Noise-Induced Transition from Translational to Rotational Motion of Swarms

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We consider a model of active Brownian agents interacting via a harmonic attractive potential in a two-dimensional system in the presence of noise. By numerical simulations, we show that this model possesses a noise-induced transition characterized by the breakdown of translational motion and the onset of swarm rotation as the noise intensity is increased. Statistical properties of swarm dynamics in the weak noise limit are further analytically investigated.

I. INTRODUCTION

In different natural and social systems, agents form groups characterized by cooperative coherent motion [1]. Such collective swarm motions have been observed and investigated in bacterial populations [2, 3, 4, 5, 6], in slime molds [7, 8, 9, 10], for the ants [11, 12] and fish [13, 14, 15], in the motion of pedestrians [16] and for the car traffic [17]. To describe these phenomena, various models of collective coherent motion in populations of self-propelled or self-driven particles have been proposed (see [18, 19]). Some of them are formulated in terms of interacting automata located on a lattice or having continuous coordinates [20, 21, 22, 23]. A different class of models is based on dynamical equations for individual particles (agents). Such kinetic transitions show the emergence of coherent collective flows starting from a disordered state with random velocity directions of individual particles (agents). Such kinetic transitions have been extensively investigated for the automata systems [24, 25, 26, 27] and in the framework of hydrodynamics [28, 29, 30]. The ordered states of swarms can represent simple translational motion or be characterized by vortex flows. Both spatially distributed and localized swarm states are possible. In a distributed state, the population fills the entire available medium. In contrast to this, the population forms compact spatial groups in a localized state. An interesting example of a localized swarm state is a rotating flock of finite extent, seen in the simulations of a discrete model of self-propelling particles and described analytically within a continuum active fluid approximation [23] (see also [17]).

Localized states of swarms may undergo transitions leading to new dynamical regimes, when the system parameters or the noise intensity are gradually changed. In the previous publication [40], a noise-induced transition from the localized state with translational motion to a state with incoherent oscillations without translational motion was investigated numerically and analytically for a one-dimensional system of interacting self-propelled particles. In the present article, we extend investigations of this system to two spatial dimensions. We study here a population of self-propelled particles interacting via a parabolic interaction potential corresponding to linear attracting forces between the pairs. In absence of noise, this dynamical system has two kinds of attractors, corresponding, respectively, to a compact traveling state of the entire population and to a state where it rotates as a vortex without any global translational motion. The aim of our study is to investigate effects of noise on translational swarm motion. We find that the system is highly sensitive to stochastic forces. When noise is present, the traveling swarm is a cloud of particles characterized by different dispersions in the directions parallel and transverse to the direction of translational motion. Our numerical simulations confirmed by an approximate analytical study show that the mean-square transverse dispersion of a swarm is proportional to the square root of the noise intensity, whereas its dispersion along the direction of motion depends linearly on the noise intensity. Therefore, for weak noises the swarm looks like a pancake oriented orthogonally to the motion direction. When the noise is increased, the swarm gradually acquires a more symmetric shape. For strong noises, we find that the translational motion of a swarm becomes suddenly impossible and is abruptly replaced by a rotational regime with a vortex flow. The detailed formulation of the model...
is presented in the next section. In Sect. II we describe
the results of numerical simulations. The statistical
properties of a traveling swarm in the weak noise limit are
approximately explained by an analytical theory which is
constructed in Sect. III. The paper ends with conclusions
and discussion of obtained results.

II. THE MODEL

We consider a swarm formed by $N$ identical self-
propelled particles of unit mass interacting via an attrac-
tive parabolic pair potential. The dynamics of the system
is given by the following set of evolution equations:

$$\dot{\mathbf{r}}_i = \mathbf{v}_i$$

(1a)

$$\dot{\mathbf{v}}_i = \mathbf{F}_i - \frac{a}{N} \sum_{j=1}^{N} (\mathbf{r}_i - \mathbf{r}_j) + \xi_i(t).$$

(1b)

