FROM FLOPS TO DIFFEOMORPHISM GROUPS
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Abstract. We exhibit many examples of closed complex surfaces whose diffeomorphism groups are not simply-connected and contain loops that are not homotopic to loops of symplectomorphisms.

1. Introduction. Let $X_0$ be a closed algebraic surface with a single ordinary double-point. Assume that $X_0$ admits a global smoothing $f : X \to \Delta$, $X_t = f^{-1}(t)$, where $\Delta \subset \mathbb{C}$ is a complex disk, such that $f$ has a single isolated singularity, modeled in local complex coordinates by

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$ 

As $X_0$ is a surface, it has a unique minimal resolution $q : \tilde{X}_0 \to X_0$. The exceptional locus of this resolution is a smooth rational curve $C \subset \tilde{X}_0$ of self-intersection number $(-2)$. Atiyah [3] proved that $\tilde{X}_0$ is diffeomorphic to the surface $X_t$ for any $t \in \Delta - \{0\}$. Suppose that

$$\pi_1(\tilde{X}_0) = 0, \quad p_g(\tilde{X}_0) > 1, \quad p_g(\tilde{X}_0) > h^0(\mathcal{O}(K_{\tilde{X}_0} - C)), \quad (1.1)$$

where $p_g(\tilde{X}_0) = h^0(\mathcal{O}(K_{\tilde{X}_0}))$ is the geometric genus of $\tilde{X}_0$; the latter inequality of (1.1) means that the canonical bundle of $\tilde{X}_0$ has a section that does not vanish identically on $C$. Suppose further that $\tilde{X}_0$ is endowed with a symplectic form $\omega$, which may or may not be Kähler. As $\text{Symp}(X, \omega)$ is a subgroup of $\text{Diff}(X)$, we have the inclusion induced homomorphisms $\pi_k(\text{Symp}(X, \omega)) \to \pi_k(\text{Diff}(X, \omega))$. This note aims to prove the following

**Theorem 1.** Under assumptions (1.1), the homomorphism $\pi_1(\text{Symp}(\tilde{X}_0, \omega)) \to \pi_1(\text{Diff}(\tilde{X}_0))$ is not surjective.

To give an example of a resolution $\tilde{X}_0$ which agrees with (1.1), we consider a smooth quintic surface in $\mathbb{CP}^3$. The geometric genus of a quintic can be computed as follows: any canonical divisor of a quintic parameterizes all quintics, there is a codimension-1 locus $\Sigma$ which parameterizes singular quintics. Each smooth point of $\Sigma$ corresponds to a quintic with a single double-point singularity. Let $\Delta$ be a small complex disk in $\mathbb{CP}^3$ that intersects $\Sigma$ transversally at a smooth point $p \in \Sigma$, and let $t : \Delta \to \mathbb{C}$ be a local parameter on $\Delta$ such that $t(p) = 0$. Denote by $X_t$ the quintic corresponding to the point $t \in \Delta$. Then, the family $\{X_t\}_{t \in \Delta}$ gives a global smoothing of $X_0$. Let $q : \tilde{X}_0 \to X_0$ be the minimal resolution of $X_0$, and let $C$ the exceptional $(-2)$-curve. Recall that $\pi_1(X_t) = 0$. Both $\pi_1$ and $p_g$ are diffeomorphism invariants. Since $X_t$ is diffeomorphic to $\tilde{X}_0$, it follows that $\tilde{X}_0$ satisfies the first two conditions of (1.1).

Let us check that $\tilde{X}_0$ obeys the third condition: Consider a hyperplane $h \subset \mathbb{CP}^3$ which does not pass through the singular point of $X_0$. The intersection $H = X_0 \cap h$ is a canonical divisor of $X_0$. The minimal resolution being crepant ($K_{\tilde{X}_0} = q^*K_{X_0}$), the divisor $q^*H$ is canonical. Since $q^*H$ is disjoint from $C$, the third inequality of (1.1) follows.

To construct a loop in $\text{Diff}(\tilde{X}_0)$ that is not represented by a loop in $\text{Symp}(\tilde{X}_0, \omega)$, we use the Atiyah flop, a birational surgery introduced in [3]. Consider the ramified double covering of $X$:

$$\mathcal{N} = \{(t, x) \in \Delta \times X \mid f(x) = t^2\} \quad (1.2)$$
The 3-fold $N$ fibers over $\Delta$ via the map $f_{N}(t,x) = t$. If $sq: \Delta \to \Delta$ is given by $sq(t) = t^2$, then $N$ corresponds to the base change

$$
\begin{array}{ccc}
N & \longrightarrow & X \\
\downarrow f_{N} & & \downarrow f \\
\Delta & \xrightarrow{sq} & \Delta,
\end{array}
$$

(1.3)

where the upper horizontal arrow is the covering map given by $(t,x) = x$. The 3-fold $N$ is a nodal 3-fold in the sense of [3], with a single double-point in the fiber over 0. Atiyah shows (see [3, §2]) that there exists a resolution $r: V \to N$ which replaces the double-point by a smooth rational $(-1, -1)$-curve $C$, that is, a curve whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Define $p: V \to \Delta$ by the diagram:

$$
\begin{array}{ccc}
V & \longrightarrow & N \\
\downarrow r & & \downarrow f_{N} \\
\Delta & \xrightarrow{p} & \Delta,
\end{array}
$$

(1.4)

Atiyah proves (see [3, §3]) that $p$ has maximal rank everywhere. Hence, $V \xrightarrow{p} \Delta$ is a holomorphic fiber bundle. In particular, for each $t \in \Delta$, the fiber $p^{-1}(t)$ is diffeomorphic to $p^{-1}(0)$. On the other hand, the restriction of $r$ to the fiber $p^{-1}(0)$ gives the minimal resolution $r|_{p^{-1}(0)}: p^{-1}(0) \to f_{N}^{-1}(0) = f^{-1}(0)$.

Thus, for the surface $X_{0} = f^{-1}(0)$, one can form a smoothing $X_{t} = f^{-1}(t)$ and the minimal resolution $\tilde{X}_{0}$, and those two are diffeomorphic. Detailed proofs of these results are given in [3]. The resolution $r: V \to N$ is called a small resolution of $N$ because the double-point of $N$ is replaced by a curve, a codimension 2 subvariety. It is well known that this process can be done in two different ways, so we obtain two different families of smooth surfaces, $V$ and $V'$, which coincide outside their central fibers. More precisely, we have:

**Theorem 2** (Burns-Rapoport, [4]). Let $p: V \to \Delta$ be a holomorphic family of smooth complex surfaces. For each $t \in \Delta$, set $X_{t} = p^{-1}(t)$. If $X_{0}$ contains a smooth rational $(-2)$-curve $C$ which is embedded in $V$ as a $(-1, -1)$-curve, then there exists another holomorphic family $p': V' \to \Delta$, $X'_{t} = p'^{-1}(t)$, and a birational isomorphism

$$
\rho_{C}: V \to V',
$$

whose indeterminacy locus is $C$, that fits into a diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\rho_{C}} & V' \\
\downarrow p & & \downarrow p' \\
\Delta & & \Delta,
\end{array}
$$

(1.5)

where the dashed arrow indicates that $\rho_{C}$ is not a map but merely a birational map. Although one cannot extend $\rho_{C}$ into the curve $C$ to make it a proper isomorphism, one can restrict it to $X_{0}$ to obtain a birational isomorphism

$$
\rho: X_{0} \to X'_{0},
$$

which then extends to a proper isomorphism between $X_{0}$ and $X'_{0}$. Under that isomorphism, the image $C' = \rho(C)$ is also a smooth rational curve which is embedded in $V'$ as a $(-1, -1)$-curve.

The family $p: V \to \Delta$ is differentiably trivial, so it provides an identification $\alpha: H_{2}(X_{0};\mathbb{Z}) \to H_{2}(X_{t};\mathbb{Z})$, where $t \neq 0$ is some fixed base-point. Similarly, we have another identification $\alpha': H_{2}(X'_{0};\mathbb{Z}) \to H_{2}(X'_{t};\mathbb{Z})$. 
corresponding to the family $p': \mathcal{V} \to \Delta$. If we identify $H_2(X_0; \mathbb{Z})$ and $H_2(X'_0; \mathbb{Z})$ via the diagram

$$
\begin{array}{ccc}
H_2(X_0; \mathbb{Z}) & \longrightarrow & H_2(X'_0; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_2(X_1; \mathbb{Z}) & \xrightarrow{\rho_C} & H_2(X'_1; \mathbb{Z}),
\end{array}
$$

(1.6)

then the formula for $\rho_*: H_2(X_0; \mathbb{Z}) \to H_2(X'_0; \mathbb{Z})$ is as follows:

$$
A \to A + (A,C)C \quad \text{for each } A \in H_2(X_0; \mathbb{Z}),
$$

(1.7)

where $(A,C)$ stands for the intersection pairing.

