DIFFERENTIAL FORMS IN POSITIVE CHARACTERISTIC
AVOIDING RESOLUTION OF SINGULARITIES

ANNETTE HUBER, STEFAN KEBEKUS, AND SHANE KELLY

Abstract. In this sequel to the recent paper arXiv:1305.7361 of Huber-Jörder, we study several notions of sheaves of differential forms in positive characteristic. We identify the universal extension of the presheaf of differential forms from smooth varieties to all varieties as a notion that is well-behaved without using assumptions on resolution of singularities. Under this assumption, it agrees with the sheafification in the cdh-topology. We present a number of (counter-)examples which highlight the difficulties that arise in trying to apply the topological methods of Huber-Jörder to positive characteristic. Particular attention is given to torsion sections.

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Sheaves of differential forms play a key rôle in many areas of mathematics, in particular in the study of birational geometry and the study of singularities. On non-singular schemes however, the theory of Kähler differentials displays a number of unpleasant behaviours, and there are a number of competing generalisations, at least in characteristic 0.

- **Torsion free Kähler differential forms** are elements of the quotient of the group of Kähler differentials modulo those differentials which vanish on a dense open subscheme.
- **Differential forms of first kind** on an irreducible variety are differential forms on a log resolution (see [vSS85, 1.2]).
- **Reflexive differential forms** $\Omega^n_X$ on a normal variety $X$ are differential forms on the regular locus $X^{\text{reg}}$ ([Kni73], [LW09], [GKKP11]).
• Using simplicial hyperresolutions Du Bois [DB81] defines complexes of coherent sheaves $\Omega^n$ suggested from Hodge theory.
• Finally $h$-differentials were introduced in [HJ13] in characteristic 0 as the sheafification of the presheaf $\Omega^n$ in Voevodsky’s $h$-topology.

The latter were shown in loc. cit. to agree with torsion-free differentials on normal crossings divisors and with reflexive differentials on klt base spaces. The Du Bois complexes is reinterpreted as the derived direct image for the morphism of sites from the $h$-topology to the Zariski-topology.

In this note we discuss analogues of these ideas in positive characteristic. We were particularly interested in obtaining results which do not rely on resolution of singularities. The obvious idea was to try to use de Jong’s theorem on alterations instead. Unfortunately, this is not possible. We identify a number of problems, but are also able to give a well-behaved definition.

In Section 2 we discuss torsion and torsion free differentials. A torsion form is a section of $\Omega^p$ which vanishes on a dense open. Over the complex numbers, the pull-back of a torsion form is again a torsion form, and so the groups of torsion free forms obtain the structure of a presheaf. Surprisingly, this does not work in positive characteristic. We provide in Example 2.6 an example of a morphism and a torsion form on the target, for which the pull-back is not a torsion form.

One of the characterisations of $\Omega^h(X)$ given in [HJ13] is as coherent sets of sections of forms on regular varieties over $X$. In Section 3.1 we consider this presheaf which we denote $\Omega_{reg}$ (Definition 3.1). The presheaf $\Omega_{reg}$ agrees with $\Omega$ on regular varieties, and under the assumption of weak resolution of singularities $\Omega_{reg}$ is isomorphic to the cdh sheafification $\Omega_{cdh}$ of $\Omega$ (Proposition 4.9). The hope is that one might be able to do everything with $\Omega_{reg}$ that one wants to do with $\Omega_{cdh}$, but without the resolution of singularities assumption. At the end of this section we use Example 2.6 again to show $\Omega_{reg}$ is not torsion free, and its torsion is not functorial either.

These results are bases on a very general result which should be of independent interest: In Proposition 3.12 we show that the extension of an unramified presheaf on the category of regular $S$-schemes (with $S$ noetherian) to all $S$-schemes of finite type is an $rh$-sheaf.

When one tries to use the $h$-topology to study differential forms in positive characteristic, the first obstacle one discovers is that $\Omega^h_0 = 0$ (Lemma 5.1). This is due to the fact that the geometric Frobenius is an $h$-cover which induces the zero morphism. However, almost all of the results of [HJ] are still valid using the coarser $eh$ topology, and these remain valid in positive characteristic if one assumes that resolution of singularities is true. This was already observed by Geisser [Gei06] under a strong form of resolution of singularities.

If one wants to avoid assuming conjectures which have resisted proof for more than half a century, there are the slightly weaker desingularisation theorems of de Jong and Gabber, but now one needs to refine the cdh topology a little so that these theorems can be used to cover singular varieties with non-singular ones. An example of the successful application of such a program is [Kel12] where the $l$dh topology [Kel12, Def. 3.2.1] is introduced ($l$ is a prime different from the characteristic) and successfully used as a replacement to the cdh topology. The $l$dh topology is unsuitable for applications in differential forms as the geometric Frobenius is still an $l$dh cover. In Section 5 we propose a couple of new, initially promising sites (Definitions 5.2 and 5.8), but then also provide an example (Example 5.5) which shows that surprisingly, the sheafification of $\Omega$ on these sites does not preserve its values on regular schemes (Proposition 5.6, Lemma 5.10).
Finally, to show that $\Omega^n_{\text{cdh}}$ agrees with $\Omega^n_{\text{reg}}$ and hence with $\Omega^n$ on smooth varieties, we can get away with a hypothesis – Hypothesis H, page 11 – which looks weaker than weak resolution of singularities. In the appendix, we discuss possible strategies to prove this hypothesis, first using a hyperplane argument, and then using the Riemann-Zariski space as a “desingularisation”.

What is missing from this paper is a discussion of cohomological descent. Under resolutions of singularities, Geisser has shown [Gei06] that cdh-homology of $\Omega^n_{\text{cdh}}$ agrees with Zariski-cohomology of $\Omega^n$ on all smooth varieties $X$. It remains open if this can be extended unconditionally to $\Omega^n_{\text{reg}}$.

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1. Notation and Conventions

1.1. Categories of schemes and presheaves. All schemes are assumed separated. Most of the material will be directed at varieties over a field, but since some works more generally, let $S$ be a noetherian scheme. We will work at one time or another with the following categories of $S$-schemes:

Let $\operatorname{Sch}(S)$ be the category of separated schemes of finite type over $S$, and let $\operatorname{Reg}(S)$ be the full subcategory of regular schemes in $\operatorname{Sch}(S)$. If $S$ is the spectrum of a field $k$, we also write $\operatorname{Sch}(k)$ and $\operatorname{Reg}(k)$. If $k$ is perfect, then $\operatorname{Reg}(k)$ is the category of (not necessarily connected) smooth $k$-varieties.

Recall that a presheaf $F$ of abelian groups on $\operatorname{Sch}(S)$ is simply a contravariant functor. It is called a presheaf of $\mathcal{O}$-modules, if every $F(X)$ has an $\mathcal{O}(X)$-module structure in way such that the $F(X) \to F(Y)$ are compatible with the $\mathcal{O}(X) \to \mathcal{O}(Y)$ for every morphism $Y \to X$ in $\operatorname{Sch}(S)$.

**Definition 1.1.** Let $F$ be a presheaf on $\operatorname{Sch}(S)$ and $X \in \operatorname{Sch}(S)$. We write

$$\operatorname{tor} F(X)$$

for the set of those sections of $F(X)$ which vanish on a dense open subscheme.

**Remark 1.2.** Note that the groups $\operatorname{tor} F(X)$ do not necessarily have the structure of a presheaf on $\operatorname{Sch}(S)$ in general! For a morphism $Y \to X$ in $\operatorname{Sch}(S)$, the image of $\operatorname{tor} F(X)$ under the morphism $F(X) \to F(Y)$ does not necessarily lie in $\operatorname{tor} F(Y)$.

We are particularly interested in the case of the presheaf of Kähler differentials. For $n \geq 0$, we denote $\Omega^n$ the presheaf $X \mapsto \Omega^n_{X/S}(X)$. Note that $\Omega^0 = \mathcal{O}$. We also abbreviate $\Omega = \Omega^1$. The notation $\Omega^n_X$ means as usual the Zariski-sheaf on $X$. That is, $\Omega^n_X = \Omega^n|_{X_{\text{Zar}}}$ where $X_{\text{Zar}}$ is the usual topological space associated to the scheme $X$.

1.2. Topologies. We are going to use various topologies on $\operatorname{Sch}/S$, which we want to introduce now. They are variants of the h-topology introduced by Voevodsky in [Voe96]. Recall that a Grothendieck topology on $\operatorname{Sch}(S)$ is defined by specifying which collections $\{U_i \to X\}_{i \in I}$ of morphisms should be considered as open covers. By definition, a presheaf $F$ is a sheaf if for any such collection, $F(X)$ is equal to the set of those elements $(s_i)_{i \in I}$ in $\prod_{i \in I} F(U_i)$ for which $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$ for every $i, j \in I$.

We refer to the ordinary topology as the Zariski topology.

