The white wall, a gravitational mirror

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Abstract. We describe the exact solution of Einstein’s equation corresponding
to a static homogenous distribution of matter with plane symmetry lying below
\( z = 0 \). We study the geodesics in it and we show that this simple spacetime
exhibits very curious properties. In particular, it has a repelling singular boundary
and all geodesics bounce off it.

1. Introduction

Given the complexity of Einstein’s field equations, one cannot find exact solutions
except in spaces of rather high symmetry, but very often with no direct physical
application. Nevertheless, exact solutions can give an idea of the qualitative features
that could arise in General Relativity and so, of possible properties of realistic solutions
of the field equations.

In this paper we want to illustrate some curious features of gravitation by means
of a very simple solution: the gravitational field of a static homogenous distribution
of matter with plane symmetry lying below \( z = 0 \).

Of course, the Newtonian exterior (i.e., \( z > 0 \)) solution \( g = -g e_z \) is very simple:
a static uniform gravitational field pointing in the negative \( z \) direction.

This uniform gravitational field has played an outstanding role in the History
of Physics and of Mankind, being perhaps the most important physical entity in our
daily life, since it is the field that we always feel and to which evolution has made us
perfectly adapted.

It is not by chance that Galileo started his studies of movement laws by analyzing
the way bodies move in its presence, or that Einstein did explicit use of it in his crucial
Equivalence Principle in 1911, the cornerstone of his wonderful travel from flat to
curved space-time.

In spite of this, the exact solution of Einstein’s field equations representing it, i.e.,
a full general relativistic (plane) homogenous gravitational field in the vacuum, seems
not to have been properly spread into the wide physics community. In fact, we haven’t
found it mentioned in any book on General Relativity. However, this simple vacuum
solution has been known for several years. It appears in Taub’s study of Mach’s
Principle in General Relativity [1], relativistic perfect fluids with plane symmetry [2],
or in the study of domain walls in the early Universe [3].

This Taub’s plane vacuum solution turns out to be the exterior gravitational field
(i.e., \( z > 0 \)) of the problem we are considering here: the solution of Einstein’s equation
corresponding to a static homogenous distribution of matter with plane symmetry lying below $z = 0$. For these reasons and for the sake of completeness, we present the solution in Sec. II, and a detailed study of the geodesics in it in Sec. III.

This very simple spacetime turns out to present some somehow astonishing properties. Namely, the attraction of the infinite amount of matter lying below $z = 0$ shrinks the space-time in such a way that it finishes high above at a singular boundary.

Throughout this paper, we adopt the convention in which the space-time metric has signature $(- + + +)$, the system of units in which the speed of light $c = 1$, and $g$ will denote gravitational field and not the determinant of the metric.

### 2. The metric

We start by determining the solution of Einstein’s equation corresponding to the exterior gravitational field of a static and homogenous distribution of matter with plane symmetry sitting somewhere at $z < 0$. That is, it should be invariant under translations in the plane and under rotations around its normal.

Since the solution should be static, we can find a time coordinate $t$ such that all metric’s components are $t$-independent, and in addition $g_{tt} = 0$ for $i = 1, 2, 3$. By the plane translation symmetry, the metric can depend only on the variable $z$, so

$$ds^2 = -A(z) dt^2 + h_{ij}(z) dx^i dx^j,$$

where $i, j = 1, 2, 3$ and $x^3 = z$. Furthermore, the rotation invariance imposes $h_{11} = h_{22}$ and $h_{12} = h_{21} = 0$. Moreover, by a change of the plane coordinates of the form $x \to x + f(z)$, and $y \to y + g(z)$ we can make $h_{13} = h_{23} = 0$. So, we can choose coordinates $x, y$ on the plane in such a way that $g_{x y} = g_{x z} = g_{y z} = 0$, and $g_{x x} = g_{y y}$.

Thus, we have found coordinates $(t, x, y, z)$ such that

$$ds^2 = -A(z) dt^2 + B(z) (dx^2 + dy^2) + dz^2,$$

where, without loss of generality, we have chosen $g_{zz} = 1$, so that the coordinate $z$ turns out to be the physical distance to the plane $z = 0$.

The non identically zero components of the Einstein tensor are

$$G_{tt} = \frac{A(z) \left( B'(z)^2 - 4 B(z) B''(z) \right)}{4 B(z)^2},$$

$$G_{xx} = G_{yy} = -\frac{A(z)^2 B'(z)^2 + B(z)^2 A'(z)^2}{4 A(z)^2 B(z)} + \frac{2 B(z) A''(z) + A'(z) B'(z) + 2 A(z) B''(z)}{4 A(z)},$$

$$G_{zz} = \frac{B'(z) \left( 2 B(z) A'(z) + A(z) B'(z) \right)}{4 A(z) B(z)^2}.$$

The vacuum Einstein equation, i.e., $G_{ab} = 0$, can be readily solved for $z > 0$. In fact, as the metric depends only on one variable $z$, it corresponds to a Kasner solution with exponents $\{-1/3, 2/3, 2/3\}$ (see, for example, [1]).

