VANISHING THEOREMS FOR THE MOD p COHOMOLOGY OF SOME SIMPLE SHIMURA VARIETIES

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Abstract. We show that the mod $p$ cohomology of a simple Shimura variety treated in Harris-Taylor’s book vanishes outside a certain nontrivial range after localizing at any non-Eisenstein ideal of the Hecke algebra. In cases of low dimensions, we show the vanishing outside the middle degree.

1. Introduction

Let $F$ be a CM filed that contains an imaginary quadratic field $K$. Let $G$ be an anisotropic similitude unitary group that is associated with a division algebra $B$ with the center $F$ of dimension $n^2$ and an involution of the second kind, so that it gives rise to Kottwitz’s simple Shimura variety $X_K$ for a fixed sufficiently small level $K$ defined over the reflex field $E$.

Let $\ell \neq p$ be a prime such that everything is unramified at $\ell$ and $\ell$ splits over $K$. Fix a prime $p$. Let $m$ be a system of Hecke eigenvalues appearing in $H^i_{\text{ét}}(X_{K,E}, \mathbb{F}_p)$ for some $i$. Caraiani and Scholze [CS17] constructed a semisimple Galois representation $\rho_m : \Gamma_F := \text{Gal}(\overline{F}/F) \to \text{GL}_n(\mathbb{F}_p)$ associated with $m$. (Our normalization of $\rho_m$ is “geometric”.) Their proof also provides a character $\chi_m : \Gamma_K \to \mathbb{F}_p^\times$ corresponding to the similitude factor; see the main text.

The main result of Caraiani-Scholze’s work [CS17] in this setting is the following vanishing theorem:

Theorem 1.1 ([CS17, 1.5, 6.3.3]). If $\rho_m$ is generic at some $v$, then $i = \text{dim } X_K$. Namely, $H^i_{\text{ét}}(X_{K,E}, \mathbb{F}_p)_m$ vanishes outside the middle degree.

Remark 1.2. In [CS17], $\rho_m$ is assumed to be decomposed generic at $v$; a slightly stronger condition than being generic. But their proof can be modified easily to cover generic ones; see also [CS], especially the proof of Corollary 5.1.3.

Now, assume the signature at $\tau$ is $(1, n - 1)$; this is essentially the Harris-Taylor case [HT01]. (There are some technical additional assumptions in [HT01].) So, the reflex field equals $F$ and $\text{dim } X_K = n - 1$. In the Harris-Taylor case, the above vanishing theorem is also proved in [Boy19, 4.7] by a different argument. In
fact, he proved the following stronger result. Note first that the Galois action on \( H^i_{\text{ét}}(X_{K, F_p}, F_p)_m \otimes \chi_m \) is unramified at \( v \).

**Theorem 1.3 (Boy19).** If \( \alpha \) is an eigenvalue of Frobenius at \( v \) acting on the cohomology \( H^i_{\text{ét}}(X_{K, F_p}, F_p)_m \otimes \chi_m \), then a multiset

\[
\alpha, \quad q_v \alpha, \ldots, \quad q_v^{(n-1)-i} \alpha
\]

is a subset of the multiset \( \{ \alpha_v, \ldots, \alpha_v, n \} \) of generalized eigenvalues of Frobenius of \( \rho_m |_{\Gamma_{F_v}} \).

**Remark 1.4.** This result is actually not clearly stated in Boy19 but follows from an argument along the line of [Boy19, 4.14] by considering the greatest integer \( i' \geq 0 \) such that \( H^{i+1}_{\text{ét}}(X_{K, F_p}, F_p)_m \neq 0 \) or (in fact, and, a posteriori) \( H^{i+1}_{\text{ét}}(X_{K, F_p}, F_p)_m \neq 0 \), where \( m' \) is the “dual” of \( m \). It can be proved using the method of [CS17] as well. This will be discussed in a forthcoming article of the author.

**Remark 1.5.** Assume that \( \ell \) splits completely in \( F \). (This assumption is often harmless when we are allowed to change \( v \) because of the Chebotarev density.) Then, any eigenvalue of Frobenius at \( v \) acting on \( H^i_{\text{ét}}(X_{K, F_p}, F_p)_m \otimes \chi_m \) is a Frobenius eigenvalue of \( \rho_m |_{\Gamma_{F_v}} \) by Wedhorn’s congruence relation [Wed00] and our normalization of \( \rho_m \).

