Light tetraquarks in a Dyson-Schwinger/Bethe-Salpeter approach

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Abstract. A number of bound states of the strong interaction with potential non-$q\bar{q}$ nature are waiting to be understood within QCD. A prominent example are the light scalars with $J^{PC} = 0^{++}$, which have been interpreted as tetraquarks already long ago. Recent four-body calculations in the Dyson-Schwinger/Bethe-Salpeter approach support the tetraquark hypothesis and find the $\sigma$ meson to have a strong pion-pion component.

1. Are the light scalar mesons tetraquarks?
The perhaps oldest tetraquark candidates are the light scalar mesons with $J^P = 0^{++}$, namely, the $\sigma/f_0(500)$ and its multiplet partners $\kappa(800)$, $a_0(980)$ and $f_0(980)$, which have been subject to intense debate for many decades \cite{1}. Combining new data with dispersive approaches to $\pi\pi$ scattering using Roy equations \cite{2,3}, the $\sigma$ is now again listed in the PDG with a pole position $(400 \ldots 550) - i(200 \ldots 350)$ MeV \cite{4}. Theoretically, the situation is complicated by the fact that the quark content of the $\sigma$ and its partners cannot be unambiguously identified, because in principle they can be mixtures of $q\bar{q}$, $q\bar{q}q\bar{q}$ and other configurations. Phenomenological evidence for their non-$q\bar{q}$ nature comes from the comparison with other mesons, cf. Fig. 1: in the non-relativistic quark model the light scalars are $p$ waves with orbital angular momentum $L = 1$, which should put them in a similar mass range as the axialvector and tensor mesons. This is clearly not the case in the experimental spectrum, where the members of the ‘first radially excited’ nonet in the $0^{++}$ channel appear to be better candidates for the actual $q\bar{q}$ ground states. But what are then the light scalars? Even more striking is the mass ordering within the $0^{++}$ multiplet. If they were $q\bar{q}$ states, the isotriplet $a_0$ and isosinglet $\sigma$ should be made of light $q\bar{q}$ states. Instead, the $a_0$ is approximately degenerate with the $f_0$, which seems rather nonsensical in a $q\bar{q}$ picture. Their decay widths are quite different as well: the $a_0$ and $f_0$ have masses just at or below $K\bar{K}$ threshold with widths below 100 MeV, whereas the $\sigma$ and $\kappa$ are broad states above the $\pi\pi$ and $K\pi$ thresholds, respectively.

These problems disappear within a tetraquark picture. In the simplest case advocated by Jaffe \cite{5}, one combines a (color and flavor antitriplet) scalar diquark with a scalar antidiquark, which produces again a flavor nonet of color singlets. However, in this case the $\sigma$ would be the lightest state with four light quarks, followed by the $\kappa$ with one strange quark and the mass-degenerate $a_0$ and $f_0$ with two strange quarks. This naturally explains the mass ordering...
Figure 1. Light meson spectrum from the PDG [4]; the bars show the experimental mass range. The light scalar mesons are magnified on the right.

Figure 2. Four-body equation for tetraquarks. The lines with circles stand for a dressed quark propagator and the yellow blobs for the tetraquark Bethe-Salpeter amplitude. Note that the figure only shows the \((q\bar{q})(\bar{q}\bar{q})\) topology; there are two further \((q\bar{q})(q\bar{q})\) topologies which all add up to the final equation.

as well as the decay widths: the dominant \(a_0\) decay into \(\pi\eta\) and its coupling to \(KK\) support an internal \(s\bar{s}\) pair, whereas the \(\sigma\) and \(\kappa\) can simply fall apart into \(\pi\pi\) and \(K\pi\) through the ‘OZI-superallowed’ mechanism, which does not require the exchange of gluons. The non-\(q\bar{q}\) interpretation of the light scalars is supported by various approaches such as QCD sum rules [6], unitarized ChPT [7,8], quark models [9], and the extended linear \(\sigma\) model [10]. What remains to be understood is the precise binding mechanism: are such tetraquarks strongly bound diquark-antidiquark states, loosely bound meson molecules, or a bit of both? If they are made of diquarks and antidiquarks, then why are their masses so unexpectedly low? And is it possible to distinguish between these scenarios without any prior knowledge of mesons or diquarks, i.e., when starting directly from four current quarks in QCD?

