SET-THEORETIC TYPE SOLUTIONS OF THE BRAID EQUATION

JORGE A. GUCCIONE, JUAN J. GUCCIONE, AND CHRISTIAN VALQUI

Abstract. In this paper we begin the study of set-theoretic type solution of the braid equation. Our theory includes set-theoretical solutions as basic examples. More precisely, the linear solution associated to a set-theoretic solution on a set $X$ can be regarded as coming from the coalgebra $kX$, where $k$ is a field and the elements of $X$ are group-like. We introduce and study a broader class of linear solutions associated in a similar way to more general coalgebras. We show that the relationships between set-theoretical solutions, $q$-cycle sets, $q$-braces, skew-braces, matched pairs of groups and invertible $1$-cocycles remain valid in our setting.

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Introduction

The Yang–Baxter equation first appeared in theoretical physics in 1967 in the paper [27] by Yang, and then in 1972 in the paper [2] by Baxter. Since then, many solutions of various forms of the Yang–Baxter equation have been constructed by physicists and mathematicians. In the past three decades this equation has been widely studied from different perspectives and attracted the attention of a wide range of mathematicians because of the applications to mathematical physics, non-commutative descent theory, knot theory, representations of braid groups, Hopf algebras and quantum groups, etc.

Let $V$ be a vector space over a field $k$ and let $r: V \otimes V \to V \otimes V$ be a linear map. We say that $r$ satisfies the Yang-Baxter equation on $V$ if

$$r_{12} r_{13} r_{23} = r_{23} r_{13} r_{12} \quad \text{in } \text{End}_k(V \otimes V \otimes V),$$

where $r_{ij}$ means $r$ acting in the $i$-th and $j$-th components. It is easy to check that this occurs if and only if $s := \tau r$, where $\tau$ denotes the flip, satisfies the braid equation $s_{12} s_{23} s_{12} = s_{23} s_{12} s_{23}$. In [8], Drinfeld raised the question of finding set-theoretical (or combinatorial) solutions; i.e. pairs $(Y, s)$, where $Y$ is a set and $s: Y \times Y \to Y \times Y$ is a map satisfying this equation. This approach was first considered by Etingof, Schedler and Soloviev [9] and Gateva-Ivanova and Van den Bergh [15], for involutive solutions, and by Lu, Yan and Zhu [21], and Soloviev [26], for non-involutive solutions. Now it is known that there

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are connections between solutions and affine torsors, Artin-Schelter regular rings, Bieberbach groups and groups of I-type, Garside structures, Hopf-Galois theory, left symmetric algebras, etc. ([1,4–7,11–15,19])

Let $Y$ be a set and let $kY$ be the free vector space with basis $Y$ endowed with the unique coalgebra structure such that $\Delta(y) = y \otimes y$ for all $y \in Y$. Each set-theoretical solution $s : Y \times Y \to Y \times Y$ of the braid equation induces a solution $\tilde{s} : kY \otimes kY \to kY \otimes kY$, where $\tilde{s}$ is the coalgebra morphism obtained by linearization of $s$. In this paper we begin the study of set-type solutions of the braid equation in the category of coalgebras. The underlying idea is to replace $kY$ by an arbitrary coalgebra $X$. Our main aim is to show that the relationships between set-theoretic solutions, $q$-cycle sets, $q$-braces, skew-braces, matched pairs of groups and invertible 1-cocycles ([3,9,16,21,24]) remain valid in our setting. Many of the results in this paper were obtained previously in [1,17] for cocommutative coalgebras. In order to remove the cocommutativity hypothesis, first we consider left non-degenerate coalgebra endomorphisms of $X \otimes X$ and $q$-magma coalgebras on $X$, and we prove that there is a one to one correspondence between these structures. By restriction this induces a one to one correspondence between left non-degenerate solutions of the braid equation and $q$-cycle coalgebras (a modification to the context of Yetter-Drinfeld braces, recently introduced in [25]).

A particular type of Hopf $q$-braces are the Hopf skew-braces. We prove that having a Hopf skew-brace with bijective antipode is the same as having a linear $q$-cycle coalgebra (an adaptation to the context of coalgebras of the concept of linear $q$-cycle set due to Rump). Recall that a braiding operator on a group $G$, as defined in [21], is a bijective map $s : G \times G \to G \times G$, satisfying several conditions. If we relax the definition, not demanding that identity [7] in [23] be satisfied, then we arrive at the definition of weak braiding operator (which we extend to the setting of Hopf algebras). Let $H$ be a Hopf algebra. One of our first results about Hopf $q$-braces is that Hopf $q$-braces on $H$ and weak braiding operators on $H$ are essentially the same. Moreover, we adapt [21, Theorem 1] to the setting of Hopf algebra and we obtain several results about the relation between the binary operations in a Hopf $q$-brace and the antipode, and we generalize some results of [25, Section 3].

A particular type of Hopf $q$-braces are the Hopf skew-braces. We prove that having a Hopf skew-brace with bijective antipode is the same as having a linear $q$-cycle coalgebra (an adaptation to the context of coalgebras of the concept of linear $q$-cycle set due to Rump), and is also the same as having a GV-Hopf skew-brace (an adaptation to the context of Hopf algebras of the original definition of skew-brace due to Guarnieri and Vendramin). Furthermore, we prove that Hopf skew-braces are equivalent to braiding operators and to invertible 1-cocycles (if the antipode of the underlying Hopf algebra is bijective), which generalizes [21, Theorem 2]. Moreover, we prove that the category of Hopf skew braces is isomorphic to the category of Yetter-Drinfeld braces, recently introduced in [10].

An ideal of a $q$-brace $H$ is a Hopf ideal $I$ of $H$ (the underlying Hopf algebra of $H$) such that the quotient $H/I$ has a structure of $q$-brace induced by the one on $H$. An example is the $q$-commutator $[H,H]$, which is the smallest ideal $I$ of $H$ such that the quotient $H/I$ is a skew-brace. Additionally to this notion, we also study the closely related concept of Hopf sub $q$-brace. An example that we analyze in detail is the socle.

Finally, we construct the universal Hopf $q$-brace with bijective antipode and the universal Hopf skew-brace with bijective antipode of a very strongly regular $q$-cycle coalgebra (Definition 3.6). This generalizes [21, Theorem 4].

### Precedence of operations

The operations precedence in this paper is the following: the operators with the highest precedence are the binary operations $\cdot$, $\ast$, $\circ$, $\circlearrowright$, $\circlearrowleft$, $\odot$, $\circlearrowright$, $\odot$, $\times$, $\times^n$, $\times^n$, $\odot^n$ and $\leftrightarrow$, that have equal precedence. Of course, as usual, this order of precedence can be modified by the use of parenthesis.

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## 1 Preliminaries

In this paper we work in the category of vector spaces over a fixed field $k$, all the maps are supposed to be $k$-linear maps, we set $\otimes := \otimes_k$ and the tensor product $V \otimes \cdots \otimes V$, of $n$ copies of a vector space $V$ is denoted by $V^n$. As usual, $\Delta$ and $\epsilon$ denote the comultiplication and counit of a coalgebra $X$, respectively. Moreover, we use the Sweedler notation $x_{(1)} \otimes x_{(2)}$ without the summation symbol, for the comultiplication of $x \in X$. As it is also usual, $X^{\text{cop}}$ stands for the opposite coalgebra of a coalgebra $X$, we let $S$ denote the antipode of a Hopf algebra $H$, and $H^{\text{cop}}$ stands for the opposite Hopf algebra of $H$.

Let $X,Y_1,\ldots,Y_n$ be coalgebras. For each map $f : X \to Y_1 \otimes \cdots \otimes Y_n$ and for each $1 \leq i \leq n$, we set $f_i := (\epsilon^{i-1} \otimes Y_i \otimes \epsilon^{n-i})f$. Moreover for each map $s : X \otimes Y \to Y \otimes X$ we let $\tilde{s}$ denote $s$ regarded as a map from $X^{\text{cop}} \otimes Y^{\text{cop}}$ to $Y^{\text{cop}} \otimes X^{\text{cop}}$, and we set $s_\tau := \tau s$, where $\tau : Y \otimes X \to X \otimes Y$ denote the flip. Note that $(s_\tau)_1 = s_2 \tau$ and $(s_\tau)_2 = s_1 \tau$. 


Remark 1.1. We will need the following facts.

1) \( \tilde{s} \) is a coalgebra morphism if and only if \( s \) is.

2) For each coalgebra morphism \( f: X \to Y \otimes \cdots \otimes Y_n \), the maps \( f_i \) are the unique coalgebra morphisms such that \( f = (f_1 \otimes \cdots \otimes f_n) \Delta_n \), where \( \Delta_n(x) = x_{(1)} \otimes \cdots \otimes x_{(n)}. \)

3) A map \( s: X \otimes Y \to Y \otimes X \) is a coalgebra morphism if and only if \( s_1 \) and \( s_2 \) are coalgebra morphisms, \( s = (s_1 \otimes s_2) \Delta_X \otimes \Delta_Y \), and, for all \( x \in X \) and \( y \in Y \),

\[
    s_1(x_{(1)} \otimes y_{(1)}) \otimes s_2(x_{(2)} \otimes y_{(2)}) = s_1(x_{(2)} \otimes y_{(2)}) \otimes s_2(x_{(1)} \otimes y_{(1)}).
\]

[1.1]

Proposition 1.2. Let \( X \) be a coalgebra and let \( \alpha: X^2 \to X \) and \( \beta: X \otimes X^{\text{cop}} \to X \) be two maps. Set \( x^y := \alpha(x \otimes y) \) and \( x : y := \beta(x \otimes y) \).

\[
    (x \cdot y_{(1)})^{y(2)} = x^{y(1)} \cdot y_{(2)} = \epsilon(y)x \quad \text{for all } x, y \in X,
\]

then \( \alpha \) is a coalgebra map if and only if \( \beta \) is.

Proof. We prove one of the implications and leave the other one to the reader. Assume that \( \alpha \) is a coalgebra map. By equality \([1.2]\), we have \( \epsilon(x) \epsilon(y) = \epsilon((x \cdot y_{(1)})^{y_{(2)})} = \epsilon(x \cdot y) \). So, \( \beta \) is counitary. Using again equality \([1.2]\), we get

\[
    \epsilon(y)x_{(1)} \otimes x_{(2)} = x_{(1)}^{y(1)} \cdot y_{(4)} \otimes x_{(2)}^{y(2)} \cdot y_{(3)} = (x^{y(1)})_{(1)} \cdot y_{(3)} \otimes (x^{y(1)})_{(2)} \cdot y_{(2)}.
\]

Since \( (x \cdot y_{(1)}) \otimes (x \cdot y_{(2)}) = \epsilon(y_{(2)})((x \cdot y_{(1)})_{(1)}) \otimes (x \cdot y_{(1)})_{(2)} \), we have

\[
    (x \cdot y_{(1)}) \otimes (x \cdot y_{(2)}) = ((x \cdot y_{(1)})^{y(1)}_{(1)}) \cdot y_{(4)} \otimes ((x \cdot y_{(1)})^{y(2)}_{(2)}) \cdot y_{(3)} = x_{(1)} \cdot y_{(2)} \otimes x_{(2)} \cdot y_{(1)},
\]

which proves that \( \beta \) is compatible with the comultiplications. \( \square \)

Remark 1.3. Note that the equalities \([1.2]\) hold if and only if the map \( x \otimes y \mapsto x \cdot y_{(1)} \otimes y_{(2)} \) is bijective with inverse \( x \otimes y \mapsto x^{y(1)} \otimes y_{(2)} \). Hence for each operation \( \cdot \), there exists at most one map \( x \otimes y \mapsto x^y \) satisfying \([1.2]\).

1.1 Left non-degenerate coalgebra endomorphisms of \( X^2 \)

Let \( X \) be a coalgebra. For each map \( s: X^2 \to X^2 \) we set \( x^y := s_1(x \otimes y) \) and \( x^y := s_2(x \otimes y) \). Also, let \( G_s \in \text{End}_k(X^2) \) denote the map defined by \( G_s(x \otimes y) := x^{y(1)} \otimes y_{(2)}. \)

Definition 1.4. A coalgebra morphism \( s: X^2 \to X^2 \) is left non-degenerate if \( G_s \) is invertible; and it is right non-degenerate if \( \tilde{s} \) is left non-degenerate. If \( s \) is left and right non-degenerate, then we say that \( s \) is non-degenerate.

Remark 1.5. Note that \( \tilde{s}_r(x \otimes y) = y_{(2)} x^{y(2)} \otimes y_{(1)} \), and so \( G_{\tilde{s}_r}(x \otimes y) = y_{(2)} x \otimes y_{(1)}. \)

Remark 1.6. A coalgebra morphism \( s: X^2 \to X^2 \) is non-degenerate if and only if \( \tilde{s}_r \) is non-degenerate.

Remark 1.7. Our notions of left non-degenerate, right non-degenerate and non-degenerate coalgebra morphism generalize the corresponding notions in the set-theoretic framework (see \([9, 23, 25]\)). In fact, if \( X = kY \) and \( s \) extends linearly a map \( s': Y^2 \to Y^2 \), then \( G_s \) is defined, on the canonical basis, by \( G_s(x \otimes y) = x^y \otimes y \), which is bijective if and only if the map \( x \mapsto x^y \), from \( Y \) to \( Y \), is bijective for all \( y \in Y \) (which means that \( s' \) is left non-degenerate according to \([25\text{, page 143}\])). Similarly, \( G_{\tilde{s}_r} \) is given by \( G_{\tilde{s}_r}(x \otimes y) = y \otimes x \), for \( x, y \in Y \). So, \( s \) is right non-degenerate if and only if the map \( x \mapsto y_x \) is bijective for all \( y \in Y \) (see \([25\text{, Definition 4}\])).

Notation 1.8. For a left non-degenerate coalgebra morphism \( s: X^2 \to X^2 \), we set \( \mathcal{X} := (X, \cdot, :) \), where \( x \cdot y := (X \otimes e) G_s^{-1}(x \otimes y) \) and \( x : y := y \cdot x(1)_x(2) \). If necessary, we will write \( x \cdot y, x : y \) and \( \mathcal{X} \) instead of \( x \cdot y, x : y \) and \( \mathcal{X} \), respectively. Arguing as in Remark 1.7 we see that these operations are generalizations of the notions of \([25\text{, Page 143}]\). However, we change the order of the elements: If we use \( \cdot \) and \( : \) to represent the \( q \)-magma operations defined in \([25]\), we have \( x \cdot y = y \cdot x \) and \( x : y = y \cdot x \). Taking this transposition into account, most of our results, when considered within the set-theoretic context, align with the results presented in \([25]\).

Next we explain our choice of laterality. Since the mapping \( x \mapsto x^y \) is a “right action” (which, in section 5 is a genuine right action), we will denote its “inverse action” as \( x \mapsto x \cdot y \). The advantage of our choice of laterality can be seen for example in cases like identities \([5, 7]\), where the actions of \( \cdot \) and \( : \) on a product are more naturally represented as right actions.
Remark 1.9. Let $s \in \text{End}_{\text{Coalg}}(X^2)$. If $s$ is left non-degenerate, then $G^s_{-1}$ is right colinear since $G_x$ is, which implies that $G^s_{-1}(x \otimes y) = x \cdot y^{G^s_{-1}(x \otimes y)}$ for all $x, y \in X$. Using this, and applying $X \otimes \epsilon$ to the identities $G_s G^s_{-1} = G^s_{-1} G_s = \text{id}_X$, we obtain that identities [1.2] are fulfilled. Conversely, if there exists a map $x \otimes y \mapsto x \cdot y$ satisfying identities [1.2], then $s$ is left non-degenerate. Note that identities [1.2] imply that $y^{(2)} : x^{y^{(1)}} = \epsilon(y) x$, for all $x, y \in X$.

Notation 1.10. For a right non-degenerate coalgebra map $s : X^2 \to X^2$, we set $x \cdot y := (X \otimes \epsilon) G^s_{-1}(x \otimes y)$ and $x \cdot y := x^{(1)} y^{x^{(2)}}$.

Note that $s$ is related with $\overline{s}$, and $*$ in the same way as $s$ is related with $s$ and $\cdot$. That is, $X_\epsilon = (X_{\text{cop}}, *; \cdot)$.

If necessary, we will write $x \cdot_s y$ and $x \cdot_s y$ instead of $x \cdot y$ and $x \cdot y$, respectively. Notice that in the set-theoretic context the operation $*$ corresponds to the one introduced in [25, Definition 2].

Remark 1.11. Let $s \in \text{End}_{\text{Coalg}}(X^2)$. If $s$ is right non-degenerate, then

$$y^{(1)}(x \cdot_s y^{(2)}) = y^{(2)} x \cdot_s y^{(1)} = \epsilon(y) x$$

for all $x, y \in X$. Conversely, if there exists a map $x \otimes y \mapsto x \cdot y$ satisfying identities [1.3], then $s$ is right non-degenerate. Clearly identities [1.3] imply that $y^{(1)} : x^{y^{(2)}} = \epsilon(y) x$, for all $x, y \in X$.

1.2 Left regular q-magma coalgebras

Let $X$ be a coalgebra and let $p, d : X^2 \to X$ be maps such that $\epsilon p = \epsilon d = \epsilon \otimes \epsilon$. For each $x, y \in X$, we set $x \cdot y := p(x \otimes y)$ and $x \cdot y := d(x \otimes y)$. Let $\overline{G}_X \subseteq \text{End}_k(X^2)$ be the map defined by $\overline{G}_X(x \otimes y) := x \cdot y^{(1)} \otimes y^{(2)}$.

Definition 1.12. We say that $\mathcal{X} = (X, \cdot, :)$ is a q-magma coalgebra if the map $h : X \otimes X_{\text{cop}} \to X_{\text{cop}} \otimes X$, defined by $h(x \otimes y) := y^{(1)} = x^{(2)} \otimes x^{(1)} \otimes y^{(2)}$, is a coalgebra morphism. If necessary, we will write $h_X$ instead of $h$. Let $Y = (Y, \cdot, :)$ be another q-magma coalgebra. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is a coalgebra map $f : X \to Y$ such that $f(x \cdot x') = f(x) \cdot f(x')$ and $f(x \cdot y) = f(x) : f(y)$, for all $x, x', y \in X$.

Remark 1.13. By Remark 1.1 and the fact that $\epsilon p = \epsilon d = \epsilon \otimes \epsilon$, we know that $X$ is a q-magma coalgebra if and only if $p, d : X \otimes X_{\text{cop}} \to X$ are coalgebra maps and

$$y^{(1)} = (x \otimes y) \cdot x^{(2)} \otimes x^{(1)} \cdot y^{(2)} = (x \otimes y) \cdot (x^{(1)} \otimes x^{(2)} \cdot y^{(1)})$$

for all $x, y \in X$. Conversely, if only if $p, d : X \otimes X_{\text{cop}} \to X$ are coalgebra maps and

$$y^{(1)} = (x \otimes y) \cdot x^{(2)} \otimes x^{(1)} \cdot y^{(2)} = (x \otimes y) \cdot (x^{(1)} \otimes x^{(2)} \cdot y^{(1)})$$

for all $x, y \in X$. Thus, these notions generalize the set-theoretic ones (see [25, Definition 1]).

Remark 1.14. If $\mathcal{X} = (X, \cdot, :)$ is a q-magma coalgebra, then $X_{\text{cop}} := (X_{\text{cop}}, \cdot, :)$ is also. We call $X_{\text{cop}}$ the opposite q-magma coalgebra of $X$. Equality [1.4] says that $h_{X_{\text{cop}}} = \tau h_X \tau$. Note that $\overline{G}_{X_{\text{op}}} = (x \otimes y) := x \cdot y^{(2)} \otimes y^{(1)}$.

Definition 1.15. A q-magma coalgebra $\mathcal{X}$ is left regular if the map $\overline{G}_X$ is invertible. If $X_{\text{cop}}$ is left regular, then we call $\mathcal{X}$ right regular. Finally, we say that $X$ is regular if it is left regular and right regular.

Remark 1.16. Note that $\mathcal{X}$ is regular if and only if $\text{opsee} \subseteq \text{opsee}$ is regular.

Remark 1.17. Let $(Y, \cdot, :)$ be a q-magma (see [25, Definition 1]), and let $\mathcal{X} = (kY, \cdot, :)$ be the linearization of $(Y, \cdot, :)$. As in Remark 1.17, we see that $\mathcal{X}$ is left regular if and only if the map $x \mapsto x \cdot y$, from $Y$ to $Y$, is bijective for all $y \in Y$. Similarly, $\mathcal{X}$ is right regular if and only if the map $x \mapsto x : y$, from $Y$ to $Y$, is bijective for all $y \in Y$. Thus, these notions generalize the set-theoretic ones (see [25, Definition 1]).

Remark 1.18. For a left regular q-magma coalgebra $\mathcal{X} = (X, \cdot, :)$, we set $G_X := \overline{G}_X^{-1}$ and we define the map $s : X^2 \to X^2$ by $s(x \otimes y) := s(x)(y) \otimes x^{(1)} \cdot y^{(1)}$, where $x^{(1)} := (X \otimes \epsilon)G_X(x \otimes y)$ and $y^{(1)} := y^{(2)} \cdot x^{(1)}$ (if necessary we will write $s_x$ instead of $s$). Arguing as in Remark 1.9, we can see that $G_X(x \otimes y) = x^{(1)} \otimes y^{(2)}$, for all $x, y \in X$, and identities [1.2] are fulfilled. Conversely, if there is a map $x \otimes y \mapsto x^{(1)}$ satisfying identities [1.2], then $\mathcal{X}$ is left regular. Note that by [1.2] and Proposition 1.2,

$$x^{y^{(1)}}(y^{(2)} = x \cdot \epsilon(x) \epsilon(y)\epsilon(x^{y^{(1)}}) \quad \text{and} \quad \epsilon(x^{y^{(1)}}) = \epsilon(y^{(2)}) \epsilon(x^{y^{(1)}}) = \epsilon(x^{(2)}) \epsilon(x^{y^{(1)}}) = \epsilon(x) \epsilon(y),$$

for all $x, y \in X$. Hence, $s_{x^{(1)}}(x \otimes y) = x^{(1)}$ and $s_{x^{(1)}}(x \otimes y) = y$, for all $x, y \in X$.

Remark 1.19. Let $\mathcal{X}$ be a left regular q-magma coalgebra. A direct computation, using identities [1.2] and that $p : X \otimes X_{\text{cop}} \to X$, is a coalgebra map (by Remark 1.13), shows that $s \overline{G}_X \tau = \overline{G}_{X_{\text{cop}}}$. Thus $s$ is invertible if and only if $\mathcal{X}$ is regular.

Remark 1.20. For a right regular q-magma coalgebra $\mathcal{X} = (X, \cdot, :)$, we define $s : (X_{\text{cop}})^2 \to (X_{\text{cop}})^2$, by $s(x \otimes y) := s(x)(y) \otimes x^{(1)} \cdot y^{(1)}$, where $x^{(1)} := (X \otimes \epsilon)G_{X_{\text{cop}}}(x \otimes y)$ and $y^{(1)} := y^{(2)} \cdot x^{(1)}$. Note that $s = s_{X_{\text{cop}}}$. So, by Remark 1.18, the equality $G_{X_{\text{cop}}}(x \otimes y) = s_{y^{(2)}}(y^{(1)})$ holds, for all $x, y \in X$, and

$$(x : y^{(2)})(y^{(1)}) = s_{x^{(2)}}(y^{(1)}) = \epsilon(y)x$$

for all $x, y \in X$. Conversely, if there exists a map $x \otimes y \mapsto x_{y^{(2)}}(y^{(1)})$ satisfying identities [1.5], then $\mathcal{X}$ is right regular.
Remark 1.21. Let \((Y, \cdot, \cdot)\) be a \(q\)-magma and let \(\mathcal{X} = (kY, \cdot, \cdot)\) be the linearization of \((Y, \cdot, \cdot)\). Assume that \(\mathcal{X}\) is right regular. Arguing as in Remark 1.7, one can see that the restriction to \(Y \times Y\) of the map \(x \otimes y \mapsto xy\) is the map introduced in [25, Definition 1]. Similarly, the restriction to \(Y \times Y\) of \(x \otimes y \mapsto xy\) is the map introduced in Remark 1 of [25, Definition 4].

Definition 1.22. A \(q\)-magma coalgebra \(\mathcal{X} = (X, \cdot, \cdot)\) is right non-degenerate if it is left regular and the map \(H_X \in \text{End}_k(X^2)\), defined by \(H_X(x \otimes y) := y(2) x \otimes y(1)\), is invertible; while it is left non-degenerate if \(\mathcal{X}_{\text{op}}\) is right non-degenerate (that is, if \(\mathcal{X}\) is right regular and the map \(x \otimes y \mapsto y(1), x \otimes y(2)\) is bijective). A \(q\)-magma coalgebra is non-degenerate if it is left and right non-degenerate.

Remark 1.23. In the set-theoretic context, the notion of non-degenerate reduces to that introduced in [25, Definition 1]. In fact, in our context a \(q\)-magma coalgebra is non-degenerate, if and only if the maps

\[ x \otimes y \mapsto x \cdot y(1) \otimes y(2) \quad x \otimes y \mapsto y(2) x \otimes y(1) \]

are bijective. Since, according to [25, Definition 1], in the set-theoretic context a \(q\)-magma is non-degenerate if and only if the maps \(x \mapsto x \cdot y\), \(x \mapsto x : y\) and \(x \mapsto x \cdot y \cdot y\) are bijective, our definition generalizes the set-theoretic one.

2 Coalgebra endomorphisms vs \(q\)-magma coalgebras

Let \(X\) be a coalgebra. In this section we establish a one-to-one correspondence between the left non-degenerate coalgebra endomorphisms of \(X^2\) and the left regular \(q\)-magma coalgebras with underlying coalgebra \(X\). From here until Proposition 2.5 inclusive, we assume that \(s\) is a left non-degenerate coalgebra endomorphism of \(X^2\) and \(\cdot\) and \(:\) are as in Notation 1.8. Note that, by Remark 1.9, equalities [1.1] and [1.2] hold. Moreover, by Proposition 1.2, the map \(p: X \otimes X_{\text{op}} \rightarrow X\), defined by \(p(x \otimes y) := x \cdot y\), is a coalgebra map.

Remark 2.1. By the fact that \(p\) is a coalgebra morphism and equalities [1.1] and [1.2],

\[ s(x \cdot y(1) \otimes y(2)) = \tau^{(2)}(x(1)) y(4) \otimes (x(1) \cdot y(2))(y(3)) = y : x(2) \otimes x(1) \quad \text{for all } x, y \in X. \]

Remark 2.2. By Remark 1.9 and equality [1.1], for each \(x, y \in X\), we have:

\[ y(3) : x(2)^{y(2)} \otimes y(4) : x(1)^{y(1)} = \tau^{(2)}(x(2)) \otimes y(3) : x(1)^{y(1)} = \tau^{(1)}(y(1)) \otimes y(3) : x(2)^{y(2)} = \tau^{(1)}(y(1)) \otimes \tau^{(2)}(y(2)). \]

Remark 2.3. Since \(\epsilon(x \cdot y) = \epsilon(x) \epsilon(y)\), we have \(\epsilon(x \cdot y) = \epsilon(y^x \cdot x(2)) = \epsilon(y \cdot x(1)) \epsilon(x(2)) = \epsilon(x) \epsilon(y)\).

Lemma 2.4. The map \(d: X \otimes X_{\text{op}} \rightarrow X\), defined by \(d(x \otimes y) := x : y\), is a coalgebra morphism.

Proof. By the previous remark we know that \(d\) is compatible with \(\epsilon\). Let \(\tilde{d}: X_{\text{op}} \otimes X \rightarrow X\) be the map given by \(\tilde{d}(x \otimes y) := y : x\). Since \(d = d\tau\), in order to finish the proof it suffices to show that \(\tilde{d}\) is compatible with the comultiplications. Since \(G_s\) is bijective, in order to check this, it suffices to prove that

\[ \left(x^{(2)} x^{(1)}\right)(1) \otimes \left(x^{(2)} x^{(1)}\right)(2) = y(3) : x(2)^{y(2)} \otimes y(4) : x(1)^{y(1)} \quad \text{for all } x, y \in X. \]

But, since \(\tau^{(1)}(y(1)) = \tau^{(2)}(x(2)) \otimes \tau^{(3)}(x(2))\), from Remarks 1.9 and 2.2, it follows that

\[ y(3) : x(2)^{y(2)} \otimes y(4) : x(1)^{y(1)} = \tau^{(2)} y(1) \otimes \tau^{(3)} y(1), \]

as desired. \(\square\)

Proposition 2.5. The triple \(\mathcal{X} = (X, \cdot, \cdot)\) is a left regular \(q\)-magma coalgebra.

