A DIRECT METHOD OF MOVING PLANES FOR A FULLY NONLINEAR NONLOCAL SYSTEM

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Abstract. In this paper we consider the system involving fully nonlinear nonlocal operators:

\[
\begin{align*}
\mathcal{F}_\alpha(u(x)) &= C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{F(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy = v^p(x) + k_1(x)u^r(x), \\
\mathcal{G}_\beta(v(x)) &= C_{n,\beta} \text{PV} \int_{\mathbb{R}^n} \frac{G(v(x) - v(y))}{|x - y|^{n+\beta}} \, dy = u^q(x) + k_2(x)v^s(x),
\end{align*}
\]

where \(0 < \alpha, \beta < 2\), \(p, q, r, s > 1\), \(k_1(x), k_2(x) \geq 0\).

A narrow region principle and a decay at infinity are established for carrying on the method of moving planes. Then we prove the radial symmetry and monotonicity for positive solutions to the nonlinear system in the whole space. Furthermore non-existence of positive solutions to the system on a half space is derived.

1. Introduction. In this paper, we consider the nonlinear system involving fully nonlinear nonlocal operators:

\[
\begin{align*}
\mathcal{F}_\alpha(u(x)) &= C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{F(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy, \\
\mathcal{G}_\beta(v(x)) &= C_{n,\beta} \text{PV} \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\beta}} \, dy,
\end{align*}
\]

where PV stands for the Cauchy principal value, \(F\) and \(G\) are at least local Lipschitz continuous, \(F(0) = 0, G(0) = 0\), \(0 < \alpha, \beta < 2\). For some constant \(M > 0\), \(\forall x \in \mathbb{R}^n\), we have \(0 \leq k_i(x) \leq M\) and \(k_i(x) = k_i(|x|)\), where \(|x| = (x_1^2 + \cdots + x_n^2)^{1/2}\). If \(x, y \in \mathbb{R}^n\) with \(|x| \leq |y|\), then \(k_i(x) \geq k_i(y), i = 1, 2\). The operator \(\mathcal{F}_\alpha\) or \(\mathcal{G}_\beta\) was introduced by Caffarelli and Silvestre in [6]. Here, we require

\[u \in C^{1,1}_{loc} \cap L_\alpha, \ v \in C^{1,1}_{loc} \cap L_\beta\]

with

\[L_\alpha = \{u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} \, dx < \infty\}\]

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and
\[ L_\beta = \{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\beta}} \, dx < \infty \}. \]

In the special case that \( F(\cdot) \) is an identity map, \( F_\alpha \) becomes the usual fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\). The nonlocal nature of fractional operators brings many new difficulties comparing with the Laplacian. To treat fractional operators, Caffarelli and Silvestre [5] raised the extension method which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions. This method has been applied successfully to deal with equations involving the fractional Laplacian and a series of fruitful results has been obtained (see [1, 12], etc.). One can also use the integral equations method, such as the method of moving planes in integral forms (see [3, 4, 26, 19, 18]) and regularity lifting to investigate equations involving fractional Laplacian by showing that they are equivalent to corresponding integral equations (see [11, 9, 10] and the references therein). For more articles concerning fractional Laplacian by showing that they are equivalent to corresponding integral forms, see [13, 15, 16, 19, 20, 21, 22, 23, 24] and the references therein.

For the fully nonlinear nonlocal equations, so far as we know, there is neither any corresponding extension method nor equivalent integral equations that one can work at. A probable reason is that very few results were obtained for the fully nonlinear nonlocal operator. In [8], Chen, Li and Li developed a new method that can probe the method of moving planes for nonlocal equations and integral equations, see [11, 9, 10] and the references therein. For more articles concerning the method of moving planes for nonlocal equations and integral equations, see [13, 15, 16, 19, 20, 21, 22, 23, 24] and the references therein.

For the fully nonlinear nonlocal systems and consider two nonlinear systems involving fully nonlinear nonlocal operators
\[
\begin{align*}
\mathcal{F}_\alpha(u(x)) &= v^p(x) + k_1(x)u^r(x), \\
G_\beta(v(x)) &= u^q(x) + k_2(x)v^s(x), & x \in \mathbb{R}^n, \\
u(x) > 0, v(x) > 0, & x \in \mathbb{R}^n, 
\end{align*}
\] (1.1)
and
\[
\begin{align*}
\mathcal{F}_\alpha(u(x)) &= v^p(x) + k_1(x)u^r(x), \\
G_\beta(v(x)) &= u^q(x) + k_2(x)v^s(x), & x \in \mathbb{R}^n, \\
u(x) \equiv 0, v(x) \equiv 0, & x \not\in \mathbb{R}^n. 
\end{align*}
\] (1.2)

