Criteria for the absence of quantum fluctuations after spontaneous symmetry breaking

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Spontaneous symmetry breaking is revered by students of high-energy physics and condensed matter alike as the important mechanism for creating order out of beautifully symmetric equations of nature. Within the collection of spontaneously broken states, the Heisenberg ferromagnet has long been known to be an outlier, for instance it is an exact eigenstate of the Hamiltonian and as such has no quantum fluctuations. It also breaks time-reversal symmetry spontaneously, and has a reduced number of Nambu–Goldstone modes. The recent explanations of the counting of Nambu–Goldstone modes in non-relativistic systems lead one to wonder whether other properties of the ferromagnet carry over to such states of matter. Here I establish criteria for the absence of quantum fluctuations and the other specialties of the ferromagnet. In particular, it is not sufficient that the order parameter commute with the Hamiltonian.

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The Heisenberg ferromagnet has always been an eccentric duckling in the flock of spontaneous symmetry breaking (SSB) states consisting of antiferromagnets, crystals, superfluids, chiral SSB, the Standard Model and many others. This is only exacerbated by being one of the earliest and simplest models demonstrating SSB, used as the archetype in a large portion of the literature. Perhaps because much of its physics can be understood by undergraduate level calculations, have its peculiarities never been put in a larger perspective. Still the subtleties are intricate enough to have sparked debates between the greatest of minds in the past century [1, 2].

Why is this state different from all other states? We talk about the following observations, clarified below:

(i) the order parameter operator commutes with the Hamiltonian, is therefore a symmetry generator and is conserved in time;

(ii) two broken symmetry generators correspond to a single, quadratically dispersing Nambu–Goldstone (NG) mode;

(iii) the classical groundstate is an exact eigenstate of the Hamiltonian, there are no quantum fluctuations;

(iv) the raising operator, a root of the symmetry algebra, annihilates the groundstate (the spin of the maximally polarized state cannot be increased);

(v) there is no thin spectrum or tower of states of vanishing energy just above the groundstate;

(vi) the groundstate is an eigenstate of the unbroken symmetry generator with non-zero eigenvalue;

(vii) time-reversal symmetry is spontaneously broken.

When a continuous symmetry is spontaneously broken, there is a manifold of degenerate lowest-potential states. The spontaneously chosen classical groundstate $|0\rangle_{cl}$ is any one of these. Feature (iii) can also be worded as that $|0\rangle_{cl}$ be equal to the true, quantum groundstate. Conversely in general $|0\rangle_{cl}$ is not an eigenstate of the Hamiltonian, and the true groundstate is a superposition of those states, not breaking any symmetry. E.g. the true groundstate of an antiferromagnet is not the classical Néel state with alternating spins, but in fact a total spin singlet. The deviation from the classical groundstate towards the true groundstate is referred to as quantum fluctuations or zero-point motion although the macroscopic system is not fluctuating at all (Fig. 1). The zero-point motion of the condensate as a whole gives rise to a tower of states almost degenerate with the true groundstate (the thin spectrum), a superposition thereof is realized as the actual groundstate $|0\rangle$. Perhaps the most profound aspect of SSB is that $|0\rangle$ is very close to $|0\rangle_{cl}$: a chair is not in a momentum but almost in a position eigenstate [3–6]. The quantum fluctuations cause the thin spectrum, which in turn stabilizes the groundstate $|0\rangle$.

In many textbooks, most explicitly by Anderson [3], it is claimed that (i) implies (iii), because then the order parameter operator and the Hamiltonian can be simultaneously diagonalized, and the groundstate will be preserved in time. However, only the total order parameter (magnetization) commutes with the Hamiltonian while the local magnetization density does not. Therefore, local deviations that keep the total magnetization conserved are in principle allowed. It is in fact property (iv) that prevents any fluctuations: because the groundstate is completely polarized, any tentative fluctuation would take one out of the Hilbert space and is therefore forbidden. It raises the question what conditions one should put on the order parameter that causes the classical groundstate to be an

FIG. 1: The spontaneously chosen, classical groundstate is indicated by the arrow. The actual groundstate is a stable wave packet of deviations around this classical groundstate.
The classical groundstate being an exact eigenstate of the Hamiltonian (iii) is the strongest condition. Only if the total order parameter in Eq. (2) commutes with the Hamiltonian, (i) implies an exact eigenstate (iii). Time-reversal breaking (vii) is an independent phenomenon.

