Sum rules of polarized photon structure functions revisited — NNLO corrections to the first moment of $g_1^\gamma(x, Q^2, P^2)$ —

T. Ueda\textsuperscript{a}, T. Uematsu\textsuperscript{b} and K. Sasaki\textsuperscript{a}

\textsuperscript{a}Department of Physics, Faculty of Engineering, Yokohama National University, Yokohama 240-8501, Japan

\textsuperscript{b}Department of Physics, Faculty of Science, Kyoto University, Kyoto 606-8501, Japan

We present the next-to-next-to-leading order ($\alpha_s^2$) corrections to the first moment of the polarized virtual photon structure function $g_1^\gamma(x, Q^2, P^2)$ in the kinematical region $Q^2 \gg P^2 \gg \Lambda^2$ in QCD. We find that the $\alpha_s^2$ corrections are about 3\% of the sum of the leading ($\alpha$) and the next-to-leading ($\alpha_s$) contributions, when $Q^2 = 30 \sim 100\text{GeV}^2$ and $P^2 = 3\text{GeV}^2$.

1. Introduction

The investigation of the photon structure has been an active field of research both theoretically and experimentally \cite{1,2,3,4}. Also there has been growing interest in the study of the spin structure of photon. In particular, the first moment of the polarized photon structure function $g_1^\gamma(x, Q^2)$ has attracted attention in connection with its relevance to the QED and QCD axial anomaly \cite{5,6,7,8,9}. The polarized photon structure functions can be measured from two-photon processes in the polarized $e^+e^-$ collider experiments as shown in Fig. 1, where $-Q^2(-P^2)$ is the mass squared of the probe (target) photon.

For a real photon ($P^2 = 0$) target, there exists only one spin-dependent structure function $g_1^\gamma(x, Q^2)$. The QCD analysis for $g_1^\gamma$ was performed in the leading order (LO) \cite{10} and in the next-to-leading order (NLO) \cite{11,12}. In the case of a virtual photon target ($P^2 \neq 0$) there appear two spin-dependent structure functions, $g_1^\gamma(x, Q^2, P^2)$ and $g_2^\gamma(x, Q^2, P^2)$. The former has been investigated up to the NLO in QCD by the present authors in \cite{13,14}, and also in the second paper of \cite{12}. In Ref. \cite{15} the structure function $g_1^\gamma(x, Q^2, P^2)$ was analysed in the kinematical region

$$\Lambda^2 \ll P^2 \ll Q^2 ,$$

where $\Lambda$ is the QCD scale parameter. The advantage to study the virtual photon target in this kinematical region is that we can calculate structure functions by the perturbative method without any experimental data input \cite{16}, which is contrasted with the case of the real photon target where in the NLO there exist nonperturbative pieces.

In this talk we focus on the photon structure function $g_1^\gamma$ and, in particular, on the sum rule

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure1.png}
\caption{Deep inelastic scattering on a virtual photon in $e^+ e^-$ collision.}
\end{figure}
and present the result of the NNLO ($\alpha_s^2$) corrections to the first moment of $g_1^\gamma(x, Q^2, P^2)$ in the case of the kinematical region [1].

2. The sum rule of $g_1^\gamma$

The polarized structure function $g_1^\gamma$ of the real photon satisfies a remarkable sum rule [9,11,13,14,15,16]

$$
\int_0^1 dx \ g_1^\gamma(x, Q^2) = 0 .
$$

(2)

In fact, the authors of Ref. [9] showed that the sum rule [9], holds to all orders in perturbation theory in both QED and QCD.

When the target photon becomes off-shell, i.e., $P^2 \neq 0$, the first moment of the corresponding photon structure function $g_1^\gamma(x, Q^2, P^2)$ does not vanish any more. Indeed, for the case $\Lambda^2 \ll P^2 \ll Q^2$, the first moment has been calculated up to the NLO ($O(\alpha_s)$) in QCD as follows [7,13]:

$$
\int_0^1 dx g_1^\gamma(x, Q^2, P^2) \equiv \int_0^1 dx g_1^\gamma(x, Q^2) = -\frac{3\alpha}{\pi} \sum_{i=1}^{n_f} e_i^4 \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) - \frac{2}{\beta_0} \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \left( \frac{\alpha_s(P^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right) ,
$$

