A classification of weakly acyclic games

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Published online: 2 May 2014
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Abstract  Weakly acyclic games form a natural generalization of the class of games that have the finite improvement property (FIP). In such games one stipulates that from any initial joint strategy some finite improvement path exists. We classify weakly acyclic games using the concept of a scheduler introduced in Simon and Apt (Choosing products in social networks, 2012). We also show that finite games that can be solved by the iterated elimination of never best response strategies are weakly acyclic. Finally, we explain how the schedulers allow us to improve the bounds on finding a Nash equilibrium in a weakly acyclic game.

Keywords  Game theory · Weakly acyclic games · Scheduler · Classification

1 Introduction

1.1 Background

Given a strategic game, when we allow the players to improve their choices on a unilateral basis, we are naturally brought to the concept of an improvement path, in

A preliminary version of this paper appeared as Apt and Simon (2012).

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which at each stage a single player who did not select a best response is allowed to select a better strategy. By definition, every finite improvement path terminates in a Nash equilibrium. This suggests the finite improvement property (FIP), introduced in Monderer and Shapley (1996), according to which every improvement path is finite. It is obviously a desired property of a game that in particular is satisfied by the congestion games, as explained in Monderer and Shapley (1996).

However, the FIP is a very strong property and many natural games do not satisfy it. In particular, Milchtaich (1996) studied the congestion games in which the payoff functions are players specific. These games do not have the FIP. Milchtaich proved that such games belong to a larger class of games (essentially introduced in Young (1993)), called weakly acyclic games, in which one only stipulates that from any initial joint strategy, some finite improvement path exists.

Weakly acyclic games have a natural appeal because the concept of an improvement path captures the idea of a possible ‘interaction’ resulting from players’ preference for better strategies, and hence provides a link with distributed computing. In particular, Engelberg and Schapira (2011) introduced a natural class of weakly acyclic games, which model the routing aspects on the Internet. In turn, Marden et al. (2007) showed that for weakly acyclic games, a modification of the traditional no-regret algorithm yields almost sure convergence to a pure Nash equilibrium. Further, Fabrikant et al. (2010) proved that the existence of a unique (pure) Nash equilibrium in every subgame implies that the game is weakly acyclic.

1.2 Our work

If we view a strategic game as a distributed system in which the players attempt to find a Nash equilibrium by means of a ‘better response (respectively, ‘best response) dynamics’, then the property of being weakly acyclic only guarantees that finding a Nash equilibrium is always possible. However, such an existence guarantee does not help the players to find it. By adding to the game a scheduler, a concept introduced in Simon and Apt (2012), we ensure that the players always reach a Nash equilibrium, by repeatedly interacting with it. A scheduler is simply a function that given a finite sequence of joint strategies selects a player who can improve his payoff in the last joint strategy. Each player interacts with the scheduler by submitting it to a strategy he selected. Subsequently, the scheduler again selects a player who did not select a best response. This interaction process leaves open, how each player selects his better (respectively, best) strategy.

In the presence of a scheduler for a strategic game $G$, we can view the resulting interaction as a ‘supergame’ between the central authority represented by the scheduler and the players of $G$. The aim of the central authority is to reach a Nash equilibrium in spite of a limited guarantee on the behaviour of the players: all it can be sure of is that each selected player will select a better response (respectively a best response). The resulting interaction results in an improvement path (respectively a best response improvement path). If all so generated improvement paths are finite, we say that the game respects the scheduler.

By providing a classification of the schedulers, we obtain a natural classification of weakly acyclic games. An advantage of such a classification is that given a weakly
acyclic game, we can determine under what adverse circumstances a Nash equilibrium still can be reached. Consequently some existing results can be improved. In particular, we show in Sect. 9 how we can strengthen our result from Simon and Apt (2012) concerning a class of social network games. In turn, Brokkelkamp and Vries (2012) strengthened the above-mentioned theorem of Milchtaich (1996) by showing that congestion games with player-specific payoff functions respect every local best response scheduler, defined below in Sect. 3. Further, as we explain in Sect. 10, the existence of a scheduler allows us to improve bounds on finding a Nash equilibrium.

In turn, a recent paper Kawald and Lenzner (2013) demonstrates the applicability of schedulers in order to improve bounds on finding a Nash equilibrium in certain classes of network creation games proposed in Luthra et al. (2003). The authors show that while an arbitrary improvement path for the class of so-called MAX swap games on trees might take \( O(n^3) \) steps to converge to an Nash equilibrium, a natural scheduler reduces the complexity to \( O(n \log n) \), and moreover, this bound is optimal.

In what follows, we introduce eight natural categories of schedulers. They yield a classification of finite weakly acyclic games that consists of nine classes games that for two player games collapse into five classes. Then, we study finite games that can be solved by the iterated elimination of never best response strategies (IENBR) and show that congestion games with player-specific payoff functions respect every local best response scheduler, defined below in Sect.3. Further, as we explain in Sect. 10, the existence of a scheduler allows us to improve bounds on finding a Nash equilibrium.

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In what follows, we introduce eight natural categories of schedulers. They yield a classification of finite weakly acyclic games that consists of nine classes games that for two player games collapse into five classes. Then, we study finite games that can be solved by the iterated elimination of never best response strategies (IENBR) and show that they are weakly acyclic. We also locate where such games fit in our classification. Further, we define a modified notion of a potential using which we characterize the games that respect a scheduler. We also use this notion to reason about a natural class of coordination games. Finally, we discuss the consequences of a fact that a game respects a scheduler.

2 Preliminaries

Assume a set \( N := \{1, \ldots, n\} \) of players, where \( n > 1 \). A strategic game for \( n \) players, written as \( (S_1, \ldots, S_n, p_1, \ldots, p_n) \), consists of a non-empty set \( S_i \) of strategies and a payoff function \( p_i : S_1 \times \cdots \times S_n \to \mathbb{R} \), for each player \( i \).

Fix a strategic game \( G := (S_1, \ldots, S_n, p_1, \ldots, p_n) \). We denote \( S_1 \times \cdots \times S_n \) by \( S \), call each element \( s \in S \) a joint strategy, denote the \( i \)th element of \( s \) by \( s_i \), and abbreviate the sequence \((s_j)_{j \neq i} \) to \( s_{-i} \). Occasionally, we write \((s_i, s_{-i})\) instead of \( s \). Finally, we abbreviate \( \times_{j \neq i} S_j \) to \( S_{-i} \).

We call a strategy \( s_i \) of player \( i \) a best response to a joint strategy \( s_{-i} \) of his opponents if \( \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}) \). If \( s_i \) is (not) a best response to \( s_{-i} \), we say that player \( i \) selected (did not select) a best response in \( s \). Next, we call a joint strategy \( s \) a (pure) Nash equilibrium if each \( s_i \) is a best response to \( s_{-i} \), that is, if \( \forall i \in N \ \forall s'_i \in S_i \ p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}) \). We also define

\[
BR(s) := \{ i \mid \text{player } i \text{ selected a best response in } s \},
\]

\[
NBR(s) := \{ i \mid \text{player } i \text{ did not select a best response in } s \}.
\]

Further, we call a strategy \( s'_i \) of player \( i \) a better response given a joint strategy \( s \) if \( p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}) \). Following Monderer and Shapley (1996) a path in \( S \) is a sequence \((s^1, s^2, \ldots)\) of joint strategies such that for every \( k > 1 \) there is a player \( i \)
such that $s^k = (s'_i, s_{i-1}^{k-1})$ for some $s'_i \neq s_{i-1}^{k-1}$. A path is called an improvement path (respectively, a best response improvement path, in short a BR-improvement path) if it is maximal and for all $k > 1$, $p_i(s^k) > p_i(s_{i-1}^{k-1})$ (respectively, $s_i^k$ is a best response to $s_{i-1}^{k-1}$), where $i$ is the player who deviated from $s_{i-1}^{k-1}$. So in an improvement path each deviating player selects a better response, while in a BR-improvement path each deviating player selects a best response. Given a finite prefix $\rho$ of a path in $S$, we denote by $\rho^*$ the infinite path generated by the repeated concatenation of $\rho$.

