G-VALUED GALOIS DEFORMATION RINGS WHEN $\ell \neq p$

JEREMY BOOHER AND STEFAN PATRIKIS

Abstract. For a smooth group scheme $G$ over an extension of $\mathbb{Z}_p$ such that the generic fiber of $G$ is reductive, we study the generic fiber of the Galois deformation ring for a $G$-valued mod $p$ representation of the absolute Galois group of a finite extension of $\mathbb{Q}_\ell$ with $\ell \neq p$. In particular, we show it admits a regular dense open locus, and that it is equidimensional of dimension $\text{dim}(G)$.

1. Introduction

Many of the deepest arithmetic properties of modular forms (or more generally algebraic automorphic representations) are encoded in the congruences between eigenforms of different level or weight. These congruences can be interpreted as congruences between the corresponding modular Galois representations, and this perspective, in combination with the development of modular lifting and potential automorphy theorems, has dramatically advanced our understanding of such congruences. The most successful approach to producing congruences is based on a method of Khare-Wintenberger [KW09] that in appropriate settings produces lifts of prescribed inertial type for a potentially modular mod $p$ representation $\overline{\rho}$; when $\overline{\rho}$ is in fact modular, the method combines with modular lifting theorems to produce congruences between modular forms.

The present paper is a contribution to the local aspect of this story for Galois representations valued in general reductive groups. Namely, the method of [KW09] depends in part on having an adequate understanding of the structure of the generic fibers of local Galois deformation rings, and we focus our attention here. Let $\ell$ and $p$ be distinct primes, $K$ be a finite extension of $\mathbb{Q}_\ell$, and $E$ be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$ and residue field $k$. Consider a smooth group scheme $G$ over $\mathcal{O}_E$ with reductive generic fiber, let $\Gamma_K = \text{Gal}(\overline{K}/K)$, and consider a continuous representation

$$\overline{\rho}: \Gamma_K \to G(k).$$

The functor of lifts of $\overline{\rho}$ to artin local $\mathcal{O}_E$-algebras with residue field $k$ is pro-represented by a complete local noetherian $\mathcal{O}_E$-algebra $R_{\overline{\rho}}^\square$. This ring may be quite singular; the object of this paper is achieve some control over its generic fiber $R_{\overline{\rho}}^\square[1/\ell]$. The main result is the following. First we recall that to each homomorphism $\tau: I_K \to G(E)$ that factors through a finite quotient, we can associate a quotient $(R_{\overline{\rho}}^\square[1/\ell])^\tau$ of $R_{\overline{\rho}}^\square[1/\ell]$ ("with inertial type $\tau$") which is a union of irreducible components of $R_{\overline{\rho}}^\square[1/\ell]$.

Theorem 1. For any $\overline{\rho}: \Gamma_K \to G(k)$ and $\tau$ as above, $(R_{\overline{\rho}}^\square[1/\ell])^\tau$ admits a regular, dense open subscheme, and it is equidimensional of dimension $\text{dim}(G)$.

We will now give some of the history behind this result. Theorem 1 generalizes a result of Gee [Gee11] in which the group $G$ is taken to be $\text{GL}_n$. His result in turn is the adaptation to the $\ell \neq p$ case of a corresponding result of Kisin [Kis08] that studies the generic fibers of potentially

\[\text{Date: September 19, 2017.}\]

We are grateful to Rebecca Bellovin and Toby Gee for encouraging us to write up our work. We also thank Brian Conrad and Bhargav Bhatt for commutative algebra consultations. S.P. was partially supported by NSF Grant DMS-1700759.

\[1\text{In particular, $G$ and its generic fiber need not be connected. As the definition of a reductive group scheme typically includes a connectedness hypothesis, we avoid that language.}\]
semi-stable deformation rings (in the case \( \ell = p \)); Kisin introduced many new ideas on which all subsequent work in this subject has been based. Later, Gee’s results were reproved and slightly strengthened in [Cho09] and [BLGGT14], and in [Bel16] Bellovin proved a version of Kisin’s \( \ell = p \) result for general \( G \). While we were preparing this paper, Bellovin and Gee posted a preprint that treats both the cases \( \ell = p \) and \( \ell \neq p \) for general groups \( G \) [BG17] . In addition to treating the more difficult case \( \ell = p \), their work gives a more refined version of our Theorem 1 (finding a dense set of “very smooth” points), motivated by the refinement of [Gee11] established and applied in [BLGGT14]. Moreover, they extensively pursue global applications to lifting results of Khare-Wintenberger type, including applications to generalizations of the Serre weight conjecture: see the theorems in the introduction of [BG17].

Because the work in [BG17] is so comprehensive, and because of the timing of our two papers, we have not attempted to push our method to achieve the most refined results; nor have we included any global applications, although they are the motivation behind Theorem 1. Our proof of Theorem 1 is different from that of the corresponding result in Bellovin-Gee, and when specialized to \( G = \text{GL}_n \), it is different from the arguments of [Gee11], [Cho09], and [BLGGT14], so we hope it may still be of independent interest. In the remainder of this introduction, we will briefly describe the approach to Theorem 1.

