Local controllability of 1D Schrödinger equations with bilinear control and minimal time

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Abstract

We consider a linear Schrödinger equation, on a bounded interval, with bilinear control.

In [10], Beauchard and Laurent prove that, under an appropriate non degeneracy assumption, this system is controllable, locally around the ground state, in arbitrary time. In [15], Coron proves that a positive minimal time is required for this controllability, on a particular degenerate example.

In this article, we propose a general context for the local controllability to hold in large time, but not in small time. The existence of a positive minimal time is closely related to the behaviour of the second order term, in the power series expansion of the solution.

1 Introduction

1.1 The problem

Let us consider the 1D Schrödinger equation

\[
\begin{aligned}
    i\partial_t \psi(t, x) &= -\partial_x^2 \psi(t, x) - u(t)\mu(x)\psi(t, x), \quad (t, x) \in \mathbb{R} \times (0, 1), \\
    \psi(t, 0) &= \psi(t, 1) = 0, \quad t \in \mathbb{R}.
\end{aligned}
\] (1.1)

Such an equation arises in the modelization of a quantum particle, in an infinite square potential well, in a uniform electric field with amplitude \( u(t) \). The function \( \mu : (0, 1) \to \mathbb{R} \) is the dipolar moment of the particle. The system (1.1) is a bilinear control system in which the state is the wave function \( \psi \), with \( \|\psi(t)\|_{L^2(0,1)} = 1 \), \( \forall t \in \mathbb{R} \) and the control is the real valued function \( u \).

In this article, we study the minimal time required for the local controllability of (1.1) around the ground state. Before going into details, let us introduce several notations. The operator \( A \) is defined by

\[
D(A) := H^2 \cap H_0^1((0,1), \mathbb{C}), \quad A\varphi := -\frac{d^2\varphi}{dx^2}.
\] (1.2)

Its eigenvalues and eigenvectors are

\[
\lambda_k := (k\pi)^2, \quad \varphi_k(x) := \sqrt{2}\sin(k\pi x), \forall k \in \mathbb{N}^*.
\] (1.3)

The family \( (\varphi_k)_{k \in \mathbb{N}^*} \) is an orthonormal basis of \( L^2((0, 1), \mathbb{C}) \) and

\[
\psi_k(t, x) := \varphi_k(x)e^{-i\lambda_k t}, \forall k \in \mathbb{N}^*
\]
is a solution of (1.1) with \( u \equiv 0 \) called eigenstate, or ground state, when \( k = 1 \). We denote by \( S \) the unit \( L^2((0,1),\mathbb{C}) \)-sphere. In this article, we consider two types of initial conditions for (1.1): the ground state

\[
\psi(0,x) = \varphi_1(x), \quad x \in (0,1),
\]

or an arbitrary one

\[
\psi(0,x) = \psi_0(x), \quad x \in (0,1).
\]

Now, let us define the concept of local controllability used in this article.

**Definition 1** Let \( T > 0 \), \( X \) and \( Y \) be normed spaces such that \( X \subset L^2((0,1),\mathbb{C}) \) and \( Y \subset L^2((0,T),\mathbb{R}) \). The system (1.1) is controllable in \( X \), locally around the ground state, with controls in \( Y \), in time \( T \), if, for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, for every \( \psi_f \in S \cap X \) with \( \| \psi_f - \psi_1(T) \|_X < \delta \), there exists \( u \in Y \) with \( \| u \|_Y < \epsilon \) such that the solution of the Cauchy problem (1.1)–(1.4) satisfies \( \psi(T) = \psi_f \).

In particular, this definition requires that arbitrarily small motions may be done with arbitrarily small controls.

### 1.2 A first previous result

First, let us introduce the normed spaces

\[
H^s_{(0)}((0,1),\mathbb{C}) := D(A^{s/2}), \quad \| \varphi \|_{H^s_{(0)}} := \left( \sum_{k=1}^{\infty} |k^s \langle \varphi, \varphi_k \rangle|^2 \right)^{1/2}, \quad \forall s > 0. \tag{1.6}
\]

The following result, proved in [10], emphasizes that the local controllability holds in any positive time when the dipolar moment \( \mu \) satisfies an appropriate non-degeneracy assumption.

**Theorem 1** Let \( T > 0 \) and \( \mu \in H^3((0,1),\mathbb{R}) \) be such that

\[
\exists c > 0 \text{ such that } \frac{c}{k^3} \leq |\langle \mu \varphi_1, \varphi_k \rangle|, \forall k \in \mathbb{N}^*.
\]

\[
\tag{1.7}
\]

There exists \( \delta > 0 \) and a \( C^1 \) map \( \Gamma: \Omega_T \to L^2((0,T),\mathbb{R}) \) where

\[
\Omega_T := \{ \psi_f \in S \cap H^0_{(0)}((0,1),\mathbb{C}); \| \psi_f - \psi_1(T) \|_{H^3} < \delta \},
\]

such that, \( \Gamma(\psi_1(T)) = 0 \) and for every \( \psi_f \in \Omega_T \), the solution of the Cauchy problem (1.1)–(1.4) with control \( u := \Gamma(\psi_f) \) satisfies \( \psi(T) = \psi_f \).

First, let us remark that the assumption (1.7) holds for example with \( \mu(x) = x^2 \). Actually, it holds generically in \( H^3((0,1),\mathbb{R}) \) (see [10] Proposition 16). Indeed, for \( \mu \in H^3((0,1),\mathbb{R}) \), 3 integrations by part and the Riemann-Lebesgue Lemma prove that

\[
\langle \mu \varphi_1, \varphi_k \rangle = 2 \int_0^1 \mu(x) \sin(\pi x) \sin(k \pi x) dx = \frac{4(-1)^{k+1} \mu'(1) - \mu'(0))}{k^3 \pi^2} + \frac{1}{k^3} \quad \text{as } k \to +\infty.
\]

\[
\tag{1.8}
\]

In particular, a necessary (but not sufficient) condition on \( \mu \) for (1.7) to be satisfied is \( \mu'(1) \pm \mu'(0) \neq 0 \).

Note that the function spaces in Theorem 1 are optimal. Indeed, they are the same as for the well posedness of the Cauchy problem (1.1)–(1.4) (see Proposition 1).
Finally, let us summarize the proof of Theorem 1 in [10]. This proof relies on the linear test (see [10] Chapter 3.1), the inverse mapping theorem and a regularizing effect. In particular, the assumption (1.4) is necessary for the linearized system to be controllable in $H^3_0((0,1),\mathbb{C})$ with controls in $L^2((0,T),\mathbb{R})$. When one of the coefficients $\langle \mu \varphi_1, \varphi_k \rangle$ vanishes, then the linearized system is not controllable anymore and the strategy of [10] fails.

1.3 A second previous result

The first article in which a positive minimal time is proved, for the local controllability of system similar to (1.1), is [15]. In this reference, Coron considers the control system

$$\begin{aligned}
\left\{ \begin{array}{ll}
\Delta \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)(x - 1/2)\psi(t,x), & (t,x) \in \mathbb{R} \times (0,1), \\
\psi(t,0) = \psi(t,1) = 0, & t \in \mathbb{R}, \\
s'(t) = u(t), & t \in \mathbb{R},
\end{array} \right.
\end{aligned} \quad (1.9)$$

where the state is $(\psi,s,d)$ and the control is the real valued function $u$. This system represents a quantum particle in a moving box: $u,s,d$ are the acceleration, the speed and the position of the box.

Note that, here, the relation (1.7) is not satisfied:

$$\langle (x-1/2)\varphi_1, \varphi_k \rangle = \begin{cases} 0 & \text{if } k \text{ is odd}, \\
\frac{8k}{\pi^2(k^2-1)} & \text{if } k \text{ is even},
\end{cases}$$

thus Theorem 1 does not apply.

On one hand, it is proved in [9] that this system is controllable in $H^3_0((0,1),\mathbb{C}) \times \mathbb{R} \times \mathbb{R}$, locally around the ground state $(\psi = \psi_1, s = 0, d = 0)$, with controls $u \in L^\infty((0,T),\mathbb{R})$, in time $T$ large enough.

On the other hand, Coron proved in [15] that this local controllability does not hold in arbitrary time: contrary to Theorem 1, a positive minimal time is required for the local controllability. Precisely, Coron proved the following statement.

**Theorem 2** There exists $\epsilon > 0$ such that, for every $d \neq 0$ and $u \in L^2((0,\epsilon),\mathbb{R})$ satisfying $|u(t)| < \epsilon, \forall t \in (0,\epsilon)$, the solution $(\psi,s,d) \in C^0([0,\epsilon],H^3_0((0,1),\mathbb{C})) \times C^1([0,\epsilon],\mathbb{R}) \times C^1([0,\epsilon],\mathbb{R})$ of (1.9) such that $(\psi,s,d)(0) = (\psi_1(0),0,0)$ satisfies $(\psi,s,d)(\epsilon) \neq (\psi_1(\epsilon),0,d)$.

The goal of this article is to go further in this analysis:

1. we propose a general context for the minimal time to be positive (in particular, the variables $s$ and $d$ are not required anymore in the state),

2. we propose a sufficient condition for the local controllability to hold in large time; this assumption is compatible with the previous context and weaker than (1.7),

3. we work in an optimal functional frame, for instance, our non controllability result requires $u$ small in $H^{-1}$-norm, not in $L^\infty$-norm as in Theorem 2

4. we perform a first step toward the characterization of the minimal time.

1.4 Main results of this article

The first result of this article is the following one.

**Theorem 3** Let $K \in \mathbb{N}^*$, $\mu \in H^3((0,1),\mathbb{R})$ be such that

$$\langle \mu \varphi_1, \varphi_K \rangle = 0 \quad \text{and} \quad A_K := \langle (\mu')^2 \varphi_1, \varphi_K \rangle \neq 0,$$

(1.10)
and \( \alpha_K \in \{-1, +1\} \) be defined by
\[
\alpha_K := \text{sign}(A_K).
\] (1.11)

There exists \( T^*_K > 0 \) such that, for every \( T < T^*_K \), there exists \( \epsilon > 0 \) such that, for every \( u \in L^2((0, T), \mathbb{R}) \) with
\[
\left( \int_0^T \left( \int_0^t |u(\tau)|^2 \, d\tau \right)^{1/2} \, dt \right)^{1/2} < \epsilon
\] (1.12)
the solution of (1.1) satisfies \( \psi(T) \neq [\sqrt{1 - \delta^2} \varphi_1 + i \alpha_K \delta \varphi_K] e^{-i \lambda_1 T} \) for every \( \delta > 0 \).

First, let us remark that the assumption (1.10) holds, for example, with \( \mu(x) = (x - 1/2) \) and \( K = 1 \). In particular, Theorem 3 applies to the particular case studied by Coron in [15].

Thus, the variables \((s, d)\) are not required in the state for the minimal time to be positive. Moreover, the control \( u \) does not need to be small in \( L^\infty \) as in Theorem 2; a smallness assumption in \( H^{-1}(0, T) \) is sufficient.

Note that the validity of the same result without the assumption \( 'A_K \neq 0' \) is an open problem (see remark 2 for technical reasons). A possible (but not optimal) value of \( T^*_K \) is given in (3.18). The proof of Theorem 3 relies on an expansion of the solution to the second order.

The second result of this article is the following one.

**Theorem 4** Let \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that
\[
\mu'(0) \pm \mu'(1) \neq 0.
\] (1.13)

Then, the system (1.1) is controllable in \( H^3_{(0)}((0, 1), \mathbb{C}), \) locally around the ground state, with controls \( u \in L^2((0, T), \mathbb{R}) \), in large enough time \( T \).

A direct consequence of Theorems 3 and 4 is the following result.

**Theorem 5** Let \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that (1.10) and (1.13) hold for some \( K \in \mathbb{N}^* \). Then, there exists \( T_{min} > 0 \) such that the controllability in \( H^3_{(0)}((0, 1), \mathbb{C}), \) locally around the ground state, with controls in \( L^2((0, T), \mathbb{R}) \) does not hold when \( T < T_{min} \), and holds when \( T > T_{min} \).

First, let us remark that the assumption (1.13) is weaker than (1.7) and that the assumptions (1.10) and (1.13) are compatible: consider, for instance \( \mu(x) := x^2 - \langle x^2 \varphi_1, \varphi_2 \rangle \varphi_2 / \varphi_1. \)

Note that an explicit upper bound \( T_1 \) for the minimal time \( T_{min} \) is proposed in the proof (see (1.13)).

Let us emphasize that, when \( \mu'(0) = \mu'(1) = 0 \), then, the appropriate functional frame stops to be \( (\psi \in H^k_{(0)}, u \in L^2) \). For instance, with the tools developed in this article, one may prove: if \( L \in \mathbb{N}, \mu \in H^{2L+3}((0, 1), \mathbb{R}) \) are such that \( \mu^{(2k+1)}(0) = \mu^{(2k+1)}(1) = 0 \) for \( k = 0, \ldots, L \) and \( \mu^{(2L+1)}(0) \pm \mu^{(2L+1)}(1) \neq 0 \), then, the system (1.1) is controllable in \( H^{2L+3}_{(0)}((0, 1), \mathbb{C}), \) locally around the ground state, with controls in \( L^2((0, T), \mathbb{R}) \), in large enough time \( T \).

Finally, let us summarize the proof of Theorem 4. Under assumption (1.13), only a finite number of the coefficients \( \langle \mu \varphi_1, \varphi_k \rangle \) vanish (see (1.13)). Thus, the linearized system around the ground state is not controllable along a finite number of directions. We will see that all
of these directions are recovered at the second order. Moreover, all these directions excepted one, present a rotation phenomena in the complex plane, for the null input solution. Thus, our proof is an adaptation of [13].

Under a weaker assumption than (1.13) and still in the framework \((\psi \in H^3(\Theta), u \in L^2)\), we prove the following result.

**Theorem 6** Let \(\mu \in H^3(0,1,\mathbb{R})\) be such that
\[
\mu'(0) = \mu'(1) \neq 0 \quad (\text{resp.} \mu'(0) = -\mu'(1) \neq 0),
\]
\(N \in \mathbb{N}^*\) and \(P_N\) be the orthogonal projection from \(L^2((0,1),\mathbb{C})\) to \(V_N := \text{Span}\{\varphi_k; k \text{ is odd and } \leq N \text{ or } k \text{ is even}\}\) (resp. \(V_N := \text{Span}\{\varphi_k; k \text{ is even and } \leq N \text{ or } k \text{ is odd}\}\)). Then, for every \(\epsilon > 0\), there exists \(T > 0\) and \(\delta > 0\) such that, for every \(\tilde{\psi}_f \in V_N \cap H^3(0,1)\) with \(\|\tilde{\psi}_f - P_N\psi_1(T)\| < \delta\), there exists \(u \in L^2(0,T)\) with \(\|u\|_{L^2} < \epsilon\) such that the solution of (1.1)-(1.4) satisfies \(P_N\psi(T) = \tilde{\psi}_f\).

Under assumption (1.14), we prove that
1. an infinite number of directions are controlled at the first order, in any positive time,
2. all the lost directions are recovered either at the second order, or at the third order,
3. any direction corresponding to vanishing first and second orders, are recovered at the third order in arbitrary time.

