Analytical electromagnetic calculations for practical superconductors

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Abstract. The current distributions of twisted long filamentary composites are studied during the current sweep and under the transverse and longitudinal external field, using the inductance matrix among superconducting helical filaments and the inductive voltage between filaments and the external coil in the circuit equation. The self- and mutual inductances of helical conductors with a uniform helical current density are approximately calculated from the analytical expressions of long helical thin conductors. In addition, the magnetic field and the vector potential distributions of a twisted superconducting composite are also obtained by the numerical integral calculation over the cross section of the analytical expressions for the magnetic field and vector potential due to an infinitely long helical conductor.

1. Introduction
The current and field distributions and AC losses of practical superconductors have been extensively investigated due to analytical and numerical methods [1,2]. The current distribution among whole filaments of a twisted filamentary composite during a current sweep has been calculated fundamentally from the electric circuit equation due to Kirchhoff’s law, reflecting the helical filamentary structure of composite [3-6].

The purpose of this article is mainly to apply the similar method due to the electric circuit equation for the effect of the transverse and longitudinal external field, as shown in Fig. 1. The inductance matrix is calculated from the analytical expression for the self- and mutual inductances among helical conductors with uniform helical current densities [7]. In addition, the influence by the longitudinal field is considered by using the mutual inductances between each helical filament within composite and a longitudinal field coil. In particular, the influence by the transverse field is considered by using the mutual inductances between a transverse field coil and an untwisted composite with straight filaments, instead of twisted composite.

2. Inductance expressions for long helical conductors
The current distribution among superconducting filaments of an untwisted superconductor during a current sweep and under the external field has been calculated due to the electric circuit equation, using the analytical inductance expression of polygonal conductors [8]. Similarly, the current distribution within a twisted filamentary composite can be calculated, using the analytical inductance expression of helical conductors. The mutual inductance $L_{12}$ between two long coaxial helical thin
conductors of winding radius \( r_1 \) and the pitch length \( l_1 = (2\pi/k_1 = 1/n_1) \), passing through \((r_1, \varphi_1, z=0)\), \( r_2 \) and \( l_2 = (2\pi/k_2 = 1/n_2) \), passing through \((r_2, \varphi_2, z=0)\), and \( k = (k_1+k_2)/2 \) respectively, can be expressed as follows [7]:

\[
L_{12} = \frac{\mu_0 l_2}{2\pi} \left( \ln \frac{2l}{r_2} - 1 \right) + \frac{\mu_0 l_2 k_1 k_2 r_1^2}{4\pi} + \delta(k_1, k_2) \frac{\mu_0 l_2}{\pi} \sum_{n=1}^{\infty} I_n(nkr_1)K_n(nkr_2) \cos\left[n(\varphi_2 - \varphi_1)\right] \\
+ \delta(k_1, k_2) \frac{\mu_0 l_2 k^2 r_1 r_2}{2\pi} \sum_{n=1}^{\infty} \left\{ I_{n+1}(nkr_1)K_{n+1}(nkr_2) + I_{n-1}(nkr_1)K_{n-1}(nkr_2) \right\} \cos\left[n(\varphi_2 - \varphi_1)\right]
\]

(1)

where

\[
\delta(k_1, k_2) = \begin{cases} 
1 & (k_1 = k_2) \\
0 & (k_1 \neq k_2)
\end{cases}
\]

Furthermore, the self-inductance \( L \) for a helical conductor of the winding radius \( r \), the conductor radius \( a \), the pitch length \( l_p = (2\pi/k = 1/n) \), and the length of \( l \) can be expressed as follows [7]:

\[
L = \frac{\mu_0 l}{2\pi} \left( \ln \sqrt{r^2 + a^2} - 1 \right) + \frac{\mu_0 l}{4\pi} k^2 \left\{ \left( r - a \right)^2 + r^2 \right\} + \frac{\mu_0 l}{8\pi} \sqrt{1 + k^2 r^2} \\
+ \frac{\mu_0 l}{4\pi} k^2 r(2a) \sum_{n=1}^{\infty} \left\{ K_{n+1}(nkr)I_{n+1}(nk(r-a)) + K_{n-1}(nk)I_{n-1}(nk(r-a)) \right\} \\
+ \frac{\mu_0 l}{4\pi} k^2 r(2a) \sum_{n=1}^{\infty} \left\{ K_{n+1}(nk)I_{n+1}(nk) + K_{n-1}(nk)I_{n-1}(nk) \right\}
\]

