Solutions to the $\sigma_k$-Loewner-Nirenberg problem on annuli are locally Lipschitz and not differentiable

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Dedicated to Alice Chang and Paul Yang on their 70th birthday

Abstract

We show for $k \geq 2$ that the locally Lipschitz viscosity solution to the $\sigma_k$-Loewner-Nirenberg problem on a given annulus $\{a < |x| < b\}$ is $C^{1,\frac{1}{k}}_{\text{loc}}$ in each of $\{a < |x| \leq \sqrt{ab}\}$ and $\{\sqrt{ab} \leq |x| < b\}$ and has a jump in radial derivative across $|x| = \sqrt{ab}$. Furthermore, the solution is not $C^{1,\gamma}_{\text{loc}}$ for any $\gamma > \frac{1}{k}$. Optimal regularity for solutions to the $\sigma_k$-Yamabe problem on annuli with finite constant boundary values is also established.

1 Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. For a positive $C^2$ function $u$ defined on an open subset of $\mathbb{R}^n$, let $A^u$ denote its conformal Hessian, namely

$$A^u = -\frac{2}{n-2} u \frac{n+2}{n-2} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I, \quad (1.1)$$

and let $\lambda(-A^u)$ denote the eigenvalues of $-A^u$. Note that $A^u$, considered as a $(0,2)$ tensor, is the Schouten curvature tensor of the metric $u^{\frac{4}{n-2}} \hat{g}$ where $\hat{g}$ is the Euclidean metric.
For $1 \leq k \leq n$, let $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ denote $k$-th elementary symmetric function
\[
\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k},
\]
and let $\Gamma_k$ denote the cone $\Gamma_k = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}$.

In [7, Theorem 1.1], it was shown that the $\sigma_k$-Loewner-Nirenberg problem
\[
\sigma_k(\lambda(-A^u)) = 2^{-k} \left( \frac{n}{k} \right), \quad \lambda(-A^u) \in \Gamma_k, \quad u > 0 \quad \text{in } \Omega, \quad (1.2)
\]
\[
u(x) \to \infty \text{ as } \text{dist}(x, \partial \Omega) \to 0. \quad (1.3)
\]
has a unique continuous viscosity solution $u$ and such $u$ belongs to $C^{0,1}_{\text{loc}}(\Omega)$. Furthermore, $u$ satisfies
\[
\lim_{\text{dist}(x, \partial \Omega) \to 0} u(x) = C(n,k) \in (0, \infty). \quad (1.4)
\]

Equation (1.2) is a fully nonlinear elliptic equation of the kind considered by Caffarelli, Nirenberg and Spruck [3]. We recall the following definition of viscosity solutions which follows Li [20, Definitions 1.1 and 1.1’] (see also [19]) where viscosity solutions were first considered in the study of nonlinear Yamabe problems.

Let
\[
\overline{S}_k := \left\{ \lambda \in \Gamma_k | \sigma_k(\lambda) \geq 2^{-k} \left( \frac{n}{k} \right) \right\}, \quad (1.5)
\]
\[
S_k := \mathbb{R}^n \setminus \left\{ \lambda \in \Gamma_k | \sigma_k(\lambda) > 2^{-k} \left( \frac{n}{k} \right) \right\}. \quad (1.6)
\]

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq k \leq n$. We say that an upper semi-continuous (a lower semi-continuous) function $u : \Omega \to (0, \infty)$ is a sub-solution (super-solution) to (1.2) in the viscosity sense, if for any $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ satisfying $(u - \varphi)(x_0) = 0$ and $u - \varphi \leq 0$ ($u - \varphi \geq 0$) near $x_0$, there holds
\[
\lambda(-A^\varphi(x_0)) \in \overline{S}_k \quad (\lambda(-A^\varphi(x_0)) \in S_k, \text{ respectively}).
\]

We say that a positive function $u \in C^0(\Omega)$ satisfies (1.2) in the viscosity sense if it is both a sub- and a super-solution to (1.2) in the viscosity sense.

Equation (1.2) satisfies the following comparison principle, which is a consequence of the principle of propagation of touching points [23, Theorem 3.2]: If $v$ and $w$ are
viscosity sub-solution and super-solution of (1.2) and if \( v \leq w \) near \( \partial \Omega \), then \( v \leq w \) in \( \Omega \); see [7, Proposition 2.2]. The above mentioned uniqueness result for (1.2)-(1.3) is a consequence of this comparison principle and the boundary estimate (1.4).

In the rest of this introduction, we assume that \( \Omega \) is an annulus \( \{a < |x| < b \} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \), unless otherwise stated. \( C^2 \) radially symmetric solutions to (1.2) were classified by Chang, Han and Yang [5, Theorems 1 and 2]. As a consequence, when \( 2 \leq k \leq n \), there is no \( C^2 \) radially symmetric function satisfying (1.2)-(1.3). On the other hand, the aforementioned uniqueness result from [7, 23] implies that the solution \( u \) to (1.2)-(1.3) is radially symmetric (since \( u(Rx) \) is also a solution for any orthogonal matrix \( R \)). Therefore, (1.2)-(1.3) has no \( C^2 \) solutions.

Our first result improves on the above non-existence of \( C^2 \) solutions to (1.2)-(1.3).

**Theorem 1.2.** Suppose that \( n \geq 3 \) and \( \Omega \) is a non-empty open subset of \( \mathbb{R}^n \). Then there exists no positive function \( u \in C^2(\Omega) \) such that \( \lambda(-Au) \in \bar{\Gamma}_2 \) in \( \Omega \) and that \( (\Omega, u^{\frac{4}{n-2}}) \) admits a smooth minimal immersion \( f : \Sigma^{n-1} \rightarrow \Omega \) for some smooth compact manifold \( \Sigma^{n-1} \).

Theorem 1.2 bears some resemblance to a result of Schoen and Yau [28] on a relation between positive scalar curvature and stable minimal surfaces.

Noting that when \( u \) is radially symmetric, \( \partial B_{r_0} \) is minimal with respect to \( u^{\frac{4}{n-2}} \) if and only if \( \frac{d}{dr}|_{r=r_0}(r^{\frac{n-2}{2}}u(r)) = 0 \), we obtain the following corollary with \( r_0 \) being a minimum point of \( r^{\frac{n-2}{2}}u(r) \).

**Corollary 1.3.** Suppose that \( n \geq 3 \). Let \( \Omega = \{a < |x| < b \} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \) be an annulus. Then there exists no radially symmetric positive function \( u \in C^2(\Omega) \) such that \( \lambda(-Au) \in \bar{\Gamma}_2 \) in \( \Omega \) and \( u(x) \rightarrow \infty \) as \( x \rightarrow \partial \Omega \).

Our next result shows that the locally Lipschitz solution \( u \) is not \( C^1 \).

**Theorem 1.4.** Suppose that \( n \geq 3 \) and \( 2 \leq k \leq n \). Let \( \Omega = \{a < |x| < b \} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \) be an annulus and \( u \) be the unique locally Lipschitz viscosity solution to (1.2)-(1.3). Then \( u \) is radially symmetric, i.e. \( u(x) = u(|x|) \),

(i) \( u \) is smooth in each of \( \{a < |x| < \sqrt{ab}\} \) and \( \{\sqrt{ab} < |x| < b\} \),

(ii) \( u \) is \( C^{1,\frac{k}{2}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in each of \( \{a < |x| \leq \sqrt{ab}\} \) and \( \{\sqrt{ab} \leq |x| < b\} \),

(iii) and the first radial derivative \( \partial_r u \) jumps across \( \{|x| = \sqrt{ab}\} \):

\[
\partial_r \ln u \big|_{r=\sqrt{ab}^-} = -\frac{n-2}{\sqrt{ab}} \quad \text{and} \quad \partial_r \ln u \big|_{r=\sqrt{ab}^+} = 0.
\]
A related problem in manifold settings is to solve on a given closed Riemannian manifold \((M, g)\) the equation

\[
\sigma_k \left( \lambda \left( - A u^{\frac{4}{n-2}} \right) \right) = 2^{-k} \left( \frac{n}{k} \right), \quad \lambda \left( - A u^{\frac{4}{n-2}} \right) \in \Gamma_k, \quad u > 0 \quad \text{in} \ M, \quad (1.7)
\]

where \(A u^{\frac{4}{n-2}}\) is the so-called Schouten tensor of the metric \(u^{\frac{4}{n-2}} g\),

\[
A u^{\frac{4}{n-2}} = -\frac{2}{n-2} u^{-1} \nabla_g^2 u + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{\frac{2n}{n-2}} |\nabla_g u|_g^2 g + A_g,
\]

and where \(\lambda \left( - A u^{\frac{4}{n-2}} \right)\) is the eigenvalue of \(-A u^{\frac{4}{n-2}}\) with respect to the metric \(u^{\frac{4}{n-2}} g\).

