Constraining f(T) gravity by dynamical system analysis

Behrouz Mirza and Fatemeh Oboudiat
Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

We investigate the cosmological solutions of the f(T) gravity theory using the method of dynamical systems. For this purpose a general form of the f(T) function is considered and three conditions are defined that they have to satisfy in order to describe the standard cosmological history. We examine five specific models of f(T) gravity and obtained the valid range of their parameters.

I. INTRODUCTION

Recent cosmological observations indicate that our universe has an accelerated expansion [1]. This is not predicted by General Relativity (GR) using ordinary baryonic matter. Either of two approaches might be adopted to describe the expansion of the universe: 1) Introducing a new kind of matter, called dark energy, into the GR theory which either raises problems such as future singularities or violates certain energy conditions; and 2) Modifying GR to some new theory of gravity like f(R) or other forms of higher derivative theories [2].

An equivalent formulation of GR is teleparallel gravity (TEGR) in which the torsion scalar is replaced with a curvatureless Weitzenböck connection [3]:

\[ \Gamma^A_{\mu
u} = e^A_{\lambda} \partial_{\nu} e^\lambda_{\mu} . \]  

(1)

The torsion tensor from this connection reads:

\[ T^A_{\mu
u} = \Gamma^A_{\nu\mu} - \Gamma^A_{\mu\nu} = e^A_{\lambda} \left( \partial_\nu e^\lambda_{\mu} - \partial_\mu e^\lambda_{\nu} \right) . \]  

(2)

Using torsion tensor (2), we can construct the contorsion tensor and the S matrix as follows:

\[ K^A_{\mu\nu} = \frac{1}{2} \left( T^A_{\nu\mu} - T^A_{\mu\nu} - T^{\lambda}_{\mu\nu} \right) , \]  

(3)

\[ S^A_{\mu\nu} = \frac{1}{2} \left( K^A_{\mu\nu} + \delta^A_{\mu} T^\alpha_{\nu\alpha} - \delta^A_{\nu} T^\alpha_{\mu\alpha} \right) , \]  

(4)

both of which are antisymmetric tensors. Using these quantities, the torsion scalar T can be defined as follows:

\[ T = S^{\mu\nu} T^\rho_{\mu\nu} , \]  

(5)

which is the Lagrangian density of the so-called TG. The idea f(T) gravity is to extend T to f(T), similar to the generalization of R to f(R) in Einstein-Hilbert action. So, the action of the f(T) gravity reads:

\[ I = \frac{1}{2k} \int d^4x \left( T + f(T) + \mathcal{L}_m + \mathcal{L}_r \right) , \]  

(6)
where, $k = 8\pi G$, $G$ is the Newtonian constant, $e = det(e^A_\mu) = \sqrt{-g}$, and $L_m$ and $L_r$ are the matter and radiation lagrangian densities. Varying the action with respect to the vierbein, we have:

$$e^{-1}\partial_\nu (e^{\rho A} S^\mu_{\rho A}) [1 + f'(T)] - e^{A\lambda} T^\rho_{\mu \lambda} S^\mu_{\rho A} [1 + f'(T)] + e^{A\lambda} T^\rho_{\mu \lambda} \partial_\mu (T^{\nu}) - \frac{1}{4} e^{A\lambda} [T + f(T)] = k \frac{e^A_{\mu} T^\rho_{\nu}}{2}, \quad (7)$$

whose prime means differentiation with respect to the label and $T^\nu_{\mu}$ is the energy momentum tensor.

We choose the vierbein:

$$e^A_{\mu} = \text{diag}(1, a, a, a), \quad (8)$$

which leads to the flat FRW universe by the metric:

$$ds^2 = dt^2 - a^2(t) dx^3. \quad (9)$$

Putting the vierbein (8) in (9), the torsion scalar will take the following form:

$$T = -6H^2, \quad (10)$$

where, $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Assuming that the matter content of the universe is a perfect fluid with the energy momentum tensor $T^\mu_{\mu} = \rho_m g_{\mu\nu} - (\rho_m + p_m) u_{\mu} u_{\nu}$, we can find the Friedmann-like equations from (7) as follows:

$$H^2 = \frac{k}{3} (\rho_m + \rho_r) - \frac{f(T)}{6} - 2f'(T)H^2, \quad (11)$$

$$\dot{H} = -\frac{k}{3} \frac{(\rho_m + p_m + \rho_r + p_r)}{1 + f'(T) - 12H^2 f''(T)}, \quad (12)$$

where, $\rho_m$ and $p_m$ are the density and pressure of the matter while $\rho_r$ and $p_r$ are the density and pressure of the radiation. The superscript dot denotes the derivative with respect to the cosmic time $t$.

### III. AUTONOMOUS DYNAMICAL SYSTEM IN ITS GENERAL CASE

We consider the $f(T)$ gravity theory in the flat isotropic and homogeneous FRW space-time. The matter content of the universe is chosen to be dust (with an energy density of $\rho_m$ and zero pressure). Radiation parameters, $\rho_r$ and $p_r$, are related to each other via the equation of state $p_r = \omega_r \rho_r$ whose constant is $\omega_r = \frac{1}{3}$. No interaction is assumed to occur between radiation and matter; so, the continuity equations for matter and radiation will read as follows:

$$\dot{\rho}_m + 3H \rho_m = 0,$$

$$\dot{\rho}_r + 4H \rho_r = 0. \quad (13)$$

Assuming that dust and radiation are present in the flat FRW universe, the Friedman equations (11) and (12) will change to (14) and (15) below:

$$H^2 = \frac{k}{3} (\rho_m + \rho_r) - \frac{f(T)}{6} - 2f'(T)H^2, \quad (14)$$

$$\dot{H} (1 + f'(T) - 12H^2 f''(T)) = -\frac{k}{2} \rho_m - \frac{2}{3} k \rho_r. \quad (15)$$

We will consider the general case of the theory with no constraint on the $f(T)$ function in the present section and some useful special cases in the subsections to follow.

