Limits on possible magnetic fields at nucleosynthesis time

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Abstract

In this paper we discuss limits on magnetic fields that could have been present at nucleosynthesis time. We considered several effects that could be relevant modifying light elements relic abundances. They include: changes in reaction rates, mass shifts due to strong and electromagnetic interactions, variation of the expansion rate of the Universe due to both the magnetic field energy density and the increasing of the electrons density in overcritical magnetic fields. We find that the latter is the main effect. It was not taken into account in previous calculations. The allowed field intensity at the end of nucleosynthesis \((T = 1 \times 10^9 \text{ oK})\) is \(B \leq 3 \times 10^{10} \text{ Gauss}\).

Accepted by Astroparticle Physisc

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1 Introduction

Amongst the many uncertainties in the Early Universe environment, the possibility that large, constant magnetic fields, existed over macroscopic scales, is a fascinating possibility. Since the Early Universe is believed to be a perfect conductor, magnetic lines get thinned out by the ratio

\[ \frac{B_1}{B_2} = \left( \frac{R_2}{R_1} \right)^2 \]

The presence of primordial fields over galactic scales, when extrapolated back can give, under different assumptions, very large fields indeed.

A typical present day galactic field \( B = 10^{-6} \text{G} \), can grow, scaled as dictated by Eq. (1) alone to be as large as \( 10^{14} \) Gauss. These extrapolations are very doubtful since dynamo effects may have enlarged significantly present fields. Nevertheless, knowing the allowed fields at a given epoch and limiting its value at another one can give important dynamical restrictions.

Recently Vachaspati [1] has shown that large magnetic fields, of the order of \( B = (m_W)^2 = 10^{24} \) Gauss, should be generated at the electroweak phase transition time. His argument seems quite general. It only depends on the finiteness of the horizon and embedding electromagnetism in a larger field.

The size of a patch at electroweak time, is of order \( \chi = N(m_W)^{-1} \). Evolving the size of the patch is model dependent. More precisely, the coefficient \( N \) in [1] is a function of the scale. If the scale is determined by today’s size then \( N = 10^{13} \). This argument predicts a field of about a Gauss at nucleosynthesis time.

The size of the patch at the nucleosynthesis time might allow for larger fields, though we have no real reliable model for the field evolution.

In this paper we address ourselves to the allowed magnetic fields at nucleosynthesis time, without discussing their origin. We give a detailed analysis of the influence of the fields on the main quantities that can act to modify the relic elements abundance ratios: reaction rates, masses of the participants, electron energy densities and magnetic field energy density. We then obtain, given the present errors in these relative abundances, the upper limits the fields can take in regions large compared to the reaction scale but possibly much smaller than the horizon at that time. In the first section we discuss briefly the impact of the magnetic field on these elements of the calculation.
In the next we describe the standard nucleosynthesis calculations in this light. In the final section we discuss the constraints and their origin and compare with existing calculations.

2 Weak reaction rates in the presence of magnetic fields

The main weak processes which act to determine the \( n/p \) ratio during the the primordial nucleosynthesis are

\[
\begin{align*}
  n + e^+ &\rightleftharpoons p + \bar{\nu}, \quad (a) \\
  n + \nu &\rightleftharpoons p + e^- , \quad (b) \\
  n &\rightleftharpoons p + e^- + \bar{\nu}. \quad (c)
\end{align*}
\]

The rate for two body scattering reactions in a medium may be written in the form

\[
\Gamma(12 \rightarrow 34) = \left( \prod_i \int \frac{d^3 p_i}{(2\pi)^3 2 E_i} \right) (2\pi)^4 \delta^4(\sum_i p_i) |\mathcal{M}|^2 f_1 f_2 (1 - f_3) (1 - f_4), \quad (2)
\]

where \( p_i \) is the four momentum, \( E_i \) is the energy and \( f_i \) is the number density of each particle species. All processes in Eq. (a,b and c) have the same amplitude

\[
\mathcal{M} = \frac{G_F}{\sqrt{2}} \cos \theta_C \bar{u}_p \gamma_\alpha (1 - \alpha \gamma_5) u_n \bar{u}_e \gamma_\alpha (1 - \gamma_5) u_\nu \quad (3)
\]

where \( \alpha = g_A/g_V \simeq -1.262 \). Without any external magnetic field the total rate of the processes that convert neutrons to protons is

