Rigid local systems and potential automorphy: The $G_2$-case.

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Abstract

For $s \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0,1\}$, we study the compatible system of Galois representations

$$\rho_s(3) : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G_2(\mathbb{Q}_\ell)$$

introduced in [4], where $G_2(\mathbb{Q}_\ell) \leq \text{GL}_7(\mathbb{Q}_\ell)$ is the simple group of type $G_2$. We prove that, under some mild condition on $s$, the image of the Tate twisted Galois representation $\rho_s(3)$ coincides with $G_2(\mathbb{Z}_\ell)$ for all $\ell$ up to a set of primes having density zero, and we show that $\rho_s$ is potentially automorphic for all $\ell$. From this, meromorphic continuation and functional equation for the $L$-function of $\rho_s$ is deduced.

Introduction

Rigid local systems have a rich history, starting with Riemann’s study of Gauss’ hypergeometric differential equations. N. Katz gave a unifying motivic approach to all rigid local systems, using the theory of the middle convolution [11]. Recently, rigid local systems appeared in the work [15] of Harris, Shepherd-Barron and Taylor on the Sato-Tate conjecture. It is the aim of this article to show that one may use other rigid local systems in a similar way to generalize the results of Harris, Shepherd-Barron and Taylor to other Galois representations.

In [6] and [4], we use Katz’ theory of the middle convolution in order to study a certain rigid motivic $\ell$-adic lisse sheaf $\mathcal{V}_\ell$ of rank 7 on $T = \mathbb{P}^1(\mathbb{Z}_p[[t]]) \setminus \{0,1,\infty\}$ of weight 6. The lisse sheaf $\mathcal{V}_\ell$ corresponds to a continuous representation

$$\rho_\ell : \pi_1(T,\overline{s}) \to \text{GL}_7(\mathbb{Q}_\ell),$$

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where $\bar{s}$ is a geometric point. It is shown in [6] and [4], that the image of the Tate twisted representation $\rho_\ell(3)$ is Zariski dense in the simple algebraic group $G_2(\mathbb{Q}_\ell)$. If $F$ is a field and if $s \in T(F)$, one may consider the specialization of $\rho_\ell$ at $s$, defined as the composition $\rho_\ell^s : \text{Gal}(\bar{F}/F) \longrightarrow \pi_1(s, \bar{s}) \longrightarrow \pi_1(T, \bar{s}) \overset{\rho_\ell}{\longrightarrow} \text{GL}_7(\mathbb{Q}_\ell)$, where $\bar{s}$ is a geometric point extending $s$, and where the middle map is given by functoriality of the functor $\pi_1$.

It is shown [4] that if $s$ is a rational point of $T$ which is contained in a certain infinite subset $T' \subset T(\mathbb{Q})$, then the specialization $\rho_\ell^s(3) : G_\mathbb{Q} \to G_2(\mathbb{Q}_\ell) \leq \text{GL}_7(\mathbb{Q}_\ell)$ has an image which is Zariski dense in the group $G_2(\mathbb{Q}_\ell)$. The set $T'$ is chosen in a way which ensures that locally, at some prime $p > 2$ which is different from $\ell$, the restriction $\rho_\ell^s|_{\text{Gal}(\bar{Q}_p/Q_p)}$ corresponds to a representation of Steinberg type under the local Langlands correspondence, i.e., the eigenvalues of the Frobenius are of the form $1, q, q^2, \ldots, q^6$.

It is natural to ask, whether the Galois representations $\rho_\ell^s$, $s \in T'$, correspond under the Langlands correspondence to an automorphic representation of $\text{GL}_7(\mathbb{A})$.

Our main result is the following (see Thm 4.2):

**0.1 Theorem.** For each $s \in T'$ and for each $\ell$, the Galois representation $\rho_\ell^s$ is potentially automorphic, i.e., its restriction to some open subgroup is automorphic. Especially, the $L$-function of $\rho_\ell^s$ has a meromorphic continuation to the complex plane and it satisfies the expected functional equation.

The main ingredient of the proof are the automorphic lifting results of Richard Taylor [19] and of Clozel, Harris and Taylor [13], which play a crucial role in the recent work of Harris, Shepherd-Barron and Taylor [15] on the Sato-Tate conjecture. These results work especially well, if at some prime $p$, the local Galois representation is of Steinberg type. We first apply the above mentioned automorphic lifting results to $\ell'$-adic Galois representations $\rho_{\ell'}$ whose reduction modulo-$\ell'$ is surjective onto an irreducible monomial subgroup $H$ of $G_2(\mathbb{F}_{\ell'})$ which is isomorphic to the semidirect product of $\mathbb{F}_8$ by $\mathbb{F}_8^*$, using the results of Arthur and Clozel [1]. Then we consider the moduli space $T_W$ of Galois representations which are modulo $\ell \cdot \ell'$ isomorphic to $\rho_{\ell'}$ and a given $\rho_\ell^s$ whose residual image is large (the existence of such $\rho_\ell^s$ is ensured by the results of [4] and a result of Larsen [14]). Using Mestre’s results as in [15], we find totally real points on $T_W$ satisfying certain extra conditions. Then one deduces potential residual automorphy for $\bar{\rho}_\ell^s$ from the automorphy of $\rho_\ell^s$ by compatibility. By again applying the automorphic lifting results to the residual representation $\bar{\rho}_\ell^s$, we obtain potential modularity of $\rho_\ell^s$ for one $\ell$ – and hence for all $\ell$, by compatibility of the system $(\rho_\ell^s)_\ell$. 

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In a forthcoming work, we deduce from Thm. 0.1 the potential automorphy of the 6-th symmetric power of elliptic curves whose $j$-invariant is not an algebraic integer.

