On non-abelian T-dual geometries with Ramond fluxes

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Abstract

We show how to implement T-duality along non-abelian isometries in backgrounds with non-vanishing Ramond fields. When the dimension of the isometry group is odd (even) the duality swaps (preserves) the chirality of the theory. In certain cases a non-abelian duality can result in a massive type-IIA background. We provide two examples by dualising $SU(2)$ isometry subgroups in $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$. The resultant dual geometries inherit the original $AdS$ factors but have transverse spaces with reduced isometry and preserve only half of the original supersymmetry. The non-abelian dual of $AdS_5 \times S^5$ has an M-theory lift which is related to the gravity duals of $\mathcal{N} = 2$ superconformal theories. We comment on a possible interpretation of this as a high spin limit.
1 Introduction

T-duality is one of the most striking features of string theory and, though its discovery dates back a quarter of a century, it continues to provide us new insight into the nature of string theory.

An interesting recent development has been the use of T-duality in conjunction with the AdS/CFT correspondence to explain the amplitude/Wilson loop connection of \( \mathcal{N} = 4 \) gauge theory [1, 2]. This application required one to perform both traditional bosonic and novel fermionic T-dualities in an \( AdS_5 \times S^5 \) background supported by...
Ramond flux. These fluxes are of immense importance since they support the gravitational backgrounds corresponding to the various Dp-branes that have been instrumental to all fundamental theoretical developments ranging from U-duality to the AdS/CFT correspondence. Given the importance of such Ramond backgrounds it is natural to ask what other applications could T-duality have in this context.

In this paper we shall discuss the extension of T-duality to the case of isometries which form a non-abelian group structure [3]. In comparison to the abelian T-duality this non-abelian generalisation has remained rather poorly understood in spite of a substantial body of work, e.g. [4]-[9]. One reason for this is that global topological issues on the world sheet make it hard to establish the duality as an exact symmetry of a Conformal Field Theory (CFT) [4]. Nonetheless, one may view these transformations as a map between related CFT’s or as generating new effective backgrounds. In that respect it has been recently shown that non-abelian T-duality can be considered as an effective theory describing consistent sectors of infinite highest weight representations, i.e. infinitely large spins, of certain parent theories involving Neveu–Schwarz (NS) fields only [10].

In this paper we move a step forward in understanding non-abelian T-duality structures in string theory by examining how Ramond–Ramond (RR) field strengths transform. Since this presents technical as well as conceptual difficulties we restrict ourselves to the case where the directly relevant part of the Neveu–Schwarz sector in the supergravity background takes the form of a Principal Chiral Model.

For abelian T-duality the extension of the Buscher rules [11] to such Ramond backgrounds was first achieved from a target space perspective [12] (see also [13, 14]), and then established from a world sheet perspective in the Green–Schwarz formalism [15, 16] and more recently in the pure spinor formalism [17] and as a canonical transformation [18].

The key to understanding the behaviour of RR-fields is the observation that left and right movers transform differently under T-duality, i.e. $\partial_{\pm} \tilde{X}^I = (A_{\pm})^I_j \partial_{\pm} X^j$, for certain matrices $A_{\pm}$ that depend on the metric and NS antisymmetric tensor [13, 14]. In particular, the left and right movers of the dual theory couple to different target space vielbeins related to each other by a local Lorentz transformation given by $\Lambda = A_{\pm}^{-1} A_{-}$. To express the dual theory in a single frame one uses this transformation to rotate the left movers back to the same frame as the right movers. Doing so induces some trans-
formation $\Omega$ on the target space spinor indices. Since the RR-field strengths, together with the dilaton, can be combined into a bispinor $P_{\alpha\beta}$, they undergo a transformation of the form $\hat{P} = P\Omega^{-1}$ from which the transformation of the individual RR-fluxes can be obtained.

We will see that the above argument essentially holds for the case of non-abelian T-duality however with some important modifications. Firstly, the induced local Lorentz transformation is somewhat more complicated since it may depend on the "internal" directions, i.e., those being dualised. Furthermore, we will see that the determinant of the Lorentz transformation $\Lambda$ may be either plus or minus one, depending on the dimensionality of the isometry group being dualised. Starting with a type-IIB string background the resultant theory will remain of type-IIB for an even-dimensional isometry group, but for an odd-dimensional isometry group the T-dual is a background of type-IIA supergravity and vice versa. When the starting background is of type-IIB and has an RR three-form with support in the directions being dualised corresponding to an $SU(2)$ isometry group, it is possible to end up with a background of massive type-IIA supergravity, a theory constructed in [19]. More generally this may happen when the T-duality transformation involves $p$-forms and $p$-dimensional groups.

We will apply these ideas to two familiar geometries; the $AdS_3 \times S^3 \times T^4$ background arising in the near horizon limit of the D1-D5 system and the $AdS_5 \times S^5$ background corresponding to the D3-brane near horizon geometry. In both cases we shall consider the dualisation along isometries forming an $su(2)$ algebra which generate the left (or right) action on a three-sphere in the geometry. The result will be that we find dual geometries in which the $AdS$ factor remains unaltered but the transverse spaces are quite different and have reduced isometry. Correspondingly we find that the dual geometries have reduced supersymmetry with, in both cases, exactly one half of the original supersymmetries preserved. We show that in the dual geometries the preserved Killing spinors obey position dependent projection conditions. We suggest that the fraction of preserved supersymmetry can be established by considering the action of the spinor-Lorentz-Lie-(Kosmann) derivative $[20, 21, 22]$ along the direction of the isometries about which we dualise. More specifically, we suggest that the number of Killing spinors of the original geometry for which this derivative vanishes correspond to the number of supersymmetries preserved under dualisation.

For the case of the non-abelian T-dual $AdS_5 \times S^5$ background we show that the dual
geometry has an M-theory lift which is related to the Gaiotto–Maldacena (GM) gravity duals of $\mathcal{N} = 2$ superconformal theories. More precisely we find a background which can be obtained as a leading order approximation to a generic GM background close to the origin. We provide a possible interpretation of this in terms of a high spin limit of a parent $\mathcal{N} = 2$ gauge theory.

The structure of this paper is as follows: in the section 2 we review the action of non-abelian T-duality in Principal Chiral Models, we establish the Lorentz transformation that relates left and right moving frames from which consequently one may derive the transformation rules for the RR fluxes. In section 3 we consider the dualisation of the D1-D5 near horizon geometry and in section 4 we examine the $AdS_5 \times S^5$ background corresponding the D3-brane near horizon geometry. We close the paper in section 5 with a short discussion and conclusion. In the appendix we provide some details of the massive type-IIA supergravity which are relevant to our discussion.

2 Non-abelian T-duality in Principal Chiral Models

In this paper we restrict to a particular class of non-abelian T-duality transformations in which the isometry appears through a Principal Chiral Model (PCM) [23]-[26]. Whilst the PCMs may not constitute a solution to string/supergravity theory on its own, they can often be naturally embedded as integrable parts, as we shall see, of true supergravity solutions involving D-brane configurations.

A PCM is a class of two-dimensional $\sigma$-model with $G_L \times G_R$ global symmetry for a group $G$. The $\sigma$-model action for a group element $g \in G$ is given by

$$S(g) = - \int d^2 \sigma \, \text{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g)$$

(2.1)

and obviously, it is invariant under a global $G_L \times G_R$ symmetry. Actually, there ia a much larger classical affine symmetry [26, 27, 28] which however will play no rôle in our considerations.
2.1 The T-dual

We would like to find the non-abelian dual of this action corresponding to a subgroup $G_L$. To achieve this one first gauges some isometry by introducing gauge fields and covariant derivatives into the $\sigma$-model. This gauged theory is supplemented with Lagrange multipliers which enforce that the gauge fields are (locally) pure gauge and means that the original $\sigma$-model is recovered upon gauge fixing them to zero. If, instead of the Lagrange multipliers, the gauge fields are integrated out one arrives at the T-dual $\sigma$-model with the Lagrange multiplier playing the rôle of T-dual coordinates.

