RESEARCH ARTICLE

Maximal order Abelian subgroups of Coxeter groups

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Received: 15 October 2021; Revised: 16 March 2022; Accepted: 25 March 2022; First published online: 2 August 2022

Keywords: Coxeter groups, Lie algebras, root systems

2020 Mathematics Subject Classification: Primary - 20F55; Secondary - 17B22, 20E28

Abstract

In this note, we give a classification of the maximal order Abelian subgroups of finite irreducible Coxeter groups. We also prove a Weyl group analog of Cartan’s theorem that all maximal tori in a connected compact Lie group are conjugate.

1. Introduction

Some years ago, colleagues working in the area of statistical mechanics asked what the maximal order Abelian subgroups of the symmetric group $S_n$ looked like. Their question arose from consideration of reducible representations constructed from tensor products of unitary representations arising in the statistical mechanics of systems of $n$ quantum spins. In particular, they wanted to understand the situation as $n \to \infty$. A complete classification can be derived from general results in [17], and a classification was given in a more general setting in [9]. An elementary classification was given in [6] using Lagrange multipliers. This method indicates that in order to maximize the product $\prod m_i$ of the prime powers $m_i$ (the Abelian invariants) subject to the constraint $\sum m_i \leq n$ (because it is an Abelian subgroup of $S_n$), all or as many as possible of the integers $m_i$ should be chosen equal (to $m$ say). The problem then amounts to maximizing $m^n$ and regarding this as a function of a real variable having a maximum at $e$, we would expect that the solution to the integer-valued problem (and therefore the maximal order of an Abelian subgroup of $S_n$) is of the form $3^k$, since $2^2 < 3^1$. This is essentially the case (see Theorem 1.1 below). In this note, we give a complete classification of the maximal order Abelian subgroups $M$ for all finite irreducible Coxeter groups. We also determine the number of conjugacy classes of maximal order Abelian subgroups, and viewing a distinguished class of these subgroups as discrete analogs of maximal tori in compact Lie groups, we obtain a Weyl group analog of Cartan’s theorem that all maximal tori in a connected compact Lie group $G$ are conjugate, namely:

**Theorem.** Let $M$ and $M'$ be discrete maximal tori of $W$, then $M' = w^{-1}Mw$ for some $w \in W$.

The first author would like to thank the Institut de Mathématiques de Marseille, Aix-Marseille Université, where part of this work was carried out and in particular Professors Oeljeklaus and Short for their hospitality.

We now recall the precise solution for $S_n$ (i.e. $W$ of type $A_{n-1}$) and then state the general result.

**Theorem 1.1** Let $M$ be an Abelian subgroup of maximal order in the symmetric group $S_n$, $n \geq 2$. Then

(i) $M \cong \mathbb{Z}^k_3$ if $n = 3k$,

(ii) $M \cong \mathbb{Z}^k_3 \times \mathbb{Z}_2$ if $n = 3k + 2$,

(iii) either $M \cong \mathbb{Z}^{k-1}_3 \times \mathbb{Z}_4$ or $M \cong \mathbb{Z}^{k-1}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ if $n = 3k + 1$. 
A natural representative for $M$ in $S_n$ is generated by a collection of disjoint 3-cycles, plus a 2-cycle or a 4-cycle if appropriate.

Since the case of the root system of type $A_r$ is settled by the above theorem, we will exclude this case from the statement of the general result.

**Theorem 1.2** Let $M$ be an Abelian subgroup of maximal order in a finite irreducible Coxeter group $W$ of rank $r$, then:

(a) $(W$ crystallographic$)$

(i) For $W$ of type $B_r$ or $C_r$, we have $M \cong \mathbb{Z}_s^1 \times \mathbb{Z}_t^1$ where $0 \leq s, t$ with $s + 2t = r$ and $|M| = 2^r$.

(ii) For $W$ of type $D_{2k}$ $(r = 2k)$, we have $M \cong \mathbb{Z}_2^k$ and $|M| = 2^r$.

(iii) For $W$ of type $D_{2k+1}$ $(r = 2k + 1)$, we have $M \cong \mathbb{Z}_2^k \times \mathbb{Z}_4^1$ where $0 \leq s, t$ with $s + 2t = r - 1$ and $|M| = 2^{r-1}$.

(iv) For $W$ of type $E_6$, we have $M \cong \mathbb{Z}_3^1$, and $|M| = 3^3$.

