MATRIX INEQUALITY FOR THE LAPLACE EQUATION

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ABSTRACT. Since Li and Yau obtained the gradient estimate for the heat equation, related estimates have been extensively studied. With additional curvature assumptions, matrix estimates that generalize such estimates have been discovered for various time-dependent settings, including the heat equation on a Kähler manifold, Ricci flow, Kähler-Ricci flow, and mean curvature flow, to name a few. As an elliptic analogue, Colding proved a sharp gradient estimate for the Green function on a manifold with nonnegative Ricci curvature. In this paper we prove a related matrix inequality on manifolds with suitable curvature and volume growth assumptions.

1. INTRODUCTION

In the seminal paper [12], Li and Yau proved a sharp estimate for the gradient of the heat kernel on a complete Riemannian manifold with Ricci curvature bounded below. It leads to a Harnack inequality on such manifolds by integration along shortest geodesics, which is sometimes referred to as a differential Harnack inequality. Later Hamilton [9] discovered a time-dependent matrix quantity that stays positive-semidefinite at all time, in the case that the manifold has nonnegative sectional curvature and parallel Ricci curvature. Taking the trace of this matrix inequality yields the Li-Yau gradient estimate.

Matrix estimates have also been developed for other settings, such as the heat equation on Kähler manifolds with nonnegative holomorphic bisectional curvature by L. Ni and H. D. Cao [2], Ricci flow by Hamilton [8], Kähler-Ricci flow by L. Ni [13], and mean curvature flow by Hamilton [10], to name a few. There are close connections between such Harnack estimates and entropy formulae, as illustrated in the excellent survey by L. Ni [14].

As an elliptic setting parallel to the aforementioned time-dependent results, Colding [3] obtained a sharp gradient estimate for the minimal positive Green function for the Laplace equation under the relatively mild assumption of nonnegative Ricci curvature. This estimate is closely related to monotonicity formulae for manifolds with Ricci curvature bounded below; for details we refer to [3], [4], and [5]. Such monotone quantities turn out to be extremely useful, as they control the distance to the nearest cone of the manifold and are used to prove the uniqueness of the tangent cone for Einstein manifolds [6]. In this paper, we show that there exists a related matrix inequality.

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Theorem 1.1. Let \((M^n, g)\) be a complete non-compact Riemannian manifold of Euclidean volume growth and dimension \(n \geq 3\). Let \(G\) be the minimal positive Green function with pole at \(x \in M\) and \(b = G^{\frac{1}{2-n}}\). Suppose that \(M\) has nonnegative sectional curvature along \(\nabla G\) and parallel Ricci curvature. If \(\text{Hess}_b\) is uniformly bounded from above on \(M \setminus \{x\}\) and asymptotically bounded from above by \(Cg\) at \(x\) where \(C \geq 10\), then \(\text{Hess}_b \leq Cg\) holds everywhere on \(M \setminus \{x\}\).

To motivate the above theorem, suppose for a moment that \(M = \mathbb{R}^n\). The minimal positive Green function \(G\) with a pole at the origin is given by \(C(n) \cdot r^{2-n}\), where \(r\) is the distance from the origin and \(C(n)\) is a dimensional constant. We observe that the first and the second order derivatives of \(G\) satisfies the following relation, which motivates a bound on the Hessian of \(G\).

\[
G_{ij} + \frac{n}{2 - n} \cdot \frac{G_i G_j}{G} = (2 - n)G^{\frac{n}{2-n}} \delta_{ij}.
\]

Another motivation, which we describe here, comes from the Hessian comparison theorem for radial functions. Let \((M^n, g)\) be a complete non-compact Riemannian manifold. Fixing a point \(x\), \(M\) is called parabolic if it does not admit a positive Green function for the Laplacian with pole at \(x\). It is called non-parabolic otherwise. If \(M\) has nonnegative Ricci curvature, a result of Varopoulos [15] states that \(M\) is non-parabolic if and only if \(\int_0^\infty \frac{1}{\text{Vol}(B_x(t))} dt < \infty\) for any positive \(s\). \(M\) is said to have Euclidean volume growth if for some \(c > 0\) it holds that \(\text{Vol}(B_x(t)) \geq c \cdot t^n\) for any \(t > 0\). The result of Varopoulos implies that a manifold of dimension \(\geq 3\) and Euclidean volume growth is non-parabolic. On such \(M\), combining the results of Li and Tam [11] and Gilbarg and Serrin [7], we see that there exists a unique minimal positive symmetric Green function \(G = G(x, y)\) such that \(G(x, y) = G_x(y) = O(r^{2-n})\), where \(r\) is the distance from \(x\). Hence, a bound on the Hessian of \(G\) would be a natural analogue for the Hessian comparison theorem. From now on, we normalize \(G\) suitably so that \(G = r^{2-n}\) on the Euclidean space \(M = \mathbb{R}^n\). Then we can define a function \(b\) as follows.

