The Equations of Motion of a Charged Particle in the
Five-Dimensional Model of the General Relativity Theory with
the Four-Dimensional Nonholonomic Velocity Space

Victor R. Krym\textsuperscript{1}, Nickolai N. Petrov

St.-Petersburg State University, Department of Mathematics and Mechanics,
St.-Petersburg, 198504 Russia

Abstract

We consider the four-dimensional nonholonomic distribution defined by the 4-
potential of the electromagnetic field on the manifold. This distribution has a metric
tensor with the Lorentzian signature \((+,-,-,-)\), therefore, the causal structure appears as in the general relativity theory. By means of the Pontryagin’s maximum
principle we proved that the equations of the horizontal geodesics for this distribution
are the same as the equations of motion of a charged particle in the general relativity
theory. This is a Kaluza – Klein problem of classical and quantum physics solved by
methods of sub-Lorentzian geometry. We study the geodesics sphere which appears in
a constant magnetic field and its singular points. Sufficiently long geodesics are not
optimal solutions of the variational problem and define the nonholonomic wavefront.
This wavefront is limited by a convex elliptic cone. We also study variational prin-
ciple approach to the problem. The Euler – Lagrange equations are the same as those
obtained by the Pontryagin’s maximum principle if the restriction of the metric tensor
on the distribution is the same.

Keywords: general relativity, electromagnetic field, nonholonomic differential geometry
AMS Subj. Class. (MSC): 37J60, 37J55, 53B30, 53B50, 53D50, 58A30, 70F25, 81S05
PACS: 02.40.Ky, 04.20.Cv, 04.20.Fy, 04.50.+h, 11.25.Mj, 41.90.+e

1 Nonholonomic model in sub-Lorentzian geometry

Distribution on a smooth manifold \(M\) is a family of subspaces \(\mathcal{A}(x) \subset T_x M\) (of the tangent
bundle of the manifold) with the same dimension, smoothly parametrized by the points of
manifold \([1, 2, 3, 4]\). For each \(x \in M\) on \(\mathcal{A}(x)\) the bilinear form \(\langle u, u \rangle_x\) is defined which
which can be equally described by the metric tensor \(g_{ij}(x)\). The metric tensor of a distribution
smoothly depends of the point of the manifold. The absolutely continous path \(x(t)\) is called
horizontal, iff \(x'(t) \in \mathcal{A}(x(t))\) for almost any \(t\). In sub-Riemannian geometry the metric
tensor \(g_{ij}(x)\) of the distribution \(\mathcal{A}\) is positively defined. In this geometry we study the
shortest horizontal curves connecting two given points. If the distribution is completely
nonholonomic and the Riemannian manifold is full, then any two points of the manifold can
be connected by the shortest horizontal geodesic. On manifolds and also on distributions

\textsuperscript{1}E-mail: vkrym2007@rambler.ru
with metric tensor the geodesic is the Euler – Lagrange solution for the length functional
\[ J(x(\cdot), u(\cdot)) = \int_{t_0}^T \langle u(t), u(t) \rangle_{x(t)}^{1/2} dt. \]

In sub-Lorentzian geometry the metric tensor \( g_{ij}(x) \) of the distribution \( \mathcal{A} \) has the signature \((+, -, \ldots, -)\). In this geometry we study the longest horizontal curves connecting two given points \[5\]. The equations of motion of a charged particle in the electromagnetic and gravitational fields can be obtained as the solutions of the variational problem for the distribution \[6\]. Consider the 5-dimensional manifold \( M^5 \) with the 4-dimensional distribution \( \mathcal{A} \) defined by means of the differential form \( \omega(x) = \sum_{i=0}^{3} A_i(x) dx^i + dx^4 \), where the covector \((A_i)_{i=0,\ldots,4}\) is the 4-potential of the electromagnetic field. Consider the length functional
\[ J(x(\cdot), u(\cdot)) = \int_{t_0}^T L(x(t), u(t)) dt, \]
where \( L(x, u) = m \langle u, u \rangle_{x}^{1/2} \) is the pseudonorm of the vector \( u \), \( m \) is the mass of the particle. Consider the problem of maximizing the length functional on the set of horizontal curves connecting two given points. Solution of this problem as we prove in this paper leads to equations
\[ \frac{d}{dt} \frac{\partial L}{\partial u^k} - \frac{\partial L}{\partial x^k} + p_4 \sum_{j=0}^{3} F_{jk} u^j = 0, \quad k = 0, \ldots, 3, \]
where \( F_{jk} \) are defined by \[5\]. These equations are the same as the equations of motion of a particle with the charge \( p_4 \) in the electromagnetic and gravitational fields of the general relativity theory. This wording of the problem is different from the classical problem of physics \[7, 8\] of minimizing the action functional on a 4-dimensional manifold:
\[ S(x(\cdot), u(\cdot)) = -mc \int_{t_0}^{t_1} \langle u(t), u(t) \rangle_{x(t)}^{1/2} dt - \frac{e}{c} \int_{t_0}^{t_1} \sum_{i=0}^{3} A_i(x(t)) u^i(t) dt, \]
where \( e \) is the charge of the particle, \( c \) is the speed of light. This paper is devoted to the problem of unification of electromagnetic, gravitational and other fundamental interactions \[9, 10, 11\].

