A THREE-DIMENSIONAL CONSTITUTIVE MODEL FOR POLYCRYSTALLINE SHAPE MEMORY ALLOYS UNDER LARGE STRAINS COMBINED WITH LARGE ROTATIONS

Lei Xu  
Dept. of Aerospace Engineering  
Texas A&M University  
College Station, Texas 77843  
Email: sdf007xulei@tamu.edu

Theocharis Baxevanis  
Dept. of Mechanical Engineering  
University of Houston  
Houston, Texas 77204

Dimitris Lagoudas  
Dept. of Aerospace Engineering  
Dept. of Material Science & Engineering  
Texas A&M University  
College Station, Texas 77843

ABSTRACT

Shape Memory Alloys (SMAs), known as an intermetallic alloys with the ability to recover its predefined shape under specific thermomechanical loading, has been widely aware of working as actuators for active/smart morphing structures in engineering industry. Because of the high actuation energy density of SMAs, compared to other active materials, structures integrated with SMA-based actuators has high advantage in terms of trade-offs between overall structure weight, integrity and functionality. The majority of available constitutive models for SMAs are developed within infinitesimal strain regime. However, it was reported that particular SMAs can generate transformation strains nearly up to 8%-10%, for which the adopted infinitesimal strain assumption is no longer appropriate. Furthermore, industry applications may require SMA actuators, such as a SMA torque tube, undergo large rotation deformation at work. Combining the above two facts, a constitutive model for SMAs developed on a finite deformation framework is required to predict accurate response for these SMA-based actuators under large deformations.

A three-dimensional constitutive model for SMAs considering large strains with large rotations is proposed in this work. This model utilizes the logarithmic strain as a finite strain measure for large deformation analysis so that its rate form hypoelastic constitutive relation can be consistently integrated to deliver a free energy based hyper-elastic constitutive relation. The martensitic volume fraction and the second-order transformation strain tensor are chosen as the internal state variables to characterize the inelastic response exhibited by polycrystalline SMAs. Numerical experiments for basic SMA geometries, such as a bar under tension and a torque tube under torsion are performed to test the capabilities of the newly proposed model. The presented formulation and its numerical implementation scheme can be extended in future work for the incorporation of other inelastic phenomena such as transformation-induced plasticity, viscoplasticity and creep under large deformations.

NOMENCLATURE

\( B \) \( B_{i,j} \) \( D \) \( D_{r} \) \( D_{tr} \) \( F \) \( L \) \( S \) \( S_A \) \( S_M \) \( \Delta S \) \( W \) \( X \) \( \Lambda_{fwd} \) \( \Lambda_{rev} \)

Left Cauchy-Green tensor  
Subordinate eigenprojections of \( B \)  
Rate of deformation tensor  
Rate of deformation tensor of elastic part  
Rate of deformation tensor of transformation part  
Deformation gradient  
Velocity gradient  
Compliance tensor for austenite phase  
Compliance tensor for martensite phase  
Compliance tensor phase difference  
Anti-symmetric part of velocity gradient  
Position vector in reference configuration  
Transformation direction tensor  
Forward transformation direction tensor  
Reverse transformation direction tensor

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Critical thermodynamic driving force weight, integrity and functionality. There has been existing fa-
taining as actuators for active/smart morphing structures in engi-
neering industry. Because of the high actuation energy density
of SMAs, structures integrated with SMA-based actuators has
high advantage in terms of trade-offs between overall structure
weight, integrity and functionality. There has been existing fa-
tigue, damage \cite{12,22} and preliminary corrosion research \cite{3} into
SMAs over the past several decades in order to realize SMA-
based actuators into extensive industry applications.

A substantial of SMAs constitutive theories at continuum
levels have been proposed, the majority of them are within
small deformation regime based on infinitesimal strain assump-
tion. Some thorough review of shape memory models can be
found from Boyd and Lagoudas \cite{4}, Birman and November \cite{5},
Raniecki and Lexcellent \cite{6,7}, Patoo et al. \cite{8,9}, Hackl and
Levitas and Preston \cite{11} etc.. The aforementioned models are able to predict SMAs response accurately within in-
finitesimal strain regime. However, recent publication has re-
ported that shape memory alloys can reversibly deform to a re-
latively large strain up to 8% \cite{12,13}, and also repeated cy-
cling loading has been reported to induce irrecoverable trans-
formation induced plasticity strains up to 20% or more \cite{14,15}. Strain regime in cracked SMA specimen can also go up to 8%
\cite{16,17,18,19}. In addition to such relatively large strains, SMAs-
based devices may also undergo large rotations during its de-
ployment. Combining above two factors, it is indispensable to
develop a constitutive model based on finite deformation frame-
work to provide accurate predictions for these SMA-based actu-
ators under large deformations.

