mathbb{H}^3 \), and in case \( \Gamma \) is Fuchsian let \( \mathbb{H}^* = \mathbb{H}^2 \). Fix \( \delta \) so that all geodesic triangles in \( \mathbb{H}^2 \) and \( \mathbb{H}^3 \) are \( \delta \)-thin. Let \( \epsilon \) be the 3-dimensional (or 2-dimensional) Margulis constant (see [6] for more information). Define the \( \epsilon \)-thin part of \( \mathbb{H}^* \) with respect to \( \Gamma \):

\[
\mathcal{U}_\epsilon(\Gamma) = \{ x \in \mathbb{H}^* \mid d(x, \gamma x) \leq \epsilon \text{ for some } \gamma \neq 1 \in \Gamma \}.
\]

The set \( \mathcal{U}_\epsilon(\Gamma) \) is a disjoint union of a collection of horoballs, balls and neighborhoods of geodesics (see, for example, [6], Theorem D.3.3, though we work in the universal cover). We refer to the neighborhoods of geodesics and geodesic segments in the thin part of \( \mathbb{H}^* \) as tubes.

**Proposition 5.1.** For any \( K > 0 \) there exists \( D > 0 \) and a \( \Gamma \)-invariant open set \( \mathcal{U}^K(\Gamma) \subseteq \mathcal{U}_\epsilon(\Gamma) \) so that \( \mathcal{U}^K(\Gamma) \) satisfies the following:

1. The components of \( \mathcal{U}^K(\Gamma) \) are tubes, horoballs and balls;
2. Distinct components of \( \mathcal{U}^K(\Gamma) \) are at least \( K \) apart;
3. Suppose \( x \in \partial U \) for some connected component of \( U \) of \( \mathcal{U}^K(\Gamma) \). Then every non-trivial \( g \in \Gamma \) moves \( x \) by at least \( D \);
4. Tubes and balls in \( \mathcal{U}^K(\Gamma) \) have radius at least \( K \); and
5. Any horoball in \( \mathcal{U}_\epsilon(\Gamma) \), and every tube and ball of radius at least \( 2K \) is contained in the \( K \)-neighborhood of \( \mathcal{U}^K(\Gamma) \).

**Proof.** The set \( \mathcal{U}^K(\Gamma) \) is defined by taking \( \mathcal{U}_\epsilon(\Gamma) \), shrinking the components until they are distance \( K \) apart, and then keeping only the tubes and balls which remain that are of radius at least \( K \).

Given this construction, Item (1) follows from the Margulis Lemma. Items (2), (4) and (5) are true by the construction of \( \mathcal{U}^K(\Gamma) \). We now prove Item (3): By definition, \( U \) is obtained by shrinking the radius of a component \( U' \) of \( \mathcal{U}_\epsilon(\Gamma) \) by \( K \). By definition of \( \mathcal{U}_\epsilon(\Gamma) \), any non-trivial \( g \in \Gamma \) preserving \( U' \) moves points on \( \partial U' \) by \( \epsilon \). Hence \( g \) moves points on \( \partial U \) by an amount \( D' \) depending only on \( K \). If \( g \in \Gamma \) does not preserve \( U' \), then \( g \) does not preserve \( U \). Then \( U \) and \( gU \) are at least \( K \) apart. Therefore we can let \( D \) be the minimum of \( \{ D', K \} \). \( \square \)
Given points $x, y$ in a tube $U$ around a geodesic $\gamma$ in $\mathbb{H}^n$, let $\pi_\gamma: U \to \gamma$ be the closest point projection and let $d_H(x, y) = d_{\mathbb{H}^n}(\pi_\gamma(x), \pi_\gamma(y))$. Suppose $K > 1$ (a specific value for $K$ is fixed later). Define an “electric” distance $d_\varepsilon$ on $\mathbb{H}^n$ by

$$d_\varepsilon(x, y) = \begin{cases} 0 & \text{if } x, y \text{ lie in the same horoball or ball of } \mathcal{H}^K(\Gamma); \\ d_H(x, y) & \text{if } x, y \text{ both lie in a tube in } \mathcal{H}^K(\Gamma); \\ d_{\mathbb{H}^n}(x, y) & \text{otherwise.} \end{cases}$$

Define a pseudo-metric $\hat{d}$ on $\mathbb{H}^n$ by setting

$$\hat{d}(x, y) = \inf_{\{x_i\}_{i=0}^n} \left\{ \sum_{i=1}^n d_\varepsilon(x_i, x_{i-1}) \right\},$$

where the infimum is taken over all finite sets $x_0 = x, \ldots, x_n = y$.

Denote by $(\mathcal{H}^K(\Gamma), \hat{d})$ the induced metric space of $(\mathbb{H}^n, \hat{d})$, and let $q^K_\Gamma: \mathbb{H}^n \to \mathcal{H}^K(\Gamma)$ be the canonical quotient map. Observe $(\mathcal{H}^K(\Gamma), \hat{d})$ is a length space. Denote the length of a path $p$ in $\mathcal{H}^K(\Gamma)$ by $\hat{\ell}(p)$.

**Definition 5.2.** The collapsing locus of $\mathcal{H}^K(\Gamma)$ is the collection of points $x \in \mathcal{H}^K(\Gamma)$ so that $|\hat{q}^{K^{-1}}(x)| > 1$.

Let $\gamma$ be a path from $x$ to $y$ in $\mathcal{H}^K(\Gamma)$ so that the intersection of $\gamma$ with each component of the collapsing locus is connected. Let the components of the intersection of $\gamma$ with the collapsing locus of $\mathcal{H}^K(\Gamma)$ be $C = \{C_1, \ldots, C_n\}$. The collapsed set associated to $\gamma$ is $\{U_i = (q^K_\Gamma)^{-1}(C_i)\}$, a disjoint collection of horoballs, balls and tubes around geodesic segments. If $C_i$ does not contain $x$ or $y$ then $\gamma$ penetrates $U_i$. Let $\{\gamma_0, \ldots, \gamma_k\}$ be the components of $\gamma \setminus C$. Note that $\gamma_0$ and/or $\gamma_k$ may be empty. Let $\tilde{\gamma}$ be the closure of $(q^K_\Gamma)^{-1}(\gamma)$ in $\mathbb{H}^n$. The entry point of $\gamma$ into $U_i$ is $e(\gamma, U_i) = \gamma_{i-1} \cap U_i$ and the exit point out of $U_i$ is $o(\gamma, U_i) = \gamma_{i} \cap U_i$. Let $\tilde{\gamma}_i$ be the geodesic between $e(\gamma, U_i)$ and $o(\gamma, U_i)$. The lift of $\gamma$ is $\{\gamma_0, \tilde{\gamma}_1, \ldots, \gamma_k, \tilde{\gamma}_k\}$. If each $\tilde{\gamma}$ is geodesic then $\gamma$ is a local collapsed geodesic.

A local collapsed geodesic $\gamma$ between $x, y \in \mathcal{H}^K(\Gamma)$ is an almost geodesic if $\hat{\ell}(\gamma) \leq \hat{d}(x, y) + 1$. Any path $[x, y]$ can be replaced by a local collapsed geodesic connecting $x, y$ without increasing its length, so an almost geodesic always exists between any two points in $\mathcal{H}^K(\Gamma)$.

**Notation 5.3.** Suppose $(X, d_X)$ is a proper metric space, and $A, B \subseteq X$ are closed. If $x \in X$ then $\pi_A(x) = \{y \in A \mid d_X(x, y) = d_X(x, A)\}$ and $\pi_A(B) = \bigcup_{b \in B} \pi_A(b)$. Note that $\pi_A(x)$ and $\pi_A(B)$ are non-empty as $X$ is proper.

Here is a standard fact about hyperbolic spaces.

**Lemma 5.4.** Let $(X, d_X)$ be a $\nu$–hyperbolic metric space and suppose that $W_1, W_2 \subseteq X$ are convex and that $d(W_1, W_2) > 6\nu$. There exists $x \in X$ with $d_X(x, W_2) \leq 3\nu$ so that $\pi_{W_1}(W_2) \subseteq \pi_{W_1}(B(x, 2\nu))$.

**Lemma 5.5.** Suppose that $U_1$ and $U_2$ are convex subsets of $\mathbb{H}^n$ that are at distance $t \geq 6\delta_{\mathbb{H}^n}$ apart. Then $\text{Diam}(\pi_{U_1}(U_2)) \leq 4\delta_{\mathbb{H}^n}e^{-t + 5\delta_{\mathbb{H}^n}}$.

**Proof.** This follows from Lemma 5.4 and the fact that for any $t'$ the length of paths outside of $N(U_2, t')$ decreases by a factor of at least $e^{-t'}$ under projection to $U_2$. \qed

The following lemma is similar to [15, Lemma 4.5].
Lemma 5.6. There exists an absolute constant \( l > 2 \) independent of \( K \) with the following property: Let \( \overline{x}, \overline{y} \in \mathbb{H}^n \) and let \( \gamma = q^K_{l_1}(\overline{x}) \). Let \( \beta \) be an almost geodesic in \( \mathcal{C}(\Gamma) \) between \( x = q^K_{l_1}(\overline{x}) \) and \( y = q^K_{l_1}(\overline{y}) \). Any subsegment of \( \beta \) which lies outside of \( N(\gamma, l) \) has length at most \( 4l + 4 \). In particular, any almost geodesic from \( x \) to \( y \) stays completely inside \( N(\gamma, 3l + 2) \).

Proof. We claim any \( l \) satisfying the following conditions suffices:

1. \( l \geq 6\delta_\mathbb{H} > 2 \); and
2. \( [1 - (4\delta_\mathbb{H} + 1)e^{-l + 5\delta_\mathbb{H}}] \geq 1/2 > 0 \).

Let \( \beta' \) be a subsegment of \( \beta \) lying completely outside \( N(\gamma, l) \), and let \( z \) and \( w \) be the first point and the last point of \( \beta' \), respectively. Since \( \beta' \) lies outside \( N(\gamma, l) \), all collapsed sets penetrated by \( \beta' \) are at least \( l \) away from \( \overline{y} \). Let \( \{U_1, \ldots, U_k\} \) be the collapsed set for \( \beta' \), and let

\[
\beta_1^p, \beta_2^p, \ldots, \beta_k^p,
\]

be the lift of \( \beta' \) to \( \mathbb{H}^n \). Then \( q^K_{l_1}(\beta_i) \) is subsegment of \( \beta' \) and hence \( \beta_i \) is at least \( l \) away from \( \overline{y} \). Since \( U_i \) is at least \( l \) away from \( \overline{y} \) and \( l \geq 6\delta_\mathbb{H} \), the projection of \( \beta_i \) to \( \overline{y} \) has length \( \ell(\beta_i) e^{-l} \). Since \( \beta_i^p \subset U_i \) and \( U_i \) is \( l \) away from \( \overline{y} \), by Lemma 5.5 the projection of \( \beta_i^p \) to \( \overline{y} \) has length at most \( 4\delta_\mathbb{H} e^{-l + 5\delta_\mathbb{H}} \). Let \( l' = l - 5\delta_\mathbb{H} \).

Let \( \tilde{z} \) and \( \tilde{w} \) be the \( q^K_{l_1} \)-pre-image of \( z \) and \( w \), respectively, and let \( \tilde{z}' \) and \( \tilde{w}' \) denote the images of \( \tilde{z} \) and \( \tilde{w} \) under orthogonal projection onto \( \overline{y} \), respectively. Let \( z' = q^K_{l_1}(\tilde{z}') \) and \( w' = q^K_{l_1}(\tilde{w}') \). Then

\[
\hat{\ell}(\beta') \leq d(z, z') + d(z', w') + d(w', w) + 1
\leq l + d(\tilde{z}', \tilde{w}') + l + 1
\leq l + \hat{\ell}(\beta') \cdot e^{-l} + 4\delta_\mathbb{H} \cdot k \cdot e^{-l} + l + 1
\]

Components are at least \( K \) apart (in \( \mathbb{H}^n \)-distance) with \( K > 1 \), so \( k - 1 \leq \hat{\ell}(\beta') \), and

\[
\hat{\ell}(\beta') \leq 2l + \hat{\ell}(\beta') \cdot e^{-l} + 4\delta_\mathbb{H} \cdot (\hat{\ell}(\beta') + 1) \cdot e^{-l} + 1
\]

Therefore

\[
\hat{\ell}(\beta') \cdot [1 - (4\delta_\mathbb{H} + 1)e^{-l}] \leq 2l + 4\delta_\mathbb{H} \cdot e^{-l} + 1
\]

We chose \( l \) so that \( [1 - (4\delta_\mathbb{H} + 1)e^{-l}] \geq 1/2 > 0 \), hence \( 4\delta_\mathbb{H} \cdot e^{-l'} \leq 1 \), so

\[
\hat{\ell}(\beta') \leq 2(2l + 4\delta_\mathbb{H} \cdot e^{-l'} + 1) \leq 4l + 4,
\]

as required. \( \square \)

Lemma 5.7. The collapsing locus of \( \mathcal{C}(\Gamma) \) has the following properties:

1. A component of the collapsing locus is either a point or (homeomorphic to) a line.
2. Different components of the collapsing locus are at least \( K \) away from each other.

Proof. Item (1) follows directly from the construction of \( \mathcal{C}(\Gamma) \). Item (2) follows the second statement of Proposition 5.1 and the construction of \( \mathcal{C}(\Gamma) \). \( \square \)

Let \( l \) be as in Lemma 5.6

Proposition 5.8. If \( K \geq 8l \) then \( (\mathcal{C}^K(\Gamma), d) \) is a geodesic metric space.
Theorem 5.9. Let $l$ be as in Lemma 5.6 and $\delta = 4(6l + \delta_H + 2)$. Then for any $K > 8l$, the space $\mathcal{C}(\Gamma, d)$ is $\delta$-hyperbolic.

Proof. In the same way that [15, Proposition 4.6] follows from [15, Lemma 4.5], one can use Lemma 5.6 to see that $\mathcal{C}(\Gamma, d)$ has $(6l + \delta_H + 2)$-slim triangles.

Notation 5.10. Let $\delta$ be as in the statement of Theorem 5.9 and fix $K = 40\delta$. We write $\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma, d)$ and call $\mathcal{C}(\Gamma)$ the collapsed space associated to $\Gamma$. We drop the superscript $K$ from other constructions also. Therefore, for example, we have the map $q^\ast : \mathbb{H}^* \rightarrow \mathcal{C}(\Gamma)$, and when $\Gamma$ is assumed we just write $q : \mathbb{H}^* \rightarrow \mathcal{C}$, etc.

Lemma 5.11. If a component of the collapsing locus is a line, then it is a geodesic in $\mathcal{C}(\Gamma)$.

Proof. Let $\gamma$ be such a component. Then $\gamma = q(\tilde{\gamma})$, where $\tilde{\gamma}$ is a geodesic in $\mathbb{H}^*$. Let $x, y \in \gamma$. Then by Proposition 5.8 any geodesic $[x, y]$ is in a $3l + 2$-neighborhood of $\gamma$. By (2) of Lemma 5.5 and the fact that $K = 40\delta$ is much bigger than $3l + 2$, we know that $[x, y]$ does not intersect other components of the collapsing locus. This implies that $[x, y]$ is exactly the subsegment of $\gamma$ connecting $x$ and $y$. Therefore $\gamma$ is a geodesic.

Lemma 5.12. Let $Z$ be a $\nu$–hyperbolic space, and suppose that $C$ is a convex subset of $Z$. Suppose that $u, v \in Z$ are so that $d_2(u, C) = d_2(v, C) = 12\nu$, and that some geodesic $\sigma$ from $u$ to $v$ intersects the $6\nu$–neighborhood of $C$. Let $w$ be the point in $\sigma$ lying at distance $2\nu$ from $u$. Then $d_2(w, C) \leq 11\frac{1}{4}\nu$.

Proof. Choose a geodesic $\rho$ in $Z$ from $u$ to $C$ so that the length of $\rho$ is at most $d(u, C) + \frac{1}{4} = 12\frac{1}{4}\nu$. Let $s \in C$ be the endpoint of $\rho$, and consider a geodesic triangle with two sides $\rho$ and $\sigma$. Since $\sigma$ intersects the $6\nu$–neighborhood of $C$, it is easy to see that the Gromov product $\langle v | s \rangle_u$ is at least $2\nu$.

Therefore, there exists a point $m \in \rho$ so that $d(u, m) = 2\nu$ and $d(m, w) \leq \nu$. Note that $d(m, C) \leq d(m, s) \leq 10\frac{1}{4}\nu$ (by traveling along $\rho$). Now,

$$d(w, C) \leq d(w, s) \leq d(w, m) + d(m, s) \leq \nu + 10\frac{1}{4}\nu = 11\frac{1}{4}\nu.$$
as required. □

Lemma 5.13. Suppose that \( p \) is the image in \( \mathcal{C}(\Gamma) \) of an \( \mathbb{H}^+ \)-geodesic. Suppose that \( C \) is a component of the collapsing locus which lies within \( 6\delta \) of \( p \), that \( r \) is a point in \( N_{6\delta}(C) \cap p \), that \( u \) is a point on \( p \) so that \( d(u,C) = 12\delta \), and that \( z \) is a point on \( p \) so that \( d(u,z) = 2\delta \), where the order of the points on \( p \) is \( (z,u,r) \).

Then \( d(z,C) \geq 13\delta \).

Proof. Lift to points \( \tilde{z}, \tilde{u}, \tilde{r}, \tilde{w} \), and a path \( \tilde{p} \), all in \( \mathbb{H}^+ \). Let \( \tilde{C} \) be the set collapsed to yield \( C \), and let \( \tilde{\tau} \) be the shortest path from \( \tilde{u} \) to \( \tilde{C} \). Let \( \tilde{r} \) be the point on \( \tilde{\tau} \) at distance \( 6\delta \) from \( \tilde{C} \), and consider the geodesic triangle with vertices \( \tilde{z}, \tilde{r}, \tilde{\tau} \). Since neighborhoods of convex sets are convex in \( \mathbb{H}^+ \), and since \( \tilde{r}, \tilde{\tau} \) lie in the \( 6\delta \)-neighborhood of \( \tilde{C} \), the geodesic \( [\tilde{r}, \tilde{\tau}] \) also lies entirely in the \( 6\delta \)-neighborhood of \( \tilde{C} \).

Therefore, there are points \( \tilde{a} \in [\tilde{z}, \tilde{r}], \tilde{b} \in [\tilde{z}, \tilde{\tau}] \) so that \( d_{\mathbb{H}^+}(\tilde{a}, \tilde{b}) \leq \delta \), \( d(\tilde{z}, \tilde{a}) = d(\tilde{z}, \tilde{b}) \), and \( d_{\mathbb{H}^+}(\tilde{a}, \tilde{C}) = 8\delta \).

Now, \( d(\tilde{b}, \tilde{C}) \leq d(\tilde{a}, \tilde{C}) + d(\tilde{a}, \tilde{b}) \leq \delta + 8\delta = 9\delta \). Since \( d(\tilde{u}, \tilde{C}) = 12\delta \), it follows that \( d(\tilde{u}, \tilde{b}) \geq 3\delta \).

We now have that

\[
\begin{align*}
d(\tilde{z}, \tilde{C}) &= d(\tilde{z}, \tilde{a}) + d(\tilde{a}, \tilde{C}) \\
&= d(\tilde{z}, \tilde{a}) + 8\delta \\
&= d(\tilde{z}, \tilde{b}) + 8\delta \\
&= d(\tilde{z}, \tilde{u}) + d(\tilde{u}, \tilde{b}) + 8\delta \\
&\geq 2\delta + 3\delta + 8\delta = 13\delta,
\end{align*}
\]

as required. □

Proposition 5.14. Let \( x, y \in \mathcal{C}(\Gamma) \) and let \( \gamma_2 = [x, y] \) be a \( \mathcal{C}(\Gamma) \)-geodesic. Let \( \tilde{x}, \tilde{y} \in \mathbb{H}^+ \) be lifts of \( x, y \), respectively. Let \( \tilde{\gamma}_1 = [\tilde{x}, \tilde{y}] \) be the \( \mathbb{H}^+ \)-geodesic and \( \gamma_1 = q_{\Gamma}(\tilde{\gamma}_1) \). The Hausdorff distance between \( \gamma_1 \) and \( \gamma_2 \) is at most \( 9\delta \).

Proof. By Lemma 5.6, \( \gamma_2 \subseteq N(\gamma_1, 3l + 2) \), and \( 3l + 2 \leq 9\delta \). So it remains to prove that \( \gamma_1 \in N(\gamma_2, 9\delta) \).

First note that any segment of \( \gamma_1 \) which lies outside the \( 4l \)-neighborhood of the collapsing locus is a geodesic. For, if not the geodesic must intersect the collapsing locus. But the geodesic cannot intersect the collapsing locus, since by Lemma 5.6 the geodesic remains within distance \( 3l + 2 \) of \( \gamma_1 \) and \( l > 2 \).

Given a convex set \( W \in \mathbb{H}^+ \) and a geodesic \( \gamma \), the distance from \( \gamma(t) \) to \( W \) is a strictly convex function, so for any \( r \) there are at most two points on \( \gamma \) lying at distance exactly \( r \) from \( W \).

Now, let \( C_1, \ldots, C_k \) denote the components of the collapsing locus that lie within \( 6\delta \) of \( \gamma_2 \). Note that \( k < \infty \) as \( K \) is much bigger than \( \delta \). Suppose that \( \tilde{\gamma}_1 = p_1 \cdot p_2 \cdots p_j \) (where \( j \in \{2k - 1, 2k, 2k + 1\} \)) is so that the endpoints of each \( p_i \) lie at distance exactly \( 12\delta \) from \( C_j \). Thus, the \( p_i \) alternate between paths which lie outside the \( 12\delta \)-neighborhood of the collapsing locus, and within \( 12\delta \) of some \( C_j \). Suppose that \( p_i \) lies within \( 12\delta \) of \( C_j \), and let \( q \) be a \( \mathcal{C}(\Gamma) \)-geodesic with the same endpoints as \( p_i \). We claim that \( p_i \) is in the \( 2\delta \)-neighborhood of \( q \). We prove this claim in the next paragraph.

First note that if \( q \) does not intersect \( C_j \) then \( q = p_i \) and the claim follows. Now suppose \( q \) intersects \( C_j \). Let \( \tilde{p}_i \) be the lift of \( p_i \) to \( \mathbb{H}^+ \) and denote the end points of \( \tilde{p}_i \) by \( \tilde{x} \) and \( \tilde{y} \). Let \( \tilde{x}' \) and \( \tilde{y}' \) be the closest point projections of \( \tilde{x} \) and \( \tilde{y} \) to \( q_{\Gamma}^{-1}(C_j) \), respectively. By
hyperbolicity of $\mathbb{H}^*$. $p_i$ is contained in the $2\delta H_\mathbb{H}$-neighborhood of $[\bar{x}, \bar{x}] \cup [\bar{x}', \bar{y}] \cup [\bar{y}, \bar{y}]$. Since $q_t$ is distance decreasing, we see that $p_i$ is within $2\delta H_\mathbb{H}$ of $q_t([\bar{x}, \bar{x}] \cup [\bar{x}', \bar{y}] \cup [\bar{y}, \bar{y}])$.

Since $q$ intersects $C_\mathcal{I}$, we can write $q$ as the $q_{t_\mathcal{I}}$ image of $[\bar{x}, \bar{x}] \cup [\bar{x}_1, \bar{y}_1] \cup [\bar{y}_1, \bar{y}]$. Consider the geodesic triangle with vertices $\bar{x}, \bar{x}_1, \bar{x}'$. Since $q_{t_\mathcal{I}}^{-1}(C_\mathcal{I})$ is convex and $\bar{x}'$ is the point in $q_{t_\mathcal{I}}^{-1}(C_\mathcal{I})$ closest to $\bar{x}$, the Gromov product of $[\bar{x}, \bar{x}']$ and $[\bar{x}, \bar{x}_1]$ at $\bar{x}$ is at most $\delta H_\mathbb{H}$. Hence $[\bar{x}, \bar{x}' \cup [\bar{x}', \bar{x}_1]$ is within $2\delta H_\mathbb{H}$ of $[\bar{x}, \bar{x}_1]$. Similarly, $[\bar{y}, \bar{y}] \cup [\bar{y}', \bar{y}_1]$ is within $2\delta H_\mathbb{H}$ of $[\bar{y}_1, \bar{y}]$. Therefore, as $q_{t_\mathcal{I}}$ is distance decreasing, we have $q_{t_\mathcal{I}}([\bar{x}, \bar{x}'] \cup [\bar{x}', \bar{x}_1] \cup [\bar{x}_1, \bar{y}_1] \cup [\bar{y}_1, \bar{y}] \cup [\bar{y}, \bar{y}])$ is within $\delta$ of $q_{t_\mathcal{I}}([\bar{x}, \bar{x}_1] \cup [\bar{x}_1, \bar{y}_1] \cup [\bar{y}_1, \bar{y}])$. This implies that $p_i$ is within $2\delta$ of $q$.