2. Tetraquarks in the Dyson-Schwinger/Bethe-Salpeter framework
To address these questions, we utilize the framework of Dyson-Schwinger and Bethe-Salpeter equations (DSEs and BSEs) [11–13]. It has been successfully applied to a range of meson and baryon properties (see e.g. [14] for a review) and first investigations of glueballs [15] and light tetraquarks [16,17] have recently been pioneered as well. In [17], the \(0^{++}\) tetraquarks were calculated from a genuine four-quark equation and the resulting masses were found to qualitatively reproduce the experimentally observed \(0^{++}\) spectrum. In the following we will summarize the methods used in that work and discuss the corresponding results. The tetraquark BSE is shown in Fig. 2. It describes a \(q\bar{q}q\bar{q}\) system that is bound by two-body interactions between quarks and antiquarks, whereas three- and four-body interactions are neglected at this point. For the two-body interactions we employ the rainbow-ladder truncation where the kernels reduce to gluon exchanges, see e.g. [18,19] for details. Combined with the solution for the quark propagator, this setup ensures chiral symmetry and its dynamical breaking. Systematic ways
of improving the truncation have been successfully applied to the light meson sector lately [20]. The four-body Bethe-Salpeter amplitude is a complicated object which contains a color (C), a flavor (F), and a Dirac (D) part:

\[ \Gamma = \Gamma^D \otimes \Gamma^C \otimes \Gamma^F, \quad \Gamma^D = \sum_{i=1}^{256} \tau_i(p, q, k, P) f_i(\Omega), \quad K(M) f(M) = \lambda(M) f(M). \quad (1) \]

Here \( p, q, \) and \( k \) are the relative momenta between the (anti-)quarks and \( P \) is the total momentum of the tetraquark. The \( \tau_i \) are the basis elements in Dirac space, the \( f_i \) are the scalar dressing functions of the tetraquark amplitude, and \( \Omega \) is the set of nine Lorentz invariants that can be built from \( p, q, k, P \) with \( P^2 = -M^2 \) onshell. The last (eigenvalue) equation follows after taking traces and discretizing the momentum variables. Here \( K(M) \) is the Bethe-Salpeter kernel matrix, \( f(M) \) is the eigenvector corresponding to the tetraquark amplitude, and \( \lambda(M) \) a mass-dependent eigenvalue which satisfies \( \lambda(M) = 1 \) at the physical bound-state mass. The practical difficulty with this equation is the enormous amount of Lorentz invariants on which the \( f_i(\Omega) \) depend: with two color tensors and \( N \) grid points in each direction, the dimension of the kernel matrix is \( 2 \times 256 \times N^0 \), which impedes a straightforward numerical solution.

The solution of this equation was made possible by two strategies. One is the restriction of the 256 Dirac basis tensors to \( s \) waves only, i.e., the Fierz-complete set of 16 tensors which depend on the total momentum \( P \) but not any relative momentum. In the analogous three-body equation for baryons (where the full set of tensors is implemented) the same approximation is reliable up to \( \sim 10\% \), which is sufficient for our present purposes. The second strategy is the extensive use of the equation’s symmetries in terms of \( S_4 \) variables [21]. It turns out to be useful to group the Lorentz invariants in the set \( \Omega \) into \( S_4 \) multiplets: a singlet \( S_0 \), a doublet \( D \) and two triplets \( T_0 \) and \( T_1 \). For example, the singlet and doublet read:

\[ S_0 = \frac{p^2 + q^2 + k^2}{4}, \quad D = \frac{1}{4S_0} \left[ \sqrt{3}(q^2 - p^2) \right]. \quad (2) \]

The singlet \( S_0 \) carries the scale dependence of the dressing functions \( f_i \) above, whereas the phase space of the multiplets \( (D, T_0, T_1) \) is restricted to geometrical objects (triangle, tetrahedron, sphere). In this way, groups of momentum variables can be switched on and off at once during a calculation without destroying the permutation symmetries of the system, thus dramatically reducing the size of the kernel matrix in Eq. (1).