The narrow region principle and decay at infinity for the systems are established, which play important roles in carrying out the method of moving planes. To state them, for \( \lambda \in R \), denote by
\[ T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \} \]
the moving plane, by
\[ \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \} \]
the left region of the plane \( T_\lambda \), by \( \bar{\Sigma}_\lambda \) the reflection of \( \Sigma_\lambda \) about \( T_\lambda \), by
\[ x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n) \]
the reflection of \( x \) about \( T_\lambda \), and denote
\[ \Sigma_\lambda^k = \mathbb{R}^n \setminus \Sigma_\lambda, \quad u_\lambda(x) = u(x^\lambda), \quad v_\lambda(x) = v(x^\lambda), \quad U_\lambda(x) = u_\lambda(x) - u(x) \]
and
\[ V_\lambda(x) = v_\lambda(x) - v(x). \]
For the simplicity of notations, we stand for \( U_\lambda(x) \) by \( U(x) \) and \( V_\lambda(x) \) by \( V(x) \) in the sequel. Throughout this and next section, we assume
\[ F, G \in C^1(R), \quad F(0) = 0, \quad G(0) = 0, \quad F'(t) \geq c_0 > 0, \quad \text{and} \quad G'(t) \geq c_1 > 0, \quad \forall \ t \in R. \]
(1.3)
Theorem 1.1 (Narrow Region Principle for system). Let $\Omega$ be a bounded narrow region in $\Sigma_\lambda$ contained in 
\[ \{x|\lambda - l < x_1 < \lambda\}\] 
with small $l > 0$. Suppose that $U(x) \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and $V(x) \in L_\beta \cap C^{1,1}_{loc}(\Omega)$ are lower semi-continuous on $\bar{\Omega}$, and satisfy
\[
\begin{cases}
\mathcal{F}_\alpha(u_\lambda(x)) - \mathcal{F}_\alpha(u(x)) + c_{11}(x)U(x) + c_{12}(x)V(x) \geq 0, \\
\mathcal{G}_\beta(v_\lambda(x)) - \mathcal{G}_\beta(v(x)) + c_{21}(x)U(x) + c_{22}(x)V(x) \geq 0, \\
U(x), V(x) \geq 0, \\
U(x^\lambda) = -U(x), V(x^\lambda) = -V(x),
\end{cases}
\]
(i) If $c_{ij}(x)$, $i,j = 1,2$, are bounded from below in $\Omega$, $c_{12}(x)$ and $c_{21}(x) < 0$, then we have for sufficiently small $l$,
\[ U(x), V(x) \geq 0 \text{ in } \Omega; \] (1.5)
(ii) if $\Omega$ is unbounded, the conclusion (1.5) still holds under the conditions
\[ \lim_{|x| \to \infty} U(x), \lim_{|x| \to \infty} V(x) \geq 0; \] (1.6)
(iii) furthermore, under the conditions of (i), if $U(x)$ or $V(x)$ attains 0 somewhere in $\Omega$, then
\[ U(x) = V(x) \equiv 0, \quad x \in R^n. \] (1.7)
Remark 1.1. If $U(x) > 0$ or $V(x) > 0$ at some point in $\Omega$, then it follows by (iii) that
\[ U(x) > 0, \quad V(x) > 0, \quad x \in \Omega. \]
We call (iii) the strong maximum principle later. As we can see from the proof, to conclude (1.7), $\Omega$ does not need to be narrow.

Theorem 1.2 (Decay at Infinity for system). Assume that $U(x) \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$, $V(x) \in L_\beta \cap C^{1,1}_{loc}(\Omega)$, $U(x)$ and $V(x)$ are lower semi-continuous on $\Omega$. If $U(x)$ and $V(x)$ satisfy
\[
\begin{cases}
\mathcal{F}_\alpha(u_\lambda(x)) - \mathcal{F}_\alpha(u(x)) + c_{11}(x)U(x) + c_{12}(x)V(x) \geq 0, \\
\mathcal{G}_\beta(v_\lambda(x)) - \mathcal{G}_\beta(v(x)) + c_{21}(x)U(x) + c_{22}(x)V(x) \geq 0, \\
U(x), V(x) \geq 0, \\
U(x^\lambda) = -U(x), V(x^\lambda) = -V(x),
\end{cases}
\]
with
\[ c_{11}(x), c_{12}(x) \sim o\left(\frac{1}{|x|^\alpha}\right), \quad c_{22}(x), c_{21}(x) \sim o\left(\frac{1}{|x|^\beta}\right), \quad \text{for } |x| \text{ large}, \] (1.9)
and
\[ c_{12}(x), \quad c_{21}(x) < 0, \]
then there exists a constant $R_0 > 0$ (depending only on $c_{ij}(x)$) such that if
\[ U(\tilde{x}) = \min_{\Omega} U(x) < 0, \quad V(\tilde{x}) = \min_{\Omega} V(x) < 0, \]
then
\[ |\tilde{x}| \leq R_0 \quad \text{or} \quad |\tilde{x}| \leq R_0. \] (1.10)
Remark 1.2. Since the $x_1$ direction can be chosen arbitrarily, $\Sigma_\lambda$ changes following the change of $x_1$ direction, so we have (1.10) holds.
Based on Theorems 1.1 and 1.2, we apply the method of moving planes to obtain symmetry and monotonicity of positive solutions to (1.1) in $R^n$, as well as non-existence of positive solutions to (1.2) on the half space.