**Definition 1.3.** A morphism $f : Y \to X$ is called cdp-morphism if it is proper and completely decomposed: for every point $x \in X$ there is a point $y \in Y$ with $f(y) = x$ and $[k(y) : k(x)] = 1$. 
These morphisms are also referred to as proper cdh covers, or envelopes. Recall that the rh- \cite{GL01} (resp. cdh- \cite[§5]{SV00}, eh- \cite{Gei06}) topology on Sch(S) is generated by the Zariski topology (resp. Nisnevich, étale) and cdp-morphisms.

We are going to need the following facts from algebraic geometry.

**Lemma 1.4.** Let $X$ be a regular noetherian scheme. Let $x \in X$ be a point of codimension $n$. Then there is point $y \in X$ of codimension $n-1$, a discrete valuation ring $R$ essentially of finite type over $X$ together with a map $\text{Spec}(R) \rightarrow X$ such that the special point of $\text{Spec}(R)$ maps to $x$ and the generic point to $y$, both inducing isomorphisms on their respective residue fields.

**Proof.** The local ring $O_{X,x}$ is a regular local ring, and as such admits a regular sequence $f_1, \ldots, f_n$ generating its maximal ideal. The quotient ring $R = O_{X,x}/(f_1, \ldots, f_{n-1})$ is then a regular local ring of dimension one (\cite[Theorem 36(3)]{Mat70}), that is, a discrete valuation ring. Let $y$ be the image of the generic point of $\text{Spec}(R)$. By construction it is a point of codimension $n-1$. \hfill $\Box$

**Proposition 1.5.** Suppose that $X$ is a regular noetherian scheme. Then every proper birational morphism is a cdp-morphism, and every cdp-morphism is refinable by a proper birational morphism.

**Remark 1.6.** This fact is well-known over a field of characteristic zero and is usually proven using strong resolution of singularities. That is, by refining a proper birational morphism by a sequence of blow-ups with smooth centres. By contrast, the proof below works for ANY regular noetherian scheme $X$, without restriction on a potential base scheme, or structural morphism.

**Proof.** Let $Y \rightarrow X$ be proper and birational (with $X$ regular and noetherian as in the statement). We must show that for every point $x \in X$ the canonical inclusion admits a factorisation $x \rightarrow Y \rightarrow X$. We proceed by induction on the codimension. In codimension zero, the factorisation is a consequence of birationality. Suppose that it is true up to codimension $n-1$ and let $x$ be a point of codimension $n$. By Lemma 1.4 we can find a discrete valuation ring $R$ and a diagram

$$
\begin{array}{ccc}
\text{Spec}(R) & \rightarrow & X \\
y & \downarrow & \\
x & \rightarrow & 
\end{array}
$$

for some $y \in X^{(n-1)}$. By the inductive hypothesis, the inclusion of $y$ into $X$ admits a factorisation through $Y$, so we have a commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(R) & \rightarrow & X \\
y & \downarrow & \\
Y & \rightarrow & 
\end{array}
$$

and now the valuative criterion for properness implies that the inclusion of $\text{Spec}(R)$ into $X$ factors trough $Y$, and therefore so does the inclusion of $x$.

Finally, it is straightforward to see that every cdp-morphism admits a refinement by a proper birational morphism: If $Y \rightarrow X$ is a proper completely decomposed morphism with $X$ connected (hence irreducible), choose a factorisation $\eta \rightarrow Y \rightarrow X$ of the inclusion of the generic point $\eta$ of $X$. Then the closure of the image of $\eta$ in $Y$ is a birational and proper over $X$. \hfill $\Box$

2. **Functoriality of torsion forms**

One very useful feature of differential forms on a smooth varieties is that they are a vector bundles, in particular torsion free. In characteristic zero, the different
candidates for a good theory of differential forms share this behaviour on all varieties. It is disappointing but true that this property fails in positive characteristic, as we are going to establish.

Throughout this section, let $k$ be a perfect field. The following notion will be used throughout.

**Definition 2.1** (Torsion differentials and torsion-free differentials, [Keb13, Sect. 2.1]). Let $X \in \text{Sch}(k)$. We define the sheaf $\tilde{\Omega}^n_X$ on $X_{\text{Zar}}$ as the cokernel of the sequence

\[ 0 \to \text{tor} \Omega^n_X \xrightarrow{\alpha_X} \Omega^n_X \xrightarrow{\beta_X} \tilde{\Omega}^n_X \to 0 \]

Sections in $\text{tor} \Omega^n_X$ are called torsion differentials. By slight abuse of language, we refer to sections in $\tilde{\Omega}^n_X$ as torsion-free differentials.

**Remark 2.2** (Torsion sheaves on reducible spaces). Much of the literature discusses torsion sheaves and torsion-free sheaves only in a setting where the underlying space is irreducible. We refer to [Keb13, Appendix A and references there] for a brief discussion of torsion sheaves on reduced, but possibly reducible spaces.

2.1. **Torsion-free forms over the complex numbers.** Given a morphism between two varieties that are defined over the complex numbers, the usual pull-back map of Kähler differentials induces pull-back maps for torsion-differentials and for torsion-free differentials, even if the image of the morphism is contained in the singular set of the target variety.

**Theorem 2.3** (Pull-back for sheaves of torsion-free differentials, [Keb13, Cor. 2.7]). Let $f : X \to Y$ be a morphism of reduced, quasi-projective schemes that are defined over the complex numbers. Then there exist unique morphisms $d_{\text{tor}} f$ and $\tilde{d} f$ such that the following diagram, which has exact rows, becomes commutative

\[ \begin{array}{ccc}
\text{tor} \Omega^n_Y & \xrightarrow{f^*} & \Omega^n_Y \\
\downarrow d_{\text{tor}} f & & \downarrow \tilde{d} f \\
\text{tor} \Omega^n_X & \xrightarrow{f^*} & \Omega^n_X \\
\end{array} \]

In other words, $\text{tor} \Omega^n$ is a presheaf on $\text{Sch}(\mathbb{C})$.

The same argument works for any field of characteristic 0.

**Remark 2.4** (Earlier results). For complex spaces, the existence of a map $\tilde{d}$ has been shown by Ferrari, [Fer70, Prop. 1.1], although it is perhaps not obvious that the sheaf discussed in Ferrari’s paper agrees with the sheaf of Kähler differentials modulo torsion.

**Warning 2.5** (Theorem 2.3 is wrong in the relative setup). One can easily define torsion differentials and torsion-free differentials in the relative setting. The proof of Theorem 2.3, however, relies on the existence of a resolution of singularities for which no analogue exists in the relative case. As a matter of fact, Theorem 2.3 becomes wrong when working with relative differentials, unless one makes rather strong additional assumptions. A simple example is given in [Keb13, Warning 2.6].

2.2. **Torsion-free forms in positive characteristic.** Now let $f : X \to Y$ be a morphism of reduced, quasi-projective schemes that are defined over a field $k$ of finite characteristic. We will see in this section that in stark contrast to the case of complex varieties, the pull-back map $d f$ of Kähler differential does generally not induce a pull-back map between the sheaves of torsion-free differential forms.
Indeed, if there exists a pull-back map \( \tilde{f} : f^*\Omega^n_Y \to \Omega^n_X \) which makes the following diagram commute,

\[
\begin{array}{ccc}
\Omega^n_Y & \xrightarrow{f^*} & \Omega^n_X \\
\downarrow{\tilde{f}} & & \downarrow{df} \\
\Omega^n_Y & \xrightarrow{\beta_x} & \Omega^n_X,
\end{array}
\]

and if \( \sigma \in \text{tor} \Omega^1_Y \) is any torsion differential, then \( df(\sigma) \) is necessarily a torsion differential on \( X \), that is \( df(\sigma) \in \text{tor} \Omega^1_X \). The following example discusses a morphism between varieties for which this property does not hold.

**Example 2.6** (Pull-back of torsion form is generally not torsion). This example works for any prime \( p \), but we choose \( p = 2 \) for concreteness. Let \( k \) be an algebraically closed field of characteristic two and let \( Y \subset \mathbb{A}^3_k \) be the Whitney umbrella. More precisely, consider the polynomial \( P(x, y, z) = y^2 - x^2z^2 \in \mathbb{k}[x, y, z] \), the ring \( R := \mathbb{k}[x, y, z]/(P) \) and the schemes

\[
X := \text{Spec } \mathbb{k}[x] \quad \text{and} \quad Y := \text{Spec } R.
\]

An elementary computation shows that the polynomial \( P \) is irreducible. As a consequence, we see that \( Y \) is reduced and irreducible and that \( z \) is not a zerodivisor in \( R \). Finally, let \( f : X \to Y \) be the obvious inclusion map, which identifies \( X \) with the \( x \)-axis in \( \mathbb{A}^3 \), and which is given by the following map of rings,

\[
f^* : \mathbb{k}[x, y, z]/(P) \to \mathbb{k}[x] \quad \text{and} \quad Q(x, y, z) \mapsto Q(x, 0, 0).
\]

Note that \( X \) is nothing but the reduced singular locus of \( Y \). We want to construct a torsion differential \( \sigma \) on \( Y \). To this end, recall that the differential form \( dP \in \Gamma(\Omega^n_Y) \) induces the zero-form on \( Y \),

\[
0 = dP = -z^2 \cdot dx + 2y \cdot dy - 2xz \cdot dz = -z^2 \cdot dx \in \Gamma(\Omega_X^1).
\]

Since \( z^2 \) is not a zerodivisor, we see that the form \( \sigma := dx \) is torsion, that is, \( \sigma \in \Gamma(\text{tor} \Omega^1_Y) \). On the other hand, the pull-back of \( \sigma \) to \( X \) is clearly given by \( df(\sigma) = dx \in \Gamma(\text{tor} \Omega^1_X) \), which is not torsion.