In general, for non-abelian isometries this process can be complicated and the gauge fixing needs to be done on a case by case basis. However, for the case at hand, i.e. for PCMs, one may give general expressions.

We choose to gauge the entire $G_L$ isometry with the corresponding action

$$S_{\text{nonab}}(g, v, A_\pm) = -\int d^2\sigma \operatorname{Tr}(g^{-1}D_- g g^{-1} D_+ g ) + i \operatorname{Tr}(v F_{+-}) ,$$

(2.2)

where $v$ is the Lagrange multiplier matrix. The covariant derivatives and the field strength for the gauged fields are given by

$$D_\pm g = \partial_\pm g - A_\pm g , \quad F_{+-} = \partial_+ A_- - \partial_- A_+ - [A_+, A_-].$$

(2.3)

The action above is invariant under the local "left" symmetry

$$g \rightarrow \lambda^{-1} g , \quad v \rightarrow \lambda^{-1} v \lambda , \quad A_\pm \rightarrow \lambda^{-1} A_\pm \lambda - \lambda^{-1} \partial_\pm \lambda , \quad \lambda (\sigma^+, \sigma^-) \in G ,$$

(2.4)

as well as the global "right" symmetry $g \rightarrow g \Lambda_R$, where $\Lambda_R \in G$.

The gauge fields in (2.2) are non-dynamical and can be eliminated via their equations of motion. We also gauge fix dim$(G)$ of the parameters. As a gauge fixing we choose the group element $g$ to be the identity which is possible since the group action has no isotropy. This gauge choice completely gets rid of the parameters in $G$ and we are left with a $\sigma$-model solely for the Lagrange multipliers $v$. The result is a dual action given by

$$S = \int d^2\sigma \partial_+ v_i (M^{-1})^{ij} \partial_- v_j , \quad M_{ij} = \delta_{ij} + f_{ij} , \quad f_{ij} \equiv f_{ij}^k v_k.$$  

(2.5)

From this we may read off the target space metric and NS two-form as the symmetric

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1For didactic purposes we ignore spectator fields (i.e. directions supplementary to those being dualised) coupled to the above PCM action, though they may easily be included as we do later.
and anti-symmetric parts of \((M^{-1})^{ij}\), respectively. The above action still processes a global \(G\)-symmetry associated to the original right action of the group that is left intact. The process of integrating out the gauge fields introduces an extra factor in the path integral measure leading to a non-trivial dilaton in the T-dual theory given by

\[ e^{-2\Phi} = \det(M) . \]  

(2.6)

Of course this is the true dilaton field only when the PCM is embedded in a full string background. This is the case in the explicit examples we consider below.

### 2.2 Transforming the RR-fluxes

In order to see how a corresponding transformation for RR-flux fields is induced, we first establish how T-duality acts on world-sheet derivatives. This is properly done in the formulation of non-abelian T-duality as a canonical transformation in phase space [5, 29]. Introducing local coordinates \(\theta^a\) on the group manifold \(G\) and defining the left-invariant one-forms as \(L = g^{-1}dg = L^i d\theta^a T_i\), we may express the transformation of world-sheet derivatives as [29, 18]

\[ \partial_+ v_l = M_{kl} L^k_b \partial_+ \theta^b , \quad \partial_- v_l = -M_{lk} L^k_b \partial_- \theta^b . \]  

(2.7)

The difference in transformation between left and right movers is given by the matrix

\[ \Lambda^i_j = - \left( MM^{-1}T \right)^i_j . \]  

(2.8)

Since \(M\) and \(M^T\) commute it can be seen that \(\Lambda^T = \Lambda^{-1}\) and \(|\det \Lambda| = 1\). Hence \(\Lambda\) defines a local Lorentz transformation however, not necessarily one connected to the identity (with \(\det \Lambda = 1\)). If the dimensionality \(\text{dim}(G)\) of \(G\) is odd, the Lorentz transformation (2.8) relating left and right movers includes the action of parity whereas if it is even it does not. This Lorentz transformation also induces an action on spinors given by a matrix \(\Omega\) obtained by requiring that

\[ \Omega^{-1} \Gamma^i \Omega = \Lambda^i_j \Gamma^j . \]  

(2.9)

Hence, in the context of type-II superstrings, this transformation will map between the type-IIA and type-IIB theories when \(\text{dim}(G)\) is odd and will preserve the chirality of the theory when \(\text{dim}(G)\) is even. One may readily check that this corresponds...
to our received wisdom in the abelianised limit by taking the structure constants to vanish. Let us illustrate this in a slightly generalised context by including some extra un-dualised spectator directions but restricting to flat backgrounds with spectator fields that have no legs or dependence on the directions being dualised. The resultant Lorentz transformation corresponds to performing a T-duality for an abelian group isomorphic to $\mathbb{U}(1)^d$ and is simply given by

$$\Lambda = \text{diag}(-1_d, 1_{10-d}).$$

(2.10)

The spinorial representation of this transformation is found by solving (2.9) and reads

$$\Omega = \prod_{i=1}^{d} (\Gamma^i \Gamma_{11}),$$

(2.11)

where $\Gamma_{11} = \Gamma^0 \Gamma^1 \cdots \Gamma^9$, obeying $\Gamma_{11} \cdot \Gamma_{11} = 1$. Due to the abelian nature of the group in this case, we may interpret this transformation as that coming from performing a succession of standard abelian T-dualities in each different direction.

It is now clear how we may include RR-fields into the discussion. In the type-IIB case we have odd RR-forms which can be combined into a bi-spinor as

$$P = \frac{e^\Phi}{2} \sum_{n=0}^{4} \frac{1}{(2n+1)!} F_{2n+1}.$$ 

(2.12)

In the type-IIA case we have the similar expression

$$\hat{P} = \frac{e^\Phi}{2} \sum_{n=0}^{5} \frac{1}{(2n)!} F_{2n},$$

(2.13)

where we have included a zero form potential corresponding to the mass parameter of massive type-IIA supergravity. In addition we have made use of the standard notation $F_p = \Gamma_{\mu_1 \cdots \mu_p} F_{\mu_1 \cdots \mu_p}$. In the definitions of $P$ and $\hat{P}$ we have used the democratic formulation of type-II supergravities [30]. In this formulation and for Minkowski signature
spacetimes the conditions\(^2\)

\[
F_{2n} = (-1)^n \star F_{10-2n}, \tag{2.15}
\]

for type-IIA and

\[
F_{2n+1} = (-1)^n \star F_{9-2n}, \tag{2.16}
\]

for type-IIB should be imposed so that one remains with the right degrees of freedom. However, when we check our solutions to supergravity we shall, in general, work with the standard formulations of type-II supergravities in which no higher forms than five appear.

The above fluxes transform according to

\[
P \rightarrow P \Omega^{-1}, \tag{2.17}
\]

which is a rule of the same form as in the abelian case. That this is right transformation rule can be seen from either the space time arguments of [13, 14] or the world sheet argument using the pure spinor superstring of [17, 18]. The details of the matrix \(\Omega\) corresponding to cases of non-abelian T-duality are considerably more complicated than the abelian case. Some of the structure of the matrix \(\Omega\) is easy to infer. When the group with respect to which we T-dualize is an odd-dimensional one, \(\Omega\) starts with \(\Gamma_{11}\) to be followed by a linear combination of products of an odd number of Gamma-matrices. If, instead, the dimensionality of the group is even, then \(\Gamma_{11}\) is omitted and the linear combination is performed with products of an even number of Gamma-matrices.\(^3\)

Finally, note that assuming that we start with a type-IIA supergravity background in which (2.15) is satisfied, then after T-duality the conditions (2.16) if we end up with a type-IIB background or again (2.15) if we remain within type-IIA, should be satisfied automatically. A similar statement holds if we start from a type-IIB background.

