Uncertainty product for Vilenkin groups

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Abstract

We study a localization of functions defined on Vilenkin groups. To measure the localization we introduce two uncertainty products $UP_\lambda$ and $UP_G$ that are similar to the Heisenberg uncertainty product. $UP_\lambda$ and $UP_G$ differ from each other by the metric used for the Vilenkin group $G$. We discuss analogs of a quantitative uncertainty principle. Representations for $UP_\lambda$ and $UP_G$ in terms of Walsh and Haar basis are given.

Keywords Vilenkin group; uncertainty product; Haar wavelet; modified Gibbs derivative; generalized Walsh function.

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1 Introduction

An uncertainty product for a function characterizes how concentrated is the function in time and frequency domain. Initially the notion of uncertainty product was introduced for $f \in L^2(\mathbb{R})$ by W. Heisenberg [6] and E. Schrödinger [12]. Later on extensions of this notion appeared for various algebraic and topological structures. For periodic functions, it was suggested by E. Breitenberger [1]. For some particular cases of locally compact groups (namely a euclidean motion groups, non-compact semisimple Lie groups, Heisenberg groups) the counterpart was derived in [11]. Uncertainty products on compact Riemannian manifolds was discussed in [1]. In [5], this concept was introduced for functions defined on the Cantor group. In this paper, we discuss localization of functions defined on Vilenkin groups.

To measure the localization we introduce a functional that is similar to the Heisenberg uncertainty product (see Definition [1]). It depends on the metric used for the Vilenkin group $G$. Two equivalent metrics are in common use for the group $G$. So we discuss two uncertainty

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where \( p \). Given for brevity we set \( I_n \).

A unique number \( k > p \) coordinatewise addition modulo \( x \) is denoted by \( \oplus \) and defined as the coordinatewise addition modulo \( p : \)

\[
(z_j) = (x_j) \oplus (y_j) \iff z_j = x_j + y_j \pmod{p} \quad \text{for} \quad j \in \mathbb{Z}.
\]

The inverse operation of \( \oplus \) is denoted by \( \ominus \). The symbol \( \ominus x \) denotes the inverse element of \( x \in G \). The sequence \( 0 = (\ldots, 0, 0, \ldots) \) is a neutral element of \( G \). If \( x \neq 0 \), then there exists a unique number \( N = N(x) \) such that \( x_N \neq 0 \) and \( x_j = 0 \) for \( j < N \). The Vilenkin group \( G_p \), where \( p = 2 \) is called the Cantor group. In this case the inverse operation \( \ominus \) coincides with the group operation \( \oplus \).

Define a map \( \lambda : G \to [0, +\infty) \)

\[
\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j-1}, \quad x = (x_j) \in G.
\]

The mapping \( x \mapsto \lambda(x) \) is a bijection taking \( G \setminus \mathbb{Q}_0 \) onto \([0, \infty)\), where \( \mathbb{Q}_0 \) is a set of all elements terminating with \( p - 1 \)'s.

Two equivalent metrics are in common use for the group \( G \). One metric is defined by \( d_1(x, y) := \lambda(x \ominus y) \) for \( x, y \in G \). To define another one \( d_2 \) we consider a map \( \| \cdot \|_G : G \to [0, \infty) \), where \( \|0\|_G := 0 \) and \( \|x\|_G := p^{-N(x)} \) for \( x \neq 0 \). Then \( d_2(x, y) := \|x \ominus y\|_G, x, y \in G \). Given \( n \in \mathbb{Z} \) and \( x \in G \), denote by \( I_n(x) \) the ball of radius \( 2^{-n} \) with the center at \( x \), i.e.

\[
I_n(x) = \{ y \in G : d(x, y) < 2^{-n} \}.
\]

For brevity we set \( I_j := I_j(0) \) and \( I := I_0 \).

We denote dilation on \( G \) by \( D : G \to G \), and set \( (Dx)_k = x_{k+1} \) for \( x \in G \). Then \( D^{-1} : G \to G \) is the inverse mapping \( (D^{-1}x)_k = x_{k-1} \). Set \( D^k = D \circ \cdots \circ D \) (\( k \) times) if \( k > 0 \), and \( D^k = D^{-1} \circ \cdots \circ D^{-1} \) (\( k \) times) if \( k < 0 \); \( D^0 \) is the identity mapping.

