NEARLY KÄHLER GEOMETRY AND (2, 3, 5)-DISTRIBUTIONS
VIA PROJECTIVE HOLONOMY

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Abstract. We show that any dimension 6 nearly Kähler (or nearly para-Kähler) geometry arises as a projective manifold equipped with a $G_2^{(1)}$ holonomy reduction. In the reverse direction we show that if a projective manifold is equipped with a parallel 7-dimensional cross product on its standard tractor bundle then the manifold is: a Riemannian nearly Kähler manifold, if the cross product is definite; otherwise, if the cross product has the other algebraic type, the manifold is in general stratified with nearly Kähler and nearly para-Kähler parts separated by a hypersurface which canonically carries a Cartan (2, 3, 5)-distribution. This hypersurface is a projective infinity for the pseudo-Riemannian geometry elsewhere on the manifold, and we establish how the Cartan distribution can be understood explicitly, and also in terms of conformal geometry, as a limit of the ambient nearly (para-)Kähler structures. Any real-analytic (2, 3, 5)-distribution is seen to arise as such a limit, because we can solve the geometric Dirichlet problem of building a collar structure equipped with the required holonomy-reduced projective structure.

Our approach is to use Cartan/tractor theory to understand all structures as arising from a curved version of the algebra of imaginary (split) octonions as a flat structure over its projectivization. The perspective is used to establish results concerning the projective compactification of nearly (para-)Kähler manifolds.

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2010 Mathematics Subject Classification. Primary 53B10, 53A20, 53C29, 53C55; Secondary 53A30, 35Q76. Key words and phrases. projective differential geometry, nearly Kähler, $G_2$ geometry, holography, Einstein metrics, conformal differential geometry.
A.R.G. and R.P. gratefully acknowledge support from the Royal Society of New Zealand via Marsden Grants 10-UOA-113 and 13-UOA-018. T.W. gratefully acknowledges support from the Australian Research Council.
Nearly Kähler geometries form one of the most important classes in the celebrated Gray-Hervella classification of almost Hermitian geometries [1, 50, 70, 76]. A Cartan (2, 3, 5)-distribution is the geometry arising from a maximally nondegenerate distribution of 2-planes in the tangent bundle of a 5-manifold [31]. These have proved to be of great interest for a host of reasons including that: they provide a first case of a geometry of distributions with interesting local invariants; they arise naturally from a class of second-order ODEs; they are linked to concrete realizations of the exceptional group $G_2$ (and its variants); they have fascinating connections to rolling ball problems; and they also have important links to conformal geometry [11, 31, 71, 53]. Our aim in
this work is to expose and study a beautiful convergence of nearly Kähler and (2,3,5) geometry mediated by projective differential geometry. This uses the algebraic structure of the imaginary (split) octonions, and indeed we use new results and ideas from the general theory of Cartan holonomy reduction to describe a point-dependent imaginary octonion structure on projective 6-manifolds. We then exploit tractor calculus to understand how the differential and algebraic structures interact, enabling, for example, a holographic program for the (2,3,5)-distribution.

Let \((M^n, J)\) be an almost complex manifold of dimension \(n \geq 4\), and \(g\) a (pseudo-)Riemannian metric on \(M\). The triple \((M, g, J)\) is said to be almost (pseudo-)Hermitian if \(J\) is orthogonal with respect to \(g\), that is \(g(JU, JV) = g(U, V)\) for all tangent vector fields \(U, V\). When this holds \(\omega(\cdot, \cdot) := g(\cdot, J \cdot)\) is a 2-form called the Kähler form. If the almost complex structure in addition satisfies

\[(\nabla_U J)U = 0, \quad \forall U \in \Gamma(TM),\]

where \(\nabla\) is the Levi-Civita connection, or equivalently that \(\nabla \omega\) is totally skew, then the almost Hermitian manifold is called a nearly Kähler geometry. In dimension \(n = 4\) the equation \((1)\) implies the structure is Kähler, but in higher dimensions it is a strictly weaker condition. Throughout the article we will assume that any nearly Kähler geometry is strictly nearly Kähler, meaning that \(\nabla \omega\) vanishes nowhere. These structures are especially important in dimension 6 \([49, 70, 77]\), the dimension which is key in this article.

A projective structure on a manifold \(M\) is an equivalence class \(p\) of torsion-free affine connections, where two connections \(\nabla\) and \(\nabla'\) are said to be equivalent if they share the same geodesics as unparameterized curves. On a nearly Kähler manifold with metric \(g\), the Levi-Civita connection \([\nabla g]\) determines a projective structure \(p = [\nabla g]\); however this is the trivial aspect of a deeper link. A projective geometry determines, and is equivalent to, a structure called a projective Cartan connection \([30, 61, 32]\); this is very easily seen using an equivalent associated bundle structure called the projective tractor connection \([9, 28]\). A critical point is that this higher order structure has a very special symmetry reduction if a nearly Kähler geometry underlies the projective structure:

**Theorem 1.1.** A nearly Kähler 6-manifold \((M^6, g, J)\) determines a holonomy reduction of the projective Cartan bundle (of \((M, [\nabla g])\)) to the holonomy group \(G_2\) if \(g\) is Riemannian, or to \(G_2^*\) if \(g\) has signature \((2,4)\).

Here \(G_2\) and \(G_2^*\) denote, respectively, the compact and noncompact real forms of the complex exceptional Lie group \(G_2^*\); we write \(G_2^{(1)}\) to indicate either one of these possibilities.\(^1\) A link between nearly Kähler geometry and these exceptional groups has been previously observed in the literature using pseudo-Riemannian constructions, namely metric cones \([14, 33, 35, 51, 59, 78]\). Here it is the connection with projective differential geometry that we wish to emphasize. This delivers far more structure than Theorem 1 at first suggests.

The nearly Kähler defining equation \((1)\) determines an equation on the Kähler form \(\omega\) called the Killing-Yano equation. This is projectively invariant; it is an equation from an important class of equations known as first BGG equations (cf. e.g. \([26]\)). We show in Theorem 1.2 that on a 6-dimensional nearly Kähler manifold, \(\omega\) is a normal solution of this equation in the sense of \([26]\). In this case, this means that (by prolongation) \(\omega\) determines, and is equivalent to, a certain tractor 3-form field \(\Phi\) that is parallel for the normal projective tractor connection, and it is this that gives the holonomy reduction. This perspective provides a natural geometric framework to extend the structure and connect to other geometries. For example an important question is how, or whether, one may compactify complete nearly Kähler geometries. A result in this direction is as follows.

\(^1\)There are actually two (connected) split real forms of \(G_2^*\): the automorphism group of the split octonions (see Subsection 2.2), which has fundamental group \(\mathbb{Z}_2\), and its universal cover \([11]\). In this article, \(G_2^*\) always refers to the former.
Theorem 1.2. Let \( M \) be an open dense submanifold in a compact connected projective 6-manifold \((N, p)\), possibly with boundary, so that one of the following two possibilities hold: either \( N \) is a manifold with boundary \( \partial N \) and \( N \setminus M = \partial N \), or \( N \) is closed and \( N \setminus M \) is contained in a smoothly embedded submanifold of \( N \) of codimension at least 2. Suppose further that \( M \) is equipped with a complete nearly Kähler structure \((g, J)\) such that the projective class \( [\nabla^3] \) of the Levi-Civita connection \( \nabla^3 \) coincides with \( p|_M \). Then, either

- \( M \) is closed and \( M = N \); or
- \( g \) has signature \((2, 4)\) (resp. signature \((3, 3)\)) and we are in the first setting with \( N \setminus M \) the smooth 5-dimensional boundary for \( N \). Furthermore the metric \( g \) is projectively compact of order 2, and the boundary has a canonical conformal structure equivalent to an (oriented) Cartan \((2, 3, 5)\)-distribution.

Here by a closed manifold we mean that is compact without boundary. The notion of projectively compact, in the last part, is a projective analogue of conformal compactification, as formulated in [24]. In the second part of the theorem the signature could of course be \((4, 2)\) instead of \((2, 4)\) (since, at the cost of changing the sign of scalar curvature, we could replace the metric by its negative), hence the qualifier “without loss of generality”. In general throughout the article we will ignore this trivial freedom. This Theorem is proved in Section 5.6.

Theorem [13] suggests an obvious converse problem. It was shown recently that a Cartan holonomy reduction determines a canonical stratification of the underlying manifold into initial submanifolds, with the different strata (called curved orbits) equipped with specific geometric structures determined by the reduction; the general theory is developed in [27] following the treatment of projective geometry [20] and a “pilot case” in conformal geometry [13]. Providing the details for this geometric stratification, specific to our current setting, resolves this converse problem and more, as in the following Theorem, and it is this that leads to Theorem 1.2. A parallel tractor 3-form \( \Phi \) is said to be generic if it determines, via a certain algebraic construction (see [41]), a metric \( H \) on the tractor bundle. According to whether \( H \) is positive definite or indefinite we say \( \Phi \) is, respectively, definite-generic or split-generic.

Theorem 1.3. Suppose that \((M, p)\) is a 6-dimensional projective manifold equipped with a parallel generic tractor 3-form \( \Phi \). Then:

- If \( \Phi \) is definite-generic then it determines a \( G_3 \) holonomy reduction of the Cartan bundle, and \((M, p, \Phi)\) is equivalent to a signature-(6,0) nearly Kähler structure on \( M \) that is positive Einstein.
- If \( \Phi \) is split-generic then it determines a \( G_2^* \) holonomy reduction of the Cartan bundle and a decomposition \( M = M_+ \cup M_0 \cup M_- \) of \( M \) into a union of 3 (not necessarily connected) disjoint curved orbits, where \( M_\pm \) are open and \( M_0 \) is closed. If \( M \) is connected and both \( M_+ \) and \( M_- \) are non-empty then \( M_0 \) is non-empty and is a smoothly embedded separating hypersurface consisting of boundary points of both \( M_+ \) and \( M_- \). From \((M, p, \Phi)\) the curved orbit components inherit canonical geometric structures as follows: \( M_+ \) has a nearly Kähler structure of signature \((2, 4)\) that is positive Einstein; \( M_- \) has a nearly para-Kähler structure of signature \((3, 3)\) that is negative Einstein; \( M_0 \) has a conformal structure of signature \((2, 3)\) equipped with a \( G_2^* \) conformal holonomy reduction, and this means that the conformal structure is equivalent to a Cartan \((2, 3, 5)\)-distribution.

Some remarks are in order: A nearly para-Kähler geometry is a pseudo-Riemannian manifold satisfying \( \nabla J = 0 \), but where \( J \) is an involution and \( g(JU, JV) = -g(U, V) \). As with our conventions for nearly Kähler, our default is that this is strict, and so here nearly para-Kähler means that \( \nabla J \) is nowhere zero, where \( \nabla \) is again the Levi-Civita connection of \( g \). It is well-known that 6-dimensional strictly nearly Kähler and strictly nearly para-Kähler structures are necessarily Einstein [19] [76] [57]. That a Cartan \((2, 3, 5)\)-distribution is equivalent to a \( G_2^* \)-reduced conformal structure is a result of Nurowski [71], with further clarification and characterization given in [53]. These results play an important role here. Note that Theorem 1.1 combined with the result here
shows that on a 6-manifold a Riemannian nearly Kähler structure is simply equivalent to a projective structure with a parallel definite-generic 3-form tractor. More generally we see that nearly Kähler geometry, its para-variant, and Cartan \((2,3,5)\)-geometry arise in a uniform way from projective geometry.

Given Theorems 1.2 and 1.3 it is natural to ask whether all \((2,3,5)\)-distributions arise this way. The answer is positive in the real-analytic setting (and in general formally), as explained in Section 6. That section uses results from [41] and [47] to treat the problem of taking a distribution as Dirichlet data for the construction of a projective manifold with a \(G_2\) holonomy reduction for which the given distribution is the induced structure on the projective infinity. See in particular Theorem 6.1 which interprets in the projective tractor setting Theorem 1.1 from [47]. The latter theorem itself generalizes to all (oriented, real-analytic) \((2,3,5)\)-distributions a result in [72, §4] about a particular finite-dimensional family of such distributions; later Leistner and Nurowski proved that metrics in an explicit subset of that family have holonomy equal to \(G_2\) [64]. Section 7 gives solutions to the Dirichlet problems for a special class of \((2,3,5)\)-distributions studied by Cartan [31] §9 (the solutions themselves are essentially equivalent to a special case of the data given in [71, §3]), and these yield a 1-parameter family of geometries \((\bf{M}, \mathbf{p}, \Phi)\) whose curved orbits are all homogeneous.

These results establish that we may study \((2,3,5)\)-geometry *holo*graphically, that is using the associated nearly Kähler and nearly para-Kähler geometries of Theorems 1.2 (also Theorem 5.22) and 1.3. This is in the spirit of Fefferman and Graham’s Poincaré-Einstein program [41] and the usual holographic principle as in e.g. [44, 58, 80, 44], except that it involves projective compactification and not conformal compactification and so the asymptotics are rather different; see Section 4.3.

A first step in such a holographic treatment is to understand how (in the notation of Theorem 1.3) the distribution on \(M_0\) arises as a limit of the ambient almost (para-)complex structure on the open curved orbits \(M_\pm\). This is treated in detail in Section 5. There it is shown that the holonomy reduction determines a smooth object \(\mathcal{J}\) (see (57)), that is essentially a tractor bundle endomorphism field on \(M\). This gives the almost (para-)complex structures on \(M_\pm\) while also determining the distribution on \(M_0\) as a quotient of its kernel, see Theorems 5.8 and 5.10.

In Section 5.3 it is also shown how many of the properties of the distribution may be deduced efficiently via \(\mathcal{J}\) and the naturally accompanying perspective.

The general theory of curved orbit decompositions from [26, 27] describes how many features of orbit decompositions of homogeneous spaces carry over to corresponding holonomy reductions of Cartan geometries modelled on the given symmetry-reduced homogeneous space. Thus we should expect to understand the results in Theorem 1.3 partly as realizing curved generalizations of features of the model. This is the case, and the model is discussed in Section 5.1. As explained there, the models for our structures are the ray projectivization \(\mathbb{P}_+(\mathbb{C})\) of the imaginary octonions \(\mathbb{I}\), in the definite signature case, and the the ray projectivization \(\mathbb{P}_+(\mathbb{I}^*)\) of the imaginary split octonions \(\mathbb{I}^*\) in the indefinite case. Both \(\mathbb{I}\) and \(\mathbb{I}^*\) are algebraically rich structures: the homogeneous geometries \(\mathbb{P}_+(\mathbb{I}^*)\) include the models for nearly Kähler geometry (of both possible signatures), nearly para-Kähler geometry, and \((2,3,5)\)-geometry, as we explain in Section 5.4.

The point of presenting the model at that late stage is that these features of the model are just specializations of results that hold in more general settings, and treating the general cases is no more difficult than treating the model from the perspective developed here.

Just as a Riemannian manifold carries a point-dependent Euclidean structure that may be viewed within the context of projective geometry as a holonomy reduction of affine geometry, the Cartan and tractor machinery enables the imaginary (split) octonionic algebraic structure of either space \(\mathbb{I}^*\) to be carried fiberwise in a point-dependent but parallel manner. Because projective geometry is a higher-order structure, this parallel algebra interacts with jets of the structure.
Thus the geometries discussed in Theorem 1.3 above are in a precise way curved analogues of $\mathbb{P}_+(\mathbb{O}^*)$. On a projective 6-manifold $(M, p)$ a tractor 3-form $\Phi$ that is pointwise generic determines an algebraic binary cross product $\times$ (see Definition 2.2) that corresponds fiberwise to the cross product on the imaginary octonions, cf. [5]. This is preserved by the tractor connection if and only if $\Phi$ is parallel, and hence we have the following paraphrasing of Proposition 5.5:

**Proposition 1.4.** Suppose that $(M, p)$ is a dimension 6 projective manifold. A generic parallel 3-form tractor $\Phi$ is equivalent to a tractor cross product $\times : T \times T \rightarrow T$ that is preserved by the tractor connection.

Thus we may take $(M, p, \times)$ as the fundamental structure. This has considerable aesthetic appeal, but it is also practically useful, and after Section 5.2 much of the development is based on this point of view. For example $J$ is defined via $\times$ and then its key properties follow easily from cross-product identities. These and similar results are developed in the next section where we introduce the tools that underlie the algebraic aspects of the article.

Since the work here involves a number of geometric structures, our aim is to make the treatment as self-contained as possible. Throughout we shall use either index-free notation or Penrose’s abstract index notation according to convenience. In Section 7 frames are also used.

The authors are grateful to Paweł Nurowski, who, at an early stage in this project, pointed out Theorem 1.1 and a proof via exterior differential systems (in fact this was in an early draft of [45]). We are also grateful to Paul-Andi Nagy who assisted greatly in the part that now forms Appendix B. It is also a pleasure to thank Robert Bryant for several comments, in particular for discussion connected to the generality of strictly N(P)K structures as in Remark 6.11 and for indicating a clever proof (not included here) of an algebraic fact used in Example 8.6. We are also thankful to Antonio Di Scala, who pointed out a gap in a first version of the proof of Theorem 1.2 and to G. Manno for allowing us to use a result from their preprint work [37]. Discussions with Michael Eastwood are also much appreciated. The explicit data for the family of examples described in §7 was produced in part using the standard Maple package DifferentialGeometry.

2. **Algebraic preliminaries**

2.1. $\varepsilon$-complex structures on vector spaces. We review some variants of the notion of a complex structure on a vector space. By applying appropriate sign changes, one can define so-called paracomplex analogues of more familiar complex structures. Both here and in Subsections 3.1-3.2 where we define related geometric structures on tangent bundles, we define both kinds of structures simultaneously using a parameter $\varepsilon \in \{\pm 1\}$: In the definitions, $\varepsilon = -1$ yields the complex version of a structure and $\varepsilon = +1$ the paracomplex version. One specializes the names of structures to particular values of $\varepsilon$ by simply omitting $-1$- and replacing $+1$- with the prefix para-. See [36] for a survey of paracomplex geometry.

**Definition 2.1.** The $\varepsilon$-**complex numbers** is the ring $\mathbb{C}_\varepsilon$ generated over $\mathbb{R}$ by the single generator $i_\varepsilon$, which satisfies precisely the relations generated by $i_\varepsilon^2 = \varepsilon$. As an $\mathbb{R}$-algebra, the ring $\mathbb{C}_{+1}$ of paracomplex numbers is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

An $\varepsilon$-**complex structure** on a real vector space $\mathbb{W}$ (of necessarily even dimension, say, $2m$) is an endomorphism $J \in \text{End}(\mathbb{W})$ such that

$$J^2 = \varepsilon \text{id}_{\mathbb{W}};$$

if $\varepsilon = +1$, we require furthermore that the $(\pm 1)$-eigenspaces of $J$ both have dimension $m$ (the analogous condition holds automatically for $\varepsilon = -1$). This identifies $\mathbb{W}$ with $\mathbb{C}_\varepsilon^m$ so that the action of $J$ coincides with scalar multiplication by $i_\varepsilon \in \mathbb{C}_\varepsilon$.

An $\varepsilon$-**Hermitian structure** on a real vector space $\mathbb{W}$ (again of necessarily even dimension) is a pair $(g, J)$, where
(a) \( g \in S^2W^\ast \) is an inner product (in this article, inner products are are not necessarily
definite unless specified otherwise), and
(b) \( J \in \text{End}(W) \) is an \( \varepsilon \)-complex structure on \( W \),
compatible in the sense that
\[
g(J \cdot, J \cdot) = -\varepsilon g(\cdot, \cdot),
\]
or, equivalently, if \( \omega := g(\cdot, J \cdot) \) is skew-symmetric.

The compatibility condition imposes restrictions on the signature of an inner product \( g \) in a
\( \varepsilon \)-Hermitian structure: For a Hermitian structure \((g, J)\) on a real vector space \( W \) of dimension
\( 2m \), \( g \) must have signature \((2p, 2q)\) for some nonnegative integers \( p, q \), and for a para-Hermitian
structure, \( g \) must have neutral signature \((m, m)\). (This compatibility condition necessitates the
eigenspace condition in the definition of a paracomplex structure.)

2.2. The octonions, 7-dimensional cross products, and \( G_2^{(4)} \). The geometries investigated
in this article will be unified by so-called cross products \( \times : V \times V \to V \) on 7-dimensional
vector spaces. In this subsection, we will review a formulation of these products and recall a
classification theorem that asserts that, up to isomorphism, there are only two in this dimension.
We construct both simultaneously, one in terms of the octonion algebra \( \mathbb{O} \), the most complicated
of the four algebras in the celebrated classification of normed division algebras over \( \mathbb{R} \), and the
other using the split octonion algebra \( \mathbb{O}^\ast \), a close analog of \( \mathbb{O} \) in which the norm is replaced
by a quadratic form that induces a split signature inner product. Then, we use the abstract
properties of these algebras to establish characteristics of the cross products, and transferring
these in Section 5 to the curved geometries under study will efficiently illuminate some of their
important features.

It turns out that \( \mathbb{O} \) can be recovered from the corresponding cross product algebra, and so
the two algebras have the same automorphism group, the compact real form \( G_2 \) of the simple
complex Lie group \( G_2 \). The other cross product is analogously related to \( \mathbb{O}^\ast \), and for
this case the common automorphism group is the split real form \( G_2^\ast \) of that complex Lie group.

**Definition 2.2.** On an inner product space \((V, \cdot)\), a (binary) cross product is a skew-symmetric
linear product \( \times : V \times V \to V \) compatible with \( \cdot \), meaning that
\[
\begin{align*}
(a) & \quad (x \times y) \cdot x = 0 \quad \text{and} \\
(b) & \quad (x \times y) \cdot (x \times y) = (x \cdot x)(y \cdot y) - (x \cdot y)^2
\end{align*}
\]
for all \( x, y \in V \).

**Theorem 2.3.** [13, Theorem 4.1] Up to isomorphism, the only nonzero binary cross products
on real inner product spaces are:

- the usual cross product on \( \mathbb{R}^3 \) with a definite inner product, which is isomorphic to the
  Lie bracket on \( \mathfrak{so}(3) \) with its Killing form,
- the Lie bracket on \( \mathfrak{so}(1, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \) with its (Lorentzian) Killing form, which can be
  identified with the polarization of the determinant on \( \mathfrak{sl}(2, \mathbb{R}) \subset M(2, \mathbb{R}) \),
- a definite cross product on \( \mathbb{R}^7 \) with a definite inner product, and
- a split cross product on \( \mathbb{R}^7 \) with an inner product of (split) signature \((3, 4)\).

We can construct the latter two cross products in terms of \( \mathbb{O} \) and \( \mathbb{O}^\ast \); for convenience we write
\( \mathbb{O}^{(\ast)} \) to indicate that a given context applies to both. See [5] for the details about \( \mathbb{O} \), and [6] for
facts about \( \mathbb{O}^\ast \), though as we will see, many of the features of the two algebras are analogous.
(The claims not proved here can be verified readily in an appropriate realization of \( \mathbb{O}^{(\ast)} \); we
recall an elegant, classical realization of both algebras near the end of this subsection.)

The algebra \( \mathbb{O}^{(\ast)} \) has an identity, which we denote 1, and it is alternative but not associating;
alternativity just means that any subalgebra generated by two elements is associative, or
equivalently that the associator \( (x, y, z) \mapsto (xy)z - x(yz) \) is totally skew.
The norm on $\mathbb{O}$ induces a positive definite inner product $\cdot$, and the inner product on $\mathbb{O}^*$, which we also denote $\cdot$, has signature $(4,4)$. In both cases, the inner product defines a natural conjugation involution $\bar{\cdot}:\mathbb{O}^*\to\mathbb{O}^*$ by orthogonal reflection through the the span $\langle 1 \rangle$ of $1$, which we identify with $\mathbb{R}$:

$$\bar{x} := 2(x \cdot 1) - x.$$  

It satisfies $\bar{\bar{x}} = x$. We call the 7-dimensional orthocomplement $\mathbb{I}^* := \langle 1 \rangle$, which is precisely the $-1$-eigenspace of the conjugation map, the imaginary (split) octonions. The (hence orthogonal) respective projections onto the summands of the decomposition $\mathbb{O}^* \cong \mathbb{R} \oplus \mathbb{I}^*$ are thus:

$$\text{Re} : x \mapsto \frac{1}{2}(x + \bar{x}) = x \cdot 1$$

$$\text{Im} : x \mapsto \frac{1}{2}(x - \bar{x}),$$

and we can recover the bilinear form $\cdot$ via:

$$x \cdot y = \text{Re}(xy).$$

Now, $\cdot$ restricts to a nondegenerate bilinear form, which we also denote $\cdot$, on $\mathbb{I}^*$ of signature $(7,0)$ or $(3,4)$, and it specializes there immediately to $x \cdot y = -\frac{1}{4}(xy + yx)$. The projection $\text{Im}$ determines the nonassociative, $\mathbb{R}$-linear, skew-symmetric (split) octonionic cross product

$$\times : \mathbb{I}^* \times \mathbb{I}^* \to \mathbb{I}^*$$

by

$$(x,y) \mapsto x \times y := -\text{Im}(xy) = \frac{1}{2}(xy - yx),$$

which realizes $\mathbb{I}^*$ as an anticommutative, nonassociative algebra over $\mathbb{R}$ without unit.

By definition, for all $x, y \in \mathbb{I}^*$, $xy$ can be decomposed into its $\mathbb{R}$ and $\mathbb{I}^*$ components as

$$(2) \quad xy = -x \cdot y + x \times y.$$  

Now, one can easily reverse this construction and recover the full algebraic structure of $\mathbb{O}^*$ just from the cross product $\times$ on $\mathbb{I}^*$; for our purposes it will be enough to know that the bilinear form on $\mathbb{I}^*$ is determined by the cross product via [3] §4.1]

$$(3) \quad x \cdot y = -\frac{1}{6} \text{tr}(x \times (y \times \cdot)).$$

So, to specify a cross product $\times$ with an underlying 7-dimensional inner product space $(V, \cdot)$, it is enough just to specify $(V, \times)$. In particular, the cross products constructed on $\mathbb{I}$ and $\mathbb{I}^*$ are nonisomorphic, and hence by the above classification, any (binary) cross product $\times$ on a 7-dimensional real vector space $V$ is isomorphic to either $(\mathbb{I}, \times)$, in which case we say that $\times$ is definite, or $(\mathbb{I}^*, \times)$, in which case we say that $\times$ is split.

A cross product on $V$ canonically also determines an orientation: The form

$$(4) \quad \epsilon_{ABCDEFG} := \frac{1}{72} \times_K [AB \times K_{CD} \times EFG] \in \Lambda^7 V^*$$

is a volume form for $\cdot$; here, indices are raised and lowered with $\cdot$.

Now, any element $x \in \mathbb{I}^*$ determines a map $J_x : \mathbb{I}^* \to \mathbb{I}^*$ defined by

$$(5) \quad J_x(y) := -x \times y,$$

which part (a) of Definition 2.2 guarantees is skew-adjoint with respect to $\cdot$. Its properties will play a key role later in establishing features of the geometric structures we study in later sections.

**Proposition 2.4.** For any $x \in \mathbb{I}^*$,

$$(6) \quad J_x^2(y) = -(x \cdot y)x + (x \cdot y)x.$$

In particular,

(a) if $x$ is null, then $J_x^2(y) = (x \cdot y)x$, and

(b) if $y \in \langle x \rangle^\perp$, then $J_x^2(y) = -(x \cdot y)y$. 

Proof. If we expand the alternativity identity \((xx)y = x(xy)\) of \(O(\cdot, \cdot)\) using the decomposition \([\mathbb{2}]\), the left-hand size becomes
\[-x \cdot x + x \times x) = -(x \cdot x)y,
and the right-hand side
\[x(x \times y - x \cdot y) = x(x \times y) - (x \cdot y)x \]
\[= [x \times (x \times y) - x \cdot (x \times y)] - (x \cdot y)x \]
\[= -x \times (-x \times y) - (x \cdot y)x \]
\[= J^2_x(y) - (x \cdot y)x.
Rearranging gives the identity.
\[\square\]

**Corollary 2.5.** If \(x \cdot x = -\varepsilon \in \{\pm 1\}\), then \(J_x|_{(x)\perp} \in \text{End}((x)\perp)\) is an \(\varepsilon\)-complex structure.

The behavior of \(J_x\) is especially rich for null \(x\). Parts \([\mathbb{3}]\) \(-\mathbb{6}\) of the following proposition are formulated and proved in \([\mathbb{6}, \text{Lemma 7}]\) in a different way.

**Proposition 2.6.** Suppose \(x \in \mathbb{I}^*\) is null and nonzero. Then,

(a) \(\langle x \rangle \subset \ker J_x;\)
(b) \(\ker J_x\) is isotropic;
(c) \(\dim \ker J_x = 3.\)

In particular, \(x\) determines a proper filtration
\[\{0\} \subset \langle x \rangle \subset \ker J_x \subset (\ker J_x)^\perp \subset \langle x \rangle ^\perp \subset \mathbb{I}^*\]
whose filtrands respectively have dimension 0, 1, 3, 4, 6, and 7.

