A Mean Field Theory for the Quantum Hall Liquid. II
— The Vortex Solution—

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ABSTRACT

In the Fractional Quantum Hall state, we introduce a bi-local mean field and get vortex mean field solutions. Rotational invariance is imposed and the solution is constructed by means of numerical self-consistent method. It is shown that vortex has a fractional charge, a fractional angular momentum and a magnetic field dependent energy. In $\nu = 1/3$ state, we get finite energy gap at $B = 10, 15, 20[T]$. We find that the gap vanishes at $B = 5.5[T]$ and becomes negative below it. The uniform mean field becomes unstable toward vortex pair production below $B = 5.5[T]$. 
1. Introduction

In previous papers\cite{1}, we introduced a bi-local mean field in two dimensional electron system in perpendicular magnetic field and discussed a uniform ground state. In this paper, we discuss a vortex solution which is a non-linear and topological excitation of the system. Experiment of the fractional quantum Hall effect shows that the excited state has the energy gap and the ground state becomes the incompressible liquid. An interesting liquid state that has these properties was proposed by Laughlin\cite{2}. In Laughlin wave function the excited state includes quasiholes and quasiparticles. They give a finite energy gap of the system. Experimentally it is measured as activation energy. The other interesting properties of Laughlin wave function are that quasiholes and quasiparticles have fractional charges and obey fractional statistics. Theoretical calculations based on Laughlin wave function\cite{2-4} and on exact diagonalization of the system with finite numbers of electrons\cite{5-10} predict that activation energy is roughly proportional to \( \sqrt{B} \). Experimentally activation energy is smaller than the theoretical values and does not follow \( \sqrt{B} \) dependence. It vanishes below \( B \sim 5[T] \) and follows roughly linear dependence of \( B \) above it\cite{11}. Laughlin wave function is a trial function, then it may not be good enough for more systematic investigation. For this purpose, we start from a microscopic many-body Hamiltonian and apply a mean field theory to it. We study the vortex solutions in our theory which may correspond to Laughlin’s quasihole and quasiparticle.

It is well known that Abelian Higgs theory has the vortex solution which is described by the classical solution of bosonic fields. We construct its counterpart in two dimensional electron system in perpendicular magnetic field based on the
bi-local mean field theory. The elementary field is fermionic in this system, which does not have classical expectation value. We treat two point function $\langle \psi^\dagger(y)\psi(x) \rangle$, instead of one point function of bosonic variables, as the mean field and find the vortex solution. This mean field is bi-local and fluctuations of the mean field includes a dynamical gauge field and a density fluctuation field. The mean field satisfies a self-consistency equation, which becomes to a non-linear Schrödinger equation for these fields. The solutions are found numerically. Vortices have similar properties as Laughlin’s quasihole and quasiparticle by having fractional charges and fractional angular momenta. The $B$ dependence of activation energy is similar to experimental data rather than Laughlin’s one and is smaller than both of Laughlin’s one and experimental one. We find that energy gap vanishes at $B = 5.5[T]$ in agreement with experimental data.

The present paper is organized in the following way. In section 2, we study the non uniform density solution of the self-consistency equation. In section 3, we calculate the vortex solution at $\nu = 1/3$ by the iteration method. The shape, the charge and the angular momentun of the vortex are presented. We calculate magnetic field dependence of gap energy and find that there exists a threshold of the magnetic field for our uniform ground state to be stable. In section 4, summary and discussion are given.
2. Mean Field and Vortex Solution

The Hamiltonian of two dimensional non-relativistic electrons in the perpendicular constant magnetic field $B$ is

$$
H = \int d^2x \overline{\psi}(x) \left( \frac{P + eA}{2m} \right)^2 \psi(x) + \frac{1}{2} \int d^2x d^2y \overline{\psi}(x) \psi^\dagger(y) V(x-y) \psi(y) \psi(x),
$$