Proof. Since \(\mathcal{T}G_s = G_s^{-1}\) (by Remark 1.9 and the definition of \(\mathcal{T}G_s\) above Definition 1.12) we know that \(\mathcal{T}G_s\) is invertible. So, by Remark 1.13, the fact that \(p\) is a coalgebra morphism and Lemma 2.4, we only must prove that

\[ y(1) : x(2) \otimes x(1) \cdot y(2) = y(2) : x(1) \otimes x(2) : y(1) \quad \text{for all } x, y \in X. \]

But, by Remark 2.1, we have

\[ y : x(2) \otimes x(1) = s(x \cdot y(1) \otimes y(2)) = \tau^{(1)}(y(2)) y(3) \otimes (x(2) \cdot y(1))(y(4)) = y(2) : x(1) \otimes (x(2) : y(1))(y(4)), \]

which implies

\[ y(2) : x(1) \otimes x(2) : y(1) = y(2) : x(1) \otimes (x(2) : y(1))(y(4)) = y(1) : x(2) \otimes x(1) : y(2), \]

as desired. \(\square\)
By Proposition 2.5, each left non-degenerate coalgebra endomorphism $s$ of $X^2$ has associated a left regular $q$-magma coalgebra. Our next aim is to prove the converse. From here to Proposition 2.7 inclusive, we assume that $\mathcal{X} = (X, \cdot, ;)$ is a left regular $q$-magma coalgebra and that $s$ is as in Remark 1.18. Note that, by Remark 1.18, equalities [1.2] hold.

**Lemma 2.6.** The maps $s_1 : X^2 \to X$ and $s_2 : X^2 \to X$ are coalgebra morphisms.

**Proof.** By Proposition 1.2 and Remark 1.13, we know that $s_2$ is a coalgebra map. Next we prove that $s_1$ is also. By Remark 1.18, we know that $s_1$ is compatible with the counits. Since $G_X$ is bijective, in order to check that $s_1$ is also compatible with the comultiplications, it suffices to prove that

$$\left( x(y_1) y_2 \right)_{(1)} \otimes \left( x(y_1) y_2 \right)_{(2)} = x(y_2) y_1 \otimes x(y_1) y_4$$

for all $x, y \in X$.

But, by equality [1.4] and Remark 1.18, we have

$$\left( x(y_1) y_2 \right)_{(1)} \otimes \left( x(y_1) y_2 \right)_{(2)} = y : x_1 \otimes y_2 : x_1 = y : x(1) \otimes x(2) y_1 y_2 = x : x(1) \otimes x(2) y_1 y_2,$$

as desired. \qed

**Proposition 2.7.** The map $s$ is a left non-degenerate coalgebra endomorphism of $X^2$.

**Proof.** By Remark 1.18 and the definition of $G_s$ above Definition 1.4, we know that $G_s = G_X^{-1}$ is invertible. So, we only must prove that $s$ is a coalgebra morphism. By Remark 1.1 and Lemma 2.6 in order to check this it suffices to prove that $x(1) y \otimes x(2) y_2 = x(2) y_1 \otimes x(1) y_4$, since this implies $s = (s_1 \otimes s_2) A_X^2$. But,

$$x(1) y \otimes x(2) = y_4 : x(1) y_1 \otimes x(2) y_2 = y_3 : x(2) y_2 \otimes x(1) y_1 \otimes y_4 = x(2) y_2 \otimes x(1) y_1 \otimes y_3,$$

and so $x(1) y \otimes x(2) y_2 = x(2) y_1 \otimes \left( x(1) y_1 \cdot y_3 \right) y_4 = x(2) y_1 \otimes x(1) y_1$, as desired. \qed

**Remark 2.8.** A direct computation shows that the correspondences $s \mapsto X_s$ and $X \mapsto s_X$ are inverse one of each other. So, we have a one-to-one correspondence between left non-degenerate coalgebra endomorphisms of $X^2$ and left regular $q$-magma coalgebras with underlying coalgebra $X$.

**Proposition 2.9.** Let $s : X^2 \to X^2$ be a left non-degenerate coalgebra morphism. The following assertions are equivalent:

1) $s$ is non-degenerate.
2) $\tilde{s}_* = \text{left non-degenerate}.$
3) $s_\tau$ is non-degenerate.
4) $X_s$ is right non-degenerate.
5) $X_{\tilde{s}_*}$ is right non-degenerate.
6) $X_{\tilde{s}}$ is left regular.

**Proof.** Since $\left( \tilde{s}_* \right)_\tau = s$, from Definition 1.4, it follows that items 1), 2) and 3) are equivalent; while, by Definition 1.1, items 2) and 4) are equivalent, since $X_s$ is left regular (by Remark 2.8) and $H_{X_s} = G_{\tilde{s}_*}$ (by Remark 1.5). Clearly item 5) implies item 6); while item 6) implies item 2), by Remark 2.8. Finally, using that item 1) implies item 4), applied to $s_\tau$, we obtain that item 5) follows from item 3). \qed

**Remark 2.10.** Assume that $s : X^2 \to X^2$ is a right non-degenerate coalgebra map and let $\ast$ and $: = \ast$ be as in Notation 1.10. Applying Remark 2.1 and Proposition 2.5 to $\tilde{s}_\tau$, we get that $\tilde{s}_\tau = (X^{\text{Cop}}, \ast, ;)$ is a left regular $q$-magma coalgebra and $s(x_1 \otimes y \ast x_2) = s \tilde{s}_\tau \tau(x_1 \otimes y \ast x_2) = y_2 y \otimes x_1 y_1$, for all $x, y \in X$.

**Proposition 2.11.** A map $s \in \text{Aut}_{\text{Coalg}}(X^2)$ is left non-degenerate if and only if $\tilde{s}^{-1}$ is. Moreover, in this case, $\overline{G_{X^{\text{Cop}}}}$ is invertible, $G_{s^{-1}} = \overline{G_{X^{\text{Cop}}}}$, and $X_{s^{-1}} = X^{\text{op}}$.

**Proof.** By symmetry it suffices to prove that if $s$ is left non-degenerate, then $\tilde{s}^{-1}$ is also, and that, in this case $\overline{G_{X^{\text{Cop}}}}$ is invertible, $G_{s^{-1}} = \overline{G_{X^{\text{Cop}}}}$, and $X_{s^{-1}} = X^{\text{op}}$. In order to prove that $\tilde{s}^{-1}$ is left non-degenerate we must check that $G_{s^{-1}}$ is invertible. Since $\overline{G_{X^{\text{Cop}}}}$ is invertible (by Remarks 1.19 and 2.8), for this is suffices to check that $G_{s^{-1}} = \overline{G_{X^{\text{Cop}}}}$ is id. But,

$$G_{\tilde{s}} G_{X^{\text{Cop}}}(y \otimes x) = G_{\tilde{s}}(y : x_2 \otimes x_1) = (\tilde{s}^{-1})_2(y : x_3 \otimes x_2) \otimes x_1 = y \otimes x,$$
because \((\bar{s}^{-1})_2(y : x(2) \otimes x(1)) = \epsilon(x)y\), by Remark 2.1. It remains to prove that \(X_{s^{-1}} = X_s^{opp}\). But, by Notation 1.8 and the definition of \(\overline{\Gamma}_{X_s^{opp}}\), we have

\[
x \cdot y = (X \otimes \epsilon)G_{s^{-1}}(x \otimes y) = (X \otimes \epsilon)\overline{\Gamma}_{X_s^{opp}}(x \otimes y) = x : y,
\]

which implies that \(\overline{s}^{-1} = \overline{\epsilon}\), because

\[
x \cdot (y_1 \otimes y_2) = \overline{s}^{-1}(y : x(2) \otimes x(1)) = \overline{s}^{-1}(y \cdot x^{-1} : x(2) \otimes x(1)) = x : \overline{s}^{-1}y_1 \otimes y_2,
\]

where the first equality follows from Remark 2.1; and the last one, from Remark 2.1 applied to \(\overline{s}^{-1}\). □

Remark 2.12. By Remark 2.8 and Proposition 2.11, if \(s\) is a left non-degenerate coalgebra automorphism, then \(X_s^{opp} = X_{s^{-1}}\) is left regular, and so \(X_s\) is right regular. Hence, by Remark 1.20, we have \(\overline{s}^{-1} = \overline{s}\).

Corollary 2.13. If \(s\) and \(\overline{s}\) are left non-degenerate coalgebra automorphisms, then \(s\) is involutive if and only if \(\overline{s} = \overline{s}\) and \(\overline{s} = \overline{s}\), as desired. □

Corollary 2.14. If \(X\) is cocommutative, then a left non-degenerate coalgebra automorphism \(s : X \to X\) is involutive if and only if, in its associated \(q\)-magma coalgebra, \(\cdot = :\).

Proof. By Corollary 2.13, since \(s = \overline{s}\). □

Example 2.15. The previous result is not true, if \(X\) is non-cocommutative. Take for example \(X\) the coalgebra with basis \(\{p, pq, q\}\) and comultiplication given by

\[
\Delta(p) := p \otimes p, \quad \Delta(q) := q \otimes q \quad \text{and} \quad \Delta(pq) := p \otimes pq + pq \otimes q.
\]

Note that \(X\) is counitary with \(\epsilon(p) = \epsilon(q) = 1\) and \(\epsilon(pq) = 0\). Consider the \(k\)-linear map \(s : X \otimes X \to X \otimes X\) defined by

\[
s(p \otimes p) := p \otimes p, \quad s(p \otimes pq) := pq \otimes p, \quad s(p \otimes q) := q \otimes p,
\]

\[
s(pq \otimes p) := pq \otimes q, \quad s(pq \otimes pq) := p \otimes p - pq \otimes pq - q \otimes q, \quad s(pq \otimes q) := -q \otimes pq,
\]

\[
s(q \otimes p) := p \otimes q, \quad s(q \otimes pq) := -pq \otimes q, \quad s(q \otimes q) := q \otimes q.
\]

This example corresponds to case 2) of Theorem 5.4 of [18] with \(\Gamma_1 := 1, \alpha_1 := 1, \alpha_3 := -1\). A direct computation proves that \(s\) and \(\overline{s}\) are left non-degenerate involutive coalgebra automorphisms. In this example \(\cdot\) and \(\cdot\) are given by

\[
p \cdot p = p, \quad p \cdot pq = 0, \quad p \cdot q = p,
\]

\[
pq \cdot p = pq, \quad pq \cdot pq = -p + q, \quad pq \cdot q = -pq,
\]

\[
q \cdot p = q, \quad q \cdot pq = 0, \quad q \cdot q = q
\]

and

\[
p \cdot p = p, \quad p : pq = 0, \quad p : q = p,
\]

\[
pq \cdot p = pq, \quad pq : pq = p - q, \quad pq : q = -pq,
\]

\[
q \cdot p = q, \quad q : pq = 0, \quad q : q = q.
\]

Note that \(pq \cdot pq \neq pq \cdot pq\), and so \(\neq \cdot\). □

Corollary 2.16. If \(s \in \text{Aut}_{\text{Coalg}}(X^2)\) is non-degenerate, then \(\overline{s}^{-1}\) is non-degenerate.

Proof. By Definition 1.4 and Proposition 2.11, the morphisms \(\overline{s}^{-1}\) and \((\overline{s}^{-1})_{-1}\) are left non-degenerate. But, we know that \(\overline{s}^{-1}\) is right non-degenerate if and only if \((\overline{s}^{-1})_\tau = s_\tau^{-1} = (\overline{s}_\tau)^{-1}\) is left non-degenerate, which finishes the proof. □

Proposition 2.17. Let \(X\) be a regular \(q\)-magma coalgebra. Then \(X\) is right non-degenerate if and only if it is left non-degenerate.

Proof. By Remarks 1.19 and 2.8 the endomorphism \(s = s_\tau\) is invertible and left non-degenerate. By Proposition 2.9, we know that \(X\) is right non-degenerate if and only if \(s_\tau\) is left non-degenerate. By Proposition 2.11 this happens if and only if \(s_\tau^{-1}\) is left non-degenerate. Since, again by Proposition 2.11, the map \(\overline{s}^{-1}\) is left non-degenerate, we can apply Proposition 2.9 to \(\overline{s}^{-1}\), in order to get that \(s_\tau^{-1}\) is left non-degenerate if and only if \(X_{s_\tau^{-1}}\) is right non-degenerate, which, by Proposition 2.11, means that \(X\) is left non-degenerate (see Definition 1.22). □
Remark 2.18. Assume that $X$ is a non-degenerate $q$-magma coalgebra. Since $X$ is regular, by Remark 1.19 the map $s := s_X$ is invertible. Furthermore, by Remark 2.8 and Proposition 2.9, the maps $s$ and $\tilde{s}_r$ are non-degenerate. Thus, by Remark 1.11, equality \([1.3]\) is satisfied. Let $Y := X^\circ := (X^\circ, \ast, \circ) \in \text{coalg}_k$ be as in Notation 1.10. Since $\tilde{s}_r$ is bijective, by Remark 1.19, we know that $Y$ is regular (and so $Y^\circ$ is also). In particular, the map $\overline{G} : X^2 \to X^2$, given by $\overline{G}(x, y) = x \ast y_1 \otimes y_2$, is bijective (see Definition 1.15). Moreover, by Proposition 2.11, we know that $G_{s_X^{-1}} = \overline{G}^{-1}$. Thus, by Remark 2.20 and 2.12, we also have

$$
\tilde{s}_r^{-1}(x \otimes y) = \tilde{s}_r^{-1}(x \otimes y) = \tau \pi \tau(x \otimes y) = \pi(1) x_2(1) \otimes y_1(x_2(1)),
$$

which implies that $G_{s_X^{-1}}(x \otimes y) = y_1(1) x \otimes y_2(2)$. Thus, arguing as in Remark 1.9, we obtain that

$$
y_2(2) x_1(1) = y_1(1) x \ast y_2(2) = \epsilon(x) x \quad \text{for all } x, y \in X.
$$

[2.1]

Remark 2.19. When $X = kY$, our $(X, \ast, \ast)$ induces by restriction a $q$-magma on $Y$, which is canonically isomorphic to the dual $q$-magma $Y^*$ (see \([25, \text{Definition 2}]\)).

Proposition 2.20. A left non-degenerate coalgebra endomorphism $s$ is invertible and non-degenerate if and only if $X_s$ is non-degenerate.

Proof. Clearly, if $X_s$ is non-degenerate, then it is regular (see Definition 2.22). Moreover, by Remark 1.19 we know that $X_s$ is regular if and only if $s$ is invertible. On the other hand, by Propositions 2.9 and 2.17, the map $s$ is non-degenerate if and only if $X_s$ is.

Proposition 2.21. Let $X$ be a left regular $q$-magma coalgebra and let $s := s_X$. Then $X_s$ is right non-degenerate if and only if it is left regular and the map $h$, introduced in Definition 1.12, is invertible.

Proof. By Remark 2.8 we know that $s$ is left non-degenerate. Thus, we have

$$
hG_s(x \otimes y) = y(3) : x_2(2) \otimes x_1(1) \otimes y_1(1) \cdot y(4) : x_1(1) \otimes x_2(2) \otimes y_3(1) : x_1(1) \otimes x_2(2) = H_{X_s}(y \otimes x);
$$

where the second equality holds by identity \([1.4]\); third one, by equalities \([1.2]\) and Remark 1.9; and the last one, by Definition 2.22. Thus, since $G_s$ is invertible, $h$ is invertible if and only if $H_{X_s}$ is. Hence, $X_s$ is right non-degenerate if and only if it is left regular and $h$ is invertible (see Definition 2.22).

Proposition 2.22. Let $X$ be a regular $q$-magma coalgebra and let $s := s_X$. Then $X_s$ is non-degenerate if and only if it is left regular and the map $h$, introduced in Definition 1.12, is invertible.

Proof. By Proposition 2.21, if $X_s$ is non-degenerate, then it is left regular and $h$ is invertible. By Propositions 2.17 and 2.21, in order prove the converse, we must only check that $X_s$ is regular. But, by Remark 1.19, this happens if and only if $s$ is invertible, which is true since $s$ is invertible (again by Remark 1.19).

Lemma 2.23. Let $X$, $s$, $\tilde{s}$ and $h$ be as in Proposition 2.22. Assume that $X_s$ is non-degenerate. Then, the maps $\tilde{s}$ and $\tilde{s}_r$ are invertible and non-degenerate. Moreover, $\tilde{s}_r^{-1}$ is left non-degenerate and $X_{\tilde{s}_r^{-1}} = X^\circ_{s^{-1}}$.

Finally, if we set $(X, \circ, \circ) := X_s$, then we have

$$
y_2(2) x \circ y_1(1) = y_1(1) (x \circ y_2(2)) = \epsilon(y)x \quad \text{and} \quad y_1(1) x \circ y_2(2) = y_2(2) (x \circ y_1(1)) = \epsilon(y)x,
$$

for all $x, y \in X$.

Proof. By Proposition 2.20, the map $\tilde{s}$ is invertible and non-degenerate. By Definition 1.14, this implies that $s_r$ is invertible and left non-degenerate. By Proposition 2.11, we also know that the map $\tilde{s}_r^{-1}$ is left non-degenerate and $(X^\circ, \ast, \ast) = (X^\circ_{s^{-1}})$. Note that, by Remarks 1.20 and 2.12,

$$
s_r(x \otimes y) = y(2) x_2(2) \otimes y_1(1) x_1(1) \quad \text{and} \quad \tilde{s}_r^{-1}(x \otimes y) = y_2(2) x_1(1) \otimes y_1(1) x_2(2),
$$

for all $x, y \in X$. Hence, identities \([1.2]\) applied to $s_r$ and $\tilde{s}_r^{-1}$ yield

$$
y_1(1) x \circ y_2(2) = y_2(2) (x \circ y_1(1)) = \epsilon(y)x \quad \text{and} \quad y_2(2) \circ y_1(1) = y_1(1) (x \circ y_2(2)) = \epsilon(y)x,
$$

for all $x, y \in X$.

Proposition 2.24. Under the conditions of Lemma 2.23, the map $\overline{h} : X^\circ \otimes X \to X \otimes X^\circ$, defined by $\overline{h}(x \otimes y) := y_1(1) x_2(2) \otimes x_1(1) \ast y_2(2)$, is the inverse of the coalgebra morphism $h_X$ (see Definition 1.12).

Proof. By Proposition 2.22, we know that $h_X$ is invertible. So, it suffices to prove that $\overline{h} h_X = \text{id}_{X^\circ \otimes X^\circ}$. Since $(X, \circ, \circ) = X_s$ is a $q$-magma coalgebra, we know that $\overline{h} = h_{X^\circ}$ is a coalgebra morphism (see Definition 1.12). Thus $\overline{h} h_X$ is a coalgebra morphism and consequently, by item 2) of Remark 1.1, to check that $\overline{h} h_X = \text{id}_{X^\circ \otimes X^\circ}$, it suffices to show that

$$
(\text{id}_X \otimes \epsilon) \overline{h} h_X(x \otimes y) = \epsilon(y)x \quad \text{and} \quad (\epsilon \otimes \text{id}_X^\circ) \overline{h} h_X(x \otimes y) = \epsilon(x)y.
$$
A direct computation shows that this is equivalent to see that

\[(x(1) \cdot y(2)) \triangleright (y(1) : x(2)) = \epsilon(y)x \quad \text{and} \quad (y(1) : x(2)) \triangleright (x(1) \cdot y(2)) = \epsilon(x)y. \tag{2.4} \]

By Remark 1.20, we have \(x(x_2)\triangleright x(1) = x(1) : (y : x(3)x(2)) = x \cdot y\). Hence, by \([2.2]\),

\[(x(1) \cdot y(2)) \triangleright (y(1) : x(2)) = y(x_2)\triangleright y(x_1) \triangleright (y(1) : x(3)) = \epsilon(y)x, \]

as desired. To prove the second equality in \([2.4]\), we note that \(y : x = x(y(1)y(2))\), and so, by \([1.4]\) and \([2.2]\),

\[(y(1) : x(2)) \triangleright (x(1) \cdot y(2)) = (y(2) : x(1)) \triangleright (x(2) \cdot y(1)) = x(1)y(2)y(3) \triangleright (x(2) \cdot y(1)) = \epsilon(x)y, \]

which finishes the proof. \(\Box\)

### 3 Very strongly regular q-magma coalgebras

In this section we begin the study of a special type of q-magma coalgebras that satisfy a very strong condition of regularity. Examples are all the cocommutative q-magma coalgebras and also the Hopf q-braces introduced in Section 5.

**Definition 3.1.** Let \(\mathcal{X} = (X, \cdot, \cdot)\) be a q-magma coalgebra and let \(p, q, d\) be as above Definition 1.12. Assume that \(\mathcal{X}\) is regular (see Definition 1.15) and let \(G_X\) and \(G_{X^q}\) be as in Remarks 1.18 and 1.20, respectively. We say that \(\mathcal{X}\) is strongly regular if, for each \(i \in \mathbb{Z}\), there exist maps \(p_i : X^2 \rightarrow X\), \(d_i : X^2 \rightarrow X\), \(gp_i : X^2 \rightarrow X\) and \(gd_i : X^2 \rightarrow X\), such that

\[p_0 = p, \quad d_0 = d, \quad gp_0 = (X \otimes \epsilon)G_X, \quad gd_0 = (X \otimes \epsilon)G_{X^q} \tag{3.1}\]

and

\[x\triangleright y = (x \cdot_1 y) \cdot_1 y_{1(1)} = \epsilon(y)x, \quad \tag{3.2}\]
\[x\triangleright y_{1(2)} = (x \cdot_1 y_{1(1)}) \cdot_1 y_{1(2)} = \epsilon(y)x, \quad \tag{3.3}\]
\[x\triangleright y_{2(1)} = (x \cdot_1 y) \cdot_1 y_{2(1)} = \epsilon(y)x, \quad \tag{3.4}\]
\[x\triangleright y_{2(2)} = (x \cdot_1 y_{2(1)}) \cdot_1 y_{2(2)} = \epsilon(y)x, \quad \tag{3.5}\]

where \(\cdot_1, \cdot, \cdot : p_i(x \otimes y), \cdot : q_i(x \otimes y), x^0 := p_i(x \otimes y)\) and \(x^0 := gd_i(x \otimes y)\). If necessary, we will write \(p_i^0, d_i^0, gp_i^0\) and \(gd_i^0\) instead of \(p_i, d_i, gp_i\) and \(gd_i\), respectively.

**Remark 3.2.** The maps \(p_i, d_i, gp_i\) and \(gd_i\) are unique. In fact, since, by their very definitions, this is true for \(p_0, d_0, gp_0\) and \(gd_0\), the result follows from the fact that, for all \(i \in \mathbb{Z}\),

1) \((gp_i \otimes X)(X \otimes \Delta_{X^q})\) is the inverse of \((p_{i-1} \otimes X)(X \otimes \Delta_{X^q})\),
2) \((d_i \otimes X)(X \otimes \Delta)\) is the inverse of \((d_{i-1} \otimes X)(X \otimes \Delta)\),
3) \((p_i \otimes X)(X \otimes \Delta)\) is the inverse of \((gp_i \otimes X)(X \otimes \Delta)\),
4) \((d_i \otimes X)(X \otimes \Delta_{X^q})\) is the inverse of \((d_{i} \otimes X)(X \otimes \Delta_{X^q})\).

Note that \(x \cdot_0 y = x \cdot y, x \cdot_0 y = x : y, x^0 = x^y\) and \(x^0 = x_y\), and that if \(X\) is cocommutative, then \(p_i = p_0, d_i = d_0, gp_i = gp_0\) and \(gd_i = gd_0\) for all \(i\).

**Remark 3.3.** If \(\mathcal{X} = (X, \cdot, \cdot)\) is strongly regular, then \(X^{op} = (X^{op}, \cdot, \cdot)\) is also. In fact, in order to see this, it suffices to note that the maps \(p_{i}^{op}, d_{i}^{op}, gp_{i}^{op}\) and \(gd_{i}^{op}\), from \(X^{cop} \otimes X^{cop}\) to \(X^{cop}\), defined by

\[p_{i}^{op} := d_{i}, \quad d_{i}^{op} := p_{i}, \quad gp_{i}^{op} := gd_{i} \quad \text{and} \quad gd_{i}^{op} := gp_{i}, \quad \text{for } i \in \mathbb{Z},\]

satisfy items 1)–4) of Remark 3.2, with \(X\) replaced by \(X^{cop}\).

**Lemma 3.4.** For all \(i \in \mathbb{Z}\), the maps \(p_i\) and \(d_i\) are coalgebra morphisms from \(X \otimes X^{cop}\) to \(X\) and the maps \(gp_i\) and \(gd_i\) are coalgebra morphisms from \(X^{2}\) to \(X\).

**Proof.** Since \(p_0, d_0, gp_0, gd_0\) are coalgebra morphisms, to prove the result it suffices to check that

\[p_i\text{ is a coalgebra map} \iff gp_{i+1}\text{ is a coalgebra map} \iff p_{i+1}\text{ is a coalgebra map} \tag{3.6}\]

and

\[gd_i\text{ is a coalgebra map} \iff d_{i+1}\text{ is a coalgebra map} \iff gd_{i+1}\text{ is a coalgebra map.} \tag{3.7}\]

But the first equivalence in \([3.6]\) follows using \([3.2]\) item 1) of Remark 1.1 and Proposition 1.2 applied to \(X^{cop}\); and the second one, using \([3.4]\) and Proposition 1.2 applied to \(X\); while the first equivalence in \([3.7]\) follows using \([3.3]\) and Proposition 1.2 applied to \(X\); and the second one, using \([3.5]\), item 1) of Remark 1.1 and Proposition 1.2 applied to \(X^{cop}\). \(\square\)
Proposition 3.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be two strongly regular $q$-magma coalgebras. If $f: \mathcal{X} \to \mathcal{Y}$ is a morphism of $q$-magma coalgebras, then, for all $i \in \mathbb{Z}$,
\[ f(x^i, y) = f(x)^i \cdot f(y), \quad f(x, y) = f(x)^i \cdot f(y), \quad f(x^i)_y = f(x)^i(f(y)), \quad f(x)_y = f(x)^i(f(y)). \]

Proof. Since the above equalities hold for $i = 0$, the result will be follow if we show that
1) $f(x^i, y) = f(x)^i \cdot f(y)$ and $f(x^i)_y = f(x)^i(f(y))$.
2) $f(x^i)_y = f(x^i(y))$.
3) $f(x, y) = f(x)^i \cdot f(y)$ and $f(x^i)_y = f(x)^i(f(y))$.
4) $f(x^i)_y = f(x)^i(f(y))$.

We prove the first two items and leave the other ones to the reader. Assume that $f(x^i, y) = f(x)^i \cdot f(y)$, for all $x, y \in X$. Then, by condition [3.4], we have
\[ f(x^i, y) = f(x^i) \cdot f(y) = (f(y))^{i+1} \cdot f(x), \]
which implies that
\[ f(x^i) = (f(x)^i)^{i+1} \cdot f(y) = (f(x)^i)^{i+1} \cdot f(x). \]
Similarly, by condition [3.2], we have
\[ f(x^i)_y = f(x^i(y)) = f(x)^i \cdot f(y) = f(x)^i \cdot f(y), \]
and hence
\[ f(x^i)_y = (f(x)^i)^{i+1} \cdot f(y) = (f(x)^i)^{i+1} \cdot f(y). \]
Thus item 1) is true. We next prove item 2). Assume that $f(x^i)_y = (f(x)^i)_y$ for all $x, y \in X$. Then, by condition [3.4], we have
\[ f(x^i, y) = f(x)^i \cdot f(y) = (f(x)^i)^{i+1} \cdot f(y), \]
and consequently,
\[ f(x^i, y) = (f(x^i)^i)_y = f(x)^i \cdot f(y), \]
Finally, by condition [3.2], we have
\[ f(x^i, y) = f(x^i) \cdot f(y) = f(x)^i \cdot f(y), \]
as desired. \[ \square \]

Let $\mathcal{X}$ be a strongly regular $q$-magma coalgebra and let $s := s_X$. By Remark 1.19, we know that $s$ is bijective. Assume that $s^{-1}_\mathcal{X}$ is left non-degenerate. By Proposition 2.11, we know that $s$ is left non-degenerate and $\mathcal{X}^\text{cop} = (\mathcal{X}^\text{cop}, \Diamond, \circ)$, where $\circ$ and $\Diamond$ are as in Lemma 2.23. Assume that $\hat{\mathcal{X}} := \mathcal{X}^\text{cop}_{-1}$ is strongly regular. For $i \in \mathbb{Z}$ and $x, y \in \mathcal{X}$, write
\[ x \Diamond y := p^x_\circ(x \otimes y), \quad x \circ y := d^x_\Diamond(x \otimes y), \quad x := g^x_\circ(x \otimes y) \quad \text{and} \quad \mathcal{B} x := g^x_\Diamond(x \otimes y) \quad \text{[3.8]} \]
Note that $x \circ_0 y = x \circ y, x \Diamond_0 y = x \Diamond y, y \circ x = y \circ x, \mathcal{B} x = y \circ x$ and the last two equalities use [2.2].

