1 Introduction

In 1958 Nagata \cite{nagata} gave an ingenious argument that demonstrated the existence of counterexamples to Hilbert’s Fourteenth Problem. Recall that the original problem is the following: Let $K = F(x_1, x_2, \ldots, x_n)$ be the function field of affine $n$-space $V = F^n$ over an algebraically closed field $F$, and suppose $L \subset K$ is any subfield. Then the question is: Is $A = L \cap F[x_1, x_2, \ldots, x_n]$ a finitely generated $F$-algebra? In most cases of interest, $L$ is the field of invariants of an algebraic group $G$ acting linearly on $V$, and $A$ becomes the ring of invariant regular functions on $V$. Certainly, if $G$ is reductive, the answer is yes, a result due to Hilbert himself for $\text{char}(F) = 0$, but it is much more subtle in positive characteristic. More generally for $G$ reductive $O(X)^G$ is a finitely generated $F$-algebra for any affine variety $X$ over $F$ on which $G$ acts. On the other hand, if $G = F^+$ is the additive group, due to more recent work of for example Daigle and Freudenburg \cite{daigle}, there are actions of $G$ on $V$ for certain values of $n = \dim V$ (e.g. $n = 5$), such that $O(V)^G$ is not finitely generated, even in characteristic zero. By a well known result of Weitzenböck these actions cannot be linear, since in characteristic zero linear actions of $F^+$ always extend to representations of $SL_2(F)$. On the other hand it was not until Nagata found his counterexample, that the general question of finite generation of invariants for a representation of an algebraic group was settled. The purpose of this paper is to investigate a byproduct of the remarkably involved methods which Nagata used to prove the following result.

\begin{theorem}[Nagata’s counterexample] There is a linear action of $(F^+)^{13}$ on $V = F^{32}$, such that the ring of invariant functions is not finitely generated.
\end{theorem}

For quite some time 13 and 32 were the lowest known dimensions. Recently, Steinberg \cite{steinberg} improved this result to 6 and 18, i.e. $(F^+)^{6}$ acting on $F^{18}$, using

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similar methods. Much of the work in the present paper has been inspired by Steinberg’s result.

The idea of the proof of theorem 1 is essentially the following: Nagata defined a sequence of ideals \( \{a_m\}_{m \geq 0} \) with \( a_m a_n \subset a_{m+n} \) of the polynomial ring in three variables \( F[x, y, z] \) which satisfies the following condition: for each positive integer \( m \), there are other positive integers \( n > m \) and \( k \) such that \( a_k \neq a_{kn} \). Moreover the invariants of the product of \( G = (F^+)^{13} \) with a certain torus \( T = (F^*)^{16} \), centralizing \( G \), are isomorphic with \( R = \bigoplus_{m \geq 0} a_m t^{-m} \subset F[x, y, z, t, t^{-1}] \). Since \( T \) is reductive, the invariants of \( G \) are finitely generated if and only if the invariants of \( G \times T \) are. The cited condition above implies the failure of an Artin-Rees type lemma for \( R \), hence \( R \) is not finitely generated.

The ideal \( a_m \) is the set of polynomials in \( F[x, y, z] \), vanishing at 16 generic lines in \( F^3 \) with multiplicity \( m \). As one sees from this discussion, certain properties of the ideals \( a_m \) are essential to Nagata’s argument. He himself devoted considerable effort to the general problem of this paper in dimension 2: to determine the (dimension of the) ideal of homogeneous polynomials vanishing at finitely many points in \( \mathbb{P}^2 \) to a certain multiplicity, and algorithmically solved the problem for less than nine points. We will present below various aspects of this problem, including our version of his algorithm.

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2 The general problem

Let \( F \) be an algebraically closed field. Let \( V = F^{n+1} \) be affine \((n + 1)\)-space over \( F \), and denote the projective space of \( V \) by \( \mathbb{P}(V) \). Let \( \mathcal{O}(V) \) be the space of regular functions (i.e. polynomials) on \( V \), equipped with the usual grading \( \mathcal{O}(V) = \bigoplus_{d \geq 0} \mathcal{O}(V)_d \). Similarly, \( D(V) \) is the graded algebra of differential operators on \( \mathcal{O}(V) \) with values in \( \mathcal{O}(V) \) and constant coefficients.

Now let \( p \in \mathbb{P}(V) \) be arbitrary. For a positive integer \( m \), we say a homogeneous function \( f \in \mathcal{O}(V) \) vanishes to order \( m \) at \( p \), if \( f \in m_p^m \), where \( m_p \subset \mathcal{O}(V) \) is the homogeneous prime ideal of height one defining \( p \).

Remark 2. Throughout the paper, we will use several different notions of vanishing to a given order at a point \( p \), which appear in the following list. Suppose \( f \in \mathcal{O}(V) \) is a homogeneous polynomial. The following statements are equivalent:

\[
i) \ \text{f vanishes to order m at p.} \\
ii) \ \text{f vanishes to order m at one point } \tilde{p} \in p \setminus \{0\} \subset V \\
iii) \ \text{f vanishes to order m at all points of } p \setminus \{0\} \subset V \\
vii) \ \text{f vanishes to order m at all points of } p \subset V. \\
v) \ \text{The hypersurface } H \subset \mathbb{P}(V) \text{ defined by } f \text{ contains } p \text{ with multiplicity m.}
\]
vi) If \( \text{char}(F) = 0 \), then for every differential operator \( D \in \mathcal{D}(V) \) of order less or equal to \( m - 1 \), we have \( D(f) \in \mathfrak{m}_p \). Or, using ii) again, for every such \( D \), \( D(f) \) vanishes at a fixed \( \tilde{p} \in p \setminus \{0\} \).

We will now describe the general problem: For \( l \) points \( p_1, p_2, \ldots, p_l \in \mathbb{P}(V) \) and nonnegative integers \( m_1, m_2, \ldots, m_l \) and \( k \) we set \( p = (p_1, \ldots, p_l) \) and \( m = (m_1, \ldots, m_l) \). Let

\[
d(p; m; k) = \dim_F \bigcap_i m_{p_i, k}^{m_i}
\]

where \( m_{p_i, k} \) denotes the degree \( k \)-part of \( m_{p_i}^{m_i} \). In other words, \( d(\ldots) \) denotes the dimension of the space of all homogeneous polynomials of degree \( k \) that vanish at each of the \( p_i \) to order \( m_i \). Let \( d(m; k) \) denote the minimum over \( d(p; m; k) \) where the \( p_1, p_2, \ldots, p_l \) vary. \( l \) points will be called generic if \( d(p; m; k) = d(m; k) \) for all \( m \). We will be studying these numbers. The ideal of functions vanishing of order \( m_i \) at \( p_i \) is denoted \( I_{p, m} \) or also \( I_m \), with the points understood. The problem of determining the numbers \( d(m; k) \) is well known, and a substantial portion of the known results are due to Nagata, as we mentioned in the introduction. In his famous papers on rational surfaces [6], [7] he developed among other things a theory on linear systems, generalizing the ordinary terminology of linear systems, which allowed him to give a complete algorithm to determine these numbers for \( n = 2 \) and \( l \leq 9 \). We will give an alternate proof of this algorithm using representation theory of \( SL_2 \). It is remarkable that up today, no general solution to the problem seems to be known except for some special cases.

There are several questions related to our problem, whose answer would lead to a partial solution. We will now assume that \( \text{char}(F) = 0 \).

