Evolution of the Truncated Mellin Moments of the Parton Distributions in QCD Analysis

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Abstract—We review evolution equations for the truncated Mellin moments of the parton distributions and some of their applications in QCD analysis. The main finding of the presented approach is that the nth truncated moment of the parton distribution obeys also the DGLAP equation but with a rescaled splitting function $P(z) = z^n P(z)$. This allows one to avoid the problem of dealing with the experimentally unexplored Bjorken-$x$ region. The evolution equations for truncated moments are universal—they are valid in each order of perturbation expansion and can be useful additional tool in analysis of unpolarized as well as polarized nucleon structure functions.

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1. INTRODUCTION

Understanding the details of the Bjorken-$x$ and $Q^2$ dependence of the nucleon structure functions is one of the most important challenges in high energy physics. Perturbative QCD provides a comprehensive framework for describing deep inelastic lepton–hadron scattering (DIS) and hadron–hadron collisions (H–H) in current and planned experiments. In this framework, where the high energy collisions proceed via partonic constituents of the hadron, a key role is played by universal parton distribution functions (PDFs). According to the factorization theorem, DIS or H–H cross section is a convolution of a short distance interaction, described by the partonic cross section $σ_f$, and a long distance structure described by the parton distribution $q_f$.

Particularly interesting is a study of polarized processes which provides knowledge about the spin structure of the nucleon. Though recent experimental data and NLO analyses suggest that valence quarks carry the expected fraction of the nucleon spin, the main questions are still open: how the nucleon spin is distributed among its constituents: quarks (particularly sea quarks with negative helicity) and gluons and how the dynamics of these constituent interactions depends on spin. New experimental data in the resonance region from Jefferson Lab together with complementary data from HERMES, COMPASS and RHIC, are a crucial step towards better understanding of not only the flavor decomposition and gluon contributions to the nucleon spin but also the quark–hadron duality. The main goal of present polarized experiments is to determine the nucleon spin structure functions $g_1(x, Q^2)$, $g_2(x, Q^2)$ and their moments which are essential in testing QCD sum rules. The theoretical approach usually used in the description of these experimental results are the QCD evolution equations for the parton densities which change with $Q^2$ according to the well-known DGLAP equations [1–4]. This standard DGLAP approach operates on the parton densities $q$; hence their moments, which are, e.g., the contributions to the proton spin and other sum rules, can be obtained by the integration of the parton densities $q$ over Bjorken-$x$.

Alternatively, one can directly study the $Q^2$ evolution of the Mellin moments of the parton densities. The moments provide a natural framework in QCD analysis, as they originate from OPE—the basic formalism of quantum field theory. The idea of truncated Mellin moments (TMM) of the parton densities in QCD analysis was introduced and developed in the late 1990s [5–8]. The authors obtained the non-diagonal differential evolution equations in which the nth truncated moment coupled to all higher ones. The evolution equations for TMM within the $ln^2x$ approximation were found in [9], and finally DGLAP-type diagonal integro-differential evolution equations for the single and double truncated moments of the parton distribution functions were derived in [10, 11] and [12]. Evolution equations for double truncated moments and their application to study the quark–hadron duality were also discussed in [13].

The main finding of the truncated moments approach is that the nth moment of the parton distribution obeys also the DGLAP equation but with a rescaled splitting function $P(z) = z^n P(z)$ [10]. This
approach allows one to restrict the analysis to the experimentally available Bjorken-x region. The evolution equations for TMM are universal—they are valid in each order of perturbation expansion (then Wilson coefficients rescale in the same way as the splitting functions) and can be an additional tool in QCD analysis of the unpolarized as well as the polarized nucleon structure functions.

In this paper, we review our main results on the truncated moments approach. In the next section, we present the evolution equations for the truncated moments of the parton distributions. Section 3 contains relations between the truncated and untruncated Mellin moments, useful for solving evolution equations. Section 4 describes determination of the parton densities from their truncated moments approach. In the next section, we present the evolution equations for truncated moments approach. In higher order analysis (eg. NLO), truncated moments of the structure functions assume importance in comparison to the small-x limit because of the rapid decrease of the parton densities as $x \to 1$; nevertheless, a comprehensive theoretical analysis requires an equal treatment of both truncated limits.

