Hydrodynamic and hydrostatic modelling of hydraulic journal bearings considering small displacement condition

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Abstract. This paper proposes modified coefficients for the dynamic model of hydraulic journal bearing system that integrates the hydrodynamic and hydrostatic properties. In recent years, design of hydraulic bearing for machine tool attracts worldwide attention, because hydraulic bearings are able to provide higher capacity and accuracy with lower friction, compared to conventional bearing systems. In order to achieve active control of the flow pressure and enhance the operation accuracy, the dynamic model of hydraulic bearings need to be developed. Modified coefficients of hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping of the dynamic model are presented in this work, which are derived referring to small displacement analysis from literature. The proposed modified coefficients and model, which consider the pressure variations, relevant geometry size, and fluid properties of the journal bearings, are able to characterise the hydrodynamic and hydrostatic properties with better precision, thus offering the following pragmatic contribution: (1) on-line prediction of the eccentricity and the position of the shaft in the face of external force that results in vibration; (2) development of active control system to regulate the supply flow pressure and to minimize the eccentricity of the shaft. Theoretical derivation and simulation results with different vibration cases are discussed to verify the proposed techniques.

1. Introduction

Hydraulic journal bearing is powered by oil supply systems. High-pressured oil is pumped into the recess though a restrictor, and thus a layer of oil film is generated between the surface of the shaft and the bearing. In the operation of hydraulic journal bearings, the measurable external force is imposed to the shaft, which results in the shaft offset and meanwhile changes the oil film thickness and pressure distribution. Thus, in order to counteract the applied force and sustain the shaft balance, additional mechanical devices in connection with the recesses are developed to compensate for the pressure variation, such as capillary.

While discussing about the characteristics of hydraulic bearings, the hydrostatic and hydrodynamic properties should be taken into account. For example, P. B. Davies [1] analyzed the influences of the induced flow between adjacent recesses and the circumferential flow from other recesses on the hydrodynamic characteristics of pressure, velocity, and load capacity for multi-recess hydrostatic journal bearings. The research shows that the performance of the bearing was determined by the pressure ratio, rotational speed, and load angle. In literature, the hydrostatic and hydrodynamic properties are established in the representation of coefficients within the dynamic model; basically, the...
coefficients are relevant to the eccentricity of shaft, oil film thickness, supply pressure, fluid film lubrication effects, etc. G. H. Jang and Y. J. Kim [2] used the finite element method and perturbation equations to calculate the stiffness and damping coefficients for five degrees-of-freedom hydrodynamic bearings; the effects of eccentricity and misalignment are also investigated. Luis San Andres [3] integrated the principle of dynamic lubrication and fundamental equation of lubrication theory to derive the static and dynamic characteristics of short length cylindrical journal bearings. The stability analysis results are utilized to determine the stiffness coefficients of rotor-bearing system. In the aforementioned literature, these coefficients are derived with ideal assumption of small displacement, and the pressure variation effects are not discussed explicitly.

Therefore, in order to precisely determine the hydrostatic and hydrodynamic properties and establish the dynamic model of journal bearing, this paper discusses the modified coefficients of hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping, for flow simulation and control development, based on the results of [4, 5]. In Section 2, the linearized dynamic model of hydraulic journal bearing with ideal assumptions will be introduced [4, 5]. Then, the modified coefficients based on the dynamic model will be developed in Section 3, in order to meet the real operation conditions. Simulation studies of the proposed coefficients and the dynamic model will be presented in Section 4. Finally, the conclusion is given in Section 5.

Figure 1. Structure and geometry of the hydraulic journal bearing

Figure 2. Representation of parameter of the hydraulic journal bearing
Table 1. Notations and parameter settings for simulation

