A numerical method for the solution of relaxed one-sided Lipschitz algebraic inclusions

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Abstract
An existing solvability result for relaxed one-sided Lipschitz algebraic inclusions is substantially improved. This enhanced solvability result allows the design of a very robust numerical method for the approximation of a solution of the algebraic inclusion. Sharp error estimates for this method, illustrative analytic examples and a numerical example are provided.

1 Introduction and notation
The solution of nonlinear equations and inclusions is one of the fundamental problems in pure and applied mathematics. A multitude of analytical concepts for the identification and localization of solutions as well as numerical methods for their approximation have been developed that exploit characteristic features of particular types of mappings. In this paper, solutions of

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the algebraic inclusion
\[ \bar{y} \in F(x) \] (1)
with given \( \bar{y} \in \mathbb{R}^d \) are considered for the class of relaxed one-sided Lipschitz (ROSL, see below) multivalued mappings \( F \) with negative one-sided Lipschitz constant. The relatively modern ROSL property was introduced and investigated in [5] and other works of the same author. It generalizes the classical one-sided Lipschitz property and is a key criterion for the analysis of differential inclusions and numerical approximations of their solution sets (see e.g. [6]), so that algebraic inclusions of type (1) with ROSL multifunction \( F \) arise in a natural way. Moreover, the ROSL property is intimately related to the notion of metric regularity, which is discussed in [7, Chapter 3].

A solvability result for the class of multivalued mappings satisfying the ROSL property was proved in [3, Corollary 3]. It states that given an initial guess \( x \), there exists a solution \( \bar{x} \) of (1) in a closed ball centered at \( x \) with radius depending on the defect \( \text{dist}(\bar{y}, F(x)) \). A substantially improved version of this result is given in Theorem 1 below, which allows to localize a solution of (1) in a smaller ball \( B \) with \( x \in \partial B \) and thus specifies not only a distance but also a direction in which a solution is to be found (see Figure 1). If the mapping \( F \) is in addition Lipschitz continuous, then the localization of the solution can once again be improved.

This information can be used to design a very robust numerical algorithm for the approximation of a solution of (1) that uses the current state as initial guess for the improved solvability theorem and defines the next iterate as the center of the ball \( B \). Proposition 5 provides error estimates for this numerical scheme, and Example 7 shows that they are sharp for dimension \( d > 1 \). The one-dimensional case is treated separately in Proposition 9. Enhancements of the numerical method for \( L \)-Lipschitz multimaps \( F \) are briefly analyzed in Propositions 10 and 11, and a numerical example is provided.

Let \( \mathbb{R}^d \) be equipped with the Euclidean norm \( | \cdot | \) and the Euclidean inner product \( \langle \cdot, \cdot \rangle \). A closed ball with radius \( R \geq 0 \) centered at \( x \in \mathbb{R}^d \) will be denoted by \( B_R(x) = B(x, R) \). The family of nonempty compact and convex subsets of \( \mathbb{R}^d \) is denoted by \( \mathcal{CC}(\mathbb{R}^d) \), the one-sided and the symmetric
Hausdorff-distances of two sets \( A, B \in \mathcal{C}(\mathbb{R}^d) \) are defined by
\[
\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b|,
\]
\[
\text{dist}_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\},
\]
and the so-called norm of a set \( A \in \mathcal{C}(\mathbb{R}^d) \) is \( \|A\| := \max_{a \in A} |a| \). The metric projection of a point \( y \in \mathbb{R}^d \) to a set \( A \subseteq \mathcal{C}(\mathbb{R}^d) \) is the unique point \( \text{Proj}(y, A) \in A \) satisfying \( |y - \text{Proj}(y, A)| = \text{dist}(y, A) \).