In other words: the assignments \( X \mapsto \text{tor}^n(X) \) and \( X \mapsto \Omega^n_X(X) \) do not define presheaves on \( \text{Sch}/k \).

### 3. The extension functor \((-)_{\text{reg}}\)

In order to define a good extension of the sheaf of differentials forms on non-singular varieties to general varieties, we first study the extension functor abstractly.

#### 3.1. Definition and first properties. Let \( S \) be a noetherian scheme.

**Definition 3.1.** Given a presheaf \( F \) on \( \text{Sch}(S) \) define the presheaf \( F_{\text{reg}} \) on \( \text{Sch}(S) \) via the limits (not colimits)

\[
F_{\text{reg}}(X) = \lim_{Y \in \text{Reg}(X)} F(Y).
\]

**Warning 3.2.** The morphism \( F_{\text{reg}}(X) \to F_{\text{reg}}(X') \) associated to a morphism \( X' \to X \) is induced by the composition functor, \( \text{Reg}(X') \to \text{Reg}(X) \). It is easy to forget this and think that it is induced by \( X' \times_X - \), which does not necessarily preserve regular schemes (unless for example, \( X' \to X \) is étale).
Remark 3.3.

1. Explicitly, a section of \( F_{\text{reg}}(X) \) is a sequence of compatible sections. That is, the data of: for every morphism \( Y \to X \) with \( Y \in \text{Reg}(S) \) an element \( s_Y \in F(Y) \). Furthermore, these sections are required to satisfy: for every triangle \( Y' \to Y \to X \) with \( Y', Y \in \text{Reg}(S) \), we must have \( s_Y|_{Y'} = s_{Y'} \).

2. The assignment \( F \mapsto F_{\text{reg}} \) could be equivalently defined as the left adjoint to the restriction functor from presheaves on \( \text{Sch}(S) \) to \( \text{Reg}(S) \).

3. For \( X \) regular, \( F(X) = F_{\text{reg}}(X) \) since the object \( X \) is final in \( \text{Reg}(X) \).

4. Let \( S = \text{Spec}(k) \) with \( k \) a field of characteristic zero. Consider the presheaf \( F = \Omega^n \). Then \( \Omega^n_{\text{reg}} = \Omega^n \) by [HJ13, Theorem 1].

Lemma 3.4. Let \( S \) be a noetherian scheme. Suppose \( \tau \) is a topology on \( \text{Sch}(S) \) equal to or coarser than the étale topology. Then if \( F \) is a \( \tau \)-sheaf (resp. \( \tau \)-separated presheaf) then so is \( F_{\text{reg}} \).

Remark 3.5. This is a consequence of \( \text{Reg}(S) \to \text{Sch}(S) \) being a cocontinuous morphism of sites for such a topology ([SGA72, Definition III.2.1, Definition II.1.2], [Sta14, Tags 00XF and 00XI]). As we do not assume the reader to be at ease with categorical constructions we give an explicit proof. The key ingredient is the fact that a \( \tau \)-cover of a regular scheme is regular when \( \tau \) is coarser than the étale topology.

Proof. Let \( U \to X \) be a \( \tau \)-cover.

\( F_{\text{reg}}(X) \to F_{\text{reg}}(U) \) is monic: Two elements \( (s_Y)_{Y \in \text{Reg}(X)}, (t_Y)_{Y \in \text{Reg}(X)} \in F_{\text{reg}}(X) \) agree in \( F_{\text{reg}}(U) \) if and only if \( s_Y = t_Y \) for every \( Y \to X \) in \( \text{Reg}(X) \) which factors through \( U \to X \). But every \( Y \to X \) in \( \text{Reg}(X) \) is \( \tau \)-covered by such a morphism, namely \( U \times_X Y \to Y \). So since \( F \) is \( \tau \)-separated, \( s_Y = t_Y \) for ALL \( Y \to X \in \text{Reg}(X) \).

\( F_{\text{reg}}(X) \) is the equaliser of \( F_{\text{reg}}(U) \rightrightarrows F_{\text{reg}}(U \times_X U) \): the two images in \( F_{\text{reg}}(U \times_X U) \) of an element \( (s_Y)_{Y \in \text{Reg}(U)} \in F_{\text{reg}}(U) \) agree if and only if for every \( Y \to X \) in \( \text{Reg}(X) \) which factors through \( U \to X \), the elements \( s_Y \) are independent of the factorisation \( Y \to U \). In particular, for such a \( Y \to X \), this applies to the two canonical morphisms \( U \times_X U \to U \). So due to the required coherencies, and the fact that \( F \) is a \( \tau \)-sheaf, the element \( s_U \in F(U \times_X Y) \) lifts uniquely to an element \( t_Y \in F(Y) \). Now if \( Y' \to Y \to X \) is a morphism in \( \text{Reg}(X) \), the uniqueness of this lifting implies that \( t_Y|_{Y'} = t_{Y'} \in F(Y') \). So we obtain a coherent sequence \( (t_Y)_{Y \in \text{Reg}(X)} \in R_{\text{reg}}(X) \). Finally, if \( Y \to X \) happens to factor through \( U \to X \), then \( U \times_X Y \to Y \) admits a section, and so

\[ t_Y = t_{Y}|_{U \times_X Y}|_Y = s_{U \times_X Y}|_Y = s_Y. \]

Definition 3.6 (cf. [Mor04, Definition 1.1, Remarks 1.2, 1.4]). A presheaf on \( \text{Reg}(S) \) is unramified if the following axioms are satisfied.

1. (Unr0) The canonical morphism \( F(X) \times Y \to F(X \times Y) \) is an isomorphism for all \( X, Y \in \text{Reg}(S) \).
2. (Unr1) If \( U \to X \) is a dense open immersion, then \( F(X) \to F(U) \) is injective.
3. (Unr2) \( F \) is a Zariski sheaf, and for every open immersion \( U \to X \) which contains all points of codimension \( \leq 1 \) the morphism \( F(X) \to F(U) \) is an isomorphism.

We will say that a presheaf \( F \) on \( \text{Sch}(S) \) is unramified if its restriction to \( \text{Reg}(S) \) is unramified.

Example 3.7.

1. The sheaf \( \mathcal{O} \) is unramified [Mat70, Theorem 38, page 124].
Lemma 3.10

(2) If $F$ is a sheaf on $\text{Reg}(S)$ whose restrictions $F|_{X_{\text{Zar}}}$ to the small Zariski sites $X|_{\text{Zar}}$ of each $X \in \text{Reg}(S)$ are locally free coherent $\mathcal{O}_X$-modules, then $F$ is also unramified.

(3) If $S = \text{Spec}(k)$ is the spectrum of a perfect field, then the $\Omega^n_{/k}$ are all unramified for all $n \geq 0$.

(4) There are other, important examples of unramified presheaves, which fall out of the scope of the article. These include the Zariski sheafifications of $K$-theory, étale cohomology with finite coefficients (prime to the characteristic), or homotopy invariant Nisnevich sheaves with transfers. More generally, all reciprocity sheaves in the sense of [KSY14] which are Zariski sheaves satisfy (Unr1) by loc. cit. Theorem 6, and conjecturally satisfy (Unr2) by loc. cit. Conjecture 1.

Say that a scheme is essentially of finite type if it is an intersection of open subschemes of a scheme in $\text{Sch}(S)$. For example, if $X \in \text{Sch}(S)$ and $x \in X$ then $\text{Spec}(\mathcal{O}_{X,x})$ is an $S$-scheme essentially of finite type. The category of such schemes will be denoted by $\text{Sch}(S)^{\text{ess}}$. For any presheaf $F$ on $\text{Sch}(S)$ there is a canonical extension to $\text{Sch}(S)^{\text{ess}}$ where a scheme of the form $\cap_{i \in I} U_i$ is assigned the colimit $\varinjlim_{i \in I} F(U_i)$. For example if we take $X \in \text{Sch}(S), x \in X$ and $\{U_i\}_{i \in I}$ to be the set of open subschemes of $X$ containing $x$, then $F(\cap_{i \in I} U_i)$ is just the usual (Zariski) stalk of $F$ at $x$. In general a presheaf $F$ on $\text{Sch}(S)^{\text{ess}}$ will not necessarily satisfy $F(\cap_{i \in I} U_i) = \varinjlim_{i \in I} F(U_i)$, but many of interest do ($\Omega^n$ for example for all $n \geq 0$).