In this model the particles are considered on the 5-dimensional manifold \( M^5 \) with the special smooth structure. Allowed are only smooth coordinate transformations for which
\[ \frac{\partial y^i}{\partial x^4} = 0, \quad i = 0, \ldots, 3, \quad \frac{\partial y^4}{\partial x^4} = 1. \]
Components \( \frac{\partial y^i}{\partial x^j} \), \( i = 0, \ldots, 3 \), can be interpreted as gauge transformations \[9\]. Since the coordinate transformations are smooth, \( \frac{\partial}{\partial x^4} \frac{\partial y^i}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial y^i}{\partial x^4} = 0, \quad i, j = 0, \ldots, 4 \). Therefore the cylindricity condition \( \frac{\partial y}{\partial x^4} = 0 \) for any tensor field \( v \) is invariant. In the tangent bundle \( TM \) the subspace \( \text{Lin} \{ \partial_4 \} \) is invariant. In the cotangent bundle \( T^*M \) the subspace \( \text{Lin} \{ dx^0, dx^1, dx^2, dx^3 \} \) is invariant. Assume that the metric tensor and the 4-potential of the electromagnetic field do not depend of \( x^4 \). This condition is invariant at the coordinate
transformations with the property (4). The 4-dimensional distribution \( A \) is the velocity space, i.e. the set of all possible velocity vectors of the particle. The metric tensor of the distribution \( A \) defines 4-dimensional convex elliptic cone, completely similar to the cone of future of the general relativity theory. Causal structure in the proposed theory is similar to the causal structure of the general relativity theory \([12, 13]\). In the presence of the electromagnetic field the manifold \( A \) is nonholonomic.

The electromagnetic field is defined in physics by the antisymmetric tensor \((F_{jk})_{j,k=0,...,3}\)

\[
(F_{jk})_{j,k=0,...,3} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}.
\] (5)

Here \( E \) is the 3-dimensional vector of the electric field tension, \( H \) is the 3-dimensional vector of the magnetic field tension. There is the 4-potential \((A_i)_{i=0,...,3}\) which defines the tensor of the electromagnetic field:

\[
F_{jk} = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \quad j, k = 0, \ldots, 3.
\] (6)

4-Potential is not defined unambiguously. It is defined with an arbitrary gauge transformation \( A_i \mapsto A_i + \frac{\partial f}{\partial x^i} \), where \( f \) is any smooth function.

Let \( U \) be some coordinate neighbourhood on the smooth manifold \( M^5 \). The 4-dimensional distribution \( A \) on \( U \) can be defined by the differential form

\[
\omega_x = \sum_{i=0}^{3} A_i(x)dx^i + dx^4.
\] (7)

This definition is correct since at the coordinate transformation with the property (4) we have

\[
\sum_{i=0}^{3} A_i dx^i + dx^4 = \sum_{j=0}^{3} \left( \sum_{i=0}^{3} A_i \frac{\partial x^i}{\partial y^j} + \frac{\partial x^4}{\partial y^j} \right) dy^j + dy^4 = \sum_{j=0}^{3} \tilde{A}_j dy^j + dy^4.
\] (8)

Therefore the coordinate transformations on the 5-manifold lead to transformations of the 4-potential

\[
\tilde{A}_j = \sum_{i=0}^{3} A_i \frac{\partial x^i}{\partial y^j} + \frac{\partial x^4}{\partial y^j},
\] (9)

that include 4-dimensional coordinate transformations of the general relativity theory and gauge transformations of the 4-potential of the electromagnetic field \([6]\) as in \([7]\). Locally the distribution \( A \) can be defined by the basis vector fields \( e_i = \frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial x^4}, \quad i = 0, \ldots, 3 \). Vector fields on a manifold can be considered as operators of differentiation. During the coordinate transformation with the property (4) we have

\[
\tilde{e}_j = \sum_{i=0}^{3} \frac{\partial x^i}{\partial y^j} e_i,
\] (10)
where \( \tilde{e}_j = \frac{\partial}{\partial y^j} - \tilde{A}_j \frac{\partial}{\partial y^4} \), \( j = 0, \ldots, 3 \). Therefore the 4-dimensional basis of the distribution \( \mathcal{A} \) is transformed \(^{10}\) exactly the same way as the coordinate vector fields \( \left( \frac{\partial}{\partial x^i} \right)_{i=0, \ldots, 3} \) in the general relativity theory. The commutators of vector fields \( e_i, i = 0, \ldots, 3 \), generate the tensor of the electromagnetic field: \([e_i, e_j] = -F_{ij} \frac{\partial}{\partial x^4} \). If \( F_{ij} \neq 0 \), then the distribution \( \mathcal{A} \) is completely nonholonomic. Allowed curves on the distribution are absolutely continuous and satisfy almost everywhere the horizontality condition

\[
\omega_{\gamma(t)}(\gamma'(t)) = 0. \tag{11}
\]

In this paper we consider the optimization problem for the length functional\(^2\) on the 5-dimensional manifold with the 4-dimensional distribution \( \mathcal{A} \). We prove that the solutions of the variational problem for this distribution satisfy the equations of motion for the charged particle of the general relativity theory. We consider also the geodesic sphere for the distribution \( \mathcal{A} \). The geodesic sphere with the radius \( r \) and center \( x_0 \) is the set of points at distance \( r \) from the point \( x_0 \). To obtain the geodesic sphere we consider the set of solutions of the variational problem starting at point \( x_0 \). We constructed the 4-dimensional projection of the geodesic sphere for the particle moving in the constant magnetic field.

