Multi-pion correlations in high energy collisions

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Abstract

Any-order pion inclusive distribution for a chaotic source in high energy collisions are given which can be used in both theory and experiment to analyze any-order pion interferometry. Multi-pion correlations effects on two-pion and three-pion interferometry are discussed.

PACS number(s): 13.85 Hd, 05.30Jp, 12.40 Ee.

Hanbury-Brown and Twiss [1] were the first who applied the Bose-Einstein (BE) correlations to measure the size of distant stars. The method was first applied to particle physics by Goldhaber et al.(GGLP) [2] in 1959. Since then the size of the interaction region has been measured by numerous experiments in high energy collisions using different types of particles. Two-pion BE correlation is widely used in high energy collisions to provide the information of the space-time structure, degree of coherence and dynamics of the region where the pions were produced [3]. Ultrarelativistic hadronic collisions provide the environment for creating dozens of pions [4–7], therefore one must take into account the effects of multi-pion BE correlations in those processes. Because the bosonic nature of the pion should affect the single and $i$-pion spectra and distort the $i(i \geq 2)$-pion correlation function, thus, it is very interesting to analyze the multi-pion BE correlation effects on $i$-pion interferometry [8–15]. On the other hand, one also want to study higher-order pion interferometry directly to see what additional information can be extracted from higher-order pion interferometry. Now this aspect has aroused great interests among physicists [6,7,13,16–18]. Unfortunately all present analyses of multi-pion correlations are based on pure “multi-pion interferometry formula” without considering the higher-order pion correlation effects on the lower-order
Thus we urgently need new multi-pion interferometry formulas which can be used in both theory and experiment to analyze any-order pion interferometry. This is the main aim of this letter. In the letter, considering multi-pion BE correlations we derive new multi-pion correlations formulas which can be used to analyze any-order pion interferometry. Those new multi-pion correlations formulas are structurally similar to the previous formulas but with a modified source functions (See Eq. (21) for details). This warrants the validity of the formulas used in earlier studies of higher-order pion interferometry. Although we only study multi-pion BE correlations in this Letter, the results presented here are also held for kaon if the final state interactions are neglected.

The general definition of the "pure" \( n \) pion correlation function \( C_n(p_1, \ldots, p_n) \) is

\[
C_n(p_1, \ldots, p_n) = \frac{P_n(p_1, \ldots, p_n)}{\prod_{i=1}^{n} P_i(p_i)},
\]

where \( P_n(p_1, \ldots, p_n) \) is the probability of observing \( n \) pions with momenta \( \{p_i\} \) all in the same \( n \) pion event. The \( n \)-pion momentum probability distribution \( P_n(p_1, \ldots, p_n) \) can be expressed as

\[
P_n(p_1, \ldots, p_n) = \sum_{\sigma} \rho_{1, \sigma(1)} \rho_{2, \sigma(2)} \cdots \rho_{n, \sigma(n)},
\]

with

\[
\rho_{i,j} = \rho_i(p_i, p_j) = \int d^4x g_w(x, (p_i + p_j)/2) e^{i(p_i - p_j) \cdot x}.
\]

Here \( \sigma(i) \) denotes the \( i \)th element of a permutation of the sequence \( 1, 2, 3, \ldots, n \), and the sum over \( \sigma \) denotes the sum over all \( n! \) permutations of this sequence. \( g_w(Y, k) \) can be explained as the probability of finding a pion at point \( Y \) with momentum \( k \) which is defined as

\[
g_w(Y, k) = \int d^4y j^*(Y + y/2) j(Y - y/2) \exp(-ik \cdot y),
\]

with

\[
\int g_w(x, k) d^4x dk = n_0.
\]
Where \( j(x) \) is the current of the pion, \( n_0 \) is the mean pion multiplicity without BE correlation \([10,11]\). From Eq. (1) and Eq. (2), the pure \( n \)-pion correlation functions can be expressed as \([16]\)

\[
C_n(p_1, \ldots, p_n) = \sum \prod_{j=1}^{n} \rho_{j, \sigma_j} \rho_{j, j}.
\] (6)

The above correlation functions are widely used in both experiment and theory to analyze multi-pion interferometry. But in high energy experiment, the pion multiplicity is so large that we must take into account the multi-pion correlations effects on lower-order pion interferometry.

