BLOW-UP CRITERION FOR THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH VACUUM

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Abstract. In this paper, the 3D compressible MHD equations with initial vacuum or infinity electric conductivity is considered. We prove that the $L^\infty$ norms of the deformation tensor $D(u)$ and the absolute temperature $\theta$ control the possible blow-up (see [5][18][20]) of strong solutions, especially for the non-resistive MHD system when the magnetic diffusion vanishes. This conclusion means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of $D(u)$ or $\theta$ as the critical time approaches. The viscosity coefficients are only restricted by the physical conditions. Our criterion (see (1.17)) is similar to [17] for 3D incompressible Euler equations and [10] for 3D compressible isentropic Navier-stokes equations.

1. Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics and plasma physics. In 3-D space, the compressible magnetohydrodynamic equations in a domain $\Omega$ of $\mathbb{R}^3$ can be written as

\[
\begin{align*}
H_t - \text{rot}(u \times H) &= -\text{rot}\left(\frac{1}{\sigma}\text{rot}H\right), \\
\text{div}H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}T + \text{rot}H \times H, \\
(\rho \theta)_t + \text{div}(\rho \theta u) - \kappa \Delta \theta + P \text{div}u &= \text{div}(uT) - u \text{div}T + \frac{1}{\sigma} |\text{rot}H|^2.
\end{align*}
\]

In this system, $x \in \Omega$ is the spatial coordinate; $t \geq 0$ is the time; $H = (H(1), H(2), H(3))$ is the magnetic field; $\text{rot}H = \nabla \times H$ denotes the rotation of the magnetic field; $0 < \sigma \leq \infty$ is the electric conductivity coefficient; $\rho$ is the mass density; $u = (u(1), u(2), u(3)) \in \mathbb{R}^3$ is the velocity. The pressure is determined from the equation $\rho \theta = \rho_u$ where $\theta$ is the temperature.
is the velocity of fluids; $\kappa > 0$ is the thermal conductivity coefficient; $P$ is the pressure satisfying

$$P = R \rho \theta, \quad (1.2)$$

where $\theta$ is the absolute temperature, $R$ is a positive constant; $T$ is the stress tensor:

$$T = 2\mu D(u) + \lambda \text{div} u I_3, \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}, \quad (1.3)$$

where $D(u)$ is the deformation tensor, $I_3$ is the $3 \times 3$ unit matrix, $\mu$ is the shear viscosity coefficient, $\lambda$ is the bulk viscosity coefficient, $\mu$ and $\lambda$ are both real constants satisfying

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0, \quad (1.4)$$

which ensures the ellipticity of the Lamé operator. Although the electric field $E$ doesn’t appear in system (1.1), it is indeed induced according to a relation

$$E = \frac{1}{\sigma} \text{rot} H - u \times H$$

by moving the conductive flow in the magnetic field.

The aim of this paper is to give a blow-up criterion of strong solutions to system (1.1) in a bounded, smooth domain $\Omega \subset \mathbb{R}^3$ with the initial condition:

$$(H, \rho, u, \theta)|_{t=0} = (H_0(x), \rho_0(x), u_0(x), \theta_0(x)), \quad x \in \Omega, \quad (1.5)$$

and the Dirichlet, Neumann boundary conditions for $(H, u, \theta)$:

$$(H, u, \partial \theta / \partial n)|_{\partial \Omega} = (0, 0, 0), \quad \text{when } 0 < \sigma < +\infty; \quad (1.6)$$

$$(u, \partial \theta / \partial n)|_{\partial \Omega} = (0, 0), \quad \text{when } \sigma = +\infty, \quad (1.7)$$

where $n$ is the unit outer normal vector to $\partial \Omega$. Actually, some similar result for $\Omega = \mathbb{R}^3$ can be also obtained via the similar argument used in this paper.

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$D^{k,r} = \{ f \in L^1_{\text{loc}}(\Omega) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty \},$$

$$D^k = D^{k,2}, \quad \| (f, g) \|_X = \| f \|_X + \| g \|_X, \quad \| f \|_{1,0} = \| f \|_{H^1_0(\Omega)},$$

$$\| f \|_s = \| f \|_{H^s(\Omega)}, \quad \| f \|_p = \| f \|_{L^p(\Omega)}, \quad \| f \|_{D^k} = \| f \|_{D^k(\Omega)}.$$  

A detailed study of homogeneous Sobolev space may be found in [8].

As has been observed in [6], when vacuum appears, in order to make sure that the IBVP (1.1)-(1.5) with (1.6) or (1.7) is well-posed, the lack of a positive lower bound of the initial mass density $\rho_0$ should be compensated with some initial layer compatibility condition on the initial data $(H_0, \rho_0, u_0, \theta_0)$, for strong solutions [6], which can be shown as

**Theorem 1.1.** [6] Let the constant $q \in (3, 6]$, and the initial data $(H_0, \rho_0, u_0, \theta_0)$ satisfy

$$\rho_0 \geq 0, \quad \rho_0 \in W^{1,q}, \quad u_0 \in H^1_0 \cap H^2, \quad \theta_0 \in H^2, \quad \text{div} H_0 = 0, \quad (1.8)$$
and the following initial layer compatibility conditions:

\[
\begin{aligned}
Lu_0 + \nabla P_0 - \text{rot}H_0 \times H_0 &= \sqrt{\rho_0}g_1, \quad \text{for some } g_1 \in L^2, \\
-\kappa \triangle \theta_0 - Q(u_0) - \frac{1}{\sigma} |\text{rot}H_0|^2 &= \sqrt{\rho_0}g_2, \quad \text{for some } g_2 \in L^2,
\end{aligned}
\]

where

\[P_0 = R\rho_0 \theta_0, \quad Lu_0 = -\mu \triangle u_0 - (\mu + \lambda) \nabla \text{div} u_0.\]

1. If \(0 < \sigma < +\infty, H_0 \in H^1_0 \cap H^2,\) then there exists a small time \(T_*\) and a unique solution \((H, \rho, u, \theta)\) to IBVP \((1.7) - (1.5)\) with \((1.6)\) satisfying:

\[
\begin{aligned}
\rho &\in C([0, T_*]; W^{1,q}), \quad (H, u) \in C([0, T_*]; H^1_0 \cap H^2) \cap L^2([0, T_*]; D^{2,q}), \\
\theta &\in C([0, T_*]; H^2) \cap L^2([0, T_*]; D^{2,q}), \\
(H_1, u_t, \theta_t) &\in L^2([0, T_*]; D^{1}), \quad (\sqrt{\rho}u_t, \sqrt{\rho} \theta_t) \in L^\infty([0, T_*]; L^2).
\end{aligned}
\]

2. If \(\sigma = +\infty, H_0 \in W^{1,q},\) then there exists a small time \(T_*\) and a unique solution \((H, \rho, u, \theta)\) to IBVP \((1.7) - (1.5)\) with \((1.6)\) satisfying

\[
\begin{aligned}
(H, \rho) &\in C([0, T_*]; W^{1,q}), \quad u \in C([0, T_*]; H^1_0 \cap H^2) \cap L^2([0, T_*]; D^{2,q}), \\
\theta &\in C([0, T_*]; H^2) \cap L^2([0, T_*]; D^{2,q}), \\
(u_t, \theta_t) &\in L^2([0, T_*]; D^{1}), \quad (\sqrt{\rho}u_t, \sqrt{\rho} \theta_t) \in L^\infty([0, T_*]; L^2).
\end{aligned}
\]

Some analogous existence theorems of local strong solutions to the compressible Navier-Stokes equations have been previously established by Choe and Kim in [2,3,4]. In 3-D space, Huang-Li-Xin obtained the well-posedness of global classical solutions with small energy but possibly large oscillations and vacuum to the Cauchy problem for isentropic flow in [9]. Some similar existence results also have been obtained for compressible MHD equations in [6,13]. However, via the similar arguments used in [5,20,21], it is reasonable to believe that the local strong solution to \((1.1) - (1.5)\) with boundary condition \((1.6)\) or \((1.7)\) may cease to exist globally.

So, naturally, we want to know the mechanism of break-up and the structure of possible singularities: what kinds of singularities will form in finite time and what is the main mechanism of possible breakdown of smooth solutions for the 3-D compressible MHD equations with thermal conductivity? The similar question has been studied for the incompressible Euler equation by Beale-Kato-Majda (BKM) in their pioneering work [1], which showed that the \(L^\infty\)-bound of vorticity \(\nabla \times u\) must blow up. Later, Ponce [17] rephrased the BKM-criterion in terms of the deformation tensor \(D(u)\). However, the same result as [17] has been proved by Huang-Li-Xin [10] for compressible isentropic Navier-Stokes equations, which can be shown: if \(0 < T < +\infty\) is the maximum time for strong solution, then

\[
\lim_{T \to T^*} \sup_{T_0} \int_{T_0}^T |D(u)|_{L^\infty(\Omega)} dt = \infty,
\]

and for the compressible non-Isentropic system, Fan-Jiang-Ou [7] proved that

\[
\lim_{T \to T^*} \left( \int_0^T |\nabla u|_{L^\infty(\Omega)} dt + |\theta|_{L^\infty([0,T];L^\infty(\Omega))} \right) = \infty
\]
under the assumption $7\mu > \lambda$ on the viscosity coefficients. Recently, the similar blow-up criterion has been obtained for the 3-D non-resistive ($\sigma = +\infty$) compressible isentropic MHD equations in Xu-Zhang [19]:

$$\limsup_{T \to T^*} \int_0^T |\nabla u|_{L^\infty(\Omega)}\,dt = \infty. \quad (1.14)$$

Therefore, it is an interesting question to ask whether $L^\infty$ norm of $D(u)$ still controls the possible blow-up for strong solutions to IBVP (1.1)–(1.7) as in [10] [17] or not? However, under the assumption:

$$0 < \sigma < +\infty, \quad \mu > 4\lambda, \quad (1.15)$$

some result has been proved by Lu-Du-Yao [16], which can be shown: if $0 < T < +\infty$ is the maximum time for strong solution, then

$$\limsup_{T \to T^*} \left( \int_0^T |\nabla u|_{L^\infty(\Omega)}\,dt + |\theta|_{L^\infty([0,T];L^\infty(\Omega))} \right) = \infty, \quad (1.16)$$

and the assumption $\mu > 4\lambda$ has been removed by Chen-Liu [15].

