One-dimensionality of the minimizers for a diffuse interface generalized antiferromagnetic model in general dimension

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Abstract

In this paper we study a diffuse interface generalized antiferromagnetic model. The functional describing the model contains Modica-Mortola type local term and a second nonlocal generalized antiferromagnetic term in competition. The competition between the two terms results in a frustrated system which is believed to lead to the emergence of wide variety of pattern formation. The sharp interface limit of our model is considered in [7, 8, 9] in the discrete and in [13] and [4] in the continuous setting. The model contains two parameter \( \tau, \varepsilon \). The parameter \( \tau \) represents the relative strength of the local term with respect to the nonlocal one, while the parameter \( \varepsilon \) describes the transition scale in the Modica-Mortola type term. If \( \tau < 0 \) one has that the only minimizers of the functional are constant functions. In any dimension \( d \geq 1 \) for small but positive \( \tau \) and small \( \varepsilon \), one expects the minimizers to be one-dimensional periodic functions. In this paper we are able to prove such a characterization of the minimizers, thus showing also the symmetry breaking in any dimension \( d > 1 \).

1 Introduction

In this paper we consider the following mean field free energy functional. For \( L, J, \varepsilon > 0, d \geq 1, p \geq d + 2 \), \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d; [0,1]) \) and \([0,L]^d\)-periodic, define

\[
\tilde{F}_{J,L,\varepsilon}(u) = \frac{J}{L^d} \left[ 3 \varepsilon \int_{[0,L]^d} \| \nabla u(x) \|_1 \, dx + \frac{3}{\varepsilon} \int_{[0,L]^d} W(u(x)) \, dx \right] - \frac{1}{L^d} \int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x + \zeta) - u(x)|^2 K(\zeta) \, dx \, d\zeta,
\]

(1.1)

where, for \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d, \|y\|_1 = \sum_{i=1}^d |y_i| \), \( W(t) = t^2(1-t)^2 \) and \( K(\zeta) = \frac{1}{(\|\zeta\|_1+1)^{p}} \).

This type of local/nonlocal interaction functionals, with suitable choices of the kernel \( K \), is used to model pattern formation in several contexts, among which the most famous and studied is the one of diblock copolymer melts [16] (where the exponent is \( p = d - 2 \), namely the Coulombic one). Periodic patterns in the ground states are expected to emerge by the competition between the first
term, short-range and attractive, and the second term, long-range and repulsive. According to the mutual strength between the two terms, modulated in this case by the constant \( J \), different patterns are expected to occur. While pattern formation is observed in experiments and simulations \([17, 8, 1, 2, 14, 10]\), a rigorous proof of the appearing of such phenomenon is still in many cases an open problem, due among others to the fact that minimizers display, in dimension \( d \geq 2 \), less symmetries than the functional itself.

Let

\[
J_c := \int_{\mathbb{R}^d} |\zeta_1| K(\zeta) \, d\zeta. \tag{1.2}
\]

One can show (see Lemma 4.3), that if \( J \geq J_c \) then the minimizers of (1.1) are the constant functions \( u \equiv 0 \) and \( u \equiv 1 \). We are interested in the structure of minimizers for \( J \in [J_c - \tau, J_c] \) where \( 0 < \tau \leq \tau < 1 \) and \( 0 < \varepsilon \ll 1 \). In analogy to what happens for the sharp interface limit of this problem as \( \varepsilon \to 0 \), which was studied in \([7, 8, 9]\) in the discrete and in \([13\) and \(4\) in the continuous setting, one expects that, for \( \varepsilon \) and \( \tau \) sufficiently small, minimizers of (1.1) are periodic one-dimensional functions, namely functions of the form \( u(x) = g(x_i) \) for some \( i \in \{1, \ldots, d\} \) with the property that, for some \( h > 0 \) and for all \( x_i \in \mathbb{R} \), \( g(x_i + 2h) = g(x_i) \), and there exists \( s \in \mathbb{R} \) such that \( g(s + (2k + 1)h + t) = 1 - g(s + (2k + 1)h - t) \) for all \( k \in \mathbb{N} \cup \{0\} \), \( t \in [0, h] \). Moreover, one expects to have such structure of minimizers also in the thermodynamic limit, namely that the \( \bar{\tau} \) below which one observes one-dimensionality and periodicity of minimizers is independent of \( L \) as \( L \to +\infty \).

In this paper, we are able to prove the above conjecture on one-dimensionality of minimizers for \( \varepsilon \) and \( \tau \) small but positive and independent of \( L \), in general dimension. In order to state our results properly, it is convenient to rescale the functional in order to have that the width of the optimal period for one-dimensional functions and their energy are of order \( O(1) \). For \( \beta = p - d - 1 \), setting

\[
x = \tau^{-1/\beta} \bar{x}, \quad \zeta = \tau^{-1/\beta} \bar{\zeta}, \quad \bar{L} = \tau^{-1/\beta} L, \quad \bar{u}(\bar{x}) = u(x), \quad \bar{F}_\varepsilon,J,\tau(u) = \tau^{1+1/\beta} F_{\tau,\varepsilon,J}(\bar{u})
\]

and finally dropping the tildas, one has that the rescaled functional has the form

\[
F_{\tau,\varepsilon,J}(u) = \frac{1}{L^d} \left[ \mathcal{M}_{\alpha,\varepsilon,J}(u;[0,L])^d \left( \int_{\mathbb{R}^d} K_\tau(\zeta_1) |d\zeta_1 - 1| - \int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x) - u(x + \zeta)|^2 K_\tau(\zeta) \, dx \, d\zeta \right) \right], \tag{1.3}
\]

where for \( \alpha > 0 \)

\[
\mathcal{M}_\alpha(u;[0,L]^d) = 3\alpha \int_{[0,L]^d} \| \nabla u(x) \|_1 \, dx + \frac{3}{\alpha} \int_{[0,L]^d} W(u(x)) \, dx,
\]

\( \alpha_{\varepsilon,J} = \varepsilon \tau^{1/\beta} \) and

\[
K_\tau(\zeta) = \frac{1}{(\|\zeta\|_1 + \tau^{1/\beta})^p}. \tag{1.5}
\]

Our main theorems are the following:

**Theorem 1.1.** Let \( L > 0 \). Then there exists \( \tau_L > 0 \), \( \varepsilon_L > 0 \) such that, for any \( 0 < \tau \leq \tau_L \) and \( 0 < \varepsilon \leq \varepsilon_L \) the minimizers of (1.1) are one-dimensional periodic functions of period \( h_{\tau,\varepsilon,L} \).
Let $h_{τ,ε}^* > 0$ be such that $2h_{τ,ε}^*$ is minimal above the periods of the functions that minimize $F_{τ,L,ε}$ as $L$ varies up to $+∞$ (for the precise definitions see Section 3).

We prove that, as in the sharp interface version of the problem, the following holds:

**Theorem 1.2.** There exists $\tau > 0$, $ε > 0$ such that, for all $0 < τ ≤ \hat{τ}$, $0 < ε ≤ \hat{ε}$, $h_{τ,ε}^*$ is finite and unique. Moreover, the optimal profile $g_{ε,h_{τ,ε}}$ is also unique.

With averaging and localization techniques similar to those used in Section 7 of [4], one can improve Theorem 1.1 in order to make $τ_L$ and $ε_L$ independent of $L$, for $L$ large.

**Theorem 1.3.** There exist $\bar{τ} > 0$, $\bar{ε} > 0$ such that, for all $0 < τ ≤ \bar{τ}$, $0 < ε ≤ \bar{ε}$, $L = 2kh_{τ,ε}^*$, $k \in \mathbb{N}$, the minimizers of (1.1) are one-dimensional periodic functions of period $2h_{τ,ε}^*$.

Moreover, it is not difficult to see that the results contained in this paper can be used to prove analogous results for the diffuse interface version of the model for colloidal systems considered in [5].

In this paper we focus mainly on the proof of Theorem 1.1. In the last section, we briefly sketch how to modify the proof of the thermodynamic limit in Section 7 of [4] (for the sharp interface version of (1.1)) in order to obtain Theorem 1.3 from Theorem 1.1. Indeed, the technical tools in order to obtain independence of $τ_L$ on $L$, based on localization of the main estimates and averaging, have been established in [4] and used also in [5]. We prefer instead to focus on the proof of Theorem 1.1 since it contains the main original ideas and contributions of this paper.

### 1.1 Scientific context

For the sharp interface limit of $F_{τ,L,ε}$ as $ε → 0$, namely the functional

$$F_{τ,L}(E) := \frac{1}{L^d} \left[ \text{Per}_1(E; [0, L]^d) \left( \int_{\mathbb{R}^d} K_τ(ζ) |ζ_1| \, dζ_1 - 1 \right) - \int_{\mathbb{R}^d} \int_{[0,L]^d} |χ_E(x) - ζ| K_τ(ζ) \, dx \, dζ \right],$$

(1.6)

and for $d ≥ 2$, the fact that minimizers are periodic unions of stripes of width $h_{τ,L} ≈ h_{τ,ε}^*$ for $τ$ sufficiently small and $L$ large has been shown in the discrete setting in [9] and then extended to the continuous setting in [3].

A periodic union of stripes of width $h$ is by definition a set which, up to Lebesgue null sets, is of the form $V_i^\perp + \hat{E} e_i$ for some $i \in \{1, \ldots, d\}$, where $V_i^\perp$ is the $(d − 1)$-dimensional subspace orthogonal to $e_i$ and $\hat{E} \subset \mathbb{R}$ with $\hat{E} = \bigcup_{k=0}^{N} (2kh + ν, (2k + 1)h + ν)$ for some $ν \in \mathbb{R}$ and some $N \in \mathbb{N}$.

Some of the most physically relevant exponents $p$ in the literature are $p = d + 1$ (thin magnetic films), $p = d − 2$ (diblock copolymer) and $p = d$ (3D micromagnetics). To our knowledge, there are no results where any type of pattern formation is shown for such model. Another very important family of kernels which is physically relevant and widely used in the literature is the Yukawa or screened Coulomb kernels (commonly used model pattern formation in colloidal suspensions and protein solutions). In a recent paper [5] the authors show that in a certain regime As for the structure of minimizers of diffuse interface functionals of the type (1.1), the best results which have been obtained in the literature so far are the following. In a low density regime and for the Ohta-Kawasaki kernel, properties of the shape of droplets of minimizers for $ε ≪ 1$ and $d = 2$ were deduced from the analysis of the sharp interface limit in [11] and [12], while results on the
periodicity of minimizers of \((1.1)\) for \(d = 1\) and more general reflection positive kernels were proved in \([6]\).

