COUPLED COINCIDENCE POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. We obtain coupled coincidence and coupled common fixed point theorems for mixed g-monotone nonlinear operators \( F : X \times X \rightarrow X \) in partially ordered metric spaces. Our results are generalizations of recent coincidence point theorems due to Lakshmikantham and Ćirić [Lakshmikantham, V., Ćirić, L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349], of coupled fixed point theorems established by Bhaskar and Lakshmikantham [T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379-1393] and also include as particular cases several related results in very recent literature.

1. INTRODUCTION AND PRELIMINARIES

A very recent trend in metrical fixed point theory, initiated by Ran and Reurings [10], and continued by Nieto and Lopez [8], [9], Bhaskar and Lakshmikantham [5], Agarwal et al. [1], Lakshmikantham and Ćirić [6], Luong and Thuan [7] and many other authors, is to consider a partial order on the ambient metric space \((X,d)\) and to transfer a part of the contractive property of the nonlinear operators into its monotonicity properties. This approach turned out to be very productive, see for example [1], [4]-[10], and the obtained results found important applications to the existence of solutions for matrix equations or ordinary differential equations and integral equations, see [5], [7], [8], [9], [10] and references therein.

In this context, the main novelty brought by Bhaskar and Lakshmikantham [5] and then continued by Lakshmikantham and Ćirić [6] and other authors, was to consider nonlinear bivariate mappings \( F : X \times X \rightarrow X \) in direct connection with their so called mixed monotone property, and to study the existence (and uniqueness) of coupled fixed points for such mappings.

To fix the context in which we are placing our results, recall the following notions. Let \((X,\leq)\) be a partially ordered set and endow the product space \(X \times X\) with the following partial order:

\[
\text{for } (x,y), (u,v) \in X \times X, (u,v) \leq (x,y) \iff x \geq u, y \leq v.
\]
We say that a mapping \( F : X \times X \to X \) has the mixed monotone property if \( F(x, y) \) is monotone non-decreasing in \( x \) and is monotone non-increasing in \( y \), that is, for any \( x, y \in X \),
\[
x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)
\]
and, respectively,
\[
y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).
\]

A pair \((x, y)\) \( \in X \times X \) is called a coupled fixed point of the mapping \( F \) if
\[
F(x, y) = x, F(y, x) = y.
\]
The next theorem is the main theoretical result in [5].

**Theorem 1** (Bhaskar and Lakshmikantham [5]). Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Let \( F : X \times X \to X \) be a continuous mapping having the mixed monotone property on \( X \). Assume that there exists a constant \( k \in [0, 1) \) with
\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for each } x \geq u, y \leq v.
\]
If there exist \( x_0, y_0 \in X \) such that
\[
x_0 \leq F(x_0, y_0) \text{ and } y_0 \geq F(y_0, x_0),
\]
then there exist \( x, y \in X \) such that
\[
x = F(x, y) \text{ and } y = F(y, x).
\]

As shown in [5], the continuity assumption of \( F \) in Theorem 1 can be replaced by the following property imposed on the ambient space \( X \):

**Assumption 1.1.** \( X \) has the property that
(i) if a non-decreasing sequence \( \{x_n\}_{n=0}^{\infty} \subset X \) converges to \( x \), then \( x_n \leq x \) for all \( n \);
(ii) if a non-increasing sequence \( \{x_n\}_{n=0}^{\infty} \subset X \) converges to \( x \), then \( x_n \geq x \) for all \( n \);

These results were then extended and generalized by several authors in the last five years, see [6], [7] and references therein, to restrict citing only the ones strictly related to our approach in this paper. Amongst these generalizations, we refer especially to the one obtained in [6], which considered instead of (1.1) a more general contractive condition and established corresponding coincidence point theorems.

The following concepts were introduced in [5].

**Definition 1.** Let \((X, \leq)\) be a partially ordered set and \( F : X \times X \to X, g : X \to X \). We say that \( F \) has the mixed \( g \)-monotone property if
$F$ is monotone $g$-non-decreasing in its first argument and is monotone $g$-non-increasing in its second argument, that is, for any $x, y \in X$,

\[ x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \]

and

\[ y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2). \]

Note that if $g$ is the identity mapping, then Definition 1 reduces to Definition 1.1 in [5] of mixed monotone property.

