Controlling Chaos using an Exponential Control

Sangeeta D. Gadre and V. S. Varma
Department of Physics and Astrophysics
University of Delhi, Delhi, India.

Abstract

We demonstrate that chaos can be controlled using a multiplicative exponential feedback control. All three types of unstable orbits - unstable fixed points, limit cycles and chaotic trajectories can be stabilized using this control. The control is effective both for maps and flows. The control is significant, particularly for systems with several degrees of freedom, as knowledge of only one variable on the desired unstable orbit is sufficient to settle the system on to that orbit. We find, that in all the cases studied, the transient time is a decreasing function of the stiffness of control. But increasing the stiffness beyond an optimum value can increase the transient time. The control can also be used to create suitable new stable attractors in a map, which did not exist in the original system.
1 INTRODUCTION

The problem of controlling chaos has recently received much attention [1-8]. A chaotic system in general cannot be made to converge to a freely evolving desired trajectory, whether periodic or chaotic, because of the inherent unpredictibility of the system. The control of chaos in this context consists of forcing the system to evolve along a desired trajectory. Pecora and Carroll [1-3] demonstrated that two identical chaotic systems driven by a common signal display asymptotic convergence of their trajectories even though they may have started from very different initial conditions provided the Lyapunov exponents of the driven system are all negative. Such behaviour was demonstrated numerically and also confirmed experimentally.

Ott, Grebogi and Yorke (OGY) [4,5] succeeded in forcing a chaotic system on to one of its own unstable periodic orbits by making a set of small time-dependent perturbations on the system parameters. They demonstrated their method numerically by controlling the Hénon map. They also pointed out that as an infinite number of unstable periodic orbits are embedded in a chaotic attractor, the stabilization of these unstable periodic orbits can lead to an enhancement of the system’s performance as well as to the adaptibility of the system to varying performance requirements since the system behaviour can be changed by perturbing the system to stabilize different unstable orbits. In the absence of chaos, separate systems would be required for each of the different responses. The effectiveness of the OGY method of control has been confirmed experimentally in several systems [6-8].

In this paper we have succeeded in implementing a novel method for controlling chaos. We have used a multiplicative exponential feedback control on a parameter of the system, with the argument of the exponential being proportional to the feedback response of the system, i.e. the difference between the desired value and the actual value of one of the
suitably chosen variables of the system. Our method is a combination of variable and parameteric control and it can stabilize all three types of unstable orbits - unstable fixed points, limit cycles and chaotic trajectories. It is found to work effectively both for maps and flows.

Consider a general N-dimensional dynamical system

\[
\dot{\vec{X}} = \vec{F} (\vec{X}; \mu; t) \tag{1}
\]

where \( \vec{X} \equiv (X_1, X_2, \ldots, X_N) \) are variables and \( \mu \equiv (\mu_1, \mu_2, \ldots, \mu_K) \) are parameters whose values determine the nature of the dynamics. The stabilization of a desired unstable attractor or a chaotic trajectory is possible by multiplying a suitably chosen parameter, say \( \mu_r \), in (1) by an exponential feedback control involving only one suitably chosen variable, say \( X_l \), with the form of control being given by

\[
\exp[\epsilon (X_l - X_l^*)]. \tag{2}
\]

Here \( X_l \) is the actual value of one of the variables of the system after applying the control, \( X_l^* \) is the desired value of that variable and \( \epsilon \) is the ”stiffness” of the control which can take both positive and negative values.

The dynamics of the modulated system in the presence of control is given by

\[
\dot{\vec{X}} = \vec{F} (\vec{X}; \mu_1, \mu_2, \ldots, \mu_r \exp[\epsilon (X_l - X_l^*)], \ldots, \mu_K; t). \tag{3}
\]

Note that the control becomes passive once the desired goal \( \vec{X}^* \) is achieved. If fluctuations drive the system off the desired goal, the control reactivates.

