Fenchel–Nielsen coordinates and Goldman brackets

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Abstract. It is explicitly shown that the Poisson bracket on the set of shear coordinates defined by V. V. Fock in 1997 induces the Fenchel–Nielsen bracket on the set of gluing parameters (length and twist parameters) for pair-of-pants decompositions of Riemann surfaces \( \Sigma_{g,s} \) with holes. These structures are generalized to the case of Riemann surfaces \( \Sigma_{g,s,n} \) with holes and bordered cusps.

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Bibliography
1. Introduction

Constructing Darboux coordinates for the moduli spaces of Riemann surfaces $\Sigma_{g,s}$ of genus $g \geq 0$ with $(s > 0)$ or without $(s = 0)$ holes has a long and successful history. It is intrinsically related to the Poincaré uniformization of Riemann surfaces in which they are represented as quotients of the hyperbolic upper half-plane $\mathbb{H}^+_2$ under the action of a finitely generated discretely acting Fuchsian subgroup $\Delta_{g,s}$ of $\text{PSL}(2,\mathbb{R})$: $\Sigma_{g,s} = \mathbb{H}^+_2/\Delta_{g,s}$. Parameterizations of this action for a given $g$ and $s$ are called the Teichmüller spaces $\mathcal{T}_{g,s}$, and the action of $\Delta_{g,s}$ can be naturally lifted to mapping-class group transformations of $\mathcal{T}_{g,s}$, so we expect any natural structure on a Teichmüller space to be consistent with the Fuchsian group action.

Historically, the first set of symplectic coordinates were Fenchel–Nielsen length-twist coordinates $\{\ell_{A_i}, \tau_{B_i}\}_{i=1}^{3g-3+s}$ based on pair-of-pants decompositions of $\Sigma_{g,s}$; the corresponding symplectic form was merely $\sum_{i=1}^{3g-3+s} d\ell_{A_i} \wedge d\tau_{B_i}$. Wolpert ([48], [49]) then used the Fenchel–Nielsen form to derive a bracket between the geodesic functions $G_{\gamma} = e^{\ell_{\gamma}/2} + e^{-\ell_{\gamma}/2}$ corresponding to two intersecting geodesic lines $\gamma_1$ and $\gamma_2$ on $\Sigma_{g,s}$: it was given locally by the sum

$$\{G_{\gamma_1}, G_{\gamma_2}\} = \sum_{P \in \gamma_1 \# \gamma_2} \cos(\varphi_P) \quad (1.1)$$

over the intersection points of these two geodesics, where $\varphi_P$ is the signed angle between the corresponding geodesic lines at their crossing point $P$. Almost simultaneously, Goldman introduced [27] his celebrated bracket on the set of geodesic functions,

$$\{G_{\gamma_1}, G_{\gamma_2}\} = \sum_{P \in \gamma_1 \# \gamma_2} \frac{1}{2}(G_{\gamma_1 \circ_P \gamma_2} - G_{\gamma_1 \circ_P \gamma_2^{-1}}), \quad (1.2)$$

where the two geodesic functions on the right-hand side correspond to the geodesic lines $\gamma_1 \circ_P \gamma_2$ and $\gamma_1 \circ_P \gamma_2^{-1}$ (possibly with self-intersections) obtained by resolving the crossing at the point $P$ in two possible ways (see Fig. 1). Throughout this text we often use the fact that every homotopy class of closed curves on $\Sigma_{g,s}$ contains a unique closed geodesic having minimum length in the hyperbolic geometry. A clear advantage of the Goldman bracket is that it is manifestly invariant under the mapping-class group, since both the geodesic lengths and the homotopy relations are invariant under the action of the mapping-class group.

The proof that the brackets (1.1) and (1.2) give the same answer is a relatively easy exercise in hyperbolic geometry (see, for example, [35]), which we reproduce in the appendix to this paper (see §6).

The inverse statement, that is, that the Goldman Poisson structure on the set of geodesic functions gives the Fenchel–Nielsen bracket, is more difficult technically, because it requires finding proper normalizations of the twist coordinates $\tau_{B_i}$ to ensure the vanishing of the Poisson bracket between them (note that the brackets between the $\ell_{A_i}$ vanish automatically because geodesics constituting a pair-of-pants decomposition do not intersect, and their geodesic functions therefore commute). We demonstrate in Theorem 3.5 that the canonical normalization of twist coordinates ensures their commutativity. The corresponding formulae coincide (modulo
fixing some typos) with those obtained by Nekrasov, Rosly, and Shatashvili in [43] for the basic constructing blocks of the pair-of-pants decomposition: \( \Sigma_{0,4} \) and \( \Sigma_{1,1} \).

Another powerful approach to the description of Teichmüller spaces \( \Sigma_{g,s} \) is due to the ideal-triangle decomposition of \( \Sigma_{g,s} \) and the related Thurston shear coordinates [47] and Penner \( \lambda \)-lengths [44] for surfaces with punctures, which were generalized by Fock [17] to surfaces with holes. A log-canonical Poisson structure invariant under the mapping-class group was introduced by Fock on the set of shear coordinates in [18].

An explicit combinatorial construction of the corresponding classical geodesic functions in terms of the shear coordinates of decorated Teichmüller spaces for Riemann surfaces with \( s > 0 \) holes was proposed in [20]: it was shown there that the geodesic functions of all closed geodesics are Laurent polynomials in the exponentiated shear coordinates, with positive integer coefficients. These results were extended to orbifold Riemann surfaces in [13], where generalized cluster transformations (cluster algebras with coefficients) were introduced. Note that the shear coordinates can be identified with the \( Y \)-type cluster variables ([25], [26]), and mapping-class morphisms can be identified with cluster mutations.

The main result of [20] is that the constant Poisson brackets on the set of shear coordinates induce Goldman brackets on the set of geodesic functions. Combined with the result of Theorem 3.5, this immediately implies that the Fock Poisson structure on the set of shear coordinates for a space \( \Sigma_{g,s} \) with \( s > 0 \) holes is the Fenchel–Nielsen Poisson structure on the set of length-twist coordinates.

The next step was to generalize results obtained for Riemann surfaces with holes to Riemann surfaces \( \Sigma_{g,s,n} \) with holes and with \( n \) marked points on the hole boundaries (geometrically, these marked points are bordered cusps decorated with horocycles). A quantitative description of surfaces with marked points on boundary components was given from different perspectives by Fock and Goncharov [21], Musiker, Schiffler, and Williams ([40]–[42]), and Fomin, Shapiro, and Thurston ([23], [24]).

In [8], a theory of quantum bordered Riemann surfaces \( \Sigma_{g,s,n} \) and the corresponding quantum Teichmüller spaces \( \Sigma^h_{g,s,n} \) was constructed. We note that having at least one bordered cusp on one of the boundary components enables us to construct an ideal-triangle decomposition of such a surface \( \Sigma_{g,s,n} \) in which all arcs start and terminate at bordered cusps. Holes without cusps and orbifold points then have to be confined in monogons (loops), and we have to restrict mapping-class morphisms to generalized cluster mutations preserving this condition. For such ideal-triangle decompositions of \( \Sigma_{g,s,n} \) we have a bijection between the extended shear coordinates and \( \lambda \)-lengths, which enables us to determine Poisson and quantum relations for \( \lambda \)-lengths; they appear to be [8] the Poisson and quantum cluster algebras, respectively, of Berenstein and Zelevinsky [2]. On the other hand, a mapping-class group invariant 2-form [45] on the set of \( \lambda \)-lengths generates an invariant 2-form on the set of extended shear coordinates ([3], [7]).

It was proved in [8] that the Poisson and quantum structures on the set of extended shear coordinates, which are also clearly invariant under the mapping-class group, induce the extended Goldman bracket on the set of geodesic functions and
λ-lengths of arcs — geodesics starting and terminating at decorated bordered cusps. Hence, the second objective of this paper, carried out in § 4, is to construct a canonical extension of the Fenchel–Nielsen coordinates to surfaces with bordered cusps on the subset of decoration-independent variables.

For integrity of the presentation, we leave aside two very interesting topics: the first is quantization of the Poisson structures under consideration. Recall that algebras of observables are representations of quantum geodesic functions constructed out of exponentiated quantum shear coordinates [19]. The pivotal observation was that quantum flips enjoy the quantum pentagon identity ([19], [30]) based on the quantum dilogarithm function [16]. The second, novel topic is a generalization of Fenchel–Nielsen coordinates to the case of monodromies of $\text{SL}(n, \mathbb{R})$ Fuchsian systems using Fock–Goncharov higher Teichmüller spaces [21]. The corresponding classical and quantum monodromies were constructed in [46] and [12] for $\Sigma_{g,s,n}$ with $n > 0$, where it was shown that arc elements of these monodromies satisfy the Goldman brackets [27] for $\text{SL}(n, \mathbb{R})$ and are subject to Fock–Rosly algebras [22] developed for $\text{SL}(n, \mathbb{R})$-monodromies in [10]. Approaches to corresponding Fenchel–Nielsen coordinates include the construction of spectral networks (see, for example, [28]) and the higher Labourie–McShane identities [32] used by Huang and Sun [29] for constructing special potentials in higher Teichmüller theory. However, a complete construction of generalizations of Fenchel–Nielsen coordinates for higher-rank algebras is still lacking and deserves further studies.

2. Classical and Poisson algebras of geodesic functions and λ-lengths (cluster variables) and combinatorial models of Teichmüller spaces

In this section, we first recall the Goldman Poisson bracket on the set of geodesic functions and λ-lengths of arcs and make a brief excursion into the combinatorial description of Teichmüller spaces $\mathcal{T}_{g,s,n}$ of Riemann surfaces of genus $g$ with $s > 0$ holes/orbifold points, and with $n > 0$ decorated bordered cusps situated on the hole boundaries.

2.1. Goldman brackets and skein relations. In Fig. 1 we present three basic relations valid for all geodesic functions $G_{\gamma}$ corresponding to closed geodesics $\gamma$ and for λ-lengths $\lambda_a$ of arcs $a$ on any Riemann surface $\Sigma_{g,s,n}$ (we allow only arcs starting and terminating at decorated bordered cusps). We can replace any (or both) $G_{\gamma_i}$ in these relations by $\lambda_{a_i}$, with corresponding natural adjustments of the right-hand sides.

We have the following pattern: the geodesic functions $G_{\gamma_1 \circ P \gamma_2}$ and $G_{\gamma_1 \circ P \gamma_2^{-1}}$ on the right-hand sides correspond to closed geodesics obtained by the corresponding resolutions of the intersection at the point $P$. These closed geodesics become arcs if we replace exactly one geodesic $\gamma_i$ by an arc $a_i$, and in this case both the geodesic functions on the right-hand sides are to be replaced by the λ-lengths of the corresponding arcs. If we replace both closed geodesics by arcs on the left-hand sides, then, instead of single closed geodesics, on the right-hand side we have products of two λ-lengths of arcs obtained by the corresponding resolutions of the crossing at the point $P$. Note also that both the geodesic functions and the λ-lengths are
insensitive to the choice of the direction of the closed geodesic/arc. The rules for multiple crossings are as follows.

(a) The classical skein relation holds at any intersection point $P$. In particular, when we replace both $G_{\gamma_1}$ and $G_{\gamma_2}$ by $\lambda_{a_1}$ and $\lambda_{a_2}$, we obtain on the right-hand side the celebrated Ptolemy relation in [44]. We can apply the skein relation recurrently. Each time we obtain an empty loop (a contractible closed curve), we set its ‘geodesic function’ to be $G_{\emptyset} := -2$, and each time we have an empty arc contractible to a cusp, we set $\lambda_{\emptyset} := 0$.

(b) The Poisson bracket between two geodesic functions/\lambda-lengths is given by the sum

$$\{G_{\gamma_1}, G_{\gamma_2}\} = \sum_{P \in \gamma_1 \neq \gamma_2} \{G_{\gamma_1}, G_{\gamma_2}\}_P$$

over all intersection points of their local resolutions, with $\{G_{\gamma_1}, G_{\gamma_2}\}_P$ depicted in Fig. 1. We can then apply the classical skein relations to terms on the right-hand side at will. For the \lambda-lengths of two arcs starting or terminating at the same bordered cusp we have to supplement (1.2) by a homogeneous Poisson relation

$$\{\lambda_{a_1}, \lambda_{a_2}\} = \frac{1}{4} \lambda_{a_1} \lambda_{a_2},$$

where the arc $a_1$ is to the left of the arc $a_2$ when looking from the cusp. We then add these relations to (2.1), considering cusps shared by two arcs as additional intersection points.

