Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials

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Introduction

Harmonic sums and harmonic polylogarithms have made many loop calculations simpler since the late 1990ies.

- adequate structures for Feynman diagrams
- iterated parameter integrals of the Volterra type
- naturally emerging in the $\varepsilon$-expansion
- massless and single mass cases 2 Loops: only usual harmonic sums/polylogarithms
- 3 Loops: first traces of generalized harmonic sums
- massive calculations: 4th, 6th root of unity weights, ...
- Generalization of harmonic sums and polylogarithms required

Build in: HarmonicSums, Thesis of J. Ablinger (2012)
The code Sigma has been extensively used and generalized.
**Introduction**

**Harmonic sums:** (Vermaseren; JB, Kurth, 1998)

\[
S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(b))^k}{k|b|} S_{\vec{a}}(k), \quad S_0(N) = 1, \quad b, a_i \in \mathbb{Z}\setminus\{0\}.
\]

The Mellin transform mapping between \(N\) and \(x\) space:

\[
\mathcal{M}[f(x)](N) = \int_0^1 dx \ x^N f(x).
\]

**Generalized harmonic sums:** (Moch, Uwer, Weinzierl, 2001)

\[
S_{b,\vec{a}}(\zeta, \vec{\xi}; N) = \sum_{k=1}^{N} \frac{\zeta^k}{k^b} S_{\vec{a}}(\vec{\xi}; k), \quad b, a_i \in \mathbb{N}^+; \zeta, \xi_i \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}
\]

Known examples are related to the second index set \(\xi_i \in \{1, -1, 1/2, -1/2, 2, -2\}\).

**Other Summands occurring:**

\[
\frac{\left(\pm 1\right)^k}{(l \cdot k + m)^n}, \quad \text{with} \quad l, m, n \in \mathbb{N}^+.
\]

**New Mellin variable:**

\[
N \to k \cdot N.
\]
Basic Formalism

\[ S\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}(s_1,\ldots,s_l;N) = \sum_{k_1=1}^{N} \frac{s_1^k}{(a_1 k_1 + b_1)c_1} S\{a_2,b_2,c_2\};\ldots;\{a_l,b_l,c_l\}(s_2,\ldots,s_l;k_1), \]

\[ S_\emptyset = 1, \quad a_i, c_i \in \mathbb{N}_+, \quad b_i \in \mathbb{N}, \quad s_i = \pm 1, \quad a_i > b_i \]

**Summand representation:**

\[ \frac{(\pm 1)^k}{ak + b} = \int_0^1 \frac{dx}{x} x^{ak+b-1}(\pm 1)^k \]

\[ \frac{(\pm 1)^k}{(ak + b)^c} = \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \ldots \int_0^{x_{c-2}} \frac{dx_{c-1}}{x_{c-1}} \int_0^{x_{c-1}} dx_c \ x_c^{ak+b-1}(\pm 1)^k, \]

\[ \sum_{k=1}^{l} (\pm 1)^k x^{ak+b-1} = x^{a+b-1} \frac{(\pm x^a)^{l+1} - 1}{(\pm x^a) - 1}. \]
Let us illustrate the principle steps in case of the following example:

\[
S_{\{3,2,2\},\{2,1,1\}}(1,-1; N) = \sum_{k=1}^{N} \frac{1}{(3k+2)^2} \sum_{l=1}^{k} \frac{(-1)^l}{(2l+1)}.
\]

\[
S_{\{3,2,2\},\{2,1,1\}}(1,-1; N) = \sum_{k=1}^{N} \int_{0}^{1} dx \frac{x^2}{x^2 + 1} \frac{(-x^2)^{k+1}}{(3k+2)^2}.
\]

