UNIFORM WEYL’S LAW AND HEAT TRACE ASYMPTOTICS ON DEGENERATING SURFACES

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Abstract. We prove an uniform Weyl law for the eigenvalues of the Dirac operator on a family of degenerating hyperbolic surfaces under a certain non-triviality condition on the spin structure. We also prove an asymptotic expansion of the heat trace in this degenerating limit. The proof is based on deriving and exploiting a Selberg trace formula for the Dirac operator on hyperbolic surfaces of finite volume.

1. Introduction

The Weyl law describes the asymptotic behaviour of the eigenvalues of an elliptic pseudo-differential operator of positive order on a compact Riemannian manifold, with or without boundary. Such a law was first obtained by Weyl in 1911 [18]. Specifically, he studied the Laplacian with Dirichlet boundary condition, for Ω ⊂ R^d (d = 2, 3) a bounded domain with smooth boundary. Consider N(λ) the number of eigenvalues of ∆ (counted with multiplicity) smaller than λ. He obtained that:

\[ \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d} \text{Vol}(\Omega), \]

where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \). After Weyl, a lot of mathematicians devoted their work to the study of spectral asymptotics. We mention Courant [7] and Carleman [4], [5], who considered the case of arbitrary dimension. Later, in 1949, Minakshisundaram and Pleijel extended the result for a generalized Laplacian in a hermitian bundle on a compact manifold of dimension d:

\[ \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{\text{Vol}(M) \text{rank}(E)}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} + 1 \right)}, \]

where \( \Gamma \) is the classical Gamma function. In 1968 Hörmander [10] obtained a Weyl law with a bound for the error term for elliptic pseudodifferential operators on compact manifolds. One can try to extend the above results in two directions. First, to find an optimal estimate for the remainder in the Weyl law. There exists a huge bibliography on this topic, see for instance the work of Ivrii [11], and also Chazarain [6] and Duistermaat-Guillemin [9]. Second, it is interesting to try to generalize the law for elliptic operators on possibly non-compact manifolds. We will use below a result of Moroianu [13] in this direction, giving a Weyl law for the Dirac operator on a class of noncompact manifold containing the hyperbolic surfaces with cusps studied here.

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It is fairly clear from the proof that such Weyl laws are uniform for continuous family of operators with parameters in a compact set. However, one can wonder what happens with the spectrum of such a family when the topology of the manifold changes. For example, the spectrum of the Laplacian typically becomes continuous when the manifold degenerates to a non-compact one. In contrast, the spectrum of the Dirac operator can remain discrete, under some topological assumptions (as it was proven in [13]). The purpose of this paper is to obtain a uniform spectral asymptotic law for the spin Dirac operator on a family of degenerating surfaces, thus making a small step in the direction of studying families of spectra of geometric operators when the topology changes.

Before stating our main results, we will explain what we understand by a family of degenerating surfaces. Any hyperbolic surface $M$ is a quotient of the Poincaré half plane, $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}, \frac{dx^2 + dy^2}{y^2}$, through $\Gamma$, a discrete subgroup of isometries. We require $M$ to be of finite area. So, $\Gamma$ either contains only hyperbolic elements or contains both hyperbolic and parabolic elements. Consider some simple, disjoint, closed geodesics $\eta_j$, with $j = 1, k \leq 3g - 3$, on $M$. Complete this to a maximal system of such geodesics, denoted $\eta_j$, with $j = 1, 3g - 3$. Now decompose the surface into pairs of pants, cutting along each $\eta_j$. Take a family of hyperbolic metrics $g_t$, with $t \in [0, 1]$ (we shall denote $l_t(c)$ the length of the curve $c$ with respect to the metric $g_t$) on $M$. This process was already studied by Schultze [16], for the Selberg zeta function associated to the scalar Laplacian. In that setting, the spectrum concentrates and becomes continuous in the limit.

Definition 1. We say that $M$ goes through a pinching process after $\eta_j$, $j = 1, k$ if:

- $g_1$ equals the initial metric on $M$;
- $l_t(\eta_j) \to 0$ as $t \to 0$ for all $j = 1, k$;
- $l_t(\eta_j) \to \alpha_j$ as $t \to 0$ for all $j = k + 1, 3g - 3$, where $\alpha_j \in (0, \infty)$;
- $\theta_j(t)$ have a limit when $t \to 0$, for all $j = 1, 3g - 3$, where $\theta_j(t)$ is the angle $\eta_j$ needs to rotate in order to glue back the surface.

Remark 2. Note that throughout a pinching process, the area of $M$ remains constant and is equal to $2\pi(2g - 2)$, and also that the genus of the limiting surface decreases by the number $k$ of pinched geodesics.

Another assumption for this work is that for each pinched geodesic, the spin structure along it is not trivial. This implies that the Dirac operator has discrete spectrum (even on the limit surface), as it was shown by Moroianu in [13]. We will encapsulate this hypothesis in a class function $\varepsilon : \Gamma \to \{\pm 1\}$ (i.e. a function that is constant along conjugacy classes). The same class function appeared in the study of the Selberg zeta function for the Dirac operator. D’Hocker and Phong in [8] and Sarnack in [15] show that the determinant of the squared Dirac operator (on compact hyperbolic surfaces) is essentially given by Selberg zeta evaluated at half integers points. They used a rather algebraic definition for the function in question. In the second section, we prove that this definition can be reinterpreted in the usual geometric fashion, correcting some minor errors from the literature along the way. Our method to tackle the problem at hand will be to derive and exploit a Selberg trace formula for the squared Dirac on hyperbolic surfaces of finite area. We will use the kernel
of a convolution operator which has the same eigenspinors as the squared Dirac. The formula we derive is already known for compact surfaces. It was obtained by Bolte and Stiepan [1] using Green functions and hypergeometric functions. The methods we employ are self contained and, we hope, easier to follow. Bounds of the number of geodesics on a hyperbolic surface will be used. Some (if not all) of them are probably known, but for completeness we will include the proofs. These bounds are a necessary ingredient to obtain the trace formula for surfaces of finite area. After that, we will see how we can obtain the formula for a non-compact surface as a limit of formulae on compact surfaces that are undergoing a pinching process. A first immediate consequence is the following:

**Theorem 3.** Let \( u : \mathbb{R} \rightarrow \mathbb{R} \) be an admissible function (in the sense of definition [24]). Consider \( M \) a hyperbolic surface, with a spin structure, which undergoes a pinching process and suppose that the spin structure along the pinched geodesics is not trivial. Then, as \( t \) converges to \( 0 \), we have that:

\[
\text{Tr} \left( u(D_t^2) \right) \rightarrow \text{Tr} \left( u(D_0^2) \right),
\]

where \( D_t \) denotes the Dirac operator on \( M \) for the metric \( g_t \).

This result is interesting in itself, as there are no easy ways to determine explicitly the spectrum of the Dirac operator on hyperbolic surfaces. However, we shall apply this theorem to obtain the first main result (theorem 4), a heat trace asymptotic for the Dirac operator on a hyperbolic surface with cusps:

**Theorem 4.** Let \( \Gamma \) be a subgroup of \( \text{PSL}_2(\mathbb{R}) \) and \( M = \Gamma \backslash \mathbb{H} \) be the corresponding hyperbolic surface, with \( k \) cusps. Suppose that \( \varepsilon(\gamma) = -1 \) for each parabolic element \( \gamma \in \Gamma \). If \( \{\lambda_j\}_{j \in \mathbb{N}} \) are the eigenvalues of \( D^2 \) on \( M \), then:

\[
\sum_{j=0}^{\infty} e^{-T\lambda_j} \sim \frac{\text{Area}(M)}{4\pi T} - \frac{k \log(2)}{\sqrt{4\pi T}} + \sum_{m=0}^{\infty} \frac{a_m \text{Area}(M)}{4\pi} T^m, \quad \text{as } T \searrow 0.
\]

Note that if \( k = 0 \) (meaning that \( M \) is compact) the non-smooth part of the asymptotic is the same for any hyperbolic surface of a fixed genus (i.e. fixed area). Following the approach of Minakshisundaram and Pleijel, we will use this asymptotic together with Karamata’s theorem to deduce the uniform Weyl law:

**Theorem 5.** Let \( M \) be a compact hyperbolic surface going through a pinching process after \( \eta \) (corresponding to the conjugacy class \([\mu_t]\), for \( \mu_t \in \Gamma_t \) and \( t \in [0, 1] \)). If we denote \( N_t(\lambda) \) the number of \( D_t^2 \)-eigenvalues (counted with multiplicity) smaller than \( \lambda \), then the following two conditions are equivalent:

i) \( \varepsilon(\mu_t) = -1 \) (i.e. the spin structure along \( \mu_t \) is non-trivial),

ii) \( \lim_{\lambda \rightarrow \infty} \frac{N_t(\lambda)}{\lambda} = \frac{\text{Area}(M)}{4\pi} \) uniformly for \( t \in [0, 1] \).

**Remark 6.** In other words, the uniform Weyl law is sensitive to the choice of the spin structure (which will be fixed during the pinching process).