The control works for those combinations of controlling parameters and variables of the system for which the largest real part of the Lyapunov exponents of the modulated
system, represented by eq. (3), is negative. The feedback function in the expression of the control involves only one suitably chosen variable $X_l$ to convert the desired repellor - whether a fixed point, a limit cycle or a chaotic orbit, into an attractor. This shows that the knowledge of only one variable on the desired unstable orbit is sufficient to settle the system on to that orbit. This makes the control particularly useful for systems with several degrees of freedom. For example, a desired unstable fixed point of the Lorenz system [9] can be stabilized using control with feedback depending on any one of the $X$, $Y$ or $Z$ variables whereas an unstable limit cycle or a chaotic trajectory of the Lorenz system can be converted into an attractor using feedback depending on the $Z$ variable only. For stabilizing most repellors, it is sufficient to multiply one chosen parameter by the exponential control; but sometimes, as in the case of one of the two fixed points of the Hénon map [10], both the parameters $a$ and $b$ of the system have to be multiplied by the exponential control. Although in all the cases we have studied, only one variable on the desired orbit is sufficient to make all the Lyapunov exponents of the system negative, it may not be possible in some other systems where more than one variable may be required for the stabilization of unstable orbits. But the fact remains that the control uses only a subset of the variables and the parameters for controlling chaos. We have tested our control for stabilizing different types of orbits for the logistic map, the Hénon map and the Lorenz system and found it to work effectively.

A quantity of obvious interest in the context of controlling chaos is the time required for the system to settle on to the desired orbit. This of course depends upon the stiffness of the control i.e. $\epsilon$. For a given $\epsilon$ we study the time $\tau$ required for the system to approach within a distance $\omega$ of the desired orbit starting from some initial point. If $\omega_o$ is the initial distance from the desired orbit, then it is clear that the length of the transient $\tau$ and $\omega$
are related by $\omega = \omega_o \exp(\lambda \tau)$, where $\lambda$ is the largest real part of the Lyapunov exponents. The slope of the plot of $\tau$ against $\ln(\omega/\omega_o)$ is nothing but $1/\lambda$. This of course has to be negative for convergence. The values of the Lyapunov exponents calculated from such points are found to be in good agreement with those obtained either numerically using the method given in [11] or analytically wherever this is possible. We have studied the transient time $\tau$ required for settling on to a desired orbit within a given accuracy $\omega$ as a function of $\epsilon$ and found that $\tau$ in general is a decreasing function of $\epsilon$. But there exists an optimum stiffness of control beyond which increase in $\epsilon$ increases $\tau$. This behaviour of $\tau$ is found to be the same as that of the $\lambda$ with $\epsilon$ for any orbit.

Finally, for discrete maps we have also tried to stabilize orbits which are not the natural fixed points (stable or unstable) of the original system. We have found that the control succeeds in creating desired new stable attractors which are not the natural attractors of the unmodulated system, depending on our requirement and use, using the discrete map. However we cannot stabilize any arbitrary orbit. The functional form of the map and the control criterion decides which orbits can be forced on to the system.

There is one drawback. The form of the control is such that for a given system the unstable fixed point represented by a null vector cannot be stabilised. The reason is obvious. The Jacobian of the modulated system given by (3) evaluated at such a point in the presence of the control is the same as that of the unmodulated system (1). So the eigenvalues of the system remain unchanged in the presence of the control. Hence such a fixed point remains unstable even under control. Such points may, however, get stabilized in the process of stabilizing other fixed points, resulting in the coexistence of more than one attractor.

The organization of this paper is as follows. In section 2 we study exponential control
for stabilizing an unstable fixed point. We deal first with 1-D discrete systems, then with 2-D discrete systems and finally with continuous dynamical systems. In section 3.1 we extend the control algorithm so as to stabilize unstable closed orbits for flows. We suggest a more effective control for stabilization of higher orbits of a discrete dynamical system in section 3.2. In section 4 we discuss the stabilization of chaotic orbits. We use our control for stabilization of arbitrary fixed points for discrete systems in section 5. Finally we summarize our results in section 6.

2 EXPONENTIAL CONTROL FOR STABILIZING AN UNSTABLE FIXED POINT

2.1 1-D Discrete System

A representative one dimensional map is the logistic map given by

\[ X_{n+1} = 4\mu X_n(1 - X_n) \equiv F(\mu, X_n) \]  

(4)

where \( 0 \leq X, \mu \leq 1 \). The map has two period one fixed points corresponding to any value of \( \mu \), which are \( X_a^* = 0 \) and \( X_b^* = 1 - 1/(4\mu) \). The fixed point \( X_a^* \) is unstable for \( \mu > 0.25 \) while \( X_b^* \) is unstable for both \( 0 \leq \mu < 0.25 \) and \( \mu > 0.75 \). As mentioned in the introduction, the control is ineffective in stabilizing the unstable fixed point \( X_a^* \).

