Modules of G-dimension zero over local rings with the cube of maximal ideal being zero

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Abstract

Let \((R, m)\) be a commutative Noetherian local ring with \(m^3 = (0)\). We give a condition for \(R\) to have a non-free module of G-dimension zero. We shall also construct a family of non-isomorphic indecomposable modules of G-dimension zero with parameters in an open subset of projective space. We shall finally show that the subcategory consisting of modules of G-dimension zero over \(R\) is not necessarily a contravariantly finite subcategory in the category of finitely generated \(R\)-modules.

1 Introduction

The notion of G-dimension of finitely generated modules are introduced by Auslander and Bridger [1]. Although various properties concerning G-dimension has been known, it still seems to lack references which show us many examples of modules of G-dimension zero. One of our aims here is to give such examples of modules. Actually we construct a family of modules of G-dimension zero with continuous parameters for certain cases.

If the ring \(R\) is a Gorenstein local ring, then an \(R\)-module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. Thus it will be natural to expect that many of the properties of the category of maximal Cohen-Macaulay modules over a Gorenstein local ring are satisfied as well for the category of modules of G-dimension zero over a general local ring. This is a main reason for us to consider modules of G-dimension zero in this paper. However, as a result of our construction of family of modules G-dimension zero, we conclude unfortunately that, for a certain kind of local ring, the category of modules of G-dimension zero is not a contravariantly finite subcategory in the category of finitely generated modules.

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We start recalling the definition and elementary properties of G-dimension in §2. We shall also remark there that if \((R, \mathfrak{m})\) is an Artinian local ring with \(\mathfrak{m}^2 = (0)\) or more generally if \(R\) is a Cohen-Macaulay local ring with minimal multiplicity, then there is no nonfree \(R\)-module of G-dimension zero. Therefore we have enough reason to consider local rings \((R, \mathfrak{m})\) with \(\mathfrak{m}^3 = (0)\) as the easiest cases of Artinian local rings which possess nonfree modules of G-dimension zero. In §3 we get a necessary condition for such \(R\) to have nonfree module of G-dimension zero. Roughly speaking, the ring is homologically unique, that is, \(R\) should be a Koszul algebra with unique Poincaré series, Bass series and Hilbert series.

In §4 we consider mainly Artinian local rings with nontrivial deformation. For such local rings, the necessary condition that we get in §3 is sufficient as well to have a nonfree module of G-dimension zero. Surprisingly enough, we can also show that such a local ring \(R\) always has the form \(S/fS\) where \(S\) is a 1-dimensional Cohen-Macaulay local ring with minimal multiplicity and \(f\) is any nonzero divisor of degree two. For the rings of this form, we shall very concretely construct a family of modules of G-dimension zero with continuous parameters in §5. And in §6 we prove that the category of modules of G-dimension zero is not a contavariantly finite subcategory in the category of finitely generated modules over a local ring of this form.

## 2 Preliminaries for G-dimension

Throughout the present paper, \(R\) denotes a commutative Noetherian local ring with maximal ideal \(\mathfrak{m}\), and by an \(R\)-module we always mean a finitely generated \(R\)-module.

In this section, let us recall the definition of G-dimension and its elementary properties.

**Definition 2.1** Let \(M\) be an \(R\)-module. We say that \(M\) has G-dimension zero if it satisfies the following conditions:

1. The natural morphism \(M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)\) is an isomorphism.
2. \(\text{Ext}_R^i(M, R) = 0\) for all \(i > 0\).
3. \(\text{Ext}_R^i(\text{Hom}_R(M, R), R) = 0\) for all \(i > 0\).

It is easy to see from the definition that if \(M\) is a module of G-dimension zero, then there is an exact sequence of free \(R\)-modules:

\[
F_\bullet : \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots
\]

with the property that the dual complex \(\text{Hom}_R(F_\bullet, R)\) is also exact, and \(M \cong \text{Coker}(F_1 \rightarrow F_0)\). We call such an exact sequence \(F_\bullet\) a complete resolution of \(M\).

**Definition 2.2** Let \(n\) be an integer. We say that an \(R\)-module \(M\) has G-dimension at most \(n\), denoted by \(\text{G-dim}_R M \leq n\), if the \(n\)-th syzygy module

\[
\text{Ext}_R^n(\text{Hom}_R(M, R), R) = 0
\]
\( \Omega^n_R(M) \) of \( M \) has G-dimension zero. If there is no such integer \( n \), we denote G-dim \( R,M = \infty \).

As it is shown in the paper of Auslander and Bridger [1], the G-dimension enjoys several good properties. We recall some of them from [1] for the later use.

**Proposition 2.3**  
1. If G-dim \( R,M = \infty \), then G-dim \( R,M = \text{depth} R - \text{depth} M \).

2. The following are equivalent for a local ring \( R \):
   - \( R \) is a Gorenstein ring.
   - G-dim \( R,M < \infty \) for every \( R \)-module \( M \).
   - G-dim \( R,R/\mathfrak{m} < \infty \).

3. Let \( x \in \mathfrak{m} \) be a non-zero divisor on both \( R \) and \( M \). Then, we have G-dim \( R,M = \text{G-dim}_R/xR M/xM \).

4. Let \( x \in \mathfrak{m} \) be a non-zero divisor on \( R \) satisfying \( xM = 0 \). Then, we have G-dim \( R,M = \text{G-dim}_R/xR M + 1 \).

Now we assume that \((R,\mathfrak{m})\) is a local ring that satisfies \( \mathfrak{m}^2 = (0) \). In this case, it can be easily seen that there is no nontrivial module of G-dimension zero, unless \( R \) is Gorenstein. More precisely, we can prove the following proposition:

**Proposition 2.4** Let \((R,\mathfrak{m})\) be a local ring with \( \mathfrak{m}^2 = (0) \) and suppose that \( R \) is not a Gorenstein ring. Then every \( R \)-module of G-dimension zero is a free \( R \)-module.

**Proof** Let \( M \) be an indecomposable \( R \)-module with G-dim \( R,M = 0 \). Considering the complete resolution of \( M \), we can embed \( M \) into a free \( R \)-module; \( M \subset F \). If there is an element \( x \) of \( M \) that does not belong to \( \mathfrak{m}F \), then \( Rx \) is a direct summand of \( F \), hence of \( M \). Since we assume that \( M \) is indecomposable, \( M \cong R \) in this case.