The birational map $\rho_C$ is called the elementary modification of $\{X_t\}_{t \in \Delta}$ with the center $C$ or the Atiyah flop. It has an explicit description in terms of blowups and blowdowns (see, eg, Reid’s paper [16, Part II]). This theorem is due to Burns-Rapoport; in the present form, however, it is a mix of Corollary 2 of Morrison’s paper [14] and Theorem (6.3) of [16].

Now, let us glue the spaces $\mathcal{V}$ and $\mathcal{V}'$ along their boundaries to form a space

$$
\mathcal{W} = \mathcal{V} \cup \mathcal{V}' / \sim \quad (t,x) \in \mathcal{V} \sim (t,x') \in \mathcal{V} \text{ iff } \rho_C(x) = x' \text{ and } t \in \partial \Delta.
$$

(1.8)

Two copies of $\Delta$, glued together by the identity map along their boundaries, form a 2-sphere $S^2 = \Delta \cup_{\text{id}} \Delta$. The manifold $\mathcal{W}$ is a fiber bundle over $S^2$ under the projection map

$$
p'^\nabla: \mathcal{W} \to S^2, \quad p'^\nabla(t,x) = \begin{cases} 
p(t,x) & \text{for } (t,x) \in \mathcal{V} \\
p'(t,x) & \text{for } (t,x) \in \mathcal{V}'.\end{cases}
$$

Note that the gluing of $\Delta$’s is orientation-reversing with respect to their natural complex orientations, so $\mathcal{W}$ will not be a holomorphic bundle, but merely a bundle of complex surfaces.

Using family Seiberg-Witten invariants, we will prove $\mathcal{W}$ cannot be realized as a Hamiltonian bundle. The idea is to use a computation from Kronheimer’s paper [9] to show that this fiber bundle has non-vanishing Seiberg-Witten invariant; then to argue that if the bundle admitted fiberwise cohomologous symplectic forms, then the family Seiberg-Witten invariant would have to vanish. This argument is in spirit close to the way Ruberman (see [17, 18]) constructed the first examples of self-diffeomorphisms of four-manifolds that are isotopic to the identity in the topological category but not smoothly so. A generalization of Ruberman’s result to higher homotopy groups will appear in a joint work of Auckly and Ruberman [19].

Watanabe (see [22]) has recently proved a related result by rather different methods. He shows that $\pi_k(\text{Diff}_c(\mathbb{R}^4))$ are infinite groups for all $k \geq 1$. On the other hand, Gromov (see [7]) proved that $\text{Symp}(\mathbb{R}^4, \omega_{st})$ is contractible. Hence, the homomorphism

$$
\pi_1(\text{Symp}(\mathbb{R}^4, \omega_{st})) \to \pi_1(\text{Diff}_c(\mathbb{R}^4))
$$

(1.9)

is not surjective, and nor are the homomorphisms for higher $\pi_k$. In addition to Watanabe’s example, the only other example the author is aware of where this non-surjectivity arises is Example 10.4.2 of [12]. Let us consider the product $(S^2 \times S^2, \omega_\lambda = (1 + \lambda)\omega_{S^2} \oplus \omega_{S^2})$, where $0 \leq \lambda \in \mathbb{R}$ and $\omega_{S^2}$ is a standard area form on $S^2$ with total area of 1. For $\lambda = 0$, Gromov proved (see [7]) that the symplectomorphism group retracts onto the isometry group $\mathbb{Z}_2 \times \text{SO}(3) \times \text{SO}(3)$. In particular, $\pi_1(\text{Symp}(S^2 \times S^2, \omega_0)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In [12], McDuff and Salamon explicitly describe an element $[\psi_1] \in \pi_1(\text{Diff}(S^2 \times S^2))$ that is of infinite order. Hence, the homomorphism

$$
\pi_1(\text{Symp}(S^2 \times S^2, \omega_\lambda)) \to \pi_1(\text{Diff}(S^2 \times S^2))
$$

(1.10)

is not surjective for $\lambda = 0$. They also claim that $\pi_1(\text{Diff}(S^2 \times S^2))$ has rank at least two, for it contains both $[\psi_1]$ and its conjugate by the involution that interchanges the two $S^2$ factors. For $\lambda > 0$, Abreu and
McDuff proved (see [1][2]) that $\pi_1(\text{Symp}(S^2 \times S^2, \omega_\lambda)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Hence, the homomorphism (1.10) is not surjective for all $\lambda$.

To the author’s knowledge, Theorem 1 provides the first examples, away from the setting of rational or ruled surfaces, of symplectic manifolds $(X, \omega)$ for which the homomorphism $\pi_1(\text{Symp}(X, \omega)) \to \pi_1(\text{Diff}(X))$ is not surjective.

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### 2. Family Seiberg-Witten invariants.

In what follows we shall pass to Sobolev completions of all of the functional spaces encountered, not mentioning the choice of those completions explicitly. A general reference that in-depth covers the involved analysis is Nicolaescu’s book [15].

One starts with a closed oriented *simply-connected* 4-manifold $X$ equipped with a Riemannian metric $g$ and a self-dual form $\eta$. After picking a spin$^C$ structure on $X$, with associated spinor bundles $W^\pm$ and determinant line bundle $\mathcal{L}$, one considers the monopole map

$$
\mu: \Gamma(W^+) \times \mathcal{A} \to \Gamma(W^-) \times i\Omega^2_+(X), \quad \mu(\varphi, A) = (\mathcal{D}^A \varphi, F^+_A - \sigma(\varphi) - i\eta),
$$

where $\varphi \in \Gamma(W^+)$ is a self-dual spinor field, $A \in \mathcal{A}$ is a $U(1)$-connection on $\mathcal{L}$, and $F^+_A \in i\Omega^2_+(X)$ stands for the self-dual part of the curvature form of $A$. Finally, $\sigma: \Gamma(W^+) \to i\Omega^2_+(X)$ is the squaring map. We also write $\mu_{(g, \eta)}$ when we want to indicate the dependence of the monopole map on the metric and perturbation. The Seiberg-Witten solution space (space of monopoles) is the zero set of the function $\mu$, while the solution moduli space $\mathcal{M}_{(g, \eta)}$ is the quotient of $\mu^{-1}(0)$ by the gauge group

$$
\mathcal{G} = \{ g: X \to S^1 \},
$$

which acts on $\Gamma(W^+) \times \mathcal{A}$ as follows: locally, or if $X$ is simply-connected, every map $g: X \to S^1$ takes the form $g = e^{if}$ for some function $f$ on $X$, and its action is given by

$$
g \cdot (\varphi, A) = (e^{i f} \varphi, A + 2i df).
$$

Pick a monopole $(\varphi, A)$ and consider the differential

$$
d\mu: T\Gamma(W^+)\times\mathcal{A}|_{(\varphi,A)} \to T\Gamma(W^-)\times i\Omega^2_+(X)|_{(0,0)} \text{ of } \mu \text{ at } (\varphi, A).
$$

Then, regarding the gauge action of $\mathcal{G}$ on $(\varphi, A)$ as the map

$$
g: \mathcal{G} \to \Gamma(W^+) \times \mathcal{A}, \quad g(g) = g \cdot (\varphi, A),
$$

we consider the differential

$$
dg: T\mathcal{G}|_{f=0} \to T\Gamma(W^+)\times\mathcal{A}|_{(\varphi,A)} \text{ of } g \text{ at } e^{i 0}.
$$

Combining (2.4) and (2.5), we obtain the complex

$$
0 \to T\mathcal{G}|_{f=0} \to T\Gamma(W^+)\times\mathcal{A}|_{(\varphi,A)} \to T\Gamma(W^-)\times i\Omega^2_+(X)|_{(0,0)} \to 0,
$$

which, assuming the standard identifications

$$
T\mathcal{G}|_{f=0} \cong i\Omega^0(X), \quad T\Gamma(W^+)\times\mathcal{A}|_{(\varphi,A)} \cong \Gamma(W^+) \times i\Omega^1(X), \quad T\Gamma(W^-)\times i\Omega^2_+(X)|_{(0,0)} \cong \Gamma(W^-) \times i\Omega^2_+(X),
$$

we can write as

$$
0 \to i\Omega^0(X) \to \Gamma(W^+) \times i\Omega^1(X) \to \Gamma(W^-) \times i\Omega^2_+(X) \to 0.
$$

The cohomology groups of (2.7) are

$$
\mathcal{H}^0_{(\varphi,A)} = \ker dg, \quad \mathcal{H}^1_{(\varphi,A)} = \ker d\mu / \text{Im} d g, \quad \mathcal{H}^2_{(\varphi,A)} = \text{Coker} d\mu.
$$
A monopole \((\varphi, A)\) is called reducible if \(\varphi\) is identically zero. The stabilizer \((\varphi, A)\) under the action of gauge group is trivial unless the monopole is reducible, in which case the stabilizer is isomorphic to \(S^1\). Thus, if \((\varphi, A)\) is irreducible, then \(\mathcal{H}_{g}^{0}(\varphi, A) = 0\), and sequence (2.6) is exact in the first term. If \((\varphi, A)\) is reducible, then the Seiberg-Witten equations reads \(F_{A}^{+} - i\eta = 0\). In that case, a solution only exists iff

\[
\langle F_{A}^{+} \rangle_{g} = i \langle \eta \rangle_{g},
\]

where the brackets in both sides denote the harmonic part of the 2-form in question. Since the harmonic part of \(F_{A}^{+}\) depends only on the cohomology class of \(F_{A}\), and not on the connection \(A\) at hand, we may restate (2.8) as

\[
\langle \eta + 2\pi c_{1}(\mathcal{L}) \rangle_{g} = 0.
\]