(c) For completeness, in Fig. 1 we also present the quantum skein relation between two quantum geodesic functions/quantum \lambda-lengths (the latter can be identified with quantum cluster variables). When there are several intersection
points, we have to apply the quantum skein relation *simultaneously* at all intersection points, thus obtaining on the right-hand side a linear combination of quantum laminations—sets of non-(self)intersecting geodesic functions and arcs. Recall that quantum geodesic functions/quantum \( \lambda \)-lengths of non-intersecting geodesics/arcs commute. The quantum geodesic function \( G^{\hbar}_{\emptyset} \) corresponding to a contractible loop is set to be \( G^{\hbar}_{\emptyset} := -q - q^{-1} \), and the quantum \( \lambda \)-length of an arc contractible to a cusp remains zero, \( \lambda^{\hbar}_{\emptyset} := 0 \); for more details, see [8].

2.2. Fat-graph description for Riemann surfaces \( \Sigma_{g,s,n} \) and Teichmüller spaces \( \mathcal{T}_{g,s,n} \). We briefly recall the combinatorial description based on ideal-triangle decompositions of Riemann surfaces \( \Sigma_{g,s,n} \), the corresponding (extended) shear coordinates, and the related geodesic functions and \( \lambda \)-lengths.

**Definition 2.1.** A fat graph (a connected labelled graph with a fixed cyclic ordering of edges incident to each vertex) \( \mathcal{G}_{g,s,n} \) is called a spine of a surface \( \Sigma_{g,s,n} \) with \( s_h > 0 \) holes each containing \( n_i > 0 \) bordered cusps (\( \sum_i n_i = n > 0 \)) and with \( s_o \geq 0 \) holes/orbifold points without bordered cusps (\( s = s_h + s_o \)) if this graph can be embedded in \( \Sigma_{g,s,n} \) without self-intersections, all its vertices are three-valent except for exactly \( n \) one-valent vertices (endpoints of \( n \) pendant edges), and there are \( n_i \) pendant edges that are oriented towards the interior of the corresponding boundary component (a hole) and that correspond to \( n_i \) marked points on the hole boundary. Furthermore, we require all holes without marked points and all orbifold points to be confined in monogons (loops) of the spine. We label by \( \alpha, j, i \) all the \( 6g - 6 + 3s + 2n \) edges of the graph. A real number \( Z_\alpha \) (a shear coordinate) corresponds to the \( \alpha \)th edge that is neither a pendant edge nor a loop. Every pendant edge carries a real number \( \pi_j \) (an extended shear coordinate), and loops carry numbers (coefficients) \( \omega_i = 2 \cosh(P_i/2) \geq 2 \) for loops circumnavigating holes with perimeters \( P_i \geq 0 \) and \( \omega_i = 2 \cos(\pi/p_i) \) for loops with orbifold points of orders \( p_i \geq 2 \) inside these loops.

We identify the numbers \( Z_\alpha \) in Definition 2.1 with the (Thurston) shear coordinates (see [47] and [4]), and the numbers \( \pi_j \) with the extended shear coordinates [13]. The coordinate set \( \{Z_\alpha, \pi_j, \omega_i\} \) parameterizes the decorated Teichmüller space \( \mathcal{T}_{g,s,n} \), and it was proved in [13] that these sets parameterize all the metrizable Riemann surfaces modulo a discretely acting groupoid of flip morphisms, and conversely, each such set corresponds to a metrizable Riemann surface.

The fat graphs \( \mathcal{G}_{g,s,n} \) are in bijection with the ideal-triangle decompositions of \( \Sigma_{g,s,n} \) [44]: monogons containing either holes without marked points or orbifold points are considered elements of this decomposition, and the remaining ideal triangles correspond to three-valent vertices of the graph \( \mathcal{G}_{g,s,n} \). The edges of this decomposition are arcs: geodesic lines starting and terminating at bordered cusps. These arcs are in bijection with edges of \( \mathcal{G}_{g,s,n} \) that are not loops. We set into correspondence to an arc \( a \) its \( \lambda \)-length, \( \lambda_a = e^{\ell_a/2} - \) the exponential of one-half of the signed length of the part of \( a \) stretched between the horocycles decorating the end cusps of the arc (the sign is negative if these horocycles intersect). Then we have non-degenerate monoidal relations between the \( \lambda \)-lengths of these arcs and the
exponentiated extended shear coordinates \( \{ Z_\alpha, \pi_j \} \) (see [1], [8]). We note that the coefficients \( \omega_i \) do not contribute to these relations.

### 2.3. The Fuchsian group \( \Delta_{g,s} \), geodesic functions, and \( \lambda \)-lengths.

A metrizable surface \( \Sigma_{g,s,n} \) is the quotient of the Poincaré hyperbolic upper half-plane under a discrete action of a Fuchsian group \( \Delta_{g,s} \subset \text{PSL}(2, \mathbb{R}) \). Note that this group is ‘almost’ insensitive to the presence of bordered cusps: addition of these cusps results only in the action of group elements corresponding to the hole boundaries at which the cusps are added becoming non-trivial. The standard statement in hyperbolic geometry is that we have sets of 1-1 correspondences

\[
\begin{array}{ccc}
\text{closed geodesics on } \Sigma_{g,s,n} & \leftrightarrow & \text{conjugacy classes of } \pi_1(\Sigma_{g,s,n}) \\
\uparrow & & \uparrow \\
\text{closed paths in } \mathcal{G}_{g,s,n} & \leftrightarrow & \text{conjugacy classes of } \Delta_{g,s}
\end{array}
\]

A principal advantage of the fat–graph description of the Teichmüller spaces \( \mathcal{T}_{g,s,n} \) is the very simple and explicit formulae expressing the main algebraic objects, geodesic functions, and \( \lambda \)-lengths, in terms of shear coordinates, and also the very simple Poisson bracket on the set of these coordinates \( Z_\alpha, \pi_j \). We begin by describing the groupoid of paths.

The groupoid of paths is the set of homotopy classes of directed paths starting and terminating at bordered cusps: we denote by \( a_{i \to j} \) a path starting at the \( i \)th cusp and terminating at the \( j \)th cusp. We endow this set with the natural partial composition law: \( a_{j \to k} \circ a_{i \to j} = a_{i \to k} \). Note that for any path \( a_{i \to j} \) on \( \Sigma_{g,s,n} \) we have a unique path without backtracking in \( \mathcal{G}_{g,s,n} \) lying in the same homotopy class. We therefore use the same notation for paths in \( \Sigma_{g,s,n} \) and paths in \( \mathcal{G}_{g,s,n} \). To each such path we set into correspondence an element of \( \text{PSL}(2, \mathbb{R}) \), which we construct as a product of elementary 2 \( \times \) 2 matrices (all matrix products go from right to left). We note that elements of the path groupoid are also known as elements of the decorated character variety \( (\text{SL}(2, \mathbb{R}))^{2g+s+n-2}/U^n \) (see [9]).

Each time a path in the graph \( \mathcal{G}_{g,s,n} \) goes along an \( \alpha \)th inner edge or starts or terminates at a pendant edge, we insert the so-called 'right' and 'left' turn matrices

\[
R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L = R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},
\]

and finally, when a path goes along an \( i \)th loop clockwise (counterclockwise), we insert the matrix \( F_{\omega_i} \) (or \(-F_{\omega_i}^{-1}\)):

\[
F_{\omega_i} := \begin{pmatrix} 0 & 1 \\ -1 & -\omega_i \end{pmatrix} \quad \text{and} \quad -F_{\omega_i}^{-1} := \begin{pmatrix} \omega_i & 1 \\ -1 & 0 \end{pmatrix},
\]
sandwiched between the two edge matrices $X_{Z_\alpha}$ of the unique edge incident to the loop.

An element $P_a \in \text{PSL}(2, \mathbb{R})$ in the groupoid of paths then has the typical structure

$$P_{a_{j_1 \to j_2}} = X_{\pi_{j_1}} X_{Z_n} R X_{Z_{n-1}} \cdots R X_{Z_k} X_{Z_{k+1}} \cdots X_{Z_1} X_{\pi_{j_2}}$$

for a path starting at the cusp $j_1$ and terminating at the cusp $j_2$. The $\lambda$-length of this path is then the upper-right matrix element of $P_a$ (and it is easy to see that it does not depend on the direction of the path). For paths $a_{j \to j}$ starting and terminating at the same cusp, we obtain the geodesic functions as traces of the corresponding path matrices,

$$G_\gamma \equiv \text{tr} P_{a_{j \to j}} = 2 \cosh \left( \frac{\ell_\gamma}{2} \right),$$

where $\ell_\gamma$ is the length of the closed geodesic that is homeomorphic to the arc $a_{j \to j}$ upon identifying its endpoints and erasing the marked point thus obtained. Note that the backtracking part that appears in this procedure in the matrix product is then cancelled under the trace sign, so we can always consider only closed paths without backtracking when evaluating geodesic functions.

This construction implies the following fundamental property: for any graph $\mathcal{G}_{g,s,n}$ all matrix elements of $P_a$ for every arc $a$ (and, correspondingly, all $\lambda$-lengths and geodesic functions $G_\gamma$) are polynomials in the exponentiated shear coordinates, with sign-definite integer coefficients:

$$\lambda_a \in \mathbb{Z}_+[[e^{\pi_j/2}, e^{\pm Z_\alpha/2}, \omega_i]] \quad \text{and} \quad G_\gamma \in \mathbb{Z}_+[[e^{\pm Z_\alpha/2}, \omega_i]].$$

### 2.4. Poisson and symplectic structures.

One of the most attractive properties of the fat-graph description is a very simple Poisson algebra on the set of coordinates $Z_\alpha, \pi_j$ [18]: let $Y_k, k = 1, 2, 3 \mod 3$, denote either $Z$- or $\pi$-variables of cyclically ordered edges incident to a three-valent vertex. The Poisson (Weil–Petersson) bi-vector field is then

$$\omega_{WP} := \sum_{3\text{-valent vertices}} \sum_{k=1}^3 \partial_{Y_k} \wedge \partial_{Y_{k+1}}.$$  

**Theorem 2.2** (see [20], [8]). The bracket (2.7) induces the Goldman bracket [27] (see Fig. 1) on the set of geodesic functions and on the set of $\lambda$-lengths of arcs.

The centre of the Poisson algebra (2.7) is generated by sums of $Z_\alpha$ and $\pi_j$ (taken with multiplicities) over all edges incident to each given hole, so that together with the coefficients $\omega_i$ we have exactly $s$ independent central elements.

On the set of $\lambda$-lengths of arcs Penner [45] introduced a symplectic structure invariant under the mapping-class group:

$$\Omega_{WP} := \sum_{\text{ideal triangles}} \sum_{k=1}^3 d \log \lambda_k \wedge d \log \lambda_{k+1}.$$  

(2.8)
In the case of $\Sigma_{g,s,n}$ with $n > 0$, the bracket (2.7) induces homogeneous Poisson brackets on the sets of $\lambda$-lengths of arcs in one and the same ideal-triangle decomposition. Quantization of these brackets leads to Berenstein–Zelevinsky quantum cluster algebras [2], and this Poisson structure was shown in [7] to be inverse to the symplectic structure (2.8).

2.5. Flip morphisms for fat graphs. Different fat-graph parameterizations of Teichmüller spaces are related by mapping-class morphisms generated by sequences of flip morphisms (mutations) of edges: any two spines from a given topological class are related by a finite sequence of flips, and every time this sequence results in a graph homotopically identical to the original graph (note that we have infinitely many copies of the moduli space $\mathcal{M}_{g,s,n}$ in the Teichmüller space $\mathcal{T}_{g,s,n}$ but only a finite number of homotopically different spines $\mathcal{G}_{g,s,n}$), the transformations of the variables $\{Z_\alpha, \pi_j\}$ describe a transformation in the mapping-class group. It is also a classical result that all such transformations are produced by Dehn twists along closed geodesics, which we use in the next section.

We distinguish between two types of flip morphisms: those induced by flips of ‘typical’ inner edges (see Fig. 2) and those induced by flips of edges that are incident to a loop (see Fig. 3). No flips can be performed on pendant edges and loops.

