On Coupled Boussinesq Equations and Ostrovsky equations free from zero-mass contradiction

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Abstract. Long weakly-nonlinear waves in a layered waveguide with an imperfect interface (soft bonding between the layers) can be modelled using coupled Boussinesq equations. Previously, we considered the case when the materials of the layers have close mechanical properties, and the system supports radiating solitary waves. Here we are concerned with a more challenging case, when the mechanical properties of the materials of the layers are significantly different, and the system supports wave packet solutions. We construct a weakly-nonlinear solution of the Cauchy problem for this system, considering the problem in the class of periodic functions on an interval of finite length. The solution is constructed for the deviation from the evolving mean value using a novel multiple-scales procedure involving fast characteristic variables and two slow time variables. By construction, the Ostrovsky equations emerging within the scope of this procedure are solved for initial conditions with zero mean while initial conditions for the original system may have non-zero mean values. Asymptotic validity of the solution is carefully examined numerically. We also discuss the application of the solution to the study of co-propagating waves generated by the solitary or cnoidal wave initial conditions, as well as the case of counter-propagating waves and the resulting wave interactions. One local and two nonlocal conservation laws are obtained and used to control the accuracy of the numerical simulations.

Submitted to: Nonlinearity
1 Introduction

The propagation of long weakly-nonlinear longitudinal bulk strain solitary waves in an elastic waveguides can be modelled using Boussinesq-type equations (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Coupled Boussinesq equations were derived as a model describing similar waves in layered waveguides with a soft bonding layer [10]. The stability of solitary waves has inspired research into their applicability for detection of delamination in some cases, in addition to existing methods [11, 12, 13, 14, 15]. Low-frequency wave propagation in solids is relevant to a large number of modern applications (see, for example, [3, 4, 7, 16, 17, 18, 19] and references therein).

Of interest in this paper is the system of coupled regularised Boussinesq (cRB) equations, presented here in non-dimensional and scaled form:

\[ u_{tt} - u_{xx} = \varepsilon \left[ \frac{1}{2} (u^2)_{xx} + u_{ttxx} - \delta (u - w) \right], \quad (1.1) \]

\[ w_{tt} - c^2 w_{xx} = \varepsilon \left[ \frac{\alpha}{2} (w^2)_{xx} + \beta w_{ttxx} + \gamma (u - w) \right]. \quad (1.2) \]

This system of equations has been derived to describe long nonlinear longitudinal bulk strain waves in a bi-layer with a sufficiently soft bonding layer [10]. In this context, \( u \) and \( w \) denote longitudinal strains in the layers, \( \alpha, \beta, \gamma, \delta \) are coefficients depending on the mechanical and geometrical properties of a waveguide, \( c \) is the ratio of the characteristic linear wave speeds in the layers and \( \varepsilon \) is a small amplitude parameter. The detection of delamination regions was considered in [14], for strain solitary waves incident on the waveguide.

In all of these contexts, the natural initial conditions have non-zero mean, while the associated uni-directional equations emerging in the construction of weakly-nonlinear solutions are typically solved numerically using pseudo-spectral schemes and periodic boundary conditions. However, recent work has shown that solving such equations on a periodic domain may require careful attention if the initial conditions have non-zero mean [20, 21]. The Boussinesq-Klein-Gordon equation can be found from the cRB equations by taking the limit \( \gamma \to 0 \) with \( w = 0 \). The weakly-nonlinear solution derived via traditional procedure leads at leading order to two uni-directional Ostrovsky equations [22] that necessarily require zero-mean initial conditions. The existence of such a formal constraint is known as the ‘zero-mass (or zero-mean) contradiction’ [21]. However, in [21] it was shown that the derivation procedure can be modified in order to develop a weakly-nonlinear solution of the Boussinesq-Klein-Gordon equation for a deviation from the evolving non-zero mean. The earlier results in [20] were developed at the level of Fourier expansions in the spatial variable, while the procedure suggested in [21] can be viewed as a nonlinear extension of the d’Alembert’s solution. The Ostrovsky equations emerging within the scope of this procedure are solved for zero-mean initial conditions by construction, and the zero-mean contradiction is avoided.

Considering our cRB system of equations, in the case when \( u = w \) the system reduces to the regularised Boussinesq equation

\[ u_{tt} - u_{xx} = \varepsilon \left[ \frac{1}{2} (u^2)_{xx} + u_{ttxx} \right], \quad (1.3) \]
which supports pure propagating solitary waves. However, these solitary waves are not supported by the coupled system. When the materials in the layers have close mechanical properties, satisfying the relation \( c - 1 = \mathcal{O}(\varepsilon) \), pure solitary waves are replaced with long-living radiating solitary waves [10, 23, 24]. When the materials are significantly different, characterised by \( c - 1 = \mathcal{O}(1) \), the solitary waves are replaced by wave packets [23] governed by the Ostrovsky equations [22]. Therefore, as we have Ostrovsky equations and coupled Ostrovsky equations, which necessarily require zero-mean initial conditions, we need to consider how a weakly-nonlinear solution can be constructed that takes account of this restriction.

The first case, when \( c - 1 = \mathcal{O}(\varepsilon) \), was constructed in [25] and is briefly overviewed at the beginning of Section 2. We will re-examine this solution in comparison with the more complicated case when \( c - 1 = \mathcal{O}(1) \), which is the main topic of the present paper. In this second case the leading-order linear wave operators of the two equations have different characteristic variables, while the previous case can be brought to the asymptotically equivalent form where they are the same.

The rest of the paper is organised as follows. In Section 2 we construct a weakly-nonlinear solution of the Cauchy problem for the cRB equations (1.1) - (1.2) in the case \( c - 1 = \mathcal{O}(1) \) on a periodic domain, considering asymptotic multiple-scales expansions for the deviation from the oscillating mean values, and using two sets of fast characteristic variables and two slow time variables.

The validity of the solutions is examined in Section 3 by comparing the constructed weakly-nonlinear solution with direct numerical simulations of the Cauchy problem, and we discuss both cases \( c - 1 = \mathcal{O}(\varepsilon) \) and \( c - 1 = \mathcal{O}(1) \). In Section 4 we use both direct numerical simulations and the constructed weakly-nonlinear solutions to study the behaviour of the waves generated by the solitary and cnoidal wave initial conditions, co- and counter-propagating, for the cases when the characteristic speeds are either close or significantly different.

In Section 5 we discuss the local and non-local conservation laws used to control the accuracy of numerical simulations and we conclude our studies in Section 6.

## 2 Weakly Nonlinear Solution

We solve the equation system (1.1) - (1.2) on the periodic domain \( x \in [-L, L] \). The initial-value (Cauchy) problem is considered, and the initial conditions are written as

\[
\begin{align*}
  u(x,0) &= F_1(x), & u_t(x,0) &= V_1(x), \\
  w(x,0) &= F_2(x), & w_t(x,0) &= V_2(x).
\end{align*}
\]

(2.1)

(2.2)

Firstly, we integrate (1.1) - (1.2) in \( x \) over the period \( 2L \) to obtain evolution equations of the form

\[
\frac{d^2}{dt^2} \int_{-L}^{L} u(x,t) \, dx + \varepsilon \delta \int_{-L}^{L} (u(x,t) - w(x,t)) = 0,
\]

(2.3)

\[
\frac{d^2}{dt^2} \int_{-L}^{L} w(x,t) \, dx - \varepsilon \gamma \int_{-L}^{L} (u(x,t) - w(x,t)) = 0.
\]

(2.4)
Denoting the mean value of $u$ and $w$ as
\[ \bar{u}(t) := \frac{1}{2L} \int_{-L}^{L} u(x, t) \, dx, \quad \bar{w}(t) := \frac{1}{2L} \int_{-L}^{L} w(x, t) \, dx, \]  
(2.5)
we can solve (2.3) - (2.4) to obtain
\[ u = d_1 + \delta d_2 \cos \omega t + d_3 t + \delta d_4 \sin \omega t, \]  
(2.6)
\[ w = d_1 - \gamma d_2 \cos \omega t + d_3 t - \gamma d_4 \sin \omega t. \]  
(2.7)
Using the initial conditions (2.1), (2.2) we can determine the values of the coefficients as
\[ d_1 = \frac{\gamma F_1 + \delta F_2}{\delta + \gamma}, \quad d_2 = \frac{F_1 - F_2}{\delta + \gamma}, \quad d_3 = \frac{\gamma V_1 + \delta V_2}{\omega (\delta + \gamma)}, \quad d_4 = \frac{V_1 - V_2}{\omega (\delta + \gamma)}, \]  
(2.8)
where we introduce the notation $\omega = \sqrt{\varepsilon (\delta + \gamma)}$ and
\[ F_i = \int_{-L}^{L} F_i(x) \, dx, \quad V_i = \int_{-L}^{L} V_i(x) \, dx, \quad i = 1, 2. \]  
(2.9)
To simplify the problem we will consider initial conditions that satisfy the condition $d_3 = d_4 = 0$, that is
\[ \frac{1}{2L} \int_{-L}^{L} V_i \, dx = 0, \quad i = 1, 2. \]  
(2.10)
This condition appears naturally in many physical applications, and we impose it here in order to simplify our derivations. As was shown in [21] for the Boussinesq-Klein-Gordon equation (1.3), such conditions can be relaxed.

We subtract (2.6) from $u$ and (2.7) from $w$ to obtain an equation with zero mean value, so we introduce $\tilde{u} = u - \bar{u}$ and $\tilde{w} = w - \bar{w}$ to obtain the problem for deviations
\[ \tilde{u}_{tt} - \tilde{u}_{xx} = \varepsilon \left[ \frac{1}{2} (\tilde{u}^2)_{xx} + (d_1 + \delta d_2 \cos \omega t) \tilde{u}_{xx} + \tilde{u}_{txx} - \delta (\bar{u} - \bar{w}) \right], \]  
(2.11)
\[ \tilde{w}_{tt} - c^2 \tilde{w}_{xx} = \varepsilon \left[ \frac{\alpha}{2} (\tilde{w}^2)_{xx} + \alpha (d_1 - \gamma d_2 \cos \omega t) \tilde{w}_{xx} + \beta \tilde{w}_{txx} + \gamma (\bar{u} - \bar{w}) \right]. \]  
(2.12)
Note that this problem has variable coefficients, and the traditional procedure used to derive unidirectional models of the Korteweg-de Vries/Ostrovsky type is no longer applicable. The initial conditions become
\[ \tilde{u}(x, 0) = \tilde{F}_1(x) = F_1(x) - \bar{u}, \quad \tilde{u}_t(x, 0) = \tilde{V}_1(x) = V_1(x), \]  
(2.13)
\[ \tilde{w}(x, 0) = \tilde{F}_2(x) = F_2(x) - \bar{w}, \quad \tilde{w}_t(x, 0) = \tilde{V}_2(x) = V_2(x), \]  
(2.14)
and, by construction, have zero mean value. We omit tildes in all functions to simplify notation in subsequent sections.
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2.1 Case 1: $c - 1 = O(\varepsilon)$

The first case when $c - 1 = O(\varepsilon)$ was considered in [25], so we only summarise the results here. In this case the waves are resonant and an initial solitary wave solution in both layers will evolve into a radiating solitary wave; a solitary wave with a co-propagating one-sided oscillatory tail [10]. We note that we have $(c^2 - 1)/\varepsilon = O(1)$ and therefore we can rearrange (2.12) to give

\[
\begin{align*}
\frac{1}{2} \left( u^2 \right)_{xx} + (d_1 + \delta d_2 \cos \omega t) u_{xx} + \beta u_{ttxx} - \delta (u - w), \tag{2.15} \\
\frac{1}{2} \left( w^2 \right)_{xx} + (d_1 - \gamma d_2 \cos \omega t + \frac{c^2 - 1}{\varepsilon}) w_{xx} + \beta w_{ttxx} + \gamma (u - w). \tag{2.16}
\end{align*}
\]

We look for a weakly-nonlinear solution with two slow time scales [21], which takes the form

\[
\begin{align*}
u(x, t) &= f^-(\xi_-, \tau, T) + f^+(\xi_+, \tau, T) + \sqrt{\varepsilon} P_1(\xi_-, \xi_+, \tau, T) + \varepsilon Q_1(\xi_-, \xi_+, \tau, T) \\
&\quad + \varepsilon^\frac{3}{2} R_1(\xi_-, \xi_+, \tau, T) + \varepsilon^2 S_1(\xi_-, \xi_+, \tau, T) + O(\varepsilon^{\frac{5}{2}}), \tag{2.17} \\
w(x, t) &= f^-_2(\xi_-, \tau, T) + f^+_2(\xi_+, \tau, T) + \sqrt{\varepsilon} P_2(\xi_-, \xi_+, \tau, T) + \varepsilon Q_2(\xi_-, \xi_+, \tau, T) \\
&\quad + \varepsilon^\frac{3}{2} R_2(\xi_-, \xi_+, \tau, T) + \varepsilon^2 S_2(\xi_-, \xi_+, \tau, T) + O(\varepsilon^{\frac{5}{2}}), \tag{2.18}
\end{align*}
\]

with characteristic and slow time variables

\[
\xi_\pm = x \pm t, \quad \tau = \sqrt{\varepsilon}t, \quad T = \varepsilon t.
\]

As we are considering the solution on the periodic domain, the functions $u$ and $w$ are $2L$-periodic functions in $x$. Therefore we require that $f^\pm_{1,2}$ are periodic in $\xi_-$ and $\xi_+$ respectively, and that all terms in the asymptotic expansions are products of the functions $f^\pm_{1,2}$ and their derivatives. This implies that all functions are periodic in $\xi_-, \xi_+$ at fixed $\xi_+, \xi_-$. Furthermore, the functions $f^\pm_{1,2}$ have zero mean, satisfying

\[
\frac{1}{2L} \int_{-L}^{L} f^\pm_{1,2} \, d\xi_\pm = 0, \tag{2.19}
\]

and therefore all functions in the expansion have zero mean. These conditions are consistent with those established in [21].