Set:

\[
\Omega_{+}^{2}(X)^{*} = \{ \eta \in \Omega_{+}^{2}(X) | \langle \eta + 2\pi c_{1}(\mathcal{L}) \rangle_{g} \neq 0 \}.
\]

Now, let us consider the parameterized monopole map

\[
\mu^{\circ} : \Omega_{+}^{2}(X)^{*} \times \Gamma(W^{+}) \times \mathcal{A} \to \Gamma(W^{-}) \times i\Omega_{+}^{2}(X), \quad \mu^{\circ}(\eta, \varphi, A) = (\mathcal{D}^{A} \varphi, F_{A}^{+} - \sigma(\varphi) - i\eta),
\]

which we can regard as a family of monopoles maps parameterized by \(\Omega_{+}^{2}(X)^{*}\). Define the universal moduli space as:

\[
\mathcal{M} = \{ (\eta, \varphi, A) \in \Omega_{+}^{2}(X)^{*} \times \Gamma(W^{+}) \times \mathcal{A} | \mu^{\circ}(\eta, \varphi, A) = 0 \} / \sim, \quad (\eta, \varphi, A) \sim (\eta, g \cdot (\varphi, A)) \text{ for all } g \in G.
\]

Set:

\[
\pi : \mathcal{M} \to \Omega_{+}^{2}(X)^{*}, \quad \pi(\eta, \varphi, A) = \eta
\]

For the proof of the following statement see [10, Lem. 5].

**Theorem 3** (Kronheimer-Mrowka, [10]). The map \(\mu^{\circ}\) is transverse to the origin of \(\Gamma(W^{-}) \times i\Omega_{+}^{2}(X)\), hence \(\mathcal{M}\) is an infinite-dimensional manifold. The projection \(\pi\) is a proper Fredholm map of index

\[
d(\mathcal{L}) = \frac{1}{4} (c_{1}(\mathcal{L})^{2} - 2\chi(X) - 3\sigma(X)),
\]

and for each point \((\eta, \varphi, A) \in \mathcal{M}\), there are natural isomorphisms

\[
\text{Ker} d\pi|_{(\eta, \varphi, A)} = \mathcal{H}_{(\varphi, A)}^{1}, \quad \text{Coker} d\pi|_{(\eta, \varphi, A)} = \mathcal{H}_{(\varphi, A)}^{2}.
\]

Following Li-Liu [11], we now consider the monopole map in the more general setting of fiber bundles. Let \(B\) be a finite-dimensional manifold, and \(X \to B\) be a smooth bundle over \(B\) with fiber \(X\). We denote the vertical tangent bundle of \(X\) by \(T_{X/B}\). Pick a metric on \(T_{X/B}\), and consider the bundle \(\mathfrak{fr}\) of orienting orthonormal frames of \(T_{X/B}\). Suppose that \(T_{X/B}\) is given a spin\(^{C}\) structure \(\mathfrak{s}\), an equivalence class of lifts of the \(SO(4)\)-bundle \(\mathfrak{fr}\) to a \(Spin^{C}(4)\)-bundle \(\mathfrak{s}\). Then, if we restrict \(\mathfrak{s}\) to a fiber \(X_{b}\) at \(b \in B\), we get a spin\(^{C}\) structure \(\mathfrak{s}_{b}\) on \(X_{b}\). (Hereafter, given any object on the total space \(X\), the object with the subscript \(b\) stands for the restriction to the fiber \(X_{b}\).) Conversely, suppose a fiber \(X_{b}\) is given a spin\(^{C}\) structure \(\mathfrak{s}_{b}\) and we want to decide whether we can extend \(\mathfrak{s}_{b}\) to some spin\(^{C}\) structure \(\mathfrak{s}\) on \(T_{X/B}\). The following well-known result (see, eg, Chapter 3 in Morgan’s book [13]) answers affirmatively this question, provided that \(X\) is simply-connected and \(B\) is a homotopy \(S^{2}\). (This is the only case we will be considering in the sequel.)

**Lemma 1.** Let \(X \xrightarrow{X_{b}} B\) be a fiber bundle whose fiber \(X_{b}\) is a smooth simply-connected manifold, and whose base \(B\) is a homotopy \(S^{2}\). Suppose we are given a spin\(^{C}\) structure \(\mathfrak{s}_{b}\) on \(X_{b}\). Then there exists a spin\(^{C}\) structure \(\mathfrak{s}\) on \(T_{X/B}\) extending the spin\(^{C}\) structure \(\mathfrak{s}_{b}\) on \(X_{b}\).
Proof. Since $B$ is a homotopy $S^2$, we have a standard exact sequence:

$$0 \to H^2(B;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to H^2(X_b;\mathbb{Z}) \to 0. \tag{2.10}$$

Letting $L_b$ be the determinant line bundle of $s_b$, this sequence provides a lift of $c_1(L_b) \in H^2(X_b;\mathbb{Z})$ to a class $[S] \in H^2(X;\mathbb{Z})$. Moreover, we can assume that $[S]\mod 2 = w_2(T_{X/B})$. Hence, there is a spin$^C$ structure on $T_{X/B}$ whose Chern class is $[S]$. For a simply-connected manifold, the Chern class will distinguish any two spin$^C$ structures. Hence, the restriction of $s$ to $X_b$ is the same as $s_b$. □

Let $X_b \to B$ be a fiber bundle as above and let $\{g_b\}_{b \in B}$ be a family of fiberwise metrics. Choose a fixed spin$^C$ structure $s$ on $T_{X/B}$. Associated to $s$, there are spinor bundles $W^\pm \to B$ and determinant line bundle $\mathcal{L} = \det W^+$, which we regard as families of bundles

$$W^\pm = \bigcup_{b \in B} W^\pm_b, \quad \mathcal{L} = \bigcup_{b \in B} \mathcal{L}_b.$$

Further, we let $\mathcal{A}_b$ denote the space of $U(1)$-connections on $L_b$, $\mathcal{G}_b$ denote the gauge group acting on $(W^\pm_b, \mathcal{A}_b)$ as stated by (2.3), and $\Omega \to B$ be the fiber bundle whose fiber $\Omega_b$ is the space of those 2-forms on $X_b$ which are $g_b$-self-dual. Define $\Xi \to B$ as follows: the fiber of $\Xi$ over $b \in B$ is the space $\Gamma(W^-_b) \times i \Omega_b$. (Here $\Xi$ is for “target space”.) Below, we denote points of $\Xi$ by tuples $(b, \psi, i\eta)$, where $b \in B$, $\psi \in \Gamma(W^-_b)$, $\eta \in \Omega_b$. Set:

$$\Omega^*_b = \{ \eta \in \Omega_b \mid \langle \eta + 2\pi c_1(L_b) \rangle_{g_b} \neq 0 \}, \tag{2.11}$$

and let $j: \Omega^* \to B$ be the bundle whose fiber over $b \in B$ is $\Omega^*_b$.

Lemma 2. The fibering $j: \Omega^* \to B$ has $(b^+(X) - 2)$-connected fibers. If $b^+(X) = 1$, then those fibers consist of two connected components, each being contractible.

Proof. This is clear as equation (2.11) cuts out an affine subspace of codimension $b^+(X)$ in $\Omega_b$. □

Define $\mathfrak{D} \to B$ as follows: the fiber of $\mathfrak{D}$ over $b \in B$ is the space $\Gamma(W^+_b) \times \mathcal{A}_b \times \Omega^*_b$. (Here $\mathfrak{D}$ is for “domain”.) Below, we denote points of $\mathfrak{D}$ by tuples $([b, \eta], \varphi, A)$, where $b \in B$, $\eta \in \Omega^*_b$, $\varphi \in \Gamma(W^+_b)$, $A \in \mathcal{A}_b$. Consider an extended version of the parameterized monopole map,

$$\mu^\circ: \mathfrak{D} \to \Xi, \quad \mu^\circ([b, \eta], \varphi, A) = (b, \mu_{(g_b, \eta)}(\varphi, A)),$$

where $\mu_{(g_b, \eta)}(\varphi, A)$ is defined by (2.1) for the metric $g_b$ and perturbation $\eta$. In this family setting, the universal moduli space $\mathcal{M}$ is defined as follows:

$$\mathcal{M} = \{ ([b, \eta], \varphi, A) \in \mathfrak{D} \mid \mu^\circ([b, \eta], \varphi, A) = (b, 0, 0) \} / \sim, \quad (\eta, \varphi, A) \sim (\eta, g \cdot (\varphi, A))$$

for $(\eta, \varphi, A) \in \Omega^*_b \times \Gamma(W^+_b) \times \mathcal{A}_b$ and $g \in \mathcal{G}_b$. Set:

$$\pi: \mathcal{M} \to \Omega^*, \quad \pi([b, \eta], \varphi, A) \to [b, \eta].$$

