Number of $A + B \neq C$ solutions in abelian groups and application to counting independent sets in hypergraphs

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Abstract

The paper deals with a problem of Additive Combinatorics. Let $G$ be a finite abelian group of order $N$. We prove that the number of subset triples $A, B, C \subset G$ such that for any $x \in A$, $y \in B$ and $z \in C$ one has $x + y \neq z$ equals

$$3 \cdot 4^N + N3^{N+1} + O((3 - c_*)^N)$$

for some absolute constant $c_* > 0$. This provides a tight estimate for the number of independent sets in a special 3-uniform linear hypergraph and gives a support for the natural conjecture concerning the maximal possible number of independent sets in such hypergraphs on $n$ vertices.

1 Introduction

The paper deals with estimating the number of special subset triples in abelian groups. Let us start with some motivation of the main result and discuss the problem concerning counting the number of independent sets in hypergraphs. Recall that if $H = (V, E)$ is a hypergraph, then its vertex subset $W \subset V$ is independent if it does not contain complete edges from $E$, i.e. for any $A \in E$, $A \not\subseteq W$. Let $i(H)$ denote the total number of independent sets in $H$.

1.1 Related work: counting independent sets in hypergraphs

The classical problem of extremal graph theory asks what is the maximal number of independent sets in a $d$-regular graph $G$ on $n$ vertices. In 1991 Alon [1, Section 5] conjectured that if $n$ is divisible by $2d$ then the number of independent sets in this class of graphs is maximized when $G$ is a union of $n/(2d)$ disjoint copies of $K_{d,d}$, the complete bipartite graph with equal size.
part \( d \). The conjecture was proved by Kahn \cite{9} for bipartite graphs and, finally, by Zhao \cite{10} for all \( d \)-regular graphs on \( n \) vertices. So, if \( G \) is a \( d \)-regular graph on \( n \) vertices, then
\[
i(G) \leq (i(K_{d,d}))^{\frac{1}{d}} = (2^{d+1} - 1)^{\frac{1}{d}}.\]
Note that the inequality holds even when \( n \) is not divisible by \( 2d \).

It is quite natural to consider the same problem for hypergraphs, especially for the class of linear hypergraphs. Recall that a hypergraph \( H = (V, E) \) is said to be linear if every two of its distinct edges do not share more than one common vertex, i.e. for any \( A, B \in E, A \neq B \), we have
\[
|A \cap B| \leq 1.
\]
In recent paper \cite{3} Cohen, Perkins, Sarantis and Tetali posed the following general question: which \( d \)-regular, \( k \)-uniform, linear hypergraph on a given number of vertices has the most number of independent sets?

Suppose that \( H = (V, E) \) is a \( d \)-regular \( k \)-uniform linear hypergraph on \( n \) vertices. The general container method (see, e.g., \cite{12}) implies the following upper bound on the number of independent sets:
\[
\log_2 \frac{i(H)}{n} \leq \frac{k - 1}{k} + O_k \left( \frac{\log_2 \frac{k^{-1/k}}{d}}{d^{1/k}} \right). \tag{1}
\]
It is known that the first order term in the exponent in (1) is correct, however the second is expected to be improved. Authors of \cite{3} formulated the following conjecture:

**Conjecture 1.** For any \( d \)-regular \( k \)-uniform linear hypergraph \( H \) on \( n \) vertices the following holds
\[
\log_2 \frac{i(H)}{n} \leq \frac{k - 1}{k} + \log_2 \frac{k}{kd}. \tag{2}
\]

As in Alon’s conjecture for graphs, authors of \cite{3} provide some example of a small hypergraph, whose appropriate number of copies will give bound (2). In the current paper we concentrate on the case of 3-uniform hypergraphs, so we will describe the construction from \cite{3} only for \( k = 3 \).

Consider the following mod hypergraph \( H_d^{mod} = (V, E) \), where \( V = V_1 \sqcup V_2 \sqcup V_3 \), every \( V_i \) is equal to the cyclic group \( \mathbb{Z}/d\mathbb{Z} \) and
\[
E = \{(x, y, z) : x \in V_1, y \in V_2, z \in V_3, x + y = z \pmod{d}\}.
\]
Clearly, this hypergraph is 3-uniform, 3-partite, \( d \)-regular and linear (for each \( x \in V_1, y \in V_2 \) there is a unique \( z \in V_3 \) such that \( (x, y, z) \in E \)). It was noticed in \cite{3} that
\[
i(H_d^{mod}) \geq 3 \cdot 4^d - 3 \cdot 2^d + 1.
\]
This lower bound estimates only the independent sets that do not touch some of the parts \( V_1, V_2, V_3 \). But can we say that this bound is almost tight? One the main results of our paper answers this question positively.
Theorem 2. We have
\[ i(H_d^\text{mod}) = 3 \cdot 4^d + O((4 - c_*)^d) \] (3)
and moreover
\[ i(H_d^\text{mod}) = 3 \cdot 4^d + d3^{d+1} + O((3 - c_*)^d) \] (4)
for some absolute constant \( c_* > 0 \).

As a consequence, we immediately obtain that if \( H \) is a union of \( n/(3d) \) disjoint copies of \( H_d^\text{mod} \), then in view of (3)
\[ \log_2 \frac{i(H)}{n} \leq \frac{1}{3d} \log_2 \left( 3 \cdot 4^d + O((4 - 4\varepsilon)^d) \right) = \frac{2}{3} + \frac{\log_2 3}{3d} + O((1 - \varepsilon)^d) \]
for some positive \( \varepsilon \). This estimate strongly supports Conjecture 1, see inequality (2). Surprisingly, that on the other hand, from our more refined asymptotic formula (4) it follows that
\[ \log_2 \frac{i(H)}{n} \geq \frac{2}{3} + \frac{\log_2 3}{3d} + \Omega \left( \frac{3^d}{d4^d} \right) \]
and hence one must add exponentially small error terms to make Conjecture 1 correct.

We also would like to mention that the problem of estimating the number of independent sets has some natural extensions. One can count the strong independent sets (see [3], [10], [5]) or general \( j \)-independent sets (see [2]).

