Generating a GHZ state in $2m$-qubit spin network

M A Jafarizadeh$^{1,2,3,4}$, R Sufiani$^{1,3}$, S F Taghavi$^{3}$, E Barati$^{5}$, F Eghbalifam$^{1}$ and M Azimi$^{1}$

1 Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran
2 Center of Excellence for Photonic, University of Tabriz, Tabriz 51664, Iran
3 Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran
4 Research Institute for Fundamental Sciences, Tabriz 51664, Iran
5 Institute of Physical Chemistry, Polish Academy of Sciences, Kasprzaka 44/52, PL-01-224 Warszawa, Poland
E-mail: jafarizadeh@tabrizu.ac.ir, sofiani@tabrizu.ac.ir, f.eghbali87@ms.tabrizu.ac.ir and m.azimi87@ms.tabrizu.ac.ir

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Abstract. We consider a pure $2m$-qubit initial state to evolve under a particular quantum mechanical spin Hamiltonian, which can be written in terms of the adjacency matrix of the Johnson network $J(2m, m)$. Then, by using some techniques such as spectral distribution and stratification associated with the graphs employed in Jafarizadeh and Sufiani (2008 Phys. Rev. A 77 022315), a maximally entangled GHZ state is generated between the antipodes of the network. In fact, an algorithm is given for determining the suitable coupling strengths of the Hamiltonian, so that a maximally entangled state can be generated between antipodes of the network. By using some known multipartite entanglement measures, the amount of entanglement of the final evolved state is calculated, and finally two examples of four-qubit and six-qubit states are considered in detail.

Keywords: spin chains, ladders and planes (theory), entanglement in extended quantum systems (theory), quantum transport in one dimension, optimization over networks

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1. Introduction

The idea to use quantum spin chains for short distance quantum communication was put forward by Bose [3]. After the work of Bose, the use of spin chains [4]–[18] and harmonic chains [19] as quantum wires has been proposed. In the previous work [1], the so-called distance-regular graphs have been considered as spin networks (in the sense that, with each vertex of a distance-regular graph, a qubit or a spin was associated) and perfect state transfer (PST) of a single-qubit state over antipodes of these networks has been investigated. In that work, a procedure for finding suitable coupling constants in some particular spin Hamiltonians has been given so that perfect and optimal transfer of a quantum state between antipodes of the corresponding networks can be achieved, respectively.

Entanglement is one of the other important tasks in quantum communication. Quantum entanglement in spin systems is an extensively studied field in recent years [20]–[23], with the advent of a growing realization that entanglement can be a resource for quantum information processing. Within this general field, entanglement of spin-1/2 degrees of freedom, qubits, has been in focus for the obvious reason of their paramount importance for quantum computers, not to mention their well-known applicability in various condensed-matter systems, optics and other branches of physics. In [24], the authors attempted to generate a Bell state between distant vertices in a permanently coupled spin network interacting via invariant stratification graphs (ISGs). At the first step, they established an upper bound over achievable entanglement between the reference site and the other vertices. Due to this upper bound they found that creation of a Bell state between the reference site and a vertex is possible if the stratum of that vertex is
a single element, e.g. antipodal ISGs. The present work focuses on the provision of the GHZ state, by using a 2\(m\)-qubit initial product state. To this end, we will consider the Johnson networks \(J(2m, m)\) (which are distance regular) as spin networks. Then, we use the algebraic properties of these networks in order to find suitable coupling constants in some particular spin Hamiltonians so that a 2\(m\)-qubit GHZ state can be achieved.

The organization of this paper is as follows. In section 2, we review some preliminary facts about graphs and their adjacency matrices, spectral distribution associated with them; in particular, some properties of the networks derived from symmetric group \(S_n\) called Johnson networks are reviewed. Section 3 is devoted to 2\(m\)-qubit GHZ state provision by using algebraic properties of Johnson network \(J(2m, m)\), where a method for finding suitable coupling constants in particular spin Hamiltonians is given so that maximal entanglement in a final state is possible. The paper ends with a brief conclusion and an appendix.

2. Preliminaries

2.1. Graphs, their adjacency matrices and stratification

A graph is a pair \(\Gamma = (V, E)\), where \(V\) is a non-empty set called the vertex set and \(E\) is a subset of \(\{(\alpha, \beta) : \alpha, \beta \in V, \alpha \neq \beta\}\) called the edge set of the graph. Two vertices \(\alpha, \beta \in V\) are called adjacent if \((\alpha, \beta) \in E\), and in that case we write \(\alpha \sim \beta\). For a graph \(\Gamma = (V, E)\), the adjacency matrix \(A\) is defined as

\[
(A)_{\alpha, \beta} = \begin{cases} 
1 & \text{if } \alpha \sim \beta \\
0 & \text{otherwise.} 
\end{cases} \tag{2.1}
\]