The method of dynamical systems is one used for investigating the whole dynamics of a system near extremum points of the theory called, fixed points. In this method, the qualitative behavior of the system is pictured in the phase space by some trajectories which depend on initial values. Based on early and late time behaviors of the system as well as radiation and matter solutions, one can choose some of the trajectories and rule out the inconsistent ones. It is appropriate at this juncture to introduce some dimensionless independent variables in order to simplify the calculations:

$$\Omega_m = \frac{k \rho_m}{3H^2}, \quad (16)$$

$$\Omega_r = \frac{k \rho_r}{3H^2}, \quad (17)$$

$$x = \frac{f(T)}{T}, \quad (18)$$

$$y = -2f'(T), \quad (19)$$

$$z = 2T f''(T). \quad (20)$$

It is clear from Equation (14) that the effective density of dark energy is equal to

$$\rho_D = \frac{3}{k} \left( -\frac{f(T)}{6} - 2f'(T)H^2 \right). \quad (21)$$

So, we can define $\Omega_D$ as follows:

$$\Omega_D = \frac{k \rho_D}{3H^2} = x + y. \quad (22)$$

We should make sure that the variables (16) to (20) are independent; if not, an independent set should be chosen from among them. It is obvious from (18) that $x = x(T)$ and, reversely, $T = T(x)$. Thus, $y(T)$ and $z(T)$ are not independent of $x$ and we have $y(x)$ and $z(x)$. So, from among the three variables $x$, $y$, and $z$, we choose just $x$ as an independent variable. By differentiating $y$ with respect to $x$, we can find an explicit relation between $z$ and $y$ as in the following:

$$y'(x) = \frac{dy}{dx} = \frac{dy/dT}{dx/dT} = \frac{-2f''(T)}{f'(T)} = \frac{z(x)}{x + \frac{1}{2} y(x)}, \quad (23)$$

where, $y'(x)$ means differentiating $y$ with respect to $x$, i.e. $\frac{dy}{dx}$, not to be confused with $\frac{dy}{dT}$.

For our purposes, we choose $x$, $\Omega_m$, and $\Omega_r$ while we eliminate $y$ and $z$. Using the Friedman Equations (11) and (15), the variables (16) to (20), and the relation (23), we can write:

$$\Omega_m + \Omega_r + x + y = 1, \quad (24)$$

$$\frac{H}{H^2} \left( 1 - \frac{1}{2} y + z \right) = -\frac{3}{2} \Omega_m - 2\Omega_r. \quad (25)$$

Based on Eq. (24), we find that there are only two independent variables and that the dynamical system is a
two-dimensional one. We choose $\Omega_r$ and $x$ as the independent dynamical variables and write the autonomous dynamical system as in the following:

$$\frac{d\Omega_r}{dN} = -2\Omega_r \left( -\frac{3}{2} - \frac{1}{2} \Omega_r + \frac{3}{2} (x + y) + 2 \right),$$

$$\frac{dx}{dN} = -\frac{3}{2} - \frac{1}{2} \Omega_r + \frac{3}{2} (x + y) \left( y + 2x \right), \quad (26)$$

where, $N = \ln a$. Moreover, both continuity (13) and Friedmann Equations (24) and (25) are used. In Ref. [20], $x$, $y$, and $\Omega_r$ are chosen as independent variables. As already explained, $y$ cannot be assumed to be independent from $x$ and the autonomous set consists of $x$ and $\Omega_r$. In order to study the autonomous system around the equilibrium points, the right hand side of the system (26) should be continuous and differentiable. Hence, the following constraint applies to all the solutions of the system:

$$1 - \frac{1}{2} y(x) + y'(x) \left( x + \frac{1}{2} y(x) \right) = 1 + f'(T) + 2Tf''(T) \neq 0. \quad (27)$$

The left hand side of Equation (27) is the denominator of the dynamical Equations (26). The special case $1 + f'(T) + 2Tf''(T) = 0$ has a simple solution that will be studied in the next section.

Extremum points of the system, called fixed or critical points, occur if $\frac{dx}{dN} = \frac{d\Omega_r}{dN} = 0$. We find four categories of fixed points as presented in Table I. An accelerated universe requires minus values to be found for the deceleration parameter represented by $q$ in Table II for the present phase of the universe. This parameter is defined as follows:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} = -1 + \frac{3}{2} + \frac{1}{2} \Omega_r - \frac{3}{2} (x + y) + 1 - \frac{1}{2} y + y' \left( x + \frac{1}{2} y \right). \quad (28)$$

It is possible to investigate the stability of the solutions through the study of the eigenvalues of the first order perturbation matrix (Jacobian matrix) near the critical points. These eigenvalues are represented by $\lambda_i$ in Table II. The stability of the regimes presented in this Table depends on the values of $y'(x_0)$, where $x_0$ is the solution of $y(x) + 2x = 0$. Depending on the eigenvalues of the Jacobian matrix, we may have different kinds of stability states. The stable, unstable, and saddle states are possible for real values of $\lambda_i$. Positive eigenvalues yield an unstable fixed point; negative ones, a stable one; while mixed positive and negative values yield a saddle fixed point. When some or all of the eigenvalues have a zero real part, the fixed point is called nonhyperbolic. Novel theories will be required to determine the stability of nonhyperbolic fixed points as the linear stability theory fails to do so.

