Fractional Dynamical Systems

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Abstract—In this paper the author presents the results of the preliminary investigation of fractional dynamical systems based on the results of numerical simulations of fractional maps. Fractional maps are equivalent to fractional differential equations describing systems experiencing periodic kicks. Their properties depend on the value of two parameters: the non-linearity parameter, which arises from the corresponding regular dynamical systems; and the memory parameter which is the order of the fractional derivative in the corresponding non-linear fractional differential equations. The examples of the fractional Standard and Logistic maps demonstrate that phase space of non-linear fractional dynamical systems may contain periodic sinks, attracting slow diverging trajectories, attracting accelerator mode trajectories, chaotic attractors, and cascade of bifurcations type trajectories whose properties are different from properties of attractors in regular dynamical systems. The author argues that discovered properties should be evident in the natural (biological, psychological, physical, etc.) and engineering systems with power-law memory.

I. INTRODUCTION

Fractional differential equations are used in applications to describe systems with long-range interactions [11, 12, 13] or systems with power-law memory. Systems with memory include: Hamiltonian systems where memory is the result of stickiness of trajectories in time to the islands of regular motion [4, 5]; dielectric materials where electromagnetic fields are described by equations with time fractional derivatives due to the universal response - the power-law frequency dependence of the dielectric susceptibility in a wide range of frequencies [6, 7, 8, 9]; viscoelastic materials and materials with rheological properties where non-integer order differential stress-strain relations give a minimal parameter set concise description of polymers and other viscoelastic materials with non-Debye relaxation and memory of strain history [10, 11, 12, 13, 14]. Examples of fractional order systems in modeling and control may be found in [15].

The intuitively obvious fact that many biological systems are systems with memory is now confirmed by rigorous research. In many cases memory obeys the power law and the corresponding systems could be described by fractional differential equations. It has been shown recently that viscoelastic properties are typical for a wide variety of biological tissues (liver, heart valve, brain, etc.) [16, 17, 18, 19, 20, 21] which implies that they can be described by differential equations with time fractional derivatives. Short- and long-term memory effects in neuronal circuits may result from synaptic plasticity [22] and from intrinsic membrane currents [23]. As it has been shown recently, even processing of external stimuli by individual neurons can be described by fractional differentiation [24, 25]. In some recent papers describing dynamics of biological systems authors used power-law adaptation [27, 28, 29, 30, 31]. Fractional differentiation is used in bioengineering research (bioelectrodes, biomechanics, bioimaging) [26]. The power-law memory $t^{-\beta}$ with $\beta \in (0, 1)$ was demonstrated in experiments on human learning and forgetting [32, 33, 34, 35, 36, 37].

Fractional differential equations are integro-differential equations. Their numerical solution requires large computer memory and long runs of numerical simulations which makes it very difficult to investigate general properties of fractional dynamical systems - systems which can be described by fractional differential equations. Investigation of general properties of nonlinear fractional dynamical systems in [38, 39, 40, 41, 42, 43, 44] has been based on fractional maps derived from fractional differential equations describing systems experiencing periodic kicks introduced in [38, 39, 42, 45, 46]. In the following sections we’ll review fractional maps (Sec. II) and some of their properties (Sec. III). In Conclusion we’ll discuss possible manifestations of fractional dynamical systems in nature.

II. FRACTIONAL MAPS

The notion of the $\alpha$-Families of Maps ($\alpha$-FM) was introduced in [42].

A. Universal Fractional Map

The derivation of the equations of the Universal $\alpha$FM starts with the equation:

$$\frac{d^\alpha x}{dt^\alpha} + G_K(x(t-\Delta)) \sum_{k=-\infty}^{\infty} \delta(t-(k+\varepsilon)) = 0, \quad (1)$$

where $\varepsilon > \Delta > 0$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\varepsilon \to 0$, and the initial conditions corresponding to the type of fractional derivative to be used.

In the case of the Riemann-Liouville fractional derivative integration of Eq. (1) with the initial conditions

$$(0D^\alpha_t x)(0^+) = c_k, \quad (2)$$

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where \( k = 1, \ldots, N \), \( N - 1 < \alpha \leq N \), and \( N \in \mathbb{N} \), produces the Riemann-Liouville Universal \( \alpha \)-FM

\[
p^{\alpha}_{n+1} = p^{\alpha}_n + \sum_{k=0}^{N-s-3} \frac{p^{\alpha}_{k+n+1}}{(k+1)!} \frac{G_K(x_n)}{(N-s-2)!},
\]

\[
x_{n+1} = x_n + \sum_{k=2}^{N-1} \frac{c_k}{\Gamma(\alpha-k+1)}(n+1)^{\alpha-k} + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{N-s-2} \frac{p^{\alpha}_{k+n+1} V^{\alpha}_1(n-k+1)}{k!},
\]

where \( s = 0, 1, \ldots, N - 2 \) and \( V^\alpha_k(m) = m^{\alpha-k} - (m-1)^{\alpha-k} \).