Much efforts has been devoted to proposing such a constitu-
tive model for SMAs at finite deformation framework. Following
finite deformation theory, two commonly accepted kinematic as-
sumptions are usually adopted for the model development. One
is the multiplicative decomposition on deformation gradient, and
the other one is the additive decomposition on the total rate of de-
formation tensor. The first one is simply based on classic crystal-
plasticity theory. In contrast, the second one is following energy
conservation principle that the total energy provided from the
outside can be additively divided into a recoverable part plus a
dissipative part.

Existing approaches based on the multiplicative decompo-
sition to formulate a finite strain constitutive model for SMAs
can be found from literature Auricchio \cite{20,21}, Arghavani et
al. \cite{22,23,24}, Thamburaja and Anand \cite{25,26}, Reese and Christ
\cite{27,28} etc.. However, the complexity of this model type and its
high computational cost hinders its wide usage for industry appli-
cation analysis that involved with a large amount of trial and error
computation. In contrast, finite strain constitutive models based
on additive composition achieves a much simpler model structure
and easier implementation procedures, which enables it gaining
a lot popularity and is commonly used in current available com-
mercial finite element analysis softwares, such as ABAQUS and
ANSYS. However, this model types requires to adopt an ob-
jective rate in its rate form hypoelastic constitutive equation to
achieve the principle of objectivity. A number of objective rates,
such as Zaremba-Jaumann-Noll rate, Green-Naghdi-Dienes rate
and Truesdell rate etc., have been proposed to meet this objectiv-
ity goal. However, those objective rates aforementioned are not
real ‘objective’ for their failure to integrate the rate form hypo-
elastic equation to deliver a free energy based hyperelastic stress-
strain relation [29]. Therefore many spurious phenomenons,
such as artificial residual stress accumulation and shear stress
oscillation, are often observed even for indissipative elastic ma-
terial undergoing closed path cyclic loadings.

It was not until the logarithmic rate proposed and the proof
conducted by Xiao et al. [30, 31], Bruhns et al. [32, 33, 34] and
Meyers et al. [35, 36], that such self-inconsistency issue related
to objective rates has been resolved. In their mathematical proof
from publication [30] in the year of 1997, it was shown that:
the logarithmic rate of logarithmic strain of its Eulerian type is
exactly identical with the rate of deformation tensor, and loga-
rithmic strain is the only one among all other strain measures
enjoying this important property. This new development in fi-
nite deformation theory not only provides consistent solutions
to classical plasticity problems for metallic material, but also
sheds lights on the development of finite strain constitutive model
for SMAs. Available yet still limited publications for SMAs
model developed along this line can be obtained from Müller
and Bruhns [37], Teeriaho [38] and Xu et al. [39].

In this article, a three-dimensional constitutive model for
SMAs considering large strains with large rotations is going to
be proposed. This model utilizes the logarithmic strain as a finite
strain measure for large deformation analysis so that its rate form
hypoelastic constitutive relation can be consistently integrated to
deliver a free energy based hyperelastic constitutive relation.
The martensitic volume fraction and the second-order transformation
strain tensor are chosen as the internal state variables to charac-
terize the inelastic response exhibited by polycrystalline SMAs.
Numerical experiments for basic geometries, such as a bar under
tension and a torque tube under torsion are performed to test the
capabilities of the newly proposed model.

**PRELIMINARY**

**Continuum Mechanics**

Let body $\mathcal{B}$ with its material point defined by a position vec-
tor $\mathbf{X}$ in the reference (undeformed) configuration at time $t_0$, let
vector $\mathbf{x}$ represent the position vector occupied by that material
point in current (deformed) configuration at time $t$, therefore the
deformation process of the material point from its initial config-
uration to current configuration can be defined through the well
known second order deformation gradient tensor $\mathbf{F}(\mathbf{x}, t)$:

$$\mathbf{F}(\mathbf{x}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

(1)

The velocity field of the material point can be defined by the
second order tensor $\mathbf{v}$ as,

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}$$

(2)

Based on the velocity tensor $\mathbf{v}$, the velocity gradient $\mathbf{L}$ can be
derived as,

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}} \mathbf{F}^{-1}$$

(3)

The following equation on polar decomposition of deformation
gradient is well known,

$$\mathbf{F} = \mathbf{U} \mathbf{R} = \mathbf{V} \mathbf{R}$$

(4)

In Eqn.(4), the second order orthogonal tensor $\mathbf{R}$ is called the ro-
tation tensor, i.e., $\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$, where $\mathbf{I}$ is second order iden-
tity tensor. Symmetric and positive definite second order tensors
$\mathbf{U}$ and $\mathbf{V}$ are called right (or Lagrangian) and left (or Eulerian)
stretch tensors, through which the right Cauchy-Green tensor $\mathbf{C}$
and the left Cauchy-Green tensor $\mathbf{b}$ are obtained,

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2$$

(5)

$$\mathbf{b} = \mathbf{F} \mathbf{F}^T = \mathbf{V}^2$$

(6)

The logarithmic (or Hencky) strain of its Lagrangian type $\mathbf{H}$ and
Eulerian type $\mathbf{h}$ are calculated as,

$$\mathbf{H} = \frac{1}{2} \ln(\mathbf{C}) = \ln(\mathbf{U})$$

(7)

$$\mathbf{h} = \frac{1}{2} \ln(\mathbf{b}) = \ln(\mathbf{V})$$

(8)

It is known velocity gradient $\mathbf{L}$ can be additively decomposed
into a symmetric part called the rate of deformation tensor $\mathbf{D}$
plus an anti-symmetric part named spin tensor $\mathbf{W}$.

