Parallel Packing Squares into a Triangle

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Abstract. Assume that $T_h$ is a triangle with the interior angles at the base of the measure not greater than $90^\circ$, with the base length 1 and the height $h$. Let $S$ be a square with a side parallel to the base of $T_h$ and let $\{S_n\}$ be a collection of the homothetic copies of $S$. A tight upper bound of the sum of the areas of squares from $\{S_n\}$ that can be parallel packed into a triangle $T_h$ is determined.

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1. Introduction

Let $P$ be a polygon and let $S_i$ be a square for $i = 1, 2, \ldots$ One side of $P$ is called the base of $P$. A collection $S_1, S_2, \ldots$ is said to be packed into $P$ if their union is contained in $P$ and if these squares have pairwise disjoint interiors. A packing is called parallel if a side of each packed square is parallel to the base of $P$.

The goal is to pack the squares into $P$ with high density. The area of a polygon $R$ is denoted by $|R|$. Let $g(P)$ be the greatest number such that any collection of squares of the total area not greater than $g(P) \cdot |P|$ can be parallel packed into $P$.

Assume that $S$ is a square and that $T_h$ is a triangle with the interior angles at the base of the measure not greater than $90^\circ$, with the base length 1 and the height $h$. There are many results concerning packing squares or rectangles.

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Moon and Moser [11] showed that \( \varrho(S) = 1/2 \); in this case the most effective usual packing is the parallel packing. In many publications in which squares are placed into a rectangle, when authors write “packing” they mean “parallel packing” (see for example [1,2,5,8–10,12]). Let us add that for packing squares or rectangles of the side lengths not greater than 1 into a large square the most effective packings are not parallel (see [3,7]).

The following results concerning parallel packing squares into a triangle were known:
- \( \varrho(T_{\sqrt{3}/2}) = 2\sqrt{3} - 3 \) for an equilateral triangle \( T_{\sqrt{3}/2} \) (see [6]),
- \( \varrho(T_{\sqrt{2}} \geq (16 - 6\sqrt{2})/23 \) for a right triangle \( T_{\sqrt{2}} \) (see [4]),
- \( \varrho(T_{\sqrt{2}/2}) = 6\sqrt{2} - 8 \) and \( \varrho(T_{\sqrt{2}/3}) = 6\sqrt{2}/(11 + 6\sqrt{2}) \) (see [13]).

The aim of this note is to show that
\[
\varrho(T_{h}) = \min \left\{ \frac{2h}{(h+1)^2}, \max \left[ \frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right] \right\}.
\]

Without loss of generality one can assume that \( \alpha \geq \beta \) (see Fig. 1).

2. Upper Bounds

Claim 1. A square of the side length not greater than \( h/(h+1) \) can be parallel packed into \( T_{h} \) while one square of the side length greater than \( h/(h+1) \) cannot.

Proof. Let \( a \) be the greatest number such that the square of the side length equal to \( a \) can be parallel packed into \( T_{h} \) (see Fig. 1). By Thales’s intercept theorem, \( \frac{1}{a} = \frac{h}{h+a} \), i.e., \( a = \frac{h}{h+1} \). \( \square \)

Corollary 1. Any square of the area not greater than \( 2h/(h+1)^2 \cdot |T_{h}| \) can be parallel packed into \( T_{h} \). Moreover, \( \varrho(T_{h}) \leq 2h/(h+1)^2 \).

Proof. If the area of a square equals \( 2h/(h+1)^2 \cdot |T_{h}| = h^2/(h+1)^2 \), then its side length is equal to \( h/(h+1) \). \( \square \)
Claim 2. Two congruent squares of the side length smaller than or equal to \( \max\left[\frac{h}{2h+1}, \frac{h}{h+2}\right] \) can be parallel packed into \( T_h \) while two squares of the side length greater than \( \max\left[\frac{h}{2h+1}, \frac{h}{h+2}\right] \) cannot.

Proof. Let \( b \) be the greatest number such that two squares of the side length equal to \( b \) can be parallel packed into \( T_h \) along the base of \( T_h \) (see Fig. 2). By Thales’s intercept theorem, \( \frac{1}{2b} = \frac{h}{h-b} \), i.e., \( b = \frac{h}{2b+1} \). Moreover, let \( d \) be the greatest number such that two squares of the side length equal to \( d \) can be parallel packed into \( T_h \) “vertically” as on Fig. 3. By Thales’s intercept theorem, \( \frac{1}{d} = \frac{h}{h-2d} \), i.e., \( d = \frac{h}{h+2} \).

