UNIFORM POLYNOMIAL STABILITY OF SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES WITH POSITIVE DEFINITE KERNELS

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ABSTRACT. We are concerned with the polynomial stability and the integrability of the energy for second order integro-differential equations in Hilbert spaces with positive definite kernels, where the memory can be oscillating or sign-varying or not locally absolutely continuous (without any control conditions on the derivative of the kernel). For this stability problem, tools from the theory of existing positive definite kernels can not be applied. In order to solve the problem, we introduce and study a new mathematical concept – generalized positive definite kernel (GPDK). With the help of GPDK and its properties, we obtain an efficient criterion of the polynomial stability for evolution equations with such a general but more complicated and useful memory. Moreover, in contrast to existing positive definite kernels, GPDK allows us to directly express the decay rate of the related kernel.

1. Introduction. The earlier theory of positive definite kernel plays an important role in the study of existence and boundedness of solutions to Volterra equations, as well as in the study of stability of Volterra equations with monotone kernels. In order to investigate the stability and the asymptotic behaviour of solutions to the Volterra equations with nonmonotone kernels, the concept of strongly positive definite kernel was introduced and its basic properties were obtained by Maccamy and Wong [21]. Since then, there have been many studies on the stability and the asymptotic property of solutions to first order integro-differential functional equations such as the Volterra operator equations and integro-partial differential equations by using the strongly positive definite kernels (cf, e.g., [28, 31, 16, 30, 11]). In the last
decade, the basic theory of strongly positive definite kernel has also been applied to the asymptotic analysis of some second order integro-differential systems. For example, the exponential stability for some second order integro-differential system is obtained in [4] as the kernels are strongly positive definite and decay exponentially at infinity. The global solutions to the equation of thermoelasticity with fading memory are presented in [29]) when the kernels are strongly positive definite. The existence and stability of solutions to the inverse problem of thermoacoustic tomography is obtained in [1] when the kernels are strongly positive definite. For more recent works involving the strongly positive definite kernels or completely positive definite kernels, we refer the reader to, e.g., [19, 20].

Since the precise decay properties at infinity of the kernels are not given directly in the definition of (strong) positive definite kernels, one needs to use other techniques to explore them. This is feasible in some cases, for example, those of exponential decay rates as in [4]. On the other hand, this seems impossible or very difficult for studying general precise decay properties at infinity by the existing theory of positive definite kernel/strongly positive definite kernel. Moreover, the theory of existing positive definite kernels is ineffective for dealing with the polynomial stability of the following second order abstract evolution equations with positive definite kernels:

\[
\begin{cases}
  u''(t) + Au(t) - \int_0^t g(t-s)Au(s)ds = f(u(t)), \\
  u(0) = u_0, \quad u'(0) = u_1,
\end{cases}
\]  

(1)

where \( A \) is a positive self-adjoint linear operator in a Hilbert space \( H \), \( g(t) \in L^1(0, +\infty) \) is the memory kernel which produces damping mechanism (memory-damping), and \( g(t) \) can be oscillating or sign-varying or not locally absolutely continuous (without any control conditions on \( g' \)). In order to change the situation, we propose a new mathematical concept—generalized positive definite kernel, and present its basic properties (in section 3). Then, with the help of GPDK and its properties derived here, we prove a criterion of the polynomial stability and the integrability of the energy for (1) with such a general but more complicated and useful memory. It is, as far as we know, the first and efficient criterion of the polynomial stability for these evolution equations.

The well-posedness of the abstract Cauchy problem (1) has been intensively studied in literature since 1960s. But few results are available about the studies of stabilization of the problem (1) with the general memory kernel \( g(t) \) (cf. [4] for the exponentially decaying case), while there has been much work for stabilization when the kernel function \( g \) is nonnegative, non-increasing, and there is some control condition on \( g' \). With the help of the generalized positive definite kernel (GPDK) and its properties newly established in section 3, we obtain a polynomial stability result for the energy of (1) (in section 4). The generalized positive definite kernel, in contrast to the positive definite kernel, allows to directly express the decay rate of the kernel; this feature leads to its efficiency in handling the problem of decay rate estimation of (1). Moreover, our method by means of the GPDK shows a new way of studying the stability of (1).

We also would like to mention that for the kernel \( g(t) \) in (1) having some good properties, there have been many developments in the study of stability for the problem (1). For instance, if the kernel \( g \) is a nonnegative, decreasing and locally
absolutely continuous function such that
\[ g'(t) \leq -kg^p(t), \quad \text{for a.e. } t \geq 0, \tag{2} \]
where \( k > 0 \) and \( 1 \leq p < \frac{3}{2} \), then it was shown in [2] that the energy of a mild solution decays exponentially or polynomially as \( t \to +\infty \). If there exists a positive, strictly increasing, convex function \( H \in C^1(R^+) \), with \( H(0) = 0 \) such that
\[ g'(t) \leq -H(g(t)), \quad t > 0, \tag{3} \]
then the decay rates of (1), as those of the solutions to a given nonlinear dissipative ODE, were obtained in [17] under some other basic conditions. In [4], by using the strongly positive definite kernel and the unique nature of the exponential function \((e^{a+b} = e^a e^b)\), Cannarsa and Sforza studied the problem (1) for singular convolution kernels. Exponential stability of weak solutions was obtained if the kernel decays exponentially at infinity and possesses strongly positive definite primitives. Recently, Jin, Liang and Xiao [13] considered the following abstract coupled system via indrected memory damping,
\[
\begin{align*}
&u''(t) + Au(t) - \int_0^t g(t-s)Au(s)ds + \alpha u(t) + Bv(t) = f(u), \\
v''(t) + Av(t) + Bu(t) = 0,
\end{align*}
\]
where \( A \) is a positive self-adjoint operator in a Hilbert space \( H \), \( B \) is a symmetric linear operator in \( H \), and \( g(t) \in L^1(0, +\infty) \) is the memory kernel. Only under some basic assumptions without the control condition on \( g' \) such as (2), (3) etc., an optimal decay rate was obtained and the results are also true for the single equation system (1). As one can see, this result gave us a prior estimate for the energy for almost all non-increasing kernels. For more related papers concerning single/coupled integro-differential equations, we refer the reader to, e.g., [4, 8, 10, 12, 16, 17, 24, 22, 23, 15, 32]. We also would like to refer the reader to the works [5, 6, 7, 18, 25, 26, 27, 33, 34], where the stability of some wave equations and hyperbolic-parabolic equations are well studied.

The remaining paper is organized as follows. In section 2, we recall the definition of (strongly) positive definite kernel, and give new properties of (strongly) positive definite kernel, which will be used in the next section. In section 3, we propose a new mathematical concept—generalized positive definite kernel and present its basic properties. Some examples are also given in the end of this section. The main stability result is stated in section 4. In section 5, the complete proof of our stability result is given by virtue of the generalized positive definite kernel. An appendix with some basic lemmas is also given.

Throughout the paper, \( C, C(\cdot), C_i, D_i \) are generic positive constants (possibly different from line to line), and \( H \) is a real Hilbert space with the scalar product \( \langle \cdot, \cdot \rangle \). For a linear operator \( A \) in \( H \), \( D(A) \) and \( [D(A)] \) stand for, respectively, its domain and the domain endowed with the graph norm.

2. Properties of (strongly) positive definite kernel.

**Definition 2.1.** Let \( h \in L^1_{loc}(0, +\infty) \). The function \( h \) is called to be a positive definite kernel if
Proof. Since Proposition 2.6. Assume that \( h \in L^\infty(0, +\infty) \). Then \( h \) is positive definite if and only if for any \( z \in \mathbb{C} \) with \( \text{Re} z > 0 \),

\[
\text{Re} \int_0^{+\infty} e^{-zt} h(t) dt \geq 0.
\]

Proposition 2.3. Assume that \( h(t) \) is twice differentiable with \( h' \neq 0 \) and

\[
(-1)^i h^{(i)}(t) \geq 0, \quad \forall t > 0, \quad i = 0, 1, 2.
\]

Then \( h(t) \) is strongly positive definite.

Proposition 2.4. Let \( h \) be a strongly positive definite kernel. Then

\[
\int_0^t \|u(s)\|^2 ds \leq \|u(0)\|^2 + \frac{2}{\delta} \left( \int_0^t (h * u(s), u(s)) ds + \int_0^t (h * u'(s), u'(s)) ds \right),
\]

for any \( t \geq 0 \), and \( u \in L^1_{\text{loc}}(0, +\infty; H) \), where \( \delta \) is the constant in Definition 2.1.

The above three propositions can be found in [4, 16, 28]. Next, we prove some new properties of positive definite kernel, which will be used later.

Proposition 2.5. Assume that \( h(t) \) is (strongly) positive definite. Then \( e^{-at} h(t) \), \( a > 0 \) is also (strongly) positive definite.

Proof. Case 1: Strongly positive definite case.

