Some Orthogonal Functions from Cauchy Determinants

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Abstract. We derive Legendre polynomials using Cauchy determinants with a generalization to power functions with real exponents greater than -1/2. We also provide a geometric formulation of Gram-Schmidt orthogonalization using the Hodge star operator.

1. Introduction

The well-known Legendre polynomials are known to be orthogonal functions for polynomial functions ([1], [2]). The orthogonality is well understood using Sturm-Liouville theory which provides a general framework for understanding orthogonal functions ([1]). On the other hand, there is also a well-known Gram-Schmidt process which provides a generic procedure for orthogonalizing a set of vectors in an inner product space ([6]). The use of Gram-Schmidt process for finding orthogonal functions is however considered as "possible but very cumbersome method of generating the Legendre polynomial" (p.645 [1]). In this note we show that for special sets of functions including polynomial functions, one can indeed apply Gram-Schmidt process fruitfully if the Gram matrix of the functions are Cauchy matrices.

The note is organized as the following. In Section 2, we recall basics for Gram-Schmidt orthogonalization and its non-recursive determinant formula. We also recall basics for Cauchy matrix. In Section 3, we apply the determinant formula to polynomial functions to derive Legendre polynomials. In Section 4, we extend the approach to power functions with real exponents greater than -1/2. In Section 5, we provide a geometric formulation of Gram-Schmidt orthogonalization using the Hodge star operator.

2. Determinant Formula and Cauchy Matrix

Let \((V, \langle \rangle)\) be a vector space with an inner product over real numbers and \(S = \{v_1, v_2, \cdots, v_n\}\) be a set of linearly independent vectors. An orthogonal set \(T = \{u_1, u_2, \cdots, u_n\}\) produced by the Gram-Schmidt process is characterized by two conditions:

\[ GS1: \langle u_i, u_j \rangle = 0, \quad 1 \leq i \neq j \leq n \]

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Because of the second condition, the first condition is equivalent to

\[ GS1': \langle u_k, v_i \rangle = 0, \ 1 \leq i < k \leq n \]

The two conditions determine the orthogonal set uniquely up to scaling
and give the well-known recursive formula [4]:

\[ u_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, u_i \rangle}{\langle u_i, u_i \rangle} u_i \]

The formula (1) is well explained in many books using orthogonal projection and allows efficient algorithm for numerical implementation ([6], [7]).

A less-known non-recursive formula in terms of determinant is given by [4]:

\[ \text{Theorem 1 (Determinant Formula).} \]

\[ u_k = \frac{1}{d_{k,k}} \begin{vmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_k, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_k, v_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, v_{k-1} \rangle & \langle v_2, v_{k-1} \rangle & \cdots & \langle v_k, v_{k-1} \rangle \\ v_1 & v_2 & \cdots & v_k \end{vmatrix} \]

with \( d_{1,1} = 1 \) and \( d_{k,k} \) is the cofactor of \( v_k \) so that the coefficient of \( v_k \) is 1.

A reference of the formula can be found in the book [4] on the theory of
matrices by Gantmacher. Gantmacher also indicated the use of the formula
for orthogonal functions. However the matrix shown in the book is not a
Cauchy matrix (see below) and details were not provided.

\( d_{k,k} \) is the determinant of Gram matrix associated with the vectors \( v_1, v_2, \ldots, v_{k-1} \).

Denote the cofactor of \( v_i \) as \( d_{k,i} \), formula (2) can be rewritten as

\[ u_k = \sum_{i=1}^{k} (-1)^{i+k} \frac{d_{k,i}}{d_{k,k}} v_i \]

\[ \text{Example 2.} \text{ Consider polynomial functions } f_k(x) = x^{k-1}, k = 1, 2, \ldots, \text{ defined over the interval } [0, 1]. \text{ The } L^2 \text{ inner product is defined as} \]

\[ \langle f, g \rangle = \int_{0}^{1} f(x)g(x)dx \]

\[ \text{The inner product for a pair of polynomial functions } f_i \text{ and } f_j:\]

