A Poisson limit theorem for Gibbs–Markov maps

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ABSTRACT

We prove for Gibbs–Markov maps that the number of visits to a sequence of shrinking sets with bounded cylindrical lengths converges in distribution to a Poisson law. Applying to continued fractions, this result extends Doeblin’s Poisson limit theorem.

1. Introduction

A Poisson limit theorem for continued fractions was proved by Doeblin [15] around 1937, which says that the number of certain large partial quotients in a continued fraction expansion converges in distribution to a Poisson law. More precisely, denoting by \( [a_1, a_2, \ldots], a_i \in \mathbb{N} \), the continued fraction expansion of an irrational number \( x \in (0, 1) \),

**Theorem 1.1:** For every \( \theta > 0 \), \( k \in \mathbb{N}_0 \)

\[
\lim_{n \to \infty} \text{Leb} \{ x \in (0, 1) : \text{there are exactly } k \text{ a } \theta' \text{s with } a_i > \theta n, \ 1 \leq i \leq n \} = \frac{1}{(\theta \log 2)^k k!} e^{-1/(\theta \log 2)}.
\]

Later Iosifescu [27] proved a Poisson limit theorem for \( \psi \)-mixing processes, which in particular can be applied to the process \( \{a_n\}_{n \in \mathbb{N}} \). Treating continued fractions as a symbolic dynamical system, one can rephrase Doeblin’s theorem as a Poisson limit theorem for return times under the Gauss map \( T : x \mapsto \{1/x\} \).

**Theorem 1.2:** For every \( \theta > 0 \), \( k \in \mathbb{N}_0 \)

\[
\lim_{n \to \infty} \text{Leb} \left\{ x \in (0, 1) : \sum_{i=0}^{n-1} 1_{(0,1/(\lfloor \theta n \rfloor + 1))} \circ T^i(x) = k \right\} = \frac{1}{(\theta \log 2)^k k!} e^{-1/(\theta \log 2)}.
\]

Compared with the Poisson limit theorem for binomial random variables, an ideal Poisson limit theorem in dynamical systems would be the following. Let \( (\Omega, \mathcal{B}, \mu) \) be
a probability space, $T$ be a $\mu$-preserving map on $\Omega$, and $\{A_n\}_{n \in \mathbb{N}_0}$ be a sequence of measurable sets. If $n\mu(A_n) \to t > 0$ as $n \to \infty$, then for every $k \in \mathbb{N}_0$

$$\lim_{n \to \infty} \mu\left( \left\{ x \in \Omega : \sum_{i=0}^{n-1} 1_{A_n} \circ T^i(x) = k \right\} \right) = \frac{t^k}{k!} e^{-t}.$$ 

The summation inside the brackets counts the number of visits of $x$ to $A_n$ until the $n\text{th}$ iteration by $T$. Note that, since $\{1_{A_n} \circ T^i\}_{i \in \mathbb{N}_0}$ in general is not an independent process, it is not clear that the limit distribution exists, let alone being Poisson. The study of Poisson limit law for return times in dynamical systems was initiated in the early 1990s by Hirata, Pitskel, Sinai, etc. Since then, there are various results confirming the Poisson limit theorem and also obtaining error estimates under appropriate assumptions; see, for example, [5, 8, 9, 12, 14, 16, 19, 24, 26, 31]. But in general, the class of possible limit distributions is rather large, more than just Poisson, as shown by Lacroix and Chaumoître and Kupsa. We refer to Haydn’s recent review on this and related topics.

Let $n\mu(A_n) \to t$ as $n \to \infty$, then the measure $\mu(A_n)$ shrinks to 0. Most of the previous results consider a sequence of shrinking sets $A_n$ which converge to a generic non-periodic point (like in Theorem 2.6). When $A_n$ shrinks to a periodic point, $A_n$ always intersects with its pre-images via small iterations of $T$. In this case, it had been pointed out by both Pitskel and Hirata that the limit distribution cannot be Poisson. Later Haydn and Vaienti showed for some mixing maps, that when $A_n$ are cylinder sets of growing cylindrical lengths (the level of the filtration in which $A_n$ lives) and converging to a periodic point, the limit distribution of the number of return times becomes compound Poisson. But none of these results covers Doeblin’s theorem, in which case $A_n = (0, \frac{1}{[a_n]+1})$ converges to a non-generic point, the endpoint 0, and has a constant cylindrical length.

In this note, we study the limit distribution of the number of visits to a sequence of shrinking sets with bounded cylindrical lengths. When the sequence converges to a single point, the accumulation point may be periodic or a compactification point of the space (like the endpoint 0 in Doeblin’s theorem). This type of sequences seems novel in the literature to our knowledge. Our main result (Theorem 4.1) shows a Poisson limit theorem for Gibbs–Markov maps with this type of sequences. Some applications to continued fractions are given at the end of the note, including an extension of Doeblin’s theorem.

Examples of Gibbs–Markov maps are given in the next section, including some Markov chains with countable states, Markov maps of the unit interval, parabolic rational maps. We use perturbation of transfer operator to study the limit distribution of the number of return times, following the idea of Hirata. It turns out that for the type of sequences in our consideration, the size of this perturbation is not asymptotically small in the Banach space where the transfer operator has good spectral property. Still, a perturbation theorem of Keller and Liverani allows us to deal with this situation using a weaker norm. We note that this perturbation theorem has been used widely by many authors to study closely-related problems, such as the limit distribution of the first return time and escape rate, for example in [6, 11, 18, 29].
2. Gibbs–Markov maps

We recall the definition of Gibbs–Markov maps from [3]. Let \((\Omega, \mathcal{B}, \mu)\) be a probability space and let \(T\) be a non-singular transformation. Consider a countable partition \(\alpha\) of \(\Omega\) mod \(\mu\), let \(\alpha = \{a_i : i \in I\}\). Denote by \(\alpha_0^{n-1}\) the refined partition \(\bigvee_{i=0}^{n-1} T^{-i} \alpha\) and by \(\sigma(\cdot)\) the \(\sigma\)-algebra generated by a partition. For a set \(A \in \sigma(\alpha_0^{n-1})\), we call the minimal \(m \in \mathbb{N}\) such that \(A \in \sigma(\alpha_0^{m-1})\) its cylindrical length.

**Definition 2.1:** A quintuple \((\Omega, \mathcal{B}, \mu, T, \alpha)\) is called a Gibbs–Markov map if it satisfies the following conditions.

