HOW MANY ADJUNCTIONS GIVE RISE TO THE SAME MONAD?

Abstract. Given an adjoint pair of functors $F, G$, the composite $GF$ naturally gets the structure of a monad. The same monad may arise from many such adjoint pairs of functors, however. Can one describe all of the adjunctions giving rise to a given monad? In this paper we single out a class of adjunctions with especially good properties, and we develop methods for computing all such adjunctions which give rise to a given monad. We provide an explicit nontrivial computation to demonstrate these methods, using a bit of Auslander-Reiten theory. We also prove a criterion, reminiscent of Beck’s monadicity theorem, for when there is essentially (in a precise sense) only a single adjunction that gives rise to a given monad.

Contents

1. Introduction. 1
2. Homological presentations of a monad are equivalent to localizations of its Eilenberg-Moore category. 3
   2.1. Preliminary definitions. 3
   2.2. Localizations presenting a monad are equivalent to its homological presentations. 5
   2.3. Coordinatization of the collection of homological presentations of a monad. 7
3. A criterion for unique homological presentability of a monad. 8
   3.1. Preliminary definitions. 8
   3.2. The criterion for unique homological presentability. 9
4. A class of explicit examples. 10
   4.1. Review of some Auslander-Reiten theory. 10
   4.2. Homological presentations of the base-change monad of an Artin algebra of finite representation type. 11
References 14

1. Introduction.

If $C, D$ are categories and $G : D \to C$ a functor with a left adjoint $F$, then the composite $GF$ gets the structure of a monad. However, given a monad $T : C \to C$, there may be many categories $D$ and adjoint pairs $F, G$ such that $GF = T$ as a monad. We will call such a choice of $D, F, G$ a presentation for $T$.

It has long been known that, among all presentations of a given monad $T$, there is an initial presentation, the Kleisli category of $T$, and a terminal presentation, the Eilenberg-Moore category of $T$. Furthermore, Beck’s monadicity theorem gives a necessary and sufficient condition on $G$ for the presentation $(D, F, G)$ to be equivalent to the Eilenberg-Moore category. (See [7] for a nice exposition of these ideas.) Beck’s result has proven very useful, e.g. in algebraic geometry where, in its dual form for comonads, it is the foundation for the general theory of descent; see [4].
The applications of Beck’s monadicity theorem have made it very clear that, given a presentation \((\mathcal{D}, F, G)\) of a monad \(T\), it is very useful to be able to tell when \((\mathcal{D}, F, G)\) is the terminal presentation (i.e., the Eilenberg-Moore category) of \(T\). However, we have long wondered about what to do when one encounters a presentation of a monad which is not the terminal (Eilenberg-Moore) presentation, and not even the initial (Kleisli) presentation. Can one describe the collection of all presentations of a monad? Even better, can one establish some kind of coordinate system on the collection of presentations of a monad, so that when one encounters a presentation of a monad (or a comonad, e.g. for applications in descent theory), one can give some kind of “coordinates” that describe where this presentation sits in relation to the initial and terminal presentations, and all the other presentations, of the same monad?

In this paper we study the collection of presentations of a given monad, but with a restriction on what presentations we are willing to consider. This is because, given a presentation \((\mathcal{D}, F, G)\), one can trivially produce many more presentations by taking the Cartesian product of \(\mathcal{D}\) with any small category. We regard these presentations as degenerate, and we want to disregard presentations with this kind of redundant information in them. Consequently, in Def. 2.2 we make the definition that a presentation \((\mathcal{D}, F, G)\) is said to be homological if every object \(X\) of \(\mathcal{D}\) can be recovered from the \(F, G\)-bar construction on \(X\) (see Def. 2.2 for the precise definition). This definition eliminates the “redundant” presentations we wanted to exclude, and has some other good properties, described in Remark 2.3. We also restrict our attention to what we call “coequalizable” monads, that is, those monads \(T\) for which the Eilenberg-Moore category has coequalizers; this property is satisfied in all cases of interest which we know of, and in Remark 2.7 we explain a bit about why that is.

Once these definitions are made, we can prove some nice theorems:

- In Thm. 2.11 we prove that, if \(T\) is coequalizable, then the category of homological presentations of \(T\) is equivalent to the partially-ordered collection of reflective full replete subcategories of the Eilenberg-Moore category \(\mathcal{C}^T\).
  This means the category of all homological presentations of \(T\) is always well-behaved in at least one way: it can’t be just any arbitrary category, rather, it is always partially-ordered (i.e., there is at most one morphism from any given object to any other given object).
- In Thm. 2.17 when \(T\) is coequalizable and \(\mathcal{C}^T\) has a biproduct and is Krull-Schmidt, we actually construct a “coordinate system” on the collection of homological presentations of \(T\)! Any homological presentation is determined uniquely by specifying a subcollection of the collection of isomorphism classes of indecomposable objects of \(\mathcal{C}^T\) —in other words, a subcollection of the vertex set of the Auslander-Reiten quiver of \(\mathcal{C}^T\). So the vertices of the Auslander-Reiten quiver of \(\mathcal{C}^T\) act as our “coordinates.”
- In Thm. 3.6 we give a simple and usable criterion for the triviality of the collection of homological presentations of \(T\), i.e., a criterion for when there exists only one homological presentation of \(T\) (necessarily the Eilenberg-Moore category of \(T\)). As an example, in Cor. 3.8 we show that the base-change monad on module categories associated to a field extension has this property of unique homological presentability.
- In Thm. 4.9 we use Thms. 2.11 and Thm. 2.17 as well as some Auslander-Reiten theory, to explicitly compute the collection of all homological presentations of each monad in one particular class of monads: let \(n\) be a positive integer, let \(A\)
be the $k$-algebra $k[x]/x^n$, and let $T : \text{fgMod}(k) \to \text{fgMod}(k)$ be the base-change monad on the category of finite-dimensional $k$-vector spaces, i.e., $T$ is the composite of the extension of scalars $\text{fgMod}(k) \to \text{fgMod}(A)$ with the restriction of scalars $\text{fgMod}(A) \to \text{fgMod}(k)$. We find that there are precisely $n$ homological presentations of $T$, and we exhibit each of them as a subset of the vertex set of the Auslander-Reiten quiver $\Gamma(A)$!

