TORSION DIVISORS OF PLANE CURVES AND ZARISKI PAIRS

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Abstract. In this paper we study the embedded topology of reducible plane curves having a smooth irreducible component. In previous studies, the relation between the topology and certain torsion classes in the Picard group of degree zero of the smooth component was implicitly considered. We formulate this relation clearly and give a criterion for distinguishing the embedded topology in terms of torsion classes. Furthermore, we give a method of systematically constructing examples of curves where our criterion is applicable, and give new examples of Zariski tuples.

Keywords. Plane curve arrangements, Torsion divisors, Splitting numbers, Zariski pairs.

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Introduction

The embedded topology of plane curves has been studied by many mathematicians since the initial work of Zariski [33]. One of the main questions about the embedded topology of plane curves is its relationship with the combinatorics of the curve (roughly speaking, decomposition in irreducible components with their degrees and topological type of singular points, see Definition 2.1). This is because, the combinatorics determine the topology of the regular neighborhood of the curve, but as is demonstrated in Zariski’s example, it is not enough to determine the embedded topology. Zariski’s example also demonstrates [33] that the geometric position of singular points have a surprisingly strong influence on the embedded topology (see [5] for a detailed survey and references therein). A finite set of plane curves with the same combinatorics but different embedded topology with each other is called a Zariski tuple (see Definition 2.7).

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The first main invariant of the embedded topology is the fundamental group of the complement (cf. [33, 23, 13]), which is very powerful but at the same time presents some issues. Its computation is usually difficult (where one aims to obtain a presentation of the fundamental group in terms of generators and relations) and sometimes getting a presentation does not always provide relevant information, due to the difficulty in solving the so called "word problem". This is why other more effective (and algebraic) invariants have been used, such as Alexander polynomials [20] (involving cyclic covers) or the existence/non-existence of certain dihedral covers [30]. Later on, other finer invariants have been used such as characteristic varieties [21] or $K^3$-lattice invariants (see [12, 26]). Recently, new kinds of invariants called linking invariants ([6, 15]) and splitting invariants ([31, 7, 27, 28]) have been introduced for reducible curves, whose definitions do not involve the fundamental group (here by splitting invariants we mean invariants derived from splitting phenomena of plane curves under Galois covers). Linking invariants are derived from geometric topology, and splitting invariants are defined in terms of algebraic geometry. Both invariants essentially represent how a component of the reducible curve "tangles up" with the other components (cf. [15], [16]).

The first interesting degree for the existence of Zariski pairs or tuples is 6; the first cases of Zariski pairs were found by several authors using Alexander polynomials but for the final classification more delicate invariants were needed, such as $K^3$-lattice invariants, see [26, 12]. The existence of arithmetic Zariski pairs (curves defined over a number field $K$ whose topology depends on the embedding $K \hookrightarrow \mathbb{C}$) shows that one needs to go beyond algebraic properties.

A particularly interesting case is given for curves having only smooth irreducible components, e.g. line arrangements. For line arrangements, the combinatorics has a simpler interpretation; a famous example of Rybnikov [24] is a Zariski pair distinguished by the fundamental group. Some other Zariski pairs of line arrangements were obtained by other authors by using braid monodromy or deep properties of the fundamental group. The Zariski pairs found in [14, 17] used a linking invariant. Conic-line arrangements and conic arrangements have also been studied [22, 32, 10].

Another family of interesting cases comes from the arrangements of smooth curves allowing positive genus. A classical example was studied in [1]: a smooth cubic $D$ and 3 inflectional tangents of $D$, later generalized to conic-cubic arrangements in [9]. In these cases, the fundamental group is the key invariant to distinguish Zariski pair candidates. The case of more inflectional tangents is studied in [8] by using the linking and splitting invariants. These Zariski pairs (or tuples) are characterized by the position of the tangent points of the smooth cubic $D$ and the inflectional tangents. The linking and splitting invariants were also applied to give Zariski tuples of one smooth quartic and its bitangents in [15, 9]. Other cases with two smooth non-linear components has been studied by I. Shimada [25], the third named author [27] and the fourth named author [31]. The case of one smooth curve $D$ of arbitrary degree and an arrangement of lines (usually with some tangency relation with $D$) has been studied by the first named author
and collaborators in [4] and by the third named author in [28]. The linking and splitting invariants have been used in these cases and an important feature is that for all of these curves (besides small degree examples) the fundamental groups of their complements are abelian, certifying the independence of the linking invariants and the fundamental group. More recently, the second and fourth named authors considered the case of multiple pairs of simple tangent lines to a cubic in [11]. As in the case of inflectional points, the study is also made in terms of the geometric group structure of the cubic.

In this paper, we consider the embedded topology of plane curves consisting of a smooth curve $D$ and some additional possibly reducible curve $B$ on the complex projective plane $\mathbb{P}^2$. The key point is to relate the topology of the arrangement with the torsion classes of the subgroup $\text{Pic}^0(D) \subset \text{Pic}(D)$ of divisor classes of degree zero of the smooth curve $D$. Our idea is to formulate a criterion to distinguish the embedded topology of curves explicitly in terms of these torsion classes. We fix a large family of cyclic covers and we construct a function associating each cyclic cover to a splitting number (one of the splitting invariants) for providing a criterion to distinguish candidate Zariski pairs.

This criterion is useful in two directions. On one side it allows us to re-explain previous results in the literature in a uniform setting, allowing us to distinguish the embedded topology of curves in a more finer way (cf. Section 4). On the other side, we are able to construct new Zariski tuples by using this criterion (Section 5).

We note that the assumption that $D$ is smooth is somewhat technical (by this assumption $\text{Pic}^0(D)$ is easier to deal with) but we plan to generalize to the case where $D$ is allowed to be singular in the future. We have already a work in progress for the special case where $D$ is a nodal cubic and we expect to deal also with the case of $D$ singular and non-rational.

We organize this paper as follows. In Section 1 we state the main results of this paper. In Section 2 we recall the definition and basic properties of combinatorics of plane curves and certain subgroups of $\text{Div}(D)$. In Section 3 we give a formula to calculate splitting numbers in terms of the order of certain elements of $\text{Pic}^0(D)$, and formulate a criterion (Corollary 1.3) to distinguish the embedded topology of plane curves by using the formula.

In Section 4 we apply the criterion to some previous results. In Section 5 we give a method of systematically constructing examples of curves which are generalizations of curves studied by Shimada [25] where the criterion is applicable, and give a new Zariski 4-tuple (Theorem 1.5).

1. Main results

The plane curves which are considered in this paper are of the following form:

$$\mathcal{C} := D + C_1 + \cdots + C_k,$$

where $D$ is a smooth curve of degree $d_0$, and $C_j$ is a possibly reducible curve of degree $d_j$ for each $j = 1, \ldots, k$. Since $C_1, \ldots, C_k$ are possibly reducible, there may be different decompositions of $\mathcal{C}$. We describe the above decomposition of $\mathcal{C}$ by
First, we describe how we obtain torsion classes of $\text{Pic}^0(D)$ from a decomposition $[C]$ as above. We denote the subgroup of $\text{Pic}^0(D)$ consisting of all $n$-torsion classes by $\text{Pic}^0(D)[n]$; for $m \in \mathbb{N}$ we denote by $\text{Div}(D)_m$ the subgroup of divisors whose degree is divisible by $m$:

$$\text{Pic}^0(D)[n] := \{ \mathcal{C} \in \text{Pic}^0(D) \mid n \mathcal{C} = 0 \}$$

$$\text{Div}(D)_m := \{ \mathcal{D} \in \text{Div}(D) \mid \deg \mathcal{D} \equiv 0 \pmod{m} \}.$$ 

Put $\mathcal{O}_D := L|_D$ for a line $L \subset \mathbb{P}^2$. Note that $\mathcal{O}_D$ is a divisor of degree $d_0$ on $D$. Put $n_{[C]} := \gcd \left( \{ (C_j \cdot D) \mid P \in C_j \cap D, \ j = 1, \ldots, k \} \cup \{ d_1, \ldots, d_k \} \right)$.

We write $n = n_{[C]}$ by omitting the subscript $[C]$ for short if there is no confusion. There uniquely exists an effective divisor $\mathcal{O}_j := \mathcal{O}_j [C]$ on $D$ such that $n \mathcal{O}_j = C_j |_D$ for each $j = 1, \ldots, k$. Let $t_j [C] := \mathcal{O}_j - \frac{d_j}{n} \mathcal{O}_D$. The divisor $t_j := t_j [C]$ satisfies $n t_j = n \left( \mathcal{O}_j - \frac{d_j}{n} \mathcal{O}_D \right) = C_j |_D - d_j \mathcal{O}_D = (C_j - d_j L) |_D \sim 0$.

By abuse of notation, we denote the divisor class by the same symbol of its representative, for example, $t_j$ will also describe the element of $\text{Pic}^0(D)[n]$ containing the divisor $t_j$.