The forces $\mathbf{F}_i$ depend on the particle velocity and are
introduced to take into account active motion of particles.
We choose them in the form

$$\mathbf{F}_i = (1 - v_i^2)\mathbf{v}_i,$$

(2)

so that, in absence of noise and interactions, the particle
acquires the unit velocity $v = 1$. Additionally, the parti-
cles are subject to stochastic white forces $\xi_i$ of strength
$D$ which are independent for different particles and are
characterized by the correlation functions

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = 2D\delta(t - t')\delta_{ij}.$$

This model has previously been introduced in [10]. It
is phenomenological, but rather general because it can be
viewed as a normal form for a population of particles near
a supercritical bifurcation corresponding to spontaneous
onset of active motion (see [10]). In this model, attract-
tive interactions between particles have an infinite range
and grow linearly with the separation between them.
Because we shall consider only spatially localized states of
the population, our results will hold, however, also for the
situations when interactions are characterized by a
finite range, but it is much larger than the mean swarm
diameter. It should be noted that, in a different context,
the model [10] has been considered already by Rayleigh
[11].

The study of the one-dimensional version of the model
[10] has shown that, as the noise intensity $D$ is in-
creased, spontaneous breakdown of translational swarm
motion takes place here [10]. Some statistical proper-
ties of translational swarm motion in the two-dimensional
model (1) (with a slightly different choice of the forces
$\mathbf{F}_i$) have subsequently been discussed [12]. For the case of
two interacting particles ($N = 2$), the rotational modes
were described in [12], where simulations for small rot-
ating clusters consisting of 20 particles have also been
reported. The aim of the present work is to perform sys-
tematic, numerical and analytical, investigations of the
behavior described by this two-dimensional model.

III. NUMERICAL SIMULATIONS

Numerical integration of equations (1) was performed
using the Euler scheme with the constant time step of
0.001. In all simulations, the total number of particles
was fixed to $N = 300$ and the coefficient $a$ specifying
the strength of interactions between the particles was set
to $a = 100$. To produce a traveling localized state of
the swarm, special initial conditions were used. At time
t $= 0$, all particles had identical positions and velocities
and the noise was switched only a little later, at time
t $= 30$.

Several statistical characteristics of the swarm were
monitored during simulations. The center of mass $\mathbf{R}$
of the swarm and its mean velocity $\mathbf{V}$ at time $t$ were defined
as $\mathbf{R}(t) = (1/N) \sum_i \mathbf{r}_i(t)$ and $\mathbf{V}(t) = (1/N) \sum_i \mathbf{v}_i(t)$, re-
spectively. Because the cloud of traveling particles in the
presence of noise was anisotropic, we also determined its
instantaneous mean-square dispersions in the directions
parallel ($S_{\parallel}$) and orthogonal ($S_{\perp}$) to the direction of its
instantaneous mean velocity $\mathbf{V}$. They were defined as

$$S_{\parallel}(t) = \frac{1}{NV^2(t)} \sum_{i=1}^{N} [(\mathbf{r}_i(t) - \mathbf{R}(t)) \cdot \mathbf{V}(t)]^2$$

(3a)

$$S_{\perp}(t) = \frac{1}{NV^2(t)} \sum_{i=1}^{N} [(\mathbf{r}_i(t) - \mathbf{R}(t)) \times \mathbf{V}(t)]^2.$$  (3b)

Additionally, angular momenta of all particles with re-
spect to the mass center were determined as

$$\mathbf{L}_i(t) = (\mathbf{r}_i(t) - \mathbf{R}(t)) \times (\mathbf{v}_i(t) - \mathbf{V}(t)).$$  (4)

Figure 1(a) shows the time dependences of the magni-
tude $V = |\mathbf{V}|$ of the mean swarm velocity for two simu-
lations with different noises. When the noise is relatively
weak ($D = 0.067$), its introduction leads to some fluctua-
tions in the instantaneous swarm velocity and a decrease
of its average level. If a stronger noise ($D = 0.07$) is ap-
plied, the swarm velocity first behaves as for the weaker
noise, but then abruptly drops down to a value close to
zero. This sudden transition corresponds to the break-
down of translational motion of the swarm. In Fig. 1(b)
we have plotted the time-averaged swarm velocity $\langle V \rangle$
as the function of the noise intensity $D$. The average veloc-
ity gradually decreases with noise, until the breakdown
occurs at $0.067 < D < 0.070$.

