ω-Operads of Coendomorphisms for Higher Structures

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Abstract

It is well known that strict ω-categories, strict ω-functors, strict natural ω-transformations, and so on, form a strict ω-category. A similar property for weak ω-categories is one of the main hypotheses in higher category theory in the globular setting. In this paper we show that there is a natural globular ω-operad which acts on the globular set of weak ω-categories, weak ω-functors, weak natural ω-transformations, and so on. Thus to prove the hypothesis it remains to prove that this ω-operad is contractible in Batanin’s sense. To construct such an ω-operad we introduce more general technology and suggest a definition of ω-operad with the fractal property. If an ω-operad $B^0_p$ has this property then one can define a globular set of all higher $B^0_p$-transformations and, moreover, this globular set has a $B^0_p$-algebra structure.

Keywords. Higher categories; ω-operads; Higher weak ω-transformations.

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Introduction

The algebraic model of weak higher transformations was undertaken for the first time in [15, 16] with respectively the Penon approach and the Batanin approach. However André Joyal has pointed out to us that the 2-coloured \(\omega\)-operads for the higher transformations that we built in [16] have two many coherence cells. In this paper we propose a new approach of contractibility for coloured \(\omega\)-operads, which agrees
with contractibility in the sense of Batanin ([2]) for monochromatic
\(\omega\)-operads, and which corrects the counterexample that André Joyal
showed us.

Overall, this paper is devoted to describing, up to a precise contractibility hypothesis (see section 3.3), the first operadic approach to
the weak \(\omega\)-category of weak \(\omega\)-categories. Other approaches have been
proposed: For example Michael Makkai in [20] has described the weak
\(\omega\)-category of weak \(\omega\)-categories by using a multitopic approach, and a
simplicial approach of the weak \(\omega\)-category of weak \(\omega\)-categories is de-
scribe by Jacob Lurie in [19], in the context of the \((\infty, 1)\)-categories. We
believe that our operadic approach has the definitive advantage of being
very explicit compared to other approaches (see also [10, 12, 23, 24]).

Such a higher category theory can then be seen as the right generalisation of strict \(\omega\)-category theory, where strict \(\omega\)-categories, strict
\(\omega\)-functors, and all higher strict transformations organised themselves
into a strict \(\omega\)-category. This strict \(\omega\)-category can be seen as a sort frac-
tal object, in the sense that its objects are themselves strict \(\omega\)-categories.

In [4] the author has shown that weak \(\omega\)-categories of Penon are alge-
bras of an \(\omega\)-operad, the \(\omega\)-operad of Penon, and thus by analogy, it
should be possible to describe higher transformations in [15] as algebras
for specifics coloured \(\omega\)-operads. So this article can be used also as a
way to see how to build the weak \(\omega\)-category of weak \(\omega\)-categories, with
the Penon approach to weak \(\omega\)-categories. Our techniques also work
very well for Leinster’s version of weak \(\omega\)-categories which is a slight
modification of the original Batanin’s approach.

The main technical difficulty was to find the most natural way to
build this algebraic model of the weak \(\omega\)-category of the weak \(\omega\)-categories.
We believe that the way we describe here, up to the contractibility hypoth-
esis in the section 3.3, is the achievement of this goal. The direction
that we propose is not only very natural but also it allows us to see
quickly how to build it (see the section 3.6).

More precisely, starting from the coglobular complex of colored $\omega$-operads $B_C^\bullet$ built in [16] (where $B_C^0$ is the Batanin’s $\omega$-operad for weak $\omega$-categories) we construct its $\omega$-operad of coendomorphisms $\text{Coend}(B_C^\bullet)$ (which we called the violet operad for reason we explain later) and show that this $\omega$-operad acts on the globular set of weak $\omega$-categories, weak $\omega$-functors, etc. We conjecture that the violet operad is contractible (see 3.3). This conjecture implies immediately that weak $\omega$-categories, weak $\omega$-functors etc. form a weak $\omega$-category. We provide some evidence that our contractibility hypothesis is correct but a full proof of it requires a development of a homotopy theory (and in particular a theory of homotopy colimits) of colored $\omega$-operads. This will be the subject of our future work.

Contractibility is a specific structure of the $\omega$-operad $B_C^0$ of Batanin, but if we conceptualise this property and the technology which allows us to see the way the weak $\omega$-category of weak $\omega$-categories is built, we can describe many kinds of higher structures with similar fractal phenomena, by using the same technology of $\omega$-operads of coendomorphisms. Let us be more precise: We start with a basic data, which is a coglobular complex of $\omega$-operads $B_P^\bullet$ in $\mathbb{T}$-Cat.c equipped with a structure $P$, where $\mathbb{T}$-Cat.c is the category of $\mathbb{T}$-categories over constant $\omega$-graphs, or in other words, the category of coloured $\omega$-operad over constant $\omega$-graphs (see section 1.3 and section 2). We say that the first $\omega$-operad $B_P^0$ (the "0-step") of this coglobular complex $B_P^\bullet$ has the fractal property, if there is a morphism of $\omega$-operads between $B_P^0$ and the corresponding $\omega$-operad of coendomorphisms $\text{Coend}(B_P^\bullet)$ associated to $B_P^\bullet$. If it happens then all algebras for all $\omega$-operads $B_P^n$ ($n \in \mathbb{N}$) organise into a single algebra of $B_P^0$. If $P = S_u$, where $P$ means strictly contractible with contractible units (see the section 3.2 and the section 3.5), then we obtain the indigo operad $\text{Coend}(B_{S_u}^\bullet)$, and up to a precise contractibility
hypothesis (see section 3.3), we can describe the strict \( \omega \)-category of the strict \( \omega \)-categories with the same technology as we do for the weak case. In this article we also build two other coglobular complex \( B^\bullet_p \) of \( \omega \)-operad which corresponding \( \omega \)-operad \( B^p_0 \) have the fractal property, without requiring any hypotheses (see section 4).

The main ideas of this article were exposed for the first time in September 2010, in the Australian Category Seminar at Macquarie University [14].

The plan of this article is as follow:

In the first section (see 1), we summarise Batanin’s theory of \( \omega \)-operads (see [2]) with the goal to extract the corollary (see 1) which is a central result for our article, and this corollary is just a consequence of proposition 7.2 in [2]. A lot of material which surrounds the corollary 1 is described in [2]: Globular categories, globular functors, monoidal globular categories (called \( MG \)-categories), monoidal globular functors (called \( MG \)-functors), augmented monoidal globular categories (called \( AMG \)-categories), globular objects of a globular category, etc. However we expose these concepts in a more modern approach, which essentially follows the work of [26]. Then we explain in detail the two most important \( MG \)-categories for Batanin’s theory of \( \omega \)-operads: The \( MG \)-category \( \text{Tree} \) of trees and the \( MG \)-category \( \text{Span} \) of spans in \( \text{Set} \) (see 1.2), which are also described with a modern approach in the works [3, 5, 25, 27]. In 1.3 we briefly describe \( T \)-categories, where \( T \) is the monad of the strict \( \omega \)-categories on \( \omega \)-graphs. \( T \)-categories are important for this article because for us an \( \omega \)-operad in the sense of Batanin is a \( T \)-category over the terminal \( \omega \)-graph (see the section 1.4).

In the second section (see 2) we state the main result of the article: By using the corollary 1 of the previous section, for each coglobular object \( W^\bullet \) in \( T-\text{Cat}_c \) we associate its standard action in \( T-\text{Cat}_1 \) which roughly speaking is a diagram in \( T-\text{Cat}_1 \) built with two morphisms
of \( \omega \)-operads. In particular each coglobular object \( W^\bullet \) shows us two important \( \omega \)-operads: The \( \omega \)-operad \( W^0 \) (the "0"-step of the coglobular complex \( W^\bullet \)), and the associated \( \omega \)-operad of coendomorphism \( Coend(W) := (HOM(W^n,W^t))_{n\in \mathbb{N}, t \in \text{Tree}} \). The \( \omega \)-operad \( W^0 \) is fractal if we can build a morphism of \( \omega \)-operads between it and \( Coend(W) \).

Then we give the application of these technology to the coglobular complex \( C^\bullet \) in the category \( T\text{-Gr}_{p,c} \) of pointed \( T \)-graphs over constant \( \omega \)-graphs. For example denote by \( B^0_C \) the \( \omega \)-operad of Batanin for weak \( \omega \)-categories, and by \( Coend(B^\bullet_C) \) the \( \omega \)-operad of coendomorphisms of the coglobular complex \( B^\bullet_C \) in \( T\text{-Cat}_c \) freely generated by \( C^\bullet \). If \( B^\bullet_C \) is fractal then there is an action of it on the globular complex in \( SET \) of the weak higher transformations, which show that the weak \( \omega \)-category of weak \( \omega \)-categories exists in a completely \( \omega \)-operadic setting. It shows also that it is a weak \( \omega \)-category in the sense of Batanin.

The third section is devoted first to describing the coglobular complex \( B^\bullet_S \) in \( T\text{-Cat}_c \) of strict higher transformations, and the coglobular complex \( B^\bullet_C \) in \( T\text{-Cat}_c \) of weak higher transformations. In particular we propose a new approach to contractibility which corrects a counterexample that André Joyal constructed to our first approach to contractibility as in the article [16] for weak higher transformations. The key points of this new approach is to bring to light some remarkable cells that we call root cells, and to take account of a specific property, the loop property, that these root cells must follow for contractibility. Then we state the following hypothesis: For each tree \( t \), the coloured \( \omega \)-operad \( B^t_S \) is strictly contractible and has contractible units, and the coloured \( \omega \)-operad \( B^t_C \) is contractible. If we accept this hypothesis then it is possible to build a composition system for each \( \omega \)-operad of coendomorphism \( Coend(B^\bullet_S) \) and \( Coend(B^\bullet_C) \), and also to show that the \( \omega \)-operads \( B^0_S \) and \( B^0_C \) are fractal. It thus show that the strict \( \omega \)-category of strict \( \omega \)-categories exists in a completely \( \omega \)-operadic setting, the weak \( \omega \)-category of weak
ω-categories exists in a completely ω-operadic setting, and this facts are proved by using the same technology related to the standard action in T-Cat
.

The fourth section gives two examples of ω-operads having the fractal property: It is easy to show that the ω-operad $B^{0}_{id}$ of ω-magmas and the ω-operad $B^{0}_{idu}$ of reflexive ω-magmas, both have the fractal property. This proves the existence of the ω-magma of ω-magmas, and the reflexive ω-magma of reflexive ω-magmas, by using the same technology related to the standard action in T-Cat
.

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I dedicate this work to Michael Batanin.

1 Batanin’s theory of ω-Operads

Thoughout this paper, if C is a category then C(0) is the class of its objects (but we often omit "0" when there is no confusion) and C(1)
is the class of its morphisms. The symbol := means "by definition is". Also Set denotes the category of sets, and $SET$ denotes the category of large sets (for instance the proper class of ordinals is an object of $SET$, but not in $Set$). Similarly $Cat$ denotes the 2-category of small categories, and $CAT$ denotes the 2-category of categories.

The theory of $\omega$-operads was developed for the first time by Michael Batanin in his seminal article [2]. More precisely, he produced a theory of $\omega$-operads in the general context of his monoidal globular categories.

In this chapter we summarise the general approach of the theory of $\omega$-operads of Michael Batanin, because it is in this general approach that the important corollary 1 was formulated. This corollary is the key result to developing the main technology of this article: It is a result about the existence of the $\omega$-operad of coendomorphisms which, as we will see, plays an important role for many kinds of higher structure. A higher structure means for us a structure based on $\omega$-graphs. For instance, $\omega$-magmas are basic example of such higher structure, but we will consider also reflexive $\omega$-magmas as an other kind of higher structure, and also other more complex higher structures like the weak $\omega$-categories.

1.1 $MG$-categories and $AMG$-categories

A lot of material which surrounds corollary 1 is described in [2]: Globular categories, globular functors, monoidal globular categories (called $MG$-categories), monoidal globular functors (called $MG$-functors), augmented monoidal globular categories (called $AMG$-categories), globular objects of a globular category, etc.

Definition 1 The globe category $G$ is defined as following: For each $n \in \mathbb{N}$, objects of $G$ are formal objects $\bar{n}$. Morphisms of $G$ are generated
by the (formal) cosource and cotarget $\bar{n} \xrightarrow{t_{n+1}^n} n + 1$ such that we have the relations $s_{n-1}^n s_n^n = s_{n+1}^{n+1} t_{n-1}^n$ and $t_{n-1}^n t_{n+1}^n = t_{n+1}^{n+1} s_{n-1}^n$. For each $0 \leq p < n$, we put $s_p^n := s_{n-1}^n \circ s_{n-2}^{n-1} \circ ... \circ s_p^p$ and $t_p^n := t_{n-1}^n \circ t_{n-2}^{n-1} \circ ... \circ t_p^{p+1}$. 

