Generalized Topic Modeling

Avrim Blum*  Nika Haghtalab†
avrim@cs.cmu.edu  nhaghtal@cs.cmu.edu

November 7, 2016

Abstract

Recently there has been significant activity in developing algorithms with provable guarantees for topic modeling. In standard topic models, a topic (such as sports, business, or politics) is viewed as a probability distribution $a_i$ over words, and a document is generated by first selecting a mixture $w$ over topics, and then generating words i.i.d. from the associated mixture $Aw$. Given a large collection of such documents, the goal is to recover the topic vectors and then to correctly classify new documents according to their topic mixture.

In this work we consider a broad generalization of this framework in which words are no longer assumed to be drawn i.i.d. and instead a topic is a complex distribution over sequences of paragraphs. Since one could not hope to even represent such a distribution in general (even if paragraphs are given using some natural feature representation), we aim instead to directly learn a document classifier. That is, we aim to learn a predictor that given a new document, accurately predicts its topic mixture, without learning the distributions explicitly. We present several natural conditions under which one can do this efficiently and discuss issues such as noise tolerance and sample complexity in this model. More generally, our model can be viewed as a generalization of the multi-view or co-training setting in machine learning.

*Supported in part by National Science Foundation grants CCF-1525971 and CCF-1535967.
†Supported in part by National Science Foundation grant CCF-1525971 and by a Microsoft Research Graduate Fellowship and an IBM Ph.D Fellowship.
1 Introduction

Topic modeling is an area with significant recent work in the intersection of algorithms and machine learning (Arora et al., 2012a,b, 2013; Anandkumar et al., 2012, 2014; Bansal et al., 2014). In topic modeling, a topic (such as sports, business, or politics) is modeled as a probability distribution over words, expressed as a vector $a_i$. A document is generated by first selecting a mixture $w$ over topics, such as 80% sports and 20% business, and then choosing words i.i.d. from the associated mixture distribution, which in this case would be $0.8a_{\text{sports}} + 0.2a_{\text{business}}$. Given a large collection of such documents (and some assumptions about the distributions $a_i$ as well as the distribution over mixture vectors $w$) the goal is to recover the topic vectors $a_i$ and then to use the $a_i$ to correctly classify new documents according to their topic mixtures.

Algorithms for this problem have been developed with strong provable guarantees even when documents consist of only two or three words each Arora et al. (2012b); Anandkumar et al. (2012); Papadimitriou et al. (1998). In addition, algorithms based on this problem formulation perform well empirically on standard datasets Blei et al. (2003); Hofmann (1999).

As a theoretical model for document generation, however, an obvious problem with the standard topic modeling framework is that documents are not actually created by independently drawing words from some distribution. Better would be a model in which sentences are drawn i.i.d. from a distribution over sentences (this would at least produce grammatical objects and allow for meaningful correlation among related words within a topic, like shooting a free throw or kicking a field goal). Even better would be paragraphs drawn i.i.d. from a distribution over paragraphs (this would at least produce coherent paragraphs). Or, even better, how about a model in which paragraphs are drawn non-independently, so that the second paragraph in a document can depend on what the first paragraph was saying, though presumably with some amount of additional entropy as well? This is the type of model we study here.

Note that an immediate problem with considering such a model is that now the task of learning an explicit distribution (over sentences or paragraphs) is hopeless. While a distribution over words can be reasonably viewed as a probability vector, one could not hope to learn or even represent an explicit distribution over sentences or paragraphs. Indeed, except in cases of plagiarism, one would not expect to see the same paragraph twice in the entire corpus. Moreover, this is likely to be true even if we assume paragraphs have some natural feature-vector representation. Instead, we bypass this issue by aiming to directly learn a predictor for documents—that is, a function that given a document, predicts its mixture over topics—without explicitly learning topic distributions. Another way to think of this is that our goal is not to learn a model that could be used to write a new document, but instead just a model that could be used to classify a document written by others. This is much as in standard supervised learning where algorithms such as SVMs learn a decision boundary (such as a linear separator) for making predictions on the labels of examples without explicitly learning the distributions $D_+$ and $D_-$ over positive and negative examples respectively. However, our setting is unsupervised (we are not given labeled data containing the correct classifications of the documents in the training set) and furthermore, rather than each data item belonging to one of the $k$ classes (topics), each data item belongs to a mixture of the $k$ topics. Our goal is given a new data item to output what that mixture is.

We begin by describing our high level theoretical formulation. This formulation can be viewed as a generalization both of standard topic modeling and of a setting known as multi-view learning or co-training Blum and Mitchell (1998); Dasgupta et al. (2002); Chapelle et al. (2010); Balcan et al. (2004); Sun (2013). We then describe several natural assumptions under which we can indeed efficiently solve the problem, learning accurate topic mixture predictors.

2 Preliminaries

We assume that paragraphs are described by $n$ real-valued features and so can be viewed as points $x$ in an instance space $X \subseteq \mathbb{R}^n$. We assume that each document consists of at least two paragraphs and denote it
by \((x^1, x^2)\). Furthermore, we consider \(k\) topics and partial membership functions \(f_1, \ldots, f_k : \mathcal{X} \rightarrow [0, 1]\), such that \(f_i(x)\) determines the degree to which paragraph \(x\) belongs to topic \(i\), and \(\sum_{i=1}^{k} f_i(x) = 1\). For any vector of probabilities \(w \in \mathbb{R}^k\) — which we sometimes refer to as mixture weights — we define \(\mathcal{X}^w = \{x \in \mathbb{R}^n | \forall i, f_i(x) = w_i\}\) to be the set of all paragraphs with partial membership values \(w\). We assume that both paragraphs of a document have the same partial membership values, that is \((x^1, x^2) \in \bigcup_w \mathcal{X}^w \times \mathcal{X}^w\), although we also allow some noise later on. To better relate to the literature on multi-view learning, we will also refer to topics as “classes” and refer to paragraphs as “views” of the document.

Much like the standard topic models, we consider an unlabeled sample set that is generated by a two-step process. First, we consider a distribution \(\mathcal{P}\) over vectors of mixture weights and draw \(w\) according to \(\mathcal{P}\). Then we consider distribution \(\mathcal{D}^w\) over the set \(\mathcal{X}^w \times \mathcal{X}^w\) and draw a document \((x^1, x^2)\) according to \(\mathcal{D}^w\). We consider two settings. In the first setting, which is addressed in Section 3, the learner receives the instance \((x^1, x^2)\). In the second setting, the learner receives samples \((\hat{x}^1, \hat{x}^2)\) that have been perturbed by some noise. We discuss two noise models in Sections 4 and 5.2. In both cases, the goal of the learner is to recover the partial membership functions \(f_i\).

More specifically, in this work we consider partial membership functions of the form \(f_i(x) = f(v_i \cdot x)\), where \(v_1, \ldots, v_k \in \mathbb{R}^n\) are linearly independent and \(f : \mathbb{R} \rightarrow [0, 1]\) is a monotonic function. For the majority of this work, we consider \(f\) to be the identity function, so that \(f_i(x) = v_i \cdot x\). Define \(a_i \in \text{span}(v_1, \ldots, v_k)\) such that \(v_i \cdot a_i = 1\) and \(v_j \cdot a_i = 0\) for all \(j \neq i\). That is, \(a_i\) can be viewed as the projection of a paragraph that is purely of topic \(i\) onto the span of \(v_1, \ldots, v_k\). Define \(\Delta = \text{CH}(\{a_1, \ldots, a_k\})\) to be the convex hull of \(a_1, \ldots, a_k\).

Throughout this work, we use \(\| \cdot \|_2\) to denote the spectral norm of a matrix or the \(L_2\) norm of a vector. When it is clear from the context, we simply use \(\| \cdot \|\) to denote these quantities. We denote by \(B_r(x)\) the ball of radius \(r\) around \(x\). For any matrix \(M\), we use \(M^+\) to denote the pseudoinverse of \(M\).

**Generalization of Standard Topic Modeling**

Let us briefly discuss how the above model is a generalization of the standard topic modeling framework. In the standard framework, a topic is modeled as a probability distribution over \(n\) words, expressed as a vector \(a_i \in [0, 1]^n\), where \(a_{ij}\) is the probability of word \(j\) in topic \(i\). A document is generated by first selecting a mixture \(w \in [0, 1]^k\) over \(k\) topics, and then choosing words i.i.d. from the associated mixture distribution \(\sum_{i=1}^{k} w_i a_i\). The document vector \(\hat{x}\) is then the vector of word counts, normalized by dividing by the number of words in the document so that the \(L_1\) norm of \(\hat{x}\) is 1.

As a thought experiment, consider infinitely long documents. In the standard framework, all infinitely long documents of a mixture weight \(w\) have the same representation \(x = \sum_{i=1}^{k} w_i a_i\). This representation implies \(x \cdot v_i = w_i\) for all \(i \in [k]\), where \(V = (v_1, \ldots, v_k)\) is the pseudo-inverse of matrix \(A = (a_1, \ldots, a_k)\). Thus, by partitioning the document into two halves (views) \(x^1\) and \(x^2\), our noise-free model with \(f_i(x) = v_i \cdot x\) generalizes the standard topic model for long documents. However, our model is substantially more general: features within a view can be arbitrarily correlated, the views themselves can be correlated with each other, and even in the zero-noise case, documents of the same mixture can look very different so long as they have the same projection to the span of \(a_1, \ldots, a_k\).