**Problem 3.** When does ‘vanishing at order \( m \) at \( l \) points’ impose independent conditions on \( \mathcal{O}(V)_k \)?

Of course, independence here refers to the question, whether the obvious equations are independent: for each point \( p \), vanishing to order \( m \) at \( p \) is the same as solving the equations \( D(f)(p) = 0 \) for all differential operators \( D \in \mathcal{D}(V) \) of degree less or equal \( m - 1 \). These equations certainly are never independent. But if we choose a complement to the trivial equations, namely the elements of the ideal \( D_p \mathcal{D}(V) \), where \( D_p \) means derivation along \( p \), then the question makes sense. Note that this ideal is independent of the choice of a representative of \( p \). So the number of equations is \( \sum_{i=1}^l \binom{m_i - 1 + n}{n} \), and the problem asks, when is this the codimension of \( I_m \) in \( \mathcal{O}(V)_k \). we also say that the conditions are independent, if they cut out everything, i.e. if \( I_m = 0 \). We will come back to this later, but note for now, it is easy to see, that for \( k \) large enough the answer to problem 3 is yes (in any characteristic).

From a more algebro-geometric point of view, one can think of our \( l \) generic points with multiplicities \( m_1, \ldots, m_l \) as a zero dimensional subscheme \( Z \) of \( \mathbb{P}(V) \), defined by the sheaf of ideals \( \mathcal{I} = \bigcap \mathcal{I}_{p_i}^{m_i} \), where \( \mathcal{I}_{p_i} \) is the ideal sheaf defining
the reduced point $p_i$. Then one has the natural exact sequence
\[ 0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_Z \to 0 \] (1)

Since $Z$ is a zero-dimensional scheme, we may write $\mathcal{O}_Z = \bigoplus_i \mathcal{O}_{Z,p_i} = \bigoplus_i F^b_i$, with $b_i = \binom{m_i-1+n}{n}$. And our problem becomes equivalent to the following:

**Problem 4.** For which $k$ has the induced map
\[ H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(k)) \to H^0(Z, \mathcal{O}_Z(k)) \] (2)
maximal rank?

Since we know, that for large $k$ $H^1(\mathbb{P}(V), \mathcal{I}(k)) = 0$, our claim above follows: the answer is yes, for $k$ sufficiently large.

In this form, the problem has been studied by Alexander and Hirschowitz, for example. Alexander showed in that for $k \geq 5$ and all $m_i$ equal to 2, the codimension of $I(k)$ is either $l(n+1)$, or $I(k) = 0$. In other words the rank in problem is always maximal.

Alexander and Hirschowitz also study the case $k = 4$. For $k = 4$ and all $m_i = 2$ the same is true, except for the following cases:
\[ n = 2, l = 2; \quad n = 3, l = 9; \quad n = 4, l = 14. \]

Moreover, recently Ciliberto and Miranda showed that whenever $n = 2$ and all multiplicities equal and less than or equal 12, then problem has an affirmative answer for $k \geq 3m$.

To conclude this section we return to the original question of Nagata. In his definition of $a_m$ all the $m_i$s are equal to $m$. So as a part of our problem, one might ask

**Problem 5.** Determine $d(l, m; k) := d(m, m, \ldots, m; k)$ for $l$ points in $\mathbb{P}(V)$.

Even in this somewhat reduced form, no general answer is known. In fact, using Nagata’s algorithm, one is naturally forced to consider different $m$s even if one starts with equal multiplicities, as we will see later. The case of equal multiplicities has another nice feature: In Nagata’s counterexample one needs conditions of the form $a_k \neq a_{km}$. Denote by $I_m = \bigoplus_k I_{m,k}$ the homogeneous ideal of polynomials vanishing at our $l$ points to equal multiplicity $m$. For $n = 2$, to show that $I^k_m \neq I_{km}$ for certain arbitrarily large values of $m, k$, it suffices to know that
\[ d(l, m; k) \neq 0 \Rightarrow k > \sqrt{l} \] (3)

Notice that $d(l, m; k) \geq \binom{k+2}{2} - l\binom{m+1}{2}$ is always true. This is just saying that the rank of the equations imposed by the vanishing conditions is at most $l\binom{m+1}{2}$, which follows for example from our discussion of problem. Using this one can show that (3) cannot hold for all $k, m$, if we have $I^k_m = I_{km}$ for $m$ sufficiently large. This leads us to another
Lemma 9. For all points \(p\), Proof.

For any \(\langle \cdot \rangle\) we think of as \(m\) integers \(S = \sum_{i=0}^{d} S^d(V) = \bigoplus_{k \geq 0} S^d\) be the symmetric algebra of \(V\), which we think of as \(\mathcal{O}(V)^* = \text{Hom}_F(\mathcal{O}(V), F)\). By our remark \(3\), we may replace the points \(p_1, \ldots, p_l\) by elements of \(V\) itself. So, for \(p \in V\) and a nonnegative integers \(m\) and \(k\), let \(I_{p,m,k} = I_{Fp,m,k}\). Then we have

Lemma 9. For all \(p \in V\) and all \(m, k \geq 0\),

\[
I_{m,p,k} = (S^{m-1}(V)p^{k-m+1})^\perp \tag{4}
\]

\[
= \{f \in \mathcal{O}(V)_k \mid \langle f, \alpha p^{k-m+1} \rangle = 0, \forall \alpha \in S^{m-1}\} \tag{5}
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the natural pairing between \(\mathcal{O}(V)\) and \(S(V)\).

Proof. For any \(v \in V\), let \(D_v\) be the associated derivation of \(\mathcal{O}(V)\). Note that \(\langle D_v(f), \lambda \rangle = \langle f, v\lambda \rangle\) for all \(v \in V, f \in \mathcal{O}(V)_k, \lambda \in S^{k-1}\). To see this,
without loss of generality we may assume that $v = e_1$ the first coordinate vector, then if $\mathcal{O}(V) = F[x_1, x_2, \ldots, x_{n+1}]$ we have $D_v = \frac{\partial}{\partial x_1}$. Since $S^{k-1}$ is spanned by $(k-1)$th powers of elements of $v$, we are reduced to the case $\lambda = w^{k-1}$ for some $w \in V$. Then $\langle D_{v_1}(f), w^{k-1} \rangle = D_{v_1}(f)(w)$. Writing $f = f_0 + f_1 x_1 + f_2 x_1^2 + \cdots + f_k x^k$ with $f_i \in F[x_2, \ldots, x_{n+1}]$, we get

$$\langle f, e_1 w^{k-1} \rangle = \langle f_0, e_1 w^{k-1} \rangle + \langle f_1 x_1, e_1 w^{k-1} \rangle + \cdots + \langle f_k x^k, e_1 w^{k-1} \rangle,$$

which is $0 + f_1(w) + 2f_2(w)x_1(w) + \cdots + k f_k(w)x^k_1(w) = D_{v_1}(f)(w)$.