(2)
3. Calculation of the transport current

3.1. Circuit equation for the transport current distribution

On the circuit consisting of $N$ resistance-less superconducting filaments together with the normal return conductor, the transport current distribution among $N$ superconducting filaments within composite, as shown in Figs. 1 and 2, during the current sweep can be calculated from the following circuit equation due to Kirchhoff's laws [3]:

$$
\begin{pmatrix}
  L_{11} + L_{n1} - 2M_{sn} & L_{12} + L_{n2} - 2M_{sn} & \cdots & L_{1m} + L_{nm} - 2M_{sn} \\
  L_{21} + L_{n2} - 2M_{sn} & L_{22} + L_{n2} - 2M_{sn} & \cdots & L_{2m} + L_{nm} - 2M_{sn} \\
  \vdots & \vdots & \ddots & \vdots \\
  L_{m1} + L_{n1} - 2M_{sn} & L_{m2} + L_{n2} - 2M_{sn} & \cdots & L_{mm} + L_{nm} - 2M_{sn}
\end{pmatrix}
\begin{pmatrix}
  \frac{dl_{1,t}}{dt} \\
  \frac{dl_{2,t}}{dt} \\
  \vdots \\
  \frac{dl_{m,t}}{dt}
\end{pmatrix}
= \begin{pmatrix}
  V - R_n I_T \\
  V - R_n I_T \\
  \vdots \\
  V - R_n I_T
\end{pmatrix}
$$

where $L_{pq}$ is the self- or mutual inductance between the $p$th and $q$th layers within a filamentary composite, $M_{sn}$ is the mutual inductance between each layer within a composite and a normal return conductor, $L_{nn}$ is the self-inductance of a return conductor, $I_{p,t}$ is the transport current of the $p$th layer, $I_T$ is the total transport current, $I_{p,t}$ is the transport current of each filament within the $p$th layer, and $V$ is the excitation voltage per the length of $l$ and $R_n$ is the normal resistance of a return conductor per the length of $l$. Furthermore, the mutual inductance between each layer within a superconductor and a return conductor can be approximated to be all identical, i.e., $M_{p,n} = M_{sn}$ (for $p = 1, 2, \ldots, m$). Then, $L_{pq}$ is related by the mutual inductance $L_{p,q(i,j)}$ between filaments of the $p$th and $q$th layers as follows:

$$
L_{pq} = \frac{1}{n_q} \sum_{j=1}^{n_q} L_{p,q(i,j)} = \frac{1}{n_p} \sum_{i=1}^{n_p} L_{p,q(i,j)} = \frac{1}{n_p n_q} \sum_{j=1}^{n_q} \sum_{i=1}^{n_p} L_{p,q(i,j)} = L_{qp}
$$

3.2. Condition of zero enclosed flux between layers

So far, the current distribution within composite has been also studied, on the condition that the enclosed magnetic flux between layers or filaments is zero or unchanged. The condition of zero flux during the current sweep can be expressed as follows:

**Figure 3.** Conceptual layout used to calculate the induced current within a filamentary composite in an external longitudinal field.

**Figure 4.** Cross-sectional view of a superconductor in an external dipole coil for the homogeneous transverse field, together with magnetic field lines.
where \( \Phi_{p,q} \) is the magnetic flux enclosed between filaments of the \( p \)th and \( q \)th layers. Then, it is seen that Eq.(5) is equivalent with Eq.(3).

### 3.3. Calculated results for superconducting filamentary composites

The current and magnetic field distributions for a superconducting filamentary composite, as shown in Fig. 2, have been calculated [3,9]. However, the inductance matrix for a twisted composite of 25 layers with 246 filaments, as shown in Fig.2(c), was not calculated, because of the overlong calculation time for the numerical integration with Mathematica [3]. In this article, the inductance matrix for a twisted composite of 25 layers with 246 filaments, is calculated, using the analytical expressions of Eqs.(1) and (2) due to Mathematica [10]. The current distribution within a twisted composite of 25 layers with 246 filaments, with the saturation from the outermost to the central region, carrying a transport current increased from zero are shown in Figs. 5 and 6. In this article, the intensity of current density for each filament is shown with the deeper color for the higher current density in Figs. 5 and 9. The current distributions within a composite from the outermost to the central region, corresponding to the gradual saturation of (a), (b), and (c) of Fig.5, carrying a transport current increased from zero are shown in Fig. 6.

