Optimal control of time and therapy
in a tumor growth model
with possibly singular potentials

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Abstract

A distributed optimal control problem for a diffuse interface model which physical context is that of tumor growth dynamics is discussed. The system we deal with consists of a Cahn-Hilliard equation for the tumor phase coupled with a reaction-diffusion for an unknown species nutrient surrounding the tumorous cells. The cost functional we are considering possesses some objective terms and it also penalizes the long-time treatment and the big aggregations of tumorous cells. Let us underline that, nowadays, the most part of the therapeutic treatments for cancer are carried out within cycles. Hence, from the model viewpoint, the time we want to be optimized is the one related to the single therapeutic cycle. Indeed, owing to the strong impact of the therapies to the patients, minimize the time exposure of the patient to the care has crucial importance.

Key words Optimal control, free terminal time, phase field model, tumor growth, evolution equations, Cahn–Hilliard equation, adjoint system, necessary optimality conditions.

AMS (MOS) Subject Classification 35K61, 35Q92, 49J20, 49K20, 35K86, 92C50.
1 Introduction

In the last years, several developments were obtained by scientists in tumor modeling. The key idea arises from realizing that the tumor tissue, as a special material, has to obey physical laws. Hence, the modeling techniques developed for engineering purposes can be reproduced. This attempt has shown significant results confirming that mathematics can provide a support tool for medical therapy (see [10]). Indeed, the advantages of mathematics are, among others, that of being able to foresee, make predictions, and also provide answers that do not interfere with the patient’s health. Moreover, since tumor development is far to be understood, the mathematics has the ability to select specific behavior we could be interested in. So, it is now more than ever clear how cooperation between the doctors and mathematicians is important. Lastly, let us also mentioned that further understanding from the mathematical point of view can also allow the doctors to tailor a personalized therapeutic pathway, which is not yet the case.

To begin with, for a positive constant $T > 0$, let us define the standard parabolic cylinder and its boundary by

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t) \quad \text{for every } t \in (0, T],$$

$$Q := Q_T, \quad \text{and} \quad \Sigma := \Sigma_T,$$  \hspace{1cm} (1.1)

where $\Omega$ denote the tissue in which the tumor evolution takes place. Hence, the diffuse interface model of Cahn-Hilliard type modeling tumor growth we are going to deal with reads as follows:

$$\alpha \frac{\partial t \mu + \partial t \varphi - \Delta \mu = P(\varphi)(\sigma - \mu)}{\varphi} \quad \text{in } Q := \Omega \times (0, T) \quad \text{(1.2)}$$

$$\mu = \beta \partial t \varphi - \Delta \varphi + F'(\varphi) \quad \text{in } Q \quad \text{(1.3)}$$

$$\partial t \sigma - \Delta \sigma = -P(\varphi)(\sigma - \mu) + u \quad \text{in } Q \quad \text{(1.4)}$$

$$\partial_n \mu = \partial_n \varphi = \partial_n \sigma = 0 \quad \text{on } \Sigma := \partial \Omega \times (0, T) \quad \text{(1.5)}$$

$$\mu(0) = \mu_0, \varphi(0) = \varphi_0, \sigma(0) = \sigma_0 \quad \text{in } \Omega, \quad \text{(1.6)}$$

for two positive constants $\alpha$ and $\beta$. This model constitutes a simplification and variation of the thermodynamically consistent model proposed by Hawkins–Daruud et al. in [27] (see also [9, 25, 26, 28]), where the velocity contributions and chemotaxis effects are neglected and two relaxation terms appear. Let us point out that the first two equations enjoy a Cahn-Hilliard structure (see, e.g., [30] and the references therein for a general introduction), where the terms $P(\varphi)(\sigma - \mu)$ stands for a source term which accounts for biological mechanisms and it is motivated by linear phenomenological laws for chemical reactions. The variable $u$ appearing in (3.30) is the so-called control variable, whereas system (1.2) - (1.6) is usually referred to as state system. Since the physical background of the above model has been extensively described in [4, 6, 8, 15], we just sketch the role covered by the occurring symbols. As the unknown $\varphi$ and $\mu$ are concerned, they stand for the phase variable and the corresponding chemical potential, respectively. Moreover, $\varphi$ ranges between $-1$ and $1$, where the two extremes represent the pure phases, say the healthy case and the tumorous one. The third equation is a diffusion equation for $\sigma$ which models the evolution of an unknown nutrient species (e.g., glucose, oxygen). The variable $\sigma$ is normalized between 0 and 1, where these values denote the poor and high concentration of the nutrient, respectively. The symbol $P$ stands for a function which models a source term accounting for biological mechanisms such as proliferation. The function $F$ denotes a double-well potential. Standard examples for $F$ are the regular potential
and the singular logarithmic potential, which is more relevant for applications. They are defined as follows

\[
F_{\text{reg}}(r) := \frac{1}{4}(r^2 - 1)^2 = \frac{1}{4}r^4 - \frac{1}{4}(2r^2 - 1) \quad \text{for } r \in \mathbb{R},
\]

\[
F_{\text{log}}(r) := ((1 - r) \log(1 - r) + (1 + r) \log(1 + r)) - \lambda r^2 \quad \text{for } |r| < 1,
\]

where the constant \( \lambda \) is taken large enough to avoid convexity. Lastly, the positive constants \( \alpha \) and \( \beta \) can be seen as relaxation terms and allow us to extend the analysis to the case of singular potentials (see [36] and also [4, 6, 8]); note that the case \( \beta > 0 \) correspond to the well-known viscous Cahn-Hilliard equation.

As far as the related literature is concerned, a brief overview follows. The well-posedness and long-time behavior, in term of the omega-limit set, of the above model have been addressed in [4] for a wide class of double-well potentials, in the case \( \alpha = \beta > 0 \). Next, in [6] and [8] the authors show in which sense the parameter \( \alpha \) and \( \beta \) can be let to zero separately. In both cases, they point out the proper functional framework in which it is possible to identify the limit systems and ensure their well-posedness. Furthermore, we have to mention the work [15], where the above system without any relaxation terms, i.e., the case \( \alpha = \beta = 0 \), has been studied. There, despite they focused on regular potentials with polynomial growth, they keep the nonlinearity \( P \) quite general postulating for it a controlled polynomial growth. Then, we refer to [31], where the authors investigate the long-time behavior of system (1.2)–(1.6) in the case \( \alpha = \beta = 0 \) in term of its global attractor (see [32] for further details on global attractors). Furthermore, let us point out the contribution [17], where a non-local model is taken into account for the challenging case of singular potentials and degenerate mobilities. As for the diffuse interface models considering the velocity field by assuming a Darcy law or a Stokes-Brinkman law, we refer to [11,16,18-22,24,40], where further mechanisms such as chemotaxis and active transport are also discussed. Let us also refer to [14] and to the reference therein, where some numerical simulations can be found.

Since the well-posedness of system (1.2)–(1.6) has already been established in [36], we are in a position to properly define the control-to-state operator as the map which assigns to a given control \( u \) the corresponding solution, that is the map

\[
S : u \mapsto S(u) := (\mu, \varphi, \sigma) = (\mu(u), \varphi(u), \sigma(u)),
\]

where \( (\mu, \varphi, \sigma) \) stands for the unique solution to system (1.2)–(1.6) associated with the control variable \( u \).

Before introducing our control problem, let us spend some words explaining how the cancer treatments are usually planned; this will motivate some of the mathematical choices we made in the following. The most common treatments include surgery, chemotherapy, radiotherapy, and immunotherapy. The last three are particularly sensitive to the time exposition of the patient to the therapy. Moreover, the therapy is divided into cycles consisting of a “short” period of treatment followed by a longer period of rest. The purpose of these strategies are to reduce as much as possible the tumor in order to achieve a proper stage which is compatible with surgery. As well known, all these techniques are very burdensome for the patients which require a long enough resting time to recover. Moreover, let us emphasize that as time pass by, the dispensed drugs start to accumulate in the body bringing additional waste items to be purified by kidneys and liver. In addition, after a long-time exposure, it may happen that the tumor cells became resistant
to the medicament. Hence, trying to minimize the time of every treatment is really significant. These are just some of the numerous motivations for which the shorter is the medical treatment, the better is for the patient.

Although the investigation of mathematical models related to biological phenomena from the viewpoint of well-posedness and long-time behavior finds numerous contributions, the corresponding optimal control theory is still at its infancy. Up to our knowledge, the first paper related to an optimal control problem governed by these kinds of systems was [4], where the state system consists of (1.2)–(1.6) with $\alpha = \beta = 0$. There, the authors proved the existence of optimal control characterizing the first-order necessary condition for optimality by considering regular double-well potentials which exhibit polynomial growth. Recently, in [36], the author handles a similar optimal control problem considering system (1.2)–(1.6) as the state system in the case of singular, while regular, potentials allowing the logarithmic potential to be considered. In this direction, the relaxation terms $\alpha \partial_t \mu$ and $\beta \partial_t \varphi$ play a crucial role since, owing to their regularizing effect, they allow proving a uniform separation principle for the phase variable which was the key argument to handle such a singular nonlinearities. Next, the same author shows in [35], following the asymptotic scheme known in the literature as to deep quench limit, how non-regular potentials like the double-obstacle potential can be taken into account. Then, in [33],[34] the author proves, by invoking proper asymptotic strategies, how the optimal control problem for the case $\alpha, \beta > 0$ can be useful to solve the optimal control problems related to the state system above in which $\alpha = 0, \beta > 0$ and $\alpha > 0, \beta = 0$, respectively, by letting the parameters $\alpha$ and $\beta$ go to zero. Besides, we are also aware of the recent [3], where, after discussing the long-time behavior of solutions, the authors show that the optimal control problem [7] can be extended to the case in which the cost functional also depends on time. Referring to different models, we mention the contribution [23], where an optimal treatment time has been performed for a slightly different state system of Cahn-Hilliard type, where the control appears in the phase equation. Moreover, we refer to [38], where an optimal control problem for the two-dimensional Cahn–Hilliard-Darcy system with mass sources is addressed. Lastly, we point out [12],[13], where the optimal control for the standard tracking-type cost functional has been tackled for the more involved Cahn–Hilliard-Brinkman model, previously investigated by [14]. There, the authors analyze the control problem and provide a complete characterization of the optimality conditions. For the interested reader, we also mention [5], where a different kind of control problem, known as sliding mode control, is performed for a different system.

