Staggered dimer order in $S=\frac{1}{2}$ quantum spin ladder system with four spin exchange

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We study the $S=\frac{1}{2}$ quantum spin ladder system with the four-spin exchange, using density matrix renormalization group method and an exact diagonalization method. Recently, the phase transition in this system and its universality class are studied. But there remain controversies whether the phase transition is second order type or the other type and the nature of order parameter. There are arguments that the massless phase appears. But this does not agree with our previous result. Analyzing DMRG data, we try a new approach in order to determine a phase which appears after the phase transition. We find that the edge state appears in the open boundary condition, investigating excitation energies of states with higher magnetizations.

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I. INTRODUCTION

$S=1/2$ quantum spin two-leg ladder systems have been studied from both theoretical and experimental points of view [1]. These systems are related to high-$T_c$ superconductors and Haldane's conjecture[2]. For example, Dagotto et al. suggested that the superconductivity occurs in hole-doped ladder systems [3].

Usually, two spin exchanges have been discussed in quantum spin systems. By the way, spin exchange interactions originate from electron exchanges. For example, Heisenberg model is derived from the second order perturbation in the strong coupling limit of the Hubbard model. In contrast, higher order perturbations give many body spin exchange interactions, for example, $(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l)$. These type interactions in two leg ladder systems are discussed by Nersesyan et al[4]. Investigating correlation functions, they referred to level $k = 2$ $SU(2)$ Wess-Zumino-Witten (WZW) model or $SU(2) c = 3/2$ conformal field theory (CFT).

Recently four spin cyclic exchanges attract attention [5–14, 16, 18–20]. And from experiments, it has been suggested that a four spin cyclic exchange interaction plays an important
role in some systems, one dimensional two-leg ladder in La$_6$Ca$_8$Cu$_{24}$O$_{41}$ [9, 10] and two dimensional square lattice in La$_2$CuO$_4$ [11]. The latter is known as a parent insulator of high-$T_c$ superconducting systems. The importance of a cyclic four spin exchange in such systems is pointed out by Honda et al [12]. In two-leg ladder systems (see Fig. 1), four spin exchanges play important roles. Sakai et al [13] and Nakasu et al [14] studied this system with magnetizations in a magnetic field, in respect of the magnetization plateau and plateau-gapless transition. This phase transition is the BKT type [15].

In our previous paper, we numerically studied $s=1/2$ spin ladder system under zero field, and we found a second order phase transition [16]. We studied the critical behavior of this system based on conformal field theory [17]. We calculated the central charge and one of the scaling dimension numerically, and we obtained $c \simeq \frac{3}{2}$, $x \simeq \frac{3}{8}$. Therefore this phase transition is described as the $c = \frac{3}{2}$ CFT or the $k = 2$ SU(2) WZW model. This is a phase transition with $\mathbb{Z}_2$ symmetry breaking. We discussed that the ordered state has the translational symmetry breaking and the rung-parity symmetry breaking, which supports the staggered dimer order (see Fig. 4). Recently this result is obtained by Müller et al [18] in analytically. They used bosonization technique and its Majorana representation in the weak coupling limit. With the help of Padé-approximation in the strong coupling limit, they obtained the critical exponent for the energy gap, $\eta \simeq 1$ numerically. This corresponds with our result [16]. Läuchli et al [19] discussed a phase diagram for this model. They calculated staggered dimer structure factor with exact diagonalization and a local $\langle S_{\alpha,i} \cdot S_{\alpha,i+1} \rangle$ expectation value on one of the two leg in the dimer long range order phase with the density matrix renormalization group (DMRG) method. As a result, they conclude that there is a phase transition between a rung-singlet phase and a staggered dimer phase. It is clear and understandable to see the order parameter directly as Läuchli. But it is needed to inspect the accuracy of numerical calculation and finite size effects since the correlation length is large. In fact, as we can see later, we need a careful analysis on the DMRG data.