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \frac{1}{2} (\mathbf{L} - \mathbf{L}^T)$$

(9)
Logarithmic Rate and Logarithmic Spin

As it was discussed in introduction, two commonly accepted kinematic assumption are usually considered in finite deformation theory. The first one is based on multiplicative decomposition of deformation gradient $F$, and the other one is based on additive decomposition of total rate of deformation tensor $D$. For a long period of time, rate form hypoelastic constitutive theory has been criticized for its failure to be exactly integrated to define an elastic material behavior \cite{40}, this includes many well known objective rates such as Zaremba-Jaumann rate, Green-Naghdi rate and Truesdell rate etc.. Because of the aforementioned issues, many spurious dissipative phenomenons, such as shear stress oscillation and accumulated artificial residual stress are observed even simple elastic material deformation. In other words, the non-integrable hypoelastic constitutive equation that uses objective rates such as Zaremba-Jaumann rate, Green-Naghdi rate and Truesdell rate etc., because of the aforementioned issues, this unique relationship between logarithmic rate of the logarithmic strain $\dot{h}$ is called logarithmic spin introduced by Xiao and coworkers \cite{30, 31, 29}.

By virtue of Eqn. (10), rate form hypoelastic constitutive theory has been criticized for its failure to be exactly integrated to define an elastic material behavior \cite{40}, this includes many well known objective rates such as Zaremba-Jaumann rate, Green-Naghdi rate and Truesdell rate etc.. Because of the aforementioned issues, this unique relationship between logarithmic strain $h$ and the rate of deformation tensor $D$ can be expressed as,

$$\dot{h}^\text{log} = \dot{h} + h\Omega^\text{log} - \Omega^\text{log} \dot{h} = D$$ \hfill (10)

Where $\Omega^\text{log}$ is called logarithmic spin introduced by Xiao and Bruhns \cite{30} with its explicit expression as:

$$\Omega^\text{log} = W + \sum_{i \neq j}^n \left( \frac{1}{1 - (\lambda_i/\lambda_j)} \right) b_i b_j$$ \hfill (11)

In which $\lambda_i, j (i, j = 1, 2, 3)$ are the eigenvalues of Left Cauchy-Green tensor $b$ and $b_{i,j}$ are the corresponding subordinate eigen-projections. Given the antisymmetric logarithmic spin tensor, the associated second order rotation tensor $R^\text{log}$ can be calculated through the following differential equation, and in most situations the initial condition can be assumed as $R^\text{log}|_{t=0} = I$.

$$\Omega^\text{log} = R^\text{log}(R^\text{log})^T$$ \hfill (12)

Following the definition of corotational integration by Khan and Huang \cite{41}, applying the logarithmic corotational integration procedure on Eqn. (10), assuming initial conditions $h|_{t=0} = 0$, yields the total logarithmic strain as follows,

$$h = \int_{\text{corot.}} D \, dt = (R^\text{log})^T \left( \int R^\text{log} D^r (R^\text{log})^T \, dt \right) R^\text{log}$$ \hfill (13)

Additive Decomposition of Logarithmic Strain

Starting from the additive decomposition of the total rate of deformation tensor $D$ into an elastic part $D^e$ plus a transformation part $D^tr$,

$$D = D^e + D^tr$$ \hfill (14)

By virtue of Eqn. (10), elastic part $D^e$ and transformation part $D^tr$ in Eqn. (14) can be rewritten as $\dot{h}^e\text{log}$ and $\dot{h}^r\text{log}$ respectively,

$$\dot{h}^e\text{log} = D^e, \quad \dot{h}^r\text{log} = D^tr$$ \hfill (15)

Combine Eqn. (14) and Eqn. (15), the following equation can be obtained.

$$h^\text{log} = \dot{h}^e\text{log} + \dot{h}^r\text{log}$$ \hfill (16)

Similar as Eqn. (13), applying logarithmic corotational integration procedure on Eqn. (16) yields,

$$h^e = \int_{\text{corot.}} D^e \, dt = (R^\text{log})^T \left( \int R^\text{log} D^r (R^\text{log})^T \, dt \right) R^\text{log}$$ \hfill (17a)

$$h^r = \int_{\text{corot.}} D^r \, dt = (R^\text{log})^T \left( \int R^\text{log} D^r (R^\text{log})^T \, dt \right) R^\text{log}$$ \hfill (17b)

Thus the following additive decomposition of total logarithmic strain can be received. Namely, the total logarithmic strain can be additively split into an elastic part plus a transformation part.

$$h = h^e + h^r$$ \hfill (18)