\(^2\)Our conventions for the Hodge dual on a \(p\)-form in a \(D\)-dimensional spacetime are that

\[
(\star F_p)_{\mu_1...\mu_D} = \frac{1}{p!} \sqrt{|g|} \epsilon_{\mu_1...\mu_D} F^\mu_1...\mu_p, \tag{2.14}
\]

where \(\epsilon_01...9 = 1\). With this we have the useful identity \(\star \star F_p = s(-1)^{p(D-p)} F_p\), where \(s\) is the signature of spacetime which we take to be mostly plus.

\(^3\)In practice, because the fermions of the theory are Weyl the \(\Omega\) that we use to perform the transformation of the fluxes is actually in the Weyl representation and \(\Gamma_{11}\) contributes only an overall sign which can be taken care of by a judicial choice of conventions.
3 Non-abelian T-dual in the D1-D5 near horizon

As a first example we consider the $AdS_3 \times S^3 \times T^4$ geometry that arises as the near horizon limit of the D1-D5 brane system. The type-IIB supergravity background consist of a metric

$$ds^2 = ds^2(AdS_3) + ds^2(S^3) + ds^2(T^4), \quad (3.1)$$

where the normalization is such $R_{\mu\nu} = \mp \frac{1}{2} g_{\mu\nu}$ for the $AdS_3$ and $S^3$ factors, respectively, supported by the Ramond flux

$$F_3 = \text{Vol}(AdS_3) + \text{Vol}(S^3), \quad (3.2)$$

whereas the dilaton $\Phi = 0$.\footnote{Due to S-duality there exists a one parameter family of solutions with both RR and NS three-form fluxes turned on. However for this paper we only wish to consider the case with just RR flux.} Note that we have completely absorbed all constant factors by appropriate rescalings. The presence of $S^3$ indicates a global $SO(4) \simeq SU(2)_L \times SU(2)_R$ isometry. We will next perform a non-abelian transformation with respect to one of these $SU(2)$ factors. As we shall see, the two cases differ by a simple sign factor in the NS three-form.

In order to perform the T-duality transformation we will need the Hodge-dual of the above three-form

$$F_7 = -(\ast F_3) = \left(\text{Vol}(S^3) + \text{Vol}(AdS_3)\right) \wedge \text{Vol}(T^4), \quad (3.3)$$

where the sign in the definition has been chosen in accordance to (2.16).

3.1 The T-dual on the NS sector

We now apply our general reasoning for PCMs to the case of interesting where the group $G = SU(2)$. The representation matrices and algebra structure constants are given by

$$t_i = \frac{\sigma_i}{\sqrt{2}}, \quad f_{ijk} = \sqrt{2} \epsilon_{ijk}, \quad (3.4)$$

where $\sigma_i$ are the standard Pauli-matrices. Setting $v_i = x_i/\sqrt{2}$ we have that

$$M_{ij} = \delta_{ij} + \epsilon_{ijk} x_k \implies (M^{-1})^{ij} = \frac{1}{1 + r^2} (\delta_{ij} + x_i x_j - \epsilon_{ijk} x_k). \quad (3.5)$$
We note that had we gauged the right $G_R$ isometry would have amounted to flipping the sign of the Lagrange multiplier $v_i$. For the case of the example of interest we should send $x_i \rightarrow -x_i$.

This dual $\sigma$-model has by construction a residual $SU(2)$ symmetry that can be made manifest by introducing spherical coordinates in place of the Cartesian ones. We obtain that the fields of the NS sector of dual model are

\[
    ds^2 = ds^2(AdS_3) + dr^2 + \frac{r^2}{1 + r^2} d\Omega_2^2 + ds^2(T^4),
\]

\[
    B = e \frac{r^3}{1 + r^2} \text{Vol}(S^2) \quad \Rightarrow \quad H = e \frac{r^2(3 + r^2)}{(1 + r^2)^2} dr \wedge \text{Vol}(S^2),
\]

\[
    \Phi = -\frac{1}{2} \ln(1 + r^2), \quad \epsilon = \pm 1.
\]

The two signs parametrized by the arithmetic parameter $\epsilon$ correspond to gauging the right (for $\epsilon = 1$) and left (for $\epsilon = -1$) $SU(2)$ isometries, respectively. The above background corresponds to a smooth space, due to the fact that the isometry acts with no isotropy, and interpolates between $\mathbb{R}^3$ and $\mathbb{R} \times S^2$ (times the $AdS_3 \times T^4$ part).\(^5\)

In what follows some calculations are most easily performed by introducing vielbeins that allow to work with tangent frame Gamma matrices. For the dualised directions we have a convenient choice of three one-forms

\[
    e^i = \frac{1}{\sqrt{1 + r^2}} (dx^i + x^i a(r) dr), \quad a(r) = \frac{\sqrt{1 + r^2} - 1}{r}.
\]

### 3.2 The T-duality transformation on the RR-fluxes

The Lorentz transformation matrix (2.8) is, in this case, explicitly given by

\[
    \Lambda_{ij} = -(MM^{-1T})_{ij} = \frac{r^2 - 1}{r^2 + 1} \delta_{ij} - \frac{2}{r^2 + 1} (x_i x_j + \epsilon_{ijk} x_k),
\]

with $r^2 = x_i x_i$ and where we have used

\[
    M_{ij} = \delta_{ij} + \epsilon_{ijk} x_k, \quad M_{ij}^{-1} = \frac{1}{r^2 + 1} (\delta_{ij} + x_i x_j - \epsilon_{ijk} x_k). \quad (3.9)
\]

\(^5\)The part of the above metric and antisymmetric tensor associated to the non-abelian of PCM for $S^3$ are essentially the same as those computed in [31, 32].
The matrix $\Omega$ that solves (2.9) is given by

$$\Omega = \Gamma_{11} \tilde{\Omega}, \quad \tilde{\Omega} = \frac{\Gamma_{123} + \mathbf{x} \cdot \Gamma}{\sqrt{1 + r^2}}. \quad (3.10)$$

The matrix $\tilde{\Omega}$ is unitary satisfying (2.9) but with a minus in the right hand side. Restricting first our attention to the directions being dualised, we take the Gamma matrices $\Gamma_i$ for $i = 1, 2, 3$ to obey

$$\Gamma_i \Gamma_j = \delta_{ij} + i \epsilon_{ijk} \Sigma_k, \quad (3.11)$$

and

$$\Sigma_i \Sigma_j = \delta_{ij} + i \epsilon_{ijk} \Sigma_k, \quad \Gamma_i \Sigma_j = -i \Gamma_{123} \delta_{ij} + i \epsilon_{ijk} \Gamma_k, \quad (3.12)$$

$$\Sigma_i \Gamma_j = -i \Gamma_{123} \delta_{ij} + i \epsilon_{ijk} \Gamma_k,$$

where $\Gamma_{123} = \Gamma_1 \Gamma_2 \Gamma_3$, obeying

$$\Gamma_{123}^2 = -I, \quad \Gamma_{123} \Gamma_i = \Gamma_i \Gamma_{123} = i \Sigma_i, \quad \Gamma_{123} \Sigma_i = \Sigma_i \Gamma_{123} = i \Gamma_i. \quad (3.13)$$

The set of matrices $\{\Gamma_i, \Sigma_i\}$ form an $SO(4)$ algebra. A useful representation in computations is

$$\Gamma_i = \sigma_3 \otimes \sigma_i, \quad \Sigma_i = I_2 \otimes \sigma_i, \quad \Gamma_{123} = i \sigma_3 \otimes I. \quad (3.14)$$

where $\sigma_i$ are the usual $2 \times 2$ Pauli matrices.\(^6\) Using this representation for the Gamma matrices we immediately see that

$$\tilde{\Omega} = \text{diag}(\omega, -\omega), \quad \omega = \frac{iI + \mathbf{x} \cdot \sigma}{\sqrt{1 + r^2}}. \quad (3.15)$$

Then we compute

$$\omega^{-1} \sigma^i \omega = \frac{1 - r^2}{r^2 + 1} \sigma_i + \frac{2}{r^2 + 1} \left[ (\mathbf{x} \cdot \sigma) x_i + \epsilon_{ijk} x_j \sigma_k \right] = -\Lambda^i_j \sigma^j, \quad (3.16)$$

thus proving our assertion. To verify the equation (2.9) for the the remaining transverse direction with $a \neq 1, 2, 3$, one simply uses $\{\Gamma_{11}, \Gamma^a\} = \{\Gamma^i, \Gamma^a\} = 0.$