1.2 Problem statement for abelian groups

By analogy with \( \mathbb{Z}/d\mathbb{Z} \) we can consider any finite abelian group \( G \) and ask the following question: what is the number of triples \( A, B, C \subset G \) such that there is no solution \( x + y = z \), where \( x \in A, y \in B \) and \( z \in C \)? If \( |G| = N \), then clearly, this number is at least
\[ 3 \cdot 4^N - 3 \cdot 2^N + 1, \]
because any triple with one empty subset always fits. If we take one of \( A, B \) or \( C \) equals a one-element set, that would add \( N \cdot 3^{N+1} \) more triples, and therefore the lower bound is
\[ 3 \cdot 4^N + N \cdot 3^{N+1} + O(2^N). \] (5)

Our first aim is to show that this lower bound is tight up to some summand exponentially smaller than \( 4^N \). Then we further elaborate our argument to find the correct exponent from (5).

The organization of this paper is the following. In section 2 we consider the case when \( G \) is the prime field. In the next section 3 we deduce the main result in the general case. The arguments in the prime case are simpler but nevertheless, the scheme of the proof is similar to the case of general abelian group. Finally, in sections 4 and 5 we will repeat the same scheme but obtain a stronger estimate for the error term, using a new result on structure of dense subsets \( A, B \) of \( \mathbb{F}_p \) with \( A + B \neq \mathbb{F}_p \), see Proposition 13 below. A similar instrument was introduced for the first time in [13], where it has found already some applications to the structure of sets with small Wiener norm. Also, it allows to estimate size of \( A(A + A) \) for any \( A \subseteq \mathbb{F}_p \), see [13]. Thus this part of the paper has an independent interest for Additive Combinatorics.
2 The case of the prime field

Let $G$ be a finite abelian group, $\hat{G}$ be the dual group. It is well–known that $\hat{G}$ is isomorphic to $G$. In this paper we use the same letter to denote a set $A \subseteq G$ and its characteristic function $A : G \to \{0, 1\}$. If $f, g : G \to \mathbb{C}$ are functions, then we write
\[
(f * g)(x) = \sum_{y \in G} f(y)g(x - y).
\]
The sumset of sets $A, B \subseteq G$ is
\[
A + B = \{a + b : a \in A, b \in B\}.
\]
Similarly, one can define the difference set of $A$ and $B$. Given a prime $p$ we write $\mathbb{F}_p$ for the prime field. All logarithms are to base 2. The signs $\ll$ and $\gg$ are the usual Vinogradov symbols.

Now suppose that $A, B \subseteq G$ are two sets and put $C = C(A, B) = G \setminus (A + B)$. By the Cauchy–Davenport inequality, see, e.g., [15, Theorem 5.4] if $G = \mathbb{F}_p$ and $A + B \neq \mathbb{F}_p$, that is, $C \neq \emptyset$, we have
\[
|A + B| \geq |A| + |B| - 1. \tag{6}
\]

We need one of the main results from [3]. Recall that for a given subset $\Gamma \subseteq \hat{G}$ and a number $\varepsilon \in (0, 1)$ the set $\text{Bohr}(\Gamma, \varepsilon)$ is called a Bohr neighbourhood (or a Bohr set), see, e.g., [15, Section 4.4] if
\[
\text{Bohr}(\Gamma, \varepsilon) = \{x \in G : |\chi(x) - 1| \leq \varepsilon, \forall \chi \in \Gamma\}.
\]
Size of $\Gamma$ is called the dimension of $\text{Bohr}(\Gamma, \varepsilon)$, $\varepsilon$ is the radius of $\text{Bohr}(\Gamma, \varepsilon)$ and it is well–known the connection of size of Bohr sets with its dimension and radius, e.g., see [15, Lemma 4.20]
\[
|\text{Bohr}(\Gamma, \varepsilon)| \geq (\varepsilon/2\pi)^{|\Gamma|}|G|. \tag{7}
\]

Now we are ready to formulate the required extracted result from [6] about shifts of Bohr sets in sumsets (actually one can check that the arguments work for composite $p$ and even for an arbitrary finite abelian group as well). The quantity $\log^{1/4} p$ below is not so important and it has chosen just for convenience.

**Theorem 3.** Let $p$ be a sufficiently large prime number and $A, B$ be sets from $\mathbb{F}_p$, $|A| \geq \alpha p$, $|B| \geq \beta p$, $\kappa = \sqrt{\alpha \beta} \geq \log^{-1/4} p$. Also, let $|C(A, B)| = \gamma p$ and $\omega \leq \gamma$ be a parameter, $\omega \geq \exp(-\log^{1/4} p))$. Then there is a shift $x \in \mathbb{F}_p$ and a Bohr set $\mathcal{B} = \text{Bohr}(\Gamma, \frac{\kappa}{64 \omega})$, $|\Gamma| = d \leq 1000\kappa^{-2}\log(1/\omega)$, such that
\[
|(A + B) \cap (\mathcal{B} + x)| \geq \left(1 - \frac{32\omega}{\kappa}\right)|\mathcal{B}|.
\]

A similar result on almost periodicity of convolutions was obtained in [4]. For the sake of the completeness we formulate a consequence of it. The dependencies on the parameters in Theorems [3, 4] are slightly different but in our regime ($\alpha, \beta \gg 1$) this is absolutely not important.
Theorem 4. Let $G$ be a finite abelian group, $N = |G|$ and $A, B$ be sets from $G$, $|A| \geq \alpha N$, $|B| \geq \beta N$, $\kappa = \sqrt{\alpha \beta}$. Also, let $q \geq 2$ and $\varepsilon \in (0, 1)$ be parameters. Then there is a shift $x \in \mathbb{Z}/N\mathbb{Z}$ and a Bohr set $B = \text{Bohr}(\Gamma, c\varepsilon)$, $c > 0$ is an absolute constant, $|\Gamma| \ll q/\varepsilon^2$, such that

$$|(A + B) \cap (B + x)| \geq \left(1 - \left(\frac{\varepsilon}{\kappa}\right)^q\right)|B|.$$  \hfill (8)

Indeed, by the main result of [10] Theorem 1.2 for almost periodicity of $f(x) := (A * B)(x)$, we have

$$\sum_{x \in G} \sum_{t \in B} ((A * B)(x + t) - (A * B)(x))^q \leq (\varepsilon \kappa N)^q |B| N$$

and hence if (8) does not hold for any $x \in G$, then we obtain a contradiction in view of the simple bound

$$\kappa^{2q} N^{q+1} \leq \sum_{x \in G} (A * B)^q(x).$$

Let us write $\#\{R \mid P(R)\}$ for the number of objects $R$, satisfying a property $P$. We simply write $\#\{R\}$, if the required property is clear from the context. Also, let us introduce the following nonstandard notation. For sets $X, Y \subseteq \mathbb{F}_p$ we write $X \neq Y$ if the equation $x = y$ has no solutions in $x \in X, y \in Y$.

