Casorati Determinant Solutions for the Discrete Painlevé-II Equation

Kenji Kajiwara†, Yasuhiro Ohta‡*, Junkichi Satsuma§, Basil Grammaticos∥ and Alfred Ramani¶

†Department of Applied Physics, Faculty of Engineering, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

‡Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

§Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan

∥LPN, Université Paris VII, Tour 24-14, 5ème étage, 75251 Paris, France

¶CPT, Ecole Polytechnique, CNRS, UPR14, 91128 Palaiseau, France

* On leave from Department of Applied Mathematics, Faculty of Engineering, Hiroshima University
Abstract

We present a class of solutions to the discrete Painlevé-II equation for particular values of its parameters. It is shown that these solutions can be expressed in terms of Casorati determinants whose entries are discrete Airy functions. The analogy between the $\tau$ function for the discrete P$_{\text{II}}$ and the that of the discrete Toda molecule equation is pointed out.
1. Introduction

The six Painlevé transcendents are of very common occurrence in the theory of integrable systems[1]. Nonlinear evolution equations, integrable through inverse scattering techniques, were shown to possess one-dimensional (similarity) reductions that are just Painlevé equations. This feature of integrable PDE’s eventually evolved into an integrability criterion[2], the Painlevé property being intimately linked to integrability. Discrete integrable systems have recently become the focus of interest and an active domain of research. The study of partition function in a 2-D model quantum gravity[3,4] led to the discovery of the discrete analogue of the Painlevé-I (P₁) equation. It was followed closely afterwards by the derivation of the discrete P₁ in both a quantum gravity setting[5] and as a similarity reduction of a lattice version of mKdV equation[6]. The remaining discrete Painlevé equations (dP₁ to dP₅) were derived[7] using a more direct approach reminiscent of the Painlevé-Gambier[8] method for the continuous ones. This method, derived in [9] and dubbed singularity confinement, is the discrete equivalent of the Painlevé approach and offers an algorithmistic criterion for discrete integrability. One important result of these investigation is that the form of the discrete Painlevé equations are not unique: there exists several discrete analogues for each continuous Painlevé equations.

The continuous Painlevé equations were shown to be transcendental in the sense that their general solution cannot be expressed in terms of elementary functions[10]. In fact, this solution can be obtained only through inverse scattering methods. However in some particular cases (for special values of parameters) the solution to the Painlevé equations can be expressed in terms of special functions[11,12,13]. For example, P₁

\[ w_{xx} - 2w^3 + 2xw + \alpha = 0 , \]  

has a solution for \( \alpha = -(2N + 1) \)

\[ w = \left( \log \frac{\tau_{N+1}}{\tau_N} \right)_x , \]
where $\tau_N$ is given by an $N \times N$ Wronskian of the Airy function,

$$\tau_N = \begin{vmatrix} 1 & Ai & \frac{d}{dx} Ai & \cdots & \left(\frac{d}{dx}\right)^{N-1} Ai \\ \frac{d}{dx} Ai & \left(\frac{d}{dx}\right)^2 Ai & \cdots & \left(\frac{d}{dx}\right)^N Ai \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{d}{dx}\right)^{N-1} Ai & \left(\frac{d}{dx}\right)^N Ai & \cdots & \left(\frac{d}{dx}\right)^{2N-2} Ai \end{vmatrix}.$$  \hspace{1cm} (3)

Note that $Ai$ is the Airy function, satisfying

$$\frac{d^2}{dx^2} Ai = x Ai.$$  \hspace{1cm} (4)

From the close analogy that is known to exist between the continuous and discrete Painlevé equations one would expect special function-like solutions to exist for the discrete Painlevé equations as well. This is indeed the case. As was shown in [14], dP$_{\text{II}}$ has elementary solutions that can be expressed in terms of the discrete equivalent to the Airy function. In [14] only the simplest of these solutions was derived explicitly. The method for the obtention of the higher ones was based on the existence of an auto-Bäcklund transform for dP$_{\text{II}}$ but it is not clear how one can get the general expression of these special function solutions following this method. In this letter we intend to present the answer to this problem. Using Hirota’s bilinear formalism we can show that these particular solutions to dP$_{\text{II}}$ can be written as Casorati determinants whose entries are the discrete analogues of the Airy function.