As $\pi^{-1}([b, \eta])$ is precisely $\mathcal{M}_{(g_b, \eta)}$, we may regard $\mathcal{M}$ as a family of moduli spaces

$$\mathcal{M} = \bigcup_{b \in B, \eta \in \Omega^*_b} \mathcal{M}_{(g_b, \eta)}.$$

From Theorem 3 we have:

Theorem 4 (Li-Liu, [11]). The projection $\pi$ is a proper Fredholm map of index

$$d(\mathcal{L}) = \frac{1}{4}(c_1(L_b))^2 - 2 \chi(X_b) - 3\sigma(X_b),$$

and for each point $([b, \eta], \varphi, A) \in \mathcal{M}$, there are natural isomorphisms

$$\text{Ker} \, d\pi|_{([b, \eta], \varphi, A)} = \mathcal{H}^1_{(\varphi, A)}, \quad \text{Coker} \, d\pi|_{([b, \eta], \varphi, A)} = \mathcal{H}^2_{(\varphi, A)}.$$
Let \( \{ \eta_b \}_{b \in B} \) be a family of fiberwise \( g_b \)-self-dual forms on \( \mathcal{X} \) satisfying
\[
\langle \eta_b + 2\pi c_1(\mathcal{L}_b) \rangle_{g_b} \neq 0.
\] (2.12)

Applying Sard-Smale theorem [20], we perturb \( \{ \eta_b \}_{b \in B} \) so that it is transverse to \( \pi \). Then the moduli space
\[
\mathcal{M}_{(g_b, \eta_b)} = \bigcup_{b \in B} \mathcal{M}_{(g_b, \eta_b)}
\]
is either empty or a compact manifold of dimension \( d(\mathcal{L}) + \dim B \). Suppose \( d(\mathcal{L}) < 0 \) and suppose \( B \) is a closed manifold of dimension \( (-d(\mathcal{L})) \). Then \( \mathcal{M}_{(g_b, \eta_b)} \) is zero-dimensional, and thus consists of finitely-many points. We call
\[
\text{FSW}_{(g_b, \eta_b)}(s) = \# \{ \text{points of } \mathcal{M}_{(g_b, \eta_b)} \} \mod 2
\]
the family \((\mathbb{Z}_2, \text{Seiberg-Witten}) \) invariant of \( \mathcal{X} \) associated to \( s \) and \( \{(g_b, \eta_b)\}_{b \in B} \).

Let \( R_b \) be the space of pairs \((g_b, \eta_b)\), where \( g_b \) is a metric on \( X_b \) and \( \eta_b \) is a \( g_b \)-self-dual 2-form, and \( R_b^* \) be the subset of \( R_b \) consisting of pairs \((g_b, \eta_b)\) that satisfy (2.12). Note that \( R_b^* \) is homotopy equivalent to \( \Omega_b^* \). Let \( R^* \to B \) be the fiber bundle whose fiber over \( b \in B \) is \( R_b^* \), and let \( \Gamma(B, R^*) \) be the space of sections for this bundle. If \( \{(g_b, \eta_b)\}_{b \in B} \), \( \{(g_b', \eta_b')\}_{b \in B} \) are two families that are in the same connected component of \( \Gamma(B, R^*) \), then Sard-Smale theorem can be applied to conclude that
\[
\text{FSW}_{(g_b, \eta_b)}(s) = \text{FSW}_{(g_b', \eta_b')}(s).
\]

See [11] for details. It follows from Lemma 2 that \( R^* \to B \) has \( (b^+(X) - 2) \)-connected fibers. Hence, \( \Gamma(B, R^*) \) is connected for \( b^+(X) > \dim B + 1 \).

**Theorem 5** (Li-Liu, [11]). If \( b^+(X) - 1 > \dim B \), then \( \text{FSW}_{(g_b, \eta_b)}(s) \) is independent of the choice of \((g_b, \eta_b)\).

See Theorem 2.1 in [11] for a more general statement. We now drop the subscript \((g_b, \eta_b)\) from \( \text{FSW}_{(g_b, \eta_b)}(s) \) and write simply \( \text{FSW}(s) \).

### 3. Seiberg-Witten for complex surfaces

A general reference for the Seiberg-Witten equations on Kähler surfaces is the book by Morgan [13] or Nicolaescu’s [15]. Assume \( X \) is a Kähler surface and \( \omega \) its Kähler form with associated Kähler metric \( g \). The complex structure of \( X \) gives rise to a canonical spin\(^C\) structure \( s_0 \) on \( X \), with determinant line bundle \( K_X^* \), and spinor bundles
\[
W^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad W^- = \Lambda^{0,1},
\]
where each term \( \Lambda^{k,p} \) stands for the bundle of complex-valued \((k,p)\)-forms on \( X \). All other spin\(^C\) structures \( s_\varepsilon \) on \( X \) are obtained by taking a line bundle \( L_\varepsilon \) with \( c_1(L_\varepsilon) = \varepsilon \) and setting the spinor bundles to be
\[
W^+ = L_\varepsilon \oplus (L_\varepsilon \otimes \Lambda^{0,2}), \quad W^- = L_\varepsilon \otimes \Lambda^{0,1}.
\] (3.1)

Then the determinant line bundle of \( s_\varepsilon \) is \( K_X^* \otimes L_\varepsilon^2 \). Reverting to the notation used in the previous section, we have
\[
\mathcal{L} = K_X^* \otimes L_\varepsilon^2, \quad c_1(\mathcal{L}) = c_1(X) + 2\varepsilon, \quad d(\mathcal{L}) = c_1(X) \cdot \varepsilon + \varepsilon^2.
\]

The Kähler metric \( g \) induces a canonical holomorphic \( U(1) \)-connection \( A_0 \) on \( K_X^* \), and any choice of \( U(1) \)-connection \( B \in \mathcal{B} \) on \( L_\varepsilon \) combines with \( A_0 \) to give a connection \( A_0 + 2B \in \mathcal{A} \) on \( K_X^* \otimes L_\varepsilon^2 \). Conversely, any \( U(1) \)-connection on \( K_X^* \otimes L_\varepsilon^2 \) is obtained that way.
For a spinor \( \varphi \in \Gamma(W^+) \), we write \( \varphi = (\ell, \beta) \), where \( \ell \in \Omega^0(L_\varepsilon) \) and \( \beta \in \Omega^{0,2}(L_\varepsilon) \). With this notation, the monopole map becomes (see, e.g., \cite{13} Ch. 7, \cite{15} §3.2):

\[
\mu: \Omega^0(L_\varepsilon) \oplus \Omega^{0,2}(L_\varepsilon) \oplus B \to \Omega^{0,1}(L_\varepsilon) \oplus i \Omega^0(X) \omega \oplus \Omega^{0,2}(X),
\]

where \( \bar{\mu} = \mu \) is the evaluation map

\[
\mu(\ell, \beta, B) = \left( \bar{\partial}_B \ell + \partial_B^* \beta, (F_A^+)_{1,1} - \frac{i}{4}(|\ell|^2 - |\beta|^2)\omega - i\eta_{1,1}^2, 2F_B^{0,2} - \frac{\ell^* \beta}{2} - i\eta_{0,2} \right), \tag{3.2}
\]

where \( \bar{\partial}_B: \Omega^{0,2}(L_\varepsilon) \to \Omega^{0,1}(L_\varepsilon) \) is the formal adjoint of \( \bar{\partial}: \Omega^{0,1}(L_\varepsilon) \to \Omega^{0,2}(L_\varepsilon) \), \( \ell^* \) is the image of \( \ell \) under the (non-complex) isomorphism \( L_\varepsilon \cong L_\varepsilon^* \) induced by the metric on \( L_\varepsilon \), \( \ell^* \beta \) is the image of \( \ell^* \otimes \beta \) under the evaluation map \( L_\varepsilon^* \otimes (L_\varepsilon \otimes \Lambda^{0,2}) \to \Lambda^{0,2} \). Here we have used the standard identification \( i\Omega^2_{\mathbb{C}}(X) \cong i\Omega^0(X) \omega \oplus \Omega^{0,2}(X) \).

Since \( \mathcal{A} = A_0 + 2B \), it is convenient to view the action of the gauge group \( \mathfrak{g} \) on \( W^+ \) and \( \mathcal{A} \) through its action on \( L_\varepsilon \) and \( B \) by the formula

\[
e^{if} \cdot (\ell, B) = (e^{-if} \ell, B + idf),
\]

which is chosen so to be consistent with formula (2.3). Put \( \eta = -\rho^2 \omega \). Let us describe the solutions to the equation

\[
\mu(\ell, \beta, B) = 0. \tag{3.3}
\]

The following two theorems are well-known; see, e.g., \cite{15} §3.2, \cite{13} Ch. 7.