**Theorem 1.3.** Assume that $u(x) \in L_\alpha(R^n) \cap C^{1,1}_{\text{loc}}(R^n)$ and $v(x) \in L_{\beta}(R^n) \cap C^{1,1}_{\text{loc}}(R^n)$ are positive solutions to system (1.1). Suppose that for some $\gamma, \tau > 0$,

$$v(x) = o\left(\frac{1}{|x|}\right), \quad u(x) = o\left(\frac{1}{|x|^{\tau}}\right), \quad \text{as } |x| \to \infty,$$

(1.11)

with

$$\alpha \leq \min\{(p-1)\gamma, (r-1)\tau\}, \quad \beta \leq \min\{(q-1)\gamma, (s-1)\tau\}. \quad (1.12)$$

Then $u(x)$ and $v(x)$ must be radially symmetric and monotone decreasing about the origin.

**Theorem 1.4.** Assume that $u(x) \in L_\alpha \cap C^{1,1}_{\text{loc}}(R^n_+)$, $v(x) \in L_{\beta} \cap C^{1,1}_{\text{loc}}(R^n_+)$ are nonnegative solutions to system (1.2) and $u, v$ are lower semi-continuous on $R^n_+$. Suppose

$$\lim_{|x| \to \infty} u(x) = 0, \quad \lim_{|x| \to \infty} v(x) = 0. \quad (1.13)$$

Then $u(x) \equiv 0, \quad v(x) \equiv 0$.

In Section 2, we prove Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4 by using the previous results and the method of moving planes.

2. **Proofs of Theorems 1.1 and 1.2.** We let

$$\mathcal{F}_\alpha(u(x)) = C_{n,\alpha}PV \int_{R^n} \frac{F(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{R^n \setminus B_\epsilon(x)} \frac{F(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy,$$

and use $c$ and $C$ for general various positive constants that are usually different in different contexts.

**Proof of Theorem 1.1.** Suppose that (1.5) does not hold, without loss of generality, we assume $U(x) < 0$ at some point in $\Omega$; then the lower semi-continuity of $U(x)$ on $\Omega$ implies that there exists $\tilde{x} \in \Omega$ such that

$$U(\tilde{x}) = \min_{\Omega} U(x) < 0,$$

and $\tilde{x}$ is in the interior of $\Omega$ from the condition (1.4). By the expression of $\mathcal{F}_\alpha$ and (2.2) in [25], we have

$$\mathcal{F}_\alpha(u_{\lambda}(\tilde{x})) - \mathcal{F}_\alpha(u(\tilde{x})) \leq 2C_{n,\alpha} cU(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} \, dy. \quad (2.1)$$

Then for $0 < r < \min\{l - \tilde{x}_1\}$, we have

$$\int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} \, dy \geq \int_{B_r(\tilde{x})} \frac{1}{|\tilde{x} - y|^{n+\alpha}} \, dy \geq \int_{B_r(\tilde{x})} \frac{1}{|\tilde{x} - y|^{n+\alpha}} \, dy \geq \frac{w_n}{3r^\alpha} \geq \frac{c}{l^\alpha}. \quad (2.2)$$

Using it into (2.1), it shows

$$\mathcal{F}_\alpha(u_{\lambda}(\tilde{x})) - \mathcal{F}_\alpha(u(\tilde{x})) \leq \frac{cU(\tilde{x})}{l^\alpha} < 0.$$
Together with (1.4), we have for \( l \) sufficiently small,
\[
\mathcal{F}_\alpha(u(\lambda \tilde{x})) - \mathcal{F}_\alpha(u(\tilde{x})) + c_{11}(\tilde{x})U(\tilde{x}) \leq \frac{c}{l^\alpha} + c_{11}(\tilde{x})U(\tilde{x}) \leq \frac{c}{2l^\alpha} U(\tilde{x}) < 0. \tag{2.3}
\]
It follows that
\[
U(\tilde{x}) \geq -cl_{12}(\tilde{x})l^\alpha V(\tilde{x}) \quad \text{and} \quad V(\tilde{x}) < 0. \tag{2.4}
\]
From (2.4), we know that there exists \( \tilde{x} \) such that
\[
V(\tilde{x}) = \min_{\Omega} V(x) < 0.
\]
Similarly to (2.3), we can derive that
\[
\mathcal{G}_\beta(u(\lambda \tilde{x})) - \mathcal{G}_\beta(u(\tilde{x})) + c_{22}(\tilde{x})V(\tilde{x}) \leq \frac{cV(\tilde{x})}{2l^\beta} < 0.
\]
Using (2.4), we have for \( l \) sufficiently small,
\[
0 \leq \mathcal{G}_\beta(u(\lambda \tilde{x})) - \mathcal{G}_\beta(v(\tilde{x})) + c_{21}(\tilde{x})U(\tilde{x}) + c_{22}(\tilde{x})V(\tilde{x})
\]
\[
\leq CV(\tilde{x}) + c_{21}(\tilde{x})U(\tilde{x})
\]
\[
\leq \frac{CV(\tilde{x})}{l^\beta} - c_{21}(\tilde{x})c_{12}(\tilde{x})l^\alpha V(\tilde{x})
\]
\[
\leq \frac{C V(\tilde{x})}{l^\beta}(1 - c_{12}(\tilde{x})c_{21}(\tilde{x})^{\alpha + \beta}) < 0.
\]
This contradiction shows that (1.5) must be true.

