A remark on the integrals of motion associated with level $k$ realization of the elliptic algebra $U_{q,p}(\hat{sl}_2)$

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Abstract

We give one parameter deformation of level $k$ free field realization of the screening current of the elliptic algebra $U_{q,p}(\hat{sl}_2)$. By means of these free field realizations, we construct infinitely many commutative operators, which we call the nonlocal integrals of motion associated with level $k$-realization of the elliptic algebra $U_{q,p}(\hat{sl}_2)$. They are given as integrals involving a product of the screening current and elliptic theta functions. This paper gives level $k$ generalization of the nonlocal integrals of motion given in [1].

1 Introduction

One of the results in V.Bazhanov, S.Lukyanov, Al.Zamolodchikov [4] is construction of field theoretical analogue of the commuting transfer matrix $T(z)$, acting on the highest weight rep-
presentation of the Virasoro algebra. Their commuting transfer matrix \( T(z) \) is the trace of the image of the universal \( R \)-matrix associated with the quantum affine symmetry \( U_q(\hat{sl}_2) \). This construction is very simple and the commutativity \( [T(z), T(w)] = 0 \) is direct consequence of the Yang-Baxter equation. They call the coefficients of the Taylor expansion of \( T(z) \) the nonlocal integrals of motion. The higher-rank generalization of [4] is considered in [5, 6]. The elliptic deformation of the nonlocal integrals of motion is considered in [1]. V.Bazhanov, S.Lukyanov, A.Zamolodchikov [4] constructed the continuous transfer matrix \( T(z) \) by taking the trace of the image of the universal \( R \)-matrix associated with \( U_q(\hat{sl}_2) \). However it is not so easy to calculate the image of the elliptic version of the universal \( R \)-matrix, which is obtained by using the twister [10]. Hence the construction method of the elliptic version [1] should be completely different from those in [4]. Instead of considering the transfer matrix \( T(z) \), the authors [1] give the integral representation of the integrals of motion directly. The commutativity of the integrals of motion is not consequence of the Yang-Baxter equation. It is consequence of the commutative subalgebra of the Feigin-Odesskii algebra [11]. The higher-rank generalization of [1] is considered in [2, 3]. This paper is a continuation of [1, 2, 3]. This paper give level \( k \) generalization of the nonlocal integrals of motion given in [1].

The organization of this paper is as following. In section 2 we give one parameter “s” deformation of the level \( k \) free field realization of the screening current of the elliptic algebra \( U_{q,p}(\hat{sl}_2) \). In section 3 we construct infinitely many commutative operators, which are called the nonlocal integrals of motion associated with the elliptic algebra \( U_{q,p}(\hat{sl}_2) \) for level \( k \). In section 3 we state main theorem and give conjecture. In appendix we summarize the normal ordering of basic operators.

2 Elliptic current

In this section we give one parameter “s” deformation of the level \( k \) free field realization of the elliptic algebra \( U_{q,p}(\hat{sl}_2) \). We fix complex numbers \( x, r, r^*, s, (|x| < 1, \Re(r), \Re(r^*) > 0, s \neq 2) \), and \( k = r - r^* \neq 0, -2 \). We use symbols

\[
[n] = \frac{x^n - x^{-n}}{x - x^{-1}}, \quad [n]_+ = x^n + x^{-n}.
\]

We set the parameter \( \tau, \tau^* \)

\[
x = e^{-\pi \sqrt{-1}/r\tau} = e^{-\pi \sqrt{-1}/r\tau^*}.
\] (2.1)

Let us use parametrization \( z = x^{2u} \). The symbol \([u]_r \) stands for the Jacobi elliptic theta function

\[
[u]_r = x^{u^2/r - u\Theta_{2r}(z)}, \quad [u]_{r^*} = x^{u^2/r^* - u\Theta_{2r^*}(z)},
\] (2.2)
The theta function \( \Theta_p(z) = (z;p)_\infty(p/z;p)_\infty(p;p)_\infty \), \((z;p)_\infty = \prod_{n=0}^{\infty} (1 - p^n z) \). (2.3)

The theta function \([u]_r\) enjoys the quasi-periodicity property
\[
[u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{\pi \sqrt{-1} r - \frac{2\pi \sqrt{-1}}{r}} [u]_r.
\] (2.4)

