Sufficient conditions for the invertibility of adapted perturbations of identity on the Wiener space

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Abstract: Let \((W, H, \mu)\) be the classical Wiener space. Assume that \(U = I_W + u\) is an adapted perturbation of identity, i.e., \(u : W \to H\) is adapted to the canonical filtration of \(W\). We give some sufficient analytic conditions on \(u\) which imply the invertibility of the map \(U\). In particular it is shown that if \(u \in \mathcal{D}_{p,1}(H)\) is adapted and if \(\exp(\frac{1}{2}\|\nabla u\|^2_2 - \delta u) \in L^q(\mu)\), where \(p^{-1} + q^{-1} = 1\), then \(I_W + u\) is almost surely invertible. With the help of this result it is shown that if \(\nabla u \in L^\infty(\mu, H \otimes H)\), then the Girsanov exponential of \(u\) times the Wiener measure satisfies the logarithmic Sobolev inequality and this implies the invertibility of \(U = I_W + u\). As a consequence, if there exists an integer \(k \geq 1\) such that \(\|\nabla^k u\|_{H \otimes (k+1)} \in L^\infty(\mu)\), then \(I_W + u\) is again almost surely invertible under the almost sure continuity hypothesis of \(t \to \nabla^i \dot{u}_t\) for \(i \leq k - 1\).

1 Introduction

This paper is devoted to the search of sufficient conditions for the invertibility of a certain class of mappings on the Wiener space. This class consists of the mappings of the form of perturbation of identity, where the perturbation part is a mapping which is absolutely continuous with respect to the Lebesgue measure and the corresponding density is adapted and almost surely square integrable. To make the things more precise, let \(W = C_0([0, 1])\) be the Banach space of continuous functions on \([0, 1]\), with its Borel sigma field denoted by \(\mathcal{F}\). We denote by \(H\) the Cameron-Martin space, namely the space of absolutely continuous functions on \([0, 1]\) with square integrable Lebesgue density:

\[
H = \left\{ h \in W : h(t) = \int_0^t \dot{h}(s) ds, \|h\|^2_H = \int_0^1 |\dot{h}(s)|^2 ds < \infty \right\}.
\]

\(\mu\) denotes the classical Wiener measure on \((W, \mathcal{F})\), \((\mathcal{F}_t, t \in [0, 1])\) is the filtration generated by the paths of the Wiener process \((t, w) \to W_t(w)\), where \(W_t(w)\) is defined as \(w(t)\) for \(w \in W\) and \(t \in [0, 1]\). Assume that \(u : W \to H\) is a measurable mapping, define \(U : W \to W\) as

\[
U = I_W + u.
\]
\( U \) can be represented as
\[
U_t(w) = W_t(w) + \int_0^t \dot{u}_s(w) ds,
\]
(1.1)

using the isometry between \( H \) and \( L^2([0,1]) \). We assume that \( \dot{u} \) is adapted to the filtration \( (\mathcal{F}_t, t \in [0,1]) \). For simplicity we consider in this paper the Banach space of continuous functions on \([0,1]\), taking values in \( \mathbb{R} \); the results, however, go over directly to the infinite dimensional case, including the Wiener space corresponding to the cylindrical Wiener process based on a Hilbert space.

To illustrate a situation where the addressed problem comes up, consider the question of the absolute continuity of the measure \( U\mu \), i.e., the image of \( \mu \) under \( U \) and the calculation of the corresponding Radon-Nikodym derivative in case of absolute continuity. The celebrated Girsanov theorem (cf.\cite{6,11}) yields the change of variables formula, i.e. setting
\[
\rho_U(w) = \exp \left( -\int_0^1 \dot{u}_s dw_s - \frac{1}{2} \int_0^1 |\dot{u}_s|^2 ds \right)
\]
and assuming that \( E[\rho_U] = 1 \), then, for smooth \( f \), it holds true that
\[
E[f \circ U \rho_U] = E[f].
\]

Hence the image measure \( U\mu \) is absolutely continuous with respect to \( \mu \). Let \( \lambda \) be the corresponding Radon-Nikodym derivative:
\[
E[f \circ U] = E[f \lambda],
\]
then under “suitable conditions”
\[
\lambda = \frac{1}{\rho_U \circ U^{-1}}
\]
(1.2)
where \( U^{-1} \) is the inverse to \( U \) (cf. e.g. Section 1.3 of \cite{10}). Therefore the invertibility of \( U \) plays a fundamental role in the evaluation of the Radon-Nikodym derivative \( \lambda \). This situation is particularly important if we want to write a probability density as the Radon-Nikodym derivative of the image of the Wiener measure under a mapping of the form of perturbation of identity; we refer the reader to \cite{5} for a quick introduction to this problem.

The second and somehow related question concerns the question of the existence and uniqueness of strong solutions of the stochastic differential equations of the following type:
\[
\begin{align*}
    dV_t &= -\dot{u}_t \circ V + dW_t \\
    V_0 &= 0,
\end{align*}
\]
(1.3)
where \( \dot{u} \) is described above. Note here the fact that, though adapted, the drift coefficient \( \dot{u}_t(w) \) may depend on the whole history of the Brownian path \((w(s), s \in [0, t])\). If one can show that the map defined by \( U = I_W + u \), where \( u \) is the primitive of \( \dot{u} \), has a left inverse \( V \), then this inverse map will be the unique solution of the equation \( (1.3) \). In fact, under the hypothesis \( E[\rho_U] = 1 \) and \( \dot{u} \in L^2(dt \times d\mu) \), we prove in Theorem 3 that, if \( U = I_W + u \) has a left inverse \( V \), then the image of \( \mu \) under \( V \) is equivalent to \( \mu \), \( V \) is also right inverse and it is of the form \( V = I_W + v \) with \( v : W \to H \), \( \dot{v} \) is adapted and finally that \( V \) is the unique strong solution of the equation \( (1.3) \). This result is quite useful and seems to be new.

As it is shown by the well-known counter example given by Tsirelson (cf. \[8\], p. 181), the usual hypothesis of integrability on \( u \) does not imply the existence of strong solutions, hence the invertibility of \( U \) either. A well-known condition for the existence of such an inverse is the case where the drift coefficient is Lipschitz continuous in the Cameron-Martin space direction. Namely

\[
\sup_{s \leq t} |\dot{u}_s(w + h) - \dot{u}_s(w + k)| \leq K \sup_{s \leq t} |h(s) - k(s)|,
\]

\( \mu \)-a.s., for any \( h, k \in H \), \( t \in [0, 1] \), where \( K \) is a constant. In this case, using the usual fixed point techniques, one can prove that the stochastic differential equation has a unique adapted solution. Then it is clear that \( V \mu \) is equivalent to \( \mu \) and that \( U \circ V = V \circ U = I_W \) almost surely. In the sequel, between other things we shall also surpass this frame.

Let us summarize the contents of the paper: the basic notions of functional analysis on the Wiener space and the stochastic calculus of variations (the Malliavin calculus) are reviewed in Section 2. Section 3 presents basic results on the invertibility of \( U = I + u \) with \( u : W \to H \) adapted\(^1\). The main results are obtained by the regularization of the drift with the Ornstein-Uhlenbeck semigroup. In fact, let \((P_\tau, \tau \geq 0)\) denote the Ornstein-Uhlenbeck semigroup (cf. the formula \((2.0)\)) and set \( e^{-\tau P} u = u_\tau \). It is shown that under reasonable assumptions on \( u \), the map \( U_\tau = I + u_\tau \) is invertible. Its inverse is of the form \( V_\tau = I_W + v_\tau \), where \( v_\tau \) is \( H \)-valued and adapted. Besides the following identities are satisfied almost everywhere:

\[
v_\tau = -u_\tau \circ V_\tau \tag{1.4}
\]

and

\[
u_\tau = -v_\tau \circ U_\tau \ . \tag{1.5}
\]

If we can show that \( v = \lim_{\tau \to 0} v_\tau \) exists in probability and if it satisfies the relations \((1.4)\) and \((1.5)\) where \( u_\tau, U_\tau \) and \( v_\tau, V_\tau \) are replaced respectively by \( u, U \) and \( v, V \), then the invertibility of \( I + u \) follows. This program, which is realized in Section 3

\(^1\)For practical reasons, we call \( u \) adapted whenever its Lebesgue density \( \dot{u} \), called sometimes the drift, is adapted.
is not so evident, in fact we need the Carleman inequality (cf. [1, 3]) to find useful sufficient conditions to show the existence of this limit. The basic result of this section proves that if $u \in D_{p,1}(H)$, $p > 1$, is adapted and if $\exp(\frac{1}{2}\|\nabla u\|^2_2 - \delta u) \in L^q(\mu)$, where $p^{-1} + q^{-1} = 1$, then $U = I_W + u$ is almost surely invertible. As an application and a further tool also, we prove a logarithmic Sobolev inequality for the measures of the type $dv = \rho_v \, d\mu$, where $U = I_W + u$ and the Sobolev derivative of $u$ is essentially bounded as a Hilbert-Schmidt operator. Using this inequality, we show that such a $U$ is almost surely invertible. Although this is not the most general sufficient condition for the invertibility that we find, as hypothesis it is strictly weaker than the Lipschitz assumption, which is generally used for the construction of the inverse mapping via the stochastic differential equations as illustrated with the formulae (1.3) (cf. Remark 4 for the details).

