On the complex of separating meridians in handlebodies

Abstract. For a handlebody of genus $g \geq 6$ it is shown that every automorphism of the complex of separating meridians can be extended to an automorphism on the complex of all meridians and, in consequence, it is geometric.

Key words. Separating meridian · Handlebody · Geometric automorphism · Meridian complex

0. Definitions and statements of results

For a compact surface $S$, the complex of curves $C(S)$, introduced by Harvey [6], has vertices the isotopy classes of essential, non-boundary-parallel simple closed curves in $S$. A collection of vertices spans a simplex exactly when any two of them may be represented by disjoint curves. The complex of curves $C(S)$ as well as several subcomplexes of $C(S)$ have played an important role in the study of the mapping class group of $S$. Ivanov (see for example [10]) was the first to prove that every automorphism of $C(S)$ is geometric, that is, it is induced by an element of the mapping class group of $S$.

A prominent subcomplex of $C(S)$ is the subcomplex $SC(S)$ whose vertices are separating curves. See below, at the end of this section, for a discussion concerning the subcomplex $SC(S)$ as well as motivation for this work. The arc complex of $S$, denoted by $A(S)$, is defined analogously, with curves replaced by arcs. The arc complex has been studied by several authors (see [7,9,10]).

Similarly, for a 3-manifold $M$, the complex $M(M)$ of essential disks in $M$ is defined. It was introduced in [15], where it was used in the study of mapping class groups of 3-manifolds. In [14] it was shown to be a quasi-convex subset of $C(\partial M)$.

By $H_g$ we denote the 3-dimensional handlebody of genus $g$. As $\pi_2(H_g)$ vanishes, the complex of essential disks mentioned above is, for $M = H_g$, the complex of meridians in $H_g$. As usual, the vertices of $M(H_g)$ are isotopy classes of meridians in $H_g$ and a collection of vertices spans a simplex exactly when any two of them
may be represented by disjoint meridians. We regard $\mathcal{M}(H_g)$ as a subcomplex of $\mathcal{C}(\partial H_g)$.

**Definition 1.** Let $\mathcal{SM}(H_g)$ be the simplicial complex with vertex set being the isotopy classes of separating meridians in $H_g$. The $k$-simplices are given by collections of $k + 1$ vertices having disjoint representatives.

Note that the dimension of $\mathcal{SM}(H_g)$ is $2g - 4$. For $g = 2$ the complex $\mathcal{SM}(H_g)$ is just an infinite set of vertices so we assume that $g \geq 3$. It is well-known that $\mathcal{SM}(H_g)$ is connected for $g \geq 3$. Although we will not need this result in the sequel, we note that an easy proof of this result can be obtained by the general technique presented in [16, Lemma 2.1] considering the action of the mapping class group of $H_g$ on $\mathcal{SM}(H_g)$ and using the specific set of generators for the mapping class group of $H_g$ given in [5, Corollary 3.4, page 99].

The aim of this paper is to show

**Theorem 2.** Every automorphism of $\mathcal{SM}(H_g)$ extends uniquely to an automorphism of the complex of all meridians $\mathcal{M}(H_g)$, provided that $g \geq 6$.

It is shown in [12] that the automorphism group of $\mathcal{M}(H_g)$ is isomorphic to the mapping class group of $H_g$. In particular, every automorphism of $\mathcal{M}(H_g)$ is geometric. Thus, we have

**Corollary 3.** Every automorphism of $\mathcal{SM}(H_g)$ is geometric i.e., it is induced by an element of the mapping class group of $H_g$, provided that $g \geq 6$.

Moreover, we have

**Corollary 4.** The mapping class group of $H_g$ is isomorphic to the automorphism group of $\mathcal{SM}(H_g)$, provided that $g \geq 6$.

The proof of Theorem 2 is based on the following basic property (see Theorem 13) of genus 1 separating meridians preserved by automorphisms of $\mathcal{SM}(H_g)$: each genus 1 separating meridian gives rise to a genus 1 handlebody which contains a unique non-separating meridian. If two genus 1 meridians determine the same non-separating meridian, so do their images under an automorphism of $\mathcal{SM}(H_g)$. This property allows for the extension of an automorphism of $\mathcal{SM}(H_g)$ to the whole complex of meridians.

In Sect. 1 we introduce terminology and prove that certain intersection properties between separating meridians are preserved by the automorphisms of $\mathcal{SM}(H_g)$.

In Sect. 2 we restrict our attention inside a genus 2 handlebody with 2 spots on the boundary and we show (see Proposition 10) that an automorphism $\phi$ of $\mathcal{SM}(H_g)$ induces an automorphism on the arc complex of the boundary surface of type $(2, 2)$ which is known to be geometric (see [9] where it is shown that the mapping class group of the boundary surface is isomorphic to the group of automorphisms of the arc complex). Using this, we show the above mentioned basic property for genus 1-meridians which live inside a genus 2 handlebody with 2 spots on the boundary. This is the point where the assumption $g \geq 6$ is required.
In Sect. 3 we show that the basic property holds in the case where the two genus 1-meridians live in the complement of a cut system, that is, a maximal collection of pairwise disjoint non-separating meridians which split the handlebody $H_g$ into a 3-ball. Finally, we extend the property to arbitrary genus 1-meridians by using the fact that the complex of cut systems is connected (see [17]).

The motivation for this work comes from [1]. In that paper $S$ is a closed surface of genus $g \geq 4$. The Torelli subgroup $I(S)$ of the extended mapping class group $Mod(S)$ of $S$ acts, by definition, trivially on the first homology group $H = H_1(S, \mathbb{Z})$ of $S$. Let $K = K(S)$ be the subgroup of $I(S)$ which is generated by Dehn twists about separating curves. The group $K$ arises naturally as the kernel of the Johnson map from $I(S)$ to $\wedge^3(H)/H$ (see [11]) and it is related to the Heegaard splittings of the homology 3-sphere (the reader may find additional information in [1], as well as, in [4]). More precisely, in [1] the commensurator group $Comm(K)$ is considered and, based on ideas of Farb and Ivanov [3] which have proved a similar result for $I(S)$, the authors proved that $Comm(K) \cong Aut(K) \cong Mod(S)$, (see Main Theorem 1 in [1]) and deduce significant corollaries. The proof of Main Theorem 1 in [1] deploys a reformulation of group theoretic statements in terms of maps of curve complexes and a basic step is Theorem 1.5 of [1] which asserts that the automorphism of the complex $SC(S)$ of separating curves of $S$ is isomorphic to $Mod(S)$.

On the other hand, in [8] the mapping class group $Mod(H_g)$ of a 3-dimensional handlebody $H_g$ of genus $g$ is studied in comparison with the mapping class group of surfaces. The boundary surface $S = \partial H_g$ is considered and many similarities and differences of $Mod(H_g)$ are presented with respect to $Mod(S)$. An important topic commented in [8] is the action of $Mod(H_g)$ on the first homology group of $H_g$. Thus, similarly to the discussion above, the Torelli group $I(H_g)$ and the group $K(H_g)$ generated by Dehn twists about separating meridians can be considered. In this frame, Corollary 4 above can be useful, as in the case of surfaces, in an effort to relate $K(H_g)$ with $Aut(K(H_g))$ and $Mod(H_g)$.

1. Definitions and properties of automorphisms of $SM(H_g)$

We first give definitions and notation. Throughout this section $\phi$ will denote an arbitrary automorphism of $SM(H_g)$. We will use arcs connecting boundaries of disjoint meridians and these arcs will always be in the boundary surface of the handlebody. Meridian disks will always be properly embedded. In particular, the boundary of a disk will be considered as a simple closed curve in the boundary surface of the handlebody. All intersections between arcs and disks are assumed transverse and minimal.

1.1. Intersection numbers between isotopy classes of disks and arcs

By definition of $SM(H_g)$ all meridians are to be considered up to isotopy. If $X$ is a separating meridian in $H_g$, the vertex containing $X$, i.e. the isotopy class containing $X$ will be denoted by $[X]$. 
Although the intersection number between isotopy classes of meridians as well as boundaries of meridians, is well known and standard we need to make precise the notion of an arc in $\partial H_g$ connecting two isotopy classes of meridians as well as the intersection number between an arc and an isotopy class of meridians in a specific setup needed in the sequel.

Recall that for arbitrary (separating) meridians $X$, $Y$ in $H_g$ the intersection number $i(X,Y)$ between their isotopy classes $[X]$, $[Y]$ is a well defined non-positive integer and equals the minimal number of components of the intersection $X \cap Y$ when $X$, $Y$ vary within their isotopy class. Similarly, the intersection number of isotopy classes of simple closed curves in $\partial H_g$ is well defined and therefore the intersection number $i(\partial X, \partial Y)$ between the isotopy classes of the boundary curves determined by $X$ and $Y$ is well defined. As the transverse intersection of two separating meridians $X$, $Y$ consists of arcs with endpoints in $\partial X \cup \partial Y$ it follows that

$$i(X,Y) = \frac{1}{2} i(\partial X, \partial Y)$$ (1)

Let $E$, $F$ be disjoint separating meridians and let $\tau$ be an arc properly embedded in $\partial H_g$ with one endpoint in $\partial E$ and the other in $\partial F$ whose interior is disjoint from both $\partial E$, $\partial F$. By the union $E \cup_\tau F$ of $E$ and $F$ along $\tau$ we mean the meridian obtained as follows: let $N$ be a regular neighborhood of $\partial E \cup_\tau \partial F$ in $\partial H_g$. Then $E \cup_\tau F$ is the separating meridian whose boundary is the unique boundary curve of $N$ that is not isotopic to either $\partial E$ or $\partial F$. We also say that $E \cup_\tau F$ is the meridian obtained by joining $E$, $F$ along $\tau$.

Let $v$, $w$ be adjacent vertices in $SM(H_g)$. An isotopy class of arcs from the vertex $v$ to the vertex $w$ is defined as follows: consider the set of smooth arcs

$$\{ \tau : [0,1] \to \partial H_g \mid \tau(0) \in \partial E, \tau(1) \in \partial F \text{ with } E \in v, F \in w, E \cap F = \emptyset \}.$$

We say that two such arcs $\tau$, $\tau'$ are equivalent if and only if the simple closed curves $\partial (E \cup_\tau F)$ and $\partial (E' \cup_{\tau'} F')$ are isotopic in $\partial H_g$. An element in the quotient space is an isotopy class of arcs from $v$ to $w$.

For adjacent vertices $[E]$, $[F]$ in $SM(H_g)$ and $[\tau]$ an isotopy class of arcs from $[E]$ to $[F]$ we obtain a new vertex by setting

$$[E] \cup_{[\tau]} [F] := [E \cup_\tau F].$$

Clearly, the right hand side is independent of the choice of representatives $E$, $F$, $\tau$.

For vertices $[D]$, $[E]$, $[F]$ forming a 2-simplex in $SM(H_g)$ and $[\tau]$ an isotopy class of arcs from $[E]$ to $[F]$ we define the intersection number of $[D]$ with $[\tau]$ by

$$i([\tau],[D]) := i(E \cup_\tau F, D) \overset{(1)}{=} \frac{1}{2} i(\partial (E \cup_\tau F), \partial D)$$ (2)

where it is assumed that the representatives $D$, $E$, $F$ are distinct pair-wise disjoint meridians and $\tau$ a simple arc with endpoints on $\partial E$, $\partial F$. Clearly, $i([\tau],[D])$ depends only on $[E]$, $[F]$, $[\tau]$ and $[D]$.

From now on we will skip the notation $i(\cdot,\cdot)$ and use the symbol $|\cdot|$ to denote the minimal number of components of intersections between the isotopy classes.
involved. Moreover, we will not carry the notation \([\cdot]\) for an isotopy class (vertex) of meridians or, simple closed curves and we will use plain letters \(D, E, F\) etc. In particular, since an arc \(\tau\) can intersect only the boundary of a meridian, we will be writing

\[
|\tau \cap \partial D| \quad \text{instead of } \ i \left( [\tau], [D] \right).
\]

Moreover, with the new notation \(|\cdot|\) the defining Eq. (2) becomes

\[
|\tau \cap \partial D| = \left| (E \cup \tau F) \cap D \right|
\]  
(3)

**Alternative definition:** Given pairwise disjoint separating meridians \(D, E, F\) and an arc \(\tau\) connecting \(\partial E\) and \(\partial F\), the intersection number \(i(\tau, D)\) is the minimal number, ranging over arcs isotopic to \(\tau\) with endpoints on \(\partial E\) and \(\partial F\), of crossing of \(\tau\) with \(\partial D\).