The better response graph (respectively, the best response graph) associated with the game $G$ is defined as $(S, \rightarrow)$, where $s \rightarrow s'$ if $(s, s')$ is an improvement path (respectively, in a BR-improvement path).

Given joint strategies $s, s' \in S$ and a player $i$ we define

$$s \overset{i}{\rightarrow} s' \iff s_{-i} = s'_{-i} \text{ and } p_i(s') > p_i(s),$$

$$s \overset{i}{\Rightarrow} s' \iff s \overset{i}{\rightarrow} s' \text{ and } s_i' \text{ is a best response to } s_{-i}' .$$

Recall that $G$ has the finite improvement property (FIP), (respectively, the finite best response property (FBRP)) if every improvement path (respectively, every BR-improvement path) is finite. Obviously, if a game has the FIP or the FBRP, then it has a Nash equilibrium—it is the last element of each path. Following Young (1993); Milchtaich (1996), we call a strategic game weakly acyclic (respectively, BR-weakly acyclic) if for any joint strategy there exists a finite improvement path (respectively, BR-improvement path) that starts at it.

In Sect. 6 we shall combine two $n$ players games $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G' := (S'_1, \ldots, S'_n, p'_1, \ldots, p'_n)$ such that $S_1 \cap S'_1 = \emptyset$ by means of the following operation

$$G \uplus G' := (S_1 \cup S'_1, \ldots, S_n \cup S'_n, r_1, \ldots, r_n),$$

where

$$r_i(s) := \begin{cases} 
  p_i(s) & \text{if } s \in S_1 \times \cdots \times S_n \\
  p'_i(s) & \text{if } s \in S'_1 \times \cdots \times S'_n \\
  0 & \text{otherwise} 
\end{cases}$$

The $\uplus$ operation is obviously associative, so it is justified to use $\uplus$ as an $k$-ary operator on $n$ players games.

### 3 Schedulers

In what follows, we introduce some classes of weakly acyclic games. They are defined in terms of schedulers. By a scheduler, we mean a function $f$ that given finite sequence $s^1, \ldots, s^k$ of joint strategies that does not end in a Nash equilibrium selects a player who did not select in $s^k$ a best response. In practice schedulers will be applied only to initial prefixes of improvement paths.
Consider an improvement path \( \rho = (s^1, s^2, \ldots) \). We say that \( \rho \) respects a scheduler \( f \) if for all \( k \) smaller than the length of \( \rho \) we have \( s^{k+1} = (s'_i, s^k_i) \), where \( f(s^1, \ldots, s^k) = i \). We say that a strategic game respects a scheduler \( f \) if all improvement paths \( \rho \) which respect \( f \) are finite. Further, we say that a strategic game respects a BR-scheduler \( f \) if all BR-improvement paths \( \rho \) which respect \( f \) are finite.

In what follows, we study various types of schedulers. We say that a scheduler \( f \) is state-based if for some function \( g : S \to \mathbb{N} \) we have \( f(s^1, \ldots, s^k) = g(s^k) \).

We say that a function \( g : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \) is a choice function if for all \( A \neq \emptyset \) we have \( g(A) \in A \). Next, we say that a scheduler \( f \) is set-based if for some choice function \( g : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \)

\[
 f(s^1, \ldots, s^k) = g(NBR(s^k)).
\]

Finally, we say that a set-based scheduler \( f \) is local if the above choice function \( g \) satisfies the following property:

\[
 \text{if } g(A) \in B \subseteq A \text{ then } g(A) = g(B). \tag{1}
\]

A simple way of producing choice functions \( g : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \) that satisfy (1) is the following. Take a permutation \( \pi \) of \( 1, \ldots, n \) and define for \( A \neq \emptyset \)

\[
 [\pi](A) := \pi(k),
\]

where \( k \) is the smallest element of \( N \) such that \( \pi(k) \in A \). That is, \([\pi](A)\) is the first element on the list \( \pi(1), \ldots, \pi(n) \) that belongs to \( A \).

To simplify notation, we often view a set-based scheduler as \( f : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \) by equating it to the corresponding choice function. In Sect. 9, we shall need the following characterization result.

**Proposition 1** A choice function \( g : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \) satisfies (1) iff it is of the form \([\pi]\) for some permutation \( \pi \) of \( 1, \ldots, n \).

**Proof** Suppose a choice function \( g : \mathcal{P}(\mathbb{N}) \to \mathbb{N} \) satisfies (1). Define a permutation \( \pi \) of \( 1, \ldots, n \) inductively as follows:

\[
 \pi(1) := g(N), \quad \pi(2) := g(N \setminus \{\pi(1)\}), \ldots, \pi(n) := g(N \setminus \{\pi(1), \ldots, \pi(n-1)\}).
\]

Take now a non-empty subset \( A \) of \( N \). Let \( \pi(k) = [\pi](A) \). By definition \( \{\pi(1), \ldots, \pi(k-1)\} \cap A = \emptyset \) and \( \pi(k) \in A \). Let \( B := N \setminus \{\pi(1), \ldots, \pi(k-1)\} \). By definition \( g(B) = \pi(k) \). Further, \( A \subseteq B \) and \( \pi(k) \in A \), so by property (1) we have \( g(A) = g(B) = [\pi](A) \).

Next, it is straightforward to check that each function \([\pi]\) satisfies (1). \( \square \)
The games that respect schedulers satisfy obvious implications that we summarize in Fig. 1. Here FIP (respectively, FBRP) stands for the class of games that have the FIP (respectively, FBRP), WA (respectively, BRWA) for the class of weakly acyclic games (respectively, BR-weakly acyclic games), Sched (respectively, SchedBR) stands for the class of games that respect a scheduler (respectively, a BR-scheduler), etc. In what follows, we clarify which implications can be reversed.

4 Schedulers versus state-based schedulers

We prove here three implications which show that the classes of games SchedBR, Sched, State, and StateBR coincide.

Theorem 1 (Sched ⇒ State) If a game respects a scheduler, then it respects a state-based scheduler.

Proof Fix a finite strategic game \( G = (S_1, \ldots, S_n, p_1, \ldots, p_n) \). Let \( Y := \bigcup_{k \in \mathbb{N}} Y_k \), where

\[
Y_0 := \{ s \in S \mid s \text{ is a Nash equilibrium} \},
\]

\[
Y_{k+1} := Y_k \cup \{ s \mid \exists i \forall s' (s_i \rightarrow s' \Rightarrow s' \in Y_k) \}.
\]

For each \( s \in Y_{k+1} \setminus Y_k \), let \( f_{\text{State}}(s) := i \), where \( i \) is such that \( \forall s' (s_i \rightarrow s' \Rightarrow s' \in Y_k) \).

If \( Y = S \), then we can view \( f_{\text{State}} \) as a state-based scheduler. We now prove two claims concerning the set \( Y \) and the scheduler \( f_{\text{State}} \).

\begin{itemize}
  \item **Claim 1** If a strategic game \( G \) respects a scheduler, then \( Y = S \).
  
  **Proof** Suppose that \( G \) respects a scheduler \( f \). Assume by contradiction that \( Y \neq S \). Take \( s^0 \in S \setminus Y \). Suppose \( f(s^0) = i_1 \). By the definition of \( Y \) there exists a joint strategy \( s^1 \) such that \( s^0 \rightarrow s^1 \) and \( s^1 \in S \setminus Y \). Suppose \( f((s^0, s^1)) = i_2 \). Again, by the definition of \( Y \) there exists a joint strategy \( s^2 \) such that \( s^1 \rightarrow s^2 \) and \( s^2 \in S \setminus Y \). Iterating this argument, we construct an infinite improvement path which respects \( f \), which yields a contradiction.

