ON THE ABUNDANCE OF NON-ZERO CENTRAL LYAPUNOV EXPONENTS, PHYSICAL MEASURES AND STABLE ERGODICITY FOR PARTIALLY HYPERBOLIC DYNAMICS

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ABSTRACT. We show that the time-1 map of an Anosov flow, whose strong-unstable foliation is $C^2$ smooth and minimal, is $C^2$ close to a diffeomorphism having positive central Lyapunov exponent Lebesgue almost everywhere and a unique physical measure with full basin, which is $C'$ stably ergodic. Our method is perturbative and does not rely on preservation of a smooth measure.

RÉSUMÉ: Nous montrons que le temps-1 d’un flux d’Anosov, dont le foliation forte-instable est $C^2$ lisse et minimal, est $C^2$ proche d’un difféomorphisme ayant exposant de Lyapunov central positif Lebesgue presque partout et une unique mesure physique avec basin plein, ce qui est $C'$-stablement ergodique. Notre méthode est perturbatif, et ne repose pas sur la préservation d’une mesure de volume.

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1. INTRODUCTION

Stable ergodicity is a desirable property for dynamical systems since it is arguably the most basic global statistical feature. This is inspired by the fundamental Boltzman Ergodic Hypothesis from Statistical Mechanics, which is the main motivation behind the celebrated Birkhoff Ergodic Theorem, ensuring the equality between temporal and spatial averages with respect to a (ergodic) probability measure $\mu$ invariant under a measurable transformation $f : M \to M$ of a compact manifold $M$, i.e. for every integrable function $\varphi : M \to \mathbb{R}$ we have

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu$$

for $\mu$ almost every point $x \in M$.

Systems nearby a stably ergodic system remain ergodic. In general we need some “natural measure” to explore stable ergodicity. D. V. Anosov in [4] was the first to establish the existence of open sets of ergodic systems on a wide class of manifolds: the geodesic flow on the unit tangent bundle of compact Riemannian manifolds with constant negative sectional curvature. Systems sharing the same features are known today as “Anosov systems”: they are globally hyperbolic and structurally stable (all dynamical properties of their perturbations are the same on all scales). Previous results on hyperbolicity and ergodicity with respect to the natural Liouville volume measure on the unit tangent bundle of compact surfaces with constant negative Gaussian curvature were obtained earlier by Hedlund [27] and Hopf [29, 30].

Pugh and Shub began a program devoted to stable ergodicity [35] for partially hyperbolic volume preserving systems. They ask how frequently is partial hyperbolicity the main reason for a dynamical system to be stably ergodic and conjectured that, among the volume preserving partially hyperbolic dynamical systems, the stably ergodic ones form an open and dense set. Many advances have been obtained in this direction recently; see e.g. Rodriguez Hertz et al [38]. The natural measure in this setting is the
Lebesgue volume measure which disintegrates as a density along the unstable foliations associated to partial hyperbolic dynamics: such measures are known as \textit{u-Gibbs states}.

In the dissipative setting the \textit{SRB/physical measures} are natural candidates to play the role of Lebesgue measure, since for these measures the time averages coincide with the space averages for orbits starting in a positive volume subset of the ambient space: points satisfying (1.1) for all continuous functions \( \phi : M \to \mathbb{R} \) (the \textit{ergodic basin} of the measure) form a subset of positive volume. Moreover, such physical measures for partially hyperbolic systems disintegrate as densities along the unstable directions also, that is, they are \textit{u-Gibbs states}. So the question raised by Pugh and Shub makes sense also in a non-conservative setting.

In [18, 19] Burns, Dolgopyat, Pesin and Pollicott studied stable ergodicity for partially hyperbolic diffeomorphims, not necessarily conservative, whose Lyapunov exponents along the center direction are all negative with respect to some \textit{u-Gibbs state} which, in this setting, becomes a physical measure. In [45, 3] the case with positive Lyapunov exponents along the center direction was considered. The main reason to consider non-zero central Lyapunov exponents in the works cited before is that in many cases hyperbolic \textit{u-Gibbs states} are physical measures [22, 17, 1, 45]. Moreover Bochi, Fayad and Pujals in [12] have noted that stably ergodic \textit{conservative} diffeomorphisms must be \( C^1 \) close to a conservative diffeomorphism with stably non-zero central Lyapunov exponents.

We recall the following problem posed by Bonatti, Díaz and Pujals in [14].

\begin{problem}
Let \( f \) be a \( C^1 \) robustly transitive diffeomorphism of class \( C^2 \) on a compact manifold \( M \). Does there exist \( g \) close to \( f \) having finitely many physical measures such that the union of their basins has total Lebesgue measure in \( M \)?
\end{problem}

This problem is connected with partial hyperbolicity since in [14] it was proved that robustly transitive diffeomorphisms in compact manifolds of dimension three are partially hyperbolic.

Hence the abundance of non-zero Lyapunov exponents along the central direction of a strongly partially hyperbolic diffeomorphism is closely related with the issue of the existence (abundance) of physical measures and its ergodicity. This work is devoted to shed some light on this subject.

The aim of our work is to show that, under certain conditions explained below, it is possible to remove zero central Lyapunov exponents by perturbation inside a class of partially hyperbolic diffeomorphisms which are not conservative. This is an extension of results from Burns, Pugh and Wilkinson [47, 20] about stable ergodicity of the time-one map of a geodesic flow, now encompassing also non-conservative perturbations.

There are several results about removing zero Lyapunov exponents either in the conservative setting, by Shub and Wilkinson [43], Ruelle [39] and Baraviera and Bonatti [6] for diffeomorphims and [10] for flows; or for partially hyperbolic diffeomorphisms with center foliation formed by compact center leaves, but relying on rigidity arguments, by Viana and Yang [46] and F. Rodriguez-Hertz, M.A. Rodriguez-Hertz, Tahzibi and
Ures [37]. On the opposite direction, that of forcing zero Lyapunov exponents in the absence of weak forms of hyperbolicity (like dominated splitting on the tangent bundle) among conservative diffeomorphisms, there are the works of Bochi [11] together with Viana [13], and the corresponding versions for flows by Bessa [9] together with one of the authors [5], always restricted to \(C^1\) perturbation techniques. However, to the authors best knowledge, there are no other general perturbation results on central Lyapunov exponents for not conservative systems.

We consider the case of the time-1 map of a \(C^2\) Anosov flow which is not a suspension flow of an Anosov diffeomorphism and whose strong-unstable foliation is smooth; see the next section for details.

Our method relies on the following motivation: if we assume that a function \(\varphi_f : M \to \mathbb{R}\) and a \(f\)-invariant probability measure \(\mu_f\) are given, the function depending continuously on the local dynamics of \(f\) and the measure depending continuously on \(f\) in the space of diffeomorphisms, then we should be able to change the value of the integral \(\int \varphi_f \, d\mu_f\) by an arbitrarily small perturbation of the map \(f\).

Our results are deduced applying the above idea to the integral of the logarithm of the central Jacobian with respect to a \(u\)-Gibbs state, for a strongly partially hyperbolic system whose strong-unstable foliation is smooth (the foliation is of class \(C^2\)) and minimal (every leaf is dense in the ambient space). By Oseledet’s Multiplicative Ergodic Theorem (see e.g. Barreira and Pesin [7]) this integral gives the central Lyapunov exponent. Therefore it is not possible to have a constant zero central Lyapunov exponent for all \(u\)-Gibbs states in a \(C^2\) neighborhood of a partially hyperbolic map with a smooth minimal strong-unstable foliation.

The continuity of \(u\)-Gibbs states under perturbations has been studied by one the authors in [44]; see also [19]. To be able to control certain features of the \(u\)-Gibbs states of the perturbed map, we need to ensure that after the perturbation the new map is \(C^2\) close to the original one. For this we were led to assume that the strong-unstable foliation is \(C^2\) smooth. In this setting, we can construct our perturbation \(C^2\) close to the original map, enabling us to use the results of smooth ergodic theory already known for \(u\)-Gibbs states.

The assumption of minimality is rather natural in this setting since Parry, in [32], showed that a linear torus automorphism is ergodic (with respect to the Haar/volume measure) if, and only if, the corresponding strong-stable foliation is minimal. Moreover by the results of Plante [34] for a transitive Anosov flow on a compact manifold, either the strong-stable and strong-unstable foliations are minimal, or the flow is the suspension of an Anosov diffeomorphism of a compact submanifold with codimension one. We note that the existence of partially hyperbolic diffeomorphisms having robustly minimal strong-unstable foliations was obtained by Bonatti, Díaz and Ures in [15].

The minimality of the unstable foliation ensures, in our setting, that the future orbit of Lebesgue almost every point has positive frequency of visits to any open subset. In the conservative setting, we have that Lebesgue almost every point has well defined time averages for the future and for the past, i.e. under iterates of the map and of the inverse map. This
is not necessarily true in general and prevents us from using arguments similar to the ones of e.g. Burns, Wilkinson [21] and F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, R. Ures [36] in the conservative setting.

The closely related results of Dolgopyat [24] where obtained using completely different techniques of a more analytic nature, and are put in the setting of perturbation along generic one-parameter families of maps through the original map \( f \). In addition the results are stated and proved in the setting where the stable and unstable subbundles are one-dimensional.

So, in this work we establish a perturbative method to remove zero Lyapunov exponents (Theorem A and its proof) in a non-conservative setting; we deduce existence and uniqueness of physical measures (Corollary B) and also stable ergodicity for the perturbed systems (Corollary C).

2. Statement of the results

Let \( M \) be a closed Riemannian manifold. We denote by \( \| \cdot \| \) the norm obtained from the Riemannian structure and by \( \text{Leb} \) the Lebesgue measure on \( M \).

If \( V, W \) are normed linear spaces, we define

\[
\| A \| = \sup \{ \| Av \|_W / \| v \|_V, v \in V \setminus \{0\} \},
\]

and

\[
m(A) = \inf \{ \| Av \|_W / \| v \|_V, v \in V \setminus \{0\} \}
\]

for a linear map \( A : V \to W \).