Setting \( x = y^3 \) one obtains

\[
S_{\{3,2,2\},\{2,1,1\}}(1,-1; N) = 12 \int_{0}^{1} dy \frac{y^8}{y^6 + 1} \sum_{k=1}^{N} \frac{(-y^6)^k - 1}{(6k+4)^2}
\]

\[
= 12 \int_{0}^{1} dy \frac{y^4}{y^6 + 1} \left\{ \int_{0}^{y} dz \int_{0}^{z} dt \frac{(-t^6)^N - 1}{t^6 + 1} 

- y^4 \int_{0}^{1} \frac{dz}{z} \int_{0}^{z} dt \frac{t^{6N} - 1}{t^6 - 1} \right\}
\]

\[
= 12 \int_{0}^{1} dy \frac{y^4}{y^6 + 1} \int_{0}^{y} \frac{dz}{z} \int_{0}^{z} dt \frac{(-t^6)^N - 1}{t^6 + 1}

- (4 - \pi) \int_{0}^{1} \frac{dz}{z} \int_{0}^{z} dt \frac{t^{6N} - 1}{t^6 - 1} \right\}.
\]
In general, the polynomials

\[ x^a - 1 \]

decompose in a product of cyclotomic polynomials, except for \( a = 1 \) for which the expression is \( \Phi_1(x) \). Moreover, the polynomials

\[ x^a + 1 = \frac{x^{2a} - 1}{x^a - 1} \]

are either cyclotomic for \( a = 2^n, n \in \mathbb{N} \) or decompose into products of cyclotomic polynomials in other cases. All factors divide \((x^a)^l - 1\), resp. \((-x^a)^l - 1\).

\[
\begin{align*}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1, \text{ etc.}
\end{align*}
\]
Cyclotomic Harmonic Polylogarithms

To account for the newly emerging sums in perturbative calculations in Quantum Field Theory we introduce Poincaré-iterated integrals over the alphabet $\mathcal{A}$

$$\mathcal{A} = \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \mid k \in \mathbb{N}_+, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the $k$th cyclotomic polynomial and $\varphi(k)$ denotes Euler’s totient function.

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d<n} \Phi_d(x)}, \quad d, n \in \mathbb{N}_+,\;$$

The alphabet $\mathcal{A}$ is an extension of the alphabet

$$\mathcal{A}_H = \left\{ \frac{1}{x}, \frac{1}{\Phi_1(x)}, \frac{1}{\Phi_2(x)} \right\},$$

generating the usual harmonic polylogarithms (Remiddi, Vermaseren, 1999).
A shorthand notation for the letters of $\Omega$:

\[ f_0^0(x) = \frac{1}{x} \]
\[ f_k^l(x) = \frac{x^l}{\Phi_k(x)}, \quad k \in \mathbb{N}_+, l \in \mathbb{N}, l < \varphi(k). \]

**Examples:** (N. Nielsen, 1906)

\[ \frac{1}{2} \beta \left( \frac{x}{2} \right) = \int_0^1 dt \frac{t^{x-1}}{t^2 + 1} \]
\[ \beta \left( \frac{x}{3} \right) = \beta(x) - \int_0^1 dt \frac{t^{x-1}}{t^2 - t + 1} \left( \frac{t - 2}{t^2 - t + 1} \right), \]
\[ \beta(x) = \frac{1}{2} \left[ \psi \left( \frac{x + 1}{2} \right) - \psi \left( \frac{x}{2} \right) \right]. \]
We form the Poincaré iterated integrals

\[ C_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}(z) = \frac{1}{m!} \ln^m(x) \]

if \((l_1,\ldots,l_m) = (0,\ldots,0), (k_1,\ldots,k_m) = (0,\ldots,0),\)

\[ C_{k_m}^m(z) = \int_0^z dx \ f_{k_m}^m(x) \]

if \(k_m \neq 0,\)

\[ C_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}(z) = \int_0^z dx \ f_{k_1}^1(x) \ C_{k_2,\ldots,k_m}^{l_2,\ldots,l_m}(x) \]

if \((k_1,\ldots,k_m) \neq (0,\ldots,0),\)