| Symbol | Description                  | Value            |
|--------|------------------------------|------------------|
| a      | axial flow land width        | 0.02 (m)        |
| b      | inter-recess land width      | 0.02 (m)        |
| \(h_{if}\) | film thickness at inter-recess land | N/A             |
| \(C_d\) | diametral clearance         | 60 (\(\mu\)m)  |
| \(C_{sq}\) | squeeze damping             | N/A             |
| D      | shaft diameter              | 0.28 (m)        |
| \(e\)  | eccentricity                | N/A             |
| \(\dot{e}\) | velocity of eccentricity   | N/A             |
| h      | film thickness              | N/A             |
| \(h_0\) | initial film thickness      | 30 (\(\mu\)m)  |
| L      | bearing length              | 0.14 (m)        |
| m      | shaft mass                  | 200 (kg)        |
| n      | number of recesses          | 4               |
| N      | rotational speed of shaft    | N/A             |
| \(p_i\) | pressure in recess \(i\), \(i=1, 2, 3, 4\) | 200 (MPa)       |
| \(P_{si}\) | supply pressure for recess \(i\) | 400 (MPa)       |
| x, y   | displacements               | N/A             |
| W      | applied load                | 1280 (N)        |
| \(\alpha\) | angle of eccentricity      | N/A             |
| \(\beta\) | concentric pressure ratio \(p/P_s\) | N/A             |
| \(\gamma\) | angle of velocity           | N/A             |
| \(\gamma_c\) | circumferential flow factor | N/A             |
| \(\eta\) | dynamic viscosity           | 32 (cP)         |
| \(\varphi\) | angle between \(W\) and the \(x\) coordinate | 40°            |
| \(\lambda_{hs}\) | hydrostatic stiffness      | N/A             |
| \(\lambda_{hd}\) | hydrodynamic stiffness      | N/A             |

2. Dynamic model of recessed journal bearings

This section introduces the dynamic model of recessed journal bearings, based upon the results of [4, 5]. Figure 1 illustrates the geometry, notation, and structure of the journal bearing under consideration, where \(\{P_{s1}, P_{s2}, P_{s3}, P_{s4}\}\) and \(\{p_1, p_2, p_3, p_4\}\) represent the input and output pressures of the four recesses, respectively. In addition, \(L\) is the bearing length, \(D\) is the shaft diameter, \(a\) is the axial flow land width, and \(h_0\) is the initial condition of oil film thickness. In Figure 2, the grey and yellow parts depict the shaft and oil film, respectively. Figure 2a displays the coordinate system of the cross section. Here, \(\alpha\) represents the angle between the \(x\) axis and the vector of eccentricity, \(e\). The angle between the velocity of eccentricity, \(\dot{e}\), and the \(x\) axis, is denoted as \(\gamma\); in addition, the external load, \(W\), acts at an angle \(\varphi\) with regard to the \(x\) axis. The notations and parameters set in this paper are summarized in Table 1.

2.1. Equation of motion

As illustrated in Figure 2b, the dynamics of oil film together with the hydraulic journal bearing system can be linearized and modelled as a mass-spring-damper system. Therefore, the equations of motion of the shaft with respect to the \(x\)- and \(y\)-axis directions are written as [5]:

\[
mx\ddot{x} + C_{xx}\dot{x} + C_{yy}\dot{y} + K_{xx}x + K_{yy}y = w_x
\]

\[
my\ddot{y} + C_{xy}\dot{x} + C_{yy}\dot{y} + K_{xx}x + K_{yy}y = w_y
\]

where \(w_x\) and \(w_y\) correspond to the \(x\)- and \(y\)-axis components of \(W\), and the following relation holds:
Equation (2.1) can be arranged in matrix form to represent the characteristics coefficients as follows:

\[
W = \sqrt{w_x^2 + w_y^2}
\]  

Equation (2.1) can be arranged in matrix form to represent the characteristics coefficients as follows:

\[
m \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + C \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + K \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}
\]

(2.3)

In equation (2.3), the mass, stiffness, and damping matrices are expressed by \(m\), \(K\), and \(C\), respectively. With the assumption of small \(e\), the elements in the \(K\) and \(C\) matrices can be simplified and re-written as:

\[
\begin{align*}
K_{xx} &= K_{xy} = \lambda_{hs} \\
K_{yx} &= -K_{xx} = -\lambda_{hd} \\
C_{xx} &= C_{yy} = C_{sq} \\
C_{xy} &= C_{yx} = 0
\end{align*}
\]  

(2.4)

where \(\lambda_{hs}\) is the hydrostatic stiffness, \(\lambda_{hd}\) is the hydrodynamic stiffness, and \(C_{sq}\) is the squeeze damping. With the substitution of equation (2.4) into (2.1), equation (2.1) becomes:

\[
m\ddot{x} + C_{sq}\dot{x} + \lambda_{hs}x + \lambda_{hd}y = w_x
\]

\[
m\ddot{y} + C_{sq}\dot{y} + \lambda_{hs}y - \lambda_{hd}x = w_y
\]

(2.5)

In comparison with equation (2.1), the representation of equation (2.5) provides the necessary advantage of distinguishing the hydrostatic and hydrodynamic properties.