We substitute (2.17) into (2.15) and (2.18) into (2.16), collecting terms at increasing powers of $\sqrt{\varepsilon}$ to determine expressions for all functions in the expansion. Matching up to powers of $\varepsilon^2$, we find that

\[
\begin{align*}
u(x, t) &= f^-_1 + f^+_1 + \sqrt{\varepsilon} (g^-_1 + g^+_1) + \varepsilon (h^-_1 + h^+_1 + f_{1c}) + O(\varepsilon^{\frac{3}{2}}), \tag{2.20} \\
w(x, t) &= f^-_2 + f^+_2 + \sqrt{\varepsilon} (g^-_2 + g^+_2) + \varepsilon (h^-_2 + h^+_2 + f_{2c}) + O(\varepsilon^{\frac{3}{2}}), \tag{2.21}
\end{align*}
\]

where the functions $f^\pm_{1,2}$ are described by

\[
\begin{align*}
\left( \mp 2f^\pm_{1T} + f^\pm_{1f^\pm_{1\xi_\pm}} + d_1 f^\pm_{1f^\pm_{1\xi_\pm \xi_\pm}} \right)_{\xi_\pm} &= \delta \left( f^+_1 - f^-_1 \right), \\
\left( \mp 2f^\pm_{2T} + \alpha f^\pm_{2f^\pm_{2\xi_\pm}} + \left( ad_1 + \frac{c^2 - 1}{\varepsilon} \right) f^\pm_{2f^\pm_{2\xi_\pm \xi_\pm}} \right)_{\xi_\pm} &= \gamma \left( f^+_2 - f^-_2 \right), \quad \text{and} \quad \text{all other terms (2.22)}
\end{align*}
\]
while the functions $g_{1,2}^\pm$ at $O(\sqrt{\varepsilon})$ can be expressed in terms of the leading order functions $f_{1,2}^\pm$ as

$$
g_1^\pm = \pm \frac{d_2 \delta}{2\tilde{\omega}} \sin(\tilde{\omega} \tau) f_1^\pm, \quad g_2^\pm = \pm \frac{\alpha d_2 \gamma}{2\tilde{\omega}} \sin(\tilde{\omega} \tau) f_2^\pm.
$$

(2.23)

At $O(\varepsilon)$ we find

$$
h_1^\pm = \frac{\tilde{\omega}^2 \rho_1^2}{2} f_1^\pm - \frac{\delta (\rho_1 + \rho_2)}{2} f_2^\pm - \frac{\tilde{\omega}^2 \rho_1^2}{2} f_{1\xi_\pm}^\pm + \phi_1^\pm (\xi_\pm, T),
$$

(2.24)

$$
h_2^\pm t = - \frac{\tilde{\omega}^2 \rho_2}{2} f_2^\pm + \frac{\gamma (\rho_1 + \rho_2)}{2} f_1^\pm - \frac{\tilde{\omega}^2 \rho_1^2}{2} f_{2\xi_\pm}^\pm + \phi_2^\pm (\xi_\pm, T),
$$

(2.25)

and

$$
h_{1c} = - \frac{1}{4} \left( f_{1\xi_-} \int_{-L}^{\xi_+} f_1^+ (\sigma) \, d\sigma + 2 f_{1}^{-} f_1^+ + f_{1\xi_+}^+ \int_{-L}^{\xi_-} f_1^- (\sigma) \, d\sigma \right),
$$

(2.26)

$$
h_{2c} = - \frac{\alpha}{4} \left( f_{2\xi_-} \int_{-L}^{\xi_+} f_2^+ (\sigma) \, d\sigma + 2 f_{2}^{-} f_2^+ + f_{2\xi_+}^+ \int_{-L}^{\xi_-} f_2^- (\sigma) \, d\sigma \right),
$$

(2.27)

where we introduced the coefficients

$$
\rho_1 = - \frac{d_2 \delta}{2\tilde{\omega}^2} \cos(\tilde{\omega} \tau), \quad \rho_2 = - \frac{\alpha d_2 \delta}{2\tilde{\omega}^2} \cos(\tilde{\omega} \tau).
$$

(2.28)

The functions $\phi_{1,2}^\pm$ are described by

$$
\left( \mp 2 \phi_{1T}^\pm + (f_1^\pm \phi_1^\pm)_{\xi_\pm} + d_1 \phi_{1\xi_\pm}^\pm + \phi_{1\xi_\pm\xi_\pm}^\pm \right)_{\xi_\pm} = \delta (\phi_1^\pm - \phi_2^\pm) + f_{1TT}^\pm + 2 f_{1\xi_\pm\xi_\pm}^\pm T
$$

$$
+ \frac{\tilde{\omega}^2 \theta_1^2}{2} f_{1\xi_\pm}^\pm - \frac{\theta_2^2}{2} (\delta + \alpha \gamma) f_{2\xi_\pm}^\pm - \frac{\tilde{\omega}^2}{2} \left( f_{1\xi_\pm}^\pm \right)_{\xi_\pm}^2 \right
$$

(2.29)

and

$$
\left( \mp 2 \phi_{2T}^\pm + \alpha (f_2^\pm \phi_2^\pm)_{\xi_\pm} + \alpha d_1 \phi_{2\xi_\pm}^\pm + \frac{\tilde{\omega}^2 - 1}{\varepsilon} \phi_{2\xi_\pm}^\pm + \beta \phi_{2\xi_\pm\xi_\pm}^\pm \right)_{\xi_\pm} = \gamma (\phi_2^\pm - \phi_1^\pm)
$$

$$
+ f_{2TT}^\pm + 2 \beta f_{2\xi_\pm\xi_\pm\xi_\pm}^\pm T + \frac{\tilde{\omega}^2 \theta_2^2}{2} f_{2\xi_\pm}^\pm + \frac{\theta_2^2}{2} (\delta + \alpha \gamma) f_{1\xi_\pm\xi_\pm}^\pm - \frac{\alpha \theta_2^2}{2} \left( f_{2\xi_\pm}^\pm \right)_{\xi_\pm}^2 \right
$$

(2.30)

where we have introduced the modified non-secular coefficients

$$
\tilde{\theta}_1 = \frac{d_2 \delta}{2\tilde{\omega}}, \quad \tilde{\theta}_2 = \frac{\alpha d_2 \gamma}{2\tilde{\omega}}.
$$

(2.31)

These are complemented with initial conditions

$$
f_{11}^\pm |_{T=0} = \frac{1}{2} \left( \bar{F}_1 (x \pm t) \pm \int_{-L}^{x_{\pm}} \bar{V}_1 (\sigma) \, d\sigma \right), \quad f_{22}^\pm |_{T=0} = \frac{1}{2} \left( \bar{F}_2 (x \pm t) \pm \int_{-L}^{x_{\pm}} \bar{V}_2 (\sigma) \, d\sigma \right).
$$

(2.32)

$$
\phi_1^\pm = \frac{1}{2} \left( J_1 \pm \int_{-L}^{\xi_\pm} K_1 (\sigma) \, d\sigma \right), \quad \phi_2^\pm = \frac{1}{2} \left( J_2 \pm \int_{-L}^{\xi_\pm} K_2 (\sigma) \, d\sigma \right),
$$

(2.33)
where
\[
J_1 = -\frac{\tilde{\omega}^2 \rho_1}{2} (f_1^- + f_1^+) + \frac{\delta (\rho_1 + \rho_2)}{2} (f_2^- + f_2^+) + \frac{\tilde{\omega}^2 \rho_1^2}{2} \left( f_{1\xi_- \xi_-}^+ + f_{1\xi_+ \xi_+}^- \right) - h_{1c},
\]
\[
K_1 = f_{1T}^- + f_{1T}^+ + \frac{\tilde{\omega}^2 \rho_1}{2} f_1^- (f_{1\xi_-}^- - f_{1\xi_+}^+) + \frac{\delta (\rho_1 + \rho_2)}{2} (f_{2\xi_-}^- - f_{2\xi_+}^+) + \frac{\tilde{\omega}^2 \rho_1^2}{2} \left( f_{1\xi_- \xi_-}^- - f_{1\xi_+ \xi_+}^+ \right),
\]
\[
J_2 = \frac{\tilde{\omega}^2 \rho_2}{2} (f_2^- + f_2^+) - \frac{\gamma (\rho_1 + \rho_2)}{2} (f_1^- + f_1^+) + \frac{\tilde{\omega}^2 \rho_1^2}{2} \left( f_{2\xi_- \xi_-}^- + f_{2\xi_+ \xi_+}^+ \right) - h_{2c},
\]
\[
K_2 = f_{2T}^- + f_{2T}^+ - \frac{\tilde{\omega}^2 \rho_2}{2} (f_{2\xi_-}^- - f_{2\xi_+}^+) - \frac{\gamma (\rho_1 + \rho_2)}{2} f_1^+ (f_{1\xi_-}^- - f_{1\xi_+}^+) + \frac{\tilde{\omega}^2 \rho_1^2}{2} \left( f_{2\xi_- \xi_-}^- - f_{2\xi_+ \xi_+}^+ \right).
\] (2.34)

### 2.2 Case 2: \( c - 1 = O(1) \)

The focus of this paper is the case when the phase speeds are distinctly different i.e. \( c - 1 = O(1) \). In this case the characteristic variables cannot be the same in each layer, and instead we have two distinct pairs of characteristic variables. Therefore we look for a weakly-nonlinear solution of the form
\[
\begin{align*}
\text{\textit{\textbf{u}(x, t) = f_1^-(\xi_-, \tau, T) + f_1^+(\xi_+, \tau, T) + \sqrt{\varepsilon} P_1 (\xi_-, \xi_+, \tau, T) + \varepsilon Q_1 (\xi_-, \xi_+, \tau, T) + \varepsilon^2 R_1 (\xi_-, \xi_+, \tau, T) + O \left( \varepsilon^\frac{5}{2} \right),} \\
\text{\textit{\textbf{w}(x, t) = f_2^-(\nu_-, \tau, T) + f_2^+(\nu_+, \tau, T) + \sqrt{\varepsilon} P_2 (\nu_-, \nu_+, \tau, T) + \varepsilon Q_2 (\nu_-, \nu_+, \tau, T) + \varepsilon^2 S_2 (\nu_-, \nu_+, \tau, T) + O \left( \varepsilon^\frac{3}{2} \right),}}
\end{align*}
\] (2.35) (2.36)

where we have the following characteristic and slow time variables
\[
\xi_\pm = x \pm c t, \quad \nu_\pm = x \pm \sqrt{\varepsilon} c t, \quad \tau = \sqrt{\varepsilon} t, \quad T = \varepsilon t.
\]

