CONJUGACY DISTINGUISHED SUBGROUPS

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To the memory of Oleg V. Mel’nikov

Abstract. Let $C$ be a nonempty class of finite groups closed under taking subgroups, homomorphic images and extensions. A subgroup $H$ of an abstract residually $C$ group $R$ is said to be conjugacy $C$-distinguished if whenever $y \in R$, then $y$ has a conjugate in $H$ if and only if the same holds for the images of $y$ and $H$ in every quotient group $R/N \in C$ of $R$. We prove that in a group having a normal free subgroup $\Phi$ such that $R/\Phi$ is in $C$, every finitely generated subgroup is conjugacy $C$-distinguished. We also prove that finitely generated subgroups of limit groups, of Lyndon groups and certain one-relator groups are conjugacy distinguished ($C$ here is the class of all finite groups).

1. Introduction

In this note we are interested in the following property of a subgroup $H$ of an abstract group $R$: whenever $y \in R$, then $y$ has a conjugate in $H$ if and only if their images in every finite quotient of $R$ in a specified class. We begin by specifying which classes of finite groups will be of interest to us.

Let $C$ be an extension-closed variety of finite groups, that is, a nonempty collection of finite groups closed under taking subgroups, homomorphic images and extensions of groups in the collection: if $1 \to A \to B \to C \to 1$ is an exact sequence of groups such that $A,C \in C$, then $B \in C$. For example $C$ can be the collection of all finite groups or the collection of all finite solvable groups. If $R$ is a group then its pro-$C$ completion $R_\hat{C}$ is defined to be

$$R_\hat{C} = \lim_{\rightarrow \ N \in \mathbb{N}_C} R/N,$$

where $\mathbb{N}_C$ is the collection of all the normal subgroups $N$ of $R$ such that $R/N \in C$. Then $R_\hat{C}$ is pro-$C$ group, i.e., a compact, Hausdorff, totally disconnected topological group such that $R_\hat{C}/U \in C$ whenever $U$ is an open normal subgroup of $R_\hat{C}$. The pro-$C$ topology of $R$ is defined to be the topology on $R$ that makes it into a topological group so that $\mathbb{N}_C$ is a fundamental system of neighborhoods of the identity element 1 of $R$. This topology is Hausdorff if and only if the natural homomorphism $R \to R_\hat{C}$ is injective; if that is the case one says that $R$ is a residually $C$ group, and we think of $R$ as being embedded in $R_\hat{C}$: $R \leq R_\hat{C}$. Then if $X$ is a subset of $R$, we denote its topological closure in $R_\hat{C}$ by $X_\hat{C}$.

We shall use the standard notation for conjugacy: if $x$ and $r$ are elements of a group $R$, then $x^r = r^{-1}xr$; and we set $x^H = \{ x^r | r \in H \}$. If $H$ is a subgroup of $R$, $N_R(H) = \{ r \in R | rH = Hr \}$ denotes as usual its normalizer in $R$, and $C_R(H) = \{ r \in R | rh = hr, \forall h \in H \}$ its centralizer in $R$.

An abstract group $R$ is called conjugacy $C$-separable if for any pair of elements $x, y \in R$, these elements are conjugate in $R$ if and only if their images in every finite quotient of $R$ which is in $C$ are conjugate, or equivalently, if $x \neq y^r$ for every $r \in R$, then there exists some $N \triangleleft R$ with $R/N \in C$ such that $xN \neq x^rN$ for every $r \in R$. If $R$ is conjugacy $C$-separable then it is residually $C$. One easily checks that a residually $C$ group $R$ is conjugacy $C$-separable if and only if for any pair of elements $x, y \in R$, if $x$ and $y$ are conjugate in $R_\hat{C}$, then they are conjugate in $R$: if $x = y^\gamma$, for some $\gamma \in R_\hat{C}$, then there exists some $r \in R$ with $x = y^r$. If $C$ is the class of all finite groups, we simply write conjugacy separable, rather than conjugacy $C$-separable.

A subgroup $H$ of an abstract residually $C$ group $R$ is said to be conjugacy $C$-distinguished if whenever $y \in R$, then $y$ has a conjugate in $H$ if and only if the same holds for the images of $y$ and $H$ in every quotient...
group $R/N \in \mathcal{C}$ of $R$, or equivalently, $y^R \cap H = \emptyset$ if and only if $y^{R_C} \cap \hat{H} = \emptyset$. If $\mathcal{C}$ is the variety of all finite groups, we simply write conjugacy distinguished.

An abstract group $R$ is said to be free-by-$\mathcal{C}$ if it contains a normal free abstract subgroup $\Phi$ such that $R/\Phi \in \mathcal{C}$.

We prove the following results.

**Theorem A.** Let $R$ be a free-by-$\mathcal{C}$ abstract group and let $H$ be a finitely generated subgroup of $R$ which is closed in its pro-$\mathcal{C}$ topology. Then $H$ is conjugacy $\mathcal{C}$-distinguished.

**Theorem B.** Let $R = \langle a_1, \ldots, a_n \mid W^n \rangle$ be a one-relator group with $n > |W|$. Then every finitely generated subgroup $H$ of $R$ is conjugacy distinguished.

A group $G$ is called fully residually free if for any finite subset $X$ of $G$ there exists a homomorphism $G$ to a free group $F$ whose restriction to $X$ is injective. A finitely generated fully residually free group is called a limit group. Limit groups have been studied extensively over the last ten years and they played a crucial role in the solution of the Tarski problem.

**Theorem C.** Let $R$ be a limit group and $H$ a finitely generated subgroup of $R$. Then $H$ is conjugacy distinguished. In particular, a finitely generated subgroup of a surface group is conjugacy distinguished.

The special case of Theorem C for surface groups follows also from Theorem 1.4 of [2].

Studying equations in free groups Lyndon [13] introduced groups $F^{Z[i]}$ (later called Lyndon groups) and proved that these groups are fully residually free; hence a finitely generated subgroup of a Lyndon group is a limit group. Lyndon groups play a very important role in algebraic geometry over groups. Kharlampovich and Myasnikov [11] proved conversely that every limit group is embeddable into a Lyndon group.

**Theorem D.** Let $H$ be a finitely generated subgroup of a Lyndon group $F^{Z[i]}$, where $F$ is a free group of arbitrary rank. Then $H$ is conjugacy distinguished in $F^{Z[i]}$.

2. **Free-by-finite groups**

In this section we prove that every finitely generated subgroup of a free-by-$\mathcal{C}$ group which is closed in its pro-$\mathcal{C}$ topology is conjugacy $\mathcal{C}$-distinguished.