Now we are ready to prove the main result of this section.

Theorem 5. Let $p$ be a sufficiently large prime number. Then the number of sets $A, B, C$ from $\mathbb{F}_p$ such that $x + y \neq z$, $x \in A$, $y \in B$, $z \in C$ equals

$$3 \cdot 4^p + O((4 - c_*)^p),$$  \hfill (9)

where $c_* > 0$ is an absolute constant.

Proof. As we have noted above, if one takes $A, B$ or $C$ equals the empty set and the rest is an arbitrary, then we obtain $3 \cdot 4^p + O(2^p)$ of the required sets. The quantity $3 \cdot 4^p$ is the main term and our task is to estimate the rest.

Suppose that all sets $A, B, C$ are non–empty. Put $a = |A|, b = |B|, c = |C|$. Clearly, $|C(A, B)| \geq c$ hence by inequality (9), we get $c \leq p - |A + B| = p - a - b + 1$ and, similar bounds hold for $a$ and $b$. We begin with a crude upper bound for the number of triples $A, B, C$ with small $a, b$ or $c$. Namely, for $M \leq p/16$, say, one has

$$\sum_{a \leq M} \binom{p}{a} \sum_{b = 1}^{p-a-b+1} \binom{p}{b} \sum_{c = a+b-1}^{p-a-b+1} \binom{p}{c} 2^{-a} \sum_{b = 1}^{p} \binom{p}{b} 2^{-b} \leq 2^{p+1} \sum_{a \leq M} \binom{p}{a} 2^{-a} \sum_{b = 1}^{p} \binom{p}{b} 2^{-b} \leq 4 \cdot 3^p \left(\frac{p}{M}\right)^2 M < 4(3.75)^p.$$  \hfill (10)

Thus $a, b, c \geq p/16$. Now we apply Theorem 3 with the parameters $\alpha = \beta = \gamma = \kappa = 1/16$ and let $\omega \leq 2^{-10}$ be a sufficiently small number, which we will choose later (an alternative way is to use Theorem

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Let $d$ and $\mathcal{B}$ be as in Theorem 3. The number of all possible shifted Bohr sets is $p^{d+1}$. We know that $C$ intersects $\mathcal{B}$ by at most $2^d \omega |\mathcal{B}| := E$ points. Let us set $C' := C \cap \mathcal{B}$. The number of all sets of size at most $E$ does not exceed $E(\frac{p}{E})$. Put $q = |\mathcal{B}| - E \geq |\mathcal{B}|/2 > 0$. Without loss of generality (since it will not affect the exponent) we assume that $0 \in B$, and therefore $A \neq C$. For fixed $a, b, c$ we obtain

$$\# \{A, B, C \mid A + B \neq C\} \leq \# \{A, B \mid A \neq C\} \leq p^{d+1} E \left( \frac{p}{E} \right) \left( \frac{p-q}{c} \right) \left( \frac{p}{a} \right) \left( \frac{p-c-a+1}{b} \right).$$

Summing it for all possible triples $a, b, c$ and considering obvious inequalities $c \leq p-q, a \leq p-c, b \leq p-c-a+1$ we see that the number of all possible triples $A, B, C$ is at most

$$\sigma := p^{d+1} E \left( \frac{p}{E} \right) \sum_{c=1}^{p-q} \left( \frac{p-q}{c} \right) \sum_{a=1}^{p-c} \left( \frac{p}{a} \right) \sum_{b=1}^{p-c-a+1} \left( \frac{p-c-a+1}{b} \right) \leq 2 E p^{d+1} \left( \frac{p}{E} \right) \left( \frac{p-q}{c} \right) \left( \frac{p}{a} \right) 2^{-a} \leq 2 E p^{d+1} \left( \frac{p}{E} \right) 3^{p-q} \left( \frac{p-q}{c} \right) 3^{-c} \leq p^{d+2} \left( \frac{e^p}{E} \right) \left( \frac{3}{4} \right)^q \cdot 4^p. \quad (12)$$

Put $l = \log(1/\omega)$. By estimate [17], we know that $|\mathcal{B}| \geq q \gg p \exp(-C l \log l)$, where $C > 0$ is an absolute constant. Taking $\omega$ sufficiently small constant, we can attain

$$\left( \frac{e^p}{E} \right)^E \ll \left( \frac{C_1 \exp(C l \log l)}{\omega} \right)^{2^{10} \omega q} \leq \left( \frac{8}{7} \right)^q,$$ \quad (13)

where $C_1 > 0$ is another absolute constant. Thus the number $d \leq 1000 \kappa^{-2} \log(1/\omega)$ is a constant and hence the multiple $p^{d+2}$ in (12) is negligible. Whence for sufficiently large $p$, we have, say,

$$\sigma \ll \left( \frac{9}{10} \right)^q \cdot 4^p \ll \left( \frac{9}{10} \right)^{p \exp(-C l \log l)} \cdot 4^p = (4 - \kappa) p.$$

This completes the proof. \qed

3 The general case

The case of an arbitrary finite abelian group $G$ requires more refined arguments and generalizations. For example, inequality (3) is a particular case of Kneser’s Theorem, see, e.g., [15, Theorem 5.5], which takes place in any abelian group.

**Theorem 6.** Let $G$ be an abelian group, and $A, B \subseteq G$ be sets. Then

$$|A + B| \geq |A + H| + |B + H| - |H|,$$

where $H := \{x \in G : A + B + x = A + B\}$. 