**Remark 7.** Clearly, a similar statement is true for surfaces where we pinch several disjoint geodesics.
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2. The Dirac operator

2.1. The spinor bundle. Given a \( n \)-dimensional vector space \( V \) and a bilinear symmetric form \( q \), the Clifford algebra associated to \( (V, q) \) is defined as the quotient of the tensor algebra by the Clifford ideal, generated by all the elements of the form \( v \otimes w + w \otimes v - 2q(v, w) \). If \( V = \mathbb{R}^n \) and \( q \) is the standard scalar product, the associated Clifford algebra is usually denoted \( \text{Cl}_n \). The spin group \( \text{Spin}(n) \) is the subgroup of \( \text{Cl}_n \) consisting of all even products of unit vectors. It follows that \( \text{Spin}(n) \) is a connected, two-sheeted covering of \( \text{SO}(n) \), the space of \( n \times n \) orthogonal matrices. Moreover, if \( n \geq 3 \), \( \text{Spin}(n) \) becomes the universal cover. If \( n \) is even, the complexified Clifford algebra \( \text{Cl}_n \otimes \mathbb{R}\mathbb{C} \) is isomorphic to the algebra of complex matrices of order \( 2^{n/2} \). Let us give a concrete example of this representation. Take \( \{e_1, \ldots, e_{2k}\} \), for \( 2k = n \), the standard basis in \( \mathbb{R}^n \). The representation \( cl(v) := 2^{-1/2}(v - iJ(v)) \wedge (\cdot) - 2^{-1/2}(v + iJ(v)) \ud(\cdot) \), where \( J \) is the standard almost complex structure on \( \mathbb{R}^n \), acts on the complex vector space \( \Sigma_n := \wedge^* W \), where \( W \) is generated by \( \{2^{-1/2}(e_1 - ie_2), \ldots, 2^{-1/2}(e_{2k-1} - ie_{2k})\} \). Clearly it extends to the tensor algebra \( T\mathbb{C}^n = \bigoplus_{j=0}^\infty (\mathbb{C}^n)^{\otimes j} \). Since \( cl \) vanishes on the Clifford ideal, it descends to \( \text{Cl}_n \otimes \mathbb{R}\mathbb{C} \):

\[
cl : \text{Cl}_n \otimes \mathbb{R}\mathbb{C} \rightarrow \text{End}_\mathbb{C}(\Sigma_n) \simeq M(2^k, \mathbb{C})
\]

With the setup fixed, we will now proceed to define the Dirac operator. Consider \( (M, g) \) an oriented Riemannian manifold. A spin structure on \( M \) is a principal \( \text{Spin}(n) \) bundle \( P_{\text{Spin}(n)}M \) together with two covering maps \( \pi_1 \) and \( \pi_2 \) making the diagram:

\[
\begin{array}{ccc}
P_{\text{Spin}(n)}M & \xrightarrow{(p, s) \mapsto ps} & \text{Spin}(n) \\
\pi_2 & & \pi_1 \\
P_{\text{SO}(n)}M & \xrightarrow{(p, A) \mapsto pA} & \text{SO}(n) \\
& & \\
M & & \\
\end{array}
\]

commutative, where \( P_{\text{SO}(n)}M \) is the oriented orthonormal frame bundle. If \( M \) has a spin structure, we can define the spinor bundle as the associated vector bundle:

\[
S := P_{\text{Spin}(n)}M \times_{cl} \Sigma_n.
\]
2.2. Dirac operator. Take $\nabla$ the Levi-Civita connection on $P_{SO(n)} M$ and lift it to a connection on $P_{\text{Spin}(n)} M$. It will induce a connection, also denoted $\nabla$, on the associated vector bundle $S$. Then, the Dirac operator acts on $C^\infty(S)$, the space of smooth sections of this bundle, in the following manner:

$$D : C^\infty(S) \longrightarrow C^\infty(S)$$

$$D := \text{cl} \circ \nabla.$$ 

Clearly, it is a differential operator of order 1, but it is well known that it is also elliptic: $\sigma_1 D(\xi) = \text{cl}(\xi i)$, and $\text{cl}(\xi)^2 = -|\xi|^2$. If $M$ is compact, since $D$ is elliptic and self-adjoint, the classical theory of pseudodifferential operators tells us that its spectrum is real and discrete. This fact remains true also when $M$ is not compact (see [13]) under certain assumptions which will hold in our setting.

2.3. Spin structures on hyperbolic surfaces. From now on we will focus on complete oriented hyperbolic surfaces. Two models of the universal cover of such a surface will be used: the unit disk

$$\mathbb{D} := \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1 \right\}, \quad g = \frac{4(dx^2 + dy^2)}{(1 - r^2)^2}$$

and the upper half plane

$$\mathbb{H} := \left\{ (x, y) \in \mathbb{R}^2 : y > 0 \right\}, \quad g = \frac{dx^2 + dy^2}{y^2}.$$ 

Recall that the group of oriented isometries of $\mathbb{H}$ is $\text{PSL}_2(\mathbb{R}) := \text{SL}_2(\mathbb{R})/\{\pm 1\}$ (we denote $\pi : \text{SL}_2(\mathbb{R}) \longrightarrow \text{PSL}_2(\mathbb{R})$ the standard projection) with the action given by:

$$\gamma z = \frac{az + b}{cz + d}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R}).$$ 

With the mapping

$$\gamma \mapsto \left( \gamma i, \left\{ \gamma_{*,i} \left( \frac{\partial}{\partial x} \right), \gamma_{*,i} \left( \frac{\partial}{\partial y} \right) \right\} \right),$$

we identify $P_{\text{SO}(2)} \mathbb{H}$ to $\text{PSL}_2(\mathbb{R})$ as principal $\text{SO}(2)$ bundles. We shall identify, from now on, isometries and vector frames. Consider $\Gamma \in \text{PSL}_2(\mathbb{R})$ a discrete subgroup. Then the action:

$$\gamma_* : P_{\text{SO}(2)} \mathbb{H} \longrightarrow P_{\text{SO}(2)} \mathbb{H}, \quad \gamma_* (z, \{v_1, v_2\}) = (\gamma z, \{\gamma_{*,z} v_1, \gamma_{*,z} v_2\}),$$

where $z \in \mathbb{H}$, becomes the left multiplication of isometries (via the previous mapping). The quotient by this action is equivalent with the existence of the frame bundle over the quotient $\Gamma \backslash \mathbb{H}$.

Moreover, we can identify the spin bundle $P_{\text{Spin}(2)} \mathbb{H}$ with $\text{SL}_2(\mathbb{R})$, the space of 2 by 2 matrices of determinant 1, because $\text{SL}_2(\mathbb{R})$ is the two sheeted universal covering for the group of oriented isometries. We are able to lift the left multiplication by $\Gamma$ to $\text{SL}_2(\mathbb{R})$ if and only if there exists a spin bundle over the quotient surface. But hyperbolic surfaces always admit spin structures (see, for example [3]), hence, the short exact sequence:
1 \rightarrow \{\pm 1\} \rightarrow \hat{\Gamma} := \pi^{-1}(\Gamma) \rightarrow \Gamma \rightarrow 1,

admits a right splitting \( \rho : \Gamma \rightarrow \hat{\Gamma}, \pi \circ \rho = \text{Id}_\Gamma \).

**Remark 8.** There are \(2g\) generators of \(\Gamma\), thus there are in total \(2^{2g}\) possible spin structures over \(\Gamma \setminus \mathbb{H}\).

From the splitting lemma, since \(\{\pm 1\}\) is in the center of \(\hat{\Gamma}\), the existence of \(\rho\) implies there exists a left splitting as well: \(\chi : \hat{\Gamma} \rightarrow \{\pm 1\}, \chi \circ \iota = \text{Id}_{\{\pm 1\}}\), where \(\iota(-1) = -I_2\). The way D’Hocker and Phong \[8\] constructed the \(\{\pm 1\}\)-valued class function we talked about in the introduction is the following. For each element \(\gamma \in \Gamma\), take \(\tilde{\gamma} \in \text{SL}_2(\mathbb{R})\) the unique lift for which \(\text{Tr}(\tilde{\gamma}) \geq 2\), and define:

\[ \nu : \Gamma \rightarrow \{\pm 1\}, \quad \nu(\gamma) = \chi(\tilde{\gamma}). \]

Note that even though \(\chi\) is a group morphism, \(\nu\) doesn’t have any reason to be multiplicative. Yet, it is constant along conjugacy classes and satisfies \(\nu(\gamma^n) = \nu^n(\gamma)\). Clearly, this algebraic definition is closely related to the spin structure. So, it should have a geometric interpretation as well. We introduce an apparently different class function and then prove that it is actually the same as \(\nu\). Pick \(\gamma \in \Gamma\). If \(\gamma\) is hyperbolic, then we denote \(\eta\) the unique geodesic in \(\mathbb{H}\) (parametrized by \(t \in \mathbb{R}\), with speed 1) fixed by \(\gamma\). Consider \(p_{\eta(t)}\) the orthonormal frame \(\{-J(\dot{\eta}(t)), \dot{\eta}(t)\}\), where \(J\) is the standard almost complex structure. Clearly \(\gamma p_{\eta(t)} = p_{\eta(\gamma(t))}\), for any \(t \in \mathbb{R}\). Thus, if \(\tilde{p} \in \text{SL}_2(\mathbb{R})\) is an arbitrary lift of \(p\), we can define:

\[ \varepsilon : \Gamma \rightarrow \{\pm 1\}, \quad \rho(\gamma)\tilde{p}_{\eta(t)} = \varepsilon(\gamma)\tilde{p}_{\eta(\gamma(t))}. \]

From the definition, we can immediately see that \(\varepsilon\) is a class function and, moreover, \(\varepsilon(\gamma^n) = \varepsilon^n(\gamma)\). The case when \(\gamma\) is parabolic is quite similar. For simplicity, suppose \(\gamma\) is the translation by 1. Consider \(t \mapsto \mu_r(rt)\) the unique horocycle preserved by \(\gamma\) which passes through \(ir\), for \(r \in (0, \infty)\). Then \(p = \{-J(\dot{\mu}_r), \dot{\mu}_r\}\) is a global orthonormal frame, and thus \(\tilde{p}\), its lifting, is constant in \(t\) and \(r\). Hence, we can define \(\varepsilon\) in the same manner. Note that this class function carries more geometric insights. In the hyperbolic case, it is exactly the holonomy of the spin bundle over the quotient \(\Gamma \setminus \mathbb{H}\) along the corresponding closed geodesic. When \(\gamma\) is parabolic, \(\varepsilon(\gamma)\) is the limit of the holonomy along horocycles “escaping” in the cusp.

