On integrability of a third-order complex nonlinear wave equation

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Abstract

We show that the new third-order complex nonlinear wave equation, introduced recently by Müller-Hoissen [arXiv:2202.04512], does not pass the Painlevé test for integrability. We find two reductions of this equation, one integrable and one non-integrable, whose solutions jointly cover all solutions of the original equation.

1 Introduction

The following new third-order complex nonlinear wave equation was introduced recently by Müller-Hoissen [1]:

\[ \left( \frac{f_x}{f} \right)_t + 2 (f f^*)_x = 0, \tag{1} \]

where \( f(x,t) \) is a complex function of two real variables, subscripts denote respective derivatives, and the asterisk stands for the complex conjugate. This nonlinear wave equation (1) was called a completely integrable partial differential equation in [1], and its multi-soliton solutions were obtained there.

Let us note, however, that this new nonlinear equation was studied in [1] not in its original form (1) but in the form of the following system of two equations:

\[ a_t = (f f^*)_x, \quad f_{xt} + 2 a f = 0, \tag{2} \]

where \( a(x,t) \) is a real function. Being the first “negative flow” of the nonlinear Schrödinger equation’s hierarchy, this nonlinear system (2) is integrable and possesses multi-soliton solutions. The integrable system (2) is obviously a reduction of the new nonlinear wave equation (1), in the sense that all solutions of (2) (except for \( f = 0 \), of course) are solutions of (1) as well, because (1) follows from (2) through elimination of the dependent variable \( a \). In other words, the new equation (1) possesses multi-soliton solutions because it possesses an integrable reduction. But is the new equation (1) integrable itself? No Lax pair of (1) is known.

In the present paper, we study the integrability of the new equation (1) by means of the singularity analysis (a.k.a. the Painlevé analysis) in its version for partial differential equations [2, 3]. In Section 2 we show that the nonlinear
equation (1) does not pass the Painlevé test for integrability. In Section 3, we find two reductions of the nonlinear equation (1), one integrable and one non-integrable, whose solutions jointly cover all solutions of the original equation. Section 4 contains concluding remarks.

2 Singularity analysis

In our experience, the Painlevé analysis is a reliable and convenient method to test the integrability of nonlinear wave equations, including high-order, non-evolutionary, multi-component and high-dimensional ones [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. The reliability of the Painlevé test for integrability has been empirically verified by the analysis of wide classes of nonlinear equations, such as fifth-order Korteweg–de Vries type equations [20], bilinear equations [21], coupled Korteweg–de Vries equations [22, 23, 24, 25], coupled higher-order nonlinear Schrödinger equations [26], generalized Ito equations [27], sixth-order nonlinear wave equations [28], seventh-order Korteweg–de Vries type equations [29], etc.

To start the singularity analysis of the nonlinear equation (1), we rewrite this equation and its complex conjugate

\[
\left(\frac{f^*}{f^{**}}\right)_t + 2 (ff^*)_x = 0
\]

as the following system of two polynomial equations:

\[
\begin{align*}
ff_{xxt} - ft f_{xt} + 2 f^2 (fg)_x &= 0, \\
gg_{xxt} - gt g_{xt} + 2 g^2 (fg)_x &= 0,
\end{align*}
\]

where \(f^*\) stands for \(f^{**}\). From now on, we consider \(f(x, t)\) and \(g(x, t)\) as two mutually independent complex functions of two complex variables \(x\) and \(t\).

A singularity manifold \(\phi(x, t) = 0\) is non-characteristic for this system (1) if \(\phi_x \phi_t \neq 0\), and we take \(\phi_t = 1\) without loss of generality,

\[
\phi = t + \psi(x), \quad \psi_x \neq 0,
\]

where \(\psi(x)\) is arbitrary. Substitution of the expansions

\[
\begin{align*}
f &= f_0(x)\phi^p + \cdots + f_r(x)\phi^{p+r} + \cdots, \\
g &= g_0(x)\phi^q + \cdots + g_r(x)\phi^{q+r} + \cdots
\end{align*}
\]

to the nonlinear system (1) determines the leading exponents \(p\) and \(q\) (i.e., the dominant behavior of solutions \(f\) and \(g\) near the singularity manifold \(\phi = 0\)) and the resonances \(r\) (i.e., the positions, where arbitrary functions can enter the expansions). In this way, we find the following one branch to study further:

\[
p = q = -1, \quad r = -1, 0, 2, 3, 4,
\]

where \(r = -1\) corresponds to the arbitrariness of \(\psi(x)\) in (5).

For this branch (7), we try to represent the general solution of (1) by the Laurent type expansions

\[
\begin{align*}
f &= \sum_{i=0}^{\infty} f_i(x)\phi^{i-1}, \\
g &= \sum_{i=0}^{\infty} g_i(x)\phi^{i-1},
\end{align*}
\]
with φ given by (5). We substitute (8) to (4), collect terms with φ^n−5, for n = 0, 1, 2, 3, ... separately, and obtain in this way the following.