We are living in a dark-energy-dominated accelerated expanding universe. Going back in time, the universe used to be denser than it is today when matter dominated the universe. Since radiation decays faster than matter in an expanding universe (based on the solution of Equations (13)), radiation must have preceded matter in dominating the world. Prior to the radiation-dominated world, an accelerated expansion, called inflation, is believed to have existed, which connected the beginning of the universe Big Bang to the radiation era after inflation. Thus, the universe experienced two eras of acceleration; early time acceleration due to inflation and late time acceleration due to dark energy. Hence, any proposed cosmological model should contain at least part of the cosmological model below [18]:

inflation $\rightarrow$ radiation $\rightarrow$ matter $\rightarrow$ dark energy.

To obtain the cosmological model above, inflation should be an unstable point for the universe to have inflation exit while radiation and matter points should be saddle ones in order for it to have long enough radiation and matter eras. Finally, dark energy era should be a stable point to have an expanding accelerated phase at the end. Let us now turn to the theory outlined here and see if it is a viable cosmological model. There are four categories of fixed points in Table II. The first category of fixed points (which are represented by $P_1$ in the Table) occur with the solutions of $y(x) + 2x = 0$. We call the roots of this equation $x_0$. Depending on the functionality of $y(x)$, it is possible to have zero, one, or more fixed points. For all these points, $\Omega_r = 1 + x_0$ and $\Omega_m = 0$. The point changes to a radiation-dominated fixed point for small values of $x_0$ because $\Omega_m = 0$ and $\Omega_D = x + y(x) = 0$. Unstable solutions will be obtained in the domain $y'(x_0) > -2$ because both the eigenvalues of the Jacobian matrix (i.e., $\lambda_1$ and $\lambda_2$) are positive. In the same way, saddle solutions are obtained in the domain $y'(x_0) > -2$ because one eigenvalue is positive and the other is negative. Finally,

## Table I. The fixed points and physical parameters of the system (26).

| fixed point | $x$ | $\Omega_r$ | $\Omega_m$ | $\Omega_D$ | $q$ | $\lambda_1$ | $\lambda_2$ | $y'(x_0) < -2$ | $y'(x_0) > -2$ |
|-------------|-----|------------|------------|------------|-----|-------------|-------------|----------------|----------------|
| $P_1$       | $x_0$ | $1 + x_0$  | $0$        | $-x_0$     | $1$ | $1, 2 (y'(x_0) + 2)$ | saddle     | unstable  |
| $P_2$       | $x_0$ | $0$        | $1 + x_0$  | $-x_0$     | $\frac{1}{2}$ | $\frac{1}{2} (y'(x_0) + 2)$ | stable     | saddle    |
| $P_3$       | $x_1$ | $0$        | $0$        | $1$        | $-1, -\frac{3}{4}$ | stable     | stable    |
| $P_4$       | $x_2$ | $0$        | $-1 - x_2 - y(x_2)$ | $x_2 + y(x_2)$ | $-1$ | $\lambda(x_2)$ | saddle |

The point $x_0$ is the solution of $y(x) + 2x = 0$, the point $x_1$ is the solution of $y(x) + x = 1$, and the point $x_2$ is the value in which $1 - \frac{1}{2} y + y' \left( x + \frac{1}{2} y \right)$ diverges. The fixed points $P_2$ for $y'(x_0) < -2$ and $P_3$ are stable. $P_4$ is the solution of late time accelerating phase of the universe.
nonhyperbolic solutions are obtained for \( y'(x_0) = -2 \).
For \( x_0 = -1 \), \( P_3 \) changes to a dark energy-dominated critical point. Since there is no stable solution or accelerating phase, \( (q > 0) \), it fails to describe the late time dark energy-dominated phase of the universe.

The second category of fixed points \( (P_2 \text{ in the Table}) \) are again the solutions of the equation \( y(x) + 2x = 0 \) with \( \Omega_e = 0 \) and \( \Omega_m = 1 + x_0 \). For small values of \( x_0 \), \( \Omega_m \) tends to one, \( \Omega_D \) equals zero, and the point will be a matter-dominated fixed point which will be a stable one for \( y'(x_0) < -2 \), a saddle one for \( y'(x_0) > -2 \), and a nonhyperbolic one for \( y'(x_0) = -2 \). Similar to \( P_1 \), \( P_3 \) changes to a dark energy-dominated fixed point for \( x_0 = -1 \), but it is stable for \( y'(x_0) < -2 \). It cannot yet describe the late time accelerating phase of the universe because the point is in the decelerating phase \( (q > 0) \).
The third category of solutions \( (P_1 \text{ in the Table}) \) are those for the equation \( y(x) + x = 1 \). We call the roots of this equation \( x_1 \). Since both \( \Omega_m \) and \( \Omega_e \) are zero and \( \Omega_D = 1 \), it will be a dark energy-dominated solution. Both of the eigenvalues of the Jacobian matrix are negative; therefore, the fixed point is always stable. Since the deceleration parameter \( q \) has a minus value for this point, it represents the accelerating phase and can successfully describe the late time accelerating stable phase of the universe.

