Averaging principles for non-autonomous two-time-scale stochastic reaction-diffusion equations with polynomial growth

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Abstract

In this paper, we develop the averaging principle for a class of two-time-scale stochastic reaction-diffusion equations driven by Wiener processes and Poisson random measures. We assume that all coefficients of the equation have polynomial growth, and the drift term of the equation is non-Lipschitz. Hence, the classical formulation of the averaging principle under the Lipschitz condition is no longer available. To prove the validity of the averaging principle, the existence and uniqueness of the mild solution are proved firstly. Then, the existence of time-dependent evolution family of measures associated with the fast equation is studied, by which the averaged coefficient is obtained. Finally, the validity of the averaging principle is verified.

Keywords. Averaging principles, stochastic reaction-diffusion equations, Poisson random measures, evolution families of measures, polynomial growth

Mathematics subject classification. 70K70, 60H15, 34K33, 37B55, 60J75

1. Introduction

Multi-scale problems are widely encountered in composites, porous media, finance and other fields [2, 3]. Moreover, in practice, the parameters of systems often depend on time, non-autonomous systems are worthy of thorough analysis. For this reason, we are concerned with the following non-autonomous two-time-scale stochastic partial differential equations (SPDEs) on a bounded domain \( \mathcal{O} \) of \( \mathbb{R}^d (d \geq 1) \):

\[
\begin{aligned}
\frac{\partial u_\epsilon}{\partial t} (t, \xi) & = A_1 (t) u_\epsilon (t, \xi) + b_1 (t, \xi, u_\epsilon (t, \xi), v_\epsilon (t, \xi)) + f_1 (t, \xi, u_\epsilon (t, \xi)) \frac{\partial \tilde{N}_1}{\partial t} (t, \xi, dz), \\
\frac{\partial v_\epsilon}{\partial t} (t, \xi) & = \frac{1}{\epsilon} [\left( A_2 (t) - \alpha \right) v_\epsilon (t, \xi) + b_2 (t, \xi, u_\epsilon (t, \xi), v_\epsilon (t, \xi))] \\
& \quad + \frac{1}{\sqrt{\epsilon}} f_2 (t, \xi, v_\epsilon (t, \xi)) \frac{\partial \tilde{N}_2}{\partial t} (t, \xi) + \int_{\mathbb{R}} g_2 (t, \xi, z, v_\epsilon (t, \xi)) \frac{\partial \tilde{N}_2}{\partial t} (t, \xi, dz), \\
u_\epsilon (0, \xi) & = x (\xi), \quad v_\epsilon (0, \xi) = y (\xi), \quad \xi \in \mathcal{O}, \\
N_1 u_\epsilon (0, \xi) & = N_2 v_\epsilon (0, \xi) = 0, \quad t \geq 0, \quad \xi \in \partial \mathcal{O},
\end{aligned}
\]  

(1.1)

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where $\omega_{Q_1}, \omega_{Q_2}$ and $\tilde{N}_1, \tilde{N}_2$ are mutually independent Wiener processes and Poisson random measures, $0 < \epsilon \ll 1$ is a small parameter and $\alpha$ is a sufficiently large fixed constant. In addition, $\mathcal{N}_i(i = 1, 2)$ are the boundary operators, which can be either the identity operator (Dirichlet boundary condition) or the first order operator (coefficients satisfying a uniform nontangentiality condition). The stochastic perturbations of the equations define on the same complete stochastic basis $\left( \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P} \right)$, the specific introduction will be given in Section 2.

The averaging principle is an effective method to analysis the slow-fast systems, which can simplify the system by constructing the averaged equation. In 1961, Bogolyubov and Mitropolskii [4] studied the averaging principle, giving the first rigorous result for the deterministic case. Since then, the averaging principle became an active area of research. Khasminskii [5] established the averaging principle for stochastic differential equations (SDEs) in 1968. Then, Givon [6], Freidlin and Wentzell [7], Duan [8], Xu and his co-workers [9–11] also studied the averaging principle of SDEs. In addition, many scholars also investigated the averaging principle of SPDEs in recent years, such as, Cerrai [12, 13], Wang and Roberts [14], Pei and Xu [15–17], Xu and Miao [18]. It should be pointed out that most of the current studies about the averaging principle are based on autonomous systems. In practical problems, the parameters of the system often depend on time. Therefore, non-autonomous system can depict some actual models better, which has made itself attract more and more attention of scholars.

In 2017, effective approximation for non-autonomous slow-fast system has been presented by Cerrai [19], and the system of this paper was driven by Gaussian noises. In our previous article [20], we study the non-autonomous slow-fast system driven by Gaussian noises and Poisson random measures. An effective approximations for the slow equation of the original system in article [20] was established by using the averaging principle, where the coefficients of the equation satisfy the Lipschitz condition and linear growth. But those conditions are too strict to study the validity of the averaging principle in many other relevant cases, such as, polynomial growth. One of the reaction-diffusion equations for the coefficients satisfy the polynomial growth is the Fitzhugh-Nagumo or Ginzburg-Landau type, those systems have appeared in the fields of biology and physics and attracted considerable attention. Therefore, we are devoted to developing the averaging principle for non-autonomous systems of reaction-diffusion equations with polynomial growth.

First, with the aid of the Sobolev embedding theorem, fixed point theorem and stopping technique, the existence and uniqueness of the mild solution is proved. That is, for any $p \geq 1$ and $T \geq s$, we prove that system (1.1) admits a unique mild solution depending on the initial datum.

Next, as in our previous work [20], assuming the operator $\mathcal{A}_2(t)$ is periodic and the functions $b_2, f_2, g_2$ are almost periodic. Analyzing the fast equation with a frozen slow component and using Kunitas first inequality to deal with the Poisson terms, we get that the evolution family of measures for the fast equation also exists, and it is almost periodic. Then, the averaged coefficient is defined through it, and the following averaged equation is obtained

$$
\begin{cases}
\frac{\partial \bar{u}}{\partial t} (t, \xi) = \mathcal{A}_1 (t) \bar{u} (t, \xi) + \tilde{b}_1 (\xi, \bar{u} (t, \xi)) + f_1 (t, \xi, \bar{u} (t, \xi)) \frac{\partial \omega_{Q_1} (t, \xi)}{\partial t} \\
\quad + \int_{\mathbb{R}} g_1 (t, \xi, u (t, \xi), z) \frac{\partial \tilde{N}_1 (t, \xi, dz)}{\partial t}, \\
\bar{u} (0, \xi) = x (\xi), \quad \xi \in \mathcal{O}, \quad \mathcal{N}_1 \bar{u} (t, \xi) = 0, \quad t \geq 0, \quad \xi \in \partial \mathcal{O},
\end{cases}
$$

(1.2)

where $\tilde{b}_1 (\xi, \bar{u} (t, \xi))$ is the averaged coefficient, which will be given in Section 5.

Finally, the validity of the averaging principle is verified by using the classical Khasminskii method. That is, for any $T > 0$ and $\eta > 0$, we have

$$
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} \| u_\epsilon (t) - \bar{u} (t) \|_{\mathcal{D} (\mathcal{O})} > \eta \right) = 0.
$$

(1.3)
where $\bar{u}$ is the solution of the averaged equation (1.2).

We will give a specific definition of the notations in Section 2. In this paper, $c > 0$ with or without subscripts represents a general constant, the value of which may vary for different cases.

2. Notations, assumptions and preliminaries

Denote $\mathbb{E}$ is the space $\mathcal{C}(\bar{O})$, endowed with the following sup-norm

$$\|x\|_{\mathbb{E}} = \sup_{\xi \in \bar{O}} |x(\xi)|,$$

and the duality $\langle \cdot, \cdot \rangle_{\mathbb{E}}$. The norm of the product space $\mathbb{E} \times \mathbb{E}$ denote as

$$\|x\|_{\mathbb{E} \times \mathbb{E}} = \left(\|x_1\|_{\mathbb{E}}^2 + \|x_2\|_{\mathbb{E}}^2\right)^{\frac{1}{2}}.$$

and the corresponding duality of the product space $\mathbb{E} \times \mathbb{E}$ is $\langle \cdot, \cdot \rangle_{\mathbb{E} \times \mathbb{E}}$.

Let $X$ be any space, denote $L^p(X)$ is the space of the bounded linear operators in $X$. For any $0 \leq s < T$ and $p \geq 1$, denote the norm of the space $L^p([s,T];X)$ is

$$\|u\|_{L^p([s,T];X)} := \mathbb{E} \sup_{t \in [s,T]} \|u(t)\|_{X}^p,$$

where $[s,T]$ denotes the space of all càdlàg path from $[s,T]$ into $X$.

For any $p \in [1, \infty]$ with $p \neq 2$, denote the norms of the space $L^p(\mathcal{O})$ and $L^p(\mathcal{O}) \times L^p(\mathcal{O})$ are both $\|\cdot\|_{p}$. When $\delta > 0$ and $p < \infty$, we denote the norm of the space $W^{\delta,p}(\mathcal{O})$ is $\|\cdot\|_{\delta,p}$:

$$\|x\|_{\delta,p} = \|x\|_{p} + \left(\int_{D} \int_{D} \left|\frac{x(\xi) - x(\eta)}{\xi - \eta}\right|^{\delta p} d\xi d\eta\right)^{\frac{1}{\delta p}}.$$

Now, we introduce some notations about subdifferential. The subdifferential of $\|x\|_{\mathbb{E}}$ is defined as

$$\partial \|x\|_{\mathbb{E}} := \{h \in \mathbb{E}^*; \|h\|_{\mathbb{E}^*} = 1, \langle h, x \rangle_{\mathbb{E}} = \|x\|_{\mathbb{E}}\},$$

where $\mathbb{E}^*$ is the dual space of $\mathbb{E}$. Due to the characterization of the subdifferential [21, Appendix D], if $u : [0,T] \to \mathbb{E}$ is any differentiable mapping, then

$$\frac{d}{dt} \|u(t)\|_{\mathbb{E}} \leq \langle u'(t), \delta \rangle_{\mathbb{E}}, \quad \text{(2.1)}$$

for any $t \in [0,T]$ and $\delta \in \partial \|u(t)\|_{\mathbb{E}}$.

Now, we assume the space dimension $d > 1$, the processes $\partial \omega^{Q_1}/\partial t(t,\xi)$ and $\partial \omega^{Q_2}/\partial t(t,\xi)$ in the slow-fast system are the Gaussian noises, assumed it is white in time and colored in space. Here, $\omega^{Q_i}(t,\xi)$ $(i = 1, 2)$ is the cylindrical Wiener processes, and it defined as

$$\omega^{Q_i}(t,\xi) = \sum_{k=1}^{\infty} Q_i e_k(\xi) \beta_k(t), \quad i = 1, 2,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is a complete orthonormal basis in $H$, $\{\beta_k(t)\}_{k \in \mathbb{N}}$ is a sequence of mutually independent standard Brownian motion defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and $Q_i$ is a bounded linear operator on $H$. 

