Phragmén-Lindelöf theorems and \( p \)-harmonic measures for sets near low-dimensional hyperplanes

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March 19, 2015

Abstract

We prove estimates of \( p \)-harmonic measure, \( p \in (n - m, \infty] \), for sets in \( \mathbb{R}^n \) which are close to an \( m \)-dimensional hyperplane \( \Lambda \subset \mathbb{R}^n \), \( m \in [0, n - 1] \). Using these estimates, we derive results of Phragmén-Lindelöf type in unbounded domains \( \Omega \subset \mathbb{R}^n \setminus \Lambda \) for \( p \)-subharmonic functions. Moreover, we give local and global growth estimates for \( p \)-harmonic functions, vanishing on sets in \( \mathbb{R}^n \), which are close to an \( m \)-dimensional hyperplane.

2010 Mathematics Subject Classification. Primary 35J25, 35J60, 35J70.

Keywords: \( p \)-subharmonic function; entire \( p \)-harmonic function; growth of \( p \)-harmonic functions; infinity Laplace equation; Phragmén-Lindelöf; global estimates

1 Introduction

The \( p \)-harmonic functions, which are natural nonlinear generalizations of the harmonic functions, are solutions to the \( p \)-Laplace equation, which yields, for \( p \in (1, \infty) \),

\[
\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0.
\]  

(1.1)

If \( p = \infty \), the equation can be written as

\[
\Delta_{\infty} u := \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,
\]  

(1.2)

which is the so called \( \infty \)-Laplace equation. We refer the reader to Section 2 for the definitions of weak solutions, viscosity solutions and \( p \)-harmonicity. The \( p \)-Laplace equation has connections to e.g., minimization problems, nonlinear elasticity theory, Hele-Shaw flows and image processing. For more on applications and the \( p \)-Laplace equation, see e.g., Lundström [15, Chapter 2] and the references therein.
A class of $p$-harmonic functions that has shown to be useful consists of the following $p$-harmonic measures, which will be estimated in this paper.

**Definition 1.1** Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $E \subseteq \partial \Omega$, $p \in (1, \infty)$ and $x \in \Omega$. The $p$-harmonic measure of $E$ at $x$ with respect to $\Omega$ is defined as $\inf_{u} u(x)$, where the infimum is taken over all $p$-superharmonic functions $u \geq 0$ in $\Omega$ such that $\liminf_{z \to y} u(z) \geq 1$, for all $y \in E$.

The $\infty$-harmonic measure is defined in a similar manner, but with $p$-superharmonicity replaced by absolutely minimizing, see Peres–Schramm–Sheffield–Wilson [50, pages 173–174]. It turns out that the $p$-harmonic measure in Definition 1.1 is a $p$-harmonic function in $\Omega$, bounded below by 0 and bounded above by 1. For these and other basic properties of $p$-harmonic measure we refer the reader to Heinonen–Kilpeläinen–Martio [25, Chapter 11]. To avoid confusion, we mention that there are at least three different definitions of $p$-harmonic measure in the literature.

Besides the $p$-harmonic measure above, we refer to the definitions given by Bennnewitz–Lewis [11] and Herron–Koskela [26].

The $p$-harmonic measure is useful when estimating solutions to the $p$-Laplace equation, see e.g., [25, Theorem 11.9]. Recently, Lundström–Vasilis [47] proved estimates for $p$-harmonic measures in the plane, which, together with a result by Hirata [27] yield properties of the $p$-Green function. The $p$-harmonic measure is also useful when studying quasiregular mappings, see [25, Chapter 14]. Moreover, the $p$-harmonic measure has a probabilistic interpretation in terms of the zero-sum two-player game tug-of-war, see Peres–Sheffield [49] and Peres–Schramm–Sheffield–Wilson [50], in which also estimates for $p$-harmonic measure are proved, e.g., for porous sets.

Let $\Lambda \subset \mathbb{R}^n$ be an $m$-dimensional hyperplane, $m \in [1, n-1]$, and introduce the notation

$$\Lambda_s = \{x \in \mathbb{R}^n : d(x, \Lambda) \leq s\}.$$  

Assume that $\Omega \subset \mathbb{R}^n$ is an unbounded domain, satisfying a Harnack chain condition, with boundary $\partial \Omega$ close to $\Lambda$ in the sense that $\partial \Omega \subseteq \Lambda_s$ and $\Lambda \subseteq \partial \Omega$ for some $s > 0$. Denote by $B(w, R)$ the open ball in $\mathbb{R}^n$ with center $w$ and radius $R$. Let $w \in \Lambda$, $p \in (n - m, \infty]$ and let $v_r$ be the $p$-harmonic measure for $\partial B(w, R) \setminus \partial \Omega$ taken with respect to $B(w, R) \cap \Omega$. In Theorem 4.1 we prove that for any $x \in \Omega$ there exists a constant $C$ such that

$$\frac{1}{C} \leq v_r(x) R^\beta \leq C,$$ 

whenever $R$ is large and $\beta = (p - n + m)/(p - 1)$ with $\beta = 1$ for $p = \infty$.

Next, we use this estimate to prove Corollary 4.3, which is an extended version of the classical result of Phragmén-Lindelöf [51]. In particular, suppose that $u$ is $p$-subharmonic in an unbounded domain $\Omega$ satisfying $\Omega \cap \Lambda = \emptyset$ and suppose that $\limsup_{z \to \partial \Omega} u(z) \leq 0$. Then either $u \leq 0$ in the whole of $\Omega$ or it holds that

$$\liminf_{R \to \infty} \frac{\sup_{\partial B(w, R) \cap \Omega} u}{R^\beta} > 0,$$ 

where $\beta$ is as in (1.4). When $\Omega = \mathbb{R}^n \setminus \Lambda_s$, the above growth rate is sharp. Corollary 4.3 generalizes a result of Lindqvist [42], who studied the borderline case $p = n$, to hold in the exponent range $p \in (n - m, \infty]$. 

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Results of Phragmén-Lindelöf type, which have connections to elasticity theory, have been frequently studied during the last century. To mention a few, Ahlfors [4] extended results from [51] to the upper half space of $\mathbb{R}^n$, Gilbarg [21] and Serrin [53] considered more general elliptic equations of second order and Vitolo [54] considered the problem in angular sectors. Kurta [38] and Jin–Lancaster [29, 30, 31] considered quasilinear elliptic equations and non-hyperbolic equations while Capuzzo–Vitolo [18] and Armstrong–Sirakov–Smart [6] considered fully nonlinear equations. Adamowicz [1] studied different unbounded domains for subsolutions of the variable exponent $p$-Laplace equation, while Bhattacharya [14] and Granlund–Marola [23] considered infinity-harmonic functions in unbounded domains.

In connection with the above Phragmén-Lindelöf result, we also prove global growth estimates for positive $p$-harmonic functions, vanishing on $\partial \Omega$, where $\Omega$ is an unbounded domain as described above (1.4). This result is given in Theorem 4.5 and implies, in analogue with (4.1), that $u(x) \approx d(x, \partial \Omega)^\beta$ whenever $x \in \mathbb{R}^n$ and $d(x, \partial \Omega)$ is large. Theorem 4.5 generalizes e.g., some results by Kilpeläinen-Shahgholian-Zhong [36] to hold in a more general geometric setting.

Our proofs rely on comparison with certain explicit $p$-subharmonic and $p$-superharmonic functions, first constructed and used in Lundström [44] to prove local estimates for $p$-harmonic functions. In this paper, we first expand this construction (Lemma 3.4), through which we obtain an extension of all the main results in [44], given for $p \in (n, \infty]$, to hold also in the wider exponent range $p \in (n-m, \infty]$, see Corollary 3.6. Next, we use the explicit $p$-subharmonic and $p$-superharmonic functions, given by Lemma 3.4, to prove local growth estimates for positive $p$-harmonic functions vanishing on a fraction of $\Lambda_s$, where $s \geq 0$ is small (Theorem 3.5). The estimates in Theorem 3.5 are crucial for the proofs of Theorems 4.1 and 4.5, which in turn implies Corollary 4.3.