**Lemma.** The above definition agrees with the definition given in Eq. (2), that is

\[
i(\tau, D) = |\tau \cap \partial D|.
\]

**Proof.** The meridian boundaries \(\partial E, \partial F\) and \(\partial (E \cup \tau F)\) form a pair of pants \(P\), a subsurface of \(\partial H_g\), containing \(\tau\). The number \(k = |\tau \cap \partial D|\) is realized by \(k\) disjoint arcs in the intersection \(D \cap (E \cup \tau F)\). Their endpoints, \(2k\) of them, belong to \(\partial D \cap \partial (E \cup \tau F)\). There exist exactly \(k\) subarcs of \(\partial D\) in \(P\) joining these \(2k\) points. As all intersections are essential, each one of the \(k\) subarcs of \(\partial D\) must intersect \(\tau\) once.

In practice, given adjacent vertices \([E], [F]\) we will be constructing new vertices by considering disjoint representatives \(E, F\) and a (usual) isotopy class of arcs with endpoints contained in \(\partial E\) and \(\partial F\) but not fixed by the isotopy.

**Definition 5.** Let \(D\) be a separating meridian splitting \(H_g\) into two handlebodies of genus, say, \(g_D\) and \(g = g_D\) where \(1 \leq g_D \leq g - 1\). Such a separating meridian will be called a \(g_D\)-meridian or, equivalently, a \((g - g_D)\)-meridian and the handlebodies will be called the genus \(g_D\) and \(g = g_D\) components of \(D\) and will be denoted by \(H_{g_D}(D)\) and \(H_{g - g_D}(D)\) respectively. We will be viewing \(H_{g_D}(D)\) and \(H_{g - g_D}(D)\) as spotted handlebodies with spot \(D\).

For every 1-meridian \(D\), the genus 1 handlebody \(H_1(D)\) contains a unique non-separating meridian disjoint from \(D\) which will be denoted by \(\delta(D)\).

Let \(A, B\) be two distinct and disjoint separating meridians. If \(N(A), N(B)\) are open tubular neighborhoods of \(A, B\) respectively, then \(H_g \setminus (N(A) \cup N(B))\) consists of 3 components: a handlebody \(H_{g_A}(A)\) of genus \(g_A\) with one spot, namely \(A\), a handlebody \(H_{g_B}(B)\) of genus \(g_B\) with one spot, namely \(B\), and a genus \(m = g - g_A - g_B\) handlebody with spots (that is, disjoint, distinguished disks) \(A\) and \(B\), denoted by \(H_m(A, B)\).

Similarly, if \(A_1, \ldots, A_l\) are distinct pairwise disjoint separating meridians, each component of \(H_g \setminus (N(A_1) \cup \cdots \cup N(A_l))\) is a spotted handlebody. The component with \(l\) spots will be denoted by \(H(A_1, \ldots, A_l)\) provided it exists (for \(l = 2\) always exists). For \(l \geq 1\) we will be writing \(\partial H(A_1, \cdots, A_l)\) for the boundary of the
spotted handlebody minus the interior of the \( l \) spots. Clearly \( \partial H (A_1, \cdots, A_l) \) is a surface of the same genus with \( l \) boundary components. By abuse of language, we will be calling \( \partial H (A_1, \cdots, A_l) \) the boundary of the spotted handlebody.

For \( D, E, F \) pair-wise disjoint and distinct meridians we also have the following

\[
D \subset H (E, F) \iff |\tau \cap \partial D| \text{ is odd for all } \tau \text{ connecting } \partial E, \partial F. \tag{4}
\]

\[
E \subset H (D, F) \quad \text{and} \quad F \subset H (D, E) \iff |\tau \cap \partial D| \text{ is even or 0 for all } \tau \text{ connecting } \partial E, \partial F. \tag{5}
\]

Clearly, if \( E \subset H (D, F) \) and \( F \nsubseteq H (D, E) \) then there is no arc \( \tau \) connecting \( \partial D \) and \( \partial F \) whose interior is disjoint from \( \partial E \).

1.2. Properties invariant under automorphisms of \( \mathcal{SM} (H_g) \)

We will use the notion of the sum of two flag complexes. Recall that a complex \( K \) is a flag complex if the following property holds: if \( \{v_0, \ldots, v_n\} \) is a set of vertices with the property the edge \((v_i, v_j)\) exists for all \( 0 \leq i, j \leq n \), then \( \{v_0, \ldots, v_n\} \) is a simplex. Observe that \( \mathcal{M} (H_g) \) as well as \( \mathcal{SM} (H_g) \) are flag complexes.

If \( K, L \) are simplicial flag complexes we will write \( K \oplus L \) to denote the (flag) complex defined as follows:

1. the vertex set of \( K \oplus L \) is the union of the vertices of \( K \) and the vertices of \( L \).
2. for every \( k \)-simplex \( \{v_0, \ldots, v_k\} \), \( k \geq 0 \) in \( K \) and \( l \)-simplex \( \{w_0, \ldots, w_l\} \), \( l \geq 0 \) in \( L \) there exists a \((k+l+1)\)-simplex \( \{v_0, \ldots, v_k, w_0, \ldots, w_l\} \) in \( K \oplus L \). In other words, by (1) and (2) the vertices and the edges are defined and then we require \( K \oplus L \) to be the (unique) flag complex generated by these.

**Definition 6.** We will say that a complex \( M \) splits, equivalently \( M \) admits a splitting, if there exist subcomplexes \( K, L \) such that \( M = K \oplus L \).

**Lemma 7.** Let \( \phi : \mathcal{SM} (H_g) \rightarrow \mathcal{SM} (H_g) \) be an automorphism.

- **(a)** \( D \) is a \( k \)-meridian in \( H_g \), \( 1 < k < g - 1 \) if and only if the link \( Lk (D) \) of \( D \) in \( \mathcal{SM} (H_g) \) splits as \( K \oplus L \) where \( \dim K = 2k - 3 \), \( \dim L = 2(g-k) - 3 \) and each of \( K, L \) does not split. \( D \) is a 1-meridian in \( H_g \) if and only if \( Lk (D) \) does not split.

- **(b)** If \( D \) is a \( k \)-meridian in \( H_g \), \( 0 < k < g \), then \( \phi (D) \) is also a \( k \)-meridian.

- **(c)** Let \( D \) be a \( k \)-meridian in \( H_g \), with \( 1 < k < g - 1 \) with splitting \( Lk (D) = K \oplus L \). If \( E, F \in K \) then \( \phi (E), \phi (F) \) belong to the same summand of \( Lk (\phi (D)) \). Moreover, \( Lk (\phi (D)) = \phi (K) \oplus \phi (L) \).

**Proof.** (a) It is straightforward to see that \( Lk (D) \) splits as \( K \oplus L \) with the given dimensions where the vertex set of \( K \) (resp. \( L \)) consists of all separating meridians in \( H_k (D) \) (resp. \( H_{g-k} (D) \)). We show that \( K \) does not split further. Assume, on the contrary, that \( K = K_1 \oplus K_2 \). Pick any meridian \( \Delta \subset H_k (D) \) which, we may assume, is a vertex of the subcomplex \( K_1 \). We will show that all vertices
of \(K\) belong to \(K_1\). If \(X\) is a meridian (vertex of \(K\)) intersecting \(\Delta\) then \(X\) is necessarily, by definition of \(K_1 \oplus K_2\), a vertex of \(K_1\). If \(Y\) is a meridian disjoint from \(\Delta\) then we may find a separating meridian \(X_Y\) intersecting both \(\Delta\) and \(Y\). Then \(X_Y \in K_1\) because \(X_Y\) intersects \(\Delta\) and \(Y \in K_1\) because \(Y\) intersects \(X_Y\). Thus, all meridians are vertices of the complex \(K_1\).

(b) An automorphism \(\phi\) maps \(\text{Lk} (D)\) onto \(\text{Lk} (\phi (D))\) isomorphically and, thus, preserves the splitting (resp. non-splitting).

(c) Pick a separating meridian \(X\) intersecting both \(E\) and \(F\) but disjoint from \(D\). As \(\phi (D)\) is separating, \(\text{Lk} (\phi (D))\) splits, say, \(\text{Lk} (\phi (D)) = K' \oplus L'\). If \(\phi (E), \phi (F)\) do not belong to the same summand of the splitting of \(\text{Lk} (\phi (D))\) then \(\phi (X)\) would have to intersect \(\phi (D)\), a contradiction.

\[\square\]

Before proceeding with further properties of an automorphism \(\phi\) we need a result concerning the curve complex of a sphere with holes (see \([9,13]\)):

If \(n \geq 5\) then all elements of \(\text{Aut} (C (S_{0,n}))\) are geometric, that is, they are induced by a homeomorphism of \(S_{0,n}\).

(6)

A cut system \(\mathcal{C}\) for the handlebody \(H_g\) is a collection \(\{C_1, \ldots, C_g\}\) of pairwise disjoint non-separating meridians such that \(H_g \setminus \bigcup_{i=1}^g C_i\) is a 3-ball with \(2g\) spots. We will denote this spotted ball by \(H_g \setminus \mathcal{C}\).

A separating cut system \(\mathcal{Z}\) for the handlebody \(H_g\) is a collection \(\{Z_1, \ldots, Z_g\}\) of pairwise disjoint 1-meridians so that the intersection \(\bigcap_{i=1}^g H_{g-1} (Z_i)\) is a 3-ball with \(g\) spots. We will denote this spotted ball by \(H_g \setminus \mathcal{Z}\). Note that given \(\mathcal{C}\) we can find \(\mathcal{Z}\) so that \(C_i\) is the unique non-separating meridian in \(H_1 (Z_i)\). By Lemma 7, \(\phi (Z_i), i = 1, \ldots, g\) is also a separating cut system, that is, \(\cap_{i=1}^g H_{g-1} (\phi (Z_i))\) is a 3-ball with \(g\) spots.

Every simple closed curve in the boundary of the spotted 3-ball \(H_g \setminus \mathcal{Z}\) is a separating meridian and vice-versa. Therefore, if \(g \geq 5\), an automorphism \(\phi\) of \(SM (H_g)\) fixing \(\mathcal{Z}\) induces an element in \(\text{Aut} (C (S_{0,g}))\). By statement (6), this element is induced by a homeomorphism of \(S_{0,g}\) which fixes the boundary components. Any such homeomorphism extends inwards to the entire handlebody, thus, we have

**Theorem A.** Let \(\mathcal{Z} = \{Z_1, \ldots, Z_g\}\) be a separating cut system for \(H_g\), \(g \geq 5\), and \(\phi\) an automorphism of \(SM (H_g)\) fixing \(\mathcal{Z}\). The action of \(\phi\) on the subcomplex \(\{D \in SM (H_g) : D \cap Z_i = \emptyset, \forall i = 1, \ldots, g\}\) is induced by a homeomorphism of the handlebody.

The next Lemma contains certain intersection properties between separating meridians which are preserved by an automorphism of \(SM (H_g)\).

**Lemma 8.** Let \(E, F, D\) be three distinct pair-wise disjoint separating meridians and \(E \cup_{\tau} F\) the separating meridian obtained by joining \(E\) and \(F\) along an embedded arc \(\tau\) which has one endpoint in \(\partial E\) and the other in \(\partial F\).

(a) \(\phi (E \cup_{\tau} F)\) is a separating meridian obtained by joining \(\phi (E)\) with \(\phi (F)\) along a unique (up to isotopy) simple arc \(\tau'\).

(b) Assume \( g \geq 5 \) and \( \tau \) intersects \( \partial D \) in 1 point. Then the arc \( \tau' \) posited in part (a) above intersects \( \partial \phi(D) \) in 1 point.

(c) Assume \( g \geq 5 \) and \( \tau \) intersects \( \partial D \) in 2 points. Moreover, assume \( D \) is a \( g_D \)-meridian with \( g - g_D \geq 3 \) and \( E, F \subset H_{g-g_D}(D) \). Then the arc \( \tau' \) posited in part (a) above intersects \( \partial \phi(D) \) in 2 points.

(d) Assume \( g \geq 6 \) and \( D \) is a \( g_D \)-meridian with \( g - g_D = 2 \) and \( E, F \subset H_{g-g_D}(D) \). Let \( D' \) be a 4-meridian with \( D \subset H_4(D') \) such that \( H_{g_D}(D) \cap H_4(D') \) is the genus 2 handlebody \( H_2(D, D') \) with two spots \( D \) and \( D' \) and, in addition, \( \tau \cap \partial D' = \emptyset \). If \( \tau \) intersects \( \partial D \) in 2 points then the arc \( \tau' \) posited in part (a) above intersects \( \partial \phi(D) \) in 2 points.