**Definition 2.** An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

\[ F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y). \]

**Definition 3.** Let $X$ be a non-empty set. We say that the mappings $F : X \times X \to X$ and $g : X \to X$ commute if

\[ g(F(x, y)) = F(g(x), g(y)), \]

for all $x, y \in X$.

Using basically these concepts, the results obtained in [6] are some coincidence theorems and coupled common fixed point theorems obtained basically by considering a more general contractive condition than condition (1.1) used in [5]. The main result in [6] is given by the next theorem.

**Theorem 2.** Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to t} \psi(r) < t$ for all $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that $F$ has the mixed $g$-monotone property and

\[ d(F(x, y), F(u, v)) \leq \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right), \quad (1.2) \]

for all $x, y, u, v \in X$ with $g(x) \geq g(u), g(y) \leq g(v)$. Suppose $F(X \times X) \subset g(X)$, $g$ is continuous and commutes with $F$ and also suppose either

(a) $F$ is continuous or

(b) $X$ satisfy Assumption [7,7].

If there exist $x_0, y_0 \in X$ such that

\[ g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \leq F(y_0, x_0), \quad (1.3) \]

then there exist $\overline{x}, \overline{y} \in X$ such that

\[ g(\overline{x}) = F(\overline{x}, \overline{y}) \quad \text{and} \quad g(\overline{y}) = F(\overline{y}, \overline{x}), \]

that is, $F$ and $g$ have a coupled coincidence.
Obviously, for \( g = \text{identity} \) and \( \varphi(t) = kt, \ 0 \leq k < 1 \), Theorem 2 reduces to Theorem 1.

Starting from the results in [6] and [5], our main aim in this paper is to obtain more general coincidence point theorems and coupled common fixed point theorems for mixed monotone operators \( F : X \times X \to X \) satisfying a contractive condition which is significantly more general than the corresponding conditions (1.2) and (1.1) in [6] and [5], respectively, thus extending many other related results in literature.

2. Main results

Let \( \Phi \) denote the set of all functions \( \varphi : [0, \infty) \to [0, \infty) \) satisfying

(i) \( \varphi(t) < t \) for all \( t \in (0, \infty) \);

(ii) \( \lim_{r \to t^+} \varphi(r) < t \), for all \( t \in (0, \infty) \).

The first main result in this paper is the following coincidence point theorem which generalizes Theorem 2.1 in [6] and Theorem 2.1 in [5].

**Theorem 3.** Let \( (X, \leq) \) be a partially ordered set and suppose there is a metric \( d \) on \( X \) such that \( (X,d) \) is a complete metric space. Let \( F : X \times X \to X \) be a mixed \( g \)-monotone mapping for which there exist \( \varphi \in \Phi \) such that for all \( x,y,u,v \in X \) with \( g(x) \geq g(u), g(y) \leq g(v) \),

\[
d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \leq 2\varphi \left( \frac{d(g(x),g(u)) + d(g(y),g(v))}{2} \right).
\]

Suppose \( F(X \times X) \subset g(X) \), \( g \) is continuous and commutes with \( F \) and also suppose either

(a) \( F \) is continuous or

(b) \( X \) satisfy Assumption [1.1]

If there exist \( x_0,y_0 \in X \) such that

\[
g(x_0) \leq F(x_0,y_0) \quad \text{and} \quad g(y_0) \geq F(y_0,x_0),
\]

or

\[
g(x_0) \geq F(x_0,y_0) \quad \text{and} \quad g(y_0) \leq F(y_0,x_0),
\]

then there exist \( \overline{x},\overline{y} \in X \) such that

\[
g(\overline{x}) = F(\overline{x},\overline{y}) \quad \text{and} \quad g(\overline{y}) = F(\overline{y},\overline{x}),
\]

that is, \( F \) and \( g \) have a coupled coincidence.

