Electroweak corrections to the toponium decay width

GEORGE SIOPISIS

Department of Physics and Astronomy
The University of Tennessee, Knoxville, TN 37996–1200
U. S. A.

ABSTRACT

We discuss one-loop electroweak corrections to the decay width of toponium. We calculate the energy-level shifts by expanding around the solution of the Bethe-Salpeter equation in the instantaneous approximation, in analogy with the positronium case. We show that first-order electroweak effects are suppressed by at least four powers of the strong coupling constant, and are therefore negligible compared with QCD corrections. The calculation is manifestly gauge invariant and takes into account the contributions to the decay rate due to both Coulomb enhancement and phase space reduction effects.

*e-mail: siopsis@panacea.phys.utk.edu
The threshold production of a $t \bar{t}$ pair at future $e^+e^-$ colliders has attracted a lot of attention \cite{1}, because the measurement of its cross-section will permit a simultaneous determination of the mass of the top quark $m_t$, its decay width $\Gamma_t$, and the strong coupling constant $\alpha_s(m_t)$ \cite{2}. The threshold regime is dominated by $t \bar{t}$ resonances which enhance the cross-section. These bound states differ from those of the other heavy quarks (b and c), because $m_t$ is so large that the dominant decay mode of the top quark is the weak decay $t \rightarrow W^+b$. Thus, the decay width of a top quark is

$$\Gamma_t \sim \frac{G_F m_t^3}{\sqrt{2} \ rac{\alpha_s}{8\pi}}. \quad (1)$$

For $m_t \simeq 174 \text{ GeV}$, we obtain a decay rate of the order of 1 GeV, which exceeds $\Lambda_{QCD}$, and permits the use of perturbation theory, because the $t \bar{t}$ pair will decay before it has time to hadronize. Moreover, the quarks are produced with little kinetic energy, so that non-relativistic techniques are applicable. Thus, the toponium energy levels are well approximated by the Bohr levels of a QCD Coulomb-like effective potential. To calculate the cross-section, one can determine the appropriate Green function which obeys a non-relativistic Lippmann-Schwinger equation with the QCD effective potential \cite{2}.

An important parameter entering the Lippmann-Schwinger equation is the decay width of toponium, $\Gamma_{t\bar{t}}$. In general, it can be a function of the momentum of the top quark, and its precise form is crucial for an accurate theoretical prediction of the cross-section $\sigma(e^+e^- \rightarrow t\bar{t})$. To zeroth order,

$$\Gamma_{t\bar{t}} = 2\Gamma_t. \quad (2)$$

This relation is modified by three effects: (a) time dilatation (the top quark lives longer in the center-of-mass frame), (b) phase space reduction (due to the binding energy of the quarks), and (c) Coulomb enhancement (the b and $\bar{b}$ quarks should be described by Coulomb wave functions rather than plane waves) \cite{1}. The first two effects can be easily accounted for and they lead to a significant modification of Eq. (2). The third effect is much harder to take into account. As was shown by Kummer and M"odritsch, it cancels the second effect to $o(\alpha_s^2)$, thus leaving time dilatation as the only effect modifying Eq. (2), in analogy with the case of muonium in nuclei \cite{3}. We therefore obtain

$$\Gamma_{t\bar{t}} = 2\Gamma_t \left(1 - \langle p^2/m_t^2 \rangle \right)^{1/2}, \quad (3)$$
which is only a small modification of Eq. (2). To arrive at this result, Mödritsch and Kummer [4] calculated one-loop graphs with insertions of bound-state wave functions, as prescribed by the rigorous Bethe-Salpeter formalism [3]. The cancellation of the two effects (Coulomb enhancement and phase space reduction) was due to non-trivial cancellations between Green functions, which went well beyond the cancellation of gauge-dependent pieces implied by Ward identities.

