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Trade-off between preamplifier noise figure and decoupling in MRI detectors

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Purpose: There is a limit to the maximum achievable preamplifier decoupling. In many cases, this level is not enough. To overcome this limit, the preamplifier noise figure can be compromised for further decoupling increase. This is useful in flexible MRI arrays where ensuring coil insensitivity to changes in other array elements is a challenge.

Methods: This work establishes the relation between the preamplifier noise figure and preamplifier decoupling using closed-form equations. These equations allow the evaluation of preamplifier decoupling properties and benchmark different preamplifiers against each other. The method to design the corresponding decoupling networks is described. The derived generalized design equations, which are not limited to 50 Ω pre-matched preamplifiers, greatly improve design flexibility and enable use of new amplifiers in MRI detectors.

Results: Using the method, the decoupling properties of three preamplifiers are studied. For demonstration, the coil decoupling is further increased by 10.8 dB using one of the preamplifiers. The noise figure is sacrificed by 0.5 dB, which is predicted by equations and verified experimentally. Although examples are shown for 3 T systems at 32.13 MHz and 127.7 MHz, the approach and equations apply to any field strength and nucleus.

Conclusion: Preamplifier decoupling can be improved beyond what is possible by traditional approaches. The derived design equations cover a wide range of cases, including inductive coils and self-resonant low-impedance and high-impedance coils.

KEYWORDS
matching networks, noise figure, preamplifier decoupling, receiver coil

1 INTRODUCTION

One of the challenges in realizing MRI receive arrays is parasitic coupling between array elements. Although in theory, SNR can be recovered with coupling present, in practice it is often convenient to decouple array elements to ensure correct noise matching to preamplifiers. For this purpose, preamplifier decoupling is widely used...
because it gives more freedom to place coils than coil overlapping.2–4

The most widely used array elements are loop coils. In general, they work in either inductive or self-resonant mode. A schematic representation of such coils with distributed parameters is shown in Figure 1. As coupling between coil elements is generally attributed to current flow on coils, decoupling implies suppressing the current flow,4 for which it is often desirable in an MRI array to present high input impedance $Z_{\text{in}} \to \infty$ to the terminals of inductive coils.4–6 Self-resonant coils can use either series7 or parallel resonance.8,9 Series resonance coils exhibit low coil impedance $Z_{\text{coil}}$ at the resonance frequency. Consequently, high $Z_{\text{in}} \to \infty$ should be presented to series resonance coils. Parallel resonance coils exhibit high impedance and are often called high-impedance coils.8 Low $Z_{\text{in}} \to 0$ preamplifiers can increase decoupling in parallel-resonance coils8,10 and have become useful components in designing many MRI receive arrays.4 The extreme requirements to $Z_{\text{in}}$ can be met by purely reactive impedance preamplifiers4–6,11 (i.e., $\text{Re}Z_{\text{amp}} = 0$), which are hardly found in practice.12 For that reason, all typical preamplifiers provide finite decoupling.

In receive arrays, dedicated matching networks transform input impedance of preamplifiers $Z_{\text{amp}}$ to low or high $Z_{\text{in}}$, and at the same time transform coil impedance $Z_{\text{coil}}$ to the optimal noise impedance of preamplifiers $Z_{\text{opt}}$ to reach the best noise performance of the preamplifier. However, sometimes it is desirable to transform $Z_{\text{coil}}$ elsewhere and sacrifice a small amount of noise figure in exchange for more decoupling, which can allow even more freedom to coil placement (e.g., construction of non-overlapping arrays).13 Up to present, there has not been any quantitative description of the trade-off between decoupling and preamplifier noise figure in MRI detectors.

In this article, a theory that quantitatively describes the trade-off between preamplifier decoupling and preamplifier noise figure is developed. Using this theory, the level of achievable decoupling as a function of preamplifier noise figure can be predicted and later used as a design parameter. The design equations for corresponding matching and decoupling networks using non-ideal preamplifiers ($\text{Re}Z_{\text{amp}} > 0$) are derived here for the first time. The equations are given for series resonance (low-impedance) and parallel resonance (high-impedance) coils and cover the design of preamplifiers exhibiting low input impedance. It is also shown that the derived design equations degenerate to the special case of ideal preamplifiers5,11 having $\text{Re}Z_{\text{amp}} = 0$.

The design approach is demonstrated by evaluating the decoupling-noise figure relation of several preamplifiers and matching these preamplifiers to chosen noise figures. Calculations are verified by decoupling simulations and measurements of the design example matched to a minimum noise figure $NF_{\text{min}}$ and to $NF_{\text{min}} + 0.5$ dB.

## 2 | THEORY

### 2.1 | Relation between decoupling level and matching network impedance

The considered receiving coil setup is shown in Figure 2A. The sample loaded coil and the preamplifier can be equivalently represented by their impedances $Z_{\text{coil}} = R_{\text{coil}} + jX_{\text{coil}}$ and $Z_{\text{amp}} = R_{\text{amp}} + jX_{\text{amp}}$, respectively. The signal from the sample induces voltage $V$, and accordingly, current $I$ on the coil. It is assumed that the matching-decoupling network is passive, lossless, reciprocal, and has input and output impedances $Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}}$ and $Z_{\text{out}} = R_{\text{out}} + jX_{\text{out}}$, respectively. The preamplifier is described by an impedance matrix $Z_{\text{amp}}$ and has output impedance $Z_{\text{out}}$. It is also assumed that $R_{\text{coil}} > 0$, $R_{\text{amp}} \geq 0$, $R_{\text{in}} \geq 0$, $R_{\text{out}} > 0$.

