Bounded convergence theorem for abstract Kurzweil-Stieltjes integral

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Abstract

In the theories of Lebesgue integration and of ordinary differential equations, the Lebesgue Dominated Convergence Theorem provides one of the most widely used tools. Available analogy in the Riemann or Riemann-Stieltjes integration is the Bounded Convergence Theorem, sometimes called also the Arzelà or Arzelà-Osgood or Osgood Theorem. In the setting of the Kurzweil-Stieltjes integral for real valued functions its proof can be obtained by a slight modification of the proof given for the σ-Young-Stieltjes integral by T.H. Hildebrandt in his monograph from 1963. However, it is clear that the Hildebrandt’s proof cannot be extended to the case of Banach space-valued functions. Moreover, it essentially utilizes the Arzelà Lemma which does not fit too much into elementary textbooks. In this paper, we present the proof of the Bounded Convergence Theorem for the abstract Kurzweil-Stieltjes integral in a setting elementary as much as possible.

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1 Introduction

In the theories of Lebesgue integration and of ordinary differential equations, the Lebesgue Dominated Convergence Theorem provides one of the most widely used tools. Available analogy in the Riemann or Riemann-Stieltjes integration is the Bounded Convergence Theorem, sometimes called also the Arzelà or Arzelà-Osgood or Osgood Theorem. In the setting of the Kurzweil-Stieltjes integral for real valued functions this result reads as follows:

If \( F : [a, b] \to \mathbb{R} \) has a bounded variation on \([a, b]\), \( g : [a, b] \to \mathbb{R} \) is regulated on \([a, b]\) and the sequence \( \{g_n\} \) of functions regulated on \([a, b]\) is such that

\[
\lim_{n \to \infty} g_n(t) = g(t) \quad \text{for} \quad t \in [a, b]
\]

and

\[
\|g_n\|_{\infty} \leq K < \infty \quad \text{for} \quad n \in \mathbb{N},
\]

then

\[
\lim_{n \to \infty} \int_{a}^{b} d[F] g_n = \int_{a}^{b} d[F] g.
\]

The proof can be easily obtained by a slight modification of the proof given for the \( \sigma \)-Young integral by Hildebrandt in [7, Theorem II.19.3.14], cf. [21, Theorem I.4.24]. However, it is clear that the proof by Hildebrandt cannot be extended to the case of Banach space-valued functions. Moreover, it essentially utilizes the Arzelà Lemma which does not fit too much into elementary text-books. Even in the case of the Riemann integral the elementary proof of the corresponding Bounded Convergence Theorem has been for a long time considered to be rather impossible as stated by Lewin in [10]:

“The bounded convergence theorem follows trivially from the Lebesgue dominated convergence theorem, but at the level of an introductory course in analysis, when the Riemann integral is being studied, how hard is the bounded convergence theorem? For an answer, we might look at Bartle and Sherbert [2], page 203: “The proof of this result is quite delicate and will be omitted”. Or we might look at Apostol [1], page 228: “The proof of Arzela’s theorem is considerably more difficult than ... and will not be given here”. Walter Rudin in [14] ignores the theorem altogether in his chapter on Riemann integration, presenting it only as a corollary to the Lebesgue dominated convergence theorem several chapters later, and in [15], in an interesting problem in Chapter Two, Rudin refers his readers to [5]. In [5], Eberlein does present a proof which from some points of view is elementary. Certainly, his proof does not require any notions of measurability, but it is hardly elementary from the point of view of a student who is first learning the Riemann integral.”
Despite that, in earlier times, many other authors have dedicated themselves to obtaining a proof independent of the theory of Lebesgue measure for such convergence result. See for instance [11] and the references therein. Of course, in [10], Lewin succeeded in his search for an elementary proof that, as he claimed, “could be included for the first time in an introductory course”.

In this paper, we utilize some of the Lewin’s ideas and present the proof of the Bounded Convergence Theorem for the abstract Kurzweil-Stieltjes integral in a setting elementary as much as possible.

2 Preliminaries

Throughout this paper $X$ is a Banach space and $L(X)$ is the Banach space of all bounded linear operators on $X$. By $\| \cdot \|_X$ we denote the norm in $X$, while $\| \cdot \|_{L(X)}$ denotes the usual operator norm in $L(X)$.

For $-\infty < a < b < \infty$, $[a, b]$ and $(a, b)$ denote the corresponding closed and open intervals, respectively. Furthermore, $[a, b)$ and $(a, b]$ are the corresponding half-open intervals while $[c]$ denotes the degenerate interval consisting of a single real number $c \in [a, b]$.

A set $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\} \subset [a, b]$ with $\nu(D) \in \mathbb{N}$ is said to be a division of $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_{\nu(D)} = b.$$  

The set of all divisions of $[a, b]$ is denoted by $\mathcal{D}[a, b]$. The symbol $\nu(D)$ will be kept for the number of subintervals $[\alpha_{j-1}, \alpha_j]$ generated by the division $D$.

A function $f : [a, b] \to X$ is called a finite step function on $[a, b]$ if there exists a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ of $[a, b]$ such that $f$ is constant on every open interval $(\alpha_{j-1}, \alpha_j)$, $j = 1, 2, \ldots, m$.

For an arbitrary function $f : [a, b] \to X$ we set

$$\|f\|_\infty = \sup_{t \in [a, b]} \|f(t)\|_X$$

and the variation of $f$ on $[a, b]$ is given by

$$\text{var}_a^b f = \sup_{D \in \mathcal{D}[a, b]} \sum_{j=1}^{\nu(D)} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$  

Sometimes, $\text{var}_a^b f$ is also called the Jordan variation of $f$ on $[a, b]$. If $\text{var}_a^b f < \infty$ we say that $f$ is of bounded variation on $[a, b]$. $BV([a, b], X)$ denotes the set of all functions $f : [a, b] \to X$ of bounded variation on $[a, b]$. 
\( C([a, b], X) \) is the set of all \( X \)-valued functions which are continuous on \([a, b]\), while \( G([a, b], X) \) denotes the set of all regulated functions \( f : [a, b] \to X \). Recall that \( f : [a, b] \to X \) is regulated on \([a, b]\) if for each \( t \in [a, b] \) there is \( f(t+) \in X \) such that

\[
\lim_{s \to t^+} \|f(s) - f(t+)\|_X = 0
\]

and for each \( t \in (a, b] \) there is \( f(t-) \in X \) such that

\[
\lim_{s \to t^-} \|f(s) - f(t-)\|_X = 0.
\]

Furthermore, for \( t \in [a, b] \) we put \( \Delta^+ f(t) = f(t+) - f(t) \), \( \Delta^- f(t) = f(t) - f(t-) \) and \( \Delta f(t) = f(t+) - f(t-) \) (where by convention \( \Delta^- f(a) = \Delta^+ f(b) = 0 \)).

Given \( F : [a, b] \to L(X) \), we say \( F \) is simply-regulated on \([a, b]\) if, for each \( x \in X \), the function \( t \in [a, b] \to F(t) x \in X \) is regulated. In this case, we write \( F \in (\mathcal{B})G([a, b], L(X)) \).

Clearly,

\[
C([a, b], L(X)) \subset G([a, b], L(X)) \subset (\mathcal{B})G([a, b], L(X)).
\]

Moreover, it is known that \( BV([a, b], L(X)) \subset G([a, b], L(X)) \), as well.

Now, let us recall the definition of the abstract Kurzweil-Stieltjes integral as introduced by Š. Schwabik in [18].

Systems \( P = \{(\tau_j, [\alpha_{j-1}, \alpha_j]) : j = 1, \ldots, m\} \), where \( \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \in \mathcal{D}[a, b] \) and \( \tau_j \in [\alpha_{j-1}, \alpha_j] \) for \( j = 1, \ldots, m \), are called tagged divisions of \([a, b]\).

Furthermore, functions \( \delta : [a, b] \to (0, \infty) \) are said to be gauges on \([a, b]\). Given a gauge \( \delta \) on \([a, b]\), the tagged division \( P = \{(\tau_j, [\alpha_{j-1}, \alpha_j]) : j = 1, \ldots, m\} \) of \([a, b]\) is said to be \( \delta \)-fine if

\[
[\alpha_{j-1}, \alpha_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) \quad \text{for} \quad j = 1, \ldots, m.
\]

We remark that for an arbitrary gauge \( \delta \) on \([a, b]\) there always exists a \( \delta \)-fine tagged division of \([a, b]\). This is stated by the Cousin lemma (see [17, Lemma 1.4]).