### 2 The equations of motion of a charged particle

The optimization problem for the length functional on the set of the horizontal curves connecting two given points is one of the main problems of the theory of optimal control \(^{15, 16}\). The velocity vector \( x'(t) \) of horizontal curves belong to the distribution \( \mathcal{A} \), hence we consider the metric tensor of the distribution \( \mathcal{A} \). If the vector fields \( e_i(x), i = 0, \ldots, 3 \) form the basis of the distribution \( \mathcal{A}(x) \), then its metric tensor is \( g_{ij}(x) = \langle e_i(x), e_j(x) \rangle_x, i, j = 0, \ldots, 3 \). Consider the length functional

\[
J(x(\cdot), u(\cdot)) = \int_{t_0}^T L(x(t), u(t)) \, dt, \tag{12}
\]

where

\[
L(x, u) = m \left( \sum_{i,j=0}^3 g_{ij}(x) u^i u^j \right)^{1/2} \tag{13}
\]

and \( m \) is the mass of the particle. Since the metric tensor has the signature \((+, -, -, -)\), the velocity vector \( \frac{dx}{dt} \) belong to the cone of future \( V(x) = \{ v \in \mathcal{A}(x) \mid \langle v, v \rangle_x > 0, v^0 > 0 \} \). This set depends of the point \( x \). Hence let us consider another problem.

All spaces \( \mathcal{A}(x) \) are linearly isomorphic. Therefore we can consider the cone \( \overline{V}(x) = \{ u \in \mathbb{R}^4 \mid \sum_{i,j=0}^3 g_{ij}(x) u^i u^j > 0, u^0 > 0 \} \). For some neighbourhood of the point \( x_0 \) the intersection of all sets \( \overline{V}(x) \) is nonempty and has a nonempty interior \( Q \). In this neighbourhood of the point \( x_0 \) consider another problem for which \( u(t) \in Q \). If \( (x(t), u(t)) \) is an optimal process

\(^2\) Note that nonholonomic variational problem is different from the nonholonomic mechanical problem.
for the problem \((12)\), for which \(u(t) \in V(x(t))\), then this pair is locally an optimal process in the case \(u(t) \in Q\). From the horizontality condition \((11)\) follows that for the velocity vector of the particle

\[
\begin{align*}
\frac{dx^k}{dt} &= u^k, \quad k = 0, \ldots, 3 \\
\frac{dx^4}{dt} &= - \sum_{j=0}^{3} A_j(x) u^j. \quad (14)
\end{align*}
\]

Let us consider the maximization problem for the length functional on the set of horizontal curves connecting two given points \(x^i(t_0) = x^i_0\) and \(x^i(T) = x^i_T\), \(i = 0, \ldots, 4\). The Hamilton–Pontrjagin function for the functional \((12)\), \((14)\) has the form

\[H(x, u, p, a_0) = -a_0 L(x, u) + \sum_{j=0}^{3} p_j u^j - p_4 \sum_{j=0}^{3} A_j(x) u^j. \quad (15)\]

Due to Pontrjagin’s maximum principle \([17, 18]\) there is \(a_0 \geq 0\) and vector-function \(p_k(t)\), \(k = 0, \ldots, 4\), \(t_0 \leq t \leq T\), satisfying the conjugate system of linear differential equations

\[
\frac{dp_k}{dt} = - \frac{\partial H}{\partial x^k} \bigg|_{(x(t), u(t), p(t), a_0)} = a_0 \frac{\partial L}{\partial x^k} + p_4 \sum_{j=0}^{3} \frac{\partial A_j}{\partial x^k} u^j(t), \quad k = 0, \ldots, 4, \quad (16)
\]

where \((x(t), u(t))\) is the optimal process for the problem \((12)\). Since the metric tensor and 4-potential of the electromagnetic field do not depend of the coordinate \(x^4\), the appropriate momentum component is the integral of motion (is conserved): \(p_4(t) = \text{const}\). From the equation \((2)\) follows that \(p_4\) is the length of the particle. For almost any \(t \in [t_0, T]\) the function \(u \mapsto H(x(t), u, p(t), a_0)\) reaches its upper bound on the set \(Q\) at \(u = u(t)\). If this maximum is reached at an inner point, then \(\frac{\partial H}{\partial u^k} = 0\), i.e.

\[p_k - p_4 A_k = a_0 \frac{\partial L}{\partial u^k}, \quad k = 0, \ldots, 3. \quad (17)\]