**Finite charge densities.** The Goldstone theorem predicts one massless mode for each spontaneously broken continuous symmetry in relativistic systems. In non-relativistic systems however, the modes could acquire a non-linear dispersion relation and have their number reduced. This was recognized early on [16, 17], almost solely to accommodate for the ferromagnet, and later generalized [18], arguing modes with even-in-momentum-exponent dispersion account for not one but two generators. The kaon condensate phase of matter in QCD was found to have a quadratically dispersing NG mode [8], suggested to be related to the finite charge density, i.e. a groundstate expectation value for a symmetry generator. This was generalized using effective Lagrangians [9] (and by Mori projector operators [13]). Let $G$ be the continuous symmetry group, the Lie algebra of which is generated by the Noether charges $Q^a$ and associated fields, such that the groundstate expectation value $\langle [Q^a, \Phi(x)] \rangle \equiv \langle [0, \Phi(x)] \rangle = 0$. Here $\langle 0 | Q^a, \Phi(x) \rangle$ is the order parameter operator and $\langle 0 \rangle$ is called the order parameter, because its non-vanishing indicates the appearance of an ordered state. Ordinarily each broken generator $Q^a$ excites one NG mode. But in the special case that $\Phi(x) = j^b(x)$, a charge density itself, and $\langle 0 | [Q^a, j^b(x = 0)] | 0 \rangle \equiv \langle [Q^a, j^b(0)] \rangle = M \neq 0$, the broken densities $j^a$ and $j^b$ actually excite the same NG mode. There appears a term in the effective Lagrangian proportional to $M$, linear in time derivatives, that causes a modified dispersion relation, typically $\omega \sim k^2$ [11, 12].

$[j^a(x), j^b(y)] = i \sum_c f^{abc} j^c(x) \delta(x - y),$ 

(1)

where $f^{abc}$ are the structure constants, we see that in general $M \sim \int d^D x (j^c) = \langle Q^c \rangle$, that is, the order parameter is the groundstate expectation value for the symmetry generator $Q^c$ itself (the case of so-called central extensions of the Lie algebra will not be discussed here [11, 12]). Thus (i) implies (ii). An example is the spontaneous magnetization of a ferromagnet in, say, the $z$-direction $\langle S^z \rangle = M$, where $S^z$ and $S^y$ are broken but excite only one magnon mode.

Ordinarily, a sufficient condition for the groundstate to be symmetric under $Q$ is that $Q|0\rangle = 0$, because then $e^{iQ}$ leaves the state invariant. But when a generator $Q$ obtains a finite charge density $Q = M \neq 0$, and is itself not broken, then the groundstate must be an eigenstate of $Q$ with eigenvalue $M$. Because then $e^{iQ}|0\rangle = e^{iM}|0\rangle$, and since quantum states are equivalent up to a phase shift, this is still a symmetry. This exceptional case is property (vi). In general, $Q$ may be broken as well, due to some additional interpolating field that is not a charge

**FIG. 2:** Schematic of implications of properties (i)–(vii). The classical groundstate being an exact eigenstate of the Hamiltonian (iii) is the strongest condition. Only if the total order parameter in Eq. (2) commutes with the Hamiltonian, (i) implies an exact eigenstate (iii). Time-reversal breaking (vii) is an independent phenomenon.
density. If this breaking is spontaneous, an additional type-A NG mode appears. If the breaking is explicit, the NG mode will be gapped. In either case the state will not be maximally polarized, and we do not regard this situation any further.