Section 4 extends these results using some localization techniques. As a corollary we prove that, for any $k \geq 1$, if the $k$-th order Sobolev derivative $\nabla^k u$ of $u$ is essentially bounded as a Hilbert-Schmidt tensor and if the Sobolev derivatives upto the order $k - 1$ of the process $t \to \dot{u}(t)$ are almost surely continuous, then $U$ is almost surely invertible. For the case $k = 1$ this continuity hypothesis is avoided using the logarithmic Sobolev inequality as explained in Section 3.

Finally we underline the fact that the results of this paper can be extended to the abstract Wiener spaces where the notion of adaptedness can be defined with respect to any continuous resolution of identity of the associated Cameron-Martin space as indicated in [15] or sections 2.6 and 3.6 of [16].

A preliminary version of these results have been announced in the note [17], however the contents of this paper are considerably stronger and more general. In particular, using a convex interpolation method we succeed to diminish the Sobolev differentiability requirements about the shift, to prove a new logarithmic Sobolev inequality and several extensions as explained in the last section of the paper.

## 2 Preliminaries

Let $W = C_0([0,1])$ be the Banach space of continuous functions on $[0,1]$, with its Borel sigma field denoted by $\mathcal{F}$. We denote by $H$ the Cameron-Martin space, namely the space of absolutely continuous functions on $[0,1]$ with square integrable Lebesgue density:

$$H = \left\{ h \in W : h(t) = \int_0^t \dot{h}(s) \, ds, \ |h|^2_H = \int_0^1 |\dot{h}(s)|^2 \, ds < \infty \right\}.$$ 

$\mu$ denotes the classical Wiener measure on $(W, \mathcal{F})$, $(\mathcal{F}_t, t \in [0,1])$ is the filtration generated by the paths of the Wiener process $(t, w) \to W_t(w)$, where $W_t(w)$ is defined as $w(t)$ for $w \in W$ and $t \in [0,1]$. We shall recall briefly some well-known functional analytic tools on the Wiener space, we refer the reader to [10, 11, 12] or to [13] for
further details: \((P_\tau, \tau \in \mathbb{R}_+)\) denotes the semi-group of Ornstein-Uhlenbeck on \(W\), defined as
\[
P_\tau f(w) = \int_W f(e^{-\tau} w + \sqrt{1 - e^{-2\tau} y}) \mu(dy)
\] (2.6)

Let us recall that \(P_\tau = e^{-\tau L}\), where \(L\) is the number operator. We denote by \(\nabla\) the Sobolev derivative which is the extension (with respect to the Wiener measure) of the Fréchet derivative in the Cameron-Martin space direction. The iterates of \(\nabla\) are defined similarly. Note that, if \(f\) is real valued, then \(\nabla f\) is a vector and if \(u\) is an \(H\)-valued map, then \(\nabla u\) is a Hilbert-Schmidt operator (on \(H\)) valued map whenever defined. If \(Z\) is a separable Hilbert space and if \(p > 1, k \in \mathbb{R}\), we denote by \(I^D_{p,k}(Z)\) the \(\mu\)-equivalence classes of \(Z\)-valued measurable mappings \(\xi\), defined on \(W\) such that
\[
(I + \mathcal{L})^{k/2} \xi \in L^p(\mu, Z)
\] (2.7)
becomes a Banach space. From the Meyer inequalities, we know that the norm defined by
\[
\sum_{i=0}^k \|\nabla^i \xi\|_{L^p(\mu, Z \otimes H^\otimes i)} , k \in \mathbb{N}
\]
is equivalent to the norm \(\|\xi\|_{p,k}\) defined by (2.7). We denote by \(\delta\) the adjoint of \(\nabla\) under \(\mu\) and recall that, whenever \(u \in I^D_{p,0}(H)\) for some \(p > 1\) is adapted\(^2\), then \(\delta u\) is equal to the Itô integral of the Lebesgue density of \(u\):
\[
\delta u = \int_0^1 \dot{u}_s dW_s.
\]

Let \(X\) be a separable Hilbert space and let \(f : W \to X\) be a measurable map. We say that \(f\) is an \(H - C\)-map if \(f\) has a modification (denoted again as \(f\)) such that the mapping \(h \to f(w + h)\) is continuous for \(\mu\)-almost all \(w \in W\). Similarly, we say that \(f\) is \(H - C^k\), \(k \geq 1\) or that it is \(H\)-real analytic, if \(h \to f(w + h)\) is \(k\)-times differentiable or real analytic \(\mu\)-almost surely. In the sequel, we shall use the same notation for the \(H\)-derivative and for the Sobolev derivative since the latter is the \(L^p\)-extension of the former. Note that the set \(A = \{w \in W : h \to f(w + h) \in C^k(H)\}\) is \(H\)-invariant, i.e., \(A + H \subset A\), hence \(A^c\) has zero capacity as soon as \(\mu(A) > 0\) (cf. \cite{13}). The following result is well-known (cf. \cite{16}, Lemma 3.3.2):

**Lemma 1** Assume that \(f \in L^p(\mu, X)\), where \(X\) is a separable Hilbert space. Then, for any \(\tau > 0\), \(P_\tau f\) has a modification \(f_\tau\), such that \(h \to f_\tau(w + h)\) is almost surely analytic on \(H\), in other words \(P_\tau f\) is \(H\)-analytic. In particular it is \(H - C^\infty\).

Another important result that we shall need is the following one (cf. \cite{16}, Theorem 3.5.3 where a more general case is treated and Theorem 4.4.1 in the \(H - C^1\)-case):

\(^2\)In the sequel the adapted elements of \(I^D_{p,k}(H)\) will be denoted by \(I^D_{p,k}(H)\).
**Theorem 1** Assume that $\xi : W \to H$ is an $H - C^1$-map and denote by $T$ the map $T = I_W + \xi$. Then

1. The set $T^{-1}\{w\}$ is countable $\mu$-almost surely. Let $N(w)$ be its cardinal.

2. For any $f, g \in C_b(W)$, we have the change of variables formula:

$$E[f \circ T \rho_T] = E\left[f \sum_{y \in T^{-1}\{w\}} g(y)\right],$$

in particular

$$E[f \circ T \rho_T] = E[f N],$$

where

$$\rho_T = \det_2(I_H + \nabla \xi) \exp\left[ -\delta \xi - \frac{1}{2} |\xi|^2_H \right]$$

and $\det_2(I_H + \nabla \xi)$ denotes the modified Carleman-Fredholm determinant.

**Remark 1** If $A$ is a nuclear operator on a separable Hilbert space, then $\det_2(I_H + A)$ is defined as

$$\det_2(I_H + A) = \prod_{i=1}^\infty (1 + \lambda_i)e^{-\lambda_i} = \det(I_H + A)e^{-\text{trace } A},$$

where $(\lambda_i)$ denotes the spectrum of $A$ and each eigenvalue is counted with respect to its multiplicity. Afterwards, one can show that $A \to \det_2(I_H + A)$ extends continuously (even analytically) to the space of Hilbert-Schmidt operators, cf. [3]. If $A$ is a quasi-nilpotent operator, then by definition the spectrum of $A$ is equal to the singleton $\{0\}$, hence, in this case we always have $\det_2(I_H + A) = 1$.

