Abstract: The Grishukhin inequality $Gr_7$ is a facet of $\text{CUT}_7^\square$, the cut polytope on seven points, which is “sporadic” in the sense that its proper generalization has not been known. In this paper, we extend $Gr_7$ to an inequality $I(G, H)$ valid for $\text{CUT}_{n+1}^\square$ where $G$ and $H$ are graphs with $n$ nodes satisfying certain conditions, and prove a necessary and sufficient condition for $I(G, H)$ to be a facet. This result combined with the triangular elimination theorem of Avis, Imai, Ito and Sasaki settles Collins and Gisin’s conjecture in quantum theory affirmatively: the $I_{mm22}$ Bell inequality is a facet of the correlation polytope $\text{COR}_{n}^\square(K_{m,m})$ of the complete bipartite graph $K_{m,m}$ for all $m \geq 1$. We also extend the $Gr_8$ facet inequality of $\text{CUT}_8^\square$ to an inequality $I'(G, H, C)$ valid for $\text{CUT}_{n+2}^\square$, and provide a sufficient condition for $I'(G, H, C)$ to be a facet.

Keywords: cut polytope, Grishukhin inequality, $I_{mm22}$ Bell inequality, correlation polytope

1 Introduction

Cut polytopes are convex polytopes which arise in many different fields [6, 7, 8]. Since testing membership in cut polytopes is NP-complete [1], it is unlikely that there exists a concise and complete description of their facial structure in general. Much efforts has been devoted to identifying classes of inequalities which are valid for cut polytopes and have good properties. Hypermetric, clique-web and parachute inequalities are examples of classes of valid inequalities for which important subclasses are facet inducing. For $N \leq 6$, all facets of $\text{CUT}_N^\square$, the cut polytope of complete graph $K_N$, are hypermetric. However, $\text{CUT}_7^\square$ has a facet called the Grishukhin inequality $Gr_7$ which is not known to belong to any such general class. Efforts have been made to relate $Gr_7$ to other inequalities. As a result, De Simone, Deza and Laurent [5] showed that $Gr_7$ is a collapse of a pure facet inequality $Gr_8$ of $\text{CUT}_8^\square$.

The cut polytope $\text{CUT}_{n}^\square(K_{1,m,m})$ of the complete tripartite graph $K_{1,m,m}$ is linearly isomorphic to the correlation polytope $\text{COR}_{n}^\square(K_{m,m})$ of the complete bipartite graph $K_{m,m}$. In quantum theory, the correlation polytope $\text{COR}_{n}^\square(K_{m,m})$ is seen as the set of possible results of a
series of Bell experiments with a non-entangled (separable) quantum state shared by two distant parties, where each party has $m$ choices of measurements. In this context, a valid inequality of $\text{COR}^{\square}(K_{m,m})$ is called a Bell inequality and if facet inducing, a tight Bell inequality. Readers are referred to [11] for further information about Bell inequalities. Collins and Gisin [4] found a class of $I_{m m 22}$ inequalities valid for $\text{COR}^{\square}(K_{m,m})$ for general $m$ and conjectured that for all $m \geq 1$, $I_{m m 22}$ inequality is tight, or equivalently, that it is a facet of $\text{COR}^{\square}(K_{m,m})$.

Avis, Imai, Ito and Sasaki [2] introduced an operation called triangular elimination to convert a facet of $\text{CUT}^{\square}_{N}$ to a facet of $\text{CUT}^{\square}_{(1,m,m)}$ for appropriate $m$. By using this operation, the tightness of the $I_{3322}$ and $I_{4422}$ Bell inequalities follows from the fact that the pure pentagonal and the Grishukhin inequalities are facets of $\text{CUT}^{\square}_{5}$ and $\text{CUT}^{\square}_{7}$, respectively. This suggests that some natural extensions of the pure pentagonal and the Grishukhin inequalities may give facets of $\text{CUT}^{\square}_{2m-1}$ for $m \geq 3$. We will prove that it is the case and that hence the conjecture by Collins and Gisin is true. More specifically, we will introduce inequalities $I(G, H)$ valid for $\text{CUT}^{\square}_{n+1}$ where $G$ and $H$ are graphs with $n$ nodes which satisfy certain conditions described later, and prove a necessary and sufficient condition for $I(G, H)$ to be a facet.

As further extensions, we apply to $I(G, H)$ an operation similar to the one used to construct $\text{Gr}_{8}$ from $\text{Gr}_{7}$. Actually this operation gives inequalities $I'(G, H, C)$ valid for $\text{CUT}^{\square}_{n+2}$ where $C$ is a cycle of length four in $G$. We will give a sufficient condition for $I'(G, H, C)$ to be a facet, generalizing the fact that $\text{Gr}_{8}$ is a facet of $\text{CUT}^{\square}_{8}$.

The rest of the paper is organized as follows. In Section 2 we review the tools used later. In Section 3 we introduce the inequality $I(G, H)$ valid for the cut polytope, which is a generalization of the $\text{Gr}_{7}$ inequality, and we prove a necessary and sufficient condition for it to be a facet. Section 4 defines the valid inequality $I'(G, H, C)$, which is a generalization of the $\text{Gr}_{8}$ inequality, and we provide a sufficient condition for it to be a facet. The proof of the sufficient condition is deferred to appendix. In Section 5 we prove the tightness of $I_{m m 22}$ Bell inequalities.

## 2 Preliminaries

### 2.1 Cut polytopes

Here we review the definition of and results on cut polytopes only briefly. Readers are referred to the book by Deza and Laurent [8] for details.