A diffeomorphism \( f : M \to M \) is strongly partially hyperbolic if there exists a continuous \( Df \)-invariant splitting of \( TM \),

\[
TM = E^s \oplus E^c \oplus E^u,
\]

and there exist constants \( C \geq 0 \) and

\[
0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3
\]

with \( \mu_1 < 1 < \lambda_3 \) such that for all \( x \in M \) and every \( n \geq 1 \) we have:

\[
C^{-1} \lambda_1^n \leq m(Df^n(x)|E^s(x)) \leq \| Df^n(x)|E^s(x) \| \leq C\mu_1^n, \quad (2.1)
\]

\[
C^{-1} \lambda_2^n \leq m(Df^n(x)|E^c(x)) \leq \| Df^n(x)|E^c(x) \| \leq C\mu_2^n, \quad (2.2)
\]

\[
C^{-1} \lambda_3^n \leq m(Df^n(x)|E^u(x)) \leq \| Df^n(x)|E^u(x) \| \leq C\mu_3^n. \quad (2.3)
\]

The expression (2.1) means that \( E^s \) is uniformly contracting, while (2.3) means that \( E^u \) is uniformly expanding. The expression (2.2) implies that \( E^u \) dominates \( E^c \) and that \( E^c \) dominates \( E^s \). We assume that the subbundles are non-trivial.

It is well known that partially hyperbolic diffeomorphisms have an unstable foliation \( \mathcal{F}^u = \{ \mathcal{F}^u(x) : x \in M \} \), whose leaves are the (strong) unstable manifolds \( W^u(x), x \in M \); and they also have a stable foliation \( \mathcal{F}^s = \{ \mathcal{F}^s(x) : x \in M \} \), whose leaves
are the (strong) stable manifolds $W^s(x), x \in M$. For $C^r$ diffeomorphisms, $r > 1$, these foliations are absolutely continuous; see Hirsch, Pugh and Shub [28].

An $f$-invariant probability measure $\mu$ is a $u$-Gibbs state if the conditional measures of $\mu$ with respect to the partition into local strong-unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local strong-unstable manifolds. If $f$ is a $C^2$-partially hyperbolic diffeomorphism, there always exists a $u$-Gibbs state; see Proposition 3.1 or see e.g. Pesin and Sinai [33] for more details.

We recall that, if $\mu$ is a $f$-invariant measure, the (ergodic) basin of $\mu$ is the set $B(\mu)$ of all points $x \in M$ such that (1.1) is satisfied for all continuous functions $\varphi : M \to \mathbb{R}$. It is well known that the set $B(\mu)$ has full measure with respect to any ergodic $f$-invariant probability measure $\mu$. The $f$-invariant measure $\mu$ is a physical or SRB (Sinai-Ruelle-Bowen) measure, if its basin $B(\mu)$ has positive Lebesgue measure (volume) on $M$.

2.1. **Standing assumptions.** We assume throughout that the unstable foliation $\mathcal{F}^u$ is minimal, that is, every leaf $\xi \in \mathcal{F}^u$ is dense in $M$. This property is satisfied by a $C^1$ open set of partially hyperbolic diffeomorphisms by the results of Bonatti, Díaz and Ures in [15], among robustly transitive strongly partially hyperbolic diffeomorphism in dimension three, that is, each subbundle is one-dimensional. It is satisfied by the time one map of any transitive Anosov flow which is not the suspension of an Anosov diffeomorphism of a codimension one submanifold, by the results of Plante [34], e.g., the geodesic flow on surfaces of constant negative curvature and by many contact Anosov flows.

This is important to ensure that certain properties of the map obtained after local perturbations are spread to the entire ambient space.

We also assume that the subbundle $E^u$ induces a smooth foliation $\mathcal{F}^u$ of class $C^2$. We remark that for the general partially hyperbolic diffeomorphism the stable and unstable laminations $\mathcal{F}^s, \mathcal{F}^u$, although having leaves as smooth as $f$, are not foliations in the usual sense of Differential Topology: their leaves do not “stack on top of each other” in a smooth way.

2.2. **Removing zero central Lyapunov exponent.** This is our main result. Abundance of non-zero central Lyapunov exponents means the existence open sets of diffeomorphisms where each one exhibits non-zero central Lyapunov exponents for Lebesgue almost every point on the manifold.

A partially hyperbolic diffeomorphism such that every $u$-Gibbs state has positive central Lyapunov exponents is called mostly expanding and their properties are studied in [3, 45]. Mostly expanding is the dual notion of mostly contracting introduced by [17, 22] and studied in [2].

For $r \geq 2$, a $C^r$ mostly expanding diffeomorphism has non-zero Lyapunov exponents Lebesgue almost everywhere in the ambient manifold. This is a $C^r$-open property and it also implies the existence of physical measures (see Proposition 3.8).

**Theorem A.** Let $f$ be the time-1 map of an Anosov flow whose strong-unstable foliation is $C^2$ smooth and minimal. For $r \geq 2$, $f$ is $C^2$-close to a $C^r$-open set of mostly expanding diffeomorphisms.
We give a brief sketch of our arguments. Let \( f \) be as in the statement of Theorem A. In what follows, for each \( u \)-Gibbs state \( \mu \) of \( f \), we denote

\[
\lambda^c_\mu(f) := \int \log \|Df|_{E^c}\| \, d\mu.
\]

We note that, since the central direction is assumed to be one-dimensional, if \( \mu \) is ergodic, then this number equals the central Lyapunov exponent. Moreover, since \( f \) is the time-1 map of an Anosov flow, we always have \( \lambda^c_\mu(f) = 0 \) for each \( u \)-Gibbs state \( \mu \).

Using the perturbative methods explained in Sections 2.6, 4, 5 and 6, we conclude that there exist a partially hyperbolic diffeomorphism \( g \) \( C^2 \)-close to \( f \) such that \( g \) is mostly expanding: for each \( u \)-Gibbs state \( \mu_g \) of \( g \) we have \( \lambda^c_{\mu_g}(g) > 0 \). The minimality of \( \mathcal{F}^u \) now ensures that there is a unique \( \mu_g \) which is a \( cu \)-Gibbs state and the unique physical measure; see Lemma 3.10 for more details. The rest of the conclusion follows from the \( C^r \) openness of the mostly expanding property (see Proposition 3.8 and Proposition 3.11).

More precisely, we show that, for \( f \in \text{Diff}^r(M) \), \( r \geq 2 \) in the setting of Theorem A, then \( \text{arbitrarily} \ C^2 \)-close of \( f \), there exists a \( C^r \)-neighborhood \( \mathcal{V} \) of mostly expanding diffeomorphisms. As consequence, for each \( g \in \mathcal{V} \) there exists a physical measure given by an ergodic \( u \)-Gibbs state \( \mu \) for \( g \) with positive central exponent, whose basin has full measure.

2.3. Abundance of physical measures with non-zero central exponent. As a consequence of Theorem A we provide a partial answer to Problem 1. As usual in smooth ergodic theory, we say that an invariant probability measure \( \mu \) is hyperbolic if the Lyapunov exponents of \( \mu \)-almost every point are never zero.

**Corollary B.** Each time-1 map of an Anosov flow, whose strong-unstable foliation is \( C^2 \) smooth and minimal, is \( C^2 \)-close to a \( C^r \)-open set of partially hyperbolic diffeomorphisms admitting a unique physical and hyperbolic measure with full basin (with \( r \geq 2 \)).

Hence Corollary B ensures that Problem 1 has an affirmative answer for time-1 maps \( f \) of an Anosov flow when the strong-unstable foliation of \( f \) is \( C^2 \) smooth and minimal.

2.4. Abundance of stable ergodicity. Now we rewrite our results from the point-of-view of stable ergodicity. A diffeomorphism \( f \) is \( C^r \)-stably ergodic if there exists a \( C^r \)-neighborhood \( \mathcal{V} \) of \( f \), where for each \( g \in \mathcal{V} \) there exists a unique physical measure \( \mu_g \) whose ergodic basin has full Lebesgue measure in the ambient space.

From the results of Section 3.4 on uniqueness of physical measures for partially hyperbolic diffeomorphisms with minimal strong-unstable foliation, we obtain the following.

**Corollary C.** Let \( f \) be the time-1 map of an Anosov flow whose strong-unstable foliation is \( C^2 \) smooth and minimal. Then \( f \) is \( C^2 \)-close to a \( C^r \)-stably ergodic diffeomorphism.

From the results [19, 44] about the continuous variation of \( u \)-Gibbs states and their (non-zero) Lyapunov exponents with the diffeomorphism in the \( C^2 \) topology, the stably ergodic diffeomorphisms we obtain are necessarily statistically stable. This means that the physical measures depend continuously on the diffeomorphism.
We now remark that Bochi, Fayad and Pujals in [12] have noted that stably ergodic conservative diffeomorphisms must be $C^1$ close to a conservative diffeomorphism for which Lebesgue measure is ergodic and hyperbolic, that is, stable ergodicity in the conservative setting implies stably non-zero central Lyapunov exponents.

In our setting a straightforward consequence of our main result reads as follows.

**Corollary D.** The set of $C^2$ stably ergodic strongly partially hyperbolic diffeomorphisms with one-dimensional central subbundle and stably zero central Lyapunov exponents, cannot contain a diffeomorphism which is the time-1 map of an Anosov flow whose strong-unstable foliation is $C^2$ smooth and minimal.

2.5. **Related open questions.** The results stated above suggest naturally the following questions/problems.

**Problem 2.** We have seen that minimal unstable foliations are helpful to get a physical measure for partially hyperbolic systems. How general or abundant are the partially hyperbolic systems with minimal unstable foliations?

**Problem 3.** Given that we have a positive or negative central Lyapunov exponent for some $u$-Gibbs measure, for partially hyperbolic diffeomorphism with minimal unstable foliations, can we ensure that we have mixed central behavior for higher dimensional central subbundles? That is, can we obtain generically or densely that there are hyperbolic $u$-Gibbs states with higher dimensional central subbundle having only non-zero Lyapunov exponents along this central direction?

**Problem 4.** A natural problem in our setting is to understand if mixed central behavior together with hyperbolicity (absence of zero Lyapunov exponents along the central direction) for some $u$-Gibbs state is sufficient in general to obtain a physical measure. If not, what extra conditions are needed to obtain a physical measure in this setting?

