BRST–Antifield–Treatment of Metric–Affine Gravity

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Abstract

The metric–affine gauge theory of gravity provides a broad framework in which gauge theories of gravity can be formulated. In this article we fit metric–affine gravity into the covariant BRST–antifield formalism in order to obtain gauge fixed quantum actions. As an example the gauge fixing of a general two–dimensional model of metric–affine gravity is worked out explicitly. The result is shown to contain the gauge fixed action of the bosonic string in conformal gauge as a special case.

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1 Introduction

In search of a satisfactory theory of quantum gravity there still exists, in addition to the new methods of canonical quantization of general relativity, supergravity, and superstring theories, the more traditional approach which is based on a gauging of non–supersymmetric extensions of the Poincaré group. It developed from the early papers of Utiyama [1], Sciama [2, 3], and Kibble [4]. The main motivation to follow this approach is provided by the fact that the spacetime symmetry which is nowadays really observed is the Poincaré symmetry. Within the concept of a relativistic field theory, it is more natural to consider local spacetime symmetries rather than keeping spacetime symmetry rigid. Therefore it is natural to gauge the Poincaré group. In this scheme, general relativity can be straightforwardly derived as a gauge theory of translations. However, in a perturbative expansion the quantization of gravity models which are based on a gauging of the Poincaré group leads to either non–unitarity or non–renormalizability\footnote{It is nevertheless interesting to note that, as already observed in [5], the presence of extra symmetries in certain models of Poincaré gauge theory [6, 7] might yield surprising features. The role of extra symmetries in the context of metric–affine gravity was recently discussed in [8].}. In order to ‘repair’ these defects, it has been suggested that the Poincaré group originates from a symmetry reduction of one of its extensions [11]. A possible and fairly general framework of such a mechanism is provided by the metric–affine gauge theory of gravity (MAG) [12]. MAG is based on a gauging of the $n$–dimensional affine group $A(n, R) = T^n \ltimes GL(n, R)$, i.e. the semidirect product of the translation group $T^n$ and the group of general linear transformations $GL(n, R)$. The affine group enlarges the transformations of the Poincaré group by dilation and shear transformations. For a particular model of MAG a spontaneous symmetry breaking mechanism was constructed [13], indicating renormalizability, but with the proof of unitarity left as an open problem.

In this article we reconsider the approach to the quantization of MAG and fit MAG into the covariant Becchi-Rouet-Stora-Tyutin–(BRST)–antifield formalism in order to demonstrate how to obtain gauge fixed quantum actions. The BRST–antifield formalism was developed by Batalin and Vilkovisky [14, 15], using earlier ideas of Zinn–Justin [16] and others [17]. It relies heavily on the concept of BRST–symmetry [18] and was mainly developed in view of the quantization of gauge theories which are characterized by open or reducible gauge algebras. However, the antifield formalism seems to become a standard tool of quantum field theory [19, 20] which is also of considerable use in the context of the closed and irreducible Yang–Mills theory. The more standard but less general Feynman–DeWitt–Faddeev–Popov method is suitable to covariantly gauge fix Yang–Mills theory [21] and general relativity [22, 23]. But it cannot straightforwardly be applied to MAG since the gauge algebra of MAG contains field dependent structure functions and thus constitutes no Lie–algebra.

In the literature we have found several BRST–formulations of specific gauge models of gravity which are included in MAG, see for example [24, 25, 26, 8]. In these gauge
models the generator of translations on the spacetime manifold is taken to be a non-gauge-covariant Lie-derivative. This yields a corresponding gauge algebra which is field independent. However, a gauge-covariant notion of translational invariance is required if matter is included which transforms non-trivially under the linear part of the external gauge group, e.g. fermionic matter. Such matter can be covariantly translated on the spacetime manifold by the use of a gauge-covariant Lie-derivative. The corresponding gauge algebra of MAG, as already mentioned above, is field dependent, and it is this general case we will deal with in this paper.

In order to put MAG into the BRST-antifield formalism we will proceed as follows: In Sec. 2 we will shortly review MAG and later derive its gauge algebra in Sec. 3. The gauge algebra is the main ingredient of MAG to be inserted into the BRST-antifield formalism. This will be done in Sec. 4 in order to display the BRST-symmetry of MAG. In Sec. 5 we will outline the general process of gauge fixing and explicitly apply it to a general two-dimensional model of MAG. This yields a corresponding gauge fixed quantum action. The quantum action of the bosonic string in conformal gauge is derived as a special case from this.

2 MAG as a classical gauge field theory

A physical theory constitutes a gauge theory if some of its dynamical fields are to be expressed with respect to a certain reference frame, the specific choice of which is pointwise determined only modulo symmetry transformations. These are the gauge transformations. In this case of a local symmetry we need to describe the equivalence of reference frames at different spacetime points in order to define the differential of a field. This requires the introduction of a gauge potential, i.e. a gauge connection, and establishes the gauging of the symmetry group. In a physical gauge theory the gauge potential is usually made a dynamical variable, for example by adding a corresponding kinetic term on the Lagrangian level.

This pattern can be followed to build up MAG: One starts from an $n$-dimensional, differential base manifold which represents spacetime. At each point $x$ on $M$ it is possible to define a tangent space $T_x M$, an affine tangent space $A_x M$, and an affine frame $(e_a, p)(x)$ [27]. If physical fields are to be described in affine frames, the postulate of local affine invariance requires the introduction of an affine gauge connection $(\Gamma^{(T)} \alpha, \Gamma_\alpha \beta)$. Here, $\Gamma^{(T)} \alpha$ denotes the translational part of the affine gauge connection which accounts for local translation invariance while $\Gamma_\alpha \beta$ denotes the linear part of the affine gauge connection which accounts for local $GL(n, R)$-invariance. This completes the gauging of the affine group.

\[2\] The linear part of the external gauge group is, in general, the group $GL(n, R)$ or, after the symmetry reduction, the Lorentz group $SO(1, n - 1)$.
To arrive at a gravity theory we next have to turn the affine gauge connection $(\Gamma^{(T)\alpha}, \Gamma_\alpha^\beta)$ into an external spacetime structure. This step has no analogue in the case of an internal gauge theory, such as ordinary Yang–Mills theory, and is still not completely understood. Roughly speaking, any affine frame has to be “soldered” to the base manifold $M$. Here, soldering means to identify the point $p$ of an affine frame with a point $x$ of the base manifold $M$. This procedure breaks the translational part of the original affine invariance and turns it into translation or diffeomorphism invariance on the base manifold. It is this transition from internal to external translation invariance which, in the gauge approach, generates gravity and should deserve future investigations. This should happen not only on a geometric level but also in the context of Higgs fields and their role as generators of mass.