**Proposition 9.** The two definitions \(\nu\) and \(\varepsilon\) of the class function associated to a spin structure coincide.

**Proof.** Consider first that \(\gamma\) is hyperbolic. There exists \(l > 0\) such that: \(\gamma = ae^l a^{-1}\), where \(e^l(z) = e^l z\). Then \(\eta\), the geodesic of unit speed preserved by \(\gamma\), is parametrized by \(\eta(t) = a(ie^t)\). Moreover, with the previous identification, \(p_{\eta(t)} = ae^t\), where \(p\) is as in the definition of \(\varepsilon\). If we fix \(A \in \text{SL}_2(\mathbb{R})\) an arbitrarily lift for \(a\), then \(A \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}\) is a lift
of \( p \). Therefore:

\[
\rho(\gamma)A \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \varepsilon(\gamma)A \begin{bmatrix}
e^{l/2} & 0 \\
0 & e^{-l/2}
\end{bmatrix};
\]

\[
A^{-1} \rho(\gamma)\varepsilon(\gamma)A = \begin{bmatrix}
e^{l/2} & 0 \\
0 & e^{-l/2}
\end{bmatrix}.
\]

Since \( \text{Tr}(\rho(\gamma)\varepsilon(\gamma)) > 2 \) and \( \chi \circ \rho = \text{Id} \), we get \( \nu(\gamma) = \chi(\rho(\gamma)\varepsilon(\gamma)) = \varepsilon(\gamma). \)

From now on, to avoid a heavy notation, the action of \( \Gamma \) on the isomorphisms of the spin bundle given by \( \gamma \mapsto \rho(g) \) will be denoted by \( \gamma \mapsto \gamma^* \).

2.4. Explicit formulae for the Dirac operator. For any point \((x, y) \in \mathbb{H}\) we consider the orthonormal frame \( p := \{y \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}\} \). If \( \tilde{p} \) is one of the two lifts of this frame to \( P_{\text{Spin}(2)} \mathbb{H} \), then the spinor bundle is generated by the global sections

\[
\sigma^+ := [\tilde{p}, 1], \quad \sigma^- := [\tilde{p}, e_1 - ie_2].
\]

In the splitting given by these sections, the Dirac operator is given by the off-diagonal symmetric matrix of operators:

\[
D = \begin{bmatrix}
0 & -y \frac{\partial}{\partial x} + iy \frac{\partial}{\partial y} - \frac{i}{2} \\
y \frac{\partial}{\partial x} + iy \frac{\partial}{\partial y} - \frac{i}{2} & 0
\end{bmatrix} : S^+ \oplus S^- \rightarrow S^+ \oplus S^-.
\]

In the case of the unit disk \( D \), we use the orthonormal frame \( \{\frac{1-r^2}{2} \frac{\partial}{\partial x}, \frac{1-r^2}{2} \frac{\partial}{\partial y}\} \), where \( r := \sqrt{x^2 + y^2} \). Using the same sections \( \sigma^\pm \) as above, the Dirac operator has the form:

\[
D = \begin{bmatrix}
0 & -\frac{1-r^2}{2} \frac{\partial}{\partial x} + i\frac{1-r^2}{2} \frac{\partial}{\partial y} + \frac{x+iy}{2} \\
\frac{1-r^2}{2} \frac{\partial}{\partial x} + i\frac{1-r^2}{2} \frac{\partial}{\partial y} - \frac{x-iy}{2} & 0
\end{bmatrix}.
\]

We denote \( D^+ \) the part of \( D \) mapping \( S^+ \) into \( S^- \) (\( D^- \) is defined similarly). Thus, rewriting the operator in polar coordinates \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial \theta} \), we get:

\[
D^- = \frac{1}{2} \left((-re^{-i\theta} - e^{-i\theta}(1-r^2) \frac{\partial}{\partial r} + \frac{i e^{-i\theta}(1-r^2)}{r} \frac{\partial}{\partial \theta})\right)
\]

\[
D^+ = \frac{1}{2} \left((re^{i\theta} + e^{i\theta}(1-r^2) \frac{\partial}{\partial r} + \frac{i e^{i\theta}(1-r^2)}{r} \frac{\partial}{\partial \theta})\right).
\]

3. Bounding the number of geodesics on a hyperbolic surface

In this section we want to bound the number of primitive geodesics on a hyperbolic surface. A lot is known in this direction, we mention Randol [14], who proved that on a fixed compact hyperbolic surface,

\[
\pi(x) = \text{Li}(x) + O(x^\alpha),
\]

where \( \frac{3}{4} \leq \alpha < 1 \), \( \text{Li}(x) := \int_2^x \frac{dt}{\log t} \) is the logarithm integral function and \( \pi(x) \) represents the number of closed primitive geodesics of length less than or equal to \( \log x \). One can
prove such a formula by applying the Selberg trace formula for the scalar Laplacian $\Delta$ to a particular class of functions. One problem with this rather exact estimate is that we don’t control how the error term behaves when we consider families of hyperbolic metrics (instead of a fixed one). In what follows, we will use the following notations:

- $L(r)$, the number of closed geodesics of length at most $r$;
- $d(p, q)$, the hyperbolic distance between $p$ and $q$;
- $\text{Diam}(M)$, the diameter of the surface $M$;
- $l(\eta)$, the length of the geodesic $\eta$.

**Lemma 10.** Let $M = \Gamma \setminus \mathbb{H}$ be a hyperbolic compact surface. Then $L(r) < Ce^r$, where $C$ depends only on the diameter and the area of $M$.

**Proof.** Fix $p$ a point on $M$, consider $\tilde{p} \in \mathbb{H}$ a lift of $p$ and $D_{\tilde{p}}$ the associated fundamental domain. Take $\eta$ a closed geodesic on $M$ and fix $q$ a point on it. If $\tilde{q} \in D_{\tilde{p}}$ is the unique lift of $q$ in $D_{\tilde{p}}$, there exists a unique $\gamma \in \Gamma$ such that the line from $\tilde{q}$ to $\gamma \tilde{q}$ is the lift of $\eta$. Hence:

$$l(\eta) = d(\tilde{q}, \gamma \tilde{q}) < 2 \text{Diam}(M) + d(\tilde{p}, \gamma \tilde{p}).$$

Since we get a different $\gamma$ for each geodesic $\eta$, it follows that

$$L(r) < |\{\gamma \in \Gamma : d(\tilde{p}, \gamma \tilde{p}) \leq r + 2 \text{Diam}(M)\}|.$$

But the right-hand side is clearly bounded by the area of a hyperbolic disk of radius $r + 2 \text{Diam}(M)$ divided by the area of $M$. \qed

When we vary the metric on $M$, its diameter can explode and in fact we consider precisely deformations of exploding diameter. Hence, we need to refine the above statement. One way to do that is by considering a compact sub-surface which remains bounded throughout the process.

**Proposition 11.** Let $M = \Gamma \setminus \mathbb{H}$ be a compact hyperbolic surface and let $\eta$ be a closed, simple, geodesic. Moreover, let $g_t$ be a family of metrics on $M$ such that $l(\eta_t)$ converges to 0 as $t$ converges to 0 (as described in [1]). Then, there exists a sub-surface $K(t)$ with boundaries, and a constant $C$ which only depends on the area of $M$ and the diameter of $K(t)$, such that:

- $\text{Diam}(K(t))$ is uniformly bounded for all $t > 0$;
- $L_{\eta_t}(r) < Ce^r$, where $L_{\eta_t}(r)$ counts those closed geodesics, which are not multiples of $\eta_t$ (i.e. $t \mapsto \eta_t(nt)$, for any $n$ a positive integer), that are smaller than $r$.

**Proof.** Consider a maximal system of $3g - 3$ simple, closed geodesics on $M$, including $\eta_t$ among them. We cut along these geodesics to decompose the surface into $2g - 2$ pairs of pants. Consider a pair whose boundary contains $\eta_t$. Denote $BC$, $DE$ and $FA$ the lines that realize the distance between the boundary components, as seen in figure [1].
The points $A$ and $B$ are diametrally opposed on $\eta$ (similarly, $C$ and $D$, $E$ and $F$ are also diametrally opposed). We take $AFEDCD'EF'A'B$ a fundamental domain such that the geodesic $BC$ corresponds to the imaginary half-line as in figure 2. Continue the line $AB$ until it meets the real axis. From that point we consider the unique geodesic perpendicular to $BC$, and denote $S$ the point of intersection. Construct similarly $R \in AF$ and $R' \in A'F'$. The union of the line segments $RS$ and $R'S$ will project onto a piecewise smooth loop on $M$. Consider the same loop for the other pair of pants which has $\eta_t$ as boundary (it might be another boundary piece of the same pair of pants). These two loops will disconnect the surface $M$ into two connected components: a cylinder containing $\eta_t$ and the rest of the surface, denoted $K(t)$.