In Eqs. (3) and (4) \( p(t) = p_0 D_t^{\alpha-N+1} x(t) \), \( p^{(s)}(t) = D_t^{\alpha-s} p(t) \), \( x_n = x(n) \), and \( p_n = p^{(s)}(n) \).

In the case of the Caputo fractional derivative integration of Eq. (1) with the initial conditions

\[
(D_t^\alpha x)(0+) = b_k, \tag{5}
\]

where \( k = 0, \ldots, N - 1 \) produces the Caputo Universal \( \alpha \)-FM

\[
x^{(s)}_{n+1} = \sum_{k=0}^{N-s-1} \frac{x_0^{(k+s)}}{k!}(n+1)^{k} - \frac{1}{\Gamma(\alpha-s)} \sum_{k=0}^{n} G_K(x_k)(n-k+1)^{\alpha-s-1},
\]

where \( s = 0, 1, \ldots, N - 1 \) and \( x^{(s)}(t) = D_t^\alpha x(t) \).

For integer \( \alpha = N \) the Universal \( \alpha \)-FM can be written in the form of the N-dimensional volume preserving (for \( N \geq 2 \)) map

\[
p^{\alpha}_{n+1} = p^{\alpha}_n + \sum_{k=0}^{N-s-3} \frac{p^{\alpha}_{k+n+1}}{(k+1)!} \frac{G_K(x_n)}{(N-s-2)!},
\]

\[
x_{n+1} = x_n + \sum_{k=0}^{N-2} \frac{p^k}{(k+1)!} \frac{G_K(x_n)}{(N-1)!}.
\]

For \( N = \alpha = 2 \) and \( G_K(x) = KG(x) \) this map gives the well-known in regular dynamics 2D Universal Map

\[
p_{n+1} = p_n - K \sin(x_n), \tag{9}
\]

\[
x_{n+1} = x_n + p_{n+1}, \tag{10}
\]

**B. Standard and Logistic Fractional Maps**

In the case \( G(x) = \sin(x) \) Eqs. (9) and (10) produce the well known Standard Map

\[
p_{n+1} = p_n - K \sin(x_n), \quad (\text{mod } 2\pi), \tag{11}
\]

\[
x_{n+1} = x_n + p_{n+1}, \quad (\text{mod } 2\pi). \tag{12}
\]

This is why the Universal \( \alpha \)-Families of Maps with \( G_K(x) = K \sin(x) \)

(13)

are called the Riemann-Liouville Standard \( \alpha \)-FM (Eqs. (3) and (4)) and the Caputo Standard \( \alpha \)-FM (Eq. 6) correspondingly.

For \( \alpha = 1 \) the Caputo Universal \( \alpha \)-FM Eq. (6)

\[
x_{n+1} = x_n - G_K(x_n) \tag{14}
\]

produces the regular Logistic Map \( x_{n+1} = Kx(1-x) \) if

\[
G_K(x) = G_{LK}(x) = x - Kx(1-x). \tag{15}
\]

This is why the Universal \( \alpha \)-FMs with \( G_K(x) = G_{LK}(x) \) are called the Logistic \( \alpha \)-FMs.

The Standard and Logistic \( \alpha \)-FMs were numerically investigated for \( 0 < \alpha \leq 2 \), the case which is most important for applications.

- For \( 0 < \alpha < 1 \) the Caputo Standard \( \alpha \)-FM can be written as

\[
x_n = x_0 - K \sum_{k=0}^{n-1} \frac{\sin(x_k)}{(n-k)^{1-\alpha}}, \quad (\text{mod } 2\pi). \tag{16}
\]

- The Caputo Logistic \( \alpha \)-FM for \( 0 < \alpha < 1 \) assumes the form

\[
x_n = x_0 + \sum_{k=0}^{n-1} \frac{Kx_k(1-x_k) - x_k}{(n-k)^{1-\alpha}}. \tag{17}
\]