After the soldering procedure one can replace the affine gauge connection $(\Gamma^{(T)\alpha}, \Gamma_\alpha^\beta)$ by a so–called Cartan connection $(\vartheta^\alpha, \Gamma_\beta^\alpha)$ [28, 27], where the relation between $\Gamma^\alpha$ and $\vartheta^\alpha$ is given by [29]

$$\vartheta^\alpha := \delta^\alpha_i dx^i + \Gamma^{(T)\alpha}.$$  \(1\)

The importance of the introduction of $\vartheta^\alpha$ is rooted in the fact that under affine transformations $\delta_{(\varepsilon, \varepsilon_\alpha^\beta)}$, generated by infinitesimal vector fields $\varepsilon = \varepsilon^i \partial_i$ (internal translation) and parameters $\varepsilon_\alpha^\beta$ (general linear transformation), it transforms linearly, i.e., $\vartheta^\alpha$ is internally translation invariant:

$$\delta_{(\varepsilon, \varepsilon_\alpha^\beta)} \vartheta^\alpha = \varepsilon_\beta^\alpha \vartheta^\beta.$$  \(2\)

Therefore, expressing physical fields by means of the, in general anholonomic, coframe $\vartheta^\alpha$, yields, by construction, an internally translation invariant theory, which typically represents a gravitation theory. However, we stress that this construction comes after the soldering procedure. What remains to be considered are the general linear transformations and the (external) translations on the manifold $M$. These symmetry transformations are the ones which, in MAG, generate the physically meaningful Noether identities.

Translations on a manifold are generated by the flow of vector fields $\varepsilon$. The effect of such translations on physical fields is measured by Lie–derivatives. For physical fields which transform trivially under $GL(n, \mathbb{R})$–transformations, for example the Maxwell field or a scalar field, the Lie–derivative can be taken as the commutator of exterior derivative and interior product, $l_\varepsilon\ldots = d(\varepsilon \ldots) + (\varepsilon d\ldots)$. Otherwise, in the case of spinning matter, e.g., it should be replaced by the gauge–covariant Lie–derivative $L_\varepsilon\ldots = D(\varepsilon \ldots) + (\varepsilon D\ldots)$, where $D$ denotes the $GL(n, R)$–covariant exterior derivative. The operator $L_\varepsilon$, in contrast to $l_\varepsilon$, allows to translate tensors into tensors, i.e. it is, as its name suggests, gauge covariant and thus independent of the orientation of linear frames at different points. Therefore it is independent of the linear part of the affine gauge transformations and leads to gauge–covariant Noether identities, a property we want to require for a proper translation generator$^3$.

\(^3\)For a more geometric discussion of this point in favor of the gauge–covariant Lie–derivative see
Let us sum up: The gauging of \( A(n, R) \) and the subsequent soldering procedure has supplemented the initial base manifold \( M \) with a Cartan connection \( (\vartheta^\alpha, \Gamma^\beta_\alpha) \). A metric \( g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta \) on \( M \) is not provided by either the gauging or the, somewhat unclear, soldering procedure. Its existence has to be postulated, a conceptual drawback which has not been resolved, yet. The set of field variables to be put in the gauge Lagrangian is then given by \( (g_{\alpha\beta}, \vartheta^\alpha, \Gamma^\beta_\alpha) \). The corresponding field strengths nonmetricity, curvature, and torsion are defined by

\[
\begin{align*}
\text{nonmetricity} & \quad Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma^\gamma_{\alpha\beta} + \Gamma^\beta_{\gamma\alpha}, \\
\text{torsion} & \quad T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma^\beta_{\alpha\gamma} \wedge \vartheta^\beta, \\
\text{curvature} & \quad R^\alpha_\beta := DD\vartheta^\alpha = d\vartheta^\alpha - \Gamma^\gamma_{\alpha\beta} \wedge \Gamma^\beta_{\gamma\alpha}.
\end{align*}
\]

Then the general form of an \( A(n, R) \)–gauge invariant first order gauge Lagrangian \( V \) becomes

\[
V = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R^\alpha_\beta).
\]

An \( A(n, R) \)–gauge invariant matter Lagrangian with matter fields \( \psi \) is of the form

\[
L_{\text{mat}} = L_{\text{mat}}(g_{\alpha\beta}, \vartheta^\alpha, \psi, D\psi),
\]

such that a general model of MAG is determined by a Lagrangian

\[
L = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R^\alpha_\beta) + L_{\text{mat}}(g_{\alpha\beta}, \vartheta^\alpha, \psi, D\psi).
\]

### 3 The gauge algebra of MAG

In order to quantize a gauge theory one needs the knowledge of either the gauge constraints (Hamiltonian formulation) or the structure of the group of gauge transformations which leave the action invariant (Lagrangian formulation). The latter is determined by the gauge algebra. The gauge algebra is the main input required by the covariant BRST-antifield formalism. It can be determined by commuting the gauge transformations of a given gauge theory: Suppose we begin with an action functional of the form

\[
S_0 = \int L(\Phi^i, d\Phi^i),
\]

where the index \( i \) numbers the field species of the theory. The gauge transformations on the fields \( \Phi^i \) can be generated by a generating set

\[
\delta_\varepsilon \Phi^i = R^i_a(\Phi)\varepsilon^a,
\]

with \( \varepsilon^a \) the spacetime-dependent gauge parameters and \( R^i_a(\Phi) \) the, in general field dependent, generators of the gauge transformations. The condensed notation used in (3) and in the following was first introduced by DeWitt [32]: A repeated discrete index

\[30, 31\].
not only implies a sum over that index but also an integration over the corresponding spacetime-variable. Thus, formula (9) has to be understood as

\[
\delta_s \Phi^i(x) = \int dy R^i_a(\Phi)(x, y)\varepsilon^a(y).
\]  

(10)

The generating set must be chosen such that it contains all the information about the Noether identities. That is, from the invariance of the action \(S_0\) under the gauge transformations of a generating set follow the Noether identities in the form

\[
S_0, R^i_a = 0.
\]  

(11)

Having determined a generating set, any gauge transformation \(\delta \Phi^i\) can be written in the form

\[
\delta \Phi^i = \mu^c_a R^i_c \varepsilon^a + \mu^{ij}_a \frac{\delta S_0}{\delta \Phi^j} \varepsilon^a, \quad \mu^{ij} = -(-1)^{\epsilon_i, \epsilon_j} \mu^{ij}_a,
\]  

(12)

where the coefficients \(\mu^c_a\) and \(\mu^{ij}_a\) are arbitrary functions which may involve the fields and \(\epsilon_i\) denotes the Grassmann parity of the field \(\Phi^i\). The transformations of the form

\[
\delta \Phi^i = \mu^{ji} \frac{\delta S_0}{\delta \Phi^j}, \quad \mu^{ji} = -(-1)^{\epsilon_i, \epsilon_j} \mu^{ji},
\]  

(13)

appearing on the right hand side of (12), are called trivial gauge transformations [17]. They leave the action \(S_0\) invariant, as is easily verified. Since gauge transformations form a group the commutator \([\delta_1, \delta_2]\) of two gauge transformations is again a gauge transformation. Hence it can be expressed in the form (12),

\[
[\delta_1, \delta_2] \Phi^i = T^c_{ab} R^i_c \varepsilon^b_{\epsilon_2} \varepsilon^a_{\epsilon_1} - E^{ij}_{ab} \frac{\delta S_0}{\delta \Phi^j} \varepsilon^b_{\epsilon_2} \varepsilon^a_{\epsilon_1},
\]  

(14)

with, possibly field dependent, structure functions \(T^c_{ab}\) and \(E^{ij}_{ab}\). To determine the complete gauge algebra one has to check for higher order structure functions and calculate higher order commutators which manifest themselves in (generalized) Jacobi identities [34, 13]. The gauge algebra is said to be open if \(E^{ij}_{ab} \neq 0\). Otherwise it is called closed. A closed gauge algebra with constant structure functions \(T^a_{ab}\) specializes to a Lie algebra.