\(^6\)Of course, one should be more exact and use a representation for these $\Gamma$ matrices corresponding to the full ten dimensional Clifford algebra. However this abbreviated representation is sufficient to demonstrate the result.
3.2.1 The transformation

We may now apply the transformation rule for RR flux (2.17). We have that

\[ P = \frac{1}{2} \left( \frac{1}{3!} F_3 + \frac{1}{7!} F_7 \right) = \frac{1}{2} \left( \Gamma^{0'1'2'} + \Gamma^{123} \right) \left( 1 + \Gamma^{4567} \right), \tag{3.17} \]

where the indices 0'1'2' correspond to the AdS_3 directions, 123 to S^3 and 4567 to T^4. Given this and the structure of \( \Omega \) we expect for \( \hat{P} \) the form

\[ \hat{P} = \frac{e^\Phi}{2} \left( IF_0 + \frac{1}{2} F_2 + \frac{1}{4!} F_4 + \frac{1}{6!} F_6 + \frac{1}{10!} F_{10} \right). \tag{3.18} \]

After some rearrangements we find that

\[ F_0 = 1, \tag{3.19} \]

and that

\[ F_{10} = -\text{Vol}(G) = -\text{Vol}(AdS_3) \wedge \text{Vol}(S^3) \wedge \text{Vol}(T^4), \]

\[ F_6 = \frac{r^2}{1 + r^2} \left[ r\text{Vol}(T^4) + dr \wedge \text{Vol}(AdS_3) \right] \wedge \text{Vol}(S^2). \tag{3.20} \]

For the 2-form we find

\[ F_2 = x_1 e^2 \wedge e^3 + x_2 e^3 \wedge e^1 + x_3 e^1 \wedge e^2 \]
\[ = \frac{1}{1 + r^2} (x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2) \tag{3.21} \]
\[ = \frac{r^3}{1 + r^2} \text{Vol}(S^2). \]

Finally, we find for the 4-form and the 8-form

\[ F_4 = \text{Vol}(AdS_3) \wedge x_i e^i + \text{Vol}(T^4) = -r dr \wedge \text{Vol}(AdS_3) + \text{Vol}(T^4), \tag{3.22} \]

and

\[ F_8 = \text{Vol}(AdS_3) \wedge x_i e^i \wedge \text{Vol}(T^4) = -r dr \wedge \text{Vol}(AdS_3) \wedge \text{Vol}(T^4). \tag{3.23} \]

One can easily verify that \( \star F_2 = F_8, \star F_4 = -F_6 \) and \( \star F_{10} = 1 \), as expected from (2.15).
3.3 Verifying the field equations of massive IIA supergravity

We have now obtained a dual background given by (3.6) supported by the RR-fluxes (3.19), (3.21) and (3.22). We would like to verify that indeed it solves the equations of field equations of the massive type-IIA supergravity.

To show this we shall assume the specific form of the NS fields in (3.6) and for the fluxes we try an ansatz of the form

\[ F_2 = A(r) \text{Vol}(S^2) , \quad F_4 = B(r) dr \wedge \text{Vol}(AdS_3) + C(r) \text{Vol}(T_4) , \tag{3.24} \]

which preserves the symmetries of the NS background fields. The functions \( A(r) \), \( B(r) \) and \( C(r) \) will be determined by satisfying the Bianchi identities and flux equations. From the Bianchi identity we immediately find that

\[ A(r) = me \frac{r^3}{1 + r^2} + c_1 , \quad C(r) = c_2 , \tag{3.25} \]

where \( c_1 \) and \( c_2 \) are constants. The latter constant must be non-zero to admit a solution, but can be set \( c_2 = \pm 1 \) without loss of generality. In fact, there is a flip symmetry that allows us to take \( c_2 = 1 \) (this is not symmetry of the flux equations (A.8) in general, but is specific our ansatz).

Next we compute the Hodge duals

\[ *H = -\frac{h(r)}{f(r)} \text{Vol}(AdS_3) \wedge \text{Vol}(T^4) , \]
\[ *F_2 = \frac{A(r)}{f(r)} \text{Vol}(AdS_3) \wedge dr \wedge \text{Vol}(T^4) , \]
\[ *F_4 = B(r) f(r) \text{Vol}(S^2) \wedge \text{Vol}(T^4) + f(r) \text{Vol}(AdS_3) \wedge dr \wedge \text{Vol}(S^2) , \tag{3.26} \]

where

\[ f(r) = \frac{r^2}{1 + r^2} , \quad h(r) = e^{r^2(3 + r^2) / (1 + r^2)^2} . \tag{3.27} \]

Then the flux equations give the conditions

\[ \frac{d}{dr}(B(r) f(r)) = -h(r) , \quad \frac{d}{dr} \left( e^{-2q(r)} h(r) \right) + B(r) = m \frac{A(r)}{f(r)} . \tag{3.28} \]
Solving them requires that
\[ m = \pm 1 , \quad B(r) = -\epsilon r + c_1 \frac{1 + r^2}{r^2} . \] (3.29)

It can be readily verified that the dilaton equation (A.7) is satisfied. It remains to check Einstein’s equations (A.6). It turns out that these are satisfied provided that the constant \( c_1 = 0 \). Hence, the RR-fluxes assume the form
\[ F_2 = m \frac{r^3}{1 + r^2} \text{Vol}(S^2) , \quad F_4 = -\epsilon dr \wedge \text{Vol}(AdS_3) + \text{Vol}(T_4) . \] (3.30)

We see that this matches the background found by the T-duality transformation if \( \epsilon = 1 \) and \( m = 1 \).

We also note that in the small \( r \) limit the spacetime is that for \( AdS_3 \times R^3 \times T^4 \), whereas for \( r \to \infty \), that for \( AdS_3 \times R \times S^2 \times T^4 \). However, these limiting behaviors cannot be promoted to full supergravity solutions on their own, since, as it turns out, keeping the leading order behaviour of the fluxes and the dilaton is not consistent with the equations of motion.

### 3.4 Supersymmetry of the T-dual background

Bosonic backgrounds will preserve a fraction of supersymmetry if the dilatino and gravitino variations with all fermions set to zero admit non-trivial spinors as solutions. These variations are given by
\[ \delta \lambda = \left( \partial \Phi + \frac{1}{2 \cdot 3!} H_{11} \right) \epsilon + \frac{1}{4} e^\Phi \sum_{n=0}^{2} \frac{5 - 2n}{(2n)!} F_{2n}(\Gamma_{11})^n \epsilon , \] (3.31)

and
\[ \delta \psi_\mu = D_\mu \epsilon - \Lambda_\mu \epsilon \equiv \left( \partial_\mu + \frac{1}{4} \omega_\mu + \frac{1}{8} \Gamma_{11} H_{\mu} \right) \epsilon + \frac{e^\Phi}{8} \sum_{n=0}^{2} \frac{1}{(2n)!} F_{2n} \Gamma_\mu (\Gamma_{11})^n \epsilon , \] (3.32)

where \( D_\mu \) is the usual covariant derivative built with the spin connection. Let us consider whether the T-dual background given by (3.6), (3.19), (3.21) and (3.22) preserves a fraction of supersymmetric of the original background which is \( \frac{1}{2} \)-supersymmetric, i.e. it preserves sixteen real supercharges. We will explicitly examine and solve the above dilation and gravitino supersymmetry variations. Since the dual geometry has
all fluxes turned on it may seem at first sight that this is an unlikely proposition. However, for spinors obeying the projector conditions

\[ \Gamma^01'2'123 \Gamma_{11} \varepsilon = \varepsilon \]  

(3.33)

and

\[ \frac{1}{\sqrt{1+r^2}} (r \Gamma^1 - \Gamma^{123} \Gamma_{11}) \varepsilon = \varepsilon , \]  

(3.34)

one can establish that the dilatino variation identically vanishes. The first projector is simply inherited from the original \( AdS_3 \times S^3 \times T^4 \) geometry whereas the second projector is more exotic. This position dependent projector can be thought of as being related to the Lorentz frame rotation induced by performing the T-duality.\(^7\) A spinor obeying these commuting projectors possesses eight real degrees of freedom indicating a \( \frac{1}{4} \)-BPS solution.

