Thermal dynamics in general relativity

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We discuss a relativistic model for heat conduction, building on a convective variational approach to multi-fluid systems where the entropy is treated as a distinct dynamical entity. We demonstrate how this approach leads to a relativistic version of the Cattaneo equation, encoding the finite thermal relaxation time that is required to satisfy causality. We also show that the model naturally includes the non-equilibrium Gibbs relation that is a key ingredient in most approaches to extended thermodynamics. Focusing on the pure heat conduction problem, we compare the variational results with the second-order model developed by Israel and Stewart. The comparison shows that, despite the very different philosophies behind the two approaches, the two models are equivalent at first-order deviations from thermal equilibrium. Finally, we complete the picture by working out the non-relativistic limit of our results, making contact with recent work in that regime.

Keywords: general relativity; thermodynamics; heat conductivity; dissipation

1. Introduction

Dissipative fluid phenomena represent a number of challenges for relativistic physics. The main reason for this is the simple fact that the classic Navier–Stokes equations, which are not hyperbolic, allow instantaneous signal propagation. This is obviously not allowed in a relativistic description, where a model must respect causality in order to be considered acceptable. Given the fundamental issues involved, it is not surprising that the problem of relativistic heat conduction continues to attract interest. The issues considered range from questions like ‘Does a moving body feel cold?’ (Landsberg 1967) to issues of stability of models for non-equilibrium thermodynamics (Hiscock & Lindblom 1983, 1985, 1988; Olson & Hiscock 1990) and whether different descriptions can be distinguished by experiment (Geroch 1995; Lindblom 1996).

In the mainstream general relativity community, the debate has, to some extent, been settled since the late 1970s. The key contribution was the work of Israel and Stewart, who developed a model analogous to Grad’s 14-moment model...
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theory, firmly grounded in relativistic kinetic theory (Stewart 1977; Israel & Stewart 1979a,b). This so-called ‘second-order’ theory, which extends the pioneering ‘first-order’ work of Eckart (1940), has since found applications in a number of contexts. In particular, in the last few years, there has been a resurgence of interest in the model arising from the need to describe highly relativistic plasmas generated in colliders like RHIC at Brookhaven and the LHC at CERN (Elze et al. 2001; Muronga 2004). However, despite the obvious successes of the Israel–Stewart model and its various attractive features, there are still dissenting views in the literature, see for example Garcia-Colin & Sandoval-Villalbazo (2006) and Garcia-Perciante et al. (2009).

Particular objections concern the complexity of the model and the large number of dissipation coefficients that are needed to complete it. This is, however, a feature that is shared by all models within the extended thermodynamics paradigm (Jou et al. 1993), and it is difficult to see how a simpler model can be developed without sacrifice of causality or stability.

Our discussion is motivated by recent efforts to model the dynamics of superfluid neutron stars. This a problem that requires a general relativistic description of ‘multi-fluid’ dynamics, and where thermal and dissipative effects are expected to impact on observations. Work in this area is motivated by the effort to detect gravitational radiation, and the need to understand various oscillation instabilities in rotating neutron stars (Andersson 2003). These instabilities tend to be counteracted by viscosity, and it is obviously important to have a quantitatively accurate description of the involved mechanisms. Most studies of this problem have been carried out in Newtonian gravity, basically because of a feeling that the relativistic problem is too ‘complicated’. There is, however, a growing body of work building towards realistic, fully relativistic, models. The present analysis should be viewed in that context.

We are not providing a truly original view of the heat conduction problem. Rather, we approach the issue within the multi-fluid paradigm that has been successfully applied to superfluid systems. A key ingredient in this analysis is the treatment of the entropy as an additional (massless) ‘fluid’ component. This idea is obviously phenomenological, but we will show that it provides a model with a number of attractive features. Most importantly, the formalism is intuitive and readily generalized to more complex settings.

We build on the convective variational formulation developed by Carter (1976, 1989, 1991); see Andersson & Comer (2007) for an introductory review. In addition to the intrinsic elegance of an action principle, an appealing feature of the variational approach is that once an equation of state is provided, the theory provides the relation between the various currents and their conjugate momenta (a point that is usually not considered in the context of the heat problem). Another key advantage of the variational derivation is that incorporating additional fluid components is straightforward.

In Carter’s macroscopic model for the heat problem, one considers two fluxes, one corresponding to the matter flux and one which is associated with the entropy. These two currents are the fundamental fields appearing in the matter sector of the Einstein–Hilbert action. The Lagrangian of the theory is a relativistic invariant and hence it should depend only on covariant combinations of the two fluxes, which includes the relative flow between them.
This encodes the so-called entrainment effect, which tilts the momenta with respect to the currents when two or more fluids are coupled, and which turns out to be a crucial feature of the multi-fluid approach to relativistic heat conduction. In Carter’s original work (Carter 1988, 1989), the aim was to keep the model as simple as possible by imposing restrictions on the way that the currents interact. On the grounds of simplicity, Carter ignored the entrainment. This omission resulted in a truncated model with severe stability issues (Olson & Hiscock 1990). Given this problem, it was suggested that Carter’s approach does not provide a viable alternative to the Israel–Stewart model. This argument is, however, flawed. A detailed comparison of the two formalisms (Priou 1991) shows that, at second order in the deviation from equilibrium, the full (entrained) variational approach is essentially equivalent to the Israel and Stewart model.

An important aspect of the present work is that it extends the recent Newtonian model of thermal dynamics discussed by Andersson & Comer (2010) to general relativity. In both cases, entrainment has a fundamental impact on the dynamics of entropy. In fact, it is an essential ingredient that preserves causality and stability. Both discussions lead to a generalization of the Cattaneo equation (Cattaneo 1948) and hence a finite speed of propagation of heat. An equivalent, although not identical, equation arises in the Israel and Stewart theory. The difference between the two approaches is in the underlying thermodynamics. Basically, the Israel and Stewart formalism is based on the standard equilibrium Gibbs relation, and therefore the thermodynamic quantities take their equilibrium values. In contrast, the covariant dynamics of the variational approach leads naturally to an extended Gibbs relation (analogous to that in many models of extended irreversible thermodynamics), giving the thermodynamic quantities a different meaning. Of course, the theories are completely equivalent in the case of thermal equilibrium.

The paper is structured as follows. Section 2 provides the two-fluid variational derivation. The general philosophy that we adopt is that of Carter (1991). We obtain the equations of motion by imposing conservative constraints on the variations of the Lagrangian density, but we use the explicit freedom in the equations of motion to include momentum exchange and entropy production while keeping the energy–momentum tensor divergence-free. Once the equations of motion are obtained, it is useful to make a choice of frame to discuss the thermodynamics and the Gibbs relation. We discuss this point in detail, arguing why the choice that we make is natural. Section 3 contains the derivation of the relativistic generalization of the Cattaneo equation that follows once we impose the second law of thermodynamics. The result is then compared with the predictions of the Israel and Stewart model. The Newtonian limit of the theory is obtained in §4, which establishes the close connection to the non-relativistic model of Andersson & Comer (2010). The paper concludes with a brief discussion of the implications of the results, possible future extensions and applications.

Throughout the paper, we use the convention that four-dimensional spacetime indices are represented by lowercase letters starting from the beginning of the alphabet, $a, b, c, \ldots$, while three-dimensional spatial indices are lowercase letters $i, j, k, \ldots$. There should be little risk of confusion. We denote the covariant derivative by a semi-colon.
2. The two-fluid model

We consider the problem of heat conduction in general relativity at the macroscopic level. This means that we assume that the particle number is large enough that the fluid approximation applies and that there is a well-defined matter current, \( n^a \). Moreover, we adopt the multi-fluid formalism developed by Carter (1989) and treat the entropy as an effective fluid with flux \( s^a \). This current is in general not aligned with the particle flux. The misalignment is associated with the heat flux and leads to entropy production.

