Neutrino pair condensate and its application to cosmology and astrophysics

M. Yoshimura
Research Institute for Interdisciplinary Science, Okayama University
Tsushima-naka 3-1-1 Kita-ku Okayama 700-8530 Japan

ABSTRACT

Left-handed neutrinos interact attractively by Z-boson exchange. The Ginzburg-Landau mean field calculation and the Bogoliubov transformation suggest that the attractive force leads to neutrino pair condensate and neutrino super-fluidity. Neutrinos, as defined by quasi-particle in the super-fluid phase, behave as massless fermions. When the result of super-fluid formation is applied to the early universe, horizon scale pair condensate may become a component of dark energy. A further accretion of other fermions from thermal cosmic medium gives a seed of primordial neutron stars consisting of proton, neutron, electron, and neutrino in beta-equilibrium, surrounded by left-handed neutrino pair condensate. This possibility may provide a mechanism of giving a part or the whole of the dark matter in the present universe, if a properly chosen small fraction condenses to neutrino super-fluid and primordial neutron star not to over-close the universe. The proposal can be verified by measuring neutrino burst at primordial neutron star formation and by detecting super-fluid relic neutrinos in atomic experiments at laboratories.

Keywords Neutrino interaction, Super-fluid condensate, Neutrino super-ball, Primordial neutron star, Dark matter
1 Introduction

Neutrinos rarely interact, but their feeble interaction with matter has been very useful to clarify the point-like structure of nucleons, namely existence of quarks, using large scale detectors in accelerator experiments. More dramatically, standard theory of cosmology predicts that neutrinos frequently interact in dense medium of early universe, and their properties are well described by thermalized momentum distribution when the universe is hot enough at temperatures above 1 MeV. This has been a basis of nucleo-synthesis calculation since the pioneering work of C. Hayashi [1]. Another place one expects neutrino equilibrium is deep inside neutron stars.

In the present work we explore a novel possibility of super-fluid phase transition due to neutrino self-interaction. The description of ideal gas distribution is thus found to require a revision from the simple description, and we may study implications of super-fluid neutrino pair condensate.

Our basic theoretical framework is the well established standard theory of particle physics, however modified by finite neutrino masses. How neutrino masses, either of Majorana or Dirac type, are generated is irrelevant to the present work. Within this framework we work out neutrino self-interaction by Z-boson exchange and find that it is given by an attractive force. It is further shown that Z-boson plays a similar role to phonon in super-conductive metal [2], and left-handed spin-singlet pair $\nu_L\nu_L$ condensates like Cooper pair. Theoretical tools we use are the Ginzburg-Landau mean field theory and the technique of Bogoliubov transformation, both of which are standard methods in the theory of superconductivity.

Application of super-fluid neutrino pair condensates to the early cosmology suggests that they give rise to a seed of dark matter, leading to formation of primordial neutron stars and primordial black holes.

We use the natural unit of $\hbar = c = 1$ throughout the present work unless otherwise stated. Moreover, we express cosmic temperature in eV unit taking the Boltzmann constant $k_B = 1$.

2 Neutrino pair condensate

2.1 Neutrino interaction

Consider a gas consisting of many left-handed neutrinos in mass eigenstates denoted by $\nu^L_i$, $i = 1, 2, 3$ (with $L, R = (1 \mp \gamma_5)/2$ the chiral projection), in the early universe at temperatures above 1 MeV [3] and below the electroweak phase transition around 250 GeV. Left-handed neutrino is a part of major constituents in this universe, and we shall later consider other constituents, adding various charged particles and gauge bosons. Neutrinos rarely interact and their aggregate is well described by an ideal gas of temperature $T$ with zero chemical potential in cosmology. Although weak, interaction however exists by Z-boson exchange of Fig.1, which is the dominant interaction in the standard electroweak theory to the leading order. The Higgs exchange interaction mixes $\nu_L$ to the right-handed $\nu_R$, and is irrelevant to the present discussion. The potential of Z-exchange interaction in the coordinate space is given by

$$\frac{g_Z^2}{32\pi} \frac{e^{-m_Z r}}{r} (u_3^\dagger u_1 u_4^\dagger u_2 - u_3^\dagger \vec{\sigma} u_1 \cdot \vec{\sigma} u_2)$$

in terms of two-component spinor wave functions of neutrinos, $u_i(p_i), i = 1 \sim 4$ with $p_i$ 4-momenta of neutrinos. This amplitude arises from t-channel Z-boson exchange in the scattering $\nu(p_1) + \nu(p_2) \rightarrow \nu(p_3) + \nu(p_4)$.

To see spin-dependence of Z-boson exchange interaction, it is convenient to exchange wave functions, $u_1$ and $u_4^\dagger$, using the identity,

$$(u_1 u_4^\dagger)_{ij} = \frac{1}{2} (u_1^\dagger u_1 + u_4^\dagger \vec{\sigma} u_1 \cdot \vec{\sigma})_{ij}.$$
Figure 1: Scattering of $\nu_L$ via t-channel Z-exchange. Straight lines are for $\nu_L$, while wavy one is for Z boson.