Local estimates of positive $p$-harmonic functions, vanishing near $(n-1)$-dimensional boundaries, have drawn a lot of attention the last decades. In the case $1 < p < \infty$, see e.g., Aikawa–Kilpeläinen–Shanmugalingam–Zhong [5] for smooth boundaries, Lewis–Nyström [39, 41], Avelin [8] and Avelin–Lundström–Nyström [9] for more general geometries including Lipschitz and Reifenberg flat boundaries. For infinity-harmonic functions, see e.g., Bhattacharya [13] and Lundström–Nyström [46]. Moreover, boundary growth estimates for solutions to the variable exponent $p$-Laplace equation in smooth domains are given by Adamowicz–Lundström [2]. Only few papers considered local estimates of positive $p$-harmonic functions vanishing near boundaries having dimension less than $n-1$. Besides results given in Theorem 3.5 and Corollary 3.6, we refer the reader to Lindqvist [42] and Lundström [44].

2 Notation and preliminary lemmas

By $\Omega$ we denote a domain, that is, an open connected set. For a set $E \subset \mathbb{R}^n$, we let $\overline{E}$ denote the closure, $\partial E$ the boundary, $\overline{E}^c$ the complement of $E$ and $E^o = E \setminus \partial E$. Further, $d(x, E)$ denotes the Euclidean distance from $x \in \mathbb{R}^n$ to $E$, and $B(x, r) = \{ y : |x - y| < r \}$ denotes the open ball with radius $r$ and center $x$. We write $N = \{1, 2, 3, \ldots \}$ for the set of natural numbers and $A \approx B$ if there exists a constant $c$ such that $c^{-1}A \leq B \leq cA$. By $c$ we denote a constant $\geq 1$, not necessarily the same at each occurrence, depending only on $p$ and $n$ if nothing else is mentioned. Moreover, $c(a_1, a_2, \ldots, a_k)$ denotes a constant $\geq 1$, not necessarily the same at
each occurrence, depending only on $a_1, a_2, \ldots, a_k$. We denote points in Euclidean $n$-space $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n) = (x', x'')$, where

$$x' = (x_1, x_2, \ldots, x_{n-m}) \quad \text{and} \quad x'' = (x_{n-m+1}, x_{n-m+2}, \ldots, x_n).$$

(2.6)

We next recall standard definitions of weak solutions, viscosity solutions and $p$-harmonicity. If $p \in (1, \infty)$, we say that $u$ is a (weak) subsolution (supersolution) to the $p$-Laplace equation in $\Omega$ provided $u \in W^{1,p}_{loc}(\Omega)$ and

$$\int_{\Omega} |\nabla u|^{p-2} (\langle \nabla u, \nabla \theta \rangle) \, dx \leq (\geq) 0$$

(2.7)

whenever $\theta \in C^0(\Omega)$ is non-negative. A function $u$ is a (weak) solution of the $p$-Laplacian if it is both a subsolution and a supersolution. Here, as in the sequel, $W^{1,p}_{loc}(\Omega)$ is the Sobolev space of those $p$-integrable functions whose first distributional derivatives are also $p$-integrable, and $C^0(\Omega)$ is the set of infinitely differentiable functions with compact support in $\Omega$. If $p = \infty$, the equation is no longer of divergence form and therefore the above definition is replaced by the definition of viscosity solutions from Crandall–Ishii–Lions [19]. Here, as in the sequel, $\Delta_\infty$ is the $\infty$-Laplace operator defined in (1.2).

An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a (viscosity) subsolution of the $\infty$-Laplacian in $\Omega$ provided that for each function $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local maximum at a point $x_0 \in \Omega$, we have $\Delta_\infty \psi(x_0) \geq 0$. A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a (viscosity) supersolution of the $\infty$-Laplacian in $\Omega$ provided that for each function $\psi \in C^2(\Omega)$ such that $u - \psi$ has a local minimum at a point $x_0 \in \Omega$, we have $\Delta_\infty \psi(x_0) \leq 0$. A function $u : \Omega \to \mathbb{R}$ is a (viscosity) solution of the $\infty$-Laplacian if it is both a subsolution and a supersolution.

If $u$ is an upper semicontinuous subsolution to the $p$-Laplacian in $\Omega$, $p \in (1, \infty]$ then we say that $u$ is $p$-subharmonic in $\Omega$. If $u$ is a lower semicontinuous supersolution to the $p$-Laplacian in $\Omega$, $p \in (1, \infty]$, then we say that $u$ is $p$-superharmonic in $\Omega$. If $u$ is a continuous solution to the $p$-Laplacian in $\Omega$, $p \in (1, \infty]$, then $u$ is $p$-harmonic in $\Omega$.

We note that for the $p$-Laplacian, $1 < p < \infty$, weak solutions are also viscosity solutions (defined as above but with $\Delta_\infty$ replaced by $\Delta_p$); see Juutinen [32, Theorem 1.29]. Moreover, under suitable assumptions, an $\infty$-harmonic function is the uniform limit of a sequence of $p$-harmonic functions as $p \to \infty$; see Jensen [28]. This fact has been used to prove results for $p = \infty$ by taking limits of problems for finite $p$, in which estimates are independent of $p$ when $p$ is large, see e.g., Bhattacharya–DiBenedetto–Manfredi [15], Lindqvist–Manfredi [33], Lewis–Nyström [40] and Lundström–Nyström [46]. As for Phragmén–Lindelöf type results, see Granlund–Marola [23]. With this in mind, we chose to keep track of the dependence of $p$ in our estimates and point out when constants are independent of $p$ when $p$ is large.

We next recall some well known results for $p$-harmonic functions.

**Lemma 2.2** (Comparison principle) Let $p \in (1, \infty]$ be given, $u$ be $p$-superharmonic and $v$ be $p$-subharmonic in a bounded domain $\Omega$. If

$$\limsup_{x \to y} v(x) \leq \liminf_{x \to y} u(x)$$

for all $y \in \partial \Omega$, and if both sides of the above inequality are not simultaneously $\infty$ or $-\infty$, then $v \leq u$ in $\Omega$.  


Proof. If \( p \in (1, \infty) \), this follows from Heinonen–Kilpeläinen–Martio [25, Theorem 7.6]. For the case \( p = \infty \), this was first proved by Jensen [28, Theorem 3.11]. A shorter proof was later presented by Armstrong–Smart [7]. \( \square \)

Lemma 2.3  (Harnack’s inequality) Let \( p \in (1, \infty] \) be given and assume that \( w \in \mathbb{R}^n \), \( r \in (0, \infty) \) and that \( u \) is a positive \( p \)-harmonic function in \( B(w, 2r) \). Then there exists \( c(n, p) \), independent of \( p \) if \( p \) is large, such that

\[
\sup_{B(w,r)} u \leq c \inf_{B(w,r)} u.
\]

Proof. For the case \( p \in (1, \infty) \), when the constant is allowed to depend on \( p \), we refer the reader to Heinonen–Kilpeläinen–Martio [25, Theorem 6.2]. For the uniform in \( p \) case, see Koskela–Manfredi–Villamor [37], Lindqvist–Manfredi [43] or Lundström–Nystrom [46, Lemma 2.3]. For the case \( p = \infty \) the result follows by taking the limit \( p \to \infty \) in the above uniform in \( p \) estimate; see [43]. Moreover, another proof concerning the case \( p = \infty \) is given by Bhattacharya [12]. \( \square \)

3  Estimates for \( p \)-harmonic functions vanishing near \( m \)-dimensional hyperplanes

We begin this section by stating, in our geometric setting, some well known basic boundary estimates, such as Hölder continuity up to the boundary and the Carleson estimate. Next, we prove a refined version of Lundström [44, Lemma 3.7] which yields explicit \( p \)-subharmonic and \( p \)-superharmonic functions which will be crucial for our proofs. Finally, we state and prove Theorem 3.5 giving estimates for \( p \)-harmonic functions, vanishing near \( m \)-dimensional hyperplanes.

In the following we let \( C_p \) denote \( p \)-capacity as defined in Heinonen–Kilpeläinen–Martio [25, Chapter 2].