For a generic two-fluid system, the starting point is the definition of a relativistic invariant Lagrangian-type master function \( \mathcal{A} \). Assuming that the system is isotropic, we take \( \mathcal{A} \) to be a function of the different scalars that can be formed by the two fluxes.\(^1\) From \( n^a \) and \( s^a \), we can form three scalars:

\[
\begin{align*}
    n^2 &= -n_a n^a, \\
    s^2 &= -s_a s^a, \\
    j^2 &= -n_a s^a.
\end{align*}
\]

Hence, we can write the master function as the density for the matter sector of the Einstein–Hilbert action:

\[
S_M = \int d\Omega \mathcal{A}(n, s, j). 
\]

An unconstrained variation of \( \mathcal{A} \) then leads to

\[
\delta \mathcal{A} = \frac{\partial \mathcal{A}}{\partial n} \delta n + \frac{\partial \mathcal{A}}{\partial s} \delta s + \frac{\partial \mathcal{A}}{\partial j} \delta j. 
\]

Using equations (2.1)–(2.3), we can change the passive density variations for dynamical variations of the world lines generated by the fluxes and the metric (Andersson & Comer 2007). That is, we use

\[
\begin{align*}
    \delta n &= -\frac{1}{2n}[2g_{ab}n^a \delta n^b + n^a n^b \delta g_{ab}], \\
    \delta s &= -\frac{1}{2s}[2g_{ab}s^a \delta s^b + s^a s^b \delta g_{ab}] \\
    \text{and} \\
    \delta j &= -\frac{1}{2j}[g_{ab}(n^a \delta s^b + s^b \delta n^a) + n^a s^b \delta g_{ab}].
\end{align*}
\]

\(^1\)It should be noted that we consider the simplest ‘convective’ model. The natural way to account for viscosity would be to allow the master function to depend also on the associated stresses, see Carter (1991) and Priou (1991). This model is, however, significantly more complex and we do not consider it here in order to keep the discussion clear.

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This means that the variation (2.5) becomes

\[ \delta A = \left[ -2 \frac{\partial A}{\partial n^2} n_a - \frac{\partial A}{\partial j^2} s_a \right] \delta n^a + \left[ -2 \frac{\partial A}{\partial s^2} s_a - \frac{\partial A}{\partial j^2} n_a \right] \delta s^a + \left[ - \frac{\partial A}{\partial n^2} n^a n^b - \frac{\partial A}{\partial s^2} s^a s^b - \frac{\partial A}{\partial j^2} n^a s^b \right] \delta g_{ab}. \] (2.9)

From equation (2.9), we can read off the conjugate momentum associated with each of the fluxes:

\[ \mu_a = \frac{\partial A}{\partial n^a} = g_{ab} \left( B_n n_b + A_{ns} s_b \right) \] (2.10)

and

\[ \theta_a = \frac{\partial A}{\partial s^a} = g_{ab} \left( B_s s_b + A_{ns} n_b \right), \] (2.11)

where we have introduced the coefficients

\[ B_n \equiv -2 \frac{\partial A}{\partial n^2}, \quad B_s \equiv -2 \frac{\partial A}{\partial s^2} \quad \text{and} \quad A_{ns} \equiv - \frac{\partial A}{\partial j^2}. \] (2.12)

The conjugate variables (2.10) and (2.11) demonstrate the fundamental role of the master function (Carter 1989). The distinct roles of the fluxes and their conjugate momenta are often not considered in the fluids literature. A key advantage of the variational approach is that the quantities are immediately determined by the form of the master function. Moreover, it is clear that the momenta are not generally aligned with the respective currents, owing to the fact that the master function depends on the relative flux. This is an important effect. Fundamentally, there is no physical argument to rule out the dependence on \( j^2 \). In fact, this coupling is associated with the entrainment effect that is known to be of central importance in multi-fluid systems.

In the case of superfluid neutron star cores, the entrainment arises owing to the strong interaction and couples the neutron and proton fluxes, see for example Comer & Joynt (2003). In the present problem, with matter and entropy, we will show that the entrainment is associated with the thermal relaxation of the system. This is an important effect that must be accounted for.

The energy–momentum tensor is obtained from equation (2.4) by noting that the displacements of the conserved currents induce a variation in the spacetime metric and therefore the variations of the fluxes, \( \delta n^a \) and \( \delta s^a \), are constrained; see Carter & Quintana (1972), Andersson & Comer (2007) and Prix (2004) for discussion. The energy–momentum is thus found to be

\[ T_a^b = \mu_a n^b + \theta_a s^b + \Psi \delta_a^b, \] (2.13)

where we define the generalized pressure, \( \Psi \), as

\[ \Psi = A - \mu_a n^a - \theta_a s^a. \] (2.14)
The equations of motion are obtained by requiring that the divergence of the energy–momentum tensor (2.13) vanishes. For an isolated system, we can express this requirement as an equation of force balance

\[ T^b_{a;b} = f^n_a + f^s_a = 0, \]  

where the individual force densities are

\[ f^n_a = 2\mu_{[a;b]} n^b + n^b_{,b} \mu_a \]  

(2.16) and

\[ f^s_a = 2\theta_{[a;b]} s^b + s^b_{,b} \theta_a. \]  

(2.17)

The square brackets represent anti-symmetrization in the usual way.

We note that, in order to obtain the energy–momentum tensor (2.13), we needed to impose the conservation of the fluxes as constraints on the variation. However, the equations of motion, (2.16) and (2.17), still allow for non-vanishing production terms. If we, for simplicity, consider a single-particle species, the matter current is conserved and we have

\[ n^a_{,a} = 0. \]  

(2.18)

This removes the second term from the right-hand side of equation (2.16). In contrast, the entropy flux is generally not conserved. We will have

\[ s^a_{,a} = \Gamma_s \geq 0, \]  

(2.19)

in accordance with the second law of thermodynamics. This suggests that, to make progress, we need to connect the variational results with the relevant thermodynamical concepts. In doing this it makes sense to make a specific choice of frame.

(a) The matter frame

The conservation law (2.18) implies that the force \( f^n_a \) is orthogonal to the matter flux, \( n^a \), and therefore has only three degrees of freedom. Furthermore, because of the force balance (2.15), we also have \( n^a f^s_a = 0 \). This suggests that it is natural to focus on observers moving with the matter frame. We associate the matter current with a four-velocity \( u^a \) such that

\[ n^a = nu^a, \]  

(2.20)

where \( u_a u^a = -1 \) and \( n \) is the number density measured in this frame. Historically, this is known as the Eckart frame. As will soon become clear, this is the natural frame in the case of a single-particle species, essentially because it simplifies the analysis. More complex settings, e.g. when dealing with additional particle species and reactions, will make the choice of frame less obvious. It may well be that the best strategy in such cases is to follow Landau & Lifshitz (1959) and work in the centre-of-mass frame. Having said that, it is worth noting that, even in the more general problem, there is a unique frame associated with the entropy/heat flow. Even though we will not discuss this problem further here, there are interesting issues that warrant more detailed thinking.
Having chosen to work in the matter frame, we can decompose the entropy current and its conjugate momentum into parallel and orthogonal components. The entropy flux is then expressed as

\[ s^a = s^* (u^a + w^a), \]  

(2.21)

where \( w^a \) is the relative velocity between the two-fluid frames, and \( u^a w_a = 0 \). Letting \( s^a = su^a \) where \( u^a \) is the four-velocity associated with the entropy flux, we see that \( s^* = s \gamma \) where

\[ \gamma = |u^a + w^a| = (1 - w^2)^{-1/2}, \]  

(2.22)

is the redshift associated with the relative motion of the two frames. In the following, we will use an asterisk to denote matter frame quantities.