Note that even if \(\mu'(0) = \mu'(1) \neq 0 \quad (\text{resp.} \mu'(0) = -\mu'(1) \neq 0)\), one may sometimes control the whole wave function \(\psi\) in large time. For instance in [6], the local controllability in \(H^7((0,1),\mathbb{C})\), with controls in \(H^1_0((0,T),\mathbb{R})\), in large time \(T\), is proved for \(\mu(x) = (x-1/2)\), with the return method.

### 1.5 A review about control of bilinear systems

The first controllability result for bilinear Schrödinger equations such as (1.1) is negative and proved by Turinici [27], as a corollary of a more general result by Ball, Marsden and Slemrod [2]. Because of this noncontrollability result, such equations have been considered as non controllable for a long time. However, progress have been made and this question is now better understood. Let us also mention that this negative result has been adapted to nonlinear Schrödinger equations in [19] by Ilner, Lange and Teismann.

Concerning exact controllability issues, local results for 1D models have been proved in [6, 7] by Beauchard; almost global results have been proved in [9], by Coron and Beauchard. In [10], Beauchard and Laurent proposed an important simplification of the above proofs. In [15], Coron proved that a positive minimal time may be required for the local controllability of the 1D model. In [8], Beauchard studied the minimal time for the local controllability of 1D wave equations with bilinear controls. In this reference, the origin of the minimal time is the linearized system, whereas in the present article, the minimal time is related to the nonlinearity of the system.

Let us emphasize that exact controllability has also been studied in infinite time by Nersesyan and Nersisian in [29, 30].

Now, let us quote some approximate controllability results. In [11] Mirrahimi and Beauchard proved the global approximate controllability, in infinite time, for a 1D model and in [23] Mirrahimi proved a similar result for equations involving a continuous spectrum. Approximate controllability, in finite time, has been proved for particular models by
Boscain and Adami in [1], by using adiabatic theory and intersection of the eigenvalues in the space of controls. Approximate controllability, in finite time, for more general models, have been studied by 3 teams, with different tools: by Boscain, Chambrion, Mason, Sigalotti [13, 28, 24], with geometric control methods; by Nersesyan [25, 26] with feedback controls and variational methods; and by Ervedoza and Puel [18] thanks to a simplified model.

Optimal control techniques have also been investigated for Schrödinger equations with a non linearity of Hartee type in [3, 4] by Baudouin, Kavian, Puel and in [17] by Cances, Le Bris, Pilot. An algorithm for the computation of such optimal controls is studied in [5] by Baudouin and Salomon.

Finally, let us quote some references concerning bilinear wave equations. In [22, 21, 20], Khapalov considers nonlinear wave equations with bilinear controls. He proves the global approximate controllability to nonnegative equilibrium states.

1.6 Notations
Let us introduce some conventions and notations valid in all this article. Unless otherwise specified, the functions considered are complex valued and, for example, we write $H^1_0((0,1), \mathbb{C})$. When the functions considered are real valued, we specify it and we write, for example, $L^2((0,T), \mathbb{R})$. The same letter $C$ denotes a positive constant, that can change from one line to another one. If $(X, \|\cdot\|)$ is a normed vector space, $x \in X$ and $R > 0$, $B_X(x,R)$ denotes the open ball $\{y \in X; \|x-y\| < R\}$ and $\overline{B}_X(x,R)$ denotes the closed ball $\{y \in X; \|x-y\| \leq R\}$. We denote by $\langle \cdot, \cdot \rangle$ the $L^2(0,1)$-scalar product
\[
\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx,
\]
and by $T_S \varphi := \{\xi \in L^2(0,1); \Re\langle \varphi, \xi \rangle = 0\}$ the tangent space to $S$ at any point $\varphi \in S$.

1.7 Structure of this article
In Section 2 we recall a well posedness result concerning system (2.1). In Section 3, we prove Theorem 3. In Section 4, we prove Theorem 4 thanks to power series expansions to the order 2 and 3. In Section 5, we perform a first step toward the characterization of the minimal time, in a favorable situation. Finally, in Section 6 we gather several concluding remarks and perspectives.

2 Well posedness
This section is dedicated to the well posedness of the Cauchy problem
\[
\begin{aligned}
i\partial_t \psi &= -\partial_x^2 \psi - u(t)\mu(x)\psi - f(t,x), & (t,x) \in (0,T) \times (0,1), \\
\psi(t,0) &= \psi(t,1) = 0, & t \in (0,T), \\
\psi(0,x) &= \psi_0(x), & x \in (0,1),
\end{aligned}
\tag{2.1}
\]
proved in [10] Proposition 3].

**Proposition 1** Let $\mu \in H^3((0,1), \mathbb{R})$, $T > 0$, $\psi_0 \in H^3_0(0,1)$, $f \in L^2((0,T), H^3 \cap H^1_0)$ and $u \in L^2((0,T), \mathbb{R})$. There exists a unique weak solution of (2.1), i.e. a function $\psi \in C^0([0,T], H^3_0(0,1))$ such that the following equality holds in $H^3_0(0,1)$ for every $t \in [0,T]$,
\[
\psi(t) = e^{-iAt}\psi_0 + i \int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau.
\tag{2.2}
\]
Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if $\|u\|_{L^2(0,T)} < R$, then this weak solution satisfies

$$\|\psi\|_{C^0([0,T], H^0_{\omega_0})} \leq C \left(\|\psi_0\|_{H^0_{\omega_0}} + \|f\|_{L^2(0,T), H^1(0,1)}\right).$$

(2.3)

If $f \equiv 0$ then

$$\|\psi(t)\|_{L^2(0,1)} = \|\psi_0\|_{L^2(0,1)}, \forall t \in [0,T].$$

(2.4)

3 Proof of Theorem 3

3.1 Heuristic

Since we are interested in small motions around the trajectory $(\psi = \psi_1, u = 0)$, with small controls, it is natural to try to do them, in a first step, with the first and the second order terms. Let us consider a control $u$ of the form $u = 0 + \varepsilon v + \varepsilon^2 w$, then, formally, the solution $\psi$ of (1.1)–(1.4) writes $\psi = \psi_1 + \varepsilon \Psi + \varepsilon^2 \xi + o(\varepsilon^2)$ where

$$\begin{aligned} i \partial_t \Psi &= -\partial_x^2 \Psi - v(t) \mu(x) \psi_1, \quad (t, x) \in (0,T) \times (0,1), \\
\Psi(t, 0) &= \Psi(t, 1) = 0, \quad t \in (0, T), \\
\Psi(0, x) &= 0, \quad x \in (0,1), \\
\xi(t, x) &= \xi(t, 1) = 0, \quad t \in (0, T), \\
\xi(0, x) &= 0, \quad x \in (0,1). \end{aligned}$$

(3.1)

From the property $\|\psi(t)\|_{L^2} \equiv 1$, we deduce that $\Re(\Psi(t), \psi_1(t)) = 0$ (i.e. $\Psi(t) \in T_S \psi_1(t)$, $\forall t$) and $\|\Psi(t)\|_{L^2}^2 + 2\Re(\xi(t), \psi_1(t)) \equiv 0$. We have

$$\Psi(T, x) = i \sum_{j=1}^{\infty} \langle \mu \varphi_1, \varphi_j \rangle \int_0^T v(t) e^{i \omega_j t} dt \psi_j(T, x).$$

(3.3)

where

$$\omega_j := \lambda_j - \lambda_1, \quad \forall j \in \mathbb{N}^*.$$ 

(3.4)

Let us assume that (1.10) holds for some $K \in \mathbb{N}^*$. By adapting the choice of $v \in L^2((0,T), \mathbb{R})$, $\Psi(T)$ can reach any target in the closed subspace $\text{Adh}_{H^0_{\omega_0}(0,1)} \text{[Span}\{\varphi_k, k \in \mathbb{N}^*, k \in J\}]$ where

$$J := \{j \in \mathbb{N}^*; \langle \mu \varphi_1, \varphi_j \rangle \neq 0\}$$

(3.5)

(see Proposition 19 in Appendix); but the complex direction $\langle \Psi(T), \psi_K(T) \rangle$ is lost. Let us show that, when $T$ is small, the second order term imposes a sign on the component along this lost direction, preventing the local exact controllability around the ground state.

Thanks to (3.2) and (3.3), we have

$$\langle \xi(T), \psi_K(T) \rangle = Q_{K,T}^2(v),$$

(3.6)

where

$$Q_{K,T}^2(v) := \int_0^T v(t) \int_0^t v(\tau) h_K^2(t, \tau) d\tau dt,$$

(3.7)

$$h_K^2(t, \tau) := -\sum_{j=1}^{\infty} \langle \mu \varphi_K, \varphi_j \rangle \langle \mu \varphi_j, \varphi_1 \rangle e^{i(\lambda_K - \lambda_j) t + (\lambda_j - \lambda_1) \tau}. $$

(3.8)

Integrations by part show that

$$|\langle \mu \varphi_K, \varphi_j \rangle| \text{ and } |\langle \mu \varphi_1, \varphi_j \rangle| \leq \frac{C}{j^3}, \forall j \in \mathbb{N}^*,$$

(3.9)
for some constant \( C = C(\mu) > 0 \), thus \( h^2 \in C^0(\mathbb{R}^2, \mathbb{C}) \) and the quadratic form \( Q^2_{K,T} \) is well defined on \( L^2((0,T), \mathbb{R}) \). In particular,

\[
\Theta[(\xi(T), \varphi_K e^{-i\lambda T})] = \tilde{Q}^2_{K,T}(v)
\]

where

\[
\tilde{Q}^2_{K,T}(v) := \int_0^T \int_0^t v(t) v(\tau) \tilde{h}^2_{K,T}(t, \tau) d\tau dt,
\]

\[
\tilde{h}^2_{K,T}(t, \tau) := \sum_{j=1}^{\infty} \langle \tilde{\varphi}_j, \varphi_j \rangle \sin[(\lambda_j - \lambda_K) t - \omega_j \tau + (\lambda_K - \lambda_1) T].
\]

Let us try to move \( e\Psi(T) + e^2 \xi(T) \) in the direction of \( +i\alpha_K \varphi_K e^{-i\lambda T} \) (see (1.11) for the definition of \( \alpha_K \)). Since \( \Psi(T) \) lives in \( \text{Adh}_{H_{(0,1)}^1}[\text{Span}\{\varphi_k; k \neq K\}] \), then, necessarily \( \Psi(T) = 0 \), i.e. \( v \) belongs to

\[
V_T := \left\{ v \in L^2((0,T), \mathbb{R}); \int_0^T v(t)e^{i\omega_j t} dt = 0, \forall j \in J \right\}
\]

and \( \xi(T) = i\delta \alpha_K \varphi_K e^{-i\lambda T} \) for some \( \delta > 0 \). Thus the sign of \( \tilde{Q}^2_{K,T}(v) \) has to be \( \alpha_K \). The following lemmas show that this is not possible when \( T \) is small,

**Lemma 1** For every \( v \in V_T \), we have \( \tilde{Q}^2_{K,T}(v) = Q_{K,T}(S) \) where \( S(t) := \int_0^t v(\tau) d\tau \) and

\[
Q_{K,T}(S) := -A_K \int_0^T S(t)^2 \cos[(\lambda_K - \lambda_1)(t-T)] dt + \int_0^T S(t) \int_0^t S(\tau) k_{K,T}(t, \tau) d\tau dt,\]

\[
k_{K,T}(t, \tau) := \sum_{j=1}^{\infty} (\lambda_j - \lambda_K) \omega_j \langle \mu \varphi_1, \varphi_j \rangle \langle \mu \varphi_K, \varphi_j \rangle \sin[(\lambda_j - \lambda_K) t - \omega_j \tau + (\lambda_K - \lambda_1) T].
\]

**Remark 1** Note that \( Q_{K,T} \) is well defined on \( L^2(0,T) \) because \( k_{K,T} \in L^\infty(\mathbb{R} \times \mathbb{R}) \) (see (3.9)).

**Proof of Lemma 1** Let \( T > 0 \) and \( v \in V_T - \{0\} \). Integrations by parts show that, for every \( j \in J \),

\[
\int_0^T \int_0^t v(t) v(\tau) e^{i(\lambda_j - \lambda_K)(t-\tau)} d\tau dt = -\int_0^T S(t) e^{i(\lambda_j - \lambda_K)(t-T)} + i(\lambda_j - \lambda_K) \int_0^t v(\tau) e^{i(\lambda_j - \lambda_K)(t-\tau)} d\tau dt
\]

\[
= -\frac{1}{2} S(T)^2 e^{i(\lambda_j - \lambda_K)T} + \frac{i(\lambda_j - \lambda_K)}{2} \int_0^T S(t)^2 e^{i(\lambda_j - \lambda_K)t} dt
\]

\[
- i(\lambda_j - \lambda_K) \int_0^T S(t) e^{i(\lambda_j - \lambda_K)(t-T)} + i \omega_j \int_0^T S(\tau) e^{i(\lambda_j - \lambda_K)(t-\tau)} d\tau dt
\]

\[
= -\frac{1}{2} S(T)^2 e^{i(\lambda_j - \lambda_K)T} - i \left( \lambda_j - \frac{\lambda_1 + \lambda_K}{2} \right) \int_0^T S(t)^2 e^{i(\lambda_j - \lambda_K)t} dt
\]

\[
+ (\lambda_j - \lambda_K) \omega_j \int_0^T S(t) e^{i(\lambda_j - \lambda_K)^T} d\tau dt.
\]

The relations

\[
\sum_{j=1}^{\infty} \langle \mu \varphi_1, \varphi_j \rangle \langle \mu \varphi_K, \varphi_j \rangle = \langle \mu \varphi_1, \mu \varphi_K \rangle,
\]

\[
\sum_{j=1}^{\infty} \left( \lambda_j - \frac{\lambda_1 + \lambda_K}{2} \right) \langle \mu \varphi_1, \varphi_j \rangle \langle \mu \varphi_K, \varphi_j \rangle = \langle (\mu')^2 \varphi_1, \varphi_K \rangle = A_K.
\]

give the conclusion.

\[ \blacksquare \]
Lemma 2  Let $\mu \in H^3((0,1), \mathbb{R})$ be such that (1.10) holds for some $K \in \mathbb{N}^*$. There exists $T_K^* > 0$ such that, for every $T < T_K^*$

$$Q_{K,T}(S) \begin{cases} \leq -\frac{4K}{4T} \int_0^T S(t)^2 dt & \text{if } A_K > 0, \\ \geq -\frac{4K}{4T} \int_0^T S(t)^2 dt & \text{if } A_K < 0, \end{cases}, \quad \forall S \in L^2((0,T), \mathbb{R}). \quad (3.17)$$

Remark 2  This statement enlightens the importance of the assumption $A_K \neq 0$ in Theorem 3. Indeed, if $A_K$ vanishes then we do not know whether the quadratic form $\tilde{Q}_{K,T}$ has a sign on $V_T$ in small time $T$. Note that another integration by parts (leading to a quadratic form in $\sigma(t) := \int_0^t S$) is not possible, because of problems of divergence in infinite sums.