Corollary 2. Any two congruent squares of the sum of the areas not greater than \( \max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right] \cdot |T_h| \) can be parallel packed into \( T_h \). Moreover, \( \varrho(T_h) \leq \max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right] \).

Proof. If the area of a square equals
\[
\frac{1}{2} \cdot \max\left[\frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2}\right] \cdot |T_h| = \max\left[\frac{h^2}{(2h+1)^2}, \frac{h^2}{(h+2)^2}\right],
\]
then its side length is equal to $\max\left(\frac{h}{2h+1}, \frac{h}{h+2}\right)$. □

It can be check that $4h/(2h+1)^2 \geq 4h/(h+2)^2$ provided $h \leq 1$. Moreover, $2h/(h+1)^2 \leq 4h/(2h+1)^2$ for $h \leq \sqrt{2}/2$ and $2h/(h+1)^2 \leq 4h/(h+2)^2$ for $h \geq \sqrt{2}$. Let

$$\varrho_h = \min\left\{ \frac{2h}{(h+1)^2}, \max\left[ \frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right] \right\}$$

$$= \begin{cases} 
\frac{2h}{(h+1)^2} & \text{for } 0 < h \leq \frac{1}{2} \sqrt{2} \\
\frac{4h}{(2h+1)^2} & \text{for } \frac{1}{2} \sqrt{2} < h \leq 1 \\
\frac{4h}{(h+2)^2} & \text{for } 1 < h \leq \sqrt{2} \\
\frac{2h}{(h+1)^2} & \text{for } h > \sqrt{2} 
\end{cases}$$

It is easy to verify that $\varrho_h \leq 1/2$.

**Corollary 3.** For any $h > 0$, $\varrho(T_h) \leq \varrho_h$. 

**Proof.** If either $h \leq \frac{1}{2} \sqrt{2}$ or $h > \sqrt{2}$, then $\varrho_h = \frac{2h}{(h+1)^2}$. By Corollary 1 we get $\varrho(T_h) \leq \frac{2h}{(h+1)^2} = \varrho_h$. Note that the worst case appears when one square is packed: one square of the area greater than $\varrho_h|T_h|$ cannot be packed into $T_h$.

If $\frac{1}{2} \sqrt{2} < h \leq 1$, then $\varrho_h = \frac{4h}{(2h+1)^2} = \max\left[ \frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right]$. By Corollary 2 we obtain $\varrho(T_h) \leq \varrho_h$. Now the worst case appears when two squares are packed: two congruent squares of the total area greater than $\varrho_h|T_h|$ cannot be packed either along the base of $T_h$ or vertically. However, it is possible to pack two congruent squares of the total area equal to $\varrho_h|T_h|$ horizontally (see Fig. 2).

If $1 < h \leq \sqrt{2}$, then $\varrho_h = \frac{4h}{(h+2)^2} = \max\left[ \frac{4h}{(2h+1)^2}, \frac{4h}{(h+2)^2} \right]$. By Corollary 2 we get $\varrho(T_h) \leq \varrho_h$. In that case the worst case appears when two squares are packed: two congruent squares of the total area greater than $\varrho_h|T_h|$ cannot be packed either along the base of $T_h$ or vertically, but it is possible to pack two congruent squares of the total area equal to $\varrho_h|T_h|$ vertically (see Fig. 3).

In Sect. 5 we will show that $\varrho(T_h) \geq \varrho_h$.

### 3. Lemmas

**Claim 3.** Any two squares of the sum of the areas not greater than $\varrho_h \cdot |T_h|$ can be parallel packed into $T_h$.

**Proof.** The conditions $\alpha \geq \beta$ and $h \cot \alpha + h \cot \beta = 1$ (see Fig. 3) imply that

$$2 \cot \alpha \leq \cot \alpha + \cot \beta = \frac{1}{h}.$$
Let $S_1$ and $S_2$ be the squares of the side lengths $a_1$, $a_2$, respectively and let

$$a_1^2 + a_2^2 \leq g_h \cdot |T_h| \leq \max \left[ \frac{2h^2}{(2h+1)^2}, \frac{2h^2}{(h+2)^2} \right].$$