Similarly, we can write the thermal momentum as

\[ q^a = q^* u_a + q^♭ w_a = (B^s s^* + A^{ns} n) u_a + B^s s^* w_a, \]  

(2.23)

where we have made use of equation (2.11). From these expressions, we readily obtain a measure of the temperature measured in the matter frame:

\[ -u^a \theta_a = \theta^* = B^s s^* + A^{ns} n. \]  

(2.24)

In essence, this measure represents the effective mass associated with the entropy component. We have also defined

\[ \theta^♭ = B^s s^*. \]  

(2.25)

It is worth noting that, if we ignore the coupling between the fluxes in the master function by taking \( A^{ns} = 0 \), then we have \( \theta^* = \theta^♭ \). This particular case was considered by Carter (1988), and from the analysis of Olson & Hiscock (1990) we know that it leads to a model that exhibits instabilities. It is useful to keep this in mind during the following developments. As we will see, the main problem with Carter’s ‘regular’ model is that it leaves no freedom to adjust the thermal relaxation time scale.

In order to express the energy–momentum tensor in terms of the matter frame quantities, we define the variables \( \sigma^a = s^* w^a \) and \( p_a = B^s s^* w_a \). Using the above expressions, the stress–energy tensor (2.13) can be written in a more familiar form:

\[ T_{ab} = -[A - p_a \sigma^a] u_a u_b + 2 u_a q_b + P_{ab}, \]  

(2.26)

where, making use of the projection orthogonal to the matter flux

\[ h_{ab} = g_{ab} + u_a u_b, \]  

(2.27)

the heat flux (energy flow relative to the matter) is given by

\[ q_a = -h_{ab} u_b T^{bc} = s^* \theta^* w_a. \]  

(2.28)

We also have

\[ P_{ab} = h_{ab} \Psi + p_a \sigma_b. \]  

(2.29)

It is worth noting at this point that the variational analysis leads to the presence of ‘shear’ terms in the energy–momentum tensor. Such terms are usually associated with viscous stresses, and it is interesting to note that they arise even
though we consider the pure heat conduction problem. Moreover, this exercise shows that the energy density measured in the matter frame can be obtained by a Legendre-type transform on the master function. That is, we have

\[ \rho^s = u_\alpha u_\beta T^{ab} = -A + p_\alpha \sigma^a. \] (2.30)

In fact, this relation informs the choice of \( \sigma^a \) and \( p_\alpha \) as key variables (Carter 1976).

\( \text{(b) The temperature problem} \)

Thermodynamic properties such as pressure and temperature are uniquely defined only in equilibrium. Intuitively, this makes sense since, in order to carry out a measurement (of say the temperature), the measuring device must have time to reach ‘equilibrium’ with the system. The measurement is obviously only meaningful as long as the time scale required to obtain a result is shorter than the evolution time for the system. Of course, this does not prevent a generalization of the various thermodynamic concepts. The procedure may not be unique, but one should at least require the generalized concepts to be internally consistent within the chosen extended thermodynamics model. As a useful demonstration of this notion, and the fact that our model satisfies this criterion, we will consider the particular case of the temperature.

When the system is out of equilibrium, we can define a number of different ‘temperatures’. It makes sense to refer to the quantity obtained from equation (2.24) as the dynamical temperature since it corresponds to the effective mass of the entropy component. Similarly, the effective mass for the particles, \( \mu \), is given by

\[ \mu = -u^a \mu_\alpha = B^n n + A^{ns} s^s. \] (2.31)

Let us now show that equation (2.24) agrees with the thermodynamical temperature that an observer moving with the matter would measure. Using the standard definition of temperature, we consider the variation of the energy with respect to the entropy in the observer’s frame (while keeping the other thermodynamic variables fixed). To do this, we note that the energy density measured in the matter frame (2.30) is a function of three independent state variables, \( \rho^s = \rho^s(n, s^s, p) \).

Determining the energy density directly from equation (2.13), we get

\[ \rho^s = \mu n + \theta^s s^s - \Psi. \] (2.32)

Using the definitions (2.24), (2.25) and (2.31), we can evaluate the generalized pressure (2.14) in the matter frame as a Legendre-type transform of the master function:

\[ \Psi(\mu, \theta^s, p) = A + \mu n + \theta^s s^s - p \sigma, \] (2.33)

and so the variation of equation (2.32),

\[ d\rho^s = \mu dn + \theta^s ds^s + \sigma dp, \] (2.34)

shows that the dynamical temperature agrees with the thermodynamical temperature provided we evaluate it in the appropriate frame. We also see that, when the system is out of equilibrium, the energy variation (2.34) depends on the
heat flux (encoded in $\sigma^a$ and $p_a$). This extended Gibbs relation is similar to that which is postulated in many approaches to extended thermodynamics (Jou et al. 1993). The main difference here is that equation (2.34) arises naturally from the variational analysis.

This result is far from trivial. The requirement that the two temperature measures agree determines the additional state parameter, $p$, to be held constant in the variation of $\rho^*$. Any other choice of the third parameter, e.g. $j^2$, will lead to the determined temperatures being different and, hence, the model less consistent. Additional evidence that we have identified natural parameters of the non-equilibrium problem is provided by the Legendre transformations (2.30) and (2.33). This important point was originally made by Carter (1976) in a work that is not widely known.

(c) Thermal equilibrium

Since the main part of our discussion concerns the dynamics of systems out of equilibrium, and the comparison of different possible models, it makes sense to make a few comments on the state of equilibrium. As usual, we take thermal equilibrium to mean that there is no heat flux. Hence, we have $q^a = 0$ and the entropy is carried along with the matter. It then makes sense to introduce the equilibrium quantities $\rho$, $s$ and $\theta$, evaluated in the matter frame, in terms of which we recover the standard Gibbs relation:

$$d\rho = \mu \, dn + \theta \, ds.$$  
(2.35)

In equilibrium, both $n^a$ and $s^a$ are conserved, which means that

$$u^a_{;a} = -\dot{n} = -\dot{s},$$  
(2.36)

where

$$\dot{n} = u^a n_{;a},$$  
(2.37)

and similar for $\dot{s}$.

It is worth noting that if we introduce $S = s/n$, then we have

$$\dot{S} = 0.$$  
(2.38)

That is, the specific entropy remains constant in the matter frame.

By adding the momentum equations (2.16)–(2.17), we find that

$$(n\mu + s\theta) \dot{u} + h^b_a (n\mu_{;b} + s\theta_{;b}) = 0.$$  
(2.39)

Making use of the fundamental relation

$$P + \rho = n\mu + s\theta,$$  
(2.40)

where $P$ is the equilibrium pressure (to be distinguished from the generalized pressure $\Psi$ that is relevant also out of equilibrium), and

$$dP = n \, d\mu + s \, d\theta,$$  
(2.41)

we have

$$h^b_a [P_{;b} + (P + \rho) \dot{u}_b] = 0 \longrightarrow (P + \rho) \dot{u}^a = -h^b_a P_{;b}.$$  
(2.42)
As expected, we have the usual relation between the acceleration and the pressure gradient. Finally, the entropy momentum equation can be cast in the form (the easiest way to see this is to set $\Gamma_s = 0$ in the analysis that follows in the next section)

$$h^{ab}(\nabla_b \theta + \theta \dot{u}_b) = 0 \longrightarrow \theta \dot{u}^a = -h^b_a \theta,.$$  

Comparing these last two expressions, we see that we must have

$$\frac{1}{P + \rho} h^b_a P_{;b} = \frac{1}{\theta} h^b_a \theta,.$$  

In other words, temperature changes lead to pressure variations and vice versa.