**Definition 2** Starting with the globe category $G$ above, we build the reflexive globe category $G_r$ as follow: For each $n \in \mathbb{N}$ we add in $G$ the formal morphism $\bar{n} \xrightarrow{1_{n+1}^n} n$ such that $1_{n+1}^n \circ s_{n+1}^n = 1_{n+1}^n \circ t_{n+1}^n = 1_n$. For each $0 \leq p < n$, we put $1_p^n := 1_{p+1}^n \circ 1_{p+2}^{p+1} \circ ... \circ 1_{n-1}^{n-1}$. 

The category of $\omega$-graphs is the category of presheaves $\omega-\text{Gr} := [G^{op}; Set]$ (which is also called the category of globular sets in the literature; see for example $[2]$), the category of large $\omega$-graphs is the category of presheaves $\omega-\text{GR} := [G^{op}; SET]$, and the 2-category of globular categories is the 2-category of prestacks $\text{GCAT} := [G^{op}; CAT]$. 

**Definition 3** Consider the terminal globular category $1$ and a globular category $C$. A globular object $(W, C)$ in $C$ is a morphism $1 \xrightarrow{W} C$ in $\text{GCAT}$. 

Let us put $\omega-\text{Grr} := [G_r^{op}; Set]$, the category of the reflexive $\omega$-graphs (see $[22]$). We have the adjunction

$$U \dashv R : \omega-\text{Grr} \longleftrightarrow \omega-\text{Gr}$$

and we call $(\mathbb{R}, \eta, \mu)$ the generated monad whose algebras are reflexive $\omega$-graphs. Objects of $\omega-\text{Grr}$ are usually denoted by $(G, (1_n^p)_{0 \leq p < n})$, where the operations $(1_n^p)_{0 \leq p < n}$ form a chosen reflexive structure on the $\omega$-graph $G$.

Let us denote by $\omega-\text{Cat}$ the category of strict $\omega$-categories. The forgetful functor $\omega-\text{Cat} \xrightarrow{U} \omega-\text{Gr}$, which associates to any strict $\omega$-category $C$ its underlying $\omega$-graph $U(C)$, is monadic. The corresponding
adjunction generates a cartesian monad $T$ which is the monad of strict $\omega$-categories on $\omega$-graphs.

Consider $\mathcal{C}AT_{pull}$, the 2-category of categories with pullbacks, with morphisms functors which preserve these pullbacks, and with 2-cells natural transformations between these functors. The functor $\mathcal{C}at(-)$ which associates to any object $C$ in $\mathcal{C}AT_{pull}$ the 2-category $\mathcal{C}at(C)$ of internal categories in it, is a 2-functor

$$ \mathcal{C}AT_{pull} \xrightarrow{\mathcal{C}at(-)} 2\text{-CAT} $$

where here 2\text{-CAT} denote the 2-category of 2-categories. Thus for the case of the monad $T$ on $\omega$-Gr we can associate the 2-monad $\mathcal{T} = \mathcal{C}at(T)$ on $\mathcal{G}\mathcal{C}AT$.

**Definition 4 ([26])** An $\mathcal{M}G$-category is a normal pseudo $\mathcal{T}$-algebra for the 2-monad $\mathcal{T}$ on $\mathcal{G}\mathcal{C}AT$, $\mathcal{M}G$-functors are strong $\mathcal{T}$-morphisms, and $\mathcal{M}G$-natural transformations are algebra 2-cells of $\mathcal{T}$. These data form the 2-category $\mathcal{M}\mathcal{G}\mathcal{C}AT$ of $\mathcal{M}G$-categories.

There is a coherence result in [2] that any $\mathcal{M}G$-category is equivalent to a strict $\mathcal{M}G$-category (a strict $\mathcal{M}G$-category is just an internal strict $\omega$-category in $\mathcal{C}AT$). Because of this coherence theorem we will not mention explicitly the coherence isomorphisms in the $\mathcal{M}G$-categories which can be found in [2]. Also, the 2-category $\mathcal{M}\mathcal{G}\mathcal{C}AT$ has a cartesian monoidal structure, which allows us to make the following definition

**Definition 5 ([26])** An $\mathcal{A}M\mathcal{G}$-category is a pseudo monoid in $\mathcal{M}\mathcal{G}\mathcal{C}AT$. An $\mathcal{A}M\mathcal{G}$-functor is an $\mathcal{M}G$-functor $f : A \to A'$ equipped with a strong monoidal structure. An $\mathcal{A}M\mathcal{G}$-natural transformation $\phi : f \Rightarrow f'$ is an $\mathcal{M}G$-natural transformation such that $\phi$ is a monoidal 2-cell. These datas form the 2-category $\mathcal{A}\mathcal{M}\mathcal{G}\mathcal{C}AT$ of the $\mathcal{A}M\mathcal{G}$-categories.
1.2 Main examples of Monoidal Globular Categories

Globular categories can be defined also as internal categories in $\omega$-Gr because of the canonical isomorphism $\mathcal{C}at(\omega$-Gr) $\simeq [G^{op}, \mathcal{C}at]$. We will use this presentation to define the strict $MG$-category of $n$-trees as a discrete internal category

This $MG$-category $\text{Tree}$ has a canonical globular object given by the unit of $T : 1 \rightarrow \text{Tree}, 1(n) \rightarrow 1(n)$, where here 1 denote the terminal globular categories, where here $1(n)^1$ denotes the $n$-linear tree. It is shown in [3, 5] that it has the following universal property: If $C$ is an $MG$-category and $(C, W)$ is a globular object in it, then there is a unique, up to isomorphism, $MG$-functor $W(-)$ which makes commutative the following triangle

Let us set up the following notation: Tensors of the monoidal globular category of $n$-trees are denoted by symbols $\star^n_p$

$$\star^n_p : \text{Tree}_n \times \text{Tree}_n \rightarrow \text{Tree}_n$$

Also an $n$-tree $t$ can be degenerate if it is of the form $t = Z^n_k(t')$ where $t'$ is a $k$-tree such that $0 \leq k < n$. In [2] the author used the letter "$Z$" to express the reflexivity of an $MG$-category, but we prefer use the notation "$1$" to express these reflexivities for the specific case of $n$-trees,
to emphasis that a degenerate tree $t = 1^k_n(t')$ is also an $n$-cell of the strict
$\omega$-category $T(1)$. For example, for the $n$-linear tree $1(n)$, the $(n+1)$-tree
t $= 1^{n+1}_n(1(n))$ of $T(1)$ is degenerate.

Each $n$-tree $t$ has a unique decomposition

$$1^k_{i_1}(1(k_1)) \star_{i_1}^{\sup(i_1, i_2)} 1^k_{i_2}(1(k_2)) \star_{i_2}^{\sup(i_2, i_3)} ... \star_{i_{m-1}}^{\sup(i_{m-1}, i_m)} 1^k_{i_m}(1(k_m))$$

where for each $1 \leq j \leq m-1$, we have $i_j' < k_{j+1} \leq i_{j+1}$ and $i_j' < k_j \leq i_j$, and if $k_j = i_j$ by convention we put $1^k_{i_j}(1(k_j)) = 1(k_j)$. From this unique
decomposition, the $n$-tree $t$ can be represented by the matrix of numbers

$$
\begin{pmatrix}
i_1 & i_2 & \ldots & i_{m-1} & i_m \\
i_1' & \ldots & \ldots & \ldots & i_{m-1}'
\end{pmatrix}
$$

which we call the Grothendieck notation for the $n$-tree $t$ (see [1, 11, 21]).
Many authors gave their own approach to $n$-trees (see for instance [2, 6, 
9, 13, 18, 25]), and all these approaches are equivalent.

The second class of important examples of $MG$-category is given
by the Span and Cospan construction. For each $n \in \mathbb{N}$, consider the
following formal partialy ordered set $Oct(n)$
Let $Oct^+(n - 1)$ be the poset obtained from $Oct(n)$ by removing $\bullet^-_{n - 1}$ and $\bullet_0^+$, and similarly Let $Oct^-(n - 1)$ be the poset obtained from $Oct(n)$ by removing $\bullet^+_{n - 1}$ and $\bullet_0^-$. We obtain the following diagram in $\text{Cat}$

\[
\begin{array}{ccc}
Oct^+(n - 1) & \xrightarrow{\epsilon_n^-} & Oct(n) \\
\downarrow{i_n^+} & & \downarrow{i_n^-} \\
Oct(n - 1) & & Oct^-(n - 1)
\end{array}
\]

such that functors $i_n^+$, $i_n^-$ are just canonical inclusions, and the functors $\epsilon_n^+$ and $\epsilon_n^-$ are obvious isomorphisms. Put $s_{n - 1}^n = i_n^+ \circ \epsilon_n^+$ and $t_{n - 1}^n = i_n^- \circ \epsilon_n^-$. The family of functors $Oct(n - 1) \xrightarrow{s_{n - 1}^n} Oct(n) \ (n \geq 1)$, defines an object of $G\text{CAT}$. Furthermore, for any category $C \in \text{CAT}$, let us call the category of $n$-spans in $C$ the following category of presheaves in $C$: $Span_n(C) := \ldots$
The previous functors $s_{n-1}^n$ and $t_{n-1}^n$ induce a family of functors $\text{Span}_n(C) \xrightarrow{s_{n-1}^n} \text{Span}_{n-1}(C)$ (that we still note by $s_{n-1}^n$ and $t_{n-1}^n$ because there is no risk of confusion), which defines an object of $\mathbb{G}\text{CAT}$. Dually for any category $C \in \mathbb{C}\text{AT}$, let us call the category of presheaves $\text{Cospan}_n(C) := [\text{Oct}(n)^{op}; C]$, the category of $n$-cospans in $C$. The functors $s_{n-1}^n$ and $t_{n-1}^n$ between the $\text{Oct}(n)$, also induce a family of functors $\text{Cospan}_n(C) \xrightarrow{s_{n-1}^n} \text{Cospan}_{n-1}(C)$, which is still an object of $\mathbb{G}\text{CAT}$. These two constructions are functorial and define the Span and Cospan constructions

$$\mathbb{C}\text{AT} \xrightarrow{\text{Span}} \mathbb{G}\text{CAT} \xrightarrow{\text{Cospan}} \mathbb{G}\text{CAT}.$$  

The case of a category $C$ with pullbacks is more interesting for the span construction, because the corresponding globular category $\text{Span}(C)$ is canonically equipped with an $MG$-structure. We have a dual result for categories with pushouts and their cospans (see example 7 of section 3 in [2]). For example consider a category $C$ with pushouts and the two 2-cospans $x$ and $y$ in $C$ ($x$ is the diagram on the left).

The 1-cospans $s_1^2(x)$ and $t_1^2(y)$ are equal to the following 1-cospans.
and $x \otimes^2 y$ is given by the following 2-cospan in $C$.

\[
\begin{array}{ccc}
A_2 \sqcup A_0 & \xleftarrow{A_1^+} & B_2 \sqcup B_0 \\
\downarrow & & \downarrow \\
A_1^+ & \xleftarrow{B_1^+} & B_0
\end{array}
\]

Consider $\text{CAT}_{\text{Push}}$ the category of categories with pushouts and with morphisms functors which preserve these pushouts. Dually consider the underlying category of the 2-category $\text{CAT}_{\text{Pull}}$ that we have introduced the section 1.1. We have the following diagram

\[
\begin{array}{ccc}
\text{CAT}_{\text{Push}} & \xleftarrow{\text{Cospan}} & \text{GMCAT} \\
(\cdot)^\text{op} & & \\
\text{CAT}_{\text{Pull}} & \xleftarrow{\text{Span}} & 
\end{array}
\]

where $(\cdot)^\text{op}$ is the basic isomorphism of categories coming from duality and where the functors $\text{Cospan}$ and $\text{Span}$ are easily defined on morphisms by construction, just because functors preserving pushouts gives $MG$-functors between their categories of cospans.

**Remark 1** If $\text{CAT}_{\text{Pull}}^*$ denotes the category of categories with pullbacks and initial objects, and morphisms functors which preserve pullbacks and initial objects, and $\text{CAT}_{\text{Push}}^*$ denotes the category of categories with pushouts and initial objects, and morphisms functors which preserve
pushouts and initial objects, then we have the following constructions

\[
\begin{array}{ccc}
\mathsf{CAT}_{\text{Push}} & \xrightarrow{\text{Cospan}} & \mathsf{AMGCAT} \\
\downarrow \downarrow & & \downarrow \\
\mathsf{CAT}_{\text{Pull}} & \xrightarrow{\text{Span}} & \\
\end{array}
\]

Now consider a category \( C \) with pushouts and a globular object \((C, W)\) in \( \text{Cospan}(C) \), which is also a coglobular object in \( C \). Thanks to the universality of the map \( 1 \xrightarrow{} \text{Tree} \) above there exist a unique map

\[
W(-) : \text{Tree} \longrightarrow \text{Cospan}(C)
\]

This map \( W(-) \) sends each \( n \)-tree \( t \) to a \( n \)-coglobular object in \( C \):

\[
W(t) = (W^0 \xrightarrow{\kappa_0^1} W^{\partial_0^{n-1}}t \xrightarrow{\delta_1^{n-1}} W^{\partial_1^{n-2}}t : : : W^{\partial_n^{n-1}}t),
\]

where the \( \partial_k^t \) denotes the truncation of the \( n \)-tree \( t \) in the level \( k \) \((1 \leq k \leq n - 1)\). In this \( n \)-coglobular object \( W(t) \), \( W^t \) denotes the colimit in \( C \) of the diagram

\[
\begin{array}{cccc}
W^{i_1} & \xleftarrow{\kappa_{i_1}^1} & W^{i_2} & \xrightarrow{\kappa_{i_2}^1} \cdots \xrightarrow{\kappa_{i_m}^1} W^{i_{m-1}} & W^{i_m-1} \\
\end{array}
\]

coming from the Grothendieck presentation of the \( n \)-tree \( t \).