Proof of Lemma 2: One may assume that $A_K > 0$, $\alpha_K = 1$. We define the quantity

$$C_K := \sum_{j=1}^\infty |(\lambda_j - \lambda_K)\omega_j(\mu \varphi_1, \varphi_j)(\mu \varphi_K, \varphi_j)|.$$

Thanks to (3.16) and (1.10), there exists $j \in \mathbb{N}^* - \{1,K\}$ such that $(\mu \varphi_1, \varphi_j)(\mu \varphi_K, \varphi_j) \neq 0$. Thus, $C_K > 0$. We introduce

$$T_K^* := \begin{cases} \frac{|A_j|}{2\epsilon_j} & \text{if } K = 1, \\ \min \left\{ \frac{|A_j|}{2\epsilon_j} : \frac{\pi}{\lambda_K - \lambda_1} \right\} & \text{if } K \geq 2. \end{cases} \quad (3.18)$$

Let $T \in (0,T_K^*)$. Thanks to the inequality

$$\cos[(\lambda_K - \lambda_1)(t-T)] \geq \frac{1}{2}, \quad \forall t \in (0,T),$$

(3.11), (3.13) and Cauchy-Schwarz inequality we get, for every $S \in L^2((0,T), \mathbb{R})$,

$$Q_{K,T}(S) \leq -\frac{4K}{4T} \int_0^T S(t)^2 dt + C_K \int_0^T \|S(t)\|_2 \|S(t)\|_2 dt \leq -\frac{1}{2} |A_K - TC_K| \int_0^T S(t)^2 dt.$$  

With additional arguments, one may prove that, for $T < T_K^*$,

$$\sup\{\tilde{Q}_{K,T}^2(v); v \in V_T, \|v\|_{L^2} = 1\} = 0.$$

The non existence of a positive constant $c(T) > 0$ such that

$$\tilde{Q}_{K,T}^2(v) \leq c(T)\|v\|_{L^2}^2, \forall v \in V_T, \forall T < T_K^*$$

prevents from proving the non controllability in the framework $(\psi \in H^2_0(0,1), \mu \in L^2)$ (such an inequality would allow to deal with the nonlinear terms). Our solution relies on the fact that, for $T$ small, the quadratic form $\tilde{Q}_{K,T}$ is coercive in $S(t) := \int_0^t v(s)ds$ (see (3.17)). Thus, for the negative result, we need to work in a framework $(\psi \in H^1_0(0,1), S \in L^2(0,T))$ or, equivalently $(\psi \in H^1(0,1), u \in H^{-1}(0,T))$. This is why our analysis relies on an auxiliary system studied in the next section.

3.2 Auxiliary system

Let us consider the system

$$\begin{cases} i \partial_t \tilde{\psi} = -\partial_x^2 \tilde{\psi} - is(t)[2\mu'(x)\partial_x \tilde{\psi} + \mu''(x)\tilde{\psi}] + s(t)^2 \mu'(x)^2 \tilde{\psi}, \quad (t,x) \in (0,T) \times (0,1), \\ \tilde{\psi}(t,0) = \tilde{\psi}(t,1) = 0, \\ \tilde{\psi}(t,0) = \psi(t,1) = 0, \quad t \in (0,T). \end{cases} \quad (3.19)$$
It is a control system in which the state is \( \tilde{\psi} \), with \( \|\tilde{\psi}(t)\|_{L^2} \equiv 1 \) and the control is the real valued function \( s \). The system (3.19) results from (1.1) through the transformation

\[
\psi(t, x) = \tilde{\psi}(t, x)e^{is(t)\mu(x)} \quad \text{where} \quad s(t) := \int_0^t u(\tau)d\tau,
\]

which is also used in [15]. We will work with initial conditions

\[
\tilde{\psi}(0, x) = \tilde{\psi}_0(x), \quad x \in (0, 1).
\]  

(3.21)

or

\[
\tilde{\psi}(0, x) = \varphi_1(x), \quad x \in (0, 1).
\]

(3.22)

The well posedness of the Cauchy problem (3.19)-(3.21) may be proved similarly to [10, Proposition 3].

**Proposition 3** Let \( \mu \in H^3((0, 1), \mathbb{R}) \), \( T > 0 \), \( \tilde{\psi}_0 \in H^1_0(0, 1) \) \( f \in L^2((0, T), H^1(0, 1)) \) and \( s \in L^2((0, T), \mathbb{R}) \). There exists a unique weak solution \( \tilde{\psi} \in C^0([0, T], H^1_0(0, 1)) \) of (3.19)-(3.21). Moreover, for every \( R > 0 \), there exists \( C = C(T, \mu, R) > 0 \) such that, if \( \|s\|_{L^2(0, T)} < R \), then this weak solution satisfies

\[
\|\tilde{\psi}\|_{C^0([0, T], H^1_0)} \leq C\left(\|\tilde{\psi}_0\|_{H^1_0} + \|f\|_{L^2((0, T), H^1)}\right).
\]

(3.23)

If \( f \equiv 0 \) then

\[
\|\tilde{\psi}(t)\|_{L^2(0, 1)} = \|\tilde{\psi}_0\|_{L^2(0, 1)}, \quad \forall t \in [0, T].
\]

(3.24)

The proof of Theorem 3 is a direct consequence of the following result.

**Theorem 7** Let \( K \in \mathbb{N}^* \), \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that (1.17) holds and \( T_K^* \) be as in Lemma 2. For every \( T < T_K^* \), there exists \( \epsilon > 0 \) such that for every \( s \in L^2((0, T), \mathbb{R}) \) with \( \|s\|_{L^2} < \epsilon \), the solution of the Cauchy problem (3.19)-(3.22) is

\[
\tilde{\psi}(T, .) \neq (\sqrt{1 - \delta^2} \varphi_1 + i\alpha K \delta \varphi_K)e^{-i\lambda_1 T}e^{i\delta \mu}, \quad \forall \delta > 0, \forall \theta \in \mathbb{R}.
\]

(3.25)

The proof of Theorem 7 requires several steps, thus, it is developed in Section 3.4.

### 3.3 Proof of Theorem 3 thanks to Theorem 7

Let \( T < T_K^* \). Let \( \epsilon > 0 \) be as in Theorem 7. Let \( u \in L^2((0, T), \mathbb{R}) \) be such that the function \( s(t) := \int_0^t u(\tau)d\tau \) satisfies \( \|s\|_{L^2(0, T)} < \epsilon \). Let us assume that the solution of the Cauchy problem (3.19)-(3.22) satisfies \( \psi(T) = (\sqrt{1 - \delta^2} \varphi_1 + i\alpha K \delta \varphi_K)e^{-i\lambda_1 T}e^{-is(T)\mu} \) for some \( \delta > 0 \). Then, the function \( \tilde{\psi} \) defined by (3.20) solves (3.19)-(3.21) and satisfies \( \psi(T) = (\sqrt{1 - \delta^2} \varphi_1 + i\alpha K \delta \varphi_K)e^{-i\lambda_1 T}e^{-is(T)\mu} \). Thanks to Theorem 7, this is impossible.

### 3.4 Proof of Theorem 7

The proof of Theorem 7 requires the following preliminary result.

**Proposition 3** Let \( T > 0 \), \( K \in \mathbb{N}^* \), \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that \( (\mu \varphi_1, \varphi_K) = 0 \). There exists \( C = C(T) > 0 \) such that, for every \( s \in L^2((0, T), \mathbb{R}) \) with \( \|s\|_{L^2} < 1 \), the solution of (3.19)-(3.22) satisfies

\[
\left|\mathcal{A}(\tilde{\psi}(T), \varphi_K e^{-i\lambda_1 T}) - \mathcal{Q}_{K,T}(s)\right| \leq C\|s\|^2_{L^2},
\]

(3.26)

\[
\left|\mathcal{A}(\tilde{\psi}(T), \psi_1(T))\right| \leq C\|s\|^2_{L^2},
\]

(3.27)

\[
\left\|\left(\tilde{\psi}(T), \psi_j(T) - \omega_j (\mu \varphi_1, \varphi_j) \int_0^T s(t)e^{i\omega_j t}dt\right)_{j \in J - \{1\}}\right\|_{L^2} \leq C\|s\|^2_{L^2}.
\]

(3.28)
Proof of Theorem 3 Let $T > 0$, $s \in L^2((0, T), \mathbb{R})$ with $\|s\|_{L^2} < 1$, $\tilde{\psi}$ be the solution of (3.19)-(3.22). Let $\tilde{\Psi}$ and $\tilde{\xi}$ be the solutions of

$$
\begin{cases}
  i\partial_t \tilde{\Psi} = -\partial_x^2 \tilde{\Psi} - is(t)\{2\mu'(x)\partial_x \psi_1 + \mu''(x)\psi_1\}, & (t, x) \in (0, T) \times (0, 1), \\
  \tilde{\Psi}(t, 0) = \tilde{\Psi}(t, 1) = 0, & t \in (0, T), \\
  \tilde{\Psi}(0, x) = 0 & x \in (0, 1),
\end{cases}
$$

and

$$
\begin{cases}
  i\partial_t \tilde{\xi} = -\partial_x^2 \tilde{\xi} - is(t)\{2\mu'(x)\partial_x \tilde{\Psi} + \mu''(x)\tilde{\Psi}\} + s(t)^2\mu'(x)^2\psi_1, & (t, x) \in (0, T) \times (0, 1), \\
  \tilde{\xi}(t, 0) = \tilde{\xi}(t, 1) = 0, & t \in (0, T), \\
  \tilde{\xi}(0, x) = 0 & x \in (0, 1).
\end{cases}
$$

Easy computations show that

$$
\tilde{\Psi}(t, x) = \sum_{j=2}^{\infty} \omega_j \langle \mu \phi_1, \phi_j \rangle \int_0^t \omega_j e^{i\omega_j \tau} d\tau \psi_j(t, x),
$$

$$
\langle \tilde{\xi}(T), \psi_K(T) \rangle = -iA_K \int_0^T S(t)^2 e^{i(\lambda_K - \lambda_j)t} dt + \sum_{j=2}^{\infty} \omega_j \langle \lambda_K - \lambda_j \rangle \langle \mu \phi_K, \phi_j \rangle \langle \mu \phi_j \rangle \int_0^T s(t) \int_0^t \omega_j e^{i\omega_j \tau + i(\lambda_K - \lambda_j)\tau} d\tau d\tau dt.
$$

Thanks to Proposition 2 we get a constant $C = C(T) > 0$ such that for every $s \in L^2((0, T), \mathbb{R})$

$$
\|\tilde{\psi} - \psi_1 - \tilde{\Psi}\|_{L^\infty((0, T), H^2_0)} \leq C\|s\|^2_{L^2}, \quad \text{and} \quad \|\tilde{\psi} - \psi_1 - \tilde{\Psi} - \tilde{\xi}\|_{L^\infty((0, T), H^2_0)} \leq C\|s\|^3_{L^2}.
$$

Thus,

$$
|\mathcal{S}(\tilde{\psi}(T), \psi_K e^{-i\lambda_1 T}) - Q_{K, T}(s)| = |\mathcal{S}(\tilde{\psi} - \psi_1 - \tilde{\Psi} - \tilde{\xi})(T), \psi_K e^{-i\lambda_1 T})| \leq C\|s\|^3_{L^2},
$$

$$
|\mathcal{S}(\tilde{\psi}(T), \psi_1(T))| = |\mathcal{S}(\tilde{\psi} - \psi_1 - \tilde{\Psi})(T), \psi_1(T))| \leq C\|s\|^2_{L^2},
$$

$$
\|\left(\frac{\tilde{\psi}(T)}{\psi_1(T)} - \omega_j \langle \mu \phi_1, \phi_j \rangle \int_0^T s(t) e^{i\omega_j \tau} d\tau \right)_{j \neq 1}\|_{H^1} \leq \|\left(\frac{\tilde{\psi} - \psi_1 - \tilde{\Psi}}{\psi_1(T)}\right)_{j \in \mathbb{N}}\|_{H^1} \leq C\|s\|^2_{L^2}.
$$

Proof of Theorem 4 One may assume that $A_K > 0$, $\alpha_K = 1$. Let $T < T_K^\ast$. Working by contradiction, we assume that, for every $\epsilon > 0$, there exists $s_\epsilon \in L^2((0, T), \mathbb{R})$ with $\|s_\epsilon\|_{L^2} < \epsilon$ such that the solution $\tilde{\psi}_\epsilon$ of (3.19)-(3.22) satisfies

$$
\tilde{\psi}_\epsilon(T, ..) = (\sqrt{1 - \frac{\partial^2}{\partial t^2}} \psi_1 + i\delta_\epsilon \psi_K e^{i\theta_\epsilon \psi_1} e^{-i\lambda_1 T})
$$

for some $\delta_\epsilon > 0$ and $\theta_\epsilon \in \mathbb{R}$. Then $\theta_\epsilon, \delta_\epsilon \to 0$ when $\epsilon \to 0$.

First step: Let us prove that

$$
|\theta_\epsilon| + |\delta_\epsilon| = O_{\epsilon \to 0}(\|s_\epsilon\|_{L^2}).
$$

Thanks to (3.23), there exists $C > 0$ such that

$$
\|\tilde{\psi}_\epsilon(T, (0, 1))\|_{L^2(0, 1)} \leq C\|s_\epsilon\|_{L^2(0, T), \forall \epsilon > 0}.
$$

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Moreover, thanks to (3.29) and the assumption \( \langle \mu \varphi_1, \varphi_K \rangle = 0 \), we have
\[
\frac{1}{2} \| \tilde{\psi}_e - \psi_1 \|^2_{L^2(0,1)} = 1 - \Re \int_0^1 \tilde{\psi}_e(T,x) \psi_1(T,x) dx = 1 - \int_0^1 \left( \sqrt{1 - \delta_2^2} \varphi_1(x) \right) \varphi_K(x) sin[\theta_e \mu(x)] dx \\
= 1 - \left( 1 - \delta_2^2 \| \mu \varphi_1 \|^2 + O(\delta_1^2) \right) \left( 1 + O(\delta_2^2) \right) = \delta_e + O(\delta_1^2 + \delta_2^2).
\]
We get (3.30) thanks to the 2 previous relations.

**Second step:** Let us prove that
\[
\Im(\tilde{\psi}_e(T), \varphi_K e^{-i\lambda_1 T}) = \delta_e + O(\|s_e\|^3_{L^2}).
\] (3.31)
Thanks to (3.29) and the assumption \( \langle \mu \varphi_1, \varphi_K \rangle = 0 \), we have
\[
\Im(\tilde{\psi}_e(T), \varphi_K e^{-i\lambda_1 T}) = \int_0^1 \left( \sqrt{1 - \delta_2^2} \varphi_1(x) \varphi_K(x) sin[\theta_e \mu(x)] + \delta_e \varphi_K(x) \right) dx \\
= \left( 1 + O(\delta_2^2) \right) O(\delta_e) \delta_e + O(\delta_2^2) \left( 1 + O(\delta_1^2) \right) \\
= \delta_e + O(\|s_e\|^3_{L^2}).
\]
We get (3.31) thanks to (3.30).

**Third step:** Conclusion. Thanks to (3.31), (3.29) and (3.17), we get
\[
0 < \delta_e = \Im(\tilde{\psi}_e(T), \varphi_K e^{-i\lambda_1 T}) + O(e^{-\|s_e\|^3_{L^2}})
= \overline{Q_{K,T}(s_e)} + O(e^{-\|s_e\|^3_{L^2}}) \\
\leq - \frac{\Delta K}{4} \|s_e\|^2_{L^2} + O(e^{-\|s_e\|^3_{L^2}})
\]
which is impossible when \( \epsilon \) is small.