\[ \langle f_i, f_j \rangle = \int_{0}^{1} x^{i-1}x^{j-1}dx = \int_{0}^{1} x^{i+j-2}dx = \frac{1}{i+j-1} \]

https://en.wikipedia.org/wiki/Gram-Schmidt_process
Therefore the cofactor \( d_{k,k} \) has the following form:

\[
d_{k,k} = \begin{vmatrix}
\frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_k} \\
\frac{1}{x_2} & \frac{1}{x_3} & \cdots & \frac{1}{x_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_k} & \frac{1}{x_k} & \cdots & \frac{1}{x_k}
\end{vmatrix}
\]

The matrix inside is called Hilbert matrix and has a determinant\(^2\)

\[
d_{k,k} = \frac{c_{k-1}^4}{c_{2k-2}}
\]

where \( c_n = \prod_{i=1}^{n-1} i! \)

Other cofactors look a little bit different, but all belong to a class of matrices called Cauchy matrices.

**Definition 3 (Cauchy Matrix).** Let \( x = \{x_1, x_2, \ldots, x_m\} \) and \( y = \{y_1, y_2, \ldots, y_n\} \), we use \( x \oplus y \) to denote the associated Cauchy matrix

\[
x \oplus y = \begin{pmatrix}
\frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\
\frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_m+y_1} & \frac{1}{x_m+y_2} & \cdots & \frac{1}{x_m+y_n}
\end{pmatrix}
\]

The determinant of a Cauchy matrix has a nice expression given by

\[
(3) \quad \det(x \oplus y) = \frac{\prod_{i=2}^{m} \prod_{j=1}^{i-1} (x_i - x_j)(y_i - y_j)}{\prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j)}
\]

The formula can be proved by row and column operations\(^4\). Intuitively whenever there is an \( x_i = x_j \) or \( y_i = y_j \), the Cauchy matrix will have two identical columns or rows, its determinant must be 0. Therefore \( (x_i - x_j) \) and \( (y_i - y_j) \) must be factors of the determinant. In later sections we’ll use this formula to evaluate various cofactors and derive orthogonal functions.

3. **Legendre Polynomials**

We continue the Example\(^2\) and derive the orthogonal functions for polynomials \( \{1, x, \ldots, x^{n-1}\} \) which are expected to be (shifted) Legendre polynomials over \([0, 1]\)

For \( d_{n,n} \), we set \( x = \{1, 2, \ldots, n - 1\} \) and \( y = \{0, 1, \ldots, n - 2\} \), then

\[
d_{n,n} = \frac{\prod_{i=2}^{n-1} \prod_{j=1}^{i-1} (i - j)(i - 1) - (j - 1))}{\prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (i + j - 1)} = \frac{\prod_{i=2}^{n-1} ((i - 1)!)^2}{\prod_{i=1}^{n-1} (i + n - 2)! (i - 1)!} = \frac{\prod_{i=2}^{n-1} (i!)^2}{\prod_{i=n-1}^{2n-3} i!} = \frac{\prod_{i=1}^{n-2} (i!)^4}{\prod_{i=1}^{2n-3} i!} = \frac{c_{n-1}^4}{c_{2n-2}}
\]

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\(^2\)https://en.wikipedia.org/wiki/Hilbert_matrix