and \(C_{\vec{a}}^{\vec{l}}(z)\) denotes cyclotomic harmonic polylogarithms of weight \(w = m.\) They form a shuffle algebra by multiplication

\[ C_{\vec{a}_2}^{\vec{b}_2}(z) \cdot C_{\vec{a}_1}^{\vec{b}_1}(z) = C_{\vec{a}_2}^{\vec{b}_2}(z) \sqcup \sqcup \ C_{\vec{b}_2}^{\vec{b}_1}(z) = \sum \ C_{\vec{c}_2}^{\vec{c}_1}(z) \]

of \(M^w\) elements at weight \(w,\) where \(M\) denotes the number of chosen letters from \(\mathcal{A}\).

\[ \mathcal{N}_{\text{basic}}(w) = \frac{1}{w} \sum_{d|w} \mu \left( \frac{w}{d} \right) M^d, \ w \geq 1 \]
| weight | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|--------|----|----|----|----|----|----|----|
| 1      | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
| 2      | 1  | 3  | 6  | 10 | 15 | 21 | 28 |
| 3      | 2  | 8  | 20 | 40 | 70 | 112| 168|
| 4      | 3  | 18 | 60 | 150| 315| 588| 1008|
| 5      | 6  | 48 | 204| 624| 1554| 3360| 6552|
| 6      | 9  | 116| 670|2580| 7735|19544| 43596|
| 7      | 18 | 312|2340|11160|39990|117648|299592|
| 8      | 30 | 810|8160|48750|209790|729300|2096640|

**Table:** Number of basic cyclotomic harmonic polylogarithms in dependence of the number of letters and weight.

\[ f_{i_a}^{j_b} \text{ representation:} \]

\[
\frac{y^4}{y^6 + 1} = \frac{1}{3} \left[ f_4^0(y) - f_{12}^0(y) + 2f_{12}^2(y) \right].
\]
With integration by parts one obtains the following Mellin transforms of argument $6N$ of cyclotomic harmonic polylogarithms $C_{k_1,\ldots,k_m}^h(x)$ weighted by the letters $f_i^k(x)$ of the alphabet $\mathcal{A}$:

$$
S_{\{3,2,2\},\{2,1,1\}}(1, -1; N) =
\frac{1}{6}(4 - \pi) \int_0^1 dx x^3(x^{6N} - 1) \left[ 6 + f_1^0(x) - f_2^0(x) - 2f_3^0(x) - f_1^1(x) - 2f_6^0(x) + f_6^1(x) \right] C_0^0(x)
$$

$$
-2 \int_0^1 dx x^3 \left[ (-1)^N x^{6N} - 1 \right] \left[ 3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x) \right] C_0^0(x)
$$

$$
-\frac{4}{3} \left[ C_{0,4}^{0,0}(1) - C_{0,12}^{0,0}(1) + 2C_{0,12}^{0,2}(1) \right] \int_0^1 dx x^3 \left[ (-1)^N x^{6N} - 1 \right]
$$

$$
\times \left[ 3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x) \right]
$$

$$
+\frac{4}{3} \int_0^1 dx x^3 \left[ (-1)^N x^{6N} - 1 \right] \left[ C_{0,4}^{0,0}(x) - C_{0,12}^{0,0}(x) + 2C_{0,12}^{0,2}(x) \right]
$$

$$
\times \left[ 3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x) \right].
$$

\[ C_{0,4}^{0,0} = -C, \quad C_{0,12}^{0,0(2)} = \alpha_1 \psi'(1/12) + \alpha_2 \psi'(5/12) \]
Cyclotomic Harmonic Sums

(i) The Single Sums:

\[ S_{l,m,n}(N) = \sum_{k=0}^{N} \frac{(\pm 1)^k}{(l \cdot k + m)^n}; \quad \phi_k(l, N) = \int_0^1 dx x^N f_k^l(x) \]