However, $D(u)$ is exactly the symmetric part of $\nabla u$:

$$\nabla u = D(u) + \frac{\nabla u - (\nabla u)^\top}{2}.$$ 
So it is clear that the blow-up criterions shown in (1.14)-(1.16) for the compressible MHD equations is much stronger than the one in (1.12). This is mainly due to the presence of magnetic momentum flux density tensor

$$\frac{1}{2}|H|^2 I_3 - H \otimes H$$

in momentum equation (1.14), and the magnetic energy flux density vector

$$E \times H = \left( \frac{1}{\sigma} \text{rot} H - u \times H \right) \times H$$

in energy equation (1.15). To deal with both these two nonlinear terms, we need to control the norms ($|H|_{L^\infty}, |\nabla H|_{L^2}$), which are difficult to be bounded by $|D(u)|_{L^1((0,T);L^\infty)}$ because of the strong coupling between $u$ and $H$ in magnetic equations (1.14), and the lack of smooth mechanism of $H$ for the case $\sigma = +\infty$. These are unlike those for $(|\rho|_{L^\infty}, |\nabla \rho|_{L^2})$, which can be totally determined by $|\text{div} u|_{L^1((0,T);L^\infty)}$ due to the simple scalar hyperbolic structure of the continuity equation (1.1). So some new arguments need to be introduced to improve the results obtained above for system (1.1) with thermal conductivity.

However, via a subtle Estimate for the magnetic field $H$ and making full use of the mathematical structure of the system (1.1), our main results in the following two theorems have successfully removed the stringent condition $0 < \sigma < +\infty$ appeared in (1.15), and instead of (1.16), replacing the term $\nabla u$ with the deformation tensor $D(u)$.

**Theorem 1.2.** Let $0 < \sigma < +\infty$ and $(H, \rho, u, \theta)$ be a strong solution to IBVP (1.1)–(1.5) with (1.6) obtained in Theorem 1.1. Then if $0 < T < +\infty$ is the maximal time for the existence of $(H, \rho, u, \theta)$, we have

$$\limsup_{T \to T^*} \left( \int_0^T |D(u)|_{L^\infty(\Omega)}\,dt + |\theta|_{L^\infty([0,T];L^\infty(\Omega))} \right) = \infty.$$ 

(1.17)
Remark 1.1. This conclusion answers our question positively for compressible isentropic flow, that is, the $L^\infty$ norm of $D(u)$ still controls the possible blow-up for the corresponding strong solutions. Moreover the assumption that $\Omega$ is bounded is not essential, and our argument can be easily applied to the Cauchy problem (see \cite{15,16}) via some slight modifications. The same blow-up criterion as (1.17) is available. Some related result on Serrin-type blow-up criterion can be seen in Huang-Li \cite{12}.

And when the magnetic diffusion vanishes:

Theorem 1.3. Let $\sigma = +\infty$ and $(H, \rho, u, \theta)$ be a strong solution to IBVP (1.1)–(1.5) with (1.7) obtained in Theorem 1.1. Then if $0 < T < \infty$ is the maximal time for the existence of $(H, \rho, u, \theta)$, we have

$$\lim \sup_{T \to T} \left( \int_0^T |D(u)|_{L^\infty(\Omega)} \, dt + |\theta|_{L^\infty([0,T];L^\infty(\Omega))} \right) = \infty. \quad (1.18)$$

Remark 1.2. If we only consider the compressible isentropic flow, Theorem 1.3 has answered exactly the same question as above that whether we can replace $\nabla u$ with the deformation tensor $D(u)$ when $\sigma = +\infty$ in the blow-up criterion (1.14) or not, which is firstly raised by Xu-Zhang in \cite{19}.

The rest of this paper is organized as follows. In Section 2, we give the proof for (1.17) when $0 < \sigma < +\infty$, which improves the results obtained in \cite{15,16} via replacing $\nabla u$ with the deformation tensor $D(u)$. In Section 3, we show that the same blow-up criterion also holds when magnetic diffusion vanishes, that is $\sigma = +\infty$, which removes the stringent condition $0 < \sigma < +\infty$. Finally, we give an appendix in Section 4, which will introduce a Poincaré type inequality (see Lemma 4.1) to deal with the absolute temperature $\theta$ under the Neumann boundary condition.

2. Blow-up criterion (1.17) for $0 < \sigma < +\infty$.

We first prove (1.17) for $0 < \sigma < +\infty$. Let $(H, \rho, u, \theta)$ be the unique strong solution to IBVP (1.1)–(1.5) with boundary condition (1.6). We assume that the opposite holds, i.e.,

$$\lim \sup_{T \to T} \left( |D(u)|_{L^1([0,T];L^\infty(\Omega))} + |\theta|_{L^\infty([0,T];L^\infty(\Omega))} \right) = C_0 < \infty. \quad (2.1)$$

Firstly, based on $\text{div} H = 0$, there are some formulas for $(H, u)$:

$$\begin{align*}
\text{rot}(u \times H) &= (H \cdot \nabla)u - (u \cdot \nabla)H - H \text{div} u, \quad \text{rot}(\text{rot} H) = -\Delta H, \\
\text{rot} H \times H &= \text{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) = -\frac{1}{2} \nabla |H|^2 + H \cdot \nabla H, \\
Q(u) &= \text{div}(u^\top - u \text{div} \mathbb{T}) = \frac{\mu}{2} |\nabla u + (\nabla u)^\top|^2 + \lambda (\text{div} u)^2.
\end{align*} \quad (2.2)$$

In the following, we will use the convention that $C$ denotes a generic finite positive constant only depending on $\mu, \lambda, \kappa, R, \Omega, |(g_1, g_2)|_2$ and $\overline{T}$, and is independent of $\sigma$. We write $C(\alpha)$ to emphasize that $C(\alpha)$ depends on $\alpha$ if it is really needed, especially for $C(\sigma)$.

Next we need to show some estimates for $(H, \rho, u, \theta)$. By assumption (2.1), we first show that both the magnetic field $H$ and the mass density $\rho$ are both uniformly bounded.
Lemma 2.1. For any constant $r \geq 2$, we have
\[
|\rho(t)\|_\infty + |H(t)\|_\infty + \frac{1}{\sigma} \int_0^T \int_\Omega |H|^{r-2} |\nabla H|^2 \, dx \, dt \leq C, \quad 0 \leq t < T,
\]
where the finite constant $C > 0$ only depends on $C_0$ and $T$ (any $T \in (0, T]$).

Proof. Firstly, multiplying (1.1) by $r|H|^{r-2}H$ ($r \geq 2$) and integrating over $\Omega$ by parts, then we have
\[
\frac{d}{dt} |H|^r + \frac{r(r-1)}{\sigma} \int_\Omega |H|^{r-2} \nabla H \cdot \nabla |H|^2 \, dx
= r \int_\Omega (H \cdot \nabla u - u \cdot \nabla H - H \div \nabla \cdot H) \cdot H |H|^{r-2} \, dx
= r \int_\Omega (H \cdot D(u) - u \cdot \nabla H - H \div \nabla \cdot H) \cdot H |H|^{r-2} \, dx.
\]
Via integrating by parts, the second term on the right-hand side of (2.3) can be written as
\[
-r \int_\Omega (u \cdot \nabla H) \cdot H |H|^{r-2} \, dx = \int_\Omega \div \nabla |H|^r \, dx
\]
which, together with (2.3), immediately yields
\[
\frac{d}{dt} |H|^r + \frac{r(r-1)}{\sigma} \int_\Omega |H|^{r-2} \nabla H \cdot \nabla |H|^2 \, dx \leq (2r + 1) |D(u)|_\infty |H|^r.
\]
So, from $r \geq 2$ and (2.5), we quickly have
\[
\frac{d}{dt} |H|^r \leq \frac{(2r + 1)}{r} |D(u)|_\infty |H|^r,
\]
hence, it follows from (2.1) and (2.5)-(2.6) that
\[
\sup_{0 \leq t \leq T} |H(t)|_r + \frac{1}{\sigma} \int_0^T \int_\Omega |H|^{r-2} |\nabla H|^2 \, dx \, dt \leq C, \quad 0 \leq t < T,
\]
where $C > 0$ is independent of $r$. Therefore, letting $r \to \infty$ in the above inequality leads to the desired estimate of $|H|_\infty$. In the same way, we also obtain the bound of $|\rho|_\infty$ which indeed depends only on $\|\div \nabla\|_{L^1([0,T];L^\infty(\Omega))}$.

\[\Box\]

Remark 2.1. According to the proof for Lemma 2.1, it is obvious that we can also obtain
\[
|\rho(t)|_\infty + |H(t)|_\infty \leq C, \quad 0 \leq t < T
\]
where $C$ is only dependent of $C_0$ and $T$ (any $T \in (0, T]$), and certainly is also independent of $\sigma$. That is to say, (2.7) also holds for the case $\sigma = +\infty$ (see Lemma 3.1).

The next estimate follows from the standard energy estimate:

Lemma 2.2.
\[
\sqrt{\rho u(t)}_2^2 + \sqrt{\rho \theta(t)}_2^2 + \int_0^T \left( \frac{1}{\sigma} |\nabla H(t)|_2^2 + |\nabla u(t)|_2^2 + |\nabla \theta(t)|_2^2 \right) dt \leq C, \quad 0 \leq t < T,
\]
where the finite constant $C > 0$ only depends on $C_0$ and $T$ (any $T \in (0, T]$).
Proof. Firstly, multiplying (1.1.4) by \(u\), (1.1.3) by \(\frac{|u|^2}{2}\) and the (1.1.1) by \(H\), then summing them together and integrating the resulting equation over \(\Omega\) by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \rho |u|^2 + H^2 \right) dx + \int_\Omega \left( \frac{1}{\sigma} |\nabla H|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\text{div}u)^2 \right) dx = \int_\Omega P \text{div}u dx,
\]

where we have used the fact:

\[
\int_\Omega \text{rot}H \times H \cdot u dx = \int_\Omega -\text{rot}(u \times H) \cdot H dx.
\]

Then according to Holder’s inequality and Young’s inequality, we have

\[
\int_\Omega P \text{div}u dx \leq C \|P\|_2 |\nabla u|_2 \leq \frac{H}{4} |\nabla u|^2_2 + C,
\]

which, together with (2.8), means that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \rho |u|^2 + H^2 \right) dx + \int_\Omega \left( \frac{1}{\sigma} |\nabla H|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\text{div}u)^2 \right) dx \leq C,
\]

(2.11)

Secondly, multiplying (1.1.5) by \(\theta\) and integrating over \(\Omega\), we have

\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \rho |\theta|^2 + \frac{1}{2} H^2 \right) dx + \int_\Omega \left( \frac{1}{\sigma} |\nabla H|^2 + \mu |\nabla u|^2 + (\lambda + \mu)(\text{div}u)^2 \right) dx \leq C.
\]

(2.11)

Remark 2.2. According to the proof for Lemma 2.2, especially for (2.12), it is obvious that we can also obtain

\[
|\sqrt{\rho}u(t)|_2^2 + |\sqrt{\rho}\theta(t)|_2^2 + \int_0^T \left( |\nabla u(t)|^2_2 + |\nabla \theta(t)|^2_2 \right) dt \leq C, \quad 0 \leq t < T,
\]

(2.13)

where \(C\) is only dependent of \(C_0\) and \(T\) (any \(T \in (0, T]\)), and certainly is also independent of \(\sigma\). That is to say, (2.13) also holds for the case \(\sigma = +\infty\) (see Lemma 3.2).

