Propagators for p-forms in $AdS_{2p+1}$ and correlation functions in the $AdS_7/(2,0)$ CFT correspondence

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Abstract

In $AdS_{2p+1}$ we construct propagators for p-forms whose lagrangians contain terms of the form $A \wedge dA$. In particular we explore the case of forms satisfying “self duality in odd dimensions”, and the case of forms with a topological mass term. We point out that the “complete” set of maximally symmetric bitensors previously used in all the other propagator papers is incomplete - there exists another bitensor which can and does appear in the formulas for the propagators in this particular case. Nevertheless, its presence does not affect the other propagators computed so far.

On the $AdS$ side of the correspondence we compute the 2 and 3 point functions involving the self-dual tensor of the maximal 7$d$ gauged supergravity (sugra), $S_{\mu\nu\rho}$. Since the 7 dimensional antisymmetric self-dual tensor obeys first order field equations ($S + *dS = 0$), to get a nonvanishing 2 point function we add a certain boundary term (to satisfy the variational principle on a manifold with boundary) to the 7$d$ action. The 3 point functions we compute are of the type $SSB$ and $SBB$, describing vertex interactions with the gauge fields $B_{\mu}$. 

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1 Introduction

During the last few years, the \(AdS - CFT\) correspondence has generated a lot of interest (see [1] for a review). However, most of the work focused on the \(AdS_5/\text{SYM}_4\) correspondence because of the interest in describing strongly coupled 4 dimensional field theories and because (non)perturbative results in these field theories were known.

An interesting case of the correspondence is the one between string theory on \(AdS_7 \times R^4\) and the mysterious 6 dimensional \((2,0)\) CFT. This theory is a fixed point of the renormalization group flow from the theory on \(N\) parallel M5 branes. Not too much is known about this CFT, and so there is no independent check of the \(AdS/CFT\) correspondence besides verifying conformal invariance and comparing the masses of the primaries with those obtained in the DLCQ description of this CFT [1].

There are however many interesting new features in this correspondence, one of which is the appearance in the 7 dimensional theory of a three-index antisymmetric tensor with a first order equation of motion (of “self-duality in odd dimensions” type - \(\ast dA = mA\)). Finding its propagator and the corresponding CFT two-point function presents some new challenges. The reason for studying it is twofold.

First, by analyzing correlators of the 3-form field we learn about CFT correlators of the self-dual 3-form in 6d. The \((2,0)\) CFT is hard to describe mostly because we don’t understand how to make a self-dual 3-form nonabelian. The study of these 3-forms via the \(AdS - CFT\) correspondence may be a step towards that. Second, two of us argued in a previous paper [2] that one needs to take a nonlinear ansatz for the embedding of AdS fields into the higher-dimensional theory (in this case M theory) for the \(AdS/CFT\) correspondence. If one were able to compute correlators independently on the CFT side and compare them with the AdS results obtained here, this would give further evidence for this statement.

We start by analyzing the propagator of \(p\)-forms in \(2p + 1\) dimensions, and for generality we look both at the case of self-duality in odd dimensions and at the propagator for the Maxwell action with a topological mass term, both of which are relevant in 7 dimensions. We extend the basis of maximally symmetric bitensors (first introduced by Allen and Jacobson [3]) with a bitensor constructed from contractions of the \(\epsilon\) symbol with derivatives of the AdS chordal distance. We express the propagator ansatz in this basis, and use the equations of motion to compute it. We also present an extra possible term in the bitensor ansatz for the propagator of \(p\)-forms in \(2p\) dimensions.

We then use the \(AdS_7\) propagator for the “self-dual” 3-form to extract information about the 6 dimensional \((2,0)\) CFT. The bulk to boundary propagator is obtained straightforwardly as a limit of the bulk to bulk propagator. Nevertheless, since the 3-form action is first order in derivatives, the quadratic action vanishes on-shell. In order to use the \(AdS - CFT\) correspondence, one has to supplement the action with a boundary term which will generate the correct CFT 2-point function. This procedure was first introduced for spin 1/2 fields [4, 5], used in a 3-point function calculation for interacting spinors-scalars by [6], and was later justified by Arutyunov and Frolov [7] and Henneaux [8] by enforcing the variational principle on a manifold with boundary. We also compute the 3-point functions of two 3-forms and a gauge field, and of two gauge fields and a 3-form. For their calculation we follow the conformal methods of
Freedman et al. The results are given in (3.41) and (3.42).

The paper is organized as follows. In section 2.1 we discuss the propagator for the case of “self-duality in odd dimensions”; in section 2.2 we discuss the propagator for \( p \)-forms with a topological mass term, while in the last section dedicated to propagators, 2.3, we investigate the effect of the extra bitensors on the other propagators computed so far. In section 3.1 we study the 2-point function of the 3-form field and in section 3.2 the 3-point functions of two gauge fields, and a 3-form and two 3-forms and a gauge field. We finish with conclusions in section 4. We give some useful identities involving the chordal distance in Appendix A.1. In Appendix A.2 we derive the limits we need when computing the 2-point function, while in Appendix A.3 we included some integrals used for the 3-point functions.

2 Propagators

In the recent years a lot of papers [10, 11, 12, 13, 14, 15, 16] have been written computing propagators of various fields in \( AdS_{d+1} \). In the cases of tensor propagators, the standard procedure for computing a propagator is to express it using a basis for maximally symmetric bitensors (first introduced by Allen and Jacobson [3]), and to use the equation of motion. Typically one obtains a system of equations which can be solved in a straightforward way.

Nevertheless, if we try to apply this procedure to a Lagrangian with a topological mass term, we run into trouble. The equations obtained by using the Allen-Jacobson (A-J) basis do not make any sense. We can also see easily that the propagator for a vector with a topological mass term in 3 flat dimensions (which can be computed in a straightforward fashion), contains a term which clearly cannot be expressed in terms of the A-J basis.

What is lacking in the A-J basis is a term which contains contractions of the \( \epsilon \) tensor with derivatives of the chordal distance. These contractions can only give a bitensor for \( d = 2p \) and for \( d + 1 = 2p \). This is consistent with the fact that we can only write a topological mass term for \( p \) forms in these dimensions (as \( m A \wedge dA \) or as \( m^2 A \wedge A \)).

For \( d = 2p \) there are 2 types of equations of motion for \( p \)-forms which we can try to solve. The first one (also known as “self duality in odd dimensions” [17]) is:

\[
\frac{m}{p!} \epsilon_{\mu_1 \mu_2 \ldots \mu_p} \mu_{p+1} \ldots \mu_{d+1} \partial_{\mu_{p+1}} A_{\mu_{p+2} \ldots \mu_{d+1}} = m^2 A_{\mu_1 \ldots \mu_p} - J_{\mu_1 \ldots \mu_p}
\] (2.1)

where \( J_{\mu_1 \ldots \mu_p} \) is a covariantly conserved current which couples to the \( p \)-form.

The second one, comes from adding a topological mass term in a Maxwell action, and it describes gauge invariant forms:

\[
\frac{1}{p!} D^\lambda D_{\lambda} A_{\mu_1 \ldots \mu_p} = m \epsilon_{\mu_1 \mu_2 \ldots \mu_p} \mu_{p+1} \ldots \mu_{d+1} D_{\mu_{p+1}} A_{\mu_{p+2} \ldots \mu_{d+1}} - J_{\mu_1 \ldots \mu_p}
\] (2.2)

For \( d + 1 = 2p \) it is possible to add to a Lagrangian, besides the normal mass term \( m^2 A^2 \), a term of the form \( \tilde{m}^2 A \wedge A \). While this is an interesting possibility, it does
not seem to arise in physical situations (like compactifications of supergravity), and so we will not explore it completely here. Nevertheless we will comment in section 2.3 on the effect of such a term on the propagator. The complete investigation should be straightforward with the methods we have.

2.1 Self duality in odd dimensions

In Euclidean \( AdS_{d+1} \) with the metric

\[
d s^2 = \frac{1}{z_0^2}(d z_0^2 + \Sigma_{i=1}^d d z_i^2) \tag{2.3}
\]

equation of motion. invariant functions and tensors are most easily expressed in terms of the chordal distance

\[
u \equiv \frac{(z_0 - w_0)^2 + (z_i - w_i)^2}{2z_0 w_0} \tag{2.4}
\]

and derivatives thereof.

As explained in \[17\], when \( p \) is odd, (2.1) can be interpreted as the square root of the equation of motion of a real massive form field, in Minkowski signature.

When \( p \) is even, in order for (2.1) to be interpreted as the square root of an equation of motion with positive mass, a factor of \( i \) has to be added to its left hand side. Equation (2.1) is now complex, and describes a complex field. As explained in \[17\], this description is equivalent to that of a real field satisfying the massive Proca equation, and therefore redundant. We will discuss however at the end of this section the propagator for this case.