In this paper we are able (Theorems \([1.1]\) and \([1.3]\)) to show one-dimensionality and periodicity of minimizers of \((1.1)\) in a regime in which, for the limit problem as \(\varepsilon \to 0\), minimizers are periodic unions of stripes \((4)\).

However, unlike the previous results on diffuse interface functionals of the type \((1.1)\), most of the lower bounds and the estimates that we find for penalizing deviations from the set of one-dimensional functions are obtained directly for the diffuse-interface functional \((1.1)\), independently on its limit behaviour as \(\varepsilon \to 0\) (see Remark \([7.1]\)).

Let us now describe some main ideas of the proof of Theorem \([1.1]\). For simplicity let us assume that \(d = 2\). The main steps are the following:

Step 1. The functional is decomposed into one-dimensional functionals and a cross interaction term.

More precisely, we decompose the original functional as follows

\[
F_{\tau,L,\varepsilon}(u) = F^{1}_{\tau,L,\varepsilon}(u) + F^{2}_{\tau,L,\varepsilon}(u) + I_{\tau,L}(u),
\]

where \(F^{i}_{\tau,L,\varepsilon}\) is a one-dimensional functional in direction \(e_i\), namely depends only on oscillations of \(u\) in direction \(e_i\), and the third term is a cross interaction term depending on the simultaneous oscillations in both directions. Roughly speaking, the purpose of the term \(I_{\tau,L}\) is to penalize not being one-dimensional. With the above decomposition, if \(u(x,y) = u_0(x)\) then only the first term on the r.h.s. is not zero. Moreover, if \(u_0\) is a minimizer of \(F^{1}_{\tau,L,\varepsilon}\) then the functional has a negative value.

Step 2. By using a \(\Gamma\)-convergence argument, we can assume that (up to taking \(\tau, \varepsilon\) sufficiently small) the minimizers are \(L^1\)-close to the minimizers of the limit functional \((1.6)\), namely to a periodic unions of stripes;

Step 3. Thus to show that once close to stripes minimizers are one dimensional in direction \(e_1\) or \(e_2\), we show that if this is not the case then respectively \(F^{2}_{\tau,L,\varepsilon}(u) + I_{\tau,L}(u) > 0\) or \(F^{1}_{\tau,L,\varepsilon}(u) + I_{\tau,L}(u) > 0\). Such inequality is obtained through slicing, one-dimensional estimates and blow-up of the cross interaction term for deviations from one dimensional profiles;

Step 4. Once we know that the minimizers are one dimensional, a one dimensional optimization is needed in order to show that the minimizers are periodic.

Let us now discuss some of the main difference compared to \([4]\). (i). In Step 1, it is fundamental that if \(u(x,y) = u_0(x)\), then \(F^{2}_{\tau,L,\varepsilon}(u) = 0\). When the 1-perimeter is replaced by the Modica-Mortola approximation, it is in principle not clear if one can mimic the decomposition of \([4]\) in order to preserve this property. This is due to the fact that the Modica-Mortola term on \(\mathbb{R}^d\) depends also on the values of \(u \in (0,1)\). Thus whenever slicing a non-constant \(u(x,y) = u_0(x)\), there will be a non-negligible amount of slices which are constant but with \(u \in (0,1)\). Thus the obvious decompositions are not null in these slices. Thus a new decomposition is needed.
(ii). In \[13, 4\], the cross interaction term \(I_{\tau, L}\) is clearly positive. In this paper a careful inspection is needed to prove positivity (see Lemma 3.1).

(iii). One other crucial difference are the small oscillations. In \[4\], it is important that whenever there is an oscillation along the slice, this amount is bounded from below. In our setting this is no longer true. In particular, it is not a priori clear whether oscillations around 0 are more convenient than being flat. More precisely it is not clear whether \(u \equiv 0\) is better to an oscillating \(u \in [0, \delta)\) for \(\delta > 0\) sufficiently small. Indeed, one can devise a non-physical potential in the Modica-Mortola term, for which oscillating is better. In such a case the minimizers are not one dimensional. On the other hand as minimizers are periodic and not flat, oscillations are expected. This implies that there is a spacial scale in which oscillations hold. In order to deal with this issue new estimates are needed (see Section 4).

(iv). In Section 5 we prove that once a minimizer of (1.1) is \(L\)-periodic. In Section 4 we give some crucial one-dimensional estimates. In Section 3 we decompose the functional (1.1) into one-dimensional terms and interaction terms. In Section 1 we prove that once a minimizer of (1.1) is \(L^1\)-close to stripes then it has to be a stripe. In Section 6 we consider the associated one-dimensional problem and, starting from the results on general diffuse interface functionals obtained in \[6\] we prove existence of a finite optimal period and its uniqueness (see Theorem 1.2).

In Section 7 we prove Theorem 1.1.

In Section 8 we define the quantities which lead to the proof of Theorem 1.3.

1.2 Structure of the paper

In Section 2 we recall the main notation and the results obtained for the sharp interface problem (1.6) in \[4\].

In Section 3 we decompose the functional (1.1) into one-dimensional terms and interaction terms. In Section 4 we give some crucial one-dimensional estimates.

In Section 5 we prove that once a minimizer of (1.1) is \(L^1\)-close to stripes then it has to be a stripe.

In Section 6 we consider the associated one-dimensional problem and, starting from the results on general diffuse interface functionals obtained in \[6\] we prove existence of a finite optimal period and its uniqueness (see Theorem 1.2).

In Section 7 we prove Theorem 1.1.

In Section 8 we define the quantities which lead to the proof of Theorem 1.3.

2 Notation and preliminary results

In the following, let \(\mathbb{N} = \{1, 2, \ldots, d\}, d \geq 1\). On \(\mathbb{R}^d\), let \(\langle \cdot, \cdot \rangle\) be the Euclidean scalar product and \(|\cdot|\) be the Euclidean norm. Let \((e_1, \ldots, e_d)\) be the canonical basis in \(\mathbb{R}^d\) and for \(y \in \mathbb{R}^d\) let \(y_i = \langle y, e_i \rangle e_i\) and \(y^1_i := y - y_i\). For \(y \in \mathbb{R}^d\), let \(|y|_1 = \sum_{i=1}^{d} |y_i|\) be its 1-norm and \(|y|_\infty = \max_i |y_i|\) its \(\infty\)-norm. While writing slicing formulas, with a slight abuse of notation we will sometimes identify \(y_i \in [0, L]^d\) with its coordinate in \(\mathbb{R}\) w.r.t. \(e_i\) and \(y^1_i : y \in [0, L]^d\) with \([0, L)^{d-1} \subset \mathbb{R}^{d-1}\). For \(r > 0\) and \(x_\pm\) we let \(Q_r^\pm(x_\pm) = \{z^\pm : |x^\pm - z^\pm|_\infty \leq r\}\) or we think of \(x^\pm \in [0, L)^{d-1}\) and \(Q_r^\pm(x^\pm)\) as a subset of \([0, L)^{d-1}\).

In Section 3 instead of integrals on \([0, L]^d\) one will also consider integrals on smaller cubes centred at other points of \([0, L]^d\). Therefore, for \(z \in [0, L)^d\) and \(r > 0\), we define \(Q_r(z) = \{x \in \mathbb{R}^d : |x - z|_\infty \leq r\}\). In the whole paper we denote by \(u\) functions in \(W^{1,2}_{\text{loc}}(\mathbb{R}^d, [0,1])\) which are \([0, L]^d\)-periodic. For every \(i \in \{1, \ldots, d\}\) and for all \(x^\pm \in [0, L)^{d-1}\), we set

\[u_{x^\pm} : \mathbb{R} \to [0,1], \quad u_{x^\pm}(s) := u(se_i + x^\pm).\]
The function $u_{x^i}$ is, for almost all $x^i \in [0, L)^{d-1}$, in $W^{1,2}_{\text{loc}}(\mathbb{R}; [0, 1])$. We denote by $\partial_i$ the partial derivatives of a function with respect to $e_i$, $i \in \{1, \ldots, d\}$.

Given a measurable set $A \subset \mathbb{R}^d$, $k \in \{1, \ldots, d\}$, we denote by $|A|$ its $k$-dimensional Lebesgue measure (or if $A$ is contained in some $k$-dimensional plane of $\mathbb{R}^d$, its Hausdorff $k$-dimensional measure), being always clear from the context which will be the dimension $k$.

Moreover, let $\chi_A : \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \in \mathbb{R}^d \setminus A.
\end{cases}
$$

(2.1)

A set $E \subset \mathbb{R}^d$ is of (locally) finite perimeter if the distributional derivative of $\chi_E$ is a (locally) finite measure. We let $\partial E$ be the reduced boundary of $E$. We call $\nu_E$ the exterior normal to $E$.

Then one can define the 1-perimeter of a set relative to $[0, L)^d$ as

$$
\text{Per}_1(E, [0, L)^d) := \int_{\partial E \cap [0, L)^d} \|\nu_E(x)\|_1 \, d\mathcal{H}^{d-1}(x)
$$

where $\mathcal{H}^{d-1}$ is the $(d - 1)$-dimensional Hausdorff measure.

By extending the classical Modica-Mortola result \cite{ModicaMortola1977} to the anisotropic norm $\| \cdot \|_1$, one has the following

**Theorem 2.1.** As $\alpha \to 0$, the functionals $M_\alpha(\cdot; [0, L)^d)$ defined in (1.4) $\Gamma$-converge in $BV([0, L)^d; [0, 1])$ to the functional $P_1(\cdot; [0, L)^d)$ defined as follows:

$$
P_1(u; [0, L)^d) := \begin{cases} 
\text{Per}_1(E; [0, L)^d) & \text{if } u = \chi_E \\
+ \infty & \text{otherwise}
\end{cases}
$$

(2.2)

being $\text{Per}_1$ the perimeter functional with respect to the anisotropic norm $\| \cdot \|_1$.