### 3. The relativistic Cattaneo equation

It is well known that the classic thermodynamics description leads to non-causal heat conduction. This is obvious if we consider the general solution to the heat equation corresponding to given initial data in an unbounded region of space. Owing to the parabolic nature of the heat equation, any initial profile evolves to predict a non-zero temperature throughout space, even at arbitrarily early times. To remedy this problem, and impose a maximal propagation speed for heat, Cattaneo (1948) proposed a modification to Fourier’s Law such that the heat flux vector, $q$, is related to the temperature gradient according to

$$\tau q + q = -k \nabla T, \quad \text{with} \quad k \geq 0.$$  

This model incorporates the (finite) thermal relaxation time $\tau$ expected to arise from the fact that, on the micro-physical scale, heat propagates owing to particle collisions. Cattaneo’s equation leads to the temperature satisfying a telegraph-type equation whose hyperbolic nature leads to a finite propagation speed of heat pulses (Jou et al. 1993).

Since any relativistic model must encode causal heat propagation, one would expect the heat flux to be described by an equation similar to equation (3.1). The anticipated form for the relativistic analogue of Cattaneo’s equation is, indeed, generally agreed upon, but its derivation and the physical interpretation of the involved variables differ among proposed models. Our main aim is to obtain the relevant heat conduction equation within the variational approach, and compare the result with the second-order model of Israel & Stewart (1979b). In doing this, we will highlight the differences between the two models.

#### (a) The variational approach

We want to formulate a relativistic analogue of Cattaneo’s equation. The basic strategy will be to use the orthogonality of the entropy force density $f_s^a$ with the matter flux, solve for the entropy production rate $\Gamma_s$ and finally impose the second law of thermodynamics.

Let us first note that by making use of equation (2.28), we can express the entropy flux in terms of the matter flow and the heat flux:

$$s^a = s^s u^a + \frac{1}{\theta^s} q^a.$$  

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Moreover, the conjugate momentum becomes
\[ \theta_a = \theta^* u_a + \beta q_a, \]  
(3.3)
where \( \beta \) is given by
\[ \beta = \left( \frac{1}{s^*} + \frac{\theta^2 - \theta^*}{s^* \theta^*} \right) = \left( \frac{1}{s^*} - \frac{A^{\text{ns}} \rho}{s^* \theta^*} \right). \]
(3.4)

It is worth noting that if we set \( A^{\text{ns}} = 0 \), as in Carter’s regular model (Carter 1988), then \( \beta \) takes the specific value \( 1/s^* \).

In terms of the variables we have introduced, we have
\[ \theta^* s^*_a = 2\theta_{[a;b]} u^a q^b \frac{1}{\theta^*}, \]
(3.5)
which leads to the entropy creation rate
\[ \Gamma_s = s^*_a = -\frac{1}{\theta^* s^*} q^a (\theta^*_a + \theta^* \dot{u}_a - \beta q_{c;a} u^c + \beta \dot{q}_a + \beta c u^c q_a). \]
(3.6)

Dots represent time derivatives in the matter frame, as before. That is, \( \dot{u}^a \) is the four-acceleration and \( \dot{q}^a = u^b q^a_{,b} \)
represents the time variation of the heat flux.

The expression given in equation (3.6) has to satisfy the second law of thermodynamics. The simplest way to achieve this is to demand that the entropy production is a quadratic in the sources. This suggest that the heat flux takes the form
\[ q^a = -\kappa h^{ab} (\theta^*_b + \theta^* \dot{u}_b + 2\beta q_{[b;c]} u^c + \dot{\beta} q_b), \]
(3.8)
where \( \kappa > 0 \) is the thermal conductivity of the fluid. This leads to a relativistic analogue to Cattaneo’s equation, which can be written as
\[ 2\tau q^{[a;c]} u_c + q^a = -\tilde{\kappa} h^{ab} (\theta^*_b + \theta^* \dot{u}_b) \]
(3.9)
or
\[ \tau (\dot{q}^a + q_{c} u^{c;a}) + q^a = -\tilde{\kappa} h^{ab} (\theta^*_b + \theta^* \dot{u}_b), \]
(3.10)
where we have used the fact that \( u_c q^{c;a} = -q_c u^{c;a} \). Moreover, we have introduced the effective thermal conductivity \( \tilde{\kappa} \):
\[ \tilde{\kappa} \equiv \frac{\kappa}{1 + \kappa \beta}, \]
(3.11)
and the thermal relaxation time
\[ \tau = \frac{\kappa \beta}{1 + \kappa \beta}. \]
(3.12)

It is worth noting that, if \( \beta \) varies on a time scale \( \tau_\beta \) (say), which is long compared with the thermal relaxation, then
\[ \kappa \beta \sim \frac{\kappa \beta}{\tau_\beta} \approx \frac{\tau}{\tau_\beta} \ll 1. \]
(3.13)
One would expect this to be the case in most situations of practical interest. Thus, we would simply have

\[ \tilde{k} \approx k \quad \text{and} \quad \tau \approx k\beta. \tag{3.14} \]

(b) A different perspective

Before we proceed to compare equation (3.9) with the corresponding equation obtained within the Israel & Stewart (1979b) approach, it is worth considering the variational result from a slightly different point of view. As in the Newtonian problem (Andersson & Comer 2010), the entropy momentum equation provides an evolution equation for the heat flux. This equation can be written in a number of ways, some of which are helpful in interpreting the results.

As a first step, we recall that \( f^a_s \) is orthogonal to \( u^a \). By contracting equation (2.16) with \( u^a \) and using the result in equation (2.17), we arrive at an elegant expression for the force \( f^a_s \):

\[ -\theta^s f^a_s = 2u^c s_b(\theta^c_\cdot \theta^b_\cdot + \theta^b_\cdot c \theta^a_\cdot). \tag{3.15} \]

Written in this form, the force is clearly orthogonal to \( u^a \). Moreover, this expression emphasizes the relevance of the entropy momentum \( q^a \). However, if we want to gain insight into the key factors that contribute to the force, then we need to expand this expression. To do this, we contract equation (2.17) with equation (3.2) to get

\[ (s^a \theta_a) \Gamma_s = s^a f^a_s = \frac{1}{\theta^*} q^a f^a_s. \tag{3.16} \]

This shows that the entropy production only depends on the piece of the force \( f^a_s \) that is parallel to the heat flux. In general, we can decompose the entropy force into two terms:

\[ f^a_s = f^\parallel q^a + f^\perp q^a, \tag{3.17} \]

where both pieces are orthogonal to \( u^a \), and \( f^\perp a \) is also orthogonal to \( q^a \). From equation (3.16), it is obvious that \( f^\perp a \) cannot contribute to the entropy production. Hence, this term is not constrained by the second law. This is an important point because there is no obvious way to distinguish the viability of models with different forms of \( f^\perp a \). Given this, it is interesting to consider the specific force terms that arise in the variational formalism.