(d) The map \(J_x\) respects the filtration in that
(i) \(J_x((\mathbb{I}^*)) = (\ker J_x)^\perp\) (that is, \(\text{im} J_x = (\ker J_x)^\perp\)),
(ii) \(J_x((x)\perp) = \ker J_x,\) and
(iii) \(J_x((\ker J_x)^\perp) = \langle x \rangle.\)

**Proof.**

(a) For any \(\lambda x \in \langle x \rangle, J_x(\lambda x) = -\lambda (x \times x) = 0.\)
(b) Pick \(y \in \ker J_x\). If \(y\) is a multiple \(\lambda x\) of \(x\), then \(y \cdot y = \lambda^2 x \cdot x = 0;\) so henceforth suppose it is not. By Proposition \([\mathbb{2}]\)
\[0 = y \times J_x(y) = y \times (y \times x) = -(y \cdot y)x + (y \cdot x)y.\]
Since \(y\) is not a multiple of \(x\), both terms in the last expression are zero, and in particular \(y \cdot y = 0.\)
(c) By Proposition \([\mathbb{2}]\) \(\ker (J^2_x) = \langle x \rangle ^\perp.\) So, \(\dim \ker J^2_x = 6\) and hence \(\dim \ker J_x \geq 3.\) On the other hand, by \([\mathbb{3}]\) \(\ker J_x\) is isotropic, and so \(\ker J_x \leq 3;\) thus, equality holds.
(d) (i) For any \(y \in \mathbb{I}^*\) and \(z \in \ker J_x,\) the skew-adjointness of \(J_x\) gives
\[J_x(y) \cdot z = -y \cdot J_x(z) = 0,\]
so \(\text{im} J_x = J_x(\mathbb{I}^*) \subseteq (\ker J_x)^\perp;\) equality holds by the Rank-Nullity Theorem.
(ii) For any \(y \in (x)\perp,\) Proposition \([\mathbb{2}]\) gives \(J_x((\langle x \rangle) = (x \cdot y)x = 0,\) that is, \(J_x((x)\perp) \subseteq \ker J_x.\) We can then view \(J_x|_{(x)\perp}\) as a map \((x)\perp \rightarrow (x)\perp.\) Then, \(\dim J_x((x)\perp) = \text{rank} J_x|_{(x)\perp}\) is at least \(\text{rank} J_x - (\dim \mathbb{I}^* - \dim (x)\perp) = 3,\) and because \(\dim \ker J_x = 3,\) the above containment is actually an equality.
(iii) Pick \(y \in (\ker J_x)^\perp.\) By \([\mathbb{3}]\), there is some \(z \in \mathbb{I}^*\) such that \(y = J_x(z),\) and Proposition \([\mathbb{2}]\) \([\mathbb{4}]\) gives that
\[J_x(y) = J_x(J_x(z)) = (x \cdot z)x.\]
So, \( \mathbb{J}_x((\ker \mathbb{J}_x)^\perp) \subseteq \langle x \rangle \). On the other hand, \( (\ker \mathbb{J}_x)^\perp \) is a proper superset of \( \ker \mathbb{J}_x \), so its image under \( \mathbb{J}_x \) cannot be trivial; hence, equality holds. \( \Box \)

The automorphism group of \( O^{(*)} \) is \( G_2^{(*)} \), a real form of the 14-dimensional complex simple Lie group of type \( G_2 \). Any automorphism fixes the identity \( 1 \), and since \( \cdot \) is derived from the algebraic structure of \( O^{(*)} \), \( G_2^{(*)} \) preserves the orthocomplement \( \langle 1 \rangle^\perp = \|^{(*)} \), which is the smallest nontrivial irreducible representation of \( G_1^{(*)} \). Since all of the algebraic structure of \( O \) can be recovered from \( \times \) or equivalently (as Subsection 2.3 will show) from the 3-form \( \Phi(x, y, z) := x \cdot (y \times z) \), \( G_2^{(*)} \) is precisely the stabilizer of both \( \times \) and \( \Phi \) in \( GL(\|^{(*)}) \) under its respective standard actions on \( (2,1) \) - and \( (3,0) \)-tensors.

Conversely, one can show that the \( G_2 \)-action on \( I \) determines the cross product uniquely up to nonzero scale. Equation (3) shows that \( \cdot \) can be recovered from \( \times \), so \( G_2 \) preserves \( \cdot \) and, in particular, this defines a homomorphism (in fact, an embedding) \( G_2 \hookrightarrow SO(\cdot) \cong SO(7, \mathbb{R}) \). Similarly, there is a canonical embedding \( G_2^* \hookrightarrow SO(3,4) \).

Though we need only the above properties of the 7-dimensional (split) cross product, for concreteness and self-containment we briefly indicate an easy way to define these algebras, the Cayley-Dickson construction. The above properties can be readily checked with this realization: Let \( \mathbb{H} \) denote the algebra of quaternions, and define the octonions \( O \) (split octonions \( O^* \)) to be the real algebra with underlying set \( \mathbb{H} \times \mathbb{H} \) and multiplication

\[
(a, b)(c, d) := (ac \mp db, ad + cb),
\]

where \( \mp \) is \( - \) for \( O \) and \( + \) for \( O^* \). Then, the conjugation is given by

\[
\overline{(a, b)} = (\bar{a}, -b).
\]

**Remark 2.7.** The two 3-dimensional cross products in the classification at the beginning of this subsection can be constructed by the same method as the 7-dimensional cross products: Instead of starting with the octonions or split octonions, one begins with the quaternions \( \mathbb{H} \) or split quaternions \( \mathbb{H}^* \). One can realize these simultaneously as the real algebra with underlying set \( \mathbb{C} \times \mathbb{C} \) and multiplication given again by (7) (where taking \( \mp \) to be \( - \) yields \( \mathbb{H} \) and \( + \) yields \( \mathbb{H}^* \)), with pairs interpreted as in that set.

If one starts with \( \mathbb{H} \), endowed with the usual norm, the conjugation and the resulting cross product on \( (1)^\perp \cong \mathbb{R}^3 \) are the usual ones, and \( (\mathbb{R}^3, \times) \) is isomorphic as a Lie algebra to \( \mathfrak{so}(3, \mathbb{R}) \); any choice of isomorphism identifies the inner product with a multiple of the Killing form.

If one starts with \( \mathbb{H}^* \cong M(2, \mathbb{R}) \), endowed with the determinant regarded as a quadratic form (the induced inner product of which has signature \( (2,2) \)), the conjugation is the adjugate map, so the imaginary split quaternions comprise the tracefree matrices \( M_0(2, \mathbb{R}) \subset M(2, \mathbb{R}) \), and the cross product is \( A \times B := \frac{1}{2i}(AB - BA) = \frac{i}{2}[A, B] \). Then, the map \( A \mapsto \frac{i}{2}A \) is a Lie algebra isomorphism \( (M_0(2, \mathbb{R}), \times) \rightarrow \mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(1, 2) \) and identifies the inner product with a multiple of the Killing form.

### 2.3. Stable forms on vector spaces

In this section we review the notion of stability, a type of genericity, for alternating forms on real vector spaces, including some constructions using such forms specific to dimensions 6 and 7: A stable 3-form on an (oriented) real vector space of dimension 6 induces an \( \varepsilon \)-complex structure (as in [50] and as extended to the para-complex case in [35]), and a stable 3-form on a real vector space of dimension 7 determines a nondegenerate symmetric bilinear form and orientation there [14]. See Appendix A for explicit realizations of the relevant objects in bases.

**Definition 2.8.** On a real vector space \( \mathbb{W} \), a \( k \)-form \( \beta \in \Lambda^k \mathbb{W}^* \) is stable (or generic [2], though we will reserve this term for the analogous notion on manifolds) iff the orbit \( GL(\mathbb{W}).\beta \) is open in \( \Lambda^k \mathbb{W}^* \), where \( . \) is the action induced by the standard action on \( \mathbb{W} \).
If a real vector space $\mathbb{W}$ of finite dimension $n$ admits a stable $k$-form, then necessarily $n^2 = \dim \operatorname{GL}(\mathbb{W}) \geq \dim \Lambda^k \mathbb{W} = \binom{n}{k}$. It turns out that stable $k$-forms exist whenever this inequality is satisfied (and $k > 0$).

**Proposition 2.9.** [35 Proposition 1.1] A real vector space $\mathbb{W}$ of finite dimension $n$ admits a stable $k$-form, $k \leq n$, iff

- $k \in \{1, 2, n - 2, n - 1, n\}$, or
- $n \leq 8$ and $k \in \{3, n - 3\}$.

2.3.1. **Stable 2-forms in 2m dimensions.** On an even-dimensional vector space $\mathbb{W}$, $\Lambda^3 \mathbb{W}^*$ has exactly one $\operatorname{GL}(\mathbb{W})$-orbit, and its elements are exactly the symplectic forms on $\mathbb{W}$.

2.3.2. **Stable 3-forms in 6 dimensions.** Let $\mathbb{W}$ be a real, oriented 6-dimensional vector space; then, $\Lambda^3 \mathbb{W}^*$ has exactly two open $\operatorname{GL}(\mathbb{W})$-orbits. (See [14 Theorem 2.1.13] and [16 Proposition 12] for detailed calculations and a complete $\operatorname{GL}(\mathbb{W})$-orbit decomposition of $\Lambda^3 \mathbb{W}^*$.)

Given a stable 3-form in $\Lambda^3 \mathbb{W}^*$, we can canonically construct an $\varepsilon$-complex structure $J \in \operatorname{End}(\mathbb{W})$. Fix $\beta \in \Lambda^3 \mathbb{W}^*$, let $\kappa : \Lambda^3 \mathbb{W}^* \to \mathbb{W} \otimes \Lambda^6 \mathbb{W}^*$ denote the canonical mapping, and define

$$\tilde{J} := \kappa((\cdot, J) \wedge \beta) \in \operatorname{End}(\mathbb{W}) \otimes \Lambda^6 \mathbb{W}^*.$$ 

To determine an endomorphism of $\mathbb{W}$, we define a volume form invariantly in terms of $\tilde{J}$: First set

$$\lambda(\beta) := \frac{1}{4} \operatorname{tr}(\tilde{J}^2) \in \otimes^2 \Lambda^6 \mathbb{V}^*.$$ 

It turns out that $\lambda(\beta) \neq 0$ iff $\beta$ stable, which we henceforth assume. Now, $\varepsilon \lambda(\beta)$ is a square of an element in $\Lambda^6 \mathbb{V}^*$ for exactly one value $\varepsilon \in \{\pm 1\}$, so let $\varepsilon$ denote the unique positively oriented element there such that $\varepsilon \otimes \varepsilon = \varepsilon \lambda(\beta)$. Then, the endomorphism $J$ characterized by

$$\tilde{J} = J \otimes \varepsilon$$

satisfies $J^2 = \varepsilon \cdot \operatorname{id}_W$. It turns out that if $\varepsilon = 1$, then the $\pm 1$-eigenspaces of $J$ both have dimension 3, and thus $J$ is an $\varepsilon$-complex structure on $\mathbb{W}$.

2.3.3. **Stable 3-forms in 7 dimensions.** Let $\mathbb{V}$ be a real 7-dimensional vector space; then, $\Lambda^3 \mathbb{V}^*$ has exactly two open $\operatorname{GL}(\mathbb{V})$-orbits. (See [14] for detailed calculations, and [24], which describes a full orbit decomposition of 3-forms on 7-dimensional vector spaces over algebraically closed fields and describes a process for generalizing it to some other base fields, including $\mathbb{R}$.)

Given a stable 3-form in $\Lambda^3 \mathbb{V}^*$, we can canonically construct an inner product $H \in S^2 \mathbb{V}^*$: Fix $\Phi \in \Lambda^3 \mathbb{V}^*$, and define the $\Lambda^7 \mathbb{V}^*$-valued symmetric bilinear form

$$\tilde{H} := \frac{1}{16}((\cdot, J) \wedge (\cdot, J) \wedge \Phi) \triangleq S^2 \mathbb{V}^* \otimes \Lambda^7 \mathbb{V}^*.$$ 

It turns out that $\tilde{H}$ is nondegenerate if $\Phi$ stable, which we henceforth assume. In particular, $\tilde{H}$ determines a real-valued bilinear form up to scale; to fix the scale naturally, we define a volume form invariantly in terms of $\Phi$. Regarding $\tilde{H}$ as a map $\mathbb{V} \to \mathbb{V}^* \otimes \Lambda^7 \mathbb{V}^*$ and taking the determinant yields a map

$$\det \tilde{H} : \Lambda^7 \mathbb{V} \to \Lambda^7 (\mathbb{V}^* \otimes \Lambda^7 \mathbb{V}^*) \cong \otimes^8 \Lambda^7 \mathbb{V}^*,$$

and dualizing again gives a map

$$\det \tilde{H} : \mathbb{R} \to \otimes^8 \Lambda^7 \mathbb{V}^*.$$ 

This map turns out to be nonzero because $\Phi$ is stable; so, there is a distinguished volume form $\varepsilon \in \Lambda^7 \mathbb{V}^*$ characterized by $\varepsilon^{\otimes 9} = (\det \tilde{H})(1)$, which in turn determines a nondegenerate bilinear form $H : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ characterized by

$$\tilde{H} = H \otimes \varepsilon.$$ 

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The normalization of $\tilde{H}$ in (8) (and hence of $H$) was chosen so that $\epsilon$ is the volume form of $H$ for the orientation it determines. Alternatively, for a 3-form $\Phi$ and the volume form $\epsilon$ it determines, we can recover $\tilde{H}$ via

$$H_{AB} = \frac{1}{4!} \Phi_{ACD} \Phi_{BEF} \Phi_{GHI} \epsilon^{CDEFGHI}.$$  

Here $\epsilon$ is normalized so that $\epsilon^{CDEFGHI} \epsilon^{CDEFGHI} = 7!$. For 3-forms in one of the two open $\text{GL}(V)$-orbits, $H$ has signature $(7,0)$; we call such 3-forms \textit{definite-stable}. For 3-forms in the other orbit, $H$ has signature $(3,4)$; we call these \textit{split-stable}.

In dimension 7, stable 3-forms can be identified with cross products, which motivates the reuse here of the terms definite and split.

\textbf{Proposition 2.10.} On any real 7-dimensional vector space $V$, raising and lowering indices with the corresponding bilinear forms establishes a natural bijection

$$\{\text{cross products } \times: V \times V \to V\} \leftrightarrow \{\text{stable 3-forms } \Phi \in \Lambda^3 V^\ast\}.$$

In particular, a cross product is definite (split) iff the corresponding 3-form is definite (split) stable.

\textit{Proof.} As observed in Subsection 2.2, given a cross product $(\times, \times)$, the 3-form $\Phi(x,y,z) := x \cdot (y \times z)$ is totally skew, and the automorphism group of $\times$ is $G_2^{(3)}$. Let $\cdot$ denote the inner product $\times$ induces via (3) and $\tilde{H} \in S^3 V^\ast \otimes \Lambda^7 V^\ast$ the bilinear form defined by (8). As a $G_2^{(3)}$-representation, $S^3 V^\ast \otimes \Lambda^7 V^\ast$ decomposes into irreducible subrepresentations as

$$(S^3 V^\ast \otimes \Lambda^7 V^\ast) \oplus (\mathbb{R} \otimes \Lambda^7 V^\ast).$$

Since $\cdot \otimes \epsilon$ is $G_2^{(3)}$-invariant and nonzero, it spans the 1-dimensional subrepresentation $\mathbb{R} \otimes \Lambda^7 V^\ast$ (here, $\epsilon$ denotes the volume form of $\cdot$ defined in Subsection 2.2). Likewise, $\tilde{H}$ is a contraction of $G_2^{(3)}$-invariant tensors, so it too is $G_2^{(3)}$-invariant, and thus it is some multiple of $\cdot \otimes \epsilon$. One can verify readily that $\tilde{H}$ is nonzero, so it is nondegenerate, and hence $\Phi$ is stable; in fact, our normalizations have been chosen so that the volume form of $\cdot$ and the volume form defined in terms of $\Phi$ above coincide, and thus so do $\cdot$ and $H$. In particular, if $\times$ is definite (split), then $\Phi$ is definite (split) stable.

Conversely, given a stable 3-form $\Phi$, let $H$ denote the bilinear form it induces, and define the product $\times : V \times V \to V$ by

$$(10) \quad \times^{C \cdots AB} := H^{CK} \Phi_{KAB}.$$ 

Immediately, $\times$ satisfies condition (6) in Definition 2.2, which is all that is needed for the map $J_x := x \times \cdot$ to be skew-adjoint. One can show that $J_x$ so defined satisfies (6) (it is enough to prove this for one definite-stable and one split-stable 3-form); then, forming the inner product of both sides of that identity with $y$ and invoking the skew-adjointness of $J_x$ gives that $\times$ satisfies condition (6) of Definition 2.2 too, and so $\times$ is a cross product. Checking directly (again, say, just for one representative of each orbit) shows that the two constructions are inverses. \hfill $\square$

\textbf{Corollary 2.11.} Let $V$ be a 7-dimensional real vector space and $\Phi \in \Lambda^3 V^\ast$ a stable 3-form. Then, under the standard action of $\text{GL}(V)$, the stabilizer of $\Phi$ is $G_2$ if it is definite-stable and $G_2^{(3)}$ if it is split-stable.

Finally, we can use a cross product identity to determine $H$ from $\tilde{H}$ without computing the determinant. Using (10) to rewrite (9) and rearranging gives

$$6H_{AD} = \Phi_{ABC} \Phi_D^{BC},$$

and contracting with $H^{AD}$ yields

$$42 = \Phi_{ABC} \Phi^{ABC}.$$ 

For a 3-form $\Phi \in \Lambda^3 V^\ast$, scaling $H$ by $\lambda^2$ scales the right-hand size by $\lambda^{-6}$, and so when using $H$ to raise and lower indices, $H$ is characterized among its positive multiples by this identity.
2.3.4. Compatible stable forms. We now formulate a natural compatibility condition on a pair of stable forms in dimension 6 under which the constituent forms can be used to construct a cross product (equivalently, a stable 3-form) in dimension 7. (See [25 §§1.2-3] for much more, including proofs of the below propositions.)

**Definition 2.12.** Let \( \mathbb{W} \) be a 6-dimensional real vector space. We say that a pair \((\omega, \beta)\) of a stable 2-form \( \omega \in \Lambda^2\mathbb{W}^* \) and a stable 3-form \( \beta \in \Lambda^3\mathbb{W}^* \) is compatible if \( \omega \wedge \beta = 0 \). A compatible pair \((\omega, \beta)\) is normalized if \( J^*\beta \wedge \beta = \frac{1}{3} \omega \wedge \omega \wedge \omega \).

Given a compatible pair \((\omega, \beta)\), we can define a nondegenerate bilinear form
\[
g := \varepsilon \omega(\cdot, J\cdot),
\]
where \( J \) is the \( \varepsilon \)-complex structure induced by \( \beta \); checking directly in a basis (see Appendix A) shows that compatibility is equivalent to the pair \((g, J)\) defining an \( \varepsilon \)-Hermitian structure on \( \mathbb{W} \).

**Proposition 2.13.** [25 Proposition 1.12] Let \( \mathbb{W} \) and \( \mathbb{L} \) be real vector spaces respectively of dimensions 6 and 1, and denote \( \mathbb{V} := \mathbb{W} \oplus \mathbb{L} \). Let \( \alpha \in \mathbb{V}^* \) be a nonzero 1-form that annihilates \( \mathbb{W} \). Let \((\omega, \beta)\) be a normalized compatible pair on \( \mathbb{V} \), and identify \( \omega \) and \( \beta \) with their respective pullbacks to \( \Lambda^2\mathbb{V}^* \) and \( \Lambda^3\mathbb{V}^* \) via the decomposition projection \( \mathbb{V} \to \mathbb{W} \); let \( \varepsilon \) be the sign determined by \( \beta \), and let \( g \) denote the bilinear form defined by \((12)\).

Then, the 3-form
\[
\Phi := \alpha \wedge \omega + \beta \in \Lambda^3\mathbb{V}^*
\]
is stable, and the bilinear form that it induces via the construction above is
\[
H = g - \varepsilon \alpha \otimes \alpha \in S^2\mathbb{V}^*;
\]
in particular \( \chi_{AB} := H^{CD}\Phi_{KAB} \) is a cross product.

Conversely, a cross product \( \chi \) on a real 7-dimensional vector space \( \mathbb{V} \), together with a choice of (pseudo-)unit vector, determines a decomposition \( \mathbb{V} = \mathbb{W} \oplus \mathbb{L} \) as above and a compatible, normalized pair on \( \mathbb{W} \).

**Proposition 2.14.** [25 Proposition 1.14] Let \( \chi \) be a cross product on a 7-dimensional real vector space \( \mathbb{V} \), let \( H \in \Lambda^3\mathbb{V}^* \) be the inner product it induces, so that \( \Phi_{ABC} = H_{CDE}\chi_{AB} \in \Lambda^3\mathbb{W}^* \) is the corresponding stable 3-form. Let \( n \in \mathbb{V} \) be a vector that satisfies \( H(n, n) = -\varepsilon \in \{ \pm 1 \} \), denote \( \mathbb{W} := \langle n \rangle^\perp \), and let \( \iota \) denote the inclusion \( \mathbb{W} \to \mathbb{V} \). Then, the pair \((\omega, \beta)\) defined by
\[
\omega := \iota^*(\iota^*\Phi) \in \Lambda^2\mathbb{W}^*, \\
\beta := \iota^*\Phi \in \Lambda^3\mathbb{W}^*,
\]
is a pair of compatible, normalized stable forms, and the bilinear form \( g \in S^2\mathbb{W}^* \) the pair determines via \((12)\) satisfies \( g = \iota^*H \).

3. Nearly (para-)Kähler geometry

In this section we first introduce some basic notions and constructions for nearly Kähler and nearly para-Kähler geometry, with an emphasis on dimension 6. Both structures are closely linked to an overdetermined natural partial differential equation which, in some contexts, is called the Killing-Yano equation. This equation and its prolongation provide the critical link with projective geometry that we take up in the next section.

3.1. Conventions for affine and (pseudo-)Riemannian geometry. It will at times be useful to use the abstract index notation \( \mathcal{E}^a \) for the tangent bundle \( TM \), and \( \mathcal{E}_a \) for its dual \( T^*M \). Given a torsion-free affine connection \( \nabla \) on an \( n \)-manifold its curvature \( R_{abc}^d \) is then defined by
\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)U^c = R_{abc}^d U^d, \quad U^d \in \Gamma(\mathcal{E}^d),
\]
The Ricci tensor of \( \nabla \) is given by \( R_{ab} := R_{abc}^a \).
In particular this applies to the Levi-Civita connection of a metric $g$ of any signature. In this case we may also define the scalar curvature $Sc = g^{ab}R_{ab}$.

3.2. Almost $\varepsilon$-Hermitian geometry.

Definition 3.1. An almost $\varepsilon$-complex structure on a (necessarily even-dimensional) manifold $M$ is a linear endomorphism $J \in \text{End}(TM)$ such that, at each $x \in M$, $J_x$ is an $\varepsilon$-complex structure on $T_xM$ (see Section 2.1).

Correspondingly, an almost $\varepsilon$-Hermitian manifold is a triple $(M, g, J)$ where $M$ is a manifold, where $g$ is a (pseudo-)Riemannian metric and $J$ is an almost complex structure so that for all $x \in M$, $(g_x, J_x)$ is an $\varepsilon$-Hermitian structure on $T_xM$, that is, if $g(J\cdot, J\cdot) = -\varepsilon g(\cdot, \cdot)$,

(14) $g(J\cdot, J\cdot) = -\varepsilon g(\cdot, \cdot),$

By the remarks after Definition 2.1, the metric of an almost Hermitian manifold must have signature $(2p, 2q)$ for some nonnegative integers $p, q$, and an almost para-Hermitian manifold must have signature $(m, m)$, where $\dim M = 2m$.

On an almost $\varepsilon$-Hermitian manifold $(M^{2m}, g, J)$, the skew-symmetric 2-form $\omega := g(\cdot, J\cdot)$ is called the fundamental 2-form or Kähler form. It satisfies the identities

$$\omega(J\cdot, J\cdot) = g(J\cdot, JJ\cdot) = -\varepsilon g(\cdot, \cdot) = -\varepsilon \omega(\cdot, \cdot).$$

The Nijenhuis tensor $N_J$ of an almost $\varepsilon$-complex structure $J$ is defined by

$$N_J(U, V) := \varepsilon[U, V] - [JU, JV] + J[JU, V] + J[U, JV]$$

(15) $$= -\nabla_{JV}JU + (\nabla_{JU}J)V + J(\nabla_{U}J)V - J(\nabla_{V}J)U,$$

for arbitrary vector fields $U, V$, where $\nabla$ is any torsion-free connection. This tensor is the complete obstruction to the integrability of $J$.

The following well-known identities are easily checked.

Proposition 3.2. The Levi-Civita connection $\nabla^g$ of an almost $\varepsilon$-Hermitian manifold $(M, g, J)$ satisfies the following identities (for arbitrary vector fields $U, V, W$):

$$\nabla_U JV = -J(\nabla_U) V,$$

and

$$g((\nabla_U J)V, W) = -(\nabla_U \omega)(V,W).$$

3.3. Nearly $\varepsilon$-Kähler geometry.

Definition 3.3. An almost $\varepsilon$-Hermitian manifold $(M, g, J)$ is nearly $\varepsilon$-Kähler iff its Levi-Civita connection $\nabla$ satisfies

$$\nabla_U JU = 0$$

(17) $$(\nabla_U J)U = 0$$

for all $U \in \Gamma(TM)$, or equivalently if the covariant derivative $\nabla \omega$ of the Kähler form $\omega$ is totally skew. It is strictly nearly $\varepsilon$-Kähler if in addition $\nabla J$ or, equivalently, $\nabla \omega$, is nowhere zero. For brevity we sometimes write nearly Kähler as NK and nearly para-Kähler as NPK, and refer to both structures simultaneously using the abbreviation N(P)K.

It turns out that if the dimension of an N(P)K manifold $(M, g, J)$ is less than 6, then (17) implies that $\nabla J = 0$ and hence that the manifold is $\varepsilon$-Kähler. The definition (15) and the second equation of (16) together give that the Nijenhuis tensor of a nearly $\varepsilon$-Kähler manifold is

(18) $$N_J(U, V) = 4J(\nabla_U J)V.$$

Next, since the Levi-Civita connection is torsion free, on any N(P)K manifold

$$d\omega = 3\nabla \omega.$$

Lemma 3.4. [69, Lemma 2.5] For any vector fields $U$ and $V$ on $M$, the vector field $(\nabla_U J)V$ is orthogonal to $U, JV, V$ and $JV$. 

Proposition 3.5. \[\text{from (18).}\]

the only admissible signatures \((dS)\) sphere \(M\) that theorem in the reference. Kähler manifold satisfies the constant type condition then its dimension must be 6, is mentioned in the proof of Proof. By skew-symmetry, \(g((\nabla_U J)V, U) = \frac{1}{2}d\omega(U, V, U) = 0\). Using \((19)\) gives that \(3g((\nabla_U J)V, JU) = d\omega(U, V, JU) = d\omega(U, JV, U) = 0\). By symmetry the same arguments apply to \(V\) and \(JV\). \(\square\)

Generalizing a well-known construction in nearly Kähler geometry we define, for any nearly \(\varepsilon\)-Kähler manifold, a canonical \(\varepsilon\)-Hermitian connection: This is the unique connection \(\nabla\) with (totally) skew symmetric torsion that preserves the metric \(g\) and the almost \(\varepsilon\)-complex structure \(J\) (see \([57, 76]\)). Explicitly, it is \(\nabla_U V = \nabla_U V + \frac{1}{2}\varepsilon J(\nabla_U J)V\), for \(U, V \in \Gamma(TM)\). The torsion of \(\nabla\) is then \(\tilde{T}(U, V) = \varepsilon J(\nabla_U J)V = \frac{1}{4}\varepsilon N_J(U, V)\), where the last identity follows from \([18]\).

Proposition 3.5. \([77]\) Corollary 3.7 For any nearly \(\varepsilon\)-Kähler structure, \(\langle \nabla J, \nabla J \rangle\) is constant.

Proof. This is a consequence of the identities \(\nabla(\nabla J) = \nabla T = 0\) and \(\nabla g = 0\). The former can be shown by directly computing the covariant derivative \(\nabla\) of the tensor \(g((\nabla_B J)U, V)\) and using some standard \(N(P)K\) identities recorded in \([76]\) and \([57]\). In particular in the last step we need the identity \(g((\nabla^2_{A,B} J)U, V)\) = \(-\frac{1}{4}g((\nabla_A J)B, (\nabla_U J)V)\). \(\square\)

3.4. Dimension six. Henceforth we restrict our discussion of nearly \(\varepsilon\)-Kähler manifolds to when \(M\) has dimension 6 and the structure is strict, for which much stronger results are available.

A nearly Kähler 6-manifold \((M^6, g, J)\) is of constant type \([49]\) Theorem 5.2\(^2\), i.e. there is a constant \(\alpha \in \mathbb{R}\) such that \(g((\nabla_U J)V, (\nabla_U J)V) = \alpha\{g(U, U)g(V, V) - g(U, V)\}^2\). It is also well-known that a Riemannian strictly nearly Kähler manifold is Einstein \([49]\), and the same holds true for pseudo-Riemannian strictly nearly Kähler \([70]\) and strictly nearly para-Kähler structures \([57]\). In particular we have \(R_{ab} = 5\alpha g_{ab}\), where \(\alpha\) is the constant in \((21)\).

For a strictly nearly \(\varepsilon\)-Kähler manifold we can have \(\langle \nabla J, \nabla J \rangle = 0\) only when \(\varepsilon = 1\), and this is equivalent to either of the following conditions \([57, 77]\):

(1) \(\nabla\omega\) is not stable,
(2) the metric \(g\) is Ricci-flat.

In order to explain why \(\langle \nabla J, \nabla J \rangle = 0\) is only possible when \(\varepsilon = 1\), we recall that for \(\varepsilon = -1\) the only admissible signatures \((p, q)\) are \((6, 0)\) and \((2, 4)\) and in either case we have \(\text{sign}(\alpha) = \text{sign}(p - q)\) \([49, 60, 70]\), and so \(\alpha \neq 0\).

