Secure Reusable Base-String in Quantum Key Distribution

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Protecting secure random key from eavesdropping in quantum key distribution protocols has been well developed. In this letter, we further study how to detect and eliminate eavesdropping on the random base string in such protocols. The correlation between the base string and the key enables Alice and Bob to use specific privacy amplification to distill and reuse the previously shared base string with unconditional security and high efficiency. The analysis of the unconditional secure reusable base string brings about new concept and protocol design technique.

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Quantum cryptography has received wide attention from people who pursue perfectly secure communication. Since the first quantum key distribution (QKD) protocol by Bennett and Brassard \cite{1}, the BB84 QKD protocol, scientists have developed various techniques to improve its security and efficiency. Recently, unconditional security of BB84 QKD protocol have been achieved \cite{2,3,4}.

Most QKD protocols focus on detecting and eliminating eavesdropping on the random key by encoding it to the qubits with different bases. The choices of the bases, namely the base string likes an encryption key. Especially in some recent works, the base string are shared before the protocol \cite{5,6,7,8,9,10}. So it is interesting to investigate how to protect such encryption key from eavesdropping.

In this Letter, we explicitly prove how much unconditional secure base string can be reused for the first time. The result not only increases the efficiency of quantum cryptography, but also contributes to the foundation of quantum information science and cryptography designs.

Suppose Alice and Bob share a common and random base string. Then Alice encodes the qubits using this shared base string and sends them to Bob. Bob receives and measures the qubits in the bases determined by the shared base string. Under ideal situation, Alice and Bob can reuse the base string if no error is found in the check step. This kind of sharing is able to greatly increase the efficiency of quantum key distribution, because no qubit is wasted due to wrong choice of the measuring basis.

However, practically the channel errors are unavoidable. Eve may steal some information about the secret base string. However, we will prove that Alice and Bob can estimate and eliminate Eve’s information on the base string by privacy amplification. The proof of our QKD protocol with shared and reusable base string is also based on the lemma \cite{2} that high fidelity implies low entropy and the similar reduction technique of Shor and Preskill \cite{5}. But we extend the security analysis from one entangled pair to a block of two entangled pairs. This extension gives correlation between the error rates of the two entangled pairs.

We start with the entanglement distillation QKD protocol. Firstly, we can also suppose that the secret and random base string is also generated from another high-fidelity EPR pairs, denoted by the base pairs, namely, Alice and Bob both measure their qubits of the base pairs respectively in the $Z$-basis. Secondly, we let Alice and Bob postpone the measurements on the base pairs. Then Alice’s random Hadamard transformation on the second qubit of each communicating pair is replaced by the controlled-Hadamard operation with her own qubit of one base pair as the source qubit and the second qubit of one communicating pair as the target qubit. Particularly, the control-Hadamard gate using the first qubit to control the second qubit is denoted as $CH_{12}$.

Following the ideas above, we obtain the following protocol:

\textbf{Protocol 1 Entanglement distillation QKD protocol with reusable shared base string}

1. Alice and Bob share $2n$ base EPR pairs in the state $|\Phi^+\rangle\otimes|\Phi^+\rangle\otimes|\Phi^+\rangle\otimes|\Phi^+\rangle$.

2. Alice prepares $2n$ communicating EPR pairs in the state $|\Phi^+\rangle\otimes|\Phi^+\rangle\otimes|\Phi^+\rangle\otimes|\Phi^+\rangle$, and groups each communicating pair with one base pair to create $2n$ blocks. Fig. \textbf{1} shows the operations on one block in the protocol, in which the 1st and the 4th qubits form a base pair and the 2nd and the 3rd qubits form a communicating pair.

3. In each block, Alice applies $CH_{13}$, as shown in phase 1 of Fig. \textbf{1}.

4. Alice sends the 3rd qubit of each block to Bob, as shown in phase 2 of Fig. \textbf{1}.

5. Bob receives the qubits. In each block, he applies $CH_{34}$, as shown in phase 3 of Fig. \textbf{1} Then he publicly announces the reception.
6. The following steps are post-processing procedure. Alice and Bob randomly permute the blocks and agree on \( n \) random blocks out of the \( 2n \) blocks as check blocks.

7. In each check block, Alice and Bob both measure their own qubits of the communicating pairs in \( Z \)-basis, and publicly compare their results to obtain the channel bit error rate \( e \). If there are too many errors, they abort the protocol.

8. By estimating the bit error rate on the communicating pairs in the code blocks from the checking process, Alice and Bob apply an entanglement purification protocol (EPP) to distill \( m \) communicating pairs with high fidelity from the \( n \) corrupted communicating pairs. Then they measure them both in \( Z \)-basis to establish an \( m \)-bit secret key.