For a shorter document \(\hat{x}\), each feature \(\hat{x}_i\) is drawn according to a distribution with mean \(x_i\), where \(x = \sum_{i=1}^{k} w_i a_i\). Therefore, \(\hat{x}\) can be thought of as a noisy measurement of \(x\). The fewer the words in a document, the larger is the noise in \(\hat{x}\). Existing work in topic modeling, such as Arora et al. (2012b); Anandkumar et al. (2014), provide elegant procedures for handling large noise that is caused by drawing only 2 or 3 words according to the distribution induced by \(x\). As we show in Section 4, our method can also tolerate large amounts of noise under some conditions. While our method cannot deal with documents that are only 2- or 3-words long, the benefit is a model that is much more general in many other respects.
3 An Easier Case with Simplifying Assumptions

We make two main simplifying assumptions in this section, both of which will be relaxed in Section 4:
1) The documents are not noisy, i.e., $x^1 \cdot v_i = x^2 \cdot v_i$; 2) There is non-negligible probability density on instances that belong purely to one class. In this section we demonstrate ideas and techniques, which we will develop further in the next section, to learn the topic vectors from a corpus of unlabeled documents.

The Setting: We make the following assumptions. The documents are not noisy, i.e., for any document $(x^1, x^2)$ and for all $i \in [k]$, $x^1 \cdot v_i = x^2 \cdot v_i$. Regarding distribution $P$, we assume that a non-negligible probability density is assigned to pure samples for each class. More formally, for some $\xi > 0$, for all $i \in [k]$, $\Pr_{w \sim P}[w = e_i] \geq \xi$. Regarding distribution $D^w$, we allow the two paragraphs in a document, i.e., the two views $(x^1, x^2)$ drawn from $D^w$, to be correlated as long as for any subspace $Z \subset \text{null}\{v_1, \ldots, v_k\}$ of dimension strictly less than $n - k$, $\Pr_{(x^1, x^2) \sim D^w}[(x^1 - x^2) \notin Z] \geq \zeta$ for some non-negligible $\zeta$. One way to view this in the context of topic modeling is that if, say, “sports” is a topic, then it should not be the case that the second paragraph always talks about the exact same sport as the first paragraph; else “sports” would really be a union of several separate but closely-related topics. Thus, while we do not require independence we do require some non-correlation between the paragraphs.

Algorithm and Analysis: The main idea behind our approach is to use the consistency of the two views of the samples to first recover the subspace spanned by $v_1, \ldots, v_k$ (Phase 1). Once this subspace is recovered, we show that a projection of a sample on this space corresponds to the convex combination of class vectors that purely belong to each class by taking the extreme points of the projected samples (Phase 2). The class vectors $v_1, \ldots, v_k$ are the unique vectors (up to permutations) that classify $a_1, \ldots, a_k$ as pure samples. Phase 2 is similar to that of Arora et al. (2012b). Algorithm 1 formalizes the details of this approach.

**Algorithm 1** ALGORITHM FOR GENERALIZED TOPIC MODELS — NO NOISE

**Input:** A sample set $S = \{(x^1_i, x^2_i) \mid i \in [m]\}$ such that for each $i$, first a vector $w$ is drawn from $P$ and then $(x^1_i, x^2_i)$ is drawn from $D^w$.

**Phase 1:**
1. Let $X^1$ and $X^2$ be matrices where the $i^{th}$ column is $x^1_i$ and $x^2_i$, respectively.
2. Let $P$ be the projection matrix on the last $k$ left singular vectors of $(X^1 - X^2)$.

**Phase 2:**
1. Let $S_H = \{P x^1_i \mid i \in [m], j \in \{1, 2\}\}$.
2. Let $A$ be a matrix whose columns are the extreme points of the convex hull of $S_1$. (This can be found using farthest traversal or linear programming.)

**Output:** Return columns of $A^+$ as $v_1, \ldots, v_k$.

In Phase 1 for recovering $\text{span}\{v_1, \ldots, v_k\}$, note that for any sample $(x^1, x^2)$ drawn from $D^w$, we have that $v_i \cdot x^1 = v_i \cdot x^2 = w_i$. Therefore, regardless of what $w$ was used to produce the sample, we have that $v_i \cdot (x^1 - x^2) = 0$ for all $i \in [k]$. That is, $v_1, \ldots, v_k$ are in the null-space of all such $(x^1 - x^2)$. So, if samples $(x^1_i - x^2_i)$ span a $n - k$ dimensional subspace, then $\text{span}\{v_1, \ldots, v_k\}$ can be recovered by taking $\text{null}\{(x^1_i - x^2_i) \mid (x^1_i, x^2_i) \in X^w \times X^w, \forall w \in \mathbb{R}^k\}$. Using singular value decomposition, this null space is spanned by the last $k$ singular vectors of $X^1 - X^2$, where $X^1$ and $X^2$ are matrices with columns $x^1_i$ and $x^2_i$, respectively.

This is where the assumptions on $D^w$ come into play. By assumption, for any strict subspace $Z$ of $\text{span}\{(x^1 - x^2) \mid (x^1, x^2) \in X^w \times X^w, \forall w \in \mathbb{R}^k\}$, $D^w$ has non-negligible probability on instances $(x^1 - x^2) \notin Z$. Therefore, after seeing sufficiently many samples we can recover the space of all $(x^1 - x^2)$. The next lemma, whose proof appears in Appendix A.1, formalizes this discussion.
Lemma 3.1. Let \( Z = \text{span}\{(x^1_i - x^2_i) \mid i \in [m]\} \). Then, \( m = O\left(\frac{n-k}{\varepsilon} \log\left(\frac{1}{\delta}\right)\right) \) is sufficient such that with probability \( 1 - \delta \), \( \text{rank}(Z) = n - k \).

Using Lemma 3.1, Phase 1 of Algorithm 1 recovers \( \text{span}\{v_1, \ldots, v_k\} \). Next, we show that pure samples are the extreme points of the convex hull of all samples when projected on the subspace \( \text{span}\{v_1, \ldots, v_k\} \). Figure 1 demonstrates the relation between the class vectors, \( v_i \), projection of samples, and the projection of pure samples \( a_i \). The next lemma, whose proof appears in Appendix A.2, formalizes this claim.

Lemma 3.2. For any \( x \), let \( x_\| \) represent the projection of \( x \) on \( \text{span}\{v_1, \ldots, v_k\} \). Then, \( x_\| = \sum_{i \in [k]} (v_i \cdot x) a_i \).

With \( \sum_{i \in [k]} (v_i \cdot x) a_i \) representing the projection of \( x \) on \( \text{span}\{v_1, \ldots, v_k\} \), it is clear that the extreme points of the set of all projected instances that belong to \( A^w \) for all \( w \) are \( a_1, \ldots, a_k \). Since in a large enough sample set, with high probability for all \( i \in [k] \), there is a pure sample of type \( i \), taking the extreme points of the set of projected samples is also \( a_1, \ldots, a_k \). The following lemma, whose proof appears in Appendix A.3, formalizes this discussion.

Lemma 3.3. Let \( m = c\left(\frac{1}{\varepsilon} \log\left(\frac{1}{\delta}\right)\right) \) for a large enough constant \( c > 0 \). Let \( P \) be the projection matrix for \( \text{span}\{v_1, \ldots, v_k\} \) and \( S_\| = \{Px_i \mid i \in [m], j \in \{1, 2\}\} \) be the set of projected samples. With probability \( 1 - \delta \), \( \{a_1, \ldots, a_k\} \) is the set of extreme points of \( \text{CH}(S_1) \).

Therefore, \( a_1, \ldots, a_k \) can be learned by taking the extreme points of the convex hull of all samples projected on \( \text{span}\{v_1, \ldots, v_k\} \). Furthermore, \( V = A^+ \) is unique, therefore \( v_1, \ldots, v_k \) can be easily found by taking the pseudoinverse of matrix \( A \). Together with Lemma 3.1 and 3.3 this proves the next theorem regarding learning class vectors in the absence of noise.

Theorem 3.4 (No Noise). There is a polynomial time algorithm for which \( m = O\left(\frac{n-k}{\varepsilon} \ln\left(\frac{1}{\delta}\right) + \frac{1}{\varepsilon} \ln\left(\frac{1}{\delta}\right)\right) \) is sufficient to recover \( v_i \) exactly for all \( i \in [k] \), with probability \( 1 - \delta \).

4 Relaxing the Assumptions

In this section, we relax the two main simplifying assumptions from Section 3. We relax the assumption on non-noisy documents and allow a large fraction of the documents to not satisfy \( v_i \cdot x^1 = v_i \cdot x^2 \). In the standard topic model, this corresponds to having a large fraction of short documents. Furthermore, we relax the assumption on the existence of pure documents to an assumption on the existence of “almost-pure” documents. We further develop the approach discussed in the previous section and introduce efficient algorithms that approximately recover the topic vectors in this setting.

The Setting: We assume that any sampled document has a non-negligible probability of being non-noisy and with the remaining probability, the two views of the document are perturbed by additive Gaussian noise, independently. More formally, for a given sample \((x^1, x^2)\), with probability \( p_0 > 0 \) the algorithm receives \((x^1, x^2)\) and with the remaining probability \( 1 - p_0 \), the algorithm receives \((\hat{x}^1, \hat{x}^2)\), such that \( \hat{x}^j = x^j + e^j \), where \( e^j \sim \mathcal{N}(0, \sigma^2 I_n) \).

We assume that for each topic the probability that a document is mostly about that topic is non-negligible. More formally, for any topic \( i \in [k] \), \( \Pr_{w \sim P}[||e_i - w||_1 \leq \varepsilon] \geq g(\varepsilon) \), where \( g \) is a polynomial function of its input. A stronger form of this assumption, better known as the dominant admixture assumption, assumes that every document is mostly about one topic and has been empirically shown to hold on several real world
data sets (Bansal et al., 2014). Furthermore, in the Latent Dirichlet Allocation model, $\Pr_{w \sim P}[\max_{i \in [k]} w_i \geq 1 - \epsilon] \geq O(\epsilon^2)$ for typical values of the concentration parameter.

