Splitting Localization and Prediction Numbers

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Abstract

In this paper the work done by Newelski and Roslanowski in [7] is revisited to solve a question done by Blass about one of the possible evasion and prediction numbers (see [2]). This led to define a variation of the $k$-localization property (the $(k+1)\omega$-localization property) and the use of a forcing notion with accelerating trees.

1 Introduction

In 1993 Newelski and Roslanowski defined the $k$-localization number, $L_k$ (see [2]), as the minimal cardinality of a family $T$ of $k$-trees such that every element $(k+1)\omega$ is a branch of a tree in $T$.

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2The $k$-localization property says “all the reals in $\omega^\omega$ of the generic extension are a branch of a $k$-tree from the ground model.”

*Work done while being supported by CONACYT scholarship for Mexican student studying abroad.
Nevertheless, the $k$-localization property is not the minimum necessary to have $\mathcal{L}_k^V = \mathcal{L}_k^{V[G]}$. The minimum that we need to have that would be the property “all the reals in $(k + 1)^\omega$ of the generic extension are a branch of a $k$-tree from the ground model” (we call this the $(k + 1)^\omega$-localization property). Does this mean that there is a cardinal characteristic that is closer to the $k$-localization property than $\mathcal{L}_k$?

There is one. In his chapter of the Handbook of Set Theory [2], Andreas Blass talks about cardinal characteristics related to the concepts of evasion and prediction. At the end of that section, he introduces 36 variations of these cardinals and left as an open question to pin down 4 of them whose identity didn’t appear to be one of the known cardinal characteristic. It turns out that the same proof of Newelski and Roslanowski shows that one of these variations, specifically the prediction number for global adaptive $k$ predictors, is not one of the known cardinal characteristics and, actually, gives countable many cardinal characteristics which, consistently, can take different values (see Theorem 2.4).

This triggers the following question: are the variation of prediction and the $k$-localization number equal? This paper shows that they are not. It is consistent to have all the prediction numbers mention above at value $\aleph_2 = \mathfrak{c}$ and all localization numbers at value $\aleph_1$ (see Theorem 4.1).

To do this I used a preservation theorem for the $3^\omega$-localization property and a forcing that Noah Schweber and I called accelerating tree forcing, we created it for a computability context in a coauthor paper still in preparation.

It has been pointed out to us that the accelerating tree forcing could be related to bushy tree forcing (as done in [6]) or other fast-growing tree forcing (as done in [3]). Although we got inspiration from them, the fact that we allowed long stretches with no split makes us believe that this forcing is conceptually of a different kind.

It is also important to remark that the $(k + 1)^\omega$-localization property could be preserved under countable support iteration and product, as the $k$-localization property (see [9]). Notice that these two properties are in the same line as the Sacks property. It is unknown to me if there is a bigger theory or theorem that handle all of them at once. This, I believe, is an interesting topic.

About the structure of the paper, it has a first section with definition and background. Then, the bulk of the work is done in Section 3.

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3 It is possible to find an early version of it at http://www.math.wisc.edu/ongay/publications.html
preservation lemmas are shown. The final section has the main theorem (Theorem 4.1) with some conclusions and open problems.

Finally, I want to thank Noah Schweber for his support and for convincing me about publishing this work; Arnold Miller, for showing me a new way to order my thoughts, and Kenneth Kunen for all the advice and guidance with this project and others.

2 Definitions and background

These first definitions will be useful during the paper:

Definition 2.1. 1. A $k$-branching tree is a tree such that every node has either 1 successor or $k$ of them.

2. A $k$-tree is a tree such that every node has at least 1 successor and no more than $k$.

Now, the following definition is due to [7] (they express it as the covering number of an ideal):

Definition 2.2. The $k$-localization number, $L_k$, is the smallest cardinal such that $(k + 1)^\omega$ can be covered by $k$-branching trees.

Notice that the definition is not trivial for $k \geq 2$. Furthermore, Newelski and Roslanowski showed in [7] that, for $k \geq 2$, $L_k \geq \max\{\text{cov}(M), \text{cov}(N)\}$, that $L_{k+1} \leq L_k$ and that it is consistent that $L_{k+1} < L_k$.

On the other hand, in Blass’s chapter of the handbook of Set Theory [2] he defines:

Definition 2.3. 1. A $k$ globally adaptive predictor is a sequence of functions $\pi = \langle \pi_n : n \in \omega \rangle$ with $\pi_n : \omega^n \rightarrow [\omega]^k$. We say that a function $f \in \omega^\omega$ is predicted by $\pi$ if there is $m \in \omega$ such that for all $n > m$, $f(n) \in \pi_n(f|n)$.

2. The $k$ globally prediction number, $\sigma_k^g$, is the minimal cardinality of a set of $k$ globally adaptive predictors that predict all functions in $\omega^\omega$.

3. The $k$ globally evasion number, $\epsilon_k^g$, is the minimal cardinality of a set of functions in $\omega^\omega$ such that the whole set is not predicted by a single $k$ globally adaptive predictor.

It is important to make some remarks about the last definition:
• The ‘adaptive’ part refers to the fact that $\pi_n$ is not constant. Non-adaptive objects are closer to slaloms (or traces).

• The ‘globally’ part of the definition refers to the fact that we have $\pi_n$ for all $n \in \omega$. It is possible to define predictors using $\pi_n$ for $n \in D \subseteq \omega$.

• Blass do not give a notation for this number, so the notation $o^g_k$ and $e^g_k$ is introduce here.

• These definitions are not trivial for $k \geq 2$.