Equations (1.2) and (1.7) are fully non-linear and non-uniformly elliptic equations of Hessian type, usually referred to as the \(\sigma_k\)-Yamabe equation in the ‘negative case’, which is a generalization of the Loewner-Nirenberg problem [24]. This equation and its variants have been studied in Chang, Han and Yang [5], Gonzalez, Li and Nguyen [7], Gurksy and Viaclovsky [12], Li and Sheng [18], Guan [9], Gursky, Streets and Warren [11], and Sui [30]. For further studies on the counterpart of (1.2) in the positive case, see [4, 6, 10, 13, 16, 17, 20, 21, 29, 31, 32] and the references therein.

We observe the following result, which is essentially due to Gursky and Viaclovsky [12]. We provide in the appendix the detail for the piece which is not directly available from [12].

**Theorem 1.5.** Suppose that \(n \geq 3\), \(2 \leq k \leq n\), and \((M^n, g)\) is a compact Riemannian manifold. If \(\lambda( - A_g) \in \Gamma_k\) on \(M\), then (1.7) has a Lipschitz viscosity solution.

Here viscosity solution is defined analogously as in Definition 1.1.

**Definition 1.6.** Let \((M^n, g)\) be a Riemannian manifold, \(1 \leq k \leq n\), and \(\mathcal{S}_k\) and \(\mathcal{S}_k\) be given by (1.5) and (1.6). We say that an upper semi-continuous (a lower semi-continuous) function \(u : M \to (0, \infty)\) is a sub-solution (super-solution) to (1.7) in the viscosity sense, if for any \(x_0 \in M\), \(\varphi \in C^2(M)\) satisfying \((u - \varphi)(x_0) = 0\) and \(u - \varphi \leq 0\) \((u - \varphi \geq 0)\) near \(x_0\), there holds

\[
\lambda \left( - A \varphi^{\frac{4}{n-2}}(x_0) \right) \in \mathcal{S}_k \quad \left( \lambda \left( - A \varphi^{\frac{4}{n-2}}(x_0) \right) \in \mathcal{S}_k, \text{ respectively} \right).
\]

We say that a positive function \(u \in C^0(M)\) satisfies (1.7) in the viscosity sense if it is both a sub- and a super-solution to (1.7) in the viscosity sense.
In both contexts, it is an interesting open problem to understand relevant conditions on $\Omega$, or on $(M, g)$, which would ensure that (1.2)-(1.3), or (1.7) respectively, admits a smooth solution. We make the following conjecture.

**Conjecture 1.7.** Suppose that $n \geq 3$, $2 \leq k \leq n$, and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Then the locally Lipschitz viscosity solution to (1.2)-(1.3) is smooth near $\partial \Omega$.

Some further questions are in order.

**Question 1.8.** Suppose that $n \geq 3$, $2 \leq k \leq n$, and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. If (1.2)-(1.3) has a smooth sub-solution, must (1.2)-(1.3) have a smooth solution?

**Question 1.9.** Suppose that $n \geq 3$, $2 \leq k \leq n$, and $\Omega \subset \mathbb{R}^n$ is a smooth strictly convex (non-empty) domain. Is the locally Lipschitz viscosity solution to (1.2)-(1.3) smooth?

If $\Omega$ is a ball, then the solution to (1.2)-(1.3) is smooth and corresponds to the Poincaré metric.

**Question 1.10.** Suppose that $n \geq 3$, $2 \leq k \leq n$, and $\Omega = \Omega_2 \setminus \bar{\Omega}_1 \neq \emptyset$ where $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ are smooth bounded strictly convex domains. Is the locally Lipschitz viscosity solution to (1.2)-(1.3) $C^2$?

In the case $\Omega_1$ and $\Omega_2$ are balls, $\Omega = \Omega_2 \setminus \bar{\Omega}_1$ is conformally equivalent to an annulus, and so, by Theorem 1.4 the solution to (1.2)-(1.3) is not $C^2$. We believe that the answer to the above question is negative. We indicate here how such statement may be proved. In view of Theorem 1.2 it suffices to show that if $u$ is a $C^2$ solution to (1.2)-(1.3), then $(\Omega, u^{\frac{4}{n-2}}\tilde{g})$ admits a smooth (immersed) minimal hypersurface. It is reasonable to expect, in view of known results in the case $k = 1$ (cf. [1, 25]) and estimate (1.4), that

$$d(x, \partial \Omega) \left| \nabla \left( u(x) d(x, \partial \Omega)^{\frac{n-2}{2}} \right) \right| \to 0 \text{ as } d(x, \partial \Omega) \to 0.$$  

If the above estimate holds for $k \geq 2$, one has that, for small $\delta > 0$, the hypersurfaces $X_\delta = \{x \in \Omega : d(x, \partial \Omega) = \delta\}$ are strictly mean-convex with respect to $u^{\frac{4}{n-2}}\tilde{g}$ and the normal pointing toward the region enclosed between these two hypersurfaces. These hypersurfaces can be used as barriers to construct a desired minimal hypersurface, at least for $n \leq 7$. For example, in dimension $n = 3$, a result of Meeks and Yau [26, Theorem 7] (see also [14, Theorem 4.2]) implies that there exists a conformal
map \( f : S^2 \to \Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) > \delta \} \) which minimizes area among all homotopically nontrivial maps from \( S^2 \) into \( \Omega_\delta \) and either \( f \) is a conformal embedding or a double covering map whose image is an embedded projective plane. Since all compact surfaces in \( \mathbb{R}^3 \) are orientable (see e.g. [27] or [15, Corollary 3.46]), \( f \) is a conformal embedding and so \( f(S^2) \) is an embedded minimal sphere in \((\Omega_\delta, u^4 \tilde{g})\). This will be followed up in a subsequent joint work with Jingang Xiong.

**Question 1.11.** Suppose that \( n \geq 3, 2 \leq k \leq n \), and \((M^n, g)\) is a Riemannian manifold such that \( \lambda(-A_g) \in \Gamma_k \) on \( M \). Does (1.7) have a unique Lipschitz viscosity solution?

It is clear that (1.7) has at most one \( C^2 \) solution by the maximum principle. In fact, if (1.7) has a \( C^2 \) solution, then that solution is also the unique continuous viscosity solution in view of the strong maximum principle [2, Theorem 3.1]. Equivalently, if (1.7) has two viscosity solutions, then it has no \( C^2 \) solution.

**Question 1.12.** Suppose that \( n \geq 3 \) and \( 2 \leq k \leq n \). Does there exist a Riemannian manifold \((M^n, g)\) such that \( \lambda(-A_g) \in \Gamma_k \) on \( M \) and (1.7) has a Lipschitz viscosity solution which is not \( C^2 \)?

