Regularity results for local solutions
to some anisotropic elliptic equations

G. di Blasio∗– F. Feo†– G. Zecca ‡

Abstract
In this paper we study the higher integrability of local solutions for a class of anisotropic equations with lower order terms whose growth coefficients lay in Marcinkiewicz spaces. A condition for the boundedness of such solutions is also given.

Mathematics Subject Classifications: 35J60, 35B65
Key words: Anisotropic nonlinear equations, higher integrability, boundedness of solutions

1 Introduction
Our aim is to obtain regularity results for the following class of anisotropic elliptic equations
\[ \sum_{i=1}^{N} \partial_{x_i}(A_i(x, \nabla u) + B_i(x,u)) = \sum_{i=1}^{N} \partial_{x_i}(|F_i|^{p_i-2}F_i) \quad \text{in } \Omega, \] (1.1)
where \( \Omega \) is a domain of \( \mathbb{R}^N \), \( N > 2 \), \( p_i > 1 \) for every \( i = 1, \ldots, N \) with \( 1 < \bar{p} < N \), denoting by \( \bar{p} \) the harmonic mean of \( p_1, \ldots, p_N \), i.e.
\[ \frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}. \] (1.2)
Throughout this paper, we make the following assumptions for any \( i = 1, \ldots, N \)
\[(H1) \quad A_i : \Omega \times \mathbb{R}^N \to \mathbb{R} \text{ is a Carathéodory function that satisfies} \]
\[ |A_i(x, \xi)| \leq \beta_i |\xi_i|^{p_i-1} \] (1.3)
\[ \alpha \sum_{i=1}^{N} |\xi_i|^{p_i} \leq \sum_{i=1}^{N} (A_i(x, \xi) \xi_i) \] (1.4)

∗Dipartimento di Matematica e Fisica, Università degli Studi della Campania “L. Vanvitelli”, Viale Lincoln, 5 - 81100 Caserta, Italy. E-mail: giuseppina.diblasio@unicampania.it
†Dipartimento di Ingegneria, Università degli Studi di Napoli “Parthenope”, Centro Direzionale Isola C4, 80143 Napoli, Italy. E-mail: filomena.feo@uniparthenope.it
‡Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli Federico II, Complesso Universitario di Monte S. Angelo, Via Cintia, 80126, Napoli, Italia. E-mail: g.zecca@unina.it
for a.e. \( x \in \Omega \) and for any vector \( \xi \) in \( \mathbb{R}^N \), where \( 0 < \alpha \leq \beta_i \) are constants;

\( (H2) \) \( B_i : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that

\[
|B_i(x,s)| \leq b_i(x)|s|^{\frac{p_i}{p_i'}} \tag{1.5}
\]

for a.e. \( x \in \Omega \) and for every \( s \in \mathbb{R} \), where \( b_i \) is a non-negative function in the Marcinkiewicz space \( L_{\frac{p_i'}{p_i}, \infty}^p(\Omega) \), with \( p' = \frac{p}{p-1} \);

\( (H3) \) \( F_i : \Omega \to \mathbb{R} \) is such that \( F_i \in L_{loc}^p(\Omega) \).

Our model operator is

\[
L(u) = \sum_{i=1}^N \partial_{x_i} \left( |\partial_{x_i} u|^{p_i-2} \partial_{x_i} u + b_i(x)|u|^{\frac{p_i}{p_i'}-1} u \right),
\]

with \( b_i \) as in \( (H2) \). The anisotropy of \( L \) is due to different power growths with respect to the partial derivatives of the unknown \( u \) and it coincides with the so-called pseudo-Laplacian operator when \( b_i = 0 \) and \( p_i = p \) for \( i = 1, \ldots, N \).

Let us point out that term anisotropy is used in various scientific disciplines and could have a different meaning when it is related to equations as well. The interest in anisotropic problems has deeply increased in the last years and many results in different directions have been obtained. We quote a list of references that is obviously not exhaustive and we refer the reader to references therein to extend it: [1, 2, 3, 4, 7, 9, 10, 12, 13, 14, 15, 18, 19, 25, 24].

Several regularity results depending on the summability of datum are well-known in literature for isotropic counterpart of (1.1)

\[
div \left( |\nabla u|^{p-2} \nabla u + b(x)|u|^{p-2} u \right) = div \left( |F|^{p-2} F \right) \quad \text{in } \Omega,
\]

where \( p > 1 \), \( b \) and \( F \) are vector fields with suitable summability. In the isotropic case for linear equations, assuming the coefficient of the lower order term in suitable Lebesgue spaces, Stampacchia in [28] proves that if the datum \( F \) belongs to \( (L^r(\Omega))^N \) with \( 2 < r < N \), then the solution \( u \) belongs to \( L^r(\Omega) \). Otherwise when \( r > N \) it follows that the solution \( u \) is bounded. Similar results have been obtained also for isotropic nonlinear operators taking the coefficient of the lower order term \( b \) in Lebesgue spaces in [3] and in the Marcinkiewicz spaces in [6, 16, 21] and [22]. In this paper we prove such kind of regularity results for anisotropic equation (1.1) dealing with local solution, whose definition is recalled in what follows.

**Definition 1.1** If \( F_i \in L_{loc}^{p_i}(\Omega) \) for \( i = 1, \ldots, N \), we say that \( u \in W_{loc}^{1,p}(\Omega) \) is a local solution to (1.1) provided

\[
\sum_{i=1}^N \int_{\Omega} \left( A_i(x, \nabla u) + B_i(x, u) \right) \partial_{x_i} \varphi \, dx = \sum_{i=1}^N \int_{\Omega} \left( |F_i|^{p_i-2} F_i \right) \partial_{x_i} \varphi \, dx \tag{1.6}
\]

\( \forall \varphi \in C_0^\infty(\Omega) \).
For the definition of the anisotropic Sobolev space $W^{1,\vec{p}}_{loc}(\Omega)$ we refer to Section 4.

To give an idea of our result, let us consider, for simplicity, equation (1.1) without the lower order terms, i.e. $B_i \equiv 0$. When the datum $F_i \in L^{p_i}_{loc}(\Omega)$ for $i = 1,\ldots,N$ with $\bar{p} < N$, a local solution $u$ belongs to $L^{\bar{p}}_{loc}(\Omega)$, where

$$\bar{p}^* = \frac{N\bar{p}}{N - \bar{p}},$$

(1.7)
as suggested by the anisotropic imbedding (see Section 4). Otherwise if $F_i \in L^{r_i}_{loc}(\Omega)$ with $r_i > p_i$ for $i = 1,\ldots,N$, we expect that the summability of $u$ improves. In order to analyze the higher summability of $u$ we consider the following minimum

$$\mu = \min_i \left\{ \frac{r_i}{p_i} \right\},$$

(1.8)

first introduced in [8], and we are able to prove that $u \in L^{(\bar{p}_\mu)^*}_{loc}(\Omega)$ if $\bar{p}\mu < N$. Then the regularity of $u$ depends on $\mu$.

These regularity results are stated in Theorem 2.1 taking into account the lower order terms under the assumption that coefficients $b_i$ have a suitable distance to $L^\infty$ sufficiently small for $i = 1,\cdots,N$.

The principal difficulties are due to the anisotropy, to the managing of local solutions and to the presence of the lower order terms. In our proof, one of the key tool is a new anisotropic Sobolev inequality in Lorentz spaces that involves the product of different powers of two functions (see Proposition 4.1). This inequality naturally appears in the anisotropic framework, it is of independent interest and gives an estimate in terms of the norm of the geometric mean of the partial derivatives instead of the geometric mean of the norms of the partial derivatives as usual in literature (see (4.2)).

Moreover in Theorem 2.4 we give a sufficient condition in terms of $\mu$ for the boundedness of the solutions to (1.1).

We also make comments on the regularity of weak solutions of Dirichlet problems in a bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. In this case when $F_i \in L^{p_i}(\Omega)$ for $i = 1,\ldots,N$ with $\bar{p} < N$ a weak solution $u$ belongs to $L^{p_\infty}(\Omega)$, where $p_\infty = \max\{\bar{p}^*, p_{\max}\}$ with $p_{\max} = \max\{p_1, \cdots, p_N\}$ and $\bar{p}$ defined as in (1.7). It is evident that the regularity of $u$ depends on how much the anisotropy is concentrated, so the situation is more diversified than the isotropic case. Otherwise if $F_i \in L^{r_i}(\Omega)$ with $r_i > p_i$ for $i = 1,\ldots,N$ and $\mu\bar{p} < N$, we get $u \in L^s(\Omega)$ with $s = \max\{(\bar{p}\mu)^*, \mu p_{\max}\}$ and the regularity again depends on how much $p_i$ are spread out and on $\mu$.

The paper is organized as follows. The main results are stated in Section 2. In Section 3 we recall same properties of Lorentz spaces and in Section 4 we introduce the anisotropic Sobolev spaces and the related inequalities. The proofs of main results (Theorem 2.1 and Theorem 2.4) are given in Section 5 and 6. We conclude the paper with a technical lemma contained in the Appendix.

## 2 Main results

The first result of this paper states the regularity of local solutions to (1.1) in terms of the summability of $F_i$, replacing the usual smallness assumption on the norm of the coefficients
of the lower order terms with a weaker one given in terms of the distance of a function $f \in L^{p,\infty}(\Omega)$ to $L^{\infty}(\Omega)$, denoted by $\text{dist}_{L^{p,\infty}(\Omega)}(f, L^{\infty}(\Omega))$ and defined by (3.11) in Section 3.

**Theorem 2.1** Assume that (1.3) - (1.5) are fulfilled. Let $1 < \bar{p} < N$, $\bar{p}^* > p_{\text{max}}$ and let $r_1, \ldots, r_N$ be such that

$$1 < \mu < \frac{N}{\bar{p}}, \quad (2.1)$$

where $\mu$ is defined in (1.8). There exists a positive constant $d = d(\bar{r}, N, \alpha, \bar{p})$ such that

$$\max_i \left\{ \text{dist}_{L^{N\mu_i}/\infty(\Omega)}(b_i, L^\infty(\Omega)) \right\} < d \quad (2.2)$$

and $u \in W_{\text{loc}}^{1,\bar{p}}(\Omega)$ is a local solution to (1.1) with $F_i \in L_{\text{loc}}^{r_i}(\Omega)$ for $i = 1, \ldots, N$, then $|u|_{\bar{r}_i(N-\bar{p})} + 1 \in W_{\text{loc}}^{1,\bar{p}}(\Omega)$. In particular $u \in L^s(\Omega)$, where

$$s = (\mu\bar{p})^* = \frac{N\mu\bar{p}}{N - \mu\bar{p}} \quad (2.3)$$

Note that if $\mu$ goes to $\left(\frac{N}{\bar{p}}\right)^+$, then $s \to \infty$ as expected. In the isotropic case condition $\bar{p}^* > p_{\text{max}}$ does not turn up and assumption (2.1) reads as isotropic assumption $r < N$ considering the datum $F = (F_1, \ldots, F_N) \in (L_{\text{loc}}^{r_i}(\Omega))^N$ (see [22] and [16]).

Some comments on the case $p_{\text{max}} \geq \bar{p}$ are contained in Remark 6.1, Remark 5.3 and Remark 6.1 where weak solutions for Dirichlet problems are taking into account.