**Proof.** (a) Without loss of generality, we may assume \( E \) is a \( g_E \)-meridian and \( F \) a \( g_F \)-meridian with \( H_{g_E}(E) \cap H_{g_F}(F) = \emptyset \). Then \( X = E \cup _\tau F \) is a \( (g_E + g_F) \)-meridian with \( X, E, F \) pair-wise disjoint with \( H(X, E, F) \) being a spotted 3-ball. By Lemma 7, all these properties hold for their images \( \phi(E), \phi(F) \) and \( \phi(X) \), and hence there is a unique (up to isotopy) arc \( \tau' \) with endpoints in \( \phi(E), \phi(F) \) not intersecting \( \phi(X) \). As \( H(\phi(E), \phi(F), \phi(X)) \) is a spotted 3-ball containing \( \tau' \) we clearly have \( \phi(X) = \phi(E) \cup _{\tau'} \phi(F) \).

(b) Observe that \( D \) is necessarily a \( g_D \)-meridian with \( 1 < g_D < g - 1 \). This is because if \( D \) is a 1-meridian then both \( E \) and \( F \) are contained in \( H_{g-1}(D) \) and, by (5), the intersection \( \partial D \cap \tau \) cannot be one point.

The meridian \( X = E \cup _\tau F \) intersects \( D \) in one arc which splits both \( X, D \) into two subdisks. By surgery along this arc we obtain four distinct and pairwise disjoint separating meridians \( W_i, i = 1, \ldots, 4 \) such that \( H(W_1, W_2, W_3, W_4) \) is a spotted 3-ball containing \( X \) and \( D \). Clearly, we may find a separating cut system \( 3 = \{Z_1, \ldots, Z_g\} \) such that \( W_i \in H_g \backslash 3 \) for all \( i = 1, 2, 3, 4 \). In particular, \( X \) and \( D \) belong to the subcomplex \( \{C \in SM(H_g) : C \cap Z_i = \emptyset, \forall i = 1, \ldots, g\} \). Since there exists a homeomorphism of \( H_g \) which sends \( \phi(Z_i) \) to \( Z_i \) we may assume that \( \phi \) fixes \( 3 \).

By Theorem A, \( \phi \) acts geometrically on \( X \) and \( D \). Therefore, \( \phi(D) \) and \( \phi(X) = \phi(E) \cup _{\tau'} \phi(F) \) intersect at a single arc and, by (3), \( \tau' \) intersects \( \partial \phi(D) \) in 1 point.

(c) We may assume that \( E \) is a \( g_E \)-meridian and \( F \) is a \( g_F \)-meridian such that

\[
H_{g-g_E}(E) \supset D, F \quad \text{and} \quad H_{g-g_F}(F) \supset D, E.
\]

By Lemma 7(c) the same properties hold for the images \( \phi(D), \phi(E), \phi(F) \) which by (5) are equivalent to

\[
|\sigma \cap \partial \phi(D)| \text{ is even or } 0. 
\]

for any arc \( \sigma \) joining \( \partial \phi(E) \) with \( \partial \phi(F) \).

We first show the result in the case \( g_E + g_D + g_F < g \). As \( E, F \subset H_{g-g_D}(D) \) the latter inequality is equivalent to \( H(E, F, D) \) being a spotted handlebody of positive genus \( g - (g_E + g_D + g_F) \). The arc \( \tau \) splits into 3 subarcs \( \tau_E, \tau_D, \tau_F \) such that \( \tau_E \cup \tau_D \cup \tau_F = \tau \) and

- \( \tau_E \) (resp. \( \tau_F \)) has endpoints on \( \partial E \) (resp. \( \partial F \)) and \( \partial D \).
• \( \tau_D \subset \partial H_{g_D}(D) \).

Set \( Y_E = E \cup_{\tau_E} D \) and \( Y_F = D \cup_{\tau_F} F \). We claim that \( Y_E \) is not isotopic to \( F \). For, if \( Y_E, F \) are isotopic then \( Y_E \) would be a \((g_E + g_D)\)-separating meridian with \( g - (g_E + g_D) = g_F \), that is \( g_E + g_D + g_F = g \), a contradiction. Thus \( Y_E \neq F \) and, similarly, \( Y_F \neq E \). Clearly \( D, E \) belong to distinct components of \( H_g \setminus Y_F \) and since \( \tau_E, \tau_F \) are disjoint subarcs of \( \tau \) we have that

(i) \( |Y_F \cap Y_E| = 1 \).
(ii) \( |\tau \cap \partial Y_F| = 1 = |\tau \cap \partial Y_E| \).

Moreover, by (ii), \( \tau_F \) induces a subarc \( \overline{\tau_F} \subset \tau_F \) with endpoints on \( \partial Y_E \) and \( \partial F \). Set

\[ Y_{E,F} = Y_E \cup_{\overline{\tau_F}} F. \]

Clearly, \( H(Y_E, F, Y_{E,F}) \) is a spotted 3-ball or, equivalently,

\[ g_{Y_E} + g_F + g_{Y_{E,F}} = g \quad \xrightarrow{Y_E = E \cup_{\tau_E} D} \quad g_D + g_E + g_F + g_{Y_{E,F}} = g. \]

As the sum of the genus of each of the four meridians \( D, E, F, Y_{E,F} \) is equal to \( g \) it follows

(iii) \( H(D, E, F, Y_{E,F}) \) is a spotted 3-ball containing both meridians \( Y_E \) and \( Y_F \). The curves \( \partial Y_E \cap \partial Y_F \) are separating and, by (i), intersect twice. A regular neighborhood of their union is the four-holed sphere \( \partial H(D, E, F, Y_{E,F}) \) and all boundary components are obtained from surgery of \( \partial Y_E \) and \( \partial Y_F \). It follows that surgery along the single arc in \( Y_E \cap Y_F \) gives rise to the spots of \( H(D, E, F, Y_{E,F}) \) none of which is isotopic to either \( Y_E \) or \( Y_F \). In other words,

(iv) surgery along \( Y_E \cap Y_F \) produces the meridians \( E, F, D \) and \( Y_{E,F} \).

Let \( \tau' \) be the arc given by part (a), that is, \( \phi(E \cup_{\tau} F) = \phi(E) \cup_{\tau'} \phi(F) \). We will show that the same properties as above hold for the images under \( \phi \) of the meridians and the arcs involved:

(i)' \( |\phi(Y_F) \cap \phi(Y_E)| = 1 \).
(ii)' \( |\tau' \cap \partial \phi(Y_E)| = 1 = |\tau' \cap \partial \phi(Y_F)| \).
(iii)' \( H(\phi(D), \phi(E), \phi(F), \phi(Y_{E,F})) \) is a spotted 3-ball containing both meridians \( \phi(Y_E) \) and \( \phi(Y_F) \).
(iv)' surgery along \( \phi(Y_E) \cap \phi(Y_F) \) produces the meridians \( \phi(D), \phi(E), \phi(F) \) and \( \phi(Y_{E,F}) \).

Both (i)' and (ii)' follow immediately from part (b) and (i), (ii). By Lemma 7, the genus of the meridians \( \phi(D), \phi(E), \phi(F) \) and \( \phi(Y_{E,F}) \) satisfy

\[ g_{\phi(D)} + g_{\phi(E)} + g_{\phi(F)} + g_{\phi(Y_{E,F})} = g \]

which shows (iii)' Property (iv)' follows from (iii)' for the same reason (iii) followed from (iv) above. Since \( E \cup_{\tau} F \) intersects \( D \), the meridian

\[ \phi(E \cup_{\tau} F) = \phi(E) \cup_{\tau'} \phi(F) \]
must intersect \( \phi(D) \). As \( \phi(E), \phi(F) \) are disjoint from \( \phi(D) \), the arc \( \tau' \) must necessarily intersect \( \partial \phi(D) \) and by (5), \(|\tau' \cap \partial \phi(D)| = 2k \) with \( k \geq 1 \).

If \( k \neq 1 \), then there would exist \( k - 1 \) subarcs of \( \tau' \) with endpoints on \( \partial \phi(D) \) contained in \( \partial H_{g-g_D} \phi(D) \). By (ii)', these subarcs must be disjoint from \( \partial \phi(Y_E) \) and \( \partial \phi(Y_F) \). Since \( \phi(D) \) is one of the 4 disks produced by surgery along the intersection \( \phi(Y_E) \cap \phi(Y_F) \) (this is by (iv)'), all these subarcs must be parallel to \( \partial \phi(D) \) because their endpoints lie on \( \partial \phi(D) \). It follows that we may perform an elementary isotopy to eliminate them, showing that \( k = 1 \).

We now examine the case \( g_E + g_D + g_F = g \) or, equivalently, \( H(D, E, F) \) is a spotted 3-ball. We work under the assumption \( g - g_D \geq 3 \), hence, at least one of \( g_E, g_F \) is \( \geq 2 \). Without loss of generality we assume that \( g_E \geq 2 \). Let \( E_1 \) be a 1-meridian in \( H_{g_E}(E) \) and \( E_2 \) a \((g_E - 1)\)–meridian in \( H_{g_E}(E) \) disjoint from \( E_1 \) so that \( E = E_1 \cup \rho E_2 \) for some simple arc \( \rho \) with endpoints on \( \partial E_1, \partial E_2 \). Clearly, \( H(E_1, D, F) \) is a spotted 3-ball. Extend \( \tau \) to a simple arc \( \tau_1 \) with endpoints on \( \partial E_1, \partial F \) such that \( \tau_1 \cap \partial E_2 = \emptyset \) (which implies that \( \tau_1 \cap \rho = \emptyset \)). Clearly \( H(E_1, D, F) \) has positive genus and we claim that \(|\tau_1 \cap \partial D| = 2 \).

To see the latter, assume on the contrary that there exists an isotopy \( (\tau_1), t \in [0, 1] \) such that \( (\tau_1)_0 = \tau_1 \) and

\[(\tau_1)_1 \cap \partial D = \emptyset. \quad (8)\]

By altering, if necessary, the isotopy we may assume that the intersection \( (\tau_1)_t \cap \partial E \) is transverse and minimal. As \( E \) is a separating meridian and \( E_1 \subset H_{g_E}(E), F \subset H_{g-g_E}(E) \) the intersection \( (\tau_1)_t \cap \partial E \) consists of a single point for all \( t \in [0, 1] \).

Denote this point by \( \chi_t \) which splits \( (\tau_1)_t \), in two adjacent subarcs. Denote by \( \mu_t \) the subarc with endpoints on \( \partial E \) and \( \partial F \). Clearly \( \mu_0 = \tau \) and \( \mu_1 \) is a subarc of \( (\tau_1)_1 \) which, by (8), is disjoint from \( \partial D \). This shows that, up to isotopy, \( \tau \) is disjoint from \( \partial D \), a contradiction.

By the previous case, the arc \( \tau'_1 \) given by part (a), that is,

\[\phi(E_1 \cup \tau_1 F) = \phi(E_1) \cup \tau'_1 \phi(F)\]

satisfies

\[|\tau'_1 \cap \partial \phi(D)| = 2. \quad (9)\]

Set \( Y(\tau) = E \cup \tau F \) and \( Y(\tau_1) = E_1 \cup \tau_1 F \). With this notation, Eq. (9) becomes

\[|\phi(D) \cap \phi(Y(\tau_1))| = 2. \quad (10)\]

Observe also that \( H(Y(\tau), Y(\tau_1), E_2) \) is a spotted 3-ball and therefore the spotted handlebody

\[H' \equiv H\left(\phi(Y(\tau)), \phi(Y(\tau_1)), \phi(E_2)\right)\]

determined by the images of \( Y(\tau), Y(\tau_1), E_2 \) (see Fig. 1) is again a 3-ball.