1. \(\alpha\) is a strong generator of \(\mathcal{B}\) under \(T\), i.e. \(\sigma(\{T^{-n}\alpha : n \in \mathbb{N}_0\}) = \mathcal{B}\) mod \(\mu\).
2. (big image property) \(\inf_{a \in \alpha} \mu(Ta) > 0\).
3. For every \(a \in \alpha\), \(Ta \in \sigma(\alpha)\) mod \(\mu\), moreover the restriction \(T|_{Ta}\) is invertible and non-singular.
4. For every \(n \in \mathbb{N}\) and \(a \in \alpha_0^{n-1}\), denote the non-singular inverse branch of \(T^{-n}\) on \(T^n a\) by \(v_a : T^n a \to a\) and its Radon–Nikodym derivative by \(v'_a\). There exist \(r \in (0, 1)\) and \(M > 0\) such that for any \(n \in \mathbb{N}\), \(a \in \alpha_0^{n-1}\) and \(x, y \in T^n a\) a.e.

\[
\left| \frac{v'_a(x)}{v'_a(y)} - 1 \right| \leq M \cdot r(x, y),
\]

where \(r(x, y)\) is the metric \(\min\{n \in \mathbb{N} : T^{-n}(x)\) and \(T^{-n}(y)\) belong to different elements of \(\alpha\}\).

We present some typical examples. The first two have already been mentioned in [3].

**Example 2.2 (Markov chain with countable states):** Let \(S\) be a countable set, \(P = (p_{x,y})_{x,y \in S}\) be an irreducible ergodic stochastic matrix on \(S\) and \(\pi\) be the stationary probability measure. Let \(\Omega = \{x = (x_0, x_1, \ldots) \in S^{\mathbb{N}_0} : p_{x_k x_{k+1}} > 0, k \in \mathbb{N}_0\}\), \(\mathcal{B}\) the \(\sigma\)-field generated by all the cylinder sets of the form \([s_0, \ldots, s_n] = \{x : x_0 = s_0, \ldots, x_n = s_n\}\), \(T\) be the shift map, \(\alpha = \{[s] : s \in S\}, \mu([s_0, \ldots, s_n]) = \pi s_0 \prod_{k=0}^{n-1} p_{s_k, s_{k+1}}\).

Suppose \(a = [s_0, \ldots, s_n]\), then \(v'_a(x) = \frac{\pi x_0 \pi_{s_0} \cdots \pi_{s_0} p_{s_n}}{\pi_{s_0} \pi_{s_0+1} \cdots \pi_{s_0+n-1} p_{s_n}}\) for \(x \in T^n a\). The system is Gibbs–Markov if and only if \(|\frac{p_{x,y}}{\pi_x} / \frac{p_{y}}{\pi_y} - 1|\) is uniformly bounded for all \(s, x, y \in S\) with \(p_{sx}, p_{sy} > 0\).

An explicit example is given by [17, XV(2,k)]:

\[
P = \begin{bmatrix}
 f_1 & f_2 & f_3 & f_4 & \cdots \\
 1 & 0 & 0 & 0 & \cdots \\
 0 & 1 & 0 & 0 & \cdots \\
 0 & 0 & 1 & 0 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots \\
 \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

where \(\sum_{i \in \mathbb{N}} f_i = 1\). Let \(r_k = \sum_{i \geq k} f_i\). Suppose \(\sum_{k \in \mathbb{N}} r_k < \infty\), then the stationary measure \(\pi = (\pi_0, \pi_0 r_1, \pi_0 r_2, \ldots)\). \(T\) is Gibbs–Markov if and only if \(C^{-1} \leq \frac{f_i}{r_i} / \frac{f_j}{r_j} \leq C\) for some constant \(C > 0\) and for all \(i, j\). For example, \(f_i = \frac{1}{i!} \frac{1}{e - 1}\).
Example 2.3 (Markov interval map, [10, § 7.4]): Let $\Omega = (0, 1)$, $\mathcal{B}$ be Borel $\sigma$-algebra, Leb be Lebesgue measure, and $\alpha = \{a_i\}$ be a partition of $\Omega \mod \mu$ into open intervals. Suppose that $T : \Omega \rightarrow \Omega$ satisfies the following conditions:

- $T|_{a_i}$ is strictly monotonic and extends to a $C^2$ function on $\bar{a}_i$ for each $i$.
- If $T(a_k) \cap a_j \neq \emptyset$, then $T(a_k) \supset a_j$.
- $T$ is expanding: there exists an $m \in \mathbb{N}$ such that
  \[
  \inf_{a_i \in \alpha} \inf_{x \in a_i} |T'(x)| > 0, \quad \inf_{a_i \in \alpha} \inf_{x \in a_i} |T''(x)| := \lambda > 1.
  \]
- $T$ satisfies the Adler property:
  \[
  \sup_{a_i \in \alpha} \sup_{x, y \in a_i} \frac{|T''(x)|}{T'(y)^2} < \infty.
  \]

Then there exists a constant $M$ such that:
\[
\frac{|T^{n''}(x)|}{T^{n'}(y)^2} \leq M, \quad \text{for all } x, y \in a \in \alpha_{0}^{n-1}, n \in \mathbb{N}.
\]

Hence for some constant $M_1$, and for all $n \in \mathbb{N}, a \in \alpha_{0}^{n-1}, x, y \in T^n a$,
\[
\left| \frac{v'_a(x)}{v'_a(y)} - 1 \right| \leq M_1 |x - y|.
\]

Let $r = (1/\lambda)^{1/m}$. There is a constant $M_2$ such that for any $x, y$, if $k \in \mathbb{N}$ is the minimum natural number such that $T^{k-1}x$ and $T^{k-1}y$ belong to different elements of $\alpha$, supposing $x, y \in b \in \alpha_{0}^{k-1}$,
\[
|x - y| \leq \text{Leb}(b) = \text{Leb}(v_b(T^k b)) \leq \|v'_b\| \leq M_2 r^k.
\]

The big image property may not always hold for $T$. If it does, then $T$ is a Gibbs–Markov map. For example, let $\alpha = \{\{a_1 = n\} : n \in \mathbb{N}\}$ be the partition by the first partial quotient in the continued fraction expansion and let $T$ be the Gauss map: $x \mapsto \{\frac{1}{x}\}$. Then it is a Gibbs–Markov map. Furthermore, let $\mu$ be the Gauss measure $\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} \, dx$, then the continued fraction map $((0, 1), \mathcal{B}, \mu, T, \alpha)$ is a probability-preserving Gibbs–Markov map.

