Einstein’s Energy-Free Gravitational Field

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Abstract

We show that Einstein’s gravitational field has zero energy, momentum, and stress. This conclusion follows directly from the gravitational field equations, in conjunction with the differential law of energy-momentum conservation $T^\mu_\nu = 0$. Einstein rejected this conservation law despite the fact that it is generally covariant. We trace his rejection to a misapplication of Gauss’ divergence formula. Finally, we derive the formula which pertains to energy-momentum conservation, viz., $\int e_\mu \sqrt{-g} T^{\mu \nu} dV_\nu = \int e_\mu T^{\mu \nu}_\nu \sqrt{-g} d^4x$. 
1. Einstein’s rejection of the conservation law $T^\mu_\nu = 0$

When he was faced with the equation

$$T^\mu_\nu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \sqrt{-g} T^\mu_\nu + \Gamma^\mu_\nu_\lambda T^\nu_\lambda = 0$$  \hspace{1cm} (1)

Einstein responded as follows: the term $\Gamma^\mu_\nu_\lambda T^\nu_\lambda$ “shows that laws of conservation of momentum and energy do not apply in the strict sense for matter alone” [1]. Quotations from the literature include Ref. [2]: the equations $T^\mu_\nu = 0$ “are not what can properly be called conservation laws”; Ref. [3]: the equation $T^\mu_\nu = 0$ “does not generally express any conservation law whatever.” These statements (and many others) constitute a very forceful rejection of the generally covariant law.

2. Proof of the conservation law $T^\mu_\nu = 0$

In the theory of special relativity, conservation of energy-momentum is expressed by the Lorentz covariant equation

$$\frac{\partial T^\mu_\nu}{\partial x^\nu} = 0$$  \hspace{1cm} (2)

Here, it is understood that flat rectangular coordinates $x^\mu = (x^0, x, y, z)$ are being used. Suppose, instead, that we choose ordinary flat spherical coordinates, $x^\mu = (x^0, r, \theta, \phi)$. What will be the law of conservation in the new coordinate system? To answer this question, we begin with equation (2) and substitute the transformed quantities

$$T^\mu_\nu = \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} T^\alpha_\beta$$  \hspace{1cm} (3)

$$\frac{\partial}{\partial x^\nu} = \frac{\partial x^\alpha}{\partial x^\nu} \frac{\partial}{\partial x^\alpha}$$  \hspace{1cm} (4)

We then make use of

$$\Gamma^\mu_\nu_\lambda = \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x^\lambda} \Gamma^\beta_\gamma + \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial^2 x^\beta}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\lambda}{\partial x^\nu}$$  \hspace{1cm} (5)

$$\frac{1}{\sqrt{-g'}} \frac{\partial \sqrt{-g'}}{\partial x^\lambda} = \Gamma^\alpha_\alpha_\lambda$$  \hspace{1cm} (6)
and arrive at the equation

\[ \frac{1}{\sqrt{-g'}} \partial \sqrt{-g'} T^{\mu \nu'} + \Gamma_{\nu \lambda'}^{\mu} T^{\nu \lambda'} = 0 \]  \hspace{1cm} (7)

This proves the differential law of energy-momentum conservation in the spherical coordinate system. Because this law is generally covariant, it must hold true for all systems of coordinates, flat or curved.

3. The energy-free gravitational field

Einstein’s gravitational field equations are

\[ R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R = \kappa T^{\mu \nu} \]  \hspace{1cm} (8)

\( T^{\mu \nu} \) is the stress-energy-momentum tensor of matter and electromagnetism. The covariant divergence of the left-hand side is identically zero, therefore

\[ T^{\mu \nu} = 0 \]  \hspace{1cm} (9)

This equation means that the energy-momentum of matter and electromagnetism is conserved, at all space-time points. In other words, there is no exchange of energy-momentum with the gravitational field. We conclude that the gravitational field has no energy, momentum, or stress \([4,5]\).

