Construction of coupled Harry Dym hierarchy and its solutions from Stäckel systems.

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Abstract

In this paper we show how to construct the coupled (multicomponent) Harry Dym (cHD) hierarchy from classical Stäckel separable systems. Both nonlocal and purely differential parts of hierarchies are obtained. We also construct various classes of solutions of cHD hierarchy from solutions of corresponding Stäckel systems.

Keywords and phrases: Stäckel separable systems, Hamilton-Jacobi theory, hydrodynamic systems, rational solutions, multicomponent Harry Dym hierarchy,

1 Introduction

Various relations between finite- and infinite-dimensional nonlinear integrable systems have been investigated since the middle of 70:s in a long sequence of papers starting from the paper [1], through papers [2]-[5] (see for example [6] for more detailed bibliography) and many others. In all these efforts, however, the main idea was to pass from infinite- to finite-dimensional integrable systems. This paper is a third paper in our series of papers showing that also an opposite way is possible: that of passing from ordinary differential equations integrable in the sense of Arnold-Liouville to infinite-dimensional integrable systems (soliton hierarchies). In paper [7] we demonstrated a way of generating commuting evolutionary flows from corresponding family of Stäckel systems (that is classical finite dimensional Hamiltonian systems quadratic in momenta and separable in the sense of Hamilton-Jacobi theory). We presented our idea in the setting of coupled (multicomponent) KdV hierarchies (for definition and properties of these hierarchies, see for example [8]). In paper [9] we systematized and developed this idea by showing how solutions of these Stäckel systems can be used for generating various classes of solutions of cKdV hierarchies. Although both papers have been written for the case of cKdV, similar constructions are possible for other hierarchies as well. In this paper we demonstrate a way of generating the coupled (i.e. multicomponent) Harry Dym (cHD) hierarchy (see [10],[11]) and various classes of its solutions from a class of Stäckel systems of Benenti type. Our method leads both to the nonlocal cHD hierarchy as well as to purely differential cHD hierarchy, that is to a multicomponent generalization of HD hierarchy discussed in [12] (see also [13]). The nonlocal part of cHD hierarchy has not been discussed in [10] at all. We also clarify and simplify some of the results given in [7],[9].

The paper is organized as follows. In Section 2 we briefly remind some basic fact about Stäckel separable systems and discuss how they are related to corresponding Killing systems (dispersionless
nonlinear PDE’s of evolutionary type defined by Killing tensors of Stäckel systems). Sections 3 and 4 are devoted to description of nonlocal multicomponent Harry Dym hierarchy and its various solutions, respectively. Sections 5 and 6 are devoted to local (purely differential) cHD hierarchy.

2 Stäckel systems and their dispersionless counterpart

Stäckel separable systems can be most conveniently obtained from an appropriate class of separation relations. Generally speaking, $n$ equations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \quad a_i \in \mathbb{R}$$

(each involving only one pair $\lambda_i, \mu_i$ of canonical coordinates on a $2n$-dimensional Poisson manifold $\mathcal{M}$) are called separation relations [14] provided that $\det \left( \frac{\partial \varphi_i}{\partial a_j} \right) \neq 0$. We can then locally resolve equations (1) with respect to $a_i$ obtaining

$$a_i = H_i(\lambda, \mu), \quad i = 1, \ldots, n.$$  

with some new functions (Hamiltonians) $H_i(\lambda, \mu)$ that in turn generate $n$ canonical Hamiltonian systems on $\mathcal{M}$:

$$\lambda_i = \frac{\partial H_i}{\partial \mu_i}, \quad \mu_i = -\frac{\partial H_i}{\partial \lambda_i}, \quad i = 1, \ldots, n.$$  

All the flows (3) mutually commute since the Hamiltonians $H_i$ Poisson commute. Moreover, Hamilton-Jacobi equations for all the Hamiltonians $H_i$ are separable in the $(\lambda, \mu)$-variables since they are algebraically equivalent to the separation relations (1).

In this article we consider a special but important class of separation relations, namely

$$\sum_{j=1}^{n} a_j \lambda_i^{n-j} = \lambda_i^m \mu_i^k + \frac{\varepsilon}{4} \lambda_i^k, \quad i = 1, \ldots, n$$

with arbitrary fixed $m, k \in \mathbb{Z}$, $\varepsilon = \pm 1$ (the constant $\frac{1}{4}$ is not essential for the construction and is only introduced for a smoother identification our systems with the hierarchy in (14)). The relations (4) are linear in the coefficients $a_i$ so that they can be (globally) solved by Cramer formulas, which yields

$$a_i = \mu^T K_i G^{(m)} \mu + \frac{\varepsilon}{4} \lambda_i^{(k)} = H_i^{n,m,k}, \quad i = 1, \ldots, n, \quad m, k \in \mathbb{Z}$$

where we denote $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and $\mu = (\mu_1, \ldots, \mu_n)^T$. Functions $H_i$ defined as the right hand sides of (5) depend on $m$ and $k$ and can be interpreted as $n$ quadratic in momenta $\mu$ Hamiltonians on the phase space $\mathcal{M} = T^* \mathcal{Q}$ cotangent to a Riemannian manifold $\mathcal{Q}$ parametrized by $(\lambda_1, \ldots, \lambda_n)$ and equipped with the contravariant metric tensor $G^{(m)}$ (depending on $m \in \mathbb{Z}$) given by:

$$G^{(m)} = \text{diag} \left( \frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n} \right) \quad \text{with} \quad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j).$$

It can be shown that $G^{(m)}$ is of zero curvature for $m = 0, \ldots, n$ and that $G^{(n+1)}$ is of non-zero constant curvature, while all other choices of $m$ lead to spaces of non-constant curvature. The Hamiltonians $H_i^{n,m,k}$ are known in literature as Stäckel Hamiltonians and the corresponding commuting Hamiltonian flows (5) are then called Stäckel systems, or more precisely, Stäckel systems of Benenti type. They are obviously separable in the sense of Hamilton-Jacobi theory since they by the very definition satisfy Stäckel relations (4). The objects $K_i$ in (5) are Killing tensors for any metric $G^{(m)}$ and are given by

$$K_i = -\text{diag} \left( \frac{\partial q_i}{\partial \lambda_1}, \ldots, \frac{\partial q_i}{\partial \lambda_n} \right), \quad i = 1, \ldots, n,$$

where $q_i = q_i(\lambda)$ are Viète polynomials (signed symmetric polynomials) in $\lambda$:

$$q_i(\lambda) = (-1)^i \sum_{1 \leq s_1 < s_2 < \ldots < s_i \leq n} \lambda_{s_1} \cdots \lambda_{s_i}, \quad i = 1, \ldots, n$$

(7)
that can also be considered as new coordinates on the Riemannian manifold \( Q \) (we will then refer to them as Viète coordinates). Notice that \( K_i \) do not depend on neither \( m \) nor \( k \). Finally, the potentials \( V_i^{(k)} \) can be constructed recursively \(^{(8)}\) by

\[
V_i^{(k+1)} = V_{i+1}^{(k)} - q_i V_{i}^{(k)}, \quad k \in \mathbb{Z}, \quad \text{with} \quad V_i^{(0)} = \delta_{in},
\]

where we put \( V_i^{(k)} = 0 \) for \( i < 0 \) or \( i > n \). The first potentials are trivial: \( V_i^{(1)} = \delta_{i,n-k} \) for \( k = 0, 1, \ldots, n - 1 \). The first nontrivial potentials are \( V_i^{(n)} = -q_i \). For \( k > n \) the potentials \( V_i^{(k)} \) become complicated polynomial functions of \( q \). The recursion \(^{(8)}\) can also be reversed

\[
V_i^{(k)} = V_{i-1}^{(k+1)} - \frac{q_i - 1}{q_n} V_i^{(n+1)}, \quad k \in \mathbb{Z}, \quad r = 1, \ldots, n,
\]

leading to potentials \( V_i^{(k)} \) with \( k < 0 \). These potentials start with \( V_i^{(-1)} = -\frac{q_i}{q_n} \) and are rather complicated rational functions of \( q \). They will be referred to as negative potentials. It can also be shown \(^{(7)}\) that

\[
g_{ij}^{(m)} = V_i^{(2n-m-i-j)}
\]

where \( g^{(m)} = (G^{(m)})^{-1} \) is the corresponding covariant metric tensor.

