ON DELIGNE’S CONJECTURE FOR SYMMETRIC SIXTH $L$-FUNCTIONS OF
HILBERT MODULAR FORMS

SHIH-YU CHEN

Abstract. In this paper, we prove Deligne’s conjecture for symmetric sixth $L$-functions of Hilbert modular forms. We extend the result of Morimoto based on a different approach. We define automorphic periods associated to globally generic $C$-algebraic cuspidal automorphic representations of $GSp_4$ over totally real number fields whose archimedean components are (limits of) discrete series representations. We show that the algebraicity of critical $L$-values for $GSp_4 \times GL_2$ can be expressed in terms of these periods. In the case of Kim–Ramakrishnan–Shahidi lifts of $GL_2$, we establish period relations between the automorphic periods and powers of Petersson norm of Hilbert modular forms. The conjecture for symmetric sixth $L$-functions then follows from these period relations and our previous work on the algebraicity of critical values for the adjoint $L$-functions for $GSp_4$.

1. Introduction

In [Del79], Deligne proposed a conjecture on the algebraicity of values of motivic $L$-functions at critical points. For example, attach to a normalized elliptic newform $f$ of weight $\kappa \geq 2$ and nebentypus $\omega$, we can define the symmetric $n$-th power $L$-function $L(s, \text{Sym}^n(f))$ for each integer $n \geq 1$. In this case, the conjectural Deligne’s period $c^\ell(\text{Sym}^n(f))$ was computed in [Del79, Proposition 7.7]. It is predicted that

$$\sigma \left( \frac{L(m, \text{Sym}^n(f))}{(2\pi \sqrt{-1})^{d_+m} \cdot c^\ell(\text{Sym}^n(f))} \right) = \frac{L(m, \text{Sym}^n(\sigma f))}{(2\pi \sqrt{-1})^{d_+m} \cdot c^\ell(\text{Sym}^n(\sigma f))}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$ and critical points $m \in \mathbb{Z}$ of $L(s, \text{Sym}^n(f))$. Here $\pm = (-1)^m$, $d_+ = \frac{n+1}{2}$ if $n$ is odd, and $d_+ = \frac{n+2}{2}$ if $n$ is even. For $n = 1$ and $n = 2$, the conjecture follows from the results of Shimura [Shi76, Shi77] and Sturm [Stu80, Stu89], respectively. In [GH93], Garrett and Harris studied the algebraicity of critical values of triple product $L$-functions in the balanced case. As a consequence, they proved the conjecture for $n = 3$ under the assumptions that $\kappa \geq 6$ and $m$ is on the right and away from the center of the critical strip. We complement the result of Garrett and Harris in [Che21c] and relaxed the assumption to $\kappa \geq 3$. Based on symmetric power functoriality and various algebraicity results on critical $L$-values in the literature, especially [GL21], Morimoto proved the algebraicity of the critical $L$-values for $n = 4, 6$ in [Mor21]. More precisely, under the assumptions that $\kappa \geq 6$, $\omega = 1$, and $\text{Aut}(\mathbb{C})$ replaced by $\text{Aut}(\mathbb{C}/\mathbb{Q})$ for some bi-quadratic extension $\mathbb{C}/\mathbb{Q}$, Morimoto proves that (1.1) holds up to square. In [Che21d], we refine the result of Grobner and Lin, and prove the conjecture for $n = 4$ assuming $\kappa \geq 3$. Actually, we have analogous conjecture for symmetric power $L$-functions associated to normalized Hilbert cusp newforms over a totally real number field $\mathbb{F}$. The conjecture holds for $n = 1, 2, 3, 4, 6$ by authors mentioned above under similar assumptions. The goal of this paper is to prove the conjecture for $n = 6$. We extend the result of Morimoto based on a different approach. In particular, specializing our main result Theorem 1.3 to $\mathbb{F} = \mathbb{Q}$, we show that (1.1) holds for $n = 6$ assuming $\kappa \geq 6$. We also refer to [KT16] §7 for numerical examples in this case.

1.1. Main result. Let $\mathbb{F}$ be a totally real number field with $[\mathbb{F} : \mathbb{Q}] = d$. Let $\Pi$ be a regular algebraic irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{F})$ with central character $\omega_{\Pi}$. We have $|\omega_{\Pi}| = \left| \frac{\kappa}{\kappa} \right|$ for some $\kappa \in \mathbb{Z}$. Let $f_{\Pi}$ be the normalized newform of $\Pi$. The Petersson norm of $f_{\Pi}$ is defined by

$$\|f_{\Pi}\| = \int_{\mathbb{A}_\mathbb{F} \backslash GL_2(\mathbb{A}_\mathbb{F})} |f_{\Pi}(g)|^2 \det(g)|_{\mathbb{A}_\mathbb{F}}^{\omega} \, dg_{\text{Tam}}.$$

Here $dg_{\text{Tam}}$ is the Tamagawa measure on $\mathbb{A}_\mathbb{F} \backslash GL_2(\mathbb{A}_\mathbb{F})$. For each archimedean place $v$ of $\mathbb{F}$, there exists $\kappa_v \geq 2$ such that $\Pi_v$ is isomorphic to the discrete series representation of $GL_2(\mathbb{R})$ with minimal weights $\pm \kappa_v$.

arXiv:2110.06261v1 [math.NT] 12 Oct 2021
and has central character $\text{sgn}^m|\chi^m$. For a finite order Hecke character $\chi$ of $\mathbb{A}_\mathbb{Q}$, let

$$L(s, \Pi, \text{Sym}^6 \otimes \chi)$$

be the twisted symmetric sixth $L$-function of $\Pi$ by $\chi$. We denote by $L^{(\infty)}(s, \Pi, \text{Sym}^6 \otimes \chi)$ the $L$-function obtained by excluding the archimedean $L$-factors. A critical point for the $L$-function $L(s, \Pi, \text{Sym}^6 \otimes \chi)$ is an integer $m$ which is not a pole of the archimedean local factors $L(s, \Pi_v, \text{Sym}^6 \otimes \chi_v)$ and $L(1 - s, \Pi_v, \text{Sym}^6 \otimes \chi_v^{-1})$ for all archimedean places $v$ of $\mathbb{F}$. More precisely, $L(s, \Pi, \text{Sym}^6 \otimes \chi)$ has no critical points if the signature of $\chi$ is not parallel. Suppose $\chi$ has parallel signature $\text{sgn}(\chi) \in \{\pm 1\}$, then the set of critical points for $L(s, \Pi, \text{Sym}^6 \otimes \chi)$ is the union of the set of right-half critical points

$$\left\{ 1 - 3w \leq m \leq \min_{v|\infty} \{ \kappa_v \} - 1 - 3w \bigg| (1)^{m+1} = \text{sgn}(\chi) \right\}$$

and the set of left-half critical points

$$\left\{ -\min_{v|\infty} \{ \kappa_v \} + 2 - 3w \leq m \leq -3w \bigg| (1)^m = \text{sgn}(\chi) \right\}.$$

The Deligne's periods of $L(s, \Pi, \text{Sym}^6)$ are given by

$$c^\pm(\Pi, \text{Sym}^6) = \left\{ \begin{array}{cl}
(2\pi \sqrt{-1})^{\sum_{v|\infty} \kappa_v} \cdot G(\omega_H)^{12} \cdot \|f_H\|_6^6 & \text{if } \pm = +, \\
|D_F|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v|\infty} \kappa_v} \cdot G(\omega_H)^{9} \cdot \|f_H\|_6^6 & \text{if } \pm = -.
\end{array} \right.$$ 

Here $D_F$ is the discriminant of $\mathbb{F}$ and $G(\omega_H)$ is the Gauss sum of $\omega_H$. We refer to [Yos94] Proposition 2.2 for the appearance of $|D_F|^{1/2}$. For $\sigma \in \text{Aut}(\mathbb{C})$, let $\sigma \Pi$ be the unique regular algebraic irreducible cuspidal automorphic representation of $\text{GL}_2(F)$ such that $\sigma \Pi_F$ is the $\sigma$-conjugate of $\Pi_F = \otimes_{v|\infty} \Pi_v$. We have the following conjecture proposed by Deligne [Del79 § 7] on the algebraicity of the critical values of $L(s, \Pi, \text{Sym}^6 \otimes \chi)$.

**Conjecture 1.1** (Deligne). Let $\chi$ be a finite order Hecke character of $\mathbb{A}_\mathbb{Q}$ and $m - 3w \in \text{Crit}(\Pi, \text{Sym}^6 \otimes \chi)$ a critical point. For all $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma \left( \frac{L^{(\infty)}(m - 3w, \Pi, \text{Sym}^6 \otimes \chi)}{(2\pi \sqrt{-1})^d_{\text{sym}} \cdot G(\chi)^d_{\text{sym}} \cdot c^\pm(\Pi, \text{Sym}^6)} \right) = \frac{L^{(\infty)}(m - 3w, \sigma \Pi, \text{Sym}^6 \otimes \sigma \chi)}{(2\pi \sqrt{-1})^d_{\text{sym}} \cdot G(\sigma \chi)^d_{\text{sym}} \cdot c^\pm(\sigma \Pi, \text{Sym}^6)}.$$ 

Here $\pm = (-1)^{m+w+1}\text{sgn}(\chi)$, $d_+ = 4$, $d_- = 3$.

**Remark 1.2.** The conjecture is true when $\Pi$ is of CM-type. In [Mor21], Conjecture 1.1 was proved (up to square when $\mathbb{F} = \mathbb{Q}$) by Morimoto under the following assumptions:

1. $\mathbb{F} \cap \mathbb{Q}(\zeta_3) = \mathbb{Q}$.
2. $\kappa_v \geq 6$ for all archimedean places $v$ of $\mathbb{F}$.
3. $\omega_H \in \mathbb{A}_\mathbb{Q}$.
4. $\text{Aut}(\mathbb{C})$ replaced by $\text{Aut}(\mathbb{C}/E_{\text{Gal}})$ for some CM-extension $E/\mathbb{F}$, where $E_{\text{Gal}}$ is the Galois closure of $E$ in $\mathbb{C}$.

The first assumption is imposed in order to apply the result [CT17] on the functoriality for $\text{Sym}^6$. For the second assumption, it is required so that we can apply the result [Mor18]. Assumptions (3) and (4) can be lifted if one can refine and generalize the result [GL21] in the case $\text{GL}_4 \times \text{GL}_3$ over CM-fields.

Following is the main result of this paper. We extend the result of Morimoto based on a different approach. Instead of using the result [GL21], we define automorphic periods for $\text{GSp}_4$ and establish period relations for Kim–Ramakrishnan–Shahidi lifts for $\text{GL}_2$.

**Theorem 1.3.** Assume $\mathbb{F} \cap \mathbb{Q}(\zeta_3) = \mathbb{Q}$ and $\kappa_v \geq 6$ for all archimedean places $v$ of $\mathbb{F}$.

1. Conjecture 1.1 holds with $\text{Aut}(\mathbb{C})$ replaced by $\text{Aut}(\mathbb{C}/E_{\text{Gal}})$.
2. If we assume further that $\kappa_v = \kappa_v$ for all archimedean places $v, w$ of $\mathbb{F}$, then Conjecture 1.1 holds.

**Remark 1.4.** For assertion (1), to descend the equivariance from $\text{Aut}(\mathbb{C}/E_{\text{Gal}})$ to $\text{Aut}(\mathbb{C})$, it suffices to improve the equivariance property in [GL10 Lemma 3.3.1] (cf. Remark 5.2).
1.2. An outline of the proof. There are two key ingredients in the proof:

1. The algebraicity of critical values of twisted standard $L$-functions for $\text{GSp}_6(\mathbb{A}_F)$ (Theorem 1.3).
2. The period relations for automorphic periods of the Kim–Ramakrishnan–Shahidi lifts and Petersson norms of Hilbert cusp newforms (Theorem 3.1).

We may assume $\Pi$ is non-CM, that is, $\Pi$ is not an automorphic induction of a Hecke character over some CM-extension of $F$. Based on the results of Arthur [Art13], Clozel–Throne [CT17], and Patrikis [Pat19], in Proposition 3.5 we show that $\Pi$ transfer weakly to a $C$-algebraic irreducible cuspidal automorphic representation of $\text{GSp}_6(\mathbb{A}_F)$ with respect to the symmetric sixth power representation of $\text{GL}_2(\mathbb{C})$. More precisely, we consider the following diagram on the Galois side:

$$
\text{Selberg}\text{-cohomology}\text{condition}:
\begin{array}{c}
\text{Novodvorsky's global zeta integral for Rankin–Selberg L-functions for GSp}_6(\mathbb{A}_F)\text{ satisfying certain conditions.}
\end{array}
$$

Here $L_\mathcal{G}$ and $\phi_\Pi$ are the conjectural Langlands group of $\mathbb{F}$ and the conjectural Langlands parameter associated to $\Pi$, respectively. We twist $\omega_\Pi^{-3}$ so that the image of $(\text{Sym}^6 \circ \phi_\Pi) \otimes \omega_\Pi^{-3}$ lies in $\text{SO}_7(\mathbb{C})$. Moreover, the transfer is strong at archimedean places. By ingredient (1), when $F \neq \mathbb{Q}$, we then conclude that Conjecture 1.1 holds for all $\chi$ and $m$ if and only if it holds for some $\chi$ and $m$. Theorem 1.3 (2) for $F = \mathbb{Q}$ follows from the cases for $F \neq \mathbb{Q}$ and a base change trick (cf. § 3.2.2). We show that Conjecture 1.1 holds for

$$
\chi = \|3^m \omega_\Pi^{-3}\|, \quad m = 1 - 3w.
$$

For this specific case, we prove the conjecture by ingredient (2). In §2.6 for a globally generic $C$-algebraic irreducible cuspidal automorphic representation $\Sigma$ of $\text{GSp}_4(\mathbb{A}_F)$ whose archimedean components are (limits of) discrete series representations, we attach to an automorphic period $p^i(\Sigma) \in \mathbb{C}^\times$ for each (admissible) subset $I$ of $S_\sigma$. Here $S_\sigma$ denote the set of archimedean places of $\mathbb{F}$. The periods are obtained by comparing rational structures on $\Sigma_I$ given by the Whittaker model and by the coherent cohomology of certain automorphic vector bundles on the Hilbert–Siegel modular variety associated to $\text{GSp}_4(\mathbb{A}_F)$. By the Serre duality for coherent cohomology and our previous result [Che21b], we prove in Theorem 2.14 the period relation that

$$
L(1, \Sigma, \text{Ad}) \sim p^i(\Sigma) \cdot p^{S_\sigma \setminus I}(\Sigma^\vee).
$$

Here $L(s, \Sigma, \text{Ad})$ is the adjoint $L$-function of $\Sigma$ and $\alpha \sim \beta$ if $\alpha = \gamma \cdot \beta$ for some $\gamma \in \pi^2 \cdot \mathbb{Q}^\times$. In particular, we take $\Sigma$ be the Kim–Ramakrishnan–Shahidi lift of $\Pi$ (cf. §3.1). In this case, ingredient (2) roughly says that

$$
p^\phi(\Sigma) \sim \|f_H\|^3, \quad p^{S_\sigma}(\Sigma) \sim \|f_H\|^4.
$$

We have the factorization of $L$-functions:

$$
L(s, \Sigma, \text{Ad}) = L(s, \Pi, \text{Sym}^6 \otimes \omega_\Pi^{-3}) \cdot L(s, \Pi, \text{Sym}^2 \otimes \omega_\Pi^{-1}).
$$

Since $L(1, \Pi, \text{Sym}^2 \otimes \omega_\Pi^{-1}) \sim \|f_H\|$, we thus conclude that

$$
L(1, \Pi, \text{Sym}^6 \otimes \omega_\Pi^{-3}) \sim \|f_H\|^6.
$$

Theorem 1.3 for the critical value $L(1, \Pi, \text{Sym}^6 \otimes \omega_\Pi^{-3})$ then follows by gathering the fudge factors appear in the above relations $\sim$. We sketch the strategy to establish the period relation 1.3. We consider auxiliary algebraic irreducible cuspidal automorphic representations $\Pi_1'$ and $\Pi_2'$ of $\text{GL}_2(\mathbb{A}_F)$ satisfying certain conditions. Based on the cohomological interpretation of Novodvorsky’s global zeta integral for Rankin–Selberg $L$-functions for $\text{GSp}_4 \times \text{GL}_2$ (cf. Proposition 4.2), we prove the following result on algebraicity of the rightmost critical $L$-values:

$$
L(m_1, \Sigma \times \Pi_1') \sim p^\phi(\Sigma) \cdot \|f_{H_1'}\|, \quad L(m_2, \Sigma \times \Pi_2') \sim p^{S_\sigma}(\Sigma).
$$

Here $m_1, m_2$ are the rightmost critical points. Note that the results are special cases of Theorem 1.3. On the other hand, when we take $\Sigma$ be the Kim–Ramakrishnan–Shahidi lift of $\Pi$, the algebraicity of these critical values can be expressed in terms of powers of $\|f_H\|$ using the results [GH93], [GL10], [Mor18], [Che21c].
We then obtain the period relation (1.3). The details for the proof of Theorem 3.1 are given in §0.

1.3. Notation. Fix a totally real number field $\mathbb{F}$ with $[\mathbb{F} : \mathbb{Q}] = d$. Let $D_{\mathbb{F}}$ be the discriminant of $\mathbb{F}$. Let $\mathbb{A}_{\mathbb{F}}$ be the ring of adeles of $\mathbb{F}$ and $\mathbb{A}_{\mathbb{F}, f}$ its finite part. Let $\mathfrak{p}_\mathbb{F}$ be the maximal compact subring of $\mathbb{A}_{\mathbb{F}, f}$. Let $\zeta_{\mathbb{F}}(s)$ be the completed Dedekind zeta function of $\mathbb{F}$. Let $\psi_\mathbb{Q} = \bigotimes_v \psi_v$ be the standard additive character of $\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}$ defined so that

$$\psi_p(x) = e^{-2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{Z}[p^{-1}],$$

$$\psi_\mathbb{R}(x) = e^{2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{R}.$$ 

Let $\psi_\mathbb{F} = \psi_\mathbb{Q} \circ \text{tr}_{\mathbb{F}/\mathbb{Q}}$ and call it the standard additive character of $\mathbb{F}\backslash \mathbb{A}_{\mathbb{F}}$. Let $S_\mathbb{F}$ be the set of archimedean places of $\mathbb{F}$. For $v \in S_\mathbb{F}$, let $\iota_v$ be the real embedding of $\mathbb{F}$ associated to $v$ and identify $\mathbb{F}_v$ with $\mathbb{R}$ via $\iota_v$. Let $v$ be a place of $\mathbb{F}$. If $v$ is a finite place, let $\mathfrak{a}_v$, $\mathfrak{w}_v$, and $q_v$ be the maximal compact subring of $\mathbb{F}_v$, a generator of the maximal ideal of $\mathfrak{a}_v$, and the cardinality of $\mathfrak{a}_v/\mathfrak{w}_v$. Let $| |_v = | |$ be the absolute value on $\mathbb{F}_v$ normalized so that $|\mathfrak{w}_v|_v = q_v^{-1}$. If $v \in S_\mathbb{F}$, let $| |_v$ be the ordinary absolute value on $\mathbb{R}$. Let $| |_{\mathbb{A}_v} = \prod_v | |_v$ be the adelic norm on $\mathbb{A}_{\mathbb{F}}$. Let $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_\mathbb{C}(s)$ be the archimedean gamma functions defined by

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(\frac{s}{2}), \quad \Gamma_\mathbb{C}(s) = 2(2\pi)^{-s} \Gamma(s).$$

Let $\chi$ be an algebraic Hecke character of $\mathbb{A}_{\mathbb{F}}$. The signature $\sigma(\chi)$ of $\chi$ at $v \in S_\mathbb{F}$ is the value $\chi_v(-1) \in \{\pm 1\}$. The signature $\text{sgn}(\chi)$ of $\chi$ is the sequence of signs $(\chi_v(-1))_{v \in S_\mathbb{F}}$. We say $\chi$ has parallel signature if it has the same signature at all real places. The Gauss sum $G(\chi)$ of $\chi$ is defined by

$$G(\chi) = |D_{\mathbb{F}}|^{-1/2} \prod_{v | \infty} \varepsilon(0, \chi_v, \psi_v),$$

where $\psi = \bigotimes_v \psi_v$ and $\varepsilon(s, \chi_v, \psi_v)$ is the $\varepsilon$-factor of $\chi_v$ with respect to $\psi_v$ defined in [Tat79]. For $\sigma \in \text{Aut}(\mathbb{C})$, let $\sigma^\mathbb{F}$ be the unique algebraic Hecke character of $\mathbb{A}_{\mathbb{F}}$ such that $\sigma(\chi)(x) = \chi(x)$ for $x \in \mathbb{A}_{\mathbb{F}, f}$. Note that $\text{sgn}(\chi) = \text{sgn}(\sigma^\mathbb{F})$. It is easy to verify that

$$\sigma(G(\chi)) = \chi(\sigma)G(\chi), \quad \sigma \left( \frac{G(\chi \chi')}{G(\chi)G(\chi')} \right) = \frac{G(\sigma^\mathbb{F} \chi \chi')}{G(\sigma^\mathbb{F})G(\sigma^\mathbb{F} \chi')}$$

for algebraic Hecke characters $\chi, \chi'$ of $\mathbb{A}_{\mathbb{F}}$, where $\sigma^\mathbb{F} \in \prod_p \mathbb{Z}_p$ is the unique element such that $\sigma(\psi_\mathbb{Q}(x)) = \psi_\mathbb{Q}(u_\sigma x)$ for $x \in \mathbb{A}_{\mathbb{Q}, f}$.

Let $\sigma \in \text{Aut}(\mathbb{C})$. Define the $\sigma$-linear action on $\mathbb{C}(X)$, which is the field of formal Laurent series in variable $X$ over $\mathbb{C}$, as follows:

$$\sigma^\mathbb{C}(X) = \sum_{\ell = 0}^{\infty} \sigma(a_n) X^n$$

for $P(X) = \sum_{\ell = -\infty}^{\infty} a_n X^n \in \mathbb{C}(X)$. For a complex representation $\Pi$ of a group $G$ on the space $V_{\Pi}$ of $\Pi$, let $\sigma^\Pi$ be the representation of $\Pi$ defined

$$\sigma^\Pi(g) = t \circ \Pi(g) \circ t^{-1},$$

where $t : V_{\Pi} \to V_{\Pi}$ is a $\sigma$-linear isomorphism. Note that the isomorphism class of $\sigma^\Pi$ is independent of the choice of $t$. We call $\sigma^\Pi$ the $\sigma$-conjugate of $\Pi$.

2. Automorphic periods for $GL_2$ and $GSp_4$

2.1. Automorphic vector bundles on Shimura varieties. Let $(G, X)$ be a Shimura datum, that is, $G$ is a connected reductive algebraic group over $\mathbb{Q}$ and $X$ is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : S \to G(\mathbb{R})$ satisfying conditions (1.1.1)-(1.1.3) in [Har55]. Here $S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ is the Deligne torus. We have the associated Shimura variety

$$\text{Sh}(G, X) = \lim_{\kappa} \text{Sh}_\kappa(G, X) = \lim_{\kappa} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathbb{Q}, f})/\kappa,$$

where $\kappa$ runs through neat open compact subgroups of $G(\mathbb{A}_{\mathbb{Q}, f})$. It is a pro-algebraic variety over $\mathbb{C}$ with continuous $G(\mathbb{A}_{\mathbb{Q}, f})$-action and admits canonical model over the reflex field $E(G, X)$ of $(G, X)$.
Fix $h \in X$. Let $K_h$ be the stabilizer of $h$ in $G(\mathbb{R})$. The Hodge decomposition induced by $\text{Ad} \circ h$ on the complexified Lie algebra $\mathfrak{g}_C$ of $G(\mathbb{R})$ is given by

$$\mathfrak{g}_C = \mathfrak{g}_C^{(-1,1)} \oplus \mathfrak{g}_C^{(0,0)} \oplus \mathfrak{g}_C^{(1,-1)}.$$ 

Here

$$\mathfrak{g}_C^{(p,q)} = \{X \in \mathfrak{g}_C | h(z)^{-1}Xh(z) = z^{-p}\tau^{-q}X \text{ for } z \in \mathbb{C}\}.$$ 

We write $\mathfrak{p}_h^\pm = \mathfrak{g}_C^{(-1,1)}$. Note that $\mathfrak{p}_h^+$ and $\mathfrak{p}_h^-$ are the holomorphic and anti-holomorphic tangent spaces of $X$ at $h$, respectively, and $\mathfrak{g}_C^{(0,0)}$ is the complexified Lie algebra $\mathfrak{t}_{h,C}$ of $K_h$. Let $\mathfrak{P}_h$ be the subalgebra of $\mathfrak{g}_C$ defined by

$$\mathfrak{P}_h = \mathfrak{t}_{h,C} \oplus \mathfrak{p}_h^+,$$

and $P_h$ the parabolic subgroup of $G(\mathbb{C})$ with Lie algebras $\mathfrak{P}_h$. We write

$$\breve{X}_h = G(\mathbb{C})/P_h.$$ 

Note that the flag variety $\breve{X}_h$ has a natural structure over $E(G,X)$. The inclusion $G(\mathbb{R}) \subset G(\mathbb{C})$ induces an embedding

$$\beta_h : X \cong G(\mathbb{R})/K_h \hookrightarrow \breve{X}_h.$$ 

For a $G_C$-vector bundle $V$ on $\breve{X}_h$, we denote by $\beta_h^*(V)$ the pullback $G(\mathbb{R})$-bundle on $X$. We say $V$ is motivic if the following condition is satisfied:

The action of $G_C$ on $V$ factors through $(G/Z_s)_C$.

Here $Z_s$ is the largest subtorus of the center $Z_G$ of $G$ that is split over $\mathbb{R}$ but that has no subtorus split over $\mathbb{Q}$. For motivic $V$, the automorphic vector bundle $[V]$ on $\text{Sh}(G,X)$ is defined by

$$[V] = \lim_{\mathcal{K}} G(\mathbb{Q})\backslash \beta_h^*(V) \times G(\mathbb{A}_{\mathbb{Q},f})/\mathcal{K},$$

where $\mathcal{K}$ runs through neat open compact subgroups of $G(\mathbb{A}_{\mathbb{Q},f})$. We have the following result due to Harris [Har85, Theorem 3.3] and Milne [Mil90, Theorem 5.1].

**Theorem 2.1.** The functor $V \mapsto [V]$, from motivic $G_C$-vector bundles on $\breve{X}_h$ to $G(\mathbb{A}_{\mathbb{Q},f})$-vector bundles on $\text{Sh}(G,X)$, is rational over $E(G,X)$.