MODEL FORMULATION

Thermodynamic Potential of Constitutive Model

In this section, the development for finite strain constitutive modeling of SMAs is going to be presented. This model formulation is based on the early established SMAs model by Lagoudas and coworkers \cite{4, 42, 43} within infinitesimal strain regime. We begin with an explicit expression for Gibbs free energy, in which Kirchhoff stress tensor $\tau$ and temperature $T$ are chosen as independent state variables for Gibbs free energy $G$. The martensitic volume fraction $\xi$ and the second order transformation strain tensor $h^\tau$ are chosen as a set of internal state variables $Y = \{\xi, h^\tau\}$ to capture the nonlinear response exhibited by polycrystalline
SMAs. The explicit Gibbs free energy is given as:
\[
G = -\frac{1}{2\rho_0} \tau : \mathcal{S} \tau - \frac{1}{\rho_0} \tau : [\mathbf{a}(T - T_0) + \mathbf{h}'] + c \left( T - T_0 \right) - T \ln \left( \frac{T}{T_0} \right) - s_0(T - T_0) + u_0 + \frac{1}{\rho_0} f(\xi)
\]
(19)

In which, \( \mathcal{S} \) is the effective fourth-order compliance tensor calculated by using rule of mixtures defined in Eqn. (20a). \( \mathcal{S}^A \) is compliance matrix for austenite phase, \( \mathcal{S}^M \) is compliance matrix for martensite phase, and \( \Delta \mathcal{S} \) is the phase difference between them. \( \mathbf{a} \) is the second order thermoelastic expansion tensor, \( c \) is effective specific heat, \( s_0 \) and \( u_0 \) are effective specific entropy and effective specific internal energy at reference state, respectively. Similar to the definition of effective compliance tensor \( \mathcal{S} \) in Eqn. (20a), all those effective material parameters are using rule of mixture for their definition; \( T \) represents the temperature at current state while \( T_0 \) is temperature at reference state. \( f(\xi) \) is a smooth hardening function as it is introduced in original infinitesimal.

\[
\mathcal{S}(\xi) = \mathcal{S}^A + \xi (\mathcal{S}^M - \mathcal{S}^A) = \mathcal{S}^A + \xi \Delta \mathcal{S}
\]
(20a)

\[
\mathbf{a}(\xi) = \mathbf{a}^A + \xi (\mathbf{a}^M - \mathbf{a}^A) = \mathbf{a}^A + \xi \Delta \mathbf{a}
\]
(20b)

\[
c(\xi) = c^A + \xi (c^M - c^A) = c^A + \xi \Delta c
\]
(20c)

\[
s_0(\xi) = s_0^A + \xi (s_0^M - s_0^A) = s_0^A + \xi \Delta s_0
\]
(20d)

\[
u_0(\xi) = \nu_0^A + \xi (\nu_0^M - \nu_0^A) = \nu_0^A + \xi \Delta \nu_0
\]
(20e)

Smooth hardening function \( f(\xi) \) is defined in Eqn. (21) to consider the hardening effects associated with the transformation process. \( a_1, a_2, a_3 \) are introduced as intermediate material parameters in hardening function, and \( n_1, n_2, n_3, n_4 \) are curve fitting parameters to treat the smooth transition in the material response curve corner.

\[
f(\xi) = \begin{cases} 
\frac{1}{2} a_1 \left( \xi + \frac{n_1 + 1}{n_2 + 1} \right) + a_3 \xi, & \xi > 0, \\
\frac{1}{2} a_2 \left( \xi + \frac{n_3 + 1}{n_4 + 1} \right) - a_3 \xi, & \xi < 0
\end{cases}
\]
(21)

Following standard Coleman-Noll procedure, the constitutive relation between stress and strain is derived as,

\[
\mathbf{h} = -\rho_0 \frac{\partial G}{\partial \tau} = \mathcal{S} \tau + \mathbf{a}(T - T_0) + \mathbf{h}'
\]
(22)

Constitutive relation between entropy \( s \) and temperature \( T \) can also be derived as,

\[
s = -\rho_0 \frac{\partial G}{\partial T} = \frac{1}{\rho_0} \tau : \mathbf{a} + c \ln \left( \frac{T}{T_0} \right) + s_0
\]
(23)

The strict form dissipation inequality can be rewritten in terms of internal state variables \( \mathbf{Y} = \{ \xi, \mathbf{h}' \} \) as:

\[
-\rho_0 \frac{\partial G}{\partial \mathbf{h}'} : \dot{\mathbf{h}}' - \rho_0 \frac{\partial G}{\partial \xi} : \dot{\xi} \geq 0
\]
(24)

**Evolution Equation of Internal State Variables**

Evolution equations between internal state variables is going to be set up in this part. Following the same assumption from the early work of Lagoudas and coworkers [4, 43, 42], the following evolution relationship between \( \mathbf{h}' \) and \( \dot{\xi} \) is proposed. It is worth to point out that the rate adopted here is not a conventional time rate but logarithmic rate instead.