**Remark 2** It is well-known (cf. [9, 16]) given an $H$-valued $H - C^1$ map, there exists a measurable partition $(M_n, n \geq 1)$ of its set of non-degeneracy $M$, i.e., the set on which the $\det_2(I_H + \nabla u)$ is non-zero, such that on each $M_n$, $I_W + u$ is equal to some invertible mapping of the form of perturbation of identity. This result implies that the notion of multiplicity $N$ is well-defined and it is equal to the

$$N(w, M) = \sum_{y \in T^{-1}\{w\}\cap M} 1_y.$$ 

Besides, using the Sard Lemma on Wiener space (cf. [16], Proposition 4.4.1), one can show that

$$N(w, M) = N(w, W)$$

almost surely. This result is extended even to $H - C^1_{\text{loc}}$-maps (cf. Definition [7]) as explained in Chapter IV of [16].
**Theorem 2** Let $\xi$ be as in Theorem 1 with $\xi \in \mathbb{D}_{p,1}(H)$ for some $p > 1$. If the Lebesgue density of $\xi$, called drift and denoted by $\dot{\xi}$ is adapted to the filtration of the canonical Wiener process, then $\rho_T$ reduces to the usual exponential martingale:

$$
\rho_T = \exp \left[ -\delta \xi - \frac{1}{2} \| \xi \|^2_H \right]
= \exp \left[ -\int_0^1 \xi_s \cdot dW_s - \frac{1}{2} \int_0^1 |\dot{\xi}_s|^2 ds \right].
$$

In this case we have always $N(w) \in \{0,1\}$ almost surely and $N = 1$ a.s. if $E[\rho_T] = 1$.

**Proof:** The proof follows from the fact that $\delta \xi$ coincides with the Itô integral of $\dot{\xi}$ if the latter is adapted. In this case, we always have, from the Fatou lemma $E[\rho_T] \leq 1$ and if $E[\rho_T] = 1$, then it follows from Theorem 1 and from the Girsanov theorem that $N = 1$ almost surely.

A simple, nevertheless important corollary of Theorems 1 and 2 is

**Corollary 1** Assume that $\xi$ is adapted and $H - C^1$. Assume moreover that $E[\rho_T] = 1$. Then, there exists a map $S$ of the form $S = I_W + \eta$ with $\eta : W \to H$ adapted, such that

$$
\mu \left( \{ w \in W : S \circ T(w) = T \circ S(w) = w \} \right) = 1.
$$

In other words $T$ is almost surely invertible.

**Proof:** Let $\tilde{W} = \{ w \in W : N(w) = 1 \}$, it follows from Theorem 1 and Theorem 2 and from the hypothesis about $\rho(-\delta \xi)$ that $\mu(\tilde{W}) = 1$. Consequently, for any $w \in \tilde{W}$, there exists a unique $S(w) \in W$ such that $T(S(w)) = w$. Let us define $S$ on $W^c$ by $I_W$. Then, for any $A \in \mathcal{F}$, we have $S^{-1}(A) = S^{-1}(A) \cap \tilde{W}$ $\mu$-almost surely. Moreover, by the very definition of $S$, we have $S^{-1}(A) \cap \tilde{W} = T(A) \cap \tilde{W}$. Since $\tilde{W}$ is a Borel set, to show the measurability of $S$ with respect to the completion of $\mathcal{F}$, it suffices to show that $T(A)$ belongs to the same sigma-algebra and this follows from Theorem 4.2.1 of [10] and this settles the measurability of $S$. Moreover, from Theorem 1 for any $f, g \in C_b(W)$, we have on the one hand

$$
E[\rho(-\delta \xi) g] = E \left[ \sum_{y \in T^{-1}(w)} g(y) \right]
= E[g \circ S],
$$

In fact $T(A)$ is a Souslin set as one can show by the help of the measurable selection theorem, hence an element of the universal completion of $\mathcal{F}$, cf. [2].
hence $S(\mu)$ is equivalent to $\mu$ and on the other hand

$$E[f \circ S \circ T \rho(-\delta\xi) g] = E \left[ f \circ S \sum_{y \in T^{-1}(\{w\})} g(y) \right]$$

$$= E[f \circ S g \circ S]$$

$$= E[f g \rho(-\delta\xi)].$$

Therefore $S \circ T = I_W$ almost surely. In particular $S$ is of the form $I_W + \eta$ and $\eta$ is an adapted and $H$-valued mapping.

If $u \in D_{2,0}(H)$ is adapted and satisfies $E[\rho(-\delta u)] = 1$, but without $H-C^1$-hypothesis, in case it has a left inverse, then this left inverse is also a right inverse and it is characterized by the following Theorem:

**Theorem 3** Assume that $U = I_W + u$, $u : W \to H$, $u(t) = \int_0^t \dot{u}_s ds$, for any $t \in [0,1]$ and that $\dot{u}$ is adapted to the Brownian filtration $(\mathcal{F}_t, t \in [0,1])$. Assume further that $u \in L^2(\mu, H)$, $E[\rho(-\delta u)] = 1$. Suppose that there exists some $V : W \to W$ such that $V \circ U = I_W$ a.s. Then

1. $V\mu$ is equivalent to $\mu$ and $V$ is also a left inverse, i.e.,

$$U \circ V = I_W$$

$\mu$-almost surely. In other words $U$ is almost surely invertible and its inverse is $V$.

2. $V$ is of the form of a perturbation of identity, i.e., $V = I_W + v$ and $v : W \to H$.

3. $\dot{v}$ is adapted to the filtration $(\mathcal{F}_t, t \in [0,1])$.

4. In particular, the process $(V(t), t \in [0,1])$ is the unique strong solution of the stochastic differential equation (1.3).

**Proof:** For any $f \in C_b(W)$, it follows from the Girsanov theorem

$$E[f \circ V] = E[f \circ V \circ U \rho(-\delta u)]$$

$$= E[f \rho(-\delta u)],$$

hence $V\mu$ is equivalent to $\mu$ and the corresponding Radon-Nikodym density is $\rho(-\delta u)$. Let

$$D = \{w \in W : V \circ U(w) = w\}.$$

Since $D \subset U^{-1}(U(D))$ and by the hypothesis $\mu(D) = 1$ we get

$$E[1_{U(D)} \circ U] = 1.$$
Since $U\mu$ is equivalent to $\mu$ we have also $\mu(U(D)) = 1$. If $w \in U(D)$, then $w = U(d)$, for some $d \in D$, hence $U \circ V(w) = U \circ V\circ U(d) = U(d) = w$, consequently $U \circ V = I_W$ $\mu$-almost surely and $V$ is the two-sided inverse of $U$. Evidently, together with the absolute continuity of $V\mu$, this implies that $V$ is of the form $V = I_W + v$, with $v : W \to H$. Moreover, $\dot{u} = \dot{v} \circ U$, hence the right hand side is adapted. We can assume that all these processes are uni-dimensional (otherwise we proceed component wise). Let $\dot{v}^n = \max(-n, \min(\dot{v}, n))$. Then $\dot{v}^n \circ U$ is adapted. Let $H \in L^2(dt \times d\mu)$ be an adapted process. Using the Girsanov theorem:

$$E \left[ \rho(-\delta u) \int_0^1 \dot{v}^n_s \circ U H_s \circ U ds \right] = E \left[ \int_0^1 \dot{v}^n_s H_s ds \right] = E \left[ \int_0^1 E[\dot{v}^n_s | F_s] H_s ds \right] = E \left[ \rho(-\delta u) \int_0^1 E[\dot{v}^n_s | F_s] \circ U H_s \circ U ds \right].$$

Consequently

$$E[\dot{v}^n_s | F_s] \circ U = \dot{v}^n_s \circ U,$$

almost surely. Since $U\mu$ is equivalent to $\mu$, it follows that

$$E[\dot{v}^n_s | F_s] = \dot{v}^n_s$$

almost surely, hence $\dot{v}^n$ and also $\dot{v}$ are adapted. It is now clear that $(V(t), t \in [0, 1])$ is a strong solution of (1.3). The uniqueness follows from the fact that, any strong solution of (1.3) would be a right inverse to $U$, since $U$ is invertible, then this solution is equal to $V$. \qed

### 2.1 Carleman inequality

In the sequel we shall use the inequality of T. Carleman which says that (cf. [1] or [3], Corollary XI.6.28)

$$\|\det_2(I_H + A)(I_H + A)^{-1}\| \leq \exp \frac{1}{2} \left( \|A\|_2^2 + 1 \right),$$

for any Hilbert-Schmidt operator $A$, where the left hand side is the operator norm, $\det_2(I_H + A)$ denotes the modified Carleman-Fredholm determinant and $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. Let us remark that if $A$ is a quasi-nilpotent operator, i.e., if the spectrum of $A$ consists of zero only, then $\det_2(I_H + A) = 1$, hence in this case the Carleman inequality reads

$$\|(I_H + A)^{-1}\| \leq \exp \frac{1}{2} \left( \|A\|_2^2 + 1 \right).$$

This case happens when $A$ is equal to the Sobolev derivative of some $u \in D_{p,1}(H)$ whose drift $\dot{u}$ is adapted to the filtration $(\mathcal{F}_t, t \in [0, 1])$, cf. [12, 13].
3 A sufficient condition for invertibility