The next lemma will give a key estimate on \(\nabla H\), \(\nabla \rho\) and \(\nabla u\).

Lemma 2.3.

\[
|\nabla u(t)|_2^2 + |\nabla \rho(t)|_2^2 + |\nabla H(t)|_2^2 + \int_0^T \left( |u|^2_{D^2} + \frac{1}{\sigma} |H|^2_{D^2} \right) dt \leq C(\sigma), \quad 0 \leq t < T,
\]

where the finite constant \(C(\sigma) > 0\) only depends on \(C_0\), \(\sigma\) and \(T\) (any \(T \in (0, T]\)).
Proof. Firstly, multiplying (1.14) by $\rho^{-1}(-Lu - \nabla P - \frac{1}{2}\nabla|H|^2 + H \cdot \nabla H)$ and integrating the resulting equation over $\Omega$, via (2.2) we have
\[
\frac{1}{2} \frac{d}{dt} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) + \int_\Omega \rho^{-1}(-Lu - \nabla P - \frac{1}{2}\nabla|H|^2 + H \cdot \nabla H)^2 \, dx \] 
\[= -\mu \int_\Omega (u \cdot \nabla u) \cdot \nabla (\text{rot} u) \, dx + (2\mu + \lambda) \int_\Omega (u \cdot \nabla u) \cdot \text{div} u \, dx 
- \int_\Omega (u \cdot \nabla u) \cdot \nabla P(\rho) \, dx - \int_\Omega (u \cdot \nabla u) \left( \frac{1}{2} \nabla|H|^2 - H \cdot \nabla H \right) \, dx 
- \int_\Omega u_t \cdot \nabla P(\rho) \, dx - \int_\Omega u_t \cdot \left( \frac{1}{2} \nabla|H|^2 - H \cdot \nabla H \right) \, dx \equiv \sum_{i=1}^{6} L_i, \tag{2.14}
\]
where we have used the fact that $\Delta u = \nabla \text{div} u - \nabla \times \text{rot} u$.

We now estimate each term in (2.14). Due to the fact that $\rho^{-1} \geq C^{-1} > 0$, from the standard $L^2$-theory of elliptic systems, we find that
\[
\int_\Omega \rho^{-1}|Lu + \nabla P + \nabla|H|^2 - H \cdot \nabla H|^2 \, dx 
\geq C^{-1}|Lu|^2 - C(|\nabla P|^2 + |H|^2|\nabla H|^2) 
\geq C^{-1}|u|^2_{L^2} - C(|\nabla \rho|^2 + |\nabla \theta|^2 + |\nabla u|^2 + |\nabla H|^2), \tag{2.15}
\]
where we have used Lemma 2.2 and $L$ is a strong elliptic operator. Next according to Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce
\[
|L_1| = \mu \left| \int_\Omega (u \cdot \nabla u) \cdot \nabla (\text{rot} u) \, dx \right| 
= \mu \left| \int_\Omega \nabla \times (u \cdot \nabla u) \cdot \text{rot} u \, dx \right| 
= \mu \left| \frac{1}{2} \int_\Omega (\text{rot} u)^2 \, dx - \int_\Omega \text{rot} u \cdot D(u) \cdot \text{rot} u \, dx \right| \leq C|D(u)|_{\infty} |\nabla u|^2, 
\]
\[
|L_2| = (2\mu + \lambda) \left| \int_\Omega (u \cdot \nabla u) \cdot \text{div} u \, dx \right| 
= (2\mu + \lambda) \left| - \int_\Omega \nabla u \cdot (\nabla u)^\top \, dx + \frac{1}{2} \int_\Omega (\text{div} u)^2 \, dx \right| \leq C|D(u)|_{\infty} |\nabla u|^2, \tag{2.16}
\]
\[
|L_3| = \left| \int_\Omega (u \cdot \nabla u) \cdot \nabla P \, dx \right| \leq C|u|_6 |\nabla u|_3 |\nabla P|_2 
\leq C(\epsilon)(|\nabla \theta|^2 + |\nabla \rho|^2 + 1)|\nabla u|^2 + \epsilon |\nabla u|^2_{D^2}, 
\]
\[
L_4 = \left| \int_\Omega (u \cdot \nabla u) \left( \frac{1}{2} \nabla|H|^2 - H \cdot \nabla H \right) \, dx \right| \leq C|\nabla H|_2 |H|_{\infty} |\nabla u|_3 |u|_6 
\leq C(\epsilon)|H|^3_{\infty} |\nabla H|_2 |\nabla u|^2 + \epsilon |\nabla u|_1^2 \leq C(\epsilon)(|\nabla H|^2 + 1)|\nabla u|^2 + \epsilon |\nabla u|^2_{D^2}, 
\]
\[ L_5 = -\int_{\Omega} u_t \cdot \nabla P \, dx = \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx - \int_{\Omega} P \text{div} u \, dx \]
\[ = \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx - R \int_{\Omega} \rho \theta \text{div} u \, dx - R \int_{\Omega} \rho \theta \text{div} u \, dx \]
\[ \leq \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + R \int_{\Omega} \nabla \rho \cdot u \theta \text{div} u \, dx + R \int_{\Omega} \rho \theta (\text{div} u)^2 \, dx \]
\[ + R \int_{\Omega} \rho u \cdot \nabla \theta \text{div} u \, dx + R^2 \int_{\Omega} \rho \theta (\text{div} u)^2 \, dx - \kappa R \int_{\Omega} \Delta \theta \text{div} u \, dx \]
\[ - R \int_{\Omega} Q(u) \text{div} u \, dx - \frac{R}{\sigma} \int_{\Omega} |\text{rot} H|^2 \text{div} u \, dx \]
\[ \leq \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + C |\theta|_\infty |\nabla \rho|_2 |\nabla u|_2 |\nabla u|_3 + C (|D(u)|_\infty + |\rho \theta|_\infty) |\nabla u|_2^2 \]
\[ + C |\rho|_\infty |\nabla \rho|_2 |\nabla u|_2 |\nabla u|_3 + C |\nabla \theta|_2 |\text{div} u|_2 + C |D(u)|_\infty \left( \frac{1}{\sigma} |\nabla H|_2 \right) \]
\[ \leq \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + C(1 + |D(u)|_\infty) \left( |\nabla u|_2^2 + \frac{1}{\sigma} |\nabla H|_2 \right) \]
\[ + C(\epsilon) |\nabla u|_2^2 (|\nabla \rho|_2^2 + |\nabla \theta|_2^2) + \epsilon |u|_{D_2}^2 + C, \]

where \( \epsilon > 0 \) is a sufficiently small constant. And for the last term on the right-hand side of \((2.14)\), we have

\[ L_6 = -\int_{\Omega} u_t \cdot \left( \frac{1}{2} \nabla |H|^2 - H \cdot \nabla H \right) \, dx \]
\[ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |H|^2 \text{div} u \, dx - \frac{d}{dt} \int_{\Omega} H \cdot \nabla u \cdot H \, dx \]
\[ \leq \int_{\Omega} \text{div} u H \cdot H_t \, dx + \int_{\Omega} H_t \cdot \nabla u \cdot H \, dx + \int_{\Omega} H \cdot \nabla u \cdot H_t \, dx, \]

where we have used the fact \( \text{div} H = 0 \). To deal with the last three terms on the right-hand side of \( L_6 \), we need to use

\[ H_t = H \cdot \nabla u - u \cdot \nabla H - H \text{div} u + \frac{1}{\sigma} \nabla H. \]

Hence, similarly to the proof of the above estimate, we also have

\[ -\int_{\Omega} \text{div} u H \cdot H_t \, dx \]
\[ = \int_{\Omega} -\text{div} u H \cdot \left( H \cdot \nabla u - u \cdot \nabla H - H \text{div} u + \frac{1}{\sigma} \nabla H \right) \, dx \]
\[ \leq C |H|_\infty^2 |\nabla u|_2^2 + C |D(u)|_\infty |\nabla H|_2 |u|_6 |H|_3 \]
\[ + C |D(u)|_\infty \left( \frac{1}{\sigma} |\nabla H|_2 \right) + C |H|_\infty |u|_D^2 \left( \frac{1}{\sigma} |\nabla H|_2 \right) \]
\[ \leq C(\epsilon)(|D(u)|_\infty + 1) \left( |\nabla u|_2^2 + \left( 1 + \frac{1}{\sigma} \right) |\nabla H|_2^2 \right) + \epsilon |u|_{D_2}^2, \]
Then integrating (2.23) over Ω, we have

\[ \int_\Omega H_t \cdot \nabla u \cdot H dx + \int_\Omega H \cdot \nabla u \cdot H_t dx \]

\[ = 2 \int_\Omega \left( H \cdot \nabla u - u \cdot \nabla H - H \text{div} u + \frac{1}{\sigma} \triangle H \right) \cdot \nabla u \cdot H dx \]

\[ \leq C |H|_\infty^2 |\nabla u|_2^2 + C |u|_\infty |\nabla u|_2 |\nabla H|_2 |H|_\infty \]

\[ + C |u|_{D^2[H]} |\nabla H|_2 \left( \frac{1}{\sigma} |\nabla H|_2^2 \right) + C |D(u)|_\infty \left( \frac{1}{\sigma} |\nabla H|_2^2 \right) \]

\[ \leq C(\epsilon) \left( (1 + \frac{1}{\sigma}) |\nabla H|_2^2 + 1 \right) \left( |\nabla u|_2^2 + |D(u)|_\infty + 1 \right) + \epsilon |u|_{D^2}^2 , \]  

where we have used the fact that

\[ \int_\Omega \triangle H \cdot \nabla u \cdot H dx = \int_\Omega \sum_{k=1}^3 \sum_{i,j=1}^3 \partial_{kk} H^i \partial_{ij} u^i H^j dx \]

\[ = - \int_\Omega \sum_{k=1}^3 \sum_{i,j=1}^3 \left( \partial_{kk} H^i \partial_{ij} u^i \partial_{ij} u^2 H^j + \partial_{kk} H^i \partial_{jk} u^j H^j \right) dx \]

\[ \leq C \left( |\nabla u|_2^2 \right) \left( (1 + \frac{1}{\sigma}) |\nabla H|_2^2 + |\nabla \theta|_2^2 + 1 \right) \left( |\nabla u|_2^2 + |\nabla \theta|_2^2 + |D(u)|_\infty + 1 \right) . \]