Thus we may assume that \( M \subset \mathfrak{m}F \). Then, since \( \mathfrak{m}^2 = (0) \), we have \( \mathfrak{m}M = 0 \), and hence \( M \cong R/\mathfrak{m} \), because \( M \) is indecomposable. As a result, we have G-dim \( R/\mathfrak{m} = 0 \). Then it follows from Proposition 2.3(2) that \( R \) must be a Gorenstein ring. \( \square \)

Recall that a Cohen-Macaulay local ring \((R,\mathfrak{m})\) has minimal multiplicity if there is a system of parameters \( \underline{x} = x_1,x_2,\ldots,x_d \) such that \( \underline{x}\mathfrak{m} = \mathfrak{m}^2 \). The above proposition automatically implies the same result for modules over a Cohen-Macaulay local ring with minimal multiplicity.

**Corollary 2.5** Let \((R,\mathfrak{m})\) be a Cohen-Macaulay local ring with minimal multiplicity. Suppose that \( R \) is not a Gorenstein ring. Then every \( R \)-module of G-dimension zero is a free \( R \)-module.
Proof Let $M$ be an $R$-module with $\text{G-dim}_RM = 0$. Note that $M$ is a maximal Cohen-Macaulay module over $R$, for depth $M = \text{depth} R$ by Proposition 2.3(1).

Now take a system of parameters $x$ such as $x^m = m^2$, and let $\overline{R}$ (resp. $\overline{M}$) denote $R/xR$ (resp. $M/xM$). Note that the square of the maximal ideal of $\overline{R}$ is zero. On the other hand, by a successive use of Proposition 2.3(3), we have $\text{G-dim}_{\overline{R}}\overline{M} = 0$. Thus $\overline{M}$ is a free $\overline{R}$-module. Now let $r = \text{dim}_k(M \otimes_R k) = \text{dim}_k(M \otimes_{\overline{R}} k)$, where $k$ denotes the residue class field $R/m$. And we can make a minimal free cover of $M$:

$$0 \to N \to R^r \xrightarrow{\pi} M \to 0,$$

satisfying $\pi \otimes_R \overline{R}$ is an isomorphism. Thus, since $\text{Tor}_1^R(M, \overline{R}) = 0$, we have $N \otimes_R \overline{R} = 0$ and hence $N = 0$. Therefore $M \cong R^r$ as desired. □

3 Local rings with $m^3 = (0)$

As we have shown in the previous section, Artinian non-Gorenstein local rings with $m^2 = (0)$ have no nontrivial modules of G-dimension zero. For the next step, we shall consider Artinian non-Gorenstein local rings with $m^3 = (0)$, in this section. If such a ring has a nonfree module of G-dimension zero, then the ring structure will be strongly restricted as we prove in the following theorem.

**Theorem 3.1** Let $(R, m)$ be a non-Gorenstein local ring with $m^3 = (0)$, but $m^2 \neq (0)$. For the simplicity we assume that $R$ contains a field $k$ isomorphic to $R/m$. And let $r$ be the type of $R$, i.e. $r = \text{dim}_k \text{Hom}_R(k, R)$.

Now assume that there is a nonfree $R$-module with $\text{G-dim}_RM = 0$. Then the following conditions hold:

1. $R$ has a natural structure of homogeneous graded ring with $R = R_0 \oplus R_1 \oplus R_2$, where $R_0 = k$ and $\text{dim}_k R_1 = r + 1$, $\text{dim}_k R_2 = r$. In particular, the Hilbert series of $R$ is

   $$H_R(t) = (1 + t)(1 + rt),$$

   and $(0 :_R m) = m^2$.

2. Under the above graded structure, $R$ is a Koszul algebra whose Poincaré series is

   $$P_R(t) = \frac{1}{(1-t)(1-rt)}.$$

3. The Bass series of $R$ is as follows:

   $$B_R(t) = \frac{r - t}{1 - rt}.$$

4. Every $R$-module $M$ of G-dimension zero has a natural structure of graded $R$-module. If $M$ has no free summand, then $M$ has only two graded pieces; $M = M_0 \oplus M_1$, where putting $b = \text{dim}_k M_0$, we see that $b$ is equal to the minimal number of generators of $M$ and $\text{dim}_k M_1 = rb$, in particular, the length of the $R$-module $M$ is $\ell_R(M) = b(1 + r)$. Furthermore, $M$ has a free resolution of the following type:

   $$\cdots \to R(-n - 1)^b \to R(-n)^b \to \cdots \to R(-1)^b \to R^b \to M \to 0.$$
Proof. Let $M$ be a nonfree $R$-module with $\text{G-dim}_R M = 0$. If necessary, taking a summand of $M$, we assume that $M$ is a nonfree indecomposable module. We prove the theorem step by step.

(Step 1) We show that $m^2 M = 0$.

In fact, as in the proof of Proposition 2.3, $M$ can be embedded into $mF$ for some free module $F$. Since $m^3 = (0)$, we have $m^2 M = 0$. □

Therefore, considering a filtration $(0) \subseteq mM \subseteq M$, and noting from the above that $m(mM) = 0$, we have the following exact sequence of $R$-modules:

$$0 \to k^a \to M \to k^b \to 0,$$

where $a = \dim_k mM$ and $b = \dim_k M/mM$. Note that the length $\ell_R(M)$ of the $R$-module $M$ is $a + b$. Note also that $a \neq 0$ and $b \neq 0$. Otherwise, $M$ would be a direct sum of $R/m$ and $R$ would be a Gorenstein ring by Proposition 2.3(2).

(Step 2) We put $\mu_i = \dim_k \text{Ext}^i_R(k, k)$ for $i \geq 0$. (3.2)

And we claim that $a \geq b$ and $a \mu_i = b \mu_{i+1}$ for $i \geq 1$.

