ENDOMORPHISMS OF THE SYMMETRIC 2-RIG OF FINITE SETS

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Abstract. Let $\mathbf{FS}$ be the groupoid of finite sets and bijections between them equipped with the canonical symmetric rig category structure given by the disjoint union and the cartesian product of finite sets. We prove that the category (in fact, groupoid) of endomorphisms of $\mathbf{FS}$ is equivalent to the terminal category, thus providing some evidence that $\mathbf{FS}$ is the right categorical analog of the commutative rig $\mathbb{N}$ of nonnegative integers. This is shown using a particular semistrict skeletal version of $\mathbf{FS}$ for which the endomorphisms can be described very explicitly.

1. Introduction

A rig (a.k.a. a semiring) is a ring without negatives, i.e. an (additive) abelian monoid $(S, +, 0)$ equipped with an additional (multiplicative) monoid structure $(\cdot, 1)$ such that $\cdot$ distributes over $+$ on both sides, and $0 \cdot x = x \cdot 0 = 0$ for each $x \in S$. A paradigmatic example is the set $\mathbb{N}$ of nonnegative integers with the usual sum and product. The rig is called commutative when $\cdot$ is abelian; for instance, $(\mathbb{N}, +, \cdot, 0, 1)$ is a commutative rig. Rigs (resp. commutative rigs) are the objects of a category $\text{Rig}$ (resp. $\text{CRig}$) having as morphisms the maps that preserve both $+$ and $\cdot$, and the corresponding neutral elements.

We are interested in the categorical analog of a (commutative) rig. It is usually called a (symmetric) rig category, or a (symmetric) bimonoidal category. The last name, however, is confusing because a bimonoid in the set-theoretic context is a set simultaneously equipped with compatible monoid and comonoid structures, and not a set with two monoid structures one of them distributing over the other.

The precise definition of a (symmetric) rig category is due to Laplaza [9], and goes back to the 1970s (see also [6]). Roughly, it is a category $S$ equipped with functorial operations analogous to those of a rig, and satisfying all rig axioms up to suitable natural isomorphisms. More precisely, $S$ must be equipped with an (additive) symmetric monoidal structure $(+, 0, a, c, l, r)$, a (multiplicative) monoidal structure $(\cdot, 1, a', l', r')$ (including a multiplicative commutator $c'$ in case the rig category is symmetric), and distributor and absorbing natural isomorphisms

$$
d_{x,y,z} : x \cdot (y + z) \xrightarrow{=} x \cdot y + x \cdot z
$$

$$
d'_{x,y,z} : (x + y) \cdot z \xrightarrow{=} x \cdot z + y \cdot z
$$

$$
n_x : x \cdot 0 \xrightarrow{=} 0
$$

$$
n'_x : 0 \cdot x \xrightarrow{=} 0
$$

making commutative the appropriate ‘coherence diagrams’. For short, we shall denote by $S$ the whole data defining a (symmetric) rig category. A paradigmatic example is the category $\mathbf{Set}$ of finite sets and maps between them, with $+$ and $\cdot$ respectively given by the disjoint union and the cartesian product of finite sets. The rig category so defined $\mathbf{FS}$ is symmetric with $c'$ given by the canonical isomorphisms of sets $S \times T \cong T \times S$. Of course, rig categories (resp. symmetric rig categories) are the objects of a 2-category $\text{RigCat}$ (resp. $\text{SRigCat}$) whose 1- and 2-cells are given by the appropriate type of functors and natural transformations between these. The precise definitions are given in Section 3.

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\[ \mathcal{F}\text{Set} \] is not just a symmetric rig category. It is a categorification of the commutative rig \( \mathbb{N} \), in the sense that the set of isomorphisms classes of objects in \( \mathcal{F}\text{Set} \) with the rig structure induced by \(+\) and \( \cdot \) is isomorphic to \( \mathbb{N} \) with its canonical rig structure; for more on the idea of categorification, see [2]. In fact, there are other non-equivalent categorifications of \( \mathbb{N} \) as a rig, such as the symmetric rig category \( \mathcal{F}\text{Vect}_k \) of finite dimensional vector spaces over any given field \( k \), with \(+\) and \( \cdot \) respectively given by the direct sum and the tensor product of vector spaces, or the rig \( \mathbb{N} \) itself viewed as a discrete category with only the identity morphisms. What is then the right categorical analog of \( \mathbb{N} \) as a rig? Of course, the answer depends on what we mean by the “right categorical analog” of \( \mathbb{N} \) as a rig. As it is well known, \( \mathbb{N} \) is an initial object in the category of commutative rigs, i.e. for every commutative rig \((S, +, \cdot, 0, 1)\) (in fact, for every rig, commutative or not) there is one and only one rig homomorphism from \( \mathbb{N} \) to \( S \), namely, the map \( \varphi : \mathbb{N} \to S \) given by \( \varphi(0) = 0 \) and \( \varphi(n) = 1 + \ldots + 1 \) for each \( n \geq 1 \). Then it is reasonable to look for the symmetric rig category having the analogous categorical property.

It is a conjecture, apparently due to John Baez, that the right categorical analog of \( \mathbb{N} \) in this sense is the groupoid of finite sets and only the bijections between them equipped with the symmetric rig category structure inherited from \( \mathcal{F}\text{Set} \). Thus if we denote by \( \mathcal{F}\text{Set} \) this symmetric rig category, \( \mathcal{F}\text{Set} \) is expected to be binitital in \( \mathcal{S}\text{RigCat} \), i.e. for every symmetric rig category \( \mathcal{S} \), the category of symmetric rig category homomorphisms from \( \mathcal{F}\text{Set} \) to \( \mathcal{S} \) is expected to be equivalent to the terminal category with only one object and its identity morphism. An immediate consequence of this conjecture is that the category of endomorphisms of \( \mathcal{F}\text{Set} \) should be equivalent to the terminal category. The purpose of this paper is to prove this assertion.

At first sight, this may look easy. Thus the underlying endofunctor of each endomorphism of \( \mathcal{F}\text{Set} \) must preserve both the singletons, and the disjoint unions, at least up to isomorphism. Thus it is essentially the identity on objects because every object in \( \mathcal{F}\text{Set} \) is a finite disjoint union of singletons. Moreover, it is not difficult to see that the preservation of disjoint unions up to isomorphism implies that the action on morphisms must necessarily be by inner automorphisms (cf. Proposition 4.2.1 below). Therefore the endofunctor will be naturally isomorphic to the identity. However, the point is that together with the underlying endofunctor, giving an endomorphism of \( \mathcal{F}\text{Set} \) also requires specifying the natural isomorphisms which take account of the preservation of the \(+\) and \( \cdot \) up to isomorphisms. As explained below, these natural isomorphisms must satisfy infinitely many coherence equations, and it is not a priori clear that all possible choices for these isomorphisms actually define equivalent endomorphisms of \( \mathcal{F}\text{Set} \). In fact, working with a particular semistriek skeletal version of \( \mathcal{F}\text{Set} \), we shall be able to describe quite explicitly these endomorphisms, and to prove then that they are all indeed equivalent to the identity.

Rig categories have been used since the late 1970s as sources of examples of non-commutative monoids or semirings (see Chapter VI of [11]), and to define a sort of ‘2-K-theory’ (see [12]). Our interest in rig categories is due to the conjecture is true, and the main theorem of this paper gives evidence for this, every symmetric monoidal category of endomorphisms of the symmetric monoidal category (cf. Example 3.3.4 below). If Baez’s conjecture is true, and the main theorem of this paper gives evidence for this, every symmetric monoidal category \( \mathcal{M} = (\mathcal{M}, \oplus, 0) \) will have a unique \( \mathcal{F}\text{Set} \)-module category structure up to equivalence, given by the essentially unique rig category homomorphism from \( \mathcal{F}\text{Set} \) to the rig category of endomorphisms of \( \mathcal{M} \). The underlying functor will map the finite set \([n] = \{1, \ldots, n\}\) to the endomorphism functor \( \oplus^n : \mathcal{M} \to \mathcal{M} \), and each permutation of \([n]\) to the corresponding natural automorphism of \( \oplus^n \). The point is that \( \mathcal{F}\text{Set} \) embeds as a rig subcategory of \( \text{Fin}\text{Vect}_k \) through the free vector space construction. Hence every \( \text{Fin}\text{Vect}_k \)-module category structure on \( \mathcal{M} \) is actually partially canonical, and this should allow one to better understand what additional data defines a \( \text{Fin}\text{Vect}_k \)-module category structure on a symmetric monoidal category \( \mathcal{M} \). More
precisely, because of the Bruhat decomposition of the general linear groups
\[ GL(n, k) = T(n, k) S_n T(n, k) \]
\((T(n, k)\) denotes the subgroup of upper triangular invertible \(n \times n\) matrices), one expects that a \(\mathcal{F}Vec_k\)-module category structure on \(\mathcal{M}\) will amount to a suitable collection of group homomorphisms
\[ T(n, k) \rightarrow Aut(\mathcal{F}^n), \quad n \geq 0, \]
identifying each upper triangular \(n \times n\) matrix with an automorphism of \(\mathcal{F}^n\), and expectedly some more data related to the additive and multiplicative monoidal structures in each rig category.

1.1. **Outline of the paper and assumed background.** In section 2 we discuss some facts about permutations needed in the sequel. In particular, we define the product of permutations, and compute the centralizer of the “product” (not direct product) of the symmetric groups \(S_m\) and \(S_n\) as a subgroup of \(S_{mn}\) for each \(m, n \geq 1\). In Section 3 the definition of (symmetric) rig category is reviewed in detail, and the corresponding notions of 1- and 2-morphism are given making them the objects of a 2-category. Some examples of rig categories are mentioned, paying special attention to the symmetric rig category ‘canonically’ associated to every distributive category (i.e. a category with finite products and coproducts and whose canonical distributivity maps are invertible). The section ends with the statement of the strictification result for (symmetric) rig categories. Finally, in Section 4 we describe a semistrict version of \(\mathcal{F}Set\), and prove that its category of endomorphisms is indeed trivial by computing explicitly the endomorphisms.

The reader is assumed to be familiar with the definitions of (symmetric) monoidal category and 2-category, and with the notions of (symmetric) monoidal functor and monoidal natural transformation between them. Good references for the basics of monoidal categories are, for instance, Chapters 1 and 3 of [1], Chapter XI of [5], or Chapter 2 of [13], and for the basics of 2-categories Chapter 7 of [3]. The reader may also take a look to the standard text by MacLane [10].

1.2. **A few conventions about notation.** Both sets and structured sets (such as monoids, rigs, etc) are denoted in the same way, namely by capital letters \(A, B, C, \ldots\). Plain categories and functors between them are denoted by \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots\) (symmetric) monoidal categories and (symmetric) monoidal functors between them by \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots\) and rig categories and rig category homomorphisms between them (both notions are defined below) by \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots\). When we refer to concrete examples of categories, the same convention will be applied to the first letters. Thus \(\mathcal{Set}\), \(\mathcal{Set}\) and \(\mathcal{Set}\) respectively denote the plain category of (small) sets, the category \(\mathcal{Set}\) equipped with a monoidal structure, and the category \(\mathcal{Set}\) equipped with a rig category structure. Finally, 2-categories are denoted by boldface letters \(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots\).

Given two monoidal categories \(\mathcal{A} = (\mathcal{A}, \otimes, 1, a, l, r)\) and \(\mathcal{A}' = (\mathcal{A}', \otimes', 1', a', l', r')\), and unless otherwise indicated, by a monoidal functor between them we mean a strong monoidal functor. The underlying functor of such a monoidal functor \(\mathcal{F} : A \rightarrow A'\) is denoted by \(F\), and the structural isomorphisms taking account of the preservation of the tensor product and unit objects up to natural isomorphism by \(\varphi\) and \(\varepsilon\), respectively. Thus \(\mathcal{F}\) consists of the functor \(F\) together with a natural isomorphism
\[ \varphi_{x,y} : F(x \otimes y) \xrightarrow{\sim} Fx \otimes' Fy \]
in \(\mathcal{A}'\) for each pair of objects \(x, y \in \mathcal{A}\), and an isomorphism \(\varepsilon : F1 \rightarrow 1'\) also in \(\mathcal{A}'\), all these isomorphisms satisfying the corresponding coherence axioms.

Finally, composition of (1-)morphisms (in a category or in a 2-category) is denoted by juxtaposition, and the identity of an object \(x\) by \(id_x\). In particular, for any objects \(A, B, C\) and morphisms \(f : A \rightarrow B\) and \(g : B \rightarrow C\), the composite is denoted by \(gf : A \rightarrow C\). Exceptionally, composition of functors is denoted with the symbol \(\circ\).

2. **Preliminaries on permutations**

For each \(n \geq 1\), we shall denote by \(S_n\) the symmetric group on the set \([n] = \{1, \ldots, n\}\). When writing cycles, and to avoid possible ambiguities on their degrees, we shall denote by \((i_1, \ldots, i_k)\) the \(k\)-cycle in \(S_n\) mapping \(i_r\) to \(i_{r+1}\) for \(r = 1, \ldots, k-1\), and \(i_k\) to \(i_1\). Thus \((1, 3, 4)\) and \((1, 3, 4)\) denote two different 3-cycles, respectively in \(S_5\) and in \(S_6\).
The purpose of this section is to discuss a few facts about permutations which will be needed in the sequel.

2.1. Perfect squares in the symmetric groups. The square of a $k$-cycle is again a $k$-cycle when $k$ is odd, and is the product of two $k/2$ cycles when $k$ is even. More precisely, for each $l \geq 1$ we have

$$(i_1, i_2, \ldots, i_{2l-1})_l^2 = (i_1, i_3, \ldots, i_{2l-1}, i_2, i_4, \ldots, i_{2l})_l,$$

$$(i_1, i_2, \ldots, i_{2l})_l^2 = (i_1, i_3, \ldots, i_{2l-1}, i_2, i_4, \ldots, i_{2l})_l.$$ Conversely, the product of any two disjoint $k$-cycles with $k$ even is the square of a $2k$-cycle obtained by mixing up both cycles in the right way. It follows that an element $\sigma \in S_n$ is a perfect square if and only if, for every even value of $k$, the cycle structure for $\sigma$ has an even number of $k$-cycles.

2.2. Product of permutations. For each $m, n \geq 1$ let $b_{m,n} : [m] \times [n] \to [mn]$ be the bijection obtained when the elements of $[m] \times [n]$ are enumerated by rows. Thus for each $(i, j) \in [m] \times [n]$ we have

$$b_{m,n}(i, j) = (j - 1)m + i.$$ Then for any permutations $\tau \in S_m$ and $\sigma \in S_n$ we shall denote by $\tau \cdot \sigma \in S_{mn}$ the permutation defined by

$$\tau \cdot \sigma = b_{m,n}(\tau \times \sigma) b_{m,n}^{-1}.$$ This means that, after identifying the elements of $[mn]$ with the points in $[m] \times [n] \subset \mathbb{R}^2$ through the previous bijection, $\tau \cdot \sigma$ is the permutation acting as $\tau$ on the $m$ columns, and as $\sigma$ on the $n$ rows of $[m] \times [n]$.

2.2.1. Example. If $\tau = (1, 3, 4)$ and $\sigma = (2, 3)$ their product $\tau \cdot \sigma \in S_{12}$ is the permutation

$$\tau \cdot \sigma = (1, 3, 4)_{12} (5, 11, 8, 9, 7, 12)_{12} (6, 10)_{12}.$$ 2.2.2. Example. For every permutation $\tau \in S_m$ the permutations $\tau \cdot id_{[2]}$, $id_{[2]} \cdot \tau \in S_{2m}$ are given by

$$(\tau \cdot id_{[2]})(k) = \begin{cases} \tau(k), & \text{if } k \in \{1, \ldots, n\}, \\ n + \tau(k - n), & \text{if } k \in \{n + 1, \ldots, 2n\}. \end{cases}$$

$$(id_{[2]} \cdot \tau)(k) = \begin{cases} 2\tau((k + 1)/2) - 1, & \text{if } k \text{ is odd}, \\ 2\tau(k/2), & \text{if } k \text{ is even}. \end{cases}$$

2.2.3. Proposition. Let $\tau, \tau' \in S_m$ and $\sigma, \sigma' \in S_n$. Then:

1. $id_{[1]} \cdot \sigma = \sigma$ and $\tau \cdot id_{[1]} = \tau$;
2. $id_{[m]} \cdot id_{[n]} = id_{[mn]}$;
3. $(\tau' \cdot \sigma')(\tau \cdot \sigma) = (\tau' \cdot \tau) \cdot (\sigma' \cdot \sigma)$;
4. $(\tau \cdot \sigma)^{-1} = \tau^{-1} \cdot \sigma^{-1}$;
5. if $\tau$ is the product of $r$ disjoint cycles of respective lengths $k_1, \ldots, k_r \geq 2$ then $\tau \cdot id_{[n]}$ is the product of $nr$ disjoint cycles of respective lengths $k_1, \ldots, k_r, \ldots, k_r$;
6. if $\sigma$ is the product of $s$ disjoint cycles of respective lengths $l_1, \ldots, l_s$, then $id_{[m]} \cdot \sigma$ is the product of $ms$ disjoint cycles of respective lengths $l_1, \ldots, l_1, \ldots, l_s, \ldots, l_s$;
7. if $\tau \neq id_{[n]}$ (resp. $\sigma \neq id_{[m]}$) then $\tau \cdot id_{[m]} \neq id_{[mn]}$ (resp. $id_{[m]} \cdot \sigma \neq id_{[mn]}$).

Proof. (1) follows from the fact that $[1]$ is a (nonstrict) unit object for the cartesian product $\times$ of sets. (2) to (4) follow from the functoriality of this cartesian product. (5) and (6) are clear from the definition of the product of permutations. For instance, $id_{[m]} \cdot \sigma$ will consist of applying $\sigma$ separately to each of the $m$ columns of $[m] \times [n] \cong [mn]$. More precisely, for each $i \in \{1, \ldots, m\}$ the $i^{th}$-column of $[m] \times [n]$ corresponds through the bijection $b_{m,n}$ to the subset $S_i = \{km + i, k = 0, 1, \ldots, n - 1 \} \subset [mn]$, and $id_{[m]} \cdot \sigma$ acts on it by $km + i \mapsto (\sigma(k + 1) - 1)m + i$. In particular, each $S_i$ remains invariant, and the corresponding permutation has the same decomposition into disjoint cycles as $\sigma$. Finally, (7) follows readily from (5) and (6).
2.3. **Centralizer of the subgroup of the product permutations for fixed orders.** We shall denote by $S_m \cdot S_n$ the set of all permutations in $S_{mn}$ of the form $\tau \cdot \sigma$ for $\tau \in S_m$ and $\sigma \in S_n$. Because of items (2)-(4) of Proposition 2.2.3 it is a subgroup of $S_{mn}$. In fact, unless $m = 1$ or $n = 1$, it is a proper subgroup of the (transitive and imprimitive) maximal subgroup $S_m \wr S_n \subset S_{mn}$ consisting of all permutations of $[mn] \cong [m] \times [n]$ acting on the $n$ rows according to some element $(\tau_1, \ldots, \tau_n) \in S_m \times [n] \times S_m$ (hence, in a possibly different way on each of row), and permuting the $n$ rows according to a given permutation $\sigma \in S_n$ (see [14], § 2.5.2.). The permutations in $S_m \cdot S_n$ are the elements in $S_m \wr S_n$ for which all $\tau_i$’s are the same permutation $\tau$.

2.3.1. **Example.** The subgroup $S_2 \cdot S_2 \subset S_4$ has as elements the four products $id_{[2]} \cdot id_{[2]}$, $id_{[2]} \cdot (1, 2)$, $(1, 2) \cdot id_{[2]}$ and $(1, 2) \cdot (1, 2)$ and hence, it is the subgroup

$$S_2 \cdot S_2 = \{ id_{[2]}, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3) \}.$$ 

Notice that it is isomorphic to the Klein four group $Z_2 \times Z_2$. In particular, it is an abelian subgroup of $S_4$.