\[
b := G^{\frac{1}{2-n}}.
\]

Then \(b\) corresponds to just \(r\), and therefore might be more intuitive than \(G\). On the Euclidean space \(\mathbb{R}^n\), the Hessian of \(b^2 = r^2\) satisfies the following equation.

\[
\text{Hess}_b = 2g.
\]

The Hessian comparison theorem would suggest a result in the direction of \(\text{Hess}_b \leq Cg\) with \(2 \leq C\), so we ask under which conditions on \(M\) such a bound could be obtained. It turns out that if \(M\) has nonnegative sectional curvature along \(\nabla G\) and parallel Ricci curvature, and if \(\text{Hess}_b\) is bounded locally near \(x\).
and also arbitrarily far out, then the bound extends globally to the region in between. The precise meaning of the last two conditions is the following. Suppose that $\text{Hess}_{b^2} \leq C g$ in the neighborhood of $x$ with the constant $C$ as in the theorem below. Then only one of the two cases can happen: either $\text{Hess}_{b^2} \leq C g$ on all of $M \setminus \{x\}$, or $\text{Hess}_{b^2}$ diverges as $r \to \infty$. Such is the content of Theorem 1.1.

**Remark 1.2.** The two curvature assumptions in Theorem 1.1 are critical in the proof, and were also taken in [9]. Note, however, that all of the arguments in the proof of Theorem 1.1 can readily be generalized to the case where sectional curvature is bounded from below by $-K \cdot G^{-\frac{n-2}{2}}$ and the first derivative of Ricci curvature is bounded as $|\nabla_i R_{jk}| \leq L \cdot G^{-\frac{3}{2} - n}$. Then we obtain an upper bound of $\text{Hess}_{b^2}$ in terms of $n, K, L$. It would be an interesting question to ask whether a similar inequality holds under scale-invariant curvature assumptions with $r$ instead of $G$, i.e. under the assumptions that the sectional curvature is bounded from below by $-K \cdot r^{-\frac{n-2}{2}}$ and the first derivative of Ricci curvature is bounded as $|\nabla_i R_{jk}| \leq L \cdot r^{-\frac{3}{2} - n}$.

As a corollary we obtain a Harnack inequality for $b$. Let $y, z \in M \setminus \{x\}$ and consider a minimal geodesic segment $yz$ parametrized by arclength $s$. Then under the assumptions of Theorem 1.1, the function $\frac{C}{2} s^2 - b^2$ is convex. Hence we obtain the following corollary.

**Corollary 1.3.** Under the assumptions of Theorem 1.1, let $w$ be the point on a minimal geodesic $yz$ such that $d(y, w) = \lambda \cdot d(y, z)$ and $d(w, z) = (1 - \lambda) \cdot d(y, z)$, $0 \leq \lambda \leq 1$. Then

\[
\begin{align*}
\frac{C}{2} s^2 - b^2
\end{align*}
\]

Note that the corollary holds whether $b(y) \neq b(z)$ or not. This compares with the fact that integrating an estimate on the scalar quantity $|\nabla b|$ can only compare values of $b$ between two points on different level sets of $b$. Also, note that in the above corollary, the Euclidean space $M = \mathbb{R}^n$ achieves the equality with $C = 2$.

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### 2. Proof of the matrix inequality

**Notation.** For the convention of the curvature tensor, we use

\[
R(X, Y, Z, W) = g(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, W).
\]

In an orthonormal frame $\{e_i\}$ we write in coordinates that $R(e_i, e_j, e_k, e_l) = R_{ijkl}$. The Ricci curvature is defined as $Ric(X, Y) = \sum_k R(X, e_k, Y, e_k)$ and denoted in coordinates as $R_{ij} = Ric(e_i, e_j)$. Repeated indices are understood as summations, unless otherwise specified. $\nabla_{e_i}$ will often be abbreviated by $\nabla_i$. 
In this section we present the proof of Theorem 1.1. The main tool is the maximum principle introduced by Calabi in [1], which we recall below.