We also make mild assumptions on the distribution over instances. We assume that the covariance of the distribution over $\langle x^1_i - x^2_i \rangle (x^1_i - x^2_i)^\top$ is significantly larger than the noise covariance $\sigma^2$. That is, for some $\delta_0 > 0$, the least significant non-zero eigen value of $E_{\langle x^1_i - x^2_i \rangle} [\langle x^1_i - x^2_i \rangle (x^1_i - x^2_i)^\top]$ is greater than $6\sigma^2 + \delta_0$. At a high level, these assumptions are necessary, because if $\|x^1_i - x^2_i\|$ is too small compared to $\|\hat{x}^1_i\|$ and $\|\hat{x}^2_i\|$, then even a small amount of noise affects the structure present in $x^1_i - x^2_i$ completely. Moreover, we assume that the $L_2$ norm of each view of a sample is bounded by some $M > 0$. We also assume that for all $i \in [k]$, $\|a_i\| \leq \alpha$ for some $\alpha > 0$. At a high level, $\|a_i\|$s are inversely proportional to the non-zero singular values of $V = (v_1, \ldots, v_k)$. Therefore, $\|a_i\| \leq \alpha$ implies that the $k$ topic vectors are sufficiently different.

**Algorithm and Results:** Our approach follows the general theme of the previous section: First, recover $\text{span}\{v_1, \ldots, v_k\}$ and then recover $a_1, \ldots, a_k$ by taking the extreme points of the projected samples. In this case, in the first phase we recover $\text{span}\{v_1, \ldots, v_k\}$ approximately, by finding a projection matrix $\hat{P}$ such that $\|P - \hat{P}\| \leq \epsilon$ for an arbitrarily small $\epsilon$, where $P$ is the projection matrix on $\text{span}\{v_1, \ldots, v_k\}$. At this point in the algorithm, the projection of samples on $\hat{P}$ can include points that are arbitrarily far from $\Delta$. This is due to the fact that the noisy samples are perturbed by $\mathcal{N}(0, \sigma^2 I_n)$, so, for large values of $\sigma$ some noisy samples map to points that are quite far from $\Delta$. Therefore, we have to detect and remove these samples before continuing to the second phase. For this purpose, we show that the low density regions of the projected samples can safely be removed such that the convex hull of the remaining points is close to $\Delta$. In the second phase, we consider projections of each sample using $\hat{P}$. To approximately recover $a_1, \ldots, a_k$, we recover samples, $\hat{x}$, that are far from the convex hull of the remaining points, when $x$ and a ball of points close to it are removed. We then show that such points are close to one of the pure class vectors, $a_i$.

**Algorithm 2**

**Algorithm for Generalized Topic Models — With Noise**

**Input:** A sample set $\{(\hat{x}^1_i, \hat{x}^2_i) \mid i \in [m]\}$ such that for each $i$, first a vector $w$ is drawn from $P$, then $(x^1_i, x^2_i)$ is drawn from $D^w$, then with probability $p_0$, $\hat{x}^1_i = x^1_i$, else with probability $1 - p_0$, $\hat{x}^1_i = x^1_i + \mathcal{N}(0, \sigma^2 I_n)$ for $i \in [m]$ and $j \in \{1, 2\}$.

**Phase 1:**
1. Take $m_1 = \Omega\left(\frac{\frac{\sigma}{\epsilon} \ln(\frac{1}{\delta}) + \frac{K}{\epsilon}\frac{4}{\epsilon} M^2 \ln(\frac{1}{\delta}) + \frac{\sigma^2 M + 4}{\epsilon^2} \ln(\frac{\frac{\sigma}{\epsilon} M}{\epsilon \delta}) + \frac{\epsilon^4 M^2}{\delta \epsilon} \ln(\frac{\frac{\sigma}{\epsilon} M}{\delta})}{\frac{M^4}{\delta \epsilon^2}}\right)$ samples.
2. Let $\hat{X}^1$ and $\hat{X}^2$ be matrices where the $i$th column is $\hat{x}^1_i$ and $\hat{x}^2_i$, respectively.
3. Let $\hat{P}$ be the projection matrix on the last $k$ left singular vectors of $\hat{X}^1 - \hat{X}^2$.

**Denoising Phase:**
4. Let $\epsilon' = \frac{\epsilon}{\sqrt{\epsilon} \sigma}$ and $\gamma = g\left(\frac{\epsilon'}{\sqrt{\epsilon} \sigma} \right)$.
5. Take $m_2 = \Omega\left(\frac{k}{p_\gamma} \ln(\frac{1}{\delta})\right)$ fresh samples and let $\hat{S}_1 = \left\{\hat{P}\hat{x}^1_i \mid \forall i \in [m_2]\right\}$.
6. Remove $\hat{x}^1_i$ from $\hat{S}_1$, for which there are less than $p_\gamma m_2 / 2$ points within distance of $\epsilon'$ in $\hat{S}_1$.

**Phase 2:**
6. For all $\hat{x}^1_i$ in $\hat{S}_1$, if $\text{dist}(\hat{x}^1_i, \text{CH}(\hat{S}_1 \setminus B_{6\gamma \epsilon'}(\hat{x}^1))) \geq 2\epsilon'$ add $\hat{x}^1_i$ to $C$.
7. Cluster $C$ using single linkage with threshold $16\epsilon'$. Assign any point from cluster $i$ as $\hat{a}_i$.

**Output:** Return $\hat{a}_1, \ldots, \hat{a}_k$.

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1For the denoising step, we use a fresh set of samples that were not used for learning the projection matrix. This guarantees that the noise distribution in the projected samples remains a Gaussian.
Theorem 4.1. Consider any $\epsilon, \delta > 0$ such that $\epsilon \leq O\left( r\sigma \sqrt{k} \right)$, where $r$ is a parameter that depends on the geometry of the simplex $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and will be defined later. There is an efficient algorithm for which an unlabeled sample set of size

$$m = O\left( \frac{n - k}{\zeta} \ln\left( \frac{1}{\delta} \right) + \frac{n\sigma^4 r^2 M^2}{\delta_0^2 \epsilon^2} \ln\left( \frac{1}{\delta} \right) + \frac{n\sigma^2 M^2 r^2}{\delta_0^2 \epsilon^2} \text{polylog}\left( \frac{n}{\epsilon \delta} \right) + \frac{M^4}{\delta_0^2} \ln\left( \frac{n}{\delta} \right) + \frac{k \ln(1/\delta)}{p_0} g(\epsilon/(k \sigma \alpha)) \right)$$

is sufficient to recover $\hat{a}_i$ such that $\|\hat{a}_i - a_i\|_2 \leq \epsilon$ for all $i \in [k]$, with probability $1 - \delta$.

The proof of Theorem 4.1 involves the next three lemmas on the performance of the phases of the above algorithm. We formally state these two lemmas here, but defer their proofs to Sections 4.1, 4.2 and 4.3.

Lemma 4.2 (Phase 1). For any $\sigma > 0$ and $\epsilon > 0$, an unlabeled sample set of size

$$m = O\left( \frac{n - k}{\zeta} \ln\left( \frac{1}{\delta} \right) + \frac{n\sigma^4}{\delta_0^2 \epsilon^2} \ln\left( \frac{1}{\delta} \right) + \frac{n\sigma^2 M^2}{\delta_0^2 \epsilon^2} \text{polylog}\left( \frac{n}{\epsilon \delta} \right) + \frac{M^4}{\delta_0^2} \ln\left( \frac{n}{\delta} \right) \right).$$

is sufficient, such that with probability $1 - \delta$, Phase 1 of Algorithm 2 returns a projection matrix $\hat{P}$, such that $\|P - \hat{P}\|_2 \leq \epsilon$.

Lemma 4.3 (Denoising). Let $\epsilon' \leq \frac{1}{3} \sigma \sqrt{k}$, $\|P - \hat{P}\|_2 \leq \epsilon' / 8M$, and $\gamma = g\left( \frac{\epsilon'}{8k \sigma} \right)$. An unlabeled sample size of $m = O\left( \frac{k}{p_0^2} \ln\left( \frac{1}{\delta} \right) \right)$ is sufficient such that for $\hat{S}_1$ defined in Step 6 of Algorithm 2 the following holds with probability $1 - \delta$: For any $x \in \hat{S}_1$, $\text{dist}(x, \Delta) \leq \epsilon'$, and, for all $i \in [k]$, there exists $\hat{a}_i \in \hat{S}_1$ such that $\|\hat{a}_i - a_i\| \leq \epsilon'$.

Lemma 4.4 (Phase 2). Let $\hat{S}_1$ be a set of points for which the conclusion of Lemma 4.3 holds with the value of $\epsilon' = \epsilon / 8r$. Then, Phase 2 of Algorithm 2 returns $\hat{a}_1, \ldots, \hat{a}_k$ such that for all $i \in [k]$, $\|a_i - \hat{a}_i\| \leq \epsilon$.

We now prove our main Theorem 4.1 by directly leveraging the three lemmas we just stated.

Proof of Theorem 4.1. By Lemma 4.2, sample set of size $m_1$ is sufficient such that Phase 1 of Algorithm 2 leads to $\|P - \hat{P}\|_2 \leq \frac{\epsilon}{32M^2}$, with probability $1 - \delta/2$. Let $\epsilon' = \frac{\epsilon}{8r}$ and take a fresh sample of size $m_2$. By Lemma 4.3, with probability $1 - \delta/2$, for any $x \in \hat{S}_1$, $\text{dist}(x, \Delta) \leq \epsilon'$, and, for all $i \in [k]$, there exists $\hat{a}_i \in \hat{S}_1$ such that $\|\hat{a}_i - a_i\| \leq \epsilon'$. Finally, applying Lemma 4.4 we have that Phase 2 of Algorithm 2 returns $\hat{a}_i$, such that for all $i \in [k]$, $\|a_i - \hat{a}_i\| \leq \epsilon$.