(2.1)

where $P_i = -i \frac{\partial}{\partial x_i}, \partial_1 A_2 - \partial_2 A_1 = B$, $V(x-y) = \frac{e^2}{|x-y|}$. We neglect inessential background terms $^{[1]}$. In the functional integral formalism, the partition function is written as

$$
Z = Tr(e^{-\beta H}) = \int D\psi^\dagger D\psi e^{-\int_0^\beta d\tau \int d^2x \psi^\dagger \partial^\tau \psi + H}. 
$$

(2.2)

We introduce the bi-local auxiliary field $U(x,y)$, and $Z$ is written as

$$
Z = \int DUD\psi^\dagger D\psi e^{-\int_0^\beta d\tau \int d^2x \psi^\dagger \partial^\tau \psi + H^U}. 
$$

(2.3)

$$
H^U = \int d^2x \overline{\psi}(x) \left( \frac{P + eA}{2m} \right)^2 \psi(x) 
+ \frac{1}{2} \int d^2x d^2y V(x-y)\left| U(x,y) \right|^2 - U(x,y) \psi^\dagger(y) \psi(y) - U^*(x,y) \psi^\dagger(x) \psi(x). 
$$

(2.4)

We take $\beta \to \infty$. $U(x,y)$ is stationary at the mean field $U_0(x,y)$ which is given by

$$
U_0(x,y) = \left\langle \psi^\dagger(y) \psi(x) \right\rangle. 
$$

(2.5)

We assume the following ansatz,

$$
U_0(x,y) = U_0 \rho(x,y) e^{-\gamma(x-y)^2} \exp[i \int_x^y \alpha_i d\xi^i], 
$$

(2.6)

where $\rho$ is real symmetric function of $x$ and $y$, and $U_0 = \nu/\pi R_0^2, \gamma = 1/2 R_0^2, R_0 = \sqrt{2/eB}$. Line integral is along a straight line between $x$ and $y$. We assume,
further, \( \lim_{x \to \infty, y \neq 0} \rho(x, y) = 1 \). These parameters are fixed in order that \( U(x, y) \) may be coincident with a uniform self-consistent solution\(^1\) at infinity. The self-consistent solution for uniform ground state is obtained by replacing \( \rho \) by 1 and set \( \alpha_i = eA_i \) at Eq.(2.6),

\[
U_0^{(g.s.)}(x, y) = U_0 e^{-\gamma(x-y)^2} \exp[i \int_x^y eA_i d\xi^i]. \tag{2.7}
\]

By substituting Eq.(2.6) into Eq.(2.4), we get the mean field Hamiltonian,

\[
H_0 = \int d^2x \lim_{\bar{x} \to x} \psi^\dagger(x) \left[ \frac{(P + eA)^2}{2m} - F((P + \alpha)^2) \rho(\bar{x}, x) \right] \psi(x) + \frac{1}{2} \int d^2x d^2y V(x-y) |U_0(x,y)|^2,
\]

\[
F(p^2) = e^2 U_0 \frac{\pi^{3/2}}{\gamma^{1/2}} e^{-p^2/8\gamma} I_0(p^2/8\gamma).
\tag{2.8}
\]

Where \( I_0 \) is modified Bessel function. From Eq.(2.8), single-body Hamiltonian reads

\[
H_s = \left[ \frac{(P + eA)^2}{2m} - F((P + \alpha)^2) \rho(\bar{x}, x) \right]_{\bar{x} \to x}. \tag{2.9}
\]

The second term is mean field’s contribution through a Coulomb interaction. The system is invariant under a gauge transformation,

\[
\psi \to e^{i\Lambda(x)} \psi,
\]

\[
A_i \to A_i - \frac{1}{e} \partial_i \Lambda(x), \tag{2.10}
\]

\[
\alpha_i \to \alpha_i - \partial_i \Lambda(x).
\]

The electron field is expanded by \( H_s \)’s eigenfunctions.