Then combining (2.14)-(2.20) and choosing \( \epsilon > 0 \) suitably small, we have

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) - \left( P + \frac{1}{2} |H|^2 \right) \text{div} u + H \cdot \nabla u \cdot H \right) dx + C |\nabla^2 u|_2^2 \]

\[ \leq C \left( |\nabla u|_2^2 + \left( 1 + \frac{1}{\sigma} \right) |\nabla H|_2^2 + |\nabla \theta|_2^2 + 1 \right) \left( |\nabla u|_2^2 + |\nabla \theta|_2^2 + |D(u)|_\infty + 1 \right) . \]

Secondly, applying \( \nabla \) to (1.13) and multiplying the resulting equations by \( 2 \nabla \rho \), we have

\[ \langle |\nabla \rho|^2 \rangle_t + \text{div} (|\nabla \rho|^2 u) + |\nabla \rho|^2 \text{div} u \]

\[ = - 2 (\nabla \rho)^\top \nabla u \nabla \rho - 2 \rho \nabla \rho \cdot \nabla \text{div} u \]

\[ = - 2 (\nabla \rho)^\top D(u) \nabla \rho - 2 \rho \nabla \rho \cdot \nabla \text{div} u . \]

Then integrating (2.23) over \( \Omega \), we have

\[ \frac{d}{dt} |\nabla \rho|_2^2 \leq C(\epsilon)(|D(u)|_\infty + 1) |\nabla \rho|_2^2 + \epsilon |\nabla^2 u|_2^2 . \]

Thirdly, applying \( \nabla \) to (1.11), due to

\[ A = \nabla (H \cdot \nabla u) = (\partial_{ij} H \cdot \nabla u^i)_{(ij)} + (H \cdot \nabla \partial_{ij} u^i)_{(ij)} , \]

\[ B = \nabla (u \cdot \nabla H) = (\partial_{ij} u \cdot \nabla H^i)_{(ij)} + (u \cdot \nabla \partial_{ij} H^i)_{(ij)} , \]

\[ C = \nabla (H \text{div} u) = \nabla H \text{div} u + H \otimes \nabla \text{div} u , \]

\[ D = \nabla \triangle u : \nabla H = \sum_{i=1}^3 \sum_{j=1}^3 \partial_{ij} \triangle H^i \partial_{ij} H^i , \]
then multiplying the resulting equation $\nabla (11)_1$ by $2\nabla H$, we have

$$(|\nabla H|^2)_t - 2A : \nabla H + 2B : \nabla H + 2C : \nabla H = \frac{2}{\sigma} D.$$  \tag{2.26}

Then integrating (2.26) over $\Omega$, due to

$$\int_\Omega A : \nabla H \, dx$$

$$= \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j H^k \partial_k u^i \partial_j H^i \, dx + \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} H^k \partial_k u^i \partial_j H^i \, dx$$

$$= \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j H^k (\partial_k u^i + \partial_i u^k) \partial_j H^i \, dx + \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} H^k \partial_k u^i \partial_j H^i \, dx$$

$$\leq C|D(u)|_\infty |\nabla H|_2 + C|H|_\infty |\nabla H|_2 |u|_{D^2},$$

$$\int_\Omega B : \nabla H \, dx$$

$$= \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j u^k \partial_k H^i \partial_j H^i \, dx + \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} u^k \partial_k H^i \partial_j H^i \, dx$$

$$= \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_k H^i (\partial_j u^k + \partial_k u^j) \partial_j H^i \, dx + \frac{1}{2} \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} u^k \partial_k (\partial_j H^i)^2 \, dx$$

$$\leq C|D(u)|_\infty |\nabla H|^2,$$

$$\int_{\Omega} C : \nabla H \, dx = \int (\text{div } u|\nabla H|^2 + H \otimes \nabla \text{div } u : \nabla H) \, dx$$

$$\leq C|D(u)|_\infty |\nabla H|_2 + C|H|_\infty |\nabla H|_2 |u|_{D^2},$$

$$\int \partial H \, dx$$

$$= \int \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j \partial_{kk} H^i \partial_j H^i \, dx = -\int \sum_{j=1}^{3} \sum_{i=1}^{3} |\partial_j H^i|^2 \, dx = -|H|_{D^2},$$

we quickly obtain the following estimate:

$$\frac{d}{dt} |\nabla H|^2 + \frac{2}{\sigma} |H|^2_{D^2} \leq C(\epsilon)(|D(u)|_\infty + 1)|\nabla H|^2 + \epsilon |\nabla u|^2.$$  \tag{2.28}

Now we denote $\Gamma = \mu|\nabla u|^2 + (\mu + \lambda)|\text{div } u|^2 + |\nabla \rho(t)|_2^2 + |\nabla H(t)|_2^2$, then adding (2.21) and (2.28) to (2.22), and choosing $\epsilon > 0$ suitably small, we deduce that

$$\frac{d}{dt} \Gamma + (|u|^2_{D^2} + \frac{\mu}{\sigma} |H|^2_{D^2})$$

$$\leq C\frac{d}{dt} \int (P + \frac{1}{2} |H|^2 ) \text{div } u - H \cdot \nabla u \cdot H \, dx$$

$$+ C \Gamma (1 + \frac{1}{\sigma} |\nabla H|^2) (|\nabla u|^2 + |\nabla \theta|^2 + |D(u)|_\infty + 1).$$  \tag{2.29}
Then from Gronwall’s inequality we immediately obtain
\[
|\nabla u(t)|^2 + |\nabla \rho(t)|^2 + |\nabla H(t)|^2 + \int_0^t (|u|^2_{D^2} + \frac{1}{\sigma}|H|^2_{D^2}) \, dt 
\leq C \exp \left( (1 + \frac{1}{\sigma}) \int_0^t (|\nabla u|^2 + |\nabla \theta|^2 + |D(u)|_\infty + 1) \right) \leq C(\sigma).
\]
(2.30)

**Remark 2.3.** According to the proof for Lemma 2.2, especially for (2.29)-(2.30), it is obvious that we can also obtain
\[
|\nabla u(t)|^2 + |\nabla \rho(t)|^2 + |\nabla H(t)|^2 + \int_0^t |u|^2_{D^2} \, dt \leq C, \quad 0 \leq t < T,
\]
where \(C\) is only dependent of \(C_0\) and \(T\) (any \(T \in (0, T]\)), and certainly is also independent of \(\sigma\). That is to say, (2.31) also holds for the case \(\sigma = +\infty\) (see Lemma 3.3).

Next, we proceed to improve the regularity of \(H, \rho, u\) and \(\theta\). To this end, we first give some estimate on the terms \(\nabla^2 H\) and \(\nabla^2 u\) based on the above estimates.

**Lemma 2.4.**
\[
|H(t)|_{D^2} \leq C(\sigma)|H_0|_{D^2} + |\text{rot}(u \times H)|_{D^2} + |\nabla H|_{D^2} 
\leq C(\sigma)(|H_0|_{D^2} + |H|_\infty|\nabla u|_{D^2} + |\nabla u|_{D^2} |\nabla H|_{D^2} + |\nabla H|_{D^2}) 
\leq C(|\rho|_{D^2} |\nabla \rho u| + |\rho|_\infty|u_0|_D |\nabla u|_{D^2}) + C(|\rho|_\infty |\nabla \rho|_{D^2} + |H|_\infty |\nabla H|_{D^2}).
\]
(2.32)

\[
|\theta(t)|_{D^2} \leq C(|\rho \theta u| + |\rho \cdot \nabla \theta| + |P\text{div} u + |Q u + |\nabla \theta|_2 + |\nabla H|_2 + \frac{1}{\sigma} |\text{rot} H|^2|_2
\leq C(|\rho|_{D^2} |\nabla \rho u| + |\rho|_\infty |u_0|_D |\nabla u|_{D^2}) + C|\nabla u|_{D^2} + C|\nabla \theta|_{D^2} + C|\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2} + |\nabla \theta|_{D^2}.
\]
(2.33)

which, together with Lemmas 2.1, 2.3 and Yong’s inequality, immediately implies that
\[
|H(t)|_{D^2} \leq C(\sigma)(|H_0|_2 + |\nabla H|_2),
|\rho|_{D^2} \leq C(\sigma)(1 + |u|_\infty) \leq C(\sigma)(1 + \|\nabla u\|_1),
|u|_{D^2} \leq C(\sigma)(|\nabla \rho u| + |\nabla \theta| + 1),
|\theta|_{D^2} \leq C(\sigma)(|\nabla \rho u| + |\nabla \theta| + |\nabla u|_D |\nabla u|_3 + |\nabla H|_6 |\nabla H|_3 + 1).
\]

□
Next differentiating (1.1) with respect to $t$, we have
\begin{equation}
\rho u_t + Lu_t = -\rho_i u_t - \rho u \cdot \nabla u - \rho u_t \cdot \nabla u - \rho u \cdot \nabla u_t - \nabla P_t + (\text{rot} H \times H)_t. \tag{2.34}
\end{equation}

Multiplying (2.34) by $u_t$ and integrating the resulting equation over $\Omega$, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu) (\text{div} u_t)^2) \, dx
= - \int_{\Omega} (\rho u \cdot \nabla |u_t|^2 - \rho u (u \cdot \nabla u) \cdot |u_t| - \rho u_t \cdot \nabla |u_t| + P_t \text{div} u_t) \, dx
+ \int_{\Omega} H \cdot H_t \text{div} u_t \, dx - \int_{\Omega} (H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H) \, dx \equiv \sum_{i=7}^{12} L_i. \tag{2.35}
\end{equation}

