The properties of certain linear and nonlinear differential equations of the fourth order arising in beam models

A Chichurin\textsuperscript{1,2} and G Filipuk\textsuperscript{3}
\textsuperscript{1}The John Paul II Catholic University of Lublin, Lublin, Poland
\textsuperscript{2}The Moscow State University of Civil Engineering, Moscow, Russia
\textsuperscript{3}University of Warsaw, Warsaw, Poland

E-mail: achichurin@gmail.com, filipuk@mimuw.edu.pl

Abstract. The purpose of this paper is to present several new results concerning relations between linear differential equations of the fourth order and nonlinear differential equations of the fourth order. These equations are involved in the description of models of building structures, where there are beams with small deflections or curved axes. We consider linear differential equations of the second, the third and the fourth order and nonlinear fourth order differential equations related via the Schwarzian derivative. As a result, we obtain new relations between the solutions of these linear and nonlinear equations. For example, assuming that we know a solution of linear differential equation of the fourth order and a solution of the third order linear differential equation, then the Schwarzian derivative of their ratio solves a certain nonlinear differential equation of the fourth order. We also present some conditions on the coefficients when this statement holds. Two more similar statements are presented. To illustrate theorems and our constructive approach we give two examples. The given method may be generalized to differential equations of higher orders.

1. Introduction

In [1–3], the Schwarzian derivative is a differential operator that is invariant under all linear fractional transformations. It plays a significant role in the theory of modular forms, hypergeometric functions, univalent functions and conformal mappings [1], [2]. It is defined by

\[(S\xi)(z) = \frac{f''(z)}{f'(z)} - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.\]

The well-known relation between a second-order linear differential equation of the form

\[y''(z) + Q(z)y(z) = 0\]

and the Schwarzian derivative of the ratio of two linearly independent solutions $y_1, y_2$ of the linear equation above is as follows:

\[(S\xi)(z) = 2Q(z),\]

where $\xi = y_1 / y_2$ and $z$ is in general, a complex variable. See [1], [2] for more details.

If we have a general second order differential equation

\[y''(z) + p(z)y'(z) + q(z)y(z) = 0,\] (1)

then substituting \( y(z) = \xi(z)y_1(z) \) with the condition that \( y_1 \) is also a solution of (1), we get an expression for \( \xi, y_1 \) and their derivatives (up to order 2 and 1 respectively). Differentiating again and eliminating \( y_1, y_1' \), we get that the function

\[
w(z) = (S\xi)(z)
\]

satisfies

\[
w(z) = \frac{1}{2} (4q(z) - p(z)^2 - 2p'(z)).
\]

We call expression (2) the invariant for the second order linear differential equation (1).

On extension of this approach for linear differential equation of the third order see in [3]–[5]. The generalization of the method for the linear differential equations of the fourth order is given in [6]. In papers [7]–[9] special classes of the fourth order linear differential equations and the nonlinear fourth order differential equations related via the Schwarzian derivative are considered and general solutions of both differential equations are found.

In paper [10] the generalization of the method for a special type of linear differential equations of the fifth order is given along with a computer realization of this method in Mathematica (www.wolfram.com).

Several questions arise. What happens if the second, the third and the fourth order linear differential equations are related? What happens if we modify the function \( \xi \) to be the ratio of solutions of two different equations? These questions were answered during the studies of linear differential equation of the third order in [11].

The main objective of this paper is to answer these questions for linear differential equation of the fourth order with coefficients that satisfy the system of two differential equations of the first order. The proofs of statements are computational, that is the results can be verified by using any computer algebra system.

The practical focus of the work is that linear differential equation of the fourth order arise in the mathematical models of deflection of beams. These equations are involved in the description of models of building structures, where there are beams with small deflections or curved axes. Such beams, which appear in many structures, deflect under their own weight or under the influence of some external forces [17–19].

2. Main Results

In this section we shall present 5 main results concerning relations between linear

\[
y''''(z) + p(z)y'''(z) + q(z)y''(z) + r(z)y'(z) + s(z)y(z) = 0,
\]

where

\[
p' = \frac{1}{12} (8q - 3p^2), \quad q' = \frac{1}{4} (6r - pq)
\]

and nonlinear differential equations.

Theorem 1. Let \( y \) be a solution of the fourth order linear differential equation (3) and \( y_1 \) be a solution of another fourth order linear differential equation

\[
y''''(z) + p_1(z)y'''(z) + q_1(z)y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0,
\]

where

\[
p_1' = \frac{1}{12} (8q_1 - 3p_1^2), \quad q_1' = \frac{1}{4} (6r_1 - p_1q_1).
\]

If the function \( w(z) = (S\xi)(z) \) with \( \xi = y_1 / y_2 \) solves the nonlinear differential equation
then conditions

\[ p_1(z) = p(z), \quad q_1(z) = q(z), \quad r_1(z) = r(z), \quad s_1(z) = s(z) \]

and

\[ \varphi = pr - 16s + 4r' \]  

hold.

