SUMS OVER GRAPHS AND INTEGRATION OVER DISCRETE GROUPOIDS

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Abstract. We show that sums over graphs such as appear in the theory of Feynman diagrams can be seen as integrals over discrete groupoids. From this point of view, basic combinatorial formulas of the theory of Feynman diagrams can be interpreted as pull-back or push-forward formulas for integrals over suitable groupoids.

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Introduction

The basic idea of the theory of Feynman diagrams is that any Gaussian integral can be expanded into a sum over suitable graphs:

$$\int_V \frac{P_T(v)}{\left|\text{Aut } T\right|} e^{S(x,v)} d\mu(v) = \sum_{\Phi \in \Gamma(0)} \frac{Z_{x^*}(\Phi)}{\left|\text{Aut } \Phi\right|}.$$

The aim of this paper is to explain the above formula using the language of integration over discrete groupoids. It should not be read as an introduction to the theory of Feynman diagrams, nor to the theory of integration over discrete groupoids. Rather, this groupoid-theoretic approach to Feynman diagrams should be thought of as something similar to the group-theoretic proof of the formula for the number of combinations of $k$ objects out of $n$. Indeed, that formula has a purely combinatorial nature and can be proved without group theory; moreover, it would be quite strange to introduce groups and group actions only to prove this formula. However, if one is already familiar with basic concepts of group theory, a group-theoretic explanation of the formula is probably clearer than a purely combinatorial

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one. Similarly, the aim of this paper is to show how combinatorial formulas which are typical of the theory of Feynman diagrams can be easily understood as pull-back or push-forward formulas for integrals over suitable groupoids.

Since we need to recall basic facts from the theory of groupoids and of Feynman diagrams, and these are apparently unrelated, we had some difficulty in organizing the material of this paper. We chose to describe all the general categorical features in the first part of the paper, and to show how they appear in the context of Feynman diagrams in the second part. There are other ways of organizing the same material: for instance one could prefer to see each categorical construction acting on Feynman diagrams soon after its abstract description. The diagram below shows how different sections of the paper depend on each other, in order to allow each reader to find his personal way through the paper.

While revising this paper, we learned of [Abd], whose principal motivation is the Jacobian conjecture, who carries out work similar to ours.

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1. Generalities on groupoids

We begin by recalling some standard facts about groupoids (see [SGA72] for details). A groupoid is a small category in which every morphism is an isomorphism. If the group Aut \( x \) is finite for any object \( x \) of the groupoid \( A \), then \( A \) is called a groupoid with finite homs. The smallest full subcategory of a groupoid \( A \) containing all the objects isomorphic to a given object \( x \) is called the connected component of \( x \).

Morphisms between groupoids are functors between them; the category of groupoids will be denoted by the symbol \( \text{Grpd} \). The isomorphism class of an object \( x \)
of the groupoid $A$ will be denoted by the symbol $[x]$, and the set of isomorphism classes of objects of $A$ by the symbol $\mathcal{A}$; any morphism $\pi: A \to B$ of groupoids induces a morphism of sets $\pi: \mathcal{A} \to \mathcal{B}$. More precisely, “isomorphism classes” is a functor $\text{Grpd} \to \text{Sets}$. By abuse of notation, we will denote by the same symbol $[x]$ also the connected component of $A$ containing the object $x$ or even the set of objects of this connected component; when not specified, the exact meaning of the symbol $[x]$ in this paper will always be clear from the context.

Finally, we will say that a groupoid $A$ has a grading on the objects if it is given a groupoid isomorphism $A \simeq \coprod_{n \in \mathbb{N}} A_n$.

A trivial example of a groupoid is a group. Indeed, any group $G$ can be seen as a groupoid with only one object. Any finite group is therefore a groupoid with finite homs. Note that, if $G$ and $H$ are two groups, groupoids morphisms between $G$ and $H$ are the same thing as group homomorphisms between them.

More significant groupoids arise in geometry, see [Bro88]. For instance, for any topological space $X$, the Poincaré groupoid $\pi_1(X)$ is defined as the groupoid having points of $X$ as objects, and paths in $X$ modulo homotopy equivalence as morphisms. For any point $x \in X$ the automorphism group of $x$ as an object of $\pi_1(X)$ is the fundamental group of $X$ based at $x$: $\text{Aut}_{\pi_1(X)}(x) = \pi_1(X, x)$. Any continuous map of topological spaces $X \to Y$ induces a groupoid morphism $\pi_1(X) \to \pi_1(Y)$. Moreover, the connected components of the Poicaré groupoid $\pi(X)$ correspond precisely to the (path) connected components of the topological space $X$.

We end this short section on groupoids by recalling that the product of two groupoids $A$ and $B$ is the groupoid $A \times B$ defined by:

$$\text{Ob}(A \times B) = \text{Ob}(A) \times \text{Ob}(B)$$

$$\text{Hom}_{A \times B}((x_1, y_1), (x_2, y_2)) = \text{Hom}_A(x_1, x_2) \times \text{Hom}_B(y_1, y_2).$$

If the objects of $A$ and $B$ are graded, then the objects of $A \times B$ are graded by

$$(A \times B)_n = \coprod_{m_1 + m_2 = n} (A_{m_1} \times B_{m_2})$$

2. **Groupoid coverings**

In this section we recall a few facts from the theory of groupoid coverings (see [SGA72, Bro88]). A morphism $\pi: A \to B$ of groupoids is called a fibration if, for any morphism $y_1 \to y_2$ in $B$ and any $x_1$ with $\pi(x_1) = y_1$ there exist a morphism $x_1 \to x_2$ in $A$ such that $\pi(x_1 \to x_2) = \{y_1 \to y_2\}$. Such a morphism is called a lifting of $y_1 \to y_2$. If the lifting is unique for any $x_1$ lying over $y_1$, then the fibration $\pi$ is called a groupoid covering. As a consequence of the uniqueness of the lifting, if $\pi: A \to B$ is a groupoid covering then the induced morphisms $\pi: \text{Aut} x \to \text{Aut} \pi(x)$ are injective, for any $x \in \text{Ob}(A)$. This terminology has a clear origin in topology. In fact, if $X$ is a topological space and $E \to X$ a Serre fibration, then $\pi_1(E) \to \pi_1(X)$ is a groupoid fibration, and if $X \to X$ is a covering, then $\pi_1(X) \to \pi_1(X)$ is a groupoid covering. In the case of groups, a groupoid fibration $\pi: G \to H$ is just a surjective group homomorphism, and a covering $\pi: G \to H$ is a group isomorphism.

If $y$ is an object of $B$, we denote by the symbol $y$ the full subcategory of $B$ having only $y$ as an object; the symbol $\pi^{-1}(y)$ denotes the full subcategory of $A$ whose objects are the objects $x$ of $A$ such that $\pi(x) = y$. Note that, if $\pi: A \to B$ is a covering, then also

$$\pi|_{\pi^{-1}(y)}: \pi^{-1}(y) \to y$$
is a covering, for any object \( y \) of \( B \). If \( y \) and \( y' \) are two objects in the same isomorphism class in \( B \), then the subcategories \( \pi^{-1}(y) \) and \( \pi^{-1}(y') \) of \( A \) are isomorphic. Moreover, there is a canonical injection \( \text{Hom}_B(y, y') \rightarrow \text{Iso}(\pi^{-1}(y), \pi^{-1}(y')) \).

Indeed, let \( \phi: y \rightarrow y' \) be an isomorphism. Since \( \pi \) is a covering, for any \( x \in \text{Ob}(\pi^{-1}(y)) \) there exist a unique lifting \( \tilde{\phi}: x \rightarrow x' \) of \( \phi \). The map \( \tilde{\phi}: \pi^{-1}(y) \rightarrow \pi^{-1}(y') \) is an isomorphism, and the map \( \phi \mapsto \tilde{\phi} \) is the claimed injection.

When \( y = y' \), we obtain an action of \( \text{Aut} y \) on \( \pi^{-1}(y) \). It is immediate to see that this action restricts to a transitive action on the set

\[
[x]_y := \text{Ob} (\pi^{-1}(y) \cap [x])
\]

for any \( x \in \text{Ob}(\pi^{-1}(y)) \). On the other hand, since \( \pi(x) = y \), any \( \phi \in \text{Aut} y \) lifts to some \( \tilde{\phi}: x \rightarrow x' \). Therefore, \( \phi \) stabilizes \( x \) if and only if it lifts to some \( \psi: x \rightarrow x \), i.e., \( \phi = \pi(\psi) \) for some \( \psi \in \text{Aut} x \). This means that \( \text{Stab} x = \pi(\text{Aut} x) \), and we have a canonical isomorphism of \( \text{Aut} y \)-sets

\[
\text{Aut} y/\pi(\text{Aut} x) \simeq [x]_y \quad (2.1)
\]

If \( p: \tilde{X} \rightarrow X \) is a covering of (path connected) topological spaces, the isomorphism \((2.1)\) above is just the well known isomorphism

\[
p^{-1}(x_0) \simeq \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, \tilde{x}_0),
\]

where \( \tilde{x}_0 \) is any point in the fibre \( p^{-1}(x_0) \).

A covering \( \pi: A \rightarrow B \) is called finite if, for any \( y \in \text{Ob}(B) \), the pre-image \( \pi^{-1}(y) \) consists of finitely many objects. Since the map \( \pi: \text{Aut} x \rightarrow \text{Aut} y \) is injective by definition of covering, if \( \text{Aut} y \) is finite the isomorphism \((2.1)\) implies

\[
\frac{|\text{Aut} y|}{|\text{Aut} x|} = |[x]_y| \quad (2.2)
\]

The degree of a finite covering \( \pi: A \rightarrow B \) is the function

\[
\deg \pi: \text{Ob}(B) \rightarrow \mathbb{N}
\]

\[
y \mapsto |\text{Ob}(\pi^{-1}(y))|
\]

It is immediate to compute

\[
\deg \pi(y) = \sum_{[x] \in \pi^{-1}([y])} |\text{Ob}(\pi^{-1}(y) \cap [x])| = \sum_{[x] \in \pi^{-1}([y])} |[x]_y| \quad (2.3)
\]

If \( y \simeq y' \), then \( \pi^{-1}(y) \) and \( \pi^{-1}(y') \) are isomorphic, so the degree of a covering is constant on isomorphism classes, and defines a map

\[
\deg \pi: \mathcal{B} \rightarrow \mathbb{N}.
\]

A finite covering of groupoids \( \pi: A \rightarrow B \) is called homogeneous if its degree is constant on \( B \).

### 3. Measures and Integration on Discrete Groupoids

Groupoids can be topologized; in this paper we will be interested only in groupoids endowed with the discrete topology, which will be called discrete groupoids. Clearly, any groupoid can be topologized as a discrete groupoid. The coarse space of a discrete groupoid \( A \) is the set \( \mathcal{A} \) of isomorphism classes of objects of \( A \), endowed with the discrete topology. Any morphism \( \pi: A \rightarrow B \) induces a map between the coarse spaces: \( \pi: \mathcal{A} \rightarrow \mathcal{B} \). The discrete measure \( \mu_\mathcal{A} \) on the coarse space of a discrete groupoid \( A \) with finite homs is defined as

\[
\mu_\mathcal{A}([x]) := \frac{1}{|\text{Aut} x|}, \quad \forall [x] \in \mathcal{A},
\]

\[(3.1)\]
where $x$ is any object in the class $[x]$. It is immediate from the definition above that the coarse space of $A \times B$ is $A \times B$ and that

$$\mu_{A \times B} = \mu_A \otimes \mu_B$$

Let now $V$ be a $C$-vector space or a module over some commutative unitary $C$-algebra $R$. We can look at $V$ as a trivial groupoid, by setting

$$\text{Hom}_V(v_1, v_2) = \begin{cases} \{\text{Id}_{v_1}\} & \text{if } v_1 = v_2 \\ \emptyset & \text{if } v_1 \neq v_2 \end{cases}$$

Clearly, a morphism $A \to V$ is just a map $\text{Ob}(A) \to V$ which is constant on isomorphism classes, i.e., a map $A \to V$.