For given functions \( F : [a, b] \to L(X) \) and \( g : [a, b] \to X \) and a tagged division \( P = \{(\tau_j, [\alpha_{j-1}, \alpha_j]) : j = 1, \ldots, m\} \) of \([a, b]\), we define

\[
S(F, dg, P) = \sum_{j=1}^{m} F(\tau_j) [g(\alpha_j) - g(\alpha_{j-1})]
\]

and

\[
S(dF, g, P) = \sum_{j=1}^{m} [F(\alpha_j) - F(\alpha_{j-1})] g(\tau_j).
\]
We say that \( I \in X \) is the Kurzweil-Stieltjes integral (or shortly KS-integral) of \( F \) with respect to \( g \) from \( a \) to \( b \) and write

\[
I = \int_a^b F \, dg
\]

if for every \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \([a, b]\) such that

\[
\left\| S(F, dg, P) - I \right\|_X < \varepsilon \quad \text{for all } \delta\text{-fine tagged divisions } P \text{ of } [a, b].
\]

Similarly, \( J \in X \) is the KS-integral of \( g \) with respect to \( F \) from \( a \) to \( b \) if for every \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \([a, b]\) such that

\[
\left\| S(dF, g, P) - J \right\|_X < \varepsilon \quad \text{for all } \delta\text{-fine tagged divisions } P \text{ of } [a, b].
\]

In this case we write \( J = \int_a^b d[F] g. \)

For the basic properties of the abstract KS-integral, we refer to [18]–[20] and [12]–[13].

### 3 Jordan decomposition

In this section we will show that, as in the case of real valued functions, any function of bounded variation on \([a, b]\) can be written as the sum of a continuous function and a break function.

Recall that a function \( f: [a, b] \to X \) is said to be a break function if there exist sequences

\[
\{s_k\} \subset [a, b], \quad \{c_k\} \subset X, \quad \{d_k\} \subset X,
\]

such that

\[
s_k \neq s_\ell \text{ if } k \neq \ell, \quad \sum_{k=1}^{\infty} (\|c_k\|_X + \|d_k\|_X) < \infty,
\]

and

\[
f(t) = \sum_{k=1}^{\infty} \left( c_k \chi_{(s_k, u_k]}(t) + d_k \chi_{[s_k, u_k)}(t) \right) \quad \text{for } t \in [a, b].\tag{3.2}
\]

Since for any \( t \in [a, b] \), the series on the right-hand side of (3.2) converges absolutely, it converges also unconditionally (see [4, Chapter VI]). Hence we can write also

\[
f(t) = \sum_{a \leq s_k < t} c_k + \sum_{a < s_k \leq t} d_k \quad \text{for } t \in [a, b].\tag{3.3}
\]
It is not difficult to see that, if $f$ is given by (3.3) (or (3.2)), then
\[ \Delta^+ f(s_k) = c_k \quad \text{and} \quad \Delta^- f(s_k) = d_k \quad \text{for} \quad k \in \mathbb{N}. \]

In particular, the one-sided limits are given by the expressions
\[
\begin{align*}
 f(t+) &= \sum_{a \leq s_k \leq t} c_k + \sum_{a < s_k \leq t} d_k \quad \text{for} \quad t \in [a, b), \\
 f(t-) &= \sum_{a \leq s_k < t} c_k + \sum_{a < s_k < t} d_k \quad \text{for} \quad t \in (a, b].
\end{align*}
\]

Moreover, $f \in BV([a, b], X)$ and
\[
\var_b^a f = \sum_{t \in [a, b]} \|\Delta^+ f(t)\|_X + \sum_{t \in (a, b]} \|\Delta^- f(t)\|_X.
\]

3.1. Theorem. If $f \in BV([a, b], X)$ then there exists a break function $f^B: [a, b] \to X$ such that $f - f^B$ is continuous on $[a, b]$. Moreover, the function $f^B$ is determined uniquely up to an additive constant.

Proof. Recall that a regulated function can have at most a countable number of points of discontinuity (see [9, Corollary I.3.2]). Let $\{s_k\}$ be the set of points of discontinuity of $f$ in $[a, b]$ and consider the function $f^B: [a, b] \to X$ given by
\[
f^B(t) = \sum_{a \leq s_k < t} \Delta^+ f(s_k) + \sum_{a < s_k \leq t} \Delta^- f(s_k) \quad \text{for} \quad t \in [a, b]. \tag{3.4}
\]

By [13, Lemma 4.1], both series on the right-hand side of (3.4) are absolutely convergent, thus $f^B$ is a break function. Moreover, for $t \in [a, b]$ we have
\[
\Delta^+ f^B(t) = \Delta^+ f(t) \quad \text{and} \quad \Delta^- f^B(t) = \Delta^- f(t).
\]

This implies that the function $f^C = f - f^B$ is continuous on $[a, b]$.

To prove uniqueness, assume that $f = \tilde{f}^C + \tilde{f}^B$, where $\tilde{f}^C$ is continuous and $\tilde{f}^B$ is a break function. Let
\[
\tilde{f}^B(t) = \sum_{a \leq t_k < t} c_k + \sum_{a < t_k \leq t} d_k \quad \text{for} \quad t \in [a, b],
\]

for some sequences $\{t_k\} \subset [a, b], \{c_k\}, \{d_k\} \subset X$ fulfilling (3.1). For each $k \in \mathbb{N}$, one can easily verify that
\[
c_k = \Delta^+ \tilde{f}^B(t_k) = \Delta^+ f(t_k) = \Delta^+ f^B(t_k). \tag{3.5}
\]
However, this has a sense only if \( t_k = s_{\ell_k} \) for some \( \ell_k \in \mathbb{N} \). Analogously, we can see that
\[
d_k = \Delta^- \tilde{f}^B(t_k) = \Delta^- f^B(s_{n_k})
\]
for some \( n_k \in \mathbb{N} \). In view of this and since the series above are unconditionally convergent (see [4, Chapter VI]), we have \( f^B(t) - \tilde{f}^B(t) = f^B(a) - \tilde{f}^B(a) \) for \( t \in [a, b] \). This completes the proof.

3.2. Remark. Let \( f \in BV([a, b], X) \) and let \( f^B \) be given as in Theorem 3.1. Let us denote \( f^C := f - f^B \). Then \( f = f^C + f^B \) and we say that \( f \) is decomposed into the sum of its continuous part \( f^C \) and its break part \( f^B \). Such a decomposition is in the classical case of real valued functions called the Jordan decomposition of \( f \).

Furthermore, note that, in view of (3.5) and (3.6), we have also
\[
f^B(t) = \sum_{k=1}^{\infty} \left[ \Delta^+ f(s_k) \chi_{(s_k,b]}(t) + \Delta^- f(s_k) \chi_{[s_k,b]}(t) \right] \quad \text{for} \quad t \in [a, b].
\]

Let us put
\[
f_n^B(t) = \sum_{k=1}^{n} \left[ \Delta^+ f(s_k) \chi_{(s_k,b]}(t) + \Delta^- f(s_k) \chi_{[s_k,b]}(t) \right] \quad \text{for} \quad t \in [a, b] \quad \text{and} \quad n \in \mathbb{N}.
\]

Then it is not difficult to prove that \( \lim_{n \to \infty} \var_a^b (f_n^B - f^B) = 0 \) (cf. [21, Lemma I.4.23] for the proof of an analogous assertion for real valued functions).

4 Variation on elementary sets

First, motivated by [6, Definition 6.1], we will introduce the definition of a variation over arbitrary intervals.

4.1. Definition. Let \( J \) be a bounded interval in \( \mathbb{R} \). We say that a finite set
\[
D = \{ \alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)} \} \subset J
\]
is a generalized division of \( J \) if
\[
\alpha_0 < \alpha_1 < \cdots < \alpha_{\nu(D)}.
\]
The set of all generalized divisions of the interval \( J \) is denoted by \( \mathcal{D}^*(J) \).

Let \( f: [a, b] \to X \) and let \( J \) be an arbitrary subinterval of \( [a, b] \). Then we define the variation of \( f \) on \( J \) by
\[
\var_J f = \sup_{D \in \mathcal{D}^*(J)} \left\{ \sum_{j=1}^{\nu(D)} \| f(\alpha_j) - f(\alpha_{j-1}) \|_X \right\}.
\]
We say that $f$ is of bounded variation on $J$ if $\text{var}_J f < \infty$. In such a case, we write $f \in BV(J, X)$. For convention we set also $\text{var}_\emptyset f = 0$ and $\text{var}_{[c]} f = 0$ for $c \in [a, b]$.