### 2.1 Bosons

We set the bosons \( \alpha^j_m, \tilde{\alpha}^j_m, (j = 1, 2; m \in \mathbb{Z}_{\neq 0}) \),
\[
[\alpha^j_m, \alpha^n_n] = -\frac{1}{m} \frac{2m[rm]}{[km][(r-k)m]} \delta_{m+n,0}, \quad (j = 1, 2),
\] (2.5)
\[
[\alpha^1_m, \alpha^2_n] = \frac{1}{m} \left( \frac{x^{(r-k)m}[sm] - [(s-2)m]}{[rm]} + \frac{x^{km}[sm] + [(s-2)m]}{[km]} \right) \delta_{m+n,0},
\] (2.6)
\[
[\tilde{\alpha}^j_m, \tilde{\alpha}^n_n] = -\frac{1}{m} \frac{2m[(r-k)m]}{[km][rm]} \delta_{m+n,0}, \quad (j = 1, 2),
\] (2.7)
\[
[\tilde{\alpha}^1_m, \tilde{\alpha}^2_n] = \frac{1}{m} \left( \frac{x^{rm}[-sm] + [(s-2)m]}{[rm]} + \frac{x^{km}[sm] + [(s-2)m]}{[km]} \right) \delta_{m+n,0},
\] (2.8)
\[
[\alpha^j_m, \tilde{\alpha}^n_n] = -\frac{1}{m} \frac{2m}{[km]} \delta_{m+n,0}, \quad (j = 1, 2),
\] (2.9)
\[
[\alpha^1_m, \tilde{\alpha}^2_n] = \frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0},
\] (2.10)
\[
[\tilde{\alpha}^1_m, \alpha^2_n] = \frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0}.
\] (2.11)

We set the bosons \( \beta^j_m, \gamma^j_m, (j = 1, 2; m \in \mathbb{Z}_{\neq 0}) \),
\[
[\beta^j_m, \beta^n_n] = \frac{2m}{m} \frac{(k+2m)}{[km]} \delta_{m+n,0}, \quad (j = 1, 2),
\] (2.12)
\[
[\beta^1_m, \beta^2_n] = \frac{1}{m} \frac{[(k+2)m][sm] + [(s-2)m]}{[km]} \delta_{m+n,0},
\] (2.13)
\[
[\gamma^j_m, \gamma^n_n] = \frac{1}{m} \frac{2m}{[km]} \delta_{m+n,0}, \quad (j = 1, 2),
\] (2.14)
\[
[\gamma^1_m, \gamma^2_n] = -\frac{1}{m} \frac{[sm] + [(s-2)m]}{[km]} \delta_{m+n,0}.
\] (2.15)

We set the zero-mode operators \( P_0, Q_0, h, \alpha \) and \( h_0, h_1, h_2, \alpha_0, \alpha_1, \alpha_2 \),
\[
[P_0, iQ_0] = 1, \quad [h, \alpha] = 2,
\] (2.16)
\[
[h_0, \alpha_0] = [h_1, \alpha_2] = [h_2, \alpha_1] = (2 - s), \quad [h_1, \alpha_1] = [h_2, \alpha_2] = 0.
\] (2.17)

We set the Fock space \( \mathcal{F}_{K,L}, (K, L \in \mathbb{Z}) \),
\[
\mathcal{F}_{K,L} = \bigoplus_{n, n_0, n_1, n_2 \in \mathbb{Z}} \mathbb{C}[\alpha^j_m, \tilde{\alpha}^j_m, \beta^j_m, \gamma^j_m, (j = 1, 2; m \in \mathbb{Z}_{\neq 0})] \otimes |K, L\rangle_{n, n_0, n_1, n_2},
\]
\[ |K, L\rangle_{n_0, n_1, n_2} = e^{\left( L \sqrt{\frac{2r}{r-k} - K \sqrt{\frac{2s}{r}}} \right)} i^Q \otimes e^{n_\alpha} \otimes e^{n_0 a_0} \otimes e^{n_1 a_1} \otimes e^{n_2 a_2}. \]  

(2.19)

Upon specialization \( s \to 2 \), simplification occurs.

\[
\begin{align*}
\alpha_m^2 &= -\alpha_m^1, & \alpha_m^1 &= \left[ \frac{(r-k)m}{rm} \right]_\alpha^1, & \alpha_m^2 &= -\left[ \frac{(r-k)m}{rm} \right]_\alpha^1, \\
\beta_m^2 &= -\beta_m^1, & \gamma_m^2 &= -\gamma_m^1, & h_0 = h_1 = h_2 = \alpha_0 = \alpha_1 = \alpha_2 = 0.
\end{align*}
\]