Proof. We substitute \( w(z) = (S\xi(z)) \) into equation (4) with unknown coefficients and then replace \( \xi \) by the ratio of \( y \) and \( y_1 \). Replacing the third and higher order derivatives of \( y \) and \( y_1 \) by using the linear equations, we collect the coefficients of \( y \), \( y_1 \) and their derivatives up to order 3. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result.

Theorem 2. Let \( y \) be a solution of equation (3) and \( y_1 \) be a solution of the third order linear differential equation of the form

\[ y^{\prime\prime\prime}(z) + q_1(z)y^{\prime\prime}(z) + r_1(z)y'(z) + s_1(z)y(z) = 0. \]  

If the function \( w(z) = (S\xi(z)) \) with \( \xi = y/y_1 \) solves the nonlinear differential equation (6), then we have the condition (7) for (6) and three additional conditions on the coefficients of the linear equation (8)

\[ q_1' = q_1 + q - pq_1 - r_1, \quad r_1' = r - pr_1 + q_1r_1 - s_1, \quad s_1' = s_1(q_1 - p) + s. \]  

Proof. We substitute \( w(z) = (S\xi(z)) \) into equation (6), (7) with unknown coefficients and then replace \( \xi \) by the ratio of \( y \) and \( y_1 \). Replacing the fourth and higher order derivatives of \( y \) and the third and higher order derivatives of \( y_1 \) by using the linear equations, we collect the coefficients of \( y \), \( y_1 \) and their derivatives up to order 3 and order 2 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result.

Example 1. Let

\[ p(z) = bz^{-1}, \quad s(z) = -\frac{b(b-8)(b-4)}{16z^4}, \]  

where \( b \) is a constant. We substitute functions (10) into equations (3), (4). Solving the obtained equations we find

\[ q(z) = \frac{3b(b-4)}{8z^2}, \quad r(z) = \frac{b(b-8)(b-4)}{16z^4}, \quad y(z) = cz, \]  

where \( c \) is an arbitrary constant. Let

\[ q_1(z) = bz^{-1}. \]  

We substitute functions (10)-(12) into equations (9). Solving the resulting equations, we find

\[ r_1(z) = \frac{b(3b-4)}{8z^2}, \quad s_1(z) = \frac{b(b^2 + 16)}{16z^3}, \quad \frac{b^3}{8} + \frac{3b^2}{4} + b = 0. \]  

From the third equation of system (13) we find

\[ b = -4, \quad b = -2, \quad b = 0. \]
Let us choose, for example, the value $b = -2$. We substitute functions (13) into equation (8) and obtain

$$2z^2y'' - 4zy'' + 5y' + 5z^{-1}y = 0. \quad (15)$$

The general solution of the equation (15) is a form

$$y_1 = C_1z^{2/\sqrt{2}} + C_2z^{2i/\sqrt{2}} + C_3z,$$

where $C_1, C_2, C_3$ are arbitrary constants. We choose, for example, the values of arbitrary constants equal to one. Then the particular solution is

$$y_1 = z^{2/\sqrt{2}} + z^{2i/\sqrt{2}} + z \quad \text{and} \quad \xi = \frac{c}{z + z^{2/\sqrt{2}} + z^{2i/\sqrt{2}}}.$$

Then

$$w = \frac{\xi^m}{\xi^*} = \frac{3\left(\frac{\xi^*}{\xi}\right)^2}{2} = -\frac{(39 + 16\sqrt{6})z^\xi + 30z^\xi - 16\sqrt{6} + 39}{2x^2(2 + \sqrt{6})z^\xi - 6 + 2)2} \quad (16)$$

Differential equation (6), (7) for coefficients (10), (11), $b = -2$ is form

$$160w^3 + 240zw^2 - 300w^{*2} - \frac{150w}{z^4}w^{(4)} - 280zw^m = 0.$$

According to Theorem 2 nonlinear equation (17) has a solution (16) which can be easily verified by substitution.

Theorem 3. Let $y$ be general solution of the third order linear differential equation (8), (9). Then this solution is a three parameter family of solutions of the fourth order linear differential equation (3), (4).

Proof. The proof is computational.