Let $G[C]$ be the subgroup of $\text{Pic}^0(D)[n]$ generated by $t_1, \ldots, t_k \in \text{Pic}^0(D)[n]$. We define $\tau_{[C]} : \mathbb{Z}^{\otimes k} \to G[C]$ as the homomorphism given by

$$\tau_{[C]}(a_1, \ldots, a_k) = a_1 t_1 + \cdots + a_k t_k.$$ 

Let $d := (d_1, \ldots, d_k)$; a permutation $\rho$ of $k$ letters is said to be admissible with respect to $d$ if $d_j = d_{\rho(j)}$ for each $j$. We consider permutations with additional conditions later (see Subsection 2.1), however the definition of admissibility above is simpler, and we use this admissibility above to keep the statement of Proposition 1.1 simpler. Note that the symmetric group $S_k$ acts on $\mathbb{Z}^{\otimes k}$ by

$$\rho(a_1, \ldots, a_k) = (a_{\rho^{-1}(1)}, \ldots, a_{\rho^{-1}(j)}, \ldots, a_{\rho^{-1}(k)})$$

for $\rho \in S_k$ and $(a_1, \ldots, a_k) \in \mathbb{Z}^{\otimes k}$, and $\rho$ can be regarded as a map $\rho : \mathbb{Z}^{\otimes k} \to \mathbb{Z}^{\otimes k}$.

**Proposition 1.1.** Let $C_i$ (for $i = 1, 2$) be two plane curves

$$C_i := D_i + C_{i1} + \cdots + C_{ik}$$

where $D_1$ and $D_2$ are smooth curves of degree $d_0$, and $C_{ij}$ are planar curves of degree $C_{i,j} = \deg C_{j} = d_j$. Put $[C_i] := (D_i; C_{i1}, \ldots, C_{ik})$. Assume $n_{[C_1]} = n_{[C_2]}$. Let $\rho$ be an admissible permutation with respect to $d$. If $\ker \tau_{[C_1]} \neq \ker \tau_{[C_2]} \circ \rho$, then there exists no homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(D_1) = D_2$ and $h(C_{ij}) = C_{2 \rho(j)}$. In particular, if $G[C_1]$ and $G[C_2]$ are not isomorphic, then there exist no homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ and no permutation $\rho$ of $k$ letters such that $h(D_1) = D_2$ and $h(C_{ij}) = C_{2 \rho(j)}$. 


In the case when $D$ has a special point such as a maximal flex, Proposition 1.1 can be formulated in a different way which will be discussed in [2]. Proposition 1.1 tells us about the existence/non-existence of certain homeomorphisms. However, in order to apply it and to give a criterion to distinguish Zariski tuples, we need to consider the combinatorics of the curves in detail which can be quite subtle. See Subsection 2.1 for the definitions and notation used in Theorem 1.2 and Corollary 1.3.

**Theorem 1.2.** Let $C_i := D_i + C_{i1} + \cdots + C_{ik}$ ($i = 1, 2$) be two plane curves satisfying the hypotheses of Proposition 1.1 and having the same combinatorics. We assume that any equivalence map $\varphi : \text{Com}(C_1) \to \text{Com}(C_2)$ is admissible to $(C_1, C_2)$. Then if $\ker \tau_{[C_1]} \neq \ker \tau_{[C_2]} \circ \rho$ for any admissible permutation $\rho$ to $(C_1, C_2)$, then $(C_1, C_2)$ is a Zariski pair.

In particular, we have the following:

**Corollary 1.3.** Assume the same hypotheses as Theorem 1.2 and furthermore put $t_{i,j} := t_{j}([C_i]) \in \text{Pic}^0(D_i)[n]$ for $i = 1, 2$ and $j = 1, \ldots, k$. Then the following statements hold:

(i) If $G([C_1])$ and $G([C_2])$ are not isomorphic, then $(C_1, C_2)$ is a Zariski pair.
(ii) If $k = 1$ and $\text{ord}(t_{1,1}) \neq \text{ord}(t_{2,1})$, then $(C_1, C_2)$ is a Zariski pair.
(iii) If $(\text{ord}(t_{1,1}), \ldots, \text{ord}(t_{1,k})) \neq (\text{ord}(t_{2,\rho(1)}), \ldots, \text{ord}(t_{2,\rho(k)}))$ for any permutation $\rho$ admissible to $(C_1, C_2)$, then $(C_1, C_2)$ is a Zariski pair.

**Remark 1.4.** Note that Corollary 1.3 can be applied to Zariski pairs previously given as follows:

- A Zariski pair for 3-Artal arrangements [1] follows from Corollary 1.3 (ii).
- A Zariski pair for a smooth cubic and its tangent lines considered in [11] follows from Corollary 1.3 (i).

We will explain these examples in more detail in Section 4.

Theorem 1.2 and Corollary 1.3 enable us to distinguish the embedded topology simply. In Section 5 we consider plane curves whose embedded topology can be distinguished by Corollary 1.3. We also give a method to systematically construct more complicated examples based on the data of simple cases. In particular, we construct the following Zariski 4-tuple.

**Theorem 1.5.** There exists a Zariski 4-tuple of plane curves of the form $C = D + C$, where $D$ and $C$ are smooth curves of degree 4 and 6, respectively, and $(D \cdot C)_P = 6$ for any $P \in D \cap C$.

2. Settings

2.1. Combinatorics and Zariski tuples. The definition of the combinatorics of a curve is given in [5], but we give it here for sake of completeness and the convenience of the reader. We also formulate the notion of equivalence of combinatorial types which was intrinsically considered in [5], but not explicitly given there.
Definition 2.1. Let $\mathcal{C}$ be a plane curve in $\mathbb{P}^2$. Let $\text{bl}_C : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the minimal embedded resolution of its set $\text{Sing}_C$ of singular points (i.e., the minimal sequence of blowing-ups which resolve $\text{Sing}_C$ and such that $\text{bl}_C^{-1}(\mathcal{C})$ is a simple normal crossing divisor). Let $\text{Irr}_C$ be the set of irreducible components of $\mathcal{C}$, and let $(\Gamma_C, \text{Str}_C, e_C)$ be the 3-tuple given by

(i) $\Gamma_C$ is the dual graph of the simple normal crossing curve $\text{bl}_C^{-1}(\mathcal{C})$ with set of vertices $\mathcal{V}_C$.
(ii) $\text{Str}_C$ is the set of vertices corresponding to the strict transforms of the irreducible components of $\mathcal{C}$ (in natural bijection with $\text{Irr}_C$).
(iii) The Euler map $e_C : \mathcal{V}_C \to \mathbb{Z}$ is the map of self-intersections.

Then $\text{Comb}(\mathcal{C}) := (\Gamma_C, \text{Str}_C, e_C)$ is the combinatorial type (or combinatorics) of the curve $\mathcal{C}$.

Remark 2.2. The degrees and genera of the irreducible components of $\mathcal{C}$ can be derived from their combinatorics. Conversely, the combinatorics of a plane curve $\mathcal{C}$ is determined by combinatorial data of $\mathcal{C}$ (degrees of curves in $\text{Irr}_C$, topological types of singularities of $\mathcal{C}$, and so on). Since $\text{Str}_C$ and $\text{Irr}_C$ have a canonical bijection we will identify them as far as no confusion arises.

Definition 2.3. Let $\mathcal{C}_1, \mathcal{C}_2$ be plane curves in $\mathbb{P}^2$.

(i) An equivalence map $\varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2)$ is an isomorphism $\hat{\varphi} : \Gamma_{\mathcal{C}_1} \to \Gamma_{\mathcal{C}_2}$ of graphs such that $\hat{\varphi}(\text{Str}_{\mathcal{C}_1}) = \text{Str}_{\mathcal{C}_2}$ and $e_{\mathcal{C}_1} = e_{\mathcal{C}_2} \circ \hat{\varphi}$.
(ii) The curves $\mathcal{C}_1, \mathcal{C}_2$ have the same combinatorial type if there exists an equivalence map $\varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2)$.
(iii) A homeomorphism $h : (\mathbb{P}^2, \mathcal{C}_1) \to (\mathbb{P}^2, \mathcal{C}_2)$ induces an equivalence map $\varphi_h : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2)$ called the equivalence map induced by $h$.

Remark 2.4. An equivalence map $\varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2)$ induces canonical bijections $\varphi_{\text{Irr}} : \text{Irr}_{\mathcal{C}_1} \to \text{Irr}_{\mathcal{C}_2}$ and $\varphi_{\text{Sing}} : \text{Sing}_{\mathcal{C}_1} \to \text{Sing}_{\mathcal{C}_2}$, where $\text{Sing}_{\mathcal{C}_i}$ is the set of singular points of $\mathcal{C}_i$. Note that $\varphi_{\text{Irr}}$ and $\varphi_{\text{Sing}}$ preserve degrees of irreducible components and topological types of singularities, respectively.