Definition 3.8. Let $\text{Dvr}(S)$ be the category of schemes essentially of finite type which are regular, local, and of dimension $\leq 1$.

That is, schemes of the form $\text{Spec}(\mathcal{O}_{X,x})$ where $X \in \text{Reg}(S)$ and $x \in X$ is a point of codimension one or zero. The latter amounts to a scheme of the form $\text{Spec}(K)$ for a field extension $K/k(s)$ of finite transcendence degree of the residue field $k(s)$ of a point $s \in S$.

Proposition 3.9. Let $S$ be a noetherian scheme. Let $F$ be an unramified presheaf. Then, for any scheme $X \in \text{Sch}(S)$ we have

$$F_{\text{reg}}(X) = \varinjlim_{Y \in \text{Dvr}(X)} F(Y).$$

The proof of this claim will take the rest of this section.

Lemma 3.10 ([Mor04, Remark 1.4]). Let $S$ be a noetherian scheme. Suppose that $F$ is an unramified presheaf on $\text{Sch}(S)$. Then for any $X \in \text{Reg}(S)$,

$$F(X) = \varinjlim_{x \in X^{(\leq 1)}} F(\text{Spec}(\mathcal{O}_{X,x}))$$

where $X^{(\leq 1)}$ is the subcategory of $\text{Dvr}(X)$ consisting of inclusions of localisations of $X$ at points of codimension $\leq 1$.

Proof. Let $X$ be a connected regular scheme. Consider the canonical map

$$F(X) \to \varinjlim_{x \in X^{(\leq 1)}} F(\text{Spec}(\mathcal{O}_{X,x})).$$

The axiom (Unr1) implies that $F(X) \to F(\eta)$ is injective (where $\eta$ is the generic point of $X$) and since this factors as $F(X) \to \varinjlim_{x \in X^{(\leq 1)}} F(\text{Spec}(\mathcal{O}_{X,x})) \to F(\eta)$ (where the second map is the canonical projection) this implies that the first map is injective.

Surjectivity: Let $(s_x)_{x \in X^{(\leq 1)}}$ be a section of $\varinjlim_{x \in X^{(\leq 1)}} F(\text{Spec}(\mathcal{O}_{X,x}))$. By definition of the groups $F(\text{Spec}(\mathcal{O}_{X,x}))$, for every $x \in X^{(\leq 1)}$ there is some open $U_x \subset X$
containing $x$ and a $t_x \in F(U_x)$ which represents $s_x$. Furthermore, by the required coherency, for every $x \in X^{(1)}$ there is an open subscheme $U_{x\eta}$ of $U_x \cap U_\eta$ such that the restrictions of $t_x$ and $t_\eta$ to $U_{x\eta}$ agree. By (Unr1) again this means that we actually have $t_x|_{U_{x\eta}} = t_\eta|_{U_{x\eta}}$ for each $x, \eta \in X^{(1)}$. Since $F$ is a Zariski sheaf by (Unr2), the sections $t_x$ therefore lift to a section $t$ on $\cup_{x \in X^{(1)}} U_x$, but by (Unr2), we have $F(\cup_{x \in X^{(1)}} U_x) = F(X)$. So the map is surjective. □

**Lemma 3.11** (cf. [Kel12, Proof of 3.6.12]). For $X$ connected regular noetherian with generic point $\eta$, the projection map

$$(2) \quad \lim_{Y \in \Dvr(X)} F(Y) \to F(\eta)$$

is injective. Consequently, Proposition 3.9 is true when $X$ is regular.

**Proof.** Recall that when $X$ is regular, $F(X) = F_{\text{reg}}(X)$ (Remark 3.3(3)). The composition

$$F(X) \to \lim_{Y \in \Dvr(X)} F(Y) \to \lim_{x \in X^{(1)}} F(\Spec(O_{X,x}))$$

is the map we have shown is an isomorphism in Lemma 3.10. So to show that the first map is an isomorphism, it suffices to show that the second map is a monomorphism. Since (2) factors through this, it suffices to show that (2) is injective.

Let $(s_y)_{Y \in \Dvr(X)}, (t_y)_{Y \in \Dvr(X)} \in \lim_{x \in X^{(1)}} F(Y)$ be two sections such that $s_\eta = t_\eta$. We wish to show that $t_Y = t_Y$ for all $Y \in \Dvr(X)$.

First we consider $Y$ of the form a point $x \in X$. This is by induction on the codimension. We already know it for the generic point so suppose it is true for points in $X^{(\leq n-1)}$ and let $x \in X^{(n)}$. By Lemma 1.4 there is a point $y \in X^{(n-1)}$ and a discrete valuation ring $R$ satisfying the diagram

$$\begin{array}{ccc}
    y & \rightarrow & \Spec(R) \\
    \downarrow \" & \rightarrow & \downarrow \" \\
    x & \rightarrow & X
\end{array}$$

By the inductive hypothesis $s_y = t_y$ and then by (Unr1) we have $s_{\Spec(R)} = t_{\Spec(R)}$. Therefore $s_{\Spec(R)|x} = t_{\Spec(R)|x}$ but by the coherency requirement on $s$ and $t$ and we have $s_{\Spec(R)|x} = s_x$ and $t_{\Spec(R)|x} = t_x$ and therefore $t_x = s_x$.

Now for an arbitrary $Y \to X$ in $\Dvr(X)$ with $Y$ of dimension zero, if $x$ is the image of $Y$ we have $t_Y = t_x|_Y = s_x|_Y = s_Y$. For an arbitrary $Y$ of dimension one and generic point $\eta$ we have $t_Y|_\eta = t_\eta = s_\eta = s_Y|_\eta$ and so by (Unr1) we have $t_Y = s_Y$. □

**Proof. of Proposition 3.9.** By Lemma 3.11 we have

$$F_{\text{reg}}(X) = \lim_{Y \in \Reg(X)} F(Y) = \lim_{Y \in \Reg(X)} \lim_{W \in \Dvr(Y)} F(W)$$

Investigating this last double limit carefully, we observe that it can be described as sequences $(s_{W \to Y \to X})$ indexed by pairs of composable morphisms $W \to Y \to X$ with $W \in \Dvr(X)$ and $Y \in \Reg(X)$, and subject to the two conditions:

1. For each $W' \to W \to Y \to X$ with $W', W \in \Dvr(X)$ and $Y \in \Reg(X)$ we have $s_{W' \to Y \to X} = s_{W \to Y \to X}$.
2. For each $W \to Y' \to Y \to X$ with $W \in \Dvr(X)$ and $Y, Y' \in \Reg(X)$ we have $s_{Y \to Y' \to X} = s_{Y \to Y \to X}$.
This group has a natural morphism, say \( \alpha \), towards \( \lim_{t \in W \in \text{Dvr}(X)} F(W) \) by forgetting the \( Y \)'s. The morphism \( \alpha \) has a natural section by sending a section \((s_{W \rightarrow X})_{\text{Dvr}(X)} \in \lim_{t \in W \in \text{Dvr}(X)} F(W)\) to the element \( t \) of \( \lim_{t \in Y \in \text{Reg}(X)} \lim_{t \in W \in \text{Dvr}(Y)} F(W) \).

To show that \( \alpha \) is a monomorphism, it suffices to notice that each \( W \in \text{Dvr}(X) \) can be “thickened” to a \( Y \in \text{Reg}(X) \): By definition \( W \) is of the form \( \cap_{i \in I} U_i \) for some set of open subsets \( U_i \) of a \( Y' \in \text{Sch}(X) \). Since \( W \) is regular of dimension one, we can take \( \{U_i\}_{i \in I} \) to be the set of opens containing a regular codimension one point of \( Y' \). Since \( Y' \) has a regular codimension one point, it has a dense open regular subscheme \( Y \). So we have found a \( W \rightarrow Y \rightarrow X \) with \( Y \in \text{Reg}(X) \).

3.2. Descent properties. Recall the notion of a cdp-cover from Definition 1.3 and the rh, cdh and eh-topologies generated by cdp-covers together with open covers, resp. Nisnevich covers, resp. étale covers.

**Proposition 3.12.** Let \( S \) be a noetherian scheme. If \( F \) is an unramified presheaf on \( \text{Sch}(S) \) then \( F_{\text{reg}} \) is an rh-sheaf. In particular, if \( F \) is an unramified Nisnevich (resp. étale) sheaf on \( \text{Sch}(S) \) then \( F_{\text{reg}} \) is a cdh-sheaf (resp. uh-sheaf).

**Example 3.13.** Let \( S = \text{Spec}(k) \) with \( k \) perfect. Then \( \Omega^n_{\text{reg}} \) is an eh-sheaf for all \( n \).

**Proof of Proposition 3.12.** By Lemma 3.4, \( F_{\text{reg}} \) is a Zariski-sheaf. It remains to establish the sheaf property for cdp-morphisms.