2 Auxiliary results

We recall necessary facts about the Vilenkin group. More details can be found in [3, 13]. The Vilenkin group \( G = G_p, p \in \mathbb{N}, p \neq 1 \), is a set of the sequences

\[
x = (x_j) = (\ldots, 0, 0, x_{-k}, x_{-k+1}, x_{-k+2}, \ldots),
\]

where \( x_j \in \{0, \ldots, p - 1\} \) for \( j \in \mathbb{Z} \). The operation on \( G \) is denoted by \( \oplus \) and defined as the coordinatewise addition modulo \( p : \)

\[
\|x\|_G := \min \{ \lambda(x), \lambda(y) \}, \quad x, y \in G.
\]

One metric is defined by \( d_1(x, y) := \lambda(x \ominus y) \) for \( x, y \in G \). To define another one \( d_2 \) we consider a map \( \| \cdot \|_G : G \to [0, \infty) \), where \( \|0\|_G := 0 \) and \( \|x\|_G := p^{-N(x)} \) for \( x \neq 0 \). Then \( d_2(x, y) := \|x \ominus y\|_G, x, y \in G \). Given \( n \in \mathbb{Z} \) and \( x \in G \), denote by \( I_n(x) \) the ball of radius \( 2^{-n} \) with the center at \( x \), i.e.

\[
I_n(x) = \{ y \in G : d(x, y) < 2^{-n} \}.
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We denote dilation on \( G \) by \( D : G \to G \), and set \( (Dx)_k = x_{k+1} \) for \( x \in G \). Then \( D^{-1} : G \to G \) is the inverse mapping \( (D^{-1}x)_k = x_{k-1} \). Set \( D^k = D \circ \cdots \circ D \) (\( k \) times) if \( k > 0 \), and \( D^k = D^{-1} \circ \cdots \circ D^{-1} \) (\( k \) times) if \( k < 0 \); \( D^0 \) is the identity mapping.
We deal with functions taking $G$ to $\mathbb{C}$. Denote $1_E$ the characteristic function of a set $E \subset G$. Given a function $f : G \to \mathbb{C}$ and a number $h \geq 0$, for every $x \in G$ we define $f_{0,h}(x) = f(x \oplus \lambda^{-1}(h))$. Finally, we set for $j \in \mathbb{Z}$

$$f_{j,h}(x) = p^{j/2} f_{0,h}(D^j x), \quad x \in G.$$ 

The functional spaces $L_q(G)$ and $L_q(E)$, where $E$ is a measurable subset of $G$, are derived using the Haar measure (see [7]).

Given $\xi \in G$, a group character of $G$ is defined by

$$\chi_{\xi}(x) = \chi(x, \xi) := \exp \left( \frac{2\pi i}{p} \sum_{j \in \mathbb{Z}} x_j \xi_{-1-j} \right).$$

The functions $w_n(x) := \chi(\lambda^{-1}(n), x)$ are called the generalized Walsh functions. If $p = 2$, then $w_n$ are called the Walsh functions.

The Fourier transform of a function $f \in L^1(G)$ is defined by

$$Ff(\omega) = \int_G f(x) \overline{\chi(x, \omega)} d\mu(x), \quad \omega \in G. \quad (1)$$

The Fourier transform is extended to $L^2(G)$ in a standard way, and the Plancherel equality takes place

$$\langle f, g \rangle := \int_G f(x) \overline{g(x)} dx = \int_G Ff(\xi) \overline{Fg(\xi)} d\xi = \langle Ff, Fg \rangle, \quad f, g \in L^2(G).$$

The inversion formula is valid for any $f \in L^2(G)$

$$F^{-1} Ff(x) = \int_G Ff(\omega) \chi(x, \omega) d\mu(\omega) = f(x).$$

It is straightforward to see that

$$F(f_{j,n})(\xi) = p^{-j/2} \chi(k, D^{-j} \xi) Ff(D^{-j} \xi), \quad n \in \mathbb{Z}_+, j \in \mathbb{Z}. \quad (2)$$

The discrete Vilenkin-Chrestenson transform of a vector $x = (x_k)_{k=0}^{p^n-1} \in \mathbb{C}^{p^n}$ is a vector $y = (y_k)_{k=0}^{p^n-1} \in \mathbb{C}^{p^n}$, where

$$y_k = p^{-n} \sum_{s=0}^{p^n-1} x_s w_k(\lambda^{-1}(s/p^n)), \quad 0 \leq k \leq p^n - 1. \quad (3)$$

The inverse transform is

$$x_k = \sum_{s=0}^{p^n-1} y_s \overline{w_k(\lambda^{-1}(s/p^n))}, \quad 0 \leq k \leq p^n - 1. \quad (4)$$
Given \( f : G_2 \to \mathbb{C} \), the function

\[
f^{[1]}(x) := \lim_{n \to \infty} \sum_{j=-n}^{n} 2^{j-1}(f(x) - f_{0,2^{-j-1}}(x))
\]

is called the Gibbs derivative of a function \( f \). The following properties hold true

\[
Ff^{[1]}(\xi) = \lambda(\xi)Ff(\xi), \quad w_n^{[1]}(x) = nw_n(x).
\] (5)

Set \( \varphi = \mathbb{1}_I \). The Haar functions \( \psi^\nu, \nu = 1, \ldots, p-1 \) are defined by

\[
\psi^\nu(x) = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi i \nu n}{p}\right) \varphi(Dx \oplus \lambda^{-1}(n)).
\] (6)

The system \( \psi^\nu_{j,k}, \nu = 1, \ldots, p-1, j \in \mathbb{Z}, k \in \mathbb{Z}_+ \), forms an orthonormal basis (Haar basis) for \( L^2(G) \), see [5, 9].