(3)

with $\beta_0 = 11 - 2n_f/3$ being the one-loop QCD $\beta$ function. Here $\alpha = e^2/4\pi$ and $\alpha_s(Q^2) = \bar{g}^2(Q^2)/4\pi$ are the QED coupling constant and the QCD running coupling constant, respectively, and $e_i$ is the electromagnetic charge of the active quark with flavour $i$ in the unit of proton charge and $n_f$ is the number of active quarks. Note that the r.h.s. of (3) does not involve any experimental data input. This is because the nonperturbative pieces, which take part in the real photon target case, can be neglected in the kinematical region [1] and the whole twist-two contributions to $g_1^\gamma(x, Q^2, P^2)$ can be computed by the perturbative method. It is also noted that the first term in the square brackets of the r.h.s. of (3) results from the QCD triangle anomaly while the second term comes from the QCD triangle anomaly.

3. The NNLO ($\alpha_s^2$) corrections

Now we calculate the NNLO ($\alpha_s^2$) corrections to the first moment of $g_1^\gamma(x, Q^2, P^2)$. First recall that for the operator product expansion (OPE) of two electromagnetic (and thus gauge-invariant) currents, only gauge-invariant operators need be included with their renormalization basis [10]. Since there is no gauge-invariant twist-two gluon and photon operators with spin one, we need to consider only quark operators, i.e., the flavour singlet $R_S$ and nonsinglet $R_{NS}$ axial currents, as follows:

$$
R_S^2 = \bar{\psi} \gamma^\gamma \gamma_5 1 \psi, \quad R_{NS}^2 = -\bar{\psi} \gamma^\gamma \gamma_5 (Q_{ch}^2 - (e^2) 1) \psi ,
$$

(4)

where 1 is an $n_f \times n_f$ unit matrix and $Q_{ch}^2$ is the square of the $n_f \times n_f$ quark-charge matrix.

Then the first moment of $g_1^\gamma(x, Q^2, P^2)$ is expressed as

$$
\int_0^1 dx g_1^\gamma(x, Q^2, P^2) \equiv \int_0^1 dx g_1^\gamma(x, Q^2) = C_S(Q^2/\mu^2, \bar{g}(\mu^2), \alpha) \langle \gamma(p) | R_S(\mu^2) | \gamma(p) \rangle + C_{NS}(Q^2/\mu^2, \bar{g}(\mu^2), \alpha) \langle \gamma(p) | R_{NS}(\mu^2) | \gamma(p) \rangle .
$$

Here $C_S$ and $C_{NS}$ are the coefficient functions corresponding to the currents $R_S^2$ and $R_{NS}^2$, respectively, and $\langle \gamma(p) | R_i(\mu^2) | \gamma(p) \rangle$ with $i = S, NS$ are the photon matrix elements of these quark axial currents. We choose the renormalization point $\mu$ at $\mu^2 = P^2$.

The coefficient functions are given by

$$
C_i(Q^2/\mu^2, \bar{g}(\mu^2), \alpha) = \exp \left[ \int_{\bar{g}(Q^2)}^{\bar{g}(P^2)} dg' \frac{\gamma_i(g')}{\beta_i(g')} \right] C_i(1, \bar{g}(Q^2)),
$$

(6)

where $i = S, NS$ and $\gamma_i(g)$ is the anomalous dimension of the quark axial current $R_i^2$ and $\beta_i(g)$ is the QCD $\beta$-function. We expand $\gamma_i(g)$ in powers of $g$ as

$$
\gamma_i(g) = \gamma_i^{(0)} g^2/16\pi^2 + \gamma_i^{(1)} \left( g^2/16\pi^2 \right)^2 + \gamma_i^{(2)} \left( g^2/16\pi^2 \right)^3 + O(g^5) .
$$

(7)