The last Corollary D naturally suggests the following

**Problem 5.** Is it possible to have stable ergodicity with zero Lyapunov exponents robustly along the center direction? That is, can we have stable ergodicity and zero central Lyapunov exponents almost everywhere simultaneously?

**Problem 6.** Since for our stably ergodic maps the physical measures vary continuously with the diffeomorphism, is it true that the physical measures also depend smoothly on the diffeomorphism? That is, can we obtain a susceptibility function for strongly partially hyperbolic diffeomorphisms in our setting along the lines of the work of Ruelle [40, 41]?

The choice of the adequate $C^r$ topologies is part of the problems above.

2.6. **Overview of the arguments.** Here we present an overview of the arguments to be detailed in what follows.

The statements of known results about $u$-Gibbs and $cu$-Gibbs states versus physical measures, together with minimality of strong-unstable foliation versus uniqueness of
$u$-Gibbs states are collected in Section 3 for convenience. We refer to them along the rest of this text when needed.

Let $f$ be the time-1 map of an Anosov flow whose strong-unstable foliation is $C^2$ smooth and minimal. We claim that we can perturb $f$ to a $C^2$ close mostly expanding map $g$ whose corresponding strong-unstable foliation $\mathcal{F}_g^u$ equals that of $f$: $\mathcal{F}_g^u = \mathcal{F}_f^u$.

If we assume this claim, then since $\mathcal{F}_g^u$ is minimal we get from Lemma 3.10 that there exists a unique $cu$-Gibbs state $\mu$ for $g$. Now we are in the setting of Proposition 3.11, and so we conclude that there exists a $C^r$ neighborhood $\mathcal{V}$ of $g$ where all maps are mostly expanding and have a unique physical measure given by an ergodic $cu$-Gibbs state.

This completes the proof of Theorem A and Corollary B, and also shows that $g$ is $C^r$ stably ergodic as in the statement of Corollary C, after we prove the claim above.

2.6.1. Strategy. To prove the claim, we perform a local perturbation of the map $f$ to a map $g$ by defining $g = f \circ H$, where $H$ is a $C^2$ diffeomorphism of $M$ such that, for some given non-periodic point $q_0 \in M$ for $f$ and sufficiently small $\varepsilon, t > 0$, we have in chosen local coordinates

- $H(B(q_0, 2\varepsilon)) = B(q_0, 2\varepsilon)$;
- $H$ is the identity map $Id$ on $M \setminus B(q_0, 2\varepsilon)$;
- $\|H - Id\|_{C^1} = t\varepsilon$ and $\|H - Id\|_{C^2} \xrightarrow{\varepsilon \to 0^+} 0$;
- $DH(q) \cdot E^c_f(q)$ is the graph of a non-zero injective linear map $L_{H(q)} : E^c_f(H(q)) \to E^u_f(H(q))$, for all $q \in B(q_0, 2\varepsilon)$;

where we write $E^*_f$ for the $Df$-invariant subbundles of $f$, $* = s, c, u$.

We present the construction of this map $H$ in Section 4, where the assumption of $C^2$ smoothness on $\mathcal{F}_f^u$ enables us to define $H$ using coordinates along the leaves of the strong-unstable foliation and, most useful, to keep the strong-unstable and center-unstable foliations unchanged, so that $\mathcal{F}_f^u$ remains a minimal foliation for the perturbed map. This perturbation $g = f \circ H$ of $f$, because it is $C^2$ close to $f$, is also a strongly partially hyperbolic diffeomorphism with $Dg$-invariant subbundles $E^*_g, * = s, c, u$, where $E^s_g$ is one-dimensional.

We claim that the center $Dg$-invariant subbundle is “tilted” towards the original $E^u_f$ subbundle in such a way that there exists a non-negative measurable function $\xi : M \to \mathbb{R}$ such that for every $q \in M$,

$$||Dg(q)|E^c_g(q)|| \geq 1 + \xi(q). \quad (2.4)$$

Moreover, the set of points $q \in M$ such that $\xi(q) > 0$ has positive $\mu_g$ measure for every $u$-Gibbs state $\mu_g$ of $g$ (see Lemma 6.2). Then, we conclude that

$$\lambda^c_{\mu_g}(g) = \int \log ||Dg|E^c_g|| d\mu_g \geq \int \log(1 + \xi(q)) d\mu_g(q) > 0.$$ 

Here $\xi(q) = \xi_{t, \varepsilon}(q) \geq 0$ depends on the domination of the action of $Df$ on $E^c_f$ over $E^c_f$ (given by (2.1), (2.2) and (2.3)) and on the $C^1$ distance between $f$ and $g$. Both claims above are proved in Sections 5 and 6 by:
(i) showing that the perturbed central subbundle $E^c_S$ has a non-zero component along the old unstable subbundle, in Section 5;

(ii) taking a Riemannian adapted norm $\| \cdot \|$ for the strongly partially hyperbolic diffeomorphisms $f$, given by [26], to estimate the expansion along $E^c_S$ in a transparent way, in Section 6.

The proof is mostly a linear algebra argument taking advantage of the robustness of the domination of the splitting. These are the main arguments in the proof of Theorem A and Corollaries B through C.

2.7. Examples of application.

2.7.1. The time-one map of the geodesic flow on surfaces of constant negative curvature. Consider a compact surface $S$ with a Riemannian metric with constant negative curvature. Then the geodesic flow $\phi_t$ on the unit tangent bundle $M = T^1S$ of $S$ is an Anosov flow whose strong stable and strong unstable foliations are $C^r$ smooth for all $r > 1$; see e.g. Benoist, Foulon and Labourie [8].

In addition, both foliations are minimal: this type of geodesic flow preserves a contact form, which coincides with the Liouville measure on the unit tangent bundle, and it is known to be ergodic with respect to this measure since the work of Hopf (see e.g. [30]). Therefore, the flow is transitive and the entire phase space is non-wandering, thus both strong stable and strong unstable foliations (those tangent to $E^s$ and $E^u$ respectively) are minimal; see Plante [34].

Hence we can apply our results to $f = \phi_1 : M \to M$ which is a strongly partially hyperbolic map and also preserves a natural volume form that is a $u$-Gibbs state and the unique physical measure for $f$ on $M$. We conclude that $f$ is $C^2$ close to a $C^r$ stably ergodic (not necessarily conservative) diffeomorphism with positive central Lyapunov exponents Lebesgue almost everywhere.

2.7.2. The time-1 map of the geodesic flow on symmetric Riemannian manifold of constant negative curvature. Anosov flows $\phi_t$ in any compact finite dimensional Riemannian manifold having smooth (at least of class $C^3$) strong stable or strong unstable foliations are essentially $C^\infty$ conjugated to the geodesic flow over a locally symmetric Riemannian manifold with constant negative sectional curvature; see Benoist, Foulon and Labourie [8]. These flows preserve a smooth volume form which is a contact form, so they are contact Anosov flows. In addition, they are transitive by the classical result of Hopf [29], and do not admit sections; see Godbillon [25, pp. 146-147]. Hence, by the work of Plante [34], the strong unstable foliation is minimal.

Hence we can apply our results to $f = \phi_1$ as in the previous class of examples. We note that now the stable and unstable directions are higher dimensional: if the dimension of the manifold is $n$, the dimension of the unit tangent bundle is $2n - 1$, and then the dimension of the stable and unstable invariant distributions equals $n - 1$. 
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3. Properties of $u$-Gibbs states and $cu$-Gibbs states

Here we overview some properties of $u$- and $cu$-Gibbs states used in the detailed arguments of the previous section.

3.1. Birkhoff regular points and ergodic decomposition. A point $z \in M$ is Birkhoff regular if the Birkhoff averages

$$
\varphi^-(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^{-k}(z)),
$$

$$
\varphi^+(z) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z));
$$

are defined and $\varphi^-(z) = \varphi^+(z)$ for every $\varphi : M \to \mathbb{R}$ continuous. The set of Birkhoff regular points of $f$ has full measure with respect to any $f$-invariant measure $\mu$.

Given a point $x$ let us denote by $\mu_x$ the probability measure given by the time average along the orbit of $x$

$$
\int \varphi \, d\mu_x = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))
$$

for every continuous $\varphi : M \to \mathbb{R}$. According to the Ergodic Decomposition Theorem $\mu_x$ is well defined and ergodic for every $x$ in a set $\Sigma(f) \subseteq M$ that has full measure with respect to any invariant measure $\mu$; see e.g. Mañé [31]. Moreover, for every bounded measurable function $\varphi : M \to \mathbb{R}$ we can write

$$
\int \varphi \, d\mu = \int \left( \int \varphi \, d\mu_x \right) \, d\mu(x).
$$

For every such $\varphi$ the integral $\int \varphi \, d\mu_x$ coincides with the time average $\mu$-almost everywhere, and $x \in \text{supp} \mu_x$ for $\mu$-almost all $x$.

3.2. $u$-Gibbs states and their properties. We assume from now on in this section that $f : M \to M$ is a $C^r$ partially hyperbolic diffeomorphism ($r \geq 2$) having a splitting of the tangent bundle given by $TM = E^s \oplus E^c \oplus E^u$. The aim of this subsection is to present some useful properties of $u$-Gibbs states.

An $f$-invariant probability measure $\mu$ is a $u$-Gibbs state if the conditional measures of $\mu$ with respect to the partition into local strong unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local unstable manifolds. The following classical result shows that in this setting there always exist $u$-Gibbs states.
Proposition 3.1. [33, by Pesin and Sinai] Denote by $m_u = \dim E^u$. If $D^u$ is an $m_u$-dimensional disk inside a strong-unstable leaf, and $\text{Leb}_{D^u}$ denote the Lebesgue measure induced on $D^u$, then every accumulation point of the sequence of probability measures

$$
\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j \left( \frac{\text{Leb}_{D^u}}{\text{Leb}_{D^u}(D^u)} \right)
$$

is a $u$-Gibbs state with densities with respect to Lebesgue measure along the strong-unstable leaves uniformly bounded away from zero and infinity. In particular, the support of $\mu$ consists of entire strong-unstable leaves.