For stabilizing \( X_b^* \), we multiply \( \mu \) by \( \exp[\epsilon(X_n - X_b^*)] \) in (4) so that the logistic map in the presence of control is given by

\[ X_{n+1} = 4\mu \exp[\epsilon(X_n - X_b^*)]X_n(1 - X_n) \equiv F(\mu, X_n). \]  

(5)

For a given \( \mu \) and for \( \epsilon \) in the range \( \epsilon_{min} < \epsilon < \epsilon_{max} \), the Lyapunov exponent is negative (i.e. \( \frac{dF}{dX}|_{X^*} \) lies in the interval \((-1, 1)\)) and hence \( X_b^* \) becomes a stable fixed point (an attractor) of the modulated map. (This is found to be a general feature of the control
for maps as well as for flows.) The actual expressions for $\epsilon_{\text{min}}$ and $\epsilon_{\text{max}}$ for (5) are:

$$
\epsilon_{\text{min}} = 4\mu(4\mu - 3)/(4\mu - 1) \quad \text{and} \quad \epsilon_{\text{max}} = 4\mu.
$$

For a given $\mu$ and $\omega$, the transient $\tau$ is found to be the decreasing function of $\epsilon$, however, there is an optimum stiffness control, beyond which increasing $\epsilon$ increases the transient time. This behaviour is also seen in the variation of the $\lambda$ with $\epsilon$. The optimum stiffness of control corresponds to that value of $\epsilon$ for which $\lambda$ is minimum. For the logistic map, $\lambda = -\infty$ for $\epsilon = 4\mu(4\mu - 2)/(4\mu - 1)$ and the system settles to $X_b^*$ more slowly both for larger and smaller $\epsilon$.

### 2.2 2-D Discrete System

We have studied the Hénon map [10] as an example of 2-D maps. It is given by

$$
X_{n+1} = Y_n + 1 - aX_n^2
$$

$$
Y_{n+1} = bX_n
$$

where $X$ and $Y$ are variables and $a$ and $b$ are the controlling parameters. The map has two critical points $(X_\pm, Y_\pm)$ where

$$
X_\pm = (2a)^{-1}(-1 + b \pm \sqrt{(1-b)^2 + 4a})
$$

$$
Y_\pm = b X_\pm
$$

These two critical points are unstable for $a = 1.4$ and $b=0.3$.

To stabilize $(X_+, Y_+)$, we multiply the parameter $b$ by $\exp[\epsilon(X_n - X_+)]$ so that the dynamics in the presence of control is given by

$$
X_{n+1} = Y_n + 1 - aX_n^2
$$

$$
Y_{n+1} = b\exp[\epsilon(X_n - X_+)]X_n.
$$

The other unstable critical point becomes stable only when both parameters $a$ and $b$ are multiplied by $\exp[\epsilon(X_n - X_-)]$, so that the map in the presence of the control for
stabilizing \((X_-, Y_-)\) is given by

\[
X_{n+1} = Y_n + 1 - a \exp[\epsilon(X_n - X_-)]X_n^2
\]

\[
Y_{n+1} = b \exp[\epsilon(X_n - X_-)]X_n.
\]

(8)

For a given value of \(a\) and \(b\), the behaviour of the transient time \(\tau\) with \(\epsilon\) for a given accuracy \(\omega\) is similar to that seen in discrete 1-D maps.