Note that we have an equivalence of categories (cf. [IP21, Remark 1.2])

$$G_C$$-vector bundles on $\breve{X}_h \leftrightarrow \text{finite-dimensional algebraic representations of } P_h \text{ over } \mathbb{C}.$$

### 2.2. Coherent cohomology groups on Shimura varieties.

We keep the notation of the previous section. Let $\mathcal{A}(G(\mathbb{A}_{\mathbb{Q}}))$, $\mathcal{A}_{(2)}(G(\mathbb{A}_{\mathbb{Q}}))$, and $\mathcal{A}_0(G(\mathbb{A}_{\mathbb{Q}}))$ be the spaces of $K_h$-finite automorphic forms, essentially square-integrable automorphic forms, and cusp forms, respectively, on $G(\mathbb{A}_{\mathbb{Q}})$. Let $(\rho, V)$ be an irreducible algebraic representation of $K_h$. Then $\rho$ extends to an algebraic representation of $P_h$ so that the action factors through the reductive quotient $K_h \cdot C$. We denote by $\mathcal{V}_{\rho}$ the corresponding $G_C$-vector bundle on $\breve{X}_h$. We say $(\rho, V)$ is motivic if $\mathcal{V}_{\rho}$ is motivic. We have the $(\mathfrak{P}_h, K_h)$-modules

$$\mathcal{A}(G(\mathbb{A}_{\mathbb{Q}})) \otimes V, \quad \mathcal{A}_{(2)}(G(\mathbb{A}_{\mathbb{Q}})) \otimes V, \quad \mathcal{A}_0(G(\mathbb{A}_{\mathbb{Q}})) \otimes V,$$

where the action of $\mathfrak{P}_h$ on $V$ factors through $\mathfrak{c}_C$. Consider the complexes with respect to the Lie algebra differential operator (cf. [BW00, Chapter I]):

$$C^\bullet_\rho = (\mathcal{A}(G(\mathbb{A}_{\mathbb{Q}})) \otimes \wedge^\bullet \mathfrak{p}_h^+ \otimes V)^{K_h},$$

$$C^\bullet_{(2),\rho} = (\mathcal{A}_{(2)}(G(\mathbb{A}_{\mathbb{Q}})) \otimes \wedge^\bullet \mathfrak{p}_h^+ \otimes V)^{K_h},$$

$$C^\bullet_{\text{cusp},\rho} = (\mathcal{A}_0(G(\mathbb{A}_{\mathbb{Q}})) \otimes \wedge^\bullet \mathfrak{p}_h^+ \otimes V)^{K_h}.$$ 

The corresponding $q$-th $(\mathfrak{P}_h, K_h)$-cohomology groups of the above complexes are denoted respectively by

$$H^q(\mathfrak{P}_h, K_h; \mathcal{A}(G(\mathbb{A}_{\mathbb{Q}})) \otimes V), \quad H^q(\mathfrak{P}_h, K_h; \mathcal{A}_{(2)}(G(\mathbb{A}_{\mathbb{Q}})) \otimes V), \quad H^q(\mathfrak{P}_h, K_h; \mathcal{A}_0(G(\mathbb{A}_{\mathbb{Q}})) \otimes V).$$

Note that $G(\mathbb{A}_{\mathbb{Q},f})$ acts on the above complexes by right translation. This in turn defines $G(\mathbb{A}_{\mathbb{Q},f})$-module structures on the cohomology groups. Suppose $V$ is motivic, by the result of Harris [Har90b, Corollary 3.4] and Jun [Su18, Theorem 6.7], the cohomology group $H^q(\mathfrak{P}_h, K_h; \mathcal{A}(G(\mathbb{A}_{\mathbb{Q}})) \otimes V)$ is canonically isomorphic as.
Let $\mathcal{V}_\rho$ be a subcanonical extension of $[\mathcal{V}_\rho]$ on the good toroidal compactification over which $[\mathcal{V}_\rho]$ is defined. We denote by $H^q(\mathcal{V}_\rho)$ the $q$-th interior cohomology group of $\mathcal{V}_\rho$. For $q \geq 0$, $H^q(\mathcal{V}_\rho)$ is the $q$-th cohomology group with the relative Lie algebra $\mathfrak{h}$-cohomology groups and write

$$H^q([\mathcal{V}_\rho]^\text{can}) = H^q(\mathcal{V}_\rho; \mathcal{A}(G(\mathbb{A}_Q)) \otimes V).$$

We also write

$$H^q_{(2)}([\mathcal{V}_\rho]) = H^q(\mathcal{V}_\rho; \mathcal{A}(G(\mathbb{A}_Q)) \otimes V), \quad H^q_{\text{cusp}}([\mathcal{V}_\rho]) = H^q(\mathcal{V}_\rho; \mathcal{A}_0(G(\mathbb{A}_Q)) \otimes V).$$

The natural inclusions $\mathcal{A}_0(G(\mathbb{A}_Q)) \subset \mathcal{A}(G(\mathbb{A}_Q)) \subset \mathcal{A}(G(\mathbb{A}_Q))$ induce the $G(\mathbb{A}_Q)$-module homomorphisms

$$H^q_{\text{cusp}}([\mathcal{V}_\rho]) \rightarrow H^q_{(2)}([\mathcal{V}_\rho]) \rightarrow H^q([\mathcal{V}_\rho]^\text{can}).$$

Let $[\mathcal{V}_\rho]^\text{sub}$ be the subcanonical extension of $[\mathcal{V}_\rho]$ on the good toroidal compactification over which $[\mathcal{V}_\rho]$ is defined. We denote by $H^q([\mathcal{V}_\rho]^\text{sub})$ the $q$-th interior cohomology group of $[\mathcal{V}_\rho]$ induced by the exact sequence $[\text{Har90b}, (2.2.4)]$. We call $H^q([\mathcal{V}_\rho])$ the $q$-th interior cohomology group of $[\mathcal{V}_\rho]$. In the following theorem, we recall some results of Harris [Har85, Har90b] and Milne [Mil83] on the coherent cohomology groups. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_f$ and a positive system of $\mathfrak{g}_C$, $\mathfrak{b}_C$ such that the set of non-compact positive roots gives the root spaces in $\mathfrak{p}_C$. Denote by $\Lambda_\rho \in \mathfrak{h}_C^*$ the corresponding highest weight of $\rho$. Let $\mathcal{A}(G(\mathbb{A}_Q), \rho)$ (resp. $\mathcal{A}_0(G(\mathbb{A}_Q), \rho)$) be the space consisting of essentially square-integrable automorphic forms (resp. cusp forms) on $G(\mathbb{A}_Q)$ which are eigenfunctions of the Casimir operator of $\mathfrak{g}_C$ with eigenvalue

$$\langle \Lambda_\rho + \delta, \Lambda_\rho + \delta \rangle_{\mathfrak{h}} - \langle \delta, \delta \rangle_{\mathfrak{h}}.$$

Here $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ is the Killing form on $\mathfrak{h}_C = \text{Hom}_{\mathbb{R}}(\mathfrak{h}, \mathbb{R})$ and $\delta$ is the half-sum of positive roots.

**Theorem 2.2.** Let $(\rho, V)$ be an irreducible motivic algebraic representation of $K_h$ defined over some finite extension $\mathbb{Q}(\rho)$ of $E(G, X)$.

1. For $\sigma \in \text{Aut}(\mathbb{C}/E(G, X))$, conjugation by $\sigma$ induces natural $\sigma$-linear $G(\mathbb{A}_Q)$-module isomorphisms

$$T_\sigma : H^q([\mathcal{V}_\rho]^\text{sub}) \rightarrow H^q([\mathcal{V}_\rho]^\text{sub}), \quad T_\sigma : H^q([\mathcal{V}_\rho]^\text{can}) \rightarrow H^q([\mathcal{V}_\rho]^\text{can}),$$

and such that the diagram

$$\begin{array}{ccc}
H^q([\mathcal{V}_\rho]^\text{sub}) & \xrightarrow{T_\sigma} & H^q([\mathcal{V}_\rho]^\text{sub}) \\
\downarrow & & \downarrow \\
H^q([\mathcal{V}_\rho]^\text{can}) & \xrightarrow{T_\sigma} & H^q([\mathcal{V}_\rho]^\text{can})
\end{array}$$

is commutative. Moreover, $H^q([\mathcal{V}_\rho]^\text{sub})$ and $H^q([\mathcal{V}_\rho]^\text{can})$ are admissible $G(\mathbb{A}_Q)$-modules and have canonical rational structures over $\mathbb{Q}(\rho)$ given by taking the Galois invariants with respect to $T_\sigma$ for $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\rho))$.

2. We have

$$H^q_{(2)}([\mathcal{V}_\rho]) = (\mathcal{A}(G(\mathbb{A}_Q), \rho) \otimes \mathbb{A}^q \mathfrak{p}_h^+ \otimes V)^K_h,$$

$$H^q_{\text{cusp}}([\mathcal{V}_\rho]) = (\mathcal{A}_0(G(\mathbb{A}_Q), \rho) \otimes \mathbb{A}^q \mathfrak{p}_h^+ \otimes V)^K_h.$$

3. The homomorphism $H^q_{\text{cusp}}([\mathcal{V}_\rho]) \rightarrow H^q([\mathcal{V}_\rho]^\text{can})$ in (2.2) is injective and its image is contained in $H^q([\mathcal{V}_\rho])$.

4. The interior cohomology group $H^q_{(2)}([\mathcal{V}_\rho])$ is contained in the image of the homomorphism $H^q_{(2)}([\mathcal{V}_\rho]) \rightarrow H^q([\mathcal{V}_\rho]^\text{can})$ in (2.2). In particular, $H^q_{(2)}([\mathcal{V}_\rho])$ is a semisimple $G(\mathbb{A}_Q)$-module.

We identify $H^q_{\text{cusp}}([\mathcal{V}_\rho])$ with a $G(\mathbb{A}_Q)$-submodule of $H^q([\mathcal{V}_\rho])$ via the injection in Theorem 2.2 (3). By Theorem 2.2 (1), we have

$$T_\sigma(H^q([\mathcal{V}_\rho])) = H^q([\mathcal{V}_\rho])$$

for all $\sigma \in \text{Aut}(\mathbb{C}/E(G, X))$. 

6
2.3. Hilbert–Siegel modular varieties. For $n \geq 1$, let $\text{GSp}_{2n}$ be the symplectic similitude group defined by
\[
\text{GSp}_{2n} = \{ g \in \text{GL}_{2n} \mid g \left( \begin{array}{cc} 0 & 1_n \\ -1_n & 0 \end{array} \right) g = \nu(g) \left( \begin{array}{cc} 0 & 1_n \\ -1_n & 0 \end{array} \right), \nu(g) \in \text{GL}_1 \}.
\]
Let $\text{GSp}^+_{2n}(\mathbb{R})$ be the closed subgroup of $\text{GSp}_{2n}(\mathbb{R})$ consisting of elements with positive similitude. Let $K_n = \mathbb{R}_+ \times \text{U}(n)$ and regard it as a closed subgroup of $\text{GSp}_{2n}(\mathbb{R})$ by the homomorphism
\[
(a, A + \sqrt{-1} B) \mapsto a\mathbf{1}_{2n} \cdot \left( \begin{array}{cc} A & B \\ -B & A \end{array} \right).
\]
We write $\mathfrak{g}_n \subset \mathfrak{gl}_{2n}$ for the Lie algebra of $\text{GSp}_{2n}(\mathbb{R}) \subset \text{GL}_{2n}(\mathbb{R})$. Define $\mathfrak{k}_n \subset \mathfrak{g}_n$ and $\mathfrak{p}_n^+ \subset \mathfrak{g}_{n,\mathbb{C}}$ by
\[
\mathfrak{k}_n = \text{Lie}(K_n), \quad \mathfrak{p}_n^+ = \left\{ \left( \begin{array}{cc} -\sqrt{-1} A & \pm A \\ \pm A & \sqrt{-1} A \end{array} \right) \mid A \in \text{Sym}_n(\mathbb{C}) \right\}, \quad \mathfrak{q}_n = \mathfrak{k}_{n,\mathbb{C}} \oplus \mathfrak{p}_n^-.
\]
We will identify $\mathfrak{p}_n^+$ with $\text{Sym}_n(\mathbb{C})$ by the map
\[
\left( \begin{array}{cc} -\sqrt{-1} A & \pm A \\ \pm A & \sqrt{-1} A \end{array} \right) \mapsto A.
\]
Let $(G_n, X_n)$ be the Shimura datum defined by
\[
G_n = \text{Res}_{\mathbb{Q}/\mathbb{R}} \text{GSp}_{2n,\mathbb{R}},
\]
and $X_n$ is the $G_n(\mathbb{R})$-conjugacy class containing the morphism $h_n : \mathbb{S} \to G_n(\mathbb{R})$ with
\[
h_n(x + \sqrt{-1} y) = \left( \begin{array}{c} x1_n \\ -y1_n \\ x1_n \\ y1_n \end{array} \right)
\]
on $\mathbb{R}$-points. The associated Shimura variety is called the Hilbert–Siegel modular variety. Note that the reflex field $E(G_n, X_n)$ is equal to $\mathbb{Q}$. Under the identification of $F_n$ with $\mathbb{R}$ for each $v \in S_\infty$, we have
\[
K_{h_n} = K_n^{S_\infty}, \quad \mathfrak{k}_{h_n} = \mathfrak{t}_n^{S_\infty}, \quad \mathfrak{p}_{h_n}^+ = (\mathfrak{p}_n^{S_\infty}), \quad \mathfrak{q}_{h_n} = \mathfrak{q}_n^{S_\infty}.
\]
Any motivic irreducible algebraic representation of $K_n^{S_\infty}$ is of the form:
\[
(\rho, V_\rho) = \left( \bigotimes_{v \in S_\infty} \rho_v, \bigotimes_{v \in S_\infty} V_{\rho_v} \right)
\]
for some irreducible algebraic representation $(\rho_v, V_\rho_v)$ of $K_n$ such that $\rho_v|_{\mathbb{R}_+} \simeq \rho_w|_{\mathbb{R}_+}$ for all $v, w \in S_\infty$. For $\sigma \in \text{Aut}(\mathbb{C})$, let $(\sigma \rho, V_{\sigma \rho})$ be the motivic irreducible algebraic representation defined by
\[
(\sigma \rho, V_{\sigma \rho}) = \left( \bigotimes_{v \in S_\infty} \rho_{\sigma^{-1} v}, \bigotimes_{v \in S_\infty} V_{\rho_{\sigma^{-1} v}} \right).
\]
2.4. Automorphic periods for $\text{GL}_2$. In this section, we recall the automorphic periods of C-algebraic irreducible cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_\mathbb{Z})$ defined by Harris [Har89] through the coherent cohomology for automorphic line bundles on $\text{Sh}(G_1, X_1)$.

2.4.1. (Limits of) discrete series representations of $\text{GL}_2(\mathbb{R})$. Let $(\kappa, w) \in \mathbb{Z} \times \mathbb{Z}$ such that $\kappa \equiv w (\text{mod} \ 2)$. Let $(\rho_{(\kappa, w)}, V_{(\kappa, w)})$ be the algebraic character of $K_1$ defined by $V_{(\kappa, w)} = \mathbb{C}$ and
\[
(\mathfrak{a}w)^{\kappa} \cdot z = a^w u^\kappa \cdot z
\]
for $a \in \mathbb{R}^\times$ and $u \in \text{U}(1)$. Note that $\rho_{(\kappa, w)} = \rho_{(\kappa, -w)}$. Assume further that
\[
\kappa \geq 1.
\]
Let $D_{(\pm \kappa, w)}$ be the (limit of) discrete series representation of $\text{GL}_2^+(\mathbb{R})$ with minimal $K_1$-type $\rho_{(\pm \kappa, w)}$. Let $H_{(\kappa, w)}$ be the (limit of) discrete series representation of $\text{GL}_2(\mathbb{R})$ defined by
\[
H_{(\kappa, w)} = D_{(\kappa, w)} \oplus D_{(-\kappa, w)},
\]
with
\[
\text{diag}(-1, 1) \cdot (v_1, v_2) = (v_2, v_1).
\]
Conversely, up to central twists, any (limit of) discrete series representation of $\text{GL}_2(\mathbb{R})$ is obtained in this way. It is well-known that $H_{(\kappa, w)}$ is generic. Let $\psi_{\mathbb{R}}$ be the standard additive character of $\mathbb{R}$, that is,
\[ \psi_r(x) = e^{2\pi\sqrt{-1} x}. \] We denote by \( W(\Pi(\kappa, w), \psi_r) \) the space of Whittaker functions of \( \Pi(\kappa, w) \) with respect to \( \psi_r \). Let \( W^+_\kappa, w \in W(\Pi(\kappa, w), \psi_r) \) be the Whittaker function of weight \( \kappa \) normalized so that
\[
W^+_\kappa, w(\text{diag}(a, 1)) = |a|^{(\kappa + w)/2} e^{-2\pi |u|} \cdot \| \kappa, w^+ \cdot (a) \).
\]

### 2.4.2. Rational structure via the Whittaker model

Let \((\Pi, V_\Pi)\) be an \( \mathcal{C} \)-algebraic irreducible cuspidal automorphic representation of \( \text{GL}_2(\mathbb{A}_F) \) with central character \( \omega\Pi \). Here we follow [BG14 Definition 5.11] for the notion of \( \mathcal{C} \)-algebraicity, it is refer to algebraic automorphic representation in [Clo90]. We assume further that the following condition is satisfied:
\[
\Pi_v \text{ is a (limit of) discrete series representation for all } v \in S_\infty.
\]
Then there exists \((\kappa, w) \in \mathbb{Z}^{S_{\infty}} \times \mathbb{Z} \) with \( \kappa = (\kappa_v)_{v \in S_\infty} \) satisfying the following conditions:
- \( |\omega\Pi| = |\omega\Pi| \); \( \kappa_v \geq 1 \) and \( \kappa_v \equiv w \pmod{2} \) for all \( v \in S_\infty \).
- \( \Pi_v = \Pi(\kappa_v, w) \) for all \( v \in S_\infty \).

We say \( \Pi \) is regular if \( \kappa_v \geq 2 \) for all \( v \in S_\infty \). We call \((\kappa, w)\) the weight of \( \Pi \). Let \( \psi \) be a non-trivial additive character of \( \mathbb{F} \mathbb{A}_F \). Let \( \Pi_f = \bigotimes_v \Pi_v \) be the finite part of \( \Pi \) and \( W(\Pi_f, \psi_f) \) the space of Whittaker functions of \( \Pi_f \) with respect to \( \psi_f \). For \( \varphi \in V_\Pi \), let \( W_{\varphi, \psi} \) be the Whittaker function of \( \varphi \) with respect to \( \psi \) defined by
\[
W_{\varphi, \psi}(g) = \int_{F \mathbb{A}_F} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(x) dx_{\text{Tam}}.
\]
Here \( dx_{\text{Tam}} \) is the Tamagawa measure on \( \mathbb{A}_F \). Let \( V^+_f \) be the space of cusp forms in \( V^+_\Pi \) of weight \( \kappa \), that is, \( \varphi \in V^+_f \) if and only if
\[
\varphi(gu) = \prod_{v \in S_\infty} u_v^{\kappa_v} \cdot \varphi(g)
\]
for all \( u = (u_v)_{v \in S_\infty} \in U(1)^{S_\infty} \) and \( g \in \text{GL}_2(\mathbb{A}_F) \). Take \( \psi = \psi_\kappa \) to be the standard additive character. For \( \varphi \in V^+_f \), let \( W_{\varphi, \psi} \in W(\Pi_f, \psi_f) \) be the unique Whittaker function so that
\[
W_{\varphi, \psi} = \prod_{v \in S_\infty} W^+_\kappa, w \cdot W_{\varphi}^f.
\]

Then the map \( \varphi \mapsto W_{\varphi}^f \) defines a \( \text{GL}_2(\mathbb{A}_F^f) \)-equivariant isomorphism from \( V^+_f \) to \( W(\Pi_f, \psi_f) \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), let
\[
\sigma \Pi = \sigma \Pi \otimes \sigma \Pi_f
\]
be the irreducible admissible representation of \( \text{GL}_2(\mathbb{A}_F^f) \) defined so that the \( v \)-component of \( \sigma \Pi_v \) is isomorphic to \( \Pi_{\sigma^{-1} v} \) for each \( v \in S_\infty \). It is known that \( \sigma \Pi \) is cuspidal automorphic. Moreover, it is clear that \( \sigma \Pi \) is \( \mathcal{C} \)-algebraic of weight \( (\sigma, \kappa \cdot \sigma^{-1}, w) \), where \( \sigma \kappa = (\kappa_{\sigma^{-1} v})_{v \in S_\infty} \). Let \( Q(\kappa) \) and \( Q(\Pi) \) be the fixed fields of \( \{ \sigma \in \text{Aut}(\mathbb{C}) \mid \sigma \kappa = \kappa \} \) and \( \{ \sigma \in \text{Aut}(\mathbb{C}) \mid \sigma \Pi = \Pi_f \} \), respectively. Note that \( Q(\kappa) \subset Q(\Pi) \) by the strong multiplicity one theorem for \( \text{GL}_2 \). Let
\[
t_{\sigma} : W(\Pi_f, \psi_f) \longrightarrow W(\sigma \Pi_f, \psi_f)
\]
be the \( \sigma \)-linear \( \text{GL}_2(\mathbb{A}_F^f) \)-equivariant isomorphism defined by
\[
t_{\sigma} W(g) = \sigma(W(\text{diag}(u_\sigma^{-1}, 1) g)).
\]
Here \( u_\sigma \in \mathbb{Z}^{S_\infty} \subset \mathfrak{d}_\kappa^{\infty} \) is the unique element such that \( \sigma(\psi(x)) = \psi(u_\sigma x) \) for all \( x \in \mathbb{A}_F^f \). Let
\[
V^+_f \longrightarrow V^+_f, \quad \varphi \longrightarrow \sigma \varphi
\]
be the \( \sigma \)-linear \( \text{GL}_2(\mathbb{A}_F^f) \)-equivariant isomorphism defined such that
\[
W_{\sigma \varphi}^f = t_{\sigma} W_{\varphi}^f.
\]
We thus obtain a \( Q(\Pi) \)-rational structure \( (V^+_f)^{\text{Aut}(\mathbb{C}/Q(\Pi))} \) on \( V^+_f \) given by taking the Galois invariants:
\[
(\Pi_f)^{\text{Aut}(\mathbb{C}/Q(\Pi))} = \{ \varphi \in V^+_f \mid \sigma \varphi = \varphi \text{ for } \sigma \in \text{Aut}(\mathbb{C}/Q(\Pi)) \}. 
\]
2.4.3. Rational structure via the coherent cohomology. Now we recall the rational structures on $V_I^+$ given by the coherent cohomology of automorphic lines bundles on $\text{Sh}(G_1, X_1)$. For $I \subset S_\kappa$, define $\kappa(I) = (\kappa_v(I))_{v \in S_\kappa} \in \mathbb{Z}^{S_\kappa}$ by

$$
\kappa_v(I) = \begin{cases} 
\kappa_v - 2 & \text{if } v \in I, \\
-\kappa_v & \text{if } v \notin I.
\end{cases}
$$

By definition, we have $\sigma(\kappa(I)) = \sigma(\kappa)$ for $\sigma \in \text{Aut}(C)$. Let $Q(I)$ be the fixed field of $\{ \sigma \in \text{Aut}(C) \mid \sigma I = I \}$. It is clear that $\kappa(I)$ is invariant by $\text{Aut}(C/Q(\kappa)/Q(I))$. We say $I$ is admissible with respect to $\kappa$ if $I$ is uniquely determined by $\kappa(I)$, that is, if $J \subset S_\kappa$ with $\kappa(J) = \kappa(I)$, then $J = I$. For instance, $I$ is admissible when $I \in \{ \mathcal{O}, S_\kappa \}$ or $\kappa_v \geq 2$ for all $v \in I$. Let

$$[V_\kappa(I); -w]$$

denote the automorphic line bundle on $\text{Sh}(G_1, X_1)$ defined by the motivic algebraic representation

$$(\rho(\kappa(I); -w), V_\kappa(I); -w) = \left( \bigotimes_{v \in S_\kappa} \rho_{\kappa_v(I)}; -w, \bigotimes_{v \in S_\kappa} V_{\kappa_v(I); -w} \right)$$

of $K_1^{S_\kappa}$. Let $H^q_c([V_\kappa(I); -w])[[I]]$ be the $I_f$-isotypic component of $H^q_c([V_\kappa(I); -w])$. We have the following result proved in [Har89] Lemmas 1.4.3 and 1.4.5 (see also [Che21a] Lemma 2.5 and Remark 2.7).

**Lemma 2.3.** Let $I \subset S_\kappa$.

1. For all $\sigma \in \text{Aut}(C)$ and $q \geq 0$, we have

$$T_\sigma(H^q_c([V_\kappa(I); -w])[[I]]) = H^q_c([V_{\kappa(\sigma(I)); -w}]^\sigma[[I]]).$$

In particular, we have a $\mathbb{Q}(\Pi)\mathbb{Q}(I)$-rational structure on $H^q_c([V_\kappa(I); -w])[[I]]$ given by taking the Galois invariants:

$$H^q_c([V_\kappa(I); -w])[[I]] = \{ c \in H^q_c([V_\kappa(I); -w])[[I]] \mid T_\sigma c = c \text{ for } \sigma \in \text{Aut}(C/\mathbb{Q}(\Pi)\mathbb{Q}(I)) \}.$$  

2. Assume $I$ is admissible, then $H^q_c([V_\kappa(I); -w])[[I]] \cong \Pi_I$. In this case, we have a $\text{GL}_2(A_{\kappa,f})$-equivariant isomorphism

$$V_\Pi^+ \longrightarrow H_{cusp}^1([V_\kappa(I); -w])[[I]] , \quad \varphi \longmapsto [\varphi]_I$$

defined by

$$[\varphi]_I = \varphi^I \otimes \bigotimes_{v \in I} X_{+,v} \otimes \bigotimes_{v \in S_\kappa} 1 \in A_0(\text{GL}_2(A_{\kappa,f})) \otimes \wedge^I (p_1^+)_{S_\kappa} \otimes V_\kappa(I; -w).$$

Here $\varphi^I$ is defined by

$$\varphi^I(g) = \varphi \left( g \cdot \prod_{v \in I} \text{diag}((-1,1)) \right),$$

and $X_+ = \left( \begin{smallmatrix} -1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{smallmatrix} \right) \in p_1^+$ and $X_{+,v} \in p_1^+_{S_\kappa}$ is defined so that its $v$-component is $X_+$ and zero otherwise. We also fix an ordering of the wedge $\bigotimes_{v \in I} X_{+,v} \in \wedge^I (p_1^+)_{S_\kappa}$ once and for all.

2.4.4. Automorphic periods and period relations. By comparing the rational structures in (2.3) and Lemma 2.3(1), we have the following lemma/definition for the automorphic periods of $\Pi$.

**Lemma 2.4.** Let $I \subset S_\kappa$ be admissible with respect to $\kappa$. There exists a sequence of non-zero complex numbers $(p^\sigma(\kappa))_{\sigma \in \text{Aut}(C)}$ such that

$$T_\sigma \left( [\varphi]_I \middle/ p^\sigma(\kappa) \right) = [\varphi]_I \middle/ p^\sigma(\kappa)$$

for all $\sigma \in \text{Aut}(C)$ and $\varphi \in V_\Pi^+$. Here $T_\sigma : H_{cusp}^1([V_\kappa(I); -w]) \rightarrow H_{cusp}^1([V_{\kappa(\sigma(I)); -w}])$ is the $\sigma$-linear isomorphism in (2.3).

For the automorphic periods, we have the following period relations proved in [Har89] Propositions 1.3.3 and 1.5.6]
Lemma 2.5. Let $f_H \in V^+_H$ be the normalized newform of $H$ and $\|f_H\|$ the Petersson norm of $f_H$ defined as in [1,2].

(1) For $\sigma \in \text{Aut}(\mathbb{C})$, we have
\[
\sigma \left( \frac{p^\sigma(II)}{(2\pi \sqrt{-1})^{-\sum c_{s,e}(\kappa_e + \omega)/2}} \right) = \frac{p^\sigma(\sigma II)}{(2\pi \sqrt{-1})^{-\sum c_{s,e}(\kappa_e + \omega)/2}}.
\]

(2) Let $I \subset S_\chi$ be admissible with respect to $\kappa$. For $\sigma \in \text{Aut}(\mathbb{C})$, we have
\[
\sigma \left( \frac{\|f_H\|}{p^{\sigma(I)} \cdot p^{\sigma \chi^{-\kappa}(\kappa \chi)}} \right) = \frac{\|f_H\|}{p^{\sigma(I)} \cdot p^{\sigma \chi^{-\kappa}(\kappa \chi)}}.
\]

2.5. Holomorphic Eisenstein series on $\text{GL}_2$. In this section, we recall the algebraicity of holomorphic Eisenstein series on $\text{GL}_2(\mathbb{A}_F)$. Let $\chi$ be an algebraic Hecke character of $\mathbb{A}_F^\times$. Let $w \in \mathbb{Z}$ be the integer such that $|\chi| = |\chi|_\mathbb{A}$.