According to Lemmas [2.1][2.3] Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that
\begin{align}
L_7 &= - \int_{\Omega} \rho u \cdot \nabla |u_t|^2 \, dx \\
&\leq C |\rho|_{\infty} |\sqrt{\rho} u|_{2} |\sqrt{\rho} u_t|_{2} \leq C |\nabla u|_{1}^2 |\sqrt{\rho} u|_{2}^2 + \frac{\mu}{10} |\nabla u_t|_{2}^2,
\end{align}
\begin{align}
L_8 &= - \int_{\Omega} \rho u \nabla (u \cdot \nabla u \cdot u_t) \, dx \\
&\leq C |\rho|_{\infty} \int_{\Omega} (|u| |\nabla u|_{2} |u_t| + |u|^2 |\nabla^2 u|_{1} |u_t| + |u|^2 |\nabla u| |\nabla u_t|) \, dx \\
&\leq C |u|_{6} |\nabla u|_{3} |u|_{6} + C |u|^2 |\nabla^2 u|_{2} |u_t|_{6} + C |u|^2 |\nabla u|_{3} |u_t|_{2} \\
&\leq C (|\nabla u|_{3}^{2} |\nabla u|_{2} + |\nabla u|_{2}^{2} |\nabla u|_{1}^{2}) |\nabla u_t|_{2} \\
&\leq C (\sigma) |\nabla u|_{1} |\nabla u_t|_{2} \leq \frac{\mu}{10} |\nabla u_t|_{2}^2 + C (\sigma) |\nabla u|_{2}^2,
\end{align}
where we have used the fact that
\begin{equation}
|u|_{2}^{2} |\nabla u|_{2} \leq C |u|_{6} |\nabla u|_{3}^{2}, \quad |\nabla u|_{3}^{2} \leq C |\nabla u|_{2} |\nabla u|_{2} \leq C |\nabla u|_{2} |\nabla u|_{1}. \tag{2.37}
\end{equation}

And similarly, we also have
\begin{align}
L_9 &= - \int_{\Omega} \rho u_t \cdot \nabla u \cdot u_t \, dx \leq C |D(u)|_{\infty} |\sqrt{\rho} u_t|_{2}^2, \\
L_{10} &= \int_{\Omega} P_t \text{div} u_t \, dx = R \int_{\Omega} (\rho_i \theta + \rho \theta_t) \text{div} u_t \, dx \\
&\leq \frac{\mu}{10} |\nabla u|_{2}^2 + C (|\rho_i|_{2}^2 + |\sqrt{\rho} u_t|_{2}^2), \tag{2.38}
\end{align}
\begin{align}
L_{11} + L_{12} &= \int_{\Omega} H \cdot H_t \text{div} u_t \, dx - \int_{\Omega} (H \cdot \nabla u_t \cdot H_t + H_t \cdot \nabla u_t \cdot H) \, dx \\
&\leq C |H|_{\infty} |H_t|_{2} |\nabla u_t|_{2} \leq \frac{\mu}{10} |\nabla u_t|_{2}^2 + C |H_t|_{2}^2.
\end{align}
Then combining the above estimates (2.36)-(2.38), from (2.35) and (2.33) we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx \leq C(\sigma)(\|\nabla u\|_1^2 + |D(u)|_\infty + 1)(|\sqrt{\rho} u_t|_2^2 + 1) + C|H_t|_2^2 + C|\sqrt{\rho} \theta|_2^2.
\] (2.39)

Secondly, multiplying (1.15) by $\theta_t$ and integrating over $\Omega$, we have
\[
\begin{align*}
\frac{\kappa}{2} \frac{d}{dt} \int_\Omega |\nabla \theta|^2 dx + \int_\Omega \rho |\theta_t|^2 dx \\
= - \int_\Omega \rho u \cdot \nabla \theta \theta_t dx - \int_\Omega P \text{div} u \theta_t dx \\
+ \int_\Omega Q(u) \theta_t dx + \frac{1}{\sigma} \int_\Omega |\text{rot} H|^2 |\theta_t| dx = \sum_{i=13}^{16} L_i.
\end{align*}
\] (2.40)

According to Lemmas 2.1-2.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that
\[
\begin{align*}
L_{13} &= - \int_\Omega \rho u \cdot \nabla \theta \theta_t dx \\
&\leq C |\rho|_\infty^2 |u|_\infty |\sqrt{\rho} \theta_t|_2 |\nabla \theta|_2 \leq \frac{1}{4} |\sqrt{\rho} \theta_t|_2 + C \|\nabla u\|_1^2 |\nabla \theta|_2^2, \\
L_{14} &= - \int_\Omega P \text{div} u \theta_t dx \\
&\leq C |\rho|_\infty^2 |\theta|_\infty |\sqrt{\rho} \theta_t|_2 |\nabla u|_2 \leq \frac{1}{4} |\sqrt{\rho} \theta_t|_2 + C |\nabla u|_2^2, \\
L_{15} &= \int_\Omega Q(u) \theta_t dx = \frac{d}{dt} \int_\Omega Q(u) \theta dx - \int_\Omega Q(u)_t \theta dx \\
&\leq \frac{d}{dt} \int_\Omega Q(u) \theta dx + C |\nabla u|_2^2 + \frac{\mu}{10} |\nabla u|_2^2, \\
L_{16} &= \frac{1}{\sigma} \int_\Omega |\text{rot} H|^2 \theta_t dx \\
&= \frac{1}{\sigma} \int_\Omega |\text{rot} H|^2 \theta dx - \frac{1}{\sigma} \int_\Omega |\text{rot} H_t|^2 \theta dx \\
&\leq \frac{1}{\sigma} \int_\Omega |\text{rot} H|^2 \theta dx + C(\sigma) |\nabla H|_2^2 + \frac{1}{10\sigma} |\nabla H_t|_2^2,
\end{align*}
\] (2.41)

which, together with (2.40), implies that
\[
\begin{align*}
\frac{\kappa}{2} \frac{d}{dt} \int_\Omega |\nabla \theta|^2 dx + \int_\Omega \rho |\theta_t|^2 dx \\
&\leq \frac{d}{dt} \int_\Omega \left( \frac{1}{\sigma} |\text{rot} H|^2 + Q(u) \right) \theta dx \\
&+ C \|\nabla u\|_1^2 |\nabla \theta_t|_2^2 + \frac{\mu}{10} |\nabla u_t|_2^2 + \frac{1}{10\sigma} |\nabla H_t|_2^2 + C(\sigma).
\end{align*}
\] (2.42)
Thirdly, differentiating (1.13) with respect to \( t \), multiplying by \( H_t \) and integrating over \( \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} |H_t|_2^2 + \frac{1}{\sigma} \int_{\Omega} |\nabla H_t|^2 dx
= \int_{\Omega} (H_t \cdot \nabla u + H \cdot \nabla u_t - u_t \cdot \nabla H) \cdot H_t dx
- \int_{\Omega} (u \cdot \nabla H_t + H_t \text{div} u + H_t \text{div} u_t) \cdot H_t dx
\]

(2.43)

Then combining (2.39), (2.42) and (2.43), we have

\[
\int_{\Omega} (\rho |u_t|^2 + |H_t|^2 + |\nabla \theta|^2) dx + \int_{\Omega} (|\nabla u_t|^2 + \rho |u_t|^2 + \frac{1}{\sigma} |\nabla H_t|^2) dx
\leq C(\sigma)(|\sqrt{\rho} u_t|_2^2 + |H_t|_2^2 + |\nabla \theta|_2^2)(|D(u)|_\infty + \|u_t\|_1^2 + 1)
+ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{\sigma} |\text{rot} H|^2 + Q(\mu) \right) dx + C(\sigma)(1 + \|u_t\|_1^2).
\]

(2.44)

From the momentum equations (1.11), for any \( \tau \in (0, T) \), we easily have

\[
|\sqrt{\rho} u_t(\tau)|_2^2 \leq C \int_{\Omega} \rho |u_t|^2 \nabla u(t)^2 \, dx + C \int_{\Omega} \frac{|\nabla P + Lu - \text{rot} H \times H|^2}{\rho} \, dx,
\]

(2.45)

due to the initial layer compatibility condition (1.9), letting \( \tau \to 0 \) in (2.45), we have

\[
\limsup_{\tau \to 0} |\sqrt{\rho} u_t(\tau)|_2^2 \leq C \int_{\Omega} \rho_0 |u_0|^2 \nabla u_0^2 \, dx + C \int_{\Omega} |\theta_0|^2 \, dx \leq C.
\]

(2.46)

Then integrating (2.44) over \( (0, T) \) with respect to \( t \), via (2.46) and Gronwall’s inequality, we deduce that

\[
(|\sqrt{\rho} u_t|_2^2 + |H_t|_2^2 + |\nabla \theta|_2^2)(t) + \int_0^T \left( |\nabla u_t|_2^2 + |\sqrt{\rho} \theta_t|_2^2 + \frac{1}{\sigma} |\nabla H_t|_2^2 \right) dt \leq C(\sigma), \quad 0 \leq t \leq T,
\]

which, together with (2.33), gives the desired conclusions.

\[\square\]

Via some Poincaré type inequality (see (2.52) or Lemma 4.1) coming from [14], we have the following estimate for \(|\theta|_{D^2}\):

**Lemma 2.5.**

\[
|\sqrt{\rho} \theta_t(t)|_2^2 + |\theta(t)|_{D^2}^2 + \int_0^T \left| \frac{\partial \theta_t}{\partial s} \right|_{D^2}^2 ds \leq C(\sigma), \quad 0 \leq t \leq T,
\]

where the finite constant \( C(\sigma) > 0 \) only depends on \( C_0, \sigma \) and \( T \) (any \( T \in (0, \overline{T}) \)).
Proof. Firstly, from (2.33) and Lemma 2.3, we quickly have
\[ |\theta|_{D^2} \leq C(\sigma)(1 + |\sqrt{\rho}\theta|_2). \]  
(2.47)

Next differentiating (1.15) with respect to \( t \), we have
\[ \rho \theta_t - \kappa \theta_t = -\rho_t \theta_t - \rho_t u \cdot \nabla \theta - \rho u \cdot \nabla \theta_t - \rho_t u \cdot \nabla \theta_t + P_1 \text{div} u + P \text{div} u_t + Q(\theta) + \frac{1}{\sigma} |\text{rot} H|_2^2. \]  
(2.48)

Multiplying (2.48) by \( \theta_t \) and integrating over \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\theta_t|^2 \, dx + \kappa \int_\Omega |\nabla \theta_t|^2 \, dx = R \int_\Omega \rho |\theta_t|^2 \text{div} u \, dx + R \int_\Omega \rho_t \text{div} u \theta_t \, dx + R \int_\Omega \rho \text{div} u_{t2} \theta_t \, dx
\]
\[
+ \int_\Omega Q(\theta) \theta_t \, dx - \int_\Omega \rho_t u \cdot \nabla \theta_t \, dx - \int_\Omega \rho_t |\theta_t|^2 \, dx
\]
\[
+ \int_\Omega \rho u \cdot \nabla \theta_t \, dx + \frac{1}{\sigma} \int_{\Omega} |\text{rot} H|_t^2 \, \theta_t \, dx \equiv \sum_{i=17}^{24} L_i.
\]  
(2.49)