In the sequel, for a given $u \in \mathbb{D}_{2,0}(H)$ adapted we shall denote $e^{-\tau}P_{\tau}u$ and $e^{-\kappa}P_{\kappa}u$ by $u_{\tau}$ and $u_{\kappa}$ respectively, the reason for that is simply the identity $P_{\tau}\delta u = \delta u_{\tau}$ is more practical for controlling the Girsanov exponential. Besides we shall suppose that $u$ satisfies always the condition that $\rho(-\delta u)$ is a probability density with respect to $\mu$. We then have

Lemma 2 Assume that $u$ is adapted and that

$$E \left[ \exp \left( -\delta u - \frac{1}{2}|u|_{H}^{2} \right) \right] = E[\rho(-\delta u)] = 1. \tag{3.8}$$

Then we have

$$E \left[ \exp \left( -\lambda\delta u_{\tau} - \frac{\lambda^{2}}{2}|u_{\tau}|_{H}^{2} \right) \right] = E[\rho(-\lambda\delta u_{\tau})] = 1,$$

for any $\lambda, \tau \in [0, 1]$.

Proof: Define the stopping time

$$T_{n} = \inf \left( t : \int_{0}^{t} |\dot{u}_{s}|^{2} ds > n \right),$$

and let $u^{n}(t) = u(t \wedge T_{n})$. Let $u_{\tau}^{n} = e^{-\tau}P_{\tau}u^{n}$, then $\lim_{n \to \infty} \rho(-\lambda\delta u_{\tau}^{n}) = \rho(-\delta u)$ in probability when $n \to \infty, \lambda \to 1$ and $\tau \to 0$. Besides they have the constant expectation which is one. Hence $\{\rho(-\lambda\delta u_{\tau}^{n}) : n \geq 1, \lambda \in [0, 1], \tau \in [0, 1]\}$ is uniformly integrable. Consequently its subset $\{\rho(-\lambda\delta u_{\tau}^{n}) : n \geq 1\}$ is also uniformly integrable and this completes the proof.

Remark 3 We note also that the hypothesis $\rho(-\delta u)$ is satisfied as soon as $u$ satisfies either Novikov or Kazamaki condition, cf. [8].

Theorem 1 implies then that $U_{\tau} = I_{W} + u_{\tau}$ and $U_{\kappa} = I_{W} + u_{\kappa}$ are invertible and their inverses are of the form $V_{\tau} = I_{W} + v_{\tau}, V_{\kappa} = I_{W} + v_{\kappa}$ respectively. Moreover $v_{\tau}$ and $v_{\kappa}$ are $H$-valued and adapted. For $\alpha \in [0, 1]$, let

$$u_{\alpha}^{\tau,\kappa} = \alpha u_{\tau} + (1 - \alpha)u_{\kappa}. $$

Then $u_{\alpha}^{\tau,\kappa}$ is again an $H - C^{1}$-map, it is adapted and it inherits all the integrability properties of $u$. Consequently the map $U_{\alpha}^{\tau,\kappa}$, defined by

$$w \to w + u_{\alpha}^{\tau,\kappa}(w)$$

...
is invertible and its inverse is of the form $V^{\tau,\kappa}_\alpha = I_W + v^{\tau,\kappa}_\alpha$ where $v^{\tau,\kappa}_\alpha$ is adapted, $H$-valued, $H - C^1$ and it satisfies the relation

$$v^{\tau,\kappa}_\alpha = -u^{\tau,\kappa}_\alpha \circ V^{\tau,\kappa}_\alpha$$

a.s. Moreover, $v_\tau = v^{\tau,\kappa}_1$ and $v_\kappa = v^{\tau,\kappa}_0$.

We need the following result:

**Lemma 3** The mapping $\alpha \rightarrow v^{\tau,\kappa}_\alpha$ is almost surely continuously differentiable on the interval $(0, 1)$.

**Proof:** Define the partial map $t^\alpha_w : H \rightarrow H$ as

$$t^\alpha_w(h) = h + u^{\tau,\kappa}_\alpha(w + h)$$

for $w \in W$ fixed. Note that from the $H - C^1$-property of $u_\tau$ and $u_\kappa$, this map is $C^1$ on $H$ for all $w \in W$ outside a set of zero capacity. Define the map $\gamma$ from $(0, 1) \times H$ to itself as $\gamma(\alpha, h) = (\alpha, t^\alpha_w(h))$. Then the differential of $\gamma$ has a Carleman-Fredholm determinant which is equal to one. Consequently it is invertible as an operator, hence the inverse function theorem implies the existence of a differentiable inverse $\gamma^{-1}$. Besides this inverse can be written as $\gamma^{-1}(\alpha, h) = (\alpha, s^\alpha_w(h))$ where $s^\alpha_w$ satisfies the identity

$$t^\alpha_w \circ s^\alpha_w = I_H,$$

where $\alpha \rightarrow s^\alpha_w(h)$ is $C^1$ on $(0, 1)$. It is easy to see that $v^{\tau,\kappa}_\alpha(w) = s^\alpha_w(0)$ and this completes the proof.

Hence, due to Lemma 1, we have the following obvious relation

$$v_\tau - v_\kappa = \int_0^1 \frac{dv^{\tau,\kappa}_\alpha}{d\alpha}d\alpha.$$

**Theorem 4** We have the following inequality:

$$E(|v_\tau - v_\kappa|_H) \leq E\left[|u_\tau - u_\kappa|_H \int_0^1 \exp \frac{1}{2}(\|\nabla u^{\tau,\kappa}_\alpha\|_2^2 + 1)\rho(-\delta u^{\tau,\kappa}_\alpha)d\alpha\right]$$

(3.9)

**Proof:** From Lemma 1 it follows immediately via the chain rule that

$$\frac{dv^{\tau,\kappa}_\alpha}{d\alpha} = -(u_\tau - u_\kappa) \circ V^{\tau,\kappa}_\alpha - \nabla u^{\tau,\kappa}_\alpha \circ V^{\tau,\kappa}_\alpha \frac{dv^{\tau,\kappa}_\alpha}{d\alpha}.$$

Therefore

$$\frac{dv^{\tau,\kappa}_\alpha}{d\alpha} = - [(I_H + \nabla u^{\tau,\kappa}_\alpha)^{-1}(u_\tau - u_\kappa)] \circ V^{\tau,\kappa}_\alpha.$$
Since
\[ \frac{dV_{\tau,\kappa,\mu}}{d\mu} = \rho(-\delta u_{\tau,\kappa}^*), \]
we have
\[ E[|v_\tau - v_\kappa|_H] \leq E \int_0^1 \left| \frac{dv_{\tau,\kappa}^*}{d\alpha} \right|_H d\alpha \]
\[ = E \int_0^1 (I_H + \nabla u_{\tau,\kappa}^*)^{-1}(u_\tau - u_\kappa)|_H \circ V_{\tau,\kappa}^* d\alpha \]
\[ = E \int_0^1 (I_H + \nabla u_{\tau,\kappa}^*)^{-1}(u_\tau - u_\kappa)|_H \rho(-\delta u_{\tau,\kappa}) d\alpha. \]

Remarking that \( \nabla u_{\tau,\kappa}^* \) is quasi-nilpotent and applying the Carleman inequality in the last line of the above inequalities, we get
\[ E[|v_\tau - v_\kappa|] \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \exp \left( \frac{1}{2} \left( \frac{1}{2} \|
abla u_{\tau,\kappa}^*\|_2^2 - \delta u_{\tau,\kappa} \right) + 1 \right) \rho(-\delta u_{\tau,\kappa}) d\alpha \right] \]
and this completes the proof.

**Theorem 5** Assume that \( u \in \mathcal{D}_{p,1}(H) \) for some \( p > 1 \) and that it is adapted with \( E[\rho(-\delta u)] = 1 \). Suppose moreover that \( u \) satisfies the following condition:
\[ E \left[ \exp q \left( \frac{1}{2} \|
abla u\|_2^2 - \delta u \right) \right] < \infty, \]
where \( p^{-1} + q^{-1} = 1 \). Then \( U = I_W + u \) is almost surely invertible.