We are now interested in defining integrable functions on the coarse space $A$ of a groupoid $A$. Since the measure we are considering is discrete, an integrable function will have to vanish outside a countable subset of $A$; to simplify our treatment, we assume the coarse space $A$ we are going to integrate on is countable. Moreover, since integrals will be defined by a limiting procedure, we will have to work with topological algebras and modules (see [AGM96] for details). The base field $C$ will always be given the Euclidean topology.

**Definition 3.1.** Let $A \simeq \bigsqcup_n A_n$ be a discrete groupoid with finite homs and graded objects, such that the coarse spaces $\{A_n\}$ are finite sets and let $V$ be a topological module over a topological commutative $C$-algebra $R$. We say that a morphism $\varphi: A \to V$ is integrable if the series

$$\sum_{n=0}^{\infty} \left( \sum_{[x] \in A_n} \frac{\varphi(x)}{\text{Aut } x} \right)$$

(3.2)

is convergent. If so, we write

$$\int_A \varphi \, d\mu_A = \sum_{[x] \in A} \frac{\varphi(x)}{\text{Aut } x}$$

For instance, if we consider the trivial groupoid $\mathbb{N}$ on the set of natural numbers and the gradation $\{\mathbb{N}_n\}$ given by $\mathbb{N}_n = \{n\}$, then a morphism $a: \mathbb{N} \to C$

$$a_n \mapsto a_n$$

is integrable if and only if the series $\sum_{n=0}^{\infty} a_n$ is convergent. We now prove a criterion for integrability of a morphism $\varphi: A \to V$.

**Lemma 3.1.** Let $V$ be a graded complete $R$-module, and let $\varphi: A \to V$ be a grading-preserving morphism. Then $\varphi$ is integrable.

**Proof.** Since $V$ is a graded complete module, $V$ is the completion of $\bigoplus_{n=0}^{\infty} V_n$ in the product topology, where $V_n$ is the submodule of $V$ consisting of elements of degree $n$. Therefore, a sequence converges in $V$ if and only if its $n$-degree components converge in $V_n$, for every $n$. Since the morphism $\varphi$ is grading-preserving and $A$ contains only finitely many elements for any fixed $n$, the $n$-degree component of the sequence of the partial sums (3.2) stabilizes for any fixed $n$, hence the statement. $\Box$

If $A$ is a graded discrete groupoid, we denote by $A_n$ the free $C$-vector space generated by the elements of $A$ of degree $n$, endowed with the Euclidean topology, and by $A$ the direct sum of these spaces:

$$A := \bigoplus_{n \in \mathbb{N}} A_n.$$
Finally, let $A$ be the completion of $A$ with respect to the product topology. Lemma 3.1 immediately implies that the natural embedding $j_A : A \to A$ is integrable, so that we can write

$$\int_A j_A d\mu_A = \sum_{[x] \in A} \frac{[x]}{|\text{Aut} x|};$$

when no confusion is possible, we will omit the subscript $A$ from $j$ and $A$ from $d\mu$, i.e., we will simply write $\int_A j d\mu$ for $\int_A j_A d\mu_A$.

The integral of $j_A$ over $A$ is called the partition function of the groupoid $A$. Note that for any integrable morphism $\varphi : A \to V$

$$\int_A \varphi d\mu_A = \varphi \left( \int_A j_A d\mu_A \right)$$

It is immediate to check that, for any two discrete groupoids with finite homs and graded objects,

$$\int_{A \sqcup B} j_{A \sqcup B} d\mu_{A \sqcup B} = \left( \int_A j_A d\mu_A \right) \oplus \left( \int_B j_B d\mu_B \right)$$

and

$$\int_{A \times B} j_{A \times B} d\mu_{A \times B} = \left( \int_A j_A d\mu_A \right) \otimes \left( \int_B j_B d\mu_B \right).$$

If $\varphi_A : A \to V$ and $\varphi_B : B \to W$ are two integrable morphisms, then also $\varphi_A \oplus \varphi_B$ is integrable, and

$$\int_{A \sqcup B} (\varphi_A \oplus \varphi_B) d\mu_{A \sqcup B} = \left( \int_A \varphi_A d\mu_A \right) \oplus \left( \int_B \varphi_B d\mu_B \right).$$

Moreover, if $\varphi_A$ and $\varphi_B$ are graded, then $\varphi_A \otimes \varphi_B$ is graded (therefore integrable) and

$$\int_{A \times B} (\varphi_A \otimes \varphi_B) d\mu_{A \times B} = \left( \int_A \varphi_A d\mu_A \right) \otimes \left( \int_B \varphi_B d\mu_B \right).$$

A pull-back formula holds for integration over discrete groupoids.

**Lemma 3.2.** Let $\pi : A \to B$ be a finite covering of discrete groupoids with finite homs. Then

$$\int_A \pi^* j_B d\mu_A = \int_B (\deg \pi \cdot j_B) d\mu_B$$

In particular, for a homogeneous covering,

$$\int_A \pi^* j_B d\mu_A = (\deg \pi) \int_B j_B d\mu_B$$

**Proof.** By formulas (2.2) and (2.3), we have

$$\int_A \pi^* j_B d\mu_A = \sum_{[x] \in A} \frac{[\pi(x)]}{|\text{Aut} x|} = \sum_{[y] \in B} \left( \sum_{[x] \in \pi^{-1}([y])} \frac{[\pi(x)]}{|\text{Aut} x|} \right)$$

$$= \sum_{[y] \in B} \left( \sum_{[x] \in \pi^{-1}([y])} \frac{1}{|\text{Aut} x|} \right) [y] = \sum_{[y] \in B} \left( \sum_{[x] \in \pi^{-1}([y])} [[x]_y] \right) \frac{[y]}{|\text{Aut} y|}$$

$$= \sum_{[y] \in B} \deg \pi(y) \frac{[y]}{|\text{Aut} y|} \square$$

As an immediate consequence we get
Proposition 3.1 (The pull-back formula). If \( \pi: A \to B \) is a homogeneous covering of discrete groupoids with finite homs, then, for any integrable morphism \( \varphi: B \to V \) one has

\[
\int_A \pi^* \varphi \, d\mu_A = (\deg \pi) \int_B \varphi \, d\mu_B
\]

We end this section by introducing the concept of push-forward of an integrable morphism and prove the Fubini theorem and the push-pull formula in this context.

Definition 3.2. Let \( \pi: A \to B \) be a finite covering of discrete groupoids with finite homs, and let \( \varphi: A \to V \) be an integrable morphism. The push-forward of \( \varphi \) is the morphism \( \pi_* \varphi: B \to V \) defined by

\[
(\pi_* \varphi)(y) = \sum_{x \in \text{Ob}(\pi^{-1}(y))} \varphi(x)
\]

Note that the elements in the sum on the right-hand side are not weighted by the factors \( 1/|\text{Aut} x| \), that is, the push-forward morphism \( \pi_* \varphi \) is not an integral over the fibre. This apparently unnatural choice has two main motivations. The first is that the combinatorial relation between a coordinate-free Feynman diagrams expression and its reformulation in terms of a fixed system of coordinates is a push-forward according to the above definition. The second is that we want Fubini’s theorem to hold.

Proposition 3.2 (Fubini’s theorem). Let \( \pi: A \to B \) be a finite covering of discrete groupoids with finite homs, and let \( \varphi: A \to V \) be an integrable morphism. Then

\[
\int_B \pi_* \varphi \, d\mu_B = \int_A \varphi \, d\mu_A \tag{3.3}
\]

Proof. By definition of push-forward,

\[
\int_B \pi_* \varphi \, d\mu_B = \sum_{[y] \in B} \frac{1}{|\text{Aut} y|} \left( \sum_{x \in \text{Ob}(\pi^{-1}(y))} \varphi(x) \right)
\]

\[
= \sum_{[y] \in B} \left( \sum_{[x] \in \pi^{-1}([y])} \frac{|[x]|}{|\text{Aut} y|} \varphi(x) \right).
\]

We now use equation (2.2) to rewrite the right-hand term as

\[
\sum_{[y] \in B} \left( \sum_{[x] \in \pi^{-1}([y])} \frac{\varphi(x)}{|\text{Aut} x|} \right) = \int_A \varphi \, d\mu_A.
\]

As a consequence, we find

Proposition 3.3 (The push-pull formula). Let \( \pi: A \to B \) be a finite homogeneous covering of discrete groupoids with finite homs, and let \( \varphi: A \to V \) be an integrable morphism. Then

\[
\int_A \pi^* \pi_* \varphi \, d\mu_A = (\deg \pi) \int_A \varphi \, d\mu_A \tag{3.4}
\]
4. Symmetric powers of groupoids

One of the basic equations of the theory of Feynman diagrams states that exponentiation changes a sum over connected diagrams into a sum over all diagrams. The proof of this relation is clearer in the general context of integration over groupoids, so let us consider an arbitrary discrete groupoid with finite homs $A$, and let $S_n$ be the simply connected groupoid having the symmetric group $\Sigma_n$ as set of objects. The $n$-th symmetric product of a groupoid $A$ with itself is the quotient groupoid $\text{Sym}_n(A) = A^{\times n}/\Sigma_n$.

More explicitly, the groupoid $\text{Sym}_n(A)$ has the same objects as $A^{\times n}$, but has additional morphisms given by the action of the permutation group. For instance, the simply connected groupoid having the symmetric group $\Sigma_n$ is freely generated by

$$\int_{\text{Sym}_n(A)} \! j_{\text{Sym}_n(A)} \, d\mu_{\text{Sym}_n(A)} = \frac{1}{n!} \int_{A^{\times n} \times S_n} \pi^* j_{\text{Sym}_n(A)} \, d\mu_{A^{\times n} \times S_n}$$

where

$$\pi: A^{\times n} \times S_n \to \text{Sym}_n(A)$$

is a groupoid covering. For any $(x_1, \ldots, x_n)$ in $\text{Sym}_n(A)$, the objects of the fibre are the $(n+1)$-ples $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \sigma^{-1})$, which bijectively correspond to the elements $\sigma$ in the symmetric group $\Sigma_n$, so that $\deg \pi = n!$. In particular, it follows that

$$\int_{\text{Sym}_n(A)} \! j_{\text{Sym}_n(A)} \, d\mu_{\text{Sym}_n(A)} = \frac{1}{n!} \int_{A^{\times n} \times S_n} \pi^* j_{\text{Sym}_n(A)} \, d\mu_{A^{\times n} \times S_n}$$

$$= \frac{1}{n!} \pi \left( \left( \int_{A^{\times n}} j_{A^{\times n}} \, d\mu_{A^{\times n}} \right) \otimes \left( \int_{S_n} j_{S_n} \, d\mu_{S_n} \right) \right)$$

$$= \frac{1}{n!} \pi \left( \int_{A^{\times n}} j_{A^{\times n}} \, d\mu_{A^{\times n}} \otimes [e] \right)$$

$$= \frac{1}{n!} \pi \left( \int_{A^{\times n}} j_{A^{\times n}} \, d\mu_{A^{\times n}} \right), \quad \text{(4.1)}$$

where $e$ is the unit element in $\Sigma_n$ and the $\iota$ in the last equation is the immersion of $A^{\times n}$ in $\text{Sym}_n(A)$.