By the claim, if we replace each of the $p_i$, which lies in the $12\delta$–neighborhood of some $C_\mathcal{I}$ by a $\mathcal{C}(\Gamma)$–geodesic, then we obtain a concatenation of $\mathcal{C}(\Gamma)$–geodesics whose $2\delta$–neighborhood contains $\gamma_i$.

Using Lemmas 5.12 and 5.13 it is straightforward to check that the Gromov product at each point of concatenation is at most $2\delta$. It is also clear that each of the paths in this concatenation, except possibly the first and last, have length greater than $12\delta$ (since $K \geq 40\delta > 24\delta$). Therefore, by Lemma 4.9 (with $l = 2\delta$), the Hausdorff distance between this concatenation and $\gamma_\mathcal{I}$ is at most $7\delta$. This together with the last statement of the previous paragraph implies that $\gamma_i \in N(\gamma_\mathcal{I}, 9\delta)$ and the proposition follows.

The following is similar to [15] Lemma 4.8.

**Lemma 5.15.** There exists $D_1$ satisfying the following: Let $x, y \in \mathcal{C}(\Gamma)$ and let $\gamma_\mathcal{I} = [x, y]$ be $\mathcal{C}(\Gamma)$–geodesic and $\gamma_\mathcal{I}$ be its lift. Let $\bar{x}, \bar{y} \in \mathbb{H}^*$ be lifts of $x, y$, respectively. Let $\gamma_\mathcal{I} = [\bar{x}, \bar{y}]$ and $\gamma_\mathcal{I} = q_{t_\mathcal{I}}(\gamma_\mathcal{I})$. If precisely one of $\{\gamma_\mathcal{I}, \gamma_\mathcal{I}\}$ penetrates a component $U$ of the collapsed set then $d_{H_\mathbb{H}}(e(\gamma_\mathcal{I}, U), o(\gamma_\mathcal{I}, U)) \leq D_1$.

**Proof.** Suppose $\gamma_\mathcal{I}$ penetrates $U$ and $\gamma_\mathcal{I}$ does not. Let $\bar{z}$ be the point on $\gamma_\mathcal{I}$ between $\bar{x}$ and $e(\gamma_\mathcal{I}, U)$ and $10l$ away from $U$, or $\bar{z} = \bar{x}$ if no such point exists. Let $\bar{w}$ be the point on $\bar{y}$ between $\bar{y}$ and $o(\gamma_\mathcal{I}, U)$ and $10l$ away from $U$, or $\bar{w} = \bar{y}$ if no such point exists. Let $z = q_{t_\mathcal{I}}(\bar{z})$ and $w = q_{t_\mathcal{I}}(\bar{w})$. By Lemma 5.6 there exist $z', w'$ on $\gamma_\mathcal{I}$ with $d(z, z')$, $d(w, w') \leq 5l$. Let $\bar{z}'$ and $\bar{w}'$ be the lifts of $z'$ and $w'$, respectively. By construction $[\bar{z}', \bar{z}]$ and $[\bar{w}', \bar{w}]$ are disjoint from the collapsing locus. Hence $d_{H_\mathbb{H}}([\bar{z}', \bar{z}])$, $d_{H_\mathbb{H}}([\bar{w}, \bar{w}']) \leq 5l$. Since these paths are in $\mathbb{H}^*$, $[\bar{z}', \bar{w}] \subseteq N_{5l}([\bar{z}, \bar{w}])$. Thus, $q_{t_\mathcal{I}}([\bar{z}', \bar{w}]) \subseteq N_{5l}([\bar{z}, \bar{w}])$, since $[\bar{z}, \bar{w}] = q_{t_\mathcal{I}}([\bar{z}, \bar{w}])$. By Lemma 5.6 the subsegment $\beta$ of $\gamma_\mathcal{I}$ between $z'$ and $w'$ lies in $N_{5l+1}(q_{t_\mathcal{I}}([\bar{z}', \bar{w}']))$, so $\beta \subseteq N_{5l+1}([\bar{z}, \bar{w}]) \subseteq N_{5l+1}(q_{t_\mathcal{I}}(U))$. Components of the collapsing locus are $40l$–separated, so $\beta$ does not penetrate any component of the collapsed set except $U$. On the other hand, since $\beta$ is a subsegment of $\gamma_\mathcal{I}$, it does not penetrate $U$. Therefore the lift of $\beta$ is $[\bar{z}', \bar{w}]$ and $[\bar{z}', \bar{w}]$ is disjoint from $U$, so the projection of $[\bar{z}', \bar{w}]$ onto $U$ is bounded by a uniform constant $D$. Since $[\bar{z}, \bar{z}']$ and $[\bar{w}, \bar{w}']$ do not penetrate any component of the collapsing locus, the projections of $[\bar{z}', \bar{z}]$ and $[\bar{w}', \bar{w}]$ onto $U$ are also bounded by $D$. By choice of $\bar{z}$ and $\bar{w}$, $d_{H_\mathbb{H}}(\pi_0(\bar{z}), e(\gamma_\mathcal{I}, U))$ and $d_{H_\mathbb{H}}(\pi_0(\bar{w}), o(\gamma_\mathcal{I}, U))$ are bounded by a uniform constant $C_2$. The above distance bounds imply $d_{H_\mathbb{H}}(e(\gamma_\mathcal{I}, U), e(\gamma_\mathcal{I}, U)) \leq 3C_1 + 2C_2$.

The case where $\gamma_\mathcal{I}$ penetrates $U$ and $\gamma_\mathcal{I}$ does not is similar, the only difference being that instead of using Lemma 5.6 we use Proposition 5.14. As a result, the constants are different but they are again uniform constants.

The following is similar to [15] Lemma 4.9.
Lemma 5.16. There exists $D_2$ satisfying: Let $x, y, x', y'$ and $x_1$ be as in Lemma 5.15. If both $x'$ and $x_1$ penetrate a component $U$ of the collapsed set then $d_{\mathbb{H}^1}(e(x_1), e(x_1')) \leq D_2$. The same is true for exit points.

Proof. Let $\tilde{z}$ be the point on $\gamma_2'$ between $\tilde{x}$ and $e(x_1, U)$ and $10l$ away from $U$, or $\tilde{z} = \tilde{x}$ if no such point exists. So we have

$$d_{\mathbb{H}^1}(\tilde{z}, U) \leq 10l$$

Note that the distance between $\tilde{z}$ and $e(x_1, U)$ cannot be much bigger than $10l$ otherwise the path $[\tilde{z}, e(x_1, U)]$ intersects $U$ in a non-degenerate segment, contradicting the definition of $e(x_1, U)$. Hence there is a number $B$ depending only on $l$ such that

$$d_{\mathbb{H}^1}(\tilde{z}, e(x_1, U)) \leq B.$$ 

By Lemma 5.4, there exists $w$ on $\gamma_1$, so that $d(w, z) \leq 5l$, where $z = q_{15}(\tilde{z})$. When $\tilde{z} = \tilde{x}$, we let $w = x = z$. Definition of $\tilde{z}$ implies that we either have $d(z, q(U)) = 10l$ or $z = x$. Hence $[w, z] \subset N(U, 15l) - N(U, 5l)$ unless $z = x = w$. Since $20l < K$, there is no other connected component of the collapsing locus in the $20l$-neighborhood of $q_{15}(U)$, so the geodesics $[z, w]$ does not intersect any component of the collapsing locus. Let $\tilde{w} = q_{15}^{-1}(w)$. The geodesic $[z, w]$ lifts to the geodesic $[\tilde{z}, \tilde{w}]$. Hence

$$d_{\mathbb{H}^1}(\tilde{z}, \tilde{w}) \leq 5l.$$ 

By (3) and (1), we have $d_{\mathbb{H}^1}(\tilde{w}, U) \leq 15l$. This implies that, in the same way as (1) implying (2), there is a number $B'$ depending only on $l$ such that

$$d_{\mathbb{H}^1}(\tilde{w}, e(x_1, U)) \leq B'.$$

Therefore, by (1), (3) and (4), the distance between $d_{\mathbb{H}^1}(e(x_1), e(x_1')) \leq B + B' + 5l$.

The proof of the statement for the exit points is similar. 

Lemma 5.17. There exist uniform constants $B_1$ and $B_2$ satisfying: Let $g \in \Gamma \setminus \{1\}$ and $x \in \mathcal{C}(\Gamma)$. If $g$ is parabolic, let $\gamma$ be the component of the collapsing locus fixed by $g$. Otherwise let $\gamma$ be the $q_{15}$-image of the minimal invariant set of $g$ in $\mathbb{H}^n$. Then $2d(x, \gamma) \leq d(x, gx) - B_1$ and if $g$ is elliptic or parabolic then $d(x, gx) - d(x, \gamma) \leq B_2$.

Proof. Let $\tilde{x} = q_{15}^{-1}(x)$. Let $U_1 = q_{15}^{-1}(\gamma)$ and $\tilde{y} = \pi_{U_1}(\tilde{x})$. Let $U_2$ be a horoball (or ball or tube) with the same center (or axis) as $U_1$ so $g$ moves points on $\partial U_2$ by $5\delta + l_g$, where $l_g$ is the translation length of $g$ on $\mathbb{H}^n$. Let $\tilde{w} = \pi_{U_2}(\tilde{x})$. We claim that $d(\tilde{w}, U_1) \leq D_1$ for some absolute constant $D_1$. The claim is trivial if $U_2 \subset U_1$, so suppose $U_1 \subset U_2$.

First suppose that $\gamma$ is a component of the collapsing locus. In this case by Proposition 5.1(3), $d(\tilde{y}, \tilde{y}') \geq \max\{D, l_g\}$, so the length of $[\tilde{w}, \tilde{y}]$ is bounded above by an absolute constant.

Now suppose $\gamma$ is not a component of the collapsing locus. In this case, $g$ is either loxodromic and $l_g$ is bounded from below by an absolute positive constant, or $g$ is elliptic and the angle of rotation is bounded below by an absolute positive constant, so $U_2$ has uniformly bounded size (tube radius).

The claim gives an absolute upper bound on $d(w, \gamma)$, where $w = q_{15}^{-1}(\tilde{w})$. Since $\tilde{w}$ is the point in $U_2$ closest to $\tilde{x}$ and $[\tilde{w}, g \cdot \tilde{w}]$ is contained in $U_2$, the Gromov product of $[\tilde{x}, \tilde{w}]$ and $[\tilde{w}, g \cdot \tilde{w}]$ at $\tilde{w}$ is at most $\delta_{\tilde{w}}$. Similarly, the Gromov product of $[g \cdot \tilde{x}, g \cdot \tilde{w}]$ and $[\tilde{w}, g \cdot \tilde{w}]$ at $g \cdot \tilde{w}$ is also at most $\delta_{\tilde{w}}$. Hence, by hyperbolicity of $\mathbb{H}^n$, the Hausdorff distance between $[\tilde{x}, g \cdot \tilde{x}]$ and $[\tilde{w}, g \cdot \tilde{w}] \cup [\tilde{w}, g \cdot \tilde{w}]$ at $g \cdot \tilde{w}$ is at most $4\delta_{\tilde{w}}$. By Lemma 5.4, $[x, g \cdot x]$ and $[g \cdot x, g \cdot w]$ are in the $\delta$-neighborhoods of $q_{15}(\tilde{x}, g \cdot \tilde{x})$, $q_{15}([\tilde{x}, \tilde{w}])$ and $q_{15}([g \cdot x, g \cdot w])$, respectively. Hence the Hausdorff distance between $[x, g \cdot x]$ and $[x, w] \cup [w, g \cdot w] \cup [g \cdot w, g \cdot w]$.
Lemma 5.18. Suppose \( g \in \Gamma \) acts loxodromically on \( \mathbb{H}^n \) with axis \( \gamma \), and let \( \gamma = q(\gamma) \). If \( \gamma \) comes within \( 5\delta \) of the collapsing locus but is not contained in the collapsing locus then \( g \) moves every point of \( \gamma \) by at least \( 30\delta \).

Proof. By assumption, there exists \( x \) on \( \gamma \) within \( 5\delta \) of a component \( C \) of the collapsing locus and \( \gamma \) is not contained in \( C \). Hence \( C \) and \( g \cdot C \) are distinct components of the collapsing locus, which are separated by \( K = 40\delta \). Hence \( x \) and \( gx \) are separated by a distance at least \( 40\delta - 2(5\delta) = 30\delta \). So the translation length of \( g \) on \( \gamma \) is at least \( 30\delta \). Let \( y \) be a point on \( \gamma \). Suppose \( d(y, gy) < 30\delta \). Since the translation length of \( g \) on \( \gamma \) is at least \( 30\delta \), \( [y, gy] \) intersects some connected component \( C' \) of the collapsing locus and we have \( d(y, C') + d(C', gy) \leq 30\delta \). Hence \( d(gy, gC') + d(C', gy) \leq 30\delta \), so \( d(C', gC') \leq 30\delta \). Thus \( gC' = C' \), so \( C' \) is a line and \( \gamma = C' \). This is a contradiction. \( \square \)

6. Sequences which are not \( \mathcal{C} \)-divergent

Throughout this section make Standing Assumption 4.18. We apply the results in the previous section to the vertex groups of the geometric decomposition of a group in \( \mathcal{M}_{\text{Gen}}^\pi \). If \( \Gamma_v \) is such a vertex group, and it is hyperbolic, then it admits a complete, finite-volume hyperbolic structure, unique by Mostow–Prasad Rigidity, and this exhibits \( \Gamma_v \) as a Kleinian group. We identify all conjugates of \( \Gamma_v \) in the ambient 3-manifold group with the same Kleinian group. If \( \Gamma_v \) is an SFH-type vertex group, let \( \overline{\Gamma_v} \) be the quotient (hyperbolic) 2–orbifold group, and fix a (complete, finite-volume) hyperbolic structure on the orbifold, witnessing \( \overline{\Gamma_v} \) as a Fuchsian group. There are many such hyperbolic structures, but any suffices for our purposes. The quotients of all conjugates of \( \Gamma_v \) in the ambient 3-manifold group (modulo their centers) are identified with the same Fuchsian group.

The purpose of this section is to prove Theorem 4.19 required for the proof of Theorem 4.19. The proof of Theorem 4.19 is contingent on the technical result Theorem 6.7 proved in Appendix A.

Recall from Assumption 4.18 that \( A_i \) is the \( \omega \)-approximation to the good relative generating set \( A \) of the vertex group \( V \) of the refined GGD \( G' \) of \( L \). By Definition 4.10 \( \omega \)-almost surely there are vertices \( v_i \) of \( T_i \) so \( A_i \) fixes \( v_i \). In the above paragraph we identified \( \Gamma_{v_i} \), the stabilizer of \( v_i \) in \( \Gamma_v \) (respectively \( \Gamma_{v_i} \), the quotient of \( \Gamma_v \) by its center) with a specific Kleinian group (resp., a Fuchsian group). Therefore, there is a collapsed space associated to \( \Gamma_{v_i} \), which we denote by \( \mathcal{C}_i \) in both cases.

Definition 6.1 (\( \mathcal{C} \)-divergent). Let \( \Gamma_{v_i} \) be the vertex stabilizer of \( v_i \) in \( \Gamma_v \) and let \( \mathcal{C} \mathcal{C}_i = \mathcal{C}_i(\Gamma_{v_i}) \) (respectively, \( \mathcal{C}_i = \mathcal{C}(\Gamma_{v_i}) \)) be the associated collapsed space. We set

\[
\|\phi\|_{\mathcal{C}, \mathcal{C}_i} = \inf_{\phi \in \mathcal{C} \mathcal{C}_i} \max_{\bar{x} \in A_i} (s \cdot x, x),
\]

which is defined \( \omega \)-almost surely. The sequence \( (\phi_i) \) is \( \mathcal{C} \)-divergent with respect to \( G \) if \( \lim_{i \to \infty} \|\phi_i\|_{\mathcal{C}, \mathcal{C}_i} = \infty \) for some vertex \( v \) of \( G' \).

We recall the statement of Theorem 4.19.
Theorem 4.19. Let \( L \) be an \( \mathcal{M}_{\text{Gen}} \)-limit group defined by a non-\( \mathcal{T} \)-divergent sequence \((\phi_i)\). If there is a GGD \( G \) of \( L \) with respect to which \((\phi_i)\) is not \( \mathcal{C} \)-divergent then all stably parabolic subgroups of \( L \) are finitely generated.

Definition 6.2. A graph of groups is star-like if its underlying graph is a bipartite graph where one color of vertices consists of a single vertex.

Recall the definition of the stable center of \( V \) from Definition 4.9. When \((\phi_i)\) is not \( \mathcal{C} \)-divergent with respect to \( G \), we obtain the following crucial information about the vertex group \( V \) of \( G \):

Proposition 6.3. Suppose \((\phi_i)\) is not \( \mathcal{C} \)-divergent with respect to \( G \). Then \( V \) admits a splitting \( D \) so:

1. \( D \) is star-like;
2. \( Z^\omega(V) \) is contained in each edge group of \( D \);
3. The non-central vertices of \( D \) are in bijection with the edges in \( G' \) adjacent to \( V \). This bijection induces isomorphisms of associated groups. In particular the vertex groups associated to the non-central vertices are abelian, and the edge groups of \( D \) are all abelian; and
4. For each the edge group \( H \) of \( D \), \( H/Z^\omega(V) \) has rank at most two.

Furthermore, if \( V \) is of hyperbolic type, and a stably parabolic subgroup \( P \) of \( V \) is not conjugate into an edge group of \( G \) adjacent to \( V \), then \( P \) is finitely generated.

Proof. Notice that if \( G_v \) has hyperbolic type then it is a vertex group in \( G' \) as well as in \( G \) (by the construction of \( G' \) in Definition 4.13), and therefore there is nothing to prove in this case. Thus, we may assume that \( G_v \) is of SFH-type type, and in particular the “furthermore” part of the statement has been proved.

Let \( R(v) \) be relative 4-Linnell decomposition of \( G_v \) (see Corollary B.4). Let \( V_1, \ldots, V_t \) be the vertex groups of \( R_v \). Notice that since the splitting \( G' \) is obtained from \( G \) by refinement, the vertex groups \( V_i \) are vertex groups of \( G' \) (recall the definition of the Linnell refinement \( G' \) from Definition 4.13). Therefore, Proposition 6.3 applies to the groups \( V_1, \ldots, V_t \). Let \( D_1, \ldots, D_t \) be the corresponding splittings given by applying Proposition 6.3.
to $V_1, \ldots, V_t$, respectively. Refine $\mathbb{R}(v)$ by $D_1, \ldots, D_t$. Notice that for each pair $(w, e)$ of vertex and edge in $\mathbb{R}(v)$, where $V_i$ is the vertex group of $w$, we obtain an edge in $D_i$ and a leaf vertex group whose associated group is the local group of $e$ in $\mathbb{R}(v)$. Thus, each edge in $\mathbb{R}(v)$ becomes a path of three edges, and we collapse all such paths. The resulting splitting is the required splitting $\mathbb{D}(v)$. \hfill \square

The following is the key definition required for our proof of Theorem 4.19.

**Definition 6.5.** A graph of groups $\mathbb{E}$ is edge-twisted if:

The underlying graph of $\mathbb{E}$ is bipartite with colors $A$ and $B$. Type $A$ vertices have valence 2, and abelian vertex groups (thus the edge groups of $\mathbb{E}$ are also abelian). Let $W$ be a Type $A$ vertex group of $\mathbb{E}$ and let $E_1$ and $E_2$ be the images in $W$ of the adjacent edge groups. There are subgroups $K_j \leq E_j$ (for $j = 1, 2$) so that

1. $K_1 \cap K_2 = \{1\}$; and
2. For $j = 1, 2$, the group $E_j/K_j$ is finitely generated.

We now define a new splitting $\mathbb{K}$ of $L$. The underlying graph of $\mathbb{K}$ is the barycentric subdivision of the underlying graph $\Lambda(G)$ of $G$, so the vertices correspond to cells in $\Lambda(G)$, and edges correspond to pairs $(w, e)$, where $w$ is a vertex, $e$ is an edge, and $e$ is adjacent to $w$. The vertex group of a vertex corresponding to a vertex of $\Lambda(G)$ is the central vertex group of the splitting of the corresponding vertex group of $G$ arising from Corollary 6.4. The vertex group of a vertex corresponding to an edge of $\Lambda(G)$ is the corresponding edge group of $G$. For an edge corresponding to the pair $(w, e)$, there is a corresponding edge group in the splitting $\mathbb{D}(w)$ of $G_w$ coming from Corollary 6.4 and this is the edge group in $\mathbb{K}$. The edge-to-vertex maps naturally come from the splittings $\mathbb{D}(w)$.

**Proposition 6.6.** The graph of groups $\mathbb{K}$ is an edge-twisted splitting, where the Type $A$ vertices correspond to edges in $\Lambda(G)$ and the Type $B$ correspond to vertices in $\Lambda(G)$.

**Proof.** It is clear from the construction of $\mathbb{K}$ that Type $A$ vertices are valence 2 and that their vertex groups are abelian (since they are the edge groups of $G$). By Lemma 3.22 and Definition 4.10(2) $Z^0(W_1) \cap Z^0(W_2) = \{1\}$ for adjacent vertex groups $W_1, W_2$ of $G$. For the $K_i$ from Definition 6.5, we choose the stable center of the vertex groups of $G$. With this choice, the first condition of Definition 6.5 is satisfied. The last condition of Definition 6.5 is satisfied because of the Property 4 from Corollary 6.4. \hfill \square

The following Theorem 6.7 is proved in Appendix A.

**Theorem 6.7.** Let $\mathbb{E}$ be a finite edge-twisted graph of groups so that $\pi_1(\mathbb{E})$ is finitely generated. The Type $B$ vertex groups of $\mathbb{E}$ are finitely generated.

We finish this subsection by proving Theorem 4.19 assuming Proposition 6.3 and Theorem 6.7.

**Proof of Theorem 4.19.** Recall that $L$ is an $\mathcal{M}_{\text{Gen}}^\mathcal{F}$–limit group defined by a sequence $(\phi_i)$ which is not $\mathcal{F}$–divergent, and that there is a GGD $G$ of $L$ with respect to which $(\phi_i)$ is not $\mathcal{C}$–divergent. We are required to prove that all stably parabolic subgroups of $L$ are finitely generated.

Let $\mathbb{K}$ be the splitting defined in the paragraph before Proposition 6.6. By Proposition 6.6 and Theorem 6.7, Type $B$ vertex groups of $\mathbb{K}$ are finitely generated. Thus, each Type $B$ vertex group which is a subgroup of an SFH-type vertex groups of $G$ is an $\mathcal{M}_{\text{SFH}}^\mathcal{F}$–limit group. By Corollary 3.18, all abelian subgroups of these vertex groups are finitely generated, so edge groups adjacent to these vertex groups are finitely generated. The other
edge groups are adjacent to a hyperbolic type vertex group of $K$ and are finitely generated by the third property of Corollary 6.4, so all vertex groups of $K$ are finitely generated also. Hence all vertex groups of $G$ are finitely generated. A stably parabolic subgroup of $L$ is finitely generated by Corollary 6.4 if it is in an SFH-type vertex group or by Corollary 6.4 if it is in a hyperbolic type vertex group.

6.2. The cut point tree. The rest of the section is devoted to the proof of Proposition 6.3.

Recall that the assumption of Proposition 6.3 is that the defining sequence $(\phi_i)$ for $L$ is not $\mathcal{C}$-divergent with respect to $G$. We consider how $V$ maps into the collapsed spaces associated to the vertex groups of the geometric decompositions of the $\Gamma_i$ (with the chosen Kleinian/Fuchsian structures). Recall that in Standing Assumption 4.18 we fixed a finite good relative generating set $A_o V$, and an $\omega$–approximation $(A_i)$ to $A$.