The general solution is

$$A(z) = C_1 \left( C_3 + z \right)^{-\frac{2}{3}}, \quad \text{and} \quad B(z) = C_2 \left( C_3 + z \right)^{\frac{2}{3}}$$

(6)
for arbitrary constants $C_1$, $C_2$ and $C_3$. Now, we fix the constants by demanding in a neighborhood of the plane $z = 0$
\[ g_{tt} = -(1 + 2\Phi(z)) = -(1 + 2gz) \] for $|gz| \ll 1$. (7)
Thus, for $z > 0$ the metric reads
\[ ds^2 = -\frac{1}{(1 - 3gz)^{3/2}}dt^2 + (1 - 3gz)^{4/3}(dx^2 + dy^2) + dz^2, \] (8)
this is the Taub’s plane vacuum solution [1]. Notice that the coordinates change
\[ \hat{z} = (1 - (1 - 3gz)^{4/3})/4g \] brings the metric [8] to the form used in references [1,2,3]
\[ ds^2 = (-dt^2 + dz^2)/(1 - 4\hat{z})^{1/2} + (1 - 4\hat{z})(dx^2 + dy^2). \] (9)

By matching the solution of Einstein’s equation for the interior of the matter (i.e., $z < 0$) with the exterior one Eq.[8], we can compute the constant $g$ from the matter properties [2]. But, apart from the above comment, matter underground is not playing any relevant role here, because the outside solution will always be the one given in Eq.[8].

Thus, this metric is the unique exact solution of vacuum Einstein’s equations satisfying the required plane symmetry up to a coordinate transformation. As it occurs with Schwarzschild’s case, it depends only on a length scale parameter $1/g$. The metric [8] becomes the Minkowskian one when $g = 0$. Otherwise, it represents a curved spacetime and it turns out to have a space-time curvature singularity on the “plane” $z = 1/3g$, since straightforward computation of the scalar quadratic in the Riemann tensor, yields
\[ R_{abcd}R^{abcd} = \frac{192g^4}{(1 - 3gz)^4}, \] (10)
where $a, b, c, d = 0, 1, 2, 3$.

Note that each spacelike slice of space-time $t = t_0$ and $z = z_0$ is a Euclidean plane with metrics
\[ dt^2 = (1 - 3gz)^{4/3}(dx^2 + dy^2), \] (11)
so its “size” contracts when $z$ increases and becomes a point at the singularity ($z = 1/3g$).

Notice that, beyond the singularity (i.e., $z > 1/3g$), a mirror copy of the empty space-time emerges. But, as we shall show below, no geodesic can go from one to the other.

In the next section we shall study the geodesics in this space time. For the sake of completeness, we shall consider the complete Taub plane spacetime, i.e., $-\infty < z < 1/3g$, disregarding the matter lying below $z = 0$. Of course, it should be understood than when a geodesic reaches the matter it is modified in some way, but here we are only interested in what happens in the vacuum ($0 < z < 1/3g$).

Notice that, the coordinates change $t' = (1 - 3gz)^{-1/3}t$, $x' = (1 - 3gz)^{2/3}x$, $y' = (1 - 3gz)^{2/3}y$, and $z' = z$ brings the vacuum metric [8] to the form
\[ ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2 + g \frac{2t'dt'dz' + 4x'dx'dz' + 4y'dy'dz'}{(1 - 3gz)} + g^2 \frac{4x'^2 + 4y'^2 - t'^2}{(1 - 3gz)^2} dz'^2, \] (12)
which shows that Taub’s plane symmetric spacetime is asymptotically flat when $z \to -\infty$. Of course, we see from [10] that it is not flat for finite values of $z$. 
3. The Geodesics

We want to study the geodesics in this space-time. Since the metric is independent of \( t, \ x \) and \( y \), the momentum covector components \( p_t, p_x \) and \( p_y \) are constant along the geodesics. For timelike geodesics, we choose \( \tau \) to be the proper time while for null ones, we choose \( \tau \) to be an affine parameter. So, we can write

\[
\left( \frac{dz}{d\tau} \right)^2 = \frac{(1 - 3gz)^{2/3} \tilde{E}^2 - \mu - \frac{(\tilde{p}_x^2 + \tilde{p}_y^2)}{(1 - 3gz)^{4/3}}}{(1 - 3gz)^{4/3}},
\]