Given curves $\mathcal{C}_i$ and decompositions $[\mathcal{C}_i] (i = 1, 2)$ as in Section 1 and a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ satisfying $h([\mathcal{C}_1]) = [\mathcal{C}_2]$, $h$ induces an equivalence map of combinatorial types $\varphi_h : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2)$. However, this equivalence need not preserve the structure of the decompositions $[\mathcal{C}_i]$ in general, i.e. $\{h(C_{i1}), \ldots, h(C_{ik_i})\} \neq \{C_{21}, \ldots, C_{2k_2}\}$ in general. Since our arguments later depend on the decompositions $[\mathcal{C}_i]$ we need to take this fact into account, which leads to the notion of admissibility of equivalence maps and permutations used in Theorem 1.2 and Corollary 1.3. They are defined as follows:

Definition 2.5. Let $\mathcal{C}_i := D_1 + C_{i,1} + \cdots + C_{i,k_i} (i = 1, 2)$ be two plane curves as in Proposition 1.1 and put $[\mathcal{C}_i] := (D_1; C_{i,1}, \ldots, C_{i,k_i})$. Assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same combinatorial type.
(i) An equivalence map \( \varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2) \) is admissible to \( (\mathcal{C}_1, \mathcal{C}_2) \) if \( \varphi_{irr}(D_1) = D_2 \) and there exists a permutation \( \rho \) of \( k \) letters such that
\[
\varphi_{irr}(\text{irr}_{c_i,j}) = \text{irr}_{c_{\rho(j)},j}
\]
for any \( j = 1, \ldots, k \).

(ii) A permutation \( \rho \) of \( k \) letters is admissible to \( (\mathcal{C}_1, \mathcal{C}_2) \) if there exists an equivalence map \( \varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_2) \) admissible to \( (\mathcal{C}_1, \mathcal{C}_2) \) such that \( \rho = \rho_\varphi \).

Example 2.6. Let \( E \) be a smooth cubic curve and \( P \in E \) be a general point of \( E \). It is known that there are four lines \( L_j \) \( (j = 1, \ldots, 4) \) passing through \( P \) and tangent to \( E \) at \( Q_j \neq P \). Let \( \mathcal{C} = E + L_1 + L_2 + L_3 + L_4 \) and consider decompositions \( [\mathcal{C}]_1 = (E; (L_1 + L_2), (L_3 + L_4)) \) and \( [\mathcal{C}]_2 = (E; (L_1 + L_3), (L_2 + L_4)) \). The identity map \( \text{id} : \mathbb{P}^2 \to \mathbb{P}^2 \) trivially induces an equivalence map of combinatorics \( \varphi_{\text{id}} : \text{Comb}(\mathcal{C}) \to \text{Comb}(\mathcal{C}) \). However, \( \varphi_{\text{id}} \) is not admissible with respect to \( ([\mathcal{C}]_1, [\mathcal{C}]_2) \).

On the other hand, let \( P_1, P_2 \in E \) be distinct general points and let \( L_{ij} \) be lines passing through \( P_i \) and are tangent to \( E \) at \( Q_{ij} \neq P_1, P_2 \). Let \( \mathcal{C} = E + L_{11} + L_{12} + L_{21} + L_{22} \) and consider a decomposition \( [\mathcal{C}] = (E; (L_{11} + L_{12}), (L_{21} + L_{22})) \). Then every equivalence map \( \varphi : \text{Comb}(\mathcal{C}) \to \text{Comb}(\mathcal{C}) \) is admissible with respect to \( ([\mathcal{C}]_1, [\mathcal{C}]_2) \) by the definition of \( \varphi \).

We recall the definition of Zariski tuples.

Definition 2.7. Let \( \mathcal{C}_1, \ldots, \mathcal{C}_N \) be \( N \) plane curves. An \( N \)-tuple \( (\mathcal{C}_1, \ldots, \mathcal{C}_N) \) is called a Zariski \( N \)-tuple if

(i) \( \mathcal{C}_1, \ldots, \mathcal{C}_N \) have the same combinatorics, and

(ii) \( \langle \mathbb{P}^2, \mathcal{C}_i \rangle \) and \( \langle \mathbb{P}^2, \mathcal{C}_j \rangle \) are not homeomorphic for any \( i \neq j \).

In the case where \( N = 2 \), a Zariski 2-tuple is called a Zariski pair.

2.2. A subgroup of \( \text{Div}(D) \) and a homomorphism \( \phi_{\sigma_D} \). Assume that \( D \) is a smooth plane curve of degree \( d_0 \). We choose a line \( L \) and set \( \sigma_D = L|_D \). Let
\[
\text{Div}(D)_{d_0} := \{ \sigma \in \text{Div}(D) \mid d_0 \text{ divides } \deg \sigma \}.
\]

Since \( \deg(\sigma_1 - \sigma_2) = \deg \sigma_1 - \deg \sigma_2 \) for any \( \sigma_1, \sigma_2 \in \text{Div}(D)_{d_0} \), \( \text{Div}(D)_{d_0} \) is a subgroup of \( \text{Div}(D) \). We define a map \( \psi_{\sigma_D} : \text{Div}(D)_{d_0} \to \text{Pic}^0(D) \) by
\[
\psi_{\sigma_D} : \sigma \mapsto \sigma - \frac{\deg \sigma}{d_0} \sigma_D.
\]

It is easy to see that \( \psi_{\sigma_D} \) is a homomorphism. We say that \( \sigma \in \text{Div}(D)_{d_0} \) is a \( \nu \)-torsion divisor with respect to \( \sigma_D \) if \( \psi_{\sigma_D} (\sigma) \neq 0 \) \( (1 \leq l \leq \nu - 1) \) and \( \psi_{\sigma_D} (\nu \sigma) = 0 \).

Let \( \sigma \in \text{Div}(D)_{d_0} \) be an effective divisor which is a \( \nu \)-torsion with respect to \( \sigma_D \). Then there exists \( \phi \in \mathbb{C}(D) \setminus \{0\} \), where \( \mathbb{C}(D) \) is the rational function field of \( D \), such that
\[
\langle \phi \rangle = \nu \left( \sigma - \frac{\deg \sigma}{d_0} \sigma_D \right).
\]
This means that there exists \( s \in H^0(D, \mathcal{O} (\nu \deg \mathfrak{d} \sigma_D)) \) such that \( \nu \sigma_D \) is the effective divisor \((s)\). Let us consider the short exact sequence of sheaves on \( \mathbb{P}^2 \):

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \left( \frac{\nu \deg \mathfrak{d}}{d_0} L - D \right) \to \mathcal{O}_{\mathbb{P}^2} \left( \frac{\nu \deg \mathfrak{d}}{d_0} L \right) \to i_* \mathcal{O}_D \left( \frac{\nu \deg \mathfrak{d}}{d_0} \sigma_D \right) \to 0,
\]

where \( i : D \hookrightarrow \mathbb{P}^2 \) is the inclusion. As \( H^1(\mathbb{P}^2, \mathcal{O} (\frac{\nu \deg \mathfrak{d}}{d_0} L - D)) = 0 \), the map

\[
H^0 \left( \mathbb{P}^2, \mathcal{O} \left( \frac{\nu \deg \mathfrak{d}}{d_0} L \right) \right) \to H^0 \left( D, \mathcal{O} \left( \frac{\nu \deg \mathfrak{d}}{d_0} \sigma_D \right) \right)
\]

is surjective. This means that there exists a homogeneous polynomial \( g \), of degree \( \frac{\nu \deg \mathfrak{d}}{d_0} \), such that the plane curve \( C \) given by \( g = 0 \) defines \( \nu \sigma_D \), i.e., \( C|_D = \nu \sigma_D \).

The curve \( C \) and the divisor \( \mathfrak{d} \) can be considered as a geometric description of a \( \nu \)-torsion with respect to \( \sigma_D \). Note that \( \mathfrak{d} \) itself is not given by any plane curve (if \( \nu > 1 \)).

**Remark 2.8.** In [2], we consider the case where the smooth curve \( D \) has a maximal flex point \( P \), and discuss a similar argument taking \( \sigma_D \) as the maximal flex point \( P \).

### 3. Splitting Numbers and Proofs of Main Results

In this section we translate the torsion properties into topological properties. Consequently, we prove the main results presented in Section I. We first define the splitting number of an irreducible curve for a cyclic cover.

Let \( \phi : X \to \mathbb{P}^2 \) be a cyclic cover of degree \( n \) branched along a projective curve \( \mathfrak{B} \) given by a surjection \( \theta : \pi_1(\mathbb{P}^2 \setminus \mathfrak{B}) \to \mathbb{Z}_n \), where \( \mathbb{Z}_n \) is the cyclic group of order \( n \). Let \( \mathfrak{B}_\theta \) be the following divisor on \( \mathbb{P}^2 \):

\[
\mathfrak{B}_\theta := \sum_{i=1}^{n-1} i \mathfrak{B}_{\theta,i},
\]

where \( \mathfrak{B}_{\theta,i} \) is the sum of irreducible components of \( \mathfrak{B} \) whose meridians are sent to \([i] \in \mathbb{Z}_n \) by \( \theta \). Then we call \( \phi : X \to \mathbb{P}^2 \) the \( \mathbb{Z}_n \)-cover (or \( n \)-cyclic cover) of type \( \mathfrak{B}_\theta \). Note that the degree of \( \mathfrak{B}_\theta \) is divisible by \( n \) since \( \phi \) is of degree \( n \).