By the choice of $A_i$, $\omega$–almost surely $A_i \subset \Gamma_{v_i}$. Recall that we fixed $\mathcal{C}_i$ to be either $\mathcal{C}(\Gamma_{v_i})$ or $\mathcal{C}(\Gamma_{v_i})$, depending on whether $v_i$ is of hyperbolic or SFH-type type. For $\omega$–almost every $i$, fix $o_i \in \mathcal{C}_i$ satisfying

$$\max_{g \in A_i} \{d_i(g \cdot o_i, o_i)\} \leq \|\phi_i\|_{\mathcal{C}_i} + \frac{1}{i}.$$ 

By Definition 4.13 $V$ is generated by $A$ and the adjacent edge groups. Let $S_v$ be a (possibly infinite) generating set for $V$ consisting of $A$ and elements from the adjacent edges groups. Choose a lift $S_v \subset G$ of $S_v$.

Let $(\mathcal{C}_\infty, o)$ be the $\omega$–limit of the collapsed spaces $(\mathcal{C}_i, o_i)$. We next show that $(\phi_i)$ induces an isometric action of $V$ on $(\mathcal{C}_\infty, o)$. This construction is standard in the case where $V$ is finitely generated. Since we do not (yet) know that $V$ is finitely generated, we need the following.

**Lemma 6.8.** Let $(X_i, d_i)$ be a sequence of metric spaces with basepoints $o_i \in X_i$. Suppose that a group $G$ acts by isometries on each $X_i$ and $G$ is generated by a (possibly infinite) subset $S$ such that for all $s \in S$,

$$\lim^\omega d_i(o_i, s \cdot o_i) < \infty.$$ 

Then $G$ acts by isometries on $\lim^\omega(X_i, o_i)$.

**Proof.** By definition, a point in $\lim^\omega(X_i, o_i)$ is the equivalence class of a sequence $(x_i)$ so that (i) each $x_i \in X_i$; and (ii) $\lim^\omega d_i(o_i, x_i) < \infty$. Two such sequences $(x_i)$ and $(x'_i)$ are equivalent if $\lim^\omega d_i(x_i, x'_i) = 0$.

Let $g, h \in G$ be such that $\lim^\omega d_i(o_i, g \cdot o_i) < \infty$ and $\lim^\omega d_i(o_i, h \cdot o_i) < \infty$. Then

$$\lim^\omega d_i(o_i, gh \cdot o_i) \leq \lim^\omega d_i(o_i, g \cdot o_i) + \lim^\omega d_i(g \cdot o_i, gh \cdot o_i)$$

$$= \lim^\omega d_i(o_i, g \cdot o_i) + \lim^\omega d_i(o_i, h \cdot o_i) < \infty.$$ 

Since $S$ generates $G$, it follows by induction on word length that for all $g \in G$, $\lim^\omega d_i(o_i, g \cdot o_i) < \infty$.

Now let $g \in G$ and let $(x_i)$ be a sequence with $x_i \in X_i$ and such that $\lim^\omega d_i(o_i, x_i) < \infty$. Then

$$\lim^\omega d_i(o_i, g \cdot x_i) \leq \lim^\omega d_i(o_i, g \cdot o_i) + \lim^\omega d_i(g \cdot o_i, g \cdot x_i)$$

$$= \lim^\omega d_i(o_i, g \cdot o_i) + \lim^\omega d_i(o_i, x_i) < \infty.$$ 

That is, the sequence $(g \cdot x_i)$ defines a point of $\lim^\omega(X_i, o_i)$. If $(x'_i)$ is a sequence which is equivalent to $(x_i)$, then

$$\lim^\omega d_i(g \cdot x_i, g \cdot x'_i) = \lim^\omega d_i(x_i, x'_i) = 0.$$
So the sequences \((g \cdot x_i)\) and \((g \cdot x'_i)\) are equivalent. Thus, \((x_i) \rightarrow (g \cdot x_i)\) is a well-defined map on \(\text{lim}^\omega \Gamma(g, o_i)\). These maps clearly define an action of \(G\) on \(\text{lim}^\omega \Gamma(g, o_i)\). Finally, we observe that the metric on \(\text{lim}^\omega \Gamma(g, o_i)\) is defined by \(d((x_i), (y_i)) = \text{lim}^\omega d_i(x_i, y_i)\). It then follows that

\[
d((g \cdot x_i), (g \cdot y_i)) = \text{lim}^\omega d_i(g \cdot x_i, g \cdot y_i) = \text{lim}^\omega d_i(x_i, y_i) = d((x_i), (y_i)).
\]

Hence, \(G\) acts on \(\text{lim}^\omega \Gamma(g, o_i)\) by isometries. 

\[\square\]

**Lemma 6.9.** \(\{\phi_i\}\) induces an isometric action of \(G\) on \(\mathcal{G}_{\infty, o}\).

**Proof.** We apply Lemma 6.8 to the generating set \(A \cup E(V)\), where \(E(V)\) are the edge groups of \(\mathbb{G}\) adjacent to \(V\).

Let \(o = \{o_i\} \in \mathcal{G}_{\infty, o}\) be the basepoint of \(\mathcal{G}_{\infty, o}\). Since \(\|\phi_i\|_{E(V)}\) has finite \(\omega\)-limit, for all \(g \in A_i\) we have

\[
(\dagger) \quad \text{lim}^\omega d_i(o_i, g \cdot o_i) < \infty.
\]

Let \(H \in E(V)\), and suppose \(h \in H\) and \(g \in A \cap H\) (note that such a \(g\) exists by Definition 4.15). Let \((h_i)\) and \((g_i)\) be \(\omega\)-approximations of \(h\) and \(g\), respectively. Then \(\omega\)-almost surely \(g_i\) and \(h_i\) are parabolic fixing the same point in \(\mathcal{G}_i\). By Lemma 5.17 there exists some constant \(D\) so that \(\omega\)-almost surely \([d_i(o_i, h_i o_i) - d_i(o_i, g_i o_i)] < D\). Therefore, by \((\dagger)\)

\[
\text{lim}^\omega d_i(o_i, h_i o_i) < \infty,
\]

as required. 

\[\square\]

Recall that we fixed \(S_v\) and \(\tilde{S}_v\) above Lemma 6.8. Let \(\text{Cay}(V, S_v)\) be the (possibly locally infinite) Cayley graph of \(V\) with respect to \(S_v\). For each \(s \in S_v\), choose a geodesic \([o, s \cdot o] \subset \mathcal{G}_{\infty, o}\) and fix a homeomorphism between \([1, s] \subset \text{Cay}(V, S_v)\) and \([o, s \cdot o]\). Let \(f_{\infty} : \text{Cay}(V, S_v) \rightarrow \mathcal{G}_{\infty, o}\) be the \(V\)-equivariant map extending the above homeomorphisms. So \(f_{\infty}\) maps \([g, g s] \subset \text{Cay}(V, S_v)\) to \(g [o, s \cdot o]\) for any \(g \in V\).

Let \(\mathcal{L}\) be the set of limits of points in \(\mathcal{G}_{\infty, o}\) in the collapsing locus which are quotients of horoballs. Let \(Q\) be the quotient of \(\text{Cay}(V, S_v)\) obtained by identifying points that are mapped by \(f_{\infty}\) to the same point in \(\mathcal{L}\). The \(V\)-action on \(\text{Cay}(V, S_v)\) descends to an action of \(V\) on \(Q\). The map \(f_{\infty}\) induces a continuous map from \(Q\) to \(\mathcal{G}_{\infty, o}\), which we still denote by \(f_{\infty}\). Note that \(f_{\infty} : Q \rightarrow \mathcal{G}_{\infty, o}\) is \(V\)-equivariant and injective on \(f_{\infty, 1}(\mathcal{L})\). Points in \(f_{\infty, 1}(\mathcal{L}) \subset Q\) are called marked points of \(Q\).

**Lemma 6.10.** The map \(f_{\infty} : Q \rightarrow \mathcal{G}_{\infty, o}\) is well-defined, continuous and \(V\)-equivariant. The map \(f_{\infty}\) restricts to an injective map from the marked points of \(Q\) to the set of limits of parabolic collapsed points in \(\mathcal{G}_{\infty, o}\). Moreover, for any \(p \in \mathcal{G}_{\infty, o}\), which is a limit of parabolic collapsed points, \(f_{\infty, 1}(p)\) is either a single marked point in \(Q\) or is empty.

Recall that \(\mathcal{H}\) is the set of stably parabolic subgroups of \(V\). Suppose \(H \in \mathcal{H}\). If \(g \in H\) and \((g_i)\) is an \(\omega\)-approximation of \(g\), then \(\omega\)-almost surely \(g_i\) is contained in \(\Gamma_{V_{\omega}}\) and fixes a parabolic collapsed point \(\bar{\xi}_i(g)\) in \(\mathcal{G}(\Gamma_{V_{\omega}})\). Note that the point \(\bar{\xi}_i(g)\) \(\omega\)-almost surely depends only on \(g\) and not on the choices of \((g_i)\). In fact, so long \(g \notin Z^\omega(V)\), by Definition 4.5 the point \(\bar{\xi}_i(g)\) depends only on \(H\) (different elements of \(H \setminus Z^\omega(V)\) must \(\omega\)-almost surely fix the same edge in the geometric decomposition of \(\Gamma_{V_{\omega}}\), and so have same parabolic fixed point).

**Lemma 6.11.** The sequence \(\{\bar{\xi}_i(g)\}\) defines a point \(\bar{\xi}(g)\) in \(\mathcal{G}_{\infty, o}\), fixed by \(g\).
Proof. By Lemma 5.14 and the fact that the translation length of \( \phi_i \) with respect to \( S_i \) and \( \omega_i \) is bounded, the distance between \( \bar{\xi}_i(g) \) and \( \omega_i \) is \( \omega \)-almost surely bounded independent of \( i \). The lemma follows. \( \square \)

**Definition 6.12.** An element \( g \in H \) has Property \( \mathcal{B} \) if for some \( x \in Q \) there is a path \( \tau \) in \( Q \) between \( x \) and \( g \cdot x \) such that \( \bar{\xi}(g) \notin f_\omega(\tau) \).

**Lemma 6.13.** Let \( H \in \mathcal{H}_i \) be so that every \( h \in H \) has Property \( \mathcal{B} \). Then \( H/H \cap Z^0(V) \) is a free abelian group of rank at most two.

**Proof.** Let \( g \in H \) and fix an \( \omega \)-approximation \( (g_i) \) of \( g \). By the definition of Property \( \mathcal{B} \), there is a path \( \tau \) in \( Q \) from \( x \) to \( g \cdot x \) such that \( \bar{\xi}(g) \notin f_\omega(\tau) \). Suppose \( f_\omega(x) \) is represented by \( \{x_i \in C_i \} \). Then \( \tau \) is the limit of a sequence of paths \( \{\tau_i\} \), where \( \tau_i \) is a path in \( C_i \) connecting \( x_i \) and \( g_i \cdot x_i \), and \( \bar{\xi}_i(g) \) is not on \( \tau_i \), \( \omega \)–almost surely. This path \( \tau_i \) can be lifted to a path \( \tilde{\tau}_i \) in \( \hat{H}^* \) connecting \( \tilde{x}_i \) and \( g_i \cdot \tilde{x}_i \), where \( \tilde{x}_i \) is the pre-image of \( x_i \) under the natural projection from \( \hat{H}^* \) to \( \hat{C}(\Gamma_{\gamma}) \). Let \( U_i \subset \hat{H}^* \) be the maximal horoball in \( \hat{H}^* \) that is collapsed to make the collapsed point \( \tilde{\xi}_i \). Let \( \tilde{y}_i \) be the closest point projection of \( \tilde{x}_i \) to \( U_i \). Then \( g_i \cdot \tilde{y}_i \) is the closest point projection of \( g_i \cdot \tilde{x}_i \) to \( U_i \). Moreover, the path \( \tilde{\tau}_i \) projects to a path in the boundary of \( U_i \) connecting \( \tilde{y}_i \) and \( g_i \cdot \tilde{y}_i \). The length of \( \tau_i \) is \( \omega \)–almost surely bounded. Therefore, \( \omega \)–almost surely there is a bound on the number of components of the pre-image of the collapsing locus which \( \tilde{\tau}_i \) intersects. Then \( \omega \)–almost surely \( \tilde{\tau}_i \) does not intersect \( U_i \). If \( C \) is a component of the pre-image of the collapsing locus other than \( U_i \) then by Lemma 5.5 the projection of \( C \) to \( U_i \) has uniformly bounded diameter. Thus there exists \( N \) so that \( \omega \)–almost surely the projection of \( \tilde{\tau}_i \) to \( U_i \) has length at most \( N \). Hence, the translation length of \( g_i \) on \( U_i \) is \( \omega \)–almost surely bounded.

Let \( P_i \) be the subgroup of \( \Gamma_{\gamma_i} \) fixing \( \bar{\xi}_i(g) \). Then \( P_i \) is isomorphic to either \( \mathbb{Z} \) or to \( \mathbb{Z}^2 \). Choose a minimal cardinality generating set (basis) for \( P_i \) so that the sum of their translation lengths on the corresponding horoball/horocycle is as small as possible. This choice gives a specified isomorphism \( \psi_i \) from \( P_i \) to \( \mathbb{Z} \) or \( \mathbb{Z}^2 \).

First suppose that \( P_i \cong \mathbb{Z}^2 \). Using the above isomorphism, \( \omega \)–almost surely we have \( \psi_i(\phi_i(g_i)) \in \mathbb{Z}^2 \). The upper bound on the translation length of \( g_i \) on \( U_i \) imply that there exist \( a, b \in \mathbb{Z} \), independent of \( i \), so that \( \psi_i(\phi_i(g_i)) = (a, b) \) \( \omega \)–almost surely. Note that \( (a, b) \) does not depend on the choice of \( \omega \)–approximation \( (g_i) \).

By Proposition 5.1(3), as long as \( g \notin Z^0(V) \) there is a lower bound, independent of \( i \), on the translation length of \( g_i \) on \( U_i \). Defining \( \Phi(g) = (a, b) \) for each \( g \in H \) induces a homomorphism \( H \to \mathbb{Z}^2 \), and the lower bound on translation length for \( g \notin Z^0(V) \) implies that this homomorphism has kernel \( Z^0(V) \).

The case where \( P_i \cong \mathbb{Z} \) is entirely similar, completing the proof. \( \square \)

**Lemma 6.14.** Let \( H \in \mathcal{H}_i \). Then either \( H/H \cap Z^0(V) \) has rank at most two or \( H \) stabilizes a marked point in \( Q \).

**Proof.** Let \( g \in H \) and \( \bar{\xi}(g) \) be the point in \( \mathcal{C}_{\omega_i} \) fixed by \( g \) as in Lemma 6.11. Then \( H \) stabilizes \( \bar{\xi}(g) \). If \( \bar{\xi}(g) \) is the image of a marked point under the map \( f_\omega \), then by Lemma 6.10 \( H \) is stabilizes a marked point of \( Q \). Now suppose \( \bar{\xi}(g) \) is not the image of any marked point under the map \( f_\omega \). Then \( \bar{\xi}(g) \) is not in the image of \( f_\omega \) by the last assertion in Lemma 6.10. Hence \( \bar{\xi}(g) \notin f_\omega(\tau) \) for any path in \( Q \), and so every element of \( H \) has Property \( \mathcal{B} \). Therefore, by Lemma 6.13 \( H/H \cap Z^0(V) \) has rank at most two. \( \square \)

By Lemma 6.14 to further understand the elements of \( \mathcal{H}_i \), we study the case when they are stabilizers of marked points in \( Q \).
Lemma 6.15. Let $H \in \mathcal{H}_v$. Suppose $H$ stabilizes a marked point $z \in Q$ which is not a cut point. Then every element of $H$ has Property $\mathcal{B}$ and hence $H/H \cap Z^0(V)$ has rank at most two.

Proof. Let $g \in H$ and $x \in Q \setminus \{z\}$. Since $z$ is not a cut point, there is a path $\tau$ in $Q \setminus \{z\}$ from $x$ to $g \cdot x$. By Lemma 6.10, $f_\omega(z) \not\in f_\omega(\tau)$. Note that $\xi = f_\omega(z)$ is the fixed point of $H$ in $\mathcal{C}_m$, given by Lemma 6.11. Hence each $g \in H$ has Property $\mathcal{B}$, and so by Lemma 6.13 $H/H \cap Z^0(V)$ has rank at most two, as required.

We now consider stably parabolic subgroups which fix marked points that are not cut points. To that end, build a new tree.

Let $M \subset Q$ be the collection of all marked points that are cut points. Consider the \textit{cut-point tree} $T_{cut}$ associated to $(Q,M)$. The vertices of $T_{cut}$ are:

1. Points in $M$; and
2. The maximum connected subsets of $Q$ not separated by an element of $M$, Vertices of the first type are cut points, and vertices of the second type are \textit{blocks}. Edges of $T_{cut}$ correspond to the inclusion of a cut point in a block. Clearly, $V$ acts on $T_{cut}$.

Lemma 6.16. Let $z \in M$ and let $B$ be a block containing $z$. Let $e = \{z \in B\}$ be the corresponding edge of $T_{cut}$, and let $H$ be the stabilizer in $V$ of $e$. Then $H/Z^0(V)$ has rank at most two.

Proof. Denote the stabilizer of $z$ by $H'$. By Lemma 6.10, $H'$ fixes $\xi = f_\omega(z) \in \mathcal{C}_m$, a limit point of parabolic points $\xi_n \in \mathcal{C}(\Gamma_v)$. In particular, $H' \in \mathcal{H}_v$, hence $H \in \mathcal{H}_v$. By Lemma 6.13, it suffices to show that $H$ consists entirely of elements satisfying Property $\mathcal{B}$.

Let $g \in H$ and $x \in B$. Since $g$ fixes $B$ as a set, we know $g \cdot x \in B$. By definition of blocks, $B \setminus \{z\}$ is connected. Hence there is a path $\tau$ connecting $x$ and $g \cdot x$ and $\tau$ does not intersect $z$. Therefore, $g$ has Property $\mathcal{B}$. 

We are now ready to prove Proposition 6.3. As we showed above, this is enough to complete the proof of Theorem 4.19 in Appendix A.

Proof of Proposition 6.3. Let $\mathbb{D}_0$ be the splitting of $V$ dual to the action of $V$ on $T_{cut}$. Since $Z^0(V)$ acts trivially on $T_{cut}$, $Z^0(V)$ is contained in all the edge groups of $\mathbb{D}_0$. The underlying graph of $\mathbb{D}_0$ is bipartite, with one type of vertex corresponding to cut point vertices in $T_{cut}$ and the other type corresponding to block vertices. For each cut point vertex of $\mathbb{D}_0$, fold all the adjacent edges together to one edge, and denote the resulting splitting by $\mathbb{D}'$. Now, each cut point vertex of $\mathbb{D}'$ has one adjacent edge and $\mathbb{D}'$ is still connected, so the underlying graph of $\mathbb{D}'$ is a star.

Collapse the cut point vertices of $\mathbb{D}'$ whose stabilizers are not conjugate into edge groups of $\mathbb{G}'$ adjacent to $V$. Add vertices to the resulting graph of groups corresponding to edge groups adjacent to $V$ not corresponding to vertices of $\mathbb{D}'$ (with identical edge and vertex groups). It follows that the resulting graph of groups $\mathbb{D}$ satisfies (3). Since $\mathbb{D}'$ satisfies (1) and (2), so does $\mathbb{D}$. Statement (4) follows from Lemmas 6.14, 6.15 and 6.16.

It remains to prove the “Furthermore” assertion of the proposition. To that end, suppose $V$ is of hyperbolic type and that $P \in \mathcal{H}_v$ is not conjugate into an adjacent edge group. Then $Z^0(V) = \{1\}$. If $P$ does not stabilize a cut point vertex of $T_{cut}$ then $P$ is finitely generated by Lemma 6.14 and Lemma 6.15. If $P$ does stabilize a cut point vertex of $T_{cut}$ then $P$ is in a valence-one vertex group of $\mathbb{D}'$ whose adjacent edge group is finitely generated by Lemma 6.16. Refining $\mathbb{G}'$ by $\mathbb{D}'$ yields a splitting $\mathbb{K}'$ of $L$. Since $P$ is not contained in the conjugate of any edge group of $\mathbb{G}'$, the corresponding vertex group in $\mathbb{K}'$ containing $P$ is
still valence-one with the same adjacent edge group, so it is finitely generated since $L$ is.
In all cases, $P$ is finitely generated and the proof of Proposition 6.3 is complete.$\square$

INTERLUDE: SUMMARY OF WHAT REMAINS

The goal of the remainder of paper (other than Appendix A) is to prove Theorem 4.20. If our defining sequence is not $C$–divergent with respect to some GGD, then we may take $k = 0$ in Theorem 4.20 and take the GGDs of $S_1 = S_0$ to be the same. Therefore, there is only anything to prove in case our defining sequence $(\phi_i)$ of $L$ is $C$–divergent with respect to some GGD $G$ of $L$.

Our approach to proving Theorem 4.20 will be familiar to the experts. In Section 7 we build limiting $R$–trees from rescaled actions on the collapsed spaces, which allows us to use the Rips machine to find splittings of $L$. In Section 8 we recall and prove the required properties of JSJ decompositions and modular automorphisms. In Section 9 we undertake the version of Sela’s “shortening argument” in this context, and finally prove Theorem 4.20.

As we stated before, the construction of the collapsed spaces was made to make these arguments work in as “standard” a way as possible. Thus, while there are many technical difficulties in the next three sections, not much in these sections will surprise the experts. An exception to this are the “squared Nielsen transformations” in Section 8 which are a new kind of move required to shorten our homomorphisms. These arise from our analysis of edge groups arising as limits of infinite dihedral subgroups of Fuchsian groups coming from an arc joining two cone points of order 2, and how to shorten in the presence of such edge groups.

7. LIMITS AND $R$–TREES

Throughout this section make Standing Assumption 4.18. Suppose further that $(\phi_i)$ is $C$–divergent with respect to $G$ (see Definition 6.1) and that the vertex $v$ of $G^\omega$ associated to $V$ is one for which $\lim_{\omega} |\phi_i|_{G^\omega} = \infty$. The space $C_{\infty,v}$ is defined in Definition 7.4 below.

The goal of this section is to prove the following theorem, which is the only result from this section needed in future sections. We remind the reader that $Z^\omega(V)$ is the stable center of $V$ (see Definition 4.9) and $\overline{V} = V/Z^\omega(V)$.

**Theorem 7.1.**

1. $C_{\infty,v}$ is an $R$–tree equipped with a nontrivial isometric $V$–action. The stable center $Z^\omega(V)$ acts trivially on this tree, and so there is an induced action of $\overline{V}$.
2. Let $H \in \mathcal{H}_V$. There exists $x_H \in C_{\infty,v}$, fixed by $H$. Each nontrivial element of $H \sim Z^\omega(V)$ fixes only the point $x_H$.
3. If $g \in V$ and $(g_i)$ is an $\omega$–approximation to $g$ so $\omega$–almost surely $g_i$ is (nontrivial and) elliptic then $g$ fixes a unique point in $C_{\infty,v}$.
4. The $\overline{V}$–action on $C_{\infty,v}$ has trivial tripod stabilizers.
5. The $\overline{V}$–action on $C_{\infty,v}$ has abelian segment stabilizers.

The above theorem, together with the Rips machine, implies that the $\overline{V}$–action on $C_{\infty,v}$ admits a graph of actions decomposition with simplicial, axial and Seifert type vertex actions, which induces an abelian splitting of $\overline{V}$. The splitting of $V$ (rel $\mathcal{H}_V$) induced by this splitting also has abelian edge groups.\[1\]

\[1\]See [19] for the definitions of graphs of actions, simplicial vertex actions, axial vertex actions and vertex actions of Seifert type, as well as a statement of the Rips machine (Theorem 5.1 in that paper).
The remainder of this section is dedicated to the proof of Theorem 7.1. Recall the definitions from Section 4.10 and 4.15.

Let \( A \) be the good relative generating set for \( V \) from Assumption 4.18 and let \( (A_i) \) be the chosen \( \omega \)-approximation of \( A \). Let \( v_i \in T_i \) be the sequence of vertices associated to \( v \). Let \( \Gamma_v \subset \Gamma_i \) be the stabilizer of \( v_i \). Then since \( \mathcal{G} \) is a GGD of \( L \) with respect to \( (\phi_i) \) \( \omega \)-almost surely \( A_i \subset \Gamma_v \). As at the beginning of Section 6, let \( C_i = \mathcal{C}(\Gamma_v) \) (respectively \( C_i = \mathcal{C}(\Gamma_v) \)) be the collapsed space associated to \( \Gamma_v \) (depending on whether \( v_i \) is of hyperbolic type or SFH-type type. For \( \omega \)-almost every \( i \), fix \( o_i \in C_i \) so that

\[
\max_{g \in A_i} \{d_i(g, o_i, o_i)\} \leq \|\phi\|_{\mathcal{E}_v} + \frac{1}{i}.
\]

**Definition 7.2.** Let \( \mathcal{E}_{m,v} \) be the ultra-limit (with respect to \( \omega \)) of the sequence \( \left( \mathcal{C}_i, \frac{1}{\|\phi\|_{\mathcal{E}_v}}d_i, o_i \right) \).