Our purpose here is to extend the results of ref. [4] to higher orders in the strong coupling constant. We shall show that electroweak corrections enter at an order higher than $o(\alpha_s^5)$, and therefore we need second-order bound state perturbation theory in order to calculate them. They are due to corrections to the vertex function, which do not contain a contribution to the magnetic moment of the top quark. This is in contrast to the positronium case, where magnetic moment effects are $o(\alpha^5)$. The discrepancy arises from the fact that $W$ only couples to a left-handed current.

\[
\begin{align*}
\chi P \quad \frac{P}{2} + p \\
\frac{P}{2} - p
\end{align*}
\]

\[
\begin{align*}
\frac{P}{2} + p \\
\frac{P}{2} - p
\end{align*}
\]

\[
\begin{align*}
\frac{P}{2} + p' \\
\frac{P}{2} - p'
\end{align*}
\]

\[
\chi P \quad P
\]

Figure 1: The homogeneous Bethe-Salpeter equation.

We start with a brief review of the Bethe-Salpeter formalism in order to fix the notation. We wish to calculate the energy levels of $t \bar{t}$ bound states. They are poles in the four-point amplitude describing $t \bar{t}$ scattering. We shall only consider scattering in the $t$-channel, because the annihilation diagrams are suppressed [2]. We shall also neglect photon and $Z^0$ exchanges because they are small effects compared to a gluon exchange. A Higgs exchange will also be neglected, but depending on the mass of the Higgs boson, it can have an appreciable effect.
A bound-state wavefunction $\chi_F(p)$ satisfies the homogeneous Bethe-Salpeter equation

$$\Pi^{(1)}(p_+)\Pi^{(2)}(p_-)\chi_F(p) + \int \frac{d^4 p'}{(2\pi)^4}V(p, p'; P)\chi_F(p') = 0,$$

where $\Pi(p)$ is the complete inverse fermion propagator,

$$\Pi(p) = p - M_t - \Sigma(p),$$

and we have defined momenta

$$p_\pm = \frac{P}{2} \pm p.$$

Eq. (4) is represented graphically in Fig. 1. $V(p, p'; P)$ is a potential function which consists of the two-fermion irreducible graphs. For our purposes, the mass is complex,

$$M_t = m_t + i\Gamma_t.$$

To lowest order in $\alpha_s$ and neglecting electroweak interactions, the potential is (Fig. 2)

$$V_0(p, p'; P) = C_F \frac{4\pi\alpha_s}{\gamma_{\mu}(1)G_{\mu\nu}(p - p')\gamma_{\nu}(2)},$$

where $C_F = 4/3$ is the Casimir operator, and $G^\mu\nu(k)$ is the lowest-order gluon propagator. In the Feynman gauge (omitting group theory factors),

$$G^\mu\nu(k) = \frac{\eta^\mu\nu}{k^2 + i\epsilon},$$

and the potential $V_0(p, p'; P)$ is independent of $P$. At threshold, the quarks move with non-relativistic velocities and the Bethe-Salpeter equation can be approximated by the non-relativistic Schrödinger equation in momentum space, and then solved. To this end, we shall
work in the total rest frame in which the overall momentum is \( P^\mu = (E, \vec{0}) \). In the instantaneous approximation, the potential becomes

\[
V_0^{\text{inst}}(p, p'; P) = C_F 4\pi\alpha_s \gamma_0^{(1)} \frac{1}{(\vec{p} - \vec{p}')^2} \gamma_0^{(2)}.
\] (10)

If we integrate over \( p^0 \), we can write the Bethe-Salpeter equation (4) in terms of the wavefunction \( \Phi(\vec{p}) = \int \frac{dp^0}{2\pi} \phi(p) \) as

\[
(H^{(1)} + H^{(2)} - E)\Phi(\vec{p}) = \left( \Lambda_+^{(1)} \Lambda_+^{(2)} - \Lambda_-^{(1)} \Lambda_-^{(2)} \right) C_F 4\pi\alpha_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2} \Phi(\vec{p}') ,
\] (11)

where \( H \) is the Dirac Hamiltonian and \( \Lambda_+ (\Lambda_-) \) is the projection operator onto positive (negative) energy states. In the non-relativistic limit, this reduces to the Schrödinger equation in momentum space

\[
\left( \frac{\vec{p}^2}{M_t} + 2M_t - E \right) \Phi(\vec{p}) = C_F 4\pi\alpha_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2} \Phi(\vec{p}') .
\] (12)