According to Lemmas 2.1,2.4, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

\[ L_{17} = R \int_\Omega \rho |\theta_t|^2 \text{div} u \, dx \leq C|D(u)|_{\infty} |\sqrt{\rho}\theta_t|_2^3, \]

\[ L_{18} = R \int_\Omega \rho_t \text{div} u \theta_t \, dx = - R \int_\Omega \rho |\text{div} u|^2 \theta_t \, dx + R \int_\Omega \theta \nabla \rho \cdot \text{div} u \theta_t \, dx \leq C|\rho|_{\infty} |\theta|_{\infty} |\nabla u|_{3} |\nabla u|_{2} |\theta_t|_{6} + C|u|_{\infty} |\theta|_{\infty} |\nabla \rho|_{2} |\nabla u|_{3} |\theta_t|_{6}
\]
\[
\leq C(\sigma)(|\sqrt{\rho}\theta_t|_2 + |\nabla \theta_t|_2) \leq \frac{\kappa}{8} |\nabla \theta_t|_2^2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2 + C(\sigma),
\]

\[ L_{19} = R \int_\Omega \rho \text{div} u_{t2} \theta_t \, dx \leq C|\theta|_{\infty} |\theta|_{3} |\nabla u|_{2} |\theta_t|_{6}
\]
\[
\leq C|\nabla u|_{2} (|\nabla \theta_t|_2 + |\sqrt{\rho}\theta_t|_2) \leq \frac{\kappa}{8} |\nabla \theta_t|_2^2 + C(\sigma)|\nabla \theta_t|_2^2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2,
\]

\[ L_{20} = \int_\Omega Q(\theta) \theta_t \, dx \leq C|\nabla u|_{3} |\nabla u|_{2} |\theta_t|_{6}
\]
\[
\leq C(\sigma)|\nabla u|_{2} (|\nabla \theta_t|_2 + |\sqrt{\rho}\theta_t|_2) \leq \frac{\kappa}{8} |\nabla \theta_t|_2^2 + C(\sigma)|\nabla \theta_t|_2^2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2,
\]

\[ L_{21} = - \int_\Omega \rho_t u \cdot \nabla \theta_t \, dx \leq C|u|_{\infty} |\nabla \theta|_{3} |\rho_t|_{2} |\theta_t|_{6}
\]
\[
\leq C(\sigma)|\nabla \theta_t|_2 |\nabla \theta_t|_6 (|\nabla \theta_t|_2 + |\sqrt{\rho}\theta_t|_2)
\]
\[
\leq \frac{\kappa}{8} |\nabla \theta_t|_2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2 + C(\sigma)|\nabla \theta_t|_2^2 \leq \frac{\kappa}{8} |\nabla \theta_t|_2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2 + C(\sigma),
\]

\[ L_{22} = - \int_\Omega \rho_t |\theta_t|^2 \, dx = - 2 \int_\Omega \rho u \cdot \nabla \theta_t \, dx
\]
\[
\leq C|\rho|_{\infty} |u|_{\infty} |\sqrt{\rho}\theta_t|_2 |\nabla \theta_t|_2 \leq \frac{\kappa}{8} |\nabla \theta_t|_2 + C(\sigma)|\sqrt{\rho}\theta_t|_2^2,
\]
Then integrating (2.58) over $\Omega$, we immediately obtain

\begin{align}
L_{23} &= -\int_{\Omega} \rho u_t \cdot \nabla \theta_t \, dx 
\leq C|\rho|^\frac{3}{2}|\nabla \theta_t|_2|\theta_t|_6|\sqrt{\rho}u_t|_3 \\
\leq & C(\sigma)|\sqrt{\rho}u_t|^\frac{1}{2}|\sqrt{\rho}u_t|^\frac{1}{2}(|\nabla \theta_t|_2 + |\sqrt{\rho}t|_2) \\
\leq & \kappa \sigma |\nabla \theta_t|^2 + C(\sigma)|\nabla u_t|^2 + C(\sigma)|\sqrt{\rho}t|_2^2; \\
\text{(2.51)}
\end{align}

and its proof can be seen in Lemma 2.1. Then according to (2.49), we deduce that

\begin{align}
\frac{1}{\sigma} \int_{\Omega} |\nabla \theta_t|^2 \, dx &= \frac{1}{\sigma} \int_{\Omega} |\nabla \theta_t|^2 \, dx \\
\leq & C |\nabla H_t|_2 |\nabla \theta_t|_2 + C(\sigma)|\nabla H_t|^2 + C(\sigma)|\sqrt{\rho}t|_2, \\
\text{(2.53)}
\end{align}

where we have used the fact (2.47) and the following Poincaré type inequality (see [14]):

\begin{align}
|\theta_t|_6 & \leq C(|\sqrt{\rho}t|_2 + (1 + |\rho|_2))|\nabla \theta_t|_2, \\
\text{(2.52)}
\end{align}

From the energy equations (1.1), for any $\tau \in (0, T)$, we easily have

\begin{align}
|\sqrt{\rho}t(\tau)|_2 & \leq C \int_{\Omega} |\rho||u_t|^2|\nabla \theta_t|^2(\tau) \, dx + C \int_{\Omega} \frac{\left|\nabla \nabla \theta_t \right|^2 + Q(u) + \frac{1}{\sigma} |\nabla H_t|^2}{\rho} \, d\tau, \\
\text{(2.54)}
\end{align}

due to the initial layer compatibility condition (1.9), letting $\tau \to 0$ in (2.45), we have

\begin{align}
\lim_{\tau \to 0} \left|\sqrt{\rho}t(\tau)|_2 \leq C \int_{\Omega_0} |\rho_0||u_0|^2|\nabla \theta_0|^2 \, dx + C \int_{\Omega} |g_2|^2 \, dx \leq C. \\
\text{(2.55)}
\end{align}

Then according to Gronwall’s inequality, (2.47) and (2.55), we immediately obtain the desired conclusions.

Finally, the following lemma gives bounds of $|\rho|_{D^{1,q}}$, $|H|_{D^{2,q}}$, $|u|_{D^{2,q}}$ and $|\theta|_{D^{2,q}}$.

**Lemma 2.6.**

\begin{align}
\| (\rho(t))_{W^{1,q}} + |\rho(t)|_q + \int_0^T \left( |H|^2_{D^{2,q}} + |u|^2_{D^{2,q}} + |\theta|^2_{D^{2,q}} \right) dt \leq C(\sigma), \quad 0 \leq t < T, \\
\text{(2.56)}
\end{align}

where the finite constant $C(\sigma) > 0$ only depends on $C_0$, $\sigma$ and $T$ (any $T \in (0, \bar{T})$).

**Proof.** Firstly, from (1.1) and (1.4), the standard regularity estimate for elliptic equations and Lemmas 2.1, 2.2, we have

\begin{align}
|\nabla^2 u|_q & \leq C|\rho u_t + \rho u \cdot \nabla u + \nabla P - \nabla H|_q + C|\nabla u|_q \\
& \leq C(\sigma)(1 + |\nabla u|_2 + |\nabla \rho|_q). \\
\text{(2.57)}
\end{align}

Next, applying $\nabla$ to (1.1), multiplying the resulting equations by $q|\nabla \rho|^{q-2} \nabla \rho$, we have

\begin{align}
(|\nabla \rho|^q)_{t} + \text{div}(|\nabla \rho|^q u) + (q - 1)|\nabla \rho|^q \text{div} u \\
= - q|\nabla \rho|^{q-2}(\nabla \rho)^\top D(u)(\nabla \rho) - q|\nabla \rho|^{q-2}\nabla \rho \cdot \nabla \text{div} u. \\
\text{(2.58)}
\end{align}

Then integrating (2.58) over $\Omega$, we immediately obtain

\begin{align}
\frac{d}{dt} |\nabla \rho|_q \leq & C|D(u)|_\infty |\nabla \rho|_q + C|\nabla^2 u|_q, \\
\text{(2.59)}
\end{align}
which means that
\[
\frac{d}{dt} |\nabla \rho|_q \leq C(\sigma)(1 + |D(u)|_\infty)|\nabla \rho|_q + C(\sigma)(1 + |\nabla u_t|_2^2).
\] (2.60)

Then from Gronwall’s inequality, we immediately have
\[
|\nabla \rho|_q + \int_0^t (1 + |D(u)|_\infty)ds \leq C(\sigma), \quad 0 \leq t \leq T.
\]

So via (2.57) and Lemmas 2.1-2.5, we easily have
\[
\int_0^t |u(t)|_{D^2,q}^2 ds \leq C(\sigma) \int_0^t (1 + |\nabla u_t(s)|_2^2)ds \leq C(\sigma), \quad 0 \leq t \leq T.
\] (2.61)

Thirdly, we consider the term $|H|_{D^2,q}$ and $|\theta|_{D^2,q}$, from (1.1) and (1.5), the standard regularity estimate for elliptic equations and Lemmas 2.1-2.5, we quickly have
\[
|H|_{D^2,q} \leq C(\sigma)(|H_t - \text{rot}(u \times H)|_q + |\nabla H|_q) \leq C(\sigma)(|\nabla H_t|_2 + 1),
\]
\[
|\theta|_{D^2,q} \leq C(|\rho \theta_t + \rho u \cdot \nabla \theta + P \text{div}u|_q + |\nabla u|^2|_q + |\nabla \theta|_q) + \frac{1}{\sigma} |\text{rot}H|^2|_q
\]
\[
\leq C(\sigma)(1 + |\theta_t|_{D^1} + |u|_{D^2,q} + |H|_{D^2,q}),
\] (2.62)

which, together with Lemma 2.4 and (2.61), implies the desired conclusions. \(\square\)

And this will be enough to extend the strong solution $(H, \rho, u, \theta)$ beyond $t \geq \overline{T}$.

In truth, in view of the estimates obtained in Lemmas 2.1-2.6, we quickly know that the functions $(H, \rho, u, \theta)$ satisfy the conditions imposed on the initial data (1.8)-(1.9) with $H_0 \in H_0^1 \cap H^2$. Therefore, we can take $(H, \rho, u, \theta)|_{t=\overline{T}}$ as the initial data and apply the local existence Theorem 1.1 to extend our local strong solution beyond $t \geq \overline{T}$. This contradicts the assumption on $\overline{T}$.

3. Blow-up criterion \((1.18)\) for $\sigma = +\infty$

Based on the estimates obtained in Section 2, now we prove (1.18) for $\sigma = +\infty$. Let $(H, \rho, u, \theta)$ be the unique strong solution to the IBVP (1.1)-(1.5) with (1.7). We assume that the opposite holds, i.e.,
\[
\limsup_{T \to \overline{T}} (|D(u)|_{L^1([0,T];L^\infty(\Omega))} + |\theta|_{L^\infty([0,T];L^\infty(\Omega))}) = C_0 < \infty.
\] (3.1)

Next we need to show some estimates for our strong solution $(H, \rho, u, \theta)$. By assumption (3.1), the proof of Lemma 2.1 and Remark 2.1 we easily show that the magnetic field $H$ and the mass density $\rho$ are both uniformly bounded.