**Remark 3.1.** Let us recall the definition of a meridian. Let \( C \) be a projective plane curve with irreducible components \( C_i \). A **meridian** of \( C_i \) in \( \pi_1(\mathbb{P}^2 \setminus C) \) is a free homotopy class defined as follows. Let \( p_0 \) be a base point and let \( p_i \in C_i \) be a smooth point of \( C \). Pick a smooth holomorphic 2-disk \( \Delta \), such that \( C \cap \Delta_i = \{p_i\} \) and the intersection is transversal. Let \( p'_i \in \partial \Delta_i \). A meridian is the homotopy class of the composition of an arbitrary path \( \alpha \) from \( p_0 \) to \( p'_i \) in \( \mathbb{P}^2 \setminus C \) with \( \partial \Delta_i \) (counterclockwise) and \( \alpha^{-1} \). As for free homotopy classes the base point is not important and it can be chosen as \( p'_i \). It is not hard to prove that with other choices the free homotopy class is not changed (the key is that \( C_i \setminus \text{Sing } C \) is connected). An **antimeridian** is constructed in the same way by running \( \partial \Delta_i \) clockwise (getting the free homotopy class of the inverses of the meridians).
Definition 3.2 ([27]). Let $D$ be an irreducible curve which is not a component of $B$. The splitting number $s_\phi(D)$ of $D$ with respect to $\phi$ is the number of irreducible components of $\phi^*D$.

Note that $0 \leq n(r - \lfloor r \rfloor) < n$ for any $r \in \mathbb{R}$, and $n\left(\frac{l}{n} - \lfloor \frac{l}{n} \rfloor\right) \equiv l \pmod{n}$ for any $l \in \mathbb{Z}$, where $\lfloor r \rfloor$ is the maximal integer not beyond $r \in \mathbb{R}$. We consider splitting numbers of $D$ for certain $\mathbb{Z}_n$-covers. The following lemma enable us to reduce the number of $\mathbb{Z}_n$-covers which should be considered.

Lemma 3.3. Let $B := \sum_{j=1}^{n-1} B_j$ be a plane curve. For $l \in \mathbb{Z}$ with $\gcd(l, n) = 1$, put the divisor $B^{(l)}$ as

$$B^{(l)} := \sum_{i=1}^{n-1} \left( \frac{il}{n} - \left\lfloor \frac{il}{n} \right\rfloor \right) B_i.$$  

The degree $\deg B^{(l)}$ is divisible by $n$ for some $l_1 \in \mathbb{Z}$ with $\gcd(l_1, n) = 1$ if and only if $\deg B^{(l)}$ is divisible by $n$ for any $l \in \mathbb{Z}$ with $\gcd(l, n) = 1$. Moreover, for any irreducible curve $D$ which is not a component of $B$, the splitting numbers $s_{\phi^{(l_1)}}(D)$ and $s_{\phi^{(l_2)}}(D)$ coincide for any $l_1, l_2 \in \mathbb{Z}$ with $\gcd(l_i, n) = 1$ ($i = 1, 2$),

$$s_{\phi^{(l_1)}}(D) = s_{\phi^{(l_2)}}(D),$$  

where $\phi^{(l)} : X \rightarrow \mathbb{P}^2$ is the $\mathbb{Z}_n$-cover of type $B^{(l)}$.

Proof. Let $l_1 \in \mathbb{Z}$ be an integer with $\gcd(l_1, n) = 1$. Suppose that $\deg B^{(l_1)}$ is divisible by $n$. Then there is a surjection $\theta^{(l_1)} : \pi(X \setminus B) \rightarrow \mathbb{Z}_n$ given by

$$\theta^{(l_1)}(m_i) = \left[ il_1 - n \left\lfloor \frac{il_1}{n} \right\rfloor \right] = [il_1] \in \mathbb{Z}_n$$  

for any meridian $m_i$ of components of $B_i$. Note that, since $\gcd(l_1, n) = 1$, there exists $[l'_1] \in \mathbb{Z}_n$ such that $[l'_1l_1] = [1] \in \mathbb{Z}_n$, and the multiplication by $l'_1l$ gives an isomorphism $m_{l'_1l} : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ for any $l \in \mathbb{Z}$ with $\gcd(l, n) = 1$. Hence we obtain the surjection $\theta^{(l_1)}_l := m_{l'_1l} \circ \theta^{(l_1)} : \pi(X \setminus B) \rightarrow \mathbb{Z}_n$ given by

$$\theta^{(l_1)}_l(m_i) = [l'_1l \cdot il_1] = [il] \in \mathbb{Z}_n.$$  

Moreover, $\theta^{(l_1)}_l$ defines the $\mathbb{Z}_n$-cover $\phi^{(l)}$ of type $B^{(l)}$, and $\deg B^{(l)}$ is divisible by $n$. Therefore, there exists a homeomorphism $h_{l_1, l} : X^{(l_1)} \setminus (\phi^{(l_1)})^{-1}(B) \rightarrow X^{(l)} \setminus (\phi^{(l)})^{-1}(B)$ with $\phi^{(l_1)} = \phi^{(l)} \circ h$ over $\mathbb{P}^2 \setminus B$ by [27] Proposition 1.3. In particular, $s_{\phi^{(l_1)}}(D) = s_{\phi^{(l)}}(D)$.

We next show a relation between the splitting numbers and the topology of a plane curve $\mathcal{C}$ of the following form:

$$\mathcal{C} := D + B \quad \left( B := \sum_{j=1}^{k} C_j \right).$$
where \( D \) is a smooth curve of degree \( d_0 \), and \( C_j \) is a curve of degree \( d_j \) for each \( j = 1, \ldots, k \). Let \( n \geq 2 \) be an integer, and put \( d := (d_1, \ldots, d_k) \). Let
\[
\Theta_d^n := \left\{ (a_1, \ldots, a_k) \in \mathbb{Z}^k \mid a_1 d_1 + \cdots + a_k d_k \equiv 0 \pmod{n} \right\}.
\]

Note that, if \( a \in \Theta_d^n \), there is a surjection \( \theta : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \to \mathbb{Z}_n \) given by \( \theta(m_j) = [a_j] \) for meridians \( m_j \) of components of \( C_j \) (\( j = 1, \ldots, k \)). Moreover, this set essentially measures \( \mathbb{Z}_n \)-covers of \( \mathbb{P}^2 \) whose ramification locus is contained in \( \mathcal{B} \) (if the coordinates are considered mod \( n \)).

**Proposition 3.4.** Let \( C_i = C_{i,1} + \cdots + C_{i,k_i} \), \( i = 1, 2 \) be two curves such that there exists a homeomorphism \( h : \mathbb{P}^2 \to \mathbb{P}^2 \), where \( h(C_{1,j}) = C_{2,j} \) for \( j = 1, \ldots, k \). Then, a meridian of \( C_{1,j} \) is sent to either a meridian or an antimeridian of \( C_{2,j} \).

**Proof.** We are going to use an alternative description of meridians and antimeridians. Let us consider a closed polydisk (in some analytic coordinates \( u, v \)) \( P_i \) centered at \( p_i \) such that \( v = 0 \) is \( P_i \cap C_i = P_i \cap C_j \). Then \( \pi_1(P_i \setminus C_i) = H_1(P_i \setminus C_i) \cong \mathbb{Z} \) since \( P_i \setminus C_i \) has the homotopy type of a circle. Meridians and antimeridians of \( C_i \) in \( P_i \) are generators of this group.

Let \( C_1, C_2 \subset \mathbb{P}^2 \) and \( h : (\mathbb{P}^2, C_1) \to (\mathbb{P}^2, C_2) \) be as in the statement. Let \( h_* : H_2(\mathbb{P}^2; \mathbb{Z}) \to H_2(\mathbb{P}^2; \mathbb{Z}) \) be the isomorphism induced by \( h \). Recall that \( \mathbb{P}^2 \) and the irreducible components of \( C_1, C_2 \) have a natural orientation inherited by the complex structure. By abuse of notation, we denote the class in \( H_2(\mathbb{P}^2; \mathbb{Z}) \) containing an irreducible curve \( C \) by the same symbol \( C \). Using the intersection form, we deduce that \( h \) must respect the orientation of \( \mathbb{P}^2 \).

Let us choose a meridian \( \gamma_{1,i} \) of \( C_{1,i} \) in some polydisk \( P_{1,i} \) centered at a point \( p_{1,i} \in C_{1,i} \) such that \( p_{1,i} \) is a smooth point of \( C_1 \). Let \( p_{2,i} := \Phi(p_{1,i}) \) a smooth point of \( C_2 \) in \( C_{2,i} \). Taking \( P_{1,i} \) small enough we can find a polydisk centered at \( p_{2,i} \) as in the previous discussion and such that \( h(P_{1,i}) \subset P_{2,i} \). Since \( h(P_{1,i}) \) is a neighborhood of \( p_{2,i} \), it is possible to find small enough polydisks \( \hat{P}_{1,i} \) (centered at \( p_{1,i} \)) and \( \hat{P}_{2,i} \) (centered at \( p_{2,i} \)) such that
\[
\hat{P}_{2,i} \subset h(\hat{P}_{2,i}) \subset h(P_{1,i}) \subset P_{2,i}
\]
and the inclusions \( \hat{P}_{1,i} \setminus C_{1,i} \subset P_{1,i} \setminus C_i \) and \( \hat{P}_{2,i} \setminus C_{2,i} \subset P_{2,i} \setminus C_{2,i} \) are homotopy equivalences. Hence, we deal with the following commutative diagram.