It follows from (1) that \( F\varphi = \varphi = \mathbb{1}_I \) and \( F\psi = \mathbb{1}_{I_0 \oplus \lambda^{-1}(p-\nu)} \). Taking into account (2), we get

\[
F\psi^\nu_{j,k}(\xi) = p^{-j/2} \chi(k, D^{-j}\xi) \mathbb{1}_{I_j \oplus \lambda^{-1}(p-\nu)p^j}.
\] (7)

Given \( f \in L_1(G) \), the modified Gibbs derivative \( \mathcal{D} \) is defined by

\[
\mathcal{D}f = \| \cdot \|_G Ff.
\] (8)

It was introduced in [2] for \( L_1(G_2) \). Such kind of operators are often called pseudo-differential.

**Proposition 1.** Suppose \( g, Fg, \| \cdot \|_G Fg \) are locally integrable on \( G \), \( j \in \mathbb{Z} \). Then the assertion \( \text{supp} \hat{g} \subset I_{-j-1} \setminus I_{-j} \) is necessary and sufficient for \( g \) to be an eigenfunction of \( \mathcal{D} \) corresponding to the eigenvalue \( p^j \).

The proof can be rewritten from Proposition 1 [10], where it is proved for the Cantor group.

**Corollary 1.** Any Haar function \( \psi^\nu_{j,k} \) is an eigenfunction of \( \mathcal{D}^\alpha \) corresponding to the eigenvalue \( p^j \).

**Proof.** The statement follows from Proposition 1 and (7). \( \square \)

### 3 Uncertainty product and metrics

Originally, the concept of an uncertainty product was introduced for the real line case in 1927. The Heisenberg uncertainty product of \( f \in L_2(\mathbb{R}) \) is the functional \( UC_H(f) := \Delta_f \Delta_{\hat{f}} \) such that

\[
\Delta_f^2 := \| f \|_{L_1^2(\mathbb{R})}^2 \int_{\mathbb{R}} (x-x_f)^2 |f(x)|^2 \, dx, \quad \Delta_{\hat{f}}^2 := \| \hat{f} \|_{L_1^2(\mathbb{R})}^2 \int_{\mathbb{R}} (t-t_f)^2 |\hat{f}(t)|^2 \, dt,
\]

4
\( x_f := \|f\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} x |f(x)|^2 \, dx, \quad t_{\hat{f}} := \|\hat{f}\|_{L^2(\mathbb{R})}^{-2} \int_{\mathbb{R}} t |\hat{f}(t)|^2 \, dt, \)

where \( \hat{f} \) denotes the Fourier transform of \( f \in L_2(\mathbb{R}) \). It is well known that \( UC_H(f) \geq 1/2 \) for a function \( f \in L_2(\mathbb{R}) \) and the minimum is attained on the Gaussian. To motivate the definition of a localization characteristic for the Vilenkin group we note that on one hand \( x_f \) is the solution of the minimization problem

\[
\min_{\tilde{x}} \int_{\mathbb{R}} (x - \tilde{x})^2 |f(x)|^2 \, dx,
\]

and on another hand the sense of the sign “-” in the definition of \( \Delta_f \) is the distance between \( x \) and \( x_f \). So we come to the main definition.

**Definition 1.** Suppose \( f : G \to \mathbb{C}, f \in L_2(G) \), and \( d \) is a metric on \( G \), then a functional

\[
UP(f) := V(f)V(Ff), \quad \text{where}
\]

\[
V(f) := \frac{1}{\|f\|_{L^2(G)}^2} \min_{\tilde{x}} \int_G (d(x, \tilde{x}))^2 |f(x)|^2 \, dx
\]

is called the uncertainty product of a function \( f \) defined on the Vilenkin group.