Consider a multivalued mapping \( F : \mathbb{R}^d \to \mathcal{C}(\mathbb{R}^d) \). It is called upper semicontinuous (usc) at \( x \in \mathbb{R}^d \) if
\[
\text{dist}(F(x'), F(x)) \to 0 \text{ as } x' \to x,
\]
usc if it is usc at every \( x \in \mathbb{R}^d \), and \( L \)-Lipschitz with \( L \geq 0 \) if
\[
\text{dist}_H(F(x), F(x')) \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R}^d.
\]
The mapping is called relaxed one-sided Lipschitz with constant \( l \in \mathbb{R} \) (l-ROSL) if for any \( x, x' \in \mathbb{R}^d \) and \( y \in F(x) \), there exists some \( y' \in F(x') \) such that
\[
\langle y - y', x - x' \rangle \leq l|x - x'|^2.
\]

## 2 Solvability of ROSL algebraic inclusions

The following theorem is the core of this paper. It is a strongly improved version of the solvability theorem given in [3, Corollary 3], and its assumptions on the mapping \( F \) can still be weakened (see Remark 3). Its statement is illustrated in Figure 1.

**Theorem 1.** Let \( F : \mathbb{R}^d \to \mathcal{C}(\mathbb{R}^d) \) be usc and ROSL with constant \( l < 0 \), and let \( \tilde{x} \in \mathbb{R}^d \) and \( \tilde{y} \in \mathbb{R}^d \) be given. Then there exists a solution \( \bar{x} \in S_F(\tilde{y}) := \{x \in \mathbb{R}^d : \tilde{y} \in F(x)\} \) satisfying
\[
|\bar{x} - (\tilde{x} + \frac{1}{2l}(\tilde{y} - \text{Proj}(\tilde{y}, F(\tilde{x}))))| \leq -\frac{1}{2l}\text{dist}(\tilde{y}, F(\tilde{x})),
\]
and the set \( S_F(\tilde{y}) \) is closed. If \( F \) is in addition \( L \)-Lipschitz, then for any \( \bar{x} \in S_F(\tilde{y}) \),
\[
|\bar{x} - \tilde{x}| \geq \frac{1}{L}\text{dist}(\tilde{y}, F(\tilde{x})).
\]
Figure 1: Schematic illustration of Theorem 1. The solvability theorem given in [3, Corollary 3] only guarantees the existence of a solution $\tilde{x}$ of $\tilde{y} \in F(x)$ in the (blue) ball of radius $-\frac{1}{4} \text{dist}(\tilde{y}, F(\tilde{x}))$ centered at $\tilde{x}$. Theorem 1 guarantees such a solution in the (red) ball with radius $-\frac{1}{2l} \text{dist}(\tilde{y}, F(\tilde{x}))$ centered at $\tilde{x}_c = x + \frac{1}{2l}(\tilde{y} - \text{Proj}(\tilde{y}, F(\tilde{x})))$, and if $F$ is $L$-Lipschitz, it states that no solution is contained in the (black) ball of radius $\frac{1}{L} \text{dist}(\tilde{y}, F(\tilde{x}))$ centered at $\tilde{x}$.

**Lemma 2.** Let $F : \mathbb{R}^d \to C(\mathbb{R}^d)$ be usc and ROSL with constant $l < 0$. Then the inclusion $0 \in F(x)$ has a solution $\bar{x}$ with

$$|\bar{x}| \leq -\frac{1}{l} \text{dist}(0, F(0))$$

that satisfies the property

$$\langle -\text{Proj}(0, F(0)), \bar{x} \rangle \leq l |\bar{x}|^2.$$  

**Proof.** Let $y_0 := \text{Proj}(0, F(0))$ be the element with minimal norm. By the ROSL property of $F$, the mapping $\Psi : \mathbb{R}^d \to C(\mathbb{R}^d)$ given by

$$\Psi(x) := F(x) \cap \{y \in \mathbb{R}^d : \langle y - y_0, x \rangle \leq l |x|^2 \}$$

has nonempty images. By [1, Theorem 1.1.1], it is usc. Define the usc mapping $G : \mathbb{R}^d \to C(\mathbb{R}^d)$ by

$$G(x) := x + \alpha \Psi(x)$$
with some $\alpha > 0$. Take $y \in \Psi(x)$ and set $z := x + \alpha y$. Then

$$|z|^2 = |x|^2 + 2\alpha \langle y, x \rangle + \alpha^2 |y|^2 \leq |x|^2 + 2\alpha \langle y - y_0, x \rangle + 2\alpha \langle y_0, x \rangle + \alpha^2 |y|^2 \leq |x|^2 + 2\alpha l |x|^2 + 2\alpha |x| \dist(0, F(0)) + \alpha^2 |y|^2.$$  