Theorem 4.1 discusses the approximation of $a_i$ for all $i \in [k]$. It is not hard to see that such an approximation also translates to the approximation of class vectors, $v_i$ for all $i \in [k]$. That is, using the properties of perturbation of pseudoinverse matrices (see Proposition B.5) one can show that $\|\hat{A}^+ - V\| \leq O(\|\hat{A} - A\|)$. Therefore, $\hat{V} = \hat{A}^+$ is a good approximation for $V$.

4.1 Proof of Lemma 4.2 — Phase 1

For $j \in \{1, 2\}$, let $X^j$ and $\hat{X}^j$ be $n \times m$ matrices with the $i^{th}$ column being $x_i^j$ and $\hat{x}_i^j$, respectively. As we demonstrated in Lemma 3.1, with high probability $\text{rank}(X^1 - X^2) = n - k$. Note that the nullspace of columns of $X^1 - X^2$ is spanned by the left singular vectors of $X^1 - X^2$ that correspond to its $k$ zero singular values. Similarly, consider the space spanned by the $k$ least left singular vectors of $\hat{X}^1 - \hat{X}^2$. We show that the nullspace of columns of $X^1 - X^2$ can be approximated within any desirable accuracy by the space spanned by the $k$ least left singular vectors of $\hat{X}^1 - \hat{X}^2$, given a sufficiently large number of samples.
Let $D = X^1 - X^2$ and $\hat{D} = \hat{X}^1 - \hat{X}^2$. For ease of exposition, assume that all samples are perturbed by Gaussian noise $\mathcal{N}(0, \sigma^2 I_n)$\footnote{The assumption that with a non-negligible probability a sample is non-noisy is not needed for the analysis and correctness of Phase 1 of Algorithm 2. This assumption only comes into play in the denoising phase.}. Since each view of a sample is perturbed by an independent draw from a Gaussian noise distribution, we can view $\hat{D} = D + E$, where each column of $E$ is drawn i.i.d from distribution $\mathcal{N}(0, 2\sigma^2 I_n)$. Then, $\frac{1}{m} \hat{D} \hat{D}^T = \frac{1}{m} DD^T + \frac{1}{m} EE^T + \frac{1}{m} ED^T + \frac{1}{m} E E^T$. As a thought experiment, consider this equation in expectation. Since $E[\frac{1}{m} EE^T] = 2\sigma^2 I_n$ is the covariance matrix of the noise and $E[EE^T + ED^T] = 0$, we have
\[
\frac{1}{m} E[\hat{D} \hat{D}^T] - 2\sigma^2 I_n = \frac{1}{m} E[DD^T].
\] (1)

Moreover, the eigen vectors and their order are the same in $\frac{1}{m} E[\hat{D} \hat{D}^T]$ and $\frac{1}{m} E[\hat{D} \hat{D}^T] - 2\sigma^2 I_n$. Therefore, one can recover the nullspace of $\frac{1}{m} E[DD^T]$ by taking the space of the least $k$ eigen vectors of $\frac{1}{m} E[DD^T]$. Next, we show how to recover the nullspace using $\hat{D} \hat{D}^T$, rather than $E[\hat{D} \hat{D}^T]$. Assume that the following properties hold:

1. Equation 1 holds not only in expectation, but also with high probability. That is, with high probability, $\|\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n - \frac{1}{m} DD^T\|_2 \leq \epsilon$.

2. With high probability $\lambda_{n-k}(\frac{1}{m} \hat{D} \hat{D}^T) > 4\sigma^2 + \delta_0/2$, where $\lambda_i(\cdot)$ denotes the $i^{th}$ most significant eigen value.

Let $D = U \Sigma V^T$ and $\hat{D} = \hat{U} \hat{\Sigma} \hat{V}^T$ be SVD representations. We have that $\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n = \hat{U} \left(\frac{1}{m} \hat{\Sigma}^2 - 2\sigma^2 I_n\right) \hat{U}^T$. By property 2, $\lambda_{n-k}(\frac{1}{m} \hat{\Sigma}^2) > 4\sigma^2 + \delta_0/2$. That is, the eigen vectors and their order are the same in $\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n$ and $\frac{1}{m} DD^T$. As a result the projection matrix, $\hat{P}$, on the least $k$ eigen vectors of $\frac{1}{m} \hat{D} \hat{D}^T$, is the same as the projection matrix, $Q$, on the least $k$ eigen vectors of $\frac{1}{m} DD^T - 2\sigma^2 I_n$.

Recall that $\hat{P}$ and $P$ and $Q$ are the projection matrices on the least significant $k$ eigen vectors of $\frac{1}{m} \hat{D} \hat{D}^T$, $\frac{1}{m} DD^T$, and $\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I$, respectively. As we discussed, $\hat{P} = Q$. Now, using the Davis and Kahan (1970) or Wedin (1972) sin $\theta$ theorem (see Proposition B.1) from matrix perturbation theory, we have,
\[
\|P - \hat{P}\|_2 = \|P - Q\| \leq \frac{\|\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n - \frac{1}{m} DD^T\|_2}{\lambda_{n-k}(\frac{1}{m} \hat{D} \hat{D}^T) - 2\sigma^2 - \lambda_{n-k+1}(\frac{1}{m} DD^T)} \leq \frac{2\epsilon}{\delta_0}
\]

where we use Properties 1 and 2 and the fact that $\lambda_{n-k+1}(\frac{1}{m} DD^T) = 0$, in the last transition.

### 4.1.1 Concentration

It remains to prove Properties 1 and 2. We briefly describe our approach for obtaining concentration results and prove that when the number of samples $m$ is large enough, with high probability $\|\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n - \frac{1}{m} DD^T\|_2 \leq \epsilon$ and $\lambda_{n-k}(\frac{1}{m} \hat{D} \hat{D}^T) > 4\sigma^2 + \delta_0/2$.

Let us first describe $\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n - \frac{1}{m} DD^T$ in terms of the error matrices. We have
\[
\frac{1}{m} \hat{D} \hat{D}^T - 2\sigma^2 I_n - \frac{1}{m} DD^T = \left(\frac{1}{m} EE^T - 2\sigma^2 I_n\right) + \left(\frac{1}{m} DE^T + \frac{1}{m} E D^T\right).
\] (2)

It suffices to show that for large enough $m > m_{\epsilon, \delta}$, $\Pr[\|\frac{1}{m} EE^T - 2\sigma^2 I_n\|_2 \geq \epsilon] \leq \delta$ and $\Pr[\|\frac{1}{m} DE^T + \frac{1}{m} E D^T\|_2 \geq \epsilon] \leq \delta$. In the former, note that $\frac{1}{m} EE^T$ is the sample covariance of the Gaussian noise matrix and $2\sigma^2 I_n$ is the true covariance matrix of the noise distribution. The next claim is a direct consequence of the convergence properties of sample covariance of the Gaussian distribution (see Proposition B.2).
Claim 4.5. For \( m > n \frac{\sigma^6}{\epsilon^2} \log(\frac{1}{\delta}) \), with probability \( 1 - \delta \), \( \| \frac{1}{m} EE^T - 2\sigma^2 I_n \|_2 \leq \epsilon. \)

We use the Matrix Bernstein inequality (Tropp, 2015), described in Appendix B, to demonstrate the concentration of \( \| \frac{1}{m} DE^T + \frac{1}{m} ED^T \|_2 \). The proof of the next Claim is relegated to Appendix C.1.

Claim 4.6. \( m = O \left( \frac{2\sigma^2 M^2}{\epsilon^2} \text{polylog} \left( \frac{n}{\epsilon^2} \right) \right) \) is sufficient so that with probability \( 1 - \delta \), \( \| \frac{1}{m} DE^T + \frac{1}{m} ED^T \|_2 \leq \epsilon \).

Next, we prove that \( \lambda_{n-k} \left( \frac{1}{m} \tilde{D} \tilde{D}^\top \right) > 4\sigma^2 + \delta_0 / 2. \) Since for any two matrices, the difference in \( \lambda_{n-k} \) can be bounded by the spectral norm of their difference (see Proposition B.4), using Equation 2, we have

\[
\left| \lambda_{n-k} \left( \frac{1}{m} \tilde{D} \tilde{D}^\top \right) - \lambda_{n-k} \left( \frac{1}{m} DD^\top \right) \right| \leq \left\| 2\sigma^2 I + \left( \frac{1}{m} EE^T - 2\sigma^2 I_n \right) - \left( \frac{1}{m} DE^T + \frac{1}{m} ED^T \right) \right\|_2 \leq 2\sigma^2 + \delta_0 / 4,
\]

where in the last transition we use Claims 4.5 and 4.6 with the value of \( \delta_0 / 8 \) to bound the last two terms by a total of \( \delta_0 / 4. \) Since \( \lambda_{n-k} \left( \mathbb{E} \left[ \frac{1}{m} DD^\top \right] \right) \geq 6\sigma^2 + \delta_0 \), it is sufficient to show that \( |\lambda_{n-k} \left( \mathbb{E} \left[ \frac{1}{m} DD^\top \right] \right) - \lambda_{n-k} \left( \mathbb{E} \left[ \frac{1}{m} DD^\top \right] \right) | \leq \delta_0 / 4. \) Similarly as before, this is bounded by \( \| \frac{1}{m} DD^\top - \mathbb{E} \left[ \frac{1}{m} DD^\top \right] \|. \) We use the Matrix Bernstein inequality (Proposition B.3) to prove this concentration result. The rigorous proof of this claim appears in Appendix C.2.

Claim 4.7. \( m = O \left( \frac{M^4}{\epsilon^2} \text{log} \frac{n}{\epsilon^2} \right) \) is sufficient so that with probability \( 1 - \delta \), \( \| \frac{1}{m} DD^\top - \mathbb{E} \left[ \frac{1}{m} DD^\top \right] \|_2 \leq \frac{\delta_0}{4}. \)

This completes the analysis of Phase 1 of our algorithm and the proof of Lemma 4.2 follows directly from the above analysis and the application of Claims 4.5 and 4.6 with the error of \( \epsilon \delta_0 \), and Claim 4.7.