Obviously, (i) \(A\) is symmetric, (ii) an element of \(A\) takes a value in \(\{0, 1\}\) and (iii) a diagonal element of \(A\) vanishes. Conversely, for a non-empty set \(V\), a graph structure is uniquely determined by such a matrix indexed by \(V\). The degree or valence of a vertex \(\alpha \in V\) is defined by

\[
\kappa(\alpha) = |\{\beta \in V : \beta \sim \alpha\}| \tag{2.2}
\]

where \(|\cdot|\) denotes the cardinality. The graph is called regular if the degree of all of the vertices is the same. In this paper, we will assume that graphs under discussion are regular. A finite sequence \(\alpha_0, \alpha_1, \ldots, \alpha_n \in V\) is called a walk of length \(n\) (or of \(n\) steps) if \(\alpha_{i-1} \sim \alpha_i\) for all \(i = 1, 2, \ldots, n\). Let \(W\) denote the vector space over \(C\) consisting of column vectors whose coordinates are indexed by vertex set \(V\) of \(\Gamma\), and whose entries are in \(C\) (i.e. \(W = C^n\), with \(n = |V|\)). For all \(\beta \in V\), let \(|\beta\rangle\) denote the element of \(W\) with a 1 in the \(\beta\)th coordinate and 0 in all other coordinates. Then, \(\{|\beta\rangle : \beta \in V\}\) becomes a complete orthonormal basis of \(W\). The adjacency matrix is considered as an operator acting on \(W\) in such a way that

\[
A|\beta\rangle = \sum_{\alpha \sim \beta} |\alpha\rangle. \tag{2.3}
\]

Now, we recall the notion of stratification for a given graph \(\Gamma\). To this end, let \(\vartheta(\alpha, \beta)\) be the length of the shortest walk connecting \(\alpha\) and \(\beta\) for \(\alpha \neq \beta\). By definition \(\vartheta(\alpha, \alpha) = 0\) for all \(\alpha \in V\). The graph becomes a metric space with the distance function \(\vartheta\). Note that
∂(α, β) = 1 if and only if α ∼ β. We fix a vertex \( o \in V \) as an origin of the graph, called the reference vertex. Then, the graph \( \Gamma \) is stratified into a disjoint union of strata (with respect to the reference vertex \( o \)) as

\[
V = \bigcup_{i=0}^{\infty} \Gamma_i(o), \quad \Gamma_i(o) := \{\alpha \in V : \partial(\alpha, o) = i\}.
\]

Note that \( \Gamma_i(o) = \emptyset \) may occur for some \( i \geq 1 \). In that case we have \( \Gamma_i(o) = \Gamma_{i+1}(o) = \cdots = \emptyset \). With each stratum \( \Gamma_i(o) \) we associate a unit vector in \( W \) defined by

\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(o)} |\alpha\rangle,
\]

(2.5)

where \( \kappa_i = |\Gamma_i(o)| \) is called the \( i \)th valency of the graph (\( \kappa_i := |\{\gamma : \partial(o, \gamma) = i\}| = |\Gamma_i(o)| \)).

One should notice that, for distance-regular graphs, the above stratification is independent of the choice of reference vertex and the vectors \( |\phi_i\rangle, i = 0, 1, \ldots, d - 1 \) form an orthonormal basis for the so-called Krylov subspace \( K_d(|\phi_0\rangle, A) \) defined as

\[
K_d(|\phi_0\rangle, A) = \text{span}\{ |\phi_0\rangle, A |\phi_0\rangle, \ldots, A^{d-1} |\phi_0\rangle \}.
\]

(2.6)

Then it can be shown that [25] the orthonormal basis \( |\phi_i\rangle \) is written as

\[
|\phi_i\rangle = P_i(A)|\phi_0\rangle,
\]

(2.7)

where

\[
P_i(A) = a_0 + a_1 A + \cdots + a_i A^i
\]

is a polynomial of degree \( i \) in indeterminate \( A \) (for more details see, for example, [25, 26]).

### 2.2. Spectral distribution associated with the graphs

Now, we recall some preliminary facts about spectral techniques used in this paper, where more details have been given in [26]–[29].

Actually the spectral analysis of operators is an important issue in quantum mechanics, operator theory and mathematical physics [30, 31]. As an example \( \mu(dx) = |\psi(x)|^2 dx \) (\( \mu(dp) = |\tilde{\psi}(p)|^2 dp \)) is a spectral distribution which is assigned to the position (momentum) operator \( \hat{X}(\hat{P}) \). The mathematical techniques such as the Hilbert space of the stratification and spectral techniques have been employed in [32, 33] for investigating a continuous time quantum walk on graphs. Moreover, in general quasi-distributions are the assigned spectral distributions of two Hermitian non-commuting operators with a prescribed ordering. For example, the Wigner distribution in phase space is the assigned spectral distribution for two non-commuting operators \( \hat{X} \) (shift operator) and \( \hat{P} \) (momentum operator) with Weyl ordering among them [34, 35]. It is well known that, for any pair \( (A, |\phi_0\rangle) \) of a matrix \( A \) and a vector \( |\phi_0\rangle \), one can assign a measure \( \mu \) as follows:

\[
\mu(x) = \langle \phi_0 | E(x)|\phi_0\rangle,
\]

(2.8)

where \( E(x) = \sum_i |u_i\rangle\langle u_i| \) is the operator of projection onto the eigenspace of \( A \) corresponding to eigenvalue \( x \), i.e.

\[
A = \int x E(x) dx.
\]

(2.9)
Then, for any polynomial $P(A)$ we have

$$P(A) = \int P(x)E(x)\,dx,$$

(2.10)

where for a discrete spectrum the above integrals are replaced by summation. Therefore, using the relations (2.8) and (2.10), the expectation value of powers of adjacency matrix $A$ over reference vector $|\phi_0\rangle$ can be written as

$$\langle \phi_0 | A^m | \phi_0 \rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \ldots.$$  

(2.11)

Obviously, the relation (2.11) implies an isomorphism from the Hilbert space of the stratification onto the closed linear span of the orthogonal polynomials with respect to the measure $\mu$. For more details about the spectral technique, see the appendix.

