The axisymmetric oscillations of a cylindrical bubble in a liquid bounded volume with free deformable interface

A A Alabuzhev¹,² and M I Kaysina²
¹Institute of Continuous Media Mechanics UB RAS, Perm 614013, Russia
²Perm State University, Perm 614990, Russia

E-mail: alabuzhev@icmm.ru

Abstract. The eigen axisymmetric oscillations of a cylindrical gas bubble surrounded by an incompressible fluid with free deformable interface are considered. The bubble has an equilibrium cylindrical shape and is bounded axially by two parallel solid surfaces. Dynamics of contact lines is taken into account by an effective boundary condition: velocity of the contact line is assumed to be proportional to contact angle deviation from the equilibrium value. The equilibrium contact angle is right. Eigen frequency decreases with liquid outer free surface radius decreasing and increases with the radius-to-height ratio increasing. It’s found that the eigen frequency can vanish in some wetting parameter interval for the volume mode of natural oscillations (which describes the radial compression of the bubble). The length of this interval increases with increasing ratio of the equilibrium bubble radius to the height. The eigen frequencies of other modes decrease with increasing Hocking’s constant. The lowest natural frequency is observed for the freely sliding bubble.

1. Introduction

Motion of triple contact line dynamics have been examined in various formulations for many physical problems [1-4]. For very fast relaxation processes at the triple line the Hocking condition [5] is widely used [6-13]. This condition assumes a linear relationship between the velocity of the contact line motion and the contact angle (for the case of right equilibrium contact angle)

\[ \frac{\partial \zeta^*}{\partial t} = \Lambda \vec{k} \cdot \vec{V} \zeta^* , \]

where \( \zeta^* \) is the deviation of the interface from the equilibrium position, \( \vec{k} \) is the external normal to the solid surface, \( \Lambda \) is a phenomenological constant (the so-called wetting parameter or Hocking parameter) having the dimension of the velocity. There are two important limit of the boundary condition(1): (a) \( \zeta^* = 0 \) – the requirement of the fixed contact line (pinned-end edge condition), (b) \( \vec{k} \cdot \vec{V} \zeta^* = 0 \) – the constant contact angle.

In the present article, we consider the axisymmetric oscillations of cylindrical bubble which surrounded by a liquid with free deformable interface. We apply condition (1) and continue the study of the cases with non-deformable interface [14]. The translation oscillations of this bubble were researched in [15]. The axisymmetric natural and forced oscillations of a compressible hemispherical bubble on a solid substrate are studied in [10]. The contact line dynamics is taken into account by application of the Hocking condition (1). Authors of [12] considered a sessile hemispherical bubble sitting on the transversally oscillating bottom of a deep liquid layer and focus on the interplay of the compressibility of the bubble and the contact angle hysteresis (by analogy [11] for hemispherical drop

¹ To whom any correspondence should be addressed.
of incompressible fluid). Authors of the above articles consider isothermal problem with assumption of adiabatic motionless bubble.

2. Problem formulation

By analogy [15] a gaseous bubble surrounded by a liquid with a free external surface are consider (see figure 1). The bubble is bounded by two parallel solid plates which separated by a distance \( h \). In equilibrium, the bubble and fluid volume have circle cylindrical with a radius \( r_0^* \) and \( R_0^* \), respectively, and contact angle is \( 90^\circ \).

![Figure 1. Problem geometry.](image)

The oscillations amplitude \( A^* \) is small compared to the equilibrium radius \( r_0^* \). The fluid motion is assumed to be incompressible: \( \omega^* r_0^* / c \ll 1 \), where \( \omega^* \) is fundamental oscillation frequency, \( c \) is the sound velocity. However \( \omega^* \) is large enough for the viscosity could be ignored: \( \delta = \sqrt{\nu / \omega^*} \ll r_0^* \) where \( \delta \) is the boundary-layer thickness.