4.2 Proof of Lemma 4.3 — Denoising Step

Having approximately recovered a projection matrix \( \hat{P} \) for \( \text{span}\{v_1, \ldots, v_k\} \), we can now use this subspace to partially denoise the samples while approximately preserving \( \Delta = \text{CH}(\{a_1, \ldots, a_k\}). \) At a high level, when considering the projection of samples on \( \hat{P} \), one can show that 1) the regions around \( a_i \) have sufficiently high density, and 2) the regions that are far from \( \Delta \) have low density.

We claim that if \( \hat{x}_i \in \hat{S}_i \) is non-noisy and corresponds almost purely to one class then \( \hat{S}_i \) also includes a non-negligible number of points within \( O(\epsilon') \) distance of \( \hat{x}_i \). This is due to the fact that a non-negligible number of points (about \( p_0 \gamma m \) points) correspond to non-noisy and almost-pure samples that using \( P \) would get projected to points within a distance of \( O(\epsilon') \) of each other. Furthermore, the inaccuracy in \( \hat{P} \) can only perturb the projections up to \( O(\epsilon') \) distance. So, the projections of all non-noisy samples that are purely of class \( i \) fall within \( O(\epsilon') \) of \( a_i \). The following lemma, whose proof appears in Appendix D.1, formalizes this claim.

In the following lemmas, let \( D \) denote the flattened distribution of the first paragraphs. That is, the distribution over \( \hat{x}^1 \) where we first take \( w \sim \mathcal{P} \), then take \( (x^1, x^2) \sim D^w \), and finally take \( \hat{x}^1 \).

Claim 4.8. For all \( i \in [k] \), \( \Pr_{x \sim D} \left[ \hat{P} x \in B_{\epsilon' / 4}(a_i) \right] \geq p_0 \gamma. \)

On the other hand, any projected point that is far from the convex hull of \( a_1, \ldots, a_k \) has to be noisy, and as a result, has been generated by a Gaussian distribution with variance \( \sigma^2 \). For a choice of \( \epsilon' \) that is small with respect to \( \sigma \), such points do not concentrate well within any ball of radius \( \epsilon' \). In the next lemma we show that the regions that are far from the convex hull have low density.

Claim 4.9. For any \( z \) such that \( \text{dist}(z, \Delta) \geq \epsilon' \), we have \( \Pr_{x \sim D} \left[ \hat{P} x \in B_{\epsilon' / 2}(z) \right] \leq \frac{p_0 \gamma}{4}. \)

\(^3\)At first sight, the dependence of this sample complexity on \( \sigma \) might appear unintuitive. But, note that even without seeing any samples we can approximate the noise covariance within \( 2\sigma^2 I_n \). Therefore, if \( \epsilon = 2\sigma^2 \) our work is done.
Proof. We first show that $B_{\epsilon'/2}(z)$ does not include any non-noisy points. Take any non-noisy sample $x$. Note that $Px = \sum_{i=1}^{k} w_i a_i$, where $w_i$ are the mixture weights corresponding to point $x$. We have,

$$\|z - \hat{P}x\| = \|z - \sum_{i=1}^{k} w_i a_i + (P - \hat{P})x\| \geq \|z - \sum_{i=1}^{k} w_i a_i\| - \|P - \hat{P}\||x| \geq \epsilon'/2$$

Therefore, $B_{\epsilon'/2}(z)$ only contains noisy points. Since noisy points are perturbed by a spherical Gaussian, the projection of these points on any $k$-dimensional subspace can be thought of points generated from a $k$-dimensional Gaussian distributions with variance $\sigma^2$ and potentially different centers. One can show that the densest ball of any radius is at the center of a Gaussian. Here, we prove a slightly weaker claim. Consider one such Gaussian distribution, $\mathcal{N}(0, \sigma^2 I_k)$. Note that the pdf of the Gaussian distribution decreases as we get farther from its center. By a coupling between the density of the points, $B_{\epsilon'/2}(c)$ has higher density than any $B_{\epsilon'/2}(c)$ with $\|c\|_2 > \epsilon'$. Therefore,

$$\sup_{c} \Pr_{x \sim \mathcal{N}(0, \sigma^2 I_k)}[x \in B_{\epsilon'/2}(c)] \leq \Pr_{x \sim \mathcal{N}(0, \sigma^2 I_k)}[x \in B_{3\epsilon'/2}(0)].$$

So, over $D$ this value will be maximized if the Gaussians had the same center (see Figure 2). Moreover, in $\mathcal{N}(0, \sigma^2 I_k)$, $\Pr[\|x\|_2 \leq \sigma \sqrt{k(1-t)}] \leq \exp(-kt^2/16)$. Since $3\epsilon'/2 \leq \sigma \sqrt{k}/2 \leq \sigma \sqrt{k(1 - \sqrt{16/kt \ln \frac{4}{p_0 \gamma}})}$ we have

$$\Pr_{x \sim D}[x \in B_{\epsilon'/2}(c)] \leq \Pr_{x \sim \mathcal{N}(0, \sigma^2 I_k)}[\|x\|_2 \leq 3\epsilon'/2] \leq \frac{p_0 \gamma}{4}.$$

The next claim shows that in a large sample set, the fraction of samples that fall within any of the described regions in Claims 4.8 and 4.9 is close to the density of that region. The proof of this claim follows from VC dimension of the set of balls.

Claim 4.10. Let $D$ be any distribution over $\mathbb{R}^k$ and $x_1, \ldots, x_m$ be $m$ points drawn i.i.d from $D$. Then $m = O\left(\frac{\gamma'}{\gamma} \ln \frac{1}{\delta}\right)$ is sufficient so that with probability $1 - \delta$, for any ball $B \subseteq \mathbb{R}^k$ such that $\Pr_{x \sim D}[x \in B] \geq 2\gamma$, $|\{x_i \mid x_i \in B\}| > \gamma m$ and for any ball $B \subseteq \mathbb{R}^k$ such that $\Pr_{x \sim D}[x \in B] \leq \gamma/2$, $|\{x_i \mid x_i \in B\}| < \gamma m$.

Therefore, upon seeing $\Omega(\frac{k}{p_0 \gamma} \ln \frac{1}{\delta})$ samples, with probability $1 - \delta$, for all $i \in [k]$ there are more than $p_0 \gamma m/2$ projected points within distance $\epsilon'/4$ of $a_i$ (by Claims 4.8 and 4.10), and, no point that is $\epsilon'$ far from $\Delta$ has more than $p_0 \gamma m/2$ points in its $\epsilon'/2$-neighborhood (by Claims 4.9 and 4.10). Phase 2 of Algorithm 2 leverages these properties of the set of projected points for denoising the samples while preserving $\Delta$: Remove any point from $\hat{S}_I$ that has fewer than $p_0 \gamma m/2$ neighbors within distance $\epsilon'/2$.

We conclude the proof of Lemma 4.3 by noting that the remaining points in $\hat{S}_I$ are all within distance $\epsilon'$ of $\Delta$. Furthermore, any point in $B_{\epsilon'/4}(a_i)$ has more than $p_0 \gamma m/2$ points within distance of $\epsilon'/2$. Therefore, such points remain in $\hat{S}_I$ and any one of them can serve as $\tilde{a}_i$ for which $\|a_i - \tilde{a}_i\| \leq \epsilon'/4$.

4.3 Proof of Lemma 4.4 — Phase 2

At a high level, we consider two balls around each projected sample point $\hat{x} \in \hat{S}_I$ with appropriate choice of radii $r_1 < r_2$ (see Figure 3a). Consider the set of projections $\hat{S}_I$ when points in $B_{r_2}(x)$ are removed from it. For points that are far from all $a_i$, this set still includes points that are close to $a_i$ for all topics $i \in [k]$. So,
the convex hull of $\hat{S}_l \setminus B_{r_2}(x)$ is close to $\Delta$, and in particular, intersects $B_{r_1}(x)$. On the other hand, for $x$ that is close to $a_i$, $\hat{S}_l \setminus B_{r_2}(x)$ does not include an extreme point of $\Delta$ or points close to it. So, the convex hull of $\hat{S}_l \setminus B_{r_2}(x)$ is considerably smaller than $\Delta$, and in particular, does not intersect $B_{r_1}(x)$.

The geometry of the simplex and the angles between $a_1, \ldots, a_k$ play an important role in choosing the appropriate $r_1$ and $r_2$. Note that when the samples are perturbed by noise, $a_1, \ldots, a_k$ can only be approximately recovered if they are sufficiently far apart and the angles of the simplex at each $a_i$ is far from being flat. That is, we assume that for all $i \neq j$, $\|a_i - a_j\| \geq 3\epsilon$. Furthermore, define $r \geq 1$ to be the smallest value such that the distance between $a_i$ and $\text{CH}((\Delta \setminus B_{r}(a_i)))$ is at least $\epsilon$. Note that such a value of $r$ always exists and depends entirely on the angles of the simplex defined by the class vectors. Therefore, the number of samples needed for our method depends on the value of $r$. The smaller the value of $r$, the larger is the separation between the topic vectors and the easier it is to identify them. See Figure 3b for a demonstration of this concept.

Claim 4.11. Let $\epsilon' = \epsilon/8r$. Let $\hat{S}_l$ be the set of denoised projections, as in step 6 of Algorithm 2. For any $\hat{x} \in \hat{S}_l \setminus B_{r_2}(x)$ such that for all $i$, $\|\hat{x} - a_i\| > 8r\epsilon'$, dist($\hat{x}, \text{CH}(\hat{S}_l \setminus B_{6\epsilon'}(\hat{x}))) \leq 2\epsilon'$. Furthermore, for all $i \in [k]$ there exists $\hat{a}_i \in \hat{S}_l$ such that $\|\hat{a}_i - a_i\| < \epsilon'$ and dist($\hat{a}_i, \text{CH}(\hat{S}_l \setminus B_{6\epsilon'}(\hat{a}_i))) > 2\epsilon'$.

