A brief review of abelian categorifications

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Abstract

This article contains a review of categorifications of semisimple representations of various rings via abelian categories and exact endo-
functors on them. A simple definition of an abelian categorification is
presented and illustrated with several examples, including categorifica-
tions of various representations of the symmetric group and its Hecke
algebra via highest weight categories of modules over the Lie alge-
bra $\mathfrak{sl}_n$. The review is intended to give non-experts in representation
theory who are familiar with the topological aspects of categorifica-
tion (lifting quantum link invariants to homology theories) an idea for
the sort of categories that appear when link homology is extended to
tangles.

1 A simple framework for categorification

Categorification. The Grothendieck group $K(B)$ of an abelian category $B$
has as generators the symbols $[M]$, where $M$ runs over all the objects of $B$, and
defining relations $[M_2] = [M_1] + [M_3]$, whenever there is a short exact
sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$ 

An exact functor $F$ between abelian categories induces a homomorphism $[F]$ between their Grothendieck groups.

Let $A$ be a ring which is free as an abelian group, and $\mathbf{a} = \{a_i\}_{i \in I}$ a basis
of $A$ such that the multiplication has nonnegative integer coefficients in this
basis:

$$a_ia_j = \sum_k c_{ij}^k a_k, \quad c_{ij}^k \in \mathbb{Z}_{\geq 0}. \quad (1)$$
Let $B$ be a (left) $A$-module.

**Definition 1** A (weak) abelian categorification of $(A, a, B)$ consists of an abelian category $\mathcal{B}$, an isomorphism $\varphi : K(\mathcal{B}) \cong B$ and exact endofunctors $F_i : \mathcal{B} \to \mathcal{B}$ such that the following holds:

(C-I) The functor $F_i$ lifts the action of $a_i$ on the module $B$, i.e. the action of $[F_i]$ on the Grothendieck group of $\mathcal{B}$ descends to the action of $a_i$ on the module $B$ so that the diagram below is commutative.

\[
\begin{array}{ccc}
K(\mathcal{B}) & \xrightarrow{[F_i]} & K(\mathcal{B}) \\
\varphi \downarrow & & \downarrow \varphi \\
B & \xrightarrow{a_i} & B \\
\end{array}
\]

(C-II) There are isomorphisms

\[F_i F_j \cong \bigoplus_k F_k^{c_{ij}^k},\]

i.e., the composition $F_i F_j$ decomposes as the direct sum of functors $F_k$ with multiplicities $c_{ij}^k$ as in (1).

If there is a categorification as above we say the action of the functors $F_i$ on the category $\mathcal{B}$ categorifies the $A$-module $B$.

In all our examples, the objects of $\mathcal{B}$ will have finite length (finite Jordan-Hölder series). Consequently, if $\{L_j\}_{j \in J}$ is a collection of simple objects of $\mathcal{B}$, one for each isomorphism class, the Grothendieck group $K(\mathcal{B})$ is free abelian with basis elements $[L_j]$. The image of any object $M \in \mathcal{B}$ in the Grothendieck group is

\[[M] = \sum_j m_j(M) [L_j]\]

where $m_j(M)$ is the multiplicity of $L_j$ in some (and hence in any) composition series of $M$.

The free group $K(\mathcal{B})$ has therefore a distinguished basis $[L_j]_{j \in J}$, and the action of $[F_i]$ in this basis has integer non-negative coefficients:

\[[F_i(L_j)] = \sum d_{ij}^k [L_k],\]
with \( d_{ij}^k \) being the multiplicity \( m_k(F_i(L_j)) \). Via the isomorphism \( \varphi \) we obtain a distinguished basis \( b = \{ b_j \}_{j \in J} \) of \( B \), and
\[
a_i b_j = \sum d_{ij}^k b_k. \tag{2}
\]

Conversely, we could fix a basis \( b \) of \( B \) with a positivity constraint for the action of \( A \). As in (2). Then our definition of a categorification of \((A, a, B)\) can be amended to a similar definition of a categorification of \((A, a, B, b)\), with the additional data being the fixed basis \( b \). Ideally the basis \( b \) corresponds then via the isomorphism \( \varphi \) to a basis \([M_j]_{j \in J}\) for certain objects \( M_j \in B \). Varying the choice of basis might give rise to an interesting combinatorial interplay between several, maybe less prominent than \([L_j]_{j \in J}\) but more interesting, families \( \{ M_j \}_{j \in J} \) of objects in \( B \). Typical examples of such an interplay can be found in [12], [25, Section 5].

Of course, any such data \((A, a, B, b)\) admits a rather trivial categorification, via a semisimple category \( B \). Namely, choose a field \( k \) and denote by \( k\text{-vect} \) the category of finite-dimensional \( k \)-vector spaces. Let
\[
B = \bigoplus_{j \in J} k\text{-vect}
\]
be the direct sum of categories \( k\text{-vect} \), one for each basis vector of \( B \). The category \( B \) is semisimple, with simple objects \( L_j \) enumerated by elements of \( J \), and
\[
\text{Hom}_B(L_j, L_k) = \begin{cases} k & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}
\]
We identify \( K(B) \) with \( B \) by mapping \([L_j]\) to \( b_j \). The functors \( F_i \) are determined by their action on simple objects, hence, given (2), we can define
\[
F_i(L_j) = \bigoplus_{k \in J} L^d_{kj}^{ij} \]
and obtain a categorification of \((A, a, B, b)\). With few exceptions, semisimple categorifications bring little or no new structure into play, and we will ignore them. More interesting instances of categorifications occur for non-semisimple categories \( B \). Here is a sample list.

1. Let \( \mathcal{A}_1 \) be the first Weyl algebra (the algebra of polynomial differential operators in one variable) with integer coefficients,
\[
\mathcal{A}_1 = \mathbb{Z}(x, \partial)/(\partial x - x \partial - 1).
\]
We fix the basis \( \{ x^i \partial^j \}_{i,j \geq 0} \) of \( A_1 \). The \( \mathbb{Z} \)-lattice \( B \subset \mathbb{Q}[x] \) with the basis \( b = \{ x^n \}_{n \geq 0} \) is an \( A_1 \)-module.

To categorify this data we consider the category \( B = \bigoplus_{n \geq 0} R_n-\text{mod} \), i.e. the direct sum of the categories of finite-dimensional modules over the nilCoxeter \( \mathbb{k} \)-algebra \( R_n \). The latter has generators \( Y_1, \ldots, Y_{n-1} \) subject to relations

\[
Y_i^2 = 0, \\
Y_i Y_j = Y_j Y_i \text{ if } |i - j| > 1, \\
Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1}.
\]

The algebra \( R_n \) has a unique, up to isomorphism, finite dimensional simple module \( L_n \), and \( K(R_n-\text{mod}) \cong \mathbb{Z} \). The Grothendieck group \( K(B) \) is naturally isomorphic to the \( A_1 \)-module \( B \), via the isomorphism \( \phi \) which maps \( [L_n] \) to \( x^n \).

The endofunctors \( X, D \) in \( B \) that lift the action of \( x \) and \( \partial \) on \( B \) are the induction and restriction functors for the inclusion of algebras \( R_n \subset R_{n+1} \). One takes \( \{ x^i \partial^j \}_{i,j \geq 0} \) as the basis \( a \) of \( A_1 \). Basis elements lift to functors \( X^i D^j \), and the isomorphisms (C-II) of definition 1 are induced by an isomorphism of functors \( DX \cong XD \oplus \text{Id} \) which lifts the defining relation \( \partial x = x \partial + 1 \) in \( A_1 \). A detailed analysis of this categorification can be found in [39].

2. The regular representation of the group ring \( \mathbb{Z}[S_n] \) of the symmetric group \( S_n \) has a categorification via projective functors acting on a regular block of the highest weight BGG category \( \mathcal{O} \) from [11] for \( \mathfrak{sl}_n \). (For an introduction to the representation theory of semisimple Lie algebras we refer to [30]).

To define category \( \mathcal{O} \), start with the standard triangular decomposition \( \mathfrak{sl}_n = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \), where the first and the last terms are the Lie algebras of strictly upper-triangular (resp. lower-triangular) matrices, while \( \mathfrak{h} \) is the algebra of traceless diagonal matrices. The highest weight category \( \mathcal{O} \) of \( \mathfrak{sl}_n \) is the full subcategory of the category of finitely-generated \( \mathfrak{sl}_n \)-modules consisting of \( \mathfrak{h} \)-diagonalisable (possibly infinite dimensional) modules on which \( U(\mathfrak{n}_+) \) acts locally-nilpotently. Thus, any \( M \in \mathcal{O} \) decomposes as

\[
M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,
\]

where \( h x = \lambda(h)x \) for any \( h \in \mathfrak{h} \) and \( x \in M_\lambda \). Here \( \mathfrak{h}^* \) is the dual vector space of \( \mathfrak{h} \), its elements are called weights.
The one-dimensional modules $C_\lambda = C_{v_\lambda}$ over the positive Borel subalgebra $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ are classified by elements $\lambda$ of $\mathfrak{h}^*$. The subalgebra $\mathfrak{n}_+$ acts trivially on $v_\lambda$, while $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}$.