Since the length functional does not depend on the parametrization of the path \(x(\cdot)\), we can assume that the optimal control belong to the pseudosphere \(L(x, u) = 1\). The normal covector for this pseudosphere has the coordinates \(\frac{\partial L}{\partial u^k}\), \(k = 0, \ldots, 3\). The projection of the impulse vector \(p_k\) on the subset \((u^0, \ldots, u^3)\) has the coordinates \(p_k - p_4 A_k\). The Hamilton function reaches its maximum when the projection of the vector \(p_k\) and the normal for the unit pseudosphere are collinear. The equation \((17)\) is exactly the condition for these two vector to be collinear. To exclude the vector \(p_k\) from the equations \((16)\) and \((17)\) we use the identity

\[A_k(x(t)) - A_k(x(t_0)) = \int_{t_0}^{t} \sum_{j=0}^{3} \frac{\partial A_k}{\partial x^j} u^j dt', \quad k = 0, \ldots, 3. \quad (18)\]

Then

\[a_0 \left( \frac{\partial L}{\partial u^k} - \int_{t_0}^{t} \frac{\partial L}{\partial x^k} dt' \right) + p_4 \int_{t_0}^{t} \sum_{j=0}^{3} F_{jk} u^j dt' + p_4 A_k(x(t_0)) - p_k(t_0) = 0. \quad (19)\]
Parameters $a_0$ and $p_k$, $k = 0, \ldots, 4$, can be multiplied by any positive constant. With $a_0 = 1$ we obtain the equations of horizontal geodesics on the distribution $A$. If we additionally assume continuity of functions $u(t)$, then

$$ \frac{d}{dt} \frac{\partial L}{\partial u^k} - \frac{\partial L}{\partial x^k} + p_4 \sum_{j=0}^{3} F_{jk} u^j = 0, \quad k = 0, \ldots, 3. \tag{20} $$

These equations are identical with the equations of motion of a particle with the charge $p_4$ of the general relativity theory.

For $a_0 = 0$ we obtain the equations of abnormal geodesics. Then at least one of the parameters $p_k$, $k = 0, \ldots, 4$, should be non-zero. Since in this case $p_k = p_4 A_k$, $k = 0, \ldots, 3$, then $p_4 \neq 0$. The solutions of the variational problem for $a_0 = 0$ has the form

$$ \sum_{j=0}^{3} F_{jk} u^j = 0, \quad k = 0, \ldots, 3. \tag{21} $$

The solutions of the variational problem [12] – [14] which do not satisfy Euler – Lagrange equations are called abnormal geodesics [19] [20].

Montgomery constructed a distribution in 3-dimensional space for which the commutator of the basis vector fields is $2x^2 \frac{\partial}{\partial x^3} [21] [22]$. In his example the abnormal geodesics are straight lines parallel with the $x^2$ axis. In our model the equations of abnormal geodesics on the distribution $A$ has the form (21). If $\det F = 0$, then this equation can possess a non-trivial solution.

## 3 The geodesic sphere for the particle in the magnetic field

The geodesic sphere has a complicated structure [2] [23] [24]. In the following part of the paper we assume that $M$ is the linear space $\mathbb{R}^5$ and the metric tensor of the distribution $A$ is diagonal, $g = \text{Diag}(1, -1, -1, -1, -1)$. The equations of motion of a charged particle in the electromagnetic field [20] in the space with the diagonal metric tensor has the form

$$ m \frac{d u^k}{dt} - q \sum_{j=0}^{3} F_{jk}^i u^j = 0, \quad k = 0, \ldots, 3, \tag{22} $$

where $m$ is the mass of the particle, $q$ is the charge of the particle, $F$ is the tensor of the electromagnetic field, $u^k$ are components of the 4-velocity vector of the particle, $\frac{dx^k}{dt} = u^k$, $k = 0, \ldots, 3$. The fifth component of the velocity vector $\frac{dx^4}{dt}$ is determined by the horizontality condition.

Let us consider the motion of the particle in the magnetic field. Assume that the tension of the magnetic field $H_x \neq 0$, and the tension of the electric field is zero. Then $F_{23} =
3 THE GEODESIC SPHERE FOR THE PARTICLE IN THE MAGNETIC FIELD

\[ -F_{32} \neq 0, \text{ and other components of the tensor of the electromagnetic field } F \text{ are zero.} \]

In this coordinates the equations of motion and the horizontality condition (11) has the form

\[ \begin{align*}
    m \frac{du^2}{dt} - qH_x u^3 &= 0 \\
    m \frac{du^3}{dt} + qH_x u^2 &= 0 \\
    \frac{dx^4}{dt} + \sum_{k=0}^{3} A_k u^k &= 0.
\end{align*} \tag{23} \]

Other components of the 4-velocity vector are constant. Assume that \( H_x = \text{const} \). Then from the equations (23) follows that in the magnetic field the charged particle moves along the helicoid line.