The arc \( \rho \) intersects \( \partial Y(\tau_1) \) once and therefore \( \rho' \) intersects \( \partial \phi(Y(\tau_1)) \) once (by part (b)). In other words,

\[\rho' \text{ induces a single arc on } \partial H' \text{ connecting } \partial \phi(Y(\tau_1)) \text{ with } \partial \phi(E_2). \quad (11)\]
We want to show that $\tau'$ intersects $\partial \phi(D)$ at 2 points or, equivalently, $\phi(D)$ intersects $\phi(Y(\tau)) = \phi(E) \cup T'$ at two arcs. Assume, on the contrary,

$$|\phi(D) \cap \phi(Y(\tau))| > 2.$$ 

As $\phi(D)$ is a separating meridian the above inequality implies (see (5)) that

$$|\phi(D) \cap \phi(Y(\tau))| \geq 4. \tag{12}$$

that is, $\partial \phi(D) \cap \partial \phi(Y(\tau))$ consists of at least 8 points. Denote them by $x_1, \ldots, x_8$. Let $\sigma_1$, be the component of $\partial \phi(D) \cap \partial H'$ with $x_1 \in \partial \sigma_1$. We claim that the other endpoint of $\sigma_1$ is necessarily contained in $\partial \phi(D) \cap \partial \phi(Y(\tau_1))$. To see this, recall that $H'$ is a 3-ball with three spots and observe that

$$\emptyset = E \cap D = \phi(D) \cap \phi(E) = \phi(D) \cap (\phi(E_1) \cup \rho' \phi(E_2))$$

which implies that $\phi(D) \cap \rho' = \emptyset$ and, in particular, $\sigma_1 \cap \rho' = \emptyset$. It follows that if both endpoints of $\sigma_1$ were contained in $\partial \phi(D) \cap \partial \phi(Y(\tau))$ then $\sigma_1$ would be boundary parallel and would be eliminated. This shows that $\sigma_1$ has one endpoint in $\partial \phi(Y(\tau_1))$ and one on $\partial \phi(Y(\tau))$. Working similarly we obtain 8 arcs $\sigma_i, i = 1, \ldots, 8$ each having one endpoint in $\partial \phi(Y(\tau_1))$ and the other on $\partial \phi(Y(\tau))$. Therefore, the intersection $\partial \phi(D) \cap \partial \phi(Y(\tau_1))$ also consists of at least 8 points. This is equivalent to

$$|\phi(D) \cap \phi(Y(\tau_1))| \geq 4$$

which contradicts (10). This completes the proof of (c).

(d) Since $g \geq 6$, we have $g - 4 \geq 2$ and we may choose disjoint separating meridians $B, B'$ in $H_{g-4}(D')$ of genus 1 and $g_{B'}$ respectively, with $1 + g_{B'} = g - 4$. Moreover, by choosing $B, B'$ so that $B$ is not contained in $H_{g_{B'}}(B')$, there exists an arc $\mu$ disjoint from $\partial D'$ such that $B \cup \mu B' = D'$. There is a unique (up to isotopy with endpoints on $\partial E, \partial F$) arc $\rho$ whose interior is disjoint from $\tau$ and $\partial D$ such that $E \cup \rho F = D$. Pick an arc $\sigma$ with endpoints on $\partial (E \cup F)$ and $\partial B$ such that its interior is disjoint from $\mu, \partial D, \partial B, \partial B'$ and $\partial (E \cup \tau F)$ intersecting $\partial D'$ at a single point. Set $\Delta$ to be the meridian

$$\Delta := (E \cup F) \cup \sigma B.$$ 

Clearly, $\Delta$ is a 3-meridian and $|\rho \cap \partial \Delta| = 2$. By parts (a) and (c)

$$\phi(D) = \phi(E \cup \rho F) = \phi(E) \cup T' \phi(F), \text{ with } |\rho' \cap \partial \phi(\Delta)| = 2. \tag{13}$$

Similarly, by parts (a) and (b),

$$\phi(D') = \phi(B \cup \mu B') = \phi(B) \cup \mu' \phi(B'), \text{ with } |\mu' \cap \partial \phi(\Delta)| = 1. \tag{14}$$

The arcs $\mu', \rho'$ are contained in $\partial H(\phi(B), \phi(B'), \phi(E), \phi(F))$ and produce disjoint meridians $\phi(D), \phi(D')$ which implies $\mu' \cap \rho' = \emptyset$. The spotted handlebody $H(\phi(\Delta), \phi(E), \phi(F), \phi(B))$ is a 3-ball and by (14) $\mu'$ induces an arc $\mu'_1$ with endpoints on $\partial \phi(\Delta), \partial \phi(B)$. Similarly, by (13), $\rho'$ induces arcs $\rho'_1, \rho'_2$ with endpoints on $\partial \phi(\Delta), \partial \phi(E)$ and $\partial \phi(\Delta), \partial \phi(F)$ respectively, which, up to isotopy,
Fig. 1. All components $\sigma_i$ of $\partial \phi(D) \cap \partial H'$ have one endpoint in $\phi(Y(\tau))$ and the other in $\phi(Y(\tau_1))$.

Fig. 2. The mutually disjoint arcs $\mu'_1, \rho'_1, \rho'_2$ all with one endpoint on $\phi(\Delta)$ and the other on $\phi(B), \phi(E), \phi(F)$ respectively are disjoint from $\mu'_1$ (see Fig. 2). The arc $\tau$ does not intersect $\partial B$ and $\partial \Delta$, hence, the arc $\tau'$ given by part (a) so that $\phi(B \cup \Delta) = \phi(B) \cup \tau' \phi(D)$ is disjoint from both $\partial \phi(B)$ and $\partial \phi(\Delta)$. Therefore, the arc $\tau'$ is contained in the boundary of the spotted 3-ball $H(\phi(\Delta), \phi(E), \phi(F), \phi(B))$. Every such arc with endpoints on $\partial \phi(E)$ and $\partial \phi(F)$ which intersects either $\rho'_1$ or $\rho'_2$ must intersect $\mu'_1$. As $\mu'_1 \cap \tau' \subset \mu' \cap \tau' = \emptyset$, it follows that

$$\tau' \cap (\rho'_1 \cup \rho'_2) = \tau' \cap \rho' = \emptyset$$

This shows that $\tau'$ intersects $\partial \phi(E) \cup_{\rho'} \partial \phi(F) = \partial \phi(D)$ at 2 points and completes the proof of the Lemma.

\[\square\]

2. The arc complex automorphism

Throughout this section $A, B$ will denote two distinct disjoint separating meridians such that $H(A, B)$ is a genus 2-handlebody $H_2(A, B)$ with two spots $A, B$. We also assume that the handlebody $H_k(A)$ not containing $B$ has genus $k \geq 2$ and $H_{k'}(B)$ not containing $A$ has also genus $k' \geq 2$. These assumptions on $k, k'$ force the assumption $g \geq 6$ on the genus of the handlebody.

Recall from Definition 5 that for a 1-separating meridian $X$, $\delta(X)$ denotes the unique non-separating meridian in $H_1(X)$ disjoint from $X$. The rest of this section is devoted to showing the following proposition.
Proposition 9. (a) Let \( X, Y \subset H_2 (A, B) \) be 1-meridians and \( \phi \) an automorphism of \( SM (H_g) \), \( g \geq 6 \). If \( \delta (X) = \delta (Y) \) then \( \delta (\phi (X)) = \delta (\phi (Y)) \). (b) Moreover, if \( A', B' \) are disjoint separating meridians which are the spots of the genus 1-handlebody \( H_1 (A', B') \), the same result as in (a) holds for any 1-meridians \( X, Y \subset H_1 (A', B') \).

(c) Let \( Z \) be a 2-meridian, \( X, Y \subset H_2 (Z) \) be 1-meridians and \( \phi \) an automorphism of \( SM (H_g) \). If \( \delta (X) = \delta (Y) \) then \( \delta (\phi (X)) = \delta (\phi (Y)) \).

For the proof of the above proposition we may assume that \( \phi \) fixes \( A, B \) and similarly for parts (b) and (c). This is because, by Lemma 7, \( \phi (A) \) and \( \phi (B) \) are meridians of the same genus as \( A \) and \( B \) respectively and \( H(\phi (A), \phi (B)) \) is again of genus 2. Therefore, there exists a homeomorphism, say \( h \), of \( H_g \) which maps \( \phi (A) \) to \( A \) and \( \phi (B) \) to \( B \). The homeomorphism \( h \) induces an automorphism \( \phi_h \) of \( SM (H_g) \) which, apparently, satisfies the conclusions of the above proposition. Therefore, it suffices to prove the proposition for the composition \( \phi_h \circ \phi \) which, clearly, fixes \( A \) and \( B \).

Moreover, if \( X \subset H_2 (A, B) \) is a separating 1-meridian, the homeomorphism \( h \) mentioned above can be chosen so that, in addition, it sends \( \phi (X) \) to \( X \) allowing us to assume that \( \phi \) fixes \( A, B \) and \( X (A', B', X \) for the proof of part (b) and \( Z, X \) for the proof of part (c)).

The boundary of \( \partial H_2 (A, B) \) of the spotted handlebody \( H_2 (A, B) \) is a genus 2 surface \( \Sigma_{2,2} \) with two boundary components \( \partial A \) and \( \partial B \) (see Definition 5). Denote by \( A \) the arc complex of the surface \( \Sigma_{2,2} \). We will use Lemma 8 to define an automorphism on the arc complex \( A (\Sigma_{2,2}) \).

Proposition 10. Let \( \phi \) be an automorphism of \( SM (H_g) \) which fixes the meridians \( A \) and \( B \). Then \( \phi \) induces an automorphism \( \overline{\phi} : A \to A \) which is geometric on \( \Sigma_{2,2} \), that is, \( \overline{\phi} \) is induced by a homeomorphism of \( \Sigma_{2,2} \).

Before we proceed with the proof we need to state the following four Lemmata \( A, B, C \) and \( D \) which are stated under the notation of the previous two propositions and the assumption that \( \phi \) fixes \( A \) and \( B \).

All arcs considered below will have endpoints on meridian boundaries and all isotopies between them are meant to be relative the boundary, that is, they keep the endpoints of the arcs not fixed but within the boundary meridian they belong. This notion of isotopy of arcs relative the boundary coincides with the corresponding notion in the context of the meridian complex \( SM (H_g) \), that is, isotopy classes of arcs between adjacent vertices in \( SM (H_g) \) (see Alternative definition and the lemma following it in Sect. 1.1).

We will be writing \( \tau_1 \cong \tau_2 \) to denote that the arcs \( \tau_1, \tau_2 \) are isotopic and \( \tau_1 \not\cong \tau_2 \) to indicate distinct isotopy classes.

Lemma A. Let \( \Sigma_{2,2} = \partial H_2 (A, B) \) be the genus 2 surface with 2 boundary components \( \partial A \) and \( \partial B \).

(a) Let \( \sigma, \rho \) be non-separating simple arcs with endpoints on \( \partial B \) which are not boundary parallel and \( \sigma \not\cong \rho \). Then there exists a simple arc \( \tau \) with endpoints on \( \partial A \) and \( \partial B \) such that \( \tau \cap \sigma = \emptyset \) and \( \tau \cap \rho \neq \emptyset \).
(b) Let $\sigma, \rho$ be separating simple arcs with endpoints on $B$ which are not boundary parallel and $\sigma \not\equiv \rho$. Then there exists a simple arc $\tau$ with endpoints on $\partial A$ and $\partial B$ such that $\tau$ intersects exactly one of the arcs $\sigma, \rho$.

Proof. (a) If $\sigma \cap \rho = \emptyset$ then the existence of $\tau$ is obvious. If $\sigma \cap \rho \neq \emptyset$ we will show that there exists a simple arc $\tau$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cap \rho \neq \emptyset$.

Cut $\Sigma_{2,2}$ along $\sigma$ to obtain a surface $\Sigma_{1,3}$ with 3 boundary components, namely, $\partial A$, $\sigma \cup \partial^+ B$ and $\sigma \cup \partial^- B$ where $\partial^+ B \cup \partial^- B = \partial B$. Assuming, as always, $\sigma, \rho$ have minimal intersection, $\rho$ induces arcs $\rho_1, \ldots, \rho_r$ on $\Sigma_{1,3}$ with endpoints on $\sigma \cup \partial^+ B$ and/or $\sigma \cup \partial^- B$. We may find an arc $\tau$ in $\Sigma_{1,3}$ with one endpoint on $\partial A$ and the other on $\partial^+ B \cup \partial^- B$ such that $\tau \cap \rho_i \neq \emptyset$ for at least one $i = 1, \ldots, r$. This arc $\tau$ can be viewed as an arc in $\Sigma_{2,2}$ and has the desired property.

(b) The proof in this case is straightforward if $\sigma, \rho$ are disjoint. If $\sigma \cap \rho \neq \emptyset$, let $\Sigma^+$ be the component of $\partial H_2 (A, B) \setminus \sigma$ containing $\partial A$. The boundary $\partial \Sigma^+$ is the disjoint union $\partial A \perp (\sigma \cup B^+)$ where $B^+$ is a subarc of $\partial B$. Then, we may find an arc $\tau$ in $\Sigma^+$ with endpoints in $A$ and $B^+$ disjoint from $\sigma$ which intersects $\rho$.