For n = 0, where we have a single resonance due to (7), we get the expression for g₀,

\[ g₀ = -\frac{1}{f₀}; \]

while the function f₀(x) remains arbitrary.

For n = 1, there is no resonance, and we get the expressions

\[ f₁ = -\frac{f₀'}{2ψ'}, \quad g₁ = -\frac{f₀'}{2f₀²ψ'}; \]

where the prime denotes the derivative with respect to x.

For n = 2, we have a double resonance, the functions f₂(x) and g₂(x) remain arbitrary, the compatibility conditions are identically satisfied by (9) and (10), and no restrictions for ψ(x) and f₀(x) appear.

For n = 3, where we have a single resonance, we get the expression for g₃,

\[ g₃ = \frac{f₃}{f₀} - \frac{g₂f₀'}{2f₀ψ'} + \frac{(f₀')³}{2f₀³(ψ')³} + \frac{f₀'}{2f₀²ψ'}
- \frac{g₂'}{ψ'} + \frac{(f₀')²ψ''}{2f₀³(ψ')³} - \frac{f₀'f₀''}{2f₀³(ψ')³}, \]

the function f₃(x) remains arbitrary, but the following nontrivial compatibility condition appears:

\[ \left( \frac{f₂}{f₀} + f₀g₂ \right)' = 0. \]

The fact that we have got a nontrivial compatibility condition at a resonance means that we have to modify our expansions of solutions by additional logarithmic terms. Consequently, the nonlinear equation (1) does not pass the Painlevé test for integrability.

### 3 Two reductions

Let us return to the nonlinear equation (1) and consider it together with its complex conjugate (3). From these two equations, we get the relation

\[ \left( \frac{fₓt}{f} - \frac{fₓt}{f'} \right)_t = 0 \]

satisfied by any solution of (1). We introduce two real functions of two real variables, u(x, t) and v(x, t), such that

\[ \frac{fₓt}{f} = u(x, t) + iv(x, t), \]
where $i^2 = -1$. Then, it follows from (13) and (14) that $v_t = 0$, and (14) takes the form

$$
\frac{f_{xt}}{f} = u(x, t) + iw(x), \quad (15)
$$

where $w(x)$ is a real function of one real variable.

It was pointed out in [1] that the nonlinear equation (1) is invariant under the transformation

$$
x \mapsto h(x), \quad (16)
$$

where $h(x)$ is an arbitrary function. In other words, if a function $f(x, t)$ is a solution of the nonlinear equation (1), then $f(h(x), t)$ is also a solution of (1), for any function $h(x)$.

Under the transformation (16), the function $w(x)$ in the relation (15) changes in the following way:

$$
w(x) \mapsto h'(x) w(h(x)), \quad (17)
$$

where the prime denotes the derivative. Therefore, for any solution $f(x, t)$ which corresponds to any nonzero function $w(x)$ in (15), the integral

$$
x = \int w(h) \, dh \quad (18)
$$

determines (at least, locally) the function $h(x)$ of the transformation (16), such that the transformed solution $f(h(x), t)$ corresponds to $w = 1$. Consequently, we can use the relation (13) in the form

$$
\frac{f_{xt}}{f} = u(x, t) + ik, \quad k = 0, 1, \quad (19)
$$

provided that, in the case of $k = 1$, every single solution $f(x, t)$ represents the whole its equivalence class $f(h(x), t)$ with any function $h(x)$.

The original nonlinear equation (1) together with the relation (19) give us the following system of two equations:

$$
\begin{align*}
  f_{xt} - (u + ik)f &= 0, \\
  u_t + 2(f^*)(_x) &= 0, \quad k = 0, 1, \quad (20)
\end{align*}
$$

where $f(x, t)$ is a complex function of two real variables, but $u(x, t)$ is a real function of two real variables. The two cases of the system (20), with $k = 0$ and with $k = 1$, are two distinct reductions of the original equation (1). Solutions of these two reductions jointly cover all solutions of (1), provided that every solution $f(x, t)$ of the system (20) with $k = 1$ is generalized as $f(h(x), t)$, with any function $h(x)$. (Note that the system (20) with $k = 1$ is not invariant under the transformation (16), whereas the system (20) with $k = 0$ is invariant.)

The system (20) with $k = 0$ is just the system (2) studied in [1], the correspondence being $u = -2a$. The system (20) with $k = 1$ is something new.

Let us apply the Painlevé test for integrability to the system (20). We combine (20) with its complex conjugate, denote $f^*$ as $g$, treat $f$ and $g$ as mutually
independent, complexify all variables, and consider the following system of three equations:

\[
\begin{align*}
    f_{xt} - (u + ik)f &= 0, \\
    g_{xt} - (u - ik)g &= 0, \\
    u_t + 2(fg)_x &= 0, \\
    &\quad k = 0, 1,
\end{align*}
\]

where \(f(x,t), g(x,t)\) and \(u(x,t)\) are complex function of two complex variables.