2.2. Stiffness and damping coefficients

As the oil film thickness varies with the shaft displacement, the thickness with respect to different location is defined in Figure 3, where \(h_i\) is the oil film thickness of the \(i^{th}\) recess, and \(h_{i,j}\) is the oil film thickness between the \(i^{th}\) and \(j^{th}\) recesses. It is to be noted that the \(i^{th}\) and \(j^{th}\) recesses are adjacent, and thus \(j = i \pm 1\). For the purpose of analysis without the burden of considering units, the dimensionless parameters of pressure and oil film thickness, i.e. \(P_i\) and \(e\), are defined as follows [4]:

\[
P_i = \frac{P_i}{P_s} \quad \text{and} \quad e = \frac{e}{h_0}, \quad i = 1, 2, 3, 4
\]  

(2.6)

Furthermore, when the shaft and bearing are concentric, the input and output pressures of the four recesses are assumed to be the same, which are expressed as \(P_1 = P_3 = P_2 = P_4\) and \(p = p_1 = p_2 = p_3 = p_4\). Thus, the ratio between the input and output pressure of each recess, \(\beta\), is written as:

\[
\beta = \frac{p}{P_s}
\]  

(2.7)

In addition, the dimensionless parameters of \(h_i\) and \(h_{i,j}\) are denoted as \(H_i\) and \(H_{i,j}\), respectively, which are given by:

\[
H_i = \frac{h_i}{h_0} \quad \text{and} \quad H_{i,j} = \frac{h_{i,j}}{h_0}, \quad i = 1, 2, 3, 4, \quad j = i \pm 1
\]  

(2.8)

In the concentric condition, because the output pressures of the four recesses are identical, the following equality holds:
\[ H_i = H_{i-j} = 1, \quad i = 1, 2, 3, 4, \quad j = i \pm 1 \]  \hspace{1cm} (2.9)

Other relevant parameters are defined as follows:

\[
\gamma_c = n d (L-a) \frac{D}{\pi D b}, \quad z = \frac{\beta}{(1-\beta)}, \quad S_h = \frac{\eta}{P_i} \left( \frac{D}{C_d} \right)^2, \quad S_s = \frac{\eta}{P_i} \left( \frac{D}{C_d} \right)^2 \]  \hspace{1cm} (2.10)

Here, \( n \) is the total amount of recess; in this case, \( n \) is four. Furthermore, \( \gamma_c \) is related to the mechanical geometry, \( S_h \) is related to the rotational speed of shaft, \( S_s \) is related to the velocity of eccentricity, \( C_d \) is the radial clearance, \( \eta \) is the dynamic viscosity, and \( z \) is related to the type of restrictors and capillary.

Figure 3. The location and definition of oil film thickness in the bearing

According to [4], the coefficients of the hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping depend on the oil film thickness and input pressure, which are represented as:

\[
\lambda_{h_0} = \frac{P_i (L-a) D}{h_0} n \left[ \frac{dP}{d\varepsilon} \right] \sin \left( \frac{\pi}{n} \right) \]

\[
\lambda_{ad} = \frac{P_i (L-a) D}{h_0} n \left[ \frac{dP}{d\varepsilon} \right] \sin \left( \frac{\pi}{n} \right) \]

\[
C_{sq} = \frac{P_i (L-a) D}{h_0} n \left[ \frac{dP}{d\varepsilon} \right] \sin \left( \frac{\pi}{n} \right) \]

Equation (2.11) takes the first recess as an example. The square-bracketed terms in equation (2.11) are the differentiation of pressure with respect to the eccentricity, which are expressed in equations (2.12) to (2.14):