We claim that the length of the boundaries of $K(t)$ are bounded from below. Indeed, pick an isometry which maps the point $A$ into $i$ and the point $B$ into $ie^{l(\eta_t)/2}$. A straightforward computation will yield the coordinates of $R$ and $S$:

$$R \left( \frac{2e^{l(\eta_t)/2}}{e^{l(\eta_t)} + 1}, \frac{e^{l(\eta_t)} - 1}{e^{l(\eta_t)} + 1} \right), \quad S \left( \frac{2e^{l(\eta_t)}}{e^{l(\eta_t)} + 1}, \frac{e^{l(\eta_t)}(2e^{l(\eta_t)} - 1)}{e^{l(\eta_t)} + 1} \right).$$

Hence, the distance between these two points satisfies:

$$2 \left( \cosh d(R, S) - 1 \right) = \frac{(e^{l(\eta_t)} + 1)^2}{e^{l(\eta_t)/2} (e^{l(\eta_t)/2} + 1)^2},$$

or, equivalently:

$$\cosh d(R, S) = \frac{2 \cosh^2 \left( \frac{l(\eta_t)}{4} \right) + \cosh^2 \left( \frac{l(\eta_t)}{2} \right)}{2 \cosh^2 \left( \frac{l(\eta_t)}{4} \right)}.$$

Since the expression in the right-hand side decreases to $3/2$ as $l(\eta_t)$ goes to 0, the distance between $R$ and $S$ remains larger than $\cosh^{-1}(3/2)$. In consequence, $\text{Diam}(K(t))$ is uniformly bounded for all $t$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{pants.png}
\caption{Pair of pants}
\end{figure}
The second part of the proposition can be deduced easily. Take \( \eta \) a closed geodesic on \( M \), different from \( \eta_t \) or its integer powers. Since \( M \setminus K(t) \) is a cylinder, \( \eta \) must intersect the interior of \( K(t) \), otherwise it would have the same free homotopy class as an integer power of \( \eta_t \). Hence, we can follow the reasoning from lemma [10] the only difference being that instead of an arbitrary point, we take \( q \in \eta \cap K(t) \).

**Lemma 12.** Let \( M = \Gamma \setminus \mathbb{H} \) be a complete hyperbolic surface of finite area (with cusps). Then \( M \) can be decomposed into a compact subsurface \( K \) and a finite number of cusps, by cutting along horocycles of length 2, exactly one for each cusp. Moreover, the infinite part of each cusp is isometric to the cylinder \( \{ z \in \mathbb{H} : 1 < \Im z \text{ and } 0 \leq \Re(z) < 2 \} \). In conclusion \( L(r) < C e^r \), where \( C \) depends only on \( \text{Diam}(K) \) and the area of \( M \).

**Proof.** For simplicity we suppose the surface has only one cusp. Consider a decomposition into pairs of pants. The one pair, say \( P \), that contains the cusp will have a boundary of length 0 and two other boundaries of positive length. We now regard \( P \) as a limit of hyperbolic pants \( P_l \). One boundary of \( P_l \) has length \( l > 0 \), and the other two are precisely the two positive lengths of the boundaries of \( P \). Glue back the pieces of the initial surface, using \( P_l \) instead of \( P \). To the resulting surface, say \( M_l \) (which has a boundary of length \( l \)), we glue a hyperbolic torus with geodesic boundary such that, in the end we get a compact hyperbolic surface \( N_l \). As \( l \) goes to 0, the compact sub-surface \( M_l \subset N_l \) will go to \( M \). To finish, we adapt the proof of proposition [11] in figure 2 we can see that the geodesic \( AF \) tends to a horizontal line as \( l \to 0 \). Moreover, at the limit, the points \( R' \), \( S \) and \( R \) will
have the same height. We redefine $K(t)$ to be bounded by the euclidean line (instead of the geodesic line) between $R', S$ and $R$. This yields the desired horocycle. Its length can be easily computed, for example we suppose that the cusp corresponds to the translation $z \mapsto z + 1$. The bound $L(r) < Ce^r$ is obtained by passing to the limit the same inequality in the previous proposition.

4. Trace formula on hyperbolic surfaces

The trace formula was first introduced by Selberg [17], in 1957. In this section we give a self-contained elementary proof of the trace formula for the Dirac operator on complete hyperbolic surfaces of finite area. We treat simultaneously both the compact and the non-compact case, paying more attention to the latter. In [1], Bolte and Stiepan prove the formula for a compact surface. However, the non-compact case is, to our knowledge, new. Note that $D^+ D^- = D^- D^+$ have the same spectrum. So we shall only present the case of positive spinors, i.e. we will look at the operator $D^- D^+$.

**Definition 13.** Let $\Gamma$ be a discrete subgroup of $\text{PSL}_2(\mathbb{R})$. We say that $\mu \in \Gamma$ is primitive if it cannot be written as $\mu = \gamma^n$, for $n \geq 2$ and $\gamma \in \Gamma$.

**Remark 14.** On a hyperbolic surface $M = \Gamma \backslash \mathbb{H}$, the set of closed, oriented geodesics is in one to one correspondence with the conjugacy classes of $\Gamma$. Hence, by $l(\gamma)$ we denote the length of the unique geodesic associated to the class $[\gamma] \subset \Gamma$.

**Theorem 15.** Let $\Gamma$ be a subgroup of $\text{PSL}_2(\mathbb{R})$ and $M = \Gamma \backslash \mathbb{H}$ be the corresponding hyperbolic surface, with $k$ cusps. Suppose that $\varepsilon(\gamma) = -1$ for each parabolic element $\gamma \in \Gamma$. Consider $\{\lambda_j\}_{j \in \mathbb{N}}$ the eigenvalues of $D^- D^+$ on $M$ and chose a sequence $\{\xi_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ such that $\lambda_j = \xi_j^2$. Let $\phi \in C^\infty_c(\mathbb{R})$ and define $a, v, u : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$a(t) = \frac{\phi(t)}{\sqrt{t+4}};$$

$$v(t) = 4 \cosh \left(\frac{1}{2}\right) \int_0^{\infty} a \left(4 \sinh^2 \left(\frac{t}{2}\right) + y^2\right) dy;$$

$$u(\xi) = \hat{v}(\xi) = \int_\mathbb{R} v(t) e^{-i\xi t} dt.$$

Then:

$$\sum_{j=0}^{\infty} u(\xi_j) = \frac{\text{Area}(M)}{4\pi} \int_\mathbb{R} \xi u(\xi) \coth(\pi \xi) d\xi + \sum_{[\mu]} \sum_{n=1}^{\infty} \frac{l(\mu) \varepsilon^n(\mu) v(nl(\mu))}{2 \sinh \left(\frac{nl(\mu)}{2}\right)} - k \log(2) v(0),$$

where $[\mu]$ runs over all conjugacy classes of primitive, hyperbolic elements in $\Gamma$.

4.1. Eigenspinors of Dirac on the hyperbolic plane. Let $\tilde{S}$ be the spinor bundle on the universal covering of $M$. Consider the following smoothing operator:

$$\Phi : C^\infty(\mathbb{D}, \tilde{S}) \longrightarrow C^\infty(\mathbb{D}, \tilde{S}) \quad \Phi s(z) := \int_\mathbb{D} \phi(2 \cosh d(z, z') - 2) \tau_{z' \rightarrow z} s(z') dg_{\mathbb{D}}(z').$$
where $\tau_{z' \to z}$ is the parallel transport in $\tilde{S}$ with respect to $\nabla$ from $z'$ to $z \in \mathbb{D}$. Note that if $\sigma(z)$ is a positive spinor, then so is $\tau_{z' \to z} \sigma(z')$. We want to show that $\Phi$ and $D^- D^+$ have the same eigenspinors. In order to do this, we need some preparations. Define $P$ to be the projector:

$$P : C^\infty(\tilde{S}) \to C^\infty(\tilde{S}) \quad P \sigma(z) = P f \sigma^+(z) := \int_0^1 f(ze^{2\pi i t}) dt \sigma^+(z),$$

where $\sigma^+$ was defined in (2). We have three immediate properties:

1) $P^2 = P$;
2) $P \circ \Phi(0) = \Phi \circ P(0)$;
3) $P \circ D^- D^+ = D^- D^+ \circ P$.

The first one is obvious. To see the second property, we parametrize the unit disk $(r, \theta) \mapsto z$ by polar coordinates, $(r, \theta) \in [0, 1) \times [0, 2\pi]$. Notice that $\sigma^+$ is parallel along the radii of the disk, hence $\tau_{z' \to z} f \sigma^+(z') = f(z') \sigma^+(0)$. The third identity follows from the fact that $P$ commutes with both $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ (see (3)).

Therefore, if $f \sigma^+$ is an eigenspinor for $D^- D^+$, then so is $P f \sigma^+$. Moreover, combining the two formulae from (3), $h = P f$ is a solution of the second order differential equation:

$$\left( -\frac{2 - r^2}{4} - \frac{(1 - r^2)^2}{4r} \frac{\partial}{\partial r} - \frac{(1 - r^2)^2}{4} \frac{\partial^2}{\partial r^2} \right) h = \lambda h,$$

where $h(r)$ is smooth on $[0, 1)$. If we denote $h'$ by $\tilde{h}$, then (4) is equivalent to the system:

$$\begin{bmatrix} 0 & \frac{1}{r} \\ -\frac{2 + 4\lambda - r^2}{(1 - r^2)^2} & -\frac{1}{r} \end{bmatrix} \begin{bmatrix} h \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} h' \\ \tilde{h}' \end{bmatrix}.$$ 

Therefore, the Wronskian is a solution of:

$$w' + \frac{w}{r} = 0,$$

with the general solution:

$$w = \frac{1}{r} \cdot c,$$

where $c$ is a constant.

