Electric-Magnetic Dualities in Supergravity†

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ABSTRACT

I review electric-magnetic duality from the perspective of extended supergravity theories in four spacetime dimensions.

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1. INTRODUCTION

In early June 1974, Frank Yang returned from the London Conference and reported during an ITP luncheon meeting in Stony Brook on the two, in his view, most exciting developments: the 't Hooft-Polyakov monopole and supersymmetry. At that point I decided to start and look more closely into the papers of Wess and Zumino, which had already intrigued me during that past year. In August I spent some time in Aspen where I was the only participant working on supersymmetry, although Max Dresden and Lochlainn O’Raifeartaigh expressed a clear interest in the subject. In the fall of that year I published the results of my first research, which I believe was the first paper on supersymmetry in four spacetime dimensions that appeared in the Physical Review. During that academic year I started working with Dan Freedman; this marked the beginning of a series of fruitful collaborations. In the summer of 1975 I returned to Leiden, where I made an excursion with Gerard 't Hooft into lattice gauge theories. But in view of the exciting discoveries made in supersymmetry and supergravity, it did not take long before I returned to these topics. In 1977 I again attended the Aspen summer program, where now everybody was working on supersymmetry. When driving towards Aspen from Stony Brook with Dan Freedman we started to discuss the construction of $N = 8$ supergravity. The first person whom we met at the Center in Aspen was Murray Gell-Mann, who, after inquiring what we were working on, expressed his amazement that we were jumping right ahead from $N = 4$ (which had barely been constructed) to $N = 8$ supergravity. We had many discussions with the participants about our work in which we derived the $N = 8$ supergravity Lagrangian and transformation rules to second order using the Noether method. Our starting point was the observation that the scalar and pseudoscalar fields had to transform under SO(8) as self-dual four-rank antisymmetric tensors. We did not know what the self-duality phases were for the scalar and pseudoscalar fields and initially assumed that they were equal. However, then we got stuck; it was Martin Roček who suggested to us that perhaps the phases had to be opposite, at which point it occurred to us that opposite phases were indeed crucial for the existence of SU(8) electric-magnetic duality. Electric-magnetic duality had already been established at that time for $N \leq 4$.

This was my first encounter with electric-magnetic duality, which would turn out to be such an important feature of extended supersymmetric field theories. When preparing my presentation for this symposium I was struck by just how many papers have dealt with this subject since that time. Clearly it is hard to do justice to all this work. I stress that this modest effort represents very much my own personal perspective, but I made an effort to collect many of the relevant references at the end.

After these reminiscences let me turn to electric-magnetic duality. It is well known that electric-magnetic duality appeared in supergravity was probably in the study of the one-loop divergences of $N = 1$ Maxwell-Einstein supergravity and the one-loop finiteness of pure $N = 2$ supergravity.
that in Maxwell theory in four-dimensional (flat or curved) Minkowski space, one can perform (Hodge) duality rotations, which commute with the Lorentz group and rotate the electric and magnetic fields and inductions according to

\[ E \leftrightarrow H , \quad B \leftrightarrow D \]  \hspace{1cm} (1)

In the following I will discuss generalizations of this Maxwell duality.

Throughout this paper I consider field theories with abelian vector gauge fields labeled by indices \( I, J, \ldots \). All fields are neutral, i.e. the gauge fields enter the Lagrangian only through their (abelian) field strengths and not through covariant derivatives. These field strengths \( F_{\mu\nu} \) are decomposed into selfdual and anti-selfdual components (which are related by complex conjugation) and so are the field strengths \( G_{\mu\nu} \) that appear in the field equations (together \( F_{\mu\nu} \) and \( G_{\mu\nu} \) comprise the electric and magnetic fields and inductions),

\[ F_{\mu\nu} \rightarrow F_{\mu\nu}^{+} + F_{\mu\nu}^{-} , \]

\[ G_{\mu\nu} \rightarrow \begin{cases} G_{\mu\nu}^{+} = 2i \frac{\partial L}{\partial F_{\mu\nu}^{+}}, \\ G_{\mu\nu}^{-} = -2i \frac{\partial L}{\partial F_{\mu\nu}^{-}}. \end{cases} \]  \hspace{1cm} (2)

The Bianchi identities and equations of motion for the abelian gauge fields take the form

\[ \partial^\mu (F^+ - F^-)_{\mu\nu} = \partial^\mu (G^+ - G^-)_{\mu\nu} = 0. \]  \hspace{1cm} (3)

The above equations are obviously invariant under the following rotations of the field strengths,

\[ \left( \begin{array}{c} F_{\mu\nu}^{+} \\ G_{\mu\nu}^{+} \end{array} \right) \rightarrow \left( \begin{array}{c} \tilde{F}_{\mu\nu}^{+} \\ \tilde{G}_{\mu\nu}^{+} \end{array} \right) = \left( \begin{array}{cc} U & Z \\ W & V \end{array} \right) \left( \begin{array}{c} F_{\mu\nu}^{+} \\ G_{\mu\nu}^{+} \end{array} \right) , \]  \hspace{1cm} (4)

where \( U^J_I, V^I_J, W^I_J \) and \( Z^{IJ} \) are constant real \( n \times n \) submatrices and \( n \) denotes the number of independent gauge potentials. The question is, however, whether the rotated equations can again follow from a Lagrangian. More precisely, does there exist a Lagrangian \( \tilde{L}(\tilde{F}) \) depending on the new field strengths \( \tilde{F}_{\mu\nu}^{I} \) such that the new tensors \( \tilde{G}_{\mu\nu}^{I} \) follow from \( \tilde{L} \) by differentiation with respect to \( F_{\mu\nu}^{I} \), i.e.,

\[ \tilde{G}_{\mu\nu}^{I} \propto \frac{\partial \tilde{L}(\tilde{F})}{\partial F_{\mu\nu}^{I}} . \]  \hspace{1cm} (5)

This condition amounts to an integrability condition, because one can derive an expression for the derivative of \( \tilde{L} \) with respect to the original field strength \( F_{\mu\nu}^{I} \), which can only have a solution (for nontrivial Lagrangians) provided that the matrix \( \tilde{L} \) is an element of the group \( \text{Sp}(2n; \mathbb{R}) \)\(^2\). The symplectic group implies the following condition on the submatrices,

\[ U^T V - W^T Z = V U^T - W Z^T = 1 , \]

\[ U^T W = W^T U , \quad Z^T V = V^T Z . \]  \hspace{1cm} (6)

The fact that the symplectic redefinitions of the field strengths constitute the group \( \text{Sp}(2n; \mathbb{R}) \) was first derived in \[ [4] \], but in the context of a duality invariance rather than of a reparametrization. In this respect our presentation is more in the spirit of the later treatment in \[ [10] \] for \( N = 2 \) vector multiplets coupled to supergravity \[ [7] \]. Given that the duality belongs to the class of symplectic reparametrizations, one can derive that the new Lagrangian is of the form,

\[ \tilde{L}(\tilde{F}) + \frac{1}{4} i (\tilde{F}_{\mu\nu}^{+} \tilde{G}_{\mu\nu}^{+} - \tilde{F}_{\mu\nu}^{-} \tilde{G}_{\mu\nu}^{-}) \]

\[ = L(F) + \frac{1}{4} i (F_{\mu\nu}^{+} G_{\mu\nu}^{+} - F_{\mu\nu}^{-} G_{\mu\nu}^{-}) , \]  \hspace{1cm} (7)

up to terms independent of \( F_{\mu\nu}^{I} \). The equation \[ [4] \] is analogous to a result known for \( N = 2 \) vector multiplet Lagrangians \[ [12] \]; this will be discussed in section 4.