Hydrostatic stiffness parameter:

\[
\frac{dP}{d\varepsilon} = \frac{P_i \frac{dH_i}{d\varepsilon} + \frac{\gamma_c}{2} (P_i - P_e) \frac{dH_i}{d\varepsilon} + \frac{\gamma_c}{2} (P_i - P_e) \frac{dH_{i+1}}{d\varepsilon}}{1 + H_i^3 + \frac{\gamma_c}{2} H_{i+1} \left( 1 - \frac{2\pi}{n} \right) - \frac{\gamma_c}{2} H_{i+1} \left( 1 - \frac{2\pi}{n} \right)} \]  \hspace{1cm} (2.12)
Hydrodynamic stiffness parameter:
\[
\frac{dP}{de} = \frac{P_1}{2} \frac{dH_1}{de} + \frac{\gamma}{2} (P_1 - P_2) \frac{dH_2}{de} + \frac{\gamma}{2} (P_1 - P_2) \frac{dH_4}{de} + 12n^2 \left( a / L - \frac{1}{L} \right) (1 - a / L) | L / D | \left\{ \frac{dH_{1+1}}{de} + \frac{dH_{4-1}}{de} \right\} \\
(2.13)
\]

Squeeze damping parameter:
\[
\frac{dP}{de} = \frac{24 n (a / L - 1 / L) (L / D)^2 \sin \left( \frac{\pi}{n} \right) \frac{dS}{de}}{z + H_i + \frac{\gamma}{2} (H_{i+1} + H_{i+1}) \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
(2.14)
\]

In equations (2.12)-(2.14), the pressure variations relate to the oil film thickness of the first recess, i.e. \( H_1 \), as well as the nearby thicknesses of the second and fourth recesses, i.e. \( H_{1+1} \) and \( H_{4-1} \). In the small \( \varepsilon \) and concentric conditions, as have been discussed in equation (2.7), the following equality holds:
\[
P_1 = P_2 = P_3 = P_4 = \beta
(2.15)
\]

The dimensionless oil film and the third-order differentiations of the oil film thickness with respect to the eccentricity are expressed as:
\[
H_i = \int_0^{\pi/n} 1 - \varepsilon \cos (\theta - \alpha) d\theta
(2.16)
\]
\[
\frac{dH_i}{de} = -\frac{3n}{\pi} \sin \left( \frac{\pi}{n} \right)
(2.17)
\]

Therefore, substituting equation (2.15)-(2.17) into equations (2.12)-(2.14), and then substituting equation (2.12)-(2.14) into equation (2.11), the coefficients of hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping yield:
\[
\lambda_{\text{st}} = \frac{P_1 LD}{h_0} \frac{3n^2 \beta (1 - a / L) \sin^2 \left( \pi / n \right)}{2\pi \left[ z + 2\gamma \sin^2 \left( \pi / n \right) \right]}
\]
\[
\lambda_{\text{sd}} = \frac{P_1 LD}{h_0} \frac{12n^2 \sin^2 \left( \pi / n \right) S_n (a / L) (L / D)^2 (1 - a / L)^2}{z + 2\gamma \sin^2 \left( \pi / n \right)}
(2.18)
\]
\[
C_{sq} = \frac{\lambda_{\text{sd}}}{\pi N}
\]

With the substitution of equation (2.18) into equation (2.5), the ideal dynamic model of recessed journal bearings is obtained.

3. Modification of coefficients for the dynamic model

The ideal coefficients for the dynamic model are presented in Section 2. These coefficients are derived with simplified and ideal assumptions of small displacement, identical supply pressure, and concentric condition, as have been introduced in equation (2.15). However, pragmatically, the supply pressure and oil film thickness of each recess should be different, i.e. \( P_1 \neq P_2 \neq P_3 \neq P_4 \) and \( H_i \neq H_{i+j} \). Therefore, the modified coefficients to take into account the pragmatic conditions are discussed in this section.

Here, the pressure and oil film thickness related to the first recess is discussed to exemplify the proposed ideas and techniques; the other recesses can be derived accordingly. In the original stiffness and damping coefficients of equations (2.12)-(2.14), the pressure is differentiated with respect to the eccentricity. In order to analyze the pressure of the first recess with respect to the \( x \)- and \( y \)-axis displacements, the dimensionless parameters of \( X \) and \( Y \) are defined as follows:
\[
X = \frac{e_x}{h_y} \quad \text{and} \quad Y = \frac{e_y}{h_y} \quad (3.1)
\]

where \(e_x\) and \(e_y\) are the components of eccentricity in the \(x\) and \(y\) directions, respectively. Therefore, the square-bracketed terms of equations (2.12)-(2.14) can be modified to:

\[
\left[ \frac{dP_1}{d\varepsilon} \right] \rightarrow \left[ \frac{dP_1}{dX} \right] \quad \text{and} \quad \left[ \frac{dP_1}{d\varepsilon} \right] \rightarrow \left[ \frac{dP_1}{dY} \right] \quad (3.2)
\]