In many cases of interest (e.g. the base-change monads on module categories associated to maps of rings or maps of schemes), the Eilenberg-Moore category $\mathcal{C}^T$ is actually 

\begin{equation}
\text{abelian, hence } T \text{ is coequalizable and } \mathcal{C}^T \text{ has a biproduct automatically, and frequently } \mathcal{C}^T \text{ is actually quite computable and understandable. Under those circumstances our results seem to be fairly useful, and as we hope Thm. 4.9 demonstrates, they are actually applicable and give explicit nontrivial results in concrete situations of interest.}
\end{equation}

2. Homological presentations of a monad are equivalent to localizations of its Eilenberg-Moore category.

2.1. Preliminary definitions. Throughout this paper, when $T$ is a monad, when convenient we will sometimes also write $T$ for the underlying functor of the monad.

**Definition 2.1.** Let $\mathcal{C}$ be a category, $T$ a monad on $\mathcal{C}$. If $\mathcal{D}$ is a category equipped with a functor $F' : \mathcal{C} \to \mathcal{D}$ and a right adjoint $G'$ for $F'$ such that the associated monad $G'F'$ is equal to $T$, we call the data $(\mathcal{D}, F', G')$ a presentation of $T$. Sometimes we shall just write $\mathcal{D}$ as shorthand for $(\mathcal{D}, F', G')$, when $F', G'$ are clear from context.

The collection of all presentations of $T$ forms a large category, whose morphisms are morphisms of adjunctions (see IV.7 of [7] for the definition of morphisms of adjunctions). We call this large category the category of presentations of $T$, and for which we will write $\text{Pres}(T)$.

We note that $\text{Pres}(T)$ is not necessarily a category, but a large category, because its hom-collections are not necessarily hom-sets. The notion of a category of presentations of a monad appears in VI.5 of [7], but was not there given a name.

For the purposes of this paper we will mostly be studying presentations of a monad which have the property that there is some minimal degree of compatibility between the category and the monad, enough compatibility to guarantee that e.g. some constructions in homological algebra can be made. Here is our definition:

**Definition 2.2.** Let $\mathcal{C}$ be a category, $T$ a monad, $(\mathcal{D}, F', G')$ a presentation of $T$. We will say that $\mathcal{D}$ is a homological presentation of $T$ if for every object $X$ of $\mathcal{D}$, the coequalizer of the two natural counit maps

\begin{equation}
\epsilon_{F'G'X}, F'G'\epsilon_X : F'G'F'G'X \to F'G'X
\end{equation}

exists, and the canonical map

\begin{equation}
\text{coeq}\{\epsilon_{F'G'X}, F'G'\epsilon_X\} \to X
\end{equation}

is an isomorphism. We will write $\text{HPres}(T)$ for the full large subcategory of $\text{Pres}(T)$ generated by the homological presentations.

**Remark 2.3.** The reason for the name “homological” for this kind of presentation is the following: if $(\mathcal{D}, F', G')$ is a homological presentation of a monad and $\mathcal{D}$ is abelian, then each object $X$ in $\mathcal{D}$ admits a canonical resolution

\begin{equation}
0 \to X \leftarrow F'G'X \leftarrow F'G' \ker(\epsilon_{F'G'X} - F'G'\epsilon_X) \leftarrow \ldots
\end{equation}
obtained by repeatedly applying \( F'G' \), forming the coequalizer \([2.1.2]\) and taking the kernel of the coequalizer map. This resolution gives us a way to compute the left-derived functors of any functor on \( \mathcal{D} \) which is acyclic on every object of the form \( F'G' \). If \((\mathcal{D}, F', G')\) fails to be homological, then at least for some objects \( X \) the chain complex \([2.1.3]\) fails to be exact and hence cannot be used to compute derived functors in this way.

The resolution \([2.1.3]\) is very familiar and commonplace in its various special cases. For example, when \( \mathcal{C} \) is the category of sets and \( T \) the monad given on a set \( S \) by taking the underlying set of the free abelian group generated by \( S \), then the category \( \text{Ab} \) is a presentation for \( T \), and it is homological (because it is the terminal presentation, i.e., the Eilenberg-Moore category of \( T \), which in Cor. \(2.12\) we prove is always homological for any coequalizable monad \( T \)). The resolution \([2.1.3]\) is the elementary resolution one uses in a first course in homological algebra to prove that free resolutions exist in the category of abelian groups: given an abelian group \( X \), one can form the direct sum \( \oplus_{x \in X} \mathbb{Z} \), one can let \( X_0 \) be the kernel of the obvious surjection \( \oplus_{x \in X} \mathbb{Z} \to X \), then iterate to form a free resolution of \( X \).

When \( \mathcal{D} \) fails to be abelian, instead of \([2.1.3]\) one forms the simplicial resolution

\[
[2.1.4] \quad F'G'X \xrightarrow{\partial_0} F'G'F'G'X \xrightarrow{\partial_1} F'G'F'G'F'G'X \rightarrow \ldots
\]

of \( X \), and one can use this resolution to compute more general kinds of derived functors (e.g. if \( \mathcal{D} \) has the structure of a model category). In every case, the condition that \((\mathcal{D}, F', G')\) is homological is really the condition that \( F', G' \) gives us a way to form a canonical resolution of any object in \( \mathcal{D} \). In some sense one should think of a homological presentation for a monad \( T \) as a category equipped with a way of forming a canonical resolution of any object by \( T \)-free objects, and that means that this paper is in some sense really about classifying various ways of forming canonical resolutions.

Finally, one more note about the map \([2.1.2]\) after applying \( G' \), the map always becomes an isomorphism, because the cofork

\[
\begin{align*}
G'F'G'F'G'X & \xrightarrow{\eta_{G'X} G'F'} G'F'G'X \\
& \xrightarrow{G'F'} G'X
\end{align*}
\]

is always split by the unit map \( \eta_{G'X} : G'X \to G'F'G'X \), hence the cofork is a split coequalizer. But the map \([2.1.2]\) can fail to be an isomorphism before applying \( G' \).

Recall that a subcategory is said to be replete if it contains every object isomorphic to one of its own objects, and reflective if the inclusion of the subcategory admits a left adjoint. (Sometimes fullness of the subcategory is also assumed in the definition of a reflective subcategory, but this seems to depend on the author. In any case, in the present paper every reflective subcategory we ever say anything about is full, but we will write “reflective full subcategory” each time, for clarity.)

**Definition 2.4.** If \( \mathcal{C} \) is a category, by a localization of \( \mathcal{C} \) we shall mean a reflective, replete, full subcategory of \( \mathcal{C} \).

Note that this definition of localization is substantially weaker than the notion of “localizing subcategory” which is already in the literature!