The summation package Sigma reduces to:

\[ S_{-1}(N), S_{-1}(3N), S_{-1}(5N), S_1(N), S_1(2N), S_1(3N), S_1(4N), S_1(5N), S_1(6N), \]

\[ \phi_3(3N)_+, \phi_3(6N)_+, \phi_4(2N), \phi_4(4N), \phi_4(6N), \phi_5(5N)_+, \phi_5(1,5N)_+, \]

\[ \phi_5(2,5N)_+, \phi_6(3N), \phi_8(4N), \phi_8(2,4N), \phi_{10}(5N), \phi_{10}(1,5N), \phi_{10}(2,5N), \phi_{12}(6N). \]

and the sums at \( N \to \infty \)

\[ \sigma\{1,0,−1\}, \sigma\{2,1,−1\}, \sigma\{3,1,−1\}, \sigma\{4,1,−1\}, \sigma\{4,3,−1\}, \sigma\{5,1,−1\}, \sigma\{5,2,−1\}, \sigma\{5,3,−1\} \sigma\{6,1,−1\} \]

Asymptotic representations \( |N| \to \infty \) are derived easily.

Needed within the summation calculus.
(ii) Relations between sums:

- Differentiation w.r.t. $N$, $(D)$
- Stufffle Algebra $(A)$
- Synchronization $(M)$
- Duplication Relations: $(H_1, H_2)$

(iii) Nested Sums: Consider:

$$\frac{1}{k^1}, \quad \frac{(-1)^k}{k^2}, \quad \frac{1}{(2k+1)^3}, \quad \frac{(-1)^k}{(2k+1)^4} \quad (*)$$

| $w$ | $N_S$ | $H_1$ | $H_1, H_2$ | $H_1, M$ | $H_1, H_2, M$ | $D$ | $H_1, H_2, M, D$ | $A$ | $H_1, H_2, M, A$ | $A, D$ | all |
|-----|-------|-------|-------------|-----------|---------------|-----|------------------|-----|-----------------|-------|-----|
| 1   | 4     | 3     | 3           | 2         | 2             | 4   | 2                | 4   | 2               | 4     | 2   |
| 2   | 20    | 18    | 17          | 16        | 15            | 16  | 13               | 10  | 8               | 6     | 2   |
| 3   | 100   | 96    | 93          | 92        | 89            | 80  | 74               | 40  | 35              | 30    | 27  |
| 4   | 500   | 492   | 485         | 484       | 477           | 400 | 388              | 150 | 142             | 110   | 107 |
| 5   | 2500  | 2484  | 2469        | 2468      | 2453          | 2000| 1976             | 624 | 607             | 474   | 465 |

Reduction of the number of cyclotomic harmonic sums $N_S$ over the elements at given weight $w$ by applying the three multiple argument relations $(H_1, H_2, M)$, differentiation w.r.t. to the external sum index $N$, $(D)$, and the algebraic relations $(A)$. A sequence of symbols corresponds to the combination of these relations.
Explicit counting relations for the number of basis elements given in the above Table can be derived:

\[
N_A(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 5^d \\
N_D(w) &= N_S(w) - N_S(w - 1) = 16 \cdot 5^{w-2} \\
N_{H_1}(w) &= N_S(w) - 2^{w-1} = 4 \cdot 5^{w-1} - 2^{w-1} \\
N_{H_1H_2}(w) &= N_S(w) - (2 \cdot 2^{w-1} - 1) = 4 \cdot 5^{w-1} - (2 \cdot 2^{w-1} - 1) \\
N_{H_1M}(w) &= N_S(w) - 2 \cdot 2^{w-1} = 4 \cdot 5^{w-1} - 2 \cdot 2^{w-1} \\
N_{H_1H_2M}(w) &= N_S(w) - (3 \cdot 2^{w-1} - 1) = 4 \cdot 5^{w-1} - (3 \cdot 2^{w-1} - 1) \\
N_{AD}(w) &= N_A(w) - N_A(w - 1) \\
N_{AH_1H_2M}(w) &= \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 5^d - \left(3 \cdot \frac{1}{w} \sum_{d|w} \mu\left(\frac{w}{d}\right) 2^d - 1\right) \\
N_{DH_1H_2M}(w) &= N_{H_1H_2M}(w) - N_{H_1H_2M}(w - 1) = 16 \cdot 5^{w-2} - 3 \cdot 2^{w-2} \\
N_{ADH_1H_2M}(w) &= N_{AH_1H_2M}(w) - N_{AH_1H_2M}(w - 1)
\]