\[
\Gamma_{n\rightarrow p}(B = 0) = \frac{1}{\tau} \int_{\tau}^\infty d\epsilon \frac{\epsilon \sqrt{\epsilon^2 - 1}}{1 + e^{\epsilon + \phi_\epsilon}} \left[ \frac{(q + \epsilon)^2 e^{\frac{(e+q)m_e}{m_\nu}}}{1 + e^{\frac{(e+q)m_e}{m_\nu}}} + \frac{(\epsilon - q)^2 e^{\frac{em_e}{m_\nu} + \phi_\epsilon}}{1 + e^{\frac{e m_e}{m_\nu}}} \right] \quad (4)
\]
where \( \frac{1}{r} \equiv \frac{G^2(1+3\alpha^2)m_e^3}{2\pi^4} \) and \( q, \epsilon \) and \( \phi_e \) are respectively the neutron-proton mass difference, the electron energy and the electron chemical potential all expressed, in units of \( m_e \). We assume the neutrinos chemical potential to be vanishing.

The total rate for the \( p \rightarrow n \) processes can be obtained changing the sign of \( q \) in Eq. (4).

An external magnetic field take us to modify Eq. (4) due to the following effects.

a) The dispersion relation of charged particles propagating through a magnetic field is modified with respect to the free-field case. In fact, their 4-momentum is in this case \( p = p(B = 0) + qA \), where \( q \) is the charge of the particle and the vector potential \( A(\mathbf{r}) \) is related to the field by \( A(\mathbf{r}) = \frac{1}{2} \mathbf{r} \times \mathbf{B} \). Assuming \( \mathbf{B} \) along the \( z \) axis, the expressions for energies of electrons, protons and neutrons are respectively

\[
E_e = \left[ p_{e,z}^2 + eB(2n + 1 + s) + m_e^2 \right]^{\frac{1}{2}} + \kappa, \quad (5)
\]

\[
E_p = \left[ p_{p,z}^2 + eB(2n + 1 - s) + m_p^2 \right]^{\frac{1}{2}} - \frac{e}{2m_p}\left(\frac{g_p}{2} - 1\right)B, \quad (6)
\]

\[
E_n = \left[ p_{n}^2 + m_n^2 \right]^{\frac{1}{2}} + \frac{e}{2m_n}g_nB. \quad (7)
\]

In the above, \( n \) denotes the Landau level, \( s = \pm 1 \) indicates whether the spin is along or opposed to the field direction, and \( g_p = 5.58 \) and \( g_n = -3.82 \) are the Landé g-factors. The QED correction to the electron energy, \( \kappa \), has been first computed by Schwinger [2]. For magnetic fields larger than \( \sim 10^{13} \) G this correction is

\[
\kappa = \frac{\alpha}{2\pi} \ln \left( \frac{2eB}{m_e^2} \right)^2. \quad (8)
\]

For smaller field intensity \( \kappa \) has negligible effects on our calculations and we disregarded it. The effects of the field on the QCD ground state have been parametrized via a field dependent nucleon mass [3] as we are going to discuss below.

Neither the neutron or the neutrino have quantized levels, though the neutron has an electromagnetic interaction energy. The neutrino is totally inert vis à vis electromagnetism.
b) The number of available states for a particle obeying Eq. (4) becomes, for every value of \( n \) and \( s \), is

\[
\frac{VeB}{(2\pi)^2} dp_z
\]

This changes the phase space of the processes we are interested to consider.

c) Since the occupation number and the energy of states with opposite spin projections is not the same in a magnetic field, the spin sum of the square amplitude needs to be weighted by the appropriate spin dependent Fermi distributions.

Nucleosynthesis take place in a range of temperatures \( 0.1 < T < 10 \text{ MeV} \), hence nucleons are nonrelativistic. Therefore nucleon distribution functions are given by

\[
f_N(s = \pm 1) = (1 + e^{\pm \frac{\mu_{n} B}{T}})^{-1}
\]

where

\[
\mu_p = \frac{e}{2m_p} \frac{g_p}{2}, \quad \mu_n = \frac{e}{2m_n} \frac{g_n}{2}.
\]

Since during the nucleosynthesis \( m_N \gg T \), and momenta are also small compared to nucleon mass, \( f_N \) can be safely approximated by \( 1/2 \). This is not the case for electrons. In this case we have

\[
f_e(s) = \left(1 + e^{E_e(s)/T}\right)^{-1}
\]

where the relativistic expression for the electron energy Eq.(5) is used.