Common to all of the results related to Theorem 1 is the translation of problems about Galois representations to problems about Weil-Deligne representations. In [Gee11], [BG17], and our Theorem 1 the main \((\ell \neq p)\) result is reduced using ideas of [Kis08] to a corresponding “unobstructedness” result in an appropriate moduli space of Weil-Deligne representations; the reduction in [Kis08] is not obvious, but it carries over essentially formally to the settings of these subsequent papers. What is new in our paper is the proof of this unobstructedness result, Theorem 5, which occupies §3. In brief, there is a moduli space \( X \rightarrow \text{Spec } E \) (see §2) of Weil-Deligne representations, and for any object \( D_A \in X(A) \) an explicit complex \( C^*(D_A) \) whose cohomology controls the deformation theory of \( D_A \). Taking \( I_D = D_{\Phi,N,T} \) to be the universal object, our problem is to show that the cohomology measuring obstructions, \( H^2(D_{\Phi,N,T}) \), a coherent sheaf on \( X \), vanishes away from a dense open subscheme of \( X \). The strategy initiated by Kisin shows this by studying an analogous moduli space \( Y \) that parametrizes only a monodromy operator and an inertial type, together with the forgetful map \( X \rightarrow Y \); the essential and non-formal content of Theorem 1 is the fact that every irreducible component of every non-empty fiber of this map contains an unobstructed point. In the papers [Kis08], [Gee11], [Bel16], and [BG17], this is achieved essentially by constructing one particular unobstructed point in each component. In contrast, our argument takes any point in such a fiber and connects it by a chain of \( G_{m,n} \)'s and \( G_a \)'s to reach an unobstructed point. It relies on a series of applications of the Jacobson-Morosov theorem.

2. G-Weil-Deligne Representations and Deformation Theory

Let \( \ell \) and \( p \) be distinct primes. Let \( K \) be an \( \ell \)-adic field, with residue field of order \( q = \ell^f \), and let \( W_K \) be the Weil group of \( K \). Let \( \| \cdot \| : W_K \rightarrow W_K/I_K \simeq q^\mathbb{Z} \) denote the homomorphism which sends any geometric Frobenius to \( q^{-1} \). Throughout this paper, we will fix a geometric Frobenius \( \phi \in W_K \). Next consider a finite extension \( E \) of \( \mathbb{Q}_p \) with ring of integers \( \mathcal{O}_E \) and residue field \( k \), and let \( G \) over \( \mathcal{O}_E \) be a smooth group scheme with reductive generic fiber; for the purposes of the next two sections, we only use the generic fiber, whereas in §1 we need a smooth group scheme over \( \mathcal{O}_E \) in order to study the deformation theory of \( \mathfrak{g} \).

**Definition 2.** For an \( E \)-algebra \( A \), a \( G \)-Weil-Deligne representation over \( A \) is a trivial \( G \)-bundle \( D_A \), a homomorphism \( r : W_K \rightarrow \text{Aut}_G(D_A) \) whose restriction to \( I_K \) factors through a finite quotient, and a nilpotent \( N \in \text{Lie } \text{Aut}_G(D_A) \) such that \( \text{Ad } r(g)N = \|x\|N \).

More concretely, if the inertial action factors through a fixed finite extension \( L/K \), a \( G \)-Weil-Deligne representation over \( A \) is given by a trivial \( G \)-bundle \( D_A \) equipped with a homomorphism
\[ \tau : I_{L/K} \to \text{Aut}_G(D_A), \text{a nilpotent } N \in \text{Lie Aut}_G(D_A), \text{and an invertible } \Phi \in \text{Aut}_G(D_A) \text{ that satisfy: } \]

1. \( \text{Ad}(\tau(g))N = N \) for any \( g \in I_{L/K} \).
2. \( \text{Ad}(\Phi)N = q^{-1}N. \)
3. \( \text{Ad}(\Phi)\tau(\gamma) = \tau(\phi\gamma\phi^{-1}) \) for all \( \gamma \in I_{L/K} \).

Let \( WD_G \) denote the category whose objects are pairs consisting of an \( E \)-algebra \( A \) and a \( G \)-Weil-Deligne representation \( (D_A, r, N) \) over \( A \); a morphism \( (A, D_A, r, N) \to (A', D_A', r', N') \) is an \( E \)-algebra map \( A \to A' \) and an isomorphism \( D_A \otimes_A A' \cong D_{A'} \) of \( G \)-bundles intertwining the actions of \( (r, N) \) and \( (r', N') \). We may regard \( WD_G \) as a category cofibered in groupoids over the category of \( E \)-algebras. We define \( WD_{G,L} \) as the analogous category in which the \( I_K \)-action factors through the fixed finite quotient \( I_{L/K} \).

**Remark 3.** We can identify \( \text{Aut}_G(D_A) \) with \( G \) by picking a trivializing section, obtaining a homomorphism \( r : W_K \to G(A) \). This is analogous to the way a free rank \( d \) \( A \)-module action with action of a group \( \Gamma \) can be identified with a homomorphism \( \Gamma \to \text{GL}_d(A) \) after a choice of basis.

We wish to study deformation theory in \( WD_{G,L} \). Let \( \mathfrak{d}A := \text{Lie Aut}_G D_A \); the adjoint action gives an action of \( I_{L/K} \) and of \( \Phi \) on \( \mathfrak{ad} \mathfrak{d}A \). Consider the anti-commutative diagram

\[
\begin{array}{ccc}
(ad \mathfrak{d}A)^{I_{L/K}} & \xrightarrow{1-\text{Ad } \Phi} & (ad \mathfrak{d}A)^{I_{L/K}} \\
\downarrow \text{ad } N & & \downarrow \text{ad } N \\
(ad \mathfrak{d}A)^{I_{L/K}} & \xrightarrow{q \text{Ad } \Phi^{-1}} & (ad \mathfrak{d}A)^{I_{L/K}}.
\end{array}
\]

Let \( C^*(\mathfrak{d}A) \) denote the total complex (indexed so as to be in degrees 0, 1, and 2) associated to this double complex, and let \( H^I(D_A) \) denote the \( I \)-th cohomology. Note that the construction of \( H^2(D_A) \) commutes with arbitrary base change.

Let \( A \) be an Artin local \( E \)-algebra with maximal ideal \( \mathfrak{m}_A \), and let \( I \subset A \) be an ideal with \( I\mathfrak{m}_A = 0 \). Let \( D_{A/I} \in WD_{G,L}(A/I) \), and define \( \overline{D} = D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A \). Two liftings \( D_A \) and \( D'_A \) are equivalent if there exists a lift \( D_A \to D'_A \) of \( G \)-torsors compatible with the Weil-Deligne structure that reduces to the identity modulo \( I \).