First we show that if $S_1$ and $S_2$ cannot be packed along the base of $T_h$, then $a_1^2 + a_2^2 > \frac{2h^2}{(2h+1)^2}$. If $S_2$ cannot be parallel packed into $T_h$ ”horizontally” as on Fig. 4, left, then $a_1 \cot \alpha + a_1 + a_2 + a_2 \cot \beta > 1$. Since

$$1 < a_1 \cot \alpha + a_1 + a_2 + a_2 \cot \beta = a_1 \cot \alpha + a_1 + a_2 + a_2 \left( \frac{1}{h} - \cot \alpha \right),$$

it follows that $a_1 + a_2 > \frac{2h}{2h+1}$, i.e., $a_2 > \frac{2h}{2h+1} - a_1$. Thus

$$a_1^2 + a_2^2 > a_1^2 + \left( \frac{2h}{2h+1} - a_1 \right)^2 = 2 \left( a_1^2 - \frac{2ha_1}{2h+1} + \frac{2h^2}{(2h+1)^2} \right) \geq 2 \left( \frac{h^2}{(2h+1)^2} - \frac{2h^2}{(2h+1)^2} + \frac{2h^2}{(h+2)^2} \right) = \frac{2h^2}{(h+2)^2}.$$

This means that $a_1^2 + a_2^2 \leq \frac{2h^2}{(h+2)^2}$. In that case we show that $S_1$ and $S_2$ can be packed into $T_h$ ”vertically” as on Fig. 4, right. The assumption $a_1^2 \leq g_h \cdot |T_h|$, by Corollary 1 implies that $S_1$ can be packed into $T_h$. Let $p = a_1^2 + a_2^2$. Hence $a_1 + a_2 = a_1 + \sqrt{p - a_1^2}$. The function $f(a_1) = a_1 + \sqrt{p - a_1^2}$ has a local maximum at $a_1 = \sqrt{p}/2$. This means that $a_1 + a_2 \leq \sqrt{p}/2 + \sqrt{p}/2 = \sqrt{2p}$. Since $a_1 + a_2 = \sqrt{2p}$ for $a_1 = a_2$, it follows that if two congruent squares of the total area $p$ can be parallel packed into $T_h$, then any two squares of the total area $p$ can be packed into $T_h$. By Corollary 2 two congruent squares of the total area greater than $2h^2/(2h+1)^2$ and not greater than $2h^2/(h+2)^2$ can be parallel packed into $T_h$ ”vertically”. \hfill \Box
Let \( sT_h \) be the image of \( T_h \) after a homothety with the ratio \( s \). Clearly, \( sT_h \) is a triangle of the base length \( s \) and the height \( sh \). Since affine transformations preserve relative areas, by Claim 1 and 3 we get the following result.

**Corollary 4.** Any square of the area not greater than \( \varrho_h \cdot |sT_h| \) can be parallel packed into \( sT_h \). Moreover, two squares of the sum of the areas not greater than \( \varrho_h \cdot |sT_h| \) can be parallel packed into \( sT_h \).

The following two inequalities will be used in the proof of Lemma 3.

**Lemma 1.** Let \( f(x, h) = (1 + x^2)/(2h + 4hx + x) \). If \( h > 0 \) and if \( x \leq 1 \), then \( f(x, h) > \varrho_h/(2h) \).

**Proof.** Assume that \( h > 0 \) and that \( x \leq 1 \). We will show that

\[
f(x, h) > \frac{\varrho_h}{2h} = \begin{cases} 
\frac{1}{(h+1)^2} & \text{for } h \in (0, \frac{1}{2}\sqrt{2}] \cup (\sqrt{2}, +\infty) \\
\frac{2}{(2h+1)^2} & \text{for } h \in (\frac{1}{2}\sqrt{2}, 1] \\
\frac{2}{(h+2)^2} & \text{for } h \in (1, \sqrt{2}] 
\end{cases}
\]

Observe that

\[
f(x, h) - \frac{1}{(h+1)^2} = \frac{x^2h^2 + 2x^2h + x^2 - 4hx - x + h^2 + 1}{(2h + 4hx + x)(h+1)^2} = \frac{(hx - 1)^2(1-x) + (x-h)^2 + x^3h^2}{(2h + 4hx + x)(h+1)^2} > 0
\]

for \( h > 0 \). Moreover, if \( h \geq \sqrt{2}/2 \), then \( 1/(h+1)^2 \geq 2/(2h+1)^2 \) and, consequently, \( f(x, h) > \frac{2}{(2h+1)^2} \). Finally observe that

\[
f(x, h) - \frac{2}{(h+2)^2} = \frac{x^2h^2 + 4x^2h + 4x^2 - 8hx - 2x + h^2 + 4}{(2h + 4hx + x)(h+2)^2} = \frac{(hx - 2)^2(1-x) + (2x-h)^2 + x^3h^2 + 2x}{(2h + 4hx + x)(h+2)^2} > 0.
\]

\[\square\]

**Lemma 2.** Let \( g(y, h) = (2 + y)/(6h + 4hy + 1) \). If \( h > 0 \) and if \( y > 0 \), then \( g(y, h) > \varrho_h/(2h) \).