Examples of nearly para-Kähler manifold are given in \([57]\), one of these is the pseudo-sphere \(S^{3,3}\). In that source the authors state that there are no known examples of Ricci-flat 6-dimensional NPK manifolds.

When \(\langle \nabla J, \nabla J \rangle \neq 0\), Lemma \([83]\) allows us to use adapted frames \((e_i)\) which are convenient for local calculations. Take \(e_1\) and \(e_3\) to be any two orthonormal local vector fields such that \(e_3 \neq \pm Je_1\) and define \(e_2 := Je_1, e_4 := Je_3, e_5 := |\alpha|^{-1/2}(\nabla e_1)Je_3, e_6 := Je_5\).\(^2\)

\(^2\)This theorem says if the dimension is 6 then \(M\) is of constant type. A converse, namely that if a nearly Kähler manifold satisfies the constant type condition then its dimension must be 6, is mentioned in the proof of that theorem in the reference.
Using this frame (and following [35]) we can easily calculate
\[ \omega = e^{12} + e^{34} + e^{56}, \]
\[ \nabla \omega = e^{135} + \varepsilon(e^{146} + e^{236} + e^{245}), \]
\[ *(\nabla \omega) = -e^{246} - \varepsilon(e^{235} + e^{145} + e^{136}), \]
\[ J^*(\nabla \omega) = e^{246} + \varepsilon(e^{235} + e^{145} + e^{136}). \]

(24)

It is straightforward to check that \( \nabla \omega \land \omega = 0 \) and \( J^*(\nabla \omega) \land \nabla \omega = 2\omega \land \omega \). The stability of \( \nabla \omega \) is proved in Appendix A. So in summary we have the following result.

**Proposition 3.6.** If \((M, g, J)\) is a strictly \(N(P)K\) manifold such that \( \langle \nabla J, \nabla J \rangle \neq 0 \), then \( (\omega, \nabla \omega) \) is a pair of stable, compatible, and normalized forms in that pointwise they satisfy Definition 2.12.

Finally, we state an algebraic identity for nearly \(\varepsilon\)-Kähler manifolds that turns out to have critical consequences in the next section; we delay the somewhat involved proof to Appendix B.

**Proposition 3.7.** Let \((M^6, g, J)\) be a 6-dimensional strictly nearly (para-)Kähler manifold. Its Kähler form \(\omega\) and its projective Weyl curvature \(W\) satisfy

\[ \omega_{k[bc} W_{d]}^k = 0. \]

(25)

3.5. **Link with the Killing equation on a 2-form.** Recall that on a pseudo-Riemannian manifold \((M, g)\), the infinitesimal isometries are precisely the solutions \(U \in \Gamma(TM)\) of the Killing equation \(L_U g = 0\). If we lower an index using \(g\), we can rewrite the equation as \(\nabla(b \omega^c) = 0\).

Similarly, lowering an index in the condition (17) on \(J\) (which partially defines a nearly Kähler structure) yields the equivalent condition \(\nabla_a \omega_{bc} + \nabla_b \omega_{ac} = 0\), which we can write as

\[ \nabla_{(b} \omega_{c)d} = 0. \]

(26)

Like the Killing equation, this is an overdetermined PDE; it is usually called the *Killing-Yano equation* (on 2-forms).

3.6. **Prolonging the Killing-Yano equation.** The Killing-Yano equation (26) depends only on a connection, so it can be regarded as an equation on general affine manifolds. So, in this subsection we work in the general setting of a manifold \(M\) of dimension \(n \geq 2\) equipped with a torsion-free affine connection \(\nabla\). For simplicity, we assume that \(\nabla\) is special, that is, that (locally) it preserves a volume form—in Subsection 4.1 we will see that for our purposes this is no restriction at all. Of course, the Levi-Civita connection of any metric is special.

In this context we shall prolong the equation

\[ \nabla_{b \omega_{cd}} + \nabla_{c \omega_{bd}} = 0 \]

where \(\omega\) is an arbitrary 2-form. Prolongation involves the introduction of new variables in a way to replace a differential equation with a simpler system. For this equation, doing so will also expose a strong link with projective geometry.

On an affine manifold \((M, \nabla)\) the curvature \(R_{ab}{}^c{}^d\) may be decomposed as

\[ R_{ab}{}^c{}^d = W_{ab}{}^c{}^d + \delta^c{}_a P_{bd} - \delta^c{}_b P_{ad}, \]

where

\[ P_{ab} := \frac{1}{n-1} R_{ab} \]

is the *projective Schouten tensor* of \(\nabla\) and

\[ W_{ab}{}^c{}^d \]

is the *projective Weyl tensor* of \(\nabla\). The Weyl tensor is totally tracefree, that is, it satisfies the identities \(\delta^a{}_c W_{ab}{}^c{}^d = 0\) and \(\delta^d{}_c W_{ab}{}^c{}^d = 0\). These objects have special roles in projective geometry, which we exploit in the next section.
As a first step toward prolonging (27) we differentiate it using the connection $\nabla_a$ to obtain
$$\nabla_a\nabla_b\omega_{cd} + \nabla_a \nabla_c\omega_{bd} = 0.$$ 
Cycling on $a, b, c$ gives
$$\nabla_b\nabla_c\omega_{ad} + \nabla_b \nabla_a\omega_{cd} = 0,$$
$$\nabla_c\nabla_a\omega_{bd} + \nabla_c \nabla_b\omega_{ad} = 0.$$ 
Now adding the first two equations and subtracting the third we have:

(29) \[ 2\nabla_a\nabla_{bcd} - |\nabla_a,\nabla_b|\omega_{cd} + |\nabla_a,\nabla_c|\omega_{bd} + |\nabla_b,\nabla_c|\omega_{ad} = 0. \]

Next, using the identity $[\nabla_a, \nabla_b]|\omega_{cd} = R_{abc}^d\omega_{cd} := -R_{ab}^k\omega_{kd} - R_{bc}^k\omega_{dk}$ and the First Bianchi Identity we can rewrite (29) as
$$2\nabla_a\nabla_{bc}d + 2R_{bc}^k\omega_{dk} - R_{bd}^k\omega_{ck} - R_{ac}^k\omega_{bk} - R_{bc}^k\omega_{ak} = 0,$$
and again cycling it on $b, c, d$ gives
$$2\nabla_a\nabla_{cd}b + 2R_{cd}^k\omega_{kb} - R_{ca}^k\omega_{ck} - R_{cd}^k\omega_{dc} = 0,$$ 
$$2\nabla_a\nabla_{bc}d + 2R_{bc}^k\omega_{dk} - R_{bd}^k\omega_{ck} - R_{bc}^k\omega_{ak} = 0.$$ 
Adding the last three equations, using $\nabla_{(\omega)c} = 0$, and again applying the First Bianchi Identity yields
$$\nabla_a\nabla_{bc}d + \frac{1}{2} \{ R_{bc}^k\omega_{dk} + R_{cd}^k\omega_{bk} + R_{db}^k\omega_{ck} \} = 0.$$ 
Using the placeholder variable $\mu_{abc} := \nabla_a\omega_{bc}$, where $\mu_{abc}$ is skew, we rewrite the above equation as the first-order system
$$\begin{cases} 
0 = \nabla_a\omega_{bc} - \mu_{abc} \\
0 = \nabla_a\mu_{bcd} + \frac{1}{2} \{ R_{bc}^k\omega_{dk} + R_{cd}^k\omega_{bk} + R_{db}^k\omega_{ck} \}.
\end{cases}$$
Thus (cf. [12]) solutions of the equation (29) correspond to pairs $\Sigma := (\omega, \mu)$ parallel with respect to the connection $\tilde{\nabla}$, where

(30) \[ \tilde{\nabla}_a \begin{pmatrix} \omega_{bc} \\ \mu_{bcd} \end{pmatrix} = \begin{pmatrix} \nabla_a\omega_{bc} - \mu_{abc} \\
\nabla_a\mu_{bcd} + \frac{1}{2} \{ R_{bc}^k\omega_{dk} + R_{cd}^k\omega_{bk} + R_{db}^k\omega_{ck} \} \end{pmatrix}. \]

This leads to the following result.

**Proposition 3.8.** On a manifold of dimension $n \geq 2$ equipped with a torsion-free special affine connection $\nabla$, solutions of the equation
$$\nabla_{\omega} + \nabla_{\omega} = 0,$$
on 2-form fields $\omega_{bc}$, are in 1-1 correspondence with sections $(\omega, \mu)$ of $\Lambda^2 T^*M \oplus \Lambda^3 T^*M$ which are parallel for the connection

(31) \[ \tilde{\nabla}_a \begin{pmatrix} \omega_{bc} \\ \mu_{bcd} \end{pmatrix} = \begin{pmatrix} \nabla_a\omega_{bc} - \mu_{abc} \\
\nabla_a\mu_{bcd} + 3P_{a[\omega_{cd}]} \end{pmatrix}. \]

**Proof.** This is just a straightforward calculation as in Appendix C. Thus solutions of (27) yield sections parallel for $\tilde{\nabla}$. On the other hand if $(\omega_{bc}, \mu_{bcd})$ is parallel for $\tilde{\nabla}$ then $\nabla_a\omega_{bc} = \mu_{abc}$; in particular $\nabla_a\omega_{bc}$ is totally skew, and so (27) holds.

4. Projective geometry and $N(P)K$-structure

We will show that strictly nearly Kähler and nearly para-Kähler structures in dimension 6 have a natural interpretation in projective geometry, and that this facilitates links to other geometries. The structures treated in this section and the subsequent sections can only exist on orientable manifolds, thus with restriction we shall assume $M$ orientable.
4.1. Projective differential geometry. As mentioned in the introduction a projective structure \( p \) on a manifold \( M \) (of dimension \( n \geq 2 \)) consists of an equivalence class of torsion-free affine connections that share the same geodesics, as unparameterized curves. The class is equivalently characterized by the fact that, acting on any \( U \in \Gamma(TM) \), any two connections \( \nabla \) and \( \check{\nabla} \) in \( p \) are related by a transformation of the form

\[
\check{\nabla}_a U^b = \nabla_a U^b + \Upsilon_a U^b + \Upsilon_a U^c \delta^b_{ac},
\]

where \( \Upsilon \) is some smooth section of \( T^* M \).

According to the usual conventions in projective geometry, we write \( \mathcal{E}(1) \) for the positive \((2n + 2)\text{nd}\) root of the bundle \( (\Lambda^n TM)^2 \), which we note is canonically oriented. A connection \( \nabla \in \mathfrak{p} \) determines a connection on \( \mathcal{E}(1) \) as well as its real powers \( \mathcal{E}(w), \ w \in \mathbb{R} \); we call \( \mathcal{E}(w) \) the bundle of projective densities of weight \( w \). Conversely, for \( w \neq 0 \), a choice of connection on \( \mathcal{E}(w) \) determines a connection \( \nabla \in \mathfrak{p} \). Among the connections in \( \mathfrak{p} \) there is a (non-empty) distinguished class consisting of those connections \( \nabla \in \mathfrak{p} \) that preserve some non-vanishing section of \( \mathcal{E}(w) \), \( w \neq 0 \) (see e.g. [15]). These are exactly the special affine connections (defined in Subsection 3.6) in \( \mathfrak{p} \), and in the following we shall work only with this subset of connections. Such a connection \( \nabla \) is often called a \textit{choice of scale}. If \( \nabla \) and \( \check{\nabla} \) are two choices of scale then \( \Upsilon_b \) is exact, meaning \( \Upsilon_b = \nabla_b \phi \) for some function \( \phi \).

As a point of notation: given any bundle \( \mathcal{B} \) we shall write \( \mathcal{B}(w) \) as a shorthand for \( \mathcal{B} \otimes \mathcal{E}(w) \).

4.2. The Killing-Yano type projective BGG equation. In Section 3.3 we introduced the Killing-Yano equation \( \nabla(\omega_{bc}) = 0 \) on a 2-form \( \omega \). We want to consider the linear operator giving this equation in the case that \( \nabla \) is a (special, torsion-free) affine connection and \( \omega_{ab} \) is any 2-form field of weight \( w \). Let \( \check{\nabla} \) and \( \nabla \) be two projectively equivalent scales. From (32), we have

\[
\check{\nabla}_a \omega_{bc} + \Upsilon_b \omega_{ac} = \nabla_a \omega_{bc} + (w - 2) \Upsilon_a \omega_{bc} - \Upsilon_b \omega_{ac} - \Upsilon_c \omega_{ba},
\]

and it is easy to see that \( \check{\nabla}(\omega_{bc}) = \nabla(\omega_{bc}) \) if and only if \( w = 3 \).

When taking \( \omega \) to have projective weight 3, equation (27) fits into the class of first BGG equations on projective manifolds; see [26, 25] and references therein for a general discussion of the class.

4.3. The projective tractor connection. On a general projective manifold \((M, \mathfrak{p})\) there is no canonical connection on the tangent bundle. There is, however, a canonical connection on related natural bundle of rank \( n + 1 \); this so-called tractor connection is the fundamental invariant object capturing the geometric structure of projective geometries. We follow here the development of [9, 28].

On any smooth manifold \( M \), the first jet prolongation \( J^1 \mathcal{E}(1) \to M \) of the projective density bundle \( \mathcal{E}(1) \) of weight 1, is a natural vector bundle. Its fiber over \( x \in M \) consists of all 1-jets \( j^1_x \sigma \) of local smooth sections \( \sigma \in \Gamma(\mathcal{E}(1)) \) defined in a neighborhood of \( x \). For two sections \( \sigma \) and \( \tilde{\sigma} \) we have \( j^1_x \sigma = j^1_x \tilde{\sigma} \) if and only if in one, or equivalently any, local chart the sections \( \sigma \) and \( \tilde{\sigma} \) have the same Taylor development in \( x \) up to first order. On the other hand sections \( \sigma \in \Gamma(\mathcal{E}(1)) \) determine smooth sections \( j^1 \sigma \) of \( J^1 \mathcal{E}(1) \) via the smooth structure on the latter space. Mapping \( j^1_x \sigma \) to \( \sigma(x) \) thus defines a smooth, surjective bundle map \( J^1 \mathcal{E}(1) \to \mathcal{E}(1) \), called the \textit{jet projection}. If \( j^1_x \sigma \) lies in the kernel of this projection, so \( \sigma(x) = 0 \), then the value \( \nabla \sigma(x) \in T^*_x M \otimes \mathcal{E}_x(1) \) is the same for all linear connections \( \nabla \) on the vector bundle \( \mathcal{E}(1) \). This identifies the kernel of the jet projection with the bundle \( T^* M \otimes \mathcal{E}(1) \). See for example [23] for a general development of jet bundles.

Using an abstract index notation, we shall write \( \mathcal{E}_A \) (in index-free notation, \( T^* \)) for \( J^1 \mathcal{E}(1) \) and \( \mathcal{E}^A \) (or \( T \)) for the dual vector bundle. Then we can view the jet projection as a canonical
section $X^A$ of the bundle $\mathcal{E}^A \oplus \mathcal{E}(1) = \mathcal{E}^A(1)$. Likewise, the inclusion of the kernel of this projection can be viewed as a canonical bundle map $\mathcal{E}_a(1) \to \mathcal{E}_a$, which we denote by $Z_a^a$. Then the composition structure of $\mathcal{E}_a$, explained above, is written as

$$0 \to \mathcal{E}_a(1) \xrightarrow{Z_a^a} \mathcal{E}_a^A \xrightarrow{X^A} \mathcal{E}(1) \to 0. \tag{33}$$

This is known as the jet exact sequence at 1-jets for the bundle $\mathcal{E}(1)$. We write $\mathcal{E}_a = \mathcal{E}(1) \xrightarrow{\nabla} \mathcal{E}_a(1)$ to summarize the composition structure in $\mathcal{E}_a$. As mentioned, any connection $\nabla \in \text{p}$ is equivalent to a connection on $\mathcal{E}(1)$. But a connection on $\mathcal{E}(1)$ is precisely a splitting of the 1-jet sequence $\mathcal{E}_a$. In particular this holds for special connections. Thus given such a choice we have the direct sum decomposition $\mathcal{E}_a \cong \mathcal{E}(1) \oplus \mathcal{E}_a(1)$ with respect to which we define a connection on $\mathcal{E}_a$ by

$$\nabla^*_a \left( \begin{array}{c} \sigma \\ \mu_b \end{array} \right) := \left( \begin{array}{c} \nabla_a \sigma - \mu_a \\ \nabla_{a \mu_b} + P_{ab} \sigma \end{array} \right), \tag{34}$$

where, recall, $P_{ab}$ is the projective Schouten tensor. A simple calculation shows that $\nabla^*$ is independent of the choice $\nabla \in \text{p}$, and so $\nabla^*$ is determined canonically by the projective structure $\text{p}$.

This cotractor connection is due to [81]. It is equivalent to the normal Cartan connection (of [32]) see [21]. We shall term $\mathcal{E}_a(T^\ast)$ the cotractor bundle, and we note the dual tractor bundle $\mathcal{E}^A(T)$ has the composition structure

$$0 \to \mathcal{E}(-1) \xrightarrow{X^A} \mathcal{E}^A \xrightarrow{Z^a} \mathcal{E}^a(-1) \to 0. \tag{35}$$

This is canonically equipped with the dual tractor connection: in terms of a splitting dual to that above this is given by

$$\nabla^T_a \left( \begin{array}{c} \nu^b \\ \rho \end{array} \right) = \left( \begin{array}{c} \nabla_a \nu^b + \rho \delta^b_a \\ \nabla_a \rho - P_{ab} \nu^b \end{array} \right). \tag{36}$$

From (33) we have invariantly the map $X^A : \mathcal{E}_a \to \mathcal{E}(1)$. As mentioned above, given a special affine connection $\nabla$ on $TM$ we also have the splitting $\mathcal{E}_a = \mathcal{E}(1) \oplus \mathcal{E}_a(1)$, and so in particular the projection $W^A_a : \mathcal{E}_a \to \mathcal{E}_a(1)$ that splits the sequence (33). By definition then $\mathcal{E}_a^A W^A_a = \delta^a_b$. This splitting, and the dual splitting of the sequence (33), are also equivalent to a map $\mathcal{Y}_a : \mathcal{E}_a \to \mathcal{E}(-1)$, that satisfies $X^A \mathcal{Y}_a = 1$. In terms of these, sections $V^A \in \Gamma(\mathcal{E}^A)$ and $U_a \in \Gamma(\mathcal{E}_a)$ which are represented by

$$V^A \equiv \left( \begin{array}{c} \nu^a \\ \rho \end{array} \right), \quad \text{and} \quad U_a \equiv \left( \begin{array}{c} \sigma \\ \mu_a \end{array} \right)$$

in the given splitting, can be written $V^A = W^A_a \nu^a + X^A \rho$, and $U_a = Y_a \sigma + Z_a^a \mu_a$. These expansions, and the analogs for tensor powers of the tractor bundles, turn out to be extremely useful for managing calculations, so we record here some basic facts.

Under a change of special affine connection from $\nabla$ to $\tilde{\nabla}$, as in (32), we have

$$V^A \equiv \left( \begin{array}{c} \nu^a \\ \rho - \mathcal{Y}_a \nu^a \end{array} \right), \quad \text{and} \quad U_a \equiv \left( \begin{array}{c} \sigma \\ \mu_a + \mathcal{Y}_a \sigma \end{array} \right)$$

[9] (where $\mathcal{Y}_a$ is exact). So for the corresponding maps $\mathcal{Y}_a : \mathcal{E}_a \to \mathcal{E}(-1)$ and $\tilde{W}^A_a : \mathcal{E}_a \to \mathcal{E}_a(1)$ we have,

$$\tilde{W}^A_a = W^A_a + X^A \mathcal{Y}_a, \quad \tilde{Y}_a = Y_a - Z_a^a \mathcal{Y}_a, \quad \tilde{X}^A = X^A, \quad \text{and} \quad \tilde{Z}_a^a = Z_a^a, \tag{37}$$

where we have also recorded the projective invariance of $X^A$ and $Z_a^a$ for convenience. Finally the data of the tractor connection is captured by how it acts on the splitting maps. From (34)
and (36) we have the following:

\[
\nabla_a X^B = W_a^B, \quad \nabla_a W_{B}^{B'} = -P_{aB}^B X^B
\]

\[
\nabla_a Y_B = P_{aB} Z_B^B, \quad \nabla_a Z_B^{B'} = -\delta^B_{B'} Y_B.
\]

In these formulae we calculate in terms of a scale \(\nabla\), and the connection in the formulae is the coupling of this special affine connection with the tractor connection \(\nabla^T\).

Finally in this section we recover the canonical tractor “volume form”. On an oriented projective manifold \((M, p)\) of dimension \(n\) there is a tautological weighted \(n\)-form \(\eta\) which gives the isomorphism

\[
\eta : \Lambda^n TM \to \mathcal{E}(n + 1),
\]

that defines \(\mathcal{E}(n + 1)\). For each affine connection \(\nabla \in p\), the isomorphism (39), applied to sections, enables the definition of \(\nabla\) as a connection on \(\mathcal{E}(n + 1)\) and hence all density bundles. It follows tautologically that for any affine connection \(\nabla \in p\) we have

\[
\nabla \eta = 0.
\]

In particular the last display applies to special affine connections in \(p\). Calculating in the scale \(\nabla\) and using the formulae (38) it is easily verified that the tractor \((n + 1)\)-form \(\epsilon\), defined by

\[
\epsilon_{AB\cdots E} := Y_{[A} Z_{B}^{B'} \cdots Z_{E]} \epsilon_{b\cdots e} \in \Gamma(\Lambda^{n+1} T^*)
\]

is parallel for the tractor connection. On the other hand using (37) it follows at once that \(\epsilon\) is independent of the choice of \(\eta\). Thus we have the following (essentially well-known) result:

**Proposition 4.1.** An oriented projective \(n\)-manifold \((M, p)\) determines a canonical parallel tractor \((n+1)\)-form \(\epsilon\) by the formula (40). So the projective Cartan geometry is of type \((\text{SL}(7, \mathbb{R}), P)\) for suitable \(P\).

### 4.4. Nearly \(\varepsilon\)-Kähler geometry in terms of the tractor connection.

The tractor connection and its dual induce projectively invariant connections on all tensor parts of tensor products of the tractor bundle, and its dual. In particular we will need the tractor connection that \(\nabla^T\) induces (and which we denote by the same symbol) on

\[
\Lambda^3 \mathcal{E}_A = \mathcal{E}_{[ABC]} = \mathcal{E}_{[ab]}(3) \oplus \mathcal{E}_{[abc]}(3)
\]

In a scale this is given by

\[
\nabla^T_a \begin{pmatrix} \sigma_{bc} \\ \mu_{bcd} \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma_{bc} - \mu_{abc} \\ \nabla_a \mu_{bcd} + 3P_{a[bc][d]} \end{pmatrix}.
\]

We see from (12) that if \(\Phi_{ABC} \in \Lambda^3 \mathcal{E}_A\) is parallel for the tractor connection \(\nabla^T\) then its top component \(\sigma_{bc}\) has projective weight 3 and solves the projective Killing-Yano–type equation \(\nabla^T_{(a} \sigma_{bc)} = 0\), cf. (27) above. Conversely from Proposition 4.3 (and using again the formula (42)) we see that if \(\omega_{bc}\) is a solution of (27) then

\[
0 = \nabla_a \begin{pmatrix} \omega_{bc} \\ \mu_{bcd} \end{pmatrix} = \nabla^T_a \begin{pmatrix} \omega_{bc} \\ \mu_{bcd} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 3 \omega_{k[b} W_{c]}^k \end{pmatrix},
\]

where \(\mu_{bcd} = \nabla_a \omega_{bc}\).

What is important for us here is that for a Kähler form \(\omega\) on a nearly Kähler or nearly para-Kähler 6-manifold the second term in (13) vanishes separately, and what is more the tractor 3-form determined by prolonging \(\omega\) is nondegenerate. More precisely we have the following result.
Theorem 4.2. Let \((M^6, g, J)\) be a 6-dimensional strictly nearly \(\varepsilon\)-Kähler manifold with \((\nabla J, \nabla J) \neq 0\) and \(T\) the \((rank-7)\) standard projective tractor bundle over \(M\). Then the 3-tractor

\[
\Phi = \left( \begin{array}{c} \omega_{bc} \\ \mu_{bd} \end{array} \right),
\]

is generic and parallel with respect to \(\nabla^T\). Its pointwise stabilizer is \(G_2\) if \(\Phi\) is definite-generic, or is \(G_2^*\) if \(\Phi\) is split-generic.

Proof. As \((M^6, g, J)\) is strictly nearly \(\varepsilon\)-Kähler, identity \((25)\) holds, and from Proposition 3.8 we have that \(\Phi\) is parallel with respect to \(\nabla^T\). In Proposition 3.6 of Subsection 3.4 we observed that \(\omega\) and \(\mu = \nabla \omega\) form a stable, compatible, and normalized pair; so, the hypotheses of Proposition 2.13 are satisfied (pointwise), and hence \(\Phi\) is nondegenerate, and pointwise its stabilizer is \(G_2^*\).

Remark 4.3. The observation below \((12)\) is a special case of a general fact that applies across the entire field of parabolic geometry. A tractor field parallel for a (normal) tractor connection always determines a solution of a first BGG equation \([27, \text{Theorem 2.7}]\). BGG operator solutions arising this way are said to be normal. (The terminology follows Leitner’s [65], where a class of conformal equations were treated). Thus part of the content of Theorem \([12]\) is that the Kähler form of a strictly \(N(P)K\) structure is a normal solution of \([27]\).

4.5. A digression on conformal tractor geometry. Shortly we shall see that conformal geometry enters our picture. Recall that on a manifold \(M_0\) a conformal structure \(c\) of signature \((p, q)\) is a conformal equivalence class \([g]\) of metrics (of signature \((p, q)\)) on \(M_0\); a metric \(\tilde{g}\) is in the conformal structure \([g]\) iff \(\tilde{g} = fg\) for some positive function \(f\). As in projective geometry, on a conformal manifold there is no invariant connection on the tangent bundle but again there is on a related higher-rank bundle that we call the standard (conformal) tractor bundle. Again this is equivalent to a Cartan bundle and a canonical connection \([21]\). Here we sketch some aspects of tractor calculus following the conventions and development of \([22, 46]\), and we refer the reader to those sources for more details. For later convenience we denote the underlying manifold \(M_0\) and assume here that this has dimension \(n - 1 \geq 3\).

In conformal geometry density bundles are important. For representation-theoretic reasons the convention for weights differs from that in projective geometry: on a conformal manifold of dimension \(n - 1\) we write \(\mathcal{E}_0[1]\) for the positive \(2(n - 1)\)th root of the canonically oriented bundle \((\wedge^n T M_0)^2\). Since a metric \(g \in c\) trivializes \((\wedge^n T M_0)^2\), and hence also \(\mathcal{E}_1[1]\), \((M_0, c)\) determines a canonical section \(g \in S^2 T^* M_0[2] = S^2 T^* M_0 \otimes \mathcal{E}[2]\) called the conformal metric; then, on the fixed conformal structure \(c\), \(g\) is equivalent to \(\sigma \in \Gamma(\mathcal{E}_0[1])\) by the relation \(g = \sigma^{-2} g\). (Here \((\mathcal{E}_0)[1]\) is the positive ray sub-bundle of \(\mathcal{E}_0[1]\)) The conformal metric and its inverse are preserved by the Levi-Civita connection of every metric in the conformal class, and they determine an isomorphism \(g : TM_0 \to T^* M_0[2]\).

The conformal standard tractor bundle will be denoted \(\mathcal{T}_0\), or in abstract index notation \(\mathcal{E}^A_0\). It has rank \(n + 1\) and a composition series

\[
\mathcal{T}_0 = \mathcal{E}_0[1] \oplus TM_0[-1] \oplus \mathcal{E}_0[-1].
\]

On \(\mathcal{T}_0\), there is an invariant signature \((p + 1, q + 1)\) tractor metric \(H_0\) (which we may alternatively denote \(H^0\)), and an invariant connection \(\nabla^{\mathcal{T}_0}\) that preserves the metric; so, we use it to lower and raise tractor indices. The canonical bundle line bundle injection is denoted

\[
X^A : \mathcal{E}_0[-1] \to \mathcal{E}_0^A.
\]

No confusion should arise with the projective analogue of \(X^A\), which shares the same abstract index notation. In fact, we shall see that they are suitably compatible in the setting below where they arise together. As a section of \(\mathcal{T}_0[1]\), \(X\) is null, meaning \(H^A_{AB} X^A X^B = 0\), and \(X_A := H^A_{AB} X^B\) gives the canonical bundle map \(X_A : \mathcal{E}_0^A \to \mathcal{E}_0[1]\).
A choice of metric $g \in \mathfrak{c}$ determines a splitting of (45) to a direct sum $\mathcal{T}_0 \cong \mathcal{E}_0[1] \oplus TM_0[1] \oplus \mathcal{E}_0[-1]$ and we denote the induced projections onto the second and third components by

$$Z_A^a : \mathcal{E}_0 \to \mathcal{E}_0[1], \quad Y_A : \mathcal{E}_0 \to \mathcal{E}_0[-1].$$

We may view these as sections $Y_A \in \Gamma((\mathcal{E}_0)_A[-1])$, $Z_A^a \in (\mathcal{E}_0)_A^a[1]$ and then the tractor metric is characterized by the identities $H_{AB}^0 Z_A^a Z_B^b = g_{ab}$ and $H_{AB}^0 X^A Y^B = 1$ and that all other tractor index contractions of pairs of these splitting maps results in zero; for example $H_{AB}^0 Y^A Y^B = 0$. (Note that here we have raised and lowered indices on $Z$ and $Y$ using the conventions described.) A section $U^A \in \Gamma((\mathcal{E}_0)_A)$ may be written $U^A = \sigma Y^A + Z_A^a \xi^a + \rho X^A$, with $(\sigma, \mu^a, \rho) \in \Gamma(\mathcal{E}_0[1] \oplus TM_0[1] \oplus \mathcal{E}_0[-1])$. If $\tilde{Y}^A$ and $\tilde{Z}_a^b$ are the corresponding tractor splitting maps in terms of the metric $\tilde{g} = \Omega^2 g \in \mathfrak{c}$ then we have

$$\tilde{Z}_{ab} = Z^{AB} + \Upsilon^b X^A, \quad \tilde{Y}^A = Y^A - \Upsilon_b Z^{ab} - \frac{1}{n} \Upsilon_{\beta} \Upsilon^b X^A, \quad \tilde{X}^A = X^A,$$

where $\Upsilon = d\Omega$, and we have recorded the invariance of $X^A$ for convenience.