**Proposition 4.** If \( H^2(\overline{D}) = 0 \), then a lift \( D_A \in WD_{G,L}(A) \) of \( D_{A/I} \) exists. The set of equivalence classes of liftings of \( D_{A/I} \) to \( A \) is a (possibly empty) torsor under \( H^1(\overline{D}) \otimes_{A/\mathfrak{m}_A} I \).

**Proof.** The proof is essentially the same as that of [Bel16, Proposition 3.2] and [Gee11, Lemma 2.1.1]. \( \square \)

We let \( X \) denote the functor on the category of \( E \)-algebras which associates to an \( E \)-algebra \( A \) the set of possible triples

\[
(\Phi, N, \tau) \in G(A) \times \mathfrak{g}_A \times \text{Rep}_A(I_{L/K})
\]

that satisfy (1)-(3) of [2]. This is represented by a finite type locally closed subscheme of the space of all possible triples obtained by imposing the conditions. Likewise, we consider a functor \( Y \) consisting of pairs \((N, \tau)\) satisfying condition (1). We also have \( X_{N,\tau}, X_\tau, \text{ and } Y_\tau \), where the subscripts denote a fixed choice of that variable. There are natural (forgetful) maps between these spaces. Any object of \( X \) defines a \( G \)-Weil-Deligne representation by viewing \( G \) as a trivial \( G \)-torsor.

We write \( D_{\Phi, N, \tau} \) for the universal triple on \( X = \text{Spec } R \), and define the sheaf \( \mathcal{H} \) on \( X \) as the cokernel of

\[
q \text{Ad } \Phi - 1 \oplus \text{ad } N : \mathfrak{g}_R^{I_{L/K}} \oplus \mathfrak{g}_R^{I_{L/K}} \to \mathfrak{g}_R^{I_{L/K}}.
\]
It is a coherent sheaf on $X$, and by semi-continuity the locus where it vanishes is open. For a closed point $x$ with residue field $A$ corresponding to a $G$-Weil-Deligne representation $D_A$, we see that $\mathcal{H}_x \simeq H^2(D_A)$. We say a point $x$ is unobstructed if $\mathcal{H}_x = 0$.

The main technical result we will prove is the following:

**Theorem 5.** The unobstructed points are dense in $X$.

The argument is inspired by the proofs of [Kis08, Lemma 3.1.5], [Gee11, Lemma 2.1.3], and [Bel16, Proposition 5.2]. We will show that the unobstructed points are dense in the fiber $X_{N,\tau}$ of $X \to Y$ over any fixed $\tau$ and $N$. As being unobstructed is an open condition, it suffices to find a single unobstructed point in each irreducible component of the fiber. The proofs in these papers (and the proof of [BG17, Theorem 2.3.6]) proceed by writing down a point in each component, either by hand or by using the theory of associated cocharacters, and then directly verifying that the point is unobstructed. In contrast, we start with an arbitrary point and connect it by a succession of $G_a$'s and $G_m$'s to an unobstructed point. We will carry this out in the next section.

3. Moduli Spaces for $G$-Weil-Deligne Representations

Let $e$ be a nilpotent element in $\text{Lie} G_{\overline{E}}$. The proof of Theorem 5 repeatedly uses cocharacters that interact well with $N$.

**Definition 6.** For a subgroup $H \subset G_{\overline{E}}$ with $e \in \text{Lie} H$, a cocharacter $\lambda : G_m \to H$ is adapted to $e$ provided that $\text{Ad} \lambda(t)e = t^2e$.

As we are in characteristic zero, the nilpotent $e$ can be extended to an $\mathfrak{sl}_2$-triple by the Jacobson-Morosov theorem. If we exponentiate the triple and restrict to the diagonal $G_m$, we obtain a cocharacter adapted to $e$. These also provide examples of the associated cocharacters used in [Bel16].

Let $H$ be a reductive subgroup of $G_{\overline{E}}$, and set $\mathfrak{h} = \text{Lie} H$. Consider a semi-simple $g \in G(\overline{E})$ such that $\text{Ad} g(\mathfrak{h}) = \mathfrak{h}$, and a nilpotent $e \in \mathfrak{h}$ such that $\text{Ad}(g)e = \alpha e$ for some $\alpha \in \overline{E}$. We will find a cocharacter adapted to $N$ that interacts well with $g$.

**Lemma 7.** There exists a cocharacter $\lambda : G_m \to H^\circ$ adapted to $e$ such that for all $t$

$$\text{Ad} \lambda(t) \text{Ad} g = \text{Ad} g \text{Ad} \lambda(t).$$

**Proof.** (We learned this argument from [GR10, Lemma 2.1].) Consider the adjoint action of $g$ on the Lie algebra $\mathfrak{h}$. We will construct the desired cocharacter by constructing an $\mathfrak{sl}_2$-triple compatible with the eigenspace decomposition

$$\mathfrak{h} = \bigoplus_{\mu} \mathfrak{h}(\mu),$$

where $\mathfrak{h}(\mu)$ is the eigenspace for $\text{Ad} g$ with eigenvalue $\mu$. (As $g$ is semi-simple, we do not need generalized eigenspaces.) The Jacobson-Morosov theorem gives an $\mathfrak{sl}_2$-triple $(e, h_0, f_0)$ in $\mathfrak{h}$. Note that $e \in \mathfrak{h}(\alpha)$. For any $h_\mu \in g(\mu)$,

$$\text{Ad}(g)[h_\mu, e] = [\text{Ad}(g)h_\mu, \text{Ad}(g)e] = \mu \alpha [h_\mu, e],$$