The double truncated moment

$$q^n(x_{\min} , x_{\max} , Q^2) = \int_{x_{\min}}^{x_{\max}} dx x^{n-1} q(x , Q^2),$$

as it is a subtraction of two single truncated ones, also satisfies the DGLAP-type evolution Eq. (4) [11–13]:

$$\frac{dq^n(x_{\min} , x_{\max} , Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left( P \otimes q^n \right)(x_{\min} , Q^2),$$

$$\times \int_{z_{\min}}^{z_{\max}} \frac{dz}{z} P'(n, z) q^n(x_{\min} ,\frac{x_{\max}}{z},\frac{z}{z}, Q^2),$$

where $P(z)$ is the splitting function, which can be expanded in a power series of $\alpha_s(Q^2)$.

In the DGLAP approach, the main role is played by the PDFs and in our TMM approach we study directly the $Q^2$ evolution of the truncated moments of the PDFs.

In [10], we found that the single truncated moments of the parton distributions $q(x , Q^2)$, defined as

$$q_n(x, Q^2) = \int_{x_{\min}}^{1} dx x^{n-1} q(x , Q^2),$$

obey the DGLAP-like equation

$$\frac{dq^n(x_{\min} , Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left( P \otimes q^n \right)(x_{\min} , Q^2).$$

A role of the splitting function is played here by $P(n, z)$:

$$P'(n, z) = z^n P(z).$$

Since the experimental data cover only a limited range of $x$, except very small $x \to 0$ as well as large $x \to 1$, it is very natural and convenient to deal with the double truncated moments. Truncation at large $x$ is less important in comparison to the small-x limit because of the rapid decrease of the parton densities as $x \to 1$; nevertheless, a comprehensive theoretical analysis requires an equal treatment of both truncated limits.

The double truncated moment

$$g^n(x_{\min} , x_{\max} , Q^2) = \int_{x_{\min}}^{x_{\max}} dx x^{n-1} g(x , Q^2),$$

as it is a subtraction of two single truncated ones, also satisfies the DGLAP-type evolution Eq. (4) [11–13]:

$$\frac{dg^n(x_{\min} , x_{\max} , Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \left( P \otimes q^n \right)(x_{\min} , Q^2),$$

$$\times \int_{z_{\min}}^{z_{\max}} \frac{dz}{z} P'(n, z) g^n(x_{\min} ,\frac{x_{\max}}{z},\frac{z}{z}, Q^2),$$

with $P$ given again by Eq. (5).

Our approach, Eqs. (4)–(7), is valid for the coupled DGLAP equations for quarks and gluons and for any approximation (LO, NLO, NNLO, etc.). For clarity, we present here only the nonsinglet and leading order part. In higher order analysis (eg. NLO), truncated moments of the structure functions assume the form

$$g_{1n}(x, Q^2) = \frac{1}{2} \sum_q e_q^2 \left[ \Delta q_n(x, Q^2) + \frac{\alpha_s(Q^2)}{2\pi} \right]$$

$$\times \left( C'_q(n) \otimes \Delta q_n + C''_q(n) \otimes \Delta g_n \right)(x, Q^2),$$

where the Wilson coefficients rescale in the same way as the splitting functions:

$$C'_q(n, x) = x^2 C_q(x).$$
Let us emphasize that the evolution equations for the double truncated moments, Eq. (7), are in fact a valuable generalization of those for the single truncated and untruncated ones. Setting $x_{\text{min}} = x_0$ or $x_{\text{min}} = 0$ and $x_{\text{max}} = 1$ one obtains Eq. (4) or the well-known renorm-group equation for the moments

$$d\tilde{q}^n(Q^2)\over d\ln Q^2 = \frac{\alpha_s(Q^2)}{2\pi} \gamma^n(Q^2)\tilde{q}^n(Q^2),$$

(10)

respectively.