(13)

\[
\frac{dt}{d\tau} = (1 - 3gz)^{2/3} \tilde{E},
\]

(14)

\[
\frac{dx}{d\tau} = (1 - 3gz)^{-4/3} \tilde{p}_x,
\]

(15)

\[
\frac{dy}{d\tau} = (1 - 3gz)^{-4/3} \tilde{p}_y,
\]

(16)

where \( \mu = 1, \tilde{E} = -\mu/m, \tilde{p}_x = p_x/m \) and \( \tilde{p}_y = p_y/m \) for timelike geodesics; and \( \mu = 0, \tilde{E} = -p_t, \tilde{p}_x = p_x \) and \( \tilde{p}_y = p_y \) for null ones.

The right hand side of (13) must be positive, so we see that there is a bouncing point for each geodesic. To get the \( z \) coordinate of such point, we solve \( (dz/d\tau)^2 = 0 \) in Eq. (13). In all cases, this yields a cubic equation with only one real root.

3.1. Null Geodesics

It follows from (13) that the maximal height that a null geodesic reaches is

\[
h_{\text{max}} = \frac{1}{3g} \left( 1 - \frac{\tilde{p}_h}{\tilde{E}} \right),
\]

(17)

where \( \tilde{p}_h = \sqrt{\tilde{p}_x^2 + \tilde{p}_y^2} \). Thus, nonvertical null geodesics bounce before getting to the singularity, whereas vertical null ones just touch it.

In this case, the geodesics equation can be integrated in a closed form. Indeed, from (13), (14) and (15) we obtain the equations governing these geodesics

\[
\frac{dt}{dz} = \pm \frac{(1 - 3gz)^{4/3}}{\sqrt{(1 - 3gz)^2 - (\tilde{p}_h/\tilde{E})^2}},
\]

(18)

\[
\frac{dx}{dz} = \pm \frac{(\tilde{p}_h/\tilde{E}) (1 - 3gz)^{-2/3}}{\sqrt{(1 - 3gz)^2 - (\tilde{p}_h/\tilde{E})^2}},
\]

(19)

\[
\frac{dy}{dz} = 0,
\]

(20)

where, without loss of generality, we have assumed \( \tilde{p}_y = 0 \). The solution of these equations can be written as

\[
|t - t_0| = \left( \frac{\tilde{p}_h}{\tilde{E}} \right)^{4/3} \frac{\sqrt{\pi} \Gamma(-2/3)}{6g \Gamma(-1/6)} + \frac{(1 - 3gz)^{4/3}}{4g} 2F_1 \left( \frac{-2}{3}, \frac{1}{3}, \frac{1}{3} \left| \frac{(\tilde{p}_h/\tilde{E})^2}{(1 - 3gz)^2} \right| \right),
\]

(21)

\[
|x - x_0| = \left( \frac{\tilde{p}_h}{\tilde{E}} \right)^{1/3} \frac{\sqrt{\pi} \Gamma(1/3)}{6g \Gamma(5/6)} - \frac{(\tilde{p}_h/\tilde{E})}{2g(1 - 3gz)^{2/3}} 2F_1 \left( \frac{1}{3}, \frac{1}{3}, \frac{4}{3} \left| \frac{(\tilde{p}_h/\tilde{E})^2}{(1 - 3gz)^2} \right| \right),
\]

(22)

\[y = y_0,\]

(23)
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where \( _2F_1(a, b; c; z) \) is the Gauss hypergeometric function (see for example [5]).

From (21) and (17) we readily see that all non-vertical null geodesics start and finish at \( z = -\infty \), and all of them reach a turning point at a finite distance from the singularity, and the smaller their horizontal momentum is, the closer they get near the singularity.

Now, let us consider two points on the plane \( z = 0 \) connected by a null geodesic, we readily get from (22) that the distance between them is

\[
\left( \frac{\tilde{p}_h}{\tilde{E}} \right)^{1/3} \frac{\sqrt{\pi} \Gamma(1/3)}{3g \Gamma(5/6)} - \frac{\left( \frac{\tilde{p}_h}{\tilde{E}} \right)}{g} _2F_1 \left( \frac{1}{3}, \frac{1}{2}, \frac{4}{3}; \left( \frac{\tilde{p}_h}{\tilde{E}} \right)^2 \right). \tag{24}
\]

This distance vanishes for \( \tilde{p}_h/\tilde{E} = 0 \) and for \( \tilde{p}_h/\tilde{E} = 1 \), furthermore we can easily compute that, for \( \tilde{p}_h/\tilde{E} = 0.302684 \ldots \), it reaches a maximum

\[
R = \frac{0.635164 \ldots}{g}. \tag{25}
\]

Therefore an observer standing on the plane can only see the points of the plane contained in a circle of radius \( R \). Of course, a similar result holds for any plane \( z = \text{constant} \).