Later on, we will need to know the centralizers $C_{S_m}(S_n \cdot S_n)$ of these product subgroups for each $m, n \geq 1$. The cases $m = 1$ or $n = 1$ are very easy to compute. Thus by item (1) of Proposition 2.2.3 both $S_1 \cdot S_n$ and $S_n \cdot S_1$ are equal to $S_n$ so that their centralizers are nothing but the center of $S_n$. Hence $C_{S_n}(S_1 \cdot S_n)$ and $C_{S_n}(S_n \cdot S_1)$ are both trivial when $n \neq 2$, and equal to $S_2$ when $n = 2$. The answer in the cases $m, n \geq 3$ is also very simple.

2.3.2. **Proposition.** For each $m, n \geq 3$ we have $C_{S_{mn}}(S_m \cdot S_n) = \{ id_{[mn]} \}$.

**Proof.** It is enough to see that for every $k, l \in [mn]$ with $k \neq l$ there exists $\tau \cdot \sigma \in S_m \cdot S_n$ such that

$$\tau \cdot \sigma(k) = k,$$

$$\tau \cdot \sigma(l) \neq l.$$ 

Thus if this is true then for every permutation $\phi \neq id_{[mn]}$ there will exist $k \neq l$ such that $\phi(k) = l$, and $\phi$ will not commute with the permutation $\tau \cdot \sigma$ satisfying (2) for this pair $k, l$. In fact, we will have

$$\phi(\tau \cdot \sigma)(k) = \phi(k) = l,$$

$$\phi(\tau \cdot \sigma)(l) = \phi(\tau \cdot \sigma)(l) \neq l.$$ 

In order to prove the existence of the permutations $\tau \in S_m$ and $\sigma \in S_n$ satisfying (2) for any given $k \neq l$, let us consider the pairs $(i, j), (r, s) \in [m] \times [n]$ defined by

$$(i, j) = b_{m,n}^{-1}(k), \quad (r, s) = b_{m,n}^{-1}(l),$$

where $b_{m,n} : [m] \times [n] \to [mn]$ is the above defined bijection. Since $k \neq l$ we have $(i, j) \neq (r, s)$ and hence, either $i \neq r$ or $j \neq s$ (or both). If $i \neq r$, it is enough to take as $\tau$ any permutation in $S_s$ such that $\tau(i) = i$ and $\tau(r) \neq r$, and as $\sigma$ any permutation in $S_s$ such that $\sigma(j) = j$. Notice that such a $\tau$ exists precisely because $m \geq 3$: if $m = 1$ it could not be $i \neq r$, and if $m = 2$ the condition $\tau(i) = i$ automatically would imply $\tau(r) = r$ because we are assuming $i \neq r$. Then by definition of $\tau \cdot \sigma$ we have

$$\tau(\sigma)(k) = (b_{m,n}(r \cdot \sigma)(b_{m,n}^{-1})(k) = b_{m,n}(\tau(i), \sigma(j)) = b_{m,n}(i, j) = k$$

while

$$\tau(\sigma)(l) = b_{m,n}(\tau(r), \sigma(s)) \neq l,$$

because $\tau(r) \neq r$. The same argument works when $j \neq s$ because we are also assuming $n \geq 3$. \hfill \Box

It remains to consider the cases $m = 2$ or $n = 2$, with $m, n \neq 1$. The answer in these cases is not so simple due to the abelian character of $S_2$. For instance, it follows from item (3) of Proposition 2.2.3 that the permutation $(1, 2)_2 \cdot id_{[2]}$ commutes with every permutation in $S_2 \cdot S_2$, and similarly $id_{[2]} \cdot (1, 2)_2$ commutes with every permutation in $S_2 \cdot S_2$. Hence both $C_{S_2}(S_2 \cdot S_2)$ and $C_{S_2}(S_2 \cdot S_2)$ are nontrivial for each $n > 1$. To describe these centralizers, let us consider separately the case $n = 2$ and the remaining cases $n > 2$. If $n = 2$ we know that $S_2 \cdot S_2$ is the abelian subgroup of $S_4$ (cf. Example 2.3.1). Hence $S_2 \cdot S_2 \subset C_{S_4}(S_2 \cdot S_2)$. In fact, a look at the Cayley table of $S_4$ shows that

$$C_{S_4}(S_2 \cdot S_2) = S_2 \cdot S_2.$$

As to the centralizers $C_{S_{mn}}(S_2 \cdot S_n)$ and $C_{S_{mn}}(S_2 \cdot S_n)$ for $n > 2$, they have the following descriptions.
2.3.3. PROPOSITION. Let be $n \geq 3$. Then:

1. $C_{S_n}(S_2 \cdot S_n) = \{(1, 2)^{2n}, (3, 4)^{2n}, \ldots, (2n - 1, 2n)^{2n}\};$
2. $C_{S_n}(S_n \cdot S_2) = \{(1, n + 1)^{2n}, (2, n + 2)^{2n}, \ldots, (n, 2n)^{2n}\}.$

PROOF. Let us prove (1). It is enough to see that for every $\psi \in C_{S_n}(S_2 \cdot S_n)$ and every $k \in [2n]$ such that $\psi(k) \neq k$ we necessarily have

$$\psi(k) = \begin{cases} k - 1, & \text{if } k \text{ is even,} \\
                     k + 1, & \text{if } k \text{ is odd} \end{cases}$$

In order to see this, let us first observe that if $\psi(k) = l$ then $k, l$ necessarily belong to the same row after identifying $[2n]$ with $[2] \times [n]$ through the bijection $b_{2n}$. Thus let $k = b_{2n}(i, j)$ and $l = b_{2n}(r, s)$, and let us suppose that $i \neq r$. Then take $\tau = id_{[3]}$ and $\sigma \in S_n$ any permutation such that $\sigma(j) = j$ and $\sigma(s) = s$. Such a $\sigma$ exists precisely because of the assumption that $n \geq 3$ (if $n = 2$ then $\sigma(j) = j$ would imply $\sigma(s) = s$). It follows that $\tau \cdot \sigma$ leaves $k$ fixed, and hence

$$\psi(\tau \cdot \sigma)(k) = \psi(k) = l,$$

while

$$(\tau \cdot \sigma)(k) = (\tau \cdot \sigma)(l) = b_{2n}(r, \sigma(s)) \neq b_{2n}(r, s) = l.$$ 

This contradicts our assumption that $\psi$ is in the centralizer of $S_2 \cdot S_n$. Therefore if $k \neq l$ are such that $\psi(k) = l$ then $k, l$ necessarily belong to the same row. Since $b_{2n}$ corresponds to enumerating the elements of $[2] \times [n]$ by rows, this means that $l = k + 1$ if $k < l$, or $l = k - 1$ if $k > l$, and the claim above follows. The proof of (2) is similar with the word “row” replaced by “column”. \hfill \Box

2.3.4. COROLLARY. For each $n \geq 3$ we have $C_{S_n}(S_2 \cdot S_n) \cap C_{S_n}(S_n \cdot S_2) = \{id_{[2n]}\}$.

PROOF. Let us assume that there exists $\psi \in C_{S_n}(S_2 \cdot S_n) \cap C_{S_n}(S_n \cdot S_2) = \{id_{[2n]}\}$ such that $\psi \neq id_{[2n]}$. Since $\psi \in C_{S_n}(S_n \cdot S_2) = \{id_{[2n]}\}$, there exists some $i \in \{1, \ldots, n\}$ such that $\psi(i) = n + i > i + 1$, and $n \geq 3$. This contradicts the assumption that $\psi$ is a non-trivial product of some of the disjoint 2-cycles $(1, 2)_{2n}, \ldots, (2n - 1, 2n)_{2n}$, because any such product will map $i$ either to $i + 1$ or to $i - 1$. \hfill \Box

3. The 2-category of (symmetric) rig categories

In this section we review the notion of (symmetric) rig category, motivating the required coherence axioms, and give various examples. Special attention is paid to the canonical symmetric rig category structure on each distributive category. Next appropriate notions of 1- and 2-cell are defined making (symmetric) rig categories the objects of a 2-category. In fact, there are various reasonable notions of 1-cell, and correspondingly various 2-categories of (symmetric) rig categories. We will stick to the strong notion of 1-cell (cf. Definition 3.5.1 below), although natural examples are given which are not strong. The section ends with the statement of the corresponding strictification theorem.

3.1. Rig categories. As recalled in the introduction, a rig (or semiring) is an abelian monoid $(S, 0, +)$ equipped with an additional monoid structure $(\cdot, 1)$ such that $\cdot$ distributes over $+$ from either side and $0 \cdot x = x \cdot 0 = 0$ for each $x \in S$. Equivalently, $(\cdot, 1)$ must be such that all left and right translation maps $L^y: y \mapsto x \cdot y$ and $R^x: y \mapsto x \cdot y$ are monoid endomorphisms of $(S, +, 0)$. It follows that

1. $L^x \circ R^y = R^y \circ L^x$ and $L^x + R^y = R^y + L^x$ for each $x, y \in S$;
2. $L^x \circ L^y = L^{xy}$ and $R^x \circ R^y = R^x \circ R^y$ for each $x, y \in S$;
3. $L^x = L^1$ and $R^x = R^0$ for each $x, y \in S$;
4. $L^1 = R^0 = 1_S$;
5. $L^0 = R^0 = 0_S$,

where $1_S$ and $0_S$ denote the identity and zero maps of $S$, respectively, and the sum of endomorphisms is pointwise defined. Notice that the first condition in (1), and each of the two conditions in (2) correspond to the associativity of $\cdot$. All of them are made explicit because they lead to different conditions in the categorified definition.

When the definition of a rig is categorified, the abelian monoid $(S, +, 0)$ must be replaced by a symmetric monoidal category $S = (S, +, 0, a, c, l, r)$, with $a$, $c$, $l$, $r$ the associativity, commutativity, and left and right
unit natural isomorphisms, the monoid structure \((\cdot, 1)\) by an additional monoidal structure \((\cdot, 1, a', l', r')\) on \(\mathcal{S}\), with \(a', l', r'\) as before, and the distributivity and ‘absorbing’ axioms by natural isomorphisms (for short, the symbol \(\cdot\) between objects is omitted from now on) 

\[
d_{x,y,z} : x(y + z) \rightarrow xy + xz, \quad d'_{x,y,z} : (x + y)z \rightarrow xz + yz
\]

making each of the left and right translation functors \(L^x, R^x : \mathcal{S} \rightarrow \mathcal{S}\) into symmetric + monoidal endofunctors \(\mathcal{L}^x = (L^x, d_{x,-,-}, n_x)\) and \(\mathcal{R}^x = (R^x, d'_{x,-,-}, n'_x)\) of \(\mathcal{S}\), in the same way as the maps \(L^x, R^x\) were monoid endomorphisms of \((\mathcal{S}, +, 0)\). In doing this, equalities (1)-(5) no longer hold strictly. We only have canonical natural isomorphisms between the corresponding pairs of functors, given by the natural isomorphisms

\[
a'_{x,-} : L^x \circ R^y \Rightarrow R^y \circ L^x, \\
c_{x,-,-} : L^x + R^y \Rightarrow R^y + L^x, \\
a'_{x,-} : L^x \circ L^y \Rightarrow L^{xy}, \\
a'_{y,x} : R^{xy} \Rightarrow R^y \circ R^x, \\
\delta_{x,-} : L^y \Rightarrow L^x, \\
\delta_{y,x} : L^x \Rightarrow L^y, \\
l' : L^0 \Rightarrow 1_S, \quad r' : R^1 \Rightarrow 1_S, \\
n : L^0 \Rightarrow 0_S, \quad n' : R^0 \Rightarrow 0_S,
\]

where \(1_S, 0_S\) respectively denote the identity and zero functors of \(\mathcal{S}\). Moreover, the domain and codomain functors of all these isomorphisms are both symmetric + monoidal. As a matter of fact, however, all these natural isomorphisms except \(c_{x,-,-}\) need not be monoidal, and this must be explicitly required. Thus we are led to the following definition of a rig category.

3.1.1. Definition. A rig category is a symmetric monoidal category \(\mathcal{S} = (\mathcal{S}, +, 0, a, c, l, r)\) together with the following data:

(S1) a multiplicative monoidal structure;

(S2) two families of isomorphisms (left and right distributors)

\[
d_{x,y,z} : x(y + z) \Rightarrow xy + xz, \\
d'_{x,y,z} : (x + y)z \Rightarrow xz + yz
\]

natural in \(x, y, z \in \mathcal{S}\);

(S3) two families of isomorphisms (absorbing isomorphisms)

\[
n_x : x0 \Rightarrow 0, \\
n'_x : 0x \Rightarrow 0
\]

natural in \(x \in \mathcal{S}\).

Moreover, these data must satisfy the following axioms:

(S4) for each \(x \in \mathcal{S}\) the triples \(\mathcal{L}^x = (L^x, d_{x,-,-}, n_x)\), \(\mathcal{R}^x = (R^x, d'_{x,-,-}, n'_x)\) are symmetric + monoidal endofunctors of \(\mathcal{S}\); more explicitly, this means that the diagrams

\[
\begin{align*}
(A1.1) & \quad x(y + (z + t)) \overset{id_a+dz}{\longrightarrow} x((y + z) + t) \quad (x + (y + z)t) \overset{id_a+dz}{\longrightarrow} \quad ((x + y) + z)t \\
& \quad d'_{x,y,z} \quad d'_{x,y,z} \\
& \quad x(y + z + t) \overset{dz}{\longrightarrow} x(y + z) + xt \quad x + (y + z)t \overset{dz}{\longrightarrow} \quad (x + y)t + zt \\
& \quad id_{x,y}+dz \quad id_{x,y}+dz \\
& \quad xy + (x + z) xt + (y + z)t \overset{a_{x,y,z}+id_z}{\longrightarrow} \quad (xy + xz) + xt \quad (xt + (y + z)t) \overset{a_{x,y,z}+id_z}{\longrightarrow} \quad (xt + y + z)t + zt
\end{align*}
\]
commute for all objects $x, y, z, t \in S$;

(SC5) for each $x, y \in S$ the natural isomorphism $\alpha'_{x,y} : L^x \circ R^y \Rightarrow R^y \circ L^x$ is a symmetric monoidal natural isomorphism $L^x \circ R^y \Rightarrow R^y \circ L^x$; more precisely, this means that the diagrams

\[\begin{align*}
\text{(A2.1)} & \quad x((z + t)y) \xrightarrow{id_x \cdot d_{iota,y}} x(zy + ty) \xrightarrow{d_{iota,x}} x(z) + x(ty) \\
& \quad (x(z + t))y \xrightarrow{d_{iota,x} \cdot id_y} (xz + xt)y \xrightarrow{\alpha'_{x,y}} (xz)y + (xt)y
\end{align*}\]

commute for all objects $x, y, z, t \in S$;

(SC6) for each $x, y \in S$ the natural isomorphisms $\alpha'_{x,y} : L^x \circ L^y \Rightarrow L^y \circ L^x$ and $\alpha'_{x,y} : R^x \circ R^y \Rightarrow R^y \circ R^x$ are symmetric monoidal natural isomorphisms $L^x \circ L^y \Rightarrow L^y \circ L^x$ and $R^x \circ R^y \Rightarrow R^y \circ R^x$, respectively; more precisely, this means that the diagrams

\[\begin{align*}
\text{(A3.1)} & \quad x(y(z + t)) \xrightarrow{id_x \cdot d_{iota,y}} x(yz + yt) \xrightarrow{d_{iota,x}} x(yz) + x(yt) \\
& \quad (xy)(z + t) \xrightarrow{d_{iota,x}} (xy)z + (xy)t
\end{align*}\]

\[\begin{align*}
\text{(A3.2)} & \quad (t + z)x \xrightarrow{id_x \cdot d_{iota,y}} (ty + zy)x \xrightarrow{d_{iota,x}} (ty)x + (zy)x \\
& \quad (t + z)(yx) \xrightarrow{d_{iota,x}} t(yx) + z(yx)
\end{align*}\]

commute for all $x, y, z, t \in S$;
For short, we shall write $S$ a rig category we shall write $2$-rig category. The objects 0 and 1 will be respectively called the symmetric monoidal structure on the category $S$. Guillou [4] under the name of one-object $\text{SMC}$-definition 3.1.2. Similarly, rig categories should correspond to one-object categories enriched over a suitable symmetric monoidal closed category of abelian monoids with the usual tensor product of abelian monoids.

Remark 3.1.3. $R$ is either left or right semistrict, and $c$ is left and right null isomorphisms $L_x \Rightarrow L$ and $R_y \Rightarrow R$ and $R_z \Rightarrow R$, respectively; more precisely, this means that the diagrams

$$D_{x,z,t} \Rightarrow (x + y)(z + t) \xrightarrow{d_{x,z,t}} x(z + t) + y(z + t) \xrightarrow{d_{x,y} + d_{z,t}} (xz + xt) + (yz + yt)$$

$$D_{x,y} \Rightarrow (x + y)z + (x + y)t \xrightarrow{d_{x,y} + d_{y,t}} (xz + yt) + (xt + yt)$$

commute for all objects $x, y, z, t \in S$.

Remark 3.1.4. $R$ is symmetric monoidal natural isomorphisms $L : L^1 \Rightarrow 1_S$ and $R : R^1 \Rightarrow 1_S$ are symmetric monoidal natural isomorphisms $L^1 \Rightarrow 1_S$ and $R^1 \Rightarrow 1_S$, respectively; more explicitly, this means that the diagrams

$$1(1 + y) \xrightarrow{d_{1,y}} 1x + 1y = x + y \xrightarrow{t_{1,y}} x + y$$

$$x + y \xrightarrow{t_{x,y} + t_{y,z}} x + y$$

$$x + y \xrightarrow{t_{x,y} + t_{y,z}} x + y$$

commute for all objects $x, y \in S$, and the following equalities hold:

$$n_1 = l_0, \quad n'_1 = l'_0;$$

Remark 3.1.5. The left and right null isomorphisms $n' : L^0 \Rightarrow 0_S$ and $n : R^0 \Rightarrow 0_S$ are symmetric monoidal natural isomorphisms $L^0 \Rightarrow 0_S$ and $R^0 \Rightarrow 0_S$, respectively; more explicitly, this means that

$$n_0 = n'_0.$$

For short, we shall write $\mathbb{S}$ to denote the whole data $(S, +, \cdot, 0, 1, a, c, l, r, a', l', r', d, d', n, n')$ defining a rig category. The objects 0 and 1 will be respectively called the zero and unit objects of $\mathbb{S}$. $\mathbb{S}$ will be called a $2$-rig when the underlying category $S$ is a groupoid.

When convenient, and in order to distinguish the additive and the multiplicative monoidal structures on a rig category we shall write $\mathbb{S}^+ = (S, +, 0, a, c, l, r)$, and $\mathbb{S}^- = (S, \cdot, 1, a', l', r')$.

3.1.2. Definition. A rig category $\mathbb{S}$ is called left (resp. right) semistrict when all structural isomorphisms except $c$ and the right distributor $d'$ (resp. the left distributor $d$) are identities. It is called semistrict when it is either left or right semistrict, and strict when $c$ and both distributors are also identities.

3.1.3. Remark. Rigs can be more compactly defined as the one-object categories enriched over the symmetric monoidal closed category of abelian monoids with the usual tensor product of abelian monoids. Similarly, rig categories should correspond to one-object categories enriched over a suitable symmetric monoidal closed category of symmetric monoidal categories. Such enriched categories are considered by Guillou [4] under the name of one-object $\text{SMC}$-categories, although he avoids describing explicitly the symmetric monoidal structure on the category $\text{SMC}$ of symmetric monoidal categories.
3.1.4. Remark. According to the categorification philosophy, all natural isomorphisms replacing the basic equalities (1)-(5) above must be coherent, i.e., they must be such that there is a unique derived natural isomorphism for each derived equality. Derived isomorphisms here mean isomorphisms built as compositions of instantiations of the natural isomorphisms appearing in axioms (SC5)-(SC9), i.e., combinations by $+$ and $\cdot$ of one such isomorphism with identity morphisms. It is not clear from the previous definition that all derived natural isomorphisms between a given finite composition or sum of left and/or right translation functors, and the corresponding reduced word (or any equivalent word) are equal. In fact, none of the axioms (SC5)-(SC9) really corresponds to such a coherence condition. The fact that this is true is a consequence of Laplaza’s coherence theorem [9]. This justifies that no more axioms need to be added to the previous list.