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1 Notation and definitions

Let us first set up some notation and definitions: If $k$ is a field, then we write $\bar{k}$ for an algebraic closure of $k$ and we set $\Gamma_k = \text{Gal}(k^{\text{sep}}/k)$. If $F$ is a number field and $\omega$ is a finite prime of $F$, then the completion of $F$ with respect to $\omega$ is denoted by $F_\omega$, and the inertia subgroup of $\Gamma_F_\omega$ is denoted by $I_\omega$. The tame inertia group at $\omega$ is denoted by $I^{\text{tame}}_\omega$ (quotient of $I_\omega$ by its $\ell$-Sylow subgroup, where $\ell$ is the characteristic of $\omega$). Remember there is a character $\omega_\ell: I^{\text{tame}}_\omega \to \mathbb{F}_\ell$, obtained by adjoining the $\ell - 1$-th roots of unity of $\ell$. Usually, a primitive $k$-th root of unity in some field is denoted by $\zeta_k$.

If $V$ is a lisse sheaf on a connected scheme $X$ and if $\bar{s}$ is a geometric point of $X$, then the corresponding monodromy representation is denoted by $\rho_V: \pi_1(X, \bar{s}) \to \text{GL}(V_{\bar{s}})$.

1.1 Definition. Let $R$ be a subring of $\mathbb{C}$, let $t$ be the standard parameter of $\mathbb{A}_R^1 \subseteq \mathbb{P}_R^1$ and identify $\mathbb{C}((\frac{1}{t}))$ with $\mathcal{O}_{\mathbb{P}_R^1, \infty(t)}$. Then one may identify

$$\pi_1(\text{Spec } \mathbb{C}((\frac{1}{t}))) \cong \lim \frac{\text{Gal}(\mathbb{C}((\frac{1}{T})) \cong \mathbb{C}((\frac{1}{t})))}{\mathbb{Z}_p} \cong \prod_p \mathbb{Z}_p.$$  

Then the restriction of the monodromy representation $\rho_V$ of a lisse sheaf $V$ on $\mathbb{A}_R^1 \setminus \{0, 1\} = \mathbb{P}_R^1 \setminus \{0, 1, \infty\}$ to the spectrum of $\mathbb{C}((\frac{1}{t}))$ is called the local monodromy of $V$ at $\infty$. This notion extends in the obvious way to the notion of local monodromy at $0, 1$. We call the local monodromy of type

$$\text{U}(n_1) \oplus \cdots \oplus \text{U}(n_k)$$

if it decomposes into $k$ indecomposable unipotent representations of lengths $n_1, \ldots, n_k$ (resp.).

Let $x_0, x_1, x_2, x_3, x'_1, x'_2, x'_3$ be a base of $\mathbb{Z}^7$ and consider the Dickson trilinear form on $\mathbb{Z}^7$:

$$x_0x_1x'_1 + x_0x_2x'_2 + x_0x_3x'_3 + x_1x_2x'_3 + x'_1x'_2x'_3.$$
The stabilizer of this trilinear form defines the group scheme $\mu_3 \times G_2$ over $\mathbb{Z}$. It is a subscheme of the group scheme $GL_7$ in a natural way. Consider the following bilinear form:

\[(2)\quad -2x_0^2 + x_1x_1' + x_2x_2' + x_3x_3'.\]

Then the simple group scheme $G_2$ may be defined as the stabilizer of both, the Dickson trilinear form (1), and of (2).

## 2 A lisse sheaf of type $G_2$ and a family of $G_2$-motives

Let $\ell$ be an odd prime, let $R = \mathbb{Z}[\frac{1}{\ell}]$, and let $T := \mathbb{A}^1_R \setminus \{0, 1\} = \text{Spec}(R[x][\frac{1}{x(x-1)}])$. The equation

\[(3)\quad Y^2 = \prod_{i=1}^{7} (X_{i+1} - X_i) \prod_{i=1,3,5,7} X_i \prod_{i=1,2,4,6} (X_i - 1)\]

defines a smooth subscheme $\text{Hyp}$ of $G_{m,R} \times (\mathbb{A}^7_{X_1,\ldots,X_7,R} \setminus D)$, where $D$ is the vanishing locus of the right hand side of Equation (3). Let $\sigma$ denote the involutory automorphism of $\text{Hyp}$, defined by sending $Y$ to $-Y$, and let $p$ be the formal linear combination $\frac{1}{\ell}(1+\sigma)$, viewed as an element in the group ring of $\langle \sigma \rangle$. Let $\pi : \text{Hyp} \to T = \mathbb{A}^7_{X_1,\ldots,X_7,R} \setminus \{0, 1\}$ denote the composition of the projection of $\text{Hyp}$ onto $\mathbb{A}^7_R$ followed by the projection onto the 7-th coordinate. By Deligne’s work on the Weil conjectures (Weil II, [3]), the higher direct image with compact supports $R^6\pi_!(\mathbb{Q}_\ell)$ is mixed of weights $\leq 6$. Moreover, the element $p$ operates idempotently on $R^6\pi_!(\mathbb{Q}_\ell)$ and therefore cuts out a subsheaf from $R^6\pi_!(\mathbb{Q}_\ell)$, denoted by $p(R^6\pi_!(\mathbb{Q}_\ell))$. As shown in [6], it is a consequence of Katz’ work on rigid local systems ([11], Chap. 8), that the weight-6-quojient $V_\ell := W^6(p(R^6\pi_!(\mathbb{Q}_\ell)))$ of $p(R^6\pi_!(\mathbb{Q}_\ell))$ is lisse on $T$. Note that then, the Tate twisted sheaf $W^6(p(R^6\pi_!(\mathbb{Q}_\ell)))$ has weight zero. We set $\rho_\ell := \rho_{V_\ell}$. The following result is proved in [6], Theorem 2.4.1:

**2.1 Proposition.** The lisse sheaf $V_\ell$ has rank 7 and its restriction to $T_C$ is irreducible with monodromy group Zariski dense in $G_2(\mathbb{Q}_\ell)$. Moreover, the local monodromy of $V_\ell$ at $0, 1, \infty$ is as follows (resp.):

- involutory, \quad $U(2)^2 \oplus U(3)$, \quad $U(7)$.  


