Semiopen and semiclosed sets in fuzzy soft topological spaces

J. Mahanta

Department of Mathematics
NIT, Silchar
Assam, 788 010, India.
mahanta.juthika@gmail.com

Abstract

In this paper, we introduce semiopen and semiclosed fuzzy soft sets in fuzzy soft topological spaces. Various properties of these sets are studied along with some characterizations. Further, we generalize the structures like interior and closure via semiopen and semiclosed fuzzy soft sets and study their various properties.

1. Introduction

Many Mathematical concepts can be represented by the notion of set theory, which dichotomize the situation into the conditions: either “yes” or “no”. Till 1965, Mathematicians were concerned only about “well-defined” things, and smartly avoided any other possibility which are more realistic in nature. For instance the set of tall persons in a room, the set of hot days in a year etc. In the year 1965, Prof. L.A. Zadeh [8] introduced fuzzy set to accommodate real life situations by giving partial membership to each element of a situation under consideration.

Keeping in view that fuzzy set theory lacks the parametrization tool, Molodtsov [4] introduced soft set as another mathematical framework to deal with real life situations. Then comes another generalization of sets, namely fuzzy soft set, which is a hybridization of fuzzy sets and soft sets, in which soft set is defined over fuzzy set. Similar generalization have also spread to topological space. The notion of topological space is defined on crisp sets and hence is affected by different generalizations of crisp sets like fuzzy sets and soft sets. C.L. Chang [3] introduced fuzzy topological space in 1968 and subsequently Çağman et al. [10] and Shabir et al. [9] introduced soft topological space independently in 2011. In the same year B. Tanay et al. [2] introduced fuzzy soft topological spaces and studied neighborhood and interior of a fuzzy soft set and then used these to characterize fuzzy soft open sets. Recently Roy et al. [12] have obtained different conditions for a subfamily of fuzzy soft sets to be a fuzzy soft basis or fuzzy soft subbasis. Levine [11] introduced the concepts of semi-open sets and semicontinuous mappings in topological spaces and were applied in the field of Digital Topology [1]. Azad [7] initiated the study of these sets in fuzzy setting and in [5], authors carried the study in soft topological spaces.

This paper aims to generalize open and closed sets in fuzzy soft topological spaces as semiopen and semiclosed fuzzy soft sets. Then various set theoretic properties related to these generalized sets are to be studied. Further, it is intended to generalize the structures like interior and closure via semiopen and semiclosed fuzzy soft sets and study their properties.

It is presumed that the basic concepts like fuzzy sets, soft sets and fuzzy soft sets etc. are known to the readers. However below are some definitions and results required in the sequel.

Definition 1.1. Let \( f_E \) be a fuzzy soft set, \( \mathcal{FS}(f_E) \) be the set of all fuzzy soft subsets of \( f_E \), \( \tau \) be a subfamily of \( \mathcal{FS}(f_E) \) and \( A, B, C \subseteq E \). Then \( \tau \) is called a fuzzy soft topology on \( f_E \) if the following conditions are satisfied

(i) \( \emptyset, f_E \) belongs to \( \tau \);

Preprint submitted to Elsevier

March 17, 2014
Remark 2.5. Every clopen set is both semiclosed and semiopen.

Remark 2.4. Theorem 2.6. Every open (closed) fuzzy soft set is a semiopen (semiclosed) fuzzy soft set but not conversely. Fuzzy soft topological space (\( A \)) are called fuzzy soft closed sets. The closure and interior are generalized via semiopen and semiclosed fuzzy soft sets. The concepts of closure and interior are generalized via semiopen and semiclosed fuzzy soft sets and study various set theoretic properties related to these structures. The concepts of closure and interior are generalized via semiopen and semiclosed fuzzy soft sets.

Definition 2.1. In a fuzzy soft topological space \((\mathcal{F}_E, \tau)\), a fuzzy soft set

i. \( g_A \) is said to be semiopen fuzzy soft set if \( \exists \) an open fuzzy soft set \( h_A \) such that \( h_A \supseteq g_A \subset \text{cl}(h_A) \);

ii. \( p_A \) is said to be semiclosed fuzzy soft set if \( \exists \) a closed fuzzy soft set \( k_A \) such that \( \text{int}(k_A) \subset p_A \subset k_A. \)