3
Next, we give the definitions of Poisson random measures $\tilde{N}_1(dt,dz)$ and $\tilde{N}_2(dt,dz)$. Let $(\mathbb{Z},\mathcal{B}(\mathbb{Z}))$ be a given measurable space and $v(dz)$ be a $\sigma$-finite measure on it. $D_{p_i}, i=1,2$ are two countable subsets of $\mathbb{R}_+$. Moreover, let $p_i^1, t \in D_{p_i}^1$ be a stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with characteristic $v$, and $p_i^2, t \in D_{p_i}^2$ be the other stationary $\mathcal{F}_t$-adapted Poisson point process on $\mathbb{Z}$ with characteristic $v/\epsilon$. Denote by $N_i(dt,dz), i=1,2$ the Poisson counting measure associated with $p_i^1$, i.e.,

$$N_i(t,A) := \sum_{s \in D_{p_i}^1, s \leq t} I_A(p_i^1), i=1,2.$$ 

Let us denote the two independent compensated Poisson measures

$$\tilde{N}_1(dt,dz) := N_1(dt,dz) - v_1(dz) dt$$

and

$$\tilde{N}_2(dt,dz) := N_2(dt,dz) - \frac{1}{\epsilon} v_2(dz) dt,$$

where $v_1(dz) dt$ and $\frac{1}{\epsilon} v_2(dz) dt$ are the compensators.

In this paper, for any $t \in \mathbb{R}$, the operators $A_i(t)$ and the operators $A_2(t)$ are the second order uniformly elliptic operators with continuous coefficients on $\mathcal{O}$. As in our previous work [20], we assume that the operator $A_i(t)$ has the following form

$$A_i(t) = \gamma_i(t) A_i + L_i(t), \quad t \in \mathbb{R}, \ i = 1,2,$$

where $A_i$ independent of $t$ is a second order uniformly elliptic operator, having continuous coefficients on $\mathcal{O}$. And the operator $L_i(t)$ is a first order differential operator has the following form

$$L_i(t,\xi) u(\xi) = \langle l_i(t,\xi), \nabla u(\xi) \rangle_{\mathbb{R}^d}, \quad t \in \mathbb{R}, \ \xi \in \mathcal{O}.$$  \hspace{1cm} (2.3)

Finally, for $i = 1,2$, denote the realization of the operators $A_i$ and $L_i$ in $\mathbb{E}$ are $A_i$ and $L_i$, and the operator $A_i$ generates an analytic semigroup $e^{tA_i}$.

Now, we give the following assumptions about the operators $A_i$ and $Q_i$ as in [20] and [13].

(A1) (a) For $i = 1,2$, the function $\gamma_i : \mathbb{R} \to \mathbb{R}$ is continuous, and there exist $\gamma_0, \gamma > 0$ such that

$$\gamma_0 \leq \gamma_i(t) \leq \gamma, \quad t \in \mathbb{R}. \hspace{1cm} (2.4)$$

(b) For $i = 1,2$, the function $l_i : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d$ is continuous and bounded.

(A2) For $i = 1,2$, there exist a complete orthonormal system $\{e_{i,k}\}_{k \in \mathbb{N}}$ of $\mathbb{E}$, and two sequences of nonnegative real numbers $\{\alpha_{i,k}\}_{k \in \mathbb{N}}$ and $\{\lambda_{i,k}\}_{k \in \mathbb{N}}$ such that

$$A_i e_{i,k} = -\alpha_{i,k} e_{i,k}, \quad Q_i e_{i,k} = \lambda_{i,k} e_{i,k}, \quad k \geq 1,$$

and

$$\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{p_i} \|e_{i,k}\|^2_\infty < \infty, \quad \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i} \|e_{i,k}\|^2_\infty < \infty,$$

for some constants $p_i \in (2, +\infty]$ and $\beta_i \in (0, +\infty)$ such that

$$[\beta_i(\rho_i - 2)]/\rho_i < 1. \hspace{1cm} (2.7)$$
About the coefficients of the system (1.1), we assume it satisfy the following conditions.

(A3) (a) The mappings \(b_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}\) is continuous and there exists \(m_1 \geq 1\) such that

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} |b_1(t,\xi,x,y)| \leq c (1 + |x|^{m_1} + |y|), \quad (x,y) \in \mathbb{R}^2. \tag{2.8}
\]

(b) There exists \(c > 0\) such that, for any \(x,h \in \mathbb{R}^2\),

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} (b_1(t,\xi,x+h) - b_1(t,\xi,x)) h_1 \leq c |h_1| (1 + |x| + |h|). \tag{2.9}
\]

(c) There exists \(\theta > 0\) such that

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} |b_1(t,\xi,x) - b_1(t,\xi,y)| \leq c \left( 1 + |x|^\theta + |y|^\theta \right) |x - y|, \quad x,y \in \mathbb{R}^2. \tag{2.10}
\]

(A4) (a) The mappings \(b_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}\) is continuous and there exists \(m_2 \geq 1\) such that

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} |b_2(t,\xi,x,y)| \leq c (1 + |x| + |y|^{m_2}), \quad (x,y) \in \mathbb{R}^2. \tag{2.11}
\]

(b) There exists \(c > 0\) such that, for any \(x,h \in \mathbb{R}^2\),

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} (b_2(t,\xi,x+h) - b_2(t,\xi,x)) h_2 \leq c |h_2| (1 + |x| + |h|). \tag{2.12}
\]

(c) The mapping \(b_2(t,\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}\) is locally Lipschitz-continuous, uniformly with respect to \((t,\xi) \in \mathbb{R} \times \mathcal{O}\).

(d) For all \(x,y_1,y_2 \in \mathbb{R}\), we have

\[
b_2(t,\xi,x,y_1) - b_2(t,\xi,x,y_2) = -\tau(t,\xi,x,y_1,y_2)(y_1 - y_2). \tag{2.13}
\]

for some measurable function \(\tau : \mathbb{R} \times \mathcal{O} \times \mathbb{R}^3 \to [0,\infty)\).

(A5) The mappings \(f_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}, g_1 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R} \), \(f_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}, g_2 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R} \), \(f_3 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}, g_3 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R} \), \(f_4 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}, g_4 : \mathbb{R} \times \mathcal{O} \times \mathbb{R} \to \mathbb{R} \) are continuous, and the mappings \(f_1(t,\xi,\cdot) : \mathbb{R} \to \mathbb{R}, g_1(t,\xi,\cdot,\cdot,\cdot) : \mathbb{R} \to \mathbb{R} \), \(f_2(t,\xi,\cdot) : \mathbb{R} \to \mathbb{R}, g_2(t,\xi,\cdot,\cdot) : \mathbb{R} \to \mathbb{R} \) are Lipschitz-continuous, uniformly with respect to \((t,\xi,z) \in \mathbb{R} \times \mathcal{O} \times \mathbb{R}\). Moreover, for all \(p \geq 1\), there exist positive constants \(c_1,c_2\), such that for all \(x,y \in \mathbb{R}\), we have

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} \int_Z |g_i(t,\xi,x,z) - g_i(t,\xi,y,z)|^p v_i(dz) \leq c |x - y|^p, \quad i = 1,2. \tag{2.14}
\]

(A6) For any \(x,y \in \mathbb{R}\), it hold that

\[
\sup_{(t,\xi) \in \mathbb{R} \times \mathcal{O}} \left( |f_i(t,\xi,x)|^p + \int_Z |g_i(t,\xi,x,z)|^p v_i(dz) \right) \leq c \left( 1 + |x|^m_i \right), \quad i = 1,2, \tag{2.14}
\]

where \(m_1\) and \(m_2\) are the constants introduced in (2.8) and (2.11).
Remark 2.1. For any \((t, \xi) \in \mathbb{R} \times \mathcal{O}\) and \(x, y, h \in \mathbb{E}, z \in \mathbb{Z}\), we shall set

\[\begin{align*}
B_1(t, x, y)(\xi) &:= b_1(t, \xi, x(\xi), y(\xi)), \\
B_2(t, x, y)(\xi) &:= b_2(t, \xi, x(\xi), y(\xi)), \\
[F_1(t, x)h](\xi) &:= f_1(t, \xi, x(\xi)) h(\xi), \\
[F_2(t, x)h](\xi) &:= f_2(t, \xi, x(\xi)) h(\xi), \\
[G_1(t, x, z)h](\xi) &:= g_1(t, \xi, x(\xi), z) h(\xi), \\
[G_2(t, x, z)h](\xi) &:= g_2(t, \xi, x(\xi), z) h(\xi).
\end{align*}\]

Due to the assumption (A3) and (A4), we know the mappings \(B_1 : \mathbb{R} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}\) and \(B_2 : \mathbb{R} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}\) are well defined and continuous. According to \((2.8)\) and \((2.11)\), for any \(x, y \in \mathbb{E}\) and \(t \in \mathbb{R}\), we have

\[\|B_1(t, x, y)\|_\mathbb{E} \leq c(1 + \|x\|^2_\mathbb{E} + \|y\|_\mathbb{E}), \quad \|B_2(t, x, y)\|_\mathbb{E} \leq c(1 + \|x\|_\mathbb{E} + \|y\|^2_\mathbb{E}). \quad \tag{2.15}\]

As a consequence of \((2.9)\) and \((2.12)\), it is immediate to check that, for any \(x, y, h, k \in \mathbb{E}\), any \(t \in \mathbb{R}\), and any \(\delta \in \partial \|h\|_\mathbb{E}\),

\[\langle B_i(t, x + h, y + k) - B_i(t, x, y), \delta \rangle_\mathbb{E} \leq c(1 + \|x\|_\mathbb{E} + \|y\|_\mathbb{E} + \|h\|_\mathbb{E} + \|k\|_\mathbb{E}) \quad \tag{2.16}\]

In view of \((2.10)\), for any \(x_1, y_1, x_2, y_2 \in \mathbb{E}\), we have

\[\|B_1(t, x_1, y_1) - B_1(t, x_2, y_2)\|_\mathbb{E} \leq c(1 + \|(x_1, 1)\|^{\theta}_{\mathbb{E} \times \mathbb{E}} + \|(x_2, 2)\|^{\theta}_{\mathbb{E} \times \mathbb{E}}) (\|x_1 - x_2\|_\mathbb{E} + \|y_1 - y_2\|_\mathbb{E}). \quad \tag{2.17}\]

In addition, from the equation \((2.13)\), for every \(\delta \in \partial \|k\|_\mathbb{E}\), we have

\[\langle B_2(t, x, y + k) - B_2(t, x, y), \delta \rangle_\mathbb{E} \leq 0\]

Due to the assumption (A5) and (A6), for any fixed \((t, z) \in (\mathbb{R}, \mathbb{Z})\), the mappings

\[F_i(t, \cdot) : \mathbb{E} \to \mathcal{L}(\mathbb{E}), \quad G_i(t, \cdot, z) : \mathbb{E} \to \mathcal{L}(\mathbb{E}), \quad i = 1, 2,\]

are Lipschitz-continuous.