**Proposition 7.3** (Limiting \( \mathbb{R} \)-tree). The space \( \mathcal{E}_{m,v} \) is a (pointed) \( \mathbb{R} \)-tree equipped with a non-trivial isometric \( V \)-action.

**Proof.** Most parts of this proposition are standard facts in the theory of ultra-limits and \( \mathbb{R} \)-trees (see, for example, \([17, \S 4]\) and the references therein). The only new thing here is to show that \( \{\phi_i\} \) induces an isometric action of \( V \) on \( \mathcal{E}_{m,v} \). This does not follow from the standard theory since \( A \) may not generate \( V \). However, \( V \) is generated by \( A \cup E(V) \), where \( E(V) \) are the edge groups of \( \mathcal{G}' \) adjacent to \( V \).

Let \( o = \{o_i\} \in \mathcal{E}_{m,v} \) be the basepoint of \( \mathcal{E}_{m,v} \). By the definition of \( \|\phi\|_{\mathcal{E}_v} \), for all \( g \in A_{v_i} \)

\[
\lim_{o} \frac{1}{\|\phi\|_{\mathcal{E}_v}}d_i(o_i, g \cdot o_i) < \infty.
\]

Let \( H \in E(V) \), and suppose \( h \in H \) and \( g \in A \cap H \) (note that such a \( g \) exists by Definition 4.15). Let \( (h_i) \) and \( (g_i) \) be \( \omega \)-approximations of \( h \) and \( g \), respectively. Then \( \omega \)-almost surely \( g_i \) and \( h_i \) are parabolic fixing the same point in \( \mathcal{C}_i \). By Lemma 5.17 there exists some constant \( D \) so that \( \omega \)-almost surely \( |d_i(o_i, h_i o_i) - d_i(o_i, g_i o_i)| < D \). Therefore, by (†)

\[
\lim_{o} \frac{1}{\|\phi\|_{\mathcal{E}_v}}d_i(o_i, h_i \cdot o_i) < \infty,
\]

as required.

**Definition 7.4.** Let \( \mathcal{E}_{m,v} \) be the minimal \( V \)-invariant subtree of \( \mathcal{E}_{m,v} \).

Theorem 7.1(1) follows immediately from Proposition 7.3.

Recall that \( V = V / Z^0(V) \), where \( Z^0(V) \) is given by Definition 4.9. Each element of \( Z^0(V) \) \( \omega \)-almost surely acts trivially on \( \mathcal{C}_i \). Thus we have the following result which completes Item (1) from Theorem 7.1.

**Corollary 7.5.** The \( V \)-action descends to a non-trivial isometric \( \overline{V} \)-action on \( \mathcal{E}_{m,v}^{0} \), with minimal \( \overline{V} \)-invariant subtree \( \mathcal{E}_{m,v} \).

Theorem 7.1(2) and (3) follow from the next lemma.

**Lemma 7.6.** Let \( g \in V \) and let \( (g_i) \) be an \( \omega \)-approximation of \( g \). Suppose that \( \omega \)-almost surely \( g_i \) is parabolic (or nontrivial elliptic). Then \( g \) fixes a unique point in \( \mathcal{E}_{m,v} \). If \( g^1, g^2 \in V \), \( (g_i^1) \) and \( (g_i^2) \) are \( \omega \)-approximations of \( g^1 \) and \( g^2 \), respectively, and \( \omega \)-almost surely \( g^1_i \) and \( g^2_i \) are parabolic with the same fixed point then \( g^1 \) and \( g^2 \) fix the same point in \( \mathcal{E}_{m,v} \).
Proof. Let \( C_t \) be the component of the collapsing locus fixed by \( g_t \) (or the projection in \( \mathcal{C}_t \) of the fixed point of the elliptic \( g_t \) in \( \mathbb{H}^* \)). Consider the geodesic triangle with vertices \( o_t, g_t \cdot o_t, \) and \( C_t \). Let \( x_t \) be the point on \([o_t, C_t] \) with \( d_t(x_t, C_t) = (o_t, g_t \cdot o_t)_{C_t} \). Note that \( 2d_t(x_t, o_t) \leq d_t(o_t, g_t \cdot o_t) + 4 \delta \). Hence

\[
\lim_{t \to \infty} \frac{1}{\|\phi_t\|_{\mathcal{C}_t}} d_t(o_t, x_t) < \infty
\]

Hence \( \{x_t\} \) defines a point in \( \mathcal{C}_t^{0} \). By hyperbolicity of \( \mathcal{C}_t \), \( d_t(x_t, g_t \cdot x_t) \leq \delta \). Hence \( g \) fixes \( x \). Therefore \( g \) fixes a point in \( \mathcal{C}_t \). We now show that \( g \) fixes exactly one point in \( \mathcal{C}_t \). Suppose \( g \) fixes \( y \in \mathcal{C}_t \) and \( (y_t) \in \mathcal{C}_t \) represents \( y \). By Lemma 5.17, \( 2d_t(y_t, C_t) - d_t(y_t, g_t \cdot y_t) \) is \( \omega \)-almost surely bounded independent of \( t \). Since \( g \) fixes \( y \),

\[
\lim_{t \to \infty} d_t(y_t, g_t \cdot y_t) / \|\phi_t\|_{\mathcal{C}_t} = 0
\]

so \( \lim_{t \to \infty} d_t(y_t, C_t) / \|\phi_t\|_{\mathcal{C}_t} = 0 \), and \( y = [C_t] \in \mathcal{C}_t \).

The second assertion follows immediately from the above argument and the assumption that \( \omega \)-almost surely \( g \) and \( g_t \) fix the same point in the collapsed space \( \mathcal{C}_t \).

Corollary 7.7. Suppose \( g \in V \setminus Z^0(V) \) fixes a non-degenerate arc in \( \mathcal{C}_t \). If \( (g_t) \) is an \( \omega \)-approximation of \( g \) then \( \omega \)-almost surely \( g \) is loxodromic.

Theorem 7.1(5) follows from Lemma 7.6 and the following lemma.

Lemma 7.8. Let \( g \in V \) and \((g_t)\) be an \( \omega \)-approximation of \( g \). Suppose that \( \omega \)-almost surely \( g_t \) is loxodromic. The fixed point set of \( g \) in \( \mathcal{C}_t \) is either empty or a single geodesic. In case it is a single geodesic, it is a limit of images in the collapsed space of geodesic axes in \( \mathbb{H}^* \).

Proof. Suppose the fixed point set of \( g \) is not empty and let \( x = [\{x_t\}] \in \mathcal{C}_t \) be a point fixed by \( g \). Let \( \mathbb{H}^* \) be the domain of the map \( q_t = q_{t,x} \). Denote the axis of \( g_t \) in \( \mathbb{H}^* \) by \( \mathcal{H}_t \). Let \( \gamma_t = q_t(\mathcal{H}_t) \). Let \( \tilde{x}_t \in \mathbb{H}^* \) be a point in \( q_t^{-1}(x_t) \). By Lemma 5.17 there is a constant \( D \) so that \( \omega \)-almost surely \( d_t(x_t, g_t \cdot x_t) \geq 2d_t(x_t, \gamma_t) - D \). Since \( x_t \) is fixed by \( g_t \), we have \( \lim_{t \to \infty} d_t(x_t, g_t \cdot x_t) / \|\phi_t\|_{\mathcal{C}_t} = 0 \), and so \( \lim_{t \to \infty} d_t(x_t, \gamma_t) / \|\phi_t\|_{\mathcal{C}_t} = 0 \). Therefore \( x_t \) is on the limit of \( \{\gamma_t\} \), which is a geodesic in \( \mathcal{C}_t \).

The goal of the rest of this section is to prove Theorem 7.1(5), which completes the proof of Theorem 7.1. To that end, fix a non-trivial segment \( I = [a, b] \) in \( \mathcal{C}_t \), and suppose that \( \mathcal{H}_I, \mathcal{H} \in V \) stabilize \( I \). Fix lifts \( g, h \in V \) of \( \mathcal{H}_I \) and \((g_t) \) and \((h_t) \) be \( \omega \)-approximations of \( g \) and \( h \), respectively. We know that \( \omega \)-almost surely \( g_t \) and \( h_t \) both lie in the same \( \Gamma_{\nu,i} \), and our goal is to show that \( \omega \)-almost surely \( [g_t, h_t] \in Z(\Gamma_{\nu,i}) \). Recall that if \( \Gamma_{\nu,i} \) is of hyperbolic type then \( Z(\Gamma_{\nu,i}) = \{1\} \), and in either case \( Z(\Gamma_{\nu,i}) \) is the kernel of the action of \( \Gamma_{\nu,i} \) on the associated hyperbolic space \( \mathbb{H}^* \).

Observe that \([g_t, h_t] \) is an \( \omega \)-approximation of \([g, h] \). We may suppose \([g_t, h_t] \) is \( \omega \)-almost surely non-trivial, or else there is nothing to prove. Because \([g, h] \) stabilizes the non-trivial segment \( I \) in \( \mathcal{C}_t \), by Corollary 7.7 \( \omega \)-almost surely \([g_t, h_t] \) corresponds to a loxodromic isometry of \( \mathbb{H}^* \). Let \( \mathcal{H}_x \) be the invariant geodesic for \([g_t, h_t] \) in \( \mathbb{H}^* \) and let \( \gamma_t = q_t(\mathcal{H}_x) \). By Lemma 7.8 \( I \) is a subsegment of the limit of \( \gamma_t \). As a result, there are \( a_t, h_t \) on \( \gamma_t \) such that \( a = [a_t] \) and \( b = [b_t] \). Let \( I_t \) be a geodesic segment between \( a_t \) and \( b_t \).

That segment stabilizers are abelian is proved in the following three lemmas.

Lemma 7.9. Suppose that \( \omega \)-almost surely \( \gamma_t \) is not entirely contained in the collapsing locus but that it comes within \( 5 \delta \) of the collapsing locus. Then \( \omega \)-almost surely \([g_t, h_t] \in Z(\Gamma_{\nu,i}) \).
Proof. Since \( a_i \) and \( b_i \) are on \( \gamma \), by Lemma 5.6 and the choice of \( \delta \), \( I_i \subset N_\delta(\gamma) \). Thus for \( w_i \in I_i \) satisfying \( d_i(a_i, w_i), d_i(b_i, w_i) \geq \frac{1}{16} d_i(a_i, b_i) \), we have \( d_i([g_i, h_i], w_i, w_i) \leq 16\delta \) (See 40 Lemma 5.7). Hence \([g_i, h_i]\) moves points in the middle part of \( \gamma \) by less than \( 30\delta \), contradicting Lemma 5.18. Hence \([g_i, h_i]\) acts trivially on \( \mathbb{H}^* \) \( \omega \)-almost surely, so \( \omega \)-almost surely \([g_i, h_i] \in Z(\gamma_{\Gamma_i}) \), as required.

Lemma 7.10. Suppose that \( \omega \)-almost surely \( \gamma \) does not come within \( 5\delta \) of the collapsing locus. Then \( \omega \)-almost surely \([g_i, h_i] \in Z(\Gamma_{\gamma_i}) \).

Proof. We consider an \( \omega \)-large set of indices \( i \) so that \( g_i \) and \( h_i \) correspond to loxodromic isometries in \( \mathbb{H}^* \), as above, and implicitly concentrate only on such indices.

Suppose \([a_i, g_i \cdot a_i]\) penetrates a component \( W \) of the collapsing locus. We claim the distance between the entry and exit points of \([a_i, g_i \cdot a_i]\) in \( W \) is bounded independent of \( i \). To that end, let \( \tilde{a}_i = q_i^{-1}(a_i) \) and \( \tilde{b}_i = q_i^{-1}(b_i) \). If \( q_i([\tilde{a}_i, g_i \cdot \tilde{a}_i]) \) does not penetrate \( W \), the claim follows from Lemma 5.15. Suppose then that \( q_i([\tilde{a}_i, g_i \cdot \tilde{a}_i]) \) does penetrate \( W \) and let \( a'_i \) and \( u'_i \) be the points of entry and exit of \( q_i([\tilde{a}_i, g_i \cdot \tilde{a}_i]) \) into \( W \), respectively. By Lemma 5.16, it suffices to show \( d_{\mathbb{H}^*}(a'_i, u'_i) \) is bounded independent of \( i \). Let \( \pi_{\tilde{W}} \) be the closest-point projection onto \( \tilde{W} = q_i^{-1}(W) \). Note that \( d_{\mathbb{H}^*}(a'_i, u'_i) \leq d_{\mathbb{H}^*}(\pi_{\tilde{W}}(\tilde{a}_i), \pi_{\tilde{W}}(g_i \cdot \tilde{a}_i)) \). Suppose \( q_i([\tilde{b}_i, g_i \cdot \tilde{b}_i]) \) penetrates \( W \). Then by Lemma 5.6 \([b_i, g_i \cdot b_i]\) \( \cap N_{\delta}(W) \neq \emptyset \). Since \( d(a_i, b_i) \) is much bigger than both \( d(b_i, g_i \cdot b_i) \) and \( d(a_i, g_i \cdot a_i) \) for large \( i \), by \( \delta \)-hyperbolicity, \([a_i, b_i]\) \( \cap N_{3\delta}(W) \neq \emptyset \). By Lemma 5.6 every point on \([a_i, b_i]\) is at most \( \delta \) away from some point on \( \gamma \). As a result, \( \gamma \cap N_{3\delta}(W) \neq \emptyset \), contradicting the assumptions of the lemma. Hence \( q_i([\tilde{b}_i, g_i \cdot \tilde{b}_i]) \) does not penetrate \( W \). It follows that the projections of \([\tilde{a}_i, b_i]\), \([\tilde{b}_i, g_i \cdot \tilde{b}_i]\) and \( g_i \cdot [\tilde{a}_i, b_i] \) to \( W \) have bounded length as none of them intersect \( \tilde{W} \). The above bounds together show that \( d_{\mathbb{H}^*}(a'_i, u'_i) \) is \( \omega \)-almost surely bounded independent of \( i \).

The number of components penetrated by \([a_i, g_i \cdot a_i]\) is at most \( d(a_i, g_i \cdot a_i) \). Hence \( d_{\mathbb{H}^*}(a_i, g_i \cdot \tilde{a}_i) \) is bounded by \( d(a_i, g_i \cdot a_i) \), where \( D \) is independent of \( i \). As a result, the geodesic rectangle \([\tilde{a}_i, b_i, g_i \cdot \tilde{a}_i, g_i \cdot b_i]\) is arbitrarily thin in the middle of \([\tilde{a}_i, b_i]\). Similarly, the same is true for the geodesic rectangle \([\tilde{a}_i, b_i, h_i \cdot \tilde{a}_i, h_i \cdot b_i]\). Hence \( \omega \)-almost surely, \([g_i, h_i]\) moves a point in the middle of \([\tilde{a}_i, b_i]\) by arbitrarily small amount. If \([g_i, h_i] \notin Z(\Gamma_{\gamma_i}) \), then by the definition of collapsed spaces \( \gamma \) is in the collapsing locus \( \omega \)-almost surely. This contradicts the assumption of the lemma. Hence \([g_i, h_i] \in Z(\Gamma_{\gamma_i}) \), as required.

Lemma 7.11. If \( \omega \)-almost surely \( \gamma \) is contained in the \( 5\delta \)-neighborhood of the collapsing locus then \( \omega \)-almost surely \([g_i, h_i] \in Z(\Gamma_{\gamma_i}) \).

Proof. We first show that \( \gamma \) is entirely contained in the collapsing locus of \( \gamma_i \). By the choice of \( K \), if \( \gamma \) is entirely contained in the \( 5\delta \)-neighborhood of the collapsing locus, then it is entirely contained in the \( 5\delta \)-neighborhood of some connected component \( C \) of the collapsing locus. So \( q_i^{-1}(C) \) is a tube around some geodesic \( \beta \) in \( \mathbb{H}^* \). The \( q_i \)-preimage of the \( 5\delta \) neighborhood of \( C \) is the \( \mathcal{D} \) neighborhood of \( \beta \) for some constant \( \mathcal{D} \). The only bi-infinite geodesic contained in this neighborhood is \( \beta \), so \( \gamma = \beta \). In particular, \( \gamma \) is contained in \( q_i^{-1}(C) \), so \( \gamma \) is contained in \( C \).

As above, choose subsegments of \( \gamma \) denoted \([a_i, b_i]_i \). Let \( C_i \) be the component of the collapsing locus containing \( \gamma \). Suppose \( g_i \cdot C_i \neq C_i \). Then \( d_{\mathbb{H}^*}(C_i, g_i \cdot C_i) \geq K \) for \( \omega \)-almost all \( i \), \([a_i, b_i]\) is much longer than both \([a_i, g_i \cdot a_i]\) and \([b_i, g_i \cdot b_i]\). By \( \delta \)-hyperbolicity of \( \gamma_i \), the middle part of \([a_i, b_i]\) is in a \( 2\delta \)-neighborhood of \( g[a_i, b_i] \). But \( K \geq 2\delta \), so \( C_i \) intersects the \( K \)-neighborhood of \( g_i \cdot C_i \), a contradiction. Thus, \( g_i \cdot C_i = C_i \), and similarly
\( h_i \cdot C_i = C_i \). Hence \( g_i \) and \( h_i \) act as translations along \( \gamma_i \), and \( \omega \)-almost surely \( [g_i, h_i] = 1 \), proving the lemma. \( \square \)

The previous three lemmas finish the proof of Theorem 7.1(5), and hence of Theorem 7.1.

8. JSJ-decompositions and Modular Automorphisms

We continue to make Standing Assumption 4.18 and use the notation from there.

8.1. Summary. In the last section we showed in the \( \mathcal{C} \)-divergent case that if \( v \) is the vertex associated to \( V \) and \( \lim_{\omega} \| \phi_i \|_{\mathcal{C}, v} = \infty \) then \( V \) admits a nontrivial action on an \( \mathbb{R} \)-tree. The results of Section 7 together with [19, Theorem 5.1] imply that this action induces a graph of groups decomposition of \( V \). Associated to any such decomposition is a group of modular automorphisms of \( V \) (see Definition 8.10).

To apply the shortening argument in the next section (see Theorem 9.12), we need to use modular automorphisms of \( V \) without a priori knowing the particular splitting of \( V \). For this we need a JSJ–decomposition of \( V \), which is a graph of groups decomposition which “sees” all possible modular automorphisms of \( V \) (see Theorem 8.12).

Recall \( Z^\omega(V) \) is the stable center of \( V \) and \( \overline{V} = V / Z^\omega(V) \). Let \( \pi_v : V \rightarrow \overline{V} \) denote the quotient map. If \( V \) is a hyperbolic-type vertex group, \( Z^\omega(V) = \{1\} \) and \( \overline{V} = V \) in this case. We first construct a JSJ–decomposition for \( \overline{V} \) which naturally induces a decomposition of \( V \), see Definition 8.7.

The JSJ–decomposition of \( \overline{V} \) we use was essentially constructed by Guirardel-Levitt in [22]. If \( \overline{V} \) is finitely generated and \( K \)-CSA, the existence of the decomposition we need follows directly from [22, Theorem 9.14]. However, in our case \( \overline{V} \) is only finitely generated relative to stably parabolic subgroups and \( \overline{V} \) is not necessarily \( K \)-CSA since it can have arbitrarily large finite subgroups. Adapting the proof of [22, Theorem 9.14] to the relatively finitely generated case is straightforward. Also, in the next section we show that \( \overline{V} \) is weakly \( K \)-CSA (see Definition 8.2 below), which means that it shares enough of the properties of \( K \)-CSA groups that the proof of [22, Theorem 9.14] holds with only minor changes. We explain these changes in Appendix B see Theorem B.5. Finally, in Section 8.3 we define modular automorphisms, prove Theorem 8.12 and show how modular automorphisms of \( \overline{V} \) are induced by modular automorphisms of \( V \) (see Lemma 8.13).

8.2. Weakly \( K \)-CSA groups.

**Definition 8.1.** A group is \( K \)-virtually abelian if it has an abelian subgroup of index at most \( K \).

**Definition 8.2.** Fix \( K \geq 1 \). A group is weakly \( K \)-CSA if (i) any element \( g \) of order greater than \( K \) is contained in a unique maximal virtually abelian subgroup \( M(g) \), so that \( M(g) \) is \( K \)-virtually abelian and equal to its normalizer, and (ii) every two infinite, virtually abelian subgroups \( \langle A, B \rangle \) not virtually abelian satisfy \( |A \cap B| \leq K \).

Note that if a group \( H \) is weakly \( K \)-CSA and all finite subgroups of \( H \) have order \( \leq K \), then \( H \) is \( K \)-CSA.

**Lemma 8.3.** Let \( A \) be a virtually abelian subgroup of \( \overline{V} \).

1. If \( A \) is not abelian, then \( A \) has an abelian subgroup \( A^+ \) of index two and an element \( \tau \in A \setminus A^+ \) so that for all \( g \in A^+ \), \( \tau^{-1}g\tau = g^{-1} \).
(2) If $B$ is another virtually abelian subgroup of $\mathbf{V}$, then either $|A \cap B| \leq 2$ or $\langle A, B \rangle$ is $2$–virtually abelian.

Proof. We first assume that $A$ and $B$ are finitely generated. Let $A_i$ and $B_i$ be subgroups of $G_i$ generated by $\omega$–approximations of fixed finite generating sets of $A$ and $B$, respectively. There exist vertex groups $V_i$ in the geometric decomposition of $G_i$ so that $\omega$–almost surely $A_i, B_i \leq V_i$. Let $\overline{A_i}$ and $\overline{B_i}$ be the images of $A_i$ and $B_i$ in $V_i/Z(V_i)$, respectively. Then $\overline{A_i}$ and $\overline{B_i}$ are either both subgroups of fundamental groups of orientable finite volume hyperbolic 3–manifolds or both subgroups of fundamental groups of orientable hyperbolic 2–orbifolds. Also, $A$ and $B$ are limit groups over the families $\{\overline{A_i}\}$ and $\{\overline{B_i}\}$, respectively.

Assume $|A| \geq 3$ and $|B| \geq 3$, otherwise the result is trivial. Since $A$ and $B$ are finitely generated and virtually abelian, $\overline{A_i}$ and $\overline{B_i}$ are $\omega$–almost surely virtually abelian. Hence each $\overline{A_i}$ is contained in a unique maximal virtually abelian subgroup $M_i$ which is either $\mathbb{Z}$, $\mathbb{Z}^2$, $D_m$, or finite cyclic. Let $M_i^+$ be an abelian subgroup of $M_i$ of index at most 2. Then we can define $A^+$ to be the set of $g \in A$ such that if $(g_i)$ is an $\omega$–approximation of $g$, then the image of $g_i$ in $V_i/Z(V_i)$ belongs to $M_i^+$ $\omega$–almost surely. Suppose $g, h \in A$ with $\omega$–approximations $(g_i)$ and $(h_i)$ respectively. Let $\overline{g_i}, \overline{h_i}$ be the images of $g_i$ and $h_i$ in $V_i/Z(V_i)$. If $g, h \in A^+$, then $\overline{g_i}, \overline{h_i} \in M_i^+$ $\omega$–almost surely. Then $\overline{g_i}$ and $\overline{h_i}$ $\omega$–almost surely commute, and hence $g$ and $h$ commute. Similarly, if $g, h \not\in A^+$, then $\overline{g_i}^{-1} \overline{h_i} \in M_i^+$ $\omega$–almost surely and hence $g^{-1}h \in A^+$. Thus, $A^+$ is an abelian subgroup of $A$ of index at most 2. Finally, if $g \in A^+$ and $h \not\in A^+$, then we must have $M_i \cong D_m$ $\omega$–almost surely. It follows that $\overline{g_i}^{-1} \overline{g_i} \omega$–almost surely, hence $h^{-1}gh = g^{-1}$.