**Proof.** Consider the functional \( d_2 : X^2 \times X^2 \to \mathbb{R}_+ \) defined by

\[
d_2(Y,V) = \frac{1}{2} \left[ d(x,u) + d(y,v) \right], \ \forall Y = (x,y), V = (u,v) \in X^2.
\]

It is a simple task to check that \( d_2 \) is a metric on \( X^2 \) and, moreover, that, if \( (X,d) \) is complete, then \( (X^2,d_2) \) is a complete metric space, too. Now consider the operator \( T : X^2 \to X^2 \) defined by

\[
T(Y) = (F(x,y),F(y,x)), \ \forall Y = (x,y) \in X^2.
\]
Assume (2.2) holds (the case (2.3) is similar). Then, there exists

Thus, by the contractive condition (2.1) we obtain that

\( d_2(T(Y), T(V)) = \frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2} \)

and

\( d_2(Y, V) = \frac{d(x,u) + d(y,v)}{2}. \)

Thus, by the contractive condition (2.1) we obtain that \( F \) satisfies the following \( \varphi \)-contractive condition:

\[ d_2(T(Y), T(V)) \leq \varphi(d_2(Y, V)), \forall Y \geq V \in X^2. \]

(4.4)

Assume (2.2) holds (the case (2.3) is similar). Then, there exists \( x_0, y_0 \in X \) such that

\( g(x_0) \leq F(x_0, y_0) \) and \( g(y_0) \geq F(y_0, x_0). \)

Denote \( Z_0 = (g(x_0), g(y_0)) \in X^2 \) and consider the Picard iteration associated to \( T \) and to the initial approximation \( Z_0 \), that is, the sequence \( \{Z_n\} \subset X^2 \) defined by

\[ Z_{n+1} = T(Z_n), \quad n \geq 0, \]

(5.5)

where \( Z_n = (g(x_n), g(y_n)) \in X^2, n \geq 0. \)

Since \( F \) is \( g \)-mixed monotone, we have

\[ Z_0 = (g(x_0), g(y_0)) \leq (F(x_0, y_0), F(y_0, x_0)) = (g(x_1), g(y_1)) = Z_1 \]

and, by induction,

\[ Z_n = (g(x_n), g(y_n)) \leq (F(x_n, y_n), F(y_n, x_n)) = (g(x_{n+1}), g(y_{n+1})) = Z_{n+1}, \]

which actually shows that

\( g(x_n) \leq g(x_{n+1}) \) and \( g(y_n) \geq g(y_{n+1}), \) for all \( n \geq 0. \)

(6.6)

Note also, in particular, that the mapping \( T \) is monotone and the sequence \( \{Z_n\}_{n=0}^\infty \) is non-decreasing. Take \( Y = Z_n \geq Z_{n-1} = V \) in (2.4) and obtain

\[ d_2(T(Z_n), T(Z_{n-1}) \leq \varphi(d_2(Z_n, Z_{n-1})), \quad n \geq 1. \]

(7.7)

This shows that the sequence \( \{\delta_n\}_{n=1}^\infty \) given by

\[ \delta_n = d_2(Z_n, Z_{n-1}) = \frac{d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))}{2}, \quad n \geq 1, \]

satisfies

\[ \delta_{n+1} \leq \varphi(\delta_n), \quad \text{for all} \quad n \geq 1. \]

(8.8)

From (2.8) and \( (i) \varphi \) it follows that the sequence \( \{\delta_n\}_{n=1}^\infty \) is non-increasing. Therefore, there exists some \( \delta \geq 0 \) such that

\[ \lim_{n \to \infty} \delta_n = \lim_{n \to \infty} D(g(x_{n+1}), g(x_n)) + D(g(y_{n+1}), g(y_n)) = \delta. \]