**4 Proof of Theorem 4**

**4.1 Preliminaries**

The goal of this section is the proof of the following result.

**Proposition 4** Let \( \mu \in H^3((0,1), \mathbb{R}) \) be such that \( \mu'(0) \pm \mu'(1) \neq 0 \).

1. Then, \( N := \sharp \{ k \in \mathbb{N}^* ; \langle \mu \varphi_1, \varphi_k \rangle = 0 \} \) is finite.

2. Let \( K_1 < \ldots < K_N \in \mathbb{N}^* \) be such that \( \langle \mu \varphi_1, \varphi_{K_j} \rangle = 0 \) for \( j = 1, \ldots, N \). Then, for every \( j \in \{ 1, \ldots, N \} \) and \( T > 0 \) there exists \( v \in V_T \) such that \( Q_{K_j,T}^2(v) \neq 0 \).

3. We have
\[
\exists c > 0 \text{ such that } |\langle \mu \varphi_1, \varphi_k \rangle| \geq \frac{c}{K^3}, \forall k \in \mathbb{N}^* - \{ K_1, \ldots, K_N \}. \tag{4.1}
\]

We recall that \( Q_{K_j,T}^2 \) is defined in (3.37) - (3.38), and \( V_T \) in (3.33). For the proof of Proposition 4, we need the following preliminary result.

**Proposition 5** Let \( \mu \in H^3((0,1), \mathbb{R}) \) and \( K \in \mathbb{N}^* \) be such that \( \langle \mu \varphi_K, \varphi_n \rangle (\mu \varphi_n, \varphi_1) \neq 0 \) for some \( n \in \mathbb{N}^* \). The following statements are equivalent.
• There exists $T^* > 0$ such that, for every $T < T^*$, $Q_{K,T}^2 \equiv 0$ on $V_T$.
• The support of the sequence $(\langle \mu \varphi_K, \varphi_j \rangle, \langle \mu \varphi_j, \varphi_1 \rangle)_{j \in \mathbb{N}}$ is contained in the finite set
  \[ \{ j_* \in J \cap [1, K] : \exists k_* \in J \cap [1, K], \lambda_{j_*} - \lambda_1 = \lambda_K - \lambda_{k_*} \}. \]
Moreover, for every $j_* , k_* \in J \cap [1, K]$ such that $\lambda_{j_*} - \lambda_1 = \lambda_K - \lambda_{k_*}$ then
\[ \langle \mu \varphi_K, \varphi_{j_*} \rangle, \langle \mu \varphi_{j_*}, \varphi_1 \rangle = \langle \mu \varphi_K, \varphi_{k_*} \rangle, \langle \mu \varphi_{k_*}, \varphi_1 \rangle. \]

**Proof of Proposition** To simplify the notation of this proof, we write $Q_T$ and $h$, instead of $Q_{K,T}^2$ and $h_K^2$. Let us assume that $Q_T \equiv 0$ on $V_T$, for every $T < T^*$. Then $\nabla Q_T(v) \perp V_T$, for every $v \in V_T$ and $T < T^*$. Easy computations show that, for $v \in V_T$,
\[ \nabla Q_T(v) : t \mapsto \int_0^t v(\tau)h(\tau, t)d\tau + \int_t^T v(\tau)h(\tau, t)d\tau = \int_t^T v(\tau)[h(\tau, t) - h(t, \tau)]d\tau. \tag{4.2} \]

**First step:** Let us prove that $\nabla Q_T(v) = 0, \forall v \in V_T$. Let $T, T_1$ be such that $0 < T < T_1 < T^*$ and $v \in V_{T_1}$ supported on $(0, T)$. Since $\nabla Q_T(v) \perp V_{T_1}$, there exists a unique sequence $(\alpha_k)_{k \in \mathbb{Z} - \{0\}} \in l^2$ such that
\[ \nabla Q_{T_1}(v) = \sum_{k=1}^{\infty} \alpha_k e^{i(\lambda_k - \lambda_1)t} + \sum_{k=2}^{\infty} \alpha_k e^{-i(\lambda_k - \lambda_1)t} \quad \text{in } L^2((0, T_1), \mathbb{R}) \]
(decomposition on a Riesz-basis). We have $\nabla Q_{T_1}(v) \equiv 0$ on $(T, T_1)$ because $v$ is supported on $(0, T)$ (see Proposition 1.2). Thanks to Ingham inequality on $(T, T_1)$ we get $\alpha_k = 0, \forall k$ (see Proposition 1.9 in Appendix).

**Second step:** Let us prove that $h(\tau, t) = h(t, \tau), \forall t, \tau \in [0, T^*)$. Thanks to the first step, we have
\[ \int_t^T v(\tau)[h(\tau, t) - h(t, \tau)]d\tau = 0, \quad \forall 0 < t < T < T^*, \forall v \in V_T. \tag{4.3} \]

Note that, if $T \in (0, T^*)$ and $t \in (0, T)$ are fixed, then $V_{T}(t, T) = L^2(t, T)$ (see Proposition 1.9). Thus, we deduce from (4.3) that $\tau \mapsto h(t, \tau) - h(\tau, t)$ vanishes in $L^2(t, T)$, for every $0 < t < T < T^*$. This gives the conclusion because the function $(t, \tau) \mapsto h(t, \tau) - h(\tau, t)$ is continuous.

**Third step:** Conclusion. Let $k^* \in \mathbb{N}^*$ be such that $b_{k^*} := \langle \mu \varphi_K, \varphi_{k^*} \rangle, \langle \mu \varphi_{k^*}, \varphi_1 \rangle \neq 0$. The equality $h(t, \tau) - h(\tau, t) = 0$ with $\tau = 0$ gives
\[ b_{k^*} e^{i(\lambda_{k^*} - \lambda_1)t} = \sum_{j \in J} b_j e^{i(\lambda_j - \lambda_1)t} - \sum_{k \in J - \{k^*\}} b_k e^{i(\lambda_K - \lambda_k)t}. \tag{4.4} \]
The equality $\frac{d}{dt}[h(t, \tau) - h(\tau, t)] = 0$ with $\tau = 0$ gives
\[ (\lambda_{k^*} - \lambda_1)b_{k^*} e^{i(\lambda_{k^*} - \lambda_1)t} = \sum_{j \in J} (\lambda_K - \lambda_j)b_j e^{i(\lambda_j - \lambda_1)t} - \sum_{k \in J - \{k^*\}} (\lambda_K - \lambda_k)b_k e^{i(\lambda_K - \lambda_k)t}. \]
In the right hand side of the 2 previous equalities, the frequencies $(\lambda_j - \lambda_1)$ are $\geq 0$ for every $j \in J$, while the frequencies $(\lambda_K - \lambda_k)$ are negative for every $k > K$. Thus, for every $k > K$ the frequency $(\lambda_K - \lambda_k)$ appears only one time in the right hand side of each equality. The uniqueness of the decomposition on a Riesz basis gives
\[ (\lambda_{k^*} - \lambda_1)b_{k^*} = (\lambda_K - \lambda_1)b_k, \forall k > K \quad \text{with } k \neq k^*. \]
Thus, $b_k = 0, \forall k \in J - \{k^*\}$ with $k > K$. Coming back to (4.3), we only have a finite sum in the right hand side, over $j \in J$ with $j \leq K$ and over $k \in J - \{k^*\}$ with $k \leq K$. We
deduce the existence of a unique \( j^* \in J \) with \( j^* \leq K \) such that \( \lambda_K - \lambda_{k^*} = \lambda_j - \lambda_1 \) and \( b_{k^*} = b_j \).

Reciprocally, let \( \alpha := \lambda_K - \lambda_{k^*} = \lambda_j - \lambda_1, \beta := \lambda_K - \lambda_{k^*} = \lambda_j - \lambda_1 \). Then \( h(t, \tau) := b_k \left[ e^{i|\alpha t + \beta \tau|} + e^{i|\beta t + \alpha \tau|} \right] \), satisfies \( h(t, \tau) = h(\tau, t) \) and \( \nabla Q_T \equiv 0 \) on \( V_T \), for every \( T > 0 \). By linearity, the same conclusion holds when \( h \) is a finite sum of such terms.

**Proof of Proposition 4.** Thanks to three integrations by part and the Riemann-Lebesgue Lemma, we get

\[
\langle \mu \varphi_K, \varphi_n \rangle = \frac{4K((-1)^{K+n}/n^2)}{\pi^2} + \frac{o}{n} \quad \text{as} \quad n \to +\infty.
\]

(4.5)

Thus, for \( k \) large enough \( \langle \mu \varphi_1, \varphi_k \rangle \neq 0 \). This proves the first and third statements of Proposition 4.

Let \( j \in \{1, \ldots, N\} \). Thanks to (4.5), we have simultaneously \( \langle \mu \varphi_1, \varphi_k \rangle \neq 0 \) and \( \langle \mu \varphi_K, \varphi_k \rangle \neq 0 \) for arbitrarily large values of \( k \). Thus, Proposition 4 gives the conclusion.

**4.2 Strategy for the proof of Theorem 4**

Until the end of Section 4, we fix \( \mu \in H^1((0,1), \mathbb{R}) \) such that \( \mu'(1) = \mu'(0) \neq 0 \), \( N \in \mathbb{N} \) and \( K_1, \ldots, K_N \in \mathbb{N}^* \) as in Proposition 4. To simplify the notations, we assume that \( K_1 = 1 \). We define the space

\[
\mathcal{H} := \text{Span}_C\left\{ \psi_k(T), k \in \mathbb{N}^* - \{K_1, \ldots, K_N\} \right\},
\]

(4.6)

and, for \( j = 1, \ldots, N \) the space

\[
M^j := \left\{ \begin{array}{ll}
\text{Span}_C(\psi_{K_j}(T)) & \text{if } K_j \neq 1, \\
\text{iSpan}_R(\psi_1(T)) & \text{if } K_j = 1.
\end{array} \right.
\]

(4.7)

Let

\[
M := \bigoplus_{j=1}^{N} M^j.
\]

(4.8)

The global strategy relies on power series expansion of the solutions to the order 2 as in [13] (see also [10]). In Section 4.3, we prove the local exact controllability in \( \mathcal{H} \), with a first order strategy. Then, in Section 4.4, we prove that any direction in \( M \) is reached with the second order term. Finally, in Section 4.5, we conclude thanks to a fixed point argument.

**4.3 Controllability in \( \mathcal{H} \) in arbitrarily small time**

Let us introduce the orthogonal projection

\[
\mathcal{P}_T : L^2(0,1) \to \mathcal{H} \quad \psi \mapsto \psi - \sum_{j=1}^{N} \langle \psi, \psi_{K_j}(T) \rangle \psi_{K_j}(T)
\]

(4.9)

The goal of this section is the proof of the following result.

**Theorem 8** Let \( T_1, T > 0 \) be such that \( T_1 < T \). There exists \( \delta_1 > 0 \) and a \( C^1 \)-map \( \Gamma|_{T_1,T} : \Omega_{T_1} \times \Omega_T \to L^2((T_1,T), \mathbb{R}) \) where

\[
\Omega_{T_1} := \{ \psi_0 \in \mathcal{S} \cap H_0^3(0,1); \| \psi_0 - \psi_1(T_1) \|_{H_0^3} < \delta_1 \},
\]

(4.10)
Proposition 7
Let \( \psi \in H^3_{(0)}(0,1) \) (it is a consequence of Proposition 19 in Appendix).

\[
\Omega_T := \{ \tilde{\psi}_f \in H \cap H^3_{(0)}(0,1); \|\tilde{\psi}_f - \mathcal{P}_T[\psi_f(T)]\|_{H^3_{(0)}} < \delta_1 \}
\]

such that \( \Gamma_{[T, T]}(\psi(T)) \) and \( \psi(T) \) with initial condition \( \psi(0) = \psi_0 \) and control \( u := \Gamma_{[T, T]}(\psi_0, \tilde{\psi}_f) \) satisfies \( \mathcal{P}_T[\psi(T)] = \psi_f \).

This theorem may be proved exactly as Theorem 1 in [10]. We recall the main steps of the proof because several intermediate results will also be used in the end of this article. To simplify the notations, we take \( T_1 = 0 \).

Thanks to Proposition 1, we consider the map

\[
\Theta_T : [S \cap H^3_{(0)}(0,1)] \times L^2((0, T), \mathbb{R}) \rightarrow [S \cap H^3_{(0)}(0,1)] \times [H \cap H^3_{(0)}(0,1)]
\]

\[
(\psi_0, u) \mapsto (\psi, \mathcal{P}_T[\psi(T)])
\]

where \( \psi \) is the solution of (1.1)-(1.3). Then Theorem 5 corresponds to the local surjectivity of the nonlinear map \( \Theta_T \) around the point \( (\varphi_1, 0) \), that will be proved thanks to the inverse mapping theorem. Thus, the first property required is the \( C^1 \)-regularity of \( \Theta_T \), which is a consequence of [10] Proposition 3.

Proposition 6
Let \( T > 0 \) and \( \mu \in H^3((0, 1), \mathbb{R}) \). The map \( \Theta_T \) defined by (4.10) is \( C^1 \). Moreover, for every \( \psi_0, \Psi_0 \in H^3_{(0)}(0,1) \), \( u, v \in L^2((0, T), \mathbb{R}) \), we have

\[
d\Theta_T(\psi_0, u)(\Phi_0, v) = (\Phi_0, P_T[\Phi(T)])
\]

where \( \Phi \) is the weak solution of the linearized system

\[
\begin{aligned}
\frac{d}{dt} \Phi &= -\partial_x^2 \Phi - u(t)\mu(x)\Phi - v(t)\mu(x)\psi, & (t, x) \in (0, T) \times (0, 1), \\
\Phi(t, 1) &= \Psi(t, 1) = 0, & t \in (0, T), \\
\Phi(0, x) &= \Psi_0, & x \in (0, 1)
\end{aligned}
\]

and \( \psi \) is the solution of (1.1)-(1.3).

The second property required for the application of the inverse mapping theorem is the existence of a continuous right inverse for \( d\Theta_T(\varphi_1, 0) \), that may be proved exactly as [10] Proposition 4 (it is a consequence of Proposition 19 in Appendix).

Proposition 7
Let \( T > 0 \) and \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that (4.1) holds. The linear map

\[
d\Theta_T(\varphi_1, 0) : [T_S\varphi_1 \cap H^3_{(0)}] \times L^2((0, T), \mathbb{R}) \rightarrow [T_S\varphi_1 \cap H^3_{(0)}] \times [H \cap H^3_{(0)}]
\]

has a continuous right inverse

\[
d\Theta_T(\varphi_1, 0)^{-1} : [T_S\varphi_1 \cap H^3_{(0)}] \times [H \cap H^3_{(0)}] \rightarrow [T_S\varphi_1 \cap H^3_{(0)}] \times L^2((0, T), \mathbb{R}).
\]

4.4 Reaching the missed directions, at the second order, in large time.

The goal of this section is the proof of the following result.