Here \(\mu\) denotes the Möbius function.
Special Values: Single Sums

\( w = 1 \)

\[
\psi\left(\frac{p}{q}\right) = -\gamma_E - \ln(2q) - \frac{\pi}{2} \cot \left( \frac{p\pi}{q} \right) + 2 \sum_{k=1}^{[(q-1)/2]} \cos \left( \frac{2\pi kp}{q} \right) \ln \left[ \sin \left( \frac{\pi k}{q} \right) \right]
\]

\[
\psi\left(\frac{1}{n}\right) = -n (\gamma_E + \ln(n)) - \sum_{k=2}^{n} \psi\left(\frac{k}{n}\right).
\]

Further simplification if the regular \(q\)-polygon is constructible

\[ q \in \{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, \ldots\} \]

Consider the above alphabet (\(*\)).
The basis elements are:

\[ \sigma_0, \quad \ln(2), \quad \pi \]

\( w \geq 2, \quad l \leq 6:\)

\[ \zeta_{2k+1}, \psi^{(2k+1)} \left( \frac{1}{3} \right), \psi^{(2k+1)} \left( \frac{1}{5} \right), \psi^{(k)} \left( \frac{1}{8} \right), \psi^{(2k)} \left( \frac{1}{12} \right) \]
\[ T_{i_1}(1) = \beta_D(l) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^l} \]

\[ T_{i_2}(1) = \mathcal{C} \]

Some new numbers occurring at \( w = 1 \):

\[ \ln(3), \ln(\sqrt{2} - 1), \ln(\sqrt{3} - 1), \ln(\sqrt{5} - 1), \ldots \]

+ algebraic (irrational) numbers
\( w > 1, \) higher depth, alphabet (\( \ast \))

| weight | \( N_S \) | \( A \) | \( S_H \) | \( A + S_H \) | \( A + sh + H_1 \) | \( A + S_H + H_1 + H_2 \) | \( A + S_H + H_1 + H_2 + M \) |
|--------|---------|-------|---------|-------------|----------------|----------------|----------------|
| 1      | 4       | 4     | 4       | 4           | 4             | 3             | 3             |
| 2      | 20      | 10    | 13      | 3           | 3             | 2             | 1             |
| 3      | 100     | 40    | 46      | 6           | 6             | 5             | 3             |
| 4      | 500     | 150   | 163     | 10          | 10            | 9             | 6             |
| 5      | 2500    | 624   | 650     | 21          | 21            | 19            | 13            |
| 6      | 12500   | 2580  | 2635    | 36          | 36            | 34            | 25            |

Some basis elements:

\[
\begin{align*}
\sigma\{2,1,-2\} &= -1 + C \\
\sigma\{1,0,-2\},\{2,1,-1\} &= \frac{\pi^2}{12} - \frac{\pi^3}{48} + \frac{1}{2} \int_0^1 dx \frac{\sqrt{x}}{x+1} \operatorname{Li}_2(x) \\
\sigma\{2,1,-2\},\{1,0,-1\} &= -C \ln(2) + \int_0^1 dx \frac{1}{1+x} \frac{\chi_2(\sqrt{x})}{\sqrt{x}} \\
\chi_\nu(x) &= \frac{1}{2} [\operatorname{Li}_\nu(x) - \operatorname{Li}_\nu(-x)]
\end{align*}
\]
Infinite Sums with more Cyclotomic Letters