Thus, we study two uncertainty products \( UP_\lambda \) and \( UP_G \) that corresponds to the metric \( d_1(x, y) := \lambda(x \ominus y) \) and \( d_2(x, y) := \|x \ominus y\|_G \). More precisely,

\[
UP_\lambda(f) := V_\lambda(f)V_\lambda(Ff), \quad \text{where}
\]

\[
V_\lambda(f) := \frac{1}{\|f\|_{L^2(G)}^2} \min_{\tilde{x}} \int_G (\lambda(x \ominus \tilde{x}))^2 |f(x)|^2 \, dx.
\]

The functional \( UP_G \) is defined as

\[
UP_G(f) := V_G(f)V_G(Ff), \quad \text{where}
\]

\[
V_G(f) := \frac{1}{\|f\|_{L^2(G)}^2} \min_{\tilde{x}} \int_G \|x \ominus \tilde{x}\|_G^2 |f(x)|^2 \, dx.
\]

The functional \( UP_\lambda \) for functions defined on the Cantor group was introduced and studied in [8]. The following results are extended from the Cantor group to the Vilenkin group without any essential changes. So we omit the proofs.

**Theorem 1.** Suppose \( f : G \to \mathbb{C}, f \in L_2(G) \). Then the following inequality holds true

\[
UP_\lambda(f) \geq C, \quad \text{where} \quad C \simeq 8.5 \times 10^{-5}.
\]

**Theorem 2.** Let \( f(x) = \mathbb{1}_{\lambda^{-1}[0,1]}(x) \sum_{k=0}^{\infty} a_k w_k(x) \) be a uniformly convergent series. Denote

\[
f_n(x) = \mathbb{1}_{\lambda^{-1}[0,1]}(x) \sum_{k=0}^{p^n-1} a_k w_k(x).
\]
Let \( V_\lambda(f) < +\infty \), \( V_\lambda(Ff) < +\infty \). Then \( UP_\lambda(f) = \lim_{n \to \infty} V_\lambda(f_n) V_\lambda(Ff_n) \), where

\[
V_\lambda(f_n) = \frac{\min_{k_0=0,p^n-1} \sum_{k=0}^{p^{n-1}} b_{k \lambda(k)} |b_{k \lambda^{-1}(k_0)}|^2 ((k+1)^3 - k^3)/3}{\sum_{k=0}^{p^{n-1}} |a_k|^2},
\]

\[
V_\lambda(Ff_n) = \frac{\min_{k_1=0,p^n-1} \sum_{k=0}^{p^{n-1}} |a_{k \lambda^{-1}(k_1)}|^2 ((k+1)^3 - k^3)/3}{\sum_{k=0}^{p^{n-1}} |a_k|^2},
\]

and \( b_k, 0 \leq k \leq p^n - 1 \), is the inverse discrete Vilenkin-Chrestenson transform \([4]\).

The following Lemma shows that the functionals \( U_P_\lambda \) and \( U_P_G \) have the same order.

**Lemma 1.** Suppose \( f \in L_2(G) \), then \( p^{-4} U_P_G(f) \leq U_P_\lambda(f) < U_P_G(f) \).

**Proof.** It is sufficient to note that \( p^{-1} \|x\|_G \leq \lambda(x) < \|x\|_G \). \( \square \)

Taking into account Theorem \([1]\) we conclude that \( U_P_G \) has a positive lower bound. So, \( U_P_G \) satisfies the uncertainty principle.

**Example 1.** Let us illustrate a definition of \( U_P_G \) for \( p = 2 \) using functions \( f_1, g_1, f_2, \) and \( g_2 \) taken from \([8]\) Example 1. Recall \( f_1(x) = 1_{\lambda^{-1}(0,1/4)}(x) \), \( g_1(x) = 1_{\lambda^{-1}(3/4,1)}(x) \), \( f_2(x) = 1_{\lambda^{-1}(0,3/8)}(x) \), and \( g_2(x) = 1_{\lambda^{-1}(3/4,9/8)}(x) \). Their Walsh-Fourier transforms are \( Ff_1 = 1_{\lambda^{-1}(0,4)/4} \), \( Fg_1 = w_3(\cdot/4) 1_{\lambda^{-1}(0,4)/4} \), \( Ff_2 = 1_{\lambda^{-1}(0,4)/4} + w_1(\cdot/4) 1_{\lambda^{-1}(0,8)/8} \), and \( Fg_2 = w_3(\cdot/4) 1_{\lambda^{-1}(0,4)/4} + w_1(\cdot) 1_{\lambda^{-1}(0,8)/8} \). Given \( \alpha \in [0, \infty) \), since the mapping \( \alpha \mapsto \|\lambda^{-1}(\alpha)\|_G \) is increasing and a measure of the set \( \lambda^{-1}(a, b) \subset \bar{x} \) does not depend on \( \bar{x} \), it follows that

\[
\min_x \int_{\lambda^{-1}(0,4)/4} \|x \ominus \bar{x}\|_G \, dx = \min_x \int_{\lambda^{-1}(0,4)/4} \|\tau\|_G \, d\tau = \int_{\lambda^{-1}(0,4)/4} \|\tau\|_G \, d\tau,
\]