With the definitions above, it is straightforward to show that

\[ f^\parallel = -\frac{1}{\theta^*} \left( \frac{\beta}{\theta^*} - \frac{s^* \theta^*}{q^2} \right) q^b \left( \theta^*_{\cdot \cdot b} + \theta^* i_{\cdot b} + \beta_{\cdot b} u^c q_b + 2 \beta q\theta_{\cdot \cdot b} u^c \right). \tag{3.18} \]

Meanwhile, from equation (3.16), we have

\[ \Gamma_s = \frac{1}{s^a \theta_a} \frac{q^2}{\theta^*} f^\parallel, \tag{3.19} \]

where

\[ s^a \theta_a = s^* \theta^* + \frac{\beta}{\theta^*} q^2. \tag{3.20} \]
As expected, this takes us back to equation (3.6). Moving on to the non-entropy producing part of the force, the Ansatz (3.8) together with a projection orthogonal to $q^a$ leads to

$$f_a^\perp = -2s^a \theta^c h^b_a q^b \left[ \left( \beta q_b \right) ;c - \left( \beta q_c \right) ;b \right]$$

$$= -2s^a \theta^c h^b_a \frac{1}{c} \left[ q^2 \beta_{,b} + \frac{1}{2} \beta (q^2)_{,b} - \beta q^d q_{b,d} \right], \quad (3.21)$$

where

$$\perp^b_c = \delta^b_c - \frac{q^b q^c}{q^2}, \quad (3.22)$$

projects out the component that is orthogonal to $q^a$. We see that the variational approach leads to the presence of terms that, even though they involve the heat flux, are not associated with entropy production. As far as we are aware, the dynamical role of these terms has not been discussed in detail in the literature even though similar terms are (as we will soon see) also included in the Israel–Stewart formalism. The variational model leads to these terms taking a specific form. In particular, the terms in equation (3.21) are all quadratic in $q^a$, the deviation from equilibrium. At this order, the most general case would allow a force of form

$$f_a^\perp = h^c_a \perp^b_c \left[ A q^2 + B (q^2)_{,a} + C q^b q_{a,b} \right], \quad (3.23)$$

with $A$, $B$ and $C$ unspecified coefficients. There may also, in principle, be first-order terms. Clearly, equation (3.21) represents a particular case where all the coefficients follow from $\beta$. Hence, the variational model is a particular example of the general class of permissible theories. The fact that all these models satisfy the second law of thermodynamics means that we cannot express a preference at this point. A very interesting question concerns whether there are situations where $f_a^\perp$ has a distinguishable effect on the dynamics of the system. If one could show that this is the case, then we may be able to narrow down the possibilities.

(c) **Remarks on the role of the four-acceleration**

It is worth commenting on the presence of the term associated with the four-acceleration on the right-hand side of (3.10). As we will see later, this term has no counterpart in the Newtonian problem. However, its presence in the relativistic heat equation has been known since the pioneering work of Eckart (1940). Formally, this term originates from the local energy balance, equation (3.31). Physically, it results from the fact that the infinitesimal 3-spaces orthogonal to the matter world lines are not parallel, but relatively tipped over because of the curvature of the world line. This leads to the interpretation of the four-acceleration contribution in terms of the effective inertia of heat (Ehlers 1973). Interestingly, Garcia-Colin & Sandoval-Villalbazo (2006) and Garcia-Perciante *et al.* (2009) have recently suggested that this term may be the origin of instabilities. We will not discuss this suggestion in detail, but note that there are varying points of view in this area of research. Most researchers seem to accept that the four-acceleration term is both inevitable and physically meaningful.
The derivation of equation (3.9) was based on the thermal momentum equation (2.17). We could, in principle, also make use of the other momentum equation (2.16). Contracting this equation with \( s^a \), we have

\[
-q^2 f^\parallel = n q^a \left[ \mu \dot{u}_a + \mu_{;a} + q_a \dot{\alpha} + \frac{2\alpha}{n} u^b q_{[a;b]} \right],
\]

where

\[
\alpha = \frac{1 - \beta s^*}{n} = \frac{\mathcal{A}^{\text{NS}}}{\theta^*}.
\]

This gives us an expression for \( q^a \dot{u}_a \) that could be used in equation (3.18). We would then arrive at a different form for \( G_s \) and as a result, the Cattaneo equation will also be different. The four-acceleration term will now be replaced by the chemical potential gradient. After some algebra, we arrive at

\[
G_s = -\left( \rho^* + \Psi - \frac{\beta q^2}{\theta^*} \right)^{-1} n \mu \frac{\theta^*}{\theta^* - \mu} \left[ \theta^*_{;b} + \frac{\theta^*}{\mu} \mu_{;b} + \left( \beta + \frac{\theta^*}{\mu} \right) q_b + 2 \left( \beta - \frac{\theta^*}{\mu} \right) u^c q_{[c;b]} \right].
\]

We could impose the second law on this result, and hence derive an alternative form for the Cattaneo equation. From a modelling point of view, there is no reason why this form would be preferred to equation (3.9). However, it may still be useful since it provides a clearer link between the second law and the relevant thermodynamic ‘forces’. Conceptually, one would expect \( G_s \) to be expressed in terms of gradients of the various state variables. In this sense, the form of equation (3.26) may be more natural. We will discuss this issue in more detail elsewhere.

(d) The Israel–Stewart approach

The most successful approach to the problem of causal heat conductivity and dissipation in relativistic fluid dynamics is due to Israel and Stewart (Stewart 1977; Israel & Stewart 1979a,b). A detailed comparison between their results and the variational model has already been carried out by Priou (1991). The key results of this comparison are: (i) The two models differ only at second order in the deviation from equilibrium. (ii) The inclusion of entrainment, \( \mathcal{A}^{\text{NS}} \neq 0 \), is essential in the variational analysis. (iii) The two models belong to a larger class of non-equilibrium thermodynamics models. We will demonstrate these results in the particular case of pure heat conductivity. Priou’s analysis includes the viscosity contributions from Carter (1991), which makes many of the equations rather complex. The message is clearer if we focus on the heat conductivity problem, and we feel that it is important to make the comparison as transparent as possible.

In order to effect the comparison, it makes sense to begin by working through the derivation of the Israel–Stewart model for a heat conducting system. We focus on the phenomenological description, and simply note that the model has a firm foundation in relativistic kinetic theory. Hence, we take as the starting point the stress–energy tensor (Hiscock & Lindblom 1983):

\[
T^{ab} = \rho u^a u^b + (P + \tau) h^{ab} + 2u^{(a} q^{b)} + \tau^{ab}.
\]
The main difference from the variational model is that the thermodynamical quantities refer to an equilibrium state. As before, \( \rho \) is the energy density, \( P \) is the pressure and \( q^a \) is the heat flow orthogonal to the matter flow. Meanwhile, \( \tau^{ab} \) and \( \tau \) are the stresses caused by viscosity in the fluid. The tensor \( \tau^{ab} \) satisfies the relations

\[
0 = u^a \tau_{ab} = \tau^a_a = \tau_{[ab]}.
\]

Motivated by kinetic theory, Israel and Stewart expand the entropy flux to include a complete set of second-order terms

\[
s_I^a = su^a + \frac{q^a}{T} - \frac{1}{2}(\beta_0 \tau^2 + \beta_1 q^b q_b + \beta_2 \tau_{bc} \tau^{bc}) \frac{u^a}{T} + \alpha_0 \frac{T q^a}{T} + \alpha_1 \frac{T^a b q^b}{T}. \tag{3.29}
\]

Here, \( T \) is the absolute temperature associated with the equilibrium state, and the coefficients \( \beta_0, \beta_1, \beta_2, \alpha_0 \) and \( \alpha_1 \) correspond to different couplings that need to be provided (i.e. obtained from the microphysics of the problem). By neglecting all viscosity contributions, we are left only with the \( \beta_1 \) term, and the entropy flux (3.29) reduces to

\[
s_I^a = su^a + \frac{1}{T} q^a - \frac{\beta_1}{2T} q^2 u^a. \tag{3.30}
\]