Let now $R$ be a commutative topological $\mathbb{C}$-algebra with unit, and let $\varphi: A \to R$ be any morphism. The product in $R$ can be seen as a tensor product on the trivial groupoid having the elements of $R$ as objects. Since $A^\times$ is freely generated by $A$ as a symmetric monoidal category, $\varphi$ uniquely extends to a tensor functor $\varphi: A^\times \to R$. Explicitly,

$$\varphi(x_1, \ldots, x_n) = \varphi(x_1) \cdots \varphi(x_n)$$

Due to the commutativity of $R$, the restriction of $\varphi$ to $A^{\times n}$ is $\Sigma_n$-invariant, i.e., $\varphi$ is a tensor functor $\text{Sym}(A) \to R$. If $\varphi: A \to R$ is graded, then also $\varphi: \text{Sym}(A) \to R$.
is. Therefore, the diagram

\[
\begin{array}{c}
A^\times n \\
\downarrow \varphi \\
\text{Sym}_n(A)
\end{array} \quad \begin{array}{c}
\varphi \\
\uparrow \\
R
\end{array}
\]

commutes, and (4.1) gives

\[
\int_{\text{Sym}_n(A)} \varphi \, d\mu_{\text{Sym}_n(A)} = \frac{1}{n!} \int_{A^\times n} \varphi \, d\mu_{A^\times n}.
\]  

(4.2)

Assume that the series \( \sum_{n=0}^{\infty} \frac{r^n}{n!} \) is convergent in \( R \) and write

\[
\exp(r) = \sum_{n=0}^{\infty} \frac{r^n}{n!}.
\]

With these notations, we have

**Proposition 4.1.** Let \( A \) be a discrete groupoid with finite homs having no objects of degree zero, and let \( R \) be any complete graded \( \mathbb{C} \)-algebra with unit. Then, for any graded functor \( \varphi: A \to R \), the following identity holds:

\[
\exp \left\{ \int_A \varphi \, d\mu_A \right\} = \int_{\text{Sym}(A)} \varphi \, d\mu_{\text{Sym}(A)}.
\]

(4.3)

**Proof.** By definition of exponential and using (4.2) we find

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_A j_A \, d\mu_A \right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi \left( \int_A j_A \, d\mu_A \right) \otimes^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \varphi \left( \int_{A^\times n} j_{A^\times n} \, d\mu_{A^\times n} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{A^\times n} \varphi \, d\mu_{A^\times n}
\]

\[
= \sum_{n=0}^{\infty} \int_{\text{Sym}_n(A)} \varphi \, d\mu_{\text{Sym}_n(A)}.
\]

\[\square\]

5. **Feynman diagrams**

A Feynman diagram is, basically, a graph with a distinguished subset of 1-valent vertices (the endpoints) and some additional structure on the other vertices (the internal vertices). Moreover Feynman diagrams can be given a splitting of the endpoints into inputs and outputs. The additional structure on the internal vertices are a colour decorating the vertex and a combinatorial datum adding some “rigidity”, i.e., reducing the automorphism group of the germs of edges stemming from a vertex (the legs of the vertex) to a proper subgroup of the symmetric group. In full generality, this combinatorial datum could be any structure on the finite set of the legs of a given vertex; on the other hand, the essence of Feynman diagrams is their being a powerful graphical tool for computing asymptotic expansions of Gaussian integrals. Therefore, we will consider only those combinatorial data that the legs of a vertex “naturally inherits by drawing a graph on a sheet of paper”. More specifically, we will consider only the following three kind of vertices: *coupon vertices*, whose legs are split in the two subsets of legs stemming on the top and
on the bottom of the vertex\(^1\), *cyclic vertices* whose legs inherit a cyclic order by the orientation of the sheet of paper, and *symmetric vertices* whose legs inherit no additional structure, so that the automorphism group of a symmetric vertex is the full symmetric group on its legs. Finally, any vertex has an additional decorating *colour*; the sets of colours we can put on coupon, cyclic and symmetric vertices will be denoted by the symbols \(\text{Co}, \text{Cy}, \text{Sy}\) respectively. In general, the colouring sets can vary with the valence, for instance we may have just one colour for 3-valent vertices, and 37 (or even infinite) colours for 4-valent vertices.

In the theory of Feynman diagrams one often deals with huge families of colours decorating the vertices,\(^2\) but only few of them generally occur. To formalize this feature, each set of colours is split into the disjoint subsets of *ordinary* and *special* colours. Moreover, given a Feynman diagram, it is important to have the possibility to distinguish one of its ordinary vertices, and to look at it as a special one. To do this, an identification of the set of ordinary colours with a subset of the special ones is given; this allows us to talk of the special colour corresponding to a given ordinary colour. Colours decorating the vertices will be denoted by Greek letters; the special colour corresponding to the ordinary colour \(\alpha\) will be denoted by \(\alpha\).

We will further assume that, for any valence \(d\), the set of ordinary colours for the \(d\)-valent vertices is finite. In many applications there is just one special and one ordinary colour for each valence, so that the datum of the coloring is actually redundant and is reduced to the label “ordinary” or “special”.

A formal definition of Feynman diagram is the following.

**Definition 5.1.** Let \(n\) be a natural number. A Feynman diagram with \(n\) legs is the following set of data:

1. a 1-dimensional CW-complex \(\Gamma\);
2. \(n\) distinguished 1-valent vertices of \(\Gamma\), called the *endpoints* of \(\Gamma\); the vertices of \(\Gamma\) which are not endpoints will be called *internal vertices*. The germs of the edges stemming from the endpoints are called *legs* of the diagram and are denoted by \(\text{Legs}(\Gamma)\); any edge which does not end in an endpoint will be called an *internal edge* of the diagram;
3. a map \(\text{Internal vertices}(\Gamma) \to \text{Co} \cup \text{Cy} \cup \text{Sy}\) called *decoration*;
4. elements in the pre-image of \(\text{Co}\) are the *coupon* vertices; for any coupon vertex \(v\) of \(\Gamma\), a splitting of the set \(\text{Legs}(v)\) — i.e., germs of edges stemming from \(v\) — into two totally ordered subsets denoted \(\text{In}(v)\) and \(\text{Out}(v)\) respectively. A coupon vertex decorated by the colour \(\alpha\) is depicted by \(\alpha\)\(\rightarrow\)

\[\begin{array}{c}
\vdots \\
\alpha \\
\vdots 
\end{array}\]

with inputs on the lower side and outputs on the upper side, the total order being that induced by the horizontal coordinate in the plane;
5. elements in the pre-image of \(\text{Cy}\) are the *cyclic* vertices; for any cyclic vertex \(v\) of \(\Gamma\), a cyclic order on the set \(\text{Legs}(v)\). A cyclic vertex decorated by the

\(^1\)The reader familiar with Joyal-Street’s tensor calculus [JS91] or Reshetikhin-Turaev’s graphical calculus [RT90, BK01] will find this splitting familiar.

\(^2\)Actually, one encounters also colours decorating the edges. Here we preferred not to work in full generality here to make the exposition clearer. However, a short digression on Feynman diagrams with coloured edges will be done later, relating the colours on the edges with the elements of a chosen basis in a given vector space.
The cyclic order being that inherited by the standard counterclockwise orientation on the plane;

(6) elements in the pre-image of $Sy$ are the symmetric vertices; no additional combinatorial constraints are imposed on a symmetric vertex; a symmetric vertex decorated by the colour $\gamma$ is depicted by

The set of colours are split into the two disjoint subsets of ordinary and special colours. A vertex will be called ordinary or special depending on the colour decorating it. Moreover, an injective map from ordinary to special colours is given and, for any valence $d$, the set of colours decorating $d$-valent ordinary vertices is finite. Isomorphisms between Feynman diagrams are homotopy classes of cellular isomorphisms of the underlying CW-complexes, respecting the additional structures on the vertices.

It is clear from this definition that Feynman diagrams are deeply related with the graphical formalism of [JS91] and [RT90]. If the only morphisms we allow between Feynman diagrams are the isomorphisms, then Feynman diagrams form a groupoid. The groupoid of Feynman diagrams with $n$ legs will be denoted by the symbol $F(n)$. It is immediate to check that the groupoids $F(n)$ are discrete groupoids with finite homs. Moreover, the objects of the groupoids $F(n)$ are graded by the sum of the valencies of the ordinary vertices.

We now add a splitting of the endpoints into “inputs” and “outputs”

**Definition 5.2.** Let $n$ and $m$ be two natural numbers. A Feynman diagram of type $(n,m)$ is the following set of data:

1. a Feynman diagram $\Gamma$ with $(m+n)$ legs;
2. a splitting of Endpoints($\Gamma$) into two subsets $\text{In}(\Gamma)$ and $\text{Out}(\Gamma)$, of cardinality $n$ and $m$ respectively. The set $\text{In}(\Gamma)$ is called the set of the *inputs* of $\Gamma$ and the set $\text{Out}(\Gamma)$ is called the set of the *outputs* of $\Gamma$;
3. two bijections

$$\iota_{\Gamma}: \{1_{\text{in}}, \ldots, n_{\text{in}}\} \to \text{In}(\Gamma)$$
$$\omega_{\Gamma}: \{1_{\text{out}}, \ldots, m_{\text{out}}\} \to \text{Out}(\Gamma)$$

inducing total orders on $\text{In}(\Gamma)$ and $\text{Out}(\Gamma)$.

Isomorphisms between Feynman diagrams of type $(n, m)$ are isomorphisms of the underlying Feynman diagrams with $(m+n)$ legs respecting the additional structure. The groupoid of Feynman diagrams of type $(n, m)$, will be denoted by the symbol $F(n,m)$. There is a natural “forget the numbering” functor

$$F(n, m) \to F(m + n).$$

If $n = m = 0$, this is an isomorphism

$$F(0, 0) \to F(0).$$
We now discuss some 2-categorical features of Feynman diagrams with inputs and outputs.