Owing to the problem symmetry, it is convenient to introduce cylindrical coordinates \( r^*, \alpha, z^* \). The azimuthal angle \( \alpha \) is reckoned from the \( x \) axis, but \( \alpha \) will be neglected the further. Let the lateral surface of the bubble and the external surface of liquid volume be described by the following equations, respectively

\[
r^* = r_0^* + \zeta^* (z^*, t^*), \quad r^* = R_0^* + \xi^* (z^*, t^*)
\]

Following [10,14], we use \( \sqrt{\rho^* r_0^* / \sigma^*}, r_0^*, h^*, A^*, A' \sigma^* / r_0^*, A^* \sqrt{\sigma^* / \rho^* r_0^*} \) as the scales for the time, length, height, deviation of bubble surface and free surface from its equilibrium position, pressure, and velocity potential, respectively (\( \sigma^* \) is the surface tension and \( \rho^*_e \) is the liquid density). Thus, the dimensionless boundary value problem is determined by (intermediate steps can be found in [10,14])

\[
p_e = -\varphi_0, \quad \Delta = 0, \quad p_0 = -2n_p P_0 r_0^* / \sigma^* \left\{ \zeta^* \right\} = -P_0 \left\{ \zeta^* \right\}, \quad (2)
\]

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \varphi}{\partial r} \right) + b^2 \frac{\partial^2 \varphi}{\partial z^2},
\]

\[
r = 1: \quad \zeta_t = \varphi_t, \quad [p] = \zeta + b^2 \zeta_{zz}, \quad (3)
\]

\[
r = R_0: \quad \zeta_t = \varphi_t, \quad [p] = \zeta + b^2 \zeta_{zz}, \quad (4)
\]

\[
z = \pm \frac{1}{2}: \quad \varphi_z = 0, \quad (5)
\]

\[
r = 1, \quad z = \pm \frac{1}{2}: \quad \zeta_t = \pm \lambda \zeta_z, \quad (6)
\]

\[
r = R_0, \quad z = \pm \frac{1}{2}: \quad \zeta_t = \pm \lambda \zeta_z, \quad (7)
\]

where \( p_e \) is the liquid pressure, \( \varphi \) is potential of liquid velocity, \( p_0 \) is the gas pressure in the bubble, \( n_p \) is polytropic (e.g., adiabatic) exponent, \( P_0^* \) is dimension gas pressure in the bubble. The
boundary-value problem (2)–(6) involves five parameters: the aspect ratio, the radius of free surface, the wetting parameter, the frequency and amplitude
\[ b = r_0^* / h^*, \quad R_0 = R_0^* / r_0^*, \quad \lambda = \Lambda \sqrt{\rho_0 r_0^* / \sigma^*}, \quad \omega = \omega^* \sqrt{\rho_0 r_0^* / \sigma^*}, \quad \varepsilon = \varepsilon^* / r_0^*. \]

3. Natural axisymmetric oscillations of the bubble

In order to investigate the problem, it is convenient to begin with a consideration of the natural oscillations of a cylindrical bubble. By the evenness of the natural oscillation modes is meant the evenness of the functions under a change of sign of the axial coordinate \( z \). In view of translational symmetry, the solution of the boundary value problem (2)–(7) without external force is written as

\[ \varphi (r, z, t) = i \sum_{n=0}^{\infty} \left( a_n^{(1)} R_n^i (r) + b_n^{(1)} R_n^i (r) \right) \sin((2n+1)\pi z) + \right] e^{i\omega t}, \]

\[ \zeta (z, t) = \left( d_1 \sin \left( \frac{z}{b} \right) + d_2 \cos \left( \frac{z}{b} \right) + \sum_{n=0}^{\infty} h_n \sin((2n+1)\pi z) + \sum_{n=0}^{\infty} c_n \cos(2n\pi z) \right) e^{i\omega t}, \]

\[ \xi (z, t) = \left( D_1 \sin \left( \frac{z}{b} \right) + D_2 \cos \left( \frac{z}{b} \right) + \sum_{n=0}^{\infty} H_n \sin((2n+1)\pi z) + \sum_{n=0}^{\infty} C_n \cos(2n\pi z) \right) e^{i\omega t}, \]

where \( \Omega \) is eigen frequency, \( R_n^i (r) = I_n((2n+1)\pi br) \), \( R_n^e (r) = K_n((2n+1)\pi br) \), \( R_n^i (r) = \text{const} \), \( R_n^e (r) = r^{-1} \), \( R_n^i (r) = K_n(2n\pi br) \), \( I_n \) and \( K_n \) are modified Bessel functions of the zero order.