(v) For $W$ of type $E_7, E_8$, we have $M \cong \mathbb{Z}_2^2$ and $|M| = 2^r$.

(vi) For $W$ of type $F_4$, we have $M \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4^1$ and $|M| = 2 \cdot 3^2$.

(vii) For $W$ of type $G_2$, we have $M \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3$, and $|M| = 2 \cdot 3$.

(b) $(W$ noncrystallographic$)$

(i) For $W$ of type $H_3$, we have $M \cong \mathbb{Z}_2^3 \times \mathbb{Z}_5$, and $|M| = 2 \cdot 5^3$.

(ii) For $W$ of type $H_4$, we have $M \cong \mathbb{Z}_2^3 \times \mathbb{Z}_5^2$, and $|M| = 2 \cdot 5^2$.

(iii) For $W$ of type $I_2(m)$, $m \geq 5$, we have $M \cong \mathbb{Z}_m$ and $|M| = m$.

For $W$ of type $B_n$, a natural representative for $M$ is generated by a collection of $m$ negative 2-cycles (having order 4) and a negative 1-cycle if $n$ is odd.

For $W$ of type $D_{2k+1}$, a natural representative for $M$ comes from a subgroup of type $W(B_2k)$ contained in $W(D_{2k+1})$.

For $W$ of type $D_{2k}$, a natural representative for $M$ is a direct product of $k$ groups of type $W(A_1) \times W(A_1)$.

2. Basic facts and definitions

All basic facts and definitions used can be found in [4] or [16]. Let $(W,S)$ be an irreducible finite Coxeter system of rank $r$ with $S = \{s_1, \ldots, s_n\}$ its set of simple reflections. When $W$ is a Weyl group (W crystallographic), we have an associated connected compact Lie group $G$ (with Lie algebra $g$), containing (a fixed) maximal torus $T$ (with Lie algebra $t$) so that the Weyl group $W = N_G(T)/T$. If $g^c = t^c \oplus \bigoplus_{\alpha \in \Phi} g^\alpha$ is the root space decomposition of the complexification of $g$ with respect to $t^c$ (the complexification of $t$), then a root $\alpha$ is an element of the dual spaces $t^*$ (pure imaginary-valued) or $it^*$ (real-valued). Since $G$ is compact, the Killing form is negative definite on $t$ and gives an $(\text{Ad}(G)$ invariant) real inner product $\langle , \rangle$ on the real vector spaces $t$ and $t^*$. For $w \in N_G(T)$, $H \in t$ and $\alpha \in t^*$ we define $w(H) = \text{Ad}(w)H = wHw^{-1}$ and $w(\alpha)(H) = \alpha(\text{Ad}(w^{-1})H)$, and since Ad$(T)$ acts trivially on $t$ we obtain (faithful) induced actions of $W$. Choosing a fundamental Weyl chamber in it, we can define positive roots $\Phi^+ \cup \{\alpha, \ldots, \alpha_r\}$ a basis of positive simple roots whose simple reflections generate $W$. The fundamental weights $\{\omega_1, \ldots, \omega_r\}$ are defined by the conditions that $\langle \omega_i, 2\alpha_j \rangle := \langle \alpha_j, \alpha_j \rangle \delta_{ij}$ for all $i, j$. We will normalize $\langle , \rangle$ so that the highest root $\tilde{\alpha}$ has length squared equal to $2$. For $\alpha, \beta \in \Phi$, we define (integers) $n(\alpha, \beta) = \frac{\langle 2\alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$.