**Remark 1** The general \( n \)-time (simultaneous) solution for Hamilton equations \(^{(4)}\) associated with all the Hamiltonians \(^{(4)}\) is given implicitly by

\[
t_i + c_i = \frac{1}{2} \sum_{r=1}^{n} \frac{\lambda_r^{n-i}}{\sqrt{\lambda_r^m \left( \sum_{j=1}^{n} q_j \lambda_r^{n-j} - \xi \lambda_r^k \right)}} d\lambda_r, \quad i = 1, \ldots, n.
\]

To see this it is enough to integrate the related Hamilton-Jacobi problem. Now, with \( n \) Hamiltonians \( H_i^{n,m,k} \) in \(^{(3)}\) we can associate, by corresponding Legendre transforms, \( n \) Lagrangians \( L_i^{n,m,k} : T Q \rightarrow \mathbb{R} \) given by

\[
L_i^{n,m,k}(\lambda, \lambda_t_i) = \frac{1}{4} \lambda_i^r g^{(m)} K_i^{-1} \lambda_t_i - \frac{\xi}{4} V_i^{(k)}, \quad i = 1, \ldots, n.
\]

Every Lagrangian \( L_i^{n,m,k} \) give rise to \( n \) systems of Euler-Lagrange equations

\[
E_j^s(t_i^{n,m,k}) = 0, \quad j = 1, \ldots, n
\]

(each for every \( s \) between 1 and \( n \)) where

\[
E_j^s = \frac{\partial}{\partial \lambda_j} - \frac{d}{dt_s} \frac{\partial}{\partial (\partial \lambda_j / \partial t_s)}, \quad j = 1, \ldots, n
\]

are components of the Euler-Lagrange operator with respect to the independent variable \( t_a \).

**Remark 2** By construction, the solutions \(^{(11)}\) are also general solutions for all the Euler-Lagrange equations \(^{(13)}\). It means that for a particular \( s \) the general solution of Euler-Lagrange equations \( E_j^s(L_i^{n,m,k}) = 0 \) is given by \(^{(11)}\) where \( t_p \) for \( p \neq s \) plays a role of a constant parameter.

Denote now the variable \( t_1 \) as \( x \) (our method works similarly with any \( t_i \) chosen as \( x \)). With every Killing tensor \( K_i \) for \( i = 2, \ldots, n \) we can associate a dispersionless evolutionary PDE of the form

\[
\lambda_{t_i} = K_i \lambda_x \equiv Z_i[\lambda] \quad i = 2, \ldots, n
\]

(where \( \lambda = (\lambda_1, \ldots, \lambda_n)^T \)). We will call PDE's in \(^{(14)}\) simply Killing systems. Here and in what follows we use the notation \( f[\lambda] \) to denote integral-differential function of \( \lambda \) i.e. a function of \( \lambda \), its \( x \)-derivatives and antiderivatives (integrals). In the case above \( Z_i[\lambda] = Z_i(\lambda, \lambda_x) \). The chosen variable \( t_1 = x \) in \(^{(14)}\) plays thus the role of a space variable while the remaining variables \( t_i \) should then be considered as evolution
parameters (times). Equations (14) constitute a set of $n-1$ integrable dispersionless equations that due to the form of $K_i$ belong to the class of weakly nonlinear semi-Hamiltonian systems, i.e. hydrodynamic-type systems that are semi-Hamiltonian in the sense of Tsarev \cite{16,17} and weakly nonlinear \cite{18}. Actually, the systems (14) are finite-component restrictions of the universal hydrodynamic hierarchy considered in \cite{19}. The variables $\lambda_i$ are Riemann invariants of all the system (14) as $K_i$ are diagonal in $\lambda$. The systems (14) can also be considered as $n-1$ dynamical systems on some infinite-dimensional function space $\mathcal{V}$ of vectors $(\lambda_1(x), \ldots, \lambda_n(x))$, with $Z_i$ being $n-1$ vector fields on $\mathcal{M}$. It can be shown \cite{18} that the vector fields $Z_i$ commute on $\mathcal{V}$:

$$[Z_i, Z_j] = 0, \quad i, j = 2, \ldots, n.$$  

Note also that since $K_1 = I$ we can complete the system of equations (14) by the equation $\lambda_x = K_1 \lambda = \lambda_x \equiv Z_1$ with the translation-invariant general solution $\lambda_i = \lambda_i(x + \tau)$. The vector field $Z_1$ also commutes with all the vector fields $Z_2, \ldots, Z_n$ \cite{18}.

**Proposition 3** Every mutual solution $\lambda(t_1, \ldots, t_n)$ (11) of all Hamiltonians of Benenti type (5) is (after replacing $t_1$ with $x$) also a particular solution of all $n-1$ corresponding Killing systems in (14).

**Proof.** Let us assume that a vector function $\lambda(t_1, \ldots, t_n)$ solves (11). Then, by construction, it also solves the spatial part of (6) with appropriate functions $\mu(t_1, \ldots, t_n)$ given by $\mu_i = \partial W(\lambda, a)/\partial \lambda_i$ ($W = W(\lambda, a)$) is a common integral of all the Hamilton-Jacobi equations for Hamiltonians $H_i^{n,m,k}$) It means that $\lambda(t_1, \ldots, t_n)$ solves

$$\lambda_i = \frac{\partial}{\partial \mu} H_i^{n,m,k} = 2K_i G^{(m)}(\mu), \quad i = 1, \ldots, n. \quad \text{(15)}$$

Since $K_1 = I$ we get from the first equation in (16) $\mu(t_1, \ldots, t_n) = \frac{1}{2} g^{(m)}(\lambda(t_1, \ldots, t_n))\lambda_1(t_1, \ldots, t_n)$. Substituting it to the remaining equations in (16) yields then

$$\lambda_i(t_1, \ldots, t_n) = K_i(\lambda(t_1, \ldots, t_n))\lambda_1(t_1, \ldots, t_n), \quad i = 2, \ldots, n$$

which concludes the proof as $t_1 = x$. Thus, all the solutions (11) also solve all $n-1$ Killing systems in (14). ■

Moreover, we have

**Theorem 4** The general (n-time) solution of all the Killing systems in (14) is given by

$$t_i + c_i = \sum_{r=1}^{n} \int \frac{\lambda_r^{n-i}}{\varphi_r(\lambda_r)} d\lambda_r, \quad i = 1, \ldots, n \quad \text{(16)}$$

(where $\varphi_r$ are arbitrary functions of one variable)

The proof of this statement can be found in \cite{18}. Obviously, (16) contains all the solutions (11).

Suppose now that a particular solution (10) of our Killing systems (14) is of the more specific form (11). Since this class of solutions - by construction - satisfies all the Euler-Lagrange equations (13), we can treat equations (13) as additional bonds that these solutions satisfy. We can therefore use these bonds to express some variables $\lambda_i$ by other $\lambda$’s. Thus, within the class (11) of solutions (10) of Killing systems (14) we can perform a variable elimination (reparametrization) that turns (13) into entirely new sets of evolutionary PDE’s. As we have demonstrated in \cite{7} and in \cite{9}, in carefully chosen cases and in a particular coordinate system (Viète coordinates (7)) this reparametrization turns systems (14) into systems with dispersion (soliton hierarchies) with the solution (11) being also a solution of these new systems with dispersion. In this paper we will produce by this method (the local and the nonlocal part of) the coupled (multicomponent) Harry Dym hierarchy.
3 Nonlocal coupled Harry Dym hierarchy

Assume now that $\varepsilon = 1$ in (4) and therefore also in (11), (12) etc.). In order to perform the elimination procedure just mentioned, let us pass to Viète coordinates as given in (7). The Killing systems (13) are tensorial so in Viète coordinates they have the form

$$q_i = K_i(q)x,$$  

$i = 2, \ldots, n$

or, explicitly

$$\frac{d}{dt}q_j = (q_{j+i-1})x + \sum_{k=1}^{j-1}(q_k(q_{j+i-k-1})x - q_{j+i-k-1}(q_k)x) = (Z^n_j[q])^j, \quad j = 1, \ldots, n$$  

(17)

(where we put $q_\alpha = 0$ for $\alpha > n$), where $i = 2, \ldots, n$ and where $(Z^n_j[q])^j$ denotes the $j$-th component of the vector field $Z_i[q]$. The superscript $n$ at $Z_i$ indicates the number of components in the vector field $Z_i$ and we will sometimes use it since we will need to switch between various $n$. From (17) one can see that $(Z^n_j[q])^j = (Z^n_j[q])^j$ for all $i, j = 1, \ldots, n$. Obviously, $G^{(m)}$, $g^{(m)}$ and $K_i$ are tensors and can thus also easily be transformed to Viète coordinates.