If \( \Omega \) is unbounded, then (1.5) is easily obtained by using (1.6).

To prove (1.7), without loss of generality, we suppose that there exists \( \eta \in \Omega \) such that
\[
U(\eta) = 0.
\]
Then \( \frac{1}{|x-y|} > \frac{1}{|x-y|^\alpha}, \forall x, y \in \Sigma_\lambda \) and
\[
\mathcal{F}_\alpha(u_\lambda(\eta)) - \mathcal{F}_\alpha(u(\eta)) + c_{11}(\eta)U(\eta)
\]
\[
= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{F(u_\lambda(\eta) - u_\lambda(y)) - F(u(\eta) - u(y))}{|\eta - y|^{n+\alpha}} \, dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{F(u_\lambda(\eta) - u_\lambda(y)) - F(u(\eta) - u(y))}{|\eta - y|^{n+\alpha}} \, dy
\]
\[
+ C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{F(u_\lambda(\eta) - u(y)) - F(u(\eta) - u(y))}{|\eta - y^\lambda|^{n+\alpha}} \, dy
\]
\[
= C_{n,\alpha} F'(\cdot) \int_{\Sigma_\lambda} \frac{(U(\eta) - U(y))(1 - \frac{1}{|\eta - y|^{n+\alpha}} - \frac{1}{|\eta - y^\lambda|^{n+\alpha}})}{2U(\eta) \, dy}
\]
\[
+ C_{n,\alpha} F'(\cdot) \int_{\Sigma_\lambda} \frac{2U(\eta)}{|\eta - y^\lambda|^{n+\alpha}} \, dy
\]
\[
\leq -Cc \int_{\Sigma_\lambda} U(y)(\frac{1}{|\eta - y|^{n+\alpha}} - \frac{1}{|\eta - y^\lambda|^{n+\alpha}}) \, dy.
\tag{2.5}
\]
If $U(x) \neq 0$, then (2.5) implies that
\[ F_\alpha(u_\lambda(\eta)) - F_\alpha(u(\eta)) + c_{11}(\eta)U(\eta) < 0.\]
Together with (1.4), it shows that $V(\eta) < 0$. This is a contradiction with (1.5).
Hence $U(x)$ must be identically 0 in $\Sigma_\lambda$. Since
\[ U(x^\lambda) = -U(x), \ x \in \Sigma_\lambda, \]
it shows that
\[ U(x) \equiv 0, \ x \in \mathbb{R}^n. \]
Again from the first equation of (1.4), we know that
\[ V(x) \leq 0, \ x \in \Sigma_\lambda. \]
Since we already know that
\[ V(x) \geq 0, \ x \in \Sigma_\lambda, \]
it must hold that
\[ V(x) = 0, \ x \in \Sigma_\lambda. \]
Combining it with $V(x^\lambda) = -V(x)$, we arrive at
\[ V(x) \equiv 0, \ x \in \mathbb{R}^n. \]
Similarly, one can show that if $V(x)$ attains 0 at one point in $\Sigma_\lambda$, then both $U(x)$ and $V(x)$ are identically 0 in $\mathbb{R}^n$. This completes the proof.

**Proof of Theorem 1.2.** By the assumptions, there exists $\tilde{x} \in \Omega$, such that
\[ U(\tilde{x}) = \min_{\Omega} U(x) < 0. \]
Using (2.1), we have
\[ F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) = C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{F(u_\lambda(\tilde{x}) - u(y)) - F(u(\tilde{x}) - u(y))}{|\tilde{x} - y|^{n+\alpha}} + c_{11}(\tilde{x}) U(\tilde{x}) \leq C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{F'(\cdot)2U(\tilde{x})}{|\tilde{x} - y|^{n+\alpha}} dy \leq 2C_{n,\alpha} c U(\tilde{x}) \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy. \]
For each fixed $\lambda$, there exists $C > 0$ such that for $\tilde{x} \in \Sigma_\lambda$ and $|\tilde{x}|$ sufficiently large (see [17]),
\[ \int_{\Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \geq \int_{B_{3|\tilde{x}|}(\tilde{x}) \setminus B_{2|\tilde{x}|}(\tilde{x}) \setminus \Sigma_\lambda} \frac{1}{|\tilde{x} - y|^{n+\alpha}} dy \sim C/|\tilde{x}|^{\alpha}. \]
Hence
\[ F_\alpha(u_\lambda(\tilde{x})) - F_\alpha(u(\tilde{x})) + c_{11}(\tilde{x}) U(\tilde{x}) \leq C U(\tilde{x}) < 0. \]
Combining (2.7) with (1.8), it is easy to deduce
\[ V(\tilde{x}) < 0, \]
and
\[ U(\tilde{x}) \geq -C_{12}(\tilde{x})|\tilde{x}|^\alpha V(\tilde{x}). \]
Using (2.8), there exists $\bar{x}$ such that

$$V(\bar{x}) = \min_{\Omega} V(x) < 0.$$  

Similarly to (2.7), we can derive

$$\mathcal{G}_\beta(v_\lambda(\bar{x})) - \mathcal{G}_\beta(v(\bar{x})) \leq \frac{CV(\bar{x})}{|\bar{x}|^\beta} < 0. \tag{2.10}$$