The conformal tractor connection is characterized by its action on the splitting maps:

$$\nabla_a X^A = Z_{Aa}, \quad \nabla_a Z_{AB} = -P_{ab} X^A - g_{ab} Y_A, \quad \nabla_a Y_A = P_{ab} Z_{A}^b.$$  \hspace{1cm} (49)

Here $P_{ab}$ is the conformal Schouten tensor, defined by

$$P_{ab} = \frac{1}{n-3} \left( R_{ab} - \frac{\text{Sc}}{2(n-2)} g_{ab} \right),$$

where $R_{ab}$ is the usual (pseudo-)Riemannian Ricci tensor and $\text{Sc}$ is its metric trace In (49) the connection used is strictly the coupling of the tractor connection $\nabla^\tau$ with the Levi-Civita connection, hence the use of the notation $\nabla$ (rather than $\nabla^\tau$).

**Remark 4.4.** As we mentioned is the case for $X$, also the notation $Y, Z$ for the objects splitting the conformal tractor (via $g \in \mathfrak{c}$) is essentially the same as that used for the corresponding objects in the projective setting. Context should prevent any confusion, and indeed there is again a degree of compatibility.

### 4.6. Projective almost Einstein structures

It will be important for us to understand the meaning of a non–Ricci-flat Einstein metric in the setting of projective geometry, following [4, 28]. In fact projective geometry motivates a natural generalization of the Einstein condition [26, 27, 28] and this is a key point for us here.

From Theorem 3.3 of [28] we have the following.

**Theorem 4.5.** Let $(M, g)$ is a pseudo-Riemannian manifold of signature $(p, q)$, and dimension $n \geq 2$. If $g$ is positive (respectively, negative) Einstein and not Ricci-flat then there is a canonical parallel projective tractor metric $H$ of signature $(p+1, q)$ (respectively, $(p, q+1)$) on the projective structure $(M, [\nabla^g])$.

Here $\nabla^g$ is the Levi-Civita connection determined by $g$. In the case of dimension 2 we take $g$ Einstein to mean that $\text{Sc}^g$ is constant, although this dimension is not important for the current article.

In the converse direction, if a projective manifold admits a parallel metric $H$ on its tractor bundle it determines a section

$$\tau := H_{AB} X^A X^B$$

of the density bundle $\mathcal{E}(2)$. On any open set where $\tau$ is nowhere zero it may be used to trivialize the density bundles, and hence it determines a connection on densities and so a splitting of the sequence (43). Taking duals we obtain a splitting of the tractor sequence (45), and so from $H$ a metric $g^\tau$ on $TM$. In the (nonvanishing) scale $\tau$, and with the corresponding splittings, we have

$$H_{AB} = \begin{pmatrix} \tau & 0 \\ 0 & -\varepsilon \tau g_{ab} \end{pmatrix},$$  \hspace{1cm} (51)
where $-\varepsilon \in \{\pm 1\}$ gives the sign of the scalar curvature. This formula follows easily from the formula (53) for the tractor connection (extended to $S^2T^*$) and the definition here of the metric $g$ via Theorem 4.6. See e.g. [24] Section 3.3 for a more detailed discussion, and expression (15) in that source for the sign of the scalar curvature.

It turns out that $g_r$ is necessarily Einstein. In detail we have the following, as obtained from different points of view in [26, Theorem 3.2] and Theorem 3.1 and Proposition 3.2 in [27].

**Theorem 4.6.** Let $(M, p)$ be a projective structure endowed with a holonomy reduction given by a parallel metric $H$ of signature $(r, s)$ on the standard projective tractor bundle $T$.

(a) The metric $H$ determines a stratification $M = M_+ \cup M_0 \cup M_-$ according to the strict sign of $\tau := H_{AB}X^AX^B$. The sets $M_+ \subset M$ and $M_- \subset M$, where $\tau$ is positive and negative, respectively, are open; $M_0$ is the zero set of $\tau$ and (if non-empty) is an embedded hypersurface. Here $M_+$, $M_0$, and $M_-$ are not necessarily connected.

(b) The structure $(M, p, H)$ induces a Cartan geometry on $M_+$ (respectively $M_-$) as follows: via (39) and (50). $H$ induces an Einstein pseudo-Riemannian metric $g_{\pm}$ of signature $(r-1, s)$, if $r \geq 1$ (respectively $(r, s-1)$ if $s \geq 1$) whose Levi-Civita connection lies in the (restriction of) projective class. The scalar curvature of $g_{\pm}$ is positive (respectively negative).

(c) If $r = 0$ (or $s = 0$) then $M_0 = \emptyset$, and $M_+ = \emptyset$ (or $M_- = \emptyset$, respectively). If $M_0$ is non-empty then it naturally inherits a conformal structure of signature $(r-1, s-1)$ via the induced Cartan geometry. In this case the standard conformal tractor bundle agrees with the restriction of the projective tractor bundle $T$ to $M_0$ and the normal conformal tractor connection of $(M_0, c)$ is naturally the corresponding restriction of the ambient projective tractor connection.

The components $M_+, M_0, M_-$ are called curved orbits since they generalize to the curved setting an orbit decomposition of a model structure, as explained in [26,27]. Since $(M, p, H)$ determines on the open sets $M_\pm$ (whose union is dense) Einstein metrics, it is natural to call this a projective almost Einstein structure (following the analogous conformal notion [33]). In the case where $M_0$ is nonempty we shall term the parts $M_\pm \cup M_0$ Klein-Einstein manifolds (or Klein-Einstein structures). This follows [26, Section 3.3] and as explained there the terminology is appropriate and useful because these are the projective geometry analogues of Poincaré-Einstein geometries; see also [7, §4] for more about both Klein- and Poincaré-Einstein metrics, and about the relationship between them. (In [26] this terminology was proposed for the negative curvature part, so we are slightly generalizing its usage here.) Concerning terminology, in the subsequent discussion it will be convenient to refer to $M_0$, which is the zero locus of $\tau$, as the zero locus of $(M, p, H)$ or simply the zero locus when the meaning is clear by context.

From part (2) of the Theorem we have that the projective class of the Levi-Civita connections $\nabla^g_{\pm}$ in $M_\pm$ extend smoothly to $M_0$. In fact more is true, as follows.

First recall that, in a manifold, a defining function for a codimension-$1$ embedded submanifold $\Sigma$ is a function $r$ such that $\Sigma$ is the zero locus of $r$, and $dr$ is nowhere zero along $\Sigma$. Following [24] we make the following definition.

**Definition 4.7.** On a manifold $M$ with boundary $\partial M$ and interior $M_{\text{int}}$, an affine connection $\nabla$ on $M_{\text{int}}$ is called projectively compact of order $\alpha \in \mathbb{R}_+$ if for any $x \in \partial M$, there is a neighborhood $U$ of $x$ in $M$ and a defining function $r : U \to \mathbb{R}$ for $U \cap \partial M$ such that the connection

(52)

$\tilde{\nabla} = \nabla + \frac{dr}{r}$

on $U \cap M_{\text{int}}$ extends to all of $U$. A metric is said to be projectively compact of order $\alpha$ if its Levi-Civita connection satisfies this condition.

This notion applies in an obvious way to the setting of the Theorem 4.6 above, by considering the manifolds with boundary $M_\pm \cup M_0$. Then from Theorem 12 of [24] we have that a non-Ricci-flat projectively compact metric must have $\alpha = 2$, and hence the following result.
Proposition 4.8. The metrics $g_{\pm}$ in Theorem 4.4 are projectively compact of order 2.

For special affine connections the behavior [52] guarantees a uniformity in the rate of asymptotic volume growth as the boundary is approached. The value $\alpha = 2$ shows, for example, that this growth rate is different from that on conformally compact manifolds, cf. [24, Section 2.2].

For later (implicit) use, we record a convenient alignment of the projective and conformal conventions for weighted bundles.

Proposition 4.9. Let $(M, p)$ be a projective manifold of dimension $n \geq 2$ and $H$ a parallel tractor metric for which the zero locus $M_0$ is nonempty. Then, using the notation in Subsection 4.3 for the objects on $M$, for all real $k$ there is a canonical identification

$$E_0[w] \cong E(w)|_{M_0}.$$ 

Proof. Recall that on an oriented conformal $(n-1)$-manifold the conformal density bundle of weight $w$, denoted $E[w]$, is the positive $w/(n-1)$ root of the oriented line bundle $\Lambda^{n-1}TM_0$. We will show that $\Lambda^{n-1}TM_0$ can be identified with $E(n-1)|_{M_0}$.

Recall that on an oriented projective manifold $(M, p)$ of dimension $n$ there is a tautological weighted $n$-form $\eta$ which gives the isomorphism

$$\eta : \Lambda^n TM \to E(n + 1),$$

that defines $E(n + 1)$, see [29]. On the other hand along $M_0$, there is a canonical section $n \in \mathcal{N} \otimes E(2)$, where $\mathcal{N}$ is the conormal bundle. If $\nabla \in p$, then $n = (\nabla \tau)|_{M_0}$, but $n$ is independent of the choice of $\nabla$, since $M_0$ is the zero locus of $\tau = H(X, X)$.

Now let $n^1$ be any section of $(TM \otimes E(-2))|_{M_0}$ satisfying $n(n^2) = 1$. Then, identifying $\Lambda^{n-1}TM_0$ with its image in $\Lambda^{n-1}TM_0(M_0)$, we obtain a surjective bundle map

$$\eta(n^1, \ldots, \cdot : \Lambda^{n-1}TM_0 \to E(n-1)|_{M_0}.$$ 

Now it is easily verified that $\eta$ is independent of the choice of $n^1 \in \Gamma((TM \otimes E(-2))|_{M_0})$ satisfying $n(n^2) = 1$. Thus $\eta$ gives a canonical isomorphism

$$E[n - 1] \to E(n - 1)|_{M_0}$$

and thus we obtain $E_0[w] \cong E(w)|_{M_0}$, for all real weights $w$. \quad \square

5. Projective 6-manifolds with a parallel tractor split cross product

We now want to consider a situation essentially converse to that of Theorem 4.2. Namely we consider a projective 6-manifold $(M, p)$ that is equipped with a parallel tractor 3-form $\Phi$ that is generic, meaning that at one, equivalently any, point $x \in M$, $\Phi_x$ is stable on $T_x$ (in the sense of Section 2.3). So the projective Cartan/tractor connection admits a holonomy reduction to either $G_2$ or $G^*_2$. We will see shortly that this is the same as a parallel cross product on the tractor bundle.

5.1. The tractor metric on an $(M, p, \Phi)$ manifold. Recall that we assume $(M, p)$ is orientable. If $M$ has dimension 3 then by Proposition 4.1 the tractor connection [30] preserves a (non-trivial) parallel tractor 7-form $\epsilon$, which for convenience we shall call the tractor volume form, although of course $\epsilon$ is not a tensor. We write $\epsilon^{-1}$, or $\epsilon^{A_1 \cdots A_7}$, for the section of $\Lambda^7 T$ satisfying

$$\epsilon^{A_1 \cdots A_7} \epsilon_{A_1 \cdots A_7} = 7!.$$ 

The first observation is that with $\Phi$ this object determines a tractor metric.

Theorem 5.1. On a projective 6-manifold let $\Phi$ be a parallel generic tractor 3-form. Then $\Phi$ determines a nondegenerate parallel tractor

$$H_{AB} := \frac{1}{144} \Phi_{AC_1C_2} \Phi_{BC_3C_4} \Phi_{C_5C_6C_7} \epsilon^{C_1 \cdots C_7}$$

of signature of either $(7, 0)$ or $(3, 4)$.  

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Proof. As $\Phi$ and $\epsilon$ are parallel, this is immediate from the algebraic result (1). □

Remark 5.2. The signatures $(7,0)$ and $(3,4)$ are compatible (in a way that e.g. $(7,0)$ and $(4,3)$ are not) in the sense that we take the inner product on both $\mathcal{O}$ and $\mathcal{O}^*$ in §2 so that the real line $\mathbb{R} \subset \mathcal{O}^{(*)}$ is positive definite.

Thus the $G_2$ holonomy reduction of the projective Cartan bundle is subordinate to a $SO(7,0)$ reduction, and similarly the $G_2^{(*)}$ reduction is subordinate to a $SO(3,4)$ reduction. Furthermore we can at once exploit Theorem 4.6 and Proposition 4.8.

Corollary 5.3. Suppose that $(M, p)$ is a projective 6-manifold equipped with a generic parallel 3-form tractor $\Phi$. Then:

- If $\Phi$ is definite-generic then it determines a signature $(6,0)$ positive Einstein metric on $M$.
- If $\Phi$ is split-generic then it determines a decomposition $M = M_+ \cup M_0 \cup M_-$ of $M$ into a union of 3 (not necessarily connected) disjoint curved orbits; $M_\pm$ are open while the zero locus $M_0$ is closed. If the zero locus $M_0$ is non-empty, then both $M_+$ and $M_-$ are non-empty. Conversely, if $M$ is connected and both $M_+$ and $M_-$ are non-empty then $M_0$ is non-empty and is a smoothly embedded separating hypersurface consisting of boundary points of both $M_+$ and $M_-$. Furthermore, $M_0$ has a conformal structure of signature $(2,3)$ with normal conformal tractor connection in agreement with the pullback of the ambient projective tractor connection. $M_+$ has canonically a positive Einstein metric $g_+$ of signature $(2,4)$, while $M_-$ has canonically a negative Einstein metric $g_-$ of signature $(3,3)$. The metrics $g_\pm$ are projectively compact of order 2.

Remark 5.4. In principle it is a little misleading at this point to describe $M_+$, $M_0$, $M_-$ as curved orbits in the spirit of [27], as at this stage we are only using the data of the holonomy reduction to $SO(H)$, rather than the full reduction to $G_2^{(*)}$. This choice of language will be justified, however, in Section 5.5.

5.2. The cross product and the $N(P)K$-structure on $M_\pm$. To obtain further details concerning the geometry of the $(M, p, \Phi)$-structure it is useful to first introduce the product structure alluded to above.

The tractor metric $H$, determined by $\Phi$ in Theorem 5.1, may be used to identify the tractor bundle $\mathcal{E}^A$ with its dual $\mathcal{E}^*_A$; since $H$ is parallel, raising and lowering of tractor indices with it commutes with covariant differentiation. Thus from a generic parallel tractor 3-form $\Phi$ we obtain $\Phi^A_{BC} \in \Gamma(\mathcal{E}^A_{BC})$, which may be interpreted as the parallel tractor cross product

$$\times : \mathcal{T} \times \mathcal{T} \to \mathcal{T}, \quad (U^B, V^C) \mapsto \Phi^A_{BC}U^BV^C.$$  

(55)

Conversely from the parallel cross product we may recover $\Phi$ and $H$ via Proposition 4.10 and (51) (or (11)). We will say the tractor cross product $\times$ is definite or split iff the corresponding tractor 3-form $\Phi$ is definite-generic or split-generic, respectively. In summary we have the following.

Proposition 5.5. A projective 6-manifold $(M, p)$ equipped with a parallel generic tractor 3-form $\Phi$ has a uniquely determined parallel cross product $\times$, as given by (55). Conversely given a parallel definite (split) generic cross product on the projective tractor bundle we obtain a unique parallel definite (respectively, split) generic tractor 3-form, and these constructions are inverses.

Thus the structure $(M, p, \Phi)$ may equivalently be viewed as a triple $(M, p, \times)$. More precisely, each fiber $T_x$ of the tractor bundle canonically carries the algebraic structure of the imaginary octonions $\mathbb{O}^{(*)}$, and furthermore we can meaningfully say that this imaginary octonion structure is parallel for the tractor connection.

This means that the algebraic results from Section 4 transfer effectively and uniformly into the tractor calculus. As a first application note that by the formula (11) we have a parallel tractor
7-form
\[ \epsilon_{ABCDEF} := \frac{1}{3!} \Phi^K_{[AB} \Phi^K_{CD} \Phi_{EFG]} . \]
We henceforth assume that \( \Phi \) is normalized so that this agrees with the canonical projective tractor volume form of Proposition 4.1.

Next from (6) of Proposition 2.4 we have the next result.

**Proposition 5.6.** The tractor cross product satisfies
\[ U \times (U \times V) = -H(U, U)V + H(U, V)U, \]
for all \( U, V \in \Gamma(\mathcal{T}) \).

The cross product determines a canonical map
\[ J : \mathcal{T} \to \mathcal{T}(1) \]
defined by \( V \mapsto -X \times V \),
where \( X \in \Gamma(\mathcal{E}^A(1)) \) is the canonical weighted tractor from (59) (cf. (5)). An easy calculation verifies that \( J \) is not parallel. In fact \( J \) is a section of the non-trivially weighted tractor bundle \( \text{End}(\mathcal{T})(1) \), whereas invariantly parallel tractor fields must have weight zero. Despite this weight, it is useful to view \( J \) as essentially an endomorphism of the tractor bundle for reasons at which the next Proposition hints.

**Proposition 5.7.** Let \((M, p, \Phi)\) be a 6-dimensional projective manifold equipped with a parallel tractor cross product \( \times \) (or equivalently, a generic parallel tractor 3-form \( \Phi \)). Then
\[ J^AXB = 0 \]
and
\[ J^ACJC_B = -\tau \delta^A_B + X^AX_B \quad \text{where} \quad \tau := H_{AB}X^AX_B. \]

**Proof.** This follows at once from the algebraic Proposition 2.4. Alternatively the results can be seen in terms the discussion immediately above as follows: As a bilinear tractor form \( \times \) is skew by construction, since \( \Phi \in \Gamma(\Lambda^3\mathcal{T}^*) \). Thus (58) is immediate from the definition (57) of \( J \). On the other hand (59) follows at once from Proposition 5.6.

The result (58) and the transformation rule (57) together imply that the weighted endomorphism component
\[ J^a_b := J^AXBZ^a_AW^B_B \in \Gamma(\text{End}(TM)(1)) \]
of \( J \) is invariant, that is, it is independent of the choice of scale. Together (59) and the identity \( \delta^A_B = \delta^a_b W^A_A Z^b_B + X^AX_B \) give that
\[ J^a_b J^b_c = -\tau \delta^a_c. \]

On any open set \( U \) on which \( \pm \tau \in \mathcal{E}(2) \) is positive, we can trivialize density bundles using \( \pm \tau \), and in particular we get a canonical unweighted endomorphism
\[ J_\pm := (\pm \tau)^{-1/2} J|_U \in \text{End}(TU); \]
formally, this trivialization has the effect of “setting \( \tau \) to \( \pm 1 \)”.

Substituting gives that the unweighted endomorphism \( J_\pm \) satisfies \( J^2_\pm = \mp \id(TM)|_U \).

Recall from Subsection 1.4 that the tractor metric \( H \) determined by \( \Phi \) via (54) also determines a metric \( g_\pm \) on \( U \). Now, \( \Phi(X, \cdot, \cdot) = -\Phi(\cdot, X, \cdot) = H(\cdot, J\cdot) \), so on \( U \) contracting both sides with \( W^A_A W^B_B \) and trivializing the involved density bundles gives that the top slot \( \omega \) (now regarded via the trivialization as an unweighted 2-form) of \( \Phi \) satisfies
\[ \omega|_U = g_\pm(\cdot, J_\pm \cdot). \]
In particular, \( J_\pm \) is \( g_\pm \)-skew, so \( (g_\pm, J_\pm) \) is a \((\tau1)\)-Hermitian structure (in particular it implies in the \( - \) case that the \((\pm1)\)-eigenspaces of \( J_- \) have the same dimension, that is, that \( J_- \) is a paracomplex structure).
Theorem 5.11. We have the following result.

tractor cross product for which the zero locus $M$ $(G_T \eta \nabla$ connection standard conformal tractor bundle $T$ of structure (split-generic tractor $3$-form which corresponds to a conformal structure $(g , J)$). Let $(M, p)$ be a $6$-dimensional projective manifold equipped with a parallel tractor cross product $\times$ (equivalently a generic, parallel tractor $3$-form $\Phi$). Then on any open set $U$ where $\tau$ is positive (respectively, negative), $\Phi$ defines a strictly nearly Kähler (resp. strictly nearly para-Kähler) structure $(g_\pm, J_\pm)$ in the projective class $p|_U$ on $U$, that is, for which $\nabla^g \in p|_U$. \[ \text{Proof.} \]

For simplicity of notation, we may as well replace $\partial$ with the given open set.

Recall from the discussion before the theorem that $\Phi$ determines an $(\mp)$-Hermitian structure $(g_\pm, J_\pm)$ on $M$. Decompose $\Phi$ with respecting to any splitting as in (11). Since $\nabla^T \Phi = 0$, (12) implies that $\mu = \nabla \omega$, and a fortiori that $\nabla \omega$ is totally skew, so $(g_\pm, J_\pm)$ is nearly $(\mp)$-Kähler.

In the scale determined by $\tau$, $\nabla \omega = \mu$ is nowhere zero, as it is easily seen that if it were zero at any point then $H$ as defined by (41) would be degenerate there, which would be a contradiction. Thus, $(g_\pm, J_\pm)$ is strictly nearly $(\mp)$-Kähler. \[ \square \]

Remark 5.9. The $\varepsilon$-Kähler structures in the Theorem are necessarily Einstein by dint of being strict and in dimension $6$, cf. Section 5.3. It is useful to see that, in our current context, this follows from Theorem 1.13 as Corollary 5.3.

Remark 5.10. We observed earlier that $J$ is not parallel. Indeed the algebraic relationship between $\times$ (which is parallel) and $X$ (which is far from parallel) varies across the manifold. When $X$ is split-generic, it is precisely this relationship which enables the single “tractor endomorphism” field $J$ to deliver a nearly Kähler structure on one part of the manifold, a nearly para-Kähler structure on another part, and yet a different structure on the separating hypersurface. We now turn our attention to the latter.

5.3. The zero locus $M_0$. In this subsection, we analyze the structure a parallel split cross product (split-generic tractor $3$-form $\Phi$) induces on and along the hypersurface zero locus $M_0$, which in this section we assume is nonempty. The geometry on $M_0$ itself is intrinsically interesting, but the common source of the induced geometries on $M_\pm$ and $M_0$—namely the holonomy reduction itself—establishes a close relationship between the $N(P)K$ structures on $M_\pm$ along $M_0$ and the geometry on $M_0$ itself. (Recall that if a parallel tractor cross product $\times$ is definite, the induced tractor metric is definite, and the zero locus is empty.) Indeed, Corollary 5.3 already shows that via projective geometry we may view the geometry on $M_0$ as a (simultaneous) limit structure at infinity of the geometries on $M_\pm$. Later, in Subsection 5.6 we exploit this relationship to formulate a suitable notion of compactification for $N(P)K$ Klein-Einstein metrics.

By that corollary, the $SO(3, 4)$ holonomy reduction to which the reduction to $G_2^\times$ is subordinate determines a normal parabolic geometry $(G_0 \to M_0, \eta_0)$ of type $(SO(3, 4), P_0)$ on $M_0$, which corresponds to a conformal structure $c$ of signature $(2, 3)$ there. We may identify the standard conformal tractor bundle $T_0 := G_0 \times_{P_0} V$ (where $V$ denotes the standard representation of $SO(3, 4)$) with the restriction $T|_{M_0}$ of the projective tractor bundle $T$. Then the normal connection $\nabla$ on $T$ restricts to a connection $\nabla_0$ on $T_0$, and the latter coincides with the normal tractor connection induced by $\eta_0$. In particular the $\nabla$-parallel cross product $\times$ that defines the $G_2^\times$ holonomy reduction of $\nabla$ restricts to give a parallel cross product on $M_0$, and so in summary we have the following result.

Theorem 5.11. Suppose a $6$-dimensional projective manifold $(M^6, p)$ admits a parallel split tractor cross product for which the zero locus $M_0$ is nonempty. Then, the conformal structure $(M_0, c)$ defined above canonically admits a $\nabla_0$-parallel split-generic cross product $\times : T_0 \times T_0 \to T_0$, and hence it defines a $G_2^\times$ holonomy reduction of $\nabla_0$.

\[ \text{3} \text{Of course, } p|_U \text{ is just the projective structure on } U \text{ consisting of the restrictions } \nabla|_U \text{ of connections } \nabla \in p. \]
The geometric meaning of a holonomy reduction to $G^*_Z$ of a normal conformal tractor connection has been analyzed previously: it corresponds to the existence of an underlying, canonically associated (oriented) $(2,3,5)$-distribution, see [53] [71]. One direction of this correspondence is as follows.

**Theorem 5.12.** [71] §5.3 **Any** $(2,3,5)$-distribution $D$ on a 5-manifold $M_0$ canonically induces a conformal structure $c_D$ on $M_0$. 

This is a relatively simple example of a so-called Fefferman construction and, as outlined in the proof sketch below, it can be framed efficiently in the language of parabolic geometry. To explain this, we describe briefly the realization of $(2,3,5)$-distributions in this context, cf. [29].

First, recall from Subsection 2.2 that the $G^*_Z$-action on $I^*$ preserves a signature-$(3,4)$ inner product $\cdot$ (and orientation) there and hence determines a canonical inclusion $G^*_Z \hookrightarrow SO(\cdot) \cong SO(3,4)$. An (oriented) $(2,3,5)$-distribution is precisely the structure underlying a parabolic geometry of type $(G^*_Z, Q)$, where $Q$ is the stabilizer in $G^*_Z$ of a null ray in $I^*$. By construction, $Q = P_0 \cap G^*_Z$, where $P_0$ is the stabilizer in $SO(3,4)$ of a null ray. As a parabolic subgroup, $Q$ determines a $\mathbb{Z}$-grading $(g_a)$ on the Lie algebra $g := g^*_Z$ for which

1. the grading respects the Lie bracket in that $[g_a, g_b] \subseteq g_{a+b}$ for all $a, b \in \mathbb{Z}$,
2. $g_a \neq \{0\}$ iff $|a| \leq k$ for some positive integer $k$ (for this particular parabolic subgroup, $k = 3$),
3. $g_{-1}$ generates (under the bracket operation) the subalgebra $g_{-k} \oplus \cdots \oplus g_{-1} < g$, and
4. the Lie algebra $q$ of $Q$ satisfies $q = g_0 \oplus \cdots \oplus g_k$.

The grading determines a natural filtration

$$g^0 := g_0 \oplus \cdots \oplus g_k$$

of $g$; by definition $q = g^0$, and by construction the adjoint action of $q$ on $g$ preserves this filtration (but not the underlying grading).

Now, given a normal, regular parabolic geometry $(G^*_Z \rightarrow M_0, \eta^G_z)$ of type $(G^*_Z, Q)$ the underlying $(2,3,5)$-distribution is just the associated bundle

$$D := G^*_Z \times_Q (g^{-1}/q) \subseteq G^*_Z \times_Q (g^*_Z/q) \cong TM_0,$$

and the derived 3-plane distribution can be recovered as

$$[D, D] = G^*_Z \times_Q (g^{-2}/q) \subseteq TM_0.$$

One can generalize these identifications to the general (that is, not necessarily orientable) case by instead starting with a parabolic geometry of type $(G_2 \times \mathbb{Z}_2, Q \times \mathbb{Z}_2)$.

Though Nurowski’s construction was not originally formulated in parabolic language, from the parabolic viewpoint it simply exploits the isomorphism $g^*_Z/q = g^*_Z/(g_2^*/p_0) \cong so(3,4)/p_0$ (as $q$-representations) at the level of associated bundles:

**Sketch of proof of Theorem 5.12.** An oriented $(2,3,5)$-distribution determines a unique parabolic geometry $(G^*_Z \rightarrow M_0, \eta^G)$ of type $(G^*_Z, Q)$ [32] [29] §4.3.2). Then, the bundle $g_0 := G^*_Z \times_Q P_0 \rightarrow M_0$, together with the form $\eta_0$ defined by extending $\eta^G_0$ $P_0$-equivariantly to all of $g_0$, comprise a parabolic geometry of type $(SO(3,4), P_0)$ (for which $\eta_0$ turns out to be normal, see [53] Proposition 4), and hence an oriented conformal structure on $M_0$.

4 Of course, the reader should not confuse the graded component $g_2$ with the compact, real Lie algebra for which we use the same symbol, or the asterisk $^*$ denoting the split real form with an indication of a vector space dual.

5 Here, normality and regularity are normalization conditions that together ensure a bijective correspondence between parabolic geometries of a given type, and underlying geometric structures of the corresponding type. Normality is a natural generalization of Cartan’s normalization condition for conformal Cartan connections, and regularity is a condition that ensures suitable compatibility between a Cartan connection and the natural filtration structure of the underlying geometry.
The construction is local, and reversing the orientation of the underlying distribution fixes the conformal structure (and simply reverses its orientation), so for a non-oriented distribution one can apply the construction to orientable sets that together cover \( M_0 \), disregard the orientation, and patch together the conformal structures. \( \square \)

Such (oriented) conformal structures are precisely characterized by the holonomy reduction we are investigating: The following restates \([53\text{, Theorem } A]\) in the language of tractor geometry and parallel split cross products, but see also \([71\text{, Theorem } 9]\).

