An Online Method for the Data Driven Stochastic Optimal Control Problem with Unknown Model Parameters

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Abstract

In this work, an efficient sample-wise data driven control solver will be developed to solve the stochastic optimal control problem with unknown model parameters. A direct filter method will be applied as an online parameter estimation method that dynamically estimates the target model parameters upon receiving the data, and a sample-wise optimal control solver will be provided to efficiently search for the optimal control. Then, an effective overarching algorithm will be introduced to combine the parameter estimator and the optimal control solver. Numerical experiments will be carried out to demonstrate the effectiveness and the efficiency of the sample-wise data driven control method.

1 Introduction

In this paper, we develop an efficient online method for solving the data driven stochastic optimal control problem with unknown model parameters. In the classic stochastic optimal control problem, we want to find an optimal "control process" that controls a stochastic dynamical system (called the "state process") to meet certain optimality conditions – such like minimizing a cost function. In many practical applications (e.g., smart grids [31, 36], nano-phase materials [18, 26, 34], and mathematical biology [11]), the controlled state model often contains unknown parameters. Therefore, we need to estimate the parameters first and then use the estimated parameters to design the optimal control. When parameter estimation and stochastic optimal control are combined together, the classic optimal control problem becomes the "data driven optimal control", which is an important research topic that requires uncertainty quantification. The concept of “data driven" in our control problem refers to an extra parameter estimation procedure [29], and the estimated parameters will be the “driver" that guides the design of the optimal control. Extensive studies have been carried out to solve data driven optimal control problems with time-invariant linear controlled systems [1, 40, 46]. In this work, we will focus on data driven methods for more general nonlinear control problems.

As a key research topic in mathematical modeling, parameter estimation has been extensively studied. The efforts in solving the parameter estimation problem aim to collect observational data about model states and use the data information to estimate parameters. There are two major approaches to estimate unknown parameters: the optimization based point estimation and the Bayesian estimation [24, 33, 35, 45], and both approaches can be implemented in either the offline manner or the online manner. For the offline parameter estimation, we use all the observational data that we have ever collected to estimate the unknown parameters. On the other hand, online parameter estimation methods dynamically estimate and update the parameters based on the newly received data. Online parameter estimation methods are well-studied, and they typically provide very accurate estimation results. However, the offline parameter estimation also has several disadvantages. First of all, when observations form a massive data set, large-scale data storage and analytics are needed, which often makes the offline parameter estimation prohibitive in practice. Especially, when a set of new observational data are received, offline methods need to go through all the historic data again to incorporate the incoming data, which can be very expensive. Secondly, although we often assume that the unknown

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parameters are static, it is also possible that the true values of the parameters change unexpectedly. For example, when estimating parameters that determine physical models in scientific experiments, the true values of the parameters may change unexpectedly due to sudden environmental changes or some unaware difference in the experimental set-up [18, 26]. In this case, offline parameter estimation methods are not suitable since the full data set may be generated by different true parameters.

The most widely used online parameter estimation method is the augmented filter [20, 28, 30]. The main idea of the augmented filter is to construct an augmented state-parameter process and let state estimation guide parameter estimation under the optimal filtering framework, which is a mathematical model that provides the best estimate for the state of a hidden stochastic dynamical system based on observational data [23, 32]. The drawback of the augmented filter is that the indirect parameter estimation strategy may not always be efficient enough in terms of usage of data. In fact, most effort in the optimal filtering procedure in the augmented filter would contribute to state estimation since the observations only describe the state, and the data only give direct evidence on the accuracy of state estimation.

In this work, we introduce a novel direct filter method to estimate the unknown parameters in the controlled state model. As an optimal filtering type approach, the direct filter defines a parameter process to model the unknown parameters. The novelty of the direct filter method is that we treat the state model as a dynamically designed “observation function”, which provides partial noisy observations on parameters. Then, we apply the particle filter method, which is a state-of-the-art optimal filtering method, to approximate the conditional probability density function (PDF) of the target parameters by using a set of random samples (called “particles”). Then the estimated parameters are characterized by the parameter particles that describe the PDF of the parameter variables [3]. The major difference between the direct filter and the augmented filter is that the direct filter does not use state estimation to drive parameter estimation. Instead, it utilizes the dynamical state model to “project” observational information to the parameter space, which would directly use observational data to estimate the unknown parameters.

The main contribution of this paper is to construct an effective overarching algorithm that dynamically combines the direct filter as an online parameter estimation method with an efficient stochastic optimal control solver and let the estimated parameters drive the design of the optimal control. Therefore, besides the direct filter, the stochastic optimal control solver is another important component in our data driven control algorithm. There are two well-known approaches to solve the stochastic optimal control problem: the dynamic programming principle and the stochastic maximum principle [16, 21, 38]. Both approaches involve numerical simulations for large (stochastic) differential systems. In this work, we choose the stochastic maximum principle approach due to its advantage in solving control problems with random coefficients in the state model and the fact that it could solve problems with finite dimensional terminal state constraints [47]. The mathematical foundation of the stochastic maximum principle approach is to derive a system of backward stochastic differential equation (BSDE) as the adjoint equation of the (forward) controlled state process, and we can use the solution of the adjoint BSDE to formulate the gradient of the cost function with respect to the control process. With the gradient process, the desired optimal control can be solved via gradient descent type optimization. There are several stochastic maximum principle based numerical methods to solve the stochastic optimal control problem [22, 44], and the primary computational cost in those methods lies in obtaining the numerical solution for the BSDE. Especially, when the dimension of the problem is high, a very large number of random samples are needed to describe the solution of the BSDE, which makes the numerical approximation for the gradient process very expensive [48, 49]. Since we aim to dynamically design control actions based on estimated parameters, efficiency of the algorithm is very important.

To achieve higher efficiency, we adopt a novel concept that the random samples used to describe the solution of the adjoint BSDE can be considered as “data samples”, and we shall provide an efficient sample-wise optimal control solver. The main theme of the sample-wise optimal control solver is that we only use one realization of random sample (or a small batch of samples) to represent the solution of the BSDE and derive numerical approximation for the solution on the single-realization of random sample. Although the solution of the BSDE is a random variable, which should be characterized by a set of statistically selected samples, the rationale that allows us to solve the BSDE by using only one realization of random sample is that the solution of the adjoint BSDE is used to formulate the gradient process, which will be used to carry out gradient descent optimization for the optimal control, and obtaining a complete approximation for the
solution of the adjoint BSDE is not needed in the stochastic optimal control problem [4]. In this connection, the methodology of applying stochastic approximation in the stochastic gradient descent optimization [41, 42] can justify the application of sample-wise approximation for the solution of the BSDE in the stochastic optimal control solver. As a result, we avoid solving the adjoint BSDE on a very large number of random samples, and we transfer the computational cost of completely solving BSDEs to the optimization procedure of searching for the desired optimal control, which would improve the efficiency of the stochastic optimal control solver.

By combining the direct filter with the sample-wise stochastic optimal control solver, in this paper we shall construct an efficient sample-wise data driven control solver. Upon receiving a set of new observational data, we first use the direct filter to estimate/update the unknown parameters in the state model. Then, we use the sample-wise stochastic optimal control solver to design control actions based on the state model determined by the estimated model parameters. To incorporate more data information, which is carried by the particles that we use to describe the parameter variables in the direct filter, we treat those particles as “data samples” as well. Therefore, instead of using the mean value of the particles as a point estimate for the parameter, at each iteration step in the stochastic gradient descent optimization, we randomly select one particle in the parameter particle cloud to represent the corresponding estimated model parameter. In this way, all the particles are considered in the iteration procedure for the optimal control, which makes our optimal control solver sufficiently data-driven.

The rest of this paper is organized as follows: In Section 2, we introduce the data driven stochastic optimal control problem with unknown parameters in the state model, and we will provide preliminary knowledge that is needed in the derivation of numerical algorithms. In Section 3, we shall provide detailed discussions to derive our sample-wise data driven control solver. Numerical experiments that demonstrate the effectiveness and efficiency of our algorithm will be presented in Section 4. Finally, we will conclude our research in Section 5.

2 Problem setting

In this section, we will first introduce the data driven stochastic optimal control problem with unknown parameters. Then, we shall provide the mathematical foundations that we need to derive the numerical methods for solving the data driven control problem.

In the complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})\) with \(\mathbb{F}^W := \{\mathbb{F}_s^W\}_{s \geq 0}\) being its natural filtration augmented by all the \(\mathbb{P}\)-null sets in \(\mathcal{F}\), we consider the following controlled dynamical system in the form of a stochastic differential equation (SDE)

\[
dX_t^\alpha = b(t, X_t^\alpha, u_t, \alpha)dt + \sigma(t, X_t^\alpha, u_t)dW_t, \quad t \in [0, T],
\]

where the vector \(\alpha \in \mathcal{A}\) with \(\mathcal{A} \subseteq \mathbb{R}^q\) being a set of unknown model parameters; \(u_t\) is the control process valued in some set \(U \subseteq \mathbb{R}^m\); \(X_t^\alpha \in \mathbb{R}^d\) is the controlled process driven by the parameter \(\alpha\) and the control process \(u\), which is usually called the “state process”; \(b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^d\) and \(\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^{d\times d}\) are suitable maps called drift and diffusion, respectively; \(W := \{W_t\}_{t \geq 0}\) is a standard \(d\)-dimensional Brownian motion in \((\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})\).

Denote

\[
\mathcal{U}[0, T] := \left\{ u : [0, T] \times \Omega \rightarrow U \mid u \text{ is } \mathbb{F}^W\text{-progressively measurable} \right\}.
\]

Under some mild conditions [47], for every choice of parameter \(\alpha \in \mathcal{A}\) and control \(u \in \mathcal{U}[0, T]\), SDE (1) admits a unique solution \(X^\alpha\). The performance of the control \(u\) is measured by the following cost functional

\[
J(u) = \mathbb{E} \left[ \int_0^T f(t, X_t^\alpha, u_t)dt + h(X_T^\alpha) \right],
\]

where \(f\) is the running cost rate and \(h\) is the terminal cost at \(T\). The goal of the stochastic optimal control problem is to find the “optimal control” \(\bar{u}^\alpha\) that minimizes the cost \(J\), i.e.

\[
J(\bar{u}^\alpha) = \inf_{u \in \mathcal{U}[0, T]} J(u).
\]
In the case that the parameter $\alpha$ is given, the classic stochastic optimal control problem can be solved via the following gradient descent type optimization.