However these results do not correspond to Honda et al [20]. They insisted that a massless phase appears in the strong four-spin coupling region. Honda et al investigated a spin gap and a spin-pair correlation function using DMRG method [21]. They found that at $J_{\text{rung}} = J_{\text{leg}} = 1$ and $J_{\text{ring}} \simeq 0.3$ a spin gap between the singlet ($S_{\text{tot}}^z = 0$) and the triplet ($S_{\text{tot}}^z = 0, \pm 1$) vanishes.

Hikihara et al [22] discussed the self-duality and the chirality using Runge-Lenz vector.
And they referred to an incommensurate character. They derived the duality transformation which leaves the form of the Hamiltonian unchanged. This self-dual line is same to an exact line which was found by Kolezhuk et al [23]. With the matrix product method, one can exactly solve this model on this line [23]. The exact ground state is a product of rung-singlets on this line.

Hikihara et al calculated spin gaps, a triplet excitation and a quintet excitation, using DMRG in open boundary condition. They found out that both gaps are quite small in $J_{\text{ring}}$ large region. They found that triplet gaps are exactly zero within numerical accuracy in some system sizes, but quintet gaps seem to be finite. They said that it is very difficult to estimate the critical point accurately due to the very slow vanishing of the spin gaps around the phase transition point.

The extended massless phase is not consistent with some gapful phase. We will reanalyze the spin ladder system with four spin cyclic exchange. Müller et al and Läuchli et al insisted that there is a phase transition from a gapped phase (rung-singlet phase) to another gapped phase (staggered dimer phase). Hikihara et al discussed that the quintet gap is quite small but it seems to be finite. Honda et al insist that there is a phase transition from a gapped phase (rung-singlet phase) to a gapless phase. But these results are not compatible. At least one interpretation is mistaken.

We think that Honda et al found the edge state, since they used DMRG method with the open boundary condition. They only calculated the energy gap between the singlet state and the triplet state. So, we calculate energy gaps between the singlet and triplet, and between the singlet and the quintet. In this paper, we try to check the consistency between the second order phase transition and DMRG data using a new method. We try to calculate a correlation length quantitatively and to discuss the finite size effect.

II. SPIN HAMILTONIAN AND FOUR SPIN CYCLIC EXCHANGE

In this paper, we discuss the following Hamiltonian,

$$H = J_{\text{leg}} \sum_n \sum_{\alpha=1,2} S_{\alpha,n} \cdot S_{\alpha,n+1} + J_{\text{rung}} \sum_n S_{1,n} \cdot S_{2,n}$$

$$+ J_{\text{ring}} \sum_i \left( P_{i,i+1} + P_{i,i+1}^{-1} \right)$$  \hspace{1cm} (1)
where $P, P^{-1}$ are four spin cyclic exchange operators (see Fig.2), $\alpha = 1, 2$ are indexes of spin chains (see Fig.1), $J_{\text{rung}}$ is an interchain coupling, $J_{\text{leg}}$ is an intrachain coupling, and $J_{\text{ring}}$ is a four spin cyclic exchange coupling.

This four spin cyclic exchange operator can be expressed using spin operators as follows,

$$P_{i,i+1} + P_{i,i+1}^{-1} = \frac{1}{4} + S_{1,i}S_{1,i+1} + S_{2,i}S_{2,i+1} + S_{1,i}S_{2,i+1}$$

$$+ S_{2,i}S_{1,i+1} + S_{1,i}S_{2,i} + S_{1,i+1}S_{2,i+1}$$

$$+ 4\{(S_{1,i}S_{2,i})(S_{1,i+1}S_{2,i+1})$$

$$+ (S_{1,i}S_{1,i+1})(S_{2,i}S_{2,i+1})$$

$$- (S_{1,i}S_{2,i+1})(S_{2,i}S_{1,i+1})\}.$$  (2)

This interaction have been known from old times [24, 25].