\[
\dot{\mathbf{h}}' = \mathbf{A} \xi , \quad \mathbf{A} = \begin{cases} 
\mathbf{A}^{\text{fwd}}, & \xi > 0, \\
\mathbf{A}^{\text{rev}}, & \xi < 0,
\end{cases}
\]
(25)

Where \( \mathbf{A}^{\text{fwd}} \) is called forward transformation direction tensor while \( \mathbf{A}^{\text{rev}} \) is called reverse transformation direction tensor. They are defined as follows respectively:

\[
\mathbf{A}^{\text{fwd}} = \frac{3}{2} H^\text{tr} \mathbf{r}' \mathbf{r}' , \quad \mathbf{A}^{\text{rev}} = \mathbf{h}^\text{tr} \mathbf{r}' \mathbf{r}'.
\]
(26)

In Eqn. (26), \( \mathbf{r}' \) is the deviatoric part of Kirchhoff stress tensor which is calculated by \( \tau' = \tau - \frac{1}{2} \text{tr}(\tau) \mathbf{I} \), where \( \mathbf{I} \) is the second order identity tensor. The effective (von Mises equivalent) stress is given by \( \bar{\tau}' = \sqrt{\frac{3}{2} \mathbf{r}' \mathbf{r}'} \). And \( \mathbf{h}^\text{tr} \mathbf{r}' \) and \( \mathbf{r}' \mathbf{r}' \) represents the transformation strain value and martensitic volume fraction value at the starting point of reverse transformation. From experimental observation, the transformation strain is not a constant value but it usually depends on the material current stress state, thereby a current transformation strain \( H^\text{tr} \tau' \) is introduced through a exponential function dependent on current stress state in Eqn. (27), where \( H^\text{max} \) is the maximum transformation strain and \( k_t \) is a curve fitting parameter, \( \bar{\tau} \) is effective stress of Kirchhoff stress.

\[
H^\text{tr} \tau'(\tau) = H^\text{max} (1 - e^{-k_t \bar{\tau}})
\]
(27)

**Transformation Function**

After the definition of evolution equation for internal state variable, the next objective is to define a proper criterion to determine whether the transformation will happen or not. Substitute the evolution Eqn. (25) into the strict form dissipation inequality Eqn. (24), we obtain the following equation:

\[
(\mathbf{r}' : \mathbf{A} - \rho_0 \frac{\partial G}{\partial \xi}) \dot{\xi} = \pi \dot{\xi} \geq 0
\]
(28)
Scalar variable $\pi$ is called general thermodynamic driving force conjugated to martensitic volume fraction $\xi$. Substitution of Gibbs free energy $G$ in Eqn. (19) into Eqn. (28) yields the expression for $\pi$:

$$\pi(\tau, T, \xi) = \tau : \mathbf{A} + \frac{1}{2} \tau : \Delta \mathbf{S} : \tau + \tau : \Delta \alpha(T - T_0) - \rho_0 \Delta c \left[ T - T_0 - T \ln \left( \frac{T}{T_0} \right) \right] + \rho_0 \Delta v_0 T - \rho_0 \Delta u_0 - \frac{\partial f}{\partial \xi}$$

(29)

Where material parameters $\Delta \mathcal{S}, \Delta \alpha, \Delta c, \Delta v_0, \Delta u_0$ have the definition from Eqn. (20a) to Eqn. (20e). We assume that whenever the thermodynamic driving force $\pi$ reaches a critical value $Y(-Y)$, the forward (reverse) phase transformation will take place. Therefore, a transformation function $\Phi$, defined as Eqn. (30), can be used as a criteria to determine the occurrence for phase transformation.

$$\Phi = \begin{cases} \pi - Y, & \xi > 0, \\ -\pi - Y, & \xi < 0, \end{cases}$$

(30)

In the continuous development of Lagoudas et al. [42] model, a stress dependent critical value $Y$ was proposed. A constant reference value $Y_0$ and an additional parameter $D$ are introduced into $Y$ such that it can better capture the smooth transition during transformation initial zone. Critical value $Y$ is defined as follows,

$$Y(\tau) = \begin{cases} Y_0 + D \tau : \mathbf{A}^{\text{rev}}, & \xi > 0, \\ Y_0 + D \tau : \mathbf{A}^{\text{rev}}, & \xi < 0, \end{cases}$$

(31)

To be satisfied with the principle of maximum dissipation, a so-called Kuhn-Tucker constraint conditions has also been placed on the evolution equation for internal state variables, which are expressed as follows for forward and reverse transformation cases respectively:

$$\dot{\xi} \geq 0; \quad \Phi(\tau, T, \xi) = \pi - Y \leq 0; \quad \Phi \dot{\xi} = 0; \quad (\mathbf{A} \Rightarrow \mathbf{M})$$

$$\dot{\xi} \leq 0; \quad \Phi(\tau, T, \xi) = -\pi - Y \leq 0; \quad \Phi \dot{\xi} = 0; \quad (\mathbf{M} \Rightarrow \mathbf{A})$$