**Proof:** From Theorem 4 using the Hölder inequality we have
\[ E[|v_\tau - v_\kappa|_H] \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \exp \left( \frac{1}{2} \|
abla u_{\tau,\kappa}^*\|_2^2 + 1 \right) \rho(-\delta u_{\tau,\kappa}) d\alpha \right] \]
\[ \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \exp \left( \frac{\alpha}{2} \|
abla u_{\tau,\kappa}^*\|_2^2 - \alpha \delta u_{\tau,\kappa} + \frac{1 - \alpha}{2} \|
abla u_{\kappa}\|_2^2 - (1 - \alpha) \delta u_{\kappa} \right) d\alpha \right] \]
\[ \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \exp q \left( \frac{\alpha}{2} \|
abla u_{\tau,\kappa}^*\|_2^2 - \alpha \delta u_{\tau,\kappa} + \frac{1 - \alpha}{2} \|
abla u_{\kappa}\|_2^2 - (1 - \alpha) \delta u_{\kappa} \right) d\alpha \right]^{1/q} \]
\[ \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \left( E \left[ \exp q \left( \frac{1}{2} \|
abla u_{\tau,\kappa}^*\|_2^2 - \delta u_{\tau,\kappa} \right) \right] \right)^{\alpha} \times E \left[ \exp q \left( \frac{1}{2} \|
abla u_{\kappa}\|_2^2 - \delta u_{\kappa} \right) \right]^{1-\alpha} d\alpha \right]^{1/q} \]
From the Jensen inequality and from the relation
\[ \delta u_\tau = P_\tau \delta u , \]
we obtain
\[ E \left[ \exp q \left( \frac{1}{2} \| \nabla u_\tau \|_2^2 - \delta u_\tau \right) \right] \leq E \left[ \exp q \left( \frac{1}{2} \| \nabla u \|_2^2 - \delta u \right) \right] . \]
Consequently
\[ E[|v_\tau - v_\kappa|_H] \leq E[|u_\tau - u_\kappa|_H^{p/2}]^{1/p} E \left[ \exp q \left( \frac{1}{2} \| \nabla u \|_2^2 - \delta u \right) \right]^{1/q} \to 0 \]
since \( E[|u_\tau - u_\kappa|_H^{p/2}] \to 0 \) as \( \kappa, \tau \to 0 \) and this implies the existence of some adapted \( v : W \to H \) which is the limit in \( L^1(\mu, H) \) of \( (v_\tau, \tau \in (0,1)) \). To complete the proof we have to show that \( v \circ U = -u \) and \( u \circ V = -v \) almost surely, where \( V = I_W + v \).

For \( c > 0 \), we have
\[ \mu \{ |v_\tau \circ U_\tau - v \circ U|_H > c \} \leq \mu \{ |v_\tau \circ U_\tau - v \circ U|_H > \frac{c}{2} \} + \mu \{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2} \} = E[\rho(\delta v_\tau)1_{|v_\tau - v|_H > c/2}] + \mu \{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2} \} \]
Since
\[ E[\rho(\delta v_\tau) \log \rho(\delta v_\tau)] = \frac{1}{2} E[|u_\tau|_H^2], \]
the set \( (\rho(\delta v_\tau), \tau \in [0,1]) \) is uniformly integrable, hence the first term (3.10) can be made arbitrarily small by the convergence of \( v_\tau \to v \) in probability. Moreover, we know that \( (\rho(-\delta u_\tau), \tau \in [0,1]) \) converges in probability to \( \rho(-\delta u) \) and they have all the same expectation which is equal to one. Consequently the set \( (\rho(-\delta u_\tau), \tau \in [0,1]) \) is also uniformly integrable. To control the term (3.11), recall that, by the Lusin theorem, given any \( \varepsilon > 0 \), there exists a compact set \( K_\varepsilon \) in \( W \) such that \( \mu(K_\varepsilon) > 1 - \varepsilon \) and that the restriction of \( v \) to \( K_\varepsilon \) is uniformly continuous. Therefore
\[ \mu \{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2} \} \leq \mu \{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2}, U_\tau \in K_\varepsilon, U \in K_\varepsilon \} + \mu \{ U \in K_\varepsilon \} \]
The last two terms (3.13) can be made arbitrarily small (uniformly w.r. to \( \tau \)) by the uniform integrability of \( (\rho(-\delta u_\tau), \tau \in [0,1]) \). To control the term (3.12), let \( \beta > 0 \)
be arbitrary. Then
\[
\mu \left\{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2}, U_\tau \in K_\varepsilon, U \in K_\varepsilon \right\} \\
\leq \mu \left\{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2}, U_\tau \in K_\varepsilon, U \in K_\varepsilon, \|U_\tau - U\| > \beta \right\} \\
+ \mu \left\{ |v \circ U_\tau - v \circ U|_H > \frac{c}{2}, U_\tau \in K_\varepsilon, U \in K_\varepsilon, \|U_\tau - U\| \leq \beta \right\}
\]
(3.14)
where \( \| \cdot \| \) denotes the norm of \( W \). Since \( v \) is uniformly continuous on \( K_\varepsilon \), the term (3.15) can be made arbitrarily small by choosing \( \beta \) small enough and the term (3.14) is bounded by
\[
\mu \{ \|U_\tau - U\| > \beta \}
\]
which can be made arbitrarily small by choosing \( \tau \) small enough and this proves the relation \( v \circ U = -u \) which implies that \( V \circ U = I_W \) almost surely. To prove \( u \circ V = -v \), recall that
\[
dV_\tau \mu = \rho(-\delta u_\tau)
\]
and as we have indicated above \( \rho(-\delta u_\tau), \tau \in [0, 1] \) is uniformly integrable. Hence we can repeat the same reasoning as above by interchanging \( u \) and \( v \) in the above lines and this completes the proof.

**Corollary 2** Assume that \( u \in \mathbb{D}_{p,1}(H) \) is adapted. If \( u \) satisfies the following condition
\[
E \left[ \exp \left( q \|\nabla u\|_2^2 + 2q^2 |u|_H^2 \right) \right] < \infty,
\]
then \( U = I_W + u \) is almost surely invertible.

**Proof:** Let \( \varepsilon > 1 \), we have, using the Hölder inequality
\[
E \left[ \exp q \left( \frac{1}{2} \|\nabla u\|_2^2 - \delta u \right) \right] = E \left[ \exp \left( \frac{q}{2} \|\nabla u\|_2^2 - q\delta u - q^2 \frac{1 + \varepsilon}{2\varepsilon} |u|_H^2 + q^2 \frac{1 + \varepsilon}{2\varepsilon} |u|_H^2 \right) \right]
\leq E \left[ \exp \left( \frac{(1 + \varepsilon)q}{2} \|\nabla u\|_2^2 + q^2 \frac{(1 + \varepsilon)^2}{2\varepsilon} |u|_H^2 \right) \right]^{1/1+\varepsilon}
\times E \left[ \exp \left( -\frac{(1 + \varepsilon)q}{\varepsilon} \delta u - q^2 \frac{(1 + \varepsilon)^2}{2\varepsilon^2} |u|_H^2 \right) \right]^{\varepsilon/1+\varepsilon}
\leq E \left[ \exp \left( \frac{(1 + \varepsilon)q}{2} \|\nabla u\|_2^2 + q^2 \frac{(1 + \varepsilon)^2}{2\varepsilon} |u|_H^2 \right) \right]^{1/1+\varepsilon},
\]
since the expectation of third line is upperbounded by one. The proof follows when we take \( \varepsilon = 1 \) for which the last line attains its minimum with respect to \( \varepsilon > 0 \). \qed
Theorem 6 Assume that $L$ is a probability density with respect to $\mu$ which has an Itô representation
$$L = \rho(-\delta u) = \exp \left[ -\delta u - \frac{1}{2} |u|^2_H \right],$$
where $u \in \mathbb{D}_{2,1}(H)$ is adapted. If $\|\nabla u\|_2 \in L^\infty(\mu)$, then the measure $\nu$, defined as
$$d\nu = L \, d\mu$$
satisfies the logarithmic Sobolev inequality, i.e.,
$$E_\nu \left[ f^2 \log \frac{f^2}{E_\nu[f^2]} \right] \leq K E_\nu[|\nabla f|^2_H],$$
for any cylindrical Wiener function $f$, where
$$K = 2 \left\| \exp \left( 1 + \|\nabla u\|^2 \right) \right\|_{L^\infty(\mu)}. $$