As a part of a mod \( p \) analogue of the Arthur-Kottwitz conjecture, one would consider hypothetical Lefschetz operators \( H^i_{\text{ét}}(X_{K, F_p}, F_p)_m \to H^{i+2}_{\text{ét}}(X_{K, F_p}, F_p)_m(1) \) inducing an isomorphism \( H^{i+2}_{\text{ét}}(X_{K, F_p}, F_p)_m \cong H^i_{\text{ét}}(X_{K, F_p}, F_p)_m(i) \). This would imply that each \( \alpha, \ldots, \ell^\alpha \) is a Frobenius eigenvalue of \( \rho_m \). The above theorem is stronger actually, and gives information of multiplicities; this may be also regarded as a part of the mod \( p \) analogue of the Arthur-Kottwitz conjecture.

The main result of this note, which is deduced from Boyer’s result, is the following:

**Theorem 1.6.** Let \( X_K \) be Harris-Taylor’s Shimura variety of dimension \( n-1 \) [HT01]. Let \( m \) be a maximal ideal of the Hecke algebra contributing to the cohomology of \( X_K \), and \( \rho_m : \Gamma_F \to \text{GL}_n(F_p) \) the associated Galois representation. If \( \rho_m \) is irreducible, then \( H^j_{\text{ét}}(X_{K, F_p}, F_p)_m \) vanishes for \( j < n/2 \) and \( j > 2(n-1) - n/2 \).

In particular, the cohomology localized at \( m \) vanishes outside the middle degree if \( \rho_m \) is irreducible and \( n \leq 3 \). While the case \( n = 4 \) is difficult as \( n \) is no longer prime, the case \( n = 5 \) can be settled:

**Theorem 1.7.** If \( n = 5 \) and \( \rho_m \) is irreducible, then \( H^j_{\text{ét}}(X_{K, F_p}, F_p)_m \) vanishes outside the middle degree.

There are previous works in this direction including the works of Shin [Shi15] (see also [Fuj06]), Emerton-Gee [EG15], and Boyer [Boy19]. The novelty here is that we only assume irreducibility of \( \rho_m \). For the proofs, we use Theorem 1.3 and also group-theoretic results from [GM12], which, in full generality, rely on the classification of finite simple groups.

It is easy to control \( \rho_m \) with large image. Let us record the following remark. The argument passing to \( F(q_p) \) is very important throughout this note, and will be frequently used later as well.
Theorem 1.8. If the restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_p)} \) is irreducible and the image of \( \Gamma_{F(\zeta_p)} \) contains a regular semisimple element of \( \text{GL}_n(\overline{F}_p) \), then \( H^*_\text{et}(X_{K,F}, F_p)_m \) vanishes outside the middle degree.

Proof. For any finite place \( w \) of \( F(\zeta_p) \) not dividing \( p, q_w \) is congruent to 1 modulo \( p \). So, if the image of \( \Gamma_{F(\zeta_p)} \) under \( \rho_m \) contains a regular semisimple element of \( \text{GL}_n(\overline{F}_p) \), there exists such \( w \) such that \( \rho_m \) is unramified at \( w \) and the Frobenius element \( \text{Frob}_w \) has a regular semisimple image in \( \text{GL}_n(\overline{F}_p) \) by the Chebotarev density. (In fact, we can make \( p_w \) split completely in \( F(\zeta_p) \).) If \( v \) denotes the restriction of \( w \) to \( F \), then \( \text{Frob}_w \) is conjugate in \( \text{GL}_n(\overline{F}_p) \) to \([k_w: k_v]-\text{th power of } \text{Frob}_v\). Therefore, the set of generalized eigenvalues of \( \text{Frob}_v \) does not contain a subset of the form of \( \{\alpha, q_v \alpha\} \). We finish by Theorem 1.3.

Example 1.9. Suppose \( n \) is an odd prime, \( p > 2n - 3 \), and the restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_p)} \) is irreducible, i.e., \( \rho_m \) is not induced from a character. Then, [GM12, 1.7] says that the image of \( \Gamma_{F(\zeta_p)} \) contains a regular semisimple element. So, \( H^*_\text{et}(X_{K,F}, F_p)_m \) vanishes outside the middle degree by Theorem 1.3.