Finally, we discuss the case where (1.3) is replaced by finite constant boundary conditions

\[
u|_{\{|x|=a\}} = c_1 \quad \text{and} \quad \nu|_{\{|x|=b\}} = c_2. \tag{1.8}
\]

We completely determine in the following theorem the regularity of the solution to (1.2) and (1.8) depending on whether \( \ln \frac{b}{a} \) is larger, equal to, or smaller than \( 2T(a, b, c_1, c_2) \) where

\[
T(a, b, c_1, c_2) := \frac{1}{2} \int_{|p_b - p_a|}^{0} \left\{ 1 + e^{-2\eta - 2\max(p_a, p_b)} \left[ 1 - e^{n\eta} \right]^{1/k} \right\}^{-1/2} d\eta, \tag{1.9}
\]

\( p_a = -\frac{2}{n-2} \ln c_1 - \ln a \) and \( p_b = -\frac{2}{n-2} \ln c_2 - \ln b \).

**Theorem 1.13.** Suppose that \( n \geq 3 \) and \( 2 \leq k \leq n \). Let \( \Omega = \{ a < |x| < b \} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \) be an annulus, and \( c_1, c_2 \) be two positive constants and let \( T(a, b, c_1, c_2) \) be given by (1.9). Then there exists a unique continuous viscosity solution to (1.2) and (1.8). Furthermore, \( u \) is radially symmetric, i.e. \( u(x) = u(|x|) \), and exactly one of the following four alternatives holds.

**Case 1:** \( \ln \frac{b}{a} < 2T(a, b, c_1, c_2) \), and \( u \) is smooth in \( \{ a \leq |x| \leq b \} \),
Case 2: \( \ln \frac{b}{a} = 2T(a,b,c_1,c_2) \), \( b^{\frac{n-2}{2}} c_2 < a^{\frac{n-2}{2}} c_1 \), and \( u \) is smooth in \( \{a \leq |x| < b\} \), is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in \( \{a \leq |x| \leq b\} \).

Case 3: \( \ln \frac{b}{a} = 2T(a,b,c_1,c_2) \), \( b^{\frac{n-2}{2}} c_2 > a^{\frac{n-2}{2}} c_1 \), and \( u \) is smooth in \( \{a < |x| \leq b\} \), is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in \( \{a \leq |x| \leq b\} \).

Case 4: \( \ln \frac{b}{a} > 2T(a,b,c_1,c_2) \), and there is some \( m \in (a,b) \) such that

(i) \( u \) is smooth in each of \( \{a \leq |x| < m\} \) and \( \{m < |x| \leq b\} \),

(ii) \( u \) is \( C^{1,\frac{1}{k}} \) but not \( C^{1,\gamma} \) with \( \gamma > \frac{1}{k} \) in each of \( \{a \leq |x| \leq m\} \) and \( \{m \leq |x| \leq b\} \),

(iii) and the first radial derivative \( \partial_r u \) jumps across \( \{|x| = m\} \):

\[
\partial_r \ln u \bigg|_{r=m^-} = -\frac{n-2}{m} \quad \text{and} \quad \partial_r \ln u \bigg|_{r=m^+} = 0.
\]

Note that when \( \ln \frac{b}{a} = 2T(a,b,c_1,c_2) \), we have in view of the definition of \( T(a,b,c_1,c_2) \), \( p_a \) and \( p_b \) that \( b^{\frac{n-2}{2}} c_2 \neq a^{\frac{n-2}{2}} c_1 \).

Remark 1.14. It is clear from Theorem 1.13 (in Cases 1–3) that if \( u \) is a \( C^1 \) and radially symmetric solution to (1.2) in the viscosity sense in some open annulus \( \Omega \), then \( u \in C^\infty(\Omega) \).

Remark 1.15. In Case 4, the exact value of \( m \) is

\[
m = \sqrt{ab} \exp \left( \frac{1}{2} \int_{p_b - p}^{p_a - p} \left( 1 + e^{-2\eta - 2p} [1 - e^{\eta}]^{1/k} \right)^{-1/2} d\eta \right)
\]

where \( p \) is the solution to

\[
\ln \frac{b}{a} = \int_{p_b - p}^{0} \left( 1 + e^{-2\eta - 2p} [1 - e^{\eta}]^{1/k} \right)^{-1/2} d\eta + \int_{p_a - p}^{0} \left( 1 + e^{-2\eta - 2p} [1 - e^{\eta}]^{1/k} \right)^{-1/2} d\eta.
\]

The following question is related to Question 1.8.

**Question 1.16.** Suppose that \( n \geq 3 \), \( 2 \leq k \leq n \) and \( \Omega = \{a < |x| < b\} \subset \mathbb{R}^n \) with \( 0 < a < b < \infty \). Does there exist constants \( c_1, c_2 \) with \( \ln \frac{b}{a} > 2T(a,b,c_1,c_2) \) such that the problem (1.2) and (1.8) has a smooth sub-solution?
Recall that by Theorem 1.13, when \( \ln \frac{b}{a} > 2T(a, b, c_1, c_2) \), the problem (1.2) and (1.8) has no smooth solution.

For comparison, we recall here a result of Bo Guan [8] on the Dirichlet \( \sigma_k \)-Yamabe problem in the so-called positive case which states that the existence of a smooth sub-solution implies the existence of a smooth solution.

We conclude the introduction with one more question.

**Question 1.17.** Let \( n \geq 3, 2 \leq k \leq n \) and \( m \neq n - 1 \). Does there exist a smooth domain \( \Omega \subset \mathbb{R}^n \) such that the locally Lipschitz solution to (1.2)-(1.3) is \( C^2 \) away from a set \( \Sigma \) which has Hausdorff dimension \( m \)?

In Section 2 we prove all the results above except Theorem 1.5, whose proof is done in the appendix. Theorem 1.2 is proved first in Subsection 2.1. We then prove a lemma on the existence and uniqueness a non-standard boundary value problem for the ODE related to (1.2) in Subsection 2.3 and use it to prove Theorem 1.4 in Subsection 2.4 and Theorem 1.13 in Subsection 2.5.

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## 2 Proofs

### 2.1 Proof of Theorem 1.2

We will use the following lemma.

**Lemma 2.1.** For every symmetric \( n \times n \) matrix \( M \) with \( \lambda(M) \in \tilde{\Gamma}_2 \) and every unit vector \( m \in \mathbb{R}^n \), it holds

\[
M_{ij}(\delta_{ij} - m_im_j) \geq 0.
\]

**Proof.** Using an orthogonal transformation, we may assume without loss of generality that \( M \) is diagonal with diagonal entries \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). Then \( (\lambda_1, \ldots, \lambda_n) \in \tilde{\Gamma}_2 \).

It is well known that this implies \( \lambda_1 + \ldots + \lambda_n \geq 0 \). Now as

\[
M_{ij}(\delta_{ij} - m_im_j) = \sum_{\ell=1}^{n} \lambda_\ell - \sum_{\ell=1}^{n} \lambda_\ell m^2_\ell \geq \sum_{\ell=1}^{n} \lambda_\ell - \lambda_n \sum_{\ell=1}^{n} m^2_\ell = \sum_{\ell=1}^{n-1} \lambda_\ell,
\]

the conclusion follows. \( \square \)
We will use the following result on the mean curvatures of an immersed hypersurface with respect to two conformal metrics. Let \( \Omega \subset \mathbb{R}^n \) be an open set. Equip \( \Omega \) with the Euclidean metric \( \tilde{g} \) and a conformal metric \( \hat{g}_u := u^{\frac{4}{n-2}} \hat{g} \) where \( u \) is \( C^2 \). Let \( f : \Sigma^{n-1} \to \Omega \) be a smooth immersion of a compact manifold \( \Sigma^{n-1} \) into \( \Omega \). Let \( \tilde{u} = u \circ f, \hat{g} = f^* \hat{g} \). For every point \( p \in \Sigma \), let \( H_\Sigma(p) \) and \( H_{\Sigma,u}(p) \) denote the mean curvature vectors associated to \( f \) at \( f(p) \) and with respect to \( \hat{g} \) and \( \hat{g}_u \), respectively. To dispel confusion, we note that, in our notation, the mean curvature is the trace of the second fundamental form. Note that if \( \nu \) is a unit vector at \( f(p) \) normal to the image of a small neighborhood of \( p \), then

\[
\partial_\nu u(f(p)) + \frac{n-2}{2(n-1)} \hat{g}(H_\Sigma(p), \nu) u(f(p)) = \frac{n-2}{2(n-1)} \hat{g}_u(H_{\Sigma,u}(p), u^{-\frac{2}{n-2}} \nu) u^{\frac{n}{n-2}}. \quad (2.1)
\]