Let \( \bar{s} \) be a complex point. Note that there is a natural injection of \( \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, \bar{s}) \) to \( \pi_1(T, \bar{s}) \). It is well known that \( \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, \bar{s}) \) is generated by three elements \( \gamma_0, \gamma_1, \gamma_\infty \), satisfying the product relation \( \gamma_0 \gamma_1 \gamma_\infty = 1 \). The composition
\[
\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, \bar{s}) \to \pi_1(T, \bar{s}) \to G_2(\mathbb{Q}_\ell)
\]
defines a local system \( \mathcal{V}_\ell^\text{an} \) on \( T(\mathbb{C}) \). It follows from the motivic construction of \( \mathcal{V}_\ell \) given in [6], Section 3, and from the comparison between étale and singular cohomology, that there exists a natural number \( N \) and a local \( \mathbb{Z}[\frac{1}{N}] \)-system \( \mathcal{V}_\ell^\text{an} \) such that \( \mathcal{V}_\ell^\text{an} = \mathcal{V}_\ell^\text{et} \otimes \mathbb{Q}_\ell \), i.e., the images \( A_0, A_1, A_\infty \) of \( \gamma_0, \gamma_1, \gamma_\infty \) under the monodromy representation of \( \mathcal{V}_\ell^\text{an} \) can be written simultaneously over \( \mathbb{Z}[\frac{1}{N}] \). It follows that there exists a natural number \( N_1 \) such that for \( \ell > N_1 \), the Jordan forms of the mod-\( \ell \) residual matrices \( A_0, A_1, A_\infty \) are again of the type involution, \( U(2)^2 \oplus U(3), U(7) \) (resp.).

It was proven by Feit, Fong and Thompson ([8], [20]) that any triple \( (g_1, g_2, g_3) \) of elements in \( G_2(\mathbb{F}_\ell) \), which satisfies the product relation \( g_1 g_2 g_3 = 1 \) and whose Jordan forms are of the above type, generates the group \( G_2(\mathbb{F}_\ell) \) if \( \ell > 5 \). Moreover, any triple \( (g_1, g_2, g_3) \) in \( \text{GL}_7 \) which satisfies the product relation and whose Jordan canonical forms are of type involution, \( U(2)^2 \oplus U(3), U(7) \) (resp.) is rigid, i.e., determined up to simultaneous conjugation by these properties (cf. [17]). Therefore, we can assume that the matrices \( A_0, A_1, A_\infty \) generate the group \( G_2(\mathbb{F}_\ell) \) if \( \ell > N_1 \). Summarizing we obtain the following result:

**2.2 Proposition.** The monodromy matrices \( A_0, A_1, A_\infty \) of \( \mathcal{V}_\ell^\text{an} \) can be written as elements in the group \( \text{GL}_7(\mathbb{Z}[\frac{1}{N}]) \) for some \( N \in \mathbb{N} \). There exists a natural number \( N_1 > N \) such that for all \( \ell > N_1 \), the Jordan forms of the reduction \( A_0, A_1, A_\infty \) modulo-\( \ell \) of the monodromy matrices \( A_0, A_1, A_\infty \) (resp.) are the reduction of the respective Jordan forms. Moreover,
\[
\langle A_0, A_1, A_\infty \rangle = G_2(\mathbb{F}_\ell).
\]

**2.3 Corollary.** For \( \ell > N_1 \), the image of the monodromy representation \( \rho_\ell(3) : \pi_1(T) \to \text{GL}_7(\mathbb{Q}_\ell) \) of the third Tate twist \( \mathcal{V}_\ell(3) \) coincides with the group \( G_2(\mathbb{Z}_\ell) \).

**Proof:** It is shown in [4] (Thm. 1) that the image of \( \rho_\ell(3) \) is contained in \( G_2(\mathbb{Q}_\ell) \) for \( \ell > 2 \). Since the group \( \pi_1(T, \bar{s}) \) is compact, the image of \( \rho_\ell(3) \) is contained in a maximal compact subgroup of \( G_2(\mathbb{Q}_\ell) \). It follows from Bruhat-Tits theory that there are 3 (=Lie rank of \( G_2 \) plus 1) distinct conjugacy classes of maximal compact subgroups in \( G_2(\mathbb{Q}_\ell) \). One is the group \( G_2(\mathbb{Z}_\ell) \), the others are labeled by the simple roots \( \Delta = \{\alpha, \beta\} \) of \( G_2 \) and denoted by \( G_\alpha(\mathbb{Z}_\ell) \) and \( G_\beta(\mathbb{Z}_\ell) \) in [9]. It is shown in [9], Thm. 1, that for \( \gamma = \alpha, \) or \( \beta, \) the group \( G_\gamma(\mathbb{F}_\ell) \) is the semidirect
product of a reduced semisimple algebraic group \( G^\text{red}_\alpha \) over \( \mathbb{F}_\ell \) by its unipotent radical. Moreover, it is shown there that the set of roots

\[ \Phi_\gamma := (\Delta \setminus \{ \gamma \}) \cup \{ -\beta_0 \} \]

is a set of simple roots for \( G^\text{red}_\alpha \), where \( \beta_0 \) denotes the highest root of \( G_2 \). Therefore, we end up with \( G^\text{red}_\alpha \) to be of type \( A_2 \) and with \( G_\beta \) to be of type \( A_1 \cup A_1 \).

Let \( \ell > N_1 \), where \( N_1 \) is as in Prop. 2.2. Then the reduction modulo \( \ell \) of the monodromy matrices generate the group \( G_2(\mathbb{F}_\ell) \) and the residual representation of \( \rho_\ell(3) \) is hence not of type \( A_2 \) or \( A_1 \cup A_1 \). Therefore, the image of \( \rho_\ell(3) \) coincides with \( G_2(\mathbb{Z}_\ell) \).

2.4 Corollary. Let \( N \in \mathbb{N}_{>1} \) be a natural number whose prime divisors are all \( > N_1 \), where \( N_1 \) is as in Prop. 2.2. Let \( \bar{s} \) be a complex point of \( T_{\tilde{R}} \). Then the monodromy representation

\[ \rho_{\mathcal{V}[N]}(3) : \pi_1(T_{\tilde{R}}, \bar{s}) \to \text{GL}_7(\mathbb{Z}/N\mathbb{Z}) \simeq \text{GL}(\mathcal{V}[N], \bar{s}) \]

of \( \mathcal{V}[N] \) respects the Dickson trilinear form modulo \( N \) on \( \mathcal{V}[N], \bar{s} \simeq (\mathbb{Z}/N\mathbb{Z})^7 \), and the composition of maps

\[ \pi_1(T(\mathbb{C}), \bar{s}) \to \pi_1(T_{\mathbb{C}}, \bar{s}) \to \pi_1(T_{\tilde{R}}, \bar{s}) \xrightarrow{\rho_{\mathcal{V}[N]}(3)} G_2(\mathbb{Z}/N\mathbb{Z}) \]

is surjective.