![Figure 2. Flip on an inner edge (labelled $e$) that is neither a loop nor incident to a loop. We indicate the correspondences between paths in the graph undergoing the flip. The dashed lines are arcs of the dual ideal-triangle decomposition.](image)

Lemma 2.3 (see [19], [20]). In the notation of Fig. 2, the transformation

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{Z}_e) = (A + \phi(Z_e), B - \phi(-Z_e), C + \phi(Z_e), D - \phi(-Z_e), -Z_e),$$

where $\phi(Z) = \log(1 + e^Z)$, preserves the path products (2.6) (thus preserving both geodesic functions and $\lambda$-lengths) while simultaneously preserving the Poisson structure (2.7) on the shear coordinates. The dual Ptolemy transformation of $\lambda$-lengths (cluster mutation), $\lambda_e\lambda_e' = \lambda_a\lambda_c + \lambda_b\lambda_d$, preserves the symplectic structure (2.8).
Because the proof of the lemma is local with respect to the graph $\mathcal{G}_{g,s,n}$ and follows from the matrix equalities

$$X_DRX_{Z_e}RX_A = X_A^{-1}RX_{\bar{D}}, \quad X_DRX_{Z_e}LX_B = X_{\bar{D}}LX_{\bar{Z}_e}RX_{\bar{B}},$$

and

$$X_CLX_D = X_{\bar{C}}LX_{\bar{Z}_e}LX_{\bar{D}},$$

each pertaining to the corresponding choice of paths in Fig. 2, the proof can be extended to the whole groupoid of $\text{SL}(2,\mathbb{R})$ monodromies. The same statement is therefore valid for the $\lambda$-lengths of the corresponding arcs. We have a similar statement for flips of inner edges incident to loops.

**Lemma 2.4** (see [13], [8]). *The transformation in Fig. 3*

$$\{\tilde{A}, \tilde{B}, \tilde{Z}_e\} := \{A + \phi(Z_e + \xi) + \phi(Z_e - \xi), B - \phi(-Z_e + \xi) - \phi(-Z_e - \xi), -Z_e\},$$

$$w = e^\xi + e^{-\xi},$$

where $\phi(x) = \log(1 + e^x)$ and $\xi \in \mathbb{C}$, is a morphism of the space $\mathcal{F}_{g,s,n}$ that preserves both the Poisson structure (2.7) and the path elements from $\text{SL}(2,\mathbb{R})$. The dual transformation (generalized cluster transformation) $\lambda_e\lambda'_e = \lambda_a^2 + w\lambda_a\lambda_b + \lambda_b^2$ preserves the symplectic structure (2.8).

3. Fenchel–Nielsen brackets and shear coordinates

3.1. Fenchel–Nielsen coordinates for Riemann surfaces with holes.

3.1.1. Fenchel–Nielsen coordinates for $\Sigma_{1,1}$. We first consider the case of a torus with one hole of perimeter $p$. Choosing a closed geodesic $A$ with length $\ell_A$ and a dual closed geodesic $B$ having a single intersection point with $A$, we construct a twist coordinate $\tau_B$ that is a function of $G_B$, $\ell_A$, and $G_P := e^{p/2} + e^{-p/2}$. Together with $B$, it is useful to consider all geodesics obtained by Dehn twists along the geodesic $A$: $A^n B$, $n \in \mathbb{Z}$. The Poisson bracket is

$$\{G_A, G_{A^n B}\} = \frac{1}{2}(G_{A^{n-1} B} - G_{A^{n+1} B}), \quad n \in \mathbb{Z}, \quad (3.1)$$
where $G_{A^{n-1}B}$ and $G_{A^{n+1}B}$ are the two solutions of the quadratic equations generated by a Markov triple,

$$G_A G_{A^nB} G_{A^{n-1}B} - G_A^2 - G_{A^nB}^2 - G_{A^{n-1}B}^2 = G_P - 2, \quad n \in \mathbb{Z}. \quad (3.2)$$

We also have the classical skein relations

$$G_A G_{A^nB} = G_{A^{n+1}B} + G_{A^{n-1}B}, \quad n \in \mathbb{Z}. \quad (3.3)$$

**Proposition 3.1.** A twist coordinate $\tau_B$ having a constant unit bracket with the length $\ell_A$ of a closed geodesic $A$, $\{\tau_B, \ell_A\} = 1$, has the form

$$\tau_{A^nB} = \log(G_{A^nB} - e^{-\ell_A/2} G_{A^nB}). \quad (3.4)$$

Then $\{\ell_A, \tau_{A^nB}\} = 1$ for all $n \in \mathbb{Z}$, and the $\tau_{A^nB}$ are all related by constant shifts:

$$\tau_{A^nB} - \tau_{A^{n+1}B} = \frac{\ell_A}{2}. \quad (3.5)$$

In particular, a Dehn twist along $A$ transforms $B$ into $A^2B$, and the corresponding modular transformation is $\tau_B \to \tau_{A^2B} - \tau_A - \ell_A$, so that, as expected, the twist coordinate (3.4) takes values between 0 and $\ell_A$ in a single copy of the moduli space labelled by $A$ in Mirzakhani’s terminology [36].

**Proof.** The proof is a direct calculation. We have to choose the sign of $\ell_A$: applying (3.3), we obtain, say, for $n = -1$ that

$$G_{A^{-2}B} - e^{-\ell_A/2} G_{A^{-1}B} = (e^{\ell_A/2} + e^{-\ell_A/2}) G_{A^{-1}B} - e^{-\ell_A/2} G_{A^{-1}B} = e^{\ell_A/2} (G_{A^{-1}B} - e^{-\ell_A/2} G_{A^{-1}B}). \quad (3.6)$$

Checking that $\{\tau_B, \ell_A\} = 1$ is a straightforward calculation:

$$\{e^{\tau_B}, G_A\} = \frac{1}{2} e^{\tau_B} (e^{\ell_A/2} - e^{-\ell_A/2}) \{\tau_B, \ell_A\}$$

$$= \frac{1}{2} [G_{A^{-2}B} - G_{A^{-1}B} - e^{-\ell_A/2} (G_{A^{-1}B} - G_{AB})]$$

$$= \frac{1}{2} [G_A G_{A^{-1}B} - 2 G_{B} - e^{-\ell_A/2} (2 G_{A^{-1}B} - G_{AB})]$$

$$= \frac{1}{2} [e^{\ell_A/2} - e^{-\ell_A/2}] (G_{A^{-1}B} - e^{-\ell_A/2} G_{B}). \quad \square$$

What if we choose $\tau_B$ in another form, say, $\tau'_B = \log(G_{A^{-1}B} - e^{\ell_A/2} G_B)$? It is easy to see, using the Markov triple and skein relations, that

$$(G_{A^{-1}B} - e^{-\ell_A/2} G_B)(G_{A^{-1}B} - e^{\ell_A/2} G_B) = e^{\ell_A} + e^{-\ell_A} + e^{p/2} + e^{-p/2}, \quad (3.7)$$

so, since $\tau_B$ is defined up to a constant shift by $f(\ell_A, p)$, we can identify $\tau'_B$ with the negative of $\tau_B$.

However, we still have substantial freedom in choosing $\tau_B$: we can add to it any function $f(\ell_A, p)$ (normalization) while preserving both the Poisson bracket with $\ell_A$
and the shift symmetries. But note that adding this function (a) changes the symmetry properties under the inversion $\ell_A \to -\ell_A$ and (b) affects the Poisson brackets between the twist coordinates $\tau_B$ and $\tau_{B'}$ dual to different cycles $\gamma_A$ and $\gamma_{A'}$ from the same pair-of-pants decomposition of $\Sigma_{g,s}$. Below we see that choosing the normalization so that $\tau_B \to -\tau_B$ under the change of the sign of $\ell_A$ allows us to solve these two problems simultaneously. Moreover, the canonical twist coordinates thus constructed have a clear geometrical sense.

Let us address the geometrical interpretation of the twist coordinate $\tau_B$. On the right-hand side of Fig. 4 we depict the triangle constituted by half-lengths of the geodesics $A$, $B$, and $A^{-1}B$. It is a well-known fact from hyperbolic geometry (see §6) that for two curves $A$ and $B$ with a single intersection point $P$, the curve $A^{-1}B$ intersects $A$ and $B$ at the points $R$ and $S$ lying exactly at distances $\ell_A/2$ and $\ell_B/2$ from $P$, respectively. The distance $|SR|$ is then exactly half the length of the geodesic $A^{-1}B$.

![Figure 4. A torus with a hole as a pair of pants.](image)

Using the standard formulae for a triangle in hyperbolic geometry (see, for instance, [6]), we have

$$\cosh(|PR|) \cosh(|SP|) - \cosh(|SR|) = \sinh(|SP|) \cosh(|PR|) \tanh(|PQ|),$$

or, upon identification of the triangle sides as in Fig. 4, we obtain

$$e^{2|PQ|} = \frac{G_AG_B/2 - G_{A^{-1}B} - G_B \sinh(\ell_A/2)}{G_AG_B/2 - G_{A^{-1}B} + G_B \sinh(\ell_A/2)} = \frac{G_{A^{-1}B} - e^{-\ell_A/2}G_B}{G_{A^{-1}B} - e^{\ell_A/2}G_B},$$

so that by multiplying the numerator and denominator by $G_A$ and again applying the skein relation (3.3), we get that

$$e^{2|PQ|} = \frac{e^{\ell_A/2}G_{A^{-1}B} - e^{-\ell_A/2}G_{AB}}{e^{-\ell_A/2}G_{A^{-1}B} - e^{\ell_A/2}G_{AB}} = \frac{e^{2\tau_B}}{e^{\ell_A} + e^{-\ell_A} + e_p^2 + e^{-p^2}},$$

(3.8)

which implies that $|PQ| = \tau_B + f(\ell_A, p)$, and the rates of change of both the quantities for fixed $\ell_A$ and $p$ are the same. We then declare the signed quantity $|PQ|$ to be a canonical twist coordinate.
Lemma 3.2. The canonical twist coordinate that changes sign under a change of the orientation of \( \ell_A (\ell_A \to -\ell_A) \) for a torus with one hole having perimeter \( p \) is

\[
\tilde{\tau}_B := \log \frac{G_A^{-1}B - e^{-\ell_A/2}G_B}{(e^{\ell_A} + e^{-\ell_A} + e^p/2 + e^{-p/2})^{1/2}}.
\]  

(3.9)

Geometrically, \( \tilde{\tau}_B \) is one-half of the signed geodesic length \( 2|PQ| \) along the geodesic \( \gamma_A \) between the endpoints of a geodesic \( H \) perpendicular to \( \gamma_A \) and homeomorphic to the \( B \)-cycle (see Fig. 4).

3.1.2. Fenchel–Nielsen coordinates for \( \Sigma_{0,4} \). After a ‘warm-up’ case of a torus with one hole, let us proceed to the more laborious case of a four-hole sphere, which can be regarded as two pairs of pants glued together along a geodesic \( A \) (see Fig. 5). We let \( G_i := e^{p_i/2} + e^{-p_i/2}, \) \( i = 1, \ldots, 4 \), denote the geodesic functions for the perimeters of the four holes. The geodesics \( A \) and \( B \) now have two intersection points.

![Figure 5. A sphere with four holes as a gluing together of two pairs of pants. Here \( H_B, h_2, \) and \( h_3 \) are perpendiculars between \( \gamma_{p_1} \) and \( \gamma_{p_4} \) (the unique perpendicular between these two cycles that does not intersect \( \gamma_B \)), between \( \gamma_{p_1} \) and \( \gamma_A \), and between \( \gamma_{p_4} \) and \( \gamma_A \), respectively. The signed geodesic length \( |PQ| \) (along the geodesic \( \gamma_A \)) is identified with the canonical twist coordinate \( \tilde{\tau}_B \) in (3.20).

The Poisson bracket is exactly the same as in the torus case (note the absence of the factor \( 1/2 \)):

\[
\{G_A, G_B\} = G_A^{-1}B - G_{AB},
\]

(3.10)

where \( G_A^{-1}B \) and \( G_{AB} \) are the two solutions of the quadratic equation generated by a Markov triple:

\[
G_AG_BG_A^{\pm 1}B - G_A^2 - G_B^2 - G_A^{\pm 1}B - G_{A^{\pm 1}B}S_{AB} - G_AS_A - G_BS_B = R,
\]

(3.11)

where

\[
S_A := G_1G_2 + G_3G_4, \quad S_B := G_1G_4 + G_2G_3, \quad S_{AB} := G_1G_3 + G_2G_4
\]

(3.12)
Proposition 3.3. A twist coordinate having a constant unit bracket with the half-length \(\tau\) compare\(\modular transformation is again the form

\[ R := G_1 G_2 G_3 G_4 + \sum_{i=1}^{4} G_i^2 - 4, \]  

(3.13)

and the same relation (3.11) holds upon replacement of the indices \(B \to A^{2k} B\) and \(A^{\pm 1} B \to A^{2k \pm 1}\) for each integer \(k\).