The numbers $o^g_k$ and $e^g_k$ are duals between them and, by the work done in [2], we know that $m_{\sigma,\text{centered}} \leq c^g_k \leq \text{add}(\mathcal{N})$. So, by duality, we know that $\text{cof}(\mathcal{N}) \leq o^g_k \leq c$. Also, from the definition, we have that $o^g_{k+1} \leq o^g_k$.

Reading the definition more carefully we can notice that all the functions that are predicted by a $k$-globally adaptive predictor are covered by $\aleph_0$ $k$-branching trees, so $o^g_k$ is also the minimum cardinal of a set of $k$-branching trees (or $k$-trees) that cover $\omega^\omega$.

Furthermore, in [7], we have the following result:

**Theorem 2.4.** Given $k \geq 2$, it is consistent to have $\text{ZFC+cof}(\mathcal{N}) = o^g_{k+1} < o^g_k = c$.

This theorem is a corollary of the proof of:

**Theorem 2.5 (Newelski, Roslanowski [7]).** Given $k \geq 2$, it is consistent to have $\text{ZFC+cof}(\mathcal{N}) = L_{k+1} < L_k = c$.

Specifically, it comes from two facts: first, that the forcings that were used have the $k$-localization property. This is that “every real in $\omega^\omega$ is a branch of a $k$-tree from the ground model”, this keeps $L_{k+1}$ and $o^g_k$ at $\aleph_1$; and the forcing adds a function in $(k+1)^\omega$ that is not the branch of any $k$-tree from the ground model. Notice that this function is also a function in $\omega^\omega$ that is not the branch of any $k$-tree from the ground model. Once you take a countable support product, this makes $L_k$ and $o^g_k$ of size $c$.

The relation between these two cardinal characteristics grows closers once we realize, from the tree definition of $L_k$ and $o^g_k$, that $L_k \leq o^g_k$. So, a natural question arises of whether $L_k = o^g_k$. The propose of this paper is to answer the question in a negative way.
3 Combinatorial and preservation Lemmas

In this setting it is better to understand some of the processes as combinatorial principles instead of parts of a forcing argument. Because of that, the following lemmas comes in pairs: one is a combinatorial statement and the following one is the forcing result.

Lemma 3.1. Given \( \{ f_i : i \in I \} \subseteq 3^\omega \) with \( |I| = 3^n \) you can find \( S \subseteq I \) with \( |S| = n \) such that \( \{ f_i : i \in S \} \) is a 2-tree.

Proof. We will do the proof by induction.

For \( n = 0 \) and \( n = 1 \) it is trivially true.

Now, assume that it is true for \( n \), we will prove it for \( n + 1 \).

Given \( \{ f_i : i \in I \} \subseteq 3^\omega \) with \( |I| = 3^{n+1} \) if all of them are the same function then take the first \( n + 1 \) of them, they make trivially a 2-tree. On the other hand, if there are two of them that are different, find the first natural number \( m \) such that two of them differ. Notice that, using a pigeon hole principle, there is a value \( k \in 3 \) such that there is \( J \subseteq I \), with \( |J| \geq 3^n \) such that for all \( i \in J \) we have \( f_i(m) = k \).

Now, take \( i_0 \in I \) such that \( f_{i_0}(m) \neq k \) and let \( S' \subseteq J \) be the index set of size \( n \) given after using the induction hypothesis over \( J \). Notice that \( \{ f_{i_0} \} \cup \{ f_i \}_{i \in S'} \) forms a 2-tree and that \( S = S' \cup \{ i_0 \} \) has size \( n + 1 \).

Lemma 3.2. There is a forcing notion that adds a function from \( \omega \) to \( \omega \) that is not predicted by any \( k \)-global adaptive predictor but such that all reals in \( 3^\omega \) are a branch of a 2-tree in the ground model.

Proof. 

Definition 3.3. We say that \( T \subseteq \Pi_{n\geq 1}^\omega \) is an accelerating tree if and only if it is a subtree of \( \Pi_{n\geq 1}^\omega \), if every node has an extension that splits and given \( \sigma \in T \) such that there are \( k_i \in \omega, i < n \), such that \( \sigma|k_i \) is a splitting node (i.e., \( \sigma \) has \( n \) splits before it) then \( \sigma \) has either 1 successor or at least \( n + 2 \).

Let \( P \) be the forcing notion whose nodes are of the form \( \langle p, T \rangle \) with \( p \in \Pi_{n\geq 1}^\omega \) and \( T \) an accelerating subtree of \( \Pi_{n\geq 1}^\omega \) extending \( p \). We say that \( \langle p', T' \rangle \leq \langle p, T \rangle \) if and only if \( p \preceq p' \), \( T' \subseteq T \) and \( p' \in T \).

\[\text{We decide to define the acceleration tree forcing using pairs to create a stronger resemblance to the effective analogue of accelerating trees of } \omega^\omega \text{ (paper in preparation with Noah Schweber). Furthermore, this will allow us to easily define } (T)^{0} \text{ in Lemma 3.5.}\]

\[\text{We denote this forcing as } P_n \text{ and denote the forcing using pairs as } P_{pa}.\]
Let

\[ T_q = \{ t \in T : t \subseteq q \lor q \subseteq t \} \]

Notice that given any \( k \)-tree \( U \subseteq \omega^\omega \) and a condition \( \langle p, T \rangle \), there is \( q \in T \) that is not a node in \( U \) (for example, go to a split with \( k + 1 \) nodes, one of them is not in \( U \)). Furthermore, if we take the condition \( \langle q, T_q \rangle \), none of the branches of \( T_q \) are branches of \( U \). This shows that forcing with accelerating tree forcings adds a function from \( \omega \) to \( \omega \) that is not predicted by any \( k \)-global adaptive predictor.