The above expression \[ [4] \] is not so useful, as it requires substituting \( \tilde{F}_{\mu\nu}^{I} \) in terms of \( F_{\mu\nu}^{I} \), or vice versa. Furthermore it is clear that the Lagrangian does not transform as a function, since

\[ \tilde{L}(\tilde{F}) \neq L(F) . \]  \hspace{1cm} (8)(8)\(^3\)

\(^2\)Note that I use the tilde to denote the new rotated field strengths, and not the Hodge-dual of the field strengths! This notation is used throughout. Furthermore, observe that electric-magnetic duality can be formulated straightforwardly in curved space; it is simply a matter of convenience that I am assuming flat space here. For an early discussion of electric-magnetic duality in curved space, the reader may consult \[ [4] \].

\(^3\)I exclude transformations with \( U = V \propto 1 \) and \( W = Z = 0 \), since this amounts to a simple rescaling of the Lagrangian and field strengths.
However, the linear combination of the Lagrangian and the $F^I_{\mu\nu}, G^I_{\mu\nu}$ terms,
\[
L(F) + \frac{i}{2}(F^I_{\mu\nu}G^I_{\mu\nu} - F^{-I}_{\mu\nu}G^{-I}_{\mu\nu}),
\]
(9)
does transform as a function.

When $L$ remains unchanged, i.e. when
\[
\tilde{L}(\tilde{F}) = L(F),
\]
(10)
then the theory is invariant under the corresponding transformations. Again this is hard to verify explicitly in this general form. A more convenient method instead, is to verify that the substitution $F^I_{\mu\nu} \rightarrow \tilde{F}^I_{\mu\nu}$ into the derivatives $\partial L(F)/\partial F^I_{\mu\nu}$ correctly induces the symplectic transformations of the field strengths $G^I_{\mu\nu}$. Furthermore, the linear combination (9) must be an invariant function under $F^I_{\mu\nu} \rightarrow \tilde{F}^I_{\mu\nu}$. Note that in the literature the word duality is used both for equivalence and invariance relations. Whenever the context is not clear I will always try and indicate specifically whether I am dealing with an equivalence or an invariance.

Observe that the presence of other fields does not play an immediate role here. Their symmetry transformations must be combined with the duality transformations. The transformations of the other fields have been incorporated already in the duality transformations [15,16], as the field strengths $G_{\mu\nu}$ may depend on other fields. In that case the variations of these other fields are crucial in that they must generate (together with the transformation of the field strengths $F^I_{\mu\nu}$) the correct transformation rules for the tensors $G^I_{\mu\nu}$. Once this is accomplished, the result (10) will still apply and the $F^I_{\mu\nu}$-independent terms that are not constrained by (9) must be separately invariant under the transformations acting on the other fields than the field strengths. I return to this issue in the next section, where I will be dealing with a more specific set of Lagrangians.

When the duality invariance is continuous, one may evaluate the effect of an infinitesimal transformation. Hence one expands,
\[
\left( \begin{array}{cc} U & Z \\ W & V \end{array} \right) \approx 1 + \left( \begin{array}{cc} B & -D \\ C & -B^t \end{array} \right),
\]
(11)
where the matrices $C_{11}$ and $D_{11}$ are symmetric. Invariance requires now that the following linear combination vanishes,
\[
C_{11}(F^{+I}_{\mu\nu}F^{+J}_{\mu\nu} - F^{-I}_{\mu\nu}F^{-J}_{\mu\nu})
-2B^I_{J}(G^{+I}_{\mu\nu}F^{+J}_{\mu\nu} - G^{-I}_{\mu\nu}F^{-J}_{\mu\nu})
+D^{IJ}(G^{+I}_{\mu\nu}G^{+J}_{\mu\nu} - G^{-I}_{\mu\nu}G^{-J}_{\mu\nu}) = 0.
\]
(12)

In case the theory has electrically and/or magnetically charged states, these would show up as source terms in the Maxwell equations. Consequently, these magnetic and electric charges, which I denote by $(\rho^I, q_I)$, should rotate under the action of the duality group. However, it is well known that these charges have to satisfy Dirac-Schwinger-Zwanziger quantization conditions and thus span some lattice. This lattice is then left invariant by an arithmetic subgroup $\text{Sp}(2n; \mathbb{Z})$ of the duality group.

A relevant example of a Lagrangian that depends exclusively on a field strength, is the Born-Infeld Lagrangian of nonlinear electrodynamics, defined by [13],
\[
L = -g^{-2}\sqrt{\det[\eta_{\mu\nu} + gF_{\mu\nu}]} + g^{-2},
\]
(13)
where $g$ is a coupling constant. It has been known for a long time that this Lagrangian has electromagnetic duality symmetries [14] (for more recent references, see [15,16]). To deal with this aspect let us first rewrite the determinant as follows,
\[
\Delta = \det[\eta_{\mu\nu} + gF_{\mu\nu}]
= 1 - \frac{1}{4}g^2\text{Tr}(F^2) - \frac{1}{4}g^4[\text{Tr}(F^2)]^2 - \frac{1}{4}g^4\text{Tr}(F^4)
+ \frac{1}{2}g^2(F^{+2} + F^{-2}) + \frac{1}{16}g^4(F^{+2} - F^{-2})^2,
\]
(14)
where $F^{\pm2} = F^{\pm}_\mu F^{\pm\mu}_\nu$. The expression for $G_{\mu\nu}$ follows then directly,
\[
G^{+\mu}_{\mu\nu} = -\frac{i}{\Delta^{1/2}}F^{+\mu}_{\mu\nu}\left[1 + \frac{1}{4}g^2(F^{+2} - F^{-2})\right].
\]
(15)
Inserting this expression into the condition (12) it follows that there is only one continuous duality symmetry characterized by $C = D$ and $B = 0$. Obviously, this leads to finite transformations corresponding to $\text{SO}(2)$. There exist no new discrete duality symmetries.

Under general electric-magnetic duality transformations corresponding to $\text{Sp}(2; \mathbb{R}) \cong$
SL(2, \mathbb{R})$, one obtains a variety of different but inequivalent Lagrangians whose parameter space is isomorphic to SL(2)/SO(2). However, it is rather cumbersome to work out this parametrization explicitly.

2. U(N) DUALITY

The Lagrangians of $N$-extended supergravity in four spacetime dimensions contain $\frac{1}{2}N(N-1)$ abelian vector gauge fields\footnote{For $N = 6$ the number is different and equal to 16; I disregard $N = 7$ which is just a relabeling of the $N = 8$ theory.} and depend at most quadratically on the field strengths. I parametrize them according to

\[
L = -\frac{1}{4} \sqrt{-g} N_{IJ} F_{\mu\nu}^{\pm I} F_{\mu\nu}^{\pm J} - \frac{1}{4} i \sqrt{-g} F_{\mu\nu}^{\dagger} O_{\mu\nu}^{I}\n
+ \text{h.c.} + F\text{-independent terms},
\]

where the $F_{\mu\nu}^{\pm I}$ are the (anti-)selfdual field strengths and $N_{IJ}$ may depend on the scalar fields. In addition there are moment couplings (e.g. to the fermions) encoded in tensors $O_{\mu\nu}^{I}$ whose form is left unspecified. For these Lagrangians one can investigate whether they are invariant under a certain class of electric-magnetic duality transformations. One obvious invariance group is SO($N$) with the vector potentials transforming in the adjoint representation. For supergravity this group can be extended to U($N$) which, in view of the reality of the vector potentials, cannot act on the vector potentials but only on the field strengths by means of electric-magnetic duality. Hence the invariance does not apply directly to the Lagrangian, but to the combined equations of motion and Bianchi identities.