  \( \square \)

  \item **Claim 2** If for a strategic game \( G \) we have \( Y = S \), then \( G \) respects \( f_{\text{State}} \).

  **Proof** We prove by induction on \( k \) that all improvements paths that start in a joint strategy from \( Y_k \) and respect \( f_{\text{State}} \) are finite.

  The claim holds vacuously for \( k = 0 \). Suppose it holds for some \( k \geq 0 \). Take some \( s \in Y_{k+1} \setminus Y_k \) and an improvement path \( \xi \) that respects \( f_{\text{State}} \) and starts in \( s \). Suppose that \( f_{\text{State}}(s) := i \). Then, for some \( s', s \rightarrow s' \) is the first step in \( \xi \). By the definition of \( f_{\text{State}} \), \( s' \in Y_k \), so by the induction hypothesis \( \xi \) is finite.

  \( \square \)
Suppose now that a game $G$ respects a scheduler. By Claim 1 $Y = S$, so $f_{State}$ is a state-based scheduler. By Claim 2, $G$ respects $f_{State}$.

The proof for arbitrary games is a minor modification of the above argument and uses a transfinite induction to construct the set of strategies $Y$ and to reason about it. We omit the details. ☐

The above proof uses a construction similar to the one used to compute the winning regions of reachability games, see, e.g. (Grädel, 2011, page 104).

**Theorem 2** \((\text{Sched}_{BR} \Rightarrow \text{State}_{BR})\) If a game respects a BR-scheduler, then it respects a state-based BR-scheduler.

**Proof** The proof is the same as that of Theorem 1 with the relation $\rightarrow$ used instead of $\Rightarrow$. ☐

**Theorem 3** \((\text{Sched}_{BR} \Rightarrow \text{Sched})\) If a finite game respects a BR-scheduler, then it respects a scheduler.

**Proof** The idea of the proof is as follows. Suppose that a game respects a BR-scheduler $f_{BR}$. We construct then a scheduler $f$ inductively by repeatedly scheduling the same player until he plays a best response, and subsequently scheduling the same player as $f_{BR}$ does.

To make it precise we need some notation. We call an initial prefix of an improvement path an **improvement sequence**. To indicate the deviating players at each step of an improvement sequence $(s_0, \ldots, s_k)$ we shall write it alternatively as

$$s_0 i_1 \rightarrow s_1 i_2 \rightarrow \ldots \rightarrow i_k s_k.$$ 

Given an improvement sequence $\xi$ we denote by $[\xi]_{BR}$ the subsequence of it obtained by deleting the joint strategies that do not result from a selection of a best response. In general $[\xi]_{BR}$ is not a improvement sequence (for example, it does not need to be a maximal sequence), but it is if every maximal subsequence of it of the form $s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_m$ ends with a selection of a best response.

Given a finite sequence of joint strategies $\xi$ we denote its last element by $\text{last}(\xi)$ and denote the extension of $\xi$ by a joint strategy $s$ by $\xi, s$. We define the desired scheduler $f$ inductively by the length of the sequences. For a sequence of length 1, so a joint strategy that is not a Nash equilibrium, we put

$$f(s) := f_{BR}(s).$$

Suppose now that we defined $f$ on all sequences of length $k$. Consider a sequence $\xi, s$ of length $k + 1$. If $\xi, s$ is not an improvement path or $\text{last}(\xi) \rightarrow s$ does not hold, then we define $f(\xi, s)$ arbitrarily. Otherwise we put

$$f(\xi, s) := \begin{cases} f_{BR}([\xi, s]_{BR}) & \text{if } s_i \text{ is a best response to } s_{-i} \\ f(\xi) & \text{otherwise} \end{cases}$$
We claim that $G$ respects the scheduler $f$. To see it take an improvement path $\xi$ that respects $f$. By the definition of $f$, $[\xi]_{BR}$ is an improvement sequence that respects $f_{BR}$. By assumption $[\xi]_{BR}$ is finite, so $\xi$ is finite, as well. 

Note that the above theorem fails to hold for infinite games. Indeed, consider a two player game $([0], [0, 1], p_1, p_2)$, where $[0, 1]$ stands for the real interval $\{r \mid 0 \leq r \leq 1\}$ and $p_1(0, s_2) = p_2(0, s_2) := s_2$. Then 1 is a unique best response of player 2 to the strategy 0, so this game respects the unique BR-scheduler. However, it does not respect the unique scheduler.

5 Two player games

For two player games more implications hold.

**Proposition 2** (Sched $\Rightarrow$ FBRP) *If a two player game respects a scheduler, then every BR-improvement path is finite.*

**Proof** Suppose that a two player game $G$ respects a scheduler. Note that the best response graph of $G$ has the property that for every node $s$ that is not a source node, the set $NBR(s)$ has at most one element. Take a BR-improvement path $\xi$. Suppose that $(s, s')$ is the first step in $\xi$ and that $\eta$ is the suffix of $\xi$ that starts at $s'$. Then, every element $s''$ of $\eta$ is such that the set $NBR(s'')$ has at most one element. Hence $\eta$ respects any scheduler and consequently is finite. So $\xi$ is finite, as well. 

We also have the following examples showing that other implications do not hold.

**Example 1** (Local $\not\Rightarrow$ FIP). Consider the following game

|    | L  | C  | R  |
|----|----|----|----|
| T  | 0, 1 | 1, 0 | 0, 1 |
| B  | 1, 0 | 0, 1 | 0, 0 |

It respects the local scheduler $f$ for which $f([1, 2]) = 2$. However, this game does not have the FIP. 

**Example 2** (State $\not\Rightarrow$ Set, FBRP $\not\Rightarrow$ Set). Consider the game

|    | A  | B  | C  |
|----|----|----|----|
| A  | 2, 2 | 2, 0 | 0, 1 |
| B  | 0, 2 | 1, 1 | 1, 0 |
| C  | 1, 0 | 0, 1 | 0, 0 |

In Fig. 2, we display the better response graph and the best response graphs associated with it.

First, we show that this game respects a state-based scheduler. To define it suffices to consider the joint strategies in which none of the players selected a best response. These are $(A, C)$, $(C, A)$, $(B, B)$ and $(C, C)$. We put

$f(A, C) := 2, \quad f(C, A) := 1, \quad f(B, B) := 1, \quad f(C, C) := 1.$
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Using Fig. 2(a) it is easy to check that any improvement path that respects this scheduler ends in \((A, A)\). Further, the graph given in Fig. 2(b) is acyclic, that is, this game has the FBRP.

However, this game does not respect any set-based scheduler. Indeed, suppose otherwise. Then, such a scheduler is defined using a choice function \(g\). If \(g([1, 2]) = 1\), then the infinite improvement path \(((B, B), (A, B), (A, C), (B, C))^\ast\) respects this scheduler. In turn, if \(g([1, 2]) = 2\), then the infinite improvement path \(((B, B), (B, A), (C, A), (C, B))^\ast\) respects this scheduler.

\(\square\)

**Example 3** (BRWA \(\not\Rightarrow\) Sched\(_{BR}\)). Consider the following game

\[
\begin{array}{ccc}
L & C & R \\
T & 0, 1 & 1, 0 & 0, 1 \\
B & 1, 0 & 0, 1 & 0, 0
\end{array}
\]

It is BR-weakly acyclic. However, this game respects no BR-scheduler. Indeed, there is only one scheduler for this game and the following infinite improvement path respects it

\(((T, L), (B, L), (B, C), (T, C))^\ast\).

\(\square\)

**Example 4** (WA \(\not\Rightarrow\) BRWA). Consider now the following game

\[
\begin{array}{ccc}
L & C & R \\
T & 0, 2 & 1, 0 & 0, 1 \\
M & 1, 0 & 0, 1 & 0, 0 \\
B & 0, 0 & 0, 0 & 1, 0
\end{array}
\]

It is weakly acyclic. However, it is not BR-weakly acyclic, because the infinite BR-improvement path \(((T, L), (M, L), (M, C), (T, C))^\ast\) is a unique BR-improvement path starting at \((T, L)\).

