An inclusion property of Orlicz-Morrey spaces

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Abstract. The Orlicz-Morrey spaces $L_{\varphi, \Phi}$ (where $\Phi$ is a Young function and $\varphi$ is a parameter for the Morrey spaces) are generalizations of Orlicz spaces and Morrey spaces. Inclusion properties between Orlicz spaces $L_{\Phi}$ and between Morrey spaces $M^p_{\psi}$ are well known. In this study we will investigate the inclusion relation between Orlicz-Morrey spaces $L_{\varphi_1, \Phi_1}$ and $L_{\varphi_2, \Phi_2}$ with respect to Young functions $\Phi_1, \Phi_2$ and parameters $\varphi_1, \varphi_2$. Also, we give sufficient and necessary conditions for the inclusion property of these spaces, which are obtained through norm estimates for the characteristic functions of balls in $\mathbb{R}^n$. In addition, we shall give a sufficient and necessary condition for generalized Hölder's inequality.

1. Introduction

Orlicz-Morrey spaces are generalizations of Orlicz spaces and Morrey spaces. There are two versions of Orlicz-Morrey spaces. One is defined by Nakai [2, 9] and another by Sawano, Sugano, and Tanaka [2, 12]. Here we are interested in studying the inclusion property of Orlicz-Morrey spaces which were introduced by Nakai.

First, we recall the definition of Young functions (see [11, 9]). A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if $\Phi$ is convex, left-continuous, $\lim_{t \to 0} = \Phi(t) = \Phi(0) = 0$, and $\lim_{t \to \infty} \Phi(t) = \infty$. Let $G_1$ be the set of all functions $\varphi : (0, \infty) \to (0, \infty)$ such that $\varphi(t)$ is nondecreasing but $\varphi(t)$ is nonincreasing. For $\varphi_1, \varphi_2 \in G_1$, we write $\varphi_1 \sim \varphi_2$ if there exists a constant $C > 1$ such that

$$ C^{-1}\varphi_1(t) \leq \varphi_2(t) \leq C\varphi_1(t) $$

for all $t > 0$.

Let $\Phi$ be a Young function and $\varphi \in G_1$. The Orlicz-Morrey space $L_{\varphi, \Phi}(\mathbb{R}^n)$ is the set of measurable functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that for every open ball $B$ in $\mathbb{R}^n$, the following

$$ \| f \|_{(\varphi, \Phi, B)} := \inf \{ b > 0 : \frac{\varphi(|B|)}{|B|} \int_B \Phi\left(\frac{|f(x)|}{b}\right)dx \leq 1 \} $$

is finite. We use the notation $\mathcal{B}$ to denote the family of all open balls $B$ in $\mathbb{R}^n$, and $|B|$ for its Lebesgue measure. $L_{\varphi, \Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$ \| f \|_{L_{\varphi, \Phi}(\mathbb{R}^n)} := \sup_{B \in \mathcal{B}} \| f \|_{(\varphi, \Phi, B)} . $$
For φ(t) = t, L_{φ,φ}(\mathbb{R}^n) = L_φ(\mathbb{R}^n) is the Orlicz space. Meanwhile, for φ(t) = \frac{1}{\psi(t^{1/n})^p} (where ψ : (0, ∞) → (0, ∞) is almost decreasing and \tilde{t}^{n}\psi(t) is almost increasing) and Φ(t) = t^p we have L_{φ,φ}(\mathbb{R}^n) = M_{φ}^p(\mathbb{R}^n), the generalized Morrey space introduced by Nakai in 1994.

Gunawan et al. [3] have proved an inclusion property of generalized Morrey spaces: If 1 ≤ p_1 ≤ p_2 < ∞, then M_{φ_2}^{p_2} ⊂ M_{φ_2}^{p_1} if and only if ψ_2(t) ≤ Cψ_1(t) for all t > 0 and some C > 0. On the other hand, inclusion properties between Orlicz spaces L_φ(\mathbb{R}^n) and between weak Orlicz spaces wL_φ(\mathbb{R}^n) are well known (see [5, 6]).