Now, for \(i = 1, 2\), we define

\[\gamma_i(t, s) := \int_s^t \gamma_i(r) dr, \quad s < t,\]

and let \(\gamma(t, s) := (\gamma_1(t, s), \gamma_2(t, s))\). For any \(\epsilon > 0\) and \(\beta \geq 0\), set

\[U_{\beta, \epsilon, i}(t, s) = e^{\frac{1}{\epsilon} \gamma_i(t, s) A_i - \frac{\beta}{\epsilon} (t-s)}, \quad s < t,\]

in the case \(\epsilon = 1\), we write \(U_{\beta, i}(t, s)\), and in the case \(\epsilon = 1\) and \(\beta = 0\), we write \(U_i(t, s)\).

Next, for any \(\epsilon > 0\), \(\beta \geq 0\) and for any \(u \in \mathbb{D}([s, t]; \mathbb{E}), r \in [s, t]\), we define

\[\psi_{\beta, \epsilon, i}(u; s)(r) = \frac{1}{\epsilon} \int_s^r U_{\beta, \epsilon, i}(r, \rho) L_i(\rho) u(\rho) d\rho, \quad s < r < t,\]

in the case \(\epsilon = 1\), we write \(\psi_{\beta, i}(u; s)(r)\), and in the case \(\epsilon = 1\) and \(\beta = 0\), we write \(\psi_i(u; s)(r)\).

We can easily get that \(\psi_{\beta, \epsilon, i}(u; s)(t)\) is the solution of

\[du(t) = \frac{1}{\epsilon} (A_i(t) - \beta) u(t) dt, \quad t > s, \quad u(s) = 0.\]
3. Existence, uniqueness of the solutions

More general, in this section, we mainly study the existence and uniqueness of solutions for the following problems

\[
du(t) = [A(t)u(t) + B(t,u(t))] dt + F(t,u(t)) dw^Q(t) + \int_Z G(t,u(t),z) \tilde{N}(dt,dz),
\]

where

\[
u(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad A(t) := \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}, \quad B(t,u(t)) = \begin{pmatrix} B_1(t,u_1(t),u_2(t)) \\ B_2(t,u_1(t),u_2(t)) \end{pmatrix},
\]

and

\[
F(t,u(t)) := \begin{pmatrix} F_1(t,u_1(t)) \\ 0 \\ F_2(t,u_2(t)) \end{pmatrix}, \quad G(t,u(t),z) := \begin{pmatrix} G_1(t,u_1(t),z) \\ 0 \\ G_2(t,u_2(t),z) \end{pmatrix},
\]

and

\[
w^Q(t) := \begin{pmatrix} w^{Q_1}(t) \\ w^{Q_2}(t) \end{pmatrix}, \quad \tilde{N}(dt,dz) := \begin{pmatrix} \tilde{N}_1(dt,dz) \\ \tilde{N}_2(dt,dz) \end{pmatrix}.
\]

According to the assumption (A5), it is easy to know that for any fixed \((t,z) \in \mathbb{R} \times \mathbb{Z}\), the mappings

\[
F(t,\cdot) : \mathbb{E} \times \mathbb{E} \to \mathcal{L}(\mathbb{E} \times \mathbb{E}), \quad G(t,\cdot,z) : \mathbb{E} \times \mathbb{E} \to \mathcal{L}(\mathbb{E} \times \mathbb{E}),
\]

are Lipschitz-continuous.

**Definition 3.1.** For any fix \((x_1,x_2) \in \mathbb{E} \times \mathbb{E}\), a process \(u(t)\) is a mild solution of the equation (3.1), if

\[
u(t) = U(t,s)x + \psi(u;s)(t) + \int_s^t U(t,r)B(r,u(r))dr + \int_s^t U(t,r)F(r,u(r))dw^Q(r) + \int_s^t \int_Z U(t,r)G(r,u(r),z)\tilde{N}(dr,dz),
\]

where

\[
U(t,s) = \begin{pmatrix} U_1(t,s) \\ U_2(t,s) \end{pmatrix}, \quad \psi(u;s)(t) = \begin{pmatrix} \psi_1(u_1;s)(t) \\ \psi_2(u_2;s)(t) \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

Now, we denote

\[
\Phi(u)(t) := \int_s^t U(t,r)B(r,u(r))dr,
\]

\[
\Gamma(u)(t) := \int_s^t U(t,r)F(r,u(r))dw^Q(r),
\]

and

\[
\Psi(u)(t) := \int_s^t \int_Z U(t,r)G(r,u(r),z)\tilde{N}(dr,dz).
\]

First, we prove that the mapping \(\Psi(u)(t)\) is a contraction in \(L^p(\Omega; \mathbb{D}([s,T];\mathbb{E}))\).
Lemma 3.2. Under the assumptions \((A1)-(A6)\), for any \(u, v \in L^p (\Omega; \mathbb{D} (\mathbb{R}^2))\) with \(p \geq 1\), the mapping \(\Psi\) maps \(L^p (\Omega; \mathbb{D} (\mathbb{R}^2))\) into itself, and we have

\[
\|\Psi (u) - \Psi (v)\|_{L^p (\Omega; \mathbb{D} (\mathbb{R}^2))} \leq c_{s,p} (t) \|u - v\|_{L^p (\Omega; \mathbb{D} (\mathbb{R}^2))},
\]

where \(c_{s,p}\) is a continuous increasing function with \(c_{s,p} (s) = 0\).

Proof: By using a factorization argument [21, Theorem 8.3], we have

\[
\Psi (u) (t) - \Psi (v) (t) = \frac{\sin \pi \lambda}{\pi} \int_s^t (t - r)^{\lambda - 1} U(t, r) \phi_\lambda (u, v) (r) dr,
\]

where

\[
\phi_\lambda (u, v) (r) := \int_s^r \int_\mathbb{R} (r - \sigma)^{-\lambda} U(r, \sigma) \left[ G(\sigma, u(\sigma), z) - G(\sigma, v(\sigma), z) \right] N(d\sigma, dz),
\]

and \(\lambda \in (0, 1/2)\).

For any \(t, \epsilon > 0\) and \(p \geq 1\), the semigroup \(e^{tA}\) maps \(L^p (\mathbb{R}^2)\) into \(W^{\epsilon,p} (\mathbb{R}^2)\) and by using the semigroup law, we can obtain

\[
\|e^{tA} x\|_{\epsilon,p} \leq c (t \wedge 1)^{-\frac{\epsilon}{2}} \|x\|_p, \quad x \in L^p (\mathbb{R}^2),
\]

for some constant \(c\) independent of \(p\). Then, according to (3.4), using the Hölder inequality, for any \(\epsilon < 2\lambda\), we have

\[
\|\Psi (u) (t) - \Psi (v) (t)\|_{\mathbb{D}} \leq \|\Psi (u) (t) - \Psi (v) (t)\|_{\epsilon,p} \\
\leq c_\lambda \int_s^t ((t - r) \wedge 1)^{\lambda - \frac{\epsilon}{2} - 1} \|\phi_\lambda (u, v) (r)\|_p dr \\
\leq c_\lambda \sup_{r \in [s, t]} \|\phi_\lambda (u, v) (r)\|_p \int_0^{t-s} \lambda - \frac{\epsilon}{2} - 1 dr,
\]

so, if we show that \(\phi_\lambda (u, v) (r) \in L^p (\mathbb{R}^2)\), we can get \(\Psi (u) - \Psi (v) \in \mathbb{D} ([s, T]; W^{\epsilon,p} (\mathbb{R}^2))\), \(P - a.s.\) Using Kunita’s first inequality [22, Theorem 4.4.23] and the Hölder inequality, because \(G\) is Lipschitz-continuous, for any \(p \geq 1\), we have

\[
\mathbb{E} \left| \phi_\lambda (u, v) (t, \xi) \right|^p \leq c_p \mathbb{E} \left( \int_s^t \int_\mathbb{R} (t - \sigma)^{-2\lambda} \left| U(r, \sigma) \left[ G(\sigma, u(\sigma), z) - G(\sigma, v(\sigma), z) \right] (\xi) \right|^2 v (dz) d\sigma \right)^{\frac{p}{2}} \\
+ c_p \mathbb{E} \left( \int_s^t \int_\mathbb{R} (t - \sigma)^{-p\lambda} \left| U(r, \sigma) \left[ G(\sigma, u(\sigma), z) (\xi) - G(\sigma, v(\sigma), z) (\xi) \right] \right|^p v (dz) d\sigma \right)^{\frac{1}{2}} \\
\leq c_p \mathbb{E} \left( \int_s^t \int_\mathbb{R} (t - \sigma)^{-2\lambda} \| U(r, \sigma) \left[ G(\sigma, u(\sigma), z) - G(\sigma, v(\sigma), z) \right] \|_\mathbb{D}^2 v (dz) d\sigma \right)^{\frac{p}{2}} \\
+ c_p \mathbb{E} \left( \int_s^t \int_\mathbb{R} (t - \sigma)^{-p\lambda} \| U(r, \sigma) \left[ G(\sigma, u(\sigma), z) - G(\sigma, v(\sigma), z) \right] \|_\mathbb{D}^p v (dz) d\sigma \right)^{\frac{1}{2}}
\]

\[
+ c_p \mathbb{E} \left( \int_s^t \int_\mathbb{R} (t - \sigma)^{-p\lambda} \| U(r, \sigma) \left[ G(\sigma, u(\sigma), z) - G(\sigma, v(\sigma), z) \right] \|_\mathbb{D}^p v (dz) d\sigma \right)^{\frac{1}{2}}.
\]
\[ \leq c_p E\left( \int_0^t (t - \sigma)^{-2\lambda} \|u(\sigma) - v(\sigma)\|_{E}^p d\sigma \right)^{\frac{1}{p}} + c_0 E \int_s^t (t - \sigma)^{-p\lambda} \|u(\sigma) - v(\sigma)\|_{E}^p d\sigma \]
\[ \leq c_{p,T} \sup_{r \in [s,T]} E \|u(r) - v(r)\|_{E}^p \left[ \left( \int_0^{t-s} \sigma^{-2\lambda} \frac{p\lambda}{p - 2} d\sigma \right)^{\frac{p-2}{p}} + \int_0^{t-s} \sigma^{-p\lambda} d\sigma \right], \]