**Definition** The cut polytope $\text{CUT}^{\square}(G)$ of a graph $G = (V, E)$ is a convex polytope in the vector space $\mathbb{R}^{E}$ defined as the convex hull of the $2^{|V|}-1$ different cut vectors $\delta_G(S)$ for $S \subseteq V$. The cut vector $\delta_G(S) \in \mathbb{R}^{E}$ is a 0/1 vector defined by $\delta_{uv}(S) = 1$ if and only if exactly one of $u$ and $v$ belongs to $S$, where $uv$ denotes the edge connecting two nodes $u$ and $v$. The cut polytope $\text{CUT}^{\square}(K_N)$ of the complete graph $K_N$ is denoted by $\text{CUT}^{\square}_N$.

Similarly, the correlation polytope $\text{COR}^{\square}(G)$ is a convex polytope in $\mathbb{R}^{V \cup E}$ defined as the convex hull of the $2^{|V|}$ correlation vectors $p_G(S)$ for $S \subseteq V$. The correlation vector $p_G(S) \in \mathbb{R}^{V \cup E}$ is a 0/1 vector defined by $p_u(S) = 1$ if and only if $u \in S$ and $p_{uv}(S) = 1$ if and only if $\{u, v\} \subseteq S$.

The correlation polytope $\text{COR}^{\square}(G)$ of a graph $G = (V, E)$ is linearly isomorphic to $\text{CUT}^{\square}(\nabla G)$, where $\nabla G$ is the suspension graph of $G$: the graph obtained by adding to $G$ a new node $Z$ adjacent to all the nodes of $G$. The linear isomorphism between them is called the covariance mapping: $p_u = x_{zu}$ for $u \in V$ and $p_{uv} = \frac{1}{2}(x_{zu} + x_{zv} - x_{uv})$ for $u, v \in V$, $u \neq v$. 

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Hypermetric inequalities Let \( N \geq 3 \) be an integer and \( \mathbf{b} \in \mathbb{Z}^N \) an integer vector with \( \sum_{i=1}^{N} b_i = 1 \). The inequality \( \sum_{1 \leq i < j \leq N} b_i b_j x_{ij} \leq 0 \) is valid for \( \text{CUT}_N^\square \) and called the hypermetric inequality defined by the vector \( \mathbf{b} \).

While an exact characterization of when a hypermetric inequality becomes a facet of \( \text{CUT}_N^\square \) is not known, many sufficient conditions are known. We review here some of them which we use later.

**Theorem 1 (Corollary 28.2.5 (i) in [8])** Let \( s \geq 1 \) be an integer, and \( \mathbf{b} \in \mathbb{Z}^N \) be an integer vector with \( s + 1 \) entries equal to 1, \( s \) entries equal to \(-1\) and the other \( N - (2s + 1) \) entries equal to 0. Then the hypermetric inequality defined by \( \mathbf{b} \) is a facet of \( \text{CUT}_N^\square \). This inequality is called a pure \((2s + 1)\)-gonal inequality, or if \( s = 1 \), simply a triangle inequality.

We define \( T(u, v; w) = x_{uv} - x_{uw} - x_{vw} \). By using this notation, a triangle inequality is written as \( T(u, v; w) \leq 0 \).

**Theorem 2 (“If” part of Theorem 28.2.4 (iiib) in [8])** The hypermetric inequality defined by \( \mathbf{b} \) with \( b_1 = \cdots = b_{N-2} = 1 \), \( b_{N-1} = -1 \) and \( b_N = -N + 4 \) is a facet of \( \text{CUT}_N^\square \).

**Switching of inequality** We mention three operations on inequalities valid for cut polytopes.

One is the switching operation. Let \( G = (V, E) \) be a graph, \( \mathbf{a} \in \mathbb{R}^E \) and \( a_0 \in \mathbb{R} \). The switching of the inequality \( \mathbf{a}^T \mathbf{x} \leq a_0 \) by the cut \( S \subseteq V \) is an inequality \( \mathbf{b}^T \mathbf{x} \leq b_0 \) with \( b_{ij} = (-1)^{\delta_{ij}(S)} \cdot a_{ij} \) and \( b_0 = a_0 - \mathbf{a}^T \mathbf{d}_G(S) \).

Switching is an automorphism of the cut polytope \( \text{CUT}^\square(G) \). Therefore \( \mathbf{b}^T \mathbf{x} \leq b_0 \) is valid (resp. a facet) if and only if \( \mathbf{a}^T \mathbf{x} \leq a_0 \) is valid (resp. a facet).

**Collapsing and lifting of inequality** The other two operations are collapsing and lifting. Let \( G = (V, E) \) be a complete graph on node set \( V \) and \( uv \in E \). Let \( G' = (V', E') \) be the complete graph on node set \( V' = (V \setminus \{u, v\}) \cup \{w\} \) with a new node \( w \).

The \((u, v)\)-collapse of a vector \( \mathbf{a} \in \mathbb{R}^E \) is a vector \( \mathbf{a}^{u,v} \in \mathbb{R}^{E'} \) defined by
\[
\begin{align*}
    a_{ij}^{u,v} &= a_{ij} \quad \text{for } i, j \in V \setminus \{u, v\}, i \neq j, \\
    a_{wi}^{u,v} &= a_{wi} + a_{vi} \quad \text{for } i \in V \setminus \{u, v\}.
\end{align*}
\]

For \( \mathbf{a} \in \mathbb{R}^E \) and \( a_0 \in \mathbb{R} \), an inequality \((\mathbf{a}^{u,v})^T \mathbf{x} \leq a_0 \) is said to be the \((u, v)\)-collapse of the inequality \( \mathbf{a}^T \mathbf{x} \leq a_0 \).

If the inequality \( \mathbf{a}^T \mathbf{x} \leq a_0 \) is valid for \( \text{CUT}^\square(G) \), its collapse \((\mathbf{a}^{u,v})^T \mathbf{x} \leq a_0 \) is valid for \( \text{CUT}^\square(G') \).