2.3 Continuous Dynamical System

We have studied the Lorenz system [9] as an example of a continuous dynamical system. This is governed by the equations

\[
\dot{X} = \sigma(Y - X)
\]

\[
\dot{Y} = -XZ + rX - Y
\]

\[
\dot{Z} = XY - bZ
\]

(9)

where \(X, Y, Z\) are variables and \(\sigma, r, b\) are the controlling parameters. The system has three critical points, \(viz\)

\(X' = 0, Y' = 0, Z' = 0\)

\(X'' = \sqrt{b(r-1)}, Y'' = \sqrt{b(r-1)}, Z'' = r - 1\)

and

\(X''' = -\sqrt{b(r-1)}, Y''' = -\sqrt{b(r-1)}, Z''' = r - 1.\)

We choose \(\sigma = 10, b = 8/3\) and \(r = 60\) for which all three critical points are unstable fixed points of the Lorenz system. The first critical point \((X', Y', Z')\) cannot be made stable using our form of control. This has been discussed in the introduction. The second and third critical points can be stabilized by multiplying \(b\) by \(\exp[\epsilon(Z - Z'')]\) and \(\exp[\epsilon(Z - Z''')]\) respectively. It is also possible to stabilize these points using other combinations of parameters and variables, provided the control criterion i.e., the largest real part of the
Lyapunov exponents, $\lambda$, is negative, is satisfied. Such combinations, with the corresponding values of $\lambda$, are listed in table I for $\epsilon = -0.001, -0.01$ and $-0.1$. Similarly, $\lambda$ can be worked out for positive values of $\epsilon$ also. The table shows that the $r$-X, $r$-Y and b-Y forms of control work for suitably chosen negative values of $\epsilon$. On the other hand, it is the $r$-Z, $b$-X and b-Z forms of control which are effective for suitably chosen positive values of $\epsilon$. For the rest of the forms $viz.$ $\sigma$-X, $\sigma$-Y and $\sigma$-Z, the value of $\lambda$ remains positive for all $\epsilon$ (the Lyapunov exponents with these forms of control are independent of $\epsilon$ and are the same as that of the unmodulated system) thereby implying that these forms of exponential control cannot stabilize the unstable fixed points of the Lorenz system.

For a fixed $\omega$, the transient time $\tau$, using b-Z exponential control, is found to be a decreasing function of $\epsilon$. However there exists an optimum value of $\epsilon$ beyond which increasing $\epsilon$ increases $\tau$ (as in the case of discrete 1-D maps). The Lyapunov exponents obtained analytically for the second and third critical points in the presence of control have the same value. We plot $\lambda$ vs $\epsilon$ in fig. (1). The variation of $\lambda$ with $\epsilon$ shows the same behaviour as that of $\tau$ with $\epsilon$. The optimum value of $\epsilon$ for which $\tau$ is minimum also corresponds to the most negative value of $\lambda$ and is 0.2 for the given values of the parameters. The value of $\lambda$ obtained analytically is $-6.8993$ for $\epsilon = 0.2$. On plotting $\tau$, as a function of $\ln(\omega/\omega_o)$ for the same value of $\epsilon$, we find that the reciprocal of the slopes for the second and third critical points are $\approx -6.92$, which is in good agreement with the value obtained analytically. To compare our results with those of Pecora and Carroll, with both $X$ and $Y$ drives, we first consider the control with the parameter $r$ multiplied by the exponential feedback function involving the $X$ variable for the same parameter values (such a choice of the control is made because Pecora and Carroll’s method uses variables $X$ and $Y$ as drives). We find that the value of $\lambda$ with this form of control is $-4.536$ for
\( \epsilon = -0.17 \). The value of \( \lambda \) for control involving the parameter \( r \) and the variable \( Y \) is 
\(-7.593 \) for \( \epsilon = -0.05 \), for the same parameter values. The corresponding values of \( \lambda \) in
the method suggested by Pecora and Carroll for the same parameter values and are \(-1.83
\) for the \( X \) drive and \(-2.85 \) for the \( Y \) drive [12]. Thus our values of \( \lambda \) are more negative
and hence the control asserts itself quicker. This can be seen in fig. (2) for the case of the
second unstable fixed point of the Lorenz system. This result is not surprising in view of
the fact that our control has an exponential form.

3 EXPONENTIAL CONTROL FOR STABILIZING
UNSTABLE HIGHER PERIOD ORBITS

3.1 STABILIZATION OF UNSTABLE CLOSED ORBITS
FOR FLOWS

The unstable limit cycle of a continuous system can be converted into a stable attractor by multiplying one suitably chosen parameter, say \( \mu_1 \), (or may be more parameters
depending on the system) by the exponential feedback function which depends only on
one of the variable say \( X_r \), i.e.

\[
\mu_1 \rightarrow \mu_1 \exp[\epsilon (X_r - X_r^u)]
\]

where \( X_r \) is the actual value of the variable of the given system with the feedback control
under consideration and \( X_r^u \) is the value of the \( X_r \) coor dinate on the desired unstable
limit cycle which is required to be stabilized. \( X_r^u \) is obtained by allowing the system to
evolve freely on the desired unstable orbit with the same parameter values but without
imposing the control.