4.2. Remark. It is easy to see that Definition 4.1 coincides with the definition of the variation in the sense of Jordan if $J$ is a compact interval, that is, for $f: [a, b] \to X$ and $J = [c, d] \subseteq [a, b]$ we have

$$\text{var}_{[c, d]} f = \text{var}_c^d f.$$ 

For this reason, in the case of a compact interval, we may always restrict ourselves to the divisions containing their end points (as defined in Section 2).

Moreover, it is easy to see that if $J$ is a bounded interval and $f \in BV(J, X)$, then $f$ is bounded on $J$.

4.3. Proposition. Let $f: [a, b] \to X$ and let $J_1$ and $J_2$ be subintervals of $[a, b]$ such that $J_2 \subseteq J_1$. Then

$$\text{var}_{J_2} f \leq \text{var}_{J_1} f.$$ 

In particular, if $J$ is a subinterval of $[a, b]$ and $f \in BV(J, X)$, then $f \in BV(I, X)$ for every interval $I \subseteq J$.

The next theorem presents an equivalent formulation of the variation on arbitrary intervals commonly found in literature.

4.4. Theorem. Let $f: [a, b] \to X$ and $c, d \in [a, b]$, with $c < d$.

(i) If $f \in BV([c, d], X)$, then $\text{var}_{[c, d]} f = \lim_{\delta \to 0^+} \text{var}_{c+\delta}^d f = \sup_{t \in [c, d]} \text{var}_t f$.

(ii) If $f \in BV((c, d], X)$, then $\text{var}_{(c, d]} f = \lim_{\delta \to 0^+} \text{var}_{c+\delta}^d f = \sup_{t \in (c, d]} \text{var}_t f$.

(iii) If $f \in BV((c, d), X)$, then $\text{var}_{(c, d)} f = \lim_{\delta \to 0^+} \text{var}_{c+\delta}^{d-\delta} f$.

Proof. We will prove the assertion (i), the remaining ones follow in a similar way.

For a fixed $\delta > 0$, consider a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ of $[c, d-\delta]$. Of course, $D$ is also a generalized division of $[c, d]$ and hence

$$\sum_{j=1}^{\nu(D)} \|f(\alpha_j) - f(\alpha_{j-1})\|_X \leq \text{var}_{[c, d]} f.$$ 

Thus, taking the supremum over all divisions of $[c, d-\delta]$, we get $\text{var}_c^{d-\delta} f \leq \text{var}_{[c, d]} f$.

Since this inequality holds for every $\delta > 0$, it follows that

$$M := \sup_{t \in [c, d]} \text{var}_t f = \lim_{\delta \to 0^+} \text{var}_c^{d-\delta} f \leq \text{var}_{[c, d]} f.$$
Now, assume that \( M < \text{var}_{[c,d]} f \). Then, for \( \varepsilon = \text{var}_{[c,d]} f - M \), there exists a division \( D = \{ \alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)} \} \) of \([c, d)\) such that

\[
M = \text{var}_{[c,d]} f - \varepsilon < \sum_{j=1}^{\nu(D)} \| f(\alpha_j) - f(\alpha_{j-1}) \|_X \leq \text{var}_{c}^{\alpha(D)} f \leq M,
\]

a contradiction. This completes the proof of (i).

Dealing with functions with values in a metric space, Chistyakov presents in [3] an extensive study of the properties of the variation over subsets of the real line. Here, we call the reader’s attention to a particular result (see [3, Corollary 4.7]) connecting the variation over arbitrary intervals and the Jordan variation over a compact interval. This will be the content of Theorem 4.6 whose proof is included for sake of completeness. To this aim, the next lemma will be useful.

4.5. Lemma. Let \( f: [a, b] \to X, \ a \leq c < d \leq b \) and \( f \in BV((c, d), X) \). Then both limits \( f(c+) \) and \( f(d-) \) exist.

Proof. Let \( \varepsilon > 0 \) and an increasing sequence \( \{ t_n \} \subset (c, d) \) tending to \( d \) be given. By Theorem 4.4 (i) there is \( \delta > 0 \) such that

\[
0 < \text{var}_{[c,d]} f - \text{var}_{c}^{d-\delta} f < \varepsilon.
\]

Choose \( n_0 \in \mathbb{N} \) in such a way that \( t_n > d - \delta \) for every \( n \geq n_0 \). Therefore, for \( n > m \geq n_0 \), we have

\[
\| f(t_n) - f(t_m) \|_X \leq \text{var}^{t_n}_{t_m} f = \text{var}^{t_n}_{c} f - \text{var}^{t_m}_{c} f
\]

\[
\leq \text{var}_{[c,d]} f - \text{var}_{c}^{d-\delta} f < \varepsilon
\]

wherefrom the existence of the limit \( f(d-) \) immediately follows.

The existence of the limit \( f(c+) \) can be proved analogously.

4.6. Theorem. Let \( f: [a, b] \to X \) and \( a \leq c < d \leq b \).

(i) If \( f \in BV([c, d), X) \), then \( f(d-) \) exists and

\[
\text{var}^{d}_{c} f = \text{var}_{[c,d]} f + \| \Delta^{-} f(d) \|_X.
\]

(ii) If \( f \in BV((c, d], X) \), then \( f(c+) \) exists and

\[
\text{var}^{d}_{c} f = \text{var}_{[c,d]} f + \| \Delta^{+} f(c) \|_X.
\]
(iii) If $f \in BV((c,d), X)$, then both limits $f(c^+)$ and $f(d^-)$ exist and

$$\text{var}_c^d f = \text{var}_{(c,d)} f + \|\Delta^+ f(c)\|_X + \|\Delta^- f(d)\|_X.$$  

Proof. The existence of all the needed limits follows by Lemma 4.5.

Assume that $f \in BV([c,d), X)$ and let $\varepsilon > 0$ and $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{m+1}\} \in \mathcal{D}[c,d]$ be given. We can choose $\xi \in [c,d]$ in such a way that

$$\alpha_m < \xi < d \quad \text{and} \quad \|f(d^-) - f(\xi)\|_X < \varepsilon.$$  

Consequently,

$$\sum_{j=1}^{m+1} \|f(\alpha_j) - f(\alpha_{j-1})\|_X$$

$$\leq \sum_{j=1}^{m} \|f(\alpha_j) - f(\alpha_{j-1})\|_X + \|f(\xi) - f(\alpha_m)\|_X + \|f(d^-) - f(\xi)\|_X + \|\Delta^- f(d)\|_X$$

$$< \text{var}_\xi f + \varepsilon + \|\Delta^- f(d)\|_X \leq \text{var}_{[c,d)} f + \varepsilon + \|\Delta^- f(d)\|_X.$$  

As $D \in \mathcal{D}[c,d]$ and $\varepsilon > 0$ were arbitrarily chosen, we conclude that

$$\text{var}_c^d f \leq \text{var}_{[c,d)} f + \|\Delta^- f(d)\|_X. \quad (4.1)$$

On the other hand, for any $\delta > 0$ we have

$$\|f(d) - f(d - \delta)\|_X \leq \text{var}_{d-\delta}^d f = \text{var}_{c}^d f - \text{var}_{c}^{d-\delta} f.$$  

Hence, letting $\delta \to 0+$ we get

$$\|\Delta^- f(d)\|_X \leq \text{var}_c^d f - \lim_{\delta \to 0^+} \text{var}_{c}^{d-\delta} f = \text{var}_c^d f - \text{var}_{[c,d)} f$$

wherefrom, due to (4.1), we conclude that $\text{var}_c^d f = \text{var}_{[c,d)} f + \|\Delta^- f(d)\|_X$. This completes the proof of (i).

Similarly, we can prove the assertions (ii) and (iii). \qed

4.7 Corollary. Let $f: [a, b] \to X$ and $c, d \in [a, b]$, with $c < d$. Then the following assertions are equivalent:

(i) $f \in BV([c,d], X),$

(ii) $f \in BV((c,d], X),$

(iii) $f \in BV([c,d), X),$
(iv) \( f \in BV((c,d), X) \).