**Definition 2.5.** Let \( \mathcal{C} \) be a category, \( T \) a monad, \( \mathcal{C}^T \) the Eilenberg-Moore category of \( T \)-algebras. If \( \mathcal{D} \) is a localization of \( \mathcal{C}^T \), we will say that \( \mathcal{D} \) presents \( T \) if \( \mathcal{D} \) contains all the free \( T \)-algebras, i.e., if \( \mathcal{D} \) contains the \( T \)-algebra \( TTX \xrightarrow{\eta_X} TX \) for every object \( X \) of \( \mathcal{C} \).
We will write \( \text{Loc}(T) \) for the partially-ordered collection of all localizations \( c^T \) which present \( T \).

We note that \( \text{Loc}(T) \) is not necessarily a set, nor even a class (we are grateful to Mike Shulman for pointing out to us that the collection of subcategories of a category is not necessarily a class!). Sometimes the term “conglomerate” is used for a collection too large to form a class. In other words, if one wants to use Grothendieck universes, one must expand the universe *twice* to go from sets to conglomerates. In practical algebraic, geometric, and topological situations, however, it seems likely that \( \text{Loc}(T) \) will form a set.

Finally, it will sometimes be convenient to have coequalizers in Eilenberg-Moore categories. We introduce a definition which describes monads which have this agreeable property:

**Definition 2.6.** Let \( C \) be a category, \( T \) a monad, \( C^T \) the Eilenberg-Moore category of \( T \)-algebras. We will say that \( T \) is coequalizable if \( C^T \) has coequalizers.

**Remark 2.7.** There are many known conditions on \( T \) which guarantee that \( T \) is coequalizable; for example, in Lemma II.6.6 in [5] it is shown that, if \( T \) preserves reflexive coequalizers, then \( T \) is coequalizable. Consequently, many interesting examples of monads \( T \) are coequalizable.

For example, suppose \( R \to S \) is a map of commutative rings. Then the base-change monad \( T : \text{Mod}(R) \to \text{Mod}(R) \), i.e., the composite of the extension of scalars functor \( \text{Mod}(R) \to \text{Mod}(S) \) with the restriction of scalars functor \( \text{Mod}(S) \to \text{Mod}(R) \), is coequalizable, since extension of scalars and restriction of scalars are both right exact, preserving all coequalizers. If \( S \) is finitely generated as an \( R \)-module then the base-change monad \( \text{fgMod}(R) \to \text{fgMod}(R) \) on the finitely generated module category is also coequalizable, for the same reason. Then the Eilenberg-Moore category \( \text{Mod}(R)^T \) is equivalent to \( \text{Mod}(S) \).

More generally, if \( f : Y \to X \) is a map of schemes and \( \text{QC Mod}(\mathcal{O}_X) \) the category of quasicoherent \( \mathcal{O}_X \)-modules, then the base-change monad \( f_*f^* \) is coequalizable if \( f \) is an affine morphism, since in that case \( f_* \) is right exact (and \( f^* \) is always right exact, regardless of whether \( f \) is affine). Then the Eilenberg-Moore category \( \text{QC Mod}(\mathcal{O}_X)^{f_*f^*} \) is equivalent to \( \text{QC Mod}(\mathcal{O}_Y) \), by the results of EGA II.1.4, [6].

Usually (e.g. in the examples above, and in our Thm. 2.9) we will have an explicit description of the category \( c^T \) and we will know that it has coequalizers; what will be interesting and new will be the description of the rest of \( \text{HPres}(T) \).

**Definition 2.8.** If \( C \) is a category with coproduct \( \oplus \), we say that an object \( X \) of \( C \) is indecomposable if \( X \cong Y \oplus Z \) implies either \( Y \cong 0 \) or \( Z \cong 0 \). We say that \( C \) is Krull-Schmidt if every object \( X \) of \( C \) admits a decomposition into a finite coproduct of indecomposable objects, and that decomposition is unique up to permutation of the summands.

2.2. Localization presentations of a monad are equivalent to its homological presentations.

**Lemma 2.9.** Let \( C \) be a category, \( T \) a monad on \( C \), \( (D, F', G') \) a presentation of \( T \). If \( D \) has coequalizers of all pairs of maps of the form \( 2.7.1 \) then the canonical comparison functor \( K : D \to c^T \) has a left adjoint. Conversely, if \( T \) is coequalizable and \( K \) is full and faithful and has a left adjoint, then \( D \) has all coequalizers (and in particular, all pairs of maps of the form \( 2.7.1 \)).
Proof. We will write \( F : \mathcal{C} \to \mathcal{C}^T \) for the canonical functor and \( G \) for its right adjoint. When \( \mathcal{D} \) has coequalizers of all parallel pairs of the form \([2.1.1]\) the comparison functor \( K \) admits a left adjoint \( V \), defined on objects as follows: if \( TX \xrightarrow{p} X \) is the structure map of a \( T \)-algebra, then \( V \) applied to that \( T \)-algebra is the coequalizer of the maps

\[
F'G'p_X, \epsilon_{F'}X : F'G'F'X \to F'X,
\]

using the fact that \( G'F' = GF = T \). (The result that \( K \) has a left adjoint if \( \mathcal{D} \) has coequalizers is an old one: it appears in Beck’s thesis \([3]\), and even appears as an exercise in VI.7 of \([7]\). But the only coequalizers one actually needs are the ones used in the construction of the left adjoint, i.e., those of the form \([2.1.1]\)).

For the converse: suppose \( T \) is coequalizable and \( K \) is full and faithful and has a left adjoint \( V \). Since left adjoints preserve colimits and since fullness and faithfulness of \( K \) is equivalent to \( VK \cong \text{id}_\mathcal{D} \), we can compute the coequalizer of any pair \( f, g : X \to Y \) in \( \mathcal{D} \) by computing the coequalizer of \( Kf, K_\mathcal{D} \) in \( \mathcal{C}^T \) (which exists since \( T \) is coequalizable) and then applying \( V \). \( \square \)

Lemma 2.10. Let \( \mathcal{C} \) be a category, \( T \) a monad on \( \mathcal{C} \), \((\mathcal{D}, F', G')\) a presentation of \( T \). Suppose \( \mathcal{D} \) has coequalizers of all pairs of maps of the form \([2.1.1]\). Then the comparison functor \( K : \mathcal{D} \to \mathcal{C}^T \) is full and faithful if and only if \((\mathcal{D}, F', G')\) is homological.