There is a number of reasons to expect that U($N$) is in fact the minimal invariance group of the field equations. First of all, U($N$) is the subgroup of the automorphism group of the supersymmetry algebra, which acts chirally on the $N$ supercharges and commutes with the Lorentz group. Therefore massless supermultiplets consist of states that are assigned to representations of U($N$)\footnote{This assignment can then be extended to the underlying fields (assuming one can find suitable field representations). It is important to realize that this by itself does not imply that the action or the equations of motion are invariant under U($N$), because the theory could allow for certain deformations that break the invariance (the most obvious deformation is a gauging of SO($N$) associated with the vector fields). However, the simplest such theory will exhibit U($N$) as an invariance group. Secondly, U($N$) is a local gauge invariance of the superconformal theories, which, at least for $N \leq 4$, govern the leading spin components of the Poincaré supergravity theories $[8, 9]$. Indeed $N$-extended supergravity has been shown to exhibit U($N$) duality for $N = 2, 3$ in [2] and for $N = 4$ in [2]. For $N = 8$ the situation was analyzed in [20]. To be more specific, let me return to the Lagrangians (16). Obviously, the corresponding tensors $G_I$ take the form,

\[
G_{\mu\nu I}^+ = N_{IJ} F_{\mu\nu}^{+ J} + O_{\mu\nu I}^+,
\]

\[
G_{\mu\nu I}^- = \bar{N}_{IJ} F_{\mu\nu}^{- J} + O_{\mu\nu I}^-.
\]

The Bianchi identities and equations of motion for the abelian gauge fields are given in (3).

From (17) and (18) one derives that $N$ must transform according to a fractional linear transformation,

\[
\hat{N}_{IJ} = (V_{i}^{K} N_{KL} + W_{iL}) [(U + ZN)^{-1}]_{iJ},
\]

and this transformation must be induced by proper changes of the scalar fields. To ensure that $N$ remains a symmetric tensor, at least in the general case, the transformation (19) must be an element of Sp(2$n$; \mathbb{R}), in accord with what was claimed in the previous section (again I disregard uniform scale transformations). Furthermore the tensor $O$ must change according to

\[
\hat{O}_{\mu\nu I}^+ = O_{\mu\nu I}^+ [(U + ZN)^{-1}]_{I},
\]

and likewise for $O^-$. The function $N$ defines (possibly field-dependent) expressions for the generalized coupling constants and $\theta$-angles, $\hat{N}_{IJ} \propto \theta_{IJ/2\pi} + i4\pi g_{IJ}$. The sign of the coupling constants does not change under electric-magnetic duality, as follows from

\[
(U - \hat{N})_{IJ} = (N - \hat{N})_{KL} \times [(U + ZN)^{-1}]_{L} [(U + ZN)^{-1}]_{KJ}.
\]
It is possible to rewrite the Lagrangian (16) as follows,

\[
L = -\frac{1}{4} \sqrt{g} F_{\mu\nu}^{I} G^{+\mu\nu} - \frac{1}{4} \sqrt{g} F_{\mu\nu} G^{I} + \text{h.c.}
\]

\[
= -\frac{1}{4} \sqrt{g} F_{\mu\nu}^{I} + F_{\mu\nu} (G^{+\mu\nu} - G^{-\mu\nu})_{IJ} + \sqrt{g} \left[ O_{\mu\nu I} [(N - \bar{N})^{-1}]_{IJ} \right. \times (G^{+}_{J\mu\nu} - \bar{N}_{JK} F^{+\mu\nu K}) + \text{h.c.} \right], \tag{21}
\]

where in the second equation I introduced an \(F_{\mu\nu}^{I}\)-independent term by hand so that the last term transforms as a scalar under symplectic reparametrizations. The first term in the last equation is \textit{not} a scalar, but it is equal to a total derivative by virtue of Maxwell’s equations. Any \(F_{\mu\nu}^{I}\)-independent terms in the Lagrangian (16) must separately transform as a scalar.

To ensure invariance under \(U(N)\) duality transformations one must first verify whether the transformations of the scalars induce the correct variation (13) of \(N_{IJ}\). Secondly, one must verify that all \(F_{\mu\nu}^{I}\)-independent terms other than the ones already included in the last term in (21), are separately invariant under \(U(N)\) transformations of the remaining fields. For Lagrangians which depend at most quadratically on the field strengths (such as the one discussed in this section), infinitesimal duality symmetries must satisfy the condition (cf. (12)),

\[
C_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu} - 2 B_{IJ} G_{\mu\nu I}^{+} F^{+J\mu\nu} + D_{IJ} G_{\mu\nu I}^{+} G^{+\mu\nu} = 0 . \tag{22}
\]

This condition is the direct analogue of the so-called consistency condition that I will present in section 4 (cf. (12)) in the context of \(N = 2\) vector multiplets coupled to supergravity (11). This condition was explored extensively in (21).

It is obvious that electric-magnetic duality invariances can be broken. This is rather straightforward for low-\(N\) supergravity where one can couple to supersymmetric matter. But in all cases one can introduce nonabelian interactions for the gauge fields so that the vector potentials (which are real and cannot transform under the full duality group) appear explicitly in the field equations and Bianchi identities. Another question is, however, whether the electric-magnetic dualities can constitute a group \textit{bigger} than \(U(N)\). With hindsight it is obvious that a bigger group is in fact natural. For \(N \geq 4\) the pure supergravity theories contain scalar fields which (see Table 1) transform under \(U(N)\). The Lagrangian depends on the scalar fields in a nonlinear fashion and their kinetic terms take the form of a nonlinear sigma model with the scalars parametrizing some target space. In view of the fact that the Lagrangians become more and more restricted for large \(N\), one expects the target space to exhibit a high degree of symmetry. Specifically, one expects the target spaces to be homogeneous, which means that every two points of the space can be related by a symmetry transformation (i.e., an isometry). Because the \(U(N)\) transformations act linearly on the scalar fields they only constitute a subgroup of the full isometry group of the scalar manifold. A homogeneous space has also nonlinearly realized symmetries, so that its symmetry group is bigger than \(U(N)\). As the \(U(N)\) subgroup is realized by means of electric-magnetic duality, also the nonlinearly realized symmetries of the scalar manifold are contained in the duality group.

3. HIDDEN SYMMETRIES

As argued above pure supergravity theories with \(N \geq 4\) have scalar fields that parametrize a homogeneous manifold. The symmetries of these manifolds include the group \(U(N)\). The structure of the scalar sector of the theory is thus based on a coset space \(G/H\), where \(G\) is the “hidden symmetry” group and \(H = U(N)\), because \(U(N)\) is the group that can act on the other fields (in particular on the fermions by means of chiral transformations), as I discussed previously.

These “hidden” symmetries were first discovered for \(N = 4\), where the complex scalar parametrizes the \(SU(1, 1)/U(1)\) coset space (22). The absence of the \(SU(4)\) group is of no concern here, because this group can be included as a factor in both \(G\) and \(H\). Hence \(G = SU(1, 1) \times SU(4)\). Interestingly enough, it turned out later that the superconformal \(N = 4\) theory also contains a complex scalar which parametrizes the same coset space (13). Subsequently it was established that the same situation arises for the complex scalar.
Table 1
The scalar fields for \( N = 4, 5, 6, 8 \). The case \( N = 7 \) coincides with \( N = 8 \).

| \( N \) | scalar field U(\( N \)) decomposition | homogeneous space |
|-------|---------------------------------|------------------|
| 4     | \( \phi \)                       | \( 1 + \overline{1} \) | SU(1,1)/U(1) |
| 5     | \( \phi^i \)                     | \( 5 + \overline{5} \) | SU(5,1)/U(5) |
| 6     | \( \phi^{ij} \)                  | \( 15 + \overline{15} \) | SO*(12)/U(6) |
| 8     | \( \phi^{ijkl} \)                | \( 35 + \overline{35} \) | E\(_{7(7)}\)/SU(8) |

in 10-dimensional IIB supergravity [23].