\[
\psi(x) = \sum_{l,n} u_{l,n}(x) a_{l,n},
\]

\[
H_s u_{l,n}(x) = E_{l,n} u_{l,n}(x), \quad \{a_{l,n}^\dagger, a_{l',n'}\} = \delta_{l,l'} \delta_{n,n'}, \tag{2.11}
\]

\[
E_{l,0} < E_{l,1} < E_{l,2} < \cdots
\]
where $l$ is angular momentum quantum number and $a_{l,n}^\dagger$, $a_{l',n'}$ are anti-commuting creation and annihilation operators. The N-electron state is made as

$$|\psi\rangle = \sum_{l_1,l_2\ldots l_N=0}^M F_{l_1 \ldots l_N} a^\dagger_{l_1,0} a^\dagger_{l_2,0} \cdots a^\dagger_{l_N,0} |0\rangle.$$  \hspace{1cm} (2.12)

We restrict the state within the lowest energy level for every $l$ and assume that $|\psi\rangle$ is the superposition of same angular momentum states. Then

$$\sum_{l_2\ldots l_N=0}^M F^*_{n,l_2\ldots l_N} F_{m,l_2\ldots l_N} = \nu_n \delta_{n,m}, \quad 0 \leq \nu_n \leq 1,$$  \hspace{1cm} (2.13)

and two point function is

$$\langle \psi | \psi^\dagger(y) \psi(x) | \psi \rangle = \sum_{l=0}^M \nu_l u^*_l(y) u_l(x).$$  \hspace{1cm} (2.14)

From Eq.(2.5), Eq.(2.6) and Eq.(2.14), we get a self-consistency condition

$$U_0 \rho(x,y) e^{-\gamma(x-y)^2} \exp[i \int_x^y \alpha_i d\xi^i] = \sum_{l=0}^M \nu_l u^*_l(y) u_l(x).$$  \hspace{1cm} (2.15)

Uniform density solution Eq.(2.7) satisfies Eq.(2.15) for $\nu_l = \nu$,

$$\sum_{l_2\ldots l_N=0}^M F^{(g.s.)*}_{n,l_2\ldots l_N} F^{(g.s.)}_{m,l_2\ldots l_N} = \nu \delta_{n,m}.$$  \hspace{1cm} (2.16)

In order that the state may be coincident with a uniform density state of filling $\nu$ at infinity, $\nu_l$ must approach to the value $\nu$ for large $l$.

Let us construct the vortex solution in this formalism.
In the Abelian Higgs theory, the vortex solution\cite{12} takes the form \( \langle \phi(r, \theta) \rangle \sim v(r)e^{in\theta} \), \( n \) is integer. \( n \) is winding number of a mapping \( : S^1 \rightarrow U(1) \). At infinity \( v \) goes to its vacuum expectation value and at \( r = 0 \), \( v \) is zero. For the finiteness of the energy, the kinetic term \( |\partial\phi + ieA|^2 \) must vanish at infinity. Then gauge potential must take the form \( A_\theta \sim -n/e \). Magnetic field localizes around the vortex.

By similar considerations, it is natural to impose the following ansatz in our case,

\[
\alpha_i(x) = eA_i(x) + n_i(x), \quad n(x) = \frac{n(r)}{r^2}(-x_2, x_1),
\]

\[
A(x) = B_2(-x_2, x_1),
\]

and boundary conditions are

\[
n(0) = n \; ; \text{integer},
\]

\[
\lim_{r \rightarrow \infty} n(r) = 0,
\]

\[
\rho(x, 0) = \rho(0, x) = 0, \quad \rho(x, x) = \rho(r).
\]

Dynamical gauge field \( \alpha_i \) has singularity at origin for \( n(0) \neq 0 \). But this singularity is harmless in Eq.(2.8), since \( \rho \) vanishes at origin. Therefore eigenfunction \( u_l \) behaves regularly as \( r^l \) near the origin. By the singular gauge transformation, \( \psi \rightarrow e^{in\theta}, \alpha_i \rightarrow \alpha_i - n\partial_i\theta \), this singularity is removed. Then \( n \) is regarded as winding number in the same way as Abelian Higgs theory.