The Verma module $M(\lambda)$ is the $\mathfrak{sl}_n$-module induced from the $\mathfrak{b}_+$-module $C_\lambda$:

$$M(\lambda) = U(\mathfrak{sl}_n) \otimes_{U(\mathfrak{b}_+)} C_\lambda.$$ 

The Verma module $M(\lambda)$ has a unique simple quotient, denoted $L(\lambda)$, and any simple object of $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. We call a weight $\lambda$ positive integral if $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ for any positive simple root $\alpha \in \mathfrak{h}^*$. The representation $L(\lambda)$ is finite-dimensional if and only if $\lambda$ is a positive integral weight.

Although most of the objects in $\mathcal{O}$ are infinite dimensional vector spaces, every object $M$ of $\mathcal{O}$ has finite length, i.e. there is an increasing filtration by subobjects $0 = M^0 \subset M^1 \subset \cdots \subset M^m = M$ such that the subsequent quotients $M^{i+1}/M^i$ are isomorphic to simple objects, hence have the form $L(\lambda)$ (where $\lambda$ may vary). The Grothendieck group of $\mathcal{O}$ is thus a free abelian group with generators $[L(\lambda)]$ for $\lambda \in \mathfrak{h}^*$.

It turns out that $\mathcal{O}$ has enough projective objects: given $M$ there exists a surjection $P \twoheadrightarrow M$ with a projective $P \in \mathcal{O}$. Moreover, isomorphism classes of indecomposable projective objects are enumerated by elements of $\mathfrak{h}^*$. The indecomposable projective object $P(\lambda)$ is determined by the property of being projective and

$$\text{Hom}_{\mathcal{O}}(P(\lambda), L(\mu)) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We should warn the reader that the $P(\lambda)$’s are not projective when viewed as objects of the category of all $\mathfrak{sl}_n$-modules, while the $L(\lambda)$’s remain simple in the latter category.

The symmetric group $S_n$, the Weyl group of $\mathfrak{sl}_n$, acts naturally on $\mathfrak{h}$ by permuting the diagonal entries and then also on $\mathfrak{h}^*$. Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots. In the study of the category $\mathcal{O}$ an important role is played by the shifted (dot) action of $S_n$,

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$ 

Two simple modules $L(\lambda), L(\lambda')$ have the same central character (i.e. are annihilated by the same maximal ideal of the centre of the universal enveloping
algebra) if and only if $\lambda$ and $\lambda'$ belong to the same $S_n$-orbit under the shifted action. Consequently, $O$ decomposes into a direct sum of categories

$$O = \bigoplus_{\nu \in h^*/S_n} O_{\nu}$$

indexed by orbits $\nu$ of the shifted action of $S_n$ on $h^*$. Here, $O_{\nu}$ consists of all modules with composition series having only simple subquotients isomorphic to $L(\lambda)$ for $\lambda \in \nu$. There is no interaction between $O_{\nu}$ and $O_{\nu'}$ for different orbits $\nu$, $\nu'$. More accurately, if $\nu \neq \nu'$ then $\text{Ext}^i_O(M, M') = 0$ for any $i \geq 0$, $M \in O_{\nu}$ and $M' \in O_{\nu'}$.

Furthermore, each $O_{\nu}$ is equivalent to the category of finite-dimensional modules over some finite-dimensional $\mathbb{C}$-algebra $A_\nu$. Here’s the catch, though: explicitly describing $A_\nu$ for $n > 3$ and interesting $\nu$ is very hard, see [67]. For an implicit description we just form $P = \bigoplus_{\lambda \in \nu} P(\lambda)$, the direct sum of all indecomposable projectives over $\lambda \in \nu$. Then $A_\nu \cong \text{Hom}_O(P, P)^{op}$.

An orbit $\nu$ (for the shifted action) is called generic if $w \cdot \lambda - \lambda$ is never integral, for $\lambda \in \nu$ and $w \in S_n, w \neq 1$. For a generic orbit $\nu$, the category $O_{\nu}$ is boring and equivalent to the direct sum of $n!$ copies of the category of finite-dimensional $\mathbb{C}$-vector spaces, one for each $\lambda \in \nu$. For such $\lambda$ we have $P(\lambda) = M(\lambda) = L(\lambda)$, i.e. the Verma module with the highest weight $\lambda$ is simple as well as projective in $O$.

We call an orbit integral if it is a subset of the integral weight lattice in $h^*$. In [64] it is shown that $O_{\nu}$ for non-integral $\nu$ reduces to those for integral weights. From now on we therefore assume that $\nu$ is integral. Then the category $O_{\nu}$ is indecomposable (unlike in the generic case). Moreover, the complexity of $O_{\nu}$ only depends on the type of the orbit. If two orbits $\nu$ and $\nu'$ contain points $\lambda \in \nu$, $\lambda' \in \nu'$ with identical stabilisers, then the categories $O_{\nu}$ and $O_{\nu'}$ are equivalent, see [10], [64]. If the stabiliser of $\nu$ under the shifted action is trivial, the category $O_{\nu}$ is called a regular block. Regular blocks are the most complicated indecomposable direct summands of $O$, for instance in the sense of having the maximal number of isomorphism classes of simple modules.

There is a natural bijection between the following three sets: positive integral weights, isomorphism classes of irreducible finite-dimensional representations of $\mathfrak{sl}_n$, and regular blocks of $O$ for $\mathfrak{sl}_n$. A positive integral weight $\lambda$ is the highest weight of an irreducible finite-dimensional representation $L(\lambda)$, determined by the weight uniquely up to isomorphism. In turn, $L(\lambda)$ belongs to the regular block $O_{\nu}$, where $\nu = S_n \cdot \lambda$ is the orbit of $\lambda$. 

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Any two regular blocks of $\mathcal{O}$ are equivalent as categories, as shown in [34]. For this reason, we can restrict our discussion to the uniquely defined regular block which contains the one-dimensional trivial representation $L(0)$ of $\mathfrak{sl}_n$. We denote this block by $\mathcal{O}_0$. It has $n!$ simple modules $L(w) = L(w \cdot 0)$, enumerated by all permutations $w \in S_n$ (with the identity element $e$ of $S_n$ corresponding to $L(0)$ which is the only finite dimensional simple module in $\mathcal{O}_0$). Thus, $K(\mathcal{O}_0)$ is free abelian of rank $n!$ with basis $\{[L(w)]\}_{w \in S_n}$. Other notable objects in $\mathcal{O}_0$ are the Verma modules $M(w) = M(w \cdot 0)$ and the indecomposable projective modules $P(w) = P(w \cdot 0)$, over all $w \in S_n$. The sets $\{[M(w)]\}_{w \in S_n}$ and $\{[P(w)]\}_{w \in S_n}$ form two other prominent bases in $K(\mathcal{O}_0)$. For the set $\{[M(w)]\}_{w \in S_n}$ this is easy to see, because the transformation matrix between Verma modules and simple modules is upper triangular with ones on the diagonal. For the set $\{[P(w)]\}_{w \in S_n}$ this claim is not obvious and relates to the fact that $\mathcal{O}_0$ has finite homological dimension, see [11].

Equivalences between regular blocks are established by means of translation functors. First note that we can tensor two $U(\mathfrak{sl}_n)$-modules over the ground field. If $V$ is a finite-dimensional $\mathfrak{sl}_n$-module it follows from the definitions that $V \otimes M$ lies in $\mathcal{O}$ whenever $M$ is in $\mathcal{O}$. Hence, tensoring with $V$ defines an endofunctor $V \otimes -$ of the category $\mathcal{O}$. Taking direct summands of the functors $V \otimes -$ provides a bewildering collection of different functors and allows one to analyse $\mathcal{O}$ quite deeply. By definition, a projective functor is any endofunctor of $\mathcal{O}$ isomorphic to a direct summand of $V \otimes -$ for some finite-dimensional $\mathfrak{sl}_n$-module $V$. Projective functors were classified by J. Bernstein and S. Gelfand [10]. Translation functors are special cases of projective functors.

Let us restrict our discussion to projective endofunctors in the regular block $\mathcal{O}_0$. Each projective endofunctor $\mathcal{O}_0 \rightarrow \mathcal{O}_0$ decomposes into a finite direct sum of indecomposable functors $\theta_w$, enumerated by permutations $w$ and determined by the property $\theta_w(M(e)) \cong P(w)$. We have $P(e) = M(e)$ and the functor $\theta_e$ is the identity functor. The composition or the direct sum of two projective functors are again projective functors. With respect to these two operations, projective endofunctors on $\mathcal{O}_0$ are (up to isomorphism) generated by the projective functors $\theta_i := \theta_{s_i}$ corresponding to the simple transpositions/reflections $s_i = (i, i + 1)$. The functor $\theta_i$ is called the translation through the $i$-th wall. The functor $\theta_w$ is a direct summand of $\theta_{i_k} \ldots \theta_{i_1}$, for any reduced decomposition $w = s_{i_1} \ldots s_{i_k}$. The induced endomorphism $[\theta_i]$ of the Grothendieck group acts (in the basis given by Verma modules)
by
\[ [\theta_i(M(w))] = [M(w)] + [M(ws_i)]. \]

Now we are prepared to explain the categorification. We first fix the unique isomorphism \( \varphi \) of groups

\[
\varphi: \quad K(\mathcal{O}_0) \longrightarrow \mathbb{Z}[S_n] \\
[M(w)] \longmapsto w
\]

and define \( C_w := \varphi([P(w)]) \). Then the \( C_w, w \in W \), form a basis \( a \) of \( \mathbb{Z}[S_n] \).

The action of \( [\theta_i] \) corresponds under \( \varphi \) to the endomorphism of \( \mathbb{Z}[S_n] \) given by right multiplication with \( C_{si} := 1 + si \).