Since \( h(t) \) is strongly positive definite, we know that \( h(t) - \delta e^{-t} (\delta > 0) \) is positive definite. In view of the second mean value theorem for integrals, we have

\[
\int_0^T \left( \int_0^t (e^{-a(t-s)} h(t-s) - \delta e^{-(a+1)(t-s)}) u(s)ds, u(t) \right) dt
\]

\[
= \int_0^T e^{-at} \left( \int_0^t (h(t-s) - \delta e^{-(t-s)}) e^{as} u(s)ds, e^{at} u(t) \right) dt
\]

\[
= \int_0^T \left( \int_0^t (h(t-s) - \delta e^{-(t-s)}) e^{as} u(s)ds, e^{at} u(t) \right) dt
\]

\[
\geq 0.
\]

Moreover, \( e^{-(a+1)t} \) is strongly positive definite. Hence, \( e^{-at} h(t) \) is strongly positive definite.

Case 2: Positive definite case.

It is clear by taking \( \delta = 0 \) in the process above.

Proposition 2.6. Assume that \( h(t) \) is (strongly) positive definite. Then \( (t + a)^{-d} h(t) \) \( (a \geq 0, d > 0) \) is (strongly) positive definite.

Proof. Since

\[
(t - s + a)^{-d} = \sum_{k=0}^{+\infty} (-1)^k \binom{-d}{k} (t + a + 1)^{-d-k} (s + 1)^k
\]
\[ \sum_{k=0}^{\infty} \left( -d^k \right) \left( t + a + 1 \right)^{-d-k} (s + 1)^k, \]

we see that
\[
\int_0^T \left\langle \int_0^t (t - s + a)^{-d} \left( h(t - s) - \delta e^{-(t-s)} \right) u(s) ds, u(t) \right\rangle dt
= \sum_{k=0}^{\infty} \left( -d^k \right) \int_0^T \frac{(t + a + 1)^{-d-k}}{(t + 1)^k} \left\langle \Upsilon_1, (t + 1)^k u(t) \right\rangle dt,
\]

where
\[
\Upsilon_1 := \int_0^t \left( h(t - s) - \delta e^{-(t-s)} \right) (s + 1)^k u(s) ds.
\]

Noting that \((t + a)^{-d} e^{-t}\) is strongly positive definite, we obtain the conclusion by the second mean value theorem for integrals. \(\square\)

**Proposition 2.7.** If \(h(t)\) is (strongly) positive definite, then \(h(t) \cos at\) \((a \in R)\) is (strongly) positive definite.

**Proof.** Let \(h(t)\) be strongly positive definite and \(a > 0\). Then, by Proposition 2.2, there is a constant \(\delta > 0\) such that, for any \(d_1 > 0, d_2 \geq 0,
\[
\int_0^{+\infty} e^{-d_1t} h(t) \cos d_2 dt \geq \delta \int_0^{+\infty} e^{-(1+d_1)t} \cos d_2 dt = \frac{1 + d_1}{1 + d_1^2 + d_2^2} \delta. \quad (4)
\]

Therefore, by (4), we obtain
\[
\int_0^{+\infty} e^{-d_1t} h(t) \cos at \cos d_2 dt
\]
\[
= \frac{1}{2} \int_0^{+\infty} e^{-d_1t} h(t) \cos (d_2 + a) dt + \frac{1}{2} \int_0^{+\infty} e^{-d_1t} h(t) \cos (d_2 - a) dt
\]
\[
\geq \frac{\delta}{2 (1 + d_1)^2 + (d_2 + a)^2}
\]
\[
\geq \frac{\delta}{4 (a + 1 + d_1)^2 + d_2^2}
\]
\[
\geq \frac{\delta}{4(1 + a)^2 (1 + d_1)^2 + d_2^2}
\]

Thus, by Proposition 2.2, we know that \(h(t) \cos at\) \((a > 0)\) is strongly positive definite. Moreover, since \(\cos at\) is an even function, we see that for \(a \in R\), \(h(t) \cos at\) is strongly positive definite.

On the other hand, if \(h(t)\) is positive definite, then taking \(\delta = 0\) in the process above, we infer that \(h(t) \cos at, a \in R,\) is positive definite. \(\square\)

**Proposition 2.8.** If \(h(t)\) is nonnegative, non-increasing and bounded, then \(h(t)\) is positive definite.
Proof. Denote \( z = a + bi, \ a, b \in R \) and \( a > 0 \). Then if \( b = 0 \), it is obvious that
\[
\text{Re} \int_0^{+\infty} e^{-zt} h(t) dt = \int_0^{+\infty} e^{-at} h(t) dt \geq 0.
\]
If \( b \neq 0 \), let us assume \( b > 0 \),
\[
\text{Re} \int_0^{+\infty} e^{-zt} h(t) dt = \int_0^{+\infty} e^{-at} h(t) \cos bt dt
= \sum_{n=0}^{+\infty} e^{(n+1)\pi/b} e^{-at} h(t) \cos bt dt
= \sum_{n=0}^{+\infty} (-1)^n \int_0^{\pi/b} e^{-a(t+n\pi/b)} h(t+n\pi/b) \cos bt dt.
\]
Noting that \( \int_0^{\pi/b} e^{-a(t+n\pi/b)} h(t+n\pi/b) \cos bt dt \) is nonnegative, decreasing and
\[
\int_0^{\pi/b} e^{-a(t+n\pi/b)} h(t+n\pi/b) \cos bt dt \to 0, \quad n \to +\infty,
\]
we get by Leibniz’s test of alternating series,
\[
\text{Re} \int_0^{+\infty} e^{-zt} h(t) dt \geq \int_0^{\pi/b} e^{-a(t)} h(t) \cos bt dt \geq 0.
\]
Therefore, by Proposition 2.2, \( h(t) \) is positive definite. \( \square \)

3. Generalized positive definite kernel.

**Definition 3.1.** Let \( h \in L^1_{\text{loc}}(0, +\infty) \), and \( \varphi \in L^\infty_{\text{loc}}(0, +\infty) \) with \( \varphi(t) > 0 \) for \( t > 0 \). The function \( h \) is called to be a \( \varphi \)-positive definite kernel if
\[
\int_0^t \varphi(s) \langle h * u(s), u(s) \rangle ds \geq 0, \quad \forall \ t \geq 0,
\]
for any \( u \in L^2_{\text{loc}}(0, +\infty; H) \). Moreover, \( h \) is said to be a strongly \( \varphi \)-positive definite kernel if there exist two constants \( \delta > 0, \ N > 0 \) such that \( h(t) - \delta e^{-Nt} \) is \( \varphi \)-positive definite. We call this type of kernel as generalized (strongly) positive definite kernel (abbreviated to GPDK and GSPDK respectively).

**Remark 3.2.** Definition 3.1 coincides with the previous Definition 2.1 whenever \( \varphi = 1 \). In Definition 3.1, \( \varphi^{-1} \) implies a decay rate of \( h \) at infinity, which gives a useful priori estimate when one applies it to the study of decay properties and asymptotic analysis of integro-differential systems.

**Proposition 3.3.** Suppose that \( \kappa \) is a real number. Then \( h(t) \) is (strongly) \( e^{\kappa t} \)-positive definite if and only if \( e^{\frac{\kappa}{2} t} h(t) \) is (strongly) positive definite.

**Proof.** Immediate from Definition 2.1 and Definition 3.1. \( \square \)

**Proposition 3.4.** Assume that \( \psi \in L^\infty_{\text{loc}}(0, +\infty) \) is a decreasing positive function, and \( h \) is a (strongly) \( \varphi(t) \)-positive definite kernel. Then \( h \) is (strongly) \( \psi \varphi \)-positive definite.

**Proof.** By the second mean value theorem for integrals, we get the conclusion. \( \square \)

**Proposition 3.5.** Assume that \( (t+a)^\alpha h(t) \) \( (a \geq 0, \ \alpha \in \mathbb{R}^+) \) is (strongly) positive definite. Then, for any \( b > 0 \), \( h(t) \) is (strongly) \( (t+a+b)^\alpha \)-positive definite.
Proof. We divide the proof into three steps.

**Step 1.** Firstly, let us treat the case of positive integer powers. Since, for \( n \in \mathbb{N}^+ \),
\[
\int_0^T (t + a + b)^n \left( \int_0^t h(t - s)u(s)ds, u(t) \right) dt
= \int_0^T \left( \int_0^t (t - s + a)^n h(t - s)u(s)ds, u(t) \right) dt
+ \cdots + \binom{n}{k} \int_0^T (t + b)^{-k} \langle \Psi_2, (t + b)^k u(t) \rangle dt
+ \cdots + \int_0^T (t + b)^{-n} \left( \int_0^t h(t - s)(s + b)^n u(s)ds, (t + b)^n u(t) \right) dt,
\]
where
\[
\Psi_2 := \int_0^t (t - s + a)^{n-k} h(t - s)(s + b)^k u(s)ds.
\]
We know, by the second mean value theorem for integrals, that
\[
\int_0^T (t + a + b)^n \left( \int_0^t h(t - s)u(s)ds, u(t) \right) dt
= \int_0^T \left( \int_0^t (t - s + a)^n h(t - s)u(s)ds, u(t) \right) dt
+ \cdots + b^{-k} \binom{n}{k} \int_0^{\xi_k(T)} \langle \Psi_2, (t + b)^k u(t) \rangle dt
+ \cdots + b^{-n} \int_0^{\xi_n(T)} \left( \int_0^t h(t - s)(s + b)^n u(s)ds, (t + b)^n u(t) \right) dt.
\]
Therefore, for positive integer powers \( n \in \mathbb{N}^+ \), Proposition 2.6 implies that \( h(t) \) is (strongly) \((t + a + b)^n\)-positive definite.