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Now we are ready to prove an analogue of Theorem 5 and we will appeal to the proof of this result.

**Theorem 7.** Let \( G \) be a finite abelian group, \( N = |G| \). Then the number of sets \( A, B, C \) from \( G \) such that \( x + y \neq z, \) \( x \in A, y \in B, z \in C \) equals

\[
3 \cdot 4^N + O((4 - c_*)^N),
\]

(14)

where \( c_* > 0 \) is an absolute constant.

**Proof.** As in the proof of Theorem 5 (see calculations in (10), (11)) we can assume that two sets from \( A, B, C \) are large, say, \( \Omega(N) \). Indeed, using the notation of the proof of this Theorem, we see the rest can be estimated as \( 2^N (N^M)^2 \leq (3.75)^N \), say. Without losing of the generality, suppose that \( A \) and \( B \) are large, i.e., \( |A|, |B| \geq c_0 N \) with an absolute constant \( c_0 > 0 \). Put \( H := \{ x \in G : A + C + x = A + C \} \leq G \), and suppose firstly that \( h := |H| \leq c_1 N \), where \( c_1 > 0 \) is a sufficiently small absolute constant. Then Kneser’s inequality gives us \( |A + C| \geq |A| + |C| - |H| \).

Hence combining this with Theorem 3 or Theorem 4 and acting as in the proof of Theorem 5, we get

\[
E \left( p^{d+1} \left( \frac{p}{E} \right)^{N-q} \left( \begin{array}{c} N-q \cr c \end{array} \right) \sum_{a=1}^{N-c} \left( \begin{array}{c} N \cr a \end{array} \right) \sum_{b=1}^{N-c-a+h} \left( \begin{array}{c} N-c-a+h \cr b \end{array} \right) \right) \leq N^{d+2} 2^h \left( \frac{eN}{E} \right)^E \left( \frac{3}{4} \right)^N . 4^N
\]

(again \( d \) is as in Theorem 3). Hence as in inequality (13) we obtain the required asymptotic formula (14) taking sufficiently small \( c_1 \). Thus \( h > c_1 N \) and we can assume that \( h < N \) because otherwise \( A + C = G \). Hence \( h \leq N/2 \). Put \( n = N/h \geq 2, n \in \mathbb{Z} \). Let \( k_A \) be the number of different cosets \( G/H \), which intersects our random set \( A \) (and, similarly, define \( k_B, k_C \) for \( B \) and \( C \)). From Kneser’s Theorem we see that \( k_A, k_B, k_C < n \) otherwise the correspondent sumset coincides with the whole \( G \). Applying Theorem 3 again, we obtain that \( b \leq (n - k_A - k_C + 1) h \).

Using the arguments as in (10), (11) one more time, combining with Theorem 5 we derive a crude upper bound for the number of possible \( A, B \) and \( C \)

\[
\sum_{k_A=1}^{n-1} \left( \begin{array}{c} n \cr k_A \end{array} \right) 2^{k_A h} \sum_{k_C=1}^{n-k_A} \left( \begin{array}{c} n-k_A \cr k_C \end{array} \right) 2^{k_C h} \sum_{k_B=1}^{n-k_C-k_A+1} \left( \begin{array}{c} n-k_A-k_C+1 \cr k_B \end{array} \right) 2^{k_B h} \leq \]

\[
\leq n 2^{3n} \sum_{k_A=1}^{n-1} \sum_{k_C=1}^{n-k_A} \sum_{k_B=1}^{n-k_C-k_A+1} 2^{k_A h} + 2^{(n-k_A-k_C+1) h} \leq n 2^{3n} 2^{N-h} \leq n 3^{2n} 2^{3N/2}.
\]

Since \( h \geq c_1 N \), it follows that that \( n \leq c_1^{-1} = O(1) \) and hence the multiple \( n 2^{3n} \) in the formula above is negligible. Also, it is easy to see that the number of subgroups in \( G \) of index at most \( n = O(1) \) is \( N^{O(1)} \) (e.g., consider the canonical homomorphism of left cosets of \( H \) in \( G \) to the symmetric group on \( n \) letters) and this latter number is also negligible. This completes the proof. \( \Box \)

4 An improvement

In this section we obtain an improvement of Theorem 5 using other new tools, e.g., the Fourier transform on \( G \). Our main argument works in the case of the prime field only although some statements hold to be true for a general finite abelian group \( G \).
We denote the Fourier transform of a function \( f : G \to \mathbb{C} \) by \( \hat{f} \), namely,
\[
\hat{f}(\chi) = \sum_{x \in G} f(x)\chi(x),
\]
(15)
where \( \chi \in \hat{G} \) is an additive character on \( G \). We rely on the following basic identities. The first one is called the Plancherel formula and its particular case \( f = g \) is called the Parseval identity
\[
\sum_{x \in G} f(x)g(x) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\hat{g}(\chi). \tag{16}
\]
Another particular case of (16) is
\[
\sum_{x \in G} |(f \ast g)(x)|^2 = \sum_{x \in G} \left| \sum_{y \in G} f(y)g(x-y) \right|^2 = \frac{1}{|G|} \sum_{\chi \in \hat{G}} |\hat{f}(\chi)|^2 |\hat{g}(\chi)|^2. \tag{17}
\]
and the identity
\[
f(x) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(x). \tag{18}
\]
is called the inversion formula.

Our proof is based on several auxiliary statements. First of all, we need a consequence of the following result of J.M. Pollard [11] (also, see [14, Corollary 1.2]). Given sets \( A, B \subseteq F_p \) and \( \varepsilon \in (0, 1) \) we write \( A + \varepsilon B \) for the set of \( x \in F_p \), having at least \( \varepsilon p \) representations as sum of \( a + b, a \in A, b \in B \).

**Theorem 8.** Let \( A, B \subseteq F_p \) be sets and \( \varepsilon \in (0, 1) \) be a real number, such that \( \sqrt{\varepsilon} p < |A|, |B| \). Then
\[
|A + \varepsilon B| \geq \min\{p, |A| + |B|\} - 2p\sqrt{\varepsilon}. \tag{19}
\]

We have the following combinatorial observation: if sets \( X, Y, Z \subseteq F_p \) satisfy \( X + Y \neq Z \), then they satisfy \( X \neq Z - Y \) as well. Let us prove the following robust version of this truism.