From the above arguments it follows that the group \( G \) must satisfy the following embedding condition,

\[
U(N) \subset G \subset \text{Sp}(2n; \mathbb{R})
\]

where, for \( N = 8 \), \( n = 28 \). The major advance came with the work of [24], where it was realized that for \( N = 8 \) one has \( G = E\(_{7(7)}\) \), which is indeed a subgroup of \( \text{Sp}(56; \mathbb{R}) \) and contains \( H = \text{SU}(8) \) as its maximal compact subgroup. The fact that the \( U(1) \) subgroup of \( U(8) \) is absent may seem surprising in view of the \( U(8) \) symmetry group of the \( N = 8 \) supergravity multiplet, but one must realize that, for this multiplet, the \( U(1) \) subgroup coincides with the \( \text{SO}(2) \) helicity group. The dimension of \( E\(_{7(7)}\) \) and \( \text{SU}(8) \) is equal to 133 and 63, respectively, which shows that the coset space parametrized by the scalar fields is of dimension 70.

It is often convenient to employ the conjectured duality invariance in constructing the full supergravity Lagrangians, as the invariance gives a handle on the nonlinear interactions with the scalar fields. There are many examples in the literature where this approach was followed. With regard to \( N = 8 \) supergravity, the first iterative results for the Lagrangian and the transformation rules appeared in [3]. The nonlinear terms were uncovered in [24], where the theory was constructed by dimensional reduction from eleven dimensional supergravity [25]. In the pioneering paper [24] many of the details of the dimensional reduction of supergravity were worked out for the first time. The \( E\(_{7(7)}\) \) duality was conjectured based on a counting argument indicating that the dimension of the duality group should be 133. This was then confirmed by working out certain terms of the Lagrangian and of the supersymmetry transformation rules. The \( \text{SU}(8) \) duality of the theory was studied in [23]. Finally complete result based on \( E\(_{7(7)}\) \) duality were presented in [24] and served as a starting point for the \( \text{SO}(8) \) gauging of \( N = 8 \) supergravity. The resulting theory was written in a form where \( \text{SO}(8) \times \text{SU}(8) \) was manifest and the \( \text{SO}(8) \) gauge group was embedded into \( E\(_{7(7)}\) \), thus breaking the electric-magnetic duality to \( \text{SO}(8) \).

From a more recent perspective the question arises what the implications of electric-magnetic duality are for eleven-dimensional supergravity and/or \text{M}-theory. In this connection I note that there exists an alternative formulation of 11-dimensional supergravity, which does not truncate the theory to the massless sector in a compactification to four spacetime dimensions while still preserving certain features of the electric-magnetic duality group [27]. It is based on replacing the \( \text{SO}(1,10) \) tangent-space symmetry of the eleven-dimensional theory according to

\[
\text{SO}(1,10) \rightarrow \text{SO}(1,3) \times \text{SO}(7) \rightarrow \text{SO}(1,3) \times \text{SU}(8).
\]

The construction thus stresses a \( 4+7 \) split of the \( d = 11 \) coordinates which leads to corresponding decompositions of the tensors and spinors. Other splits have been worked out along similar lines [28]. The alternative theory is gauge equivalent to the original one of [25] and the equivalence holds at the level of the equations of motion. The reformulation can be regarded as a first step towards fusing the bosonic degrees of freedom of \( d = 11 \) supergravity in a way which is more in harmony with the hidden symmetries of the dimensionally reduced theories.

An important quantity is the so-called general-
The constraints can thus be rephrased as the statement that the product \( e^m \otimes e^n \) only contains the 1463 and 1539 representations of \( E_{7(7)} \).

In addition to the algebraic constraints, the generalized vielbein satisfies a set of first-order differential relations, called the "generalized vielbein postulate" in [27]. In order to state them, we need suitable \( E_{7(7)} \) connections \( Q_M{}^A_B \) and \( P^{ABCD}_M \) in eleven dimensions. These are built out of the \( \text{SO}(1,10) \) coefficients of anholonomity and the four-index field strength \( F_{MNPQ} \) of \( d=11 \) supergravity in the way explained in [27]; since the explicit expressions are somewhat cumbersome I refer readers there for details. The vector \( Q_M{}^A_B \) acts as the connection for the local \( \text{SU}(8) \) transformations and is therefore in the 63 representation of that group. The tensor \( P^{ABCD}_M \) transforms in the 70 representation under the action of \( \text{SU}(8) \). Together they constitute the (adjoint) 133 representation of \( E_{7(7)} \). For the massless theory these quantities are directly related to the pull-backs to \( d=11 \) spacetime of the tangent-space connection and vielbein associated with the homogeneous space \( E_{7(7)}/\text{SU}(8) \).

The generalized vielbein postulate takes the form

\[
\mathcal{D}_\mu e^m_{AB} + \frac{1}{2} \mathcal{D}_n B_{\mu}{}^{n} e^m_{AB} + \mathcal{D}_n B_{\mu}{}^{m} e^n_{AB} + 2 Q_{[AB}{}^{C} e^{m}_{CD} + P_{nABCD} e^{mCD} = 0,
\]

\[
\mathcal{D}_n e^m_{AB} + 2 Q_{n[A}{}^{C} e^{m}_{BC]} + P_{nABCD} e^{mCD} = 0,
\]

where \( B_{\mu}{}^{m} \) are the Kaluza-Klein vector fields that originate from the \( d=11 \) metric; furthermore

\[
\mathcal{D}_\mu = \partial_\mu - B_{\mu}{}^{m} D_m,
\]

for \( \mu = 0, 1, 2, 3 \) and

\[
\mathcal{D}_m e^m_{AB} = \partial_m e^m_{AB} + \Gamma_{mp}{}^{n} e^p_{AB} + \frac{1}{2} \Gamma_{mp}{}^{n} e^n_{AB},
\]

\[
\mathcal{D}_n B_{\mu}{}^{n} = \partial_n B_{\mu}{}^{n} + \Gamma_{np}{}^{m} B_{\mu}{}^{p},
\]

for the internal indices \( m,n,\ldots = 1,\ldots,7 \). The extra term with \( \Gamma_{mp}{}^{n} \) in the above relation arises because the generalized vielbein transforms as a density. Observe that the affine connection \( \Gamma_{mn}{}^{p} \) still depends on all eleven coordinates, but is still arbitrary at this point, as it cancels between the different terms in the generalized vielbein postulate. A convenient choice is the standard Christoffel connection.
So I conclude that all the quantities introduced above comprise $E_{7(7)}$ representations. I stress once more that I am still dealing with the full $d = 11$ supergravity theory. This pattern continues. For instance, the supersymmetry variation of the generalized vielbein takes a form that closely resembles the four-dimensional transformation rule for the massless modes (in the truncation to the massless modes, $e^m_{AB}$ is proportional to the $E_{7(7)}/SU(8)$ coset representative),

$$\delta e^m_{AB} = -\sqrt{2} \Sigma_{ABCD} e^m_{CD}, \quad (34)$$

where

$$\Sigma_{ABCD} = \eta_{[A} \chi_{BCD]} + \frac{1}{24} \epsilon_{EFGH} \chi^{EFGH}, \quad (35)$$

where $\epsilon_A$ and $\chi_{ABC}$ denote the supersymmetry parameters and the spin-$1/2$ fields, respectively. Similarly, the bosonic and fermionic equations of motion can be cast into a fully $SU(8)$ covariant form.