We follow the same steps as for the previous case, by substituting \( (2.35) \) and \( (2.36) \) into \( (2.11) \) and \( (2.12) \) and comparing at increasing powers of \( \sqrt{\varepsilon} \), using the same assumptions of periodicity and zero mean for all functions in the expansion. The equations are satisfied at leading order so we move onto terms at \( O \left( \sqrt{\varepsilon} \right) \). At this order we have
\[
-4P_{1\xi_- \xi_+} - 2f_{1\xi_- \tau}^+ + 2f_{1\xi_+ \tau}^+ = 0. \tag{2.37}
\]

We average \( (2.37) \) with respect to the fast spatial variable \( x \) at constant \( \xi_- \) and \( \xi_+ \) (see [21]). Averaging \( P_{1\xi_- \xi_+} \) at constant \( \xi_- \) gives
\[
\frac{1}{2L} \int_{-L}^{L} P_{1\xi_- \xi_+} \, dx = \frac{1}{4L} \int_{-2L}^{-L} P_{1\xi_- \xi_+} \, d\xi_+ = \frac{1}{4L} \left[ P_{1\xi_- \xi_+} \right]_{-2L}^{-L} = 0, \tag{2.38}
\]
with a similar result for averaging at constant \( \xi_+ \). Applying the averaging to \( (2.37) \) we find
\[
\begin{align*}
f_{1\xi_- \tau}^- = 0 \implies f_1^- = \tilde{f}_1^- (\xi_-, T) \quad \text{and} \quad f_{1\xi_+ \tau}^+ = 0 \implies f_1^+ = \tilde{f}_1^+ (\xi_+, T).
\end{align*}
\] (2.39)
Similarly for the equation in \( w \) we obtain
\[
-4c^2 P_{2w_\nu} - 2cf_{2w_{-\tau}} + 2cf_{2w_{+\tau}} = 0,
\]
(2.40)
which, after averaging, leads to
\[
f^-_2 = \tilde{f}^-_2 (\nu_-, T) \quad \text{and} \quad f^+_2 = \tilde{f}^+_2 (\nu_+, T).
\]
(2.41)
Substituting (2.39) into (2.37) we obtain
\[
P_{1\xi_-\xi_+} = 0 \implies P_1 = g^-_1 (\xi_-, \tau, T) + g^+_1 (\xi_+, \tau, T),
\]
(2.42)
and similarly substituting (2.41) into (2.40) we obtain
\[
P_{2w_{-\nu}} = 0 \implies P_2 = g^-_2 (\nu_-, \tau, T) + g^+_2 (\nu_+, \tau, T).
\]
(2.43)
As before, we omit the tildes on \( f^\pm_{1,2} \) in subsequent steps. The initial condition for \( f^\pm_{1,2} \) is found in the same way as before, by substituting (2.35) into (2.1) and (2.36) into (2.2) and comparing terms at \( O (1) \) to obtain
\[
\begin{align*}
\left\{ \begin{array}{l}
f_1^- + f^+_1 |_{T=0} = F_1(x), \\
-f_{1\xi_-}^+ + f^+_{1\xi_+} |_{T=0} = V_1(x),
\end{array} \right. \implies f^+_1 |_{T=0} = \frac{1}{2} \left( F_1 (x \pm t) \pm \int_{-L}^{x+t} V_1 (\sigma) \ d\sigma \right),
\end{align*}
\]
(2.44)
and
\[
\begin{align*}
\left\{ \begin{array}{l}
f^-_2 + f^+_2 |_{T=0} = F_2(x), \\
-2c f_{2w_-} + c f_{2w_+} |_{T=0} = V_2(x),
\end{array} \right. \implies f^+_2 |_{T=0} = \frac{1}{2c} \left( cF_2 (x \pm ct) \pm \int_{-L}^{x+ct} V_2 (\sigma) \ d\sigma \right).
\end{align*}
\]
(2.45)
Comparing terms at \( O (\varepsilon) \), using the results from the previous order, we have
\[
-4Q_{1\xi_-\xi_+} = 2g^-_{1\xi_-\tau} + \left( 2f^-_{1\xi} + f^-_{1\xi^-} + d_1 f^-_{1\xi^-} + f^+_{1\xi_-\xi_-} \right)_{\xi_-} - \delta (f^-_1 - f^-_2)
\]
\[
-2g^+_{1\xi_+\tau} + \left( -2f^+_{1\xi} + f^+_{1\xi^+} + d_1 f^+_{1\xi^+} + f^+_{1\xi_+\xi_+} \right)_{\xi_+} - \delta (f^+_1 - f^+_2)
\]
\[
+ d_2 \delta \cos (\bar{\omega} \tau) \left( f^+_{1\xi_+\xi_-} + f^+_{1\xi_-\xi_+} \right) + f^-_{1\xi_-\xi_-} + f^+_{1\xi_+\xi_+} + 2f^-_{1\xi_-\xi_+} + f^+_{1\xi_+\xi_+} + f^+_{1\xi_+\xi_+},
\]
(2.46)
for the equation governing \( u \), and for the equation governing \( w \) we obtain
\[
-4c^2 Q_{2w_{-\nu}} = 2g^-_{2w_\nu} + \left( 2c f^-_{2\nu} + \alpha f^-_{2w_\nu} + \beta c^2 f^-_{2w_{-\nu}} \right)_{\nu_-} + \gamma (f^-_1 - f^-_2)
\]
\[
-2g^+_{2w_\nu} + \left( -2c f^+_{2\nu} + \alpha f^+_{2w_\nu} + \beta c^2 f^+_{2w_{+\nu}} \right)_{\nu_+} + \gamma (f^+_1 - f^+_2)
\]
\[
- \alpha d_2 \gamma \cos (\bar{\omega} \tau) \left( f^-_{2w_-} + f^+_{2w_-} \right) + \alpha (f^-_{2w_\nu} f^+_{2w_\nu} + f^-_{2w_\nu} f^+_{2w_\nu} + f^-_{2w_\nu} f^+_{2w_\nu}).
\]
(2.47)
Averaging (2.46) at constant \( \xi_- \) or constant \( \xi_+ \) gives
\[
\pm 2g_{1\xi_\pm\tau} = d_2 \delta \cos (\bar{\omega} \tau) f^+_{1\xi_\pm\xi_\pm} + A_1 (\xi_\pm, T),
\]
(2.48)
Similarly substituting (2.53) and (2.55) into (2.47) and integrating we obtain

\[
A_1 = \left( \mp 2 f_{1T}^\pm + f_1^\pm f_{1x}^\pm + d_1 f_{1x}^\pm f_{1x}^\pm \delta f_{1x}^\pm \right) - \delta f_{1x}^\pm. \tag{2.49}
\]

To avoid secular terms we require that \( A_1 = 0 \). Therefore we obtain an Ostrovsky equation for \( f_1^\pm \) of the form

\[
\left( \mp 2 f_{1T}^\pm + f_1^\pm f_{1x}^\pm + d_1 f_{1x}^\pm f_{1x}^\pm \delta f_{1x}^\pm \right) = \delta f_{1x}^\pm. \tag{2.50}
\]

Similarly, averaging (2.47) at constant \( \nu_- \) or constant \( \nu_+ \) leads to the equation

\[
\pm 2 c g_{2\nu_{\pm}} = -d_2 \gamma \cos (\tilde{\omega} \tau) f_{2\nu_{\pm}}^\pm + A_2 (\nu_{\pm}, T), \tag{2.51}
\]

where

\[
A_2 = \left( \mp 2 c f_{2T}^\pm + \alpha f_{2}^\pm f_{2x}^\pm + \beta c^2 f_{2x}^\pm f_{2x}^\pm \right) - \gamma f_{2x}^\pm. \tag{2.52}
\]

We again require \( A_2 = 0 \) to avoid secular terms, and therefore we obtain an Ostrovsky equation for \( f_2^\pm \) of the form

\[
\left( \mp 2 c f_{2T}^\pm + \alpha f_{2}^\pm f_{2x}^\pm + \beta c^2 f_{2x}^\pm f_{2x}^\pm \right) = \gamma f_{2x}^\pm. \tag{2.53}
\]

As before we integrate (2.48) and (2.51), taking account of (2.50) and (2.53), to find an equation for \( g_{1,2}^\pm \) of the form

\[
g_1^\pm = \pm d_2 \frac{\delta}{2 \omega} \sin (\tilde{\omega} \tau) f_{1x}^\pm + G_1^\pm (\xi_{\pm}, T) = \pm \theta_1 f_{1x}^\pm + G_1^\pm (\xi_{\pm}, T), \tag{2.54}
\]

and

\[
g_2^\pm = \mp \frac{\alpha d_2 \gamma}{2 c \omega} \sin (\tilde{\omega} \tau) f_{2x}^\pm + G_2^\pm (\nu_{\pm}, T) = \mp \theta_2 f_{2x}^\pm + G_2^\pm (\nu_{\pm}, T), \tag{2.55}
\]

where \( G_1^\pm \) and \( G_2^\pm \) are functions to be found, and we introduce

\[
\theta_1 = \frac{d_2 \delta}{2 \omega} \sin (\tilde{\omega} \tau), \quad \theta_2 = \frac{\alpha d_2 \gamma}{2 c \omega} \sin (\tilde{\omega} \tau). \tag{2.56}
\]

Substituting (2.50) and (2.54) into (2.46) and integrating we obtain

\[
Q_1 = h_1^- (\xi_-, \tau, T) + h_1^+ (\xi_+, \tau, T) + h_{1c} (\xi_-, \xi_+, T) + \hat{f}_2^- (\nu_-, T) + \hat{f}_2^+ (\nu_+, T), \tag{2.57}
\]

where

\[
h_{1c} = -\frac{1}{4} \left( f_{1x} - \int_{-L}^{\xi_+} f_1^+ (\sigma) \ d\sigma + 2 f_1^- f_1^+ + f_{1x}^\pm \int_{-L}^{\xi_+} f_1^- (\sigma) \ d\sigma \right), \tag{2.58}
\]

and

\[
\hat{f}_2^\pm = \frac{\delta}{c^2 - 1} \int_{-L}^{\nu_{\pm}} f_2^\pm (u, T) \ du. \tag{2.59}
\]

Similarly substituting (2.53) and (2.55) into (2.47) and integrating we obtain

\[
Q_2 = h_2^- (\nu_-, \tau, T) + h_2^+ (\nu_+, \tau, T) + h_{2c} (\nu_-, \nu_+, T) + \hat{f}_1^- (\xi_-, T) + \hat{f}_1^+ (\xi_+, T), \tag{2.60}
\]

where

\[
h_{2c} = -\frac{\alpha}{4 c^2} \left( f_{2u_2} - \int_{-L}^{\nu_+} f_2^+ (\sigma) \ d\sigma + 2 f_2^- f_2^+ + f_{2x}^\pm \int_{-L}^{\nu_{\pm}} f_2^- (\sigma) \ d\sigma \right), \tag{2.61}
\]
and

\[ \hat{f}_1^\pm = -\frac{\gamma}{c^2 - 1} \int_{-L}^{L} f_1^\pm (u, T) \ du \ dv. \]  

(2.62)

As before we find the initial condition for \( G_{1,2}^+ \) by substituting (2.35) and (2.36) into (2.1) and (2.2), comparing terms at \( O(\sqrt{\varepsilon}) \). Therefore, taking account of the results found for (2.54) and (2.55), and noting that \( \theta_1|_{T=0} = 0 \), we obtain

\[
\begin{align*}
\theta_1 f_{1,2}^- + \left. \theta_1 f_{1,2}^+ + G_1^- + G_1^+ \right|_{T=0} &= 0, \\
-\theta_1 f_{1,2}^- + \left. \theta_1 f_{1,2}^+ - G_1^- - G_1^+ \right|_{T=0} &= 0, \quad \Rightarrow \quad G_1^+|_{T=0} = 0. 
\end{align*}
\]

(2.63)

Similarly we have

\[
\begin{align*}
\theta_2 f_{2,2}^- + \theta_2 f_{2,2}^+ + G_2^- + G_2^+|_{T=0} &= 0, \\
-c\theta_2 f_{2,2}^- + c\theta_2 f_{2,2}^+ - cG_2^- - cG_2^+|_{T=0} &= 0, \quad \Rightarrow \quad G_2^+|_{T=0} = 0. 
\end{align*}
\]

(2.64)