We assume $\chi$ has parallel signature with $\text{sgn}(\chi) = (-1)^w$. For $s \in \mathbb{C}$, let
\[
I(\chi, s) = \text{Ind}_{B_2(\mathbb{A}_F)}^{\text{GL}_2(\mathbb{A}_F)}(\chi)^{-1/2} \otimes |\chi|^{-s+1/2}
\]
be the induced representation consisting of smooth right $(K_1^{\mathbb{A}_F} \times \text{GL}_2(\mathbb{A}_F))$-finite functions $f : \text{GL}_2(\mathbb{A}_F) \to \mathbb{C}$ such that
\[
f \left( \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} g \right) = \chi(d) \frac{a|s|}{d} \cdot f(g).
\]

Here $B_2$ is the standard Borel subgroup of $\text{GL}_2$ consisting of upper triangular matrices. We denote by $\rho$ the right translation action of $K_1^{\mathbb{A}_F} \times \text{GL}_2(\mathbb{A}_F)$ on $I(\chi, s)$. A function
\[
\mathbb{C} \times \text{GL}_2(\mathbb{A}_F) \to \mathbb{C}, \quad (s, g) \mapsto f^{(s)}(g)
\]
is called a holomorphic section of $I(\chi, s)$ if it satisfies the following conditions:

- For each $s \in \mathbb{C}$, the function $g \mapsto f^{(s)}(g)$ belongs to $I(\chi, s)$.
- For each $g \in \text{GL}_2(\mathbb{A}_F)$, the function $s \mapsto f^{(s)}(g)$ is holomorphic.
- $f^{(s)}$ is right $(K_1^{\mathbb{A}_F} \times \text{GL}_2(\mathbb{A}_F))$-finite.

A function $f^{(s)}$ on $\mathbb{C} \times \text{GL}_2(\mathbb{A}_F)$ is called a meromorphic section of $I(\chi, s)$ if there exists a non-zero entire function $\beta$ such that $\beta(s)f^{(s)}$ is a holomorphic section. For a place $v$ of $\mathbb{F}$, we define $I(\chi_v, s)$ and the notion of holomorphic and meromorphic sections in a similar way.

- When $v$ is finite, a meromorphic section $f_v^{(s)}$ of $I(\chi_v, s)$ is called a rational section if the map $s \mapsto f_v^{(s)}(g)$ is a rational function in $q_v^{-s}$ for all $g \in \text{GL}_2(\mathbb{F}_v)$. For a rational section $f_v^{(s)}$ and $\sigma \in \text{Aut}(\mathbb{C})$, let $\sigma f_v^{(s)}$ be the rational section of $I(\sigma \chi_v, s)$ defined by
\[
\sigma f_v^{(s)} \left( \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} k \right) = \sigma \chi_v(d) \frac{a|s|}{d} \cdot \sigma \chi_s(k)
\]
for $a, d \in \mathbb{F}_v^\times$ and $k \in \text{GL}_2(\mathbb{F}_v)$.

- When $v$ is finite and $\chi_v$ is unramified, let $f_v^{(s)}_{\mathbb{A}_v}$ be the $\text{GL}_2(\mathbb{A}_v)$-invariant rational section of $I(\chi_v, s)$ normalized so that
\[
f_v^{(s)}(1) = L(2s, \chi_v^{-1}).
\]

- When $v \in S_\chi$ and $\kappa \geq 1$ with $(-1)^\kappa = \text{sgn}(\chi)$, let $f_{v,\kappa}^{(s)}$ be the holomorphic section of $I(\chi_v, s)$ of weight $\kappa$ normalized so that $f_{v,\kappa}^{(s)}(1) = 1$.

Let $f^{(s)}$ be a holomorphic section of $I(\chi, s)$. We define the associated Eisenstein series $E(f^{(s)})$ on $\text{GL}_2(\mathbb{A}_F)$ by the absolutely convergent series
\[
E(g, f^{(s)}) = \sum_{\gamma \in B_2(\mathbb{F}) \setminus \text{GL}_2(\mathbb{A}_F)} f^{(s)}(\gamma g)
\]
for $\Re(s) > 1 + \frac{w}{2}$, and by meromorphic continuation otherwise. We have the following result on the algebraicity of holomorphic Eisenstein series. When $\kappa > 2$, the Eisenstein series converges absolutely at $s = \frac{2 + w}{2}$. In this case, the result is a special case of the result of Harris [Har84]. Based on explicit computation of Fourier coefficients, Shimura [Shi78] proved the algebraicity for any $\kappa \geq 1$, except for $\mathbb{F} = \mathbb{Q}$, $\kappa = 2$, and $\chi = |\chi|_{\mathbb{A}_Q}$.
Proposition 2.6 (Harris, Shimura). Let \( f^{(s)} = \bigotimes_{v \in S_x} f^{(s)}_v \) be a meromorphic section of \( I(\chi_f, s) \) and \( \kappa \geq 1 \) with \((-1)^\kappa = \text{sgn}(\chi)\). Assume the following conditions are satisfied:

- \( f^{(s)} \) is holomorphic for \( \text{Re}(s) > \frac{\kappa - 1}{2} \).
- \( f^{(s)}_v \) is a rational section for all \( v \not| \infty \).
- \( f^{(s)}_v = f^{(s)}_{\overline{v}} \) for almost all \( v \).
- If \( \mathbb{F} = \mathbb{Q} \), then \( \kappa \neq 2 \) or \( \chi \neq \mid \lambda \mid ^{\frac{\kappa}{2}} \).

Then the following assertions hold:

1. The Eisenstein series \( E \left( \bigotimes_{v \in S_x} f^{(s)}_v \otimes f^{(s)} \right) \) is holomorphic at \( s = \frac{-\kappa + \omega}{2} \).
2. The automorphic form \( E[\kappa](f^{(s)}) = E \left( \bigotimes_{v \in S_x} f^{(s)}_v \otimes f^{(s)} \right) \mid_{\kappa + \omega} \) defines a global section in the coherent cohomology \( H^0([V(-\kappa, -\omega)]^\text{can}) \) given by
   \[
   [E[\kappa](f^{(s)})] = E[\kappa](f^{(s)}) \otimes \bigotimes_{v \in S_x} 1 \in \mathcal{A}(GL_2(\mathbb{A}_F)) \otimes V(-\kappa, -\omega).
   \]
3. For \( \sigma \in \text{Aut}(\mathbb{C}) \), we have
   \[
   T_\sigma \left( \frac{[E[\kappa](f^{(s)})]}{|D_\mathbb{F}|^{1/2} \cdot (2\pi \sqrt{-1})^{d(\kappa - \omega)/2} \cdot G(\chi)^{-1}} \right) = \frac{[E[\kappa](\sigma f^{(s)})]}{|D_\mathbb{F}|^{1/2} \cdot (2\pi \sqrt{-1})^{d(\kappa - \omega)/2} \cdot G(\sigma \chi)^{-1}}.
   \]

2.6. Automorphic periods for \( \text{GSp}_4 \). In this section, we define automorphic periods of globally generic \( C \)-algebraic irreducible cuspidal automorphic representations of \( \text{GSp}_4(\mathbb{A}_F) \) through the coherent cohomology for automorphic vector bundles on \( \text{Sh}(G'_2, X_2) \). Let \( U \) be the maximal unipotent subgroup of \( \text{GSp}_4 \) defined by

\[
U = \left\{ \begin{pmatrix}
1 & * & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & 0 \\
0 & 0 & * & 1
\end{pmatrix} \in \text{GSp}_4 \right\}.
\]

For a non-trivial additive character \( \psi \) of \( \mathbb{F} \setminus \mathbb{A} \), let \( \psi_U \) be the associated additive character of \( U(\mathbb{F}) \setminus U(\mathbb{A}) \) defined by

\[
\psi_U \begin{pmatrix}
1 & x & * & * \\
0 & 1 & y & * \\
0 & 0 & 1 & 0 \\
0 & 0 & -x & 1
\end{pmatrix} = \psi(-x - y).
\]

2.6.1. Algebraic representations of \( K_2 \). Let \( \ul{\lambda} ; u \in \mathbb{Z}^2 \times \mathbb{Z} \) with \( \ul{\lambda} = (\lambda_1, \lambda_2) \) such that

\[
\lambda_1 \geq \lambda_2, \quad \lambda_1 + \lambda_2 \equiv u \, (\text{mod} \, 2).
\]

Let \( (\rho(\ul{\lambda}; u), V(\ul{\lambda}; u)) \) be the irreducible algebraic representation of \( K_2 \) defined as follows: \( V(\ul{\lambda}; u) \) is the space of homogeneous polynomials over \( \mathbb{C} \) of degree \( \lambda_1 - \lambda_2 \) in variables \( x \) and \( y \). The action is given by

\[
(\rho(\ul{\lambda}; u)(au) \cdot P)(x, y) = a^u(\det u)^{\lambda_2} \cdot P((x, y)u)
\]

for \( a \in \mathbb{R}^\times \), \( u \in U(2) \), and \( P \in V(\ul{\lambda}; u) \). Put \( \ul{\lambda}^\vee = (-\lambda_2, -\lambda_1) \). Note that \( \rho(\ul{\lambda}; u) = \rho(\ul{\lambda}^\vee; -u) \). Let \( V(\ul{\lambda}; u)_\mathbb{Q} \) be the \( \mathbb{Q} \)-structure on \( V(\ul{\lambda}; u) \) consisting of polynomials over \( \mathbb{Q} \). For example, we have an isomorphism

\[
\mathfrak{p}^+_2 \rightarrow V(2, 0, 0)
\]

given by

\[
(2.7) \quad \begin{pmatrix}1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto x^2, \quad \begin{pmatrix}0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto 2xy, \quad \begin{pmatrix}0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto y^2.
\]

Here we identify \( \mathfrak{p}^+_2 \) with \( \text{Sym}_2(\mathbb{C}_2) \) as in \( \text{(2.23)} \). In this way we also fix a \( \mathbb{Q} \)-rational structure \( (\mathfrak{p}^+_2)_\mathbb{Q} \) on \( \mathfrak{p}^+_2 \).

We write \( \rho(\ul{\lambda}) = \rho(\ul{\lambda}; u)_{|U(2)} \) and \( V(\ul{\lambda}) = V(\ul{\lambda}; u) \) when we consider only the action of \( U(2) \). Let \( c_\ul{\lambda} : V_\ul{\lambda} \rightarrow V_\ul{\lambda}^\vee \) be the \( U(2) \)-conjugate-equivalent \( \mathbb{C} \)-linear isomorphism normalized so that

\[
c_\ul{\lambda}(x^{\lambda_1 - \lambda_2}) = y^{\lambda_1 - \lambda_2}.
\]
Let $\langle \cdot, \cdot \rangle_\Delta : V_\Delta \times V_\Delta^* \to \mathbb{C}$ be the U(2)-equivariant bilinear pairing normalized so that

$$\langle x^{\lambda_1 - \lambda_2}, y^{\lambda_1 - \lambda_2} \rangle_\Delta = 1.$$ 

Then it is easy to see that

$$(2.8) \quad c_\Delta(x^{\lambda_1 - \lambda_2-i}y^i) = (-1)^i \cdot x^iy^{\lambda_1 - \lambda_2-i}, \quad \langle x^{\lambda_1 - \lambda_2-i}y^i, x^iy^{\lambda_1 - \lambda_2-i} \rangle_\Delta = (-1)^i \left( \frac{\lambda_1 - \lambda_2}{i} \right)^{-1}$$

for $0 \leq i \leq \lambda_1 - \lambda_2$. Assume further that $\lambda_1 - \lambda_2 \geq 2$. We fix U(2)-equivariant embeddings

$$(2.9) \quad \xi^+_\Delta : V_\Delta \to \lambda^2 P_2 \otimes V_{\Delta-(3,1)}, \quad \xi^-_\Delta : V_\Delta^* \to P_2^+ \otimes V_{\Delta^--(2,0)}.$$ 

The existence of $\xi^+_\Delta$ is a direct consequence of the Clebsch–Gordan formula for U(2) and they are unique up to scalar multiples. We normalize them so that

$$\xi^+_\Delta(x^{\lambda_1 - \lambda_2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes x^{\lambda_1 - \lambda_2-2}, \quad \xi^-_\Delta(x^{\lambda_1 - \lambda_2}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes x^{\lambda_1 - \lambda_2-2}.$$ 

In particular, $\xi^\pm_\Delta$ preserve the $\mathbb{Q}$-rational structures. The following lemma will be used in the proof of Proposition 12.

Lemma 2.7. Assume $\lambda_1 - \lambda_2 \geq 2$. Let $0 \leq i \leq \lambda_1 - \lambda_2 - 2$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes x^{\lambda_1 - \lambda_2-i}y^i, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes x^{\lambda_1 - \lambda_2-2}y^i$$

appear in the decompositions of $\xi^+_\Delta(x^{\lambda_1 - \lambda_2-i}y^{i+1})$ and $\xi^-_\Delta(x^{\lambda_1 - \lambda_2-i}y^i)$ into $(U(1) \times U(1))$-eigenvectors.

Proof. Let $N_- \in \mathfrak{e}_{2,\mathbb{C}}$ defined by

$$N_- = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \sqrt{-1} & 0 \\ 0 & \sqrt{-1} & 0 & 1 \\ -\sqrt{-1} & 0 & -1 & 0 \end{pmatrix}.$$ 

For $0 \leq i \leq \lambda_1 - \lambda_2$, we have

$$N_- \cdot x^{\lambda_1 - \lambda_2-i}y^i = -(\lambda_1 - \lambda_2 - i) \cdot x^{\lambda_1 - \lambda_2-i}y^{i+1}.$$ 

Note that

$$X \cdot (v \otimes w) = X \cdot v \otimes w + v \otimes X \cdot w$$

for all $X \in \mathfrak{e}_{2,\mathbb{C}}$. The assertion follows from induction on $0 \leq i \leq \lambda_1 - \lambda_2 - 2$ based on these relations. We leave the details to the readers. \qed

2.6.2. (Limits of) discrete series representations of GSp$_4(\mathbb{R})$. We regard $U(1) \times U(1)$ as the maximal torus of $U(2)$ consisting of diagonal matrices. Let $\mathfrak{h}$ be the Lie algebra of $\mathbb{R}_+ \times (U(1) \times U(1))$. Then $\mathfrak{h}$ is a Cartan subalgebra of both $\mathfrak{e}_2$ and $\mathfrak{g}_2$. Note that

$$\mathfrak{h} = \mathbb{R} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \cdot \mathbf{1}_4.$$ 

We identify $\mathfrak{h}^\ast$ with $\mathbb{C}^3$ by the map

$$a \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^* \oplus b \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^* \oplus c \cdot \mathbf{1}_4^* \mapsto (\sqrt{-1}a, \sqrt{-1}b; c).$$

Here the script $\ast$ refers to dual basis. Let $\Delta^+$ and $\Delta^+_c$ be the positive systems of $(\mathfrak{g}_{2,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{e}_{2,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, respectively, given by

$$(2.10) \quad \Delta^+ = \{(1,-1;0), (2,0;0), (0,2;0), (1,1;0)\}, \quad \Delta^+_c = \{(1,-1;0)\}.$$
Note that $\Delta^+ \setminus \Delta^+$ corresponds to the root spaces in $\mathfrak{p}^+_u$. Let $(\Delta, u) \in \mathbb{Z}^2 \times \mathbb{Z}$ with $\Delta = (\lambda_1, \lambda_2)$ such that

$$1 - \lambda_1 \leq \lambda_2 \leq 0, \quad \Delta \not\equiv (1, 0), \quad \lambda_1 + \lambda_2 \equiv u \text{ (mod 2)}.$$ 

With respect to the choice of positive systems in (2.10), let $D_{(\Delta, u)}$ and $D_{(\Delta^+, u)}$ be the generic (limit of) discrete series representations of $\text{GSp}_4^+(\mathbb{R})$ with minimal $K_2$-type $\rho_{(\Delta, u)}$ and $\rho_{(\Delta^+, u)}$, respectively, defined as in [Kna06, XII, §7]. When $\lambda_1 + \lambda_2 \geq 2$, let $D_{(\Delta, -\lambda_1 - 2; u)}$ and $D_{(\Delta^+, -\lambda_1; u)}$ be the holomorphic and anti-holomorphic (limit of) discrete series representations of $\text{GSp}_4^+(\mathbb{R})$ with minimal $K_2$-type $\rho_{(\lambda_1, -\lambda_1 + 2; u)}$ and $\rho_{(\lambda_1 - 2, -\lambda_2; u)}$, respectively. We also write

$$(2.11) \quad D_{(\Delta, u)}^{(0)} = D_{(\lambda_1, -\lambda_1 + 2; u)}, \quad D_{(\Delta, u)}^{(1)} = D_{(\Delta^+, u)}, \quad D_{(\Delta, u)}^{(2)} = D_{(\Delta^+, u)}, \quad D_{(\Delta, u)}^{(3)} = D_{(\lambda_1 - 2, -\lambda_2; u)}.$$ 

Let $\Sigma_{(\Delta, u)}^{\text{gen}}$ and $\Sigma_{(\Delta, u)}^{\text{hol}}$ be the generic and holomorphic (limit of) discrete series representations of $\text{GSp}_4(\mathbb{R})$, respectively, defined by

$$\Sigma_{(\Delta, u)}^{\text{gen}} = D_{(\Delta, u)}^{(1)} \oplus D_{(\Delta, u)}^{(2)}, \quad \Sigma_{(\Delta, u)}^{\text{hol}} = D_{(\Delta, u)}^{(0)} \oplus D_{(\Delta, u)}^{(3)},$$

with

$$\text{diag}(-1, -1, 1, 1) \cdot (v_1, v_2) = (v_2, v_1).$$

Then the following set is an $L$-packet of $\text{GSp}_4(\mathbb{R})$:

$$(2.12) \quad L_{(\Delta, u)} = \begin{cases} \{ \Sigma_{(\Delta, u)}^{\text{gen}}, \Sigma_{(\Delta, u)}^{\text{hol}} \} & \text{if } \lambda_1 + \lambda_2 \geq 2, \\ \{ \Sigma_{(\Delta, u)}^{\text{gen}} \} & \text{if } \lambda_1 + \lambda_2 = 1. \end{cases}$$

Conversely, up to similitude twists, any $L$-packet of $\text{GSp}_4(\mathbb{R})$ that contains a (limit of) discrete series representation is obtained in this way. Note that $\Sigma_{(\Delta, u)}^{\text{gen}}$ is a limit of discrete series representation if and only if $\lambda_1 + \lambda_2 = 1$ or $\lambda_2 = 0$. We have the following explicit formula for Whittaker functions on $\text{GSp}_4(\mathbb{R})$ due to Moriyama [Mor04, Proposition 7]. Let $\psi_\mathbb{R}$ be the standard additive character of $\mathbb{R}$ and denote by $\mathcal{W}(\Sigma_{(\Delta, u)}^{\text{gen}}, \psi_\mathbb{R}, \nu)$ the space of Whittaker functions of $\Sigma_{(\Delta, u)}^{\text{gen}}$ with respect to $\psi_\mathbb{R}, \nu$.

**Theorem 2.8** (Moriyama). There exists a unique $K_2$-equivariant homomorphism

$$V_{(\Delta^+, u)} : \mathcal{W}(\Sigma_{(\Delta, u)}^{\text{gen}}, \psi_\mathbb{R}, \nu) \to W^+((\Delta^+, u), i)$$

such that

$$W^+((\Delta^+, u), i)(\text{diag}(a_1, a_2, a_1, a_2^{-1}, 1)) = (2\pi \sqrt{-1})^{-i} e^{-2\pi a_1} \int_{c_1 - \sqrt{-1} \infty}^{c_1 + \sqrt{-1} \infty} ds_1 \int_{c_2 - \sqrt{-1} \infty}^{c_2 + \sqrt{-1} \infty} ds_2 \frac{2^{-s_1 - s_2}}{\Gamma_R(s_1 + \lambda_1 + 1) \Gamma_R(-s_2 - \lambda_2)} \times \frac{\Gamma_R(s_1 + s_2 + \lambda_1 - \lambda_2 + 2) \Gamma_R(s_1 + s_2 + \lambda_1 + \lambda_2 + 2)}{\Gamma_R(s_1 + \lambda_1 + 1)} \times a_1^{(-s_1 - s_2 + u)/2} \left| a_2 \right|^{-s_1 - i}$$

for $a_1, a_2 \in \mathbb{R}^\times$, $a_1 > 0$. Here $c_1, c_2 \in \mathbb{R}$ satisfy

$$c_1 + c_2 + \lambda_1 + \lambda_2 + 2 > 0, \quad c_1 + \lambda_1 + 1 > 0 > c_2 + \lambda_2.$$

In the following lemma, we specialize the result [BHR94, Theorem 3.2.1] to $\text{GSp}_4(\mathbb{R})$ (see also [Har90b, Theorem 4.6.2]).

**Lemma 2.9.** For the (limit of) discrete series representation $D_{(\Delta, u)}^{(i)}$ of $\text{GSp}_4^+(\mathbb{R})$, there exists a unique $q \geq 0$ and irreducible algebraic representation $(\rho, V_\rho)$ of $K_2$ such that

$$H^q(\mathbb{P}_2, K_2; D_{(\Delta, u)}^{(i)} \otimes V_\rho) \neq 0.$$ 

More precisely, $q = i$ and

$$\rho = \begin{cases} \rho_{(\lambda_1 - 2, -\lambda_2; u)} & \text{if } i = 0, \\ \rho_{(\lambda_1 - 2, -\lambda_1; u)} & \text{if } i = 1, \\ \rho_{(\lambda_1, -2; u)} & \text{if } i = 2, \\ \rho_{(\lambda_1 - 3, -\lambda_1 - 1; u)} & \text{if } i = 3. \end{cases}$$
In this case, $H^q(\mathfrak{g}_2, K_2; D^{(i)}_{(\mathfrak{h}_2)} \otimes V_\rho)$ is one-dimensional and

$$H^q(\mathfrak{g}_2, K_2; D^{(i)}_{(\mathfrak{h}_2)} \otimes V_\rho) = \left( D^{(i)}_{(\mathfrak{h}_2)} \otimes \wedge^q \mathfrak{p}_2^+ \otimes V_\rho \right)^{K_2}.$$  

Proof. We have four positive systems of $(\mathfrak{g}_2, \mathfrak{h}_C)$ that contain $\Delta^+_K$, which are given as follows:

$$\begin{align*}
\Delta^+_0 &= \Delta^-_0, \\
\Delta^+_1 &= \{(1, -1; 0), (2, 0; 0), (0, -2; 0), (1, 1; 0)\}, \\
\Delta^+_2 &= \{(1, -1; 0), (2, 0; 0), (0, -2; 0), (-1, -1; 0)\}, \\
\Delta^+_3 &= \{(1, -1; 0), (-2, 0; 0), (0, -2; 0), (-1, -1; 0)\}.
\end{align*}$$

The set $\mathcal{F} \subset \mathfrak{h}_C$ of algebraic differentials is given by

$$\mathcal{F} = \{(a, b; c) \in \mathbb{Z}^3 \mid a + b \equiv c \pmod{2}\}.$$  

Let $\delta = (2, 1; 0)$ be the half-sum of positive roots in $\Delta^+$. For $\Lambda \in \mathcal{F} + \delta$ and $\Delta^+_K$ such that $\Lambda$ is dominant with respect to $\Delta^+_K$ and non-singular with respect to $\Delta^+_K$, we have the (limit of) discrete series representation $D(\Lambda, \Delta^+_K)$ of $\text{GSp}_4^+ (\mathbb{R})$ defined as in [Kus96, XII, §7]. By [BHR94, Theorem 3.2.1], there exist a unique $q \geq 0$ and $(\rho, V_\rho)$ such that the assertions hold for $D(\Lambda, \Delta^+_K)$. Moreover, we have

$$q = i, \quad \rho = \rho_{\Lambda + \delta}.$$

Finally, note that $D^{(i)}_{(\mathfrak{h}_2)} = D(\Lambda, \Delta^+_K)$ with

$$\Lambda = \begin{cases} 
(\lambda_1 - 1, -\lambda_2; u) & \text{if } i = 0, \\
(\lambda_1 - 1, \lambda_2; u) & \text{if } i = 1, \\
(-\lambda_2, -\lambda_1 + 1; u) & \text{if } i = 2, \\
(\lambda_2, -\lambda_1 + 1; u) & \text{if } i = 3.
\end{cases}$$

\[\square\]

2.6.3. **Rational structure via the Whittaker model.** Let $(\Sigma, V_\Sigma)$ be a globally generic $C$-algebraic irreducible cuspidal automorphic representation of $\text{GSp}_4 (K_2)$ with central character $\omega_\Sigma$. Here we follow [BG14, Definition 5.11] for the notion of $C$-algebraicity. We assume further that the following condition is satisfied:

$$(2.13) \quad \Sigma_\nu \text{ is a (limit of) discrete series representation for all } \nu \in S_{\infty}. $$

Then there exists $(\underline{\lambda}; \underline{u}) \in (\mathbb{Z}^2)^{S_{\infty}} \times \mathbb{Z}$ with $\underline{\lambda} = (\underline{\lambda}_\nu)_{\nu \in S_{\infty}}$ and $\underline{u} = (\lambda_{1, \nu}, \lambda_{2, \nu})$ satisfying the following conditions:

- $|\underline{\lambda}_{\Sigma}| = |\underline{u}_{\Sigma}|$,
- $1 - \lambda_{1, \nu} \leq \lambda_{2, \nu} \leq 0$, $\underline{\lambda}_\nu \neq (1, 0)$, and $\lambda_{1, \nu} + \lambda_{2, \nu} \equiv u \pmod{2}$ for all $\nu \in S_{\infty}$,
- $\Sigma_\nu = \Sigma_{\text{gen}} \Sigma_{\nu}$ for all $\nu \in S_{\infty}$.