\( \text{Span} := \text{Span(Set)} \) is an important \( \mathsf{MG} \)-category. Examples of \( n \)-spans in \( \text{Set} \) are given by the \textit{HOM construction} : For each globular categories \( C \in \mathsf{GCAT} \), and for each pair of objects \( A, B \in C_n \), we
associate the following $n$-span $\text{HOM}(A, B)$ in $\text{Set}$

\[
\begin{array}{c}
\text{HOM}(A, B)_n \\
\downarrow_{s^n_{n-1}} \quad \downarrow_{t^n_{n-1}} \\
\text{HOM}(A, B)_{n-1} \\
\downarrow_{s^{n-1}_{n-2}} \quad \downarrow_{t^{n-1}_{n-2}} \\
\text{HOM}(A, B)_{n-2} \\
\downarrow_{s^1_1} \\
\text{HOM}(A, B)_1 \\
\downarrow_{s^1_0} \\
\text{HOM}(A, B)_0
\end{array}
\]

which is such that $\text{HOM}(A, B)_n := \text{hom}_{C_n}(A, B)$, and for all $0 \leq k < n$, $\text{HOM}(A, B)_k := \text{hom}_{C_k}(s^n_k(A), s^n_k(B))$, where $(s^k_{k+1})_{0 \leq k \leq n-1}$ and $(t^k_{k+1})_{0 \leq k \leq n-1}$, are given by the functor sources and functor targets of the globular category $C$.

Now consider a category $C$ with pushouts and a globular object $(C, W)$ in $\text{Cospan}(C)$. If $t$ is a $n$-tree we can associate between $W(1(n))$ and $W(t) \in \text{Cospan}(C)_n$ the $n$-span $\text{HOM}(W(1(n)), W(t))$, such that elements of the set $\text{HOM}(W(1(n)), W(t))_n$ are diagrams of the form
which commute serially, that is:

- \( f_n \delta_{n-1}^u = \delta_t f_{n-1}; \ f_n \kappa_{n-1}^u = \kappa_t f_{n-1} \)

- \( \forall 1 \leq k \leq n - 1, f_{n-(k-1)} \delta_{n-k}^u = \delta f_{n-k}; \ f_{n-(k-1)} \kappa_{n-k}^u = \kappa f_{n-k} \)

See also paragraph 9.2 in [18].

**Remark 2** Spans in sets can be seen in a conceptual way: In [25], Ross Street has shown that objects of \( \text{Span} \) are internal sets in the petit topos \( \omega \text{-Gr} \) of \( \omega \)-graphs, and in [27] Mark Weber has shown that \( \text{Span} \) is a discrete opfibration classifier in the 2-topos \( \mathbf{GCAT} \) of globular categories.
We can summarise many constructions of this section with the following diagram in 2-CAT

\[
\begin{array}{ccc}
\text{CAT}_{\text{push}} & \xrightarrow{\mathcal{MGCAT}} & \text{CAT} \\
\downarrow \mathcal{G} & \xrightarrow{j} & \downarrow \mathcal{G} \\
\text{CAT}_{\text{pull}} & \xrightarrow{i} & \text{GCAT} \\
\end{array}
\]

Recall that in this section we denote by 1 the terminal globular category, and in definition 1 we have denoted the globe category by \(\mathcal{G}\).

**Lemma 1** We have the following identifications

- (1 ↓ \(i\)) is the comma category of the globular objects \(1 \xrightarrow{W} \mathcal{C}\) such that \(\mathcal{C} \in \mathcal{MGCAT}\),

- (1 ↓ \(i \circ \text{Cospan}\)) is the comma category of the globular objects \(1 \xrightarrow{W} \text{Cospan}(\mathcal{C})\) such that \(\mathcal{C} \in \text{CAT}_{\text{push}}\),

- (\(\mathcal{G} \downarrow j\)) is the comma category of the globular objects \(\mathcal{G} \xrightarrow{W} \mathcal{C}\) in \(\mathcal{C}\) such that \(\mathcal{C} \in \text{CAT}_{\text{push}}\).

- We have the following isomorphisms of categories
  
  \[ (1 \downarrow i \circ \text{Cospan}) \xrightarrow{\sim} (\mathcal{G} \downarrow j) \quad (1 \downarrow i \circ \text{Span}) \xrightarrow{\sim} (\mathcal{G}^{\text{op}} \downarrow k) \]

\[\square\]
1.3 Digression on $T$-categories

Let us recall the approach to $\omega$-operads by Tom Leinster using $T$-categories\(^2\) (see his book \cite{18}). We recall the notions of $T$-graph and $T$-category which are also defined in \cite{16, 18}. Consider the bigcategory $\text{Span}(T)$ as defined in Leinster's book (see \cite{18}). A $T$-graph $(C, d, c)$ is a diagram of $\omega$-Gr such as

$$T(G) \xrightarrow{d} C \xrightarrow{c} G$$

$T$-graphs are endomorphisms of $\text{Span}(T)$ and they form a category $T$-$\text{Gr}$. If we fix $G \in \omega$-$\text{Gr}(0)$, the endomorphisms on $G$ in $\text{Span}(T)$ forms a subcategory of $T$-$\text{Gr}$ which is denoted $T$-$\text{Gr}_G$. The category $T$-$\text{Gr}_G$ is monoidal with tensor given by:

$$(C, d, c) \otimes (C', d', c') := (T(C) \times_{\mathcal{T}(G)} T(C'), \mu(G)T(d)\pi_0, c\pi_1),$$

and with unit given by $I(G) = (G, \eta(G), 1_G)$. The object $I(G)$ is also an identity morphism of $\text{Span}(T)$. The $\omega$-graph $G$ is called the $\omega$-graph of globular arities, or the $\omega$-graph of arities for short.

**Remark 3** A $p$-cell of $G$ is denoted by $g(p)$ and this notation has the following meaning: The symbol $g$ indicates the "colour", and the symbol $p$ point out that we must see $g(p)$ as a $p$-cell of $G$, because $G$ has to be seen as an $\omega$-graph even though it is just a set.

A $T$-graph $(C, d, c)$ equipped with a morphism $I(G) \xrightarrow{p} (C, d, c)$ is called a pointed $T$-graph. That means that one has a 2-cell $I(G) \xrightarrow{p} (C, d, c)$ of $\text{Span}(T)$ such that $dp = \eta(G)$ and $cp = 1_G$. A pointed $T$-graph is denoted $(C, d, c; p)$. We define in a natural way the category $T$-$\text{Gr}_p$ of pointed $T$-graphs, and also the category $T$-$\text{Gr}_{p,G}$ of $G$-pointed $T$-graphs: Their morphisms keep pointing in an obvious direction.

\(^2\)For an arbitrary cartesian monad $M$ on a category with pullbacks the notion of $M$-category were first suggested by Albert Burroni in 1971; see \cite{8}.
A constant $\omega$-graph is an $\omega$-graph $G$ such that $\forall n, m \in \mathbb{N}$ we have $G(n) = G(m)$ and such that source and target maps are identity. We write $\omega$-Gr$_c$ for the corresponding category of constant $\omega$-graphs. We write $T$-Gr$_c$ for the subcategory of $T$-Gr consisting of $\omega$-graphs of globular arity which are constant $\omega$-graphs, and $T$-Gr$_{p,c}$ for the subcategory of $T$-Gr$_p$ consisting of pointed $\omega$-graphs of globular arity which are constant $\omega$-graphs. Also for a given $G$ in $\omega$-Gr$_c$, we write $T$-Gr$_{p,c,G}$ for the fiber subcategory in $T$-Gr$_{p,c}$.

Definition 6 Consider a $T$-graphs $(C, d, c)$. If $k \geq 1$, two $k$-cells $x, y$ of $C$ are parallel if $s_{k-1}(x) = s_{k-1}(y)$ and if $t_{k-1}(x) = t_{k-1}(y)$. In that case we write $x \parallel y$.

A $T$-category is a monad in the bicategory $\text{Span}(T)$ or in an equivalent way a monoid of the monoidal category $T$-Gr$_G$ (for a specific $G$). The category of $T$-categories will denoted $T$-Cat, and that of $T$-categories over the same $\omega$-graph of globular arities $G$ is be denoted $T$-Cat$_G$. A $T$-category $(B, d, c; \gamma, p) \in T$-Cat is specifically given by the morphism of operadic composition $(B, d, c) \otimes (B, d, c) \xrightarrow{\gamma} (B, d, c)$ and the operadic unit $I(G) \xrightarrow{p} (B, d, c)$ satisfying axioms of associativity and unity that we can find in Leinster’s book [18]. Note that $(B, d, c; \gamma, p)$ has $(B, d, c; p)$ as natural underlying pointed $T$-graph. Algebras for a $T$-category are just algebras for its underlying monad.

1.4 $\omega$-Operads of Endomorphism and Coendomorphism

Let $C \in \mathcal{G}$CAT. Recall from [2] that the category of collections $\omega$-Coll$(C)$ in $C$ has as objects globular functors $\text{Tree} \xrightarrow{A} C$ and as morphisms, globular natural transformations between such globular func-
tors. It is straightforward to see that this defines a strict 2-functor 
\[ \text{Coll} := \text{Hom}_{\mathbf{GCAT}}(\text{Tree}, -) \]

\[ \mathbf{GCAT} \xrightarrow{\text{Coll}} \mathbf{CAT} \]

The theorem 6.1 in [2] gives criteria for finding many categories of collections with monoidal structure. Colimits commuting with the monoidal structure of an AMG-category is given in definition 5.3 in [2].

**Theorem 1** If \( C \) is an AMG-category such that colimits in \( C \) commute with its monoidal structure, then \( \omega \)-\text{Coll}(\( C \)) has a natural monoidal structure. \( \square \)

For our purpose the main example of such an AMG-category as in this theorem is \( \text{Span} \). The monoidal category \( \text{Coll}(\text{Span}) \) is equivalent to the monoidal category \( \text{T-Gr}_1 \) of \( \text{T} \)-graphs over the terminal \( \omega \)-graph \( 1 \) (see 1.3 and [18]). The category of monoids in \( \text{T-Gr}_1 \) is denoted \( \text{T-Cat}_1 \), and objects of this category are thus \( \omega \)-operads of Batanin in \( \text{Span} \). So in this article we see the \( \omega \)-operad \( K^3 \) of Batanin as a specific \( \text{T} \)-categories in \( \text{T-Cat}_1 \).

Now we are ready to express the main result of this section, which in fact is just a corollary of proposition 7.2 in [2].