Proposition 8
Let \( T > T_2 \) where

\[
T_2 := \left\{ \begin{array}{ll}
2^{N-1-T_{\min}} + \sum_{k=2}^{N} ((k - 1) + 2^{k-2}) \frac{\pi}{\lambda_{K_k}^N - \lambda_1} & \text{if } K_1 = 1, \\
\sum_{k=1}^{N} \frac{\pi}{\lambda_{K_k}^N - \lambda_1} & \text{if } K_1 \neq 1.
\end{array} \right.
\]

(4.13)
There exists a continuous map
\[ \Lambda_T : M \rightarrow L^2((0, T), \mathbb{R})^2 \]
such that, for every \( z \in M \), the solutions \( \Psi \) and \( \xi \) of (3.1) and (3.2) satisfy \( \Psi(T) = 0 \) and \( \xi(T) = z \).

In this statement, the quantity \( T_{\min}^2 \) is defined as follows.

**Lemma 3** The quantity
\[ T_{\min}^2 := \inf \{ T > 0 ; \exists v_\pm \in V_T \text{ such that } \tilde{Q}_{1,T}(v_\pm) = \pm 1 \} \]
is well defined and belongs to \( (0, 2/\pi) \).

Let us recall that \( \tilde{Q}_{1,T} \) and \( V_T \) are defined in (3.11), (3.13).

**Proof of Lemma 3** Let \( T \geq 2/\pi \). If \( v_- \in V_T - \{ 0 \} \) is supported on \( (0, T^*) \) then \( \tilde{Q}_{1,T}(v) < 0 \) (see Lemma 2). Let \( v_+(t) := \cos(\pi^2 t)1_{[0,2/\pi]}(t) \). Then, \( v_+ \in V_T \) and
\[ \tilde{Q}_{1,T}(v) = \sum_{j=2}^{\infty} (\mu_{j^1}, \varphi_j)^2 \frac{4(j^2 - 1)}{\pi^3 j^2(j^2 - 2)} > 0. \]  

\[ \blacksquare \]

### 4.4.1 Preliminaries

Our proof of Proposition 8 requires 3 preliminary results. The first one consists in proving the existence of controls such that the projections of the second order term on the lost directions are non zero.

**Proposition 9** Let \( T > 0 \). For every \( j \in \{ 1, \ldots, N \} \), there exists \( v_j, w_j \in L^2((0, T), \mathbb{R}) \) such that the associated solutions \( \Psi^j \) and \( \xi^j \) of (3.1) and (3.2) satisfy
\[ \Psi^j(T, \cdot) = 0, \]
\[ (\xi^j(T, \cdot), \psi_K(T)) \neq 0, \]
\[ (\xi^j(T, \cdot), \psi_k(T)) = 0, \quad \forall k \in \mathbb{N}^* - \{ K_1, \ldots, K_N \}. \]

**Proof of Proposition 8** Let \( j \in \{ 1, \ldots, N \} \). Thanks to Proposition 4 there exists \( v_j \in V_T \) such that \( Q_{K_j,T}(v_j) \neq 0 \). Thanks to (3.3) and (3.13), we have \( \Psi^j(T) = 0 \). Thanks to (3.6) we have \( (\xi^j(T), \psi_K(T)) = Q_{K_j,T}(v_j) \neq 0 \). The equality \( (\xi^j(T), \psi_k(T)) = 0, \quad \forall k \in \mathbb{N}^* - \{ K_1, \ldots, K_N \} \) is equivalent to the following trigonometric moment problem on \( w_j \),
\[ \int_0^T w_j(t)e^{i\omega_k t} dt = \int_0^T v_j(t)(\mu_{j^1}, \varphi_k)^2 e^{i\lambda_k t} dt, \forall k \in \mathbb{N}^* - \{ K_1, \ldots, K_N \}. \]  

(4.14)

Thanks to (4.1) and (10) Lemma 1, the right hand side belongs to \( l^2 \). Thus, Proposition 19 ensures the existence of a solution \( w_j \in L^2((0, T), \mathbb{R}) \).

The second preliminary result for the proof of Proposition 8 is a measure of the rotation of the null input solution, precised in the next statement.
Lemma 4 Let $T, T_1, \theta > 0$ be such that $0 < T < T + \theta \leq T_1$, $v, w \in L^2((0, T), \mathbb{R})$ and $v_\theta, w_\theta \in L^2((0, T_1), \mathbb{R})$ be defined by

$$ (v_\theta, w_\theta)(t) := \begin{cases} (0, 0) & \text{if } t \in (0, \theta), \\ (v, w)(t - \theta) & \text{if } t \in (\theta, \theta + T), \\ (0, 0) & \text{if } t \in (\theta + T, T_1). \end{cases} $$

We denote by $(\Psi, \xi)$ and $(\Psi_\theta, \xi_\theta)$ the associated solutions of (3.1) and (3.2). Then, for every $k \in \mathbb{N}^*$

$$ \langle \Psi_\theta(T_1), \psi_k(T_1) \rangle = e^{i(\lambda_k - \lambda_1)\theta} \langle \Psi(T), \psi_k(T) \rangle, $$

$$ \langle \xi_\theta(T_1), \psi_k(T_1) \rangle = e^{i(\lambda_k - \lambda_1)\theta} \langle \xi(T), \psi_k(T) \rangle. $$

Remark 3 Note that, for $k = 1$, there is no rotation phenomenon.

Proof of Lemma 4 We have

$$ \Psi_\theta(t) = \begin{cases} 0 & \text{for } 0 < t < \theta, \\ \Psi(t - \theta)e^{-i\lambda_1\theta} & \text{for } \theta < t < \theta + T, \\ e^{-i\lambda_1(t - \theta - T)}\Psi_\theta(\theta + T) & \text{for } \theta + T < t \leq T_1, \end{cases} $$

thus

$$ \Psi_\theta(T_1) = \sum_{k=1}^{\infty} \langle \Psi(T), \varphi_k \rangle e^{-i\lambda_1\theta} e^{-i\lambda_k(T_1 - \theta - T)} \varphi_k = \sum_{k=1}^{\infty} \langle \Psi(T), \psi_k(T) \rangle e^{i(\lambda_k - \lambda_1)\theta} \psi_k(T_1). $$

The same relations hold for $\xi_\theta$. ■

The third preliminary result for the proof of Proposition 8 is the non overlapping principle.

Proposition 10 Let $T > 0$ and $T_1 \in (0, T)$. Let $v_j \in V_T$, $w_j \in L^2((0, T), \mathbb{R})$, $\Psi_j$ and $\xi_j$ be the associated solutions of (3.1) and (3.2) for $j = 1, 2$. We assume that $v_1$ is supported on $(0, T_1)$ and $v_2$ is supported on $(T_1, T)$. Let $v := v_1 + v_2$, $w := w_1 + w_2$, $\Psi$ and $\xi$ be the associated solutions of (3.1) and (3.2). Then $\Psi(T) = 0$ and $\xi(T) = \xi_1(T) + \xi_2(T)$.

Proof of Proposition 10 We have $\Psi(T) = 0$ because $v \in V_T$ (see (3.3) and (3.13)). The control $v_1$ is supported on $(0, T_1)$ and belongs to $V_{T_1}$, thus $\Psi_1$ is supported on $(0, T_1) \times (0, 1)$ (see (3.3) and (3.13)). The function $v_2$ is supported on $(T_1, T)$ thus $\Psi_2$ is supported on $(T_1, T) \times (0, 1)$. Therefore

$$ (v_1 + v_2)\mu(\Psi_1 + \Psi_2) = v_1\mu\Psi_1 + v_2\mu\Psi_2 \text{ on } (0, T) \times (0, 1), $$

i.e. $\xi_1 + \xi_2$ and $\xi$ solve the same Cauchy problem, thus $\xi = \xi_1 + \xi_2$. ■

4.4.2 Proof of Proposition 8

The strategy for the proof of Proposition 8 is the same as in [19]. It relies strongly on the rotation of the lost directions, emphasized in Lemma 4. However, it needs to be adapted because there is no rotation phenomenon on our first lost direction. In order to simplify the notations, in a first step, we prove Proposition 8 in the case

$$ N = 2, \ K_1 = 1, \ K_2 = 2, \ T_2 = 2T_{min} + \frac{3\pi}{\lambda_2 - \lambda_1}. $$

We will explain later how it can be adapted for $N \geq 3$ and $K_1, \ldots, K_N$ arbitrary.
Proof of Proposition 5 Let \( T, T_1, T_\theta, T_c, T_c^1 > 0 \) be such that

\[
T > T_2 := 2T_{\text{min}}^2 + \frac{3\pi}{\lambda_2 - \lambda_1},
\]

\[
\frac{\pi}{\lambda_2 - \lambda_1} < T_1 < T - \frac{2\pi}{\lambda_2 - \lambda_1} - 2T_{\text{min}},
\]

\[
T_c < T_\theta, \quad T_c + T_\theta < \min \left\{ \frac{\pi}{\lambda_2 - \lambda_1} : T_1 - \frac{\pi}{\lambda_2 - \lambda_1} \right\},
\]

\[
T_{\text{min}}^2 < T_c^1 < \frac{1}{2} \left( T - T_1 - \frac{2\pi}{\lambda_2 - \lambda_1} \right).
\]

Since \( T_c^1 > T_{\text{min}}^2 \), there exists controls \((\nu_\pm, w_\pm) \in L^2((0, T_c^1), \mathbb{R})^2\), such that the associated solutions \( \Psi^\pm \) and \( \xi^\pm \) of (3.1) and (3.2) satisfy

\[
\Psi^\pm(T_c^1) = 0,
\]

\[
\langle \xi^\pm(T_c^1), \psi_1(T_c^1) \rangle = \pm i,
\]

\[
\langle \xi^\pm(T_c^1), \psi_k(T_c^1) \rangle = 0, \quad \forall k \geq 3.
\]

According to Proposition 5, there exists controls \((v^2_j, w^2_j) \in L^2((0, T_c^1), \mathbb{R})^2\) such that the associated solutions \((\Psi^2_j, \xi^2_j)\) of (4.11) and (4.22) satisfy

\[
\Psi^2(T_c^1) = 0,
\]

\[
\langle \xi^2(T_c^1), \psi_2(T_c^1) \rangle \neq 0,
\]

\[
\langle \xi^2(T_c^1), \psi_k(T_c^1) \rangle = 0, \quad \forall k \geq 3.
\]

**First step: Construction of a basis for \( M^2 \), with nonoverlapping controls.** Let

\[
\theta_1 := T - T_1, \quad \theta_2 := T - T_1 + T_\theta, \quad \theta_3 := T - T_1 + \frac{\pi}{\lambda_2 - \lambda_1}, \quad \theta_4 := T - T_1 + T_\theta + \frac{\pi}{\lambda_2 - \lambda_1}
\]

and \((v^2_j, w^2_j) := (v^\theta_j, w^\theta_j)\) for \( j = 1, ..., 4 \) with the notations of Lemma 4 (in which \((T, T_1)\) is replaced by \((T_c^1, T)\)). Then \( \text{supp}(v^2_j) \subset (\theta_j, \theta_j + T_c) \) for \( j = 1, ..., 4 \) and

\[
T - T_1 = \theta_1 < \theta_1 + T_c < \theta_2 < \theta_2 + T_c < \theta_3 < \theta_3 + T_c < \theta_4 < \theta_4 + T_c < T
\]
thanks to (4.17), thus the supports do not overlap:

\[
\forall j_1, j_2 \in \{1, 2, 3, 4\} \text{ with } j_1 \neq j_2 \text{ then Supp}(v^2_{j_1}) \cap \text{Supp}(v^2_{j_2}) = \emptyset.
\]

We denote by \((\Psi^2_j, \xi^2_j)\) the associated solutions of (3.1) and (3.2). Then, \( \Psi^2_j(T) = 0 \) and \( \xi^2_j(T) = f_j^2 + f_j^2 \) for \( j = 1, ..., 4 \) where (see Lemma 4)

\[
f_j^2 = \langle \xi^2(T_c^1), \psi_1(T_c^1) \rangle \psi_1(T), \quad f_j^2 = e^{i(\lambda_2 - \lambda_1)(T - T_1)} \langle \xi^2(T_c^1), \psi_2(T_c^1) \rangle \psi_2(T) \neq 0,
\]

\[
f_j^2 = f_j^2, \quad f_j^2 = e^{i(\lambda_2 - \lambda_1)T_\theta} f_j^2,
\]

\[
f_j^2 = f_j^2, \quad f_j^2 = e^{i(\lambda_2 - \lambda_1)(\frac{\pi}{\lambda_2 - \lambda_1} + T_\theta)} f_j^2 = -f_j^2,
\]

\[
f_j^2 = f_j^2, \quad f_j^2 = e^{i(\lambda_2 - \lambda_1)(\frac{2\pi}{\lambda_2 - \lambda_1} + T_\theta)} f_j^2 = -f_j^2.
\]

Moreover

\[
\Re\langle f_j^2, \psi_1(T) \rangle = \Re\langle \xi^2(T_c^1), \psi_1(T_c^1) \rangle = -\|\Psi^2(T_c^1)\|^2 = 0, \quad \forall j = 1, ..., 4.
\]

Note that \((\lambda_2 - \lambda_1)T_\theta \in (0, \pi)\), thus \((f_j^2, f_j^2)\) is a \( \mathbb{R} \)-basis of \( M^2 \). This leads to \( M^2 = \bigcup_{j=1}^4 M^2_j \) where

\[
M^1_1 = \{ d_1^2 f_1^2 + d_2^2 f_2^2 ; d_1^2 \geq 0, d_2^2 \geq 0 \},
\]

\[
M^1_2 = \{ d_1^2 f_1^2 + d_3^2 f_3^2 ; d_1^2 > 0, d_3^2 \geq 0 \},
\]

\[
M^1_3 = \{ d_1^2 f_1^2 + d_4^2 f_4^2 ; d_1^2 \geq 0, d_4^2 > 0 \},
\]

\[
M^1_4 = \{ d_1^2 f_1^2 + d_3^2 f_3^2 ; d_1^2 > 0, d_3^2 > 0 \}.
\]
Second step : Construction of a basis for $M^1$, with non overlapping controls. The time interval $(T^1_{c}, T - T_1 - T^1_{c})$ has length $(T - T_1 - 2T^1_{c}) > 2\pi/(\lambda_2 - \lambda_1)$ thanks to \([4.13]\), thus there exists an odd integer $k$ such that

$$A := \frac{k\pi}{\lambda_2 - \lambda_1} \in (T^1_{c}, T - T_1 - T^1_{c}).$$  

(4.22)

Let us consider the following controls

$$(V_{\pm}, W_{\pm})(t) := \begin{cases} (v_{\pm}, w_{\pm})(t) & \text{if } t \in (0, T^1_{c}), \\ (0, 0) & \text{if } t \in (T^1_{c}, A), \\ (v_{\pm}, w_{\pm})(t - A) & \text{if } t \in (A, A + T^1_{c}), \\ (0, 0) & \text{if } t \in (A + T^1_{c}, T), \end{cases}$$

Then $\text{supp}(V_{\pm}) \subset [0, T - T_1)$ thanks to \([4.22]\), thus

$$\forall j \in \{1, ..., 4\}, \text{Supp}(V_{\pm}) \cap \text{Supp}(v^2_{\pm}) = \emptyset.$$  

(4.23)

We denote by $(\Psi_{\pm}, \xi_{\pm})$ the associated solutions of $(3.1)$ and $(3.2)$. Then, $\Psi_{\pm}(T) = 0$ and

$$\xi_{\pm}(T) = \pm 2i\psi_1(T) + \langle \xi_{\pm}(T^1_{c}),\psi_2(T^1_{c})\rangle + [1 + e^{iA(\lambda_2 - \lambda_1)}] \psi_2(T) = \pm 2i\psi_1(T)$$

thanks to Lemma \([3]\) and Proposition \([10]\).