(9.9)
We shall prove that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. Then by letting $n \to \infty$ in (2.8) we have
\[
\delta = \lim_{n \to \infty} \varphi(\delta_{n+1}) \leq \lim_{n \to \infty} \varphi(\delta_n) = \lim_{\delta \to \delta} \varphi(\delta_n) < \delta,
\]
a contradiction. Thus $\delta = 0$ and hence
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \frac{d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))}{2} = 0. \tag{2.10}
\]
We now prove that $\{Z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(X^2, d_2)$, that is, $\{g(x_n)\}_{n=0}^{\infty}$ and $\{g(y_n)\}_{n=0}^{\infty}$ are Cauchy sequences in $(X, d)$. Suppose, to the contrary, that at least one of the sequences $\{g(x_n)\}_{n=0}^{\infty}$, $\{g(y_n)\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{g(x_{m(k)})\}, \{g(x_{m(k)})\}$ of $\{g(x_n)\}_{n=0}^{\infty}$ and $\{g(y_{m(k)})\}$, $\{g(y_{m(k)})\}$ of $\{g(y_n)\}_{n=0}^{\infty}$, respectively, with $n(k) > m(k) \geq k$ such that
\[
\frac{1}{2} \left[ d(g(x_{m(k)}), g(x_{m(k)})) + d(g(y_{m(k)}), g(y_{m(k)})) \right] \geq \epsilon, \quad k = 1, 2, \ldots. \tag{2.11}
\]
Note that we can choose $n(k)$ to be the smallest integer with property $n(k) > m(k) \geq k$ and satisfying (2.11). Then
\[
d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)})) < \epsilon. \tag{2.12}
\]
By (2.11) and (2.12) and the triangle inequality we have
\[
\epsilon \leq r_k := \frac{1}{2} \left[ d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)})) \right] \leq
\frac{d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1}))}{2} +
\frac{d(g(x_{n(k)-1}), g(x_{m(k)})) + d(g(y_{n(k)-1}), g(y_{m(k)}))}{2} \leq
\frac{d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(y_{n(k)}), g(y_{n(k)-1}))}{2} + \epsilon.
\]
Letting $k \to \infty$ in the above inequality and using (2.10) we get
\[
\lim_{k \to \infty} r_k := \lim_{k \to \infty} \frac{1}{2} \left[ d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \right] = \epsilon. \tag{2.13}
\]
On the other hand
\[
r_k := \frac{d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))}{2} \leq
\frac{d(g(x_{n(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)+1}), g(x_{m(k)}))}{2} +
\frac{d(g(y_{n(k)}), g(y_{n(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)}))}{2} =
\delta_{n(k)} + \frac{d(g(x_{n(k)+1}), g(x_{m(k)})) + d(g(y_{n(k)+1}), g(y_{m(k)}))}{2} \leq
\]

On the other hand, by (2.5) and commutativity of which, by (2.14), yields
\[
\delta_n(k) + \delta_m(k) + \frac{d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1}))}{2}.
\]

Since \( n(k) > m(k) \), by (2.6) we have \( g(x_{n(k)}) \geq g(x_{m(k)}) \) and \( g(y_{n(k)}) \leq g(y_{m(k)}) \) and hence by (2.7) one obtains
\[
d(g(x_{n(k)+1}), g(x_{m(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1})) =
= d\left(F(g(x_{n(k)}), g(y_{m(k)})), F(g(y_{n(k)}), g(y_{m(k)}))\right)
\leq
\leq 2\varphi\left(\frac{d(g(x_{n(k)}), g(x_{m(k)})) + d(g(y_{n(k)}), g(y_{m(k)}))}{2}\right) \leq 2\varphi(r_k),
\]

which, by (2.14), yields
\[
r_k \leq \delta_n(k) + \delta_m(k) + \varphi(r_k).
\]

Letting \( k \to \infty \) in the above inequality and using (2.13) we get
\[
\epsilon \leq \lim_{k \to \infty} \varphi(r_k) = \lim_{r_k \to 0^+} \varphi(r_k) < \epsilon,
\]
a contradiction. This shows that \( \{g(x_n)\}_{n=0}^{\infty} \) and \( \{g(y_n)\}_{n=0}^{\infty} \) are indeed Cauchy sequences in the complete metric space \((X, d)\).

This implies there exist \( \overline{x}, \overline{y} \) in \( X \) such that
\[
\overline{x} = \lim_{n \to \infty} g(x_n), \quad \overline{y} = \lim_{n \to \infty} g(y_n), \quad (2.15)
\]

By (2.15) and continuity of \( g \),
\[
\lim_{n \to \infty} g(g(x_n)) = g(\overline{x}) \quad \text{and} \quad \lim_{n \to \infty} g(g(y_n)) = g(\overline{y}). \quad (2.16)
\]

On the other hand, by (2.5) and commutativity of \( F \) and \( g \),
\[
g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \quad (2.17)
\]
\[
g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \quad (2.18)
\]

We now prove that \( g(\overline{x}) = F(\overline{x}, \overline{y}) \) and \( g(\overline{y}) = F(\overline{y}, \overline{x}) \).