Example 2.2. Let \( U = \{h^1, h^2, h^3\} \) and \( E = \{e_1, e_2, e_3\} \). Consider a fuzzy soft set \( f_E = \{(e_1, [h^1, h^2, h^3]), (e_2, [h^2, h^1, h^3]), (e_3, [h^3, h^1, h^2])\} \) defined on \( U \). Then the subfamily \( \tau = \{f_E, f_E, \{e_1, [h^1_0, h^2_0, h^3_0]\}, \{e_1, [h^1_1, h^2_0, h^3_0]\}, \{e_2, [h^1_0, h^2_1, h^3_0]\}, \{e_2, [h^1_0, h^2_0, h^3_0]\}, \{e_3, [h^1_0, h^2_0, h^3_0]\}, \{e_3, [h^1_0, h^2_0, h^3_0]\}, \{e_3, [h^1_0, h^2_0, h^3_0]\}\}

is a fuzzy soft topology on \( f_E \) and \((f_E, \tau)\) is a fuzzy soft topological space.

Here \( g_E = \{(e_1, [h^1, h^2, h^3]), (e_2, [h^2, h^1, h^3]), (e_3, [h^3, h^2, h^1])\} \) is a semiopen fuzzy soft set.

Remark 2.3. Every open (closed) fuzzy soft set is a semiopen (semiclosed) fuzzy soft set but not conversely.

Remark 2.4. \( \Phi_E \) and \( f_E \) are always semiclosed and semiopen.

Remark 2.5. Every clopen set is both semiclosed and semiopen.

From now onwards, we shall denote the family of all semiopen fuzzy soft sets (semiclosed fuzzy soft sets) of a fuzzy soft topological space \((\mathcal{F}_E, \tau)\) by \( S\text{OSFS}(f_E) \) \((S\text{CFSS}(f_E))\).
Definition 2.13. It is not true that the intersection (union) of any two semiopen (semiclosed) fuzzy soft sets need not be semiopen (semiclosed) fuzzy soft set. Even the intersection (union) of a semiopen (semiclosed) fuzzy soft set with a topological space the intersection of a semiopen set with an open set is a semiopen set \([13]\) but it doesn’t hold in fuzzy setting \([7]\). Further it should be noted that the closure of a fuzzy open set, is a fuzzy semiopen set and the interior of a fuzzy closed set is a fuzzy semiclosed set.

Remark 2.7. Arbitrary intersection of semiclosed fuzzy soft sets is a semiclosed fuzzy soft set.

Theorem 2.8. If a semiopen fuzzy soft set \(g_A\) is such that \(g_A \subseteq k_A \subseteq cl(g_A)\), then \(k_A\) is also semiopen.

**Proof.** As \(g_A\) is semiopen fuzzy soft set \(\exists\) an open fuzzy soft set \(h_A\) such that \(h_A \subseteq g_A \subseteq cl(h_A)\); then by hypothesis \(h_A \subseteq k_A\) and \(cl(g_A) \subseteq cl(h_A)\) \(\Rightarrow\) \(k_A \subseteq cl(g_A)\), hence \(k_A\) is a semiopen fuzzy soft set.

Remark 2.9. It is not true that the intersection (union) of any two semiopen (semiclosed) fuzzy soft sets need not be semiopen (semiclosed) fuzzy soft set. Even the intersection (union) of a semiopen (semiclosed) fuzzy soft set with a fuzzy soft open (closed) set may fail to be a semiopen (semiclosed) fuzzy soft set. It should be noted that in general topological space the intersection of a semiopen set with an open set is a semiopen set \([13]\) but it doesn’t hold in fuzzy setting \([3]\). Further it should be noted that the closure of a fuzzy open set, is a fuzzy semiopen set and the interior of a fuzzy closed set is a fuzzy semiclosed set.

Theorem 2.10. If a semiclosed fuzzy soft set \(m_A\) is such that \(int(m_A) \subseteq k_A \subseteq m_A\), then \(k_A\) is also semiclosed.

Following two theorems characterize semiopen and semiclosed fuzzy soft sets.

Theorem 2.11. A fuzzy soft set \(g_A \in SOFSS(f_E) \iff\) for every fuzzy soft point \(e_{g_A} \in g_A\), \(\exists\) a fuzzy soft set \(h_A \in SOFSS(f_E)\) such that \(e_{g_A} \in h_A \subseteq g_A\).

**Proof.** Take \(h_A = g_A\), this shows that the condition is necessary.

For sufficiency, we have \(g_A = \bigcup_{e_{g_A} \in g_A} h_A \subseteq g_A\).