We can also use the classical skein relations

\[ G_A G_{A^{2k}} = G_{A^{2k-1}} + G_{A^{2k+1}} + S_{AB}, \]
\[ G_A G_{A^{2k+1}} = G_{A^{2k+2}} + G_{A^{2k}} + S_{B}, \quad k \in \mathbb{Z}. \]  

(3.14)

Proposition 3.3. A twist coordinate having a constant unit bracket with the half-length \(\ell_A/2\) of a closed geodesic \(A\) for a sphere with four holes, \(\{\tau_B, \ell_A/2\} = 1\), has the form

\[
\tau_{A^{2k} B} = \log \left( G_{A^{2k-1} B} - e^{-\ell_A / 2} G_{A^{2k} B} - \frac{e^{-\ell_A / 2} S_{AB} + S_B}{2 \sinh(\ell_A / 2)} \right),
\]  

(3.15)

and

\[
\tau_{A^{2k+1} B} = \log \left( G_{A^{2k} B} - e^{-\ell_A / 2} G_{A^{2k+1} B} - \frac{e^{-\ell_A / 2} S_B + S_{AB}}{2 \sinh(\ell_A / 2)} \right).
\]  

(3.16)

All these choices of a dual coordinate are related by constant shifts:

\[
\tau_{A^n B} - \tau_{A^{n+1} B} = \frac{\ell_A}{2}.
\]  

(3.17)

In particular, a Dehn twist along \(A\) transforms \(B\) into \(A^2 B\), and the corresponding modular transformation is again \(\tau_B \to \tau_{A^2 B} = \tau_B - \ell_A\), so that the twist coordinate (3.15) assumes values between 0 and \(\ell_A\) in a single copy of the moduli space labelled by \(A\).

Proof. The proof is by direct calculation. Checking that \(\{\tau_B, \ell_A/2\} = 1\) is straightforward using only (3.14), and we leave this to the reader.

Note that shifting a twist coordinate \(\tau_B\) by any function \(f(G_A, G_i)\) preserves the Poisson brackets with all length coordinates.

Take, say, \(\tau_{A^{2k} B}\) and use the fact that \(G_{A^{2k} B} = G_A G_{A^{2k-1} B} - G_{A^{2k-2} B} - S_B\):

\[
\tau_{A^{2k} B} = \log \left( G_{A^{2k-1} B} - e^{-\ell_A / 2} (G_A G_{A^{2k-1} B} - G_{A^{2k-2} B} - S_B) - \frac{e^{-\ell_A / 2} S_{AB} + S_B}{2 \sinh(\ell_A / 2)} \right)
\]  

\[
= \log \left( e^{-\ell_A / 2} (G_{A^{2k-2} B} - e^{-\ell_A / 2} G_{A^{2k-1} B}) - \frac{e^{-\ell_A / 2} S_{AB} + e^{-\ell_A / 2} S_B}{2 \sinh(\ell_A / 2)} \right) + \frac{\ell_A}{2}
\]  

\[
= -\frac{\ell_A}{2} + \tau_{A^{2k-1} B}.
\]

We therefore again have \(\tau_{A^n B} - \tau_{A^{n+1} B} = \ell_A/2\) for any \(n \in \mathbb{Z}\).

Let us now see what happens if we choose the other sign for \(\ell_A\), that is, we compare \(\tau_{A^{2k} B}\) in (3.15) with

\[
\tau'_{A^{2k} B} := \log \left( G_{A^{2k-1} B} - e^{\ell_A / 2} G_{A^{2k} B} + \frac{S_B + e^{\ell_A / 2} S_{AB}}{2 \sinh(\ell_A / 2)} \right).
\]
Then using (3.11), we get after some algebra that
\[
\tau_{A^*B} + \tau'_{A^*B} = \log \left( \frac{S_{AB} + G_{A} S_{B} S_{AB} + S_{B}^2}{4 \sinh^2(\ell_A/2)} + G_{A}^2 + S_{A} G_{A} + R \right),
\]
that is, it does not depend on $G_B$, and hence $\tau'_{A^*B}$ and $\tau_{A^*B}$ have opposite rates of change for variations of $\ell_B$. $\square$

A more convenient way of writing the expression in (3.18) is due to the identity
\[
S_{AB} + G_{A} S_{B} S_{AB} + S_{B}^2 + (G_{A}^2 - 4)(G_{A}^2 + S_{A} G_{A} + R)
= (G_{A} G_{1} G_{2} + G_{A}^2 + G_{1}^2 + G_{2}^2 - 4)(G_{A} G_{3} G_{4} + G_{A}^2 + G_{3}^2 + G_{4}^2 - 4).
\]
From this we can guess the proper normalization for the twist coordinate.

**Lemma 3.4.** The canonical twist coordinate that changes sign under a change of the orientation of $\ell_A$ ($\ell_A \to -\ell_A$) is
\[
\hat{\tau}_B := \log \frac{(e^{\ell_A/2} - e^{-\ell_A/2})(G_{A^{-1}} - e^{\ell_A/2} G_{B}) - e^{\ell_A/2} S_{AB} - S_{B}}{(G_{A} G_{1} G_{2} + G_{A}^2 + G_{1}^2 + G_{2}^2 - 4)^{1/2}(G_{A} G_{3} G_{4} + G_{A}^2 + G_{3}^2 + G_{4}^2 - 4)^{1/2}}.
\]
Geometrically, $\hat{\tau}_B$ is the signed geodesic length along the geodesic $\gamma_A$ between the endpoints of perpendiculars to $\gamma_A$ from the holes $P_1$ and $P_4$.

Before proving this lemma, let us formulate the main statement of this section.

**Theorem 3.5.** For any pair-of-pants decomposition of a Riemann surface $\Sigma_{g,s}$, let the twist variable $\hat{\tau}_B$ for each of $3g - 3 + s$ inner geodesics $\gamma_A$ of this decomposition be defined by the formulae (3.20) and (3.9). Then $\{\hat{\tau}_B, \hat{\tau}'_B\} = 0$, that is, all the canonical twist coordinates Poisson commute.

**Remark 3.6.** The coordinates $\hat{\tau}_B$ (3.20) and (3.9) were introduced by Nekrasov, Rosly, and Shatashvili in [43] as the canonical coordinates having unit brackets with the corresponding length coordinates $\ell_A$.

### 3.1.3. Proof of Lemma 3.4.
Since the commutation relations between $\ell_A$ and $\hat{\tau}_B$ were proved above, it only remains to identify the canonical twist with a geometric object, namely, with the (signed) geodesic distance $|PQ|$ in Fig. 5. We evaluate it using hyperbolic geometry identities (a list of very useful identities can be found on p. 454 of Buser’s monograph [6]). First, note that for any pair of pants, that is, a sphere with three holes with, say, perimeters $p_1$, $p_2$, and $\ell_A$, cutting it along perpendiculars between the hole boundaries turns it into a union of two mirror-symmetrical right-angled hexagons with cyclically ordered boundary lengths \{\(h_2, p_1/2, h_A, p_2/2, h_1, \ell_A/2\)\}. We can then express $h_2$ via $p_1$, $p_2$, and $\ell_A$ by the formula
\[
\cosh(h_2) \sinh\left(\frac{p_1}{2}\right) \sinh\left(\frac{\ell_A}{2}\right) = \cosh\left(\frac{p_2}{2}\right) + \cosh\left(\frac{p_1}{2}\right) \cosh\left(\frac{\ell_A}{2}\right).
\]
An analogous formula for another pair of pants bounded by holes with perimeters $p_3, p_4,$ and $\ell_A$ expresses $h_3$:

$$\cosh(h_3) \sinh\left(\frac{p_4}{2}\right) \sinh\left(\frac{\ell_A}{2}\right) = \cosh\left(\frac{p_3}{2}\right) + \cosh\left(\frac{p_4}{2}\right) \cosh\left(\frac{\ell_A}{2}\right).$$  \hspace{1cm} (3.22)

The third formula expresses $H_B$: just note that it is a perpendicular in the sphere with three-holes bounded by $p_1, p_4$ and $B$:

$$\cosh(H_B) \sinh\left(\frac{p_1}{2}\right) \sinh\left(\frac{p_4}{2}\right) = \frac{1}{2} G_B + \cosh\left(\frac{p_1}{2}\right) \cosh\left(\frac{p_4}{2}\right).$$  \hspace{1cm} (3.23)

Finally, we get a formula for $|PQ|$: note that this is the length of a side of a right-angled hexagon with a self-intersection (in Fig. 5 its boundary edges are $h_2, h_3, H_B, PQ,$ and parts of the perimeters of the holes $p_1$ and $p_4$). The needed relation from [6] is

$$\cosh(H_B) = \sinh(h_2) \sinh(h_3) \cosh(|PQ|) + \cosh(h_2) \cosh(h_3).$$  \hspace{1cm} (3.24)

Using the above formulae, we are able to express $e^{\rho PQ}$. It happens that some of the square-root expressions appearing in the answer coincide with the expressions appearing when we solve equation (3.11) with respect to $G_{A^{-1}B}$. This enables us to simplify the expression and finally to obtain the formula (3.20) in which $\tau_B = \tau_B'$.

### 3.1.4. Proof of Theorem 3.5.

We begin by proving the technically most difficult case in which both $\tau_B$ and $\tau_B'$ are twist coordinates of type (3.20). First, these coordinates obviously commute if the four-hole spheres determining these two coordinates do not share a common three-hole sphere (a pair of pants). Thus, our basic case is the five-hole sphere depicted in Fig. 6.

![Figure 6. A five-hole sphere split into three pairs of pants: the first pair bounded by $\gamma_{p_{\alpha}}, \gamma_{p_{\beta}}$, and $\gamma_{A'}$, the second by $\gamma_{A'}, \gamma_{p_3}$, and $\gamma_A$, and the third by $\gamma_{p_{\gamma}}, \gamma_{p_{\delta}}$, and $\gamma_A$.](image)
We use the following convention in Fig. 6: \( \alpha = 1, \beta = 2 \) if \( n \in 2\mathbb{Z} \), and \( \alpha = 2, \beta = 1 \) if \( n \in 2\mathbb{Z} + 1 \); correspondingly, \( \gamma = 5, \delta = 4 \) if \( m \in 2\mathbb{Z} \), and \( \gamma = 4, \delta = 5 \) if \( m \in 2\mathbb{Z} + 1 \). Denoting by \( D_{m,n} \) the geodesic functions of closed geodesics encircling the corresponding holes (as indicated in the figure) and lying in the proper homotopy class, we apply the standard Goldman brackets and skein relations to obtain the collection of useful formulae

\[
\{G_{A^nB}, G_{A'^nB'}\} = D_{m+1,n+1} - D_{m-1,n-1}, \quad (3.25)
\]

\[
G_{A^nB}G_{A'^nB'} = D_{m+1,n+1} + D_{m-1,n-1} + G_3D_{m,n} + \beta G_{\delta}, \quad (3.26)
\]

\[
G_{A'}D_{m,n} = D_{m,n+1} + D_{m,n-1} + G_{A'^nB}\alpha + \beta G_{\gamma}, \quad (3.27)
\]

\[
G_AD_{m,n} = D_{m+1,n} + D_{m-1,n} + G_{A'^nB'}\gamma + \alpha G_{\delta}. \quad (3.28)
\]

The first (and crucial) step is to find the brackets between the non-normalized twists

\[
e^{\tau_B} := G_{A^{-1}B} - e^{-\ell_A/2}G_B = \frac{G_5G_{A'} + G_3G_A + \ell_A/2(G_G_{A'} + G_G_{G_5})}{e^{\ell_A/2} - e^{-\ell_A/2}} \quad (3.29)
\]

and

\[
e^{\tau_{B'}} := G_{A'^{-1}B'} - e^{-\ell_A'/2}G_{B'} = \frac{G_1G_A + G_3G_{A'} + \ell_A'/2(G_G_{A'} + G_G_{G_1})}{e^{\ell_A'/2} - e^{-\ell_A'/2}} \quad (3.30)
\]

Note that to preserve the mirror-like symmetry in Fig. 6, we chose opposite orientations of the \( A \)- and \( A' \)-cycles. Correspondingly, in our notation we have \( \{\tau_B, \ell_A\} = -\{\tau_{B'}, \ell_{A'}\} = -2 \). A direct calculation gives

\[
\{e^{\tau_B}, e^{\tau_{B'}}\} = \left[D_{0,0} - e^{-\ell_A/2}D_{0,1} - e^{-\ell_A/2}D_{1,0} + e^{-\ell_A/2-\ell_A'/2}D_{1,1}\right]
\]

\[
- \left[D_{-2,-2} - e^{-\ell_A'/2}D_{-2,-1} - e^{-\ell_A'/2}D_{-1,-2} + e^{-\ell_A/2-\ell_A'/2}D_{-1,-1}\right]
\]

\[
+ \left(G_1 + e^{-\ell_A'/2}G_2\right)e^{\ell_A/2} - e^{-\ell_A/2} \quad e^{\tau_B}
\]

\[
+ \left(G_5 + e^{-\ell_A/2}G_4\right)e^{\ell_A'/2} - e^{-\ell_A'/2} \quad e^{\tau_{B'}}.
\]