Definition 3.6. Under the conditions above we say that $\mathcal{X}$ is very strongly regular if
\[ y(1)_i \cdot x(2) \otimes x(1)^i \cdot y(2) = y(1)_i \cdot x(2) \otimes x(1)^i \cdot y(2) \quad \text{for all} \quad i \in \mathbb{Z} \quad \text{and} \quad x, y \in \mathcal{X}, \quad \text{[3.9]} \]
\[ x(2)_i \cdot y(2) \otimes x(1)^i \cdot y(1)_i = x(2)_i \cdot y(2) \otimes x(1)^i \cdot y(1)_i \quad \text{for all} \quad i \in \mathbb{Z} \quad \text{and} \quad x, y \in \mathcal{X}, \quad \text{[3.10]} \]
\[ y(1) \circ x(2)_i \otimes x(1)^i \cdot y(1)_i = y(1) \circ x(2)_i \otimes x(1)^i \cdot y(1)_i \quad \text{for all} \quad i \in \mathbb{Z} \quad \text{and} \quad x, y \in \mathcal{X}, \quad \text{[3.11]} \]
\[ y(1) \otimes x(2)_i \circ x(1)^i \cdot y(1)_i = y(1) \otimes x(2)_i \circ x(1)^i \cdot y(1)_i \quad \text{for all} \quad i \in \mathbb{Z} \quad \text{and} \quad x, y \in \mathcal{X}. \quad \text{[3.12]} \]

Proposition 3.7. If $\mathcal{X} = (\mathcal{X}, \cdot, \cdot)$ is very strongly regular, then $\mathcal{X}^\text{cop} = (\mathcal{X}^\text{cop}, \cdot, \cdot)$ is also.
Proof. Let \( s := s_X \) be the map associated with \( X \) according to Remark 2.8. By Proposition 2.11, the map associated with \( \mathcal{X}^{\text{op}} \), according to Remark 2.8, is \( \tilde{s}^{-1} \). As we saw above \( (\tilde{s}^{-1})_{\tilde{q}}^{-1} = s_{\tilde{r}} \), is left non-degenerate. In order to verify that \( \mathcal{X}^{\text{op}} \) is very strongly regular we must prove that

- \( \mathcal{X}^{\text{op}} \) is strongly regular,
- \( \mathcal{X}^{\text{op}} = \mathcal{X}_{s_{\tilde{r}}} = (X, \circ, \tilde{z}) \) is strongly regular,
- Identities [3.9]–[3.12] are true for \( \mathcal{X}^{\text{op}} \).

The first item is true by Remark 3.3; while the second one, follows from the same remark applied to \( \tilde{X} \). Next we check the third condition. Identities [3.9] for \( \mathcal{X}^{\text{op}} \), reads

\[
y(2) \otimes x(1) \otimes x(2) := i y(1) = y(1) \circ x(2) \otimes x(1) := i y(2).
\]

But these are true by identities [3.9] for \( \tilde{X} \). On the other hand, identities [3.10] for \( \mathcal{X}^{\text{op}} \), reads

\[
x(1) y(1) \otimes x(2) y(2) := i x(2) \otimes x(1) y(1) = i y(1) \cdot x(2).
\]

But these are true by identities [3.12] for \( \tilde{X} \). The other identities can be proved in a similar way. \( \square \)

Example 3.8. The \( q \)-magma coalgebra \( \mathcal{X}_s \) associated with the map \( s \) in Example 2.15 is a very strongly regular \( q \)-magma coalgebra. In fact, in this case we have

\[
p_i = d_{i+1} = gp_{i+1} = gd_{i+1} = d_i^\tilde{X} = gp_i^\tilde{X} = gd_i^\tilde{X} = \begin{cases} p_0 & \text{if } i \text{ is even}, \\ d_0 & \text{if } i \text{ is odd}, \end{cases}
\]

where \( p_0 \) and \( d_0 \) are the maps \( \cdot \) and \( : \) given in Example 2.15.

4 Solutions of the braid equation and \( q \)-cycle coalgebras

In this section we introduce the notions of set-theoretic type solution of the braid equation and \( q \)-cycle coalgebras, and we prove that, for each coalgebra \( X \), the correspondence between left non-degenerate endomorphisms of \( X^2 \) and left regular \( q \)-magma coalgebra structures on \( X \), established in Section 2, induces a bijective correspondence between set-theoretic type solutions of the braid equation on \( X^2 \) and \( q \)-cycle coalgebra structures on \( X \).

Definition 4.1. Let \( X \) be a vector space. We say that a map \( s \in \text{End}_k(X^2) \) is a solution of the braid equation if \( s_{12} s_{23} s_{12} = s_{23} s_{12} s_{23} \), where \( s_{12} := s \otimes X \) and \( s_{23} := X \otimes s \). If, moreover, \( X \) is a coalgebra and \( s \) is a coalgebra endomorphism of \( X^2 \), then we say that \( s \) is a set-theoretic type solution.

Remark 4.2. Let \( X \) be a vector space. A map \( s \in \text{End}_k(X^2) \) is a solution of the braid equation if and only if \( s_{\tilde{r}} \) is.

Proposition 4.3. A left non-degenerate coalgebra morphism \( s : X^2 \to X^2 \), is a set-theoretic type solution of the braid equation, if and only if, for all \( x, y, z \in X \),

1) \( (x \cdot z(1)) z(2) = (x \cdot y(2)) y(1) \),
2) \( (x \cdot y(2)) z(1) = (y(1) \cdot z(1)) z(2) = x \cdot y(2) \cdot z(1) \),
3) \( y(1) \cdot z(2) = y \cdot x \cdot z(2) \).

Proof. Let \( U := s_{12} s_{23} s_{12} \) and \( V := s_{23} s_{12} s_{23} \). Since \( U \) and \( V \) are coalgebra maps we know that \( U = V \) if and only if \( U_i = V_i \), for \( i \in \{1, 2, 3\} \). A direct computation, using that

\[
U_1 = s_1 (X \otimes s_1) s_{12}, \quad U_2 = s_2 (X \otimes s_1) s_{12}, \quad U_3 = s_2 (s_2 \otimes X),
\]

\[
V_1 = s_1 (X \otimes s_1) s_{23}, \quad V_2 = s_1 (s_2 \otimes X) s_{23}, \quad V_3 = s_2 (s_2 \otimes X) s_{23},
\]

shows that \( U_i (x \otimes y \otimes z) = V_i (x \otimes y \otimes z) \) for \( i \in \{1, 2, 3\} \), if and only if

a) \( (x \cdot y)^2 = (x \cdot y(1)) \cdot (y(2) \cdot z(1)) \),
b) \( (x \cdot y(1)) \cdot (y(2) \cdot z(1)) = (x \cdot y(2) \cdot z(1)) \cdot (y(1) \cdot z(2)) \),
c) \( x \cdot y(1) \cdot (y(2) \cdot z(1)) = x \cdot (y \cdot z(2)) \).

Since \( s \) is a left non-degenerate map, in order to verify the proposition it suffices to note that, by Remark 2.1, items 1) and 2) are items a) and b) with \( x \otimes y \otimes z \) replaced by \( x \otimes y \cdot z(1) \otimes z(2) \), while item 3) is item c) with \( x \otimes y \otimes z \) replaced by \( x \cdot y(1) \otimes y(2) \otimes z \). \( \square \)
In the set-theoretic context, the following definition and theorem correspond to Definition 3 and Proposition 1 of [25], respectively.

**Definition 4.4.** A q-cycle coalgebra is a left regular q-magma coalgebra \( \mathcal{X} = (X, \cdot, : ) \) such that:

1. \( (x \cdot y(1)) \cdot (z : y(2)) = (x \cdot z(1)) \cdot (y \cdot z(2)) \),
2. \( (x \cdot y(1)) : (z \cdot y(2)) = (x : z(1)) \cdot (y \cdot z(1)) \),
3. \( (x : y(1)) : (z \cdot y(2)) = (x : z(2)) \cdot (y \cdot z(1)) \),

for all \( x, y, z \in X \).

**Remark 4.5.** By Remark 1.16, a regular q-magma coalgebra \( X \) is a q-cycle coalgebra if and only if \( X^{op} \) is.

**Example 4.6.** The q-magma coalgebra \( X_s \) associated with the map \( s \) in Example 2.15 is a very strongly regular q-cycle coalgebra.

**Example 4.7.** Let \( X \) be the coalgebra with basis \( \{ p, pq, q \} \), considered in Example 2.15. The q-magma coalgebra structure on \( X \), defined by

\[
x \cdot p := x, \quad x \cdot q := x, \quad x \cdot pq := 0 \quad \text{and} \quad x : y := \begin{cases} x & \text{if } pq \notin \{ x, y \}, \\ 0 & \text{if } pq \in \{ x, y \}, \end{cases}
\]

for all \( x \in \{ p, pq, q \} \), yields a q-cycle coalgebra, which is not right regular.

Our next aim is to prove the following result:

**Theorem 4.8.** Let \( s \) be a left non-degenerate coalgebra endomorphism of \( X^2 \) and let \( X \) be the left regular q-magma coalgebra associated with \( s \). The map \( s \) is a set-theoretic type solution of the braid equation if and only if \( X \) is a q-cycle coalgebra.

**Lemma 4.9.** Condition 1) of Proposition 4.3 is equivalent to condition 1) of Definition 4.4.

**Proof.** A direct computation, shows that the map \( x \otimes y \otimes z \mapsto x \cdot (z(1) : y(2)) \otimes y(1) \otimes z(2) \) is invertible (with inverse \( x \otimes y \otimes z \mapsto x^{z(1)} : y^{(2)} \otimes y(1) \otimes z(2) \)). Hence, the first condition in Proposition 4.3 is fulfilled if and only if \( (x \cdot (z(1) : y(2)))^{(2)} = e(x) y^{(2)} \), which happens if and only if \( x^{(2)} \cdot z = (x \cdot (z(1) : y(2)))^{y(1)} z(2) \).

Replacing \( x \otimes y \otimes z \) by \( x \cdot y(1) \otimes y(2) \otimes z \) and using identity [1.4] we obtain that this happens if and only if

\[
e(y) x \cdot z = \left[ (x \cdot y(1)) \cdot (z(1) : y(3)) \right]^{y(1)} z(2) = \left( (x \cdot y(1)) \cdot (z(2) : y(3)) \right)^{y(3)} z(1).
\]

Therefore, if condition 1) in Proposition 4.3 is satisfied, then

\[
(x \cdot y(1)) \cdot (z : y(2)) = \left[ (x \cdot y(1)) \cdot (z(3) : y(2)) \right]^{y(3)} z(2) \cdot (y(4) \cdot z(1)) = (x \cdot z(2)) \cdot (y \cdot z(1)),
\]

as desired. On the other hand, if condition 1) of Definition 4.4 is fulfilled, then a computation reversing the last step shows that the first condition in Proposition 4.3 is satisfied. \( \square \)

**Lemma 4.10.** Assume that condition 1) of Proposition 4.3 is fulfilled. Then, condition 2) is equivalent to condition 2) of Definition 4.4.

**Proof.** Since \( x \otimes y \otimes z \mapsto x \cdot (z(1) : y(2)) \otimes y(1) \otimes z(2) \) is invertible, the second condition in Proposition 4.3 is equivalent to

\[
\left( (x(0) : y(1)) (y(2)) \cdot (z(3)) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5) = e(z) y, \]

Evaluating this in \( x \cdot y(1) \otimes y(2) \otimes z \) and using that \( x^{y(1)} y(2) = y : x \), we obtain that this happens if and only if

\[
\epsilon(z) y : x = \left( (x(2) \cdot y(1)) (z(1) : y(3)) \cdot (y(4)) \cdot z(3) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5).
\]

Hence, if condition 2) in Proposition 4.3 is satisfied, then

\[
\epsilon(z) y : x = \left( (x(2) \cdot y(1)) (z(1) : y(3)) \cdot (y(4)) \cdot z(3) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5) \quad \text{by equality [1.4]}
\]

\[
= \left( (x(2) \cdot y(1)) (z(1) : y(3)) \cdot (y(4)) \cdot z(3) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5) \quad \text{by Definition 4.4(1)}
\]

\[
= \left( (x(2) \cdot y(1)) (z(1) : y(3)) \cdot (y(4)) \cdot z(3) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5) \quad \text{by [1.2] and def. of :}
\]

\[
= \left( (x(2) \cdot y(1)) (z(1) : y(3)) \cdot (y(4)) \cdot z(3) \right) \left( \left( x(1) \cdot y(2) \cdot y(3) \right) \cdot y(4) \cdot z(4) \right) z(5) \quad \text{by Definition 4.4(1)}
\]
= \((y \cdot z(1)) \cdot (x(2) \cdot z(2))\)^{z(3)}z(1), \quad \text{by the definition of :}

and so, \((y \cdot x(2)) \cdot (z \cdot x(1)) = (y \cdot z(1)) \cdot (x \cdot z(2)), \) as desired. The converse follows by a direct computation reversing the previous steps.

\[ \square \]

**Lemma 4.11.** If conditions 1) and 2) of Proposition 4.3 are fulfilled, then condition 3) is equivalent to condition 3) of Definition 4.4.

**Proof.** A direct computation using that the map \(x \otimes y \otimes z \mapsto x \cdot z(2) \otimes y \cdot z(1) \otimes z(3)\) is invertible, shows that the third condition of Proposition 4.3 is satisfied if and only if

\[
(x \cdot z(3)) \cdot (y(1) \cdot z(2)) \cdot (y(2) \cdot z(1)) = (y \cdot z(1)) \cdot (x(2) \cdot z(2)) \cdot (x(1) \cdot z(3))z(4). \quad [4.1]
\]

Since, by Definition 4.4(1), equality \([4.1]\) and the definition of :

\[
(y \cdot z(1)) \cdot (x(2) \cdot z(2)) \cdot (x(1) \cdot z(3)) \cdot z(4) = (y \cdot z(1)) \cdot (x(2) \cdot z(2)) \cdot (x(1) \cdot z(3)) \cdot z(4) = (z \cdot y(1)) \cdot (x \cdot y(2))
\]

and, by Definition 4.4(2) and the definition of :

\[
(y \cdot z(1)) \cdot (x(2) \cdot z(2)) \cdot (x(1) \cdot z(3)) \cdot z(4) = (y \cdot z(1)) \cdot (x(2) \cdot z(2)) \cdot (x(1) \cdot z(3)) \cdot z(4) = (z \cdot x(1)) \cdot (y \cdot x(2)),
\]

this finishes the proof.

\[ \square \]

**Proof of Theorem 4.8.** By Proposition 4.3 and Lemmas 4.9, 4.10 and 4.11.

**Corollary 4.12.** There is a bijective correspondence between left non-degenerate set-theoretic type solutions of the braid equation and q-cycle coalgebras. Under this bijection, invertible solutions correspond to regular q-cycle coalgebras, while non-degenerate invertible solutions correspond to non-degenerate q-cycle coalgebras.

**Proof.** This follows immediately from Theorem 4.8, Proposition 2.20 and Remarks 1.19 and 2.8.

**Remark 4.13.** In the set-theoretic context, Corollary 4.12 yields \([25, \text{Proposition 1}]\).

**Corollary 4.14.** Let \(s : X^2 \to X^2\) be a left non-degenerate coalgebra morphism. If \(X_s\) is a non-degenerate q-cycle coalgebra, then \(X_{s\circ} = (X_{s\circ}, \ast, \cdot)\) is also (see Notation 1.10).

**Proof.** By Corollary 4.12 the coalgebra map \(s : X^2 \to X^2\) is an invertible and non-degenerate set-theoretic type solution of the braid equation. Hence, by Remark 4.2 and Proposition 2.9, the map \(\tilde{\sigma}r\) is also. Thus, again by Corollary 4.12, we conclude that \(X_{s\circ}\) is non-degenerate.

**Corollary 4.15.** If \(X\) is a non-degenerate q-cycle coalgebra, then \(X_{\text{op}}\) is also.

**Proof.** This follows immediately from Proposition 2.11 and Corollaries 2.16 and 4.12.

**Definition 4.16.** A **rack coalgebra** is a coalgebra \(X\) endowed with a coalgebra map \(\cdot : X \otimes X \to X\) such that:

1. The map \(x \otimes y \mapsto x \cdot y(1) \otimes y(2)\) is invertible,
2. \(y(2) \otimes x \cdot y(1) = y(1) \otimes x \cdot y(2)\) for all \(x, y \in X\),
3. \((x \cdot y) \cdot z = (x \cdot z(2)) \cdot (y \cdot z(1))\) for all \(x, y, z \in X\).

**Remark 4.17.** Let \(X\) be a coalgebra and let \(\cdot : X \otimes X \to X\) be a map. The pair \((X, \cdot)\) is a rack coalgebra if and only if the map \(s : X^2 \to X^2\), defined by \(s(x \otimes y) := y(2) \otimes x \cdot y(1)\) is a left non-degenerate set-theoretic type solution of the braid equation. In fact, by Remark 1.1, we know that \(s\) is a coalgebra morphism if and only if \(s\) is a coalgebra morphism and \(y(2) \otimes x \cdot y(1) = y(1) \otimes x \cdot y(2)\), for all \(x, y \in X\); \(s\) is left non-degenerate if and only if the map \(x \otimes y \mapsto x \cdot y(1) \otimes y(2)\) is invertible; and finally, a straightforward computation shows that \(s\) satisfies \(s_{12} = s_{23} = s_{12} = s_{23}\), if and only if \((x \cdot y) \cdot z = (x \cdot z(2)) \cdot (y \cdot z(1))\) for all \(x, y, z \in X\).

**Example 4.18.** If in a q-cycle coalgebra \(X = (X, \cdot, : )\) the operator : is trivial, which means that \(x : y = (y) x\), then \((X, \cdot)\) is a rack coalgebra, via \(x \cdot y := x^y\). In fact, in this case, by Remark 1.18, the solution associated with \(X\) via Corollary 4.12, is the map given by \(s(x \otimes y) := y(2) \otimes x \cdot y(1)\), which by Remark 4.17 is a rack coalgebra. In the set-theoretic context this example is \([25, \text{Example 2}]\).
5  Weak braiding operators and Hopf $q$-braces

In this section we introduce the notions of Hopf $q$-brace and weak braiding operator, and we prove that they are equivalent (see Theorem 5.18). These are adaptations of the concept of $q$-brace introduced in [25] and of a weak version of the concept of braiding operator introduced in [21] to the setting of Hopf algebras. Then, we study the structure of Hopf $q$-brace in detail. In particular we obtain Hopf algebra versions of the results in [25, Section 3] and [21, Theorem 1]. We also obtain some results that only make sense in the context of Hopf algebras.

**Definition 5.1.** A weak braiding operator is a pair $(H, s)$, consisting of a Hopf algebra $H$ and a set-theoretic type solution of the braid equation $s: H^2 \to H^2$, that satisfies the following identities:

$$
\begin{align*}
  s(\mu \otimes \mu) &= (H \otimes \mu)(s \otimes H)(H \otimes s), \quad [5.1] \\
  s(H \otimes \mu) &= (\mu \otimes H)(H \otimes s)(s \otimes H), \quad [5.2] \\
  s(\eta \otimes H) &= H \otimes \eta, \quad [5.3] \\
  s(H \otimes \eta) &= \eta \otimes H, \quad [5.4]
\end{align*}
$$

where $\mu$ is the multiplication map of $H$ and $\eta$ is the unit of $H$. A weak braiding operator $(H, s)$ is a braiding operator if

$$
\mu = \mu s. \quad [5.5]
$$

A morphism from a weak braiding operator $(H, s)$ to a weak braiding operator $(K, s')$ is a Hopf algebra morphism $f: H \to K$ such that $(f \otimes f)s = s'(f \otimes f)$.

**Remark 5.2.** If $H$ is the group algebra $kG$, then our definition corresponds to the definition of braiding operator given in [21, page 3]. In fact, our conditions [5.1]–[5.5] translate directly into the conditions 4)–7) of [21], but the authors in [21] require additionally that the map $s: G \times G \to G \times G$, induced by $s$, to be bijective.

**Remark 5.3.** If $(H, s)$ is a (weak) braiding operator, then $(H^\text{op} \circ \text{op}, s)$ is also. Moreover, if the antipode of $H$ is bijective, then $(H^\text{op}, \hat{s})$ and $(H^\text{op}, s_r)$ are also (weak) braiding operators.

Let $H$ and $L$ be bialgebras and let $\alpha: L \otimes H \to L$ and $\beta: L \otimes H \to H$ be maps. For each $h \in H$ and $l \in L$ set $l^h := \alpha(l \otimes h)$ and $l^h = \beta(l \otimes h)$. Recall from [20, Definition IX.2.2], that $(L, H, \alpha, \beta)$ is a matched pair of bialgebras if

1) $L$ is a right $H$-module coalgebra via $\alpha$,

2) $H$ is a left $L$-module coalgebra via $\beta$,

3) $l(hh') = (l^{(1)}h^{(1)})(l^{(2)}h'^{(2)}h'^{(1)})$, for all $h, h' \in H$ and $l \in L$,

4) $(l^h)^{l'} = (l^{(1)}h^{(1)})(l^{(2)}h'^{(2)})$, for all $h \in H$ and $l, l' \in L$,

5) $l^{1h} = \epsilon(l)h$ and $l^{h1} = \epsilon(l)1$, for all $h \in H$ and $l \in L$,

6) $l^{(1)}h^{(1)} \otimes l^{(2)}h^{(2)} = l^{(2)}h^{(2)} \otimes l^{(1)}h^{(1)}$, for all $h \in H$ and $l \in L$.

In what follows we will write $(H, \alpha, \beta)$ instead of $(H, H, \alpha, \beta)$. A morphism, $f: (H, \alpha, \beta) \to (K, \alpha', \beta')$, is a morphism of bialgebras, $f: H \to K$, such that $f(h^{(1)}) = f(h^{(1)})f(l)$ and $f(h^{(1)}) = f(h^{(1)})f(l)$.

**Theorem 5.4.** For each matched pair of bialgebras $(L, H, \alpha, \beta)$, there exists a unique bialgebras structure $H \bowtie L$, called the bicrossed product of $H$ and $L$, with underlying coalgebra $H \otimes L$, such that

$$
(h \otimes l)(h' \otimes l') = h^{(1)}h^{(1)} \otimes l^{(2)}h^{(2)}h'^{(1)}l'^{(1)}l'^{(2)}
$$

for all $h, h' \in H$ and $l, l' \in L$.

The bialgebras $H$ and $L$ are identified with the subbialgebras $H \otimes k$ and $k \otimes L$ of $H \bowtie L$. Every element $h \otimes l$ of $H \bowtie L$ can be uniquely written as a product $(h \otimes 1)(1 \otimes l)$ of an element $h \in H$ and $l \in L$.

Moreover, if $H$ and $L$ are Hopf algebras with antipode $S_H$ and $S_L$, then $H \bowtie L$ is a Hopf algebra with antipode $S$ given by $S(h \otimes l) := S_L(l^{(2)})(h^{(1)}) \otimes S_H(h^{(2)})(l^{(1)})$. $S_H(h^{(1)})$.

**Proof.** See [20, Theorem IX.2.3].

**Proposition 5.5.** Let $H$ be a Hopf algebra and let $s: H^2 \to H^2$ be a solution of the braid equation. Then the pair $(H, s)$ is a weak braiding operator if and only if $(H, s_2, s_1)$ is a matched pair.

**Proof.** We already know that $s$ is a coalgebra map if and only if $s_1$ and $s_2$ are coalgebra maps and condition 6) above Theorem 5.4 is satisfied. We will freely use this fact. A direct computation shows that

$$
(id \otimes \mu)(s \otimes id)(id \otimes s)(h \otimes k \otimes l) = (id \otimes \mu)(s \otimes id)(h \otimes k^{(1)}(1_{(1)} \otimes k^{(2)}l^{(2)})
$$
Proof. Let \( h \cdot l := h^S(l) \). By the previous proposition, we have
\[
(h \cdot l_{(1)})^{l_{(2)}} = h^{S(l_{(1)})} h^{l_{(2)}} = \epsilon(l) h \quad \text{and} \quad h^{l_{(1)}} \cdot l_{(2)} = h^{S(l_{(1)})} = \epsilon(l) h.
\]
Hence, Remark 1.9 shows that \( s \) is left non-degenerate, as desired.\( \square \)

Remark 5.7. Let \( H \) be a Hopf algebra and let \( s : H^2 \to H^2 \) be a \( k \)-linear map. The proof of Proposition 5.5 shows that \( s \) is a coalgebra map and equalities [5.1]–[5.4] hold if and only if \((H, s_2, s_1)\) is a matched pair. Moreover, assuming as in the proof of Corollary 5.6, we obtain that \( s \) is left non-degenerate.

Definition 5.8. Let \( H \) be a Hopf algebra and let \( \mathcal{H} = (H, \cdot, :) \) be a left regular \( q \)-cycle coalgebra. We say that \( \mathcal{H} \) is a Hopf \( q \)-brace if \((H, :\) and \( (H, :) \) are right \( H^\text{cop} \)-modules and the following equalities hold:
\[
hk = (h \cdot (l_{(1)} : k_{(2)}))(k_{(1)} \cdot l_{(2)}) \quad \text{and} \quad hk = h : (l_{(1)} \cdot k_{(2)}))(k_{(1)} : l_{(2)})
\]
for all \( h, k, l \in H \). Let \( \mathcal{K} = (K, \cdot, :) \) be another Hopf \( q \)-brace. A morphism \( f : \mathcal{H} \to \mathcal{K} \) is a Hopf algebra morphism \( f : H \to K \) such that \( f(h \cdot h') = f(h) \cdot f(h') \) and \( f(h : h') = f(h) : f(h') \), for all \( h, h' \in H \).

Lemma 5.9. Let \( \mathcal{H} = (H, \cdot, :) \) be a \( q \)-magma coalgebra such that \( H \) is a Hopf algebra. The following assertions hold:
1) If \((H, :)\) is a right \( H^\text{cop} \)-module, then \( \mathcal{H} \) is right regular. Moreover \( h_k = h : S(k) \), for all \( h, k \in H \).
2) If \((H, :)\) is a right \( H^\text{cop} \)-module and the antipode \( S \), of \( H \), is bijective, then \( \mathcal{H} \) is left regular. Moreover \( h^k = h : S^{-1}(k) \), for all \( h, k \in H \).

Proof. 1) This follows from Remark 1.20 and the identities
\[
\epsilon(k) h = \epsilon(k) : 1 = h : S(k_{(2)}) = (h : S(k_{(2)})) : k_{(1)} \quad \text{and} \quad \epsilon(k) h = (h : k_{(2)}) : S(k_{(1)}).
\]
2) By item 1) to \( H^\text{cop} = (H^\text{cop}, :, :) \), taking into account that the antipode of \( H^\text{cop} \) is \( S^{-1} \).\( \square \)

Remark 5.10. By Lemma 5.9(1) every Hopf \( q \)-brace is regular.

Proposition 5.11. Let \( \mathcal{H} = (H, \cdot, :) \) be a Hopf \( q \)-brace. We have \( 1 \cdot h = \epsilon(h) 1 = 1 : h \), for each \( h \in H \).

Proof. By the first equality in [5.7] and identity [1.4],
\[
1 \cdot h = (1 \cdot (h_{(1)} : 1))(1 \cdot h_{(2)}) = (1 \cdot (h_{(2)} : 1))(1 \cdot h_{(1)}) = (1 \cdot h_{(2)})(1 \cdot h_{(1)}).
\]
So,
\[
\epsilon(h) 1 = (1 \cdot h_{(2)}) S(1 \cdot h_{(1)}) = (1 \cdot h_{(3)})(1 \cdot h_{(2)}) S(1 \cdot h_{(1)}) = 1 \cdot h.
\]
A similar argument proves that \( 1 : h = \epsilon(h) 1 \).\( \square \)

Remark 5.12. Let \( \mathcal{H} = (H, \cdot, :) \) be a Hopf \( q \)-brace. Then, from the fact that \( \cdot \) and \( : \) are coalgebra morphism from \( H \otimes H^\text{cop} \) to \( H \), it follows that \((H, :)\) and \((H, :)\) are right \( H^\text{cop} \)-module coalgebras.
Definition 5.13. Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf $q$-brace. If $H$ has bijective antipode, then we say that $\mathcal{H}$ is a Hopf $q$-brace with bijective antipode. (By Lemma 5.9, in this case, the requirement that $\mathcal{H}$ be left regular is superfluous).