The defining relations of the generators \( 1 + si \) in \( \mathbb{Z}[S_n] \) lift to isomorphisms of functors as follows

\[
\theta_i^2 \cong \theta_i \oplus \theta_i, \\
\theta_i \theta_j \cong \theta_j \theta_i \text{ if } |i - j| > 1, \\
\theta_i \theta_{i+1} \theta_i \oplus \theta_{i+1} \cong \theta_{i+1} \theta_i \theta_{i+1} \oplus \theta_i.
\]

Here, the last isomorphism follows from the existence of decompositions of functors

\[
\theta_i \theta_{i+1} \theta_i \cong \theta_{w_1} \oplus \theta_i, \\
\theta_{i+1} \theta_i \theta_{i+1} \cong \theta_{w_1} \oplus \theta_{i+1},
\]

where \( w_1 = si_{i+1}si = si_{i+1}si_{i+1} \). In particular, \( [\theta_{w_1}] \) corresponds under \( \varphi \) to the right multiplication with \( 1 + si + si_{i+1} + si_{i+1} + si_{i+1}si + si_{i+1}si \).

By the classification theorem of projective functors, the endomorphism \([\theta_w], w \in W\), corresponds then to right multiplication with the element \( C_w \).

From this one can then actually deduce that the multiplication in the basis \( a \) has non-negative integral coefficients

\[
C_wC_{w'} = \sum_{w''} c_{w,w'}^{w''}C_{w''}, \quad c_{w,w'}^{w''} \in \mathbb{Z}_{\geq 0}. \quad (4)
\]

Hence we are in the situation of (1) and are looking for an abelian categorification of \((\mathbb{Z}[S_n], a, \mathbb{Z}[S_n])\). We already have the isomorphism \( \varphi \) and the exact endofunctor \( \theta_w \) corresponding to the generator \( C_w \) satisfying condition (C-I).
The composition of two projective functors decomposed as a direct sum of indecomposable functors $[\theta_w], w \in W$, has nonnegative integral coefficients, and the equations (4) turn into isomorphisms of functors

$$\theta_w \theta'_w \cong \bigoplus_{w''} (\theta_{w''})^{c_{ww'}}, \quad c_{ww'} \in \mathbb{Z}_{\geq 0}, \quad (5)$$

It turns out that each $[\theta_w]$ acts by a multiplication with a linear combination of $y$’s for $y \leq w$. Moreover, all coefficients are nonnegative integers. For instance, if $w \in S_4$ then $[\theta_w] = \sum_{y \leq w} y$, with two exceptions:

$$[\theta_w] = \sum_{y \leq w} y + 1 + s_2, \quad w = s_2s_1s_3s_2, \quad w = s_1s_3s_2s_1s_3.$$  

We can summarise the above results into a theorem.

**Theorem 2** The action of the indecomposable projective functors $\theta_w, w \in S_n$, on the block $O_0$ for $\mathfrak{sl}_n$ categorifies the right regular representation of the integral group ring of the symmetric group $S_n$ (in the basis $a$ of the elements $C_w, w \in S_n$).

This theorem is due to Bernstein and Gelfand, see [10], where it was stated in different terms, since the word “categorification” was not in the mathematician’s vocabulary back then. In fact, Bernstein and Gelfand obtained a more general result by considering any simple Lie algebra $\mathfrak{g}$ instead of $\mathfrak{sl}_n$ and its Weyl group $W$ in place of $S_n$.

In the explanation to the theorem we did not give a very explicit description of the basis $a$ due to the fact that there is no explicit (closed) formula for the elements $C_w$ available. However, the elements $C_w$ can be obtained by induction (on the length of $w$) using the Kazhdan-Lusztig theory [35], [36]. The Kazhdan-Lusztig theory explains precisely the complicated interplay between the basis $a$ and the standard basis of $\mathbb{Z}[S_n]$.

**3.** Parabolic blocks of $O$ categorify representations of the symmetric group $S_n$ induced from the sign representation of parabolic subgroups.

Let $\mu = (\mu_1, \ldots, \mu_k), \mu_1 + \cdots + \mu_k = n$, be a composition of $n$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ the corresponding partition. In other words, $\lambda$ is a permutation of the sequence $\mu$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Denote by $p_\mu$ the
subalgebra of \( \mathfrak{sl}_n \) consisting of \( \mu \)-block upper-triangular matrices. Consider the full subcategory \( \mathcal{O}^\mu \) of \( \mathcal{O} \) which consists of all modules \( M \) on which the action of \( U(p_\mu) \) is locally finite. The category \( \mathcal{O}^\mu \) is an example of a parabolic subcategory of \( \mathcal{O} \), introduced in \[58\]. A simple object \( L(\lambda) \) of \( \mathcal{O} \) belongs to \( \mathcal{O}^\mu \) if and only if the weight \( \lambda \) is positive integral with respect to all roots of the Lie algebra \( p_\mu \). The two extreme cases are \( \mu = (1, 1, \ldots, 1) \), in which case \( \mathcal{O}^\mu \) is all of \( \mathcal{O} \), and \( \mu = (n) \), for \( \mathcal{O}^{(n)} \) is the semisimple category consisting exactly of all finite-dimensional \( \mathfrak{sl}_n \)-modules.

The direct sum decomposition (3) induces a similar decomposition of the parabolic category:

\[
\mathcal{O}^\mu \cong \bigoplus_{\nu \in h^*/S_n} \mathcal{O}_\nu^\mu.
\]

Each category \( \mathcal{O}_\nu^\mu \) is either trivial (i.e. contains only the zero module) or equivalent to the category of finite-dimensional modules over some finite-dimensional \( \mathbb{C} \)-algebra (but describing this algebra explicitly for interesting \( \mu \) and \( \nu \) is a hard problem, see \[16\]). Unless \( \mu = (1^n) \), for generic \( \nu \) the summand \( \mathcal{O}_\nu^\mu \) is trivial. Again, the most complicated summands are the \( \mathcal{O}_\nu^\mu \) where the orbit \( \nu \) contains a dominant regular integral weight. Translation functors establish equivalences between such summands for various such \( \nu \), and allow us to restrict our consideration to the block \( \mathcal{O}_0^\mu \) corresponding to the (shifted) orbit through 0. The inclusion

\[
\mathcal{O}_0^\mu \subset \mathcal{O}_0
\]

is an exact functor and induces an inclusion of Grothendieck groups

\[
K(\mathcal{O}_0^\mu) \subset K(\mathcal{O}_0).
\]

Indeed, the Grothendieck group of \( \mathcal{O}_0 \) is free abelian with generators \([L(w)]\), \( w \in S_n \). A simple module \( L(w) \) lies in \( \mathcal{O}_0^\mu \) if and only if \( w \) is a minimal left coset representative for the subgroup \( S_\mu \) of \( S_n \) (we informally write \( w \in (S_\mu \backslash S_n)_{\text{short}} \)). The Grothendieck group of \( \mathcal{O}_0^\mu \) is then the subgroup of \( K(\mathcal{O}_0) \) generated by such \( L(w) \).

The analogues of the Verma modules in the parabolic case are the so-called parabolic Verma modules

\[
M(p_\mu, V) = U(\mathfrak{sl}_n) \otimes_{U(p_\mu)} V,
\]

where \( V \) is a finite-dimensional simple \( p_\mu \)-module. The module \( M(p_\mu, V) \) is a homomorphic image of some ordinary Verma module from \( \mathcal{O} \), in particular,
it has a unique simple quotient isomorphic to some $L(w)$ for some unique $w \in S_n$. In this way we get a canonical bijection between parabolic Verma modules in $\mathcal{O}_0^\mu$ and the set $(S_\mu \setminus S_n)_{\text{short}}$ of shortest coset representatives. Hence it is convenient to denote the parabolic Verma module with simple quotient $L(w)$, $w \in (S_\mu \setminus S_n)_{\text{short}}$, simply by $M^\mu(w)$.

Generalised Verma modules provide a basis for the Grothendieck group of $\mathcal{O}_0^\mu$. Under the inclusion (6) of Grothendieck groups the image of the generalised Verma module $M^\mu(w)$ is the alternating sum of Verma modules, see [58] and [48]:

$$[M^\mu(w)] = \sum_{u \in S_\mu} (-1)^{l(u)} [M(uw)].$$

Since the projective endofunctors $\theta_w$ preserve $\mathcal{O}_0^\mu$, the inclusion (6) is actually an inclusion of $S_n$-modules, and, in view of the formula (7), we can identify $K(\mathcal{O}_0^\mu)$ with the submodule $I^-_\mu$ of the regular representation of $S_n$ isomorphic to the representation induced from the sign representation of $S_\mu$,

$$I^-_\mu \cong \text{Ind}_{S_\mu}^{S_n} \mathbb{Z}v,$$

where we denoted by $\mathbb{Z}v$ the sign representation, so that $wv = (-1)^{l(w)}v$ for $w \in S_\mu$.

To summarise, we have:

**Theorem 3** The action of the projective functors $\theta_w$, $w \in W$, on the parabolic subcategory $\mathcal{O}_0^\mu$ of $\mathcal{O}$ categorifies the induced representation $I^-_\mu$ of the integral group ring of the symmetric group $S_n$ (with basis $a = \{C_w\}_{w \in S_n}$).

As in the previous example the Grothendieck group $K(\mathcal{O}_0^\mu)$ has three distinguished basis, given by simple objects, projective objects, and parabolic Verma modules respectively.