**Step 2.** Let \( 0 < \alpha < 1 \).
Since
\[
(t - s + a)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} (t + a + b)^{\alpha-k}(s + b)^k
= (t + a + b)^\alpha - \sum_{k=1}^{\infty} \binom{\alpha}{k} (t + a + b)^{\alpha-k}(s + b)^k,
\]
we see that,
\[
\int_0^T (t + a + b)^\alpha \left( \int_0^t h(t - s)u(s)ds, u(t) \right) dt
= \int_0^T \left( \int_0^t (t - s + a)^{\alpha} h(t - s)u(s)ds, u(t) \right) dt
+ \sum_{k=1}^{\infty} \binom{\alpha}{k} \int_0^T (t + a + b)^{\alpha-k} \left( \int_0^t h(t - s)(s + b)^k u(s)ds, u(t) \right) dt.
\]
Therefore, Proposition 2.6 implies that \( h(t) \) is (strongly) \((t + a + b)^\alpha\)-positive definite, if \( 0 < \alpha < 1 \).
Step 3. Let $n < \alpha < n + 1$.
For $0 < \alpha - n < 1$, we have
\[
(t + a + b)^\alpha = (t + a + b)^{\alpha-n}(t + a + b)^n \\
= \left( (t - s + a)^{\alpha-n} + \sum_{k=1}^{+\infty} \binom{\alpha-n}{k} (t + a + b)^{\alpha-n-k}(s + b)^k \right) \\
\times \left( (t - s + a)^n + \sum_{m=1}^{n} \binom{n}{m} (t - s + a)^{n-m}(s + b)^m \right) \\
= (t - s + a)^\alpha + \sum_{m=1}^{n} \binom{n}{m} (t - s + a)^{\alpha-m}(s + b)^m \\
+ \sum_{k=1}^{+\infty} \binom{\alpha-n}{k} (t + a + b)^{\alpha-n-k}(t - s + a)^n(s + b)^k \\
+ \sum_{k=1}^{+\infty} \sum_{m=1}^{n} \binom{\alpha-n}{k} \binom{n}{m} (t + a + b)^{\alpha-n-k}(t - s + a)^{n-m}(s + b)^{m+k}.
\]

Hence, we obtain
\[
\int_0^T (t + a + b)^\alpha \left( \int_0^t h(t-s)u(s)ds, u(t) \right) dt \\
= \int_0^T \left( \int_0^t (t + a + b)^\alpha h(t-s)u(s)ds, u(t) \right) dt \\
+ \sum_{m=1}^{n} \binom{n}{m} \int_0^T \left( \int_0^t (t + a + b)^{\alpha-m}h(t-s)(s + b)^m u(s)ds, u(t) \right) dt \\
+ \sum_{k=1}^{+\infty} \binom{\alpha}{k} \int_0^T (t + a + b)^{\alpha-n-k} \left( \int_0^t (t - s + a)^n h(t-s)(s + b)^k u(s)ds, u(t) \right) dt \\
+ \sum_{k=1}^{+\infty} \sum_{m=1}^{n} \binom{n}{m} \binom{\alpha-n}{k} \int_0^T (t + a + b)^{\alpha-n-k} \\
\times \left( \int_0^t (t - s + a)^{n-m} h(t-s)(s + b)^{m+k} u(s)ds, u(t) \right) dt.
\]

Thus, by Proposition 2.6, we know that $h(t)$ is (strongly) $(t + a + b)^\alpha$-positive definite. \hfill \Box

Proposition 3.3, Proposition 3.4 and Proposition 3.5 show us the relationships between $\varphi(t)$-positive definite kernel and positive definite kernel, and give criterions of $\varphi(t)$-positive definiteness for the polynomial/exponential case.

We want now to set up some integral inequalities of $\varphi(t)$-positive definite kernel, which are useful in studying the stability of the system with memory effects.

Proposition 3.6. Let $\varphi'(t) \leq N\varphi(t)$. Then $e^{-Nt}$ is strongly $\varphi(t)$-positive definite and
Proposition 3.7. Assume that $h(t) \in L^1_{loc}(0, \infty)$ is a strongly $\varphi(t)$-positive definite kernel. Let $\varphi(t) \leq N \varphi(t)$. Then for $u \in L^2_{loc}(0, \infty; H)$ with $u' \in L^1_{loc}(0, \infty; H)$, we have

$$
\int_0^t \varphi(s) \left\| \int_0^s e^{-N(s-\tau)} u(\tau) d\tau \right\|^2 ds 
\leq \frac{2}{N} \int_0^t \varphi(s) \left\langle \int_0^s e^{-N(s-\tau)} u(\tau), u(s) \right\rangle ds.
$$

(5)

Proof. Write

$$
H(s) := \int_0^s e^{-N(s-\tau)} u(\tau) d\tau.
$$

Then,

$$
\int_0^t \varphi(s) \left\langle \int_0^s e^{-N(s-\tau)} u(\tau), u(s) \right\rangle ds
= \int_0^t \varphi(s) \langle H(s), H'(s) + NH(s) \rangle ds
= N \int_0^t \varphi(s) \|H(s)\|^2 ds + \frac{1}{2} \int_0^t \varphi(s) \frac{d}{ds} \|H(s)\|^2 ds
= N \int_0^t \varphi(s) \|H(s)\|^2 ds + \frac{1}{2} \varphi(t) \|H(t)\|^2 - \frac{1}{2} \int_0^t \varphi'(s) \|H(s)\|^2 ds.
$$

Since $\varphi'(t) \leq N \varphi(t)$, we see the desired conclusion.

Proposition 3.7. Assume that $h(t) \in L^1_{loc}(0, \infty)$ is a strongly $\varphi(t)$-positive definite kernel. Let $\varphi(t) \leq N \varphi(t)$. Then for $u \in L^2_{loc}(0, \infty; H)$ with $u' \in L^1_{loc}(0, \infty; H)$, we have

$$
\int_0^t \varphi(s) \|u(s)\|^2 ds \leq C \|u(0)\|^2 + \frac{2N}{\delta} \int_0^t \varphi(s) \langle h * u(s), u(s) \rangle ds
+ \frac{4}{N\delta} \int_0^t \varphi(s) \langle h * u'(s), u'(s) \rangle ds, \quad t \geq 0,
$$

where $\delta, N$ are the constants in Definition 3.1.

Proof. Clearly,

$$
\int_0^t e^{-N(t-s)} u'(s) ds = u(t) - e^{-Nt} u(0) - N \int_0^t e^{-N(t-s)} u(s) ds.
$$

Therefore,

$$
\varphi(t) \|u(t)\|^2 = \left\langle \varphi(t) u(t), e^{-Nt} u(0) \right\rangle + \left\langle N \varphi(t) u(t), \int_0^t e^{-N(t-s)} u(s) ds \right\rangle
+ \left\langle \varphi(t) u(t), \int_0^t e^{-N(t-s)} u'(s) ds \right\rangle
\leq \frac{1}{4} \varphi(t) \|u(t)\|^2 + e^{-2Nt} \varphi(t) \|u(0)\|^2
+ \left\langle N \varphi(t) u(t), \int_0^t e^{-N(t-s)} u(s) ds \right\rangle + \frac{1}{4} \varphi(t) \|u(t)\|^2
+ \varphi(t) \left\| \int_0^t e^{-N(t-s)} u'(s) ds \right\|^2.
$$
So, \[
\varphi(t) \|u(t)\|^2 \leq 2e^{-2Nt} \varphi(t) \|u(0)\|^2 + 2\left\langle N\varphi(t)u(t), \int_0^t e^{-N(t-s)}u(s)ds \right\rangle \\
+ 2\varphi(t) \left\| \int_0^t e^{-N(t-s)}u'(s)ds \right\|^2.
\]

Thus, by (5), we get
\[
\int_0^t \varphi(s) \|u(s)\|^2 ds \\
\leq 2\int_0^t e^{-2Ns} \varphi(s) \|u(0)\|^2 ds + 2N \int_0^t \varphi(s) \left\langle \int_0^s e^{-N(s-\tau)}u(\tau)d\tau, u(s) \right\rangle ds \\
+ \frac{4}{N} \int_0^t \varphi(s) \left\langle \int_0^s e^{-N(s-\tau)}u'(\tau)d\tau, u'(s) \right\rangle ds.
\]

This with Definition 3.1 completes the proof of the Proposition 3.7.

Now, we give some examples.

**Proposition 3.8.**
(i) If \(h(t)\) is (strongly) \(e^{bt}\)-positive definite, then \(h(t) \cos at\) \((a \in \mathbb{R})\) is (strongly) \(e^{bt}\)-positive definite.
(ii) If \((t+1)^{1/2}h(t)\) is (strongly) positive definite, then \(h(t) \cos at\) \((a \in \mathbb{R})\) is (strongly) \((t+2)^{1/2}\)-positive definite.