Remark 5.14. If $\mathcal{H} = (H, \cdot, :)$ is a Hopf $q$-brace with bijective antipode, then $\mathcal{H}^{op} = (H^{op}, \cdot, :)$ is also. In fact, $\mathcal{H}^{op}$ is a Hopf algebra with antipode $S^{-1}$; by Remark 4.5 and 5.10, we know that $\mathcal{H}^{op}$ is a regular $q$-cycle coalgebra; and a direct computation using identity [1.4] shows that $\mathcal{H}^{op}$ also satisfies conditions [5.7].

Example 5.15. Let $n \geq 2$ and let $\xi \in \mathbb{C}$ be a root of unity of order $n$. Recall that the Taft algebra $T_n$ is the Hopf algebra generated as an algebra by elements $x$ and $g$ subject to the relations $g^n = 1$, $x^n = 0$ and $xg = \xi gx$. The comultiplication, counit and antipode are determined by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0, \quad S(g) = g^{-1}, \quad S(x) = -xg^{-1}.$$ 

A direct computation shows that $T_n$ is a Hopf $q$-brace with bijective antipode via

$$(g^jx^j) \cdot (g'^jx'^j) := \begin{cases} \xi^{-j}g^jx^j & \text{if } j' = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad (g^jx^j) : (g'^jx'^j) := \begin{cases} \xi^jg^jx^j & \text{if } j' = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Of all these algebras, only the Sweedler Hopf algebra $T_2$ is a Hopf skew-brace (see Definition 6.1).

Lemma 5.16. Let $\mathcal{H} = (H, \cdot, :)$ be a left regular $q$-magma coalgebra. Assume that $H$ is a Hopf algebra and let $s: H^2 \rightarrow H^2$ be the coalgebra map associated with $\mathcal{H}$. The following assertions hold:

1. $H$ is a right $H^{op}$-module via $\cdot$ if and only if $H$ is a right $H$-module via $k \otimes h \mapsto kh$.
2. If $h^1 = h$, then $h^1 = 1$.
3. If $h^1 = \epsilon(h)1$, for all $h \in H$, then $h^1 = \epsilon(h)1$, for all $h \in H$.
4. If $h^1 = \epsilon(h)1$, for all $h \in H$, then $h^1 = 1$.
5. If $H$ is a right $H$-module via $k \otimes h \mapsto kh$, then $h \cdot l = hS(l)$, for all $h, l \in H$.

Proof. 1) Suppose that $H$ is a right $H^{op}$-module via $\cdot$ if and only if $H$ is a right $H$-module via $k \otimes h \mapsto kh$.

2) If $h^1 = h$, then $h^1 = 1$.

3) This follows from the fact that $(1 \cdot h)^{h^2} = h^{h^1} = \epsilon(h)1$.

4) This follows from $h^1 = h^{h^1}$.

5) In fact, we have $h \cdot l = (hS(l))^{h^1} \cdot l = (hS(l))^{h^1} \cdot l = hS(l)$, as desired.

Let $H$ be a Hopf algebra and let $\mathcal{H} = (H, \cdot, :)$ be a left regular $q$-magma coalgebra. Let $s: H^2 \rightarrow H^2$ be as in Remark 1.18. By Remark 2.8, we know that $s$ is a left non-degenerate coalgebra endomorphisms of $H^2$.

Lemma 5.17. If the first identity in [5.7] holds, then $(l' \cdot h)^{l'} = (l' \cdot h)^{l'}$, for all $l, l', h \in H$.

Proof. By the first identity in [5.7], Remark 1.18 and conditions [1.1] and [1.2],

$$(l' \cdot h^{l^1}) \cdot h = (l' \cdot h^{l^1}) \cdot (l^2 \cdot h^{l^2}) \cdot h = (l' \cdot h^{l^1}) \cdot (l^2 \cdot h^{l^2}) \cdot h = l' \cdot e(h),$$

which, again by [1.2], implies the statement.

Let $H$ be a Hopf algebra, let $s: H^2 \rightarrow H^2$ be a left non-degenerate set-theoretic type solution of the braid equation and let $\cdot$ and $#$ be as in Notation 1.8. By Corollary 4.12, we know that $(H, \cdot, :)$ is a $q$-cycle coalgebra. We have the following result, which we will sometimes use without mentioning it.

Theorem 5.18. The following assertions are equivalent:

1. $(H, s)$ is a weak braiding operator.
2. $(H, s_2, s_1)$ is a matched pair.
3. $(H, \cdot, :)$ is a Hopf $q$-brace.
Proof. By Proposition 5.5 items 1) and 2) are equivalent. We next prove that items 2) and 3) are also. Assume that \((H, s_2, \kappa_1)\) is a matched pair. By item 4) above Theorem 5.4 and Lemma 5.16(1), the Hopf algebra \(H\) is a right \(H^{op}\)-module via \(\cdot\). By item 4) above Theorem 5.4, and equality [1.4], we have
\[
((h \cdot (l_1) : k_2))(k_1 \cdot l_2))^{l_3} = ((h \cdot (l_1) : k_3))^{h_{(1)}(l_3)}(k_2 \cdot l_2)^{l_3} = ((h \cdot (l_1) : k_3))^{l_3}(k_2 \cdot l_2)^{l_3} = ((h \cdot (l_1) : k_3))^{(l_2)(k_2)}(k_1 \cdot l_3)^{l_4} = h(k(l)),
\]
which by [1.2] implies that the first equality in [5.7] is fulfilled. We next prove that \(H\) is a right \(H^{op}\)-module via \(\cdot\). By Lemma 5.16(4) and items 2) and 5) above Theorem 5.4, we have \(h : 1 = h\), for all \(h \in H\). Moreover,
\[
h : kl = k(h_{(1)}h_{(2)}l) = (k(h_{(1)}l_{(2)}))(l_{(1)}h_{(2)}h_{(3)}) = k(h_{(1)}l_{(2)})(h_{(2)} : l_{(1)}) = (h : l) : k,
\]
as desired. By Corollaries 4.12 and 5.6, we know that \((H, \cdot, \cdot)\) is a \(q\)-cycle coalgebra. In order to finish the proof that \((H, \cdot, \cdot)\) is a Hopf \(q\)-brace it remains to check the second equality in [5.7]. But, by the fact that \(H\) is a right \(H^{op}\)-module via \(\cdot\), and items 3) and 6) above Theorem 5.4, we have
\[
hk : l = (l_{(k)}k_{(1)}h_{(2)}h_{(3)}) = (l_{(k)}(k_{(2)}h_{(3)})) = (k_{(2)}h_{(3)}h_{(4)}) = (h : l_{(2)} : k_{(1)}) = (h : l) : k,
\]
which by equality [1.4] gives the desired formula. Next we prove the converse implication. By Corollary 4.12, the map \(s\), associated with \((H, \cdot, \cdot)\), is a left non-degenerate set-theoretic type solution of the braid equation. Hence, by Remark 1.1, the maps \(s_1\) and \(s_2\) are coalgebra maps and condition 6) above Theorem 5.4 is satisfied. By Lemma 5.16(1), the Hopf algebra \(H\) is a right \(H\)-module via \(s_2\), and so item 1) above Theorem 5.4 also holds. By Lemma 5.17, item 4) above Theorem 5.4 holds. Next we end the proof of item 2) above Theorem 5.4. By Proposition 5.11 and items 2), 3) and 4) of Lemma 5.16, we know that \(h^1 = h = h \cdot 1\), \(h = h \cdot 1\) and \(1 = h : 1 = h\). Moreover, by Remark 1.18, item 4) above Theorem 5.4 and the fact that \(H\) is an \(H^{op}\)-module via \(\cdot\), we have
\[
khl = (h_{(2)} : (khl_{(2)}) = (h_{(3)} : (l_{(2)}hl_{(3)})) = (h_{(3)} : (l_{(2)}h_{(3)})) = (h_{(3)} : (l_{(2)}h_{(3)})) = (h_{(3)} : (l_{(2)}h_{(3)})) = (h_{(3)} : (l_{(2)}h_{(3)})),
\]
which by item 6) gives item 3) above Theorem 5.4.

Remark 5.19. By Corollary 4.12 and Theorem 5.18, the categories of weak braiding operators; Hopf \(q\)-braces; and matched pairs \((H, s_1, s_2)\), such that \(s\) is a solution of the braid equation, are isomorphic.

Remark 5.20. In the set-theoretic context, the equivalence between items 2) and 3) of Theorem 5.18 is mentioned in the remark below Definition 7 in [25].

Corollary 5.21. Each Hopf \(q\)-brace \(H\) is non-degenerate.

Proof. By Remark 5.10 and Proposition 2.17 it suffices to check that \(H\) is right non-degenerate. That is, that the map \(H \rightarrow H\), introduced in Definition 1.22, is bijective. But by Theorem 5.18 and item 2) above Theorem 5.4, the map \(H \rightarrow H\) is invertible with inverse \(h \otimes k \mapsto S(k_{(2)})h \otimes k_{(1)}\).

Remark 5.22. In the set-theoretic context, Corollary 5.21 corresponds to the last assertion in [25, Proposition 6].
Remark 5.23. Let \((H, s)\) be a weak braiding operator. By Corollary 5.6, we know that \(s\) is left non-degenerate. Let \(\mathcal{H} = (H, \cdot, \cdot)\) be the Hopf \(q\)-brane associated with \((H, s)\), according to Theorem 5.18. By Corollaries 4.12 and 5.21, the map \(s\) is bijective and non-degenerate. By Remarks 1.18 and 1.19, we know that \(s^{-1} = \mathcal{G}_H\mathcal{G}_H^{rev}\). Combining this with Remark 1.20, we obtain

\[
s^{-1}(h \otimes l) = \mathcal{G}_H(l_{(1)} \otimes h_{(1)}) = (l_{(1)} \cdot h_{(1)}{_{(2)}} \otimes h_{(2)}{_{(3)}}) = l_{(1)} \cdot (h_{(1)} : S(l_{(2)})) \otimes h_{(2)} : S(l_{(3)}),
\]

where the last equality follows from Lemma 5.9(1). Moreover, it is easy to see that \((H, s^{-1})\) is also a weak braiding operator. Consequently, \((H^{op, s}, s^{-1})\) is a weak braiding operator too (see Remark 5.3). Note that, by Remark 2.12 we have \(s^{-1} = \tilde{s}\), where \(\tilde{s}\) is as in Remark 1.20. Thus \(\tilde{s}^{-1}(h \otimes l) = l_{(2)}h_{(2)} \otimes l_{(1)}h_{(1)}\), and so, analogous properties to the ones given above Theorem 5.4 are satisfied for the maps \(h \otimes l \mapsto \tilde{h}\) and \(h \otimes l \mapsto \tilde{l}\).

Corollary 5.24. Let \(\mathcal{H} = (H, \cdot, \cdot)\) be a Hopf \(q\)-brane and let \((H, s)\) be the weak braiding operator associated with \(\mathcal{H}\) according to Theorem 5.18. If \(H\) has bijective antipode, then the solution \(\tilde{s}\) is left non-degenerate and \(\mathcal{H}_{\tilde{s}} := (H^{op, \tilde{s}}, \tilde{s})\) is a Hopf \(q\)-brane with bijective antipode.

Proof. Since \((h^{(2)})^{-1}(k_{(1)}) = h^{(2)}(k_{(1)}) = \epsilon(k)h\) for all \(h, k \in H\), the solution \(\tilde{s}\) is left non-degenerate (see Remark 1.9). Moreover, clearly \((H^{op, \tilde{s}}, \tilde{s})\) is a weak braiding operator, since \((H, s)\) is. The result follows now immediately from Theorem 5.18.

Remark 5.25. Let \(H\) be a Hopf algebra, let \(s : H^2 \to H^2\) be a left non-degenerate coalgebra endomorphism and let \(\cdot\) and \(:\) be as in Notation 1.8. By Remark 2.8, we know that \((H, \cdot, :)\) is a left regular \(q\)-magma coalgebra. The proof of Theorem 5.18 shows that the following facts are equivalent:

1) \(s\) satisfies conditions [5.1], [5.2], [5.3] and [5.4].
2) \((H, s, \Delta)\) is a matched pair.
3) \((H, \cdot)\) and \((H, :)\) are right \(H^{op}\)-modules and conditions [5.7] are satisfied.

Moreover, arguing as in the proof of Corollary 5.21, we obtain that if the conditions in items 3) are satisfied, then \((H, \cdot, :)\) is non-degenerate. Thus, by Proposition 2.20, the map \(s\) is invertible and non-degenerate. The rest of Remark 5.23 and Corollary 5.24 can also be generalized to this context.

The following proposition is taken from [10, Lemma 3.6].

Proposition 5.26. Let \(H\) be a Hopf algebra and let \(s_1, s_2 : H^2 \to H^2\) be two maps. Write \(h^{k} := s_1(h \otimes k)\) and \(h^{k} := s_2(h \otimes k)\). Assume that condition [5.5] is fulfilled. Then, we have:

1) If \(s_1\) is a coalgebra map, then \(s_2\) is a coalgebra map if and only if condition [1.1] is satisfied.
2) If \(s_2\) is a coalgebra map, then \(s_1\) is a coalgebra map if and only if condition [1.1] is satisfied.

Proof. 1) Since \(h^{(1)}k_{(1)}h^{(2)}k_{(2)} = hh\), we have \(h^{k} = S(h^{(1)}k_{(1)})h^{(2)}k_{(2)}\). Thus,

\[
e(h^{k}) = \epsilon(h)\epsilon(k) \quad \text{and} \quad (h^{k})_{(1)} \otimes (h^{k})_{(2)} = (h_{(1)}^{k_{(1)}}) \otimes h_{(2)}^{k_{(2)}}
\]

if and only if

\[
S(h^{(2)}k_{(2)})h_{(3)}k_{(3)} \otimes S(h^{(1)}k_{(1)})h_{(4)}k_{(4)} = S(h^{(1)}k_{(1)})h^{(2)}k_{(2)} \otimes S(h^{(2)}k_{(3)})h_{(4)}k_{(4)}.
\]

But the last equality is fulfilled if and only if \(h_{(2)}^{k_{(2)}} \otimes S(h^{(1)}k_{(1)}) = h_{(1)}^{k_{(1)}} \otimes S(h^{(2)}k_{(2)})\). Hence from condition [1.1] it follows that \(s_2\) is a coalgebra map. Suppose now that this happens. Then,

\[
h_{(1)}^{k_{(1)}}h_{(2)}^{k_{(2)}} \otimes h_{(3)}^{k_{(3)}}h_{(4)}^{k_{(4)}} = h_{(1)}^{k_{(1)}}h_{(2)}^{k_{(2)}}
\]

\[
= (hk)_{(1)} \otimes (hk)_{(2)}
\]

\[
= \left(h_{(1)}^{k_{(1)}}h_{(2)}^{k_{(2)}}ight)_{(1)} \otimes \left(h_{(1)}^{k_{(1)}}h_{(2)}^{k_{(2)}}ight)_{(2)}
\]

\[
= h_{(1)}^{k_{(1)}}h_{(3)}^{k_{(3)}} \otimes h_{(2)}^{k_{(2)}}h_{(4)}^{k_{(4)}}
\]

Hence \(h_{(1)}^{k_{(1)}} \otimes h_{(2)}^{k_{(2)}} = h_{(3)}^{k_{(3)}} \otimes h_{(4)}^{k_{(4)}}\), as desired.

The following is the version for Hopf algebras of [21, Theorem 1].

Theorem 5.27. Let \(H\) be a Hopf algebra, let \(s_1, s_2 : H \otimes H \to H\) two maps and let \(s := (s_1 \otimes s_2)\Delta_H\). Assume that \(H\) is a left \(H\)-module coalgebra via \(s_1\) and a right \(H\)-module coalgebra via \(s_2\). If

\[
h_{(1)}^{l_{(1)}}h_{(2)}^{l_{(2)}} = hl \quad \text{for all} \ h, l \in H,
\]

where \(h^l := s_1(h \otimes l)\) and \(h^l := s_2(h \otimes l)\), then \((H, s)\) is a braiding operator. In particular \(s\) is a bijective solution of the braid equation.
Proof. To begin note that, since \( s_1 \) and \( s_2 \) are coalgebra morphisms, \((id \circ \varepsilon) s = s_1 \) and \((\varepsilon \circ id) s = s_2 \). Moreover, by Proposition 5.26, identity \([1.1] \) is fulfilled. Hence, \( s \) is a coalgebra morphism, by Remark 1.1(3). We now prove that \( s \) is a solution of braid equation. Let \( A := s_{12} s_{23} s_{12} \) and \( B := s_{23} s_{12} s_{23} \). Since \( A \) and \( B \) are coalgebra maps we know that \( A = B \) if and only if \( A_i = B_i \), for \( i \in \{1, 2, 3\} \). A direct computation using that \( A_1 = s_1 (H \circ s_1) s_{12}, B_1 = s_1 (H \circ s_1) \) and \( \mu s = \mu \), shows that

\[
A_1 (h \circ l \circ k) = h_1 (h_2 (h_3 (k_3)) k) = \mu \mu (h_3 (k_3)) k = \mu k = h_1 (k) = B_1 (h \circ l \circ k),
\]
and a similar computation proves that \( A_3 = B_3 \). Using again that \( \mu s = \mu \), we obtain that

\[
A_1 (hl \otimes k) = h_1 (h_2 (h_3 (k_3)) k) = \mu \mu (h_3 (k_3)) k = \mu k = h_1 (k) = B_1 (h \otimes l \circ k).
\]

By definition and Lemma 5.2, these arguments prove identities \([5.30] \).

Let \( H = s(\mu \circ H) \). Hence the map \( h \otimes l \circ k \) is right inverse of \((p_i \otimes H)(H \circ \Delta H) \). Thus, by Remark 3.2(3), we have \( h^{S_2} = \frac{h}{h} \). The others claim follow similarly.

Corollary 5.29. Let \( H = (H, \cdot, \cdot) \) be a Hopf q-brace with bijective antipode. For all \( i \in \mathbb{Z} \) and \( h, k \in H \),

\[
h^{S_2} (k) = h^{S_2} (k) = h \cdot k, \quad h^{S_2} (k) = h^{S_2} (k) = h \cdot k, \quad h^{S_2} (k) = h^{S_2} (k) = h \cdot k.
\]

Proof. By Proposition 5.28 and identity \([1.2] \), hence the map \( h \circ k \mapsto h^{S_2} (k) \) is right inverse of \((p_i \otimes H)(H \circ \Delta H) \). Thus, by Remark 3.2(3), we have \( h^{S_2} = \frac{h}{h} \). Other claims follow similarly.
Proposition 5.31. Let \( \mathcal{H} = (H, \cdot, : ) \) be a Hopf \( q \)-brace. For all \( j \in \mathbb{N} \), we have
\[
S^j(h) \cdot k = S^j ( h(i_1) \cdot \left( \cdots ( k : S^j(h(i_2)) : S^j-1(h(i_3)) \cdots ) : S^2(h(i_j)) \right) : S(h(i_{j+1})) \)
and
\[
S^j(h) : k = S^j ( h(i_1) \cdot \left( \cdots ( k : S^j(h(i_2)) : S^j-1(h(i_3)) \cdots ) : S^2(h(i_j)) \right) : S(h(i_{j+1})) ,
\]
where
\[
(i_1, \ldots, i_{j+1}) := \begin{cases} (n + 1, 2n + 1, 2, 2n, 3, \ldots, n - 1, n + 3, n, n + 2) & \text{if } j = 2n, \\ (n + 1, 2n + 2, 2n + 1, 2, 2n, \ldots, n - 1, n + 3, n, n + 2) & \text{if } j = 2n + 1. \end{cases}
\]
For example, the tuples obtained from this definition for \( j = 1, 2, 3, 4, 5 \) and \( 6 \) are
\[
(1, 2), \ (2, 1, 3), \ (2, 4, 1, 3), \ (3, 1, 5, 2, 4), \ (3, 6, 1, 5, 2, 4) \quad \text{and} \quad (4, 1, 7, 2, 6, 3, 5),
\]
respectively.

Proof. By Proposition 5.11, the first equality in [5.7] and equality [1.4], we have,
\[
\epsilon(hk)1 = \epsilon(h)1 \cdot k = S(h(1))h(2) \cdot k = (S(h(1)) : (k : h(2)))(h(3) : k(1)).
\]
Therefore
\[
S(h \cdot k) = \epsilon(h(1)k(2))S(h(2) \cdot k(1)) = (S(h(1)) : (k : h(2)))(h(3) : k(2))S(h(4) : k(1)) = S(h(1)) : (k : h(2)), \quad [5.11]
\]
and so,
\[
S(h) \cdot k = S(h(1)) \cdot ((k : S(h(3)) : h(2)) = S(h(1)) : (k : S(h(2))), \quad [5.12]
\]
which is the first formula for \( j = 1 \). Applying now this equality twice we obtain that
\[
S^2(h) \cdot k = S(S(h(1)) : (k : S(h(2)))) = S(S(h(2)) : (k : S^2(h(1)))) = S^2(h) : ((k : S^2(h(1)) : S(h(3))),
\]
which is the first formula for \( j = 2 \). An inductive argument following these lines proves the general case for the first formula. The formula for \( S^j(h) : k \) can be proved in the similar way. For illustration we give the proof for \( j = 1 \). Arguing as above, from Proposition 5.11, the second equality in [5.7] and equality [1.4], we obtain
\[
S(h : k) = S(h(1)) : (k : h(2)). \quad \text{Hence, from equality [3.2] with \( i = 1 \), we obtain}
\]
\[
S(h) : k = S(h(1)) : ((k : S(h(3)) : h(2)) = S(h(1)) : (k : S(h(2))), \quad [5.13]
\]
as desired.

Proof. Apply Proposition 5.31 to \( H^{\text{cop}} \), which is a Hopf \( q \)-brace by Remark 5.14.

Proposition 5.32. Let \( \mathcal{H} \) be a Hopf \( q \)-brace with bijective antipode. For all \( j \in \mathbb{N} \), we have
\[
S^{-j}(h) \cdot k = S^{-j} ( h(i_1) \cdot \left( \cdots ( k : S^{-j}(h(i_2)) : S^{-j+1}(h(i_3)) \cdots ) : S^{-2}(h(i_j)) \right) : S^{-1}(h(i_{j+1})) \)
and
\[
S^{-j}(h) : k = S^{-j} ( h(i_1) \cdot \left( \cdots ( k : S^{-j}(h(i_2)) : S^{-j+1}(h(i_3)) \cdots ) : S^{-2}(h(i_j)) \right) : S^{-1}(h(i_{j+1})) ,
\]
where
\[
(i_1, \ldots, i_{j+1}) := \begin{cases} (n + 1, 2n + 1, 2n, 2n - 1, 3, \ldots, n + 3, n - 1, n + 2, n) & \text{if } j = 2n, \\ (n + 2, 1, 2n + 2, 2n + 1, 3, \ldots, n + 4, n, n + 3, n + 1) & \text{if } j = 2n + 1. \end{cases}
\]
For example, the tuples obtained from this definition for \( j = 1, 2, 3, 4, 5 \) and \( 6 \) are
\[
(2, 1), \ (2, 3, 1), \ (3, 1, 4, 2), \ (3, 5, 1, 4, 2), \ (4, 1, 6, 2, 5, 3) \quad \text{and} \quad (4, 7, 1, 6, 2, 5, 3),
\]
respectively.

Proof. Apply Proposition 5.31 to \( H^{\text{cop}} \), which is a Hopf \( q \)-brace by Remark 5.14.

Proposition 5.33. Let \( \mathcal{H} \) be a Hopf \( q \)-brace and let \( (H, s) \) be the weak braiding operator associated with \( \mathcal{H} \). Then \( s \) is bijective and
\[
\begin{array}{ccc}
H^{\text{cop} \otimes} H^{\text{cop}} & \xrightarrow{s^{-1}} & H^{\text{cop} \otimes} H^{\text{cop}} \\
\downarrow{s \otimes s} & & \downarrow{s \otimes s} \\
H \otimes H & \xrightarrow{s} & H \otimes H
\end{array}
\]
is a commutative diagram of coalgebra morphisms.
Proof. We already know that all the maps are coalgebra morphisms. Moreover, by Remark 5.23, the map $s$ is bijective. Since, by Remark 5.23, we know that $S^{-1}(l \otimes l) = l_{(2)}h_{(2)} \otimes l_{(1)}h_{(1)}$, to prove that the diagram commutes, we must check that

$$S(h_{(1)})S(l_{(1)})S(h_{(2)}) = S(l_{(2)})S(h_{(1)})$$

for all $h, l \in H$.

By Remark 1.1, for this it suffices to check that $S(h)S(l) = S(l)S(h)$. But,

$$S(h)S(l) = S(l)S(h) = S(l_{(1)})S(h_{(1)})S(l_{(2)})S(h_{(2)}) = S(l_{(1)})S(h_{(1)})S(l_{(2)})S(h_{(2)}) = S(l_{(1)})S(h_{(1)})S(l_{(2)})S(h_{(2)})$$

where the first equality holds by Remark 1.18; the second one, by Lemma 5.16(5); the fourth one, by the second equality in Proposition 5.31 with $j = 1$ (which is [5.13]); and the last one, by Lemma 5.9(1). Finally,

$$S(lh) = S(h_{(1)})S(l_{(2)})S(l_{(1)})S(h_{(2)}) = S(l_{(1)})S(h_{(1)})S(l_{(2)})S(h_{(2)}) = S(l_{(1)})S(h_{(1)})S(l_{(2)})S(h_{(2)})$$

where the first equality holds by Remark 1.20; the second one, by Lemma 5.9(1); the third one, by the second equality in Proposition 5.31 with $j = 1$ (which is [5.12]); and the last one, by Lemma 5.16(5).

Corollary 5.34. Let $\mathcal{H}$ be a Hopf $q$-brace and let $(H, s)$ be the weak braiding operator associated with $\mathcal{H}$. Then $s(S^2 \otimes S^2) = (S^2 \otimes S^2)s$.

Proof. Proposition 5.33, applied to $(H^{op \cdot cop}, \tilde{s}_r^{-1})$, shows that $\tilde{s}_r^{-1}(S \otimes S) = (S \otimes S)s$. The result follows easily from this and Proposition 5.33.

Proposition 5.35. Each Hopf $q$-brace $\mathcal{H} = (H, \cdot, _\ast)$, with bijective antipode, is strongly regular.

Proof. By Proposition 5.28 and Remark 5.30, both $\mathcal{H}$ and $\overline{\mathcal{H}} := (H^{op \cdot cop}, \tilde{s}_r \circ \triangleright)$ are strongly regular. To finish the proof we must check identities [3.9]–[3.12]. We prove the first two equalities and leave the remaining ones to the reader. Since $S$ is bijective, in order to check [3.9], it suffices to note that

$$(S^{2i} \otimes id)(y_{(1)} \cdot_{i} x_{(2)} \otimes x_{(1)} \cdot_{i} y_{(2)}) = (S^{2i} \otimes id)(y_{(1)} \cdot S^{-2i}(x_{(2)} \otimes x_{(1)} \cdot S^{2i}(y_{(2)})))$$

where the first and last equalities hold by Proposition 5.28; the second and fourth, by Corollary 5.34; and the third one, by [1.4]. Similarly, in order to check [3.10], it suffices to note that

$$(S^{2i} \otimes id)(y_{(2)} \otimes x_{(1)} \cdot_{i} x_{(2)} \cdot_{i} y_{(1)}) = (S^{2i} \otimes id)(S^{-2i}(x_{(2)} \otimes x_{(1)} \cdot S^{2i}(y_{(1)})))$$

where the first and last equalities hold by Remark 5.30; the second and fourth, by Corollary 5.34; and the third one, by Remark 1.1(3).