Two neutrino states in the initial and the final states are $u_1u_2$ and $u_3u_4$, which may be decomposed into projected states of total spin eigenvalues, singlet and triplet, using the formulas,

$$u_{1i}u_{2j} = \sqrt{2} \left( (S_{12} + \vec{V}_{12} \cdot \vec{\sigma}) (-i\sigma_2) \right)_{ij}, \quad S_{12} = \frac{1}{2\sqrt{2}} u_1^T i\sigma_2 u_2, \quad \vec{V}_{12} = \frac{1}{2\sqrt{2}} u_1^T i\sigma_2 \vec{\sigma} u_2. \quad (3)$$

The bracket part ($\cdots$) of eq.(1) is then equal to

$$-2 \left( S_{34}^* S_{12} + \vec{V}_{34}^* \cdot \vec{V}_{12} \right). \quad (4)$$

A positive pre-factor means repulsion, while a negative one means attraction. Thus, one may conclude that Z-boson exchange gives attractive interaction both for spin-singlet and triplet, with a strength $g_Z^2 e^{-m_Z r}/(16\pi r)$. From the point of condensed state it is more advantageous to take zero momentum, spin-singlet state, which is possible from relativistically energetic neutrino of momentum pair $\vec{p}, -\vec{p}$.

The effective strength of spin-singlet neutrino interaction at finite temperatures is given by taking a thermal average over one of neutrinos, to lead to

$$V_\nu = \frac{g_Z^2}{6\zeta(3)} \frac{1}{r} \int_0^\infty dx \frac{x \sin(xm_Z r)}{(x^2 + 1)(e^{xm_Z r}/T + 1)} = \frac{g_Z^2}{6\zeta(3)} \frac{1}{3r} \int_0^\infty dx \frac{x e^{ixm_Z r}}{x^2 + 1} \left( \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nxm_Z r}/T \right). \quad (5)$$

The integral may be estimated by the residue at a pole $x = i$ in the complex plane of $x$, to give

$$V_\nu = \frac{\pi}{12\zeta(3)} \frac{g_Z^2}{r} e^{-m_Z r}. \quad (6)$$

This is larger than the free-space value by $4\pi^2/3\zeta(3) \sim 7.3$.

Consider $i$-th massive Majorana neutrinos with its number density $n_i$. The energy density of neutrino pairs at finite densities is given by

$$2\sqrt{k^2 + m_i^2} n_i - V_\nu n_i^2. \quad (7)$$

The system is unstable for

$$n_i > 2 \frac{\sqrt{k^2 + m_i^2}}{V_\nu}, \quad \frac{m_i}{V_\nu} = 0.44 \times 10^2 \text{ MeV}^3, \quad \frac{m_i}{50\text{ meV}} = 0.31 \times 10^5 \text{ cm}^{-3}, \quad \frac{m_i}{50\text{ meV}}. \quad (8)$$

Note a typical neutrino number density in the early universe,

$$n_\nu = \frac{3\zeta(3)}{2\pi^2} T^3 \sim 2.4 \times 10^{31} \text{ cm}^{-3} \left( \frac{T}{\text{MeV}} \right)^3. \quad (9)$$
As in the theory of superconductivity \[2\] in which phonon mediated interaction for a Cooper pair of electrons of momenta, \((\vec{p}, -\vec{p})\), gives rise to pair condensation, one may expect a similar phenomenon of neutrino super-fluid. First, let us discuss bound state of neutrino pair. Higher order interaction between spin-singlet neutrino pair is given by 2Z and 2W exchange and there are still higher orders like Fig(2). Summing up all these terms gives

\[
\frac{|V_\nu| n_i^2}{1 - t_Z - t_W} \quad \text{(10)}
\]

\[
t_Z = \frac{3g_Z^2}{4\pi^2} \ln \frac{m_Z^2}{m_i^2}, \quad t_W = \frac{3g_W^2}{4\pi^2} \ln \frac{m_W^2}{m_i^2} \quad \text{(11)}
\]

We have neglected sub-leading terms proportional to \(m_i^2/m_W^2\) arising from charged leptons of mass \(m_l\). Bound pair state thus appears at

\[
2m_i - \frac{|V_\nu| n_i}{1 - t_Z - t_W} < 0, \quad \text{(12)}
\]

which gives a smallest neutrino mass value for given number density \(n_i\).

To show that bound state pairs actually condensate, one further needs to discuss global and macroscopic features of many-neutrino system, and for this purpose we analyze the problem using the Ginzburg-Landau mean field theory and Bogoliubov transformation \[4\] along the same lines of arguments as in superconductivity.