In this work, we aim at generalizing the control problem investigated in [36] taking inspiration from the techniques employed in [23] and [3]. Let us recall that the cost functional considered in [36] is independent of the time variable and reads as follows

$$
\hat{J}(\varphi, \sigma, u) = \frac{\hat{b}_1}{2} \int_Q |\varphi - \hat{\varphi}_Q|^2 + \frac{\hat{b}_2}{2} \int_\Omega |\varphi(T) - \hat{\varphi}_\Omega|^2 + \frac{\hat{b}_3}{2} \int_Q |\sigma - \hat{\sigma}_Q|^2 \\
+ \frac{\hat{b}_4}{2} \int_\Omega |\sigma(T) - \hat{\sigma}_\Omega|^2 + \frac{\hat{b}_0}{2} \int_Q |u|^2, \quad (1.10)
$$

for some non-negative constants $\hat{b}_0, ..., \hat{b}_4$ and some given target functions $\hat{\varphi}_Q, \hat{\sigma}_Q, \hat{\varphi}_\Omega, \hat{\sigma}_\Omega$ defined in proper functional spaces. Hence, the optimal control considered in [36] consists in minimizing $\hat{J}$ under the constrained that the control variable $u$ belongs to an admissible
set $\mathcal{U}_{\text{ad}}$ defined by

$$\mathcal{U}_{\text{ad}} := \{ u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q \},$$

(1.11)

where $u_*$ and $u^*$ denote some prescribed functions in $L^\infty(Q)$. Moreover, the variables $(\varphi, \sigma)$ have to obey to the state system (1.2)–(1.6). Hence, within the current work, we aim at extending what already proved to the time-dependent case by adding a free terminal time penalizing long-time treatments and an objective time to be approached. Moreover, we also introduce in the cost functional an additional penalization term for the large aggregates of tumor cells. Namely, the time-dependent objective cost functional we are going to deal with is

$$J(\varphi, \sigma, u, \tau) := b_1 \int_{Q_\tau} |\varphi - \varphi_Q|^2 + b_2 \int_\Omega |\varphi(\tau) - \varphi_\Omega|^2 + b_3 \int_{Q_\tau} |\sigma - \sigma_Q|^2$$

$$+ b_4 \int_\Omega (1 + \varphi(\tau)) + b_5 \tau + b_6 \frac{1}{2} |\tau - \tau_*|^2 + b_0 \int_{Q_\tau} |u|^2,$$

(1.12)

where the symbols $b_0, \ldots, b_6$ denote non-negative constants, while $\varphi_Q, \sigma_Q, \varphi_\Omega, \text{ and } \tau_*$ stand for the targets we want to approximate. For a thorough description, we refer to [7,33,34,36] and, here, we just point out the following comments:

(i) Despite the last term, the time integrals are performed between $\tau$ and $T$, where $\tau$ models the treatment time of the cycle which the patient undergoes the clinical therapy, while $T$ may be regarded as the maximum amount of time prescribed by some protocol that the patient is allowed to undergo the treatment. Let us claim that only minor changes are in order if one substitutes the term $b_5 \tau$ in the cost functional with a more general term like $b_5 f(\tau)$, where $f : [0, \infty) \rightarrow [0, \infty)$ is an increasing and continuously differentiable function.

(ii) Moreover, $\frac{b_6}{2} |\tau - \tau_*|^2$ forces the optimal time to be as close as possible to $\tau_*$ which models some target time to be reached. Please note that this new term is actually a convex function and, since dominant with respect to the linear term $b_5 \tau$, also their sum is indeed convex.

(iii) Minimizing the integral $\int_{Q_\tau} |\varphi - \varphi_Q|^2$ leads the phase variable $\varphi$ to be as close as possible, at the time $\tau$ and in the sense of the $L^2$-norm, to the fixed target $\varphi_Q$. In a similar fashion, it goes for the other variables. Thus, the functions $\varphi_Q, \sigma_Q, \varphi_\Omega, \text{ should be chosen as a stable configuration of the system or as some desirable configurations which are meaningful for surgery.}$

(iv) The last term $\int_Q |u|^2$ penalizes the large values of the control variable designing the side-effect that the dispensation of too many drugs to the patience may cause.

(v) The term $\frac{1 + \varphi(\tau)}{2}$ measures the size of the tumorous mass at the given time $\tau$. Hence, the corresponding term in the cost functional penalizes the strategies which do not shrink the tumor. Let us note that the presence of 1 in the numerator is due to the fact that in the healthy case $\varphi = -1$, so that in that case, the corresponding tumor mass is indeed zero.

(vi) The constants $b_0, \ldots, b_6$ can be chosen accordingly to the therapeutic goal we are interested in.
The main difference with respect to our starting point relies on the time-dependence of $\mathcal{J}$ and on the additional penalization terms $b_2 \tau$ and $b_4 \int_\Omega (1 + \varphi(\tau))$. In comparison to [23] and [3], we also introduce a target time $\tau_*$ to be approximated as best as possible. This choice has the advantage to produce a better characterization of the optimality of the time variable. Moreover, the case $\tau_* = 0$ complies with the framework there considered.

Then, one can suppress the variables $\varphi$ and $\sigma$ from the cost functional by expressing them as functions of $u$ leading to the corresponding reduced cost functional. Namely, we can define

$$
\mathcal{J}_{\text{red}}(u, \tau) := \mathcal{J}(S_2(u), S_3(u), u, \tau),
$$

(1.13)

where $S_2(u), S_3(u)$ denote the second and third components of $S$, respectively.

Although the existence result can be achieved by following similar reasoning, the corresponding first-order necessary conditions for optimality turns out to be quite different. In fact, owing to the convexity of the control-box $\mathcal{U}_{\text{ad}}$ and $[0, T]$, it follows from standard arguments (see, e.g., [29, 39]) that the optimality of $(\bar{u}, \bar{\tau}) \in \mathcal{U}_{\text{ad}} \times [0, T]$ can be characterized by the following variational inequalities

$$
\begin{cases}
D_u \mathcal{J}_{\text{red}}(\bar{u}, \bar{\tau})(v - \bar{u}, \bar{\tau}) \geq 0 & \text{for every } v \in \mathcal{U}_{\text{ad}} \\
D_\tau \mathcal{J}_{\text{red}}(\bar{u}, \bar{\tau})(s - \bar{s}, \bar{\tau}) \geq 0 & \text{for every } s \in [0, T],
\end{cases}
$$

(1.14)

where the symbol $D(\cdot) \mathcal{J}_{\text{red}}$ stands for the derivative of the reduced cost functional $\mathcal{J}_{\text{red}}$ with respect to the corresponding variable in a proper functional setting.

To obtain the necessary conditions for the time optimality, the key argument is to show that the reduced cost functional is Fréchet differentiable with respect to time, which in turn, requires higher order temporal regularity for the phase variable $\varphi$. Let us anticipate that the time derivative of the reduced cost functional will produce some terms involving $\varphi(\tau)$ and $\partial_\tau \varphi(\tau)$ (c.f. Theorem 3.7), where $\bar{\tau}$ stands for some optimal time and $\bar{\varphi}$ for some optimal state. A sufficient condition which gives meaning to the above terms is $\bar{\varphi} \in H^2(0, T; L^2(\Omega))$. In fact, owing to the canonical continuous injections of $H^2(0, T; L^2(\Omega))$ in $C^1([0, T]; L^2(\Omega))$, we deduce that both the pointwise terms $\varphi(\tau)$ and $\partial_\tau \varphi(\tau)$ are meaningful, at least in $L^2(\Omega)$. In this regards, let us also point out a limitation of our model. Despite natural, in the cost functional we do not consider any term involving $\int_\Omega |\varphi(\tau) - \sigma_\Omega|^2$, where $\sigma_\Omega$ models some target function. The remark gave above hints the reason. Indeed, if such a term is considered, in the time derivative of $\mathcal{J}_{\text{red}}$ it will appear $\partial_\tau \varphi(\tau)$ which, in turn, will require to show $\bar{\varphi} \in H^2(0, T; L^2(\Omega))$. However, the nutrient equation (3.30) is the one in which $u$ is placed. Therefore, in order to get stronger regularity for the nutrient $\sigma$, we are forced to assume the control variable to be sufficiently regular, say $u \in H^1(0, T; L^2(\Omega))$, which not significant for the applications. In the same manner, we are reduced to consider $\int_Q |u|^2$ instead of $\int_{Q_*} |u|^2$ in order avoid assuming any temporal regularity for the control variable. Regardless, a trade-off, in order to keep track of the nutrient variable at time $\tau$, could be to retrace the relaxation strategy employed by Garcke et al. in [23]. Let us refer the interested reader to Section 4 where further comments are provided in that direction.

For convenience, let us introduce the so-called admissible set by

$$
\mathcal{A}_{\text{ad}} := \left\{ (\varphi, \sigma, u, \tau) : (u, \tau) \in \mathcal{U}_{\text{ad}} \times [0, T], \text{ such that } (\varphi, \sigma) = (S_2(u), S_3(u)) \right\},
$$

(1.15)

which is the class on which the cost functional introduced above has to be minimized. Summing up, the paper is devoted to seeking for a solution to the following minimization
problem:
\[ (CP) \quad \inf_{(\varphi, \sigma, u, \tau) \in \mathcal{A}_{\text{ad}}} J(\varphi, \sigma, u, \tau) \]

and to providing the first-order necessary conditions for optimality.

**Plan of the Paper** The rest of the paper is outlined as follows. In the next section, we set our conventions, present the assumptions and our statements. The existence of optimal control and the first-order necessary conditions for optimality has been addressed in Section 3. Furthermore, in the last section, we point out some possible generalization of the work via a relaxation argument.