Example 1.10. Another example satisfying the assumption of Theorem 1.8 is the case where the image of \( \Gamma_F \) contains \( \text{SL}_n(\overline{F}_p) \). Indeed, if \( (n, p) \neq (2, 2), (2, 3), \) \( \text{SL}_n(\overline{F}_p) \) is perfect and contained in the image of \( \Gamma_{F(\zeta_p)} \). If \( p = 2 \), then \( F = F(\zeta_p) \) and there is nothing to prove. If \( p = 3 \), \([F(\zeta_p) : F] \) divides 2 and \( \text{SL}_2(\overline{F}_3) \) does not have a subgroup of index 2, so it is contained in the image of \( \Gamma_{F(\zeta_p)} \). (Note that \( \text{SL}_n(\overline{F}_p) \) contains a regular semisimple element.)

Remark 1.11. We also remark that Theorem 1.3 easily implies the following: if \( \rho_m \) is irreducible and induced from a character of \( \Gamma_F \) for some cyclic extension \( E \) of \( F \) of degree \( n \) and \([F(\zeta_p) : F] > n \), then \( H^*_\text{et}(X_{K,F}, F_p)_m \) vanishes outside the middle degree.

Indeed, such \( \rho_m \) satisfies [Boy19, 4.17] as shown in [EG15, 4.1.6]. That is, any irreducible subquotient of \( H^*_\text{et}(X_{K,F}, F_p)_m \) is isomorphic to \( \rho_m \). If \( i < n - 1 \), using Theorem 1.3 one sees that the order of \( q_v \) modulo \( p \) is \( \leq n \); this contradicts to \([F(\zeta_p) : F] > n \) when varying \( v \). The dual argument settles the case \( i > n - 1 \).

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2. Preliminaries

2.1. Setting. Let \( F = F^+ \mathcal{K} \) be a CM filed with a totally real field \( F^+ \) and an imaginary quadratic field \( \mathcal{K} \). We consider a PEL datum \((B, *, V, (\cdot , \cdot ))\) of type A such that

- \( B \) is a division algebra with center \( F \) and \( V \equiv B \), and
- the associated group \( G \) has signature \( (1, n - 1) \) at one infinite place, and \((0, n)\) at the other infinite places, where \( n^2 = \dim_F B \).

Fix a sufficiently small open compact subgroup \( K = \prod_x K_x \) of \( G(\mathbb{A}_f) \). If \( \ell \) splits in \( \mathcal{K} \), by choosing a place \( y \) of \( \mathcal{K} \) over \( \ell \), we have an isomorphism \( G(\mathbb{Q}_\ell) \equiv \mathbb{Q}_\ell^\times \times \prod_x B_x^{op} \times \), where \( x \) runs over the places of \( F \) lying over \( x \).
Let $\text{Spl}^\ur$ denote the set of unramified places $v$ of $F$ such that $v$ does not divide $p$, $p_v = v|\mathbb{Q}$ splits in $K$, $B$ is split at $v$, $K_{p_v}$, as a subgroup of $\mathbb{Q}_p^\times \times \prod_{v} B_v^{\text{op}\times}$, decomposed into a product of $\mathbb{Z}_{p_v}$ and maximal open compact subgroups $K_v$ of $B_v^{\text{op}\times}$. Let $\mathbb{T}$ denote the Hecke algebra

$$\bigotimes_{p_v \in \text{Spl}^\ur} \mathbb{Z}[G(\mathbb{Q}_{p_v})//K_{p_v}].$$

If we identify $K_v$ with $\text{GL}_n(\mathcal{O}_{F_v})$, its factor at $v$ is generated by

$$T_{v,j} = K_v \text{diag}(p_v^{-1}, \ldots, p_v^{-1}, 1, \ldots, 1)K_v.$$ 

We write $c_v$ for the element of $\mathbb{T}$ determined by $p_v^{-1} \in \mathbb{Z}_{p_v}$. Our choice of the Hecke operators is different from the one of [Wed00], [EG15], [Boy19], [CS17].