\(^4\)https://proofwiki.org/wiki/Value_of_Cauchy_Determinant
For other cofactor \( d_{n,m} \) of \( f_m \) in the determinant formula \(^2\), one has
\[
d_{n,m} = |\{1, 2, \ldots, n-1\} \oplus \{0, 1, \ldots, m-2, m, \ldots, n-1\}| = \prod_{i=2}^{n-1} \prod_{j=1}^{n-1} (i-j)(y_i - y_j) = \prod_{i=2}^{n-1} (i-1)!, \prod_{j=1}^{n-1} (y_i - y_j) \prod_{j=1}^{n-1} (n+y_j)! \prod_{i=1}^{n-1} (i+y_i)! = c_{n-1} \prod_{i=2}^{n-1} (i-1)! \prod_{j=1}^{n-1} (y_i - y_j) \prod_{j=1}^{n-1} (y_i - y_j) \prod_{j=1}^{n-1} (n+y_j)! \prod_{i=1}^{n-1} (i+y_i)! = c^2_{n-1} c_n (n+m-2)! \prod_{i=2}^{m-1} (i-1)! \prod_{i=m}^{n-1} i! (m-1)! = (n+m-2)! (n+m-2)! (m-1)! = (-1)^n c^2_{n-1} c^2_n (n-1) (m-1) = c_{2n-1} (n-1)! (m-1)! = c_{2n-1} (n-1)! (m-1)! = (n-1)! (m-1)! \]
Therefore the \( n \)-th orthogonal function \( p_n \) for polynomial functions \( f_i \)'s is
\[
p_n = (n-1)!^2 \sum_{m=1}^{n} (-1)^{n-1} \binom{n-1}{m-1} \binom{-n}{m-1} f_m
\]
Omitting the factor before the summation, we get a set of orthogonal functions \( q_n \) with integral coefficients:
\[
q_n = \sum_{m=1}^{n} (-1)^{n-1} \binom{n-1}{m-1} \binom{-n}{m-1} f_m
\]
The factor \((-1)^{n-1}\) ensures the coefficient of \( f_n \) is positive. First five functions are:
\[
q_1 = f_1 \\
q_2 = 2f_2 - f_1 \\
q_3 = 6f_3 - 6f_2 + f_1 \\
q_4 = 20f_4 - 30f_3 + 12f_2 - f_1 \\
q_5 = 70f_5 - 140f_4 + 90f_3 - 20f_2 + f_1
\]
These polynomials correspond exactly to the 'shifted' Legendre polynomials defined over \([0, 1]\). A quick calculation shows that the sum of the coefficients of each above polynomial is 1. This is true for all \( q_n \):

**Proposition 4.** The sum of the coefficients of \( f_n \) in \( q_n \) is 1, i.e.
\[
q_n(1) = 1
\]

**Proof.** Consider expansion
\[
\frac{1}{1 + x} = (1 + x)^{n-1}(1 + x)^{-n} = \left( \sum_{m=1}^{n-1} \binom{n-1}{m-1} x^{n-m} \right) \left( \sum_{k=0}^{\infty} \binom{-n}{k} x^k \right)
\]

\(^2\)https://mathworld.wolfram.com/LegendrePolynomial.html
\((-1)^{n-1}q_n(1)\) is the coefficient of \(x^{n-1}\) in expansion for \(\frac{1}{1+x}\) which is \((-1)^n\).

We show this simple observation on coefficients can provide a way to find the generating function of the Legendre polynomials:

\[
\sum_{n=1}^{+\infty} q_n(x)t^{n-1} = \sum_{n=1}^{+\infty} \sum_{m=1}^{n} (-1)^{n-1} \binom{n-1}{m-1} \binom{-n}{m-1} x^{m-1} t^{n-1} \\
= \sum_{n=0}^{+\infty} \sum_{m=0}^{n} (-1)^n \binom{n}{m} \binom{-n-1}{m} x^m t^n \\
= \sum_{m=0}^{+\infty} \sum_{n=m}^{+\infty} (-1)^n \binom{n}{m} \binom{-n-1}{m} x^m t^{n+m} \\
= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (-1)^n \binom{n+2m}{m} x^m t^{n+m} \\
= \sum_{m=0}^{+\infty} (\frac{2m)!}{m!m!} (xt)^m \sum_{n=0}^{+\infty} (-1)^n \binom{n+2m}{m} t^n \\
= \sum_{m=0}^{+\infty} (\frac{2m)!}{m!m!} (xt)^m \frac{1}{(1+t)^{2m+1}} \\
\]

In the last step we used the binomial expansion for negative integral exponent \(-(2m+1)\).