\(\square\)
By definition if a two player game respects a set-based scheduler, then it respects a local scheduler. So putting together the results obtained so far, we get the implications and equivalences depicted in Fig. 3. All implications are proper.

As an illustration consider a two player congestion game with player-specific payoff functions (actually a class of games) analyzed in (Milchtaich 1996, page 115). Each player has three strategies numbered 1, 2 and 3. We omit the description of the game and mention only its relevant characteristics. The game has two Nash equilibria, (1, 2) and (2, 1), and an infinite improvement path ((3, 2), (2, 2), (2, 3), (1, 3), (1, 1), (3, 1)). Additionally \( p_1(1, 2) > p_1(1, 1), p_1(1, 2) > p_1(2, 2) \) and \( p_2(2, 1) > p_2(2, 3) \).

This implies that the graph depicted in Fig. 4 is a subgraph of the better response graph of the game. The dotted edges are implied by the fact that (1, 2) and (2, 1) are Nash equilibria, the continuous edges correspond to the infinite improvement path, while the dashed edges are implied by the mentioned properties of the payoff functions. The relation between the nodes (2, 1) and (3, 1) and the status of edges involving (3, 3) is unspecified, though the edge (2, 1) → (3, 1) is excluded since (2, 1) is a Nash equilibrium.

Note that this game respects a set-based scheduler \( g \) such that \( g([1, 2]) = 2 \). Indeed, this choice allows us to ‘exit’ the infinite improvement path both at (1, 1), (2, 2) and (2, 3). One can also check that this choice properly takes care of any legal completion of the graph depicted in Fig. 4 to a possible better response graph. For instance, addition of the edges (3, 2) → (3, 3) and (3, 3) → (2, 3) would create another infinite improvement path that would be ‘exited’ at (1, 1). We conclude that each game with the above characteristics belongs to \( Set \setminus FIP \). Figure 3 implies that each such game also belongs to \( FBRP \).

### 6 Remaining implications

To deal with the remaining implications we need games with at least three players.
Example 5 \( (\text{Local}_{\text{BR}} \not\Rightarrow \text{FBRP}) \) We consider a three player game in which every player has two strategies, 0 and 1, and such that its best response graph is as shown in Fig. 5.

To define the corresponding payoff functions we just need to interpret each directed edge \((a, b, c) \rightarrow (a', b', c')\) as the statement \(p_2(a, b, c) = 0\) and \(p_2(a', b', c') = 1\), each dotted edge \((a, b, c) \leftrightarrow (a', b', c')\) as the statement \(p_2(a, b, c) = 0\) and \(p_2(a', b', c') = 0\), and similarly for the other edges. This yields

\[
p_1(0, 1, 1) = p_1(1, 0, 1) = p_2(0, 0, 1) = p_2(1, 1, 1) = p_3(1, 0, 0) := 1,
\]

with the remaining payoffs equal to 0.

This game respects any local BR-scheduler \( f \) for which \( f(\{2, 3\}) = 3 \). However, this game does not have the FBRP.

Example 6 \( (\text{Set} \not\Rightarrow \text{Local}, \text{Set}_{\text{BR}} \not\Rightarrow \text{Local}_{\text{BR}}) \) We first construct four three player games. Every player in each game has two strategies, 0 and 1. The better response graphs of these games are depicted in Fig. 6. To define the corresponding payoff functions we proceed as in Example 5.

Next, we make for each player his strategy sets in these four games mutually disjoint by renaming the strategies 0 and 1 in the \( i \)th game to \( 0_i \) and \( 1_i \). Denote the resulting games by \( G_1, \ldots, G_4 \). Let \( G := G_1 \cup G_2 \cup G_3 \cup G_4 \). In the better response graph of \( G \) there are precisely three loops that correspond to the ones depicted in Fig. 6. Using this figure one can check that \( G \) respects the following set-based scheduler:

\[
f(\{2, 3\}) := 3, \quad f(\{1, 3\}) := 1, \quad f(\{1, 2\}) := 2,
\]

with an arbitrary value for the input \( \{1, 2, 3\} \).

However, \( G \) does not respect any local scheduler, since to ‘exit’ each of these three loops each set-based scheduler needs to make the above selections, and then no selection for the input \( \{1, 2, 3\} \) can make the scheduler local.

The above reasoning also holds for the BR-schedulers, since every better response in \( G \) is also a best response, as all payoffs in \( G \) are either 0 or 1.
Example 7 \((\text{State}_{BR} \not\Rightarrow \text{Set}_{BR})\) We first construct two three player games. Every player in each game has two strategies, 0 and 1. The best response graphs of these games are depicted in Fig. 7. The first one coincides with the first better response graph of Fig. 6.

Denote these games by \(G_1\) and \(G_2\) and let \(G := G_1 \uplus G_2\). Then \(G\) respects the following state-based BR-scheduler (we use here the same renaming of the strategies as in Example 6):

\[
f(1_1, 0_1, 1_1) := 3, \quad f(1_2, 0_2, 1_2) := 2.
\]

However, \(G\) does not respect any set-based BR-scheduler, since in each of the above two joint strategies the set of players who did not select a best response is the same (namely \(\{2, 3\}\)), and to ‘exit’ each of the two loops in the corresponding best response graphs one needs to take the above choices. \(\Box\)

This brings us to the following final diagram depicted in Fig. 8. All implications are proper.
7 Games that can be solved by the iterated elimination of NBR

Weakly acyclic games are of natural interest because they have (pure) Nash equilibria. In the literature, another class of finite games has been studied with the same property, namely those that can be solved by the iterated elimination of strictly dominated strategies. In what follows, we focus on a more general class of finite games that can be solved by the iterated elimination of never best responses (IENBR), where we limit ourselves to never best responses to joint pure strategies of the opponents.

In this context, the joint pure strategies of the opponents are usually called point beliefs, and the outcome of IENBR is the set of rationalizable joint strategies, see Bernheim (1984). In other words, we focus on finite game that have a unique rationalizable joint strategy w.r.t. point beliefs.

Given a strategic game $G = (S_1, \ldots, S_n, p_1, \ldots, p_n)$ we say that a strategy $s_i \in S_i$ is a never best response in $G$ if for every $s_{-i} \in S_{-i}$ there is a strategy $s_i' \in S_i$ such that $p_i(s_i', s_{-i}) > p_i(s_i, s_{-i})$.

To better understand the concept of a never best response recall that given a finite strategic game $G = (S_1, \ldots, S_n, p_1, \ldots, p_n)$ a mixed strategy for player $i$ is a probability distribution over $S_i$. Further, recall that a strategy $s_i \in S_i$ is strictly dominated by a mixed strategy if there exists a mixed strategy $m_i$ such that for all
\[ s_{-i} \in S_{-i}, \ p_i(m_i, s_{-i}) > p_i(s_i, s_{-i}), \] where \( p_i(m_i, s_{-i}) \) is defined in the expected way.

Then, it is straightforward to check that if \( s_i \) is strictly dominated by a mixed strategy then \( s_i \) is a never best response. Also, the reverse implication does not hold.

Given a binary relation \( \to \) we denote by \( \to^* \) its transitive reflexive closure. In what follows, we consider a specific relation \( \to \) between games defined as follows:

\[ G \to G' \]

where \( G = (S_1, \ldots, S_n, p_1, \ldots, p_n), \ G' = (S'_1, \ldots, S'_n, p'_1, \ldots, p'_n) \) is a proper subgame of \( G \), and

\[ \forall i \in \{1, \ldots, n\} \ \forall s_i \in S_i \setminus S'_i \ \exists s_{-i} \in S_{-i} s_i \text{ is a best response to } s_{-i} \text{ in } S. \]

That is, \( G \to G' \) when \( G' \) results from \( G \) by removing from it some strategies that are never best responses.