**Theorem 5.13.** \([53\text{, Theorem } A]\) An oriented conformal structure \( c \) is induced by some (2, 3, 5)-distribution, that is, \( c = c_D \) for some distribution \( D \), iff \( \nabla_0 \) admits a holonomy reduction to \( G^*_2 \), that is, iff \( \mathcal{T}_0 \) admits a \( \nabla_0 \)-parallel split cross product \( \times \).

**Sketch of proof.** The implication \( \Rightarrow \) follows from the construction of \( c_D \) in the proof of Theorem 5.12. Conversely, given a holonomy reduction of the normal conformal tractor connection \( \nabla_0 \) and hence of the Cartan connection \( q_0 \) to \( G^*_2 \), the curved orbit decomposition formalism determines a Cartan geometry of type \( (G^*_2, Q) \) on \( M_0 \), which in turn determines a (2, 3, 5)-distribution on \( M_0 \) via \((61)\); by equivariance, this construction is the inverse of Nurowski’s construction. \( \square \)

The algebraic properties of the (split) cross product described in Subsection 2.2 let us efficiently characterize in native tractor language the (2, 3, 5)-distribution determined on a signature-(2, 3) conformal manifold \( (M_0, c) \) by a parallel tractor split cross product \( \times \). Per \([9] \), define

\[
J_0 : \mathcal{T}_0 \to \mathcal{T}_0[1] \quad \text{by} \quad J_0(V) := -X \times V.
\]

In particular, if \( M_0 \) is the zero locus determined by a split parallel tractor cross product \( \times \) on a projective 6-manifold \((M, p)\), which determines a map \( J : T \to T(1) \) via \((57)\), then \( J_0 = J|_{M_0} \).

Applying Proposition 2.5 to \( X \in \Gamma(\mathcal{T}_0[1]) \) yields a filtration

\[
\mathcal{E}_0 \cong \langle X \rangle \subset (\ker J_0)[1] \subset \text{im} J_0 \subset \ker X \cong TM_0 \cong \mathcal{E}_0,
\]

where here \( X \) is regarded as a map \( \mathcal{T}_0[1] \to \mathcal{E}_0[2] \). In particular, the \( TM_0 \) filtrand of any tractor of conformal weight 1 in \((\text{im} J_0)\) is invariant.

Using that \( X \) is null, the definition of \( J_0 \) and Proposition 2.4 give a conformal analogue of Proposition 5.7.

**Proposition 5.14.** Let \((M, c)\) be a 5-dimensional conformal manifold equipped with a (necessarily split) parallel tractor cross product, or equivalently, a generic parallel tractor 3-form. Then, \( J_0 \) satisfies

\[
(J_0)^A_B X^B = 0
\]

and

\[
(J_0)^A_C (J_0)^C_B = X^A X^B.
\]

With these identities we can produce conformal analogues of some of the objects constructed before Theorem 5.5. The transformation rule \((15)\), together with \((63)\), implies that the weighted endomorphism component

\[
(J_0)^{ab} := (J_0)^A_B Z^A_a Z^B_b : TM_0 \to TM_0[1]
\]

of \( J_0 \) is invariant, that is, independent of the choice of representative metric. Again if \( M_0 \) is the zero locus determined by a parallel tractor cross product on a projective 6-manifold \((M, p)\), then by construction \( J_0 := J|_{TM_0} \), and by construction it is the map induced by \( J_0 \) on the subquotient \( \ker X/X(1) \cong TM_0 \) of \( \mathcal{T}_0[1] \), or as the restriction of the map \( J|_{M_0} : TM|_{M_0} \to TM|_{M_0}[1] \) to \( TM_0 \) (by construction, the image of the restriction is contained in \( TM_0[1] \)). Together \((63)\) and the conformal identity \( \delta^A_B = X^A Y^B + \delta^a_b Z^A_a Z^B_b + Y^A X^B \) (or just \((60)\) with the above identity \( J_0 := J|_{TM_0} \)) imply that \( J_0^2 = 0 \).
Let \( \varpi \) denote the projection \( \ker X \to TM_0 \), which is just contraction with \( Z_A^a \), or equivalently, reduction modulo \( \langle X \rangle \). Applying \( \varpi \) to the filtration \( \mathfrak{g} \) yields a natural filtration of \( TM_0 \).

**Proposition 5.15.**

(a) \( \varpi(\ker \mathfrak{g}_0[1]) = (\mathrm{im} J_0)[-1] \)

(b) \( \varpi(\mathrm{im} \mathfrak{J}_0) = \ker J_0 \)

In particular, the filtration \( \mathfrak{g} \) determines a filtration

\[
(\mathrm{im} J_0)[-1] \subset \ker J_0 \subset TM_0
\]

Proof.

(a) By Proposition 4.11, \( \ker \mathfrak{g}_0[1] = J_0(\ker X) = \ker \mathfrak{g}_0[\ker X] \), and so by the characterization before the proposition,

\[
\varpi(\ker J_0[1]) = \varpi(\ker \mathfrak{g}_0[\ker X]) = \ker J_0[-1],
\]

which proves the claim. Since \( \dim \ker \mathfrak{g}_0 = 3 \) (by Proposition 2.6(d)) and \( \ker \varpi = \langle X \rangle \subset \ker J_0 \), \( \ker J_0 = 2 \).

(b) By (62), \( J_0(\mathrm{im} \mathfrak{g}_0) = \langle X \rangle \), so

\[
\{0\} = \varpi(J_0(\mathrm{im} \mathfrak{g}_0)) = \varpi(J_0(\ker X(\mathrm{im} \mathfrak{g}_0))) = J_0(\varpi(\mathrm{im} \mathfrak{g}_0)),
\]

and thus \( \varpi(\mathrm{im} \mathfrak{g}_0) \subseteq \ker J_0 \). Proposition 2.6(c) and the fact that \( \ker J_0 = 2 \) together give that both sides of the containment have rank 3, and hence equality holds.

\( \square \)

As one expects, this filtration coincides with the filtration \( D \subset [D, D] \subset TM_0 \) determined by \( \Phi_0 \) mentioned earlier in this subsection.

**Theorem 5.16.** Let \( (M_0, c) \) be an oriented, signature-\((2,3)\) conformal structure and \( \times \) a parallel split cross product on the standard conformal tractor bundle \( \mathcal{T}_0 \), and let \( D \) be the underlying \((2,3,5)\)-distribution described by Theorem 5.13. Then,

(a) \( D = (\mathrm{im} J_0)[-1] \), and

(b) \( [D, D] = \ker J_0 \).

Proof. Let \( \mathcal{V} \) denote the irreducible 7-dimensional representation of \( g^*_2 \), fix a null vector \( x \in \mathcal{V}[1] \), and take \( q \) to be the Lie subalgebra of \( g^*_2 \) that preserves the line \( \langle x \rangle \). Then, let \( x \cdot : g^*_2 \to \mathcal{V}[1] \) denote the map \( x \cdot \phi := \phi(x) \), where here we view \( \phi \in g^*_2 \) as an element of \( \mathrm{End}(\mathcal{V}[1]) \cong \mathrm{End}(\mathcal{V}) \). The weight is chosen so that \( x \cdot \) intertwines the \( q \)-actions on \( g^*_2 \) and \( \mathcal{V}[1] \). Checking the (representation-theoretic) weights of \( \mathcal{V} \) as a \( g^*_2 \)-representation shows that (1) the filtration \( \mathcal{V} \supset g^1 \mathcal{V} \supset g^1 (\mathcal{V}) \supset \cdots \of \mathcal{V} \) induced by \( q \) is exactly the one identified in Proposition 2.6, and in particular that \( g^1 (\ker J_0)_x = \langle x \rangle \), and (2) \( x \cdot \) satisfies

(i) \( x \cdot g^0 = \langle x \rangle \),
(ii) \( x \cdot g^{-1} = \ker(J_0)_x[1] \),
(iii) \( x \cdot g^{-2} = (\ker(J_0)_x)^{-1}[1] \), and
(iv) \( x \cdot g^{-3} = x \cdot g^2 = \langle x \rangle^{-1} \).

We prove (a) explicitly; the argument for (b) is entirely analogous. Since \( x \cdot \) is equivariant with respect to \( q = g^3 \), it is \( g^3 \)-equivariant, and hence by (ii) its restriction to \( g^{-1} \) induces a map

\[
g^{-1}/q \cong g^{-1}/(g^1 g^{-1}) \to (\ker J_0)_x[1]/(g^1 (\ker J_0)_x[1]) \cong \ker J_0)_x[1]/(x),
\]

where \( \cdot \) denotes the adjoint action; since the domain and codomain both have dimension 2, and because \( x \cdot \) maps \( g^{-1} \) onto \( \ker J_0)_x[1] \), this map is an isomorphism. Passing to associated
bundles identifies \( x \) with \( X \in \mathcal{T}_0[1] \), and using the characterization of the \((2, 3, 5)\)-distribution \( D \) earlier in the subsection gives that
\[
D = G^{G_2} \times Q (g^{-1}/q) = \ker(J_0)_\mathbb{A}[1]/\langle X \rangle = \varpi(\ker(J_0)_\mathbb{A}[1]),
\]
but by Proposition 5.13 this is exactly \( J_0[-1] \).

**Corollary 5.17.** A \((2, 3, 5)\)-distribution \( D \) satisfies \( D^\perp = [D, D] \) with respect to the conformal structure \( c_D \) it induces; in particular, \( D \) is totally null.

**Proof.** The condition is local, so by restriction we may assume \( D \) is oriented, which is a hypothesis of Theorem 5.16. The identity holds because the preimages of \( D \) and \([D, D]\) under \( \varpi \) are \( H_0 \)-orthogonal and \( \dim D^\perp = 3 = \dim[D, D] \): Pick \( U \in D \), \( V \in [D, D] \). By Theorem 5.16 there are weighted tractors \( S \in \ker(J_0)[1] \) such that \( U^a = Z_A^a S^A \) and \( T \in (\ker J_0)^-[1] \) such that \( V^b = Z_B^b T^B \); in particular, \( H_0(S,T) = 0 \), and so we have
\[
0 = (H_0)_{AB} S^A T^B = (X_A Y_B + g_{ab} Z_A^a Z_B^b + Y_A X_B) S^A T^B.
\]
Proposition 2.6 gives that \( S, T \in \langle X \rangle^+[1] \), and so distributing leaves just
\[
0 = g_{ab} (Z_A^a S^A)(Z_B^b T^B) = g_{ab} U^a V^b.
\]

So, on a projective 6-manifold \((M, p)\) (possibly with a suitable boundary, per Theorem 1.2) with a split parallel tractor cross product whose zero locus in nonempty, this characterization of the induced distribution realizes simply and concretely the \((2, 3, 5)\)-distribution on the zero locus \( M_0 \) in terms of the strictly \( \mathcal{N}(P)K \) structure on \( M_0 \). Regarded as a suitably projectively weighted endomorphism, the almost \( c \)-complex structure extends to, and degenerates along, the zero locus in a controlled way, and the distribution is precisely the image of \( J_0 = J|_{TM_0} \), regarded (via tensoring with \( E[-1] \)) as a subset of \( TM_0 \). The theorem establishes which subspaces of \( \Gamma^* \) corresponds to which tangent distributions, so one can extract further identifications of these just by proving the corresponding algebraic statements: for example, consulting Proposition 2.6 passing to the quotient, and using the above identifications gives

**Proposition 5.18.** Let \((M^6, p)\) be a projective 6-manifold with a parallel split tractor cross product for which the zero locus \( M_0 \) is nonempty, and \( J \) the corresponding weighted endomorphism field. Then, the the underlying \((2, 3, 5)\)-distribution \( D \) on \( M_0 \) described by Theorem 5.17 satisfies

\[
\begin{align*}
(a) \quad & D = \ker J|_{M_0} \text{ and } \\
(b) \quad & [D, D] = \im J|_{M_0}.
\end{align*}
\]

Using the above results, we can just as well realize the 2-plane distribution in terms of the projecting part \( \omega \) of the parallel 3-form \( \Phi \) corresponding to a parallel split tractor cross product, and hence in terms of the Kähler forms of the \( \mathcal{N}(P)K \) structures on \( M_0 \). A conformal characterization of \( D \) closely related to the following is given in [19] §4.5.

**Proposition 5.19.** Let \((M^6, p)\) be a projective 6-manifold with a parallel split tractor cross product for which the zero locus \( M_0 \) is nonempty, let \( \Phi \) be the corresponding parallel projective tractor 3-form, and denote its projecting part by \( \omega \). Then, the the underlying \((2, 3, 5)\)-distribution \( D \) on \( M_0 \) described by Theorem 5.17 satisfies

\[
\begin{align*}
(a) \quad & D = \ker \omega|_{M_0} := \{ U \in TM|_{M_0} : U \iota \omega = 0 \} \text{ and } \\
(b) \quad & [D, D] = \ker \iota^* \omega,
\end{align*}
\]

where \( \iota \) is the natural inclusion \( M_0 \hookrightarrow M \). In particular, \( \omega_0 \) is nonzero and locally decomposable, and the 2-plane distribution spanned by the decomposable (weighted) bivector field \((\omega_0)^{ab} \) is exactly \( D \).

**Remark 5.20.** By construction, \( \iota^* \omega \) coincides with the projecting part
\[
X^A Z^B Z^C_{\Phi_0} (\Phi_0)_{ABC} \in \Gamma(\Lambda^2 T^* M_0[3])
\]
of the parallel conformal tractor 3-form \( \Phi_0 := \Phi|_{M_0} \).
5.4. The model. Given a dimension-\((n+1)\) real vector space \(V\) equipped with a volume form, its ray projectivization \(S^n = \mathbb{P}^+(V)\) provides the standard projectively flat model for \(n\)-dimensional projective differential geometry. Topologically a sphere, this is a homogeneous manifold for \(\text{SL}(V) \cong \text{SL}(n+1, \mathbb{R})\) and so may be identified with \(\text{SL}(V)/P\) where \(P\) is the parabolic subgroup of \(\text{SL}(n+1, \mathbb{R})\) that stabilizes a nominated ray in \(V\).

In view of the canonical fibration \(\pi: V \setminus \{0\} \to S^n\), we may regard \(V_* := V \setminus \{0\}\) as a cone manifold over \(S^n\). The tangent bundle to this \(TV_*\) is trivial and has a global parallelization by a canonical flat torsion free affine connection \(\nabla\), namely the parallel transport arising from the affine structure of the vector space \(V\). Alternatively we may view \(V_*\) as the total space of an \(\mathbb{R}_+\)-principal bundle over \(S^n\) with the \(\mathbb{R}_+\) action given by simply scaling vectors in \(V_*: \lambda v := \lambda v\), \(\lambda \in \mathbb{R}_+\). It follows that \(V_* \to S^n\) has canonically a vertical vector field \(X\) that infinitesimally generates this action on the fibres; this is usually called the Euler vector field.

We can define an equivalence relation on vectors in \(TV_*\) by declaring \(v_x \sim v_y\) iff \(v_x\) and \(v_y\) are parallel, with also \(x\) and \(y\) being points of the same fibre. It is straightforward to check that \(T := TV_*/\sim\) is a rank-\((n+1)\) vector bundle on \(S^n\), with a connection \(\nabla\) induced from \(\nabla\). In fact it is easily verified that \(T\) is the associated vector bundle \(\text{SL}(V) \times_P V\), and that \(\nabla\) is the flat connection arising from the fact that \(T\) is canonically trivialized by the mapping \(\text{SL}(V) \times_P V \to \text{SL}(V)/P \times V\) given by \((g, v) \mapsto (gP, g v)\), where \(g v\) indicates the standard action of \(g \in \text{SL}(V)\) on \(v \in V\). Thus \((T, \nabla)\) is seen to be, by definition, the standard projective tractor bundle and connection.

It follows easily now that the pullback, via \(\pi\), of each parallel tractor field on \(S^n\) is a parallel tensor field on \(V_*\), and all parallel tensors on \(V_*\) may identified with such a pullback. More generally each tractor field \(T\) of a given weight \(w\) corresponds to a tensor field \(\tilde{T}\) on \(V_*\) that is homogeneous of weight \(w\), meaning that \(\tilde{\nabla}_X \tilde{T} = w\tilde{T}\). In particular the canonical tractor \(X\) of weight 1, defined in (33), corresponds to \(\tilde{X}\) on \(V_*\).

In Theorem 1.3 and more generally in this section, we have considered a projective 6-manifold equipped with a parallel cross product, equivalently a parallel tractor 3-form \(\Phi\). From our discussion here it follows at once that the model for this is to take \(V\), as above but of dimension 7 and equipped with a cross product as defined in Section 2.2, we denote this structure by \(\mathbb{P}^+(\mathfrak{L}^{(1)})\).

Now the general results from above apply to this setting, with some seen to hold by more direct reasoning using the above. Theorem 5.1 is in the latter category, as the inner product on \(H\) on \(\mathfrak{L}^{(1)}\) determines a metric constant on \(V\) and hence a parallel projective tractor metric \(H\). Thus we have Corollary 5.3 giving a decomposition of the sphere \(S^n = \mathbb{P}^+(\mathfrak{L}^{(1)})\) with Einstein metrics on the open orbits. Recall that these open orbits are where the projective density \(\tau = H(X, X)\) is positive (and also, respectively, negative in the split case). This density is equivalent to the homogeneous degree function \(\bar{\tau} = H(\bar{X}, \bar{X})\) on \(V_*\) and it is an elementary exercise to show that, where \(\bar{\tau}\) is non-vanishing, working in the scale \(\bar{\tau}\) as on \(S^n\) (as in e.g. the proof of Theorem 5.8) corresponds to working on the level sets where \(\bar{\tau} = 1\) and \(\bar{\tau} = -1\) [26]. In particular on the model \(\mathbb{P}^+(\mathfrak{L}^{(1)})\) the Einstein structures on \(M\) in the definite case, and on \(M_{\pm}\) in the split case, are just the induced metrics on the applicable level sets \(\bar{\tau} = \pm 1\). Further details for these facts here which follow from the inner product \(H\) on \(V\) and the corresponding projective holonomy reduction can be found in [26] [28].

All of the theorems and results earlier in this Section now specialize to the model. So we recover the result that in the definite case \(\mathbb{P}^+(\mathfrak{L}^{(1)})\) has a positive definite strictly nearly Kähler structure. In the split case we see the parts \(\mathbb{P}^+(\mathfrak{L}^{(1)})_{\pm}\) given by Corollary 5.3 topologically, \(\mathbb{P}^+(\mathfrak{L}^{(1)} )_+ = S^{2,4}\) and \(\mathbb{P}^+(\mathfrak{L}^{(1)})_- = S^{3,3}\) respectively have a signature \((2, 4)\) nearly Kähler structure and a signature \((3, 3)\) nearly para-Kähler structure. Most importantly, this realizes the the conformal structure and \((2, 3, 5)\)-distribution \(\Delta\) on \(\mathbb{P}^+(\mathfrak{L}^{(1)})_0 = S^2 \times S^3\) as a simultaneous (projective) limit of these
structures, as a special case of the treatment in Section 5.3 (\(\mathbb{P}_+(\mathbb{I}^*)_0, \Delta\)) is the flat model of the geometry of (2,3,5)-distributions.

Now the decomposition of \(\mathbb{P}_+(\mathbb{I}^*)\) given by Corollary 5.3 is exactly the orbit decomposition \(\mathbb{P}_+(V)\) under the action of \(\text{SO}(H)\) [26, 27]. An important question at this stage is whether this agrees with the orbit decomposition of \(\mathbb{P}_+(V)\) under the action of \(G_2^*\). In fact it does.

**Proposition 5.21.** \(G_2\) acts transitively on \(\mathbb{P}_+(\mathbb{I})\), while \(G_2^*\) acts transitively on \(\mathbb{P}_+(\mathbb{I}^*)_+, \text{ on } \mathbb{P}_+(\mathbb{I}^*)_-\), and on \(\mathbb{P}_+(\mathbb{I}^*)_0\).

**Proof.** This uses the classification in [83, §3.4, p.42]. It is also possible to verify it using Bryant’s argument in [15] for the \(G_2\) case, and it is not hard to extend this to treat the \(G_2^*\) variant. See also [24] for a discussion of \(G_2\) acting on \(S^2 \times S^3\).

It follows that Theorem 1.3 holds for the models \(\mathbb{P}_+(\mathbb{I}^*)\).

5.5. Theorem 1.3. We are now ready to bring together essential parts of the developments above to prove this summary Theorem from the introduction.

**Proof of Theorem 1.3.**

- \(\Phi\) is definite-generic: The result is contained in Corollary 5.3 and Theorem 5.8. However, Corollary 5.3 only gives the stratification into curved orbits determined by the holonomy reduction to \(\text{SO}(H)\), that is, to the parallel tractor metric. Thus to see that the stated results are sufficient must check that a parallel definite generic tractor 3-form has only 1 curved orbit type. The projective 6-sphere \(S^6 = \mathbb{P}_+(\mathbb{R}^7)\) is a homogeneous space for \(\text{SL}(7,\mathbb{R})\). But the subgroup \(G_2 \subset \text{SL}(7,\mathbb{R})\) also acts transitively on \(S^6\) and so there is only one orbit type in the model structure. It now follows from [27] Theorem 2.6 that there is only a single curved orbit type for the case of Cartan holonomy reduction to \(G_2\).

- \(\Phi\) is split-generic: In this case all is contained in Corollary 5.3 Theorem 5.8 Theorem 5.11 and Theorem 5.13 except that, once again, it remains to check that \(\Phi\) does not induce a finer curved orbit decomposition than that given by \(H\) in Corollary 5.3. Again considering \(S^6 = \mathbb{P}_+(\mathbb{R}^7)\), but now under the action of \(G_2^* \subset \text{SL}(7,\mathbb{R})\) we see that we obtain only 3 orbit types and these are indeed given by the strictly sign of \(H_{AB}X^AX^B\). So \(M = M_- \cup M_0 \cup M_+\) is the curved \(G_2^*\)-orbit decomposition.

5.6. Compactification of \(N(P)K\)-geometries. One may ask whether there are effective ways to find and treat compactifications of complete nearly Kähler, or nearly para-Kähler, manifolds. In view of the models for these structures (see Section 5.3), and also Theorem 4.2 it is natural to approach this via projective geometry.

Taking this view point, it is then interesting to investigate the possibilities for the topology and geometry of the set of boundary points. Theorem 1.2 is concerned with this question in the nearly Kähler case, and there we find that (under the assumptions there) one is essentially forced back into the setting of Theorem 1.3. Here we prove this after first extending the theorem to include the nearly para-Kähler case.

**Theorem 5.22.** Let \(M\) be an open dense submanifold in a compact connected projective 6-manifold \((N, p)\), possibly with boundary, so that one of the following two possibilities hold: either \(N\) is a manifold with boundary \(\partial N\) and \(N \setminus M = \partial N\), or \(N\) is closed and \(N \setminus M\) is contained in a smoothly embedded submanifold of \(N\) of codimension at least 2. Suppose further that \(M\) is equipped with a complete nearly Kähler structure \((g, J)\) or a nearly para-Kähler structure \((g, J)\) that satisfies \((\nabla J, \nabla J) \neq 0\) everywhere) such that the projective class \([\nabla^9]\) of the Levi-Civita connection \(\nabla^9\) coincides with \(p|_M\). Then, either

- \(M\) is closed and \(M = N\); or
• $g$ has signature $(2,4)$ (resp. signature $(3,3)$) and we are in the first setting with $N \setminus M$ the smooth (5-dimensional) boundary for $N$. Furthermore the metric $g$ is projectively compact of order 2, and the boundary has a canonical conformal structure equivalent to an (oriented) Cartan $(2,3,5)$-distribution.

Proof. From Theorem 4.2 $M$ has canonically a parallel generic tractor 3-form $\Phi \in \Lambda^3 T^*$. This determines a parallel tractor metric $H$ on $M$ via the formula (51). On $M$ the density field $\tau := H_{AB} X^A X^B$ is nowhere zero and $H$, $\tau$ and the Einstein metric are related as in the expression (51).

Now $p|_M$ is the restriction of a smooth projective structure $p$ on $N$. Thus the projective tractor connection on $M$ is the restriction of the smooth tractor connection on $N$. Working locally it is straightforward to use parallel transport along a congruence of curves to give a smooth extension of $\Phi$ to a sufficiently small open neighborhood of any point in $N \setminus M$, and since $M$ is dense in $N$ the extension is parallel and unique. (See [37] for this extension result in general for the case where $N \setminus M$ lies in a submanifold of dimension at least 2.) It follows that $\Phi$ extends as a parallel field to all of $N$.

Thus the tractor metric $H$ extends parallelly to $N$. It follows that $\tau := H_{AB} X^A X^B$ also extends smoothly to all of $N$. Now it must be that $\tau(x) = 0$ for all points $x \in N \setminus M$. Otherwise if $\tau(x)$ were nonzero at such a point $x$ then it would be nonzero in an open connected neighborhood of such a point $x$ and we could easily conclude (via Theorem 1.10) that the metric $g$ and its Levi-Civita connection extend to this neighborhood that includes points of $N \setminus M$ and also points of $M$. But this contradicts the assumption that $(M, g)$ is complete.

We have that $N \setminus M = \mathcal{Z}(\tau)$, where $\mathcal{Z}(\tau)$ denotes the zero locus of $\tau$. It follows that if $g$ has Riemannian signature then $N \setminus M = \emptyset$, as in that case $H_{AB}$ is positive definite (see e.g. (51)) while $X$ is nowhere zero. Let us now assume $N \setminus M \neq \emptyset$ which implies that we are in a case of signature $(2,4)$ (or $(3,3)$ in the nearly para-Kähler case), without loss of generality. Using the nondegeneracy of $H$ it is straightforward to show that, for any connection $\nabla \in p$, $\nabla \tau(x) \neq 0$ at all points $x$ in the zero locus $\mathcal{Z}(\tau)$ of $\tau$; see the proof of Theorem 12 in [24]. Thus $\mathcal{Z}(\tau)$ is a smooth codimension-1 submanifold of $N$ and it follows that we are in the first case of the Theorem with $\mathcal{Z}(\tau) = \partial N$. By [24] Theorem 12] it follows that $g$ is projectively compact of order 2, that the conformal tractor bundle on $\partial N$ may be identified with the restriction to $\partial N$ of the projective tractor bundle, and that the conformal tractor connection is the pullback of the projective tractor connection. Thus $\Phi|_{\partial N}$ is a parallel tractor for the conformal structure on $\partial N$, and in particular its holonomy is contained in $G_2$. So, the final conclusion follows from Theorem 5.13. 

Remark 5.23. The reader will notice that it is the compatibility of the Einstein metric with the projective structure that plays the main role in the proof above. In particular, although it digresses from the main aims of the article, it is worthwhile to observe that a trivial adaption of the proof establishes a corresponding result concerning the possible projective compactifications of Einstein metric:

Proposition 5.24. Let $M$ be an open dense submanifold in a compact connected projective $n$-manifold $(N, p)$, possibly with boundary, so that one of the following two possibilities hold: either $N$ is a manifold with boundary $\partial N$ and $N \setminus M = \partial N$, or $N$ is closed and $N \setminus M$ is contained in a smoothly embedded submanifold of $N$ of codimension at least 2. Suppose further that $M$ is equipped with a complete, signature-$(p,q)$ Einstein metric with nonzero scalar curvature of sign $\mp$ such that the projective class $[\nabla^0]$ of the Levi-Civita connection $\nabla^0$ coincides with $p|_M$. Then, either

• $M$ is closed and $M = N$; or
• $N \setminus M$ is the smooth boundary for $N$. Furthermore the metric $g$ is projectively compact of order 2, and the boundary is canonically equipped with a conformal structure of signature $(p - 1, q)$, resp. $(p, q - 1)$. 
6. The geometric Dirichlet problem

Theorem 6.1 shows that a parallel projective tractor cross product $\times$ determines a stratification $M = M_+ \cup M_0 \cup M_-$ into curved orbits, and each of these canonically inherits an exceptional geometric structure: The open curved orbits $M_\pm$ respectively inherit strictly nearly $(\mp 1)$-Kähler structures, and the hypersurface $M_0$ that separates them inherits an oriented $(2,3,5)$-distribution.

This raises the natural questions of

- which oriented $(2,3,5)$-distributions $(\Sigma, D)$ arise this way, that is, for which $D$ can one produce a projective structure $p$ on a collar $M$ of $\Sigma$ and a parallel projective split tractor cross product $\times$ on $(M, p)$ for which $(\Sigma, D)$ is the induced geometry on the zero locus it determines, and

- for any $D$ that admits such a structure $(M, p, \times)$, to what degree is it unique?

It turns out that one can produce such a collar essentially uniquely for any $D$, at least formally and hence also in the real-analytic category (cf. [47, Theorem 1.1], which gives an analogous result in the language of Fefferman-Graham ambient metrics):

**Theorem 6.1.** Let $(\Sigma, D)$ be an oriented, real-analytic $(2,3,5)$-distribution on a connected manifold. Then, there is a projective manifold $(M, p)$ and a parallel projective split tractor cross product $\times$ for which $\Sigma$ is the zero locus $M_0$ in the stratification of $M$ that $\times$ determines, and $D$ is the $(2,3,5)$-distribution induced there. Moreover, $(\Sigma, D)$ determines the triple $(M, p, \times)$ uniquely up to an overall nonzero constant scale of $\times$, and up to pullback by diffeomorphisms fixing $\Sigma$ pointwise. Thus $(\Sigma, D)$ determines the induced $N(P)K$ structures $(M_\pm, g_\pm, J_\pm)$ uniquely up to homothety of $g_\pm$ and up to pullback by diffeomorphism.