(2.20)

The bosons \( \alpha_m^1, \beta_m^1, \gamma_m^1 \) are the same bosons which were introduced to construct the elliptic current associated with the elliptic algebra \( U_{q,p}(sl_2) \) and the deformed Virasoro algebra \( \hat{Vir}_{q,t} \) [7, 8, 9]. In order to construct infinitely many commutative operators, we introduce one parameter \( s \) deformation of the bosons in [7, 8, 9]. This additional parameter \( s \) plays an important role in proof of the main theorem.

### 2.2 Elliptic current

We introduce the operators \( C_j(z), C_j^\dagger(z), \ (j = 1, 2) \) acting on the Fock space \( \mathcal{F}_{J,K} \).

\[
\begin{align*}
C_1(z) &= e^{-\sqrt{\frac{2r}{k(r-k)}} Q_0} e^{\frac{iQ}{k(r-k)}} P_{3 \log z} \exp \left( -\sum_{m \neq 0} \alpha_m^1 z^{-m} \right), \\
C_2(z) &= e^{\sqrt{\frac{2r}{k(r-k)}} Q_0} e^{\frac{-iQ}{k(r-k)}} P_{3 \log z} \exp \left( -\sum_{m \neq 0} \alpha_m^2 z^{-m} \right), \\
C_1^\dagger(z) &= e^{\sqrt{\frac{2r}{k(r-k)}} Q_0} e^{\frac{iQ}{k(r-k)}} P_{3 \log z} \exp \left( \sum_{m \neq 0} \beta_m^1 z^{-m} \right), \\
C_2^\dagger(z) &= e^{\sqrt{\frac{2r}{k(r-k)}} Q_0} e^{\frac{-iQ}{k(r-k)}} P_{3 \log z} \exp \left( \sum_{m \neq 0} \beta_m^2 z^{-m} \right).
\end{align*}
\]

(2.22)

(2.23)

(2.24)

(2.25)

Here : * : represents normal ordering. We set the operators \( \tilde{\Psi}_{j,I}(z), \tilde{\Psi}_{j,II}(z), \tilde{\Psi}_{j,I}^\dagger(z), \tilde{\Psi}_{j,II}^\dagger(z), \ (j = 1, 2) \) acting on the Fock space \( \mathcal{F}_{J,K} \).

\[
\begin{align*}
\tilde{\Psi}_{j,I}(z) &= \exp \left( -(x - x^{-1}) \sum_{m \geq 0} \frac{x^{km}}{[m]_+} \beta_m^j z^{-m} \right) \\
&\times \exp \left( -\sum_{m \geq 0} x^{-\frac{km}{2}} \gamma_m^j z^m \right) \exp \left( -\sum_{m \geq 0} x^{\frac{(k+1)m}{2}} \gamma_m^j z^{-m} \right), \ (j = 1, 2), \\
\tilde{\Psi}_{j,II}(z) &= \exp \left( (x - x^{-1}) \sum_{m \geq 0} \frac{x^{km}}{[m]_+} \beta_m^j z^m \right),
\end{align*}
\]

(2.26)

(2.27)
where we have set

\[ \psi_j(z) = \exp \left( - \sum_{m>0} x^{\frac{km}{m}} \gamma_m z^m \right) \exp \left( - \sum_{m>0} x^{\frac{-km}{m}} \gamma_m z^m \right), \quad (j = 1, 2), \]

\[ \bar{\psi}_{j,I}(z) = \exp \left( x - x^{-1} \sum_{m>0} x^{\frac{km}{m}} \beta_m z^m \right) \]

\[ \times \exp \left( \sum_{m>0} x^{\frac{km}{m}} \gamma_m z^m \right) \exp \left( \sum_{m>0} x^{\frac{-km}{m}} \gamma_m z^m \right), \quad (j = 1, 2), \]

\[ \bar{\psi}_{j,II}(z) = \exp \left( -(x - x^{-1}) \sum_{m>0} x^{\frac{-km}{m}} \beta_m z^m \right) \]

\[ \times \exp \left( \sum_{m>0} x^{\frac{-km}{m}} \gamma_m z^m \right) \exp \left( \sum_{m>0} x^{\frac{km}{m}} \gamma_m z^m \right), \quad (j = 1, 2). \]