Since the nonsinglet quark axial current $R_{NS}^2$ is conserved in the massless limit, it undergoes no renormalization, and thus we have

$$
\gamma_{NS}^{(0)} = \gamma_{NS}^{(1)} = \gamma_{NS}^{(2)} = \cdots = 0 .
$$

(8)
On the other hand, the singlet axial current \( R_S^g \) has a non-zero anomalous dimension \( \gamma_S(g) \) due to the axial anomaly. At the one-loop level, we know \( \gamma_S^{(0)} = 0 \). The two-loop \( [18] \) and three-loop \( [19] \) results are:

\[
\begin{align*}
\gamma_S^{(1)} &= 12 C_F n_f, \\
\gamma_S^{(2)} &= \left( \frac{284}{3} C_F C_A - 36 C_F^2 \right) n_f - \frac{8}{3} C_F n_f^2,
\end{align*}
\]

with \( C_F = \frac{4}{3} \) and \( C_A = 3 \). The result \( \gamma_S^{(2)} \) was obtained in the \( \overline{\text{MS}} \) scheme with a definition of the \( \gamma_5 \)-matrix as

\[
\gamma_\mu \gamma_5 = \frac{i}{3!} e_{\mu\nu\sigma\tau} \gamma^\nu \gamma^\sigma \gamma^\tau,
\]

being used.

The \( \beta \) function has been calculated up to the four-loop level in the \( \overline{\text{MS}} \) scheme \( [20,21] \). For numerical analysis, we will use later the QCD running coupling constant \( \alpha_s(Q^2) \) where the results up to the three-loop level are taken care of \( [22] \).

But for the present we only need the \( \beta \) function up to the two-loop level:

\[
\mu \frac{\partial g}{\partial \mu} = \beta(g)
= -\beta_0 \frac{g^3}{16 \pi^2} - \beta_1 \frac{g^5}{(16 \pi^2)^2} + \cdots,
\]

with the \( SU_C(3) \) value

\[
\begin{align*}
\beta_0 &= 11 - \frac{2}{3} n_f, \\
\beta_1 &= 102 - \frac{38}{3} n_f.
\end{align*}
\]

Using Eqs. (7) and (12) we obtain

\[
\exp \left[ \int \frac{g^2}{16 \pi^2} \frac{\gamma_S(g')}{\beta(g')} \right] = 1 + \frac{\gamma_S^{(1)}}{8 \beta_0} \left( \frac{\alpha_s(Q^2)}{\pi} - \frac{\alpha_s(P^2)}{\pi} \right) + \frac{1}{64 \beta_0} \left( \gamma_S^{(2)} - \gamma_S^{(1)} \beta_1 \right) \beta_0
\]

\[
\times \left[ \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 - \left( \frac{\alpha_s(P^2)}{\pi} \right)^2 \right]
\]

\[
+ \frac{1}{128} \left( \gamma_S^{(1)} \right)^2 \left( \frac{\alpha_s(Q^2)}{\pi} - \frac{\alpha_s(P^2)}{\pi} \right)^2 + O(\alpha_s^3).
\]

We already have results of the singlet coefficient function \( C_S(1, g(Q^2), \alpha) \) calculated up to the two-loop level \( [23,24] \) and the nonsinglet coefficient function \( C_{NS}(1, g(Q^2), \alpha) \) up to the three-loop level \( [25] \). Both were calculated in the \( \overline{\text{MS}} \) scheme and with the definition of the \( \gamma_5 \)-matrix in \( [11] \). Taking up to the two-loop level, we have

\[
C_S(1, g(Q^2), \alpha)/\langle e^2 \rangle = 1 - \frac{3}{4} C_F \frac{\alpha_s(Q^2)}{\pi}
\]

\[
+ C_F \left[ \frac{21}{32} C_F - \frac{23}{16} C_A + \frac{1}{2} \zeta_3 + \frac{13}{48} n_f \right]
\times \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2,
\]

\[
C_{NS}(1, g(Q^2), \alpha) = 1 - \frac{3}{4} C_F \frac{\alpha_s(Q^2)}{\pi}
\]

\[
+ C_F \left[ \frac{21}{32} C_F - \frac{23}{16} C_A + \frac{1}{4} n_f \right]\left( \frac{\alpha_s(Q^2)}{\pi} \right)^2.
\]

where \( \zeta_3 \) is the Riemann zeta-function \( (\zeta_3 = 1.202056903 \cdots) \).