Motivated by these results, the purpose of this study is to get the inclusion property of Orlicz-Morrey spaces L_{φ,φ}(\mathbb{R}^n).

The main results are presented in Section 2. In particular, Theorem 2.2 contains a necessary and sufficient condition for the inclusion relation between Orlicz-Morrey spaces. In Section 3, we have also given sufficient and necessary conditions for generalized Hölder’s inequality.

To prove the results, we will use the same method as in [3] and [6] which pay attention to the characteristic functions of balls in \mathbb{R}^n and use the inverse function of Φ, namely Φ^{-1}(s) := \inf\{r ≥ 0 : Φ(r) > s\}.

In the following, we recall several lemmas which will be used in the next section.

**Lemma 1.1.** [11, 6] Suppose that Φ is a Young function and Φ^{-1}(s) = \inf\{r ≥ 0 : Φ(r) > s\}. We have
1. Φ^{-1}(0) = 0.
2. Φ^{-1}(s_1) ≤ Φ^{-1}(s_2) for s_1 ≤ s_2.
3. Φ^{-1}(s) ≤ Φ^{-1}(Φ(s)) for 0 ≤ s < ∞.
4. If, for some constants C_1, C_2 > 0, we have Φ_2^{-1}(s) ≤ C_1Φ_1^{-1}(C_2s), then Φ_1(t \frac{s}{t}) ≤ C_2Φ_2(t) for t = Φ_2^{-1}(s).

**Lemma 1.2.** [2] Let Φ be a Young function, a ∈ \mathbb{R}^n and r > 0. Then ∥χ_{B(a, r)}∥_{L_φ,φ(\mathbb{R}^n)} = \frac{1}{Φ^{-1}(\frac{1}{|B(a, r)|})}, where |B(a, r)| denotes the volume of B(a, r).

**Lemma 1.3.** If f ∈ L_{φ,φ}(\mathbb{R}^n), then
\[ \frac{φ(|B|)}{|B|} \int_B Φ\left(\frac{|f(x)|}{∥f∥_{(φ,φ, B)}}\right) dx ≤ 1 \]
for any open ball B ∈ \mathcal{B}. Furthermore, ∥f∥_{(φ,φ, B)} ≤ 1 if and only if \frac{φ(|B|)}{|B|} \int_B Φ\left(\frac{|f(x)|}{b_ε}\right) dx ≤ 1 for any open ball B ∈ \mathcal{B}.

**Proof.**

Let f be an element of L_{φ,φ}(\mathbb{R}^n) and take an arbitrary ε > 0, then there exists b_ε > 0 such that b_ε ≤ ∥f∥_{(φ,φ, B)} + ε and \frac{φ(|B|)}{|B|} \int_B Φ\left(\frac{|f(x)|}{b_ε}\right) dx ≤ 1 for any open ball B ∈ \mathcal{B}. Because, Φ is increasing, we have
\[ \frac{φ(|B|)}{|B|} \int_B Φ\left(\frac{|f(x)|}{∥f∥_{(φ,φ, B)} + ε}\right) dx ≤ \int_B Φ\left(\frac{|f(x)|}{b_ε}\right) dx ≤ 1. \]
Since ε > 0 is arbitrary, we can conclude
\[ \frac{φ(|B|)}{|B|} \int_B Φ\left(\frac{|f(x)|}{∥f∥_{(φ,φ, B)}}\right) dx ≤ 1 \]
for any open ball B ∈ \mathcal{B}.
Next, if \( \|f\|_{(\phi,\Phi)} \leq 1 \) for any open ball \( B \in \mathfrak{B} \), then
\[
\frac{\phi(|B|)}{|B|} \int_B \Phi(|f(x)|)dx \leq \frac{\phi(|B|)}{|B|} \int_B \Phi\left(\frac{|f(x)|}{\|f\|_{(\phi,\Phi,B)}}\right)dx \leq 1.
\]
Now, assume that \( \frac{\phi(|B|)}{|B|} \int_B \Phi(|f(x)|)dx \leq 1 \) holds for any open ball \( B \in \mathfrak{B} \). Clearly, we have \( \|f\|_{(\phi,\Phi)} \leq 1 \). \( \square \)