We say $\Sigma$ is regular if $2 - \lambda_{1, \nu} \leq \lambda_{2, \nu} \leq -1$ for all $\nu \in S_{\infty}$. We call $(\underline{\lambda}; \underline{u})$ the weight of $\Sigma$. Let $(\rho_{(\underline{\lambda}; -u)}, V_{(\underline{\lambda}; -u)})$ be the irreducible motivic algebraic representation of $K_2^{S_{\infty}}$ defined by

$$(\rho_{(\underline{\lambda}; -u)}, V_{(\underline{\lambda}; -u)}) = \left( \bigotimes_{\nu \in S_{\infty}} \rho_{(\underline{\lambda}_\nu; -u)}, \bigotimes_{\nu \in S_{\infty}} V_{(\underline{\lambda}_\nu; -u)} \right).$$

Let $\langle \cdot, \cdot \rangle_\Sigma = \prod_{\nu \in S_{\infty}} \langle \cdot, \cdot \rangle_{\underline{\lambda}_\nu}$ be the $U(2)^{S_{\infty}}$-equidistant paring on $V_{\underline{\lambda}} \times V_{\underline{\lambda}}$. Let $\psi$ be a non-trivial additive character of $\mathbb{F}_q \setminus K_2$. Let $\Sigma_f = \bigotimes_{\nu \in S_{\infty}} \Sigma_\nu$ be the finite part of $\Sigma$ and $\mathcal{W}(\Sigma_f, \psi_U, f)$ the space of Whittaker functions of $\Sigma_f$ with respect to $\psi_U, f$. For $\varphi \in V_2$, let $W_{\varphi, \psi_U}$ be the Whittaker function of $\varphi$ with respect to $\psi_U$ defined by

$$W_{\varphi, \psi_U}(g) = \int_{U(\mathbb{F}_q) \setminus U(K_2)} \varphi(ug) \overline{\psi_U(u)} \ du_{\text{Tam}}.$$  

Here $du_{\text{Tam}}$ is the Tamagawa measure on $U(K_2)$. Let $V_2^{S_{\infty}}$ be the space of vector-valued cusp forms in $V_2$ of weight $(\underline{\lambda}; \underline{u})$, that is, $V_2^{S_{\infty}}$ consisting of functions $\varphi : \text{GSp}_4 (K_2) \rightarrow V_{(\underline{\lambda}; -u)}$ satisfying the following conditions:

- $\varphi(kg) = \rho_{(\underline{\lambda}; -u)}(k)^{-1} \varphi(g)$ for all $k \in K_2^{S_{\infty}}$ and $g \in \text{GSp}_4 (K_2)$,
- The function $g \mapsto \langle \varphi(g), v \rangle_{\underline{\lambda}}$ is a cuspid form in $V_2$ for all $v \in V_{(\underline{\lambda}; -u)}$. 

\[\text{14}\]
For $\varphi \in V_\Sigma^+$ and $\hat{\imath} = (i_v)_{v \in S_x}$ with $0 \leq i_v \leq \lambda_{1,v} - \lambda_{2,v}$, let $\text{pr}_{\hat{\imath}}(\varphi) \in V_\Sigma$ be the $\hat{\imath}$-th component of $\varphi$ defined by

$$
(2.14) \quad \text{pr}_{\hat{\imath}}(\varphi)(g) = \left( \varphi(g), \bigotimes_{v \in S_x} x^{i_v} y^{\lambda_{1,v} - \lambda_{2,v} - i_v} \right).
$$

Under the diagonal embedding $U(1) \times U(1) \to U(2)$, the weight of $\text{pr}_{\hat{\imath}}(\varphi)$ at $v \in S_x$ is equal to

$$
(-\lambda_{1,v} + i_v, -\lambda_{2,v} - i_v).
$$

It is clear that we have a $\text{GSp}_4(\mathbb{A}_F, f)$-equivariant isomorphism $V_\Sigma^+ \to (V_\Sigma \otimes V_{(\Delta; -u)})K_{2}^{S_x}$ given by

$$
(2.15) \quad \varphi \mapsto \sum_{\hat{\imath}} \prod_{v \in S_x} (-1)^{i_v} \left( \lambda_{1,v} - \lambda_{2,v} \right) \cdot \text{pr}_{\hat{\imath}}(\varphi) \otimes \bigotimes_{v \in S_x} x^{\lambda_{1,v} - \lambda_{2,v} - i_v} y^{i_v}.
$$

Take $\psi = \psi_\Sigma$ to be the standard additive character. For $\varphi \in V_\Sigma^+$, let $W_{\psi}(\varphi) \in \mathcal{W}(\Sigma_f, \psi_U, f)$ be the unique Whittaker function so that

$$
W_{\text{pr}_{\hat{\imath}}(\varphi), \psi_U} = \prod_{v \in S_x} W_{(\Delta; -u), i_v} \cdot W_{\psi}(\varphi)
$$

for all $\hat{\imath} = (i_v)_{v \in S_x}$ with $0 \leq i_v \leq \lambda_{1,v} - \lambda_{2,v}$. Then the map $\varphi \mapsto W_{\psi}(\varphi)$ defines a $\text{GSp}_4(\mathbb{A}_F, f)$-equivariant isomorphism from $V_\Sigma^+$ to $\mathcal{W}(\Sigma_f, \psi_U, f)$. For $\sigma \in \text{Aut}(\mathbb{C})$, let

$$
\sigma \Sigma = \sigma \Sigma_f \otimes \sigma \psi_f
$$

be the irreducible admissible representation of $\text{GSp}_4(\mathbb{A}_F)$ defined so that the $\nu$-component of $\sigma \Sigma_f$ is isomorphic to $\Sigma_{\sigma^{-1} \nu}$ for each $\nu \in S_x$. In Proposition 2.10 below, we will show that $\sigma \Sigma$ is cuspidal automorphic and globally generic. It is clear that $\sigma \Sigma$ is $C$-algebraic of weight $(\Delta, u)$, where $\Delta = (\Delta_{\sigma^{-1} \nu})_{\nu \in S_x}$. Let $\mathbb{Q}(\Delta)$ and $\mathbb{Q}(\Sigma)$ be the fixed fields of $\{ \sigma \in \text{Aut}(\mathbb{C}) \mid \Delta = \Delta \}$ and $\{ \sigma \in \text{Aut}(\mathbb{C}) \mid \sigma \Sigma_f = \Sigma_f \}$, respectively. Note that $\mathbb{Q}(\Delta) \subset \mathbb{Q}(\Sigma)$ by the strong multiplicity one theorem for $\text{GL}_4$. Let

$$
t_{\sigma} : \mathcal{W}(\Sigma_f, \psi_U, f) \to \mathcal{W}(\sigma \Sigma_f, \psi_U, f)
$$

be the $\sigma$-linear $\text{GSp}_4(\mathbb{A}_F, f)$-equivariant isomorphism defined by

$$
t_{\sigma} W(g) = \sigma \left( W(\text{diag}(u_\sigma^{-3}, u_\sigma^{-2}, 1, u_\sigma^{-1}) g) \right).
$$

Here $u_\sigma \in \hat{\mathbb{Z}}^\times \subset \hat{\mathbb{A}}_F^\times$ is the unique element such that $\sigma(\psi(x)) = \psi(u_\sigma x)$ for all $x \in \mathbb{A}_F$. Let

$$
V_\Sigma^+ \to V_\Sigma^+, \quad \varphi \mapsto \sigma \varphi
$$

be the $\sigma$-linear $\text{GSp}_4(\mathbb{A}_F, f)$-equivariant isomorphism defined so that

$$
W_{\psi}(\varphi) = t_{\sigma} W_{\psi}(\sigma \varphi).
$$

We thus obtain a $\mathbb{Q}(\Sigma)$-rational structure $(V_\Sigma^+)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Sigma))}$ on $V_\Sigma^+$ given by taking the Galois invariants:

$$
(2.16) \quad (V_\Sigma^+)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Sigma))} = \{ \varphi \in V_\Sigma^+ \mid \sigma \varphi = \varphi \text{ for } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Sigma)) \}.
$$

2.6.4. Rational structure via the coherent cohomology. In this section, we define the rational structures on $V_\Sigma^+$ given by the coherent cohomology of automorphic vector bundles on $\text{Sh}(G_2, X_2)$. For $I \subset S_x$, define $\Delta(I) = (\Delta(I))_{v \in S_x}$ and $\Delta^I = (\Delta^I_v)_{v \in S_x}$ in $(\mathbb{Z}^{2})^{S_x}$ by

$$
\Delta_v(I) = \begin{cases} 
\Delta_v - (3, 1) & \text{if } v \in I, \\
\Delta_v - (2, 0) & \text{if } v \notin I.
\end{cases} \quad \Delta_v^I = \begin{cases} 
\Delta_v & \text{if } v \in I, \\
\Delta_v & \text{if } v \notin I.
\end{cases}
$$

By definition, we have $\sigma(\Delta(I)) = \sigma \Delta(\sigma I)$ for $\sigma \in \text{Aut}(\mathbb{C})$. It is clear that $\Delta(I)$ is invariant by $\text{Aut}(\mathbb{C}/\mathbb{Q}(\Delta) \text{Q}(I))$. We say $I$ is admissible with respect to $\Delta$ if the following condition is satisfied: Let $0 \leq j_v \leq 3$ for $v \in S_x$. We have

$$
(2.17) \quad \mathbb{H}^{d+1} \left( \mathbb{P}^{S_x}, K_2^{S_x}, \mathbb{Q}^{S_x}; \bigotimes_{v \in S_x} D(\Delta_v; -u) \otimes V(\Delta(I); -u) \right) \neq 0
$$

15
if and only if
\[ j_v = \begin{cases} 
2 & \text{if } v \in I, \\
1 & \text{if } v \notin I.
\end{cases} \]

By Lemma 2.30, the admissibility is a combinatorial condition on \( I \). For instance, \( I \) is admissible when the following condition is satisfied:

- For \( v \in I \) (resp. \( v \notin I \)), we have \( \lambda_{1,v} + \lambda_{2,v} \geq 2 \) (resp. \( \lambda_{2,v} \leq -1 \)).

In particular, any subset of \( S_\varepsilon \) is admissible if \( \Sigma_\varepsilon \) is a discrete series representation for all \( v \in S_\varepsilon \). Also note that \( I \) is admissible if and only if \( S_\varepsilon \setminus I \) is admissible. Let
\[
[-\nu]\}
\]
denote the automorphic vector bundle on \( \text{Sh}(G_2, X_2) \) defined by the motivic algebraic representation
\[
(r(\Delta(\varepsilon); -\nu), V(\Delta(\varepsilon); -\nu)) = \bigotimes_{v \in S_\varepsilon} r(\lambda_v(\varepsilon); -\nu), \bigotimes_{v \in S_\varepsilon} V(\lambda_v(\varepsilon); -\nu)
\]
of \( K_2^{S_\varepsilon} \). Let \( H_f^i([-\nu]|\Sigma_f]) \) be the \( \Sigma_f \)-isotypic component of \( H_f^i([-\nu]|\Sigma_f]) \) for \( \varepsilon \in \{\text{cusp}, 1, 2\} \). In the following proposition, we show that being globally generic is an arithmetic property of an \( C \)-algebraic irreducible cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_F) \) satisfying condition 2.13. The result was proved in [Che21b, Lemma 3.1] when \( \Sigma_v \) is a discrete series representation for all \( v \in S_\varepsilon \).

**Proposition 2.10.** For \( \sigma \in \text{Aut}(\mathbb{C}) \), the representation \( \sigma \Sigma \) is cuspidal automorphic and globally generic.

**Proof.** By Theorem 2.2 (2) and Lemma 2.9, we have \( H_{2,cusp}^d([-\nu]|\Sigma_f]) \neq 0 \). By Theorem 2.2 (3), this implies that \( H_f^{2d}([-\nu]|\Sigma_f]) \neq 0 \). Let \( \sigma \in \text{Aut}(\mathbb{C}) \). It follows from 2.3 that \( H_f^{2d}([-\nu]|\Sigma_f]) \neq 0 \).

By Theorem 2.2 (2) and (4), there exists an irreducible discrete automorphic representation \( \Sigma' \) of \( \text{GSp}_4(\mathbb{A}_F) \) such that \( \Sigma' = \sigma \Sigma_f \) and
\[
H_f^{2d}([-\nu]|\Sigma_f]) \neq 0.
\]

In particular, by [Har90b], Proposition 4.3.2], the infinitesimal characters of \( \Sigma_x' \) and \( \sigma \Sigma_x' \) are equal. On the other hand, consider the transfer \( \Psi' \) of \( \Sigma' \) to \( \text{GL}_4(\mathbb{A}_F) \) with respect to the spin representation of \( \text{GSp}_4(\mathbb{Q}) \). Since \( \Sigma' \) is almost locally generic, it follows that \( \Psi' \) is an isobaric automorphic representation of \( \text{GL}_4(\mathbb{A}_F) \). More precisely, the global Arthur parameter associated to \( \Sigma' \) is of types (a) or (b) in the notation of [GT19, Remark 6.1.8]. The condition on infinitesimal character of \( \Sigma_x' \) then implies that \( \Psi' \) is \( C \)-algebraic. By the purity lemma [Cao10, Lemma 4.9], we deduce that \( \Psi'_x \) is essentially tempered. It then follows that \( \Sigma_x' \) is also essentially tempered. Combine with (2.18), by [Har90b, Theorem 3.5], \( \Sigma'_x \) is a (limit of) discrete series representation of \( \text{GSp}_4(\mathbb{A}_F) \). By Lemma 2.9 and (2.18) again, we conclude that
\[
\Sigma'_x \in L(\varepsilon; u)
\]
for all \( v \in S_\varepsilon \). Here \( L(\varepsilon; u) \) is the \( L \)-packet of \( \text{GSp}_4(\mathbb{R}) \) defined in (2.12). Therefore, \( \Sigma' \) and \( \sigma \Sigma \) belong to the same global Arthur packet. We then deduce that \( \sigma \Sigma \) is discrete automorphic by Arthur’s multiplicity formula for \( \text{GSp}_4(\mathbb{A}_F) \) established by Gee and Taibi [GT19, Theorem 7.4.1]. Since \( \sigma \Sigma_x \) is essentially tempered, this implies that \( \sigma \Sigma \) is cuspidal automorphic by [Wal84]. Finally, as explained in [GT19, Remark 7.4.7], among the global Arthur packet associated to \( \Sigma' \), there is a unique globally generic discrete automorphic representation, which is characterized by the condition that it is locally generic at all places. Hence \( \sigma \Sigma \) must be globally generic. This completes the proof.

Let \( I \subset S_\varepsilon \). In the following lemma, we define a \( \text{GSp}_4(\mathbb{A}_{F, I}) \)-equivariant embedding from \( V_\varepsilon^+ \) to the isotypic component \( H_{cusp}^{d+\varepsilon}([-\nu]|\Sigma_f]) \). Let
\[
\xi_f^I : V(\Delta(\varepsilon); -\nu) \longrightarrow \wedge^{d+\varepsilon} (\mathbb{P}_2^+) \otimes V(\Delta(\varepsilon); -\nu)
\]
be the \( K_2^{S_\varepsilon} \)-equivariant homomorphism defined by
\[
\xi_f^I = \bigotimes_{v \in I} \xi_{\varepsilon}^v \otimes \bigotimes_{v \notin I} \xi_{\varepsilon}^v.
\]
Here \( \xi_{\underline{\lambda}} \) are defined in \((2.14)\).  

**Lemma 2.11.** Let \( I \subset S_\infty \). We have a \( \text{GSp}_4(A_{F,I}) \)-equivariant embedding  
\[
V_\Sigma^+ \rightarrow H_{\text{cusp}}^{d+1}([V_{I(\Delta)}; -u])|\Sigma_f], \quad \varphi \mapsto [\varphi]_I
\]
defined by  
\[
[\varphi]_I = \sum_{v \in S_\infty} \left( \left( \lambda_{1,v} - \lambda_{2,v}, -i_v \right) \cdot \text{pr}_{\underline{\lambda}}(\varphi)^I \otimes \xi_{\underline{\lambda}}^I \right) \left( \bigotimes_{v \in I} \left( -1 \right)^{i_v} x^{\lambda_{1,v} - \lambda_{2,v} - i_v} y^{i_v} \right) \bigotimes \left( \bigotimes_{v \notin I} x^{i_v} y^{\lambda_{1,v} - \lambda_{2,v} - i_v} \right).
\]

Here \( \underline{i} = (i_v)_{v \in S_\infty} \in \mathbb{Z}^{S_\infty} \) with \( 0 \leq i_v \leq \lambda_{1,v} - \lambda_{2,v} \) and  
\[
\text{pr}_{\underline{\lambda}}(\varphi)^I(g) = \text{pr}_{\underline{\lambda}}(\varphi) \left( \frac{g \cdot \prod_{v \in S_\infty \setminus I} \text{diag}(-1,-1,1)}{g} \right).
\]

**Proof.** By Theorem 2.2 (2), we have  
\[
\left( V_\Sigma \otimes A^{d+1}([p_2]^+_{S_\infty}) \otimes V_{I(\Delta)}; -u \right)_{K_{2,S_\infty}} \subset H_{\text{cusp}}^{d+1}([V_{I(\Delta)}; -u])|\Sigma_f].
\]
We also have a \( \text{GSp}_4(A_{F,I}) \)-equivariant embedding  
\[
\text{id} \otimes \xi_{\underline{\lambda}}^I : (V_\Sigma \otimes V_{I(\Delta)}; -u)_{K_{2,S_\infty}} \rightarrow \left( V_\Sigma \otimes A^{d+1}([p_2]^+_{S_\infty}) \otimes V_{I(\Delta)}; -u \right)_{K_{2,S_\infty}}.
\]
By considering the minimal \( K_{2,S_\infty} \)-types of \( \Sigma_f |_{\text{GSp}_4(A_{F,I})} \), we easily see that \( (V_\Sigma \otimes V_{I(\Delta)}; -u)_{K_{2,S_\infty}} \) is one-dimensional. By \((2.15)\), we have a \( \text{GSp}_4(A_{F,I}) \)-equivariant isomorphism \( V_\Sigma^+ \rightarrow (V_\Sigma \otimes V_{I(\Delta)}; -u)^{K_{2,S_\infty}} \) given by  
\[
\varphi \mapsto \sum_{v \in S_\infty} \left( \left( \lambda_{1,v} - \lambda_{2,v}, -i_v \right) \cdot \text{pr}_{\underline{\lambda}}(\varphi)^I \otimes \left( \bigotimes_{v \in I} \left( -1 \right)^{i_v} x^{\lambda_{1,v} - \lambda_{2,v} - i_v} y^{i_v} \right) \bigotimes \left( \bigotimes_{v \notin I} x^{i_v} y^{\lambda_{1,v} - \lambda_{2,v} - i_v} \right) \right).
\]
Composing with \( \text{id} \otimes \xi_{\underline{\lambda}}^I \), we then obtain the homomorphism \( \varphi \mapsto [\varphi]_I \). This completes the proof. \( \square \)

**Lemma 2.12.** Let \( I \subset S_\infty \).  

1. For all \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( q \geq 0 \), we have  
\[
T_{\sigma}(H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f]) = H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f].
\]
In particular, we have a \( \mathbb{Q}(\Sigma)\mathbb{Q}(I) \)-rational structure on \( H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f] \) given by taking the Galois invariants:  
\[
H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Sigma)\mathbb{Q}(I))} = \{ c \in H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f] \mid T_{c,c} = c \text{ for } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Sigma)\mathbb{Q}(I)) \}.
\]

2. Assume \( I \) is admissible with respect to \( \underline{\lambda} \) then \( H_{\text{cusp}}^{d+1}([V_{I(\Delta)}; -u])|\Sigma_f] \cong \Sigma_f \).

**Proof.** By \((2.3)\) and Proposition \((2.10)\) to prove the first assertion, it suffices to show that  
\[
H^q_{\text{cusp}}([V_{I(\Delta)}; -u])|\Sigma_f] = H^q_{I}([V_{I(\Delta)}; -u])|\Sigma_f].
\]
A discrete irreducible cuspidal automorphic representation \( \Sigma' \) appears in \( H^q_{I}([V_{I(\Delta)}; -u])|\Sigma_f] \) only when \( \Sigma' = \Sigma_f \). In particular, \( \Sigma' \) and \( \Sigma \) belongs to the same global Arthur packet. Since \( \Sigma_f \) is essentially tempered and generic for all \( v \in S_\infty \), the local Arthur packets at archimedean places are actually the local \( L \)-packets \( L_{\Delta, u} \) in \((2.12)\). Hence \( \Sigma' \) is also essentially tempered for all \( v \in S_\infty \). This implies that \( \Sigma' \) is cuspidal by \((2.3)\). We conclude that the multiplicities of \( \Sigma_f \) in \( H^q_{I}([V_{I(\Delta)}; -u]) \) and \( H^q_{\text{cusp}}([V_{I(\Delta)}; -u]) \) are equal. Equality \((2.19)\) then follows from Theorem \((2.2)\) (3) and (4). Also we deduce that the multiplicity of \( \Sigma_f \) in \( H_{\text{cusp}}^{d+1}([V_{I(\Delta)}; -u]) \) is less than or equal to the number of tuples \( \underline{j} = (j_v)_{v \in S_\infty} \) with \( 0 \leq j_v \leq 3 \) and such that \((2.17)\) hold. By the definition of admissibility, this number is equal to one. Therefore, \( \Sigma_f \) appears in
$H_{\text{cusp}}^{d+\varepsilon f}([V_{(\Delta l); -u}])$ with multiplicity one by Lemma 2.11. In fact, the archimedean component of \( \text{pr}_I(\varphi)^I \) belongs to

\[
\left( \bigotimes_{v \in I} D_{\Delta_v; u}^{(2)} \right) \otimes \left( \bigotimes_{v \not\in I} D_{\Delta_v; u}^{(1)} \right).
\]

for all $\varphi \in V_{\Sigma}^+$. Let $I = (i_v)_{v \in S_e} \in \mathbb{Z}^{S_e}$ with $0 \leq i_v \leq \lambda_{1,v} - \lambda_{2,v}$. This completes the proof.

2.6.5. Autormorphic periods and period relations. By comparing the rational structures in (2.14) and Lemma 2.13 (1), we have the following lemma/definition for the automorphic periods of $\Sigma$.

Lemma 2.13. Let $I \subset S_e$ be admissible with respect to $\Lambda$. There exists a sequence of non-zero complex numbers $\left( p^I(\sigma \Sigma) \right)_{\sigma \in \text{Aut}(\mathbb{C})}$ such that

\[
T_{\sigma} \left( [\varphi]^I_{p^I(\Sigma)} \right) = [\varphi]^I_{p^I(\sigma \Sigma)}
\]

for all $\sigma \in \text{Aut}(\mathbb{C})$ and $\varphi \in V_{\Sigma}^+$. Here $T_{\sigma} : H_{\text{cusp}}^{d+\varepsilon f}([V_{(\Delta l); -u}]) \to H_{1}^{d+\varepsilon f}([V_{(\Delta l); -u}])$ is the $\sigma$-linear isomorphism in (2.15).

In the following theorem, we prove a period relation for product of automorphic periods and critical value of adjoint $L$-function. The result is an analogue of Lemma 2.14 (2). Let $L(s, \Sigma, \text{Ad})$ be the adjoint $L$-function of $\Sigma$, where $\text{Ad}$ is the adjoint representation of $\text{GSp}_4(\mathbb{C})$ on $\text{pgsp}_4(\mathbb{C})$.

Theorem 2.14. Let $I \subset S_e$ be admissible with respect to $\Lambda$. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

\[
\sigma \left( \frac{L(1, \Sigma, \text{Ad})}{\pi^{3d} \cdot p^I(\Sigma) \cdot p^{S_e \setminus I}(\Sigma^\vee)} \right) = \frac{L(1, \sigma \Sigma, \text{Ad})}{\pi^{3d} \cdot p^I(\sigma \Sigma) \cdot p^{S_e \setminus I}(\sigma \Sigma^\vee)}.
\]

Proof. The key ingredients of the proof are the Serre duality for coherent cohomology and our previous result [Che21b]. Let $J = S_e \setminus I$. Note that $\Sigma^\vee$ is $C$-algebraic of weight $(\Delta; -u)$ and $J$ is also admissible with respect to $\Lambda$. We have $K_{2}^{S_e}$-equivariant pairings

\[
\lambda^{d+f}(p^+_2)^{S_e} \times \lambda^{d+f}(p^+_2)^{S_e} \to \lambda^{d}(p^+_2)^{S_e}, (X, Y) \mapsto X \wedge Y
\]

and

\[
V_{(\Delta l); -u} \times V_{(\Delta l); u} \to \bigotimes_{v \in S_e} V_{(-3, -3, 0)}, (v, w) \mapsto \langle v, w \rangle_{(\Delta l)}.
\]

Identify $\lambda^d(p^+_2)^{S_e}$ with $\bigotimes_{v \in S_e} V_{(3, 3, 0)}$ by the isomorphism in (2.17). We then have a $K_{2}^{S_e}$-equivariant pairing

\[
\left( \lambda^{d+f}(p^+_2)^{S_e} \otimes V_{(\Delta l); -u} \right) \times \left( \lambda^{d+f}(p^+_2)^{S_e} \otimes V_{(\Delta l); u} \right) \to \mathbb{C}.
\]

When restrict to $V_{(\Delta l); -u} \times V_{(\Delta l); u}$ via the embedding $\xi^l_{\Delta} \times \xi^l_{\Delta}$, the above pairing is equal to $C \cdot \langle \cdot, \cdot \rangle_{(\Delta l)}$ for some $C \in \mathbb{Q}^\times$. We also have the Petersson bilinear pairing

\[
V_{\Sigma'} \times V_{\Sigma''} \to \mathbb{C}, (\varphi_1, \varphi_2) \mapsto \int_{A_\mathbb{R}^+ \text{GSp}_4(F) \setminus \text{GSp}_4(\mathbb{A})} \varphi_1(g) \varphi_2(g) \, dg_{\text{Tam}}.
\]

Here $dg_{\text{Tam}}$ is the Tamagawa measure on $A_\mathbb{R}^+ \setminus \text{GSp}_4(\mathbb{A})$. We thus obtain a bilinear pairing

\[
H_{\text{cusp}}^{d+\varepsilon f}([V_{(\Delta l); -u}]) \times H_{\text{cusp}}^{d+\varepsilon f}([V_{(\Delta l); u}]) \to \mathbb{C}, (c_1, c_2) \mapsto \int_{\text{Sh}(G_2 \times X_2)} c_1 \wedge c_2.
\]

The pairing satisfies the Galois equivariant property:

\[
\sigma \left( \int_{\text{Sh}(G_2 \times X_2)} c_1 \wedge c_2 \right) = \int_{\text{Sh}(G_2 \times X_2)} T_{\sigma} c_1 \wedge T_{\sigma} c_2
\]

for all $\sigma \in \text{Aut}(\mathbb{C})$. Indeed, the pairing is the restriction of the Serre duality pairing

\[
H_{1}^{d+\varepsilon f}([V_{(\Delta l); -u}]) \times H_{1}^{d+\varepsilon f}([V_{(\Delta l); u}]) \to \mathbb{C}
\]
in [Har90b] Proposition 3.8 and Remark 3.8.4. For \( \varphi \in V_+^{\lambda'} \), let \( \varphi' : \mathrm{GSp}_4(\mathbb{A}_F) \to V_{\lambda'; -u} \) be the vector-valued cusp form defined by

\[
\varphi'(g) = c_\lambda \circ \varphi \left( g \cdot \prod_{\nu \in S_F \setminus \mathcal{I}} \mathrm{diag}(-1, -1, 1, 1) \right).
\]

Here \( c_\lambda : V_\lambda \to V_{\lambda'} \) is the homomorphism given by

\[
c_\lambda = \left( \prod_{\nu \in \mathcal{I}} \mathrm{id} \right) \otimes \left( \prod_{\nu \notin \mathcal{I}} c_\nu \right).
\]

For \( \mathcal{I} = (i_\nu)_{\nu \in S_F} \) with \( 0 \leq i_\nu \leq \lambda_1, -\lambda_2, v \), let \( \mathrm{pr}_\mathcal{I}(\varphi') \in V_{\Sigma} \) be defined as in (2.14) with \( \lambda \) replaced by \( \lambda' \).