Lemma 3.1  Let \( M \subset \mathbb{R}^n \) be a manifold of dimension \( m < n \), then \( C_p(M) = 0 \) if and only if \( p \leq n - m \).

Proof. The result follows from Adams–Hedberg [3, Corollary 5.1.15]. \( \square \)

Lemma 3.2  (Hölder continuity) Suppose that \( m, n \in \mathbb{N} \) such that \( m \in [0, n - 1] \), let \( \Lambda \subset \mathbb{R}^n \) be an \( m \)-dimensional hyperplane, \( w \in \Lambda \), \( r \in (0, \infty) \) and \( p \in (n - m, \infty] \). Assume that \( u \) is a non-negative \( p \)-harmonic function in \( B(w, 2r) \setminus \Lambda \), continuous in \( B(w, 2r) \) with \( u = 0 \) on \( B(w, 2r) \cap \Lambda \). Then there exist constants \( \gamma \in (0,1] \) and \( c \), both depending only on \( p \) and \( n \), independent of \( p \) if \( p \) is large, such that if \( x,y \in \Omega \cap B(w, r) \) then

\[
|u(x) - u(y)| \leq c \left( \frac{|x - y|}{r} \right)^\gamma \sup_{\Omega \cap B(w, 2r)} u.
\]

In particular, we can take \( \gamma \to 1 \) as \( p \to \infty \) with \( \gamma = 1 \) if \( p = \infty \).
Proof. If \( p > n \), we obtain the result by a Sobolev embedding theorem, see e.g., Lundström–Nyström [46, Lemma 2.4]. If \( n - m < p \leq n \), the result follows from Heinonen–Kilpeläinen–Martio [25, Theorem 6.44] if we can prove that there exist constants \( c_0 \) and \( r_0 \) so that

\[
\frac{C_p(\Lambda \cap \overline{B}(x_0, r), B(x_0, 2r))}{C_p(\overline{B}(x_0, r), B(x_0, 2r))} \geq c_0
\]  

whenever \( 0 < r < r_0 \) and \( x_0 \in \Lambda \). To prove \((3.1)\), we observe that from Lemma 3.1 and since the \( p \)-Laplace equation is invariant through scalings and translations, we have

\[
C_p(\Lambda \cap \overline{B}(x_0, r), B(x_0, 2r)) = r^{n-p}C_p(\Lambda \cap \overline{B}(0, 1), B(0, 2)) \geq c(n, p)r^{n-p}.
\]

Since [25, Example 2.12] gives \( C_p(\overline{B}(x_0, r), B(x_0, 2r)) = c(n, p)r^{n-p} \) the inequality \((3.1)\) follows for \( r_0 = \infty \). The proof of Lemma 3.2 is complete.

Given an \( m \)-dimensional hyperplane \( \Lambda \) and \( w \in \Lambda \) we let in the following \( A_r(w) \) denote a point satisfying

\[
d(A_r(w), \Lambda) = r \quad \text{and} \quad A_r(w) \in \partial B(w, r).
\]  

Lemma 3.3 (Carleson’s estimate) Suppose that \( m, n \in \mathbb{N} \) such that \( m \in [0, n - 1] \), let \( \Lambda \subset \mathbb{R}^n \) be an \( m \)-dimensional hyperplane, \( w \in \Lambda \), \( r \in (0, \infty) \) and \( p \in (n - m, \infty] \). Assume that \( u \) is a non-negative \( p \)-harmonic function in \( B(w, r) \setminus \Lambda \) continuous in \( B(w, r) \) with \( u = 0 \) on \( B(w, r) \cap \Lambda \). Then there exists \( c(n, p) \), independent of \( p \) if \( p \) is large, such that

\[
\sup_{\Omega \cap B(w, r/c)} u \leq c u(A_{r/c}(w)).
\]

Proof. A proof for linear elliptic partial differential equations, in Lipschitz domains with \((n - 1)\)-dimensional boundary, can be found in Caffarelli–Fabes–Mortola–Salsa [16]. The proof uses only the Harnack chain condition (see e.g., [9, Definition 1.3]), analogues of Harnack’s inequality, Hölder continuity up to the boundary and the comparison principle for linear equations. In particular, the proof also applies in our situation.

The following lemma extends constructions in Lundström [44, Lemma 3.7], given for \( p \in (n, \infty) \), to hold for the wider exponent range \( p \in (n - m, \infty) \). Recall from [27,6] the notation \( x = (x', x^m) \in \mathbb{R}^n \) and the geometric definition of \( \Lambda_s \), given in \((1.3)\) as

\[
\Lambda_s = \{ x \in \mathbb{R}^n : d(x, \Lambda) \leq s \},
\]

where \( \Lambda \) is an \( m \)-dimensional hyperplane.

Lemma 3.4 Suppose that \( m, n \in \mathbb{N} \) such that \( m \in [1, n - 2] \). Let \( p \in (n - m, \infty) \), \( \beta = (p - n + m)/(p - 1) \) and suppose that \( \gamma \) satisfies \( 0 < \gamma < \beta \). Then there exists \( \delta_c \in (0, 1) \), depending only on \( n, \gamma \) and \( p \), such that \( \hat{u} \) is a supersolution, and \( \hat{u} \) is a subsolution to the \( p \)-Laplace equation in \( \Lambda_{\delta_c} \cap \{ x : |x^m| < 1 \} \), where

\[
\hat{u} = |x'|^\beta + |x^m|^{\gamma} |x'|^\alpha - \frac{1}{2} |x'|^2 \quad \text{and} \quad \hat{u} = (1 - |x^m|^2) |x'|^\beta + |x'|.
\]

Moreover, if \( \gamma > 1/2 \) then \( \delta_c \) can be chosen independent of \( p \) if \( p \) is large.
Proof. For a proof showing that \( \hat{u} \) is a subsolution, as well as for the case \( \gamma = (p-n)/(p-1) \), we refer the reader to the proof of Lemma 3.7 in [44]. It remains to show that \( \hat{u} \) is a supersolution for any \( \gamma \), \( 0 < \gamma < \beta \). To do so, it suffices to show that there exists \( \delta_c \in (0,1) \), depending only on \( \gamma, n \) and \( p \), such that

\[
\Delta_p \hat{u} = \Delta \hat{u} \| \nabla \hat{u} \|^{p-2} + (p-2) \| \nabla \hat{u} \|^{p-4} \Delta \infty \hat{u} \leq 0 \quad \text{in} \quad \Lambda_{\delta_c} \cap \{ x : |x''| < 1 \}. \tag{3.3}
\]

Here, \( \Delta_p \) is the \( p \)-Laplace operator defined in (1.1), \( \Delta := \Delta_2 \) and \( \Delta \infty \) is the \( \infty \)-Laplace operator defined in (1.2). Since \( p > n - m \geq 2 \) and \( |\nabla \hat{u}| \neq 0 \) in \( C_{r_c} \setminus \Lambda \), (3.3) equals

\[
\hat{\Delta}_p \hat{u} := \frac{\Delta \hat{u} \| \nabla \hat{u} \|^2}{p-2} + \Delta \infty \hat{u} \leq 0 \quad \text{in} \quad \Lambda_{\delta_c} \cap \{ x : |x''| < 1 \}. \tag{3.4}
\]

Following the calculations in [44] Pages 6857–6858 we obtain that

\[
\hat{\Delta}_p \hat{u} = \frac{\Delta \hat{u} \| \nabla \hat{u} \|^2}{p-2} + \Delta \infty \hat{u} = Z_0 \gamma + Z_2 |x''|^2 + Z_4 |x''|^4 + Z_6 |x''|^6,
\]

where the coefficients are given by

\[
Z_0 = -\frac{\beta^2(p + n - m)}{p-2} |x'|^{2\beta - 2} + O(|x'|^{\gamma + 2\beta - 2}),
\]

\[
\leq -\beta^2 |x'|^{2\beta - 2} + O(|x'|^{\gamma + 2\beta - 2}),
\]

\[
Z_2 = -Z \frac{\gamma \beta^2}{p-2} |x'|^{\gamma + 2\beta - 4} + -z_2 \gamma \beta |x'|^{\gamma + \beta - 2} + O(|x'|^{\gamma + 2\beta - 2}),
\]

\[
Z_4 = -Z \frac{2 \gamma^2 \beta}{p-2} |x'|^{2\gamma + \beta - 4} + -z_4 \gamma^2 |x'|^{2\gamma - 2} + O(|x'|^{3\gamma - 2}), \tag{3.5}
\]

\[
Z_6 = -Z \frac{\gamma^3}{p-2} |x'|^{3\gamma - 4},
\]

in which \( Z = p - n + m - (p-1)\gamma \). Clearly \( Z > 0 \) by the assumption \( 0 < \gamma < \beta \) and, hence, we conclude that the leading terms are negative in (3.5). It follows from (3.5) and (3.6) that there exists \( \delta_c \in (0,1) \), depending only on \( \gamma, n \) and \( p \), such that (3.4) is satisfied in \( \Lambda_{\delta_c} \cap \{ x : |x''| < 1 \} \).