For \(-p^2 = P^2 \gg A^2\), the photon matrix elements of the quark axial currents can be calculated perturbatively and they are expressed in the form as

\[
\langle \gamma(p)|R_i(\mu^2 = P^2)|\gamma(p)\rangle = \frac{\alpha}{4\pi} A_i,
\]

with \( i = S, NS \), and

\[
A_i = A_i^{(0)} + \frac{\bar{g}^2(P^2)}{16 \pi^2} A_i^{(1)} + \left( \frac{\bar{g}^4(P^2)}{16 \pi^2} \right) A_i^{(2)} + \cdots\). (18)

The leading terms \( A_i^{(0)} \) are connected with Adler-Bell-Jackiw anomaly and are given by \( [26] \)

\[
A_i^{(0)} = -12 n_f \langle e^2 \rangle, \quad A^{(0)}_S = -12 n_f \langle e^4 \rangle - \langle e^2 \rangle^2. \quad (19)
\]

In our talk given at RADCOR05, we reported that for the next-leading and next-next-leading terms \( A_i^{(1)} \) and \( A_i^{(2)} \), we have \( A_S^{(1)} = A_N^{(1)} = A_{NS}^{(1)} = A_S^{(2)} = A_N^{(2)} = A_{NS}^{(2)} = 0 \) due to the nonrenormalization theorem \( [17] \) for the triangle anomaly. It is true that we obtain

\[
A_S^{(1)} = A_N^{(1)} = A_{NS}^{(1)} = 0,
\]

but we found after the workshop that \( A_S^{(2)} \) has a non-vanishing value. For the calculation of
\[ A_S^{(2)} \text{, we need to evaluate the three-loop Feynman graphs. Instead we resort to the Adler-Bardeen theorem for the axial current } J_5^\alpha (= R_S^{(2)} \text{ in } [1]), \]

\[ \partial_\mu J_5^\mu = \frac{g^2}{16\pi^2} n_f G_{\mu\nu} \tilde{G}^{\mu\nu}, \tag{22} \]

which holds in all orders in \( \alpha_s \), and we evaluate, in the \( \overline{MS} \) scheme, the matrix element

\[ \epsilon_\lambda \epsilon^{\alpha\beta\gamma} \langle 0 | T[A_\mu A_\nu(-p) A_\rho(p)]|0 \rangle \text{amputated}, \tag{23} \]

in the two-loop level, where \( A_\mu \) and \( A_\rho \) are gluon and photon fields, respectively. Details will be reported elsewhere [27]. We found

\[ A_S^{(2)} = 24 n_f (e^2) C_F \frac{n_f}{2} \left( \frac{53}{3} - 8 \zeta_3 \right). \tag{24} \]

Putting all these equations into Eq. (5) with \( \mu^2 = P^2 \), we finally obtain the NNLO \((\alpha \alpha_s^2)\) corrections to the first moment of \( g_1^e(x, Q^2, P^2) \):

\[ \int_0^1 dx g_1^e(x, Q^2, P^2)/(-3 \alpha) \]

\[ = \sum_{i=1}^{n_f} e_i^2 \left[ 1 - \frac{\alpha_s(Q^2)}{\pi} \right] \]

\[ - \frac{2}{\beta_0} \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \left[ \frac{\alpha_s(P^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right] \]

\[ + \frac{2}{\beta_0} \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \frac{\alpha_s(Q^2)}{\pi} \left[ \frac{\alpha_s(P^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right] \]

\[ + \frac{1}{4\beta_0} \left( \frac{\beta_1}{\beta_0} - \frac{59}{3} + \frac{2}{9} n_f \right) \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \]

\[ \times \left[ \frac{\alpha_s^2(P^2)}{\pi^2} - \frac{\alpha_s^2(Q^2)}{\pi^2} \right] \]

\[ + \frac{2n_f}{\beta_0} \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \left[ \frac{\alpha_s(P^2)}{\pi} - \frac{\alpha_s(Q^2)}{\pi} \right] \]

\[ - \left( \frac{55}{12} - \frac{1}{3} n_f \right) \sum_{i=1}^{n_f} e_i^2 \frac{\alpha_s^2(Q^2)}{\pi^2} \]

\[ + \left( \frac{2}{3} \zeta_3 + \frac{1}{36} \right) \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \frac{\alpha_s^2(Q^2)}{\pi^2} \]