### 2.2 Ginzburg-Landau mean field theory and Bogoliubov transformation

We assume a homogeneous density of condensate \(n\) and calculate terms proportional to \(n^4\) (term \(\propto n^3\) does not occur). The Feynman diagram for this process is four Z vertex with neutrino pair attached to each propagating Z, as shown in Fig(3). One loop diagram of circulating neutrino and other fundamental particles coupled to four Z gives divergent contribution which gives renormalization counter term to the bare four Z vertex. In terms of renormalized four Z coupling this contribution gives repulsive term of strength,

\[
\frac{4G_F g_Z^6}{\sqrt{2} m_Z^6} n^4 > 0 \quad \text{for singlet}. \quad \text{(13)}
\]
Potential minimum when $m_\nu = 0$ is at a critical density $n_c$, which is given by

$$n_c \approx \frac{2\pi m_Z^2}{\sqrt{96\zeta(3)g_Z^3}} \sim 0.48\left(\frac{m_Z}{g_Z}\right)^3,$$

close to electroweak phase transition point. More precisely,

$$n_c = \frac{2\pi}{\sqrt{96\zeta(3)g_Z^3}} \left( \frac{m_Z}{g_Z} \right)^3 \left( 1 + \frac{16m_\nu}{G_F m_Z^3} \right).$$

Condensate energy density is given by

$$2m_\nu n_c - \frac{3G_F}{4\sqrt{2}} n_c^2 \left( -n^2 + 16\left( \frac{g_Z}{m_Z} \right)^6 n^4 \right).$$

Figure 3: Eight $\nu_L$ attached to four $Z$-vertex. Straight lines are for $\nu_L$, while wavy ones are for $Z$ bosons.

For later application we recapitulate the operator form of singlet pair hamiltonian density,

$$H_\nu = 2m_\nu n + \frac{3G_F}{4\sqrt{2}} \left( -n^2 + 16\left( \frac{g_Z}{m_Z} \right)^6 n^4 \right).$$

Due to instability of bound state formation the ground state is radically changed from the usual one at zero temperature, and one needs to find a new ground state, which is done by using a Bogoliubov transformation. The Bogoliubov transformation violates lepton-number, and is introduced by a linear combination of annihilation and creation operators of definite momentum $\vec{k}$,

$$\beta_k = u_k b_k - v_k b^\dagger_{-k}, \quad \beta_k^\dagger = u_k b^\dagger_k - v_k b_{-k}. \quad (17)$$

with real functions, $u_k, v_k$, to be determined. We anticipate that the state $|\tilde{0}\rangle$ defined by $\beta_{k'}|\tilde{0}\rangle = 0$ for all $\vec{k}'$ is the ground state made of condensates, and $|\beta_{k'}\tilde{0}\rangle$ is a quasi-particle state of single neutrino in super-fluid.

Anti-commutation relations, $\{\beta_k, \beta_{k'}^\dagger\} = \delta_{k,k'}, \{\beta_k, \beta_k^\dagger\} = 0$, requires

$$u_k^2 + v_k^2 = 1, \quad u_k v_{-k'} + v_k u_{-k'} = 0. \quad (18)$$

The latter condition is readily satisfied by $u_{-k} = u_k, v_{-k} = -v_k$. With this property, the inversion is given by

$$b_k = u_k \beta_k + v_k \beta_{-k}^\dagger, \quad b_k^\dagger = u_k \beta_k^\dagger + v_k \beta_{-k}. \quad (19)$$
In the presence of condensate one defines the order parameter \( \Psi = \sqrt{n_c} + \sum_k (b_k + b_k^\dagger) \) and the number density operator \( \tilde{n} = \Psi^\dagger \Psi \). Introducing a hermitian operator \( A \), we derive relations,

\[
A_k = (u_k - v_k)(\beta_k + \beta_k^\dagger),
\]

\[
\tilde{n} = n_c + \sum_k A_k^2 + \sqrt{n_c} \sum_k A_k, \quad (20)
\]

\[
\langle \bar{0}| n| \bar{0} \rangle = n_c + \sum_k (u_k - v_k)^2 f_k = n_c + \sum_k (u_k^2 + v_k^2) f_k = 2n_c, \quad (21)
\]

\[
\sum_k f_k = \frac{d^3k}{(2\pi)^3} f_k = n_c, \quad (22)
\]

with \( f_k \) the momentum distribution function assumed spatially homogeneous. The distribution function here refers to neutrinos within a condensate, not to be confused with the thermal distribution in surrounding medium. This is a decomposition of number operator into the mean field value of condensates and elementary excitation (or quasi-particle).