**Proof.** We proceed as in the proof of Lemma 1. Observe that

\[
g(y, h) - \frac{1}{(h+1)^2} = \frac{h^2 + (y+1)(h-1)^2}{(6h + 4hy + 1)(h+1)^2} > 0
\]

and

\[
g(y, h) - \frac{2}{(h+2)^2} = \frac{4 + 2(h-1)^2 + y(h-2)^2}{(6h + 4hy + 1)(h+2)^2} > 0.
\]

\[\square\]
4. Packing Squares into Trapezoids

In the main packing method squares will be packed in trapezoid-shape layers. In this section an algorithm for packing squares into trapezoids is described.

Assume that $S_i$ is a square of the side length $a_i$, where $a_1 \geq a_2 \geq \ldots$. Let $L$ be a trapezoid of the base length $b(L)$, with the height $h(L)$ and with the base angles measuring $\alpha$ and $\beta$, where $90^\circ \geq \alpha \geq \beta$ (see Fig. 5, left). Moreover assume that $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq b(L)$ and that $h(L) = a_1$ (see Fig. 5, right).

Description of the $L$-method for packing $S_1, S_2, \ldots$ into $L$.

- The first square $S_1$ is packed into $L$ as far to the left as possible. Let $L_1$ be the part of $L$ lying to the right of $S_1$, i.e., the trapezoid of the height $h(L_1) = a_1$, of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha)$ and with the base angles measuring $90^\circ$ and $\beta$ (see Fig. 5, right).
- If $a_2 > \frac{1}{2}a_1$ (see Fig. 5), then $S_2$ is packed into $L_1$ in the left-bottom corner; then $L_{2,1}$ is the part of $L_1$ lying to the right of $S_2$, i.e., either $L_{2,1}$ is the trapezoid of the height $h(L_1) = a_1$, of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha) - a_2$ and with the base angles measuring $90^\circ$ and $\beta$ or $L_{2,1}$ is the triangle of the base length $b(L_1) = b(L) - a_1(1 + \cot \alpha) - a_2$, with the base angles measuring $90^\circ$ and $\beta$, and of the height not greater than $h(L) = a_1$. If $a_2 \leq \frac{1}{2}a_1$, then $L_1$ is partitioned into smaller trapezoids. Let $m_2$ be an integer such that $2^{-m_2-1}a_1 < a_2 \leq 2^{-m_2}a_1$. Then $L_1$ is divided into $2^m_2$ trapezoids $L_{1,1}^+, \ldots, L_{1,2^{m_2}}^+$ of the height $h_2 = 2^{-m_2}a_1$ (see Fig. 6). The square $S_2$ is packed into $L_{1,1}^+$ as far to the left as possible, i.e., in the left-bottom corner. New trapezoids are defined as follows:
Proof. Case 1: Lemma 3. Moreover assume that $h$ by a $\varrho$ of the height $h_{k-1}$ were defined. Next square $S_k$ is placed in the following way. If $a_k > \frac{1}{2}h_{k-1}$, then let $j$ be the smallest integer such that $b(L_{k-1,j}) \geq a_k(1 + \cot \beta)$ (we find the lowest lying trapezoid into which $S_k$ can be packed). Then $S_k$ is packed into $L_{k-1,j}$ in the left-bottom corner. New trapezoids are defined: $L_{k,j}$ is the part of $L_{k-1,j}$ lying to the right of $S_k$ and $L_{k,i} = L_{k-1,i}$ for $i \neq j$ (it is possible that some trapezoids are triangles). For example, squares $S_2, S_3$ and $S_4$ are packed on Fig. 8 (left) by this rule. Moreover, squares $S_4, \ldots, S_{10}$ are packed by this rule on Fig. 9, left. If $a_k \leq \frac{1}{2}h_{k-1}$, then each trapezoid $L_{k-1,i}$ is partitioned into smaller trapezoids. Let $n_k$ be an integer such that $2^{-n_k}h_{k-1} < a_k \leq 2^{-n_k+1}h_{k-1}$. Then $L_{k-1,i}$ (for $i = 1, \ldots, 2^{m_k-1}$) is divided into $2^n$ trapezoids $L_{k-1,i}(i-1)2^{n_k+1}, \ldots, L_{k-1,i}2^{n_k}$ of the height $2^{-n_k}h_{k-1}$ (see Fig. 6, right). Let $j$ be the smallest integer such that $b(L_{k-1,j}) \geq a_k(1 + \cot \beta)$ (we find the lowest lying trapezoid into which $S_k$ can be packed). The square $S_k$ is packed into $L_{k-1,j}$ in the left-bottom corner. New trapezoids are defined: $L_{k,j}$ is the part of $L_{k-1,j}$ lying to the right of $S_k$ and $L_{k,i} = L_{k-1,i}$ for $i \neq j$ and $i \in \{1, 2, \ldots, 2^{m_k}\}$, where $m_k = m_{k-1}n_k$ (it is possible that some trapezoids are triangles). For example, squares $S_3$ and $S_9$ are packed on Fig. 9 (right) by this rule.