Remark 3.2. As can be seen in Bonatti, Díaz and Viana [16] the densities of $u$-Gibbs states with respect to Lebesgue measure along the strong unstable plaques depend only on $f$ through its derivatives and the curvature of the unstable manifolds. Consequently, the bounds on the densities of $u$-Gibbs states are also uniform for maps on a $C^2$ neighborhood of $f$.

Next results present several very useful properties of $u$-Gibbs states. First, the ergodic decomposition of $u$-Gibbs states is formed by other $u$-Gibbs states.

Proposition 3.3. [16, Lemma 11.13 and Corollary 11.14] Ergodic components of any $u$-Gibbs state $\mu$ are $u$-Gibbs states whose densities are uniformly bounded away from zero and infinity. Conversely, a convex combination of $u$-Gibbs states is an $u$-Gibbs state. The support of any $u$-Gibbs state consists of entire strong-unstable leaves.

This allows us to assume without loss of generality in many settings that $u$-Gibbs states are ergodic. The fact that the support of any $u$-Gibbs state contains a full strong-unstable leaf is very important when we assume that the strong-unstable foliation is minimal, see Section 3.4.

Next we see that the “basin of the family of all $u$-Gibbs states” is very big in the manifold.

Proposition 3.4. [16, Theorem 11.16] There exists $E \subseteq M$ intersecting every unstable disk on a full Lebesgue measure subset, such that for any $x \in E$, every accumulation point $\nu$ of $\nu_{n,x} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ is a $u$-Gibbs state.

The following two propositions are consequences of Proposition 3.4.

Proposition 3.5. [16, Section 11.2.3] If $\mu$ is an physical measure for $f$, then $\mu$ must be a $u$-Gibbs state.

Proposition 3.6. [23, by Dolgopyat] If $\mu$ is the unique $u$-Gibbs state for $f$, then $\mu$ is a physical measure for $f$. Moreover its basin $B(\mu)$ has full Lebesgue measure in $M$.

3.3. $cu$-Gibbs states and their properties. The aim of this subsection is to present some useful results on $cu$-Gibbs states.

We recall that $E^{cu} := E^c \oplus E^u$ and we denote $m_{cu} := \dim E^{cu}$. An invariant measure $\mu$ is a $cu$-Gibbs state if the $m_{cu}$ largest Lyapunov exponents are positive $\mu$-almost
everywhere and the conditional measures of $\mu$ along the corresponding local Pesin’s center-unstable manifolds are $\mu$-almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.

The notion of $cu$-Gibbs state was introduced by Alves, Bonatti and Viana in [1] in a more general context; they correspond to a non-uniform version of the $u$-Gibbs states. In what follows, we present the properties of $cu$-Gibbs states adapted to our setting. The interested reader should consult [1, 16, 44] for the properties of $cu$-Gibbs in more general contexts.

We first present a condition which guarantees the existence of $cu$-Gibbs states. We say that a diffeomorphism $f$ has non-uniform expansion along the center-unstable direction if there exists a constant $c_0 > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df^{-1} | E^{cu}(f^j(x)) \| \leq -c_0 < 0.$$  \hfill (3.5)

for all $x$ in a full Lebesgue measure subset of $M$.

**Proposition 3.7.** [1, by Alves, Bonatti, Viana] If $f$ is non-uniformly expanding along the center unstable direction, then there exist finitely many $cu$-Gibbs states $\mu_1, \ldots, \mu_k$. Moreover, they are physical measures a the union of their basins cover Lebesgue almost every point of the whole manifold $M$.

If follows from Proposition 3.4 that a $cu$-Gibbs state is a $u$-Gibbs state. The converse is not true in general, even if the $u$-Gibbs state is ergodic and it has positive central Lyapunov exponents; see for instance the example in [45].

As explained along the statements of the main results, we say that $f$ is mostly expanding if every $u$-Gibbs state of $f$ has positive central Lyapunov exponents. As for the dual notion of mostly contracting, the mostly expanding diffeomorphisms possess similar properties which are studied in [3].

**Proposition 3.8.** [3, Theorem A] The class of mostly expanding partially hyperbolic diffeomorphisms constitutes a $C^2$-open subset $Diff^r(M)$. Moreover, if $f$ is a mostly expanding partially hyperbolic diffeomorphism, then it is non uniformly expanding along the central direction so that, in particular, such $f$ has a finite number of physical measures whose basins together cover Lebesgue almost every point in $M$.

We can say more about the number of physical measures.

**Proposition 3.9.** [3, Theorem B] If $f$ is a mostly expanding partially hyperbolic $C^r$ diffeomorphism, $r \geq 2$, with a unique $cu$-Gibbs state, then every $g$ $C^r$-close to $f$ has a unique physical measure.

### 3.4. Minimal unstable foliation, uniqueness of $u$-Gibbs states and stable ergodicity.

We now assume that $f$ is a partially hyperbolic $C^2$ diffeomorphism whose unstable foliation is minimal.

From the absolute continuity of the unstable foliation, we see that the subset $E$ of $M$ given by Proposition 3.4 has full Lebesgue measure (volume) in $M$. 
So for Lebesgue almost every \( x \) and for every given continuous function \( \varphi \) on \( M \), we have that \( \varphi^+ (x) \) can have many different values, but each of them is given by 
\[
\mu (\varphi) = \int \varphi \, d\mu
\]
where \( \mu \) is some \( u \)-Gibbs state. However, even if \( f \) is a strongly partially hyperbolic \( C^2 \) diffeomorphisms with simultaneously minimal stable and unstable foliations, we cannot in general ensure that \( \varphi^\pm \) is well defined nor that \( \varphi^+ = \varphi^- \) Lebesgue almost everywhere.

These averages are well-defined Lebesgue almost everywhere and \( \varphi^+ = \varphi^- \) is true, for instance, if \( f \) preserves Lebesgue measure.

**Problem 7.** For a strongly partially hyperbolic \( C^2 \) diffeomorphims \( f \) of a compact manifold \( M \), if for any given continuous function \( \varphi \) on \( M \), both the forward \( \varphi^+ (x) \) and backward \( \varphi^- (x) \) Birkhoff averages exist and coincide for Lebesgue almost every \( x \in M \), then Lebesgue measure is invariant.

Moreover, since every strong-unstable leaf is dense by assumption, and the support of every \( u \)-Gibbs state contains some full strong-unstable leaf, we deduce from Proposition 3.4 that, for each \( x \in E \), every accumulation measure \( \mu_x \) given by (3.3) has full support. In particular \( \mu_x(U) > 0 \) for every open subset \( U \).

Altogether this ensures that each \( x \in E \) has positive frequency of visits to any open subset of the manifold. In particular, Lebesgue almost every positive orbit is dense.

We will construct a perturbed map \( g \) \( C^2 \) close to \( f \) whose strong-unstable foliation coincides with the strong-unstable foliation of \( f \), so these properties persist for all maps \( g \) obtained from \( f \) according to our perturbation scheme, to be presented in the next section.

A similar argument shows that physical \( cu \)-Gibbs states are unique, if they exist.

**Lemma 3.10.** Let \( f : M \to M \) be a (strongly) partially hyperbolic \( C^2 \) diffeomorphism whose strong-unstable foliation \( 5^u \) is minimal. If \( f \) is mostly expanding, then there can exist at most one \( cu \)-Gibbs state for \( f \) which is also a physical measure and it basin cover Lebesgue almost every point of the whole manifold \( M \).

**Proof.** Indeed, an ergodic \( cu \)-Gibbs state \( \mu \) is a physical measure whose basin \( B(\mu) \) contains some open neighborhood \( U \) of any center-unstable leaf \( \zeta \) supporting \( \mu \). These neighborhoods are given by the family of strong-stable leaves through points of \( \zeta \). Therefore any other ergodic \( cu \)-Gibbs state \( \nu \) has some center-unstable leaf \( \zeta \) is its support which crosses \( U \) (Note that in this case, the center unstable leaf \( \zeta \) is foliated by strong unstable leaves and they are dense in \( M \)). The absolute continuity of the strong-stable foliation ensures that the same argument as above implies \( \mu = \nu \).

The basin of \( \mu \) has full Lebesgue measure in the ambient space as a direct consequence of Proposition 3.4. For otherwise it would be possible to find an invariant subset of positive Lebesgue measure where (3.5) holds and then, using the proof of Proposition 3.7, we would be able to construct another \( cu \)-Gibbs state which would be a new physical measure.

Combining the previous results we have proved the following.
**Proposition 3.11.** Let \( f \) be a mostly expanding strongly partially hyperbolic diffeomorphism, with one-dimensional subbundles, having a unique cu-Gibbs state.

Then there exists a \( C^r \) neighborhood \( \mathcal{U} \) of \( f \), \( r \geq 2 \), such that each \( g \in \mathcal{U} \) is mostly expanding and admits an unique cu-Gibbs state \( \mu_g \) which is a physical measure with full basin. In particular \( f \) is \( C^r \)-stably ergodic.

4. **Perturbing the central subbundle**

Here we start the proof of Theorem A, providing the details of the construction of the perturbed diffeomorphism.

4.1. **Adapted norms for partially hyperbolic diffeomorphisms.** We note first that for a flow \( X_t \) without equilibria on a compact Riemannian manifold \( M \) with an induced norm \( \| \cdot \| \), we can define a new norm \( |u|_X := \|u\|/\|X(x)\| \) which satisfies

\[
|DX_t \cdot X(x)|_{X_t(x)} = \frac{\|DX_t \cdot X(x)\|}{\|X(X_t(x))\|} = 1, \quad x \in M, t \in \mathbb{R}.
\]

We then use the results from Gourmelon [26], which provide adapted metrics on our setting. That is, we may assume without loss of generality that our partial hyperbolic map \( f = X_1 \), where \( (X_t)_{t \in \mathbb{R}} \) is now a \( C^2 \) Anosov flow, satisfies (2.1), (2.2) and (2.3) with \( C = 1 \) and, moreover, that the norm is induced by a Riemannian metric on \( M \) such that at each \( x \in M \) the directions along the different subbundles are mutually orthogonal.