We set the operators \( \psi_{j,I}(z), \bar{\psi}_{j,I}(z), \psi_{j,II}(z), \bar{\psi}_{j,II}(z), (j = 1, 2) \) acting on the Fock space \( \mathcal{F}_{I,K} \).

\[
\begin{align*}
\psi_{1,I}(z) &= \bar{\psi}_{1,I}(z)e^{a_0 + a_1 x^\frac{h}{2} + h_0 + h_1 z - \frac{h}{k}}, \\
\psi_{1,II}(z) &= \bar{\psi}_{1,II}(z)e^{a_0 + a_1 x^\frac{h}{2} + h_0 - h_1 z - \frac{h}{k}}, \\
\psi_{2,I}(z) &= \bar{\psi}_{2,I}(z)e^{-a_0 + a_2 x^\frac{h}{2} + h_0 + h_2 z - \frac{h}{k}}, \\
\psi_{2,II}(z) &= \bar{\psi}_{2,II}(z)e^{-a_0 + a_2 x^\frac{h}{2} + h_0 - h_2 z - \frac{h}{k}}, \\
\psi_{1,I}(z) &= \bar{\psi}_{1,I}(z)e^{-a_0 + a_1 x^\frac{h}{2} - h_0 - h_1 z + \frac{h}{k}}, \\
\psi_{1,II}(z) &= \bar{\psi}_{1,II}(z)e^{-a_0 + a_1 x^\frac{h}{2} - h_0 + h_1 z + \frac{h}{k}}, \\
\psi_{2,I}(z) &= \bar{\psi}_{2,I}(z)e^{a_0 + a_2 x^\frac{h}{2} - h_0 - h_2 z + \frac{h}{k}}, \\
\psi_{2,II}(z) &= \bar{\psi}_{2,II}(z)e^{a_0 + a_2 x^\frac{h}{2} - h_0 + h_2 z + \frac{h}{k}}.
\end{align*}
\]

**Definition 2.1** \( \) We set the operators \( E_j(z), F_j(z), (j = 1, 2) \), which can be regarded as one parameter deformation of the level \( k \) elliptic currents associated with the elliptic algebra \( U_{q,p}(sl_2) \) \( [7, 9] \).

\[
E_j(z) = C_j(z)\psi_j(z), \quad F_j(z) = C_j^\dagger(z)\psi_j^\dagger(z), \quad (j = 1, 2),
\]

where we have set

\[
\psi_j(z) = \frac{1}{x - x^{-1}}(\psi_{j,I}(z) - \psi_{j,II}(z)), \quad \psi_j^\dagger(z) = \frac{-1}{x - x^{-1}}(\psi_{j,I}^\dagger(z) - \psi_{j,II}^\dagger(z)), \quad (j = 1, 2).
\]

We have following proposition as direct consequence of the normal orderings of the basic operators summarized in appendix.

**Proposition 2.2** \( \) The elliptic currents \( E_j(z), (j = 1, 2) \) satisfy the following commutation relations.

\[
[u_1 - u_2]_{r-k}[u_1 - u_2 - 1]_{r-k}E_j(z_1)E_j(z_2)
\]
The currents \( E_j(z) \), \((j = 1, 2)\) satisfy the following commutation relations.

\[
[u_1 - u_2]_r [u_1 - u_2 + 1]_r E_j(z_2) F_j(z_1), \quad (j = 1, 2),
\]

\[
[u_1 - u_2 - \frac{s}{2}]_r [u_1 - u_2 + \frac{s}{2} - 1]_r E_1(z_1) E_2(z_2)
\]

\[
= \left[ u_2 - u_1 + \frac{s}{2} \right]_r \left[ u_2 - u_1 - \frac{s}{2} + 1 \right]_r E_2(z_2) E_1(z_1).
\] (2.41)

The elliptic currents \( F_j(z) \), \((j = 1, 2)\) satisfy the following commutation relations.

\[
[u_1 - u_2]_r [u_1 - u_2 + 1]_r F_j(z_2) F_j(z_1), \quad (j = 1, 2),
\]

\[
[u_1 - u_2 - \frac{s}{2}]_r [u_1 - u_2 + \frac{s}{2} - 1]_r F_1(z_1) F_2(z_2)
\]

\[
= \left[ u_2 - u_1 - \frac{s}{2} \right]_r \left[ u_2 - u_1 + \frac{s}{2} - 1 \right]_r F_2(z_2) F_1(z_1).
\] (2.43)