**Corollary 1** For each object \( (C, W) \) in \( (\mathcal{G} \downarrow j) \) we can associate the \( \omega \)-operad \( \text{Coend}(W) \) of coendomorphisms, given by the following collection

\[ \text{Coend}(W) := (\text{HOM}(W^n, W^t))_{n \in \mathbb{N}, t \in \text{Tree}} \]

\( ^3 \)We prefer to denote it \( B^0_C \) to point out that we consider it as the first step of a sequence of \( \omega \)-operads : The 2-coloured \( \omega \)-operads \( B^n_C \) \( (n \geq 1) \) of the weak higher transformations which are object of the category \( \text{T-Cat}_{1+1} \) of the \( \text{T} \)-categories over the sum \( 1 \sqcup 1 \) of the terminal \( \omega \)-graph 1 with itself (see the section 3.6 and the article [16]). The letter "B" refer to the name "Batanin", and the subscript \( C \) means contractible.
Also for each morphism in \((G \downarrow j)\)
\[(C, W) \xrightarrow{f} (C', W')\]
we can associate a morphism of \(\omega\)-operads
\[\text{Coend}(W) \xrightarrow{\text{Coend}(f)} \text{Coend}(W')\]
Furthermore this construction is functorial; thus it defines a functor
\[(G \downarrow j) \xrightarrow{\text{Coend}} \mathbb{T}\text{-Cat}_1\]
Also for each object \((C, W)\) in \((G^{\text{op}} \downarrow k)\) we can associate the \(\omega\)-operad \(\text{End}(W)\) of endomorphisms, given by the following collection
\[\text{End}(W) := (\text{HOM}(W^t, W^n))_{n \in \mathbb{N}, t \in \text{Tree}}.\]
Also for each morphism in \((G^{\text{op}} \downarrow k)\)
\[(C, W) \xrightarrow{f} (C', W')\]
we can associate a morphism of \(\omega\)-operads
\[\text{End}(W) \xrightarrow{\text{End}(f)} \text{End}(W')\]
Furthermore this construction is functorial, thus it defines a functor
\[(G^{\text{op}} \downarrow k) \xrightarrow{\text{End}} \mathbb{T}\text{-Cat}_1\]
\[\]
**Proposition 1** If \(W \in (G^{\text{op}} \downarrow k)\) then \(\text{End}(W) \xrightarrow{\sim} \text{Coend}(W^{\text{op}})\) in \(\mathbb{T}\text{-Cat}_1\).
\[\]
**Definition 7** If \(B \in \mathbb{T}\text{-Cat}_1\) then an algebra for \(B\) in the sense of Batanin is given by a morphism in \(\mathbb{T}\text{-Cat}_1\)
\[B \xrightarrow{} \text{End}(W)\]
where \(W : G^{\text{op}} \rightarrow \text{Set}\) is an object of \(\omega\)-Gr.
\[\]
**Proposition 2** ([18]) If \(B \in \mathbb{T}\text{-Cat}_1\), then an algebra for \(B\) in the sense of Batanin, and an algebra for \(B\) in the sense of Leinster (see the section 1.3) coincide.
2 Standard actions associated to a coglobular complex in $T\text{-Cat}_c$

A $T$-category over any $\omega$-graph can be seen as a coloured $\omega$-operad (see [16, 18]), and the category $T\text{-Cat}$ of coloured $\omega$-operads is locally presentable, thus it is a category with pushouts. However it is in the context of the locally presentable category $T\text{-Cat}_c$ of $T$-categories over constant $\omega$-graphs (see paragraph 3.1 and the article [16]), that we are going to build the standard actions associated to a coglobular complex in $T\text{-Cat}_c$. This concept arises as an application of the previous section to the category $T\text{-Cat}_c$. Consider the following diagram in $\mathcal{CAT}_{push}$

$$
\begin{array}{ccc}
T\text{-Cat}_c & \xrightarrow{\text{Alg}(\cdot)} & \text{CAT}^{op} & \xrightarrow{\text{Ob}(\cdot)} & \text{SET}^{op} \\
\end{array}
$$

For each coglobular object $(T\text{-CAT}_c, W)$ in $T\text{-CAT}_c$, we have the following diagram in $(G \downarrow j)$

$$
\begin{array}{ccc}
&T\text{-Cat}_c & \xrightarrow{\text{Alg}(\cdot)} & \text{CAT}^{op} & \xrightarrow{\text{Ob}(\cdot)} & \text{SET}^{op} \\
W & \xrightarrow{A^{op}} & \xrightarrow{A^{op}_{0}} & \xrightarrow{\text{End}(A_{0})} & \\
\end{array}
$$

If we apply to this diagram the functor $\text{Coend}$ of corollary 1 and if we use proposition 1, we obtain the following diagram in $T\text{-Cat}_1$

$$
\begin{array}{ccc}
\text{Coend}(W) & \xrightarrow{\text{Coend}(\text{Alg}(\cdot))} & \text{Coend}(A^{op}) & \xrightarrow{\text{Coend}(\text{Ob}(\cdot))} & \text{End}(A_{0}) \\
\end{array}
$$

that we call the standard action in $T\text{-Cat}_1$ associated to the coglobular object $(T\text{-CAT}_c, W) \in (G \downarrow j)$ in $T\text{-CAT}_c$.

Now we are ready to explain the philosophy of the standard action associated to a coglobular complex in $T\text{-Cat}_1$ : The category $T\text{-Cat}_c$ is locally finitely presentable and the forgetful functor

$$
\begin{array}{ccc}
T\text{-Cat}_c & \xrightarrow{V} & T\text{-Gr}_{p,c} \\
\end{array}
$$
is monadic (see [18]), thus according to the proposition 5.5.6 of [7], \( V \) has rank. Let us call \( M \) its left adjoint.

Now consider a category \( PT\text{-}Cat_c \) of \( \omega \)-operads equipped with a structure that we call "\( P \)" such that it is locally finitely presentable, and equipped with a monadic forgetful functor \( U_P : PT\text{-}Cat_c \rightarrow T\text{-}Cat_c \). Various concrete choices for \( P \) will be considered later in this paper. We denote by \( F_P \) the left adjoint to \( U_P \)

\[
\begin{array}{ccc}
PT\text{-}Cat_c & \xrightarrow{U_P} & T\text{-}Cat_c \\
& \xleftarrow{F_P} &
\end{array}
\]

Thus we are in a situation where \( V \circ U_P \) is monadic and the induced monad \( T_P \) on \( T\text{-}Gr_{p,c} \) has rank. Also we get the functor

\[
P := F_P \circ M : T\text{-}Gr_{p,c} \rightarrow PT\text{-}Cat_c
\]

which assigns the free \( PT \)-categories on pointed \( T \)-graphs.

Consider also the following coglobular complex in \( T\text{-}Gr_{p,c} \), that we call the coglobular complex for the higher transformations in \( T\text{-}Gr_{p,c} \), because it is built with a combinatoric that we need for higher transformations (see [16]):

\[
\begin{array}{cccccccc}
C^0 & \xrightarrow{\delta_1^0} & C^1 & \xrightarrow{\delta_1^1} & C^2 & \cdots & C^{m-1} & \xrightarrow{\delta_{n-1}^n} & C^m \\
& \xleftarrow{\kappa_1^0} & & \xleftarrow{\kappa_1^1} & & \cdots & & \xleftarrow{\kappa_{n-1}^n} &
\end{array}
\]

Let us recall the combinatorics involved in this coglobular complex. Pointings \( p \) of each collection involved in this specific coglobular complex are denoted with the symbol \( \lambda : C^0 \) is Batanin’s system of composition, i.e. there is the collection \( T(1) \xleftarrow{\delta^0} C^0 \xrightarrow{\delta^0} 1 \) where \( C^0 \) precisely contains the symbols of the compositions of the \( \omega \)-categories \( \mu_p^m \in C^0(m)(0 \leq p < m) \), plus the operadic unary symbols \( u_m \in C^0(m) \). More specifically:
\(\forall m \in \mathbb{N}, C^0\) contains the \(m\)-cell \(u_m\) such that: \(s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}\) (if \(m \geq 1\)); \(d^0(u_m) = 1(m)(= \eta(1 \cup 2)(1(m)))\), \(c^0(u_m) = 1(m)\).

\(\forall m \in \mathbb{N} - \{0, 1\}, \forall p \in \mathbb{N},\) such that \(m > p\), \(C^0\) contains the \(m\)-cell \(\mu_p^m\) such that: If \(p = m - 1\), \(s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = u_{m-1}\). If \(0 \leq p < m - 1\), \(s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}\). Also \(d^0(\mu_p^m) = 1(m) \ast_p^m 1(m)\), and inevitably \(c^0(\mu_p^m) = 1(m)\).

Furthermore \(C^0\) contains the 1-cell \(\mu_0^1\) such that \(s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0\), \(d^0(\mu_0^1) = 1(1) \ast_0^1 1(1)\), also inevitably \(c^0(\mu_0^1) = 1(1)\).

The system of composition \(C^0\) has got a well-known pointing \(\lambda^0\) which is defined by: \(\forall m \in \mathbb{N}, \lambda^0(1(m)) = u_m\).

Firstly we will define a collection \((C, d, c)\) which will be useful to build the collections of \(n\)-transformations \((n \in \mathbb{N}^*)\). \(C\) contains two copies of the symbols of \(C^0\), each having a distinct colour: The symbols formed with the letters \(\mu\) and \(u\) are those of colour 1, and those formed with the letters \(\nu\) and \(v\) are those of colour 2. Let us be more precise:

\(\forall m \in \mathbb{N}, C\) contains the \(m\)-cell \(u_m\) such that: \(s_{m-1}^m(u_m) = t_{m-1}^m(u_m) = u_{m-1}\) (if \(m \geq 1\)) and \(d(u_m) = 1(m)\), \(c(u_m) = 1(m)\).

\(\forall m \in \mathbb{N} - \{0, 1\}, \forall p \in \mathbb{N},\) such that \(m > p\), \(C\) contains the \(m\)-cell \(\mu_p^m\) such that: If \(p = m - 1\), \(s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = u_{m-1}\). If \(0 \leq p < m - 1\), \(s_{m-1}^m(\mu_p^m) = t_{m-1}^m(\mu_p^m) = \mu_p^{m-1}\). Also \(d(\mu_p^m) = 1(m) \ast_p^m 1(m)\), \(c(\mu_p^m) = 1(m)\).

Furthermore \(C\) contains the 1-cell \(\mu_0^1\) such that \(s_0^1(\mu_0^1) = t_0^1(\mu_0^1) = u_0\) and \(d(\mu_0^1) = 1(1) \ast_0^1 1(1)\), \(c(\mu_0^1) = 1(1)\).

Besides, \(\forall m \in \mathbb{N}, C\) contains the \(m\)-cell \(v_m\) such that: \(s_{m-1}^m(v_m) = t_{m-1}^m(v_m) = v_{m-1}\) (if \(m \geq 1\)) and \(d(v_m) = 2(m)\), \(c(v_m) = 2(m)\).
\( \forall m \in \mathbb{N} - \{0, 1\}, \forall p \in \mathbb{N}, \) such that \( m > p, \) \( C \) contains the \( m \)-cell \( \nu^m_p \) such that: If \( p = m - 1, \) \( s^m_{m-1}(\nu^m_{m-1}) = t^m_{m-1}(\nu^m_{m-1}) = v_{m-1}. \)

If \( 0 \leq p < m - 1, \) \( s^m_{m-1}(\nu^m_p) = t^m_{m-1}(\nu^m_p) = \nu^{m-1}_p. \) Also \( d(\nu^m_p) = 2(m) \ast^m_p 2(m), c(\nu^m_p) = 2(m). \)

Furthermore \( C \) contains the 1-cell \( \nu^1_0 \) such that \( s^1_0(\nu^1_0) = t^1_0(\nu^1_0) = v_0 \)
and \( d(\nu^1_0) = 2(1) \ast^1_0 2(1), c(\nu^1_0) = 2(1). \)

\( C^1 \) is the system of operations of \( \omega \)-functors. It is built on the basis of \( C \) adding to it a single symbol of functor (for each cell level): \( \forall m \in \mathbb{N} \) the \( F^m \) \( m \)-cell is added, which is such that: If \( m \geq 1, \) \( s^m_{m-1}(F^m) = t^m_{m-1}(F^m) = F^{m-1}. \) Also \( d^1(F^m) = 1(m) \) and \( c^1(F^m) = 2(m). \)

\( C^2 \) is the system of operations of the natural \( \omega \)-transformations. \( C^2 \) is built on \( C, \) adding to it two symbols of functor (for each cell level) and a symbol of natural transformation. More precisely \( \forall m \in \mathbb{N} \) we add the \( m \)-cell \( F^m \) such that: If \( m \geq 1, \) \( s^m_{m-1}(F^m) = t^m_{m-1}(F^m) = F^{m-1}. \) Also \( d^2(F^m) = 1(m) \) and \( c^2(F^m) = 2(m). \)

Then \( \forall m \in \mathbb{N} \) we add the \( m \)-cell \( H^m \) such that: If \( m \geq 1, \) \( s^m_{m-1}(H^m) = t^m_{m-1}(H^m) = H^{m-1}. \) Also \( d^2(H^m) = 1(m) \) and \( c^2(H^m) = 2(m). \)

And finally we add 1-cell \( \tau \) such that: \( s^1_0(\tau) = F^0 \) and \( t^1_0(\tau) = H^0. \)
Also \( d^2(\tau) = 1_{1(0)} \) and \( c^2(\tau) = 2(1). \)

We can point out that the 2-coloured collections \( C^i \) \( (i = 1, 2) \) are naturally equipped with a pointing \( \lambda^i \) defined by \( \lambda^i(1(m)) = u_m \) and \( \lambda^i(2(m)) = v_m. \)

In order to define the general theory of the \( n \)-transformations \( (n \in \mathbb{N}^+) \), it is necessary to define the systems of operations \( C^n \) for the superior \( n \)-transformations \( (n \geq 3). \) This paragraph can be left out in the first reading. Each collection \( C^n \) is built on \( C, \) adding to it the required cells. They contain four large groups of cells: The symbols of
source and target \(\omega\)-categories, the symbols of operadic units (obtained on the basis of \(C\)), the symbols of the \(\omega\)-functors (sources and targets), and the symbols of the \(n\)-transformations (natural \(\omega\)-transformations, \(\omega\)-modification, etc). More precisely, on the basis of \(C\):

Symbols of the \(\omega\)-Functors \(\forall m \in \mathbb{N}, C^n\) contains the \(m\)-cells \(\alpha_0^m\) and \(\beta_0^m\) such as: If \(m \geq 1\), \(s_{m-1}^m(\alpha_0^m) = t_{m-1}^m(\alpha_0^m) = \alpha_0^{m-1}\) and \(s_{m-1}^m(\beta_0^m) = t_{m-1}^m(\beta_0^m) = \beta_0^{m-1}\). Furthermore \(d^n(\alpha_0^m) = d^n(\beta_0^m) = 1(m)\) and \(c^n(\alpha_0^m) = c^n(\beta_0^m) = 2(m)\).