$\square$

**Notation 11.** For a simple arc $\tau$ with endpoints on $\partial A$ and $\partial B$ denote by $N (\tau)$ a regular neighborhood of $\partial A \cup_\tau \partial B$ in $\Sigma_{2,2} \equiv \partial H_2 (A, B)$ which has three boundary components $\partial A, \partial B, \partial 3$ isotopic to $\partial A, \partial B, \partial Z (\tau)$ respectively, where $Z (\tau) = A \cup_\tau B$ is the separating 2-meridian inside $H_2 (A, B)$ obtained by joining $\partial A, \partial B$ along the arc $\tau$. Recall that $Z (\tau)$ was defined as the meridian whose boundary is not isotopic to $\partial A$ and $\partial B$.

In the sequel, we will identify the pair of pants $N (\tau)$ with the boundary of the 3-ball $H (A, B, Z (\tau))$ and its three boundary components with $\partial A, \partial B, \partial Z (\tau)$.

**Lemma B.** Let $\tau$ be a simple arc with endpoints on $\partial A$ and $\partial B$, and $\sigma$ a simple arc in $\Sigma_{2,2}$ with endpoints on $\partial B$ which is not boundary parallel. Let $E_1, E_2$ be two disjoint separating meridians inside the component $H_{k'} (B)$ of $B$ (not containing $A$) such that $E_2, B$ belong to the same component of $H_g \setminus E_1$ and $E_1, B$ belong to the same component of $H_g \setminus E_2$. Let $\sigma_1, \sigma_2$ be two disjoint arcs joining the endpoints of $\sigma$ with $\partial E_1$ and $\partial E_2$ respectively, such that the interiors of $\sigma_1, \sigma_2$ are disjoint from $\partial E_1 \cup \partial E_2$. Then for the meridians $Z (\tau) = A \cup_\tau B$ and $E (\sigma) = E_1 \cup_{\sigma_1 \cup \sigma \cup \sigma_2} E_2$ the following hold

$$\sigma \cap \tau = \emptyset \iff |E (\sigma) \cap Z (\tau)| = 0 \text{ or } 2. \quad (15)$$

and

$$\sigma \cap \tau = \emptyset \iff |\phi (E (\sigma)) \cap \phi (Z (\tau))| = 0 \text{ or } 2. \quad (16)$$

Proof. It suffices to show the equivalence (15) which can be written as

$$\sigma \cap \tau = \emptyset \iff |(\sigma_1 \cup \sigma \cup \sigma_2) \cap \partial Z (\tau)| = 0 \text{ or } 2. \quad (17)$$
This is because, applying Lemma 8(a) to the meridians $Z (\tau), E_1, E_2$ and the arc $\sigma_1 \cup \sigma \cup \sigma_2$, we obtain the arc $(\sigma_1 \cup \sigma \cup \sigma_2)'$ which either is disjoint from $\partial \phi (Z (\tau))$ or, by Lemma 8(c), satisfies

$$|(\sigma_1 \cup \sigma \cup \sigma_2)' \cap \partial \phi (Z (\tau))| = 2.$$ 

By Eq. (3), this is equivalent to

$$|\phi (E (\sigma)) \cap \phi (Z (\tau))| = 0 \text{ or } 2.$$ 

As $Z(\tau)$ is a separating meridian we have $|\sigma \cap \partial Z(\tau)| = 2k$ for a non-negative integer $k$. We show that $k = 0$ or $1$.

As $\sigma \cap \tau = \emptyset$, if $k > 1$ there must exist $k - 1$ subarcs of $\sigma$ contained in $N (\tau)$ each with endpoints on $\partial Z(\tau)$ and not intersecting $\tau \subset N$. It follows that all these subarcs must be boundary parallel to the boundary $\partial Z(\tau)$ and therefore they can be eliminated by an isotopy. This shows that $k = 0, 1$ and therefore

$$|\sigma \cap \partial Z(\tau)| = |(\sigma_1 \cup \sigma \cup \sigma_2) \cap \partial Z(\tau)| = 0 \text{ or } 2.$$ 

For the opposite direction in (17), we work similarly in the pair of pants $N (\tau)$: if $\sigma$ intersects $\partial Z(\tau)$ at 2 (resp. 0) points it induces exactly 2 (resp. 0) subarcs in $N (\tau)$ with endpoints on $\partial B$ and $\partial Z(\tau)$. As $\tau$ has endpoints on $\partial A$ and $\partial B$, we may eliminate, by an isotopy in $N (\tau)$, any intersections these arcs may have with $\tau$. This shows that $\sigma \cap \tau = \emptyset$ and completes the proof of (17) and of the Lemma. □

**Lemma C.** Let $\sigma, \rho$ be two simple non-isotopic arcs in $\Sigma_{2,2}$ with endpoints on $\partial B$. Let $E_1, E_2$ be two disjoint separating meridians inside the component $H_{g_B} (B)$ of $B$ (not containing $A$) such that $H (E_1, E_2, B)$ is a spotted 3-ball. Let $\sigma_1, \sigma_2$ (resp. $\rho_1, \rho_2$) be two disjoint arcs joining the endpoints of $\sigma$ (resp. $\rho$) with $\partial E_1$ and $\partial E_2$ respectively. Then for the meridians $E (\sigma) = E_1 \cup_{\sigma_1 \cup \sigma_2} E_2$ and $E (\rho) = E_1 \cup_{\rho_1 \cup \rho_2} E_2$ the following hold

$$\sigma \cap \rho = \emptyset \iff |E (\sigma) \cap E (\rho)| = 2. \quad (18)$$

and

$$\sigma \cap \rho = \emptyset \iff |\phi (E (\sigma)) \cap \phi (E (\rho))| = 2 \quad (19)$$

In particular,

$$\sigma \cong \rho \iff E (\sigma) \cong E (\rho) \iff \phi (E (\sigma)) \cong \phi (E (\rho)). \quad (20)$$

**Proof.** Observe that the assumption $H (E_1, E_2, B)$ being a spotted 3-ball implies that, in the pair of pants $\partial H (E_1, E_2, B)$, $\sigma_1 \cong \rho_1$ and $\sigma_2 \cong \rho_2$. Moreover, all these four arcs $\sigma_1, \sigma_2, \rho_1, \rho_2$ can be chosen to be pairwise disjoint. In particular, they satisfy $(\sigma_1 \cup \sigma_2) \cap (\rho_1 \cup \rho_2) = \emptyset$.

In a similar manner using Lemma 8(c) and 8(d) it suffices to show equivalence (18). Note that 8(d) needs to be used in the case $E_1, E_2$ are both 1-meridians so that the component $H_{g_A} (A)$ containing $B$ has genus $g_A = 4$ and $A$ is playing the role of the meridian $D'$ in the statement of Lemma 8(d). The meridians $E_1, E_2$ are contained in $H_{g_{E_1} + g_{E_2}} (E (\sigma))$ and assuming $\sigma \cap \rho = \emptyset$ we have that $\rho$ is contained
in $\partial H_{g-s_{E_1}-s_{E_2}}(E(\sigma))$. By assumptions on $\rho_i, \sigma_j$, $i = 1, 2$ the arc $(\rho_1 \cup \rho \cup \rho_2)$ intersects $\partial E(\sigma)$ at two points which, up to isotopy, cannot be eliminated. For, if
\[
|((\rho_1 \cup \rho \cup \rho_2) \cap \partial E(\sigma)| = 0
\]
the arc $\rho_1 \cup \rho \cup \rho_2$ would be contained in the pair of pants $\partial H(E_1, E_2, E(\sigma))$ with endpoints on $\partial E_1, \partial E_2$. As $\sigma_1 \cup \sigma \cup \sigma_2$ is also contained in $\partial H(E_1, E_2, E(\sigma))$ with endpoints on $\partial E_1, \partial E_2$, it would then follow that $\sigma_1 \cup \sigma \cup \sigma_2 \cong \rho_1 \cup \rho \cup \rho_2$. Since $\sigma_1 \cong \rho_1$ and $\sigma_2 \cong \rho_2$ the latter implies $\sigma \cong \rho$, a contradiction. This shows that
\[
|((\rho_1 \cup \rho \cup \rho_2) \cap \partial E(\sigma)| = 2
\]
which is equivalent to $|E(\rho) \cap E(\sigma)| = 2$.

For the opposite direction, assume $|E(\rho) \cap E(\sigma)| = 2$ which, by (3), means
\[
|((\rho_1 \cup \rho \cup \rho_2) \cap \partial E(\sigma)| = 2.
\]
It follows that $\rho_1 \cup \rho \cup \rho_2$ induces two arcs on the pair of pants $\partial H(E_1, E_2, E(\sigma))$ one with endpoints on $\partial E_1, \partial E(\sigma)$ and the other with endpoints on $\partial E_2, \partial E(\sigma)$. Each of these two arcs is, up to isotopy, disjoint from $\sigma_1 \cup \sigma \cup \sigma_2$ because the latter has endpoints on $\partial E_1, \partial E_2$ and all are contained in the pair of pants $\partial H(E_1, E_2, E(\sigma))$. It follows that
\[
(\sigma_1 \cup \sigma \cup \sigma_2) \cap (\rho_1 \cup \rho \cup \rho_2) = \emptyset
\]
which, in particular, implies $\sigma \cap \rho = \emptyset$ as desired. \hfill \Box

**Lemma D.** Let $\tau_1, \tau_2$ be two simple non-isotopic arcs in $\Sigma_{2,2}$ with endpoints on $\partial A, \partial B$. Then for the meridians $Z(\tau_1) = A \cup \tau_1 B$ and $Z(\tau_2) = A \cup \tau_2 B$ the following hold
\[
\tau_1 \cap \tau_2 = \emptyset \iff |Z(\tau_1) \cap Z(\tau_2)| = 2 \quad (21)
\]
and
\[
\tau_1 \cap \tau_2 = \emptyset \iff |\phi(Z(\tau_1)) \cap \phi(Z(\tau_2))| = 2 \quad (22)
\]
In particular,
\[
\tau_1 \cong \tau_2 \iff Z(\tau_1) \cong Z(\tau_2) \iff \phi(Z(\tau_1)) \cong \phi(Z(\tau_2)). \quad (23)
\]

**Proof.** In a similar manner using Lemma 8 it suffices to show
\[
\tau_1 \cap \tau_2 = \emptyset \iff |Z(\tau_1) \cap Z(\tau_2)| = 2. \quad (24)
\]
Assume $\tau_1 \cap \tau_2 = \emptyset$ and $\tau_1 \not\cong \tau_2$. Then, as in the proof of Lemma B, $\tau_2$ intersects $\partial Z(\tau_1)$ at an even number $2k$ of points. The possibility $k = 0$ is ruled out by the assumption $\tau_1 \not\cong \tau_2$. If $k > 1$ then $\tau_2 \cap N(\tau_1)$ would contain $k - 1$ arcs with endpoints on $\partial Z(\tau_1)$ not intersecting $\tau_1$. All these arcs would have to be parallel to $\partial Z(\tau_1)$ and therefore, by an isotopy, it would be possible to eliminate their intersection with $\partial Z(\tau_1)$. This shows that $|\tau_2 \cap \partial Z(\tau_1)| = 2 \cdot 1 = 2$ which is equivalent to $|Z(\tau_1) \cap Z(\tau_2)| = 2$.