so
\[ E\|\phi_{\lambda}(u,v)(t)\|_p = E\left( \int_D |\phi_{\lambda}(u,v)(t,\xi)|^p d\xi \right)^{\frac{1}{p}} \leq c_{p,T}|O|^{\frac{1}{p}} E\|u - v\|_{L_{s,T,p}([E])} \left[ \left( \int_0^{t-s} \sigma^{-2\lambda} \frac{p\lambda}{p - 2} d\sigma \right)^{\frac{p-2}{p}} + \int_0^{t-s} \sigma^{-p\lambda} d\sigma \right]^{\frac{1}{p}} (3.6) \]

where \(|O|\) is Lebesgue measure of the bounded domain \(O\). Because \(u, v \in L^p(\Omega; \mathbb{D}([s,T];E))\), so we know that \(\Psi(u) - \Psi(v) \in \mathbb{D}([s,T];W^{\epsilon,p}(O;\mathbb{R}^2))\), \(P-a.s.\) for any \(\lambda < \min\left(\frac{p-2}{2p}, \frac{1}{p}\right)\). In addition, we know
\[ ||\Psi(u) - \Psi(v)||_{L_{s,T,p}([E])} = \left[ \sup_{t \in [s,T]} ||\Psi(u)(t) - \Psi(v)(t)||_{[E]}^p \right]^{\frac{1}{p}} (3.7) \]

according to the equation (3.5) and (3.6), we can get that \(\Psi\) maps the space \(L^p(\Omega; \mathbb{D}([s,T];\mathbb{E}))\) into itself, and (3.3) holds with
\[ c_{s,p}(t) = \int_0^{t-s} (r \wedge 1)^{\lambda - \frac{1}{2}} d\tau \cdot \left[ \left( \int_0^{t-s} \sigma^{-2\lambda} \frac{p\lambda}{p - 2} d\sigma \right)^{\frac{p-2}{p}} + \int_0^{t-s} \sigma^{-p\lambda} d\sigma \right]^{\frac{1}{p}}. \]

\[ \square \]

**Remark 3.3.** For any \(x := (x_1, x_2) \in E \times E\), according to the assumption (A6), we know that there exists \(m := (m_1, m_2) \in \mathbb{R} \times \mathbb{R}\) and positive constants \(c\), such that
\[ \sup_{\xi \in \mathcal{O}} \int_Z |G(t, x, z) v(dz)| \leq c(1 + |x|^{\frac{m_1}{m_2}}), \quad (t, z) \in (\mathbb{R}, \mathbb{Z}) . \]

For any \(p \geq 1\), if \(u \in L^p(\Omega; \mathbb{D}([s,T];E))\), by proceeding as Lemma 3.2, we can get that \(\Psi(u) \in \mathbb{D}([s,T];W^{\epsilon,p}(\mathcal{O};\mathbb{R}^2))\), and it is easy to prove that there exists some continuous increasing function \(c_{s,p}(t)\) with \(c_{s,p}(s) = 0\), such that
\[ ||\Psi(u)||_{L_{s,T,p}([E])}^p \leq c_{s,p}(T) \left( 1 + ||u||_{E}^p \right). (3.8) \]

Moreover, as the space \(W^{\epsilon,p}(\mathcal{O};\mathbb{R}^2)\) continuously into \(C^\theta(\mathcal{O})\) for any \(\theta < \epsilon - d/p\), so we have that \(\Psi(u) \in C^\theta(\mathcal{O})\), and
\[ E \sup_{t \in [s,T]} ||\Psi(u)(t)||_{C^\theta(\mathcal{O})}^p \leq c_{s,p}(T) \left( 1 + ||u||_{E}^p \right). (3.9) \]

Now, for any \(\alpha > 0\) and \(u \in L^p(\Omega; \mathbb{D}([s,T];\mathbb{E}))\), we define
\[ \Psi_\alpha(u)(t) := \int_s^t \int_Z U_\alpha(t, r) G(r, u(r), z) \tilde{N}(dr, dz) . \]
We also can prove that \( \Psi_\alpha \) maps \( L^p (\Omega; \mathbb{D} ([s, T]; \mathbb{E})) \) into itself for any \( p \geq 1 \) and

\[
\| \Psi_\alpha (u) \|_{L_{s,T,p}(\mathbb{E})}^p \leq c_{s,p}^{\Phi,\alpha} (T) \left( 1 + \| u \|_{L_{s,T,p}(\mathbb{E})} \right),
\]

for some continuous increasing function \( c_{s,p}^{\Phi,\alpha} (t) \) and \( c_{s,p}^{\Phi,\alpha} (s) = 0. \)

Now, we prove the existence and uniqueness of the solution for system (3.1).

**Theorem 3.4.** Under the assumptions (A1)-(A6), for any \( x \in \mathbb{E} \) and \( p \geq 1 \), there exists a unique mild solution \( u_s^* \in L^p (\Omega; \mathbb{D} ([s, T]; \mathbb{E})) \) for equation (3.1). Moreover, there have

\[
\| u \|_{L_{s,T,p}(\mathbb{E})} \leq c_{s,p} (T) \left( 1 + \| x \|_\mathbb{E} \right),
\]

for some continuous increasing function \( c_{s,p} \).

**Proof:** In order to prove the existence of the solution for system (3.1), we construct the following equations. For any \( n \in \mathbb{N}, \), \( i = 1, 2 \) and \( (t, \xi) \in [0, \infty) \times \mathcal{O} \), we define

\[
b_{i,n} (t, \xi, \sigma) := \begin{cases} b_i (t, \xi, \sigma) & \text{if } |\sigma| \leq n, \\ b_i (t, \xi, n\sigma/|\sigma|) & \text{if } |\sigma| > n. \end{cases}
\]

For any \( n \in \mathbb{N} \), we can easily know that \( b_{i,n} (t, \xi, \cdot) : \mathbb{R} \to \mathbb{R} \) is Lipschitz-continuous uniformly with respect to \( \xi \in \mathcal{O} \) and \( t \in [s, T] \). For any \( x \in \mathbb{E} \), define the corresponding composition operator \( B_n \) associated with \( b_n = (b_{1,n}, b_{2,n}) \) is

\[
B_n (t, x)(\xi) := b_n (t, \xi, x(\xi)), \quad \xi \in \mathcal{O}.
\]

It is easy to get that \( B_n (t, \cdot) \) is Lipschitz-continuous. Moreover, if \( m < n \), we have

\[
\| x \|_\mathbb{E} \leq m \Rightarrow B_m (t, x) = B_n (t, x) = B (t, x).
\]

Next, we give the following problem

\[
du (t) = [A(t)u (t) + B_n (t, u (t))] dt + F (t, u (t)) dw^Q (t) + \int_\mathbb{Z} G (t, u (t), z) \tilde{N} (dt, dz).
\]

Because \( B_n (t, \cdot) \) is Lipschitz-continuous, so the mapping \( \Phi_n \)

\[
\Phi_n (u) (t) := \int_s^t U (t, r) B_n (r, u (r)) dr
\]

is Lipschitz-continuous in \( L^p (\Omega; \mathbb{D} ([s, T]; \mathbb{E})) \). By proceeding as [23, Lemma 6.1.2], we can prove that for any \( t \in [s, T], \epsilon \in (0, 1] \) and \( u, v \in L^p (\Omega; \mathbb{D} ([s, T]; \mathbb{E})) \), it yield

\[
\| \psi (u; s) (t) \|_\mathbb{E} \leq c \int_s^t ((t - r) \wedge 1)^{- \frac{\nu}{p} + \frac{\epsilon}{2}} \| u (r) \|_\mathbb{E} dr
\]

\[
\leq c \int_0^{t-s} (r \wedge 1)^{- \frac{\nu}{p} + \frac{\epsilon}{2}} dr \sup_{r \in [s, t]} \| u (r) \|_\mathbb{E},
\]

so, for any \( p \geq 1 \), we can get

\[
\| \psi (u) - \psi (v) \|_{L_{s,T,p}(\mathbb{E})} \leq \| \psi (u - v) \|_{L_{s,T,p}(\mathbb{E})} \leq c_{s,p}^{\psi} (T) \| u - v \|_{L_{s,T,p}(\mathbb{E})}.
\]

\[
(3.15)
\]
where $c_{s,p}^\psi$ is a continuous increasing function with $c_{s,p}^\psi (s) = 0$. In addition, according to [24, Theorem 4.2, Remark 4.3], we know that there exists a constant $p_s \geq 1$, such that for any $p \geq p_s$, we have

$$
\| \Gamma (u) - \Gamma (v) \|_{L_{s,T,p} (\mathbb{E})}^p \leq c_{s,p}^\Gamma (T) \| u - v \|_{L_{s,T,p} (\mathbb{E})},
$$

and

$$
\| \Gamma (u) \|_{L_{s,T,p} (\mathbb{E})}^p \leq c_{s,p}^\Lambda (T) \left( 1 + \| u \|_{L_{s,T,p} (\mathbb{E})}^p \right).
$$

where $c_{s,p}^\psi$ is a continuous increasing function with $c_{s,p}^\psi (s) = 0$.

Due to Lemma 3.2, we have got that the mapping $\Psi$ is a contraction in $L^p (\Omega; \mathbb{D} ([s,T]; \mathbb{E}))$. Moreover, because $\Phi_n (u)$ is Lipschitz-continuous and according to the equation (3.15) and (3.16), we can know that the mapping $\Phi_n, \psi$ and $\Gamma$ are contraction in $L^p (\Omega; \mathbb{D} ([s,T]; \mathbb{E}))$. So, we can get that the mild solution $u_n$ of the equation (3.13) is the unique fixed point of the following mapping

$$
u (t) \mapsto U (t, s) x + \psi (u; s) (t) + \Phi_n (u) (t) + \Gamma (u) (t) + \Psi (u) (t).$$

Next, we prove that the sequence $\{u_n\}$ is bounded in $L^p (\Omega; \mathbb{D} ([s,T]; \mathbb{E}))$.