**Lemma 3.1.**
\[
|\rho(t)|_\infty + |H(t)|_\infty \leq C, \quad 0 \leq t < T,
\]
where the finite constant $C > 0$ only depends on $C_0$ and $T$ (any $T \in (0, \overline{T})$).

The next estimate directly follows from the estimate in Lemma 2.2 and Remark 2.2.
Lemma 3.2.
\[ |\sqrt{\rho}u(t)|_2^2 + |\sqrt{\rho}\theta(t)|_2^2 + \int_0^T (|\nabla u(t)|_2^2 + |\nabla \theta(t)|_2^2) dt \leq C, \quad 0 \leq t < T, \]
where the finite constant \( C > 0 \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

The next lemma directly follows from the proof for Lemma 2.3 and Remark 2.3.

Lemma 3.3.
\[ |\nabla u(t)|_2^2 + |\nabla \rho(t)|_2^2 + |\nabla H(t)|_2^2 + \int_0^T |u|_{D^2}^2 dt \leq C, \quad 0 \leq t < T, \]
where the finite constant \( C > 0 \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

Next, we proceed to improve the regularity of \( H, \rho, u \) and \( \theta \). To this end, we first derive some bounds on \( \nabla^2 u \) based on estimates above.

Lemma 3.4.
\[ |u(t)|_{D^2}^2 + |\sqrt{\rho}u_t(t)|_2^2 + |\nabla \theta(t)|_2^2 + |\rho_t(t)|_2^2 + \int_0^T (|u_t|_{D^2}^2 + |\nabla \theta_t|_2^2 + |\theta|_{D^2}^2) dt \leq C, \quad 0 \leq t < T, \]
where the finite constant \( C > 0 \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T] \)).

Proof. From system \((1.1)\) with \( \sigma = +\infty \), \((3.1)\), Lemmas 3.1-3.3 and the similar arguments used in the derivation of \((2.32)-(2.33)\), we have
\[ |H_t|_2 \leq C|\text{rot}(u \times H)|_2 \leq C(|H|_{\infty}|\nabla u|_2 + |u|_{\infty}|\nabla H|_2) \leq C(1 + ||u||_1), \]
\[ |u|_{D^2} \leq C(|\rho u_t|_2 + |\rho u \cdot \nabla u|_2 + |\nabla P|_2 + |\text{rot}H \times H|_2 + |\nabla u|_2) \leq C(|\sqrt{\rho}u_t|_2 + |\nabla \theta|_2 + 1), \]
\[ |\theta|_{D^2} \leq C(|\rho \theta_t|_2 + |\rho u \cdot \nabla \theta|_2 + |P \text{div} u|_2 + |Q(u)|_2 + |\nabla \theta|_2) \leq C(|\sqrt{\rho} \theta_t|_2 + |\nabla \theta|_2 + |u|_6 |\nabla u|_3 + |\nabla H|_6 |\nabla H|_3 + 1), \]
\[ |\rho_t|_2 \leq C(|\rho \text{div} u|_2 + |u \cdot \nabla \rho|_2) \leq C(1 + ||u||_1). \]

Next, multiplying \((2.31)\) by \( u_t \) and integrating the resulting equation over \( \Omega \), we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2) dx \]
\[ = - \int_{\Omega} (\rho u \cdot \nabla |u_t|^2 - \rho u \nabla (u \cdot u_t) - \rho u_t \cdot \nabla u \cdot u_t + P_t \text{div} u_t) dx + \int_{\Omega} H \cdot H_t \text{div} u_t dx - \int_{\Omega} (H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H) dx \equiv \sum_{i=7}^{12} L'_i. \]

We have to point out that, compared with the relation \((2.35)\), we have
\[ L_i = L'_i, \quad \text{for} \quad i = 7, ..., 12. \]
in the sense of the form. Then similarly to the derivarion of (2.35)-(2.38), via Lemmas 3.1-3.3 and (3.2), we have

\[ L_7' \leq C \| \nabla u \|_1 \| \sqrt{\rho u} \|_2^2 + \frac{\mu}{10} |\nabla u|_2^2, \]
\[ L_8' \leq C (|\nabla u|_3^2 |\nabla u|_2 + |\nabla u|_2^2 |\nabla u|_1) |\nabla u|_2 \]
\[ \leq C \| \nabla u \|_1 |\nabla u|_2 \leq \frac{\mu}{10} |\nabla u|_2^2 + C \| \nabla u \|_1^2, \]
\[ L_9' \leq C |D(u)|_\infty \sqrt{\rho u|t|_2^2}, \]
\[ L_{10}' \leq \frac{\mu}{10} |\nabla u|_2^2 + C (|\rho|_2^2 + |\sqrt{\rho \theta}|_2^2), \]
\[ L_{11}' + L_{12}' \leq C |H|_\infty |H|_2 |\nabla u|_2 \]
\[ \leq \frac{\mu}{10} |\nabla u|_2^2 + C |H|_2^2 \leq \frac{\mu}{10} |\nabla u|_2^2 + C (1 + \| \nabla u \|_1^2). \]

Then combining the above estimates (3.3)-(3.4), we have

\[ \frac{1}{2} \frac{d}{dt} \int \rho |u|_t^2 dx + \int |\nabla u|_t^2 dx \]
\[ \leq C (\| \nabla u \|_1^2 + |D(u)|_\infty + 1)(|\sqrt{\rho u}|_2^2 + 1) + C |\sqrt{\rho \theta}|_2^2. \] (3.5)

Secondly, multiplying (1.15) with \( \sigma = +\infty \) by \( \theta_1 \) and integrating over \( \Omega \), we have

\[ \kappa \frac{d}{dt} \int |\nabla \theta_1|^2 dx + \int \rho |\theta_1|^2 dx \]
\[ = - \int \rho u \cdot \nabla \theta_1 dx - \int P \div u \theta_1 dx + \int Q(u) \theta_1 dx = \sum_{i=13}^{15} L_i'. \] (3.6)

We have to point out that, compared with the relation (2.40), we have

\[ L_i = L_i', \quad \text{for} \quad i = 13, 14, 15 \]

in the sense of the form. Then similarly to the derivarion of (2.41), via Lemmas 3.1-3.3 and (3.2), we have

\[ L_{13}' \leq C |\rho|_\infty^\frac{1}{2} |u|_\infty |\sqrt{\rho \theta}|_2 |\nabla \theta|_2 \leq \frac{1}{4} |\sqrt{\rho \theta}|_2 + C \| \nabla u \|_1^2 |\nabla \theta|_2^2, \]
\[ L_{14}' \leq C |\rho|_\infty^\frac{1}{2} |\theta|_\infty |\sqrt{\rho \theta}|_2 |\nabla u|_2 \leq \frac{1}{4} |\sqrt{\rho \theta}|_2 + C |\nabla u|_2^2, \] (3.7)
\[ L_{15}' \leq \frac{d}{dt} \int Q(u) \theta dx + C |\nabla u|_2^2 + \frac{\mu}{10} |\nabla u|_2^2, \]

which, together with (3.6), implies that

\[ \kappa \frac{d}{dt} \int |\nabla \theta|_t^2 dx + \int \rho |\theta|_t^2 dx \]
\[ \leq \frac{d}{dt} \int Q(u) \theta dx + C \| \nabla u \|_1^2 |\nabla \theta|_2^2 + \frac{\mu}{10} |\nabla u|_2^2 + C. \] (3.8)

Then combining (3.5) and (3.8), we have
\[
\frac{d}{dt} \int_\Omega \left( \rho |u_t|^2 + |\nabla \theta|^2 \right) dx + \int_\Omega |\nabla u_t|^2 + \rho |u_t|^2 dx \leq C \left( \sqrt{\rho u_t} \right)_2 + |\nabla \theta|_2^2 \left( |D(u)|_\infty + \|u\|_1^2 + 1 \right) + \frac{d}{dt} \int_\Omega Q(u) \theta dx + C \left( 1 + \|\nabla u\|_1^2 \right). \tag{3.9}
\]

And, completely same as the derivation of (2.46), we have
\[
\limsup_{\tau \to 0} \left( \sqrt{\rho u_t} \right)_2 \leq C \int_\Omega \rho_0 |u_0|^2 |\nabla u_0|^2 dx + C \int_\Omega |g_1|^2 dx \leq C. \tag{3.10}
\]

Then integrating (3.9) over \((0, T)\) with respect to \(t\), via (3.10) and Gronwall’s inequality, we deduce that
\[
\left( \sqrt{\rho u_t} \right)_2^2 + |\nabla \theta|_2^2 \left( t \right) + \int_0^T \left( |\nabla u_t|^2 + |\sqrt{\rho u_t}|^2 \right) dt \leq C, \quad 0 < t \leq T,
\]
which, together with (3.2), gives the desired conclusions. \(\square\)

The next lemma is similar to Lemma 2.5:

**Lemma 3.5.**
\[
|\sqrt{\rho \theta_t}(t)|_2^2 + |\theta(t)|_{D^2}^2 + \int_0^T |\theta_t|_{D^1}^2 ds \leq C, \quad 0 \leq t \leq T,
\]
where the finite constant \(C > 0\) only depends on \(C_0\) and \(T\) (any \(T \in (0, T])\).