3.2. Symmetric rig categories. A rig $(S, +, \cdot, 0, 1)$ is commutative when the monoid $(S, \cdot, 1)$ is abelian, or equivalently when $L^x = R^x$ for each $x \in S$. When this condition is categorified, the abelian monoid structure $(\cdot, 1)$ becomes a symmetric monoidal structure $(\cdot, 1, a', c', l', r')$ on $S$, and instead of the equality $L^x = R^x$ we now have the natural isomorphism $c'_{x, y} : L^x \Rightarrow R^y$. Once more, this isomorphism may not be a symmetric monoidal natural isomorphism $L^x \Rightarrow R^x$, and this must be required explicitly. This leads us to the following definition.

3.2.1. Definition. A symmetric rig category is a rig category $S$ together with a family of natural isomorphisms $c'_{x, y} : xy \Rightarrow yx$, for each $x, y \in S$, called the multiplicative commutators, such that

(CSC1) $(S, +, \cdot, 1, a', c', l', r')$ is a symmetric monoidal category,

(CSC2) for each $x \in S$, the natural isomorphism $c'_{x, -} : L^x \Rightarrow R^x$ is a symmetric monoidal natural isomorphism $L^x \Rightarrow R^x$; more precisely, this means that the diagrams

\[
\begin{array}{ccc}
(x + y)z & \xrightarrow{c'_{x,y}z} & xz + yz \\
\downarrow{c'_{x+y,z}} & & \downarrow{c'_{x,z} + c'_{y,z}} \\
xz + yz & \xrightarrow{d_{x,z} + d_{y,z}} & z(x + y)
\end{array}
\]

\[
\begin{array}{ccc}
x0 & \xrightarrow{c'_{x,0}} & 0x \\
\downarrow{n_x} & & \downarrow{n_x} \\
0 & \xrightarrow{0} & 0
\end{array}
\]

commute for all objects $x, y, z \in S$.

A symmetric rig category is called semistrict (resp. strict) when the underlying rig category $S$ is semistrict (resp. strict and with $c'$ trivial). \[2\] It is called a symmetric 2-rig when the underlying rig category is a 2-rig.

3.2.2. Remark. The previous definition coincides with the structure described by Laplaza in [9] except that in Laplaza’s paper the distributors are only required to be monomorphisms. The correspondence between Laplaza’s axioms and the axioms in Definitions 3.1.1 and 3.2.1 goes as follows:

(i) (A1.1)-(A1.4) correspond to Laplaza’s axioms (I), (III)-(V) and (XIX)-(XXII),
(ii) (A2.1)-(A2.2) correspond to Laplaza’s axioms (VIII) and (XVII),
(iii) (A3.1)-(A3.2) correspond to Laplaza’s axioms (VI)-(VII), (XVI) and (XVIII),
(iv) (A4.1)-(A4.2) correspond to Laplaza’s axioms (IX) and (XI)-(XII),
(v) (A5.1)-(A5.2) correspond to Laplaza’s axioms (XIII)-(XIV) and (XXIII)-(XXIV),
(vi) (SC9) corresponds to Laplaza’s axiom (X), and
(vii) (CSC2) correspond to Laplaza’s axioms (II) and (XV).

3.3. Some examples of (symmetric) rig categories. Describing a rig category requires specifying the data $S, +, \cdot, 0, 1, a, c, l, r, a', c', l', r', d, d', n, n'$, and checking that they satisfy all of the above axioms. Usually, this is long and tedious. Hence in this subsection we just mention a few standard examples of rig categories without getting into the details. The particular type of rig categories we are interested in is discussed in more detail in §3.4.

\[2\] Some people, mostly those working on the K-theory of this type of categories, call the semistrict symmetric rig categories bipermutative categories because a semistrict symmetric monoidal category is also called a permutative category.
3.3.1. Example. Every rig $S = (S, +, \cdot, 0, 1)$ can be thought of as a 2-rig with only identity morphisms, and all required isomorphisms trivial. They are symmetric 2-rigs when $S$ is commutative.

3.3.2. Example. If $S$ is a rig category, and $\hat{S}$ is the groupoid with the same objects as $S$ and only the isomorphisms between them as morphisms then $\hat{S}$ inherits by restriction a canonical rig category structure. The 2-rig so obtained is denoted by $\hat{S}$. It is a symmetric 2-rig when $S$ is a symmetric rig category, and (left,right) semistrict when $\hat{S}$ is so.

3.3.3. Example. Every symmetric monoidal closed category (in particular, every cartesian closed category) with finite coproducts is canonically a symmetric rig category with $+$ and $\cdot$ respectively given by the categorical coproduct $\sqcup$ and the tensor product $\otimes$. Closedness is necessary to ensure that the tensor product indeed distributes over coproducts. Thus denoting the internal homs by $[\cdot, \cdot]$ we have

$$
\text{Hom}(x \sqcup y \otimes z, t) \cong \text{Hom}(x, [z, t]) \otimes \text{Hom}(y, [z, t]) \\
\cong \text{Hom}(x \otimes z, t) \otimes \text{Hom}(y \otimes z, t)
$$

for every objects $x, y, z, t$. Then the existence of the isomorphism $d_{x,y,z}$ follows from the Yoneda lemma. A similar argument gives the isomorphism $d_{x',y',z'}$. Symmetric rig categories of this type include those associated to the three cartesian closed categories $\text{Set}$ of sets and maps (and its full subcategory $\mathcal{F}\text{Set}$ with objects the finite sets), $\text{Set}_G$ of $G$-sets and homomorphisms of $G$-sets for any group $G$ (and its full subcategory $\mathcal{F}\text{Set}_G$ with objects the finite $G$-sets), and $\text{Cat}$ of (small) categories and functors, and those associated to the two non-cartesian symmetric monoidal closed categories $\text{Vect}_k$ of vector spaces over a field $k$ and $k$-linear maps (and its full subcategory $\mathcal{F}\text{Vect}_k$ with objects the finite dimensional vector spaces), and $\text{Rep}_k(G)$ of $k$-linear representations of a group $G$ and homomorphisms of representations for any group $G$ (and its full subcategory $\mathcal{F}\text{Rep}_k(G)$ with objects the finite dimensional representations).

3.3.4. Example. The set of endomorphisms of every abelian monoid is canonically a rig, with the pointwise sum of monoid endomorphisms, and the product given by the composition. Similarly, for every symmetric monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, 0, \mathbf{a}, \mathbf{c}, \mathbf{l}, \mathbf{r})$ the category $\text{End}(\mathcal{M})$ of symmetric monoidal endofunctors of $\mathcal{M}$, and symmetric monoidal natural transformations between them is canonically a (non-symmetric) rig category, with the additive symmetric monoidal structure given by the pointwise sum of symmetric monoidal endofunctors, and the multiplicative monoidal structure given by the composition. It is semistrict when $\mathcal{M}$ is semistrict, and a 2-rig when $\mathcal{M}$ is a groupoid.

3.4. Distributive categories. Recall that a distributive category is a cartesian and cocartesian category such that the canonical map $x y + x z \to x(y + z)$ is invertible for each objects $x, y, z$. Distributive categories generalize the cartesian closed categories with finite coproducts of Example 3.3.3 and as these have a “canonical” symmetric rig category structure. In this paragraph this structure is described in detail. In fact, every cocartesian category $C$ (i.e. a category $C$ with all finite coproducts) has a “canonical” symmetric monoidal structure associated to the choice of a particular coproduct $(x + y, t_{x,y}, c_{x,y})$ for each ordered pair of objects $(x, y)$, and a particular initial object $0$. It is given as follows:

(D1) the tensor product $C \times C \to C$ is given on objects $(x, y)$ and morphisms $(f, g) : (x, y) \to (x', y')$ by

$$(x, y) \mapsto x + y, \quad (f, g) \mapsto f + g,$$

where $f + g$ is the morphism uniquely determined by the diagram

$$
\begin{array}{ccc}
x & \xrightarrow{t_{x,y}} & x + y \\
\downarrow f & & \downarrow \mathbf{t}\!f+g \\
x' & \xrightarrow{t_{x',y'}} & x' + y'
\end{array}
$$

and the universal property of $x + y$;
the unit object is the chosen initial object 0;

(D3) for every objects $x, y, z \in \mathcal{C}$ the associator $a_{x,y,z}$ is the morphism uniquely determined by the left hand side diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{i_{x,y}} & x + (y + z) \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & (x + y) + z \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & (x + y) + z
\end{array}
\]

and the universal properties of $x + (y + z)$ (it is indeed invertible with inverse the morphism uniquely determined by the right hand side diagram and the universal property of $(x + y) + z$);

(D4) for every objects $x, y \in \mathcal{C}$ the commutator $c_{x,y}$ is the morphism uniquely determined by the diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{i_{x,y}} & x + y \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  y & \xrightarrow{i_{y,x}} & y + x \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  y & \xrightarrow{i_{y,x}} & y + x
\end{array}
\]

and the universal property of $x + y$ (it is indeed invertible with inverse $c_{x,y}^{-1} = c_{y,x}$);

(D5) for every object $x \in \mathcal{C}$ the left and right unitors $l_x, r_x$ are the morphisms uniquely determined by the diagrams

\[
\begin{array}{ccc}
  0 & \xrightarrow{!} & 0 + x \\
  & \searrow l_x & \southwestarrow \downarrow \\
  x & \xrightarrow{id_x} & x \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x & \xrightarrow{id_x} & x
\end{array}
\quad
\begin{array}{ccc}
  x & \xrightarrow{i_x,0} & x + 0 \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  0 & \xrightarrow{!} & 0 \\
  & \searrow l_x & \southwestarrow \downarrow \\
  x & \xrightarrow{id_x} & x \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x & \xrightarrow{id_x} & x
\end{array}
\]

and the universal properties of $0 + x$ and $x + 0$ (they are indeed invertible with inverses $l_x^{-1}, r_x^{-1}$ the morphisms $c_{0,x}^{-1}, c_{x,0}^{-1}$ respectively).

It is straightforward checking that different choices of binary coproducts and initial object lead to different but equivalent symmetric monoidal structures on $\mathcal{C}$ and hence, we may indeed speak of the “canonical” symmetric monoidal structure on each cocartesian category $\mathcal{C}$. Similarly, every cartesian category $\mathcal{C}$ (i.e. a category $\mathcal{C}$ with all finite products) is “canonically” a symmetric monoidal category with the associator $a'$, commutator $c'$, and left and right unitors $l', r'$ defined by the dual diagrams for some particular choices of binary products and final object.

When $\mathcal{C}$ is both cartesian and cocartesian, these two symmetric monoidal structures $(+, 0, a, c, l, r)$ and $(\cdot, 1, a', c', l', r')$ are related by the natural left and right distributor maps

\[
\begin{align*}
\widehat{d}_{x,y,z} & : xy + xz \to x(y + z) \\
\widecheck{d}_{x,y,z} & : xz + yz \to (x + y)z
\end{align*}
\]

uniquely determined by the diagrams

\[
\begin{array}{ccc}
  xy & \xrightarrow{i_{x,y,z}} & x(y + z) \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & x + z \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & x + z
\end{array}
\quad
\begin{array}{ccc}
  xz & \xrightarrow{i_{x,z,y}} & (x + y)z \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & x + z \\
  & \searrow \downarrow & \southwestarrow \downarrow \\
  x + y & \xrightarrow{i_{x+y,z}} & x + z
\end{array}
\]

and the universal properties of the coproducts $xy + xz$ and $xz + yz$. In fact, these distributors are not independent of each other. Instead, they are related as follows.
3.4.1. Lemma. Let C be a cartesian and cocartesian category. Then for every objects \(x, y, z \in C\) the left and right distributors make the diagram
\[
\begin{array}{ccc}
xz + yz & \xrightarrow{d'_{x,y,z}} & (x + y)z \\
\downarrow{c'_{x,y,z}} & & \downarrow{c'_{x,y,z}} \\
zx + zy & \xrightarrow{d_{x,y,z}} & z(x + y)
\end{array}
\]
commute. In particular, \(d'_{x,y,z}\) is invertible if and only if \(d_{x,y,z}\) is invertible.

Proof. Left to the reader. \(\square\)

The distributors \(d_{x,y,z}, d'_{x,y,z}\) are in general non-invertible, and C is called distributive precisely when \(d_{x,y,z}\) (or equivalently, \(d'_{x,y,z}\)) is invertible for every objects \(x, y, z\). In fact, for a cartesian and cocartesian category to be distributive it is enough that there exists any natural isomorphism \(xy + xz \cong x(y + z)\) (see [8]). As it is now shown, the point is that when \(C\) is distributive there are also canonical isomorphisms \(x0 \cong 0 \cong 0x\) for each \(x \in C\) such that the whole structure makes \(C\) into a symmetric rig category.

3.4.2. Lemma. Let C be a distributive category equipped with the above additive and multiplicative symmetric monoidal structures. Then for every \(x \in C\) the maps \(\pi^1_{0,x} : 0x \to 0\) and \(\pi^2_{x,0} : x0 \to 0\) are invertible with respective inverses the unique maps \(0 \to 0x\) and \(0 \to 0x\).

Proof. Clearly, precomposition of \(\pi^1_{0,x}\) with the unique map \(0 \to 0x\) is the identity of \(0\). To prove that the other composition is the identity of \(0x\) it is enough to see that for every object \(y\) every two morphisms \(f, g : 0x \to y\). To see this notice that given \(f, g\) there is a unique morphism \(h : 0x + 0x \to y\) making the diagram
\[
\begin{array}{ccc}
0x & \xrightarrow{\iota^1_{0,0x}} & 0x + 0x \\
\downarrow{f} & & \downarrow{g} \\
0x & \xleftarrow{\iota^2_{0,0x}} & 0x \\
\end{array}
\]
commute. Now, \(\iota^1_{0,0x} = \iota^2_{0,0x}\) because of the diagram which defines \(d'_{0,0x}\) and because \(\iota^1_{0,0x} = \iota^2_{0,0x}\) (0 is initial). Hence \(f = g\). It follows that the identity of \(0x\) is equal to the composite of \(\pi^1_{0,0x}\) with the unique morphism \(0 \to 0x\). The case of \(\pi^2_{x,0}\) is argued similarly.

3.4.3. Proposition. Every distributive category \(C\) equipped with the above symmetric monoidal structures (+, 0, a, c, l, r) and (\(\cdot\), 1, \(a'\), \(c'\), \(l'\), \(r'\)), and with the distributors and absorbing isomorphisms given by
\[
d_{x,y,z} = (d'_{x,y,z})^{-1}, \quad d'_{x,y,z} = (d'_{x,y,z})^{-1}, \quad n_x = \pi^2_{x,0}, \quad n'_x = \pi^1_{0,x}
\]
is a symmetric rig category.

Proof. It is long but straightforward checking that all the coherence axioms (SC4)-(SC9) hold.

A standard example of a distributive category is the category \(\text{Set}\) of finite sets and maps between them. In Section 3 the symmetric rig category structure of a skeleton of it corresponding to a particular choice of products and coproducts is explicitly described.

3.5. The 2-category of (symmetric) rig categories. (Symmetric) rig categories are the objects of a 2-category. In fact, there are various useful notions of 1-cell between (symmetric) rig categories, associated to the various notions of 1-cell between (symmetric) monoidal categories, either lax, colax, bilax, strong or strict (symmetric) monoidal functor (see [11]). Moreover, we may also consider 1-cells whose character is different for the additive and the multiplicative monoidal structures. For instance, a 1-cell may be additively strong and multiplicatively colax, and examples of these mixed kind actually arise in some natural situations. Thus there are actually various 2-categories of (symmetric) rig categories. Although we shall define
the various types of 1-cell, and give examples of various types, at the end we shall restrict to the strong morphisms and the associated 2-categories.

Recall that, given two rigs $S = (\mathcal{S}, +, \cdot, 0, 1)$ and $\tilde{S} = (\tilde{\mathcal{S}}, +, \cdot, \tilde{0}, \tilde{1})$, a rig homomorphism from $S$ to $\tilde{S}$ is a map $f : S \rightarrow \tilde{S}$ such that $f$ is both a monoid homomorphism from $(\mathcal{S}, +, 0)$ to $(\tilde{\mathcal{S}}, +, \tilde{0})$, and a monoid homomorphism from $(\mathcal{S}, \cdot, 1)$ to $(\tilde{\mathcal{S}}, \cdot, \tilde{1})$. It follows that $f$ is such that

\begin{equation}
\begin{aligned}
f \circ L^e &= \tilde{L}^e \circ f \\
f \circ R^e &= \tilde{R}^e \circ f
\end{aligned}
\end{equation}

for each $x \in \mathcal{S}$, where $L^e, R^e$ and $\tilde{L}^e, \tilde{R}^e$ respectively denote the left and right translation maps of $S$ and $\tilde{S}$. In categorifying this definition, the map $f$ must be replaced by a functor $F : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$ together with a pair $(\varphi^+, \epsilon^+)$ making it a symmetric monoidal functor $S^+ \rightarrow \tilde{S}^+$ of some type, and a pair $(\varphi^-, \epsilon^-)$ making it a monoidal functor $\tilde{S}^+ \rightarrow \tilde{\mathcal{S}}^+$ of perhaps a different type. When done, equalities (3)-(4) no longer hold. Instead, we just have natural transformations between the involved functors. For instance, in case the multiplicative monoidal structure is lax, we have the natural morphisms

\begin{align*}
\varphi^+: \tilde{L}^e x \circ F & \Rightarrow F \circ L^e \\
\varphi^-: \tilde{R}^e x \circ F & \Rightarrow F \circ R^e
\end{align*}

Moreover, the domain and codomain functors of both transformations are always symmetric monoidal of some kind. For instance, when $F$ is additively colax, they are colax monoidal with the natural morphisms given by

\begin{align*}
F(x(y + z)) & \xrightarrow{F(d_{x,z})} F(xy + xz) \xrightarrow{\varphi^{+}_{x,z}} F(xy) + F(xz) \\
F(x + y) & \xrightarrow{id_x + \varphi^+_x} F(Fy + Fz) \xrightarrow{\tilde{d}_{Fy,Fz}} Fy Fz \xrightarrow{\varphi^{+}_{y,z}} Fy + Fz \xrightarrow{F(0x)} F0 \xrightarrow{\epsilon^+} 0 \\
F((y + z)x) & \xrightarrow{F(\tilde{d}_{y,z})} F(yx + zx) \xrightarrow{\varphi^{+}_{x,z}} F(yx) + F(zx) \\
F(0x) & \xrightarrow{F(id_x)} F0 \xrightarrow{\epsilon^+} 0 \\
F(y + z) & \xrightarrow{\varphi^{+}_{x,z}} (Fy + Fz) Fx \xrightarrow{\tilde{d}_{Fy,Fz}} Fy Fx + FzFx \xrightarrow{\varphi^{+}_{y,z}} F0 Fx \xrightarrow{\epsilon^+} 0
\end{align*}

However, these natural transformations $\varphi^+_{x,z}$ and $\varphi^-_{x,z}$ need not be monoidal. Thus we are naturally led to the following notions of 1-cell between rig categories.

3.5.1. **Definition.** Let be given two rig categories $\mathcal{S}$ and $\tilde{\mathcal{S}}$. A (colax,lax) morphism of rig categories from $\mathcal{S}$ to $\tilde{\mathcal{S}}$ is a functor $F : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$ together with the following data:

(HSC1) an additive symmetric colax monoidal structure $(\varphi^+, \epsilon^+)$ on $F$, and

(HSC2) a multiplicative lax monoidal structure $(\varphi^-, \epsilon^-)$ on $F$.