### 2.3. Johnson network $J(n, m)$

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of $n$, i.e. $\lambda_1 + \cdots + \lambda_m = n$. By considering the subgroup $S_m \otimes S_{n-m}$ of the symmetric group $S_n$ with $m \leq [n/2]$, one can assume the finite set $M^\lambda = S_n/S_m \otimes S_{n-m}$ with $|M^\lambda| = n!/m!(n-m)!$ as vertex set. In fact, $M^\lambda$ is the set of $(m-1)$-faces of the $(n-1)$-simplex (recall that the graph of an $(n-1)$-simplex is the complete graph with $n$ vertices denoted by $K_n$). By denoting the vertex $i$ by an $m$-tuple $(i_1, i_2, \ldots, i_m)$, the network can be partitioned to strata according to the following relations between the vertices:

$$R_k = \{(i, j) : \partial(i, j) = k\}, \quad k = 0, 1, \ldots, m$$



where $\partial(i, j)$ means the number of components that $i = (i_1, \ldots, i_m)$ and $j = (j_1, \ldots, j_m)$ are different (this is the same as the Hamming distance which is defined in coding theory). In other words, the adjacency matrices $A_k$, $k = 0, 1, \ldots, m$ are defined as

$$(A_k)_{i,j} = \begin{cases} 1 & \text{if } \partial(i, j) = k, \\ 0 & \text{otherwise } (i, j \in M^\lambda), \end{cases} \quad k = 0, 1, \ldots, m.$$  

(2.12)

As is seen from the above definition, for $k = 0$ ($\partial(i, j) = 0$ and so $i = j$) the adjacency matrix $A_0$ equals the identity matrix $I$. The network with adjacency matrices defined by (2.12) is known as the Johnson network and denoted by $J(n, m)$. As can be seen from the relations $R_k$ or equivalently (2.12), the network $J(n, m)$ has $m+1$ strata such that

$$\kappa_0 = 1, \quad \kappa_l = \binom{m}{m-l} \binom{n-m}{l}, \quad l = 1, 2, \ldots, m.$$  

(2.13)

It is well known that the Johnson network is distance-regular, and so the adjacency matrix $A_k$ can be written as a polynomial of degree $k$ in indeterminate adjacency matrix $A_1 \equiv A$, i.e. we have [25]

$$A_k = \sqrt{\kappa_k} P_k(A)$$  

(2.14)

for $k = 0, 1, \ldots, m$, where $P_k(A)$s are the same polynomials given in (2.7) and are determined via the intersection array of the graph and spectral techniques (see the appendix). We will use this property in order to evaluate the transition amplitudes between the first and the last stratum of the network.
Then, one can show that, for the purpose of maximal entanglement provision, we must have $\kappa_m = 1$ which is fulfilled if $n = 2m$, so we will consider the network $J(2m, m)$ (hereafter we will take $n = 2m$ so that we have $\kappa_m = 1$). If we stratify the network $J(2m, m)$ with respect to a given reference node $|\phi_0\rangle = |i_1, i_2, \ldots, i_m\rangle$, where $|i_1, i_2, \ldots, i_m\rangle \equiv |0 \cdots 0_{i_1} 0 \cdots 0_{i_2} 0 \cdots 0_{i_m} 0\rangle$ and $i_1 \neq i_2 \neq \cdots \neq i_m$. The unit vectors $|\phi_i\rangle$, $i = 1, \ldots, m$ are defined as

$$|\phi_1\rangle = \frac{1}{\sqrt{\kappa_1}} \left( \sum_{i_1 \neq i_1} |i_1', i_2, \ldots, i_m\rangle + \sum_{i_2 \neq i_2} |i_1, i_2', i_3, \ldots, i_m\rangle + \cdots + \sum_{i_m \neq i_m} |i_1, \ldots, i_{m-1}, i_m'\rangle \right),$$

$$|\phi_2\rangle = \frac{1}{\sqrt{\kappa_2}} \sum_{k \neq i} \sum_{i_k \neq i_k} |i_1, \ldots, i_{l-1}, i_k', i_{l+1}, \ldots, i_{k-1}, i_k', i_{k+1}, \ldots, i_m\rangle,$$

$$\vdots$$

$$|\phi_m\rangle = \frac{1}{\sqrt{\kappa_m}} \sum_{i_1 \neq i_1, \ldots, i_m \neq i_m} |i_1', i_2', \ldots, i_m'\rangle.$$ (2.15)

Due to the distance regularity of the network $J(2m, m)$, the above stratification is independent of the choice of reference node. From the equations (2.7) and (2.14), one can see that

$$|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} A_i |\phi_0\rangle.$$ (2.16)

The intersection array of the network is given by

$$b_l = (m - l)^2, \quad c_l = l^2.$$ (2.17)

Then, by using equation (A.3), the QD parameters $\alpha_i$ and $\omega_i$ are obtained as follows:

$$\alpha_l = 2l(m - l), \quad l = 0, 1, \ldots, m; \quad \omega_l = l^2(m - l + 1)^2, \quad l = 1, 2, \ldots, m.$$ (2.18)

Then, one can show that [27]

$$A |\phi_l\rangle = (l + 1)(m - l) |\phi_{l+1}\rangle + 2l(m - l) |\phi_l\rangle + l(m - l + 1) |\phi_{l-1}\rangle.$$ (2.19)

3. GHZ state generation by using quantum mechanical Hamiltonian in the network $J(2m, m)$

3.1. Introducing the Hamiltonian

The model we consider is the distance-regular Johnson network $J(2m, m)$ consisting of $N = C^m_{2m} = (2m)!/m!m!$ sites labeled by $\{1, 2, \ldots, N\}$ and diameter $m$. Then, we stratify the network with respect to a chosen reference site, say 1 (the discussion about stratification has been given in the appendix; in these particular networks, the first and the last strata possess only one node, i.e. $|\phi_0\rangle = |1\rangle$ and $|\phi_m\rangle = |N\rangle$). At time $t = 0$, a $2m$-qubit state is prepared in the first (reference) site of the network. We wish to provide a maximal quantum entanglement between the state of this site and the state of the $N$th site after a well-defined period of time, in which the corresponding network is evolved under a particular Hamiltonian.