It is clear that in Theorem 2.1 regularity of $u$ is related to $\mu p_1, \ldots, \mu p_N$. Indeed if $\bar{r}$ and $\bar{q}$ are two different vectors such that $\min \{ \frac{r_i}{p_i} \} = \min \{ \frac{q_i}{p_i} \} = \mu$, then the solution $u \in L_{\text{loc}}^{(\mu\bar{p})^*}(\Omega)$ if either $F_i \in L_{\text{loc}}^{r_i}(\Omega)$ or $F_i \in L_{\text{loc}}^{q_i}(\Omega)$, as in the case $F_i \in L_{\text{loc}}^{\mu p_i}(\Omega)$ for $i = 1, \ldots, N$.

A standard approach to treat the presence of lower order terms is to require a smallness on the norm of $b_i$, which is avoid using assumption (2.2), firstly introduced by [21] in the isotropic case. We stress that the value of $d$ in (2.2) really depends on $\bar{r}$ (see Example 3.1 in [22]).

For example if $\Omega$ is the ball centered at the origin with radius $R > 0$, we can take

$$b_i(x) = \gamma_i |x|^{-\frac{r_i}{\bar{p}}} + h_i(x)$$

with $\gamma_i > 0$ and $h_i \in L^\infty(\Omega)$. It is not difficult to see that $b_i \in L^{N\mu_i/\bar{r}_i}(\Omega)$ and verifies (2.2) for suitable $\gamma_i$. We emphasize that condition (2.2) is trivially satisfied whenever $b_i$ belongs to a Lebesgue space or to any Lorentz space contained in $L^{N\mu_i/\bar{r}_i}(\Omega)$, $i = 1, \ldots, N$. Then as a corollary of Theorem 2.1 we have immediately the following result.

**Corollary 2.2** Assume that (1.3) - (1.5) are fulfilled. Let $1 < \bar{p} < N$, $\bar{p}^* > p_{\text{max}}$, $1 < q < \infty$,

$$b_i \in L^{\frac{N\mu_i}{\bar{r}_i}}(\Omega) \quad i = 1, \ldots, N$$

and let $r_1, \ldots, r_N$ be such that (2.1) holds. If $F_i \in L_{\text{loc}}^{r_i}(\Omega)$ for $i = 1, \ldots, N$, then any local solution $u \in W_{\text{loc}}^{1,\bar{p}}(\Omega)$ to (1.1) belongs to $L^s_{\text{loc}}(\Omega)$, where $s$ is defined in (2.3).
Without lower order terms in [8] the authors have proved that the boundedness of a weak solution of Dirichlet problems is guaranteed under the assumption
\[ \mu > \frac{N}{p}, \]
where \( \mu \) is defined in (1.8). However if \( B_i \not\equiv 0 \) for \( i = 1, \cdots, N \) the boundedness is not assured assuming that (2.2) is in force as showed in Example 4.8 of [21] (when \( p_i = 2 \) for \( i = 1, \cdots, N \)). The smallness of \( \|b_i\|_{\frac{p_i}{p_i-1}}' \) for \( i = 1, \cdots, N \) neither is sufficient to get boundedness, as the following example shows.

**Example 2.3** Let us consider the following Dirichlet problem
\[
\begin{aligned}
\text{div}(\nabla u + bu) &= \text{div} F \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \( N > 2 \), \( \Omega = B(0,1) \) is the unit ball of \( \mathbb{R}^N \), \( b(x) = \frac{x}{|x|^2} \), \( F(x) = ((2 + \gamma)x_1, \cdots, (2 + \gamma)x_N) \) and \( \gamma > 0 \). We stress that the solution \( u(x) = |x|^{-\gamma} - |x|^2 \) is unbounded even if \( \|b\|_{L_{N,\infty}} = \gamma \omega_1^{1/N} \) can be small as we want taking \( \gamma \) small enough.

In order to obtain an \( L^\infty \)-regularity result we require extra summability on the coefficients of lower order terms.

**Theorem 2.4** Assume that (1.3)-(1.5) holds. Let \( 1 < \tilde{p} < N, \tilde{p}^* > p_{\max} \), \( b_i \in L_{loc}^{\frac{r_i}{r_i-1}}(\Omega) \) and \( F_i \in L_{loc}^r(\Omega) \) for \( i = 1, \cdots, N \) with \( r_1, \cdots, r_N \) satisfying (2.4). Then any local solution \( u \in W_{1,\tilde{p}}^{1,\tilde{p}}(\Omega) \) to equation (1.1) is locally bounded.

## 3 Some properties of Lorentz spaces

In this section we recall the definitions of Lorentz spaces and their properties (see [27] for more details). There are various definitions of Lorentz spaces but all of them manage the notion of rearrangement.

Here we assume that \( \Omega \subset \mathbb{R}^N, N > 2 \) is an open set. Let \( v \) be a measurable function defined in \( \Omega \). The distribution function \( \mu_v : [0, +\infty) \rightarrow [0, +\infty) \) of \( v \) is defined as
\[ \mu_v(\lambda) := \{ x \in \Omega : |v(x)| > \lambda \} \quad \text{for } \lambda \geq 0. \]
The decreasing rearrangement of \( v \) is the map \( v^* : [0, +\infty) \rightarrow [0, +\infty] \) given by
\[ v^*(s) = \sup \{ t \geq 0 : \mu_v(t) \geq s \} \quad \text{for } s \geq 0. \]
(3.1)
By \( v^{**} \) we denote the maximal function of \( v^* \), i.e.
\[ v^{**}(s) = \frac{1}{s} \int_0^s v^*(\sigma) \, d\sigma \quad \text{for } s > 0. \]
For $1 \leq p < +\infty$ and $0 < q \leq +\infty$ the Lorentz space $L^{p,q}(\Omega)$ consists in all measurable functions $v : \Omega \to \mathbb{R}$ such that

$$
\|v\|_{L^{p,q}(\Omega)} = \begin{cases} 
\left( \frac{1}{\Omega} \int_0^{[\Omega]} \left[ t^{1/p} v^*(t) \right]^q \frac{dt}{t} \right)^{1/q} & \text{if } q < +\infty, \\
\sup_{t \in (0,\Omega]} \left[ t^{1/p} v^*(t) \right] & \text{if } q = +\infty.
\end{cases}
$$

is finite. Notice that in general $\| \cdot \|_{L^{p,q}(\Omega)}$ is a quasinorm, but replacing $v^*$ with $v^{**}$ one obtains an equivalent norm and $L^{p,q}(\Omega)$ becomes a Banach space. We observe that it holds

$$
\|v^s\|_{L^{p,q}} = \|v\|_{L^{p,sq}}^s \quad \text{for } 0 < s < +\infty.
$$

The Lorentz spaces are a refinement of Lebesgue spaces. Indeed for $p = q$, $L^{p,p}(\Omega)$ reduces to the standard Lebesgue space $L^p(\Omega)$, and $L^{p,\infty}(\Omega)$ is also known as Marcinkiewicz space $\mathcal{M}^p(\Omega)$, or weak-$L^p(\Omega)$. If $\Omega$ is bounded, such inclusions follow

$$
L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,p}(\Omega) \equiv L^p(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega)
$$

for $1 < q < p < r < +\infty$, with continuous injections. We observe that $L^{p,\infty}(\Omega) \subset L^p(\Omega)$. For example, if $\Omega \subset \mathbb{R}^N$ contains the origin, the function $v(x) = |x|^{-\frac{N}{p}} \notin L^p(\Omega)$ but $v \in L^{p,\infty}(\Omega)$ with $\|v\|_{L^{p,\infty}} = \omega_N$, where $\omega_N$ stands for the Lebesgue measure of the unit ball of $\mathbb{R}^N$.

Now we introduce a suitable characterization of Lorentz spaces that is useful to generalize Sobolev embedding theorems in anisotropic framework.

Let $k > 1$ fixed, for any measurable function $v$ defined in $\Omega$ such that for every $\lambda > 0$

$$
\mu_v(\lambda) < +\infty,
$$

we choose the levels $a_n^v \geq 0$, $n \in \mathbb{Z}$, such that

$$
|\{x \in \Omega : |v(x)| > a_n^v\}| \leq k^{-n} \leq |\{x \in \Omega : |v(x)| \geq a_n^v\}|
$$

or equivalently

$$
a_n^v \in [v^*((k^{-n})^+); v^*((k^{-n})^-)]
$$

where $v^*$ is the decreasing rearrangement of $v$ defined in (3.1) and where plus and minus denote the right and the left-hand limit respectively.

Now we consider a particular sequence of functions $\omega_n(v)$ defined by

$$
\omega_n(v) = \begin{cases} 
0 & \text{for } 0 \leq |v| \leq a_{n-1}^v, \\
|v| - a_{n-1}^v & \text{for } a_{n-1}^v < |v| \leq a_n^v, \\
a_n^v - a_{n-1}^v & \text{for } |v| > a_n^v,
\end{cases}
$$

with the levels $a_n^v$ defined for $n \in \mathbb{Z}$ as in (3.4). We observe that

$$
\omega_n(v) \leq (a_n^v - a_{n-1}^v) \chi_{\{|v| > a_{n-1}^v\}} \quad \text{and} \quad \omega_n(v) \geq (a_n^v - a_{n-1}^v) \chi_{\{|v| \geq a_n^v\}},
$$

where $\chi$ is the characteristic function, so one finds for $0 < r < \infty$, that

$$
k^{-\frac{n}{r}}(a_n^v - a_{n-1}^v) \leq \left( \int_{\mathbb{R}^N} |\omega_n(v)|^r dx \right)^{\frac{1}{r}} \leq k^{-\frac{n-1}{r}}(a_n^v - a_{n-1}^v).
$$
Recalling that a sequence \(\{a_n\}_{n \in \mathbb{Z}}\) belongs to \(l^q(\mathbb{Z})\) for \(q > 1\) iff \(\sum_{n \in \mathbb{Z}} |a_n|^q < +\infty\), we are in position to state the following announced characterization of Lorentz spaces.

**Proposition 3.1** Let be \(1 < p < \infty\) and \(1 \leq q \leq \infty\). For any \(v\) extended by \(0\) outside \(\Omega\) satisfying (3.3), one has

\[
v \in L^{p,q}(\mathbb{R}^N) \quad \text{is equivalent to} \quad k^{-\frac{n}{p}}(a_{n+1}^v - a_n^v) \in l^q(\mathbb{Z}) \quad (3.7)
\]

and in particular

\[
\|v\|_{L^{p,q}(\Omega)} = \|v\|_{L^{p,q}(\mathbb{R}^N)} \leq C_1 \left( \sum_{n \in \mathbb{Z}} |a_n^v|^q k^{-\frac{n}{p}} \right)^{1/q} \leq C_2 \left( \sum_{n \in \mathbb{Z}} [a_n^v - a_{n-1}^v]^q k^{-\frac{n}{p}} \right)^{1/q}, \quad (3.8)
\]

with \(C_1, C_2\) are positive constants independent on \(v\) and \(a_n^v\) is defined in (3.3).