Four spin coupling terms, such as $(S_{\alpha,i} \cdot S_{\beta,j})(S_{\gamma,k} \cdot S_{\sigma,l})$, are derived from fourth-order perturbation expansions in the strong coupling limit of the Hubbard model[26]. It is well known that such many body exchange interactions play important role in Wigner crystal and $^3$He solid. Recently, low dimensional quantum spin systems with these terms are well studied. In $\frac{1}{4}J_{\text{rung}} > J_{\text{ring}} = J_{\text{leg}}$ case, this model can be solved exactly using matrix product method [23]. On this region the ground state is a product of rung-singlet [23][27].

### III. OVERVIEW OF EDGE STATES

It is well known that in finite size Haldane systems, under the open boundary condition, quasi four-hold degenerate states appear. This was pointed out by Kennedy [28], who calculated the ground state and some lower excitation energies numerically.

In the Haldane system, such as valence-bond solid (VBS) states[29], a ground state is unique in the infinite system. But under the open boundary condition, it is four-hold degenerate, which is between the ground state and triplet excitations, and other states, for example quintet excitations, have a finite energy gap.

On the other hand, in the finite system, under the open boundary condition, it is quasi four-hold degenerate, which consists of the ground state and triplet excitations. This appears universally in Haldane gap systems. These quasi energy gaps between four states decay exponentially, as increasing system sizes. This is called as edge states. This can be confirmed experimentally[30]. Edge states should appear in not only $S = 1$ chain but also $S = 1/2$
two-leg ladder system.

Usually we can confirm edge states investigating the ground state energies under the open boundary condition with higher magnetizations, for example a quintet. However, in the finite system, the energy gap between the singlet and triplet remains, which results from the finite size effect.

Therefore, when the singlet-quintet gap is small, it is difficult to investigate the edge states directly.

IV. NUMERICAL RESULTS ON PHASE TRANSITION

In this section we present our numerical results for low-lying levels of the model for various couplings and system sizes, which are following as,

- singlet: total $S_z = 0$; parity=even
- triplet: total $S_z = 0, \pm 1$; parity=odd
- quintet: total $S_z = 0, \pm 1, \pm 2$; parity=even.

Here we consider the parity about the inversion symmetry for the leg direction. In the commensurate region, the singlet is the ground state.

Now we study $J_{\text{leg}} = J_{\text{rung}} = 1$ case with verifying $J_{\text{ring}}$. We consider spin gaps between the singlet and the triplet and between the singlet and the quintet.

$$\Delta E_{st} = E(\text{triplet}) - E(\text{singlet}),$$

$$\Delta E_{sq} = E(\text{quintet}) - E(\text{singlet}).$$

We calculate spin gaps, using a finite size DMRG algorithm with open boundary condition. We use the maximum system size $L = 112$ and $m = 300$ in this paper, where $m$ is the number of the bases for the truncated left-half system.

A. RATIO OF ENERGY GAPS

Here we consider a ratio of the energy gap of the triplet-singlet and the energy gap of the quintet-singlet,

$$f(L, J_{\text{ring}}) = \frac{\Delta E_{sq}}{\Delta E_{st}}.$$
In a normal rung-singlet phase (see Fig.3), this ratio is about 2, based on standard magnon picture. If the interaction of magnons is repulsive, then this value should be 2 in the infinite system, and it approach 2 from a large value increasing the system size in the finite system. In this phase, the triplet excitation state has one magnon, and the quintet excitation state has two magnons. If two magnons are independent, this ratio is equal 2. This is true in enough large systems (see Fig.5).

In a massless phase in a finite system under open boundary condition, excited energies are given as

$$\Delta E(L) = \frac{\pi v}{L} x_s$$  \hspace{1cm} (6)

where $x_s$ is a surface critical exponent, $v$ is a Fermi velocity, and $L$ is a system size based on CFT. If a phase is massless, then this ratio converge to a finite value $x'_s/x_s$. In reality, there remains the finite size effect from logarithmic correction (see Appendix).