We have implemented this idea for controlling unstable limit cycles of the Lorenz
system. We choose parameter values \( \sigma = 10, b = 8/3, r = 28 \). For these parameters one
of the unstable limit cycles has a point with coordinates \((-12.786189, -19.364189, 24.00)\) [13]. The \(\lambda\) of the system is found to be positive. Since the limit cycle is unstable, so starting with an initial state, even very close to the limit cycle, the trajectory will diverge away from the limit cycle. But after implementation of the control, multiplying parameter \(b\) by \(\exp[\epsilon(Z - Z^u)]\), the Lyapunov exponents are found to be functions of \(\epsilon\) and for a certain range of values of \(\epsilon\), the real part of all the Lyapunov exponents become negative thereby implying the stabilization of the limit cycle. Now, nearby trajectories are found to converge to the desired limit cycle as shown in fig. (3) also. The basin of attraction of the limit cycle is not infinite implying that there is a set of initial values starting from which the trajectories converge to the desired limit cycle. Otherwise they may escape to infinity. The transient \(\tau\) shows a similar trend as in the previous sections. But for a given accuracy \(\omega\) the minimum value of \(\tau\) corresponding to the optimum \(\epsilon\) is much higher compared to the case of the unstable fixed point of the Lorenz system. Moreover, the variation of \(\tau\) with \(\epsilon\) for a given \(\omega\) is not as rapid as in the case of the unstable fixed point. Consequently the \(\tau\) vs \(\epsilon\) plot is quite flat and an optimum \(\epsilon\) cannot be obtained very accurately.

We have also tested the control for another limit cycle of the Lorenz attractor. For the same parameter values, the coordinates of a point on a different limit cycle are \((-13.917865, -21.919412, 24.00)\) [13]. The control was found to work effectively in this case with the same features as have been listed above.

### 3.2 STABILIZATION OF HIGHER PERIOD ORBITS OF A DISCRETE DYNAMICAL SYSTEM

We can extend the above control to convert unstable fixed points of higher period say \((\overline{X^1}, \overline{X^2}, \ldots, \overline{X^k})\) to stable fixed points for given values of the control parameters. What is required is a feedback that encodes as much information about the periodic
orbits as is necessary for its unique characterization. But there is a practical problem here. Since in this case the form of the control given by eq. (2) requires the convergence of the chosen variable to a set of k values of that variable, the controlling technique diminishes in utility with increase in period. For higher period orbits of a discrete dynamical system the more effective control is one which employs a logical OR structure in the feedback function. So in order to stabilize unstable fixed points of period k \(i.e. \{X^{*i}\}\) one of the parameters say \(\mu_j\) in (1) is multiplied by

\[
\text{exp}[\epsilon \prod_{i=1}^{k}((X_l)_{n} - X_l^{*i})]
\]

where \((X_l)_{n}\) is the value of one of the chosen variables \(X_l\) at time \(n\) of the modulated map after applying the control and \((X_l^{*1}, X_l^{*2}, ..., X_l^{*k})\) is the set of k values of the variable \(X_l\) on the desired unstable period k orbit.

We implemented this in the case of the logistic map for stabilizing period 2 orbit having \(X^{*1}\) and \(X^{*2}\) as the fixed points and found it to work effectively.

4 STABILIZATION OF CHAOTIC ORBIT

We have also tried to converge different trajectories to a particular desired chaotic trajectory using our control. Again taking the Lorenz system as an example we choose parameter values \(\sigma = 16, b = 4, r = 40\). The real parts of the Lyapunov exponents are \((1.37, 0.0, -22.37)\) [14] implying that this is a chaotic regime. We chose a chaotic trajectory starting with initial coordinates \((10.0, 0.0, 30.0)\). We applied our control by multiplying \(b\) by \(\text{exp}[\epsilon(Z - Z^c)]\), where \(Z^c\) is the Z coordinate of a point on the freely evolving chaotic trajectory. We found that the real parts of all the Lyapunov exponents of the modified Lorenz system become negative for a certain range of values of \(\epsilon\), showing that the desired chaotic trajectory has become a stable trajectory and different closeby
trajectories starting from different initial states converge to the desired trajectory. For a given $\epsilon$ the reciprocal of the slope of $\tau$ vs $\ln(\omega/\omega_o)$ is in good agreement with the minimum value of $\lambda$. It has been verified numerically that the system settles down on to the desired trajectory in the presence of the control. The features of the control remain the same as in the previous section. We have also tested our control for other chaotic trajectories and found it to work effectively.