Assume that the optimal control $\bar{u}^\alpha$ is in the interior of $U$. Then we can deduce by using the Gâteaux derivative of $u^\alpha$ and the maximum principle that the gradient process of the cost functional $J$ with respect to the control process on time interval $t \in [0, T]$ has the following form \[4\]

$$\nabla J_u(\bar{u}^\alpha) = \mathbb{E} \left[ b_u(t, \tilde{X}^\alpha_t, \tilde{u}^\alpha_t, \alpha) \right. \left. \nabla \tilde{Y}^\alpha_t + \sigma_u(t, \tilde{X}^\alpha_t, \tilde{u}^\alpha_t) \nabla \tilde{Z}^\alpha_t + f_u(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha) \right] ,$$

where subscripts are used to denote partial derivatives of functions. The stochastic process $(\tilde{X}^\alpha, \tilde{Y}^\alpha, \tilde{Z}^\alpha)$ is the adapted solution of the following forward backward stochastic differential equations (FBSDEs) system

$$d\tilde{X}_t^\alpha = b(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha, \alpha) dt + \sigma(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha) dW_t \quad \text{(FSDE)}$$

$$d\tilde{Y}_t^\alpha = \left( - b_x(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha, \alpha) \nabla \tilde{Y}_t^\alpha - \sigma_x(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha) \nabla \tilde{Z}_t^\alpha - f_x(t, \tilde{X}_t^\alpha, \tilde{u}_t^\alpha) \right) dt + \tilde{Z}_t^\alpha dW_t, \quad \tilde{Y}_T^\alpha = h_x(\tilde{X}_T^\alpha)^\top \quad \text{(BSDE)}$$

where the first equation in (5) is a forward SDE (FSDE, for short), which has the same form as the state equation \([38]\) in the stochastic optimal control problem; the second equation is a backward stochastic differential equation (BSDE) with given terminal condition at time $T$, and we call it the adjoint equation of the forward state equation \([38]\). Note that $\tilde{Z}^\alpha$ is the integrand in the Itô integral of the martingale representation for $\tilde{Y}^\alpha$ with respect to $W$. The existence and uniqueness of the FBSDEs \([47]\) have been well studied under some standard assumptions, and it is worthy of mentioning that the value of solution $\tilde{Y}^\alpha$ of the BSDE depends on state $\tilde{X}^\alpha$ \([47]\).

Then, with the gradient $\nabla J_u$ introduced in Eq. \([4]\) and the FBSDEs \([5]\) that provides solutions $(Y, Z)$ to describe $\nabla J_u$, we introduce the following gradient descent scheme

$$u_{l+1}^\alpha(t) = u_{l}^\alpha(t) - \rho_l \nabla J_u(u_{l}^\alpha), \quad 0 \leq t \leq T, \quad l = 0, 1, 2, \cdots, L - 1,$$

(6)

to find the optimal control, where $L$ is the total number of gradient descent iterations that satisfies certain stopping criteria, $\rho_l$ is the step-size for optimization, and we let $\{u_{l}^\alpha\}_{0 \leq l \leq T}$ be our initial guess for the optimal control.

The above mathematical framework for solving the stochastic optimal control problem requires the exact value for state $X_t^\alpha$ in Eq. \([1]\). However, in reality $X_t^\alpha$ may not always be explicitly given. Instead, we can often receive observations on $X_t^\alpha$ that provide information of the state. In this work, we let

$$M_t = X_t^\alpha + \xi_t$$

be noise perturbed direct observational data about $X_t^\alpha$ with a Gaussian error term $\xi_t \sim N(0, \Sigma)$, where $\Sigma$ is the covariance matrix, and we assume that the observations are accurate, i.e. the covariance $\Sigma$ of the error-size is small. Since the observational data are accurate, when $M_t$ is available we can use it as our estimate for $X_t^\alpha$ at time $t$. Then, we calculate the gradient process beyond time $t$ based on the estimate of the current state $X_t^\alpha$, i.e. the observation $M_t$. This allows us to adjust the control actions based on observations, which can provide real-time feedback on the controlled state. In the case that the observations are partial/indirect with relatively large errors, an extra filtering procedure is needed \([5, 8, 10]\) for the feedback control, which is out of scope of this work.

On the other hand, in this paper we assume that the parameter vector $\alpha$ in the data driven stochastic optimal control problem \([1]\) - \([3]\) is unknown. Therefore, in addition to implementing the gradient descent optimization procedure \([6]\), the primary challenge that we need to address is to design the optimal control based on the unknown model parameter $\alpha$, and we need an effective parameter estimation method to determine $\alpha$.

There are extensive studies on estimating model parameters by using observational data. Most existing methods focus on offline estimation, which needs to analyze all the historic data in order to carry out parameter estimation. This may be inefficient when large amount of data are collected, which will require large-scale data

\footnote{1In general, this might not be the case. But we only consider such a case, and leave the general case to our future investigation.}
storage and analytics. Moreover, in many practical scenarios in real-world applications, the true value of the parameter may change unpredictably. In this case, traditional offline parameter estimation methods are not suitable since the mixture of static data may be generated by different parameter values, and it may neglect the sequential order of data reception, which also contains important information about the parameter.

The standard technique to dynamically estimate unknown parameters in the online manner is the optimal filtering, which aims to find the best estimate for the state of a (hidden) dynamical system by using partial noisy observational data. The most important approach that uses the optimal filtering to estimate parameters is to construct an augmented state-parameter vector, and then estimate the augmented state vector by using the observational information. Such an approach is often called the “augmented filter”.

Specifically, the augmented filter considers the following state process

$$dS_t = B(S_t)dt + \sigma_t d\bar{W}_t,$$

where \( S_t := [X_t^\alpha, \alpha_t]^{\top} \in \mathbb{R}^{d+q} \) is a \((d+q)\)-dimensional augmented vector that combines the controlled state \( X_t^\alpha \in \mathbb{R}^d \) and the parameter \( \alpha \in \mathbb{R}^q \). Note that we treat the unknown parameter \( \alpha \) as a stochastic process with “zero” dynamics in Eq. (5). The function \( B(S_t) := [b(t, X_t^\alpha, u_t^\alpha, \alpha), 0_q]^{\top} \) augments the controlled state dynamics \( b \) with the “zero” dynamics of the parameter process \( \alpha_t \), and \( \sigma_t, \bar{W}_t \) represent the augmented diffusion coefficient and the augmented Brownian motion, respectively. Then, by using the same measurement process \( M_t \) introduced in Eq. (7), the augmented filter aims to find the best estimate for the augmented state \( S \) by using the observational information \( M_t := \sigma(M_s, 0 \leq s \leq t) \). The mathematical expression for the “best estimate” under the optimal filtering framework is the conditional expectation of \( S_t \) conditioning on \( M_t \), i.e. \( \mathbb{E}[S_t|M_t] \). In this way, the observational data \( M \) about \( X \) give partial noisy observations on \( S \), which includes the unknown parameter. The estimation procedure for state \( S \) (based on data about \( X \)) will drive the estimation procedure for parameter \( \alpha \), which would eventually generate an estimate for the parameter. Such an augmented optimal filtering problem can be solved by several effective optimal filtering methods. The most important approach for solving the optimal filtering problem is the recursive Bayesian filter, which includes the Kalman type filters [19, 25], the particle filters [2, 27], and our recently developed backward SDE filter [6, 7, 9, 12–15].

Although the idea of combining a model and its parameters in an augmented state process in the augmented filter method provides a feasible framework to dynamically estimate the unknown parameters, the strategy of using the accuracy of state estimation to determine whether estimated parameters would fit the model only provides indirect evidence to the accuracy of parameter estimation. This would make the augmented filter less accurate especially when estimating parameters in more complicated dynamical models.

In the following section, we shall introduce an effective novel sample-wise data driven control method for solving the data driven stochastic optimal control problem (1) – (3).

3 Numerical derivation for the sample-wise data driven control method

Our sample-wise data driven control solver consists two major computational components: (i) a direct filter method to implement online parameter estimation, and (ii) an efficient sample-wise optimal control solver, which is driven by parameter estimation. In what follows, we first introduce the direct filter algorithm in Section 3.1. Then, in Section 3.2 we introduce a fully calculated stochastic optimal control solver for the stochastic optimal control problem with estimated parameters. Finally, in Section 3.3 we introduce a sample-wise optimization procedure to efficiently calculate the optimal control, and we let the direct filter based parameter estimation guide the optimal control solver.

In most situations, it’s very difficult to control the diffusion coefficient \( \sigma \). So we let \( \sigma \) be independent from control \( u \). Also, in this work we assume that \( \sigma \) does not depend on the state \( X \) either (i.e. \( \sigma(t, X_t^\alpha, u_t) = \sigma(t) \)), which is a common assumption in the parameter estimation problem. Despite of the aforementioned assumptions, the main difficulties that challenge the online parameter estimation problem and the stochastic optimal control problem still remain in the computational framework that we shall introduce.
In order to derive our numerical algorithm, we introduce the following temporal partition over the time interval \([0, T]\), i.e.

\[
\Pi_T^N = \{ t_n : 0 = t_0 < t_1 < t_2 < \cdots < t_{NT} = T \},
\]

where \(N_T\) is the total number of time steps, which also represents how many sets of observational data \(\{M_{t_n}\}_{n=1}^{N_T}\) that will be received, and we let \(\Delta t_n := t_{n+1} - t_n\) be the temporal step-size. For convenience of presentation, in this work we assume that the temporal partition \(\Pi_T^N\) is uniform and denote \(\Delta t = \frac{T}{N_T}\).

### 3.1 Direct filter for parameter estimation

In the direct filter method, instead of constructing an augmented state-parameter dynamics Eq. (8) as in the augmented filter approach, we only consider a parameter process. With the temporal partition \(\Pi_T^N\), we introduce the following zero dynamics to model the unknown parameter as a stochastic process:

\[
\alpha_{t_n+1} = \alpha_{t_n} + \gamma_n, \quad n = 0, 1, 2, \cdots, N_T - 1,
\]

where \(\gamma_n\) is a Gaussian random variable with a pre-determined covariance, which is an artificial noise added to the parameter process. In this work, we present our direct filter method following the particle filter implementation \([23]\). Other effective nonlinear filtering methods may also be applied to implement the direct filter following the similar methodology.

At time instant \(t_n\), assuming we have a set of \(M\) samples ("particles"), denoted by \(\{\zeta_n^{(m)}\}_{m=1}^{M}\), that describe the conditional probability distribution of the parameter \(\alpha_{t_n}\) giving the observational information \(M_{t_n}\), we introduce the empirical distribution \(\bar{P}(\alpha_{t_n} | M_{t_n})\) as follows:

\[
P(\alpha_{t_n} | M_{t_n}) \approx \bar{P}(\alpha_{t_n} | M_{t_n}) := \frac{1}{M} \sum_{m=1}^{M} \delta_{\zeta_n^{(m)}}(\alpha_{t_n}),
\]

where \(\delta\) is a delta function.