**Lemma 3.5.** For any $n \in \mathbb{N}$ and $t \in [s,T]$, there exists a continuous increasing function $c_{s,p} (t)$ such that

$$
\| u_n \|_{L_{s,T,p} (\mathbb{E})} \leq c_{s,p} (T) \left( 1 + \| x \|_{\mathbb{E}} \right),
$$

**Proof:** Denote $\Lambda (u_n)$ is the solution of

$$
dv (t) = A(t)v (t) dt + F (t, u_n (t)) dw^Q (t) + \int_Z G (t, u_n (t), z) \tilde{N} (dt, dz), \quad v(s) = 0,
$$

we can get that $\Lambda (u_n) \in L^p (\Omega; \mathbb{D} ([s,T]; \mathbb{E}))$ is the unique fixed point of the mapping

$$
v (t) \mapsto \psi (v; s) (t) + \Gamma (u_n) (t) + \Psi (u_n) (t).
$$

So, we have

$$
\| \Lambda (u_n) (t) \|_{\mathbb{E}} \leq \| \psi (A (u_n); s) (t) \|_{\mathbb{E}} + \| \Gamma (u_n) (t) \|_{\mathbb{E}} + \| \Psi (u_n) (t) \|_{\mathbb{E}}.
$$

According to the equation (3.14) and (3.20), using the Gronwall inequality, we can get

$$
\| \Lambda (u_n) (t) \|_{\mathbb{E}} \leq e \int_s^t \left( (t - r) \wedge 1 \right)^{\frac{1}{2}} \left( \| \Gamma (u_n) (r) \|_{\mathbb{E}} + \| \Psi (u_n) (r) \|_{\mathbb{E}} \right) e^{c \int_s^r \left( (t - \sigma) \wedge 1 \right)^{\frac{1}{2}} d\sigma} dr
$$

$$
+ \| \Gamma (u_n) (t) \|_{\mathbb{E}} + \| \Psi (u_n) (t) \|_{\mathbb{E}}
$$

$$
\leq \left( \sup_{r \in [s,T]} \| \Gamma (u_n) (r) \|_{\mathbb{E}} + \sup_{r \in [s,T]} \| \Psi (u_n) (r) \|_{\mathbb{E}} \right) \times \left( e^{c \int_s^t \left. \left( (t - \sigma) \wedge 1 \right)^{\frac{1}{2}} d\sigma \right) - 1 \right) + \left( \sup_{r \in [s,T]} \| \Gamma (u_n) (r) \|_{\mathbb{E}} + \sup_{r \in [s,T]} \| \Psi (u_n) (r) \|_{\mathbb{E}} \right)
$$

$$
\leq c_5 (t) \left( \sup_{r \in [s,T]} \| \Gamma (u_n) (r) \|_{\mathbb{E}} + \sup_{r \in [s,T]} \| \Psi (u_n) (r) \|_{\mathbb{E}} \right).
$$

If we set $v_n (t) := u_n (t) - \Lambda (u_n) (t)$, we know that $v_n$ is the solution of the problem

$$
\frac{dv_n}{dt} (t) = A (t) v_n (t) dt + B_n (t, v_n (t) + \Lambda (u_n) (t)), \quad v_n (s) = x.
$$
According to the assumptions (A3) and (A4), we know that there exists \( m := (m_1, m_2) \in \mathbb{R} \times \mathbb{R} \), such that for any \( \delta_{v_n} \in \partial \| v_n (t) \|_E \), we yield

\[
\frac{d}{dt} \| v_n (t) \|_E \leq \left( \langle A (t) v_n (t), \delta_{v_n} \rangle_E + \langle B_n (t, v_n (t) + A (u_n) (t) \rangle, \delta_{v_n} \rangle_E \right) \\
\leq \left( \langle A (t) v_n (t), \delta_{v_n} \rangle_E + \langle B_n (t, v_n (t) + A (u_n) (t) \rangle - B_n (t, A (u_n) (t)), \delta_{v_n} \rangle_E \right) \\
\leq c \| v_n (t) \|_E + c \left( 1 + \| A (u_n) (t) \|_E^m \right),
\]

so

\[
\| v_n (t) \|_E \leq e^{c(t-s)} \| x \|_E + \int_s^t e^{c(t-r)} \left( 1 + \| A (u_n) (r) \|_E^m \right) dr \\
\leq c_s (t) \left( 1 + \| x \|_E + \sup_{r \in [s, T]} \| \Gamma (u_n) (r) \|_E^m + \sup_{r \in [s, T]} \| \Psi (u_n) (r) \|_E^m \right). \tag{3.23}
\]

Due to the definition of \( u_n(t) \) and the equation (3.21) and (3.23), we can get that

\[
\| u_n (t) \|_E \leq c_s (t) \left( 1 + \| x \|_E + \sup_{r \in [s, T]} \| \Gamma (u_n) (r) \|_E^m + \sup_{r \in [s, T]} \| \Psi (u_n) (r) \|_E^m \right).
\]

So, due to (3.17) and Remark 3.3, we can get that there exists a constant \( p_s \geq 1 \), such that, for any \( p \geq p_s \), we have

\[
\mathbb{E} \sup_{r \in [s, t]} \| u_n (r) \|_E^p \leq c_{s, p} (t) \left( 1 + \| x \|_E^p + \left( c_s^\Gamma (t) + c_s^\Psi (t) \right) \mathbb{E} \sup_{r \in [s, t]} \| u_n (r) \|_E^p \right),
\]

because \( c_s^\Gamma (s) = c_s^\Psi (s) = 0 \) and \( c_{s, p}^\Gamma, c_{s, p}^\Psi \) are continuous, there exists \( t_0 \), such that \( c_{s, p} (s + t_0) \).

\[
\left[ c_{s, p}^\Gamma (s + t_0) + c_s^\Psi (s + t_0) \right] \leq 1/2. \text{ For any } t \in [s, s + t_0] \text{ we have}
\]

\[
\mathbb{E} \sup_{t \in [s, s + t_0]} \| u_n (t) \|_E^p \leq c_s (p) (1 + \| x \|_E^p).
\tag{3.24}
\]

By proceeding it in the intervals \([s + t_0, s + 2t_0]\), \([s + 2t_0, s + 3t_0]\) etc., we get that for any \( T > s \) and \( p \geq p_s \), (3.18) holds. If \( p < p_s \), using the Hölder inequality, we can get (3.18) also holds. \( \square \)

Finally, through the sequence \( \{ u_n \} \), we can prove that Theorem 3.4 holds. For any \( n \in \mathbb{N} \) and \( x \in \mathbb{E} \), we define

\[
\tau_n := \inf \{ t \geq s : \| u_n (t) \|_E \geq n \},
\]

and let

\[
\tau := \sup_{n \in \mathbb{N}} \tau_n.
\]

We can prove that the sequence of stopping times \( \{ \tau_n \} \) is non-decreasing, and thanks to (3.18), we can get that \( P (\tau = +\infty) = 1 \).

Therefore, for any \( t \geq s \) and \( w \in \{ \tau = +\infty \} \), there exists \( m \in \mathbb{N} \) such that for any \( t \in [s, T] \), have \( t \leq \tau_m (w) \), and then we define

\[
u(t)(w) := u_m(t)(w).
\]

Set \( \eta := \tau_n \wedge \tau_m \), due to (3.12), we can get

\[
\| u_m (t \wedge \eta) - u_n (t \wedge \eta) \|_E = \| \psi (u_m - u_n; s) (t \wedge \eta) \|_E
\].
By proceeding as the proof of Lemma 3.2, using the factorization argument for $\Psi (u_m) (r \wedge \eta) - \Psi (u_n) (r \wedge \eta)$, we can obtain

$$
E \sup_{r \in [s,t]} \| \Psi (u_m) (r \wedge \eta) - \Psi (u_n) (r \wedge \eta) \|_E \leq c^{\psi}_{s,1} (t) E \sup_{r \in [s,t]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E .
$$

(3.26)

Then, substitute (3.15), (3.16) and (3.26) into (3.25), we have

$$
\sup_{r \in [s,t]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E \leq \left( c^{\psi}_{s,1} (t) + c^{\Gamma}_{s,1} (t) + c^{\psi}_{s,1} (t) \right) \sup_{r \in [s,t]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E \\
+ c \int_s^t \sup_{\sigma \in [s,r]} \| u_m (\sigma \wedge \eta) - u_n (\sigma \wedge \eta) \|_E d\sigma.
$$

Fix $t_0 > 0$, such that $c^{\psi}_{s,1} (t_0) + c^{\Gamma}_{s,1} (t_0) + c^{\psi}_{s,1} (t_0) \leq 1/2$, we can get

$$
E \sup_{r \in [s,s+t_0]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E \leq c \int_s^{s+t_0} \sup_{\sigma \in [s,r]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E d\sigma.
$$

According to the Gronwall lemma, we have $E \sup_{r \in [s,s+t_0]} \| u_m (r \wedge \eta) - u_n (r \wedge \eta) \|_E = 0$, that is, for any $t \in [s,s+t_0]$, we have $u_m (t \wedge \eta) = u_n (t \wedge \eta)$. Repeat it in the interval $[s+t_0, s+2t_0]$, $[s+2t_0, s+3t_0]$, etc., we obtain

$$
u_m (t) = u_n (t), \quad s \leq t \leq \tau_m \wedge \tau_n ,
$$

(3.27)

for any $n \in \mathbb{N}$. Because when $w \in \{ \tau = +\infty \}$ and $t \leq \tau_m$, we have denote $u(t) = u_m(t)$, thanks to (3.12), this yields

$$
u(t) = U(t,s)x + \psi(u;s)(t) + \int_s^t U(t,r)B(r,u(r))dr + \int_s^t U(t,r)F(r,u(r))dw^Q(r) \\
+ \int_s^t \int_\mathbb{Z} U(t,r)G(r,u(r),z)\tilde{N}(dr,dz),
$$

$P - a.s.$, that is, $u(t)$ is the mild solution of the system (3.1).
Now, we prove the solution of system (3.1) is unique. Denote another solution of system (3.1) is \( v \), by proceeding as the equation (3.27), we can get that for any \( n \in \mathbb{N} \)

\[
u(t) = v(t), \quad s \leq t \leq \tau_n.
\]

For any \( T \geq s \), we know \( \{\tau_n \leq T\} \downarrow \{\tau \leq T\} \), we get that \( u = v \).

Finally, for any \( p \geq 1 \) and \( T > s \), we have

\[
sup_{t \in [s,T]} \|u(t)\|^p_E = \lim_{n \to +\infty} sup_{t \in [s,T]} \|u(t)\|^p_E I_{\{T \leq \tau_n\}} = \lim_{n \to +\infty} sup_{t \in [s,T]} \|u_n(t)\|^p_E I_{\{T \leq \tau_n\}},
\]

according to the estimate (3.18) and the Fatou lemma, we can get (3.11).