**Lemma 2.2.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \), \( n \geq 3 \), and \( f : \Sigma^{n-1} \to \Omega \) be a smooth immersion. If \( u \in C^2(\Omega) \) satisfies \( \lambda(-A^u) \in \hat{\Gamma}_2 \) in \( \Omega \), then

\[
\Delta_{\hat{g}} \tilde{u} + \frac{n-2}{4(n-1)} |H_{\Sigma,u}|_{\hat{g}_u}^2 \tilde{u}^{\frac{n+2}{n-2}} - \frac{n-2}{4(n-1)} |H_\Sigma|^2_{\hat{g}} \tilde{u} - \frac{1}{(n-2)u} |\nabla_{\hat{g}} \tilde{u}|^2 \geq 0 \text{ on } \Sigma.
\]

**Proof.** Fix some \( p \in \Sigma^{n-1} \) and let \( \nu \) be a unit vector at \( f(p) \) normal to the image of a small neighborhood of \( p \). Recall that

\[
A^u = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \left[ |\nabla^2 u| - \frac{n}{(n-2)u} \nabla u \otimes \nabla u + \frac{1}{(n-2)u} |\nabla u|^2 \mathbf{I} \right].
\]

Applying Lemma 2.1 with \( M = -\frac{n-2}{2} u^{\frac{n+2}{n-2}} A^u(f(p)) \) and \( m = \nu \) yields

\[
0 \leq \nabla_i \nabla_j u (\delta_{ij} - \nu_i \nu_j) - \frac{1}{(n-2)u} |\nabla u|^2 + \frac{n}{(n-2)u} |\partial_\nu u|^2.
\]

This means

\[
0 \leq \Delta_{\hat{g}} \tilde{u} + \hat{g}(H_\Sigma, \nu) \partial_\nu u \circ f + \frac{n-1}{(n-2)u} |\partial_\nu u \circ f|^2 - \frac{1}{(n-2)u} |\nabla_{\hat{g}} \tilde{u}|^2 \text{ on } \Sigma.
\]

Using (2.1) yields the conclusion. \( \square \)

**Proof of Theorem 1.2.** Suppose by contradiction that \( u \in C^2(\Omega) \) is such that \( \lambda(-A^u) \in \hat{\Gamma}_2 \) in \( \Omega \) and \((\Omega, u^{-\frac{2}{n-2}} \hat{g})\) admits a smooth minimal immersion \( f : \Sigma^{n-1} \to \Omega \) for some smooth compact manifold \( \Sigma^{n-1} \). Here we have renamed \( \hat{u} \) in the statement of the
theorem as \( u \) for notational convenience. Let \( \nu \) denote a continuous unit normal along \( \Sigma \). By Lemma 2.2, we have
\[
\Delta \tilde{g} \tilde{u} - \frac{n-2}{4(n-1)} |H_{\Sigma}|^2 \tilde{u} - \frac{1}{(n-2)\tilde{u}} |\nabla \tilde{g} \tilde{u}|^2 \geq 0 \text{ on } \Sigma.
\]
Integrating over \( \Sigma \), we thus have that \( H_{\Sigma} \equiv 0 \) and \( \tilde{u} \equiv \text{const} \) on \( \Sigma \). In particular, \( f : \Sigma^{n-1} \to \Omega \) is a minimal immersion with respect to \( \hat{g} \). This is impossible as there is no smooth minimal immersion in \( \mathbb{R}^n \) with codimension one.

### 2.2 Preliminary ODE analysis

By the uniqueness result in [7, 23], the solutions \( u \) in Theorems 1.4 and 1.13 are radially symmetric, \( u(x) = u(r) \) where \( r = |x| \).

As in [5, 32], we work on a round cylinder instead of \( \mathbb{R}^n \). Namely, let
\[
t = \ln r - \frac{1}{2} \ln(ab), \quad \xi(t) = -\frac{2}{n-2} \ln u(r) - \ln r
\]
so that \( \pi^{1,2}_2 \hat{g} = e^{-2\xi}(dt^2 + g_{S^{n-1}}) \). A direct computation gives that, at points where \( u \) is twice differentiable,
\[
\sigma_k(\lambda(-A^u)) = \frac{(-1)^k}{2^{k-1}} \left( \frac{n-1}{k-1} \right) e^{2k\xi}(1 - |\xi'|^2)^{k-1}[\xi'' + \frac{n-2k}{2k}(1 - |\xi'|^2)],
\]
where here and below ' denotes differentiation with respect to \( t \).

Note that, for \( k \geq 2 \), at points where \( u \) is twice differentiable, \( \lambda(-A^u) \in \Gamma_k \) if and only if \( \sigma_k(\lambda(-A^u)) > 0 \) and \( |\xi'| > 1 \). Indeed, if \( \sigma_k(\lambda(-A^u)) > 0 \) and \( |\xi'| > 1 \), then (2.2) implies \( \sigma_i(\lambda(-A^u)) > 0 \) for \( 1 \leq i \leq k \) and so \( \lambda(-A^u) \in \Gamma_k \). Conversely, if \( \lambda(-A^u) \in \Gamma_k \) for some \( k \geq 2 \), then \( \sigma_1(\lambda(-A^u)) > 0 \) and \( \sigma_k(\lambda(-A^u)) > 0 \) and \( \sigma_k(\lambda(-A^u)) > 0 \). Using (2.2), we see that the first two inequalities imply \( |\xi'| > 1 \).

By the same reasoning, we have, at points where \( u \) is twice differentiable, if \( \lambda(-A^u) \in \tilde{\Gamma}_2 \), then \( |\xi'| \geq 1 \).

We are thus led to study the differential equation
\[
e^{2k\xi}(1 - |\xi'|^2)^{k-1}[\xi'' + \frac{n-2k}{2k}(1 - |\xi'|^2)] = \frac{(-1)^kn}{2k},
\]
under the constraint that \( |\xi'| > 1 \).

It is well known (see [5, 32]) that (2.3) has a first integral, namely
\[
H(\xi, \xi') := e^{(2k-n)\xi}(1 - |\xi'|^2)^k - (-1)^k e^{-n\xi} \text{ is (locally) constant along } C^2 \text{ solutions.} \]
Figure 1: The contours of $H$ for $k = 2$, $n = 7$. Each radially symmetric viscosity solution to (1.2) lies on a single contour of $H$ but avoid the shaded region, i.e. the dotted parts of the contours of $H$ are excluded. Every smooth solution stays on one side of the shaded region. Every non-smooth solution jumps (on one contour) from the part below the shaded region to the part above the shaded region at a single non-differentiable point.

A plot of the contours of $H$ for $k = 2, n = 7$ is provided in Figure 1. See [5] for a more complete catalog.

Before moving on with the proofs of our results, we note the following statement.

**Remark 2.3.** As a consequence of Theorem 1.13, we have in fact that $H(\xi, \xi')$ is (locally) constant along viscosity solutions.

**Proof.** Fix $\tilde{a} < \tilde{b}$ in the domain of $u$ and apply Theorem 1.13 relative to the interval $[\tilde{a}, \tilde{b}]$ with $c_1 = u(\tilde{a})$ and $c_2 = u(\tilde{b})$. If we are in cases 1–3, $u$ is $C^2(\tilde{a}, \tilde{b})$ and so $H(\xi, \xi')$ is constant in $\{\tilde{a} < r < \tilde{b}\}$. Suppose we are in case 4. We have that $u$ is $C^2$ in $(\tilde{a}, m) \cup (m, \tilde{b})$ for some $m$ and so $H(\xi, \xi')$ is constant in each of $\{\tilde{a} < r < m\}$ and $\{m < r < \tilde{b}\}$. Also, as $u$ is $C^1$ in each of $(\tilde{a}, m]$ and $[m, \tilde{b})$, we have by assertion (iii) in case 4 that

$$\lim_{r \to m^-} H(\xi(t), \xi'(t)) = H(\xi(t(m)), 1) = H(\xi(t(m)), -1) = \lim_{r \to m^+} H(\xi(t), \xi'(t)).$$
Hence \( H(\xi, \xi') \) is also constant in \( \{ \tilde{a} < r < \tilde{b} \} \).

