Data completion problems solved as Nash games

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Abstract.

The Cauchy problem for an elliptic operator is formulated as a two-player Nash game. Player (1) is given the known Dirichlet data, and uses as strategy variable the Neumann condition prescribed over the inaccessible part of the boundary. Player (2) is given the known Neumann data, and plays with the Dirichlet condition prescribed over the inaccessible boundary.

The two players solve in parallel the associated Boundary Value Problems. Their respective objectives involve the gap between the non used Neumann/Dirichlet known data and the traces of the BVP’s solutions over the accessible boundary, and are coupled through a difference term. We prove the existence of a unique Nash equilibrium, which turns out to be the reconstructed data when the Cauchy problem has a solution. We also prove that the completion algorithm is stable with respect to noise, and present two 3D experiments which illustrate the efficiency and stability of our algorithm.

1. Introduction

Let be $\Omega$ a bounded open domain in $\mathbb{R}^d$ ($d = 2$, 3) with a sufficiently smooth boundary $\partial \Omega$ composed of two connected disjoint components $\Gamma_c$ and $\Gamma_i$. We consider the following elliptic Cauchy problem:

$$\begin{cases}
\nabla \cdot (k \nabla u) = 0 & \text{in } \Omega \\
u & = f \text{ on } \Gamma_c \\
k \nabla u \cdot \nu = \Phi \text{ on } \Gamma_c
\end{cases} \tag{1}$$

The parameters $k$, $f$, and $\Phi$ are given functions, $\nu$ is the unit outward normal vector on the boundary. The Dirichlet data $f$ and the Neumann data $\Phi$ are the so-called Cauchy data, which are known on the accessible part $\Gamma_c$ of the boundary $\partial \Omega$ and the unknown field $u$ is the Cauchy solution. The Cauchy problem is also known as a data completion problem, where the data to be recovered, or missing data, are $u|_{\Gamma_i}$ and $k \nabla u \cdot \nu|_{\Gamma_i}$, which are determined as soon as one knows $u$ in the whole $\Omega$.

Completion/Cauchy problems are known to be severely ill-posed (Hadamard’s), and computationally challenging.

Many authors ([1]) have formulated the Cauchy problem (1) as an optimal control one, as follows:

Let us assume that $(\Phi, f) \in H^{-\frac{1}{2}}(\Gamma_c) \times H^{\frac{1}{2}}(\Gamma_c)$. For given $\eta \in H^{-\frac{1}{2}}(\Gamma_i)$ and $\tau \in H^{\frac{1}{2}}(\Gamma_i)$, let us define $u_1(\eta)$ and $u_2(\tau)$ as the unique solutions in $H^1(\Omega)$ of the following elliptic boundary value problems:

$$\begin{cases}
\nabla \cdot (k \nabla u_1) = 0 & \text{in } \Omega \\
u_1 & = f \text{ on } \Gamma_c \\
k \nabla u_1 \cdot \nu = \eta \text{ on } \Gamma_i \tag{SP1}
\end{cases} \quad \begin{cases}
\nabla \cdot (k \nabla u_2) = 0 & \text{in } \Omega \\
u_2 & = \tau \text{ on } \Gamma_i \\
k \nabla u_2 \cdot \nu = \Phi \text{ on } \Gamma_c \tag{SP2}
\end{cases}$$

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The optimization problem amounts to minimize, among all pairs \((\eta, \tau) \in H^{-\frac{1}{2}}(\Gamma_i) \times H^{\frac{1}{2}}(\Gamma_i)\), the following “Neumann-gap” cost:

\[
J_1(\eta, \tau) = J_1(\eta, \tau, u_1(\eta), u_2(\tau)) = \frac{1}{2} \| k \nabla u_1 . \nu - \Phi \|^2_{H^{-\frac{1}{2}}(\Gamma_c)} + \frac{1}{2} \| u_1 - u_2 \|^2_{L^2(\Gamma_i)}. \tag{3}
\]

It is known that when the Cauchy problem (1) has a solution, then solving it is equivalent to solving the minimization problem

\[
\min_{(\eta, \tau) \in H^{-\frac{1}{2}}(\Gamma_i) \times H^{\frac{1}{2}}(\Gamma_i)} J_1(\eta, \tau). \tag{4}
\]

The same conclusions above hold when a “Dirichlet-gap” cost is considered:

\[
J_2(\eta, \tau) = J_2(\eta, \tau, u_1(\eta), u_2(\tau)) = \frac{1}{2} \| u_2 - f \|^2_{L^2(\Gamma_i)} + \frac{1}{2} \| u_1 - u_2 \|^2_{L^2(\Gamma_i)}. \tag{5}
\]

Formulated in the game theory vocabulary, the Neumann and Dirichlet controls \(\eta\) and \(\tau\) do cooperate to minimize either the Neumann-gap or the Dirichlet-gap costs. These two controls could as well cooperatively minimize any convex combination of the two costs \(J_1\) and \(J_2\).

