Galvin’s problem in higher dimensions

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Ramsey’s theorem

Theorem (Ramsey)

Suppose $c : [\mathbb{N}]^2 \rightarrow 2$ is any function. Then there is an infinite $X \subseteq \mathbb{N}$ such that $c$ is constant on $[X]^2$.

Theorem (Ramsey)

Suppose $k, l \geq 1$ are natural numbers. If $c : [\mathbb{N}]^k \rightarrow l$ is any function, then there is an infinite set $X \subseteq \mathbb{N}$ such that $c$ is constant on $[X]^k$. 
A consequence of Ramsey’s theorem may be a little less well known: all finitary relations on $\mathbb{N}$ can be classified modulo restriction to an infinite subset of $\mathbb{N}$.

Recall there are “too many” binary relations on $\mathbb{N}$ to classify up to isomorphism.

For example there are continuum many pairwise non-isomorphic linear orders on $\mathbb{N}$. 
Theorem (Ramsey)

Suppose \( R \subseteq \mathbb{N}^2 \) is any relation. Then there is an infinite set \( M \subseteq \mathbb{N} \) such that \( R \cap M^2 \) is equal to one of the following relations restricted to \( M \): \( \top \), \( \bot \), =, \( \neq \), <, >, \( \leq \), \( \geq \).

- There is an analogous result for subsets of \( \mathbb{N}^k \) for any finite \( k \).
- For all finite \( k \), the relations are quantifier free definable using = and <.
Definition

Let $A$ and $B$ be structures. For natural numbers $k, l, t \geq 1$, the notation

$$B \rightarrow (A)^{k}_{l,t}$$

means that for every coloring $c : [B]^k \rightarrow l$, there exists a substructure $C$ of $B$ such that $C$ is isomorphic to $A$ and $|c''[C]^k| \leq t$. 

Suppose that $C$ is some class of structures and that $A$ is a structure that embeds into every member of $C$.

**Ramsey degree**

For a natural number $k \geq 1$, the $k$-dimensional Ramsey degree of $A$ within $C$ is the smallest natural number $t_k \geq 1$ (if it exists) such that $B \rightarrow (A)^k_{l,t_k}$, for every natural number $l \geq 1$ and for every structure $B \in C$. When no such $t_k$ exists, we say that the $k$-dimensional Ramsey degree of $A$ within $C$ is infinite or does not exist.

**Theorem (Ramsey)**

For each $k \geq 1$, the $k$-dimensional Ramsey degree of $\mathbb{N}$ within the class of all infinite sets is 1.
Suppose that $C$ is some class of structures and that $A$ is a structure that embeds into every member of $C$.

**Expansion Problem**

Suppose that $R_1, \ldots, R_m$ are finitely many finitary relations on the structure $A$. The relations $R_1, \ldots, R_m$ are said to solve the expansion problem for $A$ within the class $C$ if for every structure $B \in C$ and every finitary relation $S$ on $B$, there exists a substructure $C$ of $B$ and an isomorphism $\varphi : A \to C$ such that the restriction of $S$ to $C$ is quantifier free definable from the images of $R_1, \ldots, R_m$ under $\varphi$.

**Theorem (Ramsey)**

The relations $<$ and $=$ solve the expansion problem for $\mathbb{N}$ within the class of all infinite sets.
These are equivalent problems

- Solving the expansion problem for $A$ within $C$ for $k$-ary relations is equivalent to finding the $k$-dimensional Ramsey degree of $A$ within $C$.
- This notion of $k$-dimensional Ramsey degree is distinct from the notions of Ramsey degree in the context of Fraïssé theory.
- Special cases of this problem appear in topological dynamics in the guise of computing the universal minimal flows of various automorphism groups.
The 2-dimensional Ramsey degree of $\langle \mathbb{Q}, < \rangle$ within the class $\{\langle \mathbb{Q}, < \rangle\}$ is not 1.

### Sierpinski’s coloring

Let $<_{wo}$ be a well-ordering of $\mathbb{Q}$. Define $s : [\mathbb{Q}]^2 \rightarrow \{0, 1\}$ by

$$s(\{p, q\}) = \begin{cases} 0 & \text{if } < \text{ and } <_{wo} \text{ disagree on } \{p, q\} \\ 1 & \text{if } < \text{ and } <_{wo} \text{ agree on } \{p, q\}, \end{cases}$$

for any $\{p, q\} \in [\mathbb{Q}]^2$. 

