Relating Scattering Amplitudes in Bosonic and Topological String Theories

ROGER BROOKS

Center for Theoretical Physics,
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

ABSTRACT

A formal relationship between scattering amplitudes in critical bosonic string theory and correlation functions of operators in topological string theory is found.

* This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.
At its essence, string theory is a locally reparametrization invariant\(^1\) theory in 1 + 1 dimensions. One of its interesting features is that correlation functions on the world-sheet yield scattering amplitudes for states in the space-time theory. In this vein, we might ask if the bosonic string is the unique world-sheet theory which computes space-time scattering amplitudes. That is, might there be another (and perhaps technically simpler) reparametrization invariant theory whose correlation functions might be mathematically interpreted as the scattering amplitudes for states we identify as belonging to a string theory.

One of the crucial features of the bosonic string is a direct consequence of its world-sheet reparametrization invariance. As the metric only has three components, the three local parameters of the reparametrization symmetry makes the metric a pure gauge degree of freedom. After local gauge fixing, the moduli of the Riemann surface remain. It so happens that this is all a built-in feature of topological quantum field theories coupled to two-dimensional topological gravity.

Topological gravity theories can be constructed as a gauge fixing of the local symmetry \(\delta g_{ab} = \hat{\Psi}_{ab}\) on the metric \(g_{ab}\). This is a symmetry of the Einstein–Hilbert action in two-dimensions as the latter happens to be the density for the Euler characteristic. Thus the metric is a pure gauge degree of freedom modulo boundary terms. However, for gauge fixing conditions with zero-mode solutions, interesting interpretations can be made of the resulting construction.

Of principal importance in string theory are the maps from the Riemann surface to the target or space-time manifold; the coordinates of the string. The scattering amplitudes for the string states are given by correlation functions of two dimensional vertex operators built out of these space-time coordinates. In particular, the vertex operators are written as integrals over the Riemann surface of operators with total conformal dimension equal to two. Thus, the positions at which the states puncture the Riemann surface are integrated out in the scattering amplitudes. Formally, the simplest operator is the one which creates tachyons. It is written as \(V_T[k, x, g] = \int_\Sigma e^{ik \cdot x}\) where \(k_\mu\) is the momentum of the tachyon and \(g_{z \bar{z}}\) is the fiducial metric on the Riemann surface, \(\Sigma\). Should the scattering amplitudes

\(^1\) In this paper we will only be working with the critical bosonic string theory.
for the bosonic string be computable in another theory, the identification of the vertex operators (in that theory) should be unambiguous.

As mentioned earlier, topological theories are independent of the local degrees of freedom of the metric. As in string theory we can introduce fields which act as maps from the Riemann surface to the target manifold, $M_{TS}$. Unlike the bosonic string, this coupled theory is manifestly constructed to be topological with respect to the Riemann surface. That is, the only physical states in the theory correspond to operators whose correlation functions yield cohomological information about the moduli space of the Riemann surface and/or the moduli space of particular maps (holomorphic, for example) from $\Sigma$ to $M_{TS}$. The example of such a theory which looks most like a string theory is known in the literature as the topological string theory [1,2]. It is obtained by either “twisting” a locally $N = 2$ supersymmetric string theory or via a BRST gauge fixing of local topological symmetries on the coordinates $u^A$ ($\delta u^A = \hat{\chi}^A$), which map $\Sigma$ to $M_{TS}$, and the metric on the Riemann surface ($\delta g_{ab} = \hat{\Psi}_{ab}$). In the former case, $M_{TS}$, must be Kähler. However, in the BRST approach this condition may be relaxed and $M_{TS}$ may be taken to be an arbitrary almost complex manifold. In this paper, we will take $M_{TS}$ to be flat with real dimension $D$. We will choose a value for $D$ later\textsuperscript{\dag}. The “twisting” amounts to a declaration that the fermionic fields $\bar{\psi}$ and $\psi$ of the $N = 2$ superstring are now dimension 2 and 0 fermionic fields respectively: $(\bar{\psi}, \psi) \rightarrow (\rho, \xi)$. Similarly, the Grassmann even superghosts $(\beta, \gamma)$ will be of dimension $(2,-1)$ (we will not change their names here).