The fourth category of fixed points \( (P_4 \text{ in the Table}) \) are those in which \( 1 - \frac{1}{2} y + y'(x + \frac{1}{2} y) \) diverges and the condition \( \Omega_D = x + y \neq 1 (P_4 \text{ equals to } P_2, \text{ if } \Omega_D = 1) \) applies. We call these points \( x_2 \). Given that \( \Omega_m \neq 0 \) in the accelerating phase \( (q = -1) \), the point is related to the inflation dominated era. It is a stable fixed point if \( \lambda(x_2) < 0 \), and a saddle one if \( \lambda(x_2) > 0 \), where \( \lambda(x) \) has the following form:

\[
\lambda(x) = \frac{3}{2} \left( \frac{y' + 2}{1 - \frac{1}{2} y + y'(x + \frac{1}{2} y)} - \frac{(1 - x - y)(y + 2x)}{(1 - \frac{1}{2} y + y'(x + \frac{1}{2} y))^2} \right) \]

No situation can be imagined in which \( P_4 \) becomes unstable; it is, therefore not possible for the universe to have inflation exit, implying that this point cannot be accepted as a true inflation point.

To summarize, we have already established that there existed in the history of the universe an inflation phase, two radiation and matter-dominated eras, and a dark energy accelerating phase. As already mentioned, \( P_4 \) is not an unstable fixed point and cannot be, therefore, accepted as a true inflation point. As for the other three points, if we choose \( f(T) \) in a way that \( y'(x_0) > -2 \), we will then have an unstable radiation era followed by a matter-dominated and an accelerating dark energy one. On the other hand, if \( y'(x_0) < -2 \), there is a saddle radiation phase followed by a stable matter-dominated one and a dark energy era. In each case, one of the matter or radiation eras is not a saddle one and it will not thus be possible for the universe to remain in that situation for a long time. The case with \( y'(x_0) > -2 \), however, offers the advantage that the transition radiation→matter→dark energy will become possible. In this case matter and dark energy dominated eras will be acceptable solutions, and models of \( f(T) \) will then have to be selected that have at least one point in each of the categories \( P_3 \) and \( P_4 \) presented in Table I. In other words, each of the equations \( x + y(x) = 1 \) and \( y(x) + 2x = 0 \) should have at least one real solution. Based on the energy conditions, we should have \( 0 \leq \Omega_m \leq 1, 0 \leq \Omega_e \leq 1 \). Hence, the preferred value of \( \Omega_D \) obtained from Equations \((22)\) and \((23)\) will be \( -1 \leq \Omega_D \leq 1 \). Since at any point in Table I, one of the parameters \( \Omega_m \) and \( \Omega_e \) is zero, the preferred value of \( \Omega_D \) becomes \( 0 \leq \Omega_D \leq 1 \). This yields \(-1 \leq x_0 \leq 0 \), based on Equation \((22)\) and the fact that \( y(x_0) + 2x_0 = 0 \).

As already mentioned, the points \( P_1 \) and \( P_2 \) become dark energy dominated fixed points for \( x_0 = -1 \). Since the deceleration parameters of these points are positive, they refer to a decelerating universe in a dark energy dominated era and correspond to no physical situation. Thus, the lower bound of \(-1 \leq x_0 \leq 0 \) will be forbidden. It then follows that a viable model that contains two parts of the standard cosmological model should satisfy the following conditions:

1. The equations \( x + y(x) = 1 \) and \( y(x) + 2x = 0 \) should have at least one real solution,
2. \(-1 < x_0 \leq 0 \),
3. \( y'(x_0) > -2 \).

The study of the phase space will be exhausted by investigating the critical points at infinity, where the dynamical variables diverge. This requires the prior investigation of the valid domain of the dynamical variables, \( \Omega_r \) and \( x \) to see if they can possibly tend to infinity. Based on the energy conditions, the energy density of radiation and matter should be positive. This means that the allowed domain of the density parameters \( \Omega_r \) and \( \Omega_m \) are \( 0 \leq \Omega_r \leq 1 \) and \( 0 \leq \Omega_m \leq 1 \) and, consequently, one of the dynamical variables \( (\Omega_r) \) is always finite. Considering the allowed domain of the density parameters and using Equation \((24)\), we obtain \(-1 \leq x(T) + y(T) \leq 1 \). This inequality determines the allowed domain of \( T \). If \( x(T) \) diverges in this domain, then the study of the phase space at infinity should be performed following the Painlevé central projection method \([21]\).

IV. SPECIAL CASES

In this section, we first consider the constraint mentioned in \([27]\) before we embark on the study of some specific \( f(T) \) models with two parameters \([22]\). In each case, one of the model parameters will be free while the other is determined through our analysis. We will finally examine the conditions cited in the previous section and find the preferred values for the model parameters.
IV.1. Special case: \(1 + f'(T) + 2Tf''(T) = 0\)

The general solution to the differential Equation (27) is:

\[
f(T) = -T + c_1 \sqrt{-T} + c_2.
\]  

(30)

Based on Equation (15), this solution leads to the following relation between the densities:

\[
\rho_m = -\,\frac{4}{3} \rho_r.
\]  

(31)

Except in the case \(\rho_m = \rho_r = 0\), the above equation makes no physical sense and leads to the violation of energy conditions. Plugging \(\rho_m = \rho_r = 0\) and (30) in the first Friedman Equation (14) yields \(c_2 = 0\). This is also a trivial case and leads to no physical result.