We denote by $X_K$ the canonical model of the Shimura variety attached to $(B, *, V, (\cdot, \cdot))$ of level $K$, which is a smooth projective variety over $F$. (We use the convention that (a union of) the canonical model admits a usual moduli interpretation.) The mod $p$ cohomology of $X_{K, \mathfrak{p}}$ is naturally a module of $\mathbb{T} \times G_F$.

**Theorem 2.1** ([CS17] 6.3.1]). Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ such that, for some $i$, $H^i_{\text{ét}}(X_{K, \mathfrak{p}}, \mathcal{F}_p)_\mathfrak{m} \neq 0$. Then, there is a (unique) semisimple Galois representation $\rho_{\mathfrak{m}} \colon \Gamma_F \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$ and a character $\chi_{\mathfrak{m}} \colon \Gamma_K \rightarrow \overline{\mathbb{F}}_p^\times$, both unramified at $v \in \text{Spl}^\ur$, such that the characteristic polynomial of $\rho_{\mathfrak{m}}(\text{Frob}_v)$ for $v$ is given by

$$\sum_{j=0}^{n} (-1)^j q_v^{j(j-1)/2} T_{v,j} X^{n-j},$$

and $\chi_{\mathfrak{m}}(\text{Frob}_{p_v}) = \overline{\sigma}_v^{-1}$. Where $T_{v,j}$ and $\overline{\sigma}_v$, denote the image of $T_{v,j}$ and $c_v$ in $\mathbb{T}/\mathfrak{m} \cong \overline{\mathbb{F}}_p$ respectively.

**Proof.** The existence of $\rho_{\mathfrak{m}}$ is proved in [CS17] or [Boy19] up to normalization; our $\rho_{\mathfrak{m}}$ is a twist of the dual of the representation they constructed. The existence of $\chi_{\mathfrak{m}}$ can be proved by the same method. Namely, we find a characteristic 0 lift $\Pi$ of $\mathfrak{m}$ at first; $\Pi$ is a $C$-algebraic cuspidal automorphic representation of $G$, and its stable base change is a $C$-algebraic isobaric automorphic representation of $K^\times \times \text{GL}_n(F)$ of the form of $\psi \otimes \Pi^1$. The first factor $\psi$ gives rise to a character $\overline{\chi}_m^{-1} \colon \Gamma_K \rightarrow \overline{\mathbb{Q}}_p^\times$ via the class field theory. The reciprocal of the reduction of $\overline{\chi}_m^{-1}$ mod $p$ is $\chi_{\mathfrak{m}}$. □

Throughout this note, we regard $\chi_{\mathfrak{m}}$ as a character of $\Gamma_F (\subset \Gamma_K)$ as well.

2.2. **The congruence relation.** The congruence relation is not logically needed (in the sense that Theorem [L3] is stronger) but we give a short explanation to clarify our notation and conventions. For every $v \in \text{Spl}^\ur$, there is a canonical integral model $\mathcal{X}_K$ of $X_K$, which is smooth and projective over $\mathcal{O}_{F_v}$. The action of $\mathbb{T} \times \Gamma_{F_v}$ extends naturally to the mod $p$ cohomology of the special fiber $X_{K, \mathfrak{m}}^{\text{op}\times}$ of the canonical integral model. In particular, the Galois action on $H^i_{\text{ét}}(X_{K, \mathfrak{p}}, \mathcal{F}_p)$ is unramified at $v$. 
Assume that \( p \) splits completely in \( F \). If we look at the Frobenius action on \( H^i_{\text{et}}(X_K, \mathbf{F}_p)_m \otimes \chi_m \), the main result of [Wed00] implies the following relation:

\[
\sum_{j=0}^{n} (-1)^j q^{j(j-1)/2} T_{v,j} \text{Frob}^{n-j} = 0.
\]

The formula is stated incorrectly (or imprecisely) in [FG15, 3.3.1] and [Boy19, 4.2]:

- The Hecke correspondence in [Wed00] is a left action (as a correspondence), while the Hecke action on the Shimura variety is a right action. This is why we change the choice of the Hecke operator.
- The twist by \( \chi_m \) is needed.

3. Proof of Theorem 1.6

Suppose \( H^i_{\text{et}}(X_K, \mathbf{F}_p)_m \neq 0 \) for some \( i < n/2 \), and let \( \rho \) be an irreducible constituent of \( H^i_{\text{et}}(X_K, \mathbf{F}_p)_m \) as a representation of \( \Gamma_F \).