### 2. Special Solutions of dP$_{\text{II}}$

We consider dP$_{\text{II}},$

$$w_{n+1} + w_{n-1} = \frac{(\alpha n + \beta)w_n + \gamma}{1 - w_n^2},$$  \hspace{1cm} (5)

where $\alpha$, $\beta$ and $\gamma$ are arbitrary constants. First, let us seek a simple solution of eq.(5). It is easily shown that if $w_n$ satisfies the following Riccati-type equation,

$$w_{n+1} = \frac{w_n - (an + b)}{1 + w_n},$$  \hspace{1cm} (6)
then it gives a solution of eq.(5) with the constraint $\gamma = -\alpha/2$. In fact, we have from eq.(6)

$$w_{n-1} = \frac{w_n + (an - a + b)}{1 - w_n}.$$  \hspace{1cm} (7)

Adding eqs.(6) and (7), we obtain

$$w_{n+1} + w_{n-1} = \frac{(2an - a + 2b + 2)w_n - a}{1 - w_n^2},$$  \hspace{1cm} (8)

which is a special case of eq.(5). Now we put

$$w_n = \frac{F_n}{G_n}.$$  \hspace{1cm} (9)

Substituting eq.(9) into eq.(6) and assuming that the numerators and the denominators of both sides of eq.(6) to be equal, respectively, we have

$$F_{n+1} = F_n - (an + b)G_n,$$  \hspace{1cm} (10a)

$$G_{n+1} = G_n + F_n.$$  \hspace{1cm} (10b)

Eliminating $F_n$ from eqs.(10a) and (10b), we see that $G_n$ satisfies

$$G_{n+2} - 2G_{n+1} + G_n = -(an + b)G_n,$$  \hspace{1cm} (11)

which is considered to be the discrete version of eq.(4) and has a solution given by the discrete analogue of the Airy function. By means of the solution, $w_n$ is expressed as

$$w_n = \frac{G_{n+1}}{G_n} - 1.$$  \hspace{1cm} (12)

It is possible to construct a series of solutions expressed by the discrete analogue of the Airy function. We here give the result, leaving the derivation in the next section. We consider the \( \tau \) function,

$$\tau_n^N = \begin{vmatrix}
A_n & A_{n+2} & \cdots & A_{n+2N-2} \\
A_{n+1} & A_{n+3} & \cdots & A_{n+2N-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n+N-1} & A_{n+N+1} & \cdots & A_{n+3N-3}
\end{vmatrix},$$  \hspace{1cm} (13)
where $A_n$ satisfies
\begin{equation}
A_{n+2} = 2A_{n+1} - (pn + q)A_n.
\end{equation}

We can show that $\tau^n_N$ satisfies the following bilinear forms,
\begin{equation}
\tau^{n-1}_{N+1}\tau^{n+2}_N = \tau^{n-1}_N\tau^{n+2}_N - \tau^n_N\tau^{n+1}_N ,
\end{equation}
\begin{equation}
\tau^{n+2}_{N+1}\tau^{n+1}_N - 2\tau^{n+1}_{N+1}\tau^{n+2}_N + (pn + q)\tau^{n+1}_{N+1}\tau^{n+3}_N = 0 ,
\end{equation}
and
\begin{equation}
\tau^{n+1}_{N+1}\tau^{n+2}_N = -(p(n + 2N) + q)\tau^{n+2}_N\tau^{n+1}_N + (pn + q)\tau^n_N\tau^{n+3}_N .
\end{equation}

Applying the dependent variable transformation as
\begin{equation}
w_n = \frac{\tau^{n+1}_{N+1}\tau^n_N}{\tau^{n+1}_N\tau^{n+1}_N} - 1 ,
\end{equation}
we obtain a special case of dP$_{II}$,
\begin{equation}
w_{n+1} + w_{n-1} = \frac{(2pn + (2N - 1)p + 2q)w_n - (2N + 1)p}{1 - w^2_n} .
\end{equation}

We note that eq.(8) and its solution is recovered by putting $p = a$, $q = b + 1$, and $N = 0$. We also note that eq.(19) reduces to eq.(1) with $\alpha = -(2N + 1)$ if we choose $p = -\epsilon^3$, $q = 1$, $w_n = \epsilon w$ and $n = \frac{x}{\epsilon}$, and take the limit of $\epsilon \to 1$.

3. Derivation of the results

In this section we show that eq.(13) really gives the solution of eq.(19) through the dependent variable transformation (18).

First, let us prove that the $\tau$ function (13) satisfies the bilinear forms (15)-(17). For the purpose we show that eqs.(15)-(17) reduce to the Jacobi identity or the Plücker relations. Before doing so, we give a brief explanation on the the Jacobi identity. Let $D$ be some
shown that eq.(13) satisfies eq.(15).