In the state with translational motion at relatively
weak noises, the direction of the swarm motion does not
remain constant with time. The swarm travels along a
complex trajectory, part of which is shown in Fig. 2 (such
trajectories should correspond to the Brownian motion of
the entire swarm). In the inset in this figure, we display
the distribution of particles in the swarm at some time
moment. It can be noticed that the cloud of particles is
significantly squeezed along the direction of swarm
motion.

Figure 3 shows the computed average longitudinal ($S_{\parallel}$)
and transverse ($S_{\perp}$) dispersions as functions of the noise


FIG. 1: (a) Time evolution of the mean velocity of a swarm before (solid line, $D = 0.067$) and after (dashed line, $D = 0.070$) the stochastic breakdown of the translational mode. (b) Mean velocity of the swarm with increasing noise strength. At a critical noise strength a sharp transition in the behavior of the swarm occurs (see also Fig. 4).

The sequence of snapshots in Fig. 4 displays temporal evolution of the swarm when the noise intensity exceeds the breakdown threshold. Initially, the swarm is traveling and its shape is similar to that characteristic for the weaker noises (cf. Fig. 2). However, in the course of time the swarm slowly acquires a ring shape, with particles rotating around its center. This rotating ring structure corresponds to a state where translational motion of the entire swarm is already absent.

A different visualization of the process, accompanying the breakdown of translational motion and the transition to a rotating swarm, is chosen in Fig. 5. Here we show the trajectory of motion of the center of mass of the swarm (solid line) together with the trajectory of motion of one of its particles (dashed line). We see that, in a traveling swarm, the particles perform irregular oscillations around its instantaneous mass center. When the translational motion is terminated and the rotating ring...
FIG. 4: Several subsequent snapshots of a swarm with 300 particles during the transition from translational motion to the rotational mode; the noise intensity is $D = 0.070$.

is formed, such oscillations become transformed into rotations around the ring center.

To provide statistical description of particle motions in the traveling and rotating states of the swarm, angular momentum distributions $P(L)$ have been computed. For the state with translational motion, the distribution has a single central peak at $L = 0$ (Fig. 6(a)). In contrast to this, in the rotational state the distribution has two symmetrically placed peaks corresponding to a certain non-vanishing momentum (Fig. 6(b)). Note that the particles inside the ring are rotating both in the clockwise and counter-clockwise directions, and the numbers of particles rotating in each direction are approximately equal. Thus, the swarm does not rotate as a whole and its total angular momentum remains zero on the average. This behavior is the consequence of the fact that only long-range attractive interactions between particles are present in the considered model. It can be expected that, if short range repulsive interactions are additionally introduced, the breakdown of the rotational symmetry in the ring would occur and one of the rotation directions would be selected.

IV. THE WEAK NOISE LIMIT

Our numerical simulations have shown that, for weak noises, the swarm is strongly squeezed in the direction along its center-of-mass motion and its longitudinal and transverse dispersions are strongly different. Below we derive approximate analytical expressions for $S_{\parallel}$ and $S_{\perp}$ in the limit of the small noise intensity $D \to 0$.

First, we note that in this limit the motion of the center of mass of the swarm will remain approximately linear within very long times or, in other words, the swarm velocity $\mathbf{V}$ remains approximately constant on the short time scales characteristic for the motions of individual
because, as follows from equation (8), we have

\[ \text{time} \]  

Thus, nonlinear terms play a principal role for oscillations in \( y \) of oscillations of the transverse component. This is done because such terms are essential for the damping of transverse fluctuations and we have approximately \( V^2 = 1 - \langle \dot{y}_i^2 \rangle \). Substituting this into (10), we get

\[ \dot{y}_i - \langle \dot{y}_i^2 \rangle \dot{y}_i + ay_i = \xi_i(t) \]  

(12)

The stochastic differential equation (12) does not include longitudinal fluctuations \( \delta x_i \) and, furthermore, fluctuations for different particles \( i \) are not coupled here. For subsequent analysis of this differential equation, we drop the indices and write as

\[ \dot{y} - \langle \dot{y}^2 \rangle \dot{y} + ay = \xi(t) \]  

(13)

with \( \langle \xi(t)\xi'(t) \rangle = 2D\delta(t-t') \). Here we have assumed that statistical averaging is equivalent to averaging over the ensemble.