and \( \lambda^{-1}(0, 1/4) \) is a set of minimizing \( \bar{x} \)'s as well. So, taking into account \( \|f_1\|_{L_2(G)} = \|Ff_1\|_{L_2(G)}^2 = 1/4 \), we get

\[
V_G(f_1) = \frac{1}{\|f_1\|_{L_2(G)}^2} \min_{\bar{x}} \int \|x \ominus \bar{x}\|_G^2 |f_1(x)|^2 \, dx = 4 \min_{\bar{x}} \int \|x \ominus \bar{x}\|_G^2 \, dx
\]

\[
= 4 \int_{\lambda^{-1}(0,4)/4} \|\tau\|_G^2 \, d\tau = 4 \sum_{i=2}^\infty \int_{\lambda^{-1}(2i^{-1}/8, 2i^{-1}/4)} \|\tau\|_G^2 \, d\tau = 4 \sum_{i=2}^\infty \left( \frac{1}{2i} - \frac{1}{2i+1} \right) 2^{-2i} = \frac{1}{28}.
\]

Analogously, we obtain

\[
V_G(Ff_1) = \frac{1}{\|Ff_1\|_{L_2(G)}^2} \min_{\bar{x}} \int \|x \ominus \bar{x}\|_G^2 |Ff_1(x)|^2 \, dx = \frac{1}{4} \min_{\bar{x}} \int \|x \ominus \bar{x}\|_G^2 \, dx
\]

\[
= \frac{1}{4} \int_{\lambda^{-1}(0,4)} \|\tau\|_G^2 \, d\tau = \frac{1}{4} \sum_{i=-2}^\infty \int_{\lambda^{-1}(2i^{-1}/8, 2i^{-1}/4)} \|\tau\|_G^2 \, d\tau = \frac{1}{4} \sum_{i=-2}^\infty \left( \frac{1}{2i} - \frac{1}{2i+1} \right) 2^{-2i} = \frac{64}{7}.
\]

Thus, \( U_P_G(f_1) = 16/49 \). Using the same arguments, we calculate \( U_P_G \) for the remaining functions. We collect all the information in Table \([1]\) Values of \( U_P_\lambda \) we extract from \([8]\).
Example 1. Columns named $\tilde{x}_0(f)$ and $\tilde{t}_0(f)$ contain sets of $\tilde{x}$ and $\tilde{t}$ minimizing the functionals $V_\lambda(f)$, $V_G(f)$ and $V_\lambda(F f)$, $V_G(F f)$ respectively. With respect both uncertainty products $UP_G$ and $UP_\lambda$, functions $f_1$ and $g_1$ have the same localization, while function $f_2$ is more localized then $g_2$, that is adjusted with a naive idea of localization as a characteristic of a measure for a function support.

| $f$ | $\tilde{x}_0(f)$ | $\tilde{t}_0(f)$ | $V_\lambda(f)$ | $V_\lambda(F f)$ | $V_G(f)$ | $V_G(F f)$ | $UP_G(f)$ |
|-----|----------------|----------------|----------------|----------------|---------|-----------|----------|
| $f_1$ | $[0, 1/4)$ | $[0, 4)$ | $1/48$ | $16/3$ | $1/9$ | $1/28$ | $64/7$ | $16/49$ |
| $g_1$ | $[3/4, 1)$ | $[0, 4)$ | $1/48$ | $16/3$ | $1/9$ | $1/28$ | $64/7$ | $16/49$ |
| $f_2$ | $[0, 1/8)$ | $[0, 2)$ | $3/64$ | $8$ | $3/8$ | $4/21$ | $96/7$ | $128/49$ |
| $g_2$ | $[3/4, 7/8)$ | $[0, 4)$ | $71/64$ | $32/3$ | $71/6$ | $19/14$ | $255/14$ | $4845/196$ |

Example 2. Here we discuss a dependence of a localization for a fixed function on a parameter $p$ of the Vilenkin group $G_p$. Let us consider a function $f_1(x) = \mathbb{1}_{\lambda^{-1}[0, 1/4)}(x)$ and $p = 2^k$, $k \in \mathbb{N}$. We calculate $UP_G(f_1)$.