Consider the projection of the geodesic sphere on the coordinates \((x^2, x^3, x^4)\). In the plane \((x^2, x^3)\) the particle moves along a circle. Choose the following components of the 4-potential: \( A_0 = 0, A_1 = 0, A_2 = \varphi x^3, A_3 = 0 \). Then \( F_{23} = -\varphi, H_x = \varphi \). Designate \( p = \frac{q}{m} \varphi \). Then the equations of motion of a particle in the subspace \((x^2, x^3, x^4)\) has the form

\[ \begin{align*}
    \frac{du^2}{dt} - pu^3 &= 0 \\
    \frac{du^3}{dt} + pu^2 &= 0 \\
    \frac{dx^4}{dt} + \varphi x^3 u^2 &= 0.
\end{align*} \tag{24} \]

The norm of the velocity vector along the geodesic is constant. Assume that \((u^2)^2 + (u^3)^2 = 1\).
Then

\[
\begin{align*}
  x^2(t) &= -\frac{1}{p} \cos(\alpha + pt) + b_2 \\
  x^3(t) &= \frac{1}{p} \sin(\alpha + pt) + b_3 \\
  x^4(t) &= \frac{\varphi}{4p^2} (-2pt + \sin(2\alpha + 2pt)) + \frac{\varphi}{p} b_3 \cos(\alpha + pt) + b_4.
\end{align*}
\]

(25)

In the plane \((x^2, x^3)\) the charged particle is moving along the circle with the center \((b_2, b_3)\) and the radius \(\frac{1}{|p|}\). Let \(b_2 = (\cos \alpha)/p\) and \(b_3 = -(\sin \alpha)/p\). Choose the constant \(b_4\) so that \(x^4(0) = 0\). Then

\[
\begin{align*}
  x^2(t) &= \frac{1}{p} (\cos \alpha - \cos(\alpha + pt)) \\
  x^3(t) &= \frac{1}{p} (\sin(\alpha + pt) - \sin \alpha) \\
  x^4(t) &= \frac{\varphi}{4p^2} (-2pt + \sin(2\alpha + 2pt) - \sin 2\alpha) + \frac{\varphi}{p^2} \sin \alpha (\cos \alpha - \cos(\alpha + pt)).
\end{align*}
\]

(26)

The equations \((26)\) define mapping \((t, \alpha, p) \mapsto \gamma_{\alpha,p}(t)\). In the plane \((x^2, x^3)\) the point makes one turn around the center at \(|pt| = 2\pi\). More long geodesics \((|pt| > 2\pi)\) are not the shortest paths. Here one should consider the shortest paths, since the restriction of the metric tensor on the tangent bundle for the coordinate plane \((x^2, x^3, x^4)\) is positively defined. The geodesic sphere of the radius \(s\) can be defined by the following formula:

\[
B(s) = \{ \gamma_{\alpha,p}(t) \mid \alpha \in [-\pi, \pi], t = s, \ |pt| \leq 2\pi \}.
\]

(27)

Fig. 1 shows the projection of the geodesic sphere of the radius \(s = 1\) for the coordinates \((x^2, x^3, x^4)\) at \(\varphi = 1\), with \(p \in [-2\pi, 0]\) on the left and \(p \in [0, 2\pi]\) on the right. If \(|pt| \leq 2\pi\), the geodesics are the shortest paths. The nonholonomic geodesic sphere of any radius has a pair of points such that if we continue the geodesic after this point we enter the geodesic sphere with some lesser radius. If \(|pt| \geq 2\pi\), then the geodesic of length \(t\) ceases to be optimal path. All possible endpoints \(\gamma_{\alpha,p}(t)\) of the geodesics of length \(t\) reach the \(x^4\) axis at \(pt = 2\pi k\), \(k \in \mathbb{Z}\), this point being the intersection point for the considered surface. The density of these points increases when we approach the origin. This phenomena is called the nonholonomic wave front (fig. 2). Fig. 2 shows the projection of the surface defined by the geodesics of length \(s = 1\) for the coordinates \((x^2, x^3, x^4)\) at \(\varphi = 1\) with \(p \in [-8\pi, -\pi]\) on the left and \(p \in [\pi, 8\pi]\) on the right. These geodesics cease to be optimal paths for \(|p| > 2\pi\). For large enough \(p\) this surface belongs to some cone with the apex at the origin. This cone has the axis of symmetry \(x^4\). Indeed for \(p\) large enough from the equations \((26)\) follows that

\[
x^4(t) = -\frac{\varphi}{2p} s + O\left(\frac{1}{p^2}\right) \quad (p \to \infty).
\]

(28)

In the plane \((x^2, x^3)\) the particle moves along the circle of the radius \(\frac{1}{|p|}\). The distance from the starting point to the ending point is changing from 0 to \(\frac{2}{|p|}\). For \(p\) large enough the
geodesics of the length $s$ belong to the interior of the cone with the angle of inclination $|\varphi|/4$ and the apex at the origin.

4 Distance along the $x^4$ coordinate

In the subspace $(x^2, x^3)$ the particle moves along the circle. Let us find the values of $t$ for which the end of the geodesic is over the starting point in the subspace $(x^2, x^3, x^4)$. The end of the geodesic is over the starting point for $pt = 2\pi k$, $k \in \mathbb{Z}$. Use the Taylor expansion for the equation (26) at the point $p = 2\pi k + \theta$, $\theta \to 0$:

$$x^4(t) = -\varphi \frac{t^2}{4\pi k} + \varphi \frac{t^3}{4\pi^2 k^2} + O(\theta^2) \quad (\theta \to 0).$$

(29)

Since $t = s$, the distance for which the particle moved along the $x^4$ coordinate is of the order $\varphi s^2/4\pi k$.