We give two important examples of momentum distribution: the first is Fermi-Dirac distribution at finite temperatures applicable in the early stage of condensate formation,

\[
f_k = \frac{1}{e^{\sqrt{k^2 + m^2}/T} + 1} \approx \frac{1}{e^{k/T} + 1}, \quad n_c = \int \frac{d^3k}{(2\pi)^3} f_k \approx \frac{\zeta(3)}{4\pi^2} T^3 \sim 0.046 T^3. \quad (24)
\]

The other is degenerate distribution in beta equilibrium, \( n + (\nu_e)_L \leftrightarrow p + e_L \), applicable in the late stage of condensate evolution:

\[
f_k = \theta(k_F - |\vec{k}|), \quad n_c = \int \frac{d^3k}{(2\pi)^3} f_k = \frac{1}{6\pi^2} k_F^3. \quad (25)
\]

We state two theorems and results on super-fluid state of neutrino singlet pair.

**Theorem I  Emergence of super-fluid**

First, note the following identity:

\[
[A^2, \beta_k] = 0 \quad (26)
\]

We use normal-ordered operator product as in the Wick theorem in quantum field theory, which gives contracted factor,

\[
\langle \bar{0}| A^2| \bar{0} \rangle = \sum_k (u_k^2 + v_k^2) = n_c \quad (27)
\]

The following relations then hold:

\[
\langle \bar{0}| (A^2)^p| \bar{0} \rangle = n_c^p \quad (28)
\]

\[
\langle \bar{0}| \beta_1 \cdots \beta_l (A^2)^p \beta_1^\dagger \cdots \beta_l^\dagger| \bar{0} \rangle = n_c^p \quad (29)
\]

for \( 1, 2, \cdots l \) taken all different. Expectation values of any function of number operator \( f(n) \) are then calculated as

\[
\langle \bar{0}| \beta_1 \cdots \beta_l f(n) \beta_1^\dagger \cdots \beta_l^\dagger| \bar{0} \rangle = \frac{1}{2} \left( f(n_c^+) + f(n_c^-) \right), \quad n_c^\pm = 2n_c \pm \sqrt{n_c A}. \quad (30)
\]

This implies that macroscopic event involving neutrinos in condensate has no resistance, hence it proves the super-fluid nature of condensates.

These expectation values are independent of number of neutrinos involved, and this suggests that neutrinos in condensate behave as massless. We shall prove this in the following theorem.
Theorem II  

Condensate energy and massless quasi-particle

We consider effective hamiltonian (actually density) to order $n^4$, to derive operator form,

$$H_\nu = m_\nu n + \frac{3GF}{4\sqrt{2}}(-n^2 + 16\left(\frac{g_Z}{m_Z}\right)^6 n^4)$$  \hspace{1cm} (31)

Following the method of Theorem I, we calculate energies of condensate and many quasi-particle state:

$$E(\tilde{0}) = \langle \tilde{0} | H_\nu | \tilde{0} \rangle = 2m_\nu n^c + 3G_F \sqrt{2} (-c_2 n^2_\nu + 16c_4 \left(\frac{g_Z}{m_Z}\right)^6 n^4_\nu)$$  \hspace{1cm} (32)

$$E(s) = \langle \tilde{0} | \beta_s \cdots \beta_1 H_\nu \beta_1^\dagger \cdots \beta_s^\dagger | \tilde{0} \rangle = \langle \tilde{0} | H_\nu | \tilde{0} \rangle = E(\tilde{0}) \text{ for } (1, 2, \cdots s) \text{ all different}$$  \hspace{1cm} (33)

with $c_2 = 5, c_4 = 41$.

This energy density has two extrema for $m_\nu > 0$, a local maximum near $n^c = 0$ and a global minimum $n = n^*_c$, with

$$n^*_c = \sqrt{\frac{c_2}{32c_4}} \left(\frac{m_Z}{g_Z}\right)^3$$  \hspace{1cm} (34)

This state is highly degenerates due to excitation of massless neutrinos of arbitrary numbers.

3  Cosmological evolution of pair condensates

The critical temperature $T_c$ is estimated by equating $n^*_c$ in eq.(34) to thermal number density of neutrino $3\zeta(3)T^3/(4\pi^2)$ (for one spin component $\nu_L$ not counting $\bar{\nu}_L$), to derive

$$T_c = (\frac{\pi^2 c_2}{16\zeta(3)c_4})^{1/3} \frac{m_Z}{g_Z} \sim 0.40 \frac{m_Z}{g_Z} \sim 71 \text{ GeV}.$$  \hspace{1cm} (35)

Thus, the critical temperature is less than temperature of electroweak phase transition. Below $T_c$ the instability develops towards pair condensation.

3.1 Super-fluid phases and condensate number density

Important properties of super-fluids made of $\nu_L, \bar{\nu}_L$ pairs are described by the space-time dependent order parameter $\Phi(\vec{r}, t) = \nu_L(\vec{r}, t)\nu_L(\vec{r}, t)$ in spin-singlet combination, just as by Cooper pair in the ordinary superconductor. Main differences from there are charges and masses of constituents. We shall ignore space dependence of order parameter, and focus on cosmological time evolution. Order parameter of condensate is decomposed as a product of absolute magnitude $\sqrt{n_S} > 0$ and the phase function.