From time \(t_n\) to \(t_{n+1}\), the direct filter algorithm is composed of two steps: a prediction step and an update step:

**Prediction step:** In the prediction step, we use the Chapman-Kolmogorov formula, i.e.

\[
P(\alpha_{t_{n+1}} | M_{t_n}) = \int P(\alpha_{t_n} | M_{t_n}) P(\alpha_{t_{n+1}} | \alpha_{t_n}) d\alpha_{t_n},
\]

to generate a set of proposed particles and predict possible parameter values, where \(P(\alpha_{t_{n+1}} | \alpha_{t_n})\) is the transition probability that propagates the particles from time \(t_n\) to \(t_{n+1}\). Since there’s no prior knowledge about the parameter and we use the zero dynamics for the parameter process, the transition probability just produces random walks. In this way, for each particle \(\zeta_n^{(m)}\) we just add a version of Gaussian random variable, i.e. \(\gamma_n^{(m)}\), to inflate \(\zeta_n^{(m)}\) with some artificial noise and get

\[
\zeta_{n+1}^{(m)} = \zeta_n^{(m)} + \gamma_n^{(m)}.
\]

Then, we obtain the empirical distribution that approximates the (prior) conditional PDF \(P(\alpha_{t_{n+1}} | M_{t_n})\) as follows:

\[
P(\alpha_{t_{n+1}} | M_{t_n}) \approx \tilde{P}(\alpha_{t_{n+1}} | M_{t_n}) = \frac{1}{M} \sum_{m=1}^{M} \delta_{\zeta_{n+1}^{(m)}}(\alpha_{t_{n+1}}).
\]

**Update step:** In the update step, one needs to incorporate the observational information into the prediction and update the prior distribution to obtain a posterior distribution. This procedure is typically achieved by Bayesian inference. In the online parameter estimation problem that we consider, the observational data only provide observations on the state, which don’t directly reflect parameter states. On the other hand, since
the parameter variable is not augmented with the state variable in the direct filter method, we don’t have an estimated state to be compared with the data.

The main idea of the direct filter is to treat the dynamical model as a pseudo “observation function”. Specifically, we let the state dynamics map a proposed parameter sample \( \tilde{\zeta}_{n+1}^{(m)} \) to a state point \( X_{n+1}^{\tilde{\zeta}_{n+1}^{(m)}} \) that describes \( X^n_{t_{n+1}} \), i.e.

\[
X_{n+1}^{\tilde{\zeta}_{n+1}^{(m)}} := X^n_{t_{n+1}} + b^n(t_{n+1}, X^n_{t_{n+1}}, u^n_{t_{n+1}}, \tilde{\zeta}_{n+1}^{(m)}) \Delta t,
\]

where \( u^{\alpha}_{t_{n+1}} \) is an estimated optimal control at the \( t_{n+1} \)-th iteration step and \( X^n_{t_{n+1}} \) is an estimate for the state variable \( X^n_{t_{n+1}} \). Since data \( M_{t_{n+1}} \) directly observe \( X^n_{t_{n+1}} \), we let the direct filter estimated parameter at the time instant \( t_{n+1} \) be \( \hat{\alpha}_{t_{n+1}} \).

In this paper, we introduce the following likelihood function to measure the likelihood of the sample \( \tilde{\zeta}_{n+1}^{(m)} \) in fitting the observational data, where the dynamics Eq. (12) that we use to generate \( X_{n+1}^{\tilde{\zeta}_{n+1}^{(m)}} \) contains both the parameter and the state, and it connects the proposed parameter \( \tilde{\zeta}_{n+1}^{(m)} \) to the estimated state \( X_{n+1}^{\tilde{\zeta}_{n+1}^{(m)}} \). Note that the observational error (with covariance \( \Sigma \)) is incorporated into the likelihood together with the diffusion term of the model through function (13).

Then, we apply Bayesian inference with likelihood \( p(M_{t_{n+1}}|\tilde{\zeta}_{n+1}^{(m)}) \) and empirical prior \( \hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}}) \) (introduced in Eq. (11)) to obtain an empirical posterior distribution \( \hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}}) \) as follows

\[
\hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}}) := \frac{p(M_{t_{n+1}}|\tilde{\zeta}_{n+1}^{(m)})\hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}})}{C},
\]

where \( C \) is a normalization factor. The practical implementation of the above Bayesian inference scheme is to assign a weight \( w_{n+1}^{(m)} \) to each sample \( \tilde{\zeta}_{n+1}^{(m)} \) with the likelihood value, i.e. \( w_{n+1}^{(m)} := p(M_{t_{n+1}}|\tilde{\zeta}_{n+1}^{(m)})/C \), and the empirical distribution \( \hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}}) \) becomes

\[
\hat{\pi}(\alpha_{t_{n+1}}|M_{t_{n+1}}) = \sum_{m=1}^{M} w_{n+1}^{(m)} \delta_{\tilde{\zeta}_{n+1}^{(m)}}(\alpha_{t_{n+1}}) \tag{14}
\]

To stabilize the particle filter algorithm and to alleviate the so-called “degeneracy issue” \[43\], we resample the particle cloud \( \{ \tilde{\zeta}_{n+1}^{(m)} \}_m \) and get an equally weighted set of samples \( \{ \tilde{\zeta}_{n+1}^{(m)} \}_m \). There are several well-designed resampling methods, and we refer to \[2\|\[7\|\[8\] for some examples. As a result of resampling, we obtain an empirical distribution \( \hat{P}(\alpha_{t_{n+1}}|M_{t_{n+1}}) \) described by \( \{ \tilde{\zeta}_{n+1}^{(m)} \}_m \) as

\[
\hat{P}(\alpha_{t_{n+1}}|M_{t_{n+1}}) := \frac{1}{M} \sum_{m=1}^{M} \delta_{\tilde{\zeta}_{n+1}^{(m)}}(\alpha_{t_{n+1}}),
\]

which will be used in the next recursive prediction-update procedure. With the empirical distribution \( \hat{P}(\alpha_{t_{n+1}}|M_{t_{n+1}}) \) and the particles \( \{ \tilde{\zeta}_{n+1}^{(m)} \}_m \), we let the direct filter estimated parameter at the time instant \( t_{n+1} \) be \( \hat{\alpha}_{t_{n+1}} \), which is defined as

\[
\hat{\alpha}_{t_{n+1}} := \mathbb{E}[\alpha_{t_{n+1}}|M_{t_{n+1}}] = \frac{1}{M} \sum_{m=1}^{M} \tilde{\zeta}_{n+1}^{(m)}.
\]

It’s worthy to mention that the estimated state \( \hat{X}_{n}^{\alpha} \), which describes the state \( X_{t_{n+1}}^{\alpha} \) in Eq. (12), is needed to implement the above direct filter method. Since we do not know the exact value of \( X_{t_{n+1}}^{\alpha} \) and our observational data \( M_{t_{n+1}} = X_{t_{n+1}}^{\alpha} + \xi_{t_{n+1}} \) has high accuracy (with low noise level for \( \xi_{t_{n+1}} \)), in the direct filter method we use the data, i.e. \( M_{t_{n+1}} \), as our “best” estimate for \( X_{t_{n+1}}^{\alpha} \). In this way, the estimated state \( \hat{X}_{n}^{\alpha} \) in Eq. (12) is replaced by \( M_{t_{n+1}} \) in this work.

In the following subsection, we introduce the standard fully calculated stochastic optimal control solver to implement the gradient descent optimization scheme \[9\].
3.2 Fully calculated stochastic optimal control solver

We first assume that we have an estimated parameter \( \hat{\alpha} \). The gradient descent optimization procedure will be implemented on discrete temporal points over the temporal partition \( \Pi_T^N \), i.e.

\[
 u_{t_n}^{l,\hat{\alpha}} = u_{t_n}^{l,\hat{\alpha}} - \rho t \nabla J_u(u_{t_n}^{l,\hat{\alpha}}), \quad n = 0, 1, 2, \cdots, N_T - 1,
\]

and the gradient \( \nabla J_u(u_{t_n}^{l,\hat{\alpha}}) \) at time step \( t_n \) is given by

\[
 \nabla J_u(u_{t_n}^{l,\hat{\alpha}}) = E \left[ b_u(t_n, X_{t_n}^{l,\hat{\alpha}}, u_{t_n}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t_n}^{l,\hat{\alpha}} + f_u(t_n, X_{t_n}^{l,\hat{\alpha}}, u_{t_n}^{l,\hat{\alpha}}) \right],
\]

where \( Y_{t_n}^{l,\hat{\alpha}} \) and \( Z_{t_n}^{l,\hat{\alpha}} \) are solutions in the FBSDEs system Eq. (5) driven by the estimated parameter \( \hat{\alpha} \) and the control \( u_{t_n}^{l,\hat{\alpha}} \). Hence, numerical approximations for \( X_{t_n}^{l,\hat{\alpha}} \) and \( Y_{t_n}^{l,\hat{\alpha}} \) are needed over the time partition \( \Pi_T^N \).

In what follows, we shall introduce a standard numerical scheme to solve the FBSDEs.

Consider the FBSDEs system on the sub-interval \([t_n, t_{n+1}]\), i.e.

\[
 X_{t_{n+1}}^{l,\hat{\alpha}} = X_{t_n}^{l,\hat{\alpha}} + \int_{t_n}^{t_{n+1}} b(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}}, \hat{\alpha}) dt + \int_{t_n}^{t_{n+1}} \sigma(t) dW_t, \quad (\text{FSDE})
\]

\[
 Y_{t_n}^{l,\hat{\alpha}} = Y_{t_{n+1}}^{l,\hat{\alpha}} + \int_{t_n}^{t_{n+1}} b_z(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t}^{l,\hat{\alpha}} + f_u(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}})^T dt
 - \int_{t_n}^{t_{n+1}} Z_{t}^{l,\hat{\alpha}} dW_t, \quad (\text{BSDE})
\]

The forward SDE in Eq. (16) can be solved by the following classic Euler-Maruyama scheme

\[
 X_{t_{n+1}}^{l,\hat{\alpha}} = X_{t_n}^{l,\hat{\alpha}} + b(t_n, X_{t_n}^{l,\hat{\alpha}}, u_{t_n}^{l,\hat{\alpha}}, \hat{\alpha}) \Delta t + \sigma(t_n) \Delta W_{t_n} + R_X^n,
\]

where \( \Delta W_{t_n} := W_{t_{n+1}} - W_{t_n} \) and

\[
 R_X^n := \int_{t_n}^{t_{n+1}} b(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}}, \hat{\alpha}) dt - b(t_n, X_{t_n}^{l,\hat{\alpha}}, u_{t_n}^{l,\hat{\alpha}}, \hat{\alpha}) \Delta t + \int_{t_n}^{t_{n+1}} \sigma(t) dW_t - \sigma(t_n) \Delta W_{t_n}
\]

is the approximation error term.