As a consequence, the integral for the leptonic momentum space in neutron \( \beta \) decay is modified to

\[
\frac{1}{2\pi} \sum_{n=0}^{N_e} \int_{-\infty}^{\infty} \frac{d^3P_\mu}{2E_\mu} \int_{-p_{e,z}(n)}^{p_{e,z}(n)} \frac{dp_{e,z}}{2E_e} eB |M|^2 (1 - f_\nu)(1 - f_e)
\]

where \( N_e \) is largest integer \( n \) such that \( p_{e,z}(n)^2 = Q^2 - m_e^2 - 2neB \) is positive and \( Q^2 \equiv m_n^2 - m_p^2 \).

d) The nucleon masses are affected by very strong magnetic fields. The change in effective phase space is

\[
\Delta = 0.12\mu_{N} B - M_n + M_p + f(B).
\]
The function $f(B)$ gives the rate of mass change due to colour forces being affected by the field. For nucleons the main change is the chiral condensate growth, which because of the different quark content of protons and neutrons makes the proton mass grow faster. Though the sign is certain, vacuum pairs of zero helicity get more bound in the presence of a B field, the size of the effect is model dependent. We have calculated its influence using the weakest and strongest reasonable field dependence and we find the effect always small for fields below $10^{18}$ Gauss.

Having established that hadronic mass changes will not affect nucleosynthesis we drop these effects from the equations altogether.

Taking into account the remaining effects we computed the total rate for the weak processes converting neutrons to protons in an external magnetic field. The result is

$$\Gamma_{n \rightarrow p}(B) = \frac{\gamma}{T} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \times \int_{\gamma \ll 1}^{\infty} \frac{d\epsilon}{\sqrt{1+2(n+1)\gamma}} \frac{(\epsilon - \kappa)}{\sqrt{(\epsilon - \kappa)^2 - 1 - 2(n+1)\gamma}}$$

$$\times \frac{1}{1 + e^{\frac{m_e(\epsilon+q)}{T_N} + \phi_e}} \left[ \frac{(\epsilon + q)^2 e^{\frac{m_e(\epsilon+q)}{T_N}}}{1 + e^{\frac{m_e(\epsilon+q)}{T_N} + \phi_e}} + \frac{(\epsilon - q)^2 e^{\frac{m_e(\epsilon-q)}{T_N}}}{1 + e^{\frac{m_e(\epsilon-q)}{T_N} + \phi_e}} \right]$$

(15)

where $\gamma \equiv B/B_c$ and $B_c = m^2_e/e = 4.4 \times 10^{13} G$ is usually defined to be the critical magnetic field.

Equation (15) coincides with the result of Matese and O’Connell and Cheng et al. in the limit in which the QED correction $\kappa$ goes to zero. Although the quantitative effects of this term on the nucleosynthesis predictions are subdominant, we stress that disregarding it when the field is overcritical it leads to an unstable electron ground state, thus to unphysical results.

The main effect of the magnetic field is due to the modification of the electron phase space. Eq.(15) is correct in the weak field limit, when $\gamma \ll 1$ and $\Gamma_{n \rightarrow p}(B)$ reduces to Eq.(4) in the $B = 0$ limit.

In Fig.(1) we present the dependence of $\Gamma_{n \rightarrow p}$ as a function of temperature for some values of $\gamma$.

For large values of $\gamma$ and fixed temperature, the total rate grows like $\gamma$. Increasing the temperature the relevant contribution to the integrals in Eq.(15) comes from the high energy part of the electron spectrum. Since the limit $\epsilon \rightarrow \infty$ is equivalent to the limit $\gamma \rightarrow 0$ in Eq.(15) this explains...
why the ratio $\Gamma_{n \rightarrow p}(B)/\Gamma_{n \rightarrow p}(0)$ goes to one when $T \gg m_e$. Although the global rate of the inverse process $\Gamma_{p \rightarrow n}$ also increases with $B$, it remains suppressed by a factor $\exp(-Q(B)/T)$ with respect to $\Gamma_{n \rightarrow p}$. Thus the effect of a strong magnetic field would be to reduce the final number of neutrons in the Universe, i.e. the relic $^4$He abundance, if only the correction to the weak rates is taken into account.

3 The effects of B on the expansion rate

Owing to exponential dependence of the $(n/p)$ equilibrium ratio on the temperature, the relic relative abundances of light elements depends crucially on the freeze-out $T_F$ temperature of the weak processes that keep protons and neutrons in chemical equilibrium [7]. This temperature is essentially determined by the condition

$$\Gamma_{n \leftrightarrow p}(T_F) = H(T_F)$$

(16)

where $H$ is the expansion rate of the Universe.