## 2 Mathematical Setting

This section is completely devoted to set some notation and describe our mathematical setting. For the spatial domain \( \Omega \), we assume it to be an open, bounded, and regular domain of \( \mathbb{R}^3 \) with boundary indicated by \( \Gamma \). Throughout the paper, for an arbitrary Banach space \( X \), we use \( \| \cdot \|_X \) to denote its norm, \( X^\ast \) for its topological dual, and \( X^\ast, \langle \cdot, \cdot \rangle_X \) for the duality product between \( X^\ast \) and \( X \). Meanwhile, for every \( p \in [1, +\infty] \), we simply write \( \| \cdot \|_p \) to indicate the usual norm of \( L^p(\Omega) \). Besides, it turns out to be convenient to set

\[ H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma \}, \]

where \( \partial_n \) stands for the outward normal derivative of \( \Gamma \). Moreover, we equip these spaces with their standard norms in order to work with Banach spaces. Under these assumptions, the injections \( V \hookrightarrow H \cong H^* \hookrightarrow V^* \) turn out to be both continuous and dense, which, as a consequence, entails that \( (V, H, V^*) \) forms a Hilbert triplet. Therefore, we identify in the usual way the duality product of \( V^* \) with the inner product of \( H \) in the following sense

\[ v^\ast, \langle u, v \rangle_V = \int_{\Omega} uv \quad \text{for every } u \in H \text{ and } v \in V. \]

Since we are considering [36] as our starting point, the choice of the mathematical framework is quite natural as just some additional requirements are needed in order to manage the problem we are dealing with. As far as the general assumptions are concerned, we postulate that

\[ \alpha, \beta \text{ are positive constants.} \quad (2.1) \]
\[ b_0, b_1, b_2, b_3, b_4, b_5 \text{ are non-negative constants, but not all zero.} \quad (2.2) \]
\[ \varphi_Q, \sigma_Q \in L^2(Q), \varphi_\Omega \in L^2(\Omega), u_*, u^* \in L^\infty(Q) \text{ with } u_* \leq u^* \text{ a.e. in } Q, \tau_* \in [0, T]. \quad (2.3) \]
\[ P \in C^2(\mathbb{R}) \text{ is non-negative, bounded and Lipschitz continuous.} \quad (2.4) \]
\[ \varphi_0 \in W, \mu_0 \in V \cap L^\infty(\Omega), \sigma_0 \in V. \quad (2.5) \]
\[ F(\varphi_0) \in L^1(\Omega). \quad (2.6) \]

Moreover, we assume that the control-box \( \mathcal{U}_{\text{ad}} \) is defined by (1.11), which in turn, implies it to be a closed and convex set of \( L^2(Q) \). On the other hand, it will be sometimes convenient to work with an open set. Hence, let us define the open superset \( \mathcal{U}_R \) as follows

\[ \mathcal{U}_R \subset L^2(Q) \text{ is a non-empty, bounded and open set containing } \mathcal{U}_{\text{ad}} \]

such that \( \| u \|_2 \leq R \) for all \( u \in \mathcal{U}_R \).
Asymptotic analyses of a control problem

As for the nonlinear double-well potential $F$, we require that

$$F : \mathbb{R} \to [0, +\infty), \quad \text{with} \quad F := \hat{B} + \hat{\pi}, \quad (2.7)$$

where

$$\hat{B} : \mathbb{R} \to [0, +\infty] \text{ is convex, and lower semicontinuous, with } \hat{B}(0) = 0. \quad (2.8)$$

$$\hat{\pi} \in C^3(\mathbb{R}) \text{ and } \pi := \hat{\pi}' \text{ is Lipschitz continuous.} \quad (2.9)$$

Then, it can be shown that $B := \partial \hat{B}$ is a maximal and monotone graph $B \subseteq \mathbb{R} \times \mathbb{R}$ (see, e.g., [2, Ex. 2.3.4, p. 25]) which domain we indicate by $D(B)$. Furthermore, we assume $F$, when restricted to $D(B)$, to be a smooth function. In fact, we also require that

$$D(B) = (r_-, r_+), \quad \text{with} \quad -\infty \leq r_- < 0 < r_+ \leq +\infty. \quad (2.10)$$

It is worth noting that both the regular potential (1.7) and the logarithmic potential (1.8) do fit the above assumptions. Moreover, we additionally require that the initial data verify

$$r_- < \inf \varphi_0 \leq \sup \varphi_0 < r_+, \quad (2.11)$$

which, from the physical viewpoint, means that $\varphi_0$ is not a pure phase. The above condition, combined with (2.5), leads us to infer that also

$$1/\beta (\mu_0 + \Delta \varphi_0 - B(\varphi_0) - \pi(\varphi_0)) \in H. \quad (2.12)$$

Up to now, the above mathematical setting is exactly the same as [36]. On the other hand, as already mentioned, the first-order necessary conditions for optimality that we will point out will demand higher order temporal regularity for the phase variable in order to give meaning to some appearing pointwise terms. We are able to overcome this issue by proving more regularity properties under some enforced assumptions. Namely, we replace (2.3) and (2.12) by the stronger

$$\varphi_0, \sigma_\Omega \in H^1(0, T; H) \quad (2.13)$$

$$1/\beta (\mu_0 + \Delta \varphi_0 - B(\varphi_0) - \pi(\varphi_0)) \in V, \quad (2.14)$$

respectively. Furthermore, condition (2.14) easily follows accounting for (2.5) provided to require also that $\varphi_0 \in H^3(\Omega)$.

Lastly, we recollect some well-known results which will be useful later on. At first, let us recall the standard Sobolev continuous embedding

$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{which holds for every } q \in [1, 6]. \quad (2.15)$$

Moreover, we often make use of the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \quad (2.16)$$

From now on, we convey to use the symbol small-case $c$ for every constant which only depend on structural data of the problem such as the final time $T$, $\Omega$, $R$, the shape of the nonlinearities, the norms of the involved functions, and possibly $\alpha$ and $\beta$. On the other hand, we devote the capital letters to designate some specific constants, which we eventually will refer to in the sequel.
3 The Control Problem

3.1 The State System

First, let us recall that some of the following results have been already proved in [36]. Hence, since no repetition of the proofs is needed here, we convey to specify in the brackets where the corresponding result can be found. Here, the well-posedness of the state system follows.

**Theorem 3.1** (see [36] Thms. 2.1, 2.2, and 2.3). Assume that requirements (2.1)–(2.12) are in force and let $u \in U_R$. Then, the state system (1.2)–(1.6) admits a unique solution $(\mu, \varphi, \sigma)$ satisfying

$$
\varphi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \subset C^0([0, T]; C^0(\Omega))
$$

(3.1)

$$
\mu, \sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V)
$$

(3.2)

$$
\mu \in L^\infty(Q).
$$

(3.3)

Moreover, there exists a positive constant $C_1$, which depends on $R, \alpha, \beta$, and on the data of the system, such that

$$
\|\varphi\|_{W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)} + \|\mu\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C_1.
$$

(3.4)

In addition, it holds the so-called uniform separation result. Namely, there exists a compact subset $K \subset (r_-, r_+) = D(B)$ such that

$$
\varphi(x, t) \in K \text{ for all } (x, t) \in Q.
$$

Furthermore, the following estimate

$$
\|\varphi\|_{C^0(\Omega)} + \max_{0 \leq i \leq 3} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} + \max_{0 \leq j \leq 2} \|P^{(j)}(\varphi)\|_{L^\infty(Q)} \leq C_2
$$

(3.5)

is satisfied for a positive constant $C_2$ which depends only on $R, \alpha, \beta, K$ and on the data of the system.

As already pointed out, in order to properly handle the control problem (CP), higher temporal regularity for the phase variable has to be established.

**Theorem 3.2.** Assume that (2.1)–(2.11) and (2.14) are satisfied and let $u \in U_R$. Then, the unique solution to (1.2)–(1.6), in addition to (3.1)–(3.3), enjoys the following regularity

$$
\varphi \in W^{1,\infty}(0, T; V) \cap H^2(0, T; H) \subset C^1([0, T]; H) \cap C^0([0, T]; H^2(\Omega)).
$$

(3.6)

Moreover, there exists a positive constant $C_3$ depending on $R, \alpha, \beta$, and on the data of the system, such that

$$
\|\varphi\|_{W^{1,\infty}(0,T;V) \cap H^2(0,T;H)} \leq C_3.
$$

(3.7)
Proof. By virtue of simplicity, we perform only a formal a priori estimate which can be carried out rigorously accounting for some approximation techniques, e.g. within a Faedo-Galerkin scheme.

We differentiate (1.3) with respect to time and multiply it by \( \partial_{tt} \varphi \), integrate over \( Q_t \) and by parts to obtain that

\[
\beta \int_{Q_t} |\partial_{tt} \varphi|^2 + \frac{1}{2} \int_{Q_t} |\nabla \partial_t \varphi(t)|^2 = \frac{1}{2} \int_{Q_t} |\nabla \partial_t \varphi(0)|^2 - \int_{Q_t} F''(\varphi) \partial_t \varphi \partial_{tt} \varphi + \int_{Q_t} \partial_t \mu \partial_{tt} \varphi,
\]

where the integrals on the right-hand side are denoted by \( I_1, I_2 \) and \( I_3 \), respectively. The terms on the left-hand side are non-negative, whereas \( I_2 \) and \( I_3 \) can be dealt by means of the Young inequality, along with estimate (3.5), which \( \varphi \), as a solution to (1.2)–(1.6), has to satisfy. In fact, we easily compute that

\[
|I_2| + |I_3| \leq \frac{\beta}{2} \int_{Q_t} |\partial_{tt} \varphi|^2 + c \int_{Q_t} (|\partial_t \mu|^2 + |\partial_t \varphi|^2).
\]

Moreover, taking \( t = 0 \) in (1.3), along with the additional condition (2.14), directly implies that

\[
|I_1| \leq c.
\]

Therefore, owing to estimate (3.4), we deduce that

\[
\|\partial_{tt} \varphi\|_{L^2(0,T;H)} + \|\nabla \partial_t \varphi\|_{L^\infty(0,T;H)} \leq c,
\]

which is the estimate we are looking for. Next, we easily recover that \( \partial_t \varphi \in C^0([0,T];H) \) by the well-known embedding of \( H^1(0,T;H) \) into \( C^0([0,T];H) \). Lastly, it turns out from comparison in equation (1.3) that also

\[
\Delta \varphi \in C^0([0,T];H),
\]

which conclude the proof. \( \square \)

The well-posedness result allow us to define the control-to-state operator \( S \) as the map which assigns to every control \( u \) the corresponding solution \( (\mu, \varphi, \sigma) \) to system (1.2)–(1.6). Furthermore, a continuous dependence result for (1.2)–(1.6), with respect to the control variable \( u \), has been established as well.