For distribution with the positively defined metric tensor there is the theorem about the ball shape [25, 26]. In the coordinate neighbourhood of any point one can choose the vector fields $e_i$ which form the basis of the distribution $\mathcal{A}$. The field of planes that span commutators of all possible vector fields of the distribution $\mathcal{A}$ is designated $\mathcal{A}_2 = [\mathcal{A}, \mathcal{A}]$. One also consider $\mathcal{A}_{k+1} = [\mathcal{A}_k, \mathcal{A}]$. This sequence is stabilizing: there is the minimum $m$ such that $\mathcal{A}_m = \mathcal{A}_{m+1}$. The number $m$ which can depend of the point of the manifold is called the degree of nonholonomity of the distribution $\mathcal{A}$. Designate $n_k = \dim \mathcal{A}_k$ and define the function $\varphi(i) = j$, if $n_{j-1} < i \leq n_j$. The function $\varphi$ is defined on the set of numbers $\{1, \ldots, \dim \mathcal{A}_m\}$ with values at $\{1, \ldots, m\}$. For completely nonholonomic distribution on the Riemannian manifold there are the coordinates $x^i$ such that the ball of accessibility is bounded from above and from below by the set $|x^i| \leq \varepsilon^{\varphi(i)}$. For the two-dimensional distribution on the 3-dimensional manifold $n_0 = 0$, $n_1 = 2$, $n_2 = 3$. Therefore the function $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 2$. 

Figure 2: The non-holonomic wave front defined by non-optimal geodesics.
5 Lagrange formulation for the equations of motion

5.1 General case

Let us consider the classical problem of the calculus of variations: how to find an absolutely continuous vector-function which maximizes the functional \[ J(x(\cdot), u(\cdot)) = \int_0^T L(t, x(t), u(t)) \, dt, \] where \( u(t) = \frac{dx}{dt} \). Assume that \( x(0) = x_0, \ x(T) = x_1, \) and allowed paths satisfy the conditions \( \varphi^i(t, x, \frac{dx}{dt}) = 0, \ i = 1, \ldots k. \) Then there are \( k \) measurable and limited functions \( \lambda_i(t) \) called Lagrange multipliers and a constant \( a_0 \geq 0, \) at least one non-zero and such that the function \[ L_\lambda(t, x(t), \dot{x}(t)) = a_0L(t, x(t), \dot{x}(t)) + \sum_{i=1}^k \lambda_i(t)\varphi^i(t, x(t), \dot{x}(t)) \] almost anywhere on \([0, T]\) satisfies the Euler – Lagrange equations in the integral form \[ \frac{\partial L_\lambda(t, x(t), \dot{x}(t))}{\partial \dot{x}^i(t)} = \int_0^T \frac{\partial L_\lambda(\tau, x(\tau), \dot{x}(\tau))}{\partial \dot{x}^i(\tau)} \, d\tau + c_i, \quad i = 1, \ldots n, \] where \( c_i \) are constants [17], p. 279. The parameters \( a_0, \lambda_i \) can be multiplied by any positive constant. With \( a_0 = 1 \) we obtain regular geodesics. With \( a_0 = 0 \) we obtain abnormal geodesics.

Now assume (for the general case) that the distribution \( \mathcal{B} \) is defined by the family of differential forms \( \omega^j, \ j = m+1, \ldots n. \) The horizontality conditions have the form \[ \omega^j(u) = 0, \quad j = m+1, \ldots n, \quad \dot{x} = u. \] Then the Lagrange function for the considered problem is \[ L_\lambda(x, u) = a_0\langle u, u \rangle^{1/2} + \sum_{j=m+1}^n \lambda_j \omega^j(u), \] where \( \langle \cdot, \cdot \rangle \) is some non-degenerate bilinear form on \( TM^n. \) The equations of the horizontal geodesics are \[ a_0\langle \frac{Du}{dt}, \cdot \rangle + \sum_{j=m+1}^n \lambda_j \omega^j + \sum_{j=m+1}^n \lambda_j d\omega^j(u, \cdot) = 0, \] where \( \frac{Du}{dt} \) is the covariant derivative of the velocity vector of the particle along the path. The velocity vector can be decomposed by the basis vector fields of the distribution:\[ u(t) = \sum_{k=1}^m v_k(t)\xi_k(x(t)). \]  