Proof. We use the same notation as in the proof of Lemma 2.9. That \( K \) is full and faithful is equivalent to the counit map \( VK \to \text{id}_\mathcal{D} \) of the adjunction being an isomorphism. We recall that \( K \) is defined on objects by letting \( KX \) be the \( T \)-algebra with structure map \( G'F'X = TG'X \to G'X \) given by the counit natural transformation \( F'G' \to \text{id}_\mathcal{D} \). Now \( VKX \) is precisely the coequalizer of the two maps

\[
\epsilon_{F'G'X}, F'G'\epsilon_X : F'G'F'G'X \to F'G'X,
\]

and the map \( VKX \to X \) is precisely the map \([2.1.2]\). So the condition that \((\mathcal{D}, F', G')\) be homological is equivalent to the condition that \( VK \to \text{id}_\mathcal{D} \) be an isomorphism of functors, i.e., the condition that \( K \) be full and faithful. \( \square \)

Theorem 2.11. Let \( \mathcal{C} \) be a category, \( T \) a coequalizable monad on \( \mathcal{C} \). Then the large category \( \text{HPres}(T) \) of homological presentations of \( T \) is equivalent to the partially-ordered collection \( \text{Loc}(T) \) of localizations of \( \mathcal{C}^T \) which present \( T \).

Proof. We write \( F : \mathcal{C} \to \mathcal{C}^T \) for the canonical functor and \( G \) for its right adjoint, and we write \( F'' : \mathcal{C} \to \text{Kl}(T) \) for the canonical functor and \( G'' \) for its right adjoint. The theorem follows almost immediately from Lemma 2.10 which gives us that every homological presentation \((\mathcal{D}, F', G')\) of \( T \) has the property that \( G' \) is faithful and full, hence \( \mathcal{D} \) is canonically equivalent to a full replete subcategory of \( \mathcal{C}^T \), and Lemma 2.9 which gives us that that full replete subcategory is reflective. That reflective full replete subcategory contains the free \( T \)-algebras, i.e., the Kleisli category of \( T \), since the Kleisli category is initial among presentations of \( T \). So that reflective full replete subcategory of \( \mathcal{C}^T \) is an element of \( \text{Loc}(T) \).

Conversely, if \( \mathcal{D} \) is a reflective full replete subcategory of \( \mathcal{C}^T \) with inclusion \( K : \mathcal{D} \to \mathcal{C}^T \) having left adjoint \( V \), suppose we write \( S : \text{Kl}(T) \to \mathcal{D} \) for the inclusion of the free \( T \)-algebras. We claim that the composite \( S \circ F'' : \mathcal{C} \to \mathcal{D} \) has right adjoint \( G \circ K : \mathcal{D} \to \mathcal{C} \), and that the composite monad \( G \circ K \circ S \circ F'' \) is equal to the monad \( T \). The second claim is very easy: the composite \( G \circ K \circ S \) is equal to \( G'' \), so

\[
G \circ K \circ S \circ F'' = G'' \circ F'' = T.
\]
The first claim is also not difficult: since $F = K \circ S \circ F''$ and $V \circ K \cong \text{id}_C$, we have

$$V \circ F \cong V \circ K \circ S \circ F'' \cong S \circ F''.$$ 

Now $V$ is left adjoint to $K$ and $F$ left adjoint to $G$, so $S \circ F'' \cong V \circ F$ is left adjoint to $G \circ K$, proving our first claim. It follows that $(\mathcal{D}, S \circ F'', G \circ K)$ is a presentation of $T$.

All that remains to be proven is that $(\mathcal{D}, S \circ F'', G \circ K)$ is a homological presentation of $T$. By construction, $K$ is full and faithful, so by Lemma 2.9, $\mathcal{D}$ has coequalizers of all parallel pairs of the form $2.1.1$. So by Lemma 2.10, $(\mathcal{D}, S \circ F'', G \circ K)$ is homological. □

**Corollary 2.12.** If $T$ is coequalizable, the Eilenberg-Moore adjunction $(\mathcal{C}, F, G)$ of $T$ is a homological presentation of $T$. (And, consequently, the terminal homological presentation of $T$.)

**Corollary 2.13.** If $T$ is coequalizable, the large category $\text{HomPres}(T)$ of homological presentations of $T$ is partially-ordered, i.e., for any objects $\mathcal{A}, \mathcal{B}$ of $\text{HomPres}(T)$, there is at most one morphism $\mathcal{A} \rightarrow \mathcal{B}$.

**Corollary 2.14.** Suppose $T$ is coequalizable and $\mathcal{C}$ is Krull-Schmidt. Suppose the collection of isomorphism classes of indecomposable objects forms a set (not a proper class!), and suppose that set has cardinality $\kappa$. Then $\text{HomPres}(T)$ is equivalent to a partially-ordered set of cardinality no greater than $2^{\kappa}$.

**Proof.** The partially-ordered collection $\text{Loc}(T)$, which by Thm. 2.11 is equivalent to $\text{HomPres}(T)$, is contained in the collection of subcollections of the collection of finite formal sums of indecomposable objects. This collection is, in turn, contained in the collection of subcollections of the collection of not-necessarily-finite formal sums of indecomposable objects in which each indecomposable object appears only finitely many times. This last collection has cardinality $2^{\kappa}$. □

We greatly improve this cardinality bound in Cor. 2.18 under the assumption that $\mathcal{C}$ has a biproduct.

### 2.3. Coordination of the collection of homological presentations of a monad.

**Definition 2.15.** Recall that a category $\mathcal{C}$ is said to have a biproduct if it has finite products and finite coproducts and, for each finite family $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$, the canonical map $\prod_{i \in I} X_i \rightarrow \coprod_{i \in I} X_i$ is an isomorphism.

**Lemma 2.16.** Suppose $\mathcal{A}$ is a localization of a Krull-Schmidt category $\mathcal{C}$ with biproduct $\oplus$. If $X \cong Y \oplus Z$ in $\mathcal{C}$, then $X$ is in $\mathcal{A}$ if and only if both $Y$ and $Z$ are in $\mathcal{A}$.

**Proof.** We write $L : \mathcal{C} \rightarrow \mathcal{A}$ for the composite of the reflector functor $\mathcal{C} \rightarrow \mathcal{A}$ with the inclusion $\mathcal{A} \rightarrow \mathcal{C}$. Since $L$ is a composite of a left adjoint (the reflector functor) with a right adjoint (the inclusion functor), it preserves biproducts, since the biproduct is both the finite coproduct and the finite product. So $LX \cong LY \oplus LZ$, and if $Y, Z$ are in $\mathcal{A}$, then the unit maps $Y \rightarrow LY$ and $Z \rightarrow LZ$ are both isomorphisms. So $X \cong Y \oplus Z \cong LY \oplus LZ \cong LX$ is an isomorphism. So $X$ is in $\mathcal{A}$.