Moreover, these data must satisfy the following axioms:

(HSC3) for each $x \in \mathcal{S}$ the natural morphism $\varphi^{+}_{x,z} : \tilde{L}^e x \circ F \Rightarrow F \circ L^e$ is a symmetric monoidal natural isomorphism $\tilde{L}^e x \circ F \Rightarrow F \circ L^e$, i.e. the diagrams

\begin{align*}
\begin{array}{c}
\xymatrix{F(x(y + z)) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{F(d_{x,z})} & F(xy + xz) \\
\ar[u]_{\varphi^+_{x,z} + \varphi^+_{y,z}} \ar[r]_{\varphi^+_{x,z}} & F(xy) + F(xz) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{F(0x)} & F0 \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\epsilon^+} & 0}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\xymatrix{F(x + y) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{id_x + \varphi^+_{x,z}} & F(Fy + Fz) \\
\ar[u]_{\varphi^+_{y,z}} \ar[r]_{\tilde{d}_{Fy,Fz}} & Fy Fz \\
\ar[u]_{\varphi^+_{y,z}} \ar[r]_{\epsilon^+} & 0}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\xymatrix{F((y + z)x) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{F(\tilde{d}_{y,z})} & F(yx + zx) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\varphi^+_{x,z}} & F(yx) + F(zx) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{F(id_x)} & F0 \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\epsilon^+} & 0}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\xymatrix{F(y + z) \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\varphi^+_{x,z}} & (Fy + Fz) Fx \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\tilde{d}_{Fy,Fz}} & Fy Fx + FzFx \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\varphi^+_{x,z}} & F0 Fx \\
\ar[u]_{\varphi^+_{x,z}} \ar[r]_{\epsilon^+} & 0}
\end{array}
\end{align*}

commute for each objects $y, z \in \mathcal{S}$;
(HSC4) for each $x \in S$ the natural isomorphism $\varphi_{x, x}^*: R^F \circ F \Rightarrow F \circ R^*$ is a symmetric monoidal natural isomorphism $\tilde{R}^F \circ S^\alpha \Rightarrow S^\alpha \circ \tilde{R}^*$, i.e. the diagrams

$$
\begin{align*}
F((y + z)x) & \xrightarrow{F(d_{y,z}^* \circ \varphi_{y,z})} F(yx + zx) \quad \varphi_{y,z,x}^* \quad F((yx + zx) + F(zx)) \quad \varphi_{y,z} \quad F(0x) \xrightarrow{F(0^* \circ \varepsilon)} F0 \xrightarrow{\varepsilon^*} 0 \\
F(y + z) \quad Fx \xrightarrow{\varphi_{x}^* \circ \phi_{x}} (Fy + Fz) \quad Fx \xrightarrow{\phi_{x}^* \circ \phi_{x}} FyFz + FxFz \quad F0FX \xrightarrow{\varepsilon^* \circ \phi_{x}} 0\end{align*}
$$

commute for each objects $y, z \in S$.

For any other choices of $\alpha, \beta \in \{\text{lax, colax, strong, strict}\}$, $(\alpha, \beta)$ morphisms of rig categories are defined similarly. When $\alpha = \beta$, we shall speak of an $\alpha$ morphism. Finally, when $S$, $\tilde{S}$ are symmetric rig categories, an $(\alpha, \beta)$ morphism of symmetric rig categories from $S$ to $\tilde{S}$ is an $(\alpha, \beta)$-morphism such that $(\varphi^*, \varepsilon^*)$ is a symmetric $\beta$-monoidal structure on $F$, and an $\alpha$ morphism of symmetric rig categories is an $\alpha$-morphism such that $(\varphi^*, \varepsilon^*)$ is a symmetric $\alpha$-monoidal structure on $F$.

For short, we shall denote by $\mathbb{F}$ the whole data $(F, \varphi^*, \varepsilon^*, \varphi^*, \varepsilon)$ defining a morphism of rig categories of any type.

3.5.2. Remark. A strong morphism of rig categories corresponds to the one-object case of the more general notion of a SMC-functor between SMC-categories introduced by Guilou ([3], Definition 4.1).

3.5.3. Example. Every rig homomorphism between two rigs $S$ and $\tilde{S}$ is a strict morphism between the associated discrete 2-rigs, and conversely.

3.5.4. Example. For every rig category $S$ the inclusion functor $J: \tilde{S} \hookrightarrow S$ of the underlying groupoid $\tilde{S}$ into $S$ is a strict morphism of rig categories $\tilde{J}: \tilde{S} \hookrightarrow S$ (cf. Example 3.3.2).

3.5.5. Example. Let $Set$ and $Cat$ be the symmetric rig categories of sets and (small) categories, respectively (cf. Example 3.3.3), and let $- : Set \rightarrow Cat$ be the functor mapping each set $X$ to the corresponding discrete category $\mathcal{X}$, and $| - | : Cat \rightarrow Set$ its left adjoint, mapping each (small) category $C$ to the corresponding set of objects $|C|$. Then both functors are strict morphisms of symmetric rig categories.

3.5.6. Example. Let $Set_G$ be the symmetric rig category of $G$-sets for some group $G$ (cf. Example 3.3.3). Then the forgetful functor $U_G : Set_G \rightarrow Set$ is a strict morphism of symmetric rig categories. However, its left adjoint $\tilde{I}_G : Set \rightarrow Set_G$, mapping each set $X$ to $X \times G$ with $G$-action given by $g'(x, g) = (x, g'g)$, and each map $f : X \rightarrow Y$ to the morphism of $G$-sets $f \times \text{id}_G : X \times G \rightarrow Y \times G$, is canonically just a strong-colax morphism $\tilde{I}_G : Set \rightarrow Set_G$. The additive strong monoidal structure is given by the canonical right distributors

$$
\varphi_{X,Y}^+ = d_{X,Y}^*: (X \sqcup Y) \times G \Rightarrow (X \times G) \sqcup (Y \times G),
$$

together with the unique map $\varepsilon^+ : \emptyset \times G \Rightarrow \emptyset$, while the multiplicative colax structure is given by the canonical non-invertible morphisms of $G$-sets

$$
\varphi_{X,Y}^- : (X \times Y) \times G \Rightarrow (X \times G) \times (Y \times G)
$$

defined by $((x, y), g) \mapsto (|(x, y), (g)|, (x, g))$, together with the unique map $\varepsilon^- : \{\ast\} \times G \Rightarrow \{\ast\}$.

3.5.7. Example. Let $Vect_k$ be the symmetric rig category of vector spaces over a given field $k$ (cf. Example 3.3.3). Then the forgetful functor $U_k : Vect_k \rightarrow Set$ is a lax morphism of symmetric rig categories $U_k : Vect_k \rightarrow Set$ with additive lax monoidal structure given by the canonical maps $\varphi_{W,V}^+ : V \sqcup W \rightarrow V \times W$ defined by $v \mapsto (v, 0)$ and $w \mapsto (0, w)$, together with the canonical map $\varepsilon^+ : \emptyset \Rightarrow \{0\}$, while the multiplicative lax structure is given by the canonical maps $\varphi_{W,V}^- : V \times W \Rightarrow V \otimes_k W$ given by $(v, w) \mapsto v \otimes w$, and the map $\varepsilon^- : \{\ast\} \rightarrow k$ sending $\ast$ to the unit $1 \in k$. By constrast, its left adjoint $I_k : Set \rightarrow Vect_k$, mapping each set $X$ to the vector space $k[X]$ spanned by $X$, is canonically a strong morphism of symmetric rig categories $I_k : Set \rightarrow Vect_k$ with $\varphi_{X,Y}^+, \varphi_{X,Y}^+, \varphi_{X,Y}^-, \varepsilon^-$ the usual natural isomorphisms $k[X \sqcup Y] \cong k[X] \otimes k[Y], k[\emptyset] \cong 0, k[X \times Y] \cong k[X] \otimes k[Y]$, and $k[\{\ast\}] \cong k$. 
From now on, we shall restrict to strong morphisms of (symmetric) rig categories, and they will be called homomorphisms.

3.5.8. Definition. Let $\mathcal{S}, \tilde{\mathcal{S}}$ be two (symmetric) rig categories, and $F_1, F_2 : \mathcal{S} \to \tilde{\mathcal{S}}$ two homomorphisms between them. A rig transformation from $F_1$ to $F_2$ is a natural transformation $\xi : F_1 \Rightarrow F_2$ that is both $+$-monoidal and $\cdot$-monoidal.

3.5.9. Remark. There is a more general notion of 2-cell $F_1 \Rightarrow F_2$ corresponding to Guillo’s definition of monoidal transformation between SMC-functors whose domain and codomain SMC-categories have only one object and hence, are rig categories ([4], Definition 4.2). It consists of an object $\tilde{x}$ in $\tilde{\mathcal{S}}$ together with a family of natural morphisms $\eta_x : (F_2 \circ \tilde{x}) \Rightarrow \tilde{x} \circ (F_1 \circ \tilde{x})$ in $\tilde{\mathcal{S}}$, labelled by the objects $x$ in $\mathcal{S}$, satisfying appropriate conditions. This is analogous to the existence of a more general notion of 2-cell between (symmetric) monoidal functors, corresponding to the pseudonatural transformations between them when viewed as pseudofunctors between one-object bicategories. Then the previous notion of rig transformation is to be thought of as the analog in the rig category setting of Lack’s icons [7].

Rig categories together with the rig category homomorphisms as 1-cells, and the rig transformations between these as 2-cells constitute a 2-category $\text{RigCat}$. The various compositions of 1- and 2-cells are defined in the obvious way. Similarly, symmetric rig categories with the symmetric rig category homomorphisms, and the rig transformations as 2-cells also constitute a 2-category $\text{SRigCat}$. Notice that, unlike the category $\text{CAlg}$ of commutative rigs, which is a full subcategory of $\text{Rig}$, $\text{SRigCat}$ is not a full sub-2-category of $\text{RigCat}$ because being a symmetric rig category is not a property-like structure. A given rig category can be symmetric in various non-equivalent ways.

As in any 2-category, two objects $\mathcal{S}, \tilde{\mathcal{S}}$ in $\text{RigCat}$ (or in $\text{SRigCat}$) can be equal, isomorphic or just equivalent. Equivalence is the most general notion of ‘equality’ between the objects in a 2-category.

3.5.10. Definition. Two objects $\mathcal{S}, \tilde{\mathcal{S}}$ in $\text{RigCat}$ (or in $\text{SRigCat}$) are said to be equivalent when there exists (symmetric) rig category homomorphisms $F : \mathcal{S} \to \tilde{\mathcal{S}}$ and $\tilde{F} : \tilde{\mathcal{S}} \to \mathcal{S}$ and invertible rig transformations $\xi : \tilde{F} \circ F = 1_{\mathcal{S}}$ and $\tilde{\xi} : F \circ \tilde{F} = 1_{\tilde{\mathcal{S}}}$. It can be shown that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent if there exists a (symmetric) rig category homomorphism $F : \mathcal{S} \to \tilde{\mathcal{S}}$ whose underlying functor $F$ is an equivalence between the underlying categories. Indeed, (symmetric) rig categories structures transport along equivalences of categories.

3.6. Strictification theorem. A basic feature of categorification is that we have to replace equations by natural isomorphisms satisfying the appropriate coherence axioms. This often leads to sophisticated structures involving many natural isomorphisms and lots of required commutative diagrams, as illustrated by the previous definition of a rig category. Hence it is useful to know that some of the natural isomorphisms can be assumed to be identities because the final structure is equivalent to a similar one but with some of these isomorphisms trivial. Theorems of this type are usually known as strictification theorems. For symmetric rig categories the theorem is due to May ([11], Proposition VI.3.5), and for generic rig categories it is a consequence of the more general strictification theorem for SMC-categories due to Guillo [4]. Their statements are as follows.

3.6.1. Theorem. ([11],[4]) Every rig category (resp. symmetric rig category) $\mathcal{S}$ is equivalent in $\text{RigCat}$ (resp. in $\text{SRigCat}$) to a semistrict rig category (resp. semistrict symmetric rig category).

The choice of which distributor is made trivial in a semistrict version of a given (symmetric) rig category is logically arbitrary. In some important examples it is the right distributor which is naturally trivial (as in Example 3.3.4 above). However, the only relevant point here is that, in the absence of a strict commutativity of $+$, a common and usually unavoidable situation, it is unreasonable to demand that both distributors be identities.
4. THE CATEGORY OF ENDOMORPHISMS OF THE SYMMETRIC 2-RIG OF FINITE SETS

The purpose of this section is to show that the category of endomorphisms of the symmetric 2-rig $\mathcal{FS}et$ is equivalent to the terminal category. Actually, instead of working with $\mathcal{FS}et$ we shall consider an equivalent, skeletal version of it which we shall denote by $\mathcal{FS}et_{sk}$, and which has the advantage of being semistrict. Since equivalent objects in a 2-category have equivalent categories of endomorphisms, it is indeed enough to prove that $\mathcal{FS}et_{sk}$ has a trivial category of endomorphisms. This considerably simplifies the diagrams, and makes computations much easier.

4.1. Semistrict version of the symmetric 2-rig of finite sets. This version appears elsewhere (for instance, as Example VI.5.1 in May’s work [11]). To our knowledge, however, the complete description given here, which includes an explicit description of the distributors, is new.

Let us recall from § 3.4 that, after fixing particular binary products and coproducts, and final and initial objects every distributive category has a canonical symmetric rig category structure. In general, the resulting structural isomorphisms $a, c, l, r, a', c', l', r'$ are non trivial. In some cases, however, and for suitable choices of these binary products, coproducts and final, initial objects the associator and left and right unitor (but usually not the commutators) turn out to be trivial. The goal of this subsection is to see that this is so for the skeleton of $\mathcal{F}Set$ having as objects the sets $[n] = \{1, \ldots, n\}$ for each $n \geq 1$, and $[0] = \emptyset$. We shall denote this skeleton by $\mathcal{F}Set_{sk}$. Being a skeleton of $\mathcal{F}Set$ it has all binary products and coproducts, and $[1]$ and $[0]$ as (unique) final and initial objects, respectively. Moreover, choosing binary products and coproducts just amounts in this case to making appropriate choices of the respective projections and injections.

4.1.1. Lemma. For every objects $[m], [n] \in \mathcal{F}Set_{sk}$ let us take as coproduct of the pair $([m], [n])$ the set $[m + n]$ with the injections $i^1_{[m],[n]} : [m] \to [m + n]$ and $i^2_{[m],[n]} : [n] \to [m + n]$ defined by

\[
i^1_{[m],[n]}(i) = i, \quad i = 1, \ldots, m,
\]
\[
i^2_{[m],[n]}(j) = m + j, \quad j = 1, \ldots, n.
\]

Then for every maps $f : [m] \to [m']$, $g : [n] \to [n']$ their sum $f + g : [m + n] \to [m' + n']$ is given by

\[
(f + g)(k) = \begin{cases} f(k), & \text{if } k \in \{1, \ldots, m\}, \\ m' + g(k - m), & \text{if } k \in \{m + 1, \ldots, m + n\}, \end{cases}
\]

and the resulting symmetric $+$-monoidal structure on $\mathcal{F}Set_{sk}$ is semistrict with nontrivial commutators $c_{[m],[n]} : [m + n] \to [n + m]$ given by

\[
c_{[m],[n]}(k) = \begin{cases} n + k, & \text{if } k \in \{1, \ldots, m\}, \\ k - m, & \text{if } k \in \{m + 1, \ldots, m + n\}. \end{cases}
\]

In particular, for each $n \geq 1$ the commutators $c_{[0],[n]}$ and $c_{[n],[0]}$ are both identities while $c_{[1],[n]}$ and $c_{[n],[1]}$ are the $(n + 1)$-cycles $(1, n + 1, n - 1, \ldots, 2, 1)$ and $(1, 2, 3, \ldots, n + 1, n + 1)$, respectively.

Proof. It is immediate to check that (5) indeed makes the diagram in (D1) commute for the chosen injections. Moreover, it is immediate to check that these injections are such that

\[
i^1_{[m+p],[n+p]} i^1_{[m],[n]} = i^1_{[m],[n+p]},
\]
\[
i^2_{[m],[n+p]} i^2_{[m+p],[n]} = i^2_{[m],[n]} + id_{[p]},
\]
\[
i^2_{[0],[n]} i^1_{[m],[n]} = id_{[n]} = i^1_{[m],[0]},
\]

for each $m, n, p \geq 0$. It follows that all the isomorphisms $\delta_{[m],[n],[p]}, \iota_{[m],[n]}$, defined in § 3.4 are identities because of the uniqueness of the morphisms making the diagrams in (D3) and (D5) commute. Finally, it is easy to check that (6) makes the diagram in (D4) commute. \qed
4.1.2. **Remark.** The injections $\iota_{[m],[n]}^1$, $\iota_{[m],[n]}^2$ are nothing but the composite with the canonical injections of $[m],[n]$ into the disjoint union $[m] \bigcup [n] = (([m] \times \{0\}) \cup ([n] \times \{1\})$ with the bijection $b_{m,n}^+: [m] \bigcup [n] \to [m+n]$ given by

\[
b^+(k, \alpha) = \begin{cases} k, & \text{if } \alpha = 0, \\
 m + k, & \text{if } \alpha = 1.
\end{cases}
\]

We shall denote by $\mathcal{FS}_{sk}^+$ (resp. $\mathcal{FS}_{sk}^-$) the semistrict (additive) symmetric monoidal category referred to in this lemma (resp. the underlying groupoid equipped with the inherited semistrict symmetric monoidal structure).

The multiplicative monoidal structure on $\mathcal{FS}_{sk}$ is defined similarly. The object part of any product of $([m],[n])$ in $\mathcal{FS}_{sk}$ is necessarily the set $[mn]$, and the projections onto $[m]$ and $[n]$ can be obtained by choosing any natural bijection $b_{m,n} : [mn] \to [m] \times [n]$, and composing it with the canonical projections. The point is that by suitably choosing the bijections $b_{m,n}$ the final symmetric $\cdot$-monoidal structure is again semistrict.

