RESEARCH ON SOME THEORETICAL PROBLEMS
OF MAP DATA HANDLING USING FRACTAL APPROACH

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ABSTRACT Some theoretical problems of fractal geographical map data handling are discussed and some new methods about fractal dimension introducing, developing, comparing and estimating are proposed in this paper.

Many studies have showed that fractal theory is becoming a powerful tool to handle the complex problems and finding more and more use in the area of geography. At the same time, more and more theoretical problems involved starve for solving with the increase of applying fractal theory to practice. In this paper, some valuable theoretical problems in map data handling using fractal approach are discussed constructively.

1 About introducing new fractal dimension

It is known that one of advantages of the fractal theory is its permissibility to introduce different fractal dimensions according to different research needs. But it is a pity that there is not any strict rule to define fractal dimension. In practice, we usually use the following method based on scale measuring.

Firstly, the object set $F$ is measured by scale $s$, then the measured value $M_s(F)$ is examined under the condition of $s \to 0$, and the abnormality the scale of which less than $s$ is ignored during the process. If there are constants $C$ and $D$, making:

$$M_s(F) \propto s^{-D}$$

Then $D$ could be defined as "fractal dimension" of $F$.

Generally, when $M_s(F)$ is even with $D$, namely:

$$M_s(F) = s^DM_s(F)$$

The fractal dimension can be defined by power rule that satisfies Eq. (1). But it is not always true that all of objects with the form of power function are fractals and all of power indexes are fractal dimension.

Two points or more should be considered when a new fractal dimension is introduced: Firstly, the independent variable of the power function should be measurable, otherwise, its index can not be considered as fractal dimension; Secondly, the power relation must be kept not only at specific point but also at any point.

Theoretically speaking, a new fractal dimension: $D = \dim F$ should has the following properties:

1. Monotony: If $E \subset F$, then $\dim E \leq \dim F$;
2. Stability: $\dim (E \cup F) = \max (\dim E, \dim F)$;
3. Countable stability: $\dim \left( \bigcup_{i=1}^{m} F_i \right) = \sup_{1 \leq i \leq m} \dim F_i$;
4. Geometrical invariability: If $f$ is a transform in $R^n$, such as parallel translation, rotation, analogy and affine, then $\dim f(F) = \dim F$;
5. Lipschitz stability: If $L$ is double Lipschitz transform, then $\dim L(F) = \dim F$;
6. Open set: If $F$ is an open set in $R^n$, then $\dim F = n$;
7. Smooth manifold: If $F$ is smooth manifold with dimension $m$, then $\dim F = m$.

Above-mentioned properties are summarized on the basis of the measures of Hausdorff dimension. Further research is needed before the general sys-
2 About extension of fractal dimension

Even though existing fractal dimension can yield quantitative index for describing shape feature of object, it is still questionable and difficult to express many fractal characters using existing fractal dimension. Furthermore, fractal dimension is restricted by scale interval overly, so it is undoubtedly useful to extend the concept of fractal dimension.

From Eq. (1), we can get the fractal dimension experimentally:

\[ D = - \frac{\log M_s(F)}{\log s} \quad (3) \]

Apparently, \( M_s(F) \) is the function of \( s \), thus we can define \( N(s) = M_s(F) \) and take \( D \) as the slope of function \( \log N(s) = f(\log s) \) at point \((\log s, \log N(s))\) in the reference frame of \( \log s - \log N(s) \). Therefore, the fractal dimension can be redefined as:

\[ D(s) = - \frac{\log N(s)}{\log s} \quad (4) \]

As a result of the Eq. (4), the concept of fractal dimension can be developed in following ways:

- Fractal is not only a constant but a function of scales;
- As long as \( N(s) \) is smooth (or \( \log N(s) \) is differentiable), the corresponding fractal dimension exists for any \( s \), it can not be restricted by scale interval;
- \( N(s) \) need not taking the form of power function.

After solving Eq. (4) we can obtain:

\[ N(s_2) = N(s_1) \exp \left[ - \int_{s_1}^{s_2} \frac{D(s)}{s} \, ds \right] \quad (5) \]

Consequently, the relation of \( N(s_1), N(s_2) \) and fractal dimension \( D \) can be given by Eq. (5).

When \( F \) is a fractal curve, \( M_s(F) \) is namely curve length with scale \( s \), it can be written as \( L(s) \), so the absolute value of Eq. (4) can be expressed as:

\[ f(\log s) = i - \frac{\log L(s)}{\log s} \approx \frac{\log L(s_2) - \log L(s_1)}{\log s_2 - \log s_1} \quad (6) \]

Further research will show that curve \( \log L(s) \) appears to be the shape of reverse “s” and \( f(\log s) \) bell-like (see Fig. 1).