In Eq.(2.18), \( n \) is integer for the reason that \( U(x, y) \) must be continuous function of \( x \) and \( y \). Fig.1 shows a line integral along a infinitesimal circle \( c \) around
origin, which is
\[ \oint_C \alpha_i d\xi^i = 2\pi n(0), \]  
(2.19)
generates discontinuity when the integration path cross the origin. Therefore, 
\( n(0) = n \) must be integer for continuity.

For the finite energy, \( \alpha_i \) has to coincide with the external field \( A_i \) at infinity. See Appendix A where we present exact soluble example which shows that if \( n(\infty) \neq 0 \) or \( \alpha_i(\infty) \neq eA_i \) then a vortex excitation energy diverges. Moreover the rotational invariance implies the above form of \( \alpha_i \). See Appendix B.

Momentum and angular momentum density operator are defined by

\[ P_i(x) = \frac{1}{2} \{ \psi^\dagger(x)(-i\partial_t + eA_i(x))\psi(x) + [(i\partial_t + eA_i(x))\psi^\dagger(x)]\psi(x) \}, \]
\[ L(x) = \epsilon_{ij} x_i P_j(x). \]  
(2.20)

Using Eq.(2.6) and Eq.(2.17), the expectation values are given by

\[ \langle P_i(x) \rangle = -U_0\rho(r)n_i(x), \]
\[ \langle L(x) \rangle = -U_0\rho(r)n(r), \]  
(2.21)

We can see from Eq.(2.21) that the current is in proportion to \( \rho \) and a difference between \( A_i \) and \( \alpha_i \). This corresponds to London equation in the theory of superconductivity. For the \( n_i \) given by Eq.(2.17), the current is rotating around the vortex.
3. Numerical Solution of Vortex

In Eq.(2.8), $F(p^2)$ is difficult to calculate numerically. We expand $F(p^2)$ around $p^2 = eB$ which corresponds to energy of lowest Landau level and approximate as

$$F(p^2) = F(eB) + F'(eB)(p^2 - eB) = F_0 - \frac{p^2}{2m'}. \quad (3.1)$$

$$F_0 = e^2 U_0 \frac{\pi^{3/2}}{\gamma^{3/2}} \exp\left(-\frac{1}{2}\right)[3I_0(\frac{1}{2}) - I_1(\frac{1}{2})]/2,$$

$$\frac{1}{2m'} = \frac{e^2}{8} U_0 \frac{\pi^{3/2}}{\gamma} \exp\left(-\frac{1}{2}\right)[I_0(\frac{1}{2}) - I_1(\frac{1}{2})],$$

where $I_n$ is modified Bessel function. Then we have to solve the following non-linear Schrödinger equation,

$$\left[\frac{(P + eA)^2}{2m} + \frac{(P + \alpha)^2}{2m'}\right] \rho(\bar{x}, x) + F_0(1 - \rho(x))|_{\bar{x} \to x} u_l(x) = E_l u_l(x), \quad (3.2)$$

$$u_l(x) = v(r)e^{-il\theta}; \quad l \text{ is integer},$$

where $\alpha_i$ and $\rho$ are given by self-consistency condition Eq.(2.15). We add a constant $F_0$ to energy for simplicity. The $\rho$ looks like a scalar potential. We make a quasihole state by removing $l = 0$ state from uniform density state as

$$|\text{hole}\rangle = \sum_{l_1, l_2, \ldots, l_N = 0} F^{(g.s.)}_{l_1, \ldots, l_N} a_{l_1+1,0}^{\dagger} a_{l_2+1,0}^{\dagger} \cdots a_{l_N+1,0}^{\dagger} |0\rangle. \quad (3.3)$$