so $[h_\mu, e]$ is an element of $\mathfrak{h}(\mu \alpha)$. Now decompose $h_0 = h + h'$ with $h \in \mathfrak{h}(1)$ and $h' \in \bigoplus_{\mu \neq 1} \mathfrak{h}(\mu)$. Using that $2e = [h_0, e] = [h, e] + [h', e] \in \mathfrak{h}(\alpha)$ and considering eigenspaces, we conclude that $2e = [h, e]$. Likewise, we decompose $f_0 = f + f'$ with $f \in \mathfrak{h}(\alpha^{-1})$ and $f' \in \bigoplus_{\mu \neq \alpha^{-1}} \mathfrak{h}(\mu)$ and keep track of eigenspaces in the relation $h_0 = [e, f_0] = [e, f] + [e, f']$ to conclude that $h = [e, f]$. Likewise we see that $[h, f] = -2f$, so $(e, h, f)$ is another $\mathfrak{sl}_2$-triple; to obtain the cocharacter adapted to $e$, we exponentiate and restrict to the diagonal $G_m$. Then we see that $\text{Ad} g \text{Ad} \lambda(t) = \text{Ad} \lambda(t) \text{Ad} g$ as $h \in \mathfrak{h}(1)$. \hfill \Box

We now proceed with the proof of Theorem 5.
Proof. Fix $\tau: I_{L/K} \to G(E)$. It suffices to show every non-empty fiber $X_{N,\tau}$ of the forgetful map $X_\tau \to Y_\tau$ contains a dense open subset on which $H$ vanishes. As this is an open condition, it suffices to find a single closed point $x$ in each irreducible component of $X_{N,\tau}$ for which $H_x = 0$.

Consider the fiber $X_{N,\tau}$ over a fixed nilpotent $N \in (\text{Lie} G)(E')$ for some finite extension $E'/E$; assume it is non-empty, so that we have additionally a $\Phi \in G(E)$ such that the relations (1)-(3) of §2 hold for $\Phi, N, \tau$; in this particular fiber we are only allowed to vary $\Phi$. To find an unobstructed point in the fiber, we may work over $\overline{E}$, and to simplify notation, we change notation and let $G$ (and similarly $N...$) be defined over $\overline{E}$. Fix a square root of $q$ in $\overline{E}$.

We break into cases depending on whether $N = 0$. In both cases, the strategy is to connect $\Phi$ by $G_q$'s and $G_m$'s to an unobstructed point. A basic but important observation is that the fiber $X_{N,\tau}$ is a $Z_G(N) \cap Z_G(\tau)$-torsor. In particular, for any $\lambda: G_m \to Z_G(N) \cap Z_G(\tau)$ or $\psi: G_a \to Z_G(N) \cap Z_G(\tau)$, $\Phi\lambda(t)$ and $\Phi\psi(s)$ are in the same component of the fiber as $\Phi$. (As the fiber is a torsor under $Z_G(N) \cap Z_G(\tau)$, the irreducible components are the same as connected components.)

**Case 1:** When $N = 0$, to be unobstructed means that $q \text{Ad} \Phi - 1$ is invertible on $g^{I_{L/K}}$. Given $\Phi \in X_{0,\tau}(\overline{E})$ such that $q \text{Ad} \Phi - 1$ is not invertible, we will find a $\Phi'$ in the same component as $\Phi$ such that the generalized eigenspace of $\text{Ad} \Phi'$ with eigenvalue $q^{-1}$ has smaller dimension. By induction, this produces an unobstructed point.

By hypothesis, there is an $N' \in g^{I_{L/K}}$ such that $q \text{Ad}(\Phi)N' = N'$. Note that $N'$ is nilpotent, as it is conjugate to $q^{-1}N'$, which implies the eigenvalues of $\text{ad} N'$ are zero.

**Lemma 8.** There exists $\Phi_s$ in the same component of the fiber $X_{0,\tau}$ as $\Phi$ such that $\Phi_s$ is semi-simple. There is a cocharacter $\lambda: G_m \to Z_G(\tau)$ adapted to $N'$ such that

$$\text{Ad} \Phi_s \text{Ad} \lambda(t) = \text{Ad} \lambda(t) \text{Ad} \Phi_s.$$  

**Proof.** Consider the Jordan decomposition $\Phi = \Phi_s \Phi_u$ where $\Phi_s \in G(\overline{E})$ is semi-simple and $\Phi_u \in G(\overline{E})$ is unipotent. The relation $\text{Ad} (\Phi(\tau(\gamma))) = (\phi \gamma \phi^{-1})$ implies that there is an integer $n$ such that $\Phi^n \in Z_G(\tau)(\overline{E})$. As $\Phi_s$ and $\Phi_u$ commute, we see that $\Phi^n = \Phi_s^n \Phi_u^n$. This is also the Jordan decomposition for $\Phi^n$ in $Z_G(\tau)$, so in particular $\Phi_u^n \in Z_G(\tau)(\overline{E})$. Writing $\Phi_u^n = \exp(nY)$ for $Y \in \text{Lie} Z_G(\tau)$, we see that $\Phi_u = \exp(Y)$ lies in $Z_G(\tau)$. As the fiber is a $Z_G(\tau)$-torsor, we see that $\Phi_s \exp(\mu Y)$ lies in the fiber for every $t$, and hence that $\Phi$ and $\Phi_s$ are in the same component.

The second statement is Lemma 7 applied to $Z_G(\tau) \subset G$; note that $Z_G(\tau)$ is reductive. □

We will conclude the proof of Case 1 by showing that for a generic choice of $t$, the dimension of the $q^{-1}$-eigenspace of $\text{Ad}(\Phi_s \lambda(t))$ on $g^{I_{L/K}}$ is less than the dimension of the $q^{-1}$-eigenspace of $\text{Ad} \Phi_s$. This will suffice, as $\Phi$, $\Phi_s$, and $\Phi_s \lambda(t)$ all lie in the same component of $X_{0,\tau}$, and the $q^{-1}$-generalized eigenspace for $\text{Ad} \Phi$ has the same dimension as the $q^{-1}$-eigenspace for $\text{Ad} \Phi_s$.