Third step : Conclusion. Let $z \in M$. Let us construct controls $(v, w) \in L^2((0, T), \mathbb{R})^2$ such that the associated solutions $(\Psi, \xi)$ of $(3.1)$ and $(3.2)$ satisfy $\Psi(T) = 0$ and $\xi(T) = z$. The proof relies on the two following facts:

1. $\pm 2i\psi_1(T)$ and $f^2_j + f^2_{j+1}$ for $j = 1, 2, 3, 4$ are reachable states, with controls such that their supports do not overlap (see \([4.19]\) and \([4.23]\)).

2. any vector in $M$ is a linear combination of 3 of these vectors, with only non negative coefficients before $f^2_j + f^2_{j+1}$.

There exists a unique $j \in \{1, 2, 3, 4\}$ such that $z \in M^1 + M^2_j$ (see \([4.21]\)). Then,

$$z = ix\psi_1(T) + d_1f^2_j + d_2f^2_{j+1}$$

for some $d_1, d_2 \geq 0, x \in \mathbb{R}$

with the convention $f^2_{-1} = f^2_5$. We have

$$z = \left( ix - d_1f^2_j - d_2f^2_{j+1} \right) + d_1\left( f^2_j + f^2_{j+1} \right) + d_2\left( f^2_{j+1} + f^2_{j+1} \right).$$

As $\text{Re}(\langle f^2_j, \psi_1(T) \rangle) = 0$, for all $j = 1, \ldots, 4$ (see \([4.20]\)), there exists $\kappa \in \{+, -\}$ and $c \geq 0$ such that

$$ix - d_1f^2_j - d_2f^2_{j+1} = \kappa 2ic\psi_1(T).$$

Then,

$$z = \kappa 2ic\psi_1(T) + d_1(f^2_j + f^2_{j+1}) + d_2(f^2_{j+1} + f^2_{j+1}),$$

i.e. $z$ is a linear combination of 3 states that are reachable with non overlapping controls. Hence the map

$$\Lambda_T(z) := (v, w) := \left( \sqrt{c}V_\kappa + \sqrt{d_1}v^2_j + \sqrt{d_2}v^2_{j+1}, cW_\kappa + d_1w^2_j + d_2w^2_{j+1} \right),$$

gives the conclusion.
Now, let us explain the adaptation of this strategy for \( N \geq 3 \). As previously, we denote by \( K_1 < \cdots < K_N \) the directions missed at the first order and we explain how to reach a basis of missed directions on the second order \([72]\), iteratively.

The first step consists in reaching a \( \mathbb{R}^+ \) basis of \( M^N \), the projections on \( M^1, \ldots, M^{N-1} \) being possibly non zero. This is done as in the first step of the proof of Proposition \([8]\) by designing four controls with non overlapping supports. It is done in any time \( T_1 > \frac{\pi}{\lambda_{K_N} - \lambda_1} \).

The \((k + 1)\)th step consists in reaching a \( \mathbb{R}^+ \) basis of \( M^{N-k} \) while driving to zero the projections on \( M^j \), for \( j = N - k + 1, \ldots, N \). This can be done iteratively in the following way. Let \((v^{(0)}, w^{(0)})\) be as in Proposition \([9]\) for a sufficiently small time and for \( j = N - k \).

Then, the controls 
\[
(v^{(1)}, w^{(1)}) := (v^{(0)}, w^{(0)}) + (v_{\theta}^{(0)}, w_{\theta}^{(0)}), \quad \text{with } \theta = \frac{\pi}{\lambda_{K_N} - \lambda_1}
\]
drive the projection on \( M^N \) to zero while the projection on \( M^{N-k} \) is still non zero, thanks to Lemma \([4]\). We iterate this construction 
\[
(v^{(j+1)}, w^{(j+1)}) = (v^{(j)}, w^{(j)}) + (v_{\theta}^{(j)}, w_{\theta}^{(j)}), \quad \text{with } \theta = \frac{\pi}{\lambda_{K_N-j} - \lambda_1} \quad \text{for } j = 0, \ldots, k - 1.
\]

Then the controls \((v, w) = (v^{(k)}, w^{(k)})\) drive the projection on \( M^N, \ldots, M^{N-k+1} \) to zero while the projection on \( M^{N-k} \) is still non zero. Finally, we can find \( T_2 \) sufficiently small such that \((v, w)\) and \((v_{T_2}, w_{T_2})\) have non overlapping supports and the four pairs of control \((v, w), (v_{T_2}, w_{T_2}), (v_{T_2}, w_{T_2}), (v_{T_2}, w_{T_2})\) with \( p = \frac{\pi}{\lambda_{K_N-k} - \lambda_1} \) allows to conclude the \((k + 1)\)th step. This can be done in any time \( T > \frac{\pi}{\lambda_{K_N-k} - \lambda_1} + \cdots + \frac{\pi}{\lambda_{K_N} - \lambda_1} \).

The final step depends on the value of \( K_1 \). If \( K_1 \geq 2 \), we end with the same strategy. If \( K_1 = 1 \), the elementary brick of control cannot be designed in arbitrary small time but in time greater than \( T^2_{\min} \). This is why the expression of \( T_2 \) changes when \( K_1 = 1 \).

Figure 1 illustrates the distribution of controls during the \(4\)th step with \( p_j := \frac{\pi}{\lambda_{K_N-j} - \lambda_1} \).

\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Proof of Proposition 11:
In order to prove that \( \rho > C \), thanks to Proposition 1, there exists \( T_1 \) such that

\[
\| \psi_f - \psi_1(T) \|_{H^{3}_{(0)}} < \delta_1.
\]  
(4.24)

One may assume that \( \delta_1 \) small enough so that condition (4.24) implies \( \Re(\psi_f, \psi_1(T)) > 0 \). We introduce the map

\[
F_{\psi_f} : M \cap B_{L^2(0,1)}(0, \rho) \to M, \quad z \mapsto \mathcal{P}_M[\psi_z(T)]
\]

where

- \( \rho \in (0, 1) \) will be chosen later on,
- \( \mathcal{P}_M : L^2(0, 1) \to M \) is the \( L^2 \)-orthogonal projection on \( M \)

\[
\mathcal{P}_M(\zeta) := i\Im(\langle \zeta, \psi_1(T) \rangle) \psi_1(T) + \sum_{j=2}^{N} \langle \zeta, \psi_{K_j}(T) \rangle \psi_{K_j}(T).
\]

- \( \psi_z \) is the solution of (1.1)-(1.4) associated to the control \( u_z \) defined by

\[
u_z := \begin{cases} 
\sqrt{||v_z||^2 + ||z||^2} & \text{on } (0, T_1) , \\
\Gamma_{[T_1,T]}(\psi_z(T_1), \mathcal{P}_T[\psi_f]) & \text{on } (T_1, T).
\end{cases}
\]

where \( ||.|| \) is the \( L^2(0,1) \)-norm and

\[
(v_z, u_z) := \Lambda_{T_1}(\frac{e^{iA(T-T_1)}z}{||z||}).
\]

Note that, for every \( z \), \( \mathcal{P}_T[\psi_z(T)] = \mathcal{P}_T[\psi_f] \). Thus, our goal is to find \( z^* \) such that \( F_{\psi_f}(z^*) = \mathcal{P}_M[\psi_f] \).

First, let us check that the map \( F_{\psi_f} \) is well defined when \( \rho \) is small enough.

**Proposition 11** There exists \( \rho > 0 \) such that, for every \( \psi_f \in H^3_{(0)}(0, 1) \) with (4.24), the map \( F_{\psi_f} \) is well defined and continuous on \( M \cap B_{L^2(0,1)}(0, \rho) \).

**Proof of Proposition 11.** In order to prove that \( F_{\psi_f} \) is well defined, it is sufficient to find \( \rho > 0 \) such that

\[
||z|| < \rho \quad \Rightarrow \quad ||\psi_z(T_1) - \psi_1(T_1)||_{H^{3}_{(0)}} < \delta_1.
\]

(4.25)

Thanks to Proposition 1 there exists \( C_1, C'_1 > 0 \) such that, for every \( z \in M \),

\[
||\psi_z(T_1) - \psi_1(T_1)||_{H^{3}_{(0)}} \leq C_1 ||u_z||_{L^2(0,T_1)} \leq C'_1 \sqrt{||z||}.
\]

Thus, (4.25) holds with \( \rho := \min\{1; (\delta_1/C'_1)^2\} \). The continuity of \( F_{\psi_f} \) is a consequence of the continuity of \( \Gamma_{[T_1,T]} \) and the continuity of the solutions of (1.1)-(1.4) with respect to the control \( u \) and the initial condition \( \psi_0 \) (see (2.3)).

One may assume \( \rho \) small enough so that

\[
||z|| < \rho \quad \Rightarrow \quad \Re(\psi_z(T), \psi_1(T)) > 0.
\]

The goal of this section is the proof of the following result, which proves Theorem 8.
Proposition 12 There exists $\delta \in (0, \delta_1]$ such that, for every $\psi_f \in H^3_{(0)}(0,1)$ with

$$\|\psi_f - \psi_1(T)\|_{H^3_{(0)}(0,1)} < \delta$$

(4.26)

there exists $z_* = z_*(\psi_f) \in M \cap B_{L^2}(0, \rho)$ such that $F_{\psi_f}(z_*) = P_M[\psi_f]$. Moreover, $z_*(\psi_f) \to 0$ when $\psi_f \to \psi_1(T)$ in $H^3_{(0)}(0,1)$.

Combining $P_T[z_*(T)] = P_T[\psi_f]$, Proposition 12 and $\|\psi_z(T)\|_{L^2} = \|\psi_f\|_{L^2}$ ends the proof of Theorem 4. The proof of Proposition 12 requires the following preliminarily result.

Proposition 13 There exists $C > 0$ such that, for every $\psi_f \in H^3_{(0)}(0,1)$ with (4.24) and $z \in M \cap B_{L^2(0,1)}(0, \rho)$, we have

$$\|F_{\psi_f}(z) - z\| \leq C\|\psi_f - \psi_1(T)\|^2_{H^3_{(0)}} + \|z\|^{3/2}.$$  

Proof of Proposition 13 First step: Let us prove the existence of $C_1 > 0$ such that

$$\|\psi_z(T_1) - \psi_1(T_1) - e^{i\lambda(T-T_1)}z\|_{H^3_{(0)}} \leq C_1\|z\|^{3/2}, \quad \forall z \in M \cap B_{L^2(0,1)}(0, \rho).$$

(4.27)

Let $\Psi_z, \xi_z$ be the solution of (3.1) and (3.2) associated to the controls $v_z$ and $w_z$. Then, $\Psi_z(T_1) = 0$ and $\xi_z(T_1) = e^{i\lambda(T-T_1)}z/\|z\|$. Thanks to Proposition 1 there exists $C > 0$ such that

$$\|\psi_z - \psi_1 - \sqrt{\|z\|}\Psi_z - \|z\|\xi_z\|_{L^\infty(0,T),H^3_{(0)}} \leq C\|z\|^{3/2}, \forall z \in M \cap B_{L^2(0,1)}(0, \rho),$$

which gives (4.27).

Second step: Let us prove the existence of $C_2 > 0$ such that

$$\|u_z\|_{L^2(T_1,T)} \leq C_2\|\psi_z - \psi_1(T)\|^2_{H^3_{(0)}} + \|z\|, \quad \forall z \in M \cap B_{L^2(0,1)}(0, \rho).$$

(4.28)

The map $\Gamma[T_1,T]$ is $C^1$ and $\Gamma[T_1,T](\psi_1(T_1),0) = 0$, thus there exists $C > 0$ such that, for every $z \in M \cap B_{L^2(0,1)}(0, \rho)$,

$$\|\Gamma[T_1,T](\psi_2(T_1),P_T[\psi_f])\|_{L^2(T_1,T)} \leq C\|\psi_2(T_1) - \psi_1(T_1)\|_{H^3_{(0)}} + \|P_T[\psi_f]\|_{H^3_{(0)}}.$$

We get (4.28) thanks to (4.27).

Third step: Let us prove the existence of $C_3 > 0$ such that

$$\|\psi_z - \psi_1\|_{L^\infty((T_1,T),H^3_{(0)})} \leq C_3\|\psi_z - \psi_1(T)\|^2_{H^3_{(0)}} + \|z\|, \quad \forall z \in M \cap B_{L^2(0,1)}(0, \rho).$$

(4.29)

Thanks to (4.27) and Proposition 1 we get a constant $C > 0$ such that

$$\|\psi_z - \psi_1\|_{L^\infty((T_1,T),H^3_{(0)})} \leq C\|z\| + \|u_z\|_{L^2(T_1,T)}, \forall z \in M \cap B_{L^2(0,1)}(0, \rho),$$

which gives (4.29) thanks to (4.28).

Fourth step: Conclusion. Thanks to the Duhamel formula, we get for every $z \in M \cap B_{L^2(0,1)}(0, \rho)$

$$\|F_{\psi_f}(z) - z\| = \|P_M[\psi_z(T)] - z\|$$

$$\leq \|P_M[e^{-i\lambda(T-T_1)}\psi_1(T_1)] - z\| + \int_{T_1}^T |F_{\psi_f}(z)|d\tau$$

$$\leq \|P_M[e^{-i\lambda(T-T_1)}\psi_1(T_1) - \psi_1(T_1) + e^{i\lambda(T-T_1)}z]\| + \int_{T_1}^T |u_z(\tau)|d\tau + \int_{T_1}^T |\psi_z - \psi_1(T_1)|d\tau$$

$$\leq C_1\|z\|^{3/2} + \sqrt{T - T_1}C_2C_3\|\psi_z - \psi_1(T)\|_{H^3_{(0)}} + \|z\|^2$$

$$\leq C(\rho)\|z\|^{3/2} + \|\psi_f - \psi_1(T)\|_{H^3_{(0)}}$$

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To pass from the first line to the second one, we use the commutativity between $e^{iAt}$ and $P_M$, the isometry on $L^2(0,1)$ of $e^{iAt}$. To pass from the second line to the third one, we use the relation $P_M[\mu \psi_1(t)] \equiv 0$ (that holds because $(\mu \varphi_1, \varphi_K) = 0$ for $j = 1, \ldots, N$).