**Lemma 16.** The space of solutions of (4), smooth on $[0, 1)$, has dimension 1.

**Proof.** Since the Wronskian blows up at 0, the dimension is at most 1. Thus, it is enough to find one smooth solution. A direct computation tells us that the section $x + iy \mapsto y \xi s^+$ is an eigenspinor of $D^- D^+$ of eigenvalue $-(\frac{1}{2} - \xi)^2$ on the upper half-plane $\mathbb{H}$. Pulling this section back on $\mathbb{D}$ and applying $P$ finishes the lemma.

We are now ready to prove the following proposition:
Proposition 17. For any \( \xi \in \mathbb{C} \) there exists \( u(\xi) \in \mathbb{C} \) such that if
\[
f: \mathbb{D} \rightarrow \mathbb{C} \quad \text{then} \quad D^- D^+(f \sigma^+) = \xi^2 f \sigma^+,
\]
then
\[
\Phi(f \sigma^+) = u(\xi) f \sigma^+,
\]
for the function \( u \) appearing in the statement of theorem 15.

Proof. First, we claim that is enough to prove the equality at 0. Notice that \( \gamma^* \) preserves Clifford multiplication. Moreover, since it is an isometry, \( (\gamma^{-1})^* \) preserves the connection \( \nabla \) and thus, the parallel transport. Combining these facts we get that
\[
D^- D^+ \gamma_*(f \sigma^+) = \gamma_* D^- D^+ f \sigma^+; \quad \Phi f \sigma^+(z) = \gamma_* (\Phi \gamma^{-1}_*(f \sigma^+)(0)).
\]
We will now see why the equality occurs at the origin. Note that, from the previous lemma, \( Pf \sigma^+ \) completely determined by its value at 0. Thus:
\[
\Phi f \sigma^+(0) = P \Phi f \sigma^+(0) = \Phi P f \sigma^+(0) =: u(\xi) f \sigma^+(0).
\]
An elementary computation tells us what the parallel transport is:
\[
\tau_{z \mapsto w} = -i \frac{z - \overline{w}}{|z - \overline{w}|}.
\]
With this in mind, for the final part of the proposition, we move everything on the upper half plane \( \mathbb{H} \) and carry out the computation there. It is enough to compute the value of \( \Phi \) on a fixed eigenfunction, specifically \( x + iy \mapsto y^{-\xi+1/2} \sigma^+ \):
\[
\Phi \left( y^{-\xi+1/2} \sigma^+ \right)(i) = \int_{\mathbb{H}} \phi \left( 2 \cosh d(i, z') - 2 \right) \tau_{z' \mapsto i} \left( \Im(z')^{-\xi+1/2} \sigma^+ \right) \; dg_{\mathbb{H}}(z')
\]
\[
= \sigma^+(i) \int_0^\infty \int_{\mathbb{R}} \phi \left( \frac{x^2 + (y - 1)^2}{y} \right) y^{-3/2 - \xi} \; dx \; dy.
\]
\[
= \sigma^+(i) \int_0^\infty \int_{\mathbb{R}} \phi \left( X^2 + \frac{(y - 1)^2}{y} \right) y^{-1 - \xi} \; dX \; dy.
\]
\[
= \sigma^+(i) \int_{\mathbb{R}} \phi \left( X^2 + 4 \sinh^2 \left( \frac{t}{2} \right) \right) e^{-\xi t} \; dX \; dt.
\]
Throughout the computation we have:
- changed variables from \( \{x, y\} \) to \( \{X := \frac{x}{\sqrt{y}}, y\} \);
- changed variables \( y = e^t, t \in \mathbb{R} \).
\[\square\]
4.2. A trace formula after the group. We want proposition 17 to hold true for an operator acting on spinors on $M$ as well. Thus let us define the kernel:

\[
G(z, z') := \sum_{\gamma \in \Gamma} \phi(2 \cosh d(\gamma^{-1}z, z') - 2)\gamma_\ast \tau_{z'\rightarrow \gamma^{-1}z},
\]

for $z, z' \in \mathbb{H}$. It is locally finite, since $\phi$ has compact support. Note that the $G$ descends to $M \times M$, since for every $\alpha, \beta \in \Gamma$:

\[
G(\alpha z, \beta z') = \alpha_\ast G(z, z')\beta^{-1}_\ast.
\]

Hence, it produces an operator:

\[
\mathcal{G} : C^\infty(M, S) \rightarrow C^\infty(M, S); \quad \mathcal{G}\sigma(z) = \int_M G(z, z')\sigma(z')dg(z').
\]

If $M$ is finite, the sum becomes globally finite. Otherwise, we claim that it is bounded.

**Proposition 18.** The sum defining the kernel $G$ is uniformly bounded on $M$.

*Proof.* Let $F$ be a Dirichlet domain for $M$. Take $r$ a positive real number large enough such that $\phi(2 \cosh d(z, z') - 2) = 0$ whenever $d(z, z') \geq r$. We claim that the sum defining the kernel $G$ is uniformly bounded on $F$ (and therefore on $M$). Note that $G$ only depends on the distance between $z$ and $z'$, hence it suffices to control it on a disk of radius $r$ centred at $z$. We distinguish two cases: when $\pi(z)$ (the projection of $z$ on the surface) lies inside a compact region $K \subset M$ and when $\pi(z)$ lies inside an unbounded region (usually, one of the cusps). If $\pi(z) \in K$, then the set $\{z' \in \mathbb{H} : d(z', K) < r\}$ can be intersected only by a finite number of fundamental domains $\gamma F$, for $\gamma \in \Gamma$. Therefore, only a finite number of terms in the sum (7) do not vanish.

From lemma 12 there exist horospheres delimiting a compact region of $M$ from the cusps. We now study the case when $\pi(z)$ lies in a cusp and the distance between itself and the closest horosphere is at least $r$. Furthermore, we can also suppose that said cusp lifts to the vertical cylinder of width 1. Then, $\phi(2 \cosh d(z, \gamma z') - 2)$ does not vanish only if $z'$ is above the horosphere and $\gamma$ is a translation. Thus, writing $z = x + iy$ and $z' = x' + iy'$, $G$ reads:

\[
G(z, z') = \sum_{n \in \mathbb{Z}} \phi(2 \cosh d(z - n, z') - 2)(-1)^n\tau_{z'\rightarrow z-n} = \sum_{n \in \mathbb{Z}} a \left( \frac{(x - n - x')^2 + (y - y')^2}{yy'} \right) e^{i\pi n y + y' + i(x - n - x')} \sqrt{yy'},
\]

where $a$ is the function defined in theorem 15. A fruitful way to regard this sum is by the Poisson summation formula:

\[
(8) \quad G(z, z') = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} a \left( \frac{(x - n - x')^2 + (y - y')^2}{yy'} \right) e^{i\pi n (1-2m) y + y' + i(x - n - x')} \sqrt{yy'} dn.
\]

Next, we take each term, denoted $F(m, y, y')$, separately and change the variable from $n$ to $X := (n + x - x')(yy')^{-1/2}$:
\[ F(m, y, y') = e^{i\pi(x - x')(1 - 2m)} \int_{\mathbb{R}} a \left( X^2 + \frac{(y - y')^2}{yy'} \right) e^{i\pi X \sqrt{yy'}(1 - 2m)}(y + y' - iX \sqrt{yy'})dX. \]

Since \( y, y' \) and \( 1 - 2m \) do not vanish, we claim that \( \left| \sqrt{yy'}(1 - 2m) \right|^p \) goes to 0 as \( \left| \sqrt{yy'}(1 - 2m) \right| \) goes to \( \infty \), for any \( p \) a positive integer. This result is a particular case of the stationary phase principle. Yet, it can be easily seen by integrating by parts:

\[ F(m, y, y') = e^{i\pi(x - x')(1 - 2m)} \int_{\mathbb{R}} a \left( X^2 + \frac{(y - y')^2}{yy'} \right) \frac{\partial}{\partial X} \left( e^{i\pi X \sqrt{yy'}(1 - 2m)} \right)(y + y' - iX \sqrt{yy'})dX \]

Continue inductively \( p \) more steps, and then notice that the integral is bounded because \( a \) has compact support. Therefore, the sum \( (8) \) is uniformly bounded in the cusp as well. □

**Remark 19.** The hypothesis \( \varepsilon(\gamma) = -1 \), for each \( \gamma \) a parabolic, primitive isometry, means that the power of the exponential in the integral is a multiple of \( 1 - 2m \), which does not vanish since \( m \) is an integer. This fact was essential in our proof.