Combining (1.8) and (2.9), we have for $\lambda$ sufficiently negative,

$$0 \leq \mathcal{G}_\beta(v_\lambda(\bar{x})) - \mathcal{G}_\beta(v(\bar{x})) + c_{21}(\bar{x})U(\bar{x}) + c_{22}(\bar{x})V(\bar{x})$$

$$\leq \frac{CV(\bar{x})}{|\bar{x}|^\beta} + c_{21}(\bar{x})U(\bar{x})$$

$$\leq CV(\bar{x}) \frac{c_{12}(\bar{x})|\bar{x}|^\beta}{|\bar{x}|^\beta}$$

$$\leq CV(\bar{x})(1 - c_{12}(\bar{x})|\bar{x}|^\beta c_{21}(\bar{x})|\bar{x}|^\beta),$$

which shows $1 \leq c_{12}(\bar{x})|\bar{x}|^\beta c_{21}(\bar{x})|\bar{x}|^\beta$. However, we have from (1.9) that

$$c_{12}(\bar{x})|\bar{x}|^\beta c_{21}(\bar{x})|\bar{x}|^\beta < 1$$

for $|\bar{x}|$ and $|\bar{x}|$ sufficiently large. This contradiction shows that (1.10) must be true. \hfill \Box

3. Symmetry of solutions in the whole space $R^n$. Let us prove Theorem 1.3.

Proof of Theorem 1.3. Choose an arbitrary direction as the $x_1$-axis. Let $T_\lambda = \{x \in R^n \mid x_1 = \lambda\}$, $x^\lambda = (2\lambda - x_1, x')$, $u_\lambda(x) = u(x^\lambda)$, $\Sigma_\lambda = \{x \in R^n | x_1 < \lambda\}$,

$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

Step 1. Start moving the plane $T_\lambda$ from $-\infty$ to the right in the $x_1$-direction.

We will show that for $\lambda$ sufficiently negative,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.1}$$

To prove (3.1), for the fixed $\lambda$ and $x \in \Sigma_\lambda$, by (1.11),

$$u(x) \to 0, \quad \text{as } |x| \to +\infty.$$  

Since $|x^\lambda| \to +\infty$, as $|x| \to +\infty$, it follows

$$u_\lambda(x) = u(x^\lambda) \to 0$$

and thus for $x \in \Sigma_\lambda$,

$$U_\lambda(x) \to 0, \quad \text{as } |x| \to +\infty. \tag{3.2}$$

Similarly, one can show that for $x \in \Sigma_\lambda$,

$$V_\lambda(x) \to 0, \quad \text{as } |x| \to +\infty. \tag{3.3}$$

Since $\lambda$ sufficiently negative and the properties of $k_i(x)$, it is easy to see by the mean value theorem that

$$\mathcal{F}_\alpha(u_\lambda(x)) - \mathcal{F}_\alpha(u(x)) = v_\lambda^p(x) - v^p(x) + k_1(x^\lambda)u_\lambda^s(x) - k_1(x)u^s(x)$$