**Corollary 1.4.** If \( f \in L_{\phi,\Phi}(\mathbb{R}^n) \), then
\[
\frac{\phi(|B|)}{|B|} \int_B \Phi\left(\frac{|f(x)|}{\|f\|_{L_{\phi,\Phi}(\mathbb{R}^n)}}\right)dx \leq 1
\]
for any open ball \( B \in \mathfrak{B} \). Furthermore, \( \|f\|_{L_{\phi,\Phi}(\mathbb{R}^n)} \leq 1 \) if and only if \( \frac{\phi(|B|)}{|B|} \int_B \Phi(|f(x)|)dx \leq 1 \) for any open ball \( B \in \mathfrak{B} \).

### 2. Main Results

Now, we come to the inclusion relation between Orlicz-Morrey space \( L_{\phi_1,\Phi_1} \) and \( L_{\phi_2,\Phi_2} \) with respect to Young functions \( \Phi_1, \Phi_2 \) and parameters \( \phi_1, \phi_2 \).

**Theorem 2.1.** [7] Let \( \Phi_1, \Phi_2 \) be Young functions and \( \phi_1, \phi_2 \in G_1 \) such that \( \phi_1 \sim \phi_2 \). If \( \Phi_1(x) \leq \Phi_2(Kx) \) for some \( K > 0 \) then \( L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n) \).

We can prove the Lemma 2.1 by using similar arguments in the proof of [9, Proposition 3.2].

**Theorem 2.2.** Let \( \Phi_1, \Phi_2 \) be Young functions and \( \phi_1, \phi_2 \in G_1 \) such that \( \phi_1 \sim \phi_2 \). Then the following statements are equivalent:

(1) \( \Phi_1(t) \leq \Phi_2(Ct) \), for every \( t > 0 \).

(2) \( L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n) \).

(3) There exists a constant \( C > 0 \) such that
\[
\|f\|_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C \|f\|_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)}.
\]

for every \( f \in L_{\phi_2,\Phi_2}(\mathbb{R}^n) \).

**Proof.**

Assume that (1) holds. Let \( f \) be an element of \( L_{\phi_2,\Phi_2}(\mathbb{R}^n) \). Since \( \phi_1 \sim \phi_2 \) and \( \Phi_1(t) \leq \Phi_2(Ct) \), for every \( t > 0 \), by Theorem 2.1, we have \( L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n) \). Next, since \( (L_{\phi_2,\Phi_2}(\mathbb{R}^n), L_{\phi_1,\Phi_1}(\mathbb{R}^n)) \) is a Banach pair, it follows from [4, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1). Assume that (3) holds. By Lemma 1.2, for every \( r_0 > 0 \) we have
\[
\frac{1}{\Phi_1^{-1}\left(\frac{1}{\phi_1(|B(a,r_0)|)}\right)} = \|\chi_{B(a,r_0)}\|_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C \|\chi_{B(a,r_0)}\|_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1}\left(\frac{1}{\phi_2(|B(a,r_0)|)}\right)}
\]
whence $\Phi_2^{-1}(\frac{1}{\Phi_2((B(a,r_0))}) \leq C\Phi_1^{-1}(\frac{1}{\Phi_2((B(a,r_0))}) \leq C\Phi_1^{-1}(\frac{C_2}{\Phi_2((B(a,r_0))})$, for arbitrary $a \in \mathbb{R}^n$ and $r_0 > 0$. By Lemma 1.1 (4), $\phi_1 \sim \phi_2$, and $\Phi$ is convex, we have

$$\Phi_1(t_0) \leq C\Phi_1^1(t_0) \leq \Phi_2(t_0)$$

where $t_0 = \Phi_2^{-1}(\frac{1}{\Phi_2((B(a,r_0))})$. Since $r_0$ is an arbitrary positive real number, we get $\Phi_1(t) \leq \Phi_2(Ct)$, for every $t > 0$.