### 2.3 A lemma

**Lemma 2.4.** For any \( T > 0 \), there exists a unique classical solution \( \xi \in C^\infty(0, T) \cap C_{\text{loc}}^{1,1}(0, T) \) to \( (2.3) \) in \( (0, T) \) such that

\[
\lim_{t \to T^-} \xi(t) = -\infty, \tag{2.4}
\]

\[
\xi'(0) = -1, \quad \xi'(t) < -1 \text{ in } (0, T). \tag{2.5}
\]

Furthermore, for every \( \gamma \in (\frac{1}{k}, 1] \), \( \xi \notin C_{\text{loc}}^{1,\gamma}([0, T]) \).

**Proof.** We use ideas from [5].

**Step 1:** We start by collecting relevant facts from [5] about the classical solution \( \xi_{p,q}(t) \) to \( (2.3) \) satisfying the initial condition \( \xi_{p,q}(0) = p \) and \( \xi'_{p,q}(0) = q \) for \( p \in \mathbb{R}, q \in (-\infty, -1) \) on its maximal interval of unique existence \( I_{p,q} = (T_{p,q}^-, T_{p,q}^+) \subset \mathbb{R} \).

Note that, since \( \xi'_{p,q}(0) = q < -1 \), it follows from \( (2.3) \) that, for as long as \( \xi_{p,q} \) remains \( C^2 \), \( \xi'_{p,q} < -1 \). Thus, as \( H(\xi_{p,q}, \xi'_{p,q}) = H(p, q) \), we have in \( I_{p,q} \) that

\[
\xi'_{p,q} = -\left\{ 1 + e^{-2\xi_{p,q}} \left[ 1 + (-1)^k H(p, q) e^{n\xi_{p,q}} \right]^{1/k} \right\}^{1/2}. \tag{2.6}
\]

By [5](Theorem 1, Cases II.2 and II.3 for even \( k \) and Theorem 2, Cases II.2 and II.3 for odd \( k \)), we have that \( T_{p,q} \) is finite (corresponding to \( r_+ \) being finite in the notation of [5]). Furthermore,

\[
\lim_{t \to T_{p,q}^-} \xi_{p,q}(t) = -\infty. \tag{2.7}
\]

By \( (2.6) \) we thus have

\[
T_{p,q} = \int_{-\infty}^p \left\{ 1 + e^{-2\xi} [1 - |H(p, q)| e^{n\xi}]^{1/k} \right\}^{-1/2} d\xi. \tag{2.8}
\]

In this proof, we will only need to consider the case that \( (-1)^k H(p, q) < 0 \). Then by [5](Theorem 1, Case II.2 for even \( k \) and Theorem 2, Case II.2 for odd \( k \)), we have that \( \bar{T}_{p,q} \) is also finite (corresponding to \( r_- \) being finite in the notation of [5]) and

\[
\lim_{t \to \bar{T}_{p,q}^+} \xi_{p,q}(t) \text{ is finite,} \quad \lim_{t \to \bar{T}_{p,q}^+} \xi'_{p,q}(t) = -1, \quad \lim_{t \to \bar{T}_{p,q}^+} \xi''_{p,q}(t) = -\infty. \tag{2.9}
\]
Using (2.9) as well as the fact that \( H(\xi_{p,q}, \xi_{p,q}') = H(p, q) \) and \( \xi_{p,q} \) is decreasing, we have in \( I_{p,q} \) that
\[
\xi_{p,q} < \lim_{t \to T^+_{p,q}} \xi_{p,q}(t) = -\frac{1}{n} \ln |H(p, q)|.
\] (2.10)

Differentiating (2.6), we see that, as \( t \to T^+_{p,q} \),
\[
\lim_{t \to T^+_{p,q}} (t - T_{p,q})^{\frac{k-1}{k}} \xi_{p,q}''(t) \text{ exists and belongs to } (-\infty, 0).
\]

Thus \( \xi_{p,q} \) extends to a \( C^{1,\frac{1}{k}} \) function in a neighborhood of \( T_{p,q} \) and \( \xi_{p,q} \) does not extend to a \( C^{1,\gamma} \) function in any neighborhood of \( T_{p,q} \).

Before moving on to the next stage, we note that, in view of (2.6),
\[
T_{p,q} - T_{p,q} = \int_{-\infty}^{-\frac{1}{n} \ln |H(p,q)|} \left\{ 1 + e^{-2\xi} [1 - |H(p,q)| e^{n\xi}]^{1/k} \right\}^{-1/2} d\xi
= \int_{-\infty}^{0} \{ 1 + |H(p,q)|^{2/n} e^{-2\eta} [1 - e^{n\eta}]^{1/k} \}^{-1/2} d\eta.
\] (2.11)

In particular, then length of \( I_{p,q} \) depends only on \( n, k \) and the value of \( H(p,q) \), rather than \( p \) and \( q \) themselves.

**Step 2:** We now define for each given \( p \in \mathbb{R} \) a unique classical solution \( \xi_p \) to (2.3) in some maximal interval \((0, T_p)\) satisfying \( \xi_p(0) = p, \xi_p'(0) = -1 \) and \( \xi_p' < -1 \) in \((0, T_p)\).

It is clear that \((-1)^k H(p, -1) = -e^{-np} < 0\), and as \( \partial_p H(p, -1) = (-1)^k ne^{-np} \neq 0\).

By the implicit function theorem, there exist \( \tilde{p} \) and \( \tilde{q} < -1 \) such that \( H(\tilde{p}, \tilde{q}) = H(p, -1) \).

Note that this implies
\[-e^{-np} = (-1)^k H(\tilde{p}, \tilde{q}) > -e^{-n\tilde{p}} \text{ and so } \tilde{p} < p.\]

Let
\[\xi_p(t) = \xi_{\tilde{p}, \tilde{q}}(t + T_{\tilde{p}, \tilde{q}}) \text{ and } T_p = T_{\tilde{p}, \tilde{q}} - T_{\tilde{p}, \tilde{q}}.\]

By Step 1, it is readily seen that \( \xi_p \) is smooth in \((0, T_p)\), belongs to \( C^{1,\frac{1}{k}}_{loc}([0, T_p]) \) and
Thus, for any given $T > \xi$ and $\hat{p} < p < p$, solves (2.15), we see that, as a function of $p < p$, and satisfies $p < p$ in each of $(0, T)$ such that $\xi(T) = \xi(0) = -1$, $0 > \xi''(t) = O(t^{-\frac{k-1}{k}})$ as $t \to 0^+$, and $T_p = \int_0^1 \left\{ 1 + e^{-2\eta - 2p} \left[ 1 - e^{\eta\eta} \right]^{1/k} \right\}^{-1/2} d\eta$. (2.15)

We claim that $\xi_p$ is unique in the sense that if $\xi_p \in C^2(0, T_p) \cap C^1([0, T_p])$ is a solution to (2.3) in some maximal interval $(0, \hat{T})_p$ satisfying $\xi_p(0) = p, \hat{T}_p(0) = -1$ and $\xi'_p < -1$ in $(0, \hat{T}_p)$, then $T_p = \hat{T}_p$ and $\xi_p \equiv \hat{\xi}_p$. To see this, note that, $\hat{\xi}_p(t) = \xi_{\hat{\xi}_p(s), \xi_p(s)}(t - s)$ for all $t, s \in (0, \hat{T}_p)$, since they both satisfy the same ODE in $t$ and agree up to first derivatives at $t = s$. By Step 1, $\hat{\xi}_p(t) \to -\infty$ as $t \to \hat{T}_p^-$, and so, as $\hat{p} < p$ and $\hat{\xi}_p(0) = p$, there exists $t_0 \in (0, \hat{T}_p)$ such that $\hat{\xi}_p(t_0) = \hat{p}$. This implies that $H(\hat{p}, \hat{\xi}_p(t_0)) = H(\hat{\xi}_p(t_0)) = H(p, -1) = H(\hat{p}, \hat{q})$ and so $\hat{\xi}_p(t_0) = \hat{q}$. We deduce that $t_0 = -\hat{T}_p\hat{q}, \hat{T}_p = T_p$ and $\hat{\xi}_p \equiv \xi_{\hat{p}, \hat{q}}(-t_0) \equiv \xi_p$, as claimed.