**Proof.**
(i) It follows from Proposition 3.3 that \(e^{bt}h(t)\) is (strongly) positive definite. Hence, by Proposition 2.7, \(e^{bt}h(t) \cos at\), \(a \in \mathbb{R}\), is (strongly) \(e^{bt}\)-positive definite.

(ii) Similarly, by Proposition 3.5 and Proposition 2.7, we get the desired conclusion.

**Example 3.9.**
(i) If \(g(t) > 0\) and \(g'(t) \leq -k_0g(t)\) \((k_0 > 0)\), then \(\int_t^{+\infty} g(s)ds\) is strongly \(e^{2kt}\)-positive definite, where \(0 \leq k \leq \frac{k_0}{2}\).

(ii) If \(g(t) > 0\) and \(g'(t) \leq -\frac{k_0}{t+1}g(t)\) \((k_0 > 1)\), then \(\int_t^{+\infty} g(s)ds\) is strongly \((t+2)^{1/2}\)-positive definite, where \(0 < k < \min\{k_0 - 1, \frac{k_0}{2}\}\).

and if \(k_0 > 2\), \(k\) can be taken to be \(\frac{k_0}{2}\).

Moreover, if \(g(t) \geq k_2e^{-k_1t}\) \((k_1, k_2 > 0)\) and \(g'(t) \leq -\frac{k_0}{t+1}g(t)\) \((k_0 > 1)\), then \(\int_t^{+\infty} g(s)ds\) is strongly \((t+2)^{k_0-1}\)-positive definite,
(iii) If \(g(t) \geq \left(\frac{ka}{p}t + k_1\right)^{-p}\) and
\[g'(t) \leq -k_0 g^{1+p}(t) \quad (0 < k_0, \ 1 < p),\]
then \(\int_t^{+\infty} g(s) ds\) is strongly \((t + a)^{p-1}\)-positive definite, where \(a > \frac{k_1}{k_0} + 1\).

(iv) If \(g(t) = \frac{1}{\sqrt{t+1}}\) then \(\int_t^{+\infty} g(s) ds\) is strongly \(\sqrt{t+2}\)-positive definite.

Proof. (i) Since
\[e^{kt} \int_t^{+\infty} g(s) ds \leq e^{kt} \int_t^{+\infty} g(s) ds - e^{kt} g(t)\]
and
\[e^{kt} \int_t^{+\infty} g(s) ds \geq e^{kt} \int_t^{+\infty} g(s) ds - e^{kt} g(t)\]
we know that (i) holds in view of Proposition 2.3 and Proposition 3.3.

(ii) Since
\[(t + 1)^{k-1} \int_t^{+\infty} g(s) ds \leq -\frac{1}{k_0} (t + 1)^{k-1} \int_t^{+\infty} (s+1)g'(s) ds\]
\[\leq \frac{1}{k_0} (t + 1)^k g(t) + \frac{1}{k_0} (t + 1)^{k-1} \int_t^{+\infty} g(s) ds,\]
we see that
\[(t + 1)^{k-1} \int_t^{+\infty} g(s) ds \leq \frac{1}{k_0 - 1} (t + 1)^k g(t).\]
Hence
\[(t + 1)^k \int_t^{+\infty} g(s) ds \leq \frac{k}{k_0 - 1} (t + 1)^k g(t)\]

Moreover
\[e^{kt} \int_t^{+\infty} g(s) ds \leq e^{kt} \int_t^{+\infty} g(s) ds - e^{kt} g(t)\]
\[\geq \frac{k}{k_0 - 1} (t + 1)^k g(t) - (t + 1)^k g(t)\]
\[< 0.\]
So, it is clear that if \( k \geq 1 \), then
\[
\left( (t + 1)^k \int_t^{+\infty} g(s) ds \right)'' \geq 0.
\]
On the other hand, if \( 0 < k < 1 \), then
\[
\left( (t + 1)^k \int_t^{+\infty} g(s) ds \right)'' \geq \frac{k(k - 1)}{k_0 - 1} (t + 1)^{k-1} g(t) + \frac{(k - k_0)(k - (k_0 - 1))}{k_0 - 1} (t + 1)^{k-1} g(t) \geq 0.
\]
Thus, using Proposition 2.3 and Proposition 3.5, we know that (ii) holds.

By a similar argument, we obtain
\[
\left( \left( t + \frac{3}{2} \right)^{k_0 - 1} \int_t^{+\infty} g(s) ds - \delta e^{-2k_1 t} \right)'' \leq 2k_1 \delta e^{-2k_1 t} - \frac{1}{2} \left( t + \frac{3}{2} \right)^{k_0 - 2} g(t) \leq 2k_1 \delta e^{-2k_1 t} - \frac{1}{2} k_2 \left( t + \frac{3}{2} \right)^{k_0 - 2} e^{-k_1 t}.
\]

Therefore, there exists \( \delta > 0 \) such that
\[
\left( \left( t + \frac{3}{2} \right)^{k_0 - 1} \int_t^{+\infty} g(s) ds - \delta e^{-2k_1 t} \right)'' \leq 0.
\]

Then, Proposition 3.5 and Proposition 2.8 tell us that \( \int_t^{+\infty} g(s) ds \) is strongly \( (t + 2)^{k_0 - 1} \)-positive definite.

Thus, (ii) holds.

(iii) From the assumptions, it follows that
\[
\left( \frac{k_0}{p} t + k_1 \right)^{-p} \leq g(t) \leq \left( \frac{k_0}{p} t + g^{-\frac{1}{p}}(0) \right)^{-p}, \quad t \geq 0.
\]

Hence, taking \( a_1 = 1 + \frac{pk_1}{k_0} \) yields that
\[
\left( (t + a_1)^{p-1} \int_t^{+\infty} g(s) ds \right)'' = (p - 1)(t + a_1)^{p-2} \int_t^{+\infty} g(s) ds - (t + a_1)^{p-1} g(t) = (p - 1)(t + a_1)^{p-2} \int_t^{+\infty} g^{1 + \frac{1}{p}}(s) g^{-\frac{1}{p}}(s) ds - (t + a_1)^{p-1} g(t) \leq -\frac{1}{k_0} (p - 1)(t + a_1)^{p-2} \int_t^{+\infty} g^{1 + \frac{1}{p}}(s) g^{-\frac{1}{p}}(s) ds - (t + a_1)^{p-1} g(t) \leq \frac{1}{1 - \frac{1}{p} k_0} (p - 1)(t + a_1)^{p-2} g^{1 + \frac{1}{p}}(t) - (t + a_1)^{p-1} g(t)
\]
\[
\begin{align*}
\leq & \quad (t + a_1)^{-2} g(t) \left( \frac{p}{k_0} g^{-\frac{1}{p}}(t) - (t + a_1) \right) \\
\leq & \quad (t + a_1)^{-2} g(t) \left( \frac{pk_1}{k_0} - a_1 \right) \\
\leq & \quad -(t + a_1)^{-2} g(t) \\
\leq & \quad -(t + a_1)^{-2} \left( \frac{k_0}{p} t + k_1 \right)^{-p}.
\end{align*}
\]

Therefore, there exists \( \delta > 0 \) such that
\[
\frac{(t + a_1)^{p-1}}{\int_t^{+\infty} g(s)ds - \delta e^{-t}}' \leq 0.
\]

Then, Proposition 3.5 and Proposition 2.8 tell us that \( \int_t^{+\infty} g(s)ds \) is strongly \((t + a)^{p-1}\)-positive definite.

(iv) Notice
\[
\left( \sqrt{t+1} \int_t^{+\infty} g(s)ds \right)' = \frac{1}{2\sqrt{t+1}} (\pi - 2 \arctan \sqrt{t}) - \frac{1}{\sqrt{t+1}\sqrt{t}} \leq 0,
\]
and
\[
\left( \sqrt{t+1} \int_t^{+\infty} g(s)ds \right)'' = \frac{1}{4(t+1)^{-\frac{1}{2}}} \left( 2 \frac{2t+1}{t^2} - 2t^{-\frac{1}{2}} - \pi + 2 \arctan \sqrt{t} \right) \geq 0.
\]

Thus, using Proposition 2.3 and Proposition 3.5, we know that (iv) holds. \( \square \)

**Example 3.10.**

(i) If \( h(t) = e^{-bt} \cos at \), then \( h(t) \) is strongly \( e^{2ct} \)-positive definite, where \( 0 \leq c < b \).

(ii) If \( h(t) = (t+1)^{-b} \cos at \) \((b > 0)\), then \( h(t) \) is strongly \((t+2)^{c}\)-positive definite, where \( 0 \leq c < b \).

**Proof.**

(i) Proposition 2.3 tells us that \( e^{-dt} \) \((d > 0)\) is strongly positive definite. Moreover, it follows from Proposition 3.3 that \( e^{-bt} \) \((b > 0)\) is strongly \( e^{2ct} \)-positive definite for \( 0 \leq c < b \). Therefore, by using Proposition 3.8, we infer that \( h(t) = e^{-bt} \cos at \) is strongly \( e^{2ct} \)-positive definite for \( 0 \leq c < b \).