**4.8. Remark.** In view Theorem 4.6, we can also observe that for \( f: [a, b] \to X \) and \( c \in [a, b] \), we have

\[
\lim_{\delta \to 0^+} \var_{c-\delta} f = \| \Delta^- f(c) \|_X + \| \Delta^+ f(c) \|_X
\]

provided the one-sided limits exist at the point \( c \) (see [8, Proposition I.2.8]). Furthermore, by Theorem 4.6, if \( f \in BV([a, b], X) \cap C([a, b], X) \), then

\[
\var_{[c,d]} f = \var_{(c,d)} f = \var_{(c,d)} f = \var_d f \quad \text{for} \quad c, d \in [a, b] \quad \text{such that} \quad c < d.
\]

We need to extend the notion of a variation on intervals to elementary sets.

**4.9. Definition.** Let \( E \subset \mathbb{R} \) be bounded. We say that \( E \) is an elementary set if it is a finite union of intervals.

A collection of intervals \( \{J_k: k = 1, \ldots, m\} \) is called a minimal decomposition of \( E \) if \( E = \bigcup_{k=1}^m J_k \) and the union \( J_k \cup J_\ell \) is not an interval whenever \( k \neq \ell \).

If \( S \subseteq \mathbb{R} \), then \( \mathcal{E}(S) \) stands for the set of all elementary subsets of \( S \).

Note that the minimal decomposition of an elementary set is uniquely determined. Moreover, the intervals of such decomposition are pairwise disjoint.

Having this in mind, we extend the notion of a variation over elementary sets as follows.

**4.10. Definition.** Given a function \( f: [a, b] \to X \) and an elementary subset \( E \) of \([a, b]\), the variation of \( f \) over \( E \) is

\[
\var(f, E) = \sum_{k=1}^m \var_{J_k} f,
\]

where \( \{J_k: k = 1, \ldots, m\} \) is the minimal decomposition of \( E \).

It is worth highlighting that if \( f \in BV([a, b], X) \cap C([a, b], X) \), then \( \var(f, \cdot) \) defines a finitely additive measure on \( \mathcal{E}([a, b]) \). More precisely, we have

\[
\var(f, E) \leq \var^b_a f \quad \text{for any} \quad E \in \mathcal{E}([a, b])
\]

and

\[
\var(f, E_1 \cup E_2) = \var(f, E_1) + \var(f, E_2)
\]

whenever \( E_1, E_2 \in \mathcal{E}([a, b]) \) and \( E_1 \cap E_2 = \emptyset \).
4.11. Remark. Let us note that in [6] Definition 4.1 is applied also to arbitrary subsets $E$ of $[a, b]$. Unfortunately, such a definition is not convenient for our purposes, as the variation would lose the additivity property even for continuous functions. Indeed, let $a < c < d < b$ and $E = [a, c] \cup [d, b]$. Then, according to such a definition we would have

$$\text{var}_E f \geq \text{var}_{[a, c]} f + \text{var}_{[d, b]} f$$

whenever $f(d) \neq f(c)$. This is why for elementary subsets of $[a, b]$ we will define the variation in a different way than that used by Gordon in [6].

The following lemma will give us the crucial tool for proving our main result. It is analogous to Lemma from [10]. However, instead of the Lebesgue measure it works with the variation of the given function over elementary sets. We say that a sequence $\{A_n\}$ of subsets of $[a, b]$ is contracting if $A_{n+1} \subseteq A_n$ holds for $n \in \mathbb{N}$.

4.12. Lemma. Let $f \in BV([a, b], X) \cap C([a, b], X)$ and let $A_n$ be a contracting sequence of subsets of $[a, b]$ such that $\bigcap_n A_n = \emptyset$. For $n \in \mathbb{N}$ put

$$v_n = \sup \{ \text{var}(f, E) : E \in \mathcal{E}(A_n) \}.$$  \hfill (4.2)

Then $\lim_{n \to \infty} v_n = 0$.

Proof. First, notice that, by Definition 4.1, $v_n = 0$ whenever $A_n = \emptyset$.

Let us assume that $v_n$ does not tend to zero. Since $\{v_n\}$ is decreasing, this means that there is $\varepsilon > 0$ such that

$$v_n > \varepsilon \quad \text{for all } n \in \mathbb{N}. \hfill (4.3)$$

Consequently, for each $n \in \mathbb{N}$, there is an $E_n \in \mathcal{E}(A_n)$ such that

$$\text{var}(f, E_n) > v_n - \frac{\varepsilon}{2^n}. \hfill (4.4)$$

Since $f$ is continuous, we can assume that $E_n$ is closed (cf. Remark 4.8). Now, let $H_n = \bigcap_{j=1}^n E_j$ for $n \in \mathbb{N}$. Clearly, $H_n$ is closed and $H_n \subseteq A_n$ holds for any $n \in \mathbb{N}$. We shall show that

$$H_n \neq \emptyset \quad \text{for any } n \in \mathbb{N}.$$ 

Indeed, let $n \in \mathbb{N}$ and $M \in \mathcal{E}(A_n \setminus E_n)$ be given. Then $M \cup E_n \in \mathcal{E}(A_n)$ and, due to the additivity of the variation, we can see that

$$\text{var}(f, M) + \text{var}(f, E_n) = \text{var}(f, M \cup E_n) \leq v_n \quad \text{for all } n \in \mathbb{N}.$$
Therefore, due to (4.4), we have $\var(f, M) < \frac{\varepsilon}{2^n}$. Obviously,

$$E = \bigcup_{j=1}^{n} (E \setminus E_j) \quad \text{for any} \ E \in \mathcal{E}(A_n \setminus H_n),$$

where $(E \setminus E_j) \in \mathcal{E}(A_n \setminus E_j) \subseteq \mathcal{E}(A_j \setminus E)$ for $j = 1, \ldots, n$. Hence

$$\var(f, E) \leq \sum_{j=1}^{n} \var(f, E \setminus E_j) < \sum_{j=1}^{n} \frac{\varepsilon}{2^j} < \varepsilon \quad \text{for every} \ E \in \mathcal{E}(A_n \setminus H_n).$$

This, together with (4.3), implies that there must exist $E \in \mathcal{E}(H_n)$ with $\var(f, E) > \varepsilon$, wherefrom we conclude that $H_n \neq \emptyset$.

Since $\{H_n\}$ is a contracting sequence of non-empty, closed and bounded sets, using Cantor’s intersection theorem we get

$$\bigcap_{n} A_n \supset \bigcap_{n} H_n \neq \emptyset.$$

Of course, this contradicts our hypothesis $\bigcap_{n} A_n = \emptyset$. As a result, $\lim_{n \to \infty} v_n = 0$ and this completes the proof.

\section{Integration over elementary sets}

To our knowledge, unlike the Lebesgue-Stieltjes integral, up to now the KS-integral over sets that need not be just the compact intervals has not been discussed in literature. In the case of elementary subsets of $[a, b]$ the following definition turned out to be useful for our purposes.

\subsection{Definition}

Let $F: [a, b] \to L(X)$, $g: [a, b] \to X$ and an elementary subset $E$ of $[a, b]$ be given. The Kurzweil-Stieltjes integral (or shortly KS-integral) of $g$ with respect to $F$ over $E$ is given by

$$\int_{E} d[F]g = \int_{a}^{b} d[F] (g \chi_{E})$$

provided the integral on the right-hand side exists.

Symmetrically, we define the KS-integral of $F$ with respect to $g$ over $E$ by

$$\int_{E} F d[g] = \int_{a}^{b} (F \chi_{E}) d[g]$$

provided the integral on the right-hand side exists.
According to Definition 5.1 the existence of the integral $\int_E d[F] g$ means (see Section 1) that there is an $I \in X$ such that for every $\varepsilon > 0$ we can find a gauge $\delta$ on $[a, b]$ such that

$$\left\| S(dF, g_X, P) - I \right\|_X < \varepsilon$$

for all $\delta$-fine tagged divisions $P$ of $[a, b]$.

Due to Definition 5.1 the basic properties of the KS-integral mentioned below are immediate consequences of what is known for the abstract Kurzweil-Stieltjes integral.