Thus, if $R > -\frac{1}{l} \dist(0, F(0)), |x| \leq R$, and $\alpha$ is so small that $1 + 2\alpha l \geq 0$, then

$$|z|^2 \leq R^2 + 2\alpha (lR + \dist(0, F(0)))R + \alpha^2 |y|^2 < R^2 + \alpha^2 |y|^2. \quad (6)$$

As $F$ is usc,

$$M_R := \sup_{x \in B_R(0)} \|F(x)\| < \infty,$$

and there exists an $\alpha > 0$ such that $|z|^2 \leq R^2$ follows from (6). This means that for this fixed $\alpha$,

$$H(x) := G(x) \cap B_R(0) \neq \emptyset \text{ for all } x \in B_R(0),$$

and $H(\cdot)$ is also usc. By the Kakutani Theorem (see [2, Theorem 3.2.3]), $H$ and thus also $G$ have a fixed point $x_R$ in $B_R(0)$, which implies that $0 \in \Psi(x_R)$. 

In particular, we find elements $x_n \in B(0, -\frac{1}{l} \dist(0, F(0)) + 1/n)$ for all $n \in \mathbb{N}$ such that $0 \in \Psi(x_n)$. As $B(0, -\frac{1}{l} \dist(0, F(0)) + 1)$ is compact, there exists a convergent subsequence of $\{x_n\}_{n \in \mathbb{N}}$ with limit

$$\bar{x} \in B(0, -\frac{1}{l} \dist(0, F(0))).$$

Since $\Psi$ is usc,

$$0 \in \Psi(\bar{x}) \subset F(\bar{x}).$$

Property (5) follows from the construction of $\Psi$. \hfill \Box

Proof of Theorem 4. Consider the set-valued mapping

$$G(z) := F(z + \bar{x}) - \bar{y},$$

which is ROSL with constant $l$. By the above theorem, there exists some

$$\bar{z} \in B(0, -\frac{1}{l} \dist(0, G(0))) = B(0, -\frac{1}{l} \dist(\bar{y}, F(\bar{x})))$$
such that $0 \in G(\bar{z})$ or $\bar{y} \in F(\bar{x})$, where $\bar{x} = \tilde{x} + \bar{z}$. Property (5),

$$\langle - \text{Proj}(0, G(0)), \bar{z} \rangle = \langle - \text{Proj}(0, F(\bar{x}) - \bar{y}), \bar{x} - \bar{x} \rangle = \langle \bar{y} - \text{Proj}(\bar{y}, F(\bar{x})), \bar{x} - \bar{x} \rangle,$$

and

$$l|\bar{z}|^2 = l|\bar{x} - \tilde{x}|^2.$$

imply that

$$\langle \bar{y} - \text{Proj}(\bar{y}, F(\bar{x})), \bar{x} - \tilde{x} \rangle \leq l|\bar{x} - \tilde{x}|^2,$$

which is equivalent with (2).

The fact that $S_F(\bar{y})$ is closed follows directly from the usc property of $F$.

If $F$ is in addition $L$-Lipschitz and $\bar{x} \in S_F(\bar{y})$, then

$$\text{dist}(\bar{y}, F(\bar{x})) \leq \text{dist}(F(\bar{x}), F(\bar{x})) \leq L|\bar{x} - \tilde{x}|$$

implies

$$|\bar{x} - \tilde{x}| \geq \frac{1}{L} \text{dist}(\bar{y}, F(\bar{x})).$$

Remark 3. The assumptions of Theorem 1 can be weakened. In particular, the set-valued mapping $F$ may be defined only on $B := B(\tilde{x}, -\frac{1}{L} \text{dist}(\bar{y}, F(\bar{x})))$.

a) In order to obtain the existence of a solution and estimate (2), it is sufficient to require that $F : B \to CC(\mathbb{R}^d)$ is usc and that

$$\forall x \in B \exists y \in F(x) : \langle y - \text{Proj}(\bar{y}, F(\bar{x})), x - \tilde{x} \rangle \leq l|x - \tilde{x}|^2. \quad (7)$$