If \( \Gamma \) is a Feynman diagram of type \((n,m)\), the natural numbers \(n\) and \(m\) are called the source and the target of the Feynman diagram \( \Gamma \) and are denoted by the symbols \( \text{Src}(\Gamma) \) and \( \text{Tgt}(\Gamma) \). If \( \Gamma \) and \( \Phi \) are two Feynman diagrams such that \( \text{Src}(\Gamma) = \text{Tgt}(\Phi) \), then we can form a new Feynman diagram \( \Gamma \circ \Phi \) by identifying the point \( \nu_{\text{in}} \) of \( \Gamma \) with the point \( \nu_{\text{out}} \) of \( \Phi \) for any \( \nu \) between 1 and \( \text{Src} \Gamma = \text{Tgt} \Phi \); we have \( \text{In}(\Gamma \circ \Phi) := \text{In}(\Phi) \) and \( \text{Out}(\Gamma \circ \Phi) := \text{Out}(\Gamma) \). An example is

\[
\begin{array}{ccc}
\text{In} & \circ & \text{Out} \\
1 & = & 2 \\
\end{array}
\]

In this way we have defined a composition 
\[
F(n, m) \times F(k, n) \to F(k, m).
\]

The right way to look at this composition of diagrams is as a category structure on the set of natural numbers where Feynman diagrams of type \((n,m)\) are the morphisms between \(n\) and \(m\) (see [BD01]). Moreover, since \( F(n, m) \) is a groupoid, we have defined a category whose Hom-spaces are groupoids, i.e., if we denote by the symbol \( \mathbb{N} \) the trivial groupoid having the natural numbers as objects, then

\[
F : \mathbb{N} \times \mathbb{N} \to \text{Grpd},
\]

is an enriched category in the sense of Eilenberg-Kelly ([EK, Kel82]). In particular, \( F \) is a 2-category,\(^3\) with the natural numbers as objects. The identity morphism \( j_n : n \to n \) is clearly given by

\[
j_n =
\begin{array}{ccc}
\text{In} & \circ & \text{Out} \\
1 & = & 2 \\
\end{array}
\]

Given any two Feynman diagrams \( \Gamma \) and \( \Phi \), we can make a new Feynman diagram out of them by taking their disjoint union. Some care has to be taken in defining the new numberings on the inputs and the outputs: if we denote the new Feynman diagram by the symbol \( \Gamma \otimes \Phi \), then

\[
(1) \text{Endpoints}(\Gamma \otimes \Phi) := \text{Endpoints}(\Gamma) \cup \text{Endpoints}(\Phi);
(2) \text{In}(\Gamma \otimes \Phi) := \text{In}(\Gamma) \cup \text{In}(\Phi);
(3) \text{Out}(\Gamma \otimes \Phi) := \text{Out}(\Gamma) \cup \text{Out}(\Phi);
(4)
\nu_{\Gamma \otimes \Phi}(\nu) = \begin{cases} 
\nu_{\Gamma}(\nu) & \text{if } 1 \leq \nu \leq \text{Src} \Gamma \\
\nu_{\Phi}(\nu - \text{Src}(\Gamma)) & \text{if } \text{Src}(\Gamma) + 1 \leq \nu \leq \text{Src} \Gamma + \text{Src} \Phi
\end{cases}
(5)
\omega_{\Gamma \otimes \Phi}(\nu) = \begin{cases} 
\omega_{\Gamma}(\nu) & \text{if } 1 \leq \nu \leq \text{Tgt} \Gamma \\
\omega_{\Phi}(\nu - \text{Tgt}(\Gamma)) & \text{if } \text{Tgt}(\Gamma) + 1 \leq \nu \leq \text{Tgt} \Gamma + \text{Tgt} \Phi
\end{cases}
\]

\(^3\)The reader familiar with higher dimensional category theory will immediately notice that composition in \( F \) is not strictly associative and also identity arrows are not strict, being so only up to a natural isomorphism. That is, \( F \) is a bicategory (or weak 2-category) rather than a 2-category. On the other hand, the associativity and unital constrains of the bicategory \( F \) are quite evident, so we preferred not to specify them aiming to a sort of compromise between the complete rigour of higher dimensional category theory and an exposition enjoyable by the non-expert reader.
For instance,

\[
\begin{array}{c}
\alpha_1 \text{in} \quad \otimes \\
\downarrow \downarrow \downarrow \\
\beta \text{out} \\
\end{array}
\quad = 
\begin{array}{c}
\alpha_1 \text{in} \\
\downarrow \downarrow \downarrow \\
\beta \text{out} \\
\end{array}
\]

The choice of the symbol $\Gamma \otimes \Phi$ to denote this graph is not an accident: it actually is a tensor product of morphisms in the enriched category $F$, which is therefore a monoidal 2-category (see, for instance, [KV94, DS97]).

Moreover the Feynman diagram

\[
\sigma_{m,n} = \quad \begin{array}{cccccc}
in & 2 \text{out} & \cdots & n+1 \text{out} & 2+2 \text{out} & \cdots & n \text{out} \\
1 & 2 & \cdots & 1 & 2 & \cdots & 1
\end{array}
\]

is a braiding operator between the tensor product of $m$ with $n$ and the tensor product of $n$ with $m$. This is more evident if one draws $\sigma_{m,n}$ as follows:

\[
\sigma_{m,n} = \quad \begin{array}{cccccc}
in & 2 \text{out} & \cdots & n+1 \text{out} & 2+2 \text{out} & \cdots & n \text{out} \\
1 & 2 & \cdots & 1 & 2 & \cdots & 1
\end{array}
\]

For any $k, m, n \in \mathbb{N}$ and any $\Gamma \in F(k, m)$, there are evident isomorphisms

\[
\sigma_{m,n} \circ (\Gamma \otimes j_n) \simeq (j_n \otimes \Gamma) \circ \sigma_{k,n}
\]

\[
\sigma_{n,m} \circ (j_n \otimes \Gamma) \simeq (\Gamma \otimes j_n) \circ \sigma_{n,k}
\]

\[
(\sigma_{k,m} \otimes j_n) \circ (j_k \otimes \sigma_{m,n}) \simeq \sigma_{k+m,n}
\]

\[
(j_m \otimes \sigma_{k,n}) \circ (\sigma_{k,m} \otimes j_n) \simeq \sigma_{k,m+n}
\]

which satisfy the axioms of 2-braidings [KV94]. Moreover, there is an evident isomorphism of Feynman diagrams

\[
\sigma_{n,m} \circ \sigma_{m,n} \simeq j_{m+n}
\]

so that the braiding $\sigma$ is symmetric. Therefore $F$ is a symmetric monoidal 2-category having the natural numbers as objects; such a structure can be called 2-PROP.\footnote{The reader unfamiliar with the notion of PROP may think to it just as a symmetric monoidal category having the natural numbers as objects: this quite imprecise definition suffices to deal with the use of PROPs done in this paper; see [Ada78] for a formal definition of PROP.}

As a remark, note that any braiding $\sigma_{m,n}$ can be obtained as a composition of elementary braidings of the form

\[
j_\mu \otimes \sigma_{1,1} \otimes j_\nu,
\]

with $\mu + \nu = m + n - 2$.

Finally, if $\Gamma$ and $\Phi$ are two Feynman diagrams, then $\deg(\Gamma \otimes \Phi) = \deg(\Gamma) + \deg(\Phi)$ and, if the composition $\Gamma \circ \Phi$ is defined, $\deg(\Gamma \circ \Phi) = \deg(\Gamma) + \deg(\Phi)$, i.e., the 2-PROP $F$ is a graded 2-PROP.
6. Feynman diagrams with distinguished sub-diagrams

Sums over graphs in the theory of Feynman diagrams are usually sums over Feynman diagrams containing a distinguished sub-diagram. The aim of this section is to formally introduce the concept of Feynman diagram with a distinguished sub-diagram, and to discuss how this concept is related to coverings of groupoids.

We have seen in Section 5 that Feynman diagrams with inputs and outputs can be seen as the 1-morphisms of a 2-category; in particular a composition of Feynman diagrams with inputs and outputs is defined. Also recall that there is a natural “forget the numbering” morphism $F(m,n) \to F(m+n)$, which induces an isomorphism $F(0,0) \simeq F(0)$. It is immediate to check that the projection $F(m,n) \to F(m+n)$ is a homogeneous groupoid covering of degree $(m+n)!$: the objects in the fibre over a Feynman diagram $\Gamma$ are the $(m+n)!$ Feynman diagrams of type $(m,n)$ which are obtained by numbering the endpoints of $\Gamma$ in all possible ways, and the isomorphisms in $F(m+n)$ uniquely lift to isomorphisms in $F(m,n)$.

**Definition 6.1.** Let $\Gamma \in F(n)$ be a Feynman diagram with $n$ legs. We say that $\Gamma$ is a sub-diagram of the Feynman diagram $\Psi \in F(m)$ if there is an isomorphism of Feynman diagrams $\Psi \simeq \Gamma \circ \Phi$ for a suitable pre-image $\Gamma$ of $\Gamma$ in $F(n,0)$, a suitable pre-image $\Psi$ of $\Psi$ in $F(m,0)$ and some Feynman diagram $\Phi \in F(m,n)$.

**Definition 6.2.** Let $\Gamma$ be a Feynman diagram. The groupoid $F_{\Gamma}$ is the subgroupoid of $F$ whose objects are the Feynman diagrams $\Psi$ such that

1. $\Gamma$ is a distinguished sub-diagram of $\Psi$;
2. all the vertices of $\Psi$ outside $\Gamma$ are ordinary;

Morphisms between objects of $F_{\Gamma}$ are the morphisms in $F$ which map the distinguished sub-diagram $\Gamma$ to itself.

Note that $F_\emptyset$ is the groupoid of Feynman diagrams with only ordinary vertices. The objects of the groupoids $F_{\Gamma}$ are graded by the sum of the valencies of the ordinary vertices. Note that, for any $\Gamma$ and any fixed degree $d$ there are only finitely many isomorphism classes of degree $d$ objects of $F_{\Gamma}$.

Observe that, since any two pre-images of $\Gamma$ in $F(n,0)$ may only differ by the action of an element of the symmetric group $S_n$ on the the inputs, any Feynman diagram in $F_{\Gamma}(0)$ can be obtained as the composition of any pre-image of $\Gamma$ in $F(n,0)$ with a suitable Feynman diagram in $F_\emptyset(0,n)$. Namely, the following proposition holds.

**Proposition 6.1.** Let $\Gamma$ be a Feynman diagram with $n$ legs and denote by $\pi^{-1}(\Gamma)$ the fibre over $\Gamma$ in the projection $\pi : F(n,0) \to F(n)$. Then the composition $\circ : \pi^{-1}(\Gamma) \times F_\emptyset(0,n) \to F_{\Gamma}(0)$ is a homogeneous groupoid covering of degree $n!$.

**Proof.** As remarked above, the projection $\pi : F(n,0) \to F(n)$ is a covering of degree $n!$ and the objects in $\pi^{-1}(\Gamma)$ are the same CW-complex underlying $\Gamma$, with the additional datum of a numbering on endpoints. The composition of diagrams induces a map $\circ : \pi^{-1}(\Gamma) \times F_\emptyset(0,n) \to F_{\Gamma}(0)$, which is a degree $n!$ covering, too. Indeed, let $\varphi : \Psi_1 \to \Psi_2$ be a morphism in $F_{\Gamma}(0)$, and let $(\Gamma_1, \Phi_1)$ be a pre-image of $\Psi_1$ in $\pi^{-1}(\Gamma) \times F_\emptyset(0,n)$. The pre-image $(\Gamma_1, \Phi_1)$ is simply obtained by “cutting” $\Psi_1$ along $\Gamma$ and numbering the endpoints on the two pieces in a compatible way. Since $\varphi$ is a morphism in $F_{\Gamma}(0)$, it is a Feynman diagrams isomorphism which preserves $\Gamma$. So it will induce an automorphism of $\Gamma$ and an isomorphism between $(\Psi_1 \setminus \Gamma)$ and $(\Psi_2 \setminus \Gamma)$, where we are looking at these graphs as objects of $F(n)$. Since forgetting the numbering on endpoints is a covering, these two $F(n)$-morphisms can be lifted (uniquely) to a $\pi^{-1}(\Gamma)$-morphism $\Gamma_1 \to \Gamma_2$, and to a $F_\emptyset(0,n)$-morphism $\Phi_1 \to \Phi_2$, respectively.
i.e., the $F_\Gamma(0)$-morphism $\varphi: \Psi_1 \to \Psi_2$ can be lifted to a $(\pi^{-1}(\Gamma) \times F_\emptyset(0,n))$-morphism $(\Gamma_1, \Phi_1) \to (\Gamma_2, \Phi_2)$. Therefore $\circ: \pi^{-1}(\Gamma) \times F_\emptyset(0,n) \to F_\Gamma(0)$ is a fibration. Moreover the lifting is unique, so $\circ$ is a covering.