Proof. Recall that by Lemma 4.3, for any $\hat{x} \in \hat{S}_l$ there exists $x \in \Delta$ such that $\|\hat{x} - x\| \leq \epsilon'$ and for all $i \in [k]$, there exists $\hat{a}_i \in \hat{S}_l$ such that $\|\hat{a}_i - a_i\| \leq \epsilon'$. For the first part, let $x = \sum_i \alpha_i a_i \in \Delta$ be the corresponding point to $\hat{x}$, where $\alpha_i$'s are the coefficients of the convex combination. Furthermore, let $x' = \sum_i \alpha_i \hat{a}_i$. We have,

$$\|x' - \hat{x}\| \leq \left\| \sum_{i=1}^k \alpha_i \hat{a}_i - \sum_{i=1}^k \alpha_i a_i + x - \hat{x} \right\| \leq \max_{i \in [k]} (\hat{a}_i - a_i) + \|x - x\| \leq 2\epsilon'.$$

The first claim follows from the fact that $\|\hat{x} - a_i\| > 8r\epsilon'$ and as a result $x' \in \text{CH}(\hat{S}_l \setminus B_{5r\epsilon'}(\hat{x}))$. Next, note that $B_{4\epsilon'}(a_i) \subseteq B_{5r\epsilon'}(\hat{a}_i)$. So, by the fact that $\|a_i - \hat{a}_i\| \leq \epsilon'$,

$$\text{dist}(\hat{a}_i, \text{CH}(\Delta \setminus B_{5r\epsilon'}(\hat{a}_i))) \geq \text{dist}(a_i, \text{CH}(\Delta \setminus B_{4\epsilon'}(a_i))) - \epsilon' \geq 3\epsilon'.$$

Furthermore, we argue that if there is $\hat{x} \in \text{CH}(\hat{S}_l \setminus B_{5r\epsilon'}(\hat{a}_i))$ then there exists $x \in \text{CH}(\Delta \setminus B_{4\epsilon'}(\hat{a}_i))$, such that $\|x - \hat{x}\| \leq \epsilon'$. The proof of this claim is relegated to Appendix E.1. Using this claim, we have

$$\text{dist}(\hat{a}_i, \text{CH}(\hat{S}_l \setminus B_{6\epsilon'}(\hat{a}_i))) \geq 2\epsilon'. \qed$$
Given the above structure, it is clear that set of points in \( C \) are all within \( \epsilon \) of one of the \( a_i \)'s. So, we can cluster \( C \) using single linkage with threshold \( \epsilon \) to recover \( a_i \) up to accuracy \( \epsilon \).

5 Additional Results, Extensions, and Open Problems

5.1 Sample Complexity Lower bound

As we observed the number of samples required by our method is \( \text{poly}(n) \). However, as the number of classes can be much smaller than the number of features, one might hope to recover \( v_1, \ldots, v_k \), with a number of samples that is polynomial in \( k \) rather than \( n \). Here, we show that in the general case \( \Omega(n) \) samples are needed to learn \( v_1, \ldots, v_k \) regardless of the value of \( k \).

For ease of exposition, let \( k = 1 \) and note that in this case every sample should be purely of one type. Assume that the class vector, \( v \), is promised to be in the set \( C = \{ v^j \mid v^j = 1/\sqrt{2}, \text{if } \ell = 2j - 1 \text{ or } 2j, \text{else } v^j = 0 \} \). Consider instances \( (x^1_j, x^2_j) \) such that the \( \ell \)th coordinate of \( x^1_j \) is \( x^1_{j\ell} = -1/\sqrt{2} \) if \( \ell = 2j - 1 \) and \( 1/\sqrt{2} \) otherwise, and \( x^2_{j\ell} = -1/\sqrt{2} \) if \( \ell = 2j \) and \( 1/\sqrt{2} \) otherwise. For a given \( (x^1_j, x^2_j) \), we have that \( v^j \cdot x^1_j = v^j \cdot x^2_j = 0 \). On the other hand, for all \( \ell \neq j \), \( v^\ell \cdot x^1_j = v^\ell \cdot x^2_j = 1 \). Therefore, sample \( (x^1_j, x^2_j) \) is consistent with \( v = v^\ell \) for any \( \ell \neq j \), but not with \( v = v^j \). That is, each instance \( (x^1_j, x^2_j) \) renders only one candidate of \( C \) invalid. Even after observing at most \( n/2 - 2 \) samples of this type, at least 2 possible choices for \( v \) remain. So, \( \Omega(n) \) samples are indeed needed to find the appropriate \( v \). The next theorem, whose proof appears in Appendix F generalizes this construction and result to the case of any \( k \).

**Theorem 5.1.** For any \( k \leq n \), any algorithm that for all \( i \in [k] \) learns \( v_i' \) such that \( \|v_i - v_i'\|_2 \leq 1/\sqrt{2} \), requires \( \Omega(n) \) samples.

Note that in the above construction samples have large components in the irrelevant features. It would be interesting to see if this lower bound can be circumvented using additional natural assumptions in this model, such as assuming that the samples have length \( \text{poly}(k) \).

5.2 Alternative Noise Models

Consider the problem of recovering \( v_1, \ldots, v_k \) in the presence of agnostic noise, where for an \( \epsilon \) fraction of the samples \( (x^1_j, x^2_j) \), \( x^1 \) and \( x^2 \) correspond to different mixture weights. Furthermore, assume that the distribution over the instance space is rich enough such that any subspace other than \( \text{span}\{v_1, \ldots, v_k\} \) is inconsistent with a set of instances of non-negligible density.\(^4\) Since the VC dimension of the set of \( k \) dimensional subspaces in \( \mathbb{R}^n \) is \( \min\{k, n - k\} \), from the information theoretic point of view, one can recover \( \text{span}\{v_1, \ldots, v_k\} \) as it is the only subspace that is inconsistent with less than \( O(\epsilon) \) fraction of \( \tilde{O}(n/k) \) samples. Furthermore, we can detect and remove any noisy sample, for which the two views of the sample are not consistent with \( \text{span}\{v_1, \ldots, v_k\} \). And finally, we can recover \( a_1, \ldots, a_k \) using phase 2 of Algorithm 1.

In the above discussion, it is clear that once we have recovered \( \text{span}\{v_1, \ldots, v_k\} \), denoising and finding the extreme points of the projections can be done in polynomial time. For the problem of recovering a \( k \)-dimensional nullspace, Hardt and Moitra (2013) introduced an efficient algorithm that tolerates agnostic noise up to \( \epsilon = O(k/n) \). Furthermore, they provide an evidence that this result might be tight. It would be interesting to see whether additional structure present in our model, such as the fact that samples are convex combination of classes, can allow us to efficiently recover the nullspace in presence of more noise.

Another interesting open problem is whether it is possible to handle the case of \( p_0 = 0 \). That is, when every document is affected by Gaussian noise \( N(0, \sigma^2 I_n) \), for \( \sigma \gg \epsilon \). A simpler form of this problem is

\(^4\)This assumption is similar to the richness assumption made in the standard case, where we assume that there is enough “entropy” between the two views of the samples such that even in the non-noisy case the subspace can be uniquely determined by taking the nullspace of \( X_1 - X_2 \).
as follows. Consider a distribution induced by first drawing \( x \sim D \), where \( D \) is an arbitrary and unknown distribution over \( \Delta = \text{CH}\{\{a_1, \ldots, a_k\}\} \), and taking \( \tilde{x} = x + \mathcal{N}(0, \sigma^2 I_n) \). Can we learn \( a_i \)'s within error of \( \epsilon \) using polynomially many samples? Note that when \( D \) is only supported on the corners of \( \Delta \), this problem reduces to learning mixture of Gaussians, for which there is a wealth of literature on estimating Gaussian means and mixture weights (Dasgupta et al., 2002; Kalai et al., 2012; Moitra and Valiant, 2010). It would be interesting to see whether a more general class of similarity functions, such as kernels, can also be learned in this context.

5.3 General function \( f(\cdot) \)

Consider the general model described in Section 2, where \( f_i(x) = f(v_i \cdot x) \) for an unknown strictly increasing function \( f : \mathbb{R}^+ \rightarrow [0, 1] \) such that \( f(0) = 0 \). We describe how variations of the techniques discussed up to now can extend to this more general setting.

For ease of exposition, consider the non-noisy case. Since \( f \) is a strictly increasing function, \( f(v_i \cdot x^1) = f(v_i \cdot x^2) \) if and only if \( v_i \cdot x^1 = v_i \cdot x^2 \). Therefore, we can recover \( \text{span}(v_1, \ldots, v_k) \) by the same approach as in Phase 1 of Algorithm 1. Although, by definition of pseudoinverse matrices, the projection of \( x \) is still represented by \( x = \sum_i (v_i \cdot x)a_i \), this is not necessarily a convex combination of \( a_i \)'s anymore. This is due to the fact that \( v_i \cdot x \) can add up to values larger than 1 depending on \( x \). However, \( x \) is still a non-negative combination of \( a_i \)'s. Moreover, \( a_i \)'s are linearly independent, so \( a_i \) can not be expressed by a nontrivial non-negative combination of other samples. Therefore, for all \( i \), \( a_i/\|a_i\| \) can be recovered by taking the extreme rays of the convex cone of the projected samples. So, we can recover \( v_1, \ldots, v_k \), by taking the pseudoinverse of \( a_i/\|a_i\| \) and re-normalizing the outcome such that \( \|v_i\|_2 = 1 \). When samples are perturbed by noise, a similar argument that also takes into account the smoothness of \( f \) proves similar results.