\( w = 1 \)

\[
\frac{(±1)^k}{(lk + m)^n}, \quad 1 \leq n \leq 2, 1 \leq l \leq 20, m < l.
\]

| l  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| sums | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 |
| basis | 2 | 3 | 4 | 5 | 6 | 6 | 8 | 9 | 8 | 10 | 12 | 10 | 14 | 14 | 11 | 17 | 18 | 14 | 20 | 18 |
| new basis sums | 2 | 1 | 2 | 2 | 4 | 1 | 6 | 4 | 4 | 3 | 10 | 2 | 12 | 5 | 3 | 8 | 16 | 4 | 18 | 6 |

The number of the \( w = 1 \) cyclotomic harmonic sums (1) up to \( l = 20 \), the basis elements at fixed value of \( l \), and the new basis elements in ascending sequence.

\[
\sigma\{1,0,1\}, \sigma\{1,0,-1\}, \sigma\{2,1,-1\}, \sigma\{3,1,1\}, \sigma\{3,1,-1\}, \sigma\{4,1,-1\}, \sigma\{4,3,-1\}, \\
\sigma\{5,1,1\}, \sigma\{5,1,-1\}, \sigma\{5,2,-1\}, \sigma\{5,3,-1\}, \sigma\{6,1,-1\} , \ldots
\]
$w = 2$

| $l$ | $N_S$ | SH | $A$ | $A + SH$ | $A + SH + H_1$ | $A + SH + H_1 + H_2$ | $A + SH + H_1 + H_2 + M$ |
|-----|-------|-----|-----|----------|------------------|-------------------------|---------------------------|
| 1   | 6     | 4   | 3   | 1        | 1                | 1                       | 1                         |
| 2   | 20    | 13  | 10  | 3        | 3                | 2                       | 1                         |
| 3   | 42    | 27  | 21  | 7        | 6                | 6                       | 5                         |
| 4   | 72    | 46  | 36  | 12       | 11               | 10                      | 3                         |
| 5   | 110   | 70  | 55  | 19       | 17               | 17                      | 16                        |
| 6   | 156   | 99  | 78  | 27       | 25               | 24                      | 5                         |
| 7   | 210   | 133 | 105 | 37       | 34               | 34                      | 33                        |
| 8   | 272   | 172 | 136 | 48       | 45               | 44                      | 12                        |
| 9   | 342   | 216 | 171 | 61       | 57               | 57                      | 52                        |
| 10  | 420   | 265 | 210 | 75       | 71               | 70                      | 22                        |
| 11  | 506   | 319 | 253 | 91       | 86               | 86                      | 85                        |
| 12  | 600   | 378 | 300 | 108      | 103              | 102                     | 21                        |
| 13  | 702   | 442 | 351 | 127      | 121              | 121                     | 120                       |
| 14  | 812   | 551 | 406 | 147      | 141              | 140                     | 49                        |
| 15  | 930   | 585 | 465 | 169      | 162              | 162                     | 145                       |
| 16  | 1056  | 664 | 528 | 192      | 185              | 184                     | 50                        |
| 17  | 1190  | 748 | 595 | 217      | 209              | 209                     | 208                       |
| 18  | 1332  | 837 | 666 | 243      | 235              | 234                     | 63                        |
| 19  | 1482  | 931 | 741 | 271      | 262              | 262                     | 261                       |
| 20  | 1640  | 1030| 820 | 300      | 291              | 290                     | 74                        |