4. The slow-fast system

According to the introduced in Section 2, system (1.1) can be rewritten as:

\[
\begin{cases}
du(t) = \left[ A_1(t) u(t) + B_1(t, u(t), v(t)) \right] dt + F_1(t, u(t)) dw^{Q_1}(t) \\
+ \int_Z G_1(t, u(t), z) \tilde{N}_1(dt, dz),

dv(t) = \frac{1}{2} \left[ A_2(t) - \alpha \right] v(t) + B_2(t, u(t), v(t)) dt \\
+ \int_Z F_2(t, v(t)) dw^{Q_2}(t) + \int_Z G_2(t, v(t), z) \tilde{N}_2(dt, dz),
\end{cases}
\]

\[
u(t) = x, \quad v(t) = y.
\]

Since the coefficients under the assumptions (A1)-(A6) are uniform with respect to \( t \in \mathbb{R} \), according the prove in Section 3, we can get that there exist two unique adapted \( u \) and \( v \) in \( L^p(\Omega; \mathbb{D}([s,T]; E)) \), such that

\[
\begin{cases}
u(t) = U_1(t,s) x + \psi_1(u_e; s)(t) + \int_s^t U_1(t,r) B_1(r, u_e(r), v_e(r)) dr \\
+ \int_s^t \int_Z U_1(t,r) F_1(r, u_e(r)) dw^{Q_1}(r)
\end{cases}
\]

\[
v(t) = U_{\alpha,e,2}(t,s) y + \psi_{\alpha,e,2}(v_e; s)(t) + \frac{1}{\epsilon} \int_s^t U_{\alpha,e,2}(t,r) B_2(r, u_e(r), v_e(r)) dr \\
+ \frac{1}{\epsilon} \int_s^t \int_Z U_{\alpha,e,2}(t,r) F_2(r, v_e(r)) dw^{Q_2}(r)
\]

\[
u(t) = x, \quad v(t) = y.
\]

Under the assumptions (A1)-(A6), by proceeding as [20, Lemma 5.1] and [13, Lemma 3.1], we can get that for any \( p \geq 1 \) and \( T > 0 \), there exists a positive constant \( c_{p,T} \), such that for any \( x,y \in E \) and \( \epsilon \in (0,1) \), we have

\[
E \sup_{t \in [s,T]} \|u_e(t)\|^p_E \leq c_{p,T} \left( 1 + \|x\|^p_E + \|y\|^p_E \right),
\]

and

\[
\int_s^T E \|v(t)\|^p_E dt \leq c_{p,T} \left( 1 + \|x\|^p_E + \|y\|^p_E \right).
\]

Then, due to the equation (3.9) and the estimates (4.3) and (4.4), using the proof of [13, Proposition 3.2] to the present situation, we can prove that there exists \( \theta > 0 \), such that for any \( T > s \), \( x \in C^\theta(\bar{O}) \) with \( \theta \in [0,\theta) \) and \( y \in E \), we have

\[
\sup_{\epsilon \in (0,1]} E \|u_e(t)\|_{L^\infty(s,T; C^\theta(\bar{O}))} \leq c_{\theta,T} \left( 1 + \|x\|^m_{C^\theta(\bar{O})} + \|y\|^m_{C^\theta(\bar{O})} \right),
\]

(4.5)
where \( c_{\theta, T} > 0 \) is a positive constant.

Finally, by proceeding as the proof of [20, Lemma 5.3], we can show that for any \( \theta \in [0, \hat{\theta}] \), there also exists \( \beta (\theta) > 0 \), such that, for any \( T > 0, p \geq 1, x \in C^\theta (\mathcal{O}), y \in \mathbb{E} \) and \( s, t \in [0, T] \), we have

\[
\sup_{\epsilon \in (0, 1]} \mathbb{E} \| u_{\epsilon} (t) - u_{\epsilon} (s) \|_p^p \leq c_{p, \theta, T} (|t - s|^{\beta (\theta) p} + |t - s|) \left( 1 + \| x \|_{C^\theta (\mathcal{O})}^{m_{1, p}} + \| y \|_{C^\theta (\mathcal{O})}^{m_{1, p}} \right).
\]

(4.6)

According to the equation (4.5) and (4.6), using the Arzelà-Ascoli theorem, we know that the family \( \{ \mathcal{L} (u_{\epsilon}) \}_{\epsilon \in (0, 1]} \) is tight.

5. The averaged equation

In this section, we research the fast equation with frozen slow component \( x \in E \), we main prove that there also exists an evolution family of measures for this fast equation and define the averaged equation through it.

First, for any \( s \in \mathbb{R} \), any frozen slow component \( x \in E \) and initial condition \( y \in \mathbb{E} \), we introduce the following problem

\[
dv (t) = \left[ (A_2 (t) - \alpha) v (t) + B_2 (t, x, v (t)) \right] dt + F_2 (t, v (t)) d\tilde{\omega}^{Q_2} (t) + \int_\mathbb{Z} G_2 (t, v (t), z) \tilde{N}_2' (dt, dz), \quad v (s) = y,
\]

(5.1)

where

\[
\tilde{\omega}^{Q_2} (t) = \begin{cases} w_1^{Q_2} (t), & \text{if } t \geq 0, \\ w_2^{Q_2} (-t), & \text{if } t < 0, \end{cases}
\]

\[
\tilde{N}_2' (t, z) = \begin{cases} \tilde{N}_1' (t, z), & \text{if } t \geq 0, \\ \tilde{N}_3' (-t, z), & \text{if } t < 0, \end{cases}
\]

where \( \tilde{N}_1' (dt, dz) \) and \( \tilde{N}_3' (dt, dz) \) has the same Lévy measure. The process \( w_1^{Q_2} (t), w_2^{Q_2} (t), \tilde{N}_1' (dt, dz) \) and \( \tilde{N}_3' (dt, dz) \) are independent and the definition of which is given in Section 2.

According to the prove in Section 3, we can get that for any \( x, y \in \mathbb{E}, p \geq 1 \) and \( s < T \), there exists a unique mild solution \( v^x (\cdot; s, y) \in L^p (\Omega; \mathcal{D} ([s, T]; \mathbb{E})) \). And using the same argument as our previous work [20], we can get that there also exists \( \delta > 0 \), such that for any \( x, y \in \mathbb{E} \) and \( p \geq 1 \), we have

\[
\mathbb{E} \| v^x (t; s, y) \|_p^p \leq c_p \left( 1 + \| x \|_p^p + e^{-\delta p (t-s)} \| y \|_p^p \right), \quad s < t.
\]

(5.2)

Next, same as our previous work [20], if \( t \in \mathbb{R} \), we also giving the following problem

\[
dv (t) = \left[ (A_2 (t) - \alpha) v (t) + B_2 (t, x, v (t)) \right] dt + F_2 (t, v (t)) d\tilde{\omega}^{Q_2} (t) + \int_\mathbb{Z} G_2 (t, v (t), z) \tilde{N}_2' (dt, dz),
\]

(5.3)

for every \( s < t \).

By proceeding as [19] and using the conclusion we have proved in [20], it is easy to prove that for any \( t \in \mathbb{R} \) and \( p \geq 1 \), there exists \( \eta^x (t) \in L^p (\Omega; \mathbb{E}) \) such that for all \( x, y \in \mathbb{E} \), we have

\[
\lim_{s \to -\infty} \mathbb{E} \| v^x (t; s, y) - \eta^x (t) \|_E^p = 0.
\]

(5.4)
and we can get that $\eta^x$ is a mild solution in $\mathbb{R}$ of equation (5.3). Moreover, for any $R > 0$, there also exists $c_R > 0$ such that for any $x_1, x_2 \in \mathbb{E}$

$$x_1, x_2 \in B_{\mathbb{E}}(R) \Rightarrow \sup_{t \in \mathbb{R}} \|\eta^{x_1}(t) - \eta^{x_2}(t)\|_{\mathbb{E}}^2 \leq c_R \|x_1 - x_2\|_{\mathbb{E}}^2.$$  \tag{5.5}$$

Then, for any $t \in \mathbb{R}$ and $x \in \mathbb{E}$, we denote that the law of the random variable $\eta^x(t)$ is $\mu^x_t$. As the prove of our previous work [20], we also can get that $\{\mu^x_t\}_{t \in \mathbb{R}}$ defines an evolution family of measures on $\mathbb{E}$ for equation (5.1).

Now, we give the following assumption.

(A7) (a) The functions $\gamma_2 : \mathbb{R} \to (0, \infty)$ and $I_2 : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^d$ are periodic, with the same period. (b) The families of functions

$$B_{1,R} := \{b_1 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R)\},$$ $$B_{2,R} := \{b_2 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}^2} (R)\},$$ $$F_{R} := \{f_2 (\cdot, \xi, \sigma) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}} (R)\},$$ $$G_{R} := \{g_2 (\cdot, \xi, \sigma, z) : \xi \in \mathcal{O}, \sigma \in \mathcal{B}_{\mathbb{R}} (R), z \in \mathbb{Z}\},$$

are uniformly almost periodic for any $R > 0$.

**Remark 5.1.** Similar with the proof of [19, Lemma 6.2], we get that under the assumption (A7), for any $R > 0$, the families of functions

$$\{B_1 (\cdot, x, y) : (x, y) \in \mathcal{B}_{\mathbb{E} \times \mathbb{E}} (R)\}, \quad \{B_2 (\cdot, x, y) : (x, y) \in \mathcal{B}_{\mathbb{E} \times \mathbb{E}} (R)\},$$ $$\{F_2 (\cdot, y) : y \in \mathcal{B}_{\mathbb{E}} (R)\}, \quad \{G_2 (\cdot, y, z) : (y, z) \in \mathcal{B}_{\mathbb{E} \times \mathbb{Z}} (R)\},$$

are uniformly almost periodic.

As in [20] and [19], we can prove that under the assumptions (A1)-(A7), the mapping $t \in \mathbb{R} \mapsto \mu^x_t \in \mathcal{P}_{\mathbb{E}}$ is almost periodic. Then, due to (5.5), we also can get that for every compact set $\mathcal{K} \subset \mathbb{E}$, the family of functions

$$\left\{ t \in \mathbb{R} \mapsto \int_{\mathbb{E}} B_1 (t, x, y) \mu^x_t (dy) : \ x \in \mathcal{K} \right\}$$  \tag{5.6}$$

is uniformly almost periodic. So, we define

$$\bar{B}_1 (x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{E}} B_1 (t, x, y) \mu^x_t (dy) \ dt, \quad x \in \mathbb{E},$$  \tag{5.7}$$

we can get that the mapping $\bar{B}_1 : \mathbb{E} \to \mathbb{E}$ is locally Lipschitz-continuous. Similar with the prove of [20, Lemma 4.2] and [19, Lemma 7.2], we can conclude that the following crucial results are also established in this paper.