4. Why was the covariant law rejected in the past?

(a) In special relativity, the flow of energy-momentum is described by means of Gauss’ divergence formula

\[ \oint_{\partial R} R^{\mu \nu} \sqrt{-g} T^{\mu \nu} d^3V = \int_{\partial R} \frac{\partial T^{\mu \nu}}{\partial x^\nu} d^4x \]  \hspace{1cm} (10)

The left-hand side is a closed surface integral over the boundary of an infinitesimal region \( R \); the right-hand side is an integral over the interior of that region. Energy-momentum is conserved in region \( R \), if \( \partial T^{\mu \nu}/\partial x^\nu = 0 \).

(b) In going over to curvilinear coordinates (flat or curved), it was recognized that the surface elements \( d^3V \) must be multiplied by \( \sqrt{-g} \) in order to form true vector components. In this case, Gauss’ theorem yields

\[ \oint_{\partial R} \sqrt{-g} T^{\mu \nu} d^3V = \int_{\partial R} \frac{\partial \sqrt{-g} T^{\mu \nu}}{\partial x^\nu} d^4x \]  \hspace{1cm} (11)
This formula was used in the past to describe energy-momentum flow. It focussed attention upon the ordinary divergence, $\partial \sqrt{-g} T^{\mu\nu}/\partial x^\nu$, and thus gave rise to a non-covariant law of conservation. However, the expression $\sqrt{-g} T^{\mu\nu} d^3V_\nu$ is a vector component with index $\mu$, and its value depends upon the arbitrary choice of coordinates. Therefore, it cannot fully represent any physical quantity; formula (11) as it stands has no physical meaning.

(c) By introducing the system of basis vectors $e_\mu$, we obtain an expression which does not depend upon the choice of coordinates:

$$e_\mu \sqrt{-g} T^{\mu\nu} d^3V_\nu = e_\mu' \sqrt{-g'} T'^{\mu\nu'} d^3V'_{\nu'}$$

(12)

This is the energy-momentum vector [6]. We now form the vector sum

$$\oint_R e_\mu \sqrt{-g} T^{\mu\nu} d^3V_\nu = \int_R \left\{ e_\mu \frac{\partial \sqrt{-g} T^{\mu\nu}}{\partial x^\nu} + (\nabla_\nu e_\mu) \sqrt{-g} T^{\mu\nu} \right\} d^4x$$

(13)

over the infinitesimal region $R$. The second term on the right is due to the basis vectors, which change in magnitude and direction from point to point. The rate of change is defined in terms of connection coefficients $\Gamma^\mu_{\nu\lambda}$ [6]

$$\nabla_\nu e_\mu = e_\lambda \Gamma^\lambda_{\mu\nu}$$

(14)

Substitute this expansion, then factor $e_\mu$ and $\sqrt{-g}$, in order to obtain the formula

$$\oint_R e_\mu \sqrt{-g} T^{\mu\nu} d^3V_\nu = \int_R \left\{ e_\mu \left\{ \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} T^{\nu\lambda} \right\} \right\} \sqrt{-g} d^4x$$

$$= \int_R e_\mu T^{\mu\nu}_{\nu'} \sqrt{-g} d^4x$$

(15)

We conclude that energy-momentum is conserved in region $R$, if $T^{\mu\nu}_{\nu'} = 0$ [7,8].

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\(^1\) Equation (13) is not Gauss’ formula. The left-hand side is a vector sum over the closed boundary of region $R$. The right-hand side involves the rate of change of vectors $e_\mu$, as well as functions $\sqrt{-g} T^{\mu\nu}$. Gauss’ formula applies to functions, exclusively; it can therefore be extended to cover a finite region. Equation (13) is confined to the infinitesimal region $R$; it yields a differential law of conservation.
5. Summary

(a) The differential law of conservation, $T^{\mu\nu} = 0$, has been established:

(1) by substituting flat curvilinear coordinates into the Lorentz covariant law,
(2) by deriving the vector divergence formula for energy-momentum;
(3) it is derived elsewhere by variation of the action under uniform displacement in space-time [7].

(b) The principle consequence of this law is that Einstein’s gravitational field has zero energy, momentum, and stress.

(c) The covariant law was rejected in the past.

(d) This rejection was based upon a physically meaningless application of Gauss' divergence formula.

References

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