1. **Lemma** Let $R$ be a free-by-$\mathcal{C}$ abstract group endowed with its pro-$\mathcal{C}$ topology. Let $H$ be a finitely generated closed subgroup and let $U$ be an open normal subgroup of $R$. Then

$$U \cap \hat{H} = \hat{U} \cap \hat{H}.$$ 

**Proof.** Note that $\overline{U \cap H} = \overline{U H} = (U H) \cap \mathcal{C}$ ([23], Lemma 3.1.4) and $[R : U H] = [R : \hat{U} \hat{H}]$ (cf. [23], Proposition 3.2.2). So $[H U : U] = [\hat{U} \hat{H} : \hat{U}]$. Therefore $[H : H \cap U] = [\hat{H} : \hat{H} \cap \hat{U}]$. Since $H$ is closed and finitely generated, the pro-$\mathcal{C}$ topology of $H$ coincides with the topology induced from the pro-$\mathcal{C}$ topology of $R$ (this easily follows from [21], Corollary 3.3 (ii) ). Since $H \cap U$ is open in $H$, we can apply again Proposition 3.2.2 in [23] to get that $[\hat{H} : \hat{H} \cap \hat{U}] = [\hat{H} : \hat{H} \cap \hat{U}]$. Therefore, $[H : \hat{H} \cap \hat{U}] = [\hat{H} : \hat{H} \cap \hat{U}]$. Since $\hat{H} \cap \hat{U} \leq \hat{H} \cap \hat{U}$, we deduce that $\hat{H} \cap \hat{U} = \hat{H} \cap \hat{U}$. 

The lemma above is a special case of a more general result ([24], Proposition 2.3), where one does not require that $U$ be open. We include it here because this assumption (which is what we need in this paper) allows a much simpler proof.

The next two results sharpen Lemma 2.2 and Theorem 3.2 in [24], where they are proved only for finitely generated groups $R$.

2. **Lemma** Let $H \in \mathcal{C}$ be a group of prime order $p$. Let $R = \Phi \triangleright H$ be a semidirect product, where $\Phi$ is an abstract free group. Then there is a free factor $\Phi_1$ of $\Phi$ such that
(a) $N_R(H) = H \times \Phi_1$ and $N_{R_c}(H) = H \times (\Phi_1)_C$;  
and
(b) $C_\Phi(H) = \Phi_1$ and $C_{\Phi, c}(H) = (\Phi_1)_C$.

Consequently,
(a') $N_R(H) = N_{R_c}(H)$;
(b') $C_\Phi(H) = C_{\Phi, c}(H)$.

**Proof.** By a theorem of Dyer-Scott (cf. [5], Theorem 1) the group $R$ is a free product

$$R = \left( \bigstar_{i \in I} (C_i \times \Phi_i) \right) \ast L,$$

where $L$ and each $\Phi_i$ are free groups and the $C_i$ are groups of order $p$. Since every finite subgroup of $R$ of order $p$ is conjugate to one of the $C_i$ (cf. [15], Corollary 4.1.4), we may assume without loss of generality that $H = C_{i_1}$, for some fixed $i_1 \in I$. Then $R = (C_{i_1} \times \Phi_{i_1}) \ast R_1 = (H \times \Phi_1) \ast R_1$, where $\Phi_1 = \Phi_{i_1}$ and

$$R_1 = \left( \bigstar_{i \in I \setminus \{i_1\}} (C_i \times \Phi_i) \right) \ast L.$$

It follows that $N_R(H) = H \times \Phi_1$ (cf. [15], Corollary 4.1.5), and since $H$ is abelian, $C_R(H) = H \times \Phi_1$. Hence $\Phi_1 = N_R(H) = C_{\Phi, c}(H) \leq \Phi$. Now, $C_\Phi(H)$ is the subgroup of fixed points of $\Phi$ under the action of $H$, and so $C_\Phi(H) = \Phi_1$ is a free factor of $\Phi$ (cf. [5], Theorem 2). This implies that $(\Phi_1)_C = \overrightarrow{\Phi}_1$ (cf. [23], Corollary 3.1.6).

Finally observe that $R_1 = (H \times (\Phi_1)_C) \ast (R_1)_C \ast (R_1)$ (the free pro-$C$ product), so $N_{R_1}(H) = H \times (\Phi_1)_C$ (cf. [23], Theorem 9.1.12), and since $H$ is abelian, $C_{R_1}(H) = H \times (\Phi_1)_C$. Therefore, $N_{R_1}(H) = H \times \overrightarrow{\Phi}_1 \subseteq N_R(H)$, and $C_{R_1}(H) = H \times \overrightarrow{\Phi}_1 \subseteq C_R(H)$. Thus, $C_\Phi(H) = C_{R_1}(H) \cap \overrightarrow{\Phi}_1 = (H \times \overrightarrow{\Phi}_1) \cap \overrightarrow{\Phi}_1 = \overrightarrow{\Phi}_1 = \overrightarrow{C_R(H)}$. This concludes the proof of all parts of the lemma. 

3. **Theorem** A free-by-$C$ group $R$ is conjugacy-$C$-separable.

**Proof.** To fix the notation, say that $\Phi \trianglelefteq R$, where $\Phi$ is an abstract free group such that $R/\Phi \in C$. Then we may think of $\Phi_\mathcal{C}$ as an open subgroup of $R_\mathcal{C}$ (cf. [23], Lemma 3.1.4 (a)). Let $x, y \in R$ and let $x^\gamma = y$, where $\gamma \in R_\mathcal{C}$. We have to show that $x$ and $y$ are conjugate in $R$. We may assume that $x \neq 1$. Since $R_\mathcal{C} = R_\mathcal{C} \setminus \Phi_\mathcal{C}$, we have $\gamma = r\eta$, for some $r \in R$, $\eta \in \Phi_\mathcal{C}$. So replacing $x$ by $x^r$ and $\gamma$ by $\eta$, we may assume that $\gamma$ is in $\Phi_\mathcal{C}$. Then $\gamma = \langle x \rangle \Phi_\mathcal{C} \subseteq R = \langle x \rangle \Phi_\mathcal{C} \ast R = \langle x \rangle \Phi_\mathcal{C} \ast R = \langle x \rangle \Phi$. Hence from now on we may also assume that $R = \langle x \rangle \Phi_\mathcal{C}$. Since $R_\mathcal{C} / \Phi_\mathcal{C}$ is abelian, we have $x^{r-1}x^\gamma = x^\gamma \Phi_\mathcal{C}$, i.e., $x^\gamma = x^\gamma \Phi_\mathcal{C} \ast \Phi_\mathcal{C}$. On the other hand we have that the natural map $\rho : R/\Phi \to R_\mathcal{C}/\Phi_\mathcal{C}$ is a bijection. Since $\rho(y\Phi) = y\Phi \gamma = x^\gamma \Phi_\mathcal{C} = x^\gamma \Phi_\mathcal{C} \ast \Phi_\mathcal{C} = \rho(x\Phi)$, we deduce that $y\Phi = x\Phi$. So from now on we assume that

$$R = \langle x \rangle \Phi_\mathcal{C}, \quad y = x^\gamma \in R, \quad \text{with } \gamma \in \Phi_\mathcal{C}, \text{ and } y\Phi = x\Phi. \quad (1)$$

Now we distinguish two cases.