$$\geq p(\xi_{\lambda}^{p-1}(x))V_\lambda(x) + k_1(x)\eta_{\lambda}^{-1}(x)U_\lambda(x), \tag{3.4}$$
and
\[ G_\beta(v_\lambda(x)) - G_\beta(v(x)) = u^*_\lambda(x) - u^*(x) + k_2(x^\lambda)v^*_\lambda(x) - k_2(x)v^*(x) \]
\[ \geq q\eta^{\beta-1}_\lambda(x)U_\lambda(x) + k_2(x)s\xi^{\beta-1}_\lambda(x)V_\lambda(x), \]
where \( \xi_\lambda(x) \) is between \( v_\lambda(x) \) and \( v(x) \); \( \eta_\lambda(x) \) is between \( u_\lambda(x) \) and \( u(x) \). By
Theorem 1.2, it suffices to check the decay rate at the points where \( V_\lambda(x) \) and
\( U_\lambda(x) \) are negative respectively. In fact, since \( u_\lambda(x) < u(x) \) and \( v_\lambda(x) < v(x) \), we have
\[ 0 \leq u_\lambda(x) \leq \eta_\lambda(x) \leq u(x), \quad 0 \leq v_\lambda(x) \leq \xi_\lambda(x) \leq v(x). \]
At those points for \(|x|\) sufficiently large, the decay assumptions (1.11) and (1.12)
instantly yields that
\[ c_{11}(x) = -k_1(x)r\eta^{\beta-1}_\lambda(x) \sim o\left(\frac{1}{|x|^\rho}\right) \quad \text{and} \quad c_{22}(x) = -k_2(x)s\xi^{\beta-1}_\lambda(x) \sim o\left(\frac{1}{|x|^\rho}\right). \]
Consequently, there exists \( R_0 > 0 \), such that if \( \tilde{x} \) and \( \bar{x} \) are negative minima of
\( U_\lambda(x) \) and \( V_\lambda(x) \) in \( \Sigma_\lambda \) respectively, then it holds by Theorem 1.2 that
\[ |\tilde{x}| \leq R_0 \text{ or } |ar{x}| \leq R_0. \] (3.5)
Without loss of generality, we may assume
\[ |\tilde{x}| \leq R_0. \] (3.6)
For \( \lambda \) sufficiently negative, combining (3.2) with fact that
\[ U_\lambda(x) = 0, \quad x \in T_\lambda, \]
we know that if \( U_\lambda(x) < 0 \) in \( \Sigma_\lambda \), then \( U_\lambda(x) \) must have a negative minimum in \( \Sigma_\lambda \).
This contradicts (3.6). Hence we have for \( \lambda \) sufficiently negative,
\[ U_\lambda(x) \geq 0. \] (3.7)
It follows \( V_\lambda(x) \geq 0 \) in \( \Sigma_\lambda \). Otherwise, there exists \( \bar{x} \) in \( \Sigma_\lambda \) such that
\[ V_\lambda(\bar{x}) = \min_{\Sigma_\lambda} V_\lambda(x) < 0, \]
and then we have from (2.10) that
\[ G_\beta(v_\lambda(\bar{x})) - G_\beta(v(\bar{x})) < 0. \] (3.8)
However, combining (1.1) with \( u(x), v(x) > 0, \) it yields
\[ G_\beta(v_\lambda(\bar{x})) - G_\beta(v(\bar{x})) \geq 0. \]
This is a contradiction with (3.8) and then \( V_\lambda(x) \) cannot attain its negative value
in \( \Sigma_\lambda \). It follows that (3.1) must be true.

Step 2. Keep moving the planes to the right to the limiting position \( T_{\lambda_0} \) as long as (3.1) holds.
Let
\[ \lambda_0 = \sup\{\lambda \mid U_\mu(x), V_\mu(x) \geq 0, \ x \in \Sigma_\mu, \mu \leq \lambda\}. \]
We show that
\[ \lambda_0 = 0, \] (3.9)
and
\[ U_{\lambda_0}(x) \equiv 0, \ V_{\lambda_0}(x) \equiv 0, \ x \in \Sigma_{\lambda_0}. \]
Suppose that $\lambda_0 < 0$, we will prove that the plane $T_\lambda$ can be moved to the right a little more and (3.1) is still valid. More rigorously, there exists a small $\epsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$ we have

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \quad (3.10)$$

This is a contradiction with the definition of $\lambda_0$. Hence we must have (3.9).

The remaining task is to prove (3.10) by using Theorem 1.1 and Theorem 1.2. Since $U_{\lambda_0}(x) \geq 0$ and $V_{\lambda_0}(x) \geq 0$ but $U_{\lambda_0}(x) \neq 0$ and $V_{\lambda_0}(x) \neq 0$, from that nonnegative functions $U_{\lambda_0}(x)$ or $V_{\lambda_0}(x)$ are positive at some point in $\Sigma_{\lambda_0}$, and the strong maximum principle (iii) in Theorem 1.1, we have

$$U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}.$$

Let $R_0$ be the constant in Theorem 1.2. It follows that for any $\delta > \epsilon > 0$,

$$U_{\lambda_0}(x) \geq c_0 > 0, \quad V_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \overline{\Sigma_{\lambda_0} - \delta} \cap B_{R_0}(0).$$

By the continuity of $U_\lambda(x)$ and $V_\lambda(x)$ with respect to $\lambda$, there exists $\epsilon > 0$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we have

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \overline{\Sigma_{\lambda_0} - \delta} \cap B_{R_0}(0). \quad (3.11)$$

Suppose that (3.10) is false, then $U_\lambda(x) < 0$ or $V_\lambda(x) < 0$, $x \in \Sigma_\lambda$. We claim $U_\lambda(x) < 0$, $V_\lambda(x) < 0$, $x \in \Sigma_{\lambda_0}$. In fact, if $U_\lambda(x) < 0$, $V_\lambda(x) \geq 0$, $x \in \Sigma_\lambda$, then there exists $x_0$ such that $U_\lambda(x_0) = \min U_\lambda(x) < 0$. We have from (2.3) that

$$F_\alpha(u_\lambda(x_0)) - F_\alpha(u(x_0)) + c_{11}(x_0)U_\lambda(x_0) \leq \frac{cU_\lambda(x_0)}{l^\alpha} < 0.$$

On the other hand, it yields from (3.4) that

$$F_\alpha(u_\lambda(x_0)) - F_\alpha(u(x_0)) - k_1(x_0)r\nu^{r-1}(x_0)U_\lambda(x_0) \geq pv^{p-1}(x_0)V_\lambda(x_0).$$

It follows $V_\lambda(x_0) < 0$. This is a contradiction. Similarly, the case $V_\lambda(x) < 0$, $U_\lambda(x) \geq 0$, $x \in \Sigma_{\lambda_0}$ is impossible. So we have $U_\lambda(x) < 0$, $V_\lambda(x) < 0$, $x \in \Sigma_{\lambda_0}$.