Thus, we obtain the energy levels

\[
E_n = 2M_t - \frac{M_tC_F^2\alpha_s^2}{4n^2} + o(\alpha_s^4) ,
\] (13)

which are the Bohr levels of the Coulomb-like QCD potential \([10]\). Therefore, the first-order QCD correction to the decay rate of toponium is

\[
\Gamma_{t\bar{t}} = 2\Gamma_t \left( 1 - \frac{C_F^2\alpha_s^2}{8n^2} \right) ,
\] (14)

in agreement with Eq. (3) (see [4]). The spherically symmetric \( S = 0 \) states are given by

\[
\Phi_n(\vec{p}) = (M_tC_F\alpha_s)^{-3/2} \frac{\mathcal{L}_n(n^2y)}{(1 + n^2y)^{n+1}} , \quad y = \frac{4\vec{p}^2}{M_t^2C_F^2\alpha_s^2} ,
\] (15)

where \( \mathcal{L}_n \) is a polynomial of order \( n - 1 \) related to the Laguerre polynomials. For \( n = 1 \), we have \( \mathcal{L}_1 = 16\sqrt{2\pi} \).

Higher-order corrections can be systematically introduced by perturbing around the solution to the Schrödinger equation \([12]\). The potential to be treated perturbatively is \( V - V_0^{\text{inst}} \). There is also a contribution from the disconnected diagrams which are due to the self-energy terms in the fermion propagators (Eq. (5)), but they can be absorbed in the potential if we make use of
the Schrödinger equation. Thus, according to the Bethe-Salpeter formalism \cite{6}, the first-order energy level shift is

\[ \Delta E_n = \langle \Phi_n | D_P(p) (H^{(1)} + H^{(2)} - E_n) (V - V_{0}^{\text{inst}}) (H^{(1)} + H^{(2)} - E_n) D_P(p) | \Phi_n \rangle, \]  

where the inner product involves an integral over the four-momentum. \( H \) is the Dirac Hamiltonian, and \( D_P \) is the product of two free fermion propagators (cf. Eq. (4)), which can be expressed in terms of the projection operators \( \Lambda_{\pm} \) as

\[ D_E(p) = \sum_{\pm\pm} \frac{\Lambda^{(1)}_{\pm} \Lambda^{(2)}_{\pm}}{(E/2 + p^0 \pm (E_p - i\epsilon))[E/2 - p^0 \pm (E_p - i\epsilon)]}, \]

where \( E_p = \sqrt{p^2 + M_{t}^2} \) is the energy of the quark on the mass shell.

To lowest order, the potential is \( V_0 - V_{0}^{\text{inst}} \) (Eqs. (8) and (10)). This is analogous to the positronium case, and produces an \( o(\alpha_s^4) \) shift in the energy levels. The first-order electroweak correction is

\[ V_1(p, p'; P) = 4\pi C_F \alpha_s \alpha_W \left( \Lambda^{(1)}_{\mu}(p_+, p'_+) G^{\mu\nu}(p - p') \gamma^{(2)}_{\nu} + \gamma^{(1)}_{\mu} G^{\mu\nu}(p - p') \Lambda^{(2)}_{\nu}(p_-, p'_-) \right), \]

where \( p_{\pm} = P/2 \pm p, \ p'_{\pm} = P/2 \pm p' \), and we have made explicit the electroweak coupling constant \( \alpha_W = G_F M_{W}^2 \), where \( G_F \) is the Fermi constant and \( M_W \) is the mass of the \( W \) boson. The vertex function \( \Lambda_{\mu}(p, p') \) consists of the diagrams shown in fig. 3. It is guaranteed to give a gauge invariant contribution by the Ward identity satisfied by the one-particle irreducible function,

\[ (p - p')^{\mu} \Gamma_{\mu}(p, p') = \Pi(p) - \Pi(p'). \]
Since we are only interested in first-order corrections, we may replace $M_t$ by its real part $m_t$. The contribution of $V_1$ to the energy level shift (Eq. (16)) can then be written as

$$\Delta E_n^W = \frac{C_F^2 \alpha_s^2 \alpha_W}{16 m_t} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \frac{\eta^{\mu \nu}}{(p-p')^2 + i\epsilon} \frac{\mathcal{L}_n(n^2 y)}{(1+n^2 y)^n} \frac{\mathcal{L}_n(n^2 y')}{(1+n^2 y')^n}$$

$$\times \left\langle D_P(p') \left( k \Lambda^{(1)}_{\mu}(p_+, p'_+) \gamma^{(2)}_{\nu} + \gamma^{(1)}_{\mu} \Lambda^{(2)}_{\nu}(p_-, p'_-) \right) D_P(p) \right\rangle,$$