**Remark:** If we choose a pair $\mu, \mu'$ of decompositions giving rise to the same partition $\lambda$ of $n$, then the modules $I^-_\mu$ and $I^-_{\mu'}$ are isomorphic, and will be denoted $I^-_\lambda$. However, the categories $\mathcal{O}^\mu$ and $\mathcal{O}^{\mu'}$ are not equivalent in general, which means the two categorifications of the induced representation $I^-_\lambda$ are also not equivalent. This problem disappears if we leave the world of abelian categorifications, since the derived categories $D^b(\mathcal{O}^\mu)$ and $D^b(\mathcal{O}^{\mu'})$ are equivalent [41]. The equivalence is based on the geometric description of $\mathcal{O}^\mu$ and $\mathcal{O}^{\mu'}$ in terms of complexes of sheaves on partial flag varieties.
4. Self-dual projectives in a parabolic block categorify irreducible representations of the symmetric group.

Let $I_\mu$ be the representation of $\mathbb{Z}[S_n]$ induced from the trivial representation of the subgroup $S_\mu$. Up to isomorphism, it only depends on the partition $\lambda$ associated with $\mu$. Partitions of $n$ naturally index the isomorphism classes of irreducible representations of $S_n$ over any field of characteristic zero (we use $\mathbb{Q}$ here). Denote by $S_\mathbb{Q}(\lambda)$ the irreducible (Specht) module associated with $\lambda$. It is an irreducible representation defined as the unique common irreducible summand of $I_\lambda \otimes \mathbb{Q}$ and $I_{\lambda^*} \otimes \mathbb{Q}$, where $\lambda^*$ is the dual partition of $\lambda$. Passing to duals, we see that $S_\mathbb{Q}(\lambda^*)$ is the unique common irreducible summand of $I_{\lambda^*} \otimes \mathbb{Q}$ and $I_{\lambda^*} \otimes \mathbb{Q}$.

We have already categorified the representation $I_\lambda^-$ (in several ways) via the parabolic categories $O^\mu_0$, where $\mu$ is any decomposition for $\lambda$. It’s natural to try to realise a categorification of some integral lift $S(\lambda^*)$ of the irreducible representation $S_\mathbb{Q}(\lambda^*)$ via a suitable subcategory of some $O^\mu_0$ stable under the action of projective endofunctors.

The correct answer, presented in [43], is to pass to a subcategory generated by those projective objects in $O^\mu_0$ which are also injective. Note that these modules are neither projective nor injective in $O_0$ (unless if $O^\mu_0 = O_0$).

Any projective object in $O^\mu_0$ is isomorphic to a direct sum of indecomposable projective modules $P^\mu(w)$, for $w \in (S_\mu \setminus S_n)_{\text{short}}$. Let $J \subset (S_\mu \setminus S_n)_{\text{short}}$ be the subset indexing indecomposable projectives modules that are also injective: $w \in J$ if and only if $P^\mu(w)$ is injective. Projective endofunctors $\theta_w$, $w \in S_n$, take projectives to projectives and injectives to injectives. Therefore, they take projective-injective modules (modules that are both projective and injective, also called self-dual projective, for instance, in Irving [31]) to projective-injective modules.

The category of projective-injective modules is additive, not abelian. To remedy this, consider the full subcategory $C^\mu$ of $O^\mu_0$ consisting of modules $M$ admitting a resolution

$$P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

with projective-injective $P_1$ and $P_0$. The category $C^\mu$ is abelian and stable under all endofunctors $\theta_w$ for $w \in S_n$, see [43].

Irving [31] classified projective-injective modules in $O^\mu_0$. His results were interpreted in [43] in the language of categorification:
Theorem 4 The action of the projective endofunctors \( \theta_w, \) \( w \in S_n, \) on the abelian category \( \mathcal{C}^\mu \) categorifies (after tensoring the Grothendieck group with \( \mathbb{Q} \) over \( \mathbb{Z} \)) the irreducible representation \( S_q(\lambda^*) \) of the symmetric group \( S_n. \)

The Grothendieck group \( K(\mathcal{C}^\mu) \) is a module over the integral group ring of \( S_n, \) with \( s_i \) acting by \([\theta_i] - \text{Id}, \) and the theorem says that \( K(\mathcal{C}^\mu) \otimes_{\mathbb{Z}} \mathbb{Q} \) is an irreducible representation of the symmetric group corresponding to the partition \( \lambda^*. \) Several explicit examples of categorifications via \( \mathcal{C}^\mu \) will be given in Section 2.

Remark: Suppose \( \mu \) and \( \nu \) are two decompositions of the same partition \( \lambda. \) It’s shown in [53] (Theorem 5.4.(2)) that the categories \( \mathcal{C}^\mu \) and \( \mathcal{C}^\nu \) are equivalent, through an equivalence which commutes with the action of the projective functors \( \theta_u \) on these categories (the equivalence is given by a non-trivial composition of derived Zuckerman functors). Therefore, the categorification of \( S(\lambda) \) does not depend on the choice of the decomposition \( \mu \) that represents \( \lambda, \) and we can denote the category \( \mathcal{C}^\mu \) by \( \mathcal{C}^\lambda. \) (This should be compared with the remark after Theorem 3.)

Remark: Theorem 3 and Theorem 4 can be generalised to arbitrary semi-simple complex finite-dimensional Lie algebras, see [54]. However, in the general case Theorem 4 does not categorify simple modules for the corresponding Weyl group but rather the Kazhdan-Lusztig cell modules from [35]. This can be used to describe the so-called “rough” structure of generalised Verma modules, which shows that “categorification theoretic” ideas can lead to new results in representation theory.

Remark: The inclusion of categories \( \mathcal{C}^\mu \subset \mathcal{O}_0^\mu \) is not an exact functor; however, it is a part of a very natural filtration of the category \( \mathcal{O}_0^\mu \) which can be defined using the Gelfand-Kirillov dimension of modules, see [54, 6.9]. To get the inclusion of Grothendieck groups analogous to the inclusion of representations from the irreducible Specht module into the induced sign representation, we pass to the subgroup \( K'(\mathcal{C}^\mu) \) of \( K(\mathcal{C}^\mu) \) generated by the images of projective modules in \( \mathcal{C}^\mu. \) This additional technicality is necessary as the category \( \mathcal{C}^\mu \) does not have finite homological dimension in general. The subgroup \( K'(\mathcal{C}^\mu) \) is always a finite index subgroup, stable under the action of the \( \theta_w \)'s. We denote this subgroup by \( S'(\lambda^*) \): \[ S'(\lambda^*) \overset{\text{def}}{=} K'(\mathcal{C}^\mu) \subset K(\mathcal{C}^\mu) \cong S(\lambda^*). \]
The inclusion of categories $\mathcal{C}^\mu \subset \mathcal{O}_0^\mu$ induces the inclusion
\[ \mathcal{S}'(\lambda^*) \subset K(\mathcal{O}_0^\mu) \cong I^-_\mu \]
of $\mathbb{Z}[S_n]$-modules, hence realising the integral lift $\mathcal{S}'(\lambda^*)$ of the Specht module as a subrepresentation of $I^-_\mu$.

5. Categorification of the induced representations $I_\mu$ via projectively presentable modules.

Let $\mathcal{P}_\mu$ denote the category of all modules $M$ admitting a resolution (8) in which each indecomposable direct summand of both $P_0$ and $P_1$ has the form $P(w)$, where $w$ is a longest left coset representative for $S_\mu$ in $S_n$ (we will write $w \in (S_\mu \setminus S_n)_{long}$). Such modules are called $p_\mu$-presentable modules, see [51]. As in the previous example, the category $\mathcal{P}_\mu$ is stable under all endofunctors $\theta_w$, $w \in S_n$.

By definition, $\mathcal{P}_\mu$ is a subcategory of $\mathcal{O}_0$, but just as in the example above, the natural inclusion functor is not exact. The category $\mathcal{P}_\mu$ does not have finite homological dimension in general, so we again pass to the subgroup $K'(\mathcal{P}_\mu)$ of $K(\mathcal{P}_\mu)$, generated by the images of indecomposable projective modules in $\mathcal{P}_\mu$. The latter are (up to isomorphism) the $P(u)$, $u \in (S_\mu \setminus S_n)_{long}$. This is a finite index subgroup, stable under the action of the $[\theta_w]$'s and we have the following statement proved in [51]:

**Theorem 5** The action of projective endofunctors on the abelian category $\mathcal{P}_\mu$ categorifies (after tensoring with $\mathbb{Q}$ over $\mathbb{Z}$) the induced representation $(I_\mu)_\mathbb{Q}$ of the group algebra of the symmetric group $S_n$ (with the basis $a = \{C_w\}_{w \in S_n}$).

Consider the diagram of $\mathbb{Q}[S_n]$-modules
\[ (I_\lambda)_\mathbb{Q} \xrightarrow{\iota_1} \mathbb{Q}[S_n] \xrightarrow{p_1} (I_{\lambda^*})_\mathbb{Q}. \]
The map $\iota_1$ is the symmetrisation inclusion map, while $p_1$ is the antisymmetrization quotient map. We have
\[ \mathcal{S}_\mathbb{Q}(\lambda) \overset{\text{def}}{=} p_1\iota_1((I_\lambda)_\mathbb{Q}). \]
The map $\iota_1$ is categorified as the inclusion of $\mathcal{P}_\mu$ to $\mathcal{O}_0$. The map $p_1$ is categorified as the projection of $\mathcal{O}_0$ onto $\mathcal{O}_0^{\mu^*}$, where $\mu^*$ is some decomposition.
corresponding to $\lambda^*$. Unfortunately, the composition of the two functors categorifying these two maps will be trivial in general. To repair the situation we first project $\mathcal{P}^\mu$ onto the full subcategory of $\mathcal{P}^\mu$ given by simple objects of minimal possible Gelfand-Kirillov dimension. It is easy to see that the image category contains enough projective modules, and using the equivalence constructed in [54, Theorem I], these projective modules can be functorially mapped to projective modules in $\mathcal{C}^{\mu^*}$, where $\mu^*$ is a (good choice of) composition with associated partition $\lambda^*$. The latter category embeds into $\mathcal{O}_0^{\mu^*}$ as was explained in the previous example.