The next set of formulae pertains to index ‘shifts’ of blocks containing \( D \)-terms in the above formula:

\[
D_{-1,-1} - e^{-\ell_A'/2}D_{-1,0} - e^{-\ell_A/2}D_{0,-1} + e^{-\ell_A/2-\ell_A'/2}D_{0,0}
\]

\[
= e^{\ell_A/2+\ell_A'/2}[D_{0,0} - e^{-\ell_A'/2}D_{0,1} - e^{-\ell_A/2}D_{1,0} + e^{-\ell_A/2-\ell_A'/2}D_{1,1}]
\]

\[
- G_1[G_A^{-1}B - e^{-\ell_A/2}G_B] - G_5[G_{A^{-1}B'} - e^{-\ell_A'/2}G_{B'}]
\]

\[
- G_1G_3G_5 - e^{-\ell_A/2}G_2G_5 - e^{-\ell_A'/2}G_1G_4
\]

and

\[
D_{-2,-2} - e^{-\ell_A'/2}D_{-2,-1} - e^{-\ell_A/2}D_{-1,-2} + e^{-\ell_A/2-\ell_A'/2}D_{-1,-1}
\]

\[
= e^{\ell_A/2+\ell_A'/2}[D_{-1,-1} - e^{-\ell_A'/2}D_{-1,0} - e^{-\ell_A/2}D_{0,-1} + e^{-\ell_A/2-\ell_A'/2}D_{0,0}]
\]

\[
- G_2[e^{\ell_A/2}G_{A^{-1}B} - G_B] - G_4[e^{\ell_A'/2}G_{A^{-1}B'} - G_{B'}]
\]

\[
+ G_2G_3G_4 + e^{-\ell_A/2}G_1G_4 + e^{-\ell_A'/2}G_2G_5.
\]
Using these formulae, we find that
\[
\{e^{\tau_B}, e^{\tau_{B'}}\} = (e^{-\ell_A/2-\ell_{A'/2}} - e^{\ell_A/2+\ell_{A'/2}})
\times \left( D_{-1,-1} - e^{-\ell_{A'/2}}D_{-1,0} - e^{-\ell_A/2}D_{0,-1} + e^{-\ell_A/2-\ell_{A'/2}}D_{0,0}
- \left[ G_{A^{-1}B} - e^{-\ell_A/2}G_B \right] \frac{e^{-\ell_{A'/2}G_1} + G_2}{e^{\ell_{A'/2}} - e^{-\ell_{A'/2}}}
- \left[ G_{A'{-1}B'} - e^{-\ell_{A'/2}}G_{B'} \right] \frac{e^{-\ell_A/2}G_5 + G_4}{e^{\ell_A/2} - e^{-\ell_A/2}}
+ \frac{1}{(e^{\ell_A/2} - e^{-\ell_A/2})(e^{\ell_{A'/2}} - e^{-\ell_{A'/2}})}
\times [G_3(e^{-\ell_A/2}G_1 + G_2)(e^{-\ell_{A'/2}}G_5 + G_4)
+ 2(e^{-\ell_{A'/2}}G_2 + G_1)(e^{-\ell_A/2}G_4 + G_5)] \right)
\] (3.31)

We obtain the last necessary formula from the skein relation (3.26): it turns out that its form exactly repeats (3.31), just with a different pre-factor:
\[
e^{\tau_B}e^{\tau_{B'}} = (e^{-\ell_A/2-\ell_{A'/2}} + G_3 + e^{\ell_A/2+\ell_{A'/2}})
\times \left( D_{-1,-1} - e^{-\ell_{A'/2}}D_{-1,0} - e^{-\ell_A/2}D_{0,-1} + e^{-\ell_A/2-\ell_{A'/2}}D_{0,0}
- \left[ G_{A^{-1}B} - e^{-\ell_A/2}G_B \right] \frac{e^{-\ell_{A'/2}G_1} + G_2}{e^{\ell_{A'/2}} - e^{-\ell_{A'/2}}}
- \left[ G_{A'{-1}B'} - e^{-\ell_{A'/2}}G_{B'} \right] \frac{e^{-\ell_A/2}G_5 + G_4}{e^{\ell_A/2} - e^{-\ell_A/2}}
+ \frac{1}{(e^{\ell_A/2} - e^{-\ell_A/2})(e^{\ell_{A'/2}} - e^{-\ell_{A'/2}})}
\times [G_3(e^{-\ell_A/2}G_1 + G_2)(e^{-\ell_{A'/2}}G_5 + G_4)
+ 2(e^{-\ell_{A'/2}}G_2 + G_1)(e^{-\ell_A/2}G_4 + G_5)] \right)
\] (3.32)

We have therefore proved a technical proposition.

**Proposition 3.7.** The Poisson bracket between the non-normalized twist coordinates \(\tau_B\) and \(\tau_{B'}\) defined by (3.29) and (3.30) is
\[
\{\tau_B, \tau_{B'}\} = \frac{-\sinh(\ell_A/2 + \ell_{A'/2})}{\cosh(\ell_A/2 + \ell_{A'/2}) + \cosh(p_3/2)}
= \frac{e^{-\ell_A/2-\ell_{A'/2}-p_3/2} - e^{\ell_A/2+\ell_{A'/2}-p_3/2}}{(e^{-\ell_A/2-\ell_{A'/2}-p_3/2} + 1)(e^{\ell_A/2+\ell_{A'/2}-p_3/2} + 1)}.
\] (3.33)

To complete the proof of the first case of the theorem, we present one more useful relation: for any three geodesic functions \(G_{A_i} := e^{\ell_{A_i}/2} + e^{-\ell_{A_i}/2}, i = 1, 2, 3,\) we
have
\[
G_A G_A^2 G_A^3 + G_A^2 + G_A^2 + G_A^2 - 4 \\
= (e^{-\ell_{A_1}/2 - \ell_{A_2}/2 - \ell_{A_3}/2} + 1)(e^{\ell_{A_1}/2 + \ell_{A_2}/2 - \ell_{A_3}/2} + 1) \\
\times (e^{\ell_{A_1}/2 + \ell_{A_2}/2 - \ell_{A_3}/2} + 1)(e^{\ell_{A_2}/2 + \ell_{A_3}/2 - \ell_{A_1}/2} + 1).
\]

And thus,
\[
\frac{\partial}{\partial \ell_A} + \frac{\partial}{\partial \ell_{A'}} \log(G_A G_A^2 G_{p_3} + G_A^2 + G_A^{2'} + G_{p_3}^2 - 4) \\
= \frac{e^{\ell_A/2 + \ell_{A'}/2 - p_3/2}}{e^{\ell_A/2 - \ell_{A'}/2 - p_3/2} + 1} - \frac{e^{-\ell_A/2 - \ell_{A'}/2 - p_3/2}}{e^{-\ell_A/2 + \ell_{A'}/2 - p_3/2} + 1} \\
= \frac{e^{\ell_A/2 + \ell_{A'}/2 - p_3/2} - e^{-\ell_A/2 - \ell_{A'}/2 - p_3/2}}{(e^{\ell_A/2 + \ell_{A'}/2 - p_3/2} + 1)(e^{-\ell_A/2 + \ell_{A'}/2 - p_3/2} + 1)},
\]

which is exactly the expression on the right-hand side of the formula in Proposition 3.7. We have therefore proved the first case of the statement of the theorem.

The second case is where \( \tau_B \) is a twist coordinate of type (3.9) and \( \tau_B' \) is a twist coordinate of type (3.20). We then cut the torus along the \( A \)-cycle, thus obtaining a sphere with four holes, two of which are copies of the \( A \)-cycle. The \( B \)-cycle is then an interval joining these two copies (see Fig. 7). In this case the cycles \( A^n B \) and \( A^m B' \) have a single intersection point, and as in the first case, we introduce cycles \( D_{n,m} \) obtained by the first-type resolution at this point. In Fig. 7 we demonstrate that the second-type resolution of the crossing between \( \gamma_{A^n B} \) and \( \gamma_{A^m B'} \) is homotopically equivalent to the first-type resolution of the crossing between \( \gamma_{A^{n-2} B} \) and \( \gamma_{A^{m-1} B'} \), so that
\[
\{G_{A^n B}, G_{A^m B'}\} = \frac{1}{2}(D_{n,m} - D_{n-2,m-1}),
\]

whereas the skein relation is
\[
G_{A^n B} G_{A^m B'} = D_{n,m} + D_{n-2,m-1}.
\]

Note that in this case \( S_{AB} = S_B = G_A(G_3 + G_4) \), so we have uniform formulae for all \( \tau_{A^m B'} \).

Omitting details, we have the following proposition.

**Proposition 3.8.** The Poisson bracket between non-normalized twist coordinates \( \tau_{A^n B} \) and \( \tau_{A^m B'} \) such that
\[
e^{\tau_{A^n B}} := G_{A^n B} - e^{-\ell_{A}/2} G_{A^n B'},
\]
\[
e^{\tau_{A^m B'}} := G_{A^m B'} - e^{-\ell_{A'}/2} G_{A^m B'} - \frac{G_A(G_3 + G_4)}{e^{\ell_{A'}/2} - 1},
\]
is
\[
2\{\tau_{A^n B}, \tau_{A^m B'}\} = \frac{1 - e^{\ell_{A}/2 + \ell_{A}}}{1 + e^{\ell_{A}/2 + \ell_{A}}}. \tag{3.37}
\]
Figure 7. A torus with two holes. Cutting it along the $A$-cycle gives a sphere with four holes. The $B$-cycle then becomes a line between two copies of the $A$-cycle. The cycles $B$ and $B'$ have a single intersection point, at which there are two types of resolutions of the crossing. Denote by $D_{n,m}$ the first-type resolution of the crossing between the cycles $A^m B$ and $A' B'$. It turns out that the second-type resolution of the crossing between $B$ and $B'$ is homotopically equivalent to the first-type resolution of the crossing between $A^2 B$ and $A'^{-1} B'$, so the second term is $D_{n-2,m-1}$.

To come to the canonical twist coordinates (3.9) and (3.20), we have to replace $p$ by $\ell_{A'}$ in (3.9) and shift the above variables:

$$\hat{\tau}_{A^n B} = \tau_{A^n B} - \frac{1}{2} \log [(e^{-\ell_{A'}/2 + \ell_A} + 1)(e^{\ell_{A'}/2 + \ell_A} + 1)e^{-\ell_A}],$$

$$\hat{\tau}_{A'^m B'} = \tau_{A'^m B'} - \frac{1}{2} \log [(e^{-\ell_{A'}/2 + \ell_A} + 1)(e^{-\ell_{A'}/2 - \ell_A} + 1)(e^{\ell_{A'}/2} + 1)^2].$$

Using the constant brackets between the $\tau$- and $\ell$-variables, it is now an easy exercise to verify that adding these terms results in vanishing commutation relations between the canonical twist coordinates. The theorem is therefore proved.

4. Fenchel–Nielsen coordinates for $\Sigma_{g,s,n}$

We now generalize the Fenchel–Nielsen coordinate setting to the case of surfaces with marked points on boundaries. The standard trick making geometric lengths finite is to introduce a regularization by decorating all bordered cusps with horocycles. Note that all geodesic functions are insensitive to these decorations, which therefore affect only the $\lambda$-lengths of arcs. We restrict the considerations in this
paper to subalgebras of combinations of λ-lengths that are decoration-independent. In the language of the Teichmüller spaces $T_{g,s,n}$, this means restricting ourselves to subalgebras of shear coordinates $Z_{\alpha}$ and removing the extended shear coordinates $\pi_j$ from consideration. The basic example of such a construction is a sphere with three holes and bordered cusps situated on the boundary of one of the holes.

4.1. Fenchel–Nielsen coordinates for $\Sigma_{0,3,m}$. Consider the case of a sphere with three holes and with $m$ bordered cusps located on the boundary of one of the holes. All these cusps are endowed with horocycle decorations. If we restrict the phase space to a subspace of objects independent of the decorations, then, just as in the cases of $\Sigma_{0,4}$ and $\Sigma_{1,1}$, we can single out a canonical twist coordinate $\mathcal{t}_B$ dual to $\ell_A$, where $\ell_A$ is the length of a unique closed geodesic (the perimeter of the hole) separating the boundary component with cusps from the rest of the Riemann surface.