In spite of the fact that the theory can be formulated elegantly in terms of $E_{7(7)}$ quantities, it cannot be invariant under $E_{7(7)}$. An obvious obstacle to the invariance seems to be the presence of the seven Kaluza-Klein vector fields $B^m_\mu$. For instance, when restricting ourselves to the massless modes in the toroidal compactification, which is $E_{7(7)}$ invariant, the explicit coupling to these vector fields disappears in the generalized vielbein postulate, and evidently many of the formulas displayed above are trivially satisfied.\footnote{I remind the reader that there are 21 additional massless vector fields in the toroidal reduction, which originate from the $d = 11$ tensor field. However, those fields do not appear explicitly in the generalized vielbein postulate, and are contained in the $E_{7(7)}$ connections.} Retaining the massive Kaluza-Klein states associated with the torus, one observes that it is not possible to preserve $E_{7(7)}$, because the Kaluza-Klein charges do not cover a complete $E_{7(7)}$ (Z) invariant lattice (as they should in M-theory). In other words, neither the gauge fields nor the central charges constitute a representation of the duality group. Observe that the momentum lattice of the Kaluza-Klein states would in any case restrict the duality group $E_{7(7)}$ to an arithmetic subgroup.

An intriguing question is whether one can somehow extend the Kaluza-Klein states to a fully duality-invariant set of states. This extension to a BPS-extended supergravity theory, defined as a duality invariant completion of toroidally compactified supergravity with BPS states\footnote{Similar results have been obtained for three- and five-dimensional gauged maximal supergravity\footnote{34}.}, will incorporate more essential features of M-theory. Because I did not insist on truncating to the massless modes in the above setting, it should in principle be possible to incorporate these extra states and consider other than toroidal backgrounds. Of course, one should not only extend the configuration of BPS states, but also the gauge fields that can couple to the corresponding central charges. While this can be done for higher dimensions, in four dimensions this is difficult because the 56 central charges are related to electric and magnetic charges and those cannot simultaneously be realized in a local field theory. Nevertheless, assuming that this is a way to go, then one should also find traces of the duality group in compactifications of $d = 11$ supergravity on non-trivial internal manifolds. In that respect the above results are again promising because there no reference was made to any particular background. To date there is only one non-trivial compactification, namely the $AdS_4 \times S^7$ compactification of $d = 11$ supergravity\footnote{31} (the $AdS_7 \times S^4$ truncation of\footnote{22} could eventually provide another case, but those results remain to be analyzed from the point of view taken here) which has been analyzed in the above framework. In that case, the internal connection components $Q^m_{AB}$ and $P^m_{ABCD}$ do survive the truncation to the massless modes and are metamorphosed into the $T$-tensor describing the couplings of the scalars and the fermions in gauged supergravity\footnote{31}. The remarkable fact is that for gauged $N = 8$ supergravity in four dimensions, this $T$-tensor does show features related to the $E_{7(7)}$ symmetry group, even though $E_{7(7)}$ is not a symmetry of the gauged theory! Namely, the $T$-tensor can be assigned to the $912$ representation of $E_{7(7)}$, which branches into $SU(8)$ representations according to

$$912 \rightarrow 36 + 36 \oplus 420 + 420. \quad (36)$$

The example of the $AdS_7 \times S^4$ compactification shows that the $T$-tensor can be found in gauge theories in which the internal manifold is not compact. If one looks at the remaining case of $AdS_4 \times S^7$ compactification, one observes that the $T$-tensor does not appear straightforwardly. Nevertheless, it is possible to incorporate these extra states and consider other than toroidal backgrounds. Of course, one should not only extend the configuration of BPS states, but also the gauge fields that can couple to the corresponding central charges. While this can be done for higher dimensions, in four dimensions this is difficult because the 56 central charges are related to electric and magnetic charges and those cannot simultaneously be realized in a local field theory. Nevertheless, assuming that this is a way to go, then one should also find traces of the duality group in compactifications of $d = 11$ supergravity on non-trivial internal manifolds. In that respect the above results are again promising because there no reference was made to any particular background. To date there is only one non-trivial compactification, namely the $AdS_4 \times S^7$ compactification of $d = 11$ supergravity\footnote{31} (the $AdS_7 \times S^4$ truncation of\footnote{22} could eventually provide another case, but those results remain to be analyzed from the point of view taken here) which has been analyzed in the above framework. In that case, the internal connection components $Q^m_{AB}$ and $P^m_{ABCD}$ do survive the truncation to the massless modes and are metamorphosed into the $T$-tensor describing the couplings of the scalars and the fermions in gauged supergravity\footnote{31}. The remarkable fact is that for gauged $N = 8$ supergravity in four dimensions, this $T$-tensor does show features related to the $E_{7(7)}$ symmetry group, even though $E_{7(7)}$ is not a symmetry of the gauged theory! Namely, the $T$-tensor can be assigned to the $912$ representation of $E_{7(7)}$, which branches into $SU(8)$ representations according to

$$912 \rightarrow 36 + 36 \oplus 420 + 420. \quad (36)$$

Similar results have been obtained for three- and five-dimensional gauged maximal supergravity\footnote{34}.35.
4. MATTER COUPLINGS

Electric-magnetic duality also plays an important role in supergravity coupled to vector supermultiplets and in rigidly supersymmetric gauge theories. Of course, the possibility for performing these duality transformations is restricted in the presence of charged fields, because the Lagrangian will then depend explicitly on vector potentials. Below I will concentrate on the case of $N = 2$ abelian vector multiplets. Before doing so, I should briefly draw attention to the case of $N = 4$ supersymmetry. First of all, there is the intriguing conjecture\footnote{In the rigidly supersymmetric case I assume $n$ vector multiplets, so that $I, J = 1, \ldots, n$. In the coupling to supergravity one has to include the graviphoton. In the superconformal multiplet calculus this implies that one must include one more vector multiplet, so that the indices take the values $I, J = 0, 1, \ldots, n$. For supergravity the duality group will thus be contained in Sp(2n + 2; R).} that there exists a strong-weak coupling duality between electric charges and magnetic charges in a field theory, or, equivalently, between conventional local charges and topological charges. Obviously this is an electromagnetic duality. It is believed (see, e.g.,\footnote{The terms linear in $X I$ in (39) are associated with constant translations in $F I$ in addition to the symplectic rotation shown in (38). Likewise one may introduce constant shifts in $X I$. Henceforth I ignore these shifts, which are excluded for local supersymmetry. Constant contributions to $F(X)$ are always irrelevant.}) that this duality is realized explicitly for $N = 4$ supersymmetry. First of all, there is the inhomogeneous Yang-Mills theory. Secondly, I refer to the coupling of $N = 4$ vector multiplets to supergravity, which was worked out in\footnote{In the coupling to supergravity the holomorphic function should be homogeneous of second degree.} by coupling on-shell vector multiplets to $N = 4$ conformal supergravity\footnote{In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.} and imposing electromagnetic duality on the equations of motion.

The actions for $N = 2$ vector multiplets are based on $N = 2$ chiral superspace integrals,

$$ S \propto \text{Im} \left( \int d^4 x \, d^4 \theta \, F(W^I) \right), $$

where $F$ is an arbitrary function of reduced chiral multiplets $W^I(x, \theta)$. The indices $I, J, \ldots$ will now label the vector multiplets.\footnote{In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.} Such multiplets carry the gauge-covariant degrees of freedom of a vector multiplet, consisting of a complex scalar $X^I$, a spinor doublet $\Omega^I$, and a selfdual field strength $F^{IJ}$. and a triplet of auxiliary fields $Y_{ij}$. This Lagrangian may coincide with the effective Lagrangian associated with some supersymmetric Yang-Mills theory, but for our purposes its origin is not directly relevant. To enable coupling to supergravity the holomorphic function should be homogeneous of second degree\footnote{The terms linear in $X I$ in (39) are associated with constant translations in $F I$ in addition to the symplectic rotation shown in (38). Likewise one may introduce constant shifts in $X I$. Henceforth I ignore these shifts, which are excluded for local supersymmetry. Constant contributions to $F(X)$ are always irrelevant.}.