From this definition, it is straightforward to calculate the divergence of \( s_I^a \) to impose the second law. From the equations of motion, we also have

\[
u_b T^{ab}_{;a} = -\dot{\rho} - (\rho + P) u^a_{;a} - q^a_{;a} + \dot{q}_b u^b = 0. \tag{3.31}
\]

We now use the fundamental relation of thermodynamics in the equilibrium form

\[
P + \rho = \mu n + sT, \tag{3.32}
\]

together with the equilibrium equation of state, \( \rho = \rho(n,s) \), which leads to

\[
\rho_{;a} = \mu n_{;a} + T s_{;a}. \tag{3.33}
\]

We also need to use the fact that the flux \( n^a \) is conserved. Combining these results, we arrive at

\[
s_I^{a}_{;a} = -\frac{1}{T^2} q^b \left[ T_{;b} + T \dot{u}_b + T \beta_1 \dot{q}_b + \left( \frac{\beta_1}{2T} u^a \right)_{;a} T^2 q_b \right]. \tag{3.34}
\]

As before, we need to ensure that the entropy production is positive. This follows as long as

\[
q^a = -\kappa T h^{ab} \left[ \frac{1}{T} T_{;b} + \dot{u}_b + \beta_1 \dot{q}_b + \left( \frac{\beta_1 u^c}{T} \right)_{;c} \frac{T}{2} q_b \right]. \tag{3.35}
\]

This relation provides the Israel–Stewart version of the relativistic Cattaneo equation.
As in the variational case, one may add other terms to the heat flux as long as they do not lead to the generation of entropy. As an example, Hiscock & Lindblom (1983) included the term
\[-\gamma_2 \hat{\kappa} T h^{ab} q^c u_{[c;b]},\]  
the presence of which is motivated by kinetic theory. This term is clearly orthogonal to both \( q^a \) and \( u^a \) and, hence, does not affect \( \Gamma_s \). Technically, it is also a second-order term if the equilibrium state is uniform.

(e) The extended irreversible thermodynamics view

We have reached the point where we can compare the final equations for the heat flux from the variational approach, equation (3.9), to the Israel–Stewart model, equation (3.35). However, before we do this, let us consider the problem from the ‘extended irreversible thermodynamics’ point of view. This approach, which was first developed by Carter (1976), provides an immediate illustration of the fundamental difference between the two models that we are discussing.

As in the variational analysis, we take the heat flux to be given by
\[ s^a = s^* u^a + \frac{1}{\theta^*} q^a. \]  
The corresponding stress–energy tensor is
\[ T^{ab} = \rho^* u^a u^b + 2 u^a (q^b) + \Psi h^{ab} + \sigma^a p^a, \]  
where \( \Psi \) denotes the (generalized) pressure, and the variation of the energy density (in the matter frame) is
\[ d\rho^* = \mu dn + \theta^* ds^* + \sigma dp. \]  
Obviously, we are now referring to out-of-equilibrium quantities. Hence, we use an extended Gibbs relation that accounts for the heat flux. This immediately leads to
\[ \rho^*;_a = \mu n;_a + \theta^* s^*;_a + \frac{q}{\theta^*} (\beta q);_a. \]  

Using these relations and the particle conservation, we find that
\[ u_b T^{ab};_a = (n \mu - \rho^* - \Psi) \Theta - \theta^* s^* - \sigma \dot{p} + q_{b,a} u^b u^a - q^a_{,a} + p_{,a}^b \sigma^a u_b. \]  
At this point, we need the fundamental relation (2.32), which allows us to complete the derivation. We arrive at
\[ s^a;_a = -\frac{1}{\theta^*} q^b [\theta^*;_b + \theta^* u_b - \dot{q}_b \beta + (\beta q)_b; u^c], \]  
which is identical to equation (3.6). Hence, the heat flux will (again) be governed by equation (3.9).

(f) Comparing the results

Since the two approaches are based on different strategies, any comparison between the variational model and the Israel–Stewart results must be done carefully. Notably, the Israel–Stewart model is based on an expansion including terms up to second order in the deviation from equilibrium in the entropy flux.
Meanwhile, the variational analysis did not involve such an expansion. As a result, the final equation for the heat flux, equation (3.9), contains higher order terms while equation (3.35) is manifestly linear in $q^a$.

In order to compare the results, it makes sense to focus on the linear deviation from equilibrium. Then, we have

$$ \theta^* = T + O(q^2), $$

(3.43)

and it follows that equation (3.10) can be approximated by

$$ \tau(q^a + q_c u^{(a)}_{(c)}) + q^a \approx -\tilde{k} h^{ab}(T_{:b} + T_{:\dot{u}_b}). $$

(3.44)

Here it is worth noting that in the second term in the first bracket, we could use the standard decomposition of $u_{a:b}$ in terms of the shear, expansion etc. This would lead to terms that were explicitly excluded from the Israel–Stewart model at the point where we focused on the case with $\tau_{ab} = \tau = 0$. Basically, the variational analysis leads to the presence of terms that couple the heat flux to the shear and expansion of the flow. As these were artificially excluded from the analysis that led to equation (3.35), we cannot count this as a difference between the two models. In fact, the full comparison carried out by Priou show that these terms agree in the two descriptions.

Keeping terms up to second order (treating the shear and the divergence of $u^a$ as first-order quantities), equation (3.35) can be written as

$$ \tau_I \dot{q}^a + q^a \approx -\hat{k}_I h^{ab}(T_{:b} + T_{:\dot{u}_b}), $$

(3.45)

where

$$ \tau_I = \beta_1 \hat{k} T \left[ 1 + u^c \left( \frac{\beta_1}{T} \right) \frac{\hat{k} T^2}{2} \right]^{-1}, $$

(3.46)

and

$$ \hat{k}_I = \hat{k} \left[ 1 + u^c \left( \frac{\beta_1}{T} \right) \frac{\hat{k} T^2}{2} \right]^{-1}. $$

(3.47)

Now it is clear that the two equations for the heat flux are formally identical, and we can ‘identify’ the parameters in the two models. The upshot of this is that the models will only produce different results at higher order deviations from equilibrium. Given that this regime is hardly tested at all, we cannot at this stage comment on which of the two descriptions (if either) may be the most appropriate. Having said that, it is clear that the variational approach is formally elegant and the fact that it applies also far from equilibrium (at least in principle) may be relevant. An interesting question concerns whether there are situations where the, rather specific, set of higher order terms predicted by the variational analysis affect the nonlinear dynamics.

Before we conclude the comparison, it is worth noting that the difference between the two models was apparent already from the beginning. This is clear if we consider the stress–energy tensor. In the variational case, we have

$$ T_{ab} = \rho^* u_a u_b + 2q_{(a} u_{b)} + \Psi h_{ab} + p_a \sigma_a. $$

(3.48)
Comparing this to equation (3.27), we see that

$$\tau = \frac{1}{3} (s^*)^2 B^w w^2$$

(3.49)

and

$$\tau_{ab} = 3 (s^*)^2 B^w (w_a w_b - h_{ab} w^2).$$

(3.50)

These terms are quadratic in $w^a$ (that is, $q^a$). Hence, it is obvious that, in order to carry out detailed a comparison, we ought to include also shear- and bulk viscosity in the model, cf. the analysis of Priou (1991).