We show below that the remaining $m - 1$ coordinates can be chosen to have homogeneous constant brackets between themselves and to commute with all the canonical length and twist coordinates for the rest of the surface. The algebra of them is non-degenerate for odd $m$ and has exactly one Casimir element for even $m$.

We first single out one bordered cusp by assigning the label 0 to it, and we identify the twist coordinate. This coordinate must have a constant bracket with $\ell_A$ and has to depend only on $\ell_A$, $p_1$, $p_2$, and $\lambda_B/\lambda_0$ (see Fig. 8), where $\lambda_B$ is the signed exponentiated half-length (the λ-length) of the part of a geodesic arc $B$ confined between two intersection points with the decorating horocycle, and $\lambda_0$ is the λ-length of the geodesic arc that separates the cusp ‘crown’ region from the rest of the surface, as shown in Fig. 8. The quotient $X_B := \lambda_B/\lambda_0$ is a natural decoration-independent variable.

Figure 8. A sphere with three holes and with a ‘crown’ of $m$ decorated bordered cusps.

As in the two previous cases, the choice of $\lambda_B$ is by no means unique: besides the possibility of choosing $\lambda_B/\lambda_0$ at any of the $m$ bordered cusps on the boundary
of the hole, we have an infinite family of $\lambda_A^k B$, $k \in \mathbb{Z}$, and we show several terms of arcs from this sequence in the left-hand part of Fig. 9. However, below we show that, as in the case of the other twist coordinates, all the corresponding $\tau$-variables are related by constant shifts by integer multiples of $\ell_A/2$.

Let us define arcs $A^k B$, $k \in \mathbb{Z}$, as follows (see the left-hand part of Fig. 9): all these arcs start and terminate at the same bordered cusp; the arcs $A^{2l} B$ go around the hole $P_2$, the arcs $A^{2l-1} B$ go around the hole $P_1$, and on their way to the corresponding hole they intersect the dashed vertical line in the figure $l$ times (the intersections are counted with signs: intersections in the clockwise direction come with a negative sign, and those in the counterclockwise direction come with a positive sign). We let $G_1 = e^{p_1/2} + e^{-p_1/2}$ and $G_2 = e^{p_2/2} + e^{-p_2/2}$ be the geodesic functions of the perimeters of the corresponding holes.

In this notation the bracket has the same form as in the two previous cases,

$$\{\lambda_A^k B, G_A\} = \lambda_A^{k-1} B - \lambda_A^{k+1} B,$$ (4.1)

where $\lambda_A^{k-1} B$ and $\lambda_A^{k+1} B$ are the two solutions of the quadratic equations generated by a Markov triple:

$$G_A \lambda_A^{2l} B \lambda_A^{2l+1} B - \lambda_A^{2l+1} B - \lambda_A^{2l+1} B \lambda_0 G_2 - \lambda_A^{2l+1} B \lambda_0 G_1 - \lambda_0^2 = 0.$$ (4.2)

Note that the variable $\lambda_0$ Poisson commutes with $G_A$ and with all the $\lambda_A^k B$.

We have the classical skein relations

$$G_A \lambda_A^{2l} B = \lambda_A^{2l+1} B + \lambda_A^{2l-1} B + \lambda_0 G_2,$$
$$G_A \lambda_A^{2l+1} B = \lambda_A^{2l+2} B + \lambda_A^{2l} B + \lambda_0 G_1,$$ $l \in \mathbb{Z}$. (4.3)
Let us now introduce a variable that is independent of the decoration of the cusp:

\[ X_{A^kB} := \frac{\lambda_{A^kB}}{\lambda_0}, \quad k \in \mathbb{Z}. \] (4.4)

We can easily rewrite all the relations (4.1)–(4.3) in terms of the variables \( X_{A^kB} \); for instance, the skein relations (4.3) become

\[
G A X_{A^{2l+1}B} = X_{A^{2l+1}B} + X_{A^{2l-1}B} + G_2, \\
G A X_{A^{2l+1}B} = X_{A^{2l+2}B} + X_{A^{2l}B} + G_1, \quad l \in \mathbb{Z},
\]

and the Markov triple now gives the equation

\[
G A X_{A^{2l}B}X_{A^{2l+1}B} - X_{A^{2l+1}B}^2 - X_{A^{2l+1}B}^2 - X_{A^{2l+1}B}G_2 - X_{A^{2l}B}G_1 - 1 = 0. \] (4.5)

Since \( X_{A^{k+1}B} \) and \( X_{A^{k-1}B} \) are the two solutions of the same quadratic equation (4.5), their sum and product are

\[
X_{A^{k+1}B} + X_{A^{k-1}B} = G A X_{A^kB} - G_\beta, \\
X_{A^{k+1}B}X_{A^{k-1}B} = X_{A^kB}^2 + X_{A^kB}G_\alpha + 1, \quad \{\alpha, \beta\} = \begin{cases} 
\{1, 2\}, & k \in 2\mathbb{Z}, \\
\{2, 1\}, & k \in 2\mathbb{Z} + 1.
\end{cases} \] (4.6)

**Proposition 4.1.** A decoration-independent twist coordinate having unit bracket with the half-length \( \ell_A/2 \) of a closed geodesic \( A \) for a sphere with three holes and a bordered cusp is

\[
\tau_{A^kB} = \log[(e^{\ell_A/2} - e^{-\ell_A/2})(X_{A^{k+1}B} - e^{\ell_A/2}X_{A^kB}) - e^{-\ell_A/2}G_\alpha - G_\beta], \] (4.7)

where \( \{\alpha, \beta\} = \{1, 2\} \) for even \( k \) and \( \{\alpha, \beta\} = \{2, 1\} \) for odd \( k \). All these twist coordinates are related by constant shifts:

\[
\tau_{A^kB} - \tau_{A^{k+1}B} = \frac{\ell_A}{2}. \] (4.8)

In particular, a Dehn twist along \( A \) transforms \( B \) into \( A^2B \), and the corresponding modular transformation is \( \tau_B \to \tau_{A^2B} = \tau_B - \ell_A \), so the twist coordinate (3.15) assumes values between 0 and \( \ell_A \) in a single copy of the moduli space labelled by \( A \).

**Proof.** The proof is by direct calculation. Checking that \( \{\tau_B, \ell_A/2\} = 1 \) is straightforward using only (4.6) and (4.3):

\[
\{e^{\tau_B}, G A\} = (e^{\ell_A/2} - e^{-\ell_A/2})(X_{A^{-1}B} - X_B - e^{\ell_A/2}(X_{A^{-1}B} - X_{AB})) \\
= (e^{\ell_A/2} - e^{-\ell_A/2})(G A X_{A^{-1}B} - G_1 - 2X_B \\
- e^{-\ell_A/2}(2X_{A^{-1}B} - G A X_B + G_2)) \\
= (e^{\ell_A/2} - e^{-\ell_A/2})(e^{\ell_A/2} - e^{-\ell_A/2})(X_{A^{-1}B} - e^{-\ell_A/2}X_B) \\
- G_1 - e^{-\ell_A/2}G_2) = (e^{\ell_A/2} - e^{-\ell_A/2})e^{\tau_B}. 
\]

We leave the proof of (4.8) to the reader as an exercise. □
Choosing the other sign of $\ell_A$, we obtain
\[
\tau_{A^k B} = \log[(e^{-\ell_A/2} - e^{\ell_A/2})(X_{A^k-1 B} - e^{\ell_A/2} X_{A^k B})] - e^{\ell_A/2} G_\alpha - G_\beta.
\]

Then for the sum $\tau_{A^k B} + \tau'_{A^k B}$ we get that all the terms containing $X$-terms combine into a Markov triple, and the final result is
\[
\tau_{A^k B} + \tau'_{A^k B} = \log(G_A G_1 G_2 + G_A^2 + G_1^2 + G_2^2 - 4). \tag{4.9}
\]

This immediately implies the following analogue of Lemma 3.4.

**Lemma 4.2.** The canonical twist coordinate that changes sign under a change of the orientation ($\ell_A \to -\ell_A$) of the perimeter of the hole containing a bordered cusp is
\[
\hat{\tau}_{A^k B} := \log \left(\frac{(e^{\ell_A/2} - e^{-\ell_A/2})(X_{A^k-1 B} - e^{-\ell_A/2} X_{A^k B}) - e^{-\ell_A/2} G_\alpha - G_\beta}{(G_A G_1 G_2 + G_A^2 + G_1^2 + G_2^2 - 4)^{1/2}}\right). \tag{4.10}
\]

Geometrically, $\hat{\tau}_B$ is the signed geodesic length along the geodesic $\gamma_A$ between the endpoints of the perpendiculars to $\gamma_A$ from the selected (‘zeroth’) cusp and the hole $P_2$ (see Fig. 8).

To prove the geometric component of this statement, we again use identities from hyperbolic geometry. Note that all the identities in a right-angled pentagon with a cusp at one of the vertices can be obtained from those in a regular right-angled hexagon in the limit as the length of one of its sides goes to zero (then the lengths of two adjacent sides tend to infinity, but only the ratio of their $\lambda$-lengths enters the relations). In the geometry of Fig. 8 we have the following identities: for the pentagon,
\[
e^{h_B - h_1} = \sinh(h_2) \cosh(|PQ|) + \cosh(h_2), \tag{4.11}
\]
while in the two quadrangles,
\[
e^{\ell_B/2 - h_B} = \sinh(p_2) \quad \text{and} \quad e^{\ell_0/2 - h_1} = \sinh\left(\frac{\ell_A}{2}\right). \tag{4.12}
\]

Combining these three relations and taking into account that $X_B = e^{\ell_B/2 - \ell_0/2}$, we obtain
\[
X_B \sinh\left(\frac{\ell_A}{2}\right) = \sinh(p_2) (\sinh(h_2) \cosh(|PQ|) + \cosh(h_2)). \tag{4.13}
\]

A similar relation holds for $X_{A-1 B}$, except that we must replace $h_2$ by $h'_2$ (see Fig. 8) and $|PQ|$ by $|PR|$: since $|RQ| = \ell_A/2$, it follows that
\[
X_{A-1 B} \sinh\left(\frac{\ell_A}{2}\right) = \sinh(p_1) \left(\sinh(h'_2) \cosh\left(|PQ| + \frac{\ell_A}{2}\right) + \cosh(h'_2)\right). \tag{4.14}
\]

Here $h_2$ and $h'_2$ are the lengths of sides of a right-angled hexagon with other sides $\ell_A/2$, $p_1/2$, and $p_2/2$. The hyperbolic sine theorem gives us that
\[
\sinh(p_1) \sinh(h'_2) = \sinh(p_2) \sinh(h_2).
\]
Finally, we have the standard relations for the perpendiculars $h_2$ and $h'_2$:

$$\cosh(h_2) \sinh\left(\frac{p_2}{2}\right) \sinh\left(\frac{\ell_A}{2}\right) = \cosh\left(\frac{p_1}{2}\right) + \cosh\left(\frac{p_2}{2}\right) \cosh\left(\frac{\ell_A}{2}\right),$$

$$\cosh(h'_2) \sinh\left(\frac{p_1}{2}\right) \sinh\left(\frac{\ell_A}{2}\right) = \cosh\left(\frac{p_2}{2}\right) + \cosh\left(\frac{p_1}{2}\right) \cosh\left(\frac{\ell_A}{2}\right).$$

Using them, we get after a little algebra that

$$\sinh\left(\frac{\ell_A}{2}\right) \left( X_{A^{-1}B} - e^{-\ell_A/2} X_B \right) - e^{-\ell_A/2} \cosh\left(\frac{p_2}{2}\right) - \cosh\left(\frac{p_1}{2}\right) = 2 \sinh\left(\frac{\ell_A}{2}\right) \sinh(p_2) \sinh(h_2)e^{\vert PQ\vert}.$$

It remains only to note that

$$4 \sinh^2\left(\frac{\ell_A}{2}\right) \sinh^2(p_2) \sinh^2(h_2) = G_A G_1 G_2 + G_2^2 + G_1^2 + G_2^2 - 4.$$

We then have an extension of Theorem 3.5.

**Theorem 4.3.** For any pair-of-pants decomposition of a Riemann surface $\Sigma_{g,s,n}$, define the canonical twist variables $\tau_B$ by (3.20), (3.9), and (4.10). Then all these canonical twist variables Poisson commute.