IV.2. The Power Law Model: \(f(T) = \alpha(-T)^b\)

A simple and useful choice for \(f(T)\) is a power law function \([22]\) of the following form:

\[
f(T) = \alpha(-T)^b,
\]  

(32)

in which, \(\alpha\) and \(b\) are two model parameters. As pointed out in Refs. [10, 23], the model reduces to the ΛCDM model when \(b = 0\), and to the DGP one [23] when \(b = 1/2\). However, it leads to a redefinition of the Newton constant when \(b = 1\). In order to be consistent with the observational data, the requirement that \(|b| < 1\) should be observed [10, 14, 23, 25].

As already mentioned above, any physical function \(f(T)\) should satisfy some conditions to be a viable standard cosmological model. We will examine the power law function and see which range of \(b\) will render it into a viable model for \(f(T)\) gravity. The first condition to be met is that the two equations \(y(x) + x = 1\) and \(y(x) + 2x = 0\) should have real solutions. For \(f(T)\) in Equation (32), we have:

\[
x = \frac{f(T)}{T} = -\alpha(-T)^{b-1},
\]  

(33)

\[
y = -2f'(T) = 2ab(-T)^{b-1},
\]  

(34)

which leads to \(y(x) = -2bx\). One can check the first condition as:

\[
y(x) + x = -2bx + x = 1 \Rightarrow x_1 = \frac{1}{1 - 2b},
\]  

(35)

\[
y(x) + 2x = -2bx + 2x = 0 \Rightarrow x_0 = 0.
\]  

(36)

Except for \(b = \frac{1}{2}\), both of the above equations have real solutions; thus, the first condition is satisfied. The second condition states that \(-1 < x_0 \leq 0\); since \(x_0 = 0\), the second condition is satisfied as well. Finally, for the third condition, we have \(y'(x_0) > -2\) based on (23):

\[
y'(x_0) = -2b > -2 \Rightarrow b < 1.
\]  

(37)

This result is consistent with earlier research [10, 14, 23], stating that observational data indicate that the absolute values of \(b\) are far smaller than one. Thus, the conditions for matter and dark energy points are satisfied.

The power law function with \(b < 1\) and \(b \neq \frac{1}{2}\) leads to desirable results and viable cosmological predictions. Ref. [24] did not derive this constraint while the analysis provided was basically wrong. The fixed points of the power law function are plotted in Figure 1 for two values of \(b\). Three fixed points are clearly present in Figs. 1(a) and 1(b), indicating that and the transition from a radiation era to a matter one followed by an accelerating phase is possible. Thus, the power law function successfully describes two parts of the standard cosmological model. The last step is to consider the phase space to see if it is possible for the dynamical variables to tend to infinity and whether there exists any fixed point there. The allowed domain of the dynamical variables are \(0 \leq \Omega_r \leq 1\) and \(\frac{1}{(2b-1)} \leq x \leq \frac{1}{(2b-1)} \, (-1 \leq x + y \leq 1)\). The phase space is compact except for \(b = \frac{1}{2}\). For \(b = \frac{1}{2}\), the dynamical variable \(x\) seems to be tending to infinity and the phase space to be noncompact; this is not the case, however; rather in this case \(y = -2bx = -x\) (using Eqs. (23) and (31)). The dark components in Eqs. (23) and (25) are, therefore, totally absent; i.e. \(x + y = 0\) in Eq. (24), and \(-\frac{1}{2}y + z = \frac{1}{2}x - (x - \frac{1}{2}x) = 0\) in Eq. (25). The Friedman-like equations reduce to the typical Teleparallel theory either with no dark energy sector or no dynamical variable \(x\) to tend to infinity.

IV.3. The Logarithmic Model:

\[
f(T) = \alpha T_0 \sqrt{\frac{T}{qT_0}} \ln \left(\frac{qT_0}{T}\right),
\]  

(38)

Another model is the logarithmic one [20]:

\[
f(T) = \alpha T_0 \sqrt{\frac{T}{qT_0}} \ln \left(\frac{qT_0}{T}\right),
\]  

(38)

where, \(\alpha\) and \(q\) are two model parameters and \(T_0\) is the present value of the torsion parameter. To study the model, we calculate \(x\) and \(y\) using Eqs. (18) and (19):

\[
x(T) = \frac{f(T)}{T} = \alpha \sqrt{\frac{T_0}{qT}} \ln \left(\frac{qT_0}{T}\right),
\]  

(39)

\[
y(T) = -2f'(T) = \alpha \sqrt{\frac{T_0}{qT}} \left[2 - \ln \left(\frac{T}{T_0}\right)\right]
\]  

(40)

\[
= 2\alpha \sqrt{\frac{T_0}{qT}} - x(T),
\]  

(40)

where, \(\alpha\) can be determined using (40), and the present values of the cosmological parameters in Friedman Equation (24), as follows:

\[
\alpha = (1 - \Omega_{m_0} - \Omega_{r_0}) \frac{\sqrt{q}}{2} = 0.364 \sqrt{q},
\]  

(41)

where, \(\Omega_{m_0} = 0.272\) and \(\Omega_{r_0} = 8.0331 \times 10^{-5}\) are used [27]. According to the first condition, the following equation should have a real solution:

\[
y(T) + 2x(T) = 2\alpha \sqrt{\frac{T_0}{qT}} + \alpha \sqrt{\frac{T_0}{qT}} \ln \left(\frac{qT_0}{T}\right) = 0.
\]  