- The 1D Standard Map can be written as the Circle Map with zero driving phase

\[
x_{n+1} = x_n - K \sin(x_n), \quad (\text{mod } 2\pi). \tag{18}
\]

- For \( 1 < \alpha < 2 \) the Riemann-Liouville Standard \( \alpha \)-FM can be written as

\[
p_{n+1} = p_n - K \sin(x_n), \tag{19}
\]

\[
x_{n+1} = x_n + \sum_{i=0}^{n} p_{i+1} V_\alpha^1(n-i+1), \quad (\text{mod } 2\pi). \tag{20}
\]

- The Caputo Standard \( \alpha \)-FM for \( 1 < \alpha < 2 \) assumes the form

\[
p_{n+1} = p_n - \frac{K}{\Gamma(\alpha-1)} \left[ \sum_{i=0}^{n-1} V_\alpha^2(n-i+1) \sin x_i + \sin x_n \right], \quad (\text{mod } 2\pi), \tag{21}
\]

\[
x_{n+1} = x_n + p_0 - \frac{K}{\Gamma(\alpha)} \sum_{i=0}^{n} V_\alpha^1(n-i+1) \sin x_i, \quad (\text{mod } 2\pi). \tag{22}
\]

- For \( 1 < \alpha < 2 \) the Riemann-Liouville Logistic \( \alpha \)-FM can be written as

\[
p_{n+1} = p_n - Kx_n(1-x_n) - x_n, \tag{23}
\]

\[
x_{n+1} = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n} p_{i+1} V_\alpha^1(n-i+1). \tag{24}
\]
Fig. 1. $\alpha-K$ diagrams for the Logistic (upper figure) and the Standard (lower figure) $\alpha$-FMs. The $(0,0)$ fixed point is stable in the areas I. The $T=2$ sink is stable in the areas II. In the area III in the bottom figure the $T=4$ sink is stable. The area III in the upper figure is the area of the stable $T=2^n\ (n>1, n \in \mathbb{N})$ sinks and cascade of bifurcations type trajectories (CBTT). The area IV in the lower figure is the area of the stable $T=2^n\ (n>2, n \in \mathbb{N})$ sinks and CBTT. The area IV in the upper figure and the area V in the lower figure is the area of chaos.

- The Caputo Logistic $\alpha$-FM for $1 < \alpha < 2$ assumes the form

$$x_{n+1} = x_0 + p(n+1)^k - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} x_k - Kx_k(1-x_k)\frac{(n-k+1)^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=0}^{n} x_k - Kx_k(1-x_k)\frac{(n-k+1)^{\alpha-2}}{\Gamma(\alpha-2)}.$$  \hspace{1cm} (25)

$$p_{n+1} = p_0 - \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} x_k - Kx_k(1-x_k)\frac{(n-k+1)^{\alpha-1}}{\Gamma(\alpha-1)} \sum_{k=0}^{n} x_k - Kx_k(1-x_k)\frac{(n-k+1)^{\alpha-2}}{\Gamma(\alpha-2)}.$$  \hspace{1cm} (26)

In the following section (Sec.III) we’ll present a brief summary of the results of investigation of the Standard and the Logistic $\alpha$-FMs for $0 < \alpha < 2$ obtained in [38], [39], [40], [41], [42], [43], [44].

III. FRACTIONAL ATTRACTIONS

Our analysis of fractional maps for the most important in applications case $0 < \alpha < 2$ will be based on Fig. 1. For small values of the non-linearity parameter $K$ the fixed point $(0,0)$ (which is a sink) is stable (area I). In the 2D phase space of the fractional Standard Map for $1 < \alpha < 2$ the higher period sinks which can be traced to the corresponding islands of stability of the regular Standard Map are also present. With the increase in $K$ the fixed point becomes unstable and the stable $T=2$ point appears (area II). The further increase in $K$ leads to the cascade of bifurcations, when the period $2^n\ (n>1)$ sinks become unstable and the stable period $2^{n+1}$ sinks appear (area III on the upper figure and III and IV on the lower). Properties of fractional sinks and their differences from properties of sinks in regular dissipative dynamical systems are discussed in the above-mentioned papers.

Convergence of trajectories to fractional sinks obeys the power law. The exponents of this power law, defined by the memory parameter $\alpha$, may be different for trajectories with initial conditions from basins of attraction (fast convergence)
and those that start in chaotic areas (slow convergence). In addition, phase space of fractional maps may contain attracting slow diverging trajectories (power-law divergence with the exponents depending on $\alpha$), attracting accelerator mode trajectories, and chaotic attractors.