In a staggered dimer phase, $\Delta_{st} \to 0$ and $\Delta_{sq} \to finite$, so $f \to \infty$.

So we can distinguish a massless phase from some degenerate phases in large $J_{ring}$ region, as follows,

- Massless phase : $f \to converge \ (finite)$
- Staggered dimer phase : $f \to diverge$

We find that this ratio tend to diverge in large $J_{ring}$ region (see Fig.5,6). This shows that $f \to 2$ in the small $J_{ring}$ region, and $f \to \infty$ in the large $J_{ring}$ region. So we can deny that there exists an extended massless region in the large $J_{ring}$ region. And it shows that the interaction of magnons is repulsive in the rung-singlet phase. We can understand $f \to 2$ to consider that the rung-singlet phase appears. And we can interpret $f \to \infty$ as $\Delta_{st} \to 0$ and $\Delta_{sq} \to finite$. Considering $SU(2)$ symmetry, we can conclude that the edge state appears.

Here we consider the function $f$. This phase transition is second order, where the critical point is massless, so we can expect that this function behave

- $J_{ring} < J_{ring}^c$ : $f \to finite \sim 2$
- $J_{ring} = J_{ring}^c$ : $f \to x_s(sq)/x_s(st) \sim finite$
- $J_{ring} > J_{ring}^c$ : $f \to \infty$
as increasing system size $L$, where $x_s(st), x_s(sq)$ are scaling dimensions. This function is independent of system size $L$ at $J_{ring} = J_{ring}^c$. Therefore, when we plot this function for $J_{ring}$, we can expect that this function behaves as the traditional finite size scaling function $L \Delta E$. This is based on a phenomenological renomalization group analysis [31].

In Fig.7 we define $J_{ring}^{cross} (L)$ as

$$f \left( L + 20, J_{ring}^{cross} (L + 10) \right) = f \left( L, J_{ring}^{cross} (L + 10) \right)$$

(7)

Then, we extrapolate $J_{ring}^{cross}$ as follows,

$$J_{ring}^{cross} (L) = J_{ring}^{cross} (\infty) + a \frac{1}{L^2} + b \frac{1}{L^4} + \text{higher order}$$

(8)

Here we neglect higher order terms, so we obtain $J_{ring}^{cross} (\infty) = 0.19379$ (see Fig.8).

**B. THE CORRELATION LENGTH**

In this section, we discuss the correlation length in the rung-singlet phase. We determine the correlation length $\xi$ according to the following formula,

$$f \left( L, J_{ring} \right) = f \left( L = \infty, J_{ring} \right) + aK_0 \left( \frac{L}{\xi} \right)$$

(9)

where $a$ is an amplitude, $K_0$ is the modified Bessel function, and $f \left( L = \infty, J_{ring} \right)$ is about 2. As is well known, the $d$ dimensional quantum system can be mapped onto $d + 1$ dimensional classical system[32]. If the one-dimensional quantum system has the energy gap, it is called as massive, and its correlation function becomes of the $1 + 1$ Ornstein-Zernicke form, that is the modified Bessel function,

$$\int \frac{\exp \left[ i \mathbf{q} \cdot \mathbf{l} \right]}{q^2 + \xi^{-2}} d^d q \propto \begin{cases} \exp \left[ - \frac{1}{\xi} \right] : d = 1 \\ K_0 \left( \frac{1}{\xi} \right) : d = 2 \end{cases}$$

(10)

This form is supported by Quantum Monte Carlo simulation [33] and DMRG calculation [34]

Using the nonlinear least squares method, we determine $f \left( L = \infty, J_{ring} \right), a, \xi$. We obtain $1/\xi$ as a function of $J_{ring}$, fitting for $L = 20 \sim 112$. This figure shows clearly that the $1/\xi$ is proportionate to $J_{ring}$. Note that in the case of $J_{ring} = 0.175$, $\xi$ is larger than the system size.
The correlation length diverges at the second order phase transition point. Our result is consistent with this fact.