Theorem 2.12. If \(g_A\) is any fuzzy soft set in a fuzzy soft topological space \((f_E, \tau)\) then following are equivalent:

i. \(g_A\) is semiopen fuzzy soft set;

ii. \(int(cl(g_A)) \subseteq g_A\);

iii. \(cl(int(g'_A)) \supseteq g'_A\);

iv. \(g'_A\) is semiopen fuzzy soft set;

**Proof.** (i) \(\Rightarrow\) (ii) If \(g_A\) is semiopen fuzzy soft set, then \(\exists\) closed fuzzy soft set \(h_A\) such that \(int(h_A) \subseteq g_A \supseteq h_A \Rightarrow\) \(int(h_A) \subseteq g_A \subseteq cl(g_A) \supseteq h_A\). By the property of interior we then have \(int(cl(g_A)) \subseteq int(h_A)\), hence \(int(cl(g_A)) \subseteq g_A\);

(ii) \(\Rightarrow\) (iii) \(\Rightarrow\) \(int(cl(g_A)) \subseteq g_A \Rightarrow g'_A \supseteq cl(int(g'_A)) \Rightarrow cl(int(g'_A)) \supseteq g'_A\);

(iii) \(\Rightarrow\) (iv) As \(g'_A\) is semiopen \(\exists\) an open fuzzy soft set \(h_A\) such that \(h_A \subseteq g'_A \Rightarrow h'_A\) is a closed fuzzy soft set such that \(g'_A \subseteq h'_A\) and \(g'_A \supseteq cl(h_A)\), hence \(g'_A\) is semiclosed fuzzy soft set.

Definition 2.13. Let \((f_E, \tau)\) be a fuzzy soft topological space and \(g_A\) be a fuzzy soft set over \(U\).

i. The fuzzy soft semi closure of \(g_A\) is a fuzzy soft set \(fssclg_A = \bigcap\{s_A \mid g_A \subseteq s_A\} \in SCFSS(f_E)\);

ii. The fuzzy soft semi interior of \(g_A\) is a fuzzy soft set \(fssintg_A = \bigcup\{s_A \mid s_A \subseteq g_A\} \in SOFSS(f_E)\).

\(fssclg_A\) is the smallest semiclosed fuzzy soft set containing \(g_A\) and \(fssintg_A\) is the largest semiopen fuzzy soft set contained in \(g_A\).
Theorem 2.14. Let \((f_E, \tau)\) be a fuzzy soft topological space and \(g_A\) and \(k_A\) be two fuzzy soft sets over \(U\), then

i. \(g_A \in SCFSS(f_E) \iff g_A = f sscl(g_A)\);

ii. \(g_A \in SOFSS(f_E) \iff g_A = f ssint(g_A)\);

iii. \((f sscl(g_A))^c = f ssint(g_A)^c\);

iv. \((f ssint(g_A))^c = f sscl(g_A)^c\);

v. \(g_A \subseteq k_A \Rightarrow f ssint(g_A) \subseteq f ssint(k_A)\);

vi. \(g_A \subseteq k_A \Rightarrow f sscl(g_A) \subseteq f sscl(k_A)\);

vii. \(f sscl(\Phi_E) = \Phi_E\) and \(f sscl f_E = f_E\);

viii. \(f ssint(\Phi_E) = \Phi_E\) and \(f ssint f_E = f_E\);

ix. \(f sscl(g_A \cup k_A) = f sscl(g_A) \cup f sscl(k_A)\);

x. \(f ssint(g_A \cap k_A) = f ssint(g_A) \cap f ssint(k_A)\);

xi. \(f sscl(g_A \cap k_A) \subseteq f sscl(g_A) \cap f sscl(k_A)\);

xii. \(f ssint(g_A \cup k_A) \subseteq f ssint(g_A) \cup f ssint(k_A)\);

xiii. \(f sscl(f sscl(g_A)) = f sscl(g_A)\);

xiv. \(f ssint(f ssint(g_A)) = f ssint(g_A)\).

Proof. Let \(g_A\) and \(k_A\) be two fuzzy soft sets over \(U\).

i. Let \(g_A\) be a semiclosed fuzzy soft set. Then it is the smallest semiclosed set containing itself and hence \(g_A = f sscl(g_A)\).