Lemma 3. Assume that $S_i$ is a square of the side length $a_i$ for $i = 1, 2, \ldots$, where $a_1 \geq a_2 \geq \ldots$. Let $L$ be a trapezoid of the base length $b(L)$, with the height $h(L) = a_1$ and with the base angles measuring $\alpha$ and $\beta$ (see Fig. 5, left). Moreover assume that $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq b(L)$. If $S_1$ is the first square that cannot be packed into $L$ by the $L$-method, then $|S_1| + \ldots + |S_{z-1}| > \varrho h \cdot |L|$. 

Proof. Since any homothety preserves relative areas, without loss of generality one can assume that $b(L) = 1$.

Consider four cases.

Case 1: $z = 3$, $a_3 \geq a_1/2$. (see Fig. 7)

Let $x = a_2/a_1$. Since $a_2$ cannot be packed into $L$, it follows that $a_1 \cot \alpha + a_1 + a_2 + a_3 \cot \beta > 1$. Recall that $\cot \alpha + \cot \beta = 1/h$ and $\cot \alpha \leq \frac{1}{2h}$. By

$$1 < a_1 \cot \alpha + a_1 + a_2 + a_3 \cot \beta = a_1 \cot \alpha + a_1 + 2a_2 + a_3 \left(\frac{1}{h} - \cot \alpha\right)$$

$$= (a_1 - a_2) \cot \alpha + a_1 + 2a_2 + a_3 \cdot \frac{1}{h} \leq \frac{1}{2h} (a_1 - a_1 x) + a_1 + 2a_1 x + a_1 x \cdot \frac{1}{h}$$

$$= a_1 \left(\frac{1 + x}{2h} + 1 + 2x\right) = a_1 \cdot \frac{1 + x + 4hx + 2h}{2h}$$
we get
\[ a_1 > \frac{2h}{1 + 2h + 4hx + x}. \]

Denote by \( \varrho(L) \) the ratio of the sum of the areas of squares packed into \( L \) to the area of \( L \). Only two squares are packed in \( L \), therefore
\[
\varrho(L) = \frac{a_1^2 + a_2^2}{a_1 - \frac{1}{2}a_2^2(\cot \alpha + \cot \beta)} = \frac{a_1^2 + (xa_1)^2}{a_1 - \frac{a_2^2}{2h}} = \frac{a_1(1 + x^2) \cdot 2h}{2h - a_1}.
\]

Since the function \( f(a_1) = 2ha_1(1 + x^2)/(2h - a_1) \) is increasing, it follows that
\[
\varrho(L) > \frac{2h(1 + x^2) \cdot 2h}{1 + 2h + 4hx + x} \cdot \frac{1 + 2h + 4hx + x}{2h \cdot (2h + 4hx + x)} = 2h \frac{(1 + x^2)}{2h + 4hx + x}.
\]

By Lemma 1, \( \varrho(L) > 2h \cdot \varrho_h/(2h) = \varrho_h \).

Case 2: \( z > 3 \), \( a_z \geq a_1/2 \).

Let \( L^- \) be the trapezoid with \( b(L^-) = 1 - a_3 - \ldots - a_{z-1} \), with the height \( a_1 \) and the base angles with the same measures as the base angles of \( L \) (see Fig. 8, where \( z = 5 \)). Moreover, let \( sL^- \) be the image of \( L^- \) after a homothety with the ratio \( s \) such that the larger base of \( sL^- \) is equal to 1. By Case 1, the density of packing squares \( sS_1 \) and \( sS_2 \) into \( sL^- \) is greater than \( \varrho_h \). This implies that the density of packing squares \( S_1 \) and \( S_2 \) into \( L^- \) is greater than \( \varrho_h \). The density of packing squares \( S_3, \ldots, S_{z-1} \) into the rectangle \( R_L = (a_3 + \ldots + a_{z-1}) \times a_1 \) is greater than \( 1/2 \geq \varrho_h \). Clearly, \( |L^-| + |R_L| = |L| \). As a consequence, the density of packing squares \( S_1, S_2, \ldots, S_{z-1} \) into \( L \) is greater than \( \varrho_h \).