Notice that the constant 3 in (1.1) is chosen in such a way that

$$
6 \int_0^1 t(1 - t) \, dt = 1,
$$

so that the constant in front of the perimeter in (2.2) is equal to 1.

By continuity of the nonlocal term in (1.1) with respect to $L^1$ convergence of functions valued in $[0, 1]$, one has the following

**Corollary 2.2.** As $\varepsilon \to 0$, the functionals $\mathcal{F}_{\tau, L, \varepsilon}$ $\Gamma$-converge in $BV_{\text{loc}}(\mathbb{R}^d; [0, 1])$ to the functional

$$
\mathcal{F}_{\tau, L}(u) := \begin{cases} 
\frac{1}{L^d} \left[ \text{Per}_1(E; [0, L)^d) \left( \int_{\mathbb{R}^d} K_\tau(\zeta) \, d\zeta - 1 \right) \right] \\
- \int_{\mathbb{R}^d} \int_{[0, L)^d} |\chi_E(x) - \chi_E(x + \zeta)| K_\tau(\zeta) \, dx \, d\zeta & \text{if } u = \chi_E \\
+ \infty & \text{otherwise.}
\end{cases}
$$

(2.3)
The kernel $K_\tau$ is, as shown in [4], reflection positive, namely it satisfies the following property: the function

$$\hat{K}_\tau(t) := \int_{\mathbb{R}^{d-1}} K_\tau(t, \zeta_2, \ldots, \zeta_d) \, d\zeta_2 \cdots d\zeta_d,$$

is the Laplace transform of a nonnegative function.

Regarding the limit functional (2.3), we recall the following results, obtained in [4].

**Theorem 2.3.** Let $d \geq 1$, $p \geq d+2$, $L > 0$. Then, there exists $\tau_L > 0$ such that, for all $0 < \tau \leq \tau_L$ the minimizers of the functional $F_{\tau,L}$ in (2.3) are periodic unions of stripes of width $h_{\tau,L}$.

Moreover, for fixed $\tau > 0$, consider first for all $L > 0$ the minimal value obtained by $F_{\tau,L}$ on $[0, L)^d$-periodic stripes and then the minimal among these values as $L$ varies in $(0, +\infty)$. By the reflection positivity technique, this value is attained on periodic stripes of width and distance $h^*_\tau > 0$.

In [4] the following theorems have been proved:

**Theorem 2.4.** Let $d \geq 1$, $p \geq d+2$. Then there exists $\hat{\tau} > 0$ s.t. whenever $0 < \tau < \hat{\tau}$, $h^*_\tau$ is unique.

**Theorem 2.5.** There exists a constant $C$ such that for every $0 < \tau \leq \bar{\tau}$, one has that the width $h_{\tau,L}$ of a minimizer of $F_{\tau,L}$ satisfies

$$|h^*_\tau - h_{\tau,L}| \leq \frac{C}{L}.$$  \hfill (2.4)

**Theorem 2.6.** Let $d \geq 1$, $p \geq d+2$ and $h^*_\tau$ be the optimal stripes’ width for fixed $\tau$. Then there exists $\tau_0$, such that for every $\tau < \tau_0$, one has that for every $k \in \mathbb{N}$ and $L = 2kh^*_\tau$, the minimizers $E_\tau$ of $F_{\tau,L}$ are optimal stripes of width $h^*_\tau$.

### 3 Decomposition of the functional

In this section we provide a lower bound for the functional $F_{\tau,L,\varepsilon}$ which shares the same value on one-dimensional functions. Thus our study will reduce to show the characterization of minimizers for such lower bound.

First we notice that the Modica-Mortola term $M_{\alpha,\varepsilon,\tau}(\cdot, [0, L]^d)$ can be decomposed in the following way

$$M_{\alpha,\varepsilon,\tau}(u, [0, L]^d) = \sum_{i=1}^d \int_{[0,L)^{d-1}} M_{\alpha,\varepsilon,\tau}^{1d,i}(u; x_i^\perp; [0, L]) \, dx_i^\perp,$$

where

$$M_{\alpha,\varepsilon,\tau}^{1d,i}(u; x_i^\perp; [s, t]) := 3\alpha_{\varepsilon,\tau} \int_s^t |\partial_i u_{x_i^\perp}(\rho)||\nabla u(\rho, x_i^\perp)||_1 \, d\rho + \frac{3}{\alpha_{\varepsilon,\tau}} \int_s^t W(u_{x_i^\perp}(\rho)) \frac{|\partial_i u_{x_i^\perp}(\rho)|}{||\nabla u(\rho, x_i^\perp)||_1} \, d\rho$$

and we adopt the convention that $\frac{|\partial_i u_{x_i^\perp}(\rho)|}{||\nabla u(\rho, x_i^\perp)||_1} = 0$ whenever $||\nabla u(\rho, x_i^\perp)||_1 = 0$.

As for the nonlocal term, using the elementary inequality
(a + b)^2 = a^2 + b^2 + 2ab

with \( a = u(x) - u(x + \zeta_i) \), \( b = u(x + \zeta_i) - u(x + \zeta) \), the periodicity of \( u \) and Fubini Theorem one gets for \( d = 2 \)

\[
\int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x) - u(x + \zeta)|^2 K_\tau(\zeta) \, d\zeta = \sum_{i=1}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} |u_{x_i^+}(s) - u_{x_i^+}(s + \zeta_i)|^2 \tilde{K}_\tau(\zeta_i) \, ds \, d\zeta_i \, dx_i^+
\]

\[
+ \sum_{i=1}^2 \int_{\mathbb{R}^2} \int_{[0,L]^2} (u(x) - u(x + \zeta_i))(u(x + \zeta_i) - u(x + \zeta)) K_\tau(\zeta) \, dx \, d\zeta.
\]  \hspace{1cm} (3.3)

In our analysis the following lemma on the negativity of the last sum in (3.3) will be fundamental.

**Lemma 3.1.** Let \( u \in W^{1/2}_{\text{loc}}(\mathbb{R}^d; [0,1]) \) be a \([0,L]^d\) periodic function. Then, for all \( i \in \{1,\ldots,d\} \)

\[
- \int_{\mathbb{R}^d} \int_{[0,L]^d} (u(x) - u(x + \zeta_i))(u(x + \zeta_i) - u(x + \zeta)) K_\tau(\zeta) \, dx \, d\zeta =
\]

\[
= \frac{1}{2} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} [(u(x + \zeta_i) - u(x)) - (u(x + \zeta) - u(x + \zeta_i))|^2 K_\tau(\zeta) \, dx \, d\zeta. \]  \hspace{1cm} (3.4)

**Proof of Lemma 3.1.** One has that

\[
- \int_{\mathbb{R}^d} \int_{[0,L]^d} (u(x) - u(x + \zeta_i))(u(x + \zeta_i) - u(x + \zeta)) K_\tau(\zeta) \, dx \, d\zeta =
\]

\[
= \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} (u(x + \zeta_i) - u(x))(u(x + \zeta) - u(x + \zeta_i)) K_\tau(\zeta) \, dx \, d\zeta
\]

\[
= \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} (u(x + \zeta_i) - u(x))(-u(x + \zeta) + u(x + \zeta_i) + u(x + \zeta_i^+) - u(x)) K_\tau(\zeta) \, dx \, d\zeta
\]

\[
= \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} (u(x + \zeta_i) - u(x))[u(x + \zeta_i) - u(x)] - (u(x + \zeta) - u(x + \zeta_i^+))] K_\tau(\zeta) \, dx \, d\zeta
\]  \hspace{1cm} (3.5)

\[
= \frac{1}{2} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} (u(x + \zeta_i) - u(x))^2 - (u(x + \zeta_i^+) - u(x + \zeta))^2 K_\tau(\zeta) \, dx \, d\zeta
\]

\[
+ \frac{1}{2} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} [(u(x + \zeta_i) - u(x)) - (u(x + \zeta) - u(x + \zeta_i^+))]^2 K_\tau(\zeta) \, dx \, d\zeta \]  \hspace{1cm} (3.6)

\[
= \frac{1}{2} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} [(u(x + \zeta_i) - u(x)) - (u(x + \zeta) - u(x + \zeta_i^+))]^2 K_\tau(\zeta) \, dx \, d\zeta,
\]  \hspace{1cm} (3.7)

where from (3.5) to (3.6) we used the identity \( a(a-b) = \frac{1}{2}a^2 - b^2 + (a-b)^2 \) with \( a = u(x + \zeta_i) - u(x) \)
and \( b = u(x + \zeta) - u(x + \zeta_i^+) \) and from (3.6) to (3.7) the periodicity of \( u \) and the fact that \( \int a^2 = \int b^2 \)
for \( a, b \) as above.

\[ \Box \]
Iterating the above decomposition argument for $d = 2$ to arbitrary dimension, one has that

$$-\int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x) - u(x + \zeta)|^2 K_\tau(\zeta) \, d\zeta \geq -\sum_{i=1}^{d} \int_{\mathbb{R}^d} \int_{[0,L]^d} |u(x) - u(x + \zeta_i)|^2 K_\tau(\zeta) \, d\zeta$$

$$\frac{1}{d} \sum_{i=1}^{d} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} [(u(x + \zeta_i) - u(x)) - (u(x + \zeta) - u(x + \zeta_i^+)))]^2 K_\tau(\zeta) \, dx \, d\zeta. \quad (3.8)$$

Finally, one obtains the following lower bound for the functional $F_{\tau, L, \varepsilon}$

$$F_{\tau, L, \varepsilon}(u) \geq \sum_{i=1}^{d} \left\{ \int_{[0,L]^d} \left[ -\mathcal{M}_{\alpha_{x}, \tau}^{I, i}(u; x_i^+; [0, L]) + \mathcal{G}_{\alpha_{x}, \tau}^{I, i}(u, x_i^+; [0, L]) \right] \, dx_i^+ + \mathcal{I}_{\tau, L}(u) \right\}, \quad (3.9)$$

$$\mathcal{G}_{\alpha_{x}, \tau}^{I, i}(u, x_i^+; [0, L]) = \mathcal{M}_{\alpha_{x}, \tau}^{I, i}(u; x_i^+; [0, L]) \int_{\tau} |\zeta_i| K_\tau(\zeta_i) \, d\zeta_i - \int_{\tau} \int_{0}^{L} \left| u_{x_i^+}(x_i) - u_{x_i^+}(x_i + \zeta_i) \right|^2 K_\tau(\zeta_i) \, dx_i \, d\zeta_i$$

and

$$\mathcal{I}_{\tau, L}(u) = \frac{1}{d} \int_{\{\zeta_i > 0\}} \int_{[0,L]^d} [(u(x + \zeta_i) - u(x)) - (u(x + \zeta) - u(x + \zeta_i^+)))]^2 K_\tau(\zeta) \, dx \, d\zeta. \quad (3.11)$$