Let us now determine the gauge algebra of MAG. The generating set of MAG is spanned by the covariant Lie–derivative \(L_{\varepsilon}\) and the generator of linear transformations \(\delta_{\varepsilon_\alpha\beta}\). This defines \(R^i_a\). Thus we have to take into account the commutator of two translations generated by \(L_{\varepsilon_1}\) and \(L_{\varepsilon_2}\), the commutator of a translation \(L_{\varepsilon_1}\) and a general linear transformation \(\delta_{\varepsilon_2\alpha\beta}\), and the commutator of two general linear transformations \(\delta_{\varepsilon_1\alpha\beta}\), \(\delta_{\varepsilon_2\gamma\delta}\). After some algebra we arrive at the commutation relations

\[
[L_{\varepsilon_1}, L_{\varepsilon_2}] = L_{[\varepsilon_1, \varepsilon_2]} + \delta(\varepsilon_2(\varepsilon_1 | dG_{\alpha\beta}) + \varepsilon_1 | G_{\gamma\beta}(\varepsilon_2 | G_{\gamma\alpha}) - \varepsilon_2 | G_{\gamma\beta}(\varepsilon_1 | G_{\gamma\alpha})),
\]  

(15)

\[
[L_{\varepsilon_1}, \delta_{\varepsilon_2\alpha\beta}] = \delta_{\varepsilon_1 | dG_{\varepsilon_2\alpha\beta}},
\]  

(16)

\[
[\delta_{\varepsilon_1\alpha\beta}, \delta_{\varepsilon_2\gamma\delta}] = \delta_{\varepsilon_1 \alpha \varepsilon_2 \beta \delta - \varepsilon_2 \alpha \varepsilon_1 \gamma \delta};
\]  

(17)
where, as before, the symbol $\lfloor$ denotes the interior product.

We note that the commutator of two local translations does not only yield another local translation but also exhibits a general linear transformation involving the curvature two-form $R_{\alpha \beta} = d\Gamma_{\alpha \beta} - \Gamma_{\alpha \gamma} \wedge \Gamma_{\gamma \beta}$. Also the commutator of a local translation and a general linear transformation yields as result a general linear transformation which depends on the connection $\Gamma_{\alpha \beta}$. Therefore the gauge algebra of MAG depends on the field variable $\Gamma_{\alpha \beta}$ and forms no Lie–algebra.

The commutator of two general linear transformations constitutes a subalgebra which resembles the familiar gauge algebra of a YM-theory: Denote the $n^2$ generators of $GL(n, R)$-transformations as $L_{\alpha \beta}$. The commutation of two such generators defines, according to [32], structure constants $C_{\lambda \rho \alpha \beta \gamma \delta}$

$$[L_{\alpha \beta}, L_{\gamma \delta}] = C_{\lambda \rho \alpha \beta \gamma \delta} L_{\lambda \rho}.$$  

(18)

The structure constants are explicitly given by

$$C_{\lambda \rho \alpha \beta \gamma \delta} = \delta_{\delta}^{\alpha} \delta_{\delta}^{\beta} \delta_{\delta}^{\gamma} - \delta_{\delta}^{\gamma} \delta_{\delta}^{\beta} \delta_{\delta}^{\alpha},$$

(19)

such that the commutation relation (18) can be rewritten as

$$[L_{\alpha \beta}, L_{\gamma \delta}] = \delta_{\delta}^{\alpha} L_{\gamma \beta} - \delta_{\delta}^{\gamma} L_{\alpha \beta}.$$  

(20)

Commutation of two general linear transformations $L_1 = \varepsilon_{1\alpha} L_{\alpha \beta}$, $L_2 = \varepsilon_{2\alpha} L_{\alpha \beta}$ yields a third transformation $L_3 = \varepsilon_{3\alpha} L_{\alpha \beta}$, where the parameter $\varepsilon_{3\alpha}$ is determined by the relation

$$[\varepsilon_{1\alpha}, \varepsilon_{2\delta}] = \varepsilon_{1\alpha} \varepsilon_{2\delta} C_{\lambda \rho \alpha \beta \gamma \delta} = \varepsilon_{1\delta} \varepsilon_{2\delta} - \varepsilon_{1\gamma} \varepsilon_{2\gamma} \rho = (\varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_1) \lambda \rho = \varepsilon_{3\lambda \rho}.$$  

(21)

Now the structure functions $T_{ab}^c$ and $E_{ab}^{ij}$, which were defined in (14), can be deduced:

First we notice that the gauge algebra of affine transformations closes and thus

$$E_{ab}^{ij} = 0.$$  

(22)

Next we define the parameter $\varepsilon^a$ of an affine transformation as the pair

$$\varepsilon^a := (\varepsilon^1, \varepsilon^2) := (\varepsilon, \varepsilon_{\alpha \beta}).$$  

(23)

From (15) – (17) we find

$$T_{11}^1 = \lfloor \ldots \rfloor \lfloor d \ldots - \ldots d \ldots, \ldots \rfloor \ldots,$$

(24)

$$T_{11}^2 = \lfloor \ldots \rfloor \lfloor d \Gamma_{\alpha \beta} \ldots \rfloor \ldots + \lfloor \ldots \rfloor \lfloor \Gamma_{\alpha \beta} \ldots \Gamma_{\alpha \gamma} \ldots \rfloor \ldots,$$

(25)

$$T_{12}^1 = \lfloor \ldots \rfloor \lfloor R_{\alpha \beta} - \ldots \rfloor \ldots = 0,$$

(26)

$$T_{21}^2 = \lfloor \ldots \rfloor \lfloor \Gamma_{\alpha \beta} \ldots \rfloor \ldots,$$

(27)

$$T_{22}^1 = T_{12}^1 = \lfloor \ldots \rfloor \lfloor \Gamma_{\alpha \beta} \ldots \rfloor \ldots = C_{\lambda \rho \alpha \beta \gamma \delta}.$$  

(28)
The expressions on the right hand sides of (24) and (27) look rather cryptic. This is because condensed notation within the formalism of exterior calculus is used. In order to make contact with less condensed notation, as introduced in Ref. [32], we evaluate the commutator \([\varepsilon_1, \varepsilon_2]\), appearing on the right hand side of (13), as follows:

\[
[\varepsilon_1, \varepsilon_2] = [\varepsilon^i_1 \partial_i, \varepsilon^j_2 \partial_j]
= \varepsilon^i_1 \partial_i [\varepsilon^j_2 \partial_j] - \varepsilon^j_2 \partial_j [\varepsilon^i_1 \partial_i]
= \varepsilon^i_1 \varepsilon^j_2 \delta^k_{ij} \delta_i \partial_k
= (\varepsilon^i_1 \varepsilon^j_2 \delta^k_{ij} - \varepsilon^j_2 \varepsilon^i_1 \delta^k_{ij}) \partial_k.
\]

Therefore we obtain from (24) the structure function \(T^{k}_{ij}\) in the alternative form

\[
T^{k}_{11} \equiv T^{k}_{ij''} = \delta(x - y') \delta(x - z'') \delta^i_j - \delta(x - y') \delta(x - z'') \delta^i_j\delta^k_i.
\]