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If the network is assumed as a spin network, in which a spin-1/2 particle is attached to each vertex (node) of the network, the Hilbert space associated with the network is given by $H = (C^2)^{\otimes 2m}$. The standard basis for an individual qubit is chosen to be $|0\rangle = |\downarrow\rangle, |1\rangle = |\uparrow\rangle$. Then we consider the Hamiltonian

$$H_s = \frac{1}{2} \sum_{1 \leq i < j \leq 2m} H_{ij}$$

(3.1)

where $H_{ij} = \sigma_i \cdot \sigma_j$ and $\sigma_i$ is a vector with familiar Pauli matrices $\sigma_i^x, \sigma_i^y$ and $\sigma_i^z$. One can easily see that the Hamiltonian (3.1) commutes with the total $z$ component of the spin, i.e. $[\sigma_{\text{total}}^z, H_s] = 0$, hence the Hilbert space $H$ decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator $\sigma_{\text{total}}^z$. So the total number of up and down spins is invariant under the action of the Hamiltonian or time evolution operator.

Now, we recall that the kets $|\cdots \uparrow_i \cdots \downarrow_j \cdots \rangle$ with $i_1, i_2, \ldots, i_{2m} \in \{1, \downarrow\}$ form an orthonormal basis for Hilbert space $H$. Then, one can easily obtain

$$H_{ij} |\cdots \uparrow_i \cdots \downarrow_j \cdots \rangle = |\cdots \uparrow_i \cdots \downarrow_j \cdots \rangle$$

and

$$H_{ij} |\cdots \uparrow_i \cdots \downarrow_j \cdots \rangle = -|\cdots \uparrow_i \cdots \downarrow_j \cdots \rangle + 2|\cdots \downarrow_i \cdots \uparrow_j \cdots \rangle. \quad (3.2)$$

Equation (3.2) implies that the action of $H_{ij}$ on the basis vectors is equivalent to the action of the operator $2P_{ij} - I$, i.e. we have

$$H_{ij} = 2P_{ij} - I$$

(3.3)

where $P_{ij}$ is the permutation operator acting on sites $i$ and $j$. So

$$\frac{1}{2} \sum_{1 \leq i < j \leq 2m} \sigma_i \cdot \sigma_j = \sum_{1 \leq i < j \leq 2m} P_{ij} - \frac{1}{2} \left(\frac{2m}{2}\right) I. \quad (3.4)$$

In fact, restriction of the operator $\sum_{1 \leq i < j \leq 2m} P_{ij}$ on the $m$-particle subspace (subspace spanned by the states with $m$ spin up) which has dimension $C^2_{2m}$ is written as the adjacency matrix $A$ of the Johnson network $J(2m, m)$, as

$$\sum_{1 \leq i < j \leq 2m} P_{ij} = A + m(m - 1)I. \quad (3.5)$$

For more details see [1]. Then we stratify the network with respect to a chosen reference site, say $|\phi_0\rangle$. At time $t = 0$, the state is prepared in the $2m$-qubit state $|\psi(t = 0)\rangle = |11\cdots 1\rangle |00\cdots 0\rangle$. Now, we consider the dynamics of the system to be governed by the Hamiltonian

$$H = \sum_{k=0}^{m} J_k P_k \left(\frac{1}{2} \sum_{1 \leq i < j \leq 2m} \sigma_i \cdot \sigma_j + \frac{m}{2} I\right), \quad (3.6)$$

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Then, by using (3.3)–(3.6), the Hamiltonian can be written as

\[ H = \sum_{k=0}^{m} J_k P_k(A) \]  

(3.7)

where \( J_k \) is the coupling strength between the reference site \(|\phi_0\rangle\) and all of the sites belonging to the \( k \)th stratum with respect to \(|\phi_0\rangle\) and \( P_k(A) \) are polynomials in terms of adjacency matrix of the Johnson network.

### 3.2. The time evolution and transition amplitudes

Now, the total system is evolved under unitary evolution operator \( U(t) = e^{-iHt} \) for a fixed time interval, say \( t \). The final state becomes

\[ |\psi(t)\rangle = \sum_{j=1}^{N} f_{jA}(t)|j\rangle \]

(3.8)

where \( N \) is the number of vertices, \(|j\rangle\)s have \( 2m \) entries inclusive \( m \) entries equal to 1 and the other entries are 0 and \(|A\rangle = |11\ldots100\ldots0\rangle \) so that \( f_{jA}(t) := \langle j|e^{-iHt}|A\rangle \).