**Definition 2.1.** Let $X$ be a Riemannian manifold, $x_0 \in X$, and $\varphi : X \to \mathbb{R}$ a continuous function. We say that $\Delta \varphi \leq 0$ at $x_0$ in barrier sense if for any $\varepsilon > 0$, there is a $C^2$ function $\psi_{x_0,\varepsilon}$ on a neighborhood of $x_0$ such that $\psi_{x_0,\varepsilon}(x_0) = \varphi(x_0)$, $\Delta \psi_{x_0,\varepsilon} < \varepsilon$, and $\psi_{x_0,\varepsilon} \geq \varphi$. We say that $\Delta \varphi \leq 0$ in barrier sense if $\Delta \varphi \leq 0$ at $x_0$ in barrier sense for all $x_0 \in X$.

**Lemma 2.2** (Maximum principle for barrier subsolutions). If $\Delta \varphi \leq 0$ in barrier sense, then either $\varphi$ is constant or $\varphi$ has no weak local minimum.

Define a tensor $H$ as the following, motivated by the fact that it vanishes on the Euclidean space.

$$H = \text{Hess}_G + \frac{n}{2-n} \cdot \frac{\nabla G \otimes \nabla G}{G} + (n-2) \cdot G^\frac{n}{2} g.$$ 

By a straightforward computation, it follows that $\text{Hess}_{b^2} = -\frac{n-2}{n-2} G^\frac{n}{2-n} H + 2g$. Hence, the assumption that $\text{Hess}_{b^2} \leq Dg$ is equivalent to $0 \leq H + \frac{n-2}{n-2} (C - 2) G^\alpha g$.

Let $\alpha = -\frac{n}{2-n} = \frac{n}{n-2}$, so that our goal is to show that $0 \leq H + \frac{n-2}{n-2} (C - 2) G^\alpha g$.

For convenience call this tensor $\tilde{H}$,

$$\tilde{H} := H + \frac{n-2}{2} (C - 2) G^\alpha g = \text{Hess}_G + \frac{n}{2-n} \cdot \frac{\nabla G \otimes \nabla G}{G} + \frac{n-2}{2} C \cdot G^\alpha g.$$ 

We also define the function $\Lambda$ on $M \setminus \{x\}$ to be the lowest eigenvalue of $\tilde{H}$,

$$\Lambda(p) := \min_{V \in T_p M, g(V,V) = 1} \tilde{H}(V,V).$$ 

Then $\Lambda$ is continuous, and the assumption that $\text{Hess}_{b^2} \leq Dg$ implies that $\frac{n-2}{2} (C - D) G^\alpha \leq \Lambda$.

An ingredient we will need is the following lemma. Let $p \in M \setminus \{x\}$ and let $\{e_i\}$ be a normal frame at $p$, i.e. $g_{ij}(p) = \delta_{ij}$ and $\nabla_j e_i(p) = 0$ for any $i,j$. Denote $\tilde{H}_{ij} = \tilde{H}(e_i, e_j)$. For convenience, define the tensor $B$ as $B = \frac{\nabla G \otimes \nabla G}{G}$, or in coordinates as $B_{ij} = \frac{G_{ij}}{G}$. Then $B$ is positive-semidefinite with eigenvalues $\frac{\lvert \nabla G \rvert^2}{G}$ and $0$. We compute the following quantity in a straightforward manner.