For the uniform in \( p \) case, we note that if \( p \) is large enough, then

\[
z_2 \geq \frac{2 \gamma(p^2 - 2p + 1) + (3p - 2)(n - m - 1)}{p - 3p + 2} \geq 2 \gamma,
\]

\[
z_4 \geq \frac{2 \gamma(p - 1) - p - 2 + 3(n - m)}{p - 2} \geq 2 \gamma - 1 > 0. \tag{3.7}
\]

By following calculations in [44] Pages 6857–6858, we see that the constants in the Ordos in (3.6) will not explode as \( p \to \infty \). Therefore, from (3.6), (3.7) and the assumption \( \gamma > 1/2 \), we conclude that \( \delta_c \) can be chosen independent of \( p \) if \( p \) is large, but still depending on \( n \) and \( \gamma \). This completes the proof of Lemma 3.4. \( \square \)

We are now ready to state and prove the main theorem of this section, which gives the following upper and lower growth estimates of \( p \)-harmonic functions, \( p \in (n - m, \infty] \), vanishing near an \( m \)-dimensional hyperplane \( \Lambda \). Recall the definition of \( A_r(w) \) given in (3.2).
Theorem 3.5 Suppose that \( m, n \in \mathbb{N} \) such that \( m \in [0, n-1] \), let \( \Lambda \subset \mathbb{R}^n \) be an \( m \)-dimensional hyperplane, \( w \in \Lambda \), \( r \in (0, \infty) \), \( p \in (n - m, \infty] \) and suppose that \( \beta = (p - n + m)/(p - 1) \) with \( \beta = 1 \) if \( p = \infty \). Let \( \delta \in (0, \delta_c/2) \) where \( \delta_c \) is from Lemma 3.4 and assume that \( u \) is a positive \( p \)-harmonic function in \( B(w, 4r) \setminus \Lambda_{\delta r} \), with \( u = 0 \) continuously on \( B(w, 4r) \cap \partial \Lambda_{\delta r} \). Then there exists \( c = c(p, n) \), independent of \( p \) if \( p \) is large, such that

\[
c - 1 \left( \frac{d(x, \Lambda)}{r} \right)^\beta - \delta^\beta \leq \frac{u(x)}{u(A_r(w))} \leq c \left( \frac{d(x, \Lambda)}{r} \right)^\beta - \delta^\beta
\]

whenever \( x \in B(w, \delta_c r) \setminus \Lambda_{\delta r} \).

Before proving the theorem, we make some remarks about the result. For any \( \delta \in (0, \delta_c/2) \), Theorem 3.5 implies that, locally, the \( p \)-harmonic function \( u \) vanishes at the same rate as the distance function. In particular, Taylor-expanding the above estimates yields

\[
c - 1 \delta^{\beta-1} \frac{d(x, \Lambda)}{r} \leq \frac{u(x)}{u(A_r(w))} \leq c \delta^{\beta-1} \frac{d(x, \Lambda)}{r}
\]

whenever \( x \in B(w, 2\delta r) \setminus \Lambda_{\delta r} \) and \( c = c(p, n) \), independent of \( p \) if \( p \) is large. If \( \delta = 0 \) in Theorem 3.5 then we obtain the following corollary, in which \( C^{0,\beta}(E) \) denotes the space of Hölder continuous functions in \( E \in \mathbb{R}^n \).

Corollary 3.6 Suppose that \( m, n, \Lambda, w, r, p, \beta, \delta \) and \( u \) are as in Theorem 3.5 and assume in addition that \( \delta = 0 \). Then there exists \( c = c(p, n) \), independent of \( p \) if \( p \) is large, such that

\[
c - 1 \left( \frac{d(x, \Lambda)}{r} \right)^\beta \leq \frac{u(x)}{u(A_r(w))} \leq c \left( \frac{d(x, \Lambda)}{r} \right)^\beta
\]

whenever \( x \in B(w, \delta_c r) \setminus \Lambda \). Moreover, there exists \( c(n, p) \) such that \( u \in C^{0,\beta}(B(w, r/c)) \), and \( \beta \) is the optimal Hölder exponent for \( u \).

Corollary 3.6 retrieves the geometric setting of \cite[Theorem 1.1 and Corollary 1.2]{44}, and generalizes these theorems, given for \( p \in (n, \infty) \), to hold also in the wider exponent range \( p \in (n - m, \infty) \).

Besides the applications above and those given in Section 4, Theorem 3.5 can be useful when studying local estimates of \( p \)-harmonic functions vanishing on sets which can be trapped into \( \Lambda_s \). An example of such sets are the \( m \)-dimensional Reifenberg-flat sets, which are approximable, uniformly on small scales, by \( m \)-dimensional hyperplanes. For the definition of Reifenberg-flat sets and for some applications, involving boundary behaviour of solutions to PDEs, see e.g., Kenig–Toro \cite{33}, David \cite{20}, Guanghao–Wang \cite{24}, Capogna–Kenig–Lanzani \cite{17}, Lewis–Nyström \cite{41}, and Avelin–Lundström–Nyström \cite{9, 10}.

Proof of Corollary 3.6. Estimate (3.8) follows immediately by taking \( \delta = 0 \) in Theorem 3.5 Using Theorem 3.5 in place of \cite[Theorem 1.1]{44}, and observing from Kilpelinen–Zhong \cite{31, 35} that Lemma 2.4 in \cite{44} holds also in the wider exponent range \( p \in (n - m, \infty) \), the Hölder continuity follows by mimicking the proof of Corollary 1.2 in \cite{44}. □
Proof of Theorem 3.5. Since the $p$-Laplace equation is invariant under scalings, translations and rotations, we assume, without loss of generality, that $w = 0$, $r = 1$, $u(a_r(w)) = u(a_1(0)) = 1$ and

$$\Lambda = \{x \in \mathbb{R}^n : |x'| = 0\}.$$

In these coordinates, we will prove the existence $c = c(p, n)$, independent of $p$ if $p$ is large, such that

$$c^{-1} \{d(x, \Lambda)^\beta - \delta^\beta\} \leq u(x) \leq c \{d(x, \Lambda)^\beta - \delta^\beta\}$$

whenever $x \in B(0, \delta) \setminus \Lambda_\delta$. Scaling back then yields Theorem 3.5.