As before, I choose abelian gauge groups, so that the field strengths are subject to electric-magnetic duality transformations. Supersymmetry now dictates how the fields of a vector multiplet other than the field strength should transform under electric-magnetic duality. As a result the structure of the dualities can now be captured in terms of the complex scalars $X^I$ and the holomorphic function $F(X)$. For generic $N = 2$ vector supermultiplets it was discovered\footnote{In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.} that the dualities rotate the scalar fields $X^I$ and the derivatives $F_I$ of the holomorphic function $F(X)$ that encodes the Lagrangian, by means of the same Sp(2n; R) matrix that was introduced before (c.f.\footnote{In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.}).

$$ (X^I, F_I) \rightarrow (\tilde{X}^I, \tilde{F}_I) = \left( \begin{array}{cc} U & Z \\ V & W \end{array} \right) \left( \begin{array}{c} X^I \\ F_I \end{array} \right). \quad (38) $$

Because the matrix belongs again to Sp(2n; R), one can show that the new quantities $\tilde{F}_I$ can be written as the derivatives of a new function $\tilde{F}(\tilde{X})$. The new but equivalent set of equations of motion are obtained from the Lagrangian based on $\tilde{F}(\tilde{X})$. It is possible to integrate (38) and one finds

$$ \tilde{F}(\tilde{X}) = F(X) - \frac{1}{2} X^I F_I(X) + \frac{1}{2} (U^T W)_{IJ} X^I X^J + \frac{1}{2} (U^T V + W^T Z)_{IJ} X^I F_J + \frac{1}{2} (Z^T V)^{IJ} F_I F_J, \quad (39) $$

up to a constant and terms linear in the $\tilde{X}^I$. In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.\footnote{In the coupling to supergravity, where the function must be homogeneous of second degree, such terms are obviously excluded.} The last three terms in (33) can be written as $\frac{1}{2} \tilde{X}^I \tilde{F}_I(\tilde{X})$, so that the above result is precisely analogous to the result (6). Also the derivations proceed along similar lines.

Just as (7), the above expression (39) is not so useful, as it requires substituting $\tilde{X}^I$ in terms
ric Yang-Mills theories were obtained. Singular-
ergy effective actions for work initiated in [42] where exact solutions of low-
tions directly induce the symplectic transformations [41], i.e.,

\[ F_1(X) = -\frac{X^1(X^2)^2}{X^0}, \]

\[ F_2(X) = \frac{X^0(X^2)^2}{X^1} + (X^2)^2, \]

\[ F_3(X) = \pm \sqrt{-X^0(X^1)^2X^2} + (X^1)^2 + X^1X^2 + (X^2)^2. \]  

(40)

When \( F \) remains unchanged, \( \tilde{F}(\tilde{X}) = F(X) \), the theory is invariant under the corresponding transformations, but again it is hard to verify the invariance directly in this form. A more convenient method is to check whether the substitutions \( X^I \rightarrow \tilde{X}^I \) into the derivatives \( F_I(X) \) correctly induce the symplectic transformations [41], i.e.,

\[ F_1(\tilde{X}) = V_{IJ}F_J(X) + W_{IJ}X^J. \]  

(41)

This follows straightforwardly by differentiating with respect to \( X^I \), using the fact that \( F \) remains the same. The same remark was made in section 1, when discussing general theories with abelian gauge fields. For continuous duality symmetries, one derives a condition

\[ C_{IJ}X^I X^J - 2B^I_J F_I X^J + D^{IJ}F_{IJ} = 0, \]  

(42)

for all \( X \). This condition, derived in [11], is the direct analogue of (22).

Hence the functions \( F(X) \) decompose into equivalence classes. Two different functions may still describe the same theory, and their equivalence is effected by electric-magnetic duality. For the subgroup of the symplectic group corresponding to an invariance of the equations of motion, \( F(X) \) remains the same. These symplectic reparametrizations were at the basis of the work initiated in [42] where exact solutions of low-
energy effective actions for \( N = 2 \) supersymmetric Yang-Mills theories were obtained. Singular-
ities in these effective actions signal their breakdown due to the emergence of massless states corresponding to monopoles and dyons. Although these states are the result of nonperturbative dy-
namics, they are nevertheless accessible because near these singularities one conveniently converts to an alternative dual formulation in which local field theory is again applicable. In many of the nonperturbative solutions the quantities \( (X^I, F_J) \) can be defined as the periods of a meromorphic differential corresponding to a class of hyperelliptic curves (see e.g. [43–46]; see also the talk by Argyres at the symposium [47]). The same phenomena play a role for vector fields coupled to supergravity, for instance, in the context of heterotic string compactifications [11]. In the context of type-II string compactifications on Calabi-Yau manifolds, the \( (X^I, F_J) \) can be associated with the periods of the \((3,0)\) form of the Calabi-Yau three-fold [49]. A detailed discussion of these results is outside the scope of this review. However, I should mention that these systems of vector multiplets give rise to special geometry. Again, I refrain from giving further details and refer to the literature (see, e.g., [50–53]).

The result (39) shows immediately that \( F(X) \) does not transform as a function under duality, but the combination

\[ F(X) - \frac{1}{2}X^I F_I(X) \]  

(43)

does, i.e., it transforms according to \( \tilde{f}(\tilde{X}) = f(X) \). In the case that there are duality invariances, (43) is an invariant function under the corresponding symplectic transformations. It should be stressed that, although one generically calls quantities such as \( F(X) \) holomorphic functions, I use the term now in a much more restricted sense by insisting that certain quantities transform as functions under symplectic transformations. Quantities with this property are more rare. I stress that physical results should always be expressible in terms of such functions. The reason is that an ab initio calculation of physical quantities will always yield a function (I assume that the quantity is a scalar). For instance, suppose that one could directly calculate the mass spectrum of solitonic solutions and that the soliton mass would depend exclusively on the
charges. If this were the case, the function that expresses the mass would have to be a symplectic function. In fact, there would be a stronger result: the function should be invariant with respect to $Sp(2n;\mathbb{Z})$, simply because the solutions corresponding to these charges are equivalent up to electric-magnetic duality. Clearly, this result must be wrong: there is no symplectic invariant that depends exclusively on the charges, so the mass formula would be constant! In reality the mass will depend also on the function $F(X)$, which changes under duality as well. Hence the mass formula will depend on the charges $(p^I, q_I)$ and on $X^I$ and derivatives of $F(X)$ in such a way that the expression transforms as a function under duality. However, suppose now that we restrict ourselves to the subgroup of dualities that constitutes an invariance. In that case, the function $F(X)$ remains the same and the mass formula must be an invariant function of $X^I$, $p^I$ and $q_I$. These observations have, for instance, been put to a test for extremal black holes, where the entropy for BPS black holes is indeed a symplectic function, even when one includes terms quadratic in the Weyl tensor into the effective low-energy theory \cite{54}. At the event horizon the $X^I$ are determined by the charges according to the so-called fixed-point behaviour noted in \cite{53–72}. The equations that determine the $X^I$ in terms of the charges are symplectically covariant \cite{5}. Therefore the entropy is an invariant function of the electric and magnetic charges under the duality symmetries. Note that also the ADM mass formula for the black holes is a symplectic function and is invariant under the invariant duality subgroup, except that this function depends both on the charges and on the values of the moduli at spatial infinity. The above reasoning is not restricted to $N = 2$ supersymmetric systems and there are many examples in the literature of duality invariant entropy functions which have been discussed from a variety of viewpoints, see e.g., \cite{55,60,57,54}.