As it is a topological theory, the correlation functions of observables (members of the super (BRST) charge cohomology) in this theory are independent of local world-sheet data. Thus it is suggestive that there might be some relation between the scattering amplitudes in critical bosonic string theory and the correlation functions of topological string theory. However, we are quickly sobered by the fact that the states of the bosonic string are not members of a de Rham or Dolbeault cohomology in space-time. So it would seem that the best we can hope for is that the correlation functions of some more general operators are mathematically the same as scattering amplitudes in the bosonic string. Then, exploiting the large number of symmetries of the topological string, we might be able to relate these

\textsuperscript{\dag} There is no anomaly in the topological string with flat target space.
correlation functions to topological invariants via Ward Identities. The first step of this algorithm will be the topic of this paper. We will concentrate on tachyon scattering amplitudes in the critical bosonic string as the arena for the development of our program. We will then see how straightforward it is to generalize our result to the scatterings of other states. Our work will be performed on the sphere. However, we expect that our results may be generalized to higher genus surfaces.

The path integral expression for the scattering amplitude of \( N \) tachyons is

\[
\mathcal{A}(k) = \left\langle \prod_{I=1}^{N} V_T[k_I, x, g]\right\rangle_{BS} = \int [dg] [dx] \prod_{I=1}^{N} \int d^2 z_I \sqrt{g(z_I)} e^{i k_I \cdot x(z_I)} e^{-S_{BS}}.
\] (1)

Here, \( S_{BS} \) is the action for the critical bosonic string: \( S_{BS} = \frac{1}{2} \int d^2z \sqrt{g} g^{ab} \partial_a x \cdot \partial_b x \). As mentioned earlier, we have enough gauge freedom to locally fix the metric completely. It is convenient to work on the complex plane. Our amplitude then reads

\[
\mathcal{A}(k) = \int [dx] \prod_{I=1}^{N} \int d^2 z_I e^{i k_I \cdot x(z_I)} e^{-S_{BS}} ,
\] (2)

with \( S_{BS}[x] = \frac{1}{2} \int d^2z \partial x \cdot \bar{\partial} x \). It is customary to re-write this expression as

\[
\mathcal{A}(k) = \prod_{I=1}^{N} \int d^2 z_I \int [dx] e^{-S'_{BS}[k, x]} ,
\]

\[
S'_{BS} = \frac{1}{2} \int d^2z \partial x \cdot \bar{\partial} x - \frac{1}{2} \int d^2 z \sum_{I=1}^{N} k_I \cdot x(z) \delta^{(2)}(z - z_I) .
\] (3)

We seek an expression which is formally the same as \( \mathcal{A}(k) \) in the topological string theory. We will not fully describe the manner by which the action for the topological string is derived as this is adequately discussed in the literature [1,2]. Suffice it to say that at an intermediate stage one arrives at the following (super) scale invariant action, in the super-conformal gauge.

\[
S_{TS} = \int_{\Sigma} \left( \frac{1}{2} \partial u \cdot \bar{\partial} u^* + \rho \cdot \bar{\partial} \xi + b \bar{\partial} c + \beta \bar{\partial} \gamma + \text{c.c} \right) .
\] (4)

This is the twisted version of the \( N = 2 \) superstring but in arbitrary space–time dimensions. The field content is as follows. The \( D \) sets of fields \( \{ u, \xi, \rho \} \) form topological (super)
multiplets; so do the fields $(b, \beta)$ and $(c, \gamma)$. The dimensions of the bosonic fields are $[u] = 0$, $[\beta] = 2$, $[\gamma] = -1$ and those of the fermionic fields are $[\xi] = 0$, $[\rho] = 1$, $[b] = 2$ and $[c] = -1$.

Let us now focus our attention on the scale invariance in our action. With $\varphi \equiv \ln g_{z\bar{z}}$, the scale transformation corresponds to an arbitrary local shift of $\varphi$. As this is a purely algebraic transformation (no derivatives involved), $\varphi$ may be chosen to be of whatever form is convenient to the problem at hand. In string theory it is customary to simply set $\varphi = 0$. The counterpart of this transformation in the topological string theory reads $\delta \varphi = \hat{\psi}$. In the $N = 2$ supersymmetry language $\hat{\psi}$ becomes the superpartner of $\varphi$. In the BRST language, $\hat{\psi}$ will become a scalar but fermionic ghost field.