(32)

**Consistent Tangent Stiffness and Thermal Matrix**

In this section, a detailed derivation of consistent tangent stiffness matrix and thermal matrix is provided to complete the proposed model. In most typical displacement-based commercial finite element software, like Abaqus, when an user defined material subroutine (UMAT) is written, consistent tangent matrices are usually required from finite element program solver to achieve a fast and accurate solution for global equilibrium equations during Newton-Raphson iteration procedure. Normally, consistent tangent matrices can be expressed in a rate form as defined in Eqn. (33). $\mathcal{L}$ is consistent tangent stiffness matrix and $\Theta$ is consistent thermal matrix.

$$\dot{\mathbf{\tau}} = \mathcal{L} \mathbf{\dot{h}} + \Theta \mathbf{T}$$

(33)

Apply logarithmic rate on constitutive stress strain Eqn. (22), where $\mathcal{C}$ is the stiffness matrix, the inverse of compliance tensor $\mathcal{F}$

$$\dot{\mathbf{\tau}} = \mathcal{C} \left[ \mathbf{h} - \mathbf{\alpha} T - (\Delta \mathcal{F} \tau + \Delta \mathbf{\alpha} (T - T_0) + \mathbf{A}) \xi \right]$$

(34)

Take chain rule differentiation on transformation function Eqn. (30), and replace conventional time rate with logarithmic rate. Be noticed that logarithmic rate is equivalent to conventional rate when it is applied on a scalar.

$$\dot{\Phi} = \partial_\tau \Phi : \dot{\mathbf{\tau}} + \partial_\tau \Phi \dot{\mathbf{\tau}} + \partial_\xi \Phi \dot{\xi} = 0$$

(35)

Substitute Eqn. (34) back into Eqn. (35) to eliminate $\dot{\tau}$ and solve it for $\dot{\xi}$, the following expression for $\dot{\xi}$ can be obtained,

$$\dot{\xi} = -\frac{\partial_\tau \Phi : \mathcal{C} \mathbf{h} + (\partial_\tau \Phi - \partial_\xi \Phi : \mathcal{C} \mathbf{\alpha}) T}{\partial_\xi \Phi - \partial_\xi \Phi : \mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A})}$$

(36)

Substitute Eqn. (35) back into rate form constitutive Eqn. (34), eliminate $\dot{\xi}$ and after some tensorial manipulations, a final explicit expression corresponding to Eqn. (33) can be expressed as follows,

$$\dot{\mathbf{\tau}} = \mathcal{C} \left[ \mathbf{h} - \mathbf{\alpha} T - (\Delta \mathcal{F} \tau + \Delta \mathbf{\alpha} (T - T_0) + \mathbf{A}) \xi \right] + \left[ \mathcal{C} \mathbf{\alpha} + \frac{\mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A}) \partial_\tau \Phi - \partial_\xi \Phi : \mathcal{C} \mathbf{\alpha}}{\partial_\xi \Phi - \partial_\xi \Phi : \mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A})} \right] \dot{\xi}$$

(37)

In which consistent tangent stiffness matrix $\mathcal{L}$ is obtained as,

$$\mathcal{L} = \mathcal{C} + \frac{\mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A}) \partial_\tau \Phi - \partial_\xi \Phi : \mathcal{C} \mathbf{\alpha}}{\partial_\xi \Phi - \partial_\xi \Phi : \mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A})}$$

(38)

and consistent thermal matrix $\Theta$ is derived as,

$$\Theta = -\mathcal{C} \mathbf{\alpha} + \frac{\mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A}) (\partial_\tau \Phi - \partial_\xi \Phi : \mathcal{C} \mathbf{\alpha})}{\partial_\xi \Phi - \partial_\xi \Phi : \mathcal{C} (\Delta \mathcal{F} \tau + \mathbf{A})}$$

(39)
NUMERICAL RESULTS

In this section, Boundary value problems, including a simple extension problem for SMA bar and a shear problem for a SMA torque tube, will be investigated to test the capabilities of proposed finite strain model, the response predicated by which is going to be compared with its infinitesimal counterpart. Besides, the cyclic pseudoelastic response of the SMA torque tube is also going to be examined to evaluate the effects of artificial residual stress introduced by objective rates.

Bar Problem

The first boundary value problem analyzed is a SMA bar experiencing stress-induced phase transformation under external pressure at constant temperature, which is typically called a pseudoelastic (or superelastic) response exhibited by SMAs. Referring to Fig.1 for problem schematic, a long SMA bar with length \( L = 100 \) unit (mm) and has a square cross section with edge length \( a = 10 \) unit (mm). Mechanical boundary conditions are left face fixed in certain lines and points to remove rigid body motions and right face is free to move but with an applied external pressure in its length direction. The material parameters used in this simulation are summarized in Tab.1 referenced from literature [43].