To solve the BSDE and obtain a numerical solution for \( Y \), we use the right-point formula to approximate the deterministic integral on the right hand side of the BSDE in Eq. (16) and take the conditional expectation \( \mathbb{E}_n[.] := \mathbb{E}[\cdot | X_{t_n}^{l,\hat{\alpha}}] \) on both side of the equation to get

\[
 Y_{t_n}^{l,\hat{\alpha}} = \mathbb{E}_n[Y_{t_{n+1}}^{l,\hat{\alpha}}] + \mathbb{E}_n \left[ b_z(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t_{n+1}}^{l,\hat{\alpha}} + f_u(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}})^T \right] \Delta t + R_Y^n,
\]

where

\[
 R_Y^n := \mathbb{E}_n \left[ \int_{t_n}^{t_{n+1}} \left( b_z(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t}^{l,\hat{\alpha}} + f_u(t, X_{t}^{l,\hat{\alpha}}, u_{t}^{l,\hat{\alpha}})^T \right) dt \right]
 - \mathbb{E}_n \left[ b_z(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t_{n+1}}^{l,\hat{\alpha}} + f_u(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}})^T \right] \Delta t
\]

is the approximation error for the deterministic integral, and the stochastic integral \( \int_{t_n}^{t_{n+1}} Z_{t}^{l,\hat{\alpha}} dW_t \) is eliminated due to the martingale property of Itô integral, i.e. \( \mathbb{E}_n \left[ \int_{t_n}^{t_{n+1}} Z_{t}^{l,\hat{\alpha}} dW_t \right] = 0 \). Note that we have also used that fact that \( Y_{t_n}^{l,\hat{\alpha}} = \mathbb{E}_n \left[ Y_{t_{n+1}}^{l,\hat{\alpha}} \right] \).

By dropping the error terms \( R_X^n \) and \( R_Y^n \) in (17) and (18), respectively, we have the following numerical scheme for solving the FBSDEs system (16):

\[
 X_{t_{n+1}}^{l,\hat{\alpha}} = X_{t_n}^{l,\hat{\alpha}} + b(t_n, X_{t_n}^{l,\hat{\alpha}}, u_{t_n}^{l,\hat{\alpha}}, \hat{\alpha}) \Delta t + \sigma(t_n) \Delta W_{t_n},
\]

\[
 Y_{t_n}^{l,\hat{\alpha}} = \mathbb{E}_n[Y_{t_{n+1}}^{l,\hat{\alpha}}] + \mathbb{E}_n \left[ b_z(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}}, \hat{\alpha})^T Y_{t_{n+1}}^{l,\hat{\alpha}} + f_u(t_{n+1}, X_{t_{n+1}}^{l,\hat{\alpha}}, u_{t_{n+1}}^{l,\hat{\alpha}})^T \right] \Delta t,
\]

8
where \(X^l_{n+1}\) and \(Y^l_{n+1}\) are approximations to solutions \(X\) and \(Y\) in Eq. (16) with a given estimated parameter \(\hat{\alpha}\) and an estimated optimal control \(u^l_{\hat{\alpha}}\). Although \(Z^l_{\hat{\alpha}}\) is also a solution in Eq. (16), since it does not appear in the expression (15) for the gradient \(\nabla J_u(u^l_{\hat{\alpha}})\) and \(Z\) is not included in the numerical schemes for \(X\) and \(Y\), we don’t need to obtain a numerical solution for \(Z\) in the data driven control problem that we try to solve in this work. Then, with approximation schemes (19), the gradient \(\nabla J_u(u^l_{\hat{\alpha}})\) described in Eq. (15) can be approximated as

\[
\nabla J_u(u^l_{\hat{\alpha}}) \approx \mathbb{E} \left[ b_u(t_n, X^l_{n}, u^l_{\hat{\alpha}}) \nabla Y^l_{n} + f_u(t_n, X^l_{n}, u^l_{\hat{\alpha}}) \right].
\]

Since we apply online methods to estimate the unknown parameters in this paper, the estimated parameter \(\hat{\alpha}\) in Eq. (20) becomes \(\hat{\alpha}_{t_n}\), which is the dynamically estimated parameter generated at time \(t_n\). Note that other online parameter estimation methods can also be applied in the above stochastic optimal control solver to drive the design of the optimal control. As a result, with the estimated parameter \(\hat{\alpha}_{t_n}\) at time \(t_n\), we estimate the gradient \(\nabla J_u\) for a future time \(t_{n+1}\) (\(n \geq n\)) by

\[
\nabla J_u(u^l_{\hat{\alpha}_{t_n}}) \approx \mathbb{E} \left[ b_u(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}) \nabla Y^l_{n} + f_u(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}) \right], \quad n \leq n_s \leq N_T.
\]

When \(n = 0\), the estimated parameter \(\hat{\alpha}_{t_n}\) is chosen as our initial guess for the parameter.

In order to implement the gradient descent optimization procedure, we need to approximate the conditional expectation \(\mathbb{E}[\cdot]|\) in Eq. (19) and the expectation \(\mathbb{E}[\cdot]\) in Eq. (21) with respect to the state process \(X\). The standard approach to approximate expectations is the Monte Carlo method.

To be specific, we generate \(P\) realizations of simulated paths for \(X^l_{n+1}\) as

\[
X^l_{n+1}(p) = X^l_{n} + b(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}, \hat{\alpha}_{t_n}) \Delta t + \sigma(t_n) \Delta W^l_{n+1}(p), \quad n_s \geq n, \quad p = 1, 2, \ldots, P,
\]

where \(\Delta W^l_{n+1}(p) \sim N(0, \Delta t \cdot I_d)\) is the \(p\)-th realization of \(\Delta W_{n+1}\), and \(X^l_{n}\) is a “point of interest” in the state space at which we approximate solution \(Y^l_{n}\). Then, the Monte Carlo approximation based numerical solution \(Y^l_{n}\) corresponding to the state point \(X^l_{n}\) becomes

\[
Y^l_{n}(X^l_{n}) = \sum_{p=1}^{P} \frac{1}{P} b_x(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}, \hat{\alpha}_{t_n}) \nabla Y^l_{n} + f_u(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}) \Delta t
\]

\[
+ \sum_{p=1}^{P} \frac{1}{P} f_x(t_n, X^l_{n}, u^l_{\hat{\alpha}_{t_n}}) \Delta t.
\]

In order to approximate the expectation \(\mathbb{E}[\cdot]\) in Eq. (21), we need to simulate another set of sample paths for \(X\) as

\[
X^l_{n+1}(q) = X^l_{n}(q) + b(t_n, X^l_{n}(q), u^l_{\hat{\alpha}_{t_n}}, \hat{\alpha}_{t_n}) \Delta t + \sigma(t_n) \Delta W_{n+1}(q), \quad n_s \geq n, \quad q = 1, 2, \ldots, Q.
\]

Then, the fully calculated Monte Carlo approximation for the gradient process, denoted by \(\nabla \hat{J}_u(u^l_{\hat{\alpha}_{t_n}})\), is computed as follows:

\[
\nabla \hat{J}_u(u^l_{\hat{\alpha}_{t_n}}) = \frac{1}{Q} \sum_{q=1}^{Q} \left( b_u(t_n, X^l_{n}(q), u^l_{\hat{\alpha}_{t_n}}, \hat{\alpha}_{t_n}) \nabla Y^l_{n}(X^l_{n}(q)) + f_u(t_n, X^l_{n}(q), u^l_{\hat{\alpha}_{t_n}}) \right),
\]

and the gradient descent procedure for the optimal control becomes

\[
\hat{u}^l_{n+1}(t_{n+1}) = \hat{u}^l_{n}(t_{n+1}) - \rho \nabla \hat{J}_u(u^l_{\hat{\alpha}_{t_n}}), \quad n \leq n_s \leq N_T.
\]

We refer to [22] for more detailed discussions of the above fully calculated stochastic optimal control solver.
There are two major computational challenges to carry out the fully calculated gradient descent optimization scheme (26). First of all, there are two Monte Carlo simulations involved in the approximation of $\nabla \hat{J}_u(\tilde{u}_{\alpha^{t_n}})$: (i) Monte Carlo simulation for the conditional expectation $E_n[\cdot]$ as described in Eq. (23) (to be implemented on each spatial-state point $X_{n+1}$ at each time step $t_n$), and (ii) Monte Carlo simulation for the expectation $E[\cdot]$ as described in Eq. (25). Hence it requires a very large number of Monte Carlo samples to approximate the gradient at each iteration step. Secondly, in order to use $\hat{Y}_{n+1}^{\alpha^{t_n}}$ to calculate the gradient $\nabla \hat{J}_u(\tilde{u}_{\alpha^{t_n}})$, a complete numerical approximation for $\hat{Y}_{n+1}^{\alpha^{t_n}}$ in the entire state space is needed since $\{X_{n+1}^{\alpha^{t_n}}\}_{q=1}^{Q}$ are randomly generated sample points, which may appear anywhere in the state space. Those two computational challenges make the fully calculated stochastic optimal control solver (22 - 26) extremely computationally expensive - especially when solving high dimensional problems.

In what follows, we shall introduce a sample-wise data driven control solver to efficiently implement the gradient descent optimization scheme (26) and to incorporate the probability distribution $P(\alpha_{t_n}|\mathcal{M}_{t_n})$ for the unknown parameter (obtained by the direct filter) into the optimization procedure.

3.3 Sample-wise data driven optimal control solver

The novel concept that we introduce in the sample-wise data driven control solver is to treat the random samples generated by the state variable as “data samples”. Then, we apply stochastic approximation in the gradient descent optimization procedure to improve the efficiency of numerical implementation of the stochastic optimal control solver.