In fact, many important fractal structural natures can be described quantificationally by the parameters of these curves, such as:

1. The maximum \( D_{\text{max}} \) of the curve can show the maximal fractal dimension;
2. The mean \( D_{\text{mean}} \) of the curve can show the average fractal dimension;
3. The variance \( D_{\text{var}} \) of the curve can show the approximation degree between the curve and the fitted straight line;
4. The length \( \Delta d = d_2 - d_1 \) of the flat interzone on the curve can show the width of the scale interval \((d_1, d_2)\) are horizontal coordinate values of the intersection points of the line \(0.8D_{\text{max}}\) with the curve).

It is obvious that above results are helpful for expanding application area of fractal analysis, utilizing fractal information, making the physical natures of fractals clear and estimating fractal dimension objectively.

![Fig. 1 Extension of fractal dimension](image)

3 About comparison between different fractal dimensions

During the process of map data handling using fractal approach, we often need to compare fractal dimensions of different maps, for this reason, following points should be noticed:

1. The methods to estimate fractal dimension should be uniform, namely, the compared fractal dimensions should be provided with same nature.
2. The methods to make certain scale interval should be uniform, namely, the compared fractal dimensions should be in the same scale interval.

As different kinds of fractal dimension possess different natures, the ways to express the structure of map shape are different consequentially, so the
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mentioned point (1) is reasonable. Taking the fractal dimensions of river-net as example

\[
D_1 = 1 - \frac{M \sum_{i=1}^{M} [\lg e_i \lg L(e_i)]}{M \sum_{i=1}^{M} (\lg e_i)^2} - \frac{[\sum_{i=1}^{M} \lg e_i] [\sum_{i=1}^{M} \lg L(e_i)]}{(\sum_{i=1}^{M} \lg e_i)^2} \tag{7}
\]

\[
D_2 = -\frac{M \sum_{i=1}^{M} [\lg s_i \lg N(s_i)]}{M \sum_{i=1}^{M} (\lg s_i)^2} - \frac{[\sum_{i=1}^{M} \lg s_i] [\sum_{i=1}^{M} \lg N(s_i)]}{(\sum_{i=1}^{M} \lg s_i)^2} \tag{8}
\]

\[
D_3 = \frac{M \sum_{i=1}^{M} [\lg (L_M(e_i)/e_i)] \lg (A(e_i)^{1/2}/e_i)}{M \sum_{i=1}^{M} [\lg (A(e_i)^{1/2}/e_i)]^2 - [\sum_{i=1}^{M} \lg (L_M(e_i)/e_i)] [\sum_{i=1}^{M} \lg (A(e_i)^{1/2}/e_i)]}{[\sum_{i=1}^{M} \lg (A(e_i)^{1/2}/e_i)]^2} \tag{9}
\]

\[
D_4 = \lg R_B/\lg R_L \tag{10}
\]

where \( L(e_i) \) is the length of river-net with scale \( e_i \); \( N(e_i) \) is the number of non-empty grids when the river-net is overlaid by square grids with grid width \( e_i \); \( L_M(e_i) \) and \( A(e_i) \) are length and drainage area of main riverway with scale \( e_i \), respectively; \( R_B \) is the fork ratio of two neighboring channels; \( M \) is the sample number.

According to Ref. [7], the complexity of general shape of main riverway can be described by \( D_1 \); the capability to fill area of river-net can be described by \( D_2 \); the complexity of relation between riverway length and drainage area can be described by \( D_3 \); and the complexity of topological structure of river-net can be described by \( D_4 \).

Above example indicates that different fractal dimensions have different functions to describe nature of shape and structure. It is meaningless to compare different fractal dimensions breadthwise.

For point (2), it can be explained by the following example.

The considered test curves are shown in Fig. 2; they are provided with similar shape and structure nature but different complexity degree. According to the method of fractal analysis, the fitted straight lines of relation \( \lg s - \lg L(s) \) can be get for each curve in Fig. 2. When used linear fitting is strict, the fitted straight lines turn into subsectoin lines (see Fig. 3): for curve (a), subsection lines \( L_{a1} \) and \( L_{a2} \); for curve (b), subsection lines \( L_{b1} \) and \( L_{b2} \); for curve (c), subsection lines are \( L_{c1} \) and \( L_{c2} \). In fact, the difference between slopes of \( L_{a1} \) and 1 can be named as the textural fractal dimension, it describes the nature of fine structure and texture of the map; the difference between slopes of \( L_{a2} \) and 1 can be named as the structural fractal dimension, it describes the nature of general structure and shape of the map. Above facts are just the same and hold true for curve (b) and curve (c).