Now, the fields \(u_1(\eta)\) and \(u_2(\tau)\) are aiming at the fulfillment of a possibly antagonistic goals, namely minimizing the Neumann gap \(\| k \nabla u_1 . \nu - \Phi \|^2_{H^{-\frac{1}{2}}(\Gamma_c)}\) and the Dirichlet gap \(\| u_2 - f \|^2_{L^2(\Gamma_i)}\). This antagonism is intimately related to Hadamard’s ill-posedness character of the Cauchy problem, and rises as soon as one requires that \(u_1\) and \(u_2\) coincide, which is exactly what the coupling term \(\| u_1 - u_2 \|^2_{L^2(\Gamma_i)}\) is for. Thus, one may think of an iterative process which minimizes in a smart fashion the three terms, namely Neumann-Dirichlet-Coupling terms.

Let us define the following two costs: for any \(\eta \in H^{-\frac{1}{2}}(\Gamma_i)\) and \(\tau \in H^{\frac{1}{2}}(\Gamma_i)\),

\[
J_1(\eta, \tau) = \frac{1}{2} \| k \nabla u_1 . \nu - \Phi \|^2_{H^{-\frac{1}{2}}(\Gamma_c)} + \frac{\alpha}{2} \| u_1 - u_2 \|^2_{H^{\frac{1}{2}}(\Gamma_i)}, \tag{6}
\]

\[
J_2(\eta, \tau) = \frac{1}{2} \| u_2 - f \|^2_{L^2(\Gamma_i)} + \frac{\alpha}{2} \| u_1 - u_2 \|^2_{H^{\frac{1}{2}}(\Gamma_i)}, \tag{7}
\]

where the fields \(u_1(\eta)\) and \(u_2(\tau)\) are the unique solutions to (SP1) and (SP2), respectively and \(\alpha\) is a given positive parameter (e.g. \(\alpha = 1\)).

We shall say that there are two players, referred to as player 1 or Neumann-gap, and player 2 or Dirichlet-gap. Player 1 controls the strategy variable \(\eta\), and player 2 controls the strategy variable \(\tau\). Each of the two players tries to minimize its own cost, namely \(J_1\) for player 1, and \(J_2\) for player 2. As classical, the fact that each player controls only his own strategy, while there is a strong dependance of each player’s cost on the joint strategies \((\eta, \tau)\) justifies the use of the game theory framework (and terminology), a natural setting which may be used to formulate the negotiation between these two costs.

In order to be consistent with the initial formulation of the Cauchy problem, the relevant game theoretic framework to deal with is a static with complete information one. In this case, a commonly used solution concept (roughly speaking, in the game vocabulary, a rational and stable one) is the one of Nash Equilibria. Our main results are the following:

**Proposition 1** Consider the Nash game defined by the two costs (6) and (7).

(i) The partial mapping \(\eta \rightarrow J_1(\eta, \tau)\) (respectively \(\tau \rightarrow J_2(\eta, \tau)\)) is a quadratic strictly convex and coercive functional over \(H^{-\frac{1}{2}}(\Gamma_i)\) (resp. \(H^{\frac{1}{2}}(\Gamma_i)\)).

(ii) There always exists a unique Nash equilibrium \((\eta_N, \tau_N) \in H^{-\frac{1}{2}}(\Gamma_i) \times H^{\frac{1}{2}}(\Gamma_i)\). It is also the minimum of \(L(\eta, \tau) = \frac{1}{2} \| k \nabla u_1 . \nu - \Phi \|^2_{H^{-\frac{1}{2}}(\Gamma_c)} + \frac{1}{2} \| u_2 - f \|^2_{L^2(\Gamma_i)} + \frac{\alpha}{2} \| u_1 - u_2 \|^2_{H^{\frac{1}{2}}(\Gamma_i)}\).
(iii) When the Cauchy problem has a solution \( u \), then \( u_1(\eta_N) = u_2(\tau_N) = u \), and \((\eta_N, \tau_N)\) are the missing data, namely \( \eta_N = k\nabla u.\nu|_{\Gamma_i}\) and \( \tau_N = u|_{\Gamma_c}\).