Using the description \( F_{\text{reg}}(X) = \lim_{Y \in \text{Dvr}(X)} F(Y) \) from Proposition 3.9, to show that \( F_{\text{reg}} \) is separated for cdp morphisms, it suffices to show that for every cdp morphism \( X' \rightarrow X \), every \( Y \rightarrow X \) in \( \text{Dvr}(X) \) factors through \( X' \rightarrow X \). Since \( X' \rightarrow X \) is completely decomposed it is true for \( Y \) of dimension zero, and therefore also true for the generic points of \( Y \) of dimension one. To factor all of a \( Y \) of dimension one, we now use the valuative criterion for properness.

Now consider a cdp morphism \( X' \rightarrow X \) and a cocycle
\[
s = (s_Y)_{Y \in \text{Dvr}(X')} \in \ker \left( F_{\text{reg}}(X') \rightarrow F_{\text{reg}}(X' \times_X X') \right).
\]
We have just observed that every scheme in \( \text{Dvr}(X) \) factors through \( X' \) and by making a choice of factorisation for each \( Y \rightarrow X \) in \( \text{Dvr}(X) \), and taking \( t_Y \) to be the \( s_Y \) of this factorisation, we have a potential section \( t = (t_Y)_{Y \in \text{Dvr}(X)} \in F_{\text{reg}}(X) \), which potentially maps to \( s \).

Independence of the choice: Suppose that \( f_0, f_1 : Y \rightarrow X' \rightarrow X \) are two factorisations of some \( Y \in \text{Dvr}(X) \). There is then a unique morphism \( Y \rightarrow X' \times_X X' \) such that composition with the two projections recovers the two factorisations. Saying that \( s \) is a cocycle is to say precisely that in this situation, the two \( s_Y \) corresponding to \( f_0 \) and \( f_1 \) are equal (in \( F(Y) \)).

This independence of the choice implies both that \( t \) is actually a section of \( F_{\text{reg}}(X) \) (i.e., that \( s_Y |_{Y'} = s_{Y'} \) for any morphism \( Y' \rightarrow Y \) in \( \text{Dvr}(X) \)), and also that \( t \) is mapped to \( s \).

4. reg-Differentials

Throughout the section let \( k \) be a perfect field.

**Remark 4.1.** Perfect fields have a number of characterisations. A pertinent one for us is: the field \( k \) is perfect if and only if \( \Omega^1_{L/k} = 0 \) for every algebraic extension \( L/k \). Equivalently, a field \( k \) of characteristic \( p \) is perfect if and only if every element of \( k \) has a \( p \)th root. Examples of perfect fields include: any field of characteristic zero, any finite field, and any algebraically closed field. The standard example of a non-perfect field is \( \mathbb{F}_p(T) \) where \( p \) is a prime and \( \mathbb{F}_p \) is the field with \( p \) elements.
Definition 4.2. Let $\Omega^n_{\text{reg}}$ be the extension of the presheaf $\Omega^n$ on $\text{Reg}(k)$ to $\text{Sch}(k)$ in the sense of Definition 3.1. As section of $\Omega^n_{\text{reg}}$ is called a regular differential.

Corollary 4.3. Let $k$ be perfect. Then $\Omega^n_{\text{reg}}$ is an eh-sheaf. If $X$ is regular, then $\Omega^n_{\text{reg}}(X) = \Omega^n(X)$.

Proof. $\Omega^n$ is unramified. Hence these are special cases of Proposition 3.12 and Remark 3.3 (3).

Remark 4.4. If characteristic of $k$ is zero, this is one of the equivalent characterisations of h-differentials given in [HJ13, Theorem 1].

4.1. cdh-differentials. We introduce an alternative candidate for a good theory of differentials and show that it agrees with $\Omega^n_{\text{reg}}$ under weak resolution of singularities.

Definition 4.5. Let $k$ be a perfect field. Let $\Omega^n_{\text{cdh}}$ be the sheafification of $\Omega^n$ on $\text{Sch}(k)$ in the rh-topology or equivalently in the cdh- or eh-topology. Sections of $\Omega^n$ are called cdh-differentials.

Remark 4.6. We have decided on the terminology of cdh-differential because this topology seems better known than rh or eh. In characteristic zero, it is even an h-sheaf by [topology seems better known than rh or eh. In characteristic zero, it is even an

We want to compare $\Omega^n_{\text{cdh}}$ with the eh-sheaf $\Omega^n_{\text{reg}}$. The comparison is a question about torsion.

We will refer to the following hypothesis below.

Hypothesis H. Let $k$ be a perfect field. For every reduced $Y \in \text{Sch}(k)$ and every $\omega \in \text{tor } \Omega^n(Y)$ there is a birational proper morphism $\pi : \tilde{Y} \to Y$ such that the image of $\omega$ in $\Omega^n(\tilde{Y})$ vanishes.

Remark 4.7. The above hypothesis is a consequence of weak resolution of singularities, but a priori seems to be weaker than it. See Appendix A.1 for more extended discussion of the property.

Proposition 4.8. Let $k$ be perfect, $X \in \text{Reg}(k)$. Then

$$\Omega^n_{\text{cdh}}(X) = \Omega^n(X) \oplus \text{tor } \Omega^n_{\text{cdh}}(X)$$

(see Definition 1.1). Furthermore, under the Hypothesis H one has $\text{tor } \Omega^n_{\text{cdh}}(X) = 0$.

Proof. Recall that $\Omega_{\text{reg}}$ is a cdh-sheaf. By the universal property of cdh-sheafification, there is a canonical map $\Omega^n_{\text{cdh}} \to \Omega^n_{\text{reg}}$. When $X$ is regular, the composition $\Omega^n(X) \to \Omega^n_{\text{cdh}}(X) \to \Omega^n_{\text{reg}}(X)$ is an isomorphism by Remark 3.3(3). It remains to show that the direct complement $T_{\text{cdh}}(X)$ of $\Omega^n(X)$ is torsion. But for any field $K$ (in particular, the function field of $X$), the two morphisms $\Omega^n(K) \to \Omega^n_{\text{cdh}}(K) \to \Omega^n_{\text{reg}}(K)$ are isomorphisms, so it follows from (Unr1).

Now suppose that Hypothesis H is true. Let $\omega \in T_{\text{cdh}}(X)$. It is represented by a cocycle on a cdh-cover of $X$. After refinement (see Voevodsky) the cover is of the form $V \to Y \to X$ where the first is a Nisnevich cover and the second a proper cdh-cover. As $X$ is regular, the proper cdh-cover can be refined by a proper cdh-cover which consists only of a single birational map (Proposition 1.5). Since $\Omega^n$ is a Nisnevich sheaf, the representative in $\Omega^n(V)$ can be lifted to a representative in $\Omega^n(Y)$ (use the canonical morphism $V \times Y V \to V \times X V$ for the lifting and the Nisnevich cover $V \times X V \to Y \times X Y$ to check that the lifted section is still a cocycle). Now by Hypothesis H, there exists a proper birational morphism $Y' \to Y$ such that the image of the representative in $\Omega^n(Y')$ is zero. But $Y' \to X$ is still a cdh cover (Proposition 1.5) and so $\omega$ is zero. □
Proposition 4.9. Let $k$ be perfect. Assume that weak resolution of singularities holds. Then the natural map

$$\Omega^n_{\text{cdh}} \to \Omega^n_{\text{reg}}$$

of presheaves on $\text{Sch}(k)$ is an isomorphism.

Proof. The assertion is true in the regular case by Proposition 4.8. In the general case, we argue by induction on the dimension. In dimension 0 there is nothing to show. We may assume that $X$ is reduced and hence generically regular. Let $\tilde{X} \to X$ be a desingularization which is an isomorphism outside $Z \subset X$ with exceptional locus $E$. By cdh-descent, the sequence

$$0 \to \Omega^n_{\text{?}}(X) \to \Omega^n_{\text{?}}(\tilde{X}) \oplus \Omega^n_{\text{?}}(Z) \to \Omega^n_{\text{?}}(E)$$

is exact for $? = \text{cdh, reg}$. By inductive hypothesis the comparison map is an isomorphism for $Z$ and $E$. It is also an isomorphism for $\tilde{E}$ by the regular case. Hence it is also an isomorphism for $X$. 

In particular:

Corollary 4.10. Let $k$ be perfect. Assume that weak resolution of singularities holds over $k$. Then $\Omega^n_{\text{cdh}}(X) = \Omega^n_{\text{reg}}(X)$ for all $X \in \text{Reg}(k)$. Moreover, $\Omega^n_{\text{cdh}}$ is unramified.

This was proved by Geisser in [Gei06, Theorem 4.7] under strong resolution of singularities, i.e., under the assumption that any birational proper morphism between smooth varieties can be refined by a series of blow-ups with smooth centers.

Remark 4.11. It is not clear if Hypothesis H is enough for this consequence. It only implies that

$$\Omega^n_{\text{reg}} \to \Omega^n_{\text{cdh,reg}}$$

is an isomorphism.