We decompose

$$g^{I_{L/K}} = \bigoplus \mu g^{I_{L/K}}(\mu)$$

where $g^{I_{L/K}}(\mu)$ is the $\mu$-eigenspace of $\text{Ad} \Phi_s$. Now $\text{Ad} \lambda(t)$ preserves $g^{I_{L/K}}(\mu)$ as $\text{Ad} \Phi_s$ and $\text{Ad} \lambda(t)$ commute. On the finitely-many non-zero $g^{I_{L/K}}(\mu)$ with $\mu \neq q^{-1}$, the condition that $\text{Ad} \Phi_s \lambda(t)$ not have $q^{-1}$ as an eigenvalue is simply the condition that $\text{Ad} \lambda(t)$ not have $q^{-1} \mu^{-1}$ as an eigenvalue. This is a non-empty, open condition (consider $t = 1$). On $g^{I_{L/K}}(q^{-1})$, we know that

$$\text{Ad}(\Phi_s \lambda(t))N' = q^{-1} t^2 N'.$$

So if $t \neq \pm 1$, we see that $\text{Ad}(\Phi_s \lambda(t))$ has a smaller $q^{-1}$-eigenspace than $\text{Ad}(\Phi_s)$. Thus a generic choice of $t$ decreases the dimension of the eigenspace, and we conclude by induction.

**Case 2:** The case $N \neq 0$ follows the same strategy, but is more involved. We will first find a semi-simple point in a given component of the fiber, and then modify it using cocharacters valued in $Z_G(N) \cap Z_G(\tau)$ so it is unobstructed. A key technique is passing between points in the fiber and
points of \( Z_G(N) \): for any cocharacter \( \lambda \) adapted to \( N \) and \( \Phi \) in the fiber, we see \( \Phi \lambda(q^{1/2}) \in Z_G(N) \) as

\[
\text{Ad}(\Phi \lambda(q^{1/2}))N = q^{-1}(q^{1/2})^2 N = N
\]

**Lemma 9.** In each non-empty component of the fiber \( X_{N,\tau} \) of \( X_\tau \to Y_\tau \) above \( N \), there exists a semi-simple point \( \Phi \). There is a cocharacter \( \lambda : G_m \to Z_G(\tau) \) adapted to \( N \) such that for all \( t \)

\[
\text{Ad} \Phi \text{Ad} \lambda(t) = \text{Ad} \lambda(t) \text{Ad} \Phi.
\]

**Proof.** Let \( \Phi' \) be a point in the desired component of the fiber. Let \( \lambda' : G_m \to Z_G(\tau) \) be any cocharacter adapted to \( N \). We check that \( \Psi := \Phi' \lambda'(q^{1/2}) \in Z_G(N)(\overline{E}) \) satisfies \( \text{Ad}(\Psi) \tau(\gamma) = \tau(\phi \gamma \phi^{-1}) \) for \( \gamma \in I_{L/K} \). As before, some power \( \Psi^n \) is in \( Z_G(N) \cap Z_G(\tau)(\overline{E}) \).

Consider the Jordan decomposition \( \Psi = \Psi_u \Psi_s \) in \( Z_G(N) \) with \( \Psi_u \) and \( \Psi_s \) commuting unipotent and semi-simple elements. As \( \Psi^n = \Psi_u^n \Psi_s^n \) is a Jordan decomposition for \( \Psi^n \) and Jordan decomposition is compatible with inclusions of groups, we see that \( \Psi_u^n \in Z_G(N) \cap Z_G(\tau)(\overline{E}) \). As \( \Psi_u \) is unipotent, it follows that \( \Psi_u \in Z_G(N) \cap Z_G(\tau)(\overline{E}) \). This shows that

\[
\text{Ad}(\Psi_s \lambda'(q^{1/2}))N = q^{-1}N \quad \text{and} \quad \text{Ad}(\Psi_s \lambda'(q^{1/2})) \tau(\gamma) = \tau(\phi \gamma \phi^{-1})
\]

for \( \gamma \in I_{L/K} \). Note that \( \Phi' \) and \( \Psi_s \lambda'(q^{1/2}) = \Phi' \lambda'(q^{1/2}) \Psi_s^{-1} \lambda'(q^{-1/2}) \) lie in the same component of the fiber.

The above used an arbitrary cocharacter adapted to \( N \). Using Lemma 7 now pick a cocharacter \( \lambda : G_m \to Z_G(\tau) \) adapted to \( N \) whose adjoint action commutes with that of \( \Psi_s \). Define \( \Phi = \Psi_s \lambda(q^{-1/2}) \). As before, we check that \( \Phi \) is in \( X_{N,\tau}(\overline{E}) \). Furthermore, note that \( \Phi \) and \( \Psi_s \lambda(q^{-1/2}) \) lie in the same component of the fiber, as the family

\[
\Phi_t := \Psi_s \lambda(q^{-1/2}t) \lambda(t^{-1})
\]

interpolates between them. Finally, \( \Phi \) is semi-simple as the adjoint actions of \( \Psi_s \) and \( \lambda(q^{-1/2}) \) commute and are semi-simple, and it is clear that the adjoint actions of \( \Phi \) and \( \lambda(t) \) commute. \( \square \)

Recall that a point \( (\Phi', N) \) is unobstructed if

\[
q \text{ Ad } \Phi' - 1 \oplus \text{ad } N : g_{L/K}^{I} \oplus g_{L/K}^{I} \to g_{L/K}^{I}
\]

is surjective. If \( \Phi' \) is semi-simple, then so is \( q \text{ Ad } \Phi' - 1 \). Hence \( g_{L/K}^{I} \) is a direct sum of the kernel and the image of \( q \text{ Ad } \Phi' - 1 \). In this case, \( (\Phi', N) \) is unobstructed provided that \( \ker(q \text{ Ad } \Phi' - 1) \subset \text{Im}(\text{ad } N) \). We now continue with the \( \Phi \) and \( \lambda \) produced by Lemma 9.