On the other hand, let \(g_A = f sscl(g_A)\) and \(f sscl(g_A) \in SCFSS(f_E) \Rightarrow g_A \in SCFSS(f_E)\).

ii. Similar to (i).

iii. \(f sscl(g_A)^c = \bigcap \{s_A \mid g_A \subseteq s_A \text{ and } s_A \in SCFSS(f_E)\}\)^c

iv. Similar to (iii).

v. Follows from definition.

vi. Follows from definition.

vii. Since \(\Phi_E\) and \(f_E\) are semiclosed fuzzy soft sets so \(f sscl(\Phi_E) = \Phi_E\) and \(f sscl f_E = f_E\).

viii. Since \(\Phi_E\) and \(f_E\) are semiopen fuzzy soft sets so \(f ssint(\Phi_E) = \Phi_E\) and \(f ssint f_E = f_E\).
ix. We have $g_A \subset fsscl(g_A \cup k_A)$ and $k_A \subset fsscl(g_A \cup k_A)$. Then by (vi), $fssclg_A \subset fsscl(g_A \cup k_A)$ and $fssclk_A \subset fsscl(g_A \cup k_A) \Rightarrow fssclk_A \subset fsscl(g_A \cup k_A)$.

Now, $fssclg_A, fssclk_A \in SCFS S(f \in) \Rightarrow fssclg_A \subset fssclk_A \in SCFS S(f \in)$.

Then $g_A \subset fssclg_A$ and $k_A \subset fssclk_A$ imply $g_A \cup k_A \subset fssclg_A \cup fssclk_A$, i.e., $fssclg_A \cup fssclk_A$ is a semiclosed set containing $g_A \cup k_A$. But $fsscl(g_A \cup k_A)$ is the smallest semiclosed fuzzy soft set containing $g_A \cup k_A$. Hence $fsscl(g_A \cup k_A) \subset fssclg_A \cup fssclk_A$. So, $fsscl(g_A \cup k_A) = fssclg_A \cup fssclk_A$.

x. Similar to (ix).

xi. We have $g_A \cap k_A \subset g_A$ and $g_A \cap k_A \subset k_A$

$\Rightarrow fsscl(g_A \cap k_A) \subset fssclg_A$ and $fsscl(g_A \cap k_A) \subset fssclk_A$

$\Rightarrow fsscl(g_A \cap k_A) \subset fssclg_A \cap fssclk_A$.

xii. Similar to (xi).

xiii. Since $fssclg_A \in SCSS(U)$ so by (i), $fsscl(fssclg_A) = fssclg_A$.

xiv. Since $fssintg_A \in SOSS(U)$ so by (ii), $fssint(fssintg_A) = fssintg_A$.

\[ \square \]

Remark 2.15. If $g_A$ is semiopen fuzzy soft (semiclosed fuzzy soft) set, then int($g_A$), fssint($g_A$) ($fsscl(g_A)$ and cl($g_A$)) are semiopen fuzzy soft (semiclosed fuzzy soft) set.

3. Conclusion

In this work, we have initiated the generalization of closed and open sets in a fuzzy soft topological space as semiopen and semiclosed fuzzy soft sets. We have also discussed some characterizations of these sets. Further the topological structures namely interior and closure are also generalized and several interesting properties are studied. Several remarks are stated which give comparison between the properties of these sets in three different domains, namely general topology, fuzzy topology and fuzzy soft topology. Surely the discussions in this paper will help researchers to enhance and promote the study on fuzzy soft topology for its applications in practical life.

4. References

References

[1] A. Rosenfeld, Digital topology, The American Mathematical Monthly, 86(8), (1979) 621630, 1979.
[2] B. Tanay, M. Burcu Kandemir, Topological structure of fuzzy soft sets, Comp. Math. Appl., 61, (2011), 2952–2957.
[3] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., 24, (1968), 182–190.
[4] D. Molodtsov, Soft Set Theory-First Results, Comp. Math. Appl. 37, (1999) 19-31.
[5] J. Mahanta, P.K. Das, On soft topological space via semiopen and semiclosed soft sets, arXiv:1203.4133v1 [math.GN] 16 Mar 2012.
[6] J. Mahanta, P.K. Das, Results on fuzzy soft topological spaces, arXiv:1203.0634v1 [cs.IT] 3 Mar 2012.
[7] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, Journal of Mathematical Analysis and Applications,52(1), (1981) 1432.
[8] L.A. Zadeh, Fuzzy Sets, Information and Control, 11, (1965), 341–356.
[9] M. Shahid, M. Naz, On fuzzy soft topological spaces, Comp. Math. Appl., 61, (2011) 1786-1799.
[10] N. Cagman, S. Karatas, S, Enginoğlu, Soft Topology, Comp. Math. Appl., 62, (2011) 351-358.
[11] N. Levine, Semi-open sets and semi-continuity in topological spaces, The American Mathematical Monthly, 70, (1963) 3641.
[12] S. Roy, T.K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform., 3(2), (2012), 305–311.
[13] T. Noiri, On semi-continuous mapping, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur (8) 54 (1973), 132-136.