**Step 3:** From (2.15), we see that, as a function of $p$, $T_p$ is continuous and increasing and satisfies

$$\lim_{p \to -\infty} T_p = 0 \text{ and } \lim_{p \to \infty} T_p = \infty.$$ 

Thus, for any given $T > 0$, there is a unique $p(T)$ such that $T_{p(T)} = T$. The solution $\xi_{p(T)}$ to (2.3) gives the desired solution.

**2.4 Proof of Theorem 1.4**

Let $T = \frac{1}{2} \ln \frac{b}{a}$ and $t = \ln r - \frac{1}{2} \ln(ab)$. We need to exhibit a function $\xi : (-T, T) \to \mathbb{R}$ such that $\xi$ is smooth in each of $(0, T)$ and $(-T, 0)$, is $C^{1, \gamma}_{\text{loc}}$ but not $C^{1, \gamma}_{\text{loc}}$ for any $\gamma > \frac{1}{k}$ in each of $[0, T)$ and $(-T, 0]$, the function $u$ defined by

$$u(r) = \exp \left[ -\frac{n-2}{2} \left( \xi(t) + \ln r \right) \right]$$

solves (1.2)-(1.3) in $\{a < r = |x| < b\}$ in the viscosity sense, and

(i) $\lim_{t \to \pm T} \xi(t) = -\infty$. 

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(ii) $\xi'(0^-) = 1$, $\xi'(0^+) = -1$,
(iii) and $|\xi'| > 1$ in $(-T, 0) \cup (0, T)$.

Indeed, let $\xi^T : [0, T) \to \mathbb{R}$ be the solution obtained in Lemma 2.4 and define

$$
\xi(t) = \begin{cases} 
\xi^T(t) & \text{if } 0 \leq t < T, \\
\xi^T(-t) & \text{if } -T < t < 0.
\end{cases}
$$

It is clear that $\xi$ satisfies all the listed requirements except for the statement that $u$ satisfies (1.2) in the viscosity sense at $r = \sqrt{ab}$. It remains to demonstrate, for any given $x_0$ with $|x_0| = \sqrt{ab}$, that

(a) if $\varphi$ is $C^2$ near $x_0$ and satisfies $\varphi \geq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$, then $\lambda(-A\varphi(x_0)) \in \Gamma_k$ and $\sigma_k(\lambda(-A\varphi(x_0))) \geq 2^{-k}\left( \frac{n}{k} \right)$,

(b) and if $\varphi$ is $C^2$ near $x_0$ and satisfies $\varphi \leq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$, then either $\lambda(-A\varphi(x_0)) \notin \Gamma_k$ or $\lambda(-A\varphi(x_0)) \in \Gamma_k$ but $\sigma_k(\lambda(-A\varphi(x_0))) \leq 2^{-k}\left( \frac{n}{k} \right)$.

Without loss of generality, we may assume that $x_0 = (\sqrt{ab}, 0, \ldots, 0)$.

Since $\partial_r \ln u|_{r=\sqrt{ab}} = -\frac{n-2}{\sqrt{ab}} < 0 = \partial_r \ln u|_{r=\sqrt{ab}^+}$, there is no $C^2$ function $\varphi$ such that $\varphi \geq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$. Therefore (a) holds.

Suppose now that $\varphi$ is a $C^2$ function such that $\varphi \leq u$ near $x_0$ and $\varphi(x_0) = u(x_0)$. As $u$ is radial, this implies that

$$
-\frac{n-2}{\sqrt{ab}} = \partial_{x_1} \ln u|_{r=\sqrt{ab}^-} \leq \partial_{x_1} \ln \varphi(x_0) \leq \partial_{x_1} \ln u|_{r=\sqrt{ab}^+} = 0,
$$

(2.16)

$$
\partial_{x_i} \ln \varphi(x_0) = \ldots = \partial_{x_n} \ln \varphi(x_0) = 0,
$$

(2.17)

$$
\left( \partial_{x_i} \partial_{x_j} \varphi(x_0) - \frac{1}{\sqrt{ab}} \partial_{x_1} \varphi(x_0) \delta_{ij} \right)_{2 \leq i,j \leq n} \leq 0.
$$

(2.18)

(For (2.18), note that the matrix on the left hand side is the Hessian of $\varphi|_{\partial B_{\sqrt{ab}}}$ with respect to the metric induced on $\partial B_{\sqrt{ab}}$ by the Euclidean metric.) Now define $\bar{\varphi}(x) = \bar{\varphi}(|x|) = \varphi(|x|, 0, \ldots, 0)$, $t = \ln r - \frac{1}{2} \ln(ab)$ and $\tilde{\xi}(t) = -\frac{2}{n-2} \ln \varphi(r) - \ln r$. By (2.16), we have that $|\frac{d}{dt} \tilde{\xi}(0)| \leq 1$ and so $\lambda(-A\bar{\varphi}(x_0)) \notin \Gamma_k$.

Let $O$ denote the diagonal matrix with diagonal entries $1, -1, \ldots, -1$. Note that, in block form,

$$
\nabla^2 \varphi(x_0) + O\nabla^2 \varphi(x_0)O = 2 \left( \frac{\partial^2_{x_1} \varphi(x_0)}{0} \left| \frac{\partial_{x_1} \partial_{x_j} \varphi(x_0)}{\partial_{x_i} \partial_{x_j} \varphi(x_0)}_{2 \leq i,j \leq n} \right. \right).
$$
Thus, by (2.18),
\[ \nabla^2 \varphi(x_0) + O^t \nabla^2 \varphi(x_0)O \leq 2 \left( \frac{\partial^2_x \varphi(x_0)}{0} \right) = 2 \nabla^2 \varphi(x_0). \]

Also, \( \varphi(x_0) = \bar{\varphi}(x_0) \) and, in view of (2.17), \( \nabla \varphi(x_0) = \nabla \bar{\varphi}(x_0) \). Hence
\[ -A^\varphi(x_0) - O^t A^\varphi(x_0)O \leq -2A^\varphi(x_0). \]

As \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \), it follows that \( \lambda(-A^\varphi(x_0) - O^t A^\varphi(x_0)O) \notin \Gamma_k \). Since the set of matrices with eigenvalues belonging to \( \Gamma_k \) is a convex cone (see e.g. [22, Lemma B.1]), we thus have that \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \) or \( \lambda(-O^t A^\varphi(x_0)O) \notin \Gamma_k \). Since \( O \) is orthogonal, we deduce that \( \lambda(-A^\varphi(x_0)) \notin \Gamma_k \). We have verified (b) and thus completed the proof. \( \square \)

2.5 Proof of Theorem 1.13

As mentioned before, the uniqueness of solution follows from [7, 23]. We proceed to construct a radially symmetric solution with the indicated properties.

Let \( T = \frac{1}{2} \ln \frac{b}{a}, p_a = -\frac{2}{n+2} \ln c_1 - \ln a \) and \( p_b = -\frac{2}{n+2} \ln c_2 - \ln b \). We will only consider the case that \( p_a \geq p_b \) (which is equivalent to \( \frac{b^{n+2}}{a} c_2 \geq a^{n+2} c_1 \)). (The case \( p_a < p_b \) can be treated using an inversion about \( |x| = \sqrt{ab} \).) We then have
\[ T(a, b, c_1, c_2) = \frac{1}{2} \int_{p_b-p_a}^0 \left\{ 1 + e^{-2\eta-2p_a} \left[ 1 - e^{\eta p_a} \right]^{1/k} \right\}^{-1/2} d\eta. \]
\[ = \frac{1}{2} \int_{p_a}^{p_b} \left\{ 1 + e^{-2\xi} \left[ 1 - e^{\xi (p_a)} \right]^{1/k} \right\}^{-1/2} d\xi. \]

(i) Suppose that \( T < T(a, b, c_1, c_2) \). We show that Case 1 holds.