**Case 1.** The order of $x$ is infinite. Let $n$ be a positive integer such that $x^n \in \Phi$. So $y^n \in \Phi$ and $y^n = (x^n)^\gamma$. According to a result of Baumslag and Taylor (cf. [14], Proposition 4.8), free groups are conjugacy-$C$-separable. We deduce that $y^n$ and $x^n$ are conjugate in $\Phi$. Say $f^{-1}x^n f = y^n$, where $f \in \Phi$. Replacing $x$ with $xf^{-1}$, we may assume that $y^n = x^n$. Therefore $\gamma \in C_{\Phi, c}(x^n) = C_{\Phi, c}(x^n) \in \Phi_\mathcal{C}$. Write $\Phi = \Phi_1 \ast \Phi_2$, where $\Phi_1$ is a free subgroup of $\Phi$ of finite rank such that $x^n \in \Phi_1$. Then $\Phi_\mathcal{C} = (\Phi_1)_C \ast (\Phi_2)_C = \overrightarrow{\Phi}_1 \ast \overrightarrow{\Phi}_2$, the free pro-$C$ product (here we use Corollary 3.1.6 in [23]). Note that $C_\Phi(x^n) = C_{\Phi_1}(x^n)$ and $C_{\Phi_2}(x^n) = C_{\Phi_2}(x^n)$ (cf. [23], Theorem 9.1.12). Since $\Phi_1$ has finite rank we can use Corollary 2.8 in [24] to get that $C_{\Phi_1}(x^n) = C_{\Phi_1}(x^n)$, and so $C_{\Phi}(x^n) = C_{\Phi}(x^n)$. Therefore, $\gamma \in C_{\Phi}(x^n)$. Since $C_\Phi(x^n) \leq C_R(x^n)$, we have $C_{\Phi}(x^n) \leq C_R(x^n)$. Hence $\gamma \in C_R(x^n)$. Thus $x, y, \gamma \in C_R(x^n)$.

Since $x^n \neq 1$ and $\Phi$ is free, $C_\Phi(x^n)$ is cyclic, say $C_\Phi(x^n) = \langle z \rangle$ and $z^m = x^n$, for some natural number $m$. Using the uniqueness of $m$-th roots in $\Phi$, we get that $C_R(x^n) = C_R(z)$. Hence $x \in C_R(z)$, i.e., $x$ and $z$ commute.
Since $R = \langle x \rangle \Phi$, we obtain that $C_R(x^n) = \langle x \rangle C_\Phi(x^n) = \langle x \rangle \langle z \rangle$; therefore $C_R(x^n)$ is abelian, and hence so is $C_R(x^n)$. This implies that $x = y$; thus the result holds in this case.

Case 2. The order of $x$ is finite. Observe that $\langle x \rangle$ is isomorphic to a subgroup of $R/\Phi$, and so $\langle x \rangle \in \mathcal{C}$. We proceed by induction on the order of $x$.

Subcase 2 (a). The order of $x$ is $p$, a prime. Then $y$ is also of order $p$. By a theorem of Dyer-Scott (cf. [5], Theorem 1) the group $R$ is a free product

$$R = \langle x \rangle \Phi = \left[ \bigstar_{i \in I} (C_i \times \Phi_i) \right] * L,$$

where $L$ and each $\Phi_i$ are free groups and the $C_i$ are groups of order $p$. Suppose $x$ and $y$ are not conjugate in $R$. Since every finite subgroup of $R$ of order $p$ is conjugate in $\bar{R}$ to one of the $C_i$ (cf. [15], Corollary 4.1.4), we may assume that $C_{i_1} = \langle x \rangle$ and $C_{i_2} = \langle y \rangle$, where $i_1, i_2 \in I$ and $i_1 \neq i_2$. Hence $R = (C_{i_1} \times \Phi_{i_1}) * (C_{i_2} \times \Phi_{i_2}) * R_1$, where

$$R_1 = \left[ \bigstar_{i \in I - \{i_1, i_2\}} (C_i \times \Phi_i) \right] * L.$$

Define $\tilde{R} = C_{i_1} * C_{i_2}$ and let $\varphi : R \rightarrow \tilde{R}$ be a natural epimorphism that sends $C_{i_1}$ and $C_{i_2}$ identically to their corresponding copies in $\tilde{R}$ and sends $\Phi_{i_1}, \Phi_{i_2}$ and $R_1$ to 1. Then $x$ and $y$ are not conjugate in the free pro-$\mathcal{C}$ product $\tilde{R} = C_{i_1} \amalg C_{i_2} \amalg R_1$ (cf. [23], Theorem 9.1.12). However, the epimorphism $\varphi$ induces an epimorphism $\tilde{\varphi} : \tilde{R} \rightarrow \tilde{C}_i \amalg \tilde{C}_j \amalg \tilde{R}_1$ (cf. [23], Proposition 3.2.1), and so $x^{\tilde{\varphi}} = y$ in $\tilde{R}$.