**Lemma 9.** Let \( X, Y, Z \) be nonempty subsets of \( F_p \) and \( \delta \in (0, 1) \) be a parameter such that \( Z \neq X + \delta Y \). Also, let us assume that \( |X| \geq \varepsilon p \) for a certain \( \varepsilon \in (0, 1) \). Let \( T > 1 \) be a parameter. Then there exists \( X' \subseteq X, |X'| \leq |X|/T \) such that
\[
X \setminus X' \neq Z - \delta \varepsilon Y. \tag{20}
\]

**Proof.** Let \( \eta := \delta T/\varepsilon \). Suppose that for a certain \( X' \subseteq X \) one has \( X' \subseteq Z - \eta Y \). Then we have at least \( |X'|\eta p \) triples \((x, y, z)\) such that \( x + y = z \). By the pigeonhole principle we see that there is \( z_0 \) with at least \( |X'|\eta p/|Z| \) representations \( z_0 = x + y \). By assumption \( Z \neq X + \delta Y \) and hence we derive the inequality \( |X'|\eta p/|Z| \leq \delta p \), which gives \( |X'| \leq |Z|\delta/\eta \leq p\delta/\eta \leq |X|/T \).

\[ \square \]
Secondly, we need the well–known Chang’s Theorem see, e.g., \cite{15}. Recall that for a set \( A \subseteq G \) and \( \varepsilon \in (0, 1] \) the spectrum \( \text{Spec}_\varepsilon(A) \) is defined as

\[
\text{Spec}_\varepsilon(A) := \{ \chi \in \hat{G} : |\hat{\chi}(0)| = 0 \}.
\]

Also, the additive dimension \( \dim(A) \) of a set \( A \subseteq G \) is size of the maximal dissociated subset of \( A \), i.e., size of maximal \( \Lambda \subseteq A \) such that any equation \( \sum_{\lambda \in \Lambda} \varepsilon_\lambda \lambda = 0 \) with \( \varepsilon_\lambda \in \{0, 1, -1\} \) implies \( \varepsilon_\lambda = 0, \forall \lambda \in \Lambda \).

**Theorem 10.** Let \( G \) be a finite abelian group, \( A \subseteq G \), and \( \varepsilon \in (0, 1] \) be a real number. Then

\[
\dim(\text{Spec}_\varepsilon(A)) \leq 2\varepsilon^-2 \log(|G|/|A|).
\]

Now let us formulate a rather general and simple result on level sets of an arbitrary function.

**Lemma 11.** Let \( G \) be an abelian group, \( \delta, \alpha, \beta \) be real numbers, \( \alpha \leq \beta \), \( Y \) be a set, and \( f, g : G \to \mathbb{C} \) be functions. Put

\[
X = \{ x \in G : f(x) \in [\alpha, \beta] \},
\]

and

\[
E^- = \{ x \in G : f(x) \in (\alpha - \delta, \alpha) \}, \quad E^+ = \{ x \in G : f(x) \in (\beta, \beta + \delta) \}.
\]

Suppose that for all \( x \in G \setminus Y \) one has \( \|f - g\|_\infty \leq \delta \) and write \( g(x) = \sum_j g(x)S_j(x) \), where \( g \) differs by at most \( \delta \) on some disjoint sets \( S_j \). Then

\[
\biguplus_{j : S_j \cap (X \setminus Y) \neq \emptyset} S_j = ((X \setminus Y) \cup Y') \cup E' \cup E'', \tag{20}
\]

where \( E' \subseteq E^- \), \( E'' \subseteq E^+ \), \( Y' \subseteq Y \) are some sets.

**Proof.** Put \( X' = X \setminus Y \), \( X'' = \biguplus_{j : S_j \cap X' \neq \emptyset} S_j \). Clearly, \( X' \subseteq X'' \) because we can assume that \( \biguplus_j S_j = G \). Our task is to prove that \( X'' \) coincides with the right–hand side of (20). If \( x \in X'' \), then \( f(x) \in [\alpha - \delta, \beta + \delta] \) and hence \( X'' \subseteq X \cup E' \cup E'' \), where \( E' := E^- \cap X'' \) and \( E'' := E^+ \cap X'' \). Let \( Y' := Y \cap X'' \). Then \( X'' \subseteq ((X \setminus Y) \cup Y') \cup E' \cup E'' \) but the reverse inclusion is obvious (recall that \( X' = X \setminus Y \subseteq X'' \)). This completes the proof. \( \square \)

**Remark 12.** In follows from the proof of Lemma \ref{11} that the sets \( S_j \) do not need to be disjoint and one can consider any covering of \( G \) by a collection of sets \( S_j \). Nevertheless, then we lose disjointedness in (20) and it is important sometimes to keep it, see \cite{13}.

Theorem \ref{8} combining with Lemma \ref{11} give us a useful result.

**Proposition 13.** Let \( A, B \subseteq \mathbb{F}_p \) be sets, \( |A| = \alpha p \), \( |B| = \beta p \), \( \delta, \varepsilon \in (0, 1] \) be real parameters, and \( C = C(A, B) \neq \emptyset \). Then there are sets \( Y, W, C \setminus Y \subseteq W \) such that the following holds

\[
|Y| \leq \varepsilon^2 \alpha^2 \beta \delta^{-2} p, \quad W = \bigcup_{j=1}^t S_j.
\]
where \( S_j \) are shifts of a Bohr set \( \mathcal{B} \) of dimension \( 2\varepsilon^{-2}\log\alpha^{-1} \) and radius \( \delta(\alpha\beta)^{-1/2} \),

\[
t \leq \exp(O(\varepsilon^{-2}\log\alpha^{-1} \cdot \log(\varepsilon\delta)^{-1})),
\]

\( W \neq A + \delta B, \)

and

\[
|W| \leq p - |A| - |B| + 2p\sqrt{\delta}.
\]