Case 3: \( z \geq 3 \), \( a_1/4 < a_z \leq a_1/2 \).

Let \( k \) be the smallest integer such that \( a_k < a_1/2 \) (see Fig. 9, left). Obviously, \( k \leq z \). Let \( L_{k-1,1} \) be the part of \( L_1 \) lying to the right of \( S_{k-1} \), i.e., either the trapezoid of the height \( h_r = h(L_{k-1,1}) \), of the base length \( b_r = b(L_{k-1,1}) \) and with the base angles measuring 90° and \( \beta \) or the triangle...
of the base length $b_r = b(L_{k-1,1})$, with the base angles measuring $90^\circ$ and $\beta$, and of the the height not greater than $h_r = h(L_{k-1,1})$.

Squares $S_k, S_{k+1}, \ldots$ are packed into two rectangular trapezoids $L_{k-1,1}^+$ and $L_{k-1,2}^+$ with disjoint interiors, where $L_{k-1,1} = L_{k-1,1}^+ \cup L_{k-1,2}^+$ (it is possible that trapezoids are triangles).

Assume that $S_v$ is the first square that cannot be packed into $L_{k-1,1}^+$. Clearly $a_v \leq h_r/2$ and $a_z \leq h_r/2$ (see Fig. 10, right).

Let $p_2 = |S_k| + \ldots + |S_{z-1}|$, i.e., $p_2$ is equal to the sum of the areas of squares packed into $L_{k-1,1}^+$.

**Subcase 3A:** $b_r \geq a_v (1 + \cot \beta) + \frac{1}{2} h_r \cot \beta$. Observe that (see Fig. 10, right)

$$p_2 \geq [b_r - a_v (1 + \cot \beta)] a_v + [b_r - a_z (1 + \cot \beta) - \frac{1}{2} h_r \cot \beta] a_z.$$  

Let

$$p_1 = [b_r - \frac{1}{2} h_r (1 + \cot \beta)] \frac{1}{2} h_r.$$
Figure 10. Subcase 3A

and let \( y = (2p_1)/(a_1h_r) \). Obviously, \( p_1 \) is equal to the area of the gray rectangle and \( ya_1 \) is equal to the base length of the gray rectangle on Fig. 10, left.

Since

\[
p_2 - p_1 \geq b_r (a_v + a_z - \frac{1}{2}h_r) + (1 + \cot \beta) \left( \frac{1}{4}h_r^2 - a_v^2 - a_z^2 \right) - \frac{1}{2}h_r a_z \cot \beta
\]

\[
> \left[ a_v (1 + \cot \beta) + \frac{1}{2}h_r \cot \beta \right] (a_v + a_z - \frac{1}{2}h_r)
\]

\[
+ (1 + \cot \beta) \left( \frac{1}{4}h_r^2 - a_v^2 - a_z^2 \right) - \frac{1}{2}h_r a_z \cot \beta
\]

\[
= a_z (1 + \cot \beta) (a_v - a_z) + \frac{1}{2}h_r \left( \frac{h_r}{2} - a_v \right)
\]

and \( a_z \leq a_v \leq h_r/2 \), we get \( p_2 > p_1 \). Hence, to show that \( \sum_{i=1}^{z-1} |S_i| > \varrho_h |L| \), it suffices to verify that

\[
a_1^2 + \ldots + a_{k-1}^2 + p_1 > \varrho_h |L|.
\]

First assume that \( k = 2 \). We argue as in Case 1. By conditions \( a_1 \cot \alpha + a_1 + ya_1 + \frac{1}{2}a_1 + \frac{1}{2}a_1 \cot \beta = 1 \), \( \cot \alpha + \cot \beta = 1/h \) and \( \cot \alpha \leq 1/(2h) \), we get

\[
1 = a_1 \cot \alpha + a_1 + ya_1 + \frac{1}{2}a_1 + \frac{1}{2}a_1 \cot \beta
\]

\[
= a_1 \cot \alpha + \frac{3}{2}a_1 + ya_1 + \frac{1}{2}a_1 \left( \frac{1}{h} - \cot \alpha \right)
\]

\[
= \frac{1}{2} a_1 \cot \alpha + \frac{3}{2} a_1 + ya_1 + \frac{a_1}{2h}
\]

\[
\leq \frac{1}{2} a_1 \cdot \frac{1}{2h} + \frac{3}{2} a_1 + ya_1 + \frac{a_1}{2h} = a_1 \cdot \frac{3h + 2yh + \frac{3}{2}}{2h}.
\]