By (2.9), we have

\[
\text{pr}_\mathcal{I}(\varphi') = \prod_{\nu \notin \mathcal{I}} (-1)^{i_\nu} \cdot \text{pr}_\mathcal{I}(\varphi')^\lambda.
\]

Here \( \mathcal{I} = (i_\nu)_{\nu \in \mathcal{I}} \times (\lambda_1, -\lambda_2, v - i_\nu)_{\nu \notin \mathcal{I}} \). In particular, we have

\[
[\varphi]_{\lambda} = \prod_{\nu \in \mathcal{I}} (-1)^{i_\nu} \left( \lambda_1, -\lambda_2, v \right) \cdot \text{pr}_\mathcal{I}(\varphi') \otimes \xi_\lambda \left( \prod_{\nu \in S_F} x^{\lambda_1, v - \lambda_2, v - i_\nu} y^{i_\nu} \right).
\]

Similarly we define \( \varphi' \) for \( \varphi \in V_+^{\lambda'} \). Then it is clear that

\[
\int_{\mathrm{Sh}(G_2, X_2)} [\varphi_1]_{\lambda} \wedge [\varphi_2] = C \cdot \int_{\mathbb{A}_F^*} \mathrm{GSp}_4(\mathbb{F}) \mathrm{GSp}_4(\mathbb{A}_F) \langle \varphi_1'(g), \varphi_2'(g) \rangle_{\lambda'} dg^\Lambda \tag{8.3}
\]

for \( \varphi_1 \in V_+^{\lambda'} \) and \( \varphi_2 \in V_+^{\lambda'} \). By Schur’s orthogonal relation, we have

\[
\int_{\mathbb{A}_F^*} \mathrm{GSp}_4(\mathbb{F}) \mathrm{GSp}_4(\mathbb{A}_F) \langle \varphi_1'(g), \varphi_2'(g) \rangle_{\lambda'} dg^\Lambda = \dim V_{\lambda'} \cdot \langle \mathbf{v}, \mathbf{w} \rangle_{\Lambda'} \cdot \int_{\mathbb{A}_F^*} \mathrm{GSp}_4(\mathbb{F}) \mathrm{GSp}_4(\mathbb{A}_F) \langle \varphi_1'(g), \varphi_2'(g) \rangle_{\lambda'} dg^\Lambda
\]

for any \( \mathbf{v} \in V_{\lambda'; -u} \) and \( \mathbf{w} \in V_{\lambda'; -u} \) such that \( \langle \mathbf{v}, \mathbf{w} \rangle_{\Lambda'} \neq 0 \). We take

\[
\mathbf{v} = \left( \prod_{\nu \in \mathcal{I}} x^{\lambda_1, v - \lambda_2, v} \right) \otimes \left( \prod_{\nu \notin \mathcal{I}} x^{\lambda_1, v - \lambda_2, v} \right), \quad \mathbf{w} = \left( \prod_{\nu \in \mathcal{I}} x^{\lambda_1, v - \lambda_2, v} \right) \otimes \left( \prod_{\nu \notin \mathcal{I}} x^{\lambda_1, v - \lambda_2, v} \right).
\]

Then we have

\[
\int_{\mathrm{Sh}(G_2, X_2)} [\varphi_1]_{\lambda} \wedge [\varphi_2] = C \cdot \dim V_{\lambda'} \cdot (-1)^u J \cdot \langle \text{pr}_\mathcal{I}(\varphi_1), \text{pr}_\mathcal{I}(\varphi_2) \rangle = C \cdot \dim V_{\lambda'} \cdot (-1)^u J \cdot \langle \text{pr}_\mathcal{I}(\varphi_1), \text{pr}_\mathcal{I}(\varphi_2) \rangle
\]

Similarly, we have

\[
\int_{\mathrm{Sh}(G_2, X_2)} [\varphi_1]^\sigma_{\lambda} \wedge [\varphi_2]^\sigma_{\lambda} = C \cdot \dim V_{\lambda'} \cdot (-1)^u J \cdot \langle \text{pr}_\mathcal{I}(\varphi_1)^\sigma, \text{pr}_\mathcal{I}(\varphi_2)^{S_{\varepsilon}} \rangle
\]

for all \( \sigma \in \mathrm{Aut}(\mathbb{C}) \). We thus conclude from the Galois equivariant property of the Serre duality pairing and the definition of automorphic periods that

\[
\sigma \left( \langle \text{pr}_\mathcal{I}(\varphi_1)^\sigma, \text{pr}_\mathcal{I}(\varphi_2)^{S_{\varepsilon}} \rangle \right) = \langle \text{pr}_\mathcal{I}(\varphi_1)^\sigma, \text{pr}_\mathcal{I}(\varphi_2)^{S_{\varepsilon}} \rangle \cdot p^{-u J} \cdot p^{-u J} \cdot p^{-u J} \cdot \langle \sigma \mathbf{v}, \sigma \mathbf{w} \rangle_{\Lambda'}
\]

for all \( \sigma \in \mathrm{Aut}(\mathbb{C}) \). On the other hand, by our main result [Che21b Theorem 1.1], we have

\[
\sigma \left( \frac{\pi^{3d} \cdot \langle \text{pr}_\mathcal{I}(\varphi_1)^\sigma, \text{pr}_\mathcal{I}(\varphi_2)^{S_{\varepsilon}} \rangle}{L(1, \Pi, \mathrm{Ad})} \right) = \frac{\pi^{3d} \cdot \langle \text{pr}_\mathcal{I}(\varphi_1)^\sigma, \text{pr}_\mathcal{I}(\varphi_2)^{S_{\varepsilon}} \rangle}{L(1, \sigma \Pi, \mathrm{Ad})}
\]

for all \( \sigma \in \mathrm{Aut}(\mathbb{C}) \). This completes the proof. \( \square \)
Remark 2.15. In [Che21b], the result was proved under the assumption that $Σ_v$ is a discrete series representation for all $v ∈ S_Σ$. However, the proof goes without change for limits of discrete series representations, as long as we can extend [Che21b, Lemma 3.1]. This is done in Proposition 2.10.

3. Proof of main result

In this section, we prove our main result Theorem 1.3.

3.1. The Kim–Ramakrishnan–Shahidi lifts. Let $Π$ be a regular $C$-algebraic irreducible cuspidal automorphic representation of $GL_2(𝔸_F)$ with central character $ω$ and satisfying condition (2.5). Let $(θ, w) ∈ Z_{Σ}^{S_Σ} × Z$ be the weight of $Π$. Assume $Π$ is non-CM. Let $Sym^3(Π)$ be the functorial lift of $Π$ to $GL_4(𝔸_F)$ with respect to the symmetric cube representation of $GL_2(ℂ)$. The existence of the lift was proved by Kim and Shahidi [KS02]. Since $Π$ is non-CM, $Sym^3(Π)$ is cuspidal automorphic. We have the factorization of twisted exterior square $L$-function of $Sym^3(Π)$ by $ω^{-3}_H$:

$$L(s, Sym^3(Π), Λ^2 ⊗ ω^{-3}_H) = L(s, Π, Sym^4 ⊗ ω^{-2}_H) · ζ_p(s).$$

Thus $L(s, Sym^3(Π), Λ^2 ⊗ ω^{-3}_H)$ has a pole at $s = 1$, as the twisted symmetric fourth $L$-function is holomorphic and non-zero at $s = 1$ by [BLG11 Corollary 7.1.5]. By the result of Gan and Takeda [GT11 Theorem 12.1], $Sym^3(Π)$ strongly descend to an irreducible globally generic cuspidal automorphic representation $Σ$ of $GSp_4(𝔸_F)$. We call it the Kim–Ramakrishnan–Shahidi lift of $Π$. In this section, we prove our main result Theorem 1.3.

(1) Assume $κ_v ≥ 6$ for all $v ∈ S_Σ$. For $σ ∈ Aut(ℂ)$, we have

$$σ \left( \frac{p^\sigma(Σ)}{|D|^{1/2} · (2π√{-1})^{Σ_v} · Λ_v^{κ_v + 4 - 3ω} · Ξ_v^d · G(ω_H)^3 · ||f_H||^3}} \right) = \frac{p^\sigma(σΣ)}{|D|^{1/2} · (2π√{-1})^{Σ_v} · Λ_v^{κ_v + 4 - 3ω} · Ξ_v^d · G(σω_H)^3 · ||f_H||^3}. $$

(2) Assume $κ_v ≥ 3$ for all $v ∈ S_Σ$. For $σ ∈ Aut(ℂ/ℚ^Gal)$, we have

$$σ \left( \frac{p^{S_Σ}(Σ)}{|D|^{1/2} · (2π√{-1})^{Σ_v} · Λ_v^{κ_v + 4 - 3ω} · Ξ_v^d · G(ω_H)^3 · ||f_H||^3}} \right) = \frac{p^{S_Σ}(σΣ)}{|D|^{1/2} · (2π√{-1})^{Σ_v} · Λ_v^{κ_v + 4 - 3ω} · Ξ_v^d · G(σω_H)^3 · ||f_H||^3}. $$

Here $ℚ^Gal$ is the Galois closure of $ℚ$ in $ℂ$.

(3) Assume $κ_v = κ_w ≥ 3$ for all $v, w ∈ S_Σ$. Then the assertion in (2) holds with $Aut(ℂ/ℚ^Gal)$ replaced by $Aut(ℂ)$.

Proof. The assertions will be proved in §5 below. □
3.2. Proof of Theorem 3.3. First we recall the result due to Liu [Lin21] on the algebraicity of the critical values of the twisted standard $L$-functions for $\mathrm{GSp}_{2n}(\mathbb{A}_F)$.

**Theorem 3.2 (Liu).** Let $\Psi$ be an $C$-algebraic irreducible cuspidal automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A}_F)$ such that $\Psi$ is a holomorphic discrete series representation for each $v \in S_\infty$. There exists a sequence of non-zero complex numbers $(\Omega^{(\alpha)}(\Psi))_{\alpha \in \mathrm{Aut}(\mathbb{C})}$ satisfying the following property: Let $\chi$ be a finite order Hecke character of $\mathbb{A}_F^+$ with parallel signature and $m \in \mathbb{Z}_{\geq 1}$ be a critical point of the twisted standard $L$-function $L(s, \Psi, \text{std} \otimes \chi)$ such that $m \neq 1$ if $F = \mathbb{Q}$ and $\chi^2 = 1$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$L^S(m, \Psi, \text{std} \otimes \chi^\sigma) = \frac{\left|D_E\right|^{(n+1)/2} \cdot (2\pi \sqrt{-1})^{(n+1)dm} \cdot (\sqrt{-1})^d \cdot G(\chi)^{n+1} \cdot \Omega(\Psi)}{\left|D_E\right|^{(n+1)/2} \cdot (2\pi \sqrt{-1})^{(n+1)dm} \cdot (\sqrt{-1})^d \cdot G(\chi^\sigma)^{n+1} \cdot \Omega(\sigma \Psi)}.$$  

Here $v$ is the integer so that $|\omega_v| = |\omega_v|_v$ and $S$ is a sufficiently large set of places containing $S_\infty$.

**Remark 3.3.** The result in [Lin21, Corollary 0.0.2] is stated for $F = \mathbb{Q}$ with $\mathrm{GSp}_{2n}$ and $\mathrm{Aut}(\mathbb{C})$ replaced by $\mathrm{Sp}_{2n}$ and $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\zeta_n))$ for some cyclotomic field $\mathbb{Q}(\zeta_n)$. The generalization to arbitrary $F$ is straightforward. By considering the Hilbert–Siegel modular varieties associated to $\mathrm{GSp}_{2n,F}$ instead of the connected Shimura varieties associated to $\mathrm{Sp}_{2n,F}$, we can prove the full Galois equivariance property over $\mathrm{Aut}(\mathbb{C})$ by following the standard arguments as in [GL16, §3]. The crucial point is that Liu is able to explicitly compute the archimedean doubling local zeta integrals.

**Remark 3.4.** The other results in the literatures on the algebraicity of twisted standard $L$-functions were mainly for scalar-valued Hilbert–Siegel cusp forms (cf. [Har81], [Stu81], [Miz91], [Shi00], [BS00]). For our purpose here, we need to consider vector-valued Hilbert–Siegel cusp forms. We refer to [Koz00] for certain vector-valued cases and [PSS20] for $n = 2$ and $\text{sgn}(\chi) = -1$. We also refer to the recent result [HPSS21] for sufficiently large critical points $m$.

We have weak automorphic descent from $\mathrm{GL}_7(\mathbb{A}_F)$ to $\mathrm{GSp}_6(\mathbb{A}_F)$ with respect to the symmetric sixth representation of $\mathrm{GL}_2(\mathbb{C})$. More precisely, we have following result.

**Proposition 3.5.** Assume $\Pi$ is non-CM and $F \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. There exists an $C$-algebraic irreducible cuspidal automorphic representation $\Psi$ of $\mathrm{GSp}_6(\mathbb{A}_F)$ satisfying the following conditions:

- $\Psi_v$ is a holomorphic discrete series representation for all $v \in S_\infty$.
- $L(s, \Psi_v, \text{std} \otimes \chi_v) = L(s, \Pi_v, \text{Sym}^6 \otimes \omega_{\Pi_v}^3 \chi_v)$ for all $v \in S_\infty$.
- $L^S(s, \Psi, \text{std} \otimes \chi) = L^S(s, \Pi, \text{Sym}^6 \otimes \omega_{\Pi}^3 \chi)$ for some sufficiently large set $S$ of places containing $S_\infty$.

**Proof.** Let $\text{Sym}^6(\Pi)$ be the functorial lift of $\Pi$ to $\mathrm{GL}_7(\mathbb{A}_F)$ with respect to the symmetric sixth representation of $\mathrm{GL}_2(\mathbb{C})$. Then $\text{Sym}^6(\Pi)$ is a regular $C$-algebraic irreducible cuspidal automorphic representation of $\mathrm{GL}_7(\mathbb{A}_F)$. Note that the existence of the lift was proved by Clozel and Thorne [CT17] under the assumption $F \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$. Since $\Pi_v = \Pi \otimes \omega_{\Pi}^{-1}$, it is easy to see that $\text{Sym}^6(\Pi) \otimes \omega_{\Pi}^3$ is self-dual with trivial central character. Moreover, we have the factorization of twisted symmetric square $L$-function of $\text{Sym}^6(\Pi)$:

$$L(s, \text{Sym}^6(\Pi), \text{Sym}^2 \otimes \omega_{\Pi}^3) = L(s, \Pi, \text{Sym}^{12} \otimes \omega_{\Pi}^6) \cdot L(s, \Pi, \text{Sym}^8 \otimes \omega_{\Pi}^4) \cdot L(s, \Pi, \text{Sym}^4 \otimes \omega_{\Pi}^2) \cdot \zeta_5(s).$$

Since the twisted symmetric power $L$-functions are holomorphic and non-zero at $s = 1$ by [BLGG11, Corollary 7.1.5], we see that $L(s, \text{Sym}^6(\Pi), \text{Sym}^2 \otimes \omega_{\Pi}^3)$ has a pole at $s = 1$. By Arthur’s multiplicity formula [Art13, Theorem 1.5.2], the automorphic representation $\text{Sym}^6(\Pi) \otimes \omega_{\Pi}^3$ descend, with respect to the standard representation $\mathrm{SO}_7(\mathbb{C}) \to \mathrm{GL}_7(\mathbb{C})$, weakly to an irreducible cuspidal automorphic representation $\Psi'$ of $\mathrm{Sp}_6(\mathbb{A}_F)$ such that $\Psi'_v$ is a discrete series representation for all $v \in S_\infty$. Moreover, the descent is strong at $v \in S_\infty$, since $\text{Sym}^6(\Pi_v)$ is essentially tempered for $v \in S_\infty$. By the results of Patrikis [Pat19, Corollary 3.1.6 and Proposition 3.1.14], there exists an $C$-algebraic irreducible cuspidal automorphic representation $\Psi$ of $\mathrm{GSp}_6(\mathbb{A}_F)$ satisfying the following conditions:

- For all $v \in S_\infty$, $\Psi_v$ is a holomorphic discrete series representation such that $\Psi_v|_{\mathrm{Sp}_6(\mathbb{A}_F)}$ contains $\Psi'_v$.
- $\Psi'$ is a weak functorial lift of $\Psi$ with respect to the $L$-homomorphism $\mathrm{GSpin}_7(\mathbb{C}) \to \mathrm{SO}_7(\mathbb{C})$.

The automorphic representation $\Psi$ then clearly satisfies the conditions we want. This completes the proof. □
3.2.1. Case \( \mathbb{F} \neq \mathbb{Q} \). We begin with the case \( \mathbb{F} \neq \mathbb{Q} \). We assume \( \Pi \) is non-CM, \( \mathbb{F} \cap \mathbb{Q}(\zeta_p) = \mathbb{Q} \), and \( \kappa_v \geq 6 \) for all \( v \in S_{\mathbb{F}} \). Let \( \chi \) be a finite order Hecke character of \( \mathbb{A}_p^\times \) and \( \sigma \in \text{Crit}(\Pi, \text{Sym}^6 \otimes \chi) \) a right-half critical point. For any finite place \( v \), by the Ramanujan conjecture for \( \Pi \) proved by Blasius [Bla06], we have \( L(m, \Pi_v, \text{Sym}^6 \otimes \chi_v) \neq 0 \). Moreover, it is easy to show that (cf. [Rag10 Proposition 3.17])

\[
\sigma(L(s, \Pi_v, \text{Sym}^6 \otimes \chi_v)) = L(s, \sigma \Pi_v, \text{Sym}^6 \otimes \sigma \chi_v)
\]
as rational functions in \( q_v^{-s} \). Therefore, Conjecture \ref{conjecture:adelic} holds for \( L^{(x)}(m, \Pi, \text{Sym}^6 \otimes \chi) \) if and only if it holds for \( L^S(m, \Pi, \text{Sym}^6 \otimes \chi) \) for any finite set of places \( S \) containing \( S_{\mathbb{F}} \). Note that \( L^S(m, \Pi, \text{Sym}^6 \otimes \chi) \) is non-zero by [BLGG11 Corollary 7.1.5]. By Theorem \ref{thm:adelic} and Proposition \ref{prop:adelic}, we conclude that Conjecture \ref{conjecture:adelic} holds for all \( \chi \) and right-half critical \( m \) if and only if it holds for some \( \chi \) and some right-half critical \( m \). Now we verify the conjecture for

\[
\chi = \left| \frac{3w}{3w} \right| \hat{\omega}_{\Pi}^{-3}, \quad m = 1 - 3w.
\]

Consider the Kim–Ramakrishnan–Shahidi lift \( \Sigma \) of \( \Pi \). By the period relations in Theorem \ref{thm:period_relations}, we have

\[
\sigma \left( \frac{p^3(\Sigma) \cdot p_{\mathbb{Q}}(\Sigma^\vee)}{(2\pi)^{4d} \cdot \|f_H\|^2} \right) = \frac{p^3(\sigma \Sigma) \cdot p_{\mathbb{Q}}(\sigma \Sigma^\vee)}{(2\pi)^{4d} \cdot \|f_{\mathbb{Q}}\|^2}
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{F}_{\text{Gal}}) \). Comparing with Theorem \ref{thm:adelic}, we conclude that

\[
\sigma \left( \frac{L(1, \Sigma, \text{Ad})}{(2\pi)^d \cdot \|f_H\|^2} \right) = \frac{L(1, \sigma \Sigma, \text{Ad})}{(2\pi)^d \cdot \|f_{\mathbb{Q}}\|^2}
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{F}_{\text{Gal}}) \). Moreover, the Galois equivariance holds for \( \text{Aut}(\mathbb{C}) \) if we assume further that \( \kappa_v = \kappa_w \) for all \( v, w \in S_{\mathbb{F}} \). On the other hand, we have the factorization of adjoint \( L \)-function:

\[
L(s, \Sigma, \text{Ad}) = L(s, \Pi, \text{Sym}^6 \otimes \omega_{\Pi}^{-3}) \cdot L(s, \Pi, \text{Sym}^2 \otimes \omega_{\Pi}^{-1}).
\]

By Deligne’s conjecture for \( L^{(x)}(1, \Pi, \text{Sym}^2 \otimes \omega_{\Pi}^{-1}) \) proved by Sturm [Stu80, Stu89], we have

\[
\sigma \left( \frac{L^{(x)}(1, \Pi, \text{Sym}^2 \otimes \omega_{\Pi}^{-1})}{(2\pi)^{2d+2\sum_{v \in S_{\mathbb{F}}} \kappa_v \cdot (\sqrt{-1})^{2d} \cdot \|f_H\|}} \right) = \frac{L^{(x)}(1, \sigma \Pi, \text{Sym}^2 \otimes \omega_{\Pi}^{-1})}{(2\pi)^{2d+2\sum_{v \in S_{\mathbb{F}}} \kappa_v \cdot (\sqrt{-1})^{2d} \cdot \|f_{\mathbb{Q}}\|}}
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Note that

\[
L(1, \Sigma, \text{Ad}) \in \pi^{-3\lambda_1 + \lambda_2 - 3}, \quad \mathbb{Q}^\times = \pi^{-\kappa_v + 1}, \quad \mathbb{Q}^\times
\]

for all \( v \in S_{\mathbb{F}} \). We conclude that Conjecture \ref{conjecture:adelic} holds for \( L^{(x)}(1, \Pi, \text{Sym}^6 \otimes \omega_{\Pi}^{-3}) \) with \( \text{Aut}(\mathbb{C}) \) replaced by \( \text{Aut}(\mathbb{C}/\mathbb{F}_{\text{Gal}}) \). Moreover, it holds for \( \text{Aut}(\mathbb{C}) \) if we assume further that \( \kappa_v = \kappa_w \) for all \( v, w \in S_{\mathbb{F}} \). We thus prove Theorem \ref{thm:adelic} for the right-half critical points. As for the left-half critical points, it follows from the global functional equation for \( \text{GL}_2(\mathbb{A}_E) \) applied to the regular \( \mathbb{C} \)-algebraic irreducible cuspidal automorphic representation \( \text{Sym}^6(\Pi) \). We refer to [Che21, Prop. 4.4] for the precise statement.

3.2.2. Case \( \mathbb{F} = \mathbb{Q} \). Now we prove the case when \( \mathbb{F} = \mathbb{Q} \) by a base change trick. Let \( \kappa \geq 6 \) be the weight of \( \Pi_E \). We assume \( \Pi \) is non-CM. Let \( L \) be a totally real cyclic extension over \( \mathbb{Q} \) such that \( r = [L : \mathbb{Q}] \geq 2 \) is a prime number and \( L \cap \mathbb{Q}(\zeta_p) = \mathbb{Q} \). Let \( \omega_{L/\mathbb{Q}} \) be a non-trivial Hecke character of \( \mathbb{A}_E^\times \) that vanishing on \( \text{Gal}_L/\mathbb{Q}(\mathbb{A}_L^\times) \). Note that by class field theory we have \( \omega_{L/\mathbb{Q}} = 1 \). Let \( BC_L(\Pi) \) be the base change lift of \( \Pi \) to \( \text{GL}_2(\mathbb{A}_L) \). Then \( BC_L(\Pi) \) is regular \( \mathbb{C} \)-algebraic and cuspidal with

\[
BC_L(\Pi)_v = \Pi_{L_v}
\]

for all archimedean places \( v \) of \( L \). In particular, the assumption in Theorem \ref{thm:adelic} (2) is satisfied by \( BC_L(\Pi) \). Note that \( \omega_{BC_L(\Pi)} = \omega_{L/\mathbb{Q}} \cdot \omega_{L/\mathbb{Q}} \). Let \( f_{BC_L(\Pi)} \) be the normalized newform of \( BC_{\mathbb{F}}(\Pi) \). By [Shi78 Theorem 4.3], one can deduce that

\[
\sigma \left( \frac{\|f_{BC_L(\Pi)}\|^r}{\|f_H\|^r} \right) = \frac{\|f_{BC_L(\Pi)}\|^r}{\|f_{\mathbb{Q}}\|^r}
\]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Let \( \chi \) be a finite order Hecke character of \( \mathbb{A}_E^\times \). It follows easily from the definition of Gauss sum that

\[
\sigma \left( \frac{G(\chi \otimes \omega_{L/\mathbb{Q}})}{G(\chi)} \right) = \frac{G(\sigma \chi \otimes \omega_{L/\mathbb{Q}})}{G(\sigma \chi)}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. We have the following factorization of twisted symmetric sixth $L$-function:

$$L(s, \text{BC}_L(\Pi), \text{Sym}^6 \otimes \chi \circ N_{L/\mathbb{Q}}) = \prod_{i=1}^{r} L(s, \Pi, \text{Sym}^6 \otimes \chi \omega_{L_i/\mathbb{Q}}).$$

Assume $\chi$ is chosen so that $\text{sgn}(\chi) = 1$ and $\chi^2 \not\in \langle \omega_{L_i/\mathbb{Q}} \rangle$. Consider the critical value

$$L^{(\infty)}(1 - 3\nu, \text{BC}_L(\Pi), \text{Sym}^6 \otimes \chi \circ N_{L/\mathbb{Q}}).$$

As we have proved in §3.2.1, we then conclude from the global functional equation that Conjecture 1.1 holds for the left-half critical points. This completes the proof of Theorem 1.3.

4. Algebraicity for $L$-functions on $\text{GSp}_4 \times \text{GL}_2$

Let $(\Sigma, V_\Sigma)$ and $(\Pi', V_{\Pi'})$ be irreducible globally generic cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_F)$ and $\text{GL}_2(\mathbb{A}_F)$ with central characters $\omega_\Sigma$ and $\omega_{\Pi'}$, respectively. Let

$$L(s, \Sigma \times \Pi')$$

be the Rankin–Selberg $L$-function of $\Sigma \times \Pi'$. We recall the integral representation of the Rankin–Selberg $L$-function due to Novodvorsky [Nov79 §3] (see also [PSSS94] Theorem 1.1 and [GPSR87] Theorem A) in §4.1. Under the assumptions that $\Sigma$ and $\Pi'$ are $C$-algebraic and satisfying conditions (2.5) and (2.13), in §4.2 we cohomologically interpret the global zeta integral defining the integral representation. The main result of this section is Theorem 4.5 where we prove the algebraicity of critical values of the Rankin–Selberg $L$-function in terms of the automorphic periods defined in §2. In §4.4 we also recall the result of Morimoto [Mor18] on the algebraicity for $L(s, \Sigma \times \Pi')$ and establish a period relation in Proposition 4.10.

4.1. Integral representation of $L(s, \Sigma \times \Pi')$. Let $G$ be the reductive group over $\mathbb{Q}$ defined by

$$G = \{(g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 \mid \nu(g_1) = \nu(g_2)\}.$$ 

We regard $G$ as a closed subgroup of $\text{GSp}_4$ by the embedding

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}.$$ 

Let $N_G$ be the maximal unipotent subgroup of $G$ consisting of upper unipotent matrices. By abuse of notation, we write $G$ for the base change $G_F = G \times_{\mathbb{Q}} \mathbb{F}$ to algebraic group over $\mathbb{F}$. Let $\psi = \otimes_v \psi_v$ be a non-trivial additive character of $\mathbb{F} \setminus \mathbb{A}_F$. For a finite place $v$ such that $\Pi'_v$ and $\Sigma_v$ are unramified and $\psi_v$ has conductor $a_v$, let

$$W_{v, \psi_v} \in \mathcal{W}(\Pi'_v, \psi_v), \quad W_{v, \psi_{U,v}} \in \mathcal{W}(\Sigma_v, \psi_{U,v}).$$
be the $\text{GL}_2(\mathfrak{a}_v)$-invariant and $\text{GSp}_4(\mathfrak{a}_v)$-invariant Whittaker functions, respectively, normalized so that $W^0_{v,\psi}(1) = W^0_{v,\psi,\psi,\psi}(1) = 1$. For $\varphi' \in V_{H'}$, $\varphi \in V_{\Sigma}$, and holomorphic section $f^{(s)}$ of $I(\omega_{\Sigma}^{-1} \omega_{H'}^{-1}, s)$, we define the global zeta integral

$$Z(E(f^{(s)}), \varphi', \varphi) = \int_{K_v \mathcal{G}(\mathbb{F}) \mathcal{G}(A_v)} (E(f^{(s)}) \otimes \varphi')(g) \varphi(g) \, dg_{\text{Tam}}.$$ 

Here $dg_{\text{Tam}}$ is the Tamagawa measure on $A_v ^{\times} \backslash \mathcal{G}(A_v)$. By an unfolding argument (cf. [GPSR7] § 1.1), we have

$$Z(E(f^{(s)}), \varphi', \varphi) = \int_{K_v \mathcal{G}(\mathbb{F}) \backslash \mathcal{G}(A_v)} (f^{(s)} \otimes W_{\varphi', \psi})(g) W_{\varphi, \psi}(g) \, dg_{\text{Tam}}$$

for $\text{Re}(s)$ sufficiently large. Here $dg_{\text{Tam}}$ is the quotient of $dg_{\text{Tam}}$ by the Tamagawa measure on $N_G(A_v)$. For each place $v$, $W'_{v} \in \mathcal{W}(\Pi'_{v}, \psi_{v})$, $W_{v} \in \mathcal{W}(\Sigma_{v}, \psi_{v})$, and $f^{(s)}$ a meromorphic section of $I(\omega_{\Sigma}^{-1} \omega_{H'}^{-1}, s)$, let $Z(f^{(s)}_{v}, W'_{v}, W_{v})$ be the local zeta integral defined by

$$Z(f^{(s)}_{v}, W'_{v}, W_{v}) = \int_{K_v \mathcal{G}(\mathbb{F}) \backslash \mathcal{G}(A_v)} (f^{(s)}_{v} \otimes W'_{v})(g_{v}) W_{v}(g_{v}) \, dg_{v}.$$ 

Here $dg_{v}$ is the quotient measure on $\mathbb{F}_{v}^{\times} \mathcal{G}(\mathbb{F}_{v}) \backslash \mathcal{G}(\mathbb{F}_{v})$ defined as follows:

- If $v$ is finite, then $dg_{v}$ is the quotient of the Haar measures on $\mathbb{F}_{v}^{\times} \mathcal{G}(\mathbb{F}_{v})$ and $N_{\mathbb{G}}(\mathbb{F}_{v})$ with $\text{vol}(\mathbb{F}_{v}^{\times} \mathcal{G}(\mathbb{F}_{v})) = \text{vol}(N_{\mathbb{G}}(\mathbb{F}_{v})) = 1$.
- If $v \in S_{v}$, then

$$dg_{v} = |a_{1,v}a_{2,v}|_{v}^{-3} \, da_{1,v} \, da_{2,v} \, dk_{1,v} \, dk_{2,v}$$

for

$$g_{v} = (\text{diag}(a_{1,v}, a_{2,v}, a_{2,1}^{-1})) k_{1,v} k_{2,v}$$

with $a_{1,v}, a_{2,v} \in \mathbb{F}_{v}^{\times}$ and $k_{1,v}, k_{2,v} \in \text{SO}(2)$, where $da_{i,v}$ is the Lebesgue measure and $\text{Vol}(\text{SO}(2), dk_{i,v}) = 1$ for $i = 1, 2$.