In the next section, we present relations between truncated and untruncated moments which are useful for solving the evolution equations.

3. RELATIONS BETWEEN TRUNCATED AND UNTRUNCATED MOMENTS

The evolution equations for the truncated moments, Eq. (4), are very similar to those for the PDF. In both cases, one deals with functions of two variables $x$ and $Q^2$ (with additionally fixed index $n$ for moments), which obey the differentio-integral Volterra-like equations. The only difference lies in the splitting function, which for moments has the rescaled measure $dx$, $\alpha_s(Q^2)$ and $\gamma^n(Q^2)$, which obey the differentio-integral Volterra-like equations. The only difference lies in the splitting function, which for moments has the rescaled measure $dx$, $\frac{\alpha_s(Q^2)}{2\pi}$ and $\gamma(Q^2)$, respectively.

In our previous studies on the evolution performed in moment space, we have found useful relations involving untruncated and truncated moments. Let us discuss this in detail and introduce some useful relations involving untruncated and truncated Mellin moments.

There are in literature several methods for the solution of the integro-differential DGLAP equations. They are based on either the polynomial expansion or the Mellin transformation—for review see, e.g., [14]. In our previous studies on the evolution of the truncated moments we used the Chebyshev polynomial technique [15], earlier widely applied by Jan Kwiecinski in many QCD treatments—for details see, e.g., Appendix of [16]. Using this method one obtains the system of linear differential equations instead of the original integro-differential ones. The Chebyshev expansion provides a robust method of discretising a continuous problem.

An alternative approach is based on the Mellin transformation and the moment factorization. Taking the $s$-th moment of the evolution Eq. (4) one obtains

$$dM^s,n(Q^2)\over d\ln Q^2 = \frac{\alpha_s(Q^2)}{2\pi} \gamma^{s+n}(Q^2)M^s,n(Q^2),$$

(11)

where $M^s,n$ denotes here the $s$-th (untruncated) moment of the $n$-th truncated moment of the parton density:

$$M^s,n(Q^2) = \int_0^1 dx x^{s-1} \tilde{q}^n(x, Q^2).$$

(12)

Analogically to the well known solutions for the PDFs, we can immediately write down solutions for the truncated moments:

$$M^s,n(Q^2) = M^s,n(Q_0^2) \left[ \frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{\frac{s}{s+n}},$$

(13)

and

$$\tilde{q}^n(x, Q^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds x^{-s} M^s,n(Q^2).$$

(14)

The quantity $M^s,n$ is either exotic and has no physical meaning, can be replaced by the usual truncated moment $\tilde{q}$. In [12], we found useful relations between the truncated and untruncated moments, namely:

$$M^s,n = \frac{1}{s} \int_0^1 dz x^{s+n-1} q(z) = \frac{1}{s} \tilde{q}^{s+n},$$

(15)

$$\tilde{q}^n(x, Q^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds x^{-s} \tilde{q}^{s+n}(Q^2)$$

(16)

and

$$\tilde{q}(Q^2) = (s-n)M^{s-(n,n)}(Q^2) = (s-n) \int_0^1 dx x^{s-n-1}$$

(17)

$$\times \tilde{q}^n(x, Q^2).$$

Equations (15), (16) and (17) have a large practical meaning in solving the evolution equations of TMM. Particularly, Eq. (16) seems to be helpful when the untruncated moments are known, e.g., from lattice calculations.

In the next sections we will discuss some implications of our approach for QCD analysis.

4. DETERMINATION OF THE PARTON DISTRIBUTIONS FROM THEIR TMM

The TMM approach, which refers to the physical values—moments (not to the parton densities), allows one to study directly their evolution and the scaling violation. The solutions for truncated moments can
also be used in the determination of the parton distribution functions via differentiation
\[ q(x, Q^2) = -x^{-\gamma} \frac{\partial}{\partial x} \Delta q(x, Q^2), \]

(18)

In order to reconstruct initial parton densities at scale \( Q_0^2 \) from their truncated moments, given, e.g., by experimental data at scale \( Q^2 \), we evolve moments between these two scales down (from \( Q^2 \) to \( Q_0^2 \)) and then perform the final fit of free parameters—for details see [11].