To complete this analysis one must also ensure that the gravitino equation vanishes for spinors obeying the projectors and which are of a suitable functional form. By differentiating the gravitino equation one can form an integrability condition

\[ \frac{1}{4} R_{\mu \nu ab} \Gamma^{ab} \varepsilon = (D_\mu \Lambda_\nu - D_\nu \Lambda_\mu - [\Lambda_\mu, \Lambda_\nu]) \varepsilon . \]  

(3.35)

For our particular background and for spinors obeying the above projector conditions one can verify that this integrability condition indeed holds thereby ensuring supersymmetry. Indeed, one can also integrate the gravitino equation to give an explicit form for the Killing spinors

\[ \varepsilon = \Omega_{AdS} \cdot \Omega_r \cdot \Omega_\theta \cdot \Omega_\phi \cdot \varepsilon_0 , \]  

(3.36)

where \( \varepsilon_0 \) is a spinor obeying the projectors (3.33) and (3.34) and where

\[ \Omega_r = \exp \left( \frac{1}{2} \tan^{-1} (r \Gamma^{23}) \right) , \quad \Omega_\phi = \exp \left( \frac{\phi}{2} \Gamma^{23} \right) , \]

\[ \Omega_\theta = \exp \left( -\frac{\theta}{2(1+r^2)} \left[ -\Gamma^{12} + r^2 \Gamma^{12} \Gamma_{11} - r \Gamma^{13} - r \Gamma^{13} \Gamma_{11} \right] \right) . \]  

(3.37)

The factor \( \Omega_{AdS} \) contains the functional dependance on \( AdS_3 \) and depends on the

\(^7\)We note that similar position dependent projectors arise in considering k-symmetry and branes intersecting at angles \([33]\). In searching for supersymmetry preserving supergravity solutions similar examples were found in \([34]\).
coordinate system of our preference (in horospherical coordinate system it is given by [35]). Note that although the four-form does have legs in the $T^4$, the projection conditions obeyed by $\varepsilon_0$ ensure that the gravitino equation in these direction simply boils down to ensuring that the spinor is constant with respect to these coordinates (as would be expected from the geometry).

### 3.4.1 Remarks on the Kosmann derivative

We have seen by an explicit computation that the dual geometry preserves eight supersymmetries whereas the $AdS_3 \times S^3 \times T^4$ solution of type-IIB supergravity that we started with enjoys sixteen supersymmetries. Hence half of the supersymmetries have been destroyed through the dualisation procedure. This is a statement valid in the supergravity low energy approximation of string theory. Stringy winding modes may reestablish supersymmetry [36, 37]. The origin of this supersymmetry breaking can also be seen by considering how the $SU(2)$ symmetry along which we dualise acts on the Killing spinors of the original background.

In general, it is ambiguous to define the action of a vector on a spinor, however the action of a Killing vector, $k = k^\mu \partial_\mu$, on a spinor is well defined and is given by the Kosmann (spinor-Lorentz-Lie) derivative [20]

$$
\mathcal{L}_k \varepsilon = k^\mu D_\mu \varepsilon - \frac{1}{4} \nabla_\mu k_\nu \Gamma^{\mu\nu} \varepsilon .
$$

(3.38)

This derivation maps spinors to spinors and induces on bispinors the usual action of the Lie derivative. Additionally, it forms representations of the algebra of vector fields so that

$$
[\mathcal{L}_{k_1}, \mathcal{L}_{k_2}] \varepsilon = \mathcal{L}_{[k_1, k_2]} \varepsilon
$$

(3.39)

and for further properties the reader may consult [21, 22]. We may thus ask how many of the Killing spinors of the $AdS_3 \times S^3 \times T^4$ background are invariant under the $SU(2)_L$ symmetry in the sense that their Kosman derivative vanishes. In this case the rôle of the Killing vector $k$ above is played by the three vectors $K^a$, $a = 1, 2, 3$, with components $K^a_i$ and inverse ones $K^i_a$. These obey the useful identities

$$
\sum_{a=1}^{3} K^a_i K_j^a = \delta^i_j , \quad \sum_{i=1}^{3} K^a_i K_i^b = \delta^{ab} , \quad K^a_i K_j^b K_k^c \epsilon_{abc} = -\epsilon_{ijk} .
$$

(3.40)
as well as derivative relations

\[ \nabla_i K^a_j + \nabla_j K^a_i = 0, \quad \nabla_i K^a_j - \nabla_j K^a_i = -\epsilon_{abc} K^b_i K^c_j, \tag{3.41} \]

which are just the Killing property and the Maurer–Cartan equations, respectively. In the above \(\epsilon_{ijk}\) is a density containing \(\text{Vol}(S^3)\).

The sixteen independent Killing spinors of type-IIB supergravity obeying a projection condition arising from the dilatino variation \(\mathcal{F}_3\epsilon = 0\), which using (3.2) implies that

\[ \Gamma^{01'2'123}\epsilon = -\epsilon. \tag{3.42} \]

They also satisfy

\[ D_\mu \epsilon - \frac{i}{8 \cdot 3!} \mathcal{F}_3 \Gamma_{\mu} \epsilon^* = 0, \tag{3.43} \]

arising from the gravitino variation. This can be written in the convenient form

\[ (\mathbb{1}_2 \otimes D_\mu)\epsilon - \frac{1}{8 \cdot 3!} (\sigma_1 \otimes \mathcal{F}_3 \Gamma_\mu)\epsilon = 0, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \tag{3.44} \]

where we have reassembled the complex chiral spinor \(\epsilon = \epsilon_1 + i\epsilon_2\) into a doublet. Using the above notation and variation we may write the Kosman derivative as

\[ \mathcal{L}_a \epsilon = (\mathbb{1}_2 \otimes K^i_a D_i)\epsilon - \frac{1}{4} (\mathbb{1}_2 \otimes \nabla_i K^a_j \Gamma^{ij})\epsilon \]

\[ = \frac{1}{8 \cdot 3!} (\sigma_1 \otimes K^i_a \mathcal{F}_3 \Gamma_i)\epsilon - \frac{1}{4} (\mathbb{1}_2 \otimes \nabla_i K^a_j \Gamma^{ij})\epsilon. \tag{3.45} \]

Then, using for the first term that

\[ \frac{1}{3!} \mathcal{F}_3 \Gamma_i \epsilon = (\Gamma^{01'2'} + \Gamma^{123}) \Gamma_i \epsilon = 2 \Gamma_i \Gamma^{123} \epsilon = \epsilon_{ijk} \Gamma^{jk} \tag{3.46} \]

and for the second the above Maurer–Cartan equations, we obtain that

\[ \mathcal{L}_a \epsilon = -\frac{1}{4} \left( \mathcal{P} \otimes k^i_a \epsilon_{ijk} \Gamma^{jk} \right) \epsilon, \quad \mathcal{P} = \frac{1}{2} (\mathbb{1}_2 - \sigma_1). \tag{3.47} \]

Demanding that this vanishes leads to a condition on \(\epsilon\) which can be simplified by squaring it, contracting appropriately over the index \(a\) and by using the fact that \(\mathcal{P}\) is a projector. We find a necessary condition for the Killing spinor to be invariant is that

\[ (\mathcal{P} \otimes \mathbb{1}_{32})\epsilon = 0. \tag{3.48} \]
It can be easily seen that this is also sufficient. In conclusion, exactly half of the supersymmetries commute (in the above sense) with the $SU(2)$ action and we should only expect half the original supersymmetry to be preserved in the T-dual geometry. This is in exact accordance with the explicit result.