The opposite operation of collapsing is called lifting. The following lemma provides a sufficient condition for lifting to preserve a facet. The proof of the lemma is given below Lemma 26.5.3 in the book [8].

**Lemma 3 (Lifting lemma [8])** Let \( \mathbf{a} \in \mathbb{R}^E \). The inequality \( \mathbf{a}^T \mathbf{x} \leq 0 \) is a facet of \( \text{CUT}^\square(G) \) if the following conditions are satisfied.

(i) The inequality \( \mathbf{a}^T \mathbf{x} \leq 0 \) is valid for \( \text{CUT}^\square(G) \), and its \((u, v)\)-collapse \((\mathbf{a}^{u,v})^T \mathbf{x} \leq 0 \) is a facet of \( \text{CUT}^\square(G') \).

(ii) There exist \(|V| - 1\) subsets \( T_j \) of \( V \) with \( u \notin T_j \) and \( v \in T_j \) such that the cut vectors \( \mathbf{d}_G(T_j) \) are roots (vertices lying on the face) of \( \mathbf{a}^T \mathbf{x} \leq 0 \) and the incidence vectors of \( T_j \) are linearly independent.
Grishukhin inequality The cut polytope $\text{CUT}^\square_7$ has 11 inequivalent facets under permutation and switching symmetries [9, 5]. All but one of them belong to at least one of three general classes of valid inequalities: hypermetric, clique-web and parachute inequalities. The remaining facet is not known to belong to any classes that are as general as these classes. This “sporadic” facet is called the Grishukhin inequality $\text{Gr}_7$. The Grishukhin inequality looks like
\[
\sum_{1 \leq i < j \leq 4} x_{ij} + x_{56} + x_{57} - x_{67} - x_{16} - x_{36} - x_{27} - x_{47} - 2 \sum_{1 \leq i \leq 4} x_{5i} \leq 0
\]
and is illustrated in Figure 1 (a).

De Simone, Deza and Laurent [5] found a facet of $\text{CUT}^\square_8$ which is pure (all the coefficients are 0 or ±1) and is a lifting of $\text{Gr}_7$. This facet is called $\text{Gr}_8$ in [8] and illustrated in Figure 1 (b).

2.2 Bell inequalities

$I_{mm22}$ Bell inequalities Collins and Gisin [4] showed that the $I_{mm22}$ inequalities:
\[
-pA_1 - \sum_{1 \leq j \leq m} (m - j)pB_j - \sum_{2 \leq i,j \leq m} p_{A_iB_j} + \sum_{1 \leq i,j \leq m, i+j=m+1} p_{A_iB_j} \leq 0,
\]
are valid for $\text{COR}^\square(K_{m,m})$ for all $m \geq 1$, generalizing CHSH inequality [3] for $m = 2$ which is a facet of $\text{COR}^\square(K_{2,2})$. They conjectured that for any $m \geq 1$, the $I_{mm22}$ inequality is a facet of $\text{COR}^\square(K_{m,m})$, and showed that the conjecture is true for $m \leq 7$.

Triangular elimination Avis, Imai, Ito and Sasaki [2] proposed triangular elimination operation to convert any facet inequality of $\text{CUT}^\square_0$ other than the triangle inequality to a facet of $\text{CUT}^\square(K_{1,m,m})$ for appropriate $m$. A basic step in this conversion is described in the following theorem.

Theorem 4 ([2]) Let $G = (V, E)$ be a graph and $uu' \in E$ an edge of $G$. Let $W \subseteq N_G(u) \cap N_G(u')$ be a set of nodes that are adjacent to both $u$ and $u'$. We define a graph $G^+ = (V^+, E^+)$, the detour extension of $G$, as follows. We add a new node $v$ to $G$ in the middle of the edge $uu'$, dividing $uu'$ into two edges $uw$ and $u'v$, and add new edges $vw$ for each $w \in W$.

Let $a^T x \leq a_0$ be a facet inequality of $\text{CUT}^\square(G)$. Define $b^T x \leq a_0$, the triangular elimination of $a^T x \leq a_0$, to be the inequality obtained by combining the triangle inequality $-a_{uu'}x_{uu'} + a_{uu'}x_{uv} - |a_{uu'}|x_{u'v} \leq 0$ with $a^T p \leq a_0$.

If there exists an edge $e \in E \setminus \{uu'\}$ such that $a_e \neq 0$, then the inequality $b^T x \leq a_0$ is a facet of $\text{CUT}^\square(G^+)$. 

![Figure 1: (a) The Grishukhin inequality $\text{Gr}_7$, which is a facet of $\text{CUT}^\square_7$. (b) The $\text{Gr}_8$ inequality, which is a facet of $\text{CUT}^\square_8$.](image-url)
with $S$ and $H$ labels of nodes 1, 2, 3, 4, 5, 6, respectively. Then the resulting inequality is identical to Gr$_7$.

3 Inequality $I(G, H)$: A generalization of Gr$_7$

In this section, we define the inequality $I(G, H)$ valid for the cut polytope, and give a necessary and sufficient condition for $I(G, H)$ to be a facet.