**Lemma 2.3.** The following holds at $p$.

$$\Delta(\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij})$$

$$= R_{ik} \tilde{H}_{jk} + R_{jk} \tilde{H}_{ik} - 2 R_{ikjl} \tilde{H}_{kl} - \frac{2n}{n-2} R_{ikjl} \frac{G_{ij} G_{kl}}{G} - \frac{2n}{(n-2)G} \tilde{H}_{ij}^2$$

$$- \frac{n(n-2)}{2} C^2 G^{2\alpha - 1} g_{ij} + \frac{4n}{n-2} \left[ C \cdot G^{\alpha - 1} - \frac{2 \lvert \nabla G \rvert^2}{(n-2)^2 G^2} \right] B_{ij}$$
We observe that
\[ V < \ \text{whenever } \Lambda \]
where each \( V \) is also clear that
\[ \nabla \]
\[ \Lambda \] is constant, note that \( \Lambda \geq \frac{n-2}{2} (C - D) \cdot G^\alpha \) and \( G^\alpha = O(r^{-n}) \), therefore \( \Lambda \geq 0 \).
In the case that \( \Lambda \) is not constant, \( \Lambda \) takes its negative minimum on \( \{ \varepsilon \leq r \leq R \} \) in the boundary by the maximum principle. By the same argument as in the constant case, we have that \( \inf_{r=\varepsilon} \Lambda \rightarrow 0 \) as \( R \rightarrow \infty \), and the assumption near \( x \) implies that \( \lim_{\varepsilon \rightarrow 0} \inf_{r=\varepsilon} \Lambda \geq 0 \). Therefore it would suffice to establish that \( \Delta \Lambda \leq 0 \) whenever \( \Lambda < 0 \).

Now suppose that \( \Lambda(p) = \tilde{H}(V, V) < 0 \). Write \( V = V^i e_i \) on a neighborhood of \( p \), where each \( V^i \) is extended as a constant function. Define \( \tilde{h} = \tilde{H}(V, V) = \tilde{H}_{ij} V^i V^j \).
We observe that \( \tilde{h} \) is an upper barrier for \( \Lambda \) at \( p \). Indeed, \( \tilde{h}(p) = \Lambda(p) \) and \( \tilde{h} \geq \Lambda \) near \( p \) by definition of \( \Lambda \). It only remains to show that, for any \( \varepsilon > 0 \), if we choose the neighborhood of \( p \) small enough then \( \Delta \tilde{h} < \varepsilon \). It is enough to show that if \( \tilde{h}(p) < 0 \) then \( \Delta(\tilde{H}_{ij} V^i V^j)(p) \leq 0 \), since then \( \Delta \tilde{h} < \varepsilon \) follows by continuity. Hence in what follows, all computations are made at \( p \). Note that since \( V^i \) are constant, we have that \( \Delta \tilde{h} = \Delta(\tilde{H}_{ij}) V^i V^j = (\Delta \tilde{H}_{ij}) V^i V^j \). Thus, it suffices to estimate the terms in Lemma 2.3.

We bound the first three terms related to the curvature in the following way.
Without loss of generality we can assume that \( \{ e_i \} \) diagonalizes \( \tilde{H} \) at \( p \) and write \( \tilde{H}_{ij} = \lambda_i \delta_{ij} \). Since \( V \) is the lowest eigenvector of \( \tilde{H} \), there is \( m \) such that \( V = e_m \) with \( \lambda_m = \Lambda \). Therefore (with \( m \) fixed and \( i, j, k, l \) being summed over),
\[
(R_{ik} \tilde{H}_{jk} + R_{jk} \tilde{H}_{ik} - 2R_{ikjl} \tilde{H}_{kl}) V^i V^j = R_{ik}(\tilde{H}_{jk} V^i V^j) + R_{jk}(-\tilde{H}_{ik} V^i V^j) - 2R_{ikjl} \lambda_k \delta_{kl} V^i V^j
= R_{ik}(\lambda \cdot V^k V^i) + R_{jk}(\lambda \cdot V^k V^j) - 2R_{ikjl} \lambda_k \delta_{im} \delta_{jm}
= 2\lambda \cdot R_{ij} \delta_{im} \delta_{jm} - 2R_{mkml} \lambda_k
= 2R_{mkml}(\Lambda - \lambda_k) \leq 0,
\]
since \( \Lambda \) is the lowest eigenvalue, and \( R_{mkml} \geq 0 \).

The assumption on the sectional curvature implies that \( -R_{ikjl} \frac{G^i g^j}{G} V^i V^j \leq 0 \). It is also clear that \( -\frac{2n}{(n-2)G} (\tilde{H})_{ij} V^i V^j \leq 0 \).