**Proof.** Let \( f(x) = (A * B)(x) \) and \( g(x) = p^{-1} \sum_{r \in \text{Spec}_\varepsilon(A)} \hat{f}(r)e(rx) \). Due to formulae (16), (17), we have

\[
\sum_x (f(x) - g(x))^2 \leq p^{-1} \sum_{r \notin \text{Spec}_\varepsilon(A)} |\hat{A}(r)|^2 |\hat{B}(r)|^2 \leq \varepsilon^2 |A|^2 |B|
\]

and hence \( \|f(x) - g(x)\|_\infty \leq \delta p \) outside of a set \( Y \) of size \( |Y| \leq \varepsilon^2 \alpha^2 \beta \delta^{-2} p \). Apply Lemma 11 with the parameters \( \alpha = \beta = 0 \) to the constructed functions \( f, g \) and to the set \( Y \). Then \( E' \) is empty and \( |E''| \leq p - |A| - |B| + 2\sqrt{\delta} p \) thanks to Theorem 8. Thus we put \( W := E'' = \{ x : f(x) \leq \delta \} \). In other words, \( W \neq A + \delta B \). Further by Chang’s Theorem 10 there exists a set \( \Lambda, |\Lambda| \leq 2\varepsilon^{-2}\log\alpha^{-1} \) such that any element of \( \text{Spec}_\varepsilon(A) \) can be expressed as \( \sum_{\lambda \in \Lambda} \xi_\lambda \lambda, \xi_\lambda \in \{0, 1, -1\} \). Put \( \mathcal{B} = \text{Bohr}(\Lambda, \zeta/|\Lambda|) \) with \( \zeta = \delta(\alpha\beta)^{-1/2} \). Then for any \( b \in \mathcal{B} \) and for an arbitrary \( x \in \mathbb{F}_p \), one has

\[
|g(x + b) - g(x)| \leq p^{-1} \sum_{r \in \text{Spec}_\varepsilon(A)} |\hat{A}(r)| |\hat{B}(r)| |e(rb)| - 1 | \leq \varepsilon p^{-1} \sum_r |\hat{A}(r)||\hat{B}(r)| \leq \sqrt{\alpha\beta} \zeta p = \delta p,
\]

where we have used the triangle inequality twice, the Cauchy–Schwarz inequality and finally, formula (16). It means that the function \( g \) differs at most \( \delta p \) on any shift of \( \mathcal{B} \).

Now we want to find some \( s_1, s_2, \ldots, s_t \) so that shifts \( s_i + \mathcal{B} \) cover the whole group \( \mathbb{F}_p \). Let us consider a Bohr set \( \mathcal{B}' \), identical to \( \mathcal{B} \), but with twice smaller radius. By the definition of Bohr sets it is clear that \( \mathcal{B}' - \mathcal{B}' \subseteq \mathcal{B} \). Let \( S := \{ s_1, s_2, \ldots, s_t \} \) be a maximal set such that shifts \( s_i + \mathcal{B}' \) do not overlap. Due to its maximality, for any \( s \in \mathbb{F}_p \) there exists such \( s_i \) so that \( s + \mathcal{B}' \) intersects with \( s_i + \mathcal{B}' \), and therefore \( s \in s_i + \mathcal{B}' - \mathcal{B}' \subseteq s_i + \mathcal{B} \). Therefore, shifts \( s_i + \mathcal{B} \) cover the whole group \( \mathbb{F}_p \). Since \( s_i + \mathcal{B}' \) do not overlap, we have \( p \geq |S + \mathcal{B}'| = |S||\mathcal{B}'| \), and therefore, by estimate (17), we see that the number \( t \) of such shifts is at most \( p/|\mathcal{B}'| \leq (4\pi|\Lambda|\zeta^{-1})|\Lambda| \). This completes the proof. \( \square \)

Repeating the proof above, one can obtain the following slightly different

**Proposition 14.** Let \( A, B \subseteq \mathbb{F}_p \) be sets, \( |A| = \alpha p, |B| = \beta p, \eta, \delta, \varepsilon \in (0, 1] \) be real parameters, and \( C \) be a set, \( C \neq A + \eta B \).

Then there are sets \( Y, W, C \setminus Y \subseteq W \) such that the following holds

\[
|Y| \leq \varepsilon^2 \alpha^2 \beta \delta^{-2} p, \quad \quad W = \bigcup_{j=1}^t S_j,
\]

where \( S_j \) are shifts of a Bohr set \( \mathcal{B} \) of dimension \( 2\varepsilon^{-2}\log\alpha^{-1} \) and radius \( \delta(\alpha\beta)^{-1/2} \),

\[
t \leq \exp(O(\varepsilon^{-2}\log\alpha^{-1} \cdot \log(\varepsilon\delta)^{-1})),
\]

10
\[ W \neq A +_{(\eta + \delta)} B, \]

and

\[ |W| \leq p - |A| - |B| + 2p\sqrt{\eta + \delta}. \]

We write Chernoff Bound, according to [15]:

**Theorem 15 (Chernoff’s inequality).** Assume that \( X_1, \ldots, X_n \) are jointly independent random variables where \( |X_i - \mathbb{E}X_i| \leq 1 \) for all \( i \). Set \( X := X_1 + \ldots + X_n \) and let \( \sigma := \sqrt{\text{Var}(X)} \) be the standard deviation of \( X \). Then for any \( \lambda > 0 \)

\[ \mathbb{P}(|X - \mathbb{E}X| \geq \lambda \sigma) \leq 2 \max(e^{-\lambda^2/4}, e^{-\lambda \sigma/2}) \]

Now we are ready to improve Theorem 5.

**Theorem 16.** Let \( p \) be a sufficiently large prime number and \( \varepsilon \in (0, 1) \) be any real parameter. Then the number of sets \( A, B, C \) from \( \mathbb{F}_p \) such that \( x + y \neq z, x \in A, y \in B, z \in C \) equals

\[ 3 \cdot 4^p + 3p \cdot 3^p + O((3 - c_*)^p), \tag{23} \]

for some absolute constant \( c_* > 0 \).