We proceed with the following steps:

1. Preparing available experimental data for moments \( \Delta q_n(x_0, Q_1^2) \) as a function of \( x_{\text{min}} \leq x_0 \leq 1 \) at the same scale \( Q_1^2 \).

2. Interpolation of the given data points into the points that are Chebyshev nodes. This allows us to use the Chebyshev polynomial technique for solving the evolution equations.

3. Evolution of the truncated moments from \( Q_1^2 \) to \( Q_2^2 \), according to (1), for different \( x_{\text{min}} \leq x_0 \leq 1 \).

4. Reconstruction of the parton density \( q(x, Q_2^2) \) from its truncated moment at the same scale \( Q_2^2 \) by applying the minimizing algorithm to fit free parameters.

In [11], we tested this method on the nonsinglet function parametrized in the general form
\[ q(x, Q_0^2) = N(\alpha, \beta, \gamma) x^\alpha (1-x)^\beta (1+\gamma x), \]

(19)

and also on the original fits for HERMES [17] and COMPASS [18] data. In Figs. 1–2, we show the spin-dependent valence quark distributions reconstructed from HERMES [17] and COMPASS [18] data. From these plots one can see satisfactory agreement between the reconstructed fits and experimental data. The reconstructed combined functions \( x(\Delta u_v - \Delta d_v) \) and \( x(\Delta u_v + \Delta d_v) \) overlap HERMES and COMPASS results, respectively. For the extracted valence quark densities alone the agreement is worse but still acceptable. We have found, however, that these fits are not unique and equally good agreement with the data can be obtained with the use of other (not only BB and DNS respectively) sets of free parameters. When the number of adjustable parameters is large (>3) and there are no experimental points from the low-\( x \) region \( x < 0.001 \), one cannot distinguish which fit is the best one. Only an additional constraint for small-\( x \) behaviour of the parton densities makes the fit procedure more reliable. Let us also mention that due to its large-\( x \) sensitivity, the second moment can be helpful in the precise final reconstruction of the parton density [11].

Concluding, even for the large number of adjustable parameters (6 for HERMES and 8 for COMPASS data), the presented method of reconstruction can be a hopeful tool for determining parton densities from experimental results for their truncated moments.
5. TMM IN ANALYSIS OF THE SPIN STRUCTURE FUNCTION $g_1$

For a complete description of the nucleon spin, one needs two polarized structure functions: $g_1$ and $g_2$. Recently, a new generation of experiments with high polarized luminosity performed at the Jefferson Lab allows a more precise study of the polarized structure functions and their moments. This is crucial in our understanding of the QCD spin sum rules, higher-twist effects and quark-hadron duality.

The function $g_1$ has a simple interpretation in the parton model:

$$g_1(x) = \frac{1}{2} \sum_i e_i \Delta q_i(x),$$

(20)

Powerful tools to study the internal spin structure of the nucleon are sum rules. One of them is the Bjorken sum rule (BSR) [21] which refers to the first moment of the nonsinglet spin dependent structure function $g_1^{NS}(x, Q^2)$:

$$\Gamma_1^{g_1} = \int_0^1 dx g_1^{NS}(x, Q^2) = \int_0^1 dx (q_1^n - g_1^n).$$

(21)

Due to $SU_f(2)$ flavour symmetry, BSR is regarded as exact. Thus, all of estimations of polarized parton distributions should be performed under the assumption that the BSR is valid. In the limit of the infinite momentum transfer $Q^2$, BSR has the form

$$\Gamma_1^{g_1} = \int_0^1 dx g_1^{NS}(x, Q^2) = \frac{1}{6} \frac{g_A}{g_V},$$

(22)

where $|g_A/g_V|$ is the neutron $\beta$-decay constant

$$|g_A/g_V| = F + D = 1.28. \quad (23)$$

With the next perturbative orders and higher twists corrections, the BSR reads

$$\Gamma_1^{g_1}(Q^2) = \left[ 1 - \frac{\alpha_s}{\pi} - \frac{3.58}{\pi} \frac{\alpha_s^2}{\pi^2} - 20.2 \frac{\alpha_s^3}{\pi^3} + \ldots \right] \frac{1}{6} \frac{g_A}{g_V} + \sum_{i=2}^\infty \frac{\mu_{2i}(Q^2)}{Q^{2i-2}}$$