(42)
For the third condition, we have:

$$y'(x_0) = \frac{dy}{dx} |_{T=qT_0 e^2} = \left[ \frac{2}{2 + \ln \left( \frac{qT_0}{T} \right)} - 1 \right]_{T=qT_0 e^2} = \pm \infty.$$  

Since, $y'(T)$ diverges for $T = qT_0 e^2$, the third condition is not satisfied and $x_0 = -\frac{2\alpha}{qe}$ is not a true matter point. It should be noted that the place of the radiation point in this case is slightly different with from that in Table I. Using (39) and (40), the term $y' \left( x + \frac{1}{2} y \right)$ in the denominator of Eq. (26) is equal to:

$$y'(T) \left( x(T) + \frac{1}{2} y(T) \right) = \frac{1}{2} \left[ \frac{2}{2 + \ln \left( \frac{qT_0}{T} \right)} - 1 \right] \alpha \sqrt{\frac{T_0}{qT}} \left( 2 + \ln \left( \frac{qT_0}{T} \right) \right) = -\frac{1}{2} x(T).$$  

The coordinates of the radiation point in the $(x, \Omega_r)$ plane is, therefore, $(x_0, 1-x_0)$, rather than $(x_0, 1+x_0)$ in Table I. The dark energy dominated era is based on the solution of the following equation:

$$y(T) + x(T) = 2\alpha \sqrt{-\frac{T_0}{qT}} = 1 \Rightarrow T = \frac{4\alpha^2 T_0}{q}. \quad (47)$$  

Since the above equation has a real solution, all the conditions are satisfied. Hence, the logarithmic form of Eq. (38) with $q > 0$ can describe successfully the matter and dark energy eras of the standard model of cosmology. The relevant phase space plot is presented in Fig. 2 in which the five fixed points and the transition between them are clearly seen.

The last step is to study the infinite behavior of the dynamical variables and the related fixed points, if they ever exist. We begin with the examination of the allowed domain of the dynamical variables to see if it is possible for the dynamical variables to tend to infinity. As mentioned in the previous Section, the allowed domains of $\Omega_r$...
and $x$ are, $0 \leq \Omega_r \leq 1$ and $-1 \leq x + y \leq 1$. The latter leads to $-1 \leq 2\alpha \sqrt{\frac{q}{T}} \leq 1$, based on Eq. (40), and yields $T < \frac{4\alpha^2 T_0}{q}$. Considering Eq. (39), it is obvious that, for a finite $q$, $x$ is finite in this domain and cannot tend to infinity. Hence, there are no fixed points at infinity. This concludes the study of the phase space and fixed points.

### IV.4. The Hyperbolic-Tangent Model: $f(T) = \alpha(-T)^n \tanh \left(\frac{T_0}{T}\right)$

The next model to examine is the hyperbolic-tangent model [28] of the following form:

$$f(T) = \alpha(-T)^n \tanh \left(\frac{T_0}{T}\right),$$

(48)

where, $\alpha$ and $n$ are two model parameters. Using the same calculations as in the previous subsection, we can obtain $x$, $y$, and $\alpha$ as in the following:

$$x(T) = \frac{f(T)}{T} = -\alpha(-T)^{n-1} \tanh \left(\frac{T_0}{T}\right),$$

$$y(T) = -2f'(T)$$

$$= 2\alpha \left[(-T)^{n-2} \cosh^{-2} \left(\frac{T_0}{T}\right) + n(-T)^{n-1} \tanh \left(\frac{T_0}{T}\right)\right],$$

(50)

$$\alpha = \frac{0.728(-T_0)^{1-n}}{1.523n + 1.601},$$

(51)

To evaluate the model, we start with the matter point $x_0$, which is the solution of:

$$y(T) + 2x(T) = 2\alpha \left[(-T)^{n-2} T_0 \cosh^{-2} \left(\frac{T_0}{T}\right)\right.\left. + (n-1)(-T)^{n-1} \tanh \left(\frac{T_0}{T}\right)\right] = 0. \quad (52)$$

For $n \geq 1$, $T = 0$ ($T = -\infty$) is the solution of the above equation. Since $T = 0$ corresponds to $H = 0$ ($T = -6H^2$), it can be a solution for the late time dark energy era; however, it cannot explain inflation, radiation, or matter eras. We, therefore, need to ignore $n \geq 1$, and keep $n < 1$ as a viable model parameter. The other conditions for the matter point are $-1 < x_0 \leq 0$ and $y'(x_0) > -2$, which can be checked for $T = -\infty$ and $n < 1$ as follows:

$$x_0 = \lim_{T \to -\infty} \left[ -\alpha(-T)^{n-1} \tanh \left(\frac{T_0}{T}\right) \right] = 0,$$

$$y'(x_0) = \lim_{T \to -\infty} \frac{dy/dT}{dx/dT} \bigg|_{T=-\infty}$$

$$= \lim_{T \to -\infty} \frac{2(1-n)T \left[nT \sinh \left(\frac{2T_0}{T}\right) - 4T_0\right] + 8T_0^2 \tanh \left(\frac{2T_0}{T}\right)}{T \left[(n-1)T \sinh \left(\frac{2T_0}{T}\right) - 2T_0\right]$$

$$= 2(1-n) > -2. \quad (53)$$

Clearly the conditions for the matter point are satisfied and $T^0 = -\infty$ can be accepted as a true matter solution.