Suppose that \( \rho \) is a character \( \chi \). Then, by Theorem 1.3 \( \chi(g) \in \mathbf{F}_p \) for any \( g \in \Gamma_{F(\zeta_p)} \) appears in the set of generalized eigenvalues of \( \rho_m(g) \) with multiplicity \( \geq n-i \). Therefore, \( (\rho_m \otimes \chi^{-1})(g) \) has a generalized eigenvalue \( 1 \) with multiplicity \( \geq n-i \). Since \( n-i > n/2 \), it contradicts to the following theorem. (This discussion also works for \( i = n/2 \).)

**Theorem 3.1 ([GM12, 1.5.(a)])**. Let \( H \subset \text{GL}_n(\mathbf{F}_p) \) be a finite group whose action on \( \mathbf{F}_p^m \) is irreducible. For any nontrivial normal subgroup \( H' \) of \( H \), there exists semisimple \( h \in H' \) such that the multiplicity of \( 1 \) in the set of eigenvalues of \( h \) is less than \( n/2 \).

**Proof.** In [GM12], this is stated with \( H' = H \). The proof actually finds \( h \) in any given minimal normal subgroup \( N \) of \( H \). \( \square \)

Suppose \( \dim \rho \geq 2 \). Then, we claim that the restriction of \( \rho \) to \( \Gamma_{F(\zeta_p)} \) is not unipotent modulo scalar, namely \( \rho(h) \) is not unipotent modulo scalar for some \( h \in \Gamma_{F(\zeta_p)} \). Indeed, assume that \( \rho(h) \) is unipotent modulo scalar for every \( h \in \Gamma_{F(\zeta_p)} \). Set \( \overline{H} := \rho(\Gamma_{F(\zeta_p)})/(\text{scalar}) \): this is a \( p \)-group because the order of any element is a power of \( p \). If \( \overline{H} \) is nontrivial, \( \overline{H} \) has a nontrivial center \( Z \). If \( \overline{Z} \) denotes the inverse image of \( Z \) in \( \rho(\Gamma_{F(\zeta_p)}) \), then \( \overline{Z} \) is abelian and the restriction of \( \rho \) to \( \overline{Z} \) is semisimple. This is impossible as \( \overline{Z} \) contains an element of order \( p \). Thus \( \overline{H} \) is trivial, i.e., \( \rho \) is scalar on \( \Gamma_{F(\zeta_p)} \). Then, \( \rho(\Gamma_F) \) is abelian. Contradiction.

So, there exists \( h \in \Gamma_{F(\zeta_p)} \) such that \( \rho(h) \) has at least two distinct eigenvalues, say \( \alpha, \beta \). Each has multiplicity \( \geq n-i \) in the set of generalized eigenvalues of \( \rho_m(h) \). Thus, \( \dim \rho_m \geq 2(n-i) > n \). Contradiction.

If \( i > 2(n-1)/2 \), the vanishing of \( H^i_{\text{et}}(X_K, \mathbf{F}_p)_m \) follows from the vanishing of \( H^{2(n-1)-i}_{\text{et}}(X_K, \mathbf{F}_p)_m \) and the Poincaré duality because \( \rho_m \) is also irreducible.

4. Proof of Theorem 1.7

We may only consider cohomology below the middle degree because the duality preserves the condition that \( \rho_m \) is irreducible.

Suppose that \( H^i_{\text{et}}(X_K, \mathbf{F}_p)_m \neq 0 \) for some \( i < 4 \). We consider two cases:

1. The restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_p)} \) is irreducible.
(2) The restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_5)} \) is not irreducible.

4.1. Group-theoretic results. We will use another group-theoretic result from [GM12]:

**Theorem 4.1 ([GM12, 1.7]).** Let \( H \) be a finite nonabelian simple group and \( p \) be a prime number. Assume that \((H, p) \neq (A_5, 5)\). Then, there exist \( p' \)-elements \( x, y, z \in H \) with \( xyz = 1 \) such that \( H = \langle x, y \rangle \).