Consequently,

\[
a_1 \geq \frac{4h}{6h + 4yh + 3}.
\]
Now we estimate the packing density:

\[
\frac{a_1^2 + p_1}{|L|} = \frac{a_1^2 + \frac{1}{2} a_1 \cdot ya_1}{a_1 - \frac{a_1^2}{2h}} = h(2 + y) \cdot \frac{a_1}{2h - a_1} \\
\geq h(2 + y) \cdot \frac{\frac{4h}{6h + 4y + 3}}{\frac{4h}{6h + 4y + 3}} = 2h \cdot \frac{2 + y}{6h + 4hy + 1}.
\]

By Lemma 2, \(a_1^2 + p_1 > |L| \cdot 2h \cdot \frac{\varrho}{h} = \varrho |L|\).

If \(k > 2\), then we argue as in Case 2.

**Subcase 3B:** \(b_r < \frac{1}{2} h_r (1 + \cot \beta)\). This assumption implies that a square of the side length greater than \(\frac{h_r}{2}\) cannot be packed in \(L_{k-1,1}\). By Case 1 and Case 2, \(a_1^2 + \ldots + a_{k-1}^2 > \varrho |L|\).

**Subcase 3C:** \(\frac{1}{2} h_r (1 + \cot \beta) \leq b_r \leq a_v (1 + \cot \beta) + \frac{1}{2} h_r \cot \beta\). The total area of squares packed in two trapezoids contained in \(L_{k-1,1}\) (see Fig. 11, right) is not smaller than

\[
p_2^- = [b_r - a_v (1 + \cot \beta)] a_v.
\]

In this subcase, similarly as in Subcase 3a, it suffices to check that \(p_2^- (L_{k-1,1}) \geq p_1^- (L_{k-1,1})\), where

\[
p_1^- = [b_r - \frac{1}{2} h_r (1 + \cot \beta)] \cdot \frac{1}{2} h_r
\]

\((p_1^-\) is equal to the area of the gray rectangle on Fig. 11, left). Observe that

\[
p_2^- - p_1^- = b_r a_v - a_v^2 (1 + \cot \beta) - \frac{1}{2} h_r b_r + \frac{1}{4} h_r^2 (1 + \cot \beta) \\
= \left(\frac{1}{4} h_r^2 - a_v^2\right) (1 + \cot \beta) + b_r a_v - \frac{1}{2} b_r h_r \\
= \left(\frac{1}{2} h_r - a_v\right) \left(\frac{1}{2} h_r + a_v\right) (1 + \cot \beta) + \left(\frac{1}{2} h_r - a_v\right) b_r \\
= \left(\frac{1}{2} h_r - a_v\right) \left[\frac{1}{2} h_r \cot \beta + a_v (1 + \cot \beta) - b_r + \frac{1}{2} h_r\right].
\]

By \(\frac{1}{2} h_r - a_v \geq 0\) and \(\frac{1}{2} h_r \cot \beta + a_v (1 + \cot \beta) - b_r > 0\), we get \(p_2^- - p_1^- > 0\).

**Case 4:** \(z \geq 3\), \(a_z \leq a_1/4\) (see Fig. 9, right).
We proceed in the same way as in Case 3. The sum of the areas of squares packed into two trapezoids $L_{l,i}^+, L_{l,i+1}^+$ contained in the trapezoid $L_{l,i}^+ \cup L_{l,i+1}^+$ is greater than the area of the corresponding “gray” rectangle of the height $h(L_{l,i})$ and the base length $b(L_{l,i}) - h(L_{l,i}) \cdot (1 + \cot \beta)$ contained in $L_{l,i}^+ \cup L_{l,i+1}^+$. □

5. Main Result

**Theorem 1.** Any collection of squares of the total area not greater than $\varrho_h \cdot |T_h|$ can be parallel packed into $T_h$.

**Proof.** Let $S_1, S_2, \ldots$ be a collection of squares of the total area not greater than $\varrho_h \cdot |T_h|$. There is no loss of generality in assuming that $a_1 \geq a_2 \geq \ldots$, where $a_i$ denotes the side length of $S_i$.

Squares will be packed into trapezoid-shape layers in such a way that the packing density in each layer, i.e., the ratio of the sum of packed squares to the area of the layer, is greater than $\varrho_h$.