4 Non-abelian T-dual in the D3 near horizon

Our second example concerns the type-IIB supergravity solution describing the near horizon limit of the D3-brane background. It consists of a metric

$$ds^2 = ds^2(\text{AdS}_5) + ds^2(S^5),$$

normalized such that $R_{\mu\nu} = \mp g_{\mu\nu}$ for the $\text{AdS}_5$ and $S^5$ factors, respectively, supported by the self-dual Ramond flux

$$F_5 = 2\text{Vol}(\text{AdS}_5) - 2\text{Vol}(S^5).$$

As before we note that we have completely absorbed all constant factors by appropriate rescalings. We would like to perform a non-abelian T-duality with respect to an $SU(2) \in SO(6)$ symmetry. For that purpose we write the line element for $S^5$ in the form

$$ds^2(S^5) = 4(d\theta^2 + \sin^2 \theta \ d\phi^2) + \cos^2 \theta \ ds^2(S^3),$$

where for the $S^3$ factor the normalization is such that $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}$.

4.1 The construction of the T-dual background

To obtain the dual background we follow the same steps as before. In particular, we find that the matrix $M_{ij}$ occurring in the dual $\sigma$-model (2.5) is given by

$$M_{ij} = \cos^2 \theta \ \delta_{ij} + \epsilon_{ijk} \ x_k \ \Rightarrow \ M_{ij}^{-1} = \frac{\cos^2 \theta \ \delta_{ij} + \cos^{-2} \theta \ \ x_i x_j - \epsilon_{ijk} \ x_k}{\cos^4 \theta + r^2}. \quad (4.4)$$

Finally, we obtain that the fields of the NS sector of dual model are

$$ds^2 = ds^2(\text{AdS}_5) + 4(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dr^2}{\cos^2 \theta} + \frac{r^2 \cos^2 \theta}{\cos^4 \theta + r^2} d\Omega^2_2.$$
\[
B = \frac{er^3}{\cos^4 \theta + r^2} \text{Vol}(S^2),
\]
\[
H = \frac{er^2}{(\cos^4 \theta + r^2)^2} \left[ (3 \cos^4 \theta + r^2)dr + 4r \sin \theta \cos^3 \theta d\theta \right] \wedge \text{Vol}(S^2), 
\]
\[
\Phi = -\frac{1}{2} \ln[\cos^2 \theta (\cos^4 \theta + r^2)], \quad e = \pm 1,
\]

where once again the arithmetic parameter \(e\) indicates the two distinct possibilities in performing the non-abelian T-duality transformation. This background has an \(SO(2,4) \times SU(2) \times U(1)\) symmetry. We will comment more later on this point in relation to the amount of supersymmetry preserved by the solution and its eleven dimensional supergravity lift.

We now compute the fluxes that should support the geometry and make into a full solution of type-IIA supergravity. In this case there is no zero-form produced since there is no three-form or one-form flux in the original background. We will use again (2.17), but now with
\[
\Omega = \frac{\cos^2 \theta \Gamma_{123} + x \cdot \Gamma}{\sqrt{\cos^4 \theta + r^2}},
\]
and
\[
P = \Gamma^{0'1'2'3'4'} - \Gamma^{12345},
\]
\[
\hat{P} = \frac{e\Phi}{2} \left( \frac{1}{2} F_2 + \frac{1}{4!} F_4 + \frac{1}{6!} F_6 + \frac{1}{8!} F_8 \right),
\]

where the indices \(0'1' \cdots 4'\) correspond to the \(AdS_5\) factor; 1 to \(r\); 2, 3 to the coordinates in \(S^2\); and 4, 5 to \(\theta\) and \(\phi\). Converting all Gamma matrices to tangent frame using the drei-bein
\[
e^i = \frac{1}{\cos \theta \sqrt{\cos^4 \theta + r^2}} (\cos^2 \theta \, dx^i + x^i b(r) dr), \quad b(r) = \frac{\sqrt{\cos^4 \theta + r^2} - \cos^2 \theta}{r},
\]
we find that
\[
F_8 = -2 \frac{r^2 \cos^4 \theta}{\cos^4 \theta + r^2} \text{Vol}(AdS_5) \wedge dr \wedge \text{Vol}(S^2),
\]
\[
F_6 = -2r \, dr \wedge \text{Vol}(AdS_5),
\]
\[
F_4 = -8 \frac{r^3 \cos^3 \theta \sin \theta}{\cos^4 \theta + r^2} \, d\theta \wedge d\phi \wedge \text{Vol}(S^2),
\]
\[
F_2 = -8 \cos^3 \theta \sin \theta \, d\theta \wedge d\phi.
\]
Note that $\star F_6 = F_4$ and $\star F_8 = -F_2$ in accordance with (2.15). It is easy to check that the Bianchi identities (A.4) (with $m = 0$) are obeyed provided we choose $\epsilon = 1$. In addition, one can readily see that the equations of motion (A.8) for the fluxes (again with $m = 0$) are indeed satisfied. For the $H$-equation one uses that

$$e^{-2\Phi} \star H = -4 \sin \theta \cos \theta \left( -r \sin \theta \cos \theta \, dr + (3 \cos^4 \theta + r^2) d\theta \right) \wedge d\phi \wedge \text{Vol}(AdS_5),$$

from which we find

$$d \left( e^{-2\Phi} \star H \right) = -16r \cos^3 \theta \sin \theta \, dr \wedge d\theta \wedge d\phi \wedge \text{Vol}(AdS_5).$$

We have also checked that the Einstein’s and dilaton equations (A.6) and (A.7) are satisfied.

### 4.2 Supersymmetry of the T-dual background

We first examine the dilatino equation in (3.31). It is convenient for notational purposes to write $r = R \cos^2 \theta$. Using the flat index notation we have

$$H_{123} = \frac{R^2 + 3}{\cos \theta (R^2 + 1)}, \quad H_{234} = \frac{2R \sin \theta}{\cos \theta (R^2 + 1)},$$

$$(F_2)_{45} = -2 \cos^3 \theta, \quad (F_4)_{2345} = -2R \cos^3 \theta.$$  

Then, after multiplying (3.31) by $2 \cos \theta (R^2 + 1)/(R^2 + 3)$, the dilatino equation can be cast into the form

$$\Gamma_{123} \Gamma_{11} \epsilon = (A + B) \epsilon,$$

where the matrices

$$A = \frac{R \sqrt{R^2 + 1}}{R^2 + 3} \cos \theta \, \Gamma_{2345} - \sin \theta \, \Gamma_4,$$

$$B = 3 \frac{\sqrt{R^2 + 1}}{R^2 + 3} \cos \theta \, \Gamma_{45} \Gamma_{11} + \frac{2R}{R^2 + 3} (\Gamma_1 - \sin \theta \Gamma_{234} \Gamma_{11}) .$$

We would like to write (4.13) in the form

$$\mathcal{P} \epsilon = \epsilon, \quad \mathcal{P}^2 = 1.$$
We note the relations \((\Gamma_{123}\Gamma_{11})^2 = \mathbb{1}\) and \([\Gamma_{123}\Gamma_{11}, A] = \{\Gamma_{123}\Gamma_{11}, B\} = 0\). Using them and acting on (4.13) with \(\Gamma_{123}\Gamma_{11}\) we obtain
\[
(A^2 - B^2 + [A, B])\epsilon = \epsilon .
\] (4.16)

Then we compute
\[
A^2 = \frac{R^2(R^2 + 1)}{(R^2 + 3)^2} \cos^2 \theta + \sin^2 \theta ,
\]
\[
B^2 = \frac{1}{(R^2 + 3)^2} \left[ -9(R^2 + 1) \cos^2 \theta + 4R^2(1 + \sin^2 \theta) \right]
- 4\frac{R}{(R^2 + 3)^2} \sin \theta \left( 3\sqrt{R^2 + 1} \cos \theta \Gamma_{235} + 2R\Gamma_{1234}\Gamma_{11} \right) .
\] (4.17)
\[
[A, B] = -4\frac{R^2\sqrt{R^2 + 1}}{(R^2 + 3)^2} \sin \theta \cos \theta \Gamma_5\Gamma_{11} + 4\frac{R}{R^2 + 3} \sin \theta (\sin \theta \Gamma_{23}\Gamma_{11} + \Gamma_{14}) .
\]

We further simplify (4.16) by using (4.13) to write \(\Gamma_{1234}\Gamma_{11}\epsilon = -\Gamma_4(A + B)\epsilon\). The result can be cast in the form (4.15) with
\[
\mathcal{P} = \frac{\cos \theta}{\sqrt{r^2 + \cos^4 \theta}} (\cos^2 \theta \Gamma_{12345} - r\Gamma_{145}\Gamma_{11}) + \sin \theta \Gamma_{1234}\Gamma_{11} ,
\] (4.18)

where we have restored the original \(r\) variable. One may readily verify after some algebraic manipulations that indeed \(\mathcal{P}^2 = \mathbb{1}\).