5 STABILIZATION OF ARBITRARY FIXED POINTS FOR DISCRETE SYSTEM

With exponential control it is possible to create new stable attractors which do not exist in the unmodulated system. This allows the modulated system to settle down to arbitrary fixed points which are not the fixed points (stable or unstable) of the unmodulated map. As an example we again take the logistic map given by eq. (4). Suppose the requirement is to stabilize the system to say a period 2 attractor $(X_1, X_2)$ which does not exist in the unmodulated map, implying that $X_1$ and $X_2$ are not the fixed points of $F^2(\mu, X)$. The equations governing the dynamics in the presence of the control are

$$X_{n+1} = 4\mu_1 \exp[\epsilon(X_n - X_1)(X_n - X_2)]X_n(1 - X_n)$$

$$X_{n+2} = 4\mu_2 \exp[\epsilon(X_{n+1} - X_1)(X_{n+1} - X_2)]X_{n+1}(1 - X_{n+1})$$

(11)

where $\mu_1$ and $\mu_2$ are given by

$$\mu_1 = X_2/(4X_1(1 - X_1))$$

and

$$\mu_2 = X_1/(4X_2(1 - X_2))$$

Any arbitrary combination of $X_1$ and $X_2$ cannot be stabilized using eq. (11). Only those combinations of $X_1$ and $X_2$ can be stabilized in the logistic map using the exponential
control for which the $\lambda$ of the modulated map is negative. The coordinates of the points lying in the region enclosed by the curves and the coordinate axes in fig. (4) represent such combinations of $X_1$ and $X_2$ for which $\lambda$ is negative, and only such combinations can be stabilized using the map given by eq. (11).

This procedure can easily be extended to stabilize the system to higher period orbits even when the points on the orbit do not correspond to either stable or unstable fixed points of the higher period orbits of the original system.

6 CONCLUSION

Exponential control is found to be effective both for maps as well as for flows. Since the knowledge of only a subset of variables on the desired unstable orbit is sufficient to settle the system on to that orbit, this makes the control more significant particularly for systems with several degrees of freedom. In the presence of control, there exists a range of values of the stiffness constant $\epsilon$ for which a repellor - an unstable fixed point, a limit cycle or a chaotic trajectory, of a system can be converted into an attractor. The $\lambda$ is found to be the function of $\epsilon$. The transient time $\tau$ is a decreasing function of $\epsilon$ for a given accuracy $\omega$. However there exists an optimum stiffness control beyond which increasing $\epsilon$ can increase $\tau$. This behaviour of $\tau$ is found to hold for different types of orbits of different systems and is similar to the behaviour of $\lambda$ with $\epsilon$. The optimum $\epsilon$ corresponds to the value of $\epsilon$ where $\lambda$ is minimum.

Exponential control is found to settle the Lorenz system on to its non-zero unstable fixed points faster than the control of Pecora and Carroll. In addition, it is also possible to stabilize arbitrary fixed points, chosen according to one’s requirements, which do not correspond to either stable or unstable fixed points of the unmodulated map.

We are investigating the effect of exponential control in the presence of noise. It
would be interesting to see whether such control is capable of controlling spatio-temporal chaos in coupled map lattice systems [15]. It would also be interesting to see whether exponential control can successfully recover the original signal when the original signal has been masked by the addition of noise [16].

Acknowledgements:

We would like to thank Dr. Neelima M. Gupte for her invaluable comments and suggestions. We also thank the Inter University Center for Astronomy and Astrophysics (IUCAA) for allowing the use of their computing facility. One of the authors (SDG) also thanks Kirori Mal College, University of Delhi, Delhi, for the grant of study leave.
REFERENCES

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. 64, 821 (1990).