Since $a_1^2 + a_2^2 + \ldots \leq \varrho_h \cdot |T_h|$, by Corollary 1 the first square can be packed into $T_h$. Moreover, if there are at least two squares in the collection (i.e., if $z > 1$), then, by Claim 3, squares $S_1$ and $S_2$ can be packed into $T_h$ either horizontally or vertically, i.e., either $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq 1$ or $a_1 + a_2 \leq 2h/(h + 2)$ (see the proof of Claim 3).

If $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) \leq 1$ (see Fig. 13), then let $L^1$ be the trapezoid contained in $T_h$ with the base length 1, the height $a_1$ and the base angles measuring $\alpha$ and $\beta$. This trapezoid is called the basic layer. Squares $S_1, S_2, \ldots$ are packed into $L^1$ by the $L$-method. Denote by $S_{n_1}$ the first square that cannot be packed into $L^1$ ($n_1 = 3$ on Fig. 13). By Lemma 3, the total area of squares packed into $L^1$ is not smaller than $\varrho_h \cdot |L^1|$.

If $a_1(1 + \cot \alpha) + a_2(1 + \cot \beta) > 1$ and at the same time $a_1 + a_2 \leq \frac{2h}{2h + 1}$ (this case is possible for $h > 1$), then we pack $S_1$ and $S_2$ “vertically”. The first condition implies that $1 < (a_1 + a_2)(2h + 1)/(2h)$ (see the proof of Claim 3). Let $\lambda = a_1 + a_2$. The so-called double layer $L^1$ of the height $\lambda$ is created for packing only two squares, see Fig. 12. In that case $n_1 = 3$ ($S_{n_1}$ is the first square that is not packed into $L^1$) and $\lambda > 2h/(2h + 1)$. It is easy to verify that $a_1^2 + a_2^2 \geq (\lambda/2)^2 + (\lambda/2)^2 = \lambda^2/2$. Recall that $\varrho_h \leq 1/2$ while the packing density in the double layer is not smaller than $1/2$:

$$\frac{a_1^2 + a_2^2}{1 \cdot \lambda - \frac{\lambda^2}{2h}} \geq \frac{1}{2} \frac{\lambda^2}{\lambda - \frac{\lambda^2}{2h}} = \frac{h \lambda}{2h - \lambda} \geq \frac{\frac{2h}{2h + 1}}{2h - \frac{2h}{2h + 1}} = 1/2.$$ 

Since the sum of the areas of the squares packed into $L^1$ is greater than $\varrho_h \cdot |L^1|$, it follows that the total area of the remaining squares is smaller than $\varrho_h \cdot |T_1|$, where $T_1 = T_h \setminus L^1$.

For packing squares $S_{n_1}, S_{n_1+1}, \ldots$ the new layer $L^2$ of the base length $1 - a_1 \cot \alpha - a_1 \cot \beta$, with the base angles measuring $\alpha$ and $\beta$ and with the
height either \( a_{n_1} \) or \( a_{n_1} + a_{n_1+1} \) is created directly above \( L^1 \) (see Fig. 13, where \( L^2 \) is a double layer). If \( a_{n_1} (1 + \cot \alpha) + a_{n_1+1} (1 + \cot \beta) \leq b(L^2) \), then we place \( S_{n_1}, S_{n_1+1}, \ldots \) into the basic layer \( L^2 \) of the height \( a_{n_1} \) by the \( L \)-method (denote by \( S_{n_2} \) the first square that cannot be packed into \( L^2 \)). Otherwise, we pack \( S_{n_1} \) and \( S_{n_1+1} \) into the double layer of the height \( a_{n_1} + a_{n_1+1} \) and we take \( n_2 = n_1 + 2 \).

Assume that the layers \( L^1, \ldots, L^p \) were created and that \( S_{n_p} \) is the first square that cannot be packed into \( L^p \). The total area of squares packed into each \( L^i \) \((i = 1, \ldots, p)\) is greater than \( \varrho_p \cdot |L^i| \). This implies that \( a_{n_p+1}^2 + a_{n_p+2}^2 + \ldots \) is smaller than \( \varrho_h \) times the area of \( T_p = T_h \setminus \bigcup_{i=1}^{p} L^i \). By Corollary 4, \( S_{n_p} \) can be packed into \( T_p \). Moreover by Claim 3, \( S_{n_p} \) and \( S_{n_p+1} \) can be packed into \( T_p \) provided \( z \geq n_p + 1 \). There is an empty space in \( T_h \) to create either the basic or the double layer to pack \( S_{n_p} \) and \( S_{n_p+1} \). This means that all the squares can be packed into \( T_h \). \( \square \)

The following result is a consequence of Theorem 1 and Corollary 3.
Corollary 5. For any $h > 0$, \( \varrho(T_h) = \varrho_h \).

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