We are now ready to give a formal treatment of the following basic principle in the combinatorics of Feynman diagrams: if $F_1$, $F_2$ and $F_3$ are three families of Feynman diagrams such that any diagram in $F_1$ is the composition of a diagram in $F_2$ with a diagram in $F_3$, then summing over $F_1$ is the same thing as summing over $F_2$ and $F_3$ and then composing the two sums:

$$\int_{F_1} \! j \, d\mu = \left( \int_{F_2} \! j \, d\mu \right) \circ \left( \int_{F_3} \! j \, d\mu \right).$$

For instance, if $\Gamma$ is a Feynman diagram with $n$ legs, then any Feynman diagram in $F_\Gamma(0)$ can be obtained by the composition of a preimage of $\Gamma$ in $F(n,0)$ with some Feynman diagram in $F_\emptyset(0,n)$. This implies

$$\int_{F_\Gamma(0)} \! j \, d\mu = \left( \int_{\Gamma} \! j \, d\mu \right) \circ \left( \int_{F_\emptyset(0,n)} \! j \, d\mu \right) \quad (6.1)$$

The proof of (6.1) is almost immediate: from Proposition 6.1 and Proposition 3.1 we know that

$$\int_{F_\Gamma(0)} \! j \, d\mu = \frac{1}{n!} \left( \int_{\pi^{-1}(\Gamma)} \! j \, d\mu \right) \circ \left( \int_{F_\emptyset(0,n)} \! j \, d\mu \right)$$

Due to the lack of an ordering on the legs of $\Gamma$, a composition with $\Gamma$ is not well-defined. On the other hand, any two pre-images of $\Gamma$ in $F(n,0)$ may only differ by the action of an element of the symmetric group $S_n$. Therefore a multiplication by $\Gamma$ is well defined on the space of $S_n$-invariant elements of $F_\emptyset(0,n)$. Thus, we have an operator

$$m_\Gamma: F(0,n)^{S_n} \to F(0). \quad (6.2)$$

In particular,

$$m_\Gamma: F_\emptyset(0,n)^{S_n} \to F_\Gamma(0).$$

Extending (6.2) by linearity one obtains a composition

$$\circ: F(n) \otimes F(0,n)^{S_n} \to F(0). \quad (6.3)$$

The partition function

$$\int_{F_\emptyset(0,n)} j \, d\mu$$

is clearly $S_n$-invariant, so the composition on the right-hand side of (6.1) is well defined. If $\tilde{\Gamma}$ is any object in $\pi^{-1}(\Gamma)$, then the operators $m_\Gamma$ and $m_{\tilde{\Gamma}}$ on the space of $S_n$-invariants of $F_\emptyset(0,n)$ do coincide, so that we have a commutative diagram

$$\begin{array}{ccc}
\pi^{-1}(\Gamma) & \xrightarrow{m} & \text{Hom}(F_\emptyset(0,n)^{S_n}, F_\Gamma(0)) \\
\downarrow & \scriptstyle{\pi} & \downarrow \scriptstyle{m} \\
\Gamma & \xrightarrow{m} & \text{Hom}(F_\emptyset(0,n)^{S_n}, F_\Gamma(0)) \\
\end{array}$$

and Proposition 3.1 implies the following identity in $\text{Hom}(F_\emptyset(0,n)^{S_n}, F_\Gamma(0))$:

$$m \left( \int_{\pi^{-1}(\Gamma)} j \, d\mu \right) = n! \cdot m \left( \int_{\Gamma} j \, d\mu \right)$$

and (6.1) is proven.

---

5We are using the notations from Section 3: if $A$ is a groupoid, then $A$ denotes the set of isomorphism classes of objects in $A$ and $A'$ denotes the free $\mathbb{C}$-vector space generated by $A$. 
Another point of view on diagrams in $F\Gamma(0)$ is the following: any Feynman diagram is built up of vertices joined by edges; a Feynman diagram containing $\Gamma$ as a distinguished sub-diagram is built by joining edges and vertices to $\Gamma$. This implies
\[
\int_{\mathcal{F}\Gamma(0)} j d\mu = \left( \int_\mathcal{V} j d\mu \otimes \exp \left\{ \int_\mathcal{E} j d\mu \right\} \right) \circ \int_{\mathcal{E}(0,\ast)} j d\mu ,
\]
where $\mathcal{V}$ and $\mathcal{E}$ stand for “vertices” and “edges” respectively. To give a formal proof of equation (6.4) we need some definitions. The groupoid $\mathcal{E}$ is the subgroupoid of $\mathcal{F}$ consisting of Feynman diagrams whose connected components are edges with two distinct endpoints. It is immediate from the definition of $\mathcal{E}$ that $\mathcal{E}(0,2n+1)$ is empty. Moreover, any object of $\mathcal{E}(0,2n)$ has only trivial automorphisms, so that for any positive integer $n$
\[
\int_{\mathcal{E}(0,2n)} j d\mu = \sum_{p \in P_{2n}} \left( \prod_{i=1}^n p_{2i-1}^{p_{2i} - 1} \right) \bigotimes_{\text{out}} \bigotimes_{\text{out}} \bigotimes_{\text{out}}
\]
where $p$ ranges in the set $P_{2n}$ of all partitions $\{p_1, p_2\}, \{p_3, p_4\}, \ldots$ of $\{1, 2, \ldots, 2n\}$ in 2-element subsets. For instance, for $n = 2$ we have
\[
\int_{\mathcal{E}(0,4)} j d\mu = \bigcup_{\text{out}} \bigcup_{\text{out}} + \bigcup_{\text{out}} \bigcup_{\text{out}} + \bigcup_{\text{out}} \bigcup_{\text{out}} + \bigcup_{\text{out}} \bigcup_{\text{out}} \bigcup_{\text{out}} .
\]
Note that the element defined by equation (6.5) is $S_{2n}$-invariant. The groupoid $\mathcal{E}(0,\ast)$ is defined as the union $\cup_n \mathcal{E}(0, n)$. If we denote by $\mathcal{V}$ the groupoid whose objects are ordinary vertices, then the groupoid whose objects are disjoint unions of ordinary vertices is the symmetric power $\text{Sym}(\mathcal{V})$, so that
\[
\int_{\text{Sym}(\mathcal{V})} \varphi d\mu_{\text{Sym}(\mathcal{V})} = \exp \left\{ \int_\mathcal{V} \varphi d\mu_\mathcal{V} \right\}
\]
by Proposition 4.1. Then, if we define $\Phi_1 \circ \Phi_2$ to be zero when $|\text{Out}(\Phi_2)| \neq |\text{In}(\Phi_1)|$, equation (6.4) follows reasoning as in the proof of equation (6.1).

A variant of equation (6.4) is the following. If $\Gamma$ is a Feynman diagram, let $\Gamma$ be the subgroupoid of $F\Gamma(0)$ whose objects are the Feynman diagrams that can be obtained by joining the legs of $\Gamma$ by edges in all possible ways. Such diagrams will be called the closures of $\Gamma$. With these notations,
\[
\int_\Gamma j d\mu = \int_\mathcal{E}(0,\ast) j d\mu \circ \int_{\mathcal{E}(0,\ast)} j d\mu .
\]
Clearly, if $\Gamma$ has an odd number of legs, then both sides of (6.6) are zero.

7. FEYNMAN ALGEBRAS

This section is concerned with the linear representations of Feynman diagrams. More precisely, if we denote by $\mathcal{F}$ the PROP of sets obtained from the 2-PROP $\mathcal{F}$ by taking isomorphism classes of morphisms as morphisms, then the functor free vector space (over the field $\mathbb{C}$) changes $\mathcal{F}$ into a PROP $\mathcal{F}$ of vector spaces: explicitly, $\mathcal{F}(m, n)$ is the free $\mathbb{C}$-vector space generated by isomorphism classes of Feynman diagrams of type $(m, n)$.

**Definition 7.1.** A Feynman algebra is an algebra over the PROP $\mathcal{F}$. Equivalently, it is a symmetric monoidal functor
\[
Z : \mathcal{F} \to \text{Vect} ,
\]
where \textbf{Vect} denotes the category of \( C \)-vector spaces. The morphism \( Z \) is called \textit{graphical calculus}; the operator \( Z(\Gamma) \) corresponding to a Feynman diagram \( \Gamma \) is called \textit{amplitude} of the diagram\(^6\).

Since the category of objects of \( \mathcal{F} \) is \( \mathbb{N} \), which is generated by 1 as a symmetric monoidal category, the image of \( Z_{\text{Ob}} \) will be generated by the vector space \( V = Z_{\text{Ob}}(1) \), i.e., an \( \mathcal{F} \)-algebra is actually a representation

\[
Z : \mathcal{F} \rightarrow \text{End}(V) ,
\]

where \( \text{End}(V) \) denotes the endomorphisms PROP of \( V \). In more colloquial terms, a Feynman algebra is the datum of a family of morphisms

\[
Z_{m,n} : \mathcal{F}(m,n) \rightarrow \text{Hom}(V^\otimes m, V^\otimes n)
\]

respecting the braiding and compositions and tensor products of diagrams. For instance, the fact that \( Z \) must respect the braiding forces

\[
Z\left(\begin{array}{cc}
1_{\text{out}} & 2_{\text{out}} \\
1_{\text{in}} & 2_{\text{in}}
\end{array}\right) = \sigma_{V,V}
\]

namely

\[
Z\left(\begin{array}{cc}
1_{\text{out}} & 2_{\text{out}} \\
1_{\text{in}} & 2_{\text{in}}
\end{array}\right) (v_1 \otimes v_2) = v_2 \otimes v_1 .
\]

Informally speaking, Feynman diagrams are freely generated by vertices and edges, i.e., a Feynman diagram can be built by joining together a set of vertices by means of edges in a completely arbitrary way; so one can expect that an \( \mathcal{F} \)-algebra is equivalent to assigning in a free way a value to an edge and to each vertex. In fact, we have the following proposition, which can be read as a formal statement about the forementioned freeness of Feynman diagrams.

**Proposition 7.1.** The datum of a Feynman algebra structure on a \( C \)-vector space \( V \) is the datum of

1. a symmetric non-degenerate bilinear pairing \( \langle , \rangle : V \otimes V \rightarrow \mathbb{C} \);
2. a family \( \{T^\alpha_{m,n}\}_{(m,n) \in \mathbb{N} \times \mathbb{N}} \) of tensors \( T^\alpha_{m,n} : V^\otimes m \rightarrow V^\otimes n \), indexed by the elements \( \alpha \) of the set \( \mathcal{Co} \) of coupon colours;
3. a family \( \{C^\beta_n\}_{n \in \mathbb{N}} \) of tensors \( C^\beta_n : V^\otimes n \rightarrow \mathbb{C} \), invariant with respect to the action of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) on the inputs and indexed by the elements \( \beta \) of the set \( \mathcal{Cy} \) of cyclic colours; we call these tensors cyclic tensors of the Feynman algebra;
4. a family \( \{S^\gamma_n\}_{n \in \mathbb{N}} \) of tensors \( S^\gamma_n : V^\otimes n \rightarrow \mathbb{C} \), invariant with respect to the action of the symmetric group \( \mathcal{S}_n \) on the inputs and indexed by the elements \( \gamma \) of the set \( \mathcal{Sy} \) of cyclic colours; we call these tensors symmetric tensors of the Feynman algebra.