\(^3\) This assumption for our model is not necessary, as shown below. It is enough for the restriction of the metric tensor on the distribution to be non-degenerate.
Then the covariant derivative is
\[
\frac{Du}{dt} = \sum_{k=1}^{m} \frac{dv_k}{dt} \xi_k + \sum_{i,j=1}^{m} v_i v_j \nabla \xi_i \xi_j,
\] (37)
where \( \nabla \) is the symmetric connection defined by the bilinear form. Use the Maurer – Cartan formula
\[
d\omega(\xi, \eta) = \xi \omega(\eta) - \eta \omega(\xi) - \omega([\xi, \eta]).
\] (38)
Since \( \omega^j(\xi_i) = 0, i = 1, \ldots, m, j = m+1, \ldots, n \), then for the velocity vector and the basis of the distribution \( d\omega^j(u, \xi_i) = -\omega^j([u, \xi_i]) \). This commutator can be rewritten using the structural constants of the distribution \( B \):
\[
[u, \xi_i] = \sum_{k=1}^{m} v_k [\xi_k, \xi_i] = \sum_{k=1}^{m} v_k \sum_{s=1}^{n} c^s_{ki} \xi_s,
\] (39)
where \( [\xi_k, \xi_i] = \sum_{s=1}^{n} c^s_{ki} \xi_s \). This basis and the differential forms can be selected such that \( \omega^j(\xi_i) = \delta^j_i, i, j = m+1, \ldots, n \). Then for the distribution \( B \)
\[
a_0 \langle \frac{D\gamma}{dt}, \xi_i \rangle + \sum_{j=m+1}^{n} \lambda_j \sum_{k=1}^{m} c^j_{ik} v_k = 0, \quad l = 1, \ldots, m.
\] (40)
Since \( \omega^j(\xi_i) = \delta^j_i, i, j = m+1, \ldots, n \), then for the velocity vector and all basis vector fields of the tangent space \( d\omega^j(u, \xi_i) = -\omega^j([u, \xi_i]) \). Hence we can write the projection of the equation (35) at the basis vectors which do not belong to the distribution:
\[
a_0 \langle \frac{D\gamma}{dt}, \xi_i \rangle + \lambda_l + \sum_{j=m+1}^{n} \lambda_j \sum_{k=1}^{m} c^j_{ik} v_k = 0, \quad l = m+1, \ldots, n.
\] (41)

### 5.2 Lagrange formulation for the distribution \( A \) in our model

The distribution \( A \) is defined by the differential form \( \omega_x = \sum_{i=0}^{3} A_i(x) dx^i + dx^4 \). The Euler – Lagrange equations for the length functional (12) with the condition \( \omega_{\gamma(t)}(\gamma'(t)) = 0 \) are
\[
a_0 \langle \frac{D\dot{\gamma}}{dt}, \cdot \rangle + \dot{\lambda} \omega + \lambda d\omega(\dot{\gamma}, \cdot) = 0,
\] (42)
where the covariant derivative
\[
\frac{D\dot{\gamma}}{dt} = \sum_{k=0}^{3} \frac{dv_k}{dt} \xi_k + \sum_{i,j=0}^{3} v^i v^j \nabla \xi_i \xi_j,
\] (43)
\( \nabla \) is the symmetric connection defined by the metric tensor. Please note that in the Lagrange method we have to use the metric tensor defined on the whole tangent space \( TM \) whereas in the maximum principle we used the restriction of the metric tensor on the distribution only.
This is no problem because in the result we use the components of the metric tensor of the distribution only. We can also use the connection (Christoffel symbols) with all lower indexes and in this case the bilinear form does not have to be non-degenerate. The velocity vector

\[ \dot{\gamma}(t) = \sum_{k=0}^{3} v^k(t)e_k(\gamma(t)). \]

The Euler – Lagrange method also has the constant \( a_0 \geq 0 \) \[17\] p. 279. The parameters \( a_0, \lambda \) can be multiplied by any positive constant. With \( a_0 = 1 \) we obtain the equations of regular geodesics. With \( a_0 = 0 \) we obtain the equations of abnormal geodesics. Projecting the equations of motion on the basis of the distribution \( \mathcal{A} \) we obtain

\[ a_0\langle \frac{D\dot{\gamma}}{dt}, e_i \rangle + \lambda \sum_{j=0}^{3} c_{ij}^4 v^j = 0, \quad i = 0, \ldots, 3. \tag{44} \]

The structural constants are defined by \([\xi_i, \xi_j] = \sum_{l=0}^{4} c_{ij}^l \xi_l\), where \( \xi_j = e_j, j = 0, \ldots, 3, \xi_4 = \partial_4 \).

For the distribution \( \mathcal{A} \) \([e_i, e_j] = -F_{ij}\partial_4, [e_j, \partial_4] = 0 \). Therefore \( c_{ij}^4 = F_{ji} \), and other structural constants are zero. Since \( \omega_x(\partial_4) = 1 \), then

\[ a_0\langle \frac{D\dot{\gamma}}{dt}, \partial_4 \rangle + \dot{\lambda} = 0. \tag{45} \]

We can assume that \( \langle \cdot, \partial_4 \rangle = 0 \). Then \( \lambda = \text{const} \), and \( \lambda \) can be interpreted as the charge of the particle (or charge to mass ratio depending on the Lagrangian). Note that applying the maximum principle in section 2 we used the restriction of the metric tensor on the distribution only. Since this restriction is non-degenerate, we can rise appropriate indexes of the Christoffel symbols and get ordinary differential equations. The result does not depend on the extension of the metric tensor on the whole tangent space.

All figures published in this paper were produced using our own 3D graphics program ©V.R. Krym.

