Bifurcations and Averages in the Homoclinic Chaos of a Laser with a Saturable Absorber

Hugo L. D. de S. Cavalcante, José. R. Rios Leite*

Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, PE Brazil

Abstract

The dynamical bifurcations of a laser with a saturable absorber were calculated, with the 3-2 level model, as function of the gain parameter. The average power of the laser is shown to have specific behavior at bifurcations. The succession of periodic-chaotic windows, known to occur in the homoclinic chaos, was studied numerically. A critical exponent of 1/2 is found on the tangent bifurcations from chaotic into periodic pulsations.

Key words: Dynamical Bifurcations, Chaos
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1 Introduction

Chaos in monomode \( CO_2 \) lasers with an intracavity molecular gas as saturable absorber, \( LSA \), has been observed \( \) and modeled by a three level system to represent the amplifier and a two level system for the absorber \( \). A passive Q-switching of the laser cavity is produced by the saturation of the absorber and, for appropriate relaxation and saturation rates, the laser oscillates with periodic or chaotic pulse emission. The dynamical system behavior is explained as due to the approach to a homoclinic tangency to a saddle focus \( \), or to an unstable periodic orbit \( \), depending on the operating parameters of the system.

This work presents numerical solutions of the laser model giving the dynamical bifurcations as it approaches a homoclinic orbit to an unstable cycle which

* Corresponding author. Tel.: +55-81-32718450; fax:+55-8132710359
Email address: rios@df.ufpe.br (José. R. Rios Leite).
consists of a high spike pulse and an infinite number of undulations. Bifurcations diagrams are usually exhibited by giving the peak of the pulses and here they are given by the average power of the laser. The properties of the average power throughout the bifurcations are the main results reported.

2 The 3-2 level model

Introduced by Tachikawa et al. and thoroughly analyzed by many authors the 3 – 2 level model for the LSA predicts most dynamical observations made in these lasers. The mean intracavity intensity $I(t)$, the mean gain medium population inversion $U(t)$, the effective ground state population depletion of the gain medium, $W(t)$, and the mean population difference of the absorber $\overline{U}(t)$, are the four independent variables for the rate equations in the model. It can be shown that the LSA system may be reduced to a three dimensional flux as one considers the usual fast relaxation of the absorber, and adiabatically eliminates the variable $\overline{U}(t)$.

The three level-two level rate equations describing the LSA with the normalized variables explained above are:

$$\dot{I} = I \left( U - \overline{U} - 1 \right)$$

$$\dot{U} = \epsilon \left[ W - U \left( 1 + I \right) \right]$$

$$\dot{W} = \epsilon \left( A + bU - W \right)$$

$$\overline{U} = \tau \left[ A - \overline{U} \left( 1 + aI \right) \right]$$

The parameters in the equations are: $\epsilon$ and $\tau$ the relaxation rates of the amplifier and absorber, respectively, normalized to the cavity relaxation rate (which is the inverse time unit). $A$ and $\overline{A}$ are the pumping rates in the amplifier and absorber population differences, respectively, normalized to the cavity loss. The coefficient $b$ is the difference between the population relaxation rates of lower and upper level of the gain transition, normalized to the sum of these relaxation rates. The saturability coefficient for the absorber, normalized to the two level saturability of the gain, is $a$. More detailed explanation of these dimensionless quantities and the physical behavior contained in Eqs. (1)-(4) are given in Ref.

The flux described by those equations was studied by Lefranc et al. who calculated bifurcations diagrams typical of homoclinic chaos. Generally each laser pulse has a leading spike, named the reinjection, and $(0, 1, \ldots, n, \ldots)$ small undulations in a tail associated to the cycling around the unstable orbit to be reached at the homoclinic tangency. The notation is then $P^{(n)}$ for the
pulses with \( n \) undulations and the periodic regime made with such pulses and \( C^{(n)} \) for the chaotic pulsation composed of irregular occurrence of pulses \( P^{(m)} \) with \( m \leq n + 1 \). The maxima in the spike and the undulations of a long train of pulses can be registered to indicate the dynamical state of the LSA. A bifurcation diagram consists of a plot of the maxima in the pulses emitted versus the control parameter \( A \), which is the pump rate of the laser.

The bifurcations, as shown in Figure 1(a), consist of alternating periodic-chaotic windows well predicted (6) in the homoclinic chaos of LSA. At certain chosen parameters they also show long sequences windows separated by periodic to periodic global bifurcations from a regime \( P^{(n)} \) to a \( P^{(n+1)} \) (9).

![Bifurcation Diagram](image)

**Fig. 1.** Bifurcation diagram calculated using the 3-2 level model for the gain and the absorber, respectively. The parameter choice was made following ref (6): (a) gives the peak value of the pulses and (b) the corresponding average power. Notice significant variation of the average power at the bifurcations between periodic regimes \( P^{(2)} \) to \( P^{(3)} \) and at the Hopf bifurcations, on the end of the diagram. At the period doubling and at the tangent bifurcations the changes in the average are not visible.