Recall that in the fully calculated stochastic optimal control solver, we use a set of $P$ samples, i.e. $\{X_{n+1}^{\alpha_{t_n}(p)}\}_{p=1}^{P}$, to represent the state variable $X_{n+1}^{\alpha_{t_n}}$, and we approximate the conditional expectation $E_n[\cdot]$ in scheme (19) by using the Monte Carlo average of all $P$ samples. In addition, we simulate $Q$ realizations of sample paths, i.e. $\{X_{n,q}^{\alpha_{t_n}}\}_{q=1}^{Q}$, to approximate the expectation $E[\cdot]$ in the gradient process. Note that both sample sets $\{X_{n+1}^{\alpha_{t_n}(p)}\}_{p=1}^{P}$ and $\{X_{n,q}^{\alpha_{t_n}}\}_{q=1}^{Q}$ are generated to approximate expectations, which appear in the gradient process. Therefore, we can apply the stochastic approximation method to the gradient descent optimization procedure and approximate the expectations by using a single realization of state sample (or a small batch of samples) to represent the state process.

Specifically, we generate a sample path for the state process, denoted by $\{X_{n}^{\alpha_{t_n}}\}_{n=0}^{N_T}$, corresponding to the $l$-th gradient descent iteration step as follows

$$
\hat{X}_{n+1}^{l,\alpha_{t_n}} = \hat{X}_{n}^{l,\alpha_{t_n}} + b(t_n, \hat{X}_{n}^{l,\alpha_{t_n}}, \hat{u}_{\alpha_{t_n}})\Delta t + \sigma(t_n)\Delta W_{l,n}^{(t)} , \quad n \leq n_s \leq N_T - 1 ,
$$

(27)

where $\hat{u}_{\alpha_{t_n}}$ is an estimated optimal control and $W_{l,n}^{(t)}$ is a sample of Brownian motion corresponding to the iteration index $l$. With the simulated sample path representing the state process, we approximate the conditional expectations in the scheme (19) by using only one sample of $X$, i.e. $\{X_{n}^{\alpha_{t_n}}\}_{n=0}^{N_T}$, instead of using altogether $P$ samples in the fully calculated scheme (23). As a result, we have the following sample-wise approximation for solution $Y$ of the BSDE in Eq. (16):

$$
\hat{Y}_{n}^{l,\alpha_{t_n}}(X_{n}^{l,\alpha_{t_n}}) = \hat{Y}_{n+1}^{l,\alpha_{t_n}}(\hat{X}_{n+1}^{l,\alpha_{t_n}}) + \left(b(t_{n+1}, \hat{X}_{n+1}^{l,\alpha_{t_n}}, \hat{u}_{\alpha_{t_n}})\hat{Y}_{n+1}^{l,\alpha_{t_n}}(\hat{X}_{n+1}^{l,\alpha_{t_n}}) + f_x(t_{n+1}, \hat{X}_{n+1}^{l,\alpha_{t_n}}, \hat{u}_{\alpha_{t_n}})\right)\Delta t , \quad n \leq n_s \leq N_T - 1 .
$$

(28)

Moreover, we also use the sample path $\{X_{n}^{\alpha_{t_n}}\}_{n=0}^{N_T}$ generated in Eq. (27) to represent the simulated sample paths $\{X_{n,q}^{\alpha_{t_n}}\}_{q=1}^{Q}$ that we use in the Monte Carlo approximation for the gradient process as described in Eq. (25). In this way, we introduce the following sample-wise approximation for the gradient process

$$
\nabla \hat{J}_u(\hat{u}_{\alpha_{t_n}}) = b_u(t_n, X_{n}^{\alpha_{t_n}}, u_{\alpha_{t_n}})\hat{Y}_{n}^{l,\alpha_{t_n}}(X_{n}^{l,\alpha_{t_n}}) \\
+ f_u(t_n, X_{n}^{\alpha_{t_n}}, u_{\alpha_{t_n}}) , \quad n \leq n_s \leq N_T,
$$

(29)

10
where $\tilde{X}_{n_s}^{l,\hat{\alpha}_{t_n}}$ and $\tilde{Y}_{n_s}^{l,\hat{\alpha}_{t_n}}$ (or $\tilde{X}_{n_s}^{l,\hat{\alpha}_{t_n}}$) are calculated via Eq. (27) and Eq. (28), respectively. Then, we implement the following stochastic gradient descent optimization procedure to find the optimal control

$$u_{n_s}^{l+1,\hat{\alpha}_{t_n}} = \tilde{u}_{n_s}^{l,\hat{\alpha}_{t_n}} - \rho_t \nabla \tilde{J}_u(\tilde{u}_{n_s}^{l,\hat{\alpha}_{t_n}}), \quad n \leq n_s \leq N_T, \quad l = 0, 1, 2, \ldots, L. \quad (30)$$

The numerical schemes (27) - (30) constitute the sample-wise data driven control method, and the dynamically estimated parameter $\hat{\alpha}_{t_n}$ is the driver that guides the sample-wise optimal control solver.

It’s worthy to point out that the scheme (28) only provides a sample-wise approximation to $Y$. It does not calculate the numerical solution for $Y$ completely, and accuracy of the sample-wise approximation $Y$ is not guaranteed. Actually, approximation for $Y$ in the entire state space of $X$ is needed if we want to obtain accurate numerical solutions for BSDEs, and Monte Carlo simulation for conditional expectations cannot be avoided (unless other numerical integration methods are applied). However, since the BSDE that we consider in the stochastic optimal control problem is the adjoint equation of the (forward) state equation and $Y$ only appears in the gradient process under expectation, obtaining a complete numerical solution for $Y$ is not needed in the optimization procedure for finding the optimal control. In this connection, the justification for applying stochastic approximation in stochastic gradient descent type optimization can be used to explain the methodology of our sample-wise optimal control solver.

In this work, we use the direct filter introduced in Section 3.1 to be the online parameter estimation method to calculate the estimated parameter $\hat{\alpha}_{t_n}$. In order to sufficiently incorporate parameter distribution $P(\alpha_{t_n} | M_{t_n})$, which contains more data information than the average value of parameter particles, into the stochastic optimal control solver, we randomly select $\zeta^{(l)}_n \sim \{\zeta^{(m)}_n\}_{m=1}^M$ to replace the mean value estimate $\hat{\alpha}_{t_n}$ in each stochastic gradient descent iteration step. In this way, we rewrite the approximation scheme (29) for the gradient process as

$$\nabla \tilde{J}_u(\tilde{u}_{n_s}^{l,\zeta^{(l)}_n}) = b_u(t_{n_s}, \tilde{X}_{n_s}^{l,\zeta^{(l)}_n}, \tilde{u}_{n_s}^{l,\zeta^{(l)}_n}, \hat{\alpha}_{t_n})^\top \tilde{Y}_{n_s}^{l,\zeta^{(l)}_n}(\tilde{X}_{n_s}^{l,\zeta^{(l)}_n})$$

$$+ f_u(t_{n_s}, \tilde{X}_{n_s}^{l,\zeta^{(l)}_n}, \tilde{u}_{n_s}^{l,\zeta^{(l)}_n})^\top, \quad n \leq n_s \leq N_T. \quad (31)$$

As a result, when we run a large enough number of iteration steps, the distribution of parameter particles will be well incorporated into the stochastic optimal control solver.

With the sample-wise gradient approximator $\nabla \tilde{J}_u$ introduced in Eq. (31), we carry out the following iteration scheme to implement the optimization procedure for determining the optimal control:

$$u_{n_s}^{l+1,\zeta^{(l)}_n} = \tilde{u}_{n_s}^{l,\zeta^{(l)}_n} - \rho_t \nabla \tilde{J}_u(\tilde{u}_{n_s}^{l,\zeta^{(l)}_n}), \quad n \leq n_s \leq N_T, \quad l = 0, 1, 2, \ldots, L. \quad (32)$$

### 3.4 Summary of the algorithm

In Table 1 we provide a pseudo algorithm to summarize the computational framework of our online method for data driven stochastic optimal control with unknown model parameters.

As a summary, the sample-wise data driven control (SW-DDC) utilizes the direct filter as an online parameter estimation method to drive the design of the optimal control. Instead of implementing the fully calculated stochastic optimal control solver (introduced in Section 3.2), which requires approximating the gradient process with respect to the optimal control in the entire state space, we treat the random samples that describe the state variable (as well as the parameter variable) as “data samples” and adopt the methodology of stochastic approximation to approximate expectations in the gradient process by using a single simulated state (parameter) sample or a small batch of samples. As a result, the SW-DDC solver can efficiently solve the data driven stochastic optimal control problem with unknown parameters.

### 4 Numerical Experiments

In this section, we use two numerical examples to examine the performance of our SW-DDC method. In the first example, we solve a classic linear-quadratic (LQ) stochastic optimal control problem with unknown
Table 1: Summary of the algorithm

**Algorithm:** Sample-wise data driven control (SW-DDC)

| Description | Details |
|-------------|---------|
| Initialize the number of particles $M$, the learning rate $\rho_l$ and the number of iterations $L \in \mathbb{N}$. |  |
| Generate parameter particles $\{\xi_0^{(m)}\}_{m=1}^M$ to represent the initial guess of the unknown parameter and introduce an initial guess for the optimal control. |  |
| while $n = 0, 1, 2, \ldots, N_T - 1$, do |  |
| • Implement the sample-wise optimal control solver with $\{\xi_n^{(m)}\}_{m=1}^M$: |  |
| for $l = 1, 2, \ldots L$ |  |
| - Generate a sample path $\{\tilde{X}_{n_{s}}^{l, \xi_n^{(l)}}\}_{n_{s}=n}^{N_T}$ by using the scheme (27), where $\xi_n^{(l)} \in \{\xi_n^{(m)}\}_{m=1}^M$ is a representation for $\hat{\alpha}_{t_n}$ corresponding to the $l$-th iteration. |  |
| - Generate a sample-wise approximation $\{\tilde{Y}_{n_{s}}^{l, \xi_n^{(l)}}(\tilde{X}_{n_{s}}^{l, \xi_n^{(l)}})\}_{n_{s}=n}^{N_T}$ corresponding to $\{\tilde{X}_{n_{s}}^{l, \xi_n^{(l)}}\}_{n_{s}=n}^{N_T}$ by using the scheme (28) for solution $Y$ of the adjoint BSDE. |  |
| - Generate a sample-wise gradient process $\{\nabla \tilde{J}_{u_{n_{s}}}^{l, \xi_n^{(l)}}(\tilde{u}_{n_{s}}^{l, \xi_n^{(l)}})\}_{n_{s}=n}^{N_T}$ according to Eq. (31). |  |
| - Carry out gradient descent iteration Eq. (32) to obtain an updated optimal control $\{\tilde{u}_{n_{s}}^{l+1, \xi_n^{(l)}}\}_{n_{s}=n}^{N_T}$. |  |
| end |  |
| Implement the direct filter method to generate $\{\xi_{n+1}^{(m)}\}_{m=1}^M$: |  |
| - Prediction step: Generate a set of predicted parameter particles $\{\tilde{\xi}_{n+1}^{(m)}\}_{m=1}^M$ via Eq. (10). |  |
| - Update step: Assign weight $w_{n+1}^{(m)}$ to each particle according to the likelihood function defined by Eq. (13) to generate a posterior distribution that incorporates the new observational data $M_{t_{n+1}}$. |  |
| - Resampling: Resample parameter particles to generate an equally weighted set of particles $\{\xi_{n+1}^{(m)}\}_{m=1}^M$ to formulate the empirical distribution of the estimated parameter $\hat{\alpha}_{t_{n+1}}$. |  |
| end while |  |

parameters in the state process. Since the LQ problem can be solved analytically, we use this example as a benchmark scenario to verify the efficiency and effectiveness of our method. In the second example, we solve an unmanned aerial vehicle (drone) maneuvering problem. In order to make the control problem data driven, we assume that there’s an unknown parameter that can influence the control of the drone, and we use this example to demonstrate the applicability of our SW-DDC method in solving practical problems.