**Proof.** Equivalence (3.7) is contained in [30], Proposition 3. We prove (3.8). Since the rearrangement is non-increasing, we have

\[
\|v\|_{L^{p,q}(\Omega)}^q = \int_{k^{-n}}^{k^{-(n+1)}} \left[ s^{1/p}v^*(t) \right]^q \frac{ds}{s} \leq \sum_{n \in \mathbb{Z}} k^{-\frac{n}{p}} \left[ v^*((k^{-n})^+) \right]^q \int_{k^{-n}}^{k^{-(n+1)}} \frac{ds}{s}
\]

\[
= \sum_{n \in \mathbb{Z}} |v^*((k^{-n})^+)| k^{-\frac{n-1}{p}} \log k \leq \log k \sum_{n \in \mathbb{Z}} (k^{-\frac{n-1}{p}} a_n^v)^q = k^{-\frac{1}{p}} \log k \sum_{n \in \mathbb{Z}} (k^{-\frac{n}{p}} a_n^v)^q.
\]

Now following the idea of Tartar [30], we put \(b_n = a_n^v - a_{n-1}^v\). We observe that \(a_n^v = \sum_{m=-\infty}^{n} b_m^v\) since the measure of the level sets is finite. We have that

\[
k^{-\frac{n}{p}} a_n^v = \sum_{m=-\infty}^{n} k^{\frac{m-n}{p}} (k^{-\frac{m}{p}} b_m^v)
\]

and by the convolution Young inequality, we deduce the following inequality

\[
\left( \sum_{n \in \mathbb{Z}} |k^{-\frac{n}{p}} a_n^v|^q \right)^{\frac{1}{q}} \leq \left( \sum_{m \leq 0} k^{\frac{m}{p}} \right) \left( \sum_{n \in \mathbb{Z}} |k^{-\frac{n}{p}} (a_n^v - a_{n-1}^v)|^q \right)^{\frac{1}{q}}
\]

\[
= k^{-\frac{1}{p}} \left( \sum_{m \leq 0} k^{\frac{m}{p}} \right) \left( \sum_{n \in \mathbb{Z}} |k^{-\frac{n}{p}} (a_n^v - a_{n-1}^v)|^q \right)^{\frac{1}{q}}. \quad (3.10)
\]

Combining (3.10) and (3.9), inequality (3.8) follows with \(C_2 = (\log k)^{\frac{1}{q}} \sum_{m \leq 0} k^{\frac{m}{p}}\). \(\blacksquare\)

We remark that \(L^\infty(\Omega)\) is not dense in \(L^{p,\infty}(\Omega)\), \(p \in ]1, +\infty[\). We define the distance of a given function \(f \in L^{p,\infty}(\Omega)\) to \(L^\infty(\Omega)\) as

\[
\text{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) = \inf_{g \in L^\infty(\Omega)} \|f - g\|_{L^{p,\infty}(\Omega)}. \quad (3.11)
\]
Note that, since $\| \cdot \|_{p,\infty}$ is not a norm, $\text{dist}_{L^p,\infty}(\Omega)$ is just equivalent to a metric. In [11] is proved that

$$\text{dist}_{L^p,\infty}(\Omega) = \lim_{M \to +\infty} \| f - T_M f \|_{L^p,\infty}(\Omega),$$

where the truncation at level $M > 0$ is defined as

$$T_M(y) = \frac{y}{|y|} \min\{|y|, M\}.$$ (3.12)

At the end of this section we recall the following useful lemma.

**Lemma 3.2** (see [26, page 43]) Let $X$ be a rearrangement invariant space and let $0 \leq \theta_i \leq 1$ for $i = 1, \ldots, M$, such that $\sum_{i=1}^{M} \theta_i = 1$, then

$$\left\| \prod_{i=1}^{M} |f_i|^{\theta_i} \right\|_{X} \leq \prod_{i=1}^{M} \| f_i \|_{X}^{\theta_i} \quad \forall f_i \in X.$$ (3.13)

### 4 Anisotropic inequalities

Let $\vec{p} = (p_1, p_2, \ldots, p_N)$ with $p_i > 1$ for $i = 1, \ldots, N$ and $\Omega$ be a bounded open set. As usual the anisotropic Sobolev space is the Banach space defined as

$$W^{1,\vec{p}}(\Omega) = \{ u \in W^{1,1}(\Omega) : \partial x_i u \in L^{p_i}(\Omega), i = 1, \ldots, N \}$$

equipped with

$$\| u \|_{W^{1,\vec{p}}(\Omega)} = \| u \|_{L^1(\Omega)} + \sum_{i=1}^{N} \| \partial x_i u \|_{L^{p_i}(\Omega)}$$

and

$$W^{1,\vec{p}}_{\text{loc}}(\Omega) = \{ u \in W^{1,\vec{p}}(\Omega') : \Omega' \subset \subset \Omega \text{ open set} \}.$$ (4.1)

It is well-known that in the anisotropic setting a Poincaré type inequality holds true (see [13]). Indeed for every $u \in C_0^\infty(\Omega)$ with $\Omega$ a bounded open set with Lipschitz boundary we have

$$\| u \|_{L^{p_i}(\Omega)} \leq C_P \| \partial x_i u \|_{L^{p_i}(\Omega)}, \quad i = 1, \ldots, N$$

for a constant $C_P$ depending on the diameter of $\Omega$. Moreover, for $u \in C_0^\infty(\mathbb{R}^N)$ the following anisotropic Sobolev inequality holds true (see [30])

$$\| u \|_{L^{p,q}(\mathbb{R}^N)} \leq S_N \prod_{i=1}^{N} \| \partial x_i u \|_{L^{p_i}(\mathbb{R}^N)},$$ (4.2)

where $S_N$ is an universal constant and $p = \vec{p}^*$ and $q = \vec{p}$ whenever $\vec{p} < N$, where $\vec{p}$ is defined in (1.2). Using the inequality between geometric and arithmetic mean we can replace the right-hand-side of (4.2) with $\sum_{i=1}^{N} \| \partial x_i u \|_{L^{p_i}}$.

When $\vec{p} < N$ and $\Omega$ is a bounded open set with Lipschitz boundary, the space $W^{1,\vec{p}}_0(\Omega) = C_0^\infty(\Omega) \sum_{i=1}^{N} \| \partial x_i u \|_{L^{p_i}}$ is continued embedding into $L^{q}(\Omega)$ for $q \in [1, p_\infty]$, with $p_\infty = \max\{\vec{p}, p_{\max}\}$ as a consequence of (4.2) and (4.1).

In the following proposition we generalize the anisotropic Sobolev inequality (4.2) to the product of functions.
Proposition 4.1 Let $\alpha_i, \beta_i \geq 0$ (but not both identically zero) and $p_1, \ldots, p_N \geq 1$ be such that $1 \leq \bar{p} < N$. Then for every nonnegative functions $\phi, u \in C_0^\infty(\mathbb{R}^N)$ we have

$$
\left\| \phi^{\alpha} u^{\beta} \right\|_{L^{\bar{p}}} \leq C \left( \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} (\phi^{\alpha_i} u^{\beta_i}) \right\|_{L^{p_i}} \right)^{1/N},
$$

where $\bar{p}^*$ is defined in (1.7) and $\alpha$ and $\beta$ are defined by

$$
\alpha = \frac{1}{N} \sum_{i=1}^{N} \alpha_i, \quad \beta = \frac{1}{N} \sum_{i=1}^{N} \beta_i
$$

and $C$ is a positive constant independent on $u$ and $\phi$.

It is clear that for example for $\beta_i = 1$ the sign assumption on $u$ can be dropped.

We point out that Lemma 3.2 with $X = L^\bar{p}(\Omega)$ and $\theta_i = \frac{p_i}{\bar{p} N}$ yields that

$$
\left\| \phi^{\alpha} u^{\beta} \right\|_{L^{\bar{p}}} \leq C \left( \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} (\phi^{\alpha_i} u^{\beta_i}) \right\|_{L^{p_i}} \right)^{1/N} \forall \phi, u \in C_0^\infty(\mathbb{R}^N)
$$

with a positive constant $C$ independent on $u$ and $\phi$. We emphasize that (4.3) is sharper than (4.4) as the following example shows.

Example 4.2 Let us consider $1 < \bar{p} < N$, $\Omega = \left\{ x : \sum_{i=1}^{N} |x_i|^\theta_i < R \right\}, \phi \equiv 1, \alpha_i \equiv 1$ for $R > 0, \theta_i > 1$ for $i = 1, \cdots, N$. Lemma 7.1 in the Appendix yields $(\prod_{i=1}^{N} \frac{\partial}{\partial x_i} u)^{1/N} \in L^\bar{p}(\Omega)$ taking

$$
\gamma < \sum_{j=1}^{N} \frac{1}{\theta_j \bar{p} N}
$$

and $\frac{\partial}{\partial x_i} u \in L^{p_i}(\Omega)$ taking

$$
\gamma < \sum_{j=1}^{N} \frac{1}{\theta_j p_i} - \frac{1}{\theta_i}
$$

for $i = 1, \cdots, N$. We note that (4.6) implies (4.5) and under latest assumption we get $u \in L^{\bar{p}^*}(\Omega)$, but $u \notin L^{\bar{p}^*+\epsilon}(\Omega)$ for every $\epsilon > 0$ (using again Lemma 7.1).

In order to prove Proposition 4.1 as first step we need the following result.

Lemma 4.3 For every $g_i \in C_0^\infty(\mathbb{R}^N)$, it follows that

$$
\left\| \prod_{i=1}^{N} g_i^{1/N} \right\|_{L^{1^*}} \leq \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} g_i \right\|_{L^{1}}^{1/N}.
$$
\textbf{Proof.} The proof is a generalization to product of functions of the Troisi’s one (see \cite{31} Theorem 1.2). We have, for \( q > 0 \) that will be chosen later, the following inequality
\[
\int_{\mathbb{R}^N} \prod_{i=1}^{N} |g_i|^{q/N} \, dx \leq \int_{\mathbb{R}^N} \prod_{i=1}^{N} \sup_{x_i} |g_i|^{q/N} \, dx.
\]
Since
\[
\int_{\mathbb{R}^N} \prod_{i=1}^{N} \sup_{x_i} |g_i|^{q/N} \, dx \leq \left( \prod_{i=1}^{N} \int_{\mathbb{R}^{N-1}} \left[ \sup_{x_i} |g_i|^{q/N} \right] \, dx' \right)^{N-1},
\]
where \( x = (x, x') \) (see Lemma 4.1 of \cite{20}), we get
\[
\left( \int_{\mathbb{R}^N} \prod_{i=1}^{N} |g_i|^{q/N} \, dx \right)^{N-1} \leq \prod_{i=1}^{N} \int_{\mathbb{R}^{N-1}} |\partial_{x_i} g_i| \, dx.
\]
Now choosing \( q \) such that \( \frac{q(N-1)}{N} - 1 = 0 \), we obtain inequality (4.7).

\textbf{Proof of Proposition 4.1.} Applying inequality (4.7) to the function
\[
g_i = \omega_n(\varphi^{\alpha_i} u^{\beta_i}) + a_{n-1}^{\alpha_i u^{\beta_i}},
\]
we obtain
\[
\left\| \left( \prod_{i=1}^{N} \omega_n(\varphi^{\alpha_i} u^{\beta_i}) + a_{n-1}^{\alpha_i u^{\beta_i}} \right) \right\|^{1/N}_{L^1} \leq \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left[ \omega_n(\varphi^{\alpha_i} u^{\beta_i}) + a_{n-1}^{\alpha_i u^{\beta_i}} \right] \right\|^{1/N}_{L^1},
\]
where \( \omega_n \) and \( a_n \) are defined as in \cite{35} and \cite{34}, respectively. We stress that
\[
a_n^{\alpha_i u^{\beta_i}} = \left( \prod_{i=1}^{N} a_n^{\alpha_i u^{\beta_i}} \right)^{1/N}.
\]
Moreover using the Hardy-Lyttlewood inequality, we have
\[
\left\| \frac{\partial}{\partial x_i} \left[ \omega_n(\varphi^{\alpha_i} u^{\beta_i}) + a_{n-1}^{\alpha_i u^{\beta_i}} \right] \right\|_{L^1} = \left\| \frac{\partial}{\partial x_i} \left[ \omega_n(\varphi^{\alpha_i} u^{\beta_i}) \right] \right\|_{L^1},
\]
\[
= \int_{a_{n-1}^{\alpha_i u^{\beta_i}} < \varphi^{\alpha_i} u^{\beta_i} \leq a_n^{\alpha_i u^{\beta_i}}} \left| \frac{\partial}{\partial x_i} (\varphi^{\alpha_i} u^{\beta_i}) \right| \, dx \leq \int_{0}^{k^{-\alpha_i}} \left| \frac{\partial}{\partial x_i} (\varphi^{\alpha_i} u^{\beta_i}) \right| \ast (s) \, ds.
\]
Let us consider the left-hand side of (4.8). By (3.6) and (4.9) we get
\[
\left\| \left( \prod_{i=1}^{N} \omega_n(\varphi^{\alpha_i} u^{\beta_i}) + a_{n-1}^{\alpha_i u^{\beta_i}} \right) \right\|^{1/N}_{L^1} \geq \prod_{i=1}^{N} \left( a_n^{\alpha_i u^{\beta_i}} \right)^{1/N} \chi_{\left\{ \varphi^{\alpha_i} u^{\beta_i} > a_n^{\alpha_i u^{\beta_i}} \right\}}_{L^1} \geq \prod_{i=1}^{N} \left( a_n^{\alpha_i u^{\beta_i}} \right)^{1/N} k^{-\alpha_i} = a_n^{\alpha_i u^{\beta_i}} k^{-\alpha_i} = a_n^{\alpha_i u^{\beta_i}} k^{-\alpha_i}.
\]
We observe that by (3.2), since \( C \) denotes a positive constant that could change from line to line.