The Dynkin diagram $D$ is the (multi) graph with $r$ vertices (labeled by the positive simple roots), and $c_{ij}c_{ji}$ edges joining $\alpha_i$ to $\alpha_j$ where $c_{ij} = n(\alpha_i, \alpha_j)$. The extended Dynkin diagram $\tilde{D}$ (always labeled as in [4]) is the graph constructed from $D$ by adding a new vertex $\alpha_0 = -\tilde{\alpha}$ (the affine vertex or node) and joining it to any vertex $\alpha_i$ by (the old rule of) $n(\alpha_i, \tilde{\alpha}) \cdot n(\tilde{\alpha}, \alpha_j)$ edges. We then write the coefficient $n_i$ over the vertex $\alpha_i$ and $n_0 = 1$ over $\alpha_0$, where $\tilde{\alpha} = \sum_{i=1}^r n_i \alpha_i$. Deletion of any vertex from $\tilde{D}$ and the edges connected to it produces a new (typically non-connected) Dynkin diagram $D_1$ (with the same number
of vertices as $D$) of a semisimple Lie subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}_1$ is said to be obtained from $\mathfrak{g}$ by an elementary operation. Of course, we can perform a new elementary operation on any of the connected components of $D_1$. Continuing this process, we obtain a chain of subalgebras $\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_n$, each obtained from its predecessor by an elementary operation, and any semi-simple Lie subalgebra of maximal rank is obtained by a finite number of elementary operations (see [12,18]). Note that when a diagram of type $A_n$ occurs, an elementary operation does not change the algebra. Among the maximal rank Lie subalgebras are those corresponding to maximal subgroups of maximal rank in $G$, and we recall from [20] the following for later use:

The fundamental simplex

$$D_0 = \{h \in \text{it} : \alpha_i(h) \geq 0 \forall i, \tilde{\alpha}(h) \leq 1\}$$

has vertices $\{v_0, v_1, \ldots, v_r\}$ where $v_0 = 0$, $\alpha_i(v_i) = \frac{1}{n_i} \delta_{ij}$, and it has the property that every element of $G$ (connected and centerless) is conjugate to an element of $\exp(2\pi i D_0)$. The conjugacy classes of maximal connected subgroups of maximal rank in $G$ are obtained from it by a theorem of Borel and de Siebenthal which we now recall.

**Theorem 2.1** ([2,20], p. 278) Let $G$ be a compact centerless simple Lie group with fundamental simplex $D_0 = \{v_0, v_1, \ldots, v_r\}$ and let $1 \leq i \leq r$.

1. Suppose that $n_i = 1$, then the centralizer of the circle group $\{\exp(2\pi i t v_i) : t \in \mathbb{R}\}$ is a maximal connected subgroup of maximal rank in $G$ with

$$\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$$

as a system of simple roots.
2. Suppose that $n_i$ is a prime $p > 1$, then the centralizer of the element $\exp(2\pi i v_i)$ (of order $p$) is a maximal connected subgroup of maximal rank in $G$ with

$$\{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\}$$

as a system of simple roots.
3. Every maximal connected subgroup of maximal rank in $G$ is conjugate to one of the above groups.

Finally, the trace of a finite Abelian group $A = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ is the integer $\text{Tr}(A) = \sum_{i=1}^{k} m_i$ (see [15]).

3. Proof of Theorem 1.2

For $W$ of each possible type, we first prove the existence of an Abelian subgroup of the required order and isomorphism type. We then check that $W$ contains no Abelian subgroups of larger order, or other isomorphism types of Abelian subgroups of maximal order. For the exceptional crystallographic types (except $G_2$) and the noncrystallographic type $H_4$, the check involves computer calculations using the computer package CHEVIE for GAP [14]. In order to prove the existence, we observe that if $K$ is a connected subgroup of $G$ of maximal rank (necessarily equal to that of $G$), then any maximal torus of $K$ is also a maximal torus of $G$. The Weyl group $W(K)$ of $K$ can therefore be identified with a subgroup of $W$ the Weyl group of $G$.

For $W$ of type $B_r$ or $C_r$, the elementary operation (in the extended Dynkin diagram of $C_r$) corresponding to deletion of the vertex connected to the $\alpha_i = -\tilde{\alpha}$ vertex of $\tilde{D}$ (the vertex $\alpha_i$) gives the Dynkin diagram $D_r$ of a semisimple subalgebra $\mathfrak{g}_1$ with corresponding maximal rank subgroup of $G$ of type $A_1 \times C_{r-1}$. Repeating this process in the component of $D_1$ corresponding to $C_{r-1}$, we eventually obtain a maximal rank subgroup of type $A_1 \times A_1 \times \cdots \times A_1$ ($r$ copies) and hence a subgroup $M \cong \mathbb{Z}_r^r$ of $W$. This sequence of elementary operations (i.e. successive deletion of the vertex connected to $-\tilde{\alpha}$, in successive extended Dynkin diagrams) we will call the Wolf sequence (on account of its connection to Wolf spaces,
see \([13, 21]\), and it also produces a maximal rank subgroup of type \(A_1 \times A_1 \times \ldots \times A_1\) \((r\ \text{copies})\) for types \(D_{2k}\) and \(E_r\), \(r = 7, 8\) and hence a subgroup \(M \simeq \mathbb{Z}_r^t\) of \(W\).