| Inductive | Self-resonant |
|-----------|--------------|
| **Series resonance (low impedance)** | **Parallel resonance (high impedance)** |
| ![Illustration](image) | ![Illustration](image) |

**Figure 1** Loop coils in scope. $Z_{\text{coil}}$ is the coil input impedance. $Z_{\text{in}}$ is the expected input impedance of the matching network. Self-resonant coils can be used in series or in parallel resonance. The resonance is formed by wire inductance and capacitive coupling between wires. Note the circuits represent input impedances only, and they are not the actual circuit models.
The relation between decoupling and noise figure is derived below using these impedances as input parameters. Because decoupling is related to the current magnitude $|I|$, the problem becomes finding the range of $|I|$ for a given noise figure. The preamplifier noise figure is determined by the source impedance $Z_{\text{out}}$. The power reflection coefficient at port 1 at the input of the matching-decoupling network is \(^{14,15}\)

$$\Gamma_{\text{in}} = \frac{Z_{\text{in}} - Z_{\text{out}}}{Z_{\text{in}} + Z_{\text{out}}}.$$  \hfill (1)

where * indicates complex conjugate. The power reflection coefficient at the matching-decoupling network output, port 2, is

$$\Gamma_{\text{out}} = \frac{Z_{\text{out}} - Z_{\text{amp}}^*}{Z_{\text{out}} + Z_{\text{amp}}}.$$  \hfill (2)

For passive, lossless two-port matching networks, \(^{14}\)

$$|\Gamma_{\text{in}}| = |\Gamma_{\text{out}}|.$$  \hfill (3)

As follows from Equation (2), $\Gamma_{\text{out}}$ is uniquely defined by the inherent parameters of the preamplifier, namely, preamplifier input impedance $Z_{\text{amp}}$ and equivalent source impedance $Z_{\text{out}}$ presented to the preamplifier. Hence, $|\Gamma_{\text{in}}|$ is also uniquely defined by Equation (3). Ohm’s law relates the current $I$ and the voltage $V$, which can be expressed using Equation (1):

$$\frac{I}{V} = \frac{1}{Z_{\text{in}} + Z_{\text{out}}} = \frac{1 - \Gamma_{\text{in}}}{2R_{\text{coil}}}.$$  \hfill (4)

Obviously, when $Z_{\text{out}} = Z_{\text{amp}}^*$, the problem reduces to conventional complex conjugate matching (maximum
power transmission\(^{15}\) and \(\Gamma_{\text{in}} = \Gamma_{\text{out}} = 0.\) In that case,

\[
\left. \frac{I}{V} \right|_{\Gamma_{\text{in}} = 0} = \frac{1}{2R_{\text{coil}}}. \tag{5}
\]

Therefore, the quantity

\[
1 - \Gamma_{\text{in}} = \frac{I/V}{\left. \frac{I}{V} \right|_{\Gamma_{\text{in}} = 0}}, \tag{6}
\]

is the current \(I\) flowing along the coil normalized to the current under complex conjugate matching and the same voltage \(V\). As is widely recognized, the lower the magnitude \(|I|\), the higher the decoupling\(^{4,6}\) This is because, in arrays, loop coils couple to each other mainly by mutual inductance.\(^{4,6,16}\) The lower the magnitude \(|I|\) is on one coil, the less it interferes with other elements through mutual inductance, and the less it is sensitive to current flows in other elements. The quantity \(1 - \Gamma_{\text{in}}\) can be reasonably defined as preamplifier decoupling, and it can be converted to decibels by

\[
\text{Decoupling (dB)} = -20 \log |1 - \Gamma_{\text{in}}|, \tag{7}
\]

which is usually measured using a double-loop probe connected to a vector network analyzer (VNA).\(^5\) Maximum decoupling is achieved by minimizing \(|1 - \Gamma_{\text{in}}|\) for which high impedance \(Z_{\text{in}}\) at port 1 (refer to Figure 2A) is reached. In case of high-impedance coils, \(|1 - \Gamma_{\text{in}}|\) should be maximized leading to low impedance \(Z_{\text{in}}\) at port 1. The range of \(|1 - \Gamma_{\text{in}}|\) is

\[
1 - |\Gamma_{\text{in}}| \leq |1 - \Gamma_{\text{in}}| \leq 1 + |\Gamma_{\text{in}}|. \tag{8}
\]

The conditions for reaching the maximum and the minimum are, respectively,

\[
\begin{align*}
\text{min. (high impedance): } & \angle \Gamma_{\text{in}} = 0, \\
\text{max. (low impedance): } & \angle \Gamma_{\text{in}} = -\pi.
\end{align*} \tag{9}
\]

where the angle of \(\Gamma_{\text{in}}\) is defined as \(-\pi \leq \angle \Gamma_{\text{in}} < \pi\). By Equation (3), Equation (8) can be written as

\[
1 - |\Gamma_{\text{out}}| \leq |1 - \Gamma_{\text{in}}| \leq 1 + |\Gamma_{\text{out}}|. \tag{10}
\]