4.1.3. **Lemma.** For every objects $[m],[n] \in \mathcal{FS}_{sk}$ with $m,n \geq 1$ let us take as product of the pair $([m],[n])$ the set $[mn]$ with the projections $\pi_{[m],[n]}^1 : [mn] \to [m]$ and $\pi_{[m],[n]}^2 : [mn] \to [n]$ defined by

\[
\pi_{[m],[n]}^1(k) = \begin{cases} m, & \text{if } m \mid k, \\
 r, & \text{otherwise},
\end{cases}
\]

\[
\pi_{[m],[n]}^2(k) = \begin{cases} q, & \text{if } m \mid k, \\
 q + 1, & \text{otherwise},
\end{cases}
\]

where $q,r$ respectively denote the quotient and the remainder of the euclidean division of $k$ by $m$. Then for every maps $f : [m] \to [m']$, $g : [n] \to [n']$ with $m,m',n,n' \geq 1$, the product map $f \cdot g : [mn] \to [m'n']$ is given by

\[
(f \cdot g)(k) = \begin{cases} (g(q) - 1)m' + f(m), & \text{if } m \mid k, \\
 (g(q) + 1)m' + f(r), & \text{otherwise},
\end{cases}
\]

and the resulting symmetric $\cdot$-monoidal structure on $\mathcal{FS}_{sk}$ is semistrict with nontrivial commutators $c'_{[m],[n]} : [mn] \to [mn]$, $m,n \geq 1$, given by

\[
c'_{[m],[n]}(k) = \begin{cases} (m-1)n + q, & \text{if } m \mid k, \\
 (r-1)n + q + 1, & \text{otherwise},
\end{cases}
\]

where $q,r$ are as before. In particular, for each $n \geq 1$ the commutators $c'_{[1],[n]}$, $c'_{[n],[1]}$ are both identities while $c'_{[2],[n]}$ and $c'_{[n],[2]}$ are given by

\[
c'_{[2],[n]}(k) = \begin{cases} n + (k/2), & \text{if } k \text{ is even}, \\
 (k + 1)/2, & \text{if } k \text{ is odd},
\end{cases}
\]

\[
c'_{[n],[2]}(k) = \begin{cases} 2k - 1, & \text{if } k \in \{1,\ldots,n\}, \\
 2(k-n), & \text{if } k \in \{n+1,\ldots,2n\}.
\end{cases}
\]

**Proof.** For each $m,n \geq 1$ let $b_{m,n} : [m] \times [n] \to [mn]$ be the bijection given by enumerating the elements of $[m] \times [n]$ by rows, i.e.

\[
b_{m,n}(i,j) = (j - 1)m + i.
\]

Its inverse $(b^-)^{-1}_{m,n} : [mn] \to [m] \times [n]$ is given by

\[
(b^-)^{-1}_{m,n}(k) = \begin{cases} (m,q), & \text{if } m \mid k, \\
 (r,q+1), & \text{otherwise},
\end{cases}
\]
with $q, r$ as in the statement. Then the projections $\pi^1_{[m],[n]}, \pi^2_{[m],[n]}$ make the diagram

\[
\begin{array}{c}
\text{commute, and (8) is nothing but the composite} \\
\pi^1_{[m],[n]} \\
\pi^2_{[m],[n]} \\
\end{array}
\]

\[
f \cdot g = b_{m ', n'} (f \times g) (b')^{-1}_{m,n},
\]

with $f \times g$ the usual cartesian product map given by $(i, j) \mapsto (f(i), g(j))$. It follows that the diagram

\[
\begin{array}{c}
\text{commutes, and hence (8) makes the dual of the diagram in (D1) commute for the projections } \pi^1_{[m],[n]}, \pi^2_{[m],[n]} \\
\text{in the statement. Moreover, these projections are such that} \\
\pi^1_{[m],[n]} \pi^1_{[m],[p]} = \pi^1_{[m],[np]}, \\
\pi^2_{[m],[np]} = \pi^2_{[m],[n]} \times \text{id}_{[p]}, \\
\pi^2_{[1],[n]} = \text{id}_{[n]} = \pi^1_{[n],[1]}
\end{array}
\]

for each $m, n, p \geq 1$, which together with the uniqueness of the morphisms making the duals of the diagrams in (D3) and (D5) commute implies that all isomorphism $a'_{[m],[n],[p]}, \pi'_{[m],[n]}$ are identities. To prove these equalities, let be $l \in [mpl]$, and let

\[
l = q'(mn) + r' = q''m + r''
\]

be the euclidean divisions of $l$ by $mn$ and by $m$, respectively, and $r' = q''m + r''$ the euclidean division by $m$ of the remainder $r'$ of the first of these divisions. Clearly, we have

(i) $r'' = r''',$
(ii) $q'' = q'n + q'''$, and
(iii) $m \mid l$ if and only if $m \mid r'$.

Then, on the one hand, an easy computation shows that

\[
(\pi^1_{[m],[n]} \circ \pi^1_{[m],[p]})(l) = \begin{cases} m, & \text{if } mn \mid l, \\
m, & \text{if } mn \nmid l \text{ and } m \mid r', \\
r''', & \text{if } mn \nmid l \text{ and } m \nmid r'.
\end{cases}
\]

Since conditions $mn \mid l$ and $m \nmid r'$ can not hold simultaneously we have

\[
(\pi^1_{[m],[n]} \circ \pi^1_{[m],[p]})(l) = \begin{cases} m, & \text{if } m \mid r', \\
r''', & \text{otherwise},
\end{cases}
\]

while by definition

\[
\pi^1_{[m],[n]}(l) = \begin{cases} m, & \text{if } m \mid l, \\
r''', & \text{otherwise}.
\end{cases}
\]
Hence it follows from (i) and (iii) that both maps are indeed the same. On the other hand, for each \( l \in [mn] \) we have
\[
(p^n_{[m],[n]} \times id_{[p]})(l) = \begin{cases} (q' - 1)n + p^n_{[m],[n]}(mn), & \text{if } mn \mid l, \\ (q' + p^n_{[m],[n]}(r')), & \text{if } mn \nmid l, \end{cases}
\]
\[
= \begin{cases} q' n, & \text{if } mn \mid l, \\ q' + q'', & \text{if } mn \nmid l \text{ and } m \mid r', \\ q' + q''' + 1, & \text{if } mn \nmid l \text{ and } m \nmid r', \end{cases}
\]
\[
= \begin{cases} q''', & \text{if } m \mid r', \\ q'' + 1, & \text{otherwise} \end{cases}
\]
(in the last equality we use that \( mn \mid l \) implies that \( r' = 0 \) and hence, \( q''' = 0 \)), while by definition
\[
p^n_{[m],[np]}(l) = \begin{cases} q''', & \text{if } m \mid l, \\ q'' + 1, & \text{otherwise}. \end{cases}
\]
Hence it follows from (ii) and (iii) that both maps are again the same. Finally, it readily follows from their definitions that \( p^n_{[m],[1]} \) and \( p^n_{[1],[n]} \) are identities.

It remains to see that the map (9) makes the dual of the diagram in (D4) commute. Or, (9) is nothing but the composite
\[
c'_{[m],[n]} = b_{m,n} \circ [m] \times [n] \rightarrow [n] \times [m],
\]
where \( c'_{[m],[n]} : [m] \times [n] \rightarrow [n] \times [m] \) is the permutation map \((i, j) \mapsto (j, i)\). The commutativity of the dual of the diagram in (D4) follows then from the diagram

all of whose inner triangles commute.

4.1.4. Remark. When we think of the elements in \([mn]\) as the points of the finite lattice \([m] \times [n] \subset \mathbb{R}^2\), the map \( f \circ g \) simply corresponds to applying \( f \) to the columns and \( g \) to the rows. However, the explicit formula for \((f \circ g)(k)\) depends on the way we decide to enumerate the points in the lattice and hence, on the chosen bijection \( b_{m,n} \). The same thing happens with the commutators, ultimately defined by the maps \((i, j) \mapsto (j, i)\). The above bijections \( b_{m,n} \) correspond to enumerating the points by rows, so that the formula (9) for the commutators corresponds to doing the following. Take two sets of \( mn \) aligned points, one on the top of the other. Divide the top set into \( n \) boxes each one with \( m \) points, and the bottom set into \( m \) boxes each one with \( n \) points. Then \( c'_{[m],[n]} \) maps the successive points in the top \( j \) box, for each \( j \in \{1, \ldots, n\} \), into the \( j^{th} \) point in the successive \( m \) bottom boxes.

Next step is to describe the corresponding left and right distributors. Let \( R_k : \mathbb{N} \rightarrow [k] \) be the modified remainder function mapping each nonnegative integer \( x \) to the remainder of the euclidean division of \( x \) by \( k \) if \( k \nmid x \), and to \( k \) if \( k \mid x \). Then we have the following.

4.1.5. Lemma. Let \( \mathcal{F} \text{Set}_{ab} \) be equipped with the above additive and multiplicative semistrict symmetric monoidal structures. Then for every objects \([m],[n],[p]\) the left distributor \( \tilde{d}^l_{[m],[n],[p]} \) is the identity, while the right distributor \( \tilde{d}^r_{[m],[n],[p]} : [mp + np] \rightarrow [mp + np] \) is the identity when some of the objects \([m],[n],[p]\)
is [0], and otherwise is given by
\[
\overline{d}'_{[m],[n],[p]}(s) = \begin{cases} 
  s + \frac{s - R_m(s)}{m} n, & \text{if } s \in \{1, \ldots, mp \} \\
  s - mp + \frac{s - mp + n - R_n(s - mp)}{n} m, & \text{if } s \in \{mp + 1, \ldots, mp + np \}
\end{cases}
\]

In particular, \(\overline{d}'_{[m],[n],[p]}\) is the identity when some of the integers \(m, n, p\) is zero, and \(\overline{d}'_{[n],[1],[1]}, \overline{d}'_{[1],[n],[1]} : [n + 1] \to [n + 1]\) are both identities, and \(\overline{d}'_{[1],[1],[n]} : [2n] \to [2n]\) is the permutation given by
\[
\overline{d}'_{[1],[1],[n]}(s) = \begin{cases} 
  2s - 1, & \text{if } s \in \{1, \ldots, n \}, \\
  2(s - n), & \text{if } s \in \{n + 1, \ldots, 2n \}
\end{cases}
\]
for each \(n \geq 1\).

Proof. Using (8) it is easy to check that
\[
id_{[m]} \times \iota_{[n],[p]}^1 = \iota_{[m],[n]+mp}^1,
\]
\[
id_{[m]} \times \iota_{[n],[p]}^2 = \iota_{[mp],[mp]+mp}^2.
\]
Hence \(\overline{d}'_{[m],[n],[p]}\) is the identity. As to the right distributor \(\overline{d}'_{[m],[n],[p]}\), by definition we have
\[
\overline{d}'_{[m],[n],[p]}(s) = \begin{cases} 
  (\iota_{[m],[n]}^1 \times id_{[p]})(s), & \text{if } s \in \{1, \ldots, mp \}, \\
  (\iota_{[m],[n]}^2 \times id_{[p]})(s - mp), & \text{if } s \in \{mp + 1, \ldots, mp + np \}
\end{cases}
\]
for each \(s \in \{mp + np\}\). Hence \(\overline{d}'_{[m],[n],[p]}\) is the identity when some of the objects \([m],[n],[p]\) is [0]. Otherwise, by (8) we have
\[
(\iota_{[m],[n]}^1 \times id_{[p]})(k) = \begin{cases} 
  k + (q - 1)n, & \text{if } k = qm, \\
  k + qn, & \text{if } k = qm + r, r \neq 0,
\end{cases}
\]
for each \(k \in [mp]\), and
\[
(\iota_{[m],[n]}^2 \times id_{[p]})(l) = \begin{cases} 
  l + q'm, & \text{if } l = q'n, \\
  l + (q' + 1)m, & \text{if } l = q'n + r', r' \neq 0,
\end{cases}
\]
for each \(l \in [np]\). Hence
\[
\overline{d}'_{[m],[n],[p]}(s) = \begin{cases} 
  s + (q - 1)n, & \text{if } s \in \{1, \ldots, mp\} \text{ and } m \mid s, \\
  s + qn, & \text{if } s \in \{1, \ldots, mp\} \text{ and } m \nmid s, \\
  s + (q' - p)m, & \text{if } s \in \{mp + 1, \ldots, mp + np\} \text{ and } n \mid (s - mp), \\
  s + (q' - p + 1)m, & \text{if } s \in \{mp + 1, \ldots, mp + np\} \text{ and } n \nmid (s - mp),
\end{cases}
\]
where \(q\) is the quotient of the euclidean division of \(s\) by \(m\) for \(s \leq mp\), and \(q'\) the quotient of the euclidean division of \(s - mp\) by \(n\) when \(s > mp\). It is left to the reader checking that, in terms of the modified remainder functions, this is the permutation in the statement.

4.1.6. Example. The first few non trivial distributors \(\overline{d}'_{[1],[1],[n]} : [2n] \to [2n]\) are
\[
\overline{d}'_{[1],[1],[2]} = (2, 3)_4,
\overline{d}'_{[1],[1],[3]} = (2, 3, 5, 4)_6,
\overline{d}'_{[1],[1],[4]} = (2, 3, 5, 8, 4, 7, 6)_8,
\overline{d}'_{[1],[1],[5]} = (2, 3, 5, 9, 8, 6, 10, 4, 7)_10.
\]
Their decompositions into disjoint cycles do not seem to follow an easy pattern. Hence, unlike the commutators, all of order two, the order of the distributors \(\overline{d}'_{[1],[1],[n]}\) will be a non-trivial function of \(n\).
Theorem. The symmetric rig category structure on \( F Set_{sk} \) canonically associated to the choices of products and coproducts of Lemmas 4.1.1 and 4.1.3 is left semistrict. The left symmetric rig category so obtained \( F Set_{sk} \) is equivalent to the symmetric rig category \( F Set \).

Proof. The first assertion is a consequence of Proposition 3.4.3, the previous three lemmas, and the fact that the projections \( n_{(0),[n]} \) and \( m_{[n],[0]} \) are in this case identities. As to the equivalence between \( F Set_{sk} \) and \( F Set \), an equivalence is given by the inclusion functor \( J : F Set_{sk} \hookrightarrow F Set \) with the additive and multiplicative symmetric monoidal structures defined by the above bijections, where \( m_{[n],[0]} \) is an automorphism of the underlying additive symmetric monoidal groupoid \( F Set_{sk} \) is trivial, i.e. that all its objects are isomorphic, and all of them have the respective identity as unique endomorphism. Let us first solve the easier problem of computing the groupoid of automorphisms of the underlying additive symmetric monoidal groupoid \( F Set_{sk} \).

It follows from Theorem 4.1.7 and Example 3.3.2 that \( F Set_{sk} \) is a left semistrict symmetric 2-rig equivalent to the symmetric 2-rig \( F Set \). The rest of this section is devoted to proving that, up to equivalence, the category of endomorphisms of \( F Set_{sk} \) is trivial, i.e. that all its objects are isomorphic, and all of them have the respective identity as unique endomorphism. Let us first solve the easier problem of computing the groupoid of automorphisms of the underlying additive symmetric monoidal groupoid \( F Set_{sk} \).

2-group of automorphisms of the additive symmetric monoidal groupoid of finite sets. Let us denote by \( Aut(F Set_{sk}) \) the category whose objects are the self-equivalences (hence, automorphisms) of the additive symmetric monoidal groupoid \( F Set_{sk} \), and whose morphisms are the monoidal natural transformations between them. It is in fact a groupoid because all natural transformations are actually invertible.

Our immediate goal is to see that \( Aut(F Set_{sk}) \) is equivalent to the terminal category. In fact, \( Aut(F Set_{sk}) \) has a canonical strict monoidal structure whose tensor product is given by the composition of automorphisms, and the horizontal composition of monoidal natural transformations. More precisely, for every automorphisms \( \mathcal{F} = (F, \varphi^+ \cdot \epsilon^+) \) and \( \mathcal{F} = (\hat{F}, \hat{\varphi}^+ \cdot \hat{\epsilon}^+) \) we define

\[
\mathcal{F} \otimes \hat{\mathcal{F}} := \widehat{\mathcal{F}} \circ \mathcal{F},
\]

with \( \hat{\mathcal{F}} \circ \mathcal{F} \) the symmetric \( + \)-monoidal functor with underlying functor \( \hat{F} \circ F \), and monoidality isomorphisms \( (\hat{\varphi}^+ \cdot \hat{\epsilon}^+)_{[m],[n]} \) and \( \hat{\epsilon}^+ \cdot \hat{\epsilon}^+ \) respectively given by the composites

\[
(\hat{F} \circ F)([m] + [n]) \xrightarrow{F_{\varphi^+_{[m],[n]}} \hat{F}} \hat{F}(F [m] + F [n]) \xrightarrow{\hat{\varphi}^+_{[m],[n]} \hat{F}} (\hat{F} \circ F)[m] + (\hat{F} \circ F)[n],
\]

\[
(\hat{F} \circ F)[0] \xrightarrow{F_{\epsilon^+}} \hat{F}[0] \xrightarrow{\hat{\epsilon}^+} [0].
\]

Equipped with this monoidal structure, \( Aut(F Set_{sk}) \) is what is usually called a strict 2-group, i.e. a strict monoidal groupoid such that every object has a strict inverse for the tensor product. In particular, the set of objects of \( Aut(F Set_{sk}) \) has a canonical group structure given by the above tensor product, and we shall compute this group. More precisely, we shall see that it is isomorphic to the direct product of all symmetric groups \( S_n \) for \( n \geq 1 \). Although doing this is not necessary for our purposes, we get it almost for free.

Let us start by showing that the underlying functor of every automorphism of \( F Set_{sk} \) is the identity on objects, and that it acts on morphisms by inner automorphisms.

Proposition. Let \( \mathcal{F} = (F, \varphi^+ \cdot \epsilon^+) \) be any automorphism of \( F Set_{sk} \), and let \( \tau_k : [k] \to [k] \) for each \( k \geq 2 \) be the permutation given by

\[
\tau_k = (id_{[k-2]} + \varphi^+_{[1],[1]})(id_{[k-3]} + \varphi^+_{[1],[2]}) \cdot \cdots (id_{[1]} + \varphi^+_{[1],[k-2]}) \varphi^+_{[1],[k-1]}.
\]

Then \( F \) is the identity on objects, and maps each automorphism \( \sigma : [k] \to [k] \) for any \( k \geq 2 \) to the conjugate automorphism \( \tau_k^{-1} \sigma \tau_k \).

Proof. The underlying functor \( F \) of every automorphism of \( F Set_{sk} \) is an automorphism of the skeletal groupoid \( F Set_{sk} \) and hence, it is the identity on objects. Thus \([0]\) gets mapped to itself because it is the unique initial object, and if \( m, n \geq 1 \) with \( m \neq n \), \([m]\) can not be mapped to \([n]\) because these objects
have nonisomorphic automorphism groups. To see that the action on morphisms is by conjugation, and to identify the respective inner automorphism, notice that \( \varphi^+_{[m],[n]} \) is an automorphism of \([m + n]\). Hence the naturality of \( \varphi^+_{[m],[n]} \) in \([m],[n]\) amounts to the commutativity of the diagram

\[
\begin{array}{ccc}
[m + n] & \xrightarrow{\varphi^+_{[m],[n]}} & [m + n] \\
F_{m,n}(\rho + \sigma) & \downarrow & F_{m,n}(\rho) + F_n(\sigma) \\
[m + n] & \xrightarrow{\varphi^+_{[m],[n]}} & [m + n]
\end{array}
\]

for every permutations \( \rho \in S_m \) and \( \sigma \in S_n \). Here \( F_k : S_k \to S_k \) denotes the action of \( F \) on the automorphisms of \([k]\), which is a group homomorphism because of the functoriality of \( F \). Since \( S_k \) is spanned by the \( k - 1 \) transpositions

\[
(1, 2)_k = (1, 2)_2 + id_{[k-2]},
\]

\[
(i, i + 1)_k = id_{[1]} + (i - 1, i)_{k-1}, \quad i = 2, \ldots, k - 1,
\]

it follows that each \( F_k \) for \( k \geq 3 \) can be recursively computed from \( F_2 \) and the permutations \( \varphi^+_{[1],[n]} \), \( \varphi^+_{[2],[n]} \) for \( n \geq 1 \). More precisely, due to the above commutative diagram \( F_k \) is given on these generators by

\[
F_k((1, 2)_k) = F_{2+([k-2])]((1, 2)_2 + id_{[k-2]}) = (\varphi^+_{[2],[k-2]})(1, 2)_2 + id_{[k-2]}(2)) \varphi^+_{[2],[k-2]}
\]

and

\[
F_k((i, i + 1)_k) = F_{1+([k-1])(id_{[1]} + (i - 1, i)_{k-1}) = (\varphi^+_{[1],[k-1]}(id_{[1]} + F_{k-1}((i - 1, i)_{k-1}))) \varphi^+_{[1],[k-1]}(i)
\]

for each \( i = 2, \ldots, k - 1 \). To see that \( F_k \) is then conjugating by the permutation \( \tau_k \in S_k \) given by \([12] \) we proceed by induction on \( k \geq 2 \). The case \( k = 2 \) follows from the fact that \( c_{[1],[1]} = (1, 2)_2 \) (see Lemma 4.1.1) together with the fact that \( F \) is symmetric ++-monoidal, which in particular means that the diagram

\[
\begin{array}{ccc}
[2] & \xrightarrow{\varphi^+_{[1],[1]}} & [2] \\
\downarrow & \downarrow & \downarrow \\
[2] & \xrightarrow{\varphi^+_{[1],[1]}} & [2]
\end{array}
\]

commutes. Actually, \( F_2 \) is the identity of \( S_2 \) because \( S_2 \) is abelian. Let us now assume that \( F_{k-1} \) is conjugation by \( \tau_{k-1} \) for some \( k \geq 3 \). By \([13] \), \( F_k \) acts on the generator \((1, 2)_k \) as the conjugation by \( \varphi^+_{[2],[k-2]} \).