But now we are done: By induction on $m$ we may assume that, $f \in I_{p,m,k}$ if and only if $D_v(f)$ is orthogonal to $S^{m-2p^{k-m+1}}$ for all $v$ and $f(p) = \langle f, p^k \rangle = 0$. In other words, if and only if $f$ is orthogonal to $v S^{m-2p^{k-m+1}}$ for all $v$, which is our assertion. \hfill \Box

Corollary 10. For $l$ points we have

$$I_{m,k} = \left( \sum_{i=1}^{l} S^{m_i-1} p_i^{k-m_i+1} \right)^{1} \quad (6)$$

As before $I_{m,k}$ is the degree $k$-part of $I_m = I_{p,m}$. The point is, that if we put

$$h(p; m; k) = \dim_F \langle S^k / \sum_i S^{k-r_i} p_i^{r_i} \rangle, \quad (7)$$

then we have

$$d(m; k) = h(p; m; k) \quad (8)$$

with $r_i = k - m_i + 1$, and the points $p_i$ sufficiently general. In other words, knowing $d(m; k)$ for all $k$ and $m$ is the same as knowing

$$h(r; k) := \min \{ h(p; r; k) | p \in \mathbb{P}(V)^l \} \quad (9)$$

for all $k$ and $r = (r_1, r_2, \ldots, r_l)$.

Remark 11. It is a priori not clear whether generic points always exist. But for any finite collection of values for $r$ it is not hard to see that a minimizing choice of $p$ exists such that $h(p; r; k) = h(r; k)$ for all $k$.

For fixed $r$ let $H_r(q)$ be the formal power series given by $\sum h(r; k)q^k$. Then $H_r$ is the Hilbert series of $S_r := S / (S_{p_1^{r_1}} + S_{p_2^{r_2}} + \cdots + S_{p_l^{r_l}})$ for sufficiently generic points $p_i$. And if all $r_i$ equal $r$ ($1 \leq i \leq l$) we abbreviate $H_r$ by $H_{i,r}$ and $S_r$ by $S_i,r$.

Definition 12. Let $f = \sum_i f_i q^i \in \mathbb{Q}[[q]]$ be a formal power series. Then we set

$$\text{trunc}(f) = \sum_i \text{trunc}(f_i)q^i$$

where for $r \in \mathbb{Q}$

$$\text{trunc}(r) = \begin{cases} r & \text{if } r > 0 \\ 0 & \text{otherwise} \end{cases}$$
For \( \mathbf{r} = (r_1, \ldots, r_l) \) as above we set
\[
C_{\mathbf{r}}(q) = \sum_k c(\mathbf{r}; k)q^k
\]
where
\[
c(\mathbf{r}; k) = \text{trunc}\left(\binom{k + n}{n} - \sum_{i=1}^l \binom{k - r_i + 2}{2}\right)
\]
is the virtual dimension of \( S_k^{\mathbf{r}} \). Notice that
\[
C_{\mathbf{r}} = \text{trunc}\left(\frac{1 - \sum_i q^{r_i}}{(1 - q)^n}\right).
\]
If all the \( r_i \) equal a given \( r \) (\( 1 \leq i \leq l \)) we set \( C_{l, r} = C_{\mathbf{r}} \). Based on large scale computations for values of \( r \) between 10 and 20 it seems reasonable to

**Conjecture 13.** If \( n = 2 \) and \( l \geq 10 \), then for all \( r \), \( H_{l, r} = C_{l, r} \).

For \( l = 4 \) and \( l = 9 \) the assertion is true (despite the fact that it is stated only for \( l \geq 10 \)). This is a consequence of Nagata’s algorithm (cf. section 3).

For \( l = 1, 2 \) it fails for obvious reason: In these cases \( S_r \) is not zero dimensional as a ring, hence cannot have a polynomial as Hilbert series. For \( l = 3 \) \( S_r \) is the tensor product of three copies of \( \mathbb{C}[x]/(x^r) \), and so has \( \frac{(1 - q)r}{(1 - q)^3} \) as Hilbert series. For \( l = 5, 6, 7, 8 \) the situation is more delicate. The conjecture fails for \( l = 5 \) in degree 4 for \( r = 3 \), for \( l = 6 \) in degree 24 for \( r = 15 \). For \( l = 7 \), \( h(l, r; k) \neq c(l, r; k) \) for \( k = 42 \) and \( r = 27 \). Finally, \( l = 8 \) fails in degree 96 with \( r = 63 \).

An interesting test case for the conjecture is a situation where \( \binom{k + 2}{2} = l\binom{k - r + 2}{2} \). For example this occurs for \( l = 10 \) for \( k = 174 \), \( r = 120 \). It does occur twice before, and the answer there is 0 = \( h(10, r; k) \), which is well known.

Using Groebner basis techniques on a parallel computer system, we have shown \( h(10, r; 174) = 0 \), giving further evidence for the truth of the assertion. The computation took several days, and so far it seems to be the last number where one could actually compute the result. We have added the resulting Hilbert series in the appendix below.

Returning to the general question in dimension 2, we saw above, that if \( n = 1 \), then \( H_{\mathbf{r}} = C_{\mathbf{r}} \). From this, it is easy to see the following

**Lemma 14.** For \( n = 2 \), \( h(l, r; k) = c(l, r; k) \), whenever \( k > \sum_i m_i \).

**Proof.** It suffices to show that the lemma is true for some points \( p_1, p_2, \ldots, p_l \in \mathbb{P}^2 \). We may choose the \( p_i \) sitting on a line \( L \subset \mathbb{P}^2 \), such that they are generic when considered as points of \( L \cong \mathbb{P}^1 \). Suppose \( L \) is given by an equation \( l \in (F^2)^* \). In \( S \) we may therefore assume that \( x, y, z \) are variables (on \( (F^3)^* \)) and that \( p_i \in F[x, y]|_1 \subset S^1 \) for all \( i \). We look at the natural injective restriction \( r : S^k \to k[x, y] \), given by \( f \mapsto f(x, y, 1) \), and with image all polynomials of
degree less or equal to $k$. Under this map, the $p_i$ remain homogeneous. It follows that
\[
\sum_i S^{k-r_i} p_i^{r_i} \cong \sum_i^{k-r_i} \bigoplus_{j=0}^r r(p_i)^{r_j} F[x, y]_j
\]
The latter equals
\[
\bigoplus_{k'=0}^k \sum_i^k r(p_i)^{r_{k'}} F[x, y]_{k' - r_i}, \quad (11)
\]
Now $k > \sum m_i$ implies $k' > \sum (k' - r_i + 1)$. Since we know, that the inner sum in (11) is direct under these circumstances, we are done.

We conclude this section by proving the conjecture for $l = 4$. So suppose $p_1, p_2, p_3, p_4$ are four generic points of $V = F^3$. Because $\dim \mathcal{O}(V)_2 = 6$, any five points lie on a quadric. So our four points lie, say, on $P \subset \mathbb{P}(V)$, defined by $f_0$ with $f_0$ homogeneous of degree 2. Moreover for generic points $f_0$ may be assumed generic also, so up to the action of $GL_3$, all points may be chosen to have the form $p_i = (1, t_i, t_i^2)$ and $f_0 = xz - y^2$. We have to show, that for fixed $r > 0$, we have $h(r; k) = c(r; k)$ for all $k$. We prove this by induction on $r + j = k$. Thus, we are investigating $d(4, j + 1; j + r)$. If $k = 0$, then either $r$ or $j$ is zero. For $r = 0$ the result is trivial, and for $j = 0, r = 1$ the question is, whether $4 = \dim \sum F_p t'$, which is true for the $p_i$ generic. Now suppose $h(r, r + j) = c(r; r + j)$ for all $r, j$ with $0 \leq r + j < k$, and we prove the result for $k$. As in the case $n = 1$ let $C$ be the embedded copy of $F$, parameterized by $(1, t, t^2), t \in F$. For any $f \in I_{j+1, r+j}$, let $\phi(t) = \phi_f(t) = f(1, t, t^2) \in F[t]$ be the restriction of $f$ to $C$. Then it follows that
\[
\phi(t) = g(t) \prod_{i=1}^4 (t - t_i)^{j+1} \quad (12)
\]
with $\deg g \leq 2(r + j) - 4(j + 1)$. If $2(r + j) - 4(j + 1) < 0$, there is no such $g$ and $f |_{C} = 0$, so $f = uf_0$ with $u \in I_{j, r+j-2}$. If $r' = r - 1$ and $j' = j - 1$, it follows that $2(r' + j') - 4(j' + 1)$ still is negative, so by the very same argument $\phi_u(t) = 0$, and so on. It follows $f = c f_0$ for some $d < j + 1$ with $c$ linear or constant. But $c$ vanishes on $C$, because $d < j + 1$ and $j + 1$ is the required multiplicity for $f$, hence $f = 0$. On the other hand it is easy to see, that $c(r, r + j) = 0$.