Substituting solutions (8)–(10) into (2)–(7), we obtain a spectral-amplitude problem which eigenvalues are the values of the natural oscillation frequency \( \Omega \). From the solution of this problem it follows that the eigenvalues are found from the equations:

even modes

\[ \frac{-1}{S_0 + \omega^2} \left[ \lambda \sin \frac{1}{2b} \sum_{n=1}^{\infty} f_n \Omega_n^2 (R_0) + f_0 \left( \Omega_0^2 (R_0) + \omega^2 \right) \right] + \right] + \right]

\[ + \sum_{n=1}^{\infty} \frac{f_n \omega^2}{\omega^2 - \Omega_n^2 (1)} + \cos \frac{1}{2b} \sin \frac{1}{2b}, \]

\[ \Omega_n^2 (1) = \frac{P_n - 1}{R_n^0 \ln R_0}, \quad \Omega_n^2 (1) = \frac{4\pi^2 n^2 b^2 - 1}{R_n^0 (R_0)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)}, \]

\[ S_0 = \Omega_0^2 (R_0) - \Omega_0^2 (1), \quad f_0 = 2b \sin \frac{1}{2b}, \quad f_n = \frac{4b}{4\pi^2 n^2 b^2 - 1} - \frac{1}{2b}. \]

Odd modes

\[ \sum_{n=1}^{\infty} \frac{4b}{(2n+1)^2 \pi^2 b^2 - 1} \omega^2 - \Omega_n^2 (1) = \frac{1}{2b} \]

\[ \sum_{n=1}^{\infty} \frac{4b}{(2n+1)^2 \pi^2 b^2 - 1} \omega^2 - \Omega_n^2 (R_0) = \frac{1}{2b}, \]

\[ \Omega_n^2 (1) = \frac{(2n+1)^2 \pi^2 b^2 - 1}{R_n^0 (R_0) R_n^0 (1)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)}, \]

\[ \Omega_n^2 (R_0) = \frac{(2n+1)^2 \pi^2 b^2 - 1}{R_n^0 (R_0) R_n^0 (1)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)} - \frac{R_n^0 (1) R_n^0 (1)}{R_n^0 (R_0) - R_n^0 (R_0) R_n^0 (1)}. \]
Here \( \Omega_{0}(l) \), \( \Omega_{R}(l) \), \( \Omega_{0}(R_{0}) \), \( \Omega_{R}(R_{0}) \) are the natural oscillation frequencies of the freely moving contact line. The complex algebraic equations (11)–(13) have complex solutions (were solved numerically with the usage of the two-dimensional secant method), this leads to oscillation damping due to the dissipation on the contact line.

![Figure 2](image1.png)  
**Figure 2.** Frequency (a) and damping ratio (b) of volume natural oscillations vs wetting parameter \( \lambda \) for \( \Omega_{0} \) (\( R_{0} = 5 \), \( P_{0} = 5 \)).  
\( b = 1 \) – line 1, \( b = 2 \) – line 2, liquid – solid line, bubble – dashed line.

![Figure 3](image2.png)  
**Figure 3.** Frequency (a) and damping ratio (b) of shape natural oscillations vs wetting parameter \( \lambda \) for \( \Omega_{0} \) (\( R_{0} = 5 \), \( P_{0} = 5 \)).  
\( b = 1 \) – line 1, \( b = 2 \) – line 2, 1 liquid – solid line, bubble – dashed line.

Figures 2 and 3 show the real part of \( \text{Re}(\Omega) \) (oscillation frequency) and imaginary part \( \text{Im}(\Omega) \) (damping ratio) of the complex natural frequency \( \Omega \) for the oscillation even modes \( \Omega_{e} \) (i.e., \( k = 0 \) is the wavenumber) and \( \Omega_{s} \). Even values of \( k \) correspond to the even modes (the solution of Eqs. (11)) and odd values of \( k \) correspond to the odd modes (the solution of Eqs. (12),(13)). The zero translational mode \( k = 0 \) describes volume (radial) oscillations. In a certain range of \( \lambda \), the real part of the frequency \( \text{Re}(\Omega_{s}) \) can vanish, depending on the value of the geometrical parameter \( b \) (figure 2a). For small values of \( b \), this range is absent. The vanishing of \( \text{Re}(\Omega_{s}) \) corresponds to the bifurcation of the branch of the increment \( \text{Im}(\Omega_{s}) \) (curve 2 in figure 2b). In this case dissipation is
proportional to the length of the contact line, as it is the interaction of the contact line and the solid plate is a cause of dissipation. Therefore, growing of parameter $b$ increases the length of the contact line at constant bubble volume, i.e. it increases the dissipation (also see [10,14]). Note, that the eigen frequencies of the bubble are higher than the eigen frequencies of the fluid volume. It is the result of the influence of the surface tension force on external interface.