Since \( f = X_1 \) is the time-1 map of an Anosov flow, we have

\[
\|Df(x)v\| = 1 \quad \text{for every } v \in E^c_f(x), \text{ with } \|v\| = 1.
\]

4.2. **The choice of coordinates.** The assumption that \( f = X_1 \) is the time-1 map of a \( C^2 \) Anosov flow ensures that \( E^c_f \) is an integrable subbundle, its integral manifolds \( \mathcal{F}^c \) are the orbits of the flow \( (X_t)_{t \in \mathbb{R}} \). Moreover the strong-stable \( \mathcal{F}^s \) and strong-unstable \( \mathcal{F}^u \) foliations are smooth along the central leaves since \( \mathcal{F}^*(X_t(x)) = X_t(\mathcal{F}^*(x)) \) for \( x \in M, t \in \mathbb{R} \). This means that we have dynamical coherence: for each \( x \in M \) the sets \( \mathcal{F}^{cu}(x) = \bigcup_{t \in \mathbb{R}} X_t(\mathcal{F}^u(x)) \) and \( \mathcal{F}^{cs}(x) = \bigcup_{t \in \mathbb{R}} X_t(\mathcal{F}^s(x)) \) are \( C^2 \) immersed submanifolds of \( M \) tangent to \( E^{cu} := E^c \oplus E^u \) and \( E^{cs} := E^s \oplus E^c \) respectively.

Using the assumption of \( C^2 \) smoothness of the strong-unstable foliation \( \mathcal{F}^u \) together with the dynamical coherence we see that the center-unstable foliation \( \mathcal{F}^{cu} \) is also \( C^2 \) smooth. Hence we can define for any given \( q_0 \in M \) a \( C^2 \) local parametrization \( \psi : D \to M \) of \( M \) as follows.

We start by choosing an embedded manifold \( \Sigma \) in a neighborhood of \( q_0 \), containing \( q_0 \), such that \( \Sigma \) is transverse to \( E^s_f \) at all points \( q \in \Sigma \): we simply take \( \Sigma := \bigcup_{-\gamma < t < \gamma} X_t(W^s_f(q_0)) \) where \( W^s_f(q_0) \) is the connected component of \( \mathcal{F}^s(x) \cap B_\gamma(q_0) \) containing \( q_0 \). This is a local center-stable leaf through \( q_0 \).

We write \( W^c_\gamma : \overline{B}_1(0) \to \Sigma \) for a \( C^2 \) parametrization of \( \Sigma \), where \( \overline{B}_1(0) \) is the unit ball in the Euclidean space \( \mathbb{R}^{s+c} \), with \( s := \dim E^s_f \) and \( c := \dim E^c_f = 1 \) and \( W^c_\gamma(0) = q_0 \).
We can assume that
\[ \text{DW}^{cs}(0)(\mathbb{R}^s \times \{0^c\}) = E^s(q_0) \quad \text{and} \quad \text{DW}^{cs}(0)(\{0^s\} \times \mathbb{R}^c) = E^c(q_0). \]

In this way \( \Sigma \) is tangent to the center-stable direction at all points, that is, \( T_q\Sigma = E^{cs}(q) := E^s_q(q) \oplus E^c_q(q), q \in \Sigma. \)

The Stable/Unstable Manifold Theorem ensures that for every \( q \in \Sigma \) there exists a \( C^2 \) embedding \( W^u_\gamma(q) : \tilde{B}_1(0) \to M \) where \( \tilde{B}_1(0) \) is the unit ball in the Euclidean space \( \mathbb{R}^u \) with \( u := \dim F^u_j; W^u_\gamma(q)(\tilde{B}_1(0)) = F^u(q) \cap B_\gamma(q) \) and \( T_qW^u_\gamma(q) = E^u_q(q) \); see e.g. [42].

The \( C^2 \) smoothness of \( F^u \) ensures that the following is a \( C^2 \) map
\[ \psi : \tilde{B}_1(0) \times \tilde{B}_1(0) \to M, \quad (w,z) \mapsto W^u_\gamma(W^{un}_{\gamma}(w))(z) \]
and, since \( D\psi(w,0) : \mathbb{R}^{s+c+u} \to T_qM \) is an isomorphism for all \( w \in \tilde{B}_1(0) \) and \( \psi \) maps \( \tilde{B}_1(0) \times \{0\} \) diffeomorphically onto \( \Sigma = W^{cu}_{\gamma}(\tilde{B}_1(0)) \), then \( \psi \) is a \( C^2 \) diffeomorphism on a neighborhood of \( \tilde{B}_1(0) \times \{0\} \). Thus, setting \( \gamma > 0 \) smaller if needed, we can assume without loss that \( \psi \) is a \( C^2 \) diffeomorphism between \( D = \tilde{B}_1(0) \times \tilde{B}_1(0) \) and its image in \( M \).

This is the parametrization we need to define our perturbation. We note that
\[ D\psi(w,0)(\mathbb{R}^{s+c} \times \{0^u\}) = E^s_j(\psi(w,0)) \oplus E^c_j(\psi(w,0)) \quad \text{and} \]
\[ D\psi(w,z)(\{0^{s+c}\} \times \mathbb{R}^u) = E^u_j(\psi(w,z)) \quad \text{and also} \]
\[ D\psi(w,z)(\{0^s\} \times \mathbb{R} \times \{0^u\}) = E^c_j(\psi(w,z)), \]
for all \((w,z) \in D\). The last property is a consequence of the assumption that \( f = X_{i_1} \), since for all \((w,z) \in D\) and each small \(|t|\) there exists \( z' \) such that \( \psi(X_{i_1}(q),z) = X_{i}(\psi(q,z')). \)

So this provides a \( C^2 \) coordinate system around each point of \( M \) such that \((w,z) \mapsto \psi(w,z), (w,z) \in D\), is contained in the local strong-unstable manifold of the point \( \psi(w,0) \).

### 4.3. Construction of the local perturbation.

Let us fix \( q_0 \) a recurrent non-periodic point of \( f \). We note that since we assume \( f \) has a minimal unstable foliation, then we have plenty of points with dense orbit. We fix a neighborhood \( U \) of \( q_0 \) in \( M \) such that \( f(U) \cap U = \emptyset \) and a parametrization \( \psi : D \to U \), where \( \psi \) is the coordinate system constructed in the previous subsection. By appropriately rescaling the basis vectors in \( \mathbb{R}^d \), we can further assume without loss of generality that \( \|D\psi(0)e_i\| = 1, i = 1, \ldots, d \), for the canonical basis \( \{e_i\}_{i=1}^d \) of \( \mathbb{R}^d \), where \( d := s + c + u = \dim M \).

### 4.3.1. The choice of the bump function.

Let \( \phi : \mathbb{R} \to [0,1] \) be a bump function with the following properties:

- \( \phi \equiv 0 \) on \( \mathbb{R} \setminus (-2,2) \) and \( \phi \equiv 1 \) on \([-1,1]\);
- \( (s\phi(s))' \neq 0 \) for all \( s \in (-2,2) \) except at the points \( \{\pm s_0\} \) for a value \( s_0 \in (1,2) \).
This is easy to build: we start with

\[ \eta_0(s) := \begin{cases} e^{-1/t^2} & \text{if } t > 0 \\ 0 & \text{if } t \geq 0 \end{cases} \]

and \[ \eta_1(s) := \eta_0(s)\eta_0(1-s) \]

and then consider \[ \eta_2(s) := c^{-1} \int_{-\infty}^{s} \eta_1(t) \, dt \] where \[ c := \int_{-\infty}^{\infty} \eta_1(t) \, dt \]. This function \( \eta_2 \) is of class \( C^\infty \) and satisfies \( \eta_2 \mid (-\infty, 0] \equiv 0 \), \( \eta_2 \mid [1, +\infty) \equiv 1 \) and \( \eta_2 \mid (0, 1) > 0 \). We set \( \phi(s) := \eta_2(s + 2)\eta_2(2-s) \). See Figure 1.

![Figure 1](image1.png)

**Figure 1.** At the upper left we have \( \eta_1 \) and at the upper right \( \eta_2 \) on the interval \([0, 1]\). At the lower left we have \( \zeta(s) = s\phi(s) \) and at the lower right \( \zeta' \) on the interval \([1, 2]\).

For \( s \in (-1, 1) \) we have \( s\phi(s) = s \) and there is no zero of the derivative of \( \zeta(s) := s\phi(s) \). But \( \zeta(0) = \zeta(2) = 0 \), thus there exists a zero of \( \zeta' \) in \((1, 2)\). Since \( \zeta(-s) = \zeta(s) \) we may consider only \( s \in (1, 2) \) and clearly see in Figure 1 that there exists a unique zero of \( \zeta' \) in this interval, as we wanted.
4.3.2. The choice of the perturbation. To define the perturbation we set the multi-indexes \( x = (x_1, \ldots, x_s) \) and \( z = (z_1, \ldots, z_u) \). We then define

\[ \Phi_\varepsilon(x, y, z) := \left[ \prod_{i=1}^{s} \phi\left( \frac{x_i}{\varepsilon} \right) \right] \Phi\left( \frac{y}{\varepsilon} \right) \left[ \prod_{j=1}^{u} \phi\left( \frac{z_j}{\varepsilon} \right) \right] \]

together with the following diffeomorphism for small \( t > 0 \) (to be bounded above in the following arguments) and \( 0 < \varepsilon < 1/4 \)

\[ h = h_{t, \varepsilon} : D \to D, \quad (x, y, z) \mapsto (x, y, z + ty\Phi_\varepsilon(x, y, z)e_{s+c+1}), \quad (4.1) \]

where \( s + c + 1 \) is the first coordinate along the unstable direction in the parametrization \( \psi \). We have, with respect to the canonical basis on \( \mathbb{R}^d \) and writing \( I_k \) for the identity on \( \mathbb{R}^k \) for \( k \in \mathbb{Z}^+ \)

\[ Dh_{t, \varepsilon} = \begin{bmatrix} I_s & 0 & 0 \\ tyD_x\Phi_\varepsilon \cdot e_{s+c+1} & t(\Phi_\varepsilon + y\partial_y\Phi_\varepsilon) \cdot e_{s+c+1} & I_u + tyD_z\Phi_\varepsilon \cdot e_{s+c+1} \end{bmatrix}, \quad (4.2) \]