Subcase 2 (b). The order of $x$ is finite but not a prime. Choose a natural number $n$ such that the order of $x^n$ is a prime. By the subcase 2 (a) above, replacing $x$ by a certain conjugate in $R$, we may assume that $x^n = y^n$, and so $\gamma$ centralizes $x^n$; hence $\gamma \in C_\Phi(x^n) = C_\Phi(y^n)$ (the last equality is the content of Lemma 2 (b') above). Put $H = \langle x \rangle C_\Phi(x^n)$. Since $x$ normalizes $C_\Phi(x^n)$, $H$ is a subgroup of $R$. By Lemma 2, $C_\Phi(x^n)$ is a free factor of $H$, and so it is closed in $\Phi$. Hence $C_\Phi(x^n)$ is closed in $R$; moreover the pro-$\mathcal{C}$ topology on it induced from $R$ is its full pro-$\mathcal{C}$ topology (cf. Corollary 3.1.6 in [23]). Since $\langle x \rangle$ is finite, $H$ is closed in $R$ and $H = \hat{H}$ (this follows from Corollary 3.3 in [21]). Therefore, $H = \hat{H} = \langle x \rangle C_\Phi(x^n)$. It follows that $x, y \in H$ and $\gamma \in \hat{H}$. Hence we may assume that $R = \langle x \rangle C_\Phi(x^n)$. Moreover conditions (1) still hold, where now $C_\Phi(x^n)$ plays the role of $\Phi$. Note that then $\langle x^n \rangle$ is a central subgroup of $R$, and $R/\langle x^n \rangle = \langle \langle x \rangle / \langle x^n \rangle \rangle C_\Phi(x^n)$, where, with a certain abuse of notation, we identify $C_\Phi(x^n)$ with its isomorphic image in $R/\langle x^n \rangle$. Denote by $\tilde{x}$ and $\tilde{y}$ the images of $x$ and $y$ in $R/\langle x^n \rangle$, respectively. So $R/\langle x^n \rangle = \langle \tilde{x} \rangle C_\Phi(x^n)$. Note that the order of $\tilde{x}$ is strictly smaller than the order of $x$; $\tilde{y} = \tilde{x}^\gamma$, with $\gamma \in C_\Phi(x^n)$, and $(R/\langle x^n \rangle)/C_\Phi(x^n) \cong \langle \tilde{x} \rangle \in \mathcal{C}$. By the induction hypothesis, there exists some $f \in C_\Phi(x^n)$ such that $\tilde{y} = \tilde{x}^f$. Replacing $x$ with $x^f$ and $\gamma$ with $f^{-1} \gamma$, we may assume that $\gamma = \gamma_0$; observe that conditions (1) still hold, with $C_\Phi(\langle x^n \rangle)$ playing the role of $\Phi$. Therefore $y = xc$, for some $c \in \langle x^n \rangle$. Since $x C_\Phi(\langle x^n \rangle) = y C_\Phi(\langle x^n \rangle)$, and $C_\Phi(x^n)$ is a free group, we have $c = 1$. Thus $x = y$, and the result follows.

**Proof of Theorem 2.** A. This is equivalent to proving that if $a \in R$ and $a^{-1} = a^{-1} a^r \in \hat{H}$, where $\gamma \in \hat{R}$, then there exist $c \in R$ such that $c^{-1} a c \in H$.

It follows from a result of Scott ([5]) that $R$ is the fundamental group $\Pi^{ab}(R, \Delta)$ of a graph of groups $(R, \Delta)$ over a graph $\Delta$ such that each vertex group $\mathcal{C}(v)$ is in $\mathcal{C}$ ($v \in \Delta$).

Since $R$ is free-by-$\mathcal{C}$, there exists a subgroup $\Phi$ of $R$ which is free and open in the pro-$\mathcal{C}$ topology of $R$. The pro-$\mathcal{C}$ topology of $R$ induces on $\Phi$ its own full pro-$\mathcal{C}$ topology ([23], Lemma 3.1.4 (a)).

Case (i). The element $a$ has finite order.

The pro-$\mathcal{C}$ topology of $R$ induces on $H$ its own full pro-$\mathcal{C}$ topology, so that one can make the identification $\hat{H} = H_\mathcal{C}$ (indeed since $H$ is finitely generated, one can write $\Phi = \Phi_1 \ast \Phi_2$, where $\Phi_1$ is a free group of finite rank such that $H \cap \Phi_1 \leq \Phi_1$; since $\Phi_1$ is closed in the pro-$\mathcal{C}$ topology of $R$ and $\Phi_1 = \Phi_1$ by [23], Corollary 3.1.6, so by Lemma 3.1 in [24], $\hat{H} = H_\mathcal{C}$). Observe that $H$ is also a finitely generated free-by-$\mathcal{C}$ group, and so, using a result of Karrass, Petrowski and Solitar ([10], Theorem 1) or the result of Scott mentioned above, $H$ is the fundamental group $\Pi^{ab}(R', \Delta')$ of a graph of groups $(\mathcal{C}, (R', \Delta'))$, over a finite graph $\Delta'$; and $\hat{H} = H_\mathcal{C}$ is the pro-$\mathcal{C}$ fundamental group of $(\mathcal{T}, \Delta')$. In addition we may make the identification $\mathcal{C}(v) = \Pi^{ab}(v)$, a subgroup of $H$, for every vertex $v$ of $\Delta'$ ([24], Section 0). Since $\gamma^{-1} a \gamma \in \hat{H}$ has finite order, it is conjugate in $\hat{H} = H_\mathcal{C}$ to an element of some vertex group $\mathcal{C}(w) = \Pi^{ab}(w) \leq H$ ([29], Theorem
Therefore, since $H_{\hat{c}} \leq R_{\hat{c}}$, $a$ is conjugate in $R_{\hat{c}}$ to an element, say $b$, of $H$. Thus, by Theorem 3, there exists $c \in R$ with $c^{-1}ac = b \in H$.

Case (ii). The element $a$ has infinite order.

Since $\Phi \cap H$ is finitely generated and it is closed in the pro-$C$ topology of $\Phi$, it follows from [21], Lemma 3.2 that there exists an open subgroup $U$ of $\Phi$ containing $\Phi \cap H$ such that $\Phi = (\Phi \cap H) \ast L$, for some closed subgroup $L$ of $\Phi$. Replacing $\Phi$ with $U$ we may assume that $\Phi = (\Phi \cap H) \ast L$.

Since $R$ is dense in $R_{\Phi}$ and $\Phi$ is open in $R_{\Phi}$, we have that $R_{\Phi} = R\Phi$. So $\gamma = r\gamma_1$, for some $r \in R$, $\gamma_1 \in \Phi$. Therefore, replacing $a$ with $r^{-1}ar$, we may assume that $\gamma = \gamma_1 \in \Phi = \Phi_{\hat{c}}$. Since $\Phi$ has finite index in $R$, we have $1 \neq a^n \in \Phi$, for some natural number $n$. Observe that the pro-$C$ topology of $R$ induces on $\Phi \cap H$ (respectively, on $L$) its full pro-$C$ topology ([23], Corollary 3.1.6); therefore,

$$\bar{\Phi} = \Phi_{\hat{c}} = (\Phi \cap H)_{\hat{c}} \amalg L_{\hat{c}} = (\Phi \cap H) \amalg \bar{L},$$

the free pro-$C$ product ([23], Section 9.1). By Lemma 1, $\bar{\Phi} \cap \hat{H} = \Phi \cap \hat{H}$; so $\gamma^{-1}a^n\gamma \in (\Phi \cap H)$.