We now aim to find equations for \( h_{1,2}^+ \). Comparing terms at \( O(\varepsilon^{3/2}) \) in the expansion of (2.11) and (2.12), taking account of results at previous orders, we have

\[
-4R_{1,2}^- \xi_\tau = 2h_{1,2}^- + 2h_{1,2}^+ + \left( 2g_{1T} + (f_1^- g_1^-) \right)_{\xi_\tau} + d_1 g_{1,2}^- + g_{1,2}^- \xi_\tau - \delta (g_1^- - g_2^-)
\]

\[
- g_{1,2}^- + \left( 2g_{1T} + (f_1^- g_1^-) \right)_{\xi_\tau} + d_1 g_{1,2}^+ + g_{1,2}^+ \xi_\tau - \delta (g_1^+ - g_2^+)
\]

\[
+ d_2 \delta \cos (\tilde{\omega} \tau) \left( g_{1,2}^- \xi_\tau + g_{1,2}^+ \xi_\tau \right) + g_{1,2}^- f_1^+ + 2g_{1,2}^- f_1^+ + g_{1,2}^- f_1^+ + g_{1,2}^- f_1^+.
\]

(2.65)

and

\[
-4c^2 R_{2,2}^- \nu_\tau = 2c h_{2,2}^- + \alpha \left( g_{2T} + (f_2 g_2^-) \right)_{\nu_\tau} + \alpha d_1 g_{2,2}^- + \beta c^2 g_{2,2}^- \nu_\tau - \gamma (g_1^- - g_2^-)
\]

\[
- 2c h_{2,2}^- + \alpha \left( g_{2T} + (f_2 g_2^-) \right)_{\nu_\tau} + \alpha d_1 g_{2,2}^+ + \beta c^2 g_{2,2}^+ \nu_\tau - \gamma (g_1^+ - g_2^+)
\]

\[
- \alpha d_2 \cos (\tilde{\omega} \tau) \left( g_{2,2}^- \nu_\tau + g_{2,2}^+ \nu_\tau \right) - g_{2,2}^+ \nu_\tau - g_{2,2}^+ \nu_\tau
\]

\[
+ \alpha \left( g_{2,2}^- \nu_\tau + 2g_{2,2}^- f_{2,2}^+ + 2g_{2,2}^- f_{2,2}^+ + g_{2,2}^- f_{2,2}^+ + f_{2,2}^+ f_{2,2}^+ + f_{2,2}^+ f_{2,2}^+ \right).
\]

(2.66)

Substituting (2.54) into (2.65) and averaging at constant \( \xi_\tau \) or constant \( \xi_\tau \) leads to

\[
\pm 2h_{1,2}^- \xi_\tau = \pm \theta_1 \left( \mp 2 f_{1T}^+ + f_{1,2}^+ + d_1 f_{1,2}^+ + f_{1,2}^+ \xi_\tau \xi_\tau \right)_{\xi_\tau} \mp \theta_1 \delta f_{1,2}^+ \]

\[
+ \left( \mp 2 G_{1T}^+ + \left( f_{1,2}^+ G_1^+ \right)_{\xi_\tau} + d_1 G_{1,2}^+ + G_{1,2}^- \xi_\tau \xi_\tau \right)_{\xi_\tau} - \delta G_{1,2}^+
\]

\[
\pm \theta_1 \tilde{\omega}^2 f_{1,2}^+ \xi_\tau \pm \theta_1 d_2 \delta \cos (\tilde{\omega} \tau) f_{1,2}^+ \xi_\tau \xi_\tau.
\]

(2.67)

If we differentiate (2.50) with respect to the appropriate characteristic variable, we can eliminate the first line from (2.67) to obtain an expression for \( h_{1,2}^+ \) of the form

\[
2h_{1,2}^+ = \theta_1 \tilde{\omega}^2 f_{1,2}^+ \xi_\tau \theta_1 d_2 \delta \cos (\tilde{\omega} \tau) f_{1,2}^+ \xi_\tau \xi_\tau + G_{1}^+ \xi_\tau, T
\]

(2.68)
where
\[ \tilde{G}^\pm (\xi_\pm, T) = \left( \mp 2G_{1T}^\pm + (f_1^\pm G_1^\pm)_{\xi_\pm} + d_1 G_{1\xi_\pm}^\pm + G_{1\xi_\pm\xi_\pm}^\pm \right)_{\xi_\pm} - \delta G_1^\pm. \] (2.69)

To avoid secular terms we require that \( \tilde{G}_1^\pm = 0 \) and therefore we have an equation for \( G_1^\pm \) of the form
\[ \left( \mp 2G_{1T}^\pm + (f_1^\pm G_1^\pm)_{\xi_\pm} + d_1 G_{1\xi_\pm}^\pm + G_{1\xi_\pm\xi_\pm}^\pm \right)_{\xi_\pm} = \delta G_1^\pm. \] (2.70)

Taking account of the initial condition in (2.63) and the form of (2.70), we can clearly see that \( G_1^\pm \equiv 0 \) in this derivation and therefore in all subsequent steps we will omit all terms in \( G_1^\pm \). Integrating (2.68) we obtain
\[ h_1^\pm = -\frac{\tilde{\omega}^2 \rho_1}{2} f_1^\pm - \frac{\tilde{\omega}^2 \rho_1^2}{2} f_{1\xi_\pm}^\pm + \phi_1^\pm (\xi_\pm, T), \] (2.71)
where the functions \( \phi_1^\pm \) are to be found at the next order and
\[ \rho_1 = \partial^{-1} \theta_1 = \frac{d_2 \delta}{2\tilde{\omega}^2} \cos (\tilde{\omega} \tau). \] (2.72)

Averaging (2.66) at constant \( \nu_- \) or \( \nu_+ \), and using (2.55) and (2.53) as was done above, we obtain an expression for \( h_2^\pm \) of the form
\[ \pm 2c h_2^\pm = \mp \theta \omega^2 f_{2\nu_\pm}^\pm + \theta_2 d_2 \gamma \cos (\tilde{\omega} \tau) f_{2\nu_\pm\nu_\pm}^\pm + \tilde{G}_2^\pm (\nu_\pm, T), \] (2.73)
where
\[ \tilde{G}_2^\pm (\xi_\pm, T) = \left( \mp 2c G_{2T}^\pm + \alpha \left( f_2^\pm G_2^\pm \right)_{\nu_\pm} + \alpha d_1 G_{2\nu_\pm}^\pm + \beta c^2 G_{2\nu_\pm\nu_\pm}^\pm \right)_{\nu_\pm} - \gamma G_2^\pm. \] (2.74)

As before we require that \( \tilde{G}_2^\pm = 0 \) and \( G_2^\pm \) is governed by the equation
\[ \left( \mp 2c G_{2T}^\pm + \alpha \left( f_2^\pm G_2^\pm \right)_{\nu_\pm} + \alpha d_1 G_{2\nu_\pm}^\pm + \beta c^2 G_{2\nu_\pm\nu_\pm}^\pm \right)_{\nu_\pm} = \gamma G_2^\pm. \] (2.75)

By the same argument as for \( G_1^\pm \) we have \( G_2^\pm \equiv 0 \) and therefore omit it from all subsequent derivations. Integrating (2.73) we obtain
\[ h_2^\pm = \frac{\tilde{\omega}^2 \rho_2}{2c} f_2^\pm - \frac{\tilde{\omega}^2 \rho_2^2}{2} f_{2\nu_\pm}^\pm + \phi_2^\pm (\nu_\pm, T), \] (2.76)
where again we need to find the function \( \phi_2^\pm \) and
\[ \rho_2 = \partial^{-1} \theta_2 = \frac{\alpha d_2 \gamma}{2\tilde{\omega}^2} \cos (\tilde{\omega} \tau). \] (2.77)

Substituting (2.71) into (2.65) and integrating with respect to the appropriate characteristic variables we find
\[ R_1 = \psi_1^- (\xi_-, \tau, T) + \psi_1^+ (\xi_+, \tau, T) + \psi_{1c} (\xi_-, \xi_+, T) + \hat{g}_2^- + \hat{g}_2^+, \] (2.78)
where
\[ \psi_{1c} = -\frac{\theta_1}{4} \left[ f_{1\xi_+, \xi_+}^+ \int_{-L}^{\xi_+} f_{1}^- (\sigma) \ d\sigma - f_{1}^+ f_{1\xi_-}^- + f_{1}^- f_{1\xi_+}^+ - f_{1\xi_-}^- f_{1\xi_+}^+ \right], \] (2.79)
and \( \hat{g}_2^\pm \) can be found by replacing \( f \) with \( g \) in (2.59). In a similar way we find an expression for \( R_2 \) of the form

\[
R_2 = \psi_2^- (\nu_-, \tau, T) + \psi_2^+ (\nu_+, \tau, T) + \psi_2^c (\xi_-, \xi_+, T) + \hat{g}_1^- + \hat{g}_1^+,
\]

where

\[
\psi_2^c = -\frac{\alpha \theta_2}{4c^2} \left[ f_{2\xi_-}^- \int_{-L}^{\xi_+} f_2^+ (\sigma) \, d\sigma - f_{2\xi_+}^- f_2^+ + f_2^- f_{2\xi_+}^+ - f_{2\xi_-} f_{2\xi_+}^+ \right],
\]

and again the expression for \( \hat{g}_1^+ \) can be found by replacing \( f \) with \( g \) in (2.62). The initial condition for the function \( \phi_1^+ \) is found by again substituting (2.35) into (2.1) and comparing terms at \( O(\varepsilon) \), taking account of (2.71). Therefore we obtain for \( \phi_1^+ \)

\[
\begin{align*}
\left\{ & h_1^- + h_1^+ + h_1c + \hat{f}_2^- + \hat{f}_2^+ \bigg|_{T=0} = 0, \\
& f_1T + f_1T + g_1^- + g_1^+ - h_{1\xi_-} + h_{1\xi_+} - h_{1\xi_+} - h_{1\xi_-} - c \hat{f}_{2\xi_-}^- + c \hat{f}_{2\xi_+}^+ \bigg|_{T=0} = 0,
\end{align*}
\]

\[
\implies \phi_1^+ = \frac{1}{2} \left( J_1^+ + \int_{-L}^{\xi_+} K_1 (\sigma) \, d\sigma \right),
\]

where

\[
J_1 = \frac{\tilde{c} \rho_1}{2} \left( f_1^- + f_1^+ \right) + \frac{\tilde{c} \rho_1}{2} \left( f_{1\xi_-}^- + f_{1\xi_+}^+ \right) - h_1c
\]

\[
- \frac{\delta}{c^2 - 1} \int_{-L}^{\nu_-} \int_{-L}^{\nu_+} f_2^- (u, T) \, du \, dv - \frac{\delta}{c^2 - 1} \int_{-L}^{\nu_-} \int_{-L}^{\nu_+} f_2^+ (u, T) \, du \, dv,
\]

\[
K_1 = f_1T + f_1T + \frac{\tilde{c} \rho_1}{2} \left( f_{1\xi_-}^- - f_{1\xi_+}^+ \right) + \frac{\tilde{c} \rho_1}{2} \left( f_{1\xi_-}^- + f_{1\xi_+}^+ \right) - h_{1\xi_-} + h_{1\xi_+}
\]

\[
- \frac{\delta}{c^2 - 1} \int_{-L}^{\nu_-} f_2^- (u, T) \, du + \frac{\delta}{c^2 - 1} \int_{-L}^{\nu_+} f_2^+ (u, T) \, du.
\]