Now, we will not prove with full detail that all reals in \( 3^\omega \) are a branch of a 2-tree in the ground model. This part of the theorem is a corollary of Lemma 3.5 letting \( \kappa = 1 \). Also, the forcing is the set theoretical version of the forcing used in a paper to appear with Noah Schweber\(^5\). From that proof, translating from computability theory to set theory, we have the desired result.

Now, for a sketch of the proof. The idea is that given a condition \( p \in P \) and a \( P \)-name \( \tau \) such that \( p \vDash \tau \in 3^\omega \) then you can define, in \( V \), a 2-tree, \( A \), and a condition \( q \leq p \) such that \( q \vDash \tau \in A \). To do this, you can prune the tree of \( p \), call it \( T \), in such a way that at the nodes \( \sigma \in T \), which are the \( n \)-th split of \( T \), the condition \( \langle \sigma, T_\sigma \rangle \) force completely the values of \( \tau \restrictedto n \).

To keep everything inside a 2-tree you go to a node with \( 3^n \) successors. For each one of them, say \( \rho \), we the conditions \( \langle \rho, T_\rho \rangle \) for the values of \( \tau \restrictedto M \) to be different. Now, running the above combinatorial Lemma (3.1), select \( n \) that are in a 2-tree (definable in \( V \) using the definability lemma of forcing).

Lemma 3.4. Given \( \{ f^j_i : i \in I, j \in k \} \subseteq 3^\omega \) with \( k \in \omega \), \( |I| = N(n, k) \) a big enough number and \( m \in \omega \) that makes \( \{ f^j_i \ restrictedto m : i \in I, j \in k \} \) a 2-tree such that if \( f^j_i \ restrictedto m = f^t_i \ restrictedto m \) with \( t \neq j \) we have that \( f^j_i = f^t_i \) then you can find \( S \subseteq I \) with \( |S| = n \) such that \( \{ f^j_i : i \in S, j \in k \} \) is a 2-tree.

Proof. We will prove this by induction over \( k \).

At \( k = 1 \), we need \( N(n, 1) \geq 3^n \) so that we can use Lemma 3.1 to be done.

We will do case \( k = 2 \) to explain better how our induction will work. At \( k = 2 \), we need \( N(n, 2) \geq 3^{3^n} \), with this we can use Lemma 3.1 over \( \{ f^j_i : i \in I \} \) to get \( J \subseteq I \) such that \( |J| = 3^n \) and \( \{ f^j_i : i \in J \} \) is a 2-tree.

\(^5\)You can find an early version of it at http://www.math.wisc.edu/ ongay/publications.html.
Now, we can use our induction hypothesis over \( \{f^j_i : i \in J, j \in 1\} \) to get \( S \) of size \( n \) such that \( \{f^j_i : i \in S, j \in 1\} \) is a 2-tree.

Notice that \( \{f^1_i : i \in S\} \) is also a 2-tree but, under normal circumstances, there is no good reason for

\[
\{f^j_i : i \in S, j \in 2\} = \{f^1_i : i \in S\} \cup \{f^j_i : i \in S, j \in 1\}
\]

to be a 2-tree. Nevertheless, since there is \( m \) such that \( \{f^1_i \upharpoonright m : i \in I, j \in 2\} \) is a 2-tree and such that if \( f^0_i \upharpoonright m = f^j_i \upharpoonright m \) we have that \( f^0_i = f^1_i \) then we have that \( \{f^1_i : i \in S\} \cup \{f^0_i : i \in S\} \) is also a 2-tree.

Now, assuming we have the case for \( k \) we will prove it for \( k+1 \). We need \( N(n, k+1) \geq 3^{N(n,k)} \), with this we can use Lemma 3.1 over \( \{f^k_i : i \in I\} \) to get \( J \subseteq I \) such that \( |J| = N(n, k) \) and \( \{f^j_i : i \in J, t \in k\} \) is a 2-tree. Now, we can use our induction hypothesis over \( \{f^j_i : i \in J, t \in k\} \) to get \( S \) of size \( n \) such that \( \{f^j_i : i \in S, j \in 1\} \) is a 2-tree.

Notice that \( \{f^k_i : i \in S\} \) is also a 2-tree and using the properties of \( m \) as in case \( k = 2 \) we have that

\[
\{f^j_i : i \in S, j \in k+1\} = \{f^k_i : i \in S\} \cup \{f^j_i : i \in S, j \in k\}
\]

is a 2-tree.

\[\square\]

It is important to remark that in these combinatorial lemmas it is never used that the domain of the functions is \( \omega \), so these lemmas are also true for \( 3^n \).

**Lemma 3.5.** Having all branches of \( 3^\omega \) covered by ground model 2-trees is preserved under countable product of the accelerating tree forcing.

Newelski and Roslanowski, in [3], define the \( k \)-localization property as the fact that all branches of \( \omega^\omega \) are cover by a \( k \)-tree of the ground model. This property was deeply study later by Roslanowski, in [3], and by Zapletal, in [3]. They found that the \( k \)-localization property is preserved under most of the used countable support product and iteration of proper forcings.

Our forcing does not have the 2-localization property, it will have a version of that for \( 3^\omega \). Our proof will resemble the one did by Newelski and Roslanowski, nevertheless, it is possible that there are results in the lines of the other two papers.

**Proof.** First, given a tree and \( n > 0 \), we let \( (T)^n \) be the set of all nodes such that they are the successors of the \( n \)-th split. As a convention, given
\( p = \langle s, T \rangle \) a forcing condition, we have that \((T)^0 = \{s\}\). Now, given elements of the accelerating tree forcing we will define for \(n \geq 1\), \(p = \langle s, T \rangle \leq_n p' = \langle s', T' \rangle\) if and only if \(\langle s, T \rangle \leq \langle s', T' \rangle\) and \((T')^k = (T)^k\), for all \(1 \leq k \leq n\), and \(p \leq 0 p'\) if and only if \(p \leq p'\). Notice that, since these are subtrees of \(T_{n \geq 1} n\), these orders have the fusion property and satisfy Axiom A (as in [1]).