For any $X \subseteq \mathbb{Q}$:
- if $s$ is constantly 1 on $[X]^2$, then $X$ is well-ordered by the usual ordering $<;$
- if $s$ is constantly 0 on $[X]^2$, then $X$ is well-ordered by the reserve ordering $>.$

Thus if $\langle X, < \rangle$ contains a $\mathbb{Z}$-chain, then $s$ takes both colors on $[X]^2$. 
Theorem (Galvin)

Suppose \( l \in \mathbb{N} \). If \( c : [\mathbb{Q}]^2 \to \{0, \ldots, l\} \) is any function, then there exists \( X \subseteq \mathbb{Q} \) such that \( \langle X, < \rangle \) isomorphic to \( \langle \mathbb{Q}, < \rangle \) and \( c \) takes at most 2 values on \([X]^2\).

In other words, the 2-dimensional Ramsey degree of \( \langle \mathbb{Q}, < \rangle \) within \( \{\langle \mathbb{Q}, < \rangle\} \) is precisely 2.
Theorem (Laver; Devlin)

For every \( k \geq 1 \), the \( k \)-dimensional Ramsey degree of \( \langle \mathbb{Q}, < \rangle \) within the class of all non-empty dense linear orders without endpoints exists.

This degree \( t_k \) is given by the following formula: \( t_1 = 1 \), and for \( k > 1 \),

\[
t_k = \sum_{l=1}^{k-1} \left( \frac{2k-2}{2l-1} \right) \cdot t_l \cdot t_{k-l}.
\]

- The sequence \( \{t_k\}_{k \geq 1} \) are called the odd tangent numbers because

\[
t_k = T_{2k-1}, \text{ where } \tan(z) = \sum_{n=0}^{\infty} \frac{T_n}{n!} z^n.
\]
Corollary

Let $<_\text{wo}$ be any well-ordering of $\mathbb{Q}$. Then the relations $<$, $=$, and $<_\text{wo}$ solve the expansion problem for the structure $\langle \mathbb{Q}, < \rangle$ within the class of all non-empty dense linear orders without endpoints.
The topological structure of the rationals

- Let $\mathcal{T}_\mathbb{R}$ denote the usual topology of the real numbers, and $\mathcal{T}_X$ its restriction to any $X \subseteq \mathbb{R}$.
- It is not true that if $X \subseteq \mathbb{Q}$ is order isomorphic to $\mathbb{Q}$, then $X$ is homeomorphic to $\mathbb{Q}$.
- Easy exercise: construct $X \subseteq \mathbb{Q}$ which is order-isomorphic to $\mathbb{Q}$, but so that every point is isolated.

**Theorem (Sierpiński)**

$\langle X, \mathcal{T} \rangle$ is homeomorphic to $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ if and only if $\langle X, \mathcal{T} \rangle$ is non-empty, countable, metrizable, and dense-in-itself.
It turns out that the expansion problem for $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ within the class \{\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle \} does not have any solution.

**Theorem (Baumgartner [1])**

Suppose $\langle X, \mathcal{T} \rangle$ is any Hausdorff space with $|X| = \aleph_0$. There is a coloring $c : [X]^2 \rightarrow \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^2 = \omega$.

- For each natural number $l \geq 1$, define $d_l : [\mathbb{Q}]^2 \rightarrow l$ by $d_l(\{x, y\}) = c(\{x, y\}) \mod l$.
- If $X \subseteq \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$, then $d_l$ will take all $l$ values on $[X]^2$.
- No finite list of finitary relations on $\mathbb{Q}$ will capture all binary relations on $\mathbb{Q}$ up to shrinking to a topological copy of $\mathbb{Q}$.
Theorem (Todorcevic and Weiss)

If $\langle X, d \rangle$ is a $\sigma$-discrete metric space, then there is a coloring $c : [X]^2 \to \omega$ such that $c''[Y]^2 = \omega$ for all $Y \subseteq X$ homeomorphic to $\mathbb{Q}$.

- How about the class of all uncountable sets of reals?
Galvin’s Conjecture (1970s)

Suppose $X \subseteq \mathbb{R}$ is uncountable. For every natural number $l \geq 1$,

$$\langle X, T_X \rangle \rightarrow \left( \langle \mathbb{Q}, T_{\mathbb{Q}} \rangle \right)_{l,2}^2$$
Theorem (R. and Todorcevic [2])

If there is a Woodin cardinal, then the 2-dimensional Ramsey degree of \( \langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle \) within the class of all uncountable sets of reals is 2.