Proof: It follows from Cor. 2.3 that \( \rho_{\mathcal{V}[N]}(3) \) can be assumed to respect the Dickson trilinear form modulo \( N \). The second claim follows from Prop.2.2, using the Frattini property of the natural map \( G_2(\mathbb{Z}/\ell^k\mathbb{Z}) \to G_2(\mathbb{F}_\ell) \) (which in turn is implied by the above mentioned results of [21]).

Let \( F \) be a number field and let \( W \) be a free \( \mathbb{Z}/N\mathbb{Z} \)-module of rank 7 with a continuous action of \( \Gamma_F \) and a compatible trilinear pairing

\[ \langle , , \rangle : W \times W \times W \to \mathbb{Z}/N\mathbb{Z} \]

as well as a compatible bilinear pairing

\[ \langle , \rangle : W \times W \to \mathbb{Z}/N\mathbb{Z}. \]
One may think of $W$ as a lisse étale sheaf on $\text{Spec}(F)$. Consider the functor from the category of $T_F$-schemes to sets which maps $X \to T$ to the set of isomorphisms between the pullback of $W$ to $X$ and the pullback of $\mathcal{V}[N]$ to $X$ and which sends the trilinear form $\langle \cdot, \cdot, \cdot \rangle$ to the Dickson form (1) as well as the bilinear form $\langle \cdot, \cdot \rangle$ to the bilinear form (2). By the representability lemma, this functor is represented by a finite étale cover $T_W$ of $T_F$.

2.5 Proposition. Let $N \in \mathbb{N}_{>1}$ be a natural number whose prime divisors are all $> N_1$, where $N_1$ is as in Prop. 2.2. Then $T_W(\mathbb{C})$ is connected for any embedding $F \hookrightarrow \mathbb{C}$.

Proof: Let $x \in T(\mathbb{C})$ and let $\alpha : W \to \mathcal{V}[N] \in T_W(\mathbb{C})$. It follows from Prop. 2.4 that the isomorphism $\alpha$ can be transformed to a given isomorphism $\beta \in T_W(\mathbb{C})$ using a suitable parallel transport (since the group $G_2(\mathbb{Z}/N\mathbb{Z})$ is the set of all isomorphisms respecting the Dickson form and the bilinear form induced by Poincaré duality).

The following result is similar to [4], Prop. 4:

2.6 Theorem. Let $\ell > 7$ be a prime, let $F$ be a totally real number field and let $q, q' \neq \ell$ be odd prime numbers which split completely in $F$. Let $\omega, \omega'$ denote primes of $F$, lying over $q, q'$ (resp.). Let $s \in T(\bar{F})$ be a geometric point extending an $F$-rational point $\bar{s} \in T(F)$ such that $|\frac{1}{s}|_{\omega} < 1$ and $|1 - s|_{\omega'} < 1$. Then the following holds for the specialized Galois representation $\rho^s_3 : \Gamma_F \to \text{GL}_7(\mathbb{Q}_\ell)$.

(i) The restriction of $\rho^s_3(3)$ to the inertia subgroup $I_\omega \leq \Gamma_{F,\omega}$ is unipotent and indecomposable, i.e., of type $U(7)$. If $\ell > N_1$, where $N_1$ is the constant occurring in Prop. 2.2, then the restriction of $\bar{\rho}^s_3$ to $I_\omega$ is also unipotent and indecomposable. Moreover, semisimplification of the $\Gamma_{F,\omega}$-representation $\rho^s_3|_{\Gamma_{F,\omega}}$ (resp. $\bar{\rho}^s_3|_{\Gamma_{F,\omega}}$) is unramified and the eigenvalues of Frob,ω are of the form $q^{-3}, q^{-2}, q^{-1}, 0, q, q^2, q^3$.

(ii) The restriction of $\rho^s_{3}^{\omega'}$ to the inertia subgroup $I_{\omega'} \leq \Gamma_{F,\omega'}$ is of type $U(3) \oplus U(2) \oplus U(2)$. If $\ell > N_1$, then the restriction of $\bar{\rho}^s_3$ to $I_{\omega'}$ is again of type $U(3) \oplus U(2) \oplus U(2)$.

Proof: The proof is analogous to the proof of [15], Lemma 1.15: Let $W_q$ denote the ring of Witt vectors of $\mathbb{F}_q$ and $\bar{F}$ denote its field of fractions (the maximal unramified extension of $F_\omega$). Let $t$ denote the standard parameter of $\mathbb{A}_1$. It follows
from [7], XIII.5.3, that there is a commutative diagram

$$
\begin{array}{c}
\pi_1(\text{Spec} (\bar{F}_\omega(\frac{1}{\ell}))) \xrightarrow{\sim} \prod_p \mathbb{Z}_p \\
\downarrow \quad \text{left downarrow} \downarrow \\
\pi_1(W_q(\frac{1}{\ell}))) \xrightarrow{\sim} \prod_{p \neq q} \mathbb{Z}_p \\
\uparrow \quad \text{right uparrow} \uparrow \\
\pi_1(\text{Spec}(\tilde{F})) \longrightarrow \prod_{p \neq q} \mathbb{Z}_p,
\end{array}
$$

where the left hand up-arrow is induced by $t \mapsto s$, the right hand downarrow is the natural projection, and the right uparrow is multiplication by $\nu\omega(s)$. The restriction of the sheaves $V_\ell(3)$ and $V_{[\ell]}(3)$ to $\text{Spec} (W_q(\frac{1}{\ell})))$ correspond to representations of $\pi_1(\text{Spec} (\bar{F}_\omega(\frac{1}{\ell}))))$. It follows from Prop. 2.1 and Prop. 2.2 that the pullback of these representations to $\pi_1(\text{Spec}(\bar{F}_\omega(\frac{1}{\ell}))) \simeq \prod_p \mathbb{Z}_p$ along the left downarrow sends 1 to a unipotent matrix with minimal polynomial $(X - 1)^7$ (perhaps enlarging $N_1$ if $\nu\omega(t)$ is divided by $\ell$). Since $\text{Frob}_\omega$ acts on the inertia via the cyclotomic character and since the weight of the determinant of $\rho_{\ell,s}$ is zero, the eigenvalues of $\text{Frob}_\omega$ are of the given type, proving the first claim. The second claim follows along the same arguments, using again Prop. 2.1 and Prop. 2.2.