Proof of the upper bound. We begin with the case $m = n - 1$, in which the Theorem follows by already well know results, such as e.g., Aikawa–Kilpeläinen–Shanmugalingam–Zhong [5]. We include a proof for the sake of completeness. Since, in this case, $\Lambda$ splits $\mathbb{R}^n$ in two halves, we focus on the upper of these halves. Let $\alpha = (p - n)/(p - 1)$ with $\alpha = 1$ if $p = \infty$ and consider the $p$-harmonic function

$$\tilde{f}(x) = a|x - x_0|^\alpha + b, \quad \text{if} \quad p \neq n,$$

$$\tilde{f}(x) = a \log |x - x_0| + b, \quad \text{if} \quad p = n,$$  

(3.9)

for some $a, b$. Choose $a$ and $b$ such that $\tilde{f}$ has boundary values $\tilde{f} = 0$ on $\partial B(x_0, 1/2)$ and $\tilde{f} = 1$ on $\partial B(x_0, 1)$. From (3.9) we conclude the existence of $c(n, p)$, decreasing in $p$, such that

$$c^{-1} \leq \frac{\partial \tilde{f}}{\partial \nu} \leq c \quad \text{in} \quad B(x_0, 1) \setminus B(x_0, 1/2),$$

(3.10)

where $\nu$ denotes the outer normal to $\partial B(x_0, 1)$. Since $u(A_1(0)) = 1$ there exists, by Harnack’s inequality and the Carleson estimate, a constnt $c(n, p)$ such that

$$u(x) \leq c \quad \text{in} \quad B(0, 3) \cap \{x : x' = x_1 > \delta\}.$$

Since $u$ vanishes continuously on $\partial \Lambda_\delta \cap B(0, 4)$, we can conclude, by the comparison principle applied to the functions $u$ and $c \tilde{f}$ for some large enough $c$, and by letting $x_0$ vary with the restriction that $B(x_0, 1/2)$ is tangent to $\{x : x_1 = \delta\}$, $B(x_0, 1/2) \subset \{x : x_1 < \delta\}$ and $B(x_0, 1) \subset B(0, 3)$, that there exists $c(n, p)$, independent of $\delta$ and $p$ if $p$ is large, such that

$$u(x) \leq c (|x' - \delta|) \quad \text{whenever} \quad x \in B(0, 1) \cap \{x : x' = x_1 > \delta\}.$$

Thus, we have proved the upper bound in Theorem 3.5 in the case $m = n - 1$.

In the rest of the proof of the upper bound, we assume $m \in [0, n - 2]$. Assume first also that $p > n$ and consider the $p$-harmonic function

$$\hat{f}(x) = |x - x_0|^\alpha - \delta^\alpha,$$

(3.11)

where $x_0 \in \Lambda \cap B(0, 2)$ and $\alpha$ is the exponent defined above (3.9). Note that $\hat{f} \geq 0$ on $B(x_0, 1) \setminus \Lambda_\delta$ and $\hat{f} = 1 - \delta^\alpha$ on $\partial B(x_0, 1)$. Since $u(a_1(0)) = 1$ there exists a constant $c(n, p)$, independent of $\delta$, such that

$$u(x) \leq c \quad \text{in} \quad B(0, 3) \setminus \Lambda_\delta.$$  

(3.12)
To see this, let $\tilde{u}$ be the $p$-harmonic function in e.g., $B(0, 4 - 10^{-8}) \setminus \Lambda$, satisfying boundary values $\tilde{u} = u$ on $\partial B(0, 4 - 10^{-8}) \setminus \Lambda_\delta$ and $\tilde{u} = 0$ on $(\partial B(0, 4 - 10^{-8}) \cap \Lambda_\delta) \cup \Lambda$ continuously. Note that the boundary values for $\tilde{u}$ are continuous and that existence of $\tilde{u}$ follows from \eqref{3.1} and standard existence theorems, see e.g., Heinonen–Kilpeläinen–Martio \cite{25}. It follows by construction and by the comparison principle that $u \leq \tilde{u}$ in $B(0, 4 - 10^{-8}) \setminus \Lambda_\delta$. Applying Harnack’s inequality and the Carleson estimate to $\tilde{u}$ implies \eqref{3.12}. Since $u$ vanishes continuously on $\partial \Lambda_\delta \cap B(0, 2)$, we can conclude, by the comparison principle applied to the functions $u$ and $c\tilde{f}$ for some large enough $c(n, p)$, and by letting $x_0 \in \Lambda \cap B(0, 2)$ vary, that there exists $c(n, p)$, independent of $\delta$ and $p$ if $p$ is large, such that

$$u(x) \leq c(|x'|^\alpha - \delta^\alpha) \quad \text{whenever} \quad x \in B(0, 2) \setminus \Lambda_\delta. \quad \text{(3.13)}$$

If $p = \infty$ or if $m = 0$, then we have proved the upper bound in Theorem 3.5.

Assume now that $n - m < p \leq n$ (implying $m \geq 1$) and note that by Hölder continuity up to the boundary there exists $c(n, p)$ and $\gamma(n, p)$, independent of $\delta$ and independent of $p$ if $p$ is large, such that

$$u(x) \leq c|x'|^\gamma \quad \text{whenever} \quad x \in B(0, 2) \setminus \Lambda_\delta. \quad \text{(3.14)}$$

To ensure independence of $\delta$ in the above display, consider the auxiliary function $\tilde{u}$ defined below \eqref{3.12} but with $\tilde{u} = 0$ on $(\partial B(0, 4 - 10^{-8}) \cap \Lambda_\delta) \cup \tilde{\Lambda}$, instead of $\tilde{u} = 0$ on $\Lambda$, where $\tilde{\Lambda}$ is an $m$-dimensional hyperplane parallel to $\Lambda$ satisfying $\tilde{\Lambda} \subset \Lambda_\delta$. As before, it follows that $u \leq \tilde{u}$ in $B(0, 4 - 10^{-8}) \setminus \Lambda_\delta$. Allowing $\tilde{\Lambda}$ to move in $\Lambda_\delta$ and applying Lemma 3.2 (Hölder continuity) to $\tilde{u}$ proves \eqref{3.14}.

Using estimates \eqref{3.13} and \eqref{3.14} we will now use the supersolution given in Lemma 3.4 to complete the proof of the upper bound for the remaining cases $m \in [1, n - 2]$ and $p \in (n - m, \infty)$. To do so, we will first show that there exists $c$ such that

$$c \left(\hat{u} - \delta^\beta + \frac{1}{2}\delta^2\right) \geq u \quad \text{on} \quad \partial(\{x : |x''| \leq 1\} \cap \Lambda_{\delta_c}^\circ \setminus \Lambda_\delta). \quad \text{(3.15)}$$

Recall the assumption $2\delta < \delta_c < 1$. In particular, on this set we have either

$$|x'| = \delta, \quad \text{implying} \quad \hat{u} - \delta^\beta + \frac{1}{2}\delta^2 \geq 0, \quad \text{(3.16)}$$

$$|x'| = \delta_c, \quad \text{implying} \quad \hat{u} - \delta^\beta + \frac{1}{2}\delta^2 \geq \delta_c^\beta - \delta^\beta - \frac{1}{2}\delta_c^2 + \frac{1}{2}\delta^2 \geq \frac{1}{c},$$

$$|x''| = 1 \quad \text{and} \quad \delta < |x'| < \delta_c, \quad \text{implying} \quad$$

$$\hat{u} - \delta^\beta + \frac{1}{2}\delta^2 = |x'|^\beta + |x'|^\alpha - \frac{1}{2}|x''|^2 - \delta^\beta + \frac{1}{2}\delta^2 \geq |x'|^\alpha - \frac{1}{2}|x''|^2 + \frac{1}{2}\delta^2 \geq \frac{1}{2}|x'|^\alpha,$$

for some $c$ depending only on $\beta$ and $\delta_c$. From \eqref{3.13}, \eqref{3.14} and \eqref{3.16} we conclude \eqref{3.15} and we can thus compare these functions in the set $\partial(\{x : |x''| \leq 1\} \cap \Lambda_{\delta_c}^\circ \setminus \Lambda_\delta)$. By the comparison principle and by the definition of $\hat{u}$, it follows that

$$u(x) \leq c \left(|x'|^\beta + |x''|^2 |x'|^\alpha - \frac{1}{2}|x''|^2 - \delta^\beta + \frac{1}{2}\delta^2\right) \leq c \left(|x'|^\beta - \delta^\beta\right) \quad \text{(3.17)}$$
whenever \( x \in \{ x : |x''| = 0 \} \cap \Lambda_{\delta_c}^2 \setminus \Lambda_\delta \). The constants in (3.17) depends only on \( p, n \) and \( \delta_c \), where \( \delta_c(p, n, \gamma) \) is from Lemma 3.4. Since, by Lemma 3.2, \( \gamma = \gamma(p, n) \) and \( \gamma \to 1 \) as \( p \to \infty \), we conclude, from Lemma 3.4 that the constants in (3.17) depends only on \( p, n \), independent of \( p \) if \( p \) is large. Finally, by translating the function \( \hat{u} \) and the domain \( \{ x : |x''| \leq 1 \} \cap \Lambda_{\delta_c}^2 \setminus \Lambda_\delta \) in the \( x'' \)-direction, we finish the proof of the upper bound. In particular, as long as \( \{ x : |x''| \leq 1 \} \cap \Lambda_{\delta_c}^2 \setminus \Lambda_\delta \subset B(0, 2) \) where we have (3.13) and (3.14), we may apply the same argument. Thus we obtain that (3.17) holds true in \( B(0, \delta_c) \), which completes the proof of the upper bound in Theorem 3.5.