Suppose first that assumption (a) holds. By letting \( n \to \infty \) in (2.17) and (2.18), in view of in (2.15) and (2.16), we get
\[
g(\overline{x}) = \lim_{n \to \infty} g(g(x_{n+1})) = \lim_{n \to \infty} F(g(x_n), g(y_n)) = F(\overline{x}, \overline{y})
\]
and, similarly
\[
g(\overline{y}) = \lim_{n \to \infty} g(g(y_{n+1})) = \lim_{n \to \infty} F(g(y_n), g(x_n)) = F(\overline{y}, \overline{x}),
\]

that is, \((\overline{x}, \overline{y})\) is a coincidence point of \( F \) and \( g \).

Suppose now assumption (b) holds. Since \( \{g(x_n)\}_{n=0}^{\infty} \) is a non-decreasing sequence that converges to \( \overline{x} \), we have that \( g(x_n) \leq \overline{x} \) for all \( n \). Similarly, we obtain \( g(y_n) \geq \overline{y} \) for all \( n \).

Then, by triangle inequality and contractive condition (2.1),
\[
d(g(\overline{x}), F(\overline{x}, \overline{y})) + d(g(\overline{y}), F(\overline{y}, \overline{x})) \leq d(g(\overline{x}), g(x_{n+1}))+
\]
\[
+ d(g(x_{n+1}), F(\overline{x}, \overline{y})) + d(g(\overline{y}), g(y_{n+1}))++ d(g(y_{n+1}), F(\overline{y}, \overline{x})) =
\]
\[
= d(g(\overline{x}), g(x_{n+1}))+d(g(\overline{y}), g(y_{n+1}))+d(F(x_n, y_n), F(\overline{x}, \overline{y})).
\]
\[ +d(F(y_n, x_n), F(g(x_n, x_n), g(y_n, y_n))) \leq d(g(x_n), g(x_{n+1}))+d(g(y_n), g(y_{n+1})) + 2\varphi\left(\frac{d(g(x_n), g(y_n)) + d(g(y_n), g(y_{n+1}))}{2}\right). \]

Letting now \( n \to \infty \) in the above inequality and taking into account that, by property \((i_\varphi)\), \( \lim_{r \to 0^+} \varphi(r) = 0 \), we obtain
\[ d(x, F(x, y)) + d(y, F(y, x)) = 0 \]
which implies that \( d(g(x), F(x, y)) = 0 \) and \( d(g(y), F(y, x)) = 0 \).

**Remark 1.** Theorem 3 is more general than Theorem 2, since the contractive condition (2.1) is weaker than (1.2), a fact which is clearly illustrated by the next example.

**Example 1.** Let \( X = \mathbb{R} \) with \( d(x, y) = |x - y| \) and natural ordering and let \( g : X \to X, F : X \times X \to X \) be given by \( g(x) = \frac{5}{6} x, x \in X \) and \( F(x, y) = \frac{x - 2y}{4}, (x, y) \in X^2 \).

Then \( F \) is \( g \)-mixed monotone, \( F \) and \( g \) commute and satisfy condition (2.1) but \( F \) and \( g \) do not satisfy condition (1.2). Indeed, assume, to the contrary, that there exists \( \varphi \in \Phi \), such that (1.2) holds. This means
\[ \frac{|x - 2y|}{4} - \frac{u - 2v}{4} \leq \varphi\left(\frac{5}{6} \cdot \frac{|x - u| + |y - v|}{2}\right), x \geq u, y \leq v, \]
by which, for \( x = u, y < v \) and in view of \((i_\varphi)\) we get
\[ \frac{1}{2} |y - v| \leq \varphi\left(\frac{5}{12} |y - v|\right) < \frac{5}{12} |y - v| < \frac{1}{2} |y - v|, \]
a contradiction. Hence \( F \) and \( g \) do not satisfy condition (1.2).