2.7 Lemma. Let $F$ be a number field and let $s \in T(F)$.

(i) The system of specialized Galois representations $(\rho_{\ell,s} : \Gamma_F \rightarrow \text{GL}_7(\mathbb{Q}_\ell))_\ell$ is strictly compatible.

(ii) If $\omega$ is a prime of $F$ lying over $\ell > 2$, then the restriction of $\rho_{\ell,s}$ to $\Gamma_{F,\omega}$ is de Rham with Hodge-Tate numbers $0, 1, \ldots, 6$.

(iii) If $\ell$ is large enough and if $\omega$ is a prime of $F$ lying over $\ell$, then the restriction of $\rho_{\ell,s}$ to $\Gamma_{F,\omega}$ is crystalline.

(iv) Suppose that $\ell > 7$ and that $\ell = 1 \mod 7$. If $\omega$ is a prime of $F$ lying over $\ell$, then the action of $I_\omega$ on $Q_{\ell}^7$ under $\rho_{\ell,s}$ factors over $I_{\omega}^{\text{tame}}$ and the $I_{\omega}^{\text{tame}}$-module $Q_{\ell}^7$ is isomorphic to

$$
\omega_0^0 \oplus \omega_1^1 \oplus \cdots \oplus \omega_6^6.
$$

Proof: Claim (i) follows from [11], 5.5.4. The second claim follows from the motivic interpretation of $\rho_{\ell,s}^s$ given in [5], Cor. 2.4.2, and the comparison isomorphism. The third claim follows from the motivic interpretation of $\rho_{\ell,s}^s$, because the
The underlying projective algebraic variety is smooth over $F_\omega$ if the residue characteristic of $\omega$ is large. The proof of the last claim follows from $p$-adic Hodge theory analogous to [15], Lemma 1.14.

2.8 Theorem. Suppose that $s \in T(\mathbb{Q})$ is such that there exist two different primes $p, q \neq 2$ such that $\nu_p(s) > 0$ and $\nu_q(1 - s) < 0$. Then, for all primes $\ell$ up to a set of density zero, the image of $\rho^s_{\ell}(3)$ coincides with $G_2(\mathbb{Z}_\ell)$.

Proof: It is shown in [4], Thm. 1, that the image of $\rho^s_{\ell}(3)$ is Zariski dense in $G_2(\mathbb{Q}_\ell)$. By Lemma 2.7(i), the system $(\rho^s_{\ell}(3))$ is compatible. It follows hence from the main result in [14] that for all primes $\ell$ up to a set of density zero, the image of $\rho^s_{\ell}(3)$ is a maximal compact hyperspecial subgroup of $G_2(\mathbb{Q}_\ell)$ and is hence one of the groups $G_2(\mathbb{Z}_\ell), G_\alpha(\mathbb{Z}_\ell)$ or $G_\beta(\mathbb{Z}_\ell)$ from the proof of Cor. 2.3. This implies the claim since the image of $\rho^s_{\ell}(3)$ is contained in $G_2(\mathbb{Z}_\ell)$ by Cor. 2.3.

3 Modular Lifting Results

The following theorem is the main automorphic lifting theorem of Taylor [19] (Thm. 5.2), building up on, and completing the results of Clozel, Harris and Taylor [13] (Thm. 4.5.3):

3.1 Theorem. Let $F$ be a totally real number field. Let $n \in \mathbb{Z}_{>0}$ and let $\ell > n$ be a prime which is unramified in $F$. Let

$$r : \Gamma_F \to \text{GL}_n(\overline{\mathbb{Q}}_\ell)$$

be a continuous irreducible representation with the following properties. Let $\overline{r}$ denote the semisimplification of the reduction of $r$. Suppose that the following conditions hold:

(i) $r^\vee \simeq r^\epsilon n^{-1}$, where $\epsilon$ denotes the $\ell$-adic cyclotomic character.

(ii) $r$ ramifies at only finitely many primes.

(iii) For all places $\nu|\ell$ of $F$, $r|_{\Gamma_{F_\nu}}$ is crystalline.

(iv) There is an element $a \in (\mathbb{Z}^n)_{\text{Hom}(F, \overline{\mathbb{Q}}_\ell)}$ such that

- for all $\tau \in \text{Hom}(F, \overline{\mathbb{Q}}_\ell)$ we have

$$\ell - 1 - n + a_{r,n} \geq a_{r,1} \geq \ldots \geq a_{r,n};$$
• for all $\tau \in \text{Hom}(F, \bar{\mathbb{Q}}_\ell)$ above a prime $\nu | \ell$ of $F$,

$$\dim_{\bar{\mathbb{Q}}_\ell}\text{gr}^i(r \otimes_{\tau,F_\nu} B_{\text{DR}})^{\Gamma_{F_\nu}} = 0$$

unless $i = a_{r,j} + n - j$ for some $j = 1, \ldots, n$ in which case

$$\dim_{\bar{\mathbb{Q}}_\ell}\text{gr}^i(r \otimes_{\tau,F_\nu} B_{\text{DR}})^{\Gamma_{F_\nu}} = 1.$$  