Before proving this, we note that $r = \mu_0$. Since $\text{Ext}^i_R(M, R) = 0$ for $i \geq 1$, it follows from (1) that $\text{Ext}^i_R(k^a, R) \cong \text{Ext}^{i+1}_R(k^b, R)$ for $i \geq 1$, hence we have $a \mu_i = b \mu_{i+1}$ ($i \geq 1$). Thus,

$$\mu_1 = \left(\frac{b}{a}\right) \mu_2 = \left(\frac{b}{a}\right)^2 \mu_3 = \cdots = \left(\frac{b}{a}\right)^i \mu_{i+1} = \cdots$$

Supposed that $a < b$. Then we would have $\mu_1 < \left(\frac{b}{a}\right)^i$ for a large $i$. Since all the $\mu_i$ are nonnegative integers, we must have $\mu_i = 0$ for all $i \geq 1$. This exactly means that $R$ is a Gorenstein ring, which contradicts the assumption. □

(Step 3) We denote $M^* = \text{Hom}_R(M, R)$. Then, we have equalities

$$\mu_1 = r^2 - 1 \quad \text{and} \quad \ell_R(M^*) = \ell_R(M).$$

Applying the functor $\text{Hom}_R(\quad, R)$ to the exact sequence (1), we have an exact sequence $0 \to k^{rb} \to M^* \to k^{ra} \to k^{\mu_1b} \to 0$, hence

$$0 \to k^{rb} \to M^* \to k^{ra-\mu_1b} \to 0.$$ (3.3)

Note that, since $M^*$ is also an indecomposable module of G-dimension zero, $M^*/mM^* \cong k^{ra-\mu_1b}$ and $mM^* \cong k^{rb}$. In particular, we have

$$\ell(M^*) = r \ell(M) - \mu_1 b.$$ (3.4)

Now, since $\text{G-dim}_R M^* = 0$, this equality holds for $M^*$ and hence

$$\ell(M^{**}) = r \ell(M^*) - \mu_1 (ra - \mu_1 b).$$
Since $M^{**} \cong M$, we have

\[
\ell(M) = r \ell(M^*) - \mu_1 (ra - \mu_1 b) = r \ell(M^*) - \mu_1 (r \ell(M^*) - rb) = -\mu_1 \ell(M^*) + r^2 \ell(M).
\]

As a consequence we get

\[
\mu_1 \ell(M^*) = (r^2 - 1) \ell(M).
\]

Since this equation holds for any nonfree indecomposable module $M$ of $G$-dimension 0, one can apply this to $M^*$ and obtains

\[
\mu_1 \ell(M) = (r^2 - 1) \ell(M^*).
\]

Comparing above two equations we finally obtain that $\mu_1 = r^2 - 1$ and $\ell(M^*) = \ell(M)$, as desired. □

(Step 4) The following equations hold:

\[
a = rb, \quad \text{and} \quad \mu_i = r^{i-1} \mu_1 = r^{i+1} - r^{i-1} \quad \text{for all} \quad i \geq 1.
\]

In particular, the Bass series of $R$ is

\[
B_R(t) = \sum_{i \geq 0} \mu_i t^i = r + \sum_{i \geq 1} (r^{i+1} - r^{i-1}) t^i = \frac{r - t}{1 - rt}
\]

In fact, we have shown in Step 2 that

\[
a \mu_i = b \mu_{i+1} \quad \text{for} \quad i \geq 1,
\]

which holds for any nonfree indecomposable module $M$ of $G$-dimension zero. Now apply this to $M^*$, and we have from (3) that

\[
(rb) \mu_i = (ra - \mu_1 b) \mu_{i+1} \quad \text{for} \quad i \geq 1.
\]

Note that $\mu_i > 0$ for all $i \geq 0$, since $R$ is non-Gorenstein. See [5 Lemma (3.5)]. Hence the above two equations implies that $(rb)b = a(ra - \mu_1 b)$, equivalently $rb^2 + (r^2 - 1)ab - ra^2 = 0$ by Step 3. It follows that $t^2 (rb - a)(b - ra) = 0$. Since we have shown in Step 2 that $a \geq b$, and since $r \geq 1$, this implies that $a = rb$. Also from Step 2 we have that $\mu_{i+1} = \frac{r^i}{b} \mu_i = r \mu_i$ for $i \geq 1$. Therefore $\mu_i = r^{i-1} \mu_1 = r^{i+1} (r^2 - 1)$ for $i \geq 1$. □

(Step 5) Since $M$ is generated by $b$ elements, we have a short exact sequence:

\[
0 \to \Omega M \to R^b \to M \to 0,
\]

where $\Omega M$ is the first syzygy module of $M$. We claim here that

\[
m \Omega M = m^2 R^b.
\]
Since \( R^b \to M \) is a minimal free cover, we have \( \Omega M \subseteq mR^b \), therefore \( m\Omega M \subseteq m^2R^b \). On the other hand, since \( m^2M = 0 \) by Step 1, we have \( m^2R^b \subseteq \Omega M \). In order to prove the above equality, suppose that \( m\Omega M \neq m^2R^b \). Then there is an \( x \in m^2R^b \setminus m\Omega M \subseteq \Omega M \). Note that \( x \) is one of the minimal generators of \( \Omega M \) and \( mx = 0 \), for \( m^4 = 0 \). Thus the composition of natural maps
\[
k \cong R x \subseteq \Omega M \to \Omega M/m\Omega M \to kx \cong k
\]
is the identity map on \( k \). Hence the submodule \( R x \cong k \) is a direct summand of \( \Omega M \). It then follows that \( \text{G-dim}_k k = 0 \), hence \( R \) must be a Gorenstein ring. This is a contradiction, hence we have \( m\Omega M = m^2R^b \). □

(Step 6) Now we prove that \( \Omega M \) is minimally generated by \( b \) elements and that the following equalities hold:
\[
\dim_k m/m^2 = r + 1, \quad \dim_k m^2 = r.
\]

To prove this, we put \( e = \dim_k m/m^2 \) and \( f = \dim_k m^2 \). First of all, we note from Step 5 that we have the following exact sequence:
\[
0 \to \Omega M/m\Omega M \to (R/m^2)^b \to M \to 0 \quad (3.6)
\]
Therefore, since \( \ell_R(M) = a + b \), it follows that \( \Omega M/m\Omega M \cong k^{b-r-a} \). On the other hand, by Step 5, we have \( m\Omega M \cong m^2R^b \cong k^{b-f} \). In particular, there is an exact sequence;
\[
0 \to k^{b-f} \to \Omega M \to k^{b-r-a} \to 0.
\]
Note that \( \Omega M \) is an indecomposable module of G-dimension zero as well as \( M \), and hence the equations in Step 4 hold for \( \Omega M \). Thus we have \( bf = r(be - a) \).