Now we prove that (2.1) holds. Indeed, we have
\[ \frac{|x - 2y|}{4} - \frac{u - 2v}{4} \leq \frac{1}{4} |x - u| + \frac{1}{2} |y - v|, x \geq u, y \leq v, \]
and
\[ \frac{|y - 2x|}{5} - \frac{v - 2u}{4} \leq \frac{1}{4} |y - v| + \frac{1}{2} |x - u|, x \geq u, y \leq v, \]
and by summing up the two inequalities above we get exactly (2.1) with \( \varphi(t) = \frac{1}{4} t \). Note also that \( x_0 = -3, y_0 = 3 \) satisfy (2.2).

So by Theorem 3 we obtain that \( F \) has a (unique) coupled fixed point \((0, 0)\), but Theorem 2 cannot be applied as \( F \) and \( g \) do not satisfy condition (1.2).

The following corollary generalizes Theorem 2.1 in [5] from coupled fixed points to coincidence points.
Corollary 1. Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \(F : X \times X \to X\) be a mixed \(g\)-monotone mapping for which there exist \(k \in [0, 1)\) such that for all \(x, y, u, v \in X\) with \(g(x) \geq g(u), g(y) \leq g(v),\)
\[
    d(F(x, y), F(u, v)) +
    + d(F(y, x), F(v, u)) \leq k[d(g(x), g(u)) + d(g(y), g(v))].
\]  
(2.19)
Suppose \(F(X \times X) \subset g(X)\), \(g\) is continuous and commutes with \(F\) and also suppose either
(a) \(F\) is continuous or
(b) \(X\) satisfy Assumption [2.1]
If there exist \(x_0, y_0 \in X\) such that
\[
    g(x_0) \leq F(x_0, y_0) \land g(y_0) \leq F(y_0, x_0),
\]  
(2.20)
then there exist \(\overline{x}, \overline{y} \in X\) such that
\[
    g(\overline{x}) = F(\overline{x}, \overline{y}) \land g(\overline{y}) = F(\overline{y}, \overline{x}),
\]
that is, \(F\) and \(g\) have a coupled coincidence.

Proof. Take \(\varphi(t) = kt, 0 \leq k < 1\) in Theorem [3] \qed

Remark 2. Let us note that, as suggested by Example [1] since the contractive condition [2.1] is valid only for comparable elements in \(X^2\), Theorem [3] cannot guarantee in general the uniqueness of the coincidence point.

It is now our interest to identify additional conditions, like the ones used in Theorem 2.2 of Bhaskar and Lakshmikantham [5] or in Theorem 2.2 of Lakshmikantham and Cirić [3], to ensure that the coincidence fixed point guaranteed by Theorem [3] is unique. Such a condition is the one involved in the next theorem.

Theorem 4. In addition to the hypotheses of Theorem [3] suppose that for every \((\overline{x}, \overline{y})\), \((y^*, x^*) \in X \times X\) there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x^*, y^*), F(y^*, x^*))\) and to \((F(\overline{x}, \overline{y}), F(\overline{y}, \overline{x})).\) Then \(F\) and \(g\) have a unique coupled common fixed point, that is, there exists a unique \((\overline{z}, \overline{w}) \in X^2\) such that
\[
    \overline{z} = g(\overline{z}) = F(\overline{z}, \overline{w}) \land \overline{w} = g(\overline{w}) = F(\overline{w}, \overline{z}).
\]

Proof. From Theorem [3] the set of coupled coincidences of \(F\) and \(g\) is nonempty. Assume that \(Z^* = (x^*, y^*) \in X^2\) and \(\overline{Z} = (\overline{x}, \overline{y})\) are two coincidence points of \(F\) and \(g\). We shall prove that \(g(x^*) = g(\overline{x})\) and \(g(y^*) = g(\overline{y}).\)