**Proof.** When one of the sets \( A, B, C \) is empty we obtain the term \( 3 \cdot 4^p \), and when one of them contains just one element, we get the term \( 3p \cdot 3^p \). From now let us assume that all sets contain at least two elements. We will consider two cases: when one of the sets is small, and when all \( A, B, C \) satisfy \( |A|, |B|, |C| \gg \varepsilon p \), where \( \varepsilon \) is a sufficiently small constant.

Let us assume that \( 2 \leq |B| \ll \varepsilon p \). Without loss of generality, we can assume that \( B \) contains elements 0 and \(-1\) (by multiplying and shifting \( A, B, C \) by the same constants). Therefore, \( A \neq C \cup (C + 1) \). The number of pairs \( A, C \) is at most

\[ \sum_{c=1}^{p} \sum_{|C|=c} \# \{ A : A \neq C \cup (C + 1) \} = \sum_{c=1}^{p} \sum_{|C|=c} 2^{p-|C\cup(C+1)|}. \tag{24} \]

If we write \( 2^{p-|C|} \) instead of \( 2^{p-|C\cup(C+1)|} \), then the above sums up to \( 3^p \). However, \( |C\cup(C+1)| = |C| + |C + 1| - |C \cap (C + 1)| = 2|C| - |C \cap (C + 1)| \), and we expect \( |C \cap (C + 1)| \) to be equal roughly \( \gamma^2 p \) in average, for \( |C| = \gamma p \). Using Chernoff bound [15] we can estimate the number of cases when \( |C \cap (C + 1)| = \gamma^2 p \) (the latter has exponentially small probability), and we obtain estimate \( (3 - c_*)^p \) for sum (24) in this case.

Summing it for all \( B, 2 \leq \min(|A|, |B|, |C|) \ll \varepsilon p \) we obtain

\[ \# \{ A, B, C : A + B \neq C \} \leq \# \{ B \} (3 - c_*)^p \leq \exp(O(\varepsilon p))(3 - c_*)^p \leq (3 - c_* + O(\varepsilon))^p. \]

Now let us consider the case, when all \( A, B, C \) satisfy \( |A|, |B|, |C| \gg \varepsilon p \).

Let us apply Proposition [13] to sets \( A, B, C \) with the parameters \( \delta := \delta_C \) and \( \varepsilon := \varepsilon_C \) (\( \varepsilon_C \) and \( \delta_C \) will be defined later). We find sets \( W_C, Y_C \), so that

\[ C \setminus Y_C \subseteq W_C, \quad W_C \neq A +_{\delta_C} B. \]
Let us choose some $T > 1$. By Lemma 9 we have $A \setminus A' \neq W_C - \delta_{C T / \varepsilon} B$ for some $A'$ of size at most $|A|/T$. Let $\delta_A := \delta_{C T / \varepsilon}$. Put $A_1 := A \setminus A'$. Apply Proposition 14 to the sets $A_1, W_C, B$ with the parameters $\varepsilon := \varepsilon_A, \eta := \delta := \delta_A$ (again $\varepsilon_A$ will be defined later). We see that for some $W_A, Y_A$ one has

$$A_1 \setminus Y_A \subseteq W_A, \ W_A \neq W_C - 2\delta_A B.$$ 

In a similar way by Lemma 9 we have $B \setminus B' \neq W_C - 2\delta_{A T / \varepsilon} W_A$ for some $B'$ of size at most $|B|/T$. Let us put $\delta_B := 2\delta_A T / \varepsilon$. Let $B_1 := B \setminus B'$. By Proposition 14 applied to the sets $B_1, W_A, W_C$ with parameters $\varepsilon := \varepsilon_B$ and $\eta := \delta := \delta_B$ ($\varepsilon_B$ will be defined later) we obtain some $W_B, Y_B$, so that

$$B_1 \setminus Y_B \subseteq W_B, \ W_B \neq W_C - 2\delta_B W_A.$$ 

We know that all $Y_X$ (with $X = A, B$ or $C$) satisfy the rough inequality $|Y_X| \leq (\varepsilon_X / \delta_X)^2 p$. Let us put each $\varepsilon_X$ to be equal $\varepsilon \delta_X$. Now we have bounds $|Y_X| \leq \varepsilon^2 p$ for $X = A, B, C$. It is easy to see that

$$\# \{ Y_X \} \ll \exp(O(p \varepsilon)).$$ 

Each $W_X$ is defined by a collection of at most $t := \exp(O(\varepsilon_X^{-2} \log \varepsilon^{-1} \log(\varepsilon \delta_X)^{-1}))$ shifts of Bohr sets $S_j := B_X + s_j, s_j \in \mathbb{F}_p$. We now guarantee that all $\delta_X$ (and therefore $\varepsilon_X$) do satisfy $\delta_X \gg \varepsilon^{\Omega(1)}$, by setting $T := 1 / \varepsilon^2$ and $\delta_C := \varepsilon^{10}$ (notice that $\delta_A = \delta_C T / \varepsilon = \varepsilon^7$, and $\delta_B = 2\delta_A T / \varepsilon = 2\varepsilon^4$). From this we obtain $t \ll \exp(O(\varepsilon^{-\Omega(1)}))$. We know that the dimension of our Bohr set $B_X$ is at most $\dim(B_X), \dim(B_X) \ll \varepsilon_X^{-2} \log \varepsilon^{-1} \ll \varepsilon^{-\Omega(1)}$. Hence the number of possible sets $S_j = B_X + s_j$ is at most

$$\# \{ B_X \} \# \{ s_1, \ldots, s_t \} \ll \left( \frac{p}{\dim(B_X)} \right)^{\left( \frac{p}{\varepsilon^2 p} \right)} \ll p^{\dim(B_X) + t} \ll \exp(O(p \varepsilon)).$$ 

From here we obtain

$$\# \{ W_X \} \ll \# \{ B_X \} \# \{ s_1, \ldots, s_t \} \ll \exp(O(p \varepsilon))$$ 

for all $X = A, B, C$.