4 One-dimensional estimates

One has the following property

$$\mathcal{M}_{\alpha_{x}, \tau}^{I, i}(u; x_i^+; [s, t]) \geq 6 \int_{s}^{t} \left| \partial_i u_{x_i^+}(\rho) \right| \sqrt{W(u_{x_i^+}(\rho))} \, d\rho$$

$$= \int_{s}^{t} \left| D(\omega \circ u_{x_i^+})(\rho) \right| \, d\rho$$

$$\geq |\omega(u_{x_i^+}(s)) - \omega(u_{x_i^+}(t))|, \quad (4.1)$$

where $\omega : [0, 1] \rightarrow [0, 1]$ is defined by

$$\omega(0) = 0 \quad \omega(t) = \int_{0}^{t} 6 \sqrt{W(t)} = 3t^2 - 2t^2. \quad (4.2)$$

Notice that $\omega(t)$ is the optimal transition energy from 0 to $t$ for the Modica Mortola term. The following lemma contains an estimate relating $\omega$ and the square of the distance which will be used in Lemma 4.2 and in Proposition 5.1.

**Lemma 4.1.** The optimal energy function $\omega$ satisfies the following inequality: for $a, b \in [0, 1]$ with $a = b + t$, $t > 0$

$$\frac{\omega(a) - \omega(b)}{|a - b|^2} = \frac{6b(1 - b - t)}{t} + 3 - 2t \geq 3 - 2t. \quad (4.3)$$

and equality holds if and only if $a = 1$ and $b = 0$.  

9
In the following lemma we prove the positivity of $G^{1d,i}_{\alpha, \tau}(u, x^i_\perp; [0, L])$.

**Lemma 4.2.** For any $\zeta_i \in \mathbb{R}$,

\[
|\zeta_i| \mathcal{M}^{1d,i}_{\alpha, \tau}(u; x^i_\perp; [0, L]) - \int_0^L \left| u_{x^i_\perp}(x^i_i + \zeta_i) - u_{x^i_\perp}(x^i_i) \right|^2 \, dx_i \geq 0,
\]

in particular $G^{1d,i}_{\alpha, \tau}(u, x^i_\perp; [0, L]) \geq 0$.

**Proof of Lemma 4.2.** One has that, by Fubini Theorem, periodicity of $u$ and \eqref{eq:4.1},

\[
|\zeta_i| \mathcal{M}^{1d,i}_{\alpha, \tau}(u; x^i_\perp; [0, L]) = \int_0^L \mathcal{M}^{1d,i}_{\alpha, \tau}(u; x^i_\perp; [x^i_i, x^i_i + \zeta_i]) \, dx_i \geq \int_0^L |\omega(u_{x^i_\perp}(x^i_i)) - \omega(u_{x^i_\perp}(x^i_i + \zeta_i))| \, dx_i.
\]

Finally, thanks to the inequality \eqref{eq:4.3} in Lemma 4.1,

\[
|\omega(u_{x^i_\perp}(x^i_i)) - \omega(u_{x^i_\perp}(x^i_i + \zeta_i))| \geq |u_{x^i_\perp}(x^i_i) - u_{x^i_\perp}(x^i_i + \zeta_i)|^2,
\]

which proves the positivity of \eqref{eq:4.4}.

In particular, the following lemma holds

**Lemma 4.3.** If $J \geq J_c$, where $J_c$ is defined in \eqref{eq:1.2}, then minimizers of \eqref{eq:1.1} are either $u \equiv 1$ or $u \equiv 0$.

**Proof.** By the above decomposition of the functional, since $J \geq J_c$

\[
\tilde{F}_{J,L,\epsilon}(u) \geq \sum_{i=1}^d \int_{[0,L)^{d-1}} G^{1d,i}_{\alpha,1}(u, x^i_\perp; [0, L]) \, dx^i_\perp + I^i_{1,L}(u).
\]

By Lemma 4.2 both $G^{1d,i}_{\alpha,1}$ and $I^i_{1,L}$ are nonnegative and $G^{1d,i}_{\alpha,1}$ is zero only if $u_{x^i_\perp} \equiv 1$ or $u_{x^i_\perp} \equiv 0$. Since this holds for all $i \in \{1, \ldots, d\}$, then the right hand side of \eqref{eq:4.7} is zero only if $u \equiv 1$ or $u \equiv 0$ and for such values equality in \eqref{eq:4.7} holds, which means $\tilde{F}_{J,L,\epsilon}$ is minimized.

## 5 Stability estimates

In this section we assume that $u \in W^{1,2}_{loc}(\mathbb{R}^d; [0, 1])$ $[0, L]^d$-periodic function is such that

\[
\|u - \chi_S\|_{L^1([0,L]^d)} \leq \bar{\sigma},
\]

for some $\bar{\sigma} > 0$ small enough, to be chosen later, where $S$ is a periodic union of stripes in direction $\epsilon_i$ of width $h > 0$. This will be the case for minimizers of $\tilde{F}_{\tau,L,\epsilon}$ when $\epsilon, \tau$ are small enough, due to Corollary 2.2 and Theorem 2.3.

The main result of this section is the following stability estimate
Proposition 5.1. There exist $\tilde{\sigma} > 0$ and $\tilde{\tau} > 0$ such that, if (5.1) holds for $u \in W_{\text{loc}}^{1,2}(\mathbb{R}; [0,1])$ ($0, L)^d$-periodic function and $S$ periodic union of stripes in direction $e_i$ of width $h > 0$, then for all $j \in \{1, \ldots, d\}$, $j \neq i$

\[
\int_{[0,L)^d} \left[ -\mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_j^+; [0, L]) + \mathcal{G}_{\alpha_\varepsilon,\tau}^{1d}(u, x_j^+; [0, L]) \right] \, dx_j^+ + \mathcal{I}_{\tau,L}(u) \geq 0
\]  
(5.2)

and equality holds if and only if $u$ does not depend on $x_j$.

Proof. Without loss of generality we assume $i = 1$ and $j = 2$.

We choose $\eta_0 > 0$, $\tau_0 > 0$ such that, for every $0 < \eta \leq \eta_0$ and for every $0 < \tau \leq \tau_0$

\[
- \frac{2L}{\eta} + \frac{1}{16} \int_{|\rho| > \eta} \frac{|\rho| - \eta}{(|\rho| + \tau^{1/3})^3} \, d\rho > 0.
\]  
(5.3)

Let

\[
B_{x_2^+} := \{(s, t) : s \in [0, L), t \in \mathbb{R}, \mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_2^+; [s, t)) \geq \frac{17}{16} \},
\]  
(5.4)

and, for some $\delta > 0$ small enough (which we will see can be fixed independently of $\varepsilon$ and $\tau$),

\[
D_{x_2^+} := \{(s, t) : s \in [0, L), t \in \mathbb{R}, |u_{x_2^+}(s) - u_{x_2^+}(t)| \geq 1 - 2\delta \}.
\]  
(5.5)

Define the functions

\[
b(x_2^+) := \inf\{|s-t| : (s, t) \in B_{x_2^+} \},
\]

\[
d(x_2^+) := \inf\{|s-t| : (s, t) \in D_{x_2^+} \},
\]  
(5.6)

setting them equal to $+\infty$ if the corresponding sets are empty.

Then we define a partition $[0, L)^{d-1} = A_1(\delta_0) \cup A_2(\delta_0) \cup A_3(\delta_0)$ as follows

\[
A_1(\delta_0) = \{ x_2^+ \in [0, L)^{d-1} : b(x_2^+) \geq \eta_0, d(x_2^+) \leq \delta_0 \}
\]  
(5.7)

\[
A_2(\delta_0) = \{ x_2^+ \in [0, L)^{d-1} : b(x_2^+) \geq \eta_0, d(x_2^+) \geq \delta_0 \}
\]  
(5.8)

\[
A_3 = \{ x_2^+ \in [0, L)^{d-1} : b(x_2^+) < \eta_0 \}.
\]  
(5.9)

First of all we show that, if $x_2^+ \in A_3$, then

\[
- \mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_2^+; [0, L]) + \mathcal{G}_{\alpha_\varepsilon,\tau}^{1d}(u, x_2^+; [0, L]) > 0.
\]  
(5.10)

Indeed, by compactness of the set $[0, L] \times [-\eta_0, L+\eta_0]$ and continuity of the integral $\mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_2^+; [s, t))$ with respect to $s$ and $t$, one has that for each $x_2^+ \in A_3$ there exists $0 < \eta = \eta(x_2^+) < \eta_0$ such that $b(x_2^+) = \eta$.