Consider \( \sigma_j \in C^\infty(M) \) an eigenspinor of \( D^- D^+ \) with eigenvalue \( \xi_j^2 \). Since the following diagram commutes:

\[ P_{\text{Spin}(n)} \mathbb{H} \times_\rho \Sigma_2 \longrightarrow P_{\text{Spin}(n)} M \times_\rho \Sigma_2 \]

we can take \( \tilde{\sigma}_j \) a lift of \( \sigma_j \). Using proposition \( 17 \) and changing variables we get:

\[ u(\xi_j)\tilde{\sigma}_j(z) = \Phi \tilde{\sigma}_j(z) = \sum_{\gamma \in F} \phi(2 \cosh d(\gamma^{-1}z, z') - 2)\gamma \tau_{z' \rightarrow \gamma^{-1}z} \tilde{\sigma}_j(z')dg_{\mathbb{H}}(z') \]

\[ = \int_F G(z, z')\tilde{\sigma}_j(z')dg_{\mathbb{H}}(z') = G\sigma_j(z), \]

where \( F \) is a fundamental domain of our surface \( M \). Proposition \( 18 \) allowed us to commute the integral with the sum. The trace of \( G \) can be computed as the integral of the kernel \( G \).
along the diagonal:
\[
\sum_{j=0}^{\infty} u(\xi_j) = \int_M G(z, z) dg_M(z) = \int_F \sum_{\gamma \in \Gamma} \phi(2 \cosh d(\gamma^{-1} z, z) - 2) \gamma_* \tau_{z, z}^{-1} d\mathcal{H}(z)
\]
\[
= \phi(0) \text{Area}(M) + \sum_{\gamma \neq 1} \int_F \phi(2 \cosh d(\gamma^{-1} z, z) - 2) \gamma_* \tau_{z, z}^{-1} d\mathcal{H}(z)
\]
(9)

4.3. Proof of theorem 15. We start with the relation (9) and compute the right-hand side by rearranging the terms in conjugacy classes, following Selberg’s original idea.

Lemma 20. For functions \( \phi \) and \( u \) as in theorem 15 we have:
\[
\phi(0) = \frac{1}{4\pi} \int_\mathbb{R} \xi u(\xi) \coth(\pi \xi) d\xi
\]

Proof. We start with the function \( a \) from theorem 15. Using polar coordinates we see that:
\[
-\frac{\pi}{4} a(z) = \int_0^\infty \int_0^\infty a'(z + x^2 + y^2) dx dy.
\]
Changing the variable \( y = 2 \sinh(t) \) and evaluating at 0 we get:
\[
a(0) = -\frac{4}{\pi} \int_0^\infty \int_0^\infty \frac{\partial}{\partial t} a \left( x^2 + 4 \sinh^2 \left( \frac{t}{2} \right) \right) \frac{1}{4 \sinh \left( \frac{t}{2} \right)} dxdt
\]
\[
= -\frac{1}{2\pi} \left( \int_0^\infty \frac{v'(t)}{\sinh(t)} dt + \int_0^\infty \frac{v(t)}{4 \cosh^2 \left( \frac{t}{2} \right)} dt \right).
\]
Now we proceed to compute each integral separately. Starting with the first one, we apply the Fourier inverse formula for the derivative of the even function \( v \) and write \( \sinh(t) \) as a power series
\[
\int_0^\infty \frac{v'(t) - v'(-t)}{2 \sinh(t)} dt = \frac{1}{2\pi} \int_0^\infty \int_\mathbb{R} \sum_{m=0}^{\infty} u(\xi) i \xi e^{-t(2m+1)} (e^{i\xi t} - e^{-i\xi t}) d\xi dt
\]
\[
= \frac{1}{4} \int_\mathbb{R} \xi u(\xi) \sum_{m=0}^{\infty} \frac{i}{\pi} \left( \frac{2}{1 + 2m - i\xi} - \frac{2}{1 + 2m + i\xi} \right) d\xi
\]
\[
= -\frac{1}{4} \int_\mathbb{R} \xi u(\xi) \tanh \left( \frac{\pi \xi}{2} \right) d\xi.
\]
The last equality can be seen by applying the identity:
\[
\frac{i}{\pi} \sum_{m=0}^{\infty} \left( \frac{1}{\frac{1}{2} + m - iz} - \frac{1}{\frac{1}{2} + m + iz} \right) = \tanh(\pi z),
\]
for \( \xi = 2z \). The identity (10) is well-known, one can prove it using the Poisson summation formula for \( x \mapsto \frac{1}{\pi (x+1/2)^2 + \xi^2} \), followed by the residue theorem (see e.g. [12]). For the second
integral, we use again the fact that \(v\) is even and apply the Fourier inversion formula. Next, we compute the integral in the variable \(t\) with the residue theorem

\[
\int_{\mathbb{R}} \frac{v(t)}{8 \cosh^2 \left( \frac{t}{2} \right)} dt = \frac{1}{16\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u(\xi) e^{it\xi}}{\cosh^2 \left( \frac{t}{2} \right)} dtd\xi = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{u(\xi) \xi}{\sinh(\pi\xi)}. \]

\[\square\]

We shall now continue with the studying of the second term in the right-hand side of (9). Two classical lemmas will play an important part.

**Lemma 21.** Let \(\gamma \in \Gamma\) be a non-trivial element (either hyperbolic or parabolic). Then the commutator subgroup \(C(\gamma)\) is infinite cyclic.

**Lemma 22.** Let \(\gamma \in \Gamma\) and \(\mu\) be a primitive (in the sense of 13) element in \(C(\gamma)\) (it exists from the previous lemma). Define \(A_\mu\) to be a system of representatives for \(\Gamma/\langle \mu \rangle\) (not necessarily a subgroup). Then the conjugacy class \([\gamma]\) can be described as \(\{\alpha \gamma \alpha^{-1} : \alpha \in A_\mu\}\).

We can rewrite the second term in the right-hand side of (9) as:

\[
\sum_{\gamma \neq 1} \int_F \phi(2 \cosh d(\gamma^{-1}z, z) - 2) \gamma_* \tau_{2\rightarrow\gamma^{-1}z} dg_\mathbb{H}(z)
\]

\[
= \sum_{[\mu]} \sum_{n=1}^\infty \sum_{\alpha \in A_\mu} \int_F \phi(2 \cosh d(\alpha \mu^{-n} \alpha^{-1} z, z) - 2) (\alpha \mu^n \alpha^{-1})_* \tau_{2\rightarrow(\alpha \mu^{-n} \alpha^{-1})z} dg_\mathbb{H}(z).
\]

We only need one last result to complete the proof.

**Lemma 23.** The sum after \(\alpha\) can be computed in the following way

- If \(\mu\) is hyperbolic, then:

\[
\sum_{\alpha \in A_\mu} \int_F \phi(2 \cosh d(\alpha \mu^{-n} \alpha^{-1} z, z) - 2) (\alpha \mu^n \alpha^{-1})_* \tau_{2\rightarrow(\alpha \mu^{-n} \alpha^{-1})z} dg_\mathbb{H} = \frac{l(\mu) v(nl(\mu))}{2 \sinh \left( \frac{nl(\mu)}{2} \right)};
\]

- If \(\mu\) is parabolic, then:

\[
\sum_{\alpha \in A_\mu} \int_F \phi(2 \cosh d(\alpha \mu^{-n} \alpha^{-1} z, z) - 2) (\alpha \mu^n \alpha^{-1})_* \tau_{2\rightarrow(\alpha \mu^{-n} \alpha^{-1})z} dg_\mathbb{H}
\]

\[
+ \sum_{\alpha \in A_{\mu^{-1}}} \int_F \phi(2 \cosh d(\alpha \mu^n \alpha^{-1} z, z) - 2) (\alpha \mu^{-n} \alpha^{-1})_* \tau_{2\rightarrow(\alpha \mu^n \alpha^{-1})z} dg_\mathbb{H}
\]

\[
= \frac{v(n)}{n}.
\]
Proof. We start with the first part. By changing variables in each term of the sum \( w = \alpha^{-1}z \) we obtain an integral on \( \bigcup_{\alpha \in A_\mu} \alpha^{-1}F \). This set is in fact a fundamental domain for the quotient \( \langle \mu \rangle \setminus \mathbb{H} \). Since the integrand descends on this cylinder, we can compute it on a more convenient fundamental domain, namely the horizontal band \( C := \{ w \in \mathbb{H} : 1 \leq \Im w < e^l \} \), where \( l \) is the length of the unique geodesic associated to \( \mu \) (i.e. \( l = 2 \cosh^{-1} |\text{Tr}(\mu)/2| \)). We will also use the definition of the class function \( \varepsilon \) defining the spin structure \( (1) \), and the formula for the parallel transport \( (6) \). Thus, our initial sum becomes:

\[
\sum_{\alpha \in A_\mu} \int_F \phi(2 \cosh (\alpha \mu^{-n} \alpha^{-1} z, z) - 2)(\alpha \mu^n \alpha^{-1} z) \tau_{z \rightarrow (\alpha \mu^{-n} \alpha^{-1}) z} dg_{\mathbb{H}}(z)
\]

\[
= \int_{\langle \mu \rangle \setminus \mathbb{H}} \phi(2 \cosh (\mu^{-n} w, w) - 2)(\mu^n) \tau_{w \rightarrow \mu^{-n} w} dg_{\mathbb{H}}(w)
\]

\[
= \int_C \phi(2 \cosh (w, e^{nl} w) - 2)\varepsilon^n(\mu)(-i) \frac{w - \bar{w} e^{nl}}{|w - \bar{w} e^{nl}|} dg_{\mathbb{H}}(w)
\]

\[
= \int_1^e \int_{\mathbb{R}} \phi \left( \frac{(x^2 + y^2)(1 - e^{nl})^2}{y^2 e^{nl}} \right) \varepsilon^n(\mu)(-i) \frac{x + iy - e^{nl}(x - iy)}{|(x + iy)e^{nl} - (x - iy)|} \cdot \frac{dx dy}{y^2}.
\]