For the next two of the remaining terms, we will use the sharp gradient estimate in [3] which states that \( |\nabla b| \leq 1 \) for nonnegative Ricci curvature. This is equivalent to \( |\nabla G|^2 \leq (n-2)^2 G^{\alpha+1} \).
Therefore,
\[
- \frac{n(n-2)}{2} C^2 G^{2\alpha-1} g_{ij} V^i V^j + \frac{4n}{n-2} \left[ C \cdot G^{\alpha-1} - \frac{2|\nabla G|^2}{(n-2)^2 G^2} \right] B_{ij} V^i V^j
\leq - \frac{n(n-2)}{2} C^2 G^{2\alpha-1} + \frac{4nC \cdot G^{\alpha-1}}{n-2} B_{ij} V^i V^j
\]
\[ \leq -\frac{n(n-2)}{2} C^2 G^{2\alpha - 2} + \frac{4nC \cdot G^{\alpha - 2}}{n-2} |\nabla G|^2 \]
\[ \leq -\frac{n(n-2)}{2} C^2 G^{2\alpha - 1} + \frac{4nC \cdot G^{\alpha - 2}}{n-2} \cdot (n-2)^2 G^{\alpha + 1} \]
\[ = -\frac{n(n-2)}{2} C(C - 8) G^{2\alpha - 1}. \]

For the last group of terms, we use that the top eigenvalue of \( B \) is \( \frac{|\nabla G|^2}{G} \) and the gradient estimate \( |\nabla G|^2 \leq (n-2)^2 G^{\alpha + 1} \) to obtain that
\[
\left[ \tilde{H} \left( \frac{2}{2-n} B + \frac{n-2}{2} C \cdot G^\alpha g \right) + \left( \frac{2}{2-n} B + \frac{n-2}{2} C \cdot G^\alpha g \right) \tilde{H} \right] V_i V_j \\
= \frac{2}{2-n} [\tilde{H} B + B \tilde{H}] V_i V_j + (n-2) C \cdot G^\alpha \tilde{H} V_i V_j \\
\leq \frac{4|\nabla G|^2}{(n-2) G} \| \tilde{h} \| + (n-2) C \cdot G^\alpha \tilde{h} \\
= \left[ (n-2) C \cdot G^\alpha - \frac{4|\nabla G|^2}{(n-2) G} \right] \tilde{h} \\
= (n-2)(C-4) \cdot G^\alpha \tilde{h} + \frac{4}{(n-2) G} \left[ (n-2)^2 G^{\alpha + 1} - |\nabla G|^2 \right] \tilde{h} \leq 0. 
\]

Combining all of the above, we conclude that
\[
(\Delta(\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij})) V_i V_j \leq -\frac{n(n-2)}{2} C(C-8) G^{2\alpha - 1}. 
\]

\( G^\alpha \) can be shown to satisfy the equation \( \Delta(G^\alpha) = \frac{2n}{(n-2)^2} G^{\alpha - 2} |\nabla G|^2 \). A proof of this fact is given in Section 3, Lemma 3.2. Since \( C \geq 10 \), it follows that
\[
\Delta(\tilde{H}_{ij} V_i V_j) = \Delta \left( (\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij} ) V_i V_j \right) \\
= \Delta \left( (\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij} V_i V_j \right) + \frac{(n-2) C}{2} \cdot \Delta G^\alpha \\
= \Delta \left( (\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij} V_i V_j \right) + \frac{nC}{n-2} \cdot G^{\alpha - 2} |\nabla G|^2 \\
\leq -\frac{n(n-2)}{2} C(C-8) G^{2\alpha - 1} + \frac{nC}{n-2} \cdot G^{\alpha - 2} |\nabla G|^2 \\
\leq -\frac{n(n-2)}{2} C(C-8) G^{2\alpha - 1} + n(n-2) C \cdot G^{2\alpha - 1} \\
= -\frac{n(n-2)}{2} C(C-10) G^{2\alpha - 1} \\
\leq 0, 
\]

where the gradient estimate for \( G \) was used for the second inequality. This establishes that \( \Delta(\tilde{H}_{ij} V_i V_j) \leq 0 \) and finishes the proof of Theorem 1.1. \( \square \)
Remark 2.4. In [9] it is shown that for a positive solution \( f \) of the heat equation on a closed manifold, the matrix quantity \( \text{Hess}_f - \frac{\nabla f \otimes \nabla f}{f} + \frac{f}{2t} g \) is positive-semidefinite for all time. One could ask whether we can introduce a cutoff function to view

\[
\text{Lemma 3.1.}
\]

\( f \) a closed manifold, the matrix quantity \( \text{Hess}_f \) ill-adapted to Hamilton’s matrix maximum principle argument, as the assumption that \( \partial M = \emptyset \) is essential there.