Since we use Euclidean \( AdS \), we need to analytically continue the equation of motion. Since we want the \( \epsilon \) symbol not to change, we have to change the equation of motion, multiplying by \( i \) wherever \( \epsilon \) appears. The new equations describe a complex field, but they should describe a real field when continued back into Minkowski space.

Thus, the equation satisfied by the propagator is:

\[
im \epsilon \mu_1 \mu_2 \ldots \mu_{p+1} \mu_{p+2} \ldots \mu_{d+1} D_{\mu_{p+1}} G_{\mu_{p+2} \ldots \mu_{d+1}; \mu_1' \ldots \mu_p'} = m^2 G_{\mu_1 \ldots \mu_p; \mu_1' \ldots \mu_p'} - \delta(z, w)(g_{\mu_1 \mu_1'} g_{\mu_2 \mu_2'} \ldots g_{\mu_p \mu_p'}) \tag{2.5}
\]

where the square brackets in the source term denote antisymmetrization of unprimed indices, and \( m \) is a dimensionless parameter.

Since the propagator is a maximally symmetric bitensor, conventional wisdom is to express it in terms of 2 antisymmetric bitensors:

\[
T_{\mu_1 \ldots \mu_p; \mu_1' \ldots \mu_p'}^1 = \partial_{\mu_1} \partial_{\mu_1'} u \ldots \partial_{\mu_p} \partial_{\mu_p'} u + \text{antipermutations of primed indices} \tag{2.6}
\]

\[
T_{\mu_1 \ldots \mu_p; \mu_1' \ldots \mu_p'}^2 = \frac{1}{(p - 1)!} \left( \partial_{\mu_1} u \partial_{\mu_1'} u \partial_{\mu_2} \partial_{\mu_2'} u \ldots \partial_{\mu_p} \partial_{\mu_p'} u + \text{antipermutations of all indices} \right) \tag{2.7}
\]
Nevertheless, if we try to plug a combination of these two bitensors into (2.5) we obtain nonsense. We realized however that $G$ could contain another $p-p$ bitensor, which can only exist for $d = 2p$:

$$ T^{\mu_1...\mu_p;\mu'_1...\mu'_p} = \epsilon_{\mu_1\mu_2...\mu_p}^{\mu_{p+1}...\mu_{d+1}} \partial_{\mu_{p+1}} \partial_{\mu'_1} u ... \partial_{\mu_d} \partial_{\mu'_p} u \partial_{\mu_{d+1}} u $$ (2.8)

One may ask if there is yet another bitensor, obtained from $T^3$ by switching primed with unprimed indices. However, at a closer investigation this bitensor turns out to be proportional to $T^3$.

Thus, the propagator can be expanded as:

$$ G_{\mu_1...\mu_p;\mu'_1...\mu'_p}(u) = T^1[G(u) + pH(u)] + T^2 H'(u) + T^3 K(u) $$ (2.9)

where the splitting of the coefficient of $T^1$ has been made knowing in advance that $pH(u)T^1 + H'(u)T^2$ can be expressed as an antisymmetrized covariant derivative acting on a $(p-1,p)$ bitensor, and thus it drops out when the kinetic operator is applied on it. For $z \neq w$, the Euclidean continuation of the equation for the propagator can be expressed as $dG = -imG$. We are using $G$ as both shorthand notation for the propagator bitensor, and as a scalar function of $u$. We can easily work out the actions of $d$ and $*$ on the terms of the propagator, using the formulas in the Appendix. Thus:

$$ *dT^1 G = (-1)^p T^3 G' $$ (2.10)

$$ d(T^1 pH + T^2 H') = 0 $$ (2.11)

$$ *d(T^3 K) = T^1 [u(2 + u)K' + (d - p + 1)(1 + u)K] - T^2 [(1 + u)K' + (d - p + 1)K] $$ (2.12)

The equation for the propagator implies:

$$ (-1)^p G' = (-im)K $$ (2.13)

$$ u(2 + u)K' + (d - p + 1)(1 + u)K = (-im)(G + pH) $$ (2.14)

$$ -K'(1 + u) - (d - p + 1)K = (-im)H' $$ (2.15)

Equation (2.14) also contains a source term coming from the $\delta$ function term in (2.5). Combining (2.13) and (2.14) we obtain a relation between $G$ and $H$, which we can integrate once (fixing the integration constant so that both go to 0 as $u \to \infty$) to give:

$$ (-1)^p m^2 H = G'(1 + u) + (d - p)G $$ (2.16)

From (2.16), (2.13) and (2.14) we can obtain:

$$ u(u + 2)G'' + (d + 1)(1 + u)G' + [p(d - p) + (-1)^p m^2]G = 0 $$ (2.17)

We can observe that for the cases when “self duality in odd dimensions” holds (odd $p$), $G$ and $H$ obey the same equations as in the case of the massive form propagator $\cite{14, 15}$. This is not so surprising; after all, our equation of motion was the square root of the equation of motion for massive forms. Moreover, the source term in (2.17)
We note that for cases of interest to the AdS/CFT correspondence the hypergeometric function simplifies to 1, since \( m = 1 \) for \( AdS_5 \) and \( m = 2 \) for \( AdS_7 \).

We are expanding

\[
G = \frac{(-1)^p \Gamma \left( \frac{d-1}{2} \right)}{4 \pi^{\frac{d+1}{2}} u^{\frac{d}{2}}} \left( 2F_1 \left( m + 1/2, m + 1 - p, 2m + 1, \frac{2}{2+u} \right) \right)
\]

where \( C \) is a constant that normalizes the second fraction to 1 as \( u \to 0 \) (for the curious, \( C = 2^m \Gamma(m+1) \Gamma(p - 1/2)/[\sqrt{2} \pi \Gamma(m+p)] \). \( H \) and \( K \) are given by \((2.16)\), and respectively \((2.13)\). It is interesting to notice that \( K \) satisfies the equation \((2.17)\) for \( p+1 \) forms in \( 2p+3 \), so it is - like \( G \) - a hypergeometric function. Nevertheless, its normalization is given by \((2.13)\).

When \( p \) is even, the extra \( i \) in \((2.1)\) makes the equation real in Euclidean coordinates. The factor \((-im)\) from \((2.13,2.14,2.15)\) becomes \( m \), which has the effect of changing \((-1)^p m^2 \) to \((-1)^{p+1} m^2 \) in equations \((2.16)\) and \((2.17)\). Since \( p \) is now even, the last term of \((2.17)\) is the same as before, and \((2.18)\) is unchanged.

### 2.2 Forms with a topological mass term

Unlike the previous case, when there was no gauge invariance, equation \((2.2)\) describes gauge invariant forms. As explained in \((2.2)\), working in the subspace of covariantly conserved currents makes gauge fixing unnecessary. The equation for the propagator is obtained from \((2.2)\), remembering that Euclidean continuation introduces an \( i \) multiplying the \( \epsilon \) tensors:

\[
\frac{1}{p!} D^\lambda D\left[ \lambda \right] G_{\mu_1...\mu_p;\mu'_1...\mu'_p} - ime^{\mu_1\mu_2...\mu_{d+1}} D_{\mu_{p+1}} G_{\mu_{p+2}...\mu_{d+1};\mu'_1...\mu'_p} =
\]

\[
= -\delta(z,w) \left( g_{\mu_1\mu'_1} g_{\mu_2\mu'_2} ... g_{\mu_p\mu'_p} \right) + \frac{1}{(p-1)!} D|_{\mu'_1} S_{\mu_1...\mu_p;\mu'_2...\mu'_p} \]

where the last term is a diffeomorphism whose contribution vanishes when integrated against a covariantly conserved source. Since \( S \) can be expressed as:

\[
S_{\mu_1...\mu_p;\mu'_2...\mu'_p} = S(u) \left[ \partial_{\mu_1} \partial_{\mu_2} \partial_{\mu'_2} u ... \partial_{\mu_p} \partial_{\mu'_p} u \right]
\]

the last term can be written as \( T^1 pS + T^2 S' \). For \( z = w \), equation \((2.19)\) can be written in a more compact notation as:

\[
(-1)^p * d * dG - im * dG = T^1 pS + T^2 S'
\]

We are expanding \( G \) as in \((2.9)\), remembering that the terms containing \( H \) can be written as the total divergence, and thus they are gauge artifacts. Using the equations \((2.10, 2.11, 2.12)\) we obtain after a few straightforward steps:

\[
u(u+2)K'' + (d+3)(1+u)K' + (1+p)(d-p+1)K = (-1)^p (im) G' \]

\[
u(u+2)G'' + (d-p+1)(1+u)G' = im[u(u+2)K' + (d-p+1)(1+u)K] + pS
\]

\[
(1+u)G'' + (d-p+1)G' = im[(1+u)K' + (d-p+1)K] - S'
\]
As before, equation (2.23) contains a source term coming from the right hand side of (2.19). In order to find \( G \) and \( K \) we have to do some manipulations on the system (2.22, 2.23, 2.24). We first define \( F \), such that \( F' \equiv K \). Having done this we can integrate (2.22) and (2.24) once, setting the integration constants so that everything goes to 0 as \( u \to \infty \). We obtain

\[
\begin{align*}
u(u+2)F'' + (d+1)(1+u)F' + p(d-p)F &= (-1)^p (im)G \\
(1+u)G' + (d-p)G &= im[(1+u)F' + (d-p)F] - S
\end{align*}
\]