Clearly, since $|A_1| \leq |A| / T \ll \varepsilon^2 p, |B_1| \ll \varepsilon^2 p$, it follows that

$$\# \{ A_1 \}, \# \{ B_1 \} \ll \left( \frac{p}{\varepsilon^2 p} \right) \ll \exp(O(p \varepsilon)).$$ 

Finally, from the formula $W_B \neq W_C - 2\delta_B W_A$ and estimate (19) from Theorem 8 we obtain that

$$|W_A| + |W_B| + |W_C| \leq p + 2\sqrt{2\delta_B p} = p + O(\varepsilon^2 p)$$ 

For given sets $W_A, W_B, W_C$, satisfying $|W_A| + |W_B| + |W_C| \leq p + O(\varepsilon^2 p)$, and given $Y_A, Y_B, Y_C$ the number of triples $A, B, C$ is bounded by

$$\# \{ A' \} \# \{ A_1 : A_1 \setminus Y_A \subseteq W_A \} \# \{ B' \} \# \{ B_1 : B_1 \setminus Y_B \subseteq W_B \} \# \{ C : C \setminus Y_C \subseteq W_C \},$$ 

which is at most

$$\exp(O(p \varepsilon)) 2^{W_A} 2^{W_B} 2^{W_C} \ll 2^{p + O(p \varepsilon)}.$$ 

The number of sets $W_A, W_B, W_C, Y_A, Y_B, Y_C$ can be estimated as

$$\# \{ W_A \} \# \{ Y_A \} \# \{ W_B \} \# \{ Y_B \} \# \{ W_C \} \# \{ Y_C \} \ll \exp(O(p \varepsilon)),$$

and the final result is $2^{p + O(p \varepsilon)}$. This completes the proof. □
Remark 17. It is highly likely, that when $2 \leq |B| \ll \varepsilon p$, one can estimate $\#\{A + B \neq C\}$ as

$$\exp(O(\varepsilon p)) \max_{\gamma \in (0,1)} \left( \frac{p}{\gamma p} \right) 2^{p-2\gamma p + \gamma^2 p} = \exp(O(\varepsilon p)) \max_{\gamma \in (0,1)} e^{-\gamma \ln \gamma - (1-\gamma) \ln (1-\gamma)} + (1-\gamma)^2 \ln 2},$$

which is optimized for $\gamma = 0.2653\ldots$ and gives $O(2.5926\ldots + O(\varepsilon))p$.

Remark 18. It is also highly likely, that consider separately the cases $\min(|A|, |B|, |C|) = 2, 3, 4, \ldots, k$, we can obtain bound on $\#\{A, B, C : A + B \neq C\}$ of the form

$$3 \cdot 4^p + 3p \cdot 3^p + Q_2(p)\lambda_2^p + Q_3(p)\lambda_3^p + \ldots + Q_k(p)\lambda_k^p + O((\lambda_k - \gamma)^p),$$

where $4 > 3 > \lambda_2 > \lambda_3 > \ldots > \lambda_k > c_* > 0$ are absolute constants, and $Q_i$ are some polynomials on $p$.

5 An improvement in the general case

The case of an arbitrary finite abelian group $G$ requires more refined arguments and generalizations. For example, we have more general version of inequality (19) (see [7, Corollary 6.2]) which follows by the same argument from the generalization of Pollard Theorem (see, e.g., [7, Proposition 6.1] or [8, Theorem 1]).

Theorem 19. Let $G$ be an abelian group, and $A, B \subseteq G$ be sets. Let $\varepsilon \in (0,1)$ be such that $\sqrt{\varepsilon|G|} < |A|, |B|$. Let $H$ be a maximal proper subgroup of $G$. Then

$$|A + _\varepsilon B| \geq \min(|G|, |A| + |B| - |H|) - 3\varepsilon|G|,$$

Remark 20. We will use this Theorem just to conclude the following observation. Let $W_a, W_b, W_c$ be some subsets of $G$ of size at least $\varepsilon p$. Let us also assume they satisfy $W_a \neq W_b + \delta W_c$ for some $\delta \ll \varepsilon^2$. Then we have $|W_a| \leq |G| - |W_b + \delta W_c|$. Thanks to Theorem 19, the last quantity is at most $|G| + |H| + 3\sqrt{\delta}|G| - |W_b| - |W_c|$ and this implies

$$|W_a| + |W_b| + |W_c| \leq |G| + |H| + 3\sqrt{\delta}|G| \leq (3/2 + O(\varepsilon))|G|.$$

Now we are ready to prove an analogue of Theorem 5 and we will appeal to the proof of this result.

Theorem 21. Let $G$ be a finite abelian group, $N = |G|$. Then the number of sets $A, B, C$ from $G$ such that $x + y \neq z, x \in A, y \in B, z \in C$ equals

$$3 \cdot 4^N + 3N \cdot 3^N + O((3 - c_*)^N),$$

where $c_* > 0$ is an absolute constant.
Proof. As in the proof of Theorem 16 we split the problem into two cases, namely when one of the sets $A, B, C$ is small, and when all the sets have sizes at least $\varepsilon N$ for a certain $\varepsilon \in (0, 1)$. As in the proof of Theorem 16 we see that the first case gives us at most $3 \cdot 4^N + 3N \cdot 3^N + O((3 - c_\ast)N)\) of such triples.

In the second case we repeat the same argumentation as for $G = \mathbb{F}_p$ but we need to change it slightly, since we cannot use the inequality from Theorem 8 in the general case. Replacing usage of Theorem 8 with the more general Theorem 19 and taking into account Remark 20 we obtain for the sets $W_a, W_b, W_c$ that

$$|W_a| + |W_b| + |W_c| \leq (1.5 + O(\varepsilon))N,$$

which changes the bound in the second case from $2N + O(\varepsilon)N$ to $2^{1.5N + O(\varepsilon)N} = (2\sqrt{2} + O(\varepsilon))^N$, and the proof is complete. \hfill \square

Remark 22. Let us take a finite abelian group $G$, $N = |G|$ which has a subgroup $H$ of index 2. Then for any $A, B \subseteq H, C \subseteq G \setminus H$ we have $A + B \neq C$ and $\#\{A, B, C : A + B \neq C\} \gg 2^{|H|2^{|H|}2^{|H|}} = 2^{3N/2} = (2\sqrt{2})^N$. Therefore, one cannot obtain a smaller exponent at least in the second case of the proof.

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