Since \( a = rb \), we finally get
\[
f = r(e - r).
\]
Note that \( m^2 \subseteq (0 : m) \), since \( m^3 = 0 \). In particular, \( 0 < f \leq r \) from the definition. Therefore it follows from the above equality that
\[
f = r \quad \text{and} \quad e - r = 1.
\]
Note also that we have shown that \( m^2 = (0 : m) \). Since \( \Omega M/m\Omega M \cong k^{b-r-a} \), \( \Omega M \) is minimally generated by \( be - a \) elements, but we have \( be - a = b(r + 1) - a = b + (rb - a) = b \), by virtue of Step 4. □

(Step 7) Now we shall prove that \( R \) has natural structure of homogeneous graded ring with \( R = R_0 \oplus R_1 \oplus R_2 \), where
\[
R_0 \cong k, \quad \dim_k R_1 = r + 1 \quad \text{and} \quad \dim_k R_2 = r.
\]
Furthermore, \( M \) has natural structure of graded \( R \)-module with \( M = M_0 \oplus M_1 \), where \( \dim_k M_0 = b \) and \( \dim_k M_1 = rb \).

Since we have shown in Step 6 that \( \dim_k m/m^2 = r + 1 \), we can find a \( k \)-linear subspace \( V \) of \( m \) with \( \dim_k V = r + 1 \) in such a way that \( V \) is isomorphic
to $m/m^2$ through the natural map $m \to m/m^2$. Then clearly $V \cap m^2 = (0)$, and hence $R = k \oplus V \oplus m^2$ as a $k$-vector space. This gives a graded ring structure of $R$ with $R_0 = k$, $R_1 = V$ and $R_2 = m^2$. It is easy to see that $m^2 = V \cdot V$ by the product in $R$, and thus the graded ring $R$ is generated in degree 1, i.e. a homogeneous graded ring.

Similarly to the above, since $\dim_k M/mM = b$, we can find a linear subspace $U$ of $M$ with $\dim_k U = b$ in such a way that $U$ is isomorphic to $M/mM$ through the natural map $M \to M/mM$. Then clearly $U \cap mM = (0)$, and hence $M = U \oplus mM$ as a $k$-vector space. It is easy to see that $V \cdot U = mM$ and $V \cdot mM = 0$. Therefore $M$ has a graded $R$-module structure $M_0 \oplus M_1$ with $M_0 = U$ and $M_1 = mM$. Note that $M$ is generated in degree 0 as a graded $R$-module. Note also that $\dim_k mM = a$ and this equals $rb$ by Step 4. □

(Step 8) We show here that the graded $R$-module $M$ has the following type of linear free resolution:

$$\cdots \to R(-n-1)^b \to R(-n)^b \to \cdots \to R(-2)^b \to R(-1)^b \to R^b \to M \to 0$$

In particular, we have an equality

$$\dim_k \text{Tor}_i^R(M, k)_j = \begin{cases} b & (i = j), \\ 0 & (i \neq j), \end{cases}$$

where $\text{Tor}_i^R(M, k)_j$ is piece of degree $j$ in the graded module $\text{Tor}_i^R(M, k)$.

We have shown in Step 7 that $M$ is minimally generated by $b$ elements of $M_0$. Therefore we may take a minimal free cover $R^b \to M$ being a graded homomorphism. And since $0 \to \Omega M \to R^b \to M \to 0$, $\Omega M$ is also a graded $R$-module. Furthermore, by the dimension argument, we have

$$\dim_k (\Omega M)_1 = b \dim_k R_1 - \dim_k M_1 = b(r+1) - rb = b,$$
$$\dim_k (\Omega M)_2 = b \dim_k R_2 - \dim_k M_2 = rb$$
and
$$\dim_k (\Omega M)_i = 0 \quad \text{for} \quad i \neq 1, 2.$$

Since $\Omega M$ is also an indecomposable module of $G$-dimension zero, it follows from Step 7 that $\Omega M$ is minimally generated by $b$ elements in $(\Omega M)_1$ and that there is a minimal free cover as a graded $R$-module: $R(-1)^b \to \Omega M$. Thus one can continue this procedure to get the result stated in Step 8. □

(Step 9) Now setting $\beta_{ij} = \dim_k \text{Tor}_i^R(k, k)_j$ for all integers $i$ and $j$, we can show that

$$\beta_{ij} = \begin{cases} 1 + r + r^2 + \cdots + r^i & (i = j \geq 0), \\ 0 & (\text{otherwise}). \end{cases}$$

(3.7)

In particular, the graded ring $R$ is a Koszul algebra whose Poincaré series is

$$P_R(t) = \frac{1}{(1-t)(1-rt)}.$$

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Before proving this, we note from Step 7 that there is an exact sequence
\[ 0 \to k(-1)^{br} \to M \to k^b \to 0. \]
Tensoring \( k \) over \( R \) to this sequence, we have a long exact sequence of \( k \)-vector spaces for any integer \( i \) and \( j \);
\[
\cdots \to \text{Tor}_i^R(k,k)^{br}_{j-1} \to \text{Tor}_i^R(M,k)_j \xrightarrow{\varphi_{ij}} \text{Tor}_i^R(k,k)_j^b \to \text{Tor}_i^R(k,k)^{br}_{j-1} \to \cdots
\]
(3.8)
Now we shall compute \( \beta_{ij} \). First of all, if \( j < 0 \), then it is clear that \( \beta_{ij} = 0 \) for all \( i \). We prove the equality (7) by induction on \( j \geq 0 \). For the first case we assume \( j = 0 \). Putting \( j = 0 \) in (8), we see that \( \varphi_{i0} \) is an isomorphism, since \( \beta_{i,-1} = 0 \) for all \( i \). Therefore we have \( b\beta_{i0} = \dim_k \text{Tor}_i^R(M,k)_0 \), hence it follows from Step 8 that \( \beta_{i0} = 1 \) and \( \beta_{i0} = 0 \) if \( i > 0 \).