where \( D_x\Phi_\varepsilon(x, y, z) : \mathbb{R}^s \to \mathbb{R} \) and \( D_z\Phi_\varepsilon(x, y, z) : \mathbb{R}^u \to \mathbb{R} \). Clearly \( h \) is the identity on \( D \setminus B(0, 2\varepsilon) \). For \( (x, y, z) \in B(0, 2\varepsilon) \) and some constant \( C > 0 \) (a bound on \( |D\Phi_\varepsilon| \)) we have for \( * = \) any of the variables in \( x, y, z \)

\[ |ty\partial_x\Phi_\varepsilon| \leq t \cdot 2\varepsilon(C/\varepsilon) = 2Ct \quad \text{and} \quad |t(\Phi_\varepsilon + y\partial_y\Phi_\varepsilon)| \leq t(1 + 2C), \]

and by the definition of \( \phi \) we get \( |t(\Phi_\varepsilon + y\partial_y\Phi_\varepsilon)| \neq 0 \) except at two values \( y = \pm \epsilon s_0 \) with \( |y| \in (\epsilon, 2\epsilon) \). We note that the bottom right side block of the matrix in (4.2) has a non-zero determinant

\[ |\det Dh_{t, \varepsilon}| = |1 + ty\partial_z\Phi_\varepsilon| \geq 1 - 2Ct \geq \frac{1}{2} \quad \text{for} \quad 0 < t < 1/4C. \quad (4.3) \]

Moreover, since \( \|h - Id\| = |ty\Phi_\varepsilon| \), we see that for small \( \varepsilon > 0 \) we get \( \|h_{t, \varepsilon} - Id\|_{C^1} \leq (1 + 2C)t \). In addition, for \( 0 < t < 1/4C \) we have

\[ h(x, y, z) = h(\hat{x}, \hat{y}, \hat{z}) \implies x = \hat{x}, y = \hat{y}, z_j = \hat{z}_j \quad \text{for} \quad j = s + c + 2, \ldots, d \]

and

\[ z_{s+c+1} - \tilde{z}_{s+c+1} = ty[\Phi_\varepsilon(x, y, z) - \Phi_\varepsilon(x, y, z)]. \]

From the definition of \( \Phi_\varepsilon \) we see that \( \Phi_\varepsilon(x, y, \bar{z}) = \Phi_\varepsilon(x, y, z) = 0 \) for \( |y| > 2\varepsilon \); while for \( |y| \leq 2\varepsilon \)

\[ |z_{s+c+1} - \tilde{z}_{s+c+1}| \leq t|y| \left| \phi\left( \frac{z_{s+c+1}}{\varepsilon} \right) - \phi\left( \frac{\tilde{z}_{s+c+1}}{\varepsilon} \right) \right| \leq t \frac{|y|}{\varepsilon} (\sup |D\phi|)|z_{s+c+1} - \tilde{z}_{s+c+1}| \leq 2tC|z_{s+c+1} - \tilde{z}_{s+c+1}| \leq 2tC|z_{s+c+1} - \tilde{z}_{s+c+1}| \]

so from the assumption \( 0 < t < 1/4C \) we conclude that \( z = \bar{z} \) in all cases. Under this condition the map \( h \) is injective and thus, by (4.3), a diffeomorphism onto its image.
Remark 4.1. We can easily bound the second partial derivatives as $|\partial^2_{xx} f| \leq C e^{-2\epsilon}$ for $x, y$ the variables in $x, y, z$, so that $\|h_{t, \epsilon} - Id\|_{C^2} \leq C e^{-\epsilon}$ for a constant $C > 0$ depending on the bump function $\phi$ but independent of $\epsilon, t$. Moreover, for $r > 2$ we have $|\partial^k_{xx} f| \leq C_k e^{-k}$ for every multi-index $m = (m_1, \ldots, m_k) \in \{x_1, \ldots, x_5, y, z_1, \ldots, z_2\}$ and $2 \leq k \leq r$, where $C_k$ depends on $\phi$ only. This will be essential to estimate the $C^r$ distance of the perturbed map away from $f$.

We finally define the perturbed map $g$ as

$$g(x) = \begin{cases} f(q) & \text{if } q \in M \setminus U \\ (f \circ H)(q) & \text{if } q \in U \end{cases},$$

where we write $H = \psi \circ h \circ \psi^{-1}$ from now on.

Remark 4.2. We note that $g(U) = f(U)$ by the choice of $U$ since $H(U) = U$. Thus we see that $g(U) \cap U = \emptyset$ and that the minimum $n > 0$ so that $g^n(U) \cap U \neq \emptyset$ is at least 2 and depends only on $f$, because $g = f$ outside of $U$.

We observe that $Dh(0)(u, v, w) = (u, v, w + tv \cdot e_{s+c+1})$ so at $q_0$ we have $H(q_0) = q_0$ and the image $Dg(q_0) \cdot E^c_f(q_0)$ is the graph of a non-zero injective linear map $L_g(q_0) : E^c_f(q_0) \to E^u_f(q_0)$.

In the absence of dynamical coordinates, the splitting $E^s_f \oplus E^c_f \oplus E^u_f$ in general depends not more than Hölder continuously on the base point. Hence it is not possible in general to find a smooth coordinate change that sends the $Df$-invariant direction onto the coordinate axis everywhere in a neighborhood of $q_0$. But the choice of $\psi$ through dynamical coherence ensures that for $q \in V = V_\epsilon := \psi(B(0, 2\epsilon))$ the unstable and center-unstable directions are preserved by $DH(q)$. Hence we can ensure that there is a non-zero injective linear map $L_g(q) : E^c_f(g(q)) \to E^u_f(g(q))$ such that $Dg(q) \cdot E^c_f(q)$ is the graph of $L_g(q)$.

In particular, if $\pi^u_{g(q)} : T_{g(q)} M \to E^u_f(g(q))$ is the projection into $E^u_f(g(q))$ parallel to $E^s_f(g(q)) \oplus E^c_f(g(q))$ then it is non-zero, i.e., $\pi^u_{g(q)} \circ L_g(q) \neq 0$; but the projection $\pi^s_{g(q)} : T_{g(q)} M \to E^s_f(g(q))$ parallel to $E^c_f(g(q)) \oplus E^u_f(g(q))$ is zero, i.e., $\pi^s_{g(q)} \circ L_g(q) \equiv 0$. This means that the image of the central vectors under $Dg$ has a non-zero unstable component along the original splitting but can only have a zero stable component. We remark that

- for $q \in M \setminus V$ the old invariant directions are preserved by $Dg$, i.e. $Dg(q) \cdot E^*_f(q) = E^*_f(g(q)) = E^*_f(f(q))$ since $Dg(q) = Df(q)$, for each $* \in \{s, c, u\};$
- for $q \in V$ both the direction $E^u_f$ and $E^u_f$ are preserved by $Dg$, i.e., $Dg(q) \cdot E^u_f(q) = E^u_f(g(q)) = E^u_f(f(H(q)))$ and $Dg(q) \cdot E^u_f(q) = E^u_f(g(q)) = E^u_f(f(H(q)))$.

Consequently, the unstable and center-unstable subbundles of $f$ remain as unstable and center-unstable subbundles for $g$: $E^u_g = E^u_f$ and $E^c_g = E^c_f$. Consequently the strong unstable and center-unstable foliations of $f$ and $g$ coincide since these foliations are uniquely integrable in our setting and, in particular, $\mathcal{F}^u_g$ is minimal.
4.4. The perturbed map is $C^2$ close. We can perform all the previous constructions with a family $h_{t,\varepsilon}$ with $\varepsilon$ going to zero and a function $t = t(\varepsilon)$ which also goes to zero, but essentially arbitrary; see the next sections. This freedom of choice for $t(\varepsilon)$ enables us to control the distance of $g$ to $f$ in the $C^2$ topology.

Indeed, if the strong unstable foliation $\mathcal{F}_f^u$ of $f$ is of class $C^r$, for some $r > 2$, then we can build $H$ of class $C^r$ and, from Remark 4.1, we have $\|h_{t,\varepsilon} - Id\|_{C^r} < C_r t \varepsilon^{-r}$. So we just have to choose the appropriate function $t(\varepsilon)$. In the present scenario, we have a strong unstable foliation $\mathcal{F}_f^u$ of $f$ is of class $C^2$, and we may choose $t(\varepsilon) = \min\{\varepsilon^3, 1/(4C)\}$. With this choice of $t = t(\varepsilon)$ we have

$$\|h_{t,\varepsilon} - Id\|_{C^2} \leq C_2 \frac{\varepsilon^3}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0.$$ 

So in what follows we assume that we have performed the perturbation described in the previous subsections with the aid of the family of functions $(h_{t(\varepsilon),\varepsilon})_{\varepsilon \geq 0}$ where $t(\varepsilon)$ was defined above.

5. The perturbed central subbundle

Here we prove the following lemma.

**Lemma 5.1.** For all small enough $0 < \varepsilon < 1/4$ and $t = t(\varepsilon)$, there exists a subset $\tilde{V}$ of $V$ such that:

1. for each $u$-Gibbs state $\mu$ of $g$, $\mu(\tilde{V}) > 0$;
2. for every $q \in \tilde{V}$, $E^c_{\mathcal{S}}(q) \neq E^c(q)$.

In particular, if $q \in \tilde{V}$, $E^c_{\mathcal{S}}(q)$ can be written as the graph of a nonzero linear map $G_q : E^c_{\mathcal{S}}(q) \to E^u_{\mathcal{S}}(q)$.

**Proof.** Fix $q \in M$. For $\nu \in T_qM$, denote $\nu^u := \pi^u_q(\nu)$; $\nu^c := \pi^c_q(\nu)$ and $\nu^s := \pi^s_q(\nu)$ as defined in Section 4.3.2.