**Highest-weight state.** It has been shown that only generators that lie in the Cartan subalgebra (the largest set of mutually commuting Lie algebra elements) can obtain a finite charge density [19]. The eigenstates of such generators are called weight states with weight vector \( \mu \): \( Q^\mu |\mu\rangle = \mu |\mu\rangle \) (see any textbook, e.g. [20]). The generators not in the Cartan subalgebra are called root generators \( E_\alpha = \int d^Dx \, e_\alpha(x) \) with root vector \( \alpha \), that have the property \( E_\alpha |\mu\rangle = |\mu + \alpha\rangle \). Furthermore \( E_\alpha^\dagger = E_{-\alpha} \). Thus roots are not Hermitian but the combinations \( E_\alpha^\dagger = E_\alpha + E_{-\alpha} \) and \( E_\alpha = -i(E_\alpha - E_{-\alpha}) \) are; those combinations are symmetry generators \( Q^\mu \). The commutation relation is \([E_\alpha, E_{-\alpha}] = \sum \alpha^c Q^c\) and for their densities \( e_\alpha(x, e_{-\alpha}(y)) = \sum \alpha^c f^c(x) \delta(x - y) \). When some \( Q^c \) obtains a finite charge density but is itself unbroken, the ground state is a finite-weight state with \( \mu^c = M \) as mentioned above. Each pair of root generators \( E_\alpha, E_{-\alpha} \) for which \( \sum \alpha^c Q^c \neq 0 \) corresponds to a pair of broken symmetry generators \( E^+_{\alpha}, E^-_{-\alpha} \). If \( \mu^c, \alpha^c > 0 \), \( e_{-\alpha}(k) \) will excite a type-B gapless NG mode, and \( e_\alpha(k) \) excites a massive mode, where \( e_{-\alpha}(k) = \int d^Dx \, e^{ikx} e_{\pm\alpha}(x) \). However, if \( |\mu\rangle \) is a so-called highest-weight state, then \( e_\alpha(x)|\mu\rangle = 0 \) and the second, gapped mode is absent. For instance, in a ferromagnet \( S^-(k) \) excites the magnon, while \( S^+(k) \) takes one out of the Hilbert space. If the groundstate is a highest-weight state for all pairs of root generators (completely polarized state), then no type-B NG modes will have a gapped partner. It is also this case for which no quantum fluctuations act upon the classical order parameter. This can be seen as follows: first, the other elements of the Cartan subalgebra commute with \( Q^\mu \) and they can be simultaneously diagonalized (even locally), then the groundstate is an eigenstate for all such contributions to the Hamiltonian. As \( [Q^\mu, H] = 0 \) the total weight \( \mu \) is conserved. Then, due to the commutation relations \([Q^\mu, e_{\pm\alpha}] = \pm \alpha^c e_{\pm\alpha}\), only combinations involving \( e_\alpha(x) e_{-\alpha}(y) \) or \( e_{-\alpha}(x) e_\alpha(y) \) can appear in the Hamiltonian. At equal positions \( x = y \) the groundstate is an eigenstate for this contribution. If \( x \neq y \), then it involves at least one factor of \( e_\alpha(x) \) that annihilates the highest-weight state. Thus the groundstate is an eigenstate for all possible contributions to the Hamiltonian that can affect the order parameter, with either zero or non-zero eigenvalue.

**Gapped partners.** In the case of type-A NG modes, each broken generator excites one NG mode. When a finite charge density appears, a certain combination of two broken generators excites one type-B NG mode. The orthogonal combination may or may not excite a gapped mode. Recently, the distinction between these two cases was clarified [10] [13]. The set of interpolating fields \( \Phi(x) \) may contain, next to the charge densities \( j^a(x) \), additional operators \( \phi^i(x) \) not commuting with the Hamiltonian:

\[
\Phi(x) = (\phi^i(x) \ j^a(x)) \tag{2}
\]

If the broken generators \( Q^a \) satisfy not only \(|[Q^a, j^b]| \neq 0 \) for some \( b \), but also \(|[Q^a, \phi^i]| \neq 0 \) for some \( i \), then the pair \( j^a, j^b \) excites both a type-B NG mode and a gapped partner mode [13]. For instance, in a ferrimagnet, both magnetization and staggered magnetization are order parameters. Because of the transformation properties under the symmetry group, the expectation value \(|\langle Q^a, \phi^i \rangle| \) is always some linear combination of the expectation values of the operators \( \phi^i \) itself. In other words, they are also order parameters for this state. Thus, we can now identify the following two cases in the absence of any type-A NG modes:

1) We are in a highest-weight state for all roots; any fluctuations would take one out of the Hilbert space and are forbidden; there is no additional order parameter operator that does not commute with the Hamiltonian.