First we define the inequality. Let $n \geq 1$ be an integer, and $G = (V, E)$ and $H = (V, F)$ be two graphs with $n$ nodes. We require that the edges of $H$ are node-disjoint. Let $t = |F|$ and $k = n - t$, and we denote the connected component decomposition of $H$ by $V = V_1 \cup \cdots \cup V_k$. Note that the size of any connected component $V_i$ is one or two. Finally we require that $E$ contains exactly $\binom{k}{2}$ edges: for each $1 \leq i < j \leq k$ there is an edge $e_{ij}$ connecting a node in $V_i$ and a node in $V_j$. We consider the following inequality which we denote as $I(G, H)$:

$$\sum_{uv \in E} T(u, v; n + 1) - \sum_{uv \in F} T(u, v; n + 1) + 2 \sum_{V_i = \{u\}} x_{u,n+1} \leq 2. \quad (2)$$

For example, $I(K_2, \overline{K}_2)$ is identical to the triangle inequality and $I(K_4, \overline{K}_4)$ to the pure pentagonal inequality, where $K_n$ is the complete graph on $n$ nodes, and $\overline{K}_n$ is its complement.

It is sometimes convenient to relabel the nodes in $V$ so that $H$ is in a restricted form. For $k \geq 1$ and $0 \leq t \leq k$, let $H_{k,t} = (V, E)$ be a graph with node set $V = \{1, \ldots, k + t\}$ and edge set $E = \{(i, k + i) \mid 1 \leq i \leq t\}$. Then any graph $H$ with $n = k + t$ nodes and $t$ node-disjoint edges can be relabelled to $H_{k,t}$, and therefore we can restrict $I(G, H)$ to $I(G, H_{k,t})$ without loss of generality.

We check that the Gr$_7$ inequality is a switching of an inequality of this kind. Let $G_6 = (V, E)$ and $H_{5,1} = (V, F)$ be the graphs with six nodes shown in Figure 2(a). Then the inequality $I(G_6, H_{5,1})$ is as shown in Figure 2(b). We switch $I(G_6, H_{5,1})$ by the cut $\{1, 6\}$ and change the labels of nodes 1, 2, 3, 4, 5, 6, 7 to 6, 1, 2, 3, 4, 5, 7, respectively. Then the resulting inequality is identical to Gr$_7$.

Now we prove the validity of $I(G, H)$.

**Proposition 5** The inequality $I(G, H)$ is valid for CUT$_{n+1}$. In addition, the cut vector $\delta(S)$ with $S \subseteq V$ is a root of $I(G, H)$ if and only if one of the following conditions is satisfied.

(i) There exists a unique $i$ such that $V_i \subseteq S$, and no edge of $G$ is contained in $S$.

(ii) There exist exactly two values of $i$ (let them be $i_1$ and $i_2$) such that $V_i \subseteq S$. In addition, $e_{i_1i_2}$ is the only edge of $G$ that is contained in $S$. 

Figure 2: (a) A graph $G_6 = (V, E)$ (edges drawn as single lines) and a graph $H_{5,1} = (V, F)$ (an edge drawn as a double line). (b) The inequality $I(G_6, H_{5,1})$, which is a switching of the Gr$_7$ inequality.
Proof: We show that the cut vector $\delta(S)$ defined by any subset $S$ of $V$ satisfies the inequality $I(G, H)$. Note that with $x = \delta(S)$, each term evaluates to either to zero or two.

Let $A = \{i \mid V_i \subseteq S\}$ and $B = \{ij \mid e_{ij} \subseteq S\}$. The left hand side of $I(G, H)$ evaluated with $x = \delta(S)$ is equal to $2|A| - 2|B|$. Now $|B| \geq \binom{|A|}{2}$, since for each of the $\binom{|A|}{2}$ pairs $ij$ of elements of $A$, there is an edge $e_{ij}$ with both endpoints in $S$. Therefore we have $2|A| - 2|B| \leq 3|A| - |A|^2 = 2 - (|A| - 1)(|A| - 2) \leq 2$. So $I(G, H)$ is valid.

The condition for roots is obtained from the fact that this inequality is satisfied with equality if and only if $|A|$ is one or two and $|B| = \binom{|A|}{2}$. □

Now we consider when the inequality $I(G, H)$ becomes a facet of $\text{CUT}_{n+1}^\Box$.

Theorem 6 Assume $k \geq 3$. Then the inequality $I(G, H)$ is a facet of $\text{CUT}_{n+1}^\Box$ if and only if all nodes in $G$ have degree at least two.

Proof: As mentioned above, we can assume $H = H_{k,t}$ without loss of generality.

First we prove the “only if” part. Let $u$ be a node whose degree in $G$ is at most one. In this case $H_{k,t}$ has an edge incident to node $u$. Without loss of generality, we assume $u = k + t$. If the degree of node $k + t$ in $G$ is one, then let $v$ be the only node that is adjacent to node $k + t$ in $G$. Otherwise let $v = n + 1$. In both cases, $I(G, H_{k,t})$ is the sum of a triangle inequality $T(k + t, v; t) \leq 0$ and the inequality $I(G/(t, k + t), H_{k,t-1})$, where $G/(t, k + t)$ is a graph obtained from $G$ by identifying two nodes $t$ and $k + t$ into a node $t$. Therefore, $I(G, H_{k,t})$ is not a facet of $\text{CUT}_{n+1}^\Box$.

Now we prove the “if” part. The proof is by induction on $t$.

First we consider the case $t = 0$. In this case, $H_{k,0}$ has no edges and $G$ is the complete graph $K_n$. Switching the inequality $I(K_n, K_n)$ by the cut $\{1\}$ gives a hypermetric inequality defined by an integer vector $b$ with $b_{n+1} = -(k - 3)$, $b_1 = -1$ and $b_2 = \cdots = b_n = 1$. This hypermetric inequality is a facet of $\text{CUT}_{n+1}^\Box$ by Theorem 2.

Now we consider the case $t \geq 1$. Note that contracting the edge $(t, k + t)$ in $H_{k,t}$ gives $H_{k,t-1}$. Key facts are that the inequality $I(G, H_{k,t})$ is obtained by lifting $I(G/(t, t + k), H_{k,t-1})$, and that $I(G/(t, t + k), H_{k,t-1})$ is a facet of $\text{CUT}_{n}^\Box$ by the induction hypothesis.