1. If $k = 1$, then $UP_G(f_1) = \frac{16}{49}$ (see Example 1);
2. If $k = 2$, then

\[
V_G(f_1) = \frac{1}{\| f_1 \|_{L_2(G)}^2} \min_{\tilde{x}} \int_G \| x \oplus \tilde{x} \|_G^2 | f_1(x) |^2 dx = 4 \min_{\tilde{x}} \int_{\lambda^{-1}[0, 1/4)} \| x \oplus \tilde{x} \|_G^2 dx = 4 \sum_{i=1}^{\infty} \left( \frac{1}{4^i} - \frac{1}{4^{i+1}} \right) 4^{-2i} = \frac{1}{21}.
\]

\[
V_G(F f_1) = \frac{1}{\| F f_1 \|_{L_2(G)}^2} \min_{\tilde{x}} \int_G \| x \oplus \tilde{x} \|_G^2 | F f_1(x) |^2 dx = \frac{1}{4} \min_{\tilde{x}} \int_{\lambda^{-1}[0, 4)} \| x \oplus \tilde{x} \|_G^2 dx = \frac{1}{4} \sum_{i=1}^{\infty} \left( \frac{1}{4^i} - \frac{1}{4^{i+1}} \right) 4^{-2i} = \frac{256}{21}.
\]

Hence, $UP_G(f_1) = \frac{256}{441}$.
3. If $k > 2$, then

\[
V_G(f_1) = \frac{1}{\| f_1 \|_{L_2(G)}^2} \min_{\tilde{x}} \int_G \| x \oplus \tilde{x} \|_G^2 | f_1(x) |^2 dx = 4 \min_{\tilde{x}} \int_{\lambda^{-1}[0, 1/4]} \| x \oplus \tilde{x} \|_G^2 dx = 4 \left( \sum_{i=1}^{\infty} \left( \frac{1}{(2^k)^i} - \frac{1}{(2^k)^{i+1}} \right) (2^{k-2i}) + \left( \frac{1}{4} - \frac{1}{2^k} \right) \right) = 1 - \frac{4}{2^k} + \frac{4}{2^k(2^{2k} + 2^k + 1)}.
\]
\[ V_G(F f_1) = \frac{1}{\|F f_1\|_{L^2(G)}^2} \min_{\tilde{x}} \int_G \|x \otimes \tilde{x}\|_{G}^2 |f_1(x)|^2 dx = \frac{1}{4} \min_{\tilde{x}} \int_{\lambda^{-1}[0,1]} \|x \otimes \tilde{x}\|_{G}^2 dx \]

\[ = \frac{1}{4} \int_{\lambda^{-1}[0,1] \oplus [1,4]} \|\tau\|_{G}^2 d\tau = \frac{1}{4} \left( \sum_{i=0}^{\infty} \left( \frac{1}{(2^k)^i} - \frac{1}{(2^k)^{i+1}} \right) (2^k)^{2i} + (4 - 1) \cdot 2^{2k} \right) \]

\[ = \frac{3}{4} \cdot 2^{2k} + 1 \cdot \frac{2^{2k}}{2^{2k} + 2^k + 1}. \]

Therefore, \( U_P G(f_1) = \left( 1 - \frac{4}{2^k} + \frac{4}{2^k(2^k + 2^k + 1)} \right) \left( \frac{3}{4} \cdot 2^{2k} + 1 \cdot \frac{2^{2k}}{2^{2k} + 2^k + 1} \right). \)

It is easy to see that time variance \( V_G(f_1) \) goes to 1, and frequency variance \( V_G(F f_1) \) goes to infinity as \( k \to \infty \).

4 Uncertainty product \( U_P G \).

In this section we concentrate on the uncertainty product corresponding to the metric \( d_2 \). It turns out that the modified Gibbs derivative \( \mathcal{D} \) plays a role of a usual derivative in this case. And since the Haar functions are the eigenfunctions of \( \mathcal{D} \), it is possible to get representation for \( U_P G \) using the Haar coefficients.

**Theorem 3.** Suppose \( f \in L_2(G) \cap L_1(G) \), \( \| \cdot \|_G f \in L_2(G) \), where “dot” \( \cdot \) means the argument \( x \in G \) of a function \( f \), and \( f(x) = \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} c_{j,k}^\nu \psi_{j,k}^\nu(x) \). Then

\[ \int_G \|t\|_G |F f(t)|^2 dt = \int_G |\mathcal{D} f(t)|^2 dt = \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} |p^j c_{j,k}^\nu|^2 \]  

(9)

\[ \int_G \|x\|_G^2 |f(x)|^2 dx = \int_G |\mathcal{D} F f(x)|^2 dx = \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} |p^j d_{j,k}^\nu|^2 ; \]  

(10)

where \( d_{j,k}^\nu, j \in \mathbb{Z}, k \in \mathbb{Z}_+, \nu = 1, \ldots, p-1 \), are the coefficients in the Haar series for the function \( F f \), that is \( F f(t) = \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} d_{j,k}^\nu \psi_{j,k}^\nu(t) \).