This immediately implies natural bijective correspondences between the moduli spaces of all of the involved structures, at least in the real-analytic setting; this in particular enables holographic investigation for general $(2,3,5)$-geometry. To formulate the bijections appropriately, we give suitable notions of equivalence for the involved structures:

**Definition 6.2.** Suppose $(N_a, g_a, J_a)$, $a = 1, 2$, are 6-dimensional N(P)K Klein-Einstein manifolds with respective projective infinities $\partial N_a$, and denote. We say that $(N_1, g_1, J_1)$ and $(N_2, g_2, J_2)$ are equivalent near infinity iff for $a = 1, 2$ there are open sets $A_a \subset N_a \cup \partial N_a$ such that $A_a \supset \partial N_a$ and a diffeomorphism $\phi : A_1 \to A_2$ such that

(a) $(\phi|_{A_1 \cap N_1})^* (g_2|_{A_2 \cap N_2}) = g_1|_{A_1 \cap N_1}$, and
(b) $T\phi|_{A_1 \cap N_1} \circ J_1|_{A_1 \cap N_1} = J_2|_{A_2 \cap N_2} \circ T\phi|_{A_1 \cap N_1}$.

Similarly, if $(M_b, p_b)$, $i = 1, 2$ are 6-dimensional projective structures (with respective standard tractor bundles $T_b$) endowed respectively with parallel split tractor cross products $\times_b$ for which the hypersurface curved orbits $(M_b)_0$ are both nonempty, we say that $(M_1, p_1, \times_1)$ and $(M_2, p_2, \times_2)$ are equivalent along the zero locus iff for $b = 1, 2$ there are open sets $B_b \subset M_b$ such that $B_b \supset (M_b)_0$ and a diffeomorphism $\psi : B_1 \to B_2$ such that

(a) $\psi^* (p_2|_{B_2}) = p_1|_{B_1}$, and
(b) $\Psi \cdot (U \times_1 V) = (\Psi \cdot U) \times_2 (\Psi \cdot V)$ for all $x \in M$ and all vectors $U, V$ in the fiber $(T_1)_x$, where $\Psi : T_1 \to T_2$ is the bundle isomorphism induced by $\psi$.

**Corollary 6.3.** There are bijective correspondences:

\[
\begin{align*}
&\text{real-analytic projective parallel tractor cross products } (M, p, \times) \quad \text{with } M_0 \neq \emptyset \text{ modulo equivalence along the curved hypersurface orbit } M_0 \\
\leftrightarrow &\quad \text{real-analytic, strictly NK Klein-Einstein structures } (M_+, J_+, g_+) \quad \text{modulo equivalence near infinity} \\
\leftrightarrow &\quad \text{real-analytic, strictly NPK Klein-Einstein structures } (M_-, J_-, g_-) \quad \text{modulo equivalence near infinity}.
\end{align*}
\]
Furthermore, any structure of a type in the above correspondence determines a unique oriented, real-analytic (2,3,5)-distribution. Conversely, any such (connected) distribution determines a real-analytic projective parallel tractor cross product \((M, p, \times)\) modulo equivalence along the zero locus and a positive constant rescaling of \(\times\), and hence real-analytic, strictly \(N(P)K\) Klein-Einstein structure \((M_\pm, J_\pm, g_\pm)\) modulo equivalence near infinity and homothety.

With a view toward proving Theorem 6.1, we first recall from Subsection 5.1 that a reduction of holonomy to \(G_2^+\) determines a unique reduction of holonomy of \(\nabla^T\) to \(SO(3,4)\); this reduction is realized explicitly by \([51]\) and \([50]\), which respectively give the nondegenerate symmetric bilinear form \(H\) and the compatible tractor volume form in terms of a generic tractor cross product \(\times\); recall that per the comment after \([50]\), we assume that the volume form coincides with the canonical tractor volume \(\epsilon\). Furthermore, recall from Theorem 4.6 that the holonomy reduction afforded by \(H\) and \(\epsilon\) determines a decomposition of the underlying manifold into three orbits and canonical geometric structures on each: Two open orbits with (oriented) Klein-Einstein metrics, one of signature \((2,4)\) and the other signature \((3,3)\), and a hypersurface curved orbit with an (oriented) conformal structure of signature \((2,3)\). So, any solution to the Dirichlet problem corresponding to a \(G_2^+\) holonomy reduction must also be a solution to Dirichlet problem corresponding to the weaker holonomy reduction to \(SO(3,4)\), which suggests that to understand the former it would be helpful to investigate the latter. More explicitly, given an oriented conformal structure \((\Sigma, c)\) of signature \((2,3)\), we want to understand the existence and uniqueness of a triple \((M, p, H)\) comprising a projective structure \(p\) on a 6-dimensional collar \(M \supset \Sigma\) and a parallel projective tractor metric \(H\) (of signature \((3,4)\)) for which the zero locus and the induced conformal structure are exactly \((M, c)\).

This latter Dirichlet problem, however, is a special case of the problem addressed by the Fefferman-Graham ambient metric construction \([41]\), although it is typically formulated in pseudo-Riemannian terms rather than the projective tractor framework used here. We thus proceed as follows: First, we describe the Fefferman-Graham ambient construction and the existence and uniqueness result we need in the original language of that construction, in which the output is a pseudo-Riemannian manifold \((\tilde{\Sigma}, H)\). Next, we introduce the projective Thomas cone, which lets us identify (1) part of this output with the standard projective tractor bundle, (2) the Levi-Civita connection \(\nabla^H\) of \(H\) with the data of the normal projective tractor connection, and hence (3) \(H\) with a parallel fiber metric on the tractor bundle, which in particular gives the desired holonomy reduction to \(SO(\cdot)\); so, by construction, this identification solves the conformal Dirichlet problem in projective language. Finally, we use the equivalence of these formulations to translate the Dirichlet problem corresponding to reduction of holonomy to \(G_2^+\) into ambient language and solve it in that setting.

6.1. The Fefferman-Graham ambient construction. Given a conformal structure \((\Sigma, c)\) of signature \((r, s)\), \(r + s \geq 2\), the Fefferman-Graham ambient metric construction aims to produce a metric canonically determined, to the extent possible, by \(c\). Consider the metric bundle \(\pi : \Sigma \to \Sigma\) whose fiber over \(x \in \Sigma\) comprises all of the inner products on \(T_x\Sigma\) in \(c_x\), that is,

\[\Sigma_x := \{g_x : g \in c\} .\]

It admits a tautological, degenerate bilinear form \(h_0 \in \Gamma(S^2T^*\Sigma)\), namely,

\[(h_0)_x(U, V) := g_x(T\pi \cdot U, T\pi \cdot V) ,\]

which is, by construction, homogeneous of degree 2 with respect to the natural dilations \(\delta_s : \Sigma \to \Sigma\) s > 0, \(\delta_s(g_x) := s^2 g_x\), and which together in turn realize \(\Sigma\) as an \(\mathbb{R}_+\)-principal bundle. Though \(h_0\) is degenerate (it annihilates \(ker T\pi\)), it is natural to look for metrics on a collar \(\Sigma\) of \(\Sigma\) which pull back to \(h_0\) and then attempt to formulate suitable admissibility criteria for such metrics that guarantee uniqueness and existence. Identify \(\Sigma\) with \(\Sigma \times \{0\} \subset \Sigma \times \mathbb{R}\), and denote the inclusion by \(\iota\). We call a metric \(H\) on a dilation-invariant open neighborhood \(\Sigma\) of \(\Sigma\) in \(\Sigma \times \mathbb{R}\) a pre-ambient metric for \((\Sigma, c)\) if \(\iota^*H = h_0\) and if it is homogeneous of degree 2 under the
dilations \((z, \rho) \mapsto (\delta_z(z), \rho)\), which we also denote \(\delta_z\); necessarily \(H\) has signature \((r + 1, s + 1)\).

If \(\dim \Sigma\) has odd dimension \(n := r + s \geq 3\), we say that a \emph{pre-ambient metric} for \((\Sigma, c)\) is an \emph{ambient metric} for \((\Sigma, c)\) if it (a) is Ricci-flat, and (b) satisfies the identity \(\nabla^H X = \text{id}_{T \Sigma}\),

where \(X := \delta_0|_0\delta_s\) is the infinitesimal generator of the group of dilations \(\delta_0\). For concreteness of exposition, we state this result just for real-analytic, odd-dimensional conformal structures, the case we need.

\textbf{Theorem 6.4.} \cite{[31]} \cite[Theorem 2.3]{[26]} Suppose \(c\) is a real-analytic conformal structure of signature \((r, s)\) and odd dimension \(n = r + s \geq 3\). Then, there is a real-analytic ambient metric \((\Sigma, H)\) (necessarily of signature \((r + 1, s + 1)\)), and it is unique up to pullback by a diffeomorphism fixing \(\Sigma\) pointwise.

\textbf{6.2. The Thomas cone.} We now describe the Thomas cone construction \cite{[31], [26], [30], [41], [42]}, which lets us translate the guarantee of existence and uniqueness into the projective language in which our (conformal) Dirichlet problem is formulated. Given an \((\Re, c)\), which lets us translate the guarantee of existence and uniqueness into the projective language in which our (conformal) Dirichlet problem is formulated. Given an \(n\)-dimensional projective structure \((M, p)\), \(n \geq 3\), with associated normal Cartan geometry, say, \((\mathcal{G} \to M, \eta)\), of type \((\mathfrak{sl}(n + 1, \Re), P)\), let \(V\) denote the standard representation of \(\text{SL}(n + 1, \Re)\) so that \(T := \mathcal{G} \times_P V\) is the standard tractor bundle and \(\nabla^V\) the normal connection it induces there. Pick a nonzero vector \(e_0 \in V\) in the ray stabilized by \(P\), and define \(P_0\) to be the (closed) stabilizer of \(e_0\) in \(P\). Since \(\eta\) is \(P\)-equivariant, it is \(P_0\)-equivariant, and we may define the \emph{Thomas cone} \(\hat{M}\) to be the (fiberwise) quotient \(\mathcal{G}/P_0\) (called such because \(\hat{\pi} : \hat{M} \to M\) is a principal bundle with fiber \(P/P_0 \cong \Re_+\)) and regard \((\hat{G} \to \hat{M}, \hat{\eta})\) as a Cartan geometry of type \((\mathfrak{sl}(n + 1, \Re), P_0)\).

Now, since \(\mathfrak{sl}(n + 1, \Re)/P_0\) is isomorphic (as a \(P_0\)-module) to \(V\) itself, the Cartan connection \(\eta\) canonically induces a vector bundle connection \(\nabla\) on \(TM = \mathcal{G} \times_{P_0} V\), and the normalization condition (or just as well the formula \cite{[30]} for the tractor connection) ensures that this connection is Ricci-flat and torsion-free. Again because \(P\)-equivariance implies \(P_0\)-equivariance, sections of any associated bundle \(\mathcal{G} \times_P W \to M\) correspond to \(P\)-equivariant sections of \(\mathcal{G} \times_{P_0} W \to \hat{M}\).

So, if \(W\) is a restriction of an \(\text{SL}(n + 1, \Re)\)-representation, and hence of a subrepresentation of \(V^k \otimes (V^l)^*\) for some \(k\) and \(l\), sections of the corresponding tractor bundle \(\mathcal{G} \times_{P_0} W\) correspond to tensor fields on \(\hat{M}\) that (checking shows, using that \(\hat{\nabla}\) is torsion-free) are parallel along the fibers of \(\hat{\pi}\). By construction, the sections of \(\mathcal{E}(k)\), \(k \in \Re\), correspond under this identification to functions on \(\hat{M}\) homogeneous of degree \(k\) with respect to the \(\Re_+\)-action. A section of \(T\) corresponds to a section of \(TM\) homogeneous of degree \(-1\) with respect to the \(\Re_+\)-action, and the canonical section \(X \in \Gamma(T(1))\) corresponds itself to the infinitesimal generator of that action. Taking \(W\) to be the restriction of the \(\text{SL}(n + 1, \Re)\)-representation \(S^2V^*\), we see that a metric \(H \in \Gamma(S^2T^\ast)\) corresponds to a metric on \(\hat{M}\) homogeneous of degree 2.

We formalize the common features of the projective Thomas cone and the ambient metric construction to show that, in the presence of a holonomy reduction of the normal projective tractor connection to \(\text{SO}(r + 1, s + 1)\) that determines a stratification whose zero locus is nonempty, the constructions essentially coincide.

\textbf{Theorem 6.5.}

(a) Suppose \((M, p)\) is a projective structure of dimension \(r + s + 1 \geq 4\) whose standard tractor bundle admits a parallel fiber metric \(H\) of signature \((r + 1, s + 1)\) (equivalently, a reduction of holonomy to \(\text{SO}(H)\)) with a nonempty zero locus \(M_0\), and let \(c\) denote the conformal structure on \(M_0\). If we replace \(M\) with any open collar of \(M_0\) in \(M\) and regard \(H\) as a

\footnote{Condition (b) is called \emph{straightness}; it is convenient to include it here in the definition of an ambient metric, though this is not done in \cite{[3]}.}

\footnote{This formulation avoids two separate issues: (1) The ambient metric is a formal (power series) construction, so in the odd-dimensional case a pre-ambient metric is sometimes elsewhere (including in \cite{[31]}) called \emph{ambient} if its Ricci curvature vanishes to infinite order in \(p\) along \(G\), instead of requiring that it vanish identically on \(\Sigma\). (2) The even-dimensional case is more subtle. For more about the consequences of the latter issue in the projective setting, see Appendix \cite{[19]}.}
homogeneous parallel metric on \( \hat{M} \), then \( (\hat{M}, H) \) is an ambient metric for \((M_0, c)\), and this identifies the data of the normal projective tractor connection with the Levi-Civita connection \( \nabla^H \) of \( H \).

(b) Conversely, suppose \((\hat{\Sigma}, H)\) is an ambient metric for a conformal structure \((\Sigma, c)\) of signature \((r, s), r + s \geq 3\). Then, there is a canonically determined projective structure \( p \) on the pointwise quotient \( M := \hat{\Sigma}/\mathbb{R}_+ \) (where \( \mathbb{R}_+ \) is the dilation orbit) so that \( \hat{\Sigma} \) is the Thomas cone \( \hat{M} \), the normal connection \( \nabla \) that \((M, p)\) determines there coincides with the Levi-Civita connection \( \nabla^H \) of \( H \), and so the metric \( H \) regarded as a (parallel) fiber metric on the projective tractor bundle determines a holonomy reduction to \( \text{SO}(p + 1, q + 1) \).

**Proof of part (a).** By hypothesis, take \( M \) to be an open collar of \( M_0 \). Then the connection \( \hat{\nabla} \) is torsion-free and regarded as a connection on \( \hat{M} \) it preserves \( H \); hence, it must be the Levi-Civita connection of \( H \). It has homogeneity 2 with respect to the dilations, by Theorem 4.4 it pulls back to the tautological form \( h_0 \) on the metric cone \( \Sigma \) of \((M, c)\), and it was observed before the statement of the theorem that it is Ricci-flat, so it is an ambient metric for \( c \). \( \square \)

To prove (b), we will construct a candidate \( T \) for the tractor bundle, a linear connection on it, and an adapted frame bundle \( G \to M \). Using a characterization of Čap and Gover, we will show that the linear connection is induced by a Cartan connection \( \eta \) on \( G \) such that \((\hat{G}, \eta)\) is a parabolic geometry of type \((\mathfrak{sl}(n + 1, \mathbb{R}), P)\), and then that \( \eta \) is normal. In particular, \((G, \eta)\) will determine a projective structure \( p \) on \( M \) for which the normal tractor connection is the given linear connection on \( \Sigma \).

Take \( M \) to be the quotient \( \hat{\Sigma}/\mathbb{R}_+ \) of \( \hat{\Sigma} \) by the dilation action and denote the projection onto that space by \( \hat{\pi} : \hat{\Sigma} \to M \); in particular, the below argument will show that \( \hat{\pi} \) coincides with the map so named above. By construction, sections of the weighted bundle \( \mathcal{E}(\sigma) \to M \) (see Subsection 4.1) can be identified with smooth functions on \( \hat{\Sigma} \) homogeneous of weight \( k \) with respect to the dilation action. The discussion before the theorem motivates that we take for the tractor bundle the bundle \( \mathcal{T} \to M \) with fibers

\[
\mathcal{T}_x := \{ U \in \Gamma(T\hat{\Sigma}|_{\Sigma_x}) : U \text{ is homogeneous of degree } -1 \},
\]

where the smooth structure is characterized by the fact that a section of \( T \) is smooth iff the corresponding homogeneous vector field in \( \Gamma(T\Sigma) \) is smooth. Then the connection \( \nabla^H \) descends to a connection \( \nabla \) on \( T \), and the volume form of \( H \) descends to \( \nabla \)-parallel tractor volume form \( \epsilon \in \Gamma(L^{n+1}T^*) \).

Let \( G \) denote the principal P-bundle comprising the frames \((U^A)\) that (a) are adapted to the composition structure \( [\Sigma] \) in that \( U^0 = \sigma X \) for some section \( \sigma \) of \( \mathcal{E}(\Sigma) \) and (b) satisfy \( e(U^0, \ldots, U^n) = 1 \). By construction, \( T = G \times_{\hat{\pi}} \mathbb{V} \), and as usual we may identify the sections of \( T \) with the \( \mathbb{P} \)-equivariant maps \( G \to \mathbb{V} \). Any element \( u \in \hat{G} \) determines an isomorphism \( \hat{u} : \mathbb{V} \to T_{\pi(u)} \) defined by \( v \mapsto [u, v] \), and the \( \mathbb{P} \)-equivariant function \( \hat{t} : \hat{G} \to \hat{\mathbb{V}} \) corresponding to a tractor \( t \in \Gamma(T) \) is just \( u \mapsto \hat{u}^{-1}(t(\pi(u))) \).

To formulate Čap and Gover’s condition for a linear connection \( \nabla \) on \( T \) to be induced by a Cartan connection on \( \hat{G} \), we need the following construction: Pick \( u \in \hat{G} \) and \( \xi \in T_u \hat{G} \), and denote \( x := \pi(u) \). Then, any \( t \in \Gamma(T) \) determines an element \( (\nabla_{\nabla x} \xi^t)(x) \in T_x \), and thus the image of that point under \( \hat{u}^{-1} \) in \( \mathbb{V} \). Checking shows that the difference

\[
\hat{u}^{-1}(\nabla_{\nabla x} \xi^t)(x) - (\xi \cdot t)(u)
\]

depends only on \( t(x) \), so \( \xi \) defines a linear map \( \Psi(\xi) : \mathbb{V} \to \mathbb{V} \) characterized by

\[
\hat{u}^{-1}(\nabla_{\nabla x} \xi^t)(x) - (\xi \cdot t)(u) = \Psi(\xi)(\hat{t}(u))
\]

for (all) smooth sections \( t \).

By [21, Theorem 2.7], a linear connection \( \nabla \) on a tractor bundle \( T \) for a general parabolic geometry is induced by a Cartan connection \( \eta \) on \( G \) iff (where in our case, \( g = \mathfrak{sl}(n + 1, \mathbb{R}) \))
(A) for each $\xi \in T_\mathcal{G} \mathfrak{g}$ the linear map $\Psi(\xi) : \mathcal{V} \to \mathcal{V}$ is given by the action of some element in $\mathfrak{g}$, and

(B) for each $x \in M$ and nonzero $U \in T_x M$, there is some index $a$ and a local smooth section $t \in \Gamma(T^a)$ for which $(\nabla_U t)(x) \notin T^a_x$. (Here, $(T^a)$ is the usual natural filtration of $\mathcal{T}$ induced by the $p_+$-action on $\mathcal{V}$.)

Proof of part (B) of Theorem 6.6. We first check that conditions (A) and (B) hold for $\nabla^H$ on $\mathcal{T}$:

(A) Since $\nabla^H$ preserves $\mathfrak{e}$, it preserves the bundle of oriented, unit volume frames, and so for any $\xi$, $\Psi(\xi)$ in $\mathfrak{g}$ is given by the action of an element of $\mathfrak{sl}(n+1, \mathbb{R})$.

(B) The projective case there are only two distinct nonzero filtrations of $\mathcal{T}$, namely, $\mathcal{T}$ itself and the bundle $\mathcal{T}^1$ with fiber $\mathcal{T}^1_x := \{\mu X : \mu \in \mathcal{E}(-1)_x \} \cong \mathcal{E}(-1)_x$, and so we must have $a = 1$. Fix $y \in \Sigma$. For any $\hat{U} \in T_y \Sigma$ and any nowhere zero local section $\mu X \in \Gamma(\mathcal{T}^1)$, we have

$$(\nabla^H_U(\mu X))(y) = (\hat{U} \cdot \mu)(y)X + \mu(y)(\nabla^H_U X)(y).$$

The first term on the right is in $\mathcal{T}^1$. Regarded as an object on the ambient space, $\nabla^H X = \text{id}_\Sigma \hat{Y}$, and so as an object on $\mathcal{T}$, $\nabla^H_X A^a = Z_a^b$. Now, $\mu(x) \neq 0$, the image of $Z$ is complementary to $\mathcal{T}^1$, and $Z_a^b : TM(-1) \to \mathcal{T}$ is injective, so the second term is not in $\mathcal{T}^1$, and thus neither is $(\nabla^H_U(\mu X))(y)$.

By the result given immediately before the proof of this part, $\nabla^H$ corresponds to a Cartan connection on $\mathcal{G}$. An algebraic normality condition guarantees uniqueness of the tractor connection [21] (i.e. we have the normal tractor connection in the sense of that source), and it is satisfied because $\nabla^H$ is Ricci-flat [20].

Remark 6.6. One can easily describe the projective structure $\nabla^H$ determines on $M$: Pick any section $\sigma \in \Gamma(\check{\mathcal{P}} : \check{M} \to M)$, and define the connection $\nabla^\sigma$ on $M$ by

$$\nabla^\sigma_U V := T\check{x} \cdot \nabla^H_{T\check{x}} U(T\sigma \cdot V).$$

Checking directly shows that this indeed defines a connection, and its projective class $p := [\nabla^\sigma]$ is independent on the choice of section $\sigma$.

Now, Theorem 6.6 immediately yields a translation of Theorem 6.4 into natively projective language and hence furnishes a solution to the Dirichlet problem corresponding to a reduction of the normal projective tractor connection to $\mathrm{SO}(r+1, s+1)$ for $r + s \geq 3$ odd.

Theorem 6.7. Let $(\Sigma, \mathfrak{c})$ be a connected, real-analytic conformal structure of signature $(r, s)$ of odd dimension $n := r + s \geq 3$. Then, there is (a) a real-analytic $(n+1)$-dimensional projective structure $\mathfrak{p}$ on a collar $M$ of $\Sigma$ and (b) a holonomy reduction of the normal projective tractor connection to $\mathrm{SO}(H)$ such that geometry induced on the zero locus is $(\Sigma, \mathfrak{c})$. The solution $(M, \mathfrak{p}, H)$ is unique up to a equivalence along the zero locus and positive constant scaling of $H$.

Hence, there are Klein-Einstein metrics $(M_{\pm}, g_{\pm})$ of signature $(r+1, s)$ and $(r, s+1)$ with (common) projective infinity $(\Sigma, \mathfrak{c})$, and these are unique modulo homothety and equivalence near infinity.

The notions of equivalence in the theorem are the same as those for holonomy reductions to $\mathcal{G}_+^*$ described above, replacing the 3-forms $\Phi^a$ with tractor metrics $H^a$ and eliminating the criterion on the $\mathcal{N}(\mathcal{P})K$ endomorphism fields $J^a_{\mu \nu}$.

Proof. By Theorem 6.4 there is an ambient metric $(\check{\Sigma}, H)$ for $(\Sigma, \mathfrak{c})$, which by Theorem 6.6 determines a projective structure $(M, \mathfrak{p})$ and the claimed holonomy reduction. Subsection 5.1 recalled that this structure determines the conformal structure, which coincides with $(\Sigma, \mathfrak{c})$ by construction, and Klein-Einstein metrics $(M_{\pm}, g_{\pm})$. Unwinding definitions, the uniqueness of the ambient metric in the real-analytic, odd-dimensional case described in Theorem 6.4 implies the uniqueness conditions stated here. \qed
6.3. Normal forms for ambient metrics and Klein-Einstein metrics. When working with an ambient metric $H$ for a conformal structure $(\Sigma, c)$, it is often convenient to pick a representative metric $g \in c$. Then, with respect to such a metric $g$, $H$ admits an essentially unique normal form [41, Definition 2.7 and Theorem 2.9]. Translating into projective language using the identification in [54,2] gives a normal form for parallel projective tractor metrics with a given zero locus geometry.

**Proposition 6.8.** Let $(M, p)$ be a projective structure of dimension $n \geq 3$, $T$ its standard tractor bundle, and $\nabla^T$ its normal connection. Let $H$ be a $\nabla^T$-parallel tractor metric such that the hypersurface curved orbit $M_0 = \{x \in M : H_{AB}X^AX^B = 0\}$ is nonempty, and let $c$ be the conformal structure that $H$ induces on $M_0$. We can identify an open collar $U \subseteq M$ containing $M_0$ with an open subset of $M_0 \times \mathbb{R}$ containing $M_0 \times \{0\} \leftrightarrow M_0$. (As in Subsection 6.1, we denote by $\rho$ the standard coordinate on $\mathbb{R}$.) For any representative $g \in c$, there is a representative $\nabla$ of $p$ and a weighted bilinear form $g_\rho \in \Gamma(S^2T^*U(2))$, such that

(a) the pullback to $M_0$ of $g_\rho$ is $c$, and furthermore after trivializing density bundles with the scale $\nabla$, the pullback to $M_0$ of $g_\rho$ is $g$,

(b) in the scale determined by $\nabla$, $H$ is given by

$$H = \begin{pmatrix} 2\rho & d\rho \\ d\rho & g_\rho \end{pmatrix}$$

and

(c) $g_\rho(\rho_\rho, \cdot) = 0$.

**Remark 6.9.** The identification of $U$ with a subspace of $M_0 \times \mathbb{R}$ and condition (ii) together enable us to view $g_\rho$, as a 1-parameter family of weighted bilinear forms on $M_0$, which correspond to unweighted forms in the scale determined by the representative $g$.

Note that for any parallel tractor metric in the given normal form, $\tau = H_{AB}X^AX^B = 2\rho$ (here, $\rho$ denotes the weighted function that corresponds to $\rho$ under the trivialization of $\mathcal{E}(2)$ with respect to $\nabla$), so the curved orbits in the decomposition $M = M_+ \cup M_0 \cup M_-$ determined by $H$ are just

$$M_+ = \{(k, \rho) \in M : \rho > 0\}$$

$$M_0 = \{(k, \rho) \in M : \rho = 0\}$$

$$M_- = \{(k, \rho) \in M : \rho < 0\}.$$ 

In fact, on any projective structure with a $\text{SO}(p + 1, q + 1)$ holonomy reduction and non-empty zero locus $M_0$, in a collar neighborhood of $M_0$ we can realize the open curved orbit structures $(M_\pm, g_\pm)$ of the projective structure as pseudo-Riemannian submanifolds of the ambient metric structure $(\tilde{\Sigma}, H)$. This is because weighted (density and tractor) objects on $M$ can be identified with homogeneous objects on $\tilde{\Sigma}$: Trivializing (appropriate subsets) with respect to the scales $\tau = \pm 1$ corresponds to identifying

$$M_\pm \leftrightarrow \{y \in \tilde{\Sigma} : H_{AB}X^AX^B = \pm 1\},$$

and by construction the corresponding Klein-Einstein metrics satisfy $g_\pm = \iota_\pm^* H$, where $\iota_\pm$ denotes the inclusion $M_\pm \hookrightarrow \tilde{\Sigma}$. For an ambient metric in normal form (68), the Klein-Einstein metrics assume the form

$$g_\pm = \frac{1}{2(\pm \rho)} g_\rho \mp \frac{1}{4\rho^2} d\rho^2.$$

6.4. The proof of Theorem 6.1. With the existence and uniqueness result for the ambient metric in hand, as well as the above formulation of the relationship between the ambient metric and the projective Thomas cone, we are all but prepared to solve the (real-analytic) Dirichlet problem for a $\text{G}_2$ holonomy-reduced projective structure: A real-analytic oriented generic $(2,3,5)$-distribution $(\Sigma, D)$ induces a conformal structure $c_D$ of signature $(2,3)$ on $\Sigma$. So, by
Theorem 6.10. Let $(M, p)$ be a real-analytic projective structure of even dimension $n \geq 4$, $\mathcal{T}$ its standard tractor bundle, and $\nabla^{\mathcal{T}}$ its normal connection. Let $H$ be a (necessarily indefinite) $\nabla^{\mathcal{T}}$-parallel tractor metric such that the zero locus $M_0 = \{x \in M : H_{AB}X^AX^B = 0\}$ is nonempty, and let $c$ be the conformal structure that $H$ induces there, $\mathcal{T}_0 \subset \mathcal{T}$ its standard conformal tractor bundle, and $\nabla^{\mathcal{T}_0}$ the normal conformal tractor connection.

Suppose $\chi_0 \in \Gamma(\bigotimes^r T^*_0|U)$ is a real-analytic $\nabla^{\mathcal{T}_0}$-parallel conformal tractor tensor. Then, there is a connected open subset $U \supset M_0$ of $M$ and a real-analytic $\nabla^{\mathcal{T}}$-parallel projective tractor tensor $\chi \in \Gamma(\bigotimes^r T^*|U)$ such that $\chi|_{M_0} = \chi_0$, and any two such extensions agree on some open set containing $M_0$.