**Theorem 6.** If \((\ell, \beta, B)\) is a solution to (3.3), then

\[
F_B^{0,2} \equiv 0, \quad \bar{\partial}_B \ell \equiv 0, \quad \bar{\partial}_B^* \beta \equiv 0,
\]

and either \( \ell \equiv 0 \) or \( \beta \equiv 0 \).

The equality \( F_B^{0,2} \equiv 0 \) says that \( B \) is a holomorphic connection on \( L_\varepsilon \), while \( \bar{\partial}_B \ell \equiv 0 \) implies that \( \ell \) is a holomorphic section of \( L_\varepsilon \). Similarly, \( \bar{\partial}_B^* \beta \equiv 0 \) says that the Hodge dual \( * \beta \in \Gamma(L_\varepsilon^* \otimes \Lambda^{2,0}) \) of \( \beta \) is a holomorphic section of \( L_\varepsilon^* \otimes K_X \).

**Theorem 7.** Let \((\ell, \beta, B)\) be a solution to (3.3) for a large enough \( \rho \). Then

(a) \( \beta \equiv 0 \).

(b) \( \ell \) does not vanish identically.

Conversely, if \( \ell \) is a section of \( L_\varepsilon \) that does not vanish identically on \( X \), then the equation (3.3) admits a solution \((\ell, 0, B)\). Furthermore, there is a one-to-one correspondence between the points of \( \mathcal{M}_{(g, -\rho^2 \omega)} \) and effective divisors in the class \( \varepsilon \).

We proceed by discussing the deformation complex (2.6) associated to the monopole map for the surface \( X \). While it is a difficult problem to analyze the complex (2.6) in general, there is an explicit description of its cohomology in the case of complex surfaces. For a monopole \((\ell, 0, B)\), the middle terms of (2.6) are:

\[
i\Omega^0(X) \xrightarrow{d\mu|_{(\ell,0,B)}} \Omega^0(L_\varepsilon) \oplus \Omega^{0,2}(L_\varepsilon) \oplus i\Omega^1(X) \xrightarrow{d\mu|_{(\ell,0,B)}} \Omega^{0,1}(L_\varepsilon) \oplus i\Omega^0(X) \omega \oplus \Omega^{0,2}(X),
\]

with the maps given by

\[
d\mathfrak{g}|_{(\ell,0)}(if) = (-i f \ell, i df), \quad \text{and} \quad d\mu(\ell, \beta, B) = \left( \bar{\partial}_B \ell + \partial_B^* \beta + \bar{B}^{0,1} \ell, 2(d\bar{B}^+)_{1,1} - \frac{i}{2} \ell^* \ell \omega, 2\bar{\partial}B^{0,1} - \frac{\ell^* \beta}{2} \right), \tag{3.4}
\]

where \( \ell \in \Omega^0(L_\varepsilon) \), \( \beta \in \Omega^{0,2}(L_\varepsilon) \), and \( \bar{B} \in i\Omega^1(X) \). Here \( \ell^* \ell \) is the image of \( \ell^* \otimes \ell \in L_\varepsilon^* \otimes L_\varepsilon \) under the evaluation map \( L_\varepsilon^* \otimes L_\varepsilon \to \mathbb{C} \).
Associated to the divisor \( C = \ell^{-1}(0) \), there is a natural short exact sequence:

\[
0 \to \mathcal{O}_X \xrightarrow{x\ell} \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0,
\]

where \( \mathcal{O}_X \) is the sheaf of holomorphic functions of \( X \), \( \mathcal{O}_X(C) \) is the sheaf of holomorphic sections of \( L_\ell \), and \( \mathcal{O}_C(C) \) is the restriction of \( \mathcal{O}_X(C) \) onto \( C \). The map \( x\ell: \mathcal{O}_X \to \mathcal{O}_X(C) \) is the multiplication by \( \ell \).

From this short exact sequence, we have the associated long exact cohomology sequence:

\[
0 \to H^0(X; \mathcal{O}_X) \xrightarrow{x\ell} H^0(X; \mathcal{O}_X(C)) \to H^0(X; \mathcal{O}_C(C)) \to H^1(X; \mathcal{O}_X) \xrightarrow{x\ell} H^1(X; \mathcal{O}_X(C)) \to H^1(X; \mathcal{O}_C(C)) \to 0.
\]

(3.5)

The following result is due to Friedman-Morgan (see [3, Th. 2.1]) and Kronheimer [9, Prop. 4.2].

**Theorem 8 ([3, 9]).** If \((\ell, 0, B)\) is an irreducible solution of (3.3), then the cohomology groups \( H^1(\ell, 0, B) \) and \( H^2(\ell, 0, B) \) sit in an exact sequence:

\[
0 \to H^0(X; \mathcal{O}_X) \xrightarrow{x\ell} H^0(X; \mathcal{O}_X(C)) \to H^1(X; \mathcal{O}_X) \xrightarrow{x\ell} H^1(X; \mathcal{O}_X(C)) \to H^2(X; \mathcal{O}_X) \to H^2(X; \mathcal{O}_X(C)) \to 0.
\]

**Lemma 3.** The operator \( T: \bar{\partial}_B \bar{\partial}_B^* + \frac{|\ell|^2}{4} : \Omega^{0,2}(L_\ell) \to \Omega^{0,2}(L_\ell) \) is an isomorphism.

**Proof.** \( T \) is self-adjoint; thus it suffices to prove that \( T \) is injective. If \( \bar{\partial}_B \bar{\partial}_B^* h + \frac{|\ell|^2}{4} h = 0 \), then, by taking the inner product with \( h \), we find:

\[
\int_X \langle \bar{\partial}_B^* h, \bar{\partial}_B h \rangle + \frac{1}{4} \int_X |\ell|^2 \langle h, h \rangle = 0, \quad \text{hence} \quad h \equiv 0.
\]

\[\square\]

Define \( \delta: \Omega^{0,1}(L_\ell) \oplus i \Omega^{0}(X)\omega \oplus \Omega^{0,2}(X) \to H^2(X; \mathcal{O}_X) \) by

\[
\delta(\gamma, if\omega, \nu) = \nu + \frac{\ell^*}{2} T^{-1} \left( \bar{\partial}_B \gamma - \frac{1}{2} \nu \ell \right).
\]

The following statement is implicit in [3].

**Lemma 4.** The map \( \delta \) gives a well-defined map \( \mathcal{H}^2(\ell, 0, B) \to H^2(X; \mathcal{O}_X) \).

**Proof.** Suppose \((\gamma, if\omega, \nu)\) satisfies

\[
\gamma = \bar{\partial}_B \ell + \bar{\partial}_B^* \beta + \hat{B}^{0,1} \ell, \quad \nu = 2 \bar{\partial} \hat{B}^{0,1} - \frac{\ell^*}{2} \beta.
\]

(3.6)

Then, by differentiating \( \gamma \), we find:

\[
\bar{\partial}_B \gamma = \bar{\partial}_B \bar{\partial}_B^* \beta + \hat{B}^{0,1} \ell \equiv \bar{\partial}_B \bar{\partial}_B^* \beta + \frac{1}{2} \nu \ell + \frac{|\ell|^2}{4} \beta, \quad \text{which is equivalent to} \quad \bar{\partial}_B \gamma - \frac{1}{2} \nu \ell = T(\beta).
\]

Hence, \( \delta(\gamma, if\omega, \nu) \) is equal to \( 2 \bar{\partial} \hat{B}^{0,1} \), which is a \( \bar{\partial} \)-exact form. \[\square\]
4. Seiberg-Witten for Hamiltonian bundles. The following material is well-known; see [15] §3.3 for details. Let \((X, \omega)\) be a closed symplectic 4-manifold, \(J\) an \(\omega\)-compatible almost-complex structure, and \(g(\cdot, \cdot) = \omega(\cdot, J\cdot)\) the associated Hermitian metric. The symplectic form \(\omega\) is \(g\)-self-dual and of type \((1,1)\) with respect to \(J\). The almost-complex structure \(J\) gives a canonical spin\(^C\) structure \(s_0\) with spinor bundles
\[
W^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad W^- = \Lambda^{0,1}.
\]
There exists a special connection \(A_0\) on \(K_X = \text{det} W^+\) such that the induced Dirac operator is
\[
\mathcal{D}^A_0 : \Omega^{0,0}(X; \mathbb{C}) \oplus \Omega^{0,2}(X; \mathbb{C}) \to \Omega^{0,1}(X; \mathbb{C}) \quad \mathcal{D}^A_0 = \sqrt{2}(\partial + \partial^*)\).
\]
See Proposition 1.4.23 in [15] for the proof that \(A_0\) exists. As before, we choose the spin\(^C\) structure \(s_\varepsilon\) as in [3.1] and parameterize all connections on \(\mathcal{L} = K_X \otimes L_\varepsilon^2\) as \(A = A_0 + B\), with \(B\) being a \(U(1)\)-connection on \(L_\varepsilon\). The unperturbed Seiberg-Witten equations are:
\[
\begin{cases}
\bar{\partial} \ell + \partial^* \beta = 0, \\
F_{A_0}^{0,2} + 2 F_B^{0,2} = \ell^* \beta \\
(F_A^{+})^{1,1} + 2 (F_B^+)_{1,1} = \frac{i}{4} (|\ell|^2 - |\beta|^2) \omega,
\end{cases}
\]
Choosing the perturbing term as
\[
\iota_\eta = F_{A_0}^+ - i \rho^2 \omega, \quad (4.1)
\]
we obtain what’s called the \(\rho\)-perturbed Seiberg-Witten equations. The following result is due to Taubes in [21]:

**Theorem 9** (Taubes, [21]). Under the assumption \(\varepsilon \neq 0\) and \(\varepsilon \cdot [\omega] \leq 0\), the \(\rho\)-perturbed Seiberg-Witten equations have no solutions for \(\rho\) sufficiently large.