The mapping $F$ can then be extended as in [4, proof of Theorem 2] to a set-valued function $\bar{F} : \mathbb{R}^d \to CC(\mathbb{R}^d)$ that coincides with $F$ on $B$, is usc, and satisfies property (7) for all $x \in \mathbb{R}^d$. The proof of Theorem 1 can be applied to the mapping $\bar{F}$ without changes.

b) To show estimate (3), it is enough for $F : B \to CC(\mathbb{R}^d)$ to be $L$-Lipschitz relative to $\tilde{x}$ in the sense that

$$\text{dist}_H(F(x), F(\bar{x})) \leq L|x - \tilde{x}| \text{ for all } x \in B.$$

It follows directly that for any $\bar{x} \in S_F(\bar{y}) \cap B$ (and hence for all $\bar{x} \in S_F(\bar{y}))$

$$|\bar{x} - \tilde{x}| \geq \frac{1}{L} \text{dist}(\bar{y}, F(\bar{x})).$$

Remark 4. It is unclear if additional assumptions are needed to guarantee the connectedness of $S_F(\bar{y})$. This question is linked with the parametrization problem for ROSL multifunctions (see Lemma 12 in [3]).
3 A numerical solver for ROSL algebraic inclusions

Throughout this section, the mapping $F : \mathbb{R}^d \to \mathbb{C}^d$ will be assumed to be $l$-ROSL and $L$-Lipschitz. A numerical method for finding a solution $\bar{x}$ of the inclusion $\bar{y} \in F(x)$ can be deduced directly from Theorem 1 by defining the next iterate of the scheme as the center of the ball specified by (2).

**Proposition 5.** Let $L < -2l$, and let $x_0 \in \mathbb{R}^d$ and $\bar{y} \in \mathbb{R}^d$ be given. Then the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by

$$x_{n+1} := \Phi(x_n) := x_n + \frac{1}{2l}(\bar{y} - \text{Proj}(\bar{y}, F(x_n)))$$

converges to a solution $\bar{x}$ of the inclusion $\bar{y} \in F(x)$ and satisfies the estimates

$$\text{dist}(x_n, S_F(\bar{y})) \leq \frac{L^{n-1}}{|2l|^n} \text{dist}(\bar{y}, F(x_0))$$

and

$$|x_n - \bar{x}| \leq -\frac{1}{2l} \frac{|2l|}{1 - |2l|} \text{dist}(\bar{y}, F(x_0))$$

for $n \geq 1$.

**Proof.** Set $v_n := \bar{y} - \text{Proj}(\bar{y}, F(x_n))$ for $n \in \mathbb{N}$. Then (2) implies that there exists some $\bar{x}_n \in S_F(\bar{y})$ such that

$$\text{dist}(x_{n+1}, S_F(\bar{y})) \leq |\bar{x}_n - (x_n + \frac{1}{2l}v_n)| \leq -\frac{1}{2l}|v_n|.$$  (11)

Now

$$|v_{n+1}| = \text{dist}(\bar{y}, F(x_{n+1})) \leq \text{dist}(\bar{y}, F(\bar{x}_n)) + \text{dist}(F(\bar{x}_n), F(x_n + \frac{1}{2l}v_n))$$