The currents \( E_j(z) \) and \( F_j(z) \) satisfy

\[
[E_j(z_1), F_j(z_2)] = \frac{x^{(s-1)(s-2)} x^{-1}}{x - x^{-1}} \left( C_j(z_1) C_j^\dagger(z_2) \Psi_{j,I}(z_1) \Psi_{j,I}^\dagger(z_2) : \delta \left( \frac{x^k z_2^2}{z_1^2} \right) \right)
\]

\[
- : C_j(z_1) C_j^\dagger(z_2) \Psi_{j,I}(z_1) \Psi_{j,I}^\dagger(z_2) : \delta \left( \frac{x^{-k} z_2^2}{z_1^2} \right) \right), \quad (j = 1, 2).
\] (2.44)

Here we have used the delta-function \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \).

Upon specialization \( s = 2 \) the currents \( E_1(z), F_1(z) \) degenerate to elliptic currents in [9]. We set \( E_j^{DV}(z) = E_j(z)|_{s=2}, F_j^{DV}(z) = F_j(z)|_{s=2}, \quad (j = 1, 2). \)

\section{Integrals of motion}

In this section we construct infinitely many commutative operators \( G^*_m, G_m, (m \in \mathbb{N}) \), which we call the nonlocal integrals of motion for level \( k \).

\subsection{Nonlocal integrals of motion}

Let us set the theta function \( \vartheta^*_\alpha(u), \vartheta_\alpha(u), (\alpha \in \mathbb{C}) \) by

\[
\vartheta^*(u + 1) = \vartheta^*(u), \quad \vartheta^*(u + r^* \tau^*) = e^{-2\pi \sqrt{1 - r^*^2} \pi - 2\pi \sqrt{1 - r^*^2} \pi} P_0 \Phi(2u - \sqrt{2\pi} P_0 P_0) \vartheta^*(u),
\] (3.1)

\[
\vartheta(u + 1) = \vartheta(u), \quad \vartheta(u + r \tau) = e^{-2\pi \sqrt{1 - r^2} \pi - 2\pi \sqrt{1 - r^2} \pi} P_0 \Phi(2u - \sqrt{2\pi} P_0 P_0) \vartheta(u).
\] (3.2)

Let us use the parametrization \( z_j^{(t)} = x^{2u_j^{(t)}}, (t = 1, 2; j = 1, 2, \cdots, m). \)

\textbf{Definition 3.1} We define the operator \( G^*_m \) for the regime \( \text{Re}(r) > k \) and \( 0 < \text{Re}(s) < 2 \) by

\[
\mathcal{G}_m^* = \int \prod_{j=1}^m \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^m \frac{dz_j^{(2)}}{z_j^{(2)}} E_1(z_1^{(1)}) E_1(z_2^{(1)}) \cdots E_1(z_m^{(1)}) E_2(z_2^{(2)}) \cdots E_2(z_m^{(2)})
\]
We define the operator $G$.

We call the operators $G$ were the integral contour $C$ as below. We note that parameter $s$ analytic continuation. In the limit $G$ The definition of the operators $G$ do not hold for $m$.

\[
\prod_{t=1,2} \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(t)} - u_j^{(t)} \right]_{r-k} \left[ u_j^{(t)} - u_i^{(t)} + 1 \right]_{r-k} \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(1)} - u_j^{(1)} - \frac{s}{2} \right]_{r-k} \left[ u_j^{(2)} - u_i^{(1)} - \frac{s}{2} + 1 \right]_{r-k} \vartheta^* \left( \sum_{j=1}^{m} (u_j^{(2)} - u_j^{(1)}) \right), \tag{3.3}
\]

were the integral contour $C^*$ encircles $z_j^{(t)} = 0$, $(t = 1, 2; j = 1, 2, \cdots, m)$ in such a way that

\[ |z_j^{(t)}| = 1, \quad (t = 1, 2; j = 1, 2, \cdots, m). \]