Considering equation (3.2), the representation of equations (2.12)-(2.14) are extended to:

Local hydrostatic stiffness parameter related to pressure:

\[
\frac{dP}{dX} = \frac{P_h \frac{dH}{dx} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dx} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dx}}{z + H_1^3 + \frac{\gamma}{2} H_{12}^3 \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right] + \frac{\gamma}{2} H_{41}^3 \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

\[
\frac{dP}{dY} = \frac{P_h \frac{dH}{dy} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dy} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dy}}{z + H_1^3 + \frac{\gamma}{2} H_{12}^3 \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right] + \frac{\gamma}{2} H_{41}^3 \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

Local hydrodynamic stiffness parameter related to pressure:

\[
\frac{dP}{dX} = \frac{P_h \frac{dH}{dx} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dx} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dx} + 12n \left( \frac{a}{L} \right)^2 \left( \frac{L}{D} \right)^2 S_2 \frac{dH}{dx} - \frac{dH}{dx}}{z + H_1^3 + \frac{\gamma}{2} (H_{12}^3 + H_{41}^3) \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

\[
\frac{dP}{dY} = \frac{P_h \frac{dH}{dy} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dy} + \frac{\gamma}{2} (P_h - P_i) \frac{dH}{dy} + 12n \left( \frac{a}{L} \right)^2 \left( \frac{L}{D} \right)^2 S_2 \frac{dH}{dy} - \frac{dH}{dy}}{z + H_1^3 + \frac{\gamma}{2} (H_{12}^3 + H_{41}^3) \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

Local squeeze damping parameter related to pressure:

\[
\frac{dP}{dX} = \frac{24 n \left( \frac{a}{L} \right)^2 \left( \frac{L}{D} \right)^2 \sin \left( \frac{\pi}{n} \right) \frac{dS}{dx}}{z + H_1^3 + \frac{\gamma}{2} (H_{12}^3 + H_{41}^3) \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

\[
\frac{dP}{dY} = \frac{24 n \left( \frac{a}{L} \right)^2 \left( \frac{L}{D} \right)^2 \sin \left( \frac{\pi}{n} \right) \frac{dS}{dy}}{z + H_1^3 + \frac{\gamma}{2} (H_{12}^3 + H_{41}^3) \left[ 1 - \cos \left( \frac{2\pi}{n} \right) \right]}
\]

In equation (3.3)-(3.5), the expression for \(H_1, H_{12}, \text{and } H_{41}\) need to be obtained in order to solve the third-order differentiations of the oil film thickness with respect to \(X\) and \(Y\). First, the third-order differentiations of \(H_1\) with respect to \(X\) and \(Y\) are considered, which are expressed by:

\[
\frac{dH_1^3}{dX} \quad \text{and} \quad \frac{dH_1^3}{dY} \quad (3.6)
\]
In order to obtain the normalized expression of $H_1$ in equation (3.6), the oil film thickness of the eight locations in relation to the eccentricity need to be derived. The angle between $e$ and the $x$ axis is denoted as $\alpha$, given by:

$$\alpha = \tan^{-1}\left(\frac{e_y}{e_x}\right)$$  \hspace{1cm} (3.7)

Then, referring to the trigonometric geometry of Figure 3, the angle $\theta$ is defined as:

$$\theta = \rho - \alpha$$  \hspace{1cm} (3.8)

where $\rho$ is the angle between the $y$ axis and the position of $h_i$ or $h_{i,j}$; for example, $\rho = 90$ with respect to $h_1$ and $\rho = 135$ with respect to $h_{1,1}$. Therefore, using the trigonometric geometry techniques, the oil film thickness of the first recess is obtained as follows:

$$h_1 = \left(\frac{1}{2}D + h_0\right) - \left\{\left(e_x^2 + e_y^2\right)^{1/2} \cos(90 - \alpha) + \left[\frac{1}{4}D^2 - \left(e_x^2 + e_y^2\right)\sin^2(90 - \alpha)\right]^{1/2}\right\}$$  \hspace{1cm} (3.9)