Consider now Euler-Lagrange equations (13) with $s = 1$ (so that $t_s = t_1 = x$) associated with Lagrangians $L^{n,m,k}$ denoted further on for simplicity as $L^{n,m,k}$. Denote also $E_1^i$ as $E_i$, $i = 1, \ldots, n$ and consider the equations

$$E_i(L^{n,m,k}) = 0, \quad i = 1, \ldots, n, \quad n, \in \mathbb{N}, \ k \in \mathbb{Z},$$  

written in $q$-variables, so that now

$$E_i = \frac{\partial}{\partial q_i} - \frac{d}{dx}\frac{\partial}{\partial q_{i,x}}, \quad i = 1, \ldots, n,$$

while (since $K_1 = I$)

$$L^{n,m,k} = L^{n,m,k}(q,q_x) = \frac{1}{4}q_x^Tg^{(m)}q_x - \frac{1}{4}V_1^{(k)}.$$  

(19)

As it has been shown in (7) the following symmetry relations are satisfied for $\alpha = 1, \ldots, n - 1$

$$E_i(L^{n,m,k}) = E_{i-\alpha}(L^{n,m+\alpha,k-\alpha}), \quad i = \alpha + 1, \ldots, n,$$

(20)

that can also be written as

$$E_i(L^{n,m,k}) = E_{i+\alpha}(L^{n,m-\alpha,k+\alpha}), \quad i = 1, \ldots, n - \alpha.$$  

(21)

Due to (20) and (21) the equations (18) can be embedded in the following double-infinite multi-Lagrangian "ladder" of Euler-Lagrange equations of the form

$$E_i(L^{n,m+j-1,k+j-1}) = E_2(L^{n,m+j-2,k+j-2}) = \cdots = E_n(L^{n,m+j-n,k+j+n}) = 0, \quad j = \ldots, -1, 0, 1, \ldots$$  

(22)

with fixed $m, k \in \mathbb{Z}$ (the equations (18) fit in (22) at $j = 1, 2, \ldots, n$). For a given dimension $n$ the ladder (22) is determined by the sum $m + k$ in the sense that various choices of $m$ and $k$ with the same $m + k$ yield the same ladder.

We are now ready to present our elimination procedure leading to multicomponent integral (nonlocal) Harry Dym hierarchy. Assume that we want to produce first $s - 1$ flows of the $N$-component ($N \in \mathbb{N}$) hierarchy. Let us take $n = s + N - 1$, $m = -N$ and $k = 0$ in (12), that is, let us consider the purely kinetic Lagrangian $L^{n,-N,0}$ with $n = s + N - 1$ and the corresponding Euler-Lagrange equations (18). Due to this special choice of all parameters the last $n - N$ equations in (18) attain the form

$$E_{N+1}(L^{n,-N,0}) = \frac{1}{2}q_{n,xx} + \varphi_{-N}[q_{1}, \ldots, q_{n-1}] = 0,$$

$$E_{N+2}(L^{n,-N,0}) = \frac{1}{2}q_{n-1,xx} + \varphi_{-N}[q_{1}, \ldots, q_{n-2}] = 0,$$

$$\cdots$$

$$E_{n-1}(L^{n,-N,0}) = \frac{1}{2}q_{N+2,xx} + \varphi_{2}[q_{1}, \ldots, q_{N+1}] = 0,$$

$$E_{n}(L^{n,-N,0}) = \frac{1}{2}q_{N+1,xx} + \varphi_{1}[q_{1}, \ldots, q_{N}] = 0.$$  

(23)

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and are a part of the ladder \([22]\) with \(m + k = -N\). Now, by direct calculation of \(E_i(L^{n,-N,0})\) with the use of some identities satisfied by the potentials \(V_k^{(i)}\) it can be proved that

\[
E_N(L^{n,-N,0}) = E_{N+1}(L^{n+1,-N,0}) + \frac{1}{2}q_{n+1,xx}, \\
E_i(L^{n,-N,0}) = E_{i+1}(L^{n+1,-N,0}), \quad i = N + 1, \ldots, n.
\]

These identities lead to

**Proposition 5** The functions \(\varphi_i\) in \([23]\) do not depend on \(n\) in the sense that increasing \(n\) to \(n + 1\) (and keeping \(N\) constant) turn \([23]\) into \(n - N + 1\) equations

\[
E_{N+1}(L^{n+1,-N,0}) = -\frac{1}{2}q_{n+1,xx} + E_N(L^{n,-N,0}) = -\frac{1}{2}q_{n+1,xx} + \varphi_{n-N+1}[q_1, \ldots, q_n] = 0, \\
E_{N+2}(L^{n+1,-N,0}) = E_{N+1}(L^{n,-N,0}) = -\frac{1}{2}q_{n+1,xx} + \varphi_{n-N}[q_1, \ldots, q_{n-1}] = 0, \\
\vdots \\
E_n(L^{n+1,-N,0}) = E_{n-1}(L^{n,-N,0}) = -\frac{1}{2}q_{N+2,xx} + \varphi_{1}[q_1, \ldots, q_{N+1}] = 0, \\
E_{n+1}(L^{n+1,-N,0}) = E_n(L^{n,-N,0}) = -\frac{1}{2}q_{N+1,xx} + \varphi_{1}[q_1, \ldots, q_N] = 0.
\]

It means that increasing \(n\) to \(n + 1\) (and keeping \(N\) constant) in \([23]\) does not alter these equations except that a new equation of the form

\[
E_{N+1}(L^{n+1,-N,0}) = -\frac{1}{2}q_{n+1,xx} + \varphi_{n-N+1}[q_1, \ldots, q_n] = 0
\]

is added at the top of \([23]\). As we will see soon, this will result in the fact that our construction indeed yields an infinite hierarchy of commuting flows.

Due to their structure, equations \([23]\) can be formally solved with respect to the variables \(q_{N+1}, \ldots, q_n\), which yields \(q_{N+1}, \ldots, q_n\) as some nonlocal (integral-differential) functions of \(q_1, \ldots, q_N\):

\[
q_{N+1} = f_1[q_1, \ldots, q_N] \\
\vdots \\
q_n = f_{n-N}[q_1, \ldots, q_N],
\]

where, due to Proposition 5 the functions \(f_i\) do not depend on \(n\), so increasing \(n\) by 1 (and keeping \(N\) constant) will only result in one new equation at the bottom place in \([23]\). Let us now replace the variables \(q_{N+1}, \ldots, q_n\) in the first \(N\) components of the first \(s - 1\) Killing systems \([17]\) by the corresponding functions \(f_i\) (right-hand sides of \([23]\)). This yields equations of the form

\[
\overline{\varphi}_r = \overline{Z}_r^N[\overline{\varphi}] \quad r = 2, \ldots s
\]

where \(\overline{\varphi}\) denotes the first \(N\) entries in \(\varphi\) i.e. \(\overline{\varphi} = (q_1, \ldots, q_N)^T\). They are in general highly nonlinear autonomous systems of \(N\) evolution equations for \(q_1, \ldots, q_N\).

**Theorem 6** The vector fields \(\overline{Z}_r^N[\overline{\varphi}]\) in \([26]\) do not depend on \(s\) in the sense that if we increase \(s\) by one in our procedure then \([26]\) are unaltered and a new equation \(\overline{\varphi}_{s+1} = \overline{Z}_{s+1}^N[\overline{\varphi}]\) appears.