The maximal order Abelian subgroups of \(W\) for type \(B_r\) or \(C_t\), containing direct factors isomorphic to \(\mathbb{Z}_q\) are also realized in the extended Dynkin diagram of \(C_t\) (recalling that the Weyl group for root systems of type \(B_r\) or \(C_t\) is isomorphic to the dihedral group of order eight) as follows: as our first elementary operation, we delete the vertex \(\alpha_r\) from \(\tilde{D}\) to obtain the Dynkin diagram \(D_t\) of a semisimple algebra \(g_1\) with corresponding maximal rank subgroup of \(G\) of type \(C_2 \times C_{n-2}\). Repeating either this process or taking the Wolf sequence, in the component of \(D_t\) corresponding to \(C_{n-2}\) we can eventually obtain any subgroup \(M \simeq \mathbb{Z}_r^t \times \mathbb{Z}_s^t\) where \(0 \leq s, \ t = s + 2t = r\).

In the case of \(W\) of type \(D_{2k+1}\), we note that not all subgroups listed arise from subgroups of maximal rank, for example, the maximal order Abelian subgroup \(M \simeq \mathbb{Z}_4 \times \mathbb{Z}_4\) when \(W\) is of type \(D_3\). However, the maximal order Abelian subgroups \(M \simeq \mathbb{Z}_k^2 \times \mathbb{Z}_4\) and \(M \simeq \mathbb{Z}_4^k\) both arise from a maximal rank subgroup of type \(A_1 \times A_1 \times A_1\). The result follows, however, from the \(B_{2k}\) case by folding the \(D_{2k+1}\) diagram which comes from a regular embedding (taking a torus to a torus) of Lie groups (see \([5]\), p. 265).

For \(W\) of type \(E_5\) or \(G_2\), the elementary operation of deletion of the vertex \(\alpha_i\) such that \(n_i = 3\), where \(\tilde{\alpha} = \sum_{i=1}^{t} n_i \alpha_i\), gives a maximal rank subgroup of type \(A_2 \times A_2 \times A_2\), \(A_1 \times A_2\), or \(A_2\), respectively, giving rise to an Abelian subgroup \(M\) of \(W\) with \(M \simeq \mathbb{Z}_3^2\), \(\mathbb{Z}_2^4\) or \(\mathbb{Z}_3^2\), respectively. Since \(-1 \in W\) for \(F_4\) and \(G_2\) and the center \(Z(W) \simeq \mathbb{Z}_2 \simeq (-1)\) \((11)\), we can extend these groups by \(Z(W)\) in both these cases.

We now consider the noncrystallographic cases. For \(W\) of type \(I_2(m),\ m \geq 5\) (the Dihedral groups), the result is clear. For \(W\) of type \(H_3\) and \(H_4\), the classification of their maximal proper subroot systems in \([8, 10]\) gives rise to Abelian subgroups \(M\) of \(W\) with \(M \simeq \mathbb{Z}_5\) (from a maximal subroot system of type \(I_2(5)\)), and \(\mathbb{Z}_2^4\) (from a maximal subroot system of type \(I_2(5) \times I_2(5)\)) in \(H_3\) and \(H_4\), respectively. Extending these groups by their centers \(Z(W) \simeq \mathbb{Z}_2\) gives the required Abelian subgroups \(M\).