**Proof of Proposition 12** We introduce the map

$$G_{\psi_f} : M \cap B_{L^2(0,1)}(0, \rho) \to M, \quad z \mapsto z + P_M[\psi_f] - F(z).$$

Our goal is to prove the existence of a fixed point $z_s = z_s(\psi_f)$ to the map $G_{\psi_f}$. Thanks to Proposition 13, there exists $\mathcal{C} > 0$ (independent of $\psi_f$) such that, for every $z \in M \cap B_{L^2(0,1)}(0, \rho)$,

$$\|G_{\psi_f}(z)\| \leq \|z - F(z)\| + \|P_M[\psi_f]\| \leq \mathcal{C}[\|\psi_f - \psi_1(T)\|_{H^3_{(0)}}^2 + \|z\|^{3/2}] + \|\psi_f - \psi_1(T)\|_{H^3_{(0)}}. \quad (4.30)$$

Let $\rho' \in (0, \rho)$ be such that

$$\mathcal{C} \sqrt{\rho'} < 1/2 \quad (4.31)$$

and $\delta \in (0, \delta_1)$ be such that $\mathcal{C} \delta^2 + \delta < \rho'/2$. If $\psi_f$ satisfies (4.29), then $G_{\psi_f}$ maps continuously $M \cap B_{L^2(0,1)}(0, \rho')$ into itself. Thanks to the Brouwer fixed point theorem, there exists a fixed point $z_s = z_s(\psi_f)$ of $G_{\psi_f}$ in $M \cap B_{L^2(0,1)}(0, \rho')$. We deduce from (4.30) and (4.31) that

$$\|z_s(\psi_f)\| \leq 2[\mathcal{C}\|\psi_f - \psi_1(T)\|_{H^3_{(0)}}^2 + \|\psi_f - \psi_1(T)\|_{H^3_{(0)}}],$$

thus $z_s(\psi_f) \to 0$ when $\psi_f \to \psi_1(T)$ in $H^3_{(0)}(0,1)$.

**5 Proof of Theorem 6**

In this section, we prove Theorem 6 when $\mu'(0) = \mu'(1) \neq 0$. The case $\mu'(0) = -\mu'(1) \neq 0$ may be proved similarly. The strategy is similar to the one of the previous section, except that, for some lost directions, the second order may vanish and thus, we need to go to a higher order. We prove that the third order is sufficient.

**5.1 Heuristic**

Let us consider a control $u$ of the form $u = \psi = \psi_1 + \epsilon \Psi + \epsilon^2 \xi + \epsilon^3 \zeta + o(\epsilon^3)$, where $\Psi$ and $\zeta$ solve (3.1) and (3.2) and

$$\begin{cases}
    i \partial_t \zeta = -\partial_x^2 \zeta - v(t)\mu(x)\zeta - w(t)\mu(x)\Psi - \nu(t)\mu(x)\psi_1, & (t, x) \in (0, T) \times (0, 1), \\
    \zeta(t, 0) = \zeta(t, 1) = 0, & t \in (0, T), \\
    \zeta(0, x) = 0. & x \in (0, 1).
\end{cases} \quad (5.1)$$

Let us assume that $K \in \mathbb{N}^*$ satisfies $\langle \mu \varphi_1, \varphi_K \rangle = 0$ and that the quadratic form $Q^2_{K,T}$ vanishes on $V_T$ (see Proposition 3). Then, one may prove that for any $v \in V_T$, $
abla \langle \zeta(T), \psi_K(T) \rangle = Q^2_{K,T}(v)$ where $Q^2_{K,T}$ is the cubic form

$$Q^2_{K,T}(v) := \int_0^T v(t_1) \int_0^{t_1} v(t_2) \int_0^{t_2} v(t_3) h(t_1, t_2, t_3) dt_3 dt_2 dt_1,$$

$$h^2_{K,T}(t_1, t_2, t_3) := -i \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} B_{j_1, j_2} e^{i[\lambda_K - \lambda_{j_1}]t_1 + [\lambda_{j_1} - \lambda_{j_2}]t_2 + [\lambda_{j_2} - \lambda_1]t_3},$$

$$B_{j_1, j_2} := \langle \mu \varphi_K, \varphi_{j_1} \rangle (\mu \varphi_{j_1}, \varphi_{j_2}) (\mu \varphi_{j_2}, \varphi_1).$$
Proposition 14 Let $\mu \in H^3((0,1), \mathbb{R})$ and $K \in \mathbb{N}^*$ be such that $\mu'(0) = \mu'(1) \neq 0$ and $\langle \mu \varphi_1, \varphi_K \rangle = 0$. Then,

- either, for every $A > 0$, there exists $n \geq A$ such that $\langle \mu \varphi_K, \varphi_n \rangle / \langle \mu \varphi_n, \varphi_1 \rangle \neq 0$

- or, for every $A > 0$, there exists $n_1, n_2 \geq A$ such that $\langle \mu \varphi_K, \varphi_{n_1} \rangle / \langle \mu \varphi_{n_1}, \varphi_{n_2} \rangle / \langle \mu \varphi_{n_2}, \varphi_1 \rangle \neq 0$.

Proof: The proof relies on the equality (4.5). If $K$ is odd, we are in the first case. If $K$ is even and if we are not in the first situation, then, the second situation holds: consider $n_1$ odd, $n_2$ even, both large enough.

The previous and next propositions show that any lost direction (at the first order) is recovered either at the second order, or at the third order.

Proposition 15 Let $\mu \in H^3((0,1), \mathbb{R})$, $K \in \mathbb{N}^*$ be such that

$$
\langle \mu \varphi_K, \varphi_{n_1} \rangle / \langle \mu \varphi_{n_1}, \varphi_{n_2} \rangle / \langle \mu \varphi_{n_2}, \varphi_1 \rangle \neq 0 \text{ for some } n_1, n_2 > K.
$$

Then, $Q_{K,T}^U \neq 0$ on $V_T$, $\forall T > 0$.

Proof of Proposition 15 To simplify the notations, we write $Q_T$ and $h$ instead of $Q_{K,T}^U$ and $h'_K$. Working by contradiction, we assume that $Q_T \equiv 0$ on $V_T$, for every $T < T^*$. Then $\nabla Q_T(v) \perp V_T$, for every $v \in V_T$ and $T < T^*$. Easy computations show that, for $v \in V_T$,

$$
\nabla Q_T(v) : t_3 \mapsto \int_{(0,T)^2} v(t_1)v(t_2) [\tilde{h}(t_1,t_2,t_3) + \tilde{h}(t_1,t_3,t_2) + \tilde{h}(t_3,t_2,t_1)] dt_1 dt_2
$$

where $\tilde{h}(t_1,t_2,t_3) := h(t_1,t_2,t_3)1_{t_1 > t_2 > t_3}$. Let $v \in V_T$ with a compact support $(a,b) \subset (0,T)$. For $t_3 \in (0,a)$, we have

$$
\nabla Q_T(v)(t_3) = -\sum_{k_2=1}^{\infty} \alpha_{k_2}(v) e^{i(\lambda_{k_2} - \lambda_1)t_3}
$$

where

$$
\alpha_{k_2}(v) := \int_{(a,b)^2} v(t_1)v(t_2) \sum_{k_1=1}^{\infty} B_{k_1,k_2} e^{i[(\lambda K - \lambda k_1)t_1 + (\lambda k_1 - \lambda k_2)t_2]} dt_2 dt_1.
$$

We know that $\nabla Q_T(v)$ belongs to $\text{Adh}_{L^2(0,T)}(\text{Span}\{e^{k_i(\lambda_j - \lambda_1)t}; j \in J\})$ because $\nabla Q_T(v) \perp V_T$. The uniqueness of the decomposition on a Riesz basis ensures that

$$
\nabla Q_T(v)(t_3) = \sum_{k_2=1}^{\infty} \alpha_{k_2}(v) e^{i(\lambda_{k_2} - \lambda_1)t_3}, \forall t_3 \in (0,T).
$$

For $t_3 \in (b,T)$, we have

$$
\nabla Q_T(v)(t_3) = \sum_{k_1=1}^{\infty} \langle \mu \varphi_K, \varphi_{k_1} \rangle Q_{k_1,T}^2(v) e^{i(\lambda K - \lambda k_1)t_1}.
$$

Thus,

$$
\sum_{k_1=1}^{\infty} \langle \mu \varphi_K, \varphi_{k_1} \rangle Q_{k_1,T}^2(v) e^{i(\lambda K - \lambda k_1)t_1} = \sum_{k_2=1}^{\infty} \alpha_{k_2}(v) e^{i(\lambda_{k_2} - \lambda_1)t_3}, \forall b < t_3 < T.
$$
Notice that the frequencies \((\lambda_k - \lambda_{k_1})\) in the left hand side are < 0 when \(k_1 > K\), and the frequencies \((\lambda_{k_2} - \lambda_1)\) in the right hand side are ≥ 0. The functions compactly supported on \((0, T)\) are dense in \(V_T\), thus

\[
\langle \mu \varphi_K, \varphi_{k_1} \rangle Q_{k_1, T}^2 (v) = 0, \forall k_1 > K.
\]

But \(C^0_0(0, T) \cap V_T\) is dense in \(V_T\), thus

\[
\langle \mu \varphi_K, \varphi_{k_1} \rangle Q_{k_1, T}^2 \equiv 0, \forall k_1 > K. \tag{5.2}
\]

Let \(n_1, n_2 > K\) be such that \(\langle \mu \varphi_K, \varphi_{n_1} \rangle \langle \mu \varphi_{n_1}, \varphi_{n_2} \rangle \langle \mu \varphi_{n_2}, \varphi_1 \rangle \neq 0\). In particular, \(\langle \mu \varphi_K, \varphi_{k_1} \rangle \neq 0\) and \(Q_{n_1, T}^2 \neq 0\) on \(V_T\), for every \(T > 0\) thanks to Proposition \[5\]. This is in contradiction with (5.2).

**Remark 4** Note that the third order may be necessary. For example, with \(\mu(x) := x - \langle x \varphi_1, \varphi_K \rangle \varphi_K/\varphi_1\), where \(K \in \mathbb{N}\) is even, we have \(\langle \mu \varphi_1, \varphi_n \rangle \langle \mu \varphi_n, \varphi_K \rangle = 0, \forall n \in \mathbb{N}^*\), thus \(Q_{3, K, T}^2 \equiv 0\).

### 5.2 Changes with respect to the proof of Section 4

Let \(\mathcal{N}_N := \{k \in \mathbb{N}^*; k \text{ is odd and } k \leq N \text{ or } k \text{ is even}\}\). Using, (4.4), it comes that \(\{K \in \mathcal{N}_N ; \langle \mu \varphi_1, \varphi_K \rangle = 0\}\) is finite. Thus, there exists \(p \in \mathbb{N}^*\) and \(K_1 < \ldots < K_p \in \mathbb{N}^*\) such that for any \(j \in \{1, \ldots, p\}\), \(\langle \mu \varphi_1, \varphi_{K_j} \rangle = 0\). The estimate (4.3) also implies the existence of \(C > 0\) such that

\[
|\langle \mu \varphi_k, \varphi_{k} \rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathcal{N}_N - \{K_1, \ldots, K_p\}.
\]

For any \(T > 0\), Propositions [13] and [14] imply that for any \(j \in \{1, \ldots, p\}\), if \(Q_{K_j, T}^2\) vanishes on \(V_T\), then \(Q_{3, K_j, T}^2 \equiv 0\) on \(V_T\).

Let \(K(2) := \{j \in \{1, \ldots, p\}; Q_{K_j, T}^2 \neq 0\}\) and \(K(3) := \{1, \ldots, p\} - K(2)\). The spaces \(M^j\) and \(M\) are defined as in (4.7), (4.3). Let us define

\[
M(2) := \bigoplus_{j \in K(2)} M^j, \quad M(3) := \bigoplus_{j \in K(3)} M^j.
\]

Thus, \(M = M(2) \oplus M(3)\). Proposition [5] holds with \(M\) replaced by \(M(2)\). As \(Q_{K_j, T}^3(-v) = -Q_{K_j, T}^3(v)\), we can reach a basis of \(M(3)\) on the third order in arbitrary time. More precisely, the following proposition holds.

**Proposition 16** Let \(T > 0\). There exists a continuous map

\[
\Lambda_T : \begin{array}{c}
M(3) \\
\downarrow z
\end{array} \to \begin{array}{c}
L^2((0, T), \mathbb{R})^2 \\
\quad (v, w, \nu)
\end{array}
\]

such that, for every \(z \in M(3)\), the solutions \(\Psi, \xi, \zeta\) of (5.4), (5.2) and (5.1) satisfy \(\Psi(T) = 0, \xi(T) = 0\) and \(\zeta(T) = z\).

Finally, Theorem [5] is proved, as in Section [4.5] using a fixed point argument where the control \(u_z\) is defined by

\[
u_z := \begin{cases}
\sqrt{\|v_z\|^2 + \|z_2\|^2 + \|v_{z_2}\|^2 + \|z_3\|^2 / \lambda \nu_{z_2}} + \|z_3\|^2 / \lambda \nu_{z_3} + \|z_3\| \nu_{z_3} & \text{on } (0, T_1), \\
1/|T, T_1| (\psi z(T_1), P_T[\psi f]) & \text{on } (T_1, T).
\end{cases}
\]

where \(z_2 + z_3 = z\) with \((z_2, z_3) \in M(2) \times M(3)\) and

\[
(v_{z_2}, w_{z_2} := \Lambda_{T_1} \left( \frac{e^{iA(T-T_1)}z_2}{\|z_2\|} \right),
\]
there exists \( T \). Lemma 2 ensures that (see (3.5) for the definition of the set \( J \)).

Proposition 17 Let \( \tilde{\mu} \) associated to the initial conditions \( \tilde{\mu\phi} \) \( \in V \). Proposition 8 is the only step requiring a minimal time, it has to be noticed that if \( K^{(2)} = 0 \), Theorem 7 holds in arbitrary time.

6 A first step to the characterization of the minimal time

In this section, we focus on the system

\[
\begin{aligned}
\begin{cases}
i\partial_t\psi(t, x) &= -\partial_x^2 \psi(t, x) - u(t)\mu(x)\psi(t, x), \quad (t, x) \in \mathbb{R} \times (0, 1), \\
\psi(t, 0) &= \psi(t, 1) = 0, \quad t \in \mathbb{R}, \\
s'(t) &= u(t),
\end{cases}
\end{aligned}
\tag{6.1}
\]

associated to the initial conditions

\[
(\psi, s)(0) = (\varphi_1, 0).
\tag{6.2}
\]

We consider a dipolar moment \( \mu \in H^3((0, 1), \mathbb{R}) \) such that \( \langle \mu\varphi_1, \varphi_1 \rangle = 0 \) (for instance \( \mu(x) = (x - 1/2) \)). We use the notation \( Q_T \) instead of \( Q_{1, T} \) (see (3.14)-(3.15)), \( Q_T \) instead of \( Q_{1, T}^2 \) (see (3.7)-(3.8)), \( k(t, \tau) \) instead of \( k_{1, T}(t, \tau) \) and the spaces

\[
\begin{aligned}
V_T^1 &:= \left\{ v \in L^2(0, T); \int_0^T v(t)e^{i\omega_j t} dt = 0, \forall j \in J \cup \{1\} \right\} \\
V_T &:= \left\{ S \in L^2((0, T), \mathbb{R}); \int_0^T S(t)e^{i\omega_j t} dt = 0, \forall j \in J \setminus \{1\} \right\}
\end{aligned}
\]

(see (3.5) for the definition of the set \( J \)). Let us introduce the quantities

\[
T_{min}^1 := \sup\{ T \geq 0; \quad Q_T \leq 0 \text{ on } V_T \},
\]

\[
T_{min}^2 := \inf\{ T > 0; \exists S_{\pm} \in V_T \cap H^1_0(0, T) \text{ such that } Q_T(S_{\pm}) = \pm 1 \}.
\tag{6.3}
\]

Lemma 2 ensures that \( T_{min}^1 > 0 \) and the following proposition justifies the existence of \( T_{min}^2 \).