**Theorem 3.3** (see [36 Thms. 2.2 and 2.3]). Assume that (2.11)–(2.12) are in force. Moreover, for \( i = 1, 2 \), let \( u_i \in U_R \) and \( (\mu_i, \varphi_i, \sigma_i) \) be the corresponding states. Then, there exists a positive constant \( C_4 \), which depends only on \( R, \alpha \) and \( \beta \), and on the data of the system such that

\[
\|\alpha(\mu_1 - \mu_2) + (\varphi_1 - \varphi_2) + (\sigma_1 - \sigma_2)\|_{L^\infty(0,T;V^*)} + \|\mu_1 - \mu_2\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \|\sigma_1 - \sigma_2\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C_4\|u_1 - u_2\|_{L^2(0,T;H)}. \tag{3.8}
\]

Note that the above result can be equivalently understood as the Lipschitz continuity of the operator \( S \).
3.2 Existence of Optimal Control

With these results at disposal, we are in a position to obtain the existence for the optimal control problem we are dealing with.

**Theorem 3.4.** Assume that (2.1)–(2.12) are fulfilled. Then, the optimal control problem \((CP)\) admits at least a solution. Namely, there exists some \((\varphi, \sigma, u, \tau) \in A_{ad}\) which attains the infimum

\[ J(\varphi; \sigma, u, \tau) = \inf_{(\varphi, \sigma, u, \tau) \in A_{ad}} J(\varphi, \sigma, u, \tau). \]

The control variable \(u\) is referred to as optimal control, whereas \(\tau\) and \((\mu, \varphi, \sigma)\) as to optimal time and optimal state, respectively.

**Proof.** For the proof we rely on the well-known direct method of calculus of variations. First of all, let us check that the cost functional \(J\) is bounded from below, which, in turn, implies that the minimizing sequences are bounded. In fact, from the bound for the phase variable \(\varphi\) pointed out by (3.4)–(3.5), we infer that

\[ J(\varphi, \sigma, u, \tau) \geq b_1 \int_{\Omega} \varphi(\tau) \geq -b_1 \|\varphi\|_{C^0(Q)} \geq -b_1 C_2 > -\infty. \]

Next, we take a minimizing sequence for the cost functional \(\{u_n, \tau_n\}\) of elements of \(U_{ad} \times [0, T]\). That is, by denoting with \((\mu_n, \varphi_n, \sigma_n)\) the corresponding state, we have

\[ \lim_{n \to \infty} J(\varphi_n, \sigma_n, u_n, \tau_n) = \inf_{(\varphi, \sigma, u, \tau) \in A_{ad}} J(\varphi, \sigma, u, \tau) > -\infty. \]

On the other hand, for every \(n \in \mathbb{N}\), the bound provided by estimate (3.4) holds. Therefore, it is a standard matter to realize that, up to a not relabeled subsequence, there exist some \(u \in U_{ad}\) and a triplet \((\mu, \varphi, \sigma)\) such that, as \(n \to \infty\), we have

- \(u_n \to u\) weakly star in \(L^\infty(Q)\)
- \(\mu_n \to \mu\) weakly star in \(H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty(Q)\)
- \(\varphi_n \to \varphi\) weakly star in \(W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W)\)
- \(\sigma_n \to \sigma\) weakly star in \(H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)\)

Moreover, a compactness argument (see, e.g., [37, Sec. 8, Cor. 4]) yields that

\[ \varphi_n \to \varphi \text{ strongly in } C^0(\overline{Q}). \]

Furthermore, eventually after extracting once more, we also infer that there exists a \(\tau \in [0, T]\) such that

\[ \tau_n \to \tau \quad \text{as } n \to \infty. \]

Next, estimate (3.5), the strong convergence (3.9), and the properties of \(F\), allow us to identify the nonlinear terms in the limit realizing that

\[ F'(\varphi_n) \to F'(\varphi), \quad P(\varphi_n) \to P(\varphi) \text{ strongly in } C^0(\overline{Q}). \]

Thus, we can pass to the limit, as \(n\) goes to infinity, in the variational formulation of (1.2)–(1.6) written for \((\mu_n, \varphi_n, \sigma_n)\) obtaining that \((\mu, \varphi, \sigma) = S(\mu)\). Moreover, (3.10) ensures that, as \(n \to \infty\), we have

\[ \chi_{[0, \tau_n]}(t) \to \chi_{[0, \tau]}(t) \quad \text{for a.a. } t \in (0, T). \]
Then, let us claim that the limit \((\overline{\varphi}, \overline{\sigma}, \overline{u}, \overline{\tau})\) is indeed the minimizer we are looking for. Before showing how to pass to the limit term by term, let us point out that

\[
\int_{Q_{\tau_n}} |\cdot|^2 = \int_0^{\tau_n} \|\cdot\|_2^2 = \int_0^T \|\cdot\|_2^2 \chi_{[0, \tau_n]}.
\]

As a matter of fact, it easily follows from the above convergences that \(\varphi_n - \varphi_Q \to \overline{\varphi} - \varphi_Q\) strongly in \(L^2(0, T; H)\) which suffices to prove that

\[
\int_{Q_{\tau_n}} |\varphi_n - \varphi_Q|^2 \to \int_{Q_{\overline{\tau}}} |\overline{\varphi} - \varphi_Q|^2 \quad \text{as } n \to \infty. \tag{3.12}
\]

Indeed, we have that

\[
\int_0^T \left(\|\varphi_n - \varphi_Q\|_2^2 \chi_{[0, \tau_n]} - \|\overline{\varphi} - \varphi_Q\|_2^2 \chi_{[0, \overline{\tau}]}\right)
\leq \int_0^T \|\varphi_n - \varphi_Q\|_2^2 \left(\chi_{[0, \tau_n]} - \chi_{[0, \overline{\tau}]}\right) + \chi_{[0, \overline{\tau}]} \int_0^T \left(\|\varphi_n - \varphi_Q\|_2^2 - \|\overline{\varphi} - \varphi_Q\|_2^2\right),
\]

where both the terms on the right-hand side go to zero by combining the Lebesgue convergence theorem with the pointwise convergence \(\text{(3.11)}\) and the strong convergence of \(\varphi_n - \varphi_Q\). Next, let us claim that the second term of the cost functional verifies that

\[
\int_{\Omega} |\varphi_n(\tau_n) - \varphi_\Omega|^2 \to \int_{\Omega} |\overline{\varphi}(\tau) - \varphi_\Omega|^2 \quad \text{as } n \to \infty. \tag{3.13}
\]

In fact, considering the difference we have

\[
|\int_{\Omega} |\varphi_n(\tau_n) - \varphi_\Omega|^2 - \int_{\Omega} |\overline{\varphi}(\tau) - \varphi_\Omega|^2|
\leq \|\varphi_n(\tau_n) + \overline{\varphi}(\tau) - 2\varphi_\Omega\|_2 \|\varphi_n(\tau_n) - \overline{\varphi}(\tau)\|_2. \tag{3.14}
\]

Moreover, \((\text{3.9}) - (\text{3.10})\), along with the triangular inequality and the fundamental theorem of calculus, allow us to handle the last term as follows

\[
\|\varphi_n(\tau_n) - \overline{\varphi}(\tau)\|_2 \leq \|\varphi_n(\tau_n) - \varphi_n(\overline{\tau})\|_2 + \|\varphi_n(\overline{\tau}) - \overline{\varphi}(\overline{\tau})\|_2
\leq |\tau_n - \overline{\tau}|^\frac{1}{2} \left(\int_\tau^{\tau_n} \|\partial_t \varphi_n\|_2^2\right)^\frac{1}{2} + \|\varphi_n(\overline{\tau}) - \overline{\varphi}(\overline{\tau})\|_2
\leq |\tau_n - \overline{\tau}|^\frac{1}{2} \|\partial_t \varphi_n\|_{L^2(0,T;H)} + \|\varphi_n(\overline{\tau}) - \overline{\varphi}(\overline{\tau})\|_2.
\]

Note that the first term on the right-hand side vanishes accounting for the bound \(\text{(3.4)}\) and for the convergence \(\text{(3.10)}\), meanwhile, the second goes to zero for the strong convergence \(\text{(3.9)}\), so that \(\text{(3.13)}\) follows. Lastly, owing to \(\text{(3.10)}\), we easily infer that

\[
|\tau_n - \tau_s|^2 \to |\tau - \tau_s|^2 \quad \text{as } n \to \infty.
\]

The remaining terms can be handled arguing in a similar fashion. Hence, the weak sequential lower semicontinuity of \(J\), allow us to conclude that

\[
J(\overline{\varphi}, \overline{\sigma}, \overline{u}, \overline{\tau}) \leq \liminf_{n \to \infty} J(\varphi_n, \sigma_n, u_n, \tau_n) = \inf_{(\varphi, \sigma, u, \tau) \in A_{ad}} J(\varphi, \sigma, u, \tau),
\]

which entails that the limit \((\overline{\varphi}, \overline{\sigma}, \overline{u}, \overline{\tau})\) is indeed a minimizer for the optimal control problem \((CP)\), as we claimed. \(\square\)
Once the existence has been obtained, we aim at pointing out some first-order necessary conditions for optimality exploiting the theoretical conditions \((1.14)\). At first, we show the Fréchet differentiability of the reduced cost functional with respect to the control variable \(u\), and then with respect to the time variable.

