Matched $G^1$-constructions yield $C^1$-continuous isogeometric elements

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Abstract

The note shows how $G^1$ (geometrically continuous surface) constructions can yield $C^1$ iso-geometric elements also at irregular domain points where three or more than four elements come together.

1 Introduction

This note combines two concepts: $G^1$ construction for joining two patches and the notion of iso-geometric element. (Both ‘patches’ and ‘elements’ are pieces of maps, i.e. maps whose domain is restricted). The resulting proof of $C^1$ continuity for iso-geometric elements built from the composition of $G^1$ continuous maps forms the background and detail of the presentations by the author at the Dagstuhl workshop on Geometric Design in May 2014 and at the Conference on Curves and Surfaces in Paris, June 2014, as well as the publication [NKP14].

2 Geometric continuity and Iso-geometric elements

Geometric continuity. Two patches $p_i, p_j : \Box \subseteq \mathbb{R}^2 \to \mathbb{R}^d$ defined on standard domain $\Box$, for example the unit square, join $G^k$ along a common edge $e := p_i(s, 0) = p_j(\rho(s, 0)) = p_j(0, t)$ with change of variables $\rho : \Box \to \mathbb{R}^2$, if for the entries of the $k$-jet $\partial^k$ at every point of $e$

$$
\partial^k p_i(s, 0) = \partial^k p_j(\rho(s, 0)).
$$

That is, each coordinate of the abutting patches $p_i$ and $p_j \circ \rho$ forms a $C^k$ function. $G^k$ constructions can be used to join three or more than four patches smoothly at a point (see e.g. [Pet02]).
Iso-geometry. In the iso-geometric approach to (finite element) computations, splines $x_i, i = 1..n$ parameterize a (physical) domain region $X \equiv \bigcup_{i=1}^{n} x_i(\Box) \subset \mathbb{R}^d$ as well as represent functions $u_i$ sought (for example to approximate the solution of a partial differential equation). In two variables, the physical domain $X$ can be a region of the $xy$-plane or a surface embedded in $\mathbb{R}^3$. The domain of each spline $x_i$ is a standard domain, for example the unit square, 

$$x_i : \Box \to \mathbb{R}^d, \ d \in \{2, 3\},$$

$$(s, t) \mapsto x_i(s, t) =: (x_i(s, t), y_i(s, t)).$$

Similarly $u_i : \Box \to \mathbb{R}$. (Without loss of generality, $u_i$ is scalar-valued, since we can consider one coordinate at a time.) The compositions $u_i \circ x_i^{-1}$ are the iso-geometric elements. The goal of the iso-geometric approach is to compute the functions $u_i$ so that the iso-geometric elements $u_i \circ x_i^{-1}$ solve a problem, e.g. solve a differential equation, on the given physical domain $X$ [HCB05].

3 Smoothness of the composition

We want to show that $G^1$ constructions yield $C^1$ iso-geometric elements.

Lemma 1 ($G^1$ construction yields $C^1$ isogeometric element) If $x_i$ joins $G^1$ with $x_j$ along a common edge $e$ under a reparameterization $\rho$ and $u_i$ joins $G^1$ with $u_j$ under the same reparameterization $\rho$ then the iso-geometric maps $u_i \circ x_i^{-1}$ and $u_j \circ x_j^{-1}$ join $C^1$ in each coordinate.

Proof Since the maps agree along the edge $e$ in the physical domain, it suffices, by linearity of differentiation, to show that the derivatives $\partial_{\perp}$ perpendicular to $e$ agree for every point on $e$. Denote by $e_i^{-1}$ the pre-image of the edge $e$ under $x_i$ and by $e_j^{-1}$ the pre-image of the edge $e$ under $x_j$. By assumption

$$\partial x_i(e_i^{-1}) = \partial (x_j \circ \rho)(e_i^{-1}), \quad \partial u_i(e_i^{-1}) = \partial (u_j \circ \rho)(e_i^{-1}).$$

(2)
and \( \rho : e_i^{-1} \rightarrow e_j^{-1} \). We compute

\[
\partial_{\perp} (u_i \circ x_i^{-1})(e) = \partial u_i(e_i^{-1}) \cdot \partial_{\perp} x_i^{-1}(e) \\
= \partial u_i(e_i^{-1}) \cdot (\partial_{\perp} x_i(e_i^{-1}))^{-1} \\
= \partial (u_j \circ \rho)(e_i^{-1}) \cdot (\partial_{\perp} (x_j \circ \rho)(e_i^{-1}))^{-1} \\
= \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial_{\perp} x_j(e_j^{-1}) \partial \rho(e_i^{-1}))^{-1} \\
= \partial u_j(e_j^{-1}) \partial \rho(e_i^{-1})(\partial \rho(e_i^{-1}))^{-1} (\partial_{\perp} x_j(e_j^{-1}))^{-1} \\
= \partial u_j(e_j^{-1}) \cdot (\partial_{\perp} x_j(e_j^{-1}))^{-1} \\
= \partial_{\perp} (u_j \circ x_j^{-1})(e). \\
\]

Here \( \partial \rho(e_i^{-1}) \) is a \( 2 \times 2 \) matrix (of functions in the parameter \( s \) of \( e_i^{-1} \)).

To argue that \( G^k \) constructions yield \( C^k \) isogeometric elements we observe that the union of maps \( x_i \) and \( x_j \circ \rho \), hence the union of \( x_i^{-1} \) and \( (x_j \circ \rho)^{-1} \) form a \( C^k \) map; and so does the union of \( u_i \) and \( u_j \circ \rho \). Since the combination of chain and product rules (Faa di Bruno’s law) only involves derivatives up to \( k \)th order evaluated along \( e_i^{-1} \).

\[
\partial^k (u_i \circ x_i^{-1})(e) = \partial^k (u_j \circ \rho \circ (x_j \circ \rho)^{-1})(e) = \partial^k (u_j \circ \rho \circ \rho^{-1} \circ (x_j)^{-1})(e) \\
= \partial^k (u_j \circ x_j^{-1})(e). \\
\]

References

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