It is easy to understand from above example that these fractal dimension values of three curves are equal just in the scale range \( I_4 \) (this means that the complexity of three curves in \( I_4 \) is the same under the condition of statistical self-similarity, where the precondition is that the observed details whose measure scales are less than \( I_4 \) are neglected. Outside \( I_4 \), they are unequal to each other (namely, the complexity of curves are different). For this reason, the comparison of fractal dimensions can not depart from scale interval, otherwise it is likely to make false conclusion.

4 About estimation of fractal dimension

As we know that calculating and utilizing fractal dimension are thematic matter for application of fractal theory during map data handling using fractal approach. Therefore fractal dimension estimation is a focal problem. On the one hand, the natural fractals in geography are stochastic, their self-similarity hierarchy is numbered, and they are not as pure as mathematics fractal, so they can not strictly accord with fractal nature in mathematics. On the other hand, the mathematic definition of fractals are abstract and strict, it is difficult to apply them to natural fractals (such as implementation of the
operation of “overlay” $s \to 0$ and so on), thus we ought to find some experimental and operable methods for estimating fractal dimension concretely according to the basic principle of fractal theory and practical need of map data handling. In the process of map data handling using fractal approach, following methods to estimate fractal dimension can be adopted.

4.1 The method based on changing survey scale

Experimentally, the general fractal characteristics of natural fractal map can be described by the following model:

$$M(s) \propto s^{-D}$$  (11)

Where $s$ is the survey scale, it may be small segment, square, circle or cube in practical operation; $M(s)$ is the result surveyed from the map; $D$ is the fractal dimension of the map.

Changing survey scale $s$ one by one, we can get a series of sample points, then the value $D$ can be estimated by fitting those sample points using linear regression model according to formula (11). Generally speaking, this method is typical in common use.

4.2 The method based on measure relation

According to the basic standpoint of fractal theory, the measure of fractal map can be non-integer dimension. Let $L, S$ and $V$ be the length, area and volume of the considered fractal map respectively, we have:

$$L \propto S^{1/2} \propto V^{1/3}$$  (12)

The superficial meaning of Eq. (12) is that $S^{1/2}$ and $V^{1/3}$ shall be multiplied by $K$ when $L$ is multiplied by $K$. Now let $X$ be the quantity with measure of dimension $D$. The formula (12) can be rewritten in a general form:

$$L \propto S^{1/2} \propto V^{1/3} \propto X^{1/D}$$  (13)

Generally speaking, $L, S, V$ and $X$ are confirmable for the practical geographic maps (for example, in segmented space, $L, S, V$ and $X$ can be considered as constants). Thereby, dimension $D$ can be calculated by formula (13).

The fractal dimension of point set can also be calculated using above method.

4.3 The method based on correlation function

Let $\rho(x)$ be the density function of some quantity distributing randomly in a space, $c(s)$ be its correlation function, then we have:

$$c(s) = E(\rho(x) \rho(x+s))$$  (14)

If $c(s)$ can be represented as a power function and there is not any eigenlength, correlation would attenuate with the same rate:

$$c(s) \propto s^{-a}$$  (15)

According to basic principle of fractal theory, the power index and fractal dimension satisfy the following relation:

$$a = d - D$$  (16)

Where $d$ is the dimension of considered space.

Making Fourier transform for correlation function when $0 < d - D < 1$, we have spectrum function:

$$F(k) = 4 \int_0^\infty \cos(2\pi ks) ds,$$  (17)

$$c(s) \propto k^{d-D-1}$$

As the result, fractal dimension $D$ can be obtained from Eq. (17).

4.4 The method based on distributing function

Let $P(s)$ be the occurring probability of the graphics whose “diameter” is longer than $s$, $f(s)$ is relevant density function, we have:

$$P(s) = \int_0^s f(t) dt$$  (18)

If the types of distribution are fixed when map
scale changed, following model would come into existence for any \( \lambda > 0 \):

\[
P(s) \propto P(\lambda s)
\]  

(19)

Obviously, the conformable function which satisfies formula (19) should be provided with type of power function:

\[
P(s) \propto s^{-D}
\]  

(20)

According to formulas (19) and (20), and the method based on changing scale, the dimension of distribution can be given by \( D \).

4.5 The method based on spectrum

From the viewpoint of spectrum, what is called changing scale means changing end frequency \( f_e \). When the end frequencies are changed, namely, making transform of survey scale: \( f \rightarrow \lambda f \), if the spectrum shape is stationary, the map would have fractal nature of fractal dimension. Map spectrum shape is stationary, the map would have fractal nature of fractal dimension. In fact, map spectrum \( s(f) \) with above nature is just limited to the form of power types, namely:

\[
s(f) \propto f^{-\beta}
\]  

(21)

According to basic principle of fractal theory and the practical nature of researched problems, the relation index \( \beta \) and fractal dimension \( D \) can be found. For instance, \( \beta = 5 - 2D \) for plane curve and \( \beta = 7 - 2D \) for terrain surface.

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