For $\nu = \nu^c + \nu^u \in T_qM$, the slope of $\nu$ is defined as

$$s_c(\nu) = \frac{\|\nu^u\|}{\|\nu^c\|}.$$ 

We want to estimate the the slope of $Dg^u(q)\nu$ for, $n \geq 0$, $q \in M$ and $\nu = \nu^c + \nu^u \in E^c_{\mathcal{S}}(q) \oplus E^u_{\mathcal{S}}(q)$. Note that if $q \in V$, then $g = f \circ H$, and so

$$s_c(Dg(q)\nu) = \frac{\|Dg(q)\nu\|^u}{\|Dg(q)\nu\|^c} = \frac{\|Df(H(q))[DH(q)\nu]^u\|}{\|Df(H(q))[DH(q)\nu]^c\|} = \frac{\|Df(H(q))[DH(q)\nu]^u\|}{\|[DH(q)\nu]^u\|} \frac{\|[DH(q)\nu]^u\|}{\|Df(H(q))[DH(q)\nu]^c\|} \frac{\|Df(H(q))[DH(q)\nu]^c\|}{\|[DH(q)\nu]^c\|} = \frac{\|Df(H(q))[DH(q)\nu]^u\|}{\|[DH(q)\nu]^u\|} s_c(DH(q)\nu).$$ (5.1)
The last equality is obtained from $\|Df(q)v^c\| = \|v^c\|$ for every $q \in M$ and every $v^c \in E_f^c(q)$ (the central direction of $f$ is the flow direction).

On the other hand, if $q \in M \setminus V$, since $f = g$ then

$$s_c(Dg(q)v) = \frac{\|Dg(q)v^u\|}{\|Dg(q)v^c\|} = \frac{\|Df(q)v^u\|}{\|Df(q)v^c\|}$$

$$= \frac{\|Df(q)v^u\|}{\|v^u\|} \frac{\|v^c\|}{\|Df(q)v^c\|} = \frac{\|Df(q)v^u\|}{\|v^u\|} s_c(v). \quad (5.2)$$

If $q \in V$, $g(q), \ldots, g^n(q) \notin V$ and $g^{n+1}(q) \in V$, $n \geq 1$, combining $(5.1)$ and $(5.2)$ we obtain,

$$s_c(Dg^{n+1}(q)v) = \frac{\|Df^n(g(q))Dg(q)v^u\|}{\|Dg(q)v^u\|} s_c(Dg(q)v)$$

$$= \frac{\|Df^n(g(q))Dg(q)v^u\|}{\|Dg(q)v^u\|} \frac{\|Df(H(q))DH(q)v^u\|}{\|DH(q)v^u\|} s_c(DH(q)v)$$

$$= \frac{\|Df^{n+1}(H(q))DH(q)v^u\|}{\|DH(q)v^u\|} s_c(DH(q)v). \quad (5.3)$$

In particular, we obtain the following bounds for the slope

$$m(Df^{n+1}|E_f^u)s_c(DH(q)v) \leq s_c(Dg^{n+1}(q)v) \leq \|Df^{n+1}|E_f^u\|s_c(DH(q)v), \quad (5.4)$$

and from $(2.3)$ we can write

$$\lambda_3^{n+1}s_c(DH(q)v) \leq s_c(Dg^{n+1}(q)v) \leq \mu_3^{n+1}s_c(DH(q)v). \quad (5.5)$$

For $q = q_0$, we have $H(q_0) = q_0$ and $DH(q_0)(0, v^c, v^u) = (0, v_c, tv_c + v^u)$. If $v = v^c + v^u \in E_f^{cu}(q_0)$, then

$$s_c(DH(q_0)v) = \frac{\|tv^c + v^u\|}{\|v^c\|} \geq |t| - s_c(v). \quad (5.6)$$

In particular, if $s_c(v) < \frac{|t|}{4}$, then

$$s_c(DH(q_0)v) > \frac{3|t|}{4} > \frac{|t|}{4} > s_c(v).$$

By the continuous depeence of the splitting with respect to the point $q$ for $f$, there exist $\eta > 0$ and $a > b > 0$ such that, for any $q \in B(q_0, \eta)$ and $v \in E_f^{cu}(q)$, such that $s_c(v) < b$, we have

$$s_c(DH(q)v) > a > b > s_c(v). \quad (5.7)$$

We can reformulate $(5.7)$ in terms of cones. For $b \geq 0$, we consider the cones inside $E_f^{cu}(q)$ around the subspace $E_f^{cu}(q)$ given for $q \in M$ by

$$C_b(q) := \{v = v^c + v^u \in E_f^{cu}(q) : b \geq s_c(v)\} \cup \{0\}.$$
Then, for every $q \in B(q_0, \eta)$, $DH(q)$ carries the cone $C_b(q)$ in a cone around $DH(q)E^c_q(q)$ whose slope is at least $a > b > 0$. In particular $DH(q)C_b(q) \cap C_b(H(q)) = \{0\}$; see Figure 2.

![Figure 2. Action of $DH(q)$ over the cone $C_b(q)$.

In addition, in the case $q \in B(q_0, \eta) \subseteq V$, $g(q), \ldots, g^n(q) \notin V$ and $g^{n+1}(q) \in V$, $n \geq 1$, we have

$$s_c(Dg^{n+1}(q)v) \geq \lambda_3^{n+1}s_c(DH(q)v) > \lambda_3^{n+1}a > s_c(v).$$

This implies that, when $q \in B(q_0, \eta)$ returns to $V$ after $n + 1 \geq 2$ iterates, the cone $Dg^{n+1}C_b(q)$ is far away from the cone $C_b(g^{n+1}(q))$. In other words, for any $q \in B(q_0, \eta)$ the cone $C_b(g^{R(q)}(q))$ cannot be backward $Dg^{R(q)}$-invariant, where $R(q)$ is the first return map of $q$ to $V$ under the action of $g$. Hence, for every $q \in B(q_0, \eta)$ returning to $V$ in a future iterate of $g$, the central direction $E^c_{g^{R(q)}(q)}$ of $g$ cannot be contained in $C_b(g^{R(q)}(q)).$

Denote by $\hat{V}$ the set of points $\hat{q} \in V$ such that there are $R \geq 2$ and $q \in B(q_0, \eta)$ such that $g^R(q) = \hat{q}$ and for $1 \leq n < R$, $g^n(q) \notin V$. Thus, for every $\hat{q} \in V$, $E^c_{g^{R(q)}(q)}(\hat{q}) = E^c_g(\hat{q})$ can be written as the graph of a nonzero linear map $G_{\hat{q}} : E^c_f(\hat{q}) \to E^u_f(\hat{q}).$

As explained in Section 3.4, our assumption of minimality of the unstable foliation implies that $\mu$ almost every point visits any given open subset, the set $V$ say, with positive asymptotic frequency, for each $\mu$-Gibbs state $\mu$ of $g$. In particular,

$$\mu(\hat{V}) = \mu(B(q_0, \eta)) > 0$$

concluding the proof of the lemma. \hfill \square

6. COMPARING THE ACTION OF THE DERIVATIVES

Here we complete the proof of the main Theorem A, providing the details of (2.4) from the overview in Section 2.6.
We compare the norm of the actions of the derivatives of \( g \) and \( f \) on the new central subbundle and the old central subbundle respectively. We use the adapted norms, introduced in Section 4.1, to our advantage in the calculation that follow, together with the smoothness of the strong-unstable foliation.

The main idea comes from a simple fact of linear algebra:

**Lemma 6.1.** Consider \( A : E \oplus F \to E \oplus F \) a linear transformation where the subspaces \( E \) and \( F \) are invariant under the action of \( A \). Assume that there exists \( \lambda > 1 \) such that

(a) \( m(A|F) > \lambda \), that means \( A \) is uniformly expanding on \( F \); and,

(b) \( \|A|E\| < \lambda \), or equivalently, the splitting \( E \oplus F \) is dominated.

Let \( L : E \to F \) be a linear map, \( L \not\equiv 0 \), put \( G = \text{graph } L \) and assume that the norm above is given by an inner product on \( E \oplus F \) such that \( E \) is orthogonal to \( F \). Then there exists \( \xi > 0 \) such that

\[
\|A|G\| \geq (1 + \xi)\|A|E\|. \tag{6.1}
\]

**Proof.** Fix \( u \in G \), \( u \not\equiv 0 \). Then, there is \( v \in E \setminus \{0\} \) such that \( u = v + Lv \). Since the decomposition \( E \oplus F \) is \( A \)-invariant, then \( v, Av \in E \) and \( Lv, ALv \in F \). Moreover,

\[
\frac{\|Au\|^2}{\|u\|^2} = \frac{\|Av + ALv\|^2}{\|v + Lv\|^2} = \frac{\|Av\|^2 + \|ALv\|^2}{\|v\|^2 + \|Lv\|^2} = \frac{\|Av\|^2}{\|v\|^2} \left[ 1 + \frac{\|ALv\|^2}{\|Av\|^2} \right] \left[ 1 + \frac{\|Lv\|^2}{\|v\|^2} \right].
\]

If \( u = v + w \in E \oplus F \) with \( u \in E, v \in F \) and \( s_E(u) = \frac{\|u\|}{\|v\|} \) is the slope of \( u \), then it follows directly from the assumptions over \( A \) that

\[
s_E(Au) > s_E(u). \tag{6.2}
\]

Simple algebraic manipulation over (6.2) give us

\[
\frac{1 + \|ALv\|^2}{1 + \|Lv\|^2} \geq 1 + \xi \tag{6.3}
\]

and the statement of the lemma follows. \( \square \)

**Lemma 6.2.** There exists a measurable function \( \xi : M \to \mathbb{R} \) such that for every \( q \in M \),

\[
\|Dg(q)|E^c_q(q)\| \geq 1 + \xi(q).
\]

Moreover, the set of points \( q \in M \) such that \( \xi(q) > 0 \) has positive \( \mu \) measure for every \( u \)-Gibbs state \( \mu \) of \( g \).