Let $V_i = V_{i+k} = \{i, i + k\}$ for $1 \leq i \leq t$ and $V_i = \{i\}$ for $t + 1 \leq i \leq k$. We define $n$ subsets of $V$ as follows.

- Let $p$ and $p'$ be two distinct nodes adjacent to node $t + k$ in $G$. Then define $T^{(1)} = \{t\} \cup V_p \cup V_{p'}$.

- Let $q$ and $q'$ be two distinct nodes adjacent to node $t$ in $G$. Then define $T^{(2)} = \{t + k\} \cup V_q \cup V_{q'}$.

- For each $1 \leq i \leq k$ with $i \neq t$, we define $T^{(3)} = \{t\} \cup V_i$. Otherwise, $T^{(3)} = \{t + k\} \cup V_i$.

- For each $1 \leq i \leq t - 1$, we define a subset $T^{(4)}_i$. Let $u$ be either $i$ or $i + k$ that is an endpoint of the edge $e_{it}$, and $\bar{u}$ be either $i$ or $i + k$ that is different from $u$. Let $v$ be any node in $N_G(u) \setminus V_i$ and choose $j$ so that $V_j \supseteq v$. Let $\bar{w}$ be either $t$ or $t + k$ that is not an endpoint of the edge $e_{jt}$. Then define $T^{(4)}_i = \{\bar{u}, \bar{w}\} \cup V_j$.

It is easy to check that each of these subsets is a root of $I(G, H_{k,t})$ and contains exactly one of $t$ and $t + k$. Note that none of them contains node $n + 1$. 

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Consider again the graphs $G$ and $H_{6,1}$ (an edge drawn as a double line). The inequality $I(G, H_{6,1})$, which is proved to be a facet of $\text{CUT}_8$ by Theorem 6. A line connected to a circle enclosing nodes 1, 2 and 3 represents 3 edges with identical weights each connected to the nodes 1, 2 and 3. Similar for lines connected to the other circles.

The following claim can be proved in a straightforward way. A proof is included in Appendix A.

**Claim 7**  
The $n$ incident vectors of $T^{(1)}$, $T^{(2)}$, $T^{(3)}_i$ ($i \neq t$) and $T^{(4)}_i$ ($1 \leq i \leq t - 1$) are linearly independent.

From now on, we refer to the $n$ sets $T^{(1)}$, $T^{(2)}$, $T^{(3)}_i$ ($i \neq t$) and $T^{(4)}_i$ ($1 \leq i \leq t - 1$) as $T_1, \ldots, T_n$.

Let $a^T \mathbf{x} \leq 0$ be the switching of $I(G, H_{k,t})$ by its root $\{t, t + k\}$.

The $(t, t + k)$-collapse $(a^{t+k+1} \mathbf{x} \leq 0$ is the switching by the cut $\{t\}$ of $I(G/(t, t + k), H_{k,t-1}),$ which is a facet of $\text{CUT}_n$ by induction hypothesis.

For $1 \leq i \leq n$, let $T^{(1)}_i$ be $T_i \triangle \{t, t + k\}$ if $t \in T_i$, and $(V \cup \{n + 1\}) \setminus (T_i \triangle \{t, t + k\})$ otherwise, where $\triangle$ means the symmetric difference of two sets. Then $T^{(1)}_i$ is a root of the inequality $a^T \mathbf{x} \leq 0$ and contains $t + k$ but does not contain $t$. In addition, the $n$ vectors $T^{(1)}_1, \ldots, T^{(1)}_n$ are also linearly independent. From Lemma 8, the inequality $a^T \mathbf{x} \leq 0$ is a facet of $\text{CUT}_{n+1}$, which means $I(G, H_{k,t})$ is also a facet of $\text{CUT}_{n+1}$. ∎

For example, let us consider the graphs $G = (V, E)$ and $H_{6,1} = (V, F)$ shown in Figure 3(a). In this case the inequality $I(G, H_{6,1})$, illustrated in Figure 3(b), is a facet of $\text{CUT}_8$ by Theorem 6.

4 Inequality $I'(G, H, C)$: A generalization of $\text{Gr}_8$

Let $G = (V, E)$, $H = (V, F)$, $n = |V|$, $t = |F|$, $k = n - t$ and $V = V_1 \cup \cdots \cup V_k$ be as defined in Section 3. In this section we require an additional condition that $G$ has a cycle $C$ of length four (this condition implies $k \geq 4$). Let $V_C$ be the set of the four nodes of $C$. Then we consider an inequality for the cut polytope on $n + 2$ nodes:

$$\sum_{u \in E} T(u, v; n + 1) - \sum_{u \in F} T(u, v; n + 1) + 2 \sum_{V_i = \{u\}} x_{u,n+1} + \sum_{u \in V_C} (x_{u,n+1} - x_{u,n+2}) \leq 2. \quad (3)$$

We refer to inequality 13 by $I'(G, H, C)$. Note that the $(n + 1, n + 2)$-collapsing of $I'(G, H, C)$ is identical to $I(G, H)$.

As an example, we show that the $\text{Gr}_8$ inequality is a switching of an inequality of this kind. Consider again the graphs $G_6 = (V, E)$ and $H_{5,1} = (V, F)$ shown in Figure 2(a). Note that $G_6$
contains a cycle $C = \{23, 34, 45, 52\}$ of length four. Then the inequality $I'(G, H, C)$ is as shown in Figure 4 (a), and switching it by the cut $\{1, 6\}$ and relabelling nodes appropriately gives the $Gr_8$ inequality.