In particular,

\[
\mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_2^+; [0, L]) \leq \frac{2L}{\eta}.
\]  
(5.11)

As for $\mathcal{G}_{\alpha_\varepsilon,\tau}^{1d}(u, x_2^+; [0, L])$,

\[
\mathcal{G}_{\alpha_\varepsilon,\tau}^{1d}(u, x_2^+; [0, L]) = \int_{\mathbb{R}} \int_0^L (\mathcal{M}_{\alpha_\varepsilon,\tau}^{1d}(u; x_2^+; [s, s + \zeta_2)) - |u_{x_2^+}(s + \zeta_2) - u_{x_2^+}(s)|^2) K_\tau(\zeta_2) \, ds \, d\zeta_2
\]  
(5.12)
Let now \( s_0 \in [0, L), \ t_0 > 0 \in \mathbb{R} \) such that \(|s_0 - t_0| = \eta\) and \( G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [s_0, t_0]) \geq \frac{\tau}{10}. \) Thanks to Lemma 4.2, we have that

\[
G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \geq \int_{\{\xi_2 \geq \eta\}} \int_{s_0 - \xi_2 + \eta}^{s_0} (\mathcal{M}_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [s, s + \xi_2]) - |u_{x_2 \beta}(s + \xi_2) - u_{x_2 \beta}(s)|^2) \tilde{K}_\tau(\xi_2) \, ds \, d\xi_2. \tag{5.13}
\]

Now, since for \(|\xi_2| \geq \eta\) and \( s \in [s_0 - \xi_2 + \eta, s_0]\) the segment \([s, s + \xi_2]\) contains \([s_0, t_0]\), where \( \mathcal{M}_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [s_0, t_0]) \geq \frac{\tau}{10}, \)

\[
G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \geq \frac{1}{16} \int_{\{\xi_2 \geq \eta\}} \frac{(|\xi_2| - \eta)}{(|\xi_2| + \frac{\tau}{10})^2} \, d\xi_2. \tag{5.14}
\]

From (5.11) and (5.14) we deduce that

\[
-M_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) + G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \geq -\frac{2L}{\eta} + \frac{1}{16} \int_{\{\xi_2 \geq \eta\}} \frac{(|\xi_2| - \eta)}{(|\xi_2| + \frac{\tau}{10})^2} \, d\xi_2 \tag{5.15}
\]

and since \( \eta \leq \eta_0, (5.3) \) holds and (5.10) is proved.

Let us now assume \( x_2 \beta \in A_2(\delta_0) \) and let us show that

\[
-M_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) + G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \geq 0 \tag{5.16}
\]

provided \( 0 < \tau \leq \tau_1, \tau_1 = \tau_1(\delta_0) \), with equality if and only if \( u_{x_2 \beta} \) is constant.

By (4.3) in Lemma 4.1 with \( t = |a - b| \leq 1 - 2\delta \), for all \(|\xi_2| \leq \delta_0, x_2 \in [0, L]\) one has that

\[
\frac{1}{1 + 4\delta} |\omega(u_{x_2 \beta}(x_2)) - \omega(u_{x_2 \beta}(x_2 + \xi_2))| \geq |u_{x_2 \beta}(x_2) - u_{x_2 \beta}(x_2 + \xi_2)|^2 \tag{5.17}
\]

and then by (4.5)

\[
\int_{\{\xi_2 \leq \delta_0\}} \left[ \frac{|\xi_2|}{1 + 4\delta} \mathcal{M}_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) - \int_0^L |u_{x_2 \beta}(x_2) - u_{x_2 \beta}(x_2 + \xi_2)|^2 \, dx_2 \right] \tilde{K}_\tau(\xi_2) \, d\xi_2 \geq 0. \tag{5.18}
\]

On the other hand, by the singularity of the kernel \( \tilde{K}_\tau \), one has that for \( \tau \) sufficiently small depending on \( \delta \) and \( \delta_0 \)

\[
-M_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) + \left( 1 - \frac{1}{1 + 4\delta} \right) \mathcal{M}_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \int_{\{\xi_2 \leq \delta_0\}} |\xi_2| \tilde{K}_\tau(\xi_2) \, d\xi_2 \geq 0. \tag{5.19}
\]

As for the part of \( G_{\alpha, \tau}^{1d, 2}(u; x_2 \beta; [0, L]) \) which comes from the integral on the region \( \{\xi_2 \geq \delta_0\} \), we now that it is positive by Lemma 4.2. Thus (5.10) is proved also for the set \( A_2(\delta_0) \) provided \( \tau \) is small enough depending on \( \delta \) and \( \delta_0 \).

Now notice, by Lemma 4.1 that the estimates obtained for (5.16) give that (5.16) is equal to 0 if and only if \( u_{x_2 \beta} \) is constant up to null sets. Then, also this step of the proof is completed.
Let now $x_2^\perp \in A_1(\delta_0)$. Define the slicing of $T^2_{\tau,L}$ as follows

\[
T^2_{\tau,L}(u) = \int_{[0,L]^{d-1}} T^2_{\tau,L}(u; x_2^\perp; [0, L]) \, dx_2^\perp
\]

\[
= \frac{1}{d} \int_{[0,L]^{d-1}} \int_0^L \int_{\{z_2 > 0\}} [(u(x + z_2^\perp) - u(x)) - (u(x + z) - u(x + z_2^\perp))]^2 K_\tau(z) \, dz \, dx_2^\perp.
\]

(5.20)

Our goal is to show that, there exist $\delta > 0$ and $\tau_2 > 0$ such that if $x_2^\perp \in A_1(\delta_0)$ for some $0 < \delta_0 \leq \delta$ and $0 < \tau \leq \tau_2$, then

\[
-M_{\alpha_{x,r}}^{1d,2}(u; x_2^\perp; [0, L]) + G_{\alpha_{x,r}}^{1d,2}(u, x_2^\perp; [0, L]) + T^2_{\tau,L}(u; x_2^\perp; [0, L]) > 0.
\]

(5.21)

By definition of $A_1(\delta_0)$, for every $s \in [0, L)$, $t \in \mathbb{R}$ with $|s - t| < \eta_0$, $M_{\alpha_{x,r}}^{1d,2}(u; x_2^\perp; [s, t]) \leq \frac{1}{16}$ and there exist $s_0 \in [0, L)$, $t_0 > s_0$ with $|s_0 - t_0| \leq \delta_0$ and $|u(s_0) - u(t_0)| \geq 1 - 2\delta$. Since by (4.1) and (4.3) in Lemma 4.1 $M_{\alpha_{x,r}}^{1d,2}(u; x_2^\perp; [s_0, t_0]) \geq |u(s_0) - u(t_0)|^2 \geq (1 - 2\delta)^2$, then

\[
\max\{M_{\alpha_{x,r}}^{1d,2}(u; x_2^\perp; [s_0 - \eta_0/3, s_0]), M_{\alpha_{x,r}}^{1d,2}(u; x_2^\perp; [t_0, t_0 + \eta_0/3])\} \leq \frac{17}{16} - (1 - 2\delta)^2.
\]

(5.22)

In particular, assuming without loss of generality $u(t_0) > u(s_0)$ and choosing $\delta$ small enough dependent only on the constant $17/16$,

\[
u(t) \geq \frac{5}{1} \quad \forall t \in [t_0, t_0 + \eta_0/3],
\]

(5.23)

\[
u(s) \leq \frac{3}{1} \quad \forall t \in [s_0 - \eta_0/3, s_0].
\]

(5.24)

Now let $0 < r < \eta_0$. Choosing in (5.1) $0 < \sigma \leq r^d/192$, one can assure that either

\[
\left|\left\{ z_2^\perp \in Q_r^+(x_2^\perp): \left|\left\{ s \in (s_0 - r/3, t_0 + r/3): u(z_2^\perp, s) > \frac{7}{8}\right\}\right| > \frac{r}{2}\right\}\right| \geq \frac{3}{8} r^{d-1}
\]

(5.25)

or

\[
\left|\left\{ z_2^\perp \in Q_r^+(x_2^\perp): \left|\left\{ s \in (s_0 - r/3, t_0 + r/3): u(z_2^\perp, s) < \frac{1}{8}\right\}\right| > \frac{r}{2}\right\}\right| \geq \frac{3}{8} r^{d-1}
\]

(5.26)

Indeed, setting $\Omega = \left\{ z_2^\perp \in Q_r^+(x_2^\perp): \left|\left\{ s \in (s_0 - r/3, t_0 + r/3): u(z_2^\perp, s) - \chi_S(z_2^\perp, s) > 1/8\right\}\right| > r/6\right\}$, $\alpha = |\Omega|/r^{d-1}$, one has that, thanks to (5.1)

\[
\sigma \geq \int_{Q_r^+(x_2^\perp) \times (s_0 - r/3, t_0 + r/3)} |u - \chi_S|(z) \, dz
\]

\[
\geq \int_{\{u_{z_2^\perp}(s) - \chi_S(z_2^\perp, s) > 1/8\}} |u - \chi_S|(z) \, dz
\]

\[
\geq \frac{1}{8} \frac{r}{6} r^{d-1}.
\]

(5.27)
Hence, if \( \bar{\sigma} \leq r^d/192 \), \( \alpha \leq 1/4 \) and then either (5.25) or (5.26) holds.

Assume now without loss of generality that (5.25) is satisfied. Then one has the following lower bound

\[
I_{r,L}^2(u; x_2^1; [0, L]) \geq \frac{1}{d} \int_{r_0 - r/3}^{r_0 + r/3 - x_2} \int_{\Omega - x_2^1} [u(x + \zeta_2) - u(x) - (u(x + \zeta) - u(x + \zeta_2^1))]^2 K_\tau(\zeta) \, d\zeta_2 \, dx_2.
\]

(5.28)

If we choose \( \delta_0 \leq r/12 \), then for any \( \zeta_2^1 \in \Omega - x_2^1 \) there exist at least a set of measure \( r/12 \) of \( x_2 \in (s_0 - r/3, s_0) \) and set of measure \( r/12 \) of \( \zeta_2 \in (t_0 - x_2, t_0 + r/3 - x_2) \) such that \( u(x + \zeta_2^1) > 7/8 \) and \( u(x + \zeta) > 7/8 \). Recalling (5.23) and (5.24), on this set

\[
[u(x + \zeta_2) - u(x) - (u(x + \zeta) - u(x + \zeta_2^1))]^2 \geq (1/4 - 1/8)^2 = 1/64
\]

therefore

\[
I_{r,L}^2(u; x_2^1; [0, L]) \geq \frac{1}{d} 12 \int_{r_0 - r/3}^{r_0 + r/3 - x_2} \int_{\Omega - x_2^1} [u(x + \zeta_2) - u(x) - (u(x + \zeta) - u(x + \zeta_2^1))]^2 K_\tau(\zeta) \, d\zeta_2 \, dx_2.
\]

(5.29)

Since \( b(x_2^1) \geq \eta_0 \),

\[
-\mathcal{M}_{\alpha, r}^{1, d} (u; x_2^1; [0, L]) \geq -\frac{2L}{\eta_0},
\]

and one can choose \( \tau \leq \tau_2 \) and \( r \leq \bar{r} \) (and consequently \( \delta_0 \leq 1/12r \)) sufficiently small depending on \( \eta_0 \) such that, since \( p \geq d + 2 \), the lower bound (5.29) gives

\[
I_{r,L}^2(u; x_2^1; [0, L]) \geq \frac{2L}{\eta_0}.
\]

Since by Lemma 4.2 \( G^{1, d}_{\alpha, r} (u, x_2^1; [0, L]) \geq 0 \), also (5.21) is proved, for any \( 0 < \tau \leq \tau_2 \) and \( 0 < \delta_0 \leq \delta \).