**5.2. Proposition.** Let $E$ be an elementary subset of $[a, b]$. Assume that $F: [a, b] \to L(X)$ and $g_j: [a, b] \to X$, $j = 1, 2$, are such that the integrals

$$\int_E d[F_1] g_1 \quad \text{and} \quad \int_E d[F_2] g_2$$

exist. Then the integral $\int_E d[F] (c_1 g_1 + c_2 g_2)$ exists and

$$\int_E d[F] (c_1 g_1 + c_2 g_2) = c_1 \int_E d[F] g_1 + c_2 \int_E d[F] g_2 \quad \text{for every } c_1, c_2 \in \mathbb{R}.$$

Symmetrically, if $F_j: [a, b] \to L(X)$, $j = 1, 2$, and $g: [a, b] \to X$ are such that the integrals

$$\int_E d[F_1] g \quad \text{and} \quad \int_E d[F_2] g$$

exist, then the integral $\int_E d[F_1 + F_2] g$ exists and

$$\int_E d[F_1 + F_2] g = c_1 \int_E d[F_1] g + c_2 \int_E d[F_2] g \quad \text{for all } c_1, c_2 \in \mathbb{R}.$$

**5.3. Remark.** Note that, if $E$ is an elementary subset of $[a, b]$ and $g: [a, b] \to X$ is a function such that $g = 0$ on $E$, then

$$\int_E d[F] g = 0$$

for every $F: [a, b] \to L(X)$.

Consequently, due to Proposition 5.2, if $F: [a, b] \to L(X)$ and $g: [a, b] \to X$ are such that $\int_E d[F] g$ exists, then for any function $h: [a, b] \to X$ which coincides with $g$ on $E$, we have

$$\int_E d[F] h = \int_E d[F] g.$$
5.4. **Theorem.** Let $E_1$ and $E_2$ be elementary subsets of $[a, b]$ such that $E_1 \cap E_2 = \emptyset$. Assume that $F : [a, b] \to L(X)$ and $g : [a, b] \to X$ are such that both the integrals $\int_{E_j} d[F] g$, $j = 1, 2$, exist. Then the integral $\int_{E_1 \cup E_2} d[F] g$ exists and

$$\int_{E_1 \cup E_2} d[F] g = \int_{E_1} d[F] g + \int_{E_2} d[F] g.$$  

**Proof.** Since $g (\chi_{E_1 \cup E_2}) = g \chi_{E_1} + g \chi_{E_2}$, by Definition 5.1 and [18, Proposition 6] we have

$$\int_{E_1} d[F] g + \int_{E_2} d[F] g = \int_a^b d[F] (g \chi_{E_1}) + \int_a^b d[F] (g \chi_{E_2}) = \int_a^b d[F] (g \chi_{E_1 \cup E_2}),$$

which proves the result. \qed

The following existence result is a consequence of [18, Proposition 15].

5.5. **Proposition.** Let $F \in BV([a, b], L(X))$ and $g \in G([a, b], X)$. If $E$ is an elementary subset of $[a, b]$, then the integral $\int_E d[F] g$ exists.

**Proof.** It is enough to observe that $g \chi_E : [a, b] \to X$ is a regulated function whenever $g \in G([a, b], X)$ and $E$ is an elementary subset of $[a, b]$. \qed

The estimates presented in the sequel are, in a sense, the analogues of [18, Proposition 10].

5.6. **Theorem.** Let $J$ be a subinterval of $[a, b]$ and let $c = \inf J$ and $d = \sup J$, with $c < d$. Assume that $F : [a, b] \to L(X)$, with $F \in BV(J, L(X))$, and $g : [a, b] \to X$ are such that the integral $\int_J d[F] g$ exists.

(i) If $J = (c, d)$, then $\left\| \int_{(c,d)} d[F] g \right\|_X \leq \text{var}_{(c,d)} F \left( \sup_{t \in (c,d)} \|g(t)\|_X \right)$.

(ii) If $J = [c, d)$ and $F(c-) \text{ exists}$, then

$$\left\| \int_{[c,d)} d[F] g \right\|_X \leq \text{var}_{[c,d)} F \left( \sup_{t \in [c,d)} \|g(t)\|_X \right) + \|\Delta^- F(c)\|_{L(X)} \|g(c)\|_X.$$

(iii) If $J = (c, d]$ and $F(d+) \text{ exists}$, then

$$\left\| \int_{(c,d]} d[F] g \right\|_X \leq \text{var}_{(c,d]} F \left( \sup_{t \in (c,d]} \|g(t)\|_X \right) + \|\Delta^+ F(d)\|_{L(X)} \|g(d)\|_X.$$
(iv) If $J = [c, d]$, and both $F(c−), F(d+)$ exist, then

$$\left\| \int_{[c,d]} d[F] g \right\|_X \leq \text{var}_{[c,d]} F \left( \sup_{t \in [c,d]} \| g(t) \|_X \right) + \| \Delta^- F(c) \|_{L(X)} \| g(c) \|_X + \| \Delta^+ F(d) \|_{L(X)} \| g(d) \|_X.$$

**Proof.** Since the integral $\int_J d[F] g$ exists, for any $\varepsilon > 0$ there is a gauge $\delta = \varepsilon$ on $[a, b]$ such that

$$\left\| S(dF, g \chi_J, P) - \int_J d[F] g \right\|_X < \varepsilon$$

(5.1)

for every $\delta$-fine tagged division $P$ of $[a, b]$.

Let $\delta^*$ be a gauge on $[a, b]$ such that $\delta^* \leq \delta$ on $[a, b]$ and

$$\delta^*(t) = \min\{|t-c|, |t-d|\} \quad \text{for} \quad t \in [a, b] \setminus \{c, d\}.$$  

(5.2)

Considering a $\delta^*$-fine tagged division $P$ of $[a, b]$, put $P_J = \{ (\tau, I) \in P : I \cap [c, d] \neq \emptyset \}$. Let $a < c < d < b$. Denoting $P_J = \{ (\xi_j, [\beta_{j-1}, \beta_j]) : j = 1, \ldots, \nu(P_J) \}$, without loss of generality we may assume

$$\beta_0 < c = \beta_1 < \beta_2 < \cdots < \beta_{\nu(P_J)-1} = d = \beta_{\nu(P_J)}.$$  

(5.3)

It is easy to see that $\xi_1 = \xi_2 = c$ and $\xi_{\nu(P_J)} = \xi_{\nu(P_J)-1} = d$ must hold.

(i) For $J = (c, d)$, we have

$$S(dF, g \chi_{(c,d)}, P) = S(dF, g \chi_{(c,d)}, P_J)$$

$$= \sum_{j=1}^{\nu(P_J)} [F(\beta_j) - F(\beta_{j-1})](g \chi_{(c,d)})(\xi_j) = \sum_{j=3}^{\nu(P_J)-2} [F(\beta_j) - F(\beta_{j-1})]g(\xi_j).$$

Having in mind that $\{\beta_2, \ldots, \beta_{\nu(P_J)-2}\}$ is a division of $(c, d)$, we conclude that

$$\left\| S(dF, g \chi_{(c,d)}, P) \right\|_X \leq \text{var}_{(c,d)} F \left( \sup_{t \in (c,d)} \| g(t) \|_X \right)$$

and consequently, due to (5.1),

$$\left\| \int_{(c,d)} d[F] g \right\|_X \leq \text{var}_{(c,d)} F \left( \sup_{t \in (c,d)} \| g(t) \|_X \right) + \varepsilon.$$
(ii) Let \( J = [c, d] \) and let \( g(c) \neq 0 \). Since \( F(c^-) \) exists, we can choose \( \eta > 0 \) such that
\[
\|F(c^-) - F(s)\|_{L(X)} < \frac{\varepsilon}{\|g(c)\|_X} \quad \text{for} \quad c - \eta < s < c.
\]
Moreover, we can assume that the gauge \( \delta^* \) as in (5.2) is such that \( \delta^*(c) < \eta \).

Given a \( \delta^- \)-fine tagged division \( P \) of \([a, b]\) and considering \( P_J \) as in (5.3), we have
\[
S(dF, g \chi_{[c, d]}, P) = \sum_{j=2}^{\nu(P_J)-2} [F(\beta_j) - F(\beta_{j-1})] g(\xi_j) + [F(c) - F(\beta_0)] g(c).
\]
Hence
\[
\|S(dF, g \chi_{[c, d]}, P)\|_X \leq \var_{[c, d]} F \left( \sup_{t \in [c, d]} \|g(t)\|_X \right) + \|F(c) - F(\beta_0)\|_{L(X)} \|g(c)\|_X
\leq \var_{[c, d]} F \left( \sup_{t \in [c, d]} \|g(t)\|_X \right) + \|\Delta^- F(c)\|_{L(X)} \|g(c)\|_X
\leq \var_{[c, d]} F \left( \sup_{t \in [c, d]} \|g(t)\|_X \right) + \|\Delta^- F(c)\|_{L(X)} \|g(c)\|_X + \varepsilon.
\]
In view of this and applying (5.1), for \( J = [c, d] \) we obtain
\[
\left\| \int_{[c, d]} F \, d[g] \right\|_X \leq \var_{[c, d]} F \left( \sup_{t \in [c, d]} \|g(t)\|_X \right) + \|\Delta^- F(c)\|_{L(X)} \|g(c)\|_X + 2\varepsilon,
\]
which proves (ii).