We decompose

\[
g_{I/L/K}^{I} = \sum_{n \in \mathbb{Z}} g_n
\]

where \( g_n \) is the space where \( \text{Ad } \lambda(t) \) acts by \( t^n \). For \( n \leq 0 \), we set \( g_n^{l.w.} := \ker(\text{ad } N|_{g_n}^{l.w.}) \) and \( g'_n = \text{Im}(\text{ad } N|_{g_n}^{l.w.}) \), and decompose

\[
g_n = g_n' \oplus g_n^{l.w.}
\]

Here \( g_n^{l.w.} \) are the lowest weight vectors of the \( sl_2 \)-triple containing \( N \) that was used to define \( \lambda \). Note \( \text{ad } N \) gives an isomorphism between \( g_{n-2} \) and \( g'_n \).

**Lemma 10.** For any \( n \leq 0 \), we have \( \text{Ad } \Phi(g_n^{l.w.}) = g_n^{l.w.} \) and \( \text{Ad } \Phi(g'_n) = g'_n \).

**Proof.** First note that \( \text{Ad } (\Phi) \) preserves \( g_n \) since \( \text{Ad } (\Phi) \) and \( \text{Ad } (\lambda(t)) \) commute. For \( v \in g_n^{l.w.} \), we compute that

\[
(\text{ad } N)^{-n+1} \text{Ad } (\Phi) v = q^{-n+1} \text{Ad } \Phi ((\text{ad } N)^{-n+1} v) = 0.
\]

For the second, consider \( v \in g'_n \). Writing \( v = \text{ad } N(v') \) for \( v' \in g_{n-2} \), we compute that

\[
\text{Ad } \Phi(v) = [\text{Ad } \Phi(N), \text{Ad } \Phi(v')] = q^{-1} \text{ad } N(\text{Ad } \Phi(v')).
\]

Thus \( \text{Ad } \Phi(v) \in \text{ad } N(g_{n-2} = g'_n) \). \( \square \)
Note that all $g_n$ with $n > 0$ lie in the image of ad $N$, as do $g_n'$ for $n \leq 0$. To check that $\ker(q \text{Ad}(\Phi')) \subset \text{Im}(\operatorname{ad} N)$, by the Lemma it suffices to show that $\text{Ad} \Phi'$ does not have eigenvalue $q^{-1}$ on any of the $g_n^{l,w}$. We will modify $\Phi$ so this holds.

Consider the element $\Psi = \Phi \lambda (q^{1/2})$. There are two cases to consider, depending on whether $\text{Ad}(\Psi)$ has infinite or finite order. First suppose it has finite order, say $m$, so that $\text{Ad}(\Phi)^m = \text{Ad}(\lambda (q^{1/2}))^{-m}$. Consider any eigenvector $v \in g_n^{l,w}$ (for some $n \leq 0$) of $\text{Ad}(\Phi)$, with eigenvalue $\alpha$. Then

$$\alpha^m v = \text{Ad}(\Phi)^m(v) = \text{Ad}(\lambda(q^{-m/2}))v = q^{-mn/2}v,$$

and we clearly cannot have $\alpha = q^{-1}$ for $n \leq 0$.

Next consider the case where $\text{Ad}(\Psi)$ has infinite order. The group contains a non-trivial power of the semi-simple, infinite order, element $\Psi$. Some power $\Psi^m$, with $m > 0$, is then contained in a non-trivial torus of $Z$, and we let $\lambda'$ be any co-character of (this torus of) $Z$ whose image contains $\Psi^m$; certainly in any torus every element is in the image of some co-character. We consider the adjoint action of $\Phi_t := \Phi \lambda'(t)$ and claim that for a generic choice of $t$, the point $\Phi_t$ will be an unobstructed point of the fiber of $X_\tau \to Y_\tau$ over $N$.

As the fiber is a $ZG(N) \cap ZG(\tau)$-torsor, $\Phi_t$ lies in the desired fiber. Since $\text{Ad} \Phi$ and $\text{Ad} \lambda'(t)$ commute, they have common eigenvectors. Let $v$ be an eigenvector of $\text{Ad} \Phi$ in one of the $g_n^{l,w}$.

- In the case that $\text{Ad} \lambda'(t)v \neq v$ for some $t$, it is a non-empty open condition for $\text{Ad} \Phi_t(v) \neq q^{-1}v$.
- If $\text{Ad} \lambda'(t)v = v$ for all $t$, then as $\Psi$ is in the image of $\lambda'$ we see that

$$v = \text{Ad} \Psi(v) = \text{Ad} \Phi \lambda(q^{1/2})(v) = q^{n/2} \text{Ad} \Phi(v).$$

In particular, $\text{Ad} \Phi(v) \neq q^{-1}v$ as $n/2 \leq 0$, and hence $\text{Ad} \Phi_t(v) \neq q^{-1}v$ for all $t$.