**Proof.** This theorem is a consequence of Proposition 5. If we increase \(s\) to \(s + 1\) and keep \(N\) constant we have to take \(n + 1\) instead of \(n\) in our procedure as \(n = s + N - 1\). Due to \([17]\) we have \((\overline{Z}_r^{s+1})^j = (\overline{Z}_r^s)^j\) for \(r = 2, \ldots, s\) and for \(j = 1, \ldots, N\) i.e. the first \(N\) components of the first \(s - 1\) of Killing systems \([17]\) do not change when we increase \(n\) to \(n + 1\). Moreover, as we explained above, the \(n - N\) functions \(f_i\) in \([23]\) do not change either. So, the elimination procedure for the first \(s - 1\) vector fields \(Z_i\) is not altered leading to exactly the same vector fields \(\overline{Z}_r^N[\overline{\varphi}]\) with \(r = 2, \ldots, s\) while the vector field \(\overline{Z}_{s+1}^N[\overline{\varphi}]\) yields the vector field \(\overline{Z}_{s+1}^N[\overline{\varphi}]\) i.e. a new equation at the end of the sequence \([26]\). \(\blacksquare\)

Repeating this argument we can increase \(s\) indefinitely. Thus, our procedure leads to an infinite hierarchy of evolutionary vector fields (flows)

\[
\overline{\varphi}_r = \overline{Z}_r^N[\overline{\varphi}] \quad r = 2, 3, \ldots
\]
Theorem 7  The vector fields \( Z_i^N \), \( i \neq j \) commute i.e.
\[
\left[ Z_i^N, Z_j^N \right] = 0 \text{ for any } i, j = 2, 3, \ldots.
\]

This theorem is due to the fact that the original vector fields \( Z_i^N \) commute and that the Euler-Lagrange equations \( E_i(L) = 0 \) are invariant with respect to all the fields \( Z_i^N \). Moreover, the vector fields \( Z_i^N \) still commute with \( \overline{Z}_1^N = (q_1, q_2, \ldots, q_N)^T \). As we demonstrate below, the hierarchy (27) is the nonlocal part of the multicomponent Harry Dym soliton hierarchy as discussed in [12].

Example 8  Consider first \( N = 1 \) (one-component hierarchy as discussed in [12]). Suppose that we want to obtain the first \( s - 1 = 2 \) flows of the hierarchy. We have then to take \( n = s + N - 1 = 3 \) and consider the elimination equations (23) for these parameters. The pure kinetic Lagrangian \( L^{3,-1,0} \) has the form

\[
L^{3,-1,0} = \frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 + \frac{1}{4} q_3^2 - \frac{1}{2} q_1 q_2 q_3 + \frac{1}{2} q_2 q_3 - \frac{1}{2} q_1 q_3 - \frac{1}{2} q_1 q_1 q_3 - \frac{1}{2} q_2 q_2 q_3 - \frac{1}{2} q_1 q_2 q_3
\]

so that (13) become

\[
E_2(L^{3,-1,0}) = -\frac{1}{2} q_3 q_1 + q_2 q_2 q_1 - \frac{1}{2} q_2 q_2 q_1 - \frac{1}{2} q_1 q_2 q_2 q_1 + \frac{1}{2} q_1 q_2 q_3 q_1 + \frac{1}{2} q_2 q_2 q_3 q_1 = 0,
\]

\[
E_3(L^{3,-1,0}) = -\frac{1}{2} q_2 q_2 q_1 + q_2 q_3 q_1 + \frac{1}{2} q_2 q_2 q_3 q_1 = 0.
\]

Due to their specific structure, we can solve (28) with respect to \( q_2 \) and \( q_3 \). We will thus use (28) to eliminate variables in the corresponding \( n = 3 \)-component Killing systems (17) that have in this case the form:

\[
\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} q_2 x \\ q_3 x \end{pmatrix} = \begin{pmatrix} Z_2^3 \\ Z_3^3 \end{pmatrix},
\]

\[
\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_3 x \\ q_2 q_3 x \\ q_2 q_3 x \end{pmatrix} = \begin{pmatrix} Z_3^3 \end{pmatrix}.
\]

By the second equation in (28) we obtain

\[
q_2 = \frac{1}{2} q_1^2 + q_1 x q_1
\]

Integrating it once we obtain

\[
q_2 = \frac{1}{2} q_1 t_1 + \frac{1}{2} q_1 t_2
\]

where

\[
\partial^{-1} = \int \ldots dx + \varphi(t_2, t_3)
\]

is the integration operator with the integration parameter \( \varphi \) that has to be chosen from case to case and has therefore to be treated as a part of the solution of every integration problem. It is always possible to find such a function. Integrating \( q_2 \), we obtain

\[
q_2 = \frac{1}{4} q_1^2 + \frac{1}{2} q_1 t_2
\]

Further, the first equation in (28) yields

\[
q_3 = q_1 x q_2 + q_2 x q_1 - q_1 x q_1 q_1 - q_1 q_1 q_1 + q_1 x q_2
\]
Inserting to it \( q_2 \) and \( q_{2,x} \), as calculated above, and integrating once we obtain

\[
q_{3,x} = \frac{1}{2} \partial^{-1} q_2^2 q_{1,xx} + \frac{1}{2} q_2 \partial^{-1} q_1 q_{1,xx} + \frac{1}{4} q_1^2 q_{1,x} + \frac{1}{2} q_{1,x} \partial^{-2} q_1 q_{1,xx}
\]  

(31)

By inserting the obtained formulas for \( q_{2,x} \) and \( q_{3,x} \) into the first \( N = 1 \) components of \( Z_2 \) and \( Z_3 \) we obtain the first two flows of our nonlocal soliton hierarchy:

\[
\begin{align*}
q_{1,t_2} &= \frac{1}{2} q_1 q_{1,x} + \frac{1}{2} \partial^{-1} q_1 q_{1,xx} = Z_2, \\
q_{1,t_3} &= -\frac{1}{2} \partial^{-1} q_2^2 q_{1,xx} + \frac{1}{2} q_2 \partial^{-1} q_1 q_{1,xx} + \frac{1}{4} q_1^2 q_{1,x} + \frac{1}{2} q_{1,x} \partial^{-2} q_1 q_{1,xx} = Z_3.
\end{align*}
\]  

(32)

Observe that in this particular case we did not have to calculate \( q_3 \) since it does not enter into the first component of neither \( Z_2 \) nor \( Z_3 \). We needed however \( q_2 \) in order to calculate \( q_{3,x} \). The flows \( Z_2 \) commute due to Theorem 7.

**Example 9** Let us now take \( N = 2 \) and \( s - 1 = 1 \) so that \( n = 3 \) again. We will thus eliminate \( n - N = 1 \) variables (namely \( q_3 \)) from the first \( N = 2 \) components of the field \( Z_2^3 \). The elimination equations (13) reduce now to \( E_3 \left( L^{3,-2,0} \right) = 0 \). But, according to (20), \( E_3 \left( L^{3,-2,0} \right) = E_2 \left( L^{3,-1,1} \right) = E_2 \left( L^{3,-1,0} \right), \) the last equality due to the fact that \( L^{3,-1,1} = L^{3,-1,0} - \frac{1}{4} V_1 \) and \( L^{3,-1,0} = \frac{4}{3} \). Thus, the elimination equation \( E_3 \left( L^{3,-2,0} \right) = 0 \) coincides with the first equation in (25) and yields exactly (30). Plugging its integrated form (31) into the first two components of \( Z_2^3 \) yields the first flow of the 2-component nonlocal cHD hierarchy:

\[
\frac{d}{dt_2} \left( \begin{array}{c} q_1 \\ q_2 \\ \end{array} \right) = \left( \begin{array}{c} q_1 q_{2,x} - q_2 q_{1,x} - \frac{1}{7} \partial^{-1} q_1^2 q_{1,xx} + \frac{1}{7} q_1 \partial^{-1} q_1 q_{1,xx} + \frac{1}{4} q_1^2 q_{1,x} + \frac{1}{2} q_{1,x} \partial^{-2} q_1 q_{1,xx} \\
q_{2,x} \end{array} \right) = Z_2
\]  

(33)

The map

\[
u_i = E_{N-i+1} \left( L^{N,0,0} \right), \quad i = 1, \ldots, N
\]  

(34)

transforms the hierarchy (27) into the nonlocal part of the coupled Harry Dym hierarchy (see (10) for its local part) that is the generalization of the one-field nonlocal HD hierarchy presented in (12). For example, for \( N = 2 \) this map reads

\[
\begin{align*}
u_1 &= \frac{1}{2} q_1 q_{1,xx} \\
u_2 &= -\frac{1}{2} q_2 q_{1,xx} + \frac{1}{4} q_1^2 q_{1,x} + \frac{1}{2} q_1 q_{1,xx}
\end{align*}
\]

and applied to the field \( Z_2 \) above yields

\[
\frac{d}{dt_2} \left( \begin{array}{c} u_1 \\ u_2 \\ \end{array} \right) = -2 \left( \begin{array}{c} u_{1,x} \partial^{-2} u_1 + 2 u_1 \partial^{-1} u_1 - \frac{1}{4} u_{2,x} \\
u_{2,x} \partial^{-2} u_1 + 2 u_2 \partial^{-1} u_1 \end{array} \right).
\]

4 Solutions of the multicomponent nonlocal HD hierarchy

We will now construct a variety of solutions of the hierarchy (27).