$$\leq L|\bar{x}_n - (x_n + \frac{1}{2l}v_n)| \leq -\frac{L}{2l}|v_n|$$

by (11) for $n \in \mathbb{N}$, so that

$$|v_n| \leq \frac{L^n}{|2l|^n}|v_0|,$$

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and again by (11), we have
\[ \text{dist}(x_n, S_F(y)) \leq -\frac{1}{2l} |v_{n-1}| \leq \frac{L^{n-1}}{|2l|^n} |v_0| \] (12)
for \( n \geq 1 \), which shows (9). Since
\[ |x_{n+1} - x_n| \leq -\frac{1}{2l} |v_n| \leq -\frac{1}{2l} |v_0| \] (13)
for all \( n \in \mathbb{N} \), the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is Cauchy and converges to some \( \bar{x} \in \mathbb{R}^d \). As \( S_F(y) \) is closed, estimate (12) shows that \( \bar{x} \in S_F(y) \). Finally, for all \( n, N \in \mathbb{N} \) with \( N > n \), it follows from (13) that
\[ |x_N - x_n| \leq \sum_{j=n}^{N-1} |x_{j+1} - x_j| \leq -\frac{1}{2l} |v_0| \sum_{j=n}^{N-1} \frac{L^j}{|2l|^j} = -\frac{1}{2l} |v_0| \frac{L^n}{|2l|^n} \frac{1 - L^{N-n}}{1 - \frac{L}{|2l|}} \]
\[ \leq -\frac{1}{2l} |v_0| \frac{L^n}{|2l|^n} \frac{1 - L}{|2l|}. \]
Passing to the limit as \( N \to \infty \) yields (10).

**Remark 6.** By Theorem 7 any numerical iteration \( \{x_n\}_{n \in \mathbb{N}} \) will converge to \( S_F(y) \) provided that the sequence of defects \( v_n \) converges to zero. A simple modification of the proof of Proposition 5 shows that the defect at any point in the interval \( x_n + \left( \frac{1}{2l}, 0 \right) v_n \) is smaller than at \( x_n \) and that for \( r \in [0, -\frac{1}{2l}) \) and \( |x_{n+1} - (x_n + \frac{1}{2l} v_n)| \leq r \) for all \( n \in \mathbb{N} \), the algorithm still converges linearly with reduced speed. This means that even if \( l \) is unknown, it is possible to find a next iterate with smaller defect according to simple trust region strategies.

The following example shows that Proposition 5 is sharp (apart from statement (10)).

**Example 7.** Let \( l < 0 \) and \( L \geq -l \), and set \( F(x) := lx + \alpha x^\perp \), where \( \alpha := \sqrt{L^2 - l^2} \) and \( x^\perp := (x^{(2)}, -x^{(1)}) \) is the image of \( x \) under the rotation with angle \(-\pi/2\) around the origin. The single-valued mapping \( F \) is \( l \)-OSL and
L-Lipschitz. If the numerical method (8) is applied to the problem $0 = F(x)$, we have

$$\Phi(x) = x - \frac{1}{2l} F(x) = \frac{1}{2} \begin{pmatrix} 1 & -\alpha/l \\ \alpha/l & 1 \end{pmatrix} x.$$  

The eigenvalues of the above matrix are $\lambda_{1/2} = \frac{1}{2} \pm \frac{\alpha}{2l}$, i.e. the iteration converges if and only if $L < -2l$. Moreover,

$$\| \frac{1}{2} \begin{pmatrix} 1 & -\alpha/l \\ \alpha/l & 1 \end{pmatrix} \|_2 = \frac{L}{2l},$$ 

so that the iteration converges with rate $\frac{L}{2l}$ whenever $L < -2l$. In fact, it can be shown easily by using rotational symmetry of $F$ that estimate (9) is sharp for every initial state $x_0 \in \mathbb{R}^2$.

The following example shows that the condition $L < -2l$ is not sharp for convergence of the method (8) in $d = 1$.

**Example 8.** Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \begin{cases} -L + lx - 1, & 1 \leq x \\ -Lx, & -1 \leq x \leq 1 \\ +L + lx + 1, & x \leq -1 \end{cases}$$

with $l < 0$ and $L \geq -l$. Clearly, $F$ is $l$-OSL and $L$-Lipschitz. Let $x_n \in [-1, 1]$ be a state of the root finding method that is supposed to solve $0 = F(x)$. Then

$$x_{n+1} = x_n - \frac{F(x_n)}{2l} = x_n + \frac{Lx_n}{2l} = (1 + \frac{L}{2l})x_n,$$

so that $|x_{n+1}| < |x_n|$ if and only if $L < -4l$. Figure illustrates the global behavior of the function $F$ and the numerical method $\Phi$ for characteristic ratios $-L/l$.

The gap between the condition $L < -2l$ required for convergence in Proposition 5 and the condition $L < -4l$ observed in Example 8 is due to the fact that for multifunctions $F : \mathbb{R} \to CC(\mathbb{R})$, the ROSL property is much stronger than in $\mathbb{R}^d$ with $d > 1$. In this particular context, it is possible to derive estimates for some of the defects (see Case 1a in the following proof) that only depend on the one-sided Lipschitz constant $l$ and not on the Lipschitz constant $L$. 