6. The representation theory of groups like $GL(n, \mathbb{C})$, considered as a real Lie group, naturally leads to the notion of Harish-Chandra bimodules. A Harish-Chandra bimodule over $\mathfrak{sl}_n$ is a finitely-generated module over the universal enveloping algebra $U(\mathfrak{sl}_n)$ which decomposes into a direct sum of finite-dimensional $U(\mathfrak{sl}_n)$-modules with respect to the diagonal copy $\{(X, -X) | X \in \mathfrak{sl}_n\}$ of $\mathfrak{sl}_n$. Let $\mathcal{HC}_{0,0}$ be the category of Harish-Chandra bimodules which are annihilated, on both sides, by some power of the maximal ideal $I_0$ of the centre $Z$ of $U(\mathfrak{sl}_n)$. Here $I_0$ is the annihilator of the trivial $U(\mathfrak{sl}_n)$-module considered as a $Z$-module. Thus, $M \in \mathcal{HC}_{0,0}$ if and only if $xM = 0 = Mx$ for all $x \in I_0^N$ for $N$ large enough.

By [10, Section 5] there exists an exact and fully faithful functor

$$\mathcal{O}_0 \longrightarrow \mathcal{HC}_{0,0}.$$ 

Moreover, this functor induces an isomorphism of Grothendieck groups

$$K(\mathcal{O}_0) \cong K(\mathcal{HC}_{0,0}).$$

Since the former group is isomorphic to $\mathbb{Z}[S_n]$, we can identify the Grothendieck group of $\mathcal{HC}_{0,0}$ with $\mathbb{Z}[S_n]$ as well.

The advantage of bimodules is that we now have two sides and can tensor with a finite-dimensional $\mathfrak{sl}_n$-module both on the left and on the right. In either case, we preserve the category of Harish-Chandra bimodules. Taking all possible direct summands of these functors and restricting to endofunctors on the subcategory $\mathcal{HC}_{0,0}$ leads to two sets of commuting projective functors, $\{\theta_{r,w}\}_{w \in S_n}$ and $\{\theta_{l,w}\}_{w \in S_n}$ which induce endomorphisms on the Grothendieck group $K(\mathcal{HC}_{0,0}) \cong \mathbb{Z}[S_n]$ given by left and right multiplication with $\{C_w\}_{w \in S_n}$ respectively. Summarising, we have
Theorem 6  The action of the functors \( \{ \theta_{r,w} \}_{w \in S_n} \) and \( \{ \theta_{l,w} \}_{w \in S_n} \) on the category \( \mathcal{HC}_{0,0} \) of Harish-Chandra bimodules for \( \mathfrak{sl}_n \) with generalised trivial character on both sides categorifies \( \mathbb{Z}[S_n] \), viewed as a bimodule over itself. The functors \( \{ \theta_{r,w} \}_{w \in S_n} \) induces the left multiplication with \( C_w \) on the Grothendieck group, whereas the functors \( \{ \theta_{l,w} \}_{w \in S_n} \) induces the right multiplication with \( C_w \) on the Grothendieck group.

The first half of the theorem follows at once from [10], the second half from [65]. The category of Harish-Chandra bimodules is more complicated than the category \( \mathcal{O} \). For example, \( \mathcal{HC}_{0,0} \) does not have enough projectives, and is not Koszul with respect to the natural grading, in contrast to \( \mathcal{O}_0 \) (for the Koszulity of \( \mathcal{O}_0 \) see [8]). The study of translation functors on Harish-Chandra modules goes back to Zuckerman [74].

7. In the following we will mention several instances of categorifications of modules over Lie algebras. Our definition of categorification required an associative algebra rather than a Lie algebra, so one should think of this construction as a categorification of representations of the associated universal enveloping algebra.

Let \( V \) be the fundamental two-dimensional representation of the complex Lie algebra \( \mathfrak{sl}_2 \). Denote by \( \{e, f, h\} \) the standard basis of \( \mathfrak{sl}_2 \). The \( n \)-th tensor power of \( V \) decomposes into a direct sum of weight spaces:

\[
V^\otimes n = \bigoplus_{k=0}^{n} V^\otimes n(k),
\]

where \( h x = (2k - n)x \) for \( x \in V^\otimes n(k) \).

A categorification of \( V^\otimes n \) was constructed in [9]. The authors considered certain singular blocks \( \mathcal{O}_{k,n-k} \) of the category \( \mathcal{O} \) for \( \mathfrak{sl}_n \). The Grothendieck group of this block has rank \( \binom{n}{k} \) equal to the dimension of the weight space \( V^\otimes n(k) \), and there are natural isomorphisms

\[
K(\mathcal{O}_{k,n-k}) \otimes \mathbb{C} \cong V^\otimes n(k).
\]

The Grothendieck group of the direct sum

\[
\mathcal{O}_n = \bigoplus_{k=0}^{n} \mathcal{O}_{k,n-k}
\]

is isomorphic to \( V^\otimes n \) (after tensoring with \( \mathbb{C} \) over \( \mathbb{Z} \)). Suitable translation functors \( \mathcal{E}, \mathcal{F} \) in \( \mathcal{O}_n \) lift the action of the generators \( e, f \) of \( \mathfrak{sl}_2 \) on \( V^\otimes n \).
To make this construction compatible with Definition 1 one should switch
to Lusztig’s version $\hat{U}$ of the universal enveloping algebra $U(\mathfrak{sl}_2)$ (see [49], [9]) and set $q = 1$. Instead of the unit element, the ring $\hat{U}$ contains idempotents $1_n, n \in \mathbb{Z}$, which can be viewed as projectors onto integral weights. The Lusztig basis $\mathbb{B}$ in $\hat{U}$ has the positivity property required by Definition 1, and comes along with an integral version $V^\otimes n_Z$ of the tensor power representation. The triple $(\hat{U}, \mathbb{B}, V^\otimes n_Z)$ is categorified using the above-mentioned category $\mathcal{O}_n$ and projective endofunctors of it. In fact, each element of $\mathbb{B}$ either corre-
sponds to an indecomposable projective endofunctor on $\mathcal{O}_n$ or acts by 0 on $V_Z^\otimes n$. We refer the reader to [9] for details, to [22] for an axiomatic devel-
opment of $\mathfrak{sl}_2$ categorifications, and to [9] and [69] for a categorification of the
Temperley-Lieb algebra action on $V_Z^\otimes n$ via projective endofunctors on the
category Koszul dual to $\mathcal{O}_n$ (see [8] and [50] for details on Koszul duality).

8. A categorification of arbitrary tensor products of fundamental repre-
sentations $\Lambda^i V$, where $V$ is the $k$-dimensional $\mathfrak{sl}_k$-representation and $1 \leq i \leq k - 1$ was found by J. Sussan [73]. A tensor product $\Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_r} V$ decom-
poses into weight spaces $\Lambda^{i_1} V \otimes \cdots \otimes \Lambda^{i_r} V(\nu)$, over various integral weights $\nu$ of $\mathfrak{sl}_k$. Each weight space becomes the Grothendieck group of a parabolic-
singular block of the highest weight category for $\mathfrak{sl}_N$, where $N = i_1 + \cdots + i_r$. For the parabolic subalgebra one takes the Lie algebra of traceless $N \times N$ matrices which are $(i_1, \ldots, i_r)$ block upper-triangular. The choice of the sin-
gular block is determined by $\nu$. Translation functors between singular blocks, restricted to the parabolic category, provide an action of the generators $E_j$ and $F_j$ of the Lie algebra $\mathfrak{sl}_k$. Relations in the universal enveloping algebra lift to functor isomorphisms. Conjecturally, Sussan’s categorification satisfies
the framework of Definition 1 above, with respect to Lusztig’s completion $\hat{U}$
of the universal enveloping algebra of $\mathfrak{sl}_k$ and Lusztig’s canonical basis there.
9. Ariki, in a remarkable paper [2], categorified all finite-dimensional irreducible representations of \( \mathfrak{sl}_m \), for all \( m \), as well as integrable irreducible representations of affine Lie algebras \( \hat{\mathfrak{sl}}_r \). Ariki considered certain finite-dimensional quotient algebras of the affine Hecke algebra \( \hat{H}_{n,q} \), known as Ariki-Koike cyclotomic Hecke algebras, which depend on a number of discrete parameters. He identified the Grothendieck groups of blocks of these algebras, for generic values of \( q \in \mathbb{C} \), with the weight spaces \( V_\lambda(\mu) \) of finite-dimensional irreducible representations

\[
V_\lambda = \bigoplus_{\mu} V_\lambda(\mu)
\]

of \( \mathfrak{sl}_m \). Direct summands of the induction and restriction functors between cyclotomic Hecke algebras for \( n \) and \( n + 1 \) act on the Grothendieck group as generators \( e_i \) and \( f_i \) of \( \mathfrak{sl}_m \).