3. Laplacian of the Harnack Quantity

This section is devoted to deriving Lemma 2.3. We recall the commutators in the case of parallel Ricci curvature.

\[
\text{Lemma 3.1.}
\]

Let \( \{e_i\} \) be a normal frame at \( p \). If \( M \) has parallel Ricci curvature, i.e. \( \nabla_i R_{jk} = 0 \), then for a smooth function \( f \) on \( M \), the following identities hold at \( p \).

\[
\begin{align*}
    f_{ij} &= f_{ji}, \\
    f_{ijk} - f_{ikj} &= R_{jkl} f_l, \\
    \Delta f_i - (\Delta f)_i &= R_{ik} f_k, \\
    f_{ijkl} - f_{ijk} &= R_{klm} f_{lm} + R_{kml} f_{jm}, \\
    \Delta f_{ij} - (\Delta f)_{ij} &= R_{jk} f_k + R_{ik} f_j - 2 R_{ikj} f_{kl},
\end{align*}
\]

where \( f_{i_1 i_2 \ldots i_k} \) just means the derivative \( e_{i_k} (\cdots e_{i_2}(e_{i_1}(f)) \cdots) \).

\textbf{Proof.} The first identity is the symmetry of the Hessian of \( f \). For the second one, we compute that

\[
\begin{align*}
    f_{ijk} - f_{ikj} &= e_k (g(\nabla_j \nabla f, e_i)) - e_j (g(\nabla_k \nabla f, e_i)) \\
    &= g(\nabla_k \nabla_j, e_i) + g(\nabla_j \nabla f, \nabla_k e_i) - g(\nabla_j \nabla_k, \nabla f, e_i) - g(\nabla_k \nabla f, \nabla_j e_i) \\
    &= g(\nabla_k \nabla_j, e_i) - g(\nabla_j \nabla_k, e_i) - g(\nabla_j \nabla f, e_i) \\
    &= R(e_j, e_k, \nabla f, e_i) = R_{jkl} f_l.
\end{align*}
\]

A similar identity holds for any 1-form \( S \) in place of \( df \), namely,

\[
(\nabla^2 S)(e_i, e_j, e_k) - (\nabla^2 S)(e_j, e_i, e_k) = S(R(e_j, e_i) e_k). \tag{1}
\]

This can be checked in the same manner. We will use \( \Box \) to prove the fourth identity.

The third identity is a contraction of the one above,

\[
\begin{align*}
    f_{ikk} - f_{kki} &= f_{kik} - f_{kki} = R_{ikl} f_l = R_{ik} f_k.
\end{align*}
\]

The fourth identity is actually true for any \((0,2)\)-tensor \( T \) in the following form.

\[
(\nabla^2 T)(e_l, e_k, e_i, e_j) - (\nabla^2 T)(e_k, e_l, e_i, e_j) = R_{klm} T(e_i, e_m) + R_{kml} T(e_m, e_j).
\]

To show this, let \( T = T_1 \otimes T_2 \) for 1-forms \( T_1 \) and \( T_2 \), and compute using \( \Box \) and the normality of the coordinates, that

\[
(\nabla^2 T)(e_l, e_k, e_i, e_j) - (\nabla^2 T)(e_k, e_l, e_i, e_j)
\]