**Proof of the lower bound.** We first observe that since \( u(a_1(0)) = 1 \) we obtain \( u(x) \geq c^{-1} \) on \( B(0, 4 \cdot 10^{-8}) \setminus \Lambda_{1/2} \) by Harnack’s inequality (focusing on the upper half of \( \mathbb{R}^n \) when \( m = n - 1 \).

If \( m = 0 \), then we use comparison with the function \( \hat{f} \) from (3.11) as follows. Put \( x_0 = 0 \) and observe that then \( c^{-1} \hat{f} \leq u \) on \( \partial B(0, 1) \cup \partial \Lambda_\delta \). By the comparison principle \( c^{-1} \hat{f} \leq u \) in \( B(0, 1) \setminus \Lambda_{\delta} \) and so

\[
|x|^{\alpha} - \delta^{\alpha} \leq cu(x) \quad \text{whenever} \quad x \in B(0, 1) \setminus \Lambda_\delta,
\]

giving the theorem in the cases of \( m = 0 \).

Next, assume that \( m \geq 1 \) and consider again a \( p \)-harmonic function of the form

\[
\hat{f}(x) = \begin{cases} a |x - x_0|^\alpha + b, & \text{if} \quad p \neq n, \\ a \log |x - x_0| + b, & \text{if} \quad p = n, \end{cases}
\]

for some \( a, b \) and with \( \alpha \) as defined above (3.9). Choose \( a \) and \( b \) such that \( \hat{f} \) has boundary values \( \hat{f} = 0 \) at \( \partial B(x_0, 1) \) and \( \hat{f} = 1 \) at \( \partial B(x_0, 1/2) \). Using (3.10), we see that \( c \hat{f} \geq 1 - |x - x_0| \) in \( B(x_0, 1) \setminus B(x_0, 1/2) \) for some \( c(n, p) \) decreasing in \( p \). By using \( c^{-1} \hat{f} \) as a barrier from below for \( u \) by placing the ball \( B(x_0, 1) \) tangent to \( \Lambda_\delta \) and allowing \( x_0 \) to vary, with the restriction \( B(x_0, 1) \subset B(0, 4 \cdot 10^{-8}) \), we see that

\[
|x'| - \delta \leq cu(x) \quad \text{whenever} \quad x \in B(0, 2) \setminus \Lambda_\delta. \tag{3.18}
\]

Note that if \( m = n - 1 \) or if \( p = \infty \), then from (3.18) we are done with the lower bound in Theorem 3.5.

We assume from now on that \( m \in [1, n - 2] \). The next step is to use the subsolution \( \hat{u} - (\delta^\beta + \delta) \), derived in Lemma 3.4, as follows. On \( \partial(\{ x : |x''| \leq 1 \} \cap \Lambda_{\delta_c}^2 \setminus \Lambda_\delta) \) we have either

\[
\begin{align*}
|x'| &= \delta, & \text{implying} & \quad \hat{u} - (\delta^\beta + \delta) &= (1 - |x''|^2)\delta^\beta + \delta - (\delta^\beta + \delta) \leq 0, \\
|x'| &= \delta_c, & \text{implying} & \quad \hat{u} - (\delta^\beta + \delta) &= (1 - |x''|^2)\delta_c^\beta + \delta_c - (\delta^\beta + \delta) \leq c, \\
|x''| &= 1 \text{ and } \delta < |x'| < \delta_c, & \text{implying} & \quad \hat{u} - (\delta^\beta + \delta) &= |x'| - (\delta^\beta + \delta) \leq |x'|. \tag{3.19}
\end{align*}
\]

Therefore, it follows by (3.18) and (3.19) that

\[
\hat{u} - (\delta^\beta - \delta) \leq cu \quad \text{on} \quad \partial(\{ x : |x''| \leq 1 \} \cap \Lambda_{\delta_c}^2 \setminus \Lambda_\delta),
\]

for some \( c(n, p) \), independent of \( p \) when \( p \) is large. By the comparison principle and by the definition of \( \hat{u} \), we obtain

\[
cu \geq (1 - |x''|^2)|x'|^\beta + |x' - (\delta^\beta + \delta) \geq |x'|^\beta - \delta^\beta \tag{3.20}
\]
4 Estimates of $p$-harmonic measures and theorems of Phragmén-Lindelöf type

We first state and prove our results concerning $p$-harmonic measures. Using these results, we then conclude our Phragmén-Lindelöf-type theorems for $p$-subharmonic and $p$-harmonic functions.

In the complex plane, the harmonic measure of the semicircle $|z| = r$, $Im(z) \geq 0$, taken with respect to $|z| < r$, $Im(z) > 0$, is given explicitly by

$$v_r(z) = 2 \left(1 - \frac{1}{\pi} \arg \frac{z-r}{z+r}\right) = \frac{4}{\pi} \int_0^{[z;r]^{1/4}} \frac{t \, dt}{\sqrt{1-t^4}},$$

where $[z;r] = 4r^2y^2/(4r^2y^2 + (r^2 - |z|^2)^2)$ and $z = x + iy$, see e.g., Nevalinna [48, Page 43] or Lindqvist [42, Page 310]. Moreover, in $\mathbb{R}^n$, an explicit formula is still valid in the borderline case $p = n$. In particular, [42, Lemma 3.5] proves the following. Let $\Lambda$ be an $m$-dimensional hyperplane, $m \in [1, n-1]$, and denote

$$\kappa(n, m) = \int_0^1 t^{(2m+1-n)/(n-1)}(1 - t^4)^{-1/2} \, dt.$$  

Define

$$v_r(x) = \frac{1}{\kappa(n, m)} \int_0^{[x;r]^{1/4}} t^{(2m+1-n)/(n-1)}(1 - t^4)^{-1/2} \, dt,$$

where $r > 0$ and $[x;r] = 4r^2|x'|^2/(4r^2|x'|^2 + (r^2 - |x|^2)^2)$ for $|x''|^2 \neq r^2$. Then, $v_r(x)$ is the $n$-harmonic measure for $\partial B(w, r) \setminus \Lambda$ with respect to $B(w, r) \setminus \Lambda$. The asymptotic behaviour

$$C^{-1} \leq v_r(x) r^{m/(n-1)} \leq C,$$

as $r \to \infty$ follows, see [42, Lemma 3.6].

To the authors knowledge, no explicit formula is known in the general case $p \in (n - m, \infty]$, $p \neq n$. Nevertheless, in the below theorem, which we state and prove in more general geometry, we show that the asymptotic behaviour, as $r \to \infty$, generalizes to $p \in (n - m, \infty]$ as follows.

**Theorem 4.1** Suppose that $m, n \in \mathbb{N}$ such that $m \in [0, n-1]$, let $\Lambda \subset \mathbb{R}^n$ be an $m$-dimensional hyperplane, $w \in \Lambda$, $p \in (n - m, \infty]$ and suppose that $\beta = (p - n + m)/(p - 1)$ with $\beta = 1$ if $p = \infty$. Assume that for some $s_0$, $0 \leq 2s < s_0$, $\Omega \subset \mathbb{R}^n$ is an unbounded domain so that $\partial\Omega \subset \Lambda_s$ and $\Lambda \subset C_0$. Let $v_r$ be the $p$-harmonic measure for $\partial B(w, 5r) \setminus \Omega$ with respect to $B(w, 5r) \cap \Omega$. Then there exists $c = c(p, n)$, independent of $p$ if $p$ is large, such that

$$c^{-1} s_0^\beta \leq v_r(A_{s_0}(w)) r^{\beta} \leq c s_0^{\beta},$$

whenever $s_0/\delta_c < r$, where $\delta_c$ is from Lemma 3.4.
Before we prove the theorem, we make the following remark, which proof is immediate.