The approximate solution for the probability distribution of variable \( y \) in the statistical stationary state can be derived for this equation assuming that the parameter \( a \), determining the oscillation frequency, is large \( (a \gg 1) \). We introduce slowly varying amplitudes

\[ y(t) = \eta(t)e^{i\omega t} + \eta^*(t)e^{-i\omega t} \]  

(14)

where \( \omega = \sqrt{a} \gg 1 \). Substituting this into (13) and keeping only the resonant terms of the highest order in \( \omega \), we obtain a stochastic evolution equation for the transverse complex oscillation amplitudes

\[ \dot{\eta} = \omega^2 \langle |\eta|^2 \rangle \eta - \frac{3}{2} \omega^2 |\eta|^2 \eta + \zeta(t) \]  

(15)

where the complex-valued white noise \( \zeta(t) \) has correlation functions

\[ \langle \zeta(t) \rangle = 0, \quad \langle \zeta(t)\zeta'(t') \rangle = 0, \quad \langle \zeta(t)\zeta^*(t') \rangle = \frac{D}{2\omega^2} \delta(t-t'). \]

This stochastic Langevin equation corresponds to the following Fokker-Planck equation for the probability density \( P(\eta, \eta^*, t) \):

\[ \frac{\partial P}{\partial t} = -\frac{\partial}{\partial \eta} \left[ \omega \left( \langle |\eta|^2 \rangle \eta - \frac{3}{2} |\eta|^2 \right) \eta P \right] - \frac{\partial}{\partial \eta^*} \left[ \omega \left( \langle |\eta|^2 \rangle \eta^* - \frac{3}{2} |\eta|^2 \right) \eta^* P \right] + \frac{1}{2\omega^2} D \frac{\partial^2 P}{\partial \eta \partial \eta^*} \]  

(16)

The stationary solution \( \mathcal{P} \) of the Fokker-Planck equation reads

\[ \mathcal{P} = \frac{1}{Z} \exp \left[ -\frac{\omega^4}{D} \left( -4 \langle |\eta|^2 \rangle |\eta|^2 + 3 |\eta|^4 \right) \right] \]  

(17)

where the normalization constant \( Z \) is given by

\[ Z = \int \exp \left[ -\frac{\omega^4}{D} \left( -4 \langle |\eta|^2 \rangle |\eta|^2 + 3 |\eta|^4 \right) \right] d^2 \eta. \]  

(18)
When the probability distribution is known, the second statistical moment can be calculated as

$$\langle |\eta|^2 \rangle = \int |\eta|^2 p(\eta, \eta^*) d^2 \eta$$  \hspace{1cm} (19)

Because the stationary probability distribution depends on $\langle |\eta|^2 \rangle$, this is an equation which should be solved to determine this statistical moment. Let us substitute $\eta = \rho e^{i\phi}$ and $\omega^2 = a$. Then $|\eta| = \rho$ and $d^2 \eta = \rho d\rho d\phi$. Therefore, equation (19) takes the form

$$\langle \rho^2 \rangle = \frac{\int_0^\infty \rho^3 \exp \left[ -\frac{a^2}{2} (4 \langle |\eta|^2 \rangle \rho^2 + 3 \rho^4) \right] d\rho}{\int_0^\infty \rho \exp \left[ -\frac{a^2}{2} (4 \langle |\eta|^2 \rangle \rho^2 + 3 \rho^4) \right] d\rho}$$  \hspace{1cm} (20)

Introducing the variable $u = \rho \langle |\eta|^2 \rangle^{-1/2}$, equation (20) is transformed to

$$1 = \frac{\int_0^\infty u^3 \exp \left[ -\nu (4u^2 + 3u^4) \right] du}{\int_0^\infty u \exp \left[ -\nu (4u^2 + 3u^4) \right] du}$$  \hspace{1cm} (21)

where

$$\nu = \frac{a^2}{D} \langle |\eta|^2 \rangle^2.$$  \hspace{1cm} (22)