We define the operator $G_m$ for the regime $\text{Re}(r) > 0$ and $0 < \text{Re}(s) < 2$ by

\[
G_m = \int \cdots \int_{C, \text{Arg}} \prod_{j=1}^{m} \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^{m} \frac{dz_j^{(2)}}{z_j^{(2)}} F_1(z_1^{(1)}) F_2(z_2^{(2)}) \cdots F_1(z_1^{(m)}) F_2(z_2^{(2)}) \cdots F_2(z_2^{(2)}) \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(t)} - u_j^{(t)} \right]_{r-k} \left[ u_j^{(t)} - u_i^{(t)} - 1 \right]_{r-k} \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(1)} - u_j^{(1)} + \frac{s}{2} \right]_{r-k} \left[ u_j^{(2)} - u_i^{(1)} + \frac{s}{2} - 1 \right]_{r-k} \vartheta^* \left( \sum_{j=1}^{m} (u_j^{(1)} - u_j^{(2)}) \right), \tag{3.4}
\]

were the integral contour $C^*$ encircles $z_j^{(t)} = 0$, $(t = 1, 2; j = 1, 2, \cdots, m)$ in such a way that

\[ |z_j^{(t)}| = 1, \quad (t = 1, 2; j = 1, 2, \cdots, m). \]

We call the operators $G_m^*$ and $G_m$ the nonlocal integrals of motion for level $k$.

The definition of the operators $G_m^*$, $G_m$ for generic $s \in \mathbb{C}, (s \neq 2)$ should be understood as analytic continuation. In the limit $s \to 2$, the contour $C^*$, $C$ pinch at $z_j^{(t)} = z_i^{(t')}$. Hence the definition of $G_m^*, G_m$ do not hold for $s = 2$. We give modified definition of $G_m^*, G_m$ for $s = 2$, below. We note that parameter $s \neq 2$ plays an important role in proof of main theorem 3.3.

**Definition 3.2** We define the operator $G_m^{DV*}$ for the regime $\text{Re}(r) > k$ and $s = 2$ by

\[
G_m^{DV*} = \int \cdots \int_{C, \text{Arg}} \prod_{j=1}^{m} \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^{m} \frac{dz_j^{(2)}}{z_j^{(2)}} E_1^{DV}(z_1^{(1)}) \cdots E_1^{DV}(z_1^{(m)}) E_2^{DV}(z_2^{(1)}) \cdots E_2^{DV}(z_2^{(2)}) \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(t)} - u_j^{(t)} \right]_{r-k} \left[ u_j^{(t)} - u_i^{(t)} + 1 \right]_{r-k} \prod_{1 \leq i \leq j \leq m} \left[ u_i^{(1)} - u_j^{(1)} - 1 \right]_{r-k} \left[ u_j^{(2)} - u_i^{(1)} \right]_{r-k} \vartheta^* \left( \sum_{j=1}^{m} (u_j^{(2)} - u_j^{(1)}) \right), \tag{3.5}
\]

were the integral contour $C^*_\text{Arg}$ encircles $z_j^{(t)} = 0$, $(t = 1, 2; j = 1, 2, \cdots, m)$ in such a way that

\[ |x^2 z_m^{(2)}|, |x^{2r} z_m^{(2)}| < |z_1^{(1)}| < |z_1^{(2)}| < |z_2^{(1)}| < |z_2^{(2)}| < \cdots < |z_m^{(1)}| < |z_m^{(2)}|. \]
We define the operator $G^D_m$ for the regime $\text{Re}(r) > 0$ and $s = 2$ by

$$G^D_m = \int \ldots \int_{C_{\text{Arg}}} \prod_{j=1}^{m} \frac{d^2 z_j}{z_j^{(1)}} \prod_{j=1}^{m} \frac{d^2 z_j}{z_j^{(2)}} F^D_1(z_1^{(1)}) \cdots F^D_1(z_m^{(1)}) F^D_2(z_1^{(2)}) \cdots F^D_2(z_m^{(2)})$$

$$\times \prod_{t=1, 2} \prod_{1 \leq i < j \leq m} \left[ u_i^{(t)} - u_j^{(t)} \right] \left[ u_j^{(t)} - u_i^{(t)} \right] - 1 \prod_{1 \leq i < j \leq m} \left[ u_i^{(2)} - u_j^{(2)} \right] \left[ u_j^{(2)} - u_i^{(1)} \right] - 1 \prod_{j=1}^{m} \frac{d^2 \vartheta \left( \sum_{j=1}^{m} \left( u_j^{(1)} - u_j^{(2)} \right) \right)}{r},$$

(3.6)

were the integral contour $C_{\text{Arg}}$ encircles $z_j^{(t)} = 0$, ($t = 1, 2; j = 1, 2, \ldots, m$) in such a way that

$$|x^2 z_m^{(2)}|, |x^2 z_m^{(2)}| < |z_1^{(1)}| < |z_2^{(1)}| < |z_2^{(2)}| < \cdots < |z_1^{(2)}| < |z_1^{(2)}| < \cdots < |z_m^{(2)}| < |z_m^{(2)}|.$$

### 3.2 Main result

The following is main theorem of this paper.