References

[1] Gromov M. *Carnot-Caratheodory Spaces Seen From Within*. Preprint IHES/M/94/6 (Institut des Hautes Etudes Scientifiques, 1994)

[2] Vershik A.M., Gershkovich V.Ya. *The Nonholonomic Dynamical Systems. Geometry of Distributions and Variational Problems*. Dynamical Systems–7. Results of Science and Technique, ser. ”The Contemporary Problems of Mathematics, Fundamental Directions”, v. 16, pp. 5–85. Moscow, 1987. (Russian)

[3] Dobronravov V.V. Foundations of the Mechanics of the Nonholonomic Systems. Moscow, 1970. (Russian)

[4] Newmark Yu.I., Fufaev N.A. *The Dynamics of the Nonholonomic Systems*. Moscow, 1967. (Russian)

[5] Beem J., Ehrlich P. *Global Lorentzian Geometry*. Marcel Dekker, 1981.
REFERENCES

[6] Krym V.R. Geodesics Equations for a Charged Particle in the Unified Theory of Gravitational and Electromagnetic Interactions. // Teor. Matem. Fisika, 1999, v. 119, N 3, pp. 517–528. (Russian) // Theoretical and Mathematical Physics, 1999, 119:3, 811–820 (English)

[7] Landau L.D., Lifshitz E.M. Theoretical Physics, v. 2. The Field Theory. Moscow, 1988. (Russian)

[8] Gray C.G., Karl G., Novikov V.A. Progress in Classical and Quantum Variational Principles. Reports on Progress in Physics, 2004, v. 67, N2, pp. 159–208.

[9] Rumer Yu.B. Researches in 5-Optics. Moscow, 1956. (Russian)

[10] Bailin D., Love A. Kaluza – Klein Theories. Reports on Progress in Physics, 1987, v. 50, pp. 1087–1170.

[11] Srivastava S.K. Some Aspects of Kaluza–Klein Cosmology. Pramana–Journal of Physics, 1997, v. 49, N4, pp. 323–370.

[12] Krym V.R. Smooth Manifolds of Kinematic Type. // Teor. Matem. Fisika, 1999, v. 119, N 2, pp. 264–281. (Russian) // Theoretical and Mathematical Physics, 1999, 119:2, 605–617 (English)

[13] Krym V.R, Petrov N.N. Causal Structures on Smooth Manifolds. Vestnik Sankt-Peterburgskogo Universiteta, ser. 1, 2001, N 2, pp. 27–34. (Russian)

[14] Krym V.R, Petrov N.N. Local Ordering on Smooth Manifolds. Vestnik Sankt-Peterburgskogo Universiteta, ser. 1, 2001, N 3, pp. 32–36. (Russian)

[15] Vasiljev F.P. Numerical Methods of Solution of Extremal Problems. Moscow, 1988. (Russian)

[16] Krotov V.F., Gurman V.I. Methods and Problems of Optimal Control. Moscow, 1973. (Russian)

[17] Ponrjagin L.S., Boltjanskij V.G., Gamkrelidze R.V., Mishchenko E.F. The Mathematical Theory of Optimal Processes. Moscow, 1983. (Russian)

[18] Filippov A.F. On Some Questions of the Theory of Optimal Control. Vestnik Moskovskogo Universiteta, ser. Math. & Mech., 1959, N 2, pp. 25–32. (Russian)

[19] Petrov N.N. Existence of Abnormal Shortest Paths in sub-Riemannian Geometry. Vestnik Sankt-Peterburgskogo Universiteta, ser. 1, 1993, Iss. 3 (N 15), pp. 28–32. (Russian)

[20] Bonnard B., Chyba M. Singular Trajectories and Their Role in Control Theory. Mathematiques & Applications, v. 40. Paris: Springer, 2003.

[21] Montgomery R. A Survey of Singular Curves in sub-Riemannian Geometry. J. Dynam. Contr. Syst., 1995, v. 1, pp. 49–90.
REFERENCES

[22] Montgomery R. *Survey of Singular Geodesics*. Progress in Math., 1996, v. 144, pp. 325–339.

[23] Kupka I. *Sub-Riemannian Geometry*. Asterisque, 1997, v. 241, pp. 351–380.

[24] Kupka I., Oliva W.M. *The non-Holonomic Mechanics*. J. Differ. Equ., 2001, v. 169, N 1, pp. 169–189.

[25] Gershkovich V.Ya. *Two-side Estimations of Metrics Generated by Absolutely Nonholonomic Distributions on Riemannian Manifolds*. Doklady Akademii Nauk, 1984, v. 278, N 5, pp. 1040–1044. (Russian)

[26] Trelat E. *Non-Subanalyticity of sub-Riemannian Martinet Spheres*. C. R. Acad. Sci. Paris, Ser. I, Math., 2001, v. 332, N 6, pp. 527–532.

[27] Mitchell J. *On Carnot-Caratheodory Metrics*. J. Diff. Geom., 1985, v. 21, N 1, pp. 35–45.

[28] Jean F. *Uniform Estimation of sub-Riemannian Balls*. J. Dyn. Control Syst., 2001, v. 7, N 4, pp. 473–500.

[29] Griffiths Ph. A. *Exterior Differential Systems and the Calculus of Variations*. Birkhauser, 1983. (Progress in Mathematics, v. 25).