(iii) In order to prove the desired inequality for \( J = (c, d] \), assume \( g(d) \neq 0 \) and choose \( \gamma > 0 \) such that
\[
\|F(s) - F(d^+)\|_{L(X)} < \frac{\varepsilon}{\|g(d)\|_X} \quad \text{for} \quad d < s < d + \gamma.
\]
Without loss of generality we can assume that the gauge \( \delta^* \) in (5.2) is such that \( \delta^*(d) < \gamma \). For a \( \delta^- \)-fine tagged division \( P \) of \([a, b]\), considering \( P_J \) as in (5.3), we have
\[
S(dF, g \chi_{(c, d]}, P) = \sum_{j=3}^{\nu(P_J)-1} [F(\beta_j) - F(\beta_{j-1})] g(\xi_j) + [F(\beta_{\nu(P_J)}) - F(d)] g(d),
\]
wherefrom

\[ \|S(dF, g \chi_{[c,d]}, P)\|_X \leq \text{var}_{[c,d]} F \left( \sup_{t \in [c,d]} \|g(t)\|_X \right) + \|\Delta^+ F(d)\|_{L(X)} \|g(d)\|_X \]

\[ + \|F(\beta_{\nu(P)} - F(d))\|_{L(X)} \|g(d)\|_X \]

\[ \leq \text{var}_{[c,d]} F \left( \sup_{t \in [c,d]} \|g(t)\|_X \right) \|\Delta^+ F(d)\|_{L(X)} \|g(d)\|_X + \varepsilon. \]

Therefore, by (5.1) we obtain

\[ \left\| \int_{[c,d]} d[F] \|_X \right\| \leq \text{var}_{[c,d]} F \left( \sup_{t \in [c,d]} \|g(t)\|_X \right) + \|\Delta^+ F(d)\|_{L(X)} \|g(d)\|_X + 2 \varepsilon \]

and (iii) follows.

(iv) Similarly to the above, let us consider a \( \delta^* \)-fine tagged division \( P \) of \([a, b] \), where the gauge \( \delta^* \) satisfies (5.2), \( \delta^*(c) < \eta \) and \( \delta^*(d) < \gamma \). We can see that

\[ S(dF, g \chi_{[c,d]}, P) = \sum_{j=2}^{\nu(P) - 1} [F(\beta_j) - F(\beta_{j-1})] \|g(\xi_j)\|_X \]

\[ + [F(c) - F(\beta_0)] \|g(c)\|_X + [F(\beta_{\nu(P)}) - F(d)] \|g(d)\|_X, \]

where \( P_j \) is defined as in (5.3). Thus

\[ \|S(dF, g \chi_{[c,d]}, P)\|_X \leq \text{var}_{[c,d]} F \left( \sup_{t \in [c,d]} \|g(t)\|_X \right) + \|\Delta^- F(c)\|_{L(X)} \|g(c)\|_X + \|\Delta^+ F(d)\|_{L(X)} \|g(d)\|_X + 2 \varepsilon, \]

which, together with (5.1), leads to the desired inequality.

The case when \( c = a \) and/or \( d = b \) can be proved in a similar way, recalling that, by convention, \( \Delta^- F(a) = \Delta^+ F(b) = 0. \)

In order to better characterize the integral over elementary sets, first we investigate the integral over arbitrary intervals. The case when \( E \) is just one point corresponds to the following result borrowed from [18, Lemma 12].

**5.7. Proposition.** Let \( F \in (B)G([a,b], L(X)) \) and \( g : [a,b] \to X \) be given. If \( \tau \in [a,b] \),
then the integral \( \int_{[\tau]} d[F] g \) exists and one of the following equalities is true:

\[
\begin{align*}
\int_{[\tau]} d[F] g &= \int_{a}^{b} d[F] (g \chi_{[\tau]}) = \lim_{t \to \tau+} F(t) g(a) - F(a) g(a), \\
\int_{[\tau]} d[F] g &= \int_{a}^{b} d[F] (g \chi_{[\tau]}) = \lim_{t \to \tau+} F(t) g(\tau) - \lim_{t \to \tau-} F(t) g(\tau) \quad \text{if} \quad \tau \in (a, b), \\
\int_{[\tau]} d[F] g &= \int_{a}^{b} d[F] (g \chi_{[\tau]}) = F(b) g(b) - \lim_{t \to b-} F(t) g(b).
\end{align*}
\] (5.4)

An important question which arises regarding Definition 5.1 is whether the integral over compact intervals coincides with the abstract Kurzweil-Stieltjes integral introduced in Section 1. As we will see, the equality need not be true in general. However, given functions \( F : [a, b] \to L(X) \) and \( g : [a, b] \to X \), the equality

\[
\int_{[a,b]} d[F] g = \int_{a}^{b} d[F] g
\]

holds provided one of the integrals exists.

**5.8 Theorem.** Let \( F \in (B)\mathcal{G}([a, b], L(X)), \quad g : [a, b] \to X \) and \( a < c < d < b \). Then the integral \( \int_{[c,d]} d[F] g \) exists if and only if \( \int_{c}^{d} d[F] g \) exists. Moreover,

\[
\int_{[c,d]} d[F] g = \int_{c}^{d} d[F] g + F(c) g(c) - \lim_{t \to c-} F(t) g(c) + \lim_{t \to d+} F(t) g(d) - F(d) g(d).
\] (5.5)

**Proof.** Obviously,

\[
\int_{a}^{b} d[F] (g \chi_{[c,d]}) = \int_{a}^{c} d[F] (g \chi_{[c,d]}) + \int_{c}^{d} d[F] (g \chi_{[c,d]}) + \int_{d}^{b} d[F] (g \chi_{[c,d]})
\]

\[
= \int_{a}^{c} d[F] (g \chi_{[c]}) + \int_{c}^{d} d[F] g + \int_{d}^{b} d[F] (g \chi_{[d]}).
\]

Hence, the proof follows from (5.4).

**5.9 Remark.** Recalling that \( BV([a, b], X) \subset G([a, b], X) \), formula (5.5) is valid when \( F \in BV([a, b], L(X)) \). In the particular case \( X = \mathbb{R} \) this fact corresponds to the result given by Saks in [16, Theorem VI.(8.1) (p. 208)] which states the following:
A finite function \( f \) integrable in the Lebesgue-Stieltjes (LS) sense on an interval \([c,d]\) with respect to a function of bounded variation \( \varphi \), is also integrable in the Perron-Stieltjes (PS) sense and we have

\[
(PS) \int_c^d d[\varphi(s)] f(s) = (LS) \int_{[c,d]} d[\varphi(s)] f(s) - \Delta^- \varphi(c) f(c) - \Delta^+ \varphi(d) f(d).
\]

5.10. Theorem. Let \( F \in (B)G([a,b], L(X)), \ g: [a,b] \to X, c, d \in [a,b] \) and \( c < d \). Then the integral \( \int_{(c,d)} d[F] g \) exists if and only if the integral \( \int_{(c,d)} d[F] g \) exists. Moreover,

\[
\int_{(c,d)} d[F] g = F(c) g(c) - \lim_{t \to c^-} F(t) g(c) + \int_c^d d[F] g + \lim_{t \to d^-} F(t) g(d) - F(d) g(d). \tag{5.6}
\]

Proof. By Proposition 5.7, the integrals \( \int_{[c]} d[F] g \) and \( \int_{[d]} d[F] g \) exist. Since

\[ g \chi_{[c, d]} = g \chi_{[c]} + g \chi_{(c, d)} + g \chi_{[d]}, \]

it follows that the integral over \((c,d)\) exists if and only if the integral over \([c,d]\) exists. In addition, by Theorem 5.4 we obtain

\[
\int_{[c,d]} d[F] g = \int_{[c]} d[F] g + \int_{(c,d)} d[F] g + \int_{[d]} d[F] g,
\]

which, together with (5.4) and Theorem 5.8, completes the proof.

In a similar way, applying Proposition 5.7 and dealing with characteristic functions, we can derive the following expressions for the integral over half-open intervals.