By hypothesis, there exists \((u, v) \in X^2\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x^*, y^*), F(y^*, x^*))\) and to \((F(\overline{x}, \overline{y}), F(\overline{y}, \overline{x})).\) Put \(u_0 = u, v_0 = v\) and choose \(u_1, v_1 \in X\) so that \(g(u_1) = F(u_0, v_0), g(v_1) = g(v_0) = \ldots = g(v_{n-1}) = F(v_{n-1}, u_{n-1}),\) \(g(u_n) = g(u_{n-1}) = \ldots = g(u_0) = x^*, g(v_n) = g(v_{n-1}) = \ldots = g(v_0) = y^*\).
Let $F(v_0, u_0)$. Then, similarly to the proof of Theorem 3 we obtain the sequences \(\{g(u_n)\}, \{g(v_n)\}\) defined as follows:

\[
g(u_{n+1}) = F(u_n, v_n), \quad g(v_{n+1}) = F(v_n, u_n), \quad n \geq 0.
\]

Now construct in the same manner the sequences \(\{g(x_n)\}, \{g(y_n)\}\), \(\{g(x^*_n)\}, \{g(y^*_n)\}\), by setting \(x_0 = \overline{x}\), \(y_0 = \overline{y}\), \(x^*_0 = \overline{x}\) and \(y^*_0 = \overline{y}\), respectively. This means that, for all \(n \geq 0\),

\[
g(x_n) = F(\overline{x}, \overline{y}), \quad g(y_n) = F(\overline{y}, \overline{x}); \quad g(x^*_n) = F(x^*_0, y^*_0), \quad g(y^*_n) = F(y^*_0, x^*_0).
\]

Since \((F(\overline{x}, \overline{y}), F(\overline{y}, \overline{x})) = (g(x_1), g(y_1)) = (g(\overline{x}), g(\overline{y}))\) and \((F(u, v), F(v, u)) = (g(u_1), g(v_1))\) are comparable, it follows that \(g(\overline{x}) \leq g(u_n)\) and \(g(\overline{y}) \geq g(v_n)\).

Further, we easily show that \(g((\overline{x}), g(\overline{y}))\) and \((g(u_n), g(v_n))\) are comparable, that is, \(g(\overline{x}) \leq g(u_n)\) and \(g(\overline{y}) \geq g(v_n)\), for all \(n \geq 1\).

Thus, by the contractive condition (2.1), one get

\[
\frac{d(g(\overline{x}), g(u_{n+1})) + d(g(\overline{y}), g(v_{n+1}))}{2} = \frac{d(F(\overline{x}, \overline{y}), F(u_n, v_n)) + d(F(\overline{y}, \overline{x}), F(v_n, u_n))}{2} \leq \varphi \left(\frac{d(g(\overline{x}), g(u_n)) + d(g(\overline{y}), g(v_n))}{2}\right).
\]

Thus, by (2.21), we deduce that the sequence \(\{\Delta_n\}\) defined by

\[
\Delta_n = \frac{d(g(\overline{x}), g(u_n)) + d(g(\overline{y}), g(v_n))}{2}, \quad n \geq 0,
\]

satisfies

\[
\Delta_{n+1} \leq \varphi(\Delta_n), \quad n \geq 0. \quad (2.22)
\]

Now use \(\varphi\) to deduce by (2.22) that \(\{\Delta_n\}\) is non-decreasing, hence convergent to some \(\delta \geq 0\).

We shall prove that \(\delta = 0\). Assume, to the contrary, that \(\delta > 0\). Then, we can find a \(n_0\) such that \(\Delta_n \geq \delta > 0\), for all \(n \geq n_0\). By letting \(n \to \infty\) in (2.22) we obtain, in view of (ii \(\varphi(\cdot)\)), that

\[
\delta \leq \lim_{n \to \infty} \varphi(\Delta_n) = \lim_{\Delta_n \to \delta^+} \varphi(\Delta_n) < \delta,
\]

a contradiction. Therefore \(d(g(\overline{x}), g(u_{n+1})) + d(g(\overline{y}), g(v_{n+1})) \to 0\) as \(n \to \infty\), that is,

\[
\lim_{n \to \infty} d(g(\overline{x}), g(u_{n+1})) = 0, \quad \text{and} \quad \lim_{n \to \infty} d(g(\overline{y}), g(v_{n+1})) = 0. \quad (2.23)
\]

Similarly, we obtain that

\[
\lim_{n \to \infty} d(g(x^*_n), g(u_{n+1})) = 0, \quad \text{and} \quad \lim_{n \to \infty} d(g(y^*_n), g(v_{n+1})) = 0. \quad (2.24)
\]