(24)

leading twist

higher twists

Since experimental data cover only a restricted region of the Bjorken-$x$ variable and, in fact, provide knowledge on truncated moments of the structure functions, it is very natural to use the TMM approach in analysis of these data. The TMM approach allows for a direct study of the contributions to the BSR. In Figs. 3–5 and in table, we present the results for the evolution of the first truncated moment of $g_1^{x_0}$, namely

$$\Gamma_1^{g_1}(x_0, 1, Q^2) = \int_{x_0}^1 dx y(x, Q^2).$$

(25)

Figure 3 shows $\Gamma_1^{g_1}(x_0, 1, Q^2)$ as a function of the truncation point $x_0$ for different scales $Q^2$. Figure 4
presents the $Q^2$ evolution of $\Gamma_1^{x-n}(x_0, 1, Q^2)$ for different small-$x$ behaviour of the input parametrization $g_1^{x-n}(x, Q_0^2) = N x^\alpha (1-x)^\beta$: $\alpha = 0$ (solid), $-0.2$ (dashed) and $-0.4$ (dotted). Comparison with COMPASS data [22].

Fig. 5. First truncated moment of $g_1^{x-n}$ vs. truncation point $x_0$ at evolution scale $Q^2 = 3 \text{ GeV}^2$ for different small-$x$ behaviour of the input parametrization $g_1^{x-n}(x, Q_0^2) = N x^\alpha (1-x)^\beta$: $\alpha = 0$ (solid), $-0.2$ (dashed) and $-0.4$ (dotted). Comparison with COMPASS data [22].

Theoretical predictions for truncated contributions to the BSR for different small-$x$ behaviour of $g_1^{NS}(x_0, Q_0^2)$. Our predictions are compared with experimental HERMES [17] and COMPASS [22] data.

6. TMM IN ANALYSIS OF THE SPIN STRUCTURE FUNCTION $g_2$

Unlikely $g_1$, the structure function $g_2$ has no simple interpretation in the parton model. Due to the technical difficulties of obtaining transversely polarized targets, the structure function $g_2$ has not been a topic of investigations for a long time. Recently, new experimental data at low and intermediate momentum transfers have made $g_2$ also a valuable and hopeful tool to study the spin structure of the nucleon. The function $g_2$ provides knowledge on higher twist effects which are a reflection of the quark-gluon correlations in the nucleon.

The experimental value of the function $g_2$, measured in the small to intermediate $Q^2$ region, consists of two parts: the twist-2 (leading) and the higher twist term:

$$g_2(x, Q^2) = g_2^{LT}(x, Q^2) + g_2^{HT}(x, Q^2). \quad (27)$$

The leading-twist term $g_2^{LT}$ can be determined from the structure function $g_1$ via the Wandzura–Wilczek relation [23]

$$g_2^{LT}(x, Q^2) = g_2^{WW}(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2). \quad (28)$$

Then, from the measurements of $g_1$ and $g_2$, using the Wandzura–Wilczek approximation, Eq. (28), one is able to extract the higher-twist term $g_2^{HT}$ In [12], we

| Input | $x$-range | $Q^2$, GeV$^2$ | $\Gamma_1^{NS}$ | EXP $\Gamma_1^{NS}$ |
|-------|-----------|---------------|----------------|------------------|
| $(1-x)^3$ |           |               | 0.161          | Hermes           |
| $x^{-0.2}(1-x)^3$ | 0.021–0.9 | 5              | 0.149          | 0.1479 ± 0.0055 ± 0.0142 |
| $x^{-0.4}(1-x)^3$ |           |               | 0.131          |                  |
| $(1-x)^3$ |           |               | 0.177          | Compass          |
| $x^{-0.2}(1-x)^3$ | 0.004–0.7 | 3              | 0.173          | 0.175 ± 0.009 ± 0.015 |
| $x^{-0.4}(1-x)^3$ |           |               | 0.163          |                  |
found a generalization of the Wändzura–Wilczek relation in terms of the truncated moments:

\[ g_2^n(x_0, Q^2) = \frac{1-n}{n} g_1^n(x_0, Q^2) - x_0^0 g_1(x_0, Q^2), \]  