**Lemma 5.2.** Under the assumptions (A1)-(A7), for any $T > 0$, $s \in \mathbb{R}$ and $x, y \in \mathbb{E}$, there exist some constants $\kappa_1, \kappa_2 \geq 0$, we have

$$E \left| \frac{1}{T} \int_s^{t+T} B_1 (t, x, v^\varepsilon (t; s, y)) dr - \bar{B}_1 (x) \right| \leq \frac{c}{T} (1 + \|x\|^n_\mathbb{E} + \|y\|^n_\mathbb{E}) + \alpha (T, x) \tag{5.8}$$
for some mapping $\alpha : [0, \infty) \times \mathbb{E} \to [0, \infty)$ such that

$$\sup_{T > 0} \alpha (T, x) \leq c (1 + \|x\|^p), \quad x \in \mathbb{E}, \quad (5.9)$$

and for any compact set $\mathbb{K} \subset \mathbb{E}$, have

$$\limsup_{T \to \infty} \sup_{x \in \mathbb{E}} \alpha (T, x) = 0. \quad (5.10)$$

We introduce the following averaged equation

$$du (t) = \left[ A_1 (t) u (t) + \bar{B}_1 (u (t)) \right] dt + F_1 (t, u (t)) dw^{Q_1} (t) + \int_{\mathbb{Z}} G_1 (t, u (t), z) \bar{N}_1 (dt, dz), \quad u (0) = x \in \mathbb{E}. \quad (5.11)$$

Due to Theorem 3.4, we can prove that for any $T > 0, p \geq 1$ and $x \in \mathbb{E}$, equation (5.11) admits a unique mild solution $\bar{u}$.

### 6. Averaging principles

In this section, we will show that the validity of the averaging principle. That is, the slow motion $u_\epsilon$ will converge to the averaged motion $\bar{u}$, as $\epsilon \to 0$.

**Theorem 6.1.** Under the assumptions (A1)-(A7), fix $x \in \mathcal{C}^\theta (\mathcal{O})$ with $\theta \in [0, \bar{\theta})$, and $y \in \mathbb{E}$, for any $T > 0$ and $\eta > 0$, we have

$$\liminf_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} \|u_\epsilon (t) - \bar{u} (t)\|_\mathbb{E} > \eta \right) = 0, \quad (6.1)$$

where $\bar{u}$ is the solution of the averaged equation (5.11).

**Proof:** For any $h \in \mathcal{D} (A_1)$, we have

$$\int_{\mathcal{O}} u_\epsilon (t, \xi) h (\xi) d\xi = \int_{\mathcal{O}} x (\xi) h (\xi) d\xi + \int_{\mathcal{O}} \int_{\mathcal{O}} u_\epsilon (r, \xi) A_1 (r) h (\xi) d\xi dr + \int_{\mathcal{O}} \int_{\mathcal{O}} B_1 (u_\epsilon (r)) (\xi) h (\xi) d\xi dr + \int_{\mathcal{O}} \int_{\mathcal{O}} [F_1 (r, u_\epsilon (r)) h] (\xi) dw^{Q_1} (r, \xi) + \int_{\mathcal{O}} \int_{\mathcal{O}} [G_1 (r, u_\epsilon (r), z) h] (\xi) d\xi \bar{N}_1 (dr, dz) + R_\epsilon (t),$$

where

$$R_\epsilon (t) := \int_{\mathcal{O}} \int_{\mathcal{O}} (B_1 (r, u_\epsilon (r), v_\epsilon (r)) (\xi) - B_1 (u_\epsilon (r)) (\xi)) h (\xi) d\xi dr.$$

As the proof in [20], because we have get that the family $\{\mathcal{L} (u_\epsilon)\}_{\epsilon \in (0, 1]}$ is tight in Section 4. If we want to prove Theorem 6.1, it is sufficient to prove that for any $T > 0$, we have

$$\lim_{\epsilon \to 0} \mathbb{E} \sup_{t \in [0, T]} \|R_\epsilon (t)\|_\mathbb{E} = 0.$$

First, for any $n \in \mathbb{N}$, we define

$$b_{i,n} (t, \xi, \sigma_1, \sigma_2) := \left\{ \begin{array}{ll} b_i (t, \xi, \sigma_1, \sigma_2), & \text{if } |\sigma_1| \leq n, \\ b_i (t, \xi, \sigma_1 n/|\sigma_1|, \sigma_2), & \text{if } |\sigma_1| > n. \end{array} \right. \quad (6.2)$$

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For each $b_{i,n}$, denote the corresponding composition operator is $B_{i,n}$, and we have
\[ x \in B_E (n) \Rightarrow B_{i,n} (t, x, y) = B_i (t, x, y), \quad t \in \mathbb{R}, \ y \in E. \quad (6.3) \]

It is easy to get that the mapping $b_{1,n}$ and $b_{2,n}$ satisfy all conditions in (A3) and (A4), respectively. And for any fixed $(t, \xi) \in \mathbb{R} \times \mathcal{O}$ and $\sigma_2 \in \mathbb{R}$, the mapping $b_{i,n} (t, \xi, \cdot, \sigma_2)$ are Lipschitz-continuous.

In addition, for any $n \in \mathbb{N}$, we define
\[
\begin{align*}
  f_{1,n} (t, \xi, \sigma) := \begin{cases} 
    f_1 (t, \xi, \sigma), & \text{if } |\sigma| \leq n, \\
    f_1 (t, \xi, \sigma n / |\sigma|), & \text{if } |\sigma| > n.
  \end{cases}
\end{align*}
\]

and
\[
\begin{align*}
  g_{1,n} (t, \xi, \sigma) := \begin{cases} 
    g_1 (t, \xi, \sigma, z), & \text{if } |\sigma| \leq n, \\
    g_1 (t, \xi, \sigma n / |\sigma|, z), & \text{if } |\sigma| > n.
  \end{cases}
\end{align*}
\]

where $(t, \xi) \in \mathbb{R} \times \mathcal{O}$ and $z \in \mathbb{Z}$. The corresponding composition operator of $f_{1,n}$ and $g_{1,n}$ are denoted by $F_{1,n}$ and $G_{1,n}$, respectively.

Now, for any $n \in \mathbb{N}$, we introduce the following system
\[
\begin{align*}
  \begin{cases}
    du (t) &= \left[ A_1 (t) u (t) + B_{1,n} (t, u (t), v (t)) \right] dt + F_{1,n} (t, u (t)) d\omega^Q (t) \\
    &+ \int_{\mathbb{Z}} G_{1,n} (t, u (t), z) \tilde{N}_1 (dt, dz), \\
    dv (t) &= \frac{1}{\epsilon} \left[ (A_2 (t) - \alpha) v (t) + B_{2,n} (t, u (t), v (t)) \right] dt \\
    &+ \frac{1}{\sqrt{\epsilon}} F_2 (t, v (t)) d\omega^Q (t) + \int_{\mathbb{Z}} G_2 (t, v (t), z) \tilde{N}_2 (dt, dz), \\
    u (s) &= x, \quad v (s) = y,
  \end{cases}
\end{align*}
\]

we denote the solution of (6.4) is $(u_{\epsilon,n}, v_{\epsilon,n})$.

Then, for any $n \in \mathbb{N}$ and any frozen slow component $x \in E$, we introduce the following problem
\[
\begin{align*}
  dv (t) &= \left[ (A_2 (t) - \alpha) v (t) + B_{2,n} (t, x, v (t)) \right] dt + F_2 (t, v (t)) d\omega^Q (t) \\
  &+ \int_{\mathbb{Z}} G_2 (t, v (t), z) \tilde{N}_2 (dt, dz), \\
  v (s) &= y, \quad (6.5)
\end{align*}
\]

and denote its solution is $v_{x,n}^\epsilon (t; s, y)$. Thanks to (6.3), for any $t \geq 0$ and $x \in E$, we have
\[
\begin{align*}
  v_{x,n}^\epsilon (t; s, y) &= \begin{cases} 
    \psi (t; s, y), & \text{if } |x (\xi)| \leq n, \\
    \sigma^n \psi (t; s, y), & \text{if } |x (\xi)| > n,
  \end{cases}
\end{align*}
\]

where $x_n = n \text{sign} x (\xi)$.

Due to the coefficients of equation (6.5) satisfy the same conditions as the equation (5.3), for each $x \in E$, there exists an evolution of measures family $\{\mu^{x,n}_t\}_{t \in \mathbb{R}}$ for equation (6.5)
\[
\mu^{x,n}_t = \begin{cases} 
  \mu^{x}_{t}, & \text{if } |x (\xi)| \leq n, \\
  \mu^{\psi}_{t}, & \text{if } |x (\xi)| > n.
\end{cases}
\]

As proof of (5.2), for any $T > 0$, $x \in E$ and $p \geq 1$, we can get that there also exists $\delta > 0$, such that
\[
E \|v^\epsilon _n (t; s, y)\|_E^p \leq c_{p,n} \left( 1 + e^{-\delta p(t-s)} \|y\|_E^p \right), \quad s < t. \quad (6.6)
\]

Similarly, we can define
\[
\bar{B}_{1,n} (x) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_E B_{1,n} (t, x, y) \mu^{x,n}_t (dy) dt, \quad x \in E, \quad (6.7)
\]
and, we have
\[ \|x\|_E \leq n \Rightarrow \bar{B}_{1,n}(x) = \bar{B}_1(x). \]
Moreover, it is easy to prove that the mapping \( \bar{B}_{1,n} : E \to E \) is Lipschitz-continuous, and the results similar with (5.8)-(5.10) also can be established.