4.1 Example 1. Linear-quadratic data driven stochastic optimal control problem

Consider the following linear controlled dynamical system

$$dX_t = A(t)X_t dt + BU_t dt + CdW_t, \quad X_0 = x_0,$$  

(33)
where $A(t)$, $B$ and $C$ are given coefficients and $A(t)$ contains unknown parameters, which will influence the behavior of the dynamics, $X_t$ is the multi-dimensional state process with the given initial position $X_0$, and $U_t$ is the multi-dimensional control process. The cost function $J$ is defined by

$$J(U^M) = E \left[ \frac{1}{2} \int_0^T \left( \langle QX_t, X_t \rangle + \langle RU_t, U_t \rangle \right) dt + \frac{1}{2} \langle FX_T, X_T \rangle \right]$$  \hspace{1cm} (34)$$

The data that we use to estimate the unknown parameters is noise perturbed direct observations on $X$, i.e.

$$M_t = X_t + \xi_t,$$  \hspace{1cm} (35)

where $\xi_t \sim N(0, \Sigma)$ is a Gaussian noise with covariance $\Sigma$.

The optimal control $\tilde{U}_t$ for the above LQ stochastic optimal control problem can be derived analytically as

$$\tilde{U}_t = -R^{-1}B^TP(t)X_t,$$  \hspace{1cm} (36)

where $P(t)$ is the unique solution of the following Riccati equation

$$\frac{dP(t)}{dt} = -P(t)A(t) - A^T(t)P(t) + P(t)BR^{-1}B^TP(t) - Q, \quad P(T) = F.$$

**Case 1: 1D state dynamics with 1 unknown parameter**

We first carry out an experiment to demonstrate the necessity of using online parameter estimation methods. To this end, we solve the data driven optimal control problem over the time interval $t \in [0, 1]$ with $A(t) = \alpha$, where $\alpha$ is the unknown parameter to be estimated, and we let the true value for parameter $\alpha$ be 1. Although we don’t assume that $\alpha$ is a time dependent parameter, in practice the true parameter may change unexpectedly in some circumstances. In this experiment, we let $\alpha$ change from 1 to 5 at time instant $t = 0.5$. Since the switch of the true parameter at the time instant $t = 0.5$ is unexpected, the conventional offline parameter estimation methods cannot adjust the change of the true parameter due to the mixture data from two different true parameters. The other coefficients in the data-driven LQ control problem (33)-(35) are chosen as $B = 0.5$, $C = 0.01$, $Q = 1$, $R = 0.1$, $F = 1$, and $\Sigma = 0.001$.

In our numerical experiment, we let $\Delta t = 0.02$, i.e. we receive 50 sets of observational data, and we use altogether $M = 200$ particles in the SW-DDC algorithm. In Figure 1, we present the estimated parameter and the estimated optimal control by using the SW-DDC algorithm. The black straight lines in Figure 1(a) give the real parameter $\alpha$ over the time interval $[0, 1]$. The black curve marked by crosses in Figure 1(b) gives the optimal control computed via Eq. (36) by using the real parameter $\alpha$. Note that the first piece of the true optimal control in subplot Figure 1(b) is obtained based on the assumption that the true parameter is $\alpha = 1$.

![Figure 1](image-url)
over the entire time interval [0, 1] since the switch of the parameter occurs unexpected at the time instant \( t = 0.5 \). The red curves marked by circles in subplots Figure 1(a) and (b) are the estimated parameter and the estimated optimal control, respectively. From this figure, we can see that the direct filter quickly captured the real model parameter, and it can detect and modify its estimation to adjust the change of true parameter at the time instant \( t = 0.5 \). On the other hand, the estimated optimal control presented in subplot Figure 1(b) shows that our SW-DDC algorithm can provide accurate estimates for the optimal control, and it only takes a few steps to re-capture the analytical optimal control driven by the switched real parameter.

In the second experiment for the 1D case, we demonstrate the efficiency of our sample-wise optimal control solver by comparing with the conventional fully-calculated stochastic optimal control solver described by (22) - (26) (see [22] for details). The spatial points that we use to represent the state variable \( \hat{X}_{n,t}^{t_a,t_n} \) in the fully-calculated stochastic optimal control solver is the uniform mesh in the state space. Since we aim to compare the efficiency between two optimal control methods, we use the direct filter as our parameter estimation method to drive both optimal control solvers.

In this experiment, we let \( A(t) = 2\alpha \sin(t) \), and all the other coefficients remain the same as in the first experiment. The true parameter \( \alpha \) that we choose is \( \alpha = 1 \). In Figure 2 we show the comparison results in estimating the optimal control. In subplots Figure 2(a), (b), (c), we present the estimated optimal control obtained by the fully-calculated stochastic optimal control solver with coarse mesh (\( \Delta x = 0.8, \Delta t = 0.1 \)), medium mesh (\( \Delta x = 0.6, \Delta t = 0.075 \)) and fine mesh (\( \Delta x = 0.4, \Delta t = 0.05 \)). The black curve marked by crosses in each subplot is the true optimal control (calculated based on its corresponding temporal discretization), and the green curve marked by diamonds is the corresponding estimated optimal control. In subplot Figure 2(d), we present the estimated optimal control obtained by using the sample-wise optimal control solver in the SW-DDC framework with temporal step-size \( \Delta t = 0.05 \), where the red curve marked by circles is our estimated control, and the estimated optimal control is obtained through \( 10^4 \) iteration steps in the optimization.

Figure 2: Comparison between fully calculated control solver with sample-wise optimal control solver
procedure. We can see from this figure that by using finer meshes, the fully-calculated stochastic optimal control solver can provide better estimated optimal control, while our sample-wise optimal control solver outperforms the fully-calculated stochastic optimal control solver with the finest mesh.

In Table 2, we present the computational cost for each method, and we carry out our numerical experiments on an Apple M1 CPU with 8 cores and 16 GB memory. We can see from Table 2 and the comparison results in Figure 2 that the SW-DDC has much lower computational cost while is still more accurate compared with the the fully-calculated stochastic optimal control solver with the finest mesh.

| Method   | Coarse mesh | Medium mesh | Fine mesh | SW-DDC |
|----------|-------------|-------------|-----------|--------|
| CPU time (seconds) | 40.6 | 84.2 | 287.2 | 4.49 |

Through the numerical experiments that we carried out in solving the 1D LQ data driven stochastic optimal control problems, we have verified that the direct filter in the SW-DDC algorithm works well for unexpected changes in the true parameter, which indicates the necessity of online parameter estimation in data driven control. Moreover, the comparison experiments in this 1D case show that the sample-wise optimal control solver in our SW-DDC framework outperforms conventional fully-calculated stochastic optimal control solver in efficiency. Due to the fact that our sample-wise optimal control solver does not solve the adjoint BSDE in the entire state space, the efficiency feature of our method would be more advantageous in solving higher dimensional problems. In this way, we have demonstrated the necessity of using sample-wise optimal control solver to solve data driven control problems.

In what follows, we shall demonstrate the effectiveness and efficiency of the direct filter, and we are going to compare the direct filter with other state-of-the-art online parameter estimation methods as a “driver” to drive the sample-wise optimal control solver in the SW-DDC framework.

Case 2: 2D state dynamics with 1 unknown parameter

We consider a 2D case of the the LQ data-driven stochastic optimal control problem (33)-(35), and we let $A(t) = (\alpha \sin(t), \cos(t)) I_2$, $B = (0.5, 0.5) I_2$, and $C = 0.1 I_2$ in (33), where $\alpha$ in $A(t)$ is a scaler unknown parameter. The control term $U_t$ is chosen as a scaler in this 2D problem as well. For the cost functional, we let $Q = I_2$, $R = 1$ and $F = I_2$. In the numerical experiments, we solve this data driven stochastic optimal control problem over the time interval $[0, 1]$ by choosing the temporal step-size $\Delta t = 0.02$, i.e. $N_T = 50$, and we let the covariance matrix $\Sigma$ of the observational noise be $\sigma = (0.01)^2 I_2$. The initial state of the controlled process is chosen as $X_0 = (2, -2)^\top$.

The goal of the numerical experiments for this 2D problem is to show that the direct filter method in the SW-DDC framework will outperform the augmented particle filter (AugPF), which is a particle filter implementation of the augmented filter method, and both the direct filter and the AugPF will drive the same sample-wise optimal control solver (due to its high efficiency advantage) to determine the data driven optimal control. In Figure 3, we present the comparison between the direct filter (subplot (a)) and the AugPF (subplot (b)) in estimating the unknown parameter $\alpha$, where we have used 100 particles in the direct filter and 1000 particle in the AugPF. The black solid line in each subplot indicates the true parameter value. We can see from this figure that the direct filter captured the real parameter near the time instant $t = 0.4$, and the AugPF cannot find the target parameter in the entire time interval $[0, 1]$.

In Figure 4, we present the estimated optimal control obtained by the sample-wise optimal control solver, which is driven by estimated parameters using the direct filter (subplot (a)) and the AugPF (subplot (b)). We can see that with accurate estimation for the parameter, the direct filter driven SW-DDC can provide accurate estimation for the optimal control. On the other hand, since the AugPF failed to find the true parameter, the sample-wise optimal control solver cannot provide accurate estimates for the optimal control.