**Proposition 2** Assume there exists a unique Cauchy solution \( u \in H^1(\Omega) \) for a given compatible pair of data \((f, \Phi) \in H^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)\). Let \((f^\delta, \Phi^\delta) \in H^{\frac{1}{2}}(\Gamma_c) \times H^{-\frac{1}{2}}(\Gamma_c)\) be any sequence of noisy data such that

\[
\|f^\delta - f\|^2_{H^{\frac{1}{2}}(\Gamma_c)} + \|\Phi^\delta - \Phi\|^2_{H^{-\frac{1}{2}}(\Gamma_c)} \leq \delta^2.
\]

Then, the Nash game with the costs \(J^\delta_1\) and \(J^\delta_2\), defined for the noisy data, has a unique Nash equilibrium \((\eta^\delta_N, \tau^\delta_N) \in H^{-\frac{1}{2}}(\Gamma_i) \times H^{\frac{1}{2}}(\Gamma_i)\) which strongly converges, as \(\delta \to 0\), to the Cauchy missing data \((k\nabla u.\nu|_{\Gamma_i}, u|_{\Gamma_i})\).

Moreover, the fields \(u^\delta_1(\eta^\delta_N)\) and \(u^\delta_2(\tau^\delta_N)\), which solve respectively (SP1) and (SP2) for the noisy data, strongly converge in \(H^1(\Omega)\) to the Cauchy solution \( u \).

The proofs of the propositions lie on the classical Nash existence theorem, and on the not less classical Sobolev traces theorems and a priori estimates for elliptic BVP. The above results hold as well when considering as coupling term the difference \(\|k\nabla u_1.\nu - k\nabla u_2.\nu\|_{H^{-\frac{1}{2}}(\Gamma_i)}\).

From the computational viewpoint, in [3] the authors propose an alternating minimization algorithm to compute the Nash Equilibrium by means of the following iterative process: given an initial \((\eta^0, \tau^0)\):

\[
\begin{align*}
\eta^{(k+1)} &= \text{argmin}_{\eta}\{J_1(\eta, \tau^{(k)}) + \frac{\beta}{2} \int_{\Gamma_i}(\eta - \eta^{(k)})^2 d\Gamma_i\}, \\
\tau^{(k+1)} &= \text{argmin}_{\tau}\{J_2(\eta^{(k)}, \tau) + \frac{\beta}{2} \int_{\Gamma_i}(\tau - \tau^{(k)})^2 d\Gamma_i\},
\end{align*}
\]

(8)

where \(\beta\) is a given positive parameter (e.g. \(\beta = 1\)).

In the cited reference, the convergence of the alternating algorithm above is proved, under suitable assumptions which also hold in our case, see Proposition 1.

2. Two Numerical Experiments

The involved boundary value problems are solved by means of the classical dual Raviart-Thomas mixed finite element method, in order to get accurate approximations of the normal derivatives (which intervene as well in the adjoint state equations we used to compute the costs gradients).

2.1. Non-radial 3D test-case

We consider a thick spherical shell domain \( \Omega \) with boundary components \( \Gamma_i \) and \( \Gamma_c \), which are two spheres both centered at \((0, 0, 0)\) and with radii given by respectively \( R_i = 0.6 \) and \( R_c = 1 \).

The exact solution to the Cauchy problem is given by

\[
u(x, y, z) = \frac{1}{\sqrt{(x + 0.2)^2 + y^2 + z^2}},
\]

(9)

provided that the Cauchy data \( f \) and \( \Phi \) are defined as respectively the trace and normal derivative of the \( u \) above over the sphere \( \Gamma_c \). Remark that the function \( u \) given by (9) is the non-radial solution of \( \Delta u = \delta_{X_0} \), where the source term is now \( X_0 = (-0.2, 0, 0) \).

In this experiment we put 5% of gaussian noise in \((f, \Phi)\). The obtained results are illustrated in figure-1, where the reconstructed fields \( u_1 \) and \( u_2 \) are presented at convergence. The non radial missing data are presented in figure-2 and relative \( L^2 \)-errors on the reconstructed fields and boundary data are shown in figure-3.
2.2. Non-homogeneous test-case

The domain $\Omega$ is the open bounded set delimited by two concentric spheres in 3D, of radius $R_c = 1$ and $R_i = 0.6$. The inner boundary plays the role of $\Gamma_i$, where the trace and normal derivative are missing, and the outer one plays the one of $\Gamma_c$ where the latter information is over specified. However, a third sphere of radius $R_m = 0.8$ is added between $\Gamma_i$ and $\Gamma_c$.