Near the critical point described as \( k = 2 \) \( SU(2) \) WZW model, \( \frac{1}{\xi} \) behave

\[
\frac{1}{\xi} \propto |J_{\text{ring}} - J_{\text{ring}}^c|,
\]

where \( J_{\text{ring}}^c \) is \( J_{\text{ring}} \) at the critical point. Using the method of least squares, we obtain the \( \xi^{-1} = 0 \) at \( J_{\text{ring}}^c = 0.1943 \) (see Fig. 9). This value is close to previous results [16, 19].

C. SCALING DIMENSION

In previous section, we discussed the divergence of the correlation length. In this section, returning to the CFT analysis with the periodic boundary condition, we discuss a scaling dimension independently. In our previous paper [16], we obtained a scaling dimension \( x = \frac{3}{8} \) thus we have calculated \( k = 2 \) WZW type central charge \( c = \frac{3}{2} \). But this model, level \( k = 2 \) \( SU(2) \) WZW model, has another scaling dimension \( x = 1 \). This scaling dimension is related with the energy gaps (the correlation length \( \xi^{-1} \propto \Delta E \)) near the critical point,

\[
\Delta E \propto |J_{\text{ring}} - J_{\text{ring}}^c|^\eta
\]

where \( \eta = 1 \). Müller et al[18] got this critical exponent, using the cluster expansion and Padé approximation. This is based on rung-dimer limit \( J_{\text{rung}} \rightarrow \infty \). However we are interested in \( J_{\text{leg}} = J_{\text{rung}} \) case. Now we consider a scaling dimension based on CFT. The relation between an excitation energy and a scaling dimension, under the periodic boundary condition, is

\[
E_i - E_0 = \frac{2\pi v}{L} x_i
\]

where \( L \) is a systems size, \( v \) is a Fermi velocity in the system and \( x_i \) is a scaling dimension. In addition to the ground state energy and the excitation energy, we need to obtain \( v \) numerically. This velocity is obtained from

\[
v(L) = \frac{L}{2\pi} \left[ E \left( q = \frac{2\pi}{L} \right) - E_{gs} \right]
\]

where \( q \) is a wave number. Then we extrapolate \( v(L) \) as follows,

\[
v(L) = v_\infty + a \frac{1}{L^2} + b \left( \frac{1}{L^2} \right)^2 + \text{higher order}
\]
We neglect higher order terms and use \( v_\infty \).

Unfortunately there are logarithmic corrections from the marginal operator in \( k = 2 \) \( SU(2) \) WZW model. So we must remove corrections, when we calculate scaling dimensions. After removing them with the method in Appendix, a scaling dimension is almost independent on the system size at the critical point \( J_{\text{ring}} = 0.192 \) (see Fig.10). This result is consistent with previous results [16, 18].

\[ \text{V. CONCLUSIONS} \]

In this paper we have investigated the ordered state of a \( S = \frac{1}{2} \) quantum spin two-leg ladder system with the cyclic four spin exchange using DMRG method in the open boundary condition and an exact diagonalization method in the periodic boundary condition.

Generally speaking, it is very difficult to determine whether some value is zero or not numerically, especially near the critical point, where the correlation length becomes large compared with the system size.

Considering the ratio of energy gaps of the singlet-triplet and the singlet-quintet, we find that the singlet-triplet is degenerate and the singlet-quintet is not degenerate after the phase transition. This result supports that the staggered dimer phase appears. In the staggered dimer phase with the open boundary condition, since the edge state appears, the singlet-triplet energy gap decays exponentially. On the other hand, the singlet-quintet gap is finite in this phase.