The argument can also read from Theorem 3.3.29 in [15].

Let \(\mathcal{X} \xrightarrow{\pi} B\) be a smooth fiber bundle with fiber \(X\), where \(X\) is a simply-connected 4-manifold and \(B\) is the 2-sphere, which we regard as the union \(\Delta^+ \cup \Delta^-\) of two unit disks. We denote the equator \(\partial \Delta^+ = \partial \Delta^-\) by \(\partial\). As any bundle over a disk is trivial, we can build \(\mathcal{X}\) by taking two products \(\Delta^+ \times X\) and gluing them along the boundary \(\partial \times X\) by a loop \(g_t \in \text{Diff}_0(X),\)
\[
\mathcal{X} = (\Delta^+ \times X) \bigcup (\Delta^- \times X)/\sim. \quad (e^{it}, g_t(x))_+ \sim (e^{it}, x)_+.
\]
By definition, a Hamiltonian bundle is built from a loop \(g_t\) in \(\text{Symp}(X, \omega)\). Thus, if a smooth bundle \(\mathcal{X}\) is Hamiltonian, there exists a smooth family of cohomologous symplectic forms \(\{\omega_b\}_{b \in B}\) on the fibers \(\{X_b\}_{b \in B}\) of \(\mathcal{X}\). Choosing a family \(\{J_b\}_{b \in B}\) of \(\omega_b\)-compatible almost-complex structures, we also obtain a family of canonical Hermitian metrics \(\{g_b\}_{b \in B}\). (Recall here that the space of compatible almost-complex structures is non-empty and contractible. See, e.g., [12] Prop. 4.1.1.) Pick a class \(\varepsilon \in H^2(X; \mathbb{Z})\). Let \(s_\varepsilon\) be the spin\(^C\) structure on \(X_b\) given by [3.1]. Using Lemma [1] we can choose a spin\(^C\) structure on \(T_{\mathcal{X}/B}\) whose restriction to each fiber \(X_b\) is \(s_\varepsilon\). While such an extension is not uniquely determined by \(s_\varepsilon\), we write it \(s_\varepsilon\) for short. Considering the family of \(\rho\)-perturbed Seiberg-Witten equations parameterized by \(b \in B\), we have:

**Lemma 5.** Let \((X, \omega)\) be a closed simply-connected symplectic 4-manifold, and let \(\mathcal{X} \to B\) be a Hamiltonian fiber bundle with fiber \(X\) symplectomorphic to \((X, \omega)\), where \(B\) is the 2-sphere. Suppose that \(\varepsilon \in H^2(X; \mathbb{Z})\) satisfies
\[
\varepsilon \neq 0, \quad \varepsilon \cdot [\omega] \leq 0, \quad c_1(X) \cdot \varepsilon + \varepsilon^2 = -2.
\]
Then \(FSW_{(g_b, \eta_b)}(s_\varepsilon) = 0\), where \(\eta_b\) chosen as in [4.1] for \(\rho\) large enough. If \(b^+(X) > 3\), then \(FSW(s_\varepsilon) = 0\) for \(\mathcal{X}\).
Proof. Follows from Theorem 9.

5. The Kodaira-Spencer map. The following material is well-known; see, eg, work of Griffiths [6]. Let \( \{X_t\}_{t \in \Delta} \), where \( \Delta \subset \mathbb{C} \) is a complex disk, be a complex-analytic family of compact Kähler surfaces. More precisely, we assume given a complex 3-fold \( \mathcal{V} \), together with a proper, maximal rank holomorphic map \( p: \mathcal{V} \to \Delta \) such that \( p^{-1}(t) = X_t \). Let \( C \subset X_0 \) be a smooth rational curve which is embedded in \( \mathcal{V} \) as a \((-1,-1)\)-curve.

Let \( \mathcal{O}_\mathcal{V}(T_\mathcal{V}) \) be the sheaf of holomorphic sections of \( T_\mathcal{V} \), \( \mathcal{O}_{X_0}(T_\mathcal{V}) \) the restriction of \( \mathcal{O}_\mathcal{V}(T_\mathcal{V}) \) to \( X_0 \), and \( \mathcal{O}_{X_0}(T_{X_0}) \) the sheaf of holomorphic sections of \( T_{X_0} \). The short exact sequence

\[
0 \to \mathcal{O}_{X_0}(T_{X_0}) \to \mathcal{O}_{X_0}(T_\mathcal{V}) \to \mathcal{O} \to 0,
\]

where \( \mathcal{O} \) is regarded as the sheaf of sections of the trivial bundle \( T_\mathcal{V}/T_{X_0} \), gives the cohomology long exact sequence

\[
\ldots \to H^0(X_0; \mathcal{O}_{X_0}(T_\mathcal{V})) \to H^0(X_0; \mathcal{O}) \to H^1(X_0; \mathcal{O}_{X_0}(T_{X_0})) \to \ldots
\]  

(5.1)

Let \( \left[ \frac{\partial}{\partial t} \right] \) be a generator of \( H^0(X_0; \mathcal{O}) \cong \mathbb{C} \). By definition, the Kodaira-Spencer class \( \rho \in H^1(X_0; \mathcal{O}_{X_0}(T_{X_0})) \) of the family \( \mathcal{V} \) at \( t = 0 \) is the image of \( \left[ \frac{\partial}{\partial t} \right] \) under the connecting homomorphism of (5.1).

Lemma 6. \( \rho \) is non-trivial.

Proof. Let \( \mathcal{O}_C(T_C) \) be the sheaf of holomorphic sections of \( T_C \), \( \mathcal{O}_C(T_{X_0}) \) the restriction of \( \mathcal{O}_{X_0}(T_{X_0}) \) to \( C \), and \( \mathcal{O}_C(T_\mathcal{V}) \) the restriction of \( \mathcal{O}_{X_0}(T_\mathcal{V}) \) to \( C \). We have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{X_0}(T_{X_0}) & \longrightarrow & \mathcal{O}_{X_0}(T_\mathcal{V}) & \longrightarrow & \mathcal{O} \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_C(T_{X_0}) & \longrightarrow & \mathcal{O}_C(T_\mathcal{V}) & \longrightarrow & \mathcal{O} \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_C(N_{C|X_0}) & \longrightarrow & \mathcal{O}_C(N_{C|T_\mathcal{V}}) & \longrightarrow & \mathcal{O} \longrightarrow 0,
\end{array}
\]

(5.2)

where \( N_{C|X_0} \) is the normal bundle to \( C \) in \( X_0 \) and \( N_{C|\mathcal{V}} \) is the normal bundle to \( C \) in \( \mathcal{V} \). The cohomology diagram of (5.2) is written:

\[
\begin{array}{cccccc}
\ldots & \longrightarrow & H^0(X_0; \mathcal{O}_{X_0}(T_\mathcal{V})) & \longrightarrow & H^0(C; \mathcal{O}) & \longrightarrow & H^1(X_0; \mathcal{O}_{X_0}(T_{X_0})) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H^0(C; \mathcal{O}_C(T_\mathcal{V})) & \longrightarrow & H^0(C; \mathcal{O}) & \longrightarrow & H^1(C; \mathcal{O}_C(T_{X_0})) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & H^0(C; \mathcal{O}_C(N_{C|\mathcal{V}})) & \longrightarrow & H^0(C; \mathcal{O}) & \longrightarrow & H^1(C; \mathcal{O}_C(N_{C|X_0})) & \longrightarrow & \ldots
\end{array}
\]

(5.3)

The assumption that \( C \) is of negative self-intersection implies that \( H^0(C; \mathcal{O}_C(N_{C|\mathcal{V}})) = 0 \). Let \( \kappa \in H^1(C; \mathcal{O}_C(N_{C|X_0})) \) be the restriction of \( \rho \in H^1(X_0; \mathcal{O}_{X_0}(T_{X_0})) \) to \( H^1(C; \mathcal{O}_C(N_{C|X_0})) \). Then \( \kappa \) is also
the image of \( \left[ \frac{\partial}{\partial t} \right] \in H^0(X_0; \mathcal{O}) \) under the connecting homomorphism of the last row of (5.3). That homomorphism is injective, since \( H^0(C; \mathcal{O}_C(N_{\mathcal{C}|\mathcal{V}})) = 0 \). So \( \kappa \) is non-zero, and neither is \( \rho \).