Note that we have (cf. [IP21] § 6.1)

$$dg_{\text{Tam}} = |D_{\mathbb{F}}|^{-2} \zeta_{\mathbb{F}}(2)^{-2} \cdot \prod_{\mathfrak{p} \in \mathcal{S}} dg_{v}.$$ 

Now we recall the integral representation. Let $S$ be a finite set of places of $\mathbb{F}$ containing $S_{v}$ so that for $v \notin S$, $\Pi'_{v}$ and $\Sigma_{v}$ are unramified and $\psi_{v}$ has conductor $\mathfrak{a}_{v}$. We write

$$\Pi'_{S} = \bigotimes_{v \in S} \Pi'_{v}, \quad \Sigma_{S} = \bigotimes_{v \in S} \Sigma_{v}, \quad \psi_{S} = \bigotimes_{v \in S} \psi_{v}.$$ 

By [PSR7] Theorem 3.1, we have

$$Z(f^{(s)}_{v}, W'_{v,\psi_{v}}, W_{v,\psi_{v},\psi_{v}}) = L(s, \Sigma_{v} \times \Pi'_{v})$$

for all $v \notin S$. Thus we obtain the following

**Proposition 4.1** (Novodvorsky). Let $W_{S} \in \mathcal{W}(\Pi'_{S}, \psi_{S})$, $W_{S} \in \mathcal{W}(\Sigma_{S}, \psi_{S}, S)$, and $f^{(s)}_{S}$ a meromorphic section of $I(\omega_{\Sigma}^{-1} \omega_{H'}^{-1}, s)$. Let $\varphi' \in \mathcal{V}_{H'}$, $\varphi \in \mathcal{V}_{\Sigma}$, and $f^{(s)}$ the meromorphic section of $I(\omega_{\Sigma}^{-1} \omega_{H'}^{-1}, s)$ defined by

$$W_{\varphi', \psi} = \prod_{v \notin S} W_{v, \psi_{v}, \psi_{v}}, \quad W_{\varphi, \psi_{S}} = \prod_{v \notin S} W_{v, \psi_{v}, \psi_{v}} W_{S}, \quad f^{(s)} = \bigotimes_{v \notin S} f^{(s)}_{v} \otimes f^{(s)}_{S}.$$ 

Then we have

$$Z(E(f^{(s)}), \varphi', \varphi) = |D_{\mathbb{F}}|^{-2} \zeta_{\mathbb{F}}(2)^{-2} \cdot L^{S}(s, \Sigma \times \Pi') \cdot Z(f^{(s)}_{S}, W^{0}_{S}, W_{S}).$$
4.2. Cohomological interpretation of global zeta integral. Let $K = \mathbb{R}_+ \times U(2) \times U(2)$ and regard it as a closed subgroup of $G(\mathbb{R})$ by the homomorphism
\[
(a, a_1 + \sqrt{-1} b_1, a_1 + \sqrt{-1} b_1) \mapsto \left( a \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}, a \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \right).
\]
We write $\mathfrak{g} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_1$ for the Lie algebra of $G(\mathbb{R}) \subset GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$. Define $\mathfrak{t} \subset \mathfrak{g}$ and $\mathfrak{p}^\pm, \mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$ by
\[
\mathfrak{t} = \text{Lie}(K) = \mathbb{R} \cdot (I_2 \oplus I_2) \oplus (\mathfrak{t}_1 \oplus \mathfrak{t}_1), \quad \mathfrak{p}^\pm = \mathfrak{p}_1^\pm \oplus \mathfrak{p}_1^\pm, \quad \mathfrak{p} = \mathfrak{t}_1 \oplus \mathfrak{p}^-.
\]
For $(\nu, \kappa, \kappa_2) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with $\kappa_1 + \kappa_2 \equiv \nu (\text{mod} \, 2)$, let $(\rho_{(\nu, \kappa, \kappa_2)}, V_{(\nu, \kappa, \kappa_2)})$ be the algebraic character of $K$ defined by $V_{(\nu, \kappa, \kappa_2)} = \mathbb{C}$ and
\[
\rho_{(\nu, \kappa_1, \kappa_2)}(a(u_1, u_2)) \cdot z = a^\nu u_1^{\kappa_1} u_2^{\kappa_2} \cdot z
\]
for $a \in \mathbb{R}_+ \times \mathbb{R}$ and $u_1, u_2 \in U(1)$. Let $(G, X)$ be the Shimura datum defined by
\[
G = \text{Res}_{\mathbb{F} \mathbb{Q}} G_{\mathfrak{g}},
\]
and $X$ is the $G(\mathbb{R})$-conjugacy class containing the morphism $h : S \to G_R$ with
\[
h(x + \sqrt{-1} y) = \left( \begin{array}{ccc} x & y & \cdots \\ -y & x & \cdots \\ \cdots & \cdots & \cdots \end{array} \right) \times \left( \begin{array}{ccc} x & y & \cdots \\ -y & x & \cdots \\ \cdots & \cdots & \cdots \end{array} \right)
\]
on $\mathbb{R}$-points. Under the identification of $\mathbb{F}_v$ with $\mathbb{R}$ for each $v \in S_{\mathfrak{X}}$, we have
\[
K_h = K_{S_{\mathfrak{X}}}, \quad \mathfrak{t}_h = \mathfrak{t}^{S_{\mathfrak{X}}}, \quad \mathfrak{p}_h = (\mathfrak{p}_1^-)^{S_{\mathfrak{X}}}, \quad \mathfrak{p} = \mathfrak{p}^{S_{\mathfrak{X}}}.
\]
Let $(\nu; \kappa_1, \kappa_2) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with $\kappa_1 = (\kappa_1, v) \in S_{\mathfrak{X}}, \quad \kappa_2 = (\kappa_2, v) \in S_{\mathfrak{X}},$ and $\kappa_1, \nu + \kappa_2 \equiv \nu (\text{mod} \, 2)$, let
\[
\rho_{(\nu; \kappa_1, \kappa_2)}, V_{(\nu; \kappa_1, \kappa_2)}
\]
be the motivic algebraic character of $K_{S_{\mathfrak{X}}}$ defined by
\[
\left( \bigotimes_{v \in S_{\mathfrak{X}}} \rho_{(\nu; \kappa_1, \kappa_2, v)}, \bigotimes_{v \in S_{\mathfrak{X}}} V_{(\nu; \kappa_1, \kappa_2, v)} \right).
\]
Let
\[
[V_{(\nu; \kappa_1, \kappa_2)}]
\]
be the automorphic line bundle on $\text{Sh}(G, X)$ defined by $(\rho_{(\nu; \kappa_1, \kappa_2)}, V_{(\nu; \kappa_1, \kappa_2)})$.

We assume $\Sigma$ and $\Pi'$ are $\mathbb{C}$-algebraic and satisfying conditions (2.5) and (2.13). Let $(\xi; u)$ and $(\xi'; u')$ be the weights of $\Sigma$ and $\Pi'$, respectively. A critical point for $L(s, \Sigma \times \Pi')$ is an integer $m$ which is not a pole of the archimedean local factors $L(s, \Sigma_v \times \Pi'_v)$ and $L(1-s, \Sigma'_v \times (\Pi'_v)^\vee)$ for all $v \in S_{\mathfrak{X}}$. Note that we have
\[
L(s - \frac{u+w'}{2}, \Sigma_v \times \Pi'_v) = \Gamma_C \left( s + \frac{|\lambda_{1,v} + \lambda_{2,v} - \ell_v|}{2} \right) \Gamma_C \left( s + \frac{|\lambda_{1,v} - \lambda_{2,v} - \ell_v|}{2} \right) \times \Gamma_C \left( s + \frac{|\lambda_{1,v} + \lambda_{2,v} + \ell_v - 2|}{2} \right) \Gamma_C \left( s + \frac{|\lambda_{1,v} - \lambda_{2,v} + \ell_v - 2|}{2} \right).
\]
If we assume further that $1 \leq \ell_v \leq \lambda_{1,v}$ for all $v \in S_{\mathfrak{X}}$, then the set of critical points is given by
\[
\text{Crit}(\Sigma \times \Pi') = \left\{ m \in \mathbb{Z} \mid 1 - \frac{|\lambda_{1,v} + \lambda_{2,v} - \ell_v|}{2} \leq m + \frac{u+w'}{2} \leq \frac{|\lambda_{1,v} + \lambda_{2,v} - \ell_v|}{2} \right\}.
\]
Put
\[
\kappa = \min \{|\lambda_{1,v} + \lambda_{2,v} - \ell_v|\}.
\]
Let $I, J$ be subsets of $S_{\mathfrak{X}}$ defined by
\[
I = \{ v \in S_{\mathfrak{X}} \mid 1 \leq \ell_v \leq \lambda_{1,v} + \lambda_{2,v} - 1 \}, \quad J = \{ v \in S_{\mathfrak{X}} \mid \lambda_{1,v} + \lambda_{2,v} + 1 \leq \ell_v \leq \lambda_{1,v} \}.
\]
It is clear that $I$ and $J$ are admissible with respect to $\lambda$ and $\ell$, respectively. Note that $I \cup J = S_{\mathfrak{X}}$ if and only if $\text{Crit}(\Sigma \times \Pi')$ is non-empty. In the following proposition, we prove Galois equivariant property of the global zeta integral in terms of the automorphic periods.

**Proposition 4.2.** Assume the following conditions are satisfied:

- $I \cup J = S_{\mathfrak{X}}$.
- $\kappa = |\lambda_{1,v} + \lambda_{2,v} - \ell_v|$ for all $v \in S_{\mathfrak{X}}$.
- If $F = \mathbb{Q}$, then $\kappa \neq 2$ or $\omega_\Sigma \omega_{\Pi'} \neq |u+w'|$.
Let \( \varphi' \in V_{H_1^+}^+ \), \( \varphi \in V_{\Sigma_1^+}^+ \), and \( f^{(s)} \) be a meromorphic section of \( I(\omega_{\Sigma_1^+}^{-1} \omega_{H_1^+}^{-1}, f, s) \) satisfying conditions in Proposition 2.6. For \( \sigma \in \mathrm{Aut}(\mathbb{C}) \), we have

\[
\sigma \left( \frac{Z \left( E^{[\kappa]}(f^{(s)}), (\varphi')^J, p_{\varphi'}^{-1}\Delta (\varphi)^J \right)}{|D_{\varphi'}|^{1/2} \cdot (2\pi\sqrt{-1})^{d(\kappa + u + w)/2} \cdot G(\omega_{\Sigma_1^+}^{-1} \omega_{H_1^+}^{-1}) \cdot p^J(\Sigma) \cdot p^J(H')} \right) = \frac{Z \left( E^{[\kappa]}(\sigma f^{(s)}), (\sigma \varphi')^J, p_{\sigma \varphi'}^{-1}\Delta (\sigma \varphi)^J \right)}{|D_{\sigma \varphi'}|^{1/2} \cdot (2\pi\sqrt{-1})^{d(\kappa + u + w)/2} \cdot G(\omega_{\Sigma_1^+}^{-1} \omega_{H_1^+}^{-1}) \cdot p^J(\sigma \Sigma) \cdot p^J(\sigma H')}.
\]

**Proof.** The natural inclusion \( G \subset \mathrm{GL}_2 \times \mathrm{GL}_2 \) induces a morphism \( (G, X) \to (G_1 \times G_1, X_1 \times X_1) \) of Shimura data. Note that

\[
(\rho(\omega_{\Sigma_1^+}^{-1} \omega_{H_1^+}^{-1}))_{K^{\Sigma, \omega}} = \rho(u; -\Sigma, \Sigma(J)).
\]

Then we have a canonical homomorphism

\[
H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \to \rho(\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}))
\]

By the Künneth formula, we have a canonical isomorphism

\[
H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \cong \bigoplus_{q_1 + q_2 = d+1} H^{q_1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \otimes H^{q_2}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}).
\]

In particular, we obtain a homomorphism

\[
F_1 : H^0(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \otimes H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \to H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}).
\]

For \( \varphi' \in V_{H_1^+}^+ \) and meromorphic section \( f^{(s)} \) of \( I(\omega_{\Sigma_1^+}^{-1} \omega_{H_1^+}^{-1}, f, s) \) satisfying conditions in Proposition 2.6, it is clear that \( F_1([E^{[\kappa]}(f^{(s)})] \otimes [\varphi'])_1 \) is the class represented by

\[
\left( E^{[\kappa]}(f^{(s)}) \otimes (\varphi')^J \right) \otimes \bigwedge_{v \in J} (0 \oplus X_{+, v}) \otimes \bigotimes_{v \in S_x} 1
\]

\[
eq \mathcal{A}(G, \Delta_{\Sigma_1^+}) \otimes \bigwedge^J (p^+)_{S_x} \otimes V(u; -\Sigma, \Sigma(J)).
\]

For \( \sigma \in \mathrm{Aut}(\mathbb{C}) \), in a similarly way we obtain a homomorphism

\[
F_{1, \sigma} : H^0(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \otimes H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \to H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \})
\]

satisfying the Galois equivariant property:

\[
(4.4)
\]

\[
t_{\sigma} \circ F_1 = F_{1, \sigma} \circ (t_{\sigma} \otimes t_{\sigma}).
\]

The embedding \([4.1]\) induces a morphism \( (G, X) \to (G_2, X_2) \) of Shimura data. Note that

\[
\rho(\Delta(I); -u)|_{K^{S_x}} = \bigoplus_{I \in \mathcal{I}} \left( \bigotimes_{v \in I} \rho(-u; \lambda_1, -v - 3i_v, \lambda_2, -1 + i_v) \right) \otimes \left( \bigotimes_{v \in J} \rho(-u; -\lambda_2, -v - 2i_v, -\lambda_1, v + i_v) \right),
\]

where \( \mathcal{I} = (i_v)_{v \in S_x} \in \mathbb{Z}^{S_x} \) with \( 0 \leq i_v \leq \lambda_1, v - \lambda_2, v - 2 \). The assumption on \( \kappa \) implies that \( \rho(-u; \kappa + 2, \ell(I)) \) appears in the right-hand side of the decomposition with

\[
i_v = \begin{cases} -\lambda_2, v + \ell_v - 1 & \text{if } v \in I, \\ \lambda_1, v - \ell_v & \text{if } v \in J. \end{cases}
\]

Consider the projection

\[
\rho(\Delta(I); -u) \to \rho(-u; \kappa + 2, \ell(I)).
\]

Then we have a canonical homomorphism

\[
F_2 : H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \}) \to H^{d+1}(\{\varphi \in \mathcal{A}(G, \Delta_{\Sigma_1^+}) \mid \sigma \in \mathrm{Aut}(\mathbb{C}) \})
\]

Let \( J_0 \subset J \) be the subset consisting of \( v \in J \) such that \( \ell_v \leq \lambda_1, v + 2 \). For each \( J' \subset J_0 \), define \( \Lambda(J') \in \mathbb{Z}^{S_x} \) defined by

\[
i_v(J') = \begin{cases} \ell_v - \lambda_2, v & \text{if } v \in J', \\ \ell_v - \lambda_2, v & \text{if } v \notin J'. \end{cases}
\]
For \( \varphi \in V^+_\mathbb{C} \), by the definition of \( [\varphi]_I \) in Lemma \ref{lem:11}, \( F_2([\varphi]_I) \) is represented by
\[
\sum_{J' \subseteq J_0} \left[ C_{J'} \cdot \text{pr}_{\mathcal{I}(J')}([\varphi]_I) \right] g_{(\mathbf{H})}
\]
\[
\otimes \left( \bigwedge_{\varphi \in I}(X_{+,v} \otimes 0) \otimes (0 \oplus X_{+,v}) \right) \otimes \left( \bigwedge_{\varphi \in J \setminus J'}(X_{+,v} \otimes 0) \right) \otimes \left( \bigwedge_{\varphi \in J \setminus J_0}(0 \oplus X_{+,v}) \right) \otimes \bigotimes_{\varphi \in S_{\infty}} 1
\]
\[
eq C^\infty_{\text{rad}}(G(\mathbb{A})_{\mathbb{F}}) \otimes \Lambda^{d+1} \otimes V_{[\omega; -2, \omega_{\mathcal{I}}(\mathcal{I})]}(\mathcal{I})
\]
for some \( C_{J'} \in \mathbb{Q} \) independent of \( \varphi \). Here \( C^\infty_{\text{rad}}(G(\mathbb{A})_{\mathbb{F}}) \) is the space of smooth functions \( f \) on \( G(\mathbb{F}) \setminus G(\mathbb{A})_{\mathbb{F}} \) such that \( X \cdot f \) is rapidly decreasing for all \( X \in \mathfrak{g}^{S_{\infty}} \). Note that \( \mathcal{I}(\mathcal{I}) = \mathcal{I} - \mathcal{I}_2 \) and by Lemma \ref{lem:26} we have
\[
C_{\mathcal{I}} \neq 0.
\]
For \( \sigma \in \text{Aut}(\mathbb{C}) \), in a similarly way we obtain a homomorphism
\[
F_{2,\sigma} : H^{d+1}([V([\omega; \mathcal{I}], -\varphi)]^\text{sub}) \to H^{d+1}([V([-\omega; -2, \varphi; \mathcal{I}])^\text{sub})
\]
satisfying the Galois equivariant property:
\[
T_{\sigma} \circ F_2 = F_{2,\sigma} \circ T_{\sigma}.
\]
Now consider the Serre duality pairing
\[
H^{d'}([V([\omega; \mathcal{I}], -\varphi)]^\text{an}) \times H^{d+1}([V([-\omega; -2, \varphi; \mathcal{I}])^\text{sub}) \to \mathbb{C}, \quad (c_1, c_2) \mapsto \int_{\text{Sh}(\mathbb{G}, \mathbb{X})} c_1 \wedge c_2
\]
in \[\text{Har90b}\] Proposition 3.8 and Remark 3.8.4. Similarly as in the proof of Theorem \ref{thm:14} by \[\text{Har90b}\], \ref{thm:41}, \ref{thm:66}, and \ref{thm:77} we have
\[
\int_{\text{Sh}(\mathbb{G}, \mathbb{X})} F_1([E^{[\delta]}(f(s))] \otimes [\varphi']_I) \wedge F_2([\varphi]_I) = C' \cdot Z\left( E^{[\delta]}(f(s)), (\varphi')_I, \text{pr}_{\mathcal{I}}(\mathcal{I})(\varphi)_I \right)
\]
for some \( C' \in \mathbb{Q}^\times \) independent of \( \varphi', \varphi, f(s) \). For \( \sigma \in \text{Aut}(\mathbb{C}) \), we also have
\[
\int_{\text{Sh}(\mathbb{G}, \mathbb{X})} F_{1,\sigma}([E^{[\delta]}(\varphi'; s)] \otimes [\varphi']_I) \wedge F_{2,\sigma}([\varphi]_I) = C' \cdot Z\left( E^{[\delta]}(\varphi(s)); (\varphi')_I, \text{pr}_{\mathcal{I}}(\mathcal{I})(\varphi)_I \right).
\]
By Proposition \ref{prop:20} \[\text{[Har90b]}\], \ref{prop:35}, \ref{prop:88}, for \( \sigma \in \text{Aut}(\mathbb{C}) \) we have
\[
T_{\sigma}\left( \frac{F_1([E^{[\delta]}(f(s))] \otimes [\varphi']_I)}{|D_{\mathbb{C}}|^{1/2} \cdot (2\pi \sqrt{-1})^{d(\omega_{\mathcal{X}} + \omega_{\mathcal{Y}})} \cdot G(\omega_{\mathcal{X}} \omega_{\mathcal{Y}}) \cdot p^{d}(\mathcal{I})} \right) = \frac{F_{1,\sigma}([E^{[\delta]}(\varphi'; s)] \otimes [\varphi']_I)}{|D_{\mathbb{C}}|^{1/2} \cdot (2\pi \sqrt{-1})^{d(\omega_{\mathcal{X}} + \omega_{\mathcal{Y}})} \cdot G(\sigma \omega_{\mathcal{X}} \omega_{\mathcal{Y}}) \cdot p^{d}(\sigma \mathcal{I})}
\]
and
\[
T_{\sigma}\left( \frac{F_2([\varphi]_I)}{p^{d}(\mathcal{I})} \right) = \frac{F_{2,\sigma}([\varphi]_I)}{p^{d}(\sigma \mathcal{I})}.
\]
The assertion then follows from the Galois equivariant property of the Serre duality pairing. This completes the proof.

\[\square\]

\subsection{Local zeta integrals.}

Let \( v \) be a place of \( \mathbb{F} \). In this section, we prove Galois equivariant property of the non-archimedean local zeta integrals and compute archimedean local zeta integral.

\begin{lemma}
Assume \( v \) is finite and lying above a rational prime \( p \). Let \( W'_v, W_v \in \mathcal{W}(\mathcal{I}, \psi_v), \mathcal{W}_v, W_v \in \mathcal{W}(\Sigma_v, \psi_{\mathcal{I}, v}), \) and \( f_v^{(s)} \) be a rational section of \( I(\omega_{\mathcal{X}, v}^{-1} \omega_{\mathcal{Y}, v}^{-1}, s) \). Then \( Z(f^{(s)}_v, W'_v, W_v) \) is a rational function in \( q_v^{-s} \), and we have
\[
\sigma Z(f^{(s)}_v, W'_v, W_v) = \sigma Z(f^{(s)}_v, t_{\sigma, v} W'_v, t_{\sigma, v} W_v)
\]
for all \( \sigma \in \text{Aut}(\mathbb{C}) \). Here \( u_{\sigma, v} \in \mathfrak{a}_v^\times \subset \mathfrak{o}_v^\times \) is the unique element such that \( \sigma(\psi_v(x_v)) = \psi_v(u_{\sigma, v} x_v) \) for all \( x_v \in \mathfrak{o}_v^\times \). Moreover, there exists a triplet \( (f^{(s)}_v, W'_v, W_v) \) such that \( f^{(s)}_v \) is holomorphic for \( \text{Re}(s) > -\frac{2}{q_v} \) and
\[
Z(f^{(s)}_v, W'_v, W_v) = 1.
\]

\end{lemma}
We drop the subscript $v$ for brevity. We have

$$Z(f^{(s)},W',W) = \int_{G(o)} dk \int_{F^*} d^x a_1 \int_{F^*} d^x a_2 A(f^{(s)} \otimes W') (\text{diag}(a_1 a_2, a_1, a_2^{-1}, 1)k) W (\text{diag}(a_1 a_2, a_1, a_2^{-1}, 1)k)$$

where $\text{vol}(G(o), dk) = 1$ and

$$I_s(W',W;k) = \int_{F^*} d^x a_1 \int_{F^*} d^x a_2 W'(\text{diag}(a_1, 1)k) W (\text{diag}(a_1 a_2, a_1, a_2^{-1}, 1)k) |a_1|^{-s} |a_2|^{2s-2} \omega \Sigma \omega_{II'}(a_2)$$

for $k = (k_1, k_2) \in G(o)$. By the asymptotic behavior of the Whittaker functions (cf. [LM09 Theorem 3.1]), we see that $I_s(W',W;k)$ converges absolutely for $\text{Re}(s)$ sufficiently large. Moreover, by [Che21b Lemma 6.2], the integral is actually a generalized Tate integral as in [Gro18 Proposition A]. In particular, this implies that $I_s(W',W;k)$ defines a rational function in $q^{-s}$ and

$$\sigma I_s(W',W;k) = I_s(\sigma W',\sigma W;k)$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Making a change of variables from $(a_1, a_2)$ to $(u_{-1,0} a_1, u_{-1,0} a_2)$, we see that

$$I_s(\sigma W',\sigma W;k) = \sigma \omega_{II'}(u_{\sigma,p})^{-1} : I_s(t_\sigma W',t_\sigma W;k).$$

Hence we see that $Z(W',W,f^{(s)})$ is a rational function in $q^{-s}$ and

$$\sigma Z(f^{(s)},W',W) = Z(\sigma f^{(s)},\sigma W',\sigma W)$$

$$= \omega_{II'}(u_{\sigma,p})^{-1} : Z(\sigma f^{(s)},t_\sigma W',t_\sigma W)$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Now we verify the second assertion. Without lose of generality, we assume $\psi$ has conductor $o$. Let $W'$ be the Whittaker function of $II'$ such that

$$W'(\text{diag}(a, 1)) = I_{\sigma}(a).$$

Let $N_2$ be the upper unipotent matrices of $GL_2$ and $K(n) \subset K_1(n) \subset K_0(n)$ be open compact subgroups of $GL_2(o)$ defined by

$$K(n) = \{ k \in GL_2(o) \mid k \equiv 1 \pmod{\varpi^n} \}, \quad K_1(n) = \left\{ k \in GL_2(o) \mid k \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi^n} \right\},$$

$$K_0(n) = \left\{ k \in GL_2(o) \mid k \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\varpi^n} \right\}.$$

Let $P_3$ and $Q$ be the mirabolic subgroup and Klingen subgroup of $GL_3$ and $GSp_4$, respectively, defined by

$$P_3 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \in GL_3 \right\}, \quad Q = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \in GSp_4 \right\}.$$

We have a homomorphism $i : Q \to P_3$ defined by

$$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \nu t^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $A_3$ be the diagonal matrices of $P_3$, $N_3$ the unipotent radical of $P_3$, and $\psi_{N_3}$ the character of $N_3$ defined by $\psi_{N_3}(u) = \psi(-u_{12} + u_{23})$ for $u = (u_{ij})$. Consider the compact induction $c\text{-ind}_{N_3}^{P_3(\mathbb{F})}(\psi_{N_3})$. Since $\Sigma$ is generic, by [BZ76 Proposition 2.35] and [RS75 Theorem 2.5.3-(ii)], every function in $c\text{-ind}_{N_3}^{P_3(\mathbb{F})}(\psi_{N_3})$ can be lifted to a Whittaker function of $\Sigma$ with respect to $\psi_{U'}$ through the homomorphism $i$. We refer to the proof of [RS75 Proposition 2.6.4] for similar arguments. Now we take a sufficiently large $n$ such that the following conditions are satisfied:

- $W'$ is right $K(n)$-invariant.
• \( \omega_\Sigma \) and \( \omega_{\Pi'} \) are trivial on \( 1 + \varpi^n \sigma \).