(ii) Similarly, by using Proposition 2.3, Proposition 3.5 and Proposition 3.8, we deduce that \( h(t) = (t+1)^{-b} \cos at \) \((b > 0)\) is strongly \((t+2)^{c}\)-positive definite for \( 0 \leq c < b \). \( \square \)

**Example 3.11.**

(i) If \( g(t) = be^{-bt} \cos at + ae^{-bt} \sin at \), then \( \int_t^{+\infty} g(s)ds \) is strongly \( e^{2ct} \)-positive definite, where \( 0 \leq c < b \).

(ii) If \( g(t) = b(t+1)^{-b-1} \cos at + a(t+1)^{-b} \sin at \) \((b > 0)\), then \( \int_t^{+\infty} g(s)ds \) is strongly \((t+2)^{c}\)-positive definite where \( 0 < c < b \).
Proof. Since
\[ \begin{align*}
be^{-bt}\cos at + ae^{-bt}\sin at &= -(e^{-bt}\cos at)', \\
b(t + 1)^{-b-1}\cos at + a(t + 1)^{-b}\sin at &= -(a(t + 1)^{-b}\cos at)',
\end{align*} \]
we justify the conclusions as required by using Example 3.10.

4. Uniform stability result for the problem (1). In this section and the following one, we denote 
\[ G(t) := \int_0^t g(s)ds. \]

(H1) A is a positive self-adjoint linear operator in Hilbert space \( H \), satisfying
\[ \mu_0\langle Au, u \rangle \geq \| u \|^2, \quad u \in \mathcal{D}(A), \]
for a constant \( \mu_0 > 0 \).

(H2) \( g(t) \in L^1(0, +\infty) \) satisfies \( G(0) < 1, (t + 1)^{2\lambda_0 + 1}g^2(t) \in L^1(1, +\infty), (t + 1)^{\lambda_0}G(t) \) is bounded, and \( t \mapsto G(t) \) is strongly \( (t + 1)^{\lambda_0} - \) positive definite, where \( \lambda_0 \geq 0 \) is a nonnegative number.

(H3) \( f : [D(\sqrt{A})] \to H \) is a locally Lipschitz continuous gradient operator with \( f(0) = 0 \), and either
\[ (f(u), u) \leq 0, \quad u \in \mathcal{D}(\sqrt{A}), \quad \text{(6)} \]
or there is an increasing continuous function \( \phi : [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that
\[ |(f(u), u)| \leq \phi \left( \left\| \sqrt{A}u \right\| \right) \left\| \sqrt{A}u \right\|^2, \quad \forall u \in \mathcal{D}(\sqrt{A}). \]

Write
\[ E(t) = \frac{1}{2} \| u'(t) \|^2 + \frac{1}{2} \left\| \sqrt{A}u(t) \right\|^2 - \frac{G(0)}{2} \left\| \sqrt{A}u(t) \right\|^2 - F(u(t)), \quad \text{(7)} \]
which stands for the energy of the problem (1). Here \( F \) is functional on \( \mathcal{D} \left( \sqrt{A} \right) \) and satisfies \( F(u) = \int_0^1 \langle f(tu), u \rangle dt, \quad u \in \mathcal{D} \left( \sqrt{A} \right) \). The global existence of the problem (1) and the exponential decay of the energy can be found in [4].

**Theorem 4.1.** Let (H1), (H2) and (H3) hold. Then there exists \( \varrho > 0 \) such that for any \( u_0 \in \mathcal{D}(A), u_1 \in \mathcal{D}(\sqrt{A}) \), with \( \left\| \sqrt{A}u_0 \right\| \) and \( \| u_1 \| < \varrho \), the solution energy \( E(t) \) of the problem (1) satisfies that
\[ E(t) \leq CE(0)(t + 1)^{-\lambda_0}, \quad t \geq 0, \quad \text{(8)} \]
and
(i) if \( 1 < \lambda_0 \leq 2 \), then for \( \lambda < 2(\lambda_0 - 1), \)
\[ \int_0^{+\infty} (s + 1)^{\lambda} E(s)ds \leq CE(0); \]
(ii) if \( \lambda_0 > 2 \), then
\[ \int_0^{+\infty} (s + 1)^{\lambda_0} E(s)ds \leq CE(0); \]
(iii) if \( \lambda_0 \leq 1 \), then
\[ \int_0^{+\infty} E(s)ds \leq CE(0). \]
Moreover, if \( f = 0 \), we can take \( \rho = +\infty \) and the energy \( E(t) \) always satisfies
\[
E(t) \leq C E(0)(t + 1)^{-\lambda_0}, \quad t \geq 0;
\]
and if \( \lambda_0 > \frac{1}{2} \),
\[
\int_0^{+\infty} (s + 1)^{\lambda_0} E(s) ds \leq C E(0), \quad t \geq 0.
\]

**Remark 4.2.**
(i). Noticing the assumption (H\(_2\)) on the memory kernel \( g \), we use GPDK to make some conditions on the primitives of \( g \), without assuming nonnegativity and monotonicity of \( g \). From section 2 and section 3, we see that the result obtained here is suitable for a wide class of functions, including lots of oscillating or sign-varying functions as well as the decreasing/nonnegative functions. For instance, for suitable \( \alpha_i > 0, p_i > 0 \),
- \( \alpha_1(t + \alpha_2)^{-p_i-1} \), decreasing with locally absolutely continuous kernel,
- \( \frac{\alpha_3}{\sqrt{t(t + \alpha_4)^{p_2+1}}} \), decreasing without locally absolutely continuous kernel,
- \( p_3(t+\alpha_6)^{-p_3-1} \cos \alpha_5 t + \alpha_5(t+\alpha_6)^{-p_3} \sin \alpha_5 t \), oscillating and sign-varying kernel.

(ii). If \( g(t) > 0 \) and \( g'(t) \leq 0 \), we see from Proposition 2.3 that \( G(t) \) is a strongly positive definite kernel. Therefore, when the assumption (H\(_2\)) is replaced by \( g \) being nonnegative and non-increasing, the energy \( E(t) \) of (1) is integrable on \([0, +\infty)\). This would enable us to be much easier to get the general (exponential/polynomial) decay rate, if \( g' \) is controlled by \( g \) in addition. Limited by space the relevant problem will be considered in a forthcoming paper, for some concrete viscoelastic systems involving local memory effects.

Let us close this section by the following applications to PDEs. Let \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary.

**Example 4.3.** Define the operator \( A : D(A) \subset L^2(\Omega) \to L^2(\Omega) \) by
\[
Au(x) = -\Delta u(x), \quad u \in D(A), \quad x \in \Omega \text{ a.e.,}
\]
with \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). Take
\[
f(u) = |u|^\gamma u, \quad 0 < \gamma \leq 2.
\]
According to [2, 4], \( A, f \) satisfy Assumptions (H\(_1\)) and (H\(_3\)). Thus, (1) becomes an initial-boundary value problem for semi-linear wave equations.

**Example 4.4.** Assume that \( a_{ijkl} \in C^1(\overline{\Omega}), i, j, k, l \in \{1, 2, 3\} \) is symmetric and coercive. That is, for any real symmetric matrix \( (\xi_{ij}) \) and some constants \( a_0 > 0 \),
\[
a_{ijkl} = a_{jikl} = a_{klji}, \quad \sum_{i,j,k,l=1}^3 a_{ijkl}(x)\xi_{kl}\xi_{ij} \geq a_0 \sum_{i,j=1}^3 \xi_{ij}^2, \quad x \in \Omega.
\]
Define the operator \( A \) as in [32, Example 3.6]. Then, (1) \( (f = 0) \) becomes the linear anisotropic elasticity model. Thus, by our abstract result Theorem 4.1, we can get a variety of energy decay rates corresponding to the kernel \( g \).
5. **Proofs of the stability result.** In this section, we will apply GPDK and its properties obtained in section 3 to prove the stability theorems stated in the above section. The main idea behind is: Firstly, we use Proposition 3.7 to estimate \( \int_0^t |\psi(s)| \left\| \sqrt{A}u(s) \right\|^2 ds \) for the problem (1). Secondly, we estimate the rest of energy terms in terms of \( \int_0^t |\psi(s)| \left\| \sqrt{A}u(s) \right\|^2 ds \). For this purpose, we shall first build some lemmas.

5.1. **Some lemmas. Assumption(A).**

(A) There is a positive function \( \psi(t) \in C^2([0, +\infty)) \) such that either

(a) \( \psi(t) \) is a constant or decreasing function, or

(b) \( \psi(t) \) is a monotonic increasing function satisfying

\[
\frac{|\psi(t)g(t)|^2}{\psi'(t)} \in L^1(1, +\infty), \quad \psi(t)G(t) \text{ is bounded},
\]

\( t \mapsto G(t) \) is strongly \( \psi(t) \) – positive definite.