Similarly, we have

\[ D \left( \begin{array}{c}
  i \\
  j
\end{array} \right) D \left( \begin{array}{c}
  k \\
  l
\end{array} \right) - D \left( \begin{array}{c}
  i \\
  l
\end{array} \right) D \left( \begin{array}{c}
  k \\
  j
\end{array} \right) = D D \left( \begin{array}{c}
  i \\
  j \\
  k
\end{array} \right). \] (20)

It is easily seen that eq.(15) is nothing but the Jacobi identity. In fact, taking \( \tau_{N+1}^{n-1} \) as \( D \), and putting \( i = j = 1, k = l = N + 1 \), we find that eq.(15) reduces to eq.(20). Hence it is shown that eq.(13) satisfies eq.(15).

Let us next prove eq.(16). Notice that \( \tau_{N}^{n} \) is rewritten as

\[
\tau_{N}^{n} = \begin{vmatrix}
  A_{n} & \cdots & A_{n+2N-4} & 2A_{n+2N-3} - (p(n + 2N - 4) + q)A_{n+2N-4} \\
  A_{n+1} & \cdots & A_{n+2N-3} & 2A_{n+2N-2} - (p(n + 2N - 3) + q)A_{n+2N-3} \\
  \vdots & \cdots & \vdots & \vdots \\
  A_{n+N-1} & \cdots & A_{n+3N-5} & 2A_{n+3N-4} - (p(n + 3N - 5) + q)A_{n+3N-5}
\end{vmatrix} = \begin{vmatrix}
  A_{n} & \cdots & 2A_{n+1} & \cdots & 2A_{n+2N-3} \\
  A_{n+1} & \cdots & 2A_{n+2} - pA_{n+1} & \cdots & 2A_{n+2N-2} - pA_{n+2N-3} \\
  \vdots & \cdots & \vdots & \cdots & \vdots \\
  A_{n+N-1} & \cdots & 2A_{n+N} - (N - 1)pA_{n+N-1} & \cdots & 2A_{n+3N-4} - (N - 1)pA_{n+3N-5}
\end{vmatrix}
\]

where \( B_{n}^{(k)} \), \( k = 0, 1, \cdots \), are given by

\[ B_{n}^{(0)} = A_{n}, \quad B_{n}^{(k)} = A_{n+k} + \frac{kq}{2} B_{n}^{(k-1)} \quad \text{for} \quad k \geq 1. \] (22)

Similarly, we have

\[
(pn + q)\tau_{N}^{n} = 2^{N-1} \begin{vmatrix}
  A_{n+1} & B_{n+2}^{(0)} & A_{n+3} & \cdots & A_{n+2N-3} \\
  A_{n+2} & B_{n+2}^{(1)} & A_{n+4} & \cdots & A_{n+2N-2} \\
  \vdots & \vdots & \vdots & \cdots & \vdots \\
  A_{n+N} & B_{n+2}^{(N-1)} & A_{n+N+2} & \cdots & A_{n+3N-4}
\end{vmatrix}. \] (23)
Let us introduce the notations as

\[
\begin{align*}
\text{"}j\text{"} & = \begin{pmatrix}
A_{n+j} \\
A_{n+j+1} \\
\vdots 
\end{pmatrix}, & \quad \text{"}j'\text{"} & = \begin{pmatrix}
B^{(0)}_{n+j} \\
B^{(1)}_{n+j} \\
\vdots 
\end{pmatrix}, & \quad \phi & = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 
\end{pmatrix}.
\end{align*}
\]

For example, \(\tau^n_N\) and \((pn + q)\tau^n_N\) are rewritten by

\[
\begin{align*}
\tau^n_N & = 0, 2, \cdots, 2N - 2 = 0, 2, \cdots, 2N - 2, \phi | \\
& = 2^{N-1} | 0', 1, 3, \cdots, 2N - 3 |,
\end{align*}
\]

\[
(pn + q)\tau^n_N = 2^{N-1} | 1, 2', 3, 5, \cdots, 2N - 3 | = 2^{N-1} | 1, 2', 3, \cdots, 2N - 3, \phi |
\]

respectively. Now consider the following identity of \((2N + 2) \times (2N + 2)\) determinant,

\[
0 = \begin{vmatrix}
-1 & 0' & 1 & \cdots & 2N - 5 & 0 & 2N - 3 & \phi \\
-1 & 0' & 0 & 1 & \cdots & 2N - 5 & 2N - 3 & \phi 
\end{vmatrix}.
\]

Applying the Laplace expansion on the right hand side of eq.(25), we obtain

\[
0 = - | -1, 0', 1, \cdots, 2N - 5 | \times | 1, \cdots, 2N - 5, 2N - 3, \phi | \\
- | -1, 1, \cdots, 2N - 5, 2N - 3 | \times | 0', 1, \cdots, 2N - 5, \phi | \\
+ | -1, 1, \cdots, 2N - 5, \phi | \times | 0', 1, \cdots, 2N - 5, 2N - 3 |,
\]

which is nothing but the special case of the Plücker relations. Equation (26) is rewritten by using eqs.(21) and (23) as

\[
0 = (p(n - 2) + q) \tau^{n-2}_N \tau^{n+1}_{N-1} - 2 \tau^{n-1}_N \tau^{n}_{N-1} + \tau^n_N \tau^{n-1}_N,
\]

which is essentially the same as eq.(16).