\[
= R_{klm} T(e_i, e_m) + R_{kml} T(e_m, e_j).
\]
\[ \nabla^2 (T_1 \otimes T_2)(e_i, e_k, e_i, e_j) = \nabla^2 (T_1 \otimes T_2)(e_k, e_i, e_i, e_j) \]
\[ = e_l (e_k (T_1(e_i) T_2(e_j))) - e_l (T_1(\nabla_k e_i) T_2(e_j) + T_1(e_i) T_2(\nabla_k e_j)) \]
\[ - e_k (e_l (T_1(e_i) T_2(e_j))) + e_k (T_1(\nabla_l e_i) T_2(e_j) + T_1(e_l) T_2(\nabla_e e_j)) \]
\[ = -[e_l (T_1(\nabla_k e_i)) - e_k (T_1(\nabla_l e_i)) T_2(e_j) - T_1(e_i) [e_l (T_2(\nabla_k e_j)) - e_k (T_2(\nabla_l e_j))] \]
\[ = -[\nabla^2 T_1(e_i, e_k, e_i) - \nabla^2 T_1(e_k, e_i, e_i) T_2(e_j) - T_1(e_i) [\nabla^2 T_2(e_i, e_k, e_j) - \nabla^2 T_2(e_k, e_i, e_j)] \]
\[ = -T_1(R(e_k, e_i) T_2(e_j)) - T_1(e_i) T_2(R(e_k, e_i) e_j) \]
\[ = -R_{kljm} T(e_m, e_j) - R_{kljm} T(e_i, e_m) \]
\[ = R_{klmj} T(e_i, e_m) + R_{klmj} T(e_m, e_j). \]

Now the fourth identity follows from taking \( T = \text{Hess}_f \), and using the symmetry of the Hessian and the normality of the coordinates.

For the last identity, note that
\[ \Delta f_{ij} = f_{ijkk} = (f_{ikj} + R_{jkkl} f_{li}) k \]
\[ = f_{ikj} + (\nabla_k R_{jkli}) f_l + R_{jkli} f_{kl} \]
\[ = f_{ikj} + R_{jkkm} f_{km} + R_{jkmk} f_{mi} + (\nabla_k R_{jkli}) f_l + R_{jkli} f_{kl} \]
\[ = f_{ikj} - R_{ikjm} f_{km} + R_{jkm} f_{im} + (\nabla_k R_{jkli}) f_l - R_{ikjl} f_{kl} \]
\[ = (f_{ikj} + R_{iklj} f_{jl}) - 2R_{ikjl} f_{kl} + R_{jkfik} + (\nabla_k R_{jkli}) f_l \]
\[ = (\Delta f)_{ij} + R_{iklj} f_{jl} + R_{jkfik} - 2R_{ikjl} f_{kl} + (\nabla_k R_{jkli}) f_l. \]

The second Bianchi identity implies that
\[ \nabla_k R_{jkli} + \nabla_l R_{jkik} + \nabla_i R_{jkl} = \nabla_k R_{jkli} + \nabla_i R_{jkl} = 0. \]

Since \( M \) has parallel Ricci curvature, it follows that \( \nabla_k R_{jkli} = 0 \). Thus we arrive at
\[ \Delta f_{ij} = (\Delta f)_{ij} + R_{iklj} f_{jl} + R_{jkfik} - 2R_{ikjl} f_{kl}. \]

Changing \( k \) and \( l \) suitably, we have shown the lemma. \( \square \)

With Lemma 3.1, we compute the ingredients for \( \Delta \tilde{H}_{ij} \), additionally using only the Leibniz rule.

**Lemma 3.2.** Let \( \{e_i\} \) be a normal frame at \( p \), and suppose that \( M \) has parallel Ricci curvature. Then the following identities hold at \( p \).
\[ \Delta G_{ij} = R_{jk} G_{ik} + R_{ik} G_{jk} - 2R_{ikjl} G_{kl}, \]
\[ \Delta(G_i G_j) = R_{ik} G_{jk} - R_{jk} G_{ik} + 2G_{ik} G_{jk}, \]
\[ g(\nabla G, \nabla G) = G_i G_k G_{jk} + G_j G_k G_{ik}, \]
\[ \Delta \left( \frac{G_i G_j}{G} \right) = \frac{R_{ik} G_k - 2G_{ik} G_j}{G} + \frac{R_{jk} G_k - 2G_{jk} G_i}{G} \]
\[ + \frac{2G_{ik} G_{jk}}{G^3} + \frac{2|\nabla G|^2 G_{ij}}{G^3} - \frac{2G_k (G_i G_{jk} + G_j G_{ik})}{G^2}, \]
\[ \Delta G^a = \frac{2n}{(2 - n)^2} G^{a-2} |\nabla G|^2. \]
Proof. The first identity is immediate from Lemma 3.1 and the fact that $\Delta G = 0$, and the third identity is an application of the Leibniz rule on $G_i G_j$. For the second identity,