For the converse, suppose $X$ is in $\mathcal{A}$. Let $X \cong \oplus_{i=1}^{n} X_i$ be the decomposition of $X$ into indecomposables, given by the Krull-Schmidt property. Then the unit map

$$\oplus_{i=1}^{n} X_i \cong X \rightarrow LX \cong \oplus_{i=1}^{n} LX_i$$

is an isomorphism, and it is the sum of the component maps $X_i \rightarrow LX_i$, so we have some permutation

$$\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$$
with the property that $LX_i \cong X_{c(i)}$. However, since $A$ is a localization, $L$ is idempotent, so $\sigma \circ \sigma = \sigma$. So $\sigma$ must be the identity permutation. So each component map $X_i \to LX_i$ is an isomorphism. Hence if $X$ splits as a direct sum and $X$ is in $A$, each summand is also in $A$. □

Theorem 2.17. (Coordinatization.) Let $C$ be a category, $T$ a coequalizable monad on $C$. Suppose the Eilenberg-Moore category $C^T$ has a biproduct and is Krull-Schmidt. Write $\Gamma(C^T)$ for the collection of isomorphism classes of indecomposable objects in $C^T$. Then $H\text{Pres}(T)$ embeds by an order-preserving map into the collection of subsets of $\Gamma(C^T)$.

Proof. By Thm. 2.11 specifying an element of $H\text{Pres}(T)$ is equivalent to specifying a localization of $C^T$, hence is determined uniquely by which isomorphism classes of objects in $C^T$ are contained in the localization. But by Lemma 2.16, a localization of a Krull-Schmidt category with biproduct is determined uniquely by which indecomposables are contained in it. □

In other words: under the conditions of Thm. 2.17, a homological presentation of $T$ can be specified by specifying a subset of $\Gamma(C^T)$ (which, as we describe in the next section, is the underlying vertex set of an Auslander-Reiten quiver in many cases of interest). Since $C^T$ is often computable and understandable, Thm. 2.17—when it applies—gives a coordinatization of the collection of homological presentations of $T$, as desired.

Corollary 2.18. Suppose $T$ is coequalizable and $C^T$ is Krull-Schmidt and has a biproduct. Suppose the collection of isomorphism classes of indecomposable objects forms a set (not a proper class!), and suppose that set has cardinality $\kappa$. Then $H\text{Pres}(T)$ is equivalent to a partially-ordered set of cardinality no greater than $2^\kappa$.

3. A criterion for unique homological presentability of a monad.

3.1. Preliminary definitions. Some monads can only be homologically presented in a single way, i.e., $H\text{Pres}(T)$ is equivalent to a one-object category. Here is the relevant definition:

Definition 3.1. Suppose $T$ is a monad. If $H\text{Pres}(T)$ has only a single element, then we say that $T$ is uniquely homologically presentable.

We give a concrete algebraic class of examples (base-change monads associated to field extensions) of uniquely homologically presentable monads in Cor. 3.8.

Because we will need to make use of it, we state Beck’s monadicity theorem (see e.g. VI.7 of [7]):

Theorem 3.2. (Beck.) Suppose $C, D$ are categories, $G : D \to C$ a functor with a left adjoint $F$. Then the comparison functor $D \to C^{GF}$ is an equivalence of categories if and only if, whenever a parallel pair $f, g : X \to Y$ in $D$ is such that $Gf, Gg$ has a split coequalizer in $C$, each of the following conditions hold:

- $f, g$ has a coequalizer $\text{coeq}\{f, g\}$ in $D$.
- $G$ preserves the coequalizer of $f, g$, i.e., the natural map $\text{coeq}\{Gf, Gg\} \to G \text{coeq}\{f, g\}$ is an isomorphism.
- and $G$ reflects the coequalizer of $f, g$, i.e., if $Z$ is a cocone over the diagram $f, g : X \to Y$ such that $GZ$ is a coequalizer of $Gf, Gg$, then $Z$ is a coequalizer of $f, g$.

Here is a very classical definition:
Definition 3.3. When $G$ is a functor with left adjoint, we say that $G$ is monadic if $G$ satisfies the equivalent conditions of Thm. 3.2.

We offer a (to our knowledge, new) variant on this definition which will be essential to our criterion for unique homological presentability of a monad.

Definition 3.4. Suppose $\mathcal{C}, \mathcal{D}$ are categories, $G : \mathcal{D} \to \mathcal{C}$ a functor. We say that $G$ is absolutely monadic if $G$ has a left adjoint and, whenever a parallel pair $f, g : X \to Y$ in $\mathcal{D}$ is such that $Gf, Gg$ has a split coequalizer in $\mathcal{C}$, then:

- $f, g$ has a split coequalizer $\text{coeq}(f, g)$ in $\mathcal{D}$,
- $G$ preserves the coequalizer of $f, g$, i.e., the natural map $\text{coeq}(Gf, Gg) \to G\text{coeq}(f, g)$ is an isomorphism,
- and $G$ reflects the coequalizer of $f, g$, i.e., if $Z$ is a cocone over the diagram $f, g : X \to Y$ such that $GZ$ is a coequalizer of $Gf, Gg$, then $Z$ is a coequalizer of $f, g$.

Note that a functor that is absolutely monadic is also monadic, but the converse does not always hold.

3.2. The criterion for unique homological presentability. Now we present and prove the main result of this section.

First we will need a lemma. We suspect that this lemma is already well-known, but we do not know an already-existing reference in the literature.

Lemma 3.5. Suppose $\mathcal{D}, \mathcal{E}$ are categories, and $\mathcal{D} \xrightarrow{S} \mathcal{E}$ is a full, faithful functor with a left adjoint. Then $S$ is monadic.