**Theorem 10** For any \( \beta \in \{0, \ldots, n-1\} \) the functions \( \lambda_i = \lambda_i(t_1, \ldots, t_n) \) given implicitly by

\[
t_i + c_i = \pm \frac{1}{2} \sum_{r=1}^{n} \int \frac{\lambda_r^{-\beta} \cdot 1}{\lambda_r^{-n+\beta} \left( \sum_{j=1}^{n} a_j \lambda_n^{-j} - \frac{1}{4} \lambda_r^{-\beta} \right)} d\lambda_r, \quad i = 1, \ldots, n.
\]  

(35)

are such that the corresponding functions \( q_i = q_i(x = t_1, t_2, \ldots, t_n), i = 1, \ldots, N, \) given by (7) are solutions of the first \( n - \beta \) (\( n - 1 \) for \( \beta = 0, 1 \)) equations of the \( N \)-component integral cHD hierarchy (27). The variables \( t_2, \ldots, t_{n-\beta+1} \) (\( t_2, \ldots, t_n \) for \( \beta = 0, 1 \)) play then the role of evolution parameters (dynamical times) while the remaining \( t_i \)’s are free parameters.
For the proof of this theorem, see Appendix. We will now consider some particular, interesting classes of solutions \(4\). Assume that \(\beta = 0\) in \(4\) and that \(a_j = \frac{1}{2} \delta_{j,n} + \delta_{j,n-\gamma}\) for some \(\gamma \in \{0, \ldots, n-1\}\). Then \(4\) attain the form

\[
t_i + c_i = \pm \frac{1}{2} \sum_{r=1}^{n} \frac{\lambda_r^{n-i}}{\sqrt{\lambda_r^{N+\gamma}}} d\lambda_r, \quad i = 1, \ldots, n,
\]

that integrated yields

\[
t_i + c_i = \pm \frac{1}{2 (n - i + N/2 - \gamma/2 + 1)} \sum_{r=1}^{n} \lambda_r^{n-i+N/2-\gamma/2+1}, \quad i = 1, \ldots, n. \tag{36}
\]

The above system can be algebraically solved with respect to \(\lambda_i\) only for two choices of \(\gamma\), namely \(\gamma = N\) and \(\gamma = N - 1\), but it turns out that the case \(\gamma = N\) leads to trivial solutions (polynomial solutions not depending on \(x\)). Thus, we must assume \(\gamma = N - 1\). In this case the above equations attain the form

\[
t_i + c_i = \pm \frac{1}{2(n - i + 1/2)} \sum_{r=1}^{n} \lambda_r^{n-i+1/2}, \quad i = 1, \ldots, n. \tag{37}
\]

Note that \(37\) do not depend on \(N\). It means that for any \(N\) between 1 and \(n - 2\) (as \(\gamma = N + 1 \leq n - 1\)) the functions \(\eta_1(x, t_1, \ldots, t_n), \ldots, \eta_N(x, t_1, \ldots, t_n)\) obtained from \(37\) through \(7\) solve the first \(n - 1\) equations in \(27\). The following two examples illustrate this.

**Example 11** Assume that \(n = 3\). Then \(37\) attain the form (with \(x = t_1, c_i = 0\), we also choose only + in \(37\))

\[
\begin{align*}
x &= \frac{1}{5} \sum_{i=1}^{3} z_i^5 = \frac{1}{5} \left( \rho_1^5 - 5 \rho_1 \rho_2 - \rho_3 (\rho_1^2 - \rho_2) \right) \\
t_2 &= \frac{1}{3} \sum_{i=1}^{3} z_i^3 = \frac{1}{3} (\rho_1^3 - 3 \rho_1 \rho_2 + 3 \rho_3) \\
t_3 &= \sum_{i=1}^{3} z_i = \rho_1
\end{align*}
\]

where \(z_i = \lambda_i^{1/2}, i = 1, 2, 3\) and where \(\rho_1 = \sum_{i=1}^{3} z_i, \rho_2 = z_1 z_2 + z_1 z_3 + z_2 z_3\) and \(\rho_3 = z_1 z_2 z_3\) are elementary symmetric polynomials in \(z_i\). The right hand sides of \(38\) follow from Newton formulas:

\[
\sum_{i=1}^{n} z_i^m = \sum_{a_1 + a_2 + \cdots + a_n = m} (-1)^{a_2 + a_4 + a_6 + \cdots} m! \frac{(\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1)!}{\alpha_1! \cdots \alpha_n!} \rho_1^{a_1} \rho_2^{a_2} \cdots \rho_n^{a_n} \quad \text{for } m < n,
\]

expressing sums of powers of variables as functions of their symmetric polynomials (these formulas can easily be extended to the case \(m \geq n\) by putting all \(\rho_{a+1}, \ldots, \rho_m\) equal to zero). The system \(38\) can be solved explicitly yielding the solution \(37\) in \(\rho\)-variables:

\[
\begin{align*}
\rho_1 &= t_3 \\
\rho_2 &= \frac{-15x - 2t_5^3 + 15t_2^2 t_3}{5 (3t_2^2 - t_3^2)} \\
\rho_3 &= \frac{15t_2 t_3^3 + 45t_2^2 - t_3^5 - 45xt_3}{15 (3t_2^2 - t_3^2)}
\end{align*}\tag{40}
\]

On the other hand, according with \(7\) and with \(39\)

\[
q_1 = -(\lambda_1 + \lambda_2 + \lambda_3) = -(z_1^2 + z_2^2 + z_3^2) = -(2\rho_2 - \rho_1^2).
\]

Plugging \(40\) into the above identity we obtain

\[
q_1(x, t_2, t_3) = q_1(\rho_1(x, t_2, t_3)) = \frac{t_3^5 + 15t_2^3 t_3 - 30x}{5 (3t_2^2 - t_3^2)} \tag{41}
\]

According to Theorem \(10\), the function \(q_1(x, t_2, t_3)\) given by \(41\) yield a two-time solution to the first \(n - 1 = 2\) flows of the nonlocal 1-field (i.e. with \(N = 1\)) HD hierarchy \(27\), i.e. to both systems \(39\) (after an appropriate choice of integration constants).
Example 12 Let us now take \( n = 4 \). In this case the equations (77) read (again with all \( c_i = 0 \) and with \( + \) only and due to (39))

\[
\begin{align*}
x &= \frac{1}{7} \sum_{i=1}^{4} z_i^7 = \frac{1}{7} \left( \rho_1^7 - 7(\rho_1\rho_2 - \rho_3) \left( (\rho_2^2 - \rho_2)^2 + \rho_1\rho_3 \right) - 7\rho_4(\rho_1^2 - 2\rho_1\rho_2 + \rho_3) \right) \\
t_2 &= \frac{1}{5} \sum_{i=1}^{4} z_i^5 = \frac{1}{5} \left( \rho_1^5 - 5(\rho_1\rho_2 - \rho_3)(\rho_2^2 - \rho_2) - 5\rho_1\rho_4 \right) \\
t_3 &= \frac{1}{3} \sum_{i=1}^{4} z_i^3 = \frac{1}{3} \left( \rho_1^3 - 3\rho_1\rho_2 + 3\rho_3 \right) \\
t_4 &= \sum_{i=1}^{4} z_i = \rho_1
\end{align*}
\]

where as before \( z_i = \lambda_i^{1/2} \) and \( \rho_i \) are again symmetric polynomials of the variables \( z_1, \ldots, z_4 \). This system can again be algebraically solved for \( \rho_1, \ldots, \rho_4 \) although the solutions are too complicated to present them here. We have now, according with (77),

\[
\begin{align*}
q_1 &= -\left( \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \right) = -(z_1^2 + z_2^2 + z_3^2 + z_4^2) = -(2\rho_2 - \rho_1^2) \\
q_2 &= \lambda_1\lambda_2 + \cdots + \lambda_3\lambda_4 = z_1^2 z_2^2 + \cdots + z_3^2 z_4^2 = \rho_2^2 - 2\rho_1\rho_3 + 2\rho_4
\end{align*}
\]