For the sake of simplifying the representation and calculation of equation (3.9), new parameters, $A$, $B$, and $C$ are defined as follows:

$$A = \frac{1}{2}D + h_0$$  
$$B = \left(e_x^2 + e_y^2\right)^{1/2} \cos \theta$$  \hspace{1cm} (3.10)  
$$C = \left[\frac{1}{4}D^2 - \left(e_x^2 + e_y^2\right)\sin^2 \theta\right]^{1/2}$$

With the substitution of equation (3.10) into (3.9), equation (3.9) is arranged to:

$$h_1 = \left[A - (B + C)\right]$$  \hspace{1cm} (3.11)

Then, dividing equation (3.11) by $h_0$ for normalization, as have been shown in equation (2.8), $H_1$ is obtained, and thus the cubic of $H_1$ is given by:

$$H_1^3 = \frac{1}{h_0^3} \left(A^3 - 3A^2(B+C) + 3AB^2 + 6ABC + 3AC^2 - B^3 - 3B^2C - 3BC^2 - C^3\right)$$  \hspace{1cm} (3.12)

Therefore, the third-order differentiations of equation (3.12) with respect to $X$ and $Y$ yield:

$$\frac{dH_1^3}{dX} = \frac{1}{h_0^3} \left[-3A^2B - 3A^2C + 3A(B^2) + 6A(BC) + 3A(C^2) - (B^3) - 3(B^2C) - 3(BC^2) - (C^3)\right]$$  
$$\frac{dH_1^3}{dY} = \frac{1}{h_0^3} \left[-3A^2B' - 3A^2C' + 3A(B'^2) + 6A(BC') + 3A(C'^2) - (B'^3) - 3(B'^2C') - 3(BC'^2) - (C'^3)\right]$$  \hspace{1cm} (3.13)

In equation (3.13), the terms with subscripts $x$ or $y$ mean partial differentiation with respect to $X$ or $Y$. All the partial differentiation terms in equation (3.13) are shown as follows:

$$B_x = \left(e_x^2 + e_y^2\right)^{0.5} e_x \cos \theta$$  
$$B_y = \left(e_x^2 + e_y^2\right)^{0.5} e_y \cos \theta$$  \hspace{1cm} (3.14)

$$\left(B^2\right)_x = 2e_x \cos^2 \theta$$  
$$\left(B^2\right)_y = 2e_y \cos^2 \theta$$
\[
(B^3)_x = 3(e_x^2 + e_y^2)^{\frac{3}{2}} e_x \cos^3 \theta
\]
\[
(B^3)_y = 3(e_x^2 + e_y^2)^{\frac{3}{2}} e_y \cos^3 \theta
\]
\[
C_x = [D^2 - (e_x^2 + e_y^2) \sin^2 \theta]^{-0.5} e_x \sin^2 \theta
\]
\[
C_y = [D^2 - (e_x^2 + e_y^2) \sin^2 \theta]^{-0.5} e_y \sin^2 \theta
\]
\[
(C^2)_x = 2e_x \sin^2 \theta
\]
\[
(C^2)_y = 2e_y \sin^2 \theta
\]
\[
(C^3)_x = 3[D^2 - (e_x^2 + e_y^2) \sin^2 \theta]^{0.5} e_x \sin^2 \theta
\]
\[
(C^3)_y = 3[D^2 - (e_x^2 + e_y^2) \sin^2 \theta]^{0.5} e_y \sin^2 \theta
\]
\[
(BC)_x = \left[(e_x^2 + e_y^2)D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ e_x D^2 - 2(e_x^2 + e_y^2) \sin^2 \theta \right] \cos \theta
\]
\[
(BC)_y = \left[(e_x^2 + e_y^2)D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ e_y D^2 - 2(e_x^2 + e_y^2) \sin^2 \theta \right] \cos \theta
\]
\[
(B'C)_x = \left[(e_x^2 + e_y^2)D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ 2(e_x^2 + e_y^2)e_x D^2 - 3(e_x^2 + e_y^2) \sin^2 \theta \right] \cos^3 \theta
\]
\[
(B'C)_y = \left[(e_x^2 + e_y^2)D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ 2(e_x^2 + e_y^2)e_y D^2 - 3(e_x^2 + e_y^2) \sin^2 \theta \right] \cos^3 \theta
\]
\[
(BC^2)_x = \left[(e_x^2 + e_y^2) D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ e_x D^2 - 3(e_x^2 + e_y^2)^{\frac{3}{2}} e_x \sin^2 \theta \right] \cos \theta
\]
\[
(BC^2)_y = \left[(e_x^2 + e_y^2) D^2 - (e_x^2 + e_y^2) \sin^2 \theta \right]^{-0.5} \left[ e_y D^2 - 3(e_x^2 + e_y^2)^{\frac{3}{2}} e_y \sin^2 \theta \right] \cos \theta
\]