If for some game \( G' \) in which each player is left with exactly one strategy, we have \( G \to^* G' \), then we say that \( G \) can be solved by the iterated elimination of never best responses, in short can be solved by IENBR.

The games that can be solved by IENBR are of interest because of the following observation.

**Proposition 3** If a finite game can be solved by IENBR then it has a unique Nash equilibrium.

**Proof** It suffices to note that each step of the elimination process maintains the set of Nash equilibria. \( \square \)

We now prove that this class of games is actually contained in the class of weakly acyclic games. More precisely, the following holds.

**Theorem 4** If a finite game can be solved by IENBR, then it is in StateBR.

**Proof** We use the scheduler \( f \) which schedules each time the player who did not play a best response the longest, breaking ties in favour of the player with the smallest index. Fix a finite game \( G \) that can be solved by IENBR. By definition of the scheduler \( f \), it satisfies the following property:

(P1) for all \( i \in N \), for every BR-improvement path \( \rho = s^0, s^1, \ldots \) in \( G \) that respects \( f \) and for all \( j \geq 0 \), there exists \( k \geq j \) such that \( i \in BR(s^k) \).

Let \( \rho = s^0, s^1, \ldots \) a BR-improvement path which respects \( f \). We argue that \( \rho \) is finite.

We find it useful to introduce the following notion: a finite sequence of strategy profiles \( b = s^0, \ldots, s^k \) is called a block if for all \( i \in NBR(s^0) \), there exists \( j \in \{0, \ldots, k\} \) such that either \( i \in BR(s^j) \) or \( i \in f(s^0, \ldots, s^j) \). The sequence \( b \) is said to be minimal if there is no prefix \( b' \) of \( b \) such that \( b' \) is a block. Since \( f \) satisfies (P1), we can represent the best response improvement path \( \rho \) as a concatenation of minimal blocks \( b^0, b^1, \ldots \).
Since \( G \) can be solved by IENBR, there is a finite reduction sequence \( \xi = G^0 \rightarrow G^1 \rightarrow \ldots \rightarrow G^m \) for \( G \) such that each player is left with a single strategy in \( G^m \). By Theorem 4.2 in Apt (2005) we can assume without loss of generality that each reduction step eliminates a single strategy. Let \((i_1, s_{i_1}), (i_2, s_{i_2}), \ldots, (i_m, s_{i_m})\) be the corresponding sequence of pairs of player identity and of the strategy which are eliminated in \( \xi \). Consider the first pair \((i_1, s_{i_1})\) and the block \( b^0 = s^0, \ldots, s^l \). We have two cases:

- \( s^0_{i_1} = s_{i_1} \).

Then \( i_1 \in NBR(s^0) \). Indeed, since \( s_{i_1} \) is a never best response, \( i_1 \in NBR(s) \) for any \( s \in S \). Since \( f \) satisfies (P1) and \( s_{i_1} \) is a never best response there exists \( j \in \{0, \ldots, l\} \) such that \( i_1 \in f(s^0, \ldots, s^j) \). Therefore \( s^j_{i_1} \neq s_{i_1} \). Again, since \( s_{i_1} \) is a never best response player \( i \) never switches back to \( s_{i_1} \) in the suffix of \( \rho \) starting in \( s^j \) irrespective of the strategies chosen by the other players.

- \( s^0_{i_1} \neq s_{i_1} \).

Then since \( s_{i_1} \) is a never best response, player \( i_1 \) never switches to \( s_{i_1} \) in the improvement path \( \rho \), irrespective of the strategies chosen by the other players.

Therefore, in the suffix \( \rho^1 = b^1, b^2, b^3, \ldots \) of the improvement path \( \rho \) the strategy \( s_{i_1} \) is never chosen. This means that \( \rho^1 \) is a best response improvement path in the restricted game \( G^1 \). Now consider the pair \((i_2, s_{i_2})\) and the block \( b^1 \). By the same argument as above, we can show that in the suffix \( \rho^2 = b^2, b^3, \ldots \) of the improvement path \( \rho \), the strategy \( s_{i_2} \) is never chosen, and therefore \( \rho^2 \) is a best response improvement path in the restricted game \( G^2 \). Since \( G \) can be solved by the elimination sequence \( \xi = G^0, G^1, \ldots, G^m \), by iterating the above argument we get that there is a suffix of the improvement path \( \rho \) which involves only joint strategies in \( G^m \). Since in \( G^m \) each player is left with a single strategy, \( \rho \) is finite.

We conclude that the game \( G \) respects the BR-scheduler \( f \). Consequently, by Theorem 2, \( G \) is in \( \text{State}_{BR} \). \( \square \)

In particular, in view of the remarks at the beginning of this section, if a game can be solved by iterated elimination of strictly dominated strategies (by a mixed strategy), in short, IESDS, then it is weakly acyclic. An example of such a game is the beauty contest game due to Moulin (1986). In this game there are \( n > 2 \) players, each with the set of strategies equal \( \{1, \ldots, 100\} \). Each player submits a number, and the payoff to each player is obtained by splitting 1 equally between the players whose submitted number is closest to \( \frac{2}{3} \) of the average. For example, if the submissions are 29, 32, 29, then the payoffs are respectively \( \frac{1}{7} \), 0, \( \frac{1}{2} \). By Theorem 4 this game is weakly acyclic.

Note that the implication in the above theorem cannot be reversed. Indeed, the game given in Example 7 is in \( \text{State}_{BR} \), and it is easy to check that it cannot be solved by IENBR, as every strategy in this game is a best response. An even simpler example is the well-known Battle of Sexes game.

We now identify the smallest class used in our classification of finite weakly acyclic games that includes all finite games that can be solved by IENBR. We begin with two player games for which we have the classification presented in Fig. 3 of Sect. 5. In view of this, it suffices to note the following example.
Example 8 (IENBR $\not\Rightarrow$ Set). Consider the game:

|     | A  | B  | C  |
|-----|----|----|----|
| A   | 3,2| 1,1| 2,0|
| B   | 1,2| 3,1| 1,2|
| C   | 2,2| 2,3| 1,1|

and its better response graph displayed in Fig. 9.

It is easy to see that this game can be solved by IENBR and that the outcome is $(A, A)$. However, this game does not respect any set-based scheduler. Indeed, suppose otherwise. Then, such a scheduler is defined using a choice function $g$. If $g([1, 2]) = 1$, then the infinite improvement path $((A, B), (B, B), (B, C), (C, C))^*$ respects this scheduler. In turn, if $g([1, 2]) = 2$, then the infinite improvement path $((C, A), (C, B), (B, B), (B, A))^*$ respects this scheduler.

The case of arbitrary games is more complicated. First, we show how given two $n$-player games $G_1 := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G_2 := (S'_1, \ldots, S'_n, r_1, \ldots, r_n)$, where $S_1 \cap S'_1 = \emptyset$, that have strictly positive payoffs and both can be solved by IENBR, we can construct using an auxiliary game $G_3$ a game $G := G_1 \cup G_2 \cup G_3$ that also can be solved by IENBR.

Suppose that the outcomes of the IENBR applied to the games $G_1$ and $G_2$ are respectively the joint strategies $(s_1, \ldots, s_n)$ and $(s'_1, \ldots, s'_n)$. Further, assume that $a \notin S_1 \cup S'_1$. Let $G_3 := ([a], S_2 \cup S'_2, \ldots, S_n \cup S'_n, u_1, \ldots, u_n)$, where the $u_i$ payoff functions are defined in such a way that

- the NBR eliminations steps applied to $G_1$ remain valid in the context of $G$,
- the same holds for the game $G_2$,
- the resulting game $([a, s_1, s'_1], [s_2, s'_2], \ldots, [s_n, s'_n], u_1, \ldots, u_n)$ can be solved by IENBR.