Specialising \( q \) to a primitive \( r \)-th root of unity, Ariki obtained a categorification of integrable irreducible representations of the affine Lie algebra \( \hat{\mathfrak{sl}}_r \).

We conjecture that direct summands of arbitrary compositions of Ariki’s induction and restriction functors correspond to elements of the Lusztig canonical basis \( \mathbb{B} \) of Lusztig’s completions \( \mathbb{U} \) of these universal enveloping algebras. This conjecture would imply that Ariki’s categorifications satisfy the conditions of Definition 1.

Lascoux, Leclerc and Thibon, in an earlier paper [47], categorified level-one irreducible \( \hat{\mathfrak{sl}}_r \)-representations, by identifying them with the direct sum of Grothendieck groups of finite-dimensional Hecke algebras \( H_{n,q} \), over all \( n \geq 0 \), with \( q \) a primitive \( r \)-th root of unity. Their construction is a special case of Ariki’s. We also refer the reader to related works [3], [27]. Categorifications of the adjoint representation and of irreducible \( \mathfrak{sl}_m \)-representations with highest weight \( \omega_j + \omega_k \) are described explicitly in [28], [29] and [19].

Another way to categorify all irreducible finite-dimensional representations of \( \mathfrak{sl}_m \), for all \( m \), was found by Brundan and Kleshchev [14], via the representation theory of W-algebras. There is a good chance that their categorification is equivalent to that of Ariki, and that an equivalence of two categorifications can be constructed along the lines of Arakawa-Suzuki [1] and Brundan-Kleshchev [15].

**Biadjointness.** Definition 1 of (weak) categorifications was minimalistic. Categorifications in the above examples share extra properties, the most
prominent of which is biadjointness: there exists an involution \( a_i \to a_i' \) on the basis \( a \) of \( A \) such that the functor \( F_i \) is both left and right adjoint to \( F_i' \). This is the case in the examples 2 through 9, while in example 1 the functors are almost biadjoint. Namely, the induction functor \( F_x \) lifting the action of \( x \) is left adjoint to the restriction functor \( F_\partial \) (which lifts the action of \( \partial \)) and right adjoint to \( F_\partial \) conjugated by an involution.

A conceptual explanation for the pervasiveness of biadjointness in categorifications is given by the presence of the Hom bifunctor in any abelian category. The Hom bifunctor in \( \mathcal{B} \) descends to a bilinear form on the Grothendieck group \( B \) of \( \mathcal{B} \), via

\[
([M], [N]) \overset{\text{def}}{=} \dim \text{Hom}_\mathcal{B}(M, N),
\]

where \( M \) is projective or \( N \) is injective, and some standard technical conditions are satisfied. When a representation naturally comes with a bilinear form, the form is usually compatible with the action of \( A \): there exists an involution \( a \to a' \) on \( A \) such that \( (ax, y) = (x, a'y) \) for \( x, y \in B \). A categorification of this equality should be an isomorphism

\[
\text{Hom}(F_a M, N) \cong \text{Hom}(M, F_{a'} N)
\]

saying that the functor lifting the action of \( a' \) is right adjoint to the functor lifting the action of \( a \). If the bilinear form is symmetric, we should have the adjointness property in the other direction as well, leading to biadjointness of \( F_a \) and \( F_{a'} \).

A beautiful approach to \( \mathfrak{sl}_2 \) categorifications via biadjointness was developed by Chuang and Rouquier [22] (see also [60]). The role of biadjointness in TQFTs and their categorifications is clarified in [40, Section 6.3]. An example how the existence of a categorification with a bilinear form can be used to determine dimensions of hom-spaces can be found in [70].

**Grading and \( q \)-deformation.** In all of the above examples, the data \((A, a, B)\) that is being categorified admits a natural \( q \)-deformation \((A_q, a_q, B_q)\). Here \( A_q \) is a \( \mathbb{Z}[q, q^{-1}] \)-algebra, \( B_q \) an \( A_q \)-module, and \( a_q \) a basis of \( A_q \). We assume that both \( A_q \) and \( B_q \) are free \( \mathbb{Z}[q, q^{-1}] \)-modules, that the multiplication in \( A_q \) in the basis \( a_q \) has all coefficients in \( \mathbb{N}[q, q^{-1}] \), and that taking the quotient by the ideal \((q - 1)\) brings us back to the original data:

\[
A = A_q/(q - 1)A_q, \quad B = B_q/(q - 1)B_q, \quad a_q \xrightarrow{q=1} a.
\]
An automorphism $\tau$ of an abelian category $\mathcal{B}$ (more accurately, an invertible endofunctor on $\mathcal{B}$) induces a $\mathbb{Z}[q, q^{-1}]$-module structure on the Grothendieck group $K(\mathcal{B})$. Multiplication by $q$ corresponds to the action of $\tau$:

$$[\tau(M)] = q[M], \quad [\tau^{-1}(M)] = q^{-1}[M].$$

In many of the examples, $\mathcal{B}$ will be the category of graded modules over a graded algebra, and $\tau$ is just the functor which shifts the grading. To emphasize this, we denote $\tau$ by $\{1\}$ and its $n$-th power by $\{n\}$.

**Definition 7** A (weak) abelian categorification of $(A_q, a_q, B_q)$ consists of an abelian category $\mathcal{B}$ equipped with an invertible endofunctor $\{1\}$, an isomorphism of $\mathbb{Z}[q, q^{-1}]$-modules $\varphi : K(\mathcal{B}) \overset{\sim}{\longrightarrow} B_q$ and exact endofunctors $F_i : \mathcal{B} \longrightarrow \mathcal{B}$ that commute with $\{1\}$ and such that the following hold

1. **(qC-I)** $F_i$ lifts the action of $a_i$ on the module $B_q$, i.e. the action of $[F_i]$ on the Grothendieck group corresponds to the action of $a_i$ on $B_q$, under the isomorphism $\varphi$, in the sense that the diagram below is commutative.

2. **(qC-II)** There are isomorphisms of functors

$$F_i F_j \cong \bigoplus_k F_k \epsilon_{ij}^k,$$

i.e., the composition $F_i F_j$ decomposes as the direct sum of functors $F_k$ with multiplicities $\epsilon_{ij}^k \in \mathbb{N}[q, q^{-1}]$.

The graded versions are well-known in all of the examples above up to Example 8. In Example 1 the nilCoxeter algebra $R_n$ is naturally graded with $\deg(Y_i) = 1$. The inclusion $R_n \subset R_{n+1}$ induces induction and restriction functors between categories of graded $R_n$ and $R_{n+1}$-modules. In the graded case, induction and restriction functors satisfy the isomorphism

$$DX \cong XD\{1\} \oplus \text{Id}$$
which lifts the defining relation $\partial x = qx\partial + 1$ of the $q$-Weyl algebra (see [39] for more detail).

An accurate framework for graded versions of examples 2–8 is a rather complicated affair. To construct a canonical grading on a regular block of the highest weight category [8] requires étale cohomology, perverse sheaves [6], and the Beilinson-Bernstein-Brylinski-Kashiwara localisation theorem [5], [17]. Soergel’s approach to this grading is more elementary [64], [66], [65], but still relies on these hard results. Extra work is needed to show that translation or projective functors can be lifted to endofunctors in the graded category [68].

Ariki’s categorification of irreducible integrable representations (Example 9 above) should admit a graded version as well.

2 Four examples of categorifications of irreducible representations

In the example 4 above we categorified an integral lift of the irreducible representation $S_Q(\lambda^*)$ of the symmetric group via the abelian category $\mathcal{C}_\lambda$ built out of projective-injective modules in a parabolic block of $O$. The category $\mathcal{C}_\lambda$ is equivalent to the category of finite-dimensional representations over a finite-dimensional algebra $A_\lambda$, the algebra of endomorphisms of the direct sum of indecomposable projective-injective modules $P^\mu(w)$. Under this equivalence, projective functors $\theta_i$ turn into the functors of tensoring with certain $A_\lambda$-bimodules. It’s not known how to describe $A_\lambda$ and these bimodules explicitly, except in a few cases, four of which are discussed below.

a. The sign representation. The sign representation of the symmetric group (over $\mathbb{Z}$) is a free abelian group $\mathbb{Z}v$ on one generator $v$, with $s_i v = -v$ for all $i$. It corresponds to the partition $(1^n)$ of $n$, which in our notation is $\lambda^*$ for $\lambda = (n)$. The parabolic category $\mathcal{O}_0^{(n)}$ has as objects exactly the finite-dimensional modules from $\mathcal{O}_0$ since the parabolic subalgebra in this case is all of $\mathfrak{sl}_n$.

Actually, $\mathcal{O}_0$ has only one simple module with this property, the one-dimensional trivial representation $\mathbb{C}$. In our notation, this is the module $L(e)$, the simple quotient of the Verma module $M(e)$ assigned to the unit
element of the symmetric group.

Consequently, any object of \( \mathcal{O}_0^{(n)} \) is isomorphic to a direct sum of copies of \( L(e) \), and the category is semisimple. Furthermore, the category \( \mathcal{C}^{(n)} \) is all of \( \mathcal{O}_0^{(n)} \). Thus, \( \mathcal{C}^{(n)} \) is equivalent to the category of finite-dimensional \( \mathbb{C} \)-vector spaces. Projective functors \( \theta_w \) act by zero on \( \mathcal{C}^{(n)} \) for all \( w \in S_n, w \neq e \), while \( \theta_e \) is the identity functor.

