A NILSEQUENCE WIENER–WINTNER THEOREM FOR BILINEAR ERGODIC AVERAGES

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ABSTRACT. We deduce a nilsequence Wiener–Wintner theorem for $r$-linear ergodic averages from the corresponding pointwise ergodic theorem. By Bourgain’s bilinear pointwise ergodic theorem the nilsequence Wiener–Wintner theorem holds unconditionally for $r = 2$.

1. Introduction

Let $(X, \mu, T)$ be an ergodic measure-preserving system and $\Phi$ a Følner sequence in $\mathbb{Z}$. Call a sequence $(a_n)_n$ a good weight for the $k$-linear pointwise ergodic theorem on $X$ along $\Phi$ if for some distinct non-zero integers $b_1, \ldots, b_k$ and any bounded functions $f_1, \ldots, f_k \in L^\infty(X)$ the limit

$$\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} a_n \prod_{i=1}^k f_i(T^{b_i n} x)$$

exists pointwise almost everywhere. In [EZhK13] it has been shown that nilsequences are good weights for the 1-linear pointwise ergodic theorem on any measure-preserving system along any tempered Følner sequence in a certain uniform sense (the non-uniform result is due to Host and Kra [HK09]). The main black box in that proof was the 1-linear pointwise ergodic theorem, that is, the fact that $a_n \equiv 1$ is a good weight.

Recently, Assani, Duncan, and Moore [ADM14] showed that almost periodic sequences are good weights for the 2-linear pointwise ergodic theorem in a similarly uniform sense, again using the $a_n \equiv 1$ case as a black box.

In this note we prove the natural joint generalization of these results. We fix a Følner sequence $\Phi$ and say that the system $(X, \mu, T)$ has property $P_k$ if $a_n \equiv 1$ is a good weight for the $k$-linear pointwise ergodic theorem on $X$ along $\Phi$. It is a long-standing conjecture that every measure-preserving system satisfies $P_k$ for every $k$ along the standard Følner sequence $\Phi_N = \{1, \ldots, N\}$, and we have nothing to add on this issue. For $k = 2$ this conjecture has been proved by Bourgain [Bou90] (see also [Dem07]), and for general $k$ the best partial result up to date states that distal systems satisfy $P_k$ [HSY14].

The nilsequence Wiener–Wintner theorem referred to in the title states that generalized nilsequences are good weights for the 2-linear pointwise ergodic theorem, and the set of full measure on which the convergence holds can be taken independent of the nilsequence. This follows as in [EZhK13, §6] from Bourgain’s bilinear pointwise ergodic theorem and the following estimate (we refer to the prequel [EZhK13] for the notation).

Theorem 1.1. Suppose that the ergodic measure-preserving system $(X, \mu, T)$ has property $P_k$ with parameters $b_1, \ldots, b_k$ along the Følner sequence $\Phi$. Then

$$\int \limsup_{N \to \infty} \sup_{G/\Gamma \text{ has length } l} \sup_{G/\Gamma \in F(\Phi, G_\ast)} \sup_{g \in W^2(G/\Gamma)} \left\| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^k f_i(T^{b_i n} x) g(n \Gamma) \right\|_{l^{2+1}} \lesssim_{b_1, \ldots, b_k, l} \min \|f_i\|_{l^{2+1}},$$

where $r = \sum_{m=1}^l (d_m - d_{m+1})1_{m=1}^{l+1}$ with $d_i = \dim G_i$, and the positive constants $C_{G/\Gamma}$ depend only on the nilmanifold $G/\Gamma$, that is, the filtration $G_\ast$, the lattice $\Gamma$, and the Mal’cev basis used to define the Sobolev norms.

The case $k = 2$, $l = 1$ has been proved by Assani, Duncan, and Moore [ADM14], and the case of commutative $G$ by Assani and Moore [AM14a, AM14b]. An immediate consequence of Theorem 1.1 is an extension of [AM15, Theorem 1.4]: if $(X, \mu, T)$ has property $P_k$ along $\Phi$, then for any $f_1, \ldots, f_k \in L^\infty(X)$ and almost every $x \in X$ the sequence $(\prod_{i=1}^k f_i(T^{b_i n} x))_n$ is a good weight for polynomial multiple ergodic averages along $\Phi$ (in the sense of [EZhK13, Corollary 7.1]). See [EZhK13, §7] for the proof.