(v) There is a finite non-empty set $S$ of places of $F$ not dividing $\ell$ and for each $\nu \in S$ a square integrable representation $\rho_\nu$ of $\text{GL}_n(F_\nu)$ over $\bar{\mathbb{Q}}_\ell$ such that

$$r^{\text{ss}}_{\Gamma_{F_\nu}} = r_\ell(\rho_\nu)^\vee (1-n)^{\text{ss}},$$

where $r_\ell(\rho_\nu) : \Gamma_{F_\nu} \to \text{GL}_n(\bar{\mathbb{Q}}_\ell)$ is the Galois representation which is associated to $\rho_\nu$ by the local Langlands correspondence. If $\rho_\nu = \text{Sp}_{m_\nu}(\rho'_\nu)$ then set

$$\tilde{r}_\nu = r_\ell((\rho'_\nu)^\vee | \cdot |^{(n/m_\nu-1)(1-m_\nu)/2}).$$

We assume that $\tilde{r}_\nu$ has irreducible reduction $\bar{r}_\nu : \Gamma_F \to \text{GL}_n(\bar{\mathbb{F}}_\ell)$. Finally we suppose that for $j = 1, \ldots, m_\nu$ we have

$$\tilde{r}_\nu \not\cong \bar{r}_\nu e^j.$$

(vi) $(F)^{\ker(\text{ad}\bar{r})}$ does not contain $F(\zeta).$

(vii) Let $\text{ad}\bar{r}$ denote the adjoint representation of $\bar{r}$, viewed as representation on $\text{End}(\bar{\mathbb{F}}_\ell^n)$. Let $\text{ad}^0\bar{r}$ denote the subspace of trace zero endomorphisms, viewed as subrepresentation of $\text{ad}\bar{r}$. Then $H^i(\text{ad}\bar{r}(\Gamma_{F(\zeta)}), \text{ad}^0\bar{r}) = (0)$ for $i = 0, 1$.

(viii) For all irreducible $k[\text{ad}\bar{r}(\Gamma_{F(\zeta)})]$-submodules $W$ of $\text{ad}\bar{r} = \text{End}(\bar{\mathbb{F}}_\ell^n)$ we can find $h \in \text{ad}\bar{r}(\Gamma_{F(\zeta)})$ and $\alpha \in k$ with the following properties. The $\alpha$-generalized eigenspace $V_{h,\alpha}$ of $h$ in $\bar{r}$ is one-dimensional. Let $\pi_{h,\alpha} : \bar{r} \to V_{h,\alpha}$ (resp. $i_{h,\alpha}$) denote the $h$-equivariant projection of $\bar{r}$ to $V_{h,\alpha}$ (resp. $h$-equivariant injection of $V_{h,\alpha}$ into $\bar{r}$). Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} = (0)$.

(ix) $\bar{r}$ is irreducible and automorphic of weight $a$ and type $\{\rho_\nu\}_{\nu \in S}$ with $S \neq \emptyset$.

Then $r$ is automorphic of weight $a$ and type $\{\rho_\nu\}_{\nu \in S}$ and level prime to $\ell$.

3.2 Definition. A subgroup $H$ of $\text{GL}_n(\bar{\mathbb{F}}_\ell)$ is called big, if for the underlying representation $\bar{r} : H \to \text{GL}_n(\bar{\mathbb{F}}_\ell)$, the group $\text{ad}\bar{r}(H)$ satisfies the conditions (vii) and (viii) of Thm. 3.1 with $\text{ad}\bar{r}(H) = \text{ad}\bar{r}(\Gamma_{F(\zeta)}).$
3.3 Theorem. Let $F$ be a totally real number field and let $s \in T(F)$. Suppose that there exists a finite place $\omega$ of $F$ of odd residue characteristic $q$, such $q$ splits completely in $F$ and such that $|\frac{1}{3}|\omega < 1$. Then the following holds for the system of Galois representations $(\rho_{\ell}^s : \Gamma_F \to \GL_7(\Ql))_{\ell}$:

(a) Suppose $\ell > 7$ is a prime number such that $\ell - 1 \equiv 1$ modulo 7 and that $q^i \not\equiv 1 \mod \ell$, $i = 1, \ldots, 7$, and suppose that the image $H$ of $\bar{\rho}_{\ell}^s|_{\Gamma_F(\zeta_7)}$ is big in the sense of Def. 3.2. Suppose further that $(F^\ker(ad\bar{\rho}_{\ell}^s))$ does not contain $F(\zeta_7)$ and that the residual representation $\bar{\rho}_{\ell}^s$ is absolutely irreducible and automorphic. Then $\rho_{\ell}^s|\Gamma_F$ is automorphic.

(b) Let $s = 1 + \frac{a}{7} \in T(\Q) \subset T(F)$ (where $a, b$ are coprime integers) such that $a$ and $b$ each have at least one odd prime divisor. Let $q$ be an odd prime divisor of $b$. Then there exists a prime number $\ell$ with $\ell - 1 \equiv 1$ modulo 7 and $q^i \not\equiv 1 \mod \ell$, $i = 1, \ldots, 7$, such that the image of the residual representation $\bar{\rho}_{\ell}^s|_{\Gamma_F(\zeta_7)}$ is big. If the residual representation $\bar{\rho}_{\ell}^s|_{\Gamma_F}$ is automorphic, then $\rho_{\ell}^s|\Gamma_F$ is automorphic.

Proof: We have to prove that under our assumptions, the conditions (i)–(ix) of Thm. 3.1 hold for $\rho_{\ell}^s$ in cases (a) and (b).

Case (a): Condition (i) is clear since, by [4], Thm. 1, the representation $\rho_{\omega_\ell}$, and hence $\rho_{\ell}^s(3)$, takes values in $G_2(\Ql)$ and $G_2$ is contained in the orthogonal group of the bilinear form in Equation (2) of Section 1. Condition (ii) follows from the cohomological construction of $\rho_{\omega_\ell}$. Condition (iii) follows from Lemma 2.7(iii). Condition (iv) of Thm. 3.1 follows from Lemma 2.7(ii) with the weight

$$a = ((0, \ldots, 0), \ldots, (0, \ldots, 0)),$$

using that the filtration on $(B_{\text{dR}} \otimes_{\mathcal{O}_F} (\rho_{\ell}^s|_{\Gamma_F}))_{\Gamma_F}$ is opposite to the Hodge-Tate filtration. Let $\omega$ be a prime of $F$ lying over $q$. It follows from Thm. 2.6 that the restriction of $\rho_{\ell}^s$ to $\Gamma_{F_\omega}$ is of type $\{\Sp_7(1)\}_{\{\omega\}}$, proving Condition (v) with $\{\omega\} = S$. Conditions (vi)–(ix) hold by assumption.