We consider a non-homogeneous conductivity $k$ equal to 1 in the domain between $R_c$ and $R_m$, and is an orthotropic tensor field $I_2 \otimes (\alpha I_1)$ in the domain delimited by $R_m$ and $R_i$ ($\alpha = 0.5$).

The Cauchy data are generated via a finite element solution of the mixed boundary value problem $\nabla . k \nabla u = 0$ in $\Omega$, $k \nabla u . \nu|_{\Gamma_c} = \Phi = 0$, and $u|_{\Gamma_i} = \frac{1}{\sqrt{(x+0.2)^2+y^2+z^2}}$.

The data and reference solution are represented in figure-4. After convergence of the Nash iterations, we obtain a relative error less than 1.5% on the reconstructed fields and on the missing data, see figure-5 where the Cauchy data are perturbed with a Gaussian (5% level) noise. In figure-6, we illustrate how the Nash iterations yield, as a byproduct, a regularized noise-free approximation of noisy Cauchy data.

From a computational point of view, let us underline an interesting benefit of using a game approach, namely the possibility of using, independently, optimization algorithms of very different kinds -if necessary- to solve the respective players minimization subproblems in (8).

3. Conclusion

Let us conclude the paper with some short remarks. First of all, we have used the simplest class of games to model the completion problem, namely the class of static games with complete information. This simple game formulation yields interesting results like the existence and uniqueness of a Nash equilibrium even when the Cauchy data are not compatible, and also the fact that the Nash equilibrium is the missing data when the Cauchy problem has a solution.

Investigation of more sophisticated classes of games such as dynamical games with incomplete information may lead to new efficient data completion algorithms. It is also interesting to notice that solving the data completion problem with our method makes use of the standard computational tools, be it finite element or optimization codes. The numerical experiments presented for different test-cases prove that our method exhibits remarkable numerical stability with respect to noisy Cauchy data, and numerous additional 2D computational experiments (not presented here) do corroborate the efficiency and stability of the presented algorithm to tackle the Cauchy problem.

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Figure 1. Non radial case. Noise level is 5%. The level sets of the reconstructed fields (a) $u_1$ and (b) $u_2$ are presented at convergence. The Finite Element computations are performed with 4740 nodes and 22795 tetrahedral elements.

Figure 2. Non radial case. Reconstructed non radial Dirichlet ($\tau_N$, left) and Neumann ($\eta_N$, right) data over $\Gamma_i$. The profiles are presented at convergence and for a noise level $\sigma = 5\%$.

Figure 3. Non radial case. Relative $L^2$-errors are presented for the non radial case as a function of the overall Nash iterations. Noise level is 5%. (a) reconstructed fields error : $\|u_i^{(k)} - u\|/\|u\|$, $i = 1, 2$ and Nash strategies $ds = \|S^{(k)} - S^{(k-1)}\|$ ; (b) missing Dirichlet data : $\|\tau^{(k)} - u|_{\Gamma_i}\|/\|u|_{\Gamma_i}\|$ and Neumann data $\|\eta^{(k)} - \partial u/\partial \nu|_{\Gamma_i}\|/\|\partial u/\partial \nu|_{\Gamma_i}\|$.
Figure 4. Orthotropic case. Data (a) conductivity distribution (b) The reference solution $u$ level sets in $\Omega$ and (c) Traces of $u$ on the three interfaces $\Gamma_c$, $\Gamma_m$ and $\Gamma_i$. The FE mesh has 4560 nodes and 21685 tetrahedral elements.

Figure 5. Orthotropic case. Reconstruction. The solutions are presented at convergence and for a noise level $\sigma = 5\%$. (a) reconstructed fields $u_2$ with: $\|u_2 - u_1\|/\|u\| = 0.013$ and $\|u_2 - u\|/\|u\| = 0.014$; (b) missing Dirichlet data : $\|\tau_N - u|_{\Gamma_i}\|/\|u|_{\Gamma_i}\| = 0.015$.

Figure 6. Orthotropic case. Regularization. Noisy Cauchy data are regularized by the Nash process. (a) the smoothed profile $u_2(\tau_N)|_{\Gamma_c}$ compared to (b) the random $f^\sigma$ (noise level is $\sigma = 5\%$).