We next prove that eq.(17) holds. We have the following equation similar to eqs.(21) and (23);

\[
(p(n + 2N) + q)\tau^{n+2}_N = - | 2, \cdots, 2N - 2, 2N + 2 | + 2^{N-1} | 2', 3, \cdots, 2N - 3, 2N + 1 |.
\]
Then the right hand side of eq.(17) is rewritten as

\[
\begin{align*}
&\lfloor 2, \cdots, 2N - 2, 2N + 2 \rfloor \times \lfloor 1, 3, \cdots, 2N - 1 \rfloor \\
+ &2^{N-1}\lfloor 1, 2', 3, \cdots, 2N - 5, 2N - 3 \rfloor \times \lfloor 3, 5, \cdots, 2N - 1, 2N + 1 \rfloor.
\end{align*}
\]

(29)

From the identity

\[
0 = \begin{vmatrix}
1 & 2' & 3 & \cdots & 2N - 3 \\
1 & 2' & \emptyset & 3 & \cdots & 2N - 3 \\
\end{vmatrix} \begin{vmatrix}
\emptyset & 2N - 1 & 2N + 1 \\
2N - 1 & 2N - 1 & 2N + 1 \\
\end{vmatrix}
= \lfloor 1, 2', 3, \cdots, 2N - 3 \rfloor \times \lfloor 3, 5, \cdots, 2N - 5, 2N - 1, 2N + 1 \rfloor \\
- \lfloor 1, 3, \cdots, 2N - 3, 2N - 1 \rfloor \times \lfloor 2', 3, \cdots, 2N - 3, 2N + 1 \rfloor \\
+ \lfloor 1, 3, \cdots, 2N - 3, 2N + 1 \rfloor \times \lfloor 2', 3, \cdots, 2N - 3, 2N - 1 \rfloor,
\]

(30)

the second and third terms of eq.(29) yield

\[
-2^{N-1}\lfloor 1, 3, \cdots, 2N - 3, 2N + 1 \rfloor \times \lfloor 2', 3, \cdots, 2N - 3, 2N - 1 \rfloor \\
= -\lfloor 1, 3, \cdots, 2N - 3, 2N + 1 \rfloor \times \lfloor 2, 4, \cdots, 2N - 2, 2N \rfloor.
\]

(31)

Hence, eq.(17) is reduced to

\[
\begin{align*}
&\lfloor 2, 4, \cdots, 2N - 2, 2N + 2 \rfloor \times \lfloor 1, 3, \cdots, 2N - 3, 2N - 1 \rfloor \\
- &\lfloor 1, 3, \cdots, 2N - 3, 2N + 1 \rfloor \times \lfloor 2, 4, \cdots, 2N - 2, 2N \rfloor \\
= &\lfloor 1, 3, \cdots, 2N - 1, 2N + 1 \rfloor \times \lfloor 2, 4, \cdots, 2N - 2 \rfloor,
\end{align*}
\]

(32)

which is again nothing but the Jacobi identity (20). In fact, taking \(D = \lfloor 1, 3, \cdots, 2N - 1, 2N + 1 \rfloor, i = 1, j = N + 1, k = N \) and \(l = N + 1 \), we see that eq.(20) is the same as eq.(32). This completes the proof that the \(\tau\) function (13) satisfies the bilinear forms (15)-(17).
Finally, let us derive eq.(19) from the bilinear forms (15)-(17). We introduce the dependent variables by

\[ v_n^N = \frac{\tau_{n+1}^N}{\tau_n^N}, \quad u_n^N = \frac{\tau_{n+1}^N \tau_{n+3}^N}{\tau_n^N \tau_{n+2}^N}. \]  

Then eqs. (15)-(17) are rewritten as

\[ v_n^{n-1} = v_{n-1}^{n+2} \left( 1 - \frac{1}{u_{n-1}^n} \right), \]  

\[ v_n^{n+2} - 2v_{n+1}^n + (pn + q)u_n^n v_n^N = 0, \]  

\[ v_{n+1}^n = v_{n-1}^{n+2} \left( -(p(n + 2N) + q) + (pn + q)u_n^n \right), \]

respectively. Eliminating \( u_N \) and \( v_{N-1} \) from eqs.(34)-(36) and introducing \( w_n \) by

\[ w_n = \frac{v_{n+1}^N}{v_n^N} - 1, \]  

we obtain eq.(19).