Figure 2: Behavior of the function $F$ from Example 8 and the corresponding numerical method $\Phi$ for $l = -1$ and characteristic values of $L$. The red lines limit the central interval $[-1, 1]$ in space and image. The value $L = -4l$ is the critical threshold.
Proposition 9. Let $F : \mathbb{R} \to \mathbb{CC}(\mathbb{R})$ be $l$-ROSL and $L$-Lipschitz with $l < 0$ and $L < -4l$, and let $x_0 \in \mathbb{R}$ and $\bar{y} \in \mathbb{R}$ be given. Then the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined by

$$x_{n+1} := x_n + \frac{1}{2l}(\bar{y} - \text{Proj}(\bar{y}, F(x_n)))$$

converges to a solution $\bar{x}$ of the inclusion $\bar{y} \in F(x)$ and satisfies the estimates

$$\text{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l} \kappa^{n-1} \text{dist}(\bar{y}, F(x_0))$$

and

$$|x_n - \bar{x}| \leq -\frac{1}{2l} \frac{\kappa^n}{1 - \kappa} \text{dist}(\bar{y}, F(x_0))$$

for $n \geq 1$, where $\kappa := \max\{\frac{1}{2}, 1 + \frac{L}{2l}\}$.

Proof. Let $-2l \leq L < -4l$ and set $v_n := \bar{y} - \text{Proj}(\bar{y}, F(x_n))$ for $n \in \mathbb{N}$. Without loss of generality, $\bar{y} \notin F(x_n)$ and $\bar{y} \notin F(x_{n+1})$, because otherwise the sequences $\{v_n\}$ and $\{x_n\}$ become constant and all estimates are trivially satisfied. As $F(x_n)$ is an interval, there are only two cases.

**Case 1:** $\bar{y} > y$ for all $y \in F(x_n)$.

In particular, $v_n > 0$. If $\bar{x} \in S_F(\bar{y})$, then the ROSL property yields some $y \in F(x_n)$ such that

$$(\bar{y} - y)(\bar{x} - x_n) \leq l|\bar{x} - x_n|^2,$$

which implies $\bar{x} \leq x_n$. By Theorem [1]

$$S_n := S_F(\bar{y}) \cap [x_n + \frac{1}{l}v_n, x_n - \frac{1}{L}v_n] \neq \emptyset.$$ 

Let $\bar{x}_n := \max S_n$. Without loss of generality, $x_n \neq \bar{x}_n \neq x_{n+1}$, because otherwise the sequences $\{v_n\}$ and $\{x_n\}$ become constant. There are two subcases.

**Subcase 1a:** $\bar{x}_n \in [x_n + \frac{1}{l}v_n, x_n + \frac{1}{2l}v_n)$.

Assume that there exists some $y^* \in F(x_{n+1})$ with $\bar{y} < y^*$. Since $y < \bar{y}$ for all $y \in F(x_n)$, there exists some $x^* \in (x_{n+1}, x_n)$ with $\bar{y} \in F(x^*)$ by the set-valued intermediate value theorem (see Appendix). But then $x^* \in S_F(\bar{y})$, which contradicts the maximality of $\bar{x}_n$. Therefore,

$$\bar{y} > y \text{ for all } y \in F(x_{n+1}),$$

(16)
\[ \text{Proj}(\bar{y}, F(x_n)) = \max F(x_n), \quad \text{Proj}(\bar{y}, F(x_{n+1})) = \max F(x_{n+1}). \]

It is easy to see that if \( F \) is \( l \)-ROSL, then the single-valued function \( \max F \) is \( l \)-OSL, and hence

\[
\begin{align*}
\frac{1}{2l}v_n[\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n))] \\
= [\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n))](x_{n+1} - x_n) \\
= (\max F(x_{n+1}) - \max F(x_n)) \cdot (x_{n+1} - x_n) \\
\leq l|x_{n+1} - x_n|^2 \leq \frac{1}{4l}v_n^2,
\end{align*}
\]

which implies

\[
\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n)) \geq \frac{1}{2}v_n
\]

and thus

\[
\bar{y} - \text{Proj}(\bar{y}, F(x_n)) - \frac{1}{2}v_n \geq \bar{y} - \text{Proj}(\bar{y}, F(x_{n+1}))
\]

and

\[
\frac{1}{2}v_n \geq v_{n+1}.
\]