Similarly for \( \phi_2^+ \) we find the initial condition by substituting (2.36) into (2.2) and comparing terms at \( O(\varepsilon) \), using (2.76). This gives

\[
\begin{align*}
\left\{ & h_2^- + h_2^+ + h_2c + \hat{f}_1^- + \hat{f}_1^+ \bigg|_{T=0} = 0, \\
& f_{2T}^+ + f_{2T}^- + g_{2\tau}^- + g_{2\tau}^+ - c h_{2\nu_-}^+ + c h_{2\nu_+}^+ - c h_{2\nu_-}^- + c h_{2\nu_+}^- \bigg|_{T=0} = 0,
\end{align*}
\]

\[
\implies \phi_2^+ = \frac{1}{2c} \left( J_2^+ + \int_{-L}^{\xi_+} K_2 (\sigma) \, d\sigma \right),
\]

where

\[
J_2 = -\frac{\tilde{c} \rho_2}{2c} \left( f_2^- + f_2^+ \right) + \frac{\tilde{c} \rho_2}{2c} \left( f_{2\nu_-}^- + f_{2\nu_+}^+ \right) - h_2c
\]

\[
+ \frac{\gamma}{c^2 - 1} \int_{-L}^{\xi_-} \int_{-L}^{\nu_+} f_{1\xi_+}^- (u, T) \, du \, dv + \frac{\gamma}{c^2 - 1} \int_{-L}^{\xi_-} \int_{-L}^{\nu_+} f_{1\xi_+}^+ (u, T) \, du \, dv,
\]

\[
K_2 = f_{2T}^+ + f_{2T}^- + \frac{\tilde{c} \rho_2}{2} \left( f_{2\nu_-}^- - f_{2\nu_+}^+ \right) + \frac{\tilde{c} \rho_2}{2} \left( f_{2\nu_-}^+ - f_{2\nu_+}^- \right) - c h_{2\nu_-} + c h_{2\nu_+}
\]

\[
+ \frac{\gamma}{c^2 - 1} \int_{-L}^{\xi_-} f_{1\xi_+}^- (u, T) \, du + \frac{\gamma}{c^2 - 1} \int_{-L}^{\xi_-} f_{1\xi_+}^+ (u, T) \, du.
\]
We now have expressions up to $O(\varepsilon)$, however we still need to find an equation governing $\phi_{1,2}^\pm$, therefore we compare terms at $O(\varepsilon^2)$. The weakly-nonlinear expansions (2.35) and (2.36) are substituted into (2.15) and (2.16) respectively, taking account of all previous results. All coupling terms between left- and right-propagating functions are gathered in one function, and terms of the type of (2.59) and (2.62) are gathered in another function for convenience, as we do not require them to determine $\phi_{1,2}^\pm$. Gathering terms at $O(\varepsilon^2)$ we have

$$-4S_{1,\xi_+} = -f_{1TT}^+ - f_{1TT}^- - 2g_{1T}^+ - 2g_{1T}^- - h_{1TT}^- - h_{1TT}^+ + 2h_{1T}^- + 2h_{1T}^+ + 2\psi_{1T}^- - 2\psi_{1T}^+$$

where again $\mu_1c$ is the coupling terms at this order and $\Upsilon_1$ is the terms involving $\hat{f}_2$ or equivalent. For the second equation we have

$$-4c^2S_{2\nu_-,\nu_+} = -f_{2TT}^+ - f_{2TT}^- - 2g_{2T}^+ - 2g_{2T}^- - h_{2TT}^- - h_{2TT}^+ + 2h_{2T}^- + 2h_{2T}^+ + 2c\psi_{2T}^- - 2c\psi_{2T}^+$$

where again $\mu_2c$ is the coupling terms at this order of the expansion and $\Upsilon_2$ is the terms involving $\hat{f}_2$ or equivalent. As was done for the first case, we average (2.86) at constant $\xi_-$ or constant $\xi_+$, or average (2.87) at constant $\nu_-$ or constant $\nu_+$, integrate with respect to $\tau$ and rearrange to obtain

$$\pm 2\psi_{1\xi_\pm} = H_{1,\xi_\pm}(\xi_\pm, \tau, T) + \hat{H}_1(\xi_\pm, T) \tau \quad \text{and} \quad \pm 2c\psi_{2\xi_\pm} = H_{2,\xi_\pm}(\nu_\pm, \tau, T) + \hat{H}_2(\nu_\pm, T) \tau,$$

where the functions $H_{1,2,\xi_\pm}, \hat{H}_{1,2}$ can be found from (2.86) and (2.87). To avoid secular terms we require that $\hat{H}_{1,2} = 0$ and this allows us to find equations for $\phi_{1,2}^\pm$. Therefore we look for terms in (2.86) that depend only on $\xi_\pm$ and $T$. Similarly, we look for terms in (2.87) that only depend on $\nu_\pm$ and $T$. Following this we obtain the equations

$$\left(\mp 2\phi_{1T}^\pm + (f_{1}^\pm \phi_{1T}^\pm)_{\xi_\pm} + d_1 \phi_{1\xi_\pm}^\pm + \phi_{1\xi_\pm\xi_\pm}^\pm\right)_{\xi_\pm} = \delta \phi_{1}^\pm + f_{1TT}^\pm \mp 2f_{1T}^\pm$$

and

$$\left(\mp 2c\phi_{2T}^\pm + (f_{2}^\pm \phi_{2T}^\pm)_{\xi_\pm} + ad_1 \phi_{2\xi_\pm}^\pm + \beta c^2 \phi_{2\xi_\pm\xi_\pm}^\pm\right)_{\xi_\pm} = \gamma \phi_{2}^\pm + f_{2TT}^\pm \mp 2c\beta f_{2T}^\pm$$

where $\Omega = c/(2\varepsilon) = \varepsilon^2/(2\varepsilon)$.
where
\[\tilde{\theta}_1 = \frac{\theta_1}{\sin(\omega \tau)} = \frac{d_2 \delta_1}{2 \omega}, \quad \tilde{\theta}_2 = \frac{\theta_2}{\sin(\omega \tau)} = \frac{\alpha d_2 \gamma}{2 \omega}.\] (2.92)

We have now defined all functions up to and including \(O(\varepsilon)\) and so stop our derivation, however in theory this could be continued to any order.

### 2.3 Comparison Between Cases

The weakly-nonlinear solutions presented in Sections 2.1 and 2.2 are different, despite the governing equations being the same. Namely, we see the following key differences:

- In the case when \(c - 1 = O(\varepsilon)\) we have just one set of characteristic variables, while in the case when \(c - 1 = O(1)\) we have to develop our asymptotic expansions using two distinct sets of characteristic variables.
- In the case when \(c - 1 = O(\varepsilon)\) we derive coupled Ostrovsky equations to describe the evolution of the functions \(f_{1,2}^{\pm}\), while in the case when \(c - 1 = O(1)\) we have uncoupled Ostrovsky equations.
- Similarly, for the functions \(\phi_{1,2}^{\pm}\), we derive linearised coupled Ostrovsky equations for \(c - 1 = O(\varepsilon)\) and linearised uncoupled Ostrovsky equations for the case of \(c - 1 = O(1)\).
- In the terms at \(O(\varepsilon)\), there is a coupling term in both cases. For the case when \(c - 1 = O(\varepsilon)\) we have a linear term in \(f_2^\pm\) for \(u\) and vice-versa for \(w\). However, for the case when \(c - 1 = O(1)\), we have a double integral term instead.
- There is an additional term in the right-hand side of the equation for \(\phi_{1,2}^{\pm}\) in the case when \(c - 1 = O(\varepsilon)\), as the characteristic variables are the same so it survives the averaging.

Importantly, in both cases the constructed solutions are free from the zero-mass contradiction since the Ostrovsky equations have been derived for the zero-mean functions by construction.

### 3 Validity of Weakly-Nonlinear Solution

In Section 2 we constructed the weakly-nonlinear solution for the case of close and distinct characteristic speeds. We now confirm the validity of the constructed expansions by numerically solving the system (1.1) - (1.2) and comparing this direct numerical solution to the constructed solution (2.17) and (2.18) with an increasing number of terms included. The first case, when \(c - 1 = O(\varepsilon)\), was analysed in [25]. Here we determine the validity of the expansion in the second case, when \(c - 1 = O(1)\). This was constructed in Section 2.2, so we need to solve (2.39), (2.41) for the leading-order solution and (2.90), (2.91) for the solution up to and including terms at \(O(\varepsilon)\).

In order to numerically solve these equations, in this section and the subsequent section, we implement three pseudospectral numerical schemes. For the coupled Boussinesq equations we use the technique outlined in [25, 26], while for the case of a single Ostrovsky equation we use the method in [21]. In [25] a pseudospectral method was outlined for coupled Ostrovsky equations, however we present a modified scheme here, based upon [27], to allow for larger time steps to be taken. This is presented in Appendix A. Unless stated otherwise, we assume that \(\Delta x = 0.1, \Delta t = 0.01\) and
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$\Delta T = \varepsilon \Delta t$. In some calculations these parameters may be changed to obtain a higher accuracy result and this will be stated in the figure captions. The domain is taken as $[-L, L]$ in all cases, with $L = 300$ for most calculations with solitary waves and $3 \times L_K$ for cnoidal waves, where $L_K$ is the period of the cnoidal wave.

In all subsequent calculations, we shall refer to the solution of the system (1.1) - (1.2) as the “exact solution” and compare this to the weakly-nonlinear solution with an increasing number of terms included. We calculate the solution in the domain $x \in [-40, 40]$ and for $t \in [0, T]$ where $T = 1/\varepsilon$. The initial conditions are taken to be

$$F_1(x) = A_1 \text{sech}^2\left(\frac{x}{\Lambda_1}\right) + p, \quad F_2(x) = A_2 \text{sech}^2\left(\frac{x}{\Lambda_2}\right),$$

$$V_1(x) = 2 \frac{A_1}{\Lambda_1} \text{sech}^2\left(\frac{x}{\Lambda_1}\right) \tanh\left(\frac{x}{\Lambda_1}\right), \quad V_2(x) = 2c \frac{A_2}{\Lambda_2} \text{sech}^2\left(\frac{x}{\Lambda_2}\right) \tanh\left(\frac{x}{\Lambda_2}\right),$$

(3.1)

where $p$ is a constant and we have $A_1 = 6k_1^2$, $\Lambda_1 = \sqrt{2}/k_1$, $k_1 = 1/\sqrt{6}$, $A_2 = 6ck_2^2/\alpha$, $\Lambda_2 = \sqrt{2c}/k_2$, $k_2 = \sqrt{\alpha}/6c$. We have only added a pedestal to the initial condition for $u$ in view of the translation symmetry of the system. This also gives distinct non-zero values for $d_1$ and $d_2$.

We choose $\alpha = \beta = c = 2$, $\delta = \gamma = 0.5$ and $p = 7$ and present the comparison between the exact and weakly-nonlinear solutions at various orders of $\varepsilon$ in Figure 1. We see from the enhanced image that the leading order solution (red, dashed line) is improved with the addition of the $O(\sqrt{\varepsilon})$ terms (black, dash-dotted line), and further improved by the inclusion of $O(\varepsilon)$ terms (green, dash-dotted line). We only present one example here for brevity, but the same results have been observed for multiple values of $\delta$, $\gamma$ and pedestal height $p$.

Figure 1. A comparison of the direct numerical simulation (solid, blue) and the weakly-nonlinear solution including leading order (dashed, red), $O(\sqrt{\varepsilon})$ (dash-dot, black) and $O(\varepsilon)$ (dash-dot, green) corrections, at $t = 1/\varepsilon$, for (a) $u$ and (b) $w$. Parameters are $L = 40$, $N = 800$, $k = 1/\sqrt{6}$, $\alpha = \beta = c = 2$, $\gamma = 0.5$, $\varepsilon = 0.00277$, $\Delta t = 0.01$ and $\Delta T = \varepsilon \Delta t$. The solution agrees well to leading order and this agreement is improved with the addition of higher-order corrections.
To further confirm the validity of the solution, following the approach taken in [21], we compare the error in the solution at various values of $\varepsilon$ to show that the errors behave as expected. We denote the direct numerical solution to the system (1.1) - (1.2) as $u_{\text{num}}$, the weakly-nonlinear solution (2.35), (2.36) with only leading order terms included as $u_1$, with terms up to and including $O(\sqrt{\varepsilon})$ as $u_2$ and with terms up to and including $O(\varepsilon)$ as $u_3$. The maximum absolute error over $x$ is defined as

$$
e_i = \max_{-L \leq x \leq L} |u_{\text{num}}(x,t) - u_i(x,t)|, \quad i = 1, 2, 3.$$  \hspace{1cm} (3.2)

This error is calculated at every time step and, to smooth the oscillations in the errors we average the $e_i$ values in the final third of the calculation, denoting this value as $\hat{e}_i$. We then use a least-squares power fit to determine how the maximum absolute error varies with the small parameter $\varepsilon$. Therefore we write the errors in the form

$$\exp[\hat{e}_i] = C_i\varepsilon^{\alpha_i},$$ \hspace{1cm} (3.3)

and take the logarithm of both sides to form the error plot (the exponential factor is included so that we have $\hat{e}_i$ as the plotting variable).