Note that \( H(p_a, -1) = -(-1)^k e^{-np_a} \). Thus as \( T < T(a, b, c_1, c_2) \) and \( (-1)^k H(p_a, \cdot) \) is decreasing in \( (-\infty, -1) \), we can find \( q_a < -1 \) such that
\[ T = \frac{1}{2} \int_{p_a}^{p_b} \left\{ 1 + e^{-2\xi} \left[ 1 - (-1)^k H(p_a, q_a) e^{n\xi} \right]^{1/k} \right\}^{-1/2} d\xi. \quad (2.19) \]

Recall the solution \( \xi_{p_a, q_a} \) to (2.3) considered in the proof of Lemma 2.4. By (2.8), we have that \( 2T < T_{p_a, q_a} \). We then deduce from (2.6) and (2.19) that
\[ \xi_{p_a, q_a}(2T) = p_b. \]
It thus follows that \( \xi(t) = \xi_{p_a,q_a}(t + T) \) is smooth in \([-T,T]\), satisfies \((2.3)\) and \(\xi' < -1\) in \((-T,T)\), as well as \(\xi(-T) = p_a\) and \(\xi(T) = p_b\). Returning to \(u = \exp\left(-\frac{n-2}{2}(\xi(-ln r - \frac{1}{2}\ln(ab)) + ln r)\right)\) we obtain the conclusion.

(ii) Suppose that \(T = T(a,b,c_1,c_2)\). We show that Case 3 holds.

Recalling the definition of \(T(a,b,c_1,c_2)\), we see that as \(T > 0\), we have \(p_a \neq p_b\). As \(p_a \geq p_b\), we have \(p_a > p_b\). We can now follow the argument in (i) with \(\xi_{p_a,q_a}\) replaced by \(\xi_{p_a}\) (defined in the proof of Lemma 2.4) to reach the conclusion. We omit the details.

(iii) Suppose that \(T > T(a,b,c_1,c_2)\). We show that Case 4 holds.

In this case, we select \(p \geq p_a(\geq p_b)\) such that

\[
T = \frac{1}{2} \int_{p_b-p}^0 \left\{1 + e^{-2\eta - 2p} [1 - e^{\eta p}]^{1/k}\right\}^{-1/2} d\eta + \frac{1}{2} \int_{p_a-p}^0 \left\{1 + e^{-2\eta - 2p} [1 - e^{\eta p}]^{1/k}\right\}^{-1/2} d\eta
\]

Such \(p\) exists as the right hand side tends to \(T(a,b,c_1,c_2)\) when \(p \to p_a\) and diverges to \(\infty\) as \(p \to \infty\). Recall the solution \(\xi_p\) defined in the proof of Lemma 2.4. Let

\[
T_+ = \frac{1}{2} \int_{p_b-p}^0 \left\{1 + e^{-2\eta - 2p} [1 - e^{\eta p}]^{1/k}\right\}^{-1/2} d\eta
\]

and

\[
T_- = \frac{1}{2} \int_{p_a-p}^0 \left\{1 + e^{-2\eta - 2p} [1 - e^{\eta p}]^{1/k}\right\}^{-1/2} d\eta.
\]

Then \(2T_\pm < T_b\) and the function \(\xi_p\) satisfies \(\xi_p(2T_+) = p_b\) and \(\xi_p(2T_-) = p_a\).

We then let

\[
\xi(t) = \begin{cases} 
\xi_p(T_+ - T_- + t) & \text{if } -T_+ + T_- \leq t < T, \\
\xi_p(-T_+ + T_- - t) & \text{if } -T < t < -T_+ + T_-.
\end{cases}
\]

We can then proceed as in the proof of Theorem 1.4 to show that \(\xi\) is the desired solution. \(\square\)
We abbreviate \( u^{4-\tau} g \) as \( g_u \). For small \( \tau > 0 \), let
\[
A^\tau g_u = A_{g_u} + \tau tr_{g_u}(A_{g_u})g_u
\]
\[= -\frac{2}{n-2}u^{-1}(\nabla^2 g u + \tau \Delta g u g) + \frac{2n}{(n-2)^2}u^{-2}du \otimes du - \frac{2}{(n-2)^2}u^{-\frac{n\tau}{n-2}}|\nabla g u|^2 g
\]
\[+ A_g + \frac{\tau}{2(n-1)}R_g g.
\]

By \([12, \text{Theorem 1.4}]\), we have for all sufficiently small \( \tau > 0 \) that the problem
\[
\sigma_k \left( \lambda \left( - A^\tau_{g_{u\tau}} \right) \right) = 2^{-k} \binom{n}{k}, \quad \lambda \left( - A^\tau_{g_{u\tau}} \right) \in \Gamma_k, \quad u_{\tau} > 0 \quad \text{in } M, \tag{A.1}
\]
has a unique smooth solution \( u_{\tau} \). Furthermore, by \([12, \text{Propositions 3.2 and 4.1}]\), the family \( \{u_{\tau}\} \) is bounded in \( C^1(M) \) as \( \tau \to 0 \). \( (C^2 \text{ bounds for } u_{\tau} \text{ were also proved in } [12], \text{but these bounds are unbounded as } \tau \to 0.) \) Hence, along some sequence \( \tau_i \to 0 \), \( u_{\tau_i} \) converges uniformly to some \( u \in C^{0,1}(M) \). To conclude, we show that \( u \) is a viscosity solution to \((1.7)\).

For notational convenience, we rename \( u_{\tau} \) as \( u_i \).

Step 1: We show that \( u \) is a sub-solution to \((1.7)\) at \( \bar{x} \). More precisely, we show that for every \( \varphi \in C^2(M) \) such that \( \varphi \geq u \) on \( M \) and \( \varphi(\bar{x}) = u(\bar{x}) \) there holds that
\[
\lambda \left( - A_{g_{u\varphi}}(\bar{x}) \right) \in \left\{ \lambda \in \Gamma_k \left| \sigma_k(\lambda) \geq 2^{-k} \binom{n}{k} \right. \right\} = \overline{S}_k =: \overline{S}. \tag{A.2}
\]

Here \( d_g \) denotes the distance function of \( g \) and \( B_\delta(\bar{x}) \) denote the open geodesic ball of radius \( \delta \) and centered at \( \bar{x} \) with respect to \( g \). Fix some arbitrary small \( \delta > 0 \) so that \( \varphi_\delta := \varphi + \delta d_g(\cdot, \bar{x})^2 \) is \( C^2 \) in \( B_\delta(\bar{x}) \).

Note that
\[
\varphi_\delta = \varphi + \delta^3 \geq u + \delta^3 \quad \text{on } \partial B_\delta(\bar{x}) \quad \text{and} \quad \varphi_\delta(\bar{x}) = u(\bar{x}). \tag{A.3}
\]

Select \( x_i, \delta \in B_\delta(\bar{x}) \) such that
\[
(\varphi_\delta - u_i)(x_i, \delta) = \inf_{B_\delta(\bar{x})} (\varphi_\delta - u_i) =: m_{i, \delta}.
\]
By (A.3) and the uniform convergence of $u_i$ to $u$, we have that $x_{i,\delta} \in B_\delta(\bar{x})$. It follows that

$$\nabla g(\varphi_\delta - u_i)(x_{i,\delta}) = 0, \quad \nabla^2 g(\varphi_\delta - u_i)(x_{i,\delta}) \geq 0$$

and so

$$-A_{g_{\psi_\delta}}^{\tau i}_{g_{\psi_\delta}}(x_{i,\delta}) \geq -A_{g_{u_i}}^{\tau i}(x_{i,\delta}).$$

Recalling (A.1), we hence have

$$\lambda \left( -A_{g_{\psi_\delta}}^{\tau i}_{g_{\psi_\delta}}(x_{i,\delta}) \right) \in \mathbb{S}.$$  \hfill (A.4)

On the other hand, as $\bar{x}$ is the unique minimum point of $\varphi_\delta - u$ in $B_\delta(\bar{x})$, we have $x_{i,\delta} \to \bar{x}$ and $m_{i,\delta} \to 0$ as $i \to \infty$. We can now pass $i \to \infty$ in (A.4) to obtain

$$\lambda \left( -A_{g_{\psi_\delta}}(\bar{x}) \right) \in \mathbb{S}.$$

Since $\delta$ is arbitrary, this proves (A.2) after sending $\delta \to 0$.