Hereafter lengths are defined in units of $R_0$. We fix the form of $n(r)$ for $n(0) = 1$ as

$$n_+(r) = \frac{r^2}{e^{r^2} - 1}. \quad (3.4)$$

This form is implied by removing $l = 0$ state from the lowest Landau level. That
\[ \langle \psi^\dagger(y)\psi(x) \rangle_{\text{lowest Landau level}, l \neq 0} = \frac{\nu}{\pi} \left( 1 - e^{-z_y z_x^*} \right) e^{-\frac{1}{2}(|z_x|^2 + |z_y|^2)} e^{-1/2 |z_x - z_y|^2 + i \int_x eA_x d\xi}, \]

where \( z_x = x_1 + ix_2 \). The phase part of \( 1 - e^{-z_y z_x^*} \) is coincident with Eq.(3.4) for \( |z_x - z_y| \ll 1 \).

\[ \rho_0(x, y) = \sqrt{1 + e^{-2r_x r_y \cos \theta} - 2 \cos(r_x r_y \sin \theta)} e^{-r_x r_y \cos \theta}. \]

This has properties as \( \rho_0(x, y) = \tilde{\rho}_0(\sqrt{r_x r_y}, \theta); \tilde{\rho}_0(r, 0) = \rho_0(r) \). We impose these to the \( \rho \) and calculate \( \rho \) by numerical self-consistent method. We find that ansatz Eq.(3.4) actually satisfies self-consistency condition Eq.(2.15).

By using Eq.(3.4), we get eigenvalue \( E_l \) for \( \nu = \frac{1}{3} \) and find that \( E_0 \) is bigger than the other \( E \)'s. Then we remove the \( l = 0 \) state and calculate the \( \rho(r) \) by Eq.(2.15) and using it we calculate \( E_l \) again. By iterating this procedure, we get the self-consistent solution. Final results for eigenvalues are listed at Table I for several value of \( B \) and see Fig.2. The \( \rho(r) \) is shown in Fig.3.

Asymptotically \( \rho(r) \sim 1 - e^{-r^2} \) at infinity. This behavior is different from the vortex in Higgs field. That is \( v(r) \sim v(1 - e^{-cr}) \). And also, dynamical gauge field behaves as \( n(r) \sim e^{-r^2} \) at infinity. These properties are characteristic of the system in magnetic field.

\* We have performed the numerical calculation for \( l = 0, 1, \cdots, 30 \) and interpolate for large \( l \) to the uniform solutions.
Apparently, the charge decreases by $\nu e$ as compared with the uniform density state. Then this solution corresponds to quasihole which has fractional charge $Q_+ = -\nu e$. That is

$$Q_+ = eU_0 \int d^2x (\rho(r) - 1) = -\nu e.$$  \hspace{1cm} (3.7)

This relation is exact.

Angular momentum of vortex is given by Eq.(2.21) as

$$\langle L \rangle = \int d^2x \langle L(x) \rangle = -U_0 \int d^2x \rho(r) n(r).$$  \hspace{1cm} (3.8)

This value is calculated numerically at $\nu = 1/3$ as $-0.989/3$, $-0.991/3$, $-0.993/3$, $-0.994/3$ at $B = 5$, 10, 15, 20[T] respectively. Then the angular momentum of vortex for quasihole is approximately $-1/3$.

In order to get the gap energy, we need a quasiparticle solution which has a opposite charge to quasihole. We make a quasiparticle state by removing $l = 0$ state from uniform density state and filling $l = 1$ state fully as

$$|\text{particle} \rangle = a_{1,0}^{\dagger} \sum_{l_1, l_2, \cdots, l_N = 0}^{M} F_{l_1, \cdots, l_N}^{(g.s.)} a_{l_1 + 2,0}^{\dagger} a_{l_2 + 2,0}^{\dagger} \cdots a_{l_N + 2,0}^{\dagger} |0 \rangle.$$  \hspace{1cm} (3.9)