Now, we will introduce some extra notation. Fix an enumeration of \(\omega^{<\omega}\), given \(v = \langle n, i \rangle \in \{n\} \times |(T)^n|\) we will define \(p * v\) to be \(\langle s * v, T * v \rangle\) were \(s * v\) is the \(i\)-th element of \((T)^n\) (following the fix enumeration) and \(T * v\) is the subtree of \(T\) extending \(s * v\), i.e., \(T * v = T_{s * v}\) (following the notation of Lemma [3.2]).

Assume that we have a countable support product of the accelerating tree forcing of length \(\kappa\). Call the final partial order \(P_\kappa\), as notation we will express \(q \in P_\kappa\) as \(q = \langle r, T \rangle\) and \(q(\alpha) = \langle r(\alpha), T(\alpha) \rangle\).

Given \(F \in [\kappa]^{<\omega}\) and \(\eta : F \to \omega\), we define \(p \leq_{\eta, q} q\) if and only if \(p \leq q\) and for all \(\alpha \in F\) we have that \(p(\alpha) \leq_{\eta(\alpha)} q(\alpha)\). Furthermore, given \(\sigma \in \Pi_{\alpha \in F} \{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|\) and \(p, \eta \in P_\kappa\) we define \(p * \sigma\) to be \(p(\beta)\) if \(\beta \notin F\) and \(p(\beta) * \sigma(\beta)\) if \(\beta \in F\).

The orders \(F, \eta\) have the fusion property under the following conditions: given \(p_{n+1} \leq_{F_n, \eta_n} p_n\) with \(\bigcup_{n \in \omega} F_n = \bigcup_{n \in \omega} \text{supp}(p_n)\) and \(\lim_{n \to \infty} \eta_n(\alpha) = \infty\) for all \(\alpha \in \bigcup_{n \in \omega} F_n\) we have that there exist \(q \in P_\kappa\) such that \(q \leq_{F_n, \eta_n} p_n\) for all \(n \in \omega\).

In order to complete the proof, it is enough to define the following concept and show the following claim:

**Definition 3.6.** We say that the 5-tuple \(\langle q, F, \eta, m, A \rangle\) is consolidating \(f\) if and only if \(q = \langle r, T \rangle \in P_\kappa\), \(F \in [\kappa]^{<\omega}\), \(\eta : F \to \omega\), \(m \in \omega\) and \(A \subseteq 3^\omega\) is a 2-tree such that:

- \(q \Vdash "f \in 3^\omega \& f|m = m \in A"\).

- For each \(\sigma \in \Pi_{\alpha \in F} \{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|\) there is \(g \in A\) such that \(q * \sigma \Vdash "f|m = g"\) and for all \(g \in A\) there is \(\sigma \in \Pi_{\alpha \in F} \{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|\) such that \(q * \sigma \Vdash "f|m = g"\).

- Given \(\sigma_1 \neq \sigma_2 \in \Pi_{\alpha \in F} \{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|\) such that there are \(q_1 \leq q * \sigma_1, q_2 \leq q * \sigma_2\) and \(n_0, n_1 \in \omega\) with \(q_1 \Vdash "f(n_0) = n_1"\) but \(q_2 \Vdash "f(n_0) \neq n_1"\) then there is \(g_1 \in A\) such that \(q * \sigma_1 \Vdash "f|m = g_1"\) and \(q * \sigma_2 \Vdash "f|m \neq g_1"\).
Claim 3.7. Working in $V$, given $\langle q, F, \eta, m, A \rangle$ that is consolidating $f$, there is $M > m$, $A' \subset 3^M$ a 2-tree with $A \subset A'$ and $q' = \langle r', T' \rangle \leq_{F, \eta+1} q$ such that $\langle q', F, \eta + 1, M, A' \rangle$ is also consolidating $f$.

If we prove this claim, given $p \in \mathcal{P}_\kappa$ such that $p \models \text{``}f \in 3^\omega\text{''}$ we can define $q_n, F_n, \eta_n, A_n, m_n$ as follows:

1. $q_0 = p$, $A_0 = \emptyset$ and $m = 0$.
2. We write $\text{supp}(q_0) = \{\alpha_0^i : i \in \omega\}$ and let $F_0 = \{\alpha_0^0\}$.
3. We let $\eta_0(\alpha_0^0) = 0$. Clearly, $\langle q_0, F_0, \eta_0, m_0, A_0 \rangle$ is consolidating $f$.
4. We define $q_{n+1}, A_{n+1}$ and $m_{n+1}$ as the result of the claim using $q_n$, $A_n$, $F_n$, $\eta_n$ and $m_n$.
5. We write $\text{supp}(q_{n+1}) = \{\alpha_{n+1}^i : i \in \omega\}$ and let $F_{n+1} = F_n \cup \{\alpha_{n+1}^i\}$ with $\langle i_n, f_n \rangle$ following the usual enumeration of $\omega \times \omega$.
6. Finally, we let $\eta_{n+1}(\alpha) = \eta_n(\alpha) + 1$ for $\alpha \in F_n$ and $\eta_{n+1}(\alpha_{n+1}^i) = 0$.

Again, notice that $\langle q_{n+1}, F_{n+1}, \eta_{n+1}, m_{n+1}, A_{n+1} \rangle$ is consolidating $f$.