The corresponding errors for the cases considered in Figure 1 are plotted in Figure 2. The slope of the curves is approximately 0.77, 0.99 and 1.96 for $u$ and 0.77, 1.00 and 1.95 for $w$, which are slightly higher than the theoretical values for the inclusion of leading order and the case with terms up to and including $O(\varepsilon)$ terms. However this can be explained by the solutions in this case being wave packets, so a phase shift at the level of $O(\sqrt{\varepsilon})$ will not have as much as an effect on the errors as it would in the first case, when the solution was close to a solitary wave [25]. Furthermore, the increase in the slope is consistent with the results seen for previous studies for the Boussinesq-Klein-Gordon equation [21].

Figure 2. A comparison of error curves for varying values of $\varepsilon$, at $t = 1/\varepsilon$, for the weakly-nonlinear solution including leading order (upper, blue), $O(\sqrt{\varepsilon})$ (middle, red) and $O(\varepsilon)$ (lower, black) corrections, for (a) $u$ and (b) $w$. Parameters are $L = 40$, $N = 800$, $k = 1/\sqrt{6}$, $\alpha = \beta = c = 2$, $\gamma = 0.5$, $\Delta t = 0.01$ and $\Delta T = \varepsilon\Delta t$. The inclusion of more terms in the expansion increases the accuracy.
4 Co- and Counter-Propagating Waves

We now consider various cases for wave propagation, comparing the numerical solution to the cRB equations to the constructed weakly-nonlinear solution with the inclusion of terms up to \( O(\sqrt{\varepsilon}) \) to take account of the mass. Two types of initial condition are considered: solitary wave and cnoidal wave. In each case, the first example will be for waves propagating in the same direction (co-propagating waves) and the second example will show waves propagating in opposite directions and then interacting with each other (counter-propagating waves). These results are presented for the case when the characteristic speeds are close \((c - 1 = O(\varepsilon))\) and when the characteristic speeds are significantly different \((c - 1 = O(1))\).

4.1 Solitary Wave Initial Condition

Firstly we take solitary wave initial conditions, as was done in Section 3. These are presented in (3.1) but are reproduced here, where the pedestal term \( p \) is removed so that both waves are on a zero pedestal. Namely, for the co-propagating case, we have

\[
F_1(x) = A_1 \text{sech}^2 \left( \frac{x + x_0}{\Lambda_1} \right), \quad F_2(x) = A_2 \text{sech}^2 \left( \frac{x + x_0}{\Lambda_2} \right),
\]

\[
V_1(x) = 2 \frac{A_1}{\Lambda_1} \text{sech}^2 \left( \frac{x + x_0}{\Lambda_1} \right) \tanh \left( \frac{x + x_0}{\Lambda_1} \right), \quad V_2(x) = 2c \frac{A_2}{\Lambda_2} \text{sech} \left( \frac{x + x_0}{\Lambda_2} \right) \tanh \left( \frac{x + x_0}{\Lambda_2} \right),
\]

where \( A_1 = 6k_1^2, \quad \Lambda_1 = \sqrt{2}/k_1, \quad A_2 = 6ck_2^2/\alpha, \quad \Lambda_2 = \sqrt{2c^2}/k_2 \). We will define \( k_1 \) and \( k_2 \) for each simulation as they vary between cases. For the counter-propagating waves, we introduce a second wave, with the same parameters as the first wave, at a phase shift with a sign change in \( V \) to result in wave propagation to the left for the second wave. Explicitly we have

\[
F_1(x) = A_1 \text{sech}^2 \left( \frac{x + x_0}{\Lambda_1} \right) + A_1 \text{sech}^2 \left( \frac{x + x_1}{\Lambda_1} \right), \quad F_2(x) = A_2 \text{sech}^2 \left( \frac{x + x_0}{\Lambda_2} \right) + A_2 \text{sech}^2 \left( \frac{x + x_1}{\Lambda_2} \right),
\]

\[
V_1(x) = 2 \frac{A_1}{\Lambda_1} \text{sech}^2 \left( \frac{x + x_0}{\Lambda_1} \right) \tanh \left( \frac{x + x_0}{\Lambda_1} \right) - 2 \frac{A_1}{\Lambda_1} \text{sech}^2 \left( \frac{x + x_1}{\Lambda_1} \right) \tanh \left( \frac{x + x_1}{\Lambda_1} \right),
\]

\[
V_2(x) = 2c \frac{A_2}{\Lambda_2} \text{sech} \left( \frac{x + x_0}{\Lambda_2} \right) \tanh \left( \frac{x + x_0}{\Lambda_2} \right) - 2c \frac{A_2}{\Lambda_2} \text{sech} \left( \frac{x + x_1}{\Lambda_2} \right) \tanh \left( \frac{x + x_1}{\Lambda_2} \right).
\]

4.1.1 Close Characteristic Speeds We consider the case when the characteristic speeds in the equations are close, so we have \( c - 1 = O(\varepsilon) \). The case of co-propagating waves was explored in [25], so we exclude these from our considerations here. Instead we consider the case of a counter-propagating wave. We take the initial condition (4.2), where the waves will be well separated via choice of \( x_0 \) and \( x_1 \). The results are shown in Figure 3, where the evolution of the wave is shown for \( u \) and \( w \) in a single plot. Firstly we see that the solitons form a co-propagating radiating tail, as expected from the co-propagating case [25]. As they interact the solitons appear to emerge with only a small change in their amplitudes, and there is good agreement between the direct and
weakly-nonlinear solutions. A further test was run by comparing the result of the interaction to a corresponding co-propagating case without interaction. These comparisons are shown in Figure 4 for the left- and right-propagating radiating solitary waves independently. We can see that there is excellent agreement between the cases, with only a very minor phase shift, suggesting that interaction of radiating solitary waves behaves essentially in the same way as solitary wave interaction.

4.1.2 Distinct Characteristic Speeds
Now we consider the case when the characteristic speeds in the equations are distinct, so we have \( c - 1 = O(1) \). For the case of co-propagating waves we take the initial condition (4.1) and the results are presented in Figure 5. An Ostrovsky wave packet is formed in both layers, with the packet generated in the lower layer moving faster than the packet in the upper layer. There is a reasonable agreement between the solutions, although the solution is less accurate away from the main wave packet due to the presence of radiation, which can re-enter the domain. Furthermore, the solution for \( w \) is more accurate than the solution for \( u \) as this packet moves faster and therefore will have some coupling in the upper layer for \( u \), creating the disagreement. This is likely to be captured by the higher-order corrections, although the complexity of the calculation is increased by their inclusion. Alternatively, the errors can be reduced by decreasing the value of \( \varepsilon \).

The same case for \( \varepsilon = 0.0001 \) is shown in Figure 6. We can see that the overall errors are reduced, in comparison to the previous case, although there is less evolution of the wave packet as expected. For all subsequent results we will use the larger value of \( \varepsilon = 0.0005 \) so we can see the evolution of the solution for smaller times.

We now consider the effect of wave interaction on Ostrovsky wave packets. The initial condition is taken as (4.2) and we analyse the interaction of the generated wave packets using both the cRB equations and the constructed weakly-nonlinear solution. The results are presented in Figure 7. Following the results in Figure 5, Ostrovsky wave packets are generated in both layers with the
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Figure 4. Comparison of the cRB equations in the counter-propagating case (blue, solid line) and the corresponding co-propagating case (red, dashed line), at $t = 500$ for (a, b) $u$, and (c, d) $w$. Parameters are $L = 300$, $N = 6000$, $\varepsilon = 0.05$, $\alpha = \beta = 1.05$, $c = 1.025$, $\delta = \gamma = 1$, $k_1 = 1/\sqrt{2}$, $k_2 = \sqrt{1.05/2}$, $x_0 = 250$, $x_1 = -250$, $\Delta t = 0.01$ and $\Delta T = \varepsilon \Delta t$.

packet generated in the lower layer moving faster than the corresponding packet in the upper layer. At the point of interaction the packets move through each other and emerge with the appearance of only small changes in their shape and structure, which could be attributed to evolution of the wave packet. This was confirmed by further simulations, which are excluded for brevity, showing that interaction only causes a small phase shift in the main wave packet and some changes in the tail region due to fast-moving radiation.

4.2 Cnoidal Wave Initial Condition

A second case of interest is using a cnoidal wave as the initial condition for the cRB equations. We can derive the cnoidal wave initial condition by considering the reduced case when $u = w$ so that, in
terms of \( w \), we have

\[ w_{tt} - c^2 w_{xx} = \varepsilon \left[ \frac{\alpha}{2} (w^2)_{xx} + \beta w_{tt} \right]. \quad (4.3) \]

The leading-order weakly-nonlinear solution to this equation takes the form of the KdV equation

\[ 2c f_T + \alpha f f_\xi + \beta c^2 f_{\xi\xi\xi} = 0, \quad (4.4) \]

where we have assumed that \( \xi = x - ct \). This can be thought of as a reduction of the Ostrovsky cases derived in Section 2.2, under the assumption that the initial condition has zero mean. The cnoidal wave solution to this equation can be derived as

\[ f = -\frac{6\beta c^2}{\alpha} \left( f_2 - (f_2 - f_3) \text{cn}^2 \left( (\xi + \nu T) \sqrt{\frac{f_1 - f_3}{2} |m|} \right) \right), \quad (4.5) \]
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Figure 6. Comparison of co-propagating waves in the cRB equations (blue, solid line) and the constructed weakly-nonlinear solution (red, dashed line), at $t = 250$, for (a) $u$, and (b) $w$. Parameters are $L = 300$, $N = 6000$, $\varepsilon = 0.0001$, $\alpha = \beta = c = 2$, $\delta = \gamma = 1$, $k_1 = 1/\sqrt{2}$, $k_2 = 1$, $x_0 = 250$, $\Delta t = 0.01$ and $\Delta T = \varepsilon \Delta t$.

Figure 7. Comparison of counter-propagating waves in the cRB equations (blue, solid line) and the constructed weakly-nonlinear solution (red, dashed line), for (a) $u$ and (b) $w$. Parameters are $L = 300$, $N = 6000$, $\varepsilon = 0.0005$, $\alpha = \beta = c = 2$, $\delta = \gamma = 1$, $k_1 = 1/\sqrt{2}$, $k_2 = 1$, $x_0 = 250$, $x_1 = -250$, $\Delta t = 0.01$ and $\Delta T = \varepsilon \Delta t$.

where

$$\nu = (f_1 + f_2 + f_3)\beta c, \quad m = \frac{f_2 - f_3}{f_1 - f_3}.$$
The solution is parametrised by the three constants $f_3 < f_2 < f_1$ and the restriction that $0 < m < 1$. The wave length of the cnoidal wave is given by

$$L_K = 2K(m)\sqrt{\frac{2}{f_1 - f_3}},$$

(4.6)

where $K(m)$ is the complete elliptic integral of the first kind. This wave is an exact solution to the KdV equation and therefore will be an approximate solution to the cRB equations, to leading order, in the same way as the solitary wave solution in Section 4.1 was an approximate solution.

To reduce the number of parameters in the solution, we will assume that $f_2 = 0$, which means that the cnoidal wave is moving on a zero background. This will reduce the magnitude of the zero mean term, but it will not be zero, which is consistent with the approach for the solitary wave solution.

To simplify notation we introduce the variable $\theta = \sqrt{f_1 - f_3}$. Explicitly for the co-propagating wave we have the initial condition

$$F_1(x) = -6f_3 \, \text{cn}^2 \left[(x + x_0)\theta|m\right], \quad F_2(x) = -\frac{6\alpha c^2}{\beta}f_3 \, \text{cn}^2 \left[(x + x_0)\theta|m\right],$$

$$V_1(x) = -12f_3\theta \, \text{cn} \left[(x + x_0)\theta|m\right] \text{sn} \left[(x + x_0)\theta|m\right] \text{dn} \left[(x + x_0)\theta|m\right],$$

$$V_2(x) = -\frac{12\alpha c^2}{\beta}f_3\theta \, \text{cn} \left[(x + x_0)\theta|m\right] \text{sn} \left[(x + x_0)\theta|m\right] \text{dn} \left[(x + x_0)\theta|m\right].$$

(4.7)

### 4.2.1 Close Characteristic Speeds

As was done for the solitary wave initial condition, let us consider the case when the characteristic speeds in the equations are close, so we have $c - 1 = O(\varepsilon)$. We take the initial condition for co-propagating waves, given in (4.7), and the results are presented in Figure 8. Once again we see a good agreement between the results. As the troughs of the cnoidal wave are long, we can see the generation of a co-propagating radiating tail forming behind the cnoidal wave peaks, which then begins to interact with the preceding peak. The peaks of the wave appear to survive this interaction and maintain their shape. There is a reduction in amplitude but this can be attributed to the generation of the radiating tail.