To evaluate the last summation we set \(x=1\) at both sides and use the Proposition 4:

\[
\sum_{n=1}^{+\infty} q_n(1)t^{n-1} = \sum_{m=0}^{+\infty} \frac{(2m)!}{m!m!} t^m \frac{1}{(1+t)^{2m+1}} \\
\sum_{n=1}^{+\infty} t^{n-1} = \sum_{m=0}^{+\infty} \frac{(2m)!}{m!m!} t^m \frac{1}{(1+t)^{2m+1}} \\
\frac{1+t}{1-t} = \sum_{m=0}^{+\infty} \frac{(2m)!}{m!m!} \left(\frac{t}{(1+t)^2}\right)^m 
\]
Setting $s = \frac{r}{(1+t)^2}$, we solve $t = \frac{1-\sqrt{1-4s}}{1+\sqrt{1-4s}}$ and thus

$$t = 1 - \sqrt{1 - 4s}$$

and thus

$$(1 - 4s)^{-\frac{1}{2}} = \sum_{m=0}^{+\infty} \frac{(2m)!}{m!m!}s^m.$$ 

The equation (5) can be obviously proved using binomial expansion for $-1/2$ exponent. The above alternative derivation is based on the observation on the sum of coefficients. Apply the equation (5) to the generating function with $s = \frac{xt}{(1+t)^2}$, we get the well-known generating function for Legendre polynomials ([2]):

**Proposition 5.**

$$\sum_{n=1}^{+\infty} q_n(x)t^{n-1} = ((1 + t)^2 - 4xt)^{-\frac{1}{2}}$$

4. **Power functions with Real Exponents greater than -1/2**

In this section we derive orthogonal basis for general power functions $f^\alpha_k(x) = x^{k-1+\alpha}$, $k = 1, 2, \cdots$, defined over the interval $[0, 1]$, for some $\alpha > -\frac{1}{2}$ so that the $L^2$ norm is finite. Later we’ll look at a specific case of $\alpha = 1/2$ which corresponds to exponent decay functions over $[0, +\infty]$.

The inner product for a pair of power functions $f^\alpha_i$ and $f^\alpha_j$:

$$\langle f^\alpha_i, f^\alpha_j \rangle = \int_0^1 x^{i-1+\alpha}x^{j-1+\alpha}dx = \int_0^1 x^{i+j-2+2\alpha}dx = \frac{1}{i + j - 1 + 2\alpha}$$

The cofactors in the determinant formula (2) for functions $f_i$’s can be again computed using Cauchy matrix (3). For $d_{n,n}$, we have $x = \{1 + \alpha, 2 + \alpha, \cdots, n - 1 + \alpha\}$ and $y = \{\alpha, 1 + \alpha, \cdots, n - 2 + \alpha\}$

$$d_{n,n} = \prod_{i=2}^{n-1} \prod_{j=1}^{i-1} (i - j)((i - 1) - (j - 1)) \cdot \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (i + j - 1 + 2\alpha) = \frac{\prod_{i=2}^{n-1} ((i - 1)!)^2}{\prod_{i=1}^{n-1} \Gamma((i+n-1+2\alpha))}$$

$$= c_{n-1}^2 \prod_{i=1}^{n-1} \frac{\Gamma(i + 2\alpha)}{\Gamma(i + n - 1 + 2\alpha)}$$
where \( \Gamma \) is the Gamma function. For other cofactor \( d_{n,m} \) of \( f_m^\alpha \) in the determinant formula (2), one has

\[
d_{n,m} = \left\{ 1 + \alpha, \cdots, n - 1 + \alpha \right\} \oplus \left\{ \alpha, \cdots, m - 2 + \alpha, m + \alpha, \cdots, n - 1 + \alpha \right\}
\]

\[
d_{n,m} = \frac{\prod_{i=2}^{n-1} \prod_{j=1}^{i-1} (i-j)(y_i-y_j)}{\prod_{i=2}^{n-1} (i-1)! \prod_{j=1}^{n-1} (i+y_j+\alpha)} = \frac{\prod_{i=2}^{m-1} (i-1)! \prod_{j=1}^{m-1} (i+y_j+\alpha)}{\prod_{j=1}^{n-1} \Gamma(i+y_j+\alpha)}
\]