Following [2], we study the super-current across a boundary (called junction in solid state physics) between two super-fluid phases of pair condensates. We consider a plane boundary and two super-fluid phases given by $\Psi_1, \Psi_2$, left and right super-fluids with different chemical potentials. The relevant coupled equations are

$$i \left( \frac{\partial \Psi_1}{\partial t} + 3\frac{\dot{a}}{a} \Psi_1 \right) = \mu_1 \Psi_1 + T \Psi_2,$$  \hspace{1cm} (36)

$$i \left( \frac{\partial \Psi_2}{\partial t} + 3\frac{\dot{a}}{a} \Psi_2 \right) = \mu_2 \Psi_2 + T \Psi_1,$$  \hspace{1cm} (37)

with $\mu_1, \mu_2$ chemical potentials. For simplicity time-reversal invariance is assumed, which makes the transition amplitude $T$ real [4]. These coupled equations are identical to the junction equations in ordinary superconductivity except the effect of cosmic expansion $\propto 3\dot{a}/a$ terms and time dependences of $\mu_i, T$. One
with \( |\Psi_1|^2 = |\Psi_2|^2 = n_S \), recovering the momentum fraction factor. On the other hand, the equation for \((\partial_t + 3\dot{a}/a)(\Psi_1\Psi_2)\) leads to

\[
\frac{\partial t}{\partial t} (\chi_1 - \chi_2) = - (\mu_1 - \mu_2) \equiv \delta\mu_{12},
\]

\[
(\frac{\partial}{\partial t} + 3\dot{a}/a)n_S = -2T \left( \sin \int_t^T dt' \delta\mu_{12}(t') \right) n_S.
\]

Difference from equation of stable decoupled matter is obvious: there exists continuous flow in and out from adjacent super-fluids if both \( T \), \( \delta\mu \) are non-vanishing. This gives rise to the logarithm of number density decreasing as the third power times correction given by an integral, hence the number density of super-fluid remains constant as cosmic time increases. The equation for super-fluid number density is formally solved, to give

\[
n_S(t) = n_S(t_c) \exp \left\{ -2 \int_{t_c}^t dt_1 T(t_1) \left( \frac{a(t_1)}{a(t)} \right)^3 \left( \sin \int_{t_c}^{t_1} dt'' \delta\mu(t'') \right) \right\}.
\]

It may be better to give this solution in terms of cosmic temperature;

\[
\left( \frac{a(t_1)}{a(t)} \right)^3 = \left( \frac{T}{T_1} \right)^3,
\]

and \( dt \propto dT/T^3 \), \( \propto dT/T^{5/2} \) depending on radiation- or matter-dominated epoch.

Chemical potentials in super-fluids are identified as the free neutrino energy written as \( \mu = \sqrt{p^2 + m_\nu^2} \) plus interaction term of order \( G_F n \), both real numbers. In two adjacent phases interaction are identical, but neutrino masses may be different. The result of time evolving super-fluid number density derived above depends on difference of chemical potential in adjacent phases. Thus, there are three different types of boundaries for three massive neutrinos. We are interested in relativistic neutrinos in the early epoch, hence \( \sqrt{p^2 + m_\nu^2} \sim p + m_\nu^2/2p \), giving \( \delta\mu_{ij} = (m_i^2 - m_j^2)/2p \). We shall be content with an approximation in which neutrino momentum here is replaced by its thermal average:

\[
\frac{1}{p} = \frac{4\pi^2\zeta(3)}{3T} \int_0^\infty dx \frac{x}{e^x + 1} = \frac{\pi^4\zeta(3)}{9T} \approx \frac{19.5}{T}.
\]

This gives

\[
\int_{t_c}^{t_1} dt'' \delta\mu(t'') = -\frac{4\pi^4\zeta(3)}{18c} \int_T^{t_c} dT \delta m_{ij}^2 M_{pl} \left( \frac{1}{T^3} - \frac{1}{T_1^3} \right),
\]

\[
\frac{\delta m_{ij}^2 M_{pl}}{T^3} \approx 1.2 \times 10^{-6} \left( \frac{10\text{GeV}}{T} \right)^3,
\]

with coefficient \( \frac{\pi^4\zeta(3)}{54c} \) of order 0.1 at 1 GeV region of temperature.