For shape modes $k > 0$, the behavior resembles the case of the incompressible liquid. The frequency of shape oscillations decreases monotonically as $\lambda$ increases (see figure 3a), the damping ratio has a maximum for a finite value of the capillary parameter and tends to zero as $\lambda \to 0$ and $\lambda \to \infty$ (figure 3b). However volume frequency can decreases monotonically as $\lambda$ increases (see figure 4). It is possible when the complex frequencies of the two lowest modes become (see [10,14] for details).

Figure 5 shows the real part of $\text{Re}(\Omega_0)$ (oscillation frequency) and imaginary part $\text{Im}(\Omega)$ (damping ratio) of the complex natural frequency $\Omega$ for the oscillation odd mode $\Omega_1$. The real part of the frequency $\text{Re}(\Omega_1)$ can vanish in a certain range of $\lambda$ (figure 5a) also $\text{Re}(\Omega_2)$ (figure 2a). But this range is absent for large values of $b$. The reason for the vanishing of the frequencies associated with such large dissipation (parameter $\lambda$ is finite) that oscillations become impossible: dissipation is proportional to the area of the side surface for surface modes $k > 0$ of natural oscillations (than the surface is greater than the more energy is required for surface waves and the energy of bubble oscillations is proportional to the volume). Consequently, increase of geometrical parameter $b$ is corresponds to a decrease side surface of the bubble at a constant bubble volume, i.e. dissipation is reducing for surface oscillations (see [14,16] for details).
4. Conclusions
The eigen frequencies of the bubble are higher than the eigen frequencies of the fluid volume. Eigen frequency decreases with the decrease of free surface radius and increase with the bubble aspect ratio increasing. The radial frequency of the eigen oscillations increases at gas pressure $P_g$ growing.

The shape frequencies are independent of the gas pressure inside the bubble.

The radial frequency of the eigen oscillations vanishes at a certain value of the Hocking parameter $\lambda$. This interval of values $\lambda$ grows at increasing aspect ratio $b$. The first shape frequency also can vanish at some $\lambda$, but length of interval $\lambda$ reduces at increasing $b$.

For $b < \pi^{-1}$ and a certain characteristic value of $\lambda$, the first shape frequency vanishes and the increment becomes negative, which corresponds to the occurrence of Rayleigh instability.

Acknowledgments
This work was supported by the Russian Foundation for Basic Research (project 14-01-96017-r-ural).

5. References
[1] Voinov O V 1976 Fluid Dyn 11 714–721
[2] De Gennes P G 1985 Rev. Mod. Phys. 57 827
[3] Bonn D, Eggers J, Indekeu J, Meunier J and Rolley E 2009 Rev. Mod. Phys. 81 739–805
[4] Zhang L and Thiessen D B 2013 J. Fluid Mech. 719 295–313
[5] Hocking L M 1987 J. Fluid Mech. 179 253-66
[6] Borkar A and Tsamopoulus J 1991 Phys. Fluids A 3 2866–74
[7] Perlin M, Schultz W W and Liu Z 2004 Wave Motion 40 41–56
[8] Alabuzhev A A and Lyubimov D V 2005 Fluid Dyn 40 183-192
[9] Alabuzhev A A and Lyubimov D V 2007 J. Appl. Mech. Tech. Phys. 48 686–93
[10] Shklyaev S and Straube A V 2008 Phys. Fluids 20 052102
[11] Fayzrakhmanova I S and Straube A V 2009 Phys. Fluids 21 072104
[12] Fayzrakhmanova I S, Straube A V and Shklyaev S 2011 Phys. Fluids 23 102105
[13] Alabuzhev A A and Lyubimov D V 2012 J. Appl. Mech. Tech. Phys. 53 9–19
[14] Alabuzhev A A 2014 Computational Continuum Mechanics 7 151–61 (in Russian)
[15] Alabuzhev A A and Kaysina M I 2016 J. Phys.: Conf. Ser. 681 012043
[16] Alabuzhev A A 2016 Computational Continuum Mechanics 9 316–30 (in Russian)