9. Alice and Bob can also estimate the phase error rate on the base pairs in all blocks as not greater than \( 2e \). Then Alice and Bob apply another EPP to distill \( 2m' \) base pairs with high fidelity from the \( 2n \) corrupted base pairs. Then they measure them both in \( Z \)-basis to establish a \( 2m' \)-bit secret base string.

The key point to prove the unconditional security of Protocol \([1]\) is the estimation of the error rates on the two kinds of the EPR pairs, shown in step 8 and 9. When the channel is noisy, we suppose without loss of generality the errors of the channel all come from Eve’s manipulation of the quantum system. Assuming that Eve can perform arbitrary coherent attack on all blocks. Here we apply quantum de Finetti representation \([11]\), and we only consider the asymptotic situation of large \( n \). Thanks to the random permutation of the blocks in step 6 of Protocol \([1]\) the \( 2n \) blocks are exchangeable and satisfy the condition of quantum de Finetti representation. Therefore, the final state of the total \( 2n \) blocks is a mixture of product state, namely, the density matrix of the final state is

\[
\rho'_{\text{all}} = \int p_{\rho'} \rho'^{2n} d\rho',
\]

in which \( \rho' \) is chosen from any possible corrupted density matrix of one block and \( p_{\rho'} \) is its weight. Due to the linear sum of different \( \rho'^{2n} \) in the final density matrix, the results of any measurement on \( \rho'_{\text{all}} \) are also the linear weighted sum of measurement results on different \( \rho'^{2n} \). So we can restrict our analysis within one possible value of \( \rho' \).

Considering the case with one possible value of \( \rho' \), we find that the results of measuring any operator on each block of the final state are effectively an independent and identical distribution. The bit and phase error rates of the base pairs, denoted as \( E^\text{bit}_\text{base} \) and \( E^\text{ph}_\text{base} \), and the bit and phase error rates of the communicating pairs, denoted as \( E^\text{bit}_\text{comm} \) and \( E^\text{ph}_\text{comm} \), are the rates of obtaining \(-1\) when measuring \( Z \) on each of the blocks. Because all these four measurement operators commute to each other, we are able to apply classical probability here. By the central limit theorem, all these error rates are equal to the expected values of measurement results of corresponding operators in one block, with very large probabilities. As a result, we can only analyze the error rates of one block in one possible state of \( \rho' \).

When we only study one block, the initial state is \( \rho_0 \) defined as

\[
\rho_0 = |\psi_0\rangle\langle\psi_0|,
\]

where \( |\psi_0\rangle = \frac{1}{2}(|00\rangle + |11\rangle)_4 \otimes (|00\rangle + |11\rangle)_23 \). Any possible final state, \( \rho' \), is obtained by arbitrary operation of Eve. A general operation of Eve can be described by a superoperator on the 3rd qubit in each block \([12]\), which is transmitted via the channel. We denote the superoperator as \( \mathcal{E} \). Then from the protocol, we obtain that

\[
\rho' = CH_{43} \mathcal{E} \left( CH_{13} \rho_0 C \right) CH_{13}^\dagger.
\]

A general superoperator of Eve’s operation is described as

\[
\mathcal{E}(\rho_1) = \sum_{\mu} M_\mu \rho_1 M_\mu^\dagger,
\]

in which \( \rho_1 = CH_{13} \rho CH_{13}^\dagger \) and \( M_\mu \) is an arbitrary matrix acting on the 3rd qubit, namely,

\[
M_\mu = \left( \begin{array}{cc} a_{11}^\mu & a_{12}^\mu \\ a_{21}^\mu & a_{22}^\mu \end{array} \right) _3.
\]

Due to the linearity of the superoperator, we first cal-
To interpret the above results, we note that the controlled-Hadamard operations serves as the random Hadamard operations, which make the bit and phase error rates of the communicating pairs symmetric. As no qubit of the base pairs are transmitted through the channel, their bit error rate is zero, while their phase error rate is the result of the propagation of errors on the communicating pairs.

Moreover, because Alice and Bob only need to know \( E_{\text{comm}} \) can be best estimated from the comparison of the \( Z \)-basis measurement results of \( n \) check communicating pairs, in step 7 of Protocol 1 as well as the channel bit error rate \( e \). Thus knowing the bounds of the error rates, Alice and Bob can employ two EPPs on the communicating and base pairs respectively, shown in step 8 and 9. If both EPP are successfully, they will distill both high-fidelity base and communicating EPR pairs which implies low entropy of Eve’s information \([2]\). Therefore, not only secret shared key is established but also secret base string can be reused in the future, after \( Z \)-basis measurements on these pairs. Note that no base pairs are sacrificed in the checking process so total \( 2n \) base pairs can be used in EPP.