With this, we can use the fusion property with $q_{n+1} \leq_{F_n, \eta_n} q_n$ and get $q \in \mathcal{P}_\kappa$ such that $q \leq_{F_n, \eta_n} q_n$ for all $n$ so we have that $q \models \text{``}f \in [\bigcup_{n \in \omega} A_n]\text{''}$.

This shows that all the functions in $3^\omega$ in the extension are a branch of a ground model 2-tree.

Now, to show the claim notice that the properties gave to the 2-tree in the claim aligns with those in the hypothesis of Lemma 3.4. We will use this lemma in the proof. The hardest part of the proof is to find a condition $p$ such that $p \ast \sigma$ decides the value of $f$ up to a certain point for all $\sigma$. Finding such a $p$ that works for a single $\sigma$ is easy, nevertheless, once you have more is important to keep track of what has been done before. The proof of the claim will be centered principally in solving this issue.

Now, assume that $\langle q, F, \eta, m, A \rangle$ is consolidating $f$.

Fix $\beta \in F$, and let $\nu : F \to \omega$ such that $\nu(\alpha) = \eta(\alpha)$ if $\alpha \neq \beta$ and $\nu(\beta) = \eta(\beta) + 1$. To show this claim we will look for $q' \leq_{F, \nu} q$ (instead of $q' \leq_{F, \eta+1} q$). This is enough since, changing the $\beta$ we are using, we can go from $\eta$ to $\eta + 1$ using $|F|$ intermediate $\nu$ functions.

Using the notation of Lemma 3.4 we let $n = \nu(\beta) + 1$ and we let $k = |\prod_{\alpha \in F} \eta(\alpha)| \times |(T(\alpha))^\eta(\alpha)||$. We will also use $N(n, k)$ as define in that lemma.
Now, we enumerate $\Pi_{\alpha \in F}\{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|$. Let $\sigma_\ell \in \Pi_{\alpha \in F}\{\eta(\alpha)\} \times |(T(\alpha))^{\eta(\alpha)}|$. We look for a node of $q \cdot \sigma_\ell(\beta)$ that splits into $N(n, k)$ successors, call them $r_0^\ell, r_1^\ell, \ldots, r_{N(n,k)-1}^\ell$. We select this in such a way that if $\sigma_\ell(\beta) = \sigma_s(\beta)$ then $r_\ell^s = r_\ell^s$. From now on, we will write $r_\ell^s$ to mean both the node and the condition $\langle r_\ell^s, T_{r_\ell^s} \rangle$. This may appear confusing but will simplify the notation and, by context, it should be clear which one is used.

The following proof will have a lot of different notations that express how we focus on specific nodes of our trees. Let $F^s = \{\alpha : \exists \ell < s(\sigma_s(\alpha) = \sigma_\ell(\alpha))\}$. Given $s \in k$ and $\alpha \in F^s$ we let $t^{s}_\alpha$ to be the biggest number less than $s$ such that $\sigma_{t^{s}_\alpha}(\alpha) = \sigma_s(\alpha)$.

From now on, we let $q \cdot r$ to be the forcing condition such that $(q \cdot r)(\alpha) = q(\alpha)$ for $\alpha \neq \beta$ and $(q \cdot r)(\beta) = r$ (i.e., we kept everything the same but we change the $\beta$ coordinate for $r$).

First, we will find conditions that have some degree of congruence between them and differentiate $f$ for each $\sigma_s$ to a certain extend. For $\sigma_0$ we will look for $m_0 > m$ and conditions $q^0_0, \ldots, q^0_{N(n,k)-1}$ such that:

- $q^0_0 \leq (q \cdot \sigma_0) \cdot r^0_0$.
- For all $0 < i < N(n, k)$, $q^0_{i+1} \leq (q^0_i) \cdot r^0_{i+1}$.
- For all $i < N(n, k)$ there is $g \in 3^{m_0}$ such that $q^0_i \models \langle \hat{f} \mid m_0 = g \rangle$.
- The set $\left\{ g \in 3^{m_0} : \exists i < N(n, k) \left( q^0_i \models \langle \hat{f} \mid m_0 = g \rangle \right) \right\}$ is of maximal size.
- $m_0 > m$ is minimal with the above conditions.

From now on, for all $t$ such that $\sigma_0(\beta) = \sigma_t(\beta)$ we will have $r_1^t = q^0_0(\beta)$. Notice that, by construction, we have that $(q^0_0(\beta)) \cdot (q^0_0(\beta)) \leq q^0_0$.

Given $s \in k \setminus \{0\}$ and any condition $p$ define $p^s$ to be the forcing condition such that $p^s(\alpha) = (p \cdot \sigma_s)(\alpha)$ if $\alpha \notin F^s$ and $p^s(\alpha) = q^0_{N(n,k)-1}(\alpha)$, this notation let us change the $\sigma$ that we were using and keep all the information that was used before. Now we are looking for $m_s > m$ and conditions $q^s_0, \ldots, q^s_{N(n,k)-1}$ such that:

- $q^s_0 \leq ((q^0_{N(n,k)-1})^s) \cdot r^s_0$.
- For all $0 < i < N(n, k)$, $q^s_{i+1} \leq (q^s_i) \cdot r^s_{i+1}$.
- For all $i < N(n, k)$ there are $g \in 3^{m_s}$ such that $q^s_i \models \langle \hat{f} \mid m_s = g \rangle$.  

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• The set \( \{ g \in 3^{m_s} : \exists i < N(n,k) \left( q_i^s \vdash \text{"} \hat{f} | M = g \text{"} \right) \} \) is of maximal size.

• \( m_s > m \) is minimal with the above conditions.

Again, from now on, for all \( t \) such that \( \sigma_s(\beta) = \sigma_t(\beta) \) we will have \( r_i^t = q_i^s(\beta) \). Also, by construction, we have that \( (q_i^{s_{N(n,k)-1}})^s(\beta) \leq q_i^s \).