Proposition 17 Let \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that \( \langle \mu\varphi_1, \varphi_1 \rangle = 0 \). For every \( T > 2/\pi \), there exists \( S_{\pm} \in V_T \cap H^1_0(0, T) \) such that \( Q_T(S_{\pm}) = \pm 1 \); or, equivalently, there exists \( v_{\pm} \in V_T^1 \) such that \( Q_T(v_{\pm}) = \pm 1 \). Thus,

\[
0 < T^*_1 < T_{min}^1 \leq T_{min}^2 \leq \frac{2}{\pi},
\]

where \( T^*_1 \) is as in Lemma 2.

This Proposition may be proved as Lemma 3. The goal of this section is the proof of the following theorem.

Theorem 9 Let \( \mu \in H^3((0, 1), \mathbb{R}) \) be such that

\[
\langle \mu\varphi_1, \varphi_1 \rangle = 0 \quad \text{and} \quad \exists e > 0 \text{ such that } \frac{e}{K^3} \leq |\langle \mu\varphi_1, \varphi_k \rangle|, \forall k \in J.
\tag{6.4}
\]

1. For every \( T < T_{min}^1 \), there exists \( e > 0 \) such that, for every \( u \in L^2((0, T), \mathbb{R}) \) with \( (1.12) \) the solution of (6.1)-(6.3) satisfies \( (\psi, s)(T) \neq (|\sqrt{1 - \delta^2} + i\delta|\psi_1(T), 0) \) for every \( \delta > 0 \).
2. If, moreover \( J = \mathbb{N}^* - \{1\} \), then, for every \( T > T_{\text{min}}^2 \), the system (6.3) is controllable in \( H^1_0((0,1)) \times \mathbb{R} \), locally around the ground state \( (\psi = \psi_1, s \equiv 0) \), in time \( T \), with controls \( u \in L^2((0,T),\mathbb{R}) \).

In particular, when \( J = \mathbb{N}^* - \{1\} \), the minimal time \( T_{\text{min}} \) required for the local controllability satisfies \( T_{\text{min}} \in [T_{\text{min}}^1, T_{\text{min}}^2] \).

Remark 6 The equality between \( T_{\text{min}}^1 \) and \( T_{\text{min}}^2 \) is an open problem, equivalent to the question addressed in the next paragraph.

Let \( P_T \) be the orthogonal projection from \( L^2((0,T),\mathbb{R}) \) to the closed subspace \( \mathcal{V}_T \) and \( K_T \) be the compact self adjoint operator on \( L^2((0,T),\mathbb{R}) \) defined by

\[
K_T := P_T \left[ t \mapsto \int_0^t k(t,\tau)S(\tau)d\tau \right].
\]

We know that

- for any \( T < T_{\text{min}}^1 \) all the eigenvalues of \( K_T \) are \( < A_1 \) (see the first statement of Theorem 9),
- for any \( T > T_{\text{min}}^1 \), the largest eigenvalue of \( K_T \) is \( > A_1 \). (by definition of \( T_{\text{min}}^1 \)).

For \( T > T_{\text{min}}^1 \), does the associated eigenvector belong to \( H^1_0((0,T),\mathbb{R}) \)?

The proof of the second statement of Theorem 9 may be done exactly as in Section 4. Indeed, when \( J = \mathbb{N}^* - \{1\} \), then

1. the vector space \( M \) of lost directions (at the first order) is \( i\mathbb{R}\psi_1(T) \),
2. for any \( T_1 \in (0,T) \), the controls \( S_{\pm} \in \mathcal{V}_{T_1} \cap H^1_0((0,T)) \) allow to reach the states \( \pm i\psi_1(T) \) with the second order term; moreover, \( (i\psi_1(T), -i\psi_1(T)) \) is an '\( \mathbb{R}^+ \)-basis' of \( M \).

Thus, in this section, we focus only on the proof of the first statement of Theorem 9 which is a direct consequence of the following result.

**Theorem 10** Let \( \mu \in H^3((0,1),\mathbb{R}) \) that satisfies (6.4). For every \( T < T_{\text{min}}^1 \), there exists \( \epsilon > 0 \) such that for every \( s \in L^2((0,T),\mathbb{R}) \) with \( \|s\|_{L^2} < \epsilon \), the solution of the Cauchy problem (3.13)-(3.22) satisfies \( \psi(T) \neq (\psi_1(T) + i\delta)\psi_1(T), \forall \delta > 0 \).

In section 6.1 we state several preliminary results for the proof of Theorem 10 which is detailed in section 6.2.

### 6.1 Preliminaries

For \( T > 0 \) and \( \eta > 0 \), we introduce the sets

\[
\mathcal{V}_{T,\eta} := \left\{ S \in L^2(0,T); \left\| \int_0^T S(t)e^{i\omega_j t}dt \right\|_{L^2} \leq \eta \|S\|_{L^2(0,T)} \right\} \tag{6.5}
\]

(see (3.5) for the definition of \( J \)).

**Proposition 18** For every \( T < T_{\text{min}}^1 \), there exists \( \lambda = \lambda(T), \eta = \eta(T) > 0 \) such that

\[
Q_T(S) \leq -\lambda(T)\|S\|_{L^2(0,T)}^2, \quad \forall S \in \mathcal{V}_T, \tag{6.6}
\]

\[
Q_T(S) \leq -\frac{\lambda(T)}{2}\|S\|_{L^2(0,T)}^2, \quad \forall S \in \mathcal{V}_{T,\eta}. \tag{6.7}
\]

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Using the first step and Cauchy-Schwarz inequality, we get
\[ \|C\|_{L^2(0,T)} \leq \lambda(T). \]

In particular, let \( T_1 \in (0,T_{\min}) \),

\[ S_{\ast}(t) = \sum_{j \in J_{-1}} a_j e^{i\omega_j t} \text{ in } L^2(0,T_1). \]

However, we have
\[ \nabla Q_{T_1} S_{\ast}(t) = -AS_{\ast}(t) + \int_0^t S_{\ast}(\tau)k(t,\tau)d\tau, \forall t \in (0,T_1). \]

In particular, \( \nabla Q_{T_1} S_{\ast} \equiv 0 \) on \( (T,T_1) \) thus (Ingham) \( a_j \equiv 0 \). We have proved that
\[ S_{\ast}(t) = \frac{1}{A} \int_0^t S_{\ast}(\tau)k(t,\tau)d\tau, \forall t \in (0,T). \]

Thus, \( S_{\ast}(0) = 0, S_{\ast} \in H^1((0,T),\mathbb{R}) \) and \( S_{\ast}' \) satisfies the same relation. Iterating this result, we get \( S_{\ast}^{(n)}(0) = 0 \) and \( S_{\ast}^{(n)} \in \text{Ker}(-AI + K) \) for every \( n \in \mathbb{N} \). But \( K \) is compact, so \( \dim[\text{Ker}(-AI + K)] < +\infty \). Thus there exists \( N \in \mathbb{N}^* \) and \( a_0, \ldots, a_{N-1} \in \mathbb{R} \) such that
\[ \begin{cases} S_{\ast}^{(N)} = a_0 S_{\ast} + a_1 S_{\ast}' + \cdots + a_{N-1} S_{\ast}^{(N-1)} \\ S_{\ast}(0) = 0, \ldots, S_{\ast}^{(N-1)}(0) = 0 \end{cases} \]

Therefore \( S_{\ast} = 0 \), which is a contradiction.

**Proof of Proposition 18**

First step: Proof of (6.6):

For \( T \in (0,T_{\min}) \), we define the quantity \( \lambda(T) \geq 0 \) by
\[ -\lambda(T) := \sup\{Q_T(S); S \in V_T, \|S\|_{L^2(0,T)} = 1\}. \]

(6.8)

First, let us emphasize that, if \( \lambda(T) \leq 0 \), then there exists \( S \in V_T \) such that \( \|S\|_{L^2(0,T)} = 1 \) and \( Q_T(S) = \lambda(T) \) (consider a weak \( L^2(0,T) \)-limit, of a maximizing sequence and use the compactness of the operator \( K : L^2(0,T) \rightarrow L^2(0,T) \) defined by \( KS : t \mapsto \int_0^T S(\tau)k(t,\tau)d\tau \).

Let us assume that there exists \( T \in (0,T_{\min}) \) such that \( \lambda(T) = 0 \). Let \( T_1 \in (T,T_{\min}) \). Let \( S_{\ast} \in V_T \) such that \( \|S_{\ast}\|_{L^2(0,T)} = 1 \) and \( Q_T(S_{\ast}) = 0 \). We extend \( S_{\ast} \) on \((T,T_1)\) by zero. Then, \( S_{\ast} \in V_{T_1} \) and \( Q_{T_1}(S_{\ast}) = \max\{Q_{T_1}(S); S \in V_{T_1}\} = 0 \) thus (Euler equation) \( \nabla Q_{T_1}(S_{\ast}) \perp V_{T_1} \), i.e. there exists a unique sequence \( (a_j)_{j \in J_{-1}} \in l^2 \) such that
\[ \nabla Q_{T_1} S_{\ast}(t) = \sum_{j \in J_{-1}} a_j e^{i\omega_j t} \text{ in } L^2(0,T_1). \]

This proposition may be proved with the formalism of Legendre quadratic forms (see [12]). For this article to be self-contained, we propose an elementary proof.

Second step: Proof of (6.7):

Let \( \eta > 0 \) and \( S \in V_{T,\eta} \) with \( \|S\|_{L^2} = 1 \). Let \( d := (d_k)_{k \geq 2} \) be defined by
\[ d_k := \int_0^T S(t)e^{i\omega_k t}dt, \forall k \geq 2. \]

Then \( \|d\|_{L^2} \leq \eta \). Let \( \tilde{S} := L_T(d) \) and \( S_0 := S - \tilde{S} \), where \( L_T \) is as in Proposition 19. Let \( C(T) := \|L_T\| \). We have
\[ \|\tilde{S}\|_{L^2} \leq C(T)\eta \quad \text{and} \quad 1 - C(T)\eta \leq \|S_0\|_{L^2} \leq 1 + C(T)\eta. \]

(6.9)

Using the first step and Cauchy-Schwarz inequality, we get
\[ Q_T(S) = Q_T(S_0 + \tilde{S}) = Q_T(S_0) + \int_0^T S_0(t) \int_0^T \tilde{S}(s)k(t,s)dsdt + \int_0^T \tilde{S}(t) \int_0^T S_0(s)k(t,s)dsdt \leq -\lambda(T)\|S_0\|_{L^2}^2 + \frac{\eta}{2}\|k\|_{L^\infty}\|S\|_{L^2}^2 + 2T\|k\|_{L^\infty}\|S_0\|_{L^2}\|S\|_{L^2} \leq -\lambda(T)(1 - C(T)\eta)^2 + \frac{\eta}{2}\|k\|_{L^\infty}C(T)^2\eta^2 + 2T\|k\|_{L^\infty}(1 + C(T)\eta)C(T)\eta. \]

Thus, for \( \eta \) small enough, we get \( Q_T(S) \leq -\lambda(T)/2 < 0. \)

\[ \blacksquare \]
6.2 Proof of Theorem 10

Let \( T < T_{\text{min}} \). We proceed as in the proof of Theorem 7. Working by contradiction, we assume that, for every \( \epsilon > 0 \), there exists \( s_\epsilon \in L^2(0,T) \) with \( \|s_\epsilon\|_{L^2} < \epsilon \) such that the solution \( \tilde{\psi}_\epsilon \) of (3.19)-(3.22) satisfies

\[
\tilde{\psi}_\epsilon(T) = (\sqrt{1 - \delta_\epsilon^2} + i\delta_\epsilon)\psi_1(T),
\]

for some \( \delta_\epsilon > 0 \). The only point that needs to be clarified is that, for \( \epsilon > 0 \) small enough, \( s_\epsilon \in \mathcal{V}_{\eta,\eta} \) (with \( \eta = \eta(T) \) as in Proposition 18). Thanks to (6.4) and Proposition 3, we have

\[
\left\| \left( \int_0^T s_\epsilon(t)e^{i\omega_j t}dt \right)_{j \in J_{-\{1\}}} \right\|_{L^2} \leq C \left\| \left( \omega_j \mu_\varphi_1, \varphi_j \right) \int_0^T s_\epsilon(t)e^{i\omega_j t}dt \right\|_{H^{1}} + C \|s_\epsilon\|_{L^2}^2
\]

which gives the conclusion.

\[\blacksquare\]

7 Conclusion, open problems, perspectives

In Theorem 3, we have proposed a general context for the local controllability of the system (1.1) to require a positive minimal time. This statement extends Coron’s previous result in [15] because:

1. it does not use the variables \((s,d)\) in the state,
2. the control \(u\) has to be small in \(H^{-1}\) (not in \(L^\infty\)),
3. \(\mu(x)\) is not necessarily \((x-1/2)\).

The validity of the conclusion without the assumption \(A_K \neq 0\) is an open problem.

In Theorem 4, we have proposed a sufficient condition for the system (1.1) to be controllable around the ground state in large time. This sufficient condition is compatible with the general context of Theorem 3, thus there exists a large class of functions \(\mu\) for which local controllability holds in large time, but not in small time.

The existence of a positive minimal time for the controllability is closely related to a second order approximation of the solution. When a direction is not controllable neither at the first order, nor at the second one, then it is recovered at the third one, and no minimal time is required.

The characterization of the minimal time for the local controllability around the ground state is essentially an open problem. A first step has been done in this article, when only the first direction is lost.

In [13], Crépeau and Cerpa prove the local controllability of the KdV equation, with boundary control. When the length of the domain is critical, the linearized system is not controllable along a finite number of directions, but all of them are recovered at the second order. The existence of a positive minimal time, required for the local controllability is an open problem. The techniques developed in this article may be helpful for this question.

Acknowledgments

The authors thank Jean-Michel Coron for having attracted their attention to this problem.
A Trigonometric moment problems

In this article, we use several times the following result (see, for instance [10, Corollary 1 in Appendix B] for a proof).

Proposition 19 Let $(\omega_k)_{k \in \mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $\omega_{k+1} - \omega_k \to +\infty$ when $k \to +\infty$ and $\omega_1 = 0$. Let $l^2_2(\mathbb{N}^*, \mathbb{C}) := \{d = (d_k)_{k \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C}); d_1 \in \mathbb{R}\}$. 

1. For every $T > 0$, there exists a continuous linear map $L_T : l^2_2(\mathbb{N}^*, \mathbb{C}) \to L^2((0, T), \mathbb{R})$ $d \mapsto L_T(d)$ such that, for every $d = (d_k)_{k \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C})$, the function $v := L_T(d)$ solves $\int_0^T v(t)e^{i\omega_k t} dt = d_k, \forall k \in \mathbb{N}^*$.

2. For every $T > 0$ there exists a constant $C = C(T)$ such that $\sum_{k=1}^\infty \left|a_k\right|^2 \leq \int_0^T \left|\sum_{k=1}^\infty a_k e^{i\omega_k t}\right|^2 dt, \forall (a_k)_{k \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C})$.

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