**Proof.** By the definition of the modified Gibbs derivative and the Plancherel equality we get

\[ \int_G \|t\|_G^2 |F f(t)|^2 dt = \int_G |\mathcal{D} F f(t)|^2 dt = \int_G |\mathcal{D} f(t)|^2 dt. \]

Expanding a function in the Haar series and applying Corollary \( \square \) we get

\[ \int_G |\mathcal{D} f(t)|^2 dt = \int_G \left| \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_+} c_{j,k}^\nu \mathcal{D} \psi_{j,k}^\nu(t) \right|^2 dt \]
Vilenkin-Chrestenson transform of

\[
\delta
\]

obtain this formula in the following lemma.

\[
\square
\]

analogously to (9).

The last equality follows from the orthonormality of the Haar system. Equality (10) is proved analogously to (9).

\[
\text{Remark 1. Formally, it is possible to write } \int_G \lambda^2(x)|Ff(x)|^2\,dx = \int_G |f^{[1]}(x)|^2\,dx \text{ and to try to represent } UC_\lambda \text{ in terms of eigenfunctions of the Gibbs derivative } f^{[1]} \text{ in the case of the Cantor group. (The Gibbs derivative is defined for functions defined on the Cantor group only.) However, the Gibbs differentiation is not a local operation, that is } (f1_E)^{[1]} \neq f^{[1]}1_E, \text{ see also discussion in [10]. So, usage of Walsh functions instead of Haar basis might give interesting results for periodic functions only.}
\]

We did not found in the literature a formula expressing \(d_{j,k}^\mu\) in terms of \(c_{j,k}^\nu\). So we obtain this formula in the following lemma.

\[
\text{Lemma 2. Suppose } f \in L_2(G) \text{ and the coefficients } c_{j,k}^\nu, d_{j,k}^\mu, j \in \mathbb{Z}, k \in \mathbb{Z}_+, \nu, \mu = 1, \ldots, p-1, \text{ are defined in Theorem 3. Then}
\]

\[
d_{j,k}^\mu = \sum_{\nu=1}^{p-1} p^{\mu/2} b_{\nu}^j + p^{j/2} \sum_{\nu=1}^{p-1} c_{j-1,0}^\nu \exp \left( -\frac{2\pi i \nu \mu}{p} \right) \delta_{k,0} + p^{j/2} \sum_{i=-\infty}^{-2} \sum_{\nu=1}^{p-1} c_{i,0}^\nu \delta_{k,0},
\]

(11)

where \(b_{\nu}^j = p^{-\nu_0} \sum_{n=0}^{p^{\nu_0}-1} c_{\nu_0-j,n+(\mu-\nu)p^{\nu_0}+1}^{\nu} \chi(\lambda^{-1}(n), D^{-\nu_0} \lambda^{-1}(k))\) is the \(k\)-th term of the discrete Vilenkin-Chrestenson transform of \((c_{\nu_0-j,n+(\mu-\nu)p^{\nu_0}+1}^{\nu})_{n=0}^{p^{\nu_0}-1}\), \(q_0 = \left[ \log_p \frac{k}{p-\nu} \right] \), and \(\delta_{k,0} = 1\), and \(\delta_{k,0} = 0\) if \(k \neq 0\).

\[
\text{Proof. Using the Plancherel equality and (7), we get}
\]

\[
d_{j,k}^\mu = \int_G Ff(x)\overline{\psi_{j,k}(x)}\,dx = \int_G f(x)\overline{F\psi_{j,k}(x)}\,dx = \sum_{\nu=1}^{p-1} \sum_{i \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_+} c_{i,n}^\nu \int_G \psi_{i,n}(x)\overline{F\psi_{j,k}(x)}\,dx
\]

\[
= \sum_{\nu=1}^{p-1} \sum_{i \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_+} c_{i,n}^\nu \int_G \psi_{i,n}(x)p^{-j/2} \chi(\lambda^{-1}(k), D^{-j}x)\chi_{I_{j}}(\lambda^{-1}(n), D^{-j}x)\,dx.
\]

Since \(\text{supp } \psi_{i,n} = \lambda^{-1}((n+1)p^{-1} - (n+1)p^{-1}))\), it follows that the last expression takes the form

\[
\sum_{\nu=1}^{p-1} \sum_{i=-j}^{\infty} \sum_{n=(\mu-\nu)p^{i+j}}^{(\mu+1)p^{i+j}-1} c_{i,n}^\nu \int_G \psi_{i,n}(x)p^{-j/2} \chi(\lambda^{-1}(k), D^{-j}x)\,dx
\]

\[
+ p^{-j/2} \sum_{\nu=1}^{p-1} c_{j-1,0}^\nu \exp \left( -\frac{2\pi i \nu \mu}{p} \right) + \sum_{i=-\infty}^{p-2} \sum_{\nu=1}^{p-1} c_{i,0}^\nu
\]