We fix $n(r)$ for $n(0) = 1$ as

$$n_-(r) = \frac{r^2(3 - 2r^2)}{e^{r^2} + 2r^2 - 1}.$$  \hspace{1cm} (3.10)

This form is implied by removing $l = 0$ state from the lowest Landau level and filling $l = 1$ state fully. In Eq.(3.10), $n_-$ expresses a current which is rotating
around the vortex and the direction is reversed at \( r = \sqrt{3/2} \). We calculate the eigenvalues \( E_l \) for \( \nu = 1/3 \) and find that \( E_0 \) is bigger and \( E_1 \) is smaller than the others. Then, it is natural to remove the \( l = 0 \) state and fully fill the \( l = 1 \) state, \( i.e., \nu_0 = 0, \nu_1 = 1, \nu_l = 1/3; l \geq 2 \). This construction means that \( Q_\nu = (-1/3 + 2/3)e = (1/3)e \). This state corresponds to quasiparticle which has fractional charge \( Q_\nu = \nu e \) for \( \nu = 1/3 \). We perform the same iteration procedure as quasihole. We find that ansatz Eq.(3.10) actually satisfies self-consistency condition Eq.(2.15). Final results for eigenvalues are listed at Table II for several value of \( B \) and see also Fig.2. The \( \rho(r) \) is shown in Fig.4.

By Eq.(3.8), the angular momentum is calculated numerically as 0.965/3, 0.978/3, 0.983/3, 0.986/3 at \( B = 5, 10, 15, 20[T] \) respectively. Then the angular momentum of vortex for quasiparticle is approximately 1/3.

The gap energy \( \Delta \) is half of the pair excitation energy of a quasihole and a quasiparticle which are separated infinitely. That is

\[
\Delta = (E_{\text{hole}} + E_{\text{particle}})/2. \tag{3.11}
\]

Experimentally \( \Delta \) is measured as activation energy. It is determined from temperature dependence of diagonal resistance as \( \rho_{xx} \propto e^{-\Delta/T} \). Each energy is calculated as

\[
E_{\text{vortex}} = \sum_l [\nu_l E_l - \nu E_{l}^{(g.s.)}] + \frac{1}{2} \int d^2x d^2y V(x-y)[|U_0(x,y)|^2 - |U_0^{(g.s.)}(x,y)|^2], \tag{3.12}
\]

where \( E_l^{(g.s.)} \) is energy eigenvalue of uniform ground state and \( U_0^{(g.s.)} \) is given by Eq.(2.7). For several values of \( B \), gap energies are listed in Table III. Finite
energy gap exists at $B = 10, 15, 20\,[T]$ and the gap becomes negative at $B = 5\,[T]$. See Fig.5. The calculations based on Laughlin wave function$^{[2-4]}$ and on exact diagonalization of the finite system$^{[5-10]}$ show $\Delta(B) \propto \sqrt{B}$. Our result shows $\Delta$ increases linearly and vanishes at $B = 5.5\,[T]$. The $B$ dependence of gap energy is similar to the experiments$^{[11]}$. Experiments show that gap energy increases monotonically with $B$ and there exists threshold of finite gap at $B \sim 5\,[T]$. The theoretical calculations including the effect of disorder yeild threshold effect$^{[13,14]}$. However, our result means that threshold effect remains without the effect of disorder.

Note that the quasiparticle solution is constructed above not for $n(0) = -1$ but for $n(0) = 1$. The state contains eigenfunction $u_l$’s only for $l \geq 0$ and the phase dependence is $e^{il\theta}$, then $n(0) = -1, i.e., l = -1$, can’t be induced. The energy for $l \leq -1$ belongs to the higher Landau level. In other words, the solution for $n(0) = -1$ can not be constructed self-consistently in lowest Landau level.