Equation (10) can be illustrated on a \(\Gamma\) plane, that is, Smith chart, as shown in Figure 3A. Consider the reflection coefficient \(\Gamma_{\text{out}}\) calculated by Equation (2) for arbitrary \(Z_{\text{out}}\). The maximum and minimum of \(|1 - \Gamma_{\text{in}}|\) can also be interpreted as follows: if a line is drawn through \(\Gamma_{\text{out}}\) and the origin \(O\), and we let the line \(\overline{OP}\) intersect with the circle \(|\Gamma| = 1\) at points \(P\) and \(Q\), we have

\[
\begin{align*}
\overline{Q_{\text{out}}} = 1 + |\Gamma_{\text{out}}| &= \max |1 - \Gamma_{\text{in}}|, \\
\overline{P_{\text{out}}} = 1 - |\Gamma_{\text{out}}| &= \min |1 - \Gamma_{\text{in}}|.
\end{align*} \tag{11}
\]
For a given preamplifier, noise figure $F$ is determined by the source impedance connected to the preamplifier input terminals (in this article, $F$ denotes noise figure in linear scale, and $NF$ denotes noise figure in dB; $NF = 10 \log F$). Accordingly, for a given noise figure $F > F_{\text{min}}$, a range of $Z_{\text{out}}$ or $\Gamma_{\text{out}}$ exists with a corresponding range of $1 - \Gamma_{\text{in}}$. There must be at least one $Z_{\text{out}}$ in this range that minimizes $|1 - \Gamma_{\text{in}}|$ and achieves maximum decoupling. For a given $F$, therefore, decoupling has a maximum.

### 2.2 Relation between decoupling and noise figure

In this section, a method to quantify the relation between decoupling and noise figure is proposed based on the following steps ended by:

1. For a given $F \geq F_{\text{min}}$, determine a range of corresponding $Z_{\text{out}}(F)$. The relation between $Z_{\text{out}}$ and $F$ is

$$Z_{\text{out}}(F) = C_{\Gamma \Phi}(F) + r_{\Gamma \Phi}(F)e^{j\phi},$$

where $-\pi \leq \phi < \pi$, and $C_{\Gamma \Phi}(F)$ and $r_{\Gamma \Phi}(F)$ are the centre and radius of a constant-noise circle at a noise figure $F$, which are found from amplifier noise parameters by

$$
\begin{align*}
C_{\Gamma \Phi}(F) &= Z_{n,\text{opt}} + \frac{\Delta}{2}, \\
r_{\Gamma \Phi}(F) &= \sqrt{R_{n,\text{opt}} \Delta + \left(\frac{\Delta}{2}\right)^2},
\end{align*}
$$

where $Z_{n,\text{opt}} = Y_{n,\text{opt}}^{-1}$ is the optimal noise impedance of the preamplifier,

$$
\Delta = (F - F_{\text{min}}) \frac{|Z_{n,\text{opt}}|^2}{R_{n}},
$$

in which $F$ and $F_{\text{min}}$ are the noise figure and the minimum noise figure in linear scale, respectively. $R_{n}$ is the equivalent noise resistance.

2. For each $Z_{\text{out}}$, determine the corresponding $Z_{\text{amp}}$ when the preamplifier’s output is complex conjugate matched. We have

$$
\begin{align*}
Z_{\text{out},a}(F) &= Z_{a,22} - \frac{Z_{a,12}Z_{a,21}}{Z_{a,11} + Z_{\text{out}}(F)}, \\
Z_{\text{amp}}(F) &= Z_{a,11} - \frac{Z_{a,12}Z_{a,21}}{Z_{a,22} + Z_{\text{out}}(F)},
\end{align*}
$$

where $Z_{\text{out},a}(F)$ is the preamplifier’s output impedance and $Z_{a,11}, Z_{a,12}, Z_{a,21}, Z_{a,22}$ are the elements of the preamplifier’s impedance matrix $Z_{a}$, as shown in Figure 2A.

3. Compute a range of $\Gamma_{\text{out}}(F)$ from $Z_{\text{out}}(F)$ and $Z_{\text{amp}}(F)$ using Equation (2). The resulting constant-noise contours $\Gamma_{\text{out}}(F)$ can be drawn in the $\Gamma$ plane. It should be noted that as preamplifiers are in general not unilaterals, that is, $Z_{a,12} \neq 0$, $Z_{\text{amp}}(F)$ is not constant, which implies that the constant-noise contours $\Gamma_{\text{out}}(F)$ are not circles. As, in general, for a given $F$, maximum decoupling is desired, $Z_{\text{out}}$ is chosen so that $|\Gamma_{\text{out}}(F)| = \max |\Gamma_{\text{out}}(F)|$. The maximum achievable decoupling with respect to a given $F$ is

$$
\begin{align*}
\text{Max. decoupling (dB)} &= -20 \log \min |1 - \Gamma_{\text{in}}| \\
&= -20 \log [1 - \max |\Gamma_{\text{out}}(F)|] \quad (16)
\end{align*}
$$

At this point, the desired high or low impedance can be achieved.

On a $\Gamma$ plane, this means that $\Gamma_{\text{out}}$ is moved along a constant noise contour such that $\overline{PF_{\text{out}}}$ shrinks to its shortest length, whereas $Q\Gamma_{\text{out}}$ grows to its longest length (min $|1 - \Gamma_{\text{in}}| = \overline{PF_{\text{out}}}$ and max $|1 - \Gamma_{\text{in}}| = Q\Gamma_{\text{out}}$ according to Equation [11]). At this point $\Gamma_{\text{out}} = \Gamma_{\text{out}}^{\text{opt}}$, and decoupling is $|1 - \Gamma_{\text{in}}| = |\Gamma_{\text{out}}^{\text{opt}}|$. For this noise figure $F$, the maximum decoupling is reached. Conversely, for this particular decoupling level $\Gamma_{\text{out}}^{\text{opt}}$, the minimum $F$ is also reached. For every $F \geq F_{\text{min}}$, steps 2.2.1–2.2.3 are repeated, resulting in a curve of decoupling versus noise figure as shown in Figure 3B. This curve illustrates both the maximum decoupling at a given $F$ and minimum $F$ at a given decoupling value.