Finally, choosing first \( \bar{\sigma} \leq \bar{r}^d/192 \), then \( \delta_0 \leq \bar{\delta} = \bar{r}/12 \), and then \( \bar{r} \leq \min\{\tau_0, \tau_1, \tau_2\} \) the proof of (5.2) is concluded.

6 One-dimensional problem

Let \( u \in W^{1, 2}_{\text{loc}}(\mathbb{R}^d; [0, 1]) \) be a one-dimensional \([0, L]^d\)-periodic function with \( u(x) = g(x_i) \) for some \( i \in \{1, \ldots, d\} \). We define the one-dimensional functional \( \mathcal{F}^1_{r,L,\epsilon} \) corresponding to \( \mathcal{F}_{r,L,\epsilon} \) as

\[
\mathcal{F}^1_{r,L,\epsilon}(g) := \frac{1}{L} \left[ \mathcal{M}_{\alpha, r} (g; [0, L]) \left( \int_{\mathbb{R}} \tilde{K}_\tau(\rho) |\rho| \, d\rho - 1 \right) - \int_{\mathbb{R}} \int_{0}^{L} |g(s) - g(s + \rho)|^2 \tilde{K}_\tau(\rho) \, ds \, d\rho \right].
\]

(6.1)

where

\[
\mathcal{M}_{\alpha, r} (g; [0, L]) = 3\alpha_{\epsilon, \tau} \int_{0}^{L} \|\partial g(s)\|_2^2 \, ds + \frac{3}{\alpha_{\epsilon, \tau}} \int_{0}^{L} W(g(s)) \, ds.
\]

(6.2)

Using the identity

\[
|g(s) - g(s + \rho)|^2 = |(g(s) - \frac{1}{2}) - (g(s + \rho) - \frac{1}{2})|^2
\]

we rearrange the functional in the following way.
Definition 6.2. Given a finite interval $[a, b]$ on the real line, let $\mathcal{E}_{\alpha, \tau}^{F, a, b} : W^{1,2}([a, b]) \rightarrow \mathbb{R}$ be the finite volume functional with free boundary conditions, defined as

$$
\mathcal{E}_{\alpha, \tau}^{F, a, b}(g) = \mathcal{L}_{\alpha, \tau}^{F, a, b}(g) + \int_a^b dx \int_a^b dy \left( g(x) - \frac{1}{2} \right) \hat{K}_\tau(x-y) \left( g(y) - \frac{1}{2} \right).
$$

Further, we need the following definitions.

Periodicity of minimizers of a functional of the form (6.3) has been shown using the reflection positivity technique by Giuliani, Lebowitz and Lieb in [6]. In order to state their results precisely, we need the following definitions.

**Definition 6.1.** Let $g \in W^{1,2}([-x_M, x_N]; [0, 1])$ with $g(x_i) = \frac{1}{2}, i \in [-M + 1, N], M + N \geq 1$ and $x_0 = 0, x_i$ are the points where $g(x_i) = \frac{1}{2}$.

- Given $T_i = x_i - x_{i-1}, i \in [-M + 1, N]$, define the restrictions $g_i : [0, T_i] \rightarrow [0, 1], g_i(x-x_{i-1}) = g(x), x \in [x_{i-1}, x_i]$.

- Given $T > 0, g \in W^{1,2}([0, T]; [0, 1])$, define the reflection $\theta g(x) = 1 - g(T-x)$.

- Given $G = \{g_{-M+1}, \ldots, g_N\}$ a family of restrictions, define the compound function $\varphi[G] \in \mathcal{E}_{\alpha, \tau}^{F, a, b}$ on the real line, in such a way that, if $x_{i-1} \leq x \leq x_i$, then $\varphi[G](x) = g_i(x-x_{i-1})$, for all $i = -M + 1, \ldots, N$. If $g \in W^{1,2}([0, T])$, we define $\varphi[g] = \varphi[G_\infty(g)] \in W^{1,2}_{\text{loc}}(\mathbb{R})$, where $G_\infty = \{\ldots, g_0, g_1, \ldots\}$ is the infinite sequence with $g_n = \theta^{n-1}g$.

**Definition 6.2.** Given a finite interval $[a, b]$ on the real line, let $\mathcal{E}_{\alpha, \tau}^{F, a, b} : W^{1,2}([a, b]) \rightarrow \mathbb{R}$ be the finite volume functional with free boundary conditions, defined as

$$
\mathcal{E}_{\alpha, \tau}^{F, a, b}(g) = \mathcal{L}_{\alpha, \tau}^{F, a, b}(g) + \int_a^b dx \int_a^b dy \left( g(x) - \frac{1}{2} \right) \hat{K}_\tau(x-y) \left( g(y) - \frac{1}{2} \right).
$$
Moreover let $\mathcal{E}_{\alpha,\tau,\ell}^{D,a,b}$ be the restriction of $\mathcal{E}_{\alpha,\tau,\ell}^{F,a,b}$ to $W_{1/2}^{1,2}([a,b])$, that is the finite volume functional with boundary conditions $g(a) = g(b) = \frac{1}{2}$. Let $L > 0$ and

$$E_{\alpha,\tau,\ell}^{F,L} \equiv \inf_{g \in W^{1,2}_{1/2}([0,L])} \mathcal{E}_{\alpha,\tau,\ell}^{F,0,L}(g)$$

(6.6)

$$E_{\alpha,\tau,\ell}^{D,L} \equiv \inf_{g \in W^{1,2}_{1/2}([0,L])} \mathcal{E}_{\alpha,\tau,\ell}^{D,0,L}(g).$$

(6.7)

Then we define the infinite volume optimal energy $e_{0}^{\tau,\ell}$ corresponding to the functional $F_{\tau,L,\ell}$ to be

$$e_{0}^{\tau,\ell} = \lim_{L \to \infty} E_{\alpha,\tau,\ell}^{F,L} / L = \lim_{L \to \infty} E_{\alpha,\tau,\ell}^{D,L} / L$$

(6.8)

whenever the limits on the r.h.s. exist and are equal.

**Theorem 6.3** ([6 Theorem 1]). For any $h > 0$, let $C_h = \{g \in W_{1/2}^{1,2}([0,h]) : g \geq \frac{1}{2}\}$. The limit in (6.8) exists and is given by

$$e_{0}^{\tau,\ell} = \inf_{h} e_{h}^{\tau,\ell}, \quad e_{h}^{\tau,\ell} \equiv \inf_{g \in C_h} e_{\infty}^{\tau,\ell}(g)$$

(6.9)

where

$$e_{\infty}^{\tau,\ell}(g) = \lim_{L \to \infty} \frac{\mathcal{E}_{\alpha,\tau,L}^{F,a,L}(\varphi(g))}{2L}.$$ (6.10)

Moreover $e_{h}^{\tau,\ell}$ is a continuous function of $h$ and $\lim_{h \to \infty} e_{h}^{\tau,\ell}$ exists and equals $e_{0}^{\tau,\ell}$.

There is a function, $g_{\epsilon,h}$, that is a minimizer for $e_{h}^{\tau,\ell}$ in (6.9) and satisfies $|g_{\epsilon,h}| \leq 1$. Moreover, $g_{\epsilon,h}$ solves the Euler Lagrange equation

$$(C - 1)\alpha_{\tau}(g_{\epsilon,h})^{n}(x) = \frac{1}{2} F'(g_{\epsilon,h}(x)) + \int_{-\infty}^{+\infty} \tilde{K}_{\tau}(x - y) \varphi(g_{\epsilon,h})(y) \, dy.$$ (6.11)

If there exists $h_0$ such that $e_{h_0}^{\tau,\ell} = e_{\infty}^{\tau,\ell}(g_{\epsilon,h_0})$, then $\varphi(g_{\epsilon,h_0}) \in \arg\min_{g \in \{0,L\}-\text{periodic}} F_{\tau,L,\ell}^{1}(g)$.

Now we proceed to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By the previous theorem, one has then that minimizers of $F_{\tau,L,\ell}$ as $L > 0$ varies are periodic with period $2h_{\tau,\ell}$. For the one-dimensional version of the sharp interface functional (1.6) we know (see Theorem 2.4) that for $\tau$ sufficiently small there exists a unique $h_{\tau}^{*}$ such that, for any $L = 2kh_{\tau}^{*}$, $k \in \mathbb{N}$ large, minimizers of the sharp interface functional (1.6) are stripes of period $2h_{\tau}^{*}$, and that for any $L$ large minimizers are stripes of width $h_{\tau,L} \sim h_{\tau}^{*}$ (see Theorem 2.5). Since by the reflection positivity technique (Theorem 6.3) one has that the minimizers of $F_{\tau,L,\ell}$ can be described by functions $g_{\tau,\ell} \in W_{1/2}^{1,2}(0,h_{\tau,\ell})$ on half of the period with $g_{\tau,\ell} \geq 1/2$, and that on the whole period $2h_{\tau,\ell}$ the minimizer is obtained by reflecting $g_{\tau,\ell}$, by $L^{1}$-convergence of the $\varphi[g_{\tau,\ell}]$ to minimizers of (1.6) we have that the optimal period $h_{\tau,\ell}$ must be bounded from above and from below:

$$\exists \sigma > 0, \Lambda > 0: \quad \sigma \leq h_{\tau,\ell} \leq \Lambda.$$
Moreover, as $\varepsilon \downarrow 0$ one has the optimal periods $h^*_{\tau,\varepsilon}$ converge to $h^*_{\tau}$, namely the optimal one for the sharp interface functional.

Our aim is to show that there exists a unique such $h^*_{\tau,\varepsilon}$ provided $\varepsilon$ and $\tau$ are small enough.