The result of equations (3.6)-(3.14) derive the oil film thickness of \(H_1\) related to the first recess. However, as shown in equations (3.3)-(3.5), the pressure variations are related to the second and fourth recesses as well. Therefore, the coupled oil film thickness of \(H_{1\cdot2}\) and \(H_{4\cdot1}\) need to be derived, with a similar procedure to equations (3.6)-(3.14), where the third-order differentiations of \(H_{1\cdot2}\) and \(H_{4\cdot1}\) with respect to \(X\) and \(Y\) are written as:

\[
\frac{dH_{1\cdot2}}{dX} \quad \text{and} \quad \frac{dH_{1\cdot2}}{dY}
\]
\[
\frac{dH_{4\cdot1}}{dX} \quad \text{and} \quad \frac{dH_{4\cdot1}}{dY}
\]

The derivations of equations (3.15) and (3.16) are omitted for the sake of brevity, and the results are shown directly as follows:

\[
\frac{dH_{2\cdot1}}{dX} = \frac{1}{h_0} \left[ -3A'B_e - 3A'c + 3A(B') + 6A(BC) + 3A(C) - (B' - 3(B'C) - 3(BC^2) - (C')) \right]
\]
\[
\frac{dH_{i,3}}{dY} = \frac{1}{h_0} \left[ -3A'B_i -3A'C_i +3A(B')_1 +6A(BC)_i +3A(C')_1 -3(B'C)_i -3(C')_i \right]
\]

\[
\frac{dH_{i,3}}{dX} = \frac{1}{h_0} \left[ -3A'B_i -3A'C_i +3A(B')_1 +6A(BC)_i +3A(C')_1 -3(B'C)_i -3(C')_i \right]
\]

\[
\frac{dH_{i,3}}{dY} = \frac{1}{h_0} \left[ -3A'B_i -3A'C_i +3A(B')_1 +6A(BC)_i +3A(C')_1 -3(B'C)_i -3(C')_i \right]
\]

(3.18)

With the substitution of equations (3.13), (3.17), and (3.18) into equations (3.3)-(3.5), the pressure variations related to the first recess can be obtained. Accordingly, referring to equation (2.11), the hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping of the first recess, with respect to $X$, are given by:

\[
\lambda_{hXX} = \frac{P_1(L-a)D}{h_0} \frac{dP}{dx} \sin \left( \frac{\pi}{n} \right)
\]

\[
\lambda_{hXY} = \frac{P_1(L-a)D}{h_0} \frac{dP}{dx} \sin \left( \frac{\pi}{n} \right)
\]

\[
C_{sqX} = \frac{P_1(L-a)D}{h_0} \frac{dP}{dx} \sin \left( \frac{\pi}{n} \right)
\]

(3.19)

Similarly, the parameters with respect to $dP/dY$ can be obtained by replacing $X$ by $Y$. With reference to the similar steps of equations (3.2)-(3.18), the normalized oil film thickness and coefficient of the other three recesses can be attained; here, the oil film thicknesses are shown as follows:

\[
h_2 = \left[ A - (B + C) \right]
\]

\[
h_3 = h_4 = \left[ A + (B - C) \right]
\]

\[
h_{1-2} = h_{2-1} = h_{4-1} = h_{1-4} = \left[ A - (B + C) \right]
\]

\[
h_{2-3} = h_{3-2} = h_{3-4} = h_{4-3} = \left[ A + (B - C) \right]
\]

(3.20)

where $\rho = 0$ for $h_2$ and $h_4$, $\rho = 45$ for $h_{1-2}$ and $h_{3-4}$, $\rho = 90$ for $h_1$ and $h_3$, and $\rho = 135$ for $h_{2-3}$ and $h_{1-4}$.