Converting also the remaining components of \(T^{c}_{ab}\) yields the result

\[
T^{2}_{11} \equiv T^{2}_{ij''} = \delta(x - y') \delta(x - z'') \Gamma_{j\alpha} \delta_{i\beta} - \delta(x - y') \Gamma_{i\alpha} \delta_{j\beta}
+ \delta(x - y') \delta(x - z'') \Gamma_{i\gamma} \Gamma_{j\alpha} \delta_{i\gamma} - \delta(x - y') \Gamma_{i\gamma} \Gamma_{j\alpha} \delta_{i\gamma}.
\]

\[
T^{2}_{12} \equiv T^{2}_{\rho ij''} = \delta(x - y') \delta(x - z'') \Gamma_{\rho i\alpha} \delta_{j\beta} + \Gamma_{i\gamma} \delta(x - z'') \Gamma_{j\rho} \delta_{i\beta},
\]

\[
T^{2}_{22} \equiv T^{2}_{\lambda ij''} = \delta(x - y') \delta(x - z'') \Gamma_{\lambda \alpha} \delta_{i\beta} - \Gamma_{i\alpha} \Gamma_{\lambda \beta}.
\]

In Equation (31) the components of the curvature tensor got introduced according to

\[
R^{\rho}_{ij\alpha} := \Gamma_{i\alpha} \delta_{j\beta} - \Gamma_{i\alpha} \delta_{j\beta} + \Gamma_{i\gamma} \Gamma_{j\alpha} \delta_{i\gamma} - \Gamma_{j\gamma} \Gamma_{i\alpha} \delta_{i\gamma}.
\]

After having established the structure functions \(R^{i}_{a}, T^{c}_{ab}\), and \(E^{ij}_{ab}\), one has to consider in a next step the Jacobi identity

\[
\sum_{\text{cyclic permutations of 1,2,3}} [\delta_1, [\delta_2, \delta_3]] \Phi^i = 0
\]

in order to check if it produces any non-trivial relations. If we substitute in place of any affine gauge transformation the sum of an infinitesimal translation and a general linear transformation, \(\delta_i = L_{\varepsilon_i} + \delta_{\varepsilon_i\alpha}\), it is algebraically straightforward to show that the Jacobi identity (34) is identically satisfied. No nontrivial relations are produced such that there are no nonzero higher order structure functions besides \(T^{c}_{ab}\).

As final step in the investigation of the gauge algebra it remains to observe that the \(n\) generators of translations \(L_{\varepsilon_i}\) together with the \(n^2\) generators of general linear transformations \(\delta_{\varepsilon_i\alpha}\) are independent, i.e., they define irreducible gauge transformations.

We thus conclude from this section that MAG constitutes a closed, irreducible gauge theory with field dependent structure functions.
4 BRST–antifield symmetry of MAG

The BRST–antifield construction allows to covariantly identify physical functions which differ by (i) a gauge transformation or (ii) a term proportional to the equations of motion \([18, 19]\). Such functions are viewed as physically indistinguishable. An equivalence class of physically indistinguishable functions is called an observable. Covariant quantization requires to consider observables rather than single functions. The BRST–differential \(s\) allows to associate observables to its cohomology.

In the case of a closed, irreducible gauge theory the construction of \(s\) can shortly be summarized as follows [18]: One first introduces two differentials, the so–called longitudinal exterior derivative \(d_L\) and the Koszul–Tate differential \(\delta_{KT}\). The longitudinal exterior derivative measures the change of physical functions along the gauge orbits. Applied to a gauge invariant function, for example, the exterior longitudinal derivative yields zero. Affiliated with the introduction of \(d_L\) is the introduction of ghostfields \(\eta^a\), each of which corresponds to an infinitesimal gauge parameter \(\varepsilon^a\), and a grading called the pure ghostnumber. The Koszul–Tate differential allows to identify functions which differ by terms proportional to the equations of motion. This requires the introduction of antifields \(\Phi^*_i\) and \(\eta^*_a\), which correspond to the fields \(\Phi^i\) and \(\eta^a\), and a grading called the antighostnumber. The BRST–differential \(s\) is defined as the sum of \(d_L\) and \(\delta_{KT}\) modulo terms which make \(s\) nilpotent, \(s^2 = 0\). The grading associated to \(s\) is called the ghostnumber and given by the difference of pure ghostnumber and antighostnumber, \(gh = puregh - antigh\). The objects of this paragraph are summarized in Table 1. The action of \(d_L\) and \(\delta_{KT}\) on the various fields is given by Table 2.

With these definitions the observables are obtained as the cohomology classes of \(s\) at ghostnumber 0:

\[
H^0(s) = \{\text{gauge invariant functions on shell}\} = \{\text{observables}\}
\]  

The classical BRST–transformations of a closed and irreducible gauge theory are explicitly given by [15]

\[
s\Phi^i = R^i_a \eta^a, \tag{37}
\]

\[
s\eta^a = (-1)^{\varepsilon_b} \frac{1}{2} T^a_{bc} \eta^c \eta^b, \tag{38}
\]

\[
s\Phi^*_i = -(-1)^{\varepsilon_i} S_{0,i} - (-1)^{\varepsilon_i + \varepsilon_a} \Phi^j R^i_{a,i} \eta^a

- (-1)^{\varepsilon_b + \varepsilon_c + (\varepsilon_b + \varepsilon_c + 1)} \frac{1}{2} \eta^* c T^a_{bc,i} \eta^c \eta^b, \tag{39}
\]

\[
s\eta^*_a = (-1)^{\varepsilon_a} \Phi^*_i R^i_a + \eta^*_c T^a_{bc} \eta^b. \tag{40}
\]

In order to apply this formalism to MAG we first have to enlarge the field algebra of MAG, which is given by \(g_{\alpha \beta}, \vartheta^a, \text{ and } \Gamma^a_{\alpha \beta}\), by antifields and ghosts. In particular we need the following additional fields:
| object | antigh | puregh | gh | parity |
|--------|--------|--------|----|--------|
| $\Phi^i$ | 0 | 0 | 0 | $\epsilon_i$ |
| $\eta^\alpha$ | 0 | 1 | 1 | $\epsilon_a + 1$ |
| $\Phi_i^*$ | 1 | 0 | −1 | $\epsilon_i + 1$ |
| $\eta_a^*$ | 2 | 0 | −2 | $\epsilon_i$ |
| $\delta_{KT}$ | −1 | 0 | 1 | 1 |
| $d_L$ | 0 | 1 | 1 | 1 |
| $s$ | not applicable | not applicable | 1 | 1 |

Table 1: The different degrees of the main objects in the irreducible antifield- construction. The different gradings are the antighostnumber associated to $\delta_{KT}$, the pure ghostnumber associated to $d_L$, and the ghostnumber defined by $gh = puregh - antigh$.

1. Ghost fields $\eta^\alpha$, $\eta_\alpha^\beta$ corresponding to the gauge parameters $\varepsilon^\alpha$ and $\varepsilon_\alpha^\beta$. The parameter $\varepsilon^\alpha$ denotes the component of the vector field $\varepsilon = \varepsilon^\alpha e_\alpha$ which generates an infinitesimal external translation. Correspondingly we also introduce the notation $\eta := \eta^\alpha e_\alpha$.