This will be combined with Scott’s lemma:

**Theorem 4.2 ([Scott]).** Let \( H \) be a finite group acting on a finite-dimensional vector space \( V \) over a field \( k \). Assume that \( x, y, z \) generate \( H \) and satisfy \( xyz = 1 \). Then,
\[
\dim V + \dim V^H + \dim (V^\vee)^H \geq \dim V^x + \dim V^y + \dim V^z,
\]
where \( V^* \) denotes the space of fixed vectors under the action of \(*\).

4.2. Preparation. To study the case (1), we analyze a slightly more general situation where \( n \) is prime and the restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_n)} \) is irreducible.

Let \( N \) be a minimal noncentral normal subgroup of \( \rho_m(\Gamma_{F(\zeta_n)}) \). If \( N \) is not a quasi-simple group, the proof of [GM12, 1.6] shows that \( \rho_m(\Gamma_{F(\zeta_n)}) \) contains a regular semisimple element except when \( n = p \). In more detail, if \( N \) is not a quasi-simple group, then \( N \) is an \( r \)-group for a prime \( r \neq p \). If \( N \) acts irreducibly on \( \mathbf{F}_r^5 \), then \( N \) is an extraspecial \( n \)-group and a desired element exists, cf. [GM12, 5.1]. If \( N \) does not act irreducibly, \( N \) is an elementary abelian \( r \)-group and \( \rho_m|_{\Gamma_{F(\zeta_n)}} \) is induced from a character. If \( n \neq p \), a desired element exists by [GM12, 5.2]. Let us continue to consider the case \( n = p \), and assume \( n = 5 \). We can find a regular semisimple element inside \( N \), which becomes a subgroup of the diagonal \((\mathbf{F}_r^5)^5\) stable under a permutation of order 5; we regard it as a subrepresentation of \( \mathbf{Z}/5\mathbf{Z} \) acting on \( \mathbf{F}_r^5 \). Indeed, if \( r \neq 4 \) mod 5, then the complement of the trivial representation in \( \mathbf{F}_r^5 \) is either irreducible or the direct sum of 4 distinct characters, and it is easy to find a regular semisimple element. If \( r \equiv 4 \) mod 5, then the complement of the trivial representation is the sum of two irreducible two-dimensional subrepresentations. An element \( x \) of each subrepresentation can be written as the component-wise trace of \((a, a\zeta_5, a\zeta_5^2, a\zeta_5^3, a\zeta_5^4)\) for \( a \in \mathbf{F}_{r,2} \) and a choice of \( \zeta_5 \). It is easy to see that \( x \) corresponds to a regular semisimple element if and only if all coordinates are distinct if and only if \( \pi/a \notin \{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\} \), where \( \pi \) denotes the conjugate of \( a \). Any norm 1 element in \( \mathbf{F}_{r,2} \) has the form of \( \pi/a \). Since the number of norm 1 elements in \( \mathbf{F}_{r,2} \) is \( r + 1 \) and \( r + 1 > 5 \) by the assumption, we can find a regular semisimple \( x \).

Therefore, Theorem 1.5 gives the vanishing outside the middle degree in these cases.

4.3. The first case. Assume that the restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_5)} \) is irreducible. By the discussion above, we only need to consider the case where any minimal noncentral normal subgroup \( N \) of \( \rho_m(\Gamma_{F(\zeta_5)}) \) is quasi-simple. Let \( \rho \) be an irreducible constituent of \( H_\ell^t(X, \mathbf{F}_p, \mathbf{F}_p)_m \) as a representation of \( \Gamma_F \). We regard \( \rho \) as a representation of \( N \); this is possible by [EG15, 4.1.3]. (The action of \( \Gamma_F \) on \( \rho \) factors through \( \rho_m(\Gamma_F) \).) By Theorem 1.3 and Schur’s lemma, the center \( Z \) of \( N \) acts on \( \rho \) and \( \rho_m \) by the same character. Therefore, \( \rho_m \otimes \rho' \) becomes a representation of \( N/Z \), which is a simple nonabelian group.
Assume that \( N/Z \) is not \( A_5 \) and \( p \neq 5 \). Suppose \( \rho \) is not isomorphic to \( \rho_m \). Then, we can apply Theorem 1.11 and Scott’s lemma (Theorem 1.2), and there is an element of \( N \) whose action on \( \rho_m \otimes \rho' \) is semisimple and the dimension of its fixed space is \( \leq (5 \dim \rho)/3 \). But, the dimension of the fixed space is \( \geq 2 \dim \rho \) by Theorem 1.3. Contradiction. Next, suppose \( \rho \) is isomorphic to \( \rho_m \). Then, \( \rho_m \otimes \rho' \cong \text{End}(\rho_m) \) is self-dual and has 1-dimensional subrepresentation and quotient representation given by the scalars and the trace map respectively, and there is no other trivial subrepresentation or quotient representation. So, again by Theorem 1.11 and Scott’s lemma (Theorem 1.2), we get an inequality \((25 + 2)/3 \geq 2 \dim \rho = 10\), which is impossible.