\[
\begin{array}{ccc}
H_1(\hat{P}_{1,i} \setminus C_{1,i}; \mathbb{Z}) & \xleftarrow{h_{1,i}^{-1}} & H_1(\hat{P}_{2,i} \setminus C_{2,i}; \mathbb{Z}) \\
\downarrow{\cong} & & \downarrow{\cong} \\
H_1(P_{1,i} \setminus C_{1,i}; \mathbb{Z}) & \xrightarrow{h_*} & H_1(P_{2,i} \setminus C_{2,i}; \mathbb{Z})
\end{array}
\]

From this diagram we deduce that \( h_*(\gamma_{1,i}) \) is a generator of \( H_1(\hat{P}_{2,i} \setminus C_{2,i}; \mathbb{Z}) \). Hence, the image is \( \gamma_{2,i} \), \( \varepsilon = \pm 1 \), where \( \gamma_{2,i} \) is a meridian of \( C_{2,i} \). \( \square \)
Remark 3.5. Let $\mathcal{B}_i = \sum_{j=1}^{h_i} C_{ij}$ be the irreducible decomposition of a plane curve $\mathcal{B}_i$ for $i = 1, 2$, and let $m_{ij}$ be a meridian of $C_{ij}$. If $h : \mathbb{P}^2 \to \mathbb{P}^2$ is a homeomorphism with $h(C_{ij}) = C_{2j}$ for $j = 1, \ldots, k$, then there exists $\varepsilon \in \{\pm 1\}$ such that $h_\ast(m_{ij}) = m_{2j}^\varepsilon$ for all $j = 1, \ldots, k$.

Indeed, by Proposition 3.4 we know that $h_\ast(m_{ij}) = m_{2j}^\varepsilon$, $\varepsilon_j = \pm 1$. By Poincaré-Lefschetz duality we have that

$$H_1(\mathbb{P}^2 \setminus \mathcal{B}_i) = H^3(\mathbb{P}^2, \mathcal{B}_i) = \text{coker}(H^2(\mathbb{P}^2) \to H^2(\mathcal{B}_i)).$$

The homotopy classes of the meridians are sent to the dual of the classes of $C_{ij}$ in $H_2(\mathcal{B}_i)$ and the linking numbers are determined by the signs, hence $h_\ast(C_{ij}) = \varepsilon_j C_{2j}$ as $h_\ast : H_2(\mathbb{P}^2 ; \mathbb{Z}) \to H_2(\mathbb{P}^2 ; \mathbb{Z})$ is induced by $h$. Then for $d_j := \deg C_{ij},$

$$d_j d_{j'} = C_{ij} \cdot C_{ij'} = h_\ast(C_{ij}) \cdot h_\ast(C_{ij'}) = \varepsilon_j \varepsilon_{j'} C_{2j} \cdot C_{2j'} = \varepsilon_j \varepsilon_{j'} d_j d_{j'} \Rightarrow \varepsilon_j \varepsilon_{j'} = 1;$$

hence $\varepsilon_j = \varepsilon_{j'} = \pm 1$ for any $j, j' = 1, \ldots, k$, and $h_\ast(m_{ij}) = m_{2j}^\varepsilon$ for some $\varepsilon \in \{\pm 1\}$ since a canonical orientation of $m_{ij}$ is determined by orientations of $\mathbb{P}^2$ and $C_{ij}$.

Put $[\mathcal{C}] := (D; C_1, \ldots, C_k)$. For $a = (a_1, \ldots, a_k) \in \Theta_\mathcal{C}^n$, let

$$[\mathcal{B}_a] := (a_1 - n \frac{d_1}{n}) C_1 + \cdots + (a_k - n \frac{d_k}{n}) C_k.$$

Put $\mathcal{B}_a := [\mathcal{B}_a][\mathcal{C}]$. Then we have $\deg[\mathcal{B}_a] \equiv 0 \pmod{n}$. For $a \in \Theta_\mathcal{C}^n$, we define a map $\Phi_a : \Theta_\mathcal{C}^n \to \mathbb{Z}$ by

$$\Phi_a^\mathcal{B}(a) := s_\phi(D),$$

where $\phi_a : X_a \to \mathbb{P}^2$ is the $\mathbb{Z}_n$-cover of type $\mathcal{B}_a$. By Lemma 3.3 for any $a \in \Theta_\mathcal{C}^n$ and any $l \in \mathbb{Z}$ with $\gcd(l, n) = 1$, we obtain

$$\Phi_a^\mathcal{B}(a) = \Phi_a^\mathcal{B}(la).$$

Since an admissible permutation $\rho$ to $([\mathcal{C}], [\mathcal{C}])$ satisfies $d_{\rho(j)} = d_j$ by definition, we have

$$a_1 d_1 + \cdots + a_k d_k = a_1 d_{\rho(1)} + \cdots + a_k d_{\rho(k)} = a_{\rho^{-1}(1)} d_1 + \cdots + a_{\rho^{-1}(k)} d_k.$$

Hence $\rho$ acts on $\Theta_\mathcal{C}^n$ by $\rho(a_1, \ldots, a_k) = (a_{\rho^{-1}(1)}, \ldots, a_{\rho^{-1}(k)})$.

Proposition 3.6. Let $\mathcal{B}_i$ and $\mathcal{C}_i$ ($i = 1, 2$) be plane curves

$$\mathcal{B}_i := \sum_{j=1}^{k} C_{i,j}, \quad \mathcal{C}_i := D_i + \mathcal{B}_i$$

where $D_i$ is irreducible of degree $d_i$, $C_{i,j}$ are curves of degree $d_j$ for each $i = 1, 2$. If there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(D_1) = D_2$ and $h(C_{1,j}) = C_{2,\rho(j)}$ ($j = 1, \ldots, k$) for a permutation $\rho$ of $k$ letters, then $\rho$ is admissible to $([\mathcal{C}_1], [\mathcal{C}_2])$, and the equation

$$\Phi_{[\mathcal{C}_1]}^\mathcal{B}(a) = \Phi_{[\mathcal{C}_2]}^\mathcal{B}(\rho(a))$$

holds for any $n \geq 2$ and any $a \in \Theta_\mathcal{C}^n$, where $[\mathcal{C}] := (D_1; C_{i,1}, \ldots, C_{i,k}).$
Proof. Since the permutation \( \rho \) satisfies \( h(C_{1,j}) = C_{2,\rho(j)} \), \( \rho \) is admissible to the pair \( ([C_1], [C_2]) \). In particular, \( \rho \) acts on \( \Theta_d^n \) by
\[
\rho(a_1, \ldots, a_k) = (a_{\rho^{-1}(1)}, \ldots, a_{\rho^{-1}(k)}).
\]
Fix \( n \geq 2 \). For \( a \in \Theta_d^n \), let \( \phi_{i,a} : X_{i,a} \to \mathbb{P}^2 \) be the \( \mathbb{Z}_n \)-cover of type \( \mathcal{E}_{i,a} := \mathcal{E}_a [C_i] \). Let \( h_* : H_2(\mathbb{P}^2; \mathbb{Z}) \to H_2(\mathbb{P}^2; \mathbb{Z}) \) be the isomorphism induced by \( h \). By Remark \[3.5\]
\[
 h_*(a_1C_{1,1} + \cdots + a_kC_{1,k}) = \varepsilon (a_1C_{2,\rho(1)} + \cdots + a_kC_{2,\rho(k)})
\]
for \( \varepsilon = \pm 1 \), where \( C_{i,j} \) is regarded as the class represented by \( C_{i,j} \) with the orientation induced by the complex structure for each \( i,j \). Let \( \nu_{\alpha} = \nu_{\alpha, [C]} \) be the minimal positive integer with this property, by \[27\] Theorem 1.3 and Lemma \[3.3\]. Therefore, we obtain \( \Phi_{\alpha, [C]} = \Phi_{\alpha} \circ \rho \).

Again we consider the plane curve \( C := D + \mathcal{E} \) as above. Put \( n := n_C \). Let \( \phi_\alpha : X_\alpha \to \mathbb{P}^2 \) be the \( \mathbb{Z}_n \)-cover of type \( \mathcal{E}_\alpha := \mathcal{E}_\alpha [C] \) for \( a \in \Theta_d^n \). We consider the following divisors on \( D \): \( \mathcal{D}_D := L|_D \) for some line \( L \), and:
\[
\mathcal{D}_\alpha := \sum_{P \in \mathcal{E}_\alpha \cap D} \frac{(\mathcal{E}_\alpha \cdot D) \cdot P}{n}, \quad \left( \deg \mathcal{D}_\alpha = \frac{d_0}{n} \sum_{j=1}^k a_j d_j \right)
\]
Then \( \mathcal{D}_\alpha \) is an element of \( \text{Div}(\mathcal{D})_d \) since \( \deg \mathcal{E}_\alpha = 0 \) (mod \( n \)) by definition of \( \Theta_d^n \). Let \( \nu_\alpha := \nu_\alpha [C] \) be the positive integer such that \( \mathcal{D}_\alpha \) is a \( \nu_\alpha \)-torsion divisor with respect to \( \mathcal{D} \).