The following example appears in [2] in the study of Poincaré series of the surface $\mathbb{C} \setminus \mathbb{Z}$. Let
\[
R(x) = \begin{cases} 
\frac{x}{2}, & x \in (0, \frac{1}{3}), \\
\frac{1}{2} - x, & x \in \left(\frac{1}{3}, \frac{1}{2}\right), \\
2 - \frac{1}{x}, & x \in \left(\frac{1}{2}, 1\right).
\end{cases}
\]

Denote $A := (0, \frac{1}{3}), B := \left(\frac{1}{3}, \frac{1}{2}\right), C := \left(\frac{1}{2}, 1\right)$, and $U := A \cap R^{-1} A^c = \left(\frac{1}{3}, \frac{1}{2}\right), W := C \cap R^{-1} C^c = \left(\frac{1}{2}, \frac{2}{3}\right), J := U \cup B \cup W = \left(\frac{1}{3}, \frac{2}{3}\right)$. Induce $R$ on $J$, that is, let
\[
R_J(x) = R^T(x),
\]
where \( \tau = \min \{ k \in \mathbb{N} : R^k(x) \in J \} \). Define \( B_1 = B \cap R^{-1}J \),

\[
U_n = U \cap \bigcap_{j=1}^{n-1} R^{-j}C \cap R^{-n}(B \cup W), \quad W_n = W \cap \bigcap_{j=1}^{n-1} R^{-j}A \cap R^{-n}(U \cup B),
\]

\[
B_n^- = B \cap \bigcap_{j=1}^{n-1} R^{-j}A \cap R^{-n}(U \cup B), \quad B_n^+ = B \cap \bigcap_{j=1}^{n-1} R^{-j}C \cap R^{-n}(B \cup W),
\]

\[\alpha_n = \{ U_n, W_n, B_1, B_{n+1}^- \cup B_{n+1}^+ : n \in \mathbb{N} \}.\]

Then \( (J, B, \text{Leb}, R_f, \alpha_f) \) is Gibbs–Markov.

**Example 2.4 (Rational functions on the Riemann sphere, [13]):** Let \( S^2 \) denote the Riemann sphere. A rational function \( T : S^2 \to S^2 \) has the form \( T(z) = \frac{P(z)}{Q(z)} \) when \( T \) is restricted to \( \mathbb{C} \subset S^2 \) and where \( P \) and \( Q \) are polynomials \( (Q \neq 0) \) of maximal degree \( \geq 2 \). Restrict \( T \) to its Julia set

\[ J(T) = \{ z \in S^2 : \{ T^n \}_{n \in \mathbb{N}} \text{ is not normal at } z \}. \]

These transformations include important examples for Gibbs–Markov maps. \( T \) is called hyperbolic if \( J(T) \) does not contain any critical point or rationally indifferent point. In this case \( T : J(T) \to J(T) \) is expanding and has a finite Markov partition. For general \( T : J(T) \to J(T) \) a probabilistic measure \( \mu \) on \( J(T) \) is called conformal for a continuous potential \( \varphi : J(T) \to \mathbb{R} \) if

\[ \mu(TA) = \int_A e^\varphi(y) \, d\mu(y) \]

for every measurable \( A \) where \( T|_A : A \to TA \) is invertible. Equivalently, \( \mu \) is characterized by

\[ \mathcal{L}_\varphi \mu = \mu \]

where \( \mathcal{L}_\varphi(f)(z) = \sum_{T^m z = y} f(y) e^{-\varphi(y)} \) denotes the transfer operator. In case of a hyperbolic \( T \), for any \( \varphi \) there is \( \lambda > 0 \) such that there is a \( \lambda \)-conformal measure \( \mu \) by Sullivan’s result [34]. The system \( (J(T), m, \alpha) \) is Gibbs–Markov where \( \alpha \) is the finite Markov partition. The transformation \( T \) is called parabolic if \( J(T) \) does not contain critical points, but has rationally indifferent periodic points. In this case, it is known by [13] that a conformal measure exists if the potential \( \varphi \) satisfies \( P(T, \varphi) > \sup \varphi \) where \( P(T, \varphi) \) denotes the pressure of \( \varphi \). So such measure exists for \( \varphi = t \log |T'| \). Also \( T \) admits a countable Markov partition. \( (A, m|_A, T_A, \beta) \) is a Gibbs–Markov systems, whenever \( A \) belongs to the Markov partition and \( \beta \) is the first return time partition on \( A \) which is also a (countable) Markov partition.

It is known that a probability-preserving, topologically mixing Gibbs–Markov map \((\Omega, \mathcal{B}, \mu, T, \alpha)\) is continued-fraction mixing (exponential \( \psi \)-mixing), as stated in the next proposition. A Gibbs–Markov map is called topologically mixing if for any \( a, b \in \alpha \), there is \( n_{a,b} \in \mathbb{N} \) such that for every \( n \geq n_{a,b} \), \( b \subset T^n a \). The above examples are all topologically mixing.
Proposition 2.5 ([1, Corollary 4.7.8], see also [4]): A probability-preserving, topologically mixing Gibbs–Markov map is continued-fraction mixing. That is, there exist constants $K > 0$ and $0 < \theta < 1$ such that for any $n, k \in \mathbb{N}$, $a \in \alpha_{0}^{n-1}$ and $B \in \mathcal{B}$

$$|\mu(a \cap T^{-n-k}B) - \mu(a)\mu(B)| \leq K\theta^n \mu(a)\mu(B).$$

For a $\psi$-mixing dynamical system whose $\sigma$-algebra is generated by a countable partition, one can hope for a Poisson limit theorem around a non-periodic point, as mentioned in the introduction. For example, the following Poisson limit theorem can be obtained from [21, Corollary 1] (see also [22, 26]). Note that in these cases, the authors also obtain error estimates.

Theorem 2.6: Given a probability-preserving, topologically mixing Gibbs–Markov map. Suppose $x$ is a non-periodic point and $A_n(x) \in \alpha_{0}^{n-1}$ is the cylinder set of length $n$ that contains $x$. Then for any $t > 0$ and $k \in \mathbb{N}$

$$\lim_{n \to \infty} \mu \left( \left\{ x \in \Omega : \sum_{i=0}^{\lfloor t/\mu(A_n(x)) \rfloor - 1} 1_{A_n(x)} \circ T^i(x) = k \right\} \right) = \frac{t^k}{k!} e^{-t}.$$

However, around a periodic point or when $A_n$ always intersects with its preimages via small iterations, one can see that the error estimates in [21, 24] do not imply a Poisson limit law. Indeed, when $x$ is a periodic point and $A_n = A_n(x) \in \alpha_{0}^{n-1}$, the limit distribution is compound Poisson as shown in [23]. We note that similar results hold for the induced measure $\mu_n = \frac{\mu|_{A_n}}{\mu(A_n)}$ through the relation revealed in [37] between the statistics of successive return times and the statistics of successive hitting times.

Proposition 2.5 can be proved with the following Renyi’s property and transfer operator’s spectral gap property (to be discussed in the next section).

Proposition 2.7 (Renyi’s property, [3, Proposition 1.2]): Given a topologically mixing Gibbs–Markov map. There exists a constant $M > 0$ such that for every $n \in \mathbb{N}$, $a \in \alpha_{0}^{n-1}$, a.e. $x \in T^n a$,

$$M^{-1} \mu(a) \leq v_a(x) \leq M \mu(a).$$

A simple corollary is useful to estimate short returns.