**Theorem 3.3** For the regime $s \neq 2$ and $\text{Re}(r) > k$, we have

$$[G^n_m, G^n_n] = 0, \quad (m, n \in \mathbb{N}).$$

(3.7)

For the regime $s \neq 2$ and $\text{Re}(r) > 0$, we have

$$[G_m, G_n] = 0, \quad (m, n \in \mathbb{N}).$$

(3.8)

We sketch proof of theorem 3.3. Proof is given as the same manner as level $k = 1$ case [1, 3]. By symmetrization of the screenings $E_j(z)$, the commutation relation $[G^n_m, G^n_n] = 0$ is reduced to the following sufficient condition of the theta functions, which is shown by induction as the same manner as [1, 3]. We note that this symmetrization procedure holds only for $s \neq 2$.

$$\sum_{K \subseteq K^c = \{1, 2, \ldots, n+m\} \backslash \{K = n, L^c = m\}} \sum_{L \subseteq L^c = \{1, 2, \ldots, n+m\} \backslash \{L = n, L^c = m\}} \vartheta^*(\sum_{j \in K^c} u_j^{(2)} - \sum_{j \in L^c} u_j^{(1)}) \vartheta^*(\sum_{j \in K} u_j^{(2)} - \sum_{j \in L} u_j^{(1)})$$

$$\times \prod_{i \in K^c} \prod_{j \in K^c} \prod_{p \in K^c} \prod_{q \in K^c} \frac{u_j^{(2)} - u_p^{(1)} - s}{u_i^{(1)} - u_p^{(1)} - s} \frac{u_j^{(2)} - u_q^{(1)} - s}{u_i^{(1)} - u_q^{(1)} - s} = 0$$

(3.9)
Naively, when we take the limit \( s \to 2 \), it seems that we have \( [G_m^{DV*}, G_n^{DV*}] = 0 \). However, very precisely, in order to take the limit \( s \to 2 \), we have to consider special treatment which we call “renormalized” limit in [1]. Here we state only conjecture on the operator \( G_m^{DV*} \). Theorem 3.3 give a supporting argument of the following conjecture.

**Conjecture 3.4** For the regime \( s = 2 \) and \( \text{Re}(r) > k \) we have

\[
[G_m^{DV*}, G_n^{DV*}] = 0 \quad (m, n \in \mathbb{N}). \tag{3.10}
\]

For the regime \( s = 2 \) and \( \text{Re}(r) > 0 \) we have

\[
[G_m^{DV}, G_n^{DV}] = 0, \quad (m, n \in \mathbb{N}). \tag{3.11}
\]

In this paper we gave one parameter “\( s \)” deformation of level \( k \) free field realization of the screening current of the elliptic algebra \( U_{q,p}(\hat{sl}_2) \). By means of these free field realizations, we constructed infinitely many commutative operators, which we call the nonlocal integrals of motion associated with the elliptic algebra \( U_{q,p}(\hat{sl}_2) \) for arbitrary level \( k \neq 0, -2 \). They are given as integrals involving a product of the screening current and Jacobi elliptic theta functions. The construction of the local integrals of motion \( I_m \) for arbitrary level \( k \) is open problem. Elliptic deformation of the extended Virasoro algebra is needed for this construction.

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**A Normal Ordering**

In appendix we summarize the normal orderings of the basic operators.

\[
C_j(z_1)C_j(z_2) = :: z_1^{\frac{\beta}{k} + 1} \frac{(x^{-2+2k}z_2/z_1;x^{2r^*})_{\infty}(x^{-2}z_2/z_1;x^{2k})_{\infty}}{(x^{2+2k}z_2/z_1;x^{2r^*})_{\infty}(x^{2}z_2/z_1;x^{2k})_{\infty}}, \quad (j = 1, 2), \tag{A.1}
\]