Now, repeating the above technique as many times as needed, we will find for each \( s \) such that \( m \) is minimal with the above conditions.

Given \( s \in k \) and any condition \( p \) define \( p^{s*}_t \) to be the forcing condition such that \( p^{s*}_t(\alpha) = p(\alpha) \) if \( \alpha \notin F \), \( p^{s*}_t(\alpha) = l_{i-1}^{s_0}N(n,k)-1(\alpha) \) if \( \alpha \in F^s \) and, if \( \alpha \in F \setminus F^s \), \( p^{s*}_t(\alpha) = q_i^{s_0}N(n,k)-1(\alpha) \) where \( t \) is the biggest number less than \( k \) such that \( \sigma_t(\alpha) = \sigma_s(\alpha) \). We will call this number \( s^*_t \). Again, this notation will help us change \( \sigma \) while keeping the changes done before.

Now we look \( M > \max\{m_0, ..., m_{k-1}\} \) and conditions such that

• \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq (q_i^{s_{N(n,k)-1}})^s_0(\alpha) \) and \( l_{i-1}^{s_0} \leq q_i^{s_0} \).

• For all \( 0 < i < N(n,k) \), \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq (l_{i-1}^{s_0})^s_r(\alpha) \) and \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq q_i^{s_0} \).

• For all \( i < N(n,k) \) there is \( g \in 3^M \) such that \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq (l_i^s)^s_r(\alpha) \) and \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq q_i^{s_0} \).

• For all \( 0 < i < N(n,k) \), \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq (l_i^s)^s_r(\alpha) \) and \( l_{i-1}^{s_0}N(n,k)-1(\alpha) \leq q_i^{s_0} \).

• For all \( i < N(n,k) \) there are \( g \in 3^M \) such that \( q_i^s \vdash \text{"} \hat{f} | M = g \text{"} \).

• The set \( \{ g \in 3^M : \exists i < N(n,k), s < k \left( l_i^s \vdash \text{"} \hat{f} | M = g \text{"} \right) \} \) is of maximal size.

• \( M \) is minimal with the above conditions.

Once again, it is important to rename the \( r \)'s after each step. We may need to iterate the above technique a finite amount of times but, we can indeed find these \( M \) and conditions.

With all this we can define \( p_i^t \) to be the prune of the condition

\[
\begin{align*}
&\begin{cases}
  l_{i-1}^{s_{N(n,k)-1}}(\alpha) & \alpha \notin F \\
  l_i^{s_0}(\alpha) & \alpha \in F, \alpha \neq \beta \\
  l_i^{s_0}(\beta) & \alpha = \beta 
\end{cases}
\end{align*}
\]
such that, for $\alpha \in F$, the first split of $p^i_0(\alpha)$ is of size $\nu(\alpha) + 2$, the second of size $\nu(\alpha) + 3$ and so on (follow the left most path whenever is necessary to prune). Notice that all of these conditions extend $q$, they agree outside $F$, agree at coordinate $\alpha \in F \setminus \beta$ whenever $\sigma_q(\alpha) = \sigma_r(\alpha)$ (s, t \in k), and agree at coordinate $\beta$ when $i = j \in N(n, k)$. This means that $q'$ such that $q'(\alpha) = \langle r(\alpha), \bigcup_{i,t \in k}(p^i_0(\alpha))T \rangle$, where $(p^i_0(\alpha))^T$ is the tree portion of $p^i_0(\alpha)$, is a condition in $P_\kappa$ as long as $I \subseteq N(n, k)$ is of size $n$ (or bigger).

Define $f^1_i \in 3^M$ to be such that $p^1_i \models \text{``} f \upharpoonright M = f^1_i \text{''}$. Since $\langle q, F, \eta, m, A \rangle$ is consolidating $f$, we have that $\{ f^1_i | m : i \in N(n, k), t \in k \} = A$ is a 2-tree such that if $f^1_i | m = f^2_j | m$ with $t \neq s$ we have that $f^1_i = f^2_j$.

Furthermore, joining the above observation with the properties of the construction (maximality of the set), given $t, s \in k$, $i, j \in N(n, k)$ with $t \neq s$ we have that if there are $q_1 \leq p^1_i$, $q_2 \leq p^1_j$ and $n_0, n_1 \in \omega$ with $q_1 \models \text{``} f(n_0) = n_1 \text{''}$ but $q_2 \models \text{``} f(n_0) \neq n_1 \text{''}$ then $f^1_i \neq f^1_j$. With this, any 2-tree that comes from $\{ f^1_i : i \in N(n, k), t \in k \}$ will satisfy the requirements of the claim.

Now we can use Lemma 3.3 on $\{ f^1_i : i \in N(n, k), t \in k \}$ so we can find $S \subseteq N(n, k)$ of size $n$ such that $\{ f^1_i : i \in S, t \in k \}$ is a 2-tree.

To complete the claim, we use $M, A' = \{ f^1_i : i \in S, t \in k \}$ and $q'$ where $q'(\alpha) = q^{k-1}_{N(n,k)-1}(\alpha)$ if $\alpha \notin F$ and $q'(\alpha) = \langle r(\alpha), \bigcup_{i \in S, t \in k}(p^i_0(\alpha))^T \rangle$ where, again, $(p^i_0(\alpha))^T$ is such that $p^i_0(\alpha) = \langle (p^i_0(\alpha))^r, (p^i_0(\alpha))^T \rangle$.

\begin{definition}
A forcing notion has the $(k+1)\omega$ localization property if and only every function in $(k+1)\omega$ in the generic extension is a branch of a $k$-tree from the ground model.