Corollary 2.8: Given a probability-preserving, topologically mixing Gibbs–Markov map. There exists a constant $M_1 > 0$ such that for every $n \in \mathbb{N}$, $a \in \alpha_{0}^{n-1}$, $k \leq n$,

$$\mu(a \cap T^{-k}a) \leq M_1 \mu(a)^{1+1/(1+n)}.$$

Proof: Let $r = n - \lfloor n/k \rfloor k$. Let $b \in \alpha_{0}^{k-1}$ be the cylinder set of length $k$ that contains $a$ as a subset. Note that if $a \cap T^{-k}a \neq \emptyset$,

$$a = b \cap T^{-k}b \cap \cdots \cap T^{-\lfloor n/k \rfloor k}b \cap T^{-n}a,$$

$$a \cap T^{-k}a = b \cap T^{-k}a,$$
\[ a \cap T^{-k}a \subset a \cap T^{-(n+k-r)}T^{n-r}a = a \cap T^{-n}(T^{-(k-r)}T^{n-r}a). \]

Then it follows from Renyi’s property that there exists a constant \( M \) such that
\[
\mu(a) \geq M^{-n/k} \mu(b)[n/k] \mu(T^{n-r}a),
\]
\[
\mu(a \cap T^{-k}a) \leq M\mu(b)\mu(a),
\]
\[
\mu(a \cap T^{-k}a) \leq M\mu(a)\mu(T^{n-r}a).
\]

Raise the second inequality to \([n/k]th\) power, multiply with the third inequality, then substitute the first inequality, one gets
\[
\mu(a \cap T^{-k}a)^{[n/k]+1} \leq M^{[n/k]+1} \mu(a)^{[n/k]+1} \mu(b)^{[n/k]} \mu(T^{n-r}a)
\]
\[
\leq M^{2[n/k]+1} \mu(a)^{[n/k]+2}.
\]

Therefore,
\[
\mu(a \cap T^{-k}a) \leq M^2 \mu(a)^{(2+[n/k])/(1+[n/k])} \leq M^2 \mu(a)^{1+1/(1+n)}.
\]

\[\boxed{}\]

3. Transfer operator and perturbation

From now on, we always assume that \((\Omega, B, \mu, T, \alpha)\) is a probability-preserving, topologically mixing Gibbs–Markov system. In this section, we summarize some properties of the widely-used transfer operator. For any partition \( \rho \) of \( \Omega \), define the Hölder norm subject to \( \rho \) of a function \( f : \Omega \to \mathbb{R} \) by
\[
D_{\rho}f := \sup_{b \in \rho} \sup_{x, y \in b, x \neq y} \frac{|f(x) - f(y)|}{r(x, y)},
\]
where sup is taken \( \mu \) almost everywhere. Denote the usual \( L^q \)-norm by \( \| \cdot \|_q \), \( 1 \leq q \leq \infty \).

Set
\[
\|f\|_{\infty, \rho} := \|f\|_{\infty} + D_{\rho}f.
\]

Denote by \( L^\infty_{\rho} \) the set consisting of functions of finite \( \| \cdot \|_{\infty, \rho} \)-norm. \( L^\infty_{\rho}, \| \cdot \|_{\infty, \rho} \) is a Banach space. Recall that \( T\alpha \subset \sigma(\alpha) \). For every \( n \in \mathbb{N}, T^n(a_0^n) = T\alpha \). Fix a partition \( \beta \)

such that \( \sigma(T\alpha) = \sigma(\beta) \). Define the transfer operator \( \mathcal{L} : L^1(\mu) \to L^1(\mu) \) by
\[
\mathcal{L}(f) := \sum_{b \in \beta} 1_b \sum_{a \in \alpha, Ta \supset b} \nu_a \cdot f \circ v_a.
\]

Since \( \mu \) is \( T \)-invariant, \( \mathcal{L}(1) = 1 \). \( \mathcal{L} \) satisfies and is uniquely characterized by
\[
\int_{\Omega} \mathcal{L}(f) \cdot g \, d\mu = \int_{\Omega} f \cdot g \circ T \, d\mu, \quad \forall f \in L^1(\mu), \ g \in L^\infty(\mu).
\]

We write for simplicity
\[
L := L^\infty_{\beta}, \quad \| \cdot \| := \| \cdot \|_{\infty, \beta}.
\]

As no confusion should appear, we use the same notation \( \| \cdot \| \) for the operator norm on \( L \).

On \( L \), the transfer operator \( \mathcal{L} \) satisfies Doeblin–Fortet inequality and spectral gap property.
as stated in the next proposition. Denote by $\tau(\cdot)$ the spectral radius of a linear operator. Let $r \in (0, 1)$ be the constant in Definition 2.1 (4).

**Proposition 3.1 ([3, Propositions 1.4, 2.1, Theorem 1.6]):** There are constants $C_1, C_2 > 0$ such that the following statements are true. For all $f \in L$ and $n \in \mathbb{N}$,

$$||L^n(f)|| \leq C_1(r^nD_\beta f + ||f||_1).$$

Suppose $\omega : \Omega \to [0, 1]$ satisfies $D_\alpha \omega < \infty$. Let $L_\omega(f) := L(\omega \cdot f)$ then

$$||L_\omega^n(f)|| \leq (C_1 + C_2D_\alpha \omega)(r^nD_\beta f + ||f||_1).$$

Furthermore, the essential spectral radius $\tau_{ess}(L) \leq r$ and 1 is the simple, unique maximal eigenvalue of $L$. One can decompose $L$ on $L$ as

$$L = P + N,$$

where $P(f) = \int_{\Omega} f \, d\mu$ is the eigenprojection of $L$ with respect to 1, $PN = NP = 0$ and $\tau(N) < 1$.

Following [25], one can investigate the statistics of return times by studying the operator $L_A(f) := L(1_{A^n} \cdot f)$ as a perturbation of $L$. However, in our situation when $A^n$ has a constant cylindrical length, this perturbation is not asymptotically small under $|| \cdot ||$-norm. Take the continued fraction map for example. Recall $\alpha = \{a_1 = n : n \in \mathbb{N}\}$. The Gauss map sends every element of $\alpha$ onto the whole interval $(0, 1)$, so let $\beta = \{(0, 1)\}$. Let $A_n = (0, 1/n) \in \sigma(\alpha)$. Then $||L_{A_n} - L|| = ||L_{A_n}||$ is of the order of $D_\beta 1_{A_n} = r^{-1}$ for all $n$, not tending to 0. Usual analytic perturbation theorems are not applicable. Instead as mentioned in the introduction, we use Keller and Liverani's perturbation theorem to deal with such perturbations that are asymptotically small under a weaker norm. We restate Keller and Liverani's theorems from [28] as follows.

**Theorem 3.2:** Let $\{L_n\}_{n \in \mathbb{N}_0}$ be a sequence of bounded linear operators on a Banach space $(B, || \cdot ||)$, with a second (semi-)norm $n(\cdot) \leq || \cdot ||$. Let $0 < \rho < 1$ and let $\tau : \mathbb{N}_0 \to \mathbb{R}^+$ be a monotone function with $\lim_{n \to \infty} \tau_n = 0$. Suppose that for some constant $C > 0$, and for all $n, k \in \mathbb{N}_0$ and $f \in B$,

(i) $n(L_n^k) \leq C$,

(ii) $||L_n^k(f)|| \leq C(\rho^k||f|| + n(f))$,

(iii) $n((L_0 - L_n)(f)) \leq \tau_n||f||$.