The evolution with the adjacency matrix \( H = A \equiv A_1 \) for distance-regular networks (see section 2.1) starting in \(|\phi_0\rangle\), always remains in the stratification space. For distance-regular network \( J(2m, m) \) for which the last stratum, i.e. \(|\phi_m\rangle\), contains only one site, then maximal entanglement between the starting site \(|\phi_0\rangle \equiv |A\rangle\) and the last stratum \(|\phi_m\rangle\) (the corresponding antipodal node) is generated, by choosing suitable coupling constants \( J_k \). In fact, for the purpose of a maximally entangled GHZ state generation between the first and the last stratum of the network, we impose the constraints that the amplitudes \( \langle \phi_i|e^{-iHt}|\phi_0\rangle \) be zero for all \( i = 1, \ldots, m - 1 \) and \( \langle \phi_0|e^{-iHt}|\phi_0\rangle = f \), \( \langle \phi_m|e^{-iHt}|\phi_0\rangle = f' \) so that the final state is as the form

\[ |\psi(t)\rangle = f|11\ldots100\ldots0\rangle + f'|00\ldots011\ldots1\rangle. \]  

(3.9)

In order to evaluate the amplitudes \( f \) and \( f' \), we use the stratification and spectral distribution associated with the network \( J(2m, m) \). That is, by using equations (2.16) and (3.7), the transition amplitudes are written as

\[ \langle \phi_i|e^{-iHt}|\phi_0\rangle = \langle \phi_i|e^{-i\sum_{l=0}^{m} J_l P_l(A)}|\phi_0\rangle = \frac{1}{\sqrt{\kappa_i}} \langle \phi_0|A_i e^{-i\sum_{l=0}^{m} J_l P_l(A)}|\phi_0\rangle. \]

Let the spectral distribution of the graph be \( \mu(x) = \sum_{k=0}^{m} \gamma_k \delta(x-x_k) \) (see equation (A.7)). The Johnson network is a kind of network with a highly regular structure that has a nice algebraic description. For example, the eigenvalues of this network can be computed exactly (see, for example, the notes by Chris Godsil on association schemes [39] for the details of this calculation). Indeed, the eigenvalues of the adjacency matrix of the network \( J(2m, m) \) (that is, \( x_k \)s in \( \mu(x) \)) are given by

\[ x_k = m^2 - k(2m + 1 - k), \quad k = 0, 1, \ldots, m. \]  

(3.10)

Now, from equation (2.14), \( \langle \phi_i|e^{-iHt}|\phi_0\rangle = 0 \) implies that

\[ \sum_{k=0}^{m} \gamma_k P_l(x_k)e^{-i\sum_{l=0}^{m} J_l P_l(x_k)} = 0, \quad i = 1, \ldots, m - 1. \]
Denoting $e^{-it\sum_{i=0}^m J_i P_i(x)}$ by $\eta_k$, the above constraints are rewritten as follows:

$$\sum_{k=0}^m P_i(x_k)\eta_k \gamma_k = 0, \quad i = 1, \ldots, m - 1,$$

$$\sum_{k=0}^m P_0(x_k)\eta_k \gamma_k = f$$  (3.11)

$$\sum_{k=0}^m P_m(x_k)\eta_k \gamma_k = f'.$$

From invertibility of the matrix $P_{ik} = P_i(x_k)$ (see [4]) one can rewrite equation (3.11) as

$$\begin{pmatrix} \eta_0 \gamma_0 \\
\eta_1 \gamma_1 \\
\vdots \\
\eta_{d-1} \gamma_{d-1} \\
\eta_d \gamma_d \end{pmatrix} = P^{-1} \begin{pmatrix} f \\
0 \\
\vdots \\
0 \\
f' \end{pmatrix}. $$  (3.12)

The above equation implies that $\eta_k \gamma_k$ for $k = 0, 1, \ldots, m$ are the same as the entries in the first column of the matrix $P^{-1} = WP^t$ multiplied by $f$ and the entries in the last column multiplied by $f'$, i.e. the following equations must be satisfied:

$$\eta_k \gamma_k = \gamma_k e^{-it\sum_{i=0}^m J_i P_i(x)} = (WP^t)_{k0} f + (WP^t)_{km} f' \quad \text{for } k = 0, 1, \ldots, m,$$  (3.13)

with $W := \text{diag}(\gamma_0, \gamma_1, \ldots, \gamma_m)$. By using the fact that $\gamma_k$ and $(WP^t)_{km}$ are real for $k = 0, 1, \ldots, m$, and so we have $\gamma_k = |(WP^t)_{km}|$ and $\gamma_k = (WP^t)_{k0}$. Equation (3.7) can be rewritten as

$$\eta_k = e^{-it\sum_{i=0}^m J_i P_i(x)} = f + \sigma(k) f'$$  (3.14)

where $\sigma(k)$ is defined as

$$\sigma(k) = \begin{cases} -1 & \text{for odd } k \\ 1 & \text{otherwise.} \end{cases}$$  (3.15)

From equations (3.14) and (3.27) one can see that $\eta_0 = \eta_2 = \cdots = \eta_{2l} = f + f'$ and $\eta_1 = \eta_3 = \cdots = \eta_{2l-1} = f - f'$. By using these identities, we obtain

$$e^{-it\sum_{i=0}^m J_i P_i(x)} = e^{-it\sum_{i=0}^m J_i P_i(x_0) + J_m P_m(x_0)} = e^{-it\sum_{i=0}^m J_i P_i(x_1) + J_m P_m(x_1)}$$

$$= e^{-it\sum_{i=0}^m J_i P_i(x_2l-1) + J_m P_m(x_2l-1)},$$  (3.16)

respectively. As has shown in appendix B of [2], we have $P_m(x_i) = \pm 1$, i.e. we have $P_m(x_0) = P_m(x_2l) = +1$ and $P_m(x_1) = P_m(x_2l-1) = -1$. Then, the relations (3.16) lead respectively to the following equations:

$$e^{-it\sum_{i=0}^m J_i P_i(x_0)} = e^{-it\sum_{i=0}^m J_i P_i(x_2l)}$$

$$e^{-it\sum_{i=0}^m J_i P_i(x_1)} = e^{-it\sum_{i=0}^m J_i P_i(x_2l-1)},$$  (3.17)

with $m - 1$ unknowns $J_1, \ldots, J_{m-1}$. By choosing $m - 1$ independent equations from the relations (3.17) and solving them, one can determine $J_1, \ldots, J_{m-1}$ exactly. In section 3.4, the cases $m = 2$ and $3$ are considered in detail. As will be seen in the following sections, this method gives an algorithm for obtaining the suitable coupling constants so that the GHZ state in the final state can be achieved.