Now, we turn to the dark energy dominated era. The

![FIG. 2. The behavior of phase trajectories for the system in (26) for $f(T) = \alpha T_0 \sqrt{\frac{T}{T_0}} \ln \left(\frac{T}{T_0}\right)$ where $q = 1$. Since the function $x(T)$ is not a one to one function (it has a minimum at $T = 7.39$), it is not possible to show the whole phase space in one plot. The behavior of the phase space for $T < 7.39$ and $T > 7.39$ are shown in plot (a) and (b), respectively. The critical points $P_1$, $P_2$, and $P_3$ are at $(1,0)$, $(0,0)$ and $(0,0.32)$, respectively. The two unacceptable fixed points are $(1.27,-0.27)$ and $(0,-0.27)$ which exhibit different stability behaviors in the two plots.](image)
This concludes the study of the phase space.

We can show that the phase space of the theory is compact and there exist no infinite fixed point in the theory. As in the previous models, we may claim that the phase space is compact and there exist no infinite fixed point in the theory. This concludes the study of the phase space.

IV.5. The Linder Model: \( f(T) = \alpha T_0 (1 - e^{-p\sqrt{T/T_0}}) \)

The Linder model \( f(T) = \alpha T_0 (1 - e^{-p\sqrt{T/T_0}}) \) is captured by the following function:

\[
f(T) = \alpha T_0 (1 - e^{-p\sqrt{T/T_0}}),
\]

where, \( \alpha \) and \( p \) are two model parameters. \( x, y, \) and \( \alpha \) may be obtained as follows.

\[
x = \frac{f(T)}{T} = \alpha T_0 (1 - e^{-p\sqrt{T/T_0}}),
\]

\[
y = -2f'(T) = -\alpha p \sqrt{\frac{T_0}{T}} e^{-p\sqrt{T/T_0}},
\]

\[
\alpha = \frac{1 - \Omega_{m0} - \Omega_{\Lambda0}}{1 - (1 + p)e^{-p}} = \frac{0.728}{1 - (1 + p)e^{-p}}.
\]

According to the first condition, the following equation should have a real solution:

\[
y(T) + 2x(T) = 2\alpha \frac{T_0}{T} (1 - e^{-p\sqrt{T/T_0}}) - \alpha p \sqrt{\frac{T_0}{T}} e^{-p\sqrt{T/T_0}}
\]

\[
= 0.
\]

Numerical calculations show that the above equation has a root at \( T = \frac{2.54 T_0}{p} \) for \( p < 0 \). The other conditions for the matter point are \(-1 < x_0 \leq 0 \) and \( y'(x_0) > -2 \). Replacing \( T = \frac{2.54 T_0}{p} \) in (56) yields \( x_0 = \frac{0.287(1 - e^{-1.59 T_0})p^2}{1 - e^{-p(1 + p)}} \), which is in the desired regime for \( p < -1.162 \). For a negative \( p \), however, \( y'(x_0) \) is always less than \(-2 \). Hence, the conditions for the matter point are not satisfied for \( p < 0 \). For \( p > 0 \), \( T = -\infty \) is the solution of (59).

Since \( x(-\infty) = y'(-\infty) = 0 \), both conditions for the matter point are satisfied for \( p > 0 \). We turn to the dark energy point \( x_1 \), which is the solution of the following equation:

\[
y(T) + x(T) = \alpha p \sqrt{\frac{T_0}{T}} e^{-p\sqrt{T/T_0}} + \alpha \frac{T_0}{T} \left(1 - e^{-p\sqrt{T/T_0}}\right)
\]

\[
= 1.
\]

Numerical calculations show that the above equation has a real solution except for \( 0 \leq p < 0.5 \). It follows that the Linder model with \( p \geq 0.5 \) is capable of describing successfully the matter and dark energy dominated eras of the universe. Phase space trajectories of the Linder model for \( p = 2 \) are plotted in Fig. 4, in which radiation, matter, and dark energy fixed points and the transition between them can be identified in Fig. 4.

As in the previous models, may be claimed that the phase space is compact and there exist no infinite fixed points.
For $0 < T < 1$, the condition for the dark energy dominated era is satisfied as well. However, the existence of a real $y$ is forbidden in the universe. As the valid range of $T$ breaks up into two parts in the case, the true domain is the one that contains $x_0$; i.e., the lower value of $T$ (for $x_0$, $T = -\infty$). For $1.977 < p < 6.110$, the maximum value of $x(T) + y(T)$ is less than 1 and there is, thus, no solution at all. Hence, the valid domains of $p$, in which the theory explains both matter and dark energy dominated eras, are $0.203 \leq p \leq 1.977$, $p \geq 6.110$ and $p \neq 1.256$. In Fig. 5, the behavior of the phase trajectories and fixed points of the exponential model are plotted for two values of $p$.

It is observed that the infinite value for $x$ is forbidden and the phase space is compact in this model, too. We conclude that the study of the phase space and the fixed points of the theory is complete.