**Lemma 5.1.** For any \( \psi \) satisfying Assumption (A), we have

\[
\psi(t)E(t) + \int_0^t \psi(s) (G * Au(s), u'(s)) ds \leq C_1 E(0) + C_1 \int_0^t |\psi'(s)| \left( \|u'(s)\|^2 + \left\| \sqrt{A}u(s) \right\|^2 \right) ds + C_1 \int_0^t \psi(s) (f, u'(t)) ds,
\]

and

\[
\int_0^t \psi(s) (G * u'(s), u'(s)) ds \leq C_2 E(0) + C_2 \int_0^t |\psi'(s)| \left( \|u'(s)\|^2 + \left\| \sqrt{A}u(s) \right\|^2 \right) ds + C_2 \int_0^t \psi(s) (f, A^{-1}u'(t)) ds,
\]

**Proof.** **Step 1.** Multiplying the equation

\[
u''(t) + Au(t) - \int_0^t g(t - s)Au(s)ds = f(u(t))
\]

by \( \psi(t)u'(t) \) gives that

\[
\frac{1}{2} \psi(t) \frac{d}{dt} \left( \|u'(t)\|^2 + \left\| \sqrt{A}u(t) \right\|^2 - G(0) \left\| \sqrt{A}u(t) \right\|^2 \right) + \psi(t) (G * Au(t), u'(t)) = - \psi(t) \langle G(t)Au_0, u'(t) \rangle + \psi(t) \langle f, u'(t) \rangle.
\]

Integrating (12) on \([0, t] \) yields that

\[
\frac{1}{2} \psi(t) \left( \|u'(t)\|^2 + \left\| \sqrt{A}u(t) \right\|^2 - G(0) \left\| \sqrt{A}u(t) \right\|^2 \right) + \int_0^t \psi(s) (G * Au'(s), u'(s)) ds
\]
Thus, we complete the proof of (9).

On the other hand, if $s$ satisfies (ii) (a), (9) is obvious.

If $\psi$ satisfies (ii) (b), observe

$$
=\frac{1}{2}\psi(0)\left(\|u_1\|^2 + \|\sqrt{A}u_0\|^2 + \|G(0)\|\sqrt{A}u_0\|^2\right) + \frac{1}{2}\int_0^t \psi'(s)\left(\|u'(s)\|^2 + \|\sqrt{A}u(s)\|^2 - G(0)\|\sqrt{A}u(t)\|^2\right)ds
$$

$$
-\psi(t)G(t)\langle Au_0, u(t)\rangle + \int_0^t \psi'(s)G(s)\langle Au_0, u(s)\rangle ds
-\int_0^t \psi(s)g(s)\langle Au_0, u(s)\rangle ds + \int_0^t \psi(s)\langle f, u'(t)\rangle ds.
$$

and

$$
-\int_0^t \psi(s)g(s)\langle Au_0, u(s)\rangle ds
\leq \int_0^t \psi(s)|G(s)|\|\sqrt{A}u_0\|\|\sqrt{A}u(s)\|ds
\leq \frac{1}{4}\|\sqrt{A}u_0\|^2 + \left(\int_0^t |\psi'(s)||G(s)||\sqrt{A}u(s)||ds\right)^2
$$

$$
-\psi(t)G(t)\langle Au_0, u(t)\rangle + \int_0^t \psi'(s)G(s)\langle Au_0, u(s)\rangle ds
-\int_0^t \psi(s)g(s)\langle Au_0, u(s)\rangle ds + \int_0^t \psi(s)\langle f, u'(t)\rangle ds.
$$

and

$$
-\int_0^t \psi(s)g(s)\langle Au_0, u(s)\rangle ds
\leq \int_0^t \psi(s)|G(s)||\sqrt{A}u_0||\sqrt{A}u(s)||ds
\leq \frac{1}{2}\|\sqrt{A}u_0\|^2 + \frac{1}{2}\left(\int_0^t \psi(s)|g(s)||\sqrt{A}u(s)||ds\right)^2
$$

$$

Noticing $\psi(t)G(t)$ is bounded, we derive (9) by Young’s inequality under (ii) (b).

Thus, we complete the proof of (9).
Step 2. Multiply (11) by $\psi(t)A^{-1}u'(t)$ and integrate from 0 to $t$. Then we have
\[
\frac{1}{2} \psi(t) \left( \|u(t)\|^2 - G(0) \langle u(t), u(t) \rangle \right) + \int_0^t \psi(s) \langle G * u'(s), u'(s) \rangle \, ds
\]
\[
\leq \frac{1}{2} \psi(0) \left( \|A^{-\frac{1}{2}} u_1\|^2 + \|u_0\|^2 + G(0) \|u_0\|^2 \right) + \int_0^t \psi'(s) \left( \|A^{-\frac{1}{2}} u'(s)\|^2 + \|u(s)\|^2 - G(0) \langle u(s), u(s) \rangle \right) \, ds
\]
\[
- \psi(t)G(t) \langle u_0, u(t) \rangle + \int_0^t \psi'(s)G(s) \langle u_0, u(s) \rangle \, ds
\]
\[
- \int_0^t \psi(s)g(s) \langle u_0, u(s) \rangle \, ds
\]
\[
+ \int_0^t \psi(s) \langle f, A^{-1}u'(t) \rangle \, ds.
\]
Then (13) will lead to (10) by a similar argument as in step 1. Thus, we finish the proof.

Lemma 5.2. For any $\psi(t)$ satisfying Assumption (A), we have
\[
\int_0^t \psi(s) \langle G * Au(s), u(s) \rangle \, ds
\]
\[
\leq C_4 E(0) + C_4 \int_0^t \psi'(s) \left( \|u'(s)\|^2 + \|\sqrt{A}u(s)\|^2 \right) \, ds
\]
\[
+ C_4 \int_0^t \psi'(s) \left( \|G * \sqrt{A}u(s)\|^2 \right) \, ds
\]
\[
+ C_4 \int_0^t \psi(s) \langle f, u'(t) \rangle \, ds
\]
\[
+ C_4 \int_0^t \psi(s) \langle f, A^{-1}u'(t) \rangle \, ds
\]
\[
+ \int_0^t \psi(s) \langle G * u(s), f \rangle \, ds.
\]

Proof. From (11) and
\[
\frac{d}{dt}(G * u(t)) = G(t)u(0) + G * u'(t), g * u(t) = G(0)u(t) - \frac{d}{dt}(G * u(t)),
\]
it follows that
\[
\int_0^t \psi(s) \langle G * Au(s), u(s) \rangle \, ds
\]
\[
= \int_0^t \psi(s) \langle G * u(s), Au(s) \rangle \, ds
\]
\[
= \int_0^t \psi(s) \langle G * u(s), g * Au(s) \rangle \, ds - \int_0^t \psi(s) \langle G * u(s), u''(s) \rangle \, ds
\]
\[
+ \int_0^t \psi(s) \langle G * u(s), f \rangle \, ds
\]
Proof of Theorem 4.1.

Lemma 5.3. For any \( \psi(t) \) satisfying Assumption (A), we have

\[
\int_0^t \psi(s) \| \sqrt{A}u(s) \|^2 \, ds \\
\leq C_5 E(0) + C_5 \int_0^t |\psi'(s)| \left( \| u'(s) \|^2 + \| \sqrt{A}u(s) \|^2 \right) \, ds \\
+ C_5 \int_0^t |\psi(s)| \left( |A^{-1}u'(s)| \right)^2 \, ds + C_5 \int_0^t |\psi(s)| |A^{-1}u'(t)| \, ds + C_5 \int_0^t \psi(s) \| G * u(s) \|, f) \, ds.
\]

(15)

Proof. Using (9), (14) and Proposition 3.7 ends the proof.

5.2. Proof of Theorem 4.1.

Proof. Step 1. Take

\[ \psi(t) = (t + 1)^\lambda \quad (for \ \lambda \leq \lambda_0). \]

Then, by (9) and the energy definition (7), we have

\[
(t + 1)^\lambda E(t) \\
\leq D_3 E(0) + D_3 |\lambda| \int_0^t (s + 1)^{\lambda - 1} \left( \| u'(s) \|^2 + \| \sqrt{A}u(s) \|^2 \right) \, ds.
\]

(16)
By (15), we obtain
\[
\int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \\
\leq D_4 E(0) + D_4 |\lambda| \int_0^t (s + 1)^{\lambda - 1} \left( \left\| u'(s) \right\|^2 + \left\| \sqrt{A}u(s) \right\|^2 \right) ds \\
+ D_4 |\lambda| \int_0^t (s + 1)^{\lambda - 1} \left( |G| \right\| \sqrt{A}u(s) \right\|^2 ds \\
+ D_4 \int_0^t (s + 1)^\lambda \langle f(u(s)), A^{-1}u'(s) \rangle ds \\
+ D_4 \int_0^t (s + 1)^\lambda \langle G * u(s), f(u(s)) \rangle ds.
\]

**Step 2.** Observe
\[
\frac{d}{dt} (-\langle u'(t), u(t) \rangle) \\
= -\|u'(t)\|^2 + \left\| \sqrt{A}u(t) \right\|^2 - \langle g * \sqrt{A}u(t), \sqrt{A}u(t) \rangle - \langle f(u(s)), u(t) \rangle \\
\leq -\|u'(t)\|^2 + C \left\| \sqrt{A}u(t) \right\|^2 + \left\| g * \sqrt{A}u(t) \right\|^2 - \langle f(u(s)), u(t) \rangle,
\]
and
\[
\int_0^t (s + 1)^\lambda \left\| g * \sqrt{A}u(s) \right\|^2 ds \\
\leq D_5 \int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds, \quad \text{for any } \lambda \leq \lambda_0,
\]
according to the assumption (H_2). Then, multiplying (18) by (t + 1)^\lambda (for any \lambda \leq \lambda_0) and integrating from 0 to t yields that
\[
\int_0^t (s + 1)^\lambda \|u'(s)\|^2 ds \\
\leq D_6 E(0) + C(t + 1)^\lambda E(t) + D_6 \int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \\
- \int_0^t (s + 1)^\lambda \langle f(u(s)), u(s) \rangle ds.
\]
Thus, by (16), we have
\[
\int_0^t (s + 1)^\lambda \|u'(s)\|^2 ds \\
\leq D_7 E(0) + D_7 \int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \\
- \int_0^t (s + 1)^\lambda \langle f(u(s)), u(s) \rangle ds.
\]
Since this is true for any \lambda \leq \lambda_0, we see that
\[ \int_0^t (s + 1)^{\lambda - 1} \| u'(s) \|^2 ds \leq D_8 E(0) + D_8 \int_0^t (s + 1)^{\lambda - 1} \| \sqrt{A}u(s) \|^2 ds \]

(21)

Case (I) The linear case \( f = 0 \).