Next, we prove that the validity of the averaging principle by using the classical Khasminskii method as in [20]. For any \( \epsilon > 0 \), we divide the interval \([0,T]\) in subintervals of the size \( \delta_{\epsilon} \), where \( \delta_{\epsilon} > 0 \) is a deterministic constant. Then, we define the following auxiliary fast motion \( \hat{v}_\epsilon \) in each time interval \([k\delta_{\epsilon}, (k+1)\delta_{\epsilon}]\), \( k = 0,1,\cdots, [T/\delta_{\epsilon}] \)
\[
\begin{aligned}
\left\{ \begin{array}{l}
d\hat{v} = \frac{1}{\epsilon} [ (A_2(t) - \alpha) \hat{v}_n(t) + B_{2,n}(t, u_{\epsilon,n}(k\delta_{\epsilon}), \hat{v}_{\epsilon,n}(t))] dt \\
\hat{v}_{\epsilon,n}(k\delta_{\epsilon}) = v_{\epsilon,n}(k\delta_{\epsilon}).
\end{array} \right.
\end{aligned}
\]
Like the equation (4.4), we also can prove that for any \( p \geq 1 \), we have
\[
\int_0^T E \| \hat{v}_{\epsilon,n}(t) \|^p E dt \leq c_{p,T} (1 + \|x\|^p_E + \|y\|^p_E).
\]

**Lemma 6.2.** Under the assumptions (A1)-(A7), fix \( x \in C^6(\bar{O}) \) with \( \theta \in [0, \bar{\theta}) \), and \( y \in E \), there exists a constant \( \kappa > 0 \), such that if
\[ \delta_{\epsilon} = \epsilon \ln^{1-\kappa}, \]
and, for any fixed \( n \in \mathbb{N} \), we have
\[
\lim_{\epsilon \to 0} \sup_{t \in [0,T]} E \| \hat{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t) \|^p_E = 0.
\]

**Proof:** Fixed \( \epsilon > 0 \) and \( n \in \mathbb{N} \). For any \( t \in [k\delta_{\epsilon}, (k+1)\delta_{\epsilon}] \), \( k = 0,1,\cdots, [T/\delta_{\epsilon}] \), let \( \rho_{\epsilon,n}(t) \) be the solution of the following problem
\[
d\rho_{\epsilon,n}(t) = \frac{1}{\epsilon} (A_2(t) - \alpha) \rho_{\epsilon,n}(t) dt + \frac{1}{\sqrt{\epsilon}} K_{\epsilon,n}(t) d\omega^Q(t)
+ \int_Z H_{\epsilon,n}(t,z) \tilde{N}^\epsilon_2(dt,dz),
\]
where
\[
K_{\epsilon,n}(t) := F_2(t, \hat{v}_{\epsilon,n}(t)) - F_2(t, v_{\epsilon,n}(t)),
H_{\epsilon,n}(t,z) := G_2(t, \hat{v}_{\epsilon,n}(t), z) - G_2(t, v_{\epsilon,n}(t), z).
\]
We have
\[
\rho_{\epsilon,n}(t) = \psi_{\alpha,\epsilon,2}(\rho_{\epsilon,n}; k\delta_{\epsilon})(t) + \Gamma_{\epsilon,n}(t) + \Psi_{\epsilon,n}(t), \quad t \in [k\delta_{\epsilon}, (k+1)\delta_{\epsilon}],
\]
where
\[
\Gamma_{\epsilon,n}(t) = \frac{1}{\sqrt{\epsilon}} \int_{k\delta_{\epsilon}}^t U_{\alpha,\epsilon,2}(t,r) K_{\epsilon,n}(r) d\omega^Q(r),
\Psi_{\epsilon,n}(t) = \int_{k\delta_{\epsilon}}^t \int_Z U_{\alpha,\epsilon,2}(t,r) H_{\epsilon,n}(r,z) \tilde{N}^\epsilon_2(dr,dz).
\]
Using the same argument as [13, Lemma 6.3], we yield
\[
\mathbb{E} \| \Gamma_{\epsilon,n}(t) \|^p \leq \frac{c_p,n}{\epsilon} \int_{k\delta}^t \mathbb{E} \| \hat{\nu}_{\epsilon,n}(r) - \nu_{\epsilon,n}(r) \|^p dr. \tag{6.11}
\]
For \( \Psi_{\epsilon,n}(t) \), using Kunita's first inequality and the Hölder inequality, we can get
\[
\mathbb{E} \| \Psi_{\epsilon,n}(t) \|^p \leq \frac{c_p,n}{\epsilon^2} \int_{k\delta}^t \mathbb{E} \left( \frac{1}{\epsilon} \int_{t}^{s} \left[ \left( t - r \right) e^{-\frac{\alpha}{2}(t-r)} e^{-\frac{\alpha}{2}(t-r)} A_2 H_{\epsilon,n}(r,z) \right]^2 v_2(dz) dr \right)^{\frac{p}{2}} dr
\]
\[
+ \frac{c_p,n}{\epsilon^2} \int_{k\delta}^t \mathbb{E} \left( \frac{1}{\epsilon} \int_{t}^{s} \left( t - r \right) e^{-\frac{\alpha}{2}(t-r)} A_2 H_{\epsilon,n}(r,z) \right)^2 v_2(dz) dr
\]
\[
\leq \frac{c_p,n}{\epsilon^2} \int_{k\delta}^t \mathbb{E} \left( \int_{t}^{s} \left( t - r \right) e^{-\frac{\alpha}{2}(t-r)} A_2 H_{\epsilon,n}(r,z) \right)^2 v_2(dz) dr
\]
\[
+ \frac{c_p,n}{\epsilon^2} \int_{k\delta}^t \mathbb{E} \left( \frac{1}{\epsilon} \int_{t}^{s} \left( t - r \right) e^{-\frac{\alpha}{2}(t-r)} A_2 H_{\epsilon,n}(r,z) \right)^2 v_2(dz) dr
\]
\[
\leq c_p,n \left( \frac{\delta e}{\epsilon^2} + 1/\epsilon \right) \int_{k\delta}^t \mathbb{E} \| \hat{\nu}_{\epsilon,n}(r) - \nu_{\epsilon,n}(r) \|^p dr. \tag{6.12}
\]
Thanks to \( \alpha > 0 \) is large enough, we have
\[
\| \rho_{\epsilon,n}(t) \|_E \leq \| \psi_{\alpha,2}(\rho_{\epsilon,n}; k\delta_\epsilon) \|_E + \| \Gamma_{\epsilon,n}(t) \|_E + \| \Psi_{\epsilon,n}(t) \|_E
\]
\[
\leq c_p,n \left( \frac{\delta e}{\epsilon^2} + 1/\epsilon \right) \int_{k\delta}^t \mathbb{E} \| \hat{\nu}_{\epsilon,n}(r) - \nu_{\epsilon,n}(r) \|^p dr. \tag{6.13}
\]
If we denote \( \Lambda_{\epsilon,n}(t) := \hat{\nu}_{\epsilon,n}(t) - \nu_{\epsilon,n}(t) \) and \( \vartheta_{\epsilon,n}(t) := \Lambda_{\epsilon,n}(t) - \rho_{\epsilon,n}(t) \), we have
\[
d\vartheta_{\epsilon,n}(t) = \frac{1}{\epsilon} \left[ (A_2(t) - \alpha) \vartheta_{\epsilon,n}(t) + B_{2,n}(t, u_{\epsilon,n}(k\delta_\epsilon), \hat{\nu}_{\epsilon,n}(t)) - B_{2,n}(t, u_{\epsilon,n}(t), \nu_{\epsilon,n}(t)) \right] dt
\]
\[
- \tau(t, u_{\epsilon,n}(t), \hat{\nu}_{\epsilon,n}(t), \nu_{\epsilon,n}(t)) (\vartheta_{\epsilon,n}(t) + \rho_{\epsilon,n}(t)) dt \tag{6.14}
\]
By proceeding as [13, Lemma 6.3], we can get
\[
\| \vartheta_{\epsilon,n}(t) \|_E \leq \frac{c_n}{\epsilon} \int_{k\delta}^t e^{-\frac{\alpha}{2}(t-r)} \| u_{\epsilon,n}(k\delta_\epsilon) - u_{\epsilon,n}(r) \|_E dr + \frac{1}{\epsilon} \sup_{r \in [k\delta_\epsilon, t]} \| \rho_{\epsilon,n}(r) \|_E \int_{k\delta}^t \exp \left( - \frac{\alpha}{2} \int_r^t \tau_{\epsilon,n}(\sigma) d\sigma \right) \tau_{\epsilon,n}(r) dr
\]
\[
\leq c_p,n \left( 1 + \| x \|_{C^p(\bar{O})} \right) \left( \delta e \right) \left( \frac{\delta e}{\epsilon^2} + 1/\epsilon \right) \int_{k\delta}^t \mathbb{E} \| \hat{\nu}_{\epsilon,n}(r) - \nu_{\epsilon,n}(r) \|^p dr. \tag{6.15}
\]
where
\[
\tau_{\epsilon,n}(r) := \tau(\xi_{\epsilon,n}(t), u_{\epsilon,n}(t), \xi_{\epsilon,n}(t)), \hat{\nu}_{\epsilon,n}(t, \xi_{\epsilon,n}(t), \nu_{\epsilon,n}(t), \xi_{\epsilon,n}(t)), \nu_{\epsilon,n}(t, \xi_{\epsilon,n}(t)) \in \bar{O},
\]
and satisfi
\[
| \vartheta_{\epsilon,n}(t, \xi_{\epsilon,n}(t)) | = \| \vartheta_{\epsilon,n}(t) \|_E.
\]
Due to (6.13) and (6.15), for any \( p \geq 1 \), we have
\[
\mathbb{E} \| \hat{\nu}_{\epsilon}(t) - \nu_{\epsilon}(t) \|^p \leq c_p,n \left( 1 + \| x \|_{C^p(\bar{O})} \right) \left( \delta e \right) \left( \frac{\delta e}{\epsilon^2} + 1/\epsilon \right) \int_{k\delta}^t \mathbb{E} \| \hat{\nu}_{\epsilon,n}(r) - \nu_{\epsilon,n}(r) \|^p dr.
\]
\[ +c_{p,n}(\delta^2 + \frac{1}{\epsilon}) \int_{k_{\delta,\epsilon}}^{t} E \| \tilde{v}_{\epsilon,n}(r) - v_{\epsilon,n}(r) \|_E^p dr. \]

From the Gronwall lemma, this means

\[ E \| \tilde{v}_{\epsilon,n}(t) - v_{\epsilon,n}(t) \|_E^p \leq c_{p,n} \left( 1 + \| x \|_{C^0(\bar{O})}^p + \| y \|_E^p \right) (\delta \beta(\theta) + \delta) e^{c_{p,n}(\delta^2 + \frac{1}{\epsilon}) \delta}. \]

For \( t \in [0,T] \), selecting \( \delta = \epsilon \ln^{-\kappa} \), then if we take \( \kappa < \frac{\beta(\theta)}{\beta(\theta)p + 2c_{p,n}} \wedge \frac{1}{1 + 2c_{p,n}} \), we have (6.10). \( \Box \)

Finally, under the same assumptions as in Theorem 6.1, by proceeding as [19, Lemma 8.2] and [20, Lemma 6.4], for any \( T > 0 \), we can get

\[ \lim_{\epsilon \to 0} E \sup_{t \in [0,T]} \| R_{\epsilon}(t) \|_E = 0. \]

Through the above proof, Theorem 6.1 is established. \( \Box \)

7. Conclusions

In this paper, we study the averaging principle for a class of non-autonomous slow-fast system with polynomial growth. First, using the Sobolev embedding theorem, fixed point theorem and stopping technique, the existence and uniqueness of the mild solution is proved. Next, by means of the comparison theorem and the properties of transition operator, the existence of time-dependent evolution family of measures associated with the fast equation is studied, and the averaged coefficient is obtained. Finally, through the truncation technique, the averaging principle for a class of non-autonomous slow-fast systems with polynomial growth is presented.

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