**Remark 4.2** If one assumes also that Ω in Theorem 4.1 satisfies a Harnack chain condition, see e.g., [8, Definition 1.3], then, using Harnack’s inequality, Theorem 4.1 implies that for any x ∈ Ω there exists a constant C such that

\[ C^{-1} \leq v_r(x) r^\beta \leq C \]

whenever r is so large that x ∈ B(w, 5r) and s_0/δ_c < r. Moreover, the lower bound in Theorem 4.1 holds for any Ω ⊂ \( \mathbb{R}^n \) such that ∂Ω ⊆ \( \Lambda_s \).

**Proof of Theorem 4.1.** In the following, if m = n − 1 so that \( \Lambda \) splits \( \mathbb{R}^n \) in two halves, we focus on the upper of these halves. To prove the upper bound, let \( \hat{v} \) be the p-harmonic function in \( B(w, 5r) \setminus \Lambda \), satisfying boundary values 1 on \( \partial B(w, 5r) \) and 0 on \( B(w, 4r) \cap \Lambda \) continuously. If m ≥ 1 then we also let \( \hat{v} \) increase continuously from 0 to 1 on the set \( \Lambda \cap (B(w, 5r) \setminus B(w, 4r)) \). Note that the boundary values for \( \hat{v} \) are continuous and that existence of \( \hat{v} \) follows from (3.1) and standard existence theorems, see e.g., Heinonen–Kilpeläinen–Martio [25]. By construction and by the comparison principle we obtain

\[ v_r \leq \hat{v} \text{ in } B(w, 5r) \cap \Omega. \]

Moreover, using Harnack’s inequality and a well known Hölder continuity up to \( \partial B(w, 5r) \), near \( A_{5r}(w) \), of the p-harmonic functions \( v_r \) and \( \hat{v} \), we obtain

\[ v_r(A_r(w)) \approx \hat{v}(A_r(w)) \approx 1 \]

for constants depending only on p and n. We next apply Theorem 3.5 to \( \hat{v} \), with \( x = A_{s_0}(w) \) and \( \delta = 0 \), giving

\[ c^{-1}v_r(A_{s_0}(w)) \leq \frac{\hat{v}(A_{s_0}(w))}{\hat{v}(A_r(w))} \leq c \left( \frac{d(A_{s_0}(w), \Lambda)}{r} \right)^\beta \leq c \frac{s_0^\beta}{r^\beta} \]

whenever \( s_0 < \delta_c r \) and \( c = c(n, p) \), independent of p when p is large. This proves the upper bound in Theorem 4.1.

To prove the lower bound, let \( \tilde{v} \) be the p-harmonic function in \( B(w, 5r) \setminus \Lambda_s \), satisfying boundary values 1 on \( \partial B(w, 5r) \setminus \Lambda_{2s} \) and 0 on \( B(w, 5r) \cap \partial \Lambda_s \) continuously. If m ≥ 1 then we also let \( \tilde{v} \) increase continuously from 0 to 1 on the set \( \partial B(w, 5r) \cap (\Lambda_{2s} \setminus \Lambda_s) \). By similar reasoning as in the proof of the upper bound we have

\[ \tilde{v} \leq v_r \text{ in } B(w, 5r) \setminus \Lambda_s, \]

and

\[ v_r(A_r(w)) \approx \tilde{v}(A_r(w)) \approx 1. \]

We now apply Theorem 3.5 to \( \tilde{v} \), with \( x = A_{s_0}(w) \) and \( \delta = s/r \), to obtain

\[ c^{-1}s_0^\beta \leq c^{-1} \left\{ \frac{s_0^\beta}{r^\beta} - \frac{s_0^\beta}{r^\beta} \right\} \leq c^{-1} \left\{ \left( \frac{d(A_{s_0}(w), \Lambda)}{r} \right)^\beta - \frac{s_0^\beta}{r^\beta} \right\} \leq \frac{\tilde{v}(A_{s_0}(w))}{\tilde{v}(A_r(w))} \leq \tilde{c} v_r(A_{s_0}(w)) \]
whenever \( s_0 < \delta, r \) and \( c = c(n, p) \), independent of \( p \) when \( p \) is large. This proves the lower bound of \( v_r \) and hence the proof of Theorem 4.1 is complete.

\[ \square \]

We continue this section by using the estimates for \( p \)-harmonic measure, given in Theorem 4.1, to prove a result of Phragmen-Lindelöf type. Before stating the theorem, let us recall the classical result of Phragmén-Lindelöf [51]: If \( u(z) \), \( z = x + iy \), is subharmonic in the upper half plane \( \text{Im}(z) > 0 \), and if \( \limsup u(z) \leq 0 \) as \( z \) approaches any point on the real axis, then, either \( u \leq 0 \) in the whole upper plane or \( u \) grows so fast that

\[ \liminf_{R \to \infty} \frac{\sup_{|z|=R} u(z)}{R} > 0. \]

In the below Corollary, we expand this theorem to \( p \)-subharmonic functions, \( p \in (n - m, \infty] \), in domains in \( \mathbb{R}^n \) lying outside an \( m \)-dimensional hyperplane. We note that the borderline case \( p = n \) was considered by Lindqvist [42, Theorem 4.8 and Remark 4.9], where he used the explicit formula (stated above Theorem 4.1) for \( n \)-harmonic measure.

To formulate and prove the result we use the notation

\[ M(R) = \sup_{\partial B(w, R) \cap \Omega} u(x). \]

**Corollary 4.3** Suppose that \( m, n \in \mathbb{N} \) such that \( m \in [0, n-1] \), let \( \Lambda \subset \mathbb{R}^n \) be an \( m \)-dimensional hyperplane, \( w \in \Lambda \), \( p \in (n - m, \infty] \) and suppose that \( \beta = (p - n + m)/(p - 1) \) with \( \beta = 1 \) if \( p = \infty \). Let \( \Omega \) be an unbounded domain so that \( \Lambda \cap \Omega = \emptyset \). Suppose that \( u \) is \( p \)-subharmonic in \( \Omega \) and that

\[ \limsup_{x \to y} u(x) \leq 0 \quad \text{for all} \quad y \in \partial \Omega. \]

Then, either \( u \leq 0 \) in \( \Omega \) or

\[ \liminf_{r \to \infty} \frac{M(R)}{R^\beta} > 0. \]

**Remark 4.4** Note that if \( \Omega = \mathbb{R}^n \setminus \Lambda_s \) in Theorem 4.3, for some \( s \geq 0 \), then the \( p \)-harmonic function

\[ d(x, \Lambda)^\beta - s^\beta \]

shows that the growth condition in Corollary 4.3 is sharp.