For the converse, assume $|\tau_2 \cap \partial Z(\tau_1)| = 2$. Then $\tau_2 \cap N(\tau_1)$ consists of two arcs one with endpoints on $\partial Z(\tau_1), \partial A$ and the other with endpoints on $\partial Z(\tau_1), \partial B$. As $N(\tau_1)$ is a pair of pants, we may eliminate any intersection points these two arcs may have with $\tau_1$, making the intersection $\tau_1 \cap \tau_2$ empty. \hfill \Box
Proof of Proposition 10. CASE 1: We will first define $\phi (\tau)$ for $\tau$ being an isotopy class of simple arcs properly embedded in $\Sigma_{2,2} = \partial (H_2 (A, B))$ with one endpoint in $\partial A$ and the other in $\partial B$. The correspondence

$$\tau \xrightarrow{\psi} Z(\tau)$$

which sends the arc $\tau$ to the separating meridian $Z(\tau)$ is, by (23), well defined as a map on the set of isotopy classes of arcs with one endpoint in $\partial A$ and the other in $\partial B$. Clearly, the opposite is also true: if $X$ is a separating 2-meridian inside $H_2 (A, B)$ then $X$ splits $H_2 (A, B)$ into two components. The boundary of the genus 0 component is a pair of pants whose boundary consists of $\partial A$, $\partial B$ and $\partial X$. There is unique isotopy class of arcs from $\partial A$ to $\partial B$ (disjoint from $\partial X$) and for any arc $\tau$ in this class we have $\partial A \cup_{\tau} \partial B = \partial X$ and therefore $X = Z(\tau) = A \cup_{\tau} B$. In other words, $\psi$ is onto the set of isotopy classes of 2-m erotidions in $H_2 (A, B)$. Since $\phi$ is an automorphism of $S.M (H_g)$ we have that $\phi (Z(\tau)) \subset H_2 (A, B)$ and by Lemma 7(a) it is a 2-meridian. By the previous argument there is a unique isotopy class of arcs $\tau$ from $\partial A$ to $\partial B$ disjoint from $\partial \phi Z(\tau)$ so that $A \cup_{\tau} B = \phi (Z(\tau))$. Define $\bar{\phi} (\tau) := \tau$. Clearly, $\bar{\phi}$ is 1–1 because $\bar{\phi} (\tau) = (\psi^{-1} \circ \phi \circ \psi) (\tau)$ which (stated for later use) implies

$$\tau_1 \not\equiv \tau_2 \Leftrightarrow Z(\tau_1) \not\equiv Z(\tau_2) \Leftrightarrow \phi (Z(\tau_1)) \not\equiv \phi (Z(\tau_2)) \Leftrightarrow \bar{\phi} (\tau_1) \not\equiv \bar{\phi} (\tau_2).$$

(25)

Similarly, since every separating 2-meridian inside $H_2 (A, B)$ has a pre-image under $\phi$,

$$\bar{\phi} \text{ is onto}$$

(26)

Moreover, by equivalences (22) and (21) of Lemma D we have, for non-isotopic arcs $\tau_1, \tau_2$,

$$\tau_1 \cap \tau_2 = \emptyset \quad \Leftrightarrow \quad |\phi (Z(\tau_1)) \cap \phi (Z(\tau_2))| = 2 \quad \Leftrightarrow \quad \bar{\phi} (\tau_1) \cap \bar{\phi} (\tau_2) = \emptyset.$$

(27)

In other words

$$\tau_1 \cap \tau_2 = \emptyset \iff \bar{\phi} (\tau_1) \cap \bar{\phi} (\tau_2) = \emptyset.$$  

(28)

Observation 1: it is straightforward to extend the above property for any finite collection of pairwise disjoint simple arcs $\tau_1, \ldots, \tau_m$ from $\partial A$ to $\partial B$.

Observation 2: For every simple arc $\rho$ with both endpoints in $\partial B$ define $n_\rho$ to be the maximal number of pairwise disjoint and non-isotopic simple arcs from $\partial A$ to $\partial B$ with each being disjoint from $\rho$ (for example, if $\rho$ non-separating, then $n_\rho = 4$). If $\rho, \sigma$ are two arcs with endpoints in $\partial B$ with $\rho$ non-separating and $\sigma$ separating but not parallel to $\partial B$, then clearly $n_\rho > n_\sigma$.

CASE 2: In this case we will define $\bar{\phi} (\sigma)$ when $\sigma$ is a simple arc properly embedded in $\Sigma_{2,2} = \partial (H_2 (A, B))$ with both endpoints in $\partial B$ and which does not separate $\Sigma_{2,2}$. For this, we will use the whole collection of (isotopy classes of) simple arcs $\tau$ from $\partial A$ to $\partial B$ with the property $\tau \cap \sigma = \emptyset$.

We claim that there exists a unique non separating arc $\bar{\sigma}$ with endpoints in $\partial B$ satisfying the following property

for every arc $\tau$ from $\partial A$ to $\partial B$, $\tau \cap \sigma = \emptyset \Leftrightarrow \bar{\phi} (\tau) \cap \bar{\sigma} = \emptyset$.  

(28)
For the existence of such $\sigma$, pick disjoint separating meridians $E_1, E_2$ inside the component $H_{k'}(B)$ of $B$ (not containing $A$) such that $H(E_1, E_2, B)$ is a spotted handlebody. Extend $\sigma$ to an arc $\sigma'_{12} = \sigma_1 \cup \sigma \cup \sigma_2$ where $\sigma_1, \sigma_2$ are two disjoint arcs joining the endpoints of $\sigma$ with $\partial E_1$ and $\partial E_2$ respectively and let $E(\sigma) = E_1 \cup_{\sigma_{12}} E_2$. By Lemma 8(c,d), the image $\phi(E(\sigma))$ must be of the form

$$\phi(E_1) \cup_{\sigma_{12}'} \phi(E_2)$$

for some arc $\sigma'_{12}$ which intersects $\partial B$ at two points. The intersection $\sigma'_{12} \cap \partial H_2(A, B)$ is the desired arc $\sigma$ for which we write

$$E(\sigma) \equiv \phi(E(\sigma)) = \phi(E_1) \cup_{\sigma_{12}'} \phi(E_2)$$

To check property (28), let $\tau$ be an arc from $\partial A$ to $\partial B$ with the property $\tau \cap \sigma = \emptyset$. Then

$$\tau \cap \sigma = \emptyset \iff |\phi(E(\sigma)) \cap \phi(Z(\tau))| = 0 \text{ or } 2$$
$$\iff |E(\sigma) \cap Z(\tau)| = 0 \text{ or } 2$$
$$\iff \sigma \cap \phi^{-1}(\tau) = \emptyset$$ (29)

where the first equivalence is by (16) in Lemma B, the second is just an interpretation of $\phi(E(\sigma))$ and $\phi(Z(\tau))$ and the third is by (15) in Lemma B.

To see that such an arc $\sigma$ is unique, assume $\sigma$ is another such arc. Then by Lemma A(a) there exists an arc $\alpha$ with endpoints on $\partial A$ and $\partial B$ such that $\alpha \cap \sigma = \emptyset$ and $\alpha \cap \sigma \neq \emptyset$. Since $\sigma$ is onto (see property 26), there exists an arc $\tau$ with $\phi(\tau) = \alpha$ for which (29) implies that $\tau \cap \sigma = \emptyset$. Then,

$$\phi(\tau) \cap \sigma = \alpha \cap \sigma \neq \emptyset$$

which means that $\sigma$ does not satisfy property (28). This shows that $\sigma$ is unique with respect to property (28). Note also that by Observations 1 and 2, $\sigma$ must be non-separating. We may now define $\phi(\sigma) := \sigma$ where $\sigma$ is the above described non-separating arc uniquely determined by $\sigma$.

**CASE 3:** In this last case we will define $\phi(\sigma)$ when $\sigma$ is an arc properly embedded in $\Sigma_{2,2} = \partial (H_2(A, B))$ with both endpoints in $\partial B$ and which separates $\Sigma_{2,2}$.

As in Case 2, we will show that there exists a unique separating arc $\sigma$ with endpoints on $\partial B$ satisfying the property

$$\text{for every arc } \tau \text{ from } \partial A \text{ to } \partial B, \tau \cap \sigma = \emptyset \iff \phi(\tau) \cap \sigma = \emptyset.$$ (30)

Existence and uniqueness of such an arc $\sigma$ follows exactly as in Case 2 by using Lemma A(b) instead of A(a). By Observations 1 and 2, $\sigma$ must be separating. We may now define $\phi(\sigma) := \sigma$ where $\sigma$ is the above described separating arc uniquely determined by $\sigma$. In an identical way as in Cases 2 and 3, the image of an arc with endpoints in $\partial A$ is defined.

In order to show that $\phi$ is a well defined automorphism of the arc complex $A$ of $\Sigma_{2,2}$, it only remains to check that disjoint arcs are mapped to disjoint arcs.
For disjoint arcs with endpoints on $\partial A$, $\partial B$ this was verified by statement (27). For arcs $\sigma$, $\tau$ with $\partial \sigma \subset \partial A$ and $\tau$ having one endpoint in $\partial A$ and one in $\partial B$, the desired property follows from the if and only if statements given in (28) and (30).

For disjoint arcs $\sigma, \rho$ with $\partial \sigma \subset \partial A$ and $\partial \rho \subset \partial B$ we may find disjoint separating meridians $E_1^A, E_2^A$ contained in the component of $H_g \setminus A$ not containing $B$ such that $H \left( E_1^A, E_2^A, A \right)$ is a spotted 3-ball. The arc $\sigma$ extends uniquely, up to isotopy, to a simple arc $\sigma_{12}$ joining the boundaries of $E_1^A, E_2^A$ and we form the meridian $E(\rho) = E_1^A \cup_{\partial_1} E_2^A$. Similarly we construct $E(\rho) = E_1^B \cup_{\rho_1} E_2^B$ where $E_1^B, E_2^B$ are contained in the component of $H_g \setminus B$ not containing $A$ and $H \left( E_1^B, E_2^B, B \right)$ is a spotted 3-ball. Clearly $E(\sigma), E(\rho)$ are disjoint which implies that

$$\phi \left( E(\sigma) \right) = \phi \left( E_1^A \cup_{\sigma_{12}} E_2^A \right) \text{ and } \phi \left( E(\rho) \right) = \phi \left( E_1^B \cup_{\rho_{12}} E_2^B \right)$$

are disjoint and therefore

$$\sigma_{12} \cap \rho_{12} = \emptyset. \quad (31)$$

As explained in the definition of $\sigma$ above, $\sigma_{12}$ intersects $\partial A$ at two points (by Lemma 8(c,d)) giving rise to the image $\sigma$ of $\sigma$ which is uniquely determined by $\sigma_{12}$ and vice-versa. Similarly, $\rho_{12}$ intersects $\partial B$ at two points giving rise to the uniquely determined image $\rho$ of $\rho$. As $\sigma \subset \sigma_{12}$ and $\rho \subset \rho_{12}$, properly (31) implies that $\sigma \cap \rho = \emptyset$ as desired.

The last case $\partial \sigma \subset \partial B$ and $\partial \rho \subset \partial A$ follows similarly from statements (19), (18) of Lemma C since $\sigma, \rho$ can always be extended to arcs $\sigma_{12} = \sigma_1 \cup \sigma_2$ and $\rho_{12} = \rho_1 \cup \rho_2$ so that $\sigma_{12} \cap \rho_{12} = \emptyset$.

To check that $\phi$ is injective, by property (25), we only have to check injectivity for arcs $\sigma$ with $\partial \sigma \subset \partial A$ (the case $\partial \sigma \subset \partial B$ is treated identically). Observe that $\overline{\phi}$, by its definition, respects separating (resp. non-separating) arcs $\sigma$ with $\partial \sigma \subset \partial A$. If $\sigma, \rho$ are separating (resp. non-separating) non-isotopic arcs with $\sigma = \rho$, by Lemma A(b) (resp. A(a)), there exists an arc $\tau$ from $\partial A$ to $\partial B$ such that

$$\tau \cap \sigma = \emptyset \text{ and } \tau \cap \rho \neq \emptyset.$$ 

By statements (16), (15) of Lemma B we have

$$\tau \cap \sigma = \emptyset \by (16) \implies |\phi \left( E(\sigma) \right) \cap \phi \left( Z(\tau) \right)| = 0 \text{ or } 2 \by (15) \implies \tau \cap \sigma = \emptyset.$$

Similarly we have

$$\tau \cap \rho \neq \emptyset \by (16) \implies |\phi \left( E(\rho) \right) \cap \phi \left( Z(\tau) \right)| > 2 \by (15) \implies \tau \cap \rho \neq \emptyset$$

and therefore

$$\tau \cap \sigma = \emptyset \text{ and } \tau \cap \rho \neq \emptyset$$

which contradicts the assumption $\sigma = \rho$. It is shown in [9] that every injective simplicial map of the arc complex of a surface $S_{g,b}$ is induced by a homeomorphism of $S_{g,b}$ provided that $(g, b) \neq (0, 1), (0, 2), (0, 3), (1, 1)$. Thus, the above defined $\overline{\phi} : \mathcal{A} \to \mathcal{A}$ is geometric. This completes the proof of Proposition 10. ∎
Clearly, $\overline{\phi} : A \to A$ induces an automorphism on the curve complex $C(\Sigma_{2,2})$ denoted again by $\overline{\phi}$. We next show that $\overline{\phi}$ agrees with $\phi$ on the meridian curves in $\Sigma_{2,2} = \partial H_2(A, B)$.