So far, we have shown the unconditional security of Protocol 1. Our final goal is to derive a prepare-and-measure protocol with reusable shared base string from Protocol 1. The reduction lies on the fact that some of the final \( Z \)-basis measurements in step 8 and 9 commute to other operations and measurements in the protocol, and that can be brought forward to the beginning of the protocol without affecting the security \([4]\).

Firstly, both Alice and Bob’s final measurements on both pairs can be brought before the error correcting procedure in step 8 and 9. If we use EPP with one-way classical communications, the result effectively changes the EPP to error correction with Calderbank-Shor-Steane(CSS) codes \([4]\) on single qubits. A CSS Code \( Q(C_1, C_2) \) employs two classical linear code \( C_1 \) and \( C_2 \), in which \( C_1 \) and \( C_2 \) are used for correcting bit and phase errors respectively and \( C_2 \subset C_1 \) \([13]\). Moreover, because the bit error rate on the base pairs is always \( 0 \), we only need \( C_2 \) to correct the base pairs.

Secondly, Alice’s final measurements commute with the controlled-Hadamard gates applied in step 3. It can also be verified that Bob’s final measurements on the base pairs commute to the gates in step 5.

Thirdly, no measurement of the phase error rates are required, because we have proved that the upper bounds of the two kinds of phase error rates can be estimated using the channel bit error rate \( e \). Thus Alice and Bob’s final measurements commute to the measurements in step 7.

Finally, we have successfully brought Alice’s final measurements to the beginning of Protocol 1. We are also able to bring Bob’s final measurements on the base pairs to the beginning of the protocol, and to bring those on the communicating pairs to Bob’s reception of the qubits. We summarize the result of the equivalent transformations as the following protocol.

**Protocol 2 BB84 QKD protocol with reusable shared base string**

1. Alice and Bob share \( 2n \)-bit binary base string \( b \).
2. Alice prepares \( 2n \) qubits. If the corresponding bit value of \( b \) is \( 0 \), she randomly prepares the qubit in \( \ket{0} \) or \( \ket{1} \); otherwise she randomly prepares the qubit in \( \ket{+} \) or \( \ket{-} \).
3. Alice sends the qubits to Bob.
4. Bob receives the qubits and immediately measure them in certain basis according his \( b \). Then he publicly announces the reception.
5. Alice and Bob agree on \( n \) qubits as check qubits. The rest \( n \) qubits are code qubits.
6. Alice and Bob publicly compare the bit values on the check qubits and obtain the channel bit error rate \( e \). If there are too many errors, they abort the protocol.
7. Alice and Bob select a CSS code \( Q(C_1, C_2) \) that are capable of correcting both bit and phase errors on the code qubits, which are both \( e \). They employ \( C_1 \) to correct the bit errors in the measurement results of the code qubits. Then they use the cosset of the corrected results to \( C_2 \) as the final key.
8. Alice and Bob select another linear code \( C_2' \) that are capable of correcting the phase errors on the hypothetic base EPR pairs represented by \( b \), whose rate is at most \( 2e \). They use the cosset of \( b \) to \( C_2' \) as final reusable secret base string.
We also plot $L_k/(2n)$ on Fig. 3 and find that if the channel bit error rate is low enough, a small length of initial base string can generate much longer random key.

In conclusion, we have shown that if Alice and Bob first share a secret base string and use them in BB84 QKD protocol to encode the qubits, they are able to reuse this shared information in the future by the distillation of the base string using certain privacy amplification methods. In particular, in the part of the qubits successfully received by Bob, the generation rate of the base distillation is also related to the channel bit error rate, and higher than that of the key distillation. Furthermore, as the bit error rate of the base string is zero, we need only use a classical linear code instead of CSS codes to distill the base string, so the distillation is much simpler and more efficient. Secure reusable base string is contrasting different, and it may lead new strategy in protocol designs in quantum cryptography, hence contributes to the foundation of quantum information science.

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\[ L_k = nR_k(e)(1 + R_b(e) + R_b(e)^2 + \cdots) = \frac{nR_k(e)}{1 - R_b(e)}. \]  

FIG. 2: The generation rates of final secure key(solid line) and base string(dashed line)

FIG. 3: $L_k/(2n)$ verse the channel bit error rate $e$
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