For \( n\pi \) events, considering the \( n \) pion correlations effect, the \( i \)-pion correlation function can be defined as \([11]\)

\[
C^n_i(p_1, \ldots, p_i) = \frac{P^n_i(p_1, \ldots, p_i)}{\prod_{j=1}^{i} P^n_j(p_j)},
\] (7)

where \( P^n_i(p_1, \ldots, p_i) \) is the the normalized modified \( i \)-pion inclusive distribution in \( n \) pion events which can be expressed as

\[
P^n_i(p_1, \ldots, p_i) = \frac{\int \prod_{j=i+1}^{n} dp_j P^n(p_1, \ldots, p_n)}{\int \prod_{j=1}^{n} dp_j P^n(p_1, \ldots, p_n)}.
\] (8)

Now we define the function \( G_i(p, q) \) as \([10,11]\)

\[
G_i(p, q) = \int \rho(p, p_1) dp_1 \rho(p_1, p_2) dp_2 \ldots \rho(p_{i-2}, p_{i-1}) dp_{i-1} \rho(p_{i-1}, q).
\] (9)

From the expression of \( P^n_i(p_1, \ldots, p_n) \) (Eq. (2)), the one-pion to three-pion inclusive distribution can be expressed as \([11]\)

\[
P^n_1(p) = \frac{1}{n} \frac{1}{\omega(n)} \sum_{i=1}^{n} G_i(p, p) \cdot \omega(n - i),
\] (10)

\[
P^n_2(p_1, p_2) = \frac{1}{n(n-1)} \frac{1}{\omega(n)} \sum_{i=2}^{n} [\sum_{m=1}^{i-1} G_m(p_1, p_1) \cdot G_{i-m}(p_2, p_2) \]
\[+ G_m(p_1, p_2) \cdot G_{i-m}(p_2, p_1)] \omega(n - i),
\] (11)
Here \( \omega \) is the pion multiplicity distribution probability.

Similar expression can be given for \( i (i \leq n) \) pion inclusive distribution. From the above method the \( i \)-pion correlation function can be calculated for \( n \) pion events. Experimentally, one usually mixes all events to analyze the two-pion and higher-order pion interferometry.

Then the \( i \) pion correlation function can be expressed as \([19,20]\)

\[
G_i^\phi (p_1, \ldots, p_i) = \frac{N_i(p_1, \ldots, p_i)}{\prod_{j=1}^{i} N_1(p_j)}. \tag{14}
\]

Here the \( i \)-pion inclusive distribution, \( N_i(p_1, \ldots, p_i) \), can be expressed as

\[
N_i(p_1, \ldots, p_i) = \frac{\sum_{n=i}^{\infty} \omega(n) \cdot n(n-1) \cdots (n-i+1) P^n_i(p_1, \ldots, p_i)}{\sum_n \omega(n)}, \tag{15}
\]

with

\[
\int N_i(p_1, \ldots, p_i) \prod_{j=1}^{i} dp_j = \langle n(n-1) \cdots (n-i+1) \rangle. \tag{16}
\]

Then the one-pion to three-pion inclusive distribution read:

\[
N_1(p) = \sum_{i=1}^{\infty} G_i(p, p), \tag{17}
\]

\[
N_2(p_1, p_2) = \sum_{i=1}^{\infty} G_i(p_1, p_1) \sum_{j=1}^{\infty} G_j(p_2, p_2) + \sum_{i=1}^{\infty} G_i(p_1, p_2) \sum_{j=1}^{\infty} G_j(p_2, p_1), \tag{18}
\]
\[ N_3(p_1, p_2, p_3) = \sum_{i=1}^{\infty} G_i(p_1, p_1) \sum_{j=1}^{\infty} G_j(p_2, p_2) \sum_{k=1}^{\infty} G_k(p_3, p_3) + \sum_{i=1}^{\infty} G_i(p_1, p_2) \sum_{j=1}^{\infty} G_j(p_2, p_1) \]

\[ + \sum_{k=1}^{\infty} G_k(p_3, p_3) + \sum_{i=1}^{\infty} G_i(p_3, p_3) \sum_{j=1}^{\infty} G_j(p_1, p_1) \sum_{k=1}^{\infty} G_k(p_2, p_2) + \sum_{i=1}^{\infty} G_i(p_2, p_3) \sum_{j=1}^{\infty} G_j(p_3, p_2) \sum_{k=1}^{\infty} G_k(p_1, p_1) + \sum_{i=1}^{\infty} G_i(p_2, p_3) \sum_{j=1}^{\infty} G_j(p_3, p_2) \sum_{k=1}^{\infty} G_k(p_1, p_1) + \sum_{i=1}^{\infty} G_i(p_1, p_2) \sum_{j=1}^{\infty} G_j(p_2, p_1) \sum_{k=1}^{\infty} G_k(p_3, p_3) + \sum_{i=1}^{\infty} G_i(p_1, p_3) \sum_{j=1}^{\infty} G_j(p_3, p_2) \sum_{k=1}^{\infty} G_k(p_1, p_1) + \sum_{i=1}^{\infty} G_i(p_1, p_3) \sum_{j=1}^{\infty} G_j(p_3, p_2) \sum_{k=1}^{\infty} G_k(p_1, p_1) \]