Now assume \( j > 0 \). By the induction hypothesis, we have that \( \text{Tor}_i^R(k,k)_{j-1} = 0 \) unless \( i = j - 1 \). Hence it follows from (8) that
\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Tor}_i^R(M,k)_j & \xrightarrow{\varphi_{ij}} \text{Tor}_i^R(k,k)_j^b & \longrightarrow & \text{Tor}_i^R(k,k)^{br}_{j-1} \\
\longrightarrow & & \text{Tor}_i^R(M,k)_j & \xrightarrow{\varphi_{j-1j}} \text{Tor}_i^R(k,k)_j^b & \longrightarrow & 0,
\end{array}
\]
and that \( \text{Tor}_i^R(M,k)_j \cong \text{Tor}_i^R(k,k)_j^b \) if \( i \neq j, j-1 \). Using the result in Step 8, we get \( \text{Tor}_i^R(k,k)_j^b = 0 \) if \( i \neq j \) and \( b\beta_{jj} = b + br\beta_{j-1j-1} \). This shows that \( \beta_{ij} = 0 \) for \( i \neq j \) and \( \beta_{jj} = 1 + r\beta_{j-1j-1} \), which proves Step 9. □

From all these steps we have proved any of the statements in Theorem (3.1).
(Q.E.D.)

4 Local rings with nontrivial deformation

As we have shown in the previous section, a local ring \((R, \mathfrak{m})\) with \( \mathfrak{m}^3 = (0) \) has a nonfree module of G-dimension zero only if \( R \) is a homogeneous graded ring. Therefore, in this section, we consider only a homogeneous graded Artinian ring, that is, the ring \( R \) is a residue ring of a polynomial ring by a homogeneous ideal:
\[ R = k[X_0, X_1, \ldots, X_r]/I, \]
where all variables \( X_i \) are of degree one and \( I \) is a homogeneous ideal, and \( \mathfrak{m} \) is the maximal homogeneous ideal generated by \( X_0, X_1, \ldots, X_r \).

In this case, we make the following definition.

**Definition 4.1** we say that a graded Artinian ring \( R \) has a **nontrivial deformation** if there is a homogenous Cohen-Macaulay graded ring \( S \) of positive dimension \( d \) and a homogeneous regular sequence \( f_1, f_2, \ldots, f_d \) of \( S \) with degree \( \geq 2 \) such that \( R \cong S/(f_1, f_2, \ldots, f_d)S \).

In Theorem (3.1) we have shown several necessary conditions for \( R \) to have a nonfree module of G-dimension zero. If \( R \) has a nontrivial deformation, then we can prove they are sufficient as well.

**Theorem 4.2** Let \( R \) be a homogeneous graded ring over a field \( k \) with the graded maximal ideal \( \mathfrak{m} \). Assume that \( R \) is not a Gorenstein ring. And suppose that \( R \) has a nontrivial deformation. Then the following conditions are equivalent:
(1) $R$ has a nonfree module of $G$-dimension zero, and $m^3 = (0)$.

(2) The Hilbert series of $R$ is

$$H_R(t) = (1 + t)(1 + rt),$$

for some integer $r \geq 2$.

(3) There is a one-dimensional Cohen-Macaulay homogeneous graded ring $S$ with minimal multiplicity and a homogeneous non-zero divisor $f$ of degree 2 of $S$ such that $R \cong S/fS$.

Furthermore, under the condition (3), we have an equality

$$G\text{-dim}_R M = \text{pd}_S M - 1 \quad (\leq \infty) \quad (4.1)$$

for any $R$-module $M$. (Therefore $G\text{-dim}_R M = \text{CI}\text{-dim}_R M$.)

**Proof** (1) $\Rightarrow$ (2): We have proved in Theorem (3.1) the condition on the Hilbert series of $R$. We just note that the socle dimension $r = \dim_k \text{Hom}_R(k, R)$ is greater than 1, since $R$ is not Gorenstein.

(2) $\Rightarrow$ (3): Let $S$ be a Cohen-Macaulay homogeneous graded algebra over $k$ of positive dimension $d$ and $R \cong S/(f_1, f_2, \ldots, f_d)S$ with a regular sequence $f_1, f_2, \ldots, f_d$ in $S$ of degree $\geq 2$. Note, in general, that the Hilbert series of $S$ is of the form

$$H_S(t) = \frac{p(t)}{(1-t)^d},$$

for some polynomial $p(t)$ with nonnegative integral coefficients. See [6, Corollary 4.1.10]. And also note that $S$ has minimal multiplicity if and only if $p(t) = 1 + nt$ for some integer $n$. Putting $c_i = \deg(f_i) \geq 2$, we know that there is an equality

$$H_R(t) = \prod_{i=1}^{d} (1 - t^{c_i}) H_S(t).$$

Hence, under the condition of (2), we have

$$(1 + t)(1 + rt) = \prod_{i=1}^{d} (1 + t + t^2 + \cdots + t^{c_i-1}) p(t).$$

Since this equality holds as an elements in $\mathbb{Z}[t]$, we must have $d = 1$, $c_1 = 2$ and $p(t) = 1 + rt$. This exactly means that $S$ is of dimension one, having minimal multiplicity and $f_1$ is of degree 2.

(3) $\Rightarrow$ (1): Let $R \cong S/fS$, where $S$ is a one-dimensional Cohen-Macaulay homogeneous graded ring with minimal multiplicity and $f$ is a homogeneous non-zero divisor of degree 2 of $S$.

First we shall prove the equality (9). For this, let $M$ be an $R$-module. Then it follows from Proposition (2.3)(4) that $G\text{-dim}_R M = G\text{-dim}_S M - 1$. In particular, $G\text{-dim}_R M < \infty$ if and only if $G\text{-dim}_S M < \infty$. If this is the case, then a certain syzygy module of $M$ as an $S$-module has $G$-dimension zero,
however as we showed in Corollary 2.5, such a syzygy module has to be a free $S$-module. This shows that $G\dim_R M < \infty$ if and only if the $S$-module $M$ has finite projective dimension, i.e. $\text{pd}_S M < \infty$. On the other hand, we know from Proposition 2.3(1) that $G\dim_R M = \text{depth } R - \text{depth } M = 0$ and $\text{pd}_S M = \text{depth } S - \text{depth } M = 1$ if they are finite. Thus we have shown the equality (9) including the case that the both sides are infinite. (Note from [4] that it is known the inequality $G\dim_R M \leq CI\dim_R M \leq \text{pd}_S M - 1$ for any $R$-module $M$, hence they are all equal in this case.)

Next we note that $m^3 = (0)$. In fact, since $S$ has minimal multiplicity of dimension one, the Hilbert series $H_S(t)$ of $S$ is of the form

$$H_S(t) = \frac{1 + rt}{1 - t}.$$  

Thus,

$$H_R(t) = H_S(t)(1 - t^2) = (1 + t)(1 + rt),$$

since $f$ is a non-zero divisor of degree two. This shows that $R$ has no nontrivial homogeneous components of degree greater than two, and hence we have $m^3 = (0)$.

