Minimum-Error Discrimination of Qubit States Revisited

Mahdi Rouhbakhsh N. and Seyed Arash Ghoreishi

1Sharif University of Technology, Department of Physics, 14588 Tehran, Iran
2Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik, 72076 Tübingen, Germany
3RCQI, Institute of Physics, Slovak Academy of Sciences, Dábravská Česta 9, 84511 Bratislava, Slovakia

We study the problem of Minimum-Error (ME) discrimination for qubit states in detail using a geometric approach. We employ the Helstrom conditions in a constructive way and use the Bloch vector formulation. All POVM answers of the ME problem define a unique operator known as the Lagrange operator, and our method starts with finding this operator. We see that in an optimal strategy there might be some non-detectable quantum states with corresponding null measurement operators. To find Γ, we introduce the notion of circumsphere for equiprobable problems. For a general problem of N qubit states with arbitrary priori probabilities \( \{p_i, \rho_i\}_{i=1}^N \), we introduce a structured instruction with four steps to find Γ by using the fact that there is always a nondecomposable optimal POVM with at most four detectable states. To use this instruction, we need to know the optimal solution of two, three, and four mixed qubit states with arbitrary priori probability. Therefore, we solve the problem for these cases, including some completely new results like the ME problem of four qubit states. Moreover, we introduce some classes of POVM answers (involving the geometric of the polytope of qubit states inside the Bloch sphere), and the notion of nondecomposable POVM subsets of a given problem which provide us all we need for constructing all optimal answers.

I