We deduce from ([22], Proposition 2.9) that $a^n$ is nonhyperbolic as an element of the free product $\Phi = (\Phi \cap H) \ast L$; i.e., $a^n$ is conjugate in $\Phi$ to an element of either $\Phi \cap H$ or $L$; in fact it must be conjugate in $\Phi$ to an element of $\Phi \cap H$, since otherwise $\Phi \cap H$ would contain a conjugate in $\Phi$ of a nontrivial element of $L$, which is not possible ([23], Theorem 9.1.12). Say $c^{-1}a^n\gamma \in \Phi \cap H$, for some $c \in \Phi$. Then $(\gamma^{-1}\gamma)c^{-1}a^n\gamma c^{-1} \gamma = \bar{\Phi} \cap \hat{H}$; therefore using again Theorem 9.1.12 in [23], we have that $c^{-1}\gamma \in \Phi \cap H$. We deduce that $c^{-1}\gamma \in \hat{H}$. Since $\gamma^{-1}a\gamma \in \hat{H}$, we have $(\gamma^{-1}\gamma)c^{-1}ac(c^{-1}\gamma) \in \hat{H}$, and therefore $c^{-1}ac \in \hat{H}$. Now, since $H$ is closed in the pro-$C$ topology of $R$ by assumption, we have $\hat{H} \cap R = H$. Thus

$$c^{-1}ac \in \hat{H} \cap R = H,$$

as needed.

In the profinite topology of a free-by-finite group every finitely generated subgroup is closed (this follows easily from [7], Theorem 5.1). Hence one deduces the following result (essentially proved in [4] when the subgroup $H$ is cyclic).

**4. Corollary** Let $R$ be a free-by-finite abstract group, and let $H$ be a finitely generated subgroup of $R$. Then $H$ is conjugacy distinguished.

**5. Remark** The condition in Theorem A that $H$ is closed in the pro-$C$ topology of $R$ is necessary. For example, let $R = \mathbb{Z}$, the free group of rank 1. Let $p$ be a prime number and let $C$ consist of all finite $p$-groups, so that the pro-$C$ topology is, in this case, the pro-$p$ topology. Consider the subgroup $H = q\mathbb{Z}$ of $\mathbb{Z}$, where $q$ is a prime, $q \neq p$. Then $H$ is not closed in the pro-$p$ topology of $\mathbb{Z}$, but if $\varphi_n: \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ is the natural epimorphism ($n = 1, 2, \ldots$), then $\varphi_n(H) = \mathbb{Z}/p^n\mathbb{Z}$. Therefore, if $x \in \mathbb{Z} - H$, one has $\varphi_n(x) \in \varphi_n(H)$ for each $n$. Thus, since $\mathbb{Z}$ is abelian, $H$ is not conjugacy $C$-distinguished in $\mathbb{Z}$.

**3. Virtual retracts**

In this section we prove Theorems B and C. We first introduce the concept of ‘virtual retract’ for a subgroup $H$ of a group $R$ and show how this property helps to establish that $H$ is conjugacy distinguished.

We say that a subgroup $H$ of an abstract group $R$ is a $C$-virtual retract of $R$ if there is an open subgroup $U$ in the pro-$C$ topology of $R$ such that $H \leq U$ and there exists an epimorphism $f: U \to H$ that is the identity when restricted to $H$ (such an epimorphism is called a retract and the subgroup $H$ is called a retract of $U$); in other words $U = K \ast H$, where $K$ is a subgroup of $U$. When $C$ is the class of all finite groups, we simply say that $H$ is a virtual retract of $R$. Observe that if $H$ is a $C$-virtual retract of $R$, then $H$ is closed in the pro-$C$ topology of $R$ and $\hat{H} = H_{\hat{c}}$ ([23], Lemmas 3.1.4 and 3.1.5).

**6. Lemma** Let $R$ be a torsion-free group. Assume that every open (in the pro-$C$ topology) subgroup of $R$ is conjugacy $C$-separable. Let $H$ be a $C$-virtual retract of $R$. Suppose $C_R(h)$ is abelian for every $1 \neq h \in H$. Then $H$ is conjugacy $C$-distinguished.
Proof. As pointed out above, \( \hat{H} = H_\xi \). Let \( g \in R \), \( \gamma \in R_\xi \) be such that \( g^\gamma \in \hat{H} \). We need to show that there exists some \( c \in R \) with \( g^c \in H \). Let \( U \) be an open subgroup of \( R \) containing \( H \) such that there exists an epimorphism \( f : U \to H \) that is the identity map on \( H \). Consider the extension \( \hat{f} : \hat{U} = U_\xi \to \hat{H} = H_\xi \) of \( f \) to the pro-\( C \) completions of \( U \) and \( H \). Then the restriction of \( \hat{f} \) to \( \hat{H} \) is the identity map on \( \hat{H} \).

Since \( \hat{U} \) is open in \( R_\xi \), one has \( R_\xi = \hat{R} \hat{U} \). Write \( \gamma = r\gamma_1 \), with \( r \in R \) and \( \gamma_1 \in \hat{U} \). Then \( g^\gamma = (g^r)^{\gamma_1} \). Therefore replacing \( \gamma \) with \( \gamma_1 \), if necessary, we may assume that \( \gamma \in \hat{U} \). Let \( n \) be a natural number such that \( u = g^n \in U \). Since \( u^\gamma = (g^\gamma)^n \in \hat{H} \), one has

\[
u^\gamma = \hat{f}(u^\gamma) = f(u)^{\hat{f}(\gamma)}.
\]

Therefore, \( u \) and \( f(u) \) are conjugate in \( \hat{U} \). Since \( \hat{U} \) is conjugacy \( C \)-separable by assumption, \( u \) and \( f(u) \) are conjugate in \( U \); say \((s^{-1}gs)^n = f(g^n)\), where \( s \in U \). So \((s^{-1}gs)^n \in \hat{H} \). Then (2) can be rewritten as

\[
((g^n)^s)^{s^{-1}\gamma} = (f(g^n)^s)^{s^{-1}\hat{f}(\gamma)}
\]

since \( f(s) \) and \( \hat{f}(s^{-1}\gamma) \) are defined. Replacing \( g \) with \( g^s = s^{-1}gs \) and \( \gamma \) with \( s^{-1}\gamma \), we may assume that \( g^n \in H \). Hence the equation above reads as \((g^n)^\gamma = (g^n)^{\hat{f}(\gamma)}\). It follows that \( \gamma(\hat{f}(\gamma))^{-1} \in C_\hat{H}(g^n) \), which is an abelian group by hypothesis, because \( g^n \neq 1 \). Since we also have that \( g \in C_\hat{R}(g^n) \), one gets

\[
g^\gamma = (g^n)^{\hat{f}(\gamma)^{-1}} = g^{\hat{f}(\gamma)}.
\]

From \( g^\gamma \in \hat{H} \), we deduce that \( g^{\hat{f}(\gamma)} \in \hat{H} \). Therefore \( g \in \hat{H} \), because \( \hat{f}(\gamma) \in \hat{H} \). Since \( H \) is closed in the pro-\( C \) topology of \( R \), we obtain that \( g \in \hat{H} \cap R = H \), so that in this case we can take \( c = 1 \), proving the result.