2. Antifields $g^{*\alpha\beta}, \vartheta_{\alpha}^*, \Gamma^{*\alpha}_{\beta}$ of antighostnumber 1. In particular, it follows from Table 2 that they have to fulfill

\[
\delta_{KT} g^{*\alpha\beta} = - \frac{\delta S_0}{\delta g_{\alpha\beta}}, \quad (41)
\]

\[
\delta_{KT} \vartheta_{\alpha}^* = - \frac{\delta S_0}{\delta \vartheta_{\alpha}}, \quad (42)
\]

\[
\delta_{KT} \Gamma^{*\alpha}_{\beta} = - \frac{\delta S_0}{\delta \Gamma^{*\alpha}_{\beta}}. \quad (43)
\]
the gauge parameters, simplifies the commutator structure a bit. We obtain gauge structure functions (24) – (28). The opposite statistics of the ghosts, relative to Table 2: The action of the Koszul-Tate differential and the longitudinal exterior derivative on the fields involved in the closed, irreducible antifield-construction.

3. Antifields $\eta^*_a$, $\eta^{\alpha\beta}$ of antighost number 2. In the BRST-antifield formalism they got, in fact, introduced in order to ensure the validity of the Noether identities and to make the antifields $g^{*\alpha\beta}$, $\vartheta^*_\alpha$, $\Gamma^{*\alpha\beta}$ $\delta_{KT}$-exact \cite{18}. The corresponding transformation behavior $\delta_{KT}\eta^*_a = (-1)^{\alpha} \Phi^*_a R^a_\alpha$, see Table 2, becomes explicitly

$$
\delta_{KT}\eta^*_a = g^{*\gamma\delta} L_{e_a g_{\gamma\delta}} + \vartheta'^*_{\gamma} \wedge L_{e_a \vartheta^\gamma} + \Gamma^{*\gamma\delta} \wedge L_{e_a \Gamma_{\gamma\delta}}, \\
\delta_{KT}\eta^{*\alpha\beta} = + 2g^{*\alpha\gamma} g_{\beta\gamma} + \vartheta^*_{\beta} \wedge \vartheta^\alpha + \Gamma^{*\alpha\beta} \Gamma. 
$$

(44)

(45)

It is straightforward to obtain explicit expressions like this: Take for example the term $\Phi^*_a R^a_\alpha\eta^*_a$, where $R^a_\alpha$ is supposed to represent the generating set of $A(n, R)$-gauge transformations. Its explicit form is derived from the transformation behavior of the fields $g_{\alpha\beta}$, $\vartheta^\alpha$, and $\Gamma_{\alpha\beta}$ under external translations and $GL(n, R)$-gauge transformations: One simply replaces gauge parameters by ghosts and contracts with the corresponding antifield. This furnishes successively the contributions

$$
\Phi^i \equiv g_{\alpha\beta} \quad \rightarrow \quad \Phi^*_a R^a_\alpha \eta^*_a \equiv g^{*\gamma\delta} (L_{\eta g_{\alpha\beta}} + 2\eta_{(\alpha\beta)}) , \\
\Phi^i \equiv \vartheta^\alpha \quad \rightarrow \quad \Phi^*_a R^a_\alpha \eta^*_a \equiv \vartheta'^*_{\alpha} (L_{\eta \vartheta} + \eta_{\alpha} \vartheta^\alpha) , \\
\Phi^i \equiv \Gamma_{\alpha\beta} \quad \rightarrow \quad \Phi^*_a R^a_\alpha \eta^*_a \equiv \Gamma^{*\alpha\beta} (L_{\eta \Gamma_{\alpha\beta}} - \Gamma). 
$$

(46)

(47)

(48)

As another example we consider the term $\frac{1}{2} \eta^*_a T^a_{bc} \eta^* \eta^b$. Its explicit form follows from the gauge structure functions (24) – (28). The opposite statistics of the ghosts, relative to the gauge parameters, simplifies the commutator structure a bit. We obtain

$$
\eta^*_a \equiv \eta^* \quad \rightarrow \quad \frac{1}{2} \eta^*_a T^a_{bc} \eta^* \eta^b \equiv \frac{1}{2} \eta^* \eta = \eta^* \eta \eta d\eta , \\
\eta^*_a \equiv \eta^{*\alpha\beta} \quad \rightarrow \quad \frac{1}{2} \eta^*_a T^a_{bc} \eta^* \eta^b \equiv \eta^{*\alpha\beta} (\eta \eta^{\alpha}) + \eta^{*\alpha\beta} (\eta^{\alpha} \eta^{\gamma} \eta^{\beta}). 
$$

(49)

(50)
Now we write down the classical BRST(–antifield)–transformations of MAG according to the general formulas (37)–(40):

\[
\begin{align*}
sg_{\alpha\beta} &= L_{\eta} g_{\alpha\beta} + 2\eta_{(\alpha}\delta \eta_{\beta)}, \\
sg^{\alpha} &= L_{\eta} g^{\alpha} + \eta_{\beta}^{\alpha} g^{\beta}, \\
sg_{\Gamma^\alpha_{\beta}} &= L_{\eta} \Gamma^\alpha_{\beta} - \frac{\Gamma}{D} \eta_{\alpha}^{\beta}, \\
sg_{\eta_{\alpha}} &= -\eta \frac{\partial}{\partial \eta_{\alpha}}, \\
sg_{\eta_{\alpha}^{\beta}} &= -\eta |(\eta | R_{\alpha}^{\beta}) - \eta | D \eta_{\alpha}^{\beta} - \eta_{\gamma}^{\beta} \eta_{\alpha}^{\gamma}, \\
sg_{g^{*\alpha\beta}} &= -\frac{\delta S_0}{\delta g_{\alpha\beta}} - g^{*\alpha\beta} L_{\eta} - g^{*\alpha\gamma} \eta_{\gamma}^{\beta} - g^{*\gamma\beta} \eta_{\alpha}^{\gamma}, \\
sg_{\eta_{\alpha}^{\ast}} &= -\frac{\delta S_0}{\delta g^{\alpha}_{\ast}} - \vartheta \Gamma^\alpha_{\ast} L_{\eta} - \vartheta \eta_{\alpha}^{\beta}, \\
sg_{\Gamma^{*\alpha}_{\beta}} &= -\frac{\delta S_0}{\delta \Gamma^{*\alpha}_{\beta}} - \Gamma^{*\alpha}_{\beta} L_{\eta} + \Gamma^{*\gamma}_{\delta} C_{\gamma} \Gamma^{\delta}_{\beta} \rho \eta_{\rho}^{\lambda} \\
&+ \eta^{*\gamma}_{\delta} C_{\gamma} \Gamma^{\delta}_{\beta} \rho \eta_{\rho}^{\lambda} + \eta^{*\alpha}_{\beta} \eta | (d), \\
&+ 2 \eta^{*\gamma}_{\delta} (\eta | \Gamma^{\delta}_{\beta}) \eta_{\gamma}^{\lambda}, \\
sg_{\eta_{\alpha}^{*}} &= g^{*\gamma}_{\delta} e_{\alpha}^{\gamma} + \vartheta_{\ast}^{\gamma} \Gamma^{\gamma}_{\delta} \eta_{\delta}^{\gamma} + \vartheta_{\ast}^{\alpha}, \\
sg_{\eta_{\alpha}^{*\beta}} &= +2g^{*\gamma}_{\delta} \rho g_{\beta\gamma} + \vartheta_{\ast}^{\gamma} \vartheta_{\ast}^{\alpha} + \Gamma^{\alpha}_{\beta} \rho D \\
&+ \eta^{*\alpha}_{\beta} \eta | D + \eta^{*\gamma}_{\delta} \eta_{\gamma}^{\lambda}. \\
\end{align*}
\]