Substituting the variables \( \rho_i \) obtained by solving (42) into these expressions we obtain expressions for \( q_1(x, t_2, t_3, t_4) \) and \( q_1(x, t_2, t_3, t_4) \):

\[
q_1(x, t_2, t_3, t_4) = \frac{P_1(x, t_2, t_3, t_4)}{Q(t_2, t_3, t_4)} , \quad q_2(x, t_2, t_3, t_4) = \frac{P_2(x, t_2, t_3, t_4)}{Q(t_2, t_3, t_4)}
\]

where \( P_1 \) and \( Q \) are rather complicated, but perfectly manageable for any computer algebra program, polynomials. More specifically

\[
P_1(x, t_2, t_3, t_4) = \frac{1}{7} \left( 105t_2^3 t_4 - t_4^8 - 21t_2^5 t_3 + 630t_2 t_3 - 630xt_4 + 315t_4^2 t_2 \right)
\]

and

\[
Q(t_2, t_3, t_4) = 45t_2 t_4 + t_4^5 - 15t_3 t_4^3 - 45t_3^2,
\]

while \( P_2 \) is a quadratic in \( x \) polyom that is too complicated to present it here. Now, according to Theorem 34 and the theory above, the function \( q_1(x, t_2, t_3, t_4) \) in (43) solves the first \( n - 1 = 3 \) 1-field flows of the hierarchy (27), so in particular both the flows (33), while the vector function

\[
\begin{pmatrix}
q_1(x, t_2, t_3, t_4) \\
q_1(x, t_2, t_3, t_4)
\end{pmatrix}
\]

solves the first \( n - 1 = 3 \) flows of the \( N = 2 \)-field cHD hierarchy (27) starting with (33).

Let us also remark that formulas (30) often lead to implicit solutions of (27). We illustrate it in the following example. Choose \( N = 1, n = 2 \) and \( \gamma = 0 \) in (30). This yields (again for \( c_i = 0 \))

\[
\begin{align*}
x &= \frac{1}{5} \left( z_1^5 + z_2^5 \right) , \quad t_2 = \frac{1}{5} \left( z_1^3 + z_2^3 \right)
\end{align*}
\]

(with \( z_i = \lambda_i^{1/2} \)) that can not be algebraically solved. However, (44) can be embedded in the algebraically solvable system (35) in the sense that (35) reduces to (44) if we put \( z_3 = 0 \) or equivalently \( \rho_3 = 0 \), since \( \rho_3 = z_1 z_2 z_3 \). By virtue of Theorem 34 it means that the function

\[
q_1(x, t_2, y(x, t_2)) = \frac{y(x, t_2)^5 + 15y(x, t_2)^3 t_2 - 30x}{5(3t_2 - y(x, t_2)^3)}
\]

with the variable \( y(x, t_2) \) defined implicitly by the equation

\[
15t_2 y^3 + 45t_2^2 - y^6 - 45xy = 0
\]

(i.e. by the last equation in (40) with \( y \) instead of \( t_4 \)), also satisfies the first flow of the nonlocal HD-hierarchy i.e. the first flow in (32).
5 Differential (local) cHD hierarchy and its solutions

We will now obtain the purely differential part of cHD hierarchy as well as a class of its implicit solutions. We choose now \( \varepsilon = -1 \) in (4) in order to obtain real solutions in the local case (note that it does not influence the potentials \( V^{(m)}_r \)). Analogously to the case of nonlocal hierarchy, we will perform some variable elimination on the sequence of Killing systems (17). Suppose thus that we want to produce the first \( s \) flows of the \( N \)-component local (i.e., purely differential) Harry-Dym hierarchy. Put \( n = s + N \) and consider the first \( n - N \) Euler-Lagrange equations for the Lagrangian \( L^{n,n-N,-n} \). Using the fact that \( V^{(j)}_r = V_n^{(j)}(q_{n-j+1}, \ldots, q_n) \) it can be shown that they attain the form

\[
E_i \left( L^{n,n-N,-n} \right) = \frac{1}{4q_n^2} + \gamma_1(N) [q_1, \ldots, q_N] = 0, \tag{45}
\]

\[
E_i \left( L^{n,n-N,-n} \right) = -\frac{q_n-i+1}{2q_n^3} + \gamma_{i,1}(N) [q_1, \ldots, q_{N-i+1}] + \frac{1}{q_n^{i+1}} \gamma_{i,2}(N) [q_{n-i+2}, \ldots, q_n] = 0, \quad i = 2, \ldots, n - N.
\]

where as usual \( q_n = 0 \) for \( \alpha < 1 \). Note that (45) and (23) belong to the same ladder (22) of Euler-Lagrange equations since in both cases \( m + k = -N \).

**Proposition 13** The functions \( \gamma_{i,1}(N), \gamma_{i,2}(N), \gamma_1(N) \) do not depend on \( n \) in the sense that increasing \( n \) to \( n + 1 \) will not alter (45) except that a new equation originates at the bottom of the list (45).

The proof of this proposition resembles the proof of the analogous statement for nonlocal case i.e. Proposition 5. Note now that the structure of (45) makes it possible to eliminate (express) the variables \( q_{N+1}, \ldots, q_n \) as (purely differential now) functions of \( q_1, \ldots, q_N \) (although now, opposite to the nonlocal case, we first calculate \( q_n \), then \( q_{n-1} \) and so on up to \( q_{N+1} \)):

\[
q_n = f_1^{(N)} [q_1, \ldots, q_N],
\]

\[
q_{N+1} = f_n^{(N)} [q_1, \ldots, q_N].
\]

Now, let us replace the variables \( q_{N+1}, \ldots, q_n \) in the first \( N \) components of the last \( s \) systems in (17). That leads to \( s \) highly nonlinear (purely differential) evolutionary equations of the form

\[
\overline{q}_r = \overline{Z}_r^{N} \tag{47}
\]

where as before \( \overline{q} = (q_1, \ldots, q_N)^T \) but with new, purely differential, vector fields \( \overline{Z}_r^{N} \). These fields constitute in fact the first \( s \) fields of the local cHD hierarchy. Contrary to the nonlocal case, however, the first field of the hierarchy appears as the last equation in (47) i.e. \( \overline{q}_m = \overline{Z}_m^{N} \overline{q} \), the second field is \( \overline{q}_{n-1} = \overline{Z}_{n-1}^{N} \overline{q} \) and so on so that the fields of the hierarchy originate in (47) in the reverse order. We will therefore introduce a new notation and denote

\[
\tau_p = t_{n-p+1}, \quad \overline{x}_p^{N} = \overline{Z}_{n-p+1}^{N}, \quad p = 1, \ldots, n - 1
\]

so that \( \overline{q}_m = \overline{Z}_m^{N} \overline{q} \) reads \( \overline{q}_{\tau_1} = \overline{x}_1^{N} \overline{q} \) and so on. The sequence (47) becomes therefore

\[
\overline{q}_{\tau_r} = \overline{x}_r^{N} \overline{q}, \quad r = 1, \ldots, s. \tag{49}
\]

A theorem analogous to Theorem 14 explains that this procedure leads to a hierarchy.