### 3.3 The Linearized System

The first step toward the optimality starts from the investigation of the linearized system for system \((1.2)-(1.6)\). In this direction, let us pick a control \(\overline{u} \in \mathcal{U}_R\) with the corresponding state \((\overline{u}, \overline{\varphi}, \overline{\sigma})\), and consider an arbitrary \(h \in L^2(Q)\). Then, the linearized system for \((1.2)-(1.6)\) reads as

\[
\begin{align*}
\alpha \partial_t \eta + \partial_t \vartheta - \Delta \eta &= P'(\overline{\varphi})(\overline{\sigma} - \overline{u}) \vartheta + P(\overline{\varphi})(\rho - \eta) & \text{in } Q \quad (3.15) \\
\eta &= \beta \partial_t \vartheta - \Delta \vartheta + F''(\overline{\varphi}) \vartheta & \text{in } Q \quad (3.16) \\
\partial_t \rho - \Delta \rho &= -P'(\overline{\varphi})(\overline{\sigma} - \overline{u}) \vartheta - P(\overline{\varphi})(\rho - \eta) + h & \text{in } Q \quad (3.17) \\
\partial_h \rho = \partial_h \vartheta = \partial_h \eta &= 0 & \text{on } \Sigma \quad (3.18) \\
\rho(0) = \vartheta(0) = \eta(0) &= 0 & \text{in } \Omega. \quad (3.19)
\end{align*}
\]

The expectation is as follows: for every \(h \in L^2(Q)\), denoting with \((\eta, \vartheta, \rho)\) the corresponding solution to system \((3.15)-(3.19)\), provided to find the proper Banach space, the Fréchet derivative of \(S\) along the direction \(h\) is given by \(DS(\overline{u})h = (\eta, \vartheta, \rho)\). Note that the linearized system is the same of [36] since it is independent of the choice of the cost functional.

**Theorem 3.5** (see [36] Thm. 2.4). Assume that \((2.1)-(2.12)\) are fulfilled. Then, for every \(h \in L^2(Q)\), the linearized system \((3.15)-(3.19)\) admits a unique solution \((\eta, \vartheta, \rho)\) satisfying

\[
\eta, \vartheta, \rho \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \subset C^0([0,T];V). \quad (3.20)
\]

In addition, there exists a positive constant \(C_5\), which depends on the data of the system, and possibly on \(\alpha\) and \(\beta\), such that

\[
\begin{align*}
\|\eta\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\vartheta\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} + \|\rho\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} & \leq C_5.
\end{align*}
\]

Now, we can rigorously formulate our expectation concerning the Fréchet differentiability of the map \(S\) in the following way:

**Theorem 3.6** (see [36] Thm. 2.5). Assume that \((2.1)-(2.12)\) are satisfied and let \(\overline{u}\) and \((\overline{u}, \overline{\varphi}, \overline{\sigma})\) be a given optimal control with the corresponding state. Then, the control-to-state operator \(S\) is Fréchet differentiable at \(\overline{u}\) as a mapping from \(\mathcal{U}_R\) into the space \(\mathcal{Y}\), where

\[
\mathcal{Y} := \left( H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \right)^3. \quad (3.21)
\]

Moreover, for any \(\overline{u} \in \mathcal{U}_R\), the Fréchet derivative \(DS(\overline{u})\) is a linear and continuous operator from \(L^2(Q)\) to \(\mathcal{Y}\) which, for every \(h \in L^2(Q)\), is given by \(DS(\overline{u})h = (\eta, \vartheta, \rho)\), where \((\eta, \vartheta, \rho)\) is the unique solution to system \((3.15)-(3.19)\) corresponding to \(h\).
In order to exploit (1.14) to characterize the optimality conditions, we are left with the task of proving the Fréchet differentiability of the reduced cost functional $\mathcal{J}_{\text{red}}$ with respect to the time variable, which, taking advantage of Theorem 3.2, can be now presented in a rigorous way.

**Theorem 3.7.** Assume that (2.1)–(2.11) and (2.14) are in force, and (2.3) be replaced by (2.13). Let $(u, \tau)$ be an admissible pair and $(\mu, \varphi, \sigma)$ be the corresponding state. Then, the reduced cost functional $\mathcal{J}_{\text{red}}$ is Fréchet differentiable with respect to time and

$$D_{\tau}\mathcal{J}_{\text{red}}(u, \tau) = \frac{b_1}{2} \int_\Omega |\varphi(\tau) - \varphi_\Omega(\tau)|^2 + b_2 \int_\Omega (\varphi(\tau) - \varphi_\Omega) \partial_t \varphi(\tau)$$

$$+ \frac{b_1}{2} \int_\Omega |\sigma(\tau) - \sigma_\Omega(\tau)|^2 + \frac{b_1}{2} \int_\Omega \partial_t \varphi(\tau) + b_5 + b_6(\tau - \tau_\ast).$$

(3.22)

**Proof.** It simply follows from computing the derivative (see [23], where the authors checked how the time derivative can be performed in a general framework) which turns out to be meaningful by virtue of the regularity we have for $\varphi$ and $\sigma$. □

Here, let us emphasize that the terms $b_2 \int_\Omega (\varphi(\tau) - \varphi_\Omega) \partial_t \varphi(\tau) + \frac{b_1}{2} \int_\Omega \partial_t \varphi(\tau)$ are meaningful by virtue of Theorem 3.2. On the other hand, we will see that the final condition for the adjoint variable $q$ is $\beta q_\tau(\tau) = b_2 (\varphi_\tau(\tau) - \varphi_\Omega) + \frac{b_1}{2}$, so that the above terms can be rewritten as $\beta \int_\Omega \partial_t \varphi(\tau) q(\tau)$. Furthermore, this integral can be characterized in different ways providing that the variational formulation for (1.2)–(1.6) is satisfied pointwise for all $t \in [0, T]$. In fact, by multiplying by $\beta$ the variational formulation of equation (1.2) and from the variational formulation of (1.3) tested by $q$, we infer that

$$\beta \int_\Omega \partial_t \varphi(t) q(t) = \int_\Omega \Delta \varphi(t) q(t) - \int_\Omega F'(\varphi(t)) q(t) + \int_\Omega \mu(t) q(t) \quad \text{for all } t \in [0, T],$$

and that

$$\beta \int_\Omega \partial_t \varphi(t) q(t) = \beta \int_\Omega P(\varphi(t)) (\sigma(t) - \mu(t)) q(t) - \alpha \beta \int_\Omega \partial_t \mu(t) q(t) + \beta \int_\Omega \Delta \mu(t) q(t)$$

for all $t \in [0, T]$.

This is completely rigorous if we can ensure that

$$\sigma, q \in C^0([0, T]; H), \partial_\mu, \partial_\varphi \in C^0([0, T]; H), \text{ and } \mu, \varphi \in C^0([0, T]; H^2(\Omega)).$$

On the other hand, the regularity pointed out by (3.1)–(3.3) and Theorem 3.2 do not ensure the continuity of $\mu$ and of its derivative. However, let us claim that this can be proved arguing as in Theorem 3.2 providing to require that $\mu_0 \in H^3(\Omega)$. Indeed, we have:

**Theorem 3.8.** Assume that (2.1)–(2.11) and (2.14) are verified and let $\mu_0 \in H^3(\Omega)$. Then, the unique solution to (1.2)–(1.6), in addition to (3.1)–(3.3), satisfies

$$\mu \in W^{1, \infty}(0, T; V) \cap H^2(0, T; H) \subset C^1([0, T]; H) \cap C^0([0, T]; H^2(\Omega)).$$

(3.23)

Moreover, there exists a positive constant $C_6$ depending on $R$, $\alpha$, $\beta$, and on the data of the system, such that

$$\|\mu\|_{W^{1, \infty}(0, T; V) \cap H^2(0, T; H)} \leq C_6.$$

(3.24)
Proof. Differentiating (1.2) with respect to time, multiplying it by $\partial_{tt}\mu$, and integrating over $Q_t$ and by parts lead to obtain that
\[
\alpha \int_{Q_t} |\partial_{tt}\mu|^2 + \frac{1}{2} \int_{\Omega} |\nabla \partial_t \mu(t)|^2 = \frac{1}{2} \int_{\Omega} |\nabla \partial_t \mu(0)|^2 + \int_{Q_t} P'(\varphi)(\sigma - \mu) \partial_t \varphi \partial_{tt} \mu \\
+ \int_{Q_t} P(\varphi)(\partial_t \sigma - \partial_t \mu) \partial_{tt} \mu - \int_{Q_t} \partial_{tt} \varphi \partial_{tt} \mu,
\]
where we indicate by $I_1, \ldots, I_4$ the integrals on the right-hand side, in this order. The first term can be controlled combining assumption (2.5) with (2.14) and the new $\mu_0 \in H^3(\Omega)$. In fact, evaluating equation (1.2) at $t = 0$, and then (1.3) at $t = 0$ allow us to realize that
\[
\partial_t \mu(0) = \frac{1}{\alpha} \left[ \frac{1}{\beta} (-\mu_0 - \Delta \varphi_0 + F'(\varphi_0)) + \Delta \mu_0 + P(\varphi_0)(\sigma_0 - \mu_0) \right],
\]
which belongs to $V$ under these assumptions. Therefore, we get that
\[
|I_1| \leq c.
\]
Then, thanks to (3.4)–(3.5) and to (3.7), to Hölder inequality, and to the Sobolev embedding $V \subset L^4(\Omega)$, we have
\[
|I_2| \leq c \int_0^t (\|\sigma\|_4 + \|\mu\|_4) \|\partial_t \varphi\|_4 \|\partial_{tt} \mu\|_2 \\
\leq \delta \int_{Q_t} |\partial_{tt} \mu|^2 + c_\delta \int_{Q_t} (\|\sigma\|_V^2 + \|\mu\|_V^2) \|\partial_t \varphi\|_V^2 \\
\leq \delta \int_{Q_t} |\partial_{tt} \mu|^2 + c_\delta,
\]
where in the last line we also invoke the fact that, owing to (3.1)–(3.3) and to (3.7), we have that $\|\sigma\|_V, \|\mu\|_V$ and $\|\partial_t \varphi\|_V$ all belong to $L^\infty(0, T)$. Using the Young inequality, along with (3.5), we can easily manage the last two terms as
\[
|I_3| + |I_4| \leq \delta \int_{Q_t} |\partial_{tt} \mu|^2 + c_\delta \int_{Q_t} (|\partial_t \varphi|^2 + |\partial_t \sigma|^2 + |\partial_t \mu|^2),
\]
for a positive $\delta$ yet to be determined. Hence, adjusting $\delta \in (0, 1)$ small enough and accounting for the above estimates, we infer that
\[
\|\partial_{tt} \mu\|_{L^2(0, T; H)} + \|\nabla \partial_t \mu\|_{L^2(0, T; H)} \leq c.
\]
Arguing as above, we easily deduce that $\partial_t \mu \in C^0([0, T]; H)$ and then, by comparison in equation (1.2) we also conclude that
\[
\Delta \mu \in C^0([0, T]; H).
\]

Hence, under the framework of Theorem 3.2 and Theorem 3.8 the variational formulations written above are completely meaningful in a pointwise sense.