Suppose further that $\tau_{ess}(L_0) \leq \rho$ and $\lambda_0$ is the simple, unique maximal eigenvalue of $L_0$ with $|\lambda_0| = 1$. Decompose $L_0 = \lambda_0P_0 + N_0$ in the form of (1). Let $\rho' \in (\rho, 1)$ and $\delta > 0$ be small such that $\{z \in \mathbb{C} : |z - \lambda_0| \leq \delta\}$ is disjoint from $\{z \in \mathbb{C} : |z| \leq \rho'\}$. Then there exist $n_0 \in \mathbb{N}, \eta \in (0, 1), K > 0$ such that for every $n \geq n_0$, $\tau_{ess}(L_n) \leq \rho'$, $L_n$ has a simple, unique maximal eigenvalue $\lambda_n$ and one can decompose $L_n = \lambda_nP_n + N_n$ in the form of (1) satisfying additionally

$$|\lambda_n - \lambda_0| \leq \delta,$$
(2) $n((P_n - P_0)(f)) \leq K \tau_n^\eta \|f\| \text{ for any } f \in B,$

(3) $\|P_n(f)\| \leq Kn(P_n(f)) \text{ for any } f \in B,$

(4) $\|N_n^k\| \leq K(1 - \delta)^k \text{ for any } k \in \mathbb{N}.$

4. Main theorem and applications to continued fractions

Now we prove the main theorem in this note. We consider a sequence of shrinking sets with bounded lengths. If the sequence converges to a point, it may be a periodic point or a compactification point. To some extent, our result complements both the Poisson limit Theorem 2.6 for a sequence of cylinder sets with increasing lengths converging to a non-periodic point and the compound Poisson limit theorem for a sequence of cylinder sets with increasing lengths converging to a periodic point.

**Theorem 4.1:** Given a measure-preserving, topologically mixing Gibbs–Markov map. Assume that a sequence of measurable sets $\{A_n\}_{n \in \mathbb{N}}$ satisfies the following conditions.

(a) For some $m \in \mathbb{N}$, $A_n \in \sigma(\alpha_0^{m-1})$ for all $n \in \mathbb{N}$.

(b) For all $i \in \mathbb{N}$, $\lim_{n \to \infty} \mu(A_n \cap T^{-i}A_n)/\mu(A_n) = 0$.

If $n\mu(A_n)$ converges to a constant $t > 0$ as $n \to \infty$, then for every $k \in \mathbb{N}_0$

$$\lim_{n \to \infty} \mu\left(\left\{x \in \Omega : \sum_{i=0}^{n-1} 1_{A_n} \circ T^i(x) = k\right\}\right) = \frac{t^k}{k!} e^{-t}.$$ 

**Proof:** Let $S_{n,k} = 1_{A_n} + \cdots + 1_{A_n} \circ T^{k-1}$. We will show that the Laplace transform of $S_{n,k}$ converges to the Laplace transform of the Poisson distribution with parameter $t$, that is, $\int e^{-s S_{n,k}} d\mu \to e^{-t(1-e^{-s})}$ as $n \to \infty$ for every $s \geq 0$. Fix an arbitrary $s \geq 0$. Define $\mathcal{L}_n$ by

$$\mathcal{L}_n(f) := \mathcal{L}(e^{-s 1_{A_n} f}),$$

then for all $k \in \mathbb{N}$

$$\mathcal{L}_n^k(f) = \mathcal{L}(e^{-s S_{n,k}}).$$

Set $\mathcal{L}_0 := \mathcal{L}$. We verify the conditions (i)–(iii) in Theorem 3.2 for operators $\{\mathcal{L}_n\}_{n \in \mathbb{N}_0}$ on $(L, \|\cdot\|), a second norm n(\cdot) = \|\cdot\|_1, and \tau_n = (1 - e^{-s})\mu(A_n).$

It is clear that $\|\mathcal{L}_n^k\|_1 \leq 1$ for all $n, k \in \mathbb{N}_0$, so condition (i) is satisfied. Because $A_n \in \sigma(\alpha_0^{m-1})$, $A_n$ is a countable union of cylinder sets in $\alpha_0^{m-1}$. Let $r \in (0, 1)$ be the constant in Definition 2.1 (4). So $D_{\alpha 1} A_n = 0$ when $m = 1$, and $D_{\alpha} A_n \leq r^{-m}$ when $m \geq 2$. Now, Proposition 3.1 implies a uniform Doeblin–Fortet inequality for $\mathcal{L}_n$: there are constants $C_1, C_2 > 0$ such that for all $n, k \in \mathbb{N}_0$ and $f \in L$,

$$\|\mathcal{L}_n^k(f)\| \leq (C_1 + C_2 r^{-m})(r^k \|f\| + \|f\|_1).$$

This inequality verifies condition (ii). Condition (iii) also holds because
\[(\mathcal{L}_0 - \mathcal{L}_n)(f) \|_1 = \| \mathcal{L}((1 - e^{-s1\lambda_n}) \cdot f) \|_1 \]
\[\leq \int_{\Omega} (1 - e^{-s1\lambda_n}) |f| \, d\mu = (1 - e^{-s}) \int_{\Lambda_n} |f| \, d\mu \]
\[\leq (1 - e^{-s}) \| f \| \mu(A_n) = \tau_n \| f \| .\]

In addition, \(\mathcal{L}_0\) has a spectral gap as stated in (1). Fix \(r' \in (r, 1)\) and \(\delta > 0\) such that \(|z - 1| \leq \delta\) is disjoint from \(|z| \leq r'\). It follows from Theorem 3.2 that there exist \(n_0 \in \mathbb{N}, \eta \in (0, 1), K > 0\) such that for every \(n \geq n_0, \tau_{\text{ess}}(\mathcal{L}_n) \leq r', \mathcal{L}_n\) has a simple, unique maximal eigenvalue \(\lambda_n\) and one can decompose \(\mathcal{L}_n = \lambda_n P_n + N_n\) in the form of (1) satisfying additionally

(1) \(|\lambda_n - 1| \leq \eta, \)
(2) \(\int |P_n(f) - \int f \, d\mu| \, d\mu \leq K \mu(A_n)^\eta \| f \| \) for any \(f \in L, \)
(3) \(\|P_n(f)\| \leq K \int |P_n(f)| \, d\mu \) for any \(f \in L, \)
(4) \(\|N_n^k\| \leq K(1 - \delta)^k\) for any \(k \in \mathbb{N}. \)

Then one has, since \(\mathcal{L}_n^n(1) = \mathcal{L}_n^n(e^{-sS_{n,n}}), \)

\[\int e^{-sS_{n,n}} \, d\mu = \int \mathcal{L}_n^n(1) \, d\mu = \lambda_n^n \int P_n(1) \, d\mu + \int N_n^n(1) \, d\mu, \]

hence

\[\lim_{n \to \infty} \int e^{-sS_{n,n}} \, d\mu = \lim_{n \to \infty} \lambda_n^n \]

whenever the right-hand limit exists. The existence of the limit is implied by the following lemma.