When a preamplifier input impedance is insensitive to output load because of high reverse isolation or fixed load termination like 50Ω, the constant-noise contours $\Gamma_{\text{out}}(F)$ become circles, and Equation (16) has a relatively simple analytical form, as shown in Supporting Information A. Nonetheless, the approximate form in Supporting Information A requires preamplifier input impedance $Z_{\text{amp}}$, which, if unknown, must be calculated through step (2); to obtain the corresponding matching network output impedance $Z_{\text{out}}$, the rationale in step (3) (i.e., finding the $\Gamma_{\text{out}}(F)$ such that the decoupling reaches the maximum) is still required. Therefore, in this article, the precise forms in Section 2.2 are used.

### 2.3 Z matrices of matching/decoupling networks

Now that the relation between decoupling and noise figure has been established, the next step is to calculate the impedance transformation the matching network should perform to reach the desired matching point. The impedance matrices of possible matching-decoupling networks can be calculated from conditions (9) at port 1 (refer to Figure 2A), given that $Z_{\text{out}}$ is chosen as described in...
Section 2.2. From Equation (1), Equation (8), conditions Equation (9) can be rewritten as

$$HZ : X_{in} = -X_{coil} \land R_{in} > R_{coil},$$
$$LZ : X_{in} = -X_{coil} \land 0 \leq R_{in} < R_{coil},$$

(17)

where $\land$ means “logical AND”; “HZ” means “high impedance”, “LZ” means “low impedance”. The case $X_{in} = -X_{coil} \land R_{in} = R_{coil}$ is complex conjugate matching at both input and output ports, which has been well analyzed in textbooks and excluded from this analysis. From Equations (1), (3), and (17), the conditions of input impedance are derived as

$$HZ : Z_{in} = Z^{HZ}_{in} = \beta R_{coil} - jX_{coil},$$
$$LZ : Z_{in} = Z^{LZ}_{in} = \frac{R_{coil}}{\beta} - jX_{coil},$$

(18)
(19)

where

$$\beta = \frac{1 + |\Gamma_{out}|}{1 - |\Gamma_{out}|} > 1,$$

(20)

can be interpreted as the power standing wave ratio at port 2 (refer to Figure 2A); $\Gamma_{out}$ is defined by Equation (2).

The impedance matrix of the lossless reciprocal matching network in Figure 2A is expressed as

$$Z = j \begin{bmatrix} X_{11} & X_o \\ X_o & X_{22} \end{bmatrix},$$

(21)

where $X_{11}$, $X_{22}$, and $X_o$ are real numbers. Using the input and output impedances found above, the $Z$ matrix Equation (21) can be calculated as shown in Appendix A and Supporting Information B.

2.3.1 High-impedance case

For the high-impedance case, we have

$$\begin{cases} 
X_{11} = X^{HZ}_{11} = -X_{coil} + \beta R_{coil} \eta_{HZ}, \\
X_{22} = X^{HZ}_{22} = -X_{amp} + \beta R_{amp} \eta_{HZ}, \\
X_o^2 = X^{HZ}_{o} = R_{coil} R_{out} (1 + \beta^2 \eta_{HZ}^2),
\end{cases}$$

(22)

where $X_{amp} + X_{out} \neq 0$, or $X_{amp} + X_{out} = 0 \land R_{out} > R_{amp} \geq 0$, and $\eta_{HZ} = (X_{amp} + X_{out}) / (R_{amp} - R_{out})$.

The sign of $X^{HZ}_o$ can be positive or negative; each corresponds to one set of solutions. When $R_{amp} \to 0$, Equation (22) gives the same three-element matching networks described by Wang et al. When $R_{amp} \to 0$, $X_{out} = 0$, Equation (22) describes four-element matching networks proposed by Reykowski et al., as shown in Supporting Information C.

2.3.2 Low-impedance case

Similarly, for the low-impedance case, we have

$$\begin{cases} 
X_{11} = X^{LZ}_{11} = -X_{coil} + R_{coil} \eta_{LZ}, \\
X_{22} = X^{LZ}_{22} = -X_{amp} + \beta R_{amp} \eta_{LZ}, \\
X_o^2 = X^{LZ}_{o} = R_{coil} R_{out} (1 + \eta_{LZ}^2),
\end{cases}$$

(23)

where $X_{amp} + X_{out} \neq 0$, or $X_{amp} + X_{out} = 0 \land R_{amp} > R_{out} > 0$, and $\eta_{LZ} = (X_{amp} + X_{out}) / (\beta R_{amp} - R_{out})$.

Likewise, the sign of $X^{LZ}_o$ can be positive or negative; each corresponds to one set of solutions.

Equation (23) can be used to design impedance-transforming networks inside preamplifiers that exhibit $Z_{in,opt} = 50 \Omega$ and low $|Z_{amp}|$ to users by setting $Z_{coil} = 50 \Omega$.

Equations (22) and (23) can be converted to three-element, four-element, and other circuit topologies as needed. Methods can be found in RF textbooks.