\[
c_{n-1}(\prod_{i=2}^{n-1} \prod_{j=1}^{i-1} (y_i-y_j)) = \frac{\prod_{i=2}^{n-1} \Gamma(n+m-1+2\alpha)(i-1)}{\prod_{i=2}^{n-1} \Gamma(m+2\alpha)}
\]

\[
g_n = \frac{\Gamma(n + 2\alpha)}{\Gamma(2n - 1 + 2\alpha)} \sum_{m=1}^{n} (-1)^{m+n} \binom{n-1}{m-1} \frac{\Gamma(n + m - 1 + 2\alpha)}{\Gamma(n+2\alpha)} f_m^\alpha
\]

Therefore the \( n \)-th orthogonal function \( g_n \) for power functions \( f_i^\alpha \)'s is

\[
g_n = \frac{\Gamma(n + 2\alpha)}{\Gamma(2n - 1 + 2\alpha)} \sum_{m=1}^{n} (-1)^{m+n} \binom{n-1}{m-1} \frac{\Gamma(n + m - 1 + 2\alpha)}{\Gamma(n+2\alpha)} f_m^\alpha
\]

To align with the Legendre polynomials where \( \alpha = 0 \), we scale the coefficients to get the following orthogonal functions:

\[
h_n = \sum_{m=1}^{n} (-1)^{m+n} \binom{n-1}{m-1} \frac{\Gamma(n + m - 1 + 2\alpha)}{\Gamma(n+2\alpha)} f_m^\alpha
\]

\[
\text{Example 6.} \quad \text{The case } \alpha = 1/2 \text{ corresponds to exponential functions since the mapping } x^{k+\frac{1}{2}} \mapsto e^{-kx} \text{ preserves the } L^2 \text{ inner products where the inner product for exponential functions is defined over } (0, +\infty)
\]

\[
\langle f, g \rangle = \int_{0}^{+\infty} f(x)g(x)dx
\]

The orthogonal functions obtained for \( \alpha = 1/2 \) are orthogonal functions for exponential function \( e_k(x) = e^{-kx} \) defined over \( (0, +\infty) \) for \( k = 1, 2, \cdots \):

\[
h_n = \sum_{m=1}^{n} (-1)^{m+n} \binom{n-1}{m-1} \frac{\Gamma(n + m - 1 + 1)}{\Gamma(n+1)} e_m
\]

\[
h_n = \sum_{m=1}^{n} (-1)^{m+n} \binom{n-1}{m-1} \binom{-n}{m} e_m
\]
Note that all the coefficients are integral. The first five functions are
\[ h_1 = e_1 \]
\[ h_2 = 3e_2 - 2e_1 \]
\[ h_3 = 10e_3 - 12e_2 + 3e_1 \]
\[ h_4 = 35e_4 - 60e_3 + 30e_2 - 4e_1 \]
\[ h_5 = 126e_5 - 280e_4 + 210e_3 - 60e_2 + 5e_1 \]

We also have the following observation on the sum of the coefficients:

**Proposition 7.** The sum of the coefficients of \( e_n \) in \( h_n \) is 1, i.e.

\[ h_n(0) = 1 \]

**Proof.** Consider expansion

\[
\frac{1}{1 + x} = (1 + x)^{n-1}(1 + x)^{-n} = \left( \sum_{m=1}^{n} \binom{n-1}{m-1} x^{n-m} \right) \left( \sum_{k=0}^{\infty} \binom{-n}{k} x^k \right)
\]

\[ (-1)^n h_n(0) \text{ is the coefficient of } x^n \text{ in expansion for } \frac{1}{1+x} \text{ which is } (-1)^n. \quad \square \]

5. **A Geometric Formulation using the Hodge Star Operator**

Hodge star operator is introduced by Hodge in studying harmonic integrals on manifolds ([5]). The operator provides a duality in differential forms and subsequent de Rham cohomology on manifolds ([8]). The operator has become a basic jargon in modern differential geometry and mathematical physics. For example, it is used to define self-dual and anti-self-dual connections in gauge theory ([3]).