### 3 Averages and Bifurcations

The average of a dynamical variable in a nonlinear flux or map may manifest some of its bifurcations (8 10). Such property has been pointed out in the early days of studies of chaotic maps (11). It is our purpose here to verify how such average behaves in a flux associated to a laser system where experimental
tests are feasible \cite{8,12}.

The parameters used to numerically solve equations (1-4) were: $\bar{A} = 2.16$, $\epsilon = 0.137$, $\sigma = 1.2$, $a = 4.17$ and $b = 0.85$. The control parameter $A$ was varied between 1.4 and 2.0. The bifurcation diagram showing the maxima of the pulses is given in Figure 1(a). Lefranc et al \cite{6} have discussed this diagram in details. At each position of $A$ 400 maxima are saved from the numerical integration done with a standard fourth order Runge-Kutta routine. The calculation at each of the 300 steps of $A$ had as initial condition the last values from the previous step. For low gain, i.e. small $A$, the laser shows periodic pulsation with spike pulses $P^{(0)}$ which bifurcates into periodic pulsation of one undulation pulses. This global bifurcation consists in a transition that generally has bistability. Increasing the value of $A$ gives a cusp bifurcation from periodic $P^{(2)}$ into chaotic $C^{(2)}$ pulsation. The chaotic window ends at a tangent bifurcation, when the system recovers periodic pulsation $P^{(3)}$. This $P^{(3)}$ window develops a cascade of period doubling bifurcations into $C^{(4)}$ chaos and so on. The succession of windows is one of the signatures of the homoclinic chaos. At $A = 1.92$ an inverse cascade of period doubling leads the system to a Hopf bifurcation connected to the continuous wave laser operation, when the fixed point $I_+ \neq 0$ is stable. Figure 1(b) shows the average power of the laser calculated in same range of variation of the control parameter $A$. The transient at each step of the calculation, which is very important near the bifurcation points, was eliminated, as far as possible within computation time, by neglecting the first 25000 time units in a total of 80000 time units of integration. It can be seen that the most important manifestation of the bifurcations in the average occur at the periodic to periodic transition and at the Hopf bifurcation. The average suffers a discontinuity at the periodic to periodic transitions. Such discontinuity also occurs at the Hopf bifurcation, when the parameters make this a supercritical one. At the period doubling and at the tangent bifurcations the changes in the average are not visible in the scale resolution of Figure (1b). To search for the average variations on these bifurcations a refined calculation was done over the period doubling and the tangent bifurcation around the $C^{(3)}$ chaotic window, as shown in Figure 2. The period doubling bifurcation remains without changes for the average. At these bifurcations the only expected changes on the average is in one of its derivative with respect to the control parameter. For the flux studies here no derivative discontinuity was obtained to the resolution of the calculation. Conversely a first derivative discontinuity is obtained, analytically and numerically, for the average of one dimensional maps at period doubling bifurcations \cite{10}. Figure 2 shows a bifurcation diagram for the average still with the same parameters of Figure 1, but spanning a very narrower range of $A$ at the onset of the tangent bifurcation from chaotic regime $C^{(3)}$ to periodic $P^{(4)}$. Similar behavior occur for the other tangent bifurcations. To fit the dependence of the
average $\bar{T}$ on $A$ the expression

$$\bar{T}(A) = \bar{T}_i + B_i(A_i^c - A)^{\nu_i}$$

(5)

was used, with the values of $\bar{T}_i$ and $A_i^c$ extracted from inspection of the numerical diagram near the ($i$) bifurcation. A best fitting of eq. (5) to the data is shown in Figure (2b) in a log-log graph. The dots are the numerical calculation and the line corresponds to the fitting which verifies that the average decreases following a power law whose exponent is $\nu_i = 1/2$. The other tangent bifurcation in the diagram seems to have the same exponent but a detailed study was not pursued.
4 Conclusions

Bifurcation diagrams associated to the homoclinic tangency to an unstable periodic orbit in the 3-2 level model for a laser with saturable absorber were calculated numerically.

The numerical solutions were used to calculate the average power emitted by the chaotic laser. It was verified that the average power has a discontinuous change on the cusp bifurcations, related to the bistability between different periodic regimes. Through period doubling bifurcations, no change of the average power was obtained. The tangent bifurcations from chaos into periodic pulsation show a characteristic average power dependence, similar to the averages observed on the logistic map [10], with a critical exponent 1/2. Preliminary experiments observing bifurcations in the average power of a CO$_2$ laser with SF$_6$ as saturable absorber have been done [8] and the verification of the properties of critical exponents on the tangent bifurcations is under development [12]. Characterizing Dynamical bifurcations by averages, which can be measured experimentally by slow detectors, may be useful to study ultra fast chaotic systems.

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