Multiplying for \( k \), elevating to the power \( \pbar \) and summing, we obtain

\[
\sum_{n \in \mathbb{Z}} \left[ a_n^{\alpha \mu \beta} \right]^{- \pbar} k^{- \pbar} \leq C \sum_{n \in \mathbb{Z}} \left( \prod_{i=1}^{N} k^{\alpha_i} \int_0^{k^{- \alpha_i} (1 - 1/\pi)} \left| \frac{\partial}{\partial x_i} (\phi^{\alpha_i} u^{\beta_i}) \right|^* (s) \, ds \right)^{- \pbar/N}.
\]

(4.10)

On the other hand, by Proposition 3.3 with \( p = \pbar \) and \( q = \pbar \), it follows that

\[
\left\| \phi^{\alpha \mu \beta} \right\|_{L^{\pbar} \pbar} \leq C \left( \sum_{n \in \mathbb{Z}} \left[ a_n^{\alpha \mu \beta} \right]^{- \pbar} k^{- \pbar} \right)^{1/\pbar}.
\]

We have, by the previous inequality and (4.10), that

\[
\left\| \phi^{\alpha \mu \beta} \right\|_{L^{\pbar} \pbar} \leq C \left( \sum_{n \in \mathbb{Z}} \left( \prod_{i=1}^{N} \int_0^{k^{- \alpha_i} (1 - 1/\pi)} \left| \frac{\partial}{\partial x_i} (\phi^{\alpha_i} u^{\beta_i}) \right|^* (s) \, ds \right)^{\pbar/N} \right)^{1/\pbar}.
\]

We observe that by (3.2), since \( k > 1 \) it follows that

\[
\left\| \phi^{\alpha \mu \beta} \right\|_{L^{\pbar} \pbar} \leq C \left( \sum_{n \in \mathbb{Z}} \left( \prod_{i=1}^{N} \int_0^{k^{- \alpha_i} (1 - 1/\pi)} \left| \frac{\partial}{\partial x_i} (\phi^{\alpha_i} u^{\beta_i}) \right|^* (s) \, ds \right)^{\pbar/N} \right)^{1/\pbar}.
\]

that is (4.3).

We stress that using inequality (4.7) we can prove the following well-known anisotropic Sobolev inequalities in Lebesgue spaces. For every \( u \in C_0^\infty (\mathbb{R}^N) \) there exists two positive constant \( C_1, C_2 \) independent of \( u \) such that

\[
\left\| u \right\|_{L^{\pbar} \pbar} \leq C_1 \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} u \right\|_{L^{\pbar} \pbar}^{1/N} \quad \text{when } \pbar < N
\]
\[ \|u\|_{L^q} \leq C_2 \left[ \|u\|_{L^{p_0}} + \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} u \right\|_{L^{p_i}}^{1/N} \right] \quad \text{when } p = N \]

for every \( q \in [p_0, \infty) \) and \( p_0 \geq 1 \). Indeed the starting point is to take \( g_i = u^{\sigma_i} \) in (4.7) with \( \sum_{i=1}^{N} \sigma_i = Nt \) for suitable \( t > 0 \).

## 5 Proof of Theorem 2.1

The proof consists in two steps. First we prove our regularity result assuming that \( \|b_i\|_{L^{\infty}} \) are small enough for \( i = 1, \ldots, N \). Then, last assumption is removed thanks to (2.2).

**Step 1. Proof assuming that \( \|b_i\|_{L^{\infty}} \) are small enough.** Let us fix \( \varphi \in C_0^1(\mathbb{R}^N), \varphi \geq 0 \), supported in a ball contained in \( \Omega \). Let \( u = u - T_k(u) \), where \( T_k \) is defined as in (3.13), for all \( s \in \mathbb{R} \). We use \( w = \varphi^q v \) with \( q > 0 \) (that we will choose later) as test function in the weak formulation (1.6) and we denote by \( \Omega_b = \{|u| > k\} \). Using assumptions (1.3), (1.4) and (1.5), we get

\[
\begin{align*}
\alpha \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \right|^{p_i} \varphi^q \, dx &\leq \sum_{i=1}^{N} \beta_i \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \right|^{p_i-1} \left| \frac{\partial}{\partial x_i} \varphi \right| \varphi^q \, dx + \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx \\
&+ \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx + \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx \\
&\leq \sum_{i=1}^{N} \beta_i \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} u \right|^{p_i-1} \left| \frac{\partial}{\partial x_i} \varphi \right| \varphi^q \, dx + C \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx \\
&+ C \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx \\
&\quad + \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx + \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \varphi^q \right| \, dx \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{align*}
\]

where \( C \) is a positive constant independent of \( u \). By Young inequality, we have

\[
I_1 \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \right|^{p_i} \varphi^q + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega_k} \varphi^{q-p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} v^{p_i},
\]

where here and in what follows \( \varepsilon \) denotes a positive constant that will be chosen later.
Using Young inequality, Hölder inequality and (4.4), we get

\[ I_3 \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega_k} |b_i|^{p_i'} |v|^p \varphi^q dx \]

\[ \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + C(\varepsilon) \sum_{i=1}^{N} \|b_i||p_i'_{L-N_{p_i'}|\varphi|\infty}\|v\varphi^q\|_{L^p}^p \]

\[ \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + C(\varepsilon) \sum_{i=1}^{N} \|b_i||p_i'_{L-N_{p_i'}|\varphi|\infty}\left(\prod_{i=1}^{N} \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right|_{L^{p_i}}^{\frac{1}{p_i}} \right)^p \] (5.3)

and

\[ I_4 \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^{-p_i} dx + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^q dx + \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega_k} |F|^{p_i} \varphi^q dx. \] (5.4)

Finally Young and Hölder inequality yields

\[ I_5 + I_6 \leq \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^{-p_i} dx + \varepsilon \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right|_{L^{p_i}}^{\frac{1}{p_i}} \] (5.5)

Now putting together inequalities (5.2), (5.3), (5.4) and (5.5), rearranging and choosing \( \varepsilon \) small enough then inequality (5.1) becomes

\[ \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx \leq C \left[ \sum_{i=1}^{N} \int_{\Omega_k} |b_i|^k \varphi \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right| dx + \sum_{i=1}^{N} \|b_i||p_i'_{L-N_{p_i'}|\varphi|\infty}\left(\prod_{i=1}^{N} \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right|_{L^{p_i}}^{\frac{1}{p_i}} \right)^p \right] + \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^{-p_i} dx + \sum_{i=1}^{N} \int_{\Omega_k} |F|^{p_i} \varphi^q dx, \] (5.6)

for suitable constant \( C = C(p, \alpha, \beta, N) \) that from now on could change from line to line.

Now we note that

\[ \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right|^{p_i} dx \leq \sum_{i=1}^{N} \int_{\Omega_k} 2^{p_i-1} \left| \frac{\partial}{\partial x_i} v^i \right|^{p_i} \varphi^q dx + \sum_{i=1}^{N} \int_{\Omega_k} 2^{p_i-1} v^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^{-p_i} dx \]

\[ \leq C \left[ \sum_{i=1}^{N} \int_{\Omega_k} |b_i|^k \varphi \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right| dx + \sum_{i=1}^{N} \|b_i||p_i'_{L-N_{p_i'}|\varphi|\infty}\left(\prod_{i=1}^{N} \left| \frac{\partial}{\partial x_i} (v^i \varphi^i) \right|_{L^{p_i}}^{\frac{1}{p_i}} \right)^p \right] + \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} \varphi^{-p_i} dx + \sum_{i=1}^{N} \int_{\Omega_k} |F|^{p_i} \varphi^q dx, \]
and so
\[
\left( \int_{\Omega_k} \left| \frac{\partial}{\partial x_j} \left( v p_j \phi \right) \right|^p dx \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} \left( v p_i \phi \right) \right|^p dx \right)^{\frac{1}{p}} \]
\[
\leq C^{\frac{1}{p}} \left[ \sum_{i=1}^{N} \int_{\Omega_k} |b_i| k^{\alpha} \left| \frac{\partial}{\partial x_i} (v \phi^q) \right| dx + \sum_{i=1}^{N} \| b_i \|_{L^{Np_i}} A \right]
+ \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \phi \right|^{p_i} \phi^{q-p_i} dx + \sum_{i=1}^{N} \int_{\Omega_k} |F_i|^{p_i} \phi^q dx \right]^{\frac{1}{p}},
\]
where \( A = \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left( u p_i \phi \right) \right\|_{L^{p_i}} \). Making the product on the left and right sides of (5.7), we get
\[
A \leq C \left[ \left( \sum_{i=1}^{N} \int_{\Omega_k} |b_i| k^{\alpha} \left| \frac{\partial}{\partial x_i} (v \phi^q) \right| dx \right)^{\frac{N}{\gamma}} + \left( \sum_{i=1}^{N} \| b_i \|_{L^{Np_i}}^\alpha \right)^{\frac{N}{\gamma}} A \right]
+ \left( \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \phi \right|^{p_i} \phi^{q-p_i} dx \right)^{\frac{N}{\gamma}} + \left( \sum_{i=1}^{N} \int_{\Omega_k} |F_i|^{p_i} \phi^q dx \right)^{\frac{N}{\gamma}} \]  
(5.8)

The assumption that the norms \( \| b_i \|_{L^{Np_i}}^{p_i} \) are small than a suitable constant depending on \( \bar{p}, \alpha, \bar{\beta}, N \) and (5.8) allow us to obtain
\[
A \leq C \left[ \left( \sum_{i=1}^{N} \int_{\Omega_k} |b_i| k^{\alpha} \left| \frac{\partial}{\partial x_i} (v \phi^q) \right| dx \right)^{\frac{N}{\gamma}} + \left( \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \phi \right|^{p_i} \phi^{q-p_i} dx \right)^{\frac{N}{\gamma}} \right]
+ \left( \sum_{i=1}^{N} \int_{\Omega_k} |F_i|^{p_i} \phi^q dx \right)^{\frac{N}{\gamma}} \]  
(5.9)