**Lemma 4.2:**

\[\lim_{n \to \infty} \frac{1 - \lambda_n}{(1 - e^{-s}) \mu(A_n)} = 1. \]

**Proof:** Suppose \(n \geq n_0.\) Let \(f_n\) be an eigenfunction of \(\mathcal{L}_n\) with respect to \(\lambda_n,\) then \(P_n(f_n) = f_n\) and \(N_n(f_n) = 0.\) Note that \(\int f_n \, d\mu \neq 0,\) otherwise using the above properties (2) and (3) about \(P_n,\)

\[\|f_n\| = \|P_n f_n\| \leq K \int |P_n(f_n)| \, d\mu = K \int \left| P_n(f_n) - \int f_n \, d\mu \right| \, d\mu \leq K^2 \mu(A_n)^\eta \|f_n\|. \]

This yields a contradiction when \(n \to \infty.\) So we can assume that \(\int f_n \, d\mu = 1.\) Repeat the previous argument to see that

\[(1 - K^2 \mu(A_n)^\eta) \|f_n\| \leq K. \]

Hence we can further assume that \(\|f_n\|\) is uniformly bounded, say \(\|f_n\| \leq C\) for all \(n.\) For every \(k \in \mathbb{N},\)

\[1 - \lambda_n^k = (1 - \lambda_n^k) \int f_n \, d\mu = \int (f_n - \mathcal{L}_n^k(f_n)) \, d\mu = \int (1 - e^{-sS_{n,k}}) f_n \, d\mu. \quad (2) \]
When \( k = 1 \) this equation says
\[
1 - \lambda_n = (1 - e^{-s}) \int_{A_n} f_n \, d\mu,
\]
so
\[
|\lambda_n| = \left| 1 - (1 - e^{-s}) \int_{A_n} f_n \, d\mu \right| \geq 1 - (1 - e^{-s}) C \mu(A_n). \tag{3}
\]
In (2) replace \( k \) by \( k + 1 \) to get
\[
1 - \lambda_{n}^{k+1} = \int (1 - e^{-s_{n,k+1}}) f_n \, d\mu
\]
\[
= \int_{(T^{-k}A_n)^c} (1 - e^{-s_{n,k}}) f_n \, d\mu + \int_{T^{-k}A_n} (1 - e^{-s_{n,k}}) f_n \, d\mu
\]
\[
+ \int_{T^{-k}A_n} (e^{-s_{n,k}} - e^{-s} e^{-s_{n,k}}) f_n \, d\mu
\]
\[
= \int (1 - e^{-s_{n,k}}) f_n \, d\mu + (1 - e^{-s}) \int_{T^{-k}A_n} e^{-s_{n,k}} f_n \, d\mu
\]
\[
= 1 - \lambda_n^k + (1 - e^{-s}) \int_{T^{-k}A_n} e^{-s_{n,k}} f_n \, d\mu.
\]
Therefore,
\[
\lambda_n^k (1 - \lambda_n) = (1 - e^{-s}) \int e^{-s_{n,k}} f_n \cdot 1_{A_n} \circ T^k \, d\mu
\]
\[
= (1 - e^{-s}) \int \mathcal{L}^k (e^{-s_{n,k}} f_n) \cdot 1_{A_n} \, d\mu
\]
\[
= (1 - e^{-s}) \int_{A_n} \left( \int e^{-s_{n,k}} f_n \, d\mu + N^k (e^{-s_{n,k}} f_n) \right) \, d\mu
\]
\[
= (1 - e^{-s}) \left( \mu(A_n) \int_{A_n} ^k (f_n) \, d\mu + \int_{A_n} N^k (e^{-s_{n,k}} f_n) \, d\mu \right)
\]
\[
= (1 - e^{-s}) \left( \mu(A_n) \lambda_n^k + \int_{A_n} N^k (e^{-s_{n,k}} f_n) \, d\mu \right).
\]
We obtain that for every \( k \in \mathbb{N} \)
\[
\frac{1 - \lambda_n}{(1 - e^{-s}) \mu(A_n)} = 1 + \frac{1}{\mu(A_n) \lambda_n^k} \int_{A_n} N^k (e^{-s_{n,k}} f_n) \, d\mu. \tag{4}
\]
In view of (3), we can choose \( k = k(n) \) so that \( k(n) \to \infty \) as \( n \to \infty \) and that \( |\lambda_n|^k > \frac{1}{3} \) for large \( n \). Moreover, choose \( \ell = \ell(n) \in \mathbb{N} < k(n) \) so that \( \ell(n) \to \infty \) and \( \ell \mu(A_n) \to 0 \) when \( n \to \infty \). Now the continued-fraction mixing property (Proposition 2.5) and the
assumption $A_n \in \sigma(\alpha_0^{m-1})$ imply that there is a constant $C_3 > 0$ such that

$$\sum_{i=m+1}^{\ell} \mu(A_n \cap T^{-i}A_n) \leq C_3(\ell - m)\mu(A_n)^2.$$ 

Together with the assumption (b) $\mu(A_n \cap T^{-i}A_n) = o(\mu(A_n))$ for all $i \in \mathbb{N}$, we get that

$$\lim_{n \to \infty} \frac{1}{\mu(A_n)} \sum_{i=1}^{\ell} \mu(A_n \cap T^{-i}A_n) = 0.$$ 

Back to (4), by choice of $k$, it suffices to show

$$\int_{A_n} N^k(e^{-s_{n,k}f_n}) \, d\mu = o(\mu(A_n)).$$ 

Separate it into two parts by

$$e^{-s_{n,k}} = e^{-s_{n,k-\ell}} + e^{-s_{n,k-\ell}}(e^{-s_{n,\ell}}T^{k-\ell} - 1).$$ 