Proof of Theorem 6.10. Denote by $c_D$ the conformal structure that $D$ induces on $\Sigma$. Theorem 6.13 yields the corresponding split-generic 3-form $\Phi_0$, on the conformal tractor bundle $\mathcal{T}_0$ of $c_D$, that is parallel with respect to the normal conformal tractor connection; by naturality $\Phi_0$ is real-analytic. Then, Theorem 6.7 gives that there is a projective structure $p$, say, with normal tractor connection $\nabla^{\mathcal{T}}$, and a $\nabla^{\mathcal{T}}$-parallel tractor metric $H$ on a collar $M$ of $\Sigma$ for which the geometry on the zero locus is $(\Sigma, c_D)$. Next, Theorem 6.10 guarantees the existence of a parallel split-generic tractor 3-form $\Phi$ on $M$ such that $\Phi|_{\Sigma} = \Phi_0$ (after possibly replacing $M$ with a smaller collar of $\Sigma$). Raising an index of $\Phi$ using $H$ gives a parallel split tractor cross product $\times$ on $(M, p)$; by construction, the geometry $\times$ determines on the zero locus is $(\Sigma, D)$. Tracing the uniqueness statements in the involved theorems yields the claimed uniqueness.

Remark 6.11. The Klein-Einstein condition appears to impose severe restrictions on strictly N(P)K structures $(M_{\pm}, g_{\pm}, J_{\pm})$, and hence on those structures that admit bounding $(2, 3, 5)$-distributions. This is true in at least a naive sense: Applying the Cartan-Kähler Theorem gives that, locally, 6-dimensional strictly N(P)K structures depend, modulo diffeomorphism, on 2 arbitrary functions of 5 real variables, the same generality as for Kähler structures in this dimension; see [14] §4.3 and Remark 23] for the definite and indefinite strictly NK cases. On the other hand, Corollary 5.3 shows that near a point on the projective infinity, a (real-analytic) Klein-Einstein strictly N(P)K structure is mutually determined by the $(2, 3, 5)$-distribution it defines there, and a naive count shows that locally $(2, 3, 5)$-distributions (and hence such strictly N(P)K structures) are considerably less general: modulo diffeomorphism, they depend on just 1 function of 5 variables.

7. Examples

Section 6 suggests a method for producing explicit examples of projective structures $(M, p)$ with parallel split tractor cross products $\times$ for which the zero locus $M_0$ is nonempty:

(a) Select a (real-analytic, oriented) $(2, 3, 5)$-distribution $(\Sigma, D)$. 


(b) Compute the Nurowski conformal structure \( c_D \) that \( D \) induces.

(c) Compute the parallel split-generic conformal tractor cross product \( \times \) (or equivalently, the parallel split-generic conformal tractor 3-form \( \Phi_0 \)) on \( \Sigma \), which \( D \) determines up to constant scale. (The cross product can be normalized using a choice of oriented conformal tractor volume form, by demanding that it coincide with the volume form determined pointwise by \( \Phi_0 \).)

(d) Compute the (essentially unique) real-analytic ambient metric \( (\tilde{\Sigma}, \tilde{g}_D) \) of \( c_D \).

(e) Set \( M := \tilde{\Sigma}/\mathbb{R}_+ \); the Levi-Civita connection of \( \tilde{g}_D \) descends to a connection \( \nabla^T \) on \( T\tilde{\Sigma}/\mathbb{R}_+ \to M \), which we may view as the standard projective tractor bundle for the projective structure \( \mathbf{p} \) that \( \nabla^T \) determines.

(f) Compute the parallel extension of \( \times \) (or \( \Phi_0 \)) to a parallel split-generic projective tractor cross product \( \times \) (respectively, to a parallel split-generic projective tractor 3-form \( \Phi \)).

Theorem \[6.1\] ensures that this construction always yields such a triple \((M, \mathbf{p}, \times)\), and Corollary \[6.3\] shows that essentially all such triples \((M, \mathbf{p}, \times)\) with nonempty zero locus arise this way. Computing the indicated data explicitly for a general \((2,3,5)\)-distribution \( D \), however, is generally difficult: Step \( (b) \) amounts to solving a typically intractable system of partial differential equations—indeed, explicit ambient metrics have only been produced for a limited number of classes of conformal structures, and for only a few families of Nurowski conformal structures. The other parts of the procedure are variously less formidable: Step \( (d) \) amounts to carrying out Cartan’s normalization procedure for these geometries (giving a \( \mathfrak{g}_5^2 \)-valued Cartan connection) and exploiting the inclusion \( \mathfrak{g}_5^2 \hookrightarrow \mathfrak{so}(3,4) \), or, in the case that \( D \) is given in so-called Monge normal form (see below), simply computing using a formula of Nurowski. Step \( (g) \) is first a matter of computing the normal conformal Killing form associated to \( D \): Proposition \[6.10\] shows that simply forming the wedge product of the vectors in an oriented local basis and lowering indices using the conformal structure gives this form up to a positive real-valued function (in five local coordinates), and hence solving for the normal conformal Killing form amounts locally to solving an overdetermined PDE for this function (which always admits a solution). With that solution in hand, one can recover the corresponding parallel section of the conformal tractor bundle \( \Lambda^3 T_0^* \) of \( c_D \) by applying to that solution the so-called BGG splitting operator for that bundle; this operator was recorded for this purpose in \[53, (15)]\), and we reproduce it in \[92\] in Appendix \[D\]. Finally, Step \( (f) \) amounts to solving a system of 35 ordinary differential (parallel transport) equations in \( \rho \).

### 7.1. A Monge quasi-normal form for \((2,3,5)\)-distributions

Any ordinary differential equation \( z' = F(x, y, y', y'', z) \), where \( y \) and \( z \) are regarded as functions of \( x \) can be encoded on the jet space \( J^{2,0}_{x,y,q,z} \cong \mathbb{R}^5 \supseteq \text{dom } F \) (where \( p \) and \( q \) are jet coordinates respectively corresponding to \( y' \) and \( y'' \)) as the 2-plane distribution

\[
D_F := \ker\{dy - p \, dx, dp - q \, dx, dz - F \, dx\}. \tag{70}
\]

Checking shows that this is a \((2,3,5)\)-distribution iff \( \partial^2_x F \) vanishes nowhere, so any such smooth function \( F(x, y, p, q, z) \) specifies such a distribution. In fact, \[12\] \[76\] shows that every \((2,3,5)\)-distribution is locally equivalent at each point of the underlying manifold to \( D_F \) for some function \( F \).

Nurowski has practical (albeit complicated) formula \[71\] \[54\] that gives for general \( F \) (such that \( \partial^2_x F \) vanishes nowhere) an explicit coordinate expression for a representative of the conformal structure induced by \( D_F \) that is polynomial in the 4-jet of \( F \).

### 7.2. A family with homogeneous curved orbits

**Example 7.1.** We apply the construction at the beginning of the section to describe a 1-parameter family of deformations of the flat model described in Subsection \[5.4\] for which the geometries induced on the curved orbits are all still homogeneous.
In [62], Cartan solved the equivalence problem for (2, 3, 5)-distributions. Moreover, he showed that if such a distribution $D$ is not locally equivalent to the flat model for that geometry, then the infinitesimal symmetry algebra $\mathfrak{inf}(D)$ of $D$ has dimension at most 7 and gave (in the complex setting) an explicit coframe description of the local distributions for which equality holds.\(^5\)

We construct parallel projective split tractor cross products $(M, p, x)$ for (real versions of) these distributions using a well-known realization amenable to our purposes: Up to local equivalence (and again a suitable notion of complexification) the distributions with 7-dimensional symmetry algebra are exhausted by the distributions defined on $\mathbb{R}^5_2 := \{(x, y, p, q, z) : q > 0\}$ via (70) by $D_m := D_qm$ and $D_{\log q}$, where $m \not\in \{-1, 0, 1, \frac{1}{4}, \frac{3}{4}, 2\}$. For $m \in \{-1, \frac{1}{4}, \frac{3}{4}, 2\}$, $D_m$ is locally equivalent to the flat model for that geometry; for $m \in \{0, 1\}$, $D_m$ is not a (2, 3, 5)-distribution. Some distinct values of $m$ yield locally equivalent distributions $D_m$, but varying $m$ over the half-open interval $[\frac{1}{2}, 1)$ exhausts all such distributions without any such redundancy, and includes the flat model at $m = \frac{1}{2}$; see [62] and the minor correction thereto in [38] for details. It turns out that the conformal structures of the distributions $D_m$ and $D_{\log q}$ are all almost Einstein (in fact, almost Ricci-flat), which leaves their ambient metrics more amenable to explicit computation. We exploit this to give explicit data for the parallel projective tractor cross products determined by the distributions $D_m$; the distribution $D_{\log q}$ can be handled similarly, but we do not do so here. By Theorem 6.1 a real-analytic (2, 3, 5)-distribution determines (at least in a collar along the underlying manifold) the real-analytic parallel projective tractor cross product, so for the locally flat distributions $D_m$, $m \in \{-1, \frac{1}{4}, \frac{3}{4}, 2\}$ the corresponding parallel projective tractor cross product $(M, p, x)$ is locally equivalent to the flat model in Subsection 6.1. In particular, we may view the family of resulting structures as one of smooth deformations of that model. The ambient metrics associated to these distributions were first given in coordinates in [72].

We describe this family of geometries in terms of a well-adapted global frame (or rather, a corresponding family of global frames) on $\mathbb{R}^5_2$. All of the claims can be verified by direct (if tedious) computation, in particular using the data [64] for the representative connection of the underlying projective structure.

Fix $m \in \mathbb{R} \setminus \{0, 1\}$. The infinitesimal symmetry algebra $\mathfrak{inf}(D_m)$ of $D_m$ satisfies [62] §5

\[
\mathfrak{inf}(D_m) \geq \langle \xi_1, \ldots, \xi_7 \rangle,
\]

where

\[
\begin{align*}
\xi_1 &:= \partial_x, \\
\xi_2 &:= \partial_y, \\
\xi_3 &:= x\partial_y + \partial_p, \\
\xi_4 &:= y\partial_y + p\partial_p + q\partial_q + mz\partial_z, \\
\xi_5 &:= \partial_z, \\
\xi_6 &:= x\partial_z + 2y\partial_y + p\partial_p + z\partial_z, \\
\xi_7 &:= q^{m-1}\partial_x + (pq^{m-1} - \frac{1}{m}z)\partial_y + (1 - \frac{1}{m})q^m\partial_p + (m-1)\int q^{2m-2}dq \cdot \partial_z.
\end{align*}
\]

If $D_m$ is not flat, that is, if $m \not\in \{-1, \frac{1}{4}, \frac{3}{4}, 2\}$, then equality holds. The generators $\xi_1, \ldots, \xi_5$ together span a subalgebra $\mathfrak{s}_m$ of $\mathfrak{inf}(D_m)$ that acts infinitesimally transitively on $\mathbb{R}^5_2$. Then, analyzing the flows of the generators $\xi_i$, enables us to identify $\mathbb{R}^5_2$ with the connected, simply connected Lie group $S_m$ with Lie algebra $\mathfrak{s}_m$; it is isomorphic to the matrix group

\[
\begin{pmatrix}
a_5 & a_4 & a_3 & a_1 \\
a & a^5 & a_2 \\
\cdot & a_5 & a_3 \\
\cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1
\end{pmatrix} : a_1, a_2, a_3, a_4 \in \mathbb{R}; a_5 > 0.
\]

The infinitesimal symmetry algebra of a distribution $(M, D)$ is the vector space of all vector fields on $M$ whose flows preserve $D$, that is, the vector fields $U$ such that $\mathcal{L}_U V \in \Gamma(D)$ for all vector fields $V \in \Gamma(D)$.

Strictly speaking, he established this bound under the modest assumption of “constant root type”, which in particular holds for any homogeneous distribution. This restriction was recently lifted in [63] Theorem 5.5.2.
Consider the left-invariant frame \((E_a)\) of \(TS_m\) given by

\[
E_1 := \frac{q}{15\sqrt{10}m^4}\partial_y
\]

\[
E_2 := \frac{(3m + 14)(m + 1)}{30m^2} (\partial_x + p\partial_y + q\partial_p + q^m\partial_z) + \frac{q(m - 1)}{45\sqrt{10}m^5}\partial_y
\]

\[
E_3 := -\frac{1}{2\sqrt{5}m^2}[(m + 2)\partial_x + (m + 2)p\partial_y + (m + 1)q\partial_p + 2q^m\partial_z]
\]

\[
E_4 := \frac{m - 1}{15m^3} (\partial_x + p\partial_y + q\partial_p + q^m\partial_z) + \sqrt{10}q\partial_y
\]

\[
E_5 := -\frac{1}{5m^2}(\partial_x + p\partial_y + q\partial_p + q^m\partial_z).
\]

The nonzero bracket relations for \((E_a)\) are given in [93] in Appendix [F]. With respect to this basis, the (left-invariant) distribution \(D_m\) is

\[
D := \langle E_4, E_5\rangle,
\]

the derived 3-plane distribution is

\[
[D, D] := \langle E_3, E_4, E_5\rangle,
\]

and the conformal structure \(c_\mathcal{D}\) it induces admits the representative

\[
2e_1^2e^4 + 2e_2^2e^3 - (e^3)^2.
\]

Consider the 6-manifold \(M := S^6 \times \mathbb{R}\), identify the vector fields \(E_a \in \Gamma(TS_m), a \in \{1, \ldots, 5\}\), respectively with \((E_a, 0) \in \Gamma(TM)\), let \(\rho\) denote the standard coordinate on \(\mathbb{R}\), denote \(E_6 := \partial_\rho \in \Gamma(TM)\). Take \(p\) to be the projective class on \(M\) containing the connection \(\nabla\) whose nonzero connection forms with respect to the global frame \((E_a) = (E_1, \ldots, E_6)\) of \(M\) (and its dual coframe \((e^a)\)) are given in [93] in Appendix [F].

Consider the weighted 2-form

\[
\omega := \sqrt{2} e^1 \wedge e^2 + \sqrt{2} \left[\begin{array}{cc}
(m + 1)(m - 2)e^1 \wedge e^5 + e^4 \wedge e^3
\end{array}\right].
\]

\(\rho - e^3 \wedge dp \in \Gamma(\Lambda^2 T^*M[3])\)

(here written with respect to the scale determined by \(\nabla\)). Computing gives that

\[
\frac{1}{4}d\omega = -e^1 \wedge e^3 \wedge e^4 - e^2 \wedge e^5 \wedge e^3 + \sqrt{2} e^1 \wedge e^5 \wedge dp.
\]

Now, computing gives that the projective tractor 3-form

\[
\Phi_{ABC} := \left(\frac{\omega_{bc}}{4(\omega)_{abc}}\right) \in \Gamma(\Lambda^3 T^*),
\]

is parallel and split-generic.

The corresponding operator \(J : \mathcal{T} \rightarrow \mathcal{T}[1]\) is

\[
J^A_B = \left(\begin{array}{cc}
0 & x_b \\
0 & J^b_a
\end{array}\right),
\]

where (in the scale determined by \(\nabla\))

\[
J = -E_3 \otimes dp - \sqrt{2} E_4 \otimes e^2 + \sqrt{2} E_5 \otimes e^1
\]

\[
+ \rho[\sqrt{2} E_1 \otimes e^5 + \sqrt{2}(m + 1)(m - 2) E_2 \otimes e^1
\]

\[
+ \sqrt{2} E_2 \otimes e^4 + \sqrt{2}(m + 1)(m - 2) E_4 \otimes e^5 + 2\partial_\rho \otimes e^3]\]

\[
\chi = -e_3.
\]

In the scale \(\nabla\), the parallel tractor metric \(\Phi\) determines on \(\mathcal{T}\) is (given by the normal form of Subsection [6.3])

\[
H = \left(\begin{array}{cc}
2\rho & dp \\
0 & g_\rho
\end{array}\right),
\]

(72)
where 

\[ g_\rho = 2e^1 e^4 + 2e^2 e^5 - (e^3)^2 - 2(m + 1)(m - 2)(e^5)^2 \cdot \rho. \]

The strictly nearly \((\mp 1)\)-Kähler structure \((g_{\pm}, J_\pm)\) that the scale \(\tau = \pm 1\) determines on \(M_\pm\) is given by (cf. [69])

\[ g_{\pm} = \frac{1}{2(\pm \rho)}[2e^1 e^4 + 2e^2 e^5 - (e^3)^2 - 2(m + 1)(m - 2)(e^5)^2 \cdot \rho] \pm \frac{1}{4\rho^2}d\rho^2 \]

and

\[ J_{\pm} = \pm(\pm \rho)^{1/2}[E_1 \otimes e^5 - (m + 1)(m - 2)E_2 \otimes e^1 \]

\[ - E_2 \otimes e^4 - (m + 1)(m - 2)E_4 \otimes e^5 - \sqrt{2}\partial_\rho \otimes e^3 \]

\[ + (\pm \rho)^{-1/2} \left( \frac{1}{\sqrt{2}} E_3 \otimes d\rho + E_4 \otimes e^2 - E_5 \otimes e^1 \right); \]

the Kähler form is

\[ \omega_{\pm} = (\pm \rho)^{-3/2} \left( \frac{1}{2} e^1 \wedge e^2 - \frac{1}{2\sqrt{2}} e^3 \wedge d\rho \right) \pm (\pm \rho)^{-1/2} \cdot \left[ -\frac{1}{2}(m + 1)(m - 2)e^3 \wedge e^5 + \frac{1}{4} e^4 \wedge e^5 \right]. \]

The infinitesimal symmetries \(\xi_\alpha\) given in [71] are (interpreted as vector fields \((\xi_\alpha, 0)\) on \(M\)) symmetries of \((M, p, \Phi)\) except for \(\xi_6 := x\partial_x + y\partial_y + \rho\partial_\rho + z\partial_z; \) however, \(\xi_6 := \xi_6 + 2\rho\partial_\rho\) is a symmetry of \((M, p, \Phi)\). Define \(\xi_a = (\xi_a, 0)\) for \(a \in \{1, 2, 3, 4, 5, 7\}\).

The integral curves of \(\xi_a\) include \(t \rightarrow ((0, \Omega, 0, 1, 0), Ce^{2t}), C \in \mathbb{R}, \) so the infinitesimal symmetry algebra \(\text{inf}(M, p, \Phi)\) acts transitively on each of the three curved orbits; in particular, the underlying structures, \((M_{\pm}, g_{\pm}, J_\pm)\) and \((M_0, D_m),\) are homogeneous. In fact, \(\tilde{S}_m := (\xi_1, \ldots, \xi_6)\) is a subalgebra of \((\xi_1, \ldots, \xi_7) \cong (\xi_1, \ldots, \xi_7),\) so we may regard the structures \((g_{\pm}, J_\pm)\) as left-invariant structures on the connected, simply connected Lie group \(\tilde{S}_m\) with Lie algebra \(\tilde{\mathfrak{s}}_m.\)

7.3. Other explicit examples. There are other examples of \((2, 3, 5),\) distributions for which the ambient metrics of the induced conformal structures have been computed explicitly; the same relatively easy computations that yielded the data in Example [71] above would yield the corresponding strictly \(N(P)K\) structures.

**Example 7.2.** In [72], Nurowski found explicit ambient metrics for the (Monge normal form) \((2, 3, 5),\) distributions \(D_{F[a_0, \ldots, a_6, b]}\), \(a_0, \ldots, a_6, b \in \mathbb{R},\) where

\[ F[a_0, \ldots, a_6, b] := q^2 + \sum_{k=0}^{6} a_k p^k + b z. \]

These examples partially overlap with those in Example [71] above: The distribution \(D_{F[a_0, \ldots, a_6, b]}\) is locally equivalent to \(D_{log q}\) or \(D_{q^m}\) for some \(m \in \mathbb{R}\) iff \(a_0 = a_5 = a_4 = a_3 = 0.\) Except for these redundant examples, these distributions are not homogeneous.

**Example 7.3.** Willse recently found an explicit ambient metric for the distribution \(D_{F^*},\) where \[ F^* := y + q^{1/3}. \]

This distribution, which was recently produced by Doubrov and Govorov [88] is of independent interest: it is locally equivalent to a homogeneous (in fact, left-invariant) distribution, and the infinitesimal symmetry algebra \(\text{inf}(D_{F^*})\) has dimension 6, but it is missing from Cartan’s ostensibly local (complex) classification of \((2, 3, 5),\) distributions whose symmetry algebra is at least that size [82]. Unlike for the homogeneous distributions \(D_m\) in Example [61] the corresponding strictly \(N(P)K\) structures are not homogeneous.

\[ ^{10}\text{This infinitesimal symmetry algebra is defined as the vector space of all vector fields on } M \text{ whose flows preserve both } p \text{ and } \Phi. \]
8. Outlook: Related holonomy reductions

We briefly describe here some systems of interrelated structures, linked through the general curved orbit decomposition program of [27], that can be treated at least partially using the techniques of this article: namely, by analyzing the holonomy reduction of the given Cartan geometry in terms of the algebraic objects associated to the involved groups, and also, on each curved orbit type, through the available identifications with known geometric structures.

Each of these examples is the beginning of a rich story, the details of which would require a substantial article. So it should be emphasized that here we are deliberately sketchy with these examples. Our aim is simply to indicate the potential power of the holonomy reduction approach for relating different structures, by touching on a number of cases that are quite close to that treated above, and also to indicate the difference between the different sorts of compactifications (see Example 8.7).

8.1. Holonomy reductions of projective structures to subgroups of $\text{SO}(\cdot)$. Let $(M, p)$ be a projective manifold of dimension $n \geq 4$, and let $\nabla^T$ be the normal tractor connection. By Theorem 4.6 a $\nabla^T$-parallel tractor metric $H$ of signature, say, $(p, q)$, $p + q = n + 1$, (equivalently, a reduction of the holonomy of $\nabla^T$ to $\text{SO}(p, q)$) determines a stratification $M = M_+ \cup M_0 \cup M_-$ into curved orbits; the open sets $M_\pm$ inherit Einstein metrics $g_\pm$, and $M_0$ inherits a conformal structure. As in the example in this article (as described from this perspective in Section 5), a holonomy reduction to a proper subgroup of $\text{SO}(p, q)$ determines additional structures on each of these orbits. (Such a reduction can determine a finer stratification of $M$, but this does not occur for any of the examples we discuss in this subsection.)

In [3] Armstrong outlines some important possible holonomy reductions of the projective tractor connection and gives a few geometric consequences of some of these, but he does not consider the associated curved orbit decompositions. His list thus makes a natural starting point for investigating such reductions, and we discuss a few examples here.

Example 8.1 $(\text{SU}(\frac{q}{2}, \frac{q}{2}), n \geq 7)$. Such a reduction, which requires that $p$ and $q$ both be even, amounts precisely to the existence of a parallel complex structure $\mathbb{K}$ on $\mathcal{T}$ skew with respect to the parallel tractor metric $H$, which realizes $\mathcal{T}$ as a complex vector bundle, together with a parallel complex tractor volume form there; this is a matter of translating, e.g., [8] §5 into projective tractor language via the Thomas cone construction). In this case, $M_\pm$ inherit Sasaki structures, partially encoded as the projecting part $\mathbb{K}^A_B X^B Z_A^a$ of $\mathbb{K}^A_B$ (regarded as an unweighted vector field using the scale defined by $\nabla^T$). These are odd-dimensional analogs of Kähler structures [75], [78] §3. By part (3) of Theorem 4.6 $\mathbb{K}$ gives a parallel complex structure on the conformal tractor bundle over $M_0$, so the conformal structure determined on $M_0$ (whose non-emptyness requires that $p, q \geq 2$) arises from the classical Fefferman construction, that is, at least locally, it arises as a conformal structure (of signature $(p - 1, q - 1)$) on a circle bundle over an integrable hypersurface type CR manifold (of dimension $n - 2$) canonically determined by that structure [40], [23], [66]; such CR-manifolds are parabolic geometries of type $(\text{SU}(\frac{q}{2}, \frac{q}{2}), P_{\text{SU}})$, where $P_{\text{SU}}$ is the stabilizer of a null complex line in the standard representation $\mathbb{C}^{\frac{q}{2}}$ of $\text{SU}(\frac{q}{2}, \frac{q}{2})$.

Example 8.2 $(\text{Sp}(\frac{p}{2}, \frac{q}{2}), n \geq 11)$. This example, which requires that $p$ and $q$ both be multiples of 4, is quite similar to the last; this holonomy reduction is equivalent to the existence of three parallel, complex structures on $\mathcal{T}$ which are skew with respect to the tractor metric $H$ and which formally satisfy the algebraic relations of the imaginary quaternion units $i, j, k$. In this case, $M_\pm$ inherits a so-called 3-Sasaki structure, which entails three suitably compatible Sasaki structures [78] §3. Again using part (3) of Theorem 4.6 we conclude that the conformal structure determined on $M_0$ (again, for $p, q > 0$) arises from a quaternionic analog of the classical Fefferman construction: At least locally, it arises as a conformal structure (of signature $(p - 1, q - 1)$) on a bundle with fibers $\text{Sp}(1) \cong S^3$ over a quaternionic contact structure canonically determined by that structure [3] Theorem 1], [10]. Quaternionic contact structures are again a type of parabolic
geometry, this time of type \((\text{Sp}(\frac{\pi}{4}, \frac{\pi}{4}), P_{\text{Sp}}))\), where \(P_{\text{Sp}}\) is the stabilizer of a null quaternionic line in the standard representation \(\mathbb{H}^{\frac{n}{2}}\) of \(\text{Sp}(\frac{\pi}{4}, \frac{\pi}{4})\) \([29] \S 4.3.3\).

**Example 8.3** (Spin\((7)\) and \(\text{Spin}_h(3, 4), n = 7\)). This example is closely analogous to the main example considered in this article (namely, reduction to \(G_2^{(\ast)}\) on a projective 6-manifold). These holonomy reductions are equivalent to the existence of a parallel three-fold tractor cross product \(\chi \in \Gamma(\Lambda^3 \Psi^* \otimes \mathcal{T})\), or equivalently a parallel tractor 4-form \(\Psi\), of a suitable algebraic type. (By Proposition \([29] \S 4.3.3\) there are no stable 4-forms in dimension 8, so these 4-forms are not generic in the way that the 3-forms in the \(G_2^{(\ast)}\) example are.) The open orbits \(M_\pm\) inherit nearly parallel \(G_2^{(\ast)}\)-structure,\(^{11}\) which can be encoded by generic 3-forms \(\phi\) (given by the projecting part of \(X^A\Phi_{ABCD}\), regarded as unweighted forms by using the trivialization determined by \(\nabla g_\pm\)) whose covariant derivative \(\nabla g_\pm \phi\) is totally skew. The conformal structure on the zero locus \(M_0\) (which can only be nonempty in the \(\text{Spin}_h(3, 4)\) case) has signature (3, 3) and is induced via another so-called Fefferman construction, this time by a (3, 6)-distribution on \(M_0\) (a 3-plane field \(E\) on \(M_0\) that satisfies \(|[E, E] = TM_0|\); this Fefferman-type construction was identified by Bryant \([17]\) Proposition 1). Yet again, the underlying geometry is parabolic: (3, 6)-distributions can be identified with parabolic geometries of type \((\text{SO}(3, 4), P_3)\), where \(P_3\) is the stabilizer in \(\text{SO}(3, 4)\) of a null 3-plane in \(\mathbb{R}^{3, 4}\) \([29] \S 4.3.2\). Note that as in the \(G_2^{(\ast)}\) case (but unlike in the two examples above), the underlying geometry that determines the conformal structure on \(M_0\) is an object on \(M_0\) itself, rather than (locally, anyway) on the base space of a bundle for which \(M_0\) is the total space.

**Remark 8.4.** These three cases, and the \(G_2^{(\ast)}\) case investigated in this article, are all similar: Each holonomy reduction is governed by a parallel (tractor) \(k\)-fold cross product (cf. \([78] \S 3\)) or, after lowering indices using the tractor metric, a parallel tractor \((k + 1)\)-form \(\Psi\) of a particular algebraic type (or in the \(\text{Sp}(\cdot)\) case, a compatible triple of cross products or forms). The projecting part \(\psi\) of \(X^A;\Psi_{A_1...A_k+1}\) is a normal Killing \(k\)-form (of projective weight \((k + 1)\)). On \(M_\pm\), one can regard (the restriction to \(M_\pm\) of) \(\psi\) as an unweighted \(k\)-form using the representative connection \(\nabla g_\pm\) that together with \(g_\pm\) determines a special geometry. On \(M_0\), the conformal structure is determined via a Fefferman construction by a geometric structure (which can be identified as a parabolic geometry) on the base of a bundle (possibly \(M_0\) itself) for which \(M_0\) is the total space.

**Remark 8.5.** Even though much of the analysis of the above projective holonomy reductions should proceed similarly to the \(G_2^{(\ast)}\) reduction studied in this article, we note that the corresponding geometric Dirichlet problems are apparently more difficult, because the conformal ambient construction is considerably more involved for even-dimensional conformal structures, including those induced on \(M_0\) in the above examples. Even-dimensional analogues of Theorems \(6.7\) and \(6.10\) which were used in the solution of the \(G_2^{(\ast)}\) reduction Dirichlet problem, are given in Appendix \([2]\) below. In short, they guarantee only that the Dirichlet problem can be solved formally to order \(\frac{1}{2}(\dim M - 1)\); stronger general existence and uniqueness results will probably require separate analysis of each holonomy reduction.