**Proposition 8** The inequality $I'(G, H, C)$ is valid for $\text{CUT}_{n+2}^\Box$.

**Proof:** Let $M$ be a set of two node-disjoint edges in the cycle $C$. Note that there are two choices of $M$. No matter which set we choose as $M$, the inequality $I'(G, H, C)$ can be written as

$$\sum_{u \in E \setminus M} T(u, v; n + 1) + \sum_{u \in M} T(u, v; n + 2) - \sum_{u \in F} T(u, v; n + 1) + 2 \sum_{V_i = \{u\}} x_{u,n+1} \leq 2. \quad (4)$$

We show that the cut vector $\delta(S)$ defined by any subset $S$ of $V \cup \{n + 2\}$ satisfies (4). Let $A = \{i \mid V_i \subseteq S\}$ and $B = \{ij \mid e_{ij} \subseteq S, e_{ij} \in E \setminus M\}$. Now $|B| \geq \binom{|A|}{2} - \frac{|A|}{2}$, since for each $ij \in B$ there is an edge $e_{ij}$ with both endpoints in $A$, except for up to $\frac{|A|}{2}$ edges that may be part of $M$. The left hand side of (4) evaluated with $x = \delta(S)$ is at most $2|A| - 2|B|$. Combining inequalities we have $2|A| - 2|B| \leq 3|A| + 2\frac{|A|}{2} - |A|^2 \leq 2$ except when $|A| = 2$. So (4) is valid for all these cases. Suppose $|A| = 2$.

**Case 1:** The two nodes in $A$ do not form an edge in $M$.

In this case $|B| = 1$, the LHS of (4) is at most $2|A| - 2|B| = 2$ and the inequality is valid.

**Case 2:** The two nodes in $A$ form an edge in $M$.

In this case we replace $M$ by $C \setminus M$. This does not change the LHS of (4), and the inequality is valid by Case 1. □

Before we state a sufficient condition for $I'(G, H, C)$ to be a facet of $\text{CUT}_{n+2}^\Box$, we assume some conditions on $H$ and $C$ without loss of generality. We assume $H = H_{k,t}$, where $H_{k,t}$ is the same as that defined in the previous section, and we also assume that indices of the four nodes of $C$ are at most $k$. We say that node $i$ in $C$ is free if $1 \leq i \leq t$ and $i + k$ is incident to edge $e_{ij}$ where $j$ is the unique node in $C$ that is not adjacent to $i$ in $C$. The following theorem gives a sufficient condition for $I'(G, H_{k,t}, C)$ to be a facet.

**Theorem 9** The inequality $I'(G, H_{k,t}, C)$ is a facet of $\text{CUT}_{n+2}^\Box$ if all of the following conditions are satisfied:

(i) All nodes in $G$ have at least two neighbors.
(ii) For each $t + 1 \leq i \leq k$ except for nodes in $C$, there exists a free node $j$ in $C$ such that $e_{ij}$ is incident to $j + k$.

(iii) For each $1 \leq i \leq t$ except for nodes in $C$, either:

- Nodes $i$ and $i + k$ are incident to exactly two out of four edges $e_{ij}$ with $j \in V_C$, or
- There exists a free node $j$ in $C$ such that $e_{ij}$ is incident to $j + k$.

Since $I'(G, H, C)$ is a lifting of $I(G, H)$, we may prove Theorem 9 by combining the lifting lemma (Lemma 3) with Theorem 6. The proof is given in Appendix B. As an example of the theorem, consider the graphs $G_6$ and $H_{5,1}$ shown in Figure 2 (a), but this time let $C = \{12, 23, 34, 41\}$. In this case the inequality $I'(G, H, C)$, shown in Figure 4 (b), is a facet of $\text{CUT}^2_{k+1}$ with $\text{CUT}^2_{k+2}$ supported by the inequality $I'(K_k, \overline{K}_k, C)$ with $k \geq 5$ and $C = \{12, 23, 34, 41\}$ is contained in a triangle facet $x_{5,k+2} - x_{5,k+1} - x_{k+1,k+2} \leq 0$ and never supports a facet.

5 Tightness of the $I_{mm22}$ Bell inequalities

In this section, we prove that for any $m$, the $I_{mm22}$ inequality is a facet of $\text{COR}^2(K_{m,m})$, or in other words, a tight Bell inequality. Since the proof does not depend on the proof of validity given in [4], our proof also serves as another way to prove the validity of the $I_{mm22}$ inequality.

Let $K_{1,m,m} = (V_{1,m,m}, E_{1,m,m})$ be a complete tripartite graph with node set $V_{1,m,m} = \{Z, A_1, \ldots, A_m, B_1, \ldots, B_m\}$ and edge set $E_{1,m,m} = \{ZA_i | 1 \leq i \leq m\} \cup \{ZB_j | 1 \leq j \leq m\} \cup \{A_iB_j | 1 \leq i, j \leq m\}$. We rewrite the $I_{mm22}$ inequality to an inequality for $\text{CUT}^2(K_{1,m,m})$ by using the covariance mapping. We switch this inequality by the cut $\{A_1, \ldots, A_m\}$. After that, we change the labels of the $m$ nodes $B_1, B_2, \ldots, B_m$ to $B_{m+1}, B_{m+2}, \ldots, B_2$, respectively, both in the inequality and the graph $K_{1,m,m}$. Let us denote the resulting complete tripartite graph by $K'_{1,m,m}$. Then the inequality becomes

$$-(m-2)x_{ZA_1} - \sum_{2 \leq i \leq m} (m-i)x_{ZA_i} - (m-2)x_{ZB_{m+1}} - \sum_{2 \leq i \leq m} (j-2)x_{ZB_j} - \sum_{2 \leq i \leq m} x_{A_iB_i} + \sum_{1 \leq i < j \leq m+1} x_{A_iB_j} \leq 2. \quad (5)$$

It is easy to check that the inequality (5) is identical to the inequality $I(G, H)$ with $G = (V, E)$ and $H = (V, F)$, where $V = \{A_1, \ldots, A_m, B_2, \ldots, B_{m+1}\}$, $E = \{A_iB_j | 1 \leq i < j \leq m+1\}$, and $F = \{A_iB_i | 2 \leq i \leq m\}$. Therefore the $I_{mm22}$ inequality is a tight Bell inequality if and only if the inequality $I(G, H)$ is a facet of $\text{CUT}^2(K'_{1,m,m})$.