Substituting (5.9) in (5.6), by easy calculations it follows
\[
\sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} \phi \right|^p dx \leq C \left[ \sum_{i=1}^{N} \int_{\Omega_k} |b_i k^{\alpha} \left| \frac{\partial}{\partial x_i} (v \phi^q) \right| dx + \sum_{i=1}^{N} \int_{\Omega_k} v^{p_i} \left| \frac{\partial}{\partial x_i} \phi \right|^{p_i} \phi^{q-p_i} dx \right] + \sum_{i=1}^{N} \int_{\Omega_k} |F_i|^{p_i} \phi^q dx \]  
(5.10)

At this point we multiply both sides of previous inequality by \( k^{\gamma} \) for fixed \( \gamma > 0 \) that will be chosen later and we integrate with respect to \( k \) over \([0, K]\) for fixed \( K > 0 \). A repeated use of Fubini’s theorem gives
\[
\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \leq C \left[ \sum_{i=1}^{N} \int_{\Omega} b_i |T_K u|^{\gamma+1} \left( \left| \frac{\partial}{\partial x_i} u \right|^{q} + q \varphi^{q-1} \left| \frac{\partial}{\partial x_i} \varphi \right| \right) dx \right]^{\frac{1}{p_i}} \\
+ \sum_{i=1}^{N} \int_{\Omega} |u|^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{q-p_i} |T_K u|^{\gamma+1} dx + \sum_{i=1}^{N} \int_{\Omega} |\mathcal{F}_i|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \right]^{\frac{1}{p_i}}
\]

(5.10)

where \( C = C(\gamma, \alpha, N, \tilde{p}, \tilde{\beta}) \). Now by Hölder inequality and Young inequality we get

\[
\sum_{i=1}^{N} \int_{\Omega} b_i |T_K u|^{\gamma+1} \left( \left| \frac{\partial}{\partial x_i} u \right|^{q} + q \varphi^{q-1} \left| \frac{\partial}{\partial x_i} \varphi \right| \right) dx \leq \\
\left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} dx \right)^{\frac{1}{p_i}} \left( \sum_{i=1}^{N} \int_{\Omega} b_i^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} \varphi^{\frac{q}{p_i}} dx \right)^{\frac{1}{p_i}} \\
+ \sum_{i=1}^{N} \int_{\Omega} |u|^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{q-p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} dx \right)^{\frac{1}{p_i}} \left( \sum_{i=1}^{N} \int_{\Omega} b_i^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} \varphi^{\frac{q}{p_i}} dx \right)^{\frac{1}{p_i}}
\]

\[
\leq \epsilon \sum_{i=1}^{N} \int_{\Omega} \frac{\partial}{\partial x_i} u^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} \varphi^{\frac{q}{p_i}} dx + \epsilon \sum_{i=1}^{N} \int_{\Omega} u^{p_i} \left| \frac{\partial}{\partial x_i} \varphi \right|^{q-p_i} |T_K u|^{\gamma+1} dx \\
+ 2C(\epsilon) \|b\|_{L^{\infty}} \sum_{i=1}^{N} \left| B \right|^{\frac{1}{p_i}} \left| T_K u \right|^{\gamma + 1 + \frac{q}{p_i}} \left| \varphi \right|^{\frac{q}{p_i}} |L^{p_i}|
\]

where \( \epsilon \) is a positive constant small enough. Substituting the previous inequality in (5.10) and by (4.11), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \leq C \left[ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} dx \right]^{\frac{1}{p_i}} \\
+ \sum_{i=1}^{N} \|b\|_{L^{\infty}} \left( \prod_{i=1}^{N} \left| \frac{\partial}{\partial x_i} \left( |T_K u|^{\gamma + 1 + \frac{q}{p_i}} \varphi^{\frac{q}{p_i}} \right) \right| \right)^{\frac{1}{p_i}} \left( \sum_{i=1}^{N} \left| \mathcal{F}_i \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \right) \\
+ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} \left( |T_K u|^{\gamma + 1 + \frac{q}{p_i}} \varphi^{\frac{q}{p_i}} \right) \right| \left| \mathcal{F}_i \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \right]^{\frac{1}{p_i}}
\]

(5.11)

Denoting \( B = \prod_{i=1}^{N} \left| \frac{\partial}{\partial x_i} \left( |T_K u|^{\gamma + 1 + \frac{q}{p_i}} \varphi^{\frac{q}{p_i}} \right) \right| \left| \mathcal{F}_i \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \) it follows

\[
\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} \left( |T_K u|^{\gamma + 1 + \frac{q}{p_i}} \varphi^{\frac{q}{p_i}} \right) \right|^{p_i} dx \leq \\
\sum_{i=1}^{N} \int_{\Omega} 2^{p_i-1} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx + \sum_{i=1}^{N} \int_{\Omega} 2^{p_i-1} \left| \frac{\partial}{\partial x_i} \varphi \right|^{p_i} |T_K u|^{\gamma+1+\frac{q}{p_i}} dx \]

\[
\leq C \left[ \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} |T_K u|^{\gamma+1} dx + \sum_{i=1}^{N} \|b\|_{L^{\infty}} \left| B \right|^{\frac{1}{p_i}} + \sum_{i=1}^{N} \int_{\Omega} \left| \mathcal{F}_i \right|^{p_i} \varphi^{q} |T_K u|^{\gamma+1} dx \right].
\]

Previous inequality give us an estimate of the \( j \)-th addendum of the sum at the left-hand side of (5.12) as well. Then elevating to the power \( \frac{1}{p_j} \), making the product on the left and
right sides of \((5.12)\), we get

\[
B \leq C \left[ \left( \sum_{i=1}^{N} \int_{\Omega} \left| u \frac{\partial}{\partial x_{i}} \varphi \right|^{p_i} |T_{K}u|^{\gamma+1} \varphi dx \right) \right]^{\frac{N}{p^*}} + \left( \sum_{i=1}^{N} \| b_{i} \|_{L^\infty} \right)^{\frac{N}{p^*}} B \tag{5.13}
\]

Using again that the norms \(\| b_{i} \|^{p^*_i}_{L^\infty} \) are small than a suitable constant depending on \(\gamma, \alpha, N, \bar{p}, \bar{\beta}\) from \((5.13)\) we obtain

\[
B \leq C \left[ \left( \sum_{i=1}^{N} \int_{\Omega} \left| u \frac{\partial}{\partial x_{i}} \varphi \right|^{p_i} |T_{K}u|^{\gamma+1} \varphi dx \right) \right]^{\frac{N}{p^*}} + \left( \sum_{i=1}^{N} \int_{\Omega} |F_{i}|^{p_i} \varphi^{q} |T_{K}u|^{\gamma+1} dx \right)^{\frac{N}{p^*}}. \tag{5.14}
\]

Substituting \((5.14)\) in \((5.11)\) it follows that

\[
\sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_{i}} \left( |T_{K}u|^{\frac{\gamma+1}{p_i}} \right) \right\|^{p_i}_{L^{p_i}} \leq C \left[ \sum_{i=1}^{N} \int_{\Omega} \left| u \frac{\partial}{\partial x_{i}} \varphi \right|^{p_i} |T_{K}u|^{\gamma+1} \varphi^{q} dx + \sum_{i=1}^{N} \int_{\Omega} |F_{i}|^{p_i} \varphi^{q} |T_{K}u|^{\gamma+1} dx \right]. \tag{5.15}
\]

At this point, let us assume that \(u \in L^{\mu \max}_{\text{loc}}(\Omega)\), where \(\mu\) is defined in \((1.8)\). Recalling that \(F_{i} \in L^{p_i}_{\text{loc}}(\Omega)\) for every \(i = 1, ..., N\), then \(F_{i} \in L^{p_i}_{\text{loc}}(\Omega)\). Hence by Hölder inequality with exponents \(\mu\) and \(\mu'\), from \((5.15)\) we have

\[
\prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_{i}} \left( |T_{K}u|^{\frac{\gamma+1}{p_i}} \right) \right\|^{\frac{1}{p^*}}_{L^{p^*}} \leq C \left( \int_{\Omega} |T_{K}u|^{(\gamma+1)\frac{\mu'}{p_i - \mu}} \varphi^{1/(1-\mu)} dx \right)^{\frac{1}{p^*}} \times \left[ \sum_{i=1}^{N} \left( \int_{\Omega} \left| u \frac{\partial}{\partial x_{i}} \varphi^{\alpha} dx \right|^{p_i} \right)^{\frac{1}{p_i}} + \sum_{i=1}^{N} \left( \int_{\Omega} |F_{i}|^{p_i} \varphi^{\alpha} dx \right)^{\frac{1}{p_i}} \right]. \tag{5.16}
\]

with \(\alpha_1 = \left[ 1 - \frac{\mu}{p^*} \right] \mu q - p_i \mu\), and \(\alpha_2 = \left[ 1 - \frac{\mu}{p^*} \left( 1 - \frac{1}{\mu} \right) \right] q\mu\). Now, by \((2.21)\) we have \(\alpha_2 > 0\) and we can now choose \(q\) large enough to have \(\alpha_1 > 0\). Finally, we choose now \(\gamma = \frac{Np}{N - p\mu}(\mu - 1) - 1\) so that

\[
(\gamma + 1) \left( \frac{\mu}{\mu - 1} \right) \frac{p}{\gamma + 1 + p} = p^*. \]
Thus, using Proposition 4.1 (5.16) becomes

\[
\prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left( |T_K u|^\frac{1}{p_i} + 1 \varphi_{\gamma_i} \right) \right\|_{L^{p_i}}^{\frac{1}{p_i}} \leq C \left\| T_K u \right\|_{L^{\infty}} \left( \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \varphi \right\|_{L^{p_i}} + \left\| F_i \right\|_{L^{p_i}} \right)
\]

\[
\leq C \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left( |T_K u|^\frac{1}{p_i} + 1 \varphi_{\gamma_i} \right) \right\|_{L^{p_i}}^{\frac{1}{p_i}} \leq C \left( \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \varphi \right\|_{L^{p_i}}^{\alpha_i} + \left\| F_i \right\|_{L^{p_i}}^{\beta_i} \right)
\]

Rearranging the previous inequality it follows that

\[
\left( \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left( |T_K u|^\frac{1}{p_i} + 1 \varphi_{\gamma_i} \right) \right\|_{L^{p_i}} \right)^{\rho} \leq C \left( \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \varphi \right\|_{L^{p_i}}^{\alpha_i} + \left\| F_i \right\|_{L^{p_i}}^{\beta_i} \right)
\]

where \( \rho := \left[ 1 - \frac{\alpha_i}{p_i} \right] \) that is positive by (2.1). Finally letting \( K \rightarrow +\infty \) in (5.17), we conclude

\[
\left( \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \left( |u|^\frac{1}{p_i} + 1 \varphi_{\gamma_i} \right) \right\|_{L^{p_i}} \right)^{\rho} \leq C \left( \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} \varphi \right\|_{L^{p_i}}^{\alpha_i} + \left\| F_i \right\|_{L^{p_i}}^{\beta_i} \right)
\]

Since \( \left( \frac{1}{p} + 1 \right)^{\rho} = \frac{\rho p}{N - p} \), using again Proposition 4.1 we obtain that

\[
u \in L^{s(\mu)}(\Omega) \quad \text{with} \quad s(\mu) := (\bar{p} \mu)^{\ast}.
\]

We observe that previous argument works directly when \( \mu p_{\max} \leq \bar{p} \), since in this case \( u \in W^{1,\bar{p}}_{loc}(\Omega) \) implies that \( u \in L^{\mu_{p_i}}(\Omega) \) for every \( i = 1, ..., N \). If otherwise there exist \( i \in \{1, ..., N\} \) such that \( \mu p_i > \bar{p}_{\ast} \) we use a bootstrap procedure.