In Fig.1 we show the null geodesics that can reach a point \( O \). The region of the space that can be seen by an observer at rest at the point \( O \) is bounded by a “cupola”, and he cannot see anything that is beyond it. Moreover, any point \( P \) in this region can be seen from \( O \) by looking at two different directions. But, since both rays of light last different times to travel from \( P \) to \( O \), the observer sees what happens at \( P \) at two different times. For instance, the observer can see his neighborhood “at present” by looking horizontally, whereas he see what happened there “1/2g ago” by looking vertically.

![Figure 1. Null geodesics that reach the the point O.](image)

When \( \tilde{p} \to 0 \) we can get from (21) and (22) the limit geodesic for vertically moving photons

\[
|t - t_0| = \frac{1}{4g} (1 - 3gz)^{1/4}, \quad x = x_0 \quad \text{and} \quad y = y_0. \tag{26}
\]

Thus, we see that a photon moving upwards crossing the plane \( z = 0 \) at coordinate time \( t = t_0 - 1/4g \) reaches the singularity at time \( t_0 \), comes back to \( z = 0 \) at \( t_0 + 1/4g \),
and afterwards it follows its travel to \( z = -\infty \). Thus, every null geodesic traveling upwards is reflected at the singularity.

Therefore, we can fill the whole space-time with never-stopping future-oriented null geodesics, all of them starting and finishing at \( z = -\infty \).

Now, consider a local Galilean observer momentarily at rest at a height \( z \) measuring the energy of a photon with \( p_t = -\hat{E} \). Since the time component of the observer’s velocity must be \( U^t_{\text{obs.}} = (1 - 3gz)^{1/3} \), for \( g_{ab} U^a_{\text{obs.}} U^b_{\text{obs.}} = -1 \), the energy he measures is \(-U^a_{\text{obs.}} p_a = -U^t_{\text{obs.}} p_t = (1 - 3gz)^{1/3} \hat{E} \). Since \( g_{tt} = -1 \) at \( z = 0 \), the constant of motion \( p_t = -\hat{E} \) is the energy that an observer at \( z = 0 \) would measure. So, if a photon is emitted at height \( h \) and received at \( z = 0 \), we have

\[
\nu_{\text{rec}} = \frac{\nu_{\text{em}}}{(1 - 3gh)^{1/3}},
\]

and if the photon were emitted at the singularity, the frequency record would be infinitely shifted to the blue.

On the other hand, for “terrestrial” (i.e., \( gz \ll 1 \)) experiments, the exact result in Eq. (27) becomes the classical Einstein prediction

\[
\nu_{\text{rec}} \simeq \nu_{\text{em}}(1 + gh),
\]

measured by Pound, Rebka and Snider [6].

3.2. Timelike Geodesics

Clearly, Eq. (13) shows that no massive particle can reach the singularity. For timelike geodesics, the general expression for the maximal height reached can be explicitly written down, but it turns out to be rather involved. We show it here only for the sake of completeness

\[
h_{\text{max}} = \frac{1}{3g} - \frac{1}{9\sqrt{3}g \hat{E}^3} \left( 1 + \frac{1 + \sqrt{1 + 27 \hat{E}^4 \hat{p}^2 / 2 + 3 \hat{E}^2 \sqrt{3\hat{p}^2 + 81 \hat{E}^4 \hat{p}^4 / 4}}{2} \right)^{3/2} \sqrt{1 + 27 \hat{E}^4 \hat{p}^2 / 2 + 3 \hat{E}^2 \sqrt{3\hat{p}^2 + 81 \hat{E}^4 \hat{p}^4 / 4}}.}
\]

(29)

So, for the sake of clarity, we shall discuss two simple cases:

For particles moving vertically (i.e. \( \hat{p}_x = \hat{p}_y = 0 \)), equation (29) reduces to

\[
h_{\text{max}} = \frac{1}{3g} \left( 1 - \frac{1}{\hat{E}^3} \right).
\]

(30)

In this case, from (13) and (14) we obtain the equation governing these geodesics

\[
\frac{dt}{dz} = \pm \frac{(1 - 3gz)^{2/3}}{\sqrt{(1 - 3gz)^{2/3} - 1/\hat{E}^2}}
\]

(31)