Symbols of the Higher \(n\)-Transformations \(\forall p\), with \(1 \leq p \leq n-1\), \(C^n\) contains the \(p\)-cells \(\alpha_p\) and \(\beta_p\) which are such as: \(\forall p\), with \(2 \leq p \leq n-1\), \(s^p_{p-1}(\alpha_p) = s^p_{p-1}(\beta_p) = \alpha_{p-1}\) and \(t^p_{p-1}(\alpha_p) = t^p_{p-1}(\beta_p) = \beta_{p-1}\). If \(p = 1\), \(s^0_0(\alpha_1) = s^0_0(\beta_1) = \alpha_0^0\) and \(t^0_0(\alpha_1) = t^0_0(\beta_1) = \beta_0^0\). What’s more, \(\forall p\), with \(1 \leq p \leq n-1\), \(d^n(\alpha_p) = d^n(\beta_p) = 1^p_0(1(0))\) and \(c^n(\alpha_p) = c^n(\beta_p) = 2(p)\). Finally \(C^n\) contains the \(n\)-cell \(\xi_n\) such that \(s^n_{n-1}(\xi_n) = \alpha_{n-1}\), \(t^n_{n-1}(\xi_n) = \beta_{n-1}\) and \(d^n(\xi_n) = 1^n_0(1(0))\) and \(c^n(\xi_n) = 2(n)\).

We can see that \(\forall n \in \mathbb{N}^+, \) the 2-colored collection \(C^n\) is naturally equipped with the pointing \(1 \cup 2 \xrightarrow{\lambda^n} (C^n, d, c)\) defined as:

\[\forall m \in \mathbb{N}, \lambda^n(1(m)) = u_m\] and \(\lambda^n(2(m)) = v_m\).

The set \(\{C^n/n \in \mathbb{N}\}\) has a canonical structure of coglobular complex. This coglobular complex is generated by diagrams

\[\xymatrix{ C^n \ar@{=>}[rr]^{c^n_{n+1}} & & C^{n+1} \ar@{=>}[ll]_{s^n_{n+1}} }\]

of pointed 2-coloured collections. For \(n \geq 2\), these diagrams are defined as follows: First the \((n+1)\)-colored collection contains the same symbols of operations as \(C^n\) for the \(j\)-cells, \(0 \leq j \leq n-1\) or \(n+2 \leq j < \omega\). For the \(n\)-cells and the \((n+1)\)-cells the symbols of operations will change.
$C^n$ contains the $n$-cell $\xi_n$ whereas $C^{n+1}$ contains the $n$-cells $\alpha_n$ and $\beta_n$; in addition contains the $(n+1)$-cell $\xi_{n+1}$. If one denotes by $C^n - \xi_n$ the $n$-coloured collection obtained on the basis of $C^n$ by taking from it the $n$-cell $\xi_n$, then $\delta_{n+1}^n$ is defined as follows: $\delta_{n+1}^n|_{C^n - \xi_n}$ (i.e the restriction of $\delta_{n+1}^n$ to $C^n - \xi_n$) is the canonical injection $C^n - \xi_n \hookrightarrow C^{n+1}$ and $\delta_{n+1}^n(\xi_n) = \alpha_n$. In a similar way $\kappa_{n+1}^n$ is defined as follows: $\kappa_{n+1}^n|_{C^n - \xi_n} = \delta_{n+1}^n|_{C^n - \xi_n}$ and $\kappa_{n+1}^n(\xi_n) = \beta_n$. We can notice that $\delta_{n+1}^n$ and $\kappa_{n+1}^n$ keeps pointing, i.e we have for all $n \geq 1$ the equalities $\delta_{n+1}^n \alpha = \lambda^{n+1}$ and $\kappa_{n+1}^n \lambda = \lambda^{n+1}$.

The morphisms of 2-colored pointing collections of the diagram

$$C^0 \xrightarrow{\delta_0^0} C^1 \xrightarrow{\delta_1^0} C^2 \xrightarrow{\delta_2^0} C^3$$

have a similar definition:

We have for all integers $0 \leq p < n$ and for all $\forall m \in \mathbb{N}$:

$\delta_0^0(\mu_p^n) = \mu_p^n; \quad \delta_0^0(u_m) = u_m; \quad \kappa_0^0(\mu_p^n) = \nu_p^n; \quad \kappa_0^0(u_m) = v_m.$

Also: $\delta_1^1(\mu_p^n) = \mu_p^n; \quad \delta_1^1(u_m) = u_m; \quad \delta_1^1(\nu_p^n) = \nu_p^n; \quad \delta_2^1(v_m) = v_m; \quad \delta_2^1(F^n) = F^m$. And $\kappa_1^1(\mu_p^n) = \mu_p^n; \quad \kappa_1^1(u_m) = u_m; \quad \kappa_1^1(\nu_p^n) = \nu_p^n; \quad \kappa_2^1(v_m) = v_m; \quad \kappa_2^1(F^m) = H^m.$

Finally: $\delta_3^2(\mu_p^n) = \mu_p^n; \quad \delta_3^2(u_m) = u_m; \quad \delta_3^2(\nu_p^n) = \nu_p^n; \quad \delta_3^2(v_m) = v_m; \quad \delta_3^2(F^m) = \alpha_0^m; \quad \delta_3^2(H^m) = \beta_0^m; \quad \delta_3^2(\tau) = \alpha_0^1. \quad \text{And} \quad \kappa_3^2(\mu_p^n) = \mu_p^n; \quad \kappa_3^2(u_m) = u_m; \quad \kappa_3^2(\nu_p^n) = \nu_p^n; \quad \kappa_3^2(v_m) = v_m; \quad \kappa_3^2(F^m) = \alpha_0^m; \quad \kappa_3^2(H^m) = \beta_0^m; \quad \kappa_3^2(\tau) = \beta_1.$

The pointed 2-coloured collections $C^n$ ($n \in \mathbb{N}^*$) are the systems of operations of the $n$-transformations.

If we apply the functor $P$ to this coglobular complex we obtain a coglobular complex in $\mathbf{PT}$-Cat$_c$

$$B_p^0 \xrightarrow{\delta_0^n} B_p^1 \xrightarrow{\delta_1^n} B_p^2 \xrightarrow{\delta_2^n} B_p^{n+1} \xrightarrow{\delta_{n+1}^n} B_p^n$$
which is also, when we forget its structure "$P$", a coglobular object $W = B_p^\bullet$ of $\mathcal{T}$-$\mathcal{C}at_c$, and thus we obtain its resulting standard action

$$\text{Coend}(B_p^\bullet) \xrightarrow{\text{Coend}([\text{Alg}(\cdot)])} \text{Coend}(A_p^{\text{op}}) \xrightarrow{\text{Coend}(\text{Ob}(\cdot))} \text{End}(A_{0,p})$$

where in particular $\text{Coend}(B_p^\bullet)$ is the monochromatic $\omega$-operad of coendomorphism associated to this coglobular complex. This kind of standard actions is called a standard action for higher transformations because it is built with the coglobular complex of the higher transformations $C^\bullet$ in $\mathcal{T}$-$\mathcal{G}r_{p,c}$.

The main problem of our philosophy, is to build a morphism of $\omega$-operads between the monochromatic $\omega$-operad $B_p^0$ (the "0-step" of the coglobular object $B_p^\bullet$) and the monochromatic $\omega$-operad $\text{Coend}(B_p^\bullet)$ (built with the whole coglobular object $B_p^\bullet$). If such a morphism exists then we have a morphism of operads

$$B_p^0 \longrightarrow \text{End}(A_{0,p})$$

which shows that $B_p^0$-algebras and all its higher transformations form a $B_p^0$-algebra. In this case we say that $B_p^0$ has the fractal property.

### 3 Contractibility Hypotheses

In this paragraph we will consider two cases: When $P$ is strict with contractible units (indicated with the letters $S_u$), and when $P$ is contractible (indicated with the letter $C$). We will state the hypotheses that the $\omega$-operad $B_{S_u}^0$ of strict $\omega$-categories, and the $\omega$-operad $B_C^0$ of the weak $\omega$-categories, have the fractal property.

In the section 4 we give two examples of other higher structure such that it is possible to prove that their associated $\omega$-operads $B_p^0$ have the fractal property.
3.1 The functor of the contractible units

In this paragraph we are going to build the functor of the contractible units for $T$-categories (see 1.3). But first we must define the pointed $T$-graphs with contractible units which are for the pointed $T$-graphs what the reflexive $\omega$-graphs are for the $\omega$-graphs. In order to define it we are going first to define an intermediary structure on $T$-graphs. Consider a $T$-graph $(C, d, c)$, and for each $n \in \mathbb{N}$ we write $C(n)$ for the set of $n$-cells of the $T$-graph $(C, d, c)$. Note that $G$ is also equipped with a trivial reflexivity structure $(G, (1^n_p)_{0 \leq p < n})$ where the operations $1^n_p$ are defined by $1^n_p(g(p)) = g(n)$, and which force $c$ to be a morphism of reflexive $\omega$-graphs as well.

**Definition 8** We say that the $T$-graph $(C, d, c)$ is equipped with a reflexive structure, if its underlying $\omega$-graph $C$ is equipped with a reflexive structure in the usual sense, such that $d$ is a morphism of reflexive $\omega$-graphs.

We denote $(C, d, c; (1^n_p)_{0 \leq p < n})$ a reflexive $T$-graph where the operations $1^n_p$ are those of $C$. A morphism between two reflexive $T$-graphs are just morphism of $T$-graphs which preserve reflexivity, and the category of reflexive $T$-graphs over constant $\omega$-graphs is denoted by $T$-$\text{Gr}_r$.

A pointed $T$-graph $(C, d, c; p)$ over a constant $\omega$-graph $G$ has contractible units if it is equipped with a monomorphism $\mathbb{R}(G) \rightarrowtail C$ such that $p$ factorises as follow:

\[
\begin{array}{ccc}
\mathbb{T}(G) & \overset{d}{\rightarrow} & \mathbb{R}(G) \\
\downarrow{\eta(G)} & & \downarrow{\eta(G)} \\
G & \overset{id}{\rightarrow} & G
\end{array}
\]

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such that $p = i\eta(G)$, and such that the induced $T$-graph:

$$
\begin{align*}
T(G) & \xrightarrow{d} R(G) \xrightarrow{c} G
\end{align*}
$$

is reflexive, i.e the restriction of $d$ on $R(G)$ is a morphism of reflexive $\omega$-graphs. We let $(C, d, c; p, i, (1^p_n)_{0 \leq p < n})$ be a pointed $T$-graph with contractible units. Morphisms of pointed $T$-graphs with contractible units

$$(C, d, c; p, i, (1^p_n)_{0 \leq p < n}) \xrightarrow{(f, h)} (C', d', c'; p', i', (1^p_n)_{0 \leq p < n})$$

is given by morphisms of pointed $T$-graphs (see [18])

$$(C, d, c; p) \xrightarrow{(f, h)} (C', d', c'; p')$$

such that $fi = i'\eta(h)$. The category of pointed $T$-graphs with contractible units is denoted by $Id_{u}T\text{-Gr}_{p,c}$, and a $T$-category has contractible units if its underlying pointed $T$-graph lies in $Id_{u}T\text{-Gr}_{p,c}$. Morphisms between two $T$-categories equipped with contractible units are just morphisms of $T$-categories which preserve contractible units. Let us write $Id_{u}T\text{-Cat}_{c}$ this category. It is a locally presentable category, and also we can prove that the forgetful functor is monadic.

$UT\text{-Cat}_{c} \xrightarrow{U_{Id_{u}}} T\text{-Cat}_{c}$

Let us note $F_{Id_{u}} \dashv U_{Id_{u}}$ this adjunction. In particular we get the functor of the free $T$-categories with contractible units $Id_{u} : T\text{-Gr}_{p,c} \longrightarrow UT\text{-Cat}_{c}$ which is a left adjoint and which monad $T_{Id_{u}}$ has rank.

### 3.2 The functors of strictification and the functors of contractibility

In [16] we defined a coglobular complex of $\omega$-operads
such that algebras for $B^0$ are the weak $\omega$-categories, algebras for $B^1$ are the weak $\omega$-functors, algebras for $B^2$ are the weak $\omega$-natural transformations, etc. However André Joyal has pointed out to us that there are too many coherence cells for each $B^n$ when $n \geq 2$, and gave us a simple example of a natural transformation which cannot be an algebra for the 2-coloured $\omega$-operad $B^2$.