- Note this includes sets of reals of size \( \aleph_1 \). Recall \( \aleph_1 \rightarrow [\aleph_1]^2_{\aleph_1} \).
This result solves the expansion problem for **binary relations** for the structure \( \langle \mathbb{Q}, T_\mathbb{Q} \rangle \) within the class of all uncountable sets of real numbers.

**Theorem (R.+Todorcevic [2])**

Assume that there is a Woodin cardinal. Let \( <_{\text{wo}} \) be any well-ordering of \( \mathbb{R} \). Then for every uncountable \( X \subseteq \mathbb{R} \) and every binary relation \( M \subseteq X^2 \), there exists a set \( Y \subseteq X \), which is homeomorphic to \( \mathbb{Q} \), such that \( M \cap Y^2 \) is quantifier free definable from the restrictions of \( <_{\text{wo}}, <, \) and \( = \) to \( Y \).
We can go beyond just sets of reals.

**Theorem (R.+Todorcevic [2])**

*If there is a proper class of Woodin cardinals, then the 2-dimensional Ramsey degree of \( \langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle \) within the class of all non-\( \sigma \)-discrete metric spaces is equal to 2.*
Definition

Let $\langle X, \mathcal{T} \rangle$ be a topological space. A base $\mathcal{B} \subseteq \mathcal{T}$ is said to be point-countable if for each $x \in X$, $\{ U \in \mathcal{B} : x \in U \}$ is countable.

Definition

A topological space $\langle X, \mathcal{T} \rangle$ is said to be left-separated if there exists a well-ordering $<_{wo}$ of $X$ so that for each $x \in X$, $\{ y \in X : y <_{wo} x \}$ is a closed set.

Theorem (R. + Todorcevic [2])

If there is a proper class of Woodin cardinals, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ within the class of all regular, non-left-separated spaces with point-countable bases is at most 2.
Definition

\( \delta \) is a Woodin cardinal if for every \( f : \delta \to \delta \), there exists \( \kappa < \delta \) such that \( \kappa \) is closed under \( f \) and there exists \( j : V < M \) with \( \text{crit}(j) = \kappa \) and \( V_{j(f)(\kappa)} \subseteq M \).

- We need a \( \delta \) such that the countable stationary tower up to \( \delta \) is precipitous.
Definition

Let $\delta$ be a strongly inaccessible cardinal. As usual, $V_\delta$ denotes \{a : rank(a) < $\delta$\}. The countable stationary tower up to $\delta$, denoted $Q_{<\delta}$, is defined to be the collection of all $\langle A, S \rangle \in V_\delta$ such that $A$ is a non-empty set and $S \subseteq [A]^{<\aleph_1}$ is stationary in $[A]^{<\aleph_1}$.

An ordering on $Q_{<\delta}$ is defined as follows. For $\langle A, S \rangle, \langle B, T \rangle \in Q_{<\delta}$, define $\langle B, T \rangle \leq \langle A, S \rangle$ to mean that $B \supseteq A$ and $T \subseteq \{M \in [B]^{<\aleph_1} : M \cap A \in S\}$.
**Definition**

Define a two-player game $\mathcal{D}(\delta)$ as follows. Two players Empty and Non-Empty take turns playing conditions in $\mathbb{Q}_{<\delta}$, with Empty making the first move. When one of the players has played $\langle A_n, S_n \rangle \in \mathbb{Q}_{<\delta}$, his opponent is required to play $\langle A_{n+1}, S_{n+1} \rangle \leq \langle A_n, S_n \rangle$. Thus each run of the game produces a sequence

| Empty   | $\langle A_0, S_0 \rangle$ | $\langle A_2, S_2 \rangle$ | $\cdots$ |
|---------|---------------------------|---------------------------|---------|
| Non-Empty | $\langle A_1, S_1 \rangle$ | $\cdots$ |

such that for each $n \in \omega$, $\langle A_{2n}, S_{2n} \rangle$ has been played by Empty, $\langle A_{2n+1}, S_{2n+1} \rangle$ has been played by Non-Empty and $\langle A_{n+1}, S_{n+1} \rangle \leq \langle A_n, S_n \rangle$. Non-Empty wins this particular run of $\mathcal{D}(\delta)$ if and only if there exists a sequence $\langle N_l : l \in \omega \rangle$ such that $\forall l \in \omega \ [N_l \in S_l]$ and $\forall k \leq l \ [N_k = N_l \cap A_k]$. 
We need a $\delta$ such that Empty does not have a winning strategy in $\mathcal{E}(\delta)$. 
When $C$ is the class of all regular, non-left-separated spaces with point-countable bases, then the large cardinal hypothesis can be weakened to the following: for every ordinal $\alpha$, there exists an inner model $N$ of ZFC such that $V_\alpha \subseteq N$ and there is a Woodin cardinal greater than $\alpha$ in $N$.