Proof. Since $S$ is full and faithful, we regard it as inclusion of a subcategory $\mathcal{D}$ of $\mathcal{E}$. Then since $S$ has a left adjoint, $\mathcal{D}$ is a reflective subcategory of $\mathcal{E}$. We write $V$ for the left adjoint of $S$. Let $f, g : X \to Y$ be a pair of maps in $\mathcal{D}$ such that $Sf, Sg$ has a split coequalizer $Z$ in $\mathcal{E}$. Then we can apply $V$ together with the natural equivalence $VS \simeq \text{id}_\mathcal{D}$ to get that $VZ$ is a split coequalizer of $f, g$. Hence $S$ sends a cofork in $\mathcal{D}$ to a split coequalizer in $\mathcal{E}$ if and only if the cofork was already a split coequalizer in $\mathcal{D}$. So $S$ preserves coequalizers of all pairs in $\mathcal{D}$ with a $S$-split coequalizer, and since $S$ is faithful and injective on objects, it reflects isomorphisms; so $S$ is monadic. □

Theorem 3.6. Suppose $\mathcal{C}$ is a category, $T$ a coequalizable monad on $\mathcal{C}$. We write $F$ for the canonical functor $\mathcal{C} \to \mathcal{C}^T$ and $G$ for its right adjoint. If $G$ is absolutely monadic, then $T$ is uniquely homologically presentable.

Proof. Suppose $G$ is absolutely monadic, and suppose that $(\mathcal{D}, F', G')$ is a presentation of $T$. We have the comparison functor $K : \mathcal{D} \to \mathcal{C}^T$, and we have that $G' = G \circ K$. We are going to show that, if $(\mathcal{D}, F', G')$ is homological, then $K$ is an equivalence.

Suppose $f, g$ is a parallel pair in $\mathcal{D}$ such that $G'f, G'g$ has a split coequalizer in $\mathcal{C}$. Then $Kf, Kg$ is a parallel pair in $\mathcal{C}^T$ such that $G(Kf), G(Kg)$ has a split coequalizer in $\mathcal{C}$, and since $G$ is absolutely monadic, $Kf, Kg$ has a split coequalizer $Z$ such that $GZ$ is the given split coequalizer for $G'f, G'g$. But, by Lemma 3.5, $K$ is monadic; hence, by Thm. 3.2, $f, g$ has a coequalizer $W$ such that $KW$ is $Z$. Hence $f, g$ has a coequalizer in $\mathcal{D}$ and $G'$ preserves that coequalizer.

Now we check that $G'$ reflects appropriate coequalizers. Suppose

\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z
\end{array}
\]

(3.2.1)
is a cofork in $\mathcal{D}$ such that the cofork

\[(3.2.2) \quad G'X \underbrace{\longrightarrow}_{G'g} G'Y \longrightarrow G'Z\]

is a split coequalizer sequence in $\mathcal{C}$. Again using the fact that $G' = G \circ K$ and using that $G$ is absolutely monadic, we have that the cofork

\[(3.2.3) \quad KX \underbrace{\longrightarrow}_{Kg} KY \longrightarrow KZ\]

in $\mathcal{C}^T$ is a split coequalizer sequence; finally, by Lemma 3.5 and Thm. 3.2 $K$ reflects such coequalizers, so cofork $3.2.1$ is a coequalizer sequence in $\mathcal{D}$.

Hence $G'$ preserves and reflects coequalizers of all parallel pairs $f, g$ such that $Gf, Gg$ has a split coequalizer. Hence, by Thm. 3.2 $G'$ is monadic, and the comparison map $\mathcal{D} \to \mathcal{C}^{G'} = \mathcal{C}^T$ is an equivalence of categories.

Hence every homological presentation $\mathcal{D}$ of $T$ is equivalent to the entire Eilenberg-Moore category of $T$. Hence $\text{HPres}(T)$ consists of a single element, the Eilenberg-Moore presentation. $\square$

**Corollary 3.7.** Suppose $\mathcal{C}$ is an abelian category and $T$ a monad on $\mathcal{C}$ such that $\mathcal{C}^T$ is abelian and the canonical functor $G : \mathcal{C}^T \to \mathcal{C}$ is additive. Suppose that, if $X \to Y \to Z$ is a pair of maps in $\mathcal{C}^T$ such that $GX \to GY \to GZ \to 0$ is split exact in $\mathcal{C}$, then $X \to Y \to Z \to 0$ is split exact in $\mathcal{C}^T$. Then $T$ is uniquely homologically presentable.

**Proof.** The assumed condition on $G$ is precisely what absolute monadicity of $G$ means in the abelian setting. $\square$

**Corollary 3.8.** Suppose $L/K$ is a field extension and $T : \text{Mod}(K) \to \text{Mod}(K)$ the associated base change monad, i.e., $TM$ is the underlying $K$-module of $L \otimes_K M$. Then $T$ is uniquely homologically presentable.

4. A class of explicit examples.

4.1. **Review of some Auslander-Reiten theory.** We present some very basic constructions and ideas of Auslander-Reiten theory. A good reference is the book [2].

**Definition 4.1.** By an Artin algebra we mean an algebra $A$ over a commutative Artinian ring $k$ which is, in addition, finitely generated as a $k$-module. If the collection of isomorphism classes of finitely generated indecomposable left $A$-modules forms a finite set, then we say $A$ is of finite representation type.

Finitely-generated left (equivalently, right) modules over an Artin algebra are a good example of a Krull-Schmidt category.

**Definition 4.2.** Suppose $M, N$ are indecomposable modules over a ring $R$, and $f : M \to N$ a morphism. We say that $f$ is irreducible if $f$ is not an isomorphism and, whenever $f$ factors as a composite $g \circ h$, either $h$ is split monic or $g$ is split epic.
Definition 4.3. Suppose $A$ is an algebra over an algebraically closed field $k$. By the Auslander-Reiten quiver of $A$ we mean the quiver whose vertex set is the set of isomorphism classes of indecomposable left $A$-modules, and which has one directed edge $M \to N$ for each irreducible map from $M$ to $N$. We write $\Gamma(A)$ for this quiver.

When $k$ is not algebraically closed, one still can define the Auslander-Reiten quiver of $A$, but it is slightly more complicated (the edges get labels); see e.g. chapter VII of [2].

Theorem 4.4. (Auslander, Ringel-Tachikawa.) Let $A$ be an Artin algebra of finite representation type. Then every left $A$-module is a direct sum of finitely generated indecomposable left $A$-modules. Furthermore, every nonzero non-isomorphism $f : X \to Y$ between indecomposable modules is a $k$-linear combination of composites of irreducible maps between indecomposable modules.

The first claim is in [1] and [8]. See Prop. V.7.4 of [2] for the second claim.

4.2. Homological presentations of the base-change monad of an Artin algebra of finite representation type. When $x$ is a vertex of $\Gamma(A)$, we will write $M_x$ for the associated indecomposable $A$-module.