Precisely, if there exist \( i_j \in \{1, ..., N\} \) such that \( \mu p_{i_j} > \bar{p}_{\ast}, \ j = 1, ..., m, \) with \( m \leq N \), we repeat the previous argument (from (5.15) to (5.18)) with \( r_{i_j}, j = 1, ..., m \) replaced by \( s(1) = \bar{p}_{\ast} \) and \( \mu \) replaced by the corresponding new minimum in (1.8), i.e. \( \mu_{1} = \frac{\bar{p}_{\ast}}{p_{\max}} \). In this way we find \( u \in L^{s(\mu_1)}(\Omega) \) and

\[
s(\mu_1) - s(1) > \bar{p}^{\ast} - p_{\max}.
\]

At this point, if \( \mu p_{\max} \leq s(\mu_1) \) we can use the information \( u \in L^{s(\mu_1)}(\Omega) \) to conclude our proof as before. Otherwise we repeat the procedure again, that is: if there exist \( i_j \in \{1, ..., N\} \) such that \( \mu p_{i_j} > s(\mu_1) \), we repeat the previous argument (again, from (5.15) to (5.18)) with \( r_{i_j} \) replaced by \( s(\mu_1) \) and \( \mu \) replaced by \( \mu_2 := \frac{s(\mu_1)}{p_{\max}} \) so that we find \( u \in L^{s(\mu_2)}(\Omega) \) with \( s(\mu_2) = \left( \frac{s(\mu_1)}{p_{\max}} \right)^{\ast} \). Using the convexity of the function \( f(p) = \frac{N}{N - p} \), we find

\[
s(\mu_2) - s(\mu_1) \geq \frac{N^2}{(N - \bar{p})^2} \frac{\bar{p}}{p_{\max}} (s(\mu_1) - \bar{p}^{\ast}) + \frac{\bar{p}^{\ast}}{p_{\max}} (\bar{p}^{\ast} - p_{\max}),
\]
and so on. Since, if necessary, at the $h$-th step one has $\mu_h = \frac{s(\mu_{h-1})}{p_{\text{max}}}$ and

$$s(\mu_h) - s(\mu_{h-1}) > \left( \frac{\bar{p}^*}{p_{\text{max}}} \right)^{h-1} (\bar{p}^* - p_{\text{max}}),$$

it is now clear that, in a finite number of times, we can conclude the proof of Step 1.

**Step 2. Dropping the smallness assumption**

Now we remove the smallness assumptions on $\|b_i\|_{L^\frac{Np}{p} \to \infty}$ assuming that

$$\max_i \left\{ \text{dist}_{L^\frac{Np}{p} \to \infty} (b_i, L^\infty(\Omega)) \right\}$$

is sufficiently small. Setting

$$\theta_i(x) = \begin{cases} \frac{T_M b_i(x)}{b_i(x)} & \text{if } b_i(x) \neq 0 \\ 1 & \text{if } b_i(x) = 0 \end{cases}$$

for $M > 0$, we rewrite equation (1.6) in the following form

$$\sum_{i=1}^N \int_{\Omega} (A_i(x, \nabla u) + (1 - \theta_i(x))B_i(x, u) \partial_x \varphi \, dx = \sum_{i=1}^N \int_{\Omega} |F_i|^{p_i-2}F_i - \theta_i(x)B_i(x, u) \partial_x \varphi \, dx. \tag{5.19}$$

Assumption (1.5) yields

$$|\theta_i(x)B_i(x, u)| \leq \|b_i(x) - T_M b_i(x)\|_{L^\frac{Np}{p} \to \infty}\|u\|_{L^\frac{Np}{p} \to \infty} \tag{5.20}$$

Using (3.12) we can find a sufficiently large constant $M > 0$ independent on $i$, such that for every $i = 1, ..., N$ the norm $\|b_i - T_M b_i\|_{L^\frac{Np}{p} \to \infty}$ is small in order to verify the smallness assumption of Step 1.

Hence, we can apply the result obtained in the previous step to (5.19) whenever

$$G_i(x, u) := |F_i|^{p_i-2}F_i - \theta_i(x)B_i(x, u) \in L^{\frac{Np_i}{p_i}}_{\text{loc}}(\Omega), \text{ for every } i = 1, ..., N.$$

Using (5.20), it is enough to have

$$|u|_{L^{\frac{Np_i}{p_i}}_{\text{loc}}(\Omega)} \leq M \|u\|_{L^{\frac{Np_i}{p_i}}_{\text{loc}}(\Omega)} \tag{5.21}$$

At this point we observe that if $r_i \leq \frac{\bar{p}^*}{p} p_i$ for every $i = 1, ..., N$, then (5.21) holds directly by Sobolev inequality. If otherwise there exist $i \in \{1, ..., N\}$ such that $r_i > \frac{\bar{p}^*}{p} p_i$, we use a bootstrap procedure similar to the one used to conclude the proof of Step 1.

Precisely, if there exist $i_j \in \{1, ..., N\}$ such that $r_{i_j} > \frac{\bar{p}^*}{p} p_{i_j}$, $j = 1, ..., m$, with $m \leq N$, we replace $r_{i_j}$, $j = 1, ..., m$ with $\frac{\bar{p}^*}{p} p_{i_j}$ and $\mu$ with $\mu := \min\{\mu, \frac{\bar{p}^*}{p} \}$ respectively. Now, if $\mu_1 = \mu$ the proof is completed, since in this case we find $u \in L^{s(\mu_1)}_{\text{loc}}(\Omega)$ with $s(\mu)$ as in (5.18). Otherwise, if $\mu_1 = \frac{\bar{p}^*}{p}$ we find $u \in L^{s(\mu_1)}_{\text{loc}}(\Omega)$ with $s(\mu_1) = \bar{p}^*$ and $s(\mu_1) - s(1) > \bar{p}^* - p_{\text{max}}$.

Now, if $r_{i_j} \leq \frac{\bar{p}^*}{p} p_{i_j}$ for every $j = 1, ..., m$, then $u \in L^{s(\mu_1)}_{\text{loc}}(\Omega)$ gives (5.21). Otherwise we repeat the procedure again. It is clear that, in a finite number of times, we can conclude our proof. $\blacksquare$
Remark 5.1 In the previous proof the definition of \( \mu \) in (1.8) as a minimum obviously appears. Indeed, using Hölder inequality in (5.14) with exponent \( \zeta \) and \( \zeta' \), it is enough to require \( p_i \zeta \leq r_i \), \( i = 1, \ldots, N \). Hence, the best choice of such \( \zeta \) is \( \mu \).

Remark 5.2 We stress that assumption \( \overline{\mu} > \mu_{\text{max}} \) is essential for our technique. Indeed, at the end of the Step 1 of previous proof, in order to start with the bootstrap argument we need \( u \in L^{r_i}_{\text{loc}}(\Omega) \) with \( r_i \leq \overline{\mu} \) for every \( i = 1, \ldots, N \) that means \( p_i < \overline{\mu} \), \( i = 1, \ldots, N \).

Remark 5.3 If \( \Omega \) is bounded open set with Lipschitz boundary and we consider homogeneous Dirichlet problems we can argue as in Theorem 2.1 to obtain a regularity of solutions without restriction \( \mu_{\text{max}} < \overline{\mu} \). Precisely we have the following result. Assume that (1.3)-(1.4) are fulfilled, let \( 1 < \overline{\mu} < N \) and let \( r_1, \ldots, r_N \) be such that (2.1) holds. There exists a positive constant \( d = d(\overline{\mu}, N, \alpha, \overline{\mu}) \) such that if

\[
\max_i \left\{ \text{dist} \frac{b_i}{L^{\overline{\mu}, \infty}(\Omega)} : (b_i, L^{\infty}(\Omega)) \right\} < d
\]

and \( u \in W^{1, \overline{\mu}}(\Omega) \) is a weak solution to (1.1) with \( \mathcal{F}_i \in L^{r_i}(\Omega) \), then

\[
u \in L^s(\Omega) \quad \text{with} \quad s = \max\{\overline{\mu}_{\text{max}}, \mu_{\text{max}}\}.
\]

Indeed when the \( \max\{\overline{\mu}_{\text{max}}, \mu_{\text{max}}\} = \overline{\mu}_{\text{max}} \) the proof of (5.22) runs as Theorem 2.1 taking into account that we are managing not local solutions. Otherwise if \( \max\{\overline{\mu}_{\text{max}}, \mu_{\text{max}}\} = \mu_{\text{max}} \) we can reason as in Step 1 of Theorem 2.1 but instead of (5.14) we obtain

\[
\left\| \frac{\partial}{\partial x_j} \left[ T_{K_i} u \right] \right\|_{L^{\mu_{\text{max}}}} \leq C \sum_{i=1}^{N} \left( \int_{\Omega} |\mathcal{F}_i|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |T_{K_i} u|^{\gamma+1} \right)^{\frac{\mu}{\gamma + 1 + \mu_{\text{max}}}} \left( \frac{1}{\overline{\mu}} \right)^{\frac{\gamma}{\gamma + 1 + \mu_{\text{max}}}}.
\]

Now choosing \( \gamma > 0 \) such that

\[
(\gamma + 1) \left( \frac{\mu}{\mu - 1} \right) \frac{\mu_{\text{max}}}{\gamma + 1 + \mu_{\text{max}}} = \mu_{\text{max}},
\]

by Poincaré inequality (4.11), (5.23) becomes

\[
\left\| \frac{\partial}{\partial x_j} \left[ T_{K_i} u \right] \right\|_{L^{\mu_{\text{max}}}} \leq C \sum_{i=1}^{N} \|\mathcal{F}_i\|_{L^{r_i}},
\]

where the constant \( C \) depends now also by \( \Omega \). Letting \( K \to +\infty \) in (5.25), observing that equality (5.24) implies \( \frac{\gamma + 1}{\mu_{\text{max}}} + 1 = \mu \), and using again Poincaré inequality (4.11), we obtain (5.22) with \( s = \mu \mu_{\text{max}} \).

Remark 5.4 Let us consider the homogeneous Dirichlet problem in a bounded open set with Lipschitz boundary \( \Omega \) under the assumptions (1.3)-(1.4), when (1.5) is replaced by

\[
|B_i(x, s)| \leq b_i(x)|s|^{p_i - 1} \quad \forall i
\]

for a.e. \( x \in \Omega \) and for every \( s \in \mathbb{R} \) and \( b_i \in L^\infty(\Omega) \) for all \( i \). In this case we obtain the same regularity result as in Remark 5.3. Indeed one can obtain the analogous inequality of (5.15) using Poincaré inequality (4.11) instead of Sobolev inequalities. Starting from the obtained estimate, one can conclude the proof as before under assumption (5.26).
6 Proof of Theorem 2.4

The first step of our proof is the boundedness of a local weak solution without the lower order terms in order to adapt Stampacchia’s arguments [28] to the anisotropic case in the same spirit of Lemma 5.4 in [23] when one deals with local solutions. Finally, our assumptions on the summability of the coefficients \( b_i \) allow us to apply Corollary 1.3 concluding also when \( B_i \not\equiv 0 \).

Step 1. Proof assuming that \( B_i \) vanishes. Using the same test function \( w \) as in Theorem 2.1 (Step 1) and assumptions (1.3) and (1.4), we get

\[
\alpha \sum_{i=1}^{N} \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \right|^{p_i} \varphi dx \leq \sum_{i=1}^{N} \beta_i \int_{\Omega_k} \left| \frac{\partial}{\partial x_i} v \right|^{p_i-1} \left| \frac{\partial}{\partial x_i} \varphi \right| v dx + \sum_{i=1}^{N} \int_{\Omega_k} \left| F_i \right|^{p_i-1} \left| \frac{\partial}{\partial x_i} \varphi \right| \varphi^q dx + \sum_{i=1}^{N} \int_{\Omega_k} \left| F_i \right|^p \varphi^q dx.
\]

for suitable positive constant \( C = C(\alpha, N, \bar{p}, \bar{\beta}) \) that will change from line to line.