We are going to propose a notion of contractibility, slightly different from those used in [2, 16]. This new approach excludes the counterexample of André Joyal, but also shows that with a strict version of this corrected version, algebras for the specific strict $\omega$-operads $B^0_{S_u}$ with contractible units ($n \geq 1$) of the higher transformations that we propose (see section 3.5), follow the axioms of the strict higher transformations.

Let us speak now about our own intuition about this new notion of contractibility that we propose: Roughly speaking, it deals with notions of root cells plus the loop condition. It is first based on a basic observation about the contractibility of the Batanin’s operad $B^0_C$: The pairs of cells in $B^0_C$, say $(x, y) \in B^0_C(n) \times B^0_C(n)$, which are parallels and have same arity must be connected by a coherence cells, but notice that they also have the following extra property: $s^n_0(x) = s^n_0(y) = t^n_0(x) = t^n_0(y)$.

By extra we mean that this is extra over the usual definition of contractibility in the sense of Batanin, where normalised $\omega$-operads (which by definition follow this extra condition; see [2]), are not explicitly considered in his approach. Let us call this property the loop property.

**Definition 9** For any $T$-graph $(C, d, c)$ over a constant $\omega$-graph $G$, a pair of cells $(x, y)$ of $C(n)$ has the the loop property if: $s^n_0(x) = s^n_0(y) = t^n_0(x) = t^n_0(y)$. \[\square\]
Secondly it concerns the correct $\omega$-operads $B^i_C(i \in \mathbb{N}^*)$ of the weak higher transformations that we are looking for: In our approach these $\omega$-operads should be freely generated by the $T$-graphs $C^i(i \in \mathbb{N}^*)$ of the weak higher transformations (see section 2). We observe the important fact that all the symbols specific for the higher transformations in it, have arities which are the reflexivity of $1(0)$, where $1(0)$ denotes the unique 0-cell of colour 1 of $T(1+2)$, i.e $\forall n \geq 1$, those cells $x \in C(n)$ specific for the higher transformations are such that $d(x) = 1^0_n(1(0))$. Let us call these kind of cells the root cells.

**Definition 10** For any $T$-graph $(C, d, c)$ over a constant $\omega$-graph $G$, we call the root cells of $(C, d, c)$, those cells whose arities are the reflexivity of a 0-cell $g(0)$ of $G$, where here "g" indicates the colour (see section 3), or in other words, those cells $x \in C(n)$ ($n \geq 1$) such that $d(x) = 1^0_n(g(0))$.

These notions of root cells and loop condition are the keys for our new approach to contractibility. These observations motivate us to put the following definition of what should be a contractible $T$-graphs $(C, d, c)$.

For each integers $k \geq 1$ let us note $\tilde{C}(k) = \{(x, y) \in C(k) \times C(k) : x \parallel y$ and $d(x) = d(y)$, and if also $(x, y)$ is a pair of root cells then they also need to verify the loop property: $s^k(x) = t^k(y)\}$. Also we denote $\tilde{C}(0) = \{(x, x) \in C(0) \times C(0)\}$.

**Definition 11** A contraction on the $T$-graph $(C, d, c)$, is the datum, for all $k \in \mathbb{N}$, of a map $\tilde{C}(k) \xrightarrow{[\_, \_]} C(k + 1)$ such that

- $s([\alpha, \beta]_k) = \alpha, t([\alpha, \beta]_k) = \beta,$
- $d([\alpha, \beta]_k) = 1_{d(\alpha) = d(\beta)}.$

A $T$-graph which is equipped with a contraction will be called contractible and we use the notation $(C, d, c; ([\_, \_]_k)_{k \in \mathbb{N}})$ for a contractible
Nothing prevents a contractible \( T \)-graph from being equipped with several contractions. So here \( CT\text{-Gr}_c \) is the category of the contractible \( T \)-graphs equipped with a specific contraction, and morphisms of this category preserves the contractions. One can also refer to the category \( CT\text{-Gr}_{c,G} \), where here contractible \( T \)-graphs are only taken over a specific constant \( \infty \)-graph \( G \). A pointed contractible \( T \)-graphs (see section 1.3) is denoted \( (C, d, c; p, (\cdot)_k)_{k \in \mathbb{N}} \), and morphisms between two pointed contractible \( T \)-graphs preserve contractibilities and pointings. The category of pointed contractible \( T \)-graphs is denoted \( CT\text{-Gr}_{p,c} \). Objects of the category \( CT\text{-Gr}_{p,c} \) are examples of \( T \)-graphs equipped with contractible units (see section 3.1), and \( CT\text{-Gr}_{p,c} \) is a subcategory of \( Id_u T\text{-Gr}_{p,c} \). A \( T \)-category is contractible if its underlying pointed \( T \)-graph lies in \( CT\text{-Gr}_{p,c} \). Morphisms between two contractible \( T \)-categories are morphisms of \( T \)-categories which preserve contractibilities. Let us write \( CT\text{-Cat}_c \) for the category of contractible \( T \)-categories.

**Remark 4** It is evident that the \( \omega \)-operad \( B^0_C \) of Michael Batanin is still initial in the category of contractible \( \omega \)-operads equipped with a composition system, where our new approach of contractibility is considered.

\( CT\text{-Cat}_c \) is a locally presentable category, and also we can prove that the forgetful functor is monadic.

\[
CT\text{-Cat}_c \xrightarrow{U_C} T\text{-Cat}_c
\]

Let us write \( F_C \dashv U_C \) this adjunction. In particular we get the functor of the free contractible \( T \)-categories \( C : T\text{-Gr}_{p,c} \longrightarrow CT\text{-Cat}_c \), which is a left adjoint and whose monad \( T_C \) has rank.

Now let us explain the notion of strict contractibility on a \( T \)-graph \( (C, d, c) \). We say that \( (C, d, c) \) is strictly contractible if for each integer
$k \in \mathbb{N}$ and each $(x, y) \in \hat{C}(k)$ we have $x = y$. This notion of strict contractibility plus the notion of $T$-graphs equipped with contractible units (see section 3.1), is going to be used to define a notion of $\omega$-operads for strict higher transformations (see section 3.5). It is evident that a morphism in $T\text{-Gr}_c$ between two strictly contractible $T$-graphs preserves strict contractibility. So we write $ST\text{-Gr}_c$ for the category of strictly contractible $T$-graphs as a full subcategory of $T\text{-Gr}_c$. The definition of pointed strictly contractible $T$-graphs is obvious: It is just the strictly contractible $T$-graphs equipped with pointings. Morphisms between two pointed strictly contractible $T$-graphs are morphisms of $T\text{-Gr}_c$ which preserve pointings. We write $ST\text{-Gr}_{p,c}$ the category of pointed strictly contractible $T$-graphs. A $T$-category is strictly contractible if its underlying pointed $T$-graph is strictly contractible. Morphisms between two strictly contractible $T$-categories are morphisms of $T$-categories which preserve the strict contractibilities. Let us write $ST\text{-Cat}_c$ for the category of strictly contractible $T$-categories. It is a locally presentable category, and also we can prove that the forgetful functor is monadic.

\[
\begin{array}{ccc}
ST\text{-Cat}_c & \overset{U_S}{\longrightarrow} & T\text{-Cat}_c \\
& F_S \dashv & \\
\end{array}
\]

Denote this adjunction by $F_S \dashv U_S$. In particular we get the functor $S$ for the free strictly contractible $T$-categories

\[
S : T\text{-Gr}_{p,c} \longrightarrow ST\text{-Cat}_c
\]

which is a left adjoint and whose monad $T_S$ has rank.

Two kind of monads on $T\text{-Gr}_{p,c}$ are relevant for us: The monad $T_C$ and the coproduct of monads $T_{S_u} := T_{Id_{u}} \coprod T_S$ which also has rank, and whose category of algebras is denoted $S_u T\text{-Cat}_c$. In particular the monad $T_{S_u}$ gives the functor $S_u$ of free strict $T$-categories equipped with contractible units

\[
S_u : T\text{-Gr}_{p,c} \longrightarrow S_u T\text{-Cat}_c
\]
The functor $S_u$ is used to build the coglobular complex of the $\omega$-operads of the higher transformations for strict $\omega$-categories (see 3.5). The functor $C$ is used to build the coglobular complex of the $\omega$-operads of the higher transformations for weak $\omega$-categories (see 3.6).

### 3.3 Contractibility Hypotheses

By using functors of the previous section $S_u : T\text{-Gr}_{p,c} \longrightarrow S_u T\text{-Cat}_c$ and $C : T\text{-Gr}_{p,c} \longrightarrow C T\text{-Cat}_c$ with the coglobular complexes of the higher transformations $C^\bullet$ in $T\text{-Gr}_{p,c}$ we get two coglobular complex in $T\text{-Cat}_c$:

\[
\begin{array}{cccc}
B^0_S & \xrightarrow{\delta_0} & B^1_S & \xrightarrow{\delta_1} B^2_S & \cdots & B^{n-1}_S & \xrightarrow{\delta_{n-1}} B^n_S \\
\delta_0 & & \delta_1 & & \cdots & & \delta_{n-1} \\
\kappa_0 & & \kappa_1 & & \cdots & & \kappa_{n-1} \\
\end{array}
\]

\[
\begin{array}{cccc}
B^0_C & \xrightarrow{\delta_0} & B^1_C & \xrightarrow{\delta_1} B^2_C & \cdots & B^{n-1}_C & \xrightarrow{\delta_{n-1}} B^n_C \\
\delta_0 & & \delta_1 & & \cdots & & \delta_{n-1} \\
\kappa_0 & & \kappa_1 & & \cdots & & \kappa_{n-1} \\
\end{array}
\]

where $B^0_S$ is the $\omega$-operad for strict $\omega$-categories and $B^0_C$ is the $\omega$-operad of Batanin for weak $\omega$-categories. In particular the $\omega$-operads $B^0_P$, where $P$ is either $S_u$ for strict with contractible units (see section 3.5), or $C$ for contractible (see section 3.6), have interesting common characteristics: Each $B^0_P$ has a universal property in $PT\text{-Cat}_c$, which is to be initial among $\omega$-operads (monochromatic or even coloured) equipped with a composition system (explicitly given by the collection $C^0$), and satisfying the property $P$. Thus if we show, for a fixed property $P$, that $coend(B^*_P)$ is at the same time equipped with a composition system and verifies the property $P$, then $B^0_P$ must have the fractal property because we would obtain a unique morphism of $PT\text{-Cat}_c$,

\[
\begin{array}{c}
B^0_P \xrightarrow{1_P} coend(B^*_P) \\
\end{array}
\]
which express an action of $B^0_P$ on $\text{End}(A_{0,P})$ (see section 2).

At this stage it is fundamental to remark that if for each tree $t$, the coloured $\omega$-operad $B^t_P$ keeps the kind of contractibility which is involved (strict contractibility with contractible units, or contractibility), then $\text{Coend}(B^t_P)$ is not only equipped with a composition system (see section 3.4) but also has the property $P$ (see proposition 3 below).

We didn’t resolve yet the case $P = S_u$, $P = C$, but we believe that it is the case. For the rest of this article we accept the following hypotheses:

**Hypotheses** Each coloured $\omega$-operad $B^t_P$ ($t \in \text{Tree}$) built in $\mathbb{T}$-$\text{Cat}_c$ has the following virtue

- Each $B^t_{S_u}$ is strictly contractible with contractible units.
- Each $B^t_C$ is contractible.

We have no full proof of this hypothesis at the moment. For example consider the tree $t = 1(1) *_{0}^{1}(1)$. The corresponding operad $B^t_C$ is given by the pushout

$$
\begin{array}{ccc}
B^0_C & \xrightarrow{\kappa_0} & B^1_C \\
\delta^1_1 & & \downarrow \iota_1 \\
B^1_C & \xrightarrow{i_2} & B^1_C \cup_{B^0_C} B^1_C
\end{array}
$$

The contractibility hypothesis states that this $\omega$-operad is contractible.