**Lemma 12.** If $X$ is a separating meridian in $H_2(A, B)$ then $\partial (\phi(X)) = \overline{\phi}(\partial X)$.

**Proof.** If $X$ is a 2-meridian then, as explained in the proof of Proposition 10, Case 1, $X$ is of the form $Z(\tau) = A \cup \tau B$ for some (unique) arc $\tau$ with endpoints on $A$ and $B$ and the result follows from the definition of $\overline{\phi}$.

If $X$ is a 1-meridian, we may choose a 1-meridian $Z$ disjoint from $X$, mutually disjoint non-parallel arcs $\tau_1, \tau_2$ with endpoints on $\partial A$ and $\partial B$ not intersecting $\partial Z, \partial X$ and an arc $\tau_3$ with endpoints on $\partial A$ and $\partial B$ disjoint from $\partial X, \tau_1, \tau_2$ but with $\tau_3 \cap \partial Z \neq \emptyset$.

The meridian $Z(\tau_1) = A \cup \tau_1 B$ is disjoint from $X$ and clearly $\phi(Z(\tau_1)) = A \cup \tau_1 B$ is disjoint from $\phi(X)$. It follows, by Lemma 8, that

$$\overline{\tau}_1 \cap \partial \phi(X) = \emptyset \quad (32)$$

and similarly

$$\overline{\tau}_2 \cap \partial \phi(X) = \overline{\tau}_3 \cap \partial \phi(X) = \emptyset \quad (33)$$

$$\overline{\tau}_1 \cap \partial \phi(Z) = \overline{\tau}_2 \cap \partial \phi(Z) = \emptyset \quad (34)$$

and $\overline{\tau}_3 \cap \partial \phi(Z) \neq \emptyset$. (35)

Clearly, $\Sigma_{2,2} \setminus (\overline{\tau}_1 \cup \overline{\tau}_2)$ has two components, say $\Sigma^+$ and $\Sigma^-$, each of type $(1, 1)$. Without loss of generality, assume $\overline{\tau}_1 \subset \Sigma^-$. By (34), (35), $\partial \phi(Z)$ is also contained in $\Sigma^-$. Since $\partial \phi(X) \cap \partial \phi(Z) = \emptyset$, properties (32), (33) imply that $\partial \phi(X)$ is contained in $\Sigma^+$. By the choice of $X, Z, \tau_1, \tau_2, \tau_3$ the automorphism $\overline{\phi}$ sends $\partial X$ in $\Sigma^+$. Therefore, both curves $\overline{\phi}(\partial X)$ and $\partial \phi(X)$ are simple separating curves in the surface $\Sigma^+$. As there is only one simple separating curve in a surface of type $(1, 1)$, this completes the proof. \hfill $\Box$

**Proof of Proposition 9.** Notation. For a pair of simple closed curves $\sigma$ and $\tau$ in $\partial H_2(A, B)$ with $|\sigma \cap \tau| = 1$, denote by $N(\sigma, \tau)$ a tubular neighborhood of the union $\sigma \cup \tau$. We view $N(\sigma, \tau)$ as a genus 1 subsurface of $\Sigma_{2,2}$ with single boundary $\partial N(\sigma, \tau)$.

(a) Let $X$ be a 1-meridian in $H_2(A, B)$ and $D = \delta(X)$ the (unique) non-separating meridian in $H_1(X)$. As observed at the beginning of the present Section (after the statement of Proposition 9), we may assume that $\phi$ fixes $X$ and, by the above Lemma, $\overline{\phi}$ fixes $\partial X$ and we will show that $\overline{\phi}(\partial D) = \partial D$.

Assume, on the contrary, that $\overline{\phi}(\partial D) \neq \partial D$. We claim that we can pick a simple closed curve $\alpha \in \Sigma_{2,2}$ intersecting $\overline{\phi}(\partial D)$ once such that the boundary curve $\beta := \partial N(\alpha, \overline{\phi}(\partial D))$ is not a meridian in $H_2(A, B)$. To find such a curve, let $e_1, e_2$ be two generators of $\pi_1(H_{\delta})$ such that the free homotopy class of $e_1$ can be represented by a simple closed curve $\alpha_1$ in $\partial H_1(X)$ and $e_2$ can be represented by a simple closed curve $\alpha_2$ in $\partial H_1(X, A, B) = \partial H_{-1}(X) \cap \partial H_2(A, B)$. Let $\alpha$ be any simple closed curve in $\Sigma_{2,2}$ intersecting $\overline{\phi}(\partial D)$ once such that its free homotopy class is of the form $\alpha^k \alpha_2^\lambda$ with $k, \lambda \in \mathbb{Z}$ and $\lambda \neq 0$. As $\overline{\phi}(\partial D) \subset H_1(X)$,
its free homotopy class is of the form $\alpha^\mu_1, \mu \in \mathbb{Z}$ with $\mu \not= 0$ because we assumed $\overline{\phi} (\partial D) \not= \partial D$. The commutator $[\alpha^\kappa_1 \alpha^\lambda_2, \alpha^\mu_1]$ gives the free homotopy class of the boundary curve $\beta$. Since the above commutator is never trivial when $\lambda$ and $\mu$ are both $\not= 0$, it follows that $\beta$ is not a meridian boundary.

By Proposition 10, $\overline{\phi}$ is induced by a homeomorphism, say $\overline{f}$, of $\Sigma_{2,2} = \partial H_2 (A, B)$ which, by Lemma 12, sends separating meridian curves to separating meridian curves. Since $\overline{f}$ is a homeomorphism of $\Sigma_{2,2}$ it follows that the inverse image $(\overline{f})^{-1} (N(\alpha, \overline{\phi} (\partial D)))$ is a tubular neighborhood of the curves $(\overline{f})^{-1} (\alpha)$ and $\partial D$ whose boundary is $(\overline{\phi})^{-1} (\beta)$. Since $D$ is a (non-separating) meridian and $\partial D$ intersects $(\overline{f})^{-1} (\alpha)$ once, it follows that the curve $(\overline{\phi})^{-1} (\beta)$ is homotopically trivial, that is, it bounds a separating meridian. This contradicts Lemma 12 because $\beta$, by construction, is not homotopically trivial and $(\overline{\phi})^{-1} (\beta)$ is.

(c) This follows directly from part (a) because, given a separating 2-meridian $Z$ we may find disjoint separating meridians $A, B \subset H_{g-2} (Z)$ such that $H(A, B, Z)$ is a 3-ball. Then, $X, Y \subset H_2 (A, B)$.

(b) If $g_{A'} \geq 3$, this follows directly from part (a) because we may find a meridian $A$ such that $H_2 (A, B') \supset H_1 (A', B')$ and (a) applies to $H_2 (A, B')$. If $g_{A'} = 2$, then we may find a meridian $B$ such that $H_2 (A', B) \supset H_1 (A', B')$ and (a) applies to $H_2 (A', B)$. If $g_{A'} = 1$ then $B'$ is a 2-meridian and the result follows from part (c).

3. Extension to the complex of meridians

In order to extend an automorphism $\phi : SM(H_g) \rightarrow SM(H_g)$ to an automorphism $\mathcal{M}(H_g) \rightarrow \mathcal{M}(H_g)$ we will prove the following generalization of Proposition 9:

**Theorem 13.** Let $X, Y$ be separating 1-meridians in $H_g$ and $\phi$ an automorphism of $SM(H_g)$. Then

$$\delta(X) = \delta(Y) \Rightarrow \delta(\phi(X)) = \delta(\phi(Y))$$

provided $g \geq 6$.

Our first task is to assert that Proposition 9 can be applied in the case where the 1-meridians $X, Y$ do not intersect a (common) cut system in $H_g$.

**Lemma 14.** Let $X, Y$ be separating 1-meridians such that $\delta(X) = \Delta = \delta(Y)$. Assume both $X$ and $Y$ are disjoint from a cut system $\mathcal{C} = \{\Delta, C_2, \ldots, C_g\}$ in $H_g$. Then there exists a sequence of separating 1-meridians $X_1 = X, X_2, \ldots, X_n = Y$ such that $\delta(X_i) = \Delta$ for all $i = 1, \ldots, n$ and $X_i, X_{i+1}$ lie in an embedded handlebody of genus $\leq 2$ with at most two (meridian) spots.

**Proof.** For the meridian $X$ (resp. $Y$) there exists a simple closed curve $\sigma$ (resp. $\tau$) intersecting $\partial \Delta$ once such that $\partial X$ (resp $\partial Y$) is the boundary of a regular neighborhood of $\partial \Delta \cup \sigma$ (resp. $\partial \Delta \cup \tau$).
By cutting the handlebody along the meridians of the cut system, we obtain a 3-ball $B$ with $2g$ spots. Denote by $\Delta^+, \Delta^-$ the spots coming from $\Delta$. Then the meridians $X, Y$ in $B$ are given by

$$X = \Delta^+ \cup_{\sigma} \Delta^- \quad \text{and} \quad Y = \Delta^+ \cup_{\tau} \Delta^-$$

The boundary $\partial B$ of $B$ is a genus 0 surface with $2g$ boundary components containing the arcs $\sigma, \tau$ which have endpoints on $\partial \Delta^+, \partial \Delta^-.$

Suppose that the arcs $\sigma, \tau$ intersect transversely and minimally at $k$ points, $k > 0.$ The intersection $\sigma \cap \tau$ splits $\tau$ into $k+1$ subarcs. Exactly two of them intersect $\sigma$ at a single point (each one of the rest intersects $\sigma$ at two points). Moreover, one of them, say $\tau^+$, has one endpoint in $\partial \Delta^+$ and the other in $\sigma \cap \tau.$ We perform surgery on the intersection point $\sigma \cap \tau^+$ as follows: the point $\sigma \cap \tau^+$ cuts $\sigma$ into two subarcs and one of them, say $\sigma^-$, has one endpoint in $\partial \Delta^-.$ Set $\sigma_2 = \tau^+ \cup \sigma^-.$ Clearly, $\sigma_2$ has endpoints on $\partial \Delta^+, \partial \Delta^-$ and, as the interior of $\tau^+$ is disjoint from $\sigma$, it follows that $\sigma_2$ is disjoint from $\sigma \equiv \sigma_1.$ Moreover, $|\sigma_2 \cap \tau| < k.$ This surgery procedure is the basic tool for proving that the arc complex of a surface is connected (in fact, contractible). For details see for example [2, Theorem 1.3].

By repeating the above procedure we obtain a sequence of arcs

$$\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_n = \tau$$

with endpoints on $\partial \Delta^+, \partial \Delta^-$ such that for all $i, \sigma_i, \sigma_{i+1}$ are disjoint and each $\sigma_i$ is disjoint from $\left\{C^+_2, C^-_2, \ldots, C^+_g, C^-_g\right\}.$ Set $X_i = \Delta^+ \cup_{\sigma_i} \Delta^-.$

Clearly, $\delta(X_i) = \Delta$ for all $i = 1, \ldots, n$ and it remains to show that $X_i, X_{i+1}$ lie in an embedded handlebody of genus $\leq 2$ having at most two (meridian) spots.

Since $\sigma_i, \sigma_{i+1}$ are disjoint, the intersection $H_{g-1}(X_i) \cap \sigma_{i+1}$ is a single subarc $\overline{\sigma_{i+1}}$ of $\sigma_{i+1}$ with endpoints in $\partial X_i.$ As $\sigma_{i+1}$ is disjoint from $\left\{C^+_2, C^-_2, \ldots, C^+_g, C^-_g\right\},$ the arc $\overline{\sigma_{i+1}}$ is an arc in the boundary of the 3-ball $H_{g-1}(X_i) \setminus \left\{C_2, \ldots, C_g\right\}.$ Let $N \subset \partial H_{g-1}(X_i)$ be a regular neighborhood of $\partial X_i \cup \overline{\sigma_{i+1}}.$ Since $N$ is disjoint from $\left\{C_2, \ldots, C_g\right\}$ the three boundary curves of $\partial N$ are contained in the boundary of the 3-ball $H_{g-1}(X_i) \setminus \left\{C_2, \ldots, C_g\right\},$ and hence they bound meridians. One of them is $X_i$ and denote by $D, E$ the other two meridians. If both $D, E$ are separating then $H(E, D)$ is a genus 1 handlebody with two spots containing $X_i$ and $X_{i+1}$ as desired. If both $D, E$ are non-separating then they form a bounding pair, that is, $D \cup E$ separates $H_g.$ Using a simple arc $\tau$ in the boundary of $H_{g-1}(X_i) \setminus \left\{C_2, \ldots, C_g\right\}$ connecting $\partial D, \partial E$ and disjoint from the interior of $N$ we obtain the separating meridian $D \cup_{\tau} E$ so that $H_2(D \cup_{\tau} E)$ is a genus 2 handlebody with one spot containing both $X_i$ and $X_{i+1}.$ □

We briefly recall the notion of the cut system complex and a result due to B. Wajnryb (see [17]) which states that the complex of cut systems is connected.