Number of basis elements of the $w = 2$ cyclotomic harmonic sums up to cyclotomy $l = 20$ after applying the quasi-shuffle algebra of the sums ($A$), the shuffle algebra of the cyclotomic harmonic polylogarithms ($SH$), and the three multiple argument relations ($H_1, H_2, M$) of the sums.
Counting Relations:

\[
\begin{align*}
N_S(l) &= 2l(2l+1) \\
N_A(l) &= l(2l+1) \\
N_{SH}(l) &= \frac{(5l+3)l}{2} \\
N_{A,SH}(l) &= \frac{6l^2 + 1 - (-1)^l}{8} \\
N_{A,SH,H_1}(l) &= \frac{6l^2 - 4l + 7 - (-1)^l}{8} \\
N_{A,SH,H_1,H_2}(l) &= \frac{6l^2 - 4l + 3(1 - (-1)^l)}{8} \\
N_{A,SH,H_1,H_2}(l) &= \frac{3}{4} l^2 - 12l + \text{if } (\text{modp}(1, 2) = 0, 1, 3/4) \quad (\text{Broadhurst})
\end{align*}
\]
Generalized Harmonic Sums at Roots of Unity

\[
\lim_{N \to \infty} S_{k_1, \ldots, k_m}(x_1, \ldots, x_m; N) = \sigma_{k_1, \ldots, k_m}(x_1, \ldots, x_m), x_j \in \mathcal{C}_n, n \geq 1, k_1 \neq 1 \text{ for } x_1 = 1
\]

\[
\mathcal{C}_n \in \{ e_n | e_n^n = 1, e_n \in \mathbb{C} \}
\]

\[
\sigma_w(x) = \text{Li}_w(x), \quad w \in \mathbb{N}, w \geq 1
\]

\[
\sigma_1(x) = \text{Li}_1(x) = -\ln(1-x)
\]

\[
\sigma_{1,1}(x, y) = \text{Li}_2(x) + \frac{1}{2} \ln^2(1-x) + \text{Li}_2\left(-\frac{x(1-y)}{1-x}\right)
\]

\[
\sigma_{1,1}(x, x^*) = \text{Li}_2(x) + \frac{1}{2} \ln^2(1-x) + \zeta_2
\]

\[
\text{Li}_w(x) = \text{Li}_{w}^{*}(x^*)
\]

\[
\sigma_{1,1}(x, y) + \sigma_{1,1}(y, x) = \ln(1-x) \ln(1-y) + \text{Li}_2(xy).
\]

Root structure \( l = 12 \):

\[
\left\{ e_{12}^k \right\}_{k=1}^{12} \equiv \left\{ e_{12}, e_6, e_4, e_3, e_{12}^5, e_2, e_{12}^5^*, e_3^*, e_4^*, e_6^*, e_{12}^*, e_1 \right\}.
\]

\[
\text{Im} \left[ \text{Li}_n(e_i^k) \right] = r_{n,l,k} \pi^n \text{ for } n \text{ odd}, \quad \text{Re} \left[ \text{Li}_n(e_i^k) \right] = r_{n,l,k} \pi^n \text{ for } n \text{ even}, r_{n,l,k} \in \mathbb{Q}.
\]
\( w = 1 \)

| \( l \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| basis set | 0 | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 3 | 4 | 4 | 6 | 4 | 7 | 5 | 6 | 5 | 9 | 5 | 10 | 6 |
| Racinet '01 new elements | 0 | 1 | 2 | 0 | 2 | 0 | 3 | 1 | 2 | 0 | 5 | 1 | 6 | 0 | 2 | 2 | 8 | 0 | 9 | 2 |

The number of the basis elements spanning the \( w=1 \) cyclotomic harmonic polylogarithms at \( l \)th roots of unity up to 20.