Case (b): By Theorem 2.8, there exists a prime $\ell > 7$ such that $q^i \not\equiv 1 \mod \ell$ and $\ell \equiv 1 \mod 7$, and such that the image of $\rho_{\ell}^s$ coincides with $G_2(\Zl)$. Note that by choosing $\ell$ large enough, the field $F$ can be assumed to be linearly independent to the fixed field of $\rho_{\ell}^s(3)$ since the groups $G_2(\F_{\ell}) = \text{im}(\rho_{\ell}^s(3))$ are simple and nonisomorphic for $\ell$ large enough. Conditions (i)–(v) then already hold by Case (a). The image of the adjoint representation $ad\rho_{\ell}^s$ coincides with $G_2(\F_{\ell})$. Since $G_2(\F_{\ell})$ has no nontrivial abelian factor, Condition (vi) holds for $\rho_{\ell}^s$. Condition (ix) follows from the assumption using type $\{\Sp_7(1)\}_{\{\omega\}}$ and the fact that $G_2(\F_{\ell})$ is an absolutely irreducible subgroup of $\GL_7(\F_{\ell})$. Thus we are left to show that the
group $G_2(\mathbb{F}_\ell)$ is big. As remarked in [10], Remark to Def. 3.1, Condition (viii) holds for a group $\Delta \leq \text{GL}_n(\bar{\mathbb{F}}_\ell)$, if it holds for a subgroup of $\Delta$. Condition (viii) was shown in [15], Lemma 3.2, to hold for the image of $\text{SL}_2(\mathbb{F}_\ell)$ in $\text{GL}(\tilde{\rho}^n)$ if $\ell > 2n + 1$ (where $\tilde{\rho}^n$ denotes the $n$-th symmetric power of the standard representation of $\text{SL}_2(\mathbb{F}_\ell)$). It is well known that under the above assumptions on $\ell$, the module $\text{ad} \tilde{\rho}^6$ decomposes into irreducible representations $1 \oplus \tilde{\rho}^2 \oplus \tilde{\rho}^4 \oplus \cdots \oplus \tilde{\rho}^{12}$, where $1$ denotes the trivial representation on the scalars (recall that the image of $\text{SL}_2(\mathbb{F}_\ell)$ in $\text{GL}(\tilde{\rho}^6)$ is contained in the group $G_2(\mathbb{F}_\ell)$). Therefore, $H^0(\text{ad}\tilde{\rho}^6_\ell(\Gamma_{F(\zeta_\ell)}), \text{ad}^0\tilde{\rho}^6_\ell) = (0)$ and Condition (viii) also holds for $\tilde{\rho}^6_\ell$ if we choose $\ell > 13$. To any $\ell$-restricted weight $\lambda$ of $G_2$, there is associated an irreducible representation $L(\lambda)$ of $G_2(\mathbb{F}_\ell)$. By [2], Section 7.5, the cohomology groups $H^1(G_2(\mathbb{F}_\ell), L(\lambda))$ vanish for all $\ell$-restricted weights $\lambda$, unless $\lambda$ is of the following form:

$$(p - 5, 0), (3, p - 2), (6, p - 5), (p - 2, 3)$$

(here, a weight $(n_1, n_2)$ denotes the weight $n_1\pi_1 + n_2\pi_2$, where $\pi_1$ and $\pi_2$ are the fundamental weights). The representation $\text{ad}^0$ decomposes into the direct sum of $\ell$-restricted weights $(1, 0), (0, 1), (2, 0)$. Hence, for $\ell > 13$, $H^1(G_2(\mathbb{F}_\ell), L(\lambda)) = 0$ and therefore $H^1(\text{ad}\tilde{\rho}^6_\ell(\Gamma_{F(\zeta_\ell)}), \text{ad}^0\tilde{\rho}^6_\ell) = (0)$, proving Condition (vii).

\section{Potential modularity of $G_2$-extensions}

The following result can be found in [15], Prop. 2.1:

\textbf{4.1 Proposition.} Let $F$ be a number field and let $S = S_1 \bigsqcup S_2$ be a finite set of places of $F$ such that $S_2$ contains no infinite place. Suppose that $T/F$ is a smooth, geometrically connected variety. Suppose also that for $\nu \in S_1$, $\Omega_\nu \subseteq T(F_\nu)$ is non-empty open (for the $\nu$-topology) subset and that for $\nu \in S_2$, $\Omega_\nu \subseteq T(F_\nu^{nr})$ is non-empty open $\text{Gal}(F_\nu^{nr}/F_\nu)$-invariant subset. Suppose finally that $L/F$ is a finite extension. Then there is a finite Galois extension $F'/F$ and a point $P \in T(F')$ such that

- $F'/F$ is linearly disjoint from $L/F$;
- every place $\nu$ of $S_1$ splits completely in $F'$ and if $\omega$ is a prime of $F'$ above $\nu$ then $P \in \Omega_\nu \subseteq T(F_\omega)$;
every place $\nu$ of $S_2$ is unramified in $F'$ and if $\omega$ is a prime of $F'$ above $\nu$ then $P \in \Omega_\omega \cap T(F'_\omega)$.

Let $F$ be a number field. Let us call an irreducible Galois representation $\rho_\ell : \Gamma_F \to \GL_n(\mathbb{Q}_\ell)$ potentially automorphic if there exists a finite extension $F'/F$ such that for almost all primes of $F'$, the local $L$-function of the restriction of $\rho_\ell$ to $\Gamma_{F'}$ coincides with the local $L$-functions of a cuspidal automorphic representation of $\GL_n(\mathbb{A}_{F'})$.

4.2 Theorem. Let $a, b$ be coprime integers each possessing at least one odd prime divisor and let $s = 1 + \frac{a}{b}$. Then the Galois representations $\rho_\ell^s$ are potentially automorphic for all $\ell$.