Let $H$ be a Hopf algebra endowed with binary operations $\cdot$ and $\ast$. We write

$$h \times k := (k \cdot h_{(1)})h_{(2)} \quad \text{and} \quad h \star k := (k : h_{(2)})h_{(1)}.$$

Note that

$$1 \times k = k, \quad 1 \star k = k, \quad h \times 1 = (1 : h_{(1)})h_{(2)} \quad \text{and} \quad h \star 1 = (1 : h_{(1)})h_{(2)}.$$

Proposition 5.36. Let $\mathcal{H} = (H, \cdot, _\ast)$ be a $q$-magma coalgebra, where $H$ is a Hopf algebra and let $\times$ and $\star$ be as in [5.14]. If $\mathcal{H}$ is a Hopf $q$-brace, then $\times$ and $\star$ are associative with unit 1, and the following equalities are satisfied for all $h, k, l \in H$:

$$h \cdot kl = (h \cdot l) \cdot k, \quad (k \star l) \cdot h = (k \cdot h_{(2)}) \star (l \cdot h_{(1)}), \quad h \cdot (k \times l) = h \cdot (l \star k),$$

$$h \cdot kl = (h \cdot l) \cdot k, \quad (k \times l) : h = (k : h_{(1)}) \times (l : h_{(2)}), \quad h : (k \times l) = h : (l \star k).$$

Conversely, if $\times$ and $\star$ are associative with left unit 1, the first and third conditions in [5.16] and the three conditions in [5.17] are satisfied, then $\mathcal{H}$ is a Hopf $q$-brace.
Proof. By the definition of Hopf $q$-bracket, the first two equalities in [5.15] and Lemma 5.9(1), we can assume that $\mathcal{H}$ is a regular $q$-magma coalgebra, 1 is a left unit of $\times$ and $\rhd$, the equalities in the first column of identities [5.16] and [5.17] are satisfied, and both $(H, \cdot )$ and $(H, :)$ are right $H^{\text{op}}$-modules. Moreover, by the equalities in the first column of [5.16] and [5.17], and the very definitions of $\times$ and $\rhd$, we have

$$h \cdot (k \times l) = h \cdot (k \cdot (l_1))(k_2) = (h \cdot k_2) \cdot (l_1 k_1),$$

$$h \cdot (l \rhd k) = h \cdot (k : l_1) = (h \cdot k_1) \cdot (l_2 k_2),$$

and

$$h : (k \times l) = (h \cdot k_2) \cdot (l_1 k_1).$$

Therefore, the equalities in the third column of [5.16] and [5.17] are equivalent to items 1) and 3) of Definition 4.4. Assume now that these conditions hold. We have

$$(h \times k) \times (l \times k) = ((h \times k)(1))(h \times k)(2) = (l \cdot (h_1 \cdot k_2)h_3)(k_2 \cdot h_1)h_4$$

and

$$h \times (k \times l) = (((k \times l) \cdot h_1)h_2) = (((k \cdot l_1)(k_2) \cdot h_1)h_2).$$

Since, by the equality in the first column of [5.16] and item 1) of Definition 4.4,

$$(l \cdot (k_1 : h_2)h_3)(k_2 : h_1) = (((l \cdot h_3)(k_1 : h_2))(k_2 : h_1) = (((l \cdot h_3) \cdot (h_2 : k_2))(k_3 : h_1),$$

the operation $\times$ is associative if and only if

$$(l \cdot (k_1 : h_2)h_3)(k_2 : h_1) = (((l \cdot k_1)k_2) : h) \text{ for all } h, k, l \in H.$$ 

But, by identity [1.4] and the fact that $\mathcal{H}$ is a left regular $q$-magma coalgebra, this happens if and only if the first equality in [5.7] holds. A similar argument proves that $\rhd$ is associative if and only if the second equality in [5.7] holds. Assume now that these conditions are also satisfied. Then

$$(k : h_1 \times (l : h_2)) = ((l : h_2) \cdot (k : h_1))(h : h_2)(2) = ((l : h_3) \cdot (k_2 : h_1))(k_2 : h_1)$$

and

$$(k \times l) : h = (((l \cdot k_1)k_2) : h = (((l \cdot k_1) : (h_2 : k_3)) \cdot (k_2 : h_1) = (((l \cdot k_3) : (h_2 : k_2))(k_3 : h_1),$$

where in the last equality we have used [1.4]. So, the second equality in [5.17] is satisfied if and only if the second identity in Definition 4.4 is fulfilled (use $(k_1 : h_2)S(k_2 : h_1) = (h_1) = 1$). Finally, by the very definition of $\rhd$ and the first equality in [5.7]

$$(k \rhd l) \cdot h = ((l : k_2)(k_1) : h = ((l : h_3) : (k_1 : h_2))(k_1 : h_2)$$

and

$$(k \rhd l)(1) \cdot (k \rhd l)(2) = ((l : k_1)(k_2)(k_2) \cdot (h_1) : (k_1 : h_2))(k_1 : h_2),$$

and consequently, the second identity in Definition 4.4 implies the second equality in [5.16]. Finally, by Proposition 5.11 and the third and fourth equalities in [5.15], if $\mathcal{H}$ is a Hopf $q$-bracket, then 1 is a unit of $\times$ and $\rhd$.

Remark 5.37. When $H$ is a group algebra, Proposition 5.36 yields [25, Proposition 5], where $+, \# , 0$ correspond to $\times, \rhd, 1$, respectively.

Corollary 5.38. If $\mathcal{H} = (H, \cdot , :)$ is a Hopf $q$-bracket with bijective antipode, then

$$k \cdot (S(h_1) \cdot S^{-1}(h_2)) = k \cdot (S^{-1}(h_2) \cdot S(h_1))$$

and

$$k : (S^{-1}(h_2) : S(h_1)) = (S(h_1) \cdot S^{-1}(h_2)),$$

for all $h, k \in H$.

Proof. By Remark 5.14, it suffices to prove the first equality. Using that $H$ is a right $H^{\text{op}}$-module via $\cdot$ and the third identity in [5.16], we get

$$k \cdot (S(h_1) \cdot S^{-1}(h_2)) = k \cdot (S(h_1) \cdot S^{-1}(h_4))S^{-1}(h_3)h_2$$

$$= (k \cdot h_2) \cdot (S^{-1}(h_3) \cdot S(h_1))$$

$$= (k \cdot h_2) \cdot (S(h_1) \cdot S^{-1}(h_3))$$

$$= k \cdot (S^{-1}(h_4) : S(h_1))S(h_2)h_3$$

$$= k \cdot (S^{-1}(h_4) : S(h_1)),$$

as desired.

\[\square\]
Notation 5.39. Given a $k$-vector space $H$, endowed with a coassociative and counitary comultiplication $\Delta$ and an associative and unitary multiplication $\circ$, we let $\text{End}_{\Delta,0}(H)$ denote $\text{End}_k(H)$, endowed with the convolution product associated with $\Delta$ and $\circ$.

The following lemma is inspired by equality (6) of [10].

Lemma 5.40. If $(H,s)$ is a weak braiding operator, then $k^{S(S(h_{(1)})h_{(2)})} = k^{S(h_{(1)})h_{(2)}}$, for all $k, h \in H$. 

Proof. Since $s$ is a solution of the braid equation and $(H, s_2, s_1)$ is a matched pair (see Theorem 5.18), we have $s_2(H \otimes \mu) s_{23} = s_2(s_2 \otimes H) s_{23} = (\epsilon \otimes s_2 \otimes H) s_{12} s_{23} = (\epsilon \otimes \epsilon \otimes H) s_{12} s_{23} s_{12} = s_2(H \otimes \mu)$. In other words

$$k^{S(h_{(1)}h_{(2)})} = k^{h_{(1)}h_{(2)}},$$

and so $k^{h_{(1)}h_{(2)}} = k^{h_{(2)}h_{(2)}}$. Hence, $k^{S(h_{(1)})} = k^{S(h_{(1)})S(h_{(1)})},$ and consequently,

$$k^{S(h_{(1)})} = k^{S(h_{(3)})S^2(h_{(2)})S^{(h_{(1)})}h_{(4)}} = k^{S(h_{(1)})h_{(2)}},$$

as desired. \hfill $\square$

Proposition 5.41. If $\mathcal{H} = (H, \cdot, \circ)$ is a Hopf $q$-brace, then $\text{id}_H$ is invertible in $\text{End}_{\Delta^{op}, \circ}(H)$ and its inverse is the map $T_x(h) := S(h_{(1)})h_{(2)}$. Moreover, for all $h, k, l \in H$,

$$(k \times l) \cdot h = (k \cdot h_{(1)}) \times (l \cdot h_{(2)}) \quad \text{and} \quad (k \otimes l) \cdot h = (k \cdot h_{(2)}) \otimes (l \cdot h_{(1)}).$$

Finally, if the antipode $S$, of $H$, is bijective, then $\text{id}_H$ is invertible in $\text{End}_{\Delta, \circ}(H)$ and its inverse is the map $T_x(h) := S^{-1}(h_{(2)})h_{(1)}$.

Proof. By [5.14], we have

$$h_{(2)} \times T_x(h_{(1)}) = (S(h_{(1)})h_{(2)})h_{(3)}h_{(4)} = S(h_{(1)})h_{(2)} \in H,$$

for all $h \in H$, which proves that $T_x$ is right inverse of $\text{id}_H$ in $\text{End}_{\Delta^{op}, \circ}(H)$. On the other hand, we have

$$T_x(h_{(2)}) \times h_{(1)} = (h_{(1)} \cdot (S(h_{(2)})h_{(3)})h_{(4)}) (S(h_{(2)})h_{(5)})h_{(6)} = h_{(1)}S(h_{(3)}h_{(6)})S(h_{(2)})h_{(5)},$$

where the second equality holds by Lemma 5.16(5); and we have

$$\epsilon(h_{(1)}) = (S(h_{(2)})h_{(3)})h_{(4)} = (S(h_{(2)})h_{(3)}h_{(4)})h_{(5)} = (S(h_{(3)})h_{(4)})S(h_{(2)})h_{(5)},$$

where the first equality holds by Proposition 5.11 and Lemma 5.16(3); and the second one, by Theorem 5.18 and item 4) above Theorem 5.4. By Lemma 5.40, this implies that $T_x$ is left inverse of $\text{id}_H$ in $\text{End}_{\Delta^{op}, \circ}(H)$. In order to check the first equality in [5.18], we note that, by the first equality in [5.7] and equality [1.4],

$$(k \times l) \cdot h = ((l \cdot k_{(1)})k_{(2)} \cdot h = ((l \cdot k_{(1)}) \cdot (h_{(1)} : k_{(3)}))(k_{(2)} \cdot h_{(2)}) = ((l \cdot k_{(1)}) \cdot (h_{(2)} : k_{(2)}))(k_{(3)} \cdot h_{(1)}).$$

Thus, by item 1) of Definition 4.4, we have

$$(k \times l) \cdot h = ((l \cdot h_{(3)}) (k_{(1)} : h_{(2)}) \cdot h_{(1)}) = (k \cdot h_{(1)}) \cdot (l \cdot h_{(2)}),$$

as desired. The second equality in [5.18] follows by a similar argument, and the last assertion holds by Lemma 5.9 and Remark 15.4. \hfill $\square$

Remark 5.42. In the set-theoretic context, Proposition 5.41 together with Corollary 5.21, yield [25, Proposition 6].

Definition 5.43. A pair $(C, \cdot)$, consisting of a coalgebra $C$ and a coalgebra morphism $c \otimes d \to c \cdot d$, from $C \otimes C^{cop}$ to $C$, is a regular magma coalgebra if there exists a map $c \otimes d \to c^d$, from $C^2$ to $C$, such that the identities in [1.2] are satisfied.

Lemma 5.44. Let $(H, \cdot)$ be a regular magma coalgebra endowed with a distinguished group-like element 1 and maps $\times : H^2 \to H$ and $T_x : H \to H$, such that $\times$ is an associative operation with identity 1. Assume that, for all $h, k, l \in H$, the following equalities hold:

1) $(k \times l) \cdot h = (k \cdot h_{(1)}) \times (l \cdot h_{(2)}),$
2) $h \cdot (l_{(2)} \times k_{(1)}) = (h \cdot l) \cdot k,$
3) $h_{(2)} \times T_x(h_{(1)}) = \epsilon(h)1.$

Then, $h \cdot 1 = h^1 = h$ and $1 \cdot h = 1^h = \epsilon(h)1$, for all $h \in H$. 

Proof. Since the map \( h \mapsto h \cdot 1 \) is bijective (with inverse \( h \mapsto h^1 \)) and \( h \cdot 1 = (h \times 1) \cdot 1 = (h \cdot 1) \times (1 \times 1) \) for all \( h \in H \), we have \( 1 \cdot 1 = 1 \cdot 1 = 1 \cdot 1 \), which implies that \( 1 = 1 \). Thus, \( h \cdot 1 = h \cdot h^1 = h \cdot (1 \times 1) = (h \cdot 1) \cdot 1 \), and so, \( h \cdot 1 = h \), for all \( h \in H \) (which implies that \( h \times 1 = h \), for all \( h \in H \)). On the other hand, 
\[
1 \cdot h = (1 \times 1) \cdot h = (1 \cdot h(1)) \times (1 \cdot h(2)) \quad \text{for all } h \in H,
\]
and, consequently,
\[
\epsilon(h)1 = \epsilon(1 \cdot h)1 = (1 \cdot h(1)) \times T_s(1 \cdot h(2)) = (1 \cdot h(1)) \times (1 \cdot h(2)) \times T_s(1 \cdot h(3)) = 1 \cdot h.
\]
Therefore, we also have \( \epsilon(h)1 = (1 \cdot h(1))h^2 = 1^h \), which finishes the proof. \( \square \)

**Lemma 5.45.** Let \((H, \cdot)\) be a regular magma coalgebra endowed with a distinguished group-like element \( 1 \) and maps \( \times : H^2 \rightarrow H \) and \( T_s : H \rightarrow H \), such that:

1) the map \( h \otimes k \mapsto hk := k(2) \times h^k(1) \) is a coalgebra morphism from \( H^2 \) to \( H \);
2) \( \times \) is an associative operation with identity \( 1 \);
3) the map \( h \mapsto S(h) := T_s(h(1)) \cdot h(2) \) is a coalgebra antimorphism of \( H \).

If, for all \( h, k, l \in H \) the following equalities hold:

4) \( (k \times l) \cdot h = (k \cdot h(1)) \times (l \cdot h(2)) \),
5) \( h \cdot kl = (h \cdot l) \cdot k \),
6) \( T_s(h(2)) \times h(1) = h(2) \times T_s(h(1)) = \epsilon(h)1 \),

then \( H \) becomes a Hopf algebra via \( \mu(h \otimes k) := hk \). The unit of \( H \) is \( 1 \) and the antipode of \( H \) is \( S \).

**Proof.** By items 1) and 5), for all \( h, k, l \in H \), we have
\[
(h^{k_1}, l_1) \times (k_2, l_2) = h^{k_1, l_1} \times k_2, l_2) = h^{k_1, l_1} \times (k_2, l_2) = h \epsilon(l) \epsilon(k),
\]
and so
\[
h^{kl} = ((h^{k_1, l_1} \times (k_2, l_2))^k_1, l_2))^{l_1} = (h_k)^l.
\]
A similar computation, using item 4), yields
\[
(k \times l)^h = k^{hl} \times l^h \quad \text{for all } h, k, l \in H.
\]
Using these facts and item 1), we obtain that
\[
(hk)l = l(2) \times (hk)^l_1 = l_2 \times (k_2 \times h^{k(1)})^l_1 = l_3 \times (k_2^l_2 \times (h^{k(1)})^l_1)
\]
and
\[
h(kl) = (kl)(2) \times h^{kl}(1) = k_2^l_2 \times h^{kl}(1) = (l_3 \times k_2^l_2) \times (h^{k(1)})^l_1.
\]
Thus, from item 2) we conclude that \( \mu \) is associative. Using now that \( 1 \) is group-like, item 2) and Lemma 5.44, we obtain that
\[
h1 = 1 \times h^1 = h^1 = h \quad \text{and} \quad 1h = h(2) \times 1^{h(1)} = h \times 1 = h,
\]
which proves that \( 1 \) is the unit of \( \mu \). Since \( 1 \) is group-like and \( \mu \) is a coalgebra morphism, we conclude that \( H \) is a bialgebra.

Moreover, by item 6), we have
\[
S(h(1))h(2) = h(3) \times S(h(1))h^{2}(h(3)) = h(4) \times (T_s(h(1)) \cdot h(2))^h(3) = h(2) \times T_s(h(1)) = \epsilon(h)1.
\]
It remains to prove that \( h(1)S(h(2)) = \epsilon(h)1 \), for all \( h \in H \). Since, by the previous lemma, \( 1 \cdot h = \epsilon(h)1 \), for this it suffices to check \( (h(1)S(h(2)))^h(3) = \epsilon(h)1 \). But,
\[
(h(1)S(h(2)))^h(3) = (S(h(2)) \times h(1))^{h(3)}h(4) = S(h(2))^h(3) \times h(1) = T_s(h(2)) \times h(1) = \epsilon(h)1,
\]
where the first equality holds by item 3); the second one, by [5.19], [5.20] and [5.21]; the third one, by the definition of \( S \) in item 3); and the last one, by item 6). \( \square \)

**Theorem 5.46.** Let \( \mathcal{H} = (H, \cdot, \) be a left regular \( q \)-magma coalgebra endowed with a distinguished group-like element \( 1 \) and maps \( \times, \triangleright : H^2 \rightarrow H \) and \( T_s : H \rightarrow H \). Set \( hk := k(2) \times h^{k(1)} \) and \( S(h) := T_s(h(1)) \cdot h(2) \).

Assume that

1) The map \( h \otimes k \mapsto hk \) is a coalgebra morphism from \( H^2 \) to \( H \),
2) \( \times \) and \( \triangleleft \) are associative operations with identity \( 1 \),
3) \( S \) is a coalgebra antimorphism of \( H \).

If for all \( h, k, l \in H \) the following equalities hold:

4) \( (k \times l) \cdot h = (k \cdot h(1)) \times (l \cdot h(2)) \),
5) \((k \times l) : h = (k : h_{(1)}) \times (l : h_{(2)})\),
6) \(h \cdot kl = (h \cdot l) \cdot k\) and \(h \cdot kl = (h : l) : k\),
7) \(T_x(h_{(2)}) \times h_{(1)} = h_{(2)} \times T_x(h_{(1)}) = \epsilon(h) 1\),
8) \(h \cdot (k \times l) = h \cdot (l \otimes k)\) and \(h : (k \times l) = h : (l \otimes k)\),
9) \(h \mathbin{\otimes} k = h_{(2)} \otimes (k : h_{(3)})^{h(1)}\),

then \(H\) is a Hopf algebra via \(\mu(h \otimes k) := hk\) (with unit 1 and antipode \(S\)) and \(H\) is a Hopf \(q\)-brace. Conversely, if \(H\) is a Hopf \(q\)-brace, then \(H\) is a regular \(q\)-magma coalgebra with distinguished group-like element 1, the maps \(\times, \mathbin{\otimes}\), and \(T_x\), defined in \([5.14]\) and Proposition 5.41, satisfy equalities

\[k_{(2)} \times h^{k(1)} = hk \quad \text{and} \quad T_x(h_{(1)}) \cdot h_{(2)} = S(h),\]  \([5.22]\)

conditions 1)–9) are fulfilled, and

\[(k \otimes l) \cdot h = (k \cdot h_{(2)}) \otimes (l \cdot h_{(1)}) \quad \text{and} \quad (k \otimes l) : h = (k : h_{(2)}) \otimes (l : h_{(1)}), \quad \text{for all } h, k, l \in H.\]

**Proof.** By Lemma 5.45, we know that \(H\) is a Hopf algebra via the multiplication map \(\mu(h \otimes k) := hk\), with the unit and antipode as in the statement. Moreover, we have

\[(k \cdot h_{(1)})h_{(2)} = h_{(3)} \times (k \cdot h_{(1)})^{h(2)} = h \times k \quad \text{and} \quad (k : h_{(2)})h_{(1)} = h_{(2)} \times (k : h_{(3)})^{h(1)} = h \mathbin{\otimes} k.\]

Hence, we can apply Proposition 5.36 to obtain that \(H\) is a Hopf \(q\)-brace.

\(\Rightarrow\) Since \(H\) is a Hopf \(q\)-brace, from Remark 5.10, it follows that \(H\) is a regular \(q\)-cycle coalgebra and 1 is a group-like element. Moreover items 1) and 3) are trivial; items 2), 5), 6), 8) and the fact that \((k \otimes l) \cdot h = (k \cdot h_{(2)}) \otimes (l \cdot h_{(1)}), \) for all \(h, k, l \in H\), hold by Proposition 5.36; items 4), 7) and the fact that \((k \otimes l) : h = (k : h_{(2)}) \otimes (l : h_{(1)}), \) for all \(h, k, l \in H\), hold by Proposition 5.41; and item 9) holds since, by equalities [5.14],

\[h_{(2)} \times (k : h_{(3)})^{h(1)} = ((k : h_{(4)})^{h(1)} : h_{(2)}) h_{(3)} = (h : h_{(2)}) h_{(1)} = h \mathbin{\otimes} k.\]

Furthermore,

\[k_{(2)} \times h^{k(1)} = (k^{k(1)} \cdot k_{(2)}) k_{(3)} = hk \quad \text{and} \quad T_x(h_{(1)}) \cdot h_{(2)} = S(h_{(1)})^{h_{(2)}} \cdot h_{(3)} = S(h),\]

as desired. \(\square\)

**Remark 5.47.** From the first identity in [5.22] it follows that \(k^{h} = T_x(h_{(2)}) \times h_{(1)}\), for all \(h, k \in H\).

**Remark 5.48.** Let \(H = (H, \cdot, \mathbin{\otimes})\) be a Hopf \(q\)-brace. By the first equality in [5.14], we know that

\[(k_{(1)} \times h) S(k_{(2)}) = h \cdot k \quad \text{for all } h, k \in H.\]

Hence, the map \(h \otimes k \mapsto (k_{(1)} \times h) S(k_{(2)})\) is a coalgebra morphism from \(H \otimes H^{\text{cop}}\) to \(H\). Assume now that \(S\) is bijective. The previous argument applied to \(H^{\text{op}} = (H^{\text{cop}}, \cdot, \mathbin{\otimes})\) shows that

\[(k_{(2)} \mathbin{\otimes} h) S^{-1}(k_{(1)}) = h : k \quad \text{for all } h, k \in H,\]

and the map \(h \otimes k \mapsto (k_{(2)} \mathbin{\otimes} h) S^{-1}(k_{(1)})\) is a coalgebra morphism from \(H \otimes H^{\text{cop}}\) to \(H\).

**Proposition 5.49.** If \(H = (H, \cdot, \mathbin{\otimes})\) is a Hopf \(q\)-brace, then

\[(k \times l) h = kh_{(3)} \times T_x(h_{(2)}) \times lh_{(1)} \quad \text{for all } h, k, l \in H.\]  \([5.23]\)

Moreover, if \(S\) is bijective, then

\[(k \mathbin{\otimes} l) h = kh_{(4)} \mathbin{\otimes} T_x(h_{(2)}) \mathbin{\otimes} lh_{(3)} \quad \text{for all } h, k, l \in H.\]  \([5.24]\)

**Proof.** We have

\[(k \times l) h = (I_{S(k_{(1)})}) k_{(2)} h \quad \text{by the first equality in [5.14]} \]

\[= (k h_{(1)}) S(h_{(2)}) S(k_{(1)}) k_{(2)} h_{(3)} \]

\[= (k h(1)) S(k_{(2)} k_{(1)}) k_{(2)} h_{(3)} \]

\[= kh_{(2)} \times l h_{(1)} \quad \text{by the first equality in [5.14]} \]

\[= kh_{(3)} \times T_x(h_{(2)}) \times lh_{(1)} \quad \text{by the first equality in [5.22]},\]

which proves equality [5.23]. Applying this to \(H^{\text{op}} = (H^{\text{cop}}, \cdot, \mathbin{\otimes})\), we obtain [5.24]. \(\square\)
Definition 5.50. For a Hopf $q$-brace $H = (H, \cdot, :)$, we recursively define binary operations $\cdot^n$ on $H$, by

\[
h \cdot^n k := \begin{cases} \h k (h_{(1)}) \cdot^{n-1} h_{(2)} k_{(2)} & \text{if } n = 0, \\ h_{(1)} k_{(1)} \cdot^{n-1} h_{(2)} k_{(2)} & \text{if } n > 0. \end{cases} \tag{5.25}
\]

Then, we define $\times^n$ and $\star^n$, by $h \times^n k := (k \cdot h_{(1)}) \cdot^n h_{(2)}$ and $h \star^n k := (k : h_{(2)}) \cdot^n h_{(1)}$. Note that

\[
h \times^0 k = h \times h, \quad h \star^0 k = h \star h \quad \text{and} \quad k \times^1 h = (h \cdot k_{(1)}) \cdot^1 k_{(2)} = (k : h_{(2)}) h_{(1)} = h \star h,
\]

where the second equality holds by Remark 2.1.

Remark 5.51. In the set-theoretic context, the operations $\cdot^n$, $\times^n$ and $\star^n$ correspond to the operations $\circ_n$, $\ast_n$ and $\ast_n$, introduced above Theorem 3 of [25].

Theorem 5.52. Let $H = (H, \cdot, :)$ be a Hopf $q$-brace with bijective antipode, let $s$ be the invertible non-degenerate set-theoretic type solution of the braid equation associated with $H$, let $H^{\sqsupset}$ be the underlying coalgebra of $H$, endowed with the multiplication map $\cdot^n$, and let $H^{\sqsupset} := H^{\sqcup}$. The triples $H^{\sqsupset} := (H^{\sqsupset}, :, s)$ and $H^{\sqsupset} := (H^{\sqsupset}, s, \circ)$ are Hopf $q$-braces with bijective antipodes.

Proof. By Theorem 5.18, in order to check that $H^{\sqsupset}$ and $H^{\sqsupset}$ are Hopf $q$-braces with bijective antipode, we must prove that $H^{\sqsupset}$ and $H^{\sqsupset}$ are Hopf algebras (which automatically implies that their antipodes are bijective because they are inverses one of each other) and that $(H^{\sqsupset}, s)$ and $(H^{\sqsupset}, \circ)$ are weak braiding operators. Set $\mu^n(h \otimes k) := h \cdot^n k$. Since $s$ is a coalgebra automorphism of $H^2$ and $\mu^{n+1} = \mu^n s$, for all $n$, from the fact that $\mu^0 : H^2 \to H$ is a coalgebra morphism it follows that $\mu^n : H^2 \to H$ is a coalgebra morphism, for all $n$.

By Theorem 5.18 we know that $\mu^0$ satisfies identities [5.1] and [5.2]. This implies that $\mu^n$ satisfies the same identities for all $n$. In fact, set $s_{12} := s \otimes H$ and $s_{23} := H \otimes s$. Since $s$ is an invertible solution of the braid equation, we have

\[
s(\mu^n \otimes H) = (H \otimes \mu^n) s_{12} s_{23} \iff s(\mu^{n+1} \otimes H) = (H \otimes \mu^{n+1}) s_{12} s_{23}
\]

and

\[
s(H \otimes \mu^n) = (\mu^n \otimes H) s_{23} s_{12} \iff s(H \otimes \mu^{n+1}) = (\mu^{n+1} \otimes H) s_{23} s_{12},
\]

which implies that $\mu^n$ satisfies identities [5.1] and [5.2], for all $n$. Using this we obtain that

\[
\mu^{n+1} (H \otimes H) = \mu^n s(H \otimes H) = \mu^n s(\mu^n \otimes H) s_{12} = \mu^n (H \otimes \mu^n) s_{12} s_{23} s_{12},
\]

and

\[
\mu^{n+1} (H \otimes H) = \mu^n s(H \otimes H) = \mu^n s(H \otimes \mu^n) s_{23} = \mu^n (H \otimes \mu^n) s_{23} s_{12} s_{23},
\]

which proves that $\mu^n$ is associative for all $n$, since $\mu^0$ is associative and $s$ is a solution of the braid equation. Moreover, by Theorem 5.18 and condition 5) above Theorem 5.4, we have

\[
1 \cdot^{n+1} h = 1 h_{(1)} \cdot^{n+1} h_{(2)} = h \cdot^{n+1} 1 \quad \text{and} \quad h \cdot^{n+1} 1 = h_{(1)} 1 \cdot^{n+1} h_{(2)} = 1 \cdot^n h \quad \text{for all } h \in H,
\]

and so, 1 is the unit of $\mu_n$, for all $n$. Hence, the $H^{\sqsupset}$’s are bialgebras and identities [5.3] and [5.4] are fulfilled, for all the $H^{\sqsupset}$’s. Consequently, the $H^{\sqsupset}$’s are also bialgebras satisfying conditions [5.1]–[5.4]. In order to finish the proof, we need to check that $H^{\sqsupset}$ and $H^{\sqsupset}$ are Hopf algebras, for all $n \in \mathbb{N}_0$. Since $H$ and $H^{\sqsupset}$ are Hopf algebras, this is true for $n = 0$. Assume that it is true for a fixed $n \geq 0$. We are going to check that $H^{n+1}$ is a Hopf algebra by proving that $(\mu^{n+1} \otimes H)(H \otimes \Delta)$ is bijective (which happens if and only if $H^{n+1}$ have antipode). Write $L_{n+1} := (\mu^{n+1} \otimes H)(H \otimes \Delta) \overline{\sigma}_H$, where $\overline{\sigma}_H$ is the map introduced at the beginning of Subsection 1.2. Since $\overline{\sigma}_H$ is invertible (see Definition 1.15), in order to fulfill our task we only must show that $L_{n+1}$ is bijective. But, by Remark 2.1 and the very definitions of $\mu^{n+1}$, $\overline{\sigma}_H$ and $\star^n$, we have

\[
L_{n+1}(h \otimes k) = (h \cdot k_{(1)}) \cdot^{n+1} k_{(2)} k_{(3)} = (k_{(1)} ; h_{(2)}) \cdot^n h_{(1)} k_{(2)} = h \star^n k_{(1)} \otimes k_{(2)},
\]

and so, by Proposition 5.41, the map $L_{n+1}$ is invertible with inverse $h \otimes k \mapsto h \star^n T_{\star^n}(k_{(1)}) \otimes k_{(2)}$. The fact that $H^{n+1}$ is a Hopf algebra follows applying the same argument to $H^{\sqsupset} = (H^{\sqsupset}, \cdot, s, \circ)$. \hfill \square

Remark 5.53. By Remark 5.14, if $H$ is a Hopf $q$-brace with bijective antipode, then $H^{\sqsupset}$ is also. Applying Theorem 5.52 to $H^{\sqsupset}$, we obtain other families $H^{\sqsupset} \sqsupset (n \in \mathbb{N}_0)$ and $H^{\sqsupset} \sqsupset (n \in \mathbb{N}_0)$, of Hopf $q$-braces.
6 Hopf Skew-braces

In this section, motivated by the remark after Corollary 2 of [25, Proposition 4], we define Hopf skew-braces as a particular type of Hopf $q$-braces. Then, we prove that a Hopf $q$-brace is a Hopf skew-brace if and only if its associated weak braiding operator is a braiding operator. Moreover, we also prove that the notion of Hopf skew-brace with bijective antipode has the following equivalent avatars: a direct generalization of the concept of skew-brace as formulated originally by Guarnieri and Vendramin and a generalization of the concept of linear $q$-cycle set formulated by Rump. Furthermore, we introduce the concept of invertible 1-cocycle and we prove that the category of Hopf skew-braces with bijective antipode is equivalent to the category of invertible 1-cocycles. Finally, in Subsection 6.2, we show that the category of Hopf skew-braces is isomorphic to the category of Yetter-Drinfeld braces, recently introduced in [10].