**Proposition 3.7.** In the above setup, the following equation holds:
\[
s_{\phi_\alpha}(D) = \frac{n}{\nu_\alpha}.
\]
Proof. Since \( \nu_\alpha (\mathcal{D}_\alpha - \frac{\deg \mathcal{E}_\alpha}{n} \cdot \mathcal{D}) \sim 0 \) by the hypothesis, there is a divisor \( C \) on \( \mathbb{P}^2 \) such that \( C|_D = \nu_\alpha \mathcal{D}_\alpha \) as divisors on \( D \) by the arguments of Subsection \[2.2\]. As \( \nu_\alpha \) is the minimal positive integer with this property, by \[16\] Theorem 2.1, we obtain \( s_{\phi_\alpha}(D) = \frac{n}{\nu_\alpha} \).

We next prove a corollary of Propositions \[3.6\] and \[3.7\] which gives a method of distinguishing embedded topology of plane curves. Let \( \mathcal{C}_1 := D_1 + \sum_{i=1}^k C_{i,j} \) be two plane curves, where \( D_1 \) are smooth, and \( C_{i,j} \) are of degree \( d_i \), and put \( \mathcal{C}_\alpha := (D_1 : C_{i,1}, \ldots, C_{i,k}) \). Assume that there is an equivalence map \( \varphi : \text{Comb}(\mathcal{C}_1) \to \text{Comb}(\mathcal{C}_\alpha) \) admissible to \( ([C_1], [C_\alpha]) \). Then we can put
\[
n := n_{[C_1]} = n_{[C_\alpha]}.
\]
For \( a \in \Theta_d^n \), let \( \phi_{i,a} : X_{i,a} \to \mathbb{P}^2 \) be the \( \mathbb{Z}_n \)-cover of type \( \mathcal{E}_{i,a} := \mathcal{E}_a [C_i] \). The next corollary follows from Proposition \[3.6\] and \[3.7\].
Corollary 3.8. Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be two plane curves as above. Let \( \rho \) be an admissible permutation with respect to \( d := (d_1, \ldots, d_k) \). If there is an element \( \mathbf{a} \in \Theta_d^2 \) such that \( \nu_{\mathbf{a}}[\mathcal{C}_1] \neq \nu_{\rho(\mathbf{a})}[^r \mathcal{C}_2] \), then there exists no homeomorphism \( h : \mathbb{P}^2 \to \mathbb{P}^2 \) such that \( h(D_1) = D_2 \) and \( h(C_{1,j}) = C_{2,\rho(j)} \).

The main results in Section 1 follow from Corollary 3.8.

Proof of Proposition 1.1. Put \( t_{i,j} := t_j[\mathcal{C}_i] \). Suppose that \( \ker \tau_{\mathcal{C}_i} \nsubseteq \ker \tau_{\mathcal{C}_j} \circ \rho \). Then there is an element \((a'_1, \ldots, a'_k) \in \mathbb{Z}^{\oplus k} \) such that \( a'_1t_{1,1} + \cdots + a'_kt_{1,k} = 0 \in G[\mathcal{C}_1] \), \( a'_{\rho^{-1}(1)}t_{2,1} + \cdots + a'_{\rho^{-1}(k)}t_{2,k} \neq 0 \in G[\mathcal{C}_2] \). Since \( nt_{i,j} = 0 \), we may assume that \( 0 \leq a'_j \leq n - 1 \) for each \( j = 1, \ldots, k \). Let \( \mathbf{a} := (a_1, \ldots, a_k) = \frac{(a'_1, \ldots, a'_k)}{\gcd(a'_1, \ldots, a'_k)} \).

Since \( n \) is a divisor of \( \gcd(d_1, \ldots, d_k) \), we have \( \mathbf{a} := (a_1, \ldots, a_k) \in \Theta_d^2 \). Moreover, we have \( \text{ord}(a_1t_{1,1} + \cdots + a_kt_{1,k}) \neq \text{ord}(a'_{\rho^{-1}(1)}t_{2,1} + \cdots + a'_{\rho^{-1}(k)}t_{2,k}) \).

Since the above orders correspond to \( \nu_{\mathbf{a}}[\mathcal{C}_1] \) and \( \nu_{\rho(\mathbf{a})}[\mathcal{C}_2] \) in Corollary 3.8 there exists no homeomorphism \( h : \mathbb{P}^2 \to \mathbb{P}^2 \) such that \( h(D_1) = D_2 \) and \( h(C_{1,j}) = C_{2,\rho(j)} \) by Corollary 3.8. In the case of \( \ker \tau_{\mathcal{C}_i} \nsubseteq \ker \tau_{\mathcal{C}_j} \circ \rho \), the assertion can be proved by the same way.

We next prove Theorem 1.2.

Proof of Theorem 1.2. Suppose there exists a homeomorphism of topological pairs \( h : (\mathbb{P}^2, \mathcal{C}_1) \to (\mathbb{P}^2, \mathcal{C}_2) \). By the assumption, we have \( h(D_1) = D_2 \) and \( h(C_{1,j}) = C_{2,\rho(j)} \) for some admissible permutation \( \rho \) with respect to \( d \). This is a contradiction to Proposition 1.1. \( \square \)

We prove Corollary 1.3.

Proof of Corollary 1.3. Parts (i) and (ii) are clear. Let us prove (iii). Suppose that there is a homeomorphism \( h : (\mathbb{P}^2, \mathcal{C}_1) \to (\mathbb{P}^2, \mathcal{C}_2) \). By the assumption, there exists a permutation \( \rho \) such that \( \varphi_{h, \text{tr}(\text{Irr}_{\mathcal{C}_{1,j}})}(\text{Irr}_{\mathcal{C}_{1,j}}) = \text{Irr}_{\mathcal{C}_{2,\rho(j)}}(\text{Irr}_{\mathcal{C}_{2,\rho(j)}}) \) for \( j = 1, \ldots, k \), i.e., \( h(C_{1,j}) = C_{2,\rho(j)} \). In particular, \( h \) gives a homeomorphism \( (\mathbb{P}^2, D_1 + C_{1,j}) \to (\mathbb{P}^2, D_2 + C_{2,\rho(j)}) \) for each \( j \). However, there is \( j_0 \) such that \( \text{ord}(t_{1,j_0}) \neq \text{ord}(t_{2,\rho(j_0)}) \) by the assumption. By (iii) the homeomorphism \( h \) as above does not exist. \( \square \)

4. Remarks on previous examples

We here explain two previous examples where \( D \) is a smooth cubic \( E \) from our new viewpoint in this article. For a smooth cubic \( E \), there is a classically well-known description on \( \text{Pic}^0(E) \). In \([2]\), we reformulate our setting in this case and discuss our problem in detail.
4.1. 3-Artal arrangements. A k-Artal arrangement is a reduced plane curve whose irreducible components are a smooth cubic curve and k of its tangent lines at inflection points. A Zariski pair for 3-Artal arrangements is given in [1] and those for k-Artal arrangements are considered in [8]. We here consider 3-Artal arrangements based on our approach. Note that this is essentially considered in [8] from the viewpoint of splitting numbers.

Let $E$ be a smooth cubic curve and let $L_i$ ($i = 1, 2, 3$) be a tangent line at inflection points $P_i$ ($i = 1, 2, 3$). Put $\triangle := L_1 + L_2 + L_3$, $[C] := E + \triangle$ and $[C'] := (E; \triangle)$. We now choose a line $L$ and put $\sigma_L := L|_E$. In this case,

$$\sigma_\triangle := \sigma_L[C] = P_1 + P_2 + P_3, \quad t_\triangle := t_1[C] = \sigma_\triangle - \sigma_E.$$  

If $P_1, P_2$ and $P_3$ are collinear, i.e., there exists a line $L_\triangle$ such that $L_\triangle|_E = P_1 + P_2 + P_3$, we have $t_\triangle = 0$ in $\text{Pic}^0(E)$. On the other hand, if $P_1, P_2$ and $P_3$ are not collinear, $t_\triangle$ is a 3 torsion as an element of $\text{Pic}^0(E)$. Hence we choose $\triangle_i := L_{i1} + L_{i2} + L_{i3}$ ($i = 1, 2$), where $L_{ij}$ are the inflectional tangents of $E$, such that

(i) both of $\triangle_i$ for $i = 1, 2$ are not concurrent three lines,

(ii) the three inflection points for $\triangle_1$ are collinear, and

(iii) the three inflection points for $\triangle_2$ are not collinear.