The graded version \( \mathcal{C}_{gr}^{(n)} \) is equivalent to the category of graded finite-dimensional \( \mathbb{C} \)-vector spaces. Again, projective functors \( \theta_w, w \neq e, \) act by zero, and \( \theta_e \) is the identity functor.

Thus, our categorification of the sign representation is rather trivial.

b. The trivial representation. The trivial representation \( \mathbb{Z}z \) of \( \mathbb{Z}[S_n] \) is a free abelian group on one generator \( z \), with the action \( wz = z, w \in S_n \). The corresponding partition is \( (n) \), with the dual partition \( \lambda = (1^n) \). Only one decomposition \( \mu = (1^n) \) corresponds to the dual partition; the parabolic subalgebra associated with \( (1^n) \) is the positive Borel subalgebra, and the parabolic category \( \mathcal{O}_0^{(1^n)} \) is all of \( \mathcal{O}_0 \).

The unique self-dual indecomposable projective \( P \) in \( \mathcal{O}_0 \) is usually called the big projective module. Its endomorphism algebra \( \text{End}_{\mathcal{O}}(P) \) is isomorphic to the cohomology ring \( H_n \) of the full flag variety \( \text{Fl} \) of \( \mathbb{C}^n \), see [64].

The category \( \mathcal{C}^{(1^n)} \) is equivalent to the category of finite-dimensional \( H_n \)-modules. The unique (up to isomorphism) simple \( H_n \)-module generates the Grothendieck group \( K_0(\mathcal{H} - \text{mod}) \sim = \mathbb{Z} \).

To describe how the functors \( \theta_i \) act on \( \mathcal{C}^{\lambda} \) consider generalised flag varieties

\[
\text{Fl}_i = \{0 \subset L_1 \subset L_2 \subset \cdots \subset L_{n-1} \subset \mathbb{C}^n, L_i'\} \quad \text{dim}(L_j) = j, \text{dim}(L_i') = i, L_{i-1} \subset L_i' \subset L_{i+1}\}
\]

This variety is a \( \mathbb{P}^1 \)-bundle over the full flag variety \( \text{Fl} \) in two possible ways, corresponding to forgetting \( L_i \), respectively \( L_i' \). These two maps from \( \text{Fl}_i \) onto \( F \) induce two ring homomorphisms

\[
H_n = H(\text{Fl}, \mathbb{C}) \longrightarrow H(\text{Fl}_i, \mathbb{C})
\]

which turn \( H(F_i, \mathbb{C}) \) into an \( H_n \)-bimodule. The functor \( \theta_i : \mathcal{C}^{(1^n)} \longrightarrow \mathcal{C}^{(1^n)} \) is given by tensoring with this \( H_n \)-bimodule (under the equivalence \( \mathcal{C}^{(1^n)} \approx H_n - \text{mod} \)).
To describe functors $\theta_w$ for an arbitrary $w \in S_n$, we recall that $\operatorname{Fl} = G/B$ where $G = \operatorname{SL}(n, \mathbb{C})$ and $B$ the Borel subgroup of $G$. The orbits of the natural left action of $G$ on $\operatorname{Fl} \times \operatorname{Fl}$ are in natural bijection with elements of the symmetric group. Denote by $O_w$ the orbit associated with $w$ and by $\text{IC}(O_w)$ the simple perverse sheaf on the closure of this orbit. The cohomology of $\text{IC}(O_w)$ is an $H_n$-bimodule, and the functor

$$\theta_w : \operatorname{H}_n-\operatorname{mod} \rightarrow \operatorname{H}_n-\operatorname{mod}$$

takes a module $M$ to the tensor product

$$\text{H} (\text{IC}(O_w), \mathbb{C}) \otimes_{H_n} M.$$

Notice that all cohomology rings above have a canonical grading (by cohomological degree). The graded version of $\mathcal{C}^{(1^n)}$ is the category of finite-dimensional graded $H_n$-modules and the graded version of $\theta_w$ tensors a graded module with the graded $H_n$-bimodule $\text{H}(\text{IC}(O_w), \mathbb{C})$.

It is surprising how sophisticated the categorification of the trivial representation is, especially when compared with the categorification of the sign representation. Both the trivial and the sign representation are one-dimensional, but their categorifications have amazingly different complexities. All of the complexity is lost when we pass to the Grothendieck group, which has rank one.

c. Categorification of the Burau representation. Consider the partition $\lambda^* = (2, 1^{n-2})$ and the dual partition $\lambda = (n-1, 1)$. The category $\mathcal{C}^{(n-1,1)}$ admits an explicit description, as follows. For $n > 3$ let $A_{n-1}$ be the quotient of the path algebra of the graph from Figure 1 by the relations

$$ (i|i + 1|i + 2) = 0, $$
$$ (i|i - 1|i - 2) = 0, $$
$$ (i|i - 1|i) = (i|i + 1|i) $$

Also, let $A_1$ be the exterior algebra on one generator, and $A_2$ be the quotient of the path algebra of the graph from Figure 1 (for $n = 2$) by the relations $(1|2|1|2) = 0 = (2|1|2|1)$. The $\mathbb{C}$-algebra $A_{n-1}$ is finite-dimensional.

**Proposition 8** The category $\mathcal{C}^{(n-1,1)}$ is equivalent to the category of finite-dimensional left $A_{n-1}$-modules.
This is a well-known result, see e.g. [67] for $n = 2$ and [71] for the general case.

Denote by $P_i$ the indecomposable left projective $A_{n-1}$-module $A_{n-1}(i)$. This module is spanned by all paths that end in vertex $i$. Likewise, let $iP$ stand for the indecomposable right projective $A_{n-1}$-module $(i)A_{n-1}$. Under the equivalence between $C^{(n-1,1)}$ and the category $A_{n-1}$-mod of finite-dimensional $A_{n-1}$-modules, the functor $\theta_i$ becomes the functor of tensoring with the bimodule

$$P_i \otimes_i P.$$

The functors $\theta_w$ are zero for most $w \in S_n$. They are nonzero only when the corresponding composition of $\theta_i$’s is nonzero (which rarely happens, note that already $\theta_i \theta_j = 0$ for $|i - j| > 1$).

The algebras $A_{n-1}$, as well as the modules $P_i$, $iP$ are naturally graded by the length of paths. The categories of finite dimensional graded modules over these algebras provide a categorification of the reduced Burau representation of each of the corresponding braid groups. For more information about the algebras $A_{n-1}$ and their uses we refer the reader to the papers [45], [61], [63], [70], [28].

d. Categorification of the 2-column irreducible representation (partition $(2^n)$).

Let $\lambda^* = (2^n)$ and $\lambda = (n, n)$. The irreducible representation $S_\mathbb{Q}(\lambda^*)$ has the following explicit description. The basis of the representation consists of crossingless matchings of $2n$ points positioned on the $x$-axis by $n$ arcs lying in the lower half-plane, as depicted below.

The element $1 + s_i$ acts on a basis element by concatenating it with the
If the concatenation contains a circle, we remove it and multiply the result by 2, see figure 2.

Figure 2: The product \((1 + s_i)b\), for a basis element \(b\), is either another basis element (top diagram) or the same basis element times 2.

A categorification of this representation and of its quantum deformation was described in [40], in the context of extending a categorification of the Jones polynomial to tangles. The basis elements \(b\) corresponding to crossingless matchings become indecomposable projective modules \(P_b\) over a certain finite-dimensional algebra \(H^n\). The space of homs \(\text{Hom}_{H^n}(P_a, P_b)\) between projective modules is given by gluing crossingless matchings \(a\) and \(b\) along their endpoints and applying a 2-dimensional TQFT to the resulting 1-manifold. The TQFT is determined by a commutative Frobenius algebra, which is just the cohomology of the 2-sphere. Spaces of these homs together with compositions

\[
\text{Hom}_{H^n}(P_a, P_b) \times \text{Hom}_{H^n}(P_b, P_c) \longrightarrow \text{Hom}_{H^n}(P_a, P_c)
\]
determine $H^n$ uniquely. The above geometric action of $1 + s_i$ lifts to the action on the category of $H^n$-mod of finite-dimensional $H^n$-modules given by tensoring with a certain $H^n$-module. This results in a very explicit categorification of the 2-column irreducible representation of $S_n$ (and of the corresponding representation of the Hecke algebra) via the category of $H^n$-modules.

It was shown in [71] that $H^n$-mod is equivalent to the category $\mathcal{C}^{(n,n)}$ generated by projective-injective modules in the parabolic block $\mathcal{O}_0^{(n,n)}$. (An extension of this equivalence to functors will be treated in [72]). Subquotient algebras of $H^n$ considered in [71], [20], [19] can be used to categorify other 2-column representations of the Hecke algebra and the symmetric group. These subquotient algebras provide also a graphical description of the whole category $\mathcal{O}_{\mu}^0$ for any composition $\mu_1 + \mu_2 = n$ ([71], [72]).

3 Miscellaneous

Braid group actions. Graded versions of projective functors $\theta_w$ categorify the action of the Hecke algebra $H_{n,q}$ on its various representations. There is a homomorphism from the braid group on $n$ strands to the group of invertible elements in $H_{n,q}$. This homomorphism, too, admits a categorification. The categorification should be at least an action of the braid group on a category, and this action is indeed well-known. To define it we need to pass to one of the triangulated extensions of the highest weight category: there does not seem to exist any interesting braid group actions on abelian categories, due to the positivity imposed by the abelian structure (see discussion in Section 6 of [40]).