Definition 6.1. A Hopf $q$-brace $\mathcal{H} = (H, \cdot, :)$ is a Hopf skew-brace if

$$h \not\preceq k = k \times h \quad \text{for all } h, k \in H,$$

that is, $(k : h(2))h(1) = (h \cdot k(1))k(2)$. 

Examples 6.2. 1) Hopf skew-braces whose underlying Hopf algebra structure are group algebras can be naturally identified with skew-braces ([16]). This is clearer if we think in the GV-Hopf skew-brace avatar of Hopf skew-braces (Definition 6.18).

2) In Example 5.15 we point out that the Sweedler algebra has a Hopf skew-brace structure.

3) The dihedral group $D_{2m}$ is generated by elements $x$ and $y$ subject to the relations $x^{2m} = y^2 = xyx = 1$. It is well known that the underlying set of $D_{2m}$ is $\{0, 1, \ldots, 2m-1\}$ and that its center is $\{0, m\}$. Let $k$ be a field of characteristic different of 2 and let $H$ be the Hopf $k$-algebra dual of $k[D_{2m}]$. Thus, $H$ is the $k$-vector space with basis $\{d_a : a \in D_{2m}\}$, endowed with the product and the coproduct given by

$$d_a d_b = \delta_a^b d_a \quad \text{and} \quad \Delta(d_a) = \sum_{b \in D_{2m}} d_{ab} \otimes d_{b^{-1}},$$

where $\delta_a^b$ is the Kronecker delta. The unit, counit and antipode of $H$ are $1_H = \sum_{a \in D_{2m}} d_a$, $\epsilon(d_a) = \delta_a^0$ and $S(d_a) = d_{a^{-1}}$. There are exactly four skew-braces structures on $H$ such that $d_a \cdot d_b = d_a : d_b = 0$ for all $b \not\in \{0, m\}$. In all of them coincides with $\cdot$ and $d_a \cdot d_1 = d_a \cdot d_{x^m} = d_a$ for all $a \in D_{2m}$. They are:

$$d_x \cdot d_1 = d_x \cdot d_y,$$

$$d_x \cdot d_1 = \begin{cases} \frac{1}{2} d_{x^j y} + \frac{1}{2} d_{x^{m+j} y} & \text{if } j = 1, \\
\frac{1}{2} d_{x^j y} & \text{if } j = 0. \end{cases}$$

$$d_x \cdot d_1 = \begin{cases} \frac{1}{2} d_{x^j y} + \frac{1}{2} d_{x^{m+j} y} & \text{if } i \text{ is odd}, \\
\frac{1}{2} d_{x^j y} & \text{if } i \text{ is even}. \end{cases}$$

$$d_x \cdot d_1 = \begin{cases} \frac{1}{2} d_{x^j y} + \frac{1}{2} d_{x^{m+j} y} & \text{if } i + j \text{ is odd}, \\
\frac{1}{2} d_{x^j y} & \text{if } i + j \text{ is even}. \end{cases}$$

Remark 6.3. For Hopf skew-braces with bijective antipode there is an improvement of Corollary 5.38. In fact, in this case the monoids $\text{End}_{\Delta \circ \varphi, \times}(H)$ and $\text{End}_{\Delta, \times}(H)$, introduced in Proposition 5.41, are opposite monoids, and so $T_\varphi = T_{\varphi^-}$. In other words,

$$S(h(1)) \cdot S^{-1}(h(2)) = S^{-1}(h(2)) \cdot S(h(1)) \quad \text{for all } h \in H.$$

Recall that having a Hopf $q$-brace $\mathcal{H} = (H, \cdot, :)$ is equivalent to having a weak braiding operator $(H, s)$.

Proposition 6.4. A weak braiding operator $(H, s)$ is a braiding operator if and only if its associated Hopf $q$-brace is a Hopf skew-brace.

Proof. By Remark 2.1, we know that $(H, s)$ is a braiding operator if and only if

$$(k : h(2))h(1) = (h \cdot k(1))k(2) \quad \text{for all } h, k \in H.$$

In other words if and only if condition [6.1] is satisfied.

Remark 6.5. By Proposition 6.4, the categories of braiding operators and Hopf skew-braces are isomorphic.

Proposition 6.6. Let $H$ be a Hopf algebra and let $\mathcal{H} = (H, \cdot, :)$ be a left regular $q$-magma coalgebra. Then $\mathcal{H}$ is a Hopf skew-brace if and only if $(H, \cdot)$ and $(H, :)$ are right $H^\text{op}$-modules and condition [6.1] is fulfilled.
Proof. By definition, if $\mathcal{H}$ is a Hopf skew-brace, then $(H, \cdot)$ and $(H, :)$ are right $H^{\text{op}}$-modules and condition [6.1] is satisfied. Conversely, assume that these facts hold, and let $s: H^2 \to H^2$ be the left non-degenerate coalgebra endomorphisms of $H^2$, associated with $\mathcal{H}$ according to Remarks 1.18 and 2.8. Since $\mathcal{H}$ is left regular, from Remark 2.1 and condition [6.1], it follows that condition [5.5] is satisfied. Hence, by Theorem 5.27 and Proposition 6.4, in order to finish the proof it suffices to show that $H$ is a left $H$-module via $h \otimes l \mapsto h^l$ and a right $H$-module via $h \otimes l \mapsto h^l$. By Lemma 5.16(1), the second fact holds. We next prove the first one. Let $h, k, l \in H$ arbitrary. By Remark 1.18 and the fact that $(H, :)$ is a right $H^{\text{op}}$-module, we have
\[ h k l = l(2) : (h k) l(1) \]
and
\[ h^{k l} = h^{l(2) : (h k) l(1)} = (l(4) : k(l(1) l(2)) : h^{l(3) : (k(l(1) l(2)) l(1))} = l(3) : h^{l(2) : (k(l(1) l(2)) l(1))} \]

Hence, by equality [6.1] and Lemma 5.17, we are reduced to check that the first identity in [5.7] holds. But, by condition [6.1], the fact that $(H, :)$ is a right $H^{\text{op}}$-module and Remark 1.13, we have
\[ (h k \cdot l(1)) l(2) = (l : k(2)) h(1) k(1) = (h \cdot (l(1) : k(3))) l(2) = (h \cdot (l(1) : k(3))) k(1) = (h \cdot (l(1) : k(2))) (k(1) : l(2)) l(3) \]
from which, the first identity in [5.7] follows immediately. \qed

Remark 6.7. In the set-theoretic context, Proposition 6.6 corresponds to Corollary 4 of [25, Proposition 6].

Proposition 6.8. Let $H$ be a Hopf algebra with bijective antipode and let $p, d: H \otimes H^{\text{op}} \to H$ be two maps. Write $h \cdot k := p(h \otimes k)$ and $h : k := d(h \otimes k)$. Assume that condition [6.1] is fulfilled. Then, we have:
1) If $p$ is a coalgebra map, then $d$ is a coalgebra map if and only if condition [1.4] is satisfied.
2) If $d$ is a coalgebra map, then $p$ is a coalgebra map if and only if condition [1.4] is satisfied.

Proof. Assume that $p$ is a coalgebra map. By identity [6.1], we have $h : k = (k(2) : h(1) h(2) S^{-1}(k(1)))$. Thus, $\epsilon(h : k) = \epsilon(h) \epsilon(k)$ and $(h : k)_{(1)} \otimes (h : k)_{(2)} = h(1) : k(2) \otimes h(2) : k(1)$ if and only if
\[ (k(3) : h(2)) h(3) S^{-1}(k(2)) \otimes (k(4) : h(1)) h(4) S^{-1}(k(1)) = (k(3) : h(1)) h(2) S^{-1}(k(3)) \otimes (k(2) : h(3)) h(4) S^{-1}(k(1)) \]
But the last equality is fulfilled if and only if
\[ (k(2) : h(3)) h(2) S^{-1}(k(1)) \otimes k(3) : h(1) = (k(3) : h(1)) h(2) S^{-1}(k(3)) \otimes k(1) : h(3) \]
or, in other words, $h(2) : (k(1) \otimes k(2)) : h(1) = h(1) : k(2) \otimes k(1) : h(2)$. This ends the proof of item 1). The proof of item 2) is similar. \qed

Remark 6.9. For item 2), it is not necessary for $S$ to be bijective.

Proposition 6.10. Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf q-brace and let $H \bowtie H$ be the bicrossed product associated with $\mathcal{H}$ according to Theorems 5.4 and 5.18. Let $F: H \to H \bowtie H$ be the map given by $F(h) := S(h(1)) \otimes h(2)$. If $S$ is injective, then $\mathcal{H}$ is a Hopf skew-brace if and only if $F(H)$ is a subalgebra of $H \bowtie H$.

Proof. By Proposition 5.31, we have
\[ S(k(1)) : (h \cdot k(2)) = S(k(1)) : (h \cdot (k(3) : S(k(2)))) = S(k(1)) : (h : S(k(2)) k(3)) = S(k : h), \]
for all $h, k \in H$. Using this, the very definition of the multiplication in $H \bowtie H$, the last assertion in Remark 1.9, Lemma 5.9 and identity [1.4], we obtain that
\[ (S(h(1)) \otimes h(2))(S(k(1)) \otimes k(2)) = S(h(1)) (h(2) S(k(1))) \otimes (h(3) S(k(1))) k(3) = S(h(1)) (h(2) S(k(1))) \otimes (h(3) S(k(1))) k(4) = S(h(1)) (h(2) S(k(1))) \otimes (h(3) S(k(1))) k(4) = S(h(1)) S(k(2)) S(k(3)) \otimes (h(3) : k(1)) k(3) = S(k(1)) (h(3) h(1)) \otimes (h(2) : k(2)) k(3) \]
If identity [6.1] is fulfilled, then
\[ F(h) F(k) = S((k : h(4)) h(1)) \otimes (k(2) : h(3)) h(2) = F((k : h(2)) h(1)) = F(h \star k), \]
and consequently $F(H)$ is a subalgebra of $H \bowtie H$. Conversely, if $F(H)$ is a subalgebra of $H \bowtie H$, then, for each $h, k \in H$ there exists $l \in H$ such that $S((k : h(3)) h(1)) \otimes (h(2) : k(3)) k(3) = S(l(1)) \otimes l(2)$. So, on one hand, $(h \cdot k(1)) k(2) = l$; while, on the other hand, $S((k : h(2)) h(1)) = S(l)$, which implies that $(k : h(2)) h(1) = l$, since $S$ is injective. So $h \star k = (k : h(2)) h(1) = (h \cdot k(1)) k(2) = k \times h$. \qed
Remark 6.11. In the set-theoretic context, Proposition 6.10 corresponds to Corollary 3 of [25, Proposition 6].

Definition 6.12. A linear q-cycle coalgebra is a tuple \( \mathcal{H} := (H, \times, T_x, \cdot, 1) \), consisting of a coalgebra \( H \) with a distinguished group-like element \( 1 \), a map \( h \otimes k \mapsto h \times k \) from \( H^2 \) to \( H \), a coalgebra morphism \( h \otimes k \mapsto h \cdot k \) from \( H \otimes H^{\text{cop}} \) to \( H \), and a map \( T_x : H \to H \), such that:

1) \( (H, \cdot) \) is a regular magma coalgebra,
2) the map \( h \otimes k \mapsto hk := k(2) \times h^{k(1)} \) is a coalgebra morphism from \( H^2 \) to \( H \),
3) \( \times \) is an associative operation with identity \( 1 \),
4) the map \( h \mapsto S(h) := T_x(h(1)) \cdot h(2) \) is a coalgebra antiautomorphism of \( H \),
5) \( (k \times l) \cdot h = (k \cdot h(1)) \times (l \cdot h(2)) \) for all \( h, k, l \in H \),
6) \( h \cdot (k \times l) = (h \cdot k(2)) \cdot (l \cdot k(1)) \) for all \( h, k, l \in H \),
7) \( h(2) \times T_x(h(1)) = T_x(h(2)) \times h(1) = \epsilon(h)1 \) for all \( h \in H \),
8) the map \( h \otimes k \mapsto h : k := (k(2) \cdot h(1))h(2)S^{-1}(k(1)) \), is a coalgebra morphism, from \( H \otimes H^{\text{cop}} \) to \( H \).

Let \( \mathcal{H} := (H, \times, T_x, \cdot, 1) \) and \( \mathcal{L} := (L, \times, T_x, \cdot, 1) \) be linear q-cycle coalgebras. A morphism \( f : \mathcal{H} \to \mathcal{L} \) is a coalgebra morphism \( f : H \to L \), such that \( f(1) = 1 \) and such that for all \( h, k \in H \),

\[
\begin{align*}
(f(h \times k) &= f(h) \times f(k), \quad f(T_x(h)) = T_x(f(h)) \quad \text{and} \quad f(h \cdot k) = f(h) \cdot f(k).
\end{align*}
\]

Remark 6.13. Items 3), 5) and 7) imply that \( 1 \cdot h = \epsilon(h)1 \) for all \( h \in H \).

Remark 6.14. Assume that \( H \) is cocommutative. Then item 8) of Definition 6.12 is trivially satisfied and item 2) is fulfilled if and only if \( \times \) is a coalgebra morphism (use Proposition 1.2). Consequently, in this case \((H, \Delta, \times)\) is a Hopf algebra with antipode \( T_x \). Even more, \( \mathcal{H} = (H, \times, T_x, \cdot, 1) \) is a linear q-cycle coalgebra if and only if \((H, \Delta, \times)\) is a Hopf algebra with identity \( 1 \) and antipode \( T_x \), and items 1), 5) and 6) are fulfilled.

Remark 6.15. When \( H \) is a group algebra \( k[G] \), then a linear q-cycle coalgebra structure on \( H \) induces by restriction a linear q-cycle set structure on \( G \) in the sense of [25, Definition 6]. Conversely, each linear q-cycle set structure on \( G \) yields a unique linear q-cycle coalgebra structure on \([k[G]]\).

Lemma 6.16. Let \( H \) be a Hopf algebra with bijective antipode \( S \) and let \( h \otimes k \mapsto h \times k \) and \( h \mapsto T_x(h) \) be morphisms, from \( H^2 \) to \( H \) and from \( H \) to \( H \), respectively. If the binary operation \( \times \) is associative with unity \( 1 \) and \( h(2) \times T_x(h(1)) = T_x(h(2)) \times h(1) = \epsilon(h)1 \), for all \( h \in H \), then the maps \( k^h := T_x(h(2)) \times kh(1) \) and \( k \cdot h := (h(1) \times k)S(h(2)) \), satisfy the equalities \( k^{h(1)} \cdot h(2) = (k \cdot h(1))^{h(2)} = \epsilon(h)k \), for all \( h, k \in H \).

Proof. In fact

\[
\begin{align*}
k^{h(1)} \cdot h(2) &= (T_x(h(2)) \times kh(1)) \cdot h(2) = (h(3) \times T_x(h(2)) \times kh(1))S(h(4)) = \epsilon(h)k \\
(k \cdot h(1))^{h(2)} &= T_x(h(3)) \times (k \cdot h(1))h(2) = T_x(h(4)) \times (h(1) \times k)S(h(2))h(3) = \epsilon(h)k,
\end{align*}
\]

as desired. \( \square \)

The following lemma generalizes [16, Proposition 1.9].

Lemma 6.17. Let \( H \) be a Hopf algebra with bijective antipode \( S \) and let \( h \otimes k \mapsto h \times k \) and \( h \mapsto T_x(h) \) be morphisms, from \( H^2 \) to \( H \) and from \( H \) to \( H \), respectively. Assume that \( \times \) is associative with unity \( 1 \) and set \( k^h := T_x(h(2)) \times kh(1) \). The following assertions are equivalent:

1) \( (k \times l)h = k(h(3)) \times T_x(h(2)) \times lh(1) \), for all \( h, k, l \in H \).
2) \( h \cdot k \times T_x(h_1) = T_x(h_2) \times h_1 = \epsilon(h)1 \) and \( k^{hl} = (k^h)^l \), for all \( h, k, l \in H \).
3) \( h(2) \times T_x(h(1)) = T_x(h(2)) \times h(1) = \epsilon(h)1 \) and \( (k \times l)^h = k^{hl} \times l^{h(1)} \), for all \( h, k, l \in H \).

Proof. 1) \( \Rightarrow \) 2) The first condition in item 2) follows applying item 1) to

\[
\epsilon(h)1 = (S^{-1}(h(2)) \times 1)h(1) = (1 \times S(h(1)))h(2).
\]

Since, by definition,

\[
k^{hl} = T_x(h(2)l(2)) \times kh(1)l(1) \quad \text{and} \quad (k^h)^l = T_x(l(2)) \times k^hl(1) = T_x(l(2)) \times (T_x(h(2)) \times kh(1))l(1),
\]

the second condition in item 2) follows from the fact that, applying item 1) twice, we obtain

\[
l(2) \times T_x(h(2)l(2)) \times kh(1)l(1) = (S^{-1}(h(2)) \times k)h(1)l = (T_x(h(2)) \times kh(1))l.
\]
2) ⇒ 3) Let \( \cdot \) be as in the previous lemma. We have

\[
(k \times l)^h = ((l \cdot k(1))k(2))^h = T_x(h_1(1)) \times ((l \cdot k(1))k(2))h(1)
\]

by definition

\[
= T_x(h_1(2)) \times ((l \cdot k(1))k(2))h(1)
\]

by definition

\[
= k(3)^h(2) \times ((l \cdot k(1))k(2))h(1)
\]

by hypothesis

\[
= k(2) \times l^h(1)
\]

by Lemma 6.16

as desired.

3) ⇒ 1) This follows immediately from the fact that

\[
k(2) \times l^h(1) = T_x(h_1(4)) \times kh_3(1) \times T_x(h_2(1)) \times lh(1)
\]

and

\[
(k \times l)^h = T_x(h_1(2)) \times (k \times l)h(1),
\]

for all \( h, k, l \in H \).

\[\square\]

**Definition 6.18.** A GV-Hopf skew-brace is a tuple \( \mathcal{H} := (H, \times, T_x) \), consisting of a Hopf algebra \( H \) with bijective antipode \( S \) and maps \( h \otimes k \mapsto h \times k \) and \( h \mapsto T_x(h) \), from \( H^2 \) to \( H \) and from \( H \) to \( H \), respectively, such that:

1) \( \times \) is associative with unity 1,
2) \( (k \times l)h = kh_3(1) \times T_x(h_2(1)) \times lh(1) \) for all \( h, k, l \in H \),
3) \( h \otimes k \mapsto (k(1) \times h)S(k(2)) \) is a coalgebra morphism from \( H \otimes H^{cop} \) to \( H \),
4) \( h \otimes k \mapsto (h \times k)S^{-1}(k(1)) \) is a coalgebra morphism from \( H \otimes H^{cop} \) to \( H \).

Let \( \mathcal{K} = (K, \times, T_x) \) be another GV-Hopf skew-brace. A morphism \( f: \mathcal{H} \to \mathcal{K} \) is a morphism \( f: H \to K \) of Hopf algebras, such that \( f(h \times h') = f(h) \cdot f(h') \) for all \( h, h' \in H \).

**Remark 6.19.** By Lemma 6.17, the map \( T_x \) is the convolution inverse of \( id \) in \( \text{End}_{\Delta^{cop}, \times}(H) \). Hence each GV-Hopf skew-brace morphism \( f: \mathcal{H} \to \mathcal{K} \), satisfies \( fT_x = T_x f \).

**Remark 6.20.** If \( (H, \times, T_x) \) is a GV-Hopf skew-brace, then \( (H^{cop}, \star, T_x) \), also is (here \( \star \) is the opposite multiplication of \( \times \) and \( T_x = T_x \)). Moreover the definition \( k := T_x(h_2(1)) \times kh_3(1) \), in Lemma 6.16 for \( (H, \times, T_x) \), in \( (H^{cop}, \star, T_x) \) reads \( k := T_x(h_1(1)) \times kh_2(1) = kh_2(1) \times T_x(h_1(1)) \). Furthermore, by Lemma 6.17, we have

\[
k_h = (kh)h \quad \text{and} \quad (k \times l)_h = (l \times k)_h = l_h \quad \text{for all} \quad h, k, l \in H.
\]

**Remark 6.21.** Item 1) in Definition 6.18 can be replaced by

1') \( \times \) is associative with unity and \( T_x \) is the convolution inverse of \( id \) in \( \text{End}_{\Delta^{cop}, \times}(H) \).

In fact, by item 2), we have \( k \times 1 = (k \times 1)1 = k \times T_x(1)1 = 1 = k \), which proves that the unity of \( \times \) is 1.

**Remark 6.22.** When \( H \) is cocommutative, then items 3) and 4) are equivalent and they are fulfilled if and only if \( \times: H^2 \to H \) is a coalgebra morphism. So, by Remark 6.19, in the cocommutative case, our definition reduces to the definition of Hopf brace structure given in [1, Definition 1.1]. For the set-theoretic case see [1, Example 1.4].

**Theorem 6.23.** The three notions of Hopf skew-brace with bijective antipode, linear q-cycle coalgebra and GV-Hopf skew-brace are equivalent. More precisely:

1) If \( (H, \cdot, \cdot) \) is a Hopf skew-brace with bijective antipode, then \( (H, \times, T_x, \cdot, 1) \) is a linear q-cycle coalgebra, where 1 is the unit of \( H \), \( h \times k := (k \cdot h(1))h(2) \) and \( T_x(h) := S(h(1))h(2) = S(h(1)) \cdot S^{-1}(h(2)) \).

2) If \( (H, \times, T_x, \cdot, 1) \) is a linear q-cycle coalgebra, then the coalgebra \( H \), endowed with the multiplication \( hk := k(2) \times h^{k(1)} \), is a Hopf algebra with bijective antipode \( S(h) := T_x(h(1)) \cdot h(2) \). Moreover, the triple \( (H, \times, T_x) \) is a GV-Hopf skew-brace.

3) If \( (H, \times, T_x) \) is a GV-Hopf skew-brace, then \( (H, \cdot, \cdot) \) is a Hopf skew-brace with bijective antipode, where \( h \cdot k := (k(1) \times h)S(k(2)) \) and \( h : k := (h \times k(2))S^{-1}(k(1)) \).

**Proof.** 1) Item 1) of Definition 6.12 follows from the definition of Hopf skew-brace, items 2), 3), 4), 5) and 7), from items 1), 2), 3), 4) and 7) of Theorem 5.46, respectively. Item 6) follows from item 6) of the same theorem, noting that \( h \cdot (k \times l) = h \cdot ((l \cdot k(1))k(2)) = (h \cdot k(2)) \cdot (l \cdot k(1)) \). Finally, item 8) holds, because by [6.1], in each Hopf skew brace with bijective antipode the coalgebra morphism : satisfies

\[
h : k = (h \cdot k(3))k(2)S^{-1}(k(1)) = (k(2) \cdot h(1))h(2)S^{-1}(k(1)).
\]
2) By items 1) and 6) of Definition 6.12, we have
\[ h \cdot kl = h \cdot (l_{(2)} \times k_{(1)}) = (h \cdot l_{(3)}) \cdot (k_{(1)} \cdot l_{(2)}) = (h \cdot l) \cdot k. \]  
[6.2]

So, the hypotheses of Lemma 5.45 are satisfied. Hence \( H \), endowed with the multiplication map \( h \otimes k \mapsto h k \), is a Hopf algebra with unit 1 and bijective antipode \( S \). Moreover, item 1) of Definition 6.18 holds, since it is item 3) of Definition 6.12; while item 4) of Definition 6.18 follows from item 8) of Definition 6.12, because
\[ (k \cdot h_{(1)}) h_{(2)} = h_{(3)} \times (k \cdot h_{(1)}) h_{(2)} = h \times k. \]  
[6.3]

A direct computation using [6.3], proves that \( (k_{(1)} \times h) S(k_{(2)}) = h \cdot k \), for all \( h, k \in H \). Hence item 3) of Definition 6.18 follows from item 1) of Definition 6.12. It remains to check item 2) of Definition 6.18. By item 7) of Definition 6.12, we can apply Lemma 6.16, which shows that \( k^h = T_x(h_{(2)}) \times k h_{(1)} \) (see Remark 1.3). Moreover, by item 5) of Definition 6.12,
\[ (k \times l)^h = (((k^h_{(2)} \cdot h_{(3)}) \times (l^h_{(1)} \cdot h_{(4)}))^h_{(5)}) = (((k^h_{(2)} \times l^h_{(3)}), h_{(3)})^h_{(4)}) = k^h_{(2)} \times l^h_{(3)}, \]
for all \( h, k, l \in H \). Hence, item 2) of Definition 6.18 follows immediately from Lemma 6.17.

3) Identity [6.1] holds by the definitions of \( \cdot \) and \( : \). Moreover, by items 3) and 4) of Definition 6.18, Proposition 6.8 and Remark 1.3, we know that \( (H, \cdot, :) \) is a \( \mathbb{q} \)-magma coalgebra. Furthermore, by items 1) and 2) of Definition 6.18 and Lemma 6.17, we know that \( h_{(2)} \times T_x(h_{(1)}) = T_x(h_{(2)}) \times h_{(1)} = \epsilon(h) 1 \), for all \( h \in H \). Hence, we can apply Lemma 6.16, which shows that \( (H, \cdot, :) \) is left regular (see Remark 1.18). Consequently, by Proposition 6.6, in order to complete the proof, we only need to check that \( H \) is a right \( H^{\text{op}} \)-module via \( \cdot \) and \( : \). For \( \cdot \) this follows from item 1) of Lemma 5.16, since \( H \) is a right \( H \)-module via \( k \otimes h \mapsto k^h \) (by Lemma 16.17). For : this follows from the same argument, applied to the GV-Hopf skew-brace \( (H^{\text{op}} \times T_x, \times) \), taking into account that \( h \cdot k = (k_{(2)} \times h) S^{-1}(k_{(1)}) \).

\[ \square \]

Remark 6.24. In the set-theoretic context, Theorem 6.23 yields Corollary 2 of [25, Proposition 4].

6.1 Hopf skew-braces and invertible 1-cocycle

**Proposition 6.25.** A triple \( \mathcal{H} := (H, \times, T_x) \), consisting of a Hopf algebra \( H \) with bijective antipode \( S \) and maps \( \times : H^2 \to H \) and \( T_x : H \to H \), is a GV-Hopf skew-brace if and only if the following conditions hold:

1) \( \times \) is associative with identity 1,
2) \( T_x \) is the inverse of id in \( \text{End}_{\text{H-co}}(H) \),
3) the vector space \( H \) is a right \( H \)-module coalgebra both via \( h \otimes k \mapsto h^k := T_x(k_{(2)}) \times h k_{(1)} \) and via \( h \otimes k \mapsto h k := h k_{(2)} \times T_x(k_{(1)}) \),
4) \( (k \times l)^h = (k_{(1)}) h_{(2)} \times l_{(3)}, \) for all \( h, k, l \in H \).