4.2. Torsion of $\text{reg}$-sheaves. In characteristic 0, the sheaf $\Omega^n_{\text{cdh}} = \Omega^n_{\text{reg}}$ is torsion free. We are going to show that this fails in positive characteristic.

Example 4.12 (Existence of $\text{reg}$-torsion). Let $k$, $X$ and $Y$ be as in Example 2.6 (so $k$ is algebraically closed of characteristic 2, $Y$ is the Whitney umbrella, and $X$ is its singular locus). Let $\pi : \tilde{Y} \to Y$ be the desingularization of $Y$. It is given by $\mathbb{A}^2 = \text{Spec}(k[x, u])$ with

$$\pi^# : k[x, y, z] \to k[u, z]$$

mapping

$$x \mapsto u^2, \quad y \mapsto uz$$

Let $E \subset \tilde{Y}$ be the exceptional locus, i.e., the pre-image of $X$. It is given by $z = 0$. Note that the morphism $E \to X$ is radicial and induces the zero morphism on $\Omega^1$. We compute $\Omega^1_{\text{reg}}$ of $Y, \tilde{Y}, X, E$. The last three are regular, hence $\Omega^1 \to \Omega^1_{\text{reg}}$ is an isomorphism on these varieties. Since $\Omega^1_{\text{reg}}$ is a cdh-sheaf and $\{X, \tilde{Y}\}$ is a cdh-cover of $Y$ we have the following exact sequence:

$$0 \to \Omega^1_{\text{reg}}(Y) \to \Omega^1(\tilde{Y}) \oplus \Omega^1(X) \to \Omega^1(E)$$

This means that $\Omega^1_{\text{reg}}(Y)$ is the kernel of

$$k[u, z]du \oplus k[u, z]dz \oplus k[x]dx \to k[z]dz, \quad (\omega_1, \omega_2, \omega_3) \mapsto\omega_1|_{z=0}$$

Hence,

$$\text{tor} \Omega^1_{\text{reg}}(Y) = k[x]dx$$

However, the pullback to $X$ of every non-zero element of $\text{tor} \Omega^1_{\text{reg}}(Y)$ is non-torsion.
Corollary 4.13. Let \( k \) be a perfect field of positive characteristic. Then \( \Omega^n_{\text{reg}} \) is not torsion-free in general.

We have also shown:

Corollary 4.14. Let \( k \) be a perfect field of positive characteristic. Then \( \text{tor } \Omega^1_{\text{reg}}(\cdot) \) does not allow pull-backs. In other words, \( \text{tor } \Omega^1_{\text{reg}} \) is not presheaf on \( \text{Sch}(k) \).

Remark 4.15. The same computation works for any extension of \( \Omega^1 \) to a sheaf on \( \text{Sch} \) which has cdh-descent and agrees with Kähler differentials on regular varieties.

5. Separably decomposed topologies

In many applications, de Jong’s theorem on alterations [dJ96] and Gabber’s refinement [ILO12] have proved a very good replacement for weak resolution of singularities, which is not yet available in characteristic \( p \). We explore the possibility of passing from the eh-topology to a refinement which allows alterations as covers while still preserving the notion of a differential in the smooth case. This turns out impossible.

5.1. h-topology. For completeness, we record:

Lemma 5.1. Let \( S \) be a noetherian scheme of characteristic \( p > 0 \) and \( n \geq 0 \). Then the h-sheafification \( \Omega^n_h \) of \( \Omega^n \) on \( \text{Sch}(S) \) is zero. In fact, even the h-separated presheaf associated to \( \Omega^n \) is zero.

Proof. Since the h-topology is finer than the Zariski topology, it suffices to prove the statement for affine schemes. We claim that for any ring \( A \), and any generator \( da \in \Omega(\text{Spec}(A)) \) there exists an \( h \)-cover \( Y \to \text{Spec}(A) \) such that \( da \) is sent to zero in \( \Omega(Y) \). Indeed, it suffices to consider the finite surjective morphism \( \text{Spec}(A[T]/(T^p - a)) \to \text{Spec}(A) \) is sufficient. The image of \( da \) under this morphism is \( da = d(T^p) = pT^{p-1}dT = 0 \).

5.2. sdh-topology. To avoid such phenomena, one could try considering the following coarser topology than the h-topology. We only allow proper maps which are generically separable. By making the notion stable under base change, we are led to the following notion:

Definition 5.2. Let \( S \) be noetherian. We define the sdh-topology on \( \text{Sch}(S) \). It is generated by the étale topology, and proper morphisms \( f : Y \to X \) such that for every \( x \in X \) there exists \( y \in f^{-1}(x) \) with \([k(y) : k(x)]\) finite separable.

Example 5.3. Let \( \pi : X' \to X \) be a proper morphism and let \( Z \subset X \) be a closed subscheme such that \( \pi \) is finite étale over \( X - Z \). Then \( \{X', Z\} \) is an sdh-cover.

Remark 5.4. In characteristic zero, the h and sdh topologies are the same [Voe96, (proof of) Proposition 3.1.9].

Example 5.5. Let \( k \) be a perfect field of characteristic \( p \),

\[
S = \frac{k[x, y, z]}{z^p - zx^n - y} \cong k[x, z], \quad R = k[x, y].
\]

We consider the covering map

\[
\pi : \tilde{X} = \text{Spec}(S) \to X = \text{Spec}(R)
\]

Note that both \( X \) and \( \tilde{X} \) are regular. The covering map is finite of degree \( p \). It is étale outside the exceptional set \( Z = V(x) \subset X \). Indeed, the minimal polynomial of \( z \) over \( k[x, y] \) is \( T^p - x^nT - y \) with derivative \( x^n \). Hence \( \pi \) is an alteration and generically separable.
Let \( E = V(x) \subset \tilde{X} \) be the preimage of \( Z \). We have
\[
E = \text{Spec} \left( \frac{k[x, y, z]}{(z^p - yx^n - x, y)} \right) \cong \text{Spec} \left( \frac{k[y, z]}{z^p - y} \right) \cong \text{Spec}(k[z]).
\]
Hence \( Z \) and \( E \) are regular as well. Finally, we also need \( \tilde{X} \times_X \tilde{X} \), the spectrum of the ring
\[
\frac{k[x, y, z]}{z^p - yx^n - y} \oplus \frac{k[x, y, t]}{t^p - t^x^n - y} \cong \frac{k[x, y, z, t]}{(z^p - yx^n - y, t^p - t^x^n - y)}
\]
with \( u = t - z \). It is regular outside of
\[
Z' = V(x, u) = \text{Spec} \left( \frac{k[x, z, u]}{(u^p - x^n u, x, u)} \right) \cong \text{Spec} \left( \frac{k[z]}{x, z, t} \right) \subset \tilde{X} \times_X \tilde{X}
\]
As \( \tilde{X} \times_X \tilde{X} \) is of dimension \( \leq 2 \), there is a resolution of singularities, i.e., a proper birational map \( \pi' : X' \to \tilde{X} \times_X \tilde{X} \) which is an isomorphism outside \( Z' \) with \( X' \) regular. We now work out the modules of \( k \)-differentials.
\[
\Omega^1(X) = Rdx \oplus Rdy,
\Omega^1(\tilde{X}) = Sdx \oplus Sdy
\]
with restriction morphism mapping \( dx \mapsto dx \) and
\[
dy \mapsto d(z^p + yx^n) = x^n dy + znx^{n-1} dx.
\]
Moreover,
\[
\Omega^1(Z) = k[y]dy,
\Omega^1(E) = k[z]dz
\]
with restriction morphism
\[
dy \mapsto d(z^p) = 0.
\]
We also have injections
\[
\Omega^1_{\text{reg}}(\tilde{X} \times_X \tilde{X}) \to \Omega^1(X') \oplus \Omega^1_{\text{reg}}(Z') \to \Omega^1(k(\tilde{X} \times_X \tilde{X})) \oplus \Omega^1(k(z)).
\]

**Proposition 5.6.** In the above example, sdh-descent fails for \( \Omega^1_{\text{reg}} \) and the cover \( \{\tilde{X}, Z\} \). In other words, \( \Omega^1(X) \neq \Omega^1_{\text{sdh}}(X) \) for this regular \( X \).

**Remark 5.7.** The same example works to show that every presheaf \( F \) such that \( F|_{\text{reg}} \cong \Omega^1 \) is not an sdh-sheaf.

**Proof.** We have to consider the sequence
\[
0 \to \Omega^1(X) \to \Omega^1(\tilde{X}) \oplus \Omega^1(Z) \to \Omega^1_{\text{reg}}(\tilde{X} \times_X \tilde{X}) \oplus \Omega^1(E) \oplus \Omega^1(Z \times_X Z)
\]
Note that \( Z \times_X Z = Z \). The two restriction maps from \( Z \) agree, hence the condition is vacuous. Consider \( (0, \omega_Z) \) in the middle. It is in the kernel of the differential. On the other hand, it cannot be induced from an element in \( \Omega^1(X) \) because the restriction map \( \Omega^1(X) \to \Omega^1(\tilde{X}) \) is injective. \( \square \)

### 5.3. The site s-alt
As the problem seems to be in the non-separable locus of \( X' \to X \), one could try removing the need for \( Z \), by considering the following version of [ILO12, Expose II, Section 1.2].