First of all notice that, for all $g \in C^1_h$, the Modica Mortola term in $\mathcal{F}^1_{\tau,L,\varepsilon}$ can be rewritten after a rescaling as

$$
\int_0^1 \left( \frac{\alpha_{\varepsilon,\tau}}{h} (\bar{g}'(x))^2 + \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}(x)) \right) = 2 \int_0^1 |\bar{g}'(x)| \sqrt{W(\bar{g}(x))} + \int_0^1 \left( \sqrt{\frac{\alpha_{\varepsilon,\tau}}{h}} |\bar{g}'(x)| - \sqrt{\frac{h}{\alpha_{\varepsilon,\tau}}} W(\bar{g}(x)) \right)^2,
$$

(6.12)

where $\bar{g}(x) = g(hx)$.

From the one-dimensional estimates of Section 4 we can deduce that the last term in the r.h.s. of (6.12) is small for sufficiently small $\varepsilon$, namely

$$
\lim_{\varepsilon \downarrow 0} \int_0^1 \left( \sqrt{\frac{\alpha_{\varepsilon,\tau}}{h}} |\bar{g}'(x)| - \sqrt{\frac{h}{\alpha_{\varepsilon,\tau}}} W(\bar{g}(x)) \right)^2 = 0.
$$

(6.13)

Indeed, if this was not the case we would have that the same term would appear with a factor $\frac{1}{\tau}$ in our functional, making it strictly positive.

Since $\bar{g}$ for $\varepsilon$ small approximates the characteristic function $\chi_{[0,1]}$, one has that

$$
\int_0^1 \frac{\alpha_{\varepsilon,\tau}}{h} (\bar{g}')^2 + \int_0^1 \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}) \approx 1
$$

and thus

$$
\int_0^1 \frac{\alpha_{\varepsilon,\tau}}{h} (\bar{g}')^2 \approx \frac{1}{2}.
$$

(6.15)

Let us do now the same spatial rescaling for the whole functional $\mathcal{F}^1_{\tau,2h,\varepsilon}$. We have that

$$
\mathcal{F}^1_{\tau,2h,\varepsilon}(g) = \mathcal{F}(\bar{g}, h, \varepsilon) := -\frac{\alpha(\bar{g}, h, \varepsilon)}{h} + \beta(\bar{g}, h, \varepsilon),
$$

(6.16)

where

$$
\alpha(\bar{g}, h, \varepsilon) = \int_0^1 \left( \frac{\alpha_{\varepsilon,\tau}}{h} (\bar{g}'(x))^2 + \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}(x)) \right) \, dx,
$$

(6.17)

$$
\beta(\bar{g}, h, \varepsilon) = h \int_{\mathbb{R}} |t| \tilde{K}_{\tau,h}(t) \, dt \int_0^1 \left( \frac{\alpha_{\varepsilon,\tau}}{h} (\bar{g}'(x))^2 + \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}(x)) \right) \, dx
$$

$$
- h \int_{\mathbb{R}} \int_0^1 (\bar{g}(x + t) - \bar{g}(x))^2 \tilde{K}_{\tau,h}(t) \, dx \, dt
$$

(6.18)

and

$$
\tilde{K}_{\tau,h}(t) := \frac{C_q}{(h|t| + \tau^{1/\beta})^q}, \quad q = p - d + 1.
$$

(6.19)
The computations made in [13, Lemma 6.1] tell us that in the case of sharp interface and for \( \tau = 0 \) the above expression can be computed explicitly and is equal to

\[
\frac{-1}{h} + \frac{C_q}{h^{q-1}},
\]  

(6.20)

with

\[
C_q = \frac{4C_q(1 - 2^{-(q-3)})}{(q-2)(q-1)} \sum_{k \geq 1} \frac{1}{k^{q-2}}.
\]  

(6.21)

Because of the \( \Gamma \)-convergence of the energies \( F_{\tau,2h,\varepsilon} \) as \( \tau, \varepsilon \downarrow 0 \) one has that the optimal periods \( h_{\tau,\varepsilon}^* \) also for \( \varepsilon, \tau > 0 \) small have to be close to this value. In particular for \( \varepsilon, \tau \) sufficiently small one has that the optimal periods are close to

\[
h^* := \left( (q-1)C_q \right)^{-1/(q-1)},
\]  

(6.22)

which is the minimizer of (6.20).

For every \( \varepsilon, h \), let \( g_{\varepsilon,h} \) be a minimizer among all the \( 2h \) periodic functions obtained by reflection as in Theorem 6.3. We will consider the map \( f : h \mapsto \mathcal{F}(\bar{g}_{\varepsilon,h}, h, \varepsilon) \). In order to show that there exists a unique period for \( \varepsilon \) and \( \tau \) small enough, it is sufficient to show that \( f''(h^*) > 0 \). Since \( \bar{g}_{\varepsilon,h} \) minimizes \( \mathcal{F}(\cdot, h, \varepsilon) \), one has that

\[
f''(h) = \partial_h^2 \mathcal{F}(\bar{g}_{\varepsilon,h}, h, \varepsilon).
\]

With simple calculations one has that

\[
\partial_h \left( -\alpha(\bar{g}_{\varepsilon,h}, h, \varepsilon) \right) = \partial_h \left( - \int_0^1 \frac{\alpha_{\varepsilon,\tau}(\bar{g}_{\varepsilon,h}')^2}{h} \, dh + \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}_{\varepsilon,h}) \right)
\]

\[
= \frac{1}{h} \int_0^1 \left( \frac{\alpha_{\varepsilon,\tau}(\bar{g}_{\varepsilon,h}')^2}{h} - \frac{h}{\alpha_{\varepsilon,\tau}} W(\bar{g}_{\varepsilon,h}) \right) \approx 0
\]

\[
\partial_h^2 \left( -\alpha(\bar{g}_{\varepsilon,h}, h, \varepsilon) \right) = - \frac{2}{h^2} \int_0^1 \alpha_{\varepsilon,\tau}(\bar{g}_{\varepsilon,h}')^2 \approx \frac{1}{h^2}
\]

\[
\partial_h^2 \left( \frac{-\alpha(\bar{g}_{\varepsilon,h}, h, \varepsilon)}{h} \right) = \partial_h^2 \left( - \frac{1}{h^2} \int_0^1 \alpha_{\varepsilon,\tau}(\bar{g}_{\varepsilon,h}')^2 \right) \approx -\frac{3}{h^3}
\]

\[
\partial_h (h \hat{K}_{\tau,h}) = \hat{K}_{\tau,h} + h \partial_h \hat{K}_{\tau,h} = - \frac{C_q(q-1)ht}{(ht + \tau^{1/\beta})^q+1}
\]

\[
\partial_h^2 (h \hat{K}_{\tau,h}) = \frac{C_qq(q-1)ht^2}{(ht + \tau^{1/\beta})^{q+2}}
\]

Moreover,

\[
\partial_h^2 \beta(\bar{g}_{\varepsilon,h}, h, \varepsilon) = A_1 + A_2 + A_3 - A_4
\]

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where

\[
A_1 = \int_{\mathbb{R}} |t| \partial_h^2 (h \tilde{\mathcal{K}}_{\tau,h}(t)) \, dt \alpha(\bar{g}_{\varepsilon,h},h,\varepsilon) \approx C_q q(q-1) \int_{\mathbb{R}} \frac{h|t|^3}{(h|t| + \tau^{1/\beta})^{q+2}} \, dt
\]

\[
A_2 = 2 \int_{\mathbb{R}} |t| \partial_h (h \tilde{\mathcal{K}}_{\tau,h}(t)) \, dt \partial_h \alpha(\bar{g}_{\varepsilon,h},h,\varepsilon) \approx 0
\]

\[
A_3 = \int_{\mathbb{R}} |t|h \tilde{\mathcal{K}}_{\tau,h}(t) \, dt \partial_h^2 \alpha(\bar{g}_{\varepsilon,h},h,\varepsilon) > 0
\]

\[
A_4 = \int_{\mathbb{R}} \int_{0}^{1} (\bar{g}_{\varepsilon,h}(x) - \bar{g}_{\varepsilon,h}(x + t))^2 \partial_h^2 (h \tilde{\mathcal{K}}_{\tau,h})(t) \, dx \, dt = C_q q(q-1) \int_{0}^{1} \int_{\mathbb{R}} \frac{h t^2 (\bar{g}_{\varepsilon,h}(x+t) - \bar{g}_{\varepsilon,h}(x))^2}{(h|t| + \tau^{1/\beta})^{q+2}} \, dx \, dt
\]

For \( \tau \) is sufficiently small, we have that

\[
A_1 - A_4 \approx C_q q(q-1) \int_{\mathbb{R}} \left( |t| - \int_{0}^{1} (\bar{g}_{\varepsilon,h}(x+t) - \bar{g}_{\varepsilon,h}(x))^2 \, dx \right) \frac{h t^2}{(h t + \tau^{1/\beta})^{q+2}} \, dt.
\]

The integral in the r.h.s. for \( \tau = 0 \) has been calculated in \([13\text{ Lemma 6.3}]\) and is equal to \( C_q/h^{q+1} \).

By substituting it in the above we obtain that

\[
f''(h^*) \geq -\frac{3}{h^{3\beta}} + q(q+1) \frac{C_q}{h^{q+1}}.
\]

Recalling the expression for \( h^* \) in \((6.22)\), the expression for \( C_q \) in \((6.21)\) and the fact that \( q = p - d + 1 \geq 3 \) one easily sees that \( f''(h^*) > 0 \).