Table II summarizes the general features and the related domains of the independent model parameters of the five specific two-parameter models investigated in this section. It is straightforward to check other functional forms of the $f(T)$ theory using similar calculations. Once the $f(T)$ function is chosen, it will be possible to calculate the variables $x$ and $y$ along the lines outlined above and calculate $x_0$ and $x_1$ either analytically or numerically. The existence of a real $x_0$ is a necessary condition for describing the matter dominated era. The two other conditions, which are $-1 < x_0 \leq 0$ and $y'(x_0) > -2$, should be satisfied as well. However, the existence of a real $x_1$ suffices for describing the dark energy dominated era.

**IV.6. The Exponential Model:**

Finally, we investigate the exponential model defined as follows:

$$f(T) = \alpha T_0 (1 - e^{-p T_0}),$$

where $\alpha$ and $p$ are two model parameters. $x$, $y$, and $\alpha$ are expressed as follows:

$$x = \frac{f(T)}{T} = \alpha \frac{T_0}{T} (1 - e^{-p T_0}),$$

$$y = -2f'(T) = -2\alpha e^{-p T_0},$$

$$\alpha = \frac{1 - \Omega_{m0} - \Omega_{r0}}{1 - (1 + 2p)e^{-p}} = \frac{0.728}{1 - (1 + 2p)e^{-p}}.$$  \hspace{1cm} (64)

The matter point, $x_0$, is the solution of the following equation:

$$y(T) + 2x(T) = -2\alpha e^{-p T_0} + 2\alpha \frac{T_0}{T} (1 - e^{-p T_0}) = 0.$$  \hspace{1cm} (65)

The above equation, has two solutions at $T = 0$ and $T = -\infty$ for $p > 0$. $T = 0$ is not an acceptable solution for the matter point while for $T = -\infty$, $x_0 = y'(x_0) = 0$, and the conditions for the matter point are satisfied as well. The condition for the dark energy dominated era is that the solution for the following equation should exist:

$$y(T) + x(T) = -2\alpha e^{-p T_0} + \alpha \frac{T_0}{T} (1 - e^{-p T_0}) = 1.$$  \hspace{1cm} (66)

The function $x(T) + y(T)$ has a complicated behavior. For $0 \leq p < 0.203$, the above equation has a positive solution ($T > 0$); however, the solution is not acceptable since $T = -6H^2$. The function $x(T) + y(T)$ has a minimum and a maximum value for $0.203 \geq p < 1.256$ and $p > 1.256$, respectively. For $p = 1.256$, $\alpha$ becomes infinite. For $1.256 \leq p \leq 1.977$, the maximum value of the $x(T) + y(T)$ is greater than 1; hence, there are two solutions for $x(T) + y(T) = 1$. This leads to the question which solution is the late time attractor of the universe. As the valid range of $T$ breaks up into two parts in the case, the true domain is the one that contains $x_0$; i.e., the lower value of $T$ (for $x_0$, $T = -\infty$). For $1.977 < p < 6.110$, the maximum value of $x(T) + y(T)$ is less than 1 and there is, thus, no solution at all.

In Table II, the behavior of the phase trajectories and fixed points of the exponential model are plotted for two values of $p$.

It is observed that the infinite value for $x$ is forbidden and the phase space is compact in this model, too. We conclude that the study of the phase space and the fixed points of the theory is complete.

**V. CONCLUDING REMARKS**

The cosmological behavior of the $f(T)$ gravity theory was investigated for the perfect dust matter and radia-
FIG. 5. The behavior of the phase trajectories for the exponential model, \( f(T) = \alpha T_0 (1 - e^{-p T/T_0}) \) where, \( p = 1 \) in plot (a) and \( p = 1.5 \) in plot (b). The critical points \( P_1 \) and \( P_2 \) are located at \((1, 0)\), and \((0, 0)\), respectively. For \( 0.203 \leq p \leq 1.256 \), there is one \( P_3 \) at \((-0.59, 0)\) in plot (a). For \( 1.256 \leq p \leq 1.977 \), there are two \( P_3 \) solutions located at at \((0, 1)\) and \((0, 5.04)\) in plot (b). The late time attractor of the universe is \((0, 1)\), to which the transition of the matter point.

| The model          | \( f(T) \) function                                 | Allowed range of the independent parameter |
|--------------------|-----------------------------------------------------|--------------------------------------------|
| Power law          | \( \alpha (-T)^b \)                                 | \( b < 1, b \neq \frac{1}{2} \)            |
| Logarithmic        | \( \alpha T_0 \sqrt{\frac{T}{T_0}} \ln \left( \frac{T}{T_0} \right) \) | \( q > 0 \)                               |
| Hyperbolic-tangent | \( \alpha (-T)^n \tanh \left( \frac{T}{T_0} \right) \) | \( n < -1 \)                              |
| Linder             | \( \alpha T_0 (1 - e^{-p \sqrt{T/T_0}}) \)         | \( p \geq 0.5 \)                           |
| Exponential        | \( \alpha T_0 (1 - e^{-p \sqrt{T/T_0}}) \)         | \( 0.203 \leq p \leq 1.977, p \geq 6.110, p \neq 1.256 \) |

Based on these conditions, the theory describes successfully two periods of standard cosmological model, namely matter and dark energy dominated eras. The constraint \( 1 + f'(T) + 2T f''(T) = 0 \) was studied separately. Five special cases of the \( f(T) \) function (namely the power law, logarithmic, hyperbolic-tangent, Linder, and exponential models) were also investigated. All the models were found to satisfy the conditions for certain values of the model parameters, which revealed their capability to describe the matter and dark energy eras of the standard model of cosmology. Our method may also be used for other generalized theories of gravity.

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