4. The Newtonian limit

Having developed a model for heat conductivity in general relativity, and discussed the results in the context of the well-established Israel–Stewart model, we should also consider the implications of the model for non-relativistic systems. The close connection between the variational multi-fluid approach in Newtonian gravity and extended irreversible thermodynamics has already been discussed by Andersson & Comer (2010). Their results demonstrate that a two-fluid model based on a massive component representing the particles and a massless component representing the entropy reproduces many key results from the literature (Jou et al. 1993). In particular, the non-relativistic Cattaneo equation is obtained immediately from the momentum conservation law for the entropy. The non-relativistic analysis is, in fact, completely analogous to the discussion in this paper. To demonstrate this, and relate the two models, we will now work out the Newtonian, low-velocity, weak gravity, limit of our main equations.

Let us first return to the variation of the master function $A$. Assuming low velocities, we have

$$j^2 = -n^a s_a \approx s n \left(1 + \frac{w^2}{2c^2}\right),$$

(4.1)

where $w^2 = w_{ns}^2$ represents the (squared) magnitude of the relative velocity between matter and entropy ($w_{ns}^i = v_{ni} - v_{si}$). Note that we need to keep the speed of light, $c$, explicit in this discussion. This then leads to

$$dj^2 \approx s \left(1 + \frac{w^2}{2c^2}\right) dn + n \left(1 + \frac{w^2}{2c^2}\right) ds + \frac{sn}{2c^2} dw^2,$$

(4.2)

and

$$dA \approx - \left[ n B^n + s \left(1 + \frac{w^2}{2c^2}\right) A_{ns} \right] dn - \left[ s B^s + n \left(1 + \frac{w^2}{2c^2}\right) A_{ns} \right] ds - \frac{sn}{2c^2} A_{ns} dw^2.$$

(4.3)

To make progress it is essential to appreciate that the Newtonian limit is singular, see for example the rigorous analysis of Carter & Chamel (2004). In order to effect a direct ‘calculation’, it is useful to separate the ‘ballistic’ rest-mass
contribution to the master function. That is, we use (recalling that the entropy is taken to be massless)

\[ \Lambda = -mnc^2 - E(n, s, w^2). \] (4.4)

From the above results, it follows that

\[
\left. \frac{\partial \Lambda}{\partial n} \right|_{s, u^2} \approx - \left[ nB^n + s \left( 1 + \frac{w^2}{2c^2} \right) A^{ns} \right] = -mc^2 - \mu, \] (4.5)

defining the chemical potential, \( \mu \), and

\[
\left. \frac{\partial \Lambda}{\partial s} \right|_{n, w^2} \approx - \left[ sB^s + n \left( 1 + \frac{w^2}{2c^2} \right) A^{ns} \right] = T, \] (4.6)

which defines the temperature \( T \) (as the entropy chemical potential). Finally,

\[
\left. \frac{\partial \Lambda}{\partial w^2} \right|_{n, s} \approx - \frac{s}{2c^2} A^{ns} = - \frac{\partial E}{\partial w^2} \bigg|_{n, s} = \alpha, \] (4.7)

defines the Newtonian entropy entrainment parameter \( \alpha \) (Prix 2004). These three relations allow us to express \( B^n, B^s \) and \( A^{ns} \) in terms of the Newtonian coefficients \( \mu, T \) and \( \alpha \).

We also need the weak field spacetime metric. To the required order, we have the line element

\[
ds^2 \approx -c^2 \left( 1 + \frac{2\Phi}{c^2} \right) dt^2 + g_{ij} dx^i dx^j, \] (4.8)

where \( \Phi \) is the gravitational potential and \( g_{ij} \) is the flat space metric. Meanwhile, the four velocities have components

\[
u^n_\tau = 1 - \frac{\Phi}{c^2} + \frac{v^2_n}{2c^2}, \quad u^n_i = v^n_i, \] (4.9)

and

\[
u^s_\tau = 1 - \frac{\Phi}{c^2} + \frac{v^2_s}{2c^2}, \quad u^s_i = v^s_i. \] (4.10)

This means that the two fluxes become

\[
n^a = \frac{nu^n_\tau}{c} \rightarrow \begin{cases} n^0 = \frac{n}{c} \left( 1 - \frac{\Phi}{c^2} + \frac{v^2_n}{2c^2} \right), & n^i = \frac{nv^n_i}{c}, \\
n_0 = -nc \left( 1 - \frac{\Phi}{c^2} + \frac{v^2_s}{2c^2} \right), & n_i = \frac{nv^s_i}{c}. \end{cases} \] (4.11)

Using the different expressions in the momentum equations, keeping only terms of order unity, we obtain the Newtonian equations of motion. For the particles we then find

\[ n^a \mu^n_{\nu, a} \approx n(\partial_t + v^n_i \nabla_j) \left( mv^n_i + \frac{2\alpha}{n} w^{sn}_i \right) \] (4.12)
and
\[ n^a \mu_{a:b} \approx -n \nabla_i (\mu + m\Phi) - 2\alpha w_{sn}^j \nabla_i v^j_n. \] (4.13)

This means that the relevant momentum equation becomes
\[ n(\partial_t + v^j_n \nabla_j) \left( m v_i^n + \frac{2\alpha}{n} w_i^{sn} \right) + n \nabla_i (\mu + m\Phi) + 2\alpha w_{sn}^j \nabla_i v^j_n = 0. \] (4.14)

For the massless entropy, it is easy to see that we get
\[ s^a \mu_{b:a} \approx -s \nabla_i T - 2\alpha w_{ns}^j \nabla_i v^s_j, \] (4.16)

and the final momentum equation becomes
\[ s(\partial_t + v^j_s \nabla_j) \left( \frac{2\alpha}{s} w_i^{ns} \right) + s \nabla_i T + 2\alpha w_{ns}^j \nabla_i v^s_j = 0. \] (4.17)

In order to facilitate a direct comparison with the discussion by Andersson & Comer (2010), we now use the entropy conservation law
\[ s^a \approx \partial_i s + \nabla_j (s v^j_n) = \Gamma_s, \] (4.18)

and define
\[ \pi^s_i = 2\alpha w_i^{ns}. \] (4.19)

Then equation (4.17) can be written as
\[ \partial_i \pi^s_i + \nabla_j (v^j_n \pi^s_i) + s \nabla_i T + \pi^s_i \nabla_i v^j_n = \frac{\Gamma_s}{s} \pi^s_i = f^s_i, \] (4.20)
or
\[ (\partial_t + v^j_n) \pi^s_i - \nabla_j \left( \frac{\pi^j_s \pi^s_i}{2\alpha} \right) - \pi^s_i \nabla_i \left( \frac{\pi^s_j}{2\alpha} \right) + s \nabla_i T + \pi^s_j \nabla_i v^j_n + \pi^s_i \nabla_j v^j_n = \frac{\Gamma_s}{s} \pi^s_i. \] (4.21)

We have arrived at the Newtonian two-fluid point model that was taken as the starting point by Andersson & Comer (2010). In other words, their Newtonian model is the natural non-relativistic counterpart to the model developed in §2. That this had to be the case was, more or less, obvious given the discussion by Andersson & Comer (2007), but it is still useful to have a direct comparison. In particular, the derivation shows explicitly that the four-acceleration term in equation (3.9) is a purely relativistic effect. The comparison also clarifies the physical interpretation of the different variables, and the meaning of the various parameters.