Similarly, each $\overline{B_i}$ is contained in a unique maximal virtually abelian subgroup $N_i$. Note that in the fundamental group of an orientable finite volume hyperbolic 3–manifolds, any two maximal virtually abelian subgroups are equal or disjoint. In the fundamental group of an orientable hyperbolic 2–orbifold, two maximal virtually abelian subgroups are either equal, disjoint, or their intersection contains a single non-trivial element of order 2. In either case, we have $M_i = N_i$ or $|M_i \cap N_i| \leq 2$ whenever $M_i$ and $N_i$ both exist. If the first case happens $\omega$–almost surely, then the same argument as above gives that $\langle A, B \rangle$ is $2$–virtually abelian. If the second happens $\omega$–almost surely, $|A \cap B| \leq 2$.

Now suppose $A$ and $B$ are not finitely generated. We can apply the above construction to any finitely generated subgroup of $A$ and to produce the sequence $M_i$. Note that the sequence $M_i$ is independent of the finitely generated subgroup chosen, since any two finitely generated subgroups are both contained in a larger finitely generated subgroup. In particular, $A^+$ can defined as before and the same arguments show that (1) holds in this case.

Now if $|A \cap B| > 2$, then the above argument shows that all finitely generated subgroups of $\langle A, B \rangle$ are $2$–virtually abelian. A diagonal argument shows that this implies that $\langle A, B \rangle$ are $2$–virtually abelian, see [22, Lemma 9.6].

\begin{lemma}
Let $g \in \mathbf{V}$ be an element of order at least 3. Let

$M(g) = \{ h \in \mathbf{V} \mid \langle g, h \rangle \text{ is virtually abelian} \}$.

Then $M(g)$ is $2$–virtually abelian and $M(g)$ is the unique, maximal virtually abelian subgroup of $\mathbf{V}$ containing $g$.
\end{lemma}

Proof. By induction and Lemma 8.3, all finitely generated subgroups of $M(g)$ are $2$–virtually abelian so $M(g)$ is $2$–virtually abelian by [22, Lemma 9.6]. The maximality and uniqueness of $M(g)$ follow. \qed

\begin{thebibliography}{99}

Lemma 8.5. Let $g \in V$ have order at least 3. Then $M(g)$ is equal to its normalizer.

Proof. Suppose $x \in V$ and $x^{-1}M(g)x = M(g)$. Let $V_i$ be as in the proof of Lemma 8.3. Let $g_i$ and $x_i$ be $\omega$–approximations of $g$ and $x$ respectively, and let $\overline{g}_i$ and $\overline{x}_i$ be the images of $g_i$ and $x_i$ in $V_i/Z(V_i)$ respectively.

Then $\omega$–almost surely, $(\overline{g}_i, x_i^{-1}\overline{g}_i\overline{x}_i)$ is virtually abelian, so it is contained in a maximal virtually abelian subgroup $M_i$. If $M_i$ is finite, then it must be a finite cyclic group corresponding to a cone point on an orbifold, where this cone point has order at least 3 $\omega$–almost surely. In this case $M_i$ is malnormal, so $\overline{g}_i, x_i^{-1}\overline{g}_i, x_i \in M_i$ $\omega$–almost surely imply $\overline{x}_i \in M_i$ $\omega$–almost surely.

Now suppose $M_i$ is infinite. Since $M_i$ is a maximal virtually abelian subgroup of a hyperbolic 3–manifold group or a hyperbolic 2–orbifold group, for all $h \in V_i/Z(V_i)$ either $h \in M_i$ or $|h^{-1}M_i h \cap M_i| < \infty$. Since $M_i$ is either $\mathbb{Z}, \mathbb{Z}^2$, or $D_\infty$, $|h^{-1}M_i h \cap M_i| < \infty$ implies that $|h^{-1}M_i h \cap M_i| \leq 2$. Now, $\omega$–almost surely $\overline{x}_i^{-1}\overline{g}_i, x_i \in M_i \cap (\overline{x}_i^{-1}\overline{g}_i, x_i)$ and $\overline{x}_i^{-1}\overline{g}_i, x_i$ has order at least 3, hence $\overline{x}_i \in M_i$ $\omega$–almost surely.

Since in either case we have that $\omega$–almost surely $\overline{x}_i \in M_i$, it follows that $x \in M(g)$. □

Lemmas 8.3, 8.4, and 8.5 together imply the following.

Proposition 8.6. $V$ is weakly 2–CSA.

Using Proposition 8.6, Theorem 8.3 applies to $V$. Let $\mathcal{A}$ be the family of all virtually abelian subgroups of $V$, and $\mathcal{A}_V$ the family of all stably parabolic subgroups of $V$ (recall Definition 4.5).

Definition 8.7. Let $\overline{J}(v)$ be the tree of cylinders of the $(\mathcal{A}, \mathcal{H}_V)$–JSJ decomposition of $V$ obtained by Theorem 8.5.

Let $J(v)$ be the splitting of $V$ induced by the splitting $\overline{J}(v)$ of $V$. That is, $J(v)$ and $\overline{J}(v)$ have the same underlying graph and each vertex and edge group of $\overline{J}(v)$ is the pre-image of the corresponding vertex or edge group of $J(v)$.

By Theorem 8.5 $\overline{J}(v)$ is compatible (in the sense of [17, Definition 5.19]) with every $(\mathcal{A}, \mathcal{H}_V)$–splitting of $V$. That is, for any $(\mathcal{A}, \mathcal{H}_V)$–splitting $\mathcal{B}$ there exists an $(\mathcal{A}, \mathcal{H}_V)$–splitting $\mathcal{B}$ together with collapse maps $\mathcal{B} \to \mathcal{A}$ and $\mathcal{B} \to \overline{J}(v)$.

Next we describe the structure of certain important vertex groups of $\overline{J}(v)$ and $J(v)$, referred to as the flexible vertex groups in Theorem 8.5.

Definition 8.8. We say a group $D$ is a dihedral type group if there exist (i) has a subgroup $D^+$ of index two in $D$; and (ii) $\tau \in D \setminus D^+$ so that for all $g \in D^+, \tau^{-1}g\tau = g^{-1}$.

Note that by Lemma 8.3, every virtually abelian subgroup of $V$ is either dihedral or dihedral type. If $\overline{A}$ is a virtually abelian subgroup of $V$ and $A$ is the pre-image of $\overline{A}$ in $V$, then $A$ is either abelian by Lemma 4.12 or $A$ is a central extension of a dihedral type group.

For a general group $H$, we will again use $\mathcal{A}$ to denote the family of virtually abelian subgroups of $H$. We will also use $\mathcal{H}$ to denote an arbitrary collection of subgroups of $H$.

Definition 8.9. Let $\mathcal{H}$ be an $(\mathcal{A}, \mathcal{H})$ splitting of a group $H$ and let $U$ be a vertex group of $\mathcal{H}$. $U$ is called a QH–vertex group with fiber $F$ if there is a short exact sequence of the form

$$1 \to F \to U \to O \to 1$$

Where $O$ is the fundamental group of an orientable hyperbolic 2–orbifold of finite-type, i.e. $O \in \mathcal{H}_{\text{o-fin}}$. Moreover, we require that if $H \leq U$ is an edge group of $\mathcal{H}$ or $H$ is conjugate into an element of $\mathcal{H}$, then the image of $H$ in $O$ is either finite or is conjugate into a boundary subgroup of $O$.
By Theorem 8.5, the QH–vertex groups of \( \mathbb{J}(v) \) have fiber of size at most 2. In this case, the fiber is a central subgroup of the QH–vertex group. The QH–vertex groups of the corresponding splitting \( \mathbb{J}(v) \) of \( V \) are then central extensions of the QH–vertex groups of \( \mathbb{V} \), hence these are also QH–vertex groups with central fiber.

8.3. Modular automorphisms. We define the group of modular automorphisms relative to a particular graph of groups decomposition and a fixed family of subgroups. These definitions are standard except that some care has to be taken to ensure that the modular automorphisms of \( \mathbb{V} \) are induced by automorphisms of \( V \), especially in the case of the dihedral type vertex groups.

Suppose \( H = A \ast_C B \) and \( c \in Z_H(C) \). The Dehn twist by \( c \) is the automorphism of \( H \) defined by fixing each \( a \in A \) and mapping each \( b \in B \) to \( cbc^{-1} \). If \( H = A \ast_C B \) and \( c \in Z_H(C) \) then the Dehn twist by \( c \) is the automorphism defined by fixing each \( a \in A \) and mapping the stable letter \( t \) to \( tc \). If \( \mathcal{A} \) is a graph of groups decomposition and \( e \) is an edge of \( \mathcal{A} \), a Dehn twist over \( e \) is a Dehn twist in the one edge splitting corresponding to collapsing all edges of \( \mathcal{A} \) other than \( e \).

Now suppose \( A \) is an abelian group which is finitely generated relative to a family of subgroups \( \mathcal{E} \). Let \( P_A \) be the minimal direct factor of \( A \) which contains the subgroups in \( \mathcal{E} \) and the torsion of \( A \). The subgroup \( P_A \) can be characterized as the intersection of the kernels of all homomorphisms \( A \to \mathbb{Z} \) which are identically 0 on the elements of \( \mathcal{E} \). Write \( A = A_0 \oplus P_A \), and note that since \( A \) is finitely generated relative to \( \mathcal{E} \), \( A_0 \) is a finitely generated free abelian group. We consider the group of automorphisms of \( A \) fixing \( P_A \). In particular, this group contains automorphisms sending \( g \) to \( gh \) for some basis elements \( g \) and \( h \) of \( A_0 \) and fixing all other elements of the basis of \( A_0 \), and all elements of \( P_A \). We call such automorphisms of \( A \) Nielsen transformations relative to \( \mathcal{E} \) and note that they generate a group isomorphic to \( SL_n(\mathbb{Z}) \) where \( n \) is the rank of \( A_0 \).

Now suppose \( D \) is a dihedral type group which is finitely generated relative to a family of subgroups \( \mathcal{E} \) which are closed under conjugation. Let \( D^+ \) and \( \tau \) be as in Definition 8.8. We again define \( P_D^+ \) as the minimal direct factor of \( D^+ \) which contains the intersection of \( D^+ \) with the elements of \( \mathcal{E} \) and with the torsion of \( D^+ \). Write \( D^+ = D_0 \oplus P_D^+ \), and suppose \( g \) and \( h \) are distinct elements of a basis for \( D_0 \). A squared Nielsen transformation relative to \( \mathcal{E} \) is an automorphism of \( D \) fixing \( \tau \) and extending the automorphism of \( D^+ \) sending \( g \) to \( gh^2 \) and fixing \( P_D^+ \) and all other basis elements of \( D_0 \). As we show in Lemma 8.13, a squared Nielsen transformation of a dihedral type vertex group of a splitting of \( \mathbb{V} \) is induced by an automorphism of the pre-image of that vertex group in \( V \), which may not be true for the Nielsen transformation which sends \( g \) to \( gh \). We also show in Lemma 9.11 that the shortening argument still works using these squared Nielsen transformations.

Suppose \( \mathcal{A} \) is a graph of groups decomposition of a group \( H \) and \( v_0 \) is a vertex of \( \mathcal{A} \). Any element of \( g \) can be represented (non-uniquely) by \( [a_0, e_1, a_1, \ldots, e_n, a_n] \) where \( e_1, \ldots, e_n \) is an edge path in \( \mathcal{A} \) from \( v_0 \) to \( v_n \), \( a_i \) \( a_n \) are edges in \( \mathcal{A} \), and each \( a_i \) \( a_n \) where \( v_i \) \( v_n \) is the terminal vertex of \( e_i \) for \( 1 \leq i \leq n \). Suppose \( v \) is a vertex of \( \mathcal{A} \) and \( \sigma \in \text{Aut}(\mathcal{A}_v) \) acts by conjugation on each adjacent edge group. Then \( \sigma \) can be extended to an automorphism of \( G \) as in Definition 4.13 as follows: for each adjacent edge \( e \) there is \( \gamma_e \in \mathbb{A}_v \), so that \( \sigma(h) = \gamma_e h \gamma_e^{-1} \) for all \( h \) in the image of \( \mathbb{A}_v \) in \( \mathbb{A}_v \). Then \( \sigma \) extends to \( \overline{\sigma} \in \text{Aut}(H) \) by defining

\[
\overline{\sigma}([a_0, e_1, a_1, \ldots, e_n, a_n]) = [\overline{a}_0, e_1, \overline{a}_1, \ldots, e_n, \overline{a}_n]
\]

where

\[
\overline{a}_i = \begin{cases} a_i & \text{if } a_i \notin \mathbb{A}_v \\ \gamma_{e_i}^{-1} \sigma(a_i) \gamma_{e_{i+1}} & \text{if } a_i \in \mathbb{A}_v \end{cases}
\]
We call $\mathcal{T}$ a natural extension of $\sigma$. This natural extension is not unique: it depends on the choice of $\gamma$.

**Definition 8.10.** Let $\mathcal{H}$ be a reduced $(\mathcal{A}, \mathcal{H})$ splitting of a group $H$. The modular automorphism group of $H$ relative to $(\mathcal{A}, \mathcal{H})$, denoted $\text{Mod}^\mathcal{H}_H(H)$, is the subgroup of $\text{Aut}(H)$ generated by:

1. Inner automorphisms.
2. Dehn twists over edges of $\mathcal{A}$ with abelian edge groups.
3. Natural extensions of automorphisms of $QH$-vertex groups of $\mathcal{A}$ which fix the fiber pointwise, which act by conjugation on elements of $\mathcal{H}$, and which project to automorphisms of the associated orbifold group induced by homeomorphisms of the underlying orbifold fixing the boundary and cone points.
4. Natural extensions of Nielsen transformations of maximal abelian vertex groups of $\mathcal{A}$ relative to $\mathcal{H}$ and to the adjacent edge groups.
5. Natural extensions of automorphisms of vertex groups $U$ where $U$ is a central extension $1 \to Z \to U \to D \to 1$, $Z$ is contained in all edge groups adjacent to $U$, $D$ is a dihedral-type group and the automorphism fixes $Z$ and projects to squared Nielsen transformations of $D$ relative to the image in $D$ of the edge group of $\mathcal{A}$ adjacent to $U$ and to the elements of $\mathcal{H}$ contained in $U$.

When working with the $\mathcal{V}$, the type (5) automorphisms are squared Nielsen transformations of dihedral type vertex groups. That is, the central extension is trivial in this case.

Note that each type of modular auomorphism listed in Definition 8.10 acts by (possibly trivial) conjugation on each subgroup in $H$. Since this holds for the generators of $\text{Mod}^\mathcal{H}_H(H)$, it also holds for all the elements of $\text{Mod}^\mathcal{H}_H(H)$. We record this fact with the following lemma.

**Lemma 8.11.** If $\sigma \in \text{Mod}^\mathcal{H}_H(H)$ and $P \in \mathcal{H}$, then $\sigma(P)$ is conjugate to $P$.

The following is one of the main results of this section and is similar to [50, Proposition 4.17], [21, Theorem 5.4], and [17, Theorem 5.23].

**Theorem 8.12.** For any $(\mathcal{A}, \mathcal{H})$–splitting $\mathcal{H}$ of $\mathcal{V}$, $\text{Mod}^\mathcal{H}_\mathcal{V}(\mathcal{V}) \leq \text{Mod}^\mathcal{H}_{\mathcal{J}(v)}(\mathcal{V})$.

The proof of Theorem 8.12 is essentially the same as [17, Theorem 5.23]. There are some additional complications in [17, Theorem 5.23] which do not arise here, but the differences between the proofs are almost all notational. We provide a sketch of how to derive the proof of Theorem 8.12 from the proof of Theorem 17 Theorem 5.23. Similar arguments also appear in the proof of [50, Proposition 4.17].

**Proof of Theorem 8.12.** In our proof, the decomposition $\mathcal{J}(v)$ plays the role of $\mathcal{A}_{JSJ}$ from [17 Section 5.4]. In [17, Section 5.4], there is another splitting, $\mathcal{A}_M$, which is built as a refinement of $\mathcal{A}_{JSJ}$. This refinement is only needed because the splittings considered in [17, Section 5.4] do not allow arbitrary virtually abelian edge groups, but only certain types of virtually abelian edge groups. For our purposes, It suffices to read the proofs of [17 Section 5.4] with the additional assumption that $\mathcal{A}_{JSJ} = \mathcal{A}_M$.

Let $\mathcal{H}$ be an $(\mathcal{A}, \mathcal{H})$–splitting of $\mathcal{V}$. We consider each type of generator of $\text{Mod}^\mathcal{H}_\mathcal{V}(\mathcal{V})$ given by Definition 8.10 in turn. The type (1) generators of $\text{Mod}^\mathcal{H}_\mathcal{V}(\mathcal{V})$ clearly belong to $\text{Mod}^\mathcal{H}_{\mathcal{J}(v)}(\mathcal{V})$. The argument that the type (2) generators of $\text{Mod}^\mathcal{H}_\mathcal{V}(\mathcal{V})$ belong to $\text{Mod}^\mathcal{H}_{\mathcal{J}(v)}(\mathcal{V})$ is the same as [17, Lemma 5.25]. The argument that the type (3) generators of $\text{Mod}^\mathcal{H}_\mathcal{V}(\mathcal{V})$
belong to $\text{Mod}^\tau_0(\mathcal{V})$ is the same as [17] Lemma 5.27]. The type (4) and (5) generators of $\text{Mod}^\tau_0(\mathcal{V})$ both correspond to virtually abelian vertex groups (note that the central extensions of dihedral type vertex groups are all extensions with trivial kernel in this case, so these vertex groups are actually dihedral type), and the argument that these generators belong to $\text{Mod}^\tau_0(\mathcal{V})$ is the same as [17] Lemma 5.26].

Next we show modular automorphisms of $\mathcal{V}$ are induced by modular automorphisms of $V$. By Lemma 4.12, the pre-image in $V$ of any abelian subgroup of $\mathcal{V}$ is abelian. Recall that $\mathcal{J}(v)$ is the splitting of $V$ induced by the splitting $\mathcal{J}(v)$ of $\mathcal{V}$ and $\pi_v: V \to \mathcal{V}$ is the quotient map.

**Lemma 8.13.** Let $\alpha \in \text{Mod}^\tau_0(\mathcal{V})$ be a modular automorphism of type (1), (2), (4), or (5) according to the notation of Definition 8.70 Then there exists $\bar{\alpha} \in \text{Mod}^\tau_0(V)$ so that $\bar{\alpha}$ acts trivially on the stable center and for all $g \in V$, $\pi_v(\bar{\alpha}(g)) = \alpha(\pi_v(g))$.

**Proof.** First, the inner automorphisms of $\mathcal{V}$ are induced by inner automorphisms of $V$ since $\mathcal{V}$ is a central extension of $V$.

Suppose $\alpha$ is a Dehn twist by $c$ over an edge $e$ of $\mathcal{J}(v)$. Then $V$ has a one-edge splitting $\mathcal{B}$ with edge $f$ such that $\mathcal{B}_f = \pi^{-1}_v(\mathcal{J}(v))$, and $\mathcal{B}_f$ is abelian, by Lemma 4.12. Let $\tilde{c} \in \pi^{-1}_v(c)$. Define $\tilde{\alpha}$ to be the Dehn twist by $\tilde{c}$ over $f$ by the element.

If $\alpha$ is a type (4) automorphism for a maximal abelian vertex group $\bar{A}$ in $\mathcal{J}(v)$. Then the corresponding vertex group $A$ of $\mathcal{J}(v)$ is also maximal abelian by Lemma 4.12. Let $\mathcal{P}_\alpha$ be the subgroup from the definition of relative Nielsen transformations, so $A = A_0 \oplus \mathcal{P}_\alpha$. Let $\{a_1, \ldots, a_n\}$ be a basis for $A_0$. Then $P_\alpha = \pi^{-1}_v(\mathcal{P})$ is the minimal direct factor of $A$ which contains all adjacent edge groups, elements of $\mathcal{H}_v$, and the stable center. If we choose $a_i \in \pi^{-1}_v(\pi(a_i))$ for each $1 \leq i \leq n$, and let $A_0$ be the subgroup generated by $\{a_1, \ldots, a_n\}$ then we have that $A \cong A_0 \oplus P_\alpha$. Hence if $\alpha$ is the relative Nielsen transformation which sends $\pi$ to $\pi_\alpha$, then $\bar{\alpha}$ can be defined as the relative Nielsen transformation which sends $a_i$ to $a_i a_j$.

Suppose now that $\mathcal{D}$ is a dihedral type vertex group of $\mathcal{J}(v)$, let $D$ be the pre-image of $D$ in $V$, and let $D^+$ be the pre-image of $\mathcal{D}^+$. By Lemma 4.12 $D^+$ is abelian. Let $t$ be a pre-image of $\tau$. For each $x \in D^+$, let $c_x = t^{-1}xt$. Then $\pi_v(c_x) = 1$, hence $c_x$ belongs to $\text{Ker}(\pi_v) \cap D$, which we denote by $Z^0(D)$. Next we show that the map $D^+ \to Z^0(D)$ defined by $x \to c_x$ is a homomorphism.

Let $x, y \in D^+$. Recall that $D^+$ is abelian so $x$ and $y$ commute, and $c_x$ and $c_y$ belong to the center of $D$. It follows that

$$c_{xy} = t^{-1}xytxy = (t^{-1}xt)(t^{-1}yt)xy = x^{-1}c_xyt^{-1}c_yxy = c_xc_y$$

Define $P_{\mathcal{D}^+}$ as the minimal direct factor of $\mathcal{D}^+$ which contains the intersection of $\mathcal{D}^+$ with all edge groups adjacent to $\mathcal{D}$, all stably parabolic subgroups of $\mathcal{D}^+$ and all the torsion of $\mathcal{D}^+$. Then as in the abelian case, we can write $\mathcal{D}^+ = \mathcal{D}_0^+ \oplus P_{\mathcal{D}^+}$ and $D^+ = D_0^+ \oplus P$ with $P = \pi^{-1}_v(P_{\mathcal{D}^+})$, where $D_0^+$ is a finite rank free abelian group which maps isomorphically to $\mathcal{D}_0^+$. Suppose $\tilde{g}, \tilde{h}$ are basis elements of $\mathcal{D}^+$, $g$ and $h$ are pre-images of $\tilde{g}, \tilde{h}$ in $D^+$, and $\alpha$ is a squared Nielsen transformation of $\mathcal{D}$ of the form $\tilde{g} \to \tilde{g} \tilde{h}^2$. Let $\tilde{\alpha}$ be the map on $D$ which is the identity on $t$, $P$, and on all basis elements of $D_0^+$ except $g$ where $\tilde{\alpha}(g) = gh^2 c_h t^{-1}$. The map $\tilde{\alpha}$ is invertible and induces $\alpha$ on $\mathcal{D}$, so it only remains to verify that $\tilde{\alpha}$ is a homomorphism. It clearly induces a homomorphism on the abelian group $D^+$, so it remains
to show that it preserves the relations $t^{-1}xt = x^{-1}c_t$ for all $x \in D^+$. By our definition of $c_t$, $t^{-1}\bar{\alpha}(x)t = \bar{\alpha}(x)^{-1}c_{\bar{\alpha}(x)}$. Since $\bar{\alpha}$ fixes $t$, this means we need to show that $c_t = c_{\bar{\alpha}(x)}$ for all $x \in D^+$. Observe that

$$c_{\bar{\alpha}(x)} = t^{-1}gh^2c_{h_1}tgh^2c_{h_1}^{-1}
= t^{-1}gh^2tgh^2c_{h_1}^{-1}
= t^{-1}gt(r(t^{-1}h^2t^2)g^{-2}c_{h_1}^{-1}
= t^{-1}gtgc_{h_1}^{-2}c_{h_1}^{-1}
= c_g.$$

The map $x \mapsto c_x$ is a homomorphism and $c_{\bar{\alpha}(x)} = c_x$ for all $y$ which are either in $P$, or in a basis for $D_0^\gamma$. Therefore for all $x \in D^+$ $c_{\bar{\alpha}(x)} = c_x$, and so $\bar{\alpha}$ is an automorphism of $D$ inducing $\alpha$ on $\overline{D}$.