Since \( v_{n+1} > 0 \) by inequality (16),

\[
|v_{n+1}| \leq \frac{1}{2}|v_n|.
\]

Subcase 1b: \( \bar{x}_n \in (x_n + \frac{1}{2l}v_n, x_n - \frac{1}{L}v_n] \).

In this case,

\[
|v_{n+1}| = \text{dist}(\bar{y}, F(x_{n+1})) \leq \text{dist}(F(\bar{x}_n), F(x_{n+1})) \leq L|\bar{x}_n - x_{n+1}|
\]

\[
\leq L|(x_n - \frac{1}{L}v_n) - (x_n + \frac{1}{2l}v_n)| \leq L\left|\frac{1}{2l} + \frac{1}{L}\right| \cdot |v_n| = |1 + \frac{L}{2l}| \cdot |v_n|.
\]

Case 2: \( \bar{y} < y \) for all \( y \in F(x_n) \).

All arguments and estimates are symmetric to those in Case 1.

Summarizing Cases 1 and 2,

\[
|v_{n+1}| \leq \max\left\{\frac{1}{2}, |1 + \frac{L}{2l}|\right\}|v_n| =: \kappa|v_n|,
\]
so that by induction,
\[ |v_n| \leq \kappa^n |v_0|. \]

By estimate (2), we have
\[ \text{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}|v_n| \leq -\frac{1}{2l}\kappa^{n-1}|v_0| \] (17)
for \( n \geq 1 \), which shows (14). Since
\[ |x_{n+1} - x_n| \leq -\frac{1}{2l}|v_n| \leq -\frac{1}{2l}\kappa^n |v_0| \] (18)
for all \( n \in \mathbb{N} \), the sequence \( \{x_n\}_{n \in \mathbb{N}} \) is Cauchy and converges to some \( \bar{x} \in \mathbb{R} \). As \( S_F(y) \) is closed, estimate (17) shows that \( \bar{x} \in S_F(\bar{y}) \). Finally, for all \( N, n \in \mathbb{N} \) with \( N > n \), it follows from (18) that
\[ |x_N - x_n| \leq \sum_{j=n}^{N-1} |x_{j+1} - x_j| \leq -\frac{1}{2l}|v_0| \sum_{j=n}^{N-1} \kappa^j \]
\[ = -\frac{1}{2l}|v_0| \kappa^n \sum_{j=0}^{N-n-1} \kappa^j = -\frac{1}{2l}|v_0| \kappa^n \frac{1 - \kappa^{N-n}}{1 - \kappa} \]
\[ \leq -\frac{1}{2l}|v_0| \frac{\kappa^n}{1 - \kappa}. \]
Passing to the limit as \( N \to \infty \) yields (15).

If \( L < -2l \), then Cases 1b and 2b cannot occur, so that all estimates hold with the optimal rate \( \kappa = \frac{1}{2} \). \( \Box \)

If the Lipschitz constant \( L \) of the mapping \( F \) is known explicitly, the numerical method (8) can be refined using estimate (3) from Theorem 1. The proofs will only be sketched, because they coincide in large parts with those of the above propositions.

**Proposition 10.** If \( d > 1 \) and \( L \leq -\sqrt{2}l \), then the iteration
\[ x_{n+1} := x_n + \frac{l}{L^2}(\bar{y} - \text{Proj}(\bar{y}, F(x_n))) \]
converges to a solution \( \bar{x} \in S_F(\bar{y}) \) and satisfies
\[ \text{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}\kappa^{n-1} \text{dist}(\bar{y}, F(x_0)) \]
and
\[ |x_n - \bar{x}| \leq \frac{l}{L^2} \kappa^n \text{dist}(\bar{y}, F(x_0)), \]
where \( \kappa := \frac{\sqrt{L^2 - l^2}}{l} \).