With all the local hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping coefficients for the four recesses obtained, and referring to equation (3.19), the $x$ and $y$ components of coefficients of each recess can be collected and lumped together. Thus, the new coefficients of global hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping read:

\[
C_x = C_{sqX1} + C_{sqX2} + C_{sqX3} + C_{sqX4}
\]

\[
C_y = C_{sqY1} + C_{sqY2} + C_{sqY3} + C_{sqY4}
\]

\[
\lambda_{hXX} = \lambda_{hXX1} + \lambda_{hXX2} + \lambda_{hXX3} + \lambda_{hXX4}
\]

\[
\lambda_{hXY} = \lambda_{hXY1} + \lambda_{hXY2} + \lambda_{hXY3} + \lambda_{hXY4}
\]

\[
\lambda_{hXY} = \lambda_{hXY1} + \lambda_{hXY2} + \lambda_{hXY3} + \lambda_{hXY4}
\]

(3.21)
Therefore, substituting equation (3.21) into (2.5), the new equation of motion of the shaft is modified to:

\[ m\ddot{x} + C_x\dot{x} + \lambda_{hx} x + \lambda_{hy} y = w_x \]

\[ m\ddot{y} + C_y\dot{y} + \lambda_{hy} y \pm \lambda_{bdy} x = w_y \]

Equation (3.22)

In comparison with the original form of equation (2.5), the effects of input pressure variations on the coefficients are considered explicitly. Simulation work based on equation (3.22) is studied in the next section.

4. Simulation results and discussion

The dynamic model of hydraulic bearing system is built in MATLAB/Simulink, based upon equation (3.22) and the modified coefficients of equation (3.21). Table 1 summarizes the parameter settings for the simulation study, and the rotational speed of shaft is denoted as \( N \). Figure 4 compares a series of simulation results, and the unit for the \( x \) and \( y \) axes is in meter. The origin (0,0) in the \( x-y \) plot is the initial position of shaft centroid, which is concentric with the bearing centroid. The exertion of an external force, \( W = 1280 \text{ N} \), was considered for simulation study, which offset the shaft centre by \( e \).

Different rotational speeds of \( N = 0, 0.1, 1.0, 10, 23.15, \) and \( 24 \text{ rev/s} \) were discussed, while the shaft was rotating counterclockwise.

In the simulation results of Figure 4a-c with lower rotational speeds, the orbit always converged to a certain point, and the distance between the point and the origin is the eccentricity of the shaft, which is indicated using the black line. With higher rotational speeds, as shown in Figure 4d-f, the eccentricity decayed due to the increases of hydrodynamic stiffness. When the rotational speed reached the marginal value, \( N = 23.15 \text{ (rev/s)} \), oscillation behavior of the shaft appeared, and thus the hydraulic bearing system was in marginal stable status. As the rotational speed exceeded the marginal value, the trajectory diverged and the system becomes unstable, as seen in Figure 4f, which may severely damage the bearing system. In equation (3.22), \( \lambda_{bdx} \) and \( \lambda_{bdy} \) include the parameter \( S_h \), referred to equation (2.18), and \( S_h \) is associated with the rotational speed of shaft, as shown in equation (2.10). Therefore, as \( N \) gradually increases, the locations of poles of equation (3.22) move close to the imaginary axis in the Laplace domain, and finally result in instability. The results of Figure 4 verify equations (3.21) and (3.22), which provide a possible technique to predict the optimal and ultimate rotational speeds.

5. Conclusion

This paper proposes modified coefficients for the dynamic model of recessed journal bearing. In literature, the coefficients of hydrostatic stiffness, hydrodynamic stiffness, and squeeze damping are simplified with ideal assumptions, such as the journal bearing in concentric condition or identical supply pressure for recesses. In order to precisely model the behaviour of hydraulic bearing under realistic operation, the modified coefficients, which take into account the pressure discrepancies of recesses, are considered in this work. In the derivation process, the oil film thickness related to the eccentricity in the \( x \) and \( y \) directions are discussed. New equation of motion of shaft is re-written using the modified coefficients, and the new dynamic model is built within the MATLAB/Simulink software. The simulation work compares the trajectories of the shaft centre with different rotational speeds, and verifies that the proposed techniques provide a basis to investigate the optimal stability and ultimate rotational speed of hydraulic bearing system. Based on the present work, future work will consider online prediction and control of the eccentricity.
Figure 4. The orbits of shaft center with different rotational speeds

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