The first part

$$\left| \int_{A_n} N^k(e^{-s_{n,k-\ell}f_n}) \, d\mu \right| \leq K(1 - \delta)^{\ell} \mu(A_n)\|N^k(e^{-s_{n,k-\ell}f_n})\|$$

$$= K(1 - \delta)^{\ell} \mu(A_n) \left\| L^k(e^{-s_{n,k-\ell}f_n}) - \int e^{-s_{n,k-\ell}f_n} \, d\mu \right\|$$

$$= K(1 - \delta)^{\ell} \mu(A_n) \left\| L^k_{n}(f_n) - \int e^{-s_{n,k-\ell}f_n} \, d\mu \right\|$$

$$\leq K(1 - \delta)^{\ell}(C_1 + C_2r^{-m} + 1)C\mu(A_n) = o(\mu(A_n)).$$

The second part, letting $g := e^{-s_{n,k-\ell}}(e^{-s_{n,\ell}}T^{k-\ell} - 1)$,

$$\left| \int_{A_n} N^k(gf_n) \, d\mu \right| = \left| \int_{A_n} L^k(gf_n) \, d\mu - \mu(A_n) \int gf_n \, d\mu \right|$$

$$= \left| \int gf_n \cdot 1_{A_n} \circ T^k \, d\mu - \mu(A_n) \int gf_n \, d\mu \right|$$

$$\leq 2\|f_n\|_{\infty} \mu(\{S_n,\ell \circ T^{k-\ell} \neq 0\} \cap T^{-k}A_n) + 2\mu(A_n)\|f_n\|_{\infty} \mu(\{S_n,\ell \circ T^{k-\ell} \neq 0\})$$

$$\leq 2C\mu \left( T^{-k}A_n \cap \bigcup_{i=k-\ell}^{k-1} T^{-i}A_n \right) + 2C\mu(A_n)\mu \left( \bigcup_{i=k-\ell}^{k-1} T^{-i}A_n \right)$$

$$\leq 2C \sum_{i=1}^{\ell} \mu(A_n \cap T^{-i}A_n) + 2C(\mu(A_n))^2 = o(\mu(A_n)).$$

Hence we get the desired limit.
Back to the proof of Theorem 4.1. As a byproduct of Lemma 4.2, \( |\lambda_n| \leq 1 \) for large \( n \). Then,
\[
|\lambda_n^n - (1 - (1 - e^{-s})\mu(A_n))^n| \leq n \cdot |\lambda_n - 1 + (1 - e^{-s})\mu(A_n)|.
\]
Because \( n\mu(A_n) \to t \) and because of the lemma, the right-hand side converges to 0, then
\[
\lim_{n \to \infty} \lambda_n^n = \lim_{n \to \infty} (1 - (1 - e^{-s})\mu(A_n))^n = e^{-t(1-e^{-s})}.
\]

In the following situations, assumption (b) can be easily verified.

**Corollary 4.3:** The Poisson limit theorem holds when (1) \( A_n \in \sigma(\alpha) \) for all \( n \in \mathbb{N} \), or (2) for some \( m \in \mathbb{N} \), \( A_n \in \alpha_{m-1}^0 \) for all \( n \in \mathbb{N} \).

**Proof:** In the first situation, assumption (b) can be deduced from continued-fraction mixing. In the second situation, assumption (b) can be deduced from Corollary 2.8.

**Remark 4.4:** Here is an example when \( A_n \) satisfies assumption (a) but not assumption (b). Consider the continued fraction map. Let \( A_n = [1, n] \cup [n, 1] \in \sigma(\alpha_1^0) \). That is, \( A_n = \{ x \in (0, 1) : \text{either } a_1 = 1, a_2 = n, \text{or } a_1 = n, a_2 = 1. \} \). Then \( \mu(A_n) = O(n^{-2}) \). But \( A_n \cap T^{-1}A_n = [1, n, 1] \cup [n, 1, n] \) has measure of the order \( O(n^{-2}) \) too.

**Remark 4.5:** Applying Theorem 3.2 to \( \tilde{L}_n(f) := L(1_{A_n^c} \cdot f) \), one can again decompose \( \tilde{L}_n \) likewise with a leading eigenvalue \( \tilde{\lambda}_n \). Keller and Liverani’s formula in [29] for escape rate gives
\[
\lim_{n \to \infty} \frac{1 - \tilde{\lambda}_n}{\mu(A_n)} = 1.
\]
This formula also holds when applying Theorem 3.2 to more general cases, which require that \( \|1_{A_n}\| \) is uniformly bounded for all \( n \), like in assumption (a), and that only a small portion of \( \{A_n\} \) will return in short times, like in assumption (b). Consequently, it confirms that the distribution of the first return time tends to an exponential distribution. Note that continued-fraction mixing (or \( \psi \)-mixing) is not necessary in this line of proof, whereas a direct calculation of escape rate assuming \( \psi \)-mixing but not necessarily spectral gap can be found in [36].

For \( L_n \) considered in this note (suited to study successive return times), the same formula in [29] implies that
\[
\lim_{n \to \infty} \frac{1 - \lambda_n}{(1 - e^{-s})\mu(A_n)} = 1 - \sum_{k=0}^{\infty} q_k,
\]
where
\[
q_k = \lim_{n \to \infty} \frac{1}{(1 - e^{-s})\mu(A_n)} \int (L - L_n) L_n^k (L - L_n)(1) \, d\mu = \lim_{n \to \infty} \frac{1}{\mu(A_n)} \int e^{-s_{n,k}} \cdot (2 \cdot 1_{A_n} - 1_{T^{-1}A_n}) \circ T^k \, d\mu.
\]
It is likely that one can go on to obtain the same Poisson limit theorem by arguments similar to the second half of the proof of Lemma 4.2.

Lastly, we state some applications of Theorem 4.1 to continued fractions. Let \((0, 1), \mu, T, \alpha\) be the continued fraction map, where \(d\mu = \frac{1}{\log 2} \frac{1}{1+x} \, dx\), \(T(x) = \{1/x\}\), and \(\alpha = \{[a_i = n] : n \in \mathbb{N}\}\). As seen in Example 2.3, it is a probability-preserving, topologically mixing Gibbs–Markov map. Fix any real number \(\theta > 0\). By applying Corollary 4.3(1) to \(A_n = (0, (\theta n + 1)^{-1}) \in \sigma(\alpha)\), we recover Doeblin’s theorem in the introduction. Note that as mentioned in [27] the Gauss measure can be replaced by the Lebesgue measure, because \(|\mu(A) - \text{Leb}(A)| \leq \epsilon_m\) for \(A \in \mathcal{A}^\infty(\alpha)\), where \(\epsilon_m \to 0\) as \(m \to \infty\).

Choosing \(A_n \in \mathcal{A}^m(\alpha)\) such that \(n\mu(A_n)\) converges, one can find by Corollary 4.3 (2) limit laws for the number of occurrences of pattern \(A_n\). For example, if one chooses \(A_n = [[n^{1/4}], [n^{1/4}]] \in \mathcal{A}_0^\infty\), then

**Corollary 4.6:** For every \(m, k \in \mathbb{N}_0\)

\[
\lim_{n \to \infty} \text{Leb}\{ x \in (0, 1) : \text{there are exactly } k m\text{-tuples } a_i a_{i+1} \ldots a_{i+m-1} \text{ with all } a_i, \ldots, a_{i+m-1} > (\theta n)^{1/m}, 1 \leq i \leq n \} = \frac{1}{(\theta \log 2)^k} \frac{1}{k!} e^{-1/(\theta \log 2)}.
\]

Choosing \(A_n \in \mathcal{A}_0^{m-1}\) such that \(n\mu(A_n)\) converges, one can find by Corollary 4.3 (2) limit laws for the number of occurrences of pattern \(A_n\). For example, if one chooses \(A_n = [n^{1/4}], [n^{1/4}] \in \mathcal{A}_0^2\), then

**Corollary 4.7:** For every \(k \in \mathbb{N}_0\)

\[
\lim_{n \to \infty} \text{Leb}\{ x \in (0, 1) : \text{there are } k \text{ occurrences of } [n^{1/4}] [n^{1/4}] \text{ in } [a_1, \ldots, a_{n+1}] \} = \frac{1}{(\log 2)^k} \frac{1}{k!} e^{-1/\log 2}.
\]

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**References**

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, Mathematical Surveys and Monographs, Vol. 50, American Mathematical Society, Providence, RI, 1997.
[2] J. Aaronson and M. Denker, *The Poincaré series of C \ Z*, Ergodic Theory Dyn. Syst. 19(1) (1999), pp. 1–20.