7. Remarks

7.1 In contrast to residual finiteness and subgroup separability, the conjugacy separability property is not inherited by subgroups of finite index. So the assumption in Lemma 6 that every open subgroup of the group \( R \) is conjugacy \( C \)-separable is essential.

7.2 The assumption that \( C_\hat{R}(h) \) is abelian for every \( h \in H \) is, in principle, not easily verifiable. With this in mind we make use of the following result of Minasyan ([16], Proposition 3.2): a group \( R \) and all its subgroups of finite index are conjugacy separable if and only if \( R \) is conjugacy separable and \( C_\hat{R}(g) = C_R(g) \), for every \( g \in R \) (this result has been extended by Ferov in [6], Theorem 4.2, to a corresponding equivalence for the property of conjugacy \( C \)-separability). The idea is that in certain cases it suffices to know the abelianness of \( C_R(g) \).

Next we apply Lemma 6 to important groups of geometric nature. A group \( G \) is called virtually special if there exists a special compact cube complex \( X \) having a finite index subgroup of \( G \) as its fundamental group (see [28] for definition of special cube complex). The importance of virtually special groups was pointed out by Daniel Wise who proved that many important groups are virtually special. For example, the fundamental group of a hyperbolic 3-manifold, one-relator groups with torsion, small cancellation groups and hyperbolic Coxeter groups are virtually special. Moreover, Wise showed that quasiconvex subgroups of a virtually special group \( G \) (i.e., a subgroups that represents a quasiconvex subset in the set of vertices of the Cayley graph of \( G \)) are virtual retracts of \( G \). Thus the next corollary applies in particular to this important class of subgroups.

8. Corollary Let \( G \) be a torsion-free hyperbolic virtually special group and let \( H \) be a virtual retract of \( G \). Then \( H \) is conjugacy distinguished.

Proof. The centralizers of torsion-free hyperbolic groups are cyclic (cf. Proposition 12 in [19]). By Lemma 4.1 in [18] \( G \) is hereditarily conjugacy separable. So, using Remark 7.2, \( C_{\hat{G}}(h) \) is procyclic for every \( h \in H \). Thus the result follows from Lemma 6.
Proof of Theorem B. Combining Theorem 1.4 in [28], Theorem 1.2 in [8] and Proposition 4.3 in [1], one has that every finitely generated subgroup of \( R \) is a virtual retract of \( R \). Another important fact about \( R \), proved by Newman in [20], Theorem 2 (see also [9], p. 956), states that the centralizers of nontrivial elements in one-relator groups with torsion are cyclic. On the other hand, by Theorem 1.1 in [17], \( R \) is hereditarily conjugacy separable. Therefore, using Remark 7.2 above, we deduce that \( \hat{C}_R(g) \) is procyclic for every \( g \in R \). Thus the result follows from Lemma 6. \( \square \)

Proof of Theorem C. According to Theorem B in [27], every finitely generated subgroup of \( R \) is a virtual retract of \( R \). Now, by Lemma 3.5 in [3], the centralizers of elements of \( R \) are abelian and \( \bar{C}_R(g) = C_{\hat{R}}(g) \), for every \( g \in R \). So the \( \hat{R} \)-centralizers of elements of \( R \) are abelian. Thus the result follows from Lemma 6. \( \square \)

4. Lyndon Groups

The aim of this section is to show that every finitely generated subgroup of the Lyndon group is conjugacy distinguished. We begin by recalling the concept of Lyndon group. We then extend the proof in [3] that the Lyndon group is conjugacy separable for a more general concept of Lyndon group than the one considered there (this result follows also from the result obtained by Lioutikova [12]). The Lyndon group was first defined in [13] with the aim of enlarging the set of ‘exponents’ allowed in a group. One begins with a free group \( F \) of arbitrary rank and one wants to enlarge \( F \) to a group, usually denoted \( F[Z[t]] \), on which the ring of polynomials \( Z[t] \) operates (in a manner analogous with the way the ring of integers \( Z \) operates on any group). Myasnikov and Remeslennikov ([19]) give an explicit construction of the Lyndon group \( F[Z[t]] \) as follows.

1st step: One starts with the free group \( F^{(0)} = F \).

2nd step: We consider a tree of groups of the form

\[
\begin{align*}
F^{(0)} & \to C_1 \otimes Z[t] \\
& \downarrow \qquad \downarrow \quad \downarrow \quad \downarrow \\
C_2 \otimes Z[t] & \to C_3 \otimes Z[t] \\
& \quad \downarrow \\
& \quad \vdots \\
C_\delta \otimes Z[t] & \to C_\delta \otimes Z[t]
\end{align*}
\]

where \( \{C_i \mid 1 \leq i \leq \delta_0\} \) is a collection of infinite cyclic subgroups of \( F^{(0)} \) indexed by the ordinals less than or equal a certain ordinal number \( \delta_0 \) and \( C_i \otimes Z[t] = C_i \otimes Z \) is the usual tensor product of \( Z \)-modules (more precisely, the \( C_i \) are representatives of all centralizers of nontrivial elements of \( F^{(0)} = F \), which of course in this case are all of them maximal cyclic subgroups). The edge group \( C_i \) is embedded into the vertex group \( C_i \otimes Z[t] \) by the map \( C_i \to C_i \otimes Z[t] \) that sends \( c \in C_i \) to \( c \otimes 1 \); this is indeed an embedding because \( C_i \) is infinite cyclic. Let \( F^{(1)} \) be the fundamental group (the tree product) of this graph of groups. Then \( F^{(1)} \) is the union of a chain

\[
F = F^{(0)} = F^{(00)} \leq F^{(01)} \leq \cdots \leq F^{(0\delta_0)} \leq \cdots \leq F^{(0\delta_1)} = F^{(1)},
\]

where each \( i \) is an ordinal, \( 1 \leq i \leq \delta_0 \), and if \( i \geq 1 \) is not a limit ordinal, then \( F^{(0i)} = F^{(0i-1)} \ast_{C_i} C_i \otimes Z[t] \), while if \( i \) is a limit ordinal, then \( F^{(0i)} = \bigcup_{j < i} F^{(0j)} \).

\( n \)th step: Here we repeat the same procedure described in step 2, but with \( F^{(0)} \) replaced with \( F^{(n-1)} \) and the \( C_i \) being a set of representatives of all cyclic centralizers of nontrivial elements of \( F^{(n-1)} \) (that is to say, intuitively, those centralizers on which \( Z[t] \) does not operate yet).