Notice that the lemma above can be phrase as “the $3\omega$ localization property is preserved under countable support product of accelerating tree forcing”. Furthermore,

\begin{corollary}
For all $k \geq 2$, the $(k+1)\omega$ localization property is preserved under countable support product of accelerating tree forcing.
\end{corollary}

\begin{proof}
To prove this, it is enough to show that the $(k+1)\omega$ localization property is implied by the $(s+1)\omega$ localization property for $k \geq s \geq 2$, then, the result is a corollary of Lemma 3.5.

Fix a surjective function $f : (k+1) \rightarrow (s+1)$. Notice that this function induces a surjective function $f^* : (k+1)\omega \rightarrow (s+1)\omega$. Now, working in a generic extension given a $s$-tree $T$ from the ground model, $(f^*)^{-1}[T]$ is a $k$-tree from the ground model.
\end{proof}
Therefore, if in the generic extension \((s + 1)^\omega\) is covered by \(s\)-trees from the ground model, then \((k+1)^\omega\) is covered by \(k\)-trees from the ground model. 

This definition can let us expand our last result a little more.

**Definition 3.10.** Forcing with \(k\)-branching trees of \(k^\omega\) is the forcing notion that uses subtrees of \(k^\omega\) such that every node has either 1 or \(k\) successors.

This forcing is used in [7] where Newelski and Roslanowski showed that this forcing has the \(k\)-localization property, i.e., that every function of \(\omega^\omega\) in the generic extension is the branch of a \(k\)-tree from the ground model. Notice that this property implies the \((k + 1)^\omega\) localization property. A first step in order to investigate if the \((k + 1)^\omega\) localization property can be preserved under a bigger spectrum of forcings than the accelerating tree forcing is to show the following lemmas, that are analogues of Lemma 3.4 and 3.5:

**Lemma 3.11.** Given \(\{f^i_j : i \in I, j \in l\} \subseteq (k + 1)^\omega\) with \(l \in \omega\) and \(|I| = N(n, l)\) a big enough number and \(m \in \omega\) that makes \(\{f^i_j|_m : i \in I, j \in l\}\) a \(k\)-tree and such that if \(f^i|_m = f^t|_m\) with \(t \neq j\) we have that \(f^i_j = f^t_s\) then you can find \(S \subseteq I\) with \(|S| = n\) such that \(\{f^i_s : i \in S, j \in l\}\) is a \(k\)-tree.

*Proof.* This follows from the proofs of Lemma 3.1 and Lemma 3.4, in those lemmas we had \(k = 2\). The same reasoning will give us this lemma. 

**Lemma 3.12.** The \((k + 1)^\omega\) localization property is preserved under countable support product of alternating accelerating tree forcing and forcing with \(k\)-branching trees of \(k^\omega\).

*Proof.* Notice that the orders \(\leq_n\) also make sense when forcing with \(k\)-branching trees of \(k^\omega\).

The proof in full detail will have the same extension as the proof of Lemma 3.5. Nevertheless, here we give a sketch of how to combine the technique used in [7] and the proof of 3.5.

Everything works the same changing 2 for \(k\) and 3 for \(k + 1\). Now, to show the analogue of Claim 3.7 we will have two cases:

1. If you are extending a node that comes from an accelerating tree, then use Lemma 3.12 instead of Lemma 3.5. Everything else works the same.
2. If you are extending a node that comes from a \(k\)-branching tree instead of using Lemma 3.12, look for the next split after the node and look extension of those splits that have the properties of the \(q^t_i\) conditions in the Claim 3.7 above. Since the next split only has \(k\) successors, they naturally form a \(k\)-tree. Everything else works the same as the proof of Claim 3.7 or you can use the technique used in [7].

\[\square\]

4 Main Theorem, conclusion and open questions

**Theorem 4.1.** It is consistent with ZFC that \(\forall k \geq 2 (\mathcal{L}_k < \sigma^q_k = c).\)

**Proof.** Starting with a model of ZFC + GCH we can make a countable support product of the accelerating tree forcing describe in Lemma 3.2. Using Axiom A, as in in [1], we know that the product preserves cardinals and that \(c = \aleph_2\). Also, by Lemma 3.5, the resulting model will have \(\mathcal{L}_k = \mathcal{L}_2 = \aleph_1\). We just need to show that in the extension \(\sigma^q_k = \aleph_2 = c\).

Let \(P_{\omega_2} = \Pi_{\alpha \in \omega_2} Q_\alpha\) be the countable support product of accelerating tree forcings. Let \(G = \{c_\alpha : \alpha \in \omega_2\}\) be generic over \(P_{\omega_2}\). Now, for all \(\beta < \omega_1\) let \(T_\beta \subseteq \omega^\omega\) be a \(k(\beta)\)-tree, with \(k(\beta) \in \omega\), in \(V[G]\).

Now, in \(V\), we can find \(\dot{\tau}(\beta)\) a \(P_{\alpha(\beta)}\)-name for some \(\alpha(\beta) \in \omega_2\). So, there is \(\gamma \in \omega_2\) such that \(\alpha(\beta) < \gamma\) for all \(\beta\). Therefore, we have that \(T_\beta \in V[\{c_\alpha : \alpha < \gamma\}\] for all \(\beta \in \omega_1\).

Since \(c_\gamma\) is an accelerating forcing generic over \(V[\{c_\alpha : \alpha < \gamma\}\], then \(c_\gamma\) is not a branch of any \(k\)-tree in \(V[\{c_\alpha : \alpha < \gamma\}\], \(k \in \omega\). Therefore, \(c_\gamma\) is not a branch of any \(T_\beta\).