### 3. Results

The resulting inverse eigenvalues \( 1/\lambda(M) \) of the tetraquark BSE, Eq. (1), for different multiplet combinations are shown in Fig. 3. If only the momentum dependence in \( S_0 \) is retained in the equation, the resulting tetraquark mass is \( \sim 1500 \) MeV. Naively, this is what one expects from combining four current quarks with a dynamical mass function: the result is roughly ‘four times the constituent-quark mass’. Switching on the triplet \( T_1 \) changes almost nothing, whereas the triplet \( T_0 \) reduces the mass by a few hundred MeV. A much more drastic effect happens when the doublet \( D \) is switched on, because in that case the tetraquark mass drops to \( \sim 350 \) MeV. This is because the equation dynamically develops meson and diquark poles, which appear in the Mandelstam plane formed by the doublet triangle in Eq. (2). Each side of the triangle corresponds to one of the three topologies in the BSE – ‘diquark-antidiquark’, (12)(34), or ‘meson-meson’, (13)(24) and (14)(23). Because the diquark mass scales are much larger than the pion mass (about 800 MeV for a scalar diquark), the diquark poles have a much weaker effect on the dressing functions than the two pion poles sitting to the left and the right. Hence, the light mass of the \( \sigma \) is driven by the pion poles, which leads to a predominant ‘meson molecule’ interpretation of the tetraquark.
Figure 3. **Left panel:** dominant dressing function $f_1$ of Eq. (1) within the doublet triangle in the case of the $\sigma$, together with the pole positions of the dynamically generated pions and diquarks. **Right panels:** inverse eigenvalues of Eq. (1) for different $S_4$ multiplets.

The results for the $0^{++}$ multiplet are shown in the left panel of Fig. 4 as a function of the current-quark mass. One can see that they are in qualitative agreement with the $0^{++}$ spectrum from the PDG. The problems of the mass ordering within the multiplet and the alignment in comparison to other multiplets discussed in Sec. 1 do not appear. The dynamical features of the $\sigma$ repeat themselves in the multiplet partners $\kappa$, $a_0$ and $f_0$, which are dominated by internal $K\pi$ and $\eta\pi/K\bar{K}$ poles, respectively. For tetraquark masses $M > 2m_\pi$ the pion poles eventually move inside the integration domain and the $\sigma$ becomes a resonance. So far we performed an extrapolation to $\lambda = 1$ to estimate the tetraquark mass, whereas a proper implementation of the pole structure requires contour-deformation methods – see [22] for an analogous treatment of the $\pi^0 \rightarrow e^+e^-$ decay. In any case, for large current-quark masses the resulting $\sigma$ mass goes below threshold and thus becomes a genuine bound state. This is analogous to the case of the $\rho$ meson shown in the right panel of Fig. 4. At large current masses (large $m_\pi^2$), $m_\rho < 2m_\pi$ and the system is a bound state; at low current masses, chiral dynamics becomes important and the $\rho$ becomes a resonance. The figure compares the rainbow-ladder result for $m_\rho$ with contemporary lattice results. In contrast to the tetraquarks, the $\rho$ meson in this case is always a bound state because chiral effects are missing in rainbow-ladder. The comparison with lattice and experiment suggests, however, that the effects on the real part of the mass are rather modest. In this sense an eigenvalue extrapolation beyond the threshold should also provide a reasonable first estimate in the four-body case, where the resonance dynamics is already built into the system.

To summarize, the genuine four-quark ($q\bar{q}q\bar{q}$) equation, where the quarks interact by gluon exchange in a chiral-symmetry preserving way, can qualitatively reproduce the mass pattern of the light scalar mesons. This does not happen in the corresponding two-body ($q\bar{q}$) equation, where the resulting $a_0$ and $\sigma$ are mass-degenerate and one would require large unquenching effects to establish an (accidental) mass degeneracy between $a_0$ and $f_0$. Therefore, it provides further support for the tetraquark interpretation of the light scalar mesons. The four-body equation dynamically produces meson poles in the Bethe-Salpeter amplitude, which drive the system and are responsible for the low masses of the $0^{++}$ multiplet. In this sense the light scalar mesons are conceptually closer to ‘meson molecules’ than ‘diquark-antidiquark’ states.

4. Outlook
Recent experimental findings at BES III, Belle and LHCb point towards a number of tetraquark candidates in the charm region. Furthermore, PANDA will provide more detailed information about channels other than $1^{--}$. Therefore, in the future we would like to extend our calculations to other quantum numbers to help understand the nature of some of these states.
Figure 4. Left: masses of the $0^{++}$ multiplet members around the light $u/d$ quark mass and compared to the PDG. The error bands correspond to different extrapolation schemes. Right: $\rho$-meson mass calculated in rainbow-ladder (solid, blue) compared to lattice results (see [14] for references). The star is the PDG value and the dashed line is the $\pi\pi$ threshold.

5. Acknowledgement
This work has been supported by DFG, FI 970/11-1.

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