A proof of this statement can be found in [FM02], where techniques derived by the Reshetikhin-Turaev’s graphical calculus (see [BK01, RT90]) are used. We remark that in [FM02] coupon, cyclic and symmetric vertices are treated as distinct cases; yet, the arguments used there apply to Feynman diagrams with the three kinds of vertices occurring at the same time.

Note that no relation is required between the tensors defining the Feynman algebra structure, nor between the tensors and the bilinear pairing \( \langle , \rangle \). This should be thought as an algebraic counterpart of the fact that Feynman diagrams are freely

\(^6\)By abuse of notation we will write \( Z(\Gamma) \) for \( Z(|\Gamma|) \).
generated by the vertices (which correspond to the tensors) and by the edges (which correspond to the pairing).

In the physicists’ parlance, the tensors corresponding to vertices are called interactions and the dual of the pairing corresponding to the edges is called propagator, see for instance [DEF+99].

Recall that the set of colours were split into the two subsets of ordinary and special colours, and that an identification of ordinary colours with a subset of special colours was given. So far, we have not implemented this datum in the definition of Feynman algebra, not to make the definition too involved. Then, let us complete the definition of Feynman algebra, by requiring that an ordinary colour and the corresponding special colour define the same tensor: if \( v^\alpha_{m,n} \) is an ordinary coupon vertex of type \((m,n)\), and \( v^\alpha_{m,n} \) is the corresponding special vertex, then

\[
Z(v^\alpha_{m,n}) = T^\alpha_{m,n} = T^\alpha_{m,n} = Z(v^\alpha_{m,n})
\]

and similarly for cyclic or symmetric vertices. Actually, we will be mostly working with Feynman algebras whose ordinary tensors depend linearly on some complex parameter: if \( Z: \mathcal{F} \to \text{End}(V) \) is a Feynman algebra, we denote by \( Z_x: \mathcal{F} \to \text{End}(V)[x_*] \).

the Feynman algebra obtained by changing the ordinary tensor \( T^\alpha_{m,n} \) (resp. \( C^\beta_n \) and \( S^\gamma_n \)) of \( Z \) with the tensor \( x^\alpha_{m,n}T^\alpha_{m,n} \) (resp. \( x^\beta_nC^\alpha_n \) and \( x^\gamma_nS^\alpha_n \)), where the \( x_* \) are complex variables, and leaving the special tensors unchanged.

To make \( Z_x \) a graded morphism, we put the variables \( x^\alpha_{m,n} \) in degree \( n_m + n_n \) and the variables \( x^\beta_n \) and \( x^\gamma_n \) in degree \( n_n \).

Note that, if \( v^\alpha_n \) is an ordinary (coupon, cyclic or symmetric) \( n \)-valent vertex and \( v^\alpha_n \) is the corresponding special vertex, then by definition of \( Z_x \) we have

\[
\frac{\partial}{\partial x^\alpha_n} Z_x(v^\alpha_n) = Z_x(v^\alpha_n)
\]
i.e., the derivatives of the amplitudes of ordinary vertices with respect to the parameters \( x_* \) can be written as amplitudes of the corresponding special vertices. In particular, this implies that the derivatives of the amplitude of a Feynman diagram \( \Gamma \) with respect to the parameters \( x_* \) can be written as sums of copies of diagrams obtained by changing some ordinary vertex of \( \Gamma \) with the corresponding special vertex, which justify the convention of having the ordinary colours identified with a subset of the special ones.

If \( \Gamma \) is an object of \( \mathcal{F}(n) \) then, due to the lack of an ordering on the legs, \( \Gamma \) does not define a linear operator via the graphical calculus \( Z_x \). Anyway, two pre-images of \( \Gamma \) in \( \mathcal{F}(n,0) \) can only differ by the action of an element of the symmetric group \( \mathfrak{S}_n \), so that a linear operator \( Z_x(\Gamma) \) on the subspace of \( \mathfrak{S}_n \)-invariant vectors of \( V^\otimes n \) is well defined:

\[
Z_x(\Gamma) = Z_x(\hat{\Gamma})|_{(V^\otimes n)^{\mathfrak{S}_n}} : (V^\otimes n)^{\mathfrak{S}_n} \to \mathbb{C},
\]
where \( \hat{\Gamma} \) is any Feynman diagram in the fibre of \( \mathcal{F}(n,0) \to \mathcal{F}(n) \) over \( \Gamma \). Hence we obtain the polynomial function associated to the diagram \( \Gamma \):

\[
P_\Gamma : v \mapsto Z_x(\Gamma)(v^\otimes n)
\]
The association \( \Gamma \mapsto P_\Gamma \) extends to a linear map

\[
P : \mathcal{F}(n) \to \{\text{degree } n \text{ homogeneous polynomials on } V\}
where \( \mathcal{F}(n) \) denotes the \( \mathbb{C} \)-vector space generated by isomorphism classes of objects in \( F(n) \). We remark that sometimes the term “amplitude” is used to denote the polynomial function \( P_\Gamma \) rather than the linear operator \( Z_{x_*}(\Gamma) \).

8. EXPECTATION VALUES

Let now \( Z_{x_*}: F \to \text{End}(V)[[x_*]] \) be a Feynman algebra as defined in Section 7. Since the morphism \( Z_{x_*} \) is graded by construction, it is integrable as a morphism \( Z_{x_*}: F \to \text{End}(V)[[x_*]] \) by Lemma 3.1. The expectation value of \( \Gamma \) is the element of \( \mathbb{C}[x_*] \) defined by

\[
\langle \langle \Gamma \rangle \rangle \;
= \int_{\mathcal{F}_\emptyset(0)} Z_{x_*} \, d\mu,
\]

or, in the more familiar “sums” notation,

\[
\langle \langle \Gamma \rangle \rangle \;
= \sum_{[\Phi] \in \mathcal{F}_\emptyset(0)} \frac{Z_{x_*}(\Phi)}{|\text{Aut } \Phi|}.
\]

The expectation value with potential\(^7\) of \( \Gamma \) is the formal series in the variables \( x_* \) defined by

\[
\langle \langle \Gamma \rangle \rangle_{x_*} \;
= \int_{\mathcal{F}_\emptyset(0)} Z_{x_*} \, d\mu,
\]

or, in the more familiar “sums” notation,

\[
\langle \langle \Gamma \rangle \rangle_{x_*} \;
= \sum_{[\Phi] \in \mathcal{F}_\emptyset(0)} \frac{Z_{x_*}(\Phi)}{|\text{Aut } \Phi|}.
\]

A useful relation among expectation values is the following. Assume that \( \Gamma \) and \( \Phi_1, \ldots, \Phi_k \) are objects of \( F(n) \), for some fixed \( n \), and that, for suitable polynomials \( a_k(x_*) \)

\[
\int_\Gamma Z_{x_*} \, d\mu = \sum_k a_k(x_*) \int_{\Phi_k} Z_{x_*} \, d\mu
\]

as linear operators on \( (V^\otimes n)^{S_n} \). Then, by equations (6.4) and (6.6), it immediately follows

(8.1)

\[
\langle \langle \Gamma \rangle \rangle = \sum_k a_k(x_*) \langle \langle \Phi_k \rangle \rangle \quad \text{and} \quad \langle \langle \Gamma \rangle \rangle_{x_*} = \sum_k a_k(x_*) \langle \langle \Phi_k \rangle \rangle_{x_*}.
\]

In the more familiar “sums” notation, equation (8.1) above is the following identity among polynomials on \( V \):

\[
\frac{P_\Gamma(v)}{|\text{Aut } \Gamma|} = \sum_k a_k(x_*) \frac{P_{\Phi_k}(v)}{|\text{Aut } \Phi_k|}.
\]

9. PARTITION FUNCTIONS AND FREE ENERGY

We have seen that the graphical calculus \( Z_{x_*} \) is integrable on \( F_\Gamma \), for any Feynman diagram \( \Gamma \). The partition function of the Feynman algebra \( Z_{x_*}: F \to \text{End}(V)[[x_*]] \) is the formal series defined as the integral of \( Z_{x_*} \) on the groupoid \( F_\emptyset(0) \) of all Feynman diagrams with no legs and only ordinary vertices, namely,

\[
Z(x_*) := \int_{F_\emptyset(0)} Z_{x_*} \, d\mu = \langle \langle \emptyset \rangle \rangle_{x_*}.
\]

\(^7\)This terminology will be explained in Section 11.
The free energy of the Feynman algebra $Z_{x_*}$ is defined as the integral of $Z_{x_*}$ on the subgroupoid $F_{\emptyset,\text{conn.}}(0)$ of $F_\emptyset(0)$ consisting of connected Feynman diagrams with only ordinary vertices:

$$F(x_*) := \int_{F_{\emptyset,\text{conn.}}(0)} Z_{x_*} \, d\mu, \quad (9.2)$$

In the more familiar “sums” notation

$$Z(x_*) = \sum_{[\Phi] \in F_{\emptyset}(0)} \frac{Z_{x_*}(\Phi)}{|\text{Aut } \Phi|} \quad (9.3)$$

The following well known relation connects the partition function with the free energy:

$$Z(x_*) = \exp \{ F(x_*) \} \quad (9.3)$$

To prove it, just observe that there is an isomorphism of measure spaces

$$F_{\emptyset}(0) \simeq \text{Sym}(F_{\emptyset,\text{conn.}}(0))$$

and apply Proposition 4.1.

We now show how the derivatives of the partition function $Z(x_*) = \langle \emptyset \rangle_{x_*}$ are related to the expectation values of disjoint unions of special vertices. Let $\alpha$ be an ordinary colour for a vertex of $F$ and let $v^\alpha$ and $v^{\alpha^*}$ be the corresponding ordinary and special vertices. By definition of $Z_{x_*}$, we have

$$Z_{x_*}(v^\alpha) = x_\alpha \cdot Z_{x_*}(v^{\alpha^*}),$$

with $Z_{x_*}(v^{\alpha^*})$ which is actually independent of the variables $x_*$. So, if we apply the differential operator $\partial / \partial x^\alpha$ to the partition function $Z(x_*)$, we find

$$\frac{\partial Z(x_*)}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} \left( \left( \exp \left\{ \int_{v} Z_{x_*} \, d\mu \right\} \right) \circ \int_{E(0,*)} Z_{x_*} \, d\mu \right)$$

If $e$ is an edge, the graphical calculus $Z_{x_*}(e)$ is actually independent of $x_*$, and the right-hand side of the above equation equals

$$\left( \left( \int_{e\alpha} Z_{x_*} \, d\mu \right) \otimes \left( \exp \left\{ \int_{v} Z_{x_*} \, d\mu \right\} \right) \right) \circ \int_{E(0,*)} Z_{x_*} \, d\mu$$

$$= Z_{x_*} \left( \left( \int_{e\alpha} j \, d\mu \otimes \exp \left\{ \int_{v} j \, d\mu \right\} \right) \circ \int_{E(0,*)} j \, d\mu \right)$$

$$= Z_{x_*} \int_{F_{\emptyset}} j \, d\mu ,$$

by equation (6.4). But this is just the expectation value with potential of the vertex $v^{\alpha^*}$. So, summing up, we have proved the formula

$$\frac{\partial \langle \emptyset \rangle_{x_*}}{\partial x^\alpha} = \langle v^{\alpha^*} \rangle_{x_*} \quad (9.4)$$

which holds for any ordinary colour $\alpha$ for the vertices of $F$. More generally,

$$\frac{\partial^\ell \langle \emptyset \rangle_{x_*}}{\partial (x^\alpha_1 \cdots \partial x^\alpha_k)_{e_*}} = (e_1! \cdots e_k!) \langle v^{\alpha_1} \otimes \cdots \otimes v^{\alpha_k} \rangle_{x_*} \quad (9.5)$$

for any choice of ordinary colours $\alpha_1, \ldots, \alpha_k$ for the vertices of $F$. 