The transmission matrix element \( T \) is given by real part of forward scattering amplitude times number density of scattered target in the boundary wall. For a simple estimate we shall replace the real part by the imaginary part or absorptive part, which is expressed in terms of total cross section by the optical theorem. With this assumption we estimate

\[
T(T) \approx \sigma_{tot} n = \frac{G_F^2 T^5}{2\pi^3} \int_0^\infty dx \frac{x^4}{e^x + 1} = \frac{45\zeta(5)G_F^2 T^5}{4\pi^3} \approx 8.5 G_F^2 T^5,
\]

\[
dt_1 T(t_1) \left( \frac{a(t_1)}{a(t)} \right)^3 = -G_F^2 dT_1 \frac{M_{pl} T_1^3}{T} \frac{45\zeta(5)}{4\pi^3 c} \approx -5.75 \times 10^{11} \left( \frac{T}{10\text{GeV}} \right)^3\frac{dT_1}{T_1}.
\]
The exponent of \( n_S(t)/n_S(t_c) \) exhibits an interesting behavior: above a few times 100 MeV temperature region it is a large number independent of temperature. It starts to rapidly oscillate when the argument of sinusoidal function becomes of order unity, which is around \( 50 \sim 100 \) MeV. The amplitude is of order \( 10^5(T/50\text{MeV})^3 \) or larger. The oscillation ends around 10 MeV prior to the onset time of nucleo-synthesis, decreasing rapidly, and a large constant number density \( n_S(t_c) \) remains at later epochs.

We took here a junction between two super-fluids, but a junction between super- and normal-phases can be dealt with in a similar way.

\[
\text{Exponent function}
\]

![Exponent function graph](image)

Figure 4: Exponent function \( f(T, T_c) \) of super-fluid number density given by (50). Parameter \( \delta^2 m_{ij} \) chosen is 10 meV\(^2\) in solid black and 250 meV\(^2\) in dashed red.

We illustrate the behavior of exponent function in Fig(4). The formula we used for this is the exponent function,

\[
-\frac{45\zeta(5)G_F^2M_{\text{pl}}}{2\pi^3cT^3} \int_T^{T_c} \frac{dT_1}{T_1} \sin \left( \frac{\pi^4\zeta(3)}{54c} \delta m_{ij}^2 M_{\text{pl}} \left( \frac{1}{T_1^3} - \frac{1}{T_c^3} \right) \right)
\]

\[
\sim -1.15 \times 10^9 \left( \frac{T}{\text{1GeV}} \right)^3 \int_T^{T_c} \frac{dT_1}{T_1} \sin \left( 1.7 \times 10^{-4} \frac{\delta m_{ij}^2}{(10\text{meV})^2} \left( \frac{1\text{GeV}}{T_1} \right)^3 - (T_1 \to T_c) \right) \equiv -f(T, T_c). \quad (50)
\]

The corresponding super-fluid number density \( n_S(T)/n_S(T_c) \) is shown in Fig[5].

We call condensate neutrino pair neutrino super-ball (NSB) to distinguish this from another type of neutrino ball [5], [6]. What would be the mass distribution of NSB? Super-fluid formation starts at temperature \( T_c \) till around 50 MeV. Horizon sizes during this interval are possible sizes of super-fluid phases, thus ranging in \( 4 \sim 4 \times 10^6 \) cm. NSB’s, however, attract each other and they may further grow.

Time evolution of pair condensate as described above is valid only when neither accretion nor leakage of pair condensate occurs. But as discussed in the next subsection, accretion is likely to proceed.

Before closing this subsection, it is instructive to discuss why electron-proton plasma at similar epochs of early universe does not produce condensate. Stronger Coulomb attraction between electron and proton might suggest condensate of hydrogen atoms. But this stronger interaction produces abundant photons, to create a plasma state including thermal photons which block condensate formation. In the neutrino case mediating Z-boson is energetically impossible to be produced after the electroweak phase transition such that no blocking occurs towards neutrino pair condensation.
Figure 5: Super-fluid number density $n_s(T)/n_s(T_c)$ corresponding to the same parameter choice of Fig 4. Two parameter $\delta^2 m_{ij}$ cases of 10 meV$^2$ and 250 meV$^2$ give nearly identical $n_s(T)$ in this temperature range with this calculation resolution. Note a smaller temperature range than in Fig 4.

3.2 Accretion of other fermions

Note first that the ratio of left-handed neutrino density in super-fluid phase to that in normal phase increases rapidly, with temperature power $T^{-3}$. It thus seems that leakage of left-handed neutrinos dominantly occurs from over-dense super-state to normal state. However, accretion of charged fermions, $u, d, e$ and later $p, n, e$ may balance against this neutrino leakage.

We consider for definiteness the epoch of cosmic temperature in the range 100 MeV $\sim$ down to epoch of nucleo-synthesis. During this epoch anti-nucleon and positron annihilation occur. Below $\approx$ 20 MeV (time of anti-nucleon annihilation) the dominant components of fermions are proton, neutron, electron, positron and 3 species of neutrinos, and left-handed neutrino condensates having enormous number density $\geq (0.1 \text{GeV})^3$.