Step 3. By (17) with \( f = 0 \), we get

\[ \int_0^t (s + 1)^{\lambda - 1} \| Au'(s) \|^2 ds \leq D_9 E(0) + D_9 |\lambda| \int_0^t (s + 1)^{\lambda - 1} \| u'(s) \|^2 ds \]

(22)

Putting (21) into (22) with \( f = 0 \) gives that

\[ \int_0^t (s + 1)^{\lambda - 1} \| Au'(s) \|^2 ds \leq D_{10} E(0) + D_{10} |\lambda| \int_0^t (s + 1)^{\lambda - 1} \| G * \sqrt{A}u(s) \|^2 ds \]

(23)

Taking \( \lambda = 0 \), we get

\[ \int_0^t \| \sqrt{A}u(s) \|^2 ds \leq CE(0). \]

(24)

Thus, (21) with \( f = 0 \) implies that

\[ \int_0^t \| u'(s) \|^2 ds \leq CE(0). \]

(26)
Combining (25), (26) and (16) together yields that
\[ E(t) \leq CE(0)(t + 1)^{-\lambda_0}. \]  

**Step 4.** Notice that
\[ \int_0^t (s + 1)^{\lambda - 1} \left( |G| \ast \|\sqrt{A}u(s)\| \right)^2 \, ds \]
\[ \leq D_{13} \int_0^t (s + 1)^{\lambda - 1} \left( (s + 1)^{-\lambda_0} \ast \|\sqrt{A}u(s)\| \right)^2 \, ds \]
\[ \leq D_{13} \int_0^t (s + 1)^{\lambda - 1} \|\sqrt{A}u(s)\|^2 \, ds, \quad \text{for } \lambda \leq \lambda_0. \]

By Lemma A.4, we get
\[ \int_0^t (s + 1)^{\lambda - 1} \left( |G| \ast \|\sqrt{A}u(s)\| \right)^2 \, ds \]
\[ \leq D_{14} \int_0^t (s + 1)^{\lambda - 1} \left( (s + 1)^{\frac{1}{2} - \lambda_0} \ast \|\sqrt{A}u(s)\| \right)^2 \, ds \]
\[ \leq D_{14} \int_0^t (s + 1)^{\lambda_0} \|\sqrt{A}u(s)\|^2 \, ds, \quad \text{for } \lambda \leq \lambda_0, \]

if \( \frac{1}{2} < \lambda_0 \leq 1 \). Thus, by (23), we have, for \( \lambda_0 > \frac{1}{2} \),
\[ \int_0^t (s + 1)^{\lambda_0} \|\sqrt{A}u(s)\|^2 \, ds \leq CE(0), \]  

(28)

Therefore,
\[ \int_0^t (s + 1)^{\lambda_0} \|u'(s)\|^2 \, ds \leq CE(0), \]  

(29)

by virtue of (28) and (20). Combining (28) and (29) yields that, for \( \lambda_0 > \frac{1}{2} \),
\[ \int_0^t (s + 1)^{\lambda_0} E(s) \, ds \leq CE(0). \]  

(30)

**Case (II)** The nonlinear case \( f \neq 0 \).

**Step 5.** According to Lemma A.2, we have, for any \( 0 < \lambda \leq \lambda_0 \), \( \varepsilon > 0 \),
\[ \int_0^t (s + 1)^{\lambda} \langle f(u(s)), A^{-1}u'(s) \rangle \, ds \]
\[ \leq \varepsilon \int_0^t (s + 1)^{\lambda} \left( \|u'(s)\|^2 + \|\sqrt{A}u(s)\|^2 \right) \, ds, \]  

(31)

and
\[ \int_0^t (s + 1)^{\lambda} \langle f(u(s)), u(s) \rangle \, ds \leq \varepsilon \int_0^t (s + 1)^{\lambda} \|\sqrt{A}u(s)\|^2 \, ds, \]  

(32)

\[ \int_0^t (s + 1)^{\lambda - 1} \langle f(u(s)), u(s) \rangle \, ds \leq \varepsilon \int_0^t (s + 1)^{\lambda} \|\sqrt{A}u(s)\|^2 \, ds. \]  

(33)
Moreover, since $(t + 1)^{\lambda_0} G(t)$ is bounded, we see that for any $0 < \lambda < 2(\lambda_0 - 1)$, $\epsilon > 0$,

$$\int_0^t (s + 1)^\lambda \langle f(u(s)), G \ast u(s) \rangle ds \leq \epsilon \int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds,$$

(34)

if $\lambda_0 > 1$.

**Step 6.** According to Lemma A.3 and (16), we have

$$E(t) \leq CE(0)(t + 1)^{-\lambda_0},$$

(35)

if $\lambda_0 \leq 1$.

**Step 7.** If $1 < \lambda_0 \leq 2$, taking $\lambda < 2(\lambda_0 - 1) \leq \lambda_0$ yields that

$$\int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \leq D_{15} E(0) + D_{15} \int_0^t (s + 1)^{\lambda - 1} \left( \left\| u'(s) \right\|^2 + \left\| \sqrt{A}u(s) \right\|^2 \right) ds \tag{36}$$

by virtue of (28), (31), (32), (33) and (34). Moreover, for any $\lambda < 2(\lambda_0 - 1) \leq \lambda_0$, it is clear from (20) and (21) that

$$\int_0^t (s + 1)^\lambda \left\| u'(s) \right\|^2 ds \leq D_{16} E(0) + D_{16} \int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds, \tag{37}$$

$$\int_0^t (s + 1)^{\lambda - 1} \left\| u'(s) \right\|^2 ds \leq D_{17} E(0) + D_{17} \int_0^t (s + 1)^{\lambda - 1} \left\| \sqrt{A}u(s) \right\|^2 ds. \tag{38}$$

Combining (36), (37) and (38) yields that

$$\int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \leq D_{18} E(0) + D_{18} \epsilon \int_0^t (s + 1)^{\lambda} \left\| \sqrt{A}u(s) \right\|^2 ds. \tag{39}$$

Thus, by Lemma A.2 we can take initial data $\varrho$ small sufficiently such that $D_{18} \epsilon \leq \frac{1}{2}$. Hence, for any $\lambda < 2(\lambda_0 - 1)$,

$$\int_0^t (s + 1)^\lambda \left\| \sqrt{A}u(s) \right\|^2 ds \leq CE(0). \tag{40}$$

Putting (40) into (37) gives that, for any $\lambda < 2(\lambda_0 - 1)$,

$$\int_0^t (s + 1)^\lambda \left\| u'(s) \right\|^2 ds \leq CE(0).$$

Therefore, if $1 < \lambda_0 \leq 2$, for $\lambda < 2(\lambda_0 - 1)$,

$$\int_0^t (s + 1)^\lambda E(s) ds \leq CE(0). \tag{41}$$
Step 8. If $\lambda_0 > 2$, then $\lambda_0 < 2(\lambda_0 - 1)$. Arguing similarly to Step 7, we have
\[
\int_0^t (s + 1)^{\lambda_0} E(s) ds \leq CE(0) . \tag{42}
\]

Step 9. Since $\lambda_0 - 1 < \min\{\lambda_0, 2(\lambda_0 - 1)\}$, (16), (41) and (42) imply that, for $\lambda_0 > 1$,
\[
\frac{1}{2} (t + 1)^{\lambda_0} E(t) \leq CE(0). \tag{43}
\]
Thus, (35) and (43), (8) is true.

Then we complete the proof. \qed

Remark 5.4. Consider the following condition:
$(H_2')$ $g(t) \in L^1(0, +\infty)$ with $G(0) < 1$, and there is a constant $\lambda_0 \geq 0$ such that $(t + 1)^{\lambda_0} g(t) \in L^1(0, +\infty)$, $t \mapsto g(t)$ is strongly $(t + 1)^{\lambda_0}$ – positive definite. If replacing $(H_2)$ with $(H_2')$, one can check that Lemma 5.1–Lemma 5.3, (19), (34) still hold, and so does Theorem 4.1.