It would be interesting to see whether a more general class of similarity functions, such as kernels, can be also learned in this context.

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A Omitted Proof from Section 3 — No Noise

A.1 Proof of Lemma 3.1

For all \( j \leq n - k \), let \( Z_j = \{ (x_1^i - x_2^i) \mid i \leq \frac{j}{\xi} \ln \frac{n}{\delta} \} \). We prove by induction that for all \( j \), \( \text{rank}(Z_j) < j \) with probability at most \( \frac{j}{n} \).

For \( j = 0 \), the claim trivially holds. Now assume that the induction hypothesis holds for some \( j \). Furthermore, assume that \( \text{rank}(Z_j) \geq j \). Then, \( \text{rank}(Z_{j+1}) < j+1 \) only if the additional \( \frac{j}{\xi} \ln \frac{n}{\delta} \) samples in \( Z_{j+1} \) all belong to \( \text{span}(Z_j) \). Since, the space of such samples has rank \( < n - k \), this happens with probability at most
\[ (1 - \zeta)^{\frac{1}{2}} \ln \frac{2}{\delta} \leq \frac{\delta}{n}. \] Together with the induction hypothesis that \( \text{rank}(Z_j) \geq j \) with probability at most \( j \frac{\delta}{n} \), we have that \( \text{rank}(Z_{j+1}) < j + 1 \) with probability at most \( \frac{(j+1)^{\frac{1}{2}}}{n} \). Therefore \( \text{rank}(Z) = \text{rank}(Z_{n-k}) = n - k \) with probability at least \( 1 - \delta \).

### A.2 Proof of Lemma 3.2

First note that \( V \) is the pseudo-inverse of \( A \), so their span is equal. Hence, \( \sum_{i \in [k]} (v_i \cdot x) a_i \in \text{span} \{ v_1, \ldots, v_k \} \).

It remains to show that \( (x - \sum_{i \in [k]} (v_i \cdot x) a_i) \in \text{null} \{ v_1, \ldots, v_k \} \). We do so by showing that this vector is orthogonal to \( v_j \) for all \( j \). We have

\[
\begin{align*}
(x - \sum_{i=1}^{k} (v_i \cdot x) a_i) \cdot v_j &= x \cdot v_j - \sum_{i=1}^{k} (v_i \cdot x) (a_i \cdot v_j) \\
&= x \cdot v_j - \sum_{i \neq j} (v_i \cdot x) (a_i \cdot v_j) - (v_j \cdot x) (a_j \cdot v_j) \\
&= x \cdot v_j - x \cdot v_j = 0.
\end{align*}
\]

Where, the second equality follows from the fact when \( A = V^+ \), for all \( i, a_i \cdot v_i = 1 \) and \( a_j \cdot v_i = 0 \) for \( j \neq i \). Therefore, \( \sum_{i \in [k]} (v_i \cdot x) a_i \) is the projection of \( x \) on \( \text{span} \{ v_1, \ldots, v_k \} \).

### A.3 Proof of Lemma 3.3

Assume that \( S \) included samples that are purely of type \( i \), for all \( i \in [k] \). That is, for all \( i \in [k] \) there is \( j \leq m \), such that \( v_i \cdot x_j^i = v_i \cdot x_j^1 = 1 \) and \( v_i' \cdot x_j^1 = v_i' \cdot x_j^1 = 0 \) for \( i' \neq i \). By Lemma 3.2, the set of projected vectors form the set \( \{ \sum_{i=1}^{k} (v_i \cdot x_j) a_i \mid j \in [m] \} \). Note that \( \sum_{i=1}^{k} (v_i \cdot x_j) a_i \) is in the simplex with vertices \( a_1, \ldots, a_k \). Moreover, for each \( i \), there exists a pure sample in \( S \) of type \( i \). Therefore, \( \text{CH} \{ \sum_{i=1}^{k} (v_i \cdot x_j) a_i \mid j \in [m] \} \) is the simplex on linearly independent vertices \( a_1, \ldots, a_k \). As a result, \( a_1, \ldots, a_k \) are the extreme points of it.

It remains to prove that with probability \( 1 - \delta \), the sample set has a document of purely type \( j \), for all \( j \in [k] \). By the assumption on the probability distribution \( P \), with probability at most \( (1 - \xi)^m \), there is no document of type purely \( j \). Using the union bound, we get the final result.

### B Technical Spectral Lemmas

#### Proposition B.1 (Davis and Kahan (1970) sin \( \theta \) theorem).

Let \( B, \tilde{B} \in \mathbb{R}^{p \times p} \) be symmetric, with eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_p \) and \( \tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_p \), respectively. Fix \( 1 \leq r \leq s \leq p \) and let \( V = (v_r, \ldots, v_s) \) and \( \tilde{V} = (\tilde{v}_r, \ldots, \tilde{v}_s) \) be the orthonormal eigenvectors corresponding to \( \lambda_r, \ldots, \lambda_s \) and \( \tilde{\lambda}_r, \ldots, \tilde{\lambda}_s \). Let \( \delta = \inf \{ |\lambda - \tilde{\lambda}| : \lambda \in [\lambda_s, \lambda_r], \tilde{\lambda} \in [-\infty, \tilde{\lambda}_{r+1}) \cup (\tilde{\lambda}_{r+1}, \infty) \} > 0 \). Then,

\[
\| \sin \Theta(V, \tilde{V}) \|_2 \leq \frac{\| \tilde{B} - B \|_2}{\delta}.
\]

where \( \sin \Theta(V, \tilde{V}) = PV - \tilde{P}V \), where \( PV \) and \( \tilde{P}V \) are the projection matrices for \( V \) and \( \tilde{V} \).

#### Proposition B.2 (Corollary 5.50 (Vershynin, 2010)).

Consider a Gaussian distribution in \( \mathbb{R}^n \) with covariance matrix \( \Sigma \). Let \( A \in \mathbb{R}^{n \times m} \) be a matrix whose rows are drawn i.i.d from this distribution, and let \( \Sigma_m = \frac{1}{m} AA^\top \). For every \( \varepsilon \in (0, 1) \), and \( t, m \geq \text{cn}(t/\varepsilon)^2 \) for some constant \( c \), then with probability at least \( 1 - 2 \exp(-t^2/2) \), \( \| \Sigma_m - \Sigma \|_2 \leq \varepsilon \| \Sigma \|_2 \)

#### Proposition B.3 (Matrix Bernstein (Tropp, 2015)).

Let \( S_1, \ldots, S_n \) be independent, centered random matrices with common dimension \( d_1 \times d_2 \), and assume that each one is uniformly bounded. That is, \( \mathbb{E}S_i = 0 \) and
\[ \|S_i\|_2 \leq L \text{ for all } i \in [n]. \] Let \( Z = \sum_{i=1}^{n} S_i \), and let \( v(Z) \) denote the matrix variance:

\[
v(Z) = \max \left\{ \left\| \sum_{i=1}^{n} E[S_i S_i^\top] \right\|, \left\| \sum_{i=1}^{n} E[S_i^\top S_i] \right\| \right\}.
\]

Then,

\[
\mathcal{P}[\|Z\| \geq t] \leq (d_1 + d_2) \exp \left( \frac{-t^2/2}{v(Z) + Lt/3} \right).
\]

**Proposition B.4** (Theorem 4.10 of Stewart and Sun (1990)). Let \( \hat{A} = A + E \) and let \( \lambda_1, \ldots, \lambda_n \) and \( \lambda'_1, \ldots, \lambda'_n \) be the eigenvalues of \( A \) and \( A + E \). Then, \( \max \{ |\lambda'_i - \lambda_i| \} \leq \|E\|_2 \).

**Proposition B.5** (Theorem 3.3 of Stewart (1977)). For any \( A \) and \( B = A + E \),

\[
\|B^+ - A^+\| \leq \max \left\{ \|A^+\|^2, \|B^+\|^2 \right\} \|E\|,
\]

where \( \| \cdot \| \) is an arbitrary norm.

## C Omitted Proof from Section 4.1 — Phase 1

### C.1 Proof of Claim 4.6

Let \( e_i \) and \( d_i \) be the \( i^{th} \) row of \( E \) and \( D \). Then \( ED^\top = \sum_{i=1}^{m} e_i d_i^\top \) and \( DE^\top = \sum_{i=1}^{m} d_i e_i^\top \). Let \( S_i = \frac{1}{m} \begin{bmatrix} 0 & e_i d_i^\top \\ d_i e_i^\top & 0 \end{bmatrix} \). Then, \( \frac{1}{m} DE^\top + \frac{1}{m} ED^\top \|_2 \leq 2 \| \sum_{i=1}^{m} S_i \|_2 \). We will use matrix Bernstein to show that \( \sum_{i \in [m]} S_i \) is small with high probability.