**Sketch of proof.** Define \( S_n := B(x_n + \frac{1}{2L} v_n, -\frac{1}{2L} |v_n|) \setminus B(x_n, \frac{1}{2L} |v_n|) \). By Theorem [1] there exists some \( \bar{x}_n \in S_F(\bar{y}) \cap S_n \). By simple geometric arguments,
\[ |\bar{x}_n - x_{n+1}| \leq \text{dist}(S_n, x_{n+1}) \leq \frac{\sqrt{L^2 - l^2}}{L^2} |v_n|, \]
so that
\[ |v_{n+1}| = \text{dist}(\bar{y}, F(x_{n+1})) \leq \text{dist}(\bar{y}, F(\bar{x}_n)) + \text{dist}(F(\bar{x}_n), F(x_{n+1})) \]
\[ \leq L |\bar{x}_n - x_{n+1}| \leq \frac{\sqrt{L^2 - l^2}}{L} |v_n| =: \kappa |v_n|. \]

The case \( d = 1 \) allows more effective estimates.

**Proposition 11.** If \( d = 1 \) and \( L \leq -2l \), then the iteration
\[ x_{n+1} := x_n + \frac{1}{2L} (1 - \frac{1}{L})(\bar{y} - \text{Proj}(\bar{y}, F(x_n))) \]
converges to a solution \( \bar{x} \in S_F(\bar{y}) \) and satisfies
\[ \text{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2L} \kappa^{n-1} \text{dist}(\bar{y}, F(x_0)) \]
and
\[ |x_n - \bar{x}| \leq \frac{1}{2} (\frac{1}{L} - \frac{1}{l}) \kappa^n \text{dist}(\bar{y}, F(x_0)) \]
for \( n \geq 1 \), where \( \kappa := \frac{1}{2}(1 - \frac{L}{l}). \)

**Sketch of proof.** By Theorem [1] there exists some \( \bar{x}_n \in S_F(\bar{y}) \cap S_n \), where
\[ S_n := [x_n + \frac{1}{L} v_n, x_n] \setminus [x_n - \frac{1}{L} v_n, x_n + \frac{1}{L} v_n] = [x_n + \frac{1}{L} v_n, x_n - \frac{1}{L} v_n]. \]
Therefore,
\[ |v_{n+1}| = \text{dist}(\bar{y}, F(x_{n+1})) \leq \text{dist}(\bar{y}, F(\bar{x}_n)) + \text{dist}(F(\bar{x}_n), F(x_{n+1})) \]
\[ \leq L |\bar{x}_n - x_{n+1}| \leq \frac{L}{2} \frac{1}{L - \frac{1}{L}} |v_n| =: \kappa |v_n|. \]
The following numerical example illustrates that the algorithm indeed approximates an element of the solution set successfully for any given initial value.

**Example 12.** Consider the multivalued mapping \( F : \mathbb{R}^2 \rightarrow \mathcal{C}(\mathbb{R}^2) \) given by

\[
F(x) = -3x + A(x)Q,
\]

where

\[
A(x) = \begin{pmatrix}
\cos(|x|) & -\sin(|x|) \\
\sin(|x|) & \cos(|x|)
\end{pmatrix}
\]

and

\[
Q = \overline{\{(1,0), (0,-1), (-1,0), (0,1)\}}
\]

are a rotation matrix with angle depending on the norm of \( x \) and a square centered at the origin. It is easy to check that \( F \) is \((-2)\)-ROSL and \(3\)-Lipschitz, so that the statements of Proposition hold. The solution set \( S_F(0) \) and typical trajectories of the numerical method applied to the problem \( 0 \in F(x) \) are depicted in Figure 3.

**Appendix**

The proof of the following proposition does not differ much from that of the classical intermediate value theorem and is therefore omitted.
Proposition 13. Let $a, b \in \mathbb{R}$ with $a < b$, and let $F : [a, b] \rightarrow C^C(\mathbb{R})$ be an usc mapping such that there exists some $f_a \in F(a)$ and $f_b \in F(b)$ with $f_a < 0$ and $f_b > 0$. Then there exists some $x^* \in (a, b)$ such that $0 \in F(x^*)$.

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