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3.3. Quantification of entanglement

One attempt to provide a computationally feasible and scalable quantification of entanglement in multipartite systems was made in [40]–[42]. For a pure $n$-qubit state $|\psi\rangle$, the so-called global entanglement is defined as

$$Q(|\psi\rangle) = 2 \left( 1 - \frac{1}{N} \sum_{i=0}^{N-1} \text{Tr}[\rho_i^2] \right)$$

(3.18)

where $\rho_i$ represents the density matrix of the $i$th qubit after tracing out all other qubits. As seen from this definition, the global entanglement can be interpreted as the average over the (bipartite) entanglements of each qubit with the rest of the system. The global entanglement for the state in equation (3.9) will be

$$Q(|\psi\rangle) = 4|f|^2|f'|^2.$$  

(3.19)

From equation (3.11), and using the facts that $P_0(x_i) = 1$, $P_m(x_i) = \pm 1$ and $\sum \gamma_{2i} = \sum \gamma_{2i-1} = \frac{1}{4}$, one can obtain

$$f = \eta_0 \sum \gamma_{2i} + \eta_1 \sum \gamma_{2i-1} = \frac{1}{2}(\eta_0 + \eta_1) = \frac{1}{2} \{e^{-i[f(0)+\sum_{i=1}^{m-1} J_i P_i(0)+J_m]}$$

$$+ e^{-i[f(0)+\sum_{i=1}^{m-1} J_i P_i(1)-J_m]} \}$$

$$f' = \eta_0 \sum \gamma_{2i} - \eta_1 \sum \gamma_{2i-1} = \frac{1}{2}(\eta_0 - \eta_1) = \frac{1}{2} \{e^{-i[f(0)+\sum_{i=1}^{m-1} J_i P_i(0)+J_m]}$$

$$- e^{-i[f(0)+\sum_{i=1}^{m-1} J_i P_i(1)-J_m]} \}.$$  

Then, one can easily obtain

$$Q(|\psi\rangle) = \sin^2 \left[ \sum_{i=1}^{m-1} J_i (P_i(0) - P_i(1)) + 2J_m \right] t$$

(3.20)

where $J_1, \ldots, J_{m-1}$ are known and determined as illustrated in the previous subsection. Therefore, as is seen from the result (3.20), the global entanglement $Q(|\psi\rangle)$ is independent of $J_0$ and varies by changing the coupling strength $J_m$. The coupling $J_m$ can be chosen suitably so that the amount of entanglement be maximum. In section 3.4, we will determine $J_m$ for some special cases for $m$ so that $Q(|\psi\rangle)$ is maximal and the maximally entangled GHZ state is generated between the first and the last stratum of the network.

Also we introduce a simple multiqubit entanglement quantifier based on the idea of bipartition and the measure negativity (which is two times the absolute value of the sum of the negative eigenvalues of the corresponding partially transposed matrix of a state $\rho$) [43]. For an arbitrary $N$-qubit state $\rho_{s_1 s_2 \ldots s_N}$, a multiqubit entanglement measure can be formulated as [44]

$$\mathcal{Q} = \frac{N}{2} \sum_{k=1}^{N/2} \mathcal{Q}_{k[N-k] N}(\rho_{s_1 s_2 \ldots s_N})$$

(3.21)

where $N$ is assumed even, otherwise $N/2$ should be replaced by $(N-1)/2$ and $\mathcal{Q}_{k[N-k] N}(\rho_{s_1 s_2 \ldots s_N})$ is the entanglement in terms of negativity between two blocks of a

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the QD parameters are given by $m$. The case $m = 2$ : $\psi(t = 0) = |1100\rangle$ and the six-qubit state (the case $m = 3$ ) $\psi(t = 0) = |111000\rangle$ in detail.

The case $m = 2$ :

From equation (2.18), for $m = 2$ ($J(4, 2)$ which is also called an octahedral network), the QD parameters are given by

\[ \alpha_1 = 2, \quad \alpha_2 = 0; \quad \omega_1 = \omega_2 = 4. \]

For the state equation (3.13), one can see that

\[
\Pi^{\prime}_{q_1 \cdots q_k|q_{k+1} \cdots q_N} = \varphi^2_{q_1 \cdots q_k|q_{k+1} \cdots q_N} - \sum_{i=1}^{N} \sum_{j=k+1}^{N} \varphi^2_{q_i|q_j}. (3.23)
\]

and

\[
\Pi_{q_1 \cdots q_m|q_{m+1} \cdots q_k|q_{k+1} \cdots q_N} = \varphi^2_{q_1 \cdots q_m|q_{m+1} \cdots q_k|q_{k+1} \cdots q_N} = 4 |f|^2 |f'|^2. (3.24)
\]