Finally, we observe that $\bar{\alpha}$ acts by conjugation on the adjacent edge groups of $D$. Suppose $E$ is such an edge group. Then $E \cap D^+$ is contained in $P$ and hence fixed by $\bar{\alpha}$. If $E \subset D^+$ then we are done, so assume that $E \not\subset D^+$. The subgroup $D^+$ is index two in $D$, so there exists $d \in D^+$ such that $E = \langle td, E \cap D^+ \rangle$. Then $d = d_0p$ for some $d_0 \in D_0^\gamma$ and $p \in P$. Suppose that $g^n$ is the power of $g$ which appears when $d_0$ is written in terms of the basis of $D_0^\gamma$ given above. Then $\bar{\alpha}(d_0) = d_0h^{2n}c_{h_1}^{-1}$. Since $\bar{\alpha}(p) = p$ and $p$ commutes with $d_0$ and $c_{h_1}^{-1}$, $\bar{\alpha}(d) = d_0h^{2n}c_{h_1}^{-1}p = dh^{2n}c_{h_1}^{-1}$. Using the fact that $\bar{\alpha}(t) = t$, we have that

$$\bar{\alpha}(td) = t(h^{2n}c_{h_1}^{-1}).$$

On the other hand, the definition of $c_x$ gives that $h^{-n}th^n = th^{2n}c_{h_1}^{-1}$. Since $h^n$ commutes with $d$, we have

$$h^{-n}th^n = t(h^{2n}c_{h_1}^{-1}).$$

Since $h$ commutes with $E \cap D^+$, we have that for all $e \in E$, $\bar{\alpha}(e) = h^{-n}eh^n$. \hfill \Box

**Lemma 8.14.** For each QH-vertex group $U$ of $\overline{J}(v)$ with underlying orbifold $\overline{O}$ and each homeomorphism $\phi$ of $\overline{O}$ fixing the boundary and cone points, there exists $\bar{\alpha} \in \operatorname{Mod}^{\overline{J}(v)}(V)$ so that $\bar{\alpha}(U) = U, \bar{\alpha}$ acts trivially on the stable center and on the fiber of $U$, and $\bar{\alpha}$ induces $\phi$ on $\overline{O}$.

**Proof.** Let $U$ be a QH-vertex group of $\overline{J}(v)$ and let $\overline{U}$ be the image of $U$ in $\overline{V}$. Then $\overline{U}$ is a QH-vertex group of $\overline{J}(v)$. Let $F$ the fiber of $\overline{U}$ so $\overline{U}/F \cong O$, where $O$ is an orbifold fundamental group as in Definition [8.9]. By Theorem [B.3] we can also assume that $|F| \leq 2$. Since $F$ is normal, this means that $F$ must also be central in $\overline{U}$. Now let $\gamma \in O$ be an element which is represented by a simple closed curve on the underlying orbifold, and let $\overline{\alpha}_\gamma$ be the pre-image of $\langle \gamma \rangle$ in $\overline{U}$. Then $\overline{\alpha}_\gamma$ is a central extension of a cyclic group, hence $\overline{\alpha}_\gamma$ is abelian.

Now let $A_\gamma = \pi_{\gamma}^{-1}(\overline{\alpha}_\gamma)$. Note that $U$ has fiber $\pi_{\gamma}^{-1}(F)$, and $A_\gamma$ is an abelian subgroup of $U$ by Lemma [4.12]. Since $\gamma$ is represented by a simple closed curve, the group $O$ splits over $\langle \gamma \rangle$ and this induces a splitting of $\overline{U}$ over $A_\gamma$. The dehn twist by $\gamma$ is an automorphism of $O$ which is induced by a Dehn twist on $\overline{U}$ over any element in $A_\gamma$ which maps to $\gamma$. This Dehn twist on $\overline{U}$ is a type (3) element of $\operatorname{Mod}^{\overline{J}(v)}(V)$. Since the pure mapping class group of the orbifold is generated by Dehn twists, we are done. \hfill \Box
9. Resolutions and factoring

In this section we complete the proof of Theorem 4.20. Given the setup in the previous two sections, the purpose of this section is to undertake the version of Sela’s shortening argument which works in the setting of \( \mathbb{R} \)-trees arising from the construction in Section 7.

Throughout this section we continue to make Standing Assumption 4.18 and use the notation from there. In particular, we consider a sequence \( (\varphi_i : G \to \Gamma_i = \pi_1(M_i)) \) from \( \text{Hom}(G, \mathcal{G}_{\text{Gen}}) \), which is not \( \mathcal{T} \)-divergent. In addition, we also assume throughout this section that the sequences \( (\varphi_i) \) is \( C \)-divergent.

9.1. Shortening quotients. We first use the decompositions \( \mathcal{J}(v) \) from Definition 8.7 to refine the decomposition \( G_r \) of \( L \).

Definition 9.1. Since the edge groups of \( G_r \) are elliptic in each \( \mathcal{J}(v) \), we can refine \( G_r \) by replacing each vertex \( v \) by the splitting \( \mathcal{J}(v) \) (see \([22, \text{Lemma 4.12}]\)). Denote the resulting splitting of \( L \) by \( \mathcal{J} \).

As in \([50]\) (see also \([17]\)), we create a sequence of graphs of groups approximating \( \mathcal{J} \) so that the \( \varphi_i \) factor through the terms in the approximating graphs of groups. This is essentially the same as \([50, \text{Lemma 7.1}]\), and the proof from there works in our situation without change. The idea of approximating splittings of non-finitely generated limit groups with associated splittings of finitely presented approximations comes from \([40, \text{Section 3}]\).

Lemma 9.2. Let \( \mathcal{J} \) be the splitting of \( L \) from Definition 9.1. There exists a sequence of finitely presented groups \( G = W_0, W_1, \ldots \) and epimorphisms \( f_i : W_i \to W_{i+1} \) and \( h_i : W_i \to L \) for \( i \geq 0 \) so that:

1. \( \varphi_\infty = h_0; \)
2. for all \( i \geq 1 \) we have \( h_i = h_{i+1} \circ f_i; \)
3. \( L \) is the direct limit of the sequence \( G \to W_1 \to \ldots \). Equivalently,

\[
\ker^\omega(\phi_i) = \bigcup_{k=1}^\infty \ker(f_{k-1} \circ \ldots \circ f_0)
\]

4. Each \( W_i \) has a graph of groups decomposition \( \mathcal{A}_i \). For each \( i \) there exists morphisms \( f^i : \mathcal{A}_i \to \mathcal{A}_{i+1} \) and \( h^i : \mathcal{A}_i \to \mathcal{J} \) such that the underlying graph morphisms are isomorphisms, \( f^i \) induces \( f_i \) and \( h^i \) induces \( h_i \).
5. Let \( V_i \) be a vertex group of \( \mathcal{A}_i \) and \( V \) be the corresponding vertex group of \( \mathcal{J} \) under \( h^i \). Then we have
   (a) \( V = \bigcup_{i=1}^\infty h_i(V_i) \)
   (b) \( V_i \) has a central subgroup \( Z_i \) which is contained in all edge groups adjacent to \( V_i \) such that \( h_i \) maps \( Z_i \) injectively into \( Z^\omega(V) \) and \( Z^\omega(V) = \bigcup_{i=1}^\infty h_i(Z_i) \)
   (c) If \( V \) is QH-vertex group, then \( h_i \) induces an isomorphism from \( V_i/Z_i \) to \( V \).
   (d) If \( V \) is a virtually abelian vertex group, then \( h_i \) induces an injective map from \( V_i/Z_i \) to \( V \).
6. If \( E \) is an edge group of \( \mathcal{J} \) and \( E_i \) is the corresponding edge group of \( \mathcal{A}_i \), then \( h_i \) maps \( E_i \) injectively into \( E \) and \( E = \bigcup_{i=1}^\infty h_i(E_i) \).
7. The edge groups and vertex groups of \( \mathcal{A}_i \) are finitely generated.
Fix a vertex $v$ of $G'$. There is a collapse map $\mathbb{J} \to G'$, such that the pre-image of $v$ is $\mathbb{J}(v)$. For each $\mathbb{A}_i$, the underlying graph is isomorphic to $\mathbb{J}$. Let $\hat{\mathbb{A}}_{v_i}$ denote the subgraph (of groups) corresponding to $\mathbb{J}(v)$. Let $\mathbb{W}_i$ be the splitting of $W_i$ obtained from $\mathbb{A}_i$ by collapsing each $\hat{\mathbb{A}}_{v_i}$. Let $W_{v_i}$ be the fundamental group of $\hat{\mathbb{A}}_{v_i}$; note that $W_{v_i}$ is identified with a subgroup of $W_i$ by Section 2.2. It follows from the construction that $f_i|W_{v_i}$ maps into (though possibly not onto) $W_{v_{i+1}'}$ and that $V$ is the direct limit of the $W_{v_i}$.

We use modular automorphisms associated to these graphs of groups to “shorten” $\phi_i$ relative to the vertex $v$, as we now explain.

Let $\tilde{\xi}_i : G \to W_i$ be the natural map, that is $\tilde{\xi}_i = f_{i-1} \circ \ldots \circ f_0$. Since $W_j$ is finitely presented, for fixed $j$, $\text{Ker}(\tilde{\xi}_j) \subseteq \text{Ker}(\phi)$ for an $\omega$ large set of $i$. Hence after passing to a subsequence of $\phi_i$ and re-indexing, we may assume $\text{Ker}(\tilde{\xi}_j) \subseteq \text{Ker}(\phi)$ for all $j$ and all $i \geq j$. Thus, for all $j$ and all $i \geq j$ the map $\phi_i$ factors through $\tilde{\xi}_j$, so there is $\tilde{\lambda}_i^j : W_j \to \Gamma_i$ so

$$\phi_i = \tilde{\lambda}_i^j \circ \tilde{\xi}_j.$$ 

Associated to $v$ is a sequence $\{v_i\}$ of vertices in the geometric trees of $\{\Gamma_i\}$, and $g \in V$ if and only if for some $\omega$–approximation $(g_i)$ to $g$ $\omega$–almost surely $g_i \in \Gamma_v$. Since each $W_{v_i}$ is finitely generated, after again passing to a subsequence of $\phi_i$ and hence $\{\tilde{\lambda}_i^j\}$ and re-indexing we can assume $\tilde{\lambda}_i^j$ is in fact a morphism of graphs of groups from $\mathbb{W}_j$ to the geometric splitting of $\Gamma_i$. Let $\tilde{\lambda}_i = \tilde{\lambda}_i^j$. Summarizing, we have

**Lemma 9.3.**

1. $\phi_i = \tilde{\lambda}_i \circ \tilde{\xi}_i$;
2. $\tilde{\lambda}_i$ is a morphism of graphs of groups from $\mathbb{W}_i$ to the geometric splitting of $\Gamma_i$.

Let $\tilde{A}_i$ be a lift to $G$ of the good relative generating set $A_i$ of $V$ and for each $i$ let $\tilde{\lambda}_i = \tilde{\xi}_i(\tilde{A}_v)$. Lemma 9.3 allows us make the following definition.

**Definition 9.4.**

$$\|\tilde{\lambda}_i\|_{\mathcal{G}_i} = \inf_{x \in \mathcal{G}(\tilde{\lambda}_i)} \max_{a \in \mathbb{A}_i} d_i(\tilde{\lambda}_i(a) \cdot x, x).$$

For $\omega$–almost every $i$, choose a point $\alpha_i \in \mathcal{G}(\tilde{\lambda}_i)$ which satisfies

$$\max_{a \in \mathbb{A}_i} \{ d_i(\tilde{\lambda}_i(a) \cdot \alpha_i, \alpha_i) \} \leq \|\tilde{\lambda}_i\|_{\mathcal{G}_i} + \frac{1}{i}. $$

Define

$$\|\tilde{\lambda}_i\|_{\mathcal{G}_i, \omega} = \max_{a \in \mathbb{A}_i} \{ d_i(\tilde{\lambda}_i(a) \cdot \alpha_i, \alpha_i) \}$$

Let $\mathcal{H}_i$ denote the family of subgroups $H$ of $W_i$ such that $H$ is elliptic in $\mathbb{A}_i$ and $h_i(H)$ is a stably parabolic subgroup of $L$. For a vertex $v$ of $G'$ let $\mathcal{H}_v$ denote the family of subgroups of $W_{v_i}$ which belong to $\mathcal{H}_i$. Note that any edge group of $\mathbb{W}_i$ which belongs to $W_{v_i}$ maps to a stably parabolic subgroup of $L$ and hence belongs to $\mathcal{H}_v$. By Lemma 8.11 $\text{Mod}_{\mathbb{A}_{v_i}}(W_{v_i})$ acts on these subgroups by conjugation, and hence can be extended to automorphisms of $W_i$. Let $\text{Mod}_i(W_i)$ be the subgroup of $\text{Aut}(W_i)$ generated by these extensions. Denote by $\text{Mod}(W_i)$ the subgroup of $\text{Aut}(W_i)$ generated by all subgroups $\text{Mod}_i(W_i)$ over all vertices $v$ of $G'$. Note that $\text{Mod}(W_i)$ acts on subgroups in $\mathcal{H}_i$ by conjugation. We define an equivalence relation on $\text{Hom}(W_i, \Gamma_i)$ by setting $\tilde{\lambda} \sim \tilde{\lambda}'$ if there is an $\alpha \in \text{Mod}(W_i)$ so $\tilde{\lambda}' = \tilde{\lambda} \circ \alpha$.

**Definition 9.5.** A sequence of homomorphisms $\{\tilde{\lambda}_i\}$ is called almost short if for any vertex $v$ of $G'$ and any sequence $\{\tilde{\lambda}_i \sim \tilde{\lambda}_v\}$, we have $\|\tilde{\lambda}_i\|_{\mathcal{G}_i} \leq \|\tilde{\lambda}_i\|_{\mathcal{G}_i} + \frac{1}{i}$. 

Lemma 9.6. There exists an almost short sequence of homomorphisms \( \{ \hat{\lambda}_i \} \) such that \( \hat{\lambda}_i \sim \lambda_i \) for all \( i \).

Proof. For each vertex \( v \) of \( \Gamma' \), choose \( \alpha_i^v \in \text{Mod}(W_v) \) so that \( \| \hat{\lambda}_i \circ \alpha_i^v \|_{\mathcal{G},v} \leq \| \lambda_i^v \|_{\mathcal{G},v} + \frac{1}{i} \) for any \( \lambda_i^v \sim \lambda_i \). Since \( \text{Mod}(W_v) \) is generated by all subgroups of the form \( \text{Mod}_i(W_v) \), where \( v' \) is a vertex of \( \Gamma' \), we have \( \alpha_i^v = \alpha_i \cdots \alpha_v \), where each \( \alpha_v \in \text{Mod}_i(W_v) \) for some vertex \( v' \) of \( \Gamma' \). Note that if \( v' \neq v \), then \( \alpha_v \) acts on \( W_{i,v} \) by conjugation and hence does not change the translation length \( \| \hat{\lambda}_i \|_{\mathcal{G},v} \). Therefore we can delete all such \( \alpha_v \) in \( \alpha_i^v \) while keeping \( \| \hat{\lambda}_i \circ \alpha_i^v \|_{\mathcal{G},v} \leq \| \lambda_i^v \|_{\mathcal{G},v} + \frac{1}{i} \) true. As a result, we can assume that \( \alpha_i^v \in \text{Mod}_i(W_v) \) without loss of generality. Let \( \alpha_v \) be a product of all these \( \alpha_i^v \) and \( \hat{\lambda}_i = \hat{\lambda}_i \circ \alpha_v \). Then \( \hat{\lambda}_i \) realize the “almost minimum” of \( \| \cdot \|_{\mathcal{G},v} \) for all vertices \( v \) simultaneously. \( \square \)

Definition 9.7. Let \( \{ \hat{\lambda}_i \} \) be as in Lemma 9.6 and define \( \eta_i : = \hat{\lambda}_i \circ \tilde{\xi}_v \). We call the limit group \( S = G/\text{Ker}^\omega(\eta_i) \) associated to the sequence \( \{ \eta_i : G \to \Gamma_i \} \) a \( \mathcal{G} \)-shortening quotient of \( L \) and \( \{ \eta_i : G \to \Gamma_i \} \) the defining sequence of \( S \).

Note that \( S \) is indeed a quotient of \( L \) since by Lemma 9.2 and construction we have \( \text{Ker}^\omega(\phi_i) = \text{Ker}^\omega(\tilde{\xi}_v) \subseteq \text{Ker}^\omega(\eta_i) \), so there is a natural quotient map \( \pi : L \to S \).

Lemma 9.8. Suppose \( \{ \phi_i : G \to \Gamma_i \} \) is not \( \mathcal{T} \)-divergent. Let \( \{ \eta_i \} \) be the defining sequence of a \( \mathcal{G} \)-shortening quotient of \( G/\text{Ker}^\omega(\phi_i) \). Then \( \eta_i \) is not \( \mathcal{T} \)-divergent.

Proof. Let \( g \in G \). Represent \( \phi_i(g) \) as a reduced \( \mathbb{Z} \)-loop \( [a_0, e_1, \ldots, e_n, a_n] \) based at \( v \). For \( \omega \)-almost every \( i \), \( \tilde{\xi}_i(g) \) is a lift of \( \phi_i(g) \) in \( W_i \) and \( \tilde{\xi}_i(g) = [\tilde{a}_0, e_1, \ldots, e_n, \tilde{a}_n] \), where \( \tilde{a}_k \) is a lift of \( a_k \) in the corresponding vertex group of the geometric splitting of \( W_i \). By construction, the factoring map \( \lambda_i \) is a morphism from the geometric splitting of \( W_i \) to the geometric splitting of \( \Gamma_i \). Combining this with the definition of the \( \alpha_v \), \( \hat{\lambda}_i = \lambda_i \circ \alpha_v \) is also a morphism from the geometric splitting of \( W_i \) to the geometric splitting of \( \Gamma_i \). Hence \( \eta_i(g) = \hat{\lambda}_i(\tilde{\xi}_i(g)) \) moves \( v_i \in T_i \) by a distance not bigger than the amount by which \( \phi_i(g) \) moves \( v_i \). Therefore, \( \eta_i(g) \) moves a point in \( T_i \) independent of \( g \) by an amount independent of \( i \), so \( \eta_i \) is not \( \mathcal{T} \)-divergent. \( \square \)

Recall \( \mathcal{H}_{\mathcal{L}}(\phi_i) \) is the family of stably parabolic subgroups of \( L \) with respect to \( (\phi_i) \). This family includes all subgroups fixing edges in \( T_m \).

Lemma 9.9. Each \( P \in \mathcal{H}_{\mathcal{L}}(\phi_i) \) maps injectively into an element of \( \mathcal{H}_{\mathcal{S}}(\eta_i) \).

Proof. If \( \omega \)-almost surely \( \phi_i \) maps \( g \in G \) to a non-trivial element in a parabolic subgroup of \( \Gamma_i \), then \( \phi_i(g) \) and \( \eta_i(g) \) are conjugate \( \omega \)-almost surely, and hence \( \eta_i(g) \) is a non-trivial element in a stably parabolic subgroup of \( S \). Since modular automorphisms act by conjugation on stably parabolic subgroups, it is easy to see that \( \pi \) takes stably parabolic subgroups of \( L \) injectively to stably parabolic subgroups of \( S \). \( \square \)

We first consider the case where the quotient map \( \pi \) is injective, which implies that \( L \cong S \). Hence we can consider \( (\eta_i) \) as a defining sequence for \( L \) in this case. The following lemma is the reason why we consider GGDs instead of geometric decompositions. It follows directly from the construction of \( \eta_i \) from \( \phi_i \).

Lemma 9.10. If \( \pi : L \to S \) is injective, then \( \mathcal{G} \) is the Linnell refinement of a GGD with respect to \( (\eta_i) \).
If $\pi : L \to S$ is injective then for a vertex $v$ of $G'$ we define

$$\|\eta_i\|_{\mathcal{E}, v} = \inf_{x \in \mathcal{E}} \max_{s \in A_i} d_i(\eta_i(s)x, x).$$

Recall that in our definition of modular automorphisms, we only included squared Nielsen transformations for the dihedral-type vertex groups. The next lemma is what guarantees that this still provides enough automorphisms for the shortening argument to work. If we allowed all Nielsen transformations, then the conclusion of Lemma 9.11 below follows from the Euclidean algorithm.

For a dihedral type group $D$ which is finitely generated relative to a family of subgroups $\mathcal{E}$, let $\text{Aut}_{\mathcal{E}}^f(D)$ denote the subgroup of $\text{Aut}(D)$ generated by squared Nielsen transformations relative to $\mathcal{E}$. We also define $D^+$ and $P_{D^+}$ as in Section 4.3.

**Lemma 9.11.** Suppose $D$ is a dihedral type group which is finitely generated relative to a collection of subgroup $\mathcal{E}$ and $g_1, \ldots, g_k \in D$ project to form a basis of $D^+/P_{D^+}$. Suppose $D$ acts on the real line $T$ so that $(g_1, \ldots, g_k)$ has indiscrete orbits. Then there exists a sequence $\alpha_i \in \text{Aut}_{\mathcal{E}}^f(D)$ so that for all $1 \leq j \leq k$, the translation length of $\alpha_i(g_j)$ goes to zero as $i \to \infty$.

**Proof.** Without loss of generality, we can assume $k = 2$ and that $g_1$ and $g_2$ act on $T$ as rationally independent translations in the same direction. We choose an orientation on $T$ and for each $g \in D$ which acts as a translation on $T$, we set $\tau_g$ to be the signed translation length of $g$. That is, for any $x \in T$, $d_T(x, gx) = \tau_g$ when $g$ translates in the positive direction and $d_T(x, gx) = -\tau_g$ when $g$ translates in the negative direction, so the translation length of $g$ is $|\tau_g|$.

Without loss of generality, we can assume that $\tau_{g_1} > \tau_{g_2} > 0$. To complete the proof, it suffices to find $\alpha \in \text{Aut}_{\mathcal{E}}^f(D)$ such that $|\tau_{\alpha(g_1)}| < \frac{1}{2}|\tau_{g_1}|$ and $|\tau_{\alpha(g_2)}| \leq |\tau_{g_2}|$.

Case 1: $0 < \tau_{g_2} < \frac{1}{2} \tau_{g_1}$. Then we can choose $n \geq 1$ such that $0 < \tau_{g_1} - 2n \tau_{g_2} < \frac{1}{2} \tau_{g_1}$. Hence we can choose $\alpha = \beta^n$, where $\beta$ is the the squared Nielsen transformation which fixes $g_2$ and sends $g_1$ to $g_1 g_2^{-2}$.

Case 2: $\frac{1}{2} \tau_{g_1} \leq \tau_{g_2} < \tau_{g_1}$. In this case, we let $\beta_1$ be the squared Nielsen transformation which fixes $g_2$ and sends $g_1$ to $g_1 g_2^{-2}$. Let $x_1 = \beta_1(g_1)^{-1}$, and note that $0 < \tau_{x_1} < \tau_{g_1}$. Moreover,

$$|\tau_{x_1} - \tau_{x_1}| = \tau_{g_1} - (\tau_{g_1} + 2 \tau_{g_2}) = 2 (\tau_{g_1} - \tau_{g_2}).$$

Letting $y_1 = \beta_1(g_2) = g_2$, we have that $|\tau_{y_1} - \tau_{x_1}| = |\tau_{y_1} - \tau_{x_1}| = |\tau_{x_1} - \tau_{y_1}| = \tau_{g_1} - \tau_{g_2}$. Now, if $\tau_{x_1} > \frac{1}{2} \tau_{y_1}$, then we can let $\beta_2$ be the squared Nielsen transformation which fixes $x_1$ and sends $y_1$ to $y_1 x_1^{-2}$ and set $y_2 = \beta(y_1)^{-1}$ and $x_2 = \beta_2(x_1) = x_1$. Then we will again have $|\tau_{y_2} - \tau_{x_2}| = |\tau_{y_1} - \tau_{x_1}| = |\tau_{x_1} - \tau_{y_1}| = \tau_{g_1} - \tau_{g_2}$ by the same calculation as before. Since we are decreasing by a fixed amount each time, after finitely many steps we will obtain $x_m$ and $y_m$ where either $0 < \tau_{x_m} < \frac{1}{2} \tau_{y_m}$ or $0 < \tau_{y_m} < \frac{1}{2} \tau_{x_m}$. From the construction, we also have that $0 < \tau_{y_m} \leq \tau_{x_{m-1}} \leq \cdots \leq \tau_{x_1} < \tau_{y_1}, 0 < \tau_{y_{m-1}} \leq \cdots \leq \tau_{x_1} \leq \tau_{y_1},$ and there are squared Nielsen transformations $\beta_1, \ldots, \beta_m$ such that $\beta_m \circ \cdots \circ \beta_1(g_1) = x_m^{-1}$ and $\beta_m \circ \cdots \circ \beta_1(g_2) = y_m^{-1} > 0$.