Let us show how to see the appearance of the contraction cells in a simple example. For the following example we will use the same notation for the symbol of functor $F_1$ (which is a 1-cell of $B^1_C \cup_{B^0_C} B^1_C$, but which can have arity $1(1)$ or $2(1)$, depending on where this symbol lives for each $B^1_C$ of this pushout). Consider the 1-cells

$$
x = \gamma_1(\gamma_1(F_1; \gamma_1(\mu_0^1; u_1 *_{0}^{1} \mu_0^1)); F_1 *_{0}^{1} F_1 *_{0}^{1} F_1)
$$
and
\[
y = \gamma_1(\gamma_1(F_1; \gamma_1(\mu_0^1; \mu_0^1 \ast_0^1 u_1)); F_1 \ast_0^1 F_1 \ast_0^1 F_1) .
\]
It is not difficult to show that the arity of \(x\) and \(y\) is \(1(1) \ast_0^1 1(1) \ast_0^1 1(1)\), and also that \(x \| y\). The 2-cell
\[
\gamma_2([\gamma_1(F_1; \gamma_1(\mu_0^1; u_1 \ast_0^1 \mu_0^1)); \gamma_1(F_1; \gamma_1(\mu_0^1; \mu_0^1 \ast_0^1 u_1))]; 1_2(F_1) \ast_0^1 1_2(F_1) \ast_0^1 1_2(F_1))
\]
is a coherence cell connecting \(x\) and \(y\). At this stage we can see that such coherence cells emerge from the contractibility of \(B_C^1\) plus the operadical multiplication of \(B_C^1 \sqcup B_0^1 = B_C^1\). Unfortunately, it is an impossible task to try to generalise these calculations because the combinatorics becomes unmanageable very quickly. We believe, however, that there is a more elegant way to prove this hypothesis based on abstract homotopy theory. Indeed, according to our construction, the \(\omega\)-operad \(B_C^\omega\) is a finite (wide) pushout of contractible colored operads. If we assume that there is a nice enough model structure on the category of colored \(\omega\)-operads such that the operads \(B_C^1\) are contractible and cofibrant in the model theoretic sense, and all morphisms in the pushout are cofibrations then the contractibility of \(B_C^\omega\) will follow from the standard model theoretic argument. The existence of such a model structure is very important not only for the contractibility hypothesis but for many other applications. It will be a subject of our future paper.

**Remark 5** The reader can keep in mind a striking analogy between our operad \(\text{Coend}(B_C^\bullet)\) and the operad \(\text{Coend}(D^\bullet)\) constructed by Michael Batanin in [2]. The coglobular complex \(D^\bullet\) consists of the standard topological disks. An amazing fact is that \(\text{Coend}(D^\bullet)\) turns out to be a contractible \(\omega\)-operad which acts naturally on globular complex of points, paths, 2-paths etc. of a topological space. In this way fundamental \(\omega\)-groupoid functor is constructed in [2]. We believe that our coglobular complex \(B_C^\bullet\) plays the same role for the homotopy theory of \(\omega\)-operads, as \(D^\bullet\) does for homotopy theory of topological spaces. \(\square\)
**Proposition 3** Under the hypotheses above, $\text{Coend}(B_C^\bullet)$ is contractible and $\text{Coend}(B_{S_n}^\bullet)$ is strictly contractible with contractible units.

\[\square\]

**Proof** Consider two \((n-1)\)-cells of $\text{Coend}(B_C^\bullet)$ which are parallels and have the same arity. That means we give ourselves a diagram in $\text{T-Cat}_c$ such that this diagram commute serially. In particular we have

\[f_{n-1}^n \delta_{n-1}^{n-2} = f_{n-1}^+ \delta_{n-1}^{n-2}\]

and

\[f_{n-1}^- \kappa_{n-1}^{n-2} = f_{n-1}^+ \kappa_{n-1}^{n-2}\]

Contemplate the diagram
Call \( \alpha^{n-1} \) the principal cell in \( C^{n-1} \) and \( \alpha^{n-2} \) the principal cell in \( C^{n-2} \). By definition we have \( \delta_{n-1}^{n-2}(\alpha^{n-2}) = s_{n-2}^{n-1}(\alpha^{n-1}) \) and \( \kappa_{n-1}^{n-2}(\alpha^{n-2}) = t_{n-2}^{n-1}(\alpha^{n-1}) \).

We have \( \overline{f}_{n-1}(\alpha^{n-1}) \parallel \overline{f}^+_{n-1}(\alpha^{n-1}) \), because

\[
\begin{align*}
{s}_{n-2}^{n-1}\overline{f}_{n-1}(\alpha^{n-1}) &= \overline{f}_{n-1}(s_{n-2}^{n-1}(\alpha^{n-1})) \\
&= \overline{f}_{n-1}(\delta_{n-1}^{n-2}(\alpha^{n-2})) \\
&= \overline{f}_{n-1}(\delta_{n-1}^{n-2}f_{n-2}^{n-2}(\alpha^{n-2})) \\
&= \overline{f}^+_{n-1}(\delta_{n-1}^{n-2}f_{n-2}^{n-2}(\alpha^{n-2})) \\
&= \overline{f}^+_{n-1}(s_{n-2}^{n-1}(\alpha^{n-1})) \\
&= s_{n-2}^{n-1}\overline{f}^+_{n-1}(\alpha^{n-1})
\end{align*}
\]

In the same way we can prove that

\[
\overline{t}_{n-2}^{n-1}\overline{f}_{n-1}(\alpha^{n-1}) = \overline{t}_{n-2}^{n-1}\overline{f}^+_{n-1}(\alpha^{n-1})
\]

But \( B_C^n \) is contractible thus \( \overline{f}_{n-1}(\alpha^{n-1}) \) and \( \overline{f}^+_{n-1}(\alpha^{n-1}) \) are connected by the coherence cell

\[
[\overline{f}_{n-1}(\alpha^{n-1}); \overline{f}^+_{n-1}(\alpha^{n-1})]
\]

We then build \( C^n \xrightarrow{f_n} B_C^n \) as follow: If \( \alpha^n \) is the principal cell of \( C^n \) then we put

\[
\overline{f}_n(\alpha^n) = [\overline{f}_{n-1}(\alpha^{n-1}); \overline{f}^+_{n-1}(\alpha^{n-1})]
\]

Thus we obtain

\[
\begin{array}{ccc}
B_C^n & \xrightarrow{f_n} & B_C^n \\
\delta_{n-1} & \xrightarrow{\kappa_{n-1}} & f_{n-1} \\
B_C^{n-1} & \xrightarrow{\overline{f}_{n-1}} & B_C^n
\end{array}
\]
and finally we put $f_n := [f_{n-1}^-; f_{n-1}^+]$.

The proof of the strict contractibility with contractible units of $\text{Coend}(B_{S_n}^\bullet)$ is entirely similar. ■

3.4 Composition Systems

$B_p^\bullet$, or $B^\bullet$ for short, denotes either the coglobular complex $B_{S_n}^\bullet$, or $B_C^\bullet$ in $\mathbb{C}^{\mathcal{T}-\mathcal{Cat}}$. Also denote by $B^n \sqcup B^n$ the 3-coloured $\omega$-operad in $\mathbb{C}^{\mathcal{T}-\mathcal{Cat}}$ which is obtain by pushing out, in $\mathbb{C}^{\mathcal{T}-\mathcal{Cat}}$, the following diagram

\[
\begin{array}{ccc}
B^p & \xrightarrow{\nu_p^n} & B^n \\
\delta^n_p \downarrow & & \downarrow \\
B^n \\
\end{array}
\]

where $\delta^n_p = \delta^n_{p-1} \cdots \delta^n_0$ and $\kappa^n_p = \kappa^n_{p-1} \cdots \kappa^n_{p+1}$. For each integers $0 \leq p < n$ we are going to define a morphism in $\mathbb{C}^{\mathcal{T}-\mathcal{Gr}_{p,c}}$

\[
\begin{array}{ccc}
C^n & \xrightarrow{\nu_p^n} & B^n \sqcup B^n \\
\end{array}
\]

which, depending on the universality property required, gives us a unique morphism in $\mathbb{C}^{\mathcal{T}-\mathcal{Cat}}$, that we still call $\mu_p^n$ because there is no risk of confusion,

\[
\begin{array}{ccc}
B^n & \xrightarrow{\nu_p^n} & B^n \sqcup B^n \\
\end{array}
\]

For instance, if we accept the contractibility hypothesis 1 whose consequence is that $B^n_C \sqcup B^n_C$ is still an object of $\mathbb{C}^{\mathcal{T}-\mathcal{Cat}}$, the universal map $C^n \xrightarrow{\eta_C^n} B^n_C$ gives us such morphism $\mu_p^n$. We have similar technology for $P = S_n$. The key point to defining these morphisms $\mu_p^n$ is first to describe the different compositions $o_p^n$ of the strict higher transformations. If $0 < p < n$, we know that for two strict $n$-transformations $\sigma$ and $\tau$, we have
\[(\sigma \circ_p^n \tau)(a) := \sigma(a) \circ_{p-1}^{n-1} \tau(a)\]

whose operadic interpretation is given by the cell \(\gamma(\mu_{p-1}^{n-1}; \sigma \ast_{p-1}^{n-1} \tau)\). Then the morphism in \(\mathbb{T} \text{-} \mathcal{Gr}_{p,c}\)

\[\begin{array}{c}
C^n \\
\downarrow^\mu^p \\
B^n \sqcup_{B^p} B^n
\end{array}\]

sends the principal cell \(\tau\) of \(C^n\) to the \((n-1)\)-cell \(\gamma(\mu_{p-1}^{n-1}; \sigma \ast_{p-1}^{n-1} \tau)\) of \(B^n \sqcup_{B^p} B^n\), sends for each \(i \in \mathbb{N}\), the \(i\)-cell \(F_i\) of \(C^n\) to the \(i\)-cell \(F_i\) of \(B^n \sqcup_{B^p} B^n\), and sends the \(i\)-cell \(G_i\) of \(C^n\) to the \(i\)-cell \(H_i\) of \(B^n \sqcup_{B^p} B^n\). This morphism of \(\mathbb{T} \text{-} \mathcal{Gr}_{p,c}\) is boundary preserving in an evident sense.

If \(p = 0\) it is a bit more complex. We are in the situation of the pushout diagram below

\[\begin{array}{c}
B^0 \\
\downarrow^\delta^0 \\
B^n \sqcup_{B^0} B^n
\end{array}\]

First we describe the composition \(\circ_0^n\) for the strict case, to be able to find the cells that we need in our \(\omega\)-operad. Consider the following diagram in the strict \(\omega\)-category of the strict \(\omega\)-categories.

\[\begin{array}{c}
\mathcal{C} \\
\downarrow^\tau \\
\mathcal{D} \sqcup \mathcal{E}
\end{array}\]

Here \(\mathcal{C}, \mathcal{D}\) and \(\mathcal{E}\) are 0-cells (i.e. strict \(\omega\)-categories), \(F, G, H\) and \(K\) are 1-cells (i.e. strict \(\omega\)-functors) and \(\tau\) and \(\sigma\) are \(n\)-cells (i.e. strict \(n\)-transformations). This picture describes \(\tau\) and \(\sigma\) with 2-cells, but the reader must see them as \(n\)-cells. \(\tau\) and \(\sigma\) are such that : \(s_0^n(\sigma) = \mathcal{C}\), \(t_0^0(\sigma) = s_0^n(\tau) = \mathcal{D}\), and \(t_0^n(\tau) = \mathcal{E}\). If \(a \in \mathcal{C}(0)\), then \(F^0 \xrightarrow{\tau(a)} G^0\) is
an \((n - 1)\)-cells of \(D\) and it induces the following commutative square of \((n - 1)\)-cells in \(E\)

\[
\begin{array}{ccc}
H^0(F^0(a)) & \xrightarrow{H^{n-1}(\tau(a))} & H^0(G^0(a)) \\
\downarrow{\sigma(F^0)} & & \downarrow{\sigma(G^0)} \\
K^0(F^0(a)) & \xrightarrow{K^{n-1}(\tau(a))} & K^0(G^0(a))
\end{array}
\]

which gives

\[
(\sigma \circ_0^n \tau)(a) = \sigma(G_0(a)) \circ_0^{n-1} H_{n-1}(\tau(a)) = K_{n-1}(\tau(a)) \circ_0^{n-1} \sigma(F_0(a))
\]

and this gives the two principal \((n - 1)\)-cells of \(B^n \sqcup B^n\) that we need:

\[
\gamma^{n-1}(\mu_0^{n-1}; \gamma(\sigma; G^0) \ast_0^{n-1} \gamma(H^{n-1}; \tau))
\]

and

\[
\gamma^{n-1}(\mu_0^{n-1}; \gamma(K^{n-1}; \tau) \ast_0^{n-1} \gamma(\sigma; F^0))
\]

Then we have two choices of

\[
\begin{array}{ccc}
C^n & \xrightarrow{\mu^n_p} & B^n \sqcup B^n
\end{array}
\]

which send the principal cell \(\tau\) of \(C^n\) to \(\gamma^{n-1}(\mu_0^{n-1}; \gamma(\sigma; G^0) \ast_0^{n-1} \gamma(H^{n-1}; \tau))\) or on \(\gamma^{n-1}(\mu_0^{n-1}; \gamma(K^{n-1}; \tau) \ast_0^{n-1} \gamma(\sigma; F^0))\), and for both cases, which send for each \(i \in \mathbb{N}\), the \(i\)-cell \(F^i\) of \(C^n\) to the \(i\)-cell \(\gamma(F_i; H_i)\) of \(B^n \sqcup B^n\), and the \(i\)-cell \(G^i\) of \(C^n\) to the \(i\)-cell \(\gamma(G_i; K_i)\) of \(B^n \sqcup B^n\). These morphisms of \(T\)-\(Gr_{p,c}\) are boundary preserving in an evident sense.