The interacting case was also explored, with no visible change to the radiating solitary waves, as was seen for the solitary wave initial condition in the previous section. These are excluded as the images appear qualitatively the same as the case of solitary wave initial conditions.

### 4.2.2 Distinct Characteristic Speeds

Following on from Section 4.1, we now examine the case when the characteristic speeds in the equations are distinct, so we have $c - 1 = O(1)$. We take the initial condition (4.7) and the results are presented in Figure 9. As expected, an Ostrovsky wave packet is formed by each peak of the cnoidal wave, with the packets formed in the lower layer moving faster than the upper layer. The choice of initial condition here was such that the peaks of the cnoidal wave are distinct, so that each wave packet can be seen clearly. We can see that the agreement is good around the wave packet but becomes worse away from the main wave packet due to the radiation generated. This is consistent with the observation for the case of distinct characteristic speeds with solitary wave initial conditions, however we note that the disagreement is much smaller in this case.
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Figure 8. Comparison of co-propagating waves in the cRB equations (blue, solid line) and the constructed weakly-nonlinear solution (red, dashed line), at \( t = 0 \), for (a) \( u \) and (b) \( w \), and at \( t = 300 \) for (c) \( u \) and (d) \( w \). Parameters are \( L \approx 148.6 \), \( N = 2974 \), \( \varepsilon = 0.05 \), \( f_1 = 1 \times 10^{-12} \), \( f_2 = 0 \), \( f_3 = -\frac{1}{6} \), \( \alpha = \beta = 1 \), \( c = 1.025 \), \( \delta = \gamma = 1 \), \( x_0 = 0 \), \( \Delta t = 0.01 \) and \( \Delta T = \varepsilon \Delta t \).

In the case of counter-propagating waves, due to the close proximity of the peaks for the cnoidal wave initial condition, the wave packets interact with each other and so we cannot observe the wave packet evolution clearly. Therefore we exclude it from our considerations here.

To see the evolution of the cnoidal wave initial condition into a series of Ostrovsky wave packets, we plot the evolution of the solution to the cRB equations over time in Figure 10, at time intervals of \( t = 100 \) up to \( t = 1000 \). We have increased the value of \( \delta \) and \( \gamma \) here to increase the speed of the packet evolution. We can see that the peaks of the cnoidal wave gradually evolve into an Ostrovsky wave packet and that the evolution occurs faster for \( u \), as expected from the equations derived in
Section 2.2 due to the divisor of 2c for w. Note that the domain is periodic, so although the wave appears to have a low speed, the time between images means that the previous peak is located close to the current peak’s location.
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Figure 10. Solution to the cRB equations at intervals of \( t = 50 \), up to \( t = 1000 \), for (a) \( u \), and (b) \( w \). Parameters are \( L \approx 148.6 \), \( N = 2974 \), \( \varepsilon = 0.0005 \), \( f_1 = 1 \times 10^{-12} \), \( f_2 = 0 \), \( f_3 = -\frac{1}{6} \), \( \alpha = \beta = 2 \), \( c = 2 \), \( \delta = \gamma = 2 \), \( x_0 = 0 \) and \( \Delta t = 0.01 \).

5 Conservation laws

The system (1.1) - (1.2) is related to the system
\[
U_{tt} - U_{xx} = \varepsilon \left[ U_x U_{xx} + U_{ttxx} - \delta (U - W) \right], \quad (5.1)
\]
\[
W_{tt} - \varepsilon^2 W_{xx} = \varepsilon \left[ \alpha W_x W_{xx} + \beta W_{ttxx} + \gamma (U - W) \right], \quad (5.2)
\]
by differentiation \( u = U_x \) and \( w = W_x \). The system (5.1) - (5.2) is Lagrangian, and it has three local conservation laws for the mass, energy and momentum [10]:
\[
\left( U_t + \frac{\delta}{\gamma} W_t \right)_t - \left( U_x + \frac{\delta c^2}{\gamma} W_x + \frac{\varepsilon}{2} U_x^2 + \frac{\varepsilon \alpha \delta}{2 \gamma} W_x^2 + U_{tx} + \frac{\varepsilon \beta \delta}{\gamma} W_{tx} \right)_x = 0, \quad (5.3)
\]
\[
\frac{1}{2} \left\{ U_t^2 + \frac{\delta}{\gamma} W_t^2 + U_x^2 + \frac{\delta c^2}{\gamma} W_x^2 + \varepsilon U_x^3 + \frac{\varepsilon \alpha \delta}{\gamma} W_x^3 \right\}_t + \left( U_t U_x + \frac{\delta c^2}{\gamma} W_t W_x + \varepsilon U_t U_x^2 + \frac{\varepsilon \alpha \delta}{\gamma} W_t W_x^2 + \varepsilon U_t U_{tx} + \frac{\varepsilon \beta \delta}{\gamma} W_{tx} W_t \right)_x = 0, \quad (5.4)
\]
\[
\left( U_t U_x + \frac{\delta}{\gamma} W_t W_x + \varepsilon U_t U_{xx} + \frac{\varepsilon \beta \delta}{\gamma} W_{tx} W_{xx} \right)_t - \left\{ \varepsilon U_t U_{tx} + \frac{\varepsilon \beta \delta}{\gamma} W_x W_{tx} \right\}_x + \frac{1}{2} \left[ U_t^2 + \frac{\delta}{\gamma} W_t^2 + U_x^2 + \frac{\delta c^2}{\gamma} W_x^2 + \frac{2 \varepsilon}{3} U_x^3 + \frac{\varepsilon \alpha \delta}{\gamma} W_x^3 \right]_t + \left( U_{tx}^2 + \frac{\varepsilon \beta \delta}{\gamma} W_{tx}^2 - \varepsilon \delta (U - W)^2 \right)_x = 0. \quad (5.5)
\]
Using these known conservation laws, we can find the conservation laws for our system (1.1) - (1.2) in the form of one local (mass) and two non-local (energy and momentum) conservation laws. Indeed, differentiating (5.3) with respect to $x$ and rewriting (5.4) and (5.5) in terms of $u$, $w$ instead of $U_x$, $W_x$ we obtain

$$
\left( u_t + \frac{\delta}{\gamma} w_t \right)_t - \left( u_x + \frac{\delta c^2}{\gamma} w_x + \varepsilon \alpha \frac{\partial}{\partial x} \left( w_x + u_{tx} + \frac{\varepsilon \beta}{\gamma} w_{tx} \right) \right)_x = 0, \quad (5.6)
$$

$$
\frac{1}{2} \left\{ U_t^2 + \frac{\delta}{\gamma} W_t^2 + u^2 + \frac{\delta c^2}{\gamma} w^2 + \frac{\varepsilon}{3} \left( u^3 + \alpha \frac{\partial}{\partial t} w^3 \right) + \varepsilon u_t^2 + \frac{\varepsilon \beta}{\gamma} \frac{\partial}{\partial t} w_t^2 + \varepsilon \delta (U - W)^2 \right\}_t 
- \left( U_t u + \frac{\delta c}{\gamma} W_t w + \frac{\varepsilon}{2} U_t u^2 + \frac{\varepsilon \alpha}{\gamma} W_t w^2 + \varepsilon U_t u_{tt} + \frac{\varepsilon \beta}{\gamma} W_t w_t \right) = 0, \quad (5.7)
$$

$$
+ \frac{1}{2} \left[ U_t^2 + \frac{\delta}{\gamma} W_t^2 + u^2 + \frac{\delta c^2}{\gamma} w^2 + \frac{2 \varepsilon}{3} \left( u^3 + \alpha \frac{\partial}{\partial t} w^3 \right) + \varepsilon u_t^2 + \frac{\varepsilon \beta}{\gamma} \frac{\partial}{\partial t} w_t^2 - \varepsilon \delta (U - W)^2 \right] \right\}_x = 0. \quad (5.8)
$$

It is natural to assume that $U(-L) = W(-L) = 0$, and we make this assumption in what follows. Integrating these conservation laws with respect to $x$ from $-L$ to $L$, using the periodicity of $u$ and $w$ on $[-L, L]$, we obtain one conserved quantity (mass)

$$
\frac{d}{dt} \left( \int_{-L}^{L} \frac{\delta}{\gamma} w \right) = \text{const}, \quad (5.9)
$$

and two non-local conservation laws (energy and momentum, respectively)

$$
\frac{1}{2} \frac{d}{dt} \int_{-L}^{L} \left\{ U_t^2 + \frac{\delta}{\gamma} W_t^2 + u^2 + \frac{\delta c^2}{\gamma} w^2 + \frac{\varepsilon}{3} \left( u^3 + \alpha \frac{\partial}{\partial t} w^3 \right) + \varepsilon u_t^2 + \frac{\varepsilon \beta}{\gamma} \frac{\partial}{\partial t} w_t^2 + \varepsilon \delta (U - W)^2 \right\} dx 
- \int_{-L}^{L} \left[ U_t(L) \left( u(L) + \frac{\varepsilon}{2} u^2(L) + \varepsilon u_t(L) \right) + W_t(L) \left( \frac{\delta c^2}{\gamma} w(L) + \frac{\varepsilon \alpha}{\gamma} w^2(L) + \frac{\varepsilon \beta}{\gamma} w_t(L) \right) \right] \right\}, \quad (5.10)
$$

$$
\frac{d}{dt} \int_{-L}^{L} \left( U_t u + \frac{\delta}{\gamma} W_t w + \varepsilon u_t u_x + \frac{\varepsilon \beta}{\gamma} w_t w_x \right) dx = \frac{1}{2} \left[ U_t^2(L) + \frac{\delta}{\gamma} W_t^2(L) - \varepsilon \delta (U(L) - W(L))^2 \right], \quad (5.11)
$$

where

$$
U_t = \dot{U}_t + \frac{d\tilde{u}}{dt}, \quad W_t = \dot{W}_t + \frac{d\tilde{w}}{dt}, \quad (5.12)
$$

are the known periodic functions of time and $\dot{U}_t$, $\dot{W}_t$ are zero-mean functions. As we impose that $U(-L) = W(-L) = 0$ it follows that $U_t(-L) = W_t(-L) = 0$. Therefore, we adjust the value of $U_t$ and $W_t$ to satisfy the zero boundary condition at $x = -L$.

Note that if $\tilde{u} = \tilde{w} = 0$ (i.e. $d_1 = d_2 = 0$), then the nonlocal conservation laws (5.10) and (5.11) yield the usual (local) conservation of energy and momentum. More generally, the nonlocal
conservation laws are applicable when the waves propagate on the background of some initially pre-strained basic state characterised by non-zero mass of \( u \) and \( w \).

We now calculate the energy to confirm the derived relations (5.6) - (5.8). The parameters for the simulation are chosen as \( \varepsilon = 0.01 \), \( c = 1.025 \) and \( \alpha = \beta = \delta = \gamma = 1 \), with solitary wave initial conditions as taken in (3.1) with \( p = 1 \) to provide a significant non-zero mean value. The energy and momentum relations are plotted in Figure 11, where we note that the left-hand side of the mass equation (5.9) is equal to zero in this case and therefore is omitted from the plot.

![Figure 11](image_url)

(a) Energy for cRB equations.  

(b) Momentum for cRB equations.

**Figure 11.** Conserved quantities in the cRB equations, where the blue, solid line is the left-hand side of the relation and the red, dashed line is the right-hand side of the relation, for (a) energy and (b) momentum. Parameters are \( L = 300 \), \( N = 60000 \), \( c = 1.025 \), \( \varepsilon = 0.01 \), \( \alpha = \beta = \delta = \gamma = 1 \), \( k_1 = k_2 = 1/\sqrt{2} \), \( \Delta t = 0.001 \) and \( \Delta T = \varepsilon \Delta t \).

We see that the energy and momentum oscillate with period \( f = 2\pi/\omega \), although the energy relation alternates between two different peaks. To verify the conservation laws, the peaks and troughs were tracked for both the left-hand side and right-hand side of the conservation laws and the absolute percentage error between peaks or troughs was calculated as \( 4.24 \times 10^{-6}\% \) for the energy and \( 3.60 \times 10^{-10}\% \) for the momentum. It is of note that the left-hand side of these laws require a time derivative on discrete data and so the accuracy to which they are conserved is hampered by this limitation. In our case the time step was taken as \( \Delta t = 0.001 \).