Now we want to show that there exists a nonfree $R$-module of $G$-dimension zero. If $k$ is an algebraically closed field, then we shall construct a continuous family of such modules in the next section. And the same proof shows the existence of such a module when $k$ is an infinite field. Here we just prove the existence of such a module by another idea using matrix factorizations, which can be applied to any case including the case that $k$ is a finite field.

Since $f \in S$ is an element of degree 2, we can write

$$f = \sum_{i=1}^n x_i y_i,$$

where $x_i, y_i$ are elements of $S$ of degree 1. We set graded free $S$-modules $F$ and $\wedge F$ as follows:

$$F = \bigoplus_{i=1}^n S e_i, \quad \wedge F = \bigoplus_{i=1}^n \wedge F$$

And we define homogeneous $S$-linear maps $\phi$ and $\psi : \wedge F \to \wedge F$ by

$$\phi(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{j=1}^s x_{i_j} (e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_s}),$$

$$\psi(\omega) = \left(\sum_{i=1}^n y_i e_i\right) \wedge \omega.$$

Then it is easy to verify the equalities $\phi^2 = \psi^2 = 0$ and $\phi \cdot \psi + \psi \cdot \phi = f \cdot 1_{\wedge F}$, hence $(\phi + \psi)(\phi + \psi) = f \cdot 1_{\wedge F}$. This means that the pair of linear maps $(\phi + \psi, \phi + \psi)$ on $\wedge F$ gives a matrix factorization of $f$. See [11, p.66]. Hence, if we define an $S$-module $M$ by the exact sequence :

$$0 \longrightarrow \wedge F \xrightarrow{\phi + \psi} \wedge F \longrightarrow M \longrightarrow 0,$$

it is easy to see that $f$ annihilates $M$, hence $M$ is actually an $R$-module and $\text{pd}_S M = 1$. Hence $G\dim_R M = 0$ by (9). On the other hand, the Hilbert series of $M$ can be computed as

$$H_M(t) = H_{\wedge F}(t)(1 - t) = 2^n H_S(t)(1 - t) = 2^n(1 + rt),$$

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and thus $M$ has only two graded pieces, hence $M$ never be a free $R$-module. □

**Example 4.3** For any integer $r \geq 2$, we can construct examples of rings $R$ satisfying the conditions in Theorem 4.2. In fact, we set

$$S = k[X_0, X_1, \ldots, X_r]/I_2\left(\begin{array}{cccc} X_0 & X_1 & X_2 & \cdots & X_{r-1} & X_r \\ X_1 & X_2 & X_3 & \cdots & X_r & X_0 \end{array}\right),$$

where $I_2(A)$ denotes the ideal generated by all the 2-minors of the matrix $A$. Then it is easy to see that $S$ is a Cohen-Macaulay ring of dimension one with the Hilbert series

$$H_S(t) = \frac{1 + rt}{1 - t},$$

hence $S$ has minimal multiplicity. To get an example, just set $R = S/fS$ for any homogeneous element $f$ of $S$ of degree 2.

**Example 4.4** In her paper [8], Veliche considers a ring

$$R = k[x, y, z, w]/(x^2, xy - zw, xy - w^2, xz - yw, xw - y^2, xw - yz, xw - z^2),$$

and an $R$-module defined by the following exact sequence

$$\cdots \rightarrow R^2 \left(\begin{array}{cc} z & x \\ w & y \end{array}\right) \rightarrow R^2 \left(\begin{array}{cc} y & -x \\ -w & z \end{array}\right) \rightarrow R^2 \left(\begin{array}{cc} z & x \\ w & y \end{array}\right) \rightarrow R^2 \rightarrow M \rightarrow 0.$$

She proved that $\text{G-dim}_R M = 0$, but $\text{CI-dim}_R M = \infty$. Note that $m^3 = (0)$ for the graded maximal ideal $m$ of $R$, hence $R$ is a Koszul algebra by Theorem 3.1. However we also conclude from Theorem 4.2 that the ring $R$ has no nontrivial deformations.

### 5 Construction of continuous family of modules of G-dimension zero

In this section, we consider the case that the conditions in Theorem 4.2 hold, i.e. we always denotes that $S$ is a homogeneous Cohen-Macaulay graded ring over a field $k$ with minimal multiplicity of dimension one, $f$ is an homogeneous non-zero divisor of $S$ of degree two and that $R \cong S/fS$. Recall that the Hilbert series of $R$ is given as

$$H_R(t) = (1 + t)(1 + rt),$$

where $r$ is the socle dimension of $R$.

In such a case, we have shown in the previous section that there exists a nonfree $R$-module of G-dimension zero.

The purpose of this section is to show that under a further condition that $k$ is an algebraically closed field, we can actually construct a continuous family of isomorphism classes of indecomposable modules of G-dimension zero. The main result is the following.
Theorem 5.1 Let $R$ be as above with $k$ being an algebraically closed field and let $V$ be the $k$-vector space that consists of elements of degree one of $R$. Note that $\dim_k V = r + 1$. Now denote by $\mathbb{P}(V)$ the projective space over $V$, i.e. the set of $k$-linear subspaces of dimension one in $V$. Under these circumstances, there are a non-empty open subset $O$ of $\mathbb{P}(V)$ and $R$-modules $M(p,n)$ for each $(p,n) \in O \times \mathbb{N}$ satisfying the following conditions:

1. Each $M(p,n)$ is an indecomposable $R$-module that is minimally generated by $n$ elements.

2. $G\dim_R M(p,n) = 0$ for any $(p,n) \in O \times \mathbb{N}$.

3. If $(p,n) \neq (p',n') \in O \times \mathbb{N}$, then $M(p,n)$ and $M(p',n')$ are non-isomorphic.

Proof (Step 1) First we note that there is a homogeneous element of degree one $x \in V$ with $x^n = n^2$ where $n$ is the graded maximal ideal of $S$, since $S$ is a one-dimensional Cohen-Macaulay ring with minimal multiplicity and since $k$ is an infinite field. Now define the subset $O_1$ of $\mathbb{P}(V)$ as

$$O_1 = \{ [x] \in \mathbb{P}(V) \mid x \in V, \ x^n = n^2 \},$$

where $[x]$ denotes the $k$-linear subspace of $V$ generated by $x$. We claim that $O_1$ is a non-empty open subset of $\mathbb{P}(V)$.