Now, let us move to the optimality conditions noting that Theorem 3.6 and Theorem 3.7 allow us to express (1.14) in a convenient way to obtain the first-order necessary condition for the optimal control problem (CP).
Theorem 3.9. Assume that (2.1)–(2.11) and (2.14) are fulfilled. Furthermore, let (2.3) be replaced by (2.13), and \( \overline{\upsilon} \) be an optimal control. Then, \( \overline{\upsilon} \) satisfies the following variational inequality
\[
b_1 \int_{Q_r} (\varphi - \varphi_Q) \psi + b_2 \int_{\Omega} (\varphi(\tau) - \varphi_\Omega) \partial(\tau) + b_3 \int_{Q_r} (\sigma - \sigma_Q) \rho + b_4 \int_{\Omega} \psi(\tau) + b_5 \int_{\Omega} \varphi(\tau) + b_6 \int_{\Omega} \sigma(\tau) \geq 0 \quad \text{for every} \quad \psi \in \mathcal{U}_{ad},
\]
where \( \varphi \) and \( \rho \) are the solutions to the linearized system (3.15)–(3.19) corresponding to \( h = v - \overline{u} \). Moreover, we have that
\[
D_T\mathcal{J}_{red}(\overline{\upsilon}, \tau) \begin{cases} 
\geq 0 & \text{if} \ \tau = 0 \\
= 0 & \text{if} \ \tau \in (0, T), \\
\leq 0 & \text{if} \ \tau = T
\end{cases}
\]
where \( D_T\mathcal{J}_{red}(\overline{\upsilon}, \tau) \) is given by (3.22) evaluated at the optimum pair \( (\overline{\upsilon}, \tau) \). In addition, denoting by
\[
\Lambda(\overline{\upsilon}, \tau) := \frac{b_1}{2} \int_{\Omega} |\varphi(\tau) - \varphi_Q(\tau)|^2 + b_2 \int_{\Omega} (\varphi(\tau) - \varphi_\Omega) \partial_t \varphi(\tau)
\]
\[
+ b_3 \int_{\Omega} |\sigma(\tau) - \sigma_Q(\tau)|^2 + b_4 \int_{\Omega} \partial_t \varphi(\tau) + b_5,
\]
it follows that \( D_T\mathcal{J}_{red}(\overline{\upsilon}, \tau) = \Lambda(\overline{\upsilon}, \tau) + b_6(\tau - \tau_*) \). Hence, whenever \( b_6 \neq 0 \), condition (3.26) can be implicitly characterized as follows
\[
\begin{cases}
\Lambda(\overline{\upsilon}, 0) \geq b_6 \tau_* & \text{if} \ \tau = 0 \\
\tau = \tau_* - b_6^{-1} \Lambda(\overline{\upsilon}, \tau) & \text{if} \ \tau \in (0, T), \\
\Lambda(\overline{\upsilon}, T) \leq b_6(\tau_* - T) & \text{if} \ \tau = T.
\end{cases}
\]

Proof. As already mentioned, (3.25) and (3.26) can be carried out by exploiting the formal conditions (1.14). As the first variational inequality is concerned, let us claim that, loosely speaking, \( \mathcal{J}_{red} \) is nothing but the composition of \( \mathcal{J} \) with \( S \). So, it suffices to combine the Fréchet differentiability of the two operators with the chain rule to get (3.25). In this direction, let us introduce the auxiliary function \( \tilde{S} : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{U}_R \), where
\[
\mathcal{X} := (H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W))^2,
\]
defined by \( \tilde{S}(u) := (S_2(u), S_3(u), u) \). Therefore, Theorem 3.6 allows us to realize that
\[
D_T\tilde{S}(u) : h \mapsto (\varphi, \rho, h) \quad \text{for every} \quad h \in \mathcal{U}_R,
\]
where \( (\eta, \varphi, \rho) \) is the solution to the linearized system (3.15)–(3.19) corresponding to \( h \). Hence, we infer that \( \mathcal{J}_{red}(u, \tau) = \mathcal{J}(\tilde{S}(u), \tau) \). Moreover, it is straightforward to realize that \( \mathcal{J} \) is Fréchet differentiable with respect to \( u \) and that for every \( \tau \in [0, T] \) we have that
\[
[D_u\mathcal{J}(\varphi, \sigma, u, \tau)](\Phi, \Psi, h, \tau) = b_1 \int_{Q_r} (\varphi - \varphi_Q) \Phi + b_2 \int_{\Omega} (\varphi(\tau) - \varphi_\Omega) \Phi(\tau)
\]
\[
+ b_3 \int_{Q_r} (\sigma - \sigma_Q) \Psi + \frac{b_4}{2} \int_{\Omega} \Phi(\tau) + b_5 \int_{Q} uh
\]
for every \( (\Phi, \Psi, h) \in \mathcal{X} \times \mathcal{U}_R \).
Thus, we invoke the chain rule to obtain that

\[
[D_u g(\bar{u}, \bar{\tau})](h, \tau) = \left[D_u g\left(\bar{S}(\bar{u}), \bar{\tau}\right)\right]\left(\left[D\bar{S}(\bar{u})\right](h, \tau) = \left[D_u g\left(\bar{\sigma}, \bar{\tau}, \bar{u}, \bar{\tau}\right)\right]\left(\partial, \rho, h, \tau\right)
\]

\[
= b_1 \int_{Q_{\bar{\tau}}} (\bar{\varphi} - \varphi_Q) \vartheta + b_2 \int_{Q_{\bar{\tau}}} (\bar{\varphi} - \varphi_\Omega) \vartheta(\bar{\tau}) + b_3 \int_{Q_{\bar{\tau}}} (\bar{\varphi} - \sigma_Q) \rho
\]

\[
+ b_4 \int_{Q_{\bar{\tau}}} \vartheta(\bar{\tau}) + b_5 \int_{Q_{\bar{\tau}}} \pi h,
\]

so that \((1.14)\) can be written as \((3.25)\).

Furthermore, \((3.26)\) directly follows from the second of \((1.14)\) along with \((3.22)\). Indeed, the first and the last conditions of \((3.26)\) are consequences of the fact that we cannot exclude the cases \(s = 0\) and \(s = T\), while the middle one follows from the fact that, whenever \(\bar{\tau} \in (0, T)\), we can simply take \(s = \bar{\tau} \pm \zeta\), with \(\zeta > 0\), to argue that \(D_{\tau} g(\bar{u}, \bar{\tau}) = 0\).

\[
\text{Let us emphasize that the above \((3.27)\) is new with respect to \((3.23)\), where only condition \((3.26)\) was obtained. Actually, the introduction of the tracking-type term \(\frac{1}{2} |\tau - \tau_s|^2\) in the cost functional was motivated by the fact that it will allow us to write condition \((3.26)\) in a more explicit way leading to \((3.27)\).}
\]

### 3.4 Adjoint System and First-order Optimality Condition

In this section, we face the adjoint system which is the key argument to simplify the variational formulation \((3.25)\). Only straightforward modifications have to be performed with respect to \([36]\) and it can be easily shown that the formal adjoint problem reads as

\[
-\beta \partial_t q - \partial_t p - \Delta q + F'(\bar{\tau}) q + P'(\bar{\varphi})(\bar{\varphi} - \bar{\pi})(r - p) = b_1 (\bar{\varphi} - \varphi_Q) \quad \text{in } Q_{\bar{\tau}} \quad (3.28)
\]

\[
-\alpha \partial_t p - \Delta p - q + P(\bar{\varphi})(p - r) = 0 \quad \text{in } Q_{\bar{\tau}} \quad (3.29)
\]

\[
-\partial_t r - \Delta r + P(\bar{\varphi})(r - p) = b_3 (\bar{\varphi} - \sigma_Q) \quad \text{in } Q_{\bar{\tau}} \quad (3.30)
\]

\[
\partial_n q = \partial_n p = \partial_n r = 0 \quad \text{on } \Sigma_{\bar{\tau}} \quad (3.31)
\]

\[
\beta q(\bar{\tau}) = b_2 (\bar{\varphi}(\bar{\tau}) - \varphi_\Omega) + \frac{b_4}{2}, \quad p(\bar{\tau}) = 0, \quad r(\bar{\tau}) = 0 \quad \text{in } \Omega. \quad (3.32)
\]

Let us emphasize that we also perform a change of variable passing from \(q\) to \(-q\) since in this framework it is more convenient. The main differences with respect to \([36]\) are that we have to consider the above system for \(a.a.t \in (0, \bar{\tau})\) and that in the final condition \(\beta q(\bar{\tau})\) there appears a new term \(b_4/2\) which is due to the presence of \(\frac{1}{2} \int_{Q_{\bar{\tau}}} (1 + \varphi(\tau))\) in the cost functional. Moreover, the final condition for \(r\) is zero since we are not considering any term involving \(\int_{Q_{\bar{\tau}}} |\sigma(\tau) - \sigma_\Omega|^2\) in the cost functional. Note that the above system is a backward-in-time problem with terminal data for \(q\) only belonging to \(L^2(\Omega)\) (see \((2.3)\)). Therefore, for that variable we cannot expect to recover strong regularity and equation \((3.28)\) should be considered in terms of the corresponding variational formulation.