The loading history of SMA bar is as following: increase the pressure on right face from zero value to a maximum value 1200 (MPa), during which the SMA bar will experience a stress-induced extension due to forward transformation from austenite phase to detwinned martensite phase; Decrease the pressure linearly from its maximum value to zero, during which the SMA bar will contract to its original length due to the reverse transformation from detwinned martensite phase to austenite phase. The temperature is kept constant as 360 K throughout the whole loading procedure. The material response predicted by proposed finite strain model are compared against its infinitesimal counterpart.

As the result is shown in Fig.2, when the maximum transformation is strain 1%, the difference of pseudoelastic response curve predicted by the two models is almost negligible. However, with the increasing of maximum transformation strain from 1% to 3%, and more to 5% and 10%, the difference of predicted pseudoelastic response is becoming predominant. Specifically, with max transformation strain \( H_{\text{max}} = 3\% \) in Fig.3 the difference is becoming perceivable at the end of forward transformation, normalized displacement has larger value predicted by proposed finite strain model in contrast to infinitesimal strain model. This response difference is growing substantial in the cases of \( H_{\text{max}} \) to be 5% and 10%. Since there is no rotation involved here, the difference for the result comes from the fact that infinitesimal strain neglect the higher order terms in strain measure while logarithmic strain used in proposed model doesn’t.

Cyclic Response of SMA Torque Tube

As it was discussed in the section of introduction, based on the additive decomposition in finite deformation theory, a rate form hypoelastic constitutive equation is often seen as stress and strain relationship, in which a objective rate is usually required on the stress tensor in order to meet the principle of objectivity. There are many different objective rates proposed by different researchers along this line, among which several famous ones are Zaremba-Jaumann rate, Green-Naghdi rate and New-dell rate etc.. However, for a long period of time, rate form hypoelastic constitutive theory has been criticized for its inconsistent choices on objective rates [40], and also the rate form hypoelastic constitutive equation is not able to be integrated to deliver a algebraic constitutive equation, thereby many spurious phenomenons, such as shear stress oscillation, dissipative phe-
FIGURE 1. SCHEMATIC OF A SMA BAR EXTENSION PROBLEM UNDER EXTERNAL PRESSURE AT CONSTANT TEMPERATURE

TABLE 1. SUMMARY OF USED MATERIAL PARAMETERS FOR NUMERICAL SIMULATIONS

| Type                  | Parameter | Value       | Parameter | Value       |
|-----------------------|-----------|-------------|-----------|-------------|
| Material Constants    |           |             |           |             |
|                       | $E_A$     | 90 [GPa]    | $C_A$     | 16 [MPa/K]  |
|                       | $E_M$     | 63 [GPa]    | $C_M$     | 10 [MPa/K]  |
|                       | $v_A = v_M$ | 0.3        | $M_f$     | 242 [K]     |
|                       | $\alpha_A = \alpha_M$ | $2.2 \times 10^{-5}$ [K$^{-1}$] | $A_s$     | 288 [K]     |
|                       |           |             | $A_f$     | 342 [K]     |
| Smooth Hardening      |           |             |           |             |
|                       | $H_{max}^{\text{max}}$ | 1%, 3%, 5%, 10% | $n_1$     | 0.5         |
|                       | $k_t$     | 0.0075      | $n_2$     | 0.5         |
|                       |           |             | $n_3$     | 0.5         |
|                       |           |             | $n_4$     | 0.5         |

FIGURE 2. COMPARISON OF PSEUDOElastic RESPONSE FOR A SMA BAR PREDICTED BY PROPOSED FINITE STRAIN MODEL AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H_{max}^{\text{max}} = 1\%$

nomenon or artificial residual stress are observed in even simple elastic deformation. The aforementioned self-inconsistent issues of hypoelastic constitutive model are resolved by the logarithmic rate proposed by Xiao et al. [30,31,29], Bruhns et al. [32,33,34], Meyers et al. [35,36].

In this section, a segment torsion tube is analyzed by proposed finite strain model and its original infinitiesimal counterpart with Abaqus NLGEOM (nonlinear geometry) option to extend it for large deformation analysis. The loading conditions and material parameters used are the same as previous segment torsion tube. It should be clear, when the Abaqus NLGEOM option is activated for infinitiesimal strain based SMA model during implicit

FIGURE 3. COMPARISON OF PSEUDOElastic RESPONSE FOR A SMA BAR PREDICTED BY PROPOSED FINITE STRAIN MODEL AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H_{max}^{\text{max}} = 3\%$
analysis, the strain measure is logarithmic strain and Jaumann rate is the utilized objective rate to capture rotation deformation. As it is mentioned about with objective rates, artificial residual stresses will be introduced by using other objective rates except for the logarithmic one.

Pseudoelastic response of the SMA torque tube are examined by proposed finite strain model and its infinitesimal counterpart with Abaqus NLGEOM option. At the end of the first loading cycle, the Von Mises stress are checked in both cases. As it is shown in Fig.11, the artificial residual stress is almost zero predicted by proposed finite strain model, while the residual stress obtained from infinitesimal model with Abaqus NLGEOM option is around 8 MPa in contrast. The artificial residual stresses from those two cases are summarized in principal components representation at Tab.2.