So suppose $2(r + j) \geq 4(j + 1)$. In this case we have
\[
\dim I_{j+1, r+j} |_{C} \leq 2(r + j) - 4(j + 1) + 1.
\]
Using our induction hypothesis this means that $h(4, r; r + j) \leq c(4, r - 1; r + j - 2) + 2(r + j) - 4(j + 1) + 1$, because the space of functions in $I_{j+1, r+j}$ restricting to zero on $C$ is just $f_0 P_j, r + j - 2$. But $c(4, r - 1; r + j - 2)$ is now nonnegative.
Indeed $2(r + j) - 4(j + 1) \geq 0$, so $r + j \geq 2j + 2$, and

$$\frac{(r + j)}{2} = \frac{(r + j)(r + j - 1)}{2} \geq \frac{(2j + 2)(2j + 1)}{2}$$

$$= 4 \frac{(j + 1)(j + \frac{1}{2})}{2} > 4 \frac{(j + 1)j}{2} = \left(\frac{j + 1}{2}\right).$$

It follows that $c(4, r - 1; r + j - 2) = \text{trunc} \left(\frac{(r + j - 2 + 2)}{2} - 4 \frac{(j - 1 + 2)}{2}\right)$ is actually positive. Hence,

$$h(r; r + j) \leq 2(r + j) - 4(j + 1) + 1 + \frac{(r + j)}{2} - 4 \frac{(j + 1)}{2}$$

$$= \left(\frac{r + j + 2}{2}\right) - 4 \frac{(j + 2)}{2} = c(r; r + j). \quad (14)$$

But certainly $h(4, r; r + j) \geq c(4, r; r + j)$ and the claim follows.

It should be remarked, that exactly the same argument works also for nine points, except that one has to use a cubic instead of our quadric $xz - y^2$ here.

We will outline a proof in section 6. And the argument definitely does not work for 5, 6, 7 points.

4 $n + 2$ points in $\mathbb{P}^n$ and representations of $SL_2$

Now we turn to a different perspective. Suppose we are given $p_1, p_2, \ldots, p_l$ generic points in $\mathbb{P}(V) = \mathbb{P}^{n+1}$, and assume that $l \geq n + 1$. It is clear that $p_1, \ldots, p_{n+1}$ may be taken to be linearly independent. In this case, if we take $e_i \in p_i$ for $i = 1, 2, \ldots, n + 1$ as a basis for $V$, the elements $e_1^*, e_2^*, \ldots, e_{n+1}^*$ are a system of parameters, and also a regular sequence of the ring $S(V)$. Set

$$M = S(V)/\left(\sum_{i=1}^{n+1} S(V)e_i^*\right) \quad (15)$$

and view $M$ as an $S(V) \cong F[e_1] \otimes F[e_2] \otimes \cdots \otimes F[e_{n+1}]$-module, we get

$$M = F[e_1]/e_1^* F[e_1] \otimes F[e_2]/e_2^* F[e_2] \otimes \cdots \otimes F[e_{n+1}]/e_{n+1}^* F[e_{n+1}] \quad (16)$$

with $e_i$ acting on the $i$th factor by multiplication and fixing the other factors identically. On $F[e_i]/e_i^* F[e_i]$ we choose a basis $v_{i,j}$ with $v_{i,j} = e_i^{r_i-j-1}$, $j = 0, 2, \ldots, r_i - 1$. Then $e_i$ acts relative to this bases by

$$\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix} \quad (17)$$
i.e. one nilpotent Jordan block. Define an endomorphism \( h_i \) by \( h_i(v_{i,j}) = (r_i - 1 - 2j)v_{i,j} \), then there is exactly one nilpotent operator \( f_i \) with \([e_i, f_i] = h_i\) and \([h_i, f_i] = -2f_i\). In other words \( e_i, f_i, h_i \) form an \( \mathfrak{sl}_2 \)-triple and give rise to an \( SL_2 \)-action on \( M \), by extending the action on the \( i \)th factor by the trivial representation to all others. It follows \( M \) is a representation of the product of \( n + 1 \) copies of \( SL_2 \). Moreover

\[
M = L^{r_1} \otimes L^{r_2} \otimes \cdots \otimes L^{r_{n+1}}
\]

where for each nonnegative integer \( r \), \( L^r \) is the irreducible \( SL_2 \)-module of dimension \( r \). To avoid clumsy notation we define \( L^0 = 0 \), which corresponds to the case that one of \( r_i = 0 \) (and so \( M = 0 \)).

For the rest of this section, we will assume that \( l = n + 2 \), and so we have one additional point \( p = p_{n+2} \in \mathbb{P}(V) \). Since we are interested in dimensions only it is clear, that we may replace \( p_1, p_2, \ldots \) by \( gp_1, gp_2, \ldots \) where \( g \in GL_{n+1} \) is any element. Furthermore, since the \( p_i \) are generic, we may assume that \( p = [a_1e_1 + a_2e_2 + \cdots + a_{n+1}e_{n+1}] \in \mathbb{P}(V) \) with all the \( a_i \) nonzero. The common stabilizer of our first \( n + 1 \) points is the usual diagonal torus of \( GL_{n+1} \), so we may replace \( p \) by \( [e_1 + e_2 + \cdots + e_{n+1}] \), and set \( e = e_1 + e_2 + \cdots + e_{n+1} \). This last definition makes sense also in \( \text{Lie}((SL_2)^{n+1}) \), so put \( f = f_1 + f_2 + \cdots + f_{n+1} \) and \( h = h_1 + h_2 + \cdots + h_{n+1} \). Therefor we get another \( \mathfrak{sl}_2 \)-triple corresponding to the diagonal \( G \) in \( (SL_2)^{n+1} \). Moreover the action of \( e \in \text{Lie}(G) \) on \( M \) is the same as the action of the point \( e \in V \).

For any vectorspace \( Z \), on which \( h \) acts, and any integer \( \lambda \), we define \( Z[\lambda] \) to be the \( \lambda \)-eigenspace of \( h \) on \( Z \). Moreover for any degree \( k \), \( M_k \) is the weight space \( M[\lambda] \) for \( h \) for a suitable integer \( \lambda \). Thus, finding \( H_r \) is equivalent to finding the decomposition of \( M/e^rM \) as an \( h \)-module, where we put \( r = r_{n+2} \) for short.