**Remark 4.4.** Although for brevity we omit the proof of commutativity of canonical twist variables of type (4.10) with other types of canonical twist variables and with canonical twist variables of the same type, we remark that either we can carry out this proof directly by slightly modifying the proof of Theorem 3.5, or, more preferably, we can use the fact that all surfaces with bordered cusps can be obtained as reductions of surfaces with holes by using the ‘chewing gum’ moves in [8] and [9]. Correspondingly, there should be a way to obtain canonical twist variables of type (4.10) from canonical twist variables of types (3.20) and (3.9).

### 4.2. Local variables on $\Sigma_{0,2,m}$ with $m \geq 2$.

After identifying the twist coordinate $\tau_B$ (4.10), we are going to construct the remaining $m - 1$ coordinates on the selected boundary component. This is done in the following way. Fixing the zeroth cusp used to construct $\tau_B$, we enumerate all the other cusps in the clockwise direction counted from the selected zeroth cusp. Then let $\lambda_i^S$ and $\lambda_i^N$, $i = 1, \ldots, m - 1$, denote the $\lambda$-lengths of arcs starting at the zeroth cusp and terminating at the $i$th cusp, going in the clockwise and counterclockwise directions, respectively (see the right-hand part of Fig. 9).

Note first that all the variables $\lambda_i^N$ and $\lambda_i^S$ for a given hole commute with the $X_B$ and $\ell_A$ constructed for this hole and obviously commute with similar variables constructed for other holes and with all canonical length-twist coordinates, so that the algebra of these variables separates completely from the ‘big’ algebra of Fenchel–Nielsen coordinates.
We introduce the $m - 1$ decoration-independent variables

$$x_i := \frac{\lambda_i^N}{\lambda_i^S}.$$  (4.15)

Using the skein relations and Goldman brackets, it is an easy exercise to find the Poisson relations between the $x_i$:

$$\{x_i, x_j\} = x_i^2 - x_i x_j, \quad 1 \leq i < j \leq m - 1.$$  (4.16)

We next consider combinations of these variables:

$$r_1 := x_1, \quad r_i := x_i - x_{i-1}, \quad 2 \leq i \leq m - 1.$$  (4.17)

Then we have the following lemma.

**Lemma 4.5.** The decoration-independent variables $r_i$ in (4.17) have the homogeneous Poisson brackets

$$\{r_i, r_j\} = r_i r_j, \quad 1 \leq i < j \leq m - 1.$$  (4.18)

The algebra of them is non-degenerate for odd $m$ and it has the unique Casimir element

$$C = \frac{r_1 r_3 \cdots r_{m-3} r_{m-1}}{r_2 r_4 \cdots r_{m-2}}$$  (4.19)

for even $m$.

Note that the variables $r_i$ take all values in $\mathbb{R}^{m-1}_+$, and for any given set of values of $r_i$ we have a unique configuration of cusps on the corresponding boundary component.

**Remark 4.6.** Since the Fenchel–Nielsen Poisson structure was derived solely using the Goldman bracket, and the Goldman bracket, in turn, follows from the Fock bracket on the set of shear coordinates, the constructed Fenchel–Nielsen Poisson structure is derived from the Fock bracket. Since the latter is manifestly invariant under the mapping-class group, so is the Fenchel–Nielsen bracket. Moreover, since we restrict consideration to decoration-independent variables, all length and twist coordinates and all $r$-variables are functions only of the shear coordinates $Z_\alpha$ and the coefficients $\omega_j$. The only Casimir elements of the subalgebra of coordinates $Z_\alpha$ are those for holes containing even numbers $n_i$ of bordered cusps. For each of the $n_i$ windows (in the Kaufmann–Penner terminology [31]) constituting together the boundary of the corresponding hole, we consider the sum $\sum_\alpha Z_\alpha^{(j)}$ of the shear coordinates of edges incident to the $j$th window, and the logarithm of the Casimir element (4.19) is then given by the alternating sum

$$\sum_{j=1}^{n_i} (-1)^j \sum_\alpha Z_\alpha^{(j)}.$$
4.3. Special cases. To complete our description of the Fenchel–Nielsen coordinates, it remains to consider two special cases: $\Sigma_{0,2,n_1+n_2}$ — a cylinder with $n_1 > 0$ and $n_2 > 0$ bordered cusps on its two boundaries, and $\Sigma_{0,1,n}$ — a disk with $n \geq 3$ bordered cusps on the single boundary.

4.3.1. The twist coordinate for $\Sigma_{0,2,n_1+n_2}$. In the case of a cylinder containing bordered cusps on both its boundaries, we have a unique closed geodesic $A$, and we can select two cusps marked by $0$ and $0'$ on the two different boundary components and consider a $\mathbb{Z}$-labelled set of arcs $a_{AkB}$ joining these two cusps. We let $\lambda_{AkB}$ denote the $\lambda$-lengths of the arcs in this set. As above, we also have two special arcs $a_0$ and $a_0'$ (see Fig. 10), and we let $\lambda_0$ and $\lambda_0'$ denote their $\lambda$-lengths. We now construct the twist coordinate dual to $\ell_A$ and determine its geometric origin.

![Figure 10. A cylinder with $n_1 > 0$ and $n_2 > 0$ decorated bordered cusps on its two boundaries.](image)

We have the following Poisson and skein relations for $G_A$ and the $\lambda$-lengths:

\[
\{G_A, \lambda_{AkB}\} = \frac{1}{2}(\lambda_{Ak^{-1}B} - \lambda_{Ak^{+1}B}),
\]

\[
G_A\lambda_{AkB} = \lambda_{Ak^{-1}B} + \lambda_{Ak^{+1}B}, \quad \lambda_{Ak^{-1}B}\lambda_{Ak^{+1}B} = \lambda_{AkB}^2 + \lambda_0\lambda_0'.
\]

We now introduce the decoration-independent variables

\[
X_{AkB} := \frac{\lambda_{AkB}}{\sqrt{\lambda_0\lambda_0'}} = \exp\left\{\frac{\ell_{AkB}}{2} - \frac{\ell_{\lambda_0}}{4} - \frac{\ell_{\lambda_0'}}{4}\right\}.
\]

In terms of these variables the above relations take the form

\[
\{G_A, X_{AkB}\} = \frac{1}{2}(X_{Ak^{-1}B} - X_{Ak^{+1}B}),
\]

\[
G_AX_{AkB} = X_{Ak^{-1}B} + X_{Ak^{+1}B}, \quad X_{Ak^{-1}B}X_{Ak^{+1}B} = X_{AkB}^2 + 1,
\]

from which we get the Markov relation

\[
G_AX_{AkB}X_{Ak^{-1}B} - X_{AkB}^2 - X_{Ak^{-1}B} - 1 = 0, \quad k \in \mathbb{Z}.
\]
As before, a twist coordinate having unit bracket with $\ell_A$ is

$$
\tau_{A^k B} := \log(X_{A^{k-1} B} - e^{-\ell_A/2} X_{A^k B}),
$$

and it turns out to also be a canonical twist coordinate. Indeed, for $\ell_A \to -\ell_A$ we obtain

$$
\tau'_{A^k B} := \log(X_{A^{k-1} B} - e^{\ell_A/2} X_{A^k B}),
$$

and

$$
\exp\{\tau_{A^k B}\} \exp\{\tau'_{A^k B}\} = (X_{A^{k-1} B} - e^{-\ell_A/2} X_{A^k B})(X_{A^{k-1} B} - e^{\ell_A/2} X_{A^k B})
$$

$$
= -G_A X_{A^k B} X_{A^{k-1} B} + X^2_{A^k B} + X^2_{A^{k-1} B} = -1. \quad (4.24)
$$

Therefore, we have the following lemma.

**Lemma 4.7.** For a cylinder with bordered cusps 0 and 0′ on its two boundary components, the canonical twist coordinate that has unit Poisson bracket with $\ell_A$ and changes sign under a change of orientation ($\ell_A \to -\ell_A$) of the diameter of the cylinder is given by

$$
\bar{\tau}_{A^k B} := \log(X_{A^{k-1} B} - e^{-\ell_A/2} X_{A^k B}),
$$

where the $X_{A^k B}$ are decoration-independent variables (4.22). Geometrically, $\bar{\tau}_B$ is half the signed geodesic length $|PQ|$ along the geodesic $\gamma_A$ between the endpoints of the perpendiculars to $\gamma_A$ from the selected (‘zeroth’) cusps (see Fig. 10).

Let us prove the geometric part of the statement of the lemma. Taking the limit as $h_2 \to \infty$ in (4.11) and accounting for decorations of the cusps, we get that

$$
e^{\ell_B-h_1-h_2} = \frac{1}{2} (\cosh(|PQ|) + 1) = \cosh^2\left(\frac{|PQ|}{2}\right).
$$

Together with the standard relations (4.12) in the quadrangles, which are

$$
e^{\ell_{\lambda_0}/2-h_1} = e^{\ell_{\lambda_0}/2-h_2} = \sinh\left(\frac{\ell_A}{2}\right),
$$

we obtain

$$
X^2_B = e^{\ell_B-\ell_{\lambda_0}/2-\ell_{\lambda_0}/2} = \frac{\cosh^2(|PQ|/2)}{\sinh^2(\ell_A/2)},
$$

or

$$
(e^{\ell_A/2} - e^{-\ell_A/2}) X_B = e^{\text{P}Q}/2 + e^{-|PQ|/2}. \quad (4.26)
$$

We now use (4.24), which implies that $e^{-\tau_B} = -(X_{A^{-1} B} - e^{\ell_A/2} X_B)$, to obtain

$$
e^{\tau_B} + e^{-\tau_B} = X_B (e^{\ell_A/2} - e^{-\ell_A/2}) = e^{\text{P}Q}/2 + e^{-|PQ|/2},
$$

which immediately gives us the lemma.
4.3.2. Fenchel–Nielsen coordinates for $\Sigma_{0,1,n}$. The last remaining case is a disk $\Sigma_{0,1,n}$ with $n \geq 3$ bordered cusps, which we enumerate from 0 to $n - 1$ moving counterclockwise. The cusps labelled 0, 1, and 2 play a special role in the construction below. We now have unique arcs $a_{i,j}$ joining the cusps with labels $i$ and $j$, and since arcs are not directed, we can always assume that $i < j$. Consider arcs $a_{0,i}$ and $a_{1,i}$ with the corresponding $\lambda$-lengths $\lambda_{0,i}$ and $\lambda_{1,i}$. We have the Poisson relations

\[
\{\lambda_{0,i}, \lambda_{0,j}\} = \frac{1}{4} \lambda_{0,i} \lambda_{0,j}, \quad \{\lambda_{1,i}, \lambda_{1,j}\} = \frac{1}{4} \lambda_{1,i} \lambda_{1,j}, \quad i < j;
\]

\[
\{\lambda_{0,i}, \lambda_{0,j}\} = 0, \quad \{\lambda_{0,i}, \lambda_{1,j}\} = \frac{1}{2} (\lambda_{1,i} \lambda_{0,j} - \lambda_{j,i} \lambda_{0,1}), \quad i < j;
\]

and the skein relation

\[
\lambda_{0,i} \lambda_{1,j} = \lambda_{1,i} \lambda_{0,j} + \lambda_{j,i} \lambda_{0,1},
\]

the use of which makes it possible to express the last bracket in the form

\[
\{\lambda_{0,i}, \lambda_{1,j}\} = \lambda_{1,i} \lambda_{0,j} - \frac{1}{2} \lambda_{0,i} \lambda_{1,j}, \quad i < j.
\]

We introduce the (still not completely decoration-independent) combinations

\[
x_j := \frac{\lambda_{1,j}}{\lambda_{0,j}}, \quad j = 2, \ldots, n - 1.
\]

The brackets between the $x_j$ are

\[
\{x_i, x_j\} = x_i x_j - x_j^2, \quad 2 \leq i < j \leq n - 1,
\]

that is, they have exactly the same form as brackets in the subalgebra of 'local' variables for a selected boundary component. Then, taking

\[
r_i = \{x_2, i = 2; \ x_i - x_{i-1}, \ 3 \leq i \leq n - 1\},
\]

we obtain homogeneous Poisson relations $\{r_i, r_j\} = r_i r_j$ for $2 \leq i < j \leq n - 1$ and, finally, we introduce the completely decoration-independent variables

\[
\hat{r}_i := \frac{r_i}{r_2}, \quad 3 \leq i \leq n - 1,
\]

for which we have the following proposition.