Build on the results for one loading cycle, let us now examine whether this accumulation of artificial residual stresses will continue to grow, and how it will affect the material response of SMAs torque tube under cyclic loadings. Referring to Fig.12 for material response predicted by the infinitesimal model using Abaqus NLGEOM option, it can be seen that there is a clear response shifting from initial to final loading cycle due to the accumulation of artificial residual stress. Specifically, the stress levels required to start the forward phase transformation is decreasing, maximum shear stress achieved is increasing and the shear stress at end of each loading cycle is deviating more and more from the initial zero value. In contrast, material response predicted by proposed finite strain model in Fig.13, the initial material response is nearly overlapping with the material response from subsequent loading cycles.

Based on the results for segment SMA torque tube under cyclic loading, it is demonstrated that, because of the small amount of artificial residual stress introduced by using Abaqus NLGEOM option, the infinitesimal strain based constitutive model for SMAs will predict an shifting, instead of stable, cyclic pseudoelastic response for a torque tube. It is also shown that the proposed finite strain SMA constitutive model, based on logarithmic strain and logarithmic rate, can effectively rule out the effects of artificial residual stress during cyclic loadings. Nevertheless, it is worthy of pointing out for general metallic material within small strain regime during low loading cycles, the amount of artificial residual stresses have quite limited effects on the material response, and it is still a practical and relatively accurate way to utilize the Abaqus NLGEOM option to consider deformation involved with large rotations. At the same time, it is crucial to realized that such undesired artificial stresses need be taken into consideration when the material is undergoing a large number of cyclic loadings, for which a finite strain constitutive model for SMAs based on logarithmic strain and logarithmic rate is indispensible in order to deliver an stable and accurate material response.

CONCLUSION

Based on the infinitesimal strain based constitutive model for shape memory alloys from Lagoudas and coworkers [4,42], a three dimensional phenomenological constitutive model for martensitic transformation in polycrystalline shape memory alloys considering large strains and large rotations is proposed in
this work. Numerical simulations considering basic SMA component geometries, such as a bar and a torque tube under stress induced phase transformation, are performed to test the capabilities of the proposed model. For numerical examples of SMA bar, relatively large discrepancies are observed between the responses predicted by the proposed model and its infinitesimal counterpart when strain is no longer considered small. In the numerical simulations of a SMA torque tube with large rotations, it is demonstrated that the difference of predicted material response will be perceivable in moderate strain regime of 5% and continues to grow substantially in large strain regime at 10%. From the results of SMA torque tube under cyclic loading, it has been shown that infinitesimal strain based constitutive model will predict an shifting cyclic pseudoelastic response as a result of the artificial residual stress introduced by using Abaqus NLGEOM option. In comparison, the proposed finite strain SMA constitutive model, based on logarithmic strain and logarithmic rate, can effectively rule out the effects of artificial residual stress during cyclic loadings.

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FIGURE 7. COMPARISON OF PSEUODELASTIC RESPONSE FOR A SHORT SMA TORSION TUBE PREDICTED BY PROPOSED FINITE STRAIN AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H^{\text{max}} = 1\%$, GEOMETRY RATIO $L/r = 2$

FIGURE 8. COMPARISON OF PSEUODELASTIC RESPONSE FOR A SHORT SMA TORSION TUBE PREDICTED BY PROPOSED FINITE STRAIN AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H^{\text{max}} = 3\%$, GEOMETRY RATIO $L/r = 2$

FIGURE 9. COMPARISON OF PSEUODELASTIC RESPONSE FOR A SHORT SMA TORSION TUBE PREDICTED BY PROPOSED FINITE STRAIN AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H^{\text{max}} = 5\%$, GEOMETRY RATIO $L/r = 2$

FIGURE 10. COMPARISON OF PSEUODELASTIC RESPONSE FOR A SHORT SMA TORSION TUBE PREDICTED BY PROPOSED FINITE STRAIN AND INFINITESIMAL STRAIN MODEL WITH TRANSFORMATION STRAIN $H^{\text{max}} = 10\%$, GEOMETRY RATIO $L/r = 2$

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FIGURE 11. COMPARISON OF ARTIFICIAL RESIDUAL STRESS CONTOUR PREDICTED BY PROPOSED FINITE STRAIN MODEL AND INFINITESIMAL MODEL WITH ABAQUS NLGEOM OPTION FOR A SMA TORQUE TUBE AFTER ONE LOADING CYCLE.

FIGURE 12. CYCLIC PSEUDELASTIC RESPONSE OF A SMA TORQUE TUBE PREDICTED BY INFINITESIMAL MODEL WITH ABAQUS NLGEOM OPTION UNDER 90 CYCLIC LOADINGS

FIGURE 13. CYCLIC PSEUDELASTIC RESPONSE OF A SMA TORQUE TUBE PREDICTED BY PROPOSED FINITE STRAIN MODEL UNDER 90 CYCLIC LOADINGS

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