If matter fields \( \psi \) are present we also have to introduce an antifield \( \psi^* \) corresponding to any matter field \( \psi \). The BRST–transformations of these fields are of the form

\[
\begin{align*}
sg_{\psi} &= (L_{\eta} + \delta_{\eta_{\alpha}^{\beta}}) \psi, \\
sg_{\psi^*} &= -(-1)^{\epsilon_{\psi}} \frac{\delta S_0}{\delta \psi} - \frac{\delta (s \psi)}{\delta \psi}. \\
\end{align*}
\]

In the antifield formalism the BRST–transformations are generated by means of the antibracket \( (\cdot, \cdot) \) and the extended action \( S \). That is, the BRST–transformation \( s \mathcal{F} \) of a functional \( \mathcal{F} = \mathcal{F}[\Phi^i, \Phi^*_i, \eta^a, \eta^*_a] \) are given by

\[
s \mathcal{F} = (\mathcal{F}, S).
\]

The antibracket is explicitly defined by the requirements

\[
\begin{align*}
(\Phi^i(x), \Phi^*_j(x')) &= \delta^j_i \delta(x - x'), \quad \text{i.e.} \quad (\Phi^i, \Phi^*_j) = \delta^j_i, \\
(\eta^a(x), \eta^*_b(x')) &= \delta^a_b \delta(x - x'), \quad \text{i.e.} \quad (\eta^a, \eta^*_b) = \delta^a_b.
\end{align*}
\]
such that its action on functionals $\mathcal{F}$, $\mathcal{G}$ reads:

\[
(\mathcal{F}, \mathcal{G}) = \frac{\delta^r \mathcal{F}}{\delta \Phi^i} \frac{\delta^l \mathcal{G}}{\delta \Phi^i} - \frac{\delta^r \mathcal{F}}{\delta \Phi^i} \frac{\delta^l \mathcal{G}}{\delta \Phi^i} + \frac{\delta^r \mathcal{F}}{\delta \eta^a} \frac{\delta^l \mathcal{G}}{\delta \eta^a} - \frac{\delta^r \mathcal{F}}{\delta \eta^a} \frac{\delta^l \mathcal{G}}{\delta \eta^a}.
\] (66)

The nilpotency of the BRST–transformation, $s^2 = 0$, is equivalent to the (classical) Master equation:

\[
(S, S) = 0,
\] (67)

which can also be taken as the starting point of the BRST–antifield construction.

In the case of a closed, irreducible gauge theory the BRST–transformations (37), (40) are generated according to (63) if the extended action $S$ takes the form

\[
S = S_0 + \Phi^*_i R^a \eta^a + (-1)^{\epsilon_1} \frac{1}{2} \Phi^*_a T^{a}_{bc} \eta^b.
\] (68)

This is easily proven by taking in (63) the functional $F$ successively as $\Phi$, $\Phi^*$, $\eta^a$, and $\eta^*_a$. Due to the nilpotency of the BRST–transformations the extended action $S$ satisfies automatically the Master equation (67), i.e. the extended action is BRST–invariant. It is a proper BRST–invariant extension of the gauge invariant, classical action $S_0$.

According to (68), the BRST–transformations of MAG, (51) – (62), are generated by the extended action

\[
S = S_0 + \int \left( g^{*\alpha\beta} (L_\eta g_{\alpha\beta} + 2\eta_{(\alpha\beta)}) + \vartheta^*_\alpha \wedge (L_\eta \vartheta^\alpha + \eta^\alpha \vartheta^\beta) + \Gamma^{*\alpha\beta} \wedge (L_\eta \Gamma^\alpha_{\beta} - D\eta^\alpha_{\beta}) + \psi^* \wedge (L_\eta \psi + \delta \eta_{\alpha\beta} \psi) + \eta^*_\alpha (\eta^\beta D\eta^\alpha) + \eta^* \alpha_{\beta} (\eta^\beta R^\alpha_{\beta}) + \eta^\beta D\eta^\alpha_{\beta} + \eta^\alpha \gamma R^\beta_{\gamma}) \right).
\] (69)

5 Gauge fixing of two–dimensional MAG

The BRST–invariant action (68) is not yet suitable to be used in a generating functional of the form

\[
Z[J] \sim \int [D\Phi D\eta D\Phi^* D\eta^*] \exp \left( \frac{i}{\hbar} S(\Phi, \eta, \Phi^*, \eta^*, J) \right) \quad \text{(not well defined)}.
\] (70)

This is because differentiating the Master equation yields an unwanted set of gauge transformation under which $S$ is invariant [18]. Also one would like to eliminate the antifields before deriving Green’s functions in a perturbative expansion, simply because there exists no satisfying physical interpretation of the antifields, yet.

\footnote{The indices $r$ and $l$ denote right and left differentiation, respectively. So far we used right differentiation without an index.}
Following [14] we can eliminate both the unwanted gauge invariances and the antifields by introducing a \textit{gauge fixing fermion} $\Psi$ which is defined to be a functional of fields $\Phi^A$. With an appropriate gauge fixing fermion a gauge fixed extended action $S_{\text{fix}}$ can be reached by the replacement of antifields by fields according to
\[ \Phi^*_A = - \frac{\delta \Psi}{\delta \Phi^A}. \] (71)
Since $\Phi^*_A$ and $\Phi^A$ are of different parity, $\Psi$ must be of odd parity, i.e. fermionic.

A field $\Phi^A$, i.e., an original field or a ghost field, possesses a ghost number $n \geq 0$. The corresponding antifield $\Phi^*_A$ is of ghostnumber $-(n+1) < 0$. Thus, according to equation (71), $\Psi$ must be of ghostnumber $-1$. Since $\Psi$ is supposed to be a functional of fields only it is inevitable to introduce auxiliary fields with negative ghostnumber (plus their corresponding antifields in order to maintain the symplectic structure on $M$). This can be done straightforwardly since it is always possible to add within the BRST-formalism \textit{cohomologically trivial pairs} that do not change the physical content of the theory: Consider the auxiliary fields $\eta^a, b^a$ which are defined to satisfy
\[ s \eta^a = b^a, \quad sb^a = 0. \] (72)
The field $\eta^a$ is not an element of the kernel $\text{Ker}(s)$ while the field $b^a$ is both an element of $\text{Ker}(s)$ and the image $\text{Im}(s)$. Thus both fields $\eta^a$ and $b^a$ are not contained in $H(s) = \text{Ker}(s) \cap \text{Im}(s)$ and do not contribute to the spectrum of observables.