In fact, the multiplication of $S$ induces a $k$-linear map $\phi : V \to \text{Hom}_k(V,S_2)$. Fixing a $k$-basis $\{e_0, \ldots, e_r\}$ of $V$ and writing $x = \sum_{i=0}^{r} a_i e_i$ ($a_i \in k$), we see that $\phi(x) : V \to S_2$ is a surjective map if and only if the rank of the matrix $\sum_{i=0}^{r} a_i \phi(e_i)$ is not smaller than $\dim_k S_2$. Thus it is easy to see that the set consisting of elements $x \in V$ with $\phi(x)$ being surjective is an open subset of $V$. Note that an element $[x] \in \mathbb{P}(V)$ belongs to $O_1$ if and only if $\phi(x)$ is surjective, and that there is at least one such $x$ as remarked above. Therefore $O_1$ is a non-empty open subset of $\mathbb{P}(V)$. Compare with the proof of [7, Theorem 14.14]. □

(Step 2) For $[x] \in O_1$, define an $S$-module $M([x],1)$ as

$$M([x],1) = S/xS.$$ 

Then, $M([x],1)$ is actually an $R$-module and $G\dim_R M([x],1) = 0$.

Note that any $x \in O_1$ is a non-zero divisor on $S$, since it is a parameter of the Cohen-Macaulay ring $S$. Thus we have the following exact sequence of $S$-modules:

$$0 \longrightarrow S(-1) \xrightarrow{x} S \longrightarrow M([x],1) \longrightarrow 0.$$ 

In particular, the projective dimension of $M([x],1)$ as an $S$-module is one. Since $x^n = n^2$, and since $f$ is of degree two, we have that $f \in n^2 \subseteq xS$, hence that $fM([x],1) = 0$. This implies that $M([x],1)$ is actually an $R$-module. Thus it follows from (9) in Theorem [12] that $G\dim_R M([x],1) = 0$. □
Note that $M([x], 1)$ is indecomposable, because it is generated by a single element. Also note that $M([x], 1) = R/xR$.

Now we fix a nonzero element $z \in V$ and we set

$$O = O_1 \setminus \{z\},$$

which is also a non-empty open subset of $P(V)$. For an element $([x], n) \in O \times \mathbb{N}$, we define an $R$-module $M([x], n)$ by the following exact sequence:

$$R(-1)^n \xrightarrow{\Phi_x} R^n \longrightarrow M([x], n) \longrightarrow 0,$$

where $\Phi_x$ is a $n \times n$ matrix

$$(5.1) \begin{pmatrix} 1 & z & z & \cdots & 0 \\ x & z & x & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \vdots & \vdots & x & z \\ \end{pmatrix}.$$  

(Step 3) We shall prove that $G\dim_R M([x], n) = 0$ for all $([x], n) \in O \times \mathbb{N}$.

From the definition, we have a filtration of $M([x], n)$ by submodules;

$$M_0 = 0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M([x], n),$$

such that $M_i/M_{i-1} \cong R/xR$ for $1 \leq i \leq n$. Since $G\dim_R R/xR = 0$, we have $G\dim_R M([x], n) = 0$ as desired. $\square$

(Step 4) Next we show that the modules $M([x], n)$ are indecomposable.

Let $\Lambda = \text{End}_R(M([x], n))$ be the endomorphism ring of $M([x], n)$. Note that $\Lambda$ is a nonnegatively graded (noncommutative) finitely dimensional $k$-algebra. It is enough to prove that $\Lambda$ is a local ring. For this, let $\varphi$ be an element of $\Lambda$ of degree 0. Then $\varphi$ induces a commutative diagram

$$\begin{array}{ccc} R(-1)^n & \xrightarrow{\Phi_x} & R^n \\ Q \downarrow & & \downarrow \varphi \\ R(-1)^n & \xrightarrow{\Phi_x} & R^n \end{array} \longrightarrow M([x], n) \longrightarrow 0,$$

where $P$ and $Q$ are $n \times n$ square matrices whose entries are in $k$, for $\varphi$ is of degree 0. By the commutativity of the diagram, we have $\Phi_xQ = P\Phi_x$. Putting

$$E = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & \vdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 \end{pmatrix}, \quad (5.2)$$

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we have \( xQ + zJQ = xP + zPJ \). Since \( x \) and \( z \) are linearly independent as elements of the \( k \)-vector space \( V \), it follows that \( Q = P \) and \( JQ = PJ \), hence \( JP = PJ \). Then it is easy to see that

\[
P = a_0E + a_1J + a_2J^2 + \cdots + a_{n-1}J^{n-1},
\]

for some \( a_i \in k \). Thus all the elements in \( \Lambda \) of degree 0 are induces by matrices of the form (12). Note that if \( i > 0 \), then \( J^i \) induces a nilpotent element of \( \Lambda \), hence an element in the radical \( \text{rad}(\Lambda) \). Since any elements of \( \Lambda \) of positive degree are nilpotent as well, they are also belonging to \( \text{rad}(\Lambda) \). Thus we conclude that \( \Lambda/\text{rad}(\Lambda) \cong k \) whose nontrivial element is represented by \( a_0E \) in (12). As a result, \( \Lambda \) is a local ring. \( \square \)

(Step 5) Finally we show that if \( ([x], n) \neq ([x'], n') \in \mathcal{O} \times \mathbb{N} \), then \( M([x], n) \not\cong M([x'], n') \). This will complete the proof of the theorem.

Since \( n = \dim_k(M([x], n) \otimes_R \mathfrak{m}) \), the above claim will be obvious if \( n \neq n' \). Thus we may assume that \( n = n' \).

Recall, in general, if an \( R \)-module \( M \) has a presentation

\[
\begin{array}{cccccc}
R^n & \xrightarrow{A} & R^m & \xrightarrow{} & M & \xrightarrow{} & 0
\end{array}
\]

with a matrix \( A \), then the ideal generated by all the entries of \( A \) is an invariant of the isomorphism class of the module \( M \). That is actually an invariant called the Fitting invariant.