Hence we have seen that the dilatino equation is solved by supersymmetry parameters obeying a single, albeit somewhat complicated, field-dependent projector condition, projecting out half of the components leaving a possible of sixteen surviving supersymmetries. The complicated nature of the background makes it rather difficult to give an explicit formula for the Killing spinors in this case. However, one can check, for instance using the Mathematica computer programme, that for such a restricted spinor the integrability condition for the gravitino equation is obeyed. Hence this T-dual background is found to be supersymmetric, preserving precisely sixteen supercharges.

As before we may also infer the amount of supersymmetry that will be preserved after the T-duality is performed by examining the action of the Kosmann derivative on the Killing spinor of the original background for \(AdS_5 \times S^5\). This satisfies
\[
D_\mu \epsilon - \frac{i}{4 \cdot 5!} \Phi_5 \Gamma_\mu \epsilon = 0 ,
\] (4.19)
arising from the gravitino variation. There is of course no associated projection since the dilatino equations is identically satisfied, i.e. the background is maximally supersymmetric. The above Killing spinor equation can be written in the convenient form

\[(\mathbb{1}_2 \otimes D_\mu) \epsilon + \frac{1}{4 \cdot 5!} (i \sigma_2 \otimes \mathcal{F}_5 \Gamma_\mu) \epsilon = 0, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (4.20)\]

where as before we have assembled the two chiral spinors \(\epsilon_i\) into a doublet. Using the above notation and variation we may write the Kosmann derivative as

\[\mathcal{L}_a \epsilon = (\mathbb{1}_2 \otimes K^a_\mu D_\mu) \epsilon - \frac{1}{4} (\mathbb{1}_2 \otimes \nabla_\mu K^a_\nu \Gamma^{\mu \nu}) \epsilon
= -\frac{1}{4 \cdot 5!} (i \sigma_2 \otimes K^a_\mu \mathcal{F}_5 \Gamma_\mu) \epsilon - \frac{1}{4} (\mathbb{1}_2 \otimes \nabla_\mu K^a_\nu \Gamma^{\mu \nu}) \epsilon. \quad (4.21)\]

This is quite complicated to analyze analytically in full detail, the reason being that the three-dimensional part obtained by T-duality is coupled to the rest of the coordinates, in particular \(\theta\). Nevertheless, we have verified, on computer, that demanding vanishing Kosmann derivative places constraints on the spinor \(\epsilon\) which project out exactly half of the degrees of freedom. Hence from this perspective we also recover the expectation that the dual background should be \(\frac{1}{2}\)-BPS, which, of course, has been explicitly shown above.

### 4.3 M-theory lift and its interpretation

The solution found above in (4.5) and (4.9) has a natural eleven-dimensional origin. In [38] the gravity duals for a large class of generalized quiver \(\mathcal{N} = 2\) superconformal field theories [39] were presented. These eleven dimensional geometries contain \(AdS_5\) factors, they possess \(SO(2, 4) \times SU(2) \times U(1)\) isometry, as required by \(\mathcal{N} = 2\) superconformal invariance and fall into the general ansatz of [40] (for related earlier work see also [41]). In general these geometries depend on a single function satisfying the continual Toda equation [42]. However, if this solution has an additional \(U(1)\) symmetry then the geometry simplifies and takes the form

\[ds^2_{11} = \left(\frac{\check{V} \check{\Delta}}{2V''} \right)^{1/3} \left[ ds^2_{AdS_5} + \frac{2V'' \check{V}}{\check{\Delta}} d\Omega_2^2 + \frac{2V''}{V} \left( d\rho^2 + \frac{2V}{2V - V} \rho^2 d\phi^2 + d\eta^2 \right) + \frac{2(2V - \check{V})}{\check{V} \Delta} \left( d\beta + \frac{2\check{V} \check{V}'}{2V - V} d\phi \right)^2 \right], \]

22
\[ C_3 = 2 \left[ -2 \frac{V''}{\Delta} d\phi + \left( \frac{V'}{\Delta} - \eta \right) d\beta \right] \wedge \text{Vol}(S_2), \quad (4.22) \]

where
\[ \tilde{\Delta} = (2\dot{V} - \ddot{V}) V'' + (\dot{V}')^2 \quad (4.23) \]

and where we used the definitions
\[ \rho \partial_{\rho} V = \dot{V}, \quad \partial_{\eta} V = V'. \quad (4.24) \]

The function \( V = V(\rho, \eta) \) is a rotational invariant solution of the Poisson equation in three dimensions with cylindrical polar coordinates \( \rho, \phi, \eta \), i.e.
\[ \ddot{V} + \rho^2 V'' = \lambda(\eta) \rho \delta(\rho). \quad (4.25) \]

By Gauss’ law, equivalently by dividing this equation by \( \rho \) and then integrating, one can determine that the charge density is given by
\[ \lambda(\eta) = \dot{V}(\eta, \rho = 0). \quad (4.26) \]

For physically acceptable backgrounds the basic requirement for the line density is that it should be composed of segments linear in \( \eta \) with integer slopes.

Since the background (4.22) is also isometric around the periodic \( \beta \) direction one can perform a Kaluza-Klein reduction to arrive at a ten-dimensional type-IIA geometry as detailed in [43]. The result for the metric is
\[ ds_{10}^2 = \left( \frac{2\dot{V} - \ddot{V}}{V''} \right) \left( ds_{\text{AdS}_5}^2 + \frac{2V'' \dot{V}}{\Delta} d\Omega_2 + \frac{2V''}{V} \left( d\rho^2 + d\eta^2 \right) + \frac{4V''}{2\dot{V} - \ddot{V}} \rho^2 d\phi^2 \right). \quad (4.27) \]

By making the coordinate transformations
\[ \rho = \sin \theta, \quad \eta = \frac{r}{2}, \quad (4.28) \]

we can bring the solution found in (4.5) and (4.9) into exactly this form with the potential given by
\[ V = \eta \ln \rho + \eta \left( \frac{\eta^2}{3} - \frac{\rho^2}{2} \right). \quad (4.29) \]

This potential satisfies the Poisson equation (4.25) and gives rise to a linear charge density \( \lambda(\eta) = \eta \). In fact, the first term produces the charge density and the second
is the first harmonic in an expansion of the general solution for $V$ in monomials of $\rho \eta$. Without either of these terms the supergravity solution constructed from (4.22) wouldn’t be possible to even be defined. In addition, one easily checks that (4.29) is the unique potential with the property that, besides satisfying (4.25), it makes the overall prefactor in (4.27) equal to unity and hence the type-IIA supergravity solution contains the $AdS_5$ factor with no warping.\footnote{A solution for $V$ equal to the cubic term in $\eta$ and $\rho$ appeared also in [40] and gives rise to a pp-wave background. However, it corresponds to a reduction of the continual Toda equation to the Laplace equation corresponding to a translational and not to a rotational isometry which is our case.}

Despite being of the general form (4.22) it does not seem appropriate to directly identify our geometry as being dual one of the $N = 2$ quiver gauge theories of [39]. One reason for this is that our geometry contains a singularity for $\theta = \pi/2$ (or $\rho = 1$).\footnote{The reason for this singularity can be traced back to the fact that the $S^3$ inside the $S^5$ which we dualised actually shrinks at $\theta = \pi/2$. This is similar to the appearance of a singularity when one performs an abelian T-duality in $\mathbb{R}^2$ about its polar angular direction.}

Instead, we should think of the geometry we have found as capturing just part of the general solution to (4.22).