According to (63) one can impose the BRST-transformations (72) by adding the auxiliary term $S_{\text{aux}} = \int \eta^a \wedge b^a$ to the action $S$:
\[ S \longrightarrow S_{\text{non-min}} = S + S_{\text{aux}} = S + \int \eta^a \wedge b^a. \] (73)
The extended action together with this supplementary term is an example of a non-minimal solution of the master equation. The antibracket $( , ) = \frac{\delta}{\delta \Phi^A} \frac{\delta}{\delta \Phi^*_A} - \frac{\delta}{\delta \Phi^*_A} \frac{\delta}{\delta \Phi^A}$ now also contains derivatives involving the field-antifield pairs $\eta^a, \eta^*_a$ and $b^a, b^*_a$.

Finally one has to gauge fix the action $S_{\text{non-min}}$ by actually choosing a gauge fixing fermion $\Psi$. Not all choices are meaningful. The trivial choice $\Psi = 0$, for example, sets all antifields to zero and leads back to the classical action $S_0$. There are no definite rules how to choose an appropriate gauge fixing fermion.

In the following we will illustrate gauge fixing of MAG by means of a general two-dimensional model. Gauge fixing procedures for higher dimensional models follow the same pattern but quickly get algebraically more complicated.

We start from the extended action
\[ S = S_0 + \int \left( g^{\alpha \beta} (L_\eta g_{\alpha \beta} + 2 \eta_{(\alpha \beta)}) + \partial^*_\alpha \wedge (L_\eta \partial^\alpha + \eta^\alpha \partial^\beta) + \Gamma^*_\alpha \wedge (L_\eta \Gamma^\alpha - \bar{D} \eta^\beta) + \psi^* \wedge (L_\eta \psi + \delta_\alpha \psi) + \eta^*_a \eta^a \partial^\alpha + \eta^*_a \eta^\beta \eta^a \right). \] (74)
The indices $\alpha, \beta$ run from 0 to 1. Here and in the following we will use 0, 1 to indicate anholonomic indices and $\tau, \sigma$ to indicate holonomic indices. A convention like this is necessary in view of partial derivatives or frames which could be understood as holonomic ($\partial_i, dx^i$) or anholonomic ($\partial^\alpha, \vartheta^\alpha = e^i_\alpha dx^i$, with, in general, coordinate dependent tetrad coefficients $e^i_\alpha, e^\alpha_i$).

Next we introduce the following $4 \times 2 = 8$ auxiliary fields:

translating: \( \eta^\alpha, b_\alpha, \eta^*\alpha, b^*\alpha \),  
linear transformations: \( \eta^\alpha_\beta, b_\alpha^\beta, \eta^{*\alpha}_\beta, b^{*\alpha}_\beta \).  

We impose the correct BRST–transformation behavior of these auxiliary fields by adding the auxiliary term 

\[ S_{aux} = \int (\eta^\alpha b_\alpha + \frac{1}{2} \eta^{\alpha\beta} b_\alpha^\beta) \]  

(77)

to the extended action.

Now we have to think of an appropriate gauge fixing fermion. In two dimensional MAG we have six gauge parameters, i.e., two parameters $\varepsilon_\alpha$ of translation invariance and four parameters $\varepsilon^\alpha_\beta$ of general linear invariance. We can use four of these degrees of gauge freedom to fix the coframe $\vartheta^\alpha$ to the conformal gauge 

\[ \vartheta^0 = d\tau, \quad \vartheta^1 = d\sigma. \]  

(78)

However, this does not fix the coframe to be orthonormal since the metric components are independent fields of the theory. In two dimensions, the metric tensor has three independent components. We can use the remaining two degrees of gauge freedom to fix the metric to be diagonal with one remaining degree of freedom, \( g_{01} = g_{10} = 0, g_{00} = -g_{11} =: (1/2) \exp(\rho) = (1/2) \exp(\rho(\tau, \sigma)) \). (We put $g_{00} = -g_{11}$ since we assume a Minkowskian signature of the metric. The following procedure works also for Euclidean signature, though.) This gauge can be reached if we choose the gauge fixing fermion as 

\[ \Psi = \int \left( *\eta^0_0 (\partial_\tau \vartheta^1 - \partial_\sigma \vartheta^0) + *\eta^1_0 (\partial_\tau \vartheta^0 - \partial_\sigma \vartheta^1) + *\eta^1_0 (\partial_\tau \vartheta^0 + \partial_\sigma \vartheta^1 - 2) + *\eta_0 g_{01} + *\eta_1 (g_{00} + g_{11}) \right). \]  

(79)

We remove the antifields via the rule $\Phi^*_A = -\frac{\delta\Psi}{\delta\Phi^*_A}$ and obtain the explicit replacements

\[ \eta^0_0 = -*(\partial_\tau \vartheta^1 - \partial_\sigma \vartheta^0), \]  

(80)
\[ \eta^1_0 = -*(\partial_\tau \vartheta^0 - \partial_\sigma \vartheta^1), \]  

(81)
\[ \eta^0_1 = -*(\partial_\tau \vartheta^1 + \partial_\sigma \vartheta^0), \]  

(82)
\[ \eta^1_0 = -*(\partial_\tau \vartheta^0 + \partial_\sigma \vartheta^1 - 2), \]  

(83)
\[ \eta^0 = -*g_{01}. \]  

(84)
The single terms that appear in the integral of (97) turn out to be

$$\eta^1 \eta = - (g_{00} + g_{11})$$  \hspace{1cm} (85)

$$\vartheta^*_0 \varphi = - \partial_\sigma [\vartheta^*_0 + \partial_\tau] \varphi^*_1 + \partial_\sigma \varphi^*_1 + \partial_\tau \varphi^*_1$$  \hspace{1cm} (86)

$$\varphi^*_1 = + \partial_\sigma \varphi^*_0 - \partial_\sigma \varphi^*_0 + \partial_\tau \varphi^*_0 + \partial_\sigma \varphi^*_0$$  \hspace{1cm} (87)

$$g^{01} = - \varphi^*_0$$  \hspace{1cm} (88)

$$g^{00} = \varphi^*_1 = - \varphi^*_1$$  \hspace{1cm} (89)

$$\Gamma^\alpha_{\beta \gamma} = \eta^{\alpha \beta} = \eta^{\alpha \beta} = 0$$  \hspace{1cm} (90)

Within a path integral we can integrate out the auxiliary variables $b_\alpha$, $b_\alpha^\beta$ and also $\varphi^\alpha$, $g_{01}$, and $g_{00}$. This leads to the gauge conditions

$$\partial_\tau \varphi^0 = \partial_\sigma \varphi^1$$  \hspace{1cm} (91)

$$\partial_\tau \varphi^0 = \partial_\sigma \varphi^1$$  \hspace{1cm} (92)

$$\partial_\tau \varphi^0 = - \partial_\sigma \varphi^0$$  \hspace{1cm} (93)

$$\partial_\tau \varphi^0 + \partial_\sigma \varphi^1 = 2$$  \hspace{1cm} (94)

$$g_{01} = 0$$  \hspace{1cm} (95)

$$g_{00} = - g_{11}$$  \hspace{1cm} (96)

and the gauge fixed action reduces to

$$S_{fix} = S_0 + \int \left( g^{\alpha \beta} (\mathcal{L}_\eta g_{\alpha \beta} + 2 \eta_{(\alpha \beta)}) + \varphi^* \eta^\alpha \wedge (b_\eta \varphi^\alpha + \eta_\beta \varphi^\beta) \right)$$  \hspace{1cm} (97)