Another useful entanglement measure was introduced in [45, 46] for an $n$-qubit state $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i |i\rangle$ with even $n$ as

\[
\tau(\psi) = 2 |\chi^*(a, n)| (3.25)
\]

where

\[
\chi^*(a, n) = \sum_{i=0}^{2^n-1} \text{sgn}^*(n, i) (a_{2^i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}), (3.26)
\]

\[
\text{sgn}^*(n, i) = \begin{cases} (-1)^{N(i)} & 0 \leq i \leq 2^n-3-1 \\ (-1)^{N(i)+n} & 2^n-3 \leq i \leq 2^n-2-1 \end{cases} (3.27)
\]

where $N(i)$ is the number of the occurrences of 1 in the $n$-bit binary representation of $i$ as $i_{n-1} \cdots i_1 i_0$ (in binary representation, $i$ is written as $i = i_{n-1} 2^{n-1} + \cdots + i_1 2^1 + i_0 2^0$). For the state equation (3.13), one can see that

\[
\tau(\psi) = 2 |\chi^*(a, n)| = 2 |a_{2^m-1} a_{2^{m-2}}| = 2 |f f'|. (3.28)
\]

In order to achieve maximal entanglement (GHZ state), we should have

\[
|f| = |f'| = \frac{1}{\sqrt{2}}. (3.29)
\]

Then $Q(|\psi\rangle) = \Pi_{q_1 \cdots q_m|q_{m+1} \cdots q_k|q_{k+1} \cdots q_N} = \Pi^\prime_{q_1 \cdots q_k|q_{k+1} \cdots q_N} = \tau(\psi) = 1$.
Generating a GHZ state in $2m$-qubit spin network

Then by using the recursion relations (A.2) and (A.5), we obtain

$$Q_2^{(1)}(x) = x^2 - 2x - 4, \quad Q_3(x) = x(x - 4)(x + 2),$$

so that the Stieltjes function (see the appendix) is given by

$$G_{\mu}(x) = \frac{Q_2^{(1)}(x)}{Q_3(x)} = \frac{x^2 - 2x - 4}{x(x - 4)(x + 2)}.$$ 

Then the corresponding spectral distribution is given by

$$\mu(x) = \sum_{l=0}^{2} \gamma_l \delta(x - x_l) = \frac{1}{6}\{\delta(x - 4) + 3\delta(x) + 2\delta(x + 2)\},$$

with $x_0 = 4, x_1 = 0$ and $x_2 = -2$. In order to obtain the suitable coupling constants, we need also the eigenvalue matrix $P$ with entries $P_{ij} = P_i(x_j) = (1/\sqrt{\omega_1 \ldots \omega_l})Q_l(x_j)$. By using the recursion relations (A.2), one can obtain $P_0(x) = 1, P_1(x) = x/2$ and $P_2(x) = \frac{1}{4}(x^2 - 2x - 4)$, so that

$$P = \begin{pmatrix}
1 & 1 & 1 \\
2 & 0 & -1 \\
1 & -1 & 1
\end{pmatrix}.$$ 

Then, equation (3.17) leads to

$$e^{-2itJ_1} = e^{itJ_1}$$

which implies that

$$J_1 = \frac{2c_0 \pi}{3t}.$$ 

Now, from equation (3.20) the global entanglement is given by

$$Q(|\psi\rangle) = \sin^2 \left( \left[ \frac{4\pi c_0}{3t} + 2J_2 \right] t \right).$$

Therefore, in order to achieve maximum entanglement, the coupling $J_2$ should be chosen as

$$J_2 = \frac{-2\pi c_0}{3t} + \frac{\pi c_1}{2t} \pm \frac{\pi}{4t}.$$ 

Now, by considering $c_0 = c_1 = 0$ we obtain

$$J_1 = 0, \quad J_2 = \pm \frac{\pi}{4t}.$$ 

Also by considering $c_0 = 1, c_1 = 0$ the coupling constants will be

$$J_1 = \frac{2\pi}{3t}, \quad J_2 = \frac{-5\pi}{12t} \quad \text{or} \quad \frac{-11\pi}{12t}.$$ 

The case $m = 3$:

From equation (2.18), for $m = 3$ ($J(6,3)$), the QD parameters are given by

$$\alpha_1 = 4, \quad \alpha_2 = 4, \quad \alpha_3 = 0; \quad \omega_1 = \omega_3 = 9, \quad \omega_2 = 16.$$ 

Then by using the recursion relations (A.2) and (A.5), we obtain

$$Q_3^{(1)}(x) = x^3 - 8x^2 - 9x + 36, \quad Q_4(x) = (x^2 - 9)(x - 9)(x + 1),$$

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so that the Stieltjes function is given by

\[ G_\mu(x) = \frac{Q_4^{(1)}(x)}{Q_4(x)} = \frac{x^3 - 8x^2 - 9x + 36}{(x^2 - 9)(x - 9)(x + 1)}. \]

Then the corresponding spectral distribution is given by

\[ \mu(x) = \sum_{i=0}^{3} \gamma_i \delta(x - x_i) = \frac{1}{20} \{ \delta(x - 9) + 5\delta(x - 3) + 9\delta(x + 1) + 5\delta(x + 3) \}, \]

with \( x_0 = 9, x_1 = 3, x_2 = -1 \) and \( x_3 = -3 \). By using the recursion relations (A.2), one can obtain \( P_0(x) = 1, P_1(x) = x/3, P_2(x) = \frac{1}{12}(x^2 - 4x - 9) \) and \( P_3(x) = \frac{1}{36}(x^3 - 8x^2 - 9x + 36) \), so that

\[ P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -\frac{1}{3} & -1 \\ 3 & -1 & -\frac{1}{3} & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}. \]