**Step 2:** We show that $u$ is a super-solution to (1.7) at $\bar{x}$, i.e. if $\varphi \in C^2(M)$ is such that $\varphi \leq u$ on $M$ and $\varphi(\bar{x}) = u(\bar{x})$, then

$$\lambda \left( -A_{g_{\varphi}}(\bar{x}) \right) \in \mathbb{R}^n \setminus \left\{ \lambda \in \Gamma_k \bigg| \sigma_k(\lambda) > 2^{-k} \left( \frac{n}{k} \right) \right\} = \mathbb{S}_k =: \mathbb{S}.$$  \hfill (A.5)

The proof is analogous to that in Step 1. Fix some arbitrary small $\delta > 0$ so that $\hat{\varphi}_\delta := \varphi - \delta = \varphi - \delta d_g(\cdot, \bar{x})^2$ is $C^2$ in $B_\delta(\bar{x})$. Clearly

$$\hat{\varphi}_\delta \leq u - \delta^3$$

on $\partial B_\delta(\bar{x})$ and $\hat{\varphi}_\delta(\bar{x}) = u(\bar{x})$.

We next select $\hat{x}_{i,\delta} \in B_\delta(\bar{x})$ such that

$$(\hat{\varphi}_\delta - u_i)(\hat{x}_{i,\delta}) = \sup_{B_\delta(\bar{x})} (\hat{\varphi}_\delta - u_i) =: \hat{m}_{i,\delta}.$$

As before, we have $\hat{x}_{i,\delta} \in B_\delta(\bar{x})$, $\nabla g(\hat{\varphi}_\delta - u_i)(\hat{x}_{i,\delta}) = 0$, $\nabla^2 g(\hat{\varphi}_\delta - u_i)(\hat{x}_{i,\delta}) \leq 0$ and

$$-A_{g_{\hat{\varphi}_\delta}}^{\tau i}_{g_{\hat{\varphi}_\delta}}(\hat{x}_{i,\delta}) \leq -A_{g_{u_i}}^{\tau i}(\hat{x}_{i,\delta}).$$

By (A.1), we hence have

$$\lambda \left( -A_{g_{\hat{\varphi}_\delta}}^{\tau i}_{g_{\hat{\varphi}_\delta}}(\hat{x}_{i,\delta}) \right) \in \mathbb{S}.$$  \hfill (A.6)

Also, as $\hat{x}_{i,\delta} \to \bar{x}$ and $\hat{m}_{i,\delta} \to 0$ as $i \to \infty$, we can first pass $i \to \infty$ and then $\delta \to 0$ in (A.6) to reach (A.5).
References

[1] L. Andersson, P. T. Chruściel, and H. Friedrich, On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein’s field equations, Comm. Math. Phys., 149 (1992), pp. 587–612.

[2] L. Caffarelli, Y. Y. Li, and L. Nirenberg, Some remarks on singular solutions of nonlinear elliptic equations III: viscosity solutions including parabolic operators, Comm. Pure Appl. Math., 66 (2013), pp. 109–143.

[3] L. Caffarelli, L. Nirenberg, and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, Acta Math., 155 (1985), pp. 261–301.

[4] S.-Y. A. Chang, M. J. Gursky, and P. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. of Math. (2), 155 (2002), pp. 709–787.

[5] S.-Y. A. Chang, Z.-C. Han, and P. Yang, Classification of singular radial solutions to the $\sigma_k$ Yamabe equation on annular domains, J. Differential Equations, 216 (2005), pp. 482–501.

[6] Y. Ge and G. Wang, On a fully nonlinear Yamabe problem, Ann. Sci. École Norm. Sup. (4), 39 (2006), pp. 569–598.

[7] M. d. M. González, Y. Li, and L. Nguyen, Existence and uniqueness to a fully nonlinear version of the Loewner-Nirenberg problem, Commun. Math. Stat., 6 (2018), pp. 269–288.

[8] B. Guan, Conformal metrics with prescribed curvature functions on manifolds with boundary, Amer. J. Math., 129 (2007), pp. 915–942.

[9] ———, Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary, Int. Math. Res. Not. IMRN, (2008), pp. Art. ID rnm 105, 25.

[10] P. Guan and G. Wang, Local estimates for a class of fully nonlinear equations arising from conformal geometry, Int. Math. Res. Not., (2003), pp. 1413–1432.

[11] M. Gursky, J. Streets, and M. Warren, Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary, Calc. Var. Partial Differential Equations, 41 (2011), pp. 21–43.
[12] M. J. Gursky and J. A. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J., 52 (2003), pp. 399–419.

[13] ———, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, Ann. of Math. (2), 166 (2007), pp. 475–531.

[14] J. Hass and P. Scott, *The existence of least area surfaces in 3-manifolds*, Trans. Amer. Math. Soc., 310 (1988), pp. 87–114.

[15] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[16] A. Li and Y. Y. Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., 56 (2003), pp. 1416–1464.

[17] ———, *On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe*, Acta Math., 195 (2005), pp. 117–154.

[18] J. Li and W. Sheng, *Deforming metrics with negative curvature by a fully nonlinear flow*, Calc. Var. Partial Differential Equations, 23 (2005), pp. 33–50.

[19] Y. Y. Li, *Local gradient estimates of solutions to some conformally invariant fully nonlinear equations*, https://arxiv.org/abs/math/0605559v2, (2006).

[20] ———, *Local gradient estimates of solutions to some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., 62 (2009), pp. 1293–1326.

[21] Y. Y. Li and L. Nguyen, *A compactness theorem for fully nonlinear Yamabe problem under a lower Ricci curvature bound*, J. Funct. Anal., 266 (2014), pp. 2741–3771.

[22] ———, *Existence and uniqueness of Green’s function to a nonlinear Yamabe problem*, (2020). https://arxiv.org/abs/2001.00993.

[23] Y. Y. Li, L. Nguyen, and B. Wang, *Comparison principles and Lipschitz regularity for some nonlinear degenerate elliptic equations*, Calc. Var. Partial Differential Equations, 57 (2018), pp. Art. 96, 29.

[24] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, in Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 245–272.
[25] R. Mazzeo, *Regularity for the singular Yamabe problem*, Indiana Univ. Math. J., 40 (1991), pp. 1277–1299.

[26] W. H. Meeks, III and S. T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2), 112 (1980), pp. 441–484.

[27] H. Samelson, *Orientability of hypersurfaces in $R^n$*, Proc. Amer. Math. Soc., 22 (1969), pp. 301–302.

[28] R. Schoen and S. T. Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2), 110 (1979), pp. 127–142.

[29] W.-M. Sheng, N. S. Trudinger, and X.-J. Wang, *The Yamabe problem for higher order curvatures*, J. Differential Geom., 77 (2007), pp. 515–553.

[30] Z. Sui, *Complete conformal metrics of negative Ricci curvature on Euclidean spaces*, J. Geom. Anal., 27 (2017), pp. 893–907.

[31] N. S. Trudinger and X.-J. Wang, *On Harnack inequalities and singularities of admissible metrics in the Yamabe problem*, Calc. Var. Partial Differential Equations, 35 (2009), pp. 317–338.

[32] J. A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J., 101 (2000), pp. 283–316.