**Corollary 2.3.** [6] Let $\Phi_1, \Phi_2$ be Young functions and $\phi_1(t) = \phi_2(t) = t$ for $t > 0$. Then the following statements are equivalent:

1. $\Phi_1(t) \leq \Phi_2(Ct)$, for every $t > 0$.
2. $L_{\Phi_2}(\mathbb{R}^n) \subseteq L_{\Phi_1}(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that

$$\| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \leq C \| f \|_{L_{\Phi_2}(\mathbb{R}^n)}$$

for every $f \in L_{\Phi_2}(\mathbb{R}^n)$.

**Remark.** Corollary 2.3 refines Corollary 2.11 in [10] which only states that (1) implies (2) for $w_1(x) = w_2(x) = 1$ and $X = \mathbb{R}^n$.

### 3. Generalized Hölder Inequality

In the following, we will give sufficient and necessary conditions for the generalized Hölder inequality on Orlicz-Morrey spaces. To get the result we will give attention to estimate the norm of the characteristic function of ball in $\mathbb{R}^n$.

**Theorem 3.1.** For $m \geq 2$. Let $\Phi_i$ be Young functions and $\phi_i \in G_i$, for $i = 1, 2, 3, ..., m$. If there exists a constant $C > 0$ such that

$$\prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{1}{\phi_i(t)}) \leq C\Phi_i^{-1}(\frac{1}{\phi_i(t)})$$

for $s, t > 0$, then for $f_i \in L_{\phi_i, \Phi_i}(\mathbb{R}^n)$, $i = 1, 2, 3, ..., m - 1$ we have

$$\prod_{i=1}^{m-1} f_i \in L_{\Phi_i, \phi_i}(\mathbb{R}^n)$$

with

$$\| \prod_{i=1}^{m-1} f_i \|_{L_{\phi_i, \Phi_i}(\mathbb{R}^n)} \leq (m - 1)C \prod_{i=1}^{m-1} \| f_i \|_{L_{\phi_i, \Phi_i}(\mathbb{R}^n)} .$$

**Proof.**

Let $f_i$ be an element of $L_{\phi_i, \Phi_i}(\mathbb{R}^n)$ for $i = 1, 2, 3, ..., m - 1$. Then

$$\frac{\phi_i(|B|)}{|B|} \int_B \Phi_i\left(\frac{|f_i(x)|}{\| f_i \|_{\phi_i, \Phi_i(B)}}\right) dx \leq 1$$

for any ball $B \in \mathfrak{B}$ and $i = 1, 2, 3, ..., m - 1$. Now fix $B$. For each $x \in B$, let

$$M(x) = \max\left\{ \phi_i(|B|)\Phi_i\left(\frac{|f_i(x)|}{\| f_i \|_{\phi_i, \Phi_i(B)}}\right) : 1 \leq i \leq m - 1 \right\} .$$

From $\Phi_i\left(\frac{|f_i(x)|}{\| f_i \|_{\phi_i, \Phi_i,B}}\right) \leq \frac{M(x)}{\phi_i(|B|)}$ and Lemma 1.1(3), we have

$$\frac{|f_i(x)|}{\| f_i \|_{\phi_i, \Phi_i,B}} \leq \Phi_i^{-1}\left(\frac{M(x)}{\phi_i(|B|)}\right) \leq \Phi_i^{-1}\left(\frac{1}{\phi_i(|B|)}\right) .$$
for $i = 1, 2, 3, \ldots, m - 1$. Hence

$$
\prod_{i=1}^{m-1} \frac{|f_i(x)|}{\| f_i \|_{(\phi_i, \Phi_i, B)}} \leq \prod_{i=1}^{m-1} \Phi_i^{-1}(M(x)_{\phi_i(|B|)}) \leq C \Phi_m^{-1}(\frac{M(x)}{\phi_m(|B|)})
$$

and

$$
\Phi_m\left( \frac{1}{(m-1)C} \prod_{i=1}^{m-1} \frac{|f_i(x)|}{\| f_i \|_{(\phi_i, \Phi_i, B)}} \right) \leq \frac{\Phi_m(\frac{M(x)}{\phi_m(|B|)})}{(m-1)} \leq \frac{M(x)}{(m-1)\phi_m(|B|)}
$$