5.11. Theorem. Let \( F \in (B)G([a,b], L(X)), \ g: [a,b] \to X, c, d \in [a,b] \) and \( c < d \). Then the following assertions are true:

(i) The integral \( \int_{(c,d)} d[F] g \) exists if and only if the integral \( \int_{c}^{d} d[F] g \) exists. Moreover,

\[
\int_{(c,d)} d[F] g = F(c) g(c) - \lim_{t \to c^-} F(t) g(c) + \int_{c}^{d} d[F] g + \lim_{t \to d^-} F(t) g(d) - F(d) g(d). \tag{5.7}
\]
(ii) The integral \( \int_{(c,d]} d[F] g \) exists if and only if the integral \( \int_c^d d[F] g \) exists. Moreover,

\[
\int_{(c,d]} d[F] g = F(c) g(c) - \lim_{t \to c^+} F(t) g(c) + \int_c^d d[F] g + \lim_{t \to d^+} F(t) g(d) - F(d) g(d). \tag{5.8}
\]

5.12. Remark. According to Theorems 5.8, 5.10 and 5.11, for \( F \in (B)G([a, b], L(X)) \), \( g: [a, b] \to X \) and \( a \leq c < d \leq b \), if one of the integrals

\[
\int_{[c,d]} d[F] g, \quad \int_{(c,d]} d[F] g, \quad \int_{(c,d)} d[F] g, \quad \int_{(c,d)} d[F] g \quad \text{and} \quad \int_c^d d[F] g
\]

exists, then all the others exist as well.

If we assume in addition \( F \in C([a, b], L(X)) \), then

\[
\int_{[c,d]} d[F] g = \int_{(c,d]} d[F] g = \int_{(c,d)} d[F] g = \int_c^d d[F] g = \int_{[c,d]} d[F] g.
\]

We are now ready to deduce an expression for the integral over elementary sets.

5.13. Theorem. Let \( E \) be an elementary subset of \([a, b]\). If \( F \in (B)G([a, b], L(X)) \) and \( g: [a, b] \to X \) are such that the integral \( \int_E d[F] g \) exists, then

\[
\int_E d[F] g = \sum_{k=1}^m \int_{J_k} d[F] g, \tag{5.9}
\]

where \( \{J_k: k = 1, \ldots, m\} \) is the minimal decomposition of \( E \).

Proof. For \( k = 1, \ldots, m \), let \( c_k \) and \( d_k \) be such that \([c_k, d_k]\) is the closure of \( J_k \). By the hypothesis the integral

\[
\int_E d[F] g := \int_{a}^{b} d[F] (g \chi_E)
\]

exists, and hence, by [18, Proposition 8], so does the integral \( \int_{c_k}^{d_k} d[F] (g \chi_{J_k}) \). Moreover,

\[
\int_{c_k}^{d_k} d[F] (g \chi_{J_k}) = \int_{c_k}^{d_k} d[F] (g \chi_{J_k}) \quad \text{for} \quad k = 1, \ldots, m.
\]
Notice that, for each \( k = 1, \ldots, m \), the integrals

\[
\int_a^{c_k} d[F] (g \chi_{J_k}) \quad \text{and} \quad \int_{d_k}^b d[F] (g \chi_{J_k})
\]

are either zero or integrals over one point sets (which exist by Proposition 5.7). Therefore, the integral \( \int_{J_k} d[F] g \) exists.

Having in mind that the intervals of minimal decomposition are pairwise disjoint, (5.9) follows from Theorem 5.4.

As a consequence of Definition 4.10, Theorem 5.6 and Theorem 5.13 we have the following.

5.14. Corollary. Let \( E \) be an elementary subset of \([a, b]\). If \( F \in BV([a, b], L(X)) \cap C([a, b], L(X)) \) and \( g: [a, b] \to X \) are such that the integral \( \int_E d[F] g \) exists, then

\[
\left\| \int_E d[F] g \right\|_X \leq \var(F, E) \left( \sup_{t \in E} \|g(t)\|_X \right).
\]

Theorem 5.13 also means that once the integral over an elementary set exists so does the integral over subintervals of the minimal decomposition. This implies immediately that the following assertions hold.

5.15. Corollary. Let \( E \) be an elementary subset of \([a, b]\). If \( F \in (B)G([a, b], L(X)) \) and \( g: [a, b] \to X \) are such that the integral \( \int_E d[F] g \) exists, then the integral \( \int_T d[F] g \) exists for every elementary subset \( T \) of \([a, b]\), with \( T \subseteq E \).

5.16. Corollary. Let \( E_1 \) and \( E_2 \) be elementary subsets of \([a, b]\), \( F \in (B)G([a, b], L(X)) \) and \( g: [a, b] \to X \). If both the integrals

\[
\int_{E_1} d[F] g \quad \text{and} \quad \int_{E_2} d[F] g
\]

exist, then the integral \( \int_{E_1 \cup E_2} d[F] g \) exists and

\[
\int_{E_1 \cup E_2} d[F] g = \int_{E_1} d[F] g + \int_{E_2} d[F] g - \int_{E_1 \cap E_2} d[F] g. \quad (5.10)
\]

Similarly, if the integral \( \int_{E_1 \cup E_2} d[F] g \) exists, then both the integrals \( \int_{E_1} d[F] g \) and \( \int_{E_2} d[F] g \) exist and the equality (5.10) holds.
5.17. Remark. Of course, results analogous to those given in this section for integrals of the form \[\int_E d[F]g\] can be obtained in a similar way also for the symmetrical case \[\int_E F d[g]\].

6 Bounded convergence theorem

Now we can prove our main result, which is Theorem 6.3. To this aim, we will first treat the case when the integrator is a function of bounded variation which is also continuous. This is the content of the following theorem whose proof uses the integration over elementary sets discussed in the previous section.

6.1. Theorem. Let \(F \in BV([a, b], L(X)) \cap C([a, b], L(X))\) and assume that the sequence \(\{g_n\} \subset G([a, b], X)\) is such that

\[
\lim_{n \to \infty} g_n(t) = 0 \quad \text{for} \quad t \in [a, b]
\]

and

\[
\|g_n\|_\infty \leq K < \infty \quad \text{for} \quad n \in \mathbb{N}.
\]

Then the integral \(\int_a^b d[F]g_n\) exists for each \(n \in \mathbb{N}\) and

\[
\lim_{n \to \infty} \int_a^b d[F]g_n = 0. \quad (6.1)
\]

Proof. If \(\text{var}_a^b F = 0\), then \(\int_a^b d[F]g_n = \int_a^b d[F]g = 0\) for all \(n \in \mathbb{N}\). Therefore, without loss of generality we may assume that \(\text{var}_a^b F \neq 0\). Let \(\varepsilon > 0\) be given. For any \(n \in \mathbb{N}\) define

\[
A_n = \{t \in [a, b]: \|g_m(t)\|_X \geq \frac{\varepsilon}{6 \text{ var}_a^b F} \text{ for some } m \geq n\}.
\]

Clearly, \(A_{n+1} \subseteq A_n\) for \(n \in \mathbb{N}\) and \(\bigcap_n A_n = \emptyset\). Let \(v_n\) be given as in (4.2) with \(F\) in the place of \(f\). Recall that \(v_n = 0\) if \(A_n = \emptyset\). By Lemma 4.12 we have \(\lim_{n \to \infty} v_n = 0\). In particular, there exists \(n_\varepsilon \in \mathbb{N}\) such that

\[
v_n < \frac{\varepsilon}{6K} \quad \text{for} \quad n \geq n_\varepsilon. \quad (6.2)
\]

As a consequence, we have

\[
\text{var}(F, E) < \frac{\varepsilon}{6K} \quad \text{for any} \quad n \geq n_\varepsilon \quad \text{and any} \quad E \in \mathcal{E}(A_n). \quad (6.3)
\]
Now, fix $n \geq n_\varepsilon$. Since $g_n$ is regulated, by [9, Theorem I.3.1] there exists a step function $h_n : [a, b] \to X$ such that
\[ \|h_n - g_n\|_\infty < \varepsilon', \] (6.4)
where $\varepsilon' = \min \left\{ \frac{\varepsilon}{6 \ \text{var}_a^b F}, K \right\}$. In particular, we have
\[ \|h_n\|_\infty < K + \varepsilon' \leq 2K. \] (6.5)

Let us consider the sets
\[ U_n = \left\{ t \in [a, b] : \|h_n(t)\|_X \geq \frac{\varepsilon}{3 \ \text{var}_a^b F} \right\} \quad \text{and} \quad V_n = [a, b] \setminus U_n. \]