[3] J. Aaronson and M. Denker, *Local limit theorems for partial sums of stationary sequences generated by Gibbs–Markov maps*, Stoch. Dyn. 1(2) (2001), pp. 193–237.

[4] J. Aaronson, M. Denker, and Mariusz Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Am. Math. Soc. 337(2) (1993), pp. 495–548.

[5] M. Abadi and N. Vergne, *Sharp errors for point-wise Poisson approximations in mixing processes*, Nonlinearity 21(12) (2008), pp. 2871–2885.

[6] H. Bruin, M.F. Demers, and M. Todd, *Hitting and escaping statistics: Mixing, targets and holes*, Adv. Math. 328 (2018), pp. 1263–1298.

[7] V. Chaumoître and M. Kupsa, *k-limit laws for return and hitting times*, Discrete Contin. Dyn. Syst. 15(1) (2006), pp. 73–86.

[8] J.-R. Chazottes and P. Collet, *Poisson approximation for the number of visits to balls in non-uniformly hyperbolic dynamical systems*, Ergodic Theory Dyn. Syst. 33(1) (2013), pp. 49–80.

[9] Z. Coelho and P. Collet, *Asymptotic limit law for the close approach of two trajectories in expanding maps of the circle*, Probab. Theory Relat. Fields 99(2) (1994), pp. 237–250.

[10] I.P. Cornfeld, S.V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Graduate Texts in Mathematics, Vol. 245, Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinski.

[11] M.F. Demers and P. Wright, *Behaviour of the escape rate function in hyperbolic dynamical systems*, Nonlinearity 25(7) (2012), pp. 2133–2150.

[12] M. Denker, *Remarks on weak limit laws for fractal sets*, in Fractal Geometry and Stochastics (Finsterbergen, 1994), Christoph Bandt, Siegfried Graf and Martina Zähle eds., Progr. Probab., Vol. 37, Birkhäuser, Basel, 1995, pp. 167–178.

[13] M. Denker and M. Urbański, *Ergodic theory of equilibrium states for rational maps*, Nonlinearity 4(1) (1991), pp. 103–134.

[14] M. Denker, M. Gordin, and A. Sharova, *A Poisson limit theorem for toral automorphisms*, Illinois J. Math. 48(1) (2004), pp. 1–20.

[15] W. Doeblin, *Remarques sur la théorie métrique des fractions continues*, Compos. Math. 7 (1940), pp. 353–371.

[16] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Trans. Am. Math. Soc. 356(4) (2004), pp. 1637–1689 (electronic).

[17] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 1, 3rd ed., John Wiley & Sons Inc., New York, 1968.

[18] A. Fergusson and M. Pollicott, *Escape rates for Gibbs measures*, Ergodic Theory Dyn. Syst. 32(3) (2012), pp. 961–988.

[19] N. Haydn, *Statistical properties of equilibrium states for rational maps*, Ergodic Theor. Dyn. Syst. 20(5) (2000), pp. 1371–1390.

[20] N.T.A. Haydn, *Entry and return times distribution*, Dyn. Syst. 28(3) (2013), pp. 333–353.

[21] N.T.A. Haydn and Y. Psiloyenis, *Return times distribution for Markov towers with decay of correlations*, Nonlinearity 27(6) (2014), pp. 1323–1349.

[22] N. Haydn and S. Vaienti, *The limiting distribution and error terms for return times of dynamical systems*, Discrete Contin. Dyn. Syst. 10(3) (2004), pp. 589–616.

[23] N. Haydn and S. Vaienti, *The compound Poisson distribution and return times in dynamical systems*, Probab. Theory Relat. Fields 144(3–4) (2009), pp. 517–542.

[24] N. Haydn and F. Yang, *Entry times distribution for mixing systems*, J. Stat. Phys. 163(2) (2016), pp. 374–392.

[25] M. Hirata, *Poisson law for Axiom A diffeomorphisms*, Ergodic Theory Dyn. Syst. 13(3) (1993), pp. 533–556.

[26] M. Hirata, B. Saussol, and Sandro Vaienti, *Statistics of return times: A general framework and new applications*, Commun. Math. Phys. 206(1) (1999), pp. 33–55.

[27] M. Iosifescu, *A Poisson law for $\psi$-mixing sequences establishing the truth of a Doeblin's statement*, Rev. Roumaine Math. Pures Appl. 22(10) (1977), pp. 1441–1447.
[28] G. Keller and C. Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28(1) (1999), pp. 141–152.

[29] G. Keller and C. Liverani, *Rare events, escape rates and quasistationarity: Some exact formulae*, J. Stat. Phys. 135(3) (2009), pp. 519–534.

[30] Y. Lacroix, *Possible limit laws for entrance times of an ergodic aperiodic dynamical system*, Israel J. Math. 132 (2002), pp. 253–263.

[31] F. Pène and B. Saussol, *Poisson law for some non-uniformly hyperbolic dynamical systems with polynomial rate of mixing*, Ergodic Theory Dyn. Syst. 36(8) (2016), pp. 2602–2626.

[32] B. Pitskel’, *Poisson limit law for Markov chains*, Ergodic Theory Dyn. Syst. 11(3) (1991), pp. 501–513.

[33] Ya. G. Sinai, *Some mathematical problems in the theory of quantum chaos*, Physics A 163(1) (1990), pp. 197–204. Statistical physics (Rio de Janeiro, 1989).

[34] D. Sullivan, *Conformal dynamical systems*, in Geometric Dynamics (Rio de Janeiro, 1981), J. Palis, eds., Lecture Notes in Mathematics, Vol. 1007, Springer, Berlin, 1983, pp. 725–752.

[35] X. Zhang, *Studies on the weak convergence of partial sums in Gibbs–Markov dynamical systems*, PhD thesis, The Pennsylvania State University, 2015.

[36] X. Zhang, *Note on limit distribution of normalized return times and escape rate*, Stoch. Dyn. 16(3) (2016), p. 1660014.

[37] R. Zweimüller, *The general asymptotic return-time process*, Israel J. Math. 212(1) (2016), pp. 1–36.