We end this section with a digression on $\Gamma$-reduced diagrams. An object $\Phi$ of $F_\Gamma$ is called $\Gamma$-reduced if no connected component of $\Phi$ has empty intersection with $\Gamma$. In particular, the only $\emptyset$-reduced Feynman diagram is the empty Feynman diagram. We denote the subgroupoid of $\Gamma$-reduced Feynman diagrams by the symbol $F_{\Gamma, \text{red}}$.

The isomorphism of measure spaces $F_\Gamma(0) \simeq F_{\Gamma, \text{red}}(0) \times F_{\emptyset}(0)$ gives

$$\int_{F_{\Gamma, \text{red}}(0)} Z_{x_\ast} d\mu = \frac{1}{Z(x_\ast)} \int_{F_\Gamma(0)} Z_{x_\ast} d\mu,$$

i.e., in the “sums” notation,

$$\sum_{[\Phi] \in F_{\Gamma, \text{red}}(0)} Z_{x_\ast}(\Phi) \frac{1}{|\text{Aut } \Phi|} = \frac{1}{Z(x_\ast)} \sum_{[\Phi] \in F_\Gamma(0)} Z_{x_\ast}(\Phi) \frac{1}{|\text{Aut } \Phi|}.$$  

10. **Gaussian integrals**

Let $V$ be a finite dimensional real Hilbert space, with inner product $(-\mid-)$. If $\{e_i\}$ is a basis of $V$, we denote the coordinate maps relative to this basis as $e_i: V \to \mathbb{R}$, and write $v^i = e_i(v)$, for any vector $v$ of $V$. Via the inner product of $V$, we can identify the functionals $\{e^i\}$ with vectors of $V$ that we will denote by the same symbols. The vectors $\{e^i\}$ are a basis for $V$, called the dual basis with respect to $\{e_i\}$. The matrix associated to $(-\mid-)$ with respect to $\{e^i\}$ is the matrix $(g_{ij})$ defined by

$$g_{ij} := (e_i|e_j).$$

As customary, we set $g^{ij} := (g^{-1})_{ij} = (e^i|e^j)$.

Let now $dv$ be a (non trivial) translation invariant measure on $V$. The function $e^{-\frac{1}{2} (v|v)}$ is positive and integrable with respect to $dv$. The probability measure on $V$ defined by

$$d\mu(v) = e^{-\frac{1}{2} (v|v)} dv,$$

is called the **Gaussian measure** on $V$. Since a non-trivial translation invariant measure on $V$ is unique up to a scalar factor, $d\mu$ is actually independent of the chosen $dv$.

The inner product $(-\mid-)$ extends uniquely to a symmetric $\mathbb{C}$-bilinear pairing on the complex vector space $V_\mathbb{C} := V \otimes \mathbb{C}$; this pairing, which we shall denote by the same symbol $(-\mid-)$, is clearly non-degenerate. Identify $V$ with the subspace $V \otimes \{1\}$ of real vectors in $V_\mathbb{C}$. Polynomial functions on $V_\mathbb{C}$ are integrable with respect to the Gaussian measure; for any polynomial function $f: V_\mathbb{C} \to \mathbb{C}$ we set

$$\langle f \rangle = \int_V f(v) d\mu(v).$$

The complex number $\langle f \rangle$ is called the **average** of $f$ with respect to the Gaussian measure.

Since the vectors $\{e_i\}$ are a basis for the complex vector space $V_\mathbb{C}$, the vectors $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ are a basis for the vector space $V_\mathbb{C}^{\otimes n}$: any element $v_{[n]}$ of $V_\mathbb{C}^{\otimes n}$ can be uniquely written as

$$v_{[n]} = \sum_{i_1, \ldots, i_n} v_{[n]}^{i_1, \ldots, i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}.$$
For any $v \in V_C$, the vector $v^\otimes n$ is an element of $V_C^\otimes n$, and the following identity holds:

$$v^\otimes n = \sum_{i_1,\ldots,i_n} v^{i_1} \otimes \cdots \otimes v^{i_n}. $$

The functions $v \mapsto v^{i_1} \otimes \cdots \otimes v^{i_n}$ are polynomials on $V_C$ and we can define the average of $v^\otimes n$ as

$$\langle v^\otimes n \rangle := \sum_{i_1,\ldots,i_n} \langle v^{i_1} \otimes \cdots \otimes v^{i_n} \rangle e^{i_1} \otimes \cdots \otimes e^{i_n};$$

it is clearly independent of the basis $\{e_i\}$ chosen.

The golden bridge between Gaussian integrals and Feynman diagrams is the following Lemma, due to Gian Carlo Wick. In its original formulation it is stated in terms of momenta of the Gaussian measure, i.e., averages of monomials in the coordinates $v^i$; see for instance [BIZ80]. Here, using the notion of average of $v^\otimes n$, we recast it in a coordinate-free way, which is more suitable for a reinterpretation through the graphical formalism of the previous sections. Recently, Robert Oeckl has proven that a Wick-type lemma holds in the wider context of general braided tensor categories, see [Oec01].

**Lemma 10.1 (Wick).** Tensor powers of vectors are integrable with respect to the Gaussian measure $d\mu$ and:

1. $\langle v^\otimes 2n+1 \rangle = 0$, (10.1)
2. $\langle v^\otimes 2 \rangle = \sum_{i,j} g^{ij} e_i \otimes e_j$, (10.2)
3. $\langle v^\otimes 2n \rangle = \sum_{i_1,\ldots,i_{2n}, s} g^{i_1s_1} \cdots g^{i_{2n}s_{2n}} e^{i_1} \otimes \cdots \otimes e^{i_{2n}}$, (10.3)

where $P$ denotes the set of all distinct pairings of the set of indices $\{i_1,\ldots,i_{2n}\}$, i.e., over the set of all partitions $\{\{i_{s_1},i_{s_2}\},\{i_{s_3},i_{s_4}\},\ldots\}$ of $\{i_1,i_2,\ldots,i_{2n}\}$ into 2-element subsets.

### 11. Feynman diagrams expansion of Gaussian integrals

We now show how a Gaussian integral can be expanded into a sum of Feynman diagrams, to be evaluated according to the rules of graphical calculus. Historically, Gaussian integrals are the context where Richard Feynman originally introduced the diagrams that nowadays bear his name. The key point will be a graphical interpretation of Wick’s lemma.

Let $Z_{x_*} : F \rightarrow \text{End}(V_C)[x_*]$ be a Feynman algebra compatible with the bilinear pairing $\langle -| - \rangle$, i.e., such that

$$\bigcup_x = (x|y), \quad \forall x, y \in V_C.$$ 

Since the right-hand side of (10.2) in Wick’s lemma is the co-pairing relative to the pairing $\langle -| - \rangle$ on $V_C$, we can rewrite (10.2) as

$$\langle v^\otimes 2 \rangle = Z_{x_*} \left( \bigcup^{2n} \right) = \int_{E(0,2)} Z_{x_*} d\mu.$$ 

Also (10.3) can be expressed as an integral over Feynman diagrams: by (6.5), we have the following recasting of Wick’s lemma in terms of integrals on the groupoid of edges.
Lemma 11.1 (Wick’s lemma via graphical calculus). Tensor powers of vectors are integrable with respect to the Gaussian measure $d\mu$ and:

$$\langle v^{\otimes m} \rangle = \int_{\mathcal{E}(0,m)} Z_{x_*} d\mu.$$  \hspace{1cm} (11.1)

We now introduce the potential of a Feynman algebra. Recall that $V$ denotes the groupoid whose objects are ordinary vertices with no numbering on the legs. The potential of the Feynman algebra $Z_{x_*} : F \to \End(V)[x_*]$ is defined as the formal series

$$S(x_*) := \int_V P d\mu,$$  \hspace{1cm} (11.2)

where $P$ denotes the polynomial function associated to a diagram. In more familiar “sums” notations, equation (11.2) above reads

$$S(x_*; v) := \sum_{[v] \in V} P_{\Gamma}(v) |\text{Aut } v|$$

$$= \sum_{m,n \in \mathbb{N}} \sum_{\alpha \in C_0} x^\alpha_{m,n} \langle T_{m,n}(v^{\otimes m}), v^{\otimes n} \rangle + \sum_{n \in \mathbb{N}} \sum_{\beta \in C_y} x^\beta_n C_n^\beta (v^{\otimes n}) n!$$

$$+ \sum_{n \in \mathbb{N}} \sum_{\gamma \in S_y} S_n^\gamma (v^{\otimes n}) n!$$

If $f$ is a polynomial function on $V$, the average of $f$ with potential $S(x_*)$ is the formal series in the variables $x_*$ defined by

$$\langle f \rangle_{x_*} = \left\langle f \cdot e^{S(x_*)} \right\rangle$$  \hspace{1cm} (11.3)

If $\Gamma$ is a Feynman diagram, the function $P_{\Gamma}$ is a polynomial on $V$. Therefore we can consider the average of $P_{\Gamma}$ with respect to the Gaussian measure. The following result shows that our notations are consistent.