Now we choose the cut-off function \( \varphi \). Let \( \sigma > \tau > 0 \), we fix two concentric balls \( B_{\tau} \subset B_{\sigma} \subset \subset \Omega \) of radii \( \tau \) and \( \sigma \) respectively. We consider \( \varphi \) such that \( 0 \leq \varphi \leq 1, \varphi \equiv 1 \) in \( B_{\tau} \), \( \varphi \equiv 0 \) in \( \Omega \setminus B_{\sigma} \) and \( |\nabla \varphi| \leq \frac{2}{\sigma-\tau} \). We put \( \Omega_{k,\sigma} = \Omega_k \cap B_{\sigma} \). By (6.2), taking \( q > \max_i p_i \) and using Hölder inequality we get

\[
\sum_{i=1}^{N} \int_{\Omega_{k,\sigma}} \left| \frac{\partial}{\partial x_i} v \right|^{p_i} dx \leq C \sum_{i=1}^{N} \frac{1}{(\sigma-\tau)^{p_i}} \int_{\Omega_{k,\sigma}} (|u| - k)^{p_i} dx + C \sum_{i=1}^{N} \| F_i \|_{L^{p_i}(\Omega_{k,\sigma})}^{p_i} \Omega_{k,\sigma} |1 - \frac{1}{q}|.
\]

We stress that since \( F_i \in L^{p_i}_{\text{loc}}(\Omega) \), then \( F_i \in L^{p_i\sigma}(\Omega) \) for every \( i = 1, \ldots, N \). Moreover there exists \( \tilde{k} > 0 \) such that \( |\Omega_{k,\sigma}| < 1 \) for \( k \geq \tilde{k} \), and then

\[
|\Omega_{k,\sigma}|^{1 - \frac{1}{q}} \leq |\Omega_{k,\sigma}|^{1 - \frac{1}{q}}
\]

under the assumption (2.4). Then, for \( k > \tilde{k} \)

\[
\sum_{i=1}^{N} \int_{\Omega_{k,\sigma}} \left| \frac{\partial}{\partial x_i} v \right|^{p_i} dx \leq C \sum_{i=1}^{N} \left[ (\sigma-\tau)^{-p_i} \int_{\Omega_{k,\sigma}} (|u| - k)^{p_i} dx + (\sigma-\tau)^{-p_i} \| F_i \|_{L^{p_i}(\Omega_{k,\sigma})}^{p_i} \right],
\]

where \( \delta \) is a positive constant that we choose later. Inequality (6.3) is the key ingredient to conclude that we can choose \( k_0 > 0 \) such that

\[
\int_{\Omega_{2k_0,\sigma}}^{} (u - 2k_0) \, dx = 0,
\]

(6.4)
for a fixed $0 < \sigma_0 < \sigma$, i.e. $u$ is locally bounded when the norms of $b_i$ are small enough.

The claim (6.4) will be proved in the next step following the idea of Lemma 5.4 of [23].

**Step 2. Proof of (6.4).**

Let us consider the sequence of concentric balls $B_{\rho h}$, with $\rho h = \sigma - \sigma_0 + \frac{\sigma_0}{2\pi}$ and the sequence $k_h = 2k_0 - \frac{k_0}{2\pi}$, where $k_0 > 0$ will be fixed later. We will denote for $h = 0, 1, \ldots$

$$J_h^i = \int_{\Omega_{k_h, \rho h}} (|u| - k_h)^{p_i} \, dx \quad \text{and} \quad \zeta_h(x) = \zeta \left(2^{h+1}(|x| - \sigma + \sigma_0)\right),$$

where $\zeta(s)$ is a continuously differentiable nonincreasing function on $\mathbb{R}$ that is equal to 1 for $s \leq \sigma_0$ and equal to 0 for $s \geq \frac{3}{2}\sigma_0$. We stress that $\zeta_n \equiv 1$ inside the ball $B_{\rho h+1}$ and $\zeta_n \equiv 0$ outside the ball $B_{\tau h}$, where $\tau h = \frac{1}{2}(\rho h+1 + \rho h)$. We have

$$J_{h+1}^i \leq \int_{\Omega_{k_{h+1}, \tau h}} (|u| - k_{h+1})^{p_i} \zeta_h^i \, dx.$$

Moreover by Poincaré inequality obtained by Sobolev and Hölder inequalities we get

$$J_{h+1}^i \leq C|\Omega_{k_{h+1}, \tau h}|^{1 - \frac{N}{p_i} + \frac{p_i}{N}} \left( \prod_{j=1}^N \int_{\Omega_{k_{h+1}, \tau h}} \left( \frac{\partial}{\partial x_j} (|u| - k_{h+1}) \zeta_h \right)^{p_j} \right)^{\frac{1}{p_j}}^{p_i},$$

$$\leq C|\Omega_{k_{h+1}, \tau h}|^{1 - \frac{N}{p_i} + \frac{p_i}{N}} \left( \prod_{j=1}^N \int_{\Omega_{k_{h+1}, \tau h}} \left| \frac{\partial}{\partial x_j} u \right|^{p_j} \, dx + \nu_j 2^{p_j} J_h^j \right)^{\frac{1}{p_j}}^{p_i},$$

where $C = C(\alpha, N, \bar{p}, \bar{\beta})$ and

$$\nu_j = \max_{s \in [\sigma_0, 2\sigma_0]} [\zeta'(s)]^{p_i}.$$

Now we write (6.3) with $k = k_{h+1}$ and $\sigma = \rho h$ and $\tau = \bar{\rho} h$:

$$\sum_{i=1}^N \int_{\Omega_{k_{h+1}, \tau h}} \left| \frac{\partial}{\partial x_i} u \right|^{p_i} \, dx \quad (6.6)$$

$$\leq C \left[ \frac{1}{(\rho h - \bar{\rho} h)^{p_i}} \int_{\Omega_{k_{h+1}, \rho h}} (|u| - k_{h+1})^{p_i} \, dx + \rho h^{-\delta N} |\Omega_{k_{h+1}, \rho h}|^{1 - \frac{N}{p_i} + \delta} \sum_{i=1}^N \|F_i\|_{L^{p_i}} \right]$$

$$\leq C \left( \rho h^{-\delta N} |\Omega_{k_{h+1}, \rho h}|^{1 - \frac{N}{p_i} + \delta} + \sum_{i=1}^N 2^{(h+3)p_i} J_h^i \right),$$

where $C = C(\alpha, \|F_i\|_{L^{p_i}}, N, \bar{p}, \bar{\beta})$. Let us find a bound for the measure of the set $\Omega_{k_{h+1}, \rho h}$.

We have

$$J_h^i \geq \int_{\Omega_{k_{h+1}, \rho h}} (|u| - k_h)^{p_i} \geq (k_{h+1} - k_h)^{p_i} |\Omega_{k_{h+1}, \rho h}| = 2^{-(h+1)p_i} k_0^{p_i} |\Omega_{k_{h+1}, \rho h}|.$$
Putting together (6.7), (6.6), inequality (6.5) can be rewrite as

\[ J_{h+1}^i \leq C[\Omega_{h_{k+1}, \pi}]^{1-\frac{p_i}{p_m}} \left\{ \prod_{j=1}^{N} \left[ \sum_{m=1}^{N} \int_{\Omega_{h_{k+1}, \pi}} \left| \frac{\partial}{\partial x_m} u \right|^{p_m} \ dx + \sum_{m=1}^{N} \nu_m 2^{p_m h J_m^i} \right]^{\frac{1}{p_m}} \right\}^{p_i} \]

\[ \leq C \left( 2^{(h+1)p_i k_0^{-p_i} J_h^i} \right)^{1-\frac{p_i}{p_m}} \left[ \sum_{m=1}^{N} \left( 2^{(h+3)p_m J_m^i} + \nu_m 2^{p_m h J_m^i} \right) \right]^{\frac{p_i}{p_m}} \]

\[ \leq C \left( 2^{(h+1)p_i k_0^{-p_i} J_h^i} \right)^{1-\frac{p_i}{p_m}} \left[ \sum_{m=1}^{N} \left( 2^{(h+3)p_m J_m^i} + \nu_m 2^{p_m h J_m^i} \right) \right]^{\frac{p_i}{p_m}} \]

\[ \leq C 2^{h p_{\text{max}} k_0^{-p_{\text{min}} + \frac{p_{\text{max}}}{p_m}}} \left( 2^{h \frac{p_{\text{max}}^2}{p_m}} (J_h^i)^{1+\frac{p_i}{p_m}} \sum_{m=1}^{N} J_m^i \right)^{\frac{p_i}{p_m}} \]

\[ + 2^{h \frac{p_{\text{max}}^2}{p_m}} \left( \max_{m} \nu_m \right)^{\frac{p_i}{p_m}} (J_h^i)^{1+\frac{p_i}{p_m}} \sum_{m=1}^{N} J_m^i \right]^{\frac{p_i}{p_m}} \]

\[ \leq C 2^{h (p_{\text{max}} + \frac{p_{\text{max}}^2}{p_m}) k_0^{-p_{\text{min}} + \frac{p_{\text{max}}}{p_m}}} \left( Y_h^i \right)^{1+\frac{p_i}{p_m}} \left[ (Y_0)^{\frac{p_i}{p_m} - \frac{\delta}{p_m} + 1} \right], \]

where \( p_{\text{min}} = \min_i \{p_1, \ldots, p_N\} \). Since \( p_{\text{max}} > p_{\text{max}} \) the exponent \( 1 + \frac{p_i}{p_m} - \frac{\delta}{p_m} > 0 \) and setting \( Y_h = \sum_{m=1}^{N} J_m^i \), one can rewrite the previous inequality as follows

\[ J_{h+1}^i \leq C 2^{h (p_{\text{max}} + \frac{p_{\text{max}}^2}{p_m}) k_0^{-p_{\text{min}} + \frac{p_{\text{max}}}{p_m}}} \left( Y_h \right)^{1+\frac{p_i}{p_m}} \left[ (Y_0)^{\frac{p_i}{p_m} - \frac{\delta}{p_m} + 1} \right] \]

(6.8)

where the last inequality follows taking \( \delta < \frac{p_m}{N} \) and observing that \( J_h^i \) are decreasing. Putting \( \delta' = \frac{\delta}{p_m} \) and summarizing left and right side of (6.8), we get

\[ Y_{h+1} \leq NC 2^{h (p_{\text{max}} + \frac{p_{\text{max}}^2}{p_m}) k_0^{-p_{\text{min}} + \frac{p_{\text{max}}}{p_m}}} (Y_h)^{1+\delta'} \max_i \left[ (Y_0)^{\frac{p_i}{p_m} - \delta' + 1} \right]. \]

Denoting \( C_1 = NC \max_i \left[ (Y_0)^{\frac{p_i}{p_m} - \delta' + 1} \right], \omega = p_{\text{min}} - \frac{p_{\text{max}}^2}{p_m} \) and \( b = 2^{(p_{\text{max}} + \frac{p_{\text{max}}^2}{p_m})} \) the previous inequality became

\[ Y_{h+1} \leq C_1 b^h k_0^{-\omega} (Y_h)^{1+\delta'}. \]

Choosing \( k_0 = \max(\tilde{k}, 1, C_1^{1/\omega} b^{1/\left[\omega(1+\delta')\right]} a^{\delta'/\omega}) \), we obtain

\[ Y_1 \leq C_1 k_0^{-\omega} Y_0^{1+\delta'} \leq k_0^{\omega/\delta'} C_1^{-1/\delta'} b^{-1/(\delta')^2}. \]

Now we are in position to apply Lemma 4.7 of [23] in order to obtain (6.4).
Step 3. Dropping the assumption that $B_i \equiv 0$.