Define \( f_n \in \text{c-ind}^{P_3(\mathbb{F})}_{N_3(\mathbb{F})}(\psi_N) \) by

\[
f_n(g) = \begin{cases} 
0 & \text{if } g \notin N_3(\mathbb{F})K_1(n), \\
\psi_N(u) & \text{if } g = u \cdot k \in N_3(\mathbb{F})K_1(n).
\end{cases}
\]

Here we regard \( \text{GL}_2(\mathbb{F}) \) as subgroup of \( P_3(\mathbb{F}) \) by the map \( g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \). Note that \( f_n \) is well-defined as we assume \( \psi \) has conductor \( \sigma \). Fix a lift \( W \in W(\Sigma, \psi_U) \) of \( f_n \). Let \( n' \geq n \) be another sufficiently large integer such that \( W \) is right invariant by \( (K(n') \times K(n')) \cap G(\sigma) \). Let \( f^{(s)} = f^{(s)}_\Phi \) be the rational section of \( I(\omega_\Sigma^{-1}, \omega_{\Pi'}^{-1}, s) \) defined by

\[
f^{(s)}(g) = |\det g|^s \int_{\mathbb{R}^\times} \Phi((0, t)g)\omega_\Sigma \omega_{\Pi'}(t)| t |^{2s} \, dt,
\]

where \( \Phi \) is the Schwartz function on \( \mathbb{F}^2 \) given by

\[
\Phi(x, y) = \omega_\Sigma^{-1}(y)\|x^s\|_\sigma(1)\|y^s\|_\sigma(y).
\]

It is clear that \( f^{(s)} \) is holomorphic for \( \Re(s) > \frac{-d + 2}{2} \). Let \( k = (k_1, k_2) \in G(\sigma) \). Note that \( f^{(s)}(k_1) = 0 \) unless \( k_1 \in K_0(n') \). Assume \( k_1 \in K_0(n') \) and write

\[
k_1 \in \text{diag}(t_1, t_2)N_2(\sigma)K(n')
\]

for some \( t_1, t_2 \in \sigma^\times \) with \( t_1 t_2 = \det k_1 \). Then \( f^{(s)}(k_1) = \omega_\Sigma^{-1}(t_2) \) and we have

\[
W(\text{diag}(a_1a_2, a_1, a_2^{-1}, 1)k) = \omega_\Sigma(a_2^{-1}t_2)f_n(\text{diag}(a_1a_2t_2^{-1}, a_2t_2^{-1}, 1)k).
\]

In particular, \( W(\text{diag}(a_1a_2, a_1, a_2^{-1}, 1)k) = 0 \) unless \( a_2 \in \sigma^\times \) and \( k_2 \in K_0(n) \). Assume \( k_2 \in K_0(n) \) and write

\[
k_2 \in \text{diag}(u_1, u_2)N_2(\sigma)K(n)
\]

for some \( u_1, u_2 \in \sigma^\times \) with \( u_1 u_2 = \det k_2 \). Then \( W'(\text{diag}(a_1, 1)k_2) = \omega_{\Pi'}(u_2)\|x^s\|_\sigma(a_1) \) and we have

\[
I_s(W', W; k) = \omega_\Sigma(t_2)\omega_{\Pi'}(u_2) \int_{\mathbb{F}^\times} d^x a_1 \int_{\mathbb{F}^\times} d^x a_2 \|x^s\|_\sigma(a_1)\|y^s\|_\sigma(a_2) \omega_{\Pi'}(a_2)
\]

\[
= \left[ \sigma^\times : 1 + \varpi^n \sigma \right]^{-1} \omega_\Sigma \omega_{\Pi'}(t_2).
\]

We conclude that

\[
Z(f^{(s)}, W', W) = \left[ \sigma^\times : 1 + \varpi^n \sigma \right]^{-1} [G(\sigma) : (K_0(n') \times K_0(n)) \cap G(\sigma)]^{-1}.
\]

This completes the proof. \( \square \)

**Lemma 4.4.** Assume \( v \in S_\Sigma, \kappa = |\lambda_1 + \lambda_2 - \ell_v| \geq 1 \), and \( \psi_v \) is the standard additive character of \( \mathbb{F}_v \).

Let \( \epsilon_v = 1 \) if \( v \in I \) and \( \epsilon_v = -1 \) if \( v \in J \). We have

\[
Z \left( \rho(\text{diag}(\epsilon_v, 1)) f^{(s)} \right), W^+_{(\sigma, \ell_1, \sigma, -\lambda_2, v)} \right) \right)
\]

\[
= 2^{-s+2+3(\lambda_1 + \lambda_2 - \ell_v - \epsilon_v - \ell_v)} \cdot (\sqrt{-1})^{\lambda_2, v - \ell_v} \cdot \Gamma_C \left(s + \frac{\lambda_1 + \lambda_2 + \ell_v - 2}{2} + \frac{u + w'}{2} \right).
\]

\[
\times \Gamma_C \left(s + \frac{\lambda_1 + \lambda_2 + \ell_v - 2}{2} + \frac{u + w'}{2} \right).
\]

29
Proof. We identify \( \mathbb{F}_v = \mathbb{R} \) and drop the subscript \( v \) for brevity. By explicit formulas (2.41) and Theorem 2.8, we have

\[
Z \left( \rho(\text{diag}(\epsilon, 1)) f^{(s)}_\kappa, W^{+}_\ell, W^{+}_{(\Delta, u)}, \ell - \lambda_2 \right)
\]

\[
= \int_{\text{SO}(2)} \int_{\text{SO}(2)} dk_1 \int_{\mathbb{R}^2} d^\times a_1 \int_{\mathbb{R}^2} d^\times a_2 \int_{\mathbb{R}^2} d^\times k_1 f^{(s)}_\kappa \left( \text{diag}((-1)^s a_1 a_2, a_2^{-1}) k_1 \right) W^{+}_\ell \left( \text{diag}(a_1, 1) k_2 \right) \times W^{+}_{(\Delta, u), \ell - \lambda_2} \left( \text{diag}(a_1 a_2, a_1, a_2^{-1}, 1)(k_1, k_2) \right)
\]

\[
= 2(2\pi \sqrt{-1})^\lambda_2 - \infty \int_0^\infty d^\times a_1 \int_0^\infty d^\times a_2 a_1^{s+(u+w'+\ell-4)/2} a_2^{2s+u+w'+\ell-2} \delta_{-4\pi a_1}
\]

\[
\times \Gamma^2(s_1 - s_2)(s_1 + \lambda_1 + 1) \Gamma^2(-s_2 - \lambda_2) \Gamma^2(s_1 + s_2 + \lambda_1 - \lambda_2 + 2) \Gamma^2(s_1 + s_2 + \lambda_1 + \lambda_2 + 2)
\]

\[
\times \Gamma^2(2s + u + w' + \ell - s_1 - s_2 - 4).
\]

Put \( t = s + \frac{u + w'}{2} \). Then we have

\[
Z \left( \rho(\text{diag}(\epsilon, 1)) f^{(s)}_\kappa, W^{+}_\ell, W^{+}_{(\Delta, u)}, \ell - \lambda_2 \right)
\]

\[
= 2^{2t - \ell + 5}(2\pi \sqrt{-1})^\lambda_2 - \ell \int_0^\infty d^\times a_2 a_2^{2t + \lambda_2 - \ell - 2} \int_{c_1 - \sqrt{-1}}^{c_1 + \sqrt{-1}} \frac{ds_1}{2\pi \sqrt{-1}} a_2^{-s_1} \Gamma^2(s_1 + \lambda_1 + 1) \Gamma^2(s_1 + \lambda_1 + 1) \Gamma^2(s_1 + s_2 + \lambda_1 - \lambda_2 + 2) \Gamma^2(s_1 + s_2 + \lambda_1 + \lambda_2 + 2) \Gamma^2(s_1 + s_2 + \lambda_1 + \lambda_2 + 2)
\]

\[
\times \Gamma^2(s_1 + s_2 + \lambda_1 - \lambda_2 + 2) \Gamma^2(2t + \ell - s_1 - s_2 - 4).
\]

By Barne's first lemma, the above integration in \( s_2 \) is equal to

\[
2 \cdot \Gamma^2(s_1 + \lambda_1 - 2\lambda_2 + 2) \Gamma^2(2t + \lambda_1 - \lambda_2 + \ell - 2) \Gamma^2(s_1 + \lambda_1 + 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + 2)
\]

\[
\Gamma^2(2t + s_1 + 2\lambda_1 - \lambda_2 + \ell).
\]

Therefore, we have

\[
Z \left( \rho(\text{diag}(\epsilon, 1)) f^{(s)}_\kappa, W^{+}_\ell, W^{+}_{(\Delta, u)}, \ell - \lambda_2 \right)
\]

\[
= -2^{2t - \ell + 6}(2\pi \sqrt{-1})^\lambda_2 - \ell \cdot \Gamma^2(2t + \lambda_1 - \lambda_2 + \ell - 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + \ell - 2)
\]

\[
\times \Gamma^2(s_1 + \lambda_1 - 2\lambda_2 + 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + 2)
\]

\[
\times \Gamma^2(2t + s_1 + 2\lambda_1 - \lambda_2 + \ell)
\]

\[
= -2^{4t - \ell + 7} \cdot (\sqrt{-1})^\lambda_2 \cdot \Gamma^2(2t + \lambda_1 - \lambda_2 - \ell) \Gamma^2(2t + \lambda_1 + \lambda_2 + \ell - 2) \Gamma^2(2t + \lambda_1 + \lambda_2 + \ell - 2).
\]

Here the last equality follows from the Mellin inversion formula. Note that \( \Gamma^2(2s) = 2^{s-1} \Gamma^2(s) \). This completes the proof. \( \square \)

4.4. Algebraicity of critical values. We keep the notation and assumptions as in §4.2. Let \( S \) be a finite set of places containing \( S_\infty \) such that \( \Pi'_v \) and \( \Sigma_v \) are unramified for \( v \notin S_\infty \). Let \( L^S(s, \Sigma \times \Pi') \) be the partial \( L \)-function with respect to \( S \). In the following theorem, we prove algebraicity of the critical values for \( L^S(s, \Sigma \times \Pi') \) in terms of the automorphic periods \( p^f(\Sigma) \) and \( p^f(\Pi') \).

**Theorem 4.5.** Let \( I, J \) be subsets of \( S_\infty \) defined in (4.3). Assume the following conditions are satisfied:

- \( I \cup J = S_\infty \).

---

30
\[ \kappa = |\lambda_1,v + \lambda_2,v - \ell_v| \text{ for all } v \in S_\infty. \]

- If \( F = \mathbb{Q} \), then \( \kappa \neq 2 \) or \( \omega_{\Sigma'\Pi'} \neq |\frac{u+w}{2}| \).

Let
\[ p(\Sigma \times \Pi') = |D \xi|^{1/2} \cdot (2\sqrt{-1})^{\sum_{v \in S_\infty} (3\lambda_1,v + \lambda_2,v + 5u + 5w')/2} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in I} \lambda_v + \sum_{v \in J} (\lambda_v + \ell_v)} \times G(\omega_{\Sigma\Pi'}) \cdot p''(\Sigma) \cdot p''(\Pi') \]

1. Let \( m = \frac{u + w'}{2} \) be the rightmost critical point. We have
   \[ \sigma \left( \frac{L^S(m, \Sigma \times \Pi')}{(2\sqrt{-1})^{4dm} \cdot p(\Sigma \times \Pi')} \right) = \frac{L^S(m, \sigma \Sigma \times \Pi')}{(2\sqrt{-1})^{4dm} \cdot p'(\sigma \Sigma \times \sigma \Pi')} \]
   for all \( \sigma \in \text{Aut}(\mathbb{C}) \). In particular, we have
   \[ \frac{L^S(m, \Sigma \times \Pi')}{(2\sqrt{-1})^{4dm} \cdot p(\Sigma \times \Pi')} \in \mathbb{Q}(\Sigma)\mathbb{Q}(\Pi') \mathbb{Q}(l). \]

2. Assume \( \Sigma_v \) and \( \Pi'_v \) are discrete series representations for all \( v \in S_\infty \). Then the assertion in (1) holds for all critical points.

**Proof.** We begin with the first assertion. Let \( \psi = \psi_F \) be the standard additive character of \( F \) and write \( T = S \setminus S_\infty \). By Lemma 4.3 there exist \( W^\sigma_{\Sigma,\Pi} \in W(H'_\Sigma, \psi) \), \( W_{\Sigma} \in W(S_{\Sigma}, \psi_{\Pi,\Sigma}) \), and a rational section \( f^{(s)}_{\Sigma} \) of \( I(\omega_{\Sigma',\Pi'}^{-1}, \omega_{\Pi'}^{-1}, s) \) holomorphic for \( \text{Re}(s) > -\frac{u+w'}{2} \) such that
\[ Z(f^{(s)}_{\Sigma}, W^\sigma_{\Sigma,\Pi}, W_{\Sigma}) = 1. \]

Moreover, combine with (4.1), for all \( \sigma \in \text{Aut}(\mathbb{C}) \) we have
\[ Z(\sigma f^{(s)}_{\Sigma}, W^\sigma_{\Sigma,\Pi}, W_{\Sigma}) = \frac{\sigma(G(\omega_{\Pi'}))}{G(\omega_{\Pi'})}. \]

Let \( \varphi' \in V^+_\Sigma, \varphi \in V^+_\Pi \), and \( f^{(s)} \) the rational section of \( I(\omega_{\Sigma',\Pi'}^{-1}, f^{(s)}, s) \) defined so that
\[ W^{(\Sigma)}_{\varphi'} = \prod_{v \in S} W^\circ_{v, \varphi v} \cdot W_T, \quad W^{(\Pi)}_{\varphi} = \prod_{v \in S} W^\circ_{v, \varphi v} \cdot W_T, \quad f^{(s)} = \bigotimes_{v \in S} f^{(s)}_{v, \varphi v} \otimes f^{(s)}_{\Sigma,\Pi}. \]

For \( v \in S_\infty \), let \( \epsilon_v = 1 \) if \( v \in I \) and \( \epsilon_v = 1 \) otherwise. By Proposition 4.1 we have
\[ Z \left( \begin{array}{c} E^{[s]}(f^{(s)}), \varphi' \end{array} \right)^J \cdot \text{pr}_{\Sigma - \Delta_v} \left( \begin{array}{c} \varphi \end{array} \right)^J = |D \xi|^{-2} \cdot \frac{\sqrt{-1}}{2^{2dm}} \cdot \frac{L^S(m, \Sigma \times \Pi')}{L^S(m, \Sigma' \times \Pi')} \times \prod_{v \in S_\infty} \left( \rho(\text{diag}(\epsilon_v, 1)) f^{(s)}_{v, \varphi v} \right)^v \cdot W^+_{\Sigma,\Pi}(u, \ell_v - \lambda_v) \bigg|_{s = m}. \]

By Lemma 4.3 up to a non-zero rational number, the above product of archimedean local zeta integrals is equal to
\[ \pi^{-3dm - \sum_{v \in S_\infty} (3\lambda_1,v + \lambda_2,v + 3u + 3w - 4)/2} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in I} \lambda_v + \sum_{v \in J} (\lambda_v + \ell_v)} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v}. \]

Also note that \( \zeta(2) \in \pi^d \cdot \mathbb{Q}^\times \). Therefore we can rewrite the above equality as follows:
\[ Z \left( \begin{array}{c} E^{[s]}(f^{(s)}), \varphi' \end{array} \right)^J \cdot \text{pr}_{\Sigma - \Delta_v} \left( \begin{array}{c} \varphi \end{array} \right)^J = C \cdot \pi^{-3dm - \sum_{v \in S_\infty} (3\lambda_1,v + \lambda_2,v + 3u + 3w - 4)/2} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot L^S(m, \Sigma \times \Pi'). \]

for some \( C \in \mathbb{Q}^\times \) depending only on \( F, S_\infty, \) and \( \Pi'_\infty \). Similarly, together with (4.9), we have
\[ Z \left( \begin{array}{c} E^{[s]}(f^{(s)}), \varphi' \end{array} \right)^J \cdot \text{pr}_{\Sigma - \Delta_v} \left( \begin{array}{c} \varphi \end{array} \right)^J = C \cdot \pi^{-3dm - \sum_{v \in S_\infty} (3\lambda_1,v + \lambda_2,v + 3u + 3w - 4)/2} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot \sigma(G(\omega_{\Pi'})) \cdot \left( \sqrt{-1} \right)^{\sum_{v \in S_\infty} \lambda_v} \cdot L^S(m, \Sigma \times \Pi'). \]

for all \( \sigma \in \text{Aut}(\mathbb{C}) \). The first assertion then follows from Proposition 4.2.
is holomorphic for \( \Re(s) \geq 1 - \frac{u+w'}{2} \). In particular \( L\left( \frac{1}{2} - \frac{u+w'}{2}, \Sigma_v \times \Pi'_v \right) \neq 0 \). Also note that (cf. [Mor14, Proposition 5.4])

\[
\sigma L(s, \Sigma_v \times \Pi'_v) = L(s, \sigma \Sigma_v \times \sigma \Pi'_v)
\]
as rational functions in \( q_v^{-s} \) for all \( s \in \Aut(\mathbb{C}) \). Therefore, assertion (1) actually holds for \( L^{(\infty)}\left( \frac{1}{2} - \frac{u+w'}{2}, \Sigma \times \Pi' \right) \). Since we have assumed \( \Sigma_v \) is a discrete series representation for all \( v \in S_x \), the functorial lift of \( \Sigma \) to \( \GL_4(\mathbb{A}_F) \) is a regular \( C \)-algebraic isobaric automorphic representation of \( \GL_4(\mathbb{A}_F) \). Together with the assumption on \( \Pi'_v \) for \( v \in S_x \), by the result of Harder and Raghuram [HR20, Theorem 7.21], we have

\[
\sigma \left( \frac{L^{(\infty)}(m-1, \Sigma \times \Pi')}{(2\pi \sqrt{-1})^{-4d} \cdot L^{(\infty)}(m, \Sigma \times \Pi')} = \frac{L^{(\infty)}(m-1, \sigma \Sigma \times \sigma \Pi')}{(2\pi \sqrt{-1})^{-4d} \cdot L^{(\infty)}(m, \sigma \Sigma \times \sigma \Pi')}
\]
for all \( \sigma \in \Aut(\mathbb{C}) \) and \( m \in \text{Crit}(\Sigma \times \Pi') \) such that \( L^{(\infty)}(m, \Sigma \times \Pi') \neq 0 \). By [Sha81, Theorem 5.2], the non-vanishing condition is satisfied for all critical points \( m \geq 1 - \frac{u+w'}{2} \). We thus conclude from (4.10) that assertion (1) holds for all right-half critical points \( m \), that is, \( m \geq \frac{1}{2} - \frac{u+w'}{2} \). For the left-half critical points, the assertion follows from the global functional equation. This completes the proof.

4.5. A period relation. We keep the notation and assumptions as in §4.2. Assume further the following conditions are satisfied:

- The functorial lift of \( \Sigma \) to \( \GL_4(\mathbb{A}_F) \) is cuspidal.
- \( \Sigma_v \) is a discrete series representation for all \( v \in S_x \).

For \( v \in S_x \), let \( \Sigma_v^{\text{hol}} \) be the holomorphic discrete series representation of \( \GSp_4(\mathbb{F}_v) \) which belongs to the \( L \)-packet containing \( \Sigma_v \). Let \( \Sigma^{\text{hol}} \) be the irreducible admissible representation of \( \GSp_4(\mathbb{A}_F) \) defined by

\[
\Sigma^{\text{hol}} = \bigotimes_{v \in S_x} \Sigma_v^{\text{hol}} \otimes \Sigma_f.
\]

Since the transfer of \( \Sigma \) to \( \GL_4(\mathbb{A}_F) \) is cuspidal, it follows from Arthur’s multiplicity formula proved by Gee and Taïbi [GT19, Theorem 7.4.1] that \( \Sigma^{\text{hol}} \) appears in the automorphic discrete spectrum of \( \GSp_4(\mathbb{A}_F) \). It then follows from the tempernedness of \( \Sigma_v^{\text{hol}} \) for all \( v \in S_x \) and the result of Wallach [Wal83, Theorem 4.3] that \( \Sigma^{\text{hol}} \) is cuspidal. Recall the period \( \Omega(\Sigma^{\text{hol}}) \in \mathbb{C}^* \) in Theorem 3.2. In this section, we establish a period relation between the periods \( p^\Omega(\Sigma) \) and \( \Omega(\Sigma^{\text{hol}}) \) in Proposition 4.10 below. The period relation is a consequence of Theorem 4.5 for \( I = \emptyset \) and a result of Morimoto [Mor18, Theorem 4.6] recalled in the following theorem.

**Theorem 4.6 (Morimoto).** Let \( \Pi' \) be an irreducible cuspidal automorphic representation of \( \GL_2(\mathbb{A}_F) \). Assume \( \Pi' \) is regular \( C \)-algebraic of weight \((\frac{3}{2}, w')\) and

\[
\lambda_{1,v} + \lambda_{2,v} + 5 \leq \ell_v \leq \lambda_{1,v} - \lambda_{2,v} + 5
\]

for all \( v \in S_x \). Let \( m \) be a critical point of \( L(s, \Sigma \times \Pi') \) such that \( m + \frac{u+w'}{2} > 2 \). For \( \sigma \in \Aut(\mathbb{C}) \), we have

\[
\sigma \left( \frac{L^{(\infty)}(m, \Sigma \times \Pi')}{(2\pi \sqrt{-1})^{-4d+m+\Sigma_{v \in S_x}(2u+(\ell_v+5w')/2)} \cdot \sqrt{-1}^{du+dw} \cdot G(\omega_{\Sigma} \omega_{\Pi'})^2 \cdot \Omega(\Sigma^{\text{hol}}) \cdot p^{\sigma}(\Pi')} = \frac{L^{(\infty)}(m, \sigma \Sigma \times \sigma \Pi')}{(2\pi \sqrt{-1})^{-4d+m+\Sigma_{v \in S_x}(2u+(\ell_v+5w')/2)} \cdot \sqrt{-1}^{du+dw} \cdot G(\sigma_{\omega_{\Sigma}} \sigma_{\omega_{\Pi'}})^2 \cdot \Omega(\sigma^{\text{hol}}) \cdot p^{\sigma}(\sigma \Pi')}.
\]

**Remark 4.7.** In the notation of [Mor18, Theorem 1], we have

\[
\Omega(\Sigma^{\text{hol}}) = \langle \Phi, \Phi \rangle, \quad p^{\sigma}(\Pi') = (2\pi \sqrt{-1})^{-\Sigma_{v \in S_x}(\ell_v+w')/2} \cdot \langle \Phi, \Phi \rangle.
\]

The result of Morimoto was proved when \( u \) is even. Proceeding similarly as in [Mor18, §6], we see that in order to extend the result to arbitrary \( u \), it suffices to generalize [Mor18 Proposition 6.2]. More precisely, we have the following period relation for arbitrary lifts on \( \GSp_4(\mathbb{A}_Q) \).

**Proposition 4.8.** Let \( \Psi \) be a \( C \)-algebraic irreducible cuspidal automorphic representation of \( \GSp_4(\mathbb{A}_Q) \) such that \( \Psi_{\Sigma} \) is a holomorphic discrete series representation. Assume there exist non-isomorphic irreducible cuspidal automorphic representations \( \Pi_1 \) and \( \Pi_2 \) of \( \GL_2(\mathbb{A}_Q) \) such that \( \omega_{\Pi_1} = \omega_{\Pi_2} = \omega_{\Psi} \) and the functorial lift of \( \Psi \) to \( \GL_4(\mathbb{A}_Q) \) is the isobaric sum \( \Pi_1 \boxplus \Pi_2 \). Let \( (\kappa_1; u) \) and \( (\kappa_2; u) \) be the weights of \( \Pi_1 \) and \( \Pi_2 \), respectively, with \( \kappa_1 \geq \kappa_2 \). For \( \sigma \in \Aut(\mathbb{C}) \), we have

\[
\sigma \left( \frac{\Omega(\Psi)}{(2\pi \sqrt{-1})^{\kappa_1} \cdot \|f_{\Pi_1}\|} = \frac{\Omega(\sigma \Psi)}{(2\pi \sqrt{-1})^{\kappa_1} \cdot \|f_{\sigma \Pi_1}\|}.
\]
Here $\Omega(\Psi) \in \mathbb{C}^\times$ is the period in Theorem 3.2.

Proof. We have the factorization of $L$-functions:

$$L(s, \Psi, \text{std} \otimes \chi) = L(s, \Pi_1 \times \Pi_2' \otimes \chi) \cdot L(s, \chi)$$

for any Hecke character $\chi$ of $\mathbb{A}^\times_f$. Here $L(s, \Pi_1 \times \Pi_2' \otimes \chi)$ is the Rankin–Selberg $L$-function of $\Pi_1 \times \Pi_2' \otimes \chi$. Assume $\chi$ has finite order with signature $-1$. Then $m = 1$ is a critical point for $L(s, \Psi, \text{std} \otimes \chi)$. For the Dirichlet $L$-function $L(s, \chi)$, we have the well-known algebraicity result that

$$\sigma \left( \frac{L^{(\infty)}(1, \chi)}{(2\pi \sqrt{-1})^2 \cdot G(\chi)} \right) = \frac{L^{(\infty)}(1, \sigma\chi)}{(2\pi \sqrt{-1})^2 \cdot G(\sigma\chi)}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Note that the condition $\Psi_{\sigma\chi}$ is a discrete series representation implies that $\kappa_1 > \kappa_2$. We have the algebraicity result proved by Shimura [Shi76, Theorem 3] that

$$\sigma \left( \frac{L^{(\infty)}(1, \Pi_1 \times \Pi_2' \otimes \chi)}{(2\pi \sqrt{-1})^2 \cdot (\sqrt{-1})^u \cdot G(\chi)^2 \cdot \|f_{\Pi_1}\|} \right) = \frac{L^{(\infty)}(1, \sigma\Pi_1 \times \sigma\Pi_2' \otimes \sigma\chi)}{(2\pi \sqrt{-1})^{k_2+2} \cdot (\sqrt{-1})^{\sigma u} \cdot G(\sigma\chi)^2 \cdot \|f_{\Pi_1}\|}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Since $\Pi_1$ and $\Pi_2 \otimes \chi^{-1}$ are non-isomorphic and $\chi$ is non-trivial, it follows that $L^{(\infty)}(1, \chi)$ and $L^{(\infty)}(1, \Pi_1 \times \Pi_2' \otimes \chi)$ are non-zero. The period relation thus follows from Theorem 3.2 by taking $\chi$ so that $\chi^2 \neq 1$. This completes the proof. ∎

Remark 4.9. When $\omega_{\Pi_1} = \omega_{\Pi_2} = 1$, and $\Pi_1$ and $\Pi_2$ have square-free levels, the period relation follows from the explicit inner product formula for Yoshida lifts established by Böcherer–Dummigan–Schulze-Pillot [BDSP12 Corollary 8.8]. Based on [Mor14], Saha proved the period relation in [Sah15, Theorem 5.1] assuming $\kappa_1 \geq 12$ and $\kappa_2 = 2$.

Proposition 4.10. Assume the following conditions are satisfied:

- The functorial lift of $\Sigma$ to $\text{GL}_4(\mathbb{A}_E)$ is cuspidal.
- $\Sigma_v$ is a discrete series representation for all $v \in S_{\infty}$.
- $\lambda_{2, v} \leq -5$ for all $v \in S_{\infty}$.