Then the Lyndon group is

\[
F[Z[t]] = \bigcup_{m=0}^{\infty} F^{(m)}
\]
Therefore one can describe $F^\mathbf{Z}[t]$ as a union of a chain of groups

$$F = F(0) \leq F(1) \leq \cdots \leq F(i) \leq \cdots \leq F(\delta) = F^\mathbf{Z}[t]$$

indexed by the ordinals $i$ less than or equal to a certain ordinal $\delta$ and such that $F(i) = F(i-1) \ast C_i \{C_i \otimes \mathbf{Z}[t]\}$, when $i$ is a nonlimit ordinal, while if $i$ is limit ordinal, then $F(i) = \bigcup_{j<i} F(j)$.

Observe that the image $A_i = C_i \otimes 1$ of $C_i$ in $C_i \otimes \mathbf{Z}[t]$ is a direct summand of $C_i \otimes \mathbf{Z}[t]$. Say $C_i \otimes \mathbf{Z}[t] = A_i \oplus A_i$, where $A_i \cong C_i \cong \mathbf{Z}$ and $A_i$ is a free abelian group of infinite rank. We identify $A_i$ with $C_i$.

9. Lemma Let $0 \leq i \leq s \leq \delta$. Then

(a) there exists a group epimorphism $\varphi_{s,i} : F(s) \to F(i)$ which is the identity on the subgroup $F(i)$ of $F(s)$;

(b) $F(s) = K_{s,i} \rtimes F(i)$, for some normal subgroup $K_{s,i}$ of $F(s)$.

Proof.

(a) We shall use transfinite induction to define a group homomorphism $\varphi_{s,i}$ from $F(s)$ onto $F(i)$ which is the identity on the subgroup $F(i)$ of $F(s)$, and such that the restriction of $\varphi_{s,i}$ to $F(k)$ is $\varphi_{k,i}$ whenever $i \leq k \leq s$. Define $\varphi_{s,i}$ to be the identity map and assume that $\varphi_{r,i}$ has already been defined for all $r < s$ ($i \leq r \leq s \leq \delta$). Then we define $\varphi_{s,i} : F(s) \to F(i)$ as follows:

- if $s$ is a limit ordinal, put $\varphi_{s,i} = \bigcup_{r<s} \varphi_{r,i}$, and

- if $s$ is a nonlimit ordinal, then define first

$$\psi : F(s) = F(s-1) \ast C_{s-1} (C_{s-1} \oplus \hat{A}_{s-1}) \to F(s-1)$$

by sending $F(s-1)$ identically to $F(s-1)$ and sending $\hat{A}_{s-1}$ to 1. Then define $\varphi_{s,i} = \varphi_{s-1,i} \psi$.

(b) This is clear from (a).

The next task is to show that $F^\mathbf{Z}[t]$ is a conjugacy separable group. In fact we will prove more generally that the Lyndon group belongs to a class $\mathcal{X}$ of abstract groups that satisfy a series of properties including that of being conjugacy separable. This class $\mathcal{X}$ was introduced in [25] and we describe it briefly here. An abstract group $R$ is in $\mathcal{X}$ if

(a) $R$ is conjugacy separable (so that in particular $R \leq \hat{R}$);

(b) $R$ is quasi-potent (i.e., for every cyclic subgroup $H$ of $R$, there exists a subgroup $K$ of finite index in $H$ such that every subgroup of finite index of $K$ is of the form $K \cap N$, for some normal subgroup $N$ of finite index in $R$);

(c) whenever $A$ and $B$ are cyclic subgroups of $R$, the set $AB$ is closed in the profinite topology of $R$;

(d) every cyclic subgroup of $R$ is conjugacy distinguished, i.e., if $C$ is a cyclic subgroup of $R$ and $a \in R$, then $a^R \cap C = \emptyset$ if and only if $a^\hat{R} \cap \hat{C} = \emptyset$;

(e) if $A$ and $B$ are cyclic subgroups of $R$, then $A \cap B = 1$ if and only if $\hat{A} \cap \hat{B} = 1$; and

(f) if $A = \langle a \rangle$ is an infinite cyclic subgroup of $R$, and $\gamma \in \hat{R}$ with $\gamma \in N_{\hat{R}}(\hat{A})$, then $\gamma \in N_{\hat{R}}(A)$, i.e., $\gamma a \gamma^{-1} \in \{a, a^{-1}\}$.

10. Proposition The Lyndon group $F^\mathbf{Z}[t]$ is in the class $\mathcal{X}$, and in particular it is conjugacy separable.

Proof. We continue with the above notation. We shall prove inductively that in fact each $F(s)$ is in the class $\mathcal{X}$, for all $0 \leq s \leq \delta$. It is well-known that the free group $F(0) = F$ is in class $\mathcal{X}$. Assume that $F(j) \in \mathcal{X}$ for $0 \leq j < s$. If $s$ is a nonlimit ordinal, then $F(s) = F(s-1) \ast C_{s-1} \{C_{s-1} \otimes \mathbf{Z}[t]\}$ is in $\mathcal{X}$ according to Theorem A in [RSZ], since $C_{s-1} \otimes \mathbf{Z}[t]$ is free abelian of infinite rank, and so both $C_{s-1} \otimes \mathbf{Z}[t]$ and $F(s-1)$ are in the class $\mathcal{X}$.

Let now $s$ be a limit ordinal. Then $F(s) = \bigcup_{j<s} F(j)$. We have to verify that $F(s)$ satisfies properties (a)-(f) of class $\mathcal{X}$. Observe first that $F(s)$ is residually finite: indeed, let $1 \neq x \in F(s)$; then $x \in F(j)$, for some $i < s$, and since $F(i)$ is residually finite, there exists some $N \triangleleft F(i)$ with $x \not\in N$; so, if $\varphi_{s,i} : F(s) \to F(i)$
denotes the epimorphism defined in Lemma 9, we have that $\varphi_{s,i}^{-1}(N)$ is a normal subgroup of finite index in $F_s$ that misses $x$. It follows from this and part (b) of Lemma 9 that $F_i$ is closed in the profinite topology of $F_s$, and moreover the profinite topology of $F_s$ induces on $F_i$ its full profinite topology (cf. Lemma 3.1.5 in [23]). Using this one easily verifies that $F_s$ satisfies properties (c) and (e) of class $\mathcal{X}$.

To verify property (a) of class $\mathcal{X}$ (i.e., that $F_s$ is conjugacy separable) let $x, y \in F_s$ and assume that $y = x^\gamma = \gamma^{-1}x\gamma$, where $\gamma \in \hat{F}_s$. Then there exists some ordinal $i, 0 \leq i < s$ with $x, y \in F_i$. Let $\varphi_{s,i} : F_s \to F_i$ be an epimorphism such that $\varphi_{s,i} = 1$ on $F_i$ (see Lemma 9). Let $\varphi_{s,i} : \hat{F}_s \to \hat{F}_i$ be the continuous homomorphism induced by $\varphi_{s,i}$ (cf. [23], Lemma 3.2.3). Put $\tilde{\gamma} = \varphi_{s,i}(\gamma)$. Then $y = x^{\tilde{\gamma}}$. Since, by assumption, $F_i$ is conjugacy separable, there exists some $c \in F_i$ such that $y = xc$, as needed.