The first two items can be achieved by ensuring that for each $i \in \{2, \ldots, n\}$ and each eliminated strategy $s''_i \in (S_i \cup S'_i) \setminus \{s_i, s'_i\}$ is not a best response to a joint strategy.
\[ s_{-i} \text{ in } G \text{ with } s_1 = a. \] We thus stipulate that
\[ u_i(a, s_{-1}) := 0, \quad \text{when } s_i \in (S_i \cup S'_i) \setminus \{s_i, s'_i\} \quad \text{for some } i \in \{2, \ldots, n\}. \]

To solve the game \(|\{a, s_1, s'_1\}, \{s_2, s'_2\}, \ldots, \{s_n, s'_n\}, u_1, \ldots, u_n|| by IENBR we first stipulate that the strategies \(s_1\) and \(s'_1\) are strictly dominated in it by \(a\). We thus stipulate that
\[ u_1(a, s_{-1}) := \max +1, \quad \text{when } s_{-1} \in \times_{j \neq 1} \{s_j, s'_j\}, \]
where max is the maximum payoff used in \(G_1\) or \(G_2\).

To ensure that the game \(|\{a, s_1, s'_1\}, \{s_2, s'_2\}, \ldots, \{s_n, s'_n\}, u_1, \ldots, u_n|| can be solved by IENBR we further stipulate that for \(i \in \{2, \ldots, n\}\)
\[ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}), \quad \text{where } s_{-i} \in \{a\} \times_{j \neq i} \times_{j \neq 1} \{s_j, s'_j\}. \]

Then for any payoff functions \(u_1, \ldots, u_n\) that satisfy the above conditions the outcome of the IENBR applied to above game \(G\) yields the joint strategy \((a, s_2, \ldots, s_n)\).

We now proceed with the following example.

**Example 9** (IENBR \(\not\Rightarrow\) LocalBR)] First, consider Fig. 10.

It depicts the best response graph of a three player game \(G_1\). We define the payoff functions analogously to Example 5. Note that this game can be solved by IENBR with the outcome being the joint strategy \((0, 0, 0)\).

Further, this game respects a set-based BR-scheduler \(f\) iff
\[ f(\{1, 3\}) = 3 \quad \text{or} \quad f(\{2, 3\}) = 3. \]

So it does not respect the local BR-schedulers that correspond to the permutations 1, 2, 3 and 2, 1, 3.

By renaming the players we obtain two other games, \(G_2\) and \(G_3\), that both can be solved by IENBR and such that \(G_2\) does not respect the local BR-schedulers that correspond to the permutations 1, 3, 2 and 3, 1, 2, and \(G_3\) does not respect the local BR-schedulers that correspond to the permutations 3, 2, 1 and 2, 3, 1. Using the construction explained prior to this example twice we construct then a game \(G\) that can be solved by IENBR and that does not respect any local BR-scheduler.

So this example shows that the class of finite games that can be solved by IENBR is not contained in LocalBR. We suspect that in fact this class is not contained in SetBR, however, were not able to prove it.

**8 Potentials**

To characterize the finite games that have the FIP Monderer and Shapley (1996) introduced the concept of a (generalized ordinal) potential. We now introduce an appropriately modified notion to characterize the games that respect a scheduler. We shall use it in the next section to reason about a natural class of games.
Consider a game \((S_1, \ldots, S_n, p_1, \ldots, p_n)\) and a scheduler \(f\). We say that a function \(F: S \rightarrow \mathbb{R}\) is an \(f\)-potential (respectively, an \(f\)-BR-potential) if for every initial prefix of an improvement path (respectively, an BR-improvement path) \((s^1, \ldots, s^k, s^{k+1})\) that respects \(f\) we have \(F(s^{k+1}) > F(s^k)\).

Note that when \(f\) is a state-based scheduler, then a function \(F\) is an \(f\)-potential iff for all \(i, s'_i\) and \(s\)

\[
\text{if } f(s) = i \quad \text{and} \quad p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i}), \quad \text{then } F(s'_i, s_{-i}) > F(s_i, s_{-i}),
\]

and similarly for the \(f\)-BR-potentials. In the proof below we use the following classic result of König (1927).

**Lemma 1** (König’s Lemma) Any finitely branching tree is either finite or it has an infinite path.

**Theorem 5** Consider a finite game \(G\).

(i) \(G\) respects a scheduler \(f\) iff an \(f\)-potential exists.

(ii) \(G\) respects a BR-scheduler \(f\) iff an \(f\)-BR-potential exists.

**Proof** (i) (\(\Rightarrow\)) Consider a branching tree the root of which has all joint strategies as successors, the non-root elements of which are joint strategies, and whose branches are the improvement paths that respect \(f\). Because the game is finite this tree is finitely branching. By König’s Lemma this tree is finite, so we conclude that the number of improvement paths that respect \(f\) is finite. Given a joint strategy \(s\) define \(F(s)\) to be the number of improvement sequences (in the sense of the proof of Theorem 3) that respect \(f\) and that terminate in \(s\). Clearly \(F\) is an \(f\)-potential.

(\(\Leftarrow\)) Let \(F\) be an \(f\)-potential. Suppose by contradiction that an infinite improvement path that respects \(f\) exists. Then the corresponding values of \(F\) form a strictly increasing infinite sequence. This is a contradiction, since there are only finitely many joint strategies.

The proof of (ii) is analogous. \(\square\)
The argument given in (i) follows the proof of Milchtaich (1996) of the fact that every game that has the FIP has a generalized ordinal potential. Note that when the range of the $f$-potential is finite the implications $(\Leftarrow)$ in (i) and (ii) also hold for infinite games.

9 Cyclic coordination games

In coordination games, the players need to coordinate their strategies in order to choose among multiple pure Nash equilibria. Here we consider a natural set up according to which the players are arranged in a directed simple cycle and the payoff functions can yield three values: 0 if one chooses a ‘noncommitting’ strategy, 1 if one coordinates with the neighbour and $-1$ otherwise. We call such games cyclic coordination games. They are special cases of strategic games introduced in Simon and Apt (2012) that are naturally associated with social networks built over arbitrary weighted directed graphs.

More precisely, let $G_{\text{coord}} = (S_1, \ldots, S_n, p_1, \ldots, p_n)$ be a (possibly infinite) strategic game in which there is a special strategy $t_0 \in \bigcap_{i \in N} S_i$ common to all the players. For $i \in N$, let $i \oplus 1$ and $i \ominus 1$ denote the increment and decrement operations done in cyclic order within $\{1, \ldots, n\}$. That is, for $i \in \{1, \ldots, n-1\}$, $i \oplus 1 = i+1$, $n \oplus 1 = 1$, for $i \in \{2, \ldots, n\}$, $i \ominus 1 = i-1$, and $1 \ominus 1 = n$. The payoff functions are defined as,

$$p_i(s) := \begin{cases} 
0 & \text{if } s_i = t_0, \\
1 & \text{if } s_i = s_i \ominus 1 \text{ and } s_i \neq t_0, \\
-1 & \text{otherwise}.
\end{cases}$$

Thus for each player $i$, and joint strategy $s$, the best response of $i$ is to choose $s_i \ominus 1$ if $s_i \ominus 1 \in S_i$ and $t_0$ otherwise. Therefore, $s$ is a Nash equilibrium in the game $G_{\text{coord}}$ iff it is of the form $(t, \ldots, t)$.

We first show that cyclic coordination games are weakly acyclic. More precisely, we prove the following stronger result.

**Theorem 6** (i) The game $G_{\text{coord}}$ has the FIP iff $n = 2$ or $\bigcap_{i \in N} S_i = \{t_0\}$.
(ii) In $G_{\text{coord}}$, starting from each joint strategy there exists an improvement path of length at most $n$ and a BR-improvement path of length at most $2n - 2$.