To see how this can arise more explicitly, let us consider as a prototypical example the Maldacena–Nunez (MN) solution [44] described by the following potential [43]

\[
2V_{MN} = \sqrt{\rho^2 + (N + \eta)^2} - (N + \eta) \sin^{-1} \left( \frac{N + \eta}{\rho} \right) - \sqrt{\rho^2 + (N - \eta)^2} + (N - \eta) \sin^{-1} \left( \frac{N - \eta}{\rho} \right),
\]

(4.30)

which is reproduced by the standard electrostatic solution in free three-dimensional space with the line density along the $\eta$-axis given by

\[
\lambda_{MN} = \begin{cases} 
-N, & \eta \leq -N, \\
\eta, & |\eta| \leq N, \\
N, & \eta \geq N.
\end{cases}
\]

(4.31)

On physical grounds we expect that if we concentrate on the part of space near the line distribution that will be equivalent to having a linear charge distribution everywhere. Indeed, if we expand the potential for small $\eta$ and $\rho$ we find that the leading order behaviour reproduces that in (4.29). In fact, this behaviour is somewhat universal; by performing a further rescaling of $\eta$ and $\rho$ one can always tune the relative coefficient between the two harmonics to become precisely that appearing in (4.29) without altering the background geometry. This is possible since nothing in the geometry depends
on $V'$ and a linear term in $\eta$ (with constant coefficient) can be always be added at no cost.

This sort of “zooming in” that transforms the MN geometry to the one we have found seems to be quite characteristic of non-abelian T-duality. Recall that, the non-abelian T-dual of a $G_k$ WZW model with respect to a subgroup $H$ gives rise to a gravitational background corresponding to the gauged WZW model for the coset $(G_k \times H_\ell)/H_{k+\ell}$ in the limit $\ell \to \infty$ [9] which precisely corresponds to a zooming in part of the manifold. At the level of states this limiting procedure can be understood as a large spin limit [10]. In addition, this is also reminiscent of the Penrose limit that transforms the $AdS_5 \times S^5$ solution to the plane wave [45] and its corresponding understanding in terms of states of parametrically large angular momentum in the gauge theory side [46]. Due to these analogies it is natural to suggest that the geometry we have found corresponds to a high spin sector of some parent $\mathcal{N} = 2$ superconformal theory.

5 Concluding remarks

In this paper we have shown how to implement non-abelian T-duality in supergravity backgrounds supported by RR-fluxes which has been an open issue for a long time. We worked out in detail two specific examples by taking advantage of an $SU(2)$ subgroup of their full isometry group. The first example was the type-IIB supergravity background arising in the near horizon limit of the D1-D5 brane system corresponding to the $AdS_3 \times S^3 \times T^4$ geometry. We found that its non-abelian T-dual is of the form $AdS_3 \times X_3 \times T^4$ and is a regular solution of massive IIA supergravity. Solutions of massive IIA supergravity exist in the literature [19] and more recently in [47]. However, to our best knowledge the solution we presented here is novel. Similarly, in the second example we computed the non-abelian T-dual corresponding to the near horizon limit of D3-branes with $AdS_5 \times S^5$ geometry and found a dual solution of the form $AdS_5 \times X_5$ in type-IIA supergravity. In both examples we have focused to the near horizon geometry of the corresponding brane systems. It will be interesting to extend our construction to the full solution expecting to obtain one-eighth and one-quarter supersymmetric solutions for the two cases, respectively. Equally it would be desirable to understand whether the dual geometries we find can also be understood as near-horizon limits of other brane systems.
A rather generic feature of non-abelian T-duality is the appearance of non-compact variables in the T-dual background even if the isometry group we dualise with is compact. For the case of NS backgrounds it was shown in [10] that this is related to the fact that the non-abelian T-dual effectively describes high spin sectors of some parent theory that involves as an essential ingredient the gauged WZW model action. One might hope to attempt a similar interpretation when RR-fluxes are turned on. An intriguing feature that suggests some similar description may be possible is displayed by the non-abelian dual of the $AdS_5 \times S^5$ background. We demonstrated that it has an eleven-dimensional origin which falls into the general class of $\mathcal{N} = 2$ superconformal backgrounds. We argued that it must effectively describe, within the AdS/CFT correspondence, a large spin sector of the gauge theory. Needless to say, understanding this more precisely, as well as other related issues, will be important.

Given that this is the first work in which non-abelian T-duality is implemented in Ramond backgrounds, it would be very interesting to provide additional examples, especially ones in which the isometry group action is different than the special kind we focused on in this work.

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A Brief review of massive IIA supergravity

In the conventions of [30], the action of the massive type-IIA supergravity [19] is given by

$$S_{\text{Massive IIA}} = \frac{1}{2\kappa^2} \int_{M_{10}} \left[ e^{-2\Phi} \left( R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left( \frac{m^2 + F_2^2}{2} + \frac{F_4^2}{4!} \right) - \frac{1}{2} \left( dC_3 \wedge dC_3 \wedge B + \frac{m}{3} dC_3 \wedge B^3 + \frac{m^2}{20} B^5 \right) \right], \quad (A.1)$$

where the field strengths are defined as

$$H = dB, \quad F_2 = dC_1 + mB, \quad F_4 = dC_3 - H \wedge C_1 + \frac{m}{2} B \wedge B, \quad (A.2)$$

and where $m$ is the mass parameter. Note that, the presence of the $\frac{1}{2}m^2$ term in the action reveals that $m$ plays the rôle of a zero-form $F_0$. The relative coefficients have been fixed so that the field strengths are invariant under the gauge transformations

$$\delta B = d\Lambda, \quad \delta C_1 = -m\Lambda, \quad \delta C_3 = -m\Lambda \wedge B, \quad (A.3)$$

where $\Lambda$ is a one-form. The Bianchi identities are

$$dH = 0, \quad dF_2 = mH, \quad dF_4 = H \wedge F_2. \quad (A.4)$$

The topological term in the action can be written as

$$-\frac{1}{2} \int_{M_{11}} dC_3 \wedge dC_3 \wedge B + \frac{m}{3} dC_3 \wedge B^3 + \frac{m^2}{20} B^5 = -\frac{1}{2} \int_{M_{11}} F_4 \wedge F_4 \wedge H, \quad (A.5)$$

where $\partial M_{11} = M_{10}$, so that gauge invariance under (A.3) becomes manifest.

The equations of motions that follow from varying the metric are

$$R_{\mu\nu} + 2D_\mu D_\nu \Phi - \frac{1}{4} H_{\mu\nu}^2 = e^{2\Phi} \left[ \frac{1}{2} (F_2^2)_{\mu\nu} + \frac{1}{12} (F_4^2)_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \left( \frac{1}{2} F_2^2 + \frac{1}{24} F_4^2 + m^2 \right) \right], \quad (A.6)$$

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whereas the dilaton equation is

\[ R + 4D^2\Phi - 4(\partial\Phi)^2 - \frac{1}{12}H^2 = 0. \]  \tag{A.7}

From varying the fluxes we obtain (after simplifying using Bianchi identities)

\[ d \left( e^{-2\Phi} * H \right) - F_2 \wedge * F_4 - \frac{1}{2}F_4 \wedge F_4 = m * F_2 , \]

\[ d * F_2 + H \wedge * F_4 = 0 , \]  \tag{A.8}

\[ d * F_4 + H \wedge F_4 = 0 . \]

This set of equations is consistent with the Bianchi identities as it can be seen by applying to each one of them the exterior derivative. In particular, we note the necessity of the term proportional to \( m \) in the right hand side of the first of (A.8).

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