We now show that the obtained lower bounds on \(|M|\) are also upper bounds, and that there are no other isomorphism types of maximal order Abelian subgroups. Before doing so, we remark (see \([3]\), p. 134) that a subgroup of \(W\) isomorphic to \(\mathbb{Z}_p^t\) \((p\ \text{a prime})\) admits a faithful real representation of dimension \(r\) \((on\ t)\) and therefore \(s \leq 2\) if \(p = 2\) and \(s \leq \frac{r}{2}\) if \(p \neq 2\). There is therefore no larger order elementary Abelian 2-group in \(W(B_2), W(D_2), W(E_7),\) or \(W(E_8)\), and no larger order elementary Abelian 3-group in \(W(E_6)\) than obtained above. In ruling out other possibilities, we begin with the infinite families (i.e. classical types). We embed \(M\) in a symmetric group \(S_N\) and use the fact that we must then have \(\text{Tr}(M) \leq N\). Here, the vertices of the extended Dynkin diagram \(\tilde{D}\) with \(n_i = 1\) and the maximal subgroup of maximal rank \(K\) corresponding to part (i) of Theorem 2.1, play a role. This subgroup is the isotropy subgroup of an Hermitian symmetric space \(H = G/K\). Taking a maximal torus \(T\) of \(G\) to lie in \(K\), the Weyl group \(W\) acts transitively and faithfully on the fixed point set \(F(T,H)\) of the action of \(T\) on \(H\). This set has cardinality equal to the Euler number \(\chi(H)\) of \(H\) which is equal to \(2r\) when \(W\) is of type \(B_r\) or \(D_r\) (and its elements are pairwise antipodal on totally geodesic two-dimensional spheres in \(H\)) see \([19]\), so that \(\text{Tr}(M) \leq 2r\). Alternatively, instead of \(F(T,H)\), we can take the weights \(\pm \lambda_1, \ldots, \pm \lambda_r\) of the vector representation of \(\mathfrak{g}\) for the simple Lie algebras of type \(C_r\) and \(D_r\). Now for \(B_r\) and \(D_r\) \((r\ \text{even})\), the center \(Z(W) \simeq (-1)\) is contained in \(M\) so that the orbits \(\Omega_{\pm}\) of \(M\) have even cardinality and \(M\) is the direct product of its restrictions to the orbits \(\Omega_{\pm}\). Since a transitive Abelian permutation group has order equal to its degree, we can rule out elements of order three in \(M\) since they must contribute at least six to \(\text{Tr}(M)\) and \(|M|\), whereas \(\mathbb{Z}_3^r\) contributes six to \(\text{Tr}(M)\) and eight to \(|M|\). The argument for ruling out higher torsion elements of \(M\) other than four is similar, so that \(2^{r}\) is the maximal order of an Abelian subgroup in these cases. Again, the case of \(D_{2k+1}\) follows from folding to \(B_{2k}\). The remaining large order cases are in the exceptional families and were verified by computer calculations.

**Definition 3.1.**

(i) The 2-rank of \(W\) is equal to the integer \(r_2\) such that the maximal order of an elementary Abelian 2-subgroup of \(W\) is \(2^{r_2}\).
Similarly, the Weyl group of Type \( \text{maximal Abelian subgroup of } G \) is called a discrete maximal torus of \( W \).

We now have the following corollary to Theorem 1.2.

**Corollary 3.1.** Let \( W \) be the Weyl group of an irreducible root system, then the 2-rank \( (r_2) \) of \( W \) is equal to the maximal cardinality of a set of strongly orthogonal roots and \( r_2 = r \) if and only if \(-1 \in W\).

**Proof:** As in the proof of Theorem 1.2, the Wolf sequence of elementary operations corresponding to successive deletion of the vertex connected to \(-\tilde{a}\) (in successive extended Dynkin diagrams) produces a maximal rank subgroup of \( G \) of type \( A_1 \times A_1 \times \ldots \times A_1 \) (with \( r \) copies), and corresponding maximal order elementary 2-subgroup \( M \cong \mathbb{Z}_2^r \) of \( W \), unless we start with or encounter a diagram of type \( A_s \) with \( 2 \leq s \), in which case \( r_2 < r \). This will occur only in diagrams of type \( A_r, D_r, r \text{ odd}, \text{ or } E_6 \), namely in those cases where \(-1 \notin W\), and then \( r_2 \) takes the values \([(r+1)/2], r - 1 \text{ and } 4\), respectively. That the corresponding elementary 2-groups are of maximal order was checked by computer for \( E_6 \), follows from Theorem 1.2 for \( D_r, r \text{ odd} \) and by induction for \( A_r \). That these procedures also produce a set of s.o. roots of maximal cardinality \( r_2 \) follows from orthogonality in the simply laced cases and from the classification of maximal sets of strongly orthogonal roots in [1], p. 121 and p. 127 otherwise.

**Definition 3.2.** A maximal order Abelian subgroup \( M \) (of a finite irreducible Coxeter group \( W \)) with minimal number of Abelian invariants is called a discrete maximal torus of \( W \).