Next, we change variables two more times, from \( \{ x, y \} \) to \( \{ X := \frac{x}{e^l}, t := \log y \} \) and then to \( \{ z := 2X \sinh(\frac{nl}{2}) \} \):

\[
= \int_{\mathbb{R}} \int_0^l \phi \left( 4(X^2 + 1) \sinh^2 \left( \frac{nl}{2} \right) \right) \varepsilon^n(\mu)(-i) \frac{X(1 - e^{nl}) + i(1 + e^{nl})}{\sqrt{X^2(e^{nl} - 1)^2 + (e^{nl} + 1)^2}} \cdot dX \cdot dt
\]

\[
= \frac{l \varepsilon^n(\mu)}{2 \sinh \left( \frac{nl}{2} \right)} \int_{\mathbb{R}} a \left( z^2 + 4 \sinh^2 \left( \frac{nl}{2} \right) \right) dz \cdot 2 \cosh \left( \frac{nl}{2} \right) = \frac{l \varepsilon^n(\mu) v(nl)}{2 \sinh \left( \frac{nl}{2} \right)}.
\]

For parabolic elements, we proceed similarly. Since \( \langle \mu \rangle \setminus \mathbb{H} = \langle \mu^{-1} \rangle \setminus \mathbb{H} \), we can suppose that \( A_\mu = A_{\mu^{-1}} \). After changing the variables (the same changes for both \( \mu \) and \( \mu^{-1} \)), we can compute the integrals on the cylinder \( C' := \{ w \in \mathbb{H} : 0 \leq \Re w < 1 \} \), which is a fundamental domain of the translation by 1 (or its inverse). Moreover, notice that integrand does not depend on the real part of \( w \). Thus, the right-hand side becomes:

\[
\int_{C'} \left( \phi(2 \cosh (w - n, w) - 2)\varepsilon^n(\mu)(-i) \frac{w - \bar{w} + n}{|w - \bar{w} + n|} \right) \cdot d\mu
\]

\[
+ \phi(2 \cosh (w + n, w) - 2)\varepsilon^n(\mu)(-i) \frac{w - \bar{w} - n}{|w - \bar{w} - n|} \right) d\mu
\]

\[
= \int_0^\infty \phi \left( \frac{n^2}{y^2} \right) \varepsilon^n(\mu)(-i) \left( \frac{2iy + n}{|2iy + n|} \right) + \frac{2iy - n}{|2iy - n|} \right) \frac{dy}{y^2}
\]

\[
= \varepsilon^n(\mu) 4 \int_0^\infty \frac{a \left( \frac{n^2}{y^2} \right) dy}{y^2} = \frac{\varepsilon^n(\mu) v(0)}{n}.
\]
At this point, the proof of theorem \ref{thm:15} is immediate. We start from the equality \eqref{eq:9}, and then apply both lemma \ref{lem:20} and lemma \ref{lem:23}.

4.4. The trace formula for a larger class of functions. We can prove the trace formula for more general functions \( v \) by approximations with compactly supported functions.

Definition 24. We say that \( v \) is an admissible function if:

- it has exponential decay: \( |v(x)| \leq ce^{-|x|(1/2+\epsilon)} \), for some \( c > 0 \) and some \( \epsilon > 0 \);
- There exists \( \epsilon' > 0 \) such that \( \xi^{2+\epsilon'} u(\xi) \in L^1(\mathbb{R}) \).

Remark 25. The second condition is equivalent with the fact that \( u \) belongs to the Sobolev space \( W^{2+\epsilon,1}(\mathbb{R}) \).

Theorem 26. Theorem \ref{thm:15} holds true for admissible functions \( v \).

Proof. Take \((v_m)_{m \in \mathbb{N}}\) an increasing sequence of compactly supported functions converging pointwise to \( v \). It follows that \( \hat{v}_m \) also converges point-wise to \( \hat{v} = u \). From the trace formula \ref{thm:15}, we get a sequence of equalities which we would like to pass to the limit. To do so, we will show that each side is bounded. Let us start with the left-hand side. The derivatives of \( v \) are integrable, hence \( u \in \mathcal{O}(\xi^{-2-\epsilon'}) \). If \( M \) is compact, we can use Weyl’s law to deduce that that \( \xi_j \sim j^{1/2} \), for large \( j \). Otherwise, since the spin structures along the cusp is not trivial, we can use the generalisation of Weyl’s law (see \cite{13}) to deduce the same approximation. In conclusion

\[
\sum_{j=0}^{\infty} u(\xi_j) \sim \sum_{j=1}^{\infty} j^{-1-\epsilon'/2} < \infty.
\]

To bound the right-hand side we first rewrite the two sums as a sum after all closed geodesics on \( M \) (not necessarily primitive). Let \( l_k \) denote the length of the \( k \)-th smallest geodesic, counted with multiplicity. From lemma \ref{lem:10} we get that \( l_k \sim \log k \), for large \( k \). Thus, using the exponential decay of \( v \), we get:

\[
\sum_{l \in \mathcal{L}} \sum_{n=1}^{\infty} \frac{l|v(nl)|}{2 \sinh \left( \frac{m}{2} \right)} \leq \sum_{k=1}^{\infty} \frac{l_k|v(l_k)|}{2 \sinh \left( \frac{1}{2} \right)} \sim \sum_{k=1}^{\infty} \log(k) k^{-1-\epsilon} < \infty.
\]

5. The non-compact case as a limit of compact cases

Now we have theorem \ref{thm:15} for both compact and non-compact surfaces of finite area. We use the same notations as before. One may wonder if the formula for the compact surfaces converges to the one for surfaces with cusps when \( M \) goes through a pinching process (definition \ref{def:1}). In this section we prove that this is indeed so. For simplicity, we present the case when the length of only one geodesic converges to 0.
Theorem 27. Let $M$ be a compact hyperbolic surface going through a pinching process after $\eta$ (corresponding to the conjugacy class $[\mu_t]$), for $\mu_t \in \Gamma_t$ and $t \in [0, 1])$. If $\varepsilon(\mu_t) = -1$ (i.e. the spin structure along $\eta$ is not trivial), then:

$$\sum_{[\gamma]} \sum_{n=1}^{\infty} \frac{l_t(\gamma)\varepsilon^n(\gamma)v(nl_t(\gamma))}{2\sinh\left(\frac{nl_t(\gamma)}{2}\right)} \rightarrow \sum_{[\gamma]} \sum_{n=1}^{\infty} \frac{l_0(\gamma)\varepsilon^n(\gamma)v(nl_0(\gamma))}{2\sinh\left(\frac{nl_0(\gamma)}{2}\right)} - 2\log(2)v(0),$$

as $t \to 0$, where $[\gamma]$ varies over all conjugacy classes of primitive, hyperbolic elements. In consequence, we get theorem 3.

Proof. There are three types of closed geodesics on $M$: those who do not intersect $\eta$, those who do intersect it transversally, and $\eta$ itself. The length of the geodesics in the first category can vary during the process, but remains bounded and does not vanish. The length of those in the second category increases to $\infty$. The sum after the conjugacy classes can be split as follows:

$$\sum_{[\gamma]} \sum_{n=1}^{\infty} \frac{l_t(\gamma)\varepsilon^n(\gamma)v(nl_t(\gamma))}{2\sinh\left(\frac{nl_t(\gamma)}{2}\right)} = 2\sum_{n=1}^{\infty} \frac{l_t(\mu_t)\varepsilon^n(\mu_t)v(nl_t(\eta))}{2\sinh\left(\frac{nl_t(\eta)}{2}\right)} + \sum_{[\gamma]\neq[\mu_t]} \sum_{n=1}^{\infty} \frac{l_t(\gamma)\varepsilon^n(\gamma)v(nl_t(\gamma))}{2\sinh\left(\frac{nl_t(\gamma)}{2}\right)}.$$

We study each term separately. Let us consider the function:

$$f : [0, \infty) \longrightarrow \mathbb{R}, \quad f(x) := \frac{xv(2x)}{\sinh(x)}.$$

In terms of $f$, the first term becomes:

$$2\sum_{n=1}^{\infty} \frac{l_t(\eta)\varepsilon^n(\mu_t)v(nl_t(\eta))}{2\sinh\left(\frac{nl_t(\eta)}{2}\right)} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}f\left(\frac{nl_t(\eta)}{2}\right)$$

$$= 2\sum_{j=1}^{\infty} \frac{f(jl_t(\eta)) - f\left(\frac{2j-1}{2}l_t(\eta)\right)}{2j-1} - \frac{f(jl_t(\eta))}{2j(2j-1)}$$

$$= 2\sum_{j=1}^{\infty} \frac{jl_t(\eta)f'(c_{j,t})}{2j(2j-1)} - \frac{f(jl_t(\eta))}{2j(2j-1)},$$

where $c_{j,t} \in [(2j-1)l_t(\eta)/2, jl_t(\eta)]$ is given by the mean value theorem. Since both $|f(x)|$ and $|xf'(x)|$ are bounded, the above sum is bounded uniformly in $t$, and hence:

$$2\sum_{n=1}^{\infty} \frac{l_t(\eta)\varepsilon^n(\mu_t)v(nl_t(\eta))}{2\sinh\left(\frac{nl_t(\eta)}{2}\right)} \rightarrow -2\log(2)v(0) \quad \text{as } t \to 0.$$