\[ - \frac{1}{12} \left( \frac{53}{3} - 8 \zeta_3 \right) \left( \sum_{i=1}^{n_f} e_i^2 \right)^2 \frac{\alpha_s^2(P^2)}{\pi^2} \], \tag{25} \]

where the third to 8th terms are the NNLO contributions. In the case of \( n_f = 4 \), for an example, we have

\[ \int_0^1 dx g_1^e(x, Q^2, P^2) \]

\[ = -\frac{3n_f}{\pi} \left\{ \frac{0.4198 - 0.1235 \alpha_s(Q^2)}{\pi} - 0.2963 \frac{\alpha_s(P^2)}{\pi} \right. \]

\[ - 0.02731 \frac{\alpha_s(Q^2)}{\pi} \right. \left. \right) \]

\[ + \left. 0.01183 \frac{\alpha_s(Q^2)}{\pi} \frac{\alpha_s(P^2)}{\pi} \right. \]

\[ - 1.153 \left( \frac{\alpha_s(P^2)}{\pi} \right)^2 \right\} \]. \tag{26} \]

To estimate the sizes of the NLO \((\alpha \alpha_s)\) and NNLO \((\alpha \alpha_s^2)\) corrections compared to the LO \((\alpha)\) term, we take, for instance, \( Q^2 = 30 \) and \( 100 \text{ GeV}^2 \), and \( P^2 = 3 \text{ GeV}^2 \). The corresponding values of \( \alpha_s \) are obtained from Ref. [22]. We get \( \alpha_s(Q^2 = 30 \text{ GeV}^2) = 0.2048, \alpha_s(Q^2 = 100 \text{ GeV}^2) = 0.1762, \) and \( \alpha_s(P^2 = 3 \text{ GeV}^2) = 0.3211 \). The results are given in Table I.

4. Summary

We have investigated the next-to-next-to-leading order \((\alpha \alpha_s^2)\) corrections to the first moment of the polarized virtual photon structure function \( g_1^e(x, Q^2, P^2) \) in the kinematical region \( Q^2 \gg P^2 > A^2 \) in QCD. All the necessary information on the coefficient functions and anomalous dimensions corresponding to the quark axial currents has been already known, except for the three-loop-level photon matrix element (the finite term) of the flavour singlet quark axial current \( R_S^{(2)} \). Instead of evaluating the relevant three-loop Feynman diagrams, we resort to the Adler-Bardeen theorem for the axial current [22]. Then calculation reduces to the one in the two-loop level. We evaluate in effect the two-loop diagrams for the photon matrix element of the gluon operator [23].

The \( \alpha \alpha_s^2 \) corrections are found to be about 3% of the sum of the leading \((\alpha)\) and the next-to-leading \((\alpha \alpha_s)\) contributions, when \( Q^2 = 30 \sim 100 \text{ GeV}^2 \) and \( P^2 = 3 \text{ GeV}^2 \).
NNLO corrections to the first moment of $g_1(x, Q^2, P^2)$

Table 1
The NLO and NNLO contributions relative to LO in the first moment of $g_1(x, Q^2, P^2)$.

| $Q^2/\text{GeV}^2$ | $P^2/\text{GeV}^2$ | $n_f = 3$ | $n_f = 4$ | $n_f = 5$ | $n_f = 3$ | $n_f = 4$ | $n_f = 5$ | $n_f = 3$ | $n_f = 4$ | $n_f = 5$ |
|------------------|------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 30               | 3                | 1        | -0.0816  | -0.0766  | -0.0913  | -0.0986  | -0.0986  | -0.0977  | -0.0319  | -0.0343  |
| 100              | 3                | 1        | -0.0250  | -0.0234  | -0.0288  | -0.0309  | -0.0319  | -0.0315  | -0.0343  | -0.0354  |
| 30               | 3                | 1        | -0.0272  | -0.0254  | -0.0317  | -0.0309  | -0.0315  | -0.0343  | -0.0354  | -0.0354  |
| 100              | 3                | 1        | -0.0272  | -0.0254  | -0.0317  | -0.0309  | -0.0315  | -0.0343  | -0.0354  | -0.0354  |

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