If $\bar{x}$ and $\tilde{x}$ are the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ in $\Sigma_\lambda$ respectively, we consider two possibilities.

**Case 1.** One of the negative minima of $U_\lambda(x)$ and $V_\lambda(x)$ lies in $B_{R_0}(0)$, i.e. in the narrow region $\Sigma_{\lambda_0 + \epsilon} \setminus \Sigma_{\lambda_0 - \delta}$, and the other is outside of $B_{R_0}(0)$. Without loss of generality, we may assume the negative minimum of $U_\lambda(x)$ lies in $B_{R_0}(0)$. From (2.4), we have

$$U_\lambda(\bar{x}) \geq -cc_{12}(\bar{x})l^pV_\lambda(\bar{x}). \quad (3.12)$$

and

$$0 \leq G_\beta(v_\lambda(\bar{x})) - G_\beta(v(\bar{x})) + c_{21}(\bar{x})U_\lambda(\bar{x}) + c_{22}(\bar{x})V_\lambda(\bar{x})$$

$$\leq \frac{CV_\lambda(\bar{x})}{|\bar{x}|^\beta} + c_{21}(\bar{x})U_\lambda(\bar{x})$$

$$\leq C\left\{ \frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta} - c_{21}(\bar{x})c_{12}(\tilde{x})l^pV_\lambda(\tilde{x}) \right\}$$

$$\leq C\left\{ \frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta} - c_{21}(\bar{x})c_{12}(\tilde{x})l^pV_\lambda(\tilde{x}) \right\}$$

$$\leq C\frac{V_\lambda(\bar{x})}{|\bar{x}|^\beta} \left[ 1 - c_{12}(\bar{x})l^p c_{21}(\bar{x})|\bar{x}|^{\beta} \right].$$

Hence

$$1 \leq c_{12}(\bar{x})l^p c_{21}(\bar{x})|\bar{x}|^{\beta}. \quad (3.13)$$
We know by (1.9) that $c_{21}(\bar{x})|\bar{x}|^{\beta}$ is small for $|\bar{x}|$ sufficiently large. Since $l = \epsilon + \delta$ is very narrow and $c_{12}(\bar{x})$ is bounded from below in $\Sigma_{\lambda_{0}+\epsilon} \setminus \Sigma_{\lambda_{0}-\delta}$, it derives that $c_{12}(\bar{x})l^{n}$ is small. Consequently, it sees that $c_{12}(\bar{x})l^{n}c_{21}(\bar{x})|\bar{x}|^{\beta} < 1$. This is a contradiction with (3.13) and so (3.10) is proved.

**Case 2.** The negative minima of $U_{\lambda}(x)$ and $V_{\lambda}(x)$ lie all in $B_{R_{0}}(0)$, i.e. in the narrow region $\Sigma_{\lambda_{0}+\epsilon} \setminus \Sigma_{\lambda_{0}-\delta}$.

By (2.3),
\[
\mathcal{F}_{\alpha}(u_{\lambda}(\bar{x})) - \mathcal{F}_{\alpha}(u(\bar{x})) \leq \frac{CU_{\lambda}(\bar{x})}{l^{\alpha}} < 0, \tag{3.14}
\]
where $l = \delta + \epsilon$. Together with (1.4), it implies
\[
U_{\lambda}(\bar{x}) \geq -cc_{12}(\bar{x})l^{n}V_{\lambda}(\bar{x}). \tag{3.15}
\]

Similarly to (3.14), we derive
\[
\mathcal{G}_{\beta}(v_{\lambda}(\bar{x})) - \mathcal{G}_{\beta}(v(\bar{x})) \leq \frac{CV_{\lambda}(\bar{x})}{l^{\beta}} < 0.
\]

Noting (3.15), we have for $l$ sufficiently small,
\[
0 \leq \mathcal{G}_{\beta}(v_{\lambda}(\bar{x})) - \mathcal{G}_{\beta}(v(\bar{x})) + c_{21}(\bar{x})U_{\lambda}(\bar{x}) + c_{22}(\bar{x})V_{\lambda}(\bar{x})
\]
\[
\leq \frac{CV_{\lambda}(\bar{x})}{l^{\beta}} + c_{21}(\bar{x})U_{\lambda}(\bar{x})
\]
\[
\leq C\{V_{\lambda}(\bar{x}) - c_{21}(\bar{x})c_{12}(\bar{x})l^{n}V_{\lambda}(\bar{x})\}
\]
\[
\leq C\{\frac{V_{\lambda}(\bar{x})}{l^{\beta}} - c_{21}(\bar{x})c_{12}(\bar{x})l^{n}V_{\lambda}(\bar{x})\}
\]
\[
\leq C\frac{V_{\lambda}(\bar{x})}{l^{\beta}}[1 - c_{12}(\bar{x})c_{21}(\bar{x})l^{\alpha+\beta}]
\]
\[
< 0.
\]

This contradiction shows that (3.10) has to be true.