(29)

where

\[ g_{1,2}^n(Q^2) = \int_0^1 dx^n x^{n-1} g_{1,2}(x, Q^2), \]  

(30)

\[ g_{1,2}(x_0, Q^2) = \int_{x_0}^1 dx^n x^{n-1} g_{1,2}(x, Q^2), \]  

(31)

and

\[ g_1^0(x_0, Q^2) = \int_{x_0}^1 dx x g_1(x, Q^2). \]  

(32)

It is easy to see that for the untruncated moments, Eq. (29), takes the well-known form

\[ g_1^n(Q^2) = \frac{1-n}{n} g_1^0(Q^2). \]  

(33)

From Eq. (29), setting \( n = 1 \) and \( x_0 \to 0 \), one can automatically obtain the Burkhardt–Cottingham sum rule (BC) [24] for \( g_2^{WW} \):

\[ \int_0^1 dx g_2(x, Q^2) = 0. \]  

(34)

Using the generalization of the Wändzura–Wilczek relation, Eq. (29), for \( n = 1 \) at two different points of the truncation and applying the BC sum rule, Eq. (34), we obtain an interesting relation:

\[ \int_{x_1}^{x_2} dx g_2^{WW}(x, Q^2) = (x_2 - x_1) \int_{x_1}^{x_2} dx g_1(x, Q^2) - x_1 \]

\[ \times \int_{x_1}^{x_2} dx g_1(x, Q^2), \]  

(35)

which can be very useful in determination of the partial twist-2 contribution to the BC sum rule. Namely, setting \( x_1 = 0 \) and \( x_2 = x_0 \), when \( x_0 \to 0 \), one can get the small-\( x \) contribution to the BC sum rule:

\[ \int_0^{x_0} dx g_2^{WW}(x, Q^2) = x_0 \int_{x_0}^{x_1} dx g_1(x, Q^2). \]  

(36)

Now we would like to discuss the problem of the \( Q^2 \) evolution of \( g_2 \) [12]. While a general DGLAP-type equation for \( g_2 \) does not exist, for the twist-3 component of \( g_2 \) suitable evolution equation is known (see eg. [25]). In the leading twist-2 approximation, the \( Q^2 \) evolution of \( g_2 \) is governed by the evolution of \( g_1 \), according to the Wändzura–Wilczek relation. Since the second term on the r.h.s. of Eq. (28) is the \( n = 0 \)th truncated moment of the function \( g_1 \), Eq. (32), we can rewrite the Wändzura–Wilczek relation in the form

\[ g_2^{WW}(x, Q^2) = -g_1(z, Q^2) + g_1^0(z, Q^2) \]  

(37)

and obtain the evolution equation for \( g_2^{WW} \):

\[ \frac{d g_2^{WW}(x, Q^2)}{d \ln Q^2} = \frac{d g_1(z, Q^2)}{d \ln Q^2} + \frac{d g_1^0(z, Q^2)}{d \ln Q^2}. \]  

(38)

It is worth noting that according to Eqs. (4), (5), the \( n = 0 \)th truncated moment of the parton distribution \( q \) evolves in the same way as \( q \) itself (\( P(0, z) = P(z) \)). Taking this into account in the case of \( g_1 \) we obtain from Eqs. (38), (37) the evolution equation

\[ \frac{d g_2^{WW}(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{z_0}^{z} dz P(z) g_2^{WW}(z, Q^2) \]  

(39)

or finally

\[ \frac{d g_2^{WW}(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{z_0}^{z} dz P(z) g_2^{WW}(z, Q^2). \]  