### 8.2 Parallel tractor cross products on geometries of type \((\text{SL}(7, \mathbb{R}), P_r)\).

There are many types of \((\text{SL}(7, \mathbb{R}), P_r)\) parabolic geometries, some of which have been studied outside the parabolic context, and for each of them we can consider the consequences of the existence of a parallel tractor cross product to unveil a new uniform treatment of (and in some cases connections among) some \(G_2^{(\ast)}\)-related geometries. (In a few cases the types are rigid in the sense of Yamaguchi \([34]\), that is, the normality and regularity conditions force local equivalence of any structure of the given type to the corresponding flat model, but most types are not rigid.) Since a copy of \(G_2^{(\ast)} < \text{SL}(V)\) determines a group \(\text{SO}(\cdot) > G_2^{(\ast)}\) per Subsection \(2.3.3\) such holonomy reductions are mediated by reductions to \(\text{SO}(7, \mathbb{R})\) and \(\text{SO}(3, 4)\), and as is the case for the geometry studied in this article, it would likely be useful to investigate these weaker reductions as an intermediate step in understanding the full reduction to \(G_2^{(\ast)}\).

\(^{11}\) Semmelmann calls the definite version of these weak \(G_2\)-manifolds.
Example 8.6 (Almost Grassmannian structures in dimension 10). Let $P_2$ be the stabilizer in $\text{SL}(V)$ of a 2-plane in $V$, which turns out to be parabolic; the geometric structures corresponding to parabolic geometry of type $(\text{SL}(7, \mathbb{R}), P_2)$ are so-called almost Grassmannian (or Segré) structures: In this case, such a geometry, defined on a 10-dimensional manifold, say, $N$, corresponds to a choice of auxiliary vector bundles $E$ of rank 2 and $F$ of rank 5 together with a trivialization $\Lambda^2 E \otimes \Lambda^3 F \to N \times \mathbb{R}$ and an isomorphism $TN \to \text{Hom}(E, F)$ [29, §4.1.3]. Now, the group $G_2 < \text{SO}(7, \mathbb{R})$ acts transitively on 2-planes, so the orbit decomposition determined by a definite parallel cross product is trivial. On the other hand, the action of $\text{SO}(3, 4)$ on $Gr_2(V)$ has 6 orbits (partitioned according to the signature of the bilinear form induced on the planes via pullback), the most degenerate of which is the 7-dimensional projective variety $Gr_2(V)_0$ consisting of the null 2-planes. The stabilizer of the $SO(3, 4)$-action on $Gr_2(V)_0$ is parabolic, so this null Grassmannian is the model space of a parabolic geometry, which turns out to be the geometry of Lie contact structures of signature $(3, 4)$ [29 §4.2.5]. So, given an almost Grassmannian manifold $N$ as above, a fiber metric of signature $(3, 4)$ on the standard tractor bundle parallel with respect to the normal tractor connection determines a curved orbit decomposition of $N$, and the closed, 7-dimensional orbit inherits a Lie contact structure of that signature: Such a structure consists of a 7-manifold $N'$, a contact structure $H' \subset TN'$, auxiliary vector bundles $E' \to N'$ of rank 2 and $F' \to N'$ of rank 3, a bundle metric on $F'$ of signature $(1, 2)$, and an isomorphism $H' \cong \text{Hom}(E', F')$ such that the Levi bracket—the map $H' \times H' \to TN'/H'$ induced by the Lie bracket—is invariant under the $O(F_u)$-action on $H_u$.

Now, in contrast to the holonomy reduction to $G_2^*$ of projective geometry discussed in this article, the stratification of an almost Grassmannian structure of the indicated type determined by a holonomy reduction to $G_2$ is strictly finer than that determined by $SO(3, 4)$: The closed $SO(3, 4)$-orbit $Gr_2(V)_0$ decomposes further under the restriction to the $G_2^*$-action into two orbits, according to whether the restriction of the cross product to each plane is zero [15]. The stabilizer of such a plane in $G_2^*$ is parabolic, so it is the model space of a parabolic geometry. This geometry turns out to be the exotic parabolic contact structure associated to $G_2$ [29 §4.2.8]: So, given an almost Grassmannian manifold $N$ as above, a parallel (necessarily split) tractor cross product determines a curved orbit decomposition of $N$, and the closed, 5-dimensional curved orbit inherits such a structure, which consists of a contact structure $H''$ on a 5-manifold $N''$, an auxiliary rank-2 bundle $E'' \to N''$, and an isomorphism $S^3 E'' \to H''$ such that the Levi bracket $H'' \times H'' \to TN''/H''$ is invariant under the action of $\mathfrak{sl}(E'')$.[12]

Note that a priori the Cartan connections induced in the closed orbits associated to the holonomy reductions considered here need not be normal—this remains to be checked for these reductions.

8.3. A special holonomy reduction of conformal geometry in dimension 6.

Example 8.7 (Almost Einstein (3,6)-distributions). In some cases a given geometry can be realized as the induced structure on a curved orbit for more than one type of ambient geometry and holonomy reduction. For example, let $(N, E)$ be an oriented (3,6)-distribution (see Example 5.3) whose induced Bryant conformal structure $c_E$ (with normal conformal tractor connection $\nabla^7_0$) admits an almost Einstein scale with nonzero Einstein constant. The distribution $E$ determines a parallel tractor 4-form $\Psi \in \Gamma(\Lambda^4 T^* N)$ stabilized by $\text{Spin}(3, 4)$ [17], and the almost Einstein scale determines a parallel tractor $I \in \Gamma(T_0)$ [13], which is non-null because the Einstein constant is nonzero. Then, by [27] Theorem 3.5(1) or [43], $I$ determines a curved orbit decomposition $N = N_+ \cup N_0 \cup N_-$ according to the sign of $\sigma := H_{AB} X^A I^B \in \mathfrak{e}(1)$, where $H$ is the parallel conformal tractor metric (see Subsection 4.5) and $X$ is the weighted conformal tractor defined in [16]; the corresponding reduction of the holonomy of $\nabla^7$ from $SO(4, 4)$ to the stabilizer of $I$ (namely, $SO(3, 4)$, after possibly replacing the tractor metric with its negative) determines the

---

12Note that as $\mathfrak{sl}(2, \mathbb{R})$-representations, $\Lambda^2(\mathbb{R}^4) \cong S^4 \mathbb{R}^2 \oplus \mathbb{R}$, so up to scale $S^4 \mathbb{R}^2$ admits a unique $\mathfrak{sl}(2, \mathbb{R})$-invariant skew bilinear form.
Einstein structure $\sigma^{-2}g|_{N \setminus N_0} \in c_E|_{N \setminus N_0}$ on $N \setminus N_0$, and a conformal structure $c'$ of signature $(2,3)$ on the zero locus $N_0$.

A suitable split analogue of [50, Lemma 8] guarantees that at each point $u \in N$ the restriction of $I^D \Psi_{DABC}$ to $(I_u)^+ \subset (T_0)_u$ is a split-stable 3-form on that vector space, and a general-signature version of the arguments in [43] §3 identifies $(I)^-|_{N_0}$ with the standard conformal tractor bundle $T_0'$ of $c'$, and so the restriction $\Phi_{ABC} := I^D \Psi_{DABC}|_{N_0}$ takes values in the sub-bundle $\Lambda^3(T_0')^* \subset \Lambda^3 T_0^*$, that is, it can be identified as a conformal tractor 3-form on $(N_0, c')$. Furthermore, the normal conformal tractor connection on $c'$ is essentially given by restricting $\nabla^T$, so the split-generic 3-form $\Phi$ is parallel because $I$ and $\Psi$ are. Then applying Theorem 5.13 to $\Phi$ gives that, in fact, $c' = c_D$ for some oriented $(2,3,5)$-distribution $D$ on $N_0$. Thus, $(2,3,5)$-distributions serve as the natural boundary structures for two natural compactifications of two very different geometries. In the example here it is a conformal compactification in contrast to Theorem 5.22 which involves projective compactification.

**Appendix A. Stable forms via bases**

In this appendix we realize in explicit bases some of the constructions of Section 2.3.

**A.1. 3-forms in dimension 6.** The following proposition describes the orbit decomposition of the standard $\text{GL}(\mathbb{W})$-action on the space $\Lambda^3 \mathbb{W}^*$ of 3-forms on a real 6-dimensional vector space $\mathbb{W}$ (see [7, Theorem 2.1.13] and [16, Proposition 12]).

**Proposition A.1.** Let $\mathbb{W}$ be a real vector space of dimension 6, and let $\rho \in \Lambda^3(\mathbb{W}^*)$ be any element. Then there exists a basis $(e^\alpha)$ of $\mathbb{W}^*$ in which $\rho$ is given by one of the following normal forms:

$$
\begin{align*}
\beta_1 &= e_{123} + e_{456} \\
\beta_2 &= e_{135} - e_{146} - e_{236} - e_{245} \\
\beta_3 &= e_{156} + e_{204} + e_{345} \\
\beta_4 &= e_{125} + e_{345} \\
\beta_5 &= e_{123} \\
\beta_6 &= 0.
\end{align*}
$$

Here, $e^{ijk} := e^i \wedge e^j \wedge e^k$. Moreover, these six forms are mutually inequivalent. The $\text{GL}(\mathbb{W})$-orbits of the first two 3-forms are open in $\Lambda^3 \mathbb{W}^*$, the $\text{GL}(\mathbb{W})$-orbit of the third 3-form is a hypersurface in $\Lambda^3 \mathbb{W}^*$, and the $\text{GL}(\mathbb{W})$-orbits of the remaining forms are of higher codimension.

By definition $\beta_1$ and $\beta_2$ are stable, and they are non-degenerate in the sense that $v \cdot \rho = 0 \implies v = 0$.

Note that this property does not characterize $\beta_1$ and $\beta_2$, as it is easily checked that $\beta_3$ satisfies it too.

Now if we let

$$
A = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
$$

then $A^* \beta_1 = e_{135} + e_{146} + e_{236} + e_{245}$, and so the form

$$
\beta_\epsilon = e_{135} + \epsilon(e_{146} + e_{236} + e_{245})
$$

(75)
is equivalent to $\beta_1$ for $\varepsilon = +1$ and is (exactly) $\beta_2$ for $\varepsilon = -1$.

Using the classification above it can be easily check that $\lambda(\beta) \neq 0$ if and only of $\beta$ is stable, and from basic linear algebra the sign of $\lambda(\beta)$ is constant on the $GL(W)$-orbits of $\beta$. Finally, direct computing using the form $\beta_i$ in (75) shows that $\lambda(\beta_i) = 4\varepsilon(e^{1\ldots-6})^{3/2}$ and for $i \in \{1, 3, 5\}$ $J_{\beta_i}(e_i) = e_i e_{i+1}$, $J_{\beta_i}(e_{i+1}) = e_i$. The last part gives that $J_{\beta_i}^2 = \varepsilon \text{id}_W$.

### A.2. 3-forms in dimension 7.

Let $V$ be a real, 7-dimensional vector space. As we already observed, any 3-form $\Phi \in A^3V^*$ defines a symmetric bilinear form with values in $A^2V^*$ by

$$\tilde{H}(X, Y) := \frac{1}{5}(X \cdot \Phi) \wedge (Y \cdot \Phi) \wedge \Phi, \quad \forall X, Y \in V,$$

and this is non-degenerate if and only if $\Phi$ is stable. We check that the 3-forms Bryant uses in [14] to define $G_2$ and $G_2^*$ are stable, and then we compute $\tilde{H}$.

Bryant sets $G_2 := \{A \in GL(V) | A^* \Phi = \Phi\}$, where (in an arbitrary basis $(e^a)$ of $V$)

$$\Phi := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

and $G_2^* := \{A \in GL(V) | A^* \Phi^* = \Phi^*\}$, where

$$\Phi^* := e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.$$  

We can describe both forms simultaneously by

$$\Phi_\xi = e^{123} + \xi(e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}), \quad \xi \in \{\pm 1\}. $$

One can check by expanding in the above basis the action of an arbitrary element in $\mathfrak{gl}(V)$ on $\Phi_\xi$ that $\dim G_2(\xi) = 14$ (alternatively, see [14] Theorems 1, 2]). On the other hand, $\dim(A^3V^*) = 35$ and $\dim(GL(V)) = 49$, so by the Orbit-Stabilizer Theorem the orbit of $\Phi_\xi$ has maximal dimension, that is, $\Phi_\xi$ is stable. Finally it is easy to check that $v \cdot \Phi_\xi = 0$ implies $v = 0$.

Using the form $\Phi_\xi$ we can compute the symmetric bilinear form in (76)

$$\tilde{H}_{\Phi_\xi}(X, Y) = (x^1 y^1 + x^2 y^2 + x^3 y^3 + \xi(x^4 y^4 + x^5 y^5 + x^6 y^6 + x^7 y^7))e^{1234567},$$

where $X = x^ae_a$, $Y = y^ae_a$.

A straightforward calculation shows that $(\det \tilde{H}_{\Phi_\xi})^\frac{1}{4} = e^{1234567}$, and substituting gives that in the basis $(e^a)$ $H$ is diagonal, with entries $(1, 1, 1, 1, 1, 1, 1)$. It has signature $(7, 0)$ for $\xi = 1$ and signature $(3, 4)$ for $\xi = -1$.

Finally here we give an explicit table of multiplication for the imaginary (split-)octonions, this table can be used to verify the unproved claims in subsection 2.2 (and to give alternative proofs of many results that are proved there).

For $\Phi$ and $H$ as above the multiplication $\times$ characterized by $\Phi(X, Y, Z) = H(X, Y \times Z)$ has the following table:

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| $e_1$ | $-e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $e_7$ | $-e_6$ |
| $e_2$ | $e_3$ | $e_1$ | $e_6$ | $-e_7$ | $-e_4$ | $e_5$ |
| $e_3$ | $-e_2$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $e_5$ |
| $e_4$ | $e_5$ | $-e_6$ | $e_7$ | $\xi e_1$ | $\xi e_2$ | $-\xi e_3$ |
| $e_5$ | $e_4$ | $e_7$ | $-e_6$ | $-\xi e_1$ | $-\xi e_2$ | $-\xi e_3$ |
| $e_6$ | $-e_7$ | $-e_4$ | $-e_5$ | $\xi e_1$ | $\xi e_2$ | $\xi e_3$ |
| $e_7$ | $e_6$ | $e_5$ | $-e_4$ | $\xi e_1$ | $\xi e_2$ | $\xi e_3$ |

This is the multiplication table of the imaginary (split-)octonions for $\xi = 1$ ($\xi = -1$). Note that the multiplication $\times$ is skew and nonassociative, and satisfies $H(X, Y \times Y) = 0$ for all $X, Y \in V$. Proposition 2.4 for example, is easily verified via this table.
Appendix B. Proof of Proposition 3.7

In this appendix we prove Proposition 3.7 that any six-dimensional $N(P)K$ manifold $(M, g, J)$, with Kähler form, say, $\omega$, satisfies the identity $\omega_{[k} W_{c[d]}^k = 0$, where $W$ is the projective Weyl tensor.

First we recall that on a pseudo-Riemannian manifold the Riemann curvature tensor $R_{abcd}$ may be decomposed into the totally tracefree conformal curvature $C_{abcd}$ and a remaining part described by the symmetric conformal Schouten tensor $P_{ab}$, according to

$$R_{abcd} = C_{abcd} + 2g_{[a} P_{b]d} + 2g_{[d} P_{a]c},$$

where $\{\cdots\}$ indicates the antisymmetrization over the enclosed indices. We may write $R = C + P \Lambda g$ to indicate this in index-free notation (and where density bundles have been trivialized via the scale $g$).

Since $(M, g, J)$ is 6-dimensional $N(P)K$, it is Einstein. Now an Einstein manifold the projective and conformal Schouten tensors agree; we have $W_{ab} = C_{ab}$. For example this follows from [39, Proposition 5.5]. It follows that (in index-free notation) using the almost $\varepsilon$-complex structure $J$ instead of the Kähler form $\omega$, the identity to be established is equivalent to

$$C_{S,T,JU,V} + C_{T,U,JS,V} + C_{U,S,JT,V} = 0,$$

and we can write this more compactly by encoding the cyclic sum in $S, T, U$ by $
abla_{S,T,U} C_{S,T,JU,V}$.

In order to prove the identity we need essentially two formulae. The first is

$$R_{S,T,JU,JV} = -\varepsilon R_{S,T,U,V} + \varepsilon g((\nabla_S J)T, (\nabla_U J)V);$$

this is proven in the Riemannian nearly Kähler case in [15] (3)], the pseudo-Riemannian nearly Kähler case in [76] (1.1), and in the nearly para-Kähler case in [57, Proposition 5.2].

The second is the polarization of the constant type formula (see [39, 21]):

$$g((\nabla_S J)T, (\nabla_U J)V) = \alpha \{g(S, U)g(T, JV) - g(JV, T)g(S, U)
+ g(S, JU)g(T, V) - g(V, J)g(S, T)\}.$$  

Replacing $V$ with $JV$ in (80) gives

$$R_{S,T,JU,V} = -R_{S,T,U,V} + g((\nabla_S J)T, (\nabla_U J)V),$$

and by cycling in $S, T, U$ and applying the First Bianchi Identity we obtain

$$\nabla_{S,T,U} R_{S,T,JU,V} = \nabla_{S,T,U} g((\nabla_S J)T, (\nabla_U J)V).$$

Once again using that $M$ is 6-dimensional and Einstein, we have $Ric = \frac{8}{60} g$, and hence $P = \frac{8}{60} g$. On the other hand $Ric = 5ag$, and so $\alpha = \frac{5}{60}$. Substituting in the decomposition

$$R_{S,T,JU,V} = C_{S,T,JU,V} + P(S, JU)g(T, V) + P(T, JU)g(S, V) + P(T, V)g(S, JU) - P(S, V)g(T, JU),$$

gives

$$R_{S,T,JU,V} = C_{S,T,JU,V} + \frac{8}{60} \{g(S, JU)g(T, V) - g(JV, T)g(S, U)
+ g(T, V)g(S, JU) - g(V, J)g(S, T)\},$$

and simplifying the right-hand side yields

$$R_{S,T,JU,V} = C_{S,T,JU,V} + \alpha \{g(S, JU)g(T, V) - g(T, JU)g(S, V)\}. $$
Now, cycling in $S, T, U$ and using (83) gives
\[
\mathcal{G}_{S,T,U} \{ g((\nabla S)T, (\nabla T)JV) - g(S, JU, V) + \alpha \mathcal{G}_{S,T,U} \{ g(S, JU)g(T, V) - g(S, V)g(T, JU) \} \}
\]
(85)
Comparing the last formula with (81) yields
\[
\mathcal{G}_{S,T,U} \{ g(S, U)g(T, JV) - g(S, JV)g(T, U) \}
\]
(86)
Finally, expanding the right-hand side and canceling gives
\[
\mathcal{G}_{S,T,U} \{ g(S, U)g(T, JV) - g(S, JV)g(T, U) \}
\]
\[
= g((S, U)g(T, JV) - g(S, JV)g(T, U) + g(T, S)g(U, JV)
\]
\[
- g(T, JV)g(U, S) + g(U, T)g(S, JV) - g(U, JV)g(S, T)
\]
\[
= 0.
\]

**APPENDIX C. PROOF OF PROPOSITION 3.8**

Expanding the quantity
\[
\frac{1}{2} \left\{ R_{bc}^k a \omega_{dk} + R_{cd}^k a \omega_{bk} + R_{db}^k a \omega_{ck} \right\}
\]
(87)
using the decomposition
\[
R_{ab}^k d = W_{ab}^k d + \delta^k_b P_{ad} - \delta^k_b P_{ad}
\]
(88)
gives
\[
\frac{1}{2} \left\{ (W_{bc}^k a + \delta^k_b P_{ca} - \delta^k_b P_{ca}) \omega_{dk}
\right. \]
\[
+ (W_{cd}^k a + \delta^k_c P_{da} - \delta^k_c P_{da}) \omega_{bk}
\]
\[
+ (W_{db}^k a + \delta^k_d P_{ba} - \delta^k_d P_{ba}) \omega_{ck} \right\}.
\]
We can write the sum of the terms involving the Weyl curvature as
\[
\frac{1}{2} \left\{ W_{bc}^k a \omega_{dk} + W_{cd}^k a \omega_{bk} + W_{db}^k a \omega_{ck} \right\} = -\frac{3}{2} \omega_{k[b} W_{cd}]^k a.
\]
(89)
Also, we can simplify the two terms involving the symbol $P_{ab}$ together as
\[
P_{ab} (\delta^k_d \omega_{ck} - \delta^k_c \omega_{dk}) = P_{ab} (\omega_{cd} - \omega_{dc}) = 2P_{ab} \omega_{cd}
\]
(90)
and simplify similarly the other terms involving the Schouten tensor. Rearranging yields the desired identity:
\[
P_{ab} \omega_{cd} + P_{bc} \omega_{db} + P_{ad} \omega_{bc} = 3P_{a[b} \omega_{cd]}.
\]

**APPENDIX D. THE SPLITTING OPERATOR FOR $\Lambda^3 T^*_0$**

On a 5-dimensional conformal structure $(M, c)$, the BGG splitting operator $L_0 : \mathcal{E}_{ab}[3] \rightarrow \Lambda^3 T^*_0$ is given in any scale (say, that given by the representative metric $g \in e$) by [53] (15)
\[
L_0 : (\omega_0)_{bc} \mapsto \begin{pmatrix}
(\omega_0)_{bc} \\
-\frac{1}{3} (\omega_0)_{bc,k}^k \\
-\frac{1}{6} (\omega_0)_{bc,k}^k - \frac{1}{12} (\omega_0)_{k[b,c]}^k + \frac{1}{6} k \left( P_{k(b,c)]k}^k \omega_0 \right)_{bc}
\end{pmatrix}
\]
(92)
Here, $P$ is the conformal Schouten tensor of $g$, and indices are raised using $c$, regarded as section of $\Gamma(S^2 T^* M[2])$. 
Appendix E. Analogues of results for odd-dimensional projective structures

Though the geometry we consider in this article involves projective structures in an even dimension (correspondingly, conformal structures in an odd dimension), Section 8 indicates that there are many interesting odd-dimensional projective geometries that correspond to holonomy reductions of a normal projective tractor connection. With this in mind, we record here odd-dimensional versions of the general parity-dependent results stated in Section 8.

These analogues turn out to be more subtle than their even-dimensional counterparts; to formulate them, we briefly review a notion of order along the projective infinity. Given a projective structure \((M, \rho)\) of dimension \(n \geq 4\) with a holonomy reduction to \(SO(r + 1, s + 1),\) \(r + s + 1 = n,\) with nonempty zero locus \(M_0\), via Theorem 6.5 we may transfer the ambient coordinate \(\rho\) in Subsection 6.1 to \(M\). We may frame the role of \(\rho\) here invariantly: We say that a function (or tensor or tractor) \(f\) vanishes to order \(k\) along \(M_0\), where \(k\) is a nonnegative integer, if there is a smooth function (or tensor or tractor) \(f\) such that \(f = \sigma^k f\) for some defining function \(\sigma\) of \(M_0\) (see Subsection 4.6); this condition is independent of the choice of defining function \(\sigma\). A function (or tensor or tractor) \(f\) vanishes to infinite order along \(M_0\) if it vanishes to order \(k\) for all nonnegative integers \(k\). By construction, \(\rho\) is a defining function for \(M_0\) in \(M\).

With this language available, we can now state the even-dimensional analogue of the general solution to the (conformal) Dirichlet problem given in Theorem 6.7, which again is just a translation of the cited result in [11] into projective language via Theorem 6.5.

**Theorem E.1.** Let \((\Sigma, c)\) be a conformal structure of signature \((r, s)\) of signature \((r, s)\) of even dimension \(n := r + s \geq 4\). Then, there is (1) an \((n + 1)\)-dimensional projective structure \(\rho\) on a collar \(M\) of \(\Sigma\) and a tractor metric \(H\) such that the restriction \(H_{|M_0}\) induces the conformal structure \(c\) on \(M_0\) and \(\nabla H\) vanishes to order \(\frac{n}{2} - 1\) along \(M_0\), where \(\nabla\) is the normal tractor connection determined by \(\rho\).

Here the appropriate existence and uniqueness statement is more subtle that in the odd-dimensional case, so we avoid stating it precisely (and consequently drop the real-analyticity hypothesis), but roughly (a) for a general even-dimensional conformal structure \((M_0, c)\) there is an obstruction to the existence of a compatible triple \((M, \rho, H)\) for which \(\nabla H\) vanishes to order \(\frac{n}{2}\), and when the obstruction vanishes, one can find a structure \((M, \rho, H)\) for which \(\nabla H\) vanishes to infinite order along \(M_0\) (and so in the real-analytic case, for which \(H\) determines a bona fide holonomy reduction of the projective tractor connection of \(\rho\)), and (b) when solutions exist, in general they are not unique.

Theorem 6.10 extends to the odd-dimensional case as follows. (This statement remains true if we also discard the real-analyticity conditions from both the hypothesis and the conclusion.)

**Theorem E.2.** Let \((M, \rho)\) be a real-analytic projective structure of odd dimension \(n \geq 5\), \(T\) its standard tractor bundle, and \(\nabla^T\) its normal connection. Let \(H\) be a \(\nabla^T\)-parallel tractor metric such that the zero locus \(M_0 = \{x \in M : H_{AB}X^AX^B = 0\}\) is nonempty, and let \(c\) be the conformal structure that \(H\) induces there, \(\mathcal{T}_0 \subset T\) its standard conformal tractor bundle, and \(\nabla^T\) the normal conformal tractor connection.

Suppose \(\chi_0 \in \Gamma(\otimes^r \mathcal{T}_0)\) is a real-analytic \(\nabla^T\)-parallel conformal tractor tensor. Then, there is an open subset \(U \supset M_0\) of \(M\) and a real-analytic projective tractor tensor \(\chi \in \Gamma(\otimes^r \mathcal{T}_1|U)\) such that (a) \(\chi|_{M_0} = \chi_0\) and (b) \(\nabla^T \chi\) vanishes to order \(\frac{n-3}{2}\) along \(M_0\).

Appendix F. Some data for Example 7.1

In this appendix we give some explicit data for the Example 7.1 in Section 7.
In the given basis \((E_a)\), the nonzero bracket relations of \(s_m\) are generated by\(^{13}\)

\[(93)\]

\[
\begin{align*}
[E_1, E_3] &= -\frac{1}{2\sqrt{2}}(m + 1)(7m + 6)E_2 \\
[E_1, E_4] &= -\sqrt{10}(m + 1)E_2 \\
[E_1, E_5] &= \sqrt{10}E_1 + \frac{\sqrt{10}(m-1)^2}{3m}E_2 + \sqrt{2}(m + 1)(m - 2)E_3 - \frac{\sqrt{2}}{3\sqrt{2}}(3m^2 + 3m + 2)(m + 1)E_4 \\
[E_2, E_3] &= -\sqrt{10}E_2 \\
[E_2, E_4] &= -\frac{3}{\sqrt{2}}E_2 \\
[E_3, E_5] &= \frac{3}{2\sqrt{2}}E_1 - \frac{1}{2\sqrt{2}}(3m + 14)(m + 1)E_4 \\
[E_4, E_5] &= 2\sqrt{2}E_3 - \sqrt{10}(m + 1)E_4.
\end{align*}
\]

In this frame, the representative connection \(\nabla\) of the projective structure \(p\) has connection forms

\[
\begin{align*}
\omega^2 &= \frac{\sqrt{5}}{3\sqrt{2}}(3m^2 + 3m + 2)(m + 1)e^1 \\
\omega^4 &= -\frac{\sqrt{5}}{3\sqrt{2}}(3m^2 + 3m + 2)(m + 1)e^5 \\
\omega^5 &= e^4 \\
\omega^6 &= -e^5 \\
\omega^7 &= \frac{1}{\sqrt{2}}e^6 \\
\omega^8 &= \frac{1}{2\sqrt{2}}(m + 1)(7m + 6)e^1 - \frac{1}{\sqrt{2}}e^4 \\
\omega^9 &= -\frac{1}{2\sqrt{2}}(m + 1)(7m + 6)e^5 \\
\omega^{10} &= e^1 + \sqrt{2}e^3 \\
\omega^{11} &= \sqrt{2}e^5 \\
\omega^{12} &= -\sqrt{10}(m + 1)e^5 \\
\omega^{13} &= -e^1 \\
\omega^{14} &= -\sqrt{10}e^1 - \sqrt{2}e^3 \\
\omega^{15} &= -\frac{\sqrt{10}(m-1)^2}{3m}e^1 + \sqrt{10}e^2 - 2\sqrt{10}(m + 1)(m - 2)e^5 - \rho - (m + 1)(m - 2)d\rho \\
\omega^{16} &= -\sqrt{2}(m + 1)(m - 2)e^1 - \sqrt{2}e^4 \\
\omega^{17} &= -\sqrt{2}(m + 1)(m - 2)e^3 + \sqrt{10}e^4 + \frac{\sqrt{10}(m-1)^2}{3m}e^5 \\
\omega^{18} &= -\sqrt{10}e^5 \\
\omega^{19} &= -e^2 \\
\omega^{20} &= -(m + 1)(m - 2)e^5.
\end{align*}
\]

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\(^{13}\)Of course, \(s_m\) admits a basis for which the generating relations are much simpler; for example, for one basis \((E'_a)\), the nonzero bracket relations are generated by \([E'_1, E'_3] = E'_1, [E'_2, E'_3] = mE'_2, [E'_4, E'_5] = E'_4, [E'_5, E'_6] = E'_5\), and \([E'_4, E'_5] = E'_5\). The indicated basis \((E_a)\) is instead optimized first for simplicity of the data describing the geometries in this example.
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