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2. Proof of Theorem 1.1

By induction on \( l \). For \( l = 0 \) the group \( G \) is trivial, so the nilsequences are constant and the averages over \( n \) converge pointwise almost everywhere by property \( P_k \). Hence the left-hand side of (1.2) equals

\[
\lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n} x) \bigg|_{L^2(G/\Gamma)},
\]

the limit now being taken in \( L^2 \). The estimate for this limit follows by [HK05, Theorem 12.1], which is formulated for the standard Følner sequence but holds for any Følner sequence.

Suppose now that the claim holds for \( l - 1 \). Writing the function \( F \) as a vertical Fourier series \( F = \sum_x F_x \) as in [EZK13 (3.5)] we obtain for the supremum on the left-hand side of (1.2)

\[
\sup_{g \in \mathcal{P}(\mathbb{Z}, G), \ F \in \mathcal{W}^{l, 2}(G/\Gamma)} \left| \frac{\| F \|_{L^2(G/\Gamma)}}{|\Phi_N|} \sum_{n \in \Phi_N} \prod_{i=1}^{k} f_i(T^{b_i n} x) F(g(n)\Gamma) \right|^{2^{l+1}},
\]

where we have used [EZK13, Lemma 3.7] in the last line. We incorporate the \( G/\Gamma \)-dependent constant into \( C_{G/\Gamma} \), which may change from line to line. Next we apply the van der Corput lemma [EZK13, Lemma 2.7] and estimate the integrand on the left-hand side of (1.2) by

\[
\lim_{m \to \infty} \frac{1}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} f_i(T^{b_i n} x) F(g(n)\Gamma) \left| \frac{\| f_i \|_{L^2(G/\Gamma)}}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} f_i(T^{b_i n} x) F(g(n)\Gamma) \right|^{2^{l+1}},
\]

where \( \lim_{m \to \infty} a_m := \lim_{M \to \infty} \left| \frac{1}{M^2} \sum_{m=-M}^{M} (M - |m|) a_m \right| \). With the notation for the cube construction from [EZK13, § 3] this becomes

\[
\lim_{m \to \infty} \frac{1}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} f_i(T^{b_i n} x) F(g(n)\Gamma) \left| \frac{\| f_i \|_{L^2(G/\Gamma)}}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} (f_i(T^{b_i n} \hat{g}_m(n)\hat{\Gamma})) \right|^{2^{l+1}}.
\]

By [EZK13, Lemma 3.2] this is bounded by

\[
\lim_{m \to \infty} \frac{1}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} f_i(T^{b_i n} x) F(g(n)\Gamma) \left| \frac{\| f_i \|_{L^2(G/\Gamma)}}{|\Phi_m|} \sum_{n \in \Phi_m} \prod_{i=1}^{k} (f_i(T^{b_i n} \hat{g}_m(n)\hat{\Gamma})) \right|^{2^{l+1}}.
\]

Integrating the last display over \( X \) and applying Fatou’s lemma and the inductive hypothesis we obtain

\[
\lim_{m \to \infty} \min \| f_i \|_{L^{2^{l+1}}(G/\Gamma)} \cdot \min_{l \leq \Gamma} \min_{m \leq \Gamma} \| f_i \|_{L^{2^{l+1}}(G/\Gamma)}.
\]

By Hölder’s inequality and the inductive construction of the Gowers–Host–Kra seminorms this is bounded by

\[
\min_{l \leq \Gamma} \lim_{m \to \infty} \min \| f_i \|_{L^{2^{l+1}}(G/\Gamma)} \cdot \min_{l \leq \Gamma} \min_{m \leq \Gamma} \| f_i \|_{L^{2^{l+1}}(G/\Gamma)}.
\]

as required.

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