Obviously, $U_n$ and $V_n$ are elementary sets (possibly empty). Furthermore, by (6.4) we obtain
\[ \frac{\varepsilon}{6 \ \text{var}_a^b F} = \frac{\varepsilon}{3 \ \text{var}_a^b F} - \frac{\varepsilon}{6 \ \text{var}_a^b F} \leq \|h_n(t)\|_X - \varepsilon' < \|g_n(t)\|_X \quad \text{for} \quad t \in U_n. \]

This means that $U_n$ is an elementary subset of $A_n$, which implies that
\[ \text{var}(F, U_n) < \frac{\varepsilon}{6K}. \] (6.6)

Moreover, by [18, Proposition 10] and (6.4) we deduce that
\[ \left\| \int_a^b d[F] g_n \right\|_X \leq \left\| \int_a^b d[F] (g_n - h_n) \right\|_X + \left\| \int_a^b d[F] h_n \right\|_X \]
\[ \leq \text{var}_a^b F \|g_n - h_n\|_\infty + \left\| \int_a^b d[F] h_n \right\|_X < \frac{\varepsilon}{6} + \left\| \int_a^b d[F] h_n \right\|_X, \]
i.e.
\[ \left\| \int_a^b d[F] g_n \right\|_X \leq \frac{\varepsilon}{6} + \left\| \int_a^b d[F] h_n \right\|_X. \] (6.7)

On the other hand, it follows from Theorem 5.4, Corollary 5.14 and relations (6.5), (6.6), and from the definition of $V_n$ that
\[ \left\| \int_a^b d[F] h_n \right\|_X \leq \left\| \int_{U_n} d[F] h_n \right\|_X + \left\| \int_{V_n} d[F] h_n \right\|_X \]
\[ \leq \text{var}(F, U_n) \|h_n\|_\infty + \text{var}(F, V_n) \left( \sup_{t \in V_n} \|h_n(t)\|_X \right) \]
\[ \leq \frac{\varepsilon}{6} 2K + \text{var}_a^b F \frac{\varepsilon}{3 \ \text{var}_a^b F} < \frac{2\varepsilon}{3}. \]
Inserting this into (6.7), we conclude that
\[ \left\| \int_a^b d[F]g_n \right\|_X < \varepsilon \quad \text{for} \quad n \geq n_\varepsilon, \]
wherefrom the desired relation (6.1) follows. \( \square \)

The following assertion will be useful in the proof of Theorem 6.3 and, to our knowledge, is not available in literature.

6.2. Lemma. Let \( F \in BV([a,b], L(X)) \) and let \( F^B \) be the break function from the Jordan decomposition of \( F \). Then, for every \( g \in G([a,b], X) \), we have
\[ \int_a^b d[F^B]g = \sum_{t \in W} \Delta F(t)g(t) \]
where \( W \) stands for the set of points of discontinuity of \( F \) in \([a,b]\).

Proof. Since \( F \) is regulated, we know that \( W \) is a countable set (cf. [9, Corollary I.3.2]). Moreover, the sum \( \sum_{t \in W} \Delta F(t)g(t) \) is well-defined due to [12, Lemma 4.1].

Let \( W = \{s_k\} \) and note that the function \( F^B \) is defined as in (3.4) (with \( F \) in place of \( f \)). For each \( n \in \mathbb{N} \), consider the function
\[ F^B_n(t) = \sum_{k=1}^n [\Delta^+ F(s_k) \chi_{(s_k,b)}(t) + \Delta^- F(s_k) \chi_{[s_k,b]}(t)] \quad \text{for} \quad t \in [a,b]. \]
According to Remark 3.2, we have \( \lim_{n \to \infty} \text{var}_a^b (F^B_n - F^B) = 0 \). This, together with [18, Proposition 10], implies that
\[ \lim_{n \to \infty} \left\| \int_a^b d[F^B_n - F^B]g \right\|_X = 0, \]
that is,
\[ \int_a^b d[F^B]g = \lim_{n \to \infty} \int_a^b d[F^B_n]g. \]
To complete the proof it is enough to determine explicitly the integrals \( \int_a^b d[F^B_n]g \), \( n \in \mathbb{N} \). Applying [22, Proposition 2.3.3] (with an obvious extension to Banach space-valued functions), we obtain
\[ \int_a^b d[F^B_n]g = \sum_{k=1}^n [\Delta^+ F(s_k)g(s_k) + \Delta^- F(s_k)g(s_k)] = \sum_{k=1}^n \Delta F(s_k)g(s_k) \]
wherefrom the result follows. \( \square \)
6.3. Theorem. Let $F \in BV([a, b], X)$ and let a function $g \in G([a, b], X)$ and a sequence $\{g_n\} \subset G([a, b], X)$ be such that

$$\lim_{n \to \infty} g_n(t) = g(t) \quad for \quad t \in [a, b]$$

and

$$\|g_n\|_\infty \leq K < \infty \quad for \quad n \in \mathbb{N}.$$  

Then the integrals $\int_a^b d[F] g$ and $\int_a^b d[F] g_n$ exist for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \int_a^b d[F] g_n = \int_a^b d[F] g. \quad (6.8)$$

Proof. Let $F = F^C + F^B$ be the decomposition of $F$ as in Theorem 3.1, that is, $F^C$ is continuous on $[a, b]$ and $F^B$ is the break function given by

$$F^B(t) = \sum_{a \leq s_k < t} \Delta^+ F(s_k) + \sum_{a < s_k \leq t} \Delta^- F(s_k) \quad for \quad t \in [a, b],$$

where $W = \{s_k\} \subset [a, b]$ is the set of discontinuities of $F$ in $[a, b]$.

By [18, Proposition 15], we know that $\int_a^b d[F] g_n$ exists for each $n \in \mathbb{N}$, and from the linearity of the integral it follows that

$$\int_a^b d[F] g_n = \int_a^b d[F^C] g_n + \int_a^b d[F^B] g_n.$$ 

Put $h_n = g_n - g$ for $n \in \mathbb{N}$. Therefore $\lim_{n \to \infty} h_n(t) = 0$, and $\|h_n\|_\infty \leq K + \|g\|_\infty$ for every $n \in \mathbb{N}$. By Lemma 6.1, we obtain

$$\lim_{n \to \infty} \int_a^b d[F^C] h_n = 0,$$

that is,

$$\lim_{n \to \infty} \int_a^b d[F^C] g_n = \int_a^b d[F^C] g.$$

Now, it is enough to prove the convergence of the sequence $\int_a^b d[F^B] g_n$. To this aim, note that by Lemma 6.2 we have

$$\int_a^b d[F^B] g_n = \sum_{t \in W} \Delta F(t) g_n(t) \quad and \quad \int_a^b d[F^B] g = \sum_{t \in W} \Delta F(t) g(t).$$
If $W$ is finite, it is clear that
\[ \lim_{n \to \infty} \sum_{t \in W} \Delta F(t) g_n(t) = \sum_{t \in W} \Delta F(t) g(t). \]
and consequently (6.8) holds.

In case $W$ is countable, noting that \( \sum_{k=1}^{\infty} \| \Delta F(s_k) \|_X \leq \text{var}_a F < \infty \) (see [13, Lemma 4.1]), there exists \( p_\varepsilon \in \mathbb{N} \) such that
\[ \sum_{k=p_\varepsilon+1}^{\infty} \| \Delta F(s_k) \|_X < \frac{\varepsilon}{2(K + \| g \|_\infty)}, \]
and thus
\[ \left\| \sum_{k=p_\varepsilon+1}^{\infty} \Delta F(s_k) [g_n(s_k) - g(s_k)] \right\|_X \leq (K + \| g \|_\infty) \sum_{k=p_\varepsilon+1}^{\infty} \| \Delta F(s_k) \|_X < \frac{\varepsilon}{2}. \]
On the other hand, we may choose \( n_\varepsilon \in \mathbb{N} \) so that
\[ \left\| \sum_{k=1}^{p_\varepsilon} \Delta F(s_k) [g_n(s_k) - g(s_k)] \right\|_X < \frac{\varepsilon}{2} \quad \text{for} \quad n \geq n_\varepsilon. \]
With this in mind, we obtain
\[ \left\| \int_a^b d[F^B] (g_n - g) \right\|_X = \left\| \sum_{k=1}^{\infty} \Delta F(s_k) [g_n(s_k) - g(s_k)] \right\|_X < \varepsilon, \quad \text{for} \quad n \geq n_\varepsilon \]
which completes the proof.

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