(40)

The above formula shows that the twist-2 component of the function \( g_2 \) obeys the standard DGLAP evolution with the same evolution kernel as \( g_1 \). In this way, we obtained a system of evolution equations for

\[ g_2 = g_2^{EXP} + g_2^{pert-3}. \]  

(41)

\[ \frac{d [g_2^{EXP}(x, Q^2) - g_2^{WW}(x, Q^2)]}{d \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{z_0}^{z} dz P'(z) \left[ g_2^{EXP}(z, Q^2) - g_2^{WW}(z, Q^2) \right]. \]  

(42)

\[ \frac{d g_2^{WW}(x, Q^2)}{d \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_{z_0}^{z} dz P(z) g_2^{WW}(z, Q^2). \]  

(43)

In Figs. 6, 7, we present numerical solutions of Eq. (40) calculated in LO, for different low-\( x \) behaviour of \( g_1^{NS} \sim x^{\alpha} \) and for different \( Q^2 \). Figure 6 shows the predictions for \( x g_2^{NS} \) vs. \( x \) for different scales of \( Q^2 \): 1, 10 and 100 GeV\(^2\). In the input parametrization of \( g_1^{NS} \sim x^{\alpha} \) we assume at the scale \( Q_0^2 \alpha = -0.4 \). Note that \( x g_2^{NS} \) is positive for low-\( x \), at about \( x = 0.1–0.2 \) changes sign and becomes negative for larger \( x \). This is
in agreement with the BC sum rule. One can see also that with increasing $Q^2$, an $x$-intercept of $g_1^{NS}$ occurs at smaller values of $x$. In Fig. 7, we compare the predictions for $g_2^{NS}$ for different small-$x$ behaviour of the $g_1^{NS}$ parametrization: $\alpha = 0, -0.4, -0.8$. We find that more singular small-$x$ behaviour of $g_1$ implies smaller value of the $x$-intercept of $g_2$.

7. SUMMARY

This paper is a review of our studies on the truncated Mellin moments of the parton distributions. We presented the evolution equations for the single and double truncated n-th moments and useful relations between the truncated and untruncated moments. We gave examples of application of our approach to the determination of the PDFs and to QCD analysis of the spin-dependent structure functions $g_1$ and $g_2$. We presented the Wandzura–Wilczek relation in terms of the truncated moments, which implies the truncated sum rules. We also discussed the system of the DGLAP evolution equations for $g_2$. We presented the numerical predictions for the evolution of the truncated moments of $g_1^{NS}$ and the contributions to the BSR. We tested different small-$x$ behaviour of the initial parametrization of $g_1^{NS}$ and compared our results with COMPASS data.

The method of the truncated moments enables one direct, efficient study of the evolution of the moments (and hence sum rules) for non-spin as well as for spin-dependent parton distributions and can be used in all orders of perturbative theory. The adaptation of the evolution equations for the available experimentally $x$-region provides a new, additional tool for analysis of the nucleon structure functions.

Finally, let us list possible future applications of the TMM approach:

• Study of the fundamental properties of the nucleon structure, concerning moments of $F_1$, $F_2$ and $g_1$. These are: the momentum fraction carried by quarks, quark helicities contributions to the spin of nucleon and, what is particularly important, estimation of the polarized gluon contribution $\Delta G$ from COMPASS and RHIC data.

• Determination of Higher Twist (HT) effects from the moments of $g_2$ in the restricted $x$-region, which will be measured at JLab and can provide information on the quark–hadron duality.

• Test of Burkhardt–Cottingham and Efremov–Leader–Teryaev sum rules [26], also for their truncated contributions together with comparison to experimental data.

• Predictions for the generalized parton distributions (GPDs). Moments of the GPDs can be related to the total angular momentum (spin and orbital) carried by various quark flavors. Measurements of DVCS, sensitive to GPDs, will be carried out at JLab. This would be an important step towards a full understanding of the nucleon spin.

Concluding, in light of the recent progress in experimental program, theoretical efforts in improving our knowledge of the nucleon structure functions and their moments are of great importance.
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