2) There is a finite charge density, but the state is not maximally polarized; the roots will excite a NG mode and a gapped partner mode; there is an additional order parameter operator not commuting with the Hamiltonian.

This settles the issue mentioned above: even if the naive order parameter—that nevertheless determines the massless NG modes—commutes with the Hamiltonian, there may be other order parameters that do not. In that case, quantum fluctuations are present. Note that if one were to look only at the order parameters \( \phi^i \), one would correctly conclude the spontaneous breaking of symmetry generators \( Q^a \), yet one would not recognize that they actually excite type-B and not type-A NG modes. Thus one needs the complete information contained in \( \Phi(x) \) of Eq. (2) to characterize the low-energy spectrum and the presence of quantum fluctuations.

**Thin spectrum.** If quantum fluctuations are present after spontaneous symmetry breaking, there will be a thin spectrum or tower of extremely low-lying states that hardly contribute to the free energy, associated with the \( k = 0 \) fluctuations of the condensate as a whole [3][6]. The energy spacing between these states is inversely proportional to the number of degrees of freedom of the system. The actual groundstate \(|0\rangle\) is a combination of such states and very close to \(|0\rangle_{cl}\). As the energy spacing is tiny, this wave packet is very long-lived, and nothing actually fluctuates. The thin spectrum actually limits the maximum coherence time of any quantum superposition of many-body states [6][21]. Clearly, if there are no quantum fluctuations possible as argued above, then there are also no \( k = 0 \) fluctuations and no thin spectrum. This is property (v). The absence of quantum fluctuations has therefore measurable differences in the coherence time of superpositions. This observation implies that, ceteris
paribus, SSB states without quantum fluctuations are on paper more suitable to fabricate qubits, as they do not have this fundamental limit in their coherence time.

Time-reversal symmetry. It is claimed by several authors [3][11][15] that the spontaneous breaking of time-reversal symmetry is responsible for the peculiarities of the ferromagnet, such as the quadratic NG dispersion relation. This is not the case. Obviously Lorentz invariance is broken in such systems, but there is no fundamental reason time-reversal should be as well. Consider the Lie algebra relations Eq. (1); for simplicity, take the Lie group SU(2) such that $f^{abc} = e^{abc}$, $a,b,c \in x,y,z$. Under time reversal, $i \rightarrow -i$. There are now two possibilities: all three of $j^x, j^y, j^z$ are odd under time-reversal; or two are even and one is odd. In the latter case, a charge density that is even under time-reversal may obtain an expectation value, leading to a type-B NG mode. The groundstate, an eigenstate of the charge density, is then time-reversal invariant as well. The first case is realized in Heisenberg magnets, where the charge densities are spin rotations, which are all odd under time-reversal. Examples of the second case [22] include isospin, which are spin rotations, which are all odd under time-reversal symmetry. Consider spin-S Heisenberg magnets on bipartite lattices with Hamiltonian

$$\mathcal{H} = J \sum_{(j,l)} \mathbf{S}_j \cdot \mathbf{S}_l + K \sum_{(j,l)} (\mathbf{S}_j \cdot \mathbf{S}_l)^2$$

The operators $S^a_j$, $a = x,y,z$ are generators of SU(2) in the spin-$S$ representation at lattice site $j$. $J$ is the exchange parameter, $K$ the biquadratic exchange parameter, and the sums are over nearest-neighbor lattice sites. The Hamiltonian is invariant under SU(2)-rotations of all spins simultaneously, generated by $S^a = \sum_j S^a_j$.

- Antiferromagnet, $J > 0, K = 0$. The spins prefer to anti-align, in a spontaneously chosen direction which we can take to be the z-axis. The classical groundstate is the Néel state, with order parameter operator the staggered magnetization $\mathbf{S}^z = \sum_j (-1)^j S^z_j$. Rotations around the z-axis are unbroken, while the other two generators are spontaneously broken: $\langle [S^a, \mathbf{S}^z_j] \rangle = i e^{abc} (\mathbf{S}^b_j) \neq 0$, $SU(2) \rightarrow U(1)$. The expectation value of charge density ($S^z$) is zero. There are two type-A NG modes. For low dimension $D$ and low $S$, quantum fluctuations are pronounced, most extreme for $S = \frac{1}{2}$, $D = 2$, where the staggered magnetization is reduced to about 60%.