Finally, since \( g_{\varepsilon,h_{\tau,\varepsilon}} \) solves \((6.11)\) and \( |g_{\varepsilon,h_{\tau,\varepsilon}}| \leq 1 \), it is unique.

\[\square\]

**Remark 6.4.** The fact that for any given \( L > 0 \) there exist \( \varepsilon_L > 0 \) and \( \tau_L > 0 \) such that for any \( 0 < \varepsilon \leq \varepsilon_L \) and \( 0 < \tau \leq \tau_L \) minimizers of \( F_{\tau,\varepsilon,L}^1 \) are periodic of period \( 2h_{\tau,\varepsilon,L} \) follows from the above as e.g. in \([13]\) and \([3]\) with the estimate

\[
|h_{\tau,\varepsilon,L} - h_{\tau,\varepsilon,\varepsilon}| \lesssim \frac{1}{L}.
\]

### 7 Proof of Theorem 1.1

By the \( \Gamma \)-convergence result of Corollary 2.2 and Theorem 2.3 there exist \( \tilde{\varepsilon} > 0 \) and \( \tilde{\tau} > 0 \) such that, for all \( \varepsilon \leq \tilde{\varepsilon}, \tau \leq \tilde{\tau} \), then minimizers \( u \) of \( F_{\tau,\varepsilon,L} \) satisfy

\[
\|u - \chi S\|_{L^1([0,L]^d)} \leq \tilde{\sigma},
\]

with \( \tilde{\sigma} \) as in Proposition 5.1 and \( S \) periodic union of stripes in direction \( i \) for some \( i \in \{1, \ldots, d\} \).

Without loss of generality, let us assume that \( i = 1 \).

Recall now the lower bound for the functional \((1.1)\) given by

\[
F_{\tau,\varepsilon,L}(u) \geq \int_{[0,L]^d} \left[ -\mathcal{M}_{\alpha_{\tau,\varepsilon}}^{1,1}(u;x_1^+;[0,L]) + \mathcal{G}_{\alpha_{\tau,\varepsilon}}^{1,1}(u,x_1^+;[0,L]) \right] \, dx_1^+
\]

\[
+ \sum_{i=2}^{d} \left[ \int_{[0,L]^d} \left[ -\mathcal{M}_{\alpha_{\tau,\varepsilon}}^{1,i}(u;x_i^+;[0,L]) + \mathcal{G}_{\alpha_{\tau,\varepsilon}}^{1,i}(u,x_i^+;[0,L]) \right] \, dx_i^+ + \mathcal{T}_{\tau,L}^i(u) \right].
\]

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By the results of Section 6 (see Theorem 6.3, Theorem 1.2 and Remark 6.4), we know that for \( \varepsilon \) and \( \tau \) eventually smaller but positive the functional in (7.2) is minimized by periodic one-dimensional functions of period \( 2h_{\tau,\varepsilon,L} \) in direction \( e_1 \) and by Proposition 6.1 we know that, if additionally \( \tau \leq \hat{\tau} \), each of the \( d \) terms of the sum in (7.3) is zero if \( u(x) = g(x_1) \) and strictly positive otherwise. Therefore, up to translations in direction \( e_1 \), \( u(x) = \bar{u}_{\varepsilon,\tau}(x_1) \) with \( \bar{u}_{\varepsilon,\tau} \) periodic minimizers of the one-dimensional functional (6.1).

**Remark 7.1.** Notice that the \( \Gamma \)-convergence result of Corollary 2.2 and Theorem 2.3 were only used in establishing the \( L^1 \)-estimate (7.1). All the other estimates giving that minimizers of \( \mathcal{F}_{\tau,L,\varepsilon} \) must be exactly one-dimensional (namely Lemma 4.2 and Proposition 5.1) are based only on the functional \( \mathcal{F}_{\tau,L,\varepsilon} \) and do not use the estimates found in [4] to prove that minimizers in the limit event that are stripes.

### 8 Remarks on the proof of Theorem 1.3

Analogously to what has been done in Section 7 of [4] for the proof of the fact that \( \hat{\tau} > 0 \) defining the regime in which minimizers of \( \mathcal{F}_{\tau,L} \) are periodic and one-dimensional can be chosen independently on \( L \), we define the following quantities:

\[
\begin{align*}
  r_{i,\alpha,\varepsilon,\tau}(u, x_i^+, (a, b)) &= -\mathcal{M}_{\alpha,\varepsilon,\tau}^{1d,i}(u; x_i^+, [a, b]) + \int_a^{\infty} \int_a^b \mathcal{M}_{\alpha,\varepsilon,\tau}^{1d,i}(u; x_i^+, [s, s + z]) \hat{K}_\tau(z) \, dz \, ds \\
  &\quad - \int_0^{+\infty} \int_a^b |u_{x_i^+}(s) - u_{x_i^+}(s + z)|^2 \hat{K}_\tau(z) \, dz \, ds \\
  &\quad + \int_{-\infty}^0 \int_a^b (\mathcal{M}_{\alpha,\varepsilon,\tau}^{1d,i}(u; x_i^+, [s + z, s]) - |u_{x_i^+}(s) - u_{x_i^+}(s + z)|^2) \hat{K}_\tau(z) \, dz \, ds \\
  f_u(x_i^+, x_i, \zeta_i^+, \zeta_i) &= |u(x_i^+ + x_i + \zeta_i) - u(x_i^+ + x_i) - (u(x_i^+ + x_i + \zeta_i) - u(x_i^+ + x_i + \zeta_i))|^2 \\
  v_i,\tau(u, x_i^+, (a, b)) &= \frac{1}{2d} \int_a^b \int_{\mathbb{R}^d} f_u(x_i^+, x_i, \zeta_i^+, \zeta_i) K_\tau(\zeta) \, d\zeta \, dx_i \\
  w_i,\tau(u, x) &= \frac{1}{2d} \int_{\mathbb{R}^d} f_u(x_i^+, x_i, \zeta_i^+, \zeta_i) K_\tau(\zeta) \, d\zeta \\
  \tilde{F}_{i,\tau,\varepsilon}(u, Q_\ell(z)) := \frac{1}{L} \left[ \int_{Q_\ell(z)} r_{i,\alpha,\varepsilon,\tau}(u, x_i^+, (z_i - l/2, z_i + l/2)) + v_{i,\tau}(u, x_i^+, (z_i - l/2, z_i + l/2)) \, dx_i^+ \right. \\
  &\quad + \left. \int_{Q_\ell(z)} w_i,\tau(u, x) \, dx \right] \\
  \tilde{F}_{\tau,\varepsilon}(u, Q_\ell(z)) := \sum_{i=1}^d \tilde{F}_{i,\tau,\varepsilon}(u, Q_\ell(z))
\end{align*}
\]

where \( a, b \in \mathbb{R}, \, 0 < l < L \).
One has that
\[ \mathcal{F}_{\tau, L, \varepsilon}(u) \geq \frac{1}{L^d} \int_{[0,L)^d} \tilde{F}_{\tau, \varepsilon}(u, Q_l(z)) \, dz \]
and since equality holds for one-dimensional functions, if we show that minimizers of the r.h.s. are one-dimensional, the same claim holds for $\mathcal{F}_{\tau, L, \varepsilon}$. The essential thing is that this fact holds for $\varepsilon, \tau$ depending only on the smaller scale $l$ and not on $L$, so that Theorem 1.3 follows. For the interested reader, we refer to Section 7 in [4]. Here we make a couple of remarks.

The proof consists in localizing the rigidity estimates and stability estimates, namely proving them for the functionals (8.5) for some $\varepsilon, \tau$ depending only on $l$. In order to do so, one has to implement lemmas analogous to those presented in Section 7 of [4], now for functions $u$ instead of sets $E$ and with the quantities defined above. The proof of such lemmas is very similar to those given in [4], so that here we limit ourselves to point out the less obvious variations. In the same way as in [4], one can define an $L$-distance of $u$ on a cube $Q$ from stripes in direction $i$ and of width at least $\eta$, $D_{\eta}^i(u, Q)$. Then one partitions $[0, L)^d$ into sets $A_0$, $A_{-1}$, $A_1$, ..., $A_d$, where $z \in A_i$, $i \in \{1, \ldots, d\}$ if such distance of $u$ on $Q_l(z)$ in direction $i$ is smaller than some fixed $\delta$, $z \in A_{-1}$ if there is more than one $i$ for which this holds, and $z \in A_0$ if the distance from all the stripes is larger than $\delta$. The aim is to prove that for $\tau$ and $\varepsilon$ small depending on $l$, there exists only one $A_i$ with $i \in \{1, \ldots, d\}$. If this is proved, then a local version of the stability Proposition 5.1 shows that it is not convenient to deviate from being one-dimensional in direction $\tau$ on every $Q_l(z)$, then on all $[0, L)^d$. Inspecting the proof of Proposition 5.1 one has the following generalization of the local stability Lemma 7.8 in [4]:

**Lemma 8.1.** Let $z \in [0, L)^d$, $0 < l < L$. Then, there exist $\eta_0, \tau_0, \sigma_0, \varepsilon_0$ possibly depending on $l$ such that for every $\tau < \tau_0$, $\varepsilon < \varepsilon_0$, $\sigma < \sigma_0$ the following holds: assume that for all $s < t \in [z - l/2, z + l/2]$ such that $|u(s) - u(t)| \geq 1 - 2\delta$, then for the infimum of such $t$ there exists such an $s$ with $s - z + l/2 > \eta_0$ and for the supremum of such $s$ we have such a $t$ with $z + l/2 - \eta_0 > t$ (roughly speaking, almost jump points between 0 and 1 do not happen close to the boundary of $[z - l/2, z + l/2]$); moreover, assume $D_{\eta}^j(u, Q_l(z)) \leq \frac{\sigma^d}{16\tau}$ for some $\eta > 0$ for some $j \neq i$. Then,

\[ r_{i, \alpha, \tau, \varepsilon}(u, x_i^+, (z - l/2, z + l/2)) + \nu_{i, \tau}(u, x_i^+, (z - l/2, z + l/2)) \geq 0. \]  

(8.7)

Otherwise,

\[ r_{i, \alpha, \tau, \varepsilon}(u, x_i^+, (z - l/2, z + l/2)) + \nu_{i, \tau}(u, x_i^+, (z - l/2, z + l/2)) \geq -2 \cdot \frac{17}{16}. \]  

(8.8)

Estimate (8.8) substitutes the estimate $r_{i, \tau}(E, x_i^+, s) \geq -2$ due to the possible presence of points in the boundary of $E$ close to the boundary of $[z - l/2, z + l/2]$. Now, in comparison to Lemma 7.8 in [4], $\eta_0$ and $\tau_0$ might depend on $l$, but this is not a restriction because in the proof $l$ is fixed independently on $L$ and depending on the optimal value of the functional. The other lemmas analogous to those in Section 7 of [4] has a very similar proof, therefore we omit them.

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