We also want to mention that the failure of the AugPF in the above experiment could be caused by inefficiency of the particle filter in estimating the augmented state process. For a relatively low dimensional
Figure 3: Example 1. Comparison between the direct filter and the AugPF in parameter estimation (2D case)

Figure 4: Example 1. Comparison of estimated optimal controls corresponding to each parameter estimation method (2D case)

problem, the parameter estimation performance of the AugPF may improve by increasing the particle-size. In Figure 5, we present the parameter estimation performance (subplot (a)) and the optimal control estimation performance (subplot (b)) of the AugPF driven SW-DDC by using 20000 particles in the AugPF procedure. We can see that in this experiment the AugPF driven SW-DDC could provide much better estimation for both the parameter and the optimal control when significantly more particles are used.

To further compare the performance of the direct filter and the AugPF in driving the sample-wise optimal control solver under our SW-DDC framework, we carry out the above comparison experiments repeatedly 20 times (with different particle-sizes in the AugPF) and present the root mean square errors (RMSEs) of each implementation in Figure 6, where Figure 6 (a) shows the RMSEs of parameter estimation and Figure 6 (b) shows the RMSEs of optimal control estimation. We can see from this figure that by using more and more particles, the AugPF can provide more and more accurate parameter estimation results, which will result better and better estimated optimal control. On the other hand, the direct filter can provide the best estimated parameter, which resulted the best estimated optimal control. In Table 3, we present the comparison of (average) computational cost between the AugPF (with 1, 000 particles, 20, 000 particles and 40, 000 particles) driven SW-DDC and the direct filter (with only 100 particles) driven SW-DDC. We can see from this table that the computational cost for the direct filter is the lowest since it only used 100 particles to
estimate the unknown parameter.

Case 3: 4D state dynamics with 2 unknown parameters

The third case that we consider in the LQ data driven stochastic control example has 4D state dynamics with 2 unknown parameters. Specifically, we let $A(t) = (\sin(t), \cos(t), \alpha, \beta) \mathbb{I}_4$, $B = (0.5, 0; 0.5, 0; 1, 0; 1, 0)^\top$, $C = 0.1 \mathbb{I}_4$, where $\alpha$ and $\beta$ in $A(t)$ are 2 unknown parameters to be estimated, and $U_t$ is a 2D control vector. For the cost functional, we let $Q = I_4$, $R = I_2$ and $F = I_4$. In the numerical experiments, we solve this
data driven control problem over the time interval $[0, 1]$ by choosing the temporal step-size $\Delta t = 0.025$, i.e. $N_T = 40$, and we let the covariance matrix $\Sigma$ of the observational noise be $\sigma = (0.01)^2 I_2$. The initial state of the controlled process is chosen as $X_0 = (1, 2, -1, 2)^\top$.

Figure 7: Example 1. Performance of the SW-DDC algorithm driven by the direct filter (4D case)

In Figure 7, we present the performance of the SW-DDC algorithm driven by the direct filter, and we used 500 particles in the direct filter method to estimate the parameters. Figure 7 (a) and (b) provide the performance of the direct filter in estimating the unknown parameters $\alpha$ and $\beta$, respectively, and Figure 7 (c) shows the estimation accuracy for the optimal control. We can see from this figure that the direct filter can capture the true target parameters quickly, and once accurate estimates for parameters are obtained, the sample-wise optimal control solver starts to provide good estimates for the optimal control.

As a higher dimensional problem (with 4D state and 2D parameter), we observe that the AugPF suffers more severe degeneracy problem. In Figure 8, we show the comparison of RMSEs between the direct filter driven SW-DDC and the AugPF driven SW-DDC, where subplot (a) shows comparison of accuracy in estimating the unknown parameters and subplot (b) shows comparison of accuracy in estimating the optimal control. We can see from this figure that although we have used 20,000, 50,000 and 80,000 particles, the AugPF provided similar parameter estimation performance, which resulted similar performance in estimating the optimal control. This indicates that increasing particles from 20,000 to 80,000 would not be enough to
let AugPF accurately capture the true parameters in this experiment, and this is caused by the well-known “degeneracy problem” in the particle filter when solving higher dimensional problems (see [2, 17, 39, 43]). On the other hand, by using only 500 particles, the direct filter provided very accurate estimated parameters, which resulted very accurate estimates for the optimal control in this experiment.

Table 4: Example 1. 4D computational cost comparison

| Method             | AugPF (20K) | AugPF (50K) | AugPF (80K) | Direct filter (500) |
|--------------------|-------------|-------------|-------------|---------------------|
| CPU time (seconds) | 81.3        | 159.8       | 283.1       | 58.3                |

In Table 4, we present comparison of computational costs corresponding to each experiment in Figure 8, and we can see that the direct filter method can provide the most accurate parameter/control estimation while using the lowest amount of CPU time.

Moreover, we want to mention that although the dimension of the problem was increased from 2D state/1D parameter to 4D state/2D parameter, the computational cost for the direct filter driven SW-DDC method only increased from 1.35 seconds/step to 1.46 seconds/step. It indicates that our SW-DDC algorithm is not very sensitive to dimensions, and it can alleviate the so-called “curse of dimensionality” in moderately high dimensional problems.

4.2 Example 2: Data driven control of an unmanned aerial vehicle

In this example, we consider a more practical scenario and solve an unmanned aerial vehicle maneuvering problem. The controlled state process is given by the following nonlinear dynamical system

\[
\begin{align*}
    dX_t &= \sin(\theta_t)dt + \sigma dW_t, \\
    dY_t &= \cos(\theta_t)dt + \sigma dW_t, \\
    dZ_t &= (F_t - mg)dt + \sigma dW_t, \\
    d\theta_t &= (\alpha_t - \mu m)dt + \sigma^2 dW_t,
\end{align*}
\] (37)

where \((X, Y, Z)^\top\) describes the position of an unmanned aerial vehicle (drone) moving in the 3D space, \(\theta\) is the steering angle that controls the horizontal moving direction of the drone, which is governed by a steering control factor \(\alpha_t\), and \(\sigma\) is the noise that perturbs both the motions and control actions. The unknown parameter \(m\) is the total mass (weight) of the drone with its load, \(g = 9.8\) is the standard gravity constant. Then, we let \(F_t\) be another control factor that controls the elevating force. Therefore, the control term \(u_t := (\alpha_t, F_t)\) becomes a 2D process. The term \(\mu m\) in the equation of \(d\theta_t\) (the fourth equation in Eq. (37)) can be interpreted as the resistance for the steering control factor \(\alpha_t\), which is influenced by the weight, and we choose \(\mu = 0.1\) in this example.

The cost functional that we aim to minimize is defined as

\[
J^*(u) = E \left[ \int_0^T \frac{1}{2} ((F_t)^2 + (\alpha_t)^2)dt + 10 \left( (X_T - X_P)^2 + (Y_T - Y_P)^2 + (Z_T - Z_P)^2 + (Y_T - Y_P)^2 \right) \right],
\] (38)

where \((X_P, Y_P, Z_P)^\top\) is the pre-designated target platform, and we let \((X_P, Y_P, Z_P)^\top = (6, 7, 8)^\top\) in our numerical experiments. The goal of the optimal control problem is to let the controlled drone be as close as to the target at terminal time \(T\) while trying to keep the cost of control actions low.

The observational data that we use to design data driven control actions \(\alpha_t\) and \(F_t\) is

\[
M_t = (X_t, Y_t, Z_t, \theta_t)^\top + (R, R, R, 0.1R)\xi_t,
\]

where \(\xi_t \sim N(0, I_4)\) is a standard multi-variate Gaussian random variable and \(R\) determines the size of observation errors.
In our numerical experiments, we choose $\sigma = 0.2$, $R = 0.01$, and we let the designated terminal time be $T = 1$ with each observation/estimation step-size $\Delta t = 0.02$, i.e. $N_T = 50$. The initial condition of the drone is $(X_0, Y_0, Z_0, \theta_0)^\top = (0, 0, 5, 0)^\top$. In Figure 9, we present the controlled trajectory of the drone in the $X_1 - X_2$ plane (subplot (a)) and in the $X_3$ direction (subplot (b)) based on the control actions designed by the SW-DDC algorithm, and the parameter estimation method was chosen as the direct filter method with 200 particles in estimating the unknown parameter. The red curves marked by circles are drone trajectories, and the black box (in subplot (a)) and the black solid line (in subplot (b)) indicate the target platform in the $X_1 - X_2$ plane and the height ($X_3$-coordinate), respectively. Also, we plot the 3D trajectory of the drone in Figure 10 to provide a more direct viewing for the performance of our algorithm, where the black box is the 3D position of the target platform. We can see from Figure 9 and Figure 10 that the SW-DDC can effectively guide the drone to the target platform at the pre-designated time. The overall computational time to design altogether 50 data driven control actions in this experiment is 23.99 seconds, which shows that our algorithm is potentially feasible to implement real-time data analysis and optimal control when solving practical application problems. In Figure 11, we present the performance of parameter estimation by using the direct filter method. We can see that after receiving only a few sets of observational data, the direct filter (by using only 200 particles) obtained accurate estimation for the unknown parameter.

To further demonstrate the advantageous performance of our SW-DDC method, we use the augmented ensemble Kalman filter (AugEnKF) (the ensemble Kalman filter implementation for the augmented filter method) to drive the sample-wise optimal control solver, where the AugEnKF is well known as the most widely
Figure 11: Example 2. Parameter estimation performance of the direct filter used method in solving the online parameter estimation problem. In Figure 12 we present the parameter estimation performance of the AugEnKF with 50 realizations of Kalman filter samples (in subplot (a)) and the 3D trajectory of the controlled drone (in subplot (b)), where the black solid line in subplot (a) is the real parameter and the black box in subplot (b) shows the target platform in the 3D space. Note that 50 realizations of Kalman filter samples is already a very large number of samples for the ensemble Kalman filter in this relatively low dimensional problem. From this figure, we can see that the AugEnKF made unreliable estimation for the unknown parameter, which is mainly because of the nonlinearity of the dynamical model and the nonlinear parameter-state relation. As a result, the control actions designed for the drone are based on inaccurate parameter, which made the drone fail to arrive at the target platform (as presented in Figure 12 (b)).