Let $\Sigma_{\text{hol}}$ be the $C$-algebraic irreducible cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_E)$ defined in (4.11). For $\sigma \in \text{Aut}(\mathbb{C})$, we have

$$\sigma \left( \frac{p^{\sigma}(\Sigma)}{D_E^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_{\infty}}(-3\lambda_{1, v} + \lambda_{2, v} - u)/2 \cdot (\sqrt{-1})^{\sum_{v \in S_{\infty}}(\lambda_{1, v} \cdot G(\omega_{\Sigma}) \cdot \Omega(\Sigma_{\text{hol}}))}} \right) = \frac{p^{\sigma}(\sigma\Sigma)}{D_E^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_{\infty}}(-3\lambda_{1, v} + \lambda_{2, v} - u)/2 \cdot (\sqrt{-1})^{\sum_{v \in S_{\infty}}(\lambda_{1, v} \cdot G(\omega_{\Sigma}) \cdot \Omega(\sigma\Sigma_{\text{hol}}))}}$$

Proof. Let $\Pi'$ be an $C$-algebraic irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_E)$ with weight $(\ell; w')$ such that

$$(4.12) \lambda_{1, v} + \lambda_{2, v} + 5 \leq \ell_v \leq \lambda_{1, v}$$

for all $v \in S_{\infty}$. We refer to [Wei09, Theorem 1.1] for the existence of $\Pi'$. Note that the rightmost critical value of $L(s, \Sigma \times \Pi')$ is non-zero by condition (4.12). Therefore, the period relation follows immediately from Theorem 4.5 (1) and Theorem 4.6. ∎

5. Period relations

We keep the notation of §3.1. The purpose of this section is to prove the period relations in Theorem 3.1. The assertions are proved based on Theorem 4.3, Proposition 4.10 and algebraicity results in the literature. Let $\Sigma$ be the Kim–Ramakrishnan–Shahidi lift of $\Pi$. Recall $\Sigma$ is regular $C$-algebraic, satisfying condition (2.13) with weight $(\lambda; u)$, and has central character $\omega_{\Pi}'$, where

$$\lambda_v = (2\kappa_v - 1, 1 - \kappa_v), \quad u = 3w$$

for $v \in S_{\infty}$. 33
5.1. Proof of Theorem 3.1 (1). In this section, we prove assertion (1) of Theorem 3.1. We assume $\kappa_v \geq 6$ for all $v \in S_x$. By Deligne’s conjecture for symmetric fourth $L$-function proved in [Che21d, Theorem 1.4], for all critical points $m \geq 1$ of $L(s, I, \text{Sym}^4 \otimes \omega_f^{-2})$ (see also [Mor21]), we have

$$\sigma \left( \frac{L^{(\infty)}(m, I, \text{Sym}^4 \otimes \omega_f^{-2})}{|D_f|^{1/2} \cdot (2\pi \sqrt{-1})^{3dn+3\sum_{w \mid \kappa_w} \kappa_w \cdot (\sqrt{-1})^{\sigma_w} \cdot \|f_f\|^3}} \right) = \frac{L^{(\infty)}(m, \sigma I, \text{Sym}^4 \otimes \omega_f^{-2})}{|D_f|^{1/2} \cdot (2\pi \sqrt{-1})^{3dn+3\sum_{w \mid \kappa_w} \kappa_w \cdot (\sqrt{-1})^{\sigma_w} \cdot \|f_f\|^3}}$$

for all $\sigma \in \text{Aut} (\mathbb{C})$. Note that the critical values $L^{(\infty)}(m, I, \text{Sym}^4 \otimes \omega_f^{-2})$ are non-zero for all critical points $m$ as we explained in §3 for symmetric sixth $L$-functions. Therefore, it follows from (3.2) and Theorem 3.2 for $L(s, \Sigma, \text{std}) = L(s, \Sigma_{\text{hol}}, \text{std})$ that

$$\sigma \left( \frac{\Omega^{(\text{hol})}}{(2\pi \sqrt{-1})^{3\sum_{v \mid \kappa_v} \kappa_v \cdot \|f_f\|^3}} \right) = \frac{\Omega^{(\text{hol})}}{(2\pi \sqrt{-1})^{3\sum_{v \mid \kappa_v} \kappa_v \cdot \|f_f\|^3}}$$

for all $\sigma \in \text{Aut} (\mathbb{C})$. Note that the assumption $\kappa_v \geq 6$ is equivalent to $\lambda_{2,v} \leq -5$ for $v \in S_x$. We thus conclude (1) of Theorem 3.1 from the above period relation and Proposition 4.10.

5.2. Proof of Theorem 3.1 (2) and (3). In this section, we prove assertions (2) and (3) of Theorem 3.1. We assume $\kappa_v \geq 3$ for all $v \in S_x$. Firstly we explain the idea of the proof. We consider the Rankin–Selberg $L$-function $L(s, \Sigma \times I')$ for some auxiliary $C$-algebraic irreducible cuspidal automorphic representation $I'$ of $GL_2(\mathbb{A}_F)$ with weight $(\ell_w, w)$ which satisfies the following conditions:

(i) $I'$ is the automorphic induction $I'_{\mathbb{K}}(\chi')$ of some Hecke character $\chi'$ of $\mathbb{A}_K$ for some CM-extension $\mathbb{K}/\mathbb{F}$.

(ii) The set $I$ in (4.3) is equal to $S_x$, that is, $1 \leq \ell_v \leq \kappa_v - 1$ for all $v \in S_x$.

(iii) We have $\kappa_v - \ell_v = \kappa_w - \ell_w$ for all $v, w \in S_x$.

(iv) If $F = \mathbb{Q}$, then $\omega_{\Sigma} \omega_f \neq \omega_{\Sigma} \omega_f$.

By conditions (ii)-(iv), the algebraicity of the rightmost critical value of $L(s, \Sigma \times I')$ can be expressed in terms of $p^{S_x} (\Sigma)$ and some fudge factors by Lemma 4.3 (1) and Theorem 4.3 (1). On the other hand, consider the functorial lift $\text{Sym}^3 (I)$ of $I$ to $GL_4(\mathbb{A}_F)$ with respect to the symmetric cube representation of $GL_2(\mathbb{C})$. Note that $\text{Sym}^3 (I)$ is cuspidal since $I$ is non-CM. Let $BC_{\mathbb{K}} (\text{Sym}^3 (I))$ be the base change lift of $\text{Sym}^3 (I)$ to $GL_4(\mathbb{A}_K)$. By condition (i) and the adjointness property between automorphic induction and base change, we have

$$L(s, \Sigma \times I') = L(s, BC_{\mathbb{K}} (\text{Sym}^3 (I))) \otimes \chi').$$

We then use results in the literature [GL16, Har21, and IST21], together with Deligne’s conjecture for symmetric cube $L$-functions of $I$ [GH93 and Che21c] to study the algebraicity of the $L$-function on the right-hand side of (5.1). As a consequence, we obtain period relation between $p^{S_x} (\Sigma)$ and $\|f_f\|^2$.

Let $K$ be a totally imaginary quadratic extension over $F$, $c \in \text{Gal}(K/F)$ the non-trivial automorphism, and $\omega_{K/F}$ the quadratic Hecke character of $\mathbb{A}_K^c$ associated to $K/F$ via class field theory. For each $v \in S_x$, fix a complex embedding $j_v$ of $K$ lying over $v$ and we identify $K_v$ with $\mathbb{C}$ via $j_v$. Then $\Phi = \{j_v, v \in S_x\}$ is a CM-type of $K$. Let $\chi$ be an algebraic Hecke character of $\mathbb{A}_K^c$. There exist $\ell = (\ell_v)_{v \in S_x} \in Z^{S_x}$ and $w' \in Z$ such that

$$\chi_v (z) = z^{(\ell_v + w')/2} (\ell_v + w')/2, \quad \ell_v \equiv w' \pmod{2}$$

for all $v \in S_x$. Let $I'_{\mathbb{F}} (\chi |_{\mathbb{A}_K^{c}})^{1/2}$ be the automorphic induction of $\chi |_{\mathbb{A}_K^{c}}^{1/2}$ to $GL_2(\mathbb{A}_F)$. It is an isobaric automorphic representation of $GL_2(\mathbb{A}_K)$ with central character $\chi |_{\mathbb{A}_K^{c}}^{1/2}$. If we assume further that $\chi \neq \chi^c$ and $\ell \in Z^{S_x}$, then $I'_{\mathbb{F}} (\chi |_{\mathbb{A}_K^{c}})^{1/2}$ is cuspidal, C-algebraic, and satisfying condition (2.5) of weight $(\ell, w')$. We have the following result on the algebraicity of critical values of Rankin–Selberg $L$-functions on $GL_4(\mathbb{A}_K) \times GL_1(\mathbb{A}_K)$, which is a special case of the results of Guerberoff and Lin [GL15, Theorem 2] and Harris [Har21, Theorem 7.1] for $n = 4$.

Theorem 5.1 (Guerberoff–Lin, Harris). Let $\Psi$ be a regular C-algebraic conjugate self-dual irreducible cuspidal automorphic representation of $GL_4(\mathbb{A}_K)$. For each $v \in S_x$, let $\{z^{a_i} \chi^{k_i} \ell_i \}_{1 \leq i \leq 4}$ be the infinity type
of $\Psi$ at $v$ arranged so that $a_{1,v} > a_{2,v} > a_{3,v} > a_{4,v}$. There exists a sequence of non-zero complex numbers $(P(\sigma(\Psi)))_{\sigma \in \text{Aut}(\mathbb{C})}$ satisfying the following property: Let $\chi$ be an algebraic Hecke character of $\mathbb{A}_K^\times$ such that

$$2a_{2,v} > 1 - \ell_v > 2a_{3,v}$$

for all $v \in S_\infty$, where $\ell$ is determined as in (5.2). For any critical point $m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ of $L(s, \Psi \otimes \chi)$, we have

$$\sigma \left( \frac{L^S(m + \frac{1}{2}, \Psi \otimes \chi)}{(2\pi - 1)^{4dm + 2d'w} \cdot G(\chi|_{A_v^2})^2 \cdot P(\Psi)} \right) = \frac{L^S(m + \frac{1}{2}, \sigma \Psi \otimes \sigma \chi)}{(2\pi - 1)^{4dm + 2d'w} \cdot G(\sigma \chi|_{A_v^2})^2 \cdot P(\sigma \Psi)}$$

for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{K}^\text{Gal})$. Here $\mathbb{K}^\text{Gal}$ is the Galois closure of $\mathbb{K}$ in $\mathbb{C}$ and $S$ is a sufficiently large set of places containing $S_\infty$.

Proof. In order to apply [GL16 Theorem 2] in our case under assumption (5.3), we need to verify that $\Psi$ can be descend to a cohomological irreducible cuspidal automorphic representation of the quasi-split unitary group $U_2(2, \mathbb{A}_F)$ and so that the descent is strong at $v \in S_\infty$ (cf. the assumption in the beginning of [GL16 § 4.3]). The existence of descent is guaranteed by Arthur’s multiplicity formula [Mok15]. By [GL16 Theorem 2] and [Har21 Theorem 7.1], we have

$$\sigma \left( \frac{L^S(m + \frac{1}{2}, \Psi \otimes \chi)}{(2\pi - 1)^{4dm + 2d'w} \cdot P(\Psi) \cdot p(\tilde{\chi}, \Phi)^2 \cdot p(\tilde{\chi}, \Phi')^2} \right) = \frac{L^S(m + \frac{1}{2}, \sigma \Psi \otimes \sigma \chi)}{(2\pi - 1)^{4dm + 2d'w} \cdot P(\sigma \Psi) \cdot p(\sigma \tilde{\chi}, \sigma \Phi)^2 \cdot p(\sigma \tilde{\chi}, \sigma \Phi')^2}$$

for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{K}^\text{Gal})$ and critical points $m + \frac{1}{2}$. Here $\tilde{\chi} = (\chi)^{-1}$, and $p(\tilde{\chi}, \Phi)$ and $p(\tilde{\chi}, \Phi')$ are the CM-periods of $\tilde{\chi}$ with respect to the CM-types $\Phi$ and $\Phi'$, respectively, defined in [Har93 § 1]. By the properties of CM-periods [Har93 Proposition 1.4 and Lemma 1.6] and [Har97 (1.10.9) and (1.10.10)], we have

$$\sigma \left( \frac{p(\tilde{\chi}, \Phi) \cdot p(\tilde{\chi}, \Phi')}{(2\pi - 1)^{4d'(w'-1)} \cdot G(\chi|_{A_v^2})} \right) = \frac{p(\sigma \tilde{\chi}, \sigma \Phi) \cdot p(\sigma \tilde{\chi}, \sigma \Phi')}{(2\pi - 1)^{4d'(w'-1)} \cdot G(\sigma \chi|_{A_v^2})}$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. Thus we can replace $p(\tilde{\chi}, \Phi)^2 \cdot p(\tilde{\chi}, \Phi')^2$ by $(2\pi - 1)^{2d'(w'-1)} \cdot G(\chi|_{A_v^2})^2$ in (5.5). \hfill \Box

Remark 5.2. As explained in [GL16 Remarks 3.7.1 and 3.7.2], in order to prove the Galois equivariance property (5.4) over $\text{Aut}(\mathbb{C})$, it suffices to refine the Galois equivariance property in [GL16 Lemma 3.3.1] from $\text{Aut}(\mathbb{C}/\mathbb{K}^\text{Gal})$ to $\text{Aut}(\mathbb{C})$.

We prove assertion (2) first. Assume $\kappa_v \geq 3$ for all $v \in S_\infty$. Assume $\mathbb{K}$ is chosen so that $\text{BC}_\mathbb{K}(\text{Sym}^3(P))$ is cuspidal. Fix an algebraic Hecke character $\eta$ of $\mathbb{A}_\mathbb{K}^\times$ such that the following conditions are satisfied:

- $\eta_\kappa(z) = z^{-3\omega}$ for all $v \in S_\infty$.
- $\eta|_{A_v^3} = \omega_H^{-3}$.

We refer to [CHT08 Lemma 4.1.4] for the existence of $\eta$. Let

$$\Psi = \text{BC}_\mathbb{K}(\text{Sym}^3(P)) \otimes \eta.$$ 

Then $\Psi$ is $C$-algebraic and conjugate-self dual. Indeed, we have $\text{Sym}^3(P)^{\vee} = \text{Sym}^3(P) \otimes \omega_H^{-3}$. Note that $\text{BC}_\mathbb{K}(\text{Sym}^3(P))^c = \text{BC}_\mathbb{K}(\text{Sym}^3(P))$. Thus we have

$$\text{BC}_\mathbb{K}(\text{Sym}^3(P))^c = \text{BC}_\mathbb{K}(\text{Sym}^3(P)) \otimes \omega_H^{-3} \circ N_{\mathbb{K}/\mathbb{R}}.$$ 

Also the infinity type of $\Psi$ at $v \in S_\infty$ is given by $\{ z^{a_v} \cdot z^{-a_v} \}_{1 \leq a_v \leq 4}$ with

$$a_{1,v} = \frac{3\kappa_v - 3 - 3\omega}{2}, \quad a_{2,v} = \frac{\kappa_v - 1 - 3\omega}{2}, \quad a_{3,v} = \frac{1 - \kappa_v - 3\omega}{2}, \quad a_{4,v} = \frac{3 - 3\kappa_v - 3\omega}{2}.$$

Fix $\ell \in \mathbb{Z}^{S_\infty}$ and $w' \in \mathbb{Z}$ such that the following conditions are satisfied:

- $\ell_v \equiv w' \mod 2$ for all $v \in S_\infty$.
- $1 \leq \ell_v \leq \kappa_v - 2$ for all $v \in S_\infty$.
- $\kappa_v - \ell_v = \kappa_w - \ell_w$ for all $v, w \in S_\infty$.

Let $\chi$ be an algebraic Hecke character of $\mathbb{A}_\mathbb{K}^\times$ such that $\chi_v$ is given by (5.2) for $v \in S_\infty$ and $\chi \neq \chi^c$. Let $P' = \text{I}_F(\chi|_{A_v^2})^{S_\infty}$. We assume further that $\omega_H \omega_H^{-1} \neq |_{A_v^3}$ if $F = \mathbb{Q}$. By (5.1), we have

$$L(s, \Sigma \times P') = L(s + \frac{1}{2}, \Psi \otimes \eta^{-1} \chi).$$
Let $m \in \mathbb{Z}$ be the rightmost critical point of $L(s, \Sigma \times H')$. The second assumption on $\nabla$ implies that the Rankin–Selberg $L$-function has more than one critical point. In particular, we have $L^{(\psi)}(m, \Sigma \times H') \neq 0$. It is clear that the pair $(\Psi, \eta^{-1}\chi)$ satisfies condition (5.3). By Lemma 2.5 (1), Theorem 4.5 (1), and Theorem 5.1 applied to the rightmost critical value $L^{(\psi)}(m + \frac{1}{2}, \Psi \otimes \eta^{-1}\chi)$, we obtain the following period relation:

$$
(5.6) \quad \sigma \left( \frac{p^{S(x)}(\Sigma)}{|D_F|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\omega_H)^3 \cdot P(\Psi)} \right) = \frac{p^{S(x)}(\sigma \Sigma)}{|D_F|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\sigma \omega_H)^3 \cdot P(\sigma \Psi)}
$$

for all $\sigma \in \text{Aut}(\mathbb{C}/K_{\text{Gal}})$. On the other hand, we have

$$
L(s, \Psi \otimes \eta^{-1}) = L(s, \Pi, \text{Sym}^3) \cdot L(s, \Pi, \text{Sym}^3 \otimes \omega_{K/F}).
$$

By Deligne’s conjecture for symmetric cube $L$-function proved in [GH93, Theorem 6.2] and [Cle21, Theorem 1.6], together with Theorem 5.1 applied to the rightmost critical point $L(s, \Psi \otimes \eta^{-1})$, we obtain the following period relation:

$$
(5.7) \quad \sigma \left( \frac{P(\Psi)}{(2\pi \sqrt{-1})^{\sum_{v \in S_K} 4\kappa_v \cdot \|f_H\|^4}} \right) = \frac{P(\sigma \Psi)}{(2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\omega_H)^3 \cdot \|f_H\|^4}
$$

for all $\sigma \in \text{Aut}(\mathbb{C}/K^F_{\text{Gal}})$. It follows from the period relations (5.6) and (5.7) that

$$
(5.8) \quad \sigma \left( \frac{p^{S(x)}(\Sigma)}{|D_F|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\omega_H)^3 \cdot \|f_H\|^4} \right) = \frac{p^{S(x)}(\sigma \Sigma)}{|D_F|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\sigma \omega_H)^3 \cdot \|f_H\|^4}
$$

for all $\text{Aut}(\mathbb{C}/K_{\text{Gal}})$. Let $K'/F$ be another CM-extension such that $BC_{K'}(\text{Sym}^3(\Pi))$ is cuspidal and $K_{\text{Gal}} \cdot Q(\Pi) \cap (K')_{\text{Gal}} \cdot Q(\Pi) = F_{\text{Gal}} \cdot Q(\Pi)$. Then (5.8) also holds for $\text{Aut}(\mathbb{C}/(K')_{\text{Gal}})$. In particular, we have

$$
(5.9) \quad \frac{p^{S(x)}(\Sigma)}{|D_E|^{1/2} \cdot (2\pi \sqrt{-1})^{\sum_{v \in S_K} (\kappa_v + 4 - 3\omega)/2} \cdot (\sqrt{-1})^d \cdot G(\omega_H)^3 \cdot \|f_H\|^4} \in F_{\text{Gal}} \cdot Q(\Pi).
$$

We assume further that $F_{\text{Gal}} \cdot Q(\Pi) \cap K_{\text{Gal}} = F_{\text{Gal}}$. Then it is easy to see that (2) of Theorem 3.1 follows from (5.8) and (5.9).

Now we prove assertion (3). Assume $\kappa_v = \kappa_v \geq 3$ for all $v, w \in S_K$. We also assume $K$ is chosen so that $BC_{K}(\text{Sym}^3(\Pi))$ is cuspidal. We have the factorization of twisted exterior square $L$-function of $BC_{K}(\text{Sym}^3(\Pi))$ by $\omega_H^{-3} \circ N_{K/F}$:

$$
L(s, BC_{K}(\text{Sym}^3(\Pi)), \lambda^2 \otimes \omega_H^{-3} \circ N_{K/F}) = L(s, BC_{K}(\Pi), \text{Sym}^4 \otimes \omega_H^{-2} \circ N_{K/F} \cdot \rho_K(s)).
$$

Thus $L(s, BC_{K}(\text{Sym}^3(\Pi)), \lambda^2 \otimes \omega_H^{-3} \circ N_{K/F})$ has a pole at $s = 1$. Let $\chi$ be a finite order Hecke character of $A_K$, such that $\chi \neq \chi^c$. Let $m + \frac{1}{2}$ be the rightmost critical point of $L(s, BC_{K}(\text{Sym}^3(\Pi)))$. Note that $L(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\Pi))) \neq 0$ by the assumption $\min_{v \in S_K} \{\kappa_v\} \geq 3$. By the result of Jiang, Sun, and Tian [JST21, Theorem 1.1], we have

$$
(5.10) \quad \sigma \left( \frac{L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\Pi)) \otimes \chi)}{G(\chi | \lambda^2) \cdot L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\Pi)))} \right) = \frac{L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\Pi)) \otimes \chi)}{G(\chi | \lambda^2) \cdot L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\sigma \Pi)))}
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. On the other hand, by Deligne’s conjecture for symmetric cube $L$-function, we have

$$
(5.11) \quad \sigma \left( \frac{L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\Pi)))}{(2\pi \sqrt{-1})^{4dm + 2d + \sum_{v \in S_K} (4\kappa_v + 6\omega) \cdot G(\omega_H)^6 \cdot \|f_H\|^4}} \right) = \frac{L^{(\chi)}(m + \frac{1}{2}, BC_{K}(\text{Sym}^3(\sigma \Pi)))}{(2\pi \sqrt{-1})^{4dm + 2d + \sum_{v \in S_K} (4\kappa_v + 6\omega) \cdot G(\sigma \omega_H)^6 \cdot \|f_H\|^4}}.
$$
for all $\sigma \in \text{Aut}(\mathbb{C})$. Let $II' = I_\Sigma^I(\chi | \phi^{1/2})$. Then $II'$ is $C$-algebraic of weight $(1 \cdot 1)$. We assume further that $\chi$ is chosen so that $\omega_1 I_\Sigma' \neq | \omega_{3I+1}$ if $F = \mathbb{Q}$. In particular, the assumptions in Theorem 1.3 are satisfied for the pair $(\Sigma, II')$ with $I = S_\Sigma$. We then easily verify that assertion (3) of Theorem 3.1 follows from (5.10), (5.11). Lemma (1, and Theorem 4.5). This completes the proof of Theorem 3.1.

REFERENCES

[Art13] J. Arthur. The endoscopic classification of representations: orthogonal and symplectic groups, volume 61 of Colloquium Publications. American Mathematical Society, 2013.

[BDS12] S. Böcherer, N. Dummigan, and R. Schulze-Pillot. Yoshida lifts and Selmer groups. J. Math. Soc. Japan, 64(4):1353–1405, 2012.

[BG14] K. Buzzard and T. Gee. The conjectural connections between automorphic representations and Galois representations. In Automorphic Forms and Galois Representations, Volume 1, volume 414 of London Mathematical Society Lecture Note Series, pages 135–187. Cambridge University Press, 2014.

[BHR94] D. Blasius, M. Harris, and D. Ramakrishnan. Cohomology, limits of discrete series, and Galois conjugation. Duke Math. J., 73(3):647–685, 1994.

[Blau06] D. Blasius. Hilbert modular forms and the Ramanujan conjecture. In Noncommutative geometry and number theory, volume E37 of Aspects of Mathematics, pages 35–56. Friedr. Vieweg, Wiesbaden, 2006.

[BLGG11] T. Barnet-Lamb, T. Gee, and D. Geraghty. The Sato–Tate conjecture for Hilbert modular forms. J. Amer. Math. Soc., 24(2):411–469, 2011.

[BS00] S. Böcherer and C.-G. Schmidt. $p$-adic measures attached to Siegel modular forms. Ann. de l’Institut Fourier, 50(5):1375–1443, 2000.

[BW00] A. Borel and N. Wallach. Continuous cohomology, discrete subgroups, and representations of reductive groups, second edition, volume 67 of Mathematical Surveys and Monographs. Amer. Math. Soc., 2000.

[BZ76] I. N. Bernstein and A. V. Zelevinsky. Representations of the group $GL(n, F)$ where $F$ is a non-archimedean local field. Russian Math. Surveys, 31(3):1–68, 1976.

[Che21a] S.-Y. Chen. Algebraicity of the central critical values of twisted triple product $L$-functions. Ann. Math. QuÆ., 2021. DOI:10.1007/s40316-021-00169-3.

[Che21b] S.-Y. Chen. Algebraicity of critical values of adjoint $L$-functions for $GSp_4$. 2021. arXiv:2102.11197

[Che21c] S.-Y. Chen. Algebraicity of critical values of triple product $L$-functions in the balanced case. 2021. arXiv:2108.02111

[Che21d] S.-Y. Chen. On Deligne’s conjecture for symmetric fourth $L$-functions of Hilbert modular forms. 2021. arXiv:2101.07507

[CHT08] L. Clozel, M. Harris, and R. Taylor. Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations. Publ. Math. Inst. Hautes Études Sci., 108:1–181, 2008.

[Clo90] L. Clozel. Motifs et Formes Automorphes: Applications du Principe de Fonctorialité. In Automorphic Forms, Shimura Varieties, and $L$-functions, Vol. I, Perspectives in Mathematics, pages 77–159, 1990.

[CT17] L. Clozel and J. Thorne. Level-raising and symmetric power functoriality, III. Duke Math. J., 166(2):325–402, 2017.

[Del79] P. Deligne. Valeurs de fonctions $L$. Ann. of Math., 19(II):431–452, 1979.

[GPSR87] S. Gelbart, I. I. Piatetski-Shapiro, and S. Rallis. Part B: Automorphic Forms and Galois Representations, Volume 33, Part 2 of Aspects of Mathematics, pages 77–159, 1990.

[GL16] L. Guerberoff and J. Lin. Galois equivariance of critical values of $L$-functions. Amer. J. Math., 138(1):107–144, 2016.

[GL21] H. Grobner and J. Lin. Special values of $L$-functions and the refined Gan-Gross-Prasad conjecture. Amer. J. Math., 2021. To appear.

[GPSR87] S. Gelbart, I. I. Piatetski-Shapiro, and S. Rallis. Part B: Automorphic Forms and Galois Representations, Volume 33, Part 2 of Aspects of Mathematics, pages 77–159, 1990.

[Gro87] H. Grobner. Rationality results for the exterior and the symmetric square $L$-function (with an appendix by Nadir Matringe). Math. Ann., 370:1639–1679, 2018.

[GT11] W. T. Gan and S. Takeda. The local Langlands conjecture for GSp(4). Ann. of Math., 173:1841–1882, 2011.

[GT19] T. Gee and O. Taibi. Arthur’s multiplicity formula for GSp4 and restriction to Sp4. J. Éc. polytech. Math., 6:499–535, 2019.

[Har81] M. Harris. Special values of zeta functions attached to Siegel modular forms. Ann. Sci. de l’Ecole Norm. Superieure, 14(1):77–120, 1981.

[Har84] M. Harris. Eisenstein series on Shimura varieties. Ann. of Math., 119(1):59–94, 1984.

[Har85] M. Harris. Arithmetic vector bundles and automorphic forms on Shimura varieties. I. Invent. Math., 82:151–189, 1985.

[Har89] M. Harris. Period invariants of Hilbert modular forms. I: Trilinear differential operators and $L$-functions. In Cohomology of Arithmetic Groups and Automorphic Forms, volume 1447 of Lecture Notes in Mathematics, pages 155–202. Springer-Verlag, 1989.

[Har90a] M. Harris. Automorphic forms and the cohomology of vector bundles on Shimura varieties. In Automorphic Forms, Shimura Varieties, and $L$-functions, Vol. II, Perspectives in Mathematics, pages 41–91, 1990.

[Har90b] M. Harris. Automorphic forms of $\mathfrak{g}$-cohomology type as coherent cohomology classes. J. Differential Geom., 32:1–63, 1990.
[Stu89] J. Sturm. Evaluation of the symmetric square at the near center point. Amer. J. Math., 111(4):585–598, 1989.

[Su18] J. Su. Coherent cohomology of Shimura varieties and automorphic forms. 2018. arXiv:1810.12056.

[Tat79] J. Tate. Number theoretic background. In Automorphic forms, representations, and L-functions, volume 33, Part 2 of Proceedings of Symposia in Pure Mathematics, pages 3–26. Amer. Math. Soc., 1979.

[Wal84] N. Wallach. On the constant term of a square integrable automorphic form. In Operator algebras and group representations, volume 18 of Monographs and Studies in Mathematics, pages 227–237, 1984.

[Wei09] J. Weinstein. Hilbert modular forms with prescribed ramification. Int. Math. Res. Not., pages 1388–1420, 2009.

[Yos94] H. Yoshida. On the zeta functions of Shimura varieties and periods of Hilbert modular forms. Duke Math. J., 75(1):121–191, 1994.

Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan, ROC

Email address: sychen0626@gate.sinica.edu.tw