The only remaining case is \( N/Z = A_5 \) and \( p = 5 \). Note that the only such quasi-simple group is \( A_5 \) itself or \( SL_2(F_5) \), which is a double covering of \( A_5 \).

- Suppose \( N = A_5 \). There are only three isomorphism classes of irreducible representations in characteristic 5, and one of them has dimension 5; it must be \( \rho_m \). The other two are the trivial representation and a faithful 3-dimensional representation defined over \( F_5 \). Whatever \( \rho \) is, any element \( g \) of order 3 has an eigenvalue 1. However, \( \rho_m(g) \) has the eigenvalues \( \{1, \zeta_3, \zeta_3^2, \zeta_3^2 \} \) and 1 has the multiplicity one. This contradicts to Theorem 1.3.

- Suppose \( N = SL_2(F_5) \). Any irreducible representations in characteristic 5 is given by the symmetric power \( \text{Sym}^k F_5^2 \) of the standard representation of dimension 2 for an integer \( k \in [0, 4] \). The action of any (elliptic) element \( g \) of order 6 on \( \rho_m \) has eigenvalues \( \{1, \zeta_3, \zeta_3^2, \zeta_3^2 \} \) and 1 has the multiplicity one. If \( \rho = F_5^2 \) or \( \text{Sym}^3 F_5^2 \), then any eigenvalue of \( \rho(g) \) does not appear in the eigenvalues of \( \rho_m(g) \). Otherwise, \( \rho(g) \) has an eigenvalue 1, but the consideration of multiplicity gives the contradiction to Theorem 1.3.

4.4. The second case. Assume that the restriction of \( \rho_m \) to \( \Gamma_{F(\zeta_p)} \) is not irreducible. Then, \( \rho_m \) is induced from a character, and if \( [F(\zeta_p): F] > 5 \) we can apply Remark 1.11. Suppose \( [F(\zeta_p): F] = 5 \), in which case \( \rho_m \) is induced from a character \( \psi \) of \( \Gamma_{F(\zeta_p)} \). Take a lift \( g \in G_F \) of a generator of \( \text{Gal}(F(\zeta_p)/F) \). The restriction of \( \rho_m \) is the direct sum of \( \psi^i \) for \( i = 0, 1, 2, 3, 4 \), and these characters are all distinct because \( \rho_m \) is irreducible. So, we can find an element \( g' \in \Gamma_{F(\zeta_p)} \) such that \( \rho_m(g') \) is not a scalar.

Any irreducible constituent of \( H^1_{\text{dR}}(X_K, \mathbf{F}_p)_{\text{nm}} \) as a representation of \( \Gamma_F \) is isomorphic to \( \rho_m \) by [EG15]. Therefore, by Theorem 1.13 each eigenvalue of \( \rho_m(g') \) has the multiplicity \( > 1 \), hence 2 or 3 since \( \rho_m(g') \) is not a scalar. By permutation, we may assume that \( \psi(g') = \psi^0(g') = \psi^2(g') \) or \( \psi(g') = \psi^3(g') = \psi^3(g') \). In the former case, \( \rho_m(g'(g^2g'g^{-2})) \) has an eigenvalue \( \psi(g')^2 \) with multiplicity one. In the latter case, \( \rho_m(g'(gg'g^{-1})) \) has an eigenvalue \( \psi(g')^2 \) with multiplicity one. These contradict to Theorem 1.13 as \( g'(g^2g'g^{-2}), g'(gg'g^{-1}) \in \Gamma_{F(\zeta_p)} \).

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