- Ferromagnet, $J < 0, K = 0$. The groundstate has all spins aligned in a spontaneously chosen direction $z$. The order parameter is the generator $S^z$ itself. The symmetry breaking pattern is the same $SU(2) \rightarrow U(1)$, but now a finite charge density $\langle S^z \rangle = M \neq 0$ is present, such that the two broken densities $S^z_k$ and $S^z_k$ excite the same type-B NG mode with quadratic dispersion. If $M > 0$, $S^z_k = S^z_k - i S^y_k$ excites the NG mode, while $S^z_k = S^z_k + i S^y_k$ takes one out of the Hilbert space: there is no gapped partner mode. The state is maximally polarized and there are no quantum fluctuations.

- Ferrimagnet, $J > 0, K = 0$, but the spin $S_A$ on the A-sublattice is different from $S_B$ on the B-sublattice. For instance, let $A$ have spin-$1$ and $B$ spin-$\frac{1}{2}$. For $J > 0$, the spins on the different sublattices prefer to anti-align in chosen direction $z$. However, because of the imbalance in spin, the expectation value of the generator $S^z$ is non-zero. Thus we have a single type-B NG mode for the broken densities $S^z_k, S^z_k$. The staggered magnetization, not commuting with the Hamiltonian, is also an order parameter for this state. We find ourselves in case 2) and there is a gapped partner mode (cf. [23]).

- Canted magnet, $J < 0, K > 0$. If the biquadrartic term is non-zero, it penalizes a too-high degree of magnetization. This will induce canting of the spins, i.e. classically they will make an angle with the magnetization axis $z$. The rotation symmetry about the $z$-axis is also spontaneously broken SU(2) → 1, leading to a type-A NG mode in addition to the type-B NG mode of the ferromagnet. The state is not maximally polarized and there are quantum fluctuations, for which it is sufficient to recognize the presence of the type-A NG mode.

- Intermediately-polarized magnet, $J < 0, K > 0$. For $S > 1$, it is possible that the parameters $J$ and $K$ are fine-tuned such that the average magnetization value is precisely equal to an allowed magnetic quantum number $S > m > 0$. It could be surmised that the groundstate will have all spins in the state $|S,m\rangle$. Such intermediately-polarized states are identified in spin-2 and spin-3 spinor BECs, and can be stable in the latter case [21][24]. They follow again the SU(2) → U(1) pattern, and have finite magnetization $M = N m$, with $N$ the number of sites. However, the state is clearly not maximally polarized, and $S^z_k$ excite a type-B NG mode and $S^z_k$ a gapped partner mode. In this case the nematic tensor $N^{ab} = \sum_j \frac{1}{2} (S^a_j S^b_j + S^b_j S^a_j)$ can be seen to obtain an expectation value in the $N^{zz}$-component. The expectation values $\langle [S^x, N^{yx}] \rangle$ and $\langle [S^y, N^{yz}] \rangle$ do not vanish and indicate the existence of the gapped partner mode. Interestingly, the nematic tensor is also an order parameter for the ferromagnet, but then it can be shown that

FIG. 3: Cartoons of magnetically-ordered phases: antiferromagnet, ferromagnet, ferrimagnet, canted magnet, intermediately-polarized magnet.
the mode associated with these expectation values is linearly dependent on the NG mode, and cannot be taken as an independent, gapped, mode (cf. [24]).

Other states. After these considerations, it is fair to say that SSB states without quantum fluctuations will be exceedingly rare. For instance, any model involving a Higgs-type potential of a complex scalar or vector field $\Psi(x)$, will always include at least one $U(1)$-symmetry breaking associated with the global phase of the order parameter leading to a type-A NG mode (for real fields, such a phase is absent). In spinor-BEC ferromagnets, density fluctuations are still present [7]. However the examples are not limited to the $SU(2)$ ferromagnet only. Indeed, Heisenberg-type Hamiltonians for any $SU(N)$ would feature a completely polarized ‘ferromagnetic’ state if the exchange $J$ is negative.

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