**Proof.** Assume first that \( \mathcal{H} \) is a GV-Hopf skew-brace. Item 1) holds by item 1) of Definition 6.18, while items 2), 3) and 4) follow from Lemma 6.17 and Remark 6.20. We now assume that items 1)–4) are satisfied and prove the converse. Item 1) of Definition 6.18 is trivial. Item 2) of Definition 6.18 follows from the equivalence between items 1) and 2) of Lemma 6.17. Consequently,
\[ (k_{(1)} \times h) S(k_{(2)}) = k_{(1)} S(k_{(2)}) \times T_x(S(k_{(1)})) \times h S(k_{(2)}) = T_x(S(k_{(1)})) \times h S(k_{(2)}) = h S(k_{(1)}). \]

Since \( S \); \( H^{\text{op}} \to H \) is a coalgebra map, this combined with item 3) shows that item 3) of Definition 6.18 is fulfilled. Finally, we have
\[ (h \times k_{(2)}) S^{-1}(k_{(1)}) = h S^{-1}(k_{(1)}) \times T_x(S^{-1}(k_{(2)})) \times k_{(4)} S^{-1}(k_{(3)}) = h S^{-1}(k_{(2)}) \times T_x(S^{-1}(k_{(1)})) = h S^{-1}(k_{(1)}), \]
which, combined with item 3), shows that item 4) of Definition 6.18 is also fulfilled, because \( S^{-1} : H^{\text{op}} \to H \) is a coalgebra map. \( \square \)

Let \( H \) be a Hopf algebra with bijective antipode and let \((L, \cdot', T_{x'}, \leftrightarrow)\) be a coalgebra \( L \) endowed with a binary operation \( \cdot' : L^2 \to L \), a map \( T_{x'} : L \to L \) and a right action \( \leftrightarrow : L \otimes H \to L \) such that \( \cdot' \) is associative with unit \( 1_L \), \( T_{x'} \) is the inverse of id in \( \text{End}_{\text{L-co}}(x'(L)) \), \( L \) is a right \( H \)-module coalgebra via \( \leftrightarrow \) and \((k \cdot' l) \leftrightarrow h = (k \leftrightarrow h_{(2)}) \cdot' (l \leftrightarrow h_{(1)}) \), for all \( k, l \in L \) and \( h \in H \).

**Definition 6.26.** A bijective 1-cocycle of \( H \) with values in \((L, \cdot', T_{x'}, \leftrightarrow)\) is a coalgebra isomorphism \( \pi : H \to L \), such that:

1) \( L \) is a right \( H \)-module coalgebra via \( l \otimes h \mapsto T_{x'}(\pi(h_{(3)})) \cdot' (l \leftrightarrow h_{(2)}) \cdot' \pi(h_{(1)}), \)
2) \( \pi(hk) = (\pi(h) \leftrightarrow k_{(2)}) \times' \pi(k_{(1)}), \) for all \( h, k \in H \) (cocycle condition).
Let $\xi: K \to J$ be another bijective 1-cocycle. A morphism from $\pi$ to $\xi$ is a pair $(f, g)$, where $f: H \to K$ is a Hopf algebra morphism and $g: L \to J$ is a coalgebra morphism, such that $\xi f = g \pi$, $g(1_L) = 1_f$, $g(l \times' k) = g(l) \times' g(k)$ and $g(l \leftrightarrow h) = g(l) \leftrightarrow f(h)$, for all $h \in H$ and $l, k \in L$.

**Remark 6.27.** In the set-theoretic context, Definition 6.26 reduces to the definition above [16, Proposition 1.11] (Note that they use left actions).

**Remark 6.28.** If $(f, g): \pi \to \xi$ is a morphism of bijective 1-cocycles, then $gT_{\pi'} = T_{\xi'}g$.

**Remark 6.29.** Note that $\pi(1_H) = 1_L$. In fact, the cocycle condition implies that $\pi(1_H) \times' \pi(1_H) = \pi(1_H)$, which combined with the fact that $\pi$ is a coalgebra map and $T_{\pi'}$ is the inverse of id in $\text{End}_{\Delta_{\text{cop}, \times}(L)}$ yields

$$\pi(1_H) = \pi(1_H) \times' \pi(1_H) \times' T_{\pi'}(\pi(1_H)) = \pi(1_H) \times' T_{\pi'}(\pi(1_H)) = \epsilon(\pi(1_H))1_L = 1_L,$$

as desired.

**Proposition 6.30.** If $\mathcal{H} = (H, \times, T_{\pi})$ is a GV-Hopf skew-brace, then $\text{id}_H$ is a bijective 1-cocycle of $H$ with values in $(H, \times, T_{\pi}, \leftarrow)$, where $h \leftarrow k := h_k$. Conversely, if $\pi: H \to L$ is an bijective 1-cocycle of $H$ with values in $(L, \times', T_{\pi'}, \leftarrow)$, then $\mathcal{H} = (H, \times, T_{\pi})$ is a GV-Hopf skew-brace, where $h \times k := \pi^{-1}(\pi(h) \times' \pi(k))$ and $T_{\pi}(h) := \pi^{-1}(T_{\pi'}(\pi(h)))$.

**Proof.** Let $\mathcal{H} = (H, \times, T_{\pi})$ be a GV-Hopf skew-brace and let $(H, \times, T_{\pi}, \leftarrow)$ be as in the statement. By Proposition 6.25, the conditions above Definition 6.26 are satisfied. Item 2) of Definition 6.26 follows from the equality $h_k = hh_{k(2)} \times T_{\pi}(k(1))$ in Remark 6.20. By this and the definition of $l^h$ given in Lemma 6.16, we obtain that

$$T_{\pi'}(\pi(h(3))) \times' (l \leftrightarrow h_k) \times' \pi(h(1)) = T_{\pi'}(h(3)) \times h_{k(2)} \times h(1) = T_{\pi'}(h(2)) \times h(1) = l^h,$$

which yields item 1) (with $\pi = \text{id}_H$, $\times' = \times$ and $l \leftrightarrow h = h_k$), since $l \otimes h \mapsto l^h$ is a coalgebra morphism.

We next take a bijective 1-cocycle $\pi: H \to L$, of $H$ with values in $(L, \times', T_{\pi'}, \leftarrow)$, and prove the converse. For this it suffices to show that conditions 1)–4) of Proposition 6.25 are satisfied. Clearly the operation $\times$ is associative, $1_H$ is its unit (by Remark 6.29) and $T_{\pi}$ is the convolution inverse of $\text{id}_H$ in $\text{End}_{\Delta_{\text{cop}, \times}(H)}$. So, conditions 1) and 2) are satisfied. We now prove that conditions 3) and 4) are also.

Since, by the cocycle condition,

$$h_k := hh_{k(2)} \times T_{\pi}(k(1)) = \pi^{-1}(\pi(h) \times k(3)) \times' \pi(k(2)) \times T_{\pi}(k(1)) = \pi^{-1}(\pi(h) \leftarrow k),$$

$H$ is a right $H$-module coalgebra via $h \otimes k \mapsto h_k$; while, by the last condition above Definition 6.26, item 4) of Proposition 6.25 is fulfilled. Finally, by the cocycle condition and item 2) of Definition 6.26,

$$h_k := T_{\pi}(k(2)) \times h_{k(1)} = \pi^{-1}(T_{\pi'}(\pi(h(3))) \times' \pi(h(1))) = \pi^{-1}(T_{\pi'}(\pi(1_H)) \times' (\pi(h) \leftrightarrow k(2)) \times' \pi(k(1))),$$

and so, by item 1) of Definition 6.26, $H$ is a right $H$-module coalgebra via $h \otimes k \mapsto h^k$, which completes the proof of condition 3). □

**Remark 6.31.** In the set-theoretic context, Proposition 6.30 yields [16, Proposition 1.11].

**Remark 6.32.** The correspondences in Proposition 6.30 yield an equivalence between the categories of GV-Hopf skew-braces and bijective 1-cocycles.

### 6.2 Comparison between Hopf skew-braces and Yetter-Drinfeld brances

Let $H$ be a Hopf algebra and let $s_1, s_2: H \otimes H \to H$ be maps. For each $h, l \in H$, set $h^l := s_1(h \otimes l)$ and $h^l := s_2(h \otimes l)$. Recall that $(H, s_1, s_2)$ is a matched pair if conditions 1)–6) above Theorem 5.4 are fulfilled. Assume that $(H, s_1, s_2)$ is a matched pair and let $s: H^2 \to H^2$ be the map defined by

$$s(h \otimes l) := s_1(h_1) \otimes s_2(l_2).$$

As we saw at the beginning of the proof of Theorem 5.27,

$$(\text{id} \otimes \epsilon) s = s_1, \quad (\epsilon \otimes \text{id}) s = s_2 \text{ and } s \text{ is a coalgebra morphism.}$$

We say that a matched pair $(H, s_1, s_2)$ is commutative if $s$ is a left non-degenerate solution of the braid equation and condition 5.8 is satisfied. We let $\mathcal{MP}1$ denote the category of commutative matched pairs.

Let $H$ and $s_1, s_2: H \otimes H \to H$ be as above. Recall from [10, Definition 2.1], that the triple $(H, s_1, s_2)$ is a matched pair of actions if condition 5.8 and conditions 1)–5) above Theorem 5.4 are satisfied. The definition of morphisms of matched pairs of actions is the same as that of matched pair morphisms. We let $\mathcal{MP}2$ denote the category of matched pairs of actions. Clearly $\mathcal{MP}1$ is a full subcategory of $\mathcal{MP}2$, but by Theorem 5.27, actually $\mathcal{MP}1 = \mathcal{MP}2$. 
Let $H$ be a Hopf algebra. A (left-left) Yetter–Drinfeld module on $H$ is the datum of a left $H$-module $X$, via an action \( h \otimes x \mapsto h \cdot x \), which is also a left $H$-comodule, via a coaction $\rho: X \to H \otimes X$, such that, for all $h \in H$ and $x \in X$,

$$\rho(h \cdot x) = (h_{(1)} x_{(-1)}) S(h_{(3)}) \otimes h_{(2)} x_{(2)} \quad \text{where } x_{(-1)} \otimes x_{(0)} := \rho(x).$$

A morphism of Yetter–Drinfeld modules is a morphism of both left $H$-modules and left $H$-comodules. It is well known that the category $\mathcal{H}_H^\mathcal{YD}$, of Yetter–Drinfeld modules on $H$, is a braided tensor category.

The following Definition is taken from [10, Definition 3.16].

**Definition 6.33.** A Yetter–Drinfeld brace is a triple $(H, \times, T)$, consisting of a Hopf algebra $H$, an associative internal operation $\times$, of $H$, and a map $T: H \to H$, such that:

- The underlying coalgebra of $H$, endowed with the multiplication $\times$, the same unit of $H$, the left action $h^l := T(h_{(1)}) \times h_{(2)} l$ and the coaction $\rho(h) := h_{(1)} S(h_{(3)}) \otimes h_{(2)}$, is a Hopf algebra in $\mathcal{H}_H^\mathcal{YD}$, with antipode $T$,

- if we define $h^l := S(h_{(1)} h_{(2)}) h_{(3)} l$, then $h^{l(1)} \otimes h^{l(2)} (a) = h^{l(2)} l(2) \otimes h^{l(1)} l(1)$, for all $h, l \in H$,

- $h(k \otimes l) = h_{(1)} k \times T(h_{(2)}) \times h_{(3)} l$, for all $h, k, l \in H$.

A morphism from a Yetter–Drinfeld brace $(H, \times, T)$ to a Yetter–Drinfeld brace $(K, \times, T)$, is a Hopf algebra morphism $f: H \to K$, such that $f(h \times l) = f(h) \times f(l)$ and $T f = f T$.

**Theorem 6.34.** The categories of braiding operators, Hopf-skew braces and Yetter–Drinfeld braces are isomorphic.

**Proof.** From Remark 5.19, it follows easily that the category $\mathcal{MP}1$, of commutative matched pairs, and the category of braiding operators are isomorphic. On the other hand, in [10, Theorem 3.25] it was proved that the categories of Yetter-Drinfeld braces and $\mathcal{MP}2$, matched pair of actions, are also isomorphic. Since $\mathcal{MP}1 = \mathcal{MP}2$, we conclude that the categories of braiding operators and Yetter-Drinfeld braces are isomorphic. By Remark 6.6, this finishes the proof. □

**Remark 6.35.** By Theorems 6.23 and 6.34, the categories of GV-Hopf skew-braces and Yetter-Drinfeld braces with bijective antipode are isomorphic.

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### 7 Ideals of Hopf q-braces

In this section we introduce the notions of an ideal and of a Hopf sub q-brace of a Hopf q-brace with bijective antipode, and we begin the study of these notions.

**Definition 7.1.** An ideal of a Hopf $q$-brace $\mathcal{H} = (H, \cdot, ;)$ is a Hopf ideal $I$ of $H$ such that $h \cdot k, k \cdot h, h : k$ and $k : h$ belong to $I$, for all $h \in H$ and $k \in I$.

**Proposition 7.2.** Let $\mathcal{H} = (H, \cdot, ;)$ be a Hopf $q$-brace. A Hopf ideal $I$ of $H$ is an ideal of $\mathcal{H}$ if and only if $h \cdot k$ and $k \cdot h$ belong to $I$, for all $h \in H$ and $k \in I$. Moreover, if the antipode of $H$ is bijective, then this happens if and only if $h \cdot k$ and $k \cdot h$ belong to $I$, for all $h \in H$ and $k \in I$.

**Proof.** Let $h \in H, k \in I$ and assume that $h \cdot k$ and $k \cdot h$ belong to $I$. Since, by identity [6.1],

$$h \cdot k = (h_{(1)} : h_{(2)}) h_{(1)} S(h_{(2)}) \quad \text{and} \quad k \cdot h = (h_{(1)} : k_{(2)}) h_{(1)} S(h_{(2)}),$$

we obtain that $h \cdot k$ and $k \cdot h$ belong to $I$. The proof of the last assertion is similar. □

**Remark 7.3.** Let $\mathcal{H} = (H, \cdot, ;)$ be a Hopf $q$-brace, and let $I$ be a Hopf ideal of $H$. The relation $h \simeq k$ if $h \in I$, is a congruence with respect to $\cdot$ and $;$ if and only if $I$ is an ideal of $\mathcal{H}$.

**Remark 7.4.** By Remark 7.3, each ideal $I$ of a Hopf $q$-brace $\mathcal{H} = (H, \cdot, ;)$ gives rise to a quotient Hopf $q$-brace $\mathcal{H}/I := (H/I, \cdot, ;)$ and a canonical surjective morphism $p: \mathcal{H} \to \mathcal{H}/I$, with the evident universal property. In the sequel we let $[h]$ denote the class of $h \in H$ in $\mathcal{H}/I$.

**Example 7.5.** Let $\mathcal{H} = (H, \cdot, ;)$ be a Hopf $q$-brace with bijective antipode and let $F, G: H \otimes H^{\text{cop}} \to H$ be the maps defined by $F(l \otimes h) = l \times h = (h \cdot l_{(1)}) l_{(2)}$ and $G(l \otimes h) := h \times l = (l : h_{(2)}) h_{(1)}$, respectively. Let $*$ be the usual convolution product in $\text{Hom}_{\mathcal{H}}(H \otimes H^{\text{cop}}, H)$. Note that $G$ is convolution invertible with convolution inverse $G^*(l \otimes h) = S^{-1}(h_{(2)}) S(l : h_{(1)})$. We define the $q$-commutator $[h, l]_q$, of $h, l \in H$, by

$$[h, l]_q := F * G^*(l \otimes h) = (h_{(3)} : l_{(1)}) l_{(2)} S^{-1}(h_{(2)}) S(l_{(3)} : h_{(1)})$$

(note that $[h, l]_q = \epsilon(h l) 1$, for all $h, l \in H$ if and only if $\mathcal{H}$ is a skew-brace). By definition, the $q$-commutator ideal $[\mathcal{H}, \mathcal{H}]_q$ of $\mathcal{H}$ is the ideal of $\mathcal{H}$ generated by the set $\{[h, l]_q - \epsilon(h l) 1 : h, l \in H\}$. Then, the quotient
Hopf q-brace $\mathcal{H}/[\mathcal{H}, \mathcal{H}]_q$ is a skew-brace and the canonical map $p: \mathcal{H} \to \mathcal{H}/[\mathcal{H}, \mathcal{H}]_q$ is universal in the following sense: given a Hopf q-brace morphism $f: \mathcal{H} \to \mathcal{K}$, with $\mathcal{K}$ a skew-brace with bijective antipode, there exists a unique Hopf skew-brace morphism $f: \mathcal{H}/[\mathcal{H}, \mathcal{H}]_q \to \mathcal{K}$ such that $f = fp$.

Next we adapt the notion of ideal of a q-brace, introduced in [25, Definition 8], to the context of Hopf q-braces.

**Definition 7.6.** Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf q-brace with bijective antipode. A Hopf sub q-brace of $\mathcal{H}$ is a normal Hopf subalgebra $A$ of $\mathcal{H}$ such that $a \cdot h, a : h, S^{-1}(h_{(2)})((h_{(1)} : a)$ and $S(h_{(1)})(h_{(2)} : a)$ belong to $A$, for all $a \in A$ and $h \in H$.

**Proposition 7.7.** Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf q-brace with bijective antipode. If $A$ is a Hopf sub q-brace of $\mathcal{H}$, then $A^+ H$ is an ideal of $\mathcal{H}$, where $A^+ := A \cap \ker \epsilon$.

**Proof.** By [22, Lemma 3.4.2] we know that $A^+ H$ is a Hopf ideal of $H$ and $H A^+ = A^+ H$. Let $a \in A^+$ and $h, k \in H$. By the first identity in [5.7], the fact that $\epsilon$ is compatible with $\cdot$, and the hypotheses,

$$ah \cdot k = ((a \cdot (k_{(1)} : h_{(2)})))(h_{(1)} : k_{(2)}) \in A^+ H,$$

and a similar computation proves that also $ah : k \in A^+ H$. Since $k \cdot ah = (k \cdot h) \cdot a$, in order to prove that $k \cdot ah \in A^+ H$, it suffices to check that $k \cdot a \in A^+ H$, for all $k \in H$ and $a \in A^+$. But this follows immediately from the fact that

$$k \cdot a = k_{(1)}(k_{(2)})(k_{(3)} : a) \in H A^+ = A^+ H.$$

Finally, a similar computation proves that $k \cdot ah \in A^+ H$. \hfill $\square$

**Proposition 7.8.** Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf skew-brace with bijective antipode. A normal Hopf subalgebra $A$ of $\mathcal{H}$ is a Hopf sub q-brace of $\mathcal{H}$ if and only if $a \cdot h \in A$ and $a : h \in A$, for all $a \in A$ and $h \in H$.

**Proof.** $\Rightarrow$ This is clear.

$\Leftarrow$ By identity [6.1], we have

$$S(h_{(1)})(h_{(2)} : a) = S(h_{(1)})(h_{(2)} : a_{(1)})a_{(2)}S(a_{(3)}) = S(h_{(1)})(a_{(1)} : h_{(3)})h_{(2)}S(a_{(2)})$$

and

$$S^{-1}(h_{(2)})(h_{(1)} : a) = S^{-1}(h_{(2)})(h_{(1)} : a_{(3)})a_{(2)}S^{-1}(a_{(1)}) = S^{-1}(h_{(3)})(a_{(2)} : h_{(1)})h_{(2)}S^{-1}(a_{(1)}).$$

Hence $S(h_{(1)})(h_{(2)} : a), S^{-1}(h_{(2)})(h_{(1)} : a) \in A$, because $A$ is a normal Hopf subalgebra. \hfill $\square$

**Proposition 7.9.** Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf q-brace with bijective antipode. If $A$ is a Hopf subalgebra of $H$ and $k \cdot h \in A$, for all $k, h \in A$, then $A$ is closed under $\times$ and $T_x$.

**Proof.** This is true since, by definition, $h \times k = (k \cdot h_{(2)})h_{(1)}$ and $T_x(h) = S(h_{(1)})S^{-1}(h_{(2)})$. \hfill $\square$

**Proposition 7.10.** Let $\mathcal{H} = (H, \cdot, :)$ be a Hopf skew-brace with bijective antipode and let $A$ be a normal Hopf subalgebra of $H$. Assume that $a \cdot h_{(1)} \otimes h_{(2)} = a \cdot h_{(2)} \otimes h_{(1)}$, for all $a \in A$ and $h \in H$. Then $A$ is a Hopf sub q-brace of $\mathcal{H}$ if and only if the following conditions are fulfilled:

a) $a \cdot h \in A$, for all $a \in A$ and $h \in H$,

b) $h_{(2)} \times a \times T_x(h_{(1)}) \in A$, for all $a \in A$ and $h \in H$.

**Proof.** By identity [6.1], the fact that $H$ is an $H^\text{op}$-module via $\cdot$ and Remark 1.13, we have

$$(a \cdot h_{(2)}): S(h_{(1)}) = ((a \cdot h_{(3)}): S(h_{(1)})h_{(1)}h_{(2)})$$

and

$$= (S(h_{(1)}) \cdot (a_{(1)} : h_{(4)}))(a_{(2)} : h_{(3)}h_{(2)})$$

$$= S(h_{(1)}) \cdot (h_{(3)}S^{-1}(h_{(2)})) \cdot (a_{(1)} : h_{(6)}))(a_{(2)} : h_{(5)}h_{(4)})$$

$$= ((S(h_{(1)}) \cdot S^{-1}(h_{(2)})) \cdot (a_{(1)} : h_{(6)}h_{(3)}))(a_{(2)} : h_{(5)}h_{(4)})$$

$$= (T_x(h_{(1)})(a_{(1)} : h_{(6)}h_{(3)}))(a_{(2)} : h_{(5)}h_{(4)})$$

$$= (T_x(h_{(1)})(a_{(1)} : h_{(6)}h_{(3)}))(a_{(2)} : h_{(5)}h_{(4)})$$

By [7.1] it is clear that if $A$ is a Hopf sub q-brace of $\mathcal{H}$, then $A$ is a normal Hopf subalgebra of $H$ and items a) and b) hold. Conversely, if $A$ is a normal Hopf subalgebra of $H$ and items a) and b) are satisfied, then again by [7.1]

$$a \cdot S(h) = ((a \cdot S(h_{(3)})) \cdot h_{(2)}): S(h_{(1)}) = h_{(2)} \times (a \cdot S(h_{(3)})) \times T_x(h_{(1)}) \in A,$$

and so, by Proposition 7.8 and the fact that $S$ is bijective, $A$ is a Hopf sub q-brace of $\mathcal{H}$. \hfill $\square$
7.1 The socle

Throughout this subsection we assume that $H = (H, \cdot, \varepsilon)$ is a Hopf $q$-brace with bijective antipode.

**Definition 7.11.** The socle of $\mathcal{H}$ is the set

$\text{Soc}(\mathcal{H}) := \{ h \in H : h_{(1)} \otimes k \cdot h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes k : h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes k \otimes h_{(2)} \; \forall k \in H \}.$

**Remark 7.12.** In the set-theoretic context, Definition 7.11 corresponds to [25, formula (31)].

**Remark 7.13.** Let $\ell$ denote any of the operations $\cdot$ or $\triangleright$. Note that if $h \in \text{Soc}(\mathcal{H})$, then

$$k \ell h_{(1)} \otimes h_{(2)} = k \otimes h, \; h_{(1)} \otimes k \ell h_{(2)} = h \otimes k \quad \text{and} \quad k \ell h = k\varepsilon(h),$$

for all $k \in H$. Note also that if $k \ell h_{(1)} \otimes h_{(2)} = k \ell h_{(2)} \otimes h_{(1)}$, for all $h, k \in H$ (which happens, for instance, if $H$ is cocommutative or if $\cdot$ and $\triangleright$ are the trivial actions), then the first two conditions in (7.2) are equivalent. Moreover, under this hypothesis, if the first condition is true for $\ell = \cdot$ and $\ell = \triangleright$, then $h \in \text{Soc}(\mathcal{H})$.

**Proposition 7.14.** The socle of $\mathcal{H}$ is a Hopf subalgebra of $H$. Moreover

$$S(m_{(1)})(m_{(2)} \cdot h), S^{-1}(m_{(2)}) \cdot m_{(1)}, h \cdot m, h : m \in \text{Soc}(\mathcal{H}), \quad \text{for all } h \in \text{Soc}(\mathcal{H}) \text{ and } m \in H.$$

**Proof.** It is clear that $1 \in \text{Soc}(\mathcal{H})$ and $\text{Soc}(\mathcal{H})$ is a linear subspace of $H$. Let $\ell$ denote any of the operations $\cdot$ or $\triangleright$. Let $h, k \in \text{Soc}(\mathcal{H})$ and $k \in H$. Since

$$h_{(1)}k \triangleright l_{(1)} \otimes k \triangleright l_{(2)} \otimes h_{(3)}l_{(3)} = h_{(1)}k \triangleright (k \triangleright l_{(2)}) \triangleright h_{(3)}l_{(3)} = h_{(1)}k \triangleright k \otimes h_{(2)}l_{(2)},$$

the socle of $\mathcal{H}$ is closed under products. We claim that $\Delta(\text{Soc}(\mathcal{H})) \subseteq \text{Soc}(\mathcal{H}) \otimes \text{Soc}(\mathcal{H})$. Take $h \in \text{Soc}(\mathcal{H})$ and write $\Delta(h) = \sum_{i=1}^{r} U_i \otimes V_i$ with $r$ minimal. For all $k \in H$, we have

$$\sum_{i=1}^{r} U_{(1)}^i \otimes k \triangleright U_{(2)}^i \otimes U_{(3)}^i \otimes V^i = h_{(1)} \otimes k \triangleright h_{(2)} \otimes \Delta(h_{(2)}) = h_{(1)} \otimes k \otimes \Delta(h_{(2)}) = \sum_{i=1}^{r} U_{(1)}^i \otimes k \otimes U_{(2)}^i \otimes V^i.$$

Since the $V_i$s are linearly independent, this implies that $U_{(1)}^i \otimes k \triangleright U_{(2)}^i \otimes U_{(3)}^i = U_{(1)}^i \otimes k \otimes U_{(2)}^i$ for all $i$. Thus, $U_i \in \text{Soc}(\mathcal{H})$, for all $i$, and similarly for $V_i$. This finishes the proof of the claim. Let $h \in \text{Soc}(\mathcal{H})$ and $k \in H$. From the equality $h_{(1)} \otimes k \triangleright h_{(2)} = h_{(1)} \otimes k \cdot h_{(2)} \otimes h_{(3)}$, we obtain

$$h_{(1)} \otimes k \cdot S(h_{(2)}) \otimes h_{(3)} = h_{(1)} \otimes (k \cdot h_{(3)}) \triangleright S(h_{(2)}) \otimes h_{(3)} = h_{(1)} \otimes k \cdot S(h_{(2)}) \otimes h_{(3)} \otimes h_{(4)} = h_{(1)} \otimes k \cdot h_{(2)},$$

and so, $S(\text{Soc}(\mathcal{H})) \subseteq \text{Soc}(\mathcal{H})$. Let $h \in \text{Soc}(\mathcal{H})$ and $m \in H$. By the third equality in (7.2), we have

$$S(m_{(1)})(m_{(2)} \cdot h) = S(m_{(1)})m_{(2)} \varepsilon(h) = 1\varepsilon(mh) = S^{-1}(m_{(2)})m_{(1)}\varepsilon(h) = S^{-1}(m_{(2)})(m_{(1)} : h),$$

and so $S(m_{(1)})(m_{(2)} \cdot h), S^{-1}(m_{(2)})(m_{(1)} : h) \in \text{Soc}(\mathcal{H})$. We next prove that $h \cdot m, h : m \in \text{Soc}(\mathcal{H})$. For this it suffices to check that, for all $k \in H$,

$$h_{(1)} \cdot m_{(3)} \otimes k \cdot (h_{(2)} \cdot m_{(2)}) \otimes h_{(3)} \cdot m_{(1)} = h_{(1)} \cdot m_{(2)} \otimes k \otimes h_{(2)} \cdot m_{(1)},$$

and

$$h_{(1)} : m_{(3)} \otimes k \cdot (h_{(2)} : m_{(2)}) \otimes h_{(3)} : m_{(1)} = h_{(1)} : m_{(2)} \otimes k \otimes h_{(2)} : m_{(1)}.$$
Proof. By the previous proposition we know that \( \text{Soc}(H) \) is a Hopf subalgebra of \( H \) and that condition [7.3] is satisfied. It remains to prove that \( \text{Soc}(H) \) is a normal Hopf subalgebra of \( H \), which means that \( \text{S}(h_{(1)})ah_{(2)}, h_{(1)}aS(h_{(2)}) \in \text{Soc}(H) \), for all \( a \in \text{Soc}(H) \) and \( h \in H \). Let \( \ell \) denote any of the operations \( \cdot \) or \( : \) and let \( a \in \text{Soc}(H) \) and \( h, k \in H \) arbitrary. Then
\[
\ell \left( \text{S}(h_{(1)})a_{(1)}h_{(2)}) \right) \otimes \left( \text{S}(h_{(1)})a_{(1)}h_{(2)}) \right) = \ell \left( \text{S}(h_{(2)})a_{(1)}h_{(3)} \otimes \text{S}(h_{(1)})a_{(1)}h_{(4)} \right)
\]
and similarly
\[
\ell \left( \text{S}(h_{(1)})a_{(1)}h_{(2)}) \right) \otimes \left( \text{S}(h_{(1)})a_{(1)}h_{(2)}) \right) = h_{(1)}a_{(1)}S(h_{(4)}) \otimes k \ell \left( h_{(2)}a_{(2)}h_{(3)}S(h_{(1)})a_{(1)}h_{(4)} \right)
\]
Hence, by Remark 7.13, \( \text{S}(h_{(1)})ah_{(2)}, h_{(1)}aS(h_{(2)}) \in \text{Soc}(H) \), as desired.