Proof: We shall use a reasoning analogous to that of [14]. Let $\alpha$ be a positive, smooth function of compact support with $\alpha(0) = 1$ and $|\alpha'(t)| \leq c$ for any $t \geq 0$. Define $\dot{u}_t^n$ and $u_t^n$ as
$$\dot{u}_t^n = \alpha\left( \frac{1}{n} \dot{u}_t \right) \dot{u}_t, \quad u_t^n = \int_0^t \dot{u}_s^n \, ds.$$

$u_t^n$ is bounded and
$$\|\nabla u_t^n\|^2_2 \leq 2(1 + c^2)\|\nabla u\|^2_2 \in L^\infty(\mu).$$

From Theorem \ref{thm:uniform-parallel} $U_t^n = I_W + u_t^n$ is a.s. invertible and its inverse $V_t^n$ is of the form $I_W + v_t^n$ such that $v_t^n$ is in $\mathbb{D}_{2,1}(H)$ and adapted. Consequently
$$\frac{dV_t^n\mu}{d\mu} = \rho(-\delta u_t^n).$$

Let $\nu_n$ be the probability measure defined as $d\nu_n = \rho(-\delta u_t^n)\, d\mu$. Using the log-Sobolev inequality of L. Gross for $\mu$, cf. [7], and the Carleman inequality we get
$$E_{\nu_n} \left[ f^2 \log \frac{f^2}{E_{\nu_n}[f^2]} \right] = E \left[ (f \circ V_t^n)^2 \log \frac{f^2 \circ V_t^n}{E_{\nu_n}[f^2]} \right]$$
$$\leq 2 E \left[ \|\nabla (f \circ V_t^n)\|^2_H \right]$$
$$\leq 2 E \left[ \|\nabla f \circ V_t^n\|^2_H \|I_W + \nabla v_t^n\|^2_2 \right]$$
$$= 2 E \left[ \rho(-\delta u_t^n)\|\nabla f\|^2_H \|(I_W + \nabla u_t^n)^{-1}\|^2_2 \right]$$
$$\leq E \left[ \rho(-\delta u_t^n)\|\nabla f\|^2_H \exp \left( 1 + \|\nabla u_t^n\|^2_2 \right) \right]$$
$$\leq 2 \left\| \exp \left( 1 + \|\nabla u_t^n\|^2_2 \right) \right\|_{L^\infty(\mu)} E \left[ \rho(-\delta u_t^n)\|\nabla f\|^2_H \right]$$
$$\leq 2 E_{\nu_n}[\|\nabla f\|^2_H] \exp \left( 1 + 2(1 + c^2)\|\nabla u\|^2_2 \right) \right\|_{L^\infty(\mu)} \cdot$$

To complete the proof, take first the limit of this inequality as $n \to \infty$ and remark that $(\rho(-\delta u_t^n), n \in \mathbb{N})$ is uniformly integrable. Finally it suffices to take the infimum of the right hand side with respect to $c > 0$. \qed
Theorem 7 \ Under the hypothesis of Theorem 6, the mapping \( U = I_W + u \) is almost surely invertible.

Proof: With the notations of Theorem 5 we have

\[
E[|v_\tau - v_\kappa|_H] \leq E \left[ |u_\tau - u_\kappa|_H \int_0^1 \rho(-\delta u_\tau^{\tau,\kappa})d\alpha \right].
\]

Let us denote \( \rho(-\delta u_\tau^{\tau,\kappa}) \) by \( \rho_\alpha^{\tau,\kappa} \) and let \( \nu_\alpha^{\tau,\kappa} \) be the measure whose Radon-Nikodym derivative with respect to \( \mu \) is given by \( \rho_\alpha^{\tau,\kappa} \). We have

\[
E[\rho_\alpha^{\tau,\kappa}|u_\tau|_H^2] \leq 2E[\rho_\alpha^{\tau,\kappa}|u_\tau - E_{\nu_\alpha^{\tau,\kappa}}[u_\tau]|_H^2] + 2|E_{\nu_\alpha^{\tau,\kappa}}[u_\tau]|_H^2
= I_{\tau,\kappa,\alpha} + II_{\tau,\kappa,\alpha}.
\]

From Theorem 6 and from the fact that logarithmic Sobolev inequality implies the Poincaré inequality, the first terms at the right hand side of the above inequality is bounded:

\[
I_{\tau,\kappa,\alpha} \leq 2C E_{\nu_\alpha^{\tau,\kappa}}[\|\nabla u_\tau\|_H^2] \leq 2C\|\nabla u\|_{L^\infty(\mu_H^{\otimes H})},
\]
where \( C \) is independent of \( \alpha, \kappa \) and \( \tau \); in fact it is the constant of the logarithmic Sobolev inequality of Theorem 5. Moreover

\[
u_\alpha^{\tau,\kappa}(|u_\tau - E_{\nu_\alpha^{\tau,\kappa}}[u_\tau]|_H > c) \to 0
\]

as \( c \to \infty \). Besides, from the uniform integrability of \( (\rho_\alpha^{\tau,\kappa} : \tau, \kappa, \alpha \in [0,1]) \), we also have

\[
\sup_{\tau,\kappa,\alpha} \nu_\alpha^{\tau,\kappa}(|u_\tau|_H > c) \leq \sup_{\tau,\kappa,\alpha} E[\rho_\alpha^{\tau,\kappa}1_{(|u_\tau|_H > 0)}] \to 0
\]
as \( c \to \infty \). Consequently, there exists a constant \( c > 0 \) such that

\[
\sup_{\tau,\kappa,\alpha \in [0,1]} |E_{\nu_\alpha^{\tau,\kappa}}[u_\tau]|_H \leq c.
\]

This implies the uniform integrability of the family

\[
\left( |u_\tau - u_\kappa|_H \int_0^1 \rho(-\delta u_\tau^{\tau,\kappa})d\alpha : \tau, \kappa \in [0,1] \right).
\]

Since it converges already in probability to zero, the convergence in \( L^1(\mu) \) holds also. The rest of the proof follows the same lines as the proof of Theorem 5 hence the it is completed. \( \square \)
Remark 4 In terms of the stochastic differential equations, the question of finding an inverse to \( U = I_W + u \), where \( u \in \mathbb{D}_{2,0}(H) \) is adapted, amounts to solving the following stochastic differential equation

\[
\begin{align*}
    dV_t(w) &= -\dot{u}_t(V(w))dt + dW_t(w) \\
    V_0(w) &= 0,
\end{align*}
\]

and this problem is solved only under a Lipschitz hypothesis imposed to \( \dot{u} \), which can be expressed as follows

\[
\sup_{s \leq t} |\dot{u}_s(w + h) - \dot{u}_s(w + k)| \leq K \sup_{s \leq t} |h(s) - k(s)|, \tag{3.18}
\]

\( \mu \)-a.s., for any \( h, k \in H, t \in [0,1] \), where \( K \) is a constant. Since

\[
\sup_{s \leq t} |h(s) - k(s)| \leq |h - k|_H,
\]

the Lipschitz condition (3.18) implies that

\[
|\nabla \dot{u}_s|_H \leq K,
\]

hence

\[
\|\nabla u\|_2^2 = \int_0^1 |\nabla \dot{u}_s|_H^2 ds \leq K^2
\]

\( \mu \)-almost surely. Therefore, the Lipschitz condition (3.18) is stronger than the hypothesis of Theorem 7. For example, assume that the drift \( \dot{u} \) has a Sobolev derivative which satisfies

\[
|\nabla \dot{u}_s|_H \leq K s^{-\alpha},
\]

almost surely, where \( 0 \leq \alpha < 1/2 \). Then the Lipschitz property may fail although the stochastic differential equation (3.17) has a unique solution by the theorem since \( \|\nabla u\|_2 \in L^\infty(\mu) \).