4. Concluding Remarks

In this letter, we have discussed the solution of dP\( _{II} \), (for semi-integer values of the parameter \( \gamma/\alpha \)) and shown that it is expressed as a Casorati determinant of the discrete Airy function. The most remarkable result is the structure of the \( \tau \) function (13). The subscript of \( A_n \) does not vary in the same way in the horizontal and vertical directions: it increases by one with each new row and by two with each new column. This is a feature which has not been encountered before in other discrete integrable systems.

Before concluding let us point out the relation with the Toda molecule equation. It is known in general that the \( \tau \) function of P\( _{II} \) satisfies the Toda molecule equation[11],

\[ \frac{d^2}{dx^2} \tau_N \cdot \tau_N - \left( \frac{d}{dx} \tau_N \right)^2 = \tau_{N+1} \tau_{N-1}, \quad N = 0, 1, 2, \ldots, \]

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whose solution is expressed as

\[ \tau_N = \begin{vmatrix} f & \frac{d}{dx} f & \cdots & \left(\frac{d}{dx}\right)^{N-1} f \\ \frac{d}{dx} f & \left(\frac{d}{dx}\right)^2 f & \cdots & \left(\frac{d}{dx}\right)^N f \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{d}{dx}\right)^{N-1} f & \left(\frac{d}{dx}\right)^N f & \cdots & \left(\frac{d}{dx}\right)^{2N-2} f \end{vmatrix} , \quad (39) \]

where \( f \) is an arbitrary function. It is clear that eq.(3) is a special case of eq.(39). Hence, we may expect that the \( \tau \) function of dP_{II} satisfies the discrete Toda molecule equation proposed by Hirota[15],

\[ \Delta^2 \tau_n^N \cdot \tau_n^N - \left(\Delta \tau_n^N\right)^2 = \tau_{n+1}^N \tau_{n-1}^{n+2} , \quad N = 0, 1, 2 \cdots , \quad (40a) \]

or

\[ \tau_n^{n+2} \tau_n^N - \left(\tau_{n+1}^{n+1}\right)^2 = \tau_{n+1}^N \tau_{n-1}^{n+2} , \quad N = 0, 1, 2, \cdots , \quad (40b) \]

whose solution is given by

\[ \tau_N = \begin{vmatrix} f_n & \Delta f_n & \cdots & \Delta^{N-1} f_n \\ \Delta f_n & \Delta^2 f_n & \cdots & \Delta^N f_n \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{N-1} f_n & \Delta^N f_n & \cdots & \Delta^{2N-2} f_n \end{vmatrix} \quad (41) \]

for arbitrary \( f_n \), where \( \Delta \) is a forward difference operator in \( n \) defined by

\[ \Delta \tau_n^N = \tau_{n+1}^N - \tau_n^N . \]

However, because of the difference of the structure of the \( \tau \) function mentioned above, that of dP_{II} does not satisfy the discrete Toda molecule equation (40) itself. In fact, eq.(15) may be regarded as an alternative of eq.(40), which is rewritten as

\[ \left(\Delta \Delta' \tau_n^N\right) \cdot \tau_n^N - \left(\Delta \tau_n^N\right) \cdot \left(\Delta' \tau_n^N\right) = \tau_{n+1}^N \tau_{n-1}^{n+3} \quad (43) \]

where \( \Delta' \) is given by

\[ \Delta' \tau_n^N = \tau_{n+2}^N - \tau_n^N . \]
Indeed, eq.(43) also reduces to the ordinary Toda molecule equation (38) in the continuum limit.

It is expected that the other discrete Painlevé equations have also the solutions expressed by Casorati determinants whose entries are the discrete special functions. In particular for dP_{III} it was shown in [16] that solutions in terms of discrete Bessel functions exist for some values of the parameters, while for dP_{IV} the particular solutions are in terms of discrete parabolic cylinder (Weber-Hermite) functions[17]. In a forthcoming paper we intend to present Casorati determinant-type solutions for these discrete Painlevé equations. One more interesting point concerns the existence of rational solutions. Both continuous and discrete Painlevé equations possess such solutions and it should be in principle to obtain general expressions for them in terms of Casorati determinants.

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