**Theorem 3.10.** Assume that the assumptions \((2.1)\)–\((2.12)\) are verified. Then, the system \((3.28)\)–\((3.32)\) admits a unique solution \((q, p, r)\) that satisfies the following regularity

\[
q \in H^1(0, \bar{\tau}; V^*) \cap L^\infty(0, \bar{\tau}; H) \cap L^2(0, \bar{\tau}; V) \subset C^0([0, \bar{\tau}]; H) \quad (3.33)
\]

\[
p, r \in H^1(0, \bar{\tau}; H) \cap L^\infty(0, \bar{\tau}; V) \cap L^2(0, \bar{\tau}; W) \subset C^0([0, \bar{\tau}]; V). \quad (3.34)
\]
Proof. Below, we proceed formally since the approach is quite standard and we just point out that within a Faedo-Galerkin scheme the argument can be made rigorous.

First estimate First we add to both sides of (3.29) and (3.30) the terms $p$ and $r$, respectively. Then, we multiply (3.28) by $q$, the new (3.29) by $\partial_r p$, the new (3.30) by $\partial_r r$, add the resulting equalities, integrate over $Q_T$ and by parts. After some rearrangements and a cancellation, we obtain that

\[
\frac{\beta}{2} \|q(t)\|_H^2 + \int_{Q_T} |\nabla q|^2 + \alpha \int_{Q_T} |\partial_r p|^2 + \frac{1}{2} \|p(t)\|_H^2 + \frac{1}{2} \|\nabla p(t)\|_H^2 \\
+ \int_{Q_T} |\partial_t r|^2 + \frac{1}{2} \|r(t)\|_H^2 + \frac{1}{2} \|\nabla r(t)\|_H^2 \\
= \frac{\beta}{2} \|q(\tau)\|_H^2 + \int_{Q_T} b_1(\varphi - \varphi q) - \int_{Q_T} b_3(\sigma - \sigma q) \partial_r r \\
- \int_{Q_T} F''(\varphi)|q|^2 - \int_{Q_T} F'(\varphi)(\sigma - \mu)(r - p)q + \int_{Q_T} P(\varphi)(p - r) \partial_t p \\
- \int_{Q_T} p \partial_r p + \int_{Q_T} P(\varphi)(r - p) \partial_r r - \int_{Q_T} r \partial_r r,
\]

where we denote by $I_1, ..., I_6$ the integrals on the right-hand side. Moreover, invoking the Young inequality and recalling (2.5) and (3.5), we easily obtain that

\[
|I_1| + |I_2| + |I_3| + |I_4| + |I_5| + |I_6| \\
\leq \delta \int_{Q_T} (|\partial_r p|^2 + |\partial_r r|^2) + c_8 \int_{Q_T} (|q|^2 + |p|^2 + |r|^2) + c,
\]

for a positive $\delta$ yet to be determined and for a positive constant $c_8$ which only depends on $\delta$. Furthermore, using the Hölder and Young inequalities we obtain that

\[
|I_5| + |I_6| + |I_8| \leq c \int_{Q_T} (|\varphi|_4 + |\varphi|_4)(||r||_4 + ||p||_4)||q||_2 \\
+ c \int_{Q_T} ((|p| + |r|)(|\partial_r p| + |\partial_r r|) \\
\leq c \int_{Q_T} (||\varphi||_4 + ||\varphi||_4)(||r||_4 + ||p||_4)||q||_H \\
+ \delta \int_{Q_T} (|\partial_r p|^2 + |\partial_r r|^2) + c_8 \int_{Q_T} (|p|^2 + |r|^2) \\
\leq c \int_{Q_T} (||\varphi||^2_4 + ||\varphi||^2_4)(||r||^2_4 + ||p||^2_4) + c \int_{Q_T} |q|^2 \\
+ \delta \int_{Q_T} (|\partial_r p|^2 + |\partial_r r|^2) + c_8 \int_{Q_T} (|p|^2 + |r|^2),
\]

where we also invoked the embedding $V \subset L^4(\Omega)$ and that $\varphi$ and $\varphi$, as solutions to (1.2)–(1.6), have to satisfy (3.4), verifying that $(||\varphi||^2_4 + ||\varphi||^2_4) \in L^\infty(0, T)$. Lastly, adjusting $\delta \in (0, 1)$ small enough, a Gronwall argument yields that

\[
||q||_{L^\infty(0, T; V)} + ||p||_{H^1(0, T; V)} + ||r||_{H^1(0, T; V)} \leq c.
\]
Second estimate Next, comparison in (3.29) and then in (3.30), along with the above estimate, and elliptic regularity theory allow us to deduce that

$$\|p\|_{L^2(0,\tau;W)} + \|r\|_{L^2(0,\tau;W)} \leq c.$$  

Third estimate Lastly, by comparison in (3.28) we immediately realize that

$$\|\partial_t q\|_{L^2(0,\tau;V^*)} \leq c,$$

which conclude the proof since the uniqueness directly follows from existence since system (3.28)–(3.32) is linear.

Next, accounting for the adjoint variables, we can eliminate the linearized variables from the variational inequality (3.25) in order to produce a simpler form for the first-order necessary condition for optimality.

Theorem 3.11. Suppose that (2.1)–(2.11) and (2.14) are satisfied. Let $(\pi, \tau)$, $(\widehat{\pi}, \widehat{\tau}, \widehat{\tau})$ and $(q, p, r)$ be an optimum pair with the corresponding optimal state and adjoint state, respectively. Then, the necessary condition for optimality is carried out by the following variational inequality

$$\int_Q r(v - \pi) + b_0 \int_Q \pi(v - \pi) \geq 0 \quad \text{for every } v \in U_{ad}. \tag{3.35}$$

Corollary 3.12. Assume that (2.1)–(2.11) and (2.14) are fulfilled. Moreover, let us set $\widehat{r}$ as the zero extension of $r$ in the whole of $[0, T]$. Then, the above variational inequality became

$$\int_Q (\widehat{r} + b_0 \pi)(v - \pi) \geq 0 \quad \text{for every } v \in U_{ad}. \tag{3.36}$$

Then, whenever $b_0 \neq 0$, the optimal control $\pi$ is nothing but the $L^2(0, T; H)$–orthogonal projection of $-b_0^{-1} \widehat{r}$ onto the closed subspace $U_{ad}$.

Note that the case $\tau = 0$ covers a special, since trivial, role. As a matter of fact, in this case the above variational inequality (3.35) reduces to

$$b_0 \int_Q \pi(v - \pi) \geq 0 \quad \text{for every } v \in U_{ad},$$

which, whenever $b_0 > 0$, yields that $\pi$ is nothing but the orthogonal projection of 0 onto the closed subspace $U_{ad}$. The same consequence can be drawn from evaluating inequality (3.25) at $\tau = 0$ and using that $\vartheta(0) = 0$.

It is worth noting that, as a consequence of (3.36), we can identify, via Riesz’s representation theorem, the gradient of the reduced cost functional as below

$$\nabla J_{\text{red}}(\pi, \tau) = \widehat{r} + b_0 \pi.$$  

This fact is extremely important from the numerical viewpoint since it implies the possibility to analyze the optimal control problem $\text{CP}$ as a constrained minimization problem via standard techniques (e.g. with the conjugate gradient method).

Lastly, let us conclude by checking Theorem 3.11.
Proof of Theorem 3.11. By virtue of simplicity we proceed formally by employing the formal Lagrangian method (see, e.g., [29, 39]). Comparing the two variational inequalities (3.25) and (3.35), we realize that it suffices to check that

$$\int_{Q_\tau} rh = b_1 \int_{Q_\tau} (\overline{\sigma} - \varphi_Q) \vartheta + b_2 \int_\Omega (\overline{\sigma}(\tau) - \varphi_\Omega) \vartheta(\tau) + b_3 \int_{Q_\tau} (\overline{\sigma} - \sigma_Q) \rho + \frac{b_4}{2} \int_\Omega \vartheta(\tau),$$

where \(h\) is taken as \(h = v - \overline{u}\) and \((\eta, \vartheta, \rho)\) is the unique corresponding solution to (3.15)–(3.19). Hence, we multiply (3.15) by \(p\), (3.16) by \(q\), (3.17) by \(r\), and integrate over \(Q_\tau\) to get

$$0 = \int_{Q_\tau} p[\alpha \partial_t \eta + \partial_t \vartheta - \Delta \eta - P'(\overline{\sigma})(\overline{\sigma} - \overline{u}) \vartheta - P(\overline{\sigma})(\rho - \eta)]$$

$$+ \int_{Q_\tau} q[\beta \partial_t \vartheta - \Delta \vartheta + F'(\overline{\sigma}) \vartheta - \eta]$$

$$+ \int_{Q_\tau} r[\partial_t \rho - \Delta \rho + P'(\overline{\sigma})(\overline{\sigma} - \overline{u}) \vartheta + P(\overline{\sigma})(\rho - \eta) - h].$$

Then, we move the last term to the left-hand side and integrate by parts to obtain that

$$\int_{Q_\tau} rh = \int_{Q_\tau} \eta[-\alpha \partial_t p - \Delta q - q + P(\overline{\sigma})(\rho)]$$

$$+ \int_{Q_\tau} \vartheta[-\beta \partial_t q - \partial_t p - \Delta q + F'(\overline{\sigma})q + P'(\overline{\sigma})(\overline{\sigma} - \overline{u})(r - p)]$$

$$+ \int_{Q_\tau} \rho[-\partial_t r - \Delta r + P(\overline{\sigma})(r - p)]$$

$$+ \int_\Omega [\alpha \eta(\overline{\sigma}) p(\overline{\sigma}) + \vartheta(\overline{\sigma}) p(\overline{\sigma}) + \beta \vartheta(\overline{\sigma}) q(\overline{\sigma}) + \rho(\overline{\sigma}) r(\overline{\sigma})],$$

where we also owe to the homogeneous Neumann boundary conditions for the linearized and adjoint variables, and to the initial condition for the linearized variables. Finally, accounting for the adjoint system (3.28)–(3.32), we conclude that the above equation reduces to

$$\int_{Q_\tau} rh = b_1 \int_{Q_\tau} (\overline{\sigma} - \varphi_Q) \vartheta + b_2 \int_\Omega (\overline{\sigma}(\tau) - \varphi_\Omega) \vartheta(\tau) + b_3 \int_{Q_\tau} (\overline{\sigma} - \sigma_Q) \rho + \frac{b_4}{2} \int_\Omega \vartheta(\tau),$$

which conclude the proof allowing to obtain (3.35) from (3.25).

\[\square\]