Let $H_g$ be a handlebody of genus $g$ with a finite number of spots on its boundary. The complex of cut systems is a 2-dimensional complex with vertices being cut systems of $H_g$ and two cut systems are connected by an edge if they have $g - 1$ meridians in common and the other two are disjoint. The cut system complex $CS(H_g)$ of $H_g$ is defined to be the 2-dimensional flag complex determined by the
above mentioned vertices and edges, that is, if \( \{ \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 \} \) is a set of vertices with the property the edge \( (\mathcal{C}_i, \mathcal{C}_j) \) exists for all \( i \neq j, 0 \leq i, j \leq 2 \), then \( \{ \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2 \} \) is a 2-simplex. The following is shown in [17]:

**Theorem B.** The cut system complex \( \mathcal{CS}(H_g) \) is connected and simply connected.

Our next task is to assert that Proposition 9 can be applied in the case where the 1-meridian \( X \) (resp. \( Y \)) does not intersect a cut system \( \mathcal{C}_X \) (resp. \( \mathcal{C}_Y \)) in \( H_g \) and \( \mathcal{C}_X, \mathcal{C}_Y \) are connected by an edge in the cut system complex of \( H_g \).

**Lemma 15.** Let \( X, Y \) be separating 1-meridians with \( \delta(X) = \Delta = \delta(Y) \) and

\[
\mathcal{C}_X = \{ \Delta, C_X, C_3, \ldots, C_8 \}, \quad \mathcal{C}_Y = \{ \Delta, C_Y, C_3, \ldots, C_8 \}
\]

cut systems connected by an edge in the cut system complex \( \mathcal{CS}(H_g) \) such that \( X \subset H_g \setminus \mathcal{C}_X \) and \( Y \subset H_g \setminus \mathcal{C}_Y \). Then there exists a sequence of separating 1-meridians \( X_1 = X, X_2, \ldots, X_n = Y \) such that \( \delta(X_i) = \Delta \) for all \( i = 1, \ldots, n \) and \( X_i, X_{i+1} \) lie in an embedded handlebody of genus \( \leq 2 \) with at most two (meridian) spots.

**Proof.** Cutting \( H_g \) along \( \mathcal{C}_X \cup \{C_Y\} \) we obtain two components (3-balls) with a total of \( 2g + 2 \) spots denoted by \( B^+, B^- \). Observe that cutting \( H_g \) along \( \mathcal{C}_Y \) and then along \( C_X \) has the same result as cutting along \( \mathcal{C}_X \) and then along \( C_Y \), namely, the balls \( B^+ \) and \( B^- \). In other words,

\[
(H_g \setminus \mathcal{C}_Y) \setminus C_X = B^+ \perp B^- = (H_g \setminus \mathcal{C}_X) \setminus C_Y. \tag{36}
\]

Case I: the 2 spots corresponding to \( \Delta \) lie on the same component. In this case we may find a simple closed curve \( \sigma \subset \partial H_g \) intersecting \( \partial \Delta \) at a single point and, in addition,

\[
\sigma \cap (C_X \cup C_Y \cup C_3 \cup \ldots \cup C_8) = \emptyset.
\]

Then the separating meridian \( Z_{X,Y} = \Delta \cup \sigma \Delta \) clearly has \( \delta(Z_{X,Y}) = \Delta \) and satisfies

\[
Z_{X,Y} \subset H_g \setminus \mathcal{C}_X \quad \text{and} \quad Z_{X,Y} \subset H_g \setminus \mathcal{C}_Y.
\]

Lemma 14, provides the desired sequence of 1-meridians from \( X \) to \( Z_{X,Y} \) as well as a sequence of 1-meridians from \( Y \) to \( Z_{X,Y} \). This completes the proof in the case the 2 spots corresponding to \( \Delta \) lie on the same component of \( H_g \setminus (\mathcal{C}_X \cup \{C_Y\}) \).

Case II: the spots \( \Delta^+ \subset B^+ \) and \( \Delta^- \subset B^- \).

The left hand side equality in (36) asserts that one of the spots \( C_X^+, C_X^- \) corresponding to \( C_X \) lies on \( B^+ \) and the other on \( B^- \) and similarly for the spots \( C_Y^+, C_Y^- \). Therefore, in this case \( B^+ \) has spots \( \Delta^+, C_X^+ \) and \( C_Y^+ \) corresponding to \( \Delta, C_X \) and \( C_Y \) respectively and similarly for \( B^- \).

Pick disjoint arcs \( \sigma_X^+ \) and \( \sigma_Y^+ \) in \( \partial B^+ \) with endpoints on \( \partial \Delta^+, \partial C_X^+ \) and \( \partial \Delta^+, \partial C_Y^+ \) respectively. Pick disjoint arcs \( \sigma_X^- \) and \( \sigma_Y^- \) in \( \partial B^- \) so that \( \sigma_X^- \) (resp. \( \sigma_Y^- \)) has the same endpoints with \( \sigma_X^+ \) (resp. \( \sigma_Y^+ \)). Then \( \sigma_X = \sigma_X^+ \cup \sigma_X^- \) and \( \sigma_Y = \sigma_Y^+ \cup \sigma_Y^- \) are simple closed curves in

\[
H_g \setminus (C_Y \cup C_3 \cup \ldots \cup C_g), \quad \text{and} \quad H_g \setminus (C_X \cup C_3 \cup \ldots \cup C_g)
\]
respectively, each intersecting $\partial \Delta$ once. Set $X_0, Y_0$ to be the 1-meridians

$$X_0 = \Delta \cup_{\sigma_Y} \Delta \quad \text{and} \quad Y_0 = \Delta \cup_{\sigma_X} \Delta$$

which clearly satisfy

$$X_0 \subset H_g \setminus \mathcal{C}_X \quad \text{and} \quad Y_0 \subset H_g \setminus \mathcal{C}_Y.$$ 

By Lemma 14, the conclusion of the Lemma holds for the pair $X, X_0$ as they are disjoint from $\mathcal{C}_X$ and similarly for the pair $Y, Y_0$. To complete the proof we will show that the pair $X_0, Y_0$ is contained in an embedded handlebody of genus 2 with one meridian spot.

Set $D^+$ and $D^-$ to be the meridians $\Delta^+ \cup_{\sigma_X} C_X^+$ and $\Delta^- \cup_{\sigma_X} C_X^-$ respectively. Viewed as meridian in $B^+$, $D^+$ is separating and so is $D^-$ in $B^-$. One of the two components of $B^+ \setminus D^+$ is a 3-ball, denoted by $B_1^+$ with spots $\Delta^+, C_X^+, D^+$ and similarly $D^-$ cuts off a 3-ball $B_1^-$ with spots $\Delta^-, C_X^-, D^-$. It follows that one of the components of $H_g \setminus (D^+ \cup D^-)$ is $B_1^+ \cup B_1^-$ which is a handlebody of genus 1 with spots $D^+, D^-$. In particular, $D^+, D^-$ is a bounding pair.

The curve $\sigma_Y$ intersects $\partial D^+ \cup \partial D^-$ in two points giving rise to two subarcs of $\sigma_Y$. Let $\rho$ be the subarc of $\sigma_Y$ disjoint from $\partial \Delta$. The meridian $Z = D^+ \cup_\rho D^-$ is a separating 2-meridian because $B_1^+ \cup B_1^-$ is a genus 1 handlebody with two spots $D^+, D^-$ which form a bounding pair. Moreover, $H_2(Z)$ contains $\sigma_X, \sigma_Y$ and $\Delta$ and therefore it contains the pair $X_0, Y_0$ and the proof of the Lemma is completed.

$\square$

**Proof of Theorem 13.** The proof follows immediately from Proposition 9, Lemma 14, Lemma 15 and Theorem B.

We may now extend an arbitrary automorphism $\phi : \mathcal{SM}(H_g) \to \mathcal{SM}(H_g)$ to an automorphism $\phi_M : \mathcal{M}(H_g) \to \mathcal{M}(H_g)$ on the whole complex of meridians $\mathcal{M}(H_g)$.

**Definition 16.** Let $D \in \mathcal{M}(H_g)$ be a non-separating meridian. Pick any separating 1-meridian $Z$ with $\delta(Z) = D$. Define

$$\phi_M : \mathcal{M}(H_g) \to \mathcal{M}(H_g)$$

by $\phi_M(D) = \delta(\phi(Z))$. By Theorem 13, $\phi_M$ is, as a map, well defined.

Surjectivity of $\phi$ clearly implies surjectivity of $\phi_M$. The inverse automorphism $\phi^{-1}$ can be extended as above to an automorphism $(\phi^{-1})_M : \mathcal{M}(H_g) \to \mathcal{M}(H_g)$ which satisfies $(\phi^{-1})_M \circ \phi_M = \text{Id}_{\mathcal{M}(H_g)}$. This shows that the map $\phi_M$ is injective and surjective. In the sequel we will suppress the lower index in $\phi_M$.

**Proposition 17.** The map $\phi : \mathcal{M}(H_g) \to \mathcal{M}(H_g)$ is the unique complex automorphism of $\mathcal{M}(H_g)$ extending the given automorphism of $\mathcal{SM}(H_g)$. 
Proof. Let \( D, E \) be any two meridians. We must show
\[
D \cap E = \emptyset \iff \phi(D) \cap \phi(E) = \emptyset. \tag{37}
\]
We will only deal with one direction because, as \( \phi \) has an inverse, the converse follows in an identical way. Clearly, if both \( D, E \) are separating we have nothing to show. We need to examine two cases:

Case I: \( D \) is non-separating and \( E \) separating. If \( E \) is a separating 1-meridian with \( \delta(E) = D \) then, by definition of \( \phi : \mathcal{M}(H_g) \to \mathcal{M}(H_g) \) we have \( \phi(D) = \delta(\phi(E)) \) and, clearly, \( \phi(D) \cap \phi(E) = \emptyset \). If either \( \delta(E) \neq D \) or, \( E \) is a separating \( k \)-meridian with \( 2 \leq k \leq g-1 \) we may find a 1-meridian \( X \) with \( \delta(X) = D \) and \( X \cap E = \emptyset \). Then \( \phi(X) \cap \phi(E) = \emptyset \) which implies that \( \phi(D) \cap \phi(E) = \emptyset \).

Case II: \( D \) and \( E \) are both non-separating. We may find two disjoint simple closed curves \( \sigma, \tau \) in \( \partial H_g \) such that \( \sigma \) (resp. \( \tau \)) intersects \( D \) (resp. \( E \)) and does not intersect \( E \) (resp. \( D \)). Then the separating 1-meridians \( Z(\sigma) = D \cup \sigma \) and \( Z(\tau) = E \cup \tau \) are disjoint and so are \( \phi(Z(\sigma)) \), \( \phi(Z(\tau)) \). In particular,
\[
\delta(\phi(Z(\sigma))) \cap \delta(\phi(Z(\tau))) = \emptyset
\]
which, by definition of \( \phi \), implies that \( \phi(D) \cap \phi(E) = \emptyset \).

For uniqueness, let \( \phi' : \mathcal{M}(H_g) \to \mathcal{M}(H_g) \) be a complex automorphism such that \( \phi' = \phi \) on \( \mathcal{S} \mathcal{M}(H_g) \) and assume \( \phi'(D) \neq \phi(D) \) for some non-separating meridian \( D \). Choose a simple closed curve \( \alpha \) with \( |\alpha \cap \phi(D)| = 1 \). Then, for the separating 1-meridian \( Z = \phi(D) \cup_\alpha \phi(D) \) we clearly have \( \delta(Z) = \phi(D) \) and \( Z \cap \phi'(D) \neq \emptyset \). The latter implies that the meridians \( (\phi')^{-1}(Z) = (\phi^{-1})^{-1}(Z) \) and \( (\phi')^{-1}(\phi'(D)) = D \). This is a contradiction because, by definition of \( \phi \), \( \delta(\phi^{-1}(Z)) = D \). \( \square \)

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Declarations

Availability of data All data supporting the conclusions of this article are included within the article.

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