2.4 A corollary: disconnect-and-resonate

It is interesting to note that, for the high-impedance case, when the preamplifier is disconnected from a well-tuned matching network, $Z_{in}$ becomes

$$\lim_{Z_{amp} \to \infty} Z^{HZ}_{in} = j X^{HZ}_{11} = j (-X_{coil} + \beta R_{coil} \eta_{HZ})$$

(24)

If $X_{amp} + X_{out} = 0$,

$$\lim_{Z_{amp} \to \infty} Z^{HZ}_{in} = -j X_{coil}.$$
METHODS

To verify the theory and to demonstrate a design process using matching networks’ Z matrices formulated in Section 2, the following validation process is adopted: numerical examples for both high- and low-impedance cases at several frequencies are made (Section 3.1). Decoupling-noise figure trade-off and values for corresponding matching networks for the numerical examples in Section 3.1 are calculated (Section 3.2).

Decoupling improvement is verified experimentally. For this purpose, two receive coils are fabricated; decoupling and SNRs are measured and compared against calculated and electromagnetic (EM) co-simulated results (Section 3.3).

3.1 | Numerical examples

The chosen design frequencies are 32.13 MHz and 127.7 MHz. These are the resonance frequencies of 13C and 1H at 3 T. The design examples A to F include HZ and LZ cases and a range of noise figures ("Target") as listed in Table 1.

Three preamplifiers are used: WMA32C (WanTCom), ElCry1-u (datasheet available in ElCry Electronics), and a transistor BFP740 (Infineon Technologies). ElCry1-u and WMA32C are used at 32.13 MHz, and the BFP740-based preamplifier is used at 127.7 MHz. Details of the BFP740 transistor’s biasing network are shown in Figure 4. The loop coil in cases A, B, D is made of a flat conductor of inner diameter 70 mm, outer diameter 82 mm and thickness 17 µm. The coil is loaded with a phantom based on saltwater solution emulating a human thigh. Zcoil is measured by a double-loop probe. The Zcoil value at 127.7 MHz is derived from simulations of 62.5 mm diameter wire coil with 2.5 mm wire diameter. Zcoil values of self-resonant coils (“LZ”, cases C, F) in Table 1 are taken from the literature.8

3.2 | Design steps

To illustrate the design procedure, the case B in Table 1 is described below in detail. All other cases are analyzed using the same procedure. The procedure is also summarized in Figure 2B.

| TABLE 1 | Example cases |
|----------|---------------|
|          | WMA32C | ElCry1-u | BFP740 |
| f0, MHz | 32.13  | 32.13  | 127.7  |
| Z12,amp, Ω | 2.45∠−22.5° | 551∠−84.1° | 414∠−25.2° |
| Z22,amp, Ω | 0.013 3∠56.8° | 0.320 6 10° | 1.62∠12.5° |
| Z22,amp, Ω | 1338∠−161° | 2573∠99.2° | 6760∠98.8° |
| Zopt, Ω | 51.9∠5.09° | 12.5∠13.6° | 37.0∠67.3° |
| Zopt, Ω | 52.0∠0.80 0° | 114∠10.2° | 84.4∠4.67° |
| Ropt, Ω | 4.00 | 1.80 | 5.62 |
| NFmin, dB | 0.680 | 0.135 | 0.583 |
| Case no. | (A) | (B) | (C) | (D) | (E) | (F) |
| Target | HZ | HZ | LZ | HZ | HZ | LZ |
| Zout, Ω | 52.0∠0.800° | 186∠−2.51° | 114∠10.2° | 35.8∠0.361° | 45.5∠0.143° |
| Zout, Ω | 2.43∠−18.5° | 2.43∠−18.5° | 583∠−83.6° | 551∠−47.0° |
| Rout, Ω | 0.382 | 0.382 | 2.0×10³ | 0.382 | 0.536 | 2.0×10³ |
| Xout, Ω | 33.9 | 33.9 | 0 | 33.9 | −285 | 0 |
| X11, Ω | −3.39 | −3.39 | −396 | −27.7 | 294 | −121 |
| X22, Ω | 0.770 | 0.770 | −1.95 | 587 | 751 | −2.64 |
| | 4.56 | 8.43 | 483 | 60.6 | 81.7 | 302 |

Note: For convenience, the BFP740-based preamplifier is referred to as “BFP740.” In the row “Case,” “HZ” means “maximum decoupling (high impedance),” and “LZ” means “low impedance.” “HZ” columns are calculated by Equation (22) and “LZ” columns by Equation (23). The cases are: (A) a loaded coil to WMA32C for HZ at minimum noise figure NFmin (B) and at NFmin + 0.5 dB; (C) a high impedance coil for LZ to ElCry1-u at NFmin; (D) the same coil as in (A) and (B) to ElCry1-u for HZ at NFmin + 0.1 dB; (E) another coil at 127.7 MHz to the BFP740-based preamplifier for HZ at NFmin + 0.1 dB; (F) a high impedance coil to the BFP740-based preamplifier at NFmin + 0.1 dB. Refer to Section 3.1 for more information on coils and preamplifiers.
1 Extract the preamplifier’s Z matrix and noise parameters \((R_n, NF_{\text{min}}, Z_{\text{out,opt}})\) at 32.13 MHz. These parameters are provided by the preamplifier manufacturer typically in a datasheet or extracted from the preamplifier model. In this case, Z matrix is obtained by transforming the scattering S matrix\(^\text{15}\) available. The extracted parameters are listed in Table 1, column “WMA32C”.

2 For each noise figure \(F \geq F_{\text{min}}\), calculate \(Z_{\text{amp}}(F)\) and \(Z_{\text{out}}(F)\) that maximize decoupling, i.e., \(|I_{\text{out}}(F)| = \max |I_{\text{out}}(F)|\) using Equations (12–14) for \(Z_{\text{amp}}(F)\), Equation (15) for \(Z_{\text{out}}(F)\), Equation (2) for \(I_{\text{out}}(F)\), and Equation (16) for maximum decoupling. In this example, the NF ranges from \(NF_{\text{min}} = 0.68\) dB to 1.28 dB in 0.01 dB steps. The computation is performed using MATLAB (The MathWorks).