Following the BRST approach, impose that the curvature of the super-conformal gauge metric is equal to a fixed curvature, $\hat{R}$. Our only condition on $\hat{R}$ is that it be of the topological class of the Riemann surface the theory is defined on. So we take $\frac{1}{2\pi} \int_\Sigma \hat{R} = \chi$ to be the Euler characteristic of $\Sigma$. In order to relate $\mathcal{A}(k)$ in equation (3) to a correlation function in the topological string, we will find it useful to choose the gauge [3]

$$\partial \bar{\partial} \ln g_{z\bar{z}} = -g_{z\bar{z}} \hat{R} \equiv \sum_{I=1}^{N} \kappa_I \delta^{(2)}(z - z_I) .$$

(5)

It then follows that

$$\sum_{I=1}^{N} \kappa_I = 2\pi \chi .$$

(6)

Then the partition function for the gauge fixing of the scale invariance reads

$$Z_{TW} = \int [dg_{z\bar{z}}][d\lambda][d\eta][d\psi] e^{-S_{TW}} ,$$

$$S_{TW} = \int d^2 z \lambda [\partial \bar{\partial} \varphi - \sum_{I=1}^{N} \kappa_I \delta^{(2)}(z - z_I)] + \int d^2 z \eta \partial \bar{\partial} \psi .$$

(7)

In the action $S_{TW}$, the first term is the gauge fixing term and the second is the corresponding ghost action. Note that it is quadratic in derivatives. The field $\lambda$ is a Lagrange multiplier (BRST auxiliary field) which imposes the constraint (5) while $\eta$ and $\psi$ are dimensionless fermionic fields. These fields form the following topological multiplets: $(\varphi, \psi)$
and \((\eta, \lambda)\). To compute this partition function we must first remove \(N\) \(\varphi\) zero-modes which we label by \(\varphi^{(I)}\). Their equations are

\[
\bar{\partial} \partial \varphi^{(I)} = \kappa_I \delta^{(2)}(z - z_I) .
\]  

(8)

We know this as the equation for the scalar Green’s function with solution

\[
\varphi^{(I)} = \tilde{\kappa}_I \ln |z - z_I| \quad \Rightarrow \quad g^{(I)}_{zz} = |z - z_I| \tilde{\kappa}_I ,
\]  

(9)

where \(\tilde{\kappa}_I = \frac{\kappa_I}{2\pi}\). Having removed these zero-modes, the integrals over the Lagrange multiplier, the \(\varphi\) non-zero-mode and the ghost fields may be done. The resulting determinants cancel each other. As the \(\varphi\) zero-modes are in one-to-one correspondence with the locations of the singularities we then replace \(\int [d g_{zz}]\) by \(\prod_{I=1}^{N} \int d^2 z_I\) up to a Jacobian factor. Altogether, the full partition function of the topological string now reads

\[
Z_{TS} = \prod_{I=1}^{N} \int d^2 z_I J(\kappa_I, z_I) \int [du][dp][d\xi][db][dc][d\beta][d\gamma] e^{-S_{TS}} ,
\]  

(11)

where \(J\) is the Jacobian for \(g^{(I)}_{zz} \rightarrow z_I\). It is clear that the integrals over the locations of the singularities are needed as these points are, after all, gauge artifacts.

Now, when we bosonize the fields which appear in first order form in \(S_{TS}\) we find

\[
S_{TS} = \int_{\Sigma} \left( \frac{1}{2} \partial u \cdot \bar{\partial} u^* + \frac{1}{2} \partial \phi \cdot \bar{\partial} \phi + \frac{i}{8} \sum_{I=1}^{N} \kappa_I \delta^{(2)}(z - z_I) q \cdot \phi(z) \right) .
\]  

(12)

Here we have replaced the set of \((\rho, \xi)\) fields by a set of \(D\) real bosons \(\phi^i\), the \((b, c)\) system by the single real boson \(\phi_{D+1}\) and \((\beta, \gamma)\) by \(\phi_{D+2}\). Due to the fact that the \((\beta, \gamma)\) system is bosonic, the metric on the space \((\phi^i, \phi_{D+1}, \phi_{D+2})\) is \(	ext{diag}(+, \ldots, +, -)\) with the minus corresponding to the \(\phi_{D+2}\) direction. Whereas the bosonization of the fermionic partners of the target manifold’s coordinates do not change the Euclidean signature, the superghosts give the new target manifold a Minkowskian signature with \(\phi_{D+2}\) being the time direction. Note that the dimension of this new space is \(D + 2\). The background charges are \((q_i, q_{D+1}, q_{D+2}) = (-1, -3, 3)\). The condition (5) has been imposed.