We combine (2.23) with (2.26), and obtain:

\[
u(u+2)G'' + (d+1)(1+u)G' + p(d-p)G = im[u(u+2)F'' + (d+1)(1+u)F' + p(d-p)F]
\]

which by using (2.25) gives

\[
u(u+2)G'' + (d+1)(1+u)G' + [p(d-p) + (-1)^p m^2]G = 0
\]

Both (2.27) and (2.28) contain a source term at the right hand side. For odd \( p \), (2.28) is the massive form propagator equation, and \( G \) is given by (2.18). We will discuss the even \( p \) situation later. Equation (2.27) implies that \( G - imF \) satisfies the massless propagator equation, with the same source term, so \( F \) will be proportional to the difference between the massive and the massless propagator. This makes sense; as \( m \to 0 \) we expect the propagator to approach the massless one, which does not contain \( K \). Thus:

\[
K = \frac{-i}{m}(G' - G'_{m=0})
\]

where \( G_{m=0} \) is given by setting \( m = 0 \) in (2.18).

As promised before, we now investigate the situation of even \( p \). Naively, our theory can be defined for any \( p \). However, if we square the equation satisfied by the field, \((-1)^p \ast d \ast dA = im \ast dA\) and call \( B \equiv \ast dA \), we obtain \((-1)^p \ast d \ast dB = (-1)^{p+1}m^2B\) which describes a field of positive mass only in odd dimensions. One way to remedy this is to modify (2.19) by making the mass complex \((m \to im)\). This equation now describes a complex Minkowski field of positive mass.

### 2.3 Odds and ends

There are two questions which we have not yet addressed. The first one has to do with the effect of the extra term in the bitensor ansatz on the other propagators computed so far. We can investigate the massive propagator, from which the massless one can be obtained as a limit. We can plug the ansatz (2.9) into the equation for the propagator:

\[
(-1)^p \ast d \ast dG = m^2G
\]

and obtain the equations for \( G \) and \( H \), in the normal fashion. The equation for \( K \) is decoupled from the equations for \( G \) and \( H \). \( K \) satisfies the same equation as \( G \), but \emph{without} a source term. Therefore \( K \) is zero. In general, when the equation of motion
contains an even number of *s (or ϵ symbols) the equation for \( K \) decouples from the other equations, and \( K \) vanishes because of the lack of a source term.

The second issue to address is the case \( d + 1 = 2p \). Adding to the massive form Lagrangian a term of the form \( m^2 A \wedge A \) is legitimate. The equation of motion becomes:

\[
(-1)^p \ast d \ast dA = m^2 A + \tilde{m}^2 \ast A
\]

and the new ansatz for the propagator is:

\[
G_{\mu_1 \ldots \mu_p;\mu'_1 \ldots \mu'_p}(u) = T^1[G(u) + pH(u)] + T^2 H'(u) + T^4 K(u)
\]

where

\[
T^4_{\mu_1 \ldots \mu_p;\mu'_1 \ldots \mu'_p} = \epsilon_{\mu_1 \mu_2 \ldots \mu_p}^{\mu_{p+1} \ldots \mu_{d+4}} \partial_{\mu_{p+1}} \partial_{\mu'_1} u \ldots \partial_{\mu_{d+4}} \partial_{\mu'_p} u
\]

For \( \tilde{m} \neq 0 \) we obtain a set of coupled differential equations involving \( G, H \) and \( K \). We do not solve this case here, since the presence of an \( \tilde{m} \) term is more a logical possibility than something which arises naturally in a physical theory.

Also, this extra term has no effect on the other propagators computed so far. When \( \tilde{m} = 0 \) the equation for \( K \) decouples from the equations for \( G \) and \( H \), and thus \( K = 0 \), in absence of a source term.

### 3 Correlators

We begin this section dedicated to computing correlators on \( AdS_7 \) involving the self-dual 3-form \( S_{\mu
\nu\rho} \) with a brief review of the correspondence between the fields of supergravity and the \( CFT \) operators. The fields of maximally 7d gauged \( (SO(5)_g) \) sugra couple on the boundary (via the \( AdS - CFT \) correspondence) to the operators of the 6d \((2,0)\) \( CFT \). It is well known that the 6d \((2,0)\) \( CFT \) with gauge group \( SU(N) \) has no known lagrangian formulation, but the abelian version corresponds to the tensor multiplet of \((2,0)\) supersymmetry. The \( CFT \) operators (characterized by their Lorentz \( (Spin(6)) \) and by their R-symmetry \( Sp(2)_R \sim SO(5) \) quantum numbers) are built out of the primary gauged invariant operators: \( \phi^A \) (a vector under the R-symmetry group), \( \psi \) (a spinor under the R-symmetry), \( H_{ijk} \) (a singlet under the R-symmetry). These operators, which are all in the adjoint representation of \( SU(N) \) transform under supersymmetry as a \((2,0)\) tensor multiplet \([13]\). Below we list the supergravity fields and the \( CFT \) operators to which they couple:

- the scalars \( \Pi^A \) in the coset \( SL(5,R)/SO(5)_g \) (with the group \( SL(5,R) \) broken after gauging to \( SO(5)_c \)) and in the 14 of \( SO(5)_g \rightarrow tr(\phi^A) \)
- the spin 1/2 fields \( \lambda_i \) in the 16 of \( SO(5)_g \rightarrow tr(\psi \phi) \)
- the self-dual 3-form \( S^A_{\mu \nu \rho} \) in the 5 of \( SO(5)_g \rightarrow tr(H_{ijk} \phi^A) \equiv O^A_{ijk} \)
- the gauge fields \( B^A_{\mu \nu} \) in the adjoint 10 of \( SO(5)_g \) couple to the R-current \( \rightarrow tr(\psi \gamma_i \psi) \equiv J_i \)
- the four gravitini \( \psi_\mu \) in the spinor 4 representation of \( SO(5)_g \rightarrow tr(\gamma^k \psi H_{ijk}) \)
- the graviton \( g_{\mu \nu} \), which is singlet under \( SO(5)_g \), couples to the stress-energy tensor \( \rightarrow tr(H_{mjk} H_{njk}) + \ldots \)


3.1 2-point function

In the first section we derived the propagator for a self-dual form in odd dimensions. In particular, in 7 dimensions we have:

\[ G(u) = c \cdot u^{-5/2}(2 + u)^{-m-1/2} F_1(m + 1/2, m - 2; 2m + 1; 2/(2 + u)) \]  

(3.1)

with \( c \) a constant. From the bulk propagator one can derive easily the bulk-to-boundary propagator:

\[ S_{\mu_1\mu_2\mu_3}(w^0, w) = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} d^6z \sqrt{h(z)} G_{\mu_1\mu_2\mu_3;\mu'_1\mu'_2\mu'_3}(w, z)(z^0)^{-m} s_{\mu'_1\mu'_2\mu'_3}(z) \]  

(3.2)

where \( d^6z \sqrt{h(z)} \) is the invariant volume element on the hypersurface \( M_{\epsilon} = \{ z^0 = \epsilon \} \) and \( G(w, z) \) is the bulk-to-bulk propagator. The field \( S_{\mu_1\mu_2\mu_3}(w) \) as defined by (3.2) satisfies its field equation for any finite \( s_{\mu_1\mu_2\mu_3}(z) \). Moreover, the only nonvanishing components of the \( s(z)_{\mu_1\mu_2\mu_3} \) are the ones with indices "on the boundary" \( \{ i, j, \ldots \} \). In the limit \( \epsilon \to 0 \) we have \( u \to \infty \), \( G(u) \to c u^{-m-3} \), \( H(u) \to -(c/m) u^{-m-3} \) and \( K(u) \to (i(m + 3)c/m) u^{-m-4} \). So,

\[ S_{ijk}(w) = \lim_{\epsilon \to 0} \int_{M_{\epsilon}} d^6z G_{ijkl;ij'kl'}(w, z)s_{ij'kl'}(z) \]  