3 Plot the decoupling versus noise figure. The result is illustrated in Figure 3B, curve “WMA32C, 32.13 MHz” in a solid line. For comparison, the decoupling versus noise figure with fixed \(Z_{\text{amp}} = Z_{\text{amp}}(F_{\text{min}})\) is also shown in Figure 3B with dashed lines.

4 Choose the desired decoupling improvement and read the corresponding noise figure (i.e., target) from Figure 3B. For example, compared with \(NF_{\text{min}} = 0.68\) dB, 10.8 dB more decoupling can be achieved at \(NF = NF_{\text{min}} + 0.5\) dB = 1.18 dB. In this example, \(NF_{\text{min}} + 0.5\) dB becomes the matching target, as listed in Table 2. This noise figure is achieved when \(Z_{\text{out}} = 186\angle -2.51^\circ\Omega\), \(Z_{\text{amp}} = 2.43\angle -18.5^\circ\Omega\) as follows from the steps in Section 2.2.

5 Find the coil impedance by measurement or simulation. In this case, the measured coil impedance is \(Z_{\text{coil}} = 0.382 + j3.39\) \(\Omega\) at 32.13 MHz.

6 Use this Z matrix to design the circuit. For this high impedance case, Equation (22) is used. For low impedance, Equation (23) should be used. For this case, the condition after Equation (22), that is, \(X_{\text{amp}} + X_{\text{out}} \neq 0\), or \(X_{\text{amp}} + X_{\text{out}} = 0 \land R_{\text{amp}} > R_{\text{amp}} \geq 0\), is satisfied, and solutions exist. It should be noted that if the condition after Equation (22) or Equation (23) is not satisfied, there is no solution, and the reader should stop this design process and change a preamplifier.

The design steps (1) to (3) in Section 3.2 are repeated for all three amplifiers in Section 3.1. The resulting curves are drawn in Figure 3B. For comparison, the decoupling curves versus noise figure with fixed \(Z_{\text{amp}} = Z_{\text{amp}}(F_{\text{min}})\) are also shown in Figure 3B with dashed lines. For WMA32C, the highest decoupling possible is 21.4 dB at \(NF_{\text{min}}\) and 32.2 dB at \(NF_{\text{min}} + 0.5\) dB, respectively. For ElCry1-u and the BFP740-based preamplifier, the highest decoupling that preserves minimum noise figure \(NF_{\text{min}}\) are 27.3 dB and 14.6 dB, respectively. The general trend is that, as NF increases, the decoupling increases non-linearly. For all three preamplifiers in Section 3.1, the decoupling curves versus noise figure with fixed \(Z_{\text{amp}} = Z_{\text{amp}}(F_{\text{min}})\) overlap the curves with varying \(Z_{\text{amp}}\).

### 3.3 Experimental demonstration

To experimentally demonstrate that the same preamplifier can be manipulated to further increase the decoupling in a controlled manner, cases A and B from Table 1 were fabricated and tested. Both cases are based on the same
FIGURE 5  (A) A circuit diagram of the designed matching networks. Coils are integrated on PCBs and are equivalently represented by series connection of $R_{L1}$ and $L_{L1}$ on the circuit diagram. $J_1$ is a jumper. This matching network structure can match the coil to $N_{F_{\text{min}}}$ and $N_{F_{\text{min}}} + 0.5$ dB of WMA32C. $Z_{\text{out}}$ is measured at Pin 1 of $J_1$ by SProbe1. (B) Complex conjugate matching to 50 $\Omega$. Coils are represented by $R_{L1}$ and $L_{L1}$. To reduce capacitor loss, capacitors $C_{II}$ in (A) and $C_{M1}$, $C_{M2}$ in (B) are formed by combining fixed low loss capacitors (PFI1111C, Passive Plus). $C_{I}$, $C_{II}$ contain trimmer capacitors (SGC3S series, EW Electronics) for precise tuning. (C) The decoupling measurement setup. A white PCB is fixed by screws onto a support that holds a phantom underneath. A double-loop probe is placed $\sim 5$ cm above the coil.

Matching network topology in Figure 5A. The photograph of the printed circuit board (PCB) and the schematic of the implemented circuit are shown in Figure 5A,C. The experimental setup allows measuring the decoupling, SNR, and $Z_{\text{out}}$ (using SProbe1 as shown in Figure 5A).

A double-loop probe connected to a VNA$^5$ (Keysight E5062A, Keysight Technologies) is used to measure the decoupling. To increase the dynamic range of measurement and compensate for parasitic cross-talk between probe loops, the probe is first characterized in free space. The $S_{21}$ measured and averaged over 360 sweeps is used for calibration. This complex $S_{21}$ is deducted from all following decoupling measurements. The measured $|S_{21}|$ of a complex conjugate matching to 50 $\Omega$—of which the circuit is shown in Figure 5B—is used as a reference for decoupling evaluation.

For SNR measurements, the preamplifier is mechanically fixed in a shielded box. The preamplifier output is connected to a spectrum analyzer (Keysight E4440A). A single loop connected to a signal generator (Rohde and Schwarz SMC100A) is used as a signal source. SNR values are averaged and fitted by normal distribution.

For comparison, Advanced Design System (Keysight Technologies) is used for EM co-simulation of the PCB, together with component models from Passive Plus and Coilcraft. The simulated decoupling, $Z_{\text{out}}$, noise figure and output SNR are listed in Table 2.