Theorem 11.1 (Feynman-Reshetikhin-Turaev). For any Feynman diagram $\Gamma$ the following equations hold:

$$\langle \Gamma \rangle = \left\langle \frac{P_{\Gamma}(v)}{|\text{Aut } \Gamma|} \right\rangle = \int_V \frac{P_{\Gamma}(v)}{|\text{Aut } \Gamma|} d\mu_V(v);$$  \hspace{1cm} (11.4)

$$\langle \Gamma \rangle_{x_*} = \left\langle \frac{P_{\Gamma}(v)}{|\text{Aut } \Gamma|} \right\rangle_{x_*} = \int_V \frac{P_{\Gamma}(v)}{|\text{Aut } \Gamma|} e^{S(x_*; v)} d\mu_V(v).$$  \hspace{1cm} (11.5)

Proof. We prove only (11.4), equation (11.5) being completely analogous. Let $n$ be the number of legs of $\Gamma$; by linearity

$$\frac{1}{|\text{Aut } \Gamma|} (P_{\Gamma}) = \frac{Z_{x_*}(\Gamma)}{|\text{Aut } \Gamma|} \langle v^{\otimes n} \rangle$$

If $n$ is odd, the right-hand side of the above equation is zero by (11.1). On the other hand, a Feynman diagram with an odd number of legs cannot be closed by joining its endpoints by edges, since a disjoint union of edges always has an even number of endpoints; so equation (11.4) is verified for odd $n$. For $n = 2m$ we find, again by (11.1) and by (6.6),

$$\frac{Z_{x_*}(\Gamma)}{|\text{Aut } \Gamma|} \langle v^{\otimes n} \rangle = \int_\Gamma Z_{x_*} d\mu \circ \left( \int_{\mathcal{E}(0,2m)} Z_{x_*} d\mu \right)$$

$$= Z_{x_*} \left( \int_\Gamma j d\mu \circ \int_{\mathcal{E}(0,2m)} j d\mu \right)$$

$$= Z_{x_*} \int_\Gamma j d\mu = \langle \Gamma \rangle$$

□
As a particular case of (11.5), corresponding to $\Gamma = \emptyset$, we have

$$Z(x_\ast) = \int_V e^{S(x_\ast, v)} d\mu_V(v).$$

We will conclude with three examples of equation (11.4) involving a coupon, a cyclic or a symmetric vertex. Equation (11.1), for $n = 2$ gives

$$\langle v^{\otimes 4} \rangle = Z_{x_\ast} \left( \begin{array}{c} 1 \text{out} \ 2 \text{out} \ 3 \text{out} \ 4 \text{out} \\ + \end{array} \right),$$

Consider now a 4-valent coupon vertex decorated by the colour $\alpha$. By linearity,

$$\langle P_{\nu_{\alpha, 0}} \rangle = Z_{x_\ast} \left( \begin{array}{c} \alpha \\ \end{array} \right) \langle v^{\otimes 4} \rangle$$

$$= Z_{x_\ast} \left( \begin{array}{c} \alpha \\ \end{array} \right) \circ Z_{x_\ast} \left( \begin{array}{c} 1 \text{out} \ 2 \text{out} \ 3 \text{out} \ 4 \text{out} \\ + \end{array} \right)$$

$$= Z_{x_\ast} \left( \begin{array}{c} \alpha \\ \end{array} \right) + Z_{x_\ast} \left( \begin{array}{c} \alpha \\ \end{array} \right) + Z_{x_\ast} \left( \begin{array}{c} \alpha \\ \end{array} \right) = \langle \begin{array}{c} \alpha \\ \end{array} \rangle$$

If we consider a 4-valent cyclic vertex decorated by the colour $\beta$ instead, we have

$$\langle P_{\nu_{\beta, 4}} \rangle = \frac{1}{4} Z_{x_\ast} \left( \begin{array}{c} \beta \\ \end{array} \right)$$

$$= Z_{x_\ast} \left( \begin{array}{c} \beta \\ \end{array} \right) + \frac{1}{4} Z_{x_\ast} \left( \begin{array}{c} \beta \\ \end{array} \right) = \langle \begin{array}{c} \beta \\ \end{array} \rangle$$

Finally, if we consider a symmetric 4-valent vertex decorated by the colour $\gamma$, we find

$$\langle P_{\nu_{\gamma, 24}} \rangle = \frac{1}{24} Z_{x_\ast} \left( \begin{array}{c} \gamma \\ \end{array} \right)$$

$$= Z_{x_\ast} \left( \begin{array}{c} \gamma \\ \end{array} \right) = \langle \begin{array}{c} \gamma \\ \end{array} \rangle$$
Appendix: Working with coordinates

In the main body of the paper we have only worked with coordinate-free formulas; since the pairing $v \otimes w \mapsto \langle v, w \rangle$ of a Feynman algebra is a nondegenerate symmetric pairing on a $\mathbb{C}$-vector space $V$, it admits orthonormal bases $\{e_i\}_{i=1 \ldots N}$, and it is possible to write the coordinate version of these formulas with respect to an orthonormal basis $\{e_i\}$ in terms of Feynman diagrams with coloured edges. In particular, when the Feynman algebra is related to the Feynman diagram expansion of a Gaussian integral, the vector space $V$ is the complexification of a real Hilbert space $V_R$ and one can choose $\{e_i\}$ to be the complexification of an orthonormal basis of $V_R$.

By definition, a Feynman diagram with edges coloured by the symbols $\{1, \ldots, N\}$ is a pair $(\Gamma, \eta)$, where $\Gamma$ is a Feynman diagram and $\eta : \text{Edges}(\Gamma) \to \{1, \ldots, N\}$. We represent an edge coloured by the symbol “$i$” by writing an “$i$” near it. The groupoid of Feynman diagrams with edges coloured by $\{1, \ldots, N\}$ is denoted by the symbol $\{1, \ldots, N\}_F$; clearly, automorphisms of Feynman diagrams with coloured edges are required to preserve the colouring. It is immediate to check that the “forget the colouring on the edges” map is a covering

$$\pi : \{1, \ldots, N\}_F \to F$$

We now have to define the amplitudes $Z_{x*}(\Gamma, \eta)$. Let $\pi_i : V \to V$ be the orthogonal projection on the subspace spanned by $e_i$. We make the graphical assignment

$$\begin{array}{c|c}
\text{in} & \text{out} \\
1 & i \\
i & \pi_i \\
\text{in} & \text{out}
\end{array}$$

Since the basis $\{e_i\}$ is orthonormal with respect to the pairing $(-, -)$, this graphical assignment is consistent with the other rules of graphical calculus and so a well-defined amplitude $Z_{x*}$ is induced on Feynman diagrams with coloured edges.

The equation $\text{Id}_V = \bigoplus_{i=1}^N \pi_i$ is translated in graphical terms into

$$\begin{array}{c|c}
\text{in} & \text{out} \\
1 & i \\
i & \pi_i \\
\text{in} & \text{out}
\end{array} = \bigoplus_{i=1}^N \begin{array}{c|c}
\text{in} & \text{out} \\
1 & i \\
i & \pi_i \\
\text{in} & \text{out}
\end{array}$$

so that the amplitude of a Feynman diagram is expanded into a sum of amplitudes of Feynman diagrams with coloured edges:

$$Z_{x*}(\Gamma) = \sum_{\eta : \text{Edges}(\Gamma) \to \{1, \ldots, N\}} Z_{x*}(\Gamma, \eta) \quad \text{(11.6)}$$

But this is just a push-forward formula:

$$Z_{x*} = \pi_* Z_{x*}$$

where $\pi$ is the “forget the colouring on the edges” map. So we can apply Fubini’s theorem (Proposition 3.2) and find

$$\int_{F} Z_{x*} \, d\mu_F = \int_{\{1, \ldots, N\}_F} Z_{x*} \, d\mu_{\{1, \ldots, N\}_F} \quad \text{(11.7)}$$

that is, summing over Feynman diagrams is the same thing as summing over Feynman diagrams with coloured edges (obviously, with the right weights).

A classical example of (11.7) is the following. Let $\phi$ be an analytic function defined on a neighborhood of 0 in $V$. The derivatives $D^n \phi$ define a family of
symmetric tensors on $V$. We make the graphical assignment

\[ \begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\nwarrow \nwarrow \nwarrow \nwarrow \nwarrow \\
\cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\n_{i \text{int}} \\
\rightarrow
\end{array}
\end{array} \rightarrow D^n \phi \bigg|_0 \]

and call this an $n$-valent black vertex. Any vector $v \in V$ can be seen as a morphism $v: \mathbb{C} \to V$; we make the graphical assignment

\[ \begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\nwarrow \nwarrow \nwarrow \nwarrow \nwarrow \\
\cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\n_{i \text{int}} \\
\rightarrow
\end{array}
\end{array} \mapsto v \]

and call this a $v$-vertex. Both types of vertices will be considered as special. Finally, for a fixed $v \in V$, we denote by $\text{Stars}(v)$ the groupoid of Feynman diagrams whose objects are the diagrams with no legs and exactly one black vertex such that all the edges stemming from the black vertex end in some $v$-vertex. If $v$ is a vector in the domain of $\phi$, the Taylor formula for $\phi$ can then be written as a sum over Feynman diagrams:

\[ \phi(v) = \int_{\text{Stars}(v)} Z_x \, d\mu = \int_{(1, \ldots, N) \text{Stars}(v)} Z_x \, d\mu. \quad (11.8) \]

Written out explicitly, the above equation is just the well-known identity

\[ \phi(v) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n \phi \bigg|_0 (v^\otimes n) = \sum_{d_1, \ldots, d_N = 0}^{\infty} \frac{1}{d_1! \cdots d_N!} \cdot \frac{\partial^{d_1 + \cdots + d_N} \phi}{(\partial v^1)^{d_1} \cdots (\partial v^N)^{d_N}} \bigg|_0 \cdot (v^1)^{d_1} \cdots (v^N)^{d_N}, \quad (11.9) \]

where $(v^1, \ldots, v^N)$ are the coordinates of the vector $v$ with respect to the basis $\{e_i\}$.

References

[Abd] Abdelmalek Abdesselam. Feynman diagrams in algebraic combinatorics. Sém. Lothar. Combin., 49:Art. B49c, 45 pp. (electronic), 2002/04 E-print: math.CO/0212121.

[Ada78] John Frank Adams. Infinite loop spaces. Princeton University Press, Princeton, N.J., 1978.

[AGM96] V. I. Arnautov, S. T. Glavatsky, and A. Y. Mikhalev. Introduction to the theory of topological rings and modules. Marcel Dekker Inc., New York, 1996.

[BD01] John C. Baez and James Dolan. From finite sets to Feynman diagrams. In Mathematics unlimited—2001 and beyond, pages 29–50. Springer, Berlin, 2001.

[BIZ80] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum field theory techniques in graphical enumeration. Adv. in Appl. Math., 1(2):109–157, 1980.

[BK01] Bojko Bakalov and Aleksandr A. Kirillov. Lectures on Tensor Categories and Modular Functors. Number 21 in University Lecture Series. American Mathematical Society, Providence, RI, 2001.

[Bro88] Ronald Brown. Topology. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester, 1988.

[DEF99] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, editors. Quantum fields and strings: a course for mathematicians. Vol. 1, 2. American Mathematical Society, Providence, RI, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997.

[DS97] Brian Day and Ross Street. Monoidal bicategories and Hopf algebroids. Adv. Math., 129(1):99–157, 1997.

[EK] Samuel Eilenberg and Gregory Maxwell Kelly. Closed categories. In Proceedings of the Conference on Categorical Algebra, (La Jolla, 1965).

[FM02] D. Fiorenza and R. Murri. Feynman diagrams via graphical calculus. J. Knot Theory Ramif., 11(7):1095–1131, 2002, E-print: math.QA/0106001.
[JS91] André Joyal and Ross Street. The geometry of tensor calculus. I. Adv. Math., 88(1):55–112, 1991.

[Kei82] Gregory Maxwell Kelly. Basic concepts of enriched category theory. Cambridge University Press, Cambridge, 1982.

[KV94] M. M. Kapranov and V. A. Voevodsky. 2-categories and Zamolodchikov tetrahedra equations. In Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), volume 56 of Proc. Sympos. Pure Math., pages 177–259. Amer. Math. Soc., Providence, RI, 1994.

[Oec01] Robert Oeckl. Braided quantum field theory. Comm. Math. Phys., 217(2):451–473, 2001, E-print: hep-th/9906225.

[RT90] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys., 127(1):1–26, 1990.

[SAG72] SGA1. Séminaire de géométrie algébrique du Bois Marie, 1. Groupe fondamentale et revêtements étales. I.H.E.S., 1972.

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