First note that the distribution of \( e_i \) is a Gaussian centered at 0, therefore, \( E[S_i] = 0 \). Furthermore, for each \( i \), with probability \( 1 - \delta \), \( \|e_i\|_2 \leq \sigma \sqrt{\log \frac{1}{\delta}} \). So, with probability \( 1 - \delta \), for all samples \( i \in [m] \), \( \|e_i\|_2 \leq \sigma \sqrt{\log \frac{m}{\delta}} \). Moreover, by assumption \( \|d_i\| = \|x_i^1 - x_i^2\| \leq 2M \). Therefore, with probability \( 1 - \delta \),

\[
L = \max_i \|S_i\|_2 = \frac{1}{m} \max_i \|e_i\| \|d_i\| \leq \frac{2}{m} \sigma \sqrt{nM \log \frac{n}{\delta}}.
\]

Note that, \( \|E[S_i S_i^\top]\| = \frac{1}{m^2} \|E[(e_i d_i^\top)^2]\| \leq L^2 \). Since \( S_i \) is Hermitian, the matrix covariance defined by Matrix Bernstein inequality is

\[
v(Z) = \max \left\{ \left\| \sum_{i=1}^{m} E[S_i S_i^\top] \right\|, \left\| \sum_{i=1}^{m} E[S_i^\top S_i] \right\| \right\} = \left\| \sum_{i=1}^{m} E[S_i S_i^\top] \right\| \leq mL^2.
\]

If \( \epsilon \leq v(Z)/L \) and \( m \in \Omega(\frac{n \sigma^2 M^2}{\epsilon^2} \log \frac{n}{\delta^2}) \) or \( \epsilon \geq v(Z)/L \) and \( m \in \Omega(\frac{\sqrt{n} \sigma M}{\epsilon} \log \frac{n}{\delta}) \), using Matrix Bernstein inequality (Proposition B.3), we have

\[
\Pr \left[ \left\| \frac{1}{m} DE^\top + \frac{1}{m} ED^\top \right\| \geq \epsilon \right] = \Pr \left[ \left\| \sum_{i=1}^{m} S_i \right\| \geq \frac{\epsilon}{2} \right] \leq \delta.
\]

### C.2 Proof of Claim 4.7

Let \( d_i \) be the \( i^{th} \) row of \( D \). Then \( DD^\top = \sum_{i=1}^{m} d_i d_i^\top \). Let \( S_i = \frac{1}{m} d_i d_i^\top - \frac{1}{m} E[d_i d_i^\top] \). Then, \( \frac{1}{m} DD^\top - E \left[ \frac{1}{m} DD^\top \right] \|_2 = \left\| \sum_{i=1}^{m} S_i \right\|_2 \). Since, \( d_i = x_i^1 - x_i^2 \) and \( \|x_i^j\| \leq M \), we have that for any \( i \), \( \|d_i d_i^\top - E[d_i d_i^\top]\| \leq 4M^2 \). Then,

\[
L = \max_i \|S_i\|_2 = \frac{1}{m} \max_i \|d_i d_i^\top - E[d_i d_i^\top]\|_2 \leq \frac{4}{m} M^2,
\]

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and \( \|E[S_i S_i^T]\| \leq L^2 \). Note that \( S_i \) is Hermitian, so the matrix covariance is

\[
v(Z) = \max \left\{ \left\| \sum_{i=1}^{m} E[S_i S_i^T] \right\|, \left\| \sum_{i=1}^{m} E[S_i^T S_i] \right\| \right\} = \left\| \sum_{i=1}^{m} E[S_i S_i^T] \right\| \leq mL^2.
\]

If \( \delta_0 \leq 4M^2 \) and \( m \in \Omega(\frac{M^3}{\delta_0^2} \log \frac{n}{\delta_0}) \) or \( \delta_0 \geq 4M^2 \) and \( m \in \Omega(\frac{M^2}{\delta_0} \log \frac{n}{\delta_0}) \), then by Matrix Bernstein inequality (Proposition B.3), we have

\[
\Pr \left[ \left\| \sum_{i=1}^{m} S_i \right\| \geq \frac{\delta_0}{2} \right] \leq \delta.
\]

### D Omitted Proof from Section 4.2 — Denoising

#### D.1 Proof of Claim 4.8

Recall that for any \( i \in [k] \), with probability \( \gamma = \frac{\epsilon}{(8k\alpha)} \) a nearly pure weight vector \( w \) is generated from \( P \), such that \( \|w - e_i\| \leq \epsilon/(8k\alpha) \). And independently, with probability \( p_0 \) the point is not noisy. Therefore, there is \( p_0 \gamma \) density on non-noisy points that are almost purely of class \( i \). Note that for such points, \( x \),

\[
\|Px - a_i\| = \left\| \sum_{j=1}^{k} w_j a_j - a_i \right\| \leq k(\epsilon/(8k\alpha))(\alpha) \leq \frac{\epsilon}{8}.
\]

Since \( \|P - \hat{P}\| \leq \epsilon/8M \), we have

\[
\|a_i - \hat{P}x\| = \|a_i - Px\| + \|Px - \hat{P}x\| \leq \frac{\epsilon}{8} + \frac{\epsilon}{8} \leq \frac{\epsilon}{4}.
\]

The claim follows immediately.

### E Omitted Proof from Section 4.3 — Phase 2

#### E.1 Omitted proof from Claim 4.11

Here, we prove that \( \hat{x} \in \text{CH}(\hat{S}_0 \setminus B_{d+\epsilon'}(\hat{a}_i)) \) then there exists \( x \in \text{CH}(\Delta \setminus B_{d}(a_i)) \), such that \( \|x - \hat{x}\| \leq \epsilon' \).

Let \( x = \sum_i \alpha_i z_i \) be the convex combination of \( z_1, \ldots, z_t \in \hat{S}_0 \setminus B_{d+\epsilon'}(\hat{a}_i) \). By Claim 4.3, there are \( z_1, \ldots, z_t \in \Delta \), such that \( \|z_i - \hat{z}_i\| \leq \epsilon' \) for all \( i \in [k] \). Furthermore, by the proximity of \( z_i \) to \( \hat{z}_i \) we have that \( z_i \notin B_d(\hat{a}_i) \). Therefore, \( z_1, \ldots, z_t \in \Delta \setminus B_d(\hat{a}_i) \). Then, \( x = \sum_i \alpha_i z_i \) is also within distance \( \epsilon' \).

### F Proof of Theorem 5.1 — Lower Bound

For ease of exposition assume that \( n \) is a multiple of \( k \). Furthermore, in this proof we adopt the notion \( (x_i, x'_i) \) to represent the two views of the \( i^{th} \) sample. For any vector \( u \in \mathbb{R}^n \) and \( i \in [k] \), we use \((u)_i\) to denote the \( i^{th} \) \( \frac{n}{k} \)-dimensional block of \( u \), i.e., coordinates \( u_{(i-1)\frac{n}{k}+1}, \ldots, u_{i\frac{n}{k}} \).

Consider the \( \frac{n}{k} \)-dimensional vector \( u_j \), such that \( u_{j,\ell} = 1 \) if \( \ell = 2j - 1 \) or \( 2j \), and \( u_{j,\ell} = 0 \), otherwise. And consider \( \frac{1}{k} \)-dimensional vectors \( z_i' \) and \( z'_i' \), such that \( z_{i,\ell} = 1 \) if \( \ell = 2j - 1 \) and \( z'_{i,\ell} = 1 \) otherwise, and \( z'_{i,\ell} = 1 \) otherwise. Consider a setting where \( v_i \) is restricted to the set of candidate \( C_i = \{v_i^j \mid (v_i^j)_i = u_{j,\ell} / \sqrt{2} \text{ and } (v_i^j)_{i'} = 0 \text{ for } i' \neq i\} \). In other words, the \( \ell^{th} \) coordinate of \( v_i^j \) is \( 1/\sqrt{2} \) if \( \ell = (i-1)\frac{n}{k} + 2j - 1 \) or \( (i-1)\frac{n}{k} + 2j \), else 0. Furthermore, consider instances \( (x_i^j, x''_i^j) \) such that \( (x_i^j)_i = z_j/\sqrt{2} \) and \( (x''_i^j)_i = z'_j/\sqrt{2} \) and for all \( i' \neq i \), \( (x_i^j)_{i'} = (x''_i^j)_{i'} = 0 \). In other words,

\[
(x_i^j)_{(i-1)\frac{n}{k}+2j-1(i-1)\frac{n}{k}+2j} = \frac{1}{\sqrt{2}} (0, \ldots, 0, 1, \ldots, 1, \frac{1}{\sqrt{2}}, 1, \ldots, 1, 0, \ldots, 0).
\]
First note that, for any $i, i' \in [k]$ and any $j, j' \in \left[ \frac{n}{2k} \right]$, $v_i^j \cdot x_i^j = v_i^j \cdot x_i^{j'}$. That is, the two views of all instances are consistent with each other with respect to all candidate vectors. Furthermore, for any $i$ and $i'$ such that $i \neq i'$, for all $j, j'$, $v_i^j \cdot x_i^j = 0$. Therefore, for any observed sample $(x_i^j, x_i^{j'})$, the sample should be purely of type $i$.

For a given $i$, consider all the samples $(x_i^j, x_i^{j'})$ that are observed by the algorithm. Note that $v_i^j \cdot x_i^j = v_i^j \cdot x_i^{j'} = 0$. And for all $j' \neq j$, $v_i^j \cdot x_i^j = v_i^{j'} \cdot x_i^{j'} = 1$. Therefore, observing $(x_i^j, x_i^{j'})$ only rules out $v_i^j$ as a candidate, while this sample is consistent with candidates $v_i^{j'}$ for $j' \neq j$. Therefore, even after observing $\leq \frac{n}{2k} - 2$ samples of this types, at least 2 possible choices for $v_i$ remain valid. Moreover, the distance between any two $v_i^j, v_i^{j'} \in C_i$ is $\sqrt{2}$. Therefore, $\frac{n}{2k} - 1$ samples are needed to learn $v_i$ to an accuracy better than $\sqrt{2}/2$.

Note that consistency of the data with $v_{i'}$ is not affected by the samples of type $x_i^j$ that are observed by the algorithms when $i' \neq i$. So, $\Omega(k \frac{n}{k}) = \Omega(n)$ samples are required to approximate all $v_i$'s to an accuracy better than $\sqrt{2}/2$.  

$$x_i^j = \frac{1}{\sqrt{2}} (0, \ldots, 0, 1, \ldots, 1, -1, 1, \ldots, 1, 0, \ldots, 0),$$
$$v_i^j = \frac{1}{\sqrt{2}} (0, \ldots, 0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0).$$

$i$th block