**Proposition 4.8.** In a disk with $n \geq 4$ bordered cusps there are $n - 3$ Fenchel–Nielsen coordinates $\hat{r}_i$ (4.30) that are decoration-independent and have homogeneous Poisson relations:

\[
\{\hat{r}_i, \hat{r}_j\} = \hat{r}_i \hat{r}_j, \quad 3 \leq i < j \leq n - 1.
\]

This algebra is non-degenerate for odd $n$, and it has the unique Casimir element

\[
C = \frac{\hat{r}_3 \cdots \hat{r}_{n-3} \hat{r}_{n-1}}{\hat{r}_4 \cdots \hat{r}_{n-2}}
\]

for even $n$. 
4.4. Mirzakhani’s volumes of the spaces \( \mathcal{M}_{g,s,n} \). We can now define the Fenchel–Nielsen symplectic form to be an inverse to the Poisson brackets: there are \( 3g - 3 + s + s_h \) pairs of canonical length-twist coordinates, and for every hole with \( n_i > 0 \) bordered cusps on its boundary there is an \( (n_i - 1) \)-dimensional Poisson algebra of \( r \)-variables associated with this hole. These algebras of \( r \)-variables are non-trivial provided that \( n_i \geq 3 \). The corresponding Fenchel–Nielsen (or Weil–Petersson) 2-form is then

\[ \omega_{\text{WP}} = \sum_{k=1}^{3g-3+s+s_h} d\ell_{A_k} \wedge d\tau_{B_k} + \sum_{j=1}^{s_h} \omega_j, \]

where the \( \omega_j \) denote the ‘local’ 2-forms obtained by inverting the Poisson structures (4.18) (in the orthogonal complements of the Casimir elements (4.19) if \( n_i \) is even). The volume element is the appropriate power of the form \( \omega_{\text{WP}} \).

A fundamental result of Mirzakhani in [36] enables one to compute the volumes of the moduli spaces \( \mathcal{M}_{g,s} \) (for \( s > 0 \)) obtained by factoring the Teichmüller spaces \( \mathcal{F}_{g,s} \) by the action of the mapping-class group. Mirzakhani’s theorem states that for fixed perimeters \( p_i \) of all the holes, these volumes are finite and are polynomials in the \( p_i \).

This assertion was based on identities of McShane ([33], [34]) and Mirzakhani [36] for the lengths of simple (that is, without self-intersections) closed geodesics on \( \Sigma_{g,s} \). The recursion relation derived in [37] was shown by Mulase and Safnuk [39] and Do and Norbury [14] to satisfy the Virasoro algebra relations, and this eventually resulted in the construction by Eynard and Orantin [15] of a topological recursion model giving a generating function for these volumes. This generating function was immediately identified with a KdV hierarchy \( \tau \)-function.

We note that the Fenchel–Nielsen coordinates were instrumental in Mirzakhani’s derivation of his volume formulae. An advantage of this set of coordinates is that the integration over the twist coordinate goes from 0 to \( \ell_A \) in one copy of the corresponding moduli space, so that the net effect of this integration is multiplication by \( \ell_A \). For example, for a torus with a hole of perimeter \( p \) we have the McShane–Mirzakhani identity ([33], [36]), which, after differentiation with respect to \( p \) takes the convenient form

\[ \sum_{\gamma} \left[ \frac{1}{e^{\ell_{\gamma} + p/2} + 1} + \frac{1}{e^{\ell_{\gamma} - p/2} + 1} \right] = 1, \]

where the sum is over all simple closed curves, each of which can be identified with an \( A \)-cycle in the corresponding copy of the moduli space, and the volume of the latter space is, on the one hand, just the integral of the constant function 1 and, on the other hand, the integral from zero to infinity (the range of the variable \( \ell_{\gamma} \)) of the function on the left-hand side multiplied by \( \ell_{\gamma} \), because of the integration over the twist coordinate. The result is then

\[ \text{Vol}_{1,1} \simeq \int_0^\infty \left[ \frac{1}{e^{\ell_{\gamma} + p/2} + 1} + \frac{1}{e^{\ell_{\gamma} - p/2} + 1} \right] \ell_{\gamma} \, d\ell_{\gamma} = \frac{\pi^2}{6} + \frac{p^2}{8}, \]
and amending for the volumes of the discrete automorphism groups for the two terms on the right-hand side (2 and 6 in this case), we finally get that

\[ \text{Vol}_{1,1} = \frac{\pi^2}{12} + \frac{p^2}{48}. \]

Since the mapping-class group is not affected by the presence of bordered cusps, and all the closed curves that are the perimeters of the holes are preserved under the action of this group, we can fix all these perimeters \( p_i, i = 1, \ldots, s \), exactly as in Mirzakhani’s approach. We now have twist coordinates \( \tau_{B_i} \) for the \( s_h \) holes that contain bordered cusps. We note that the normalizations of the \( \tau_{B_i} \) are not invariant under the action of the mapping-class group (since they depend on the choice of the internal \( \Lambda \)-cycles), so that only the differentials \( d\tau_{B_i} \) are objects invariant under this action, and adding the integrations over these twist coordinates amounts to multiplying the corresponding Mirzakhani volumes by \( \prod_{j=1}^{s_h} p_j \). Adding one more cusp to a boundary component results in the appearance of one more Casimir element and does not affect the volume, but if there are boundary components with more than two cusps, then an integration over the whole plane \( \mathbb{R}^2 \) is added, making the corresponding volumes infinite.

5. Concluding remarks and perspectives

We have shown in this paper that the Poisson bracket introduced by Fock on the set of shear coordinates induces, via the Goldman bracket, the Fenchel–Nielsen Poisson bracket on the set of length-twist coordinates, and we can identify the canonical twist coordinates with geometric structures on pair-of-pants decompositions of Riemann surfaces \( \Sigma_{g,s} \) with \( s \geq 0 \) holes. Of course, the proof based on the Goldman bracket remains valid in the case of smooth surfaces \( (g \geq 2, s = 0) \), but in this case we lack the first ingredient — the Fock bracket and also the shear coordinate description, so we lack the cluster algebra description of the corresponding Teichmüller spaces. We have generalized the Fenchel–Nielsen coordinates to the case of \( \Sigma_{g,s,n} \) — the Riemann surfaces with \( n \geq 0 \) bordered cusps on boundary components. Particular cases of such surfaces played a pivotal role in the description of the monodromies of the Painlevé equations in [9]. It is therefore natural to expect that results of this paper will find applications in the description of symplectic structures on the corresponding manifolds. In particular, the example in Fig. 8 in the case of two cusps on the boundary component describes the monodromy manifold for the Painlevé V equation.

A prospective direction for the development of the method of this paper is related to the dynamics of continued fractions based on Markov triples and their higher-dimensional generalizations. This direction addresses the hard problem of finding a way to ‘stabilize’ geometric structures underlying continued-fraction expansions of projectivized real numbers in order to produce a variant of the Poisson or quantum theory of Thurston. The first attempt in this direction was undertaken in [11], and quite recently other methods emerged (see [38] and [5]) based on representations of \( \text{SL}(2, \mathbb{C}) \)-monodromies of Fuchsian equations.
6. Appendix: Wolpert’s form and the Goldman bracket

Consider two oriented closed geodesics $\gamma_1$ and $\gamma_2$ intersecting at a point $P \in \gamma_1 \neq \gamma_2$. They may have other intersection points. We lift this construction to the whole of the Poincaré upper half-plane: with each closed geodesic $\gamma$ we associate a hyperbolic element $S_\gamma \in \text{SL}(2, \mathbb{R})$ whose invariant axis, that is, the unique geodesic that is invariant under the action of $S_\gamma$, becomes a closed geodesic upon identification by the action of $S_\gamma$, which acts as a shift by $\ell_\gamma$ along this invariant axis. The endpoints on the absolute of this axis are two distinct stable points of the hyperbolic element $S_\gamma$. Consider two such elements $S_{\gamma_1}$ and $S_{\gamma_2}$. Let the invariant axis of $S_{\gamma_1}$ be the vertical half-line starting at the origin. Then the action of $S_{\gamma_1}$ is a dilatation, $z \to Rz$, with $R = e^{\ell_{\gamma_1}} > 1$. Let the stable points of the second element $S_{\gamma_2}$ be $-x_1$ and $x_2$ (we assume that $x_i > 0$); see Fig. 11. Then the (Euclidean) height $h$ of the intersection point of the axes is connected with $x_i$ by the simple relation $h^2 = x_1 x_2$, and the $\text{SL}(2, \mathbb{R})$ element $S_{\gamma_2}$ has the form $z \to (\alpha z - h^2)/(z - \beta)$ with arbitrary $\alpha$ and $\beta$. Consider now the composition

$S_{\gamma_1} \circ S_{\gamma_2} : z \to R \frac{\alpha z - h^2}{z - \beta} = \frac{R \alpha z - Rh^2}{z - \beta}$.

By the same consideration, the intersection point of the invariant axis of this element with the invariant axis of $S_{\gamma_1}$ is now at height $h' = \sqrt{R} h$, which means that the geodesic distance between these two points (along the geodesic $\gamma_1$) is exactly $\ell_{\gamma_1}/2$. We therefore have the following general fact: given two hyperbolic elements $S_{\gamma_1}$ and $S_{\gamma_2}$ of a Fuchsian group $\Delta_{g,s} \subset \text{PSL}(2, \mathbb{R})$ such that the corresponding closed geodesics $\gamma_1$ and $\gamma_2$ intersect at a point $P \in \gamma_1 \neq \gamma_2$, the closed geodesic $\gamma_1 \circ_P \gamma_2$ (and $\gamma_1^{-1} \circ_P \gamma_2$) intersects the geodesics $\gamma_1$ and $\gamma_2$ at the respective points $Q$ and $R$ (at the respective points $Q'$ and $R'$) situated at geodesic distances that are exactly the halves of the lengths of the corresponding closed geodesics: $|PQ| = |PQ'| = \ell_{\gamma_1}/2$ and $|PR| = |PR'| = \ell_{\gamma_2}/2$. Note that we then have $|RQ| = \ell_{\gamma_1 \circ_P \gamma_2}/2$ and $|RQ'| = \ell_{\gamma_1^{-1} \circ_P \gamma_2}/2$.

![Figure 11](image-url)
Let $\psi$ be the angle between $\gamma_1$ and $\gamma_2$ at the intersection point $P$. For a triangle $\{P, Q, R\}$ we have the hyperbolic cosine formula

$$\cosh(|QR|) = -\sinh(|PQ|)\sinh(|PR|)\cos(\psi) + \cosh(|PQ|)\cosh(|PR|).$$

Applying this to the two triangles $\{P, Q, R\}$ and $\{P, Q', R\}$ (and replacing $\psi$ by $\pi - \psi$ in the second triangle), we obtain

$$\cosh(\ell_{\gamma_1 \circ P \gamma_2}) = -\sinh\left(\frac{\ell_{\gamma_1}}{2}\right)\sinh\left(\frac{\ell_{\gamma_2}}{2}\right)\cos(\psi) + \cosh\left(\frac{\ell_{\gamma_1}}{2}\right)\cosh\left(\frac{\ell_{\gamma_2}}{2}\right);$$

$$\cosh(\ell_{\gamma_1^{-1} \circ P \gamma_2}) = \sinh\left(\frac{\ell_{\gamma_1}}{2}\right)\sinh\left(\frac{\ell_{\gamma_2}}{2}\right)\cos(\psi) + \cosh\left(\frac{\ell_{\gamma_1}}{2}\right)\cosh\left(\frac{\ell_{\gamma_2}}{2}\right).$$

Adding these two equalities, we obtain the classical skein relation

$$G_{\gamma_1 \circ P \gamma_2} + G_{\gamma_1^{-1} \circ P \gamma_2} = G_{\gamma_1}G_{\gamma_2},$$

and subtracting one from the other, we obtain

$$2\sinh\left(\frac{\ell_{\gamma_1}}{2}\right)\sinh\left(\frac{\ell_{\gamma_2}}{2}\right)\cos(\psi) = \cosh(\ell_{\gamma_1^{-1} \circ P \gamma_2}) - \cosh(\ell_{\gamma_1 \circ P \gamma_2}),$$

which connects Wolpert’s formula with the Goldman bracket: we can write the right-hand side as the Poisson bracket

$$\{G_{\gamma_1}, G_{\gamma_2}\} = \sinh\left(\frac{\ell_{\gamma_1}}{2}\right)\sinh\left(\frac{\ell_{\gamma_2}}{2}\right)\{\ell_{\gamma_1}, \ell_{\gamma_2}\},$$

which gives us that $\{\ell_{\gamma_1}, \ell_{\gamma_2}\} = 2\cos(\psi)$.

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