**Theorem 14** The vector fields in (49) do not depend on \( s \) in the sense that if we increase \( s \) to \( s + 1 \) then the above elimination procedure produces the same sequence (49) of evolutionary systems plus a new system \( \overline{q}_{\tau_{s+1}} = \overline{x}_{s+1}^{N} \overline{q} \) at the end of the sequence (49) (i.e. at the beginning of the sequence (49)).
Proof. Consider the s systems \((17)\) and increase \(s\) to \(s + 1\) keeping \(N\) constant. We have then to take \(n + 1\) instead of \(n\) in our elimination procedure. Since, according to Proposition 13 the functions \(\gamma_{n+1}^{(N)}, \gamma_{n+2}^{(N)}, \gamma_{n+1}^{(N)}\) do not depend on \(n\) the functions \(f_{i}^{(N)}\) do not depend on \(n\) either. It means that increasing \(n\) to \(n + 1\) (and keeping \(N\) constant) turns the equations \((16)\) into

\[
q_{n+1} = f_{1}^{(N)}[q_1, \ldots, q_N]
\]

\[
q_{N+2} = f_{n}^{(N)}[q_1, \ldots, q_N]
\]

\[
q_{N+1} = f_{n+1}^{(N)}[q_1, \ldots, q_N]
\]

and at the same time the the structure of the last \(s\) equations in \((17)\) changes so that \(q_n\) is replaced by \(q_{n+1}, q_{n-1}\) is replaced by \(q_n\) and so on until \(q_{N+2}\). It means that the last \(s\) equations in the (extended to \(n + 1\)) sequence \((17)\) will after elimination remain the same while a new equations originates - this time before (with lowest \(r\)) the other \(s\) ones. ■

Thus, by taking appropriate \(s\) we can produce on demand an arbitrary (finite) number of evolutionary vector fields

\[
\overline{\mathbf{q}}_{r} = X^N_{r}[\overline{q}], \quad r = 1, 2 \ldots
\]

and due to same argument as in the nonlocal case, these vector fields all mutually commute:

\[
[X^N_i, X^N_j] = 0 \quad \text{for all } i, j = 1, 2 \ldots
\]

The described procedure leads in fact to multicomponent local Harry Dym hierarchy.

**Example 15** Let us first produce the first \(s = 2\) flows of the standard Harry Dym hierarchy i.e. with \(N = 1\). We have \(n = s + N = 3\). Consider the Lagrangian

\[
L^{n,n-N,-n} = L^{3,2,-3} = \frac{1}{4}q_1^2 \cdot \frac{q_2^2}{2q_3} + \frac{q_3^2q_2}{4q_3} + \frac{q_1}{4q_3} - \frac{q_2^2}{4q_3}
\]

and the corresponding Euler-Lagrange equations \((45)\). They attain the form

\[
E_1 \left( L^{3,2,-3} \right) = \frac{1}{4q_3^2} - \frac{1}{2q_{1,xx}} = 0
\]

\[
E_2 \left( L^{3,2,-3} \right) = -\frac{q_2}{2q_3} \cdot \frac{q_3^2}{4q_3} + \frac{q_{3,xx}}{2q_3} = 0
\]

and can thus easily be solved with respect to \(q_2\) and \(q_3\) yielding \((46)\) in the explicit form

\[
q_3 = q_3[q_1] = (2q_{1,xx})^{-1/2}
\]

\[
q_2 = q_2[q_1] = \frac{1}{2} \left( 5q_{1,xxx} - 4q_{1,xxx} q_{1,xxxx} \right) (2q_{1,xx})^{-7/2}
\]

Substituting these expressions to the first (since \(N = 1\)) component of the last \(s = 2\) Killing systems of the sequence \((17)\) we obtain the following two commuting flows:

\[
q_{1,t_2} = (q_2[q_1])_x \quad q_{1,t_3} = (q_3[q_1])_x
\]

or

\[
q_{1,\tau_1} = (q_3[q_1])_x = X^1_1 \quad q_{1,\tau_2} = (q_2[q_1])_x = X^1_2
\]

(with the differential functions \(q_2[q_1]\) and \(q_3[q_1]\) given as above) i.e. the first two members of the well known local Harry Dym hierarchy.
Example 16 Let us now produce the first $s = 2$ flows of the $N = 2$-component Harry Dym hierarchy, we need therefore $n = s + N = 4$. The Euler-Lagrange equations \[45\] for the Lagrangian $L^{n,n−N,−n} = L^{4,2,−4}$ attain the form

\[E_1(L^{4,2,−4}) = \frac{1}{4q_1^2} + \frac{1}{2}q_1q_{1,xx} + \frac{1}{4}q_1^2 - \frac{1}{2}q_2xx\]

\[E_2(L^{4,2,−4}) = -\frac{q_3^4}{2q_2^2} - \frac{1}{2}q_1xx\]

that is soluble with respect to $q_3$ and $q_4$ yielding

\[q_4 = q_4[q_1, q_2] = −w^{−1/2} = −(2q_{2,xx} − q_1^2x − 2q_1q_{1,xx})^{−1/2}\]

\[q_3 = q_4[q_1, q_2] = −q_{1,xx}w^{−3/2}\]

Substituting these functions to the first $N = 2$ components of the last $s = 2$ Killing systems of the sequence \[47\] (with $n = 4$) yields the desired flows

\[\frac{d}{d\tau_1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathcal{X}_1^2 \equiv \begin{pmatrix} (w^{−1/2})_x - w^{−1/2}q_{1,xx} \\ q_{1,xx}w^{−3/2}q_1 - q_1(q_{1,xx}w^{−3/2}/2 + (w^{−1/2})_x) \end{pmatrix}\]

and

\[\frac{d}{d\tau_2} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathcal{X}_2^2 \equiv \begin{pmatrix} \frac{1}{2}q_{1,xx}w^{−3/2}q_1 - q_1(q_{1,xx}w^{−3/2}/2) + (w^{−1/2})_x \\ (w^{−1/2})_x - w^{−1/2}q_{1,xx} \end{pmatrix}\]

Our parametrization of Harry Dym hierarchy differs from the parametrization given in \[10\]. Generally speaking, the hierarchy \[50\] is transformed into the multicomponent Harry Dym hierarchy presented in \[10\] through a complex version of the map \[44\]

\[u_r = −iE_{N−r+1}(L^{N,0,0}), \quad r = 1, \ldots, N, \quad i^2 = −1.\]

For example, in the $u$-variables the system \[51\] attains the form

\[\frac{d}{d\tau_1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathcal{X}_1^2 [u] \equiv \begin{pmatrix} \frac{1}{4}u_2^{−1/2}x^{xx} + \frac{1}{2}u_2^{−1/2}u_{1,xx} \\ \frac{1}{4}u_2^{−1/2}x^{xx} + \frac{1}{2}u_2^{−1/2}u_{1,xx} \end{pmatrix}\]

that is exactly the flow \[24a\] in \[10\].

We will now formulate a theorem corresponding to Theorem \[10\]. i.e. we will generate a wide class of solutions of the hierarchy \[50\].

Theorem 17 For any $\beta \in \{0, \ldots, n − 1\}$ the functions $\lambda_i = \lambda_i(t_1, \ldots, t_n)$ given implicitly by

\[t_i + c_i = \pm \frac{1}{2} \int_{r=1}^{n} \frac{\lambda_r^{n−i}}{\lambda_r^{−N+\beta} \left(\sum_{j=1}^{n} a_j \lambda_r^{n−j} + \frac{1}{4} \lambda_r^{−\beta}\right)} d\lambda_r, \quad i = 1, \ldots, n. \]

are such that the corresponding functions $q_i = q_i(x = t_1, t_2, \ldots, t_n)$, $i = 1, \ldots, N$, given by \[7\] are solutions of the first $n − \beta$ ($n = 0, 1$) equations of the $N$-component integral cHD hierarchy \[7\]. The variables $t_{j+1} = \tau_{n−\beta}, \ldots, t_n = \tau_1$ ($t_2, \ldots, t_n$ for $\beta = 0, 1$) are evolution parameters (dynamical times) while the remaining $t_i$’s are free parameters.

We will not prove this theorem here as its proof resembles the proof of Theorem \[10\]. Comparing Theorems \[10\] and \[17\] we can see that the solutions \[53\] and \[54\] are for $\beta = 1, \ldots, n − 1$ related through the transformation $\beta \rightarrow n − \beta$, $\varepsilon \rightarrow −\varepsilon$, i.e. every solution \[53\] for $\beta = 1, \ldots, n − 1$ coincides, after changing $\varepsilon \rightarrow −\varepsilon$, with the solution \[54\] with $\beta' = n − \beta$. It also means that the nonlocal flow \[55\] and the local flow \[56\] share the same family of solutions, namely \[53\] (or \[54\] with $\beta' = n − \beta$ and with $\varepsilon' = −\varepsilon$).