Definition 4.5. Let $k$ be an algebraically closed field, $A$ an Artin algebra over $k$ of finite representation type. By a distinguished subset of $\Gamma(A)$ we will mean a subset $U$ of the set of vertices of $\Gamma(A)$ satisfying the following properties:

- Every vertex of $\Gamma(A)$ corresponding to a projective $A$-module is contained in $U$.
- For each vertex $x$ of $\Gamma(A)$ not contained in $U$, there exists an $A$-module $L_x$ and an $A$-module map $M_x \to L_x$ such that:
  - $L_x$ is a direct sum of indecomposables modules associated to vertices of $U$,
  - for every vertex $y \in U$ and every map $f : M_x \to M_y$, there exists a unique factorization of $f$ through the map $M_x \to L_x$.

Remark 4.6. The condition in Def. 4.5 that every map $M_x \to M_y$ factors through $M_x \to L_x$ does not seem to be expressible in terms of the Auslander-Reiten quiver $\Gamma(A)$ in any simple way. While one would like this condition to be equivalent to knowing that the map $M_x \to M_y$ associated to each chain of edges

$$(4.2.1) \quad x \to x_1 \to \cdots \to x_n \to y$$

in $\Gamma(A)$ factors uniquely through $M_x \to L_x$, the problem is uniqueness: while any map $M_x \to M_y$ can be written as a $k$-linear combination of maps $M_x \to M_y$ associated to chains of edges of the form $4.2.1$, it is not generally true that any map $M_x \to M_y$ can be written uniquely as a sum of maps associated to chains of edges.

This means that, if $U$ is a subset of the vertex set of $\Gamma(A)$, checking that $U$ is distinguished takes some work. We do this very explicitly in some special cases in Thm. 4.9.

Proposition 4.7. Let $k$ be an algebraically closed field, $A$ an Artin algebra over $k$ of finite representation type. Let $T$ be the associated base-change monad $T : \text{fgMod}(k) \to \text{fgMod}(k)$ on the category of finite-dimensional $k$-vector spaces, i.e., $TM$ is the underlying $k$-vector space of $A \otimes_k M$. Then $\text{HPres}(T)$ is equivalent to the set of distinguished subsets of $\Gamma(A)$, ordered by inclusion.

Proof. First, it is elementary that $C^T$ is the category of finitely generated (left) $A$-modules. By Thm. 2.11 the homological presentations of $T$ are equivalent to localizations of $C^T$ which present $T$. By Lemma 2.16 and the fact that $C^T$ is Krull-Schmidt, a localization of
\(C^T\) is uniquely determined by specifying which indecomposable objects of \(C^T\) are in the localization, i.e., a subset of the set of vertices of \(\Gamma(A)\). For the localization to present \(T\) is equivalent to every indecomposable summand of the free \(T\)-algebras, i.e., free \(A\)-modules, to be in the localization; but summands of free \(A\)-modules are precisely the projective \(A\)-modules. So \(\text{HPres}(T)\) embeds into the set of subsets of \(\Gamma(A)\) containing all the vertices whose associated indecomposables \(A\)-modules are projective.

Suppose \(U\) is a subset of the vertex set of \(\Gamma(A)\), and \(\mathcal{B}_U \subseteq C^T\) is the full subcategory generated by all direct sums of modules \(M_i\) with \(x \in U\). We write \(G\) for the inclusion functor \(G : \mathcal{B}_U \to C^T\). Then Kan’s necessary and sufficient condition for existence of a left adjoint to \(G\) (see section X.3 of [7]) is that, for each object \(X\) of \(C^T\), the category of pairs \((Y, f)\), where \(Y\) is an object of \(\mathcal{B}_U\) and \(f : X \to Y\) is a map in \(C^T\), has an initial object. But this is precisely the factorization condition in Def. 4.5. So a subset \(U\) of the vertex set of \(\Gamma(A)\) which contains all the vertices associated to projective indecomposable modules is distinguished if and only if the subcategory \(\mathcal{B}_U \subseteq C^T\) it generates is reflective.

**Corollary 4.8.** Let \(k\) be an algebraically closed field, \(A\) an Artin algebra over \(k\) of finite representation type. Let \(T\) be the associated base-change monad \(T : \text{fgMod}(k) \to \text{fgMod}(k)\) on the category of finite dimensional \(k\)-vector spaces, i.e., \(T\) is the underlying \(k\)-vector space of \(A \otimes_k M\). Then \(\text{HPres}(T)\) is skeletally finite, i.e., \(T\) has (up to equivalence) only finitely many homological presentations.

We now offer an example computation:

**Theorem 4.9.** Let \(k\) be an algebraically closed field and let \(A\) be the \(k\)-algebra \(A = k[x]/x^n\) for \(n\) a positive integer. Let \(T\) be the associated base-change monad \(T : \text{fgMod}(k) \to \text{fgMod}(k)\) on the category of finite-dimensional \(k\)-vector spaces, i.e., \(T\) is the underlying \(k\)-vector space of \(A \otimes_k M\). Then \(\Gamma(A)\) is of the form

\[
\begin{array}{c}
\cdots \xrightarrow{x_{n-2}} x_n \xrightarrow{x_{n-1}} x_{n-2} \xrightarrow{x_{n-3}} \cdots \\
\end{array}
\]

with \(n\) vertices, and \(\text{HPres}(T)\) is equivalent to the partially-ordered set of subsets of the vertex set of \(\Gamma(A)\) of the form

\[
U_j = \{x_i : i \geq j\}
\]

for \(j = 0, \ldots, n - 1\).

So the category \(\text{HPres}(T)\) of homological presentations of \(T\) is equivalent to the partially-ordered set of the natural numbers less than \(n\). In particular, \(T\) has (up to equivalence) exactly \(n\) homological presentations.

**Proof.** We write \(M_i\) for the \(A\)-module \(k[x]/x^{i+1}\). Then the usual module theory for finitely-generated modules over principal ideal domains gives us that \(M_0, \ldots, M_{n-1}\) is a full set of (isomorphism classes of) indecomposable finitely generated \(A\)-modules. We will use the symbols \(x_0, \ldots, x_{n-1}\) as labels for the vertices of \(\Gamma(A)\) corresponding to the modules \(M_0, \ldots, M_{n-1}\), respectively. The almost-split exact sequences of finitely generated \(A\)-modules are all of the form

\[
0 \to M_i \to M_{i-1} \oplus M_{i+1} \to M_i \to 0
\]

and since the irreducible maps are the components of the injections and surjections in the almost-split exact sequences, the Auslander-Reiten quiver \(\Gamma(A)\) is as in [4.2.2]. The maps of \(A\)-modules \(M_i \to M_{i+1}\) corresponding to the edges are the obvious inclusions and the maps of \(A\)-modules \(M_{i+1} \to M_i\) are the obvious surjections. The only vertex corresponding to a projective \(A\)-module is \(x_{n-1}\), since it corresponds to the free \(A\)-module \(M_{n-1}\). (This
example is, so far, an elementary and standard exercise in Auslander-Reiten theory. The remainder of the theorem and its proof, however, is new, to the best of our knowledge."