Remarks and Examples: The Weyl group of Type \( A_1 \) is the symmetric group \( S_3 \), and it already hints at the definition of a maximal torus (of \( W \)). \( S_3 \) has three conjugacy classes of maximal order Abelian subgroups, those of \( M_1 = \langle 1234 \rangle \), \( M_2 = \langle 12 \rangle \times \langle 34 \rangle \), and \( M_3 = \langle 12 \rangle \langle 34 \rangle \). Whereas \( M_2 \) and \( M_3 \) are isomorphic as abstract groups they are not as permutation groups, that is, they are not conjugate in \( S_3 \). On the other hand, the cycle \( (1234) \) is a Coxeter element and it generates (for \( W \) of any type) a maximal Abelian subgroup of \( W \) (in this case also of maximal order) and a distinguished conjugacy class. Similarly, the Weyl group of Type \( B_2 \) has three conjugacy classes of maximal order Abelian subgroups, two of which are isomorphic to \( \mathbb{Z}_4^2 \) and the other (isomorphic to \( \mathbb{Z}_4 \)) is corresponding to the Coxeter element. Whereas the Abelian subgroup generated by the Coxeter element (although maximal) is no longer of maximal order in higher rank, by Theorem 1.2, the above conjugacy class phenomenon persists for classical types (excluding \( D_3 \)).

We now prove an analog (for \( W \)) of Cartan’s theorem that all maximal tori of a compact connected Lie group \( G \) are conjugate in \( G \).

**Theorem 3.1.** Let \( M \) and \( M' \) be discrete maximal tori of \( W \), then \( M' = w^{-1}Mw \) for some \( w \in W \).

**Proof:** For \( W \) of type \( A_r \), we note that Theorem 1.1 and the definition of a maximal torus \( M \) of \( W \) imply that

1. \( M \cong \mathbb{Z}_2^k \) if \( r + 1 = 3k \),
2. \( M \cong \mathbb{Z}_2^k \times \mathbb{Z}_2 \) if \( r + 1 = 3k + 2 \), and
3. \( M \cong \mathbb{Z}_2^{k-1} \times \mathbb{Z}_4 \) if \( r + 1 = 3k + 1 \).

Since all direct factors correspond to disjoint cycles of length 2, 3, or 4 (with the sum of all lengths equal to \( r + 1 \)) and at most one transposition occurring, the result follows from the fact that permutations of the same cycle type are conjugate in \( S_r \).

For \( W \) of type \( B_r \), viewed as all signed permutations of \( \{1, 2, \ldots, r\} \), that is, injective maps from \( \{1, 2, \ldots, r\} \) to \( \{\pm 1, \pm 2, \ldots, \pm r\} \), with either \( i \) or \(-i \) in the image, elements can again be expressed in cyclic form, and the above argument generalizes. Cycles either contain both \( i \) and \(-i \) (called negative
cycles) and are of the form \((i_1 i_2 \ldots i_k - i_1 - i_2 \ldots - i_k)\) or do not contain both \(i\) and \(-i\) for any \(i\) (called positive cycles), and they occur in pairs of the form \((i_1 i_2 \ldots i_k)(-i_1 - i_2 \ldots - i_k)\). Since (as with ordinary permutations) conjugation by \(w\) of a signed permutation in cyclic form sends \(i\) to \(w(i)\), two signed permutations are conjugate if and only if they have the same number of positive and negative cycles of every length. We now recall that a maximal torus \(M\) of \(W\) (by Theorem 1.2) is of the form:

\[
\begin{align*}
(i) & \quad M \simeq \mathbb{Z}_4^k \text{ if } r = 2k \\
(ii) & \quad M \simeq \mathbb{Z}_4^k \times \mathbb{Z}_2 \text{ if } r = 2k + 1.
\end{align*}
\]

When \(r = 2k\), the \(k\) commuting \(\mathbb{Z}_4\) factors are negative cycles \((i_1 i_2, -i_1 - i_2)\) (Coxeter elements of a \(B_2\) or \(C_2\) system), and we have that all maximal tori are conjugate. When \(r = 2k + 1\), the argument is the same because the additional \(\mathbb{Z}_2\) factor must be a negative 1-cycle. The case of \(D_r\) (\(r\) odd) is similar.