Theorem 4. Let $y$ be a solution of equation (3) and $y_1$ be a solution of the second order linear differential equation of the form

$$y''(z) + r_1(z)y'(z) + s_1(z)y(z) = 0. \quad (18)$$

If the function $w(z) = (S\xi)(z)$ with $\xi = y / y_1$ solves a nonlinear differential equation (6), then we have condition (7) for (6) and two additional conditions on the coefficients of the linear equation (10)

$$r_1'' = -pr_1 - pr_1' - ps_1 - qr_1 + r + 3r_1r_1' + 2r_1s_1 - r_1^2 - 2s_1, \quad \text{and} \quad s_1'' = pr_1s_1 - ps_1 + 2s_1r_1' + r_1s_1' - r_1^2s_1 + s + s_1^2. \quad (19)$$
Proof. We substitute \( w(z) = (S\xi)(z) \) into equation (6), (7) with unknown coefficients and then replace \( \xi \) by the ratio of \( y \) and \( y_1 \). Replacing the fourth and higher order derivatives of \( y \) and the second and higher order derivatives of \( y_1 \) by using the linear equations, we collect the coefficients of \( y, y_1 \) and their derivatives up to order 3 and order 1 respectively. In the result we obtain a system of equations on the coefficients of linear and nonlinear equations, from which we get the desired result.

Example 2. Let

\[
p(z) = z^{-4}, \quad s(z) = -\frac{16b^4 - 80b^3 + 110b^2 - 25b}{16z^4}.
\]

We substitute functions (20) into equations (3), (4). Solving the resulting equations, we find

\[
q(z) = -\frac{9}{8z^2}, \quad r(z) = \frac{21}{16z^4}, \quad y(z) = z^b,
\]

where \( b \) is a constant. Let

\[
r_1(z) = \frac{1}{2z}
\]

We substitute the relations (20)-(22) into (19). After simplifications we obtain the system

\[
s_1''(z) = \frac{b(-16b^3 + 80b^2 - 110b + 25) - 8z^4 s_1''(z)}{16z^4} + \frac{3s_1(z)}{8z^2} + s_1'(z), \quad 8z^2 s_1'(z) - \frac{3}{z} = 0.
\]

We find the following function from the second equation of system (23) by integrating:

\[
s_1(z) = C_1 - \frac{3}{16z^2},
\]

where \( C_1 \) is an arbitrary constant. We substitute the function \( s_1(z) \) from (24) into the first equation of system (23) and find condition for parameters \( C_1 \) and \( b \):

\[
256C_1^2 z^4 - 256b^4 + 1280b^3 - 1760b^2 + 400b + 231 = 0.
\]

From this condition we find the value of \( C_1 \) and four values of \( b \):

\[
C_1 = 0, \quad b = -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{11}{4}.
\]

We substitute the relations (22), (24), (25) into equation (18) and integrate it. We write the general solution in the form

\[
y_1 = C_2 z^{3/4} + C_2 z^{-1/4},
\]

where \( C_1, C_2 \) are arbitrary constants. We choose, for example, the values of arbitrary constants equal to one. Then we obtain

\[
y_1 = z^{3/4} + z^{-1/4} \quad \text{and} \quad \xi = \frac{z^b}{z^{3/4} + z^{-1/4}}, \quad \text{where} \ b \ \text{takes one of the values} \ (25). \ \text{Let} \ b = 7/4. \ \text{Then we find the solution}
\]

\[
w = \frac{\xi^n}{\xi^2} = -\frac{6}{\xi^n (z + 2)^2}.
\]

Differential equation (6), (7) for coefficients (20), (21), \( b = 7/4 \) has the form

\[
20(8w^3 + 12ww - 15w^2)w^{(4)} - 2800ww - 280w^3w^3 + 192w^2w^2 + 8(56w^4 + 255w^2)w^3 - 1275w^2 - 560w^3w^3 + 64w^6 = 0.
\]

According to Theorem 4 equation (27) has the solution (26) that can be easily verified by the direct substitution.
Theorem 5. Let \( y \) be general solution of the second order linear differential equation (18), (19). Then this solution is a two parameter family of solutions of the fourth order linear differential equation

\[
(3), (4).
\]

Proof. The proof is computational.

The invariant property. The symmetric cube for the linear equation of second order is the fourth order linear differential equation for which solutions are the products of the solutions of the second order equation. More precisely, if \( y_1, y_2, \) and \( y_3 \) satisfy

\[
y''(z) + q_4(z)y(z) = 0,
\]

then \( y = y_1y_2y_3 \) satisfies

\[
y''' + 6q_1y'' + (11q_1^2 + 10q_2 + 4q_3)y'' + (6q_1^3 + 30q_1q_2 + 7q_4q_2 + 10q_4q_2 + q_4^2)y' + 3(6q_1^2q_2 + 3q_2^2 + 2q_4q_2 + 5q_4q_2 + q_4^2)y = 0.
\]

The second order linear equation has the invariant defined above. The fourth order linear equation is connected with the nonlinear fourth order equation. The invariant \( w = 2q_2 - q_3^2 / 2 - q_4 \) related to equation (28) gives trivial solution of the fourth order nonlinear equation (27) for the symmetric cube equation (28).