Now we have shown that $U_{0}(x) \equiv 0$, $V_{0}(x) \equiv 0$, $x \in \Sigma_{0}$. Since the $x_{1}$ direction can be chosen arbitrarily, we actually prove that $u(x)$ and $v(x)$ must be radially symmetric about the origin. Also the monotonicity follows easily from the argument.

This completes the proof of Theorem 1.3. \qed

4. **Non-existence of positive solutions on a half space $R_{+}^{n}$.** We investigate the system (1.2).

**Proof of Theorem 1.4.** Based on (1.13), one can see from the proof of Lemma 2.1 in [25] that

- either $u(x) > 0$, $v(x) > 0$ or $u(x) \equiv 0$, $v(x) \equiv 0$, for $x \in R_{+}^{n}$,

where $R_{+}^{n} = \{x \in R_{+}^{n} \mid x_{n} \geq 0\}$. In fact, assume $u(x) \neq 0$, there exists $x^{0}$ such that $u(x^{0}) = 0$, and

\[
\mathcal{F}_{\alpha}(u(x^{0})) = c_{n,\alpha}PV \int_{R^{n}} \frac{F(u(x^{0}) - u(y))}{|x^{0} - y|^{n+\alpha}} dy < 0,
\]

i.e. $0 \leq v^{p}(x) + k_{1}(x)u^{n}(x) = \mathcal{F}_{\alpha}(u(x)) < 0$, which is impossible. Hence if $u(x)$ or $v(x)$ attains 0 somewhere in $R_{+}^{n}$, then $u(x) = v(x) \equiv 0$, $x \in R_{+}^{n}$.

Now we always assume that $u(x) > 0$ and $v(x) > 0$ in $R_{+}^{n}$. Let us carry on the method of moving planes to the solution $u$ along the $x_{n}$ direction.
Denote \( T_\lambda = \{ x \in \mathbb{R}^n | x_n = \lambda \} \), \( \lambda > 0 \), \( \Sigma_\lambda = \{ x \in \mathbb{R}^n | 0 < x_n < \lambda \} \). Let \( x^\lambda = (x_1, \cdots, x_{n-1}, 2\lambda - x_n) \) be the reflection of \( x \) about the plane \( T_\lambda \), and \( U_\lambda(x) = u_\lambda(x) - u(x), \ V_\lambda(x) = v_\lambda(x) - v(x) \).

Using (2.1) in this proof of Theorem 1.1 to the situation here, we only need to take \( \Sigma = \Sigma_\lambda \cup \mathbb{R}^n_+ \), where \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n | x_n \leq 0 \} \).

Step 1. It is obvious that, for \( \lambda \leq 0 \), we have
\[
U_\lambda(x) \geq 0, \ V_\lambda(x) \geq 0, \ x \in \mathbb{R}^n_+.
\] (4.1)

For \( \lambda \) sufficiently small, we have immediately
\[
U_\lambda(x) \geq 0, \ V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda,
\] (4.2)

since \( \Sigma_\lambda \) is a narrow region.

Step 2. Since (4.2) provides a starting point, we move the plane \( T_\lambda \) upward as long as (4.2) holds. Define
\[
\lambda_0 = \sup \{ \lambda > 0 | U_\mu(x) \geq 0, \ V_\mu(x) \geq 0, \ x \in \Sigma_\mu, \ \mu \leq \lambda \}.
\] We show that
\[
\lambda_0 = \infty.
\] (4.3)

Otherwise, if \( \lambda_0 < \infty \), we show that the plane \( T_\lambda \) can be moved further up. To be more rigorous, there exists some \( \epsilon > 0 \) such that, for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),
\[
U_\lambda(x) \geq 0, \ V_\lambda(x) \geq 0, \ x \in \Sigma_\lambda.
\] (4.4)

This is a contradiction with the definition of \( \lambda_0 \). Hence, (4.3) holds.

Now we prove (4.4) by using Theorem 1.1, Theorem 1.2 and the similar arguments as in Section 3 that
\[
U_{\lambda_0} \equiv 0, \ V_{\lambda_0} \equiv 0, \ x \in \Sigma_{\lambda_0}, \ \lambda_0 = \infty,
\]
which implies
\[
u(x_1, \cdots, x_{n-1}, 2\lambda_0) = u(x_1, \cdots, x_{n-1}, 0) = 0,
\]
\[v(x_1, \cdots, x_{n-1}, 2\lambda_0) = v(x_1, \cdots, x_{n-1}, 0) = 0.
\]
This is impossible, because we have assumed that \( u(x), v(x) > 0 \) in \( R^n_+ \).

Therefore, (4.3) must be valid and the solutions \( u(x), v(x) \) are increasing with respect to \( x_n \). This contradicts (1.13) and completes the proof of Theorem 1.4. \( \square \)

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