We have also investigated the correlation length near the critical point. That result is consistent with previous results [16, 18, 19].

Honda et al [20] concluded that the massless phase appeared, using DMRG with the open boundary condition. But we can consider that they found the small edge state. We obtain the unified interpretation which is consistent with previous results [16, 18–20, 22].

This problem is related to electron systems on a two-leg ladder. Our result is consistent with Tsuchiiizu et al [35]. They discussed the ground state phase diagram of half filled two-leg Hubbard ladder with inter-site Coulomb repulsions and exchange coupling. They insisted that the universality class of the phase transition between the rung-singlet phase and the staggered dimer phase is \( c = 3/2 \) CFT or the first order phase transition. Our result is consistent with their result.
According to Hikihara et al this model has a self-duality \((J_{\text{ring}} = J_{\text{leg}}/2)\), which is between a spin and a vector chirality [22]. In the case that the system has duality and there exists a phase transition, if a phase transition point is not the self-dual point, there must be another phase transition. Now Hamiltonian is not invariant for the duality transformation exactly. But we can expect that there is another phase transition reflected in the duality transformation. This should be a future problem.

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APPENDIX A: WESS-ZUMINO-WITTEN MODEL AND LOGARITHMIC CORRECTION [17, 36, 37]

The minimal conformal field theory with the smallest spectrum containing currents obeying the Kac-Moody algebra with central charge \(c\) is the Wess-Zumino-Witten non-linear \(\sigma\) model, with topological coupling constant \(k\). Thus the relation between \(c\) and \(k\) is

\[
c = \frac{k \dim G}{k + \frac{C_A}{2}}.
\]

where \(\dim G\) is dimension for an arbitrary representation and \(\frac{C_A}{2}\) is called as the dual Coxeter number. In the present case, since the system has SU(2) symmetry, we have \(\dim G = 3\), and \(\frac{C_A}{2} = 2\).

Now we think of the case with the SU(2) symmetry. In this case, primary fields can be classified according to their left and right moving spin. There are operators with \(s_L = s_R = 0, \frac{1}{2}, ..., \frac{k}{2}\). Their scaling dimensions are

\[
x = \frac{2s_L (s_L + 1)}{2 + k}.
\]

The operators with \(s_L\) half-odd integer (integer) are odd (even) under translation by one site; thus they correspond to states with momentum \(\pi\) (zero).
Here we think of the case of the $k = 2$ SU(2) WZW model. In the case of $s_L = s_R = \frac{1}{2}$, which forms $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, the scaling dimension is $x = \frac{3}{8}$. In the case of $s_L = s_R = 1$ which forms $1 \otimes 1 = 0 \oplus 1 \oplus 2$, the scaling dimension is $x = 1$. This is related to the critical exponent of an energy gap.

Besides Kac-Moody primary operators, there is a marginal operator for all $k$, namely $\vec{J}_L \cdot \vec{J}_R$ (which is a primary field with respect to the Virasoro algebra). Current operator $\vec{J}_L$ has the conformal weight $(h, \bar{h}) = (1, 0)$ and $\vec{J}_R$ has the conformal weight $(h, \bar{h}) = (0, 1)$. Thus the $\vec{J}_L \cdot \vec{J}_R$ operator has the conformal weight $(h, \bar{h}) = (1, 1)$ which has the scaling dimension $x = h + \bar{h} = 2$ and the conformal spin 0, i.e. wave number $q = 0$.

Current operators $\vec{J}_L, \vec{J}_R$ themselves have conformal spin $\pm 1$, corresponding to the wave number $q = \pm 2\pi/L$, thus they are related with the spin wave velocity.