Remark 1. The general solution of equation (27) was found in the elementary functions, see [9].

Remark 2. The given method may be generalized to equation (3) without restriction (4). In [6] a nonlinear differential equation that generalizes equation (6) with (7) was found.

3. Discussion

It is interesting to obtain a discrete analogue of the main results of this paper. It is an open problem to obtain a difference operator that has similar to the Schwarzian derivative invariance properties. One more research direction is to replace linear differential equations with nonlinear equations of second and higher order and to consider the Schwarzian derivative of the ratio of 2 solutions. This might give a new insight into the theory of some nonlinear special functions.

Taking into account the obtained results for the known solutions of the fourth and the second order linear equations, we can formulate the corresponding theorems for the known solutions of the fourth order linear equation and the Riccati equation, to which the second order linear equation reduces. Here it seems appropriate to use the results of [11] and the method of V. Orlov [12-15] for the study of the Riccati equation and nonlinear differential equations of the second order.

From the point of view of programming algorithms for solving the considered problems, the opportunities of Wolfram Research technologies described in [16] are essential. They significantly complement the set of tools for creating, maintaining and distributing dynamic content when constructing and studying solutions of differential equations.

Linear ordinary differential equations of the fourth order have important applications in materials science, particularly speaking about the case of the beam deflection. We also note that this equation describes the deformations of an elastic beam [17, 18]. Therefore, the problems studied in the work can be used in modeling of building structures with the presence of beams.

GVF acknowledges the support of the Alexander von Humboldt Foundation and the support of National Science Center (Narodowe Centrum Nauki NCN) OPUS grant 2017/25/B/BST1/00931 (Poland).
References

[1] Ahlfors L V 1981 *Mobius transformations in several dimensions* (Minneapolis: Lecture notes at the University of Minnesota)
[2] Dobrovolsky V A 1974 *Essays on the development of the analytic theory of differential equations* (Kiev: Vishcha Shkola) (in Russian)
[3] Lukashevich N A 1999 *Differ. Equ. 35* 1384
[4] Lukashevich N A and Martynov I P 1998 Proc. Int. Conf. Differential Equations and Their Applications (Grodno: Grodno State University) p 78 (in Russian)
[5] Lukashevich N A and Chichurin A V 1999 *Differential equations of the first order* (Minsk: BSU) (in Russian)
[6] Chichurin A V 2003 *Chazy equation and linear equations of the Fuchs class* (Moscow: RUDN) (in Russian)
[7] Chichurin A and Stepaniuk G 2014 *Studia i Materialy EWSIE* 8 17
[8] Chichurin A V and Stepaniuk G P 2014 *Bulletin of Taras Shevchenko National University of Kyiv Series: Physics and Mathematics* 31 29
[9] Chichurin A 2014 *Studia i Materialy EWSIE* 7 39
[10] Chichurin A 2016 *Recent Developments in Mathematics and Informatics, Contemporary Mathematics and Computer Science* (Lublin: KUL) I 19
[11] Filipuk G and Chichurin A 2019 *Advances in Mechanics and Mathematics* (Springer) 41, 193
[12] Orlov V N 2015 *Approximate method for solving scalar and matrix Riccati differential equations* (Cheboksary: I.Yakovlev Chuvash State Pedagological University) (in Russian)
[13] Orlov V and Kovalchuk O 2018 *IOP Conf. Series: Materials Science and Engineering* (FORM IOP Publishing) 365 042045
[14] Orlov V, Kovalchuk O, Linnik E and Linnik I 2018 Vestn. Mosk. Gos. Tekh. Univ. im. N.E. Baumana, Estestv. Nauki 4 24
[15] Orlov V and Kovalchuk O 2018 *IOP Conf. Series: Materials Science and Engineering* (FORM IOP Publishing) 456 012122
[16] Taranchuk V B and Zhuravkov M A 2016 *Vestnik BSU. Ser. 1, Fiz. Mat. Inform.* 3 97
[17] Bonanno G, Di Bella B and O’Regan D 2011 *Computers & Mathematics with Applications* 62 (4) 1862
[18] Zamorska I 2014 *Journal of Applied Mathematics and Computational Mechanics* 13(4) 157
[19] Saker S H, Agarwal R P and O’Regan D 2013 *Journal of Inequalities and Applications* 278