It is evident that besides the rate of the weak processes we need to pay attention to the effects of the magnetic field on $H$. If no cosmological constant is present, the expansion rate is determined by the Einstein equation

$$H^2(T) = \frac{8\pi G_N}{3} \rho(T).$$

(17)

where $\rho(T)$ is the total energy density of the Universe. In the case that no magnetic field is present $\rho(T)$ is given by the sum of the energy density of all the particle species in thermal equilibrium with the primordial plasma

$$\rho(T) = \rho_\gamma(T) + \rho_e(T) + \rho_\nu(T) + \rho_b(T)$$

(18)

where the subscripts $\gamma$, $e$, $\nu$, $b$ stand, respectively, for photons, electrons, the three species of neutrinos and baryons, including their respective anti-particles. In our case, since the magnetic field has energy density $\rho_B(T) = B(T)^2/8\pi$ this term also needs to be added to Eq.(18). Since we have magnetic flux conservation in the plasma

$$B \propto R^{-2} \propto T^2$$
the energy density of the magnetic field has the same temperature dependence as the energy density of the radiation.

This new contribution to $\rho(T)$ will dominate over the other terms in Eq.(18) if

$$B(T = 10^{11}\text{ oK}) \gtrsim 10^{16}\text{ G.}$$

We assumed the pressure associated to the random magnetic field to be zero in average. Although a nonvanishing mean pressure is also possible for random magnetic fields [8], our final conclusions are not affected also taking this pressure into account.

The presence of a nonvanishing $\rho_B$ is not the only effect that modifies the expansion rate of the Universe. The energy density of charged particles in the primordial plasma is also affected. In the previous section we have showed how the electron dispersion relation and the electron phase-space are modified by the magnetic field. Using Eqs.(5) and (9) we get the electron energy density in function of $\gamma$

$$\rho_e(T) = \frac{eB}{2\pi^2} \sum_{n=0}^{\infty} \int dp_e E_e(s) f_e =$$

$$= \frac{\gamma}{2\pi^2} \sum_{n=0}^{\infty} (2 - \delta_{n0}) \int d\epsilon \frac{(\epsilon - \kappa)}{\sqrt{(\epsilon - \kappa)^2 - 1} - 2(n + 1)\gamma + 1 + e^{\frac{2\pi + \phi_e}{\epsilon}}} .$$

The reader can see from Fig.(2) that the effect of an overcritical magnetic given by Eq.(19) is to increase the electron energy density roughly linearly with the field intensity. The same of Eq.(20) is valid for positrons once the sign of the chemical potential $\phi_e$ has been changed. In analogy with what happen for the reaction rates, this effect becomes less important at high temperatures if $B$ is left fixed. However it is a very relevant effect when $T \lesssim 1$ MeV. Although the field intensity, hence the correction to $\rho_e$, decreases like $T^2$, we are going to show that this is the main effect on the primordial nucleosynthesis predictions.

4 Results and conclusions

In the previous sections we have shown that the existence of large magnetic fields during the primordial nucleosynthesis affects the final light elements
Table I. Predictions of light element abundances at the end of the primordial nucleosynthesis are given for several values of the magnetic field intensity given at the temperature $T = 10^{11}\,\text{oK}$.

| $B(T = 10^{11}\,\text{oK})$ | $^4\text{He}$ | $(\text{D}+^3\text{He})/\text{H}$ | $^7\text{Li}/\text{H}$ |
|-----------------------------|----------------|-------------------------------|-----------------|
| 0                           | 0.236          | $1.14 \times 10^{-4}$         | $1.11 \times 10^{-10}$ |
| $1 \times 10^{12}$          | 0.236          | $1.14 \times 10^{-4}$         | $1.11 \times 10^{-10}$ |
| $5 \times 10^{12}$          | 0.236          | $1.14 \times 10^{-4}$         | $1.11 \times 10^{-10}$ |
| $1 \times 10^{13}$          | 0.237          | $1.13 \times 10^{-4}$         | $1.11 \times 10^{-10}$ |
| $5 \times 10^{13}$          | 0.240          | $1.08 \times 10^{-4}$         | $1.14 \times 10^{-10}$ |
| $1 \times 10^{14}$          | 0.242          | $1.05 \times 10^{-4}$         | $1.15 \times 10^{-10}$ |
| $5 \times 10^{14}$          | 0.247          | $9.99 \times 10^{-5}$         | $1.20 \times 10^{-10}$ |
| $1 \times 10^{15}$          | 0.250          | $9.71 \times 10^{-5}$         | $1.23 \times 10^{-10}$ |
| $5 \times 10^{15}$          | 0.257          | $9.15 \times 10^{-5}$         | $1.32 \times 10^{-10}$ |
| $1 \times 10^{16}$          | 0.348          | $8.92 \times 10^{-5}$         | $1.35 \times 10^{-10}$ |

relative abundances via two main effects:

a) the increasing of the weak reaction rates and

b) the increasing of the expansion rate of the Universe.