It turns out that \[53\] cannot be explicitly solved. However, by taking all $a_i = 0$ in \[54\] (which yields the so called zero-energy solutions) we can obtain interesting implicit solutions to our hierarchy \[50\].
Example 18 Consider the solutions (54) with $N = 2, n = 3$ and with all $a_i = 0$. They have the form

\[ t_i + c_i = \pm \sum_{r=1}^{3} \int \lambda_r^{4-i} d\lambda_r, \quad i = 1, 2, 3. \]  

(55)

(the same for all $\beta$ since $\beta$-terms cancel after inserting $a_i = 0$) and according to Theorem 17 they solve the first $n - 1 = 2$ flows of the $N = 2$-component cHD hierarchy (60) i.e. both the flows (61) and (62). Equations (62) after integrating yield (remember that $t_1 = x$; we also put all $c_i = 0$ for simplicity of the formulas)

\[ x = \frac{1}{4} \sum_{i=1}^{3} \lambda_i^4, \quad t_2 = \frac{1}{3} \sum_{i=1}^{3} \lambda_i^3, \quad t_3 = \frac{1}{2} \sum_{i=1}^{3} \lambda_i^2 \]  

(56)

which can not be algebraically solved. However, similarly as in the nonlocal case, we can embed (60) in the system

\[
\begin{align*}
&x = \frac{1}{4} \sum_{i=1}^{4} \lambda_i^4, \quad t_2 = \frac{1}{3} \sum_{i=1}^{4} \lambda_i^3, \quad t_3 = \frac{1}{2} \sum_{i=1}^{4} \lambda_i^2 \\
&t_4 = \sum_{i=1}^{4} \lambda_i = \rho_4
\end{align*}
\]

(57)

(where $\rho_i$ are symmetric polynomials in $\lambda_i$ so that $q_i = (-1)^i \rho_i$) in the sense that putting $\lambda_4 = 0$ (so that $\rho_4 = 0$ since $\rho_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4$; the righthand sides of (57) are again due to (56)) in (57) we obtain (60). The equations (57) can be explicitly solved yielding,

\[
\begin{align*}
q_1 &= -\rho_1 = -t_4 \\
q_2 &= \rho_2 = -t_3 + \frac{1}{2} t_4^2 \\
q_3 &= -\rho_3 = -t_2 - \frac{1}{6} t_4^3 + t_3 t_4 \\
q_4 &= \rho_4 = -x + \frac{1}{24} t_4^4 - \frac{1}{2} t_4^2 t_3 + \frac{1}{2} t_3^2 + t_2 t_4
\end{align*}
\]

Thus, the functions $q_i(x, t_2, t_3)$ given implicitly by

\[
\begin{align*}
q_1 &= -\rho_1 = -t_4(x, t_1, t_2) \\
q_2 &= \rho_2 = -t_3(x, t_1, t_2) + \frac{1}{2} t_4(x, t_1, t_2)^2
\end{align*}
\]

where $t_3(x, t_1, t_2)$ and $t_4(x, t_1, t_2)$ are any pair of functions identically satisfying the condition

\[ 0 = -x + \frac{1}{24} t_4^4 - \frac{1}{2} t_4^2 t_3 + \frac{1}{2} t_3^2 + t_2 t_4 \]

solve both (61) and (62).

6 Conclusions

In this article we presented a novel method of obtaining multicomponent Harry Dym hierarchy (both its local and nonlocal part) as well as wide classes of its solutions, from a family of finite dimensional separable systems (Stäckel systems of Benenti type). This method has been previously applied to coupled Kortevge-de Vries hierarchy where it produced novel rational solutions and also a family of implicit solutions. In the case of cHD hierarchy discussed here, the method produces among others rational and implicit solutions in case of nonlocal hierarchy and implicit solutions of the local part. In addition, the method produces wide families of other solutions that are to be exploited elsewhere. It also indicates the existence of common solutions of local and nonlocal cHD systems.

Our method can hopefully be extended to other systems, for example by taking more general separation relations than relations (44).
7 Appendix

We prove here Theorem 11. We start with the case $\beta = 0$. For $\beta = 0$ the solutions (55) are just solutions (11) with our choice of $m$ and $k$, namely $m = -N, k = 0$. The functions

$$q_1(x = t_1,t_2, \ldots , t_n), \ldots , q_s(x = t_1,t_2, \ldots , t_n)$$

(58)

obtained from (55) (with $\beta = 0$) through (7) satisfy thus all $n - 1$ Killing systems (17). Moreover they satisfy all the equations (23) and thus also all the equations (25) used in our elimination procedure. This means that we are free to use any part of (23) or (25) to perform an elimination of variables in (17). Such elimination thus leads to new equations that are satisfied by those functions from the set (58) that survive the elimination. Now, we know that replacing the variables $q_{N+1}, \ldots , q_n$ in the first $N$ components of the first $s - 1 = n - N$ equations (17) by the functions given by (25) leads to the first $s - 1$ flows of the hierarchy (27). That means precisely that the first $N$ functions in (55)

$$q_1(x = t_1,t_2, \ldots , t_n), \ldots , q_N(x = t_1,t_2, \ldots , t_n)$$

(59)

satisfy the first $s - 1 = n - N$ equations in (27). We will now show that they actually solve the first $n - 1$ equation in (27). Consider the next flow $\mathbf{t}_{t_{i+1}} = Z_{s+1}^N[q]$ in (27). In order to obtain this flow, we have to perform the elimination of variables $q_{N+1}, \ldots , q_n$, $q_{n+1}$ in the flow $\mathbf{t}_{t_{i+1}} = Z_{s+1}^N[q]$ through (25) written for $n + 1$ instead of $n$ i.e. obtained from solving (24). This elimination is therefore performed with the help of the same functions $q_1 = q_1[q_1, \ldots , q_N]$ as for $n$ plus a new function $q_{n+1} = f_{n-N+2}[q_1, \ldots , q_N]$. However, $(Z_{s+1}^N[q])^j = (Z_{s+1}^N[q])^j$ for all $j = 1, \ldots , N - 1$ (it follows from (17)) while $(Z_{s+1}^N[q])^N$ contains the additional variable $q_{n+1}$ not present in $(Z_{s+1}^N[q])^N$. It means that solutions (59) will certainly satisfy the first $N$ - 1 components in $\mathbf{t}_{t_{i+1}} = Z_{s+1}^N[q]$. Further, since $E_{N+1}(L^{n+1,-N,0}) = - \frac{1}{2} q_{n+1,xx} + E_N(L^{n,-N,0})$, the function $q_{n+1} = f_{n-N+2}[q_1, \ldots , q_N]$ is (after choosing both integration constants equal to zero) identically equal to zero on the solutions (59). That means that on the solutions (59) we have $(Z_{s+1}^N[q])^N = (Z_{s+1}^N[q])^N$ which means indeed that (59) solves $\mathbf{t}_{t_{i+1}} = Z_{s+1}^N[q]$. By expanding this argument, the functions $q_{n+1}, q_{n+2}, \ldots , q_{n+N-1}$ obtained from (25) with $n$ replaced by $n' = n + N - 1$ i.e. from the $n' - N = n - 1$ equations

$$E_{N+1}(L^{n',-N,0}) = - \frac{1}{2} q_{n+N-1,xx} + \varphi_{n-1}[q_1, \ldots , q_{n+N-2}] = 0,$n+2}(L^{n',-N,0}) = - \frac{1}{2} q_{n+N-2,xx} + \varphi_{n-2}[q_1, \ldots , q_{n+N-3}] = 0,$n+1}(L^{n',-N,0}) = - \frac{1}{2} q_{n+2,xx} + \varphi_2[q_1, \ldots , q_{N+1}] = 0,$n'}(L^{n',-N,0}) = - \frac{1}{2} q_{n+1,xx} + \varphi_1[q_1, \ldots , q_N] = 0.$

(60)

(which are necessary to obtain the first $n - 1$ flows of (27)) are identically zero on the solutions (59) which leads to the conclusion that (59) indeed solve the first $n - 1$ equations of (27).

Assume finally that $0 < \beta \leq n - 1$. The functions (55) are then the complete solution (as usual, through the map (7)) of all the Euler-Lagrange equations $E_i(L^{n,\beta-N,-\beta})$ associated with the Lagrangian $L^{n,\beta-N,-\beta}$. As such, they still must solve all the Killing systems (17). However, since $E_i(L^{n,\beta-N,-\beta}) = E_{i+\beta}(L^{n,0})$ for $i = 1, \ldots , n - \beta$ due to (20), for any $\beta > 1$ we lose the first $\beta - 1$ equations in (60) which means that our proof works only for the first $n - \beta$ flows in (27) - we simply can not "blow up" $n$ to $n' = n + N - 1$ but only to $n'' = n + N - 1 - \beta$.

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