4. Summary and Discussion

We construct the vortex solution in the bi-local mean field theory by numerical method. By dynamical gauge field which has singularity at core of vortex, energy level becomes non-degenerate for small $l$ and charge density decreases or increases near the vortex. Their charges are exactly $\pm e/3$ at $\nu = 1/3$ and both of them have winding number $n = 1$. Their angular momenta are approximately $\mp 1/3$. Then statistics of the vortex is supposed to be fractional.$^{[15]}$

We get finite energy gap at $B > 5.5\,[T]$ and $\nu = 1/3$, then the ground state is incompressible. Below $B = 5.5\,[T]$, the energy gap takes negative value. This result
means that the uniform mean field solution becomes unstable and activation energy vanishes at $B < 5.5[T]$. Thus we have obtained magnetic field threshold effect without disorder for the first time. Experiment of mobility dependence of threshold effect will discriminate our calculation from other disorder theory.\cite{16} Other theoretical calculations without disorder did not show threshold effect. They include only lowest Landau level and next Landau level at most. On the other hand our method includes Landau levels higher than next Landau level. Especially at low magnetic fields, higher Landau levels become important. Furthermore, since vortex solutions are highly non-linear objects, the projection on the lowest Landau level may not be good approximation. Our results suggest that the threshold effect may be related to higher Landau levels.

For large $B$, our energy gap is much smaller than the value of experiments. However experimental data suggest the existence of second activation energy at lower temperature.\cite{17-19} Our results are consistent with the second activation energy. But hopping conduction\cite{20,21} is another candidate for lower temperature behavior of $\rho_{xx}$. Thus we wish more experiments to be done at lower temperature.

We construct a excited state by Eq.(3.3) and Eq.(3.9). In addition to these state, our formalism must include other excited states. Then more than one gap energies may exist.

The following improvements for our method may be possible. The first is to take higher derivative terms in Eq.(3.1). Higher derivative terms may be irrelevant in long distance physics. However these terms affect Eq.(3.2) and energy eigenvalues may change. The second is to calculate $n(r)$ by numerical iteration method self-consistently.
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APPENDIX A

We consider here the soluble eigenvalue problem of following Hamiltonian,

$$H = \frac{(P + A)^2}{2m} + \frac{(P + A' + n)^2}{2m'},$$  \hspace{1cm} (A.1)

$$A = \frac{B}{2}(-x_2, x_1), \quad A' = \frac{B'}{2}(-x_2, x_1), \quad n = \frac{n}{r^2}(-x_2, x_1),$$  \hspace{1cm} (A.2)

where $n$ is constant value. The eigenfunction is

$$\psi_{N,l} = \text{Const} \cdot r^l e^{-i\theta - (\tilde{B}/4)r^2} u_{N,l}(r), \quad l = 0, \pm 1, \pm 2, \cdots,$$

$$u_{N,l}(r) = L_N^{(i)}(\frac{\tilde{B}}{2}r^2),$$

$$\tilde{l} = \sqrt{M(\tilde{l}^2/m + (l - n)^2/m')}^{1/2}, \quad 1/M = 1/m + 1/m',$$

$$\tilde{B} = \sqrt{M(B^2/m + B'^2/m')}^{1/2},$$

where $L_N^{(i)}$ is associated Laguerre function. The eigenvalue is

$$E_{N,l} = \frac{\tilde{B}}{M}(N + \frac{1}{2}) + \left[nB'/m' - (B/m + B'/m')l + \tilde{l}B/M\right]/2.$$  \hspace{1cm} (A.4)

(a) $B \neq B'$ case

$$E_{N,l} - E_{N,l}^{n=0} \sim C \cdot l,$$  \hspace{1cm} (A.5)

$$C = \left[\sqrt{(B/m + B'/m')}^2 + (B - B')^2/m'm' - (B/m + B'/m')\right]/2 > 0.$$
Energy difference diverges in proportion to \( l \).