Then $[C_i] := E + \triangle_i$ for $i = 1, 2$ have the same combinatorics, and any equivalence map $\varphi : \text{Combi}(C_1) \to \text{Combi}(C_2)$ is admissible to $(C_1, C_2)$ since $\deg \varphi_{\text{Irr}}(C_1) = \deg C_1$ for each $C_1 \in \text{Irr}(C_i)$. Therefore $(C_1, C_2)$ is a Zariski pair by Corollary [8, 13]

4.2. A smooth cubic and 4 tangent lines. In [11], we study Zariski tuples for a smooth cubic $E$ and its tangent lines. We distinguish the embedded topology by counting the number of basic subarrangements consisting of $E$ and 4 of its tangent lines as follows:

Choose two distinct points $P_1$ and $P_2$ of $E$. For each $P_j$ ($j = 1, 2$), we choose two tangent lines $L_{P_j, 1}$ and $L_{P_j, 2}$ tangent at $Q_{j, 1}$ and $Q_{j, 2}$, respectively. Put

$$\Lambda_j = L_{P_j, 1} + L_{P_j, 2}.$$  

We consider plane curves of the form $[C] := E + \Lambda_1 + \Lambda_2$, and put $[C] := (E; \Lambda_1, \Lambda_2)$. Then we have

$$\sigma_{\Lambda_j} := \sigma_j[C] = P_j + Q_{j, 1} + Q_{j, 2}.$$  

Now we choose a line $L$ and put $\sigma_L := L|_E$. Since $P_j, Q_{j, 1}, Q_{j, 2}$ cannot be collinear,

$$t_{\Lambda_j} := t_j[C] = P_j + Q_{j, 1} + Q_{j, 2} - \sigma_L \not\sim 0.$$  

Hence $t_{\Lambda_j} \in \text{Pic}^0(E)[2]$ as $2t_{\Lambda_j} = 2\sigma_{\Lambda_j} - 2\sigma_L = 0$ in $\text{Pic}^0(E)$. Note that $t_{\Lambda_j}$ depends on the choice of $L_{P_j, k}$ ($k = 1, 2$) as there exist 4 tangent lines passing through $P_j$. Choose $\{\Lambda_1, \Lambda_2\}$ and $\{\Lambda_1', \Lambda_2'\}$ such that $t_{\Lambda_j}$ and $t_{\Lambda_j'}$ as elements of $\text{Pic}^0(E)$ satisfy

$$t_{\Lambda_1} = t_{\Lambda_2} \quad \text{and} \quad t_{\Lambda_1'} \neq t_{\Lambda_2'} \quad \text{in} \quad \text{Pic}^0(E)[2] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$
Put

\[
\mathcal{C} := E + \Lambda_1 + \Lambda_2, \quad [\mathcal{C}] := (E; \Lambda_1, \Lambda_2), \\
\mathcal{C}' := E + \Lambda'_1 + \Lambda'_2, \quad [\mathcal{C}'] := (E; \Lambda'_1, \Lambda'_2).
\]

Then we obtain \(G[\mathcal{C}] \cong \mathbb{Z}_2\) and \(G[\mathcal{C}'] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\), and \(\mathcal{C}\) and \(\mathcal{C}'\) have the same combinatorics.

Let \(\varphi : \text{Comb}(\mathcal{C}) \to \text{Comb}(\mathcal{C}')\) be any equivalence map. We prove that \(\varphi\) is admissible to \(([\mathcal{C}],[\mathcal{C}'])\). Put \(\Lambda'_j := L_{P'_j,1} + L_{P'_j,2}\), and \(E \cap L_{P'_j, k} = \{P'_j, Q'_{j,k}\}\), where \(Q'_{j,k}\) is the tangent point of \(E\) and \(L_{P'_j, k}\). Note that

\[
\text{Sing}_E = \{P_1, P_2, Q_{1,1}, Q_{1,2}, Q_{2,1}, Q_{2,2}\}, \\
\text{Sing}_{\mathcal{C}'} = \{P'_1, P'_2, Q'_{1,1}, Q'_{1,2}, Q'_{2,1}, Q'_{2,2}\}.
\]

Since \(\deg E > 1\) for any \(j, k\), we have \(\varphi_{\text{irr}}(E) = E\). Since \(\sharp_{\mathcal{C}}(P_j) = \sharp_{\mathcal{C}'}(P'_j) = 3\) and \(\sharp_{\mathcal{C}}(Q_{j,k}) = \sharp_{\mathcal{C}'}(Q'_{j,k}) = 2\) for each \(j, k\), \(\varphi_{\text{Sing}}(P_j) = P_{\rho(j)}\) if \(j = 1, 2\) for a certain permutation \(\rho\) of two letters. Moreover, there is a permutation \(\rho'\) of two letters such that \(\varphi_{\text{Sing}}(Q_{j,k}) = Q'_{\rho(j),\rho'(k)}\) since \(\varphi\) satisfies \(\beta_{\mathcal{C}_2, \varphi_{\text{Sing}}(P)} \circ \varphi_P = \varphi_{\text{irr}} \circ \beta_{\mathcal{C}_1, P}\).

Hence we have \(\varphi_{\text{irr}}(\text{Irr}_{\mathcal{C}_1}) = \text{Irr}_{\mathcal{C}'_{1 \rho(j)}}\), and \(\varphi\) is admissible to \(([\mathcal{C}],[\mathcal{C}'])\). Therefore, by Corollary 1.3 [13] \((\mathcal{C}, \mathcal{C}')\) is a Zariski pair.

5. Generalizations of Shimada’s curves and \(n\)-torsion divisors

In this section, we construct certain plane curves consisting of two smooth curves, which are generalizations of Shimada’s curves in [25].

**Definition 5.1.** Let \(\mathcal{C} := D + C\) be a plane curve, where \(D\) and \(C\) are smooth curves of degrees \(d_0\) and \(d_1\) with \(d_1 \geq d_0\). For a divisor \(n\) of \(d_1\), we call \(\mathcal{C}\) a plane curve of type \((d_0, d_1; n)\) if the local intersection number \((D \cdot C)_P = n\) for each \(P \in C \cap D\).

**Remark 5.2.** A plane curve of type \((3, d_1; n)\) is called a plane curve of type \((d_1, n)\) by Shimada in [25].

Let \(\mathcal{C} := D + C\) be a plane curve of type \((d_0, d_1; n)\), and put \([\mathcal{C}] := (D; C)\). The divisors \(\varnothing_D\) and \(\varnothing := \varnothing_1[\mathcal{C}]\) on \(D\) are defined as follows:

\[
\varnothing_D := L|D, \quad \varnothing := \sum_{P \in C \cap D} P,
\]

where \(L\) is a general line on \(\mathbb{P}^2\). Since \(n\varnothing = C|D\) as a divisor on \(D\), we have that \(n\varnothing \sim d_1\varnothing_D\), \(n(\varnothing - \frac{d_1}{n}\varnothing_D)\) vanishes in Pic\(^0\)(\(D\)) and the order of \(\varnothing - \frac{d_1}{n}\varnothing_D\) is a divisor of \(n\).

**Definition 5.3.** Let \(\mathcal{C} := D + C\) be a plane curve of type \((d_0, d_1; n)\). We call \(\mathcal{C}\) a plane curve of type \((d_0, d_1; n, \nu)\) if the torsion order of \(\varnothing - \frac{d_1}{n}\varnothing_D\) is exactly \(\nu\).

By Corollary 1.3 we obtain the following corollary.
Corollary 5.4. Let \( C_i := D_i \cap C_1 \) \((i = 1, 2)\) be plane curves of type \((d_0, d_1; n, \nu)\). Assume that \( C_1 \) and \( C_2 \) satisfy the following conditions:

(i) \( \deg D_i = d_0 \) and \( \deg C_i = d_1 \) for \( i = 1, 2 \),

(ii) \( d_0 < d_1 \), and

(iii) \( \nu_1 \neq \nu_2 \).

Then \( (C_1, C_2) \) is a Zariski pair.

Proof. By the definition of plane curves of type \((d_0, d_1; n)\), we have \( \text{Sing}_{C_i} = D_i \cap C_i \), \#\( \text{Sing}_{C_i} = \frac{d_0 d_1}{n} \), and all singularities of \( C_i \) have the same topological types. Hence the bijections \( \varphi_{\text{irr}} : \text{Irr}_{C_i} \rightarrow \text{Irr}_{C_2} \) and \( \varphi_{\text{sing}} : \text{Sing}_{C_i} \rightarrow \text{Sing}_{C_2} \) induce an equivalence map \( \varphi : \text{Comb}(C_1) \rightarrow \text{Comb}(C_2) \), where \( \varphi_{\text{sing}} \) is a bijective map, and \( \varphi_{\text{irr}} \) is given by

\[ \varphi_{\text{irr}}(D_1) := D_2, \quad \varphi_{\text{irr}}(C_1) := C_2. \]

Thus \( C_1 \) and \( C_2 \) have the same combinatorics. Furthermore, any equivalence map \( \varphi : \text{Comb}(C_1) \rightarrow \text{Comb}(C_2) \) satisfies \( \varphi_{\text{irr}}(D_1) = D_2 \) and \( \varphi_{\text{irr}}(C_1) = C_2 \) since \( d_0 < d_1 \). Therefore \( (C_1, C_2) \) is a Zariski pair by Corollary 1.3 (ii) since \( \nu_1 \neq \nu_2 \). \( \square \)

The following proposition provides a method for the construction of a new plane curve of type \((d_0', d_1'; n, \nu')\) from a given plane curve \( C := D + C \) of type \((d_0, d_1; n, \nu)\). Let \( f_0 \in H^0(\mathbb{P}^2, \mathcal{O}(d_0)) \) be a homogeneous polynomial defining \( D \). From [27] Theorem 2.7 we deduce that \( \nu \) is the minimal number for which there are homogeneous polynomials \( g \) and \( h \) of degree \( d_1 - d_0 \) and \( \frac{d_0}{\nu} \), respectively, such that \( C \) is defined by \( f_1 := h^\nu + f_0 g = 0 \), where \( \mu := \frac{\nu}{\nu} \).