4 | RESULTS

EM co-simulated and measured $|S_{21}|$ (decoupling) SNR are shown in Figure 6, wherein Figure 6B also shows $\sigma$ intervals, among which the widest is $-0.126$ dB to $+0.122$ dB. Calculated, EM co-simulated, and measured $Z_{\text{out}}$, decoupling, noise figure, and output SNR, are listed in Table 2. The highest possible decoupling values at $N_{F_{\text{min}}}$ and $N_{F_{\text{min}}} + 0.5$ dB reported by EM co-simulation drop to 18.3 dB and 29.0 dB, respectively. Measured decoupling values are, however, 20.2 dB and 31.3 dB, closer to theoretical values. As can be observed from the measurement results in Table 2, using the design with the target $N_{F_{\text{min}}} + 0.5$ dB, the SNR drops by 0.27 dB, whereas the decoupling rises by 11.1 dB, which is reasonable, taking into account measurement uncertainties.

5 | DISCUSSION

5.1 | Theory in Section 2

According to Equations (12–16), decoupling is explicitly defined by the impedance parameters and noise parameters of the given preamplifier. The matching and decoupling networks can be designed with Equations (22) and (23) using preamplifiers that do not have low $R_{\text{amp}}$ or
purely real $Z_{n,\text{opt}}$, as long as the boundary conditions are fulfilled. This opens up for much greater flexibility in matching network design and widens a range of applicable preamplifiers compared with traditional design schemes.\(^5\)

For a given coil and a given preamplifier, once $Z_{\text{out}}$ and case (“HZ” or “LZ”) are chosen, the $Z$ matrices of matching networks are defined by Equations (22) or (23). Designers can transform the $Z$ matrices to circuit structures as needed.\(^15\) The designer could also change the target noise figure (e.g., from $NF_{\text{min}} + 0.1$ dB to $NF_{\text{min}} + 0.2$ dB), which corresponds to noise sacrifice of 2.33%, so that the $Z$ matrices can give suitable values. This makes network design even more flexible.

### 5.2 On Z matrices of matching/decoupling networks in Section 2.3

Equations (22) and (23) match between two complex impedances. As matching between two complex impedances has been used for noise matching for a long time, at first glimpse, Equations (22) and (23) tell nothing beyond general matching methods. However, general matching between complex impedances lacks condition Equations (18) or (19) and therefore, cannot apply to preamplifier decoupling. In general complex matching, a designer aims at transforming real and imaginary parts of a source impedance to an amplifier noise impedance. It requires two degrees of freedom and can be done with two reactive components.\(^21\) In preamplifier decoupling, an additional condition Equations (18) or (19) that minimizes source current is needed, which requires one more degree of freedom. The circuit, therefore, needs three components at least.

Consider the setup in Figure 7A,B. The design input is $Z_{\text{coil}} = 0.382 + j33.9 \Omega$, $Z_{\text{out}} = Z_{n,\text{opt}} = 51.99 + j0.07 \Omega$, $Z_{\text{amp}} = 2.304 – j0.771 \Omega$. The common practice of noise matching transforms $Z_{\text{coil}}$ to $Z_{n,\text{opt}}$, which is described by Bahl.\(^21\) The resulting circuit is shown in Figure 7A. For comparison, the matching/decoupling network designed by Equation (22) is shown in Figure 7B. It is seen that both networks present $Z_{n,\text{opt}}$ to the amplifier terminals, which ensures minimum noise. The impedance presented to the coil, $Z_{\text{in}}$, is different, resulting in disparate preamplifier decoupling levels.

### 5.3 Experimental results in Section 4

The maximum decoupling by EM co-simulation is lower than calculated values by 3.1 dB and 3.2 dB for $NF_{\text{min}}$ and $NF_{\text{min}} + 0.5$ dB, respectively. Although measured decoupling is closer to theory at 32.13 MHz, the values are still lower by 1.2 dB at $NF_{\text{min}}$ and 0.9 dB at $NF_{\text{min}} + 0.5$ dB, respectively. Apart from this, 0.5 dB noise figure sacrifice in theory leads to output SNR decrease by 0.27 dB in the experiment.

These differences are attributable to the matching network being assumed perfectly lossless in the theoretical analysis, whereas in reality, all passive components and substrates carry loss. Obviously, the lower the loss in the components of the fabricated matching network, the closer the measurement results will be to the theoretically predicted values. The simulated decoupling value deviates more from the theoretical value than measured because simulation of decoupling depends on $R_{\text{coil}}$ estimation, but acquisition of $R_{\text{coil}}$ is less accurate when coil $Q$ is high and $R_{\text{coil}}$ is low. In this case, $R_{\text{coil}} = 0.382 \Omega$, which means that an error of 0.05 $\Omega$ results in 13% change in $R_{\text{coil}}$. The error accumulates when simulated $|S_{21}|$ of complex conjugate matching to 50 $\Omega$ by lossy components (Figure 5B) is used as a reference for decoupling evaluation.