This shows that, in \(V[G]\), \(\omega^\omega\) is not cover by \(\{T_\beta : \beta \in \omega_1\}\). Since this was an arbitrary collection we have that \(\sigma^q_k = \aleph_2\) for all \(k \in \omega\).

\[\square\]

This theorem proves that it is consistent that \(\sigma^q_k \neq \mathcal{L}_k\) and answers the question from Blass about the identity of \(\sigma^q_k\): they indeed are a different cardinal characteristic from the ones that are known.

Furthermore, we can see that there are more ways to do this split:

**Theorem 4.2.** For all \(s \geq 2\) it is consistent with ZFC that \(\forall k \geq 2 (\mathcal{L}_{s+1} < \mathcal{L}_s = \sigma^q_k = c).\)

**Proof.** Following the same strategy as above, starting with a model of ZFC + GCH we can make a countable support product of the accelerating tree forcing alternated with forcing with \(s + 1\)-branching trees of \((s + 1)^\omega\). Just as
before, we know that the product preserves cardinals and that \( \kappa = \aleph_2 \). Also, by Lemma 3.12, the resulting model will have \( \mathcal{L}_{s+1} = \aleph_1 \). We just need to show that, in the extension, \( \mathcal{L}_s = \sigma_k^g = \aleph_2 = \kappa \).

Let \( \mathbb{P}_{\omega_2} = \prod_{\alpha \in \omega_2} Q_\alpha \) be the countable support product of accelerating tree forcings, when \( \alpha \) is even and forcing with \( s+1 \) subtrees of \( (s+1)^\omega \) when \( \alpha \) is odd. Let \( G = \{c_\alpha : \alpha \in \omega_2\} \) be generic over \( \mathbb{P}_{\omega_2} \).

To see that \( \sigma_k^g = \aleph_2 = \kappa \), we can do the same as above. Now, showing that \( \mathcal{L}_s = \aleph_2 = \kappa \) can be found in [7]. Nevertheless, for convenience to the reader, we give an argument here:

For all \( \beta < \omega_1 \) let \( T_\beta \subseteq (s+1)^\omega \) be a s-tree in \( V[G] \). In \( V \), we can find \( \dot{T}(\beta) \) a \( \mathbb{P}_{\alpha(\beta)} \)-name for some \( \alpha(\beta) \in \omega_2 \). So, there is \( \gamma \in \omega_2 \) such that \( \alpha(\beta) < 2 \cdot \gamma + 1 \) for all \( \beta \). Therefore, we have that \( T_\beta \in V[\{c_\alpha : \alpha < 2 \cdot \gamma + 1\}] \) for all \( \beta \in \omega_1 \).

Since \( c_{2 \cdot \gamma + 1} \) is a generic for the forcing using \( s+1 \)-branching trees of \( (s+1)^\omega \) over \( V[\{c_\alpha : \alpha < \gamma\}] \), then \( c_{2 \cdot \gamma + 1} \) is not a branch of any s-tree in \( V[\{c_\alpha : \alpha < \gamma\}] \). Therefore, \( c_{2 \cdot \gamma + 1} \) is not a branch of any \( T_\beta \).

This shows that, in \( V[G] \), \( (s+1)^\omega \) is not cover by \( \{T_\beta : \beta \in \omega_1\} \). Since this was an arbitrary collection we have that \( \sigma_k^g = \aleph_2 = \kappa \) for all \( k \in \omega \).

\[ \Box \]

**Question 4.3.** What is the value of \( \text{cof}(\mathcal{N}) \) in the above models?

Theorem 4.2 shows that in order to have different values for \( \sigma_k^g \) and \( \mathcal{L}_k \) it is not necessary that every \( \mathcal{L}_s \) have the same value. In this same venue, we can wonder if it is necessary that all \( \sigma_k^g \) have the same value. In other words:

**Question 4.4.** Can we have \( \mathcal{L}_k = \sigma_k^g < \sigma_3^g \) for all \( k \geq 2 \)?

For this question, it is not possible to use neither the accelerating tree forcing nor the forcing with 3-branching trees of \( 3^\omega \), which are the two ways used in this paper to make \( \sigma_3^g = \kappa \). Another approach will be to use trees of \( \omega^\omega \) that branches more than 3 times at each split. Nevertheless, I do not see any good reason for that forcing to have the \( 3^\omega \)-localization property. Maybe a modification of it can do the trick.

Now, during the paper, the \( (k+1)^\omega \) localization property played a really important role. In order to show that it was preserved the proofs showed above are really case specific. This is useful for our purposes, but a question arises:

**Question 4.5.** Can we show that the \( (k+1)^\omega \)-localization property is preserved under countable support iteration and products?
This is likely to be possible. In [9], Zapletal showed that the \( n \)-localization property is preserved under countable support product and iteration of a broad variety of forcings (some kind definable proper forcings).

Finally, notice that \( \omega_k \) is a cardinal characteristic that is usually really closed to \( c \). This is not true in cardinal arithmetic, but it is true in the Chicon Diagram: all of these numbers are above \( \text{cof}(N) \). So, in order to work with them, it is important to use forcing notions that are tame somehow (they cannot add Cohen or random reals, for example). In this case, we used a forcing notion with the \((k + 1)^\omega\) localization property but, in the literature, there are examples of properties like the Sacks property, the \( n \)-localization property and, most recently, the shrink wrapping property (see [5]) that are also tame with reals. It is important to notice that most of these 'tameness' properties relates to the idea of keeping the new reals inside a tree of some sort.

**Question 4.6.** Is there an underlying theorem (or meta theorem) that relates all (of some) of this tameness properties?

One possible result could be that all of them are preserved under countable support product of a variety of forcings, but I do not have any good guess of whether this is possible or not.
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