Note that we cannot use Theorem 6 directly to prove that $I(G, H)$ is a facet, since the graph $G$ does not satisfy the condition of Theorem 6. However, if we assume $m \geq 3$, the inequality $I(G, H)$ is the triangular elimination of another inequality $I(G', H')$, where $G'$ (resp. $H'$) is the graph obtained from $G$ (resp. $H$) by identifying node $B_2$ to $A_2$ and $A_m$ to $B_m$. The inequality $I(G', H')$ is proved to be a facet of $\text{CUT}^2_{m-1}$ by Theorem 6. Now, as was pointed out in [1] and [10], we can apply triangular elimination twice to the facet inequality $I(G', H')$ to obtain $I(G, H)$. The first application is done with $uu' = A_1A_2$, $v = B_2$ and $W = \{Z, A_3, \ldots, A_{m-1}, B_m, B_{m+1}\}$. The second application is done with $uu' = B_mB_{m+1}$, $v = A_m$ and $W = \{Z, B_2, \ldots, B_{m-1}\}$. Therefore, from Theorem 6 $I(G, H)$ is a facet of $\text{CUT}^2(K'_{1,m,m})$.

Since it is easy to check the cases $m = 1$ and 2, we obtain the following theorem.
Theorem 10  For any $m \geq 1$, the $I_{m^222}$ inequality is a tight Bell inequality.

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A  Proof of Claim \[7\]

**Proof of Claim \[7\]** First let \(1 \leq i \leq t - 1\). In these \(n\) sets, \(T_i^{(4)}\) is the only one that contains exactly one of \(i\) and \(i + k\). This means that the linear independence of the incident vectors of \(k + 1\) sets \(T_i^{(1)}, T_i^{(2)}\) and \(T_i^{(3)}\) \((i \neq t)\) implies the linear independence of all the \(n\) incident vectors.

Next let \(1 \leq i \leq k\), \(i \neq t, p, p', q, q'\). In these \(k + 1\) sets, \(T_i^{(3)}\) is the only one that contains \(i\). This means that the linear independence of the incident vectors of \(6\) sets \(T^{(1)}, T^{(2)}, T_p^{(3)}, T_{p'}^{(3)}, T_q^{(3)}, T_{q'}^{(3)}\) implies the linear independence of all the \(n\) incident vectors.

Finally, these \(6\) incident vectors are linearly independent since they form an \((n + 1) \times 6\) matrix containing \(6\) rows which form a nonsingular matrix:

\[
\begin{pmatrix}
(t) & 1 & 1 & 1 & 0 & 0 & 0 \\
(t + k) & 0 & 0 & 0 & 1 & 1 & 1 \\
(p) & 1 & 1 & 0 & 0 & 0 & 0 \\
(p') & 1 & 0 & 1 & 0 & 0 & 0 \\
(q) & 0 & 0 & 0 & 1 & 1 & 0 \\
(q') & 0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

\[\Box\]

B  Proof of Theorem \[9\]

First we note that from the proof of Proposition \[8\] some of the roots of \(I'(G, H, C)\) are characterized as follows.

**Proposition 11** A cut vector \(\delta(S)\) with \(S \subseteq V \cup \{n + 2\}\) is a root of \(I'(G, H, C)\) if one of the following conditions is satisfied.

(i) \(S\) does not contain node \(n + 2\), and \(\delta(S)\) is a root of \(I(G, H)\).

(ii) \(S\) contains node \(n + 2\) and exactly two out of four nodes of \(C\) (possibly along with other nodes), and \(\delta(S \setminus \{n + 2\})\) is a root of \(I(G, H)\).

(iii) \(S = V_{c_1} \cup V_{c_2} \cup V_{c_3} \cup \{n + 2\}\), where each \(V_{c_i} (i = 1, 2, 3)\) contains a node of \(C\).

By using this characterization, we prove Theorem \[9\] as follows.

**Proof of Theorem \[9\]** The \((n + 1, n + 2)\)-collapse of \(I'(G, H_{k,t}, C)\) is the inequality \(I(G, H_{k,t})\), and it is a facet of \(\text{CUT}_{n+1}\) by Theorem \[6\].

Let \(C = \{c_1 c_2, c_2 c_3, c_3 c_4, c_4 c_1\}\). Note that \(1 \leq c_1, c_2, c_3, c_4 \leq k\). For \(1 \leq \lambda, \mu \leq 4\), we denote by \(\lambda \oplus \mu\) the unique integer \(\nu\) such that \(1 \leq \nu \leq 4\) and \(\nu \equiv \lambda + \mu \pmod{4}\). Let \(V_i = \{i, i + k\}\) for \(1 \leq i \leq t\) and \(V_t = \{i\}\) for \(t + 1 \leq i \leq k\).

To use Lemma \[8\] we define \(n + 1\) subsets of \(V \cup \{n + 2\}\) as follows.

- We define \(T_1^{(1)} = V_{c_1} \cup V_{c_2} \cup \{n + 2\}, T_2^{(1)} = V_{c_1} \cup V_{c_3} \cup \{n + 2\}, T_3^{(1)} = V_{c_1} \cup V_{c_4} \cup \{n + 2\}, T_4^{(1)} = V_{c_2} \cup V_{c_3} \cup \{n + 2\}\) and \(T_5^{(1)} = V_{c_1} \cup V_{c_2} \cup V_{c_3} \cup \{n + 2\}\).