**Definition 5.8.** Let \( S \) be noetherian, \( X \in \text{Sch}(S) \). We define the site \( \text{s-alt}(X) \) as follows. The objects are those morphisms \( f : X' \to X \) in \( \text{Sch} \) such that \( X' \) is reduced, and for every generic point \( \xi \in X' \), the point \( f(\xi) \) is a generic point of \( X \) and moreover, \( k(\xi)/k(f(\xi)) \) is finite and separable. The topology is generated by the étale topology, and morphisms of \( \text{s-alt}(X) \) which are proper (they are automatically generically separable by virtue of being in \( \text{s-alt}(X) \)).
The category $s$-alt($X$) admits fibre products in the categorical sense, which can be calculated as follows: For morphisms $Y' \to Y$ and $Y'' \to Y$ in $s$-alt($X$) let $Y' \times_Y Y''$ be the union of the reduced irreducible components of $Y' \times_Y Y''$ (the usual fibre product of schemes) which dominate an irreducible component of $X$. A presheaf $\mathcal{F}$ on $s$-alt($X$) is a $s$-alt-sheaf if the following sequence is exact

$$0 \to \mathcal{F}(X_1) \to \mathcal{F}(X_2) \to \mathcal{F}(X_2 \times_{X_1} X_2)$$

for all covers $X_2 \to X_1$.

By de Jong’s theorem on alterations [dJ96], the system of covers $Y \to X$ with $Y$ regular is cofinal.

**Lemma 5.9.** Let $k$ be perfect, $X \in \text{Sch}(k)$. For general $X$, the presheaf $\Omega^1_{\text{reg}}$ is not an $s$-alt-sheaf on $X$. In, fact it is not separated.

**Proof.** Assume that $\Omega^1_{\text{reg}}$ is separated, i.e., the map $\Omega^1_{\text{reg}}(Y) \to \Omega^1_{\text{reg}}(Y')$ is injective for all separable alterations $Y' \to Y$ over $X$. Let $X$ be irreducible. Choose $Y \to X$ a separable alteration with $Y$ regular. Then we have injective maps

$$\Omega^1_{\text{reg}}(X) \to \Omega^1_{\text{reg}}(Y) = \Omega^1(Y) \to \Omega^1(k(Y)).$$

The composition factors via $\Omega^1(k(X)) \to \Omega^1(k(Y))$. This map is also injective, because $k(Y)/k(X)$ is separable. In total, the map

$$\Omega^1_{\text{reg}}(X) \to \Omega^1(k(X))$$

is injective. This contradicts Corollary 4.13. \qed

Torsion only occurs for singular $X$. The following example shows that $s$-alt-descent also fails for regular $X$.

**Lemma 5.10.** Let $k$ be perfect. $\Omega^1_{\text{reg}}$ on $\text{Sch}(k)$ does not have $s$-alt-descent for the $\pi : \tilde{X} \to X$ of Example 5.5 if $n \geq 2$. In particular, $\Omega^1_{\text{s-alt}}(X) \neq \Omega^1(X)$ for this particular $X$.

**Proof.** We consider the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1(X) & \longrightarrow & \Omega^1(\tilde{X}) & \longrightarrow & \Omega^1_{\text{reg}}(\tilde{X} \times_X \tilde{X}) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1(k(X)) & \longrightarrow & \Omega^1(k(\tilde{X})) & \longrightarrow & \Omega^1(k(X) \times_X \tilde{X} \times_X \tilde{X})
\end{array}
$$

We wish to show that the top row is not exact. As $X$ and $\tilde{X}$ are smooth, the left two vertical morphisms are monomorphisms. Moreover, since $\tilde{X} \to X$ is generically étale and $\Omega^1$ is an étale sheaf, the lower row is exact. Consequently, the top row is exact at $\Omega^1(X)$. It also follows from these observations that the top row is exact at $\Omega^1(\tilde{X})$ if and only if $\Omega^1(X)$ is the intersection of $\Omega^1(k(X))$ and $\Omega^1(\tilde{X})$ inside $\Omega^1(k(\tilde{X}))$. That is, if and only if the inclusion

$$k[x,y]dx \oplus k[x,y]dy \subseteq \left( k(x,y)dx \oplus k(x,y)dy \right) \cap \left( k[x,z]dx \oplus k[x,z]dz \right)$$

is an equality inside $k(x,z)dx \oplus k(x,z)dz$, where $y = z^p + zx^n$.

However, the element

$$x^{-1}dy = n z x^{n-2} dx + x^{n-1} dz$$

is in the intersection on the right, but cannot come from an element on the left, since for any element coming from the left the coefficient of $dz$ is divisible by $x^n$. Hence, the inclusion is strict. \qed
Appendix A. Hypothesis H

We have seen in the main text that $\Omega^e_{\text{cdh}}$ has good properties under the assumption of resolution of singularities. Actually, the (possibly) weaker Hypothesis H was enough. In this Appendix we study this hypothesis further. The considerations are independent of the main text.

Hypothesis H. Let $k$ be a perfect field. For every $Y \in \mathcal{S}(k)$ and every $\omega \in \text{tor } \Omega^n(Y)$ there is a birational proper morphism $\pi : \tilde{Y} \to Y$ such that the image of $\omega$ in $\Omega^n(\tilde{Y})$ vanishes.

Lemma A.1. Let $k$ be perfect. Assume weak resolution of singularities holds over $k$, i.e., for every reduced $Y$ there is a proper birational morphism $X \to Y$ with $X$ smooth. Then Hypothesis H holds true.

Proof. Let $\omega \in \text{tor } \Omega^n(Y)$. By definition there is a dense open subsect $U \subset Y$ such that $\omega|_U$ vanishes. Let $\pi : X \to Y$ a desingularization. Then $\pi^*\omega \in \Omega^n(X)$ is a torsion from because it vanishes on $\pi^{-1}U$. As $X$ is smooth, this implies that $\pi^*\omega = 0$. \qed

A.1. Hyperplane section criterion. We first give a criterion for testing the vanishing of torsion forms. The idea behind the criterion is use some kind of induction on the dimension in attacking a proof of Hypothesis H. In this section $k$ is assumed algebraically closed.

Lemma A.2 (Hyperplane section criterion). Let $X$ be a quasi-projective variety, defined over an algebraically closed field $k$. Let $H \subset X$ be any reduced, irreducible hyperplane that is not contained in the singular locus of $X$. Then, the natural map $(\text{tor } \Omega^1_X)|_H \to \Omega^1_H$ is injective.

In particular, if $U \subset X$ is open and $\sigma \in (\text{tor } \Omega^1_X)(U)$ is any torsion-form with induced form $\sigma_H \in \Omega^1_H(U \cap H)$, then $\text{supp } \sigma \cap H \subseteq \text{supp } \sigma_H$.

Proof. Consider the restriction $(\text{tor } \Omega^1_X)|_H$. Since $H$ is not contained in the singular locus of $X$, this is a torsion sheaf on $H$. Recalling the exact sequence that defines torsion- and torsion-free forms, Sequence (1) on page 5, and using that $\Omega^1_X$ is torsion-free and $\text{Tor}^{\mathcal{O}_X}_1(\mathcal{O}_H, \Omega^1_X)$ hence zero, observe that this sequence restricts to an exact sequence of sheaves on $H$,

$$0 \longrightarrow (\text{tor } \Omega^1_X)|_H \longrightarrow \Omega^1_X|_H \longrightarrow \Omega^1_H|_H \longrightarrow 0.$$  

In particular, $\alpha$ is injective. It injects the torsion sheaf $(\text{tor } \Omega^1_X)|_H$ into the middle term of the second fundamental sequence for differentials, [Har77, II. Prop. 8.12],

$$(\text{tor } \Omega^1_X)|_H \xrightarrow{\text{injection}} \mathscr{F}_H / \mathscr{G}^2_H \xrightarrow{\alpha} \Omega^1_X|_H \xrightarrow{\beta} \Omega^1_H|_H \longrightarrow 0.$$  