The translation through the wall functors $\theta_i$, $i \leq 1 \leq n - 1$, for the regular block $\mathcal{O}_0$ are compositions of two projective endofunctors (on and off the wall) of $\mathcal{O}$, which are biadjoint to each other. This results in natural transformations $\theta_i \rightarrow \text{Id}$ and $\text{Id} \rightarrow \theta_i$. Let $D(\mathcal{O}_0)$ be the bounded derived category of $\mathcal{O}_0$. The complexes $R_i$ and $R'_i$ of functors $0 \rightarrow \theta_i \rightarrow \text{Id} \rightarrow 0$ and $0 \rightarrow \text{Id} \rightarrow \theta_i \rightarrow 0$ can be viewed as endofunctors of $D(\mathcal{O}_0)$ (we normalise the above functors so that $\text{Id}$ sits in cohomological degree 0).

**Proposition 9** The functors $R_i$ define a braid group action on $D(\mathcal{O}_0)$. The functor $R'_i$ is the inverse of $R_i$. 

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We distinguish between weak and genuine group actions; the terminology can be found in [45], [59], [46]. That $R'_i$ and $R_i$ are inverses of each other follows from a more general result of J. Rickard. That $R_i$ define a weak braid group action follows from [51], [52, Proposition 10.1].

The Koszul duals of the functor $R_i$ and its inverse are described in [50] in terms of the so-called twisting and completion functors on $\mathcal{O}_0$. A geometric description of these functors can be found in [7] and [59].

The functors $\theta_i$ restrict to exact endofunctors of the parabolic categories $\mathcal{O}_0^\mu$ and of the categories $\mathcal{C}^\mu$. Hence, the functors $R_i$ and $R'_i$ first induce endofunctors on $D^b(\mathcal{O}_0^\mu)$, and define braid group actions there and then restrict to endofunctors of the subcategory given by complexes of projective-injective modules in $\mathcal{O}_0^\mu$.

The braid group acts by functors respecting the triangulated structure of the involved categories, resulting in a categorification of parabolic braid group modules as well as those irreducible representations of the braid group that factor through the Hecke algebra. The two commuting actions of projective functors on the category of Harish-Chandra bimodules as described in example 6 of Section 1 give rise to two commuting actions on the braid group on the derived category of the category of Harish-Chandra bimodules. For more examples of braid group actions on triangulated categories and a possible framework for these actions see [46].

**Invariants of tangle cobordisms.** In several cases, braid group actions on triangulated categories can be extended to representations of the 2-category of tangle cobordisms. The objects of this 2-category (when 2-tangles are not decorated) are non-negative integers, morphisms from $n$ to $m$ are tangles with $n$ bottom and $m$ top boundary components, and 2-morphisms are isotopy classes of tangle cobordisms. A representation of the 2-category of tangle cobordisms associated a triangulated category $\mathcal{K}_n$ to the object $n$, an exact functor $\mathcal{K}_n \longrightarrow \mathcal{K}_m$ to a tangle, and a natural transformation of functors to a tangle cobordism. Such representations can be derived from examples 7 and 8 of Section 1 (see [70], [73]) and from example d of Section 2 (see [40]). Example 6 is related to at least braid cobordisms (if not tangle cobordisms) via the construction of [42]. We expect that a categorification of tensor products of representations of quantum $\mathfrak{sl}_2$, mentioned at the end of example 7 extends (after passing to derived categories, suitable functors, and natural transformations) to a representation of the 2-category of tangle
cobordisms coloured by irreducible representations of quantum $\mathfrak{sl}_2$. Such an extension would give a categorification of the coloured Jones polynomial.

The Cautis-Kamnitzer invariant of tangle cobordisms [18] is based on a similar framework, but their version of the category $\mathcal{K}_n$ is the derived category of coherent sheaves on a certain iterated $\mathbb{P}^1$-bundle. The Grothendieck group of their category is isomorphic to $V^\otimes n$, where $V$ is the fundamental representation of quantum $\mathfrak{sl}_2$, just like in the example 7, but these two categorifications of $V^\otimes n$ are noticeably different. For instance, in the example 7 the category decomposes into the direct sum matching the weight decomposition of the tensor product, while the category in [18] is indecomposable. When the parameter is even, the two categorifications of $V^\otimes 2n$ appear to have a common “core” subcategory, a categorification of the invariants in $V^\otimes 2n$ (the latter isomorphic to $S((2^n))$) briefly reviewed in the example d above.

In the matrix factorization invariant of tangle cobordisms [44], the abelian category remains hidden inside the triangulated category of matrix factorisations.

**Other categorifications, abelian and triangulated.** Our list of examples of abelian categorifications is very far from complete. Many great results in the geometric representation theory can be interpreted as categorifications via abelian or triangulated categories. This includes the early foundational work of Beilinson-Bernstein and Brylinsky-Kashiwara on localisation [5], [17], [55], the work of Kazhdan and Lusztig on geometric realisation of representations of affine Hecke algebras [37], [21], Lusztig’s geometric construction of the Borel subalgebras of quantum groups [49], Nakajima’s realisation of irreducible Kac-Moody algebra representations as middle cohomology groups of quiver varieties [56], and various constructions related to Hilbert schemes of surfaces [26], [57], quantum groups at roots of unity [4], geometric Langlands correspondence [24], etc.

**Determinant of the Cartan matrix.** With $\lambda$ and $\mu$ as in example 4, let $\{P_a\}_{a \in I}$ be a collection of indecomposable projectives in $\mathcal{C}^\mu$, one for each isomorphism class. The Cartan matrix of $\mathcal{C}^\mu$ is an $I \times I$ matrix $C$ with the $(a, b)$-entry being the dimension of $\text{Hom}(P_a, P_b)$, the space of homomorphisms between projective modules $P_a$ and $P_b$. Since $\text{End}(P, P)$ is a symmetric algebra by [53], where $P = \bigoplus_{a \in I} P_a$, the Cartan matrix is symmetric, $c_{a,b} = c_{b,a}$. These algebras are not commutative, but the centre has a nice geometrical
description as the cohomology of some Springer fibre ([41], and more general [13], [71]).

What is the determinant of this Cartan matrix? Since $C^\mu$ depends (up to equivalence) on the partition $\lambda$ only ([53]), so does the determinant. The answer to the question is obvious in each of the first three cases considered in the previous section: the determinant is equal to 1 for $\lambda = (n)$, to $n!$ for $\lambda = (1^n)$, and to $n$ for $\lambda = (n-1, 1)$. The fourth case, when $\lambda = (2^n)$, requires more work, and follows from the results of [23] and [38]. The determinant equals

$$
\prod_{i=1}^{n}(i+1)^{r_{n,i}}, \quad r_{n,i} = \binom{2n}{n-i} - 2\binom{2n}{n-i-1} + \binom{2n}{n-i-2},
$$

(9)

with the convention $\binom{j}{s} = 0$ if $s < 0$. The answer for an arbitrary $\lambda$ is more complicated. However, we want to point out that this determinant of the Cartan matrix is the determinant of the Shapovalov form ([62]) on a certain weight space of some irreducible $\mathfrak{sl}_n$-module, as can be obtained from instance from [15]. It can be computed using the so-called Jantzen-Schaper formula [33, Satz 2].

The absolute value of the determinant has an interesting categorical interpretation. $C^\mu$ is equivalent to the category of finite-dimensional modules over some symmetric $C$-algebra $A^\mu$. Given any symmetric $C$-algebra $A$ (an algebra with a nondegenerate symmetric trace $A \to C$), the stable category $A-\text{mod}$ is triangulated. Objects of $A-\text{mod}$ are finite-dimensional $A$-modules and the set of morphisms from $M$ to $N$ is the quotient vector space of all module maps modulo those that factor through a projective module. If $\det(C) \neq 0$ then the Grothendieck group of the stable category is finite abelian of cardinality equal to the absolute value of the determinant.

The graded version of this problem makes sense as well. Modules $P_a$ are naturally graded, and to a pair $(a, b)$ we can assign the Laurent polynomial in $q$ which is the graded dimension of the graded vector space $\text{Hom}(P_a, P_b)$. Arrange these polynomials into an $I \times I$ matrix (the graded Cartan matrix of $C^\mu_{gr}$).

**Problem:** Find the determinant of the graded Cartan matrix of $C^\mu$.

The determinant depends only on $\lambda$. Again, the answer is known in the above four cases. In the last case, the determinant of the graded Cartan
matrix is given by formula (9), with the quantum integer \([i + 1] = 1 + q^2 + \cdots + q^{2i}\) in place of \((i + 1)\) in the product (the proof follows by combining results of [23] and [38]).

The determinant is algorithmically computable, since the entries of the graded Cartan matrix can be computed from the Kazhdan-Lusztig polynomials of the symmetric group. We are almost tempted to conjecture that, for any \(\lambda\), the determinant (up to a power of \(q\)) is a product of quantum integers \([j] = q^{i-1} + q^{i-3} + \cdots + q^{1-j}\), for small \(j\), with some multiplicities.

In [32], a \(q\)-analogue of the Jantzen-Schaper formula is obtained. Generalising [13] by working out a graded or \(q\)-version, should imply that the determinant is equal to the determinant of the \(q\)-analogue of the Shapovalov form on a suitable weight space of an irreducible \(\mathfrak{sl}_m\)-module.

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