**Lemma 15.** Suppose \( r = (r_1, r_2, \ldots, r_{n+2}) \), then

\[
q^{-(r_1+r_2+\cdots+r_{n+1}-n+1)}H_r(q^2) = \sum_{\lambda} q^\lambda \dim(M/e^{r_{n+2}}M)[\lambda] \]

where \( \mathfrak{sl}_2 \) acts via the tensor product action on \( M = L^{r_1} \otimes L^{r_2} \otimes \cdots \otimes L^{r_{n+1}} \).

**Proof.** To keep things short, set \( \lambda(k) = 2k - r_1 - r_2 - \cdots - r_{n+1} + (n + 1) \). Then \( M[\lambda(k)] = M_k \) in our discussion of \( M \) as an \( S(V) \)-module above, and of course also \( (M/e^{r_{n+2}}M)[\lambda(k)] = (M/e^{r_{n+2}}M)_k \). It follows that

\[
q^{-(r_1+r_2+\cdots+r_{n+1}-n+1)}H_r(q^2) = \sum_k h(r; k)q^{\lambda(k)},
\]

and the latter is also the righthand side of \([19]\). \( \square \)

For \( n = 1 \) this gives a geometric interpretation of the Clebsch-Gordan formula and for \( n = 2 \) it gives an interpretation of the \( 6j \)-symbol. We also note
that if \( r = (r_1, r_2, \ldots, r_{n+1}, 1) \), then finding \( H_r \) is equivalent to finding the decomposition of \( M \) into irreducibles. Once this is done, it is a simple matter to replace the 1 in the \((n + 2)\)th position by \( r_{n+2} \).

If all \( r_i \) are equal, say, to \( r \), then it is also possible to deduce the multiplicities when \( H_r \) is known. For example, what we proved above for four points amounts to:

**Lemma 16.**

\[
L^r \otimes L^r \otimes L^r = \bigoplus_{j=0}^{r-1} (j + 1)L_3^{(r-1)-2j+1} \oplus \bigoplus_{j=1}^{\lceil \frac{r}{2} \rceil} (r - 2j)L_r^{T-2j}
\]

In the rest of this section we write \( n \), the dimension of \( \mathbb{P}(V) \), as a subscript to avoid any confusion. Note that \( H_{n, (r_1, r_2, \ldots, r_{n+1}, 1)} = H_{n, (r_1, \ldots, r_{n+1})} \) and also \( S/\mathbb{C}S_i = F[\epsilon_1, \ldots, \epsilon_n] \). Thus we get

\[
H_{n, (r_1, \ldots, r_{n+1}, 1)} = H_{n-1, (r_1, \ldots, r_{n+1})}.
\]

Here on the right hand side everything takes place in one dimension less. In the special case of \( n = 2 \), it follows, that for the coefficients of \( H_{2, (r_1, r_2, r_3, 1)} \), we have

\[
h_2((r_1, r_2, r_3, 1); k) = \text{trunc}(k + 1 - \sum_{k-r_i \geq 0} (k - r_i + 1))
\]

(21)

On the other hand, the left hand side of (21) is

\[
\dim((L^{r_1} \otimes L^{r_2} \otimes L^{r_3})/e(L^{r_1} \otimes L^{r_2} \otimes L^{r_3})[-r_1 - r_2 - r_3 + 3 + 2k])
\]

Summarizing, this implies the following generalization to the Clebsch-Gordan formula.

**Proposition 17.** Let \( r_1, r_2, r_3, l \) be positive integers. If \( l \equiv r_1 + r_2 + r_3 \mod 2 \), then with

\[
k = \frac{r_1 + r_2 + r_3 - 3 - l}{2},
\]

we have

\[
\dim \text{Hom}_{SL_3}(L^{L+1}, L^{r_1} \otimes L^{r_2} \otimes L^{r_3}) = \text{trunc}\left(k + 1 - \sum_{k-r_i \geq 0} (d - r_i + 1)\right).
\]

Otherwise \( \text{Hom}_{SL_3}(L^{L+1}, L^{r_1} \otimes L^{r_2} \otimes L^{r_3}) \) is \((0)\).

In the last section of this paper, we will give an algorithm, that computes \( H_r \) for at most nine points in \( \mathbb{P}^2 \). By what was said above, to compute the multiplicities in \( L^{r_1} \otimes \cdots \otimes L^{r_4} \), one has to know \( H_{3, (r_1, r_2, \ldots, r_{4}, 1)} = H_{2, (r_1, r_2, \ldots, r_{4})} \).

We may assume that \( r_1 \leq r_2 \leq r_3 \leq r_4 \), and we borrow the following result from section 6.

**Proposition 18.** Suppose \( 2k + 3 \leq r_1 + r_2 + r_3 \) and \( 0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \leq k \), then \( h_2(r_1, r_2, r_3, r_4; k) = c_2(r_1, r_2, r_3, r_4; k) > 0 \).
As we will see now, this is almost everything one has to know for decomposing a fourfold tensor product. We are immediately reduced to the case where \(2k + 3 > r_1 + r_2 + r_3\) or \(r_4 > k\). In all other cases, proposition 3 gives the correct answer. As before set \(\lambda(k) = -(r_1 + r_2 + r_3) + 3 + 2k\), the corresponding weight of \(h\) in degree \(k\). If \(r_4 \leq \lambda(k)\), then the transformation rule in the next section asserts that \(h_2(r_1, r_2, r_3, r_4; k) = 0\). On the other hand, if \(r_4 > k\), the answer is given by \(h_2(r_1, r_2, r_3; k)\), the coefficient of \(q^k\) in

\[
\frac{(1 - q^r_1)(1 - q^{r_2})(1 - q^{r_3})}{(1 - q)\lambda}.
\]

Finally, if \(\lambda(k) \leq r_4 \leq k\), the dimension of \((e^{r_4}M)[\lambda(k)]\) is the sum of multiplicities of \(L^p\) in \(M\), where \(p\) ranges over the set \(p - 1 \equiv \lambda(k) \mod 2, p > 2r_4 - \lambda(k)\). If we write \(p = \lambda(k) + 1 + 2j\), this multiplicity is \(h_1(r_1, r_2, r_3; k + j + 1)\). Thus, in this case

\[
h_2(r_1, r_2, r_3, r_4; k) = h_2(r_1, r_2, r_3; k) - \sum_{j=r_4-\lambda(k)}^{\infty} h_1(r_1, r_2, r_3; k + j + 1).
\]

## 5 The transformation rule

We now turn to the central point of our discussion. Our aim is to redevelop Nagata’s algorithm from our point of view. There is a universal rule which simply says that \(h(r; k) = h(r', k')\) for certain \(r', k'\). The algorithm depends on the fact that for nine or less points it is possible to see exactly when \(k' < k\), and then to handle the case \(k' \geq k\) directly. We will come to that in a moment.

First we adapt our approach from the last section to more than \(n + 2\) points. Suppose we are given \(l\) generic points in \(\mathbb{P}(V)\). As before we may identify the first \(n + 1\) points with the coordinate lines and choose representatives \(e_1, e_2, \ldots, e_{n+1}\).