Suppose \( U \) is a subset of the vertex set of \( \Gamma(A) \) which contains \( x_{n-1} \). Then \( U \) is specified by specifying a subset of the set \( S \) of nonnegative integers less than \( n - 1 \), i.e., specifying the numbers \( i \) such that \( x_i \in U \). We claim that \( U \) is distinguished if and only if, whenever \( x_i \in U \), then \( x_{i+1} \in U \) as well. In other words, we claim that \( U \) is distinguished if and only if \( U \) consists of all vertices \( \{x_i : i \geq j\} \subseteq \Gamma(A) \) for some \( j \). Clearly this claim, once we have proven it, implies the theorem as stated.

To prove our claim, first we assume that \( U_j = \{x_i : i \geq j\} \subseteq \Gamma(A) \) for some \( j \). We show that \( U_j \) is distinguished as follows: for each vertex \( x_i \) with \( i < j \), let \( L_i \) denote the \( A \)-module \( M_j \), and let \( M_i \rightarrow L_i \) be the composite of the obvious inclusion maps, i.e., the shortest path from \( x_i \) to \( x_j \) in \( \Gamma(A) \). Then every \( A \)-module map \( M_i \rightarrow M_j \), for \( \iota \geq j \), factors uniquely through \( M_j = L_i \). So \( L_i \) is the desired left adjoint construction on \( M_i \) in Def. 4.5.

So \( U_j \) is distinguished.

For the converse, we must show that every distinguished subset of \( \Gamma(A) \) is of the form \( U_j \) for some \( j \). Suppose \( U \subseteq \Gamma(A) \) is not of the form \( U_j \) for any \( j \). Then there must exist natural numbers \( \ell, m \) such that \( \ell < m < n - 1 \) and such that \( x_\ell \in U \) but \( x_m \notin U \). Let \( m' \) be the least integer such that \( m' > m \) and \( x_{m'} \in U \). If \( U \) were distinguished, then there would be an \( A \)-module \( L_m \) and an \( A \)-module map \( f : M_m \rightarrow L_m \) satisfying the left adjoint condition of Def. 4.5. We now aim to show that this is impossible. If \( L_m \) and \( f \) existed with these properties, then in particular, the obvious surjective \( A \)-module map \( s : M_m \rightarrow M_r \), i.e., the shortest path from \( x_m \) to \( x_r \) in \( \Gamma(A) \), would have to factor uniquely through \( f \). Let \( H \) denote the direct sum of all summands of \( L_m \) of the form \( M_{r'} \) for \( \ell' < \ell \). Then the unique factor map \( \tilde{s} : L_m \rightarrow M_r \) such that \( \tilde{s} \circ f = s \) must be zero when restricted to the summand \( H \subseteq L_m \), since the image of \( 1 \in M_m \) in \( M_r \) is an \( x^m \)-torsion element and not an \( x^{m'} \)-torsion element, and there are no such elements in \( H \). So the inclusion \( H \subseteq L_m \) cannot be an isomorphism, i.e., there exists a summand of \( L_m \) of the form \( M_r \) or of the form \( M_{r'} \) with \( \ell'' \geq m' \). Furthermore, the unique factor map \( \tilde{s} : L_m \rightarrow M_r \) then factors through the quotient map \( q : L_m \rightarrow L_m/H \). We will write \( \tilde{s} : L_m/H \rightarrow M_r \) for this new factor map.

Now the element \( 1 \in M_m \) must go to an \( x^{m+1} \)-torsion element of \( L_m/H \) which is not divisible by \( x \), in order to have \( \tilde{s} \circ q \circ f = s \). But if \( L_m/H \) has no summand of the form \( M_r \), then every summand of \( L_m/H \) is of the form \( M_{r''} \) for \( \ell'' > m' > m \), and hence \( L_m/H \) has no \( x^{m+1} \)-torsion elements which are not divisible by \( x \). So \( L_m/H \) must have \( M_r \) as a summand. So \( L_m \cong H \oplus (L_m/H) \) must have \( M_r \) as a summand.

Now we turn to the obvious inclusion \( i : M_m \rightarrow M_{m'} \), i.e., the shortest path from \( x_m \) to \( x_{m'} \) in \( \Gamma(A) \). The map \( i \) must also factor uniquely through the putative map \( f : M_m \rightarrow L_m \). Hence \( L_m \) must have \( M_{m'} \) as a summand, with the composite map \( M_m \rightarrow L_m \rightarrow M_{m'} \) being \( i \). Hence \( L_m \) has \( M_r \oplus M_{m'} \) as a summand, with the component maps \( M_m \rightarrow M_r \oplus M_{m'} \) of \( f \) being \( s \) on the first factor and \( i \) on the second factor. We will write \( (s, i) : M_m \rightarrow M_r \oplus M_{m'} \) for this map. But \( i \) factors non-uniquely through \( (s, i) \): we have the map \( p_2 : M_r \oplus M_{m'} \rightarrow M_{m'} \) which is projection to the second factor, and we have the map \( g : M_r \oplus M_{m'} \rightarrow M_{m'} \) given by \( g(1, 0) = x^m \) and \( g(0, 1) = 1 - x^m \). We have \( p_2 \circ (s, i) = i \), but we also have \( g \circ (s, i) = i \). So \( i \) cannot factor uniquely through \( f \).

This proves our claim that every distinguished subset of \( \Gamma(A) \) is of the form \( U_j \) for some \( j \). \( \square \)
References

[1] Maurice Auslander. Representation theory of Artin algebras. I, II. *Comm. Algebra*, 1:177–268; ibid. 1 (1974), 269–310, 1974.

[2] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1995 original.

[3] Jon Beck. Triples, algebras, and cohomology. *Reprints in Theory and Applications of Categories*, 2:1–59, 2003.

[4] Jean Bénabou and Jacques Roubaud. Monades et descente. *C. R. Acad. Sci. Paris Sér. A-B*, 270:A96–A98, 1970.

[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[6] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[7] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[8] Claus Michael Ringel and Hiroyuki Tachikawa. QF 3 rings. *J. Reine Angew. Math.*, 272:49–72, 1974.