However, the coherence axiom on the isomorphisms \( \varphi^+_{[m],[n]} \) says that the diagram

\[
\begin{array}{ccc}
[k] & \xrightarrow{\varphi^+_{[1],[k-1]}} & [k] \\
\downarrow & \downarrow & \downarrow \\
[k] & \xrightarrow{\varphi^+_{[1],[k-1]} + id_{[k-2]}} & [k]
\end{array}
\]

commutes. Thus we have

\[
F_k((1, 2)_k) = (\varphi^+_{[2],[k-2]})(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[2],[k-2]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(\varphi^+_{[1],[k-1]} + id_{[k-2]}(2)) + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]

\[
= (\varphi^+_{[1],[k-1]}(id_{[1]} + \varphi^+_{[2],[k-2]}))(1, 2)_2 + id_{[k-2]}(2)(\varphi^+_{[1],[k-1]})
\]
(in the third equality we have used the functoriality of \(+\), and in the fourth one that \(S_2\) is abelian). Hence \(F_k\) also acts on \((1,2)_k\) as the conjugation by \((id_{[1]} + \varphi^+_{[1],[k-2]} ) \varphi^+_{[1],[k-1]}\). The point now is that conjugating \((1,2)_k\) by \((id_{[1]} + \varphi^+_{[1],[k-2]} ) \varphi^+_{[1],[k-1]}\) is the same as conjugating it by \(\tau_k\). Indeed, for each \(i = 2, \ldots, k-2\) we have

\[
(id_{[1]} + \varphi^+_{[1],[k-1]} )^{-1} (1,2)_k (id_{[1]} + \varphi^+_{[1],[k-1]} ) = (id_{[2]} + (id_{[1]} + \varphi^+_{[1],[k-1]} )^{-1} (id_{[2]} + (id_{[1]} + \varphi^+_{[1],[k-1]} )) = (1,2)_k + (id_{[1]} + \varphi^+_{[1],[k-1]} )^{-1} (id_{[1]} + \varphi^+_{[1],[k-1]} ) = (id_{[2]} + (id_{[1]} + \varphi^+_{[1],[k-1]} ))^{-1} (id_{[2]} + (id_{[1]} + \varphi^+_{[1],[k-1]} ))
\]

(we have used the functoriality of \(+\) and its strict associative character). Therefore \(F_k\) indeed acts on the generator \((1,2)_k\) via conjugation by \(\tau_k\). As to the action on the remaining generators, it follows from \([14]\) and the induction hypothesis that

\[
F_k((i,i+1)_k) = (\varphi^+_{[1],[k-1]} )^{-1} (id_{[1]} + \tau_{k-1}^{-1} (i-1, i)_k \tau_{k-1}^{-1} ) (\varphi^+_{[1],[k-1]} )^{-1} = (\varphi^+_{[1],[k-1]} )^{-1} (id_{[1]} + \tau^{-1}_{k-1}^{-1} (id_{[1]} + \tau_{k-1}^{-1} ) ) (id_{[1]} + \tau_{k-1}^{-1} ) (\varphi^+_{[1],[k-1]} )^{-1} = \tau_{k}^{-1} (id_{[1]} + (i-1, i)_k \tau_{k-1}^{-1} ) \tau_{k} = \tau_{k}^{-1} (i,i+1)_k \tau_{k},
\]

where we have used that \(\tau_k = (id_{[1]} + \tau_{k-1}^{-1} ) \varphi^+_{[1],[k-1]} \). Thus \(F_k\) acts on every generator of \(S_k\) as the conjugation by \(\tau_k\) and hence, \(F_k\) is conjugation by \(\tau_k\).

4.2.2. Remark. It is well known that every symmetric group \(S_k\) with \(k \neq 6\) has no outer automorphisms. Therefore every automorphism of the groupoid \(\mathcal{F}\text{Set}_{\tilde{\mathcal{F}}}\) necessarily acts on all but the automorphisms of \([6]\) by inner automorphisms. What we have proved is that when the automorphism of \(\mathcal{F}\text{Set}_{\tilde{\mathcal{F}}}\) is symmetric ++-monoidal, it also acts by inner automorphisms on the automorphisms of \([6]\), and we have identified the inner automorphism corresponding to each object in terms of the ++-monoidal structure. It is also worth emphasizing that the above argument actually works for every endomorphism of \(\mathcal{F}\text{Set}_{\tilde{\mathcal{F}}}^+\) which acts as the identity on objects.

4.2.3. Corollary. Every automorphism \(\mathcal{F} = (F, \varphi^+, e^+)\) of \(\mathcal{F}\text{Set}_{\tilde{\mathcal{F}}}^+\) is completely determined by the family \(\varphi^+ = \{\varphi^+_{[m],[n]}\}_{m,n \geq 1}\). Moreover, such a family of permutations defines an automorphism if and only if it makes the following diagrams commute:

(A1) for each \(m, n, p \geq 1\)

\[
\begin{array}{ccc}
[m + n + p] & \xrightarrow{\varphi^+_{[m],[n+p]}} & [m + n + p] \\
\varphi^+_{[m],[n+p]} & & \varphi^+_{[m],[n+p]} \\
[m + n + p] & \xrightarrow{id_{[m]} + \varphi^+_{[n,p]}} & [m + n + p]
\end{array}
\]

(A2) for each permutations \(\rho \in S_n\) and \(\sigma \in S_m\), and each \(m, n \geq 1\)

\[
\begin{array}{ccc}
[m + n] & \xrightarrow{\varphi^+_{[m],[n]}} & [m + n] \\
\varphi^+_{[m],[n]} & & \varphi^+_{[m],[n]} \\
[m + n] & \xrightarrow{F_n \rho + F_{m+n} \sigma} & [m + n]
\end{array}
\]

where \(F_k : S_k \rightarrow S_k\) for each \(k \geq 2\) is conjugation by the element \(\tau_k \in S_k\) in \([12]\).
(A3) for each \( m, n \geq 1 \)

\[
\begin{array}{c}
\begin{array}{c}
[m + n] \\
\downarrow
\end{array}
\xrightarrow{\varphi^{+}_{m,[n]}}
[m + n] \\
\downarrow
\\underline{\tau_{m,[n]}}
\end{array}
\xrightarrow{\rho_{m,[n],1}}
\begin{array}{c}
[m + n] \\
\downarrow
\end{array}
\xrightarrow{\varphi^{+}_{[n],m}}
[m + n]
\]

**Proof.** It follows from Proposition 4.2.1 that the underlying functor \( F \) is completely given by the isomorphisms \( \varphi^{+}_{[1],[n]} \), \( m \geq 1 \). Moreover, the permutations \( \varphi^{+}_{[0],[0]}, \varphi^{+}_{[1],[1]}, \varepsilon^{+} \) are necessarily identities because their domains and codomains are either \([0]\) or \([1]\), and the permutations \( \varphi^{+}_{[n],[0]} \) for each \( n \geq 2 \) are also identities because of the coherence diagrams

\[
\begin{aligned}
F[0] + F[n] & \xrightarrow{\varepsilon^{+} + id_{F[n]}} [0] + F[n] \\
\varphi^{+}_{[n],[0]} & \downarrow \quad \downarrow \quad \varphi^{+}_{[0],[n]} \\
F([0] + [n]) & \xrightarrow{\rho_{[n],1}} F[n] \\
F([n] + [0]) & \xrightarrow{\rho_{[0],1}} F[n]
\end{aligned}
\]

and Lemma 4.1.1, which in particular means that the unit isomorphisms \( l, r \) of \( \mathcal{F}\text{Set}_{sk} \) are trivial. Finally, conditions (A1) and (A3) are nothing but the coherence axioms on the remaining isomorphisms, and (A2) is the naturality of these isomorphisms in \([m],[n]\). \( \square \)

It follows from Corollary 4.2.3 that the problem of finding all automorphisms of \( \mathcal{F}\text{Set}_{sk}^{+} \) amounts to solving the infinitely many equations (A1)-(A3) in the infinitely many unknown permutations \( \varphi^{+}_{m,[n]} \), \( m, n \geq 1 \). In spite of its apparent complexity, the solution to this problem turns out to be amazingly simple. Let us start by describing a generic solution to (A1). It turns out to be parametrized by an element in the cartesian product of all the symmetric groups.

4.2.4. **Lemma.** Let \( \{\varphi^{+}_{[m],[n]} : [m + n] \to [m + n]\}_{m,n \geq 1} \) be a family of permutations that satisfies (A1). Then

\[
\varphi^{+}_{m,[n]} = (\tau_{m} + \tau_{n})^{-1} \tau_{m+n}, \quad m, n \geq 1,
\]

with \( \tau_{k} \) given by (12) for each \( k \geq 2 \), and \( \tau_{1} = id_{[1]} \). In particular, if \( \{\varphi^{+}_{[m],[n]} \}_{m,n \geq 1} \) satisfies (A1), each permutation \( \varphi^{+}_{[m],[n]} \) is completely determined by the basic permutations \( \{\varphi^{+}_{[1],[n]} \}_{n \geq 1} \). Moreover, for any choices of the permutations \( \{\tau_{k}\}_{k \geq 1} \), with \( \tau_{k} \in S_{k} \), the family \( \{\varphi^{+}_{[m],[n]} \}_{m,n \geq 1} \) given as before is a solution of (A1).

**Proof.** Let the family \( \{\varphi^{+}_{[m],[n]} \}_{m,n \geq 1} \) satisfy (A1). Then (15) is shown by induction on \( m \geq 1 \). The case \( m = 1 \) follows from the relation \( \tau_{n+1} = (id_{[1]} + \tau_{n}) \varphi^{+}_{[1],[n]} \), valid for each \( n \geq 1 \), and the fact that \( \tau_{1} = id_{[1]} \). Let us now assume that (15) is true for some \( m \geq 1 \) and every \( n \geq 1 \). Then

\[
\begin{aligned}
\varphi^{+}_{m+[1],[n]} &= (\varphi^{+}_{[1],[m]} + id_{[m]}^{-1} (id_{[1]} + \varphi^{+}_{[m],[n]})) \varphi^{+}_{[1],[m+n]} \\
&= (\varphi^{+}_{[1],[m]} + id_{[m]}^{-1} (id_{[1]} + (\tau_{m} + \tau_{n})^{-1} \tau_{m+n})) \varphi^{+}_{[1],[m+n]} \\
&= (\varphi^{+}_{[1],[m]} + id_{[m]}^{-1} (id_{[1]} + \tau_{m} + \tau_{n}^{-1} (id_{[1]} + \tau_{m+n}))) \varphi^{+}_{[1],[m+n]} \\
&= ((\varphi^{+}_{[1],[m]}^{-1} (id_{[1]} + \tau_{m})^{-1} + \tau_{n}^{-1} (id_{[1]} + \tau_{m+n}))) \varphi^{+}_{[1],[m+n]} \\
&= (\tau_{m+1} + \tau_{n}^{-1}) \tau_{m+n+1},
\end{aligned}
\]

for each \( n \geq 1 \), where in the first equality we have used (A1), in the second the induction hypothesis, in the third and fourth the functoriality of +, and in the last one the relation \( \tau_{k+1} = (id_{[1]} + \tau_{k}) \varphi^{+}_{[1],[k]} \).
To prove the last assertion, let the family \( \{\phi^+_{[m],[n]}\}_{m,n \geq 1} \) be given by (15) for any choices of the permutations \( \tau_k, k \geq 1 \). Then we have

\[
(\phi^+_{[m],[n]} + id_{[p]})(\phi^+_{[m+n],[p]}) = ((\tau_m + \tau_n)^{-1} \tau_{m+n} + id_{[p]})(\tau_m + \tau_p)^{-1} \tau_{m+n+p} = (\tau_m + \tau_n + id_{[p]})(\tau_m + \tau_p)^{-1} \tau_{m+n+p} = ((\tau_m + \tau_n)^{-1} \tau_{m+n} + \tau_p^{-1}) \tau_{m+n+p} = (\tau_m + \tau_n + \tau_p)^{-1} \tau_{m+n+p},
\]

and

\[
(id_{[m]} + \phi^+_{[n],[p]})(\phi^+_{[m],[n]} + id_{[p]}) = (id_{[m]} + (\tau_n + \tau_p)^{-1} \tau_{n+p})((\tau_m + \tau_n + \tau_p)^{-1} \tau_{m+n+p} = (\tau_m + \tau_n + id_{[p]})(\tau_m + \tau_n + \tau_p)^{-1} \tau_{m+n+p} = (\tau_m + \tau_n + \tau_p)^{-1} \tau_{m+n+p}.
\]

Hence, both permutations are the same, and the family satisfies (A1). \( \square \)

The point now is the somewhat surprising fact that conditions (A2)-(A3) readily follow from (A1) in the following sense.

4.2.5. Lemma. Every family of permutations \( \{\phi^+_{[m],[n]}\}_{m,n \geq 1} \) satisfying (A1) automatically satisfies axioms (A2) and (A3).

Proof. Let \( \{\phi^+_{[m],[n]}\}_{m,n \geq 1} \) satisfy (A1). By Lemma 4.2.4 each \( \phi^+_{[m],[n]} \) is of the form (15) with the \( \tau_k \)'s for \( k \geq 2 \) given by (12), and \( \tau_1 = id_{[1]} \). Then for each \( m, n \geq 1 \) and each permutations \( \rho \in S_m, \sigma \in S_n \) we have

\[
(F_m(\rho) + F_n(\sigma))\phi^+_{[m],[n]} = (\tau_m^{-1} \rho \tau_m + \tau_n^{-1} \sigma \tau_n)(\tau_m + \tau_n)^{-1} \tau_{m+n} = (\tau_m + \tau_n)^{-1} (\rho \+ \sigma) \tau_{m+n} = (\tau_m + \tau_n)^{-1} (\rho \+ \sigma) \tau_{m+n} = \phi^+_{[m],[n]} F_{m+n}(\rho \+ \sigma).
\]

Hence (A2) holds. Similarly for each \( m, n \geq 1 \), and using again the functoriality of +, as well as the naturality of \( c_{[m],[n]} \) in \([m],[n]\) we have

\[
c_{[m],[n]} \phi^+_{[m],[n]} = c_{[m],[n]} (\tau_m + \tau_n)^{-1} \tau_{m+n} = (\tau_n + \tau_m)^{-1} c_{[m],[n]} \tau_{m+n} = (\tau_n + \tau_m)^{-1} c_{[m],[n]} \tau_{m+n} = \phi^+_{[m],[n]} F_{m+n}(c_{[m],[n]})
\]

and (A3) also holds. \( \square \)

Let us denote by \( Aut(\mathcal{F}Set^+_sk) \) the set of automorphisms of \( \mathcal{F}Set^+_sk \). Together with Corollary 4.2.3 the previous two lemmas allow us to define a bijection between this set and the cartesian product \( \prod_{n \geq 1} S_n \). In fact, we have two such bijections. Thus we can identify \( \mathcal{F} = (F, \phi^+, \varepsilon^+) \) with the sequence of basic permutations \( \{\phi^+_{[1],[n-1]}\}_{n \geq 1} \), or with the corresponding sequence \( \{\tau_n\}_{n \geq 1} \) given by (12). Both bijections just differ by the composition of the first one with the bijection \( T : \prod_{n \geq 1} S_n \rightarrow \prod_{n \geq 1} S_n \) defined by

\[
T(\sigma)_n = (id_{[n-2]} + \sigma_2)(id_{[n-3]} + \sigma_3) \cdots (id_{[1]} + \sigma_{n-1}) \sigma_n,
\]

whose inverse \( T^{-1} \) is given by

\[
T^{-1}(\sigma)_n = (id_{[1]} + \sigma_{n-1})^{-1} \sigma_n
\]

for \( n \geq 2 \), and \( \sigma \in \prod_{n \geq 1} S_n \). In fact, as pointed out before, \( Aut(\mathcal{F}Set^+_sk) \) is a group with the composition of automorphisms, and parametrizing the elements of \( Aut(\mathcal{F}Set^+_sk) \) by the sequences \( \tau = (\tau_n)_{n \geq 1} \) is better because, as it is next shown, it defines not only a set theoretic bijection, but a group isomorphism when \( \prod_{n \geq 1} S_n \) is equipped with the usual direct product group structure.
4.2.6. Proposition. Let $\Psi : \prod_{n \geq 1} S_n \to \text{Aut}(\mathcal{FSET}_{sk})$ be the map given on each sequence $\tau = (\tau_n)_{n \geq 1}$ by

$$\Psi(\tau) = (F(\tau), \varphi(\tau)^+, id_{[0]}),$$

where $F(\tau) : \mathcal{FSET}_{sk} \to \mathcal{FSET}_{sk}$ is the functor acting as the identity on objects, and as conjugation by $\tau_n$ on the automorphisms of $[n]$, and $\varphi(\tau)^+$ is given by

$$\varphi(\tau)^+_{[m],[n]} = (\tau_m + \tau_n)^{-1} \tau_{m+n}$$

for each $m,n \geq 1$. Then $\Psi$ is a group isomorphism whose inverse maps an arbitrary automorphism $(F, \varphi^+, id_{[0]})$ to the sequence $(\tau_n)_{n \geq 1}$ given by (12).

Proof. We already know that $\Psi$ is a bijection, and it clearly preserves the unit elements because $\Psi(\tau)$ is the identity of $\mathcal{FSET}_{sk}$ when $\tau = (id_{[n]})_{n \geq 1}$. Moreover, for any sequences $\tau, \tau'$ we have

$$F(\tau') = F(\tau) \circ F(\tau')$$

(conjugating by $\tau_n'$ is the same as conjugating first by $\tau_n'$ and next by $\tau_n$), and it follows from (10) that the symmetric +-monoidal structure on the composite functor $F(\tau) \circ F(\tau')$ is given by the permutations

$$\varphi(\tau)^+_{[m],[n]} F(\tau)(\varphi(\tau')^+_{[m],[n]}) = (\tau_m + \tau_n)^{-1} \tau_{m+n} (\tau_m' + \tau_n')^{-1} \tau_{m+n}$$

$$\varphi(\tau)^+_{[m],[n]} = (\tau_m' + \tau_n')^{-1} \tau_{m+n} \tau_{m+n}$$

Hence

$$\Psi(\tau \cdot \tau') = \Psi(\tau) \circ \Psi(\tau') = \Psi(\tau') \otimes \Psi(\tau).$$

as required. \qed

For short, we shall denote by $\mathcal{F}_\tau$ the automorphism $\Psi(\tau) = (F(\tau), \varphi(\tau)^+, id_{[0]})$ parametrized by the sequence $\tau = (\tau_n)_{n \geq 1}$.

4.2.7. Examples. (1) Let $\tau \in \prod_{n \geq 1} S_n$ be given by $\tau_2 = (1,2)_2$, and $\tau_n = id_{[n]}$ for every $n \geq 3$. Then $\mathcal{F}_\tau$ is the non-strict automorphism whose underlying functor is the identity, and whose symmetric +-monoidal structure is given by

$$\varphi(\tau)^+_{[m],[n]} = \begin{cases} (1,2)_2, & \text{if } m = n = 1, \\ (1,2)_4 (3,4)_4, & \text{if } m = n = 2, \\ (1,2)_{n+2}, & \text{if } m = 2 \text{ and } n \neq 2, \\ (n+1,n+2)_{n+2}, & \text{if } m \neq 2 \text{ and } n = 2, \\ id_{[m+n]}, & \text{otherwise}. \end{cases}$$

(2) Let $\tau \in \prod_{n \geq 1} S_n$ be given by $\tau_n = (n-1,n)_n$ for each $n \geq 2$. Then $\mathcal{F}_\tau$ is the non-strict automorphism mapping each morphism $f : [n] \to [n]$ to its conjugate by $(n-1,n)_n$, and whose symmetric +-monoidal structure is given by

$$\varphi(\tau)^+_{[m],[n]} = \begin{cases} id_{[n+1]}, & \text{if } m = 1, \\ (m-1,m,m+1)_{m+1}, & \text{if } m = 1, \\ (m-1,m)_{m+n}, & \text{otherwise}. \end{cases}$$

Thus in this case $\mathcal{F}_\tau$ is the unique automorphism whose basic permutations $\varphi^+_{[1],[n]}$ are identities for each $n \geq 2$, but $\varphi^+_{[1],[1]} = (1,2)_2$.