Moreover, we may assume that all the other \(p_i\) have nonzero coordinates, i.e. \(p_i = [a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{in+1}e_{n+1}]\), with all the \(a_{ij}\) nonzero. As before we interpret \(e_i\) as a certain element of the \(i\)th factor in \((\mathfrak{s}\mathfrak{l}_2)^{n+1}\). Also \(e, f, h\) are defined as in the last section. Given this suppose \(t\) is an element of the maximal torus \(T \subset (\mathfrak{s}\mathfrak{l}_2)^{n+1}\) consisting of diagonal matrices. Then \(t\) corresponds to an \(n + 1\)-tuple \((t_1, t_2, \ldots, t_{n+1})\) of points in \(F^*\), and we have

\[
\text{Ad}(t)e = t_1^2e_1 + t_2^2e_2 + \cdots + t_{n+1}^2e_{n+1}.
\]

Taking residue classes in

\[
M = S(V)/e_1^rS(V) + e_2^rS(V) + \cdots + e_{n+1}^rS(V)
\]

we see that \(e_j = \sum a_{ji}e_i\) for \(j > n + 1\) is of the form \(\text{Ad}(t_j)e\) for a suitable \(t_j \in T\). Thus,

\[
e_{j}^rM = t_{j}e_{r}^jM \quad j = n + 2, n + 3, \ldots, l.
\]
Of course, $t_je^\gamma M$ and also $e^\gamma M$ are $h$-stable, and even isomorphic as $h$-modules. Notice also, that $e_i, f_i, h_i = h$ form an $\mathfrak{sl}_2$-triple where $f_i = \text{Ad}(t_i)f$ (and $h = \text{Ad}(t_i)h$).

**Theorem 19.** If $l > n + 1$ then,

$$h(r; k) = h(r'; k')$$

with $k' = r_1 + r_2 + \cdots + r_{n+1} - n - 1 - k$, $r' = (r_1, r_2, \ldots, r_{n+1}, r'_{n+2}, r'_{n+3}, \ldots, r'_l)$, and $r'_i = r_1 + r_2 + \cdots + r_{n+1} - n - 1 + r_i - 2k$.

**Remark 20.** As mentioned before, in the case $n = 2$ the theorem is equivalent to a result of Nagata ([7]), which he proves by methods of classical projective geometry.

On the other hand, the proof we present here uses only elementary representation theory of $SL_2$.

We will need the following Lemma in the proof.

**Lemma 21.** Let $L^p$ be an irreducible $\mathfrak{sl}_2$-representation with highest weight $p-1$ and let $e, f, h$ be a standard triple inside $\mathfrak{sl}_2$. Then for all integers $r, \lambda$ with $r \geq 0$ we have: If $r > \lambda$, then

$$(e^rL^p)[\lambda] = (f^{r-\lambda}L^p)[\lambda].$$

If on the other hand $r \leq \lambda$, then

$$L^p[\lambda] = (e^rL^p)[\lambda].$$

Here for an $h$-module $V$, $V[\lambda]$ is the $h$-weight space of weight $\lambda$.

**Proof.** First we deal with the case $r > \lambda$. Notice that in (24) both sides of the equation are either zero or one-dimensional and in the latter case they equal $L^p[\lambda]$. It therefore is enough to show that the dimensions agree, i.e. that they are nonzero for the same values of $r$ and $\lambda$. If $\lambda$ and $p-1$ do not have the same parity, then $L^p[\lambda]$ is zero and therefore (24) is obviously true. Hence it is safe to assume that $\lambda$ and $p-1$ both are simultaneously odd or even.

In this case the left hand side of (24) is nonzero if and only if

$$-p + 1 + 2r \leq \lambda \leq p - 1.$$ (26)

Similarly the right hand side is nonzero if and only if

$$-p + 1 \leq \lambda \leq p - 1 - 2(r - \lambda).$$ (27)

Of course both these equations are equivalent: assuming (26), it follows immediately that $-p + 1 \leq \lambda$ because $r \geq 0$. And $p - 1 - 2(r - \lambda) = p - 1 - 2r + 2\lambda \geq -\lambda + \lambda = 0$ due to the left hand side of (26), hence (27) holds. A completely analogous argument shows the other implication, that is, if (27) holds, (26) is true as well.

Finally assume that $r \leq \lambda$. In this case, if $\lambda$ is a weight of $L^p$, so is $\lambda - 2r$. But then $L^p[\lambda] = e^r(L[\lambda - 2r]) = (e^rL^p)[\lambda]$, hence the claim. □
Using this we are able to prove the theorem.

**Proof of theorem** \[\text{(19)}\]. We are interested in

\[
\dim \left( \frac{M}{\sum_{i>n+1} e_i^r M} \right)[\lambda(k)]
\]

where \(\lambda(k) = -r_1 - r_2 - \cdots - r_{n+2} + n + 1 + 2k\) is the \(h\)-weight associated to degree \(k\). First we will consider the trivial cases. Suppose \(k'\) is negative. Thus, the right hand side of \((23)\) is zero (by convention). On the other hand, this means that \(k\) is greater than the highest weight of \(M\), and so \(\lambda(k)\) is greater than this weight, too. Thus \(M[\lambda(k)] = (0)\), and \((23)\) holds.

We may therefore assume, that \(k' \geq 0\). Notice that \(r_i' = r_i - \lambda(k')\). If there is an index \(i > n+1\) such that \(r_i' \leq 0\) then \(\lambda(k) \geq r_i\). Consider the action of the \(i\)th \(\mathfrak{sl}_2\)-triple on \(M\). For each irreducible submodule \(L_p \subset M\) we are in the second case of Lemma \((21)\) which then asserts that \(L_p[\lambda(k)] = (e_i^r M)[\lambda(k)]\). Thus \(M[\lambda(k)] = e_i^r M[\lambda(k)]\) and the left hand side of \((23)\) is zero. But the right hand side is zero as well because if \(r_i' = 0\) this is obvious and if \(r_i' < 0\) this is convention.

It remains to treat the case that all \(r_i'\) are strictly positive. For each \(i > n+1\) we are then in the first case of Lemma \((21)\). That is, looking at the \(i\)th \(\mathfrak{sl}_2\)-triple acting on \(M\), then for each irreducible \(L_p \subset M\) we have

\[
(e_i^r M)[\lambda(k)] = (f_i^r L_p)[\lambda(k)].
\]

Of course, this applies to all of \(M\) as well and we conclude, that for each \(i\)

\[
(e_i^r M)[\lambda(k)] = (f_i^r M)[\lambda(k)].
\]

Moreover,

\[
\lambda(k) = r_1 + r_2 + \cdots + r_{n+1} - (n + 1) - 2k' = -\lambda(k').
\]

For any \(\mathfrak{sl}_2\)-module \(L\) there is an involution \(\theta\) satisfying \(\theta(ex) = f\theta(x)\), and \(\theta(hx) = -h\theta(x)\). Thus there is an involution \(\Theta\) of \(M\), satisfying \(\Theta(e_i x) = f_i \Theta(x)\) and \(\Theta(h_i x) = -h_i \Theta(x)\) for \(1 \leq i \leq n + 1\) (\(\Theta\) is obtained by tensoring the \(\theta_i\)s of the individual factors \(L^{e_i}\)). Clearly this implies

\[
\Theta \left( \sum_{i>n+1} (f_i^r M)[\lambda(k)] \right) = \sum_{i>n+1} (e_i^r M)[\lambda(k')]
\]

because \(\lambda(k') = -\lambda(k)\), and the theorem now follows.

It is quite remarkable that one is able to deduce this transformation rule using only elementary representation theory of \(SL_2\). In contrast to that, Nagata’s method for \(n = 2\) was to show that under a quadratic transformations \(T\) of \(\mathbb{P}^2\), the global sections of a certain linear system \(L\) are in one to one correspondence with the sections of \(TL\).
6  9 points in \( \mathbb{P}^2 \) and Nagata’s algorithm

In this section we will develop our version of Nagata’s algorithm to determine the numbers \( h(r; k) \) for less than nine points in \( \mathbb{P}^2 \).