We just noted that $s$ is a Nash equilibrium in the game $G_{\text{coord}}$ iff it is of the form $(t, \ldots, t)$. So we can alternatively state item (i) as: The game $G_{\text{coord}}$ has the FIP iff $n = 2$ or it has exactly one Nash equilibrium.

**Proof** (i)$(\Rightarrow)$ As already mentioned when $n = 2$, $G_{\text{coord}}$ has the FIP. If $n > 2$, then the above example implies that $\bigcap_{i \in N} S_i = \{t_0\}$.

$(\Leftarrow)$ Suppose that $G_{\text{coord}}$ does not have the FIP. Consider an infinite improvement path $\xi$. Some player, say $i$, is selected in $\xi$ infinitely often. This means that player $i$ selects in $\xi$ some strategy $t \neq t_0$ infinitely often. Indeed, otherwise from some moment on in each joint strategy in $\xi$ his strategy would be $t_0$, which is not the case.
Each time player $i$ switches in $\xi$ to the strategy $t$, the strategy of his predecessor $i \ominus 1$ is necessarily $t$, as well. So also player $i \ominus 1$ switches in $\xi$ to $t$ infinitely often. Iterating this reasoning, we conclude that each player selects in $\xi$ the strategy $t$ infinitely often.

In particular $t \in \cap_{i \in N} S_i$.

(ii) Take a joint strategy $s$. Note that if all payoffs in $s$ are $\geq 0$, then $s$ is a Nash equilibrium. Suppose that some payoff in $s$ is $< 0$. Then repeatedly select the first player in the cyclic order whose payoff is negative and let him switch to $t_0$. After at most $n$ steps the Nash equilibrium $(t_0, \ldots , t_0)$ is reached.

For the BR-improvement path, we use the local scheduler $f$ associated with the identity permutation, i.e. we repeatedly schedule the first player in the cyclic order who did not select a best response.

Consider a joint strategy $s$ taken from a BR-improvement path. Observe that for all $k$ if $s_k \neq t_0$ and $p_k(s) \geq 0$ (so in particular if $s_k$ is a best response to $s_{-k}$), then $s_k = s_{k \ominus 1}$. So for all $i > 1$, the following property holds:

$$Z(i) : \text{if } f(s) = i \text{ and } s_{i-1} \neq t_0 \text{ then for all } j \in \{n, 1, \ldots , i-1\}, s_j = s_{i-1}.$$  

In words: if $i$ is the first player who did not select a best response and player $i - 1$ strategy is not $t_0$, then this is a strategy of every earlier player and of player $n$.

Along each BR-improvement path that respects $f$ the value of $f(s)$ strictly increases until the path terminates or at certain stage $f(s) = n$. Note that then $s_{n-1} = t_0$ since otherwise on the account of property $Z(n)$ all players’ strategies are equal, so $s$ is a Nash equilibrium and hence $f(s)$ is undefined. So the unique best response for player $n$ is $t_0$. This switch begins a new round with player 1 as the next scheduled player. Player 1 also switches to $t_0$ and from now on every consecutive player switches to $t_0$, as well. The resulting path terminates once player $n - 2$ switches to $t_0$.

Consequently the length of the generated BR-improvement path is at most $2n - 2$. \hfill $\square$

The proof of (ii) shows that each cyclic coordination game respects a specific state-based scheduler and the local BR-scheduler associated with the identity permutation. However, as the following theorem shows, a much stronger result holds.

**Theorem 7** Each coordination game $G_{\text{coord}}$ respects every local scheduler.

**Proof** For $n = 2$, it is easy to see that $G_{\text{coord}}$ has the FIP and hence the result follows. Therefore, assume that $n > 2$. We prove the result by showing that for every local scheduler $f$, it is possible to associate an $f$-potential with the game $G_{\text{coord}}$.

Let $f$ be a local scheduler. By Proposition 1, the choice function $g$ associated with $f$ is of the form $[\pi]$ for some permutation $\pi$ of $1, \ldots , n$. Let $l = \pi(n)$ be the last element in the permutation $\pi$ (this will be the only information about $\pi$ that we shall rely upon). Let $U := \{-1, 0, 1\}^n$ and let $F : S \to U$ be a function defined by $F(s) := (p_l(s), p_{l \ominus 1}(s), p_{l \ominus 2}(s), \ldots , p_{l \ominus (n - 1)}(s))$.

For $x \in U$ and $i \in \{1, \ldots , n\}$, $x_i$ denotes the $i$-th entry in $x$ and as before, $x_{-1} = (x_2 , \ldots , x_n)$. We also use the notation $F(s)[i]$ to denote the $i$-th entry in the $n$-tuple $F(s)$.
Let $<_L$ be the strict counterpart of the lexicographic ordering over the $(n - 1)$-tuples of $-1, 0, 1$, where $-1 <_L 0 <_L 1$. We extend $<_L$ to a relation $<_L \subseteq U \times U$. For $x, y \in U$ such that $x \neq y$, $x <_L y$ if one of the following mutually exclusive conditions holds:

C1 $x_1 \in \{-1, 1\}$ and $y_1 = 0$,
C2 $x_1 = y_1 = 0$ and $x_{-1} <_L y_{-1}$,
C3 $x_1, y_1 \in \{-1, 1\}$ and $x_{-1} <_L y_{-1}$,
C4 $x_1, y_1 \in \{-1, 1\}$, $x_{-1} = y_{-1}$ and $x_1 <_L y_1$.

In other words, if the first entry of $y$ is 0 and that of $x$ is not 0, then $x <_L y$. If the first entry of both $x$ and $y$ is 0, then to order $x$ and $y$ we use the lexicographic ordering over the $(n - 1)$-tuples $x_{-1}$ and $y_{-1}$. If the first entry of both $x$ and $y$ is not 0, then again to order $x$ and $y$ we use the lexicographic ordering over $x_{-1}$ and $y_{-1}$, the exception being when $x_{-1} = y_{-1}$. In this case, to determine the ordering we use the lexicographic ordering over $x_1$ and $y_1$.

**Claim 3** The relation $<_L$ is a strict total ordering over $U$.

Assuming Claim 3, consider an initial prefix $\xi_{k+1} = (s^1, \ldots, s^k, s^{k+1})$ of an improvement path $\xi$ that respects $f$. We claim that $F(s^k) <_L F(s^{k+1})$. We have the following cases:

- $f(s^k) = l \oplus i$ where $i \in \{1, \ldots, n - 1\}$. Since $\xi$ respects $f$, we have $p_{l \oplus i}(s^k) < p_{l \oplus i}(s^{k+1})$, so $F(s^k)[i + 1] <_L F(s^{k+1})[i + 1]$. Since $i \neq n$, if $i > 1$, then by the definition of the payoff functions, for all $j \in \{1, \ldots, i - 1\}$, $p_{l \oplus j}(s^k) = p_{l \oplus j}(s^{k+1})$. If $i \neq n - 1$, then $p_{l}(s^k) = p_{l}(s^{k+1})$ and it immediately follows that $F(s^k) <_L F(s^{k+1})$. Therefore, the interesting case is when $i = n - 1$. Here we show that the first entry in $F(s^{k+1})$ remains 0 after the update by player $n - 1$ iff the first entry in $F(s^k)$ is 0.

- If $F(s^k)[1] = 0$, then $F(s^{k+1})[1] = 0$. Indeed, suppose $F(s^k)[1] = 0$. Since $f(s^k) \neq l$, we have $s^k_l = s^{k+1}_l$. By the definition of the payoff function, for any joint strategy $s$, $p_l(s) = 0$ iff $s^0_l = 0$. Thus irrespective of the choice of $l \oplus (n - 1)$ we have $p_l(s^{k+1}) = 0$, so $F(s^{k+1})[1] = 0$.