5. DUALITY AND HOLOMORPHY

I now extend the discussion by considering supersymmetric Yang-Mills theories in the presence of a chiral background field \cite{2}. To couple supersymmetric vector multiplets to (scalar) chiral background fields is straightforward. One simply incorporates additional chiral fields $\Phi$ into the function $F$ that appears in the integrand of \cite{27}.

$$S \propto \text{Im} \left( \int d^4x \ d^4\theta \ F(W^I, \Phi) \right).$$

Also the coupling to conformal supergravity is known \cite{67}. I draw attention to the fact that the $W^I$ are reduced, while the $\Phi$ can be either reduced or general chiral fields.

Let me briefly review situations where such chiral backgrounds are relevant. In supersymmetric theories many of the parameters (coupling constants, masses) can be regarded as background fields that are frozen to constant values (so that supersymmetry is left intact). Because these background fields correspond to certain representations of supersymmetry, the way in which they appear in the theory – usually both perturbatively as well as nonperturbatively – is restricted by supersymmetry. In this way one may derive restrictions on the way in which parameters can appear. An example is, for instance, the coupling constant and $\theta$-angle of a supersymmetric gauge theory, which can be regarded as a chiral field frozen to a complex constant $iS = \theta/2\pi + i4\pi/g^2$. Supersymmetry now requires that the function $F(X)$ depends on $S$, but not on its complex conjugate. This strategy of introducing so-called spurion fields is not new. In the context of supersymmetry it has been used, for instance in \cite{58,70}, to derive nonrenormalization theorems and even exact results. Another option is to not restrict the spurion superfields to constant values, in order to introduce an explicit breaking of supersymmetry \cite{71,72}.

The above approach is very natural from the point of view of string theory, where the moduli fields, which characterize the parameters of the (supersymmetric) low-energy physics, reside in supermultiplets. In heterotic $N = 2$ compactifications the background field $S$ introduced
above coincides with the complex dilaton field, which comprises the dilaton and the axion, and belongs to a vector multiplet. The dilaton acts as the loop-counting parameter for string perturbation theory. Although the full supermultiplet that contains the dilaton is now physical, the derivation of nonrenormalization theorems can proceed in the same way [73, 41]. I should stress here that when restricting the background to a reduced chiral multiplet, one can just treat it as an additional (albeit external) vector multiplet. Under these circumstances one may consider extensions of the symplectic transformations that involve also the background itself. Of course, when freezing the background to constant values, one must restrict the symplectic transformations accordingly. The above strategy is especially useful when dealing with anomalous symmetries. By extending anomalous transformations to the background fields, the variation of these fields can compensate for the anomaly. The extended non-anomalous symmetry becomes again anomalous once the background is frozen to a constant value.

Another application of chiral backgrounds, which is relevant later on in this section, concerns the coupling to the Weyl multiplet, which leads to interactions of vector multiplets to the square of the Riemann tensor. In this case the scalar chiral background is not reduced and is proportional to the square of the Weyl multiplet. Here the strategy is not to freeze the background to a constant, but one is interested in more general couplings with conformal supergravity. This application is relevant for considering actions with higher-derivative interactions involving the square of the Weyl tensor in supergravity coupled to matter (see, e.g. [24, 25, 26]).

In supergravity the expression for the superfields $W^I$ and on the background field $\Phi$ in a way that is a priori unrestricted. Then one can proceed exactly as before and examine the symplectic equivalence classes in the presence of the background. The transformation rules, however, will also depend on the background fields, because $X^I$ transforms into $F_I$ and the latter is background dependent. This does not affect the derivation, although there are a number of new features.

The analysis of electric-magnetic duality in the presence of the chiral background proceeds in the same way as before (for more details, see [23]). Denoting the lowest $\theta$-component of $\Phi$ by $\hat{A}$, one is dealing with functions $F(X, \hat{A})$. One starts from the same transformation rule (38) and obtains the same expression (39) for the new function after the symplectic transformation. But now the relation between $X$ and $\hat{X}$ involves $\hat{A}$. In the same notation as used previously I note the relation\(^{10}\)

$$\mathcal{N}_{IJ} = \hat{F}_{IJ},$$

(45)

It is convenient to introduce the following definitions,

$$\frac{\partial \hat{X}^I}{\partial X^J} = \mathcal{S}^I_J(X, \hat{A}) = U^I_J + Z^{IK} F_{KJ},$$

$$Z^{IJ} = [S^{-1}]^{I}_K Z^{KJ},$$

$$N_{IJ} = 2 \text{Im} F_{IJ}, \quad N^{IJ} = [N^{-1}]^{IJ},$$

(46)

which all depend on the background. The quantity $Z^{IJ}$ is symmetric in $I$ and $J$, because $Z U^T$ is a symmetric matrix as a consequence of the fact that $U$ and $Z$ are submatrices of the symplectic matrix.

On the quantities $\mathcal{N}_{IJ}$ and $\mathcal{O}^{\pm}_{IJ}$ that were introduced in (16), the symplectic reparametrizations act according to (18, 19), which, in the above notation, read as follows,

$$\mathcal{N}_{IJ} = (V_I^K \mathcal{N}_{KL} + W_{IL}) [S^{-1}]^{L}_J,$$

\(^{10}\)In supergravity the expression for $\mathcal{N}$ changes when integrating out the auxiliary tensor field. This is not possible in theories with higher-derivative couplings, so I refrain from doing this. Therefore I must insist that the function $F$ exists. After integrating out the auxiliary tensor it is possible to reformulate the theory in such a way that the function $F$ no longer needs to exist, as long as the periods $(X^I, F_I)$ can be written down consistently [23].
\[ \mathcal{O}_{\mu \nu J}^+ = \mathcal{O}_{\mu \nu J}^{+} [S^{-1}]^J_I, \]  
irrespective of the presence of the background. Also the following result,

\[ \tilde{F}(\tilde{X}, \tilde{A}) - \frac{1}{2} \tilde{X}^I \bar{F}_I(\tilde{X}, \tilde{A}) = F(X, \bar{A}) - \frac{1}{2} X^I F_I(X, \bar{A}), \]  

still holds, so that there is a holomorphic function that transforms as a function under symplectic transformations. In the coupling to supergravity this result is still relevant, provided that the background field \( \bar{A} \) has a nonzero scaling weight. Other results which hold irrespective of the background, are

\[ \tilde{N}_{IJ} = N_{KL} [\tilde{S}^{-1}]^K_I [S^{-1}]^L_J, \]
\[ \tilde{N}^{IJ} = N^{KL} S^K_I S^J_L, \]
\[ \tilde{F}_{JK} = F_{MNP} [\tilde{S}^{-1}]^M_I [S^{-1}]^N_J [S^{-1}]^{P}_{K}, \]  

where all quantities depend on both the fields \( X^I \) and \( \bar{A} \). The symmetry in \( I \) and \( J \) of the first two quantities is preserved owing to the symplectic nature of the transformation. Results that specifically refer to the background are obtained by taking derivatives of \( \tilde{F} \) (cf. (39)), keeping \( \tilde{X}^I \) fixed in partial differentiations of \( \tilde{F} \) with respect to \( \tilde{A} \), and/or using already known transformations. In this way one obtains, for instance,

\[ \tilde{F}_A(\tilde{X}, \tilde{A}) = F_A(X, \bar{A}), \]
\[ \tilde{F}_{AI} = F_{AJ} [S^{-1}]^J_I, \]
\[ \tilde{F}_I - \tilde{F}_{IJ} \tilde{X}^J = [F_J - F_{JK} X^K] [S^{-1}]^J_I, \]
\[ \tilde{F}_I - \tilde{F}_{IJ} \tilde{X}^J = [F_J - F_{JK} X^K] [S^{-1}]^J_I, \]
\[ \tilde{F}_{AA}(\tilde{X}, \tilde{A}) = F_{AA}(X, \bar{A}) \tilde{Z}^I_J. \]  

Observe that we have now identified two functions that are both symplectic and holomorphic. No other functions of this type are known. In the coupling to supergravity, these functions are actually related, provided that \( \bar{A} \) has nonzero scaling weight, because in that case \( F(X, \bar{A}) \) must be homogeneous. Note that the generalized \( N = 2 \) black hole entropy formula (42) turns out to consist of two terms, \( \tilde{X}^I F_I - \tilde{F}_I X^I \) and \( F_{\bar{A}} \), which are both symplectic functions (we suppress details related to the supergravity coupling). When dealing with a duality symmetry, both functions are invariant. The result that \( F_{\bar{A}} \) is invariant is somewhat analogous to a result of (39), according to which the derivative of the Lagrangian with respect to a duality invariant parameter is invariant.