9
\[ \chi(\lambda^{-1}(k), D^{-j}x) \mathbb{1}_{I_{-j} \oplus \lambda^{-1}((p-\mu)p^j)}(x) \, dx =: S_1 + S_2. \]

For the first sum by (7) we note that
\[ \int_G \psi_{i,n}(x) \chi(\lambda^{-1}(k), D^{-j}x) \, dx = F \psi_{i,n}^\nu(D^{-j} \chi^{-1}(k)) \]
\[ = p^{-i/2} \chi(n, D^{-i-j} \chi^{-1}(k)) \mathbb{1}_{I_{-j} \oplus \lambda^{-1}((p-\nu)p^j)}(D^{-j} \chi^{-1}(k)). \]

Therefore, the first sum takes the form

\[
S_1 = \sum_{\nu=1}^{p-1} \sum_{i=-j}^{\infty} \sum_{n=(p-\mu)p^{i+j}} p^{-j+i/2} c_{i,n}^\nu \chi(\lambda^{-1}(n), D^{-i-j} \lambda^{-1}(k)) \mathbb{1}_{I_{-j} \oplus \lambda^{-1}((p-\nu)p^{i+j})}(\lambda^{-1}(k)) = \sum_{\nu=1}^{p-1} p^{-q/2} \sum_{n=0}^{p^q-1} c_{q-j,n+(p-\mu)p^q}^\nu \chi(\lambda^{-1}(n), D^{-q} \chi^{-1}(k)) \mathbb{1}_{I_{-q} \oplus \lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)).
\]

Since \( \mathbb{1}_{I_{-q} \oplus \lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)) = 1 \) for \((p-\nu)p^q \leq k < (p-\nu+1)p^q \) and \( \mathbb{1}_{I_{-q} \oplus \lambda^{-1}((p-\nu)p^q)}(\lambda^{-1}(k)) = 0 \) for the remaining \( k \), and since the inequality \((p-\nu)p^q \leq k < (p-\nu+1)p^q \), \( q \in \mathbb{Z}_+ \) is equivalent to \( q = \left\lfloor \log_p \frac{k}{p-\nu} \right\rfloor \), it follows that the only nonzero term in the sum \( \sum_{q=0}^{\infty} \) has the number \( q_0 := \left\lfloor \log_p \frac{k}{p-\nu} \right\rfloor \). So

\[
S_1 = \sum_{\nu=1}^{p-1} p^{-q_0/2} \sum_{n=0}^{p^{q_0}-1} c_{q_0-j,n+(p-\mu)p^{q_0}}^\nu \chi(\lambda^{-1}(n), D^{-q_0} \chi^{-1}(k)).
\]

By (8) we notice that up to the multiplication by a constant the inner sum in the last expression is the \( k \)-th term of the discrete Vilenkin-Chrestenson transform of the vector \( \{c_{q_0-j,n+(p-\mu)p^{q_0}}^\nu\}_{n=0}^{p^{q_0}-1} \). Denote this term by \( b_k^\nu \). Finally, for \( S_1 \) we get

\[
S_1(x) = \sum_{\nu=1}^{p-1} p^{q_0/2} b_k^\nu.
\]

Thus, the first sum takes the desired form. To conclude the proof it remains to calculate the following part of the second sum

\[
\int_G \chi(k, D^{-j}x) \mathbb{1}_{I_{-j} \oplus \lambda^{-1}((p-\mu)p^j)}(x) \, dx = p^j \int_G \chi(k, x) \mathbb{1}_{I \oplus \lambda^{-1}(p-\mu)}(x) \, dx
\]
\[
= p^j \int_I \chi(k, x) \, dx = p^j \int_I \chi(k, x) \, dx = p^j \delta_{k,0},
\]

where \( \delta_{0,0} = 1 \), and \( \delta_{k,0} = 0 \), if \( k \neq 0 \).

It is easy to see from (9) that \( \min \int_G \|t\|_{C^2}^2 |F f(t)|^2 \, dt = 0 \) and \( \max \int_G \|t\|_{C^2}^2 |F f(t)|^2 \, dt = \infty \) under the restriction \( \|f\|_{L_2(G)} = 1 \).

Formulas (9) and (10) allow for the following result on estimation of Fourier-Haar coefficients for functions defined on the Vilenkin group.
Corollary 2. Suppose $\| \cdot \|_{G} F f \in L_{2}(G)$, and $f(x) = \sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{+}} c_{j,k}^{\nu} \psi_{j,k}^{\nu}(x)$. Then the series

$$\sum_{\nu=1}^{p-1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{+}} |p^{j} c_{j,k}^{\nu}|^{2}$$

is convergent.

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