By (2.23) and (2.24) and the triangle inequality, we have

\[
\frac{d(g(\overline{x}), g(x^*_n)) + d(g(u_{n+1}), g(x^*_n))}{2} \leq d(g(\overline{x}), g(u_{n+1})) \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
\frac{d(g(\overline{y}), g(y^*_n)) + d(g(v_{n+1}), g(y^*_n))}{2} \leq d(g(\overline{y}), g(v_{n+1})) \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence
\[ g(\overline{x}) = g(x^*) \text{ and } g(\overline{y}) = g(y^*), \quad (2.25) \]
that is, \( F \) and \( g \) have a unique coupled coincidence. Now we shall prove that actually \( F \) and \( g \) have a unique coupled common fixed point. Since
\[ g(\overline{x}) = F(\overline{x}, \overline{y}) \text{ and } g(\overline{y}) = F(\overline{y}, \overline{x}), \]
and \( F \) and \( g \) commutes, we have
\[ g(g(\overline{x})) = g(F(\overline{x}, \overline{y})) = F(g(\overline{x}), g(\overline{y})), \quad (2.26) \]
and
\[ g(g(\overline{y})) = g(F(\overline{y}, \overline{x})) = F(g(\overline{y}), g(\overline{x})). \quad (2.27) \]
Denote \( g(\overline{x}) = \overline{z} \) and \( g(\overline{y}) = \overline{w} \). Then, by (2.26) and (2.27) one gets
\[ g(g(\overline{z})) = g(F(\overline{z}, \overline{w})) \text{ and } g(g(\overline{w})) = g(F(\overline{w}, \overline{z})). \quad (2.28) \]
Now from (2.25) and (2.28) we get
\[ \overline{z} = g(\overline{z}) = F(\overline{z}, \overline{w}) \text{ and } \overline{w} = g(\overline{w}) = F(\overline{w}, \overline{z}). \]
Therefore \((\overline{z}, \overline{w})\) is a coupled common fixed point of \( F \) and \( g \).

To prove the uniqueness, assume \((p, q)\) is another coupled common fixed point of \( F \) and \( g \). Then by (2.28) we have
\[ p = g(p) = g(\overline{z}) = \overline{z} \text{ and } q = g(q) = g(\overline{w}) = \overline{w}. \]
\[ \square \]

**Corollary 2.** In addition to the hypotheses of Corollary 1, suppose that for every \((\overline{x}, \overline{y}), (y^*, x^*) \in X \times X\) there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x^*, y^*), F(y^*, x^*))\) and to \((F(\overline{x}, \overline{y}), F(\overline{y}, \overline{x}))\). Then \( F \) and \( g \) have a unique coupled common fixed point.

**Corollary 3.** Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) on \(X\) such that \((X, d)\) is a complete metric space. Let \( F : X \times X \to X \) be a mixed monotone mapping for which there exist \( k \in [0, 1) \) such that for all \( x, y, u, v \in X \) with \( x \geq u, y \leq v \),
\[ d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k [d(x, u) + d(y, v)]. \quad (2.29) \]
Suppose either
(a) \( F \) is continuous or
(b) \( X \) satisfy Assumption [1.1].
If there exist \( x_0, y_0 \in X \) such that (2.2) is satisfied, then \( F \) has a coupled fixed point.

Proof. Take \( g(x) = x \) and \( \varphi(t) = kt, 0 \leq k < 1 \) in Theorem 8. \[ \square \]
Remark 3. Corollary 3 is a generalization of Theorem 1 (Theorem 2.1 in [5]). Note also that in [5] and [6] the authors use only condition (2.2), although the alternative assumption (2.3) is also applicable.

Remark 4. As a final conclusion, we note that our results in this paper improve all coincidence point theorems and coupled fixed point theorems in [6] and [5], and also many other related results: [3]-[7], for coupled fixed point results and [1], [8]-[10], for fixed point results, by considering the more general (symmetric) contractive condition (2.1). By replacing the commutativity of $F$ and $g$ by the more general property "$F$ and $g$ are compatible", we can also extend the results in [4]. This will be done in a forthcoming paper.

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