The solution of this equation can be written as

\[
|t - t_0| = \frac{1}{4g\hat{E}^4} \left[ (1 - 3gz)^{1/3} \hat{E} \sqrt{(1 - 3gz)^{2/3} \hat{E}^2 - 1} - \left( (1 - 3gz)^{2/3} \hat{E}^2 + \frac{3}{2} \right) \right.
\]

\[
+ \frac{3}{2} \log \left( (1 - 3gz)^{1/3} \hat{E} + \sqrt{(1 - 3gz)^{2/3} \hat{E}^2 - 1} \right).}
\]

(32)
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Thus, since $|t| \to \infty$ when $z \to -\infty$, we see that vertical timelike geodesics start at $z = -\infty$, reach the highest point $z = h_{\text{max}}$ at coordinate time $t_0$, and after that they follow their return travel to $z = -\infty$. Notice that (32) readily shows how timelike geodesics tend to null ones (26) for high-energy particles (i.e., $\tilde{E} \gg 1$).

If we call $v(z)$ the (tri)velocity of the particle measured by a local inertial observer momentarily at rest at a height $z$, we have

$$U_{\text{obs}}^a U_a = g_{tt} U_{\text{obs}}^t \frac{dt}{d\tau} = -(1 - 3gz)^{1/3} \tilde{E} = \frac{-1}{\sqrt{1 - v(z)^2}}, \quad (33)$$

where $U_{\text{obs}}^a$ and $U^a = \frac{dx^a}{d\tau}$ are the velocities of the observer and the particle respectively. So, it holds

$$v(z) = \pm \sqrt{1 - \frac{1}{(1 - 3gz)^{2/3} \tilde{E}^2}}, \quad (34)$$

which shows that $|v(z)| \to 1$ when $z \to -\infty$.

For nonrelativistic particles, (i.e., $\tilde{p}_x^2 + \tilde{p}_y^2 < \tilde{p}_z^2 \ll 1$), equation (29) reduces to

$$h_{\text{max}} = \frac{1}{3g} \left( 1 - \frac{1}{\tilde{E}^3} - \frac{3}{2} \tilde{E}(\tilde{p}_x^2 + \tilde{p}_y^2) + O(\tilde{p})^4 \right). \quad (35)$$

Now, by calling $v_x, v_y, v_z$ the velocity components at the plane $z = 0$, we can write $\tilde{E} \simeq 1 + (v_x^2 + v_y^2 + v_z^2)/2, \tilde{p}_x \simeq v_x$ and $\tilde{p}_y \simeq v_y$, getting the Galilean result

$$h_{\text{max}} \simeq \frac{v_z^2}{2g}. \quad (36)$$

As all geodesics bounce, we see that the singularity at $z = 1/3g$ acts as a gravitational mirror, so we call it a white wall and we are able to think of it by removing the point $z = 1/3g$ as the boundary of the space-time.

Hence, the attraction of the infinite amount of matter lying below $z = 0$, shrinks the space-time in such a way that it finishes high above at the singular boundary.

Notice that, from the Newtonian point of view, we can think of a plane as a sphere with infinite radius. Here, in contrast, we cannot obtain the exact solution as a limit of Schwarzschild’s metric, since globally these space-times are completely different.

4. Concluding remarks

We have described the general external solution of Einstein’s equation produced by a static and homogenous distribution of matter with plane symmetry sitting somewhere at $z < 0$. The solution turns out to have a repelling boundary high above and all geodesics bounce off it. This singularity is not the source of the field, but it arises owing to the attraction of distant matter.

Of course, we don’t think there is place in our universe for an object with such a symmetry, but this simple example shows some peculiar properties that perhaps could help us understand relevant facts in Nature.

Regarding the interior solutions, since as long as they exist, the exterior solution inexorably is the Taub’s one, they don’t play a relevant role in our analysis. However, as already mentioned, given an equation of state $\rho = (p)$, by matching both solutions (Taub’s plane and the interior one, e.g., at $z = 0$) $g$ can be computed from the matter properties [2]. For instance, it can be shown that, for matter with constant
density $\rho_0 > 0$, it can be done, and the space-time also finishes down below at another singularity at a finite depth [7].

After we finished this work we learned about reference [8], where some of the results here presented can be found. However, these authors consider a very different space-time: the mirror-symmetric matching of two asymptotic flat Taub’s domains to the “left” and “right” of a planar shell with infinite negative mass density sitting at the singularity, whereas we are arguing here that Taub’s plane solution is also the external gravitational field of ordinary ($\rho$ and $p \geq 0$) matter sitting at negative values of $z$, and the singularity that arises high above is due to the attraction of the distant matter.

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