Then, the relations (3.17) are

\[ e^{-it(3J_1 + 3J_2)} = e^{-it(-J_1/3 - J_2/3)}, \]

\[ e^{-it(J_1 - J_2)} = e^{-it(-J_1 + J_2)} \]

which lead to

\[ J_1 = \frac{c_1 \pi}{2t} + \frac{3\pi c_0}{10t}; \]

\[ J_2 = -\frac{c_1 \pi}{2t} + \frac{3\pi c_0}{10t}. \]

Then, from equation (3.20), the global entanglement is given by

\[ Q(|\psi\rangle) = \sin^2 \left( \left[ \frac{9\pi c_0}{5t} - \frac{\pi c_1}{t} + 2J_3 \right] t \right). \]

In order to achieve maximum entanglement, the coupling \( J_3 \) should be chosen as

\[ J_3 = -\frac{9\pi c_0}{10t} + \frac{\pi(c_1 + c_2)}{2t} \pm \frac{\pi}{4t}. \]

By considering \( c_0 = c_1 = c_2 = 0 \) we obtain

\[ J_1 = J_2 = 0, \quad J_3 = \pm \frac{\pi}{4t}. \]
4. Conclusion

A 2m-qubit initial state was prepared to evolve under a particular spin Hamiltonian, which could be written in terms of the adjacency matrix of the Johnson graph \( J(2m, m) \). By using spectral analysis methods and employing algebraic structures of the Johnson networks, such as distance regularity and stratification, a method for finding a suitable set of coupling constants in the Hamiltonians associated with the networks was given so that, in the final state, the maximal entanglement of the form GHZ state could be generated. In this work we imposed a constraint so that all amplitudes in the final state were equal to zero except to two amplitudes corresponding to the first and the final strata (any pair of antinodes of the network), where for \( J(2m, m) \) these strata contain only one vertex, then the GHZ state was generated. We hope to generalize this method to arbitrary Johnson networks \( J(n, m) \) and other various graphs, in order to investigate the entanglement of such systems by using some multipartite entanglement measures.

Appendix. Spectral distribution associated with the graphs

In this section we recall some facts about spectral techniques used in this paper. From orthonormality of the unit vectors \( |\phi_i\rangle \) given in equation (2.5) (with \( |\phi_0\rangle \) as the unit vector assigned to the reference node) we have

\[
\delta_{ij} = \langle \phi_i | \phi_j \rangle = \int_R P_i(x) P_j(x) \mu(dx). \tag{A.1}
\]

By rescaling \( P_k \) as \( Q_k = \sqrt{\omega_1 \cdots \omega_k} P_k \), the spectral distribution \( \mu \) under question will be characterized by the property of orthonormal polynomials \( \{Q_k\} \) defined recurrently by

\[
Q_0(x) = 1, \quad Q_1(x) = x, \quad xQ_k(x) = Q_{k+1}(x) + \alpha_k Q_k(x) + \omega_k Q_{k-1}(x), \quad k \geq 1. \tag{A.2}
\]

The parameters \( \alpha_k \) and \( \omega_k \) appearing in (A.2) are defined by

\[
\alpha_0 = 0, \quad \alpha_k = \kappa - b_k - c_k, \quad \omega_k \equiv \beta_k^2 = b_{k-1} c_k, \quad k = 1, \ldots, d, \tag{A.3}
\]

where \( \kappa \equiv \kappa_1 \) is the degree of the networks, and \( b_i \)s and \( c_i \)s are the corresponding intersection numbers. Following [34], we will refer to the parameters \( \alpha_k \) and \( \omega_k \) as QD (quantum decomposition) parameters (see [26]–[28], [34] for more details). If such a spectral distribution is unique, the spectral distribution \( \mu \) is determined by the identity

\[
G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \frac{1}{x-\alpha_0 - \sum_{l=0}^{d} \frac{\gamma_l}{x-x_l}} = \frac{Q^{(1)}_d(x)}{Q_{d+1}(x)} = \sum_{l=0}^{d} \frac{\gamma_l}{x-x_l}, \tag{A.4}
\]

where \( x_l \) are the roots of the polynomial \( Q_{d+1}(x) \). \( G_\mu(x) \) is called the Stieltjes/Hilbert transform of spectral distribution \( \mu \) and polynomials \( \{Q^{(1)}_k\} \) are defined recurrently as

\[
Q^{(1)}_0(x) = 1, \quad Q^{(1)}_1(x) = x - \alpha_1, \quad xQ^{(1)}_k(x) = Q^{(1)}_{k+1}(x) + \alpha_{k+1} Q^{(1)}_k(x) + \omega_{k+1} Q^{(1)}_{k-1}(x), \quad k \geq 1, \tag{A.5}
\]
respectively. The coefficients $\gamma_l$ appearing in (A.4) are calculated as
\[
\gamma_l := \lim_{x \to x_l} [(x - x_l)G_\mu(x)] .
\] (A.6)

Now let $G_\mu(z)$ is known, then the spectral distribution $\mu$ can be determined in terms of $x_l, l = 1, 2, \ldots$ and Gauss quadrature constants $\gamma_l, l = 1, 2, \ldots$ as
\[
\mu = \sum_{l=0}^{d} \gamma_l \delta(x - x_l)
\] (A.7)

(for more details see [35]–[38]).

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