To verify property (b) of class $\mathcal{X}$ (i.e., that $F_s$ is quasi-potent) let $H$ be a cyclic subgroup of $F_s$. Then $H \leq F_i$ for some ordinal $i < s$. By Lemma 9, $F_s = K_{s,i} \triangleright F_i$. Since $F_i$ is quasi-potent, there exists a subgroup $K$ of $H$ of finite index such that every subgroup of finite index of $K$ has the form $K \cap U$, for some $U \leq F_i$. Since $K \cap K_{s,i}U = K \cap U$ and $K_{s,i}U \trianglelefteq F_s$, we deduce that $F_s$ is quasi-potent.

For property (d) of class $\mathcal{X}$, let $C$ be a cyclic subgroup of $F_s$ and let $a^{\hat{F}_s} \cap C = \emptyset$, where $a \in F_s$. Assume that $a^s = a \in \hat{C}$, for some $\gamma \in \hat{F}_s$, where $\hat{C}$ is the closure of $C$ in $\hat{F}_s$. Let $i$ be an ordinal, $0 \leq i < s$ such that $a \in F_i$, let $\varphi_{s,i} : F_s \to F_i$ be the epimorphism described in Lemma 9, and let $\hat{\varphi}_{s,i} : \hat{F}_s \to \hat{F}_i$ be the induced epimorphism. Note that $\hat{C}$ is also the closure of $C$ in $\hat{F}_i$, and hence $\hat{\varphi}_{s,i}(\hat{C}) = \hat{C}$. Put $\tilde{\gamma} = \varphi_{s,i}(\gamma)$ and $\tilde{\alpha} = \varphi_{s,i}(\alpha)$. Then $a^s = \tilde{\alpha} \in \hat{C}$. Since $F_i$ has property (d), there exists $c \in F_i \leq F_s$ with $a^c \in C$, a contradiction. Hence $a^{\hat{F}_s} \cap C = \emptyset$, showing that $F_s$ has property (d).

Finally, we check that $F_s$ has property (f) of class $\mathcal{X}$. Let $A = \langle a \rangle$ be an infinite cyclic subgroup of $F_s$ and assume that $\gamma^{-1}a\gamma \in A$, for some $\gamma \in \hat{F}_s$. Let $i$ be an ordinal, $0 \leq i < s$ such that $a \in F_i$, let $\varphi_{s,i} : F_s \to F_i$ be the epimorphism described in Lemma 9 and let $\hat{\varphi}_{s,i} : \hat{F}_s \to \hat{F}_i$ be the induced epimorphism. Put $\tilde{\gamma} = \varphi_{s,i}(\gamma)$ and observe that $\gamma^{-1}a\gamma \in \hat{A}$ in $\hat{F}_i$, since $\hat{A}$ is the closure of $A$ in both $F_s$ and $F_i$. Since $F_s$ has property (f), we have that $\tilde{\gamma}^{-1}a\tilde{\gamma}$ is either $a$ or $a^{-1}$. Since $\varphi_{s,i}$ is the identity on $\hat{A}$, we deduce that also $\gamma^{-1}a\gamma \in \{a,a^{-1}\}$.

11. Proposition Let $H = \langle h_1, \ldots, h_n \rangle$ be a finitely generated subgroup of the Lyndon group $F^Z[t]$. Then

(a) there exists a finitely generated subgroup $K$ of $F^Z[t]$ such that $H \leq K$ and $K$ is a retract of $F^Z[t]$; and

(b) $H$ is a virtual retract of $F^Z[t]$.

Proof. We continue with the above description and notation. Since $H$ is finitely generated, there exists a smallest ordinal $s$ with $s < \delta$ such that $H \leq F_s$. Note that $s$ cannot be a limit ordinal.

(a) We prove this by induction on $s$. If $s = 0$, this is clear since $F_{(0)}$ is a free group. Suppose $s > 0$ and that for every ordinal $i$, with $i < s$, every finitely generated subgroup of $F_{(i)}$ is contained in a finitely generated retract of $F_{(i)}$. Since $s$ is not a limit ordinal, one has

$$F_s = F_{(s-1)} \ast_C (C \otimes Z[t]).$$

Write each $h_i$ as a product of elements of $F_{(s-1)}$ and $C \otimes Z[t]$. Say $f_1, \ldots, f_t$ are all the elements of $F_{(s-1)}$ involved in those products, and let $b_1, \ldots, b_r$ be all the elements of $C \otimes Z[t]$ involved in those products. By hypothesis there exists a finitely generated retract $K_1$ of $F_{(s-1)}$ such that $K_1 \geq \langle f_1, \ldots, f_t, C \rangle$. Let $B$ be a finitely generated direct summand of the free abelian group $C \otimes Z[t]$ such that $b_1, \ldots, b_r \in B$. Then $H \leq K_1 \ast_C B$, $K_1 \ast_C B$ is finitely generated and a retract of $F_s$. This proves part (a).

(b) Let $K$ be as in part (a). Since $K$ is a limit group and $H \leq K$, there is a subgroup $U$ of finite index in $K$ with $U \geq H$ and an epimorphism $\varphi : U \to H$ which is the identity on $H$ (cf. [27], Theorem B). Let $\psi : F^Z[t] \to K$ be a retraction and put $V = \psi^{-1}(U)$. Then $V$ has finite index in $F^Z[t]$ and the composite $\varphi \psi |_V : V \to H$ is a retraction, as needed.

Proof of Theorem D. Let $H$ be a finitely generated subgroup of $F^Z[t]$ and assume that $g^c \in \hat{H}$, where $g \in F^Z[t]$ and $\gamma \in F^Z[t]$. One needs to show that there exists some $c \in F^Z[t]$ with $g^c \in H$. By Proposition 11
there exists a retraction $\varphi : F^Z[t] \rightarrow K$, where $K$ is a finitely generated subgroup of $F^Z[t]$ containing $(H,g)$. Then $\varphi$ extends to a retraction $\hat{\varphi} : \hat{F}^Z[t] \rightarrow \hat{K} = \overline{K}$. Put $\gamma' = \hat{\varphi}(\gamma)$. Then $g\gamma' \in \overline{H}$. Since every finitely generated subgroup of $F^Z[t]$ is a limit group, the result follows from Theorem C.

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