However, higher than first derivatives of \( F \) with respect to \( \bar{A} \) do not transform as symplectic functions, with the exception of \( F_A \). This result may be somewhat disturbing especially when considering symplectic transformations that constitute an invariance. In that situation one has \( \tilde{F}(\tilde{X}, \tilde{A}) = F(\tilde{X}, \tilde{A}) \). In spite of that, this does not imply that the coefficient functions (i.e. multiple derivatives with respect to the background) are invariant functions under the corresponding transformations. This is only the case for the first one corresponding to \( F_A \).

One may wonder whether there exist modifications of the multiple-\( \bar{A} \) derivatives of \( F \) that do transform as functions under symplectic transformations. Such functions should be expected to arise when evaluating the coefficient functions directly on the basis of some underlying theory, such as string theory. Modifications seem possible in view of the fact that the combination

\[ F_{AA} + i N^{IJ} F_{AJ} F_{AI} \]

does indeed transform as a function under symplectic transformations. Likewise, one may verify by explicit calculation that there is a generalization of the third derivative,

\[ F_{AAA} + 3i N^{IJ} F_{AAJ} F_{AJ} \]
\[ - 3 N^{IK} N^{JL} F_{AIJ} F_{AK} F_{AL} \]
\[ - i N^{IL} N^{JM} N^{KN} F_{IJK} F_{AL} F_{AM} F_{AN}, \]

which also transforms as a symplectic function.

It turns out that these functions can be generated systematically. Assume that \( G(X, \bar{A}) \) transforms as a function under symplectic transformations. Then one readily proves that also \( \mathcal{D} G(X, \bar{A}) \) transforms as a symplectic function,
where \( D \equiv \frac{\partial}{\partial A} + iF_{A}N^{IJ} \frac{\partial}{\partial X^{J}}. \) (51)

Consequently one can write down a hierarchy of functions which are modifications of multiple derivatives \( F_{A}...A \).

\[ F^{(n)}(X, \hat{A}) \equiv \frac{1}{n!} D^{n-1} F_{A}(X, \hat{A}), \]  

where I included a normalization factor. All the \( F^{(n)} \) transform as functions under symplectic functions. However, except for the first one, they are not holomorphic. The lack of holomorphy is governed by the following equation \((n > 1)\),

\[ \frac{\partial F^{(n)}}{\partial X^{I}} = \frac{1}{2} \hat{F}_{IJK} \sum_{r=1}^{n-1} \frac{\partial F^{(r)}}{\partial X^{J}} \frac{\partial F^{(n-r)}}{\partial X^{K}}, \]  

where \( \hat{F}_{IJK} = \hat{F}_{ILM} N^{LJ} N^{MK} \).

Interestingly enough, this equation is reminiscent of the holomorphic anomaly equation of \([3]\).

To explain the relation let us discuss the coupling to supergravity and associate the chiral background field \( \Phi \) with the square of the Weyl multiplet \( \mathcal{W} \), thus obtaining higher-derivative couplings of vector multiplets with conformal supergravity. The square of the Weyl multiplet constitutes a scalar chiral field of scaling weight 2. Its lowest component is equal to \((\varepsilon_{IJ}T_{ab})^2\), where \( T_{ab} \) is sometimes (incorrectly) referred to as the graviphoton field strength; the highest-\( \theta \) component contains the square of the selfdual components of the Riemann tensor. I refer to \([19]\) for details. Now assume that the function \( F \) can be expanded as a power series,

\[ F(X, W^2) = \sum_{g=0}^{\infty} F^{(g)}(X) (W^2)^g. \]  

Because it must be homogeneous of second degree with scaling weights of \( X \) and \( W \) that are both equal to unity, the coefficient functions \( F^{(g)}(X) \) are homogeneous of degree \( 2(1-g) \).

In supergravity the \( X^I \) are not independent scalar fields, but are defined projectively; in more mathematical terms they can be regarded as sections of a complex line bundle. These sections can be expressed holomorphically in terms of independent complex fields \( z^A \), which describe the physical scalars of the vector multiplets. The original quantities \( X^I \) and the holomorphic sections \( X^I(z) \) differ by a factor \( m_P \exp(K/2) \), where \( K \) is the Kähler potential and \( m_P \) is the Planck mass. In view of the projective nature of the \( X^I \), there is thus always one more physical vector field than there are physical scalars. The extra vector corresponds to the graviphoton. The Lagrangian encoded by \([3]\) gives rise to terms proportional to the square of the Riemann tensor times \((\varepsilon_{IJ}T_{ab})^2\). After extracting the scale factor \( m_P \exp(K/2) \), the coefficient functions \( F^{(g)}(X) \) give rise to holomorphic functions \( F^{(g)} \) of the \( z \) (or rather sections of a line bundle). Now consider the case where the function \( F^{(4)} \) encodes the \( N = 2 \) supersymmetric effective low-energy field theory corresponding to a type-II string compactification on a Calabi-Yau manifold. In that case one can show that the \( F^{(g)} \) represent \( g \)-loop contributions in string perturbation theory. The coefficient functions can be determined in string theory from certain type-II string amplitudes \([2]\) and indeed arise in the appropriate orders in string perturbation theory. An interesting feature is that the \( F^{(g)} \) can be identified with the topological partition function of a twisted nonlinear sigma model on a Calabi-Yau target space, defined on a two-dimensional base space equal to a genus-\( g \) Riemann surface. The partition function is obtained by integrating appropriately over all these Riemann surfaces \([4]\).

However, the partition functions \( F^{(g)} \) do not depend holomorphically on the Calabi-Yau moduli. They exhibit a holomorphic anomaly due to the propagation of massless states, or equivalently, due to certain contributions from the boundary of the

\[^{15}\text{In terms of the holomorphic sections the Kähler potential takes the form} \]

\[ K(z, \bar{z}) = -\log \left( i \bar{X}^I(z) F_I(X(z)) - i X^I(z) \bar{F}_I(\bar{X}(\bar{z})) \right). \]

The Kähler metric is defined as \( g_{AB} = \partial_A \partial_B K(z, \bar{z}) \). Under projective transformations of the holomorphic sections, \( X^I \to \exp(f(z)) X^I \), the Kähler potential transforms by a Kähler transformation, so that the metric remains invariant.

\[^{19}\text{In } N = 2 \text{ compactifications of the heterotic string the counting of string loops runs differently.}\]
the moduli space $M_g$ associated with the genus-$g$ Riemann surfaces.

The holomorphic anomaly of \[76\] receives contributions from two terms. One is precisely as in \[53\] and arises from pinchings that separate the Riemann surface into two disconnected surfaces. A second term, absent in \[53\], corresponds to Riemann surfaces where a closed loop is pinched such that the genus is lowered by one unit. We should stress that \[53\] was obtained in a very general context and applies to both rigid and local $N = 2$ supersymmetry. The conclusion is that part of the holomorphic anomaly can thus be viewed as arising from a conflict between the requirements of holomorphy and of a proper (covariant) behaviour under symplectic transformations. The nonholomorphic modifications exhibited above can be regarded as (part of) the threshold corrections that arise due to the propagation of massless states \[77\].

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