We first recall some basic properties of the Hodge star operator from Warner’s book ([8]). Let \((W, \langle \rangle)\) be an \(m\)-dimensional oriented vector space with an inner product that naturally extends to its exterior algebra \(\wedge W\) by setting

\[
\langle w_1 \wedge w_2 \wedge \cdots \wedge w_p, w'_1 \wedge w'_2 \wedge \cdots \wedge w'_p \rangle = \det(\langle w_i, w'_j \rangle)_{1 \leq i,j \leq p}
\]

on \(p\)-form. The Hodge star operator \(* : \wedge^p W \to \wedge^{m-p} W\) is determined by

\[
\langle *\alpha, \beta \rangle \Omega = \alpha \wedge \beta, \forall \alpha \in \wedge^p W, \beta \in \wedge^{m-p} W
\]

where \(\Omega \in \wedge^m W\) is the volume form associated with the inner product and orientation. The Hodge star operator has the following properties:

1. \(*1 = \Omega\)
2. \(* * \alpha = (-1)^{p(m-p)}, \forall \alpha \in \wedge^p W\)
3. \(\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle, \forall \alpha, \beta \in \wedge^p W\)
4. \(\langle \alpha, \beta \rangle = *\langle \beta \wedge *\alpha \rangle = *\langle \alpha \wedge *\beta \rangle, \forall \alpha, \beta \in \wedge^p W\)

Back to the set up in the Gram-Schmidt process, for \(1 \leq k \leq n\) we denote \(V_k\) be the subspace of \(V\) spanned by \(v_1, v_2, \cdots, v_k\) with induced inner product
and orientation determined by $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, and define $u'_k \in V_k$ using the Hodge star operator
\begin{equation}
(10) \quad u'_k = *(v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1})
\end{equation}

**Lemma 8.**
\[ \langle u'_k, v_i \rangle = 0, 1 \leq i < k \leq n \]

**Proof.** Indeed by the properties of the Hodge star operator we have
\[ \langle u'_k, v_i \rangle = \langle *((v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}) \wedge v_i) \rangle = *((v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}) \wedge v_i) \]
which is 0 for $1 \leq i < k$ since $v_i$ appears twice in the wedge product. 
\[ \square \]

From the lemma we have $u'_k \in V_k \setminus V_{k-1}$, therefore $u'_k$ is linearly independent to $u'_1, u'_2, \ldots, u'_{k-1}$ and span\{u'_1, u'_2, \ldots, u'_k\} = $V_k$. This proves the main result:

**Theorem 9.** The set $T' = \{u'_1, u'_2, \ldots, u'_n\}$ is an orthogonal set satisfying Gram-Schmidt conditions $GS1'$ and $GS2$.

To see how $u'_k$ is related to the $u_k$ obtained using the recursive formula (2), we notice that
\[ \langle u'_k, u'_k \rangle = \langle *((v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}) \wedge v_k) \rangle = \frac{\det(\langle v_i, v_j \rangle)}{d_{k,k}} \]
\[ = \frac{\sqrt{\det(\langle v_i, v_j \rangle)}}{d_{k,k+1}} = \sqrt{d_{k+1,k+1}} \]

With these results we have

**Proposition 10.**
\[ u'_k = \frac{d_{k,k}}{\sqrt{d_{k+1,k+1}}} u_k \]

**Proof.** Since both sets $T$ and $T'$ satisfy Gram-Schmidt conditions, $u'_k = \lambda_k u_k$ for some scalar $\lambda_k$. Plug this into the equation
\[ \langle u_k, u_k \rangle = \langle u_k, v_k \rangle \]

We have
\[ \langle u'_k, u'_k \rangle = \langle \lambda_k u'_k, \lambda_k u'_k \rangle \]
\[ \lambda_k = \frac{\langle u'_k, u'_k \rangle}{\langle u'_k, v_k \rangle} = \frac{d_{k,k}}{\sqrt{d_{k+1,k+1}}} \]