4.2.8. Remark. The sequence $\tau = (id_{[n]})_{n \geq 1}$ is the only one such that $\mathcal{F}_\tau$ is a strict automorphism of $\mathcal{FSET}_{sk}$. In other words, the identity is the unique strict automorphism. Indeed, let $\mathcal{F}_\tau$ be strict. In particular, all isomorphisms $\varphi(\tau)^+_{[1],[n]}$ are identities and hence,

$$\tau_{n+1} = (id_{[1]} + \tau_n)^{-1} \varphi(\tau)^+_{[1],[n]} = (id_{[1]} + \tau_n)^{-1}$$

for each $n \geq 1$. By induction, it follows that $\tau_n$ is the identity for each $n \geq 1$. Therefore all the permutations $\varphi(\tau)^+_{[m],[n]}$ are actually identities.

As said before, we are actually interested in the groupoid (in fact, 2-groupoid) $\text{Aut}(\mathcal{FSET}_{sk})$. Until now, we have described its objects. It remains now to find all monoidal natural isomorphisms between two arbitrary
Extending the additive automorphisms to automorphisms of the symmetric 2-rig.

4.3.□ has the identity as unique automorphism.

Corollary 4.2.10. C and (B1), and this is an easy check left to the reader □ whose solution is (16). It remains to see that this family of permutations really satisfies (B2) for any components satisfy the recursive relation

\[ FS_n \]

\[ \hat{\phi} \]

natural isomorphism 

Notice that we have omitted the additional monoidality condition

\[ \varepsilon(\hat{\tau})^+ \xi_{[0]} = \varepsilon(\tau)^+ \]

on \( \xi \) because it holds automatically (all maps in the equation are identities).

4.2.9. Proposition. For every automorphisms \( \mathcal{F}_s, \mathcal{F}_t \) of \( \overline{\text{Set}}_{sk}^+ \) there exists one and only one +-monoidal natural isomorphism \( \xi : \mathcal{F}_s \Rightarrow \mathcal{F}_t \). Its components are given by

\[ \xi_{([n])} = \hat{\tau}_n^{-1} \tau_n, \quad n \geq 1. \]

Proof. Because of (B2) with \( m = 1 \), if there exists a monoidal natural transformation \( \xi : \mathcal{F} \Rightarrow \mathcal{F}_t \) its components satisfy the recursive relation

\[ \xi_{([n+1])} = (\varphi(\hat{\tau})_{[1],[n]}^+)^{-1}(\varepsilon_{[1]} + \xi_{[n]}^+ \varphi(\tau)_{[1],[n]}^+) \quad n \geq 1, \]

whose solution is (16). It remains to see that this family of permutations really satisfies (B2) for any \( m, n \geq 1 \) and (B1), and this is an easy check left to the reader □

4.2.10. Corollary. The underlying groupoid of the strict 2-group \( Aut(\overline{\text{Set}}_{sk}^+) \) is trivial up to equivalence.

Proof. It follows from Proposition 4.2.9 that all objects in \( Aut(\overline{\text{Set}}_{sk}^+) \) are isomorphic, and that each object has the identity as unique automorphism. □

4.3. Extending the additive automorphisms to automorphisms of the symmetric 2-rig. Let \( \mathcal{F}_s \) be the automorphism of \( \overline{\text{Set}}_{sk}^+ \) defined by \( \tau \in \prod_{n \geq 1} S_n \). We want to determine if this automorphism extends to an automorphism of the symmetric 2-rig \( \overline{\text{Set}}_{sk}^+ \), and when this is so, if it can be done in more than one way.

According to Definition 3.5.1 and being \( \overline{\text{Set}}_{sk}^+ \) left semistrict, extending \( \mathcal{F}_s \) to an automorphism of \( \overline{\text{Set}}_{sk}^+ \) means finding a pair \( (\varphi, \varepsilon) \), with \( \varphi : [\varphi_{[m],[n]} : [mn] \rightarrow [mn]]_{m,n \geq 0}, \) and \( \varepsilon : [1] \rightarrow [1] \), such that:

(C1a) \( \varphi_{[1],[n]} = \varepsilon \cdot id_{[n]} \) and \( \varphi_{[n],[1]} = id_{[n]} \cdot \varepsilon \) for each \( n \geq 0 \);

(C1b) for each \( m, n, p \geq 0 \) the following diagram commutes

\[ \begin{array}{ccc} [mnp] & \xrightarrow{\varphi_{[mnp],[n]}} & [mnp] \\
\varphi_{[mnp],[p]} & \downarrow & \varphi_{[mnp],[p]} \\
[mnp] & \xrightarrow{id_{[n]} \cdot \varphi_{[mnp],[p]}} & [mnp]; \end{array} \]
(C2) for each permutations \( \rho \in S_n \) and \( \sigma \in S_m \), and each \( m, n \geq 0 \) the following diagram commutes

\[
\begin{array}{ccc}
[mm] & \xrightarrow{\psi_{mn}[\rho]} & [mm] \\
F(_\rho \cdot \sigma) & \| & F(_\rho \cdot \sigma) \\
[mm] & \xrightarrow{\psi_{mn}[\sigma]} & [mm]
\end{array}
\]

where \( F_k : S_k \to S_k \) for each \( k \geq 2 \) is conjugation by the element \( \tau_k \in S_k \) in \( \{1\} \);

(C3) for each \( m, n \geq 0 \) the following diagram commutes

\[
\begin{array}{ccc}
[mm] & \xrightarrow{\psi_{mn}[\rho]} & [mm] \\
F(_\rho \cdot \sigma) & \| & F(_\rho \cdot \sigma) \\
[mm] & \xrightarrow{\psi_{mn}[\sigma]} & [mm]
\end{array}
\]

(C4a) for each \( m, n, p \geq 0 \) the following diagram commutes

\[
\begin{array}{ccc}
[mm + mp] & \xrightarrow{\psi_{mn}[\rho]} & [mm + mp] \\
\varphi_{mn}[\rho \cdot \sigma] & \| & \varphi_{mn}[\rho \cdot \sigma] \\
[mm + mp] & \xrightarrow{\varphi_{mn}[\sigma]} & [mm + mp]
\end{array}
\]

(C4b) for each \( m, n, p \geq 0 \) the following diagram commutes

\[
\begin{array}{ccc}
[mp + np] & \xrightarrow{\psi_{mn}[\rho]} & [mp + np] \\
\varphi_{mn}[\rho \cdot \sigma] & \| & \varphi_{mn}[\rho \cdot \sigma] \\
[mp + np] & \xrightarrow{\varphi_{mn}[\sigma]} & [mp + np]
\end{array}
\]

Conditions (C1a)-(C1b) correspond to the coherence axioms of the multiplicative monoidal structure \((\phi, \varepsilon)\), (C2) is the naturality of \( \varphi_{[m,n]} \) in \([m],[n]\), (C3) is the symmetry condition, and (C4a)-(C4b) correspond to the compatibility between the additive and the multiplicative monoidal structures.

In fact, \( \varepsilon \) is necessarily the identity of \([1]\). Hence (C1a) just says that both \( \varphi_{[1],[n]} \) and \( \varphi_{[n],[1]} \) are the identity of \([n]\) for each \( n \geq 0 \). Similarly, \( \varphi_{[0],[n]} : [0] \to [0] \) are also identities for each \( n \geq 1 \). It follows that we are looking for a family of permutations \( \{ \varphi_{[m],[n]} \}_{m,n \geq 2} \) satisfying the infinitely many equations (C1b)-(C4b) for the given family of permutations \( \{ \varphi(\tau)_{mn} \}_{m,n \geq 1} \) or equivalently, the associated sequence \( \tau = (\tau_n)_{n \geq 1} \) given by \( \{1\} \).

The first observation is that, when it exists, the solution is unique. This is a consequence of (C4a). More precisely, we have the following.

4.3.1. LEMMA. The system of equations (C4a) has one and only one solution. It is given by the permutations

\[
\varphi(\tau)_{mn} = \tau_{mn}^{-1}(\tau_m \cdot \tau_n), \quad m, n \geq 2.
\]

Moreover, this solution automatically satisfies (C4b).
Proof. An easy computation shows that the permutations \([\mathcal{I}]\) indeed satisfy (C4a) for every sequence \(\tau\).

Thus for each \(m, n, p \geq 1\) we have

\[
(\varphi_{[m],n})^+ + \varphi((\tau)_{[m],p}) (id_{[m]} \cdot \varphi((\tau)_{[n],p})) = [\tau_{mn}^{-1} (\tau_m \cdot \tau_n + \tau_{mp}^{-1} (\tau_m \cdot \tau_p)] [id_{[m]} \cdot ((\tau_n + \tau_p)^{-1} \tau_{n+p})]
\]

and

\[
(\varphi_{[m],n})^+ + \varphi((\tau)_{[m],p}) (id_{[m]} \cdot \varphi((\tau)_{[n],p})) = (\tau_{mn}^{-1} + \tau_{mp}^{-1}) (\tau_m \cdot \tau_n + \tau_{mp}) (id_{[m]} \cdot (\tau_n + \tau_{mp})^{-1} (id_{[m]} \cdot \tau_{n+p})
\]

where the second equality we used again the triviality of the left distributors. Then an easy induction on \(n \geq 1\).

Thus for each \(P\) and \(\tau\) and \(\cdot\), we get

\[
\varphi_{[m],n}^+ (\tau_{mn}) (id_{[m]} \cdot \varphi((\tau)_{[n],p})) = \varphi((\tau)_{[m],p}) (id_{[m]} \cdot \tau_{n+p})
\]

The second, fourth and sixth equalities follow by functoriality of \(+\) and \(\cdot\), and in the third and fifth ones we have used that \(F \varphi\) is left semistict, in particular, that the left distributors are trivial (in fact, in the fifth equality we also make use of the functoriality of \(+\)).

To prove that \([17]\) is the only solution to (C4a), notice that (C4a) with \(p = 1\) implies that for any solution the family \(\varphi_{[m],n}[1\leq n\leq 1]\) for any given \(m \geq 1\) can be recursively computed using

\[
\varphi_{[m],n+1} = \varphi((\tau)_{[m],n})^{-1} (\varphi_{[m],n} + id_{[m]}) (id_{[m]} \cdot \varphi((\tau)_{[n],1}))
\]

where in the second equality we have used that \(\varphi(\tau)^+\) satisfies (A1) and Lemma 4.2.4 and in the third equality we used again the triviality of the left distributors. Then an easy induction on \(n \geq 1\) shows that the solution to the recursive relation \([18]\) for the given \(m \geq 1\) is indeed \([17]\). Thus let us assume that for some \(n \geq 1\) the permutation \(\varphi_{[m],n}^+\) is given by \([17]\). Then it follows from \([18]\) that

\[
\varphi_{[m],n+1} = \tau_{mn}^{-1} (\tau_m + \tau_m) (id_{[m]} \cdot (id_{[m]} \cdot \tau_{n+1}))
\]

Using now the triviality of both the left distributors and the multiplicative unitors (together with the functoriality of \(+\)) we have

\[
(\tau_m \cdot id_{[n]}) + \tau_m = (\tau_m \cdot (id_{[1]} + \cdots + id_{[1]})) + \tau_m
\]

Therefore

\[
\varphi_{[m],n+1} = \tau_{mn}^{-1} (\tau_m \cdot id_{[n+1]})
\]

as claimed.

Finally, in order to see that this solution automatically satisfies (C4b), it is enough to replace the components of \(\varphi(\tau)^+\) and \(\varphi(\tau)^-\) by their respective expressions in terms of \(\tau\). After simplifying, (C4b) turns out to be equivalent to the condition that

\[
d'_{[m],n,p} (\tau_{mn} + \tau_p) = (\tau_m + \tau_n) d'_{[m],n,p} ((\tau_m + \tau_n)^{-1} \cdot id_{[p]}) (\tau_{mn} \cdot id_{[p]})
\]

where

\[
d'_{[m],n,p} (\tau_{mn} + \tau_n) = d'_{[m],n,p} ((\tau_m + \tau_n)^{-1} \cdot id_{[p]}) (\tau_{mn} \cdot id_{[p]})
\]

\[
d'_{[m],n,p} (\tau_{mn} \cdot \tau_p)
\]
where in the second equality we used the naturality of $d'_{[m],n,[p]}$.

4.3.2. **Corollary.** If an automorphism $F_{\tau}$ of $\tilde{\mathbf{FS}et}_{sk}^{+}$ extends to an automorphism $\mathcal{F}_{\tau}$ of $\tilde{\mathcal{F}set}_{sk}$, the extension is unique.

**Proof.** Every extension of $\mathcal{F}_{\tau}$ is in particular a solution to (C4a) and hence, it is unique.

A sequence $\tau = (\tau_n)_{n \geq 1}$ will be called admissible when the corresponding automorphism $\mathcal{F}_{\tau}$ of $\tilde{\mathbf{FS}et}_{sk}^{+}$ extends to an automorphism $\mathcal{F}_{\tau}$ of $\tilde{\mathcal{F}set}_{sk}$. Of course, the sequence given by $\tau_n = id_{[p]}$ for each $n \geq 1$ is admissible. Next example gives a non-trivial admissible sequence.

4.3.3. **Example.** Let be $\tau_n$ any square root of $id_{[p]}$ if $n$ is prime, and the identity otherwise. Then the sequence $\tau = (\tau_n)_{n \geq 1}$ is admissible. Indeed, (C1b) automatically holds when some of the integers $m,n,p$ is either 0 or 1 (in the second case, because $[1]$ is a strict unit). Otherwise, the products $mn$ are identities in these cases. If $\tau$ is equal to 0 or 1 because the corresponding commutators, as well as the monoidality isomorphisms are identities in these cases. If $m,n \geq 2$, $F(\tau)_{mn}$ is again the identity, and the commutativity of the diagram is nothing but the naturality of $\epsilon'_{[m],[n]}$. Finally, when proving Lemma 4.3.1 we already checked that both (C4a) and (C4b) hold for every sequence $\tau$ when the monoidality isomorphisms $\phi'_{[m],[n]}$ are given by (17).

Although non-trivial admissible sequences exist, as this example shows, not every sequence $\tau$ is admissible. In fact, even if the components of $\varphi$ are given by (17) in terms of $\tau$, (C1b)-(C3) need not hold unless $\tau$ satisfies some additional conditions. It turns out that the required additional conditions are more easily expressed in terms of the square permutations $\Phi(\tau)_{m,n} \in S_{mn}$ defined by

$$\Phi(\tau)_{m,n} = [(\tau_m \cdot \tau_n)^{-1}]^2, \quad m,n \geq 1.$$  

In terms of these permutations (C1b), (C2) and (C3) respectively amount to the following more transparent conditions.

4.3.4. **Lemma.** Let the components of $\varphi$ be given by (17) in terms of $\tau$. Then (C1b) holds if and only if

$$\Phi(\tau)_{m,n} \cdot id_{[p]} \cdot \Phi(\tau)_{mn,p} = (id_{[m]} \cdot \Phi(\tau)_{n,p}) \Phi(\tau)_{m,n,p}$$

for each $m,n,p \geq 2$.

**Proof.** In terms of $\tau$, (C1b) requires that

$$((\tau_m^{-1} \cdot id_{[p]}) (\tau_m \cdot \tau_n \cdot id_{[p]}) \tau_{mn,p}^{-1} (\tau_{mn} \cdot \tau_p) = (id_{[m]} \cdot \tau_{np}^{-1}) (id_{[m]} \cdot \tau_n \cdot \tau_p) \tau_{mn,p}^{-1} (\tau_m \cdot \tau_{np})$$

for each $m,n,p \geq 2$ (the condition when some of the integers $m,n,p$ is either 0 or 1 holds for any sequence $\tau$ because $\tau_1 = id_{[1]}$ and $[1]$ is a strict unit). To see that this is equivalent to (20), we just need to do the following:

(i) in the left hand side, insert the identity between $\tau_m \cdot \tau_n \cdot id_{[p]}$ and $\tau_{mn,p}^{-1}$ but written as

$$id_{[mp]} = (\tau_{mn}^{-1} \cdot id_{[p]}) (\tau_{mn} \cdot id_{[p]}) (\tau_{mn} \cdot \tau_p)^{-1} (\tau_{mn} \cdot \tau_p);$$
(ii) on the right hand side, insert the identity between \( id_{[m]} \cdot \tau_n \cdot \tau_p \) and \( \tau_m^{-1} \) but written as
\[
\Phi = (id_{[m]} \cdot \tau_n^{-1}) (id_{[m]} \cdot \tau_p) (\tau_m \cdot \tau_{np})^{-1} (\tau_m \cdot \tau_p);
\]

(iii) compose both the left and right hand sides of the equation, on the left with
\[
\tau_m \cdot \tau_n^{-1} \cdot \tau_p = (id_{[m]} \cdot \tau_n \cdot \tau_p) (\tau_m \cdot \tau_n \cdot id_{[p]}) = (\tau_m \cdot \tau_n \cdot id_{[p]}) (id_{[m]} \cdot \tau_n \cdot \tau_p),
\]
and on the right with \( \tau_m^{-1} \).

Then using the functoriality of the multiplication the left hand side gives
\[
(id_{[m]} \cdot \tau_n \cdot \tau_p) (\Phi(\tau)_{m,n} \cdot id_{[p]}) (\tau_m \cdot \tau_{np})^{-1} (\Phi(\tau)_{mn,p}) = (id_{[m]} \cdot \tau_n \cdot \tau_p) (\Phi(\tau)_{m,n} \cdot id_{[p]}) (\tau_m \cdot \tau_{np})^{-1} (\Phi(\tau)_{mn,p})
\]
and the right hand side
\[
(\tau_m \cdot \tau_n^{-1} \cdot \tau_p) (id_{[m]} \cdot \Phi(\tau)_{n,p}) (id_{[m]} \cdot \tau_p) (\Phi(\tau)_{mn,p}) \cdot (id_{[m]} \cdot \tau_n \cdot id_{[p]}) (id_{[m]} \cdot \Phi(\tau)_{n,p}) (\Phi(\tau)_{mn,p});
\]
Then (20) follows because both sides are still the same permutation. \( \square \)

4.3.5. Lemma. Let the components of \( \varphi \) be given by [17] in terms of \( \tau \). Then (C2) holds if and only if
\[
(\Phi(\tau)_{m,n} \in C_{S_{mn}}(S_m \cdot S_n))
\]
for each \( m, n \geq 2 \).

Proof. In terms of \( \tau \), (C2) says that
\[
((\tau_m^{-1} \cdot \tau_m) \cdot (\tau_n^{-1} \cdot \tau_n)) \tau_{mn}^{-1} (\tau_m \cdot \tau_n) = (\tau_m^{-1} \cdot \tau_m) \tau_{mn}^{-1} (\tau_n \cdot \tau_n), \quad m, n \geq 2
\]
for every permutations \( \rho \in S_m \) and \( \sigma \in S_n \). Notice that the cases \( m = 1 \) and \( n = 1 \) hold for any sequence \( \tau \) because \( \tau_1 = id_{[1]} \) and \( [1] \) is a strict unit object (the cases \( m = 0 \) or \( n = 0 \) are even more obvious). Then an easy computation using the functoriality of the multiplication shows that the previous condition is equivalent to
\[
\Phi(\tau)_{m,n} (\rho \cdot \sigma) = (\rho \cdot \sigma) \Phi(\tau)_{m,n}
\]
and hence, to the condition \( \Phi(\tau)_{m,n} \in C_{S_{mn}}(S_m \cdot S_n) \) for each \( m, n \geq 2 \). \( \square \)

4.3.6. Lemma. Let the components of \( \varphi \) be given by [17] in terms of \( \tau \). Then (C3) holds if and only if
\[
(\Phi(\tau)_{m,n} \cdot c'_{[m],[n]} = c'_{[m],[n]} \Phi(\tau)_{m,n})
\]
for each \( m, n \geq 2 \).

Proof. In terms of \( \tau \), (C3) says that
\[
\tau_{mn}^{-1} (\tau_n \cdot \tau_m) \tau_{mn}^{-1} c'_{[m],[n]} \cdot \tau_{mn} \cdot \tau_n = c'_{[m],[n]} \tau_{mn}^{-1} (\tau_m \cdot \tau_n), \quad m, n \geq 2.
\]
As before, the case \( m = 1 \) or \( n = 1 \) hold for any \( \tau \) because the commutators \( c'_{[1],[n]} \), \( c'_{[m],[1]} \) are identities (cf. Lemma 4.1.1.3). Composing both sides of this equality with \( \tau_n \cdot \tau_m \) on the left, and using the naturality of \( c'_{[m],[n]} \), which in particular means that \( (\tau_n \cdot \tau_m) c'_{[m],[n]} = c'_{[m],[n]} (\tau_m \cdot \tau_n) \), we obtain (22). \( \square \)
4.3.7. Corollary. A sequence \( \tau \) is admissible if and only if the associated family of square permutations \( \Phi(\tau) = (\Phi(\tau_{m,n}))_{m,n \geq 1} \) satisfies conditions (20) to (22).

Proof. It is a consequence of the previous three lemma together with Lemma 4.3.1. \( \Box \)

Let us provisionally forget that these permutations have the form (19) for some sequence \( \tau \), and let us just look for the families of permutations \( \Phi = (\Phi_{m,n} : [mn] \rightarrow [mn])_{m,n \geq 2} \) such that for each \( m, n \geq 2 \):

\begin{enumerate}
  \item[(E1)] \( (\Phi_{m,n} : id_{[m]}(n)) \Phi_{m,n,p} = (id_{[m]} \cdot \Phi_{n,p}) \Phi_{m,n,p} \).
  \item[(E2)] \( \Phi_{m,n} \in C_{S_m}(S_n \cdot S_n) \).
  \item[(E3)] \( \Phi_{m,n} c_{[m],[n]} = c'_{[m],[n]} \Phi_{m,n} \).
\end{enumerate}

These families may be grouped into two types, depending on which permutation is identical to \( \Phi \): Either \( \Phi = id_{[2]} \) or \( \Phi = id_{[2]} \cdot (23) \).

The amazing point is that there is exactly one family \( \Phi \) of each type, the trivial one when \( \Phi = id_{[2]} \), and another non-trivial with \( \Phi = id_{[2]} \cdot (23) \).

\( \Box \)

4.3.8. Lemma. Let \( \Phi = (\Phi_{m,n} : [mn] \rightarrow [mn])_{m,n \geq 2} \) be a family of permutations satisfying (E1)-(E3) with \( \Phi_{2,2} = id_{[2]} \). Then \( \Phi_{m,n} = id_{[mn]} \) for every \( m, n \geq 2 \).

Proof. It follows from (E2) and Proposition 2.3.2 that \( \Phi_{m,n} \) is the identity when \( m, n \geq 3 \). Therefore we only need to prove the cases \( m = 2 \) or \( n = 2 \). In fact, (E3) readily implies that

\[ \Phi_{m,n} = id_{[2n]} \leftrightarrow \Phi_{n,n} = id_{[2n]} \]

for each \( n \geq 2 \). Hence it is enough to prove that one of the permutations is the identity.

Let us first see that \( \Phi_{2,2} = id_{[2]} \) for each \( k \geq 1 \). This is shown by induction on \( k \). The case \( k = 1 \) holds by hypothesis. Suppose now that \( \Phi_{2,2} = id_{[2]} \) for some \( k \geq 1 \). Thus we also have \( \Phi_{2,2} = id_{[2]} \). Then taking \( m = p = 2 \) and \( n = 2^k \) in (E1) gives that

\[ \Phi_{2,2} \cdot id_{[2]} = (id_{[2]} \cdot \Phi_{2,2}) \Phi_{2,2} \cdot id_{[2]} \]

and hence, by the induction hypothesis we conclude that \( \Phi_{2,2} = id_{[2]} \) and \( \Phi_{2,2} = id_{[2]} \) are the same permutation of \( S_{2^k+1} \). However, by (E2) the first of these permutations belongs to \( C_{S_{2^k+1}}(S_{2^k+1} \cdot S_2) \) while the second one must belong to \( C_{S_{2^k+1}}(S_2 \cdot S_{2^k+1}) \). Since \( 2^{k+1} > 2 \), it follows from Corollary 2.3.4 that both permutations are the identity of \( [2^{k+1}] \).

Let us now prove that \( \Phi_{m,n} = id_{[2n]} \) for any \( m \geq 3 \), wether a power of 2 or not. Indeed, it is enough to take in (E1) the values \( n = 2 \) and \( p = 2^k \) for some \( k \geq 2 \). We obtain that

\[ \Phi_{m,n} = id_{[2n]} \]

Since \( m \geq 3 \) and \( 2^k \geq 4 \), both \( \Phi_{2m,2n} \) and \( \Phi_{2m,2n} \) are identities, and we have also proved that \( \Phi_{2m,2n} \) is an identity. Hence \( \Phi_{m,n} = id_{[2n]} \) is the identity of \( [m2^{k+1}] \). Then it follows from item (7) in Proposition 2.2.3 that \( \Phi_{m,n} \) is the identity of \( [2m] \).

\( \Box \)

4.3.9. Lemma. Let \( \Phi = (\Phi_{m,n} : [mn] \rightarrow [mn])_{m,n \geq 2} \) be a family of permutations satisfying (E1)-(E3) with \( \Phi_{2,2} = (1,4)_{(2)}(3,4)_{(2)} \). Then \( \Phi_{m,n} = id_{[mn]} \) for every \( m, n \geq 3 \), and

\begin{align*}
  (23) & \quad \Phi_{2,n} = (1,2)_{2n}(3,4)_{2n} \cdots (2n-1,2n)_{2n}, \\
  (24) & \quad \Phi_{n,2} = (1, n + 1)_{2n}(2, n + 2)_{2n} \cdots (n, 2n)_{2n}
\end{align*}

for each \( n \geq 3 \).

Proof. As before, we have that \( \Phi_{m,n} \) is the identity when \( m, n \geq 3 \) because of (E2) and Proposition 2.3.2. Moreover, using (E3) and the expression of the commutators \( c'_{[2i],[n]} \) and \( c'_{[n],[2]} \) given in Lemma 4.1.3, it is immediate to check that (23) holds if and only if (24) holds. Hence it is enough to prove for a given \( n \geq 3 \) either (23) or (24).
As before, let us first prove that (23) or equivalently, (24) hold when $n$ is $2^k$ for some $k \geq 2$. This is shown by induction on $k$. To prove the case $k = 2$, let us consider (E1) with $m = n = p = 2$. This gives that

$$(\Phi_{2,2} \cdot id_{[2]} ) \Phi_{4,2} = (id_{[2]} \cdot \Phi_{2,2} ) \Phi_{2,4}.$$  

Substituting (1, 4), (2, 3) for $\Phi_{2,2}$ in this equation, and computing we obtain

$$\Phi_{4,2} = (1, 6)_8(2, 5)_8(3, 8)_8(4, 7)_8 \Phi_{2,4}.$$ 

Now, we know from Lemma 2.3.3 that $\Phi_{2,4}$ belongs to the subgroup $\langle (1, 2)_8, (3, 4)_8, (5, 6)_8, (7, 8)_8 \rangle$, and $\Phi_{4,2}$ to $\langle (1, 5)_8, (2, 6)_8, (3, 7)_8, (4, 8)_8 \rangle$. The point is that if some of the transpositions $(1, 2)_8, (3, 4)_8, (5, 6)_8$ or $(7, 8)_8$ does not appear in $\Phi_{2,4}$ then this equation does not hold. For instance, if $(1, 2)_8$ is not present in $\Phi_{2,4}$ then 1 remains fixed by $\Phi_{2,4}$ and hence, the permutation in the right hand side maps 1 to 6. However, $\Phi_{4,2}$ necessarily leaves 1 fixed or maps it to 5. Similarly, one checks that $(3, 4)_8, (5, 6)_8, (7, 8)_8$ must also be present in $\Phi_{2,4}$, so that $\Phi_{2,4}$ is indeed given by (23) with $n = 4$, and in this case $\Phi_{4,2}$ is given by (24) with $n = 4$. Let us now assume that (23) and consequently, also (24) hold when $n = 2^k$ for some $k \geq 2$. (E1) with $m = p = 2$ and $n = 2^k$ says that

$$(\Phi_{2,2^k} \cdot id_{[2^k]} ) \Phi_{2^k+1,2} = (id_{[2^k]} \cdot \Phi_{2,2^k} ) \Phi_{2^k+1,2},$$

and hence

$$\Phi_{2^k+1,2} = (\Phi_{2,2^k} \cdot id_{[2^k]} ) (id_{[2^k]} \cdot \Phi_{2,2^k} ) \Phi_{2,2^k+1}$$

because $\Phi_{2,2^k}$ is inverse of itself. Since $\Phi_{2,2^k}$ belongs to $\langle (1, 2)_{2^k}, (3, 4)_{2^k}, \ldots, (2^{k+1} - 1, 2^{k+1})_{2^k} \rangle$ and $\Phi_{2^k+1,2}$ to $\langle (1, 2^{k+1} + 1)_{2^k}, (2, 2^{k+1} + 2)_{2^k}, \ldots, (2^{k+1}, 2^{k+2})_{2^k} \rangle$ (Lemma 2.3.3), it follows from this equality that $\Phi_{2^k+1,2}$ is necessarily given by (23) (and $\Phi_{2^k+1,2}$ by (24)) with $n = 2^{k+1}$. For instance, let us suppose that for some odd $i$ with $1 \leq i \leq 2^{k+1} - 1$ the transposition $(i, i + 1)_{2^{k+1}}$ does not appear in the cycle decomposition of $\Phi_{2^k+1,2}$. Then the permutation in the left hand side will leave $i$ fixed (when $(i, 2^{k+1} + i)_{2^{k+1}}$ is present in $\Phi_{2^k+1,2}$) or map it to $2^{k+1} + i$ (when it is not). However, $i$ will remain fixed by $\Phi_{2,2^k+1}$ and hence, the permutation in the right hand side will map it to (cf. Example 2.2.2)

$$(\Phi_{2,2^k} \cdot id_{[2^k]} ) (id_{[2^k]} \cdot \Phi_{2,2^k} ) (i) = (\Phi_{2,2^k} \cdot id_{[2^k]} ) (2\Phi_{2^k,2}(i + 1)/2 - 1)
= (\Phi_{2,2^k} \cdot id_{[2^k]} ) (2(2^k + (i + 1)/2 - 1)
= (\Phi_{2,2^k} \cdot id_{[2^k]} ) (2^{k+1} + i)
= 2^{k+1} + \Phi_{2,2^k}(i)
= 2^{k+1} + i + 1,$$

where in the last equality we have used the hypothesis that $\Phi_{2,2^k}$ contains the transposition $(i, i + 1)_{2^{k+1}}$ for every odd $i \leq 2^{k+1} - 1$. Since $2^{k+1} + i + 1$ is equal neither to $i$ nor to $2^{k+1} + i$ we conclude that $\Phi_{2^k+1,2}$ necessarily contains the transposition $(i, i + 1)_{2^{k+1}}$ for every odd $i \leq 2^{k+1} - 1$. A similar argument shows that it also contains this transposition for every odd $i$ with $2^{k+1} + 1 \leq i \leq 2^{k+2} - 1$. It remains to prove that (23) or (24) is also true when $n \geq 3$ is not a power of 2. Let us prove (24). To prove this, let us consider again (E1) but with $m = n = 2$ and $p = 2^k$ for some $k \geq 2$. This gives that

$$\Phi_{n,2} \cdot id_{[2^k]} = id_{[p]} \cdot \Phi_{2,2^k},$$

where, as already shown, $\Phi_{2,2^k}$ is given by (23). An argument completely similar to the previous ones shows now that this equation holds only if $\Phi_{n,2}$ is given by (24). For instance, let us suppose that $(i, n + i)_{2^k}$ does not appear in $\Phi_{n,2}$ for some $i \in \{1, \ldots, n\}$, so that $\Phi_{n,2}$ leaves $i$ fixed. Then the permutation on the left hand side will also leave $i$ (and in fact, every element in $[n2^{k+1}]$ of the form $i + 2nt$ for $t = 0, 1, \ldots, 2^{k+1} - 1$) fixed. However, the permutation in the right hand side will map $i$ to $n + i$ because $\Phi_{2,2^k}$ maps 1 to 2. □

Coming back to our problem of identifying the admissible sequences $\tau$, we may now prove the following necessary and sufficient condition of admissibility.
4.3.10. Proposition. Let be \( \tau = (\tau_n)_{n \geq 1} \in \prod_{n \geq 1} S_n \). Then the corresponding automorphism \( F_\tau \) of \( \widehat{\text{Set}}_k \) extends to an automorphism \( F_\tau \) of \( \widehat{\text{Set}}_{sk} \) (i.e. \( \tau \) is admissible) if and only if \( \Phi(\tau)_{m,n} = id_{[mn]} \) for each \( m,n \geq 2 \), and in this case the extension is unique.

Proof. Let \( \tau \) be admissible. Then the associated family of permutations \( \Phi(\tau) \) given by (19) satisfies equations (E1)-(E3). Therefore it must be one of the two solutions to these equations described by Lemmas 4.3.8 and 4.3.9. Now, it follows from (19) that each component \( \Phi(\tau)_{m,n} \) is a perfect square in the corresponding symmetric group \( S_{mp} \). In particular, its cycle structure must have an even number of \( k \)-cycles for every even value of \( k \) (cf. §2.1), and this excludes the non-trivial solution, whose component \( \Phi_{3,2} \), for instance, is the product of three \( 2 \)-cycles. Hence \( \Phi(\tau)_{m,n} = id_{[mn]} \) for each \( m,n \geq 2 \). Conversely, let \( \tau \) be such that \( \Phi(\tau)_{m,n} = id_{[mn]} \) for each \( m,n \geq 2 \). Then the permutations \( \varphi_{[m],[n]} \) defined given by (17) define an extension of \( F_\tau \) to an automorphism \( F_\tau \) of \( \widehat{\text{Set}}_{sk} \). Indeed, they satisfy (C4a)-(C4b) because of Lemma 4.3.1 and (C1b)-(C3) because of Lemmas 4.3.4, 4.3.5 and 4.3.6. The last assertion about the uniqueness is also a consequence of Lemma 4.3.1.

4.4. Endomorphisms of the symmetric 2-rig of finite sets. We are now able to prove that the category (in fact, groupoid) of endomorphism of \( \widehat{\text{Set}}_k \) is equivalent to the terminal category. Since \( \widehat{\text{Set}}_k \) is equivalent to the left semistrict symmetric 2-rig \( \widehat{\text{Set}}_{sk} \), and equivalent objects in a 2-category have equivalent categories of endomorphisms, it is enough to prove that the category of endomorphisms of \( \widehat{\text{Set}}_{sk} \) is trivial. This is done in two steps. Firstly, we prove that every endomorphism of \( \widehat{\text{Set}}_{sk} \) is in fact an automorphism and secondly, that the category of automorphisms of \( \widehat{\text{Set}}_{sk} \) is trivial.

4.4.1. Lemma. Let \( \text{End}(\widehat{\text{Set}}_{sk}) \) be the category of endomorphisms of \( \widehat{\text{Set}}_{sk} \), and let \( \text{Aut}(\widehat{\text{Set}}_{sk}) \) be the full subcategory having as objects the automorphisms of \( \widehat{\text{Set}}_{sk} \). Then

\( \text{End}(\widehat{\text{Set}}_{sk}) = \text{Aut}(\widehat{\text{Set}}_{sk}) \).

Proof. The underlying groupoid being skeletal, every endomorphism \( F = (F, \varphi, id_{[1]} \) of \( \widehat{\text{Set}}_{sk} \) maps the zero and unit objects \([0],[1]\) to themselves. Moreover, \( F \) is \( * \)-monoidal and hence, it also maps the object \([n] = [1] + \cdots + [n] \) to itself for each \( n \geq 2 \). Finally, the same argument made to prove Proposition 4.2.1 proves now (cf. Remark 4.2.2) that \( F \) acts on morphisms by conjugation, so that it is fully faithful.

4.4.2. Proposition. \( \text{Aut}(\widehat{\text{Set}}_{sk}) \) is equivalent to the terminal category.

Proof. Let \( U : \text{Aut}(\widehat{\text{Set}}_{sk}) \rightarrow \text{Aut}(\widehat{\text{Set}}_{sk}) \) be the forgetful functor mapping every object \( F = (F_\tau, \varphi_\tau, id_{[1]} \) of \( \widehat{\text{Set}}_{sk} \) to \( \tau \) an admissible sequence, to the underlying symmetric \( * \)-monoidal automorphism \( F_\tau \), and being the identity on morphisms. Since the extension of a symmetric \( * \)-monoidal automorphism of \( \widehat{\text{Set}}_{sk} \) to an automorphism of \( \widehat{\text{Set}}_{sk} \), when it exists, is unique (cf. ...) \( U \) is injective on objects. We claim that it is a full embedding. Faithfulness is clear. To see that it is full, it is enough to prove that for every admissible sequences \( \tau, \tilde{\tau} \) the unique morphism \( \xi : F_\tau \Rightarrow F_\tilde{\tau} \) of Proposition 4.2.9 is in fact a rig transformation \( \xi : F_\tau \Rightarrow F_\tilde{\tau} \), i.e. \( \xi \) is also \( * \)-monoidal. More explicitly, this means proving that the components of \( \xi \), given by (16), are such that the diagrams

\[
\begin{array}{ccc}
[mn] & \xrightarrow{\xi_{[mn]}} & [mn] \\
\varphi(\tau)_{[m],[n]} & \downarrow & \varphi(\tilde{\tau})_{[m],[n]} \\
[mn] & \xrightarrow{\xi_{[mn]}} & [mn]
\end{array}
\]

commute for every \( m,n \geq 1 \). However, an easy computation shows that this is nothing but the condition

\( \Phi(\tau)_{m,n} = \Phi(\tilde{\tau})_{m,n} \),

which holds because both \( \tau \) and \( \tilde{\tau} \) are admissible, so that both sides of the equality are \( id_{[mn]} \) (cf. Proposition 4.3.10). The statement follows then from Corollary 4.2.10.

In summary, we may now conclude the following, which is the main result of the paper.
4.4.3. **Theorem.** The groupoid $\mathcal{E}nd(\hat{\mathbb{F}Set})$ is equivalent to the terminal category.

**Proof.** It follows from Lemma 4.4.1 and Proposition 4.4.2.

5. **Concluding remarks**

Our ultimate goal is to have a more handy description of a $\mathbb{F}Vect_k$-module category, and to develop the theory of these higher structures. In fact, $\mathbb{F}Vect_k$-module categories, or a particular type of them, are expected to be good categorical analogs of the vector spaces and hence, good candidates to represent (possibly non-finite) 2-groups.

As explained in the introduction, the symmetric 2-rig $\hat{\mathbb{F}Set}$ canonically embeds into $\mathbb{F}Vect_k$ and hence, also into $\mathbb{F}Vect_k$. It follows that part of the structure of a $\mathbb{F}Vect_k$-module category is canonically given if Baez’s conjecture is true. However, proving this conjecture is still work in progress.

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