Now suppose \( M([x], n) \cong M([x'], n) \) for \( [x], [x'] \in \mathcal{O} \). Then by this remark we must have \( (x, z)R = (x', z)R \). Looking at the degree one part of this, we have \( [x, z] = [x', z] \) as a two-dimensional subspace of \( V \). Thus we can write

\[
x' = \alpha x + \beta z \quad (\alpha \neq 0 \in k, \ \beta \in k).
\]

As in the same way as the proof of Step 4, we have \( P \) and \( Q \in GL(n, k) \) such that \( \Phi xQ = P \Phi x' \), since \( M([x], n) \cong M([x'], n) \). Consequently, it follows that

\[
xQ + zJQ = x'P + zPJ = \alpha xP + \beta zP + zPJ,
\]

where \( J \) is the matrix defined in (11). Since \( x \) and \( z \) are linearly independent, we have \( Q = \alpha P \) and \( JQ = \beta P + PJ \). Hence, \( \alpha P^{-1}JP = \beta E + J \). Note that the left hand side is nilpotent and that the right hand side is of the Jordan standard form with eigen value \( \beta \). Therefore we get \( \beta = 0 \), hence \( x' = \alpha x \). This implies \( [x] = [x'] \) as an element of \( \mathcal{O} \). \( \square \)

6 Modules of G-dimension zero are not contravariantly finite

For a general local ring \( (R, \mathfrak{m}) \), we denote by \( \mathcal{R}-\text{mod} \) the category of finitely generated \( \mathcal{R} \)-modules and we define a full subcategory \( \mathcal{G}(\mathcal{R}) \) of \( \mathcal{R}-\text{mod} \) as follows:

\[
\mathcal{G}(\mathcal{R}) = \{ M \in \mathcal{R}-\text{mod} \mid \text{G-dim}_R M = 0 \}.
\]

According to Auslander, we call \( \mathcal{G}(\mathcal{R}) \) a contravariantly finite subcategory of \( \mathcal{R}-\text{mod} \) if it satisfies the following condition:
(*) For any $M \in R$-mod, there is an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow \pi \longrightarrow M \longrightarrow 0, \quad (6.1)$$

with $X \in \mathcal{G}(R)$ such that any morphism from any object of $\mathcal{G}(R)$ to $M$ factors through the morphism $\pi$.

Note that Wakamatsu’s lemma [9] or [10, Lemma 2.1.1] says that the condition (*) is satisfied for a sequence of the form (13) if and only if $\text{Ext}_R^1(X', Y) = 0$ for any $X' \in \mathcal{G}(R)$.

The morphism $\pi : X \to M$ with the property (*) is often called a (right) $\mathcal{G}(R)$-approximation of $M$. When $R$ is a Gorenstein ring, then $\mathcal{G}(R)$ coincides with the full subcategory consisting of all maximal Cohen-Macaulay modules, and it is known by [2] that $\mathcal{G}(R)$ is contravariantly finite. Therefore any module has a $\mathcal{G}(R)$-approximation if $R$ is Gorenstein.

In this section we claim that $\mathcal{G}(R)$ may not be a contravariantly finite subcategory of $R$-mod.

**Theorem 6.1** Let $S$ be a one-dimensional Cohen-Macaulay homogeneous graded ring over a field $k$, which is non Gorenstein and has minimal multiplicity. Taking a minimal reduction $x$ of the maximal ideal $n$ of $S$, i.e. $xn = n^2$, we set $R = S/x^2S$. Then the $R$-module $k$ has no $\mathcal{G}(R)$-approximation.

**Proof** To prove the theorem, suppose that there were an exact sequence:

$$0 \longrightarrow Y \longrightarrow i \longrightarrow X \longrightarrow \pi \longrightarrow k \longrightarrow 0, \quad (6.2)$$

with the property (*). We may take such a sequence minimal, i.e. there are no common direct summands of $Y$ and $X$ through $i$. Now we decompose $X$ into indecomposable modules: $X = \bigoplus_{i=1}^{n} X^{(i)}$. Note that each $\pi|_{X^{(i)}} : X^{(i)} \to k$ is nontrivial, because of the minimality of (14). Recall from Theorem 3.1 that all the modules $X^{(i)}$’s are graded modules, hence it follows that $\pi$ is a graded homomorphism of degree 0. Now assume that $X^{(i)} \cong R$ for $1 \leq i \leq u$ and that $X^{(j)}$ are not free for $u + 1 \leq j \leq n$. Then from the results of Theorem 3.1 we have the graded structure $X_0 \oplus X_1 \oplus X_2$ on $X$ with

$$\dim_k X_0 = u + \sum_{j=u+1}^{n} s_j, \quad \dim_k X_1 = u(r+1) + \sum_{j=u+1}^{n} rs_j, \quad \dim_k X_2 = ru,$$

where $s_j$ is the minimal number of generators of $X^{(j)}$. Now since $\pi$ is graded, $Y$ has also a graded structure; $Y = Y_0 \oplus Y_1 \oplus Y_2$ and it follows from (14) that

$$\dim_k Y_0 = u + \sum_{j=u+1}^{n} s_j - 1, \quad \dim_k Y_1 = u(r+1) + \sum_{j=u+1}^{n} rs_j, \quad \dim_k Y_2 = ru. \quad (6.3)$$

As in the proof of Theorem 6.1, $R/xR$ is an $R$-module of G-dimension zero. In fact, it has a complete resolution of the form:

$$\cdots \longrightarrow R \longrightarrow R \longrightarrow R \longrightarrow R \longrightarrow \cdots.$$
Hence if follows from Wakamatsu’s lemma that $\text{Ext}^1_R(R/xR, Y) = 0$, and that the sequence

$$\cdots \xrightarrow{x} Y \xrightarrow{x} Y \xrightarrow{x} Y \xrightarrow{x} \cdots$$

is an exact sequence of graded $R$-modules. Taking a graded piece of this, we have an exact sequence of $k$-vector spaces:

$$0 \xrightarrow{x} Y_0 \xrightarrow{x} Y_1 \xrightarrow{x} Y_2 \xrightarrow{x} 0$$

Thus it follows from (15) that

$$\left( u + \sum_{j=u+1}^{n} s_j - 1 \right) + ru = u(r + 1) + \sum_{j=u+1}^{n} rs_j,$$

equivalently,

$$(1 - r)(\sum_{j=u+1}^{n} s_j) = 1.$$ 

This is impossible, because $r$ and the all $s_j$’s are positive integers. And we conclude that there is no such exact sequence as in (14). □

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