To tackle the second term, let us denote $l'_m(t)$ the length of the $m$–th smallest closed, oriented geodesic on $M$, with respect to the metric $g_t$, excluding $\eta$ and its multiples. Using the estimates from proposition 11, we get that $l'_m(t) \geq \log(m) - C$, where $C$ is bounded
in $t$. Taking the sum after every closed, oriented geodesic (not only primitive ones), we obtain:
\[
\sum_{[\gamma] \neq [\mu]} \sum_{n=1}^{\infty} \frac{l_n(\gamma)\varepsilon_n(\gamma)\nu(nl_n(\gamma))}{2 \sinh \left(\frac{nl_n(\gamma)}{2}\right)} \leq \sum_{m=1}^{\infty} \frac{\nu_m(t)|v'(\nu_m(t))|}{2 \sinh \left(\frac{\nu_m(t)}{2}\right)} \leq C \sum_{m=1}^{\infty} \log(m)m^{-1-\epsilon}.
\]
Therefore we can pass to the $t$-limit inside the initial two sums. The geodesic $\eta$ will produce the logarithmic term, the geodesics that intersect $\eta$ will vanish ($v$ has exponential decay), and all other geodesics will eventually stabilise. Thus, the right-hand side of the trace formula converges, and so we obtain the convergence of the left-hand side as well, i.e. theorem 3.

\[\Box\]

6. Applications

Two results can be obtained from theorems 15 and 27. The first one is the heat trace asymptotic for the squared Dirac on a surface with cusps. The second one is a uniform Weyl law for a surface going through a pinching process. Both these results make use of the following family of functions:

\[v_T : \mathbb{R} \rightarrow \mathbb{R}, \quad v_T(x) := \frac{e^{-x^2/4T}}{\sqrt{4\pi T}},\]

where $T > 0$. Clearly $v_T$ is admissible. Moreover, its Fourier transform is given by:

\[u_T : \mathbb{R} \rightarrow \mathbb{R}, \quad u_T(\xi) := e^{-T\xi^2}.\]

6.1. Heat trace asymptotic. Consider $M$ a hyperbolic surface with $k$ cusps. Suppose we pick a spin structure which satisfies the hypothesis of theorem 15.

Proof of theorem 15. The trace formula reads:

\[\sum_{j=0}^{\infty} e^{-T\lambda_j} = \frac{\text{Area}(M)}{4\pi} \int e^{-T\xi^2} \xi \coth(\pi \xi) d\xi + \sum_{[\mu]} \sum_{n=1}^{\infty} \frac{l_\mu n(\mu)\varepsilon_n(\mu)\nu(nl_n(\mu))/4T}{2 \sinh \left(\frac{nl_n(\mu)}{2}\right) \sqrt{4\pi T}} - k \log(2) \sqrt{4\pi T},\]

where $[\mu]$ runs over all classes of primitive, hyperbolic elements. But the length of these elements is bounded from below by a positive constant. Thus, the sum in second term from the right-hand side decreases to 0 exponentially fast as $T \rightarrow 0$. Regarding the integral, we can compute it’s asymptotic directly. Denote:

\[I(T) := \int e^{-T\xi^2} \xi \coth(\pi \xi) d\xi.\]

Changing the variables to $\xi' = \xi \sqrt{T}$ one can easily see that $\lim_{T \rightarrow 0} TI(T) = 1$. Now consider the difference:

\[I(T) - \frac{1}{T} = \int e^{-T\xi^2} (\xi \coth(\pi \xi) - |\xi|) d\xi.\]
Note that the map $\xi \mapsto \coth(\pi \xi) - |\xi|$ vanishes exponentially fast at infinity. Hence:

$$I(T) - \frac{1}{T} \sim \sum_{m=0}^{\infty} a_m T^m; \quad a_m := \int_{\mathbb{R}} e^{-T\xi^2} (-1)^j \xi^{2m} (\coth(\pi \xi) - |\xi|) d\xi,$$

meaning that

$$\left| I(T) - \frac{1}{T} - \sum_{m=1}^{n} a_m T^m \right| \in O(T^{n+1}).$$

Therefore:

$$\sum_{j=0}^{\infty} e^{-T\lambda_j} \sim \frac{\text{Area}(M)}{4\pi T} - \frac{k \log(2)}{\sqrt{4\pi T}} + \sum_{m=0}^{\infty} a_m \frac{\text{Area}(M)}{4\pi} T^m, \quad \text{as } T \searrow 0,$$

\[\square\]

**Remark 28.** The only term in the asymptotic that depends on the surface is \(\frac{k \log(2)}{\sqrt{4\pi T}}\). The other terms come from the action of \(D\) on the hyperbolic plane and are the same for any surface.

### 6.2. Uniform Weyl law

As before, consider \(M\) a compact hyperbolic surface, and \(\eta\) a simple closed geodesic on it. Suppose that \(M\) goes through a pinching process (definition 1) after \(\eta\) (corresponding to the class \([\mu_t] \in \Gamma_t\), for \(t \in [0, 1]\)). This means that at \(t = 0\) we obtain a hyperbolic surface with 2 cusps.

**Proof of theorem** We start with the direct implication, hence \(\epsilon(\mu_t) = -1\). By \(\lambda_j(t)\) we denote the \(j\)-th largest eigenvalue (counted with multiplicity) of \(D^2_t\), the Dirac operator corresponding to the metric \(g_t\). Again, applying the trace formula for \(u_T(\xi) = e^{-T \xi^2}\), we get:

$$\sum_{j=0}^{\infty} e^{-T\lambda_j(t)} = \frac{\text{Area}(M)}{4\pi} \int_{\mathbb{R}} \xi e^{-T\xi^2} \coth(\pi \xi) d\xi + 2 \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{l(\mu_t)(-1)^n e^{-n^2l^2(\mu)/4T}}{2 \sinh \left(\frac{nl(\mu)}{2}\right) \sqrt{4\pi T}}$$

$$+ \sum_{[\mu] \neq [\mu_t]} \sum_{n=1}^{\infty} l(\mu) \epsilon_{l(\mu)} e^{-n^2l^2(\mu)/4T} 2 \sinh \left(\frac{nl(\mu)}{2}\right) \sqrt{4\pi T}.$$
From here, we finish using Karamata’s theorem. However, for completeness, we will include a sketch of the proof as well. Denote \( \alpha := \frac{\text{Area}(M)}{4\pi} \). From (11) one can easily see that:

\[
\lim_{T \to 0} T \sum_{j=0}^{\infty} e^{-T\lambda_j(t)} P(e^{-T\lambda_j(t)}) = \alpha \int_0^1 P(s) ds,
\]

where \( P(s) := \sum_{m=0}^{r} c_m s^m \) is an arbitrary polynomial. Clearly this convergence is uniform in \( t \). Since polynomials are dense in continuous functions on \([0, 1]\), we obtain:

\[
(12) \quad \lim_{T \to 0} T \sum_{j=0}^{\infty} e^{-T\lambda_j(t)} f(e^{-T\lambda_j(t)}) = \alpha \int_0^1 f(s) ds,
\]

uniformly in \( t \), where \( f : [0, 1] \to \mathbb{R} \) is the continuous function defined by:

\[
f(s) = \begin{cases} 
0 & s \in [0, a]; \\
\frac{s-a}{b-a} & s \in [a, b]; \\
\frac{1}{s} & s \in [b, 1],
\end{cases}
\]

where \( a < b \) are two numbers between 0 and 1. Hence, for \( T \) small enough, (12) implies:

\[
TN_t \left( \frac{-\log a}{T} \right) \geq T \sum_{j=0}^{\infty} e^{-T\lambda_j(t)} f(e^{-T\lambda_j(t)}) \geq TN_t \left( \frac{-\log b}{T} \right).
\]

Recall that \( N_t(\lambda) \) is the number of \( D_t^2 \)-eigenvalues smaller or equal than \( \lambda \). Subtracting \( \alpha \int_0^1 f(s) ds \) in the middle term of the inequality above, we get:

\[
TN_t \left( \frac{-\log a}{T} \right) + \alpha \log b \geq T \sum_{j=0}^{\infty} e^{-T\lambda_j(t)} f(e^{-T\lambda_j(t)}) - \alpha \int_0^1 f(s) ds
\]

\[
\geq TN_t \left( \frac{-\log b}{T} \right) + \alpha \log a,
\]

which, dividing by \(- \log b\) can be rewritten:

\[
\frac{T}{-\log a \log b} N_t \left( \frac{-\log a}{T} \right) - \alpha \geq \frac{1}{-\log b} \left( T \sum_{j=0}^{\infty} e^{-T\lambda_j(t)} f(e^{-T\lambda_j(t)}) - \alpha \int_0^1 f(s) ds \right)
\]

\[
\geq \frac{T}{-\log b} N_t \left( \frac{-\log b}{T} \right) - \alpha \frac{\log a}{\log b}.
\]

Since (12) was uniformly in \( t \in [0, 1] \), when \( T \searrow 0 \), we obtain two inequalities:

\[
\frac{\log a}{\log b} \liminf_{\lambda \to 0} \frac{N_t(\lambda)}{\lambda} \geq \alpha,
\]

\[
\alpha \frac{\log a}{\log b} \geq \limsup_{\lambda \to 0} \frac{N_t(\lambda)}{\lambda},
\]

both of them uniformly in \( t \in [0, 1] \). Finally, making \( b \to a \) finishes the proof of the direct implication.
Conversely, let us suppose that $\varepsilon(\mu_t) = 1$. Then, the sum:

$$
2 \sum_{n=1}^{\infty} l(\mu_t)(-1)^n e^{-n^2t^2(\mu_t)/4T} \frac{2}{\sinh \left( \frac{nt}{2} \right)} \sqrt{\frac{n}{4\pi T}},
$$

diverges as $t \to 0$, meaning that the right-hand side of the trace formula, $\sum_{j=0}^{\infty} e^{-T\lambda_j(t)}$, diverges as well. It follows that $N_t(\lambda)$ grows faster than any polynomial as $t \to 0$, therefore the Weyl law is no longer uniform.

\[\square\]

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