The single terms that appear in the integral of (97) turn out to be

$$g^{\alpha \beta} \mathcal{L}_\eta g_{\alpha \beta} = 0$$  \hspace{1cm} (98)

$$2 g^{\alpha \beta} \eta_{(\alpha \beta)} = - \varphi^* \exp(\rho) (\eta_0^0 + \eta_1^1) - \varphi^* \exp(\rho) (\eta_0^1 + \eta_1^0)$$  \hspace{1cm} (99)

$$\varphi^* \wedge \mathcal{L}_\eta \varphi^\alpha = - \varphi^* \exp(\rho) (\eta_0^0 - \eta_1^1) + \varphi^* \exp(\rho) (\eta_0^1 - \eta_1^0)$$

$$\varphi^* \wedge \eta_\beta^0 \varphi^\beta = - (\varphi^* \exp(\rho) (\eta_0^0 - \eta_1^1) - \varphi^* \exp(\rho) (\eta_0^1 - \eta_1^0))$$

The expression (100) can be written in a more compact way: We first note that, due to the conformal gauge,

$$(\mathcal{L}_\eta \varphi^\alpha) (\partial_\tau) = \partial_\tau \eta^\alpha + (\eta_0^0 \eta^1 + \eta_1^0 \eta^1) =: D_\tau \eta^\alpha$$  \hspace{1cm} (102)

An example of (102) is $$(\mathcal{L}_\eta \varphi^0) (\partial_\tau) = \partial_\tau \eta^0 + \eta_0^0 \eta^1 = D_\tau \eta^0$$. With this notation we can write the term (100) in the form

$$\varphi^* \wedge \mathcal{L}_\eta \varphi^\alpha = \varphi^* \exp(\rho) (\eta_0^0 \eta^1 - D_\tau \eta^1) + \varphi^* \exp(\rho) (\eta_0^1 \eta^0 - D_\tau \eta^0)$$

$$- \eta_0^0 (\eta_0^1 + \eta_1^0 - \eta_1^1) - \eta_0^1 (\eta_0^0 + \eta_1^1)$$  \hspace{1cm} (103)
We collect all pieces and obtain the gauge fixed action

\[ S_{fix} = S_0 + \int \left( *\eta_0^0 (D_\sigma \eta^0 - D_\tau \eta^1) + \ast\eta_1^1 (D_\sigma \eta^1 - D_\tau \eta^0) \right. \]
\[ \quad + \left. *\eta_0^1 (D_\sigma \eta^0 + D_\sigma \eta^1) + *\eta_1^0 (D_\tau \eta^0 + D_\sigma \eta^1) \right. \]
\[ \quad - (\ast\eta_1^1 + *\eta_0^1 + *\eta_1 \exp(\rho)) \eta_0^0 - (\ast\eta_0^0 + *\eta_0 \exp(\rho)) \eta_0^1 \]
\[ \quad \left. - (\ast\eta_0^0 - \ast\eta_0^1 + *\eta_0 \exp(\rho)) \eta_1^0 - (\ast\eta_1^1 - \ast\eta_0 \exp(\rho)) \eta_1^1 \right). \]

(104)

Integrating out the ghosts \( \eta_0^0, \eta_0^1, \eta_1^0, \) and \( \eta_1^1 \) yields finally

\[ S_{fix} = S_0 + \exp(\rho) \int \left( \ast\eta_0^0 (D_\tau \eta^1 - D_\sigma \eta^0) + \ast\eta_1^1 (D_\tau \eta^0 - D_\sigma \eta^1) \right). \]

(105)

This is a general result which does not refer to any particular form of the initial action \( S_0 \).

We see that in (105) the ghosts not only couple to the conformal factor \( \exp(\rho) \) but, via (102), also to the connection \( \Gamma^\alpha_{\beta\gamma} \). This feature could have been expected by the use of the gauge–covariant Lie–derivative as generator of translations. It makes it no longer possible to straightforwardly quantize a model based on (105). The coupling to the connection forbids to write down plane–wave solutions for the (anti–)ghosts.

As a particular example of (105) we can consider the action of the bosonic string [34] which is given by the two–dimensional integral

\[ S_0[X_\mu, g_{\alpha\beta}, \vartheta^\alpha] = -\frac{1}{4} \int dX_\mu \wedge \ast dX^\mu = -\frac{1}{4} \int_{\tau=+\infty}^{\tau=-\infty} \int_{\sigma=0}^{\sigma=\pi} dX_\mu(\tau, \sigma) \wedge \ast dX^\mu(\tau, \sigma). \]

(106)

The integration area, commonly called the world–sheet, is parameterized by the timelike coordinate \( \tau \) and the spatial coordinate \( \sigma \). The spatial region is supposed to be finite. This is indicated by letting \( \sigma \) range from 0 to \( \pi \). The fields \( X^\mu \) are defined to be scalar fields on the world–sheet. The index \( \mu \) is a priori unrelated to the two–dimensional integration area and can be seen as merely numbering the scalar fields \( X^\mu \). The integrand \( -\frac{1}{4} dX_\mu \wedge \ast dX^\mu \) of the action (106) is not written in a transparent form since it mixes derivatives of the scalar fields with the integration measure. We can write more explicitly

\[ -\frac{1}{4} dX_\mu \wedge \ast dX^\mu \quad = \quad -\frac{1}{4} \partial_\alpha X_\mu \partial_\beta X^\mu \vartheta^\alpha \wedge \ast \vartheta^\beta \]
\[ \quad = \quad -\frac{1}{4} \partial_\alpha X_\mu \partial_\beta X^\mu g^{\beta\delta} \eta_\delta \vartheta^\alpha \wedge \vartheta^\gamma \]
\[ \quad = \quad -\frac{1}{2} \partial_\alpha X_\mu \partial_\beta X^\mu g^{\alpha\beta} \eta, \]

(107)

where we introduced the two–dimensional volume element \( \eta = \frac{1}{2} \eta_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta \) with \( \eta_{\alpha\beta} = \sqrt{\left| \det g \right|} \epsilon_{\alpha\beta}, \epsilon_{01} = 1, \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \). The absence of a connection \( \Gamma^\alpha_{\beta\gamma} \) in (106) reduces
the general formula (105) to the well-known quantum action of the bosonic string in conformal gauge,

\[
S_{fix} = \int \left( \frac{-1}{2} (\partial_\tau X_\mu \partial_\tau X^\mu - \partial_\sigma X_\mu \partial_\tau X^{\mu}) \eta + \exp(\rho)(\ast \eta_0 (\partial_\sigma \eta^0 - \partial_\tau \eta^1) + \ast \eta_1 (\partial_\sigma \eta^1 - \partial_\tau \eta^0)) \right). \tag{109}
\]

Now the ghost are only coupled to the conformal factor \(\exp(\rho)\) which is the origin of the conformal anomaly.

In the context of string theory, models based on (105) with nontrivial connection \(\Gamma^\alpha_{\beta\gamma}\) seem to have not been investigated, yet. Also the quantization of some other two-dimensional model of MAG, based on the general solution (105), seems to have never been conducted.

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