Now let us come back to the specific case of the coglobular complex \(B^n_{S_\bullet}\) or \(B^n_{C}\) in \(T\)-\(Cat_c\). Suppose we accept the fractality hypothesis (see section 1) for the \(\omega\)-operads \(Coend(B^n_p)\), where \(P\) can be either \(S_n\), or \(C\). In that case, thanks to the universal property of \(\eta^n\), we get the following unique morphisms of \(\omega\)-operads \(\mu^n_p\) and \(\mu^n_0\) (the dotted arrows)

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With the identity morphisms of operads $B^n \xrightarrow{1_{B^n}} B^n$

$$C^0 \xrightarrow{coend(B^*)} B^n \sqcup B^n$$

$$\mu_p^n \xrightarrow{} \mu_p^n$$

$$u_n \xrightarrow{} 1_{B^n}$$

we thus have the following conclusion:

**Proposition 4** The $\omega$-operads of coendomorphisms $Coend(B^s_u)$ and $Coend(B^c_C)$ have composition systems.

\[
\begin{array}{c}
B^n \\
\downarrow \eta^n \\
C^n \\
\end{array}
\xrightarrow{\mu_p^n} \begin{array}{c}
B^n \\
\downarrow \eta^n \\
C^n \\
\end{array} \quad \begin{array}{c}
B^n \\
\downarrow \eta^n \\
C^n \\
\end{array}
\]

3.5 The strict $\omega$-category of strict $\omega$-categories

Consider the case $P = S_u$ ("Strict with contractible units"), i.e we deal with the category $S_u T\text{-Cat}_c$ of strict $\omega$-operads with contractible units (see section 3.2). The coglobular complex $B^s_u$ of the section 3.3 produces the following globular complex in $\text{CAT}$

$$B^s_u-\text{Alg} \xrightarrow{\sigma_{n-1}^n} B^{n-1}_u-\text{Alg} \xrightarrow{\beta_{n-1}^n} B^1_u-\text{Alg} \xrightarrow{\sigma_1^0} B^0_u-\text{Alg}$$

Also it is possible to prove the following proposition:

**Proposition 5** Objects of $B^1_u-\text{Alg}$ are strict $\omega$-functors, and for each integer $n \geq 2$, objects of $B^n_u-\text{Alg}$ are strict $n$-transformations.

The standard action of the coglobular complex $B^s_u$ is given by the following diagram in $T\text{-Cat}_1$
Coend\((B^\bullet_{S_u})\) \xrightarrow{\text{Coend}(\text{Alg}(\cdot))} \text{Coend}(A^0_{S_u}) \xrightarrow{\text{Coend}(\text{Ob}(\cdot))} \text{End}(A_{0,S_u})

It is another specific standard action of the higher transformations. The monochromatic $\omega$-operad of coendomorphism $\text{Coend}(B^\bullet_{S_u})$ plays a central role for strict $\omega$-categories. We call it the *indigo operad*\(^4\). According to the hypotheses 1, the indigo operad has a composition system (see the proposition 4) and is strictly contractible with contractible units. Thus we have a unique morphism in $\text{T-Cat}_1$

\[
B^0_{S_u} \xrightarrow{\text{1}_u} \text{Coend}(B^\bullet_{S_u})
\]

and we obtain a morphism of $\omega$-operads

\[
B^0_{S_u} \xrightarrow{\text{E}_u} \text{End}(A_{0,S_u})
\]

which expresses an action of the $\omega$-operad $B^0_{S_u}$ of strict $\omega$-categories on the globular complex $B^\bullet_{S_u}-\text{Alg}(0)$ in $\text{SET}$ of strict higher transformations, and thus gives a strict $\omega$-category structure on strict higher transformations.

### 3.6 The weak $\omega$-category of weak $\omega$-categories

Consider the case $P = C$ ("Contractible"), i.e., we deal with the category $\text{CT-Cat}$ of contractible $\omega$-operads (see 3.2). The coglobular complex $B^\bullet_C$ of section 3.3 produces the following globular complex in $\text{CAT}$

\(^4\)We use Newton’s 7 primary colours of the Rainbow to denote 4 relevant $\omega$-operads of coendomorphism of this article. We don’t mention in this article the red operad, the orange operad, and the blue operad, which are respectively specific for $\omega$-graphs, reflexive $\omega$-graphs, and semi-strict $\omega$-categories, because they are very similar to others monochromatic $\omega$-operad of coendomorphism of this article, and we don’t need them to reached the main idea of this article.
and in the article [16] it was proved, with the old notion of contractibility, that we have the proposition

**Proposition 6**  
Algebras of dimension 2 of $B^n_C$-$\mathsf{Alg}$ are pseudo-2-functors, and algebras of dimension 2 of $B^n_C$-$\mathsf{Alg}$ are pseudo-2-natural transformations.

However, with our new notion of contractibility (see section 3.2), this proposition remains true, and the proof is exactly the same as the proof in the article [16]. The standard action of the coglobular complex $B^\bullet_C$ is given by the following diagram in $\mathbb{T}$-$\mathsf{Cat}$

$$
\begin{array}{cccc}
\mathsf{Coend}(B^\bullet_C) & \xrightarrow{\mathsf{Coend}(\mathsf{Alg}(\cdot))} & \mathsf{Coend}(A_{\otimes}^0) & \xrightarrow{\mathsf{Coend}(\mathsf{Ob}(\cdot))} & \mathsf{End}(A_{0,C})
\end{array}
$$

It is an other specific standard action of the higher transformations. The monochromatic $\omega$-operad of coendomorphism $\mathsf{Coend}(B^\bullet_C)$ plays a central role for weak $\omega$-categories. We call it the *violet operad*. Batanin’s $\omega$-operad $B^0_C$ of weak $\omega$-categories is initial among contractible $\omega$-operads which have a composition system. According to the contractibility hypothesis 1, the violet operad has a composition system (see proposition 4) and is contractible. Thus we have a unique morphism in $\mathbb{T}$-$\mathsf{Cat}$

$$
B^0_C \xrightarrow{!_C} \mathsf{Coend}(B^\bullet_C)
$$

and we obtain a morphism of $\omega$-operads

$$
B^0_C \xrightarrow{c} \mathsf{End}(A_{0,C})
$$

which expresses an action of the $\omega$-operad $B^0_C$ of weak $\omega$-categories on the globular complex $B^\bullet_C$-$\mathsf{Alg}(0)$ in $\mathsf{SET}$ of weak higher transformations, and thus gives a weak $\omega$-category structure on the weak higher
transformations. It is not difficult to prove that under this weak \( \omega \)-category structure on \( B^\bullet_{\omega} - Alg(0) \), the composition of weak \( \omega \)-functors is associative up to weak natural \( \omega \)-transformations.

4 Examples of \( \omega \)-operads with fractal property

Consider now the case \( P = Id \) ("Magmatic"), i.e. we deal with the category \( T - Cat_\infty \) of \( \omega \)-operads (see section 3.1). We apply the free functor (see section 3.1)

\[
\begin{array}{ccc}
T - Gr_{p,c} & \overset{M}{\longrightarrow} & T - Cat_\infty \\
\end{array}
\]

to the coglobular complex of the higher transformations \( C^\bullet \) in \( T - Gr_{p,c} \) and we obtain a coglobular complex \( B^\bullet_{Id} \) of \( \omega \)-operads in \( T - Cat_\infty \)

\[
\begin{array}{cccccc}
B^0_{Id} & \overset{\delta^1_0}{\longrightarrow} & B^1_{Id} & \overset{\delta^1_1}{\longrightarrow} & B^2_{Id} & \overset{\delta^{n-1}_n}{\longrightarrow} & B^n_{Id} \\
\sigma^0_{b} & \sigma^1_{a} & \sigma^2_{a} & \sigma^{n-1}_a & \sigma^a_{b} \\
\end{array}
\]

which produces the following globular complex in \( CAT \).

\[
\begin{array}{cccccc}
B^n_{Id} - Alg & \overset{\sigma_{b}^{n-1}}{\longrightarrow} & B^{n-1}_{Id} - Alg & \overset{\beta_{b}^{n-1}}{\longrightarrow} & B^{n}_{Id} - Alg \\
\end{array}
\]

In particular \( B^0_{Id} \) is the \( \omega \)-operad for \( \omega \)-magmas (see [17]). The standard action associated to \( B^\bullet_{Id} \) is given by the following diagram in \( T - Cat_1 \)

\[
\begin{array}{cccccc}
Coend(B^\bullet_{Id}) & \overset{Coend(Alg(\_))}{\longrightarrow} & Coend(A^\bullet_{Id}) & \overset{Coend(Object(\_))}{\longrightarrow} & End(A_{0,Id}) \\
\end{array}
\]

that we call the standard action of \( \omega \)-magmas, thus which is a specific standard action of higher transformations. The monochromatic \( \omega \)-operad \( Coend(B^\bullet_{Id}) \) of coendomorphism plays a central role for \( \omega \)-magmas. We call it the yellow operad. Also we have the following proposition
Proposition 7  $B^0_{Id}$ has the fractal property.

If we compose the morphism $!_{Id}$

$$B^0_{Id} \xrightarrow{!_{Id}} \text{Coend}(B^\bullet_{Id})$$

with the standard action associated to $B^\bullet_{Id}$ we obtain a morphism of $\omega$-operads

$$B^0_{Id} \xrightarrow{3d} \text{End}(A_{0,Id})$$

which expresses an action of the $\omega$-operad $B^0_{Id}$ of the $\omega$-magmas on the globular complex $B^\bullet_{Id}\text{-Alg}(0)$ in $SET$ of the $(n,\omega)$-magmas ($n \in \mathbb{N}$), and thus gives an $\omega$-magma structure on the $(n,\omega)$-magmas ($n \in \mathbb{N}$).

Consider the case $P = Id_u$ ("Magmatic with contractible units"), i.e. we deal with the category $Id_u\text{-}\mathcal{C}at_c$ of $\omega$-operads with contactible units (see section 3.1). We apply the free functor (see section 3.1)

$$\mathcal{T}\text{-}\text{Gr}_{p,c} \xrightarrow{Id_u} Id_u\text{-}\mathcal{C}at_c$$

to the coglobular complex of higher transformations $C^\bullet$ in $\mathcal{T}\text{-}\text{Gr}_{p,c}$ and we obtain a coglobular complex $B^\bullet_{Id_u}$ of $\omega$-operads in $\mathcal{T}\text{-}\mathcal{C}at_c$

$$B^0_{Id_u} \xrightarrow{\delta^1_0} B^1_{Id_u} \xrightarrow{\delta^1_2} B^0_{Id_u} \xrightarrow{\delta^2_{n-1}} B^{-1}_{Id_u} \xrightarrow{\delta^2_{n-1}} B^n_{Id_u} \xrightarrow{\kappa^1_{n-1}} B^n_{Id_u} \xrightarrow{\kappa^1_{n-1}} B^{-1}_{Id_u}$$

which produces the following globular complex in $\mathcal{C}AT$.

$$\mathcal{C}AT \xrightarrow{B^0_{Id_u}\text{-Alg}} B^0_{Id_u}\text{-Alg} \xrightarrow{\sigma^n_{n-1}} B^{n-1}_{Id_u}\text{-Alg} \xrightarrow{\beta^n_{n-1}} B^1_{Id_u}\text{-Alg} \xrightarrow{\sigma^1_{0}} B^{0}_{Id_u}\text{-Alg}$$

In particular $B^0_{Id_u}$ is the $\omega$-operad for reflexive $\omega$-magmas (see [17]). The standard action associated to $B^\bullet_{Id_u}$ is given by the following diagram in $\mathcal{T}\text{-}\mathcal{C}at_1$.  

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It is a specific standard action of the higher transformations. The monochromatic $\omega$-operad $\text{Coend}(B_{Id_u}^\bullet)$ of coendomorphism plays a central role for reflexive $\omega$-magmas. We call it the \textit{green operad}. Also we have the following proposition

\textbf{Proposition 8} $B_{Id_u}^0$ has the fractal property. \hfill $\square$

If we compose the morphism $!_{Id_u}$

$$
B_{Id_u}^0 \xrightarrow{!_{Id_u}} \text{Coend}(B^{\bullet}_{Id_u})
$$

with the standard action associated to $B^{\bullet}_{Id_u}$ we obtain a morphism of $\omega$-operads

$$
B_{Id_u}^0 \xrightarrow{\varepsilon_u} \text{End}(A_{0,Id_u})
$$

which expresses an action of the $\omega$-operad $B_{Id_u}^0$ of reflexive $\omega$-magmas on the globular complex $B_{Id_u}^\bullet - \text{Alg}(0)$ in $\text{SET}$ of the reflexive $(n, \omega)$-magmas ($n \in \mathbb{N}$), and thus gives a reflexive $\omega$-magma structure on the reflexive $(n, \omega)$-magmas ($n \in \mathbb{N}$).

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