This weakening is implied by each of the following: existence of one strongly compact cardinal, PFA, PID.

The weakening does not even imply the existence of an inaccessible cardinal in $V$.

Woodin showed that this weakening is equivalent to the statement that $\Sigma^1_2$-determinacy holds in $V$ and all of its set generic extensions.
When $C$ is the class of all uncountable sets of reals, then the large cardinal hypothesis can be weakened to the following: there is an inner model containing all sets of reals with at least one Woodin cardinal in it.

Actually, if one is only interested in consistency strength, then an upper bound in this case is one measurable cardinal.

If there is a precipitous ideal on $\omega_1$, then the 2-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ within the class of all uncountable sets of reals is 2.
A generalization of Sierpinski’s coloring shows that the number of unavoidable colors in dimension $k$ (on a topological copy of $\mathbb{Q}$) is $k!(k - 1)!$.

Do large cardinals imply that this number can always be achieved for any coloring of $[\mathbb{R}]^3$?
Theorem (R.+Todorcevic [3])

Let $n \in \omega$. Let $\langle X, \mathcal{T} \rangle$ be any Hausdorff space with $|X| = \aleph_n$. There is a coloring $c : [X]^{n+2} \to \omega$ such that for any subspace $R \subseteq X$ that is homeomorphic to $\mathbb{Q}$, $c''[R]^{n+2} = \omega$.

- The case $n = 0$ is precisely Baumgartner’s theorem.
- So the $n + 2$-dimensional expansion problem for the space $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ within the class of sets of real numbers of size at most $\aleph_n$ does not have any solution.
Corollary

Let $n \in \omega$. Suppose $C$ is any class of topological spaces. If $C$ contains any Hausdorff space of cardinality at most $\aleph_n$, then the $n + 2$-dimensional Ramsey degree of $\langle \mathbb{Q}, T_\mathbb{Q} \rangle$ within $C$ does not exist.

Corollary

If $\langle \mathbb{R}, T_\mathbb{R} \rangle \to \left( \langle \mathbb{Q}, T_\mathbb{Q} \rangle \right)_{l,12}^3$, for all $1 \leq l < \omega$, then CH fails. For any $k \geq 1$, if for every $1 \leq l < \omega$, $\langle \mathbb{R}, T_\mathbb{R} \rangle \to \left( \langle \mathbb{Q}, T_\mathbb{Q} \rangle \right)_{l,k!(k-1)!}^k$, then $|\mathbb{R}| \geq \aleph_{k-1}$. If the $k$-dimensional Ramsey degree of $\langle \mathbb{Q}, T_\mathbb{Q} \rangle$ in $\{\langle \mathbb{R}, T_\mathbb{R} \rangle\}$ exists for every natural number $k \geq 1$, then $2^{\aleph_0} \geq \aleph_{\omega+1}$. 
A key combinatorial aspect of the proof is a classical set mapping theorem of Kuratowski.

**Lemma (Kuratowski)**

*For each $n \in \omega$, there exists $f_n : [\omega_n]^{n+1} \to [\omega_n]^{<\aleph_0}$ such that:*

1. $\forall s \in [\omega_n]^{n+1} \left[ f_n(s) \subseteq \max(s) \right]$;
2. $\forall t \in [\omega_n]^{n+2} \exists \alpha \in t \left[ \alpha < \max(t) \text{ and } \alpha \in f_n(t \setminus \{\alpha\}) \right]$.
Question

What is the largest class of topological spaces within which the $k$-dimensional Ramsey degree of $\langle \mathbb{Q}, \mathcal{T}_\mathbb{Q} \rangle$ is equal to $k!(k-1)!$?
J. E. Baumgartner, *Partition relations for countable topological spaces*, J. Combin. Theory Ser. A 43 (1986), no. 2, 178–195.

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