We find from (21) that, near any non-characteristic singularity manifold \(\phi(x,t) = 0\) with \(\phi\) given by (5) and arbitrary \(\psi(x)\), the leading exponents of \(f, g\) and \(u\) are \(-1, -1\) and \(-2\), respectively, and the resonances are

\[
r = -1, 0, 2, 3, 4,
\]

where \(r = -1\) corresponds to the arbitrariness of \(\psi(x)\) in (5). Then we substitute the expansions

\[
\begin{align*}
    f &= \sum_{i=0}^{\infty} f_i(x)\phi^{i-1}, &
    g &= \sum_{i=0}^{\infty} g_i(x)\phi^{i-1}, \\
    u &= \sum_{i=0}^{\infty} u_i(x)\phi^{i-2}
\end{align*}
\]

(23) to the system (21), collect terms with \(\phi^{n-3}\) for \(n = 0, 1, 2, 3, \ldots\) separately, and obtain in this way the following.

For \(n = 0\), we get the expressions

\[
\begin{align*}
    g_0 &= -\frac{1}{f_0}, &
    u_0 &= 2\psi', \\
    f_0 &= \sum_{i=0}^{\infty} f_i(x)\phi^{i-1}, &
    g_0 &= \sum_{i=0}^{\infty} g_i(x)\phi^{i-1},
\end{align*}
\]

(24)

the function \(f_0(x)\) remains arbitrary, and no nontrivial compatibility condition appears at this resonance.

For \(n = 1\), where we have no resonance, we get the expressions

\[
\begin{align*}
    f_1 &= -\frac{f_0'}{2\psi'}, &
    g_1 &= -\frac{f_0'}{2f_0\psi'}, &
    u_1 &= 0.
\end{align*}
\]

(25)

For \(n = 2\), we get the expressions

\[
\begin{align*}
    f_2 &= -\frac{f_0(u_2 + ik)}{2\psi'}, &
    g_2 &= \frac{u_2 - ik}{2f_0\psi'},
\end{align*}
\]

(26)

the function \(u_2(x)\) remains arbitrary, and no nontrivial compatibility condition appears at this resonance.

For \(n = 3\), we get the expressions

\[
\begin{align*}
    u_3 &= -\frac{u_2'}{2\psi'} + \frac{(u_2 + ik)\psi''}{2(\psi')^2}, \\
    g_3 &= \frac{f_3}{f_0} - \frac{ikf_0'}{2f_0(\psi')^2} + \frac{(f_0')^3}{2f_0(\psi')^4} - \frac{3u_2'}{4f_0(\psi')^2} \\
    &\quad - \frac{ik\psi''}{4f_0(\psi')^2} + \frac{3u_2\psi'''}{4f_0(\psi')^2} + \frac{(f_0')^2}{2f_0(\psi')^2} - \frac{f_0'f_0''}{2f_0(\psi')^2}.
\end{align*}
\]

(27)
and the function $f_3(x)$ remains arbitrary. However, the following compatibility condition appears at this resonance:

$$k\psi'' = 0,$$

which is not satisfied identically if $k = 1$. Consequently, the system (20) with $k = 1$ does not pass the Painlevé test for integrability.

If $k = 0$, the compatibility condition (28) is satisfied identically, and we continue computations. For $n = 4$, we obtain explicit expressions for $u_4$ and $g_4$ (cumbersome ones, therefore omitted here), the function $f_4(x)$ remains arbitrary, and no nontrivial compatibility condition appears at this resonance. Consequently, the system (20) with $k = 0$ has passed the Painlevé test for integrability (and this is an expected result, of course).

4 Conclusion

In this paper, we studied the integrability of the third-order complex nonlinear wave equation (1), introduced recently in [1]. We used the Painlevé test for integrability and obtained the following results.

The nonlinear equation (1) fails to pass the Painlevé test for integrability. Therefore we believe that one cannot find any good Lax pair for this equation.

The nonlinear equation (1) possesses two reductions, one integrable and one non-integrable, whose solutions jointly cover all solutions of this equation.

The integrable reduction is the system (20) with $k = 0$, studied in [1] in the form (2). This reduction corresponds to the condition $f_{xt}/f = f_{x*}/f^*$ imposed on solutions of (1). This integrable reduction is just what provides the original equation (1) with multi-soliton solutions.

The non-integrable reduction is the system (20) with $k = 1$. It corresponds to the condition $f_{xt}/f \neq f_{x*}/f^*$ imposed on solutions of (1). Solutions of this non-integrable reduction must be generalized by the arbitrary coordinate transformation (16), in order to cover all those solutions of (1) which are not solutions of the integrable reduction.

It seems to be an interesting future problem to find any explicit solutions of the new (non-integrable) system (20) with $k = 1$. We believe that there are no multi-soliton solutions, and that the travelling waves (if there are any) interact inelastically.

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