Case 2a: $0 < \tau_{y_m} < \frac{1}{2} \tau_{x_m}$. Then choosing $n$ and $\beta$ as in Case 1, and setting $x_{m-1} = \beta^n(x_m)$ we have that $\tau_{x_{m-1}} < \frac{1}{2} \tau_{x_m}$. Then we can choose $\alpha = \beta^n \circ \beta_m \circ \cdots \circ \beta_1$ since $|\tau_{\alpha(g_1)}| = |\tau_{x_m}| \leq |\tau_{g_2}|$ and

$$|\tau_{\alpha(g_1)}| = \frac{1}{2} |\tau_{x_m} | < \frac{1}{2} |\tau_{x_m} |.$$
Case 2b: $0 < \tau_{m} < \frac{1}{2} \tau_{m+1}$. Then we can again choose $\alpha$ and $n$ such that $0 < \tau_{m+1} < \frac{1}{2} \tau_{m}$ where $y_{m+1} = \beta^n(y_m)$ and $x_{m+1} = \beta^n(x_m) = x_m$. In this case, we have

$$0 < \tau_{y_m} < \frac{1}{2} \tau_{y_m} \leq \frac{1}{2} \tau_{y_1} < \frac{1}{2} \tau_{y_1}.$$ 

Hence, if $\tau_{o_m} < \tau_{o_m+1}$, we can set choose $\alpha = \beta^n \circ \beta_m \circ \ldots \circ \beta_1$. If $\frac{1}{2} \tau_{o_m+1} < \tau_{o_m+1}$, then, similar the beginning of Case 2, we can choose a squared Nielsen transformation $\theta$ which fixes $y_{m+1}$ sends $x_{m+1}$ to $x_{m+1} y_{m+1}^{-1}$ and set $x_{m+2} = \theta(x_{m+1})^{-1}$ and $y_{m+2} = \theta(y_{m+1}) = y_{m+1}$. Then we have

$$0 < \tau_{x_m} < \tau_{x_m+1} = \tau_{y_m+1} < \frac{1}{2} \tau_{y_1}.$$ 

So we can choose $\alpha = \theta \circ \beta^n \circ \beta_m \circ \ldots \circ \beta_1$.

Finally, if $0 < \tau_{o_m+1} < \frac{1}{2} \tau_{o_m+1}$, arguing as in Case 1 we can find $\theta$ such that $0 < \tau_{\theta(x_{m+1})} < \frac{1}{2} \tau_{x_{m+1}}$ and $\theta$ is a power of a squared Nielsen transformation which fixes $y_{m+1}$.

Since $\tau_{o_m+1} < \tau_{y_1}$, we can choose $\alpha = \theta \circ \beta^n \circ \beta_m \circ \ldots \circ \beta_1$ in this case.

\[ \Box \]

**Theorem 9.12.** If $\pi$ is injective then for every vertex $v$ of $G$, $\| \eta_v \|_{\mathcal{E},v}$ does not diverge.

**Proof.** The proof is based on Sela’s shortening argument, which is now well understood by the experts in the field. We give a sketch. For sake of contradiction, suppose $\| \eta_v \|_{\mathcal{E},v}$ diverges for some vertex $v$ of $G'$, and let $o_i \in \mathcal{E}(\Gamma_v)$ be as in Definition 9.4. Note that instead of each $o_i$ being centrally located and each $\hat{A}_i$ being short we only have $\{ o_i \}$ being a sequence of almost centrally located points and $\{ \hat{A}_i \}$ being an almost short sequence. As a result, we need the shortening automorphisms to shorten the translation length by an amount bounded below independently of $i$. Fortunately, the shortening automorphisms in Sela’s shortening argument do have this property.

By Theorem 7.11, $(\eta_i)$ induces an action of $\mathcal{V}$ on the $\mathbb{R}$-tree $\mathcal{G}_{v,\infty}$ of $\mathcal{E}(\Gamma_v)$. Let $T$ be the minimal $\mathcal{V}$-invariant subtree of $\mathcal{G}_{v,\infty}$, and $o \in \mathcal{G}_{v,\infty}$ the point defined by $\{ o_i \}$.

**Claim 1.** For some $s \in \hat{A}$, the geodesic segment $[o,s \cdot o]$ intersects $T$ in a non-degenerate segment.

**Proof of Claim 1** If $o \in T$, this follows by construction, as the action of $\langle \hat{A} \rangle$ on $T$ is non-trivial. Now suppose $o \notin T$. Let $o$ be the projection of $o$ in the closure (in $\mathcal{G}_{v,\infty}$) of $T$. Observe $\langle \hat{A} \rangle$ does not fix $o$. Let $s \in \hat{A}$ such that $s \cdot o \not\in o$. Then $[s \cdot o, o]$ has non-degenerate intersection with $T$ and hence $[s \cdot o, o] = [s \cdot o, s \cdot o] \cup [s \cdot o, o] \cup [o, o]$ also does. This completes the proof of Claim 1.

It follows from Theorem 7.11 and 5 and a standard argument that the action of $\mathcal{V}$ on $T$ is super-stable. Since $\mathcal{V}$ is freely indecomposable, by [19] Theorem 5.1, there is a graph of actions decomposition of the $\mathcal{V}$-tree $T$, whose vertex actions are either simplicial, axial, or Seifert type. Let $\mathcal{A}$ be the splitting of $\mathcal{V}$ induced by this decomposition. Given a generator $s \in A$, consider the path $[o, s \cdot o]$. Choosing a generator for which this path is the longest, it has a non-trivial intersection with at least one piece in the graph of actions decomposition. Note that stably parabolic subgroups of $V$ fix points in $T$ by Lemma 7.6.

Let $\mathcal{H}$ be the collection of images of stably parabolic subgroups of $V$ in $\mathcal{V}$. For the axial and Seifert type pieces, we can directly find a modular automorphism in $\text{Mod}_{\mathcal{V}}(\mathcal{V})$ which shortens the segment of $[o, s \cdot o]$ in this piece without changing the lengths of the segments in any other pieces and without changing the lengths of any other generators.
case, the shortening automorphisms in $\text{Mod}_{\mathcal{H}}(V)$ are more restrictive than those used in other applications of the shortening argument. However, Lemma 9.11 ensures these automorphisms suffice. In the proof of [50, Theorem 5.8], it is shown that the shortening automorphism shortens the translation length on $T$ by at least $\xi$. Here $r$ is the length of the shortest non-degenerate segment of the form $[0, s \cdot o] \cap T'$, where $T'$ is some axial piece and $s \in A$. Note that even though $r$ depends on the sequence $(\eta_i)$, it is independent of $i$. The same proof works verbatim in our setting using Lemma 9.11 in place of the use of the Euclidean algorithm in the proof of [50, Proposition 5.14]. In the Seifert case, the shortening automorphism is completely standard, and also reduces the translation length by an amount independent of $i$. By Theorem 8.12 this modular automorphism belongs to $\text{Mod}_{\mathcal{H}}(V)$ and by Lemma 8.13 it can be lifted to an automorphism in $\text{Mod}_{\mathcal{H}}(V)$, where $\mathcal{H}$ is the collection of stably parabolic subgroups of $V$. By construction $\omega$–almost surely this automorphism can be lifted to an element of $\text{Mod}_{\mathcal{H}}(V_i)$ since the scaled (down) action of $W_i$ on $G_i(\Gamma_{v_i})$ approximates the action of $V$ on $T$, whose translation length is reduced by an amount independent of $i$ by the shortening automorphism, this automorphism of $W_i$ $\omega$–almost surely shortens $\hat{\lambda}_i$ by an amount bounded below independent of $i$, a contradiction. In case $[o, s \cdot o]$ intersects a simplicial piece, we cannot find a automorphism which shortens the action of $V$ on $T$. However, by Theorem 7.12 any non-trivial element in a segment stabilizer is stably loxodromic. Thus, similarly as in [50, Proposition 5.19], we can find an automorphism which shortens the action of $W_{v_i}$ on $G_i(\Gamma_{v_i})$. One can see at the end of the proof of [50, Proposition 5.19] that the shortening automorphism reduces the translation length by at least $l(\varepsilon)/\|\eta_i\|_{G_i(\Gamma_{v_i})}$, up to an error $o(i)$. Here $l(\varepsilon)$ is the length of an edge in a simplicial piece, which is independent of $i$. So we again reach a contradiction.

9.2. The descending chain condition. The proof of the following result is inspired by Sela’s proof of [41, Theorem 1.12]. Note that if $\{\phi_i : G \to \Gamma_i\}$ is a $\mathcal{T}$–divergent sequence of homomorphisms, then there is an analogous construction of shortening quotients of the limit group $G/\text{Ker}^0(\phi_i)$ using the actions of $\Gamma_i$ on $T_i$, where $T_i$ is the tree dual to the geometric decomposition of $\Gamma_i$. We refer to these types of shortening quotients as $\mathcal{T}$–shortening quotients, see [17, Section 6] for the details of their construction.

**Theorem 9.13.** There is no infinite sequence of $\mathcal{M}^\pi_{\text{Gen}}$–limit groups

$$L_1 \overset{a_1}{\to} L_2 \overset{a_2}{\to} \ldots$$

such that each $a_i$ is a proper quotient map.

**Proof.** Towards a contradiction, suppose there is an infinite descending sequence of $\mathcal{M}^\pi_{\text{Gen}}$–limit groups as above. Choose a sequence of $\mathcal{M}^\pi_{\text{Gen}}$–limit groups $R_1 \overset{\beta_1}{\to} R_2 \overset{\beta_2}{\to} \ldots$ with each $R_n = F_k/\text{Ker}^0(\phi^n)$ and $\phi^n \circ \beta_n = \phi^{n+1}$ for all $n$, and further assume that, for each $n \geq 1$, $R_{n+1}$ is chosen such that if $R_n \to S \to \ldots$ is any infinite descending sequence of $\mathcal{M}^\pi_{\text{Gen}}$–limit groups with $S = F_k/\text{Ker}^0(\hat{\rho})$, then

$$|\text{Ker}(\hat{\rho}_n) \cap B_n| \leq |\text{Ker}(\phi^{n+1}) \cap B_n|$$

where $B_n$ is the ball of radius $n$ in $F_k$ with respect to the word metric.

Choose a diagonal sequence $\{\psi_n = \phi^n_{j_n}\}$, where $j_n$ is so that $\text{Ker}(\psi_n) \cap B_n = \text{Ker}(\phi^n) \cap B_n$ and for some $g \in \text{Ker}(\phi^{n+1})$, $g \notin \text{Ker}(\psi_n)$. Let $R_\infty = G/\text{Ker}^0(\psi_0)$; by construction, $\text{Ker}(\psi_\infty) = \bigcup_{j=1}^{\infty} \text{Ker}(\phi^j)$, so $R_\infty$ is also the direct limit of the sequence $F_k \to R_1 \to R_2 \to \ldots$. 
We claim every descending sequence of $\mathcal{M}_\text{Gen}^\pi$--limit groups

$$R_\infty \rightarrow L_1 \rightarrow L_2 \ldots$$

with proper quotient maps is finite. Indeed, if some element $g \in B_n$ maps trivially to $L_1$ but not to $R_\infty$, then $g$ maps non-trivially to $R_{n+1}$. But then there is an infinite descending sequence $R_n \rightarrow L_1 \rightarrow \ldots$, so this contradicts our choice of $R_{n+1}$.

Since there is no infinite descending chain starting at $R_\infty$, any sequence of proper $\mathcal{M}_\text{Gen}^\pi$--limit quotients of $R_\infty$ is finite. Consider the case where the defining sequence of homomorphisms for $R_\infty$ is $\mathcal{T}$--divergent. In this case, by [17 Lemmas 6.4 and 6.6] $R_\infty$ has a proper $\mathcal{T}$--shortening quotient $S_1$. Since $R_\infty$ does not admit an infinite descending sequence of proper $\mathcal{M}_\text{Gen}^\pi$--limit quotients, repeating this argument finitely many times we obtain a sequence of proper quotients

$$R_\infty \rightarrow S_1 \ldots \rightarrow S_q$$

where the defining sequence of homomorphisms for $S_q$ is not $\mathcal{T}$--divergent. If the defining sequence of homomorphisms for $S_q$ is $\mathcal{C}$--divergent then by applying the shortening argument above finitely many times we obtain a sequence of quotient maps

$$S_q \rightarrow U_1 \ldots \rightarrow U_i$$

of $\mathcal{M}_\text{Gen}^\pi$--limit quotients, each of whose defining sequence of homomorphisms is not $\mathcal{T}$--divergent (by Lemma [9.8] and terminating in an $\mathcal{M}_\text{Gen}^\pi$--limit group $U_i$ which has a GGD $G$ so that the defining sequence of homomorphisms for $U_i$ is not $\mathcal{C}$--divergent with respect to $G$ (a sequence of proper quotients terminates by the construction of $R_\infty$, and once a $\mathcal{C}$--shortening quotient is not a proper quotient, the defining sequence of the quotient is not $\mathcal{C}$--divergent by Theorem [9.12]. Each map $U_i \rightarrow U_{i+1}$ maps the stably parabolic subgroups of $U_i$ injectively into the stably parabolic subgroups of $U_{i+1}$, by Lemma [9.9] by Theorem [4.19] and the construction of $U_i$, all of the stably parabolic subgroups of $U_i$ are finitely generated.

It follows that all of the stably parabolic subgroups of $S_q$ are finitely generated. It now follows from Theorem [2.7] that the defining sequence of homomorphisms for $S_q$ $\omega$--factors through the limit. We now apply the arguments of [17 §6]. Since $R_\infty$ does not admit an infinite descending sequence of proper $\mathcal{M}_\text{Gen}^\pi$--limit quotients, and hence neither do any of the $S_i$, the hypotheses of [17 Lemmas 6.5 and 6.6] are satisfied for $R_\infty$ and each of the $S_i$, and so these lemmas may be applied inductively from the bottom of the sequence to prove that the defining sequence for $R_\infty$ $\omega$--factors through the limit. Then, from the construction of $R_\infty$, there exists $i$ such that $\ker(\psi_n) \subseteq \ker(\psi_{n+1}) \subseteq \ker(\phi_{n+1})$, contradicting our construction of $\psi_n$.

Finally, we prove Theorem [4.20] from Section [H]

As explained in Section [A] except for the results proved in the appendices, this completes the proof of Theorem [4.20] and hence the paper.

**Theorem 4.20.** Suppose that $L$ is an $\mathcal{M}_\text{Gen}^\pi$--limit group whose defining sequence of homomorphisms is not $\mathcal{T}$--divergent. There exists $k \geq 0$ and a sequence of epimorphisms:

$$L = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_k \rightarrow S_{k+1},$$

so that

1. Each $S_i$ is an $\mathcal{M}_\text{Gen}^\pi$--limit group;
2. For each $1 \leq i \leq k$, the map $\eta_i : S_{i-1} \rightarrow S_i$ is a proper quotient map and $\eta_{k+1}$ is an isomorphism;
3. For each $1 \leq i \leq k+1$, $\eta_i$ injectively maps the stably parabolic subgroups of $S_{i-1}$ into stably parabolic subgroups of $S_i$; and
that $\eta$ happens when there are no proper quotient maps and $\eta$ is not a proper quotient map, i.e. $\langle g \rangle$ is not finitely generated. By Lemma 9.9, any sequence of proper quotients of $\langle g \rangle$–shortening quotients $\eta_1 : L \to S_1$ as defined in Definition 9.7. We first assume that $\eta_1$ is a proper quotient map. By Lemma 9.8, $S_1$ has a defining sequence of homomorphisms which is not $\mathcal{T}$–divergent. With respect to this defining sequence of $S_1$, we can construct the geometric decomposition of $S_1$ and its Linnell refinement as in Definitions 4.16 and 4.13. We can now apply the constructions of Sections 8 and 9 to $S_1$ equipped with this refined geometric decomposition to produce a $\mathcal{C}$–shortening quotient $\eta_2 : S_1 \to S_2$.

We now inductively continue this process to produce a sequence of shortening quotients $L = S_0 \to S_1 \to S_2 \to \ldots$, each of which is an $\mathcal{M}_\text{Gen} \Gamma$–limit group by construction. By Theorem 9.13, any sequence of proper quotients of $\mathcal{M}_\text{Gen} \Gamma$–limit groups eventually terminates. This means that we eventually produce a $\mathcal{C}$–shortening quotient $\eta_{k+1} : S_k \to S_{k+1}$ which is not a proper quotient map, i.e. $\eta_{k+1}$ is an isomorphism. We choose the index $k \geq 0$ such that $\eta_{k+1}$ is the first map in this sequence which is not a proper quotient map, which means that $\eta_i$ is a proper quotient map for each $1 \leq i \leq k$. Note that we may have $k = 0$, which happens when there are no proper quotient maps and $\eta_1$ is an isomorphism. By Lemma 9.9, for each $1 \leq i \leq k+1$, each stably parabolic subgroup of $S_{i-1}$ maps injectively into a stably parabolic subgroup of $S_i$. Finally, since $\eta_{k+1}$ is injective, it follows from Lemmas 9.10 and 9.12 that there is a GGD for $S_{k+1}$ with respect to which $S_{k+1}$ is not $\mathcal{C}$–divergent. □

Appendix A. Edge-Twisted Graphs of Groups

In this appendix we prove the following result from Section 6.

Theorem 6.7. Let $\mathcal{E}$ be a finite edge-twisted graph of groups so that $\pi_1(\mathcal{E})$ is finitely generated. The Type B vertex groups of $\mathcal{E}$ are finitely generated.

We repeat the definition of an edge-twisted splitting for convenience.

Definition 6.5. A graph of groups $\mathcal{E}$ is edge-twisted if:

The underlying graph of $\mathcal{E}$ is bipartite with colors $A$ and $B$. Type A vertices have valence 2, and abelian vertex groups (thus the edge groups of $\mathcal{E}$ are also abelian). Let $W$ be a Type A vertex group of $\mathcal{E}$ and let $E_1$ and $E_2$ be the images in $W$ of the adjacent edge groups. There are subgroups $K_j \leq E_j$ (for $j = 1, 2$) so that

1. $K_1 \cap K_2 = \{1\}$; and
2. For $j = 1, 2$, the group $E_j/K_j$ is finitely generated.

A.1. Abelian vertex groups in graphs of groups. We begin with some results about graphs of groups with abelian vertex groups.

Lemma A.1. Suppose $A$ is an abelian group and $K, L \leq A$ satisfy $K \cap L = \{1\}$. Let $N \leq A$ be finitely generated. Then $\langle N, K \rangle \cap L$ is finitely generated.

Proof. Let $N_0 = \{g \in N |$ there exists $g' \in K$, such that $gg' \in L\}$

Then $N_0$ is a subgroup of the finitely generated abelian group $N$, so $N_0$ is finitely generated. Define $\phi : N_0 \to \langle N, K \rangle \cap L$ by $\phi(g) = gg'$ where $g' \in K$ is so that $gg' \in L$. If $gg' \in L$ and $gg'' \in L$ for $g', g'' \in K$ then $g'(g'')^{-1} \in L$. But $g'(g'')^{-1} \in K$ and $K \cap L = \{1\}$. So $g' = g''$, and $\phi$ is well-defined. It is easy to check that $\phi$ is a homomorphism. For any $g_0 \in \langle N, K \rangle \cap L$, we have $g_0 = gg'$ for some $g \in N$ and $g' \in K$. Then $g_0 = \phi(g)$, so $\phi$ is surjective. Since $\langle N, K \rangle \cap L$ is the image of the finitely generated group $N_0$, it is finitely generated, as required. □
**Lemma A.2.** Let $G = M *_k A$, where $A$ is abelian. Let $S_1 \subset M$, $S_2 \subset A$ and let $S_3$ be a generating set of $K \cap \langle S_2 \rangle$. Suppose $g$ is a word in $S_1 \cup S_2$. Then

1. If $g \in M$, then $g$ can be written as a word in $S_1 \cup S_3$.
2. If $g \in A$, then $g = ma$, where $m \in K \cap \langle S_1 \cup S_3 \rangle$ and $a \in \langle S_2 \rangle$.

In particular, if $G$ is finitely generated, then $M$ is finitely generated.

**Proof.** By the assumptions of the lemma, we have

$$g = r_1s_1 \cdots r_ks_k$$

where each $r_i \in \langle S_1 \rangle$ (and hence in $\langle S_1 \cup S_3 \rangle$) and each $s_i \in \langle S_2 \rangle$. If some $i > 1$ we have $s_i \in K$ then $s_i \in \langle S_3 \rangle$, so $r_is_{i-1} \in \langle S_1 \cup S_3 \rangle$. We can repeat this reduction until no $s_i$ lies in $K$. If $r_i \in K$ for some $i > 1$ then $r_i \in A$. Since $A$ is abelian, we can write $r_{i-1}s_{i-1}r_is_i$ as $r_{i-1}r_is_{i-1}s_i$, and reduce the number of syllables in our description of $g$. After repeating finitely many times, this result is an expression:

$$g = p_1q_1 \cdots p_lq_l$$

where each $p_i \in \langle S_1 \cup S_3 \rangle$, each $q_i \in \langle S_2 \rangle$. Moreover, none of the $p_i$ or $q_i$ lies in $K$, except possibly that $q_i$ is trivial, and we cannot control $p_1$ since the above reduction on $s_i$ required $i > 1$.

Consider the case where $g \in M$. In this case, the above expression cannot contain any $q_i$, so $g = p_1 \in \langle S_1 \cup S_3 \rangle$, as required.

Now suppose that $g \in A$. If $g \in K$ then $g = g \in M$ and we are in the first case. In this case, we can take $a = 1$ and $m = g$ in the conclusion of the lemma. Suppose then that $g \not\in K$. In this case, we have $l = 1$, since otherwise it is not possible that $g \in A$. This proves (2).

For the last assertion of the lemma, take a finite generating set for $G$, and let $S_1, S_2$ be all of the elements of $M, A$ that appear in normal forms for these generators. Then $S_1$ and $S_2$ are finite, and generate $G$. It follows that $S_3$ can be chosen to be finite, since $K \cap \langle S_2 \rangle$ is a subgroup of the finitely generated abelian group $\langle S_2 \rangle$. It follows from Item (1) that $M$ is finitely generated, as required.

The next two results deal with the simplest cases of Theorem 6.7, those $\mathbb{E}$ with two edges. The general case follows quickly.

**Lemma A.3.** Suppose $\mathbb{E}$ is an edge-twisted graph of groups so $G = \pi_1(\mathbb{E})$ is finitely generated. Suppose further that $\mathbb{E}$ has two edges and two vertices. Let $M$ be the type B vertex group of $\mathbb{E}$. Then $M$ is finitely generated.

**Proof.** Let $K$ and $L$ be the edge groups of $\mathbb{E}$, corresponding to edges $e_k$ and $e_L$ respectively. Let $A$ be the type A vertex group of $\mathbb{E}$. Choose the edge associated to $e_k$ as a maximal tree, which allows us to consider $K$ as a subgroup of both $A$ and $M$. Consider $L$ to be a subgroup of $A$, let $t$ be the stable letter of the splitting, and let $L' := t^{-1}Lt \leq M$.

Let $K_0 \leq K$ and $L_0 \leq L$ be the subgroups of $K$ and $L$ guaranteed by Definition 6.5. Thus,

1. $K_0 \cap L_0 = \{1\}$; and
2. There are finite sets $T_K \subset K$ and $T_L \subset L$ so that $K = \langle T_K, K_0 \rangle$ and $L = \langle T_L, L_0 \rangle$.

Choose finite subsets $S_A \subset A$ containing $T_K \cup T_L$ and $S_M \subset M$ containing $T_K$ so that $G = \langle S_A, S_M, t \rangle$. Let $N_0 = \langle S_A \rangle$. Since $K_0 \cap L_0 = \{1\}$, by Lemma A.1, $\langle N_0, K_0 \rangle \cap L_0$ and $\langle N_0, L_0 \rangle \cap K_0$ are finitely generated, by $R_1$ and $R_2$, respectively, say. Let $N_1 = \langle N_0, R_1, R_2 \rangle$.