Numerical solution of equation (21) yields $\nu \approx 0.22$. When $\nu$ is known, equation (22) determines $\langle |\eta|^2 \rangle$ as

$$\langle |\eta|^2 \rangle = \nu^{1/2} D^{1/2} a^{-1}.$$  \hspace{1cm} (23)

Using the definition (14) of variable $\eta$, we find that $S_\perp = \langle y^2 \rangle = 2 \langle |\eta|^2 \rangle$. Thus, we finally obtain the analytical estimate for the transverse dispersion of the swarm in the weak noise limit,

$$S_\perp = \kappa \frac{D^{1/2}}{a}$$  \hspace{1cm} (24)

where the numerical coefficient is $\kappa = 2\nu^{1/2} \approx 0.94$.

The longitudinal dispersion $S_\parallel = \langle \delta x^2 \rangle$ is approximately determined by the linear stochastic differential equation (6). A straightforward derivation for $a \gg 1$ yields for this property the analytical estimate

$$S_\parallel = \frac{D}{2a}.$$  \hspace{1cm} (25)

We see that in the limit $D \to 0$ the longitudinal dispersion is indeed much smaller than the transverse dispersion of the traveling swarm, as assumed in the above derivation.

Note that, for $a \gg 1$, statistical dispersions of transverse and longitudinal velocity fluctuations $W_\perp = \langle y^2 \rangle$ and $W_\parallel = \langle \delta x^2 \rangle$ are $W_\perp = aS_\perp$ and $W_\parallel = aS_\parallel$. Therefore, in the considered weak noise limit they are given by $W_\perp = \kappa D^{1/2}$ and $W_\parallel = D/2$. Comparing these analytical estimates with the simulations in Fig. 4, we find that they agree well with the respective numerical data. For the transverse dispersion, the agreement is found in a wide interval of noise intensities. The analytical expression for the longitudinal dispersion holds, on the other hand, only for very weak noises.

V. CONCLUSIONS

We have studied statistical properties of localized swarms with long-range attractive interactions in one-dimensional media. Our numerical simulations show that the swarm is highly sensitive to the action of noise. Even very weak noises lead to strong dispersion of the swarm along the direction orthogonal to the direction of its translational motion. The approximate analytical theory predicts that the transverse dispersion of aswarm increases as $\sqrt{D}$ with the noise intensity $D$, whereas its longitudinal dispersion depends linearly on $D$ in the limit $D \to 0$ and remains therefore much smaller in this limit. Hence, for weak noises the traveling swarm is strongly squeezed along the direction of its mass motion. This analytical result is confirmed by numerical simulations.

Increasing the noise intensity $D$, we find that translational motion breaks down when a certain critical intensity is reached. After the breakdown, the translational
motion is stopped and instead the swarm goes into a rotational mode where the center of mass of the swarm shows only weak random motion. This behavior resembles the breakdown of translational motion which was previously seen for the one-dimensional system \[40\]. In contrast to the one-dimensional case, we could not however analytically treat this transition, because of the strong fluctuations in the transverse direction.

Though our results are obtained in the model with harmonic attractive interactions, they are also be applicable for the models with finite-range attractive interactions between the particles, provided that the size of a localized swarm (i.e., the statistical dispersion of the coordinates of its particles with respect to the mass center) is much smaller than the interaction radius. In this situation, a harmonic approximation of the interaction potential can be applied. Moreover, similar effects can be expected for swarms in three-dimensional media. It would be interesting to see whether the discussed behavior is indeed characteristic for real biological swarms with long-range interactions between individual organisms, such as bird flocks or fish schools.

**Acknowledgments**

We are grateful to L. Schimansky-Geier and D. Zanette for useful discussions. This study has been performed in the framework of the Collaborative Research Center “Complex Nonlinear Processes” of the Deutsche Forschungsgemeinschaft (DFG-SFB 555).

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