Similar expression for \( i(i > 3) \) pion inclusive distribution can be given. Following Ref. [14,13,21], we define the following function

\[ H_{ij} = H(p_i, p_j) = \sum_{k=1}^{\infty} G_k(p_i, p_j). \]  

Then the \( n \) pion inclusive distribution can be expressed as

\[ N_n(p_1, \cdots, p_n) = \sum_{\sigma} H_{1,\sigma(1)} H_{2,\sigma(2)} \cdots H_{n,\sigma(n)}. \]  

Here \( \sigma(i) \) denotes the \( i \)th element of a permutation of the sequence 1, 2, 3, \( \cdots, n \), and the sum over \( \sigma \) denotes the sum over all \( n! \) permutations of this sequence. One of the interesting things about Eq. (21) is that it is very similar to Eq. (2). The only difference is that Eq. (2) only contains the first term of \( H_{ij} (\rho(p_i, p_j) = G_1(p_i, p_j)) \). With the help of Eq. (21), the general form of \( i \)-pion correlation function (Eq. (14)) can be re-expressed as

\[ C_i^\phi(p_1, \cdots, p_i) = \sum_{\sigma} \prod_{j=1}^{i} \frac{H_{i,\sigma(j)}}{H_{j,j}}. \]  

In the derivation of Eq.(22), we mention nothing about the structure of the source, so our results are in principle independent on the detail form of the source, i.e., one can use any kind source function (which may contain resonance and flow) to study higher-order pion interferometry. Assumed that \( H_{ij} = |H_{ij}| \exp(i\phi_{ij}) \), it is clear that two-pion interferometry does not depend on phase \( \phi_{ij} \) while which exists in higher order pion interferometry. So higher-order interferometry can be used to extract the information of phase [17]. If the source distribution function is symmetric in the coordinate space we have \( \phi_{ij} = 0 \). Then multi-pion correlations contain the same information as two-pion interferometry does. In
the following, we will use a simple model to study the multi-pion correlations effects on two-pion and three-pion interferometry. Similar to ref. [9–11], we assume the source distribution function \( g(x, p) \) as
\[
g(x, p) = n_0 \cdot \left( \frac{1}{\pi R^2} \right)^{3/2} \exp(-r^2/R^2) \left( \frac{1}{\pi \Delta^2} \right)^{3/2} \exp(-p^2/\Delta^2).
\]
(23)

Here \( R \) and \( \Delta \) are parameters which represents the radius of the chaotic source and the momentum range of pions respectively. Using Eq.(9), Eq.(14) and Eq.(15), one can calculate any-order pion correlation function for any kind source distribution \[22\]. Cs"orgő and Zimáyi have found an analytically solution for above special source distribution \[14,15\] which are quoted here:
\[
G_n(p_1, p_2) = j_n \exp\left\{ -b_n^2 \left[ \left( \frac{\gamma_n}{2} + p_1 - \frac{\gamma_n}{2} - p_2 \right)^2 + \left( \frac{\gamma_n}{2} + p_2 - \frac{\gamma_n}{2} - p_1 \right)^2 \right] \right\}
\]
(24)
\[
j_n = n_0^2 \left( \frac{b_n}{\pi} \right)^{3/2}, \quad b_n = \frac{1}{\Delta^2} \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}
\]
(25)
with
\[
\gamma_\pm = \frac{1}{4} (x \pm 1)^2, \quad x = R\Delta.
\]
(26)

Using this solution, one can calculate any-order pion interferometry for above source distribution. The analytical solution has the advantage over the previous method \[10,11\] that it can be used in theory analyses.

The two-pion interferometry for different mean multiplicity \(< n >\) is shown in Fig.1. It is clear that multi-pion correlation make the radius to become smaller. For three-pion interferometry, we choose variable \( Q^2 = (p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 = q_{12}^2 + q_{23}^2 + q_{31}^2 \) and integrate the other eight variables, then we have three-pion correlation function \( C_3^\phi(Q) \) as
\[
C_3^\phi(Q) = \frac{\int N_3(p_1, p_2, p_3) \delta(Q^2 - \mathbf{q}_{12}^2 - \mathbf{q}_{23}^2 - \mathbf{q}_{31}^2) dp_1 dp_2 dp_3}{\int N_1(p_1) N_1(p_2) N_1(p_3) \delta(Q^2 - \mathbf{q}_{12}^2 - \mathbf{q}_{23}^2 - \mathbf{q}_{31}^2) dp_1 dp_2 dp_3}.
\]
(27)