**Basis Elements:**

\[
\ln(2); \ln(3), \pi; \text{Re}(\text{Li}_1(e_5)), \text{Re}(\text{Li}_1(e_5^2)); \text{Re}(\text{Li}_1(e_7^k))\bigg|_{k=1}^3; \text{Re}(\text{Li}_1(e_8));
\]

\[
\text{Re}(\text{Li}_1(e_9)), \text{Re}(\text{Li}_1(e_9^2)); \text{Re}(\text{Li}_1(e_{11}^k))\bigg|_{k=1}^5; \text{Re}(\text{Li}_1(e_{12})); \text{Re}(\text{Li}_1(e_{13}))\bigg|_{k=1}^6;
\]

\[
\text{Re}(\text{Li}_1(e_{15})), \text{Re}(\text{Li}_1(e_{15}^2)); \text{Re}(\text{Li}_1(e_{16})), \text{Re}(\text{Li}_1(e_{16}^3)); \text{Re}(\text{Li}_1(e_{17}))\bigg|_{k=1}^8;
\]

\[
\text{Re}(\text{Li}_1(e_{19}))\bigg|_{k=1}^9; \text{Re}(\text{Li}_1(e_{20})), \text{Re}(\text{Li}_1(e_{20}^3)); \ldots
\]
$w = 2$

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Li$_2$ basis | 1 | 1 | 2 | 2 | 3 | 2 | 4 | 3 | 4 | 3 | 6 | 3 | 7 | 4 | 5 | 5 | 9 | 4 | 10 | 5 |
| Li$_2$ new basis | 1 | 0 | 1 | 1 | 2 | 0 | 3 | 1 | 2 | 0 | 5 | 0 | 6 | 0 | 1 | 2 | 8 | 0 | 9 | 1 |
| Racinet '01 new elements | 1 | 2 | 3 | 3 | 5 | 5 | 8 | 7 | 10 | 10 | 16 | 12 | 21 | 17 | 21 | 21 | 33 | 23 | 40 | 29 |
| Racinet '01 new elements | 1 | 1 | 1 | 1 | 2 | 2 | 4 | 4 | 7 | 6 | 10 | |

The number of the basis elements spanning the dilogarithms resp. $w = 2$ cyclotomic harmonic sums at $l$th roots of unity up to 20.

**Basis Elements:**

$\pi; \text{Im}(\text{Li}_2(e_3)); C; \text{Im}(\text{Li}_2(e_5)), \text{Im}(\text{Li}_2(e_5^2)); \text{Li}_4(1/2); \text{Im}(\text{Li}_2(e_7^k)) \bigg|_{k=1}^3$, $\sigma_{1,1}(e_7, e_7^2)$; $\text{Im}(\text{Li}_2(e_8)), \sigma_{1,1}(e_8, e_4), \sigma_{1,1}(e_8, e_8^3); \text{Im}(\text{Li}_2(e_9)), \text{Im}(\text{Li}_2(e_9^2)), \sigma_{1,1}(e_9, e_9^2)$, $\sigma_{1,1}(e_9, e_3), \sigma_{1,1}(e_9^2, e_3); \sigma_{1,1}(e_5, e_2), \sigma_{1,1}(e_5^2, e_2), \sigma_{1,1}(e_{10}, e_5), \sigma_{1,1}(e_{10}, e_{10}^3); \text{Im}(\text{Li}_2(e_{11}^k)) \bigg|_{k=1}^5$, $\sigma_{1,1}(e_{11}, e_{11}^k) \bigg|_{k=2}^4$, $\sigma_{1,1}(e_{11}^2, e_{11}^k) \bigg|_{k=3}^4$, ...
Conclusions

- The cyclotomic polynomials provide a natural extensions of the letters used with iterated integrals leading to **harmonic polylogarithms**. Corresponding terms occur in **massive** higher order calculations.

- Via a Mellin transform the **cyclotomic harmonic sums** are associated, together with the corresponding values for $N \to \infty$. All these systems obey (quasi)shuffle algebras.

- Different special relations of cyclotomic harmonic sums were derived, including differentiation to $N$. In all cases the asymptotic representations can be generated analytically.

- A large number of **new constants** emerges for the **infinite sums**, beyond the MZVs. They were derived and reduced to bases for the real representations.

- Infinite harmonic sums with weights at roots of unity were investigated. They obey more special relations than the case for real representations. There are relations of these quantities to the so-called **motivic numbers** (P. Deligne); Goncharov, Racinet.