- For each \(1 \leq \lambda \leq 4\) such that \(1 \leq c_\lambda \leq t\), we define \(T_\lambda^{(2)} = V_{c_\lambda} \cup V_{c_\lambda k} \cup \{c_\lambda + k, n + 2\}\).

- For each \(t + 1 \leq i \leq k\) that is not incident to \(C\), by condition \[8\], there exists a free node \(c_\lambda\) of \(C\) such that \(e_{ic_\lambda}\) is incident to \(c_\lambda + k\). Then we define \(T_i^{(3)} = V_{c_\lambda} \cup \{c_\lambda, i, n + 2\}\).
Claim 12

The \( n+1 \) incident vectors of \( T^{(1)}_i, T^{(2)}_i, T^{(3)}_i, T^{(4)}_i \) and \( T^{(5)}_i \) are linearly independent.

Proof: The proof goes similarly to that of Claim 11.

For \( t + 1 \leq i \leq k \) such that node \( i \) is not incident to \( C \), \( T^{(3)}_i \) is the only set that includes \( i \). Therefore all we have to prove is linear independence of the incidence vectors of sets \( T^{(1)}_i, T^{(2)}_i, T^{(4)}_i \) and \( T^{(5)}_i \).

For \( 1 \leq i \leq t \) such that node \( i \) is not incident to \( C \), there are two possibilities. If nodes \( i \) and \( i + k \) are incident to exactly two out of four edges \( e_{ic} \) with \( c \in V_C \), then define \( \{ \lambda, \lambda', \mu, \mu' \} = \{1, 2, 3, 4 \} \) such that \( e_{i\lambda} \) and \( e_{i\mu} \) are incident to \( i \) and \( e_{i\lambda'} \) and \( e_{i\mu'} \) are incident to \( i + k \). In this case, we define \( T^{(4)}_i = \{i, n+2 \} \cup V_{C_{\lambda}} \cup V_{C_{\mu}} \) and \( T^{(5)}_i = \{i+k, n+2 \} \cup V_{C_{\lambda}} \cup V_{C_{\mu}} \).

If not, let \( u \) be either \( i \) or \( i + k \) that is incident to at most one of \( e_{ic} \) with \( c \in V_C \). By condition 11, there exists a free node \( c_{\lambda} \) such that \( e_{i\lambda} \) is incident to \( c_{\lambda} + k \). Then we define \( T^{(4)}_i = V_{c_{\lambda}} \cup \{c_{\lambda}, i, i+k, n+2 \} \) and \( T^{(5)}_i = V_{\mu} \cup V_{\mu+2} \cup \{u, n+2 \} \), where \( \mu \) is either 1 or 2 such that neither \( e_{i\mu} \) nor \( e_{i\mu+2} \) is incident to \( u \).

Each of these subsets contains \( n+2 \) but not \( n+1 \). By using Propositions 5 and 11 it is easy to check they are roots of \( I'(G, H_{k,t}, C) \).

Now we prove the following claim.

Claim 12

The \( n+1 \) incident vectors of \( T^{(1)}_i, T^{(2)}_i, T^{(3)}_i, T^{(4)}_i \) and \( T^{(5)}_i \) are linearly independent.

Proof: The proof goes similarly to that of Claim 11.

For \( t + 1 \leq i \leq k \) such that node \( i \) is not incident to \( C \), \( T^{(3)}_i \) is the only set that includes \( i \). Therefore all we have to prove is linear independence of the incidence vectors of sets \( T^{(1)}_i, T^{(2)}_i, T^{(4)}_i \) and \( T^{(5)}_i \).

For \( 1 \leq i \leq t \) such that node \( i \) is not incident to \( C \), there are two possibilities. If nodes \( i \) and \( i + k \) are incident to exactly two out of four edges \( e_{ci} \) with \( c \in V_C \), the sets \( T^{(4)}_i \) and \( T^{(5)}_i \) are the only ones that include \( i \) and \( i+k \), respectively. Otherwise, the set \( T^{(5)}_i \) is the only one that includes exactly one of two nodes \( i \) and \( i+k \), and \( T^{(4)}_i \) is the only one that includes the other node in \( i \) and \( i+k \). Therefore, linear independence of the incidence vectors of sets \( T^{(1)}_i \) and \( T^{(2)}_i \) will imply linear independence of all the \( n+1 \) incidence vectors.

Let \( 1 \leq \lambda \leq 4 \) such that \( 1 \leq c_{\lambda} \leq t \). Among the remaining sets \( T^{(1)}_i \) and \( T^{(2)}_i \), the set \( T^{(2)}_i \) is the only one that includes exactly one of \( c_{\lambda} \) and \( c_{\lambda} + k \).

Finally, the five incidence vectors of sets \( T^{(1)}_1, T^{(2)}_1, T^{(3)}_1, T^{(4)}_1 \) and \( T^{(5)}_1 \) are linearly independent since \( (n+2) \times 5 \) matrix formed by these five vectors contains five rows with the pattern:

\[
\begin{pmatrix}
(c_1) & 1 & 1 & 1 & 0 & 1 \\
(c_2) & 1 & 0 & 0 & 1 & 1 \\
(c_3) & 0 & 1 & 0 & 1 & 1 \\
(c_4) & 0 & 0 & 1 & 0 & 0 \\
(n+2) & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

which is nonsingular. \( \square \)

By Claim 12 and Lemma 3, \( I'(G, H_{k,t}, C) \) is a facet of \( \text{CUT}_{n+2}^\square \). \( \square \)