So let \( p_1, \ldots, p_l \) be generic points of \( \mathbb{P}^2 \) with \( 3 \leq l \leq 9 \). First we state the algorithm and then prove that it terminates after finitely many steps. If we consider points in another dimension than 2, we will indicate that with a subscript, e.g. we will write \( h_1(r; k) \) for \( l \) generic points in \( \mathbb{P}^1 \).

**Algorithm 22.** With the notation above, if the test at stage \( i \) fails we go to \( i + 1 \), otherwise we are done or start at 1 again.

1. If \( l = 3 \), \( h(r_1, r_2, r_3; k) \) is the coefficient of \( q^k \) in

\[
\prod_{i=1}^{3} \frac{1 - q^{r_i}}{1 - q}.
\]

2. Put \( r_1, r_2, \ldots, r_l \) in increasing order. If \( r_1 \leq 0 \), \( h(r; k) = 0 \).

3. If \( r_1 = 1 \), then

\[
h(r; k) = h_1(r_2, r_3, \ldots, r_l; k) = \text{trunc}(k + 1 - \sum_{i=2}^{l} \text{trunc}(k - r_i + 1)).
\]

4. If \( r_1 > k \), then \( h(r_1, \ldots, r_l; k) = h(r_1, r_2, \ldots, r_{l-1}; k) \).

5. If \( 2k + 3 > r_1 + r_2 + r_3 \), we set \( r'_1 = r_1 + r_2 + r_3 - 3 + r_i - 2k \) and \( k' = r_1 + r_2 + r_3 - 3 - k \), and

\[
h(r; k) = h(r_1, r_2, r_3, r'_4, \ldots, r'_l; k').
\]

6. \( h(r; k) = h(r_1 - 2, r_2 - 2, \ldots, r_l - 2; k - 3) + 3k - \sum_{i}(k - r_i + 1) \).

We assert that this is indeed an algorithm. Suppose, 1 applies. Then it is well known that the Hilbert series of \( F[x_1, x_2, x_3]/(x_1^{r_1}, x_2^{r_2}, x_3^{r_3}) \) has the asserted form, since it is the tensor product of the \( F[x]/(x^{r_i}) \), as noted above. Hence, we may assume that the \( r_i \) are ordered, and we are in step 2. If \( r_1 \) is less than zero, we already observed that by convention \( h(r; k) \) is zero. If \( r_1 = 0 \), then \( \sum S_j p_j^{r_1} \subset S^k \), and it also follows that \( h(r; k) = 0 \). Hence, we are in step 3. But if \( r_1 = 1 \), the claimed equation is obvious. And in step 4, if \( r_1 > k \), \( S_j p_j^{r_1} \) does not contribute to degree \( k \), so \( r_1 \) may be dropped. We are now at step 5. Here, \( l \geq 4 \), and if \( 2k + 3 > r_1 + r_2 + r_3 \) then \( k' < k \) and \( r'_i < r_i \). Hence step 5 reduces the degree and the \( r_i \) for \( i > 3 \), and because of the transformation rule, it preserves the value of \( h \).

It thus remains to look at step 6. For this we may assume that \( l \geq 4 \), \( 2k + 3 \leq r_1 + r_2 + r_3 \), and \( 2 \leq r_1 \leq r_2 \leq r_3 \leq \cdots \leq r_l \leq k \). Note that we must have \( r_1 \geq 3 \), otherwise \( 2k + 3 \leq r_1 + r_2 + r_3 \) implies \( r_3 > k \). If \( r_1 = 3 \), then
all of the $r_i$ equal $k$ for $i > 1$, by the same arguments. If we can prove that in these circumstances we have

$$h(r; k) = c(r; k) = \binom{k+2}{2} - \sum_{i=1}^{l} \binom{k-r_i+2}{2},$$

then step 6 will yield a correct answer. For this we will need the following lemma.

**Lemma 23.** Assume that $2k + 3 \leq r_1 + r_2 + r_3$, and $r_1 \leq r_2 \leq \cdots \leq r_l \leq k$. Then $c(r; k) > 0$.

**Proof.** Without loss of generality, $l = 9$. Put $m_i = k - r_i + 1$. Then $m_1 \geq m_2 \geq \cdots \geq m_l \geq 0$. Moreover,

$$m_1 + m_2 + m_3 = 3k + 3 - r_1 - r_2 - r_3 \leq k.$$  

We have to show that

$$\binom{k+2}{2} \geq \sum_i \binom{m_i+1}{2} + 1 \quad (28)$$

The right hand side of (28) is

$$\sum_i \frac{m_i(m_i+1)}{2} + 1 = \frac{1}{2} \sum_i (m_i^2 + m_i) + 1 \leq \frac{m_1^2 + m_2^2 + 7m_3^2 + m_1 + m_2 + 7m_3 + 2}{2}.$$  

The left hand side is at least

$$\frac{(m_1 + m_2 + m_3 + 1)(m_1 + m_2 + m_3 + 2)}{2} = \frac{m_1^2 + m_2^2 + m_3^2 + 2m_1m_2 + 2m_1m_3 + 2m_2m_3 + 3m_1 + 3m_2 + 3m_3 + 2}{2} \geq \frac{m_1^2 + m_2^2 + 7m_3^2 + m_1 + m_2 + 7m_3 + 2}{2}.$$  

The last inequality is because $m_3 \leq m_2, m_1$. The lemma follows.

Returning to step 6 of the algorithm, we will prove a slightly stronger statement:

**Proposition 24.** Suppose $0 \leq r_1 \leq r_2 \leq \cdots \leq r_l \leq k$ with $2k + 3 \leq r_1 + r_2 + r_3$, then $h(r; k) = c(r; k) > 0$.

Obviously, proposition 18 in section 4 is a special case. The will be by induction on $k$.  

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Proof. Restricting our attention to points of the form \((1, t_i, t_i^2)\) with \(t_1 + t_2 + \cdots + t_l \neq 0\) and using arguments similar to those in section 3 for four points, we see with \(r_0 = (r_1 - 2, r_2 - 2, \ldots, r_l - 2)\) and \(k_0 = k - 3\) that

\[
h(\mathbf{p}; \mathbf{r}; k) \leq 3k - \sum_{i=1}^{l} (k - r_i + 1) + h(\mathbf{p}; r_0; k_0). \tag{29}
\]

Note that we are using special points here. But these are as generic as any others, as we will see in a moment. One might expect the right hand side of (29) to be too small by one. However, any homogeneous polynomial of degree \(k\) has, when restricted to the curve \((1, t, t^2)\), a zero coefficient in front of \(t^{3k-1}\). Since our points satisfy \(\sum t_i \neq 0\) it is easy to see that this is really a non trivial condition for polynomials vanishing to the given orders. Thus, the possible dimension of the restricted space is one less.