Figure 12: Example 2. Performance of SW-DDC driven by AugEnKF estimated parameter

To further compare the performance between the direct filter and the AugEnKF in driving the sample-wise optimal control solver and show the advantage of our SW-DDC algorithm introduced in Algorithm 1, we repeat the above experiments 20 times. In Figure 13 (a), we show the comparison of RMSEs in parameter estimation, and in Figure 13 (b) we plot the average distance to the target platform. Specifically, the red curves marked by circles are RMSEs of parameter estimation (in subplot (a)) and average distance to the target platform (in subplot (b)) obtained by the direct filter and the direct filter driven SW-DDC, respectively. The blue curves marked by crosses are RMSEs of parameter estimation (in subplot (a)) and average distance to the target platform (in subplot (b)) obtained by the AugEnKF and the AugEnKF driven optimal control solver,
Figure 13: Example 2. Performance comparison with repeated experiments

respectively. From this figure, we can see that the direct filter significantly outperforms the AugEnKF in this experiment, and therefore the controlled state designed by using the direct filter estimated parameter also outperforms the controlled state driven by the AugEnKF.

5 Concluding remarks

In this paper, we introduced a novel sample-wise data driven control (SW-DDC) solver to solve the stochastic optimal control problem with unknown model parameters. In the computational framework for SW-DDC, we utilize the direct filter as an effective online parameter estimation method to drive the design of the optimal control through an efficient sample-wise optimal control solver. The main novelty in our methodology is to treat samples that characterize state variables and parameter variables in the controlled dynamical model as pseudo-data, and then we can use stochastic approximation to accelerate the gradient descent optimization procedure for finding the data driven optimal control. The primary contribution of the SW-DDC framework is that it allows us to efficiently generate accurate estimates for both the unknown parameters and the optimal control in the online manner. Numerical experiments are carried out to demonstrate the efficiency advantage of the sample-wise optimal control solver and the accuracy advantage of the direct filter, and both components combining together resulted the advantageous overall performance of the SW-DDC solver.

References

[1] Shahrokh Akhlaghi, Ning Zhou, and Zhenyu Huang. A multi-step adaptive interpolation approach to mitigating the impact of nonlinearity on dynamic state estimation. *IEEE Transactions on Smart Grid*, 9(4), 7 2018.

[2] C. Andrieu, A. Doucet, and R. Holenstein. Particle markov chain monte carlo methods. *J. R. Statist. Soc. B*, 72(3):269–342, 2010.

[3] R. Archibald, F. Bao, and X. Tu. A direct filter method for parameter estimation. *J. Comput. Phys.*, 398:108871, 17, 2019.

[4] R. Archibald, F. Bao, and J. Yong. A stochastic gradient descent approach for stochastic optimal control. *East Asian Journal on Applied Mathematics, to appear*, 2020.
[5] R. Archibald, F. Bao, J. Yong, and T. Zhou. An efficient numerical algorithm for solving data driven feedback control problems. Journal of Scientific Computing, 85(51), 2020.

[6] F. Bao, Y. Cao, and H. Chi. Adjoint forward backward stochastic differential equations driven by jump processes and its application to nonlinear filtering problems. International Journal of Uncertainty Quantification, 9(2):143–159, 2019.

[7] F. Bao, Y. Cao, and P. Maksymovych. Backward sde filter for jump diffusion processes and its applications in material sciences. Communications in Computational Physics, 27:589–618, 2020.

[8] F. Bao, Y. Cao, and J. Yong. Data informed solution estimation for forward backward stochastic differential equations. Analysis and Applications, to appear, 2020.

[9] F. Bao, Y. Cao, and W. Zhao. Numerical solutions for forward backward doubly stochastic differential equations and zakai equations. International Journal for Uncertainty Quantification, 1(4):351–367, 2011.

[10] F. Bao, Y. Cao, and W. Zhao. A backward doubly stochastic differential equation approach for nonlinear filtering problems. Commun. Comput. Phys., 23(5):1573–1601, 2018.

[11] F. Bao, N. Cogan, A. Dobreva, and R. Paus. Data assimilation of synthetic data as a novel strategy for predicting disease progression in alopecia areata. Mathematical Medicine and Biology: A Journal of the IMA, 2021.

[12] Feng Bao, Yanzhao Cao, and Xiaoying Han. Forward backward doubly stochastic differential equations and optimal filtering of diffusion processes. Communications in Mathematical Sciences, 18(3):635–661, 2020.

[13] Feng Bao, Yanzhao Cao, Amnon Meir, and Weidong Zhao. A first order scheme for backward doubly stochastic differential equations. SIAM/ASA J. Uncertain. Quantif., 4(1):413–445, 2016.

[14] Feng Bao, Yanzhao Cao, and Weidong Zhao. A first order semi-discrete algorithm for backward doubly stochastic differential equations. Discrete and Continuous Dynamical Systems-Series B, 5(2):1297 – 1313, 2015.

[15] Feng Bao and Vasileios Maroulas. Adaptive meshfree backward SDE filter. SIAM J. Sci. Comput., 39(6):A2664–A2683, 2017.

[16] R. Bellman. Dynamic Programming. 1957.

[17] A. J. Chorin and X. Tu. Implicit sampling for particle filters. Proc. Nat. Acad. Sc. USA, 106:17249–17254, 2009.

[18] O. Dyck, M. Ziatdinov, S. Jesse, F. Bao, A. Yousefzadi Nobakht, A. Maksov, B.G. Sumpter, R. Archibald, K.J.H. Law, and S.V. Kalinin. Probing potential energy landscapes via electron-beam-induced single atom dynamics. Acta Materialia, 203:116508, 2021.

[19] G. Evensen. Data assimilation: the ensemble Kalman filter. Springer, 2006.

[20] G. Evensen. The ensemble Kalman filter for combined state and parameter estimation: Monte Carlo techniques for data assimilation in large systems. IEEE Control Syst. Mag., 29(3):83–104, 2009.

[21] Xiaobing Feng, Roland Glowinski, and Michael Neilan. Recent developments in numerical methods for fully nonlinear second order partial differential equations. SIAM Rev., 55(2):205–267, 2013.

[22] Bo Gong, Wenbin Liu, Tao Tang, Weidong Zhao, and Tao Zhou. An efficient gradient projection method for stochastic optimal control problems. SIAM J. Numer. Anal., 55(6):2982–3005, 2017.

[23] N.J Gordon, D.J Salmond, and A.F.M. Smith. Novel approach to nonlinear/non-gaussian bayesian state estimation. IEE PROCEEDING-F, 140(2):107–113, 1993.
[24] Ajay Jasra, Kengo Kamatani, Kody Law, and Yan Zhou. Bayesian static parameter estimation for partially observed diffusions via multilevel Monte Carlo. *SIAM J. Sci. Comput.*, 40(2):A887–A902, 2018.

[25] S.J. Julier and J.K. Uhlmann. Unscented filtering and nonlinear estimation. *Proceedings of the IEEE*, 92:401–422, 2004.

[26] S. Kalinin, A. Borisevich, and S. Jesse. Fire up the atom forge. *Nature*, 22 November 2016.

[27] Kai Kang, Vasileios Maroulas, Ioannis Schizas, and Feng Bao. Improved distributed particle filters for tracking in a wireless sensor network. *Comput. Statist. Data Anal.*, 117:90–108, 2018.

[28] N. Kantas, A. Doucet, S. Singh, J. Maciejowski, and N. Chopin. On particle methods for parameter estimation in state-space models. *Statistical Science*, 30(3):328–351, 2015.

[29] Dzmitry Katsiukevich and Natalia Dmitruk. Data-driven optimal control of linear time-invariant systems. *IFAC-PapersOnLine*, 53(2):7191–7196, 2020. 21st IFAC World Congress.

[30] G. Kitagawa. A self-organizing state-space model. *J. Amer. Statist. Assoc.*, 93:1203–1215, 1998.

[31] R. V. Gamkrelidze E. F. Mishchenko L. S. Pontryagin, V. G. Boltyanskii. *Power system stability and control*. 1994.

[32] J. Liu and M. West. Combined parameter and state estimation in simulation-based filtering. In *Sequential Monte Carlo methods in practice*, Stat. Eng. Inf. Sci., pages 197–223. Springer, New York, 2001.

[33] Matthias Morzfeld, Marcus S. Day, Ray W. Grout, George Shu Heng Pau, Stefan A. Finsterle, and John B. Bell. Iterative importance sampling algorithms for parameter estimation. *SIAM J. Sci. Comput.*, 40(2):B329–B352, 2018.

[34] Ali Yousefzadi Nobakht, Ondrej Dyck, David B Lingerfelt, Feng Bao, Maxim Ziatdinov, Artem Maksov, Bobby G Sumpter, Richard Archibald, and Sergei V Kalinin Stephen Jesse, and Kody JH Law. Reconstruction of effective potential from statistical analysis of dynamic trajectories. *AIP Advances*, 10:065034, 2020.

[35] D. Oliver, A. Reynolds, and N. Liu. *Inverse theory for petroleum reservoir characterization and history matching*. Cambridge University Press, 2008.

[36] N. J. Balu P. Kundur and M. G. Lauby. *The Mathematical Theory of Optimal Processes*. 1962.

[37] E. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[38] S. Peng. A general stochastic maximum principle for optimal control problems. *SIAM J. Control Optim.*, pages 966–979, 1990.

[39] Michael K. Pitt and Neil Shephard. Filtering via simulation: auxiliary particle filters. *J. Amer. Statist. Assoc.*, 94(466):590–599, 1999.

[40] Mario Sassano and Alessandro Astolfi. A local separation principle via dynamic approximate feedback and observer linearization for a class of nonlinear systems. *IEEE Trans. Automat. Control*, 64(1):111–126, 2019.

[41] I. Sato and H. Nakagawa. Convergence analysis of gradient descent stochastic algorithms. *Proceedings of the 31st International Conference on Machine Learning*, pages 982–990, 2014.

[42] A. Shapiro and Y. Wardi. Convergence analysis of gradient descent stochastic algorithms. *Journal of Optimization Theory and Applications*, pages 439–454, 1996.
[43] C. Snyder, T. Bengtsson, P. Bickel, and J. Anderson. Obstacles to high-dimensional particle filtering. *Mon. Wea. Rev.*, 136:4629–4640, 2008.

[44] Shanjian Tang. The maximum principle for partially observed optimal control of stochastic differential equations. *SIAM J. Control Optim.*, 36(5):1596–1617, 1998.

[45] S. Wilks. *Mathematical Statistics*. New York: John Wiley & Sons, 1962.

[46] W. M. Wonham. On the separation theorem of stochastic control. *SIAM J. Control*, 6:312–326, 1968.

[47] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1999. Hamiltonian systems and HJB equations.

[48] Jianfeng Zhang. A numerical scheme for BSDEs. *Ann. Appl. Probab.*, 14(1):459–488, 2004.

[49] Weidong Zhao, Yu Fu, and Tao Zhou. New kinds of high-order multistep schemes for coupled forward backward stochastic differential equations. *SIAM J. Sci. Comput.*, 36(4):A1731–A1751, 2014.