On the other hand, we have

$$
\frac{M(x)}{(m-1)\phi_m(|B|)} \leq \frac{\sum_{i=1}^{m-1} \phi_i(|B|) \Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i, \Phi_i, B)}})}{(m-1)\phi_m(|B|)}
$$

and

$$
\sum_{i=1}^{m-1} \phi_i(|B|) \int_B \Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i, \Phi_i, B)}}) dx \leq (m-1)|B|.
$$

Therefore

$$
\int_B \Phi_m\left( \frac{1}{(m-1)C} \prod_{i=1}^{m-1} \frac{|f_i(x)|}{\| f_i \|_{(\phi_i, \Phi_i, B)}} \right) dx \leq \frac{(m-1)|B|}{(m-1)\phi_m(|B|)} = \frac{|B|}{\phi_m(|B|)}.
$$

This shows that

$$
\| \prod_{i=1}^{m-1} f_i \|_{(\phi_m, \Phi_m, B)} \leq (m-1)C \prod_{i=1}^{m-1} \| f_i \|_{(\phi_i, \Phi_i, B)}
$$

for every open ball $B \in \mathcal{B}$. Hence, we conclude that

$$
\| \prod_{i=1}^{m-1} f_i \|_{L_{\phi_m}(\mathbb{R}^n)} \leq (m-1)C \prod_{i=1}^{m-1} \| f_i \|_{L_{\phi_i}(\mathbb{R}^n)}.
$$

\[\square\]

**Remark 3.2.** For $m = 3$, Theorem 3.1 reduces to Theorem 4.1 (p.200) in [7].

**Corollary 3.3.** (*Hölder’s inequality on Orlicz spaces*) Let $\Phi_i$ be Young functions, $i = 1, 2, 3$. If there exists a constant $C > 0$ such that $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq C\Phi_3^{-1}(t)$ for $t > 0$, then for $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$ we have $f \cdot g \in L_{\Phi_3}(\mathbb{R}^n)$ with

$$
\| f \cdot g \|_{L_{\Phi_3}(\mathbb{R}^n)} \leq 2C \| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}.
$$
Theorem 3.4. For $m \geq 2$. Let $\Phi_i$ be Young functions and $\phi_i \in G_1$, for $i = 1, 2, 3, \ldots, m$. If there exists a constant $K > 0$ such that $\left\| \prod_{i=1}^{m-1} f_i \right\|_{L_{\Phi_m}^m(\mathbb{R}^n)} \leq K \prod_{i=1}^{m-1} \| f_i \|_{L_{\Phi_i}^s(\mathbb{R}^n)}$ for $f_i \in L_{\Phi_i}(\mathbb{R}^n)$, $i = 1, 2, 3, \ldots, m - 1$, then we have $\prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{s}{\phi_i(t)}) \leq K \Phi_m^{-1}(\frac{s}{\phi_m(t)})$ for every $s, t > 0$.

Proof.

Observe that, for every open ball $B(a, r)$ we have

$$\left\| g \chi_{B(a, r)} \right\|_{L_{\frac{1}{\phi_m}(\mathbb{R}^n)}} = \left\| \prod_{i=1}^{m-1} \chi_{B(a, r)} \right\|_{L_{\frac{1}{\phi_m}}(\mathbb{R}^n)} \leq K \prod_{i=1}^{m-1} \| g \chi_{B(a, r)} \|_{L_{\frac{1}{\phi_i}}(\mathbb{R}^n)}.$$  

By Lemma 1.2 we have

$$\frac{1}{\Phi_m^{-1}(\frac{s}{\phi_m(||B(a, r)||)})} \leq K \prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{s}{\phi_i(||B(a, r)||)})$$

or

$$\prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{s}{\phi_i(||B(a, r)||)}) \leq K \Phi_m^{-1}(\frac{s}{\phi_m(||B(a, r)||)}).$$