5. Discussion

We have discussed a relativistic model for heat conduction. The model builds on the convective variational approach to multi-fluid systems designed by Carter (1989), and focuses on the role of the entropy as a dynamic entity. The model
assumes that the entropy can be treated as a fluid distinct from the matter flow. We have demonstrated how this approach leads to a relativistic version of the Cattaneo equation, encoding the thermal relaxation time that is needed to satisfy causality. Moreover, we have shown that the model naturally includes the non-equilibrium Gibbs relation that is a key ingredient in most approaches to extended thermodynamics. By focusing on the pure heat conduction problem, neglecting other ‘dissipation channels’, we compared the variational results with the celebrated second-order model developed by Israel & Stewart (1979b). The comparison showed that, despite the very different philosophies behind the two approaches, the two models are equivalent at first-order deviations from thermal equilibrium. This was not surprising. In fact, Priou (1991) has already carried out a similar comparison of the corresponding second-order models (including viscosity). His results show that the two models contain the same key elements, and that they belong to a wider class of permissible models. The simpler context of our analysis serves to clarify the main points. Finally, we worked out the non-relativistic limit of our results, making contact with the recent work of Andersson & Comer (2010). This essentially completes the picture, and we now have a consistent framework for discussing causal heat conductivity in both Newtonian and relativistic dynamics.

Our discussion obviously only scratched the surface of what is a very difficult problem. We did not address foundational issues concerning the link between this kind of phenomenological model (e.g. the entropy fluid) and the relevant microphysics/statistical physics. This is a rich and challenging area, where many issues remain to be resolved, and one can imagine several interesting directions in which the current work may be developed. This work may also be applied in a number of exciting contexts, ranging from high-energy collisions probed by, for example, RHIC and the LHC, to neutron star dynamics and issues relevant for multi-messenger astronomy, and even cosmology and the evolution of the Universe itself.

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References

Andersson, N. 2003 Gravitational waves from instabilities in relativistic stars. *Class. Quantum Grav.* **20**, R105–R144. (doi:10.1088/0264-9381/20/7/201)

Andersson, N. & Comer, G. L. 2007 Relativistic fluid dynamics: physics for many different scales. *Living Rev. Relativity* **10**. (http://arxiv.org/abs/gr-qc/0605010v2)

Andersson, N. & Comer, G. L. 2010 Variational multi-fluid dynamics and causal heat conductivity. *Proc. R. Soc. A* **466**, 1373–1387. (doi:10.1098/rspa.2009.0423)

Carter, B. 1976 Regular and anomalous heat conduction: the canonical diffusion equation in relativistic thermodynamics, *Journées Relativistes* (eds M. Cahen, R. Debever & J. Geheniau), pp. 12–27. Université Libre de Bruxelles.

Carter, B. 1988 Conductivity with causality in relativistic hydrodynamics: the regular solution to Eckart’s problem. In *Highlights in gravitation and cosmology* (eds B. R. Iyer, A. Kembhavi, J. V. Narlikar & C. V. Vishveshwara). Cambridge, UK: Cambridge University Press.

Carter, B. 1989 Covariant theory of conductivity in ideal fluid or solid media. In *Relativistic fluid dynamics*, vol. 1385 (eds A. Anile & M. Choquet-Bruhat), Springer Lecture Notes in Mathematics, pp. 1–64. Heidelberg, Germany: Springer.

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Carter, B. 1991 Convective variational approach to relativistic thermodynamics of dissipative fluids. Proc. R. Soc. Lond. A 433, 45–62. (doi:10.1098/rspa.1991.0034)

Carter, B. & Chamel, N. 2004 Covariant analysis of Newtonian multi-fluid models for neutron stars I. Int. J. Mod. Phys. D 13, 291–325. (doi:10.1142/S0218271804004542)

Carter, B. & Quintana, H. 1972 Foundations of general relativistic high-pressure elasticity theory. Proc. R. Soc. Lond. A 331, 57–83. (doi:10.1142/S0218271804004542)

Cattaneo, C. 1948 Sulla conduzione del calore. Atti Semin. Mat. Fis. Univ. Modena 3, 83–101.

Comer, G. L. & Joynt, R. 2003 Relativistic mean field model for entrainment in general relativistic superfluid neutron stars. Phys. Rev. D 68, 023002. (doi:10.1103/PhysRevD.68.023002)

Eckart, C. 1940 The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid. Phys. Rev. 58, 919–924. (doi:10.1103/PhysRev.58.919)

Ehlers, J. 1973 Survey of general relativity theory in relativity, astrophysics and cosmology. In Astrophysics and space science library, vol. 38 (ed. W. Israel), pp. 1–125. Dordrecht, The Netherlands: Reidel.

Elze, H.-Th., Rafelski, J. & Turko, L. 2001 Entropy production in relativistic hydrodynamics. Phys. Lett. A 506, 123–130. (doi:10.1016/S0370-2693(01)00378-1)

Garcia-Colín, L. S. & Sandoval-Villalbazo, A. 2006 Relativistic non-equilibrium thermodynamics revisited. J. Non-Equilibrium Thermodyn. 31, 11–22. (doi:10.1515/JNETDY.2006.002)

Garcia-Perciante, A. L., Garcia-Colín, L. S. & Sandoval-Villalbazo, A. 2009 On the nature of the so-called generic instabilities in dissipative relativistic hydrodynamics. Gen. Rel. Grav. 41, 1645–1654. (doi:10.1007/s10714-008-0735-0)

Geroch, R. 1995 Relativistic theories of dissipative fluids. J. Math. Phys. 36, 4226–4242. (doi:10.1063/1.530958)

Hiscock, W. A. & Lindblom, L. 1983 Stability and causality in dissipative relativistic fluids. Ann. Phys. 151, 466–496. (doi:10.1016/0003-4916(83)90288-9)

Hiscock, W. A. & Lindblom, L. 1985 Generic instabilities in first-order dissipative relativistic fluid theories. Phys. Rev. D 31, 725–733. (doi:10.1103/PhysRevD.31.725)

Hiscock, W. A. & Lindblom, L. 1988 Nonlinear pathologies in relativistic heat-conducting fluid theories. Phys. Lett. A 131, 509–513. (doi:10.1016/0375-9601(88)90679-2)

Israel, W. & Stewart, J. M. 1979a On transient relativistic thermodynamics and kinetic theory II. Proc. R. Soc. Lond. A 365, 43–52. (doi:10.1098/rspa.1979.0005)

Israel, W. & Stewart, J. M. 1979b Transient relativistic thermodynamics and kinetic theory. Ann. Phys. 118, 341–372. (doi:10.1002/9783527617494.ch16)

Jou, D., Casas-Vázquez, J. & Lebon, G. 1993 Extended irreversible thermodynamics. Berlin, Germany: Springer.

Landau, L. D. & Lifshitz, E. M. 1959 Fluid mechanics. Oxford, UK: Butterworth Heinemann.

Landsberg, P. T. 1967 Does a moving body appear cool? Nature 214, 903–904. (doi:10.1038/214903a0)

Lindblom, L. 1996 The relaxation effects in dissipative relativistic fluid theories. Ann. Phys. 247, 1–18. (doi:10.1016/aphy.1996.0036)

Muronga, A. 2004 Causal theories of dissipative relativistic fluid dynamics for nuclear collisions. Phys. Rev. C 69, 034903. (doi:10.1103/PhysRevC.69.034903)

Olson, T. S. & Hiscock, W. A. 1990 Stability, causality, and hyperbolicity in Carter’s ‘regular’ theory of relativistic heat-conducting fluids. Phys. Rev. D 41, 3687–3695. (doi:10.1103/PhysRevD.41.3687)

Priou, D. 1991 Comparison between variational and traditional approaches to relativistic thermodynamics of dissipative fluids. Phys. Rev. D 43, 1223. (doi:10.1103/PhysRevD.43.1223)

Prix, R. 2004 Variational description of multifluid hydrodynamics: uncharged fluids. Phys. Rev. D 69, 043001. (doi:10.1103/PhysRevD.69.043001)

Stewart, J. M. 1977 On transient relativistic thermodynamics and kinetic theory. Proc. R. Soc. Lond. A 357, 59–75. (doi:10.1098/rspa.1977.01550)