(3.3)

where the metric "on the boundary" (used here for raising and lowering the primed indices) is flat, \( g_{ij'}(z) = \eta_{ij'} \), while the bulk metric is AdS, \( g_{\mu_1\mu_2}(z) = (1/z^0)^2 \eta_{\mu_1\mu_2} \) and \( G(w, z) \) is the bulk-to-boundary propagator. Substituting the propagator expression into (3.3), and taking into account the various limits we get:

\[
S_{ijk}(w) = c \int d^6z \left[ \frac{m + 3}{m} s^{3 + m} \frac{w^0}{[\epsilon^2 + (z - w)^2]^{3 + m}} s_{ijk}(z) \right. \\
+ \frac{i(3 + m)}{2m} \frac{w^0}{w^0 + (z - w)^2} \frac{w^0}{m} \epsilon_{ijkl'jl'} s_{ijkl'}(z) \\
- \frac{i(3 + m)}{m} \frac{w^0}{w^0 + (z - w)^2} \frac{w^0}{m} \epsilon_{ijkl'jl'} s_{ijkl'}(z) \\
- \frac{18m}{m} \frac{(w - z)_i}{w^0 + (z - w)^2} \frac{w^0}{w^0 + (z - w)^2} \frac{w^0}{m} s_{ijkl}(z) \\
+ \frac{3i(3 + m)}{m} \frac{(w - z)_i}{w^0 + (z - w)^2} \frac{w^0}{w^0 + (z - w)^2} \frac{w^0}{m} \epsilon_{ijkl'jl'} s_{ijkl'}(z) \right]
\]  

(3.4)

Use now the Schouten identity to rewrite the last term as

\[
\epsilon_{ijkl'jl'}(w - z)_i(w - z)_l = \epsilon_{ijkl'jl'}(w - z)_i(w - z)_l \\
+ \frac{1}{3} \epsilon_{ijkl'jl'}(w - z)^2
\]  

(3.5)
and by rearranging the terms of (3.4) we finally get

\[
S_{ijk}(w) = 6c \int d^6z \left( \frac{3 + m}{m} \frac{2^{3+m}w^0m}{[w^0 + (z - w)^2]^{m+3}} \right) \\
\left[ (s_{ijk} + \frac{i}{3!} \epsilon_{ijk'j'k'}s_{ij'j'k'}) - 6 \frac{(w - z)_i(w - z)_j'}{[w^0 + (z - w)^2]} \left( s_{ij'k} + \frac{i}{3!} \epsilon_{ij'k}l_{mn}s_{l_{mn}} \right) \right]
\]

(3.6)

The value which \( S_{ijk}(w) \) takes on the boundary is obtained from (3.6) with \( w^0 \rightarrow 0 \). Using further the limits \( [\ref{A.11}][\ref{A.14}] \) we get the on-shell boundary value of the 7d self-dual tensor \( S_{ijk}(w) \):

\[
\lim_{\epsilon \rightarrow 0} \epsilon^m S_{ijk}(\epsilon, w) = 6c \pi^3 m^{m+3} \frac{m(m + 1)(m + 2)}{m(m + 1)(m + 2)} \left( s_{ijk}(w) + \frac{i}{3!} \epsilon_{ijk'j'k'}s_{ij'j'k'}(w) \right)
\]

(3.7)

We are not surprised to see that on the boundary \( S_{ijk}(w) \) is self-dual (after all, we knew that \( S_{ijk}(w) \) is source for a self-dual CFT operator). It is interesting to note however, that even if we didn’t impose any restriction on \( s_{ijk}(z) \), the form of the bulk-to-bulk propagator determined that only the self-dual part of \( s_{ijk}(z) \) propagates in the bulk. Alternatively, this can be seen in the derivation of the bulk-to-boundary propagator in the footnote on page 11, as a constraint which relates the anti-self-dual part to the self-dual part.

The other on-shell components of \( S_{\mu_1\mu_2\nu_3}(w) \), namely \( S_{0ij}(w) \), can be determined similarly, and we obtain the following bulk-to-boundary propagator:

\[
G_{\mu_\nu\rho\mu_3\nu_3}(w; z)^A:B = c_1 \left[ \frac{w^0}{(w - z)^2} \right]^m \partial_{\mu_\nu} \left( \frac{w - z)_\nu}{[(w - z)^2]^3} \delta_{l_{mn}} \delta_B \right)
\]

(3.8)

where the Kronecker delta symbols are antisymmetrized with strength one, and the indices \( l_{mn} \) are in the self-dual representation of the Lorentz group \( SO(6) \). We normalize the bulk-to-boundary propagator such that for \( w^0 \rightarrow 0 \), \( S_{ijk}(w)^m \rightarrow s_{ijk}(w) \), with \( s_{ijk}(w) = \frac{1}{2} (s_{ijk}(w) + \frac{i}{3!} \epsilon_{ijk'j'k'}s_{ij'j'k'}(w)) \) self-dual. Thus \( c_1 = (m + 1)(m + 2) / 2 \). Furthermore, using a notation which was first introduced in a paper by Freedman et al. [4], we can rewrite the bulk-to-boundary propagator in a manifestly conformal-covariant form:

\[
G_{\mu_\nu\rho\mu_3\nu_3}(w; z)^A:B = c_1 \left[ \frac{w^0}{(w - z)^2} \right]^m \frac{J_{\mu\nu}(w - z)}{(w - z)^2} \frac{J_{\mu\nu}(w - z)}{(w - z)^2} \delta_B
\]

(3.9)

where the symmetry in the indices \( \mu_\nu \) and \( l_{mn} \) is the same on both sides. The tensor \( J_{\mu\nu}(w) \) is related to the inversion \( w^\mu \rightarrow (1/w^2)w^\mu \) Jacobian

\[
\frac{\partial w^\mu}{\partial w^\nu} = w^2 J_{\mu\nu}(w')
\]

(3.10)

Clearly, for a field obeying a first order differential equation, the bulk action is vanishing on-shell. This is a situation which was first encountered for spin 1/2 fields [5][6]. The resolution which was proposed for obtaining non-zero 2-point functions (in
accord with the CFT calculations) was to supplement the bulk action with a boundary term which does not break the $AdS_{d+1}$ isometry group $O(d + 1, 1)$. This proposal was justified afterwards \[\text{[7, 8]}\] by imposing the variational principle on a manifold with boundary. On the boundary, we need to specify only the self-dual piece of $S^+_{ijk}$, and so, the anti-self-dual part is free to vary off-shell. Therefore, in order to cancel the remaining boundary term $-m/2 \oint S^+ \delta S^-$ in the variation of the action, we will add the following boundary term to the 7d gauged sugra action:

$$S_1 = \frac{m}{4} \lim_{\epsilon \to 0} \int_{M_\epsilon} d^6 z \sqrt{h(z)} S_{\mu_1 \mu_2 \mu_3}(z) S^{\mu_1 \mu_2 \mu_3}(z) \tag{3.11}$$

where, as before, $d^6 z \sqrt{h(z)}$ is the invariant volume element on the hypersurface $M_\epsilon = \{z^0 = \epsilon\}$ infinitesimally close to the boundary of the $AdS$ space.

On-shell, $S_1$ becomes:

$$S_1 = \frac{m}{4} c_1 \lim_{\epsilon \to 0} \int_{M_\epsilon} d^6 z S_{ijk}(z) \epsilon^m \int d^6 w \frac{1}{(\epsilon^2 + |w|^2)^{m+3}} \left( s^+_{ijk}(w) - 6 \frac{(w-z)_i(w-z)_j}{|w|^2} s^+_i(w) \right) \tag{3.12}$$

where we used that $\lim_{\epsilon \to 0} \epsilon^m S_{\mu_1 \mu_2 \mu_3}(z)$ is nonvanishing only for the $S_{ijk}(z)$ components. The limit was already evaluated in \[\text{[7, 8]}\], and so we get:

$$S_1 = \frac{\pi^3 \epsilon^7 m}{4(m+1)(m+2)(m+3)} \int d^6 z \int d^6 w s^+_{ijk}(z) \frac{1}{((z-w)^2)^{m+3}} \left( s^+_{ijk}(w) - 6 \frac{(w-z)_i(w-z)_j}{|w|^2} s^+_i(w) \right) \tag{3.13}$$

\footnote{We can understand the result \[\text{[8, 9]}\] from a different perspective. The (linearized) field equation in momentum space

$$[(x^0)^2 \partial_0^2 - x^0 \partial_0 - m(m \pm 2) - (x^0)^2 k^2] S^+_{ijk}(x^0, k) = 0$$

has a unique solution (which falls off at infinity) written in terms of the modified Bessel function

$$S^+_{ijk}(x^0, k) = \frac{x^0 K_{m+1}(x^0 k)}{\epsilon K_{m+1}(\epsilon k)} s^+_{ijk}(k)$$

However, as observed in \[\text{[20]}\], there is a constraint (obtained from another field equation) which relates the self-dual to the anti-self-dual boundary values. In 7d, the constraint yields

$$s^-_{ijk}(k) = -\frac{K_{m-1}(\epsilon k)}{K_{m+1}(\epsilon k)} \left( s^+_{ijk}(k) - \frac{6 k_i k_j k^k s^+_{kjk}(k)}{k^2} \right)$$

Note that $s^-_{ijk}$ vanishes if $s^+_{ijk}$ is kept finite for $\epsilon \to 0$.