Combining these conditions for each eigenvector in some $g_n^{l,w}$, for a generic choice of $t$ we see $\text{Ad} \Phi_t$ does not have eigenvalue $q^{-1}$ on any of the $g_n^{l,w}$. In that case $\Phi_t$ is an unobstructed point of the $X_{N,\tau}$, completing the case that $N \neq 0$. \hfill $\square$

### 4. Analysis of Local Galois Deformation Rings

As before, let $\ell$ and $p$ be distinct primes. Let $K$ be an $\ell$-adic field. In this section we will apply the local monodromy theorem, so we fix a compatible collection of $p$-power roots of unity in $\overline{K}$, yielding as usual a surjection $I_K^{(1)} \to \mathbb{Z}_p(1)$ from the tame inertia group of $K$ (note this will only be used implicitly). Let $E$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$ and residue field $k$ of size $q = \ell^f$. Let $\mathcal{C}_{\mathcal{O}_E}$ denote the category of complete local noetherian $\mathcal{O}_E$-algebras with residue field $k$. Let $G$ be a smooth group scheme over $\mathcal{O}_E$ such that $G_{\mathcal{O}_E}$ is reductive. Fix a continuous homomorphism $\overline{\rho} : \text{Gal}(\overline{K}/K) \to G(k)$. Consider the morphism $D_{\overline{\rho}}^\square \to D_{\overline{\rho}}$ of (categories cofibered in) groupoids over $\mathcal{C}_{\mathcal{O}_E}$, where $D_{\overline{\rho}}^\square(R)$ is the category (set) of lifts of $\overline{\rho}$ to $G(R)$, and $D_{\overline{\rho}}(R)$ is the category (set) of objects of $\overline{\rho}$ that are lifts of $\overline{\rho}$, and where a morphism between lifts $\rho$ and $\rho'$ is an element $g \in \hat{G}(R)$ such that $\rho = \rho'$. We are interested in the generic fiber $R_{\overline{\rho}}^{\square}[1]$ of the universal lifting ring $R_{\overline{\rho}}^\square$ (representing $D_{\overline{\rho}}^\square$); when the corresponding deformation functor for $\overline{\rho}$ is also representable, we obtain analogous results for $R_{\overline{\rho}}^{[1]}$. We will analyze $R_{\overline{\rho}}^{[1]}$ by means of $G$-Weil-Deligne representations.

For the remainder of this section, fix a finite extension $L/K$ and an homomorphism $\tau : I_{L/K} \to G(E)$ that arises as the restriction to $I_K$ of a $G$-Weil-Deligne representation. We call a $G(E)$-conjugacy class of such $\tau$ an inertial type. For anyartin local $E$-algebra $B$, with residue field
some finite extension $E'$ of $E$, we say a continuous homomorphism $\rho: \Gamma_K \to G(B)$ is type $\tau$ if the associated $G$-Weil-Deligne representation $(r, N)$ has inertial restriction with reduction $I_{L/K} \xrightarrow{f} G(B) \to G(E')$ being $G(E)$-conjugate to $\tau$. This condition is equivalent to $\tau: I_{L/K} \to G(B)$ being $G(B \otimes E')^{-}$-conjugate to $\tau$: this follows from standard deformation theory and the fact that $H^i(I_{L/K}, \text{ad}(\rho)) = 0$ for $i = 1, 2$. (We extend the definition of type to the case where $B$ is any finite $E$-algebra by imposing the above condition on each local factor of this artin ring.)

For any complete local noetherian $O_E$-algebra $A^0$, with generic fiber $A = A^0 \left[ \frac{1}{p} \right]$, and equipped with a continuous homomorphism $\rho: \Gamma_K \to G(A^0)$, there is a quotient $A \to A^\tau$, equal to a union of irreducible components of Spec $A$, such that for any finite $E$-algebra $B$, an $E$-algebra map $A \xrightarrow{f} B$ factors through $A^\tau$ if and only if $f \circ \rho$ has inertial type $\tau$ (see the proof of [Bal12 Proposition 4.0.12]). In particular, we can form the quotient $(R^\tau(\frac{1}{p}))^\tau$. Our main result is the following:

**Theorem 11.** Spec$(R^\tau(\frac{1}{p}))^\tau$ admits a regular, dense open subscheme, and it is equidimensional of dimension $\dim(G)$.

**Proof.** Let $A = (R^\tau(\frac{1}{p}))^\tau$. The universal lift of $\overline{\rho}$, composed into $G(A)$, and the usual procedure for associating a Weil-Deligne representation to a Galois representation together yield a $G$-Weil-Deligne representation (with the $G$-bundle canonically trivialized) $D_A \in WD_{G,L}(A)$. For any closed point $x$ of Spec $A$, write $m_x$ for the maximal ideal at $x$, $A_x$ for the completion of $A$ at $m_x$, $E_x$ for the residue field, $\rho_x$ for the associated homomorphism $\Gamma_K \to G(E_x)$, and $D_x$ for the associated object of $WD_{G,L}(E_x)$. The argument of [Kis08 Proposition 3.3.1] implies that the associated morphism (of groupoids over $E$-algebras)

$$\text{Spec } A_x \to WD_G$$

is formally smooth; the hypothesis of [Kis08 Proposition 3.3.1] is just the assertion that $D^\tau(\frac{1}{p}) \to D^\tau$ is formally smooth, which is clear since $G$ is smooth. Then the argument of [Kis08 Proposition 3.1.6] shows that if $U_x$ denotes the complement of the support of $H^2(D_{A_x})$ in Spec $A_x$, then $U_x$ is formally smooth and open dense in Spec $A_x$: here is where we crucially invoke our main result, Theorem 5.

Now let $U$ denote the complement of the support of $H^2(D_A)$. As $U_x$ is the base change of $U$ to $A_x$, the density of $U_x$ in $A_x$ for all $x$ implies that $U$ is dense in $A$. Furthermore, for any closed point $x \in U$, we know that $U_x = A_x$ is formally smooth, hence regular. Since $A$ is noetherian, this implies that the localization of $A$ at any such $x$ is regular, and hence that $U$ is regular.