Call the bosonic fields \(\Phi \equiv \{u, u^*, \phi\}\) collectively. The \(\Phi\)’s are real bosonic fields which we now interpret as the coordinates of a Minkowski target manifold. The dimension of this
manifold is $3D + 2$. In order to relate our correlation functions we must set this number equal to the space-time dimension of the bosonic string: $3D + 2 = 26$ or $D = 8$.

Consider computing the following correlation function in the topological string

$$\langle \prod_{I=1}^{N} J^{-1}(\kappa_I, z_I) e^{ip_I \cdot \Phi(z_I)} \rangle_{TS} = \prod_{I=1}^{N} \int d^2z_I \int [d\Phi] e^{ip_I \cdot \Phi(z_I)} e^{-S_{TS}}$$

$$= \prod_{I=1}^{N} \int d^2z_I \int [d\Phi] e^{-S'_{TS}[p, \Phi]} , (13)$$

$$S'_{TS} = \frac{1}{2} \int d^2z \partial \Phi \cdot \partial \Phi - i \int d^2z \sum_{I=1}^{N} p'_I \cdot \Phi(z) \delta^{(2)}(z - z_I) ,$$

where $p'_I = p_I - \frac{1}{8} \kappa_I q_\mu$ and $q_\mu$ is zero in the $u$ directions, $-1$ in the $\phi^i$ (or $(\rho, \xi)$) direction, $-3$ in the $\phi_{D+1}$ (or $bc$) direction and $3$ in the $\phi_{D+2}$ (or $(\beta, \gamma)$) direction. We can now formally equate

$$\langle \prod_{I=1}^{N} \int d^2z_I e^{ik \cdot x(z_I)} \rangle_{BS} = \langle \prod_{I=1}^{N} J^{-1}(\kappa_I, z_I) e^{ip_I \cdot \Phi(z_I)} \rangle_{TS} , (14)$$

by declaring $p_I = k_I + \frac{1}{8} \kappa_I q_\mu$. This result may be generalized to any bosonic string vertex operator of the form

$$V[k, \zeta, x] = \int d^2z P_\zeta(\partial x, \partial^2 x, \ldots) e^{ikx} , (15)$$

where $P_\zeta$ is a polynomial in powers of derivatives of $x$ and $\zeta$ is a polarization tensor. Then we obtain

$$\langle \prod_{I=1}^{N} \int d^2z_I P_\zeta(\partial x, \partial^2 x, \ldots) e^{ik \cdot x(z_I)} \rangle_{BS} = \langle \prod_{I=1}^{N} J^{-1}(\kappa_I, z_I) P_\zeta(\partial \Phi, \partial^2 \Phi, \ldots) e^{ip_I \cdot \Phi(z_I)} \rangle_{TS} , (16)$$

with $p_I = k_I + \frac{1}{8} \kappa_I q_\mu$.

Now that we have stated the formal relationship between the two theories, it is left to the future to carry out the computations of the right hand side of eqn. (16) in the topological string theory. The first step would be to exploit Ward Identities or a Hodge decomposition of the inserted operators in the topological string to relate the correlation functions to cohomology classes.
We summarize by painting the following heuristic picture of what we have done. We have given the Riemann surface of the topological string a curvature which vanishes everywhere except for $N$ points, $z_I$, where it is singular. Then bosonizing the fermions we have found that the background charge term mimicked the source term in the bosonic string's scattering amplitudes. The remaining hurdle of taking the target manifold with Euclidean signature and making it Minkowskian was scaled by bosonizing the topological gravity ghost systems. The $\beta$-$\gamma$ system was bosonic and so in bosonized form, its propagator appeared with a sign opposite to the rest of the bosons. Finally, operators analogous to the vertex operators of the bosonic string were introduced. An operator was inserted at each of these points. By construction, the manifold had a curvature singularity wherever there is an operator insertion. As these points were gauge artifacts, we have found that we must integrate over them. This was obtained by exchanging the integral over metric zero-modes for these integrals as the zero-modes were in one to one correspondence with the $z_I$ points. The price of this is a Jacobian factor which may be absorbed into the definition of the operators in the topological string.
REFERENCES

1. E. Witten, *Commun. Math. Phys.* 118 (1988) 411.

2. D. Montano and J. Sonnenschein, *Nucl. Phys.* B313 (1989) 258.

3. E. Verlinde and H. Verlinde, *Nucl. Phys.* B348 (1991) 457.