Some of left-handed fermions in surrounding medium are attracted to super-fluid left-handed neutrinos. Relative attractive or repulsive (given by negative numbers) strengths of spin singlets to $\nu_L^i$ are

$$p : \frac{1}{4} - \frac{1}{2} \sin^2 \theta_w, \quad n : -\frac{1}{4}, \quad e : |U_{ie}|^2 - \frac{1}{2}, \quad \nu_j : \frac{1}{4},$$ 

with positive numbers acting attractively. $\theta_w$ is the weak mixing angle measured to be $\sin^2 \theta_w \sim 0.238$, and $U_{ie}$ is a neutrino mass mixing matrix element of order $|U_{ie}|^2 \sim 0.7$ [11]. Interaction between $\nu_e$ and $e_L$ is mediated by W-exchange. Numerically, these numbers are $0.13, -0.25, 0.2, 0.25$, respectively. Except neutrons, all fermions are attracted by $\nu_L$ condensate.

Consider a neutrino pair condensate and calculate effective interaction hamiltonian of $p, n, e, \nu$ in surrounding thermal bath. Extending the previous calculation, one finds the effective hamiltonian is given by

$$H_a = \frac{3G_F}{4\sqrt{2}} \left( -\sum_{ij} n_i n_j + 16 \left( \frac{gZ}{m_Z} \right)^6 \sum_{ijkl} n_{i} n_{j} n_{k} n_{l} \right),$$ 

with $Z$ being nuclear charge.
where \( n_i, i = p, n, e, \nu \) are thermal number densities, and their weight factors relative to neutrino \( \xi_i \) are given by \( \xi_p = 1 - 2 \sin^2 \theta_w, \xi_n = -1, \xi_e = 4|U_{ei}|^2 - 2, \xi_\nu = 1 \). Hence the bracket quantity in eq.(52) is

\[
- \sum_{ij} n_i n_j + 16 \frac{g_Z}{m_Z} \sum_{ijkl} n_i n_j n_k n_l = - \left( \sum_i \xi_i \right)^2 n_\nu^2 + 16 \frac{g_Z}{m_Z} \left( \sum_i \xi_i \right)^4 n_\nu^4,
\]

with \( n_\nu \approx 3 \zeta(3)T^3/(4\pi^2) \sim 0.137T^3 \) using the number density appropriate to the early stage of condensate formation. Thus one may estimate the temperature \( T_a \) at which accretion starts:

\[
T_a = \left( \frac{1}{16(\sum_i \xi_i)^2} \right)^{1/6} \frac{m_Z}{g_Z} \sim 69 \text{ GeV},
\]

with \( \sum_i \xi_i = 5 - 2 \sin^2 \theta_w \sim 4.52 \) adding three neutrino species, and using \( m_Z/g_Z \sim 180\text{GeV} \). Thus, accretion of surrounding fermions becomes possible slightly below, but very close to, the critical temperature \( T_c \).

Accretion is, in its detail, likely to proceed in a different way due to stronger Coulomb attraction between electron and proton. The first accreted fermion is presumably proton due to its positive \( \xi_i \) followed by Coulomb attraction of electron. Neutron is likely to be created by inverse beta process, \( p + e \rightarrow n + \nu_e \) inside super-fluid. The accretion starts at \( \sim 100\text{MeV} \), when super-fluid settles to the number density \( n(T_1) \) with \( T_1 = 100 \sim 0.1\text{GeV} \), and proceeds until number densities of \( p, n, e \) inside become comparable to their thermal density outside the bubble.

Higgs boson exchange may also contribute to the attractive force: one can use the following Fierz identity for four component Dirac or Majorana fields,

\[
\bar{u}_1(1 - \gamma_5)u_2 \bar{u}_3(1 - \gamma_5)u_4 = -2 \left( \bar{u}_1 \gamma^\mu \frac{1 + \gamma_5}{2} u_4 \bar{u}_3 \gamma_\mu \frac{1 + \gamma_5}{2} u_2 \right) \\
\sim 2u_1^\dagger S u_4 \cdot u_3^\dagger S u_2 + \cdots
\]

Observed sign change of Higgs exchange gives spin-triplet condensate pairs of the following type,

\[
(u_L \bar{u}_R)_{S=1}, \ (d_L \bar{d}_R)_{S=1}, \ (e_L \bar{e}_R)_{S=1}
\]

and their hermitian conjugates. These are charge neutral, and readily accreted into super neutrino balls. As single charged fermions, \( \bar{u}_R, \bar{d}_R, \bar{e}_R \) are also attracted to \( \nu_L \nu_L \) pairs.

The created aggregate of beta-equilibrium may be called primordial neutron star (PNS), but its internal structure, in particular, the stratified onion skin structure, \( [7] \) of ordinary neutron stars is not realized, since ordinary neutron stars are gravitationally bound. Gravity affects NSB’s and PNS at later epochs, and may form bosonic stars \( [5] \), or axion stars \( [9], [10] \), made of neutrino-pair condensate this time. Important corrections may arise due to that neutrino pairs are fragile.