6 Conclusions

In this paper we developed an asymptotic procedure for the construction of the d’Alembert-type solution of the Cauchy problem for a system of coupled Boussinesq equations, which could describe long longitudinal strain waves in a bi-layer with an imperfect interface [10]. Two cases are presented, when the characteristic speeds in the equations are close or significantly different. An asymptotic expansion is constructed to the case of significantly different characteristic speeds and the asymptotic expansions were compared.

We examined the accuracy of the constructed solution numerically, using direct numerical simulations for the coupled Boussinesq equations and our constructed semi-analytical solution, and
showed that the constructed solution is a good approximation to the direct numerical simulations, with improving accuracy for the inclusion of additional terms.

We then studied the case of co- and counter-propagating waves within the context of close or distinct characteristic speeds, for solitary wave initial condition. In each case, we confirmed that the weakly-nonlinear solution agrees with the direct solution. Furthermore, we showed that radiating solitary waves can interact with each other in the counter-propagating case, with very little effect on their structure. In the case of distinct characteristic speeds, the generated Ostrovsky wave packets interact and emerge with the main wave packet relatively unchanged and only a small disagreement in the generated fast-moving radiation.

Next we studied the behaviour of a cnoidal wave initial condition, in the same cases as considered for the solitary wave initial condition. For close characteristic speeds we see that radiating solitary waves are generated from each cnoidal wave peak. The tails interact with the preceding peak, however the main wave structure is preserved. For distinct characteristic speeds, Ostrovsky wave packets are generated by each peak as expected and we can clearly see the evolution of each peak into an Ostrovsky wave packet, joined to its neighbours.

Finally, we showed that modified conservation laws can be derived to take account of an initial condition that is not necessarily zero-mean. These conservation laws consist of one local (mass) and two nonlocal (energy and momentum) relations, with the conversed quantity oscillating with a frequency determined by the non-zero mass of the initial data, confirmed by numerical simulations. These nonlocal conservation laws are applicable when the waves propagate on the background of some initially pre-strained basic state characterised by non-zero mass of $u$ and $w$.

The constructed solution can find useful applications in the studies of the scattering of radiating solitary waves by delamination [14], where the existing solutions do not take account of the zero-mean constraint. The propagation of cnoidal waves in these structures is also of interest.

Acknowledgements

MRT is grateful to the UK QJMAM Fund for Applied Mathematics for the support of his travel to the ICoNSoM 2019 conference in Rome, Italy, where discussions of this work have taken place.

Appendix A Numerical Methods

To solve the equations derived in Section 2 and its subsections, we will make use of the pseudo-spectral methods for coupled Boussinesq equations in [25, 26] and single Ostrovsky equation in [21]. For the coupled Ostrovsky equations we use the method outlined below, which also includes a brief overview of the pseudo-spectral method.

In the following method we use the Discrete Fourier Transform (DFT) to calculate the Fourier transform of numerical data [27]. Let us consider a function $u(x, t)$ on a finite domain $x \in [-L, L]$ and we discretise the domain into $N$ equally spaced points, so we have the spacing $\Delta x = 2L/N$. We scale the domain from $x \in [-L, L]$ to $\tilde{x} \in [0, 2\pi]$ via the transform $\tilde{x} = sx + \pi$, where $s = \pi/L$. 
Denoting \( x_j = -L + j \Delta x \) for \( j = 0, \ldots, N \), we define the DFT for the function \( u(x,t) \) as

\[
\hat{u}(k,t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} u(x_j,t) e^{-ikx_j}, \quad -\frac{N}{2} \leq k \leq \frac{N}{2} - 1, \tag{A.1}
\]

and similarly the IDFT is defined as

\[
u(x,t) = \frac{1}{\sqrt{N}} \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t) e^{ikx_j}, \quad j = 1, 2, \ldots, N, \tag{A.2}\]

where we have discretised and scaled wavenumber \( k \in \mathbb{Z} \). The transforms are implemented using the FFTW3 algorithm in C [28].

We now consider the solution to the coupled Ostrovsky equations, where we can extend the Runge-Kutta method that was used for a single Ostrovsky equation in [21]. We present an example for the equation governing \( \phi \), which can be reduced to the equation for \( f \). Explicitly we consider the equations (2.29), (2.30), as this method can be reduced to solve the system (2.22). We change the variables to \( x \) and \( t \) for simplicity, and introducing coefficients \( \mu, \omega \) for the terms in \( \phi_{1x} \) and \( \phi_{2x} \) respectively, we obtain

\[
(2\phi_{1t} + \mu \phi_{1x} + (f_1 \phi_1)_x + \phi_{1xxx})_x = \delta (\phi_1 - \phi_2) + H_1(f_1(x), f_2(x)), \tag{A.3}
\]

\[
(2\phi_{2t} + \omega \phi_{2x} + \alpha (f_2 \phi_2)_x + \beta \phi_{2xxx})_x = \gamma (\phi_2 - \phi_1) + H_2(f_1(x), f_2(x)), \tag{A.4}
\]

where \( \alpha, \beta, \mu, \omega, \delta \) and \( \gamma \) are constants, and the functions \( f_1, f_2 \) are known. We consider the equation on the domains \( t \in [0,T] \) and \( x \in [-L, L] \). Here \( H_{1,2} \) are functions of \( f_{1,2} \) and are therefore known at the given time step. Their form can be found in Section 2.1. Taking the Fourier transform of (A.3) gives (using the transform to \( \tilde{x} \))

\[
2\hat{\phi}_{1t} + (isk\mu - is^3k^3) \hat{\phi}_1 + iskF \{ f_1 \phi_1 \} = -i\frac{\delta}{sk} (\hat{\phi}_1 - \hat{\phi}_2) - i\frac{\delta}{sk} \hat{H}_1, \tag{A.5}
\]

\[
2\hat{\phi}_{2t} + (isk\omega - is^3k^3\beta) \hat{\phi}_2 + iskF \{ f_2 \phi_2 \} = -i\frac{\gamma}{sk} (\hat{\phi}_2 - \hat{\phi}_1) - i\frac{\gamma}{sk} \hat{H}_2. \]

Following the method of [27, 21] to remove the stiff term from this equation, we multiply through by a multiplicative factor \( M_{1,2} \) and introduce a new function \( \Phi_{1,2} \), where \( M_{1,2} \) and \( \Phi_{1,2} \) take the form

\[
M_1 = e^{-\frac{i}{2}(s^3k^3 - \mu sk - \frac{i}{sk})}, \quad \Phi_1 = e^{-\frac{i}{2}(s^3k^3 - \mu sk - \frac{i}{sk})t} \hat{\phi}_1 = M_1 \hat{\phi}_1, \tag{A.5}
\]

\[
M_2 = e^{-\frac{i}{2}(\beta s^3k^3 - \omega sk - \frac{i}{sk})}, \quad \Phi_2 = e^{-\frac{i}{2}(\beta s^3k^3 - \omega sk - \frac{i}{sk})t} \hat{\phi}_2 = M_2 \hat{\phi}_2. \tag{A.5}
\]

Substituting this into (A.4) leads to two ODEs for \( \Phi_{1,2} \) of the form

\[
\dot{\Phi}_{1t} = -\frac{isk}{2} M_1 F \left\{ F^{-1} \begin{bmatrix} \frac{\Phi_1}{M_1} \\ \frac{\Phi_2}{M_1} \end{bmatrix} \right\} - \frac{i}{2sk} M_1 \left( \hat{S} - \delta \frac{\Phi_2}{M_2} \right), \tag{A.6}
\]

\[
\dot{\Phi}_{2t} = -\frac{isk}{2} M_2 F \left\{ F^{-1} \begin{bmatrix} \frac{\Phi_2}{M_2} \\ \frac{\Phi_1}{M_1} \end{bmatrix} \right\} - \frac{i}{2sk} M_2 \left( \hat{S} - \gamma \frac{\Phi_1}{M_1} \right). \tag{A.6}
\]
This form of the equation contains less terms than the standard discretisation and therefore we can use an optimised 4th order Runge-Kutta algorithm. We discretise the time domain as \( t_j = j \Delta t \) and the functions as \( \Phi_{i,j} = \Phi_i(k,t_j) \), \( \phi_{i,j} = \phi_i(k,t_j) \) and \( \hat{f}_{i,j} = \hat{f}_i(k,t_j) \), \( i = 1,2 \). Introducing the additional functions

\[
E_1 = e^{\frac{i}{\Delta t} \left( \beta s k^3 - \omega s k - \pi \right)} \Delta t, \quad E_2 = e^{\frac{i}{\Delta t} \left( \beta s k^3 - \omega s k - \pi \right)} \Delta t, \tag{A.7}
\]

we can introduce the optimised Runge-Kutta algorithm in the original variables \( \phi_{1,2} \), taking the form

\[
\begin{align*}
\hat{\phi}_{1,j+1} &= E_1^2 \hat{\phi}_1 + \frac{1}{6} \left[ E_1^2 k_1 + 2 E_1 (k_2 + k_3) + k_4 \right], \quad \hat{\phi}_{2,j+1} = E_2^2 \hat{\phi}_2 + \frac{1}{6} \left[ E_2^2 l_1 + 2 E_2 (l_2 + l_3) + l_4 \right], \\
k_1 &= -i \frac{s k}{2} \Delta t \left\{ \hat{f}_{1,j} F^{-1} \left[ \hat{\phi}_{1,j} \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_1 - \delta \hat{\phi}_{2,j} \right), \\
l_1 &= -i \frac{\alpha s k}{2} \Delta t \left\{ \hat{f}_{2,j} F^{-1} \left[ \hat{\phi}_{2,j} \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_2 - \gamma \hat{\phi}_{1,j} \right), \\
k_2 &= -i \frac{s k}{2} \Delta t \left\{ \hat{f}_{1,j} F^{-1} \left[ E_1 \left( \hat{\phi}_{1,j} + \frac{k_1}{2} \right) \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_1 - \delta \hat{E}_2 \left[ \phi_{2,j} + \frac{l_1}{2} \right] \right), \\
l_2 &= -i \frac{\alpha s k}{2} \Delta t \left\{ \hat{f}_{2,j} F^{-1} \left[ E_2 \left( \hat{\phi}_{2,j} + \frac{l_1}{2} \right) \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_2 - \gamma \hat{E}_1 \left[ \phi_{1,j} + \frac{k_1}{2} \right] \right), \\
k_3 &= -i \frac{s k}{2} \Delta t \left\{ \hat{f}_{1,j} F^{-1} \left[ E_1 \hat{\phi}_{1,j} + \frac{k_2}{2} \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_1 - \delta \hat{E}_2 \left[ \phi_{2,j} + \frac{l_2}{2} \right] \right), \\
l_3 &= -i \frac{\alpha s k}{2} \Delta t \left\{ \hat{f}_{2,j} F^{-1} \left[ E_2 \hat{\phi}_{2,j} + \frac{l_2}{2} \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_2 - \gamma \hat{E}_1 \left[ \phi_{1,j} + \frac{k_2}{2} \right] \right), \\
k_4 &= -i \frac{s k}{2} \Delta t \left\{ \hat{f}_{1,j} F^{-1} \left[ E_1 \hat{\phi}_{1,j} + E_1 k_3 \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_1 - \delta \hat{E}_2 \left[ \phi_{2,j} + E_2 k_3 \right] \right), \\
l_4 &= -i \frac{\alpha s k}{2} \Delta t \left\{ \hat{f}_{2,j} F^{-1} \left[ E_2 \hat{\phi}_{2,j} + E_2 k_3 \right] \right\} - i \frac{\Delta t}{2 s k} \left( \hat{H}_2 - \gamma \hat{E}_1 \left[ \phi_{1,j} + E_1 k_3 \right] \right). \tag{A.8}
\end{align*}
\]

When calculating the solution using this algorithm, the functions \( k_i, l_i \) must be calculated “in pairs” as the functions \( k_1 \) and \( l_1 \) are required when evaluating \( k_2 \) and \( l_2 \), and so on. We can apply this algorithm to the case of a homogeneous Ostrovsky equation by setting \( \hat{H}_{1,2} = 0 \) and replacing the term \( f_i \phi_i \) with \( f_i^2 / 2 \), and therefore it can be applied to all derived coupled Ostrovsky equations in Section 2.2.

We note that this is an extension of the method for a single Ostrovsky equation, as discussed in [21]. Therefore we can reduce the method for the coupled Ostrovsky equations to a single Ostrovsky equation as required.

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