These are competing effects. In fact, whereas the former tend to reduce the $(n/p)$ freeze-out temperature, hence the final abundance of $^4\text{He}$, the latter acts in the opposite direction.

We modified the standard nucleosynthesis code [9] to take into account all the relevant effects, as well as other effects that eventually we neglected as irrelevant.

Since our aim is to get an upper limit to the magnetic field intensity, we adjusted the value of the baryon photon ratio $\eta$ in order to get the minimal $^4\text{He}$ relic abundance prediction compatible with observations [10] in the free-field case. In Tab.I we present our predictions for some light elements relic abundances. As it is evident at a glance, limits on magnetic fields are totally controlled by the $^4\text{He}$ abundance.

Other elements reach forbidden values only at very high fields, in that case the effect of mass changes due to colour forces will also be important.

The increase of the $^4\text{He}$ relic abundance with the field intensity reveals that the effect of $B$ on the expansion rate is the most relevant. Regarding this point we agree with the qualitative conclusion of Matese and O’Connell [5].
and disagree with the opposite conclusion of Cheng et al. [11]. Mainly, we do not understand how they can reconcile the claim that the dominant effects of the magnetic field are those arising from modification of the reaction rates, with the growing of the relic $^4\text{He}$ that they get increasing $B$.

Furthermore, in both refs. [5] and [11] the effect of the magnetic field on the electron and positron energy density was not considered. Leaving only this effect on in our code, we checked that this is indeed the most relevant.

We showed that this is indeed the main effect as long as the field intensity at the beginning of the nucleosynthesis is smaller than $10^{16}$ G.

Since the observational upper bound for the $^4\text{He}$ relic abundance is $Y_p \leq 0.245$ we conclude that the average intensity of a random magnetic field at the temperature of $T = 1 \times 10^{11}$ oK (beginning of nucleosynthesis) must be less than $3 \times 10^{14}$ G or, equivalently, $B(T = 10^9 \text{oK}) < 3 \times 10^{10}$ G (end of nucleosynthesis). Vachaspati [1] predicts a magnetic field strength of $\sim 10^{11}$ G, on the smallest coherence region of the field ($N = 1$), at the end of nucleosynthesis. This extreme assumption ($N = 1$) is ruled out by our limits.

Assuming the field continue to rescale according to Eq.(1) (perhaps not a reasonable assumption), our results imply that the intergalactic field is less than $\sim 3 \times 10^{-7}$ G at present.

The other light elements relic abundances are less affected than $^4\text{He}$ by the magnetic field and we do not use them to get constraints. However, it is interesting to observe the behaviour of Deuterium and $^3\text{He}$ abundances versus the initial magnetic field. Although they increase with the field at the beginning of nucleosynthesis their relic abundances follow the opposite behaviour. This can be understood since the rates of the processes converting Deuterium and $^3\text{He}$ to $^4\text{He}$ are proportional to the initial abundances $Y_D$ or $Y_{^3\text{He}}$ [7]. The greater are the rates smaller are the freeze-out temperatures for these light elements. Thus we expect smaller D and $^3\text{He}$ relic abundances even if the relic $^4\text{He}$ increases.

**Acknowledgments**

One of the authors (D.G.) is grateful to D. Fargion for pleasent and useful discussions.

This work was supported in part by the Swedish Research Council and a
Twinning EEC contract.
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Figure Caption

Fig. 1. The neutron-depletion rate $\Gamma_{n\rightarrow p}$, normalized to the free-field rate, is plotted as a function of the temperature for some constant values of $\gamma$.

Fig. 2. Electron energy densities are displayed as a function of $\gamma$. The lower curve corresponds to the temperature value $T = 0.5$ MeV; the middle one to $T = 1$ MeV and the upper one to $T = 5$ MeV.

Fig. 3. Electron and magnetic field energy densities temperature dependence are compared for two different values of the initial magnetic field.
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$B_{\text{in}} = 5 \times 10^{16} \text{ G}$

Plot showing $\rho_B$ and $\rho_e$ as functions of $T$(MeV).