Number and energy density densities of NSB stays constant with cosmological time evolution, hence they behave as dark energy. On the other hand, the energy density of PNS may be dominated by mass density of nucleons after nucleo-synthesis, hence they are classified and behave as dark matter. Masses of individual PNS’s may vary, although they have a mass range, quite unlike a definite mass assigned to particle physics dark matter; WIMP and axion. Thus, the strategy of searching PNS should keep in mind this feature. It is an attractive idea that these two different forms of constituents of universe originate from a single event, since the dark energy and the dark matter are roughly comparable energetically at the present epoch. It must be verified, however, that the major fraction of neutrinos at earlier epochs does not proceed to formation on NSB not to over-close the universe at late epochs.

4 Summary and Outlook

We studied the nature of neutrino self-interaction in the standard electroweak theory, and found that the force is attractive. This led us further to investigate the possibility of neutrino pair condensate using the
standard tools in statistical mechanics and condensed matter physics, the Ginzburg-Landau mean field theory and the Bogoliubov transformation. Results suggest that super-fluid formation is likely to occur in much the same way as condensate formation of the electron Cooper pair in superconductivity of metals. It is likely that the early universe at temperatures below 100 GeV is made of a mixed phase of ideal gas and a small fraction of super-fluid neutrino pair condensates.

Accretion of other fermions from surrounding cosmic medium starts immediately after neutrino-pair condensate formation, and primordial neutron stars (PNS) of varying masses are presumably formed. An interesting scenario to late-time cosmology is that condensed neutrino super-balls (NSB) provide the dark energy and PNS becomes the dark matter. Estimate of PNS dark matter not to over-close the universe is to be established, and PNS mass spectrum is crucial to determine their search strategy. These important issues are left to future works.

It is still premature to judge whether the suggested scenario in the present work is viable or not. Nevertheless, we may list a few items on how the proposed scenario may be verified in future. Detection of relic neutrino may directly probe the momentum distribution of condensed neutrino pairs. The idea of detecting relic neutrino proposed in [12] is to use atomic de-excitation, radiative neutrino pair emission, $|i\rangle \rightarrow |f\rangle + \gamma + \nu_i\bar{\nu}_j$ from macro-coherently state $|i\rangle$ excited by high-quality lasers, and to experimentally investigate how ambient relic neutrinos Pauli-block neutrino pair emission such that this is reflected in observed photon energy spectrum, or its accumulated parity violating magnetization that helps to reject possible QED backgrounds [13]. A large neutrino degeneracy created by NSB formation gives a large deviation from the thermal distribution of zero chemical potential.

Discovery of gravitational wave from black hole and neutron star merger enhanced the possibility that some day one may be able to measure the mass spectrum of relevant neutron stars. In this respect, simultaneous detection of gravitational wave and neutrino from mergers of black hole and PNS gives important insights on our scenario. Crucial question on observable consequences of super-fluid neutrinos in PNS remains to be studied, but the proposal can be verified by high statistics data of neutrino burst at PNS formation.

Acknowledgements

This research was partially supported by Grant-in-Aid 21K03575 from the Japanese Ministry of Education, Culture, Sports, Science, and Technology.

References

[1] C. Hayashi, Progr. Theor. Phys. 5, 224 (1950).
[2] J.M. Ziman, Principles of the Theory of Solids, Chapter 11, Cambridge University Press (1979).
[3] A standard textbook of modern cosmology is S. Weinberg, Cosmology, Oxford University Press, New York (2008).
[4] E.M. Lifshitz and L.P. Pitaevskii, Course of Theoretical Physics Volume 9 Statistical Physics: Part 2, 3rd edition. Chapter V, Pergamon Press (1980).
[5] B. Holdom, Phys.Rev. D36, 1000 (1987): Phys.Rev. D49, 3844 (1994).
[6] A.D. Dolgov and O. Yu. Markin, Sov. Phys. JETP 71, 207 (1990): Prog. Theor. Phys. 85, 1091 (1991).
[7] S.L. Shapiro and S.A. Teukolsky, Black holes, White dwarfs, and Neutron stars, John Wiley and Sons (New York), (1983).
[8] R. Ruffini and S. Bonazzola, Phys. Rev. 187, 1767 (1969).
[9] E. Takasugi and M. Yoshimura, Z.Phys.C 26, 241 (1984).

[10] E. W. Kolb and I. I. Tkachev, Phys. Rev. Lett. 71, 3051 (1993).

[11] For a review of measurements on the weak mixing angle and neutrino mass mixing matrix, see Particle Data Group Collaboration, M. Tanabashi et al., Phys. Rev. D98, 030001 (2018).

[12] M. Yoshimura, N. Sasao, and M. Tanaka, Phys. Rev.D90, 013022 (2014); Experimental method of detecting relic neutrino by atomic de-excitation, arXiv; 1403.6546[hep-ph].

[13] H. Hara, A. Yoshimi, and M. Yoshimura, Phys.Rev. D104, 115006(2021). Parity violating magnetization at neutrino pair emission using trivalent lanthanoid ions, arXiv; 2105.1114[hep-ph].