We finally get that on shell

$$S_{ijk}(x^0, k) \propto k^m \left( m K_m(x^0 k) s^+_{ijk}(k) + 3 x^0 k_i k_j k^k s^+_{jik}(k) K_{m-1}(x^0 k) \right)$$

which is nothing else but the Fourier transform of \[\text{[8, 9]}\].}
The generating functional of the CFT operators is equal, by the $AdS - CFT$ correspondence, to the supergravity partition function which in the classical limit is

$$Z_{AdS^7}[s_{ijk}] = \exp(-S_1 - S_{7d\text{ sugra}}) \quad \text{(3.14)}$$

Thus, the 2-point function of the self-dual CFT operator $O_{ijk}^A = \text{tr}(\phi^A H_{ijk})$ is

$$\langle O_{ijk}(z) O_{lmn}(w) \rangle = \frac{\pi^3 c^2 m}{2(m+1)(m+2)(m+3)} \frac{1}{\delta^B \delta^C} \left( \frac{\delta^B_{lmn} - 6 (\frac{w-z)^i}{(w-z)^2} \delta^i_{lmn}}{(z-w)^2} \right) \quad \text{(3.15)}$$

where the symmetry in the indices $ijk$ respectively $lmn$ is the same on both sides of (3.15). This is the 2-point function for a CFT operator with scaling dimension

$$\Delta = p + m = 3 + m \quad \text{(3.16)}$$

in the self-dual tensorial representation of the (Euclidean) Lorentz group $O(6)$. In general for a CFT operator with scaling dimension $\Delta$ and in some arbitrary representation of the Lorentz group, conformal invariance fixes the 2-point function to be [3, 19]

$$\langle O(z) O(w) \rangle = \frac{1}{((w-z)^2)^\Delta} R(w, z) \quad \text{(3.17)}$$

where $R(w, z)$ is the representation matrix of the $O(d)$ element

$$R^i_j(w, z) = \delta^i_j - \frac{2}{(w-z)^2} (w-z)^i (w-z)_j \quad \text{(3.18)}$$

### 3.2 3-point functions: $\langle O_{ijk} O_{lmn} J_p \rangle$ and $\langle O_{ijk} J_m J_n \rangle$

For this section, the relevant part of the action of maximal ($N = 4$) gauged 7d sugra is:

$$\int d^7 w \left[ \frac{im}{48} \epsilon_{\mu_1 \ldots \mu_7} S^A_{\mu_1 \mu_2 \mu_3} F^A_{\mu_4 \ldots \mu_7} + \frac{1}{16\sqrt{3}} \epsilon_{\mu_1 \ldots \mu_7} \epsilon_{ABCDEF} S^A_{\mu_1 \mu_2 \mu_3} F^{BC}_{\mu_4 \mu_5} F^{DE}_{\mu_6 \mu_7} + \ldots \right] \quad \text{(3.19)}$$

where $F^A_{\mu_1 \ldots \mu_7} = 4(D_{[\mu_1} S^A_{\mu_2 \mu_3 \mu_4]} + g B_{[\mu_1}^A S^B_{\mu_2 \mu_3 \mu_4]})$ is the gauge covariant field strength of $S^A_{\mu_1 \mu_2 \mu_3}$, and $F^{AB}_{\mu_1 \mu_2}$ is the gauge covariant field strength of the gauge fields $B^{AB}_{\mu_1 \mu_2}$. The coupling constant is $g$ and it equals $2m$.

To compute the 3-point functions, inspired by [3], we use a conformal-covariant bulk-to-boundary propagator for the gauge fields

$$G_{\mu_i}^{AB;CD}(w, z) = c_2 \left( \frac{w_0}{(w-z)^2} \right) 4 J_{\mu_i}(w-z) \frac{\delta_{AB}}{(w-z)^2} \quad \text{δ}_{CD} \quad \text{(3.20)}$$

---

3 Euclidean signature: $\epsilon \rightarrow i \epsilon$
Finally, we will combine the factors of $J$ example) to be -1. Substituting the various propagators into (3.21), and jumping the into det $J$ $(1 \cdots w$ dependence reads:

$$
\langle O_{ijk}(z_1)O_{lmn}(z_2)J^C_D(z_3) \rangle = \frac{igm c^2 c_2}{12} \int d^7 w \epsilon^{\mu_1 \cdots \mu_7} G_{\mu_1 \mu_2 \mu_3 \cdots \mu_7}(w; z_1) G^{EF,CD}(w; z_3) G^{FB}_{\mu_5 \mu_6 \mu_7; lmn}(w; z_2) + \text{perm.} (ijk \leftrightarrow lmn, z_1 \leftrightarrow z_2, A \leftrightarrow B) \quad (3.21)
$$

Following [9] we will use conformal invariance to fix $z_3 = 0$, and we will make a change of variable by inverting the bulk and the boundary $z_1, z_2$ points: $w^\mu \to (1/w^2) w^\mu$, $z_1 \to (1/z_1^2) z_2$, $z_2 \to (1/z_2^2) z_3$. The inversion property which the tensors $J_{\mu \nu}$ satisfy will also be used:

$$
J_{\mu \nu}(w - z) = J_{\mu \rho}(w') J_{\rho \nu}(w' - z') J_{ji}(z') \quad (3.22)
$$

Finally, we will combine the factors of $J_{\mu_1 i}(w')$, ... with the “flat space” epsilon symbol into det $[J(w')]_{\epsilon_{ijkl}}$. The determinant can be easily evaluated (by induction, for example) to be -1. Substituting the various propagators into (3.21), and jumping the intermediary steps described above, the integral in the 3-point function becomes:

$$
-\frac{igm c^2 c_2}{12} \int d^7 w' \delta^{AB} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} (w')^{2m+4} \frac{z'_1^{2(m+3)}}{(w' - z'_1)^2} \frac{z'_2^{2(m+3)}}{(w' - z'_2)^2} 
$$

$$
J_{\nu_1 \nu_2}(w' - z'_1) J_{\nu_2 \nu_3}(w' - z'_2) J_{\nu_3 \nu_4}(w' - z'_1) \frac{J_{\nu_4 \nu_5}(w' - z'_1)}{J_{\nu_5 \nu_6}(w' - z'_2) J_{\nu_6 \nu_7}(w' - z'_2) J_{\nu_7 \nu_8}(w' - z'_2) + \text{perm.}} \quad (3.23)
$$

where the factors of $w'$ canceled as expected. Furthermore, we shift the integration variable such that all dependence on $z'_1$ and $z'_2$ appears through $z'_1 - z'_2$, and we use the integrals listed in Appendix 3. In the last step we restore the $z_3$ dependence using the translational invariance of the 3-point function: $z'_1 \to z'_1 \equiv (z_1 - z_3)$, $z'_1 - z'_2 \to t \equiv (z_1 - z_3) - (z_2 - z_3)$, etc. Thus the 3-point function $\langle J^C_D(z_3)O_{ijk}(z_1)O_{lmn}(z_2) \rangle$ is

$$
-\frac{igm c^2 c_2}{12} \frac{\pi^3}{10(m+3)^2(m+2)(m+1)} \delta^{AB} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5} (z_1 z_2 z_3)^{-2(m+3)}
$$

$$
J_{\nu_1}(z_1^3) J_{\nu_2}(z_1^3) J_{\nu_3}(z_1^3) J_{\nu_4}(z_2^3) J_{\nu_5}(z_2^3) J_{\nu_6}(z_2^3) t^{2m} \left( (\delta_{\nu' \nu} t_{\nu} + \delta_{\nu' \nu} t_{\nu}) 2m - 6 (\delta_{\nu' \nu} t_{\nu} + \delta_{\nu' \nu} t_{\nu}) \right) - \frac{t_{ij} t_{ji} t_{ij}}{t^2} (2m+2) \quad (3.24)
$$

This expression can be further simplified by using the self-duality in the indices $(ijk)$, $(llmn)$, and whenever the case, by rearranging indices with the Schouten identity. We obtain:

$$
-\frac{gm c^2 c_2}{2} \frac{\pi^3}{10(m+3)^2(m+2)(m+1)} \delta^{AB} (z_1 z_2 z_3)^{-2(m+3)}
$$

$$
J_{\nu_1}(z_1^3) J_{\nu_2}(z_1^3) J_{\nu_3}(z_1^3) J_{\nu_4}(z_2^3) J_{\nu_5}(z_2^3) t^{2m-2} \left( (J_{\nu_1}(z_1^3) J_{\nu_2}(z_2^3) t_{\nu} + J_{\nu_1}(z_1^3) J_{\nu_2}(z_2^3) t_{\nu}) (2m-2
$$