(b) \( B = B' \) case

\[
E_{N,l} - E_{N,l}^0 \sim \frac{|B|n^2}{4(m + m')} \cdot \frac{1}{l} \tag{A.6}
\]

Energy difference does not diverge, but the sum \( \sum \frac{1}{l} \) diverges.

**APPENDIX B**

In the uniform magnetic field, the system has translational and rotational invariance. But, in the quantum theory, magnetic field affects the system in the form of gauge potential. Therefore the symmetries seem to be lost in the system. Since the system also has gauge symmetry, translational and rotational invariance remain in the quantum theory\(^{[22]}\).

At first, we consider the translation \( x'_i = x_i + \epsilon_i \). Vector potential is transformed as

\[
A_i \rightarrow A'_i(x') = A_i(x) + \partial_i \Lambda(x, \epsilon) \tag{B.1}
\]

Translational invariance means \( A'_i(x) = A_i(x) \), then

\[
\partial_i \Lambda(x, \epsilon) = A_i(x + \epsilon) - A_i(x). \tag{B.2}
\]

If the magnetic field \( B(x) = B(x + \epsilon) \) then \( \Lambda \) exists and given by

\[
\Lambda(x, \epsilon) = \int^x [A_i(\xi + \epsilon) - A_i(\xi)]d\xi^i \tag{B.3}
\]

Our Hamiltonian is made from two gauge field, that is, \((P + A)^2/2 + F(P + \alpha)\). Hence the \( \Lambda \) made by \( A_i \) and \( \Lambda' \) made by \( \alpha_i \) must be same. From Eq.(B.3), this
means that translational invariance only allows the difference of a constant vector between $A_i$ and $\alpha_i$. In fact, we know that when $A_i = \alpha_i$, uniform self-consistent solution exists.

Next we consider the rotation, $x'_1 = x_1 \cos \theta + x_2 \sin \theta$, $x'_2 = -x_1 \sin \theta + x_2 \cos \theta$. Vector potential is transformed as

$$
\begin{align*}
A_1 &\rightarrow A'_1(x') = A_1(x) \cos \theta + A_2(x) \sin \theta + \partial_1 \Lambda(x, \theta) \cos \theta + \partial_2 \Lambda(x, \theta) \sin \theta \\
A_2 &\rightarrow A'_2(x') = -A_1(x) \sin \theta + A_2(x) \cos \theta - \partial_1 \Lambda(x, \theta) \sin \theta + \partial_2 \Lambda(x, \theta) \cos \theta
\end{align*}
$$

(B.4)

If $B(x) = B(r)$ then $\Lambda$ exists and given by

$$
\Lambda(x, \epsilon) = \int^x \left[ \cos \theta A_1(\xi') - \sin \theta A_2(\xi') - A_1(\xi) \right] d\xi^1 + \left[ \sin \theta A_1(\xi') + \cos \theta A_2(\xi') - A_2(\xi) \right] d\xi^2
$$

(B.5)

In order that the $\Lambda$ made by $A_i$ and $\Lambda'$ made by $\alpha_i$ may be same, their difference must be the following form

$$
\mathbf{A}(x) - \mathbf{\alpha}(x) = f(r)(x_1, x_2) + g(r)(-x_2, x_1).
$$

(B.6)

One can easily see that Eq.(2.17) satisfies this condition.
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    see Appendix A therein.
FIGURE CAPTIONS

1) The path of line integral in $U_0$ is drawn.
   (a) Fixing a point $x$, a point $y$ is moved around origin.
   (b) Infinitesimal circle $c$ around origin gives a discontinuity $2\pi n$.

2) Eigenvalues at $B = 15[T]$, $\nu = 1/3$.
   • represents eigenvalue for quasihole and ◦ represents eigenvalue for quasiparticle.

3) $\rho(r)$ for quasihole at $B = 15[T]$, $\nu = 1/3$.

4) $\rho(r)$ for quasiparticle at $B = 15[T]$, $\nu = 1/3$.

5) The gap energy vanishes at $B = 5.5[T]$ and increases monotonically above it.