[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. 67, 645 (1991).

[3] L. M. Pecora and T. L. Carroll, Phys. Rev. A 44, 2374 (1991).

[4] E. Ott, C. Grebogi and J. A. Yorke, Phys. Rev. Lett. 64, 1196 (1990).

[5] T. Shinbrot, E. Ott, C. Grebogi and J. A. Yorke, Phys. Rev. Lett. 65, 3215 (1990).

[6] W. L. Ditto, S. N. Rauseo and M. L. Spano, Phys. Rev. Lett. 65, 3211, (1990).

[7] E. R. Hunt, Phys. Rev. Lett. 67, 1953 (1991).

[8] C. ReyI, L. Flepp, R. Badii and E. Brun Phys. Rev. E 47, 267 (1993).

[9] E. N. Lorenz, J. Atoms. Sci. 20, 130 (1976).

[10] M. Hénon, Commun. Math. Phys. 50, 69 (1976).

[11] C. Sparrow, The Lorenz equations, Bifurcations and Chaos (Springer Verlag, New York, 1982).

[12] N. Gupte and R. E. Amritkar, Phys. Rev. E 48, R1620 (1993).

[13] J. H. Curry, in Global Theory of Dynamical Systems, Lect. Notes in Mathematics, vol. 819, Ed. Z. Nitecki and C. Robinson (Springer Verlag, Berlin Heidelberg, 1980) p. 111-120.

[14] I. Shimada and T. Nagashima, Prog. Theor. Phys. 69, 1605 (1979).

[15] G. Hu and Z. Qu, Phys. Rev. Lett. 72, 68 (1994).

[16] K. M. Cuomo and A. V. Oppenheim, Phys. Rev. Lett. 71, 65 (1994).
FIGURE CAPTIONS

Fig.1 The variation of $\lambda$ with $\epsilon$ for the Lorenz system while stabilizing the unstable fixed point $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ using the exponential control involving the parameter $b$ and the variable $Z$, with $\sigma = 10$, $b = 8/3$ and $r = 60$.

Fig.2 Plot of the transient time $\tau$ vs $\ln(\omega/\omega_0)$ for the Lorenz system with $\sigma = 10$, $b = 8/3$ and $r = 60$ while stabilizing the unstable fixed point $(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ using Pecora and Carroll’s method with $X = \sqrt{b(r-1)}$ drive (triangles), with $Y = \sqrt{b(r-1)}$ drive (squares), and also with exponential control involving the parameter $r$ and the variable $X$ (circles), and $r$ and $Y$ (stars).

Fig.3 The dotted trajectory shows the convergence of a nearby trajectory to the unstable closed orbit, represented by the bold curve, projected on to the $(X,Z)$ plane of the Lorenz system with $\sigma = 10$, $b = 8/3$, and $r = 28$ and a point with coordinates $(-12.786189, -19.364189, 24.00)$.

Fig.4 The coordinates of the points lying in the region enclosed by the plotted curves and the coordinate axes represent the combinations of $X_1$ and $X_2$ which can be stabilized in the logistic map using exponential control for $\epsilon = 0.1$ and 1.0.
TABLE CAPTION

The values of \( \lambda \), the largest real part of the Lyapunov exponents, of the Lorenz attractor
under exponential control while stabilizing the fixed point \((\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)\)
for \( \sigma = 10, b = 8/3 \) and \( r = 60 \) for different combinations of parameters and variables.
| Control form | \( \epsilon = -0.001 \) | \( \epsilon = -0.01 \) | \( \epsilon = -0.1 \) |
|--------------|----------------|----------------|----------------|
| r-X          | 0.616          | -0.267         | -3.950         |
| r-Y          | 0.394          | -2.381         | -7.180         |
| r-Z          | 0.767          | 1.117          | 2.250          |
| b-X          | 0.739          | 0.927          | 2.425          |
| b-Y          | 0.707          | 0.611          | -0.113         |
| b-Z          | 0.786          | 1.403          | 7.884          |
| \( \sigma \)-X | 0.717          | 0.717          | 0.717          |
| \( \sigma \)-Y | 0.717          | 0.717          | 0.717          |
| \( \sigma \)-Z | 0.717          | 0.717          | 0.717          |