13
\[-(2m + 2) + \left( \frac{t_1 t_2 t_2}{\ell^2} J_{ir}(z_{13})J_{ir}(z_{23}) \right. \\
+ \left. \frac{t_1 t_2 t_2}{\ell^2} J_{ir}(z_{13})J_{ir}(z_{23}) \right) (2m + 2) \right]

(3.25)

Let’s now turn to the 3-point function \( \langle O^A_{ijk}(z_1)J^{BC}_m(z_2)J^{DE}_m(z_3) \rangle \). The AdS correlator is given by

\[
\langle O^A_{ijk}(z_1)J^{BC}_m(z_2)J^{DE}_m(z_3) \rangle = \frac{1}{4\sqrt{3}} c_1 c_2^2 \epsilon_{FGHIJ} \int d^{d+1} \epsilon_{\alpha \beta \gamma \delta \epsilon \eta \zeta} G^F_{\alpha \beta \gamma ; i j k} (w; z_1) \\
\partial_\delta G_{\epsilon ; l}^{GH; BC} (w; z_2) \partial_\eta G_{\zeta ; m}^{IJ; DE} (w; z_2) \\
+ \text{perm.} (BC \leftrightarrow DE, l \leftrightarrow m, z_2 \leftrightarrow z_3) \tag{3.26}
\]

In a manner entirely analogous to the previous calculation, we use conformal invariance to fix \( z_1 = 0 \), and then invert bulk and boundary points. We use the inversion relation (3.22) and the fact that the curl of the gauge field propagator also transforms covariantly, according to

\[
\partial_\mu G_{\nu \alpha \beta}(w, z) = w^\mu J_{\nu \rho}(w') w^\rho J_{\beta \sigma}(w') J_{\alpha \beta}(z') z^{2(d-1)} \partial_\rho G_{\sigma \beta}(w', z') \tag{3.27}
\]

and again combine all the resulting factors of \( J_{\mu \nu}(w') \) together with the \( \epsilon \) symbol into a determinant which gives \(-1\). The factors of \( w' \) cancel again, and we are left with the 3-point function

\[
-\frac{1}{4\sqrt{3}} c_1 c_2^2 \epsilon_{ABCD} \epsilon_{EFGHIJ} \frac{1}{(z_2^2)^{(d-1)}(z_3^2)^{(d-1)}} J_{\nu \nu}(z_2) J_{\mu \mu}(z_3) \\
\int d^{d+1} w e^{ijkl \epsilon \eta \zeta} \eta^m \partial_\nu \eta^m \left( \frac{w^d - 1}{w - z_2^2} J_{\nu \nu}(w', z_2') \right) \\
\partial_\nu^l \left( \frac{w^d - 1}{w - z_3^2} J_{\nu \nu}(w', z_3') \right) + \text{perm.} \tag{3.28}
\]

We then rewrite the summation over \( \delta' \epsilon' \eta' \zeta' \) as summation over \( l', m', n', \) and 0 and notice that we can replace the partial derivatives \( \partial_\nu^l \) and \( \partial_\nu^l \), w.r.t. \( w' \) with partial derivatives w.r.t. \( z_2 \) and \( z_3 \) respectively, if \( \delta' \) and \( \eta' \) are not zero. In the case neither is zero, both derivatives get out of the integral and then the antisymmetry kills them (the integral is a function of \( z_2 - z_3 \), so the derivatives are the same). Thus only the case when one of the derivatives is w.r.t. \( w' \) remains, and we get

\[
-\frac{1}{4\sqrt{3}} c_1 c_2^2 \epsilon_{ABCD} \epsilon_{EFGHIJ} \frac{1}{(z_2^2)^{(d-1)}(z_3^2)^{(d-1)}} J_{\nu \nu}(z_2) J_{\mu \mu}(z_3) \\
\epsilon^{ijkl \mu \nu \rho \sigma} \frac{\partial}{\partial y_{\rho \sigma}} \int d^{d+1} w' \left( \frac{w^m + 2d - 3}{w' - z_2^2} \right)^{2(d-1)} J_{\nu \nu}(w', z_2) \\
\left( (d - 2) w_0^{m + 2d - 3} J_{\mu \nu}(w', z_3) - \frac{2(d - 1)}{(w' - z_3^2)^2} J_{\mu \nu}(w', z_3) \right) \\
+ 4 \frac{(w' - z_3^2) m'}{(w' - z_3^2)^4} (3.29)
\]
Further using the identity

\[
\frac{(u' - z')^i (u' - z')^j}{(u' - z')^{2a}} = \frac{1}{4(a-1)(a-2)} \frac{\partial}{\partial z'^i} \frac{\partial}{\partial z'^j} (u' - z')^{2(a-2)} + \frac{1}{2(a-1)} \frac{\delta_{ij}}{(u' - z')^{2(a-1)}}
\]

(3.30)

and the fact that, by the same antisymmetry argument as above, all extra partial derivatives w.r.t. \( z_2' \) and \( z_3' \) vanish, we get

\[
\begin{align*}
\epsilon^{ijkl} & \frac{\partial}{\partial w^m} t^{m-d} \frac{d-2}{d-1} \left[ C(m + 2d - 5, d - 1, d - 1) \right] \\
-2C(m + 2d - 3, d - 1, d - 1)
\end{align*}
\]

(3.31)

where the constant \( C(a, b, c) \) is defined in (A.16). In the last step we restore the \( z_1 \) dependence in \( \langle OJJ \rangle \).

\[
\langle O^A_{ijk}(z_1) J^B_{lm'}(z_2) J^D_{m'}(z_3) \rangle =
\frac{1}{4\sqrt{3}} c_2^2 \frac{\pi^{d/2}}{2} \frac{m(d-2)^2}{\Gamma(\frac{d}{2})} \frac{1}{\Gamma(\frac{d+2}{2})} \frac{1}{\Gamma(\frac{d+m-2}{2})} \epsilon^{ijkl} \epsilon^{m'n'n'}
\]

(3.32)

where \( t = (z_2 - z_1)/(z_2 - z_1)^2 - (z_3 - z_1)/(z_3 - z_1)^2 \).

However, our 3-point function computation was too naive, since we didn’t carefully impose that the sugra fields satisfy their field equations. By doing so, we will discover that the boundary term (3.11) will give contributions to both correlators \( \langle OJJ \rangle \) and \( \langle OJJ \rangle \).

To be able to appreciate this point, we will look at a simple example of a \( \lambda \phi^3 \) scalar field theory, (understanding that this will apply to the case of general fields)

\[
S = \int d^{d+1}x \left[ \frac{1}{2} \left( \partial_\mu \phi \right)^2 + m^2 \phi^2 \right] + \frac{1}{3} \lambda \phi^3
\]

(3.33)

with equation of motion \((-\Box + m^2)\phi = \lambda \phi^2\). The solution which takes the boundary value \( \phi_0(y) \) is of the type

\[
\phi(x) = \phi^{(0)}(x) + \phi^{(1)}(x) + ...
\]

\[
\phi^{(0)}(x) = \int d^4y G(x, y) \phi_0(y)
\]

(3.34)

where \( G(x, y) \) is the bulk-to-boundary propagator and \( \phi^{(1)} \) is zero on the boundary. In the \( AdS - CFT \) correspondence, the 3-point function will a priori have a contribution not only from the cubic term, \( \int \lambda \phi^{(0)} \phi \phi \), but also from the kinetic piece, because \( \phi^{(1)}(x) = \lambda \int d^{d+1}y G(x, y) \phi^{(0)}(y)^2 \) \( G(x, y) \) is the bulk-to-bulk propagator. If we take into account the fact that \( \phi^{(0)} \) satisfies the free equation of motion the bulk
piece in the kinetic term vanishes. Moreover, the boundary term generated by partial integrations vanishes because $\phi^{(1)}$ is zero on the boundary. Hence the only contribution to the 3-point function comes from the cubic term in the action. Since we didn’t need the explicit form of $\phi^{(1)}$ for this argument, the same argument goes through for 3-point couplings to other fields (still no contribution from the $\phi$ kinetic term).

Let us carry the same analysis when the kinetic action is linear in derivatives. All the terms in the action quadratic in $\phi$ (including the interactions with other fields) don’t contribute to the 3-point functions if we use the complete equation of motion (unlike the previous case, there is no need for partial integration, so no boundary terms). In fact now we can use the complete equation of motion to replace the kinetic terms with interaction terms. So the net effect of the kinetic piece will be to modify the coefficients of the interaction terms. It would appear that in this case, all 3-point functions quadratic in $\phi$ vanish! This is not so in general, since to make the 2-point function nonzero we need to add a (pseudo)boundary term to the action. There are two ways that this term can contribute. Either it is of the type $\int d^{d+1}x \partial_{\mu}(\partial^\nu \phi \cdots)$ (real boundary term, but with an extra derivative on $\phi$, so that we get a nonzero contribution if we replace $\phi$ by $\phi^{(1)}$, or it is a pseudo-boundary term of the type $\int d^{d+1}\delta(z_0 - \epsilon)(\cdots)$, defined on the “regulated” boundary $z_0 = \epsilon$. The latter happens in our case.