\[
C_1(z_1)C_2(z_2) = :: z_1^{-\frac{2s-2}{k}} \frac{(x^sz_2/z_1;x^{2r^*})_{\infty}(x^{-s}z_2/z_1;x^{2s})_{\infty}}{(x^{-s}z_2/z_1;x^{2s})_{\infty}(x^sz_2/z_1;x^{2r^*})_{\infty}}.
\]
\[ C_2(z_1)C_1(z_2) = \frac{z_1 \tilde{\Psi}}{z_2} \frac{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}, \quad (A.2) \]

\[ C_j^\dagger(z_1)C_j^\dagger(z_2) = \frac{z_1 \tilde{\Psi}_j}{z_2} \frac{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.4) \]

\[ C_j(z_1)C_j(z_2) = \frac{z_1 \tilde{\Psi}_j}{z_2} \frac{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2^2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2^2/z_1; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.7) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.9) \]

\[ \bar{\Psi}_{2,f}(z_1)\bar{\Psi}_{1,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.10) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.11) \]

\[ \bar{\Psi}_{2,f}(z_1)\bar{\Psi}_{1,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.12) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.13) \]

\[ \bar{\Psi}_{2,f}(z_1)\bar{\Psi}_{1,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.14) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.15) \]

\[ \bar{\Psi}_{2,f}(z_1)\bar{\Psi}_{1,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.16) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.17) \]

\[ \bar{\Psi}_{2,f}(z_1)\bar{\Psi}_{1,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.18) \]

\[ \bar{\Psi}_{1,f}(z_1)\bar{\Psi}_{2,f}(z_2) = \frac{(x^{s-2} z_2/z_1; x^{2k})_\infty (x^{s-2+2k} z_2/z_1; x^{2k})_\infty}{(x^{s+2k} z_2/z_1; x^{2k})_\infty (x^{s-2} z_2/z_1; x^{2k})_\infty}, \quad (A.19) \]
\[ \tilde{\Psi}_{2,I}(z_1)\tilde{\Psi}_{1,II}(z_2) = \frac{(x^{-s+2k}z_2/z_1^2; x^{2k})_\infty (x^{2-s+2k}z_2/z_1^2; x^{2k})_\infty}{(x^{s+2k}z_2/z_1^2; x^{2k})_\infty (x^{s-2+2k}z_2/z_1^2; x^{2k})_\infty}, \quad (A.20) \]

\[ \tilde{\Psi}_{1,I}(z_1)\tilde{\Psi}_{2,II}(z_2) = \frac{(x^{-s}z_2/z_1^2; x^{2k})_\infty (x^{2-s}z_2/z_1^2; x^{2k})_\infty}{(x^s z_2/z_1^2; x^{2k})_\infty (x^{s-2}z_2/z_1^2; x^{2k})_\infty}, \quad (A.21) \]

\[ \tilde{\Psi}_{2,II}(z_1)\tilde{\Psi}_{1,II}(z_2) = \frac{(x^{-s+2k}z_2/z_1 II; x^{2k})_\infty (x^{2-s+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{s+2k}z_2/z_1 II; x^{2k})_\infty (x^{s-2+2k}z_2/z_1 II; x^{2k})_\infty}, \quad (A.22) \]

\[ \tilde{\Psi}_{1,II}(z_1)\tilde{\Psi}_{2,II}(z_2) = \frac{(x^{-s}z_2/z_1 II; x^{2k})_\infty (x^{2-s}z_2/z_1 II; x^{2k})_\infty}{(x^s z_2/z_1 II; x^{2k})_\infty (x^{s-2}z_2/z_1 II; x^{2k})_\infty}, \quad (A.23) \]

\[ \tilde{\Psi}_{2,II}(z_1)\tilde{\Psi}_{1,II}(z_2) = \frac{(x^{-s}z_2/z_1 II; x^{2k})_\infty (x^{2-s}z_2/z_1 II; x^{2k})_\infty}{(x^s z_2/z_1 II; x^{2k})_\infty (x^{s-2}z_2/z_1 II; x^{2k})_\infty}, \quad (A.24) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.25) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.26) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.27) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.28) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.29) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.30) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.31) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.32) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.33) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.34) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.35) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.36) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.37) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.38) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2), \quad (A.39) \]

\[ \tilde{\Psi}_{j,II}(z_1)\tilde{\Psi}_{j,II}(z_2) = \frac{1}{(1-z_2/z_1^2)} \frac{(x^{2+2k}z_2/z_1 II; x^{2k})_\infty}{(x^{-2}z_2/z_1 II; x^{2k})_\infty}, \quad (j = 1, 2). \quad (A.40) \]
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