According to the non-Abelian bosonization \cite{38}, in SU(2) symmetric gapless system, the excitation energies (for $q = 0$ states) in the periodic boundary condition, including log corrections, are

$$
\Delta E = E_i - E_0 \approx \frac{2\pi v}{L} \left( x_i - \frac{\langle S_L \cdot S_R \rangle}{\ln L} \right).
$$

(A3)

where $L$ is a system size, and $v$ is a spin wave velocity. Here $\mathbf{S} = \mathbf{S}_L + \mathbf{S}_R$ is total spin which is a conserved quantity. Note that

$$
\mathbf{S}_L \cdot \mathbf{S}_R = \frac{1}{2} (\mathbf{S}_L + \mathbf{S}_R)^2 - \frac{1}{2} \mathbf{S}_L^2 - \frac{1}{2} \mathbf{S}_R^2
$$

(A4)

$$
= \frac{1}{2} s(s + 1) - \frac{1}{2} s_L(s_L + 1) - \frac{1}{2} s_R(s_R + 1).
$$

(A5)

In the case of $s_L = s_R = 1$, since $s_L \otimes s_R=1 \otimes 1=0 \oplus 1 \oplus 2$, it forms a quintet $s = 2$ triplet $s = 1$ and a singlet $s = 0$. The states $s_L = s_R = 1$ correspond to a wave vector $q = 0$.

First $s = 0$ (singlet) case,

$$
\langle \text{singlet} \mid \mathbf{S}_L \cdot \mathbf{S}_R \mid \text{singlet} \rangle = -2
$$

(A6)

Second $s = 1$ (triplet) case,

$$
\langle \text{triplet} \mid \mathbf{S}_L \cdot \mathbf{S}_R \mid \text{triplet} \rangle = -1
$$

(A7)

Third $s = 2$ (quintet) case,

$$
\langle \text{quintet} \mid \mathbf{S}_L \cdot \mathbf{S}_R \mid \text{quintet} \rangle = 1
$$

(A8)
Therefore we obtain
\[
\Delta E(s = 2) = E_i - E_0 \approx \frac{2\pi v}{L} \left( x_i - \frac{1}{\ln L} \right),
\]  
(A9)
and
\[
\Delta E(s = 1) = E_i - E_0 \approx \frac{2\pi v}{L} \left( x_i + \frac{1}{\ln L} \right),
\]  
(A10)
and
\[
\Delta E(s = 0) = E_i - E_0 \approx \frac{2\pi v}{L} \left( x_i + 2 \frac{1}{\ln L} \right),
\]  
(A11)
thus we can remove logarithmic corrections of energy gap.

\[
\Delta E(s = 2) + \Delta E(s = 1) = 2 \frac{2\pi v}{L} x_i,
\]
or
\[
x_i = \frac{L}{4\pi v} \left[ \Delta E(s = 2) + \Delta E(s = 1) \right].
\]  
(A12)

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FIG. 1: Two-leg spin ladder with four spin exchange.

FIG. 2: Four spin cyclic exchange on the two-leg ladder.

FIG. 3: A rung singlet state which consists of two spins $S_{1,i}, S_{2,i}$ enclosed by a dotted line.

FIG. 4: A staggered dimer order state which consists of two spins $S_{\alpha,i}, S_{\alpha,i+1}$ enclosed by a dotted line, in the open boundary condition. This phase has an order parameter $\langle S_{\alpha,i} \cdot S_{\alpha,i+1} \rangle - \langle S_{\alpha,i} \cdot S_{\alpha,i+1} \rangle$.
FIG. 5: Ratios of energy gaps as a function of the system size $L$

FIG. 6: Ratios of energy gaps as a function of the system size $1/L$
FIG. 7: Ratios of energy gaps as a function of the system size $J_{\text{ring}}$

FIG. 8: $J_{\text{ring}}^{\text{cross}}$ as function of the system size $1/L$. 
FIG. 9: The correlation length as a function of $J_{\text{ring}}$ is computed using the modified Bessel function. The dot line was computed using a least-squares method.

FIG. 10: Size dependence of the scaling dimension $x = 1$ after removing logarithmic correction.