It is clear that \(2k_0 + 3 \leq (r_1 - 2) + (r_2 - 2) + (r_3 - 2)\), hence we are in a similar situation as when we started. There are three cases to consider: First, assume that \(r_1 < k\) for all \(i\). Then also \(r_i - 2 \leq k_0\), and induction yields the result: \(c(\mathbf{r}; k) = 3k - \sum_i (k - r_i + 1) + c(\mathbf{r}_0; k_0) > 0\), which one sees easily by straightforward computations. Thus, we may consider the second case: there is \(i > 3\) with \(r_i = k\). Since the \(r_i\)s are ordered, this holds starting from a \(j\) throughout to \(l\). When computing \(h(\mathbf{r}_0; k_0)\) the points \(p_j, \ldots, p_l\) thus may be dropped. Again, induction and the last lemma assert that \(h(\mathbf{p}; r_0; k) = c(\mathbf{r}_0^*; k_0) > 0\) where \(r_0^* = (r_1 - 2, r_2 - 2, \ldots, r_{j-1} - 2)\). On the other hand, obviously \(0 < c(\mathbf{r}; k) = c(\mathbf{r}^*; k) - \sum_{i=j}^{l} 1\). But the latter is just the right hand side of (29) together with the induction hypothesis. Again, \(\mathbf{r}^*\) is gotten by dropping \(r_j, \ldots, r_l\).

Finally, we have to consider the case when \(j \leq 3\). We are now reduced to prove the following:

\[
3k - (k - r_1 + 1) - (k - r_2 + 1) - 1 + h(\mathbf{p}; r_1 - 2, r_2 - 2, k - 2; k - 3) - (l - 3) \leq c(\mathbf{r}; k)
\]

This follows easily from the fact, that two points always generate independent conditions, if \(r_1 + r_2 \geq k + 3 \geq k\):

\[
(x_1^* F[x_1, x_2, x_3])_k \cap (x_2^* F[x_1, x_2, x_3])_k = (0)
\]

if \(r_1 + r_2 > k\). Thus,

\[
h(\mathbf{p}; r_1 - 2, r_2 - 2, k - 2; k - 3) = h(\mathbf{p}; r_1 - 2, r_2 - 2; k - 3)
\]

and the latter equals

\[
\binom{k - 1}{2} - \binom{k - r_1 + 1}{2} - \binom{k - r_2 + 1}{2},
\]

because \(r_1 + r_2 \geq k + 3\), hence \(r_1 - 2 + r_2 - 2 \geq k - 3\), and the claim follows.

In particular, our \(l\) points are generic, and we may drop \(\mathbf{r}\) in \(h\). \qed
We can now prove:

**Theorem 25.** Suppose \( l = 4 \) or \( 9 \). Then \( H_r = C_r \). In other words \( h(r; k) = c(r; k) \) for all \( r \) and \( k \).

**Proof.** We have already seen the case \( l = 4 \), so we stick with the case \( l = 9 \). All multiplicities are equal now. If we are able to prove that \( 2k + 3 > 3r \) implies that \( h(r; k) = 0 \), we are done, by what we have said above, since we know that the conjectural formula is true if \( 2k + 3 \leq 3r \) and \( r \leq k \).

So suppose \( 2k + 3 > 3r \). This means also that \( 3k \leq 9(k - r + 1) \). If we look at our curve of the form \((1, t, t^3)\) and assume our points sitting on it, it follows that all restrictions vanish. Thus, \( h(r, k) = h(r - 2, k - 3) \). But also \( 2(k - 3) + 3 > 3(r - 2) \), and we may conclude that \( h(r - 2, k - 3) = 0 \), provided \( r < k \). If \( r = k \), then \( 2k + 3 > 3k \) implies \( k = 0, 1, 2 \), and so \( h(k; k) = 0 \). \( \square \)

**Remark 26.** The theorem is also a consequence of the results of Steinberg in [8].
Appendix: The Hilbert series $H_{10,120}$ of $S_r$ for 10 sufficiently generic points in $\mathbb{P}^2$ with $r = 120$

\[
375q^{73} + 741q^{72} + 1098q^{71} + 1446q^{70} + 1785q^{69} + 2115q^{68} + 2436q^{67} + 2748q^{66} + 3051q^{65} + 3345q^{64} + 3630q^{63} + 3906q^{62} + 4173q^{61} + 4431q^{60} + 4680q^{59} + 4920q^{58} + 5151q^{57} + 5373q^{56} + 5586q^{55} + 5790q^{54} + 5985q^{53} + 6171q^{52} + 6348q^{51} + 6516q^{50} + 6675q^{49} + 6825q^{48} + 6966q^{47} + 7098q^{46} + 7221q^{45} + 7335q^{44} + 7440q^{43} + 7536q^{42} + 7623q^{41} + 7701q^{40} + 7770q^{39} + 7830q^{38} + 7881q^{37} + 7923q^{36} + 7956q^{35} + 7980q^{34} + 7995q^{33} + 8001q^{32} + 7998q^{31} + 7986q^{30} + 7965q^{29} + 7935q^{28} + 7896q^{27} + 7848q^{26} + 7791q^{25} + 7725q^{24} + 7650q^{23} + 7566q^{22} + 7473q^{21} + 7371q^{20} + 7260q^{19} + 7140q^{18} + 7021q^{17} + 6903q^{16} + 6786q^{15} + 6670q^{14} + 6555q^{13} + 6441q^{12} + 6328q^{11} + 6216q^{10} + 6105q^{9} + 5995q^{8} + 5886q^{7} + 5778q^{6} + 5671q^{5} + 5565q^{4} + 5460q^{3} + 5356q^{2} + 5253q + 5151q^{1} + 5050q^{0} + 4950q^{9} + 4851q^{8} + 4753q^{7} + 4656q^{6} + 4560q^{5} + 4465q^{4} + 4371q^{3} + 4278q^{2} + 4186q^{1} + 4095q^{0} + 4005q^{8} + 3916q^{7} + 3828q^{6} + 3741q^{5} + 3655q^{4} + 3570q^{3} + 3486q^{2} + 3403q^{1} + 3321q^{0} + 3240q^{9} + 3160q^{8} + 3081q^{7} + 3003q^{6} + 2926q^{5} + 2850q^{4} + 2775q^{3} + 2701q^{2} + 2628q^{1} + 2556q^{0} + 2485q^{9} + 2415q^{8} + 2346q^{7} + 2278q^{6} + 2211q^{5} + 2145q^{4} + 2080q^{3} + 2016q^{2} + 1953q^{1} + 1891q^{0} + 1830q^{9} + 1770q^{8} + 1711q^{7} + 1653q^{6} + 1596q^{5} + 1540q^{4} + 1485q^{3} + 1431q^{2} + 1378q^{1} + 1326q^{0} + 1275q^{9} + 1225q^{8} + 1176q^{7} + 1128q^{6} + 1081q^{5} + 1035q^{4} + 990q^{3} + 946q^{2} + 903q^{1} + 861q^{0} + 820q^{9} + 780q^{8} + 741q^{7} + 703q^{6} + 666q^{5} + 630q^{4} + 595q^{3} + 561q^{2} + 528q^{1} + 496q^{0} + 465q^{9} + 435q^{8} + 406q^{7} + 378q^{6} + 351q^{5} + 325q^{4} + 300q^{3} + 276q^{2} + 253q^{1} + 231q^{0} + 210q^{9} + 190q^{8} + 171q^{7} + 153q^{6} + 136q^{5} + 120q^{4} + 105q^{3} + 91q^{2} + 78q^{1} + 66q^{0} + 55q^{9} + 45q^{8} + 36q^{7} + 28q^{6} + 21q^{5} + 15q^{4} + 10q^{3} + 6q^{2} + 3q^{1} + 1
\]
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