The family \( \mathcal{V} \) is trivial as a \( C^\infty \)-family; i.e. we can find a smooth fiber-preserving diffeomorphism

\[
\Delta \times X_0 \longrightarrow \mathcal{V} \\
\downarrow \quad \quad \quad \quad \quad \downarrow \rho \\
\Delta \quad \quad \quad \quad \quad \Delta,
\]

(5.4)

and then, using this trivialization, we regard the complex structures on \( \{X_t\}_{t \in \Delta} \) as a family \( \{J_t\}_{t \in \Delta} \) of complex structures on \( X_0 \). Let \( L \to X_0 \) be the holomorphic line bundle corresponding to the divisor \( C \), and let \( B \) be any \( U(1) \)-connection on \( L \) which agrees with the holomorphic structure on \( L \). To shorten notation, we set:

\[
\partial_t \ell = \frac{1}{2} (d_B \ell - i d_B \circ J_t), \quad \bar{\partial}_t \ell = \frac{1}{2} (d_B \ell + i d_B \circ J_t),
\]

and

\[
\partial = \partial_0, \quad \bar{\partial} = \bar{\partial}_0.
\]

Let \( F_B \) be the curvature of \( B \), and let \( (F_B^{0,2})_t \) be the \((0,2)\)-component of \( F_B \) with respect to \( J_t \). We have:

\[
(F_B^{0,2})_t = 0 + \hat{F}_B^{0,2} t + O(t^2),
\]

where the class \([\hat{F}_B^{0,2}] \in H^2(X_0; \mathcal{O}_X)\) does not depend on the choice of \( B \) nor it depends on the trivialization (5.4).

**Lemma 7.** If \( p_g(X_0) > h^0(X_0; \mathcal{O}_X(K_{X_0} - C)) \), then \([\hat{F}_B^{0,2}]\) is non-trivial.

**Proof.** Define \( \tilde{J} \) by

\[
J_t = J_0 + \tilde{J} t + O(t^2).
\]

The “almost-complex” condition \( J_t^2 = -\text{id} \) implies that \( \tilde{J} \) and \( J_0 \) anti-commute, hence \( \tilde{J} \) can be thought as an element of \( \Omega^{0,1}(X_0; T_{X_0}) \); the “integrability” condition \( \bar{\partial}_t \partial_t = 0 \) implies (see [6, Lem. (1.8)]) that \( \tilde{J} \) is \( \bar{\partial} \)-closed as an element of \( H^1(X_0; T_X) \). Under the Dolbeault isomorphism, \( \tilde{J} \) corresponds, up to a scalar multiple, to the Kodaira-Spencer class \( \rho \). See Lemma (1.10) in [6] for the proof. Restricting \( \tilde{J} \) onto the curve \( C \) and composing it with the natural projection \( T_{X_0}|_C \to N_{\mathcal{C}|X_0} \), we get the class \( \kappa \in H^1(C; \mathcal{O}_C(N_{\mathcal{C}|X_0})) \) defined in Lemma [6].

We shall obtain an explicit Dolbeault representative of \( \kappa \). Letting \( \ell \) be a holomorphic section of \( L \) that vanishes along \( C \), we get:

\[
d_B \ell = \partial \ell.
\]

(5.5)

Let \( (d_B \ell)|_C \) be the restriction of \( d_B \ell \) to \( C \). By (5.5), \( (d_B \ell)|_C \) vanishes along \( T_C \). It follows that \( (\partial \ell)|_C \) gives an isomorphism between \( \mathcal{O}_C(N_{\mathcal{C}|X_0}) \) and \( \mathcal{O}_C(C) \). Under this isomorphism, the element \( \kappa \) becomes

\[
\kappa = [\partial \ell \circ \tilde{J}]|_C \in H^1(X_0; \mathcal{O}_C(C)).
\]

Hence, \( \kappa \) sits in diagram (3.5),

\[
\ldots \to H^1(X_0; \mathcal{O}_{X_0}(C)) \to H^1(X_0; \mathcal{O}_C(C)) \to H^2(X_0; \mathcal{O}_0) \to H^2(X_0; \mathcal{O}_{X_0}(C)) \to 0.
\]

(5.6)
Let us compute the image of \( \kappa \in H^1(X_0; \mathcal{O}_C(C)) \) under the connecting homomorphism of (5.6). To that end, we pick an extension of \( \kappa \) to a \( C^\infty \)-section of \( \Lambda^{0,1} \otimes L \). Specifically, we pick \( \partial \ell \circ \bar{J} \in \Omega^{0,1}(X_0; L) \). Differentiating \( \partial \ell \circ \bar{J} \) gives
\[
\bar{\partial} \left( \partial \ell \circ \bar{J} \right) = \bar{Q} \ell \quad \text{for some} \quad Q \in \Omega^{0,2}(X_0; \mathbb{C}).
\]
The element \([Q] \in H^2(X_0; \mathcal{O}_X)\) is the desired image. On the other hand, using the standard identities
\[
2 \bar{\partial}_t \ell = i \left( \partial \ell \circ \bar{J} \right) t + O(t^2), \quad \bar{\partial}_t \ell = \bar{F}_B^{0,2} t + O(t^2), \quad \bar{\partial}_t \ell - \bar{\partial}_t \ell = O(t^2), \quad (5.7)
\]
we get
\[
\bar{\partial} \left( \partial \ell \circ \bar{J} \right) = -2i \bar{F}_B^{0,2} \ell, \quad \text{hence} \quad Q = -2i \bar{F}_B^{0,2}.
\]
By the adjunction formula, the restriction of \( K_{X_0} \) to \( C \) is trivial. Thus, by restricting the sections of \( K_{X_0} \) to \( C \), we do not obtain more than a one-dimensional space of sections; this gives either
\[
h^0(\mathcal{O}_K(X_0 - C)) = p_g(X_0) - 1 \quad \text{or} \quad h^0(\mathcal{O}_K(X_0 - C)) = p_g(X_0).
\]
From the assumptions made about \( h^0(\mathcal{O}_K(X_0 - C)) \), we get:
\[
h^0(\mathcal{O}_K(X_0 - C)) = p_g(X_0) - 1. \quad (5.8)
\]
We have
\[
h^0(\mathcal{O}_K(C)) - h^1(\mathcal{O}_K(C)) + h^2(\mathcal{O}_K(C)) = p_g(X_0), \quad (5.9)
\]
using the Riemann-Roch formula. Applying Serre duality to (5.9) gives
\[
h^0(\mathcal{O}_K(C)) - h^1(\mathcal{O}_K(C)) + h^0(\mathcal{O}_K(K_{X_0} - C)) = p_g(X_0).
\]
Substituting (5.8) into (5.9) gives
\[
h^0(\mathcal{O}_K(C)) - h^1(\mathcal{O}_K(C)) = 1.
\]
Since \( h^0(\mathcal{O}_K(C)) = 1 \), we have \( h^1(\mathcal{O}_K(C)) = 0 \). It follows that the connecting homomorphism of (5.6) is injective, and hence \([\bar{F}_B^{0,2}] \in H^2(X_0; \mathcal{O}_X)\) is non-trivial. \( \square \)

6. Proof of Theorem 1 Let \( \{X_t\}_{t \in \Delta} \) where \( \Delta \) is a complex disk, be a complex-analytic family of compact simply-connected surfaces. Let \( C \subset X_0 \) be a smooth rational curve of self-intersection number \((-2)\). We construct a smooth family \( \{\omega_t\}_{t \in \Delta} \) of \( K\lambda h \)ler forms on \( \{X_t\}_{t \in \Delta} \). To that end, recall that the Kodaira classification of complex surfaces asserts that a complex surface is \( K\lambda h \)ler iff the first Betti number is even. Hence, each \( X_t \) is \( K\lambda h \)ler, meaning that there exists some \( K\lambda h \)ler form on each fiber \( X_t \). The existence of \( \{\omega_t\}_{t \in \Delta} \) then requires a partition of unity argument, combined with the following classical result of Kodaira and Spencer. See [8 Th. 15].

Theorem 10 (Kodaira- Spencer, [8]). If \( X_{t_0} \) carries a \( K\lambda h \)ler form, then, for a sufficiently small neighbourhood \( U \) of \( t_0 \) in \( \Delta \), the fiber \( X_t \) over any point \( t \in U \) admits a \( K\lambda h \)ler form. Moreover, given any \( K\lambda h \)ler form on \( X_{t_0} \), we can choose a \( K\lambda h \)ler form on each fiber \( X_t \), which depends differentiably on \( t \) and which coincides for \( t = t_0 \) with the given \( K\lambda h \)ler form on \( X_{t_0} \).