Proposition 5.5. Let \( C := D + C \) be a plane curve of type \((d_0, d_1; n, \nu)\), where \( D \) is defined by \( f_0 = 0 \) and \( C \) is defined by \( f_1 := h^\nu + f_0 g = 0 \) with \( \mu := \frac{\nu}{\nu} \). Let \( k \) be an integer satisfying \( kd_0 \geq d_1 \).

Then the curve \( B_{D,k} \) defined by \( f_0^k + f_1 g' = 0 \) is smooth for a general homogeneous polynomial \( g' \) of degree \( \mu k d_0 - d_1 \). Moreover, \( B_{D,k} + C \) is a plane curve of type

\[ \begin{cases} (d_1, kd_0; kn, n) & \text{if } d_0 \equiv 0 \pmod{n} \text{ and } d_0 < d_1, \\ (d_1, kd_0; kn, \nu) & \text{if } d_0 = d_1. \end{cases} \]

Proof. Let \( \Lambda \) be the linear system on \( \mathbb{P}^2 \) consisting of divisors defined by \( f_0^k + f_1 g' \) for \( g' \in H^0(\mathbb{P}^2, \mathcal{O}(kd_0 - d_1)) \). The base points of \( \Lambda \) is just the divisors points of \( C \cap D \).

By Bertini’s theorem (see [13]), a general member of \( \Lambda \) is smooth outside \( C \cap D \).

Moreover, if \( g' \in H^0(\mathbb{P}^2, \mathcal{O}(kd_0 - d_1)) \) does not vanish at each \( P \in C \cap D \), \( f_1 g' = 0 \) defines a plane curve which is smooth at each \( P \in C \cap D \). Hence \( f_0^k + f_1 g' = 0 \) defines a smooth curve \( B_{D,k} \) for a general \( g' \). It is clear that \( C \cap D = C \cap B_{D,k} \). Moreover, since \( C \) and \( D \) intersect with multiplicity \( n \) at each \( P \in C \cap D \), we have \( (B_{D,k} + C)_P = k n \). Thus \( C_{D,k} := B_{D,k} + C \) is a plane curve of type \((d_1, kd_0; kn)\) if \( d_0 \equiv 0 \pmod{n} \).
Let $D_{D,k}$ and $d_{D,k}$ be the divisors on $C$ for the plane curve $C_{D,k}$ as given before Definition 5.6. Since $B_{D,k}$ is defined by $f_0^k + f_1 g' = 0$, $n$ is a divisor of the order of $d_{D,k} - d_{D,k}$. By the short exact sequence

$$0 \rightarrow H^0(P^2, \mathcal{O}(d_0 - d_1)) \rightarrow H^0(P^2, \mathcal{O}(d_0)) \rightarrow H^0(C, \mathcal{O}(d_0)) \rightarrow 0,$$

$D$ is the unique divisor on $P^2$ satisfying $D|_C = nD_{D,k}$ if $d_0 < d_1$. Since $D$ is reduced, $C_{D,k}$ is of type $(d_1, k\nu; \nu, \nu)$ if $d_0 < d_1$.

Suppose that $d_0 = d_1$. Then, since $C \neq D$, $g$ is a non-zero complex number, and we have $f = g^{-1}(f_1 - h\nu)$. By the minimality of $\nu$, $D$ induces a divisor of order $\nu$ on Pic$^0(C)$ as before Definition 5.6. Since $D|_C = B_{D,k}|_C$, $C_{D,k}$ is a plane curve of type $(d_1, k\nu; \nu, \nu)$.

**Definition 5.6.** In the situation of Proposition 5.5, we say that the plane curve of type $(d_1, k\nu; \nu, \nu)$ is constructed from a curve of type $(d_0, d_1; n, \nu)$ by power of $k$, and write $(d_0, d_1; n, \nu) \xrightarrow{k} (d_1, k\nu; \nu, \nu)$, where

- $\nu' = \nu$ if $d_0 = d_1$,
- $\nu' = n$ if $d_0 \equiv 0 \pmod{n}$ and $d_0 < d_1$.

**Example 5.7** (Plane curves of type $(4,6;6,1)$ and $(4,6;6,2)$). Since $D + C$ is a plane curve of type $(d_0, d_1; 1, 1)$ if $C$ and $D$ intersect transversally, there are plane curves of type $(d_0, d_1; 1, 1)$ for any $d_0 \leq d_1$. We have

- $(1,4;1,1) \xrightarrow{3} (4,6;6,1)$,
- $(2,2;1,1) \xrightarrow{2} (2,4;2,1) \xrightarrow{3} (4,6;6,2)$.

Therefore, there are plane curves $C_1$ and $C_2$ of type $(4,6;6,1)$ and $(4,6;6,2)$, respectively.

**Example 5.8** (A plane curve of type $(4,6;6,3)$). We construct a plane curve of type $(4,6;6,3)$. Let $E$ be a smooth cubic, let $O \in E$ be an inflectional point, and let $P_1, P_2, P_3 \in E$ be general points. There is a conic $C_2$ through $P_1, P_2, P_3$ tangent to $E$ at $O$, and $E$ and $C_2$ intersect at $P_4$ other than $P_1, P_2, P_3, O$. Since $\sum_{i=1}^4 P_i + 2O \sim 2L|_E$ as divisors on $E$, we obtain

$$3(P_1 + P_2 + P_3 + P_4) \sim 12O \sim 4L|_E,$$

where $L$ is a general line on $P^2$. Hence there exists a quartic $C_4'$ such that $C_4'|_E = 3 \sum_{i=1}^4 P_i$. Let $f_1, f_2$ be homogeneous polynomials defining $E, C_4'$, respectively. We can choose a linear polynomial $g_1$ so that the curve $C_4$ defined by $f_4 := f_1 + f_2 g_1 = 0$ is smooth at $C_4 \cap E$. By the same argument of the proof of Proposition 5.5, $f_3^2 + f_4 g = 0$ defines a smooth sextic $B$ for a general homogeneous polynomial $g$ of degree $2$, and $C_3 := C_4 + B$ is of type $(4,6;6)$. Note that $B \cap C_4 = C_4 \cap E = \{P_1, \ldots, P_3\}$. Since $P_1, P_2, P_3 \in B \cap C_4$ are not collinear, $C_3$ is a plane curve of type $(4,6;6,1)$. If $C_3$ is of type $(4,6;6,2)$, then there is a conic $C_2'$ tangent to $C_4$ with multiplicity 2 at each $P \in B \cap C_4$; this implies that $C_2', E \geq 8$, which is a contradiction. Therefore, $C_3$ is of type $(4,6;6,3)$.
Example 5.9 (A plane curve of type (4, 6; 6, 6)). It is more difficult to construct a plane curve of type (4, 6; 6, 6). Let us sketch how to construct such a curve. Let \( C_3 \) be a nodal cubic and \( O \in C_3 \) an inflectional point. We consider the additive group structure on \( C_3 \) with \( O \) being the zero element. We denote the addition on \( C_3 \) by \( + \). Let us consider an irreducible quartic \( C_4 \) such that \( C_3 \cap C_4 = \{ P_1, P_2 \} \), where \( P_1, P_2 \) are not torsion points of \( C_3 \), \((C_3 : C_4)_{P_1} = 6\) and \( \text{Sing}(C_4) = \{ P_0 \} \) where \((C_4, P_0)\) is of type \( A_3 \). Such a curve exists, however we skip the details since the actual example has coefficients in a number field of degree 4. Then \( Q := P_1 + P_2 \) is an element of order 6. These curves satisfy the following properties: there is no smooth conic \( C_2 \) such that \((C_2 : C_4)_{P_1} = (C_2 : C_4)_{P_2} = 2 \) and \((C_2 : C_4)_{P_0} = 4\) (We omit the details due to long computation.); moreover, there is no smooth conic \( C'_2 \) such that \((C'_2 : C_4)_{P_1} = (C'_2 : C_4)_{P_2} = 3 \) and \((C'_2 : C_4)_{P_0} = 2\) (from the group law of the cubic).

Let us choose coordinates such that \( P_0 = [0 : 1 : 0] \) and its tangent line is \( Z = 0 \). Let \( \sigma : \mathbb{P}^2 \to \mathbb{P}^2 \) be the rational map defined by \( \sigma([x : y : z]) = [xz : y^2 : z^2] \). Let \( C_6, D_4 \) be the strict transforms by \( \sigma \) of \( C_3, C_4 \) respectively. Here, the subindices indicate their degrees. Choose a generic conic \( D_2 \), and a generic element \( D_6 \) of the pencil generated by \( C_6 \) and \( D_4 + D_2 \). The above properties imply that \( C_5 := D_4 + D_6 \) is of type \((4, 6; 6, 6)\). These computations can be found at https://github.com/enriqueartal/TorsionDivisorsZariskiPairs and can be checked using Sagemath [29] or Binder [18].

Finally we prove Theorem 1.5.

Proof of Theorem 1.5. By Examples 5.7, 5.8 and 5.9 there are plane curves of type \((4, 6; 6, 1), (4, 6; 6, 2), (4, 6; 6, 3)\) and \((4, 6; 6, 6)\). Therefore there exists a Zariski 4-tuple by Corollary 5.3. \( \square \)

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