**Proof.** We subdivide the argument in a number of cases for clarity.

**CASE A:** \( E^c_q(q) \) is the graph of the non-zero linear map \( G_q : E^c_f(q) \to E^u_f(q) \).

In this case

\[
E^c_q(q) = \{u + G_q(u) : u \in E^c_f(q)\} = (I + G_q)(E^c_f(q)).
\]
CASE A.1: for \( q \in M \setminus V \), we can apply the previous Lemma 6.1 to \( A = Dg(q) = Df(q) \), \( E = E_g^c(q) \), \( F = E_g^u(q) = E_f^c(q) \), \( G = E_g^c(q) \) and \( L = G_q \) obtaining
\[
\|Df(q)|E_g^c(q)\| \geq (1 + \xi(q))\|Df(q)|E_f^c(q)\| = 1 + \xi(q) \tag{6.4}
\]
for some \( \xi(q) > 0 \), which is clearly true by our assumptions (2.2) and (2.3).

CASE A.2: for \( q \in V \), we apply Lemma 6.1 to \( A = Dg(q) = Df(H(q)) \), \( E = DH(q)E_g^c(q) \), \( F = DH(q)E_f^u(q) \), \( G = DH(q)E_g^c(q) \) and \( L = DH(q)G_q \) obtaining
\[
\|Dg(q)|E_g^c(q)\| = \|Df(H(q))|DH(q)E_g^c(q)\|
\geq (1 + \xi(q))\|Df(H(q))|E_f^c(H(q))\| = 1 + \xi(q) \tag{6.5}
\]
and this is clearly true again from assumptions (2.2) and (2.3).

CASE B: \( E_g^c(q) = E_f^c(q) \) (that is, we assume that \( G_q \equiv 0 \)).

CASE B.1: for \( q \in M \setminus V \) we have
\[
\|Dg(q)|E_g^c(q)\| = \|Df(q)|E_f^c(q)\| = 1. \tag{6.6}
\]

CASE B.2: otherwise, for \( q \in V \) we have
\[
\|Dg(q)|E_g^c(q)\| = \|Df(H(q))|DH(q)E_g^c(q)\| = \|Df(H(q))|Dg(q)|E_f^c(q)\|.
\]

Now we have the following two alternatives.

CASE B.2a: If \( DH(q)E_f^c(q) = E_f^c(q) \), then \( \|Dg(q)|E_g^c(H(q))\| = 1. \)

CASE B.2b: Otherwise, \( DH(q)E_f^c(q) \) is the graph of a non-zero linear maps \( G_{H(q)} : E_f^c(H(q)) \to E_q^u(H(q)) \) so we apply Lemma 6.1 to \( A = Dg(q) = Df(H(q)) \), \( E = E_q^c(H(q)) \), \( F = E_f^u(H(q)) \), \( G = DH(q)E_f^c(q) \) and \( L = G_{H(q)} \) obtaining
\[
\|Dg(q)|E_g^c(q)\| = \|Df(H(q))|DH(q)E_g^c(q)\|
\geq (1 + \xi(q))\|Df(H(q))|E_f^c(H(q))\| = 1 + \xi(q) \tag{6.7}
\]

Finally, putting together (6.4), (6.5), (6.6) and (6.7) we obtain a measurable function \( \xi : M \to \mathbb{R} \) such that for every \( q \in M \), \( \xi(q) \geq 0 \) and
\[
\|Dg(q)|E_g^c(q)\| \geq 1 + \xi(q).
\]

It follows from Lemma 5.1 that the set of points \( q \in M \) such that \( \xi(q) > 0 \) has positive \( \mu \) measure for every \( u \)-Gibbs state \( \mu \) of \( g \).

As explained in the outline of the proof, in Section 2.6, the main theorem and corollaries follow easily from this estimates and known results on mostly expanding partially hyperbolic diffeomorphisms. This completes the proof of the main theorem.


REFERENCES

[1] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140(2):351–398, 2000.

[2] M. Andersson. Robust ergodic properties in partially hyperbolic dynamics. *Trans. Amer. Math. Soc.*, 362(4):1831–1867, 2010.

[3] M. Andersson and C. H. Vásquez. Partially hyperbolic diffeomorphisms whose central direction is mostly expanding. In preparation, 2010.

[4] D. V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Proc. Steklov Math. Inst.*, 90:1–235, 1967.

[5] V. Araújo and M. Bessa. Dominated splitting and zero volume for incompressible three flows. *Nonlinearity*, 21(7):1637, 2008.

[6] A. T. Baraviera and C. Bonatti. Removing zero Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 23(6):1655–1670, 2003.

[7] L. Barreira and Y. B. Pesin. *Lyapunov exponents and smooth ergodic theory*, volume 23 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2002.

[8] Y. Benoist, P. Foulon, and F. Labourie. Flots d’Anosov à distributions stable et instable différentiables. *J. Amer. Math. Soc.*, 5(1):33–74, 1992.

[9] M. Bessa. The Lyapunov exponents of generic zero divergence 3-dimensional vector fields. *Ergodic Theory and Dynamical Systems*, 27(5):1445–1472, 2007.

[10] M. Bessa and J. Rocha. Removing zero Lyapunov exponents in volume-preserving flows. *Nonlinearity*, 20(4):1007–1016, 2007.

[11] J. Bochi.Genericity of zero Lyapunov exponents. *Ergodic Theory Dynam. Systems*, 22(6):1667–1696, 2002.

[12] J. Bochi, B. R. Fayad, and E. Pujals. A remark on conservative diffeomorphisms. *C. R. Math. Acad. Sci. Paris*, 342(10):763–766, 2006.

[13] J. Bochi and M. Viana. The Lyapunov exponents of generic volume-preserving and symplectic maps. *Ann. of Math. (2)*, 161(3):1423–1485, 2005.

[14] C. Bonatti, L. J. Díaz, and E. Pujals. A C1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Math.*, 157(2):355–418, 2003.

[15] C. Bonatti, L. J. Díaz, and R. Ures. Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms. *J. Inst. Math. Jussieu*, 1(4):513–541, 2002.

[16] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[17] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[18] K. Burns, D. Dolgopyat, and Y. Pesin. Partial hyperbolicity, Lyapunov exponents and stable ergodicity. *J. Statist. Phys.*, 108(5-6):927–942, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[19] K. Burns, D. Dolgopyat, Y. Pesin, and M. Pollicott. Stable ergodicity for partially hyperbolic attractors with negative central exponents. *J. Mod. Dyn.*, 2(1):63–81, 2008.

[20] K. Burns, C. Pugh, and A. Wilkinson. Stable ergodicity and Anosov flows. *Topology*, 39(1):149–159, 2000.

[21] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math. (2)*, 171(1):451–489, 2010.

[22] D. Dolgopyat. On dynamics of mostly contracting diffeomorphisms. *Comm. Math. Phys.*, 213(1):181–201, 2000.

[23] D. Dolgopyat. Limit theorems for partially hyperbolic systems. *Trans. Amer. Math. Soc.*, 356(4):1637–1689, 2004.
[24] D. Dolgopyat. On differentiability of SRB states for partially hyperbolic systems. *Invent. Math.*, 155(2):389–449, 2004.

[25] C. Godbillon. *Géométrie différentielle et mécanique analytique*. Hermann, Paris, 1969.

[26] N. Gourmelon. Adapted metrics for dominated splittings. *Ergodic Theory Dynam. Systems*, 27(6):1839–1849, 2007.

[27] G. A. Hedlund. On the metrical transitivity of the geodesics on closed surfaces of constant negative curvature. *Ann. of Math. (2)*, 35(4):787–808, 1934.

[28] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lect. Notes in Math*. Springer Verlag, New York, 1977.

[29] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 91:261–304, 1939.

[30] E. Hopf. Ergodic theory and the geodesic flow on surfaces of constant negative curvature. *Bull. Amer. Math. Soc.*, 77:863–877, 1971.

[31] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer Verlag, New York, 1987.

[32] W. Parry. Dynamical systems on nilmanifolds. *Bull. London Math. Soc.*, 2:37–40, 1970.

[33] Y. Pesin and Y. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.*, 2:417–438, 1982.

[34] J. F. Plante. Anosov flows. *Amer. J. Math.*, 94:729–754, 1972.

[35] C. Pugh and M. Shub. Stable ergodicity. *Bull. Amer. Math. Soc. (N.S.*), 41(1):1–41 (electronic), 2004. With an appendix by Alexander Starkov.

[36] F. Rodríguez Hertz, M. A. Rodríguez Hertz, A. Tahzibi, and R. Ures. A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms. *Electron. Res. Announc. Math. Sci.*, 14:74–81, 2007.

[37] F. Rodríguez Hertz, M. A. Rodríguez Hertz, A. Tahzibi, and R. Ures. Maximizing measures for partially hyperbolic systems with compact center leaves. *Preprint arXiv:1010.3372*, 2010.

[38] F. Rodríguez Hertz, M. A. Rodríguez Hertz, A. Tahzibi, and R. Ures. A survey of partially hyperbolic dynamics. In *Partially hyperbolic dynamics, laminations, and Teichmüller flow*, volume 51 of *Fields Inst. Commun.*, pages 35–87. Amer. Math. Soc., Providence, RI, 2007.

[39] D. Ruelle. Perturbation theory for Lyapunov exponents of a toral map: extension of a result of Shub and Wilkinson. *Israel J. Math.*, 134:345–361, 2003.

[40] D. Ruelle. Differentiation of SRB states for hyperbolic flows. *Ergodic Theory Dynam. Systems*, 28(2):613–631, 2008.

[41] D. Ruelle. A review of linear response theory for general differentiable dynamical systems. *Nonlinearity*, 22(4):855–870, 2009.

[42] M. Shub. *Global stability of dynamical systems*. Springer Verlag, 1987.

[43] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139(3):495–508, 2000.

[44] C. Vásquez. Statistical stability for diffeomorphisms with dominated splitting. *Ergodic Theory Dynamical Systems*, 27(1):253–283, 2007.

[45] C. H. Vásquez. Stable ergodicity for partially hyperbolic attractors with positive central Lyapunov exponents. *J. Mod. Dyn.*, 3(2):233–251, 2009.

[46] M. Viana and J. Yang. Physical measures and absolute continuity for one-dimensional center direction. http://w3.impa.br/viana/out/VYSRBs.pdf, July 2010.

[47] A. Wilkinson. Stable ergodicity of the time-one map of a geodesic flow. *Ergod. Th. & Dynam. Sys.*, 18:1545–1587, 1998.
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