We next consider those cases where a maximal torus is of the form \(M \simeq \mathbb{Z}_4^2\), that is, \(D_{2k}\) and \(E_r, \ r \in \{7, 8\}\). Using the fact that for these cases \((-1) = Z(W)\) must be contained in \(M\), it is not hard to prove that \(M\) has a set of \(r\) generators, none of which is a nontrivial product of commuting reflections. These generating reflections therefore yield a maximal set of orthogonal roots that are in fact strongly orthogonal as the root systems are simply laced in the cases at hand ([1], p. 117). However, by [1], p. 119, all maximal subsets of strongly orthogonal roots are in the same Weyl group orbit for simply laced root systems (the number of such \(W\)-orbits is the number of short simple roots), and therefore the corresponding stabilizing sets of reflection generators of the discrete maximal tori are conjugate in \(W\).

Similarly in the case of \(E_6\), there is a unique \(W\) orbit of sets of three orthogonal roots ([7] p. 14), and therefore (as we now show) there is a unique \(W\) orbit of subroot systems of type \(3A_2 = A_3 + A_2\), and therefore all stabilizers (the \(M\)’s) are conjugate. Deletion of the branch node in the extended Dynkin diagram gives a subroot system of type \(3A_2\) with simple roots \(\{\alpha_1, \alpha_2\}, \{\alpha_5, \alpha_6\}\), and \(\tilde{\alpha}, \alpha_2\). Let \(\{\alpha_1', \alpha_2', \alpha_5', \alpha_6'\}\) be the simple roots of another \(3A_2\), and by [7] p. 14 we may assume that \(w(\alpha_1') = \alpha_1, w(\alpha_2') = \alpha_5, \) and \(w(\tilde{\alpha}') = \tilde{\alpha}\) for some \(w \in W\). We now show that \(w(\tilde{\alpha}') \in \{\alpha_2, \tilde{\alpha} - \alpha_2\}\). Since \(\langle \tilde{\alpha}, w(\alpha_2') \rangle = \langle w(\tilde{\alpha}'), w(\alpha_2') \rangle = \langle \tilde{\alpha}', \alpha_2' \rangle \neq 0\), we have that \(b_2 \neq 0\), where \(w(\alpha_2') = \sum_{i=1}^r b_i \alpha_i\), because \(\tilde{\alpha} = \omega_2\). Similarly \(w(\alpha_2')\) must be orthogonal to \(\alpha_1 = 2\omega_1 - \omega_3\) and \(\alpha_3 = -\omega_1 + 2\omega_3 - \omega_4\) so that \(2b_1 = b_3\) and \(b_1 + b_2 + 2b_3 = 4b_1\) and therefore \(3b_1 = b_4\). As there are only two positive roots with \(\alpha_3\)-coefficient equal to \(3\), namely \(\tilde{\alpha} = \omega_2\) and \(s_{\omega_2}(\omega_2) = -\omega_2 + \omega_3 = \tilde{\alpha} - \alpha_2\) (because the next highest root \(s_{\omega_2}(\omega_2) = -\omega_4 + \omega_3 + \omega_5\) has \(\alpha_3\)-coefficient equal to \(2\)), we have that either \(b_1 = 1\) and \(w(\alpha_2') = \tilde{\alpha} - \alpha_2\) or \(b_1 = 0 = b_4 = b_3\) and therefore \(w(\alpha_2') = \alpha_2\) as the only positive root of the system of type \(A_2 + A_1 + A_2\) (as \(b_4 = 0\)) with nonzero \(\alpha_2\)-coefficient. An identical argument gives \(w(\alpha_2') \in \{\alpha_6, \alpha_5 + \alpha_6\}\) and \(w(\tilde{\alpha}') \in \{\alpha_3, \alpha_1 + \alpha_3\}\). That all maximal order Abelian subgroups are conjugate for the cases \(G_2, F_4, H_3,\) and \(H_4\) follows from Sylow’s Second Theorem, as the groups \(M/\langle -1 \rangle\) are Sylow \(p\)-subgroups of \(W/\langle -1 \rangle\) with \(p = 3\) or \(5\).

**Remark:** The definition of a discrete maximal torus as a maximal order Abelian subgroup with minimal number of Abelian invariants applies to any finite group. However, in general, there is more than one conjugacy class of them, as illustrated by \(Q_8\). A computer search of groups of small order indicates that groups with a single conjugacy class of maximal tori are the exception rather than the rule.

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