Overall, the theory predicts the trend in the applications where decoupling has to be further improved or where better design flexibility is required and the noise figure can be used as an additional degree of freedom.
5.4 Effects in arrays for MR imaging

It has been demonstrated experimentally by Sánchez et al.\textsuperscript{22} that the trade-off between decoupling and noise figure can be used to increase decoupling, which, in that case, also results in better SNR as arrays become more robust to detuning. The authors use the same preamplifier as in Section 3.3 (WanTCom WMA32C) and the same matching network topology as in Figure 5A (please also refer to Figure 1 of the Sánchez et al.\textsuperscript{22} paper) and build 2 two-channel arrays.\textsuperscript{22} In array 1, coils are matched to 50 Ω with preamplifier noise figure 0.68 dB (in the paper of Sánchez-Heredia et al.\textsuperscript{22} this preamplifier noise figure is measured as 0.55 dB; here, the 0.68 dB noise figure as extracted from the S parameter file of WMA32C is used for consistency); in array 2, coils are matched to 100 Ω with preamplifier noise figure 0.82 dB (refer to Table 1 of the Sánchez et al. paper\textsuperscript{22}). The preamplifier noise figure in array 2 is, therefore, 0.14 dB higher than in array 1, or 3.2% more noise figure in linear scale. The SNR will drop by ∼3.2%, ceteris paribus. However, phantom imaging shows that array 2 offers maximum SNR of 197, 21.6% higher than the maximum SNR of array 1, 162 (refer to Figure 4A,B of the Sánchez et al. paper\textsuperscript{22}). The results are summarized in Table 3. Although preamplifier noise figure degrades slightly, the extra decoupling earned makes the array more robust to detuning, leading to better SNR on images.
6 | CONCLUSION

The theory presented quantifies the trade-off between preamplifier decoupling and preamplifier noise figure and applies to inductive and self-resonant coils. The design equations for matching networks that satisfy such conditions are derived. These equations significantly relax restrictions on the preamplifier parameters, which enables more freedom in the design of receive elements of MRI arrays. The approach is illustrated using a design example.

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SUPPORTING INFORMATION

Additional supporting information may be found in the online version of the article at the publisher’s website.
APPENDIX A. DERIVING MATCHING NETWORKS’ IMPEDANCE MATRICES

Consider the setup in Figure 2A. Assuming the matching circuit is passive, lossless, and reciprocal, the impedance matrix $Z$ is expressed in the form\textsuperscript{15}

$$Z = j \begin{bmatrix} X_{11} & X_0 \\ X_0 & X_{22} \end{bmatrix}. \quad (A1)$$

The input and output impedance can then be written as

$$Z_{\text{in}} = jX_{11} + \frac{X_0^2}{jX_{22} + Z_{\text{amp}}}. \quad (A2)$$

and

$$Z_{\text{out}} = jX_{22} + \frac{X_0^2}{jX_{11} + Z_{\text{coil}}}. \quad (A3)$$

The input impedances for the high impedance and low impedance cases are described by Equations (18) and (19). We begin with the high impedance case. Let $Z_{\text{in}} = Z_{\text{in}}^{HZ} = \beta R_{\text{coil}} - jX_{\text{coil}}$, and we have

$$\beta R_{\text{coil}} - jX_{\text{coil}} = jX_{11} + \frac{X_0^2}{R_{\text{amp}} + j(X_{22} + X_{\text{amp}})}, \quad (A4)$$

which is equivalent to $X_0^2 = [\beta R_{\text{coil}} - j(X_{\text{coil}} + X_{11})] \times [R_{\text{amp}} + j(X_{22} + X_{\text{amp}})]$ and $X_0^2$ is a positive real number. This means

$$-\frac{X_{\text{coil}} + X_{11}}{\beta R_{\text{coil}}} = -\frac{X_{22} + X_{\text{amp}}}{R_{\text{amp}}}. \quad (A5)$$

Equation (A–3) can be written as

$$R_{\text{out}} + jX_{\text{out}} = jX_{22} + \frac{X_0^2}{R_{\text{coil}} + j(X_{11} + X_{\text{coil}})} \quad (A6)$$

which is equivalent to $X_0^2 = [R_{\text{out}} + j(X_{\text{out}} - X_{22})] \times [R_{\text{coil}} + j(X_{11} + X_{\text{coil}})]$ and similarly,

$$\frac{X_{\text{out}} - X_{22}}{R_{\text{out}}} = \frac{X_{11} + X_{\text{coil}}}{R_{\text{coil}}}. \quad (A7)$$

If $\beta = 1$, we have $R_{\text{out}} = R_{\text{amp}}$, $X_{\text{out}} = -X_{\text{amp}}$. It is seen that Equations (A–5) and (A–7) become the same. This case is complex conjugate matching, which is ruled out in the following derivation process.

From Equations (A–5) and (A–7), $X_{11}^{HZ}$ and $X_{22}^{HZ}$ are solved and given by Equation (22). Substitute $X_{11}$ and $X_{22}$ by $X_{11}^{HZ}$ and $X_{22}^{HZ}$ in Equation (A–4), and we get

$$X_{0}^{HZ} = R_{\text{coil}}R_{\text{out}}(1 + \beta^2 R_{\text{amp}}^2). \quad (A8)$$

as has been given in Equation (22). Note that by substituting $X_{11}$, $X_{22}$ in Equation (A–6) with $X_{11}^{HZ}$, $X_{22}^{HZ}$ in Equation (22), another form of $X_{0}^{HZ}$ comes out. Further calculation shows that when $X_{\text{amp}} + X_{\text{out}} \neq 0$, or $X_{\text{amp}} + X_{\text{out}} = 0 \land R_{\text{out}} > R_{\text{amp}} \geq 0$ these two forms are equal, and, accordingly, a solution can exist; see Supporting Information B.

For the low impedance case, we follow a similar procedure and we get Equation (23). When $X_{\text{amp}} + X_{\text{out}} \neq 0$, or $R_{\text{amp}} > R_{\text{out}} > 0 \land X_{\text{amp}} + X_{\text{out}} = 0$, a solution exists, as shown in Supporting Information B.