4 Extensions

In this section we give some variations and extensions of the results proven in the last section. We start with

Theorem 8 Let \( u : W \to H \) be adapted with \( u \in \mathbb{D}_{2,0}(H) \) such that \( E[\rho(-\delta u)] = 1 \). Assume that \( (\Omega_n, n \geq 1) \) is a measurable covering of \( W \) and that \( u = u_n \) a.s. on \( \Omega_n \), where \( u_n : W \to H \) is in \( \mathbb{D}_{2,0}(H) \), adapted, \( E[\rho(-\delta u_n)] = 1 \) and \( U_n = I_W + u_n \) is almost surely invertible with the inverse denoted by \( V_n = I_W + v_n \). Then \( U = I_W + u \) is almost surely invertible and the Radon-Nikodym derivative of \( U \mu \) with respect to \( \mu \) belongs to the space \( L\log L(\mu) \).
Proof: Without loss of generality we can assume that the sets \((\Omega_n, n \geq 1)\) are disjoint. Note also that by the hypothesis, \(U \mu\) is equivalent to \(\mu\). We have

\[
\frac{dU_n \mu}{d\mu} = \rho(-\delta v_n).
\]

Then, for any \(f \in C_b(W)\)

\[
E[f \circ U \rho(-\delta u)] = \sum_{n=1}^{\infty} E[f \circ U_n 1_{\Omega_n} \rho(-\delta u_n)]
\]

\[
= \sum_{n=1}^{\infty} E[f 1_{U_n(\Omega_n)}].
\]

By the Girsanov theorem we also have

\[
E[f \circ U \rho(-\delta u)] = E[f],
\]

hence

\[
\sum_{n=1}^{\infty} 1_{U_n(\Omega_n)} = 1
\]

almost surely. This means that \((U_n(\Omega_n), n \geq 1)\) is an almost sure partition of \(W\). Define \(v\) on \(U_n(\Omega_n)\) as to be \(v_n\) and let \(V = I_W + v\). Then \(V\) is defined almost everywhere, moreover

\[
E[f \circ V] = \sum_n E[f \circ V_n 1_{U_n(\Omega_n)}] = E[f \rho(-\delta u)],
\]

therefore \(V \mu\) is equivalent to \(\mu\) and \(V\) is well-defined. Evidently, for almost all \(w \in \Omega_n\),

\[
V \circ U(w) = V \circ U_n(w) = V_n \circ U_n(w) = w,
\]

hence \(V \circ U = I_W\) almost surely, i.e., \(V\) is a left inverse of \(U\). Since

\[
\mu \left( \bigcup_{n=1}^{\infty} U_n(\Omega_n) \right) = 1,
\]

and since

\[
\{w \in W : U \circ V(w) = w\} \supset \bigcup_{n=1}^{\infty} U_n(\Omega_n),
\]

we also have

\[
\mu \left( \{w \in W : U \circ V(w) = w\} \right) = 1
\]
and this completes the proof of the invertibility of $U$. Clearly, the map $v$ is adapted to the filtration of the Wiener space, hence the stochastic integral of the drift $(\dot{v}_t, t \in [0, 1])$, with respect to the Wiener process is well-defined (using the localization techniques with the help of the stopping times) and we shall denote its value at $t = 1$ by $\delta^0 v$. Since $U$ is adapted, we have

\begin{align*}
(\delta^0 v) \circ U &= \delta^0 (v \circ U) + (v \circ U, u)_H \\
&= -\delta^0 u - |u|^2_H \\
&= -\delta u - |u|^2_H,
\end{align*}

hence

$$\rho(-\delta^0 v) \circ U \rho(-\delta u) = 1,$$

and similarly

$$\rho(-\delta u) \circ V \rho(-\delta^0 v) = 1$$

almost surely. Therefore, on the one hand

$$\frac{dU_\mu}{d\mu} = \rho(-\delta^0 v)$$

and on the other hand

$$E \left[ \rho(-\delta^0 v) \log \rho(-\delta^0 v) \right] = \frac{1}{2} E[|u|^2_H] < \infty$$

by the hypothesis and the proof is completed.

Here is another situation which is encountered in the applications:

**Theorem 9** Assume that $u \in D^a_2(H)$ with $E[\rho(-\delta u)] = 1$. Suppose that there exists a sequence of stopping times $(T_n, n \geq 1)$ increasing to infinity and a sequence $(u_n, n \geq 1) \subset D^a_2(H)$ such that

$$\dot{u}(t) = \dot{u}_n(t) \text{ for } t < T_n(w).$$

Suppose further that $U_n = I_W + u_n$ are almost surely invertible with inverse $V_n = I_W + v_n$ for any $n \geq 1$. Then $U = I_W + u$ is almost surely invertible with inverse $V = I_W + v$, $v \in L^0(\mu, H)$, with $\dot{v}$ adapted and

$$\frac{dU_\mu}{d\mu} = \rho(-\delta^0 v).$$

**Proof:** For any $n \geq 1$, $(V_n(t), t \in [0, 1])$ is the unique solution of the equation

$$V_n(t) = W_i - \int_0^t \dot{u}_n(s, V_n) ds.$$
By hypothesis, for \( m \leq n \), we have
\[
\dot{u}_m(t \wedge T_m, w) = \dot{u}_n(t \wedge T_m, w) = \dot{u}(t \wedge T_m, w)
\]
d\( t \times d\mu \)-almost surely. Hence
\[
V_n(t) = W_t - \int_0^t \dot{u}_m(s, V_n)ds
\]
for \( t \leq T_m(V_n(w)) \). Hence, by the uniqueness
\[
\forall t \leq T_m(V_n(w)) \forall n \geq m
\]
\[
V_n(t) = V_m(t)
\]
(4.19)

Similarly, for \( t \leq T_n(V_m) \) we have
\[
V_m(t) = W_t - \int_0^t \dot{u}_n(s, V_m)ds,
\]
hence again by the uniqueness
\[
V_m(t) = V_n(t) \quad \forall t \leq T_n(V_m(w)) \quad (4.20)
\]
Consequently, combining the relations (4.19) and (4.20), we get
\[
V_m(t) = V_n(t) \quad \forall t \leq T_n(V_m(w)) \quad (4.21)
\]
In particular, for any \( m \) and for any \( n \geq m \), we have
\[
V_m(t) = V_n(t) \quad \forall t \leq T_n(V_m(w)) \quad (4.22)
\]
Let \( \tilde{T}_n = \sup_{k \leq n} T_k(V_k) \), we have
\[
\mu\{T_n(V_n) > t\} = E[1_{\{T_n > t\}}\rho(-\delta u_n)]
\]
\[
= E[1_{\{T_n > t\}}E[\rho(-\delta u_n) | F_{T_n}]]
\]
\[
= E[1_{\{T_n > t\}}\rho(-\delta u)] \to 1,
\]
therefore \( (\tilde{T}_n, n \geq 1) \) increases to infinity. Define now \( (V(t), t \in [0, 1]) \) as follows:
\[
V(t) = V_n(t) \text{ if } t \leq \tilde{T}_n(w).
\]
In fact, for any \( t \in [0, 1] \) there exists some \( k \leq n \) such that \( t \leq T_k(V_k) \), hence by the relation (4.22),
\[
V_k(t) = V_n(t) = V_{n+l}(t) \quad \text{for } t \leq T_k(V_k),
\]
for any \( l \geq 1 \). In particular, for \( m \leq n \) and \( t \leq \tilde{T}_m(w) \), we have
\[
V_m(t) = V_n(t),
\]

hence $V$ is well-defined. Moreover, $V$ can be written as

$$V(t) = W_t + \int_0^t \dot{v}_s ds ,$$

with $\dot{v} \in L^0(\mu, L^2([0, 1]))$ adapted. Besides, for $t \leq \tilde{T}_n$, since $\dot{u}_n$ is adapted

$$V(t) = W_t - \int_0^t \dot{u}_n(s, V_n) ds = W_t - \int_0^t \dot{u}_n(s, V) ds ,$$

moreover, from the hypothesis

$$\dot{u}_n(s, V) = \dot{u}(s, V) \text{ for } s < T_n(V) .$$

Since $V\mu$ is absolutely continuous with respect to $\mu$, $(T_n(V), n \geq 1)$ increases to infinity almost surely. Consequently, for any $t \in [0, 1]$,

$$V(t) = W_t - \int_0^t \dot{u}(s, V) ds ,$$

almost surely. This means that $U \circ V = I_W$ almost surely, in other words $V$ is a right inverse to $U$.

Let us show now that the mapping $V$ constructed above is also a left inverse: we have

$$v_m(t, w) = v_n(t, w) = v(t, w) \text{ if } t \leq T_m(V_m), m \leq n .$$

Hence, for $t \leq T_m(w)$ and by the adaptedness of $\dot{v}$,

$$v_m(t, U_m(w)) = v_n(t, U_m(w)) = v(t, U_m(w)) = v(t, U(w)) .$$

By the hypothesis, for $t \leq T_m(w)$, we also have

$$v_m(t, U_m(w)) = -u_m(t, w) = -u(t, w) .$$

Since $(T_m, m \geq 1)$ increases to infinity, we obtain

$$v \circ U + u = 0$$

almost surely. This implies that $V$ is also a left inverse and that

$$\rho(-\delta^0v) \circ U \rho(-\delta u) = 1$$

almost surely, hence $E[\rho(-\delta^0v)] = 1$ and $\rho(-\delta^0v)$ is the Radon-Nikodym derivative of $U\mu$ with respect to $\mu$. \qed
Corollary 3  Assume that $u \in \mathcal{D}_{2,k}(H)$ is adapted with $k \geq 1$ such that $E[\rho(-\delta u)] = 1$ and that
\[
\|\nabla^k u\|_{H \otimes (k+1)} \in L^\infty(\mu).
\]
Assume moreover that $H \otimes i$-valued process $t \rightarrow \nabla^i \dot{u}_t$ is almost surely continuous for $0 \leq i \leq k-1$. Then the mapping $U = I_W + u$ is almost surely invertible.