In the general case where $b_i \in L^r_{\text{loc}}(\Omega)$, with $r_1, \cdots, r_N$ satisfying (2.4), we rewrite equation (1.6) as

$$\sum_{i=1}^{N} \int_{\Omega} (A_i(x, \nabla u)) \partial_{x_i} \varphi \, dx = \sum_{i=1}^{N} \int_{\Omega} (|F_i|^{p_i-2} F_i - B_i(x, u)) \partial_{x_i} \varphi \, dx. \quad (6.9)$$

Then, we can apply the result obtained in the previous steps to (6.9). In fact, by (2.4) and by Theorem 2.1, we have that $u \in L^q_{\text{loc}}(\Omega)$ for every $q < +\infty$. Hence we can find $p_i < s_i < r_i$ such that $\min_i \left\{ \frac{s_i}{p_i} \right\} > \frac{N}{p}$ and

$$G_i(x,u) := \left[ |F_i|^{p_i-2} F_i(x) - B_i(x,u) \right] \in L^{s_i-1}_{\text{loc}}(\Omega), \text{ for every } i = 1, \ldots, N.$$

Remark 6.1 If $\Omega$ is a bounded open set with Lipschitz boundary and we consider homogeneous Dirichlet problems we can argue as in Theorem 2.4 to obtain the boundedness of solutions. Precisely, instead of (6.2), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial x_i}{\partial x_j} \right|^{p_i} \, dx \leq C \sum_{i=1}^{N} \int_{\Omega} |F_i|^{p_i} \, dx,$$

where $C = C(\alpha, \vec{p}, \vec{\beta}, N) > 0$. At this point, we can proceed as in Theorem 2 in [29] and one can conclude the proof using the Stampacchia’s Lemma (see [28]) instead of Step 2 of Theorem 2.4.

7 Appendix

In this appendix we prove a technical lemma (see also [17]).

Lemma 7.1 Let $R > 0, \alpha \geq 0, \beta \geq 0$ and $\theta_i > 0$ for all $i = 1, \cdots, N$ and $\Omega = \{ \sum_{i=1}^{N} |x_i|^{\theta_i} < R \}$. If

$$\sum_{i=1}^{N} \frac{1}{\theta_i} + \frac{\beta}{\theta_j} > \alpha, \quad (7.1)$$

then $\left( \sum_{i=1}^{N} |x_i|^{\theta_i} \right)^{-\alpha} |x_j|^\beta \in L^1(\Omega)$. Otherwise $\left( \sum_{i=1}^{N} |x_i|^{\theta_i} \right)^{-\alpha} \notin L^1(\Omega)$.

Proof. Putting $\theta = \max \theta_i$ and $x_i = |y_i|\frac{\theta_i}{\theta} \text{sign} y_i$ we have

$$\int_{\{ \sum_{i=1}^{N} |x_i|^{\theta_i} < R \}} \left( \sum_{i=1}^{N} |x_i|^{\theta_i} \right)^{-\alpha} |x_j|^\beta \, dx = \int_{\{ \sum_{i=1}^{N} |y_i|^{\theta} < R \}} \left( \sum_{i=1}^{N} |y_i|^{\theta} \right)^{-\alpha} |y_j|^\beta \prod_{i=1}^{N} \theta_i \frac{\theta_i}{\theta_i - 1} \, dy$$

$$\leq C_1 \int_{\{ |y|^{\theta} < R \}} |y|^{-\theta \alpha + \frac{\theta}{\theta} \beta + \sum_{i=1}^{N} \frac{\theta_i}{\theta_i - 1} - N} \, dy$$

$$\leq C_1 \int_{\{ |y| < C_2 R \}} |y|^{-\theta \alpha + \frac{\theta}{\theta} \beta + \sum_{i=1}^{N} \frac{\theta_i}{\theta_i - 1} - N} \, dy$$

23
for suitable positive constants $C_1, C_2$. The last integral is finite if (7.1) holds. Otherwise putting $\theta = \min \theta_i$ and $x_i = |y_i|_{\theta_i}^\theta \text{sign} y_i$ we have

$$
\int_{\sum_{i=1}^{N} |x_i|_{\theta_i}^\theta < R} \left( \sum_{i=1}^{N} |x_i|_{\theta_i}^\theta \right)^{-\alpha} |x_j|^\beta \, dx = \int_{\sum_{i=1}^{N} |y_i|^\theta < R} \left( \sum_{i=1}^{N} |y_i|^\theta \right)^{-\alpha} |y_j|^\beta \prod_{i=1}^{N} \theta_i |y_i|_{\theta_i}^{-1} \, dy
$$

$$
\geq C_3 \int_{\{|y|^\theta < R/N\}} |y|^{-\theta \alpha + \sum_{i=1}^{N} \theta_i - N} |y_1|^{-\theta \beta} \, dy
$$

where $C_3$ is a positive constant. Now we use spherical coordinates, in which the coordinates consist of a radial coordinate $\rho$ and $N-1$ angular coordinates $\psi_1, \psi_2, \ldots, \psi_{N-1}$, where the angles $\psi_1, \psi_2, \ldots, \psi_{N-2}$ range over $[0, \pi]$ and $\psi_{N-1}$ ranges over $[0, 2\pi)$. For example $y_1$ coordinate becomes $y_1 = \rho \cos \psi_1$ and then

$$
\int_{\{|y|^\theta < R/N\}} |y|^{-\theta \alpha + \sum_{i=1}^{N} \theta_i - N} |y_1|^{-\theta \beta} \, dy
$$

$$
= \int_{0}^{2\pi} \int_{0}^{R/N} \rho^{-\theta \alpha + \sum_{i=1}^{N} \theta_i - N} \rho \cos \psi_1 |\rho \cos \psi_1|^\beta \rho^{N-1} \prod_{i=1}^{N-2} \sin^{N-1-i} \psi_1 \, d\rho \, d\psi_1 \cdots d\psi_{N-1}
$$

$$
= C(N) \int_{0}^{R/N} \rho^{-\theta \alpha + \sum_{i=1}^{N} \theta_i - N + \theta \beta + N-1} \, d\rho.
$$

The last integral is finite if (7.1) does not hold and for $j = 2, \ldots, N$ the proof runs similarly. ■

Acknowledgements

The authors are members of Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Research partially supported by project Vain-Hopes within the program VALERE: VAn-viteLli pEr la RicErca.

References

[1] Alberico A., di Blasio G., Feo F., Estimates for fully anisotropic elliptic equations with a zero order term, Nonlinear Analysis 181 (2019), 249-264.

[2] Alberico A., di Blasio G., Feo F., An eigenvalue problem for the anisotropic $\Phi$-Laplacian, J. Differential Equations 269 (2020), 4853-4883.

[3] Antontsew S., Chipot M., Anisotropic Equations: Uniqueness and existence results, Diff. Int. Eq. 21 (2008), 401-419.

[4] Bendahmane M., Karlsen K. H, Nonlinear anisotropic elliptic and parabolic equations in $\mathbb{R}^N$ with advection and lower order terms and locally integrable data, Potential Anal. 22 (2005), 207–227.
[5] Boccardo L., Finite energy solutions of nonlinear Dirichlet problems with discontinuous coefficients, Boll. Un. Mat. It. 5 (2012), 357-368.

[6] Boccardo L., Dirichlet problems with singular convection terms and applications, J. Differential Equations, 258 (2015), 2290–2314.

[7] Boccardo L., Gallouet T., Marcellini P., Anisotropic Equations in $L^1$, Diff. Int. Eq. 9 (1996), 209-212.

[8] Boccardo L., Marcellini P., Sbordone C., $L^\infty$– regularity for variational problems with sharp non-standard growth conditions, Boll. Un. Mat. Ital. A, 4 (1990), 219-225.

[9] Bousquet P., Brasco L., Lipschitz regularity for orthotropic functionals with nonstandard growth conditions, Rev. Mat. Iberoam. 36 (2020), 1989–2032.

[10] Brandolini B., Cirstea F. C., Anisotropic elliptic equations with gradient-dependent lower order terms and $L^1$ data, (2020) arXiv:2001.02754.

[11] Carozza M., Sbordone C., The distance to $L^\infty$ in some function spaces and applications, Diff. Int. Equ. 10 (1997), 599-607.

[12] Cianchi A., Local boundedness of minimizers of anisotropic functionals, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 17 (2000), 147-168.

[13] Cianchi A., Symmetrization in anisotropic elliptic problems, Comm. Partial Differential Equations 32 (2007), 693-717.

[14] Cupini G., Marcellini P., Mascolo E., Regularity under sharp anisotropic general growth conditions, Discrete Contin. Dyn. Syst. Ser. B 11 (2009), 67-86.

[15] Di Castro A., Existence and regularity results for anisotropic elliptic problems, Adv. Nonlinear Stud. 9 (2009), 367-393.

[16] Farroni F., Greco L., Moscariello G., Zecca G., Noncoercive quasilinear elliptic operators with singular lower order terms, Calc. Var. and Partial Differential Equations (2021), DOI: 10.1007/s00526-021-1965-z.

[17] Feo F., Vázquez J. L., Volzone B., Anisotropic Fast Diffusion Equations, arXiv:2007.00122.

[18] FragaÌ. I., Gazzola F., Kawohl B., Existence and nonexistence results for anisotropic quasilinear elliptic equations, Ann. I. H. Poincaré 21 (2004), 715-734.

[19] FragaÌ. I., Gazzola F., Lieberman G., Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains, Disc. Cont. Dyn. Syst. (2005), 280-286.

[20] Gagliardo E., Proprietà di alcune classi di funzioni in più variabili, Matematika 5 (1961), 87-116.
[21] Giannetti F., Greco L., Moscariello G., Linear elliptic equations with lower order terms, Diff. Int. Equ. 26 (2013), 623–638.

[22] Greco L., Moscariello G., Zecca G., Regularity for solutions to nonlinear elliptic equations, Diff. Int. Equ. 26 (2013), 1105-1113.

[23] Ladyzhenskaya O. A., Linear and Quasilinear Elliptic Equations, Academic Press (1968).

[24] Leonetti F., Innamorati A., Global integrability for weak solutions to some anisotropic elliptic equations, Nonlinear Analysis 113 (2015), 430-434.

[25] Moscariello G., Regularity results for quasi minima of functionals with nonpolynomial growth, J. Math. Anal. Appl. 168 (1992), 500–512.

[26] Krein S. G., Petunin Yu. I., Semenov E. M., Interpolation of linear operators, Transl. Math. Monogr. Amer. Math. Soc. 54 (1982).

[27] Pick L., Kufner A., John O., Fucik S., Function spaces, Walter de Gruyter, Berlin - New York (2013).

[28] Stampacchia G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Annales de l'Institut Fourier 15 (1965), 189-257.

[29] Stroffolini B., Global boundedness of solutions of anisotropic variational problems, Boll. Un. Mat. Ital. A 5 (1991), 345-352.

[30] Tartar L., Imbedding theorems of Sobolev spaces into Lorentz spaces, BUMI Serie 8 (1998), 479-500.

[31] Troisi M., Teoremi di inclusione per spazi di Sobolev non isotropi, Ricerche Mat. 18 (1969), 3-24.