We claim $\langle N_1, K_0 \rangle \cap L_0 \leq N_1$. Suppose $g \in \langle N_1, K_0 \rangle \cap L_0$. Then $g = nk_1r_2$ where $n \in N_0$, $k \in K_0$, $r_1 \in \langle R_1 \rangle$ and $r_2 \in \langle R_2 \rangle$. Since $g \in L_0$ and $\langle R_1 \rangle \leq L_0$, we have $nk_2 \in L_0$. But $k, r_2 \in K_0$, so $nk_2 \in \langle N_0, K_0 \rangle \cap L_0 = \langle R_1 \rangle$. Thus, $g = (nk_2)r_1 \in \langle R_1 \rangle \leq N_1$. A similar argument shows $\langle N_1, L_0 \rangle \cap K_0 \leq \langle R_2 \rangle \leq N_1$. **
We further claim that \( \langle N_1, K \rangle \cap L \leq N_1 \). Indeed, since \( T_K \subset S_A \subset N_1 \) we have \( \langle N_1, K \rangle \cap L_0 = \langle N_1, K_0 \rangle \cap L_0 \leq N_1 \). Now suppose that \( g \in \langle N_1, K \rangle \cap L \), and write \( g = n g_0 \) where \( n \in \langle T_K \rangle \leq N_1 \) and \( g_0 \in L_0 \). We have \( g \in \langle N_1, K \rangle \) and \( n \in N_1 \), so \( g_0 \in \langle N_1, K \rangle \cap L_0 \leq N_1 \). Therefore, \( g = n g_0 \in N_1 \), as required. Similarly, we have \( \langle N_1, L \rangle \cap K \subset N_1 \).

Since \( N_1 \) is a finitely generated abelian group, \( N_1 \cap K \) and \( N_1 \cap L \) are both finitely generated. So

\[
M' = \langle S_M, N_1 \cap K, t^{-1}(N_1 \cap L)t \rangle
\]

is finitely generated. We claim that \( M = M' \), which proves the lemma. We clearly have \( M' \leq M \), so we have to prove the opposite inclusion.

To that end, let

\[
N_2 = \langle N_1, M' \cap K, t(M' \cap L)t^{-1} \rangle \subset A.
\]

(Recall that \( L' = t^{-1}Lt \leq M \).)

We claim that

1. \( N_2 \cap K = \langle N_1 \cap K, M' \cap K \rangle \); and
2. \( N_2 \cap L = \langle N_1 \cap L, t(M' \cap L)t^{-1} \rangle \).

From the definition of \( N_2 \), it is clear that \( \langle N_1 \cap K, M' \cap K \rangle \leq N_2 \cap K \). For the reverse inclusion, since \( t(M' \cap L)t^{-1} \leq L \), we have

\[
\langle N_1, t(M' \cap L)t^{-1} \rangle \cap K \leq \langle N_1, L \rangle \cap K \leq N_1 \cap K.
\]

Since \( A \) is abelian, for any subgroup \( J \leq K \) we have \( \langle N, J \rangle \cap K = \langle N \cap K, J \rangle \). Therefore,

\[
N_2 \cap K = \langle N_1, t(M' \cap L)t^{-1}, M' \cap K \rangle \cap K
= \langle \langle N_1, t(M' \cap L)t^{-1} \rangle \cap K, M' \cap K \rangle
\leq \langle N_1 \cap K, M' \cap K \rangle,
\]

as required. The second equality of the claim is entirely similar.

As a result, we have \( M' = \langle S_M, N_2 \cap K, t^{-1}(N_2 \cap L)t \rangle \). To finish the proof, we now show that \( M \leq M' \). Choose sets \( S_1 \supset S_M \) and \( S_2 \supset S_A \) so that \( \langle S_1 \rangle = M' \) and \( \langle S_2 \rangle = N_2 \). Let \( S_3 \) be a generating set for \( K \cap N_2 \). By the above \( \langle S_3 \rangle \leq M' \).

Consider the subgroup \( \langle M, A \rangle \geq G \) (which is isomorphic to \( M *_K A \)). Since \( S_M \subset S_1 \) and \( S_A \subset S_2 \) we have \( G = \langle S_1, S_2, t \rangle \).

Suppose that \( g \in M \), and write \( g = w_1 \ldots w_j \) where each \( w_i \) is either \( t \) or \( t^{-1} \), in \( \langle S_1 \rangle \), or in \( \langle S_2 \rangle \). Since \( g \in M \), if there are any occurrences of \( t \) or \( t^{-1} \) then by Britton’s Lemma there is a subword of one of the following two forms:

1. \( t w t^{-1} \) where \( w \in \langle S_1 \cup S_2 \rangle \cap L \); or
2. \( t^{-1} u t \) where \( u \in \langle S_1 \cup S_2 \rangle \cap L \).

In the first case, \( w \in L' \leq M \), so by Lemma A.2 \( w \in \langle S_1 \cup S_3 \rangle \), and so \( w \in M' \). Then \( t w t^{-1} \in t(M' \cap L)t^{-1} \leq N_2 \), so \( t w t^{-1} \) can be replaced by an element in \( \langle S_2 \rangle \). In the second case, \( u \in L \leq A \). By Lemma A.2 \( u \) can be written as \( u = ma \) where \( m \in K \cap \langle S_1 \cup S_3 \rangle \) and \( a \in \langle S_2 \rangle \). So \( m \in K \cap M' \leq N_2 \) and \( a \in N_2 \), so \( u \in N_2 \cap L \). Therefore, \( t^{-1} u t \in t^{-1}(N_2 \cap L)t \leq M' \), so we can replace \( t^{-1} u t \) by an element of \( \langle S_1 \rangle \).

Repeating until there are no occurrences of \( t \) or \( t^{-1} \), we obtain \( g \in \langle S_1 \cup S_2 \rangle \). By Lemma A.2 \( g \in \langle S_1 \cup S_3 \rangle = M' \), so \( M = M' \) as required.

The proof of the following lemma is very similar to that of Lemma A.3 and we omit it.
Lemma A.4. Let $E$ be an edge-twisted splitting with two edges, two type B vertices and one type A vertex so that $\pi_1(E)$ is finitely generated. Then the type B vertex groups are finitely generated.

We now give the proof of the main result of this appendix.

Theorem 6.7. Let $E$ be a finite edge-twisted graph of groups so that $\pi_1(E)$ is finitely generated. The Type B vertex groups of $E$ are finitely generated.

Proof. We proceed by induction on the number of type A vertex groups of $E$. If there are none, then $E$ is a single type B vertex group, and the result is trivial. If there is a single type A vertex group, the result follows from Lemmas A.3 and A.4.

Suppose now that $E$ has $k$ type A vertices for $k > 1$ and that the result is true for any finite edge-twisted graph of groups with finitely generated fundamental group and at most $k - 1$ type A vertices. Let $A$ be a type A vertex group of $E$. Let $E_0$ be the graph of groups obtained from $E$ by collapsing all of the edges that are not adjacent to $A$. It is clear that $E_0$ is an edge-twisted graph of groups with a single type A vertex group, and so the by induction the type B vertex groups of $E_0$ are finitely generated by the case $k = 1$. However, the type B vertex groups of $E_0$ are the fundamental groups of (edge-twisted) sub-graphs of groups of $E$, with fewer type A vertex groups than $E$. Since these sub-graphs of groups have finitely generated fundamental group, the inductive hypothesis applies to them to prove that their type B vertex groups are finitely generated. But these are the type B vertex groups of $E$, so the result is proved. □

APPENDIX B. RELATIVE LINNELL AND JSJ DECOMPOSITIONS

The purpose of this section is to prove Corollary B.4 about the existence of the relative $C$–Linnell decomposition which was used in Definition 4.13 and to prove Theorem B.5 which is used in Section 8.

B.1. Relative acylindrical accessibility. We first construct the decomposition $L$ from Theorem 8.12. We refer to this decomposition as a relative Linnell decomposition. The existence of such a decomposition is proved in [27] when $G$ is finitely generated, and it can also be derived from Weidmann’s version of acylindrical accessibility [49]. Here we modify Weidmann’s argument to the case where $G$ is finitely generated relative to a finite collection of subgroups $\mathcal{H}$ and all splittings considered are rel $\mathcal{H}$.

Lemma B.1. Let $T$ be a minimal $G$–tree. Let $G_0$ be a subgroup of $G$ with no global fixed point in $T$ and let $T_0$ be the minimal $G_0$–invariant subtree of $T$. Let $H$ be a subgroup of $G$ which fixes a point $x$ such that $d(x,T_0) = k$. Let $E = \langle G_0,H \rangle$ and let $T_E$ be the minimal $E$–invariant subtree of $T$. The number of edges in $T_E/E$ is at most the number of edges in $T_0/G_0$ plus $k$.

Proof. Let $T'$ be the union of $T_0$ and the path from $x$ to $T_0$. The $E$–orbit of $T'$, denoted by $T''$, is connected since $gT' \cup T''$ is connected for each $g \in G_0 \cup H$, and $T''/E$ has at most $k$ more edges than $T_0/G_0$. On the other hand, $T_E \subset T''$, so the lemma follows. □

Definition B.2. A $G$–action on a tree is $(k,C)$–acylindrical (for $k \geq 0$ and $C \geq 1$) if the stabilizer of all paths of length $\geq k+1$ has size at most $C$.

Proposition B.3. Fix $k \geq 0$ and $C \geq 1$. Suppose $G$ is finitely generated relative to a finite collection of subgroups $\mathcal{H}$. There is a bound on the number of edges in minimal $(k,C)$–acylindrical splittings of $G$ rel $\mathcal{H}$. 
Proof. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$. It suffices to assume that each $H_i$ is infinite. Let $S \subseteq G$ be a finite set such that $S \cup H_1 \cup \cdots \cup H_n$ generates $G$ and $|S \cap H_i| \geq C + 1$ for each $i$. Let $G_0 = (S)$. By Weidmann’s $(k,C)$-acylindrical-accessibility [49, Theorem 1], the number of edges of any $(k,C)$-acylindrical splitting of $G_0$ is bounded by a constant $B$. Let $\mathcal{H}$ be a $(k,C)$-acylindrical splitting of $G$ rel $\mathcal{H}$, and let $T$ be the corresponding Bass-Serre tree. First suppose that $G_0$ does not fix a point in $T$, and let $T_0$ be the minimal $G_0$ invariant subtree. For each $i$, let $x_i$ be a fixed point of $H_i$ and let $y_i$ be the closest point in $T_0$ to $x_i$. Since $S \cap H_i$ fixes $[x_i,y_i]$ and $|S \cap H_i| \geq C + 1$, we have $d_T(x_i, T_0) = d_T(x_i, y_i) \leq k$

Let $G_i$ be the subgroup of $G$ generated by $G_0$ and $H_1, \ldots, H_i$. Then $G_n = G$. Let $T_i$ be the $G_i$-minimal invariant subtree. Clearly we have $T_i \subseteq T_{i+1}$. Each $H_i$ fixes a point at distance at most $k$ from $T_{i-1}$, so by Lemma B.1 $T_i/G_i$ has at most $k$ more edges than $T_{i-1}/G_{i-1}$. Then $T/G = T_n/G_n$ has at most $B + kn$ edges.

In case $G_0$ does fix a point in $T$ the above argument shows that if $y \in \text{Fix}(G_0)$ and $x_i \in \text{Fix}(H_i)$ then for each $i$ $d(y, x_i) \leq k$. Let $\mathcal{F}$ be the union of the geodesics $[y, x_i]$ and note $\mathcal{F}$ contains at most $kn$ edges. Since the $G$--orbit of $\mathcal{F}$ covers $T$, we are done in this case also. $\square$

Corollary B.4. Suppose $G$ is finitely generated relative to a finite collection of subgroups $\mathcal{H}$. For all $C \geq 1$, $G$ has a splitting rel $\mathcal{H}$ in which all edge groups have size $\leq C$ and no vertex group splits rel $\mathcal{H}$ over a subgroup of size $\leq C$.

B.2. JSJ-decompositions for relatively finitely generated groups with torsion.

The purpose of this section is to prove Theorem B.5. We largely follow Guirardel–Levitt [22, 20], and explain the changes in our situation. We refer to [22] for terminology.

The JSJ decomposition we use is essentially [22, Theorem 9.14], but we need to weaken hypotheses. First, we need to accommodate relatively finitely generated groups. In addition, we allow groups that are not K–CSA in the sense of [22] but instead satisfy are weakly K–CSA as in Definition 8.2. We repeat the definition here for convenience.

Definition 8.2. Fix $K \geq 1$. A group is weakly K–CSA if (i) any element $g$ of order greater than $K$ is contained in a unique maximal virtually abelian subgroup $M(g)$, so that $M(g)$ is $K$–virtually abelian and equal to its normalizer, and (ii) every two infinite, virtually abelian subgroups $A$ and $B$ with $\langle A, B \rangle$ not virtually abelian satisfy $|A \cap B| \leq K$.

If $G$ is finitely generated and K–CSA the next result follows from [22, Theorem 9.14]. We briefly sketch how to modify their proof.

Theorem B.5. Let $G$ be a weakly K–CSA group, let $\mathcal{H}$ be a finite collection of subgroups of $G$, and let $\mathcal{A}$ be the collection of virtually abelian subgroups of $G$. Suppose $G$ is finitely generated rel $\mathcal{H}$ and $G$ does not split over a subgroup of order $\leq 2K$ rel $\mathcal{H}$. The $(\mathcal{A}, \mathcal{H})$–JSJ decomposition of $G$ exists and all flexible vertex stabilizers are virtually abelian or QH with fiber of size at most $K$. Its tree of cylinders is compatible with every $(\mathcal{A}, \mathcal{H})$–tree.

Proof. The proof of [22, Theorem 9.14] is based on [22, Sections 7 and 8]. In [22, Section 7] the finite generation assumption is never used. While it is used in [22, Section 8], there is a remark at the beginning of that section that it suffices to assume $G$ is finitely generated rel $\mathcal{H}$ [22, p. 80]. Generalizing [22, Section 8] to the relatively finitely generated setting is straightforward and we omit the details.

Now we discuss replacing K–CSA groups with weakly K–CSA groups in the proof of [22, Theorem 9.14]. In [22, Section 7], they construct the tree of cylinders of a given $(\mathcal{A}, \mathcal{H})$–tree $T$. In [22] all groups in $\mathcal{A}$ are infinite, whereas our $\mathcal{A}$ may contain finite groups. However, the construction works the same way in this case, as can be seen in [20].
We next discuss this construction and verify it has the properties we need even when $\mathcal{A}$ contains finite groups.

For $A, B \in \mathcal{A}$, define $A \sim B$ if $(A, B)$ is virtually abelian. Since $G$ is weakly $K$–CSA, this is an admissible equivalence relation on $\mathcal{A}$ (see [22, Lemma 9.13]), so for any $(\mathcal{A}, \mathcal{H})$–tree $T$, we can form the tree of cylinders $T_c$ (see [20]). If $A$ and $B$ are not equivalent, then by assumption $|A \cap B| \leq K$, so the action of $G$ on $T_c$ is $(2, K)$–acylindrical (See the proof of [22, Lemma 7.7] or [20, Proposition 5.13]). Now, $T$ dominates $T_c$, and vertex stabilizers of $T_c$ which are not elliptic in $T$ are (maximal) non-(virtually cyclic) virtually abelian subgroups. That is, $T$ smallly dominates $T_c$. Then $\mathcal{A}$ contains all virtually cyclic and all finite subgroups of $G$, so by [22, Theorem 8.7] there exists an $(\mathcal{A}, \mathcal{H})$–JSJ tree whose flexible vertices are either virtually abelian or QH with fiber of size at most $K$.

If $T_{JSJ}$ is the JSJ-tree of its cylinders is compatible with every $(\mathcal{A}, \mathcal{H})$–tree. This follows from [22, Lemmas 7.14 and 7.15]. Note that [22, Lemma 7.15] assumes one-endedness. Here is how this is used: Suppose $S \to T$ collapses a single $G$–orbit of edges into the $G$–orbit of a vertex $v$ of $T$. Let $H = \text{Stab}_G(v)$. In case $H$ is small in $S$ the proof from [22] works as written. Assume that $H$ is QH with fiber $F$, where $|F| \leq K$. Let $S_v$ be the minimal subtree of $S$ for the stabilizer $H$. Since $G$ does not split over any subgroup of order $\leq 2K$, by [22, Lemma 5.16] all boundary components of the associated orbifold are used. Hence by [22, Lemma 5.18] the splitting of $H$ corresponding to the action on $S_v$ is dual to a family of geodesics on the orbifold.

The claim is that any cylinder of $S$ containing an edge of $S_v$ is contained in $S_v$. Suppose there are edges $e$ and $f$ with stabilizers $A = \text{Stab}_G(e)$ and $B = \text{Stab}_G(f)$ so that $e$ is in $S_v$ and $f$ has exactly one endpoint in $S_v$. One-endedness would imply that $B$ is infinite, and in this case the proof in [22] works as written. So assume that $B$ is finite. Since $B/F$ is a subgroup of the orbifold corresponding to the QH vertex $v$, it is in fact cyclic and contained (or conjugate into) a cone point subgroup of the orbifold (it does not come from a mirror since $|B/F| \geq 3$). By [22, Proposition 5.4], $A/F$ is cyclic subgroup corresponding to a geodesic on the orbifold. Hence $(A, B)$ is not virtually abelian, so these groups are not equivalent.

The rest of the proof of [22, Theorem 9.14] works verbatim in our situation, proving Theorem B.5.

\section*{References}

1. I. Agol, D. Groves, and J. F. Manning. An alternate proof of Wise’s malnormal special quotient theorem. \textit{Forum Math. Pi}, 4:e1, 54, 2016.
2. I. Agol and Y. Liu. Presentation length and Simon’s conjecture. \textit{J. Amer. Math. Soc.}, 25(1):151–187, 2012.
3. M. Aschenbrenner, S. Friedl, and H. Wilton. 3-manifold groups. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2015.
4. G. Baumslag, A. Myasnikov, and V. Remeslennikov. Algebraic geometry over groups. I. Algebraic sets and ideal theory. \textit{J. Algebra}, 219(1):16–79, 1999.
5. G. Baumslag, A. Myasnikov, and V. Roman’kov. Two theorems about equationally Noetherian groups. \textit{J. Algebra}, 194(2):654–664, 1997.
6. R. Benedetti and C. Petronio. \textit{Lectures on hyperbolic geometry}. Universitext. Springer-Verlag, Berlin, 1992.
7. M. Boileau, S. Boyer, and S. Wang. Roots of torsion polynomials and dominations. In \textit{The Zieschang Gedenkschrift}, volume 14 of \textit{Geom. Topol. Monogr.}, pages 75–81. Geom. Topol. Publ., Coventry, 2008.
8. M. Boileau, J. H. Rubinstein, and S. Wang. Finiteness of 3-manifolds associated with non-zero degree mappings. \textit{Comment. Math. Helv.}, 89(1):33–68, 2014.
9. M. Boileau and S. Wang. Non-zero degree maps and surface bundles over $S^1$. \textit{J. Differential Geom.}, 45(4):789–806, 1996.
10. M. R. Bridson and A. Haefliger. \textit{Metric Spaces of Non–Positive Curvature}, volume 319 of \textit{Grundlehren der mathematischen Wissenschaften}. Springer–Verlag, Berlin, 1999.
[11] D. Calegari. Review of “epimorphism sequences between hyperbolic 3-manifold groups”. Mathematical Reviews, 2002.
[12] H.-D. Cao and X.-P. Zhu. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. Asian J. Math., 10(2):165–492, 2006.
[13] P. Derbez. Volume-convergent sequences of Haken 3-manifolds. C. R. Math. Acad. Sci. Paris, 336(10):833–838, 2003.
[14] P. Derbez. Nonzero degree maps between closed orientable three-manifolds. Trans. Amer. Math. Soc., 359(8):3887–3911, 2007.
[15] B. Farb. Relatively hyperbolic groups. Geom. Funct. Anal., 8(5):810–840, 1998.
[16] F. González-Acuña and A. Ramírez. Epimorphisms of knot groups onto free products. Topology, 42(6):1205–1227, 2003.
[17] D. Groves and M. Hull. Homomorphisms to acylindrically hyperbolic groups I: Equationally noetherian groups and families. Trans. Amer. Math. Soc., 372(10):7141–7190, 2019.
[18] D. Groves, J. F. Manning, and H. Wilton. Recognizing geometric 3-manifold groups using the word problem. arxiv.org/abs/1210.2101, 2012.
[19] V. Guirardel. Actions of finitely generated groups on $\mathbb{R}$-trees. Ann. Inst. Fourier (Grenoble), 58(1):159–211, 2008.
[20] V. Guirardel and G. Levitt. Trees of cylinders and canonical splittings. Geom. Topol., 15(2):977–1012, 2011.
[21] V. Guirardel and G. Levitt. Splittings and automorphisms of relatively hyperbolic groups. Groups Geom. Dyn., 9(2):599–663, 2015.
[22] V. Guirardel and G. Levitt. JSJ decompositions of groups. Astérisque, (395):vii+165, 2017.
[23] C. Hayat-Legranda, S. Wang, and H. Zieschang. Any 3-manifold 1-dominates at most finitely many 3-manifolds of $S^3$-geometry. Proc. Amer. Math. Soc., 130(10):3117–3123, 2002.
[24] B. Kleiner and J. Lott. Notes on Perelman’s papers. Geom. Topol., 12(5):2587–2855, 2008.
[25] M. Kreck and W. Lück. Topological rigidity for non-aspherical manifolds. Pure Appl. Math. Q., 5(3, Special Issue: In honor of Friedrich Hirzebruch. Part 2):873–914, 2009.
[26] H. Liang. Discrete representations of finitely generated groups into $\text{PSL}(2,\mathbb{R})$. Preprint, arXiv:2010.16008, 2020.
[27] P. A. Linnell. On accessibility of groups. J. Pure Appl. Algebra, 30(1):39–46, 1983.
[28] Y. Liu. Finiteness of nonzero degree maps between 3-manifolds. J. Topol., 13(1):237–268, 2020.
[29] J. Morgan and G. Tian. The geometrization conjecture, volume 5 of Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014.
[30] F. Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. Invent. Math., 94(1):53–80, 1988.
[31] G. Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.
[32] G. Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds, 2003.
[33] G. Perelman. Ricci flow with surgery on three-manifolds, 2003.
[34] A. W. Reid and S. Wang. Non-Haken 3-manifolds are not large with respect to mappings of non-zero degree. Com. Anal. Geom., 7(1):105–132, 1999.
[35] A. W. Reid, S. C. Wang, and Q. Zhou. Generalized Hopfian property, a minimal Haken manifold, and epimorphisms between 3-manifold groups. Acta Math. Sin. (Engl. Ser.), 18(1):157–172, 2002.
[36] Y. W. Rong. Degree one maps between geometric 3-manifolds. Trans. Amer. Math. Soc., 332(1):411–436, 1992.
[37] G. P. Scott. Compact submanifolds of 3-manifolds. J. London Math. Soc. (2), 7:246–250, 1973.
[38] G. P. Scott. Finitely generated 3-manifold groups are finitely presented. J. London Math. Soc. (2), 6:437–440, 1973.
[39] P. Scott. The geometries of 3-manifolds. Bulletin of the London Mathematical Society, 15:401–487, 1983.
[40] Z. Sela. Diophantine geometry over groups. I. Makanin-Razborov diagrams. Publ. Math. Inst. Hautes Études Sci., 93:31–105, 2001.
[41] Z. Sela. Diophantine geometry over groups. VII. The elementary theory of a hyperbolic group. Proc. Lond. Math. Soc. (3), 99(1):217–273, 2009.
[42] T. Soma. Non-zero degree maps to hyperbolic 3-manifolds. J. Differential Geom., 49(3):517–546, 1998.
[43] T. Soma. Sequences of degree-one maps between geometric 3-manifolds. Math. Ann., 316(4):733–742, 2000.
[44] T. Soma. Epimorphism sequences between hyperbolic 3-manifold groups. Proc. Amer. Math. Soc., 130(4):1221–1223, 2002.
[45] W. P. Thurston. Geometry and topology of three-manifolds. Princeton lecture notes available at http://www.msri.org/publications/books/gt3m/ 1980.
[46] L. van den Dries and A. J. Wilkie. Gromov’s theorem on groups of polynomial growth and elementary logic. *J. Algebra*, 89(2):349–374, 1984.

[47] S. Wang. Non-zero degree maps between 3-manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 457–468. Higher Ed. Press, Beijing, 2002.

[48] S. Wang and Q. Zhou. Any 3-manifold 1-dominates at most finitely many geometric 3-manifolds. *Math. Ann.*, 322(3):525–535, 2002.

[49] R. Weidmann. On accessibility of finitely generated groups. *Q. J. Math.*, 63(1):211–225, 2012.

[50] R. Weidmann and C. Reinfeldt. Makanin-Razborov diagrams for hyperbolic groups. *Ann. Math. Blaise Pascal*, 26(2):119–208, 2019.

[51] H. Wilton and P. Zalesskii. Profinite properties of graph manifolds. *Geom. Dedicata*, 147(1):29–45, 2010.