Proof: Let $q$ be the odd prime divisor of $b$. Note that since the system $(\rho_\ell^s : \Gamma_Q \to G_2(\mathbb{Q}_\ell))_{\ell}$ is compatible by Lemma 2.7 (i), it suffices to prove the claim for one $\ell$. We will assume that $\ell$ is as in Thm. 3.3 (b), so we are left to prove that $\rho_\ell^s$ is residually potential automorphic. Let $\ell'$ be an auxiliary prime which is congruent to $1 \mod 7$ so that $F_{\ell'}$ contains a full set $\{\zeta_i^j | i = 1, \ldots, 6\}$ of primitive 7-th roots of unity and such that $q \equiv \ell' \equiv \zeta_7 \equiv F_{\ell'}$. Note also that, by the classification of maximal subgroups of $G_2(F_{\ell'})$ given in [12], the group $G_2(F_{\ell'})$ contains the group $\PSL_2(8)$ which in turn contains the semidirect product $H := F_8 \cdot F_8^*$, where $F_8^* \simeq \mathbb{Z}/7\mathbb{Z}$ acts as multiplicative subgroup on the additive group $F_8 \simeq (\mathbb{Z}/2\mathbb{Z})^3$. Since the group $H$ is solvable, there exists a surjective homomorphism $\alpha : \Gamma_Q \to H$ by the theorem of Shafarevich [16] (or by an easy argument on inducing a quadratic character of $F_8$). Let $W = (\mathbb{Z}/(\ell' \cdot \ell)\mathbb{Z})^7 \simeq F_{\ell'}^7 \oplus F_7^l$ be the $\Gamma_Q$-module where $\Gamma_Q$ acts on the first summand via $\alpha$ and on the second summand by $\hat{\rho}_\ell^s(3)$, respecting the mod-$\ell'\ell$ Dickson trilinear form (1) and the mod-$\ell'\ell$ bilinear form (2). Let $T_W$ be the étale cover of $T_Q = A_1^3 \setminus \{0, 1\}$, parametrizing isomorphisms between the pullback of $W$ and the pullback of $\mathfrak{V}(\ell'\ell)$ to $T$-schemes $X$ (see Section 2). Let $L/Q$ denote the Galois extension which is the compositum of $Q(\zeta_{\ell'})$ with the fixed field of the kernel of the module homomorphism $\Gamma_Q \to \text{Aut}(W)$.

By Prop. 4.1, there exists a totally real number field $F$ which is linearly disjoint from $L/Q$ and a point $P \in T_W(F)$ such that $q$ splits completely in $F$. Moreover, we can assume that if $\omega$ is a prime of $F$ above $q$ then $P$ is contained in $\Omega_\omega \subset T_W(F_\omega)$, where $\Omega_\omega$ is the inverse image of the set $\{t \in T(F_\omega) | \ |t|_\omega < 1\}$.

Let $\tilde{s} \in T(F)$ be the image of $P$. Since $F$ is linearly disjoint from $Q(\zeta_{\ell'})$ and since the group order $|H| = 56 = 7 \cdot 8$ is prime to $\ell'$, the fixed field of $\ker \text{ad}(\hat{\rho}_\ell^s)|_{T'}$ does not contain $F(\zeta_{\ell'})$. A computation, using the elementary properties of representations of finite groups, shows that the adjoint $H$-module $\text{End}(F_{\ell'}^l)$ decomposes into a direct sum of the seven different irreducible 1-dimensional representations.
of $H$ and a 6-fold copy of the absolutely irreducible 7-dimensional representation of $H$. Since $\ell'$ is prime to 56, it follows from a restriction argument on group cohomology that $H$ is a big subgroup of $\GL_7(\mathbb{F}_{\ell'})$. Also, the group $H$ acts absolutely irreducible on $\mathbb{F}_{\ell'}^7$.

There is an embedding of $H$ into $\GL_7(\mathbb{Q}_{\ell'})$ such that the embedding of $H$ into $\GL_7(\mathbb{F}_{\ell'})$ is given by the reduction mod $\ell'$ of the coefficients. By composing the embedding $H \to \GL_7(\mathbb{Q}_{\ell'})$ with the surjection $\Gamma_F \to H$ induced by the surjection $\Gamma_{\mathbb{Q}} \to H$ ($L$ is linearly disjoint from $F$) we obtain an irreducible Galois representation $\hat{\rho}_{\ell'} : \Gamma_F \to \GL_7(\mathbb{Q}_{\ell'})$

which is cyclically induced from the abelian character $\chi : \mathbb{F}_8 \simeq (\mathbb{Z}/2\mathbb{Z})^3 \to \mathbb{Q}_{\ell'}$ which maps any element $\neq 0$ to $-1$. It follows then from the results of Arthur and Clozel [1] that $\hat{\rho}_{\ell'}$ is automorphic. Consequently, the reduced representation $\bar{\rho}_{\ell'}|_{\Gamma_F}$ is automorphic. Since $|\frac{1}{2}|\omega < 1$, we can now apply Theorem 3.3 (a) which implies that $\bar{\rho}_{\ell'}|_{\Gamma_F}$ is automorphic. Since the Galois representations $\bar{\rho}_{\ell'}|_{\Gamma_F}$ and $\rho_{\ell'}|_{\Gamma_F}$ are compatible by Lemma 2.7 (i), the residual representation $\bar{\rho}_{\ell'}|_{\Gamma_F}$ is also automorphic. Since $\bar{\rho}_{\ell'}|_{\Gamma_F}$ and $\bar{\rho}_{\ell'}|_{\Gamma_F}$ coincide by construction, this implies that $\bar{\rho}_{\ell'}|_{\Gamma_F}$ is automorphic. We can now apply Theorem 3.3 (b) which implies that $\bar{\rho}_{\ell'}|_{\Gamma_F}$ is automorphic.

\[ \hat{\rho}_{\ell'} : \Gamma_F \to \GL_7(\mathbb{Q}_{\ell'}) \]

4.3 Corollary. The $L$-function of $\rho_{\ell'}$ converges in some half plane $\Re(s) > c$ and has a meromorphic continuation. Moreover, it satisfies the expected functional equation.

Proof: This follows from Brauer’s theorem using the same arguments as in the proof of [18], Cor. 2.2.

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