- If $F(s^k)[1] \neq 0$, then $F(s^{k+1})[1] \neq 0$. Suppose $F(s^k)[1] \neq 0$. By the definition of the payoff functions, $s^k_l \neq 0$. Since $f(s^k) \neq l$, we have $s^k_l = s^{k+1}_l$.

Therefore, irrespective of the choice of $l \oplus (n - 1)$ we have $p_l(s^{k-1}) \neq 0$, so $F(s^{k-1})[1] \neq 0$.

Thus, by conditions C2 and C3 in the definition of $<_L$, and the fact that $(F(s^k))_{-1} <_L (F(s^{k+1}))_{-1}$, we indeed have $F(s^k) <_L F(s^{k+1})$.

- $f(s^k) = l$. Since $\xi$ respects $f$, for all $i \in \{1, \ldots, n - 1\}$ we have $l \oplus i \in BR(s^k)$. We claim that in this case, $s^k_l \neq 0$ and $s^0_{l \oplus 1} = 0$. Suppose not. If $s^k_l = 0$, then for all $i \in \{1, \ldots, n - 1\}$, $l \oplus i \in BR(s^k)$ implies that $s^k_{l \oplus i} = 0$. This in turn implies that $l \in BR(s^k)$, which is a contradiction. If $s^0_{l \oplus 1} = 0$, then we have the following two possibilities:

  - $s^k_{l \oplus 1} = s^0_l$. This implies $l \in BR(s^k)$, which is a contradiction.
  - $s^k_{l \oplus 1} \neq s^0_l$. Then there exists $j \in \{1, \ldots, n - 1\}$ such that $s^k_{l \oplus j} = s^k_{l \oplus 1}$ and $s^0_{l \oplus (j - 1)} \neq s^0_{l \oplus j}$. Since $s^k_{l \oplus j} \neq 0$, this implies that $l \oplus j \notin BR(s^k)$, which is a contradiction.
Now, if \( s^k \neq t_0 \), \( s^k_{\ominus 1} = t_0 \), and \( p_i(s^k) < p_i(s^{k+1}) \), then \( s^{k+1}_i = t_0 \). By C1 in the definition of \(<\), it then follows that \( F(s^k) < F(s^{k+1}) \).

Finally, since the set \( U \) which is the range of the function \( F \) is finite and \(<\) is a strict total order on \( U \), we can use an appropriate encoding \( e : U \rightarrow \mathbb{R} \) such that \( u_1 < u_2 \iff e(u_1) < e(u_2) \). Then \( e(F(s^k)) < e(F(s^{k+1})) \). So \( e \circ F \) is an \( f \)-potential. By the remark following Theorem 5 the result follows. \( \square \)

Proof of Claim 3 Let \( x, y \in U \) such that \( x \neq y \). We have the following cases.

- \( x_1 \in \{-1, 1\} \) and \( y_1 = 0 \). Then by C1, \( x < y \).
- \( x_1 = 0 \) and \( y_1 = 0 \). Then by C2, if \( x_{-1} <_{\text{L}} y_{-1} \) then \( x < y \) else \( y < x \).
- \( x_1 = 0 \) and \( y_1 \in \{-1, 1\} \). Then by C1, \( y < x \).
- \( x_1, y_1 \in \{-1, 1\} \) and \( x_{-1} \neq y_{-1} \). Then by C3, if \( x_{-1} <_{\text{L}} y_{-1} \) then \( x < y \) else \( y < x \).
- \( x_1, y_1 \in \{-1, 1\} \) and \( x_{-1} = y_{-1} \). Then by C4, if \( x_1 <_{\text{L}} y_1 \) then \( x < y \) else \( y < x \).

Further, it can be verified that the relation \(<\) is transitive by a straightforward case analysis. \( \square \)

Note that Theorem 7 cannot be extended to set-based schedulers. Indeed, suppose that \( n > 2 \), and for some \( t \neq t_0 \) we have \( t \in \cap_{i \in N} S_i \). Consider the joint strategy \( s := (t, t_0, \ldots, t_0) \) and a set-based scheduler \( f \) such that for all \( k \in \{1, \ldots, n\} \), \( f(\{k, k \oplus 1\}) := k \), with arbitrary values for other inputs. Then the following infinite improvement path respects this scheduler. For the sake of readability, we underlined the strategies that are not best responses.

\[
(t, t_0, \ldots, t_0), (t, t, t_0, \ldots, t_0), (t_0, t, t_0, \ldots, t_0), \ldots
\]

10 Bounds on finding a Nash equilibrium

Schedulers, apart of allowing us to classify weakly acyclic games are also useful to improve bounds on finding a Nash equilibrium. We start with the following simple observation.

Proposition 4 Consider a weakly acyclic game \( G \) for \( n \) players in which each player has at most \( k \) strategies. Then starting from any joint strategy there exists a finite improvement path of length at most \( k^n \).

Proof Given a joint strategy \( s \) consider a finite improvement path \( \xi \) that starts from it. If \( \xi \) has a fragment of the form \( s^1, \ldots, s^k, s^1 \), we can safely delete from it the fragment \( s^1, \ldots, s^k \). Repeating this process we get a finite improvement path that starts from \( s \) in which each joint strategy appears at most once. By assumption there are at most \( k^n \) joint strategies, which yields the claim. \( \square \)

In what follows, given a joint strategy \( s \) in a finite game \( G \) we focus on a possibly infinite tree \( T(G, s) \) the root of which is \( s \) and the nodes of which are joint strategies such that \( s'' \) is a child of \( s' \) if \((s', s'') \) is a step in an improvement path starting from \( s \).

We have the following result in which each step consists of traversing an edge in the tree \( T(G, s) \).
Table 1  

| Type of scheduler | Number of inputs |
|-------------------|------------------|
| General           | infinite         |
| State-based       | $k^n$            |
| Set-based         | $2^n$            |
| Local             | $n$              |

**Theorem 8** Consider a weakly acyclic game $G$ for $n$ players in which each player has at most $k$ strategies.

(i) Suppose that in each joint strategy each player has at most one better response. Then in each tree $T(G, s)$ a Nash equilibrium can be found in $O(n^{kn})$ steps.

(ii) If $G$ respects a scheduler, then in each tree $T(G, s)$ a Nash equilibrium can be found in $O(k^n)$ steps.

**Proof** (i) By assumption each node in the tree $T(G, s)$ has at most $n$ successors. Consequently, the subtree that consists of the nodes that lie at a level at most $m$, where $m > 0$, has at most $\sum_{i=0}^{m} n^i$ nodes. So to find a specific node in this tree takes $O(n^m)$ steps. Putting $m = kn$, we get the claim on the account of Proposition 4.

(ii) Suppose $G$ respects a scheduler. By Theorem 1 $G$ respects a state-based scheduler $f$. Take an arbitrary joint strategy $s$ and follow an improvement path $\xi$ that starts in $s$ and respects $f$. By definition $\xi$ is a finite path in $T(G, s)$ that starts from the root.

Suppose that some strategy appears more than once in $\xi$, i.e. $\xi$ starts with a fragment of the form $s^1, \ldots, s^i, \ldots, s^j, s^i$, where $1 \leq i \leq j$. Then, since $f$ is a state-based scheduler, the infinite improvement path $s^1, \ldots, s^{i-1}, (s^i, \ldots, s^j)^\ast$ also respects $f$, which contradicts the assumption.

So each joint strategy appears at most once in $\xi$. Consequently, $\xi$ is of length at most $k^n$, which establishes the claim. □

In total we introduced four classes of schedulers, the general ones, state-based, set-based and local. To define a scheduler, we need to describe its value on all its inputs. This yields the progression given in Table 1. The last entry is justified by Proposition 1 that allows us to describe a local scheduler by specifying a single permutation of the numbers $1, \ldots, n$.

**Acknowledgments** We thank Ruben Brokkelkamp and Mees de Vries for helpful discussions and Mona Rahn and the reviewers for useful comments.

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