Let \( L \to X_0 \) be the holomorphic line bundle corresponding to the divisor \( C \), let \( B \) be any \( U(1) \)-connection on \( L \) which agrees with the holomorphic structure on \( L \). Let \( F_B \in \Omega^{1,1}(X_0; \mathbb{C}) \) be the curvature form of \( B \), and let \([\bar{F}_B^{0,2}] \in H^2(X_0; \mathcal{O}_X)\) be as in §3. Assume that
\[
[\bar{F}_B^{0,2}] \neq 0.
\]
Then there is a smaller disk \( U \subset \Delta \) such that for each \( t \in U - \{0\} \), \([C] \in H^2(X_t; \mathbb{Z})\) is not a \((1,1)\)-class. By passing to the smaller disk \( U \) if necessary, we assume that \( \Delta = U \).
Lemma 8. Let \( \{X_t\}_{t \in \Delta} \) be the family of surfaces described above. Then
(a) If \( t \in \Delta - \{0\} \), then \( X_t \) contains no effective divisors representing \( \pm [C] \).
(b) \( X_0 \) contains a single effective divisor representing \([C]\) and contains no effective divisors representing \((-[C])\).

Proof. For each \( t \in \Delta - \{0\} \), \([C] \notin H^1(X_t; \mathbb{R})\), and (a) follows. To prove (b), recall that \( C \) is smooth and has negative self-intersection number; thus there exists at most one divisor equivalent to \( C \). Since
\[
\int_C \omega_0 > 0,
\]
it follows that \((-C)\) cannot be effective.

Let \( \{g_t\}_{t \in \Delta} \) be the family of Kähler metrics corresponding to \( \{\omega_t\}_{t \in \Delta} \). For \( \varepsilon \in H^2(X_0; \mathbb{Z}) \), let \( s_{\varepsilon} \) be the spin\(^c\) structure on \( X_0 \) given by (3.1). Choose a spin\(^c\) structure on \( \{X_t\}_{t \in \Delta} \) extending \( s_{\varepsilon} \).

Lemma 9. (The notation are as in §2) If \( \varepsilon = -[C] \), then the parameterized moduli space
\[
\bigcup_{t \in \Delta} \mathcal{M}_{(g_t, -\rho^2 \omega_t)}
\]
over the disk
\[
\Delta \to \Omega^*, \quad t \to (t, -\rho^2 \omega_t)
\]
is empty for \( \rho \) large enough.

Proof. Follows from Theorem 7 and Lemma 8.

On the other hand, we have:

Theorem 11 (Kronheimer, [9]). If \( \varepsilon = [C] \), the parameterized moduli space
\[
\bigcup_{t \in \Delta} \mathcal{M}_{(g_t, -\rho^2 \omega_t)}
\]
consists of a single point corresponding to the divisor \( C \); the image of this point under \( \pi \) is \((0, -\rho^2 \omega_0)\). Furthermore, \( \pi \) is transverse to the disk
\[
\Delta \to \Omega^*, \quad t \to (t, -\rho^2 \omega_t)
\]
at \((0, -\rho^2 \omega_0)\).

Proof. This is a special case of Proposition 3.2 of [9], proved in full in §4 in the same paper. The only thing to prove is the transversality of \( \pi \), as the rest follows by Lemma 8. Let \(([0, -\rho^2 \omega_0], \varphi, A) \in \mathcal{D}\) be a gauge representative of the only point of \( \mathcal{M}_{(g_0, -\rho^2 \omega_0)} \). By Theorem 8, \( \mathcal{H}^1_{(\varphi, A)} = 0 \); thus it suffices to show that the image of \( \pi \) is not tangent to (6.1). Choose a map \(([0, -\rho^2 \omega_0], \varphi, A): \Delta \to \mathcal{D} \) with \( \varphi(0) = \varphi, A(0) = A \). We shall prove that the element
\[
\left[ \frac{d}{dt} \bigg|_{t=0} \mu^{\circ}(t, g_t, -\rho^2 \omega_t, \varphi(t), A(t)) \right] \in \mathcal{H}^2_{(\varphi, A)}
\]
is non-trivial. With the notation of §3 and §6, we have:
\[
(\varphi, A) = (\ell, 0, B), \quad (\varphi(t), A(t)) = (\ell(t), \beta(t), B(t)),
\]
\[
\mu_{(g_t, -\rho^2 \omega_t)}(\ell(t), \beta(t), B(t)) = \left( \bar{\partial}_{\ell} \ell(t) + \bar{\partial}_{\beta} \beta(t), \ldots, 2(F_B^{0,2})_t - \frac{\ell^x(t)\beta(t)}{2} \right)
\]
Putting, without loss of generality, \( \ell(t) \equiv \ell, \beta(t) \equiv 0 \), we get:

\[
\mu_{(g_t, -\rho^2 \omega_t)}(\ell(t), \beta(t), B(t)) = \left( \partial_t \ell, \ldots, 2 \hat{F}^{0,2}_B \right) + O(t^2).
\]

We now compute \( \delta(\partial_t \ell, \ldots, 2 \hat{F}^{0,2}_B) \), where \( \delta \) is the map defined in §3. Using (5.7), we get:

\[
\delta(\partial_t \ell, \ldots, 2 \hat{F}^{0,2}_B) = 2 [\hat{F}^{0,2}_B] \in H^2(X_0; \mathcal{O}_X).
\]

The assumption that \( [\hat{F}^{0,2}_B] \neq 0 \) implies that (6.2) is a non-zero, and the theorem follows.

Now suppose further that \( C \) is a \((-1, -1)\)-curve. Suppose further that \( p_g(X_0) > h^0(X_0; \mathcal{O}_X(K_{X_0} - C)) \). It follows from Lemma 7 that \( [\hat{F}^{0,2}_B] \neq 0 \), and Theorem 11 can be applied.

Let \( \{X'_t\}_{t \in \Delta} \) be the complex-analytic family obtained from \( \{X_t\}_{t \in \Delta} \) by the elementary modification with center \( C \). Let \( \delta = \rho(C) \) be the \((-1, -1)\) curve as in Theorem 2. We furnish \( \{X'_t\}_{t \in \Delta} \) with a family of Kähler forms \( \{\omega'_t\}_{t \in \Delta} \). Recall that the set of Kähler forms on a Kähler manifold is a convex cone. Hence, we may deform \( \{\omega'_t\}_{t \in \Delta} \) so that

\[
\rho_C : (X_t, \omega_t) \to (X'_t, \omega'_t) \text{ is a symplectomorphism for each } t \in \partial \Delta,
\]

(6.3)

Considering the family of Kähler metrics \( \{g'_t\}_{t \in \Delta} \) associated to \( \{\omega'_t\}_{t \in \Delta} \), we have:

**Lemma 10.** Lemma 8 holds for \( \{X'_t\}_{t \in \Delta} \) with \( C \) replaced by \( C' \) and \( \{g_t, \omega_t\}_{t \in \Delta} \) replaced by \( \{g'_t, \omega'_t\}_{t \in \Delta} \); and likewise for Theorem 11 and Lemma 9.

Gluing the families \( \{X_t\}_{t \in \Delta} \) and \( \{X'_t\}_{t \in \Delta} \) by the map \( \rho_C \),

\[
(\Delta \times X_t) \cup (\Delta \times X'_t) / \sim, \quad (t, x) \sim (t, \rho_C(x)), \quad t \in \partial \Delta,
\]

we obtain a (not complex-analytic but merely differentiable) family \( \{X_s\}_{s \in S^2} \) of Kähler surfaces parameterized by the sphere \( S^2 = \Delta \cup \partial_\Delta \). From (6.3) we see that the families \( \{\omega_t\}_{t \in \Delta} \) and \( \{\omega'_t\}_{t \in \Delta} \) form a family of Kähler forms \( \{\omega_s\}_{s \in S^2} \).

Regarding \( \{X_t\}_{t \in \Delta} \) and \( \{X'_t\}_{t \in \Delta} \) as subfamilies of \( \{X_s\}_{s \in S^2} \), we let \( X_{s_0} \) be \( X_0 \), and let \( X'_{s_0} \) be \( X'_0 \). As \( \pi_1(S^2) = 0 \), we may canonically identify \( H_2(X_{s_0}; \mathbb{Z}) \) with \( H_2(X'_{s_0}; \mathbb{Z}) \). With this identification at hand, we apply formula (1.7) to get:

\[
[C] = -[C'].
\]

Let us consider the family Seiberg-Witten invariants of \( \{X_s\}_{s \in S^2} \). Let \( s_{[C]} \) be the spin\(^c \) structure on \( X_{s_0} \) given by (3.1) with \( \varepsilon = [C] \). Similarly, let \( s_{-[C]} \) be the spin\(^c \) structure on \( X_{s_0} \) with \( \vareferences{C} = -[C] \). Choose spin\(^c \) structures on \( \{X_s\}_{s \in S^2} \) extending \( s_{[C]} \) and \( s_{-[C]} \). We use \( s_{[C]}, s_{-[C]} \) to denote these extensions, also. Combining Theorem 11 and Lemma 10 we get:

\[
FSW(s_{[C]}) = FSW(s_{-[C]}) = 1.
\]

If \( p_g(X_0) > 1 \), then \( b^+(X) = 2 p_g + 1 > 3 \), and the family invariants are well defined. The family of surfaces \( \{X_s\}_{s \in S^2} \) cannot carry a family of cohomologous symplectic forms, as this would contradict Lemma 5. This completes the proof.

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