Consider the part of 7d gauged sugra action containing only the self-dual 3-form and the gauge fields:

$$\int d^7z \sqrt{G^{-1}} \left( -\frac{1}{4}(\mathcal{F}^{A\mu
u})^2 + \frac{m^2}{2}(\mathcal{S}^{A\mu\nu\rho})^2 + \frac{im}{48\sqrt{G^{-1}}} \epsilon_{\mu\nu\rho\sigma\tau\lambda\xi} \mathcal{S}^{A\mu\nu\rho\sigma\tau\lambda\xi} \mathcal{S}^{A\mu\nu\rho\sigma\tau\lambda\xi} + \frac{3}{4m} \epsilon_{A\mu \nu \rho \sigma \tau \lambda} \mathcal{S}^{A\mu \nu \rho \sigma \tau \lambda} \right) + \frac{m}{4} \int_{M_6} d^6z \sqrt{h(z)}(\mathcal{S}^{A\mu\nu\rho})^2 \tag{3.35}$$

Imposing that $S_{\mu\nu\rho}$ satisfies its field equation (to linear order in the coupling constant, and to second order in fields, since we are interested in 3-point functions) we obtain

$$S_{\mu\nu\rho} = S^{(0)}_{\mu\nu\rho} + S^{(1)}_{\mu\nu\rho} \tag{3.36}$$

where $S^{(0)}$ satisfies the linearized field equation, while $S^{(1)}$ is given by

$$S^{(1)}_{\mu\nu\rho}(z) = -\int d^7w G^{(0)}_{\mu\nu\rho\sigma\tau\lambda\xi}(z, w) \epsilon^{\mu\nu\rho\sigma\tau\lambda\xi} \frac{im}{6} B^{(0)AB}_{\mu\nu}(w) S^{(0)B\mu\nu\rho\sigma\tau\lambda\xi}(w) - \int d^7w \epsilon^{\mu\nu\rho\sigma\tau\lambda\xi} G^{(0)}_{\mu\nu\rho\sigma\tau\lambda\xi} (z, w) \frac{1}{16\sqrt{3}} \epsilon^{A\mu \nu \rho \sigma \tau \lambda \xi} \mathcal{F}^{(0)A\mu \nu \rho \sigma \tau \lambda \xi}(w) \mathcal{F}^{(0)BC\mu \nu \rho \sigma \tau \lambda \xi}(w) \mathcal{F}^{(0)DE\mu \nu \rho \sigma \tau \lambda \xi}(w) \tag{3.37}$$

Substituting (3.37) into (3.35) we see that the SBB bulk term gets modifies by a factor $1/2$ and becomes

$$\int d^7w \frac{1}{32\sqrt{3}} \epsilon^{A\mu \nu \rho \sigma \tau \lambda \xi} S^{(0)A\mu \nu \rho \sigma \tau \lambda \xi} F^{(0)BC\mu \nu \rho \sigma \tau \lambda \xi} F^{(0)DE\mu \nu \rho \sigma \tau \lambda \xi} \tag{3.38}$$

whereas the SSB term gets killed, as we mentioned in the general discussion. The kinetic term for the $B$’s doesn’t contribute to the 3-point functions, because it has two
derivatives, and the previous discussion applies. Finally, the \( S \) boundary term yields a contribution to the 3-point functions from its \( S^{(0)}S^{(1)} \) piece
\[
\frac{m}{2} \int_{\mathcal{M}_t} d^6 z \sqrt{h(z)} S^{(0)}_{\mu \nu \rho} S^{(1) \mu \nu \rho} \tag{3.39}
\]
which becomes a bulk integral after substituting (3.37)
\[
-\frac{m}{2} \int d^7 z e^{\mu_1 + \mu_7} S^{(0)}_{\mu_1 \mu_2 \mu_3} (z) \left( \frac{img}{6} B^{(0)A}_{\mu_4} (z) S^{(0)B}_{\mu_5 \mu_6 \mu_7} (z) \right) + \frac{1}{16 \sqrt{3}} \epsilon^{ABCDE} F^{(0)BC}_{\mu_4 \mu_5} (z) F^{(0)DE}_{\mu_6 \mu_7} (z) \tag{3.40}
\]
We note here that it is crucial this was a pseudo-boundary term (defined at \( z_0 = \varepsilon \)). If it was a real boundary term, one could have used that it doesn’t contribute. Adding (3.38) to (3.40) we get the corrected coefficients of the 3-point functions on the AdS side. Thus the (corrected) 3-point functions are:
\[
\langle J_p^{CD} (z_3) O_{ijk}^A (z_1) O_{lmn}^B (z_2) \rangle =
2gm^2 c_1 c_2 \pi^3 10(m + 3)^2(m + 2)(m + 1) \delta^{AB}_{CD} (z_{13} z_{23})^{-2(m+3)}
\]
\[
J_{ij'} (z_{13}) J_{kk'} (z_{13}) J_{m'j'} (z_{23}) J_{nk'} (z_{23}) p^{2m-2}
\]
\[
- J_{ip} (z_{13}) J_{ii'} (z_{23}) t_{i1} + J_{ii'} (z_{13}) J_{ip} (z_{23}) t_{i1} + (m + 1) \frac{t_{i1} t_{i1} t_{p1}}{t^2} J_{ii'} (z_{13}) J_{ip} (z_{23}) \tag{3.41}
\]
\[
\langle O_{ijk}^A (z_1) J_{l}^{BC} (z_2) J_{m}^{DE} (z_3) \rangle =
- \frac{c_1 c_2}{\sqrt{3}} (-m + 1) \pi^3 \delta^{m+8} \Gamma \left( \frac{m + 8}{2} \right) \Gamma \left( \frac{4 - m}{2} \right) \Gamma \left( \frac{4 + m}{2} \right)
\]
\[
\epsilon^{ABCDE} \left( \frac{(z_2 - z_1)^2}{(z_3 - z_1)^2} \right)^{(d-1)} \left( \frac{(z_3 - z_1)^2}{(z_3 - z_1)^2} \right)^{(d-1)} J_{ii'} (z_2 - z_1) J_{mm'} (z_3 - z_1)
\]
\[
\epsilon^{ijklm'n''} \frac{t^{m''}}{t^{m-d+2}} \tag{3.42}
\]
where in (3.41) we defined \( t = z'_{13} - z'_{23} \), while in (3.42) we defined \( t = z'_{21} - z'_{31} \).

Although we have worked with a generic \( m \), the 7d gauged sugra action determines it exactly in units of \( AdS \) radius. In the sugra action, the purely gravitational piece is \(-1/2 \int (R + \frac{3}{2} m^2) \), whereas for unit \( AdS_7 \) the gravitational contribution is \( \int (R + 30) \). This implies \( m = 2 \).

Finally, there is one more check on our correlators. The \( < J O O > \) correlator must obey the Ward identity
\[
\frac{\partial}{\partial z_3^{\mu}} \langle J_p^{CD} (z_3) O_{ijk}^A (z_1) O_{lmn}^B (z_2) \rangle = g \delta^{CD}_{AE} \delta (z_{13}^\mu (z_1) O_{ijk}^E (z_1) O_{lmn}^B (z_2)) + g \delta^{CD}_{BE} \delta (z_{23}^\mu (z_1) O_{ijk}^A (z_1) O_{lmn}^E (z_2)) \tag{3.43}
\]
Hence, the coefficients of \( < J O O > \) and \( < O O > \) correlators are interdependent. To simplify our calculations, we will verify the Ward identity for \( z_1 = 0, z_2 \equiv y \) at infinity...
and \( z_3 \equiv x \) at finite distance. Substituting the 3-point function (3.41) into the l.h.s. of (3.43) we get (for \( m = 2 \))

\[
\frac{\partial}{\partial x^p} \langle J^{CD}_{p}(x) \mathcal{O}^A_{ijk}(0) \mathcal{O}^B_{lmn}(y) \rangle = 4gmc_1^2c_2 \frac{\pi^3}{10(m+3)(m+2)(m+1)} \delta_{CD}^{AB}
\]

Further using that

\[
\frac{\partial}{\partial x^p} \left( \frac{x^p x_i x_p}{x^8} \right) = \frac{\pi^3}{6} \delta(x) \delta_{i1}
\]

and by comparing with the r.h.s. of (3.43) we confirm the identity.