**Proof of Corollary 4.3.** The following argument is standard, see e.g., Lindqvist [12, Principle 4.3] or Heinonen–Kilpelinen–Martio [25, Section 11.11]. Assume that \( u(x_0) > 0 \) for some \( x_0 \in \Omega \). By the maximum principle for \( p \)-subharmonic functions we obtain

\[ M(R) = \sup_{\partial B(w, R) \cap \Omega} u(x) = \sup_{B(w, R) \cap \Omega} u(x). \]
Let $v_R$ be the $p$-harmonic measure for $\partial B(w, R) \setminus \Lambda$, taken with respect to $B(w, R) \setminus \Lambda$. Existence follows by Heinonen–Kilpelinen–Martio [25, Theorem 9.2]. Then

$$\lim \sup_{x \to z} u(x) \leq M(R) v_R(z) \quad \text{for all} \quad z \in \partial (B(w, R) \cap \Omega),$$

and the comparison principle implies that $u \leq M(R)v_R$ in $B(w, R) \cap \Omega$. Using Theorem 4.1 and Remark 4.2 we have, for any $x \in \Omega$, the existence of a constant $C$ such that $u(x) \leq CM(r)R^{-\beta}$ whenever $R$ is so large that $x \in B(w, R)$. Therefore

$$0 < u(x_0) \leq C \frac{M(R)}{R^\beta}$$

which proves the result. \hfill \box

We finally state and prove, using a similar approach as in the proofs of Theorems 4.1 and 4.3, the following growth estimates for $p$-harmonic functions in unbounded domains:

**Theorem 4.5** Suppose that $m, n \in \mathbb{N}$ such that $m \in [0, n-1]$, let $\Lambda \subset \mathbb{R}^n$ be an $m$-dimensional hyperplane, $w \in \Lambda$, $p \in (n-m, \infty]$ and suppose that $\beta = (p-n+m)/(p-1)$ with $\beta = 1$ if $p = \infty$. Assume that for some $s_0$, $0 \leq 2s < s_0$, $\Omega \subset \mathbb{R}^n$ is an unbounded domain so that $\partial \Omega \subseteq \Lambda_s$ and $\Lambda \subseteq \mathcal{C}\Omega$. Suppose that $u$ is a positive $p$-harmonic function in $\Omega$, satisfying $u = 0$ continuously on $\partial \Omega$. Then there exists a constant $c(p,n)$, independent of $p$ when $p$ is large, such that

$$c^{-1} s_0^{-\beta} d(x, \Lambda)^\beta \leq \frac{u(x)}{u(A_{s_0}(w))} \leq c s_0^{-\beta} d(x, \Lambda)^\beta$$

whenever $x \in \mathbb{R}^n \setminus \Lambda_{2s}$.

Theorem 4.5 generalizes parts of Kilpelainen–Shahgholian–Zhong [36] to more general geometries. In particular, in [36, Lemma 3.2] it is proved that if $u$ is a non-negative $p$-harmonic function on $\mathbb{R}^n \setminus \Lambda$, where $\Lambda$ is an $(n-1)$-dimensional hyperplane, with $u = 0$ continuously on $\Lambda$, then $u(x) = O(|x|)$ as $|x| \to \infty$. Theorem 4.5 above yields $u(x) \approx d(x, \Lambda)^\beta$ whenever $x \in \mathbb{R}^n$, and $\Lambda$ is an $m$-dimensional hyperplane, $m \in [0, n-1]$. We also remark that this result was stated already in [44], however, the proof was not complete due to a typo (the factor $r^\beta$ is missing) in [44, Theorem 1.1].

**Proof of Theorem 4.5**. We begin with the lower bound. Proceeding as in the proof of Theorem 4.3 (from beginning to (4.1)) we obtain, in place of (4.1),

$$u(A_{s_0}(w)) \leq c s_0^{-\beta} \sup_{B(w, 4r) \cap \Omega} u$$

whenever $s_0/\delta_c < r$. Let $\tilde{c}$ be the constant from Carlesons estimate and define $\tilde{u}$ as the $p$-harmonic function in $B(w, 4\tilde{c}r) \setminus \Lambda$, satisfying the boundary values $u$ on $\partial B(w, 4\tilde{c}r) \cap \Omega$ and 0 on $(\partial B(w, 4\tilde{c}r) \setminus \Omega) \cup \Lambda$ continuously. Then $u \leq \tilde{u}$ in $B(0, 4\tilde{c}r)$ by the comparison principle and $\tilde{u}$ satisfies the assumptions of the Carleson estimate. Applying the Carleson estimate to
Using Harnack’s inequality and a well known Hölder continuity we obtain
\[
\sup_{B(w, 4r) \cap \Omega} u \leq \sup_{B(w, 4r) \cap \Omega} \tilde{u} \leq c \tilde{u}(A_{4r}(w)) \leq cu(A_r(w)).
\]

By (4.2) and (4.3) we conclude that
\[
u(A_{s_0}(w)) \leq c_0 \beta \frac{u(A_r(w))}{r^\beta}
\]
whenever \(s_0 / \delta_c < r\), for \(c = c(p, n)\), independent of \(p\) if \(p\) is large. Let \(\tilde{v}\) be the \(p\)-harmonic function in \(B(w, 4r) \setminus \Lambda_s\) satisfying the boundary values \(u\) on \(\partial B(w, 4r) \setminus \Lambda_{2s}\) and \(0\) on \(B(w, 4r) \cap \partial \Lambda_s\) continuously. If \(m \geq 1\) then we also let \(\tilde{v}\) increase continuously from \(0\) to \(u\) on the set \(\partial B(w, 4r) \cap (\Lambda_{2s} \setminus \Lambda_s)\). By the comparison principle we obtain \(\tilde{v} \leq u \in B(w, 4r) \setminus \Lambda_s\). Moreover, using Harnack’s inequality and Hölder continuity we obtain \(u(A_r(w)) \approx \tilde{v}(A_r(w))\) for constants depending only on \(p\) and \(n\). Applying Theorem 3.5 to \(\tilde{v}\) with \(\delta = s / r\) gives \(c = c(n, p)\) so that
\[
c^{-1} \left\{ \frac{\beta}{} \right\} \leq \frac{\tilde{v}(x)}{u(A_r(w))} \leq \frac{\beta}{\beta} \frac{u(x)}{u(A_r(w))},
\]
whenever \(2s / \delta_c < r\) and \(x \in B(w, \delta_c r) \setminus \Lambda_s\). Using (4.4) and (4.5) we obtain
\[
\frac{1}{c_0} \beta \frac{d(x, \Lambda)}{r} \leq \frac{1}{c_0} \beta \{d(x, \Lambda) - s^\beta\} \leq \frac{u(x)}{u(A_{s_0}(w))},
\]
whenever \(x \in B(w, \delta_c r) \setminus \Lambda_{2s}\) and \(c(n, p)\), independent of \(p\) when \(p\) is large. Sending \(r \to \infty\) gives the lower bound in Theorem 4.5.

To prove the upper bound, put \(x = A_{s_0}(w)\) in (4.5) to obtain
\[
c^{-1} \beta \frac{s^\beta}{r^\beta} \leq c^{-1} \left\{ \frac{s^\beta}{r^\beta} - \frac{s^\beta}{r^\beta} \right\} \leq \frac{\tilde{v}(A_{s_0}(w))}{u(A_r(w))} \leq \frac{u(A_{s_0}(w))}{u(A_r(w))},
\]
whenever \(s_0 < \delta_c r\). Let \(\hat{v}\) be the \(p\)-harmonic function in \(B(w, 5r) \setminus \Lambda\) satisfying boundary values \(sup_{B(w, 5r) \cap \Omega} u\) on \(\partial B(w, 5r)\) and \(0\) on \(B(w, 4r) \cap \Lambda\) continuously. If \(m \geq 1\) then we also let \(\hat{v}\) increase continuously from \(0\) to \(sup_{B(w, 5r) \cap \Omega} u\) on the set \(\Lambda \cap (B(w, 5r) \setminus B(w, 4r))\). By the comparison principle we obtain \(u \leq \hat{v} \in B(w, 5r) \cap \Omega\). Moreover, using similar reasoning as in (4.3) we obtain \(u(A_r(w)) \approx \hat{v}(A_r(w))\) for constants depending only on \(p\) and \(n\). Another application of Theorem 3.5 to \(\hat{v}\) with \(\delta = 0\), gives
\[
c^{-1} \frac{u(x)}{u(A_r(w))} \leq \tilde{v}(x) \leq \frac{d(x, \Lambda)^\beta}{r^\beta},
\]
whenever \(x \in B(w, \delta_c r) \cap \Omega\) and \(2s / \delta_c < r\). Using (4.6) and (4.7) we obtain
\[
\frac{u(x)}{u(A_{s_0}(w))} \leq c_0 \beta \frac{d(x, \Lambda)^\beta}{r^\beta},
\]
whenever \(x \in B(w, \delta_c r) \cap \Omega\) and \(c = c(n, p)\), independent of \(p\) when \(p\) is large. Sending \(r \to \infty\) gives the upper bound and hence the proof of Theorem 4.5 is complete. \(\square\)
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