$$\Delta (G_i G_j) = \Delta (G_i) G_j + G_j \Delta (G_i) + 2 G_{ik} G_{jk}$$

$$= [\Delta (G_i)] + R_{ik} G_k G_j + [\Delta (G_j)] + R_{jk} G_k G_i + 2 G_{ik} G_{jk}$$

$$= R_{ik} G_k G_k + R_{jk} G_i G_k + 2 G_{ik} G_{jk}.$$ We also derive that for any $\beta$,

$$\Delta G^\beta = \text{div}(\beta \cdot G^{\beta-1} \nabla G) = \beta (\beta - 1) G^{\beta-2} |\nabla G|^2,$$

from which the last identity is immediate and it follows that $\Delta (G^{-1}) = 2 G^{-3} |\nabla G|^2$.

We use this and the third identity to check the fourth identity,

$$\Delta \left( \frac{G_i G_j}{G} \right) = \frac{\Delta (G_i G_j)}{G} + \Delta (G^{-1}) G_i G_j - \frac{2}{G G^2} g (\nabla G, \nabla (G_i G_j))$$

$$= \frac{\Delta (G_i G_j)}{G} + \frac{2|\nabla G|^2 G_i G_j}{G^3} - \frac{2 G_k (G_i G_{jk} + G_j G_{ik})}{G^2}.$$ 

□

Lemma 3.3. Let $B = \frac{\nabla G \otimes \nabla G}{G}$, or equivalently in coordinates, $B_{ij} = \frac{G_i G_j}{G}$ for an orthonormal frame $\{e_i\}$. Then $B^2 = \frac{|\nabla G|^2}{G} B$.

Proof.

$$(B^2)_{ij} = \frac{G_i G_k \cdot G_j G_k}{G^2} = \frac{|\nabla G|^2 G_i G_j}{G^2} = \frac{|\nabla G|^2}{G} B_{ij}.$$ 

□

We are now ready to prove Lemma 2.3.

Proof of Lemma 2.3. By Lemma 3.2 we have that

$$\Delta (\tilde{H}_{ij} - \frac{n - 2}{2} C \cdot G^\alpha g_{ij})$$

$$= \Delta \left( G_{ij} + \frac{n}{2 - n} \frac{G_i G_j}{G} \right)$$

$$= R_{ik} G_{ik} + R_{ik} G_{jk} - 2 R_{ikj} G_{kl} + \frac{n}{2 - n} \left( \frac{R_{ik} G_j G_k + R_{jk} G_i G_k}{G} \right)$$

$$+ \frac{2 G_{ik} G_{jk}}{G} + \frac{2 |\nabla G|^2 G_i G_j}{G^3} - \frac{2 G_k (G_i G_{jk} + G_j G_{ik})}{G^2}$$

$$= R_{ik} \left( G_{jk} + \frac{n}{2 - n} \frac{G_j G_k}{G} \right) + R_{jk} \left( G_{ik} + \frac{n}{2 - n} \frac{G_i G_k}{G} \right) - 2 R_{ikj} G_{kl}$$

$$+ \frac{2 n}{(2 - n) G} \left[ \text{Hess}_G - B \right]_{ij}^2.$$
Substituting the derivatives of $G$ with expressions in $\tilde{H}$, we obtain that
\[
\Delta(\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij}) = R_{ik} \tilde{H}_{jk} + R_{jk} \tilde{H}_{ik} - 2R_{ikjl} \tilde{H}_{kl} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} \frac{(\tilde{H})^2}{(n-2)G}.
\]

We expand the square term and rearrange as follows.
\[
\Delta(\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij}) = R_{ik} \tilde{H}_{jk} + R_{jk} \tilde{H}_{ik} - 2R_{ikjl} \tilde{H}_{kl} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} \frac{(\tilde{H})^2}{(n-2)G}.
\]

Replacing $B^2$ with $\frac{\nabla G^2}{G} B$ by Lemma 3.3 and rearranging, it follows that
\[
\Delta(\tilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij}) = R_{ik} \tilde{H}_{jk} + R_{jk} \tilde{H}_{ik} - 2R_{ikjl} \tilde{H}_{kl} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} - \frac{2n}{n-2} R_{ikjl} \frac{G_i G_j}{G} \frac{(\tilde{H})^2}{(n-2)G}.
\]

This finishes the proof of Lemma 2.3. □
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