Just below the border with chaos (the upper curves in Fig. 1) all trajectories converge to cascade of bifurcations type trajectories (CBTT). The examples of CBTT are presented in Figs. 2 and 3. In CBTT after a small number of iterations a trajectory converges to a period one trajectory (fixed point) which later bifurcates and becomes a $T = 2$ trajectory and then follows the period doubling scenario typical for cascades of bifurcations in regular dynamics. The difference is that in regular dynamics a cascade of bifurcations is the result of a change in the non-linearity parameter and in CBTT a cascade of bifurcations occurs on a single attracting trajectory. $T > 1$ trajectories may also follow the cascade of bifurcations scenario which leads to a stable high-period sink. In the fractional Logistic Map with $1 < \alpha < 2$ instead of CBTT only inverse CBTT (lower figure in Fig. 3) were found.

An interesting phenomenon is intermittent CBTT (Fig. 4). In this case a trajectory demonstrates intermittent CBTT and chaotic behavior. The time spent in CBTT may vary (see lower figure in Fig. 4). This behavior is similar to the behavior of chaotic trajectories of regular dynamics in the presence of sticky islands. The difference is that in the latter case the trajectories spend some time in the neighborhood of the islands of regular motion and in the former case there are no stable regular trajectories.

The analysis presented in this section is far from being exhaustive, but it is enough to make some conclusions presented in the following Sec. IV. More detailed analysis of the properties of fractional maps can be found in [38], [39], [40], [41], [42], [43], [44].
IV. Conclusion

The thing that interests the author of this paper the most is how general properties of systems with power-law memory (non-uniqueness of solutions, cascade of bifurcations type trajectories, etc.) reveal themselves in nature, especially in biological systems which are almost always systems with memory (see Sec. I). Natural systems are very complicated with many complex processes going on all the time, but looking at a system as a whole (a neuron, the brain, the heart, a human being, a society, ...) one may consider it as a system with memory (in many cases power-law) and many periodic processes. In many circumstances one may separate the most important process and then a fractional differential equation with periodic kicks could be an appropriate model for the considered system.

Certainly, there is a lot of noise and randomness in a natural system. How will it modify the system’s behavior? This is one of the problems to solve. The conjecture is that up to a certain level of noise trajectories would wander in the proximity of the unperturbed solutions and cascade of bifurcations type trajectories would turn into sets of bands in which a system will evolve.

Most of the beautiful features one could expect considering biological systems with memory, including cascade of bifurcations type behavior, would exist only if the system is sufficiently non-linear. Non-linearity is characterized by the value of a certain parameter. If the value of this parameter is small, the system is stable. When the value reaches a critical point a cascade of bifurcations type behavior appears: system first exists in the vicinity of a stable fixed point; after some time (which depends on the value of the non-linearity parameter and memory) it bifurcates and may stay in a new period two state forever or may bifurcate again and so on. The process may end in chaos. For certain values of the system parameters the system may reemerge again in a stable state after some random time. Could this behavior be related to the evolution of chronic diseases, epileptic seizures, and evolution of the human society? The authors believe that this question is worthy of investigation.

In this paper the author considered the notion of the universal fractional $\alpha$-families of maps depending on the non-linearity parameter $K$ and the memory parameter $\alpha > 0$ which is the order of the fractional derivative in the corresponding non-linear fractional differential equation describing a system experiencing periodic kicks. $\alpha$-families of maps correspond to a very general form of multi-dimensional non-linear maps with power-law memory in which the present state at time $t$ depends on the previous state at time $t_1$ with a weight proportional to $(t - t_1)^{\alpha-1}$.

Two important implementations of this notion are the Standard and Logistic $\alpha$-families of maps. Behavior of these maps is investigated for $\alpha \in [0, 2]$. This value of the memory parameter is relevant to the systems with memory which appear in biology and psychology. The author shows that considered maps with memory exhibit some universality in the dependence of their properties on $K$ and $\alpha$. Phase space of the considered maps, depending on the parameters, may contain periodic sinks, attracting slow diverging trajectories, attracting accelerator mode trajectories, chaotic attractors, and cascade of bifurcations type trajectories whose properties are different from properties of attractors in regular dynamical systems [38], [39], [40], [41], [42], [43], [44].

All natural systems are open systems. An example of an open system with memory, fractional dissipative Standard Map, was considered in [29]. The authors of this paper show that open systems with memory demonstrate more complicated properties and more complicated cascade of bifurcations type trajectories than corresponding conservative systems.

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