### 4. Calculation of the shielding current

#### 4.1. Shielding current in the longitudinal field

The shielding or screening current of each helical filament within a composite under the external longitudinal field, as shown in Fig. 3, can be expressed as follows [8]:

**Figure 5.** Calculated current distributions among each filament within a filamentary composite of 25 layers with 246 filaments and the twist pitch of \( l_p = 20 \) mm, with the gradual saturation of (a), (b), and (c), from the outermost to the central region, carrying a transport current increased from zero.

**Figure 6.** Current distributions among each layer within a filamentary composite of 25 layers with 246 filaments and the twist pitch of \( l_p = 20 \) mm, with the gradual saturation, corresponding to (a), (b) and (c) of Fig.5.
where the mutual inductance $L_{p,e}$ ($p = 1, 2, \ldots, m$) between one filament of the $p$th layer and a longitudinal external coil, and the internal magnetic field $B_{l,e}$ due to the longitudinal external coil with the current $I_e$ and the pitch length $l_e (= 2\pi/k_e = 1/n_e)$ are calculated as follows:

$$L_{p,e} = \frac{\mu_0 l}{2\pi} \left( \ln \frac{2l}{r_e} - 1 \right) + \frac{\mu_0 l}{4\pi} \frac{k_e}{k_p} r_p^2 \quad (p = 1, 2, \ldots, m)$$

(7)

$$B_{l,e} = \mu_0 n_e I_e = \frac{\mu_0 l_e}{2\pi} k_e I_e$$

(8)

### 4.2. Shielding current in the transverse field

The shielding currents among $N (= n_1 + n_2 + \ldots + n_m)$ helical filaments within a composite under the external transverse field with $m$ layers with an external coil, as shown in Fig. 4, can be calculated by the similar equation with Eq.(6) [8]. On the external transverse field, the inductance $L_{i,j}$ ($i, j = 1, 2, \ldots, m$) between two layers of Eq.(6) is replaced by the inductance $L_{i,j}$ ($i, j = 1, 2, \ldots, N$) between two filaments [8]. In addition, the mutual inductance $L_{p,e}$ ($p = 1, 2, \ldots, N$) is calculated as the sum of the mutual inductance between the $p$th straight (not helical) filament within an untwisted composite and each conductor of a external coil with 4 infinitely long conductors of $e_1$, $e_2$, $e_3$, and $e_4$ shown by #1-#4 of external coil as shown in Fig. 4 as follows:

$$L_{p,e} = L_{p,e_2} + L_{p,e_3} - L_{p,e_4} - L_{p,e_1}$$

(9)
4.3. Calculated results for the longitudinal and transverse fields

The calculated current and magnetic field distributions for a filamentary composite in the longitudinal magnetic field of $B_{el} = 1$ T, as shown in Figs. 2(b) and 3, are shown in Figs. 7 and 8.

On the other hand, the shielding current distribution along a helical filament within a twisted composite in the transverse field is induced due to the effect of matrix. Generally, the shielding current in the helical filaments within a twisted composite is greatly suppressed than that within an untwisted composite. For a twisted composite with a long twist pitch, however, it is thought that the current distribution can be roughly approximated by the rotation around the central axis of an untwisted composite. The calculated current and magnetic field distributions for an untwisted filamentary composite in the transverse magnetic field of $B_{et} = 1$ T, as shown in Figs. 2(c) and 4, are shown in Figs. 9 and 10.

![Figure 9. Calculated magnetic field distribution of a composite of 7 layers with 54 filaments in the external transverse field of $B_{et}=1$ T, with two rotation angles of 0 and $\pi/6$ rad, under the assumption of untwisted composite.](image)

![Figure 10. Calculated magnetic field distribution along x-axis under the external transverse field of $B_{et}=1$ T, corresponding to two rotation angles of 0 and $\pi/6$ rad of Fig. 9 (untwisted composite).](image)

5. Conclusion

The current, magnetic field and vector potential distributions of twisted filamentary composites during the current sweep and under the external fields are calculated using the conventional circuit equation with the inductance matrix among superconducting filaments within a composite and the inductive coupling between filaments and the external coil. As a result, it is recognized that the circuit theory due to Kirchhoff’s laws, combined with the analytical calculation methods of the inductance, magnetic field and vector potential for long helical conductors is a simple and powerful method for twisted filamentary composites.

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