4 Some Possible Generalizations

In the remainder of the work, we aim at providing some indications regarding some possible generalizations. First, we will show how to possibly overcome the issue already mentioned regarding the control of the nutrient at the given time \(\tau\). Next, we will spend some words concerning a similar control problem in which the role of the control variable differs from our model.
4.1 A Relaxation Argument

From the mathematical viewpoint, a natural term to be considered in the cost functional is $\int_\Omega |\sigma(\tau) - \sigma_\Omega|^2$. However, as already emphasized, in order to give meaning to the necessary conditions that will eventually appear, further temporal regularity for $\sigma$ has to be established. This will demand the control $u$ to be more regular, say $H^1(0,T;H)$, which is far to be satisfied in the application. Anyhow, a possible way to overcome this issue could be to follow the relaxation strategy employed in [23]. To this aim, let us fix a positive constant $r$ and define the relaxed cost functional $J_r$ as follows

$$J_r(\varphi, \sigma, u, \tau) := J(\varphi, \sigma, u, \tau) + \frac{\gamma}{2r} \int_{\tau-r}^\tau \int_\Omega |\sigma - \sigma_\Omega|^2,$$

for a non-negative constant $\gamma$ and for a given target function $\sigma_\Omega$. Note that the factor $1/r$ is due to normalization since $\frac{1}{r} \int_{\tau-r}^\tau = 1$. In this way, we are also able to take into account the final configuration of the nutrient, without demanding additional regularity for the nutrient variable $\sigma$. This choices directly lead to the minimization problem:

$$(CP)_r \inf_{(\varphi,\sigma,u,\tau) \in A_{ad}} J_r(\varphi, \sigma, u, \tau).$$

Clearly, the most part of the results follow the same lines as above. Hence, we proceed quite schematically, just pointing out the main differences.

**Existence** The first arrangements are concerned with the existence of optimal control. The proof can be reproduced provided to explain how the new term of the cost functional can be handled. In this direction, let us point out that, along with (3.11), we also have

$$\chi_{[\tau_n,\tau_n]}(\cdot) \to \chi_{[\tau,\tau]}(\cdot) \quad \text{for a.a. } t \in (0,T).$$

Hence, by similar reasoning, we also conclude that

$$\frac{\gamma}{2r} \int_{\tau_n}^{\tau} \int_\Omega |\sigma_\Omega(\tau_n) - \sigma_\Omega|^2 \to \frac{\gamma}{2r} \int_{\tau-r}^\tau \int_\Omega |\sigma(\tau) - \sigma_\Omega|^2$$

as $n \to \infty$,

while the rest of the proof is exactly the same.

**Fréchet differentiability of the reduced cost functional** As expected, the main differences are related to the Fréchet differentiability of the corresponding reduced cost functional. In fact, the corresponding of (3.25) becomes

$$b_1 \int_0^\tau \int_\Omega (\varphi - \varphi_Q) \vartheta + b_2 \int_\Omega (\varphi(\tau) - \varphi_\Omega) \vartheta(\tau) + b_3 \int_0^\tau \int_\Omega (\sigma - \sigma_Q) \rho$$

$$+ \frac{\gamma}{r} \int_{\tau-r}^\tau \int_\Omega (\sigma - \sigma_\Omega) \rho + b_4 \int_\Omega \vartheta(\tau) + b_0 \int_\tau^Q (v - \bar{v}) \geq 0 \quad \text{for every } v \in U_{ad}. $$

As the time derivative is concerned, we have to adjust a little the framework by assuming that $\sigma_\Omega \in H^1(-r,T;H)$ and that the variable $\sigma$ is meaningful for negative time. Hence, we simply postulate that $\sigma(t) := \sigma_0$ if $t < 0$. Thus, the corresponding
Fréchet derivative with respect to time reads as

\[
D_r \mathcal{J}_\text{red} (u, \tau) = \frac{b_1}{2} \int_\Omega |\varphi(\tau) - \varphi_Q(\tau)|^2 + b_2 \int_\Omega (\varphi(\tau) - \varphi_\Omega) \partial_t \varphi(\tau)
+ \frac{b_3}{2} \int_\Omega |\sigma(\tau) - \sigma_Q(\tau)|^2
+ \frac{\gamma}{2r} \left( \int_\Omega |(\sigma - \sigma_\Omega)(\tau)|^2 - \int_\Omega |(\sigma - \sigma_\Omega)(\tau - r)|^2 \right)
+ \frac{b_4}{2} \int_\Omega \partial_t \varphi(\tau) + b_5 + b_6(\tau - \tau_\ast).
\]

The adjoint system Lastly, the adjoint system slightly differs and becomes

\[
\begin{align*}
- \beta \partial_t q - \partial_t p - \Delta q + F''(\varphi)q + P'(\varphi)(\sigma - \mu)(r - p) &= b_1(\varphi - \varphi_Q) & \text{in } Q_{\varphi} \\
- \alpha \partial_t p - \Delta p - q + P(\varphi)(p - r) &= 0 & \text{in } Q_{\varphi} \\
- \partial_t r - \Delta r + P(\varphi)(r - p) &= b_3(\varphi - \varphi_Q) + \frac{\gamma}{r} \chi_{(\varphi - r, \varphi)}(\varphi - \sigma_\Omega) & \text{in } Q_{\varphi} \\
\partial_n q = \partial_n p = \partial_r r &= 0 & \text{on } \Sigma_{\varphi} \\
p(\varphi) + \beta q(\varphi) &= b_2(\varphi - \varphi_Q) + \frac{b_4}{2}, & \alpha p(\varphi) = 0, \ r(\varphi) = 0 & \text{in } \Omega.
\end{align*}
\]

Note that the only difference is the right-hand side of the third equation which, however, still belongs to \(L^2(0, T; H)\) and the then Theorem\[\text{3.10}\] can be applied with minor changes. It now suffices to fill the details arguing as above.

### 4.2 A similar Control Problem

Let us conclude the paper by emphasizing that in [12, 13, 23] the control variable \(u\) has a different role. In fact, it appears in the phase equation and it models the elimination of tumor cells by the effect of a cytotoxic drug. Accounting for a similar choice, we can accordingly adjust our state system and consider the following

\[
\begin{align*}
\alpha \partial_t \mu + \partial_t \varphi - \Delta \mu &= P(\varphi)(\sigma - \mu) - k u h(\varphi) & \text{in } Q \\
\mu &= \beta \partial_t \varphi - \Delta \varphi + F'(\varphi) & \text{in } Q \\
\partial_t \sigma - \Delta \sigma &= -P(\varphi)(\sigma - \mu) & \text{in } Q \\
\partial_n \mu = \partial_n \varphi = \partial_n \sigma &= 0 & \text{on } \Sigma \\
\mu(0) = \mu_0, \ \varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0 &= 0 & \text{in } \Omega,
\end{align*}
\]

where \(k\) stands for a positive constant, whereas the symbol \(h\) stands for an interpolation function which vanishes at \(-1\) and attains value 1 at 1. Moreover, the control \(u\) is taken between 0 and 1 in order to model no dosage and full dosage of the drug, respectively. So, when \(\varphi = -1\) no drug is dispensed, when \(\varphi = 1\) there is a full dosage of the drug, and in between there is a medium supply.

It is worth noting that \(h(\varphi)u \in L^\infty(Q)\), so the same arguments employed in [36] can be reproduced in the same manner to obtain the corresponding of Theorem\[\text{3.1}\] On the other hand, Proposition\[\text{3.2}\] cannot be reproduced since now the control variable is imposed in
the phase equation, so that it will demand \( u \in H^1(0,T;H) \) to be performed. Therefore, one is reduced to consider a relaxed cost functional as follows

\[
J_r(\varphi, \sigma, u, \tau) := \frac{a_1}{2} \int_{Q_r} |\varphi - \varphi_0|^2 + \frac{a_2}{2r} \int_{\tau-r}^\tau \int_\Omega |\varphi - \varphi_\Omega|^2 + \frac{a_3}{2} \int_{Q_r} |\sigma - \sigma_0|^2
\]

\[
+ \frac{a_4}{2} \int_\Omega |\sigma - \sigma_\Omega|^2 + \frac{a_5}{2r} \int_{\tau-r}^\tau \int_\Omega (1 + \varphi) + a_6 \tau + \frac{a_7}{2} |\tau - \tau_*|^2 + \frac{a_0}{2} \int_Q |u|^2,
\]

for some non-negative constants \( a_0, \ldots, a_7 \). Following the same lines as above, one cannot hope to prove higher order temporal regularity for the phase variable without assuming more regularity for the control. On the other hand, it is now very easy to check that, providing to require some natural assumptions, the variable \( \sigma \) may enjoy a better regularity, so that the third term in the cost functional above can be considered without any relaxation arguments. Let us claim that, after proving higher temporal regularity for the variable \( \sigma \), the expected optimality conditions are the following

\[
a_7 \int_Q \pi(v - \bar{\pi}) - k \int_{Q_{\tau}} h(\bar{\pi}) p(v - \bar{\pi}) \geq 0 \quad \text{for every } v \in U_{\text{ad}},
\]

and

\[
\frac{a_1}{2} \int_\Omega |\varphi(\bar{\tau}) - \varphi_\Omega(\bar{\tau})|^2 + \frac{a_2}{2r} \int_\Omega (|\varphi - \varphi_\Omega(\bar{\tau})|^2 - |\varphi - \varphi_\Omega(\bar{\tau} - r)|^2)
\]

\[
+ \frac{a_3}{2} \int_\Omega |\sigma(\bar{\tau}) - \sigma_\Omega(\bar{\tau})|^2 + a_4 \int_\Omega (\sigma(\bar{\tau}) - \sigma_\Omega) \partial_t \sigma(\bar{\tau})
\]

\[
+ \frac{a_5}{2r} \int_\Omega (\varphi(\bar{\tau}) - \varphi(\bar{\tau} - r)) + a_6 + a_7 (\tau - \tau_*) \begin{cases} 
\geq 0 & \text{if } \bar{\tau} = 0 \\
= 0 & \text{if } \bar{\tau} \in (0,T) \\
\leq 0 & \text{if } \bar{\tau} = T
\end{cases}
\]

The details are left to the reader.

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