(20)

where $y = 4\vec{p}^2 / m_t^2 C_F^2 \alpha_s^2$ and $y' = 4\vec{p}'^2 / m_t^2 C_F^2 \alpha_s^2$. A simple scaling argument shows that the lowest-order contribution to the integral comes from the small three-momentum region. Momentum insertions contribute additional powers of $\alpha_s$. At low momentum transfer, the three-point vertex $\Lambda_{\mu}$ may be written in general as

$$\Lambda_{\mu}(p, p') = k^2 F_1(k^2) + \sigma_{\mu \nu} k^\nu F_2(k^2),$$

(21)

where $k = p - p'$, and $\sigma_{\mu \nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}]$. The form factors $F_1$ and $F_2$ are regular as $k^2 \to 0$. In the positronium case, $F_2$ gives an $o(\alpha^5)$ contribution to the energy level shift, and is due to the magnetic moment interaction. In our case, we need to multiply the gamma matrices by the projection operator $\frac{1}{2}(1 - \gamma_5)$, due to parity violation of weak interactions. A straightforward explicit calculation shows that the form factor $F_2(k^2)$ vanishes to lowest order in $k^2$. It follows that the three-point vertex is proportional to $(p - p')^2$ (recall that $p_+ - p'_+ = p_- - p'_- = p - p'$).

Having established the leading-order behavior of $\Lambda_{\mu}$, we can now estimate the integral in Eq. (20). As we just showed, $\Lambda_{\mu}$ contributes a factor $(p - p')^2$. This factor cancels the gluon propagator. Then the integral over $p_0$ and $p'_0$ can be easily done, because of the respective poles in the operators $D(p)$ and $D(p')$. The resulting expression contains six three-momentum factors implying that the integral is $o(\alpha_8^8)$. Therefore, the electroweak correction to the decay width is negligible. Of course, no conclusion can be drawn regarding the exact value of the electroweak correction, because such a high order is beyond the scope of first-order perturbation theory.

In conclusion, we have calculated electroweak corrections to the decay rate of toponium in the dominant mode $t \to W^+ b$. Our results extend those of Kummer and M"odritsch [4], who used the Coulomb gauge to obtain a perturbative expansion of the Bethe-Salpeter equation. We employed a covariant gauge and perturbed around the instantaneous solution to the Bethe-Salpeter equation, in analogy with the abelian case of positronium. We found that the
electroweak corrections are suppressed by at least six powers of $\alpha_s$. This takes us beyond the realm of first-order perturbation theory, therefore we cannot calculate the precise value of the correction. It is important to extend our results by including other electroweak effects, such as a Higgs boson exchange. The development of an efficient systematic method for such calculations involving bound states would be of great interest, in view of the significance of the cross-section for threshold $t\bar{t}$ production at future colliders [1].

**Acknowledgements**

I wish to thank Bennie Ward for illuminating discussions.
References

[1] M. Jeżabek, preprint TTP94-09, hep-ph/9406411 (1994), and references therein.

[2] V. S. Fadin and V. A. Khoze, JETP Lett. 46 (1987) 525; Sov. J. Nucl. Phys. 48 (1988) 309;
M. J. Strassler and M. E. Peskin, Phys. Rev. D43 (1991) 1500;
Y. Sumino, K. Fujii, K. Hagiwara, H. Murayama, and C.-K. Ng, Phys. Rev. D47 (1992) 56.
M. Jeżabek, J. H. Kühn, and T. Teubner, Zeit. Phys. C56 (1992) 653;
M. Jeżabek and T. Teubner, Zeit. Phys. C59 (1993) 669.

[3] H. Überall, Phys. Rev. 119 (1960) 365;
R. W. Huff, Ann. Phys. (NY) 16 (1961) 288.

[4] W. Mödritsch and W. Kummer, preprints TUW-94-06 and TUW-94-16, hep-ph/9408221 (1994).

[5] W. Kummer and W. Mödritsch, preprint TUW-94-14, hep-ph/9408216 (1994).

[6] C. Itzykson and J.-B. Zuber, “Quantum Field Theory,” McGraw-Hill Inc. (1980).