We claim that the morphism $\alpha$ is also injective. To this end, recall that $\mathscr{F}_H$ is locally principal. In particular, $\mathscr{F}_H / \mathscr{G}^2_H$ is a locally free sheaf of $\mathcal{O}_H$-modules and that the morphism $\alpha$ is injective at the generic point of $H$, where $X$ is smooth, [Eis95, Ex. 17.12]. It follows that $\ker \alpha$ is a torsion-subsheaf of the torsion-free sheaf $\mathscr{F}_H / \mathscr{G}^2_H$, hence zero. The image $\text{img } \alpha$ is hence isomorphic to $\mathscr{F}_H / \mathscr{G}^2_H$, and in particular torsion-free. As a consequence, note that the img $\alpha$, which is the image of a torsion-sheaf and hence itself torsion, intersects $\text{img } \alpha = \ker \beta$ trivially. The composed morphism $b \circ \alpha$ is thus injective. This proves the main assertion of Lemma A.2.
To prove the “In particular ...”-clause, let \( U \subseteq X \) be any open set, \( \sigma \in (\text{tor} \Omega^1_X)(U) \) be any torsion-form and \( x \in \text{supp} \sigma \) be any closed point. It follows from Nakayama’s lemma that \( x \in \text{supp}(\sigma|_{U'}) \subseteq \text{supp}(\text{tor} \Omega^1_X|_{U'}) \). Since \( H \) is not contained in the singular locus, the sheaf \( \Omega^1_X \) is locally free at the generic point of \( H \), and \( \sigma_H \) is thus a torsion form on \( H \cap U \). Since \( b \circ \alpha \) is injective, its support contains \( x \) as claimed. This finishes the proof of Lemma A.2. \( \square \)

From this lemma and Flenner’s Bertini-type theorems, [Fle77] we get the following theorems:

**Theorem A.3.** Let \( X \) be a quasi-projective variety of dimension \( \dim X \geq 3 \), defined over an algebraically closed field \( k \), and let \( x \in X \) be a closed, normal point. Then, there exists a hyperplane section \( H \) through \( x \) such that \( H \) is irreducible and reduced at \( x \) and such that the following holds: if \( U \subseteq X \) is an open neighbourhood of \( x \) and if \( \tau \in \text{tor} \Omega^1_X(U) \) is any torsion form whose induced form \( \sigma_H \) vanishes at \( x \), then \( \sigma \) vanishes at \( x \).

In particular, \( \Omega^1_X \) is torsion-free at \( x \) if \( \Omega^1_H \) is torsion-free at \( x \).

**Proof.** It follows from normality of \( x \in X \) that the local ring \( O_{X,x} \) satisfies Serre’s condition \( (R_1) \), has depth \( O_{X,x} \geq 2 \) and that it is analytically irreducible, [Zar48, Thm. on page 352]. We can thus apply [Fle77, Korollar 3.6] and find a hyperplane section \( H \) through \( x \) that is irreducible and reduced at \( x \). Shrinking \( X \) if need be, we can assume without loss of generality that \( H \) is irreducible and reduced. Lemma A.2 then yields the claim. \( \square \)

**Theorem A.4.** Let \( X \) be a quasi-projective variety of dimension \( \dim X \geq 3 \), smooth in codimension one and defined over an algebraically closed field. Let \( H \) be a finite-dimensional, basepoint-free linear system. Then, there exists a dense, open subset \( H^0 \subseteq H \) such that any hyperplane section \( H \subseteq X \) which corresponds to a closed point of \( H^0 \) is irreducible, reduced, and satisfies the following additional property: if \( U \subseteq X \) is open and \( \tau \in \text{tor} \Omega^1_X(U) \) is any torsion form with induced form \( \sigma_H \in \Omega^1_H(U \cap H) \), then \( \text{supp} \sigma \cap H \subseteq \text{supp} \sigma_H \).

In particular, if \( \Omega^1_X \) is torsion-free, then \( \text{supp} \text{tor} \Omega^1_X \) is finite and disjoint from \( H \).

**Proof.** Recall from Flenner’s version of Bertini’s first theorem, [Fle77, Satz 5.2] that any general hyperplane \( H \) is irreducible and reduced. The main assertion of Theorem A.4 thus follows from Lemma A.2.

If \( \Omega^1_H \) is torsion-free, the support of \( \text{tor} \Omega^1_X \) necessarily avoids \( H \). Since general hyperplanes can be made to intersect any positive-dimensional subvariety, we obtain the finiteness of \( \text{supp} \text{tor} \Omega^1_X \). \( \square \)

### A.2. Valuation rings

We give a reformulation of hypothesis \( H \) in terms of vanishing of differential forms on (non-discrete) valuation rings. In this section, let \( k \) be a perfect field.

Let \( A \) be an integral \( k \)-algebra of finite type. Recall that the Riemann-Zariski space \( RZ(A) \) (called the Riemann surface in [ZS75, §17, page 110]) as a set is the set of valuation rings of \( \text{Frac}(A) \) which contain \( A \). To a finitely generated sub-\( A \)-algebra \( A' \) is associated the set \( E(A') = \{ R \in RZ(A) : A' \subseteq R \} \) and one defines a topology on \( RZ(A) \) taking the \( E(A') \) as a basis. This topological space is quasi-compact, in the sense that every open cover admits a finite subcover [ZS75, Theo. 40].

Consider the following:

**Hypothesis V.** For every finitely generated extension \( K/k \) and every \( k \)-valuation ring \( R \) of \( K \) the map \( \Omega^n(R) \to \Omega^n(K) \) is injective.

**Proposition A.5.** Let \( k \) be perfect. The hypotheses \( V \) and \( H \) for \( k \) are equivalent.
Proof. First we treat the integral affine case. Let $X = \text{Spec}(A) \in \text{Sch}(k)$ be integral and $\omega \in \Omega^n(Y)$ an element which vanishes on a dense open, that is, the image of $\omega$ in $\Omega^n(\text{Frac}(A))$ is zero. We wish to find a proper birational morphism $Y \to X$ such that $\omega|_Y = 0$. Since $A$ is of finite type over a field, the normalisation is a finite morphism, and we can therefore assume that $X$ is normal.

Hypothesis V implies then that the image of $\omega$ in $\Omega^n(R)$ is zero for every valuation ring $R$ of $\text{Frac}(A)$ which contains $A$. As each $R$ is the union of its finitely generated sub-$A$-algebras, for each such $R$ there is a finitely generated sub-$A$-algebra, say $A_R$, for which $\omega$ vanishes in $\Omega^n(A_R)$. The $E(A_R)$ then form an open cover of $\text{RZ}(A)$ and so since it is quasi-compact, there exists a finite subcover. That is, there is a finite set $\{A_i\}_{i=1}^n$ of finite generated sub-$A$-algebras of $\text{Frac}(A)$ such that $\omega$ is zero in each $\Omega^n(A_i)$, and every valuation ring of $\text{Frac}(A)$ which contains $A$, also contains one of the $A_i$'s.

Now for each $i$, choose a factorisation $\text{Spec}(A_i) \to Y_i \to X$ as a dense open immersion followed by a proper morphism and define $Y = Y_1 \times_X \cdots \times_X Y_n$ (or better, define $Y$ to be the closure of the image of $\text{Spec}(\text{Frac}(A))$ in this product) so that $Y \to X$ is now a proper birational morphism. Since $\omega|_{A_i} = 0$ for each $i$, it suffices now to show that the set of open immersions $\{\text{Spec}(A_i) \times_Y Y \to Y\}_{i=1}^n$ is an open cover of $Y$ to conclude that $\omega|_Y = 0$. But for every point $y \in Y$, there exists a valuation ring $R_y$ of $\text{Frac}(A)$ such that $\text{Spec}(R_y) \to Y$ sends the closed point of $\text{Spec}(R_y)$ to $y$, and since $R_y$ contains some $A_i$, there is a factorisation $\text{Spec}(R_y) \to \text{Spec}(A_i) \to Y$, and we see that $y \in \text{Spec}(A_i) \times_Y Y$.

For the case of a general $X \in \text{Sch}(k)$ we use the same trick. Since we are looking for a proper birational morphism which kills $\omega \in \Omega^n(X)$, we can assume that $X$ is integral. Now take an affine cover $\{U_i\}_{i=1}^n$ of $X$. We have just seen that there exist proper birational morphisms $V_i \to U_i$ such that $\omega|_{V_i} = 0$ for each $i$. Choose compactifications $\bar{V}_i \to Y_i \to X$ and set $Y = Y_1 \times_X \cdots \times_X Y_n$ so that $Y \to X$ is proper birational, and the same argument as above shows that $\{V_i \times_Y Y \to Y\}_{i=1}^n$ is an open cover. Since $\omega|_{V_i} = 0$ for each $i$, this implies that $\omega|_Y = 0$.

Proof. Let $K$ be a finitely generated extension of $k$, let $R$ be a $k$-valuation ring of $K$, and let $\omega$ be in the kernel of $\Omega^n(R) \to \Omega^n(K)$. There is some finitely generated sub-$k$-algebra $A$ of $R$ and $\omega' \in \Omega^n(A)$ such that $\text{Spec}(R) \to \text{Spec}(A)$ is birational, and $\omega'|_R = \omega$. Now by Hypothesis H there is a proper birational morphism $\bar{Y} \to \text{Spec}(A)$ such that $\omega'|_{\bar{Y}}$ is zero. But by the valuative criterion for properness, there is a factorisation $\text{Spec}(R) \to \bar{Y} \to \text{Spec}(A)$, and so $\omega = 0$. 

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