**Proof.** Firstly, from (3.2) and Lemmas 3.1-3.4, we quickly have
\[
|\theta|_{D^2} \leq C \left( 1 + |\sqrt{\rho \theta_t}|_2 \right). \tag{3.11}
\]

Next differentiating (1.15) with respect to \(t\) when \(\sigma = +\infty\), we have
\[
\rho \theta_{tt} - \kappa \theta_t = -\rho_t \theta_t - \rho_t u \cdot \nabla \theta - \rho u \cdot \nabla \theta - \rho u \cdot \nabla \theta_t + P_t \text{div} u + P \text{div} u_t + Q(u)_t. \tag{3.12}
\]

Multiplying (3.12) by \(\theta_t\) and integrating over \(\Omega\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\theta_t|^2 dx + \kappa \int_\Omega |\nabla \theta_t|^2 dx = R \int_\Omega \rho |\theta_t|^2 \text{div} u dx + R \int_\Omega \rho_t \theta_t \text{div} u dx + R \int_\Omega \rho t \theta_t \text{div} u dx + \int_\Omega \rho \theta_t \text{div} u dx \tag{3.13}
\]
\[
- \int_\Omega \rho t u \cdot \nabla \theta \theta_t dx + \int_\Omega \rho u \cdot \nabla \theta dx \equiv \sum_{i=17}^{23} L_i'.
\]

We have to point out that, compared with the relation (2.49), we have
\[
L_i = L_i', \quad \text{for} \quad i = 17, ..., 23.
\]
in the sense of the form. Then similarly to the derivation of (2.50), via Lemmas 3.1–3.4 and (3.2), we have

\[ L_{17} \leq C \left| D(u) \right|_{\infty} |\sqrt{\rho} \theta|_{2}^{2}, \]

\[ L_{18} \leq C \left( |\sqrt{\rho} \theta|_{2} + |\nabla \theta|_{2} \right) \leq \frac{\kappa}{8} |\nabla \theta|_{2}^{2} + C(\sigma) |\sqrt{\rho} \theta|_{2}^{2} + C, \]

\[ L_{19} \leq C |\nabla u|_{2} (|\nabla \theta|_{2} + |\sqrt{\rho} \theta|_{2}) \leq \frac{\kappa}{8} |\nabla \theta|_{2}^{2} + C |\nabla u|_{2}^{2} + C |\sqrt{\rho} \theta|_{2}^{2}, \]

\[ L_{20} \leq C |\nabla u|_{2} (|\nabla \theta|_{2} + |\sqrt{\rho} \theta|_{2}) \leq \frac{\kappa}{8} |\nabla \theta|_{2}^{2} + C |\nabla u|_{2}^{2} + C |\sqrt{\rho} \theta|_{2}^{2}, \]

\[ L_{22} \leq C |\sqrt{\rho} \theta|_{2}^{2} \leq \frac{k}{8} |\nabla \theta|_{2}^{2} + C |\sqrt{\rho} \theta|_{2}^{2}, \]

\[ L_{23} \leq C |\nabla \theta|_{2}^{2} \leq C |\nabla \theta|_{2}^{2} + C |\sqrt{\rho} \theta|_{2}^{2}, \]

(3.14)

where we have used the fact (3.11) and the following Poincaré type inequality (see [14]):

\[ |\theta|_{6} \leq C |\sqrt{\rho} \theta|_{2} + C(1 + |\rho|_{2}) |\nabla \theta|_{2}, \]

(3.15)

and its proof can be seen in Lemma 4.1. Then according to (3.13)–(3.14), we deduce that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho|\theta|_{2}^{2} dx + \kappa \int_{\Omega} |\nabla \theta|_{2}^{2} dx \]

\[ \leq C \left( |D(u)|_{\infty} + 1 \right) |\sqrt{\rho} \theta|_{2}^{2} + C (|u|_{D^{1, q}}^{2} + 1). \]

(3.16)

From the energy equations (1.15), for any \( \tau \in (0, T) \), we easily have

\[ |\sqrt{\rho} \theta(\tau)|_{2}^{2} \leq C \int_{\Omega} \rho|\theta|_{2}^{2} \nabla \theta^{2}(\tau) dx + C \int_{\Omega} \frac{|\kappa \Delta \theta + Q(u)|_{2}^{2}}{\rho}(\tau) dx, \]

due to the initial layer compatibility condition (1.9), letting \( \tau \to 0 \) in (3.17), we have

\[ \limsup_{\tau \to 0} |\sqrt{\rho} \theta(\tau)|_{2}^{2} \leq C \int_{\Omega} \rho|u|_{2}^{2} |\nabla \theta|_{2}^{2} dx + C \int_{\Omega} |g_{2}|^{2} dx \leq C. \]

(3.18)

Then according to Gronwall’s inequality, (3.11) and (3.18), we immediately obtain the desired conclusions. \( \square \)

Finally, the following lemma gives bounds of \( |\rho|_{D^{1, q}}, |H|_{D^{1, q}}, \nabla^{2} u \) and \( \nabla^{2} \theta \).

**Lemma 3.6.**

\[ \| (H, \rho)(t) \|_{W^{1, q}} + \| (H_{t}, \rho_{t})(t) \|_{q} + \int_{0}^{T} (|u|_{D^{2, q}}^{2} + |\theta|_{D^{2, q}}^{2}) dt \leq C, \quad 0 \leq t < T, \]

where the finite constant \( C > 0 \) only depends on \( C_{0} \) and \( T \) (any \( T \in (0, T) \)).
Proof. Firstly, from (1.1) and (1.5), Lemmas 3.1 and 3.5 we easily have
\[
|u|_{D^2,q} \leq C(|ru|_q + |ru \cdot \nabla u|_q + |\nabla P|_q + |\text{rot}H \times H|_q + |\nabla u|_q) \\
\leq C(1 + |u|_{D^1} + |\nabla \theta|_q + |\nabla H|_q),
\]
\[
|\theta|_{D^2,q} \leq C(|\theta|_q + |ru \cdot \nabla \theta|_q + |P\text{div}u|_q + |Q(u)|_q + |\nabla \theta|_q) \\
\leq C(1 + |\theta|_{D^1} + |u|_{D^2,q}).
\]
(3.19)

According to the proof in Lemma 2.6 we immediately obtain
\[
\frac{d}{dt} |\nabla \theta|_q \leq C|D(u)|_\infty \|\nabla \theta\|_q + C|\nabla^2 u|_q,
\]
(3.20)
which means that
\[
\frac{d}{dt} |\nabla \theta|_q \leq C(1 + |D(u)|_\infty) (|\nabla \theta|_q + |\nabla H|_q) + C(1 + |\nabla u|_q^2).
\]
(3.21)

Applying \( \nabla \) to (1.1), multiplying the resulting equation by \( q \nabla H |\nabla H|^{q-2} \), we have
\[
(\nabla H)^2 q - qA : \nabla H |\nabla H|^{q-2} + qB \nabla H |\nabla H|^{q-2} + qC : \nabla H |\nabla H|^{q-2} = 0.
\]
(3.22)

Then integrating (3.22) over \( \Omega \), due to
\[
\int_{\Omega} A : \nabla H |\nabla H|^{q-2} dx \\
= \int_{\Omega} \sum_{j=1}^{3} \left( \sum_{i,k=1}^{3} \partial_j H^k \partial_k u^i \partial_j H^i \right) |\nabla H|^{q-2} dx + \int_{\Omega} \sum_{j=1}^{3} \sum_{i,k=1}^{3} H^k \partial_k u^i \partial_j H^i |\nabla H|^{q-2} dx \\
= \int_{\Omega} \sum_{j=1}^{3} \left( \sum_{i,k=1}^{3} \partial_j H^k (\partial_k u^i + \frac{1}{2} \partial_i u^k) \partial_j H^i \right) |\nabla H|^{q-2} dx + \int_{\Omega} \sum_{i,j,k=1}^{3} H^k \partial_k u^i \partial_j H^i |\nabla H|^{q-2} dx \\
\leq C|D(u)|_\infty |\nabla H|_q + C|H|_\infty |\nabla H|_q^{q-1} |u|_{D^2,q},
\]
(3.23)
\[
\int_{\Omega} C : \nabla H |\nabla H|^q - 2 dx = \int_{\Omega} (\text{div} \nabla H)^q + (H \otimes \text{div} u) : \nabla H |\nabla H|^q - 2 dx
\]
(3.24)

\[
\leq C |D(u)|_\infty |\nabla H|^q + C |H|_\infty |\nabla H|^{q-1}_q |u|_{D^2,q},
\]
then we quickly obtain the following estimate

\[
\frac{d}{dt} |\nabla H|_q \leq C (|D(u)|_\infty + 1) |\nabla H|_q + C |u|_{D^2,q},
\]
(3.25)

which means that

\[
\frac{d}{dt} |\nabla H|_q \leq C (1 + |D(u)|_\infty) (|\nabla \rho|_q + |\nabla H|_q) + C (1 + |\nabla u_t|^2).
\]
(3.26)

Then from (3.21) and (3.26), via Gronwall’s inequality and (3.19) we quickly have

\[
|\nabla H(t)|_q + |\nabla \rho(t)|_q + \int_0^T (|\nabla^2 u|^2_q + |\nabla^2 \theta|^2_q) dt \leq C, \quad 0 < t \leq T.
\]
(3.27)

And this will be enough to extend the strong solutions of \((H, \rho, u, \theta)\) beyond \(t \geq T\).

In truth, in view of the estimates obtained in Lemmas 3.1-3.6, we quickly know that the functions \((H, \rho, u, \theta)\big|_{t=T} = \lim_{t \to T}(H, \rho, u, \theta)\) satisfy the conditions imposed on the initial data (1.8)-(1.9) with \(H_0 \in W^{1,q}\). Therefore, we can take \((H, \rho, u, \theta)\big|_{t=T}\) as the initial data and apply the local existence Theorem 1.1 to extend our local strong solution beyond \(t \geq T\). This contradicts the assumption on \(T\).

4. Appendix

We introduce some Poincaré type inequality (see Chapter 8 in [14]):

**Lemma 4.1.** There exists a constant \(C\) depending only on \(\Omega\) and \(|\rho|_r\) \((r \geq 1)\) (\(\rho \geq 0\) is a real function satisfying \(|\rho|_1 > 0\)), such that for every \(F \geq 0\) satisfying

\[
\rho F \in L^1(\Omega), \quad \sqrt{\rho} F \in L^2(\Omega), \quad \nabla F \in L^2(\Omega),
\]
we have

\[
|F|_6 \leq C (|\sqrt{\rho} F|_2 + (1 + |\rho|_2)|\nabla F|_2).
\]

**Proof.** We first denote that

\[
\mathcal{F} = \frac{1}{|\Omega|} \int_{\Omega} F(y) dy,
\]
then via the classical Poincaré inequality, we quickly deduce that

\[
\mathcal{F} \int_{\Omega} \rho dx = \int_{\Omega} \rho (F - \mathcal{F}) dx + \int_{\Omega} \rho F dx
\leq C (|\rho F|_1 + |\rho|_2|\nabla F|_2) \leq C (|\rho F|_1^\frac{1}{2} + |\rho|_2^\frac{1}{2} + |\rho|_2|\nabla F|_2),
\]
which implies that

\[
\mathcal{F} \leq C (|\sqrt{\rho} F|_2 + |\rho|_2|\nabla F|_2).
\]
(4.1)
Second, we consider that

$$\|F\|_1 = |\nabla F|_2 + |F|_2 \leq |\nabla F|_2 + |F - F|_2 + |\Omega|_1^{\frac{1}{2}} \leq C\left(|\sqrt{\rho F}|_2 + (1 + |\rho|_2)|\nabla F|_2\right),$$

(4.2)

then according to (4.1)-(4.2) and the classical Sobolev imbedding theorem, we easily obtain

the following inequality:

$$|F|_6 \leq C\|F\|_1 \leq C\left(|\sqrt{\rho F}|_2 + (1 + |\rho|_2)|\nabla F|_2\right).$$

□

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