Since $B(a, r)$ is arbitrary, we get

$$\prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{s}{\phi_i(t)}) \leq K \Phi_m^{-1}(\frac{s}{\phi_m(t)})$$

for $s, t > 0$. \qed

Corollary 3.5. Let $p_i \geq 1$ be real numbers, $i = 1, 2, 3, \ldots, m$. If $f_i \in L_{p_i}(\mathbb{R}^n)$, $i = 1, 2, 3, \ldots, m$ such that

$$\left\| \prod_{i=1}^{m} f_i \right\|_{L_1(\mathbb{R}^n)} \leq \prod_{i=1}^{m} \| f_i \|_{L_{p_i}(\mathbb{R}^n)},$$

then $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$.

Proof. By Theorem 3.4, we have $|x|^{\frac{1}{p_1}} |x|^{\frac{1}{p_2}} |x|^{\frac{1}{p_3}} |x|^{\frac{1}{p_m}} \leq |x|$ for every $x > 0$. Since $x$ is an arbitrary positive real number, we conclude that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$. \qed

Remark. Corollary 3.5 completes Theorem 2.11 in [1]. Theorem 2.1 states that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$ is a sufficient condition for generalized Hölder’s inequality on Lebesgue spaces, while Corollary 3.5 states that it is also a necessary condition.

Corollary 3.6. Let $\Phi_1, \Phi_2, \Phi_3$ be Young functions. If there exists a constant $C > 0$ such that

$$\| f \cdot g \|_{L_{\Phi_3}(\mathbb{R}^n)} \leq C \| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}$$
for \( f \in L_{\Phi_1}(\mathbb{R}^n) \) and \( g \in L_{\Phi_2}(\mathbb{R}^n) \), then
\[
\Phi^{-1}_1(t)\Phi^{-1}_2(t) \leq C\Phi^{-1}_3(t)
\]
for every \( t > 0 \).

For each Young function \( \Phi \), we can define another convex function \( \tilde{\Phi} : \mathbb{R} \to (0, \infty) \) having similar properties, by \( \tilde{\Phi}(y) = \sup_{x > 0} \{ x|y| - \Phi(x) \} \), \( y \in \mathbb{R} \). Then \( \tilde{\Phi} \) is called the complementary function to \( \Phi \).

**Corollary 3.7.** If \( \Phi \) is Young function and \( \tilde{\Phi} \) is complementary function of \( \Phi \), then
\[
\Phi^{-1}_1(t)\tilde{\Phi}^{-1}(t) \leq 2t
\]
for every \( t > 0 \).

**Proof.**

It follows from [7, Theorem 2.3] that, for any \( f \in L_{\Phi}(\mathbb{R}^n) \) and \( g \in L_{\tilde{\Phi}}(\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n}|f \cdot g|dx \leq 2 \| f \|_{L_{\Phi}(\mathbb{R}^n)} \| g \|_{L_{\tilde{\Phi}}(\mathbb{R}^n)}.
\]
By Corollary 3.5, we have \( \Phi^{-1}(t)\tilde{\Phi}^{-1}(t) \leq 2t \).

**Remark.** We can prove the Corollary 3.7 directly by definition of \( \Phi^{-1} \) and \( \tilde{\Phi}^{-1} \). What we showed here is that we can obtain the result through the lens of Orlicz-Morrey spaces.

**4. Conclusion**

In this paper, we have discussed the inclusion relation between Orlicz-Morrey spaces for Nakai version. By estimating the norm of characteristic function of balls in \( \mathbb{R}^n \), we obtain sufficient and necessary conditions for inclusion relation between Orlicz-Morrey spaces (Theorem 2.2), which were generalized for inclusion property of Orlicz spaces in [6]. Furthermore, by estimating the characteristic function of balls in \( \mathbb{R}^n \) we obtain sufficient and necessary conditions for generalized Hölder’s inequality on Orlicz-Morrey spaces (Theorem 3.1 and Theorem 3.4). Theorem 3.4 generalized Hölder’s inequality in [7].

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