**Proposition 7.16.** If \( h \in \text{Soc}(H) \), then \( h \times k = h \times k \) for all \( k \in H \).

Proof. If \( h \in \text{Soc}(H) \), then, by Remark 7.13 and Proposition 7.14,
\[
h \times k = (k \cdot h_{(1)})h_{(2)} = kh = (k \cdot h_{(2)})h_{(1)} = h \times k,
\]
for all \( k \in H \), as desired.

**Remark 7.17.** Proposition 7.16 implies Corollary 1 of [25, Proposition 8].

**Proposition 7.18.** Assume that \( k \cdot h_{(1)} \otimes h_{(2)} = k \cdot h_{(2)} \otimes h_{(1)} \) and \( k \cdot h_{(1)} \otimes h_{(2)} = k \cdot h_{(2)} \otimes h_{(1)} \) for all \( h, k \in H \). Then \( \mathcal{H}/\text{Soc}(\mathcal{H})^+\mathcal{H} \) is a Hopf skew-brace.

Proof. Let \( F, G \) and \( G^* \) be as in Example 7.5. By the universal property mentioned in that example, in order to prove the result it suffices to check that the convolution product \( F \ast G^* \) takes its values in \( \text{Soc}(\mathcal{H}) \). By Remark 7.13 and our hypothesis, it suffices to check that for all \( k, l, h \in H \),
\[
k \cdot F \ast G^*(l \otimes h)_{(1)} \otimes F \ast G^*(l \otimes h)_{(2)} = k \cdot F \ast G^*(l \otimes h)_{(1)} \otimes F \ast G^*(l \otimes h)_{(2)} = k \otimes F \ast G^*(l \otimes h).
\]
But, by the fact that \( H \) is an \( H^{\text{op}} \)-module via \( \cdot \), the hypothesis and the first condition in Definition 4.4,
\[
k \cdot F \ast G^*(l \otimes h)_{(1)} \otimes F \ast G^*(l \otimes h)_{(2)}
\]
follows by a similar computation.

**Remark 7.19.** Proposition 7.14, together with Proposition 7.18, yields [25, Proposition 8].

8 The free Hopf q-brace over a very strongly regular q-cycle coalgebra

In this section we construct the universal Hopf q-brace with bijective antipode and the universal Hopf skew-brace with bijective antipode of a very strongly regular q-cycle coalgebra.

Let \( X \) be a very strongly regular q-cycle coalgebra, let \( Y := X^{(2)} \) and let \( i_j : X \to Y \) be the \( j \)-th canonical injection. We set \( X_j := i_j(X) \), and we write \( x_j := i_j(x) \) for each \( x \in X \). Let \( S : Y \to Y \) be the bijective map defined by \( S(x_j) := x_{j+1} \). We consider \( Y \) endowed with the unique coalgebra structure
such that \(i_0: X \to Y\) is a coalgebra morphism and \(S\) is an coalgebra antiautomorphism. Note that \(i_j\) is a coalgebra morphism if \(j\) is even and a coalgebra antiautomorphism if \(j\) is odd. In this section we will freely use the notations introduced in Definition 3.1 and above Definition 3.6. Motivated by Propositions 5.28 and 5.33, we introduce bilinear operations \(x \otimes y \mapsto x \cdot y\) and \(x \otimes y \mapsto x : y\) on \(Y\) as follows: first we define \(\cdot\) and \(\cdot:\) on \(X_0\) by translation of structure through \(i_0\). Next, for each \(x, y \in X_0\) and all \(m, n \in \mathbb{Z}\), we define

\[
S^m(x) \cdot S^n(y) := \begin{cases} 
S^m(x \cdot y) & \text{if } m \text{ is even and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(x) & \text{if } m \text{ is even and } n - m = 2i - 1 \text{ with } i \in \mathbb{Z}, \\
S^m(x \circ_i y) & \text{if } m \text{ is odd and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(y) & \text{if } m \text{ is odd and } n - m = 2i - 1 \text{ with } i \in \mathbb{Z}
\end{cases}
\]

and

\[
S^m(x) : S^n(y) := \begin{cases} 
S^m(x \cdot y) & \text{if } m \text{ is even and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(x) & \text{if } m \text{ is even and } n - m = 2i + 1 \text{ with } i \in \mathbb{Z}, \\
S^m(x \circ_i y) & \text{if } m \text{ is odd and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(y) & \text{if } m \text{ is odd and } n - m = 2i + 1 \text{ with } i \in \mathbb{Z}.
\end{cases}
\]

**Remark 8.1.** A direct computation using Lemma 3.4 proves that the maps \(\cdot\) and \(\cdot:\) defined above are coalgebra morphisms from \(Y \otimes Y^{\text{cop}}\) to \(Y\).

**Remark 8.2.** We introduce bilinear operations \(x \otimes y \mapsto x^y\) and \(x \otimes y \mapsto x_y\) on \(Y\) as follows: first we define \(x^y\) and \(x_y\) on \(X_0\) by translation of structure through \(i_0\); next, for each \(x, y \in X_0\) and all \(m, n \in \mathbb{Z}\), we define

\[
S^m(x)^{S^n(y)} := \begin{cases} 
S^m(x^y) & \text{if } m \text{ is even and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(x \cdot y^i) & \text{if } m \text{ is even and } n - m = 2i - 1 \text{ with } i \in \mathbb{Z}, \\
S^m(x) & \text{if } m \text{ is odd and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(y) & \text{if } m \text{ is odd and } n - m = 2i - 1 \text{ with } i \in \mathbb{Z}.
\end{cases}
\]

and

\[
S^m(x)S^n(y) := \begin{cases} 
S^m(x^y) & \text{if } m \text{ is even and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(x \cdot y^{i+1}) & \text{if } m \text{ is even and } n - m = 2i + 1 \text{ with } i \in \mathbb{Z}, \\
S^m(x) & \text{if } m \text{ is odd and } n - m = 2i \text{ with } i \in \mathbb{Z}, \\
S^m(y) & \text{if } m \text{ is odd and } n - m = 2i + 1 \text{ with } i \in \mathbb{Z}.
\end{cases}
\]

Equalities [3.2], [3.3], [3.4] and [3.5] applied to \(X\) and \(\hat{X}\) (see the discussion above Definition 3.6) show that \((Y, \cdot)\) satisfies identities [1.2] and \((Y, : )\) satisfies identities [1.5].

**Remark 8.3.** By Proposition 1.2 the maps \(x \otimes y \mapsto x^y\) and \(x \otimes y \mapsto x_y\), introduced in Remark 8.2, are coalgebra morphisms from \(Y^2\) to \(Y\).

Motivated by Proposition 2.11 and Remarks 1.18 and 1.20, we define bilinear operations \(x \otimes y \mapsto x^y\) and \(x \otimes y \mapsto x_y\) on \(Y\), by \(y^y := y \circ y\) and \(x_y := y(1) \cdot x y(2)\), respectively.

**Proposition 8.4.** The maps \(\cdot\) and \(\cdot:\) defined above Remark 8.1 satisfy identity [1.4]

**Proof.** With must prove that

\[
S^m(x)(1) : S^n(y)(2) \otimes S^m(y)(1) \cdot S^n(x)(2) = S^m(x)(2) : S^n(y)(1) \otimes S^m(y)(2) \cdot S^n(x)(1)
\]

for all \(x, y \in X\) and \(m, n \in \mathbb{Z}\). We consider separately four cases according to the parity of \(m\) and \(n\). Suppose that \(m\) and \(n\) are even and write \(n - m = 2i\). We have

\[
S^m(x)(1) : S^n(y)(2) \otimes S^m(y)(1) \cdot S^n(x)(2) = (S^m \otimes S^n)(x(1) \cdot y(2) \otimes y(1) \cdot y(2) \cdot x(1)) = (S^m \otimes S^n)(x(2) \cdot y(1) \otimes y(2) \cdot x(1)) = S^m(x)(2) : S^n(y)(1) \otimes S^m(y)(2) \cdot S^n(x)(1)
\]

where the second equality holds by [3.9]. The other cases are similar.

By Proposition 8.4 and Remarks 1.13 and 8.1, the tuple \(\hat{Y} := (Y, \cdot, : )\) is a \(q\)-magma coalgebra. Moreover, by Remark 8.2 we know that \(\hat{Y}\) is regular. Thus, by Remark 1.20 and Proposition 2.7, the maps

\[
s: Y^2 \to Y^2 \quad \text{and} \quad \hat{s}: (Y^{\text{cop}})^2 \to (Y^{\text{cop}})^2,
\]

defined by \(s(x \otimes y) := x(1) y(1) \otimes x(2) y(2)\) and \(\hat{s}(x \otimes y) := x(1) y(1) \otimes x(2) y(2)\), respectively, are left non-degenerate coalgebra endomorphism.
Consider the tensor algebra $T(Y)$, endowed with the unique bialgebras structure such that the canonical map $i: Y \to T(Y)$ becomes a coalgebra morphism.

**Lemma 8.5.** There exist unique extensions of the binary operations $x \otimes y \mapsto x \cdot y$ and $x \otimes y \mapsto x : y$ on $Y$ to coalgebra morphisms from $T(Y) \otimes T(Y)^{op}$ to $T(Y)$ such that $(T(Y), \cdot)$ and $(T(Y), :)$ are $T(Y)^{op}$-modules and identities [5.7] are fulfilled. Moreover, $1 : h = 1 : e = (h)1$ for all $h \in T(Y)$.

**Proof.** Recall that $T(Y) = k \oplus Y \oplus Y^2 \oplus \cdots$. We say that an element $h$ of $T(Y)$ has length $n$ if $h \in Y^n$. We extend the definitions of $\cdot$ and $: T(Y) \otimes Y$ by setting $1 : y = e(y)1$ and recursively defining

$$(hx) \cdot y := (h \cdot (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)}) \quad \text{and} \quad (hx) : y := (h : (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)}),$$

for all $h \in Y^n$ and $x, y \in Y$; and then, we extend $\cdot$ and $:$ to $T(Y)^2$ by setting

$$h \cdot 1 = h : 1 = h, \quad h \cdot kx := (h : x) \cdot k \quad \text{and} \quad h : kx := (h : x) : k,$$

for all $h \in T(Y), k \in Y^n$ and $x \in Y$. Since these definitions are forced by the conditions required in the statement, they are unique. We leave to the reader to check that these maps are compatible with the counits and that $1 \cdot h = h = e(h)1$. Next we prove that

$$(h \cdot k)_{(1)} \otimes (h \cdot k)_{(2)} = h_{(1)} \cdot k_{(2)} \otimes h_{(2)} \cdot k_{(1)} \quad \text{and} \quad (h : k)_{(1)} \otimes (h : k)_{(2)} = h_{(1)} : k_{(2)} \otimes h_{(2)} : k_{(1)} \quad [8.1]$$

for all $h, k \in T(Y)$. Note that for $h, k \in Y$, these equalities are true by Remark 8.1. Since both identities can be seen in a similar way, we only prove the first one. Assume that it is true for $h \in Y^n$ and $k \in Y$. Then, by the inductive hypothesis and the fact that [1.4] holds on $Y$, we have:

$$(hx \cdot y)_{(1)} \otimes (hx \cdot y)_{(2)} = \begin{array}{c}
\left( \begin{array}{c}
(h \cdot (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)}) \\
(h \cdot (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)})
\end{array} \right) \\
\left( \begin{array}{c}
(h \cdot (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)}) \\
(h \cdot (y_{(1)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(2)})
\end{array} \right)
\end{array}$$

$$= (h_{(1)} \cdot (y_{(2)} \cdot x_{(3)})) (x_{(1)} \cdot y_{(4)}) \otimes (h_{(2)} \cdot (y_{(1)} \cdot x_{(4)})) (x_{(2)} \cdot y_{(3)})$$

$$= (h_{(1)} \cdot (y_{(3)} \cdot x_{(2)})) (x_{(1)} \cdot y_{(4)}) \otimes (h_{(2)} \cdot (y_{(1)} \cdot x_{(4)})) (x_{(3)} \cdot y_{(2)})$$

$$= h_{(1)} x_{(1)} \cdot y_{(2)} \otimes h_{(2)} x_{(2)} \cdot y_{(1)},$$

for all $x, y \in Y$. This proves the equality for all $h \in T(Y)$ and $k \in Y$. Assume now that this identity is true for all $h \in T(Y)$ and $k \in Y^n$. Then, we have

$$(h \cdot kx)_{(1)} \otimes (h \cdot kx)_{(2)} = (h_{(1)} \cdot x_{(2)}) \cdot k_{(2)} \otimes (h_{(2)} \cdot x_{(1)}) \cdot k_{(1)} = h_{(1)} \cdot (kx)_{(2)} \otimes h_{(2)} \cdot (kx)_{(1)},$$

for all $x \in Y$, which finishes the proof of the first equality in [8.1]. We next check the first identity in [5.7], and leave the second one, which is similar, to the reader. Clearly, this is fulfilled when $h = 1, k = 1$ or $l = 1$. Assume that it is true for all $h \in T(Y), k \in Y^n$ and $l \in Y$. Then,

$$h(kx) \cdot l = ((hk) \cdot (l_{(1)} : x_{(2)})) (x_{(1)} \cdot l_{(2)})$$

$$= (h \cdot (l_{(1)} : x_{(3)})) (k_{(2)})) (k_{(1)} \cdot (l_{(2)} : x_{(2)})) (x_{(1)} \cdot l_{(3)})$$

$$= (h \cdot (l_{(1)} \cdot (kx)_{(2)})) ((kx)_{(1)} \cdot l_{(2)}).$$

for all $x \in Y$. So, the first identity in [5.7] holds for all $h, k \in T(Y)$ and $l \in Y$. Assume now that it holds for all $h \in T(Y), k \in Y$ and $l \in Y^n$. Then, for all $x \in Y$,

$$hk \cdot xl = ((h \cdot (l_{(1)} : k_{(2)})) (k_{(1)} \cdot l_{(2)})) \cdot x$$

$$= ((h \cdot (l_{(1)} : k_{(3)})) \cdot (x_{(1)} : (k_{(2)} \cdot l_{(2)}))) ((k_{(1)} \cdot l_{(3)}) \cdot x_{(2)})$$

$$= (h \cdot (x_{(1)} : (k_{(2)} \cdot l_{(2)})) (l_{(1)} : k_{(3)})) (k_{(1)} \cdot x_{(2)} l_{(3)})$$

$$= (h \cdot ((x_{(1)} l_{(1)} : k_{(2)})) (l_{(1)} : x_{(2)} l_{(2)}),$$

which proves the first identity in [5.7] for all $h, l \in T(Y)$ and $k \in Y$. Finally, assume that this is true for all $h, l \in T(Y)$ and $k \in Y^n$. Then, for all $x \in Y$, we have

$$h(kx) \cdot l = (hk \cdot (l_{(1)} : x_{(2)})) (x_{(1)} \cdot l_{(2)})$$

$$= (h \cdot (l_{(1)} : x_{(3)})) (k_{(2)})) (k_{(1)} \cdot (l_{(2)} : x_{(2)})) (x_{(1)} \cdot l_{(3)})$$

$$= (h \cdot (l_{(1)} \cdot (kx)_{(2)})) ((kx)_{(1)} \cdot l_{(2)}),$$

which finishes the proof of the first identity in [5.7].

**Lemma 8.6.** $T(Y) := (T(Y), \cdot, :)$ is a regular $q$-magma coalgebra. \qed
Proof. First we prove that \( T(Y) \) is a \( q \)-magma coalgebra. By Remark 1.13 and Lemma 8.5, for this it suffices to show that identity \([1, 4]\) holds for all \( x, y \in T(Y) \). By hypothesis we know that this identity holds for all \( x, y \in Y \). Moreover, a direct computation shows that it also holds when \( x = 1 \) and \( y \in T(Y) \), and when \( y = 1 \) and \( x \in T(Y) \). Assume that

\[
h_{(1)} : y_{(2)} \otimes y_{(1)} \cdot h_{(2)} = h_{(2)} : y_{(1)} \otimes y_{(2)} \cdot h_{(1)}
\]

for all \( h \in Y^n \) and \( y \in Y \). Then, by Lemma 8.5 and the inductive hypothesis, for all \( h \in Y^n \) and \( x, y \in Y \), we have

\[
h_{(1)} x_{(1)} : y_{(2)} \otimes y_{(1)} \cdot h_{(2)} x_{(2)} = (h_{(1)} : (y_{(2)} \cdot x_{(2)}) \cdot (x_{(1)} : y_{(3)}) \otimes (y_{(1)} \cdot x_{(3)}) \cdot h_{(2)})
\]

\[
= (h_{(2)} : (y_{(1)} \cdot x_{(3)}) \cdot (x_{(1)} : y_{(3)}) \otimes (y_{(2)} : x_{(2)}) \cdot h_{(1)})
\]

\[
= (h_{(2)} : (y_{(1)} \cdot x_{(3)}) \cdot (x_{(2)} : y_{(2)}) \otimes (y_{(3)} : x_{(1)}) \cdot h_{(1)})
\]

\[
= h_{(2)} x_{(2)} : y_{(1)} \otimes y_{(2)} \cdot h_{(1)} x_{(1)},
\]

which proves \([8, 2]\) for all \( h \in T(Y) \) and \( y \in Y \). Finally, assume that \( h_{(1)} : k_{(2)} \otimes k_{(1)} \cdot h_{(2)} = h_{(2)} : k_{(1)} \otimes k_{(2)} \cdot h_{(1)} \) for all \( h \in T(Y) \) and \( k \in Y^n \). Then, for all \( h \in T(Y), k \in Y^n \) and \( y \in Y \),

\[
(h_{(1)} : k_{(2)} y_{(2)} \otimes k_{(1)} y_{(1)} \cdot h_{(2)} = (h_{(1)} : k_{(2)} \otimes k_{(1)} \cdot (h_{(2)} : y_{(2)}))(y_{(1)} : h_{(3)})
\]

\[
= (h_{(2)} : y_{(2)} \cdot k_{(1)} \otimes (h_{(1)} : k_{(2)} \cdot y_{(2)}))(y_{(1)} : h_{(3)})
\]

\[
= (h_{(2)} : y_{(2)} \otimes k_{(1)} \otimes (h_{(1)} : k_{(2)} \cdot y_{(2)}))(y_{(1)} : h_{(2)}
\]

\[
= h_{(2)} k_{(1)} y_{(1)} \otimes k_{(2)} y_{(2)} \cdot h_{(1)},
\]

for all \( h \in T(Y), k \in Y^n \) and \( y \in Y \), which proves \([1, 4]\) on \( T(Y) \). For the purpose of checking that \( T(Y) \) is regular, we first extend the maps introduced in Remark 8.2 to \( T(Y) \otimes Y \), by setting \( 1^y = 1_y = \epsilon(y)1 \) and recursively defining

\[
(hx)^y := \left( h^{(1)} x^{(1)} \right)(x^{(2)} \cdot y^{(2)}) \quad \text{and} \quad (hx)^y := \left( h^{(1)} y^{(1)} \right)(x^{(2)} \cdot y^{(2)}),
\]

for all \( h \in Y^n \) and \( x, y \in Y \); and then we extend these maps to \( T(Y)^2 \) by setting \( h^1 = h_{(1)} = h_{(2)} = h_{(3)}^x = h_{(3)}^y \) and \( h_{(k)} := (h_{(k)})^y, k \in Y^n \) and \( x, y \in Y \).

Next we prove that \( T(Y) \) is left regular, and hence that it is right regular, which is similar, to the reader. By Remark 8.2 we know that \( x^{(1)} : y_{(2)} = \epsilon(y)x \) for all \( x, y \in Y \). Moreover, \( h^1 \cdot 1 = h = \epsilon(1)h \) and \( 1h_{(1)} = 1h_{(2)} = \epsilon(h)1 \) for all \( h \in T(Y) \). Assume that \( h^{(1)} : y_{(2)} = \epsilon(h)y \) for all \( h \in Y^n \) and \( y \in Y \). Then, by the first identity in \([5, 7]\) and the discussion above Proposition 8.4, for all \( h \in Y^n \) and \( x, y \in Y \), we have

\[
(hx)^{y(1)} : y_{(2)} = \left( (h^{(1)} : y_{(2)})(x^{(1)} \cdot y^{(1)}) \right) y_{(3)} = \left( (h^{(1)} : x^{(3)})(y^{(3)} \cdot x^{(2)})(x^{(1)} \cdot y^{(6)}) \right) y_{(6)} = \epsilon(y)hx,
\]

and so \( h^{(1)} : y_{(2)} = \epsilon(y)h \) for all \( h \in T(Y) \) and \( y \in Y \). Assume now that \( h^{(1)} : k_{(2)} = \epsilon(h)k \) for all \( h \in T(Y) \) and \( k \in Y^n \). Then, for all \( h \in T(Y), k \in Y^n \) and \( y \in Y \), we have

\[
h^{(1)} : k_{(2) : y_{(2)}} = \left( (h^{(1)} : k^{(2)})(y^{(1)} \cdot k^{(1)}) \right) k_{(2)} = \epsilon(h)k^{(1)} \cdot k_{(2)} = \epsilon(ky)h,
\]

and hence \( h^{(1)} : k_{(2)} = \epsilon(k)h \) for all \( h, k \in T(Y) \). Again by Remark 8.2 we know that \( x \cdot y_{(1)}^{(2)} = \epsilon(y)h \) for all \( x, y \in Y \). Moreover, \( (1 \cdot h_{(1)}^{(2)} = \epsilon(h)1 \) for all \( h \in T(Y) \). Assume that \( h : y_{(1)}^{(2)} = \epsilon(y)h \) for all \( h \in Y^n \) and \( y \in Y \). Then, for all \( h \in Y^n \) and \( x, y \in Y \), we have

\[
(hx : y_{(1)}^{(2)}) = \left( (h \cdot (y_{(1)} : x_{(2)}))(x_{(1)} \cdot y_{(2)}) \right)^{(y_{(3)})}
\]

\[
= \left( (h \cdot (y_{(1)} : x_{(2)}))(y_{(3)}^{(1)} y_{(4)})(x_{(2)} \cdot y_{(2)}) \right)^{(y_{(5)})}
\]

\[
= \left( (h \cdot (y_{(1)} : x_{(2)}))(y_{(2)}^{(1)} x_{(2)})(x_{(1)} \cdot y_{(3)}) \right)^{(y_{(6)})}
\]

\[
= \epsilon(y)hx,
\]

and hence \( h : y_{(1)}^{(2)} = \epsilon(y)h \) for all \( h \in T(Y) \) and \( y \in Y \). Assume now that \( h : k_{(1)}^{(2)} = \epsilon(k)h \) for all \( h \in T(Y) \) and \( k \in Y^n \). Then, for all \( h \in T(Y), k \in Y^n \) and \( y \in Y \), we have

\[
(h : k_{(1)}^{(2)})(y_{(2)}) = \left( (h : k_{(1)}^{(2)})(y_{(1)} \cdot k_{(2)}) \right) y_{(2)} = \epsilon(k)(h : y_{(1)}) y_{(2)} = \epsilon(ky)h,
\]

and so \( h : k_{(1)}^{(2)} = \epsilon(k)h \) for all \( h, k \in T(Y) \), which finishes the proof that \( T(Y) \) is regular. \( \square \)

Theorem 8.7. Let \( \mathcal{X} \) be a very strongly regular \( q \)-cycle coalgebra. There exists a Hopf \( q \)-brace \( \mathcal{H}_X \) with bijective antipode and a \( q \)-cycle coalgebra morphism \( \iota_X: \mathcal{X} \rightarrow \mathcal{H}_X \), which is universal in the following sense: given a Hopf \( q \)-brace with bijective antipode \( \mathcal{L} \) and a \( q \)-cycle coalgebra morphism \( f: \mathcal{X} \rightarrow \mathcal{L} \) there exists a unique Hopf \( q \)-brace morphism \( \bar{f}: \mathcal{H}_X \rightarrow \mathcal{L} \) such that \( \bar{f} \circ \iota_X = f \).
Proof. By Lemmas 8.5 and 8.6 we know that $T(Y) = \langle T(Y), \cdot, \cdot \rangle$ is a regular $q$-magma coalgebra, that $T(Y)$ is a bialgebra, that $(T(Y), \cdot)$ and $(T(Y), \cdot)$ are $T(Y)^{op}$-right modules and that $\cdot$ and $\cdot$ satisfy identities [5,7]. Moreover, it is clear that the coalgebra isomorphism $S: Y \to Y$ can be extended in a unique way to a bialgebra isomorphism (also called $S$) from $T(Y)$ to $T(Y)^{op}$. Let $I$ be the minimal bi-ideal of $T(Y)$ such that:

- $h \cdot k, k \cdot h, h : k$ and $k : h$ belong to $I$, for all $h \in T(Y)$ and $k \in I$,
- $S(h(1))h(2) - \epsilon(h)1 \in I$ and $h(1)S(h(2)) - \epsilon(h)1 \in I$, for all $h \in T(Y)$,
- $(h \cdot k)(1)(l \cdot k)(2) - (h \cdot l)(1)(k \cdot l)(1) \in I$, for all $h, l, k \in T(Y)$,
- $(h \cdot k)(1)(l \cdot k)(2) - (h \cdot l)(1)(k \cdot l)(2) \in I$, for all $h, l, k \in T(Y)$,
- $(h \cdot k)(1)(l \cdot k)(2) - (h \cdot l)(1)(k \cdot l)(1) \in I$, for all $h, l, k \in T(Y)$.

Let $H_X : T(Y)/I$. Then $H_X$ is a Hopf algebra with bijective antipode and the operations $\cdot$ and $\cdot$ induce operations $\cdot$ and $\cdot$ on $T(Y)$ $H_X$, such that $H_X : T(Y)/I$. By the universal property of the tensor algebra, there is a unique extension of $f$ to a coalgebra morphism $f_1: Y \to L$ such that $f_1(S(x)) = S_1(f(x))$ for all $x \in X$ and $\epsilon \in \epsilon$. By the definitions of $\cdot$ and $\cdot$ on $Y$ and Proposition 3.5, applied to $X$ and $Y$ (see Definition 3.6), the map $f$ is compatible with the operations $\cdot$ and $\cdot$. By the universal property of the tensor algebra, $f_1$ extends to a unique bialgebra map $f_2: T(Y) \to L$, which, by the fact that identities [5,7] are valid on $T(Y)$ and $L$, is also compatible with the operations $\cdot$ and $\cdot$. Finally, since $L$ is a Hopf algebra and the identities in Definition 4.4 are satisfied in $L$, this map factorizes through $H_X$. 

Remark 8.8. Let $X$ be a very strongly regular $q$-cycle coalgebra. The quotient $H_X/[H_X, H_X]_q$ is a Hopf skew-brace with bijective antipode (Example 7.5) and the map $\iota: X \to Y$ is the canonical surjection, is universal in the following sense: given a Hopf skew-brace with bijective antipode $L$ and a $q$-cycle coalgebra morphism $f: X \to L$ there exists a unique Hopf skew-brace morphism $\bar{f}: H_X \to L$, such that $f = \bar{f}\iota_X$.

Remark 8.9. Let $s: Y \times X \to Y$ be a set theoretic bijection non-degenerate solution of the braid equation, let $k$: $kY \otimes kY$ be the linearization of $s$, and let $X$ be the non-degenerate $q$-cycle coalgebra associated with $k$ according to Corollary 4.12. In [2], Theorem 4], the authors construct a group $G(Y, s)$, a braiding operator $s^G$ on $G(Y, s)$ and a map $\iota: Y \to G(Y, s)$, such that $s^G(\iota \times \iota) = (\iota \times \iota)$, which is universal. By Remark 6.5 the linearization $ks^G$ of $s^G$ is the braiding operator associated with the skew-brace $H_X/[H_X, H_X]_q$ introduced in Remark 8.8. Thus the underlying Hopf algebra of $H_X/[H_X, H_X]_q$ is the group algebra of $G(Y, s)$.

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