Proof: Let $\theta_n$ be a smooth function on $\mathbb{R}$ which is equal to one on $[0, n]$ and zero on the complement of $[-1, n + 1]$. Define $u_n, n \geq 1$ as
\[
\dot{u}_n(t) = \dot{u}(t) \prod_{i=0}^{k-1} \theta_n \left( \|\nabla^i \dot{u}_t\|_{H \otimes i}^2 \right)
\]
for $t \in [0, 1]$. Then it is easy to see that $\|\nabla u_n\|_2 \in L^\infty(\mu)$, hence, from Theorem 7 $U_n = I_W + u_n$ is almost surely invertible for any $n \geq 1$. Define the stopping times $(T_n, n \geq 1)$ as
\[
T_n = \inf \left( \sum_{i=0}^{k-1} \|\nabla^i \dot{u}_t\|_{H \otimes i} > n \right)
\]
By the continuity hypothesis, $(T_n, n \geq 1)$ increases to infinity, besides for $t < T_n(w)$, $\dot{u}(t, w) = \dot{u}_n(t, w)$, hence the proof follows from Theorem 9. \qed

In Theorem 8 we have supposed that $u = u_n$ on a set $\Omega_n$, where $u_n$ is also adapted. However, we can construct easily examples where $u_n$ is not adapted but still $I_W + u_n$ is invertible and equal to $I_W + u$ almost surely on $\Omega_n$ such that the union of the sets $(\Omega_n, n \geq 1)$ is equal to $W$ almost surely. In such a situation the hypothesis of Theorem 8 are not satisfied. To study this kind of situations, we need to define some more regularity concepts which are studied in detail in [16]:

Definition 1  Let $X$ be a separable Hilbert space, then

1. a measurable map $\xi : W \rightarrow X$ is called $H - C^1_{\text{loc}}$ if there exists a measurable $q : W \rightarrow \mathbb{R}_+, q > 0$ a.s., such that the map $h \rightarrow \xi(w + h)$ is a $C^1$-map on the ball $\{h \in H : |h|_H < q(w)\}$.

2. $\xi$ is called representable by locally $H - C^1$-functions or $RH - C^1_{\text{loc}}$ in short, if there is a sequence of measurable sets $(B_n, n \geq 1)$ whose union is of full measure and a sequence of $H - C^1_{\text{loc}}$-functions $(u_n, n \geq 1)$ such that $u = u_n$ on $B_n$ almost surely.

Let us recall Theorem 3.5.3 of [16] which is valid for not neccessarily adapted perturbations of identity:
**Theorem 10** If \( u : W \to H \) is \( RH - C^1_{loc} \), then, there exists a set \( \tilde{W} \) of full-measure such that, for any \( f, g \in C_b(W) \), one has

\[
E[f \circ U | \Lambda_u] g = E \left[ f(w) \sum_{y \in U^{-1}\{w\}} g(y) 1_{M \cap \tilde{W}}(y) \right],
\]

where \( M = \{ w \in W : \det 2(I_H + \nabla u(w)) \neq 0 \} \) and

\[
\Lambda_u = \det 2(I_H + \nabla u) \exp \left( -\delta u - \frac{1}{2} |u|_H^2 \right).
\]

In particular, the multiplicity of \( U \) on the set \( M \cap \tilde{W} \) is almost surely (atmost) countable.

The next theorem answers to the question that we have asked above:

**Theorem 11** Let \( u \in D_{2,1}(H) \) be adapted and assume \( E[\rho(-\delta u)] = 1 \). Suppose that there exists \( (\Omega_n, n \geq 1) \subset B(W) \), whose union is of full measure and a sequence \( (u_n, n \geq 1) \) of \( RH - C^1_{loc} \)-functions such that \( u = u_n \) almost surely on \( \Omega_n \) for any \( n \geq 1 \). Then \( U = I_W + u \) is almost surely invertible.

**Proof:** Without loss of generality, we can assume that the sets \( (\Omega_n, n \geq 1) \) are disjoint. Then the following change of variables formula is a consequence of Theorem 10 and of the fact that \( u \) is adapted:

\[
E[f \circ U \rho(-\delta u) g] = \sum_{n=1}^{\infty} E \left[ f(w) \sum_{y \in U^{-1}_{-1}\{w\}} g(y) 1_{\Omega_n \cap W_n}(y) \right].
\]

In particular, taking \( g = 1 \), we see that

\[
\sum_{n=1}^{\infty} \sum_{y \in U^{-1}_{-1}\{w\}} 1_{\Omega_n \cap W_n}(y) = 1
\]

almost surely. Let us denote the set \( \Omega_n \cap W_n \) by \( \Omega'_n \) and the double sum above by \( N_n(w, \Omega'_n) \). Since each \( N_n \) is an integer and since their sum is equal to one almost surely, we should have \( N_n(w, \Omega'_n) \in \{0,1\} \) almost surely. Let

\[
\tilde{\Omega}_n = \{ w : N_n(w, \Omega'_n) = 1 \}.
\]

If \( w \in \tilde{\Omega}_n \), then \( N_n(w, \Omega'_n) = 1 \), i.e., the cardinal of the set, denoted by \( |U^{-1}_{-1}\{w\} \cap \Omega'_n| \) is equal to one. Consequently, there exists a unique \( y \in \Omega'_n \) such that \( U_n(y) = w \). This means that \( U_n : \Omega'_n \to \tilde{\Omega}_n \) is surjectif. Denote the map \( w \to y \) by \( V_n(w) \), hence \( V_n(\tilde{\Omega}_n) \subset \Omega'_n \). Define \( V \) on \( \cup_n \tilde{\Omega}_n \) as \( V = V_n \) on \( \tilde{\Omega}_n \). Since the sets \( \Omega'_n \) and \( \tilde{\Omega}_n \) are
measurable, $V$ is measurable with respect to the completed Borel sigma algebra of $W$. Taking $g = 1$ in the change of variables formula, we get

$$E[g \rho(-\delta u)] = E \left[ \sum_n \sum_{y \in U_n^{-1}\{w\}} g(y) 1_{\Omega_n}(y) \right]$$

$$= E \left[ \sum_n 1_{\tilde{\Omega}_n}(w) g \circ V_n(w) \right]$$

$$= E[g \circ V].$$

This implies in particular that the measure $V(\mu)$ is equivalent to $\mu$. To show that $V$ is also a left inverse, choose any two $f, g \in C_b(W)$. Then

$$E[f \circ V \circ U \rho(-\delta u) g] = \sum_n E \left[ f \circ V \sum_{y \in U_n^{-1}\{w\}} g(y) 1_{\Omega_n}(y) \right]$$

$$= \sum_n E \left[ f \circ V(w) 1_{\tilde{\Omega}_n}(w) g \circ V_n(w) 1_{\Omega_n'} \circ V_n(w) \right]$$

$$= \sum_n E \left[ f \circ V(w) 1_{\tilde{\Omega}_n}(w) g \circ V_n(w) \right]$$

$$= \sum_n E \left[ f \circ V(w) 1_{\tilde{\Omega}_n}(w) g \circ V(w) \right]$$

$$= E[f \circ V g \circ V]$$

$$= E[f g \rho(-\delta u)],$$

where the second line follows from the fact that the sum on the set $U_n^{-1}\{w\}$ is zero unless $w \in \tilde{\Omega}_n$, in which case $1_{\Omega_n'} \circ V_n(w) = 1$ since $V_n(\tilde{\Omega}_n) \subset \Omega_n'$ by the construction of $V_n$. Consequently $V \circ U = I_W \mu$-almost surely, hence $V$ is a two sided inverse, it is of the form $V = I_W + v$ and $v : W \to H$ is adapted to the (completed) filtration of the Wiener space. Moreover $\delta v$ is well-defined as local martingales final value (using the stopping techniques), in particular the Radon-Nikodym density of $U\mu$ with respect to $\mu$ is $\rho(-\delta v).$

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