We observe that the set $T'$ is not homogeneous in the sense that if the original set $S$ is scaled by a common scalar $c$, vectors in $T'$ are scaled by powers of $c$ with different exponent for different vector. Note that the vectors in $T$ are all scaled by $c$. To make a set obtained using the Hodge star operator homogeneous, we consider the following variant of $u'_k$:
Definition 11. Define \( u''_k \in V_k \) so that
\[
\langle u''_k, v \rangle = \frac{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v}{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_k}, \quad \forall \ v \in V_k
\]
The RHS of the equation (11) is well-defined since \( \dim \Lambda^k V_k = 1 \).

Although \( u''_k \) are not explicitly defined using the Hodge star operator, they are intrinsically linked to the Hodge star operator. Indeed \( \forall v \in V_k \)
\[
\langle u''_k, v \rangle = \frac{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v}{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_k} = \frac{\langle u'_k, v \rangle}{\sqrt{\det((v_i, v_j)_{1 \leq i, j \leq k})}} = \frac{\langle u'_k, v \rangle}{\sqrt{d_{k+1,k+1}}}
\]
Thus
\[
u''_k = \frac{u'_k}{d_{k+1,k+1}} = \frac{d_{k,k}}{d_{k+1,k+1}} u_k
\]

We therefore have

**Theorem 12.** The set \( T''_n = \{ u''_1, u''_2, \ldots, u''_n \} \) is also an orthogonal set satisfying Gram-Schmidt conditions GS1' and GS2.

Note that the set \( T''_n \) is homogeneous: if the original set \( S \) is scaled by a common scalar \( c \), vectors in \( T''_n \) are all scaled by \( 1/c \).

We conclude by using \( u''_k \) to derive the determinant formula (2). Consider the inner product of \( u''_k \) with any vector \( v \in V_k \):
\[
\langle u''_k, v \rangle = \frac{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v}{v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_k} = \frac{\langle v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v, v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_k \rangle}{\langle v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v, v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_k \rangle}
\]
\[
\begin{vmatrix}
\langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_k, v_1 \rangle \\
\langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_k, v_2 \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v_1, v_{k-1} \rangle & \langle v_2, v_{k-1} \rangle & \cdots & \langle v_k, v_{k-1} \rangle \\
\langle v_1, v \rangle & \langle v_2, v \rangle & \cdots & \langle v_k, v \rangle
\end{vmatrix}
\]
\[
\text{det}((v_i, v_j)_{1 \leq i, j \leq k})
\]
Therefore
\[
u''_k = \frac{1}{d_{k+1,k+1}} \begin{vmatrix}
\langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_k, v_1 \rangle \\
\langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_k, v_2 \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle v_1, v_{k-1} \rangle & \langle v_2, v_{k-1} \rangle & \cdots & \langle v_k, v_{k-1} \rangle \\
v_1 & v_2 & \cdots & v_k
\end{vmatrix}
\]
These gives the formula \[ (2) \] since

\[
u_k = \frac{d_{k+1,k+1}}{d_{k,k}} u_k''
\]

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REFERENCES

[1] G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, 6th Edition, Elsevier Academic Press 2005.

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, 1st English Edition, Interscience Publishers, INC., New York 1937.

[3] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs, Clarendon Press, Oxford 1990.

[4] F. Gantmacher, *Theory of Matrices*, AMS Chelsea Publishing, 1959.

[5] W.V.D. Hodge, *The Theory and Application of Harmonic Integrals*, Cambridge Univ. Press 1952.

[6] S.J. Leon, A. Björck, W. Gander, *Gram-Schmidt orthogonalization: 100 years and more*, Numerical Linear Algebra with Applications (Volume 20 Issue 3), Wiley 2013.

[7] G. Strang, *Linear Algebra and Its Applications*, 4th Edition, Cengage Learning, 2005.

[8] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics (Volume 94), Springer 1983.

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