### 4 Conclusions

In this paper we have studied \( p \)-forms in \( AdS_{2p+1} \). We have constructed the propagators for \( p \)-forms with topological mass terms and topological kinetic terms in \( AdS_{2p+1} \) (the latter being the case of self-dual forms in odd dimensions). We have also found that the basis for maximally symmetric bitensors previously found by Allen and Jacobson is not complete is some situations, and found the missing bitensors.

For the \( AdS-CFT \) correspondence, we have analyzed the case of \( AdS_7 - 6d (2,0) \) CFT (similar computations go through for \( AdS_5 - 4d SYM \)). We have computed the two point function of the “self-dual” 3-form by adding a boundary term to the 7d gauged sugra action. We have also computed 3-point functions of two 3-forms and a gauge field, and two gauge fields and a 3-form by using methods similar to the ones of Freedman et al. \[8\]. It is the hope of the authors that this calculation can shed some light on the 6d (2,0) CFT.

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### Appendix A

#### A.1 Conventions and useful identities involving the chordal distance

We used the symbols \(*\) and \(d\) acting on forms, or on the unprimed indices of propagators, with the following normalization:

\[
(*A)^{\mu_1 \ldots \mu_p} = \frac{1}{(d-p)!} e^{\mu_1 \mu_2 \ldots \mu_{d+1}} A_{\mu_{p+1} \ldots \mu_{d+1}}
\]

\[
(dA)_{\mu_1 \ldots \mu_{p+1}} = \frac{1}{p!} D_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \quad (A.1)
\]
where the square brackets denote antisymmetrization of all unprimed indices.

In the computations the following identities were useful:

\[
\partial_\mu \partial_{\nu'} u = \frac{-1}{z_0 w_0} [\delta_{\mu\nu'} + \frac{(z - w)_\mu \delta_{\nu'0}}{w_0} + \frac{(w - z)_\nu \delta_{\mu0}}{z_0} - u \delta_{\mu0} \delta_{\nu'0}] \tag{A.2}
\]

\[
\partial_\nu u = \frac{1}{z_0} \left( (z - w)_\mu / w_0 - u \delta_{\mu0} \right) \tag{A.3}
\]

\[
\partial_{\nu'} u = \frac{1}{w_0} \left[ (w - z)_\nu / z_0 - u \delta_{\nu'0} \right] \tag{A.4}
\]

\[
D_\mu \partial_\mu u = (d + 1)(u + 1) \tag{A.5}
\]

\[
\partial_\mu u \partial_\mu u = u(u + 2) \tag{A.6}
\]

\[
D_\mu \partial_{\nu} u = g_{\mu\nu}(u + 1) \tag{A.7}
\]

\[
(\partial^\mu u)(D_\mu \partial_{\nu'} u) = \partial_{\nu'} u \partial_{\nu'} u \tag{A.8}
\]

\[
(\partial^\mu u)(\partial_\mu \partial_{\nu'} u) = (u + 1) \partial_{\nu'} u \tag{A.9}
\]

\[
D_\mu \partial_{\nu} \partial_{\nu'} u = g_{\mu\nu} \partial_{\nu'} u \tag{A.10}
\]

### A.2 Limits

In this paragraph we present the arguments leading to the various limits taken in the body of this paper. First, we consider the limit

\[
\lim_{\epsilon \to 0} \epsilon^{2m} / (x^2 + \epsilon^2)^{3+m} = \frac{1}{m(m+1)(2+m)} \text{Area}(S_6) \delta^6(x) \tag{A.11}
\]

where \(\text{Area}(S_6)\) is the area of a 6d sphere of radius 1. To derive this limit, use a scaling argument to show that \(\int d^6x \epsilon^{2m} / (x^2 + \epsilon^2)^{3+m}\) (with positive integrand) is independent of \(\epsilon\). It is also obvious that as long as the denominator is nonzero, the limit is vanishing. Thus, the limit is proportional to a delta function. The proportionality constant is determined by evaluating the integral \(\int d^6x \epsilon^{2m} / (x^2 + \epsilon^2)^{3+m}\) in spherical coordinates.

The limit

\[
\lim_{\epsilon \to 0} \epsilon^{2m+2} / (x^2 + \epsilon^2)^{4+m} = \frac{1}{(m+1)(m+2)(m+3)} \text{Area}(S_6) \delta^6(x) \tag{A.12}
\]

follows from (A.11), by redefining \(m \to m + 1\).

Finally, we derive the limit

\[
\lim_{\epsilon \to 0} \epsilon^{2m} x^2 / (x^2 + \epsilon^2)^{4+m} = \frac{3}{m(m+1)(m+2)(m+3)} \text{Area}(S_6) \delta^6(x) \tag{A.13}
\]

For the proof of this limit we consider again the integral \(\int d^6x \epsilon^{2m} x^2 / (x^2 + \epsilon^2)^{4+m}\). By the same scaling argument, it is clear that the result is independent of \(\epsilon\). The limit is zero, as long as the denominator is non-zero. By adding and subtracting \(\epsilon^2\) in the limit numerator and by making use of (A.11, A.12), we determine the proportionality constant on the r.h.s. (A.13).
As a consequence of (A.13) we obtain the following result:

\[
\lim_{\varepsilon \to 0} \int d^6x \varepsilon^{2m} \frac{x^i x^j}{(x^2 + \varepsilon^2)^{4+m}} S_{jkl}(x + w)
\]

\[
= \lim_{\varepsilon \to 0} \int_{S_6} d\Omega \int_0^\infty dr \varepsilon^{2m} \frac{r^2}{(r^2 + \varepsilon^2)^{4+m}} n_i n_j S_{jkl}(x + w)
\]

\[
= \frac{6}{m(m + 1)(m + 2)(m + 3)} \int_{S_6} d\Omega \int_0^\infty dr \delta(r) n_i n_j S_{jkl}(x + w)
\]

\[
= \frac{1}{2m(m + 1)(m + 2)(m + 3)} \text{Area}(S_6) S_{ikl}(w)
\]

(A.14)

where \( n_i = x_i / |x| \).

### A.3 Integrals

In this paragraph we record the integrals needed in the computation of 3-point functions. We begin (for completeness) with the integral (9)

\[
I(a, b, c)(t) \equiv \int d^7w \frac{(w^0)^a}{(w - t)^{2b} w^2c}
\]

\[
= C(a, b, c)|t|^{7 + a - 2b - 2c}
\]

(A.15)

where the constant \( C(a, b, c) \) is:

\[
C(a, b, c) = \frac{\pi^3}{2} \frac{\Gamma\left(a + \frac{1}{2}\right) \Gamma(1 + a + b - c - 3)}{\Gamma(b) \Gamma(c) \Gamma\left(1 + \frac{3}{2} - b\right) \Gamma\left(1 + \frac{3}{2} - c\right) \Gamma\left(1 + \frac{3}{2} - b - c\right) \Gamma\left(7 + a - b - c\right)}
\]

(A.16)

Then, the following integrals can be obtained from (A.15) by differentiating with respect to the vector \( t \):

\[
J_i(a, b + 1, c) \equiv \int d^7w \frac{(w^0)^a w_i}{(w - t)^{2(b+1)} w^2c}
\]

\[
= \partial_i I(a, b, c) \frac{2b}{2b} + t_i I(a, b + 1, c)
\]

(A.17)

\[
J_{i_1i_2}(a, b + 2, c) \equiv \int d^7w \frac{(w^0)^a w_i w_{i_2}}{(w - t)^{2(b+2)} w^2c}
\]

\[
= \partial_i \partial_{i_2} I(a, b, c) \frac{2^2b(b+1)}{2^2b(b+1)} + t_{i_1} t_{i_2} I(a, b + 2, c) + \delta_{i_1i_2} I(a, b + 1, c) \frac{2}{2(b+1)}
\]

(A.18)

\[
J_{i_1i_2i_3}(a, b + 3, c) \equiv \int d^7w \frac{(w^0)^a w_i w_{i_2} w_{i_3}}{(w - t)^{2(b+3)} w^2c}
\]

\[
= \partial_i \partial_{i_2} \partial_{i_3} I(a, b, c) \frac{2^3b(b+1)(b+2)}{2^3b(b+1)(b+2)} + t_{i_1} t_{i_2} t_{i_3} I(a, b + 3, c)
\]

20
\[
\frac{(t_{i_1} t_{i_2} \partial_{i_3} + 2 \text{more}) I(a, b + 2, c)}{2(b+2)} + \frac{(\delta_{i_1 i_2} t_{i_3} + 2 \text{more}) I(a, b + 2, c)}{2(b+2)} + \frac{(t_{i_1} \partial_{i_3} + 2 \text{more}) I(a, b + 1, c)}{2^2(b + 1)(b + 2)} + \frac{(\delta_{i_1 i_2} \partial_{i_3} I(a, b + 1, c) + 2 \text{more})}{2^2(b + 1)(b + 2)}
\]

(A.19)

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