Appendix A. Appendix.

Lemma A.1. (See [4]) Let $(H_1)$, $(H_2)$ and $(H_3)$ hold. Then there exists $\varrho > 0$ such that for $u_0 \in D(A)$, $u_1 \in D(\sqrt{A})$, with $\|u_1\|$, $\|\sqrt{A}u_0\| < \varrho$, (1) has a unique solution $u(t)$ on $[0, \infty)$, and its energy satisfies that for $t \geq 0$,
\[
E(t) \geq \frac{1}{2} \|u'(t)\|^2 + c_1 \left\| \sqrt{A}u(t) \right\|^2, \quad E(t) \leq CE(0),
\]
where $C, c_1 > 0$ are constants with $c_1 < \frac{1-G(0)}{2}$. Moreover, one can take $\varrho = +\infty$ in the case of (6).

Lemma A.2. (See [4]) Assume that $(H_1)$, $(H_2)$ and $(H_3)$ hold. For any $\varepsilon > 0$, there exists $\varrho(\varepsilon) > 0$ satisfying $\left\| \sqrt{A}u_0 \right\|$ and $\|u_1\| < \varrho(\varepsilon)$ for $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$ such that the solution of the problem (1) satisfies that for any $t \geq 0$,
\[
\phi \left( \left\| \sqrt{A}u(t) \right\| \right) < \varepsilon.
\]

Lemma A.3. (See [4]) Let $(H_1)$, $(H_2')$ and $(H_3)$ hold. There exists $\varrho > 0$ such that for any $u_0 \in D(A)$, $u_1 \in D(\sqrt{A})$, with $\left\| \sqrt{A}u_0 \right\|$, $\|u_1\| < \varrho$, the solution energy $E(t)$ of the problem (1) satisfies
\[
E(t) \to 0 \text{ as } t \to +\infty, \quad \int_0^t E(s) ds \leq CE(0) \quad (t \geq 0),
\]
where $C > 0$ is a constant. In addition, one can take $\varrho = +\infty$ if $f = 0$.

The proofs of the above three lemmas are similar to those of [4, Theorem 3.6, Corollary 3.7 and Theorem 4.3], and we omit them here.

Lemma A.4. For any $\alpha, \beta > 0$,
\[
\int_0^t (t - s + 1)^{\alpha-1}(s + 1)^{\beta-1} ds \leq c_3 t^{\alpha+\beta-1}, \quad t \geq 0,
\]
where $c_3 > 0$ is a constant.

Proof. Immediate from the property of the Euler integral. \qed
REFERENCES

[1] S. Acosta and B. Palacios, Thermoacoustic tomography for an integro-differential wave equation modeling attenuation, *J. Differential Equations*, 264 (2010), 1984–2010.

[2] F. Alabau-Boussouira, P. Cannarsa and D. Sforza, Decay estimates for second order evolution equations with memory, *J. Funct. Anal.*, 254 (2008), 1342–1372.

[3] P. Cannarsa and D. Sforza, Semilinear integrodifferential equations of hyperbolic type: Existence in the large, *Mediterr. J. Math.*, 1 (2004), 151–174.

[4] P. Cannarsa and D. Sforza, Integro-differential equations of hyperbolic type with positive definite kernels, *J. Differential Equations*, 250 (2011), 4289–4335.

[5] M. M. Cavalcanti, F. R. Dias Silva, V. N. Domingos Cavalcanti and A. Vicente, Stability for the mixed problem involving the wave equation, with localized damping, in unbounded domains with finite measure, *SIAM J. Control Optim.*, 56 (2018), 2802–2834.

[6] M. M. Cavalcanti, V. N. Domingos Cavalcanti, M. A. Jorge Silva and A. Y. de Souza Franco, Exponential stability for the wave model with localized memory in a past history framework, *J. Differential Equations*, 264 (2018), 6535–6584.

[7] M. M. Cavalcanti, I. Lasiecka and D. Toundykov, Wave equation with damping affecting only a subset of static Wentzell boundary is uniformly stable, *Trans. Amer. Math. Soc.*, 364 (2012), 5693–5713.

[8] C. M. Dafermos and J. A. Nokel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, *Comm. Partial Differential Equations*, 4 (1979), 219–278.

[9] B. de Andrade and A. Viana, Abstract Volterra integrodifferential equations with applications to parabolic models with memory, *Math. Ann.*, 369 (2017), 1131–1175.

[10] V. Georgiev, B. Rubino and R. Sampalmieri, Global existence for elastic waves with memory, *Arch. Ration. Mech. Anal.*, 176 (2005), 303–330.

[11] G. Gripenberg, S. O. Londen and O. J. Staffans, *Volterra Integral and Functional Equations*, Encyclopedia Math. Appl., vol. 34, Cambridge Univ. Press, Cambridge, 1990.

[12] W. J. Hrusa, Global existence and asymptotic stability for a semilinear Volterra equation with large initial data, *SIAM J. Math. Anal.*, 16 (1985), 110–134.

[13] K.-P. Jin, J. Liang and T.-J. Xiao, Coupled second order evolution equations with fading memory: Optimal energy decay rate, *J. Differential Equations*, 257 (2014), 1501–1528.

[14] K.-P. Jin, J. Liang and T.-J. Xiao, Uniform stability of semilinear wave equations with arbitrary local memory effects versus frictional dampings, *J. Differential Equations*, 266 (2019), 7230–7263.

[15] K.-P. Jin, J. Liang and T.-J. Xiao, Asymptotic behavior for coupled systems of second order abstract evolution equations with one infinite memory, *J. Math. Anal. Appl.*, 475 (2019), 554–575.

[16] S. Kawashima, Global solutions to the equation of viscoelasticity with fading memory, *J. Differential Equations*, 101 (1993), 388–420.

[17] I. Lasiecka, S. A. Messaoudi and M. Mustafa, Note on intrinsic decay rates for abstract wave equations with memory, *J. Math. Phys.*, 54 (2013), 031504.

[18] C. Li, J. Liang and T.-J. Xiao, Long-term dynamical behavior of the wave model with locally distributed frictional and viscoelastic damping, *Comm. Nonlinear Sci. Numer. Simulat.*, 92 (2021), 105472, 22 pp.

[19] S.-O. Londen and W. M. Ruess, Linearized stability for nonlinear Volterra equations, *J. Evol. Equ.*, 17 (2017), 473–483.

[20] P. Loreti and D. Sforza, *A Semilinear Integro-Differential Equation: Global Existence and Hidden Regularity*, in *Trends in Control Theory and Partial Differential Equations* (eds. F. Alabau-Boussouira, F. Ancona, A. Porretta and C. Sinestrari), Springer INdAM Series, vol 32, Springer, Cham., 2019.

[21] R. C. Maccamy and J. S. W. Wong, Stability theorems for some functional equations, *Trans. Amer. Math. Soc.*, 164 (1972), 1–37.

[22] J. E. Muñoz Rivera and E. C. Lapa, Decay rates of solutions of an anisotropic inhomogeneous n-dimensional viscoelastic equation with polynomial decaying kernels, *Comm. Math. Phys.*, 177 (1996), 583–602.

[23] J. E. Muñoz Rivera and H. D. Fernández Sare, Stability of Timoshenko systems with past history, *J. Math. Anal. Appl.*, 339 (2008), 482–502.

[24] M. Nakao, $L^p$ estimates for the linear wave equation and global existence for semilinear wave equations in exterior domains, *Math. Ann.*, 320 (2001), 11–31.
[25] S. Nicaise and C. Pignotti, Stability of the wave equation with localized Kelvin-Voigt damping and boundary delay feedback, *Discrete Contin. Dyn. Syst. Ser. S*, 9 (2016), 791–813.

[26] S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Discrete Contin. Dyn. Syst. Ser. S*, 3 (2011), 693–722.

[27] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, *Discrete Contin. Dyn. Syst. Ser. S*, 2 (2009), 559–581.

[28] J. A. Nohel and D. F. Shea, Frequency domain methods for Volterra equations, *Adv. Math.*, 22 (1976), 278–304.

[29] M. Okada and S. Kawashima, Global solutions to the equation of thermoelasticity with fading memory, *J. Differential Equations*, 263 (2017), 338–364.

[30] O. J. Staffans, On a nonlinear hyperbolic Volterra equation, *SIAM J. Math. Anal.*, 11 (1980), 793–812.

[31] J. S. W. Wong, Positive definite functions and Volterra integral equations, *Bull. Amer. Math. Soc.*, 80 (1974), 679–682.

[32] T.-J. Xiao and J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, *J. Differential Equations*, 254 (2013), 2128–2157.

[33] H. Zhan and Z. Feng, Stability of hyperbolic-parabolic mixed type equations with partial boundary condition, *J. Differential Equations*, 264 (2018), 7384–7411.

[34] H. Zhan and Z. Feng, Stability of hyperbolic-parabolic mixed type equations, *Dyn. Partial Differ. Equ.*, 16 (2019), 253–272.

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