Now let $x$ be any closed point of $U \subset \text{Spec } A$. We know the completion $A_x$ is a regular local ring, and we will be done once we compute the $E_x$-dimension of its tangent space $\text{Hom}_{E_x}(m_x/m_x^2, E_x)$. By a standard deformation theory argument (see [Kis09 Proposition 2.3.5]), $A_x$ pro-represents the functor of lifts of $\rho_x$ to artin local $E_x$-algebras with residue field $E_x$, and this tangent space is then isomorphic to the space of continuous 1-cocycles $Z^1(\Gamma_K, \text{ad}(\rho_x))$, and therefore has dimension

$$\dim A_x = \dim_{E_x} H^1(\Gamma_K, \text{ad}(\rho_x)) + \dim(G) - \dim_{E_x}(\text{ad}(\rho_x)^{\Gamma_K}).$$

Now note that from the construction of $G$-Weil-Deligne representations from Galois representations, we obviously have that $H^0(D_x) = (\text{ad}(\rho_x))^{\Gamma_K}$. Moreover, $H^1(\Gamma_K, \text{ad}(\rho_x))$ classifies $E_x[\ell]$-deformations of $\rho_x$, and (by Proposition 3), $H^1(D_x)$ classifies equivalence classes of lifts of $D_x$ to $WD_{G,L}(E_x[\ell])$. Since $D_x$ arises from the Galois representation $\rho_x$, it is a bounded $G$-Weil-Deligne representation (i.e., in any finite-dimensional representation of $G$, the eigenvalues of $\Phi$ are $p$-adic units). This implies that for any lift of $D_x$ to $WD_{G,L}(E_x[\ell])$, the usual formula associating a Galois representation to a $G$-Weil-Deligne representation applies to yield a continuous lift of $\rho_x$ to $G(E_x[\ell])$. Conversely, such a lift can be by the usual argument of the monodromy theorem be converted into a lift of $D_x$. These two procedures are inverses, identifying equivalence classes of
$E_x[e]$-deformations, and they therefore identify the $E_x$-vector spaces $H^1(D_x)$ and $H^1(\Gamma_K, \text{ad}(\rho_x))$. Now combining the local Euler-characteristic formula and the fact that the Euler characteristic of the complex $C^\bullet(D_x)$ obviously vanishes, we see that

$$\dim A_x = \dim(G) + \dim_{E_x}(H^2(D_x)) = \dim(G),$$

since $H^2(D_x) = 0$ for $x \in U$. \hfill \Box

**Remark 12.** We note as a consequence of the proof that for all $x \in U$, the Galois cohomology group $H^2(\Gamma_K, \text{ad}(\rho_x))$ in fact vanishes.

**Remark 13.** The results of [Kis08 Theorem 3.3.4], and later papers like [Gee11 Theorem 2.1.6] that adapt its arguments, state that $U$ is formally smooth over $\mathbb{Q}_p$, not just regular. This distinction is irrelevant for the applications to computing the dimension of the generic fiber, as all that is used is regularity, but regularity does not imply formal smoothness in this setting. For example, consider $A = \mathbb{Z}_p[[t]]\left\lbrack \frac{1}{p} \right\rbrack$. As $\mathbb{Z}_p[[t]]$ is regular, so is its localization $A$. However, $A$ is not formally smooth over $\mathbb{Q}_p$ in the discrete topology. We thank Bhargav Bhatt for suggesting the following argument.

Let $B = \mathbb{Z}_p[t]\left\lbrack \frac{1}{p} \right\rbrack$. The natural map $B \to A$ is regular: it is flat because $\mathbb{Z}_p[[t]]$ is flat over $\mathbb{Z}_p[t]$, and the fibers are points and hence geometrically regular. Then the Jacobi-Zariski exact sequence for the maps $\mathbb{Q}_p \to B \to A$ gives

$$0 \to A \otimes_B \Omega_B/\mathbb{Q}_p \to \Omega_A/\mathbb{Q}_p \to \Omega_{A/B} \to 0.$$ 

Now $A \otimes_B \Omega_B/\mathbb{Q}_p$ is a free $A$-module with basis $dt$, which can be identified with the continuous Kähler differentials $\widehat{\Omega}_A/\mathbb{Q}_p$. The sequence is split by the element of $\text{Hom}_A(\Omega_A/\mathbb{Q}_p, \widehat{\Omega}_A/\mathbb{Q}_p) = \text{Der}_{\mathbb{Q}_p}(A, \widehat{\Omega}_A/\mathbb{Q}_p)$ given by sending $f(t) \in A$ to $f'(t)dt$. If $A$ were formally smooth over $\mathbb{Q}_p$ in the discrete topology, $\Omega_A/\mathbb{Q}_p$ would be projective over $A$. Together with the splitting, this would imply that $\Omega_{A/B}$ embeds into a free $A$-module. But the non-zero $\Omega_{A/B}$ is $t$-divisible since $B/t \to A/t$ is an isomorphism, so it cannot embed into a free $A$-module. This contradiction shows that $A$ cannot be formally smooth over $\mathbb{Q}_p$ in the discrete topology.

**REFERENCES**

[Bal12] Sundeep Balaji, *G-valued potentially semi-stable deformation rings*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–The University of Chicago. MR 3152673

[Bel16] Rebecca Bellovin, *Generic smoothness for G-valued potentially semi-stable deformation rings*, Ann. Inst. Fourier (Grenoble) 66 (2016), no. 6, 2565–2620. MR 3580181

[BG17] R. Bellovin and T. Gee, *G-valued local deformation rings and global lifts*, ArXiv e-prints (2017).

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) 179 (2014), no. 2, 501–609. MR 3152941

[Cho09] Suh-Hyun Choi, *Local deformation lifting spaces of mod l Galois representations*, Ph.D. thesis, Harvard University, Cambridge, Massachusetts, 2009. MR 2713395

[Gee11] Toby Gee, *Automorphic lifts of prescribed types*, Math. Ann. 350 (2011), no. 1, 107–144. MR 2785764

[GR10] Benedict Gross and Mark Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. 154 (2010), no. 3, 431-508. MR 2730575

[Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. 21 (2008), no. 2, 513–546. MR 2373358

[Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) 170 (2009), no. 3, 1085–1180. MR 2600871

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *On Serre’s conjecture for 2-dimensional mod p representations of Gal($\overline{\mathbb{Q}}$/Q)*, Ann. of Math. (2) 169 (2009), no. 1, 229–253. MR 2480604