Three-pion interferometry for different \( \langle n \rangle \) is shown in Fig.2. It is clear that as \( \langle n \rangle \) becomes larger, the deviations of three-pion correlation from ”pure three-pion” interferometry become larger.
In the following, we will discuss the effects of resonance, flow and energy conservation on the above calculations. The effects of resonance on pure two-pion interferometry is not a new subject which has been extensive studied by different authors [23]. Effects of the resonance on higher-order pion interferometry based on pure multi-pion interferometry formulas are recently studied by Csönögő in Ref. [16]. Because Eq.(21) is similar to Eq.(2), all analyses in Ref. [16] can be applied here. In principle, resonance should not affected the multi-pion correlation formulas presented here, but due to the limited momentum resolution of the data we will have a modified n-pion BE correlation functions as presented in Ref. [16]. The basic idea is that due to the limited resolution of the data, the contributions from the long lived resonance to the correlator are concentrated at lower relative momentum region which is not resoloved by two-pion interferometry. Thus the interference term $(G_1(p_i, p_j), i \neq j)$ now mainly contains the contributions from directly emitted pions and pions from short lived resonances. While the single particle spectrum $(G_1(p_i, p_i))$ is not affected by the two-particle momentum resolution, so the intercept of n-pion interferometry is smaller than the ideal value $n!$. Based on pure two-pion interferometry formula, one has found that flow make the apparent radius derived from two-pion interferometry to become smaller [24]. It is expected that due to the multi-pion BE correlation effects, the apparent radius derived from two-pion interferometry will become more smaller. Considering the effects of the energy constraint on the multi-pion interferometry, I have derived a new multi-pion inclusive distribution according to the method presented in Ref. [25]. This new formula is similar to Eq.(21). If we integrated the energy from zero to infinity we will have Eq. (21) again. According to the definition of Eq.(14), in the calculation of Eq.(22), we actually mix all events (with different multiplicities and different energies). It means that in the experiment, one has already integrated all the possible energy of the pions, so it is not necessary for us to consider energy constraint effects here.

Conclusion: In this letter, we have derived the $i$ pion inclusive distribution and $i$ pion correlation function for a chaotic source which can be used in experiment and theory to analyze multi-pion interferometry. For a simple model, multi-pion correlation effects on two-
pion and three-pion interferometry were discussed. It was shown that for larger mean pion multiplicity, the deviations of three-pion and two-pion correlation from the "pure" three-pion and two-pion correlation became larger. The effects of resonance, flow and energy constraint on the multi-pion correlation formulas were discussed.

Acknowledgement

The author would like to express his gratitude to Drs. U. Heinz, Y. Pang, H. Feldmeier, J. Knoll, D. Miśkowiec and V. Toneev for helpful discussions. This work was partly supported by the Alexander von Humboldt foundation in Germany.
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[20] There are two kinds of definition of \(i\)-pion correlation function [14,19], one is the definition given in Eq.(14), the another one is

\[
C_2 = \frac{P_{\text{nor}}(p_1, \ldots, p_i)}{\prod_{j=1}^{N} P_{\text{nor}}(p_j)}
\]

with

\[
P_{\text{nor}}(p_1, \ldots, p_i) = \frac{N(p_1, \ldots, p_i)}{<N_{N-i+1}>}.
\]

In Ref. [19], the author discussed those two definitions for two pion interferometry case and found that the first definition was better than the second one.

[21] This definition can be found in the preprint version of Ref. [10] and also can be found in Ref. [14,15] where the author defined it as \(G(i,j)\).

[22] The detail information about the calculation can be found in Ref. [10,11,14,15] which is originated from Pratt while derived in detail in Ref. [14]. On the other hand, Zajc gave Monte-Carlo methods [9] which enable us to study multi-pion correlation effects on lower-order pion interferometry. But to my best knowledge, the amount of calculational work for a real model will increase astronomically for all methods. So a method which enable us to calculate quickly multi-pion correlations is still a debate for physicist.

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**Figure Captions**

1. Multi-pion correlations effects on two-pion interferometry. The solid line corresponds to the result of pure two-pion interferometry. The dashed line and dotted line correspond to $<n> = 20, 146$ respectively. The input value of $R$ and $\Delta$ are $5\text{fm}$ and $0.25\text{GeV}$.

2. Multi-pion correlations effects on three-pion interferometry. The solid line corresponds to pure three-pion interferometry. The dashed line and dotted line correspond to $<n> = 11, 45$ respectively. The input value of $R$ and $\Delta$ are $3\text{fm}$ and $0.36\text{GeV}$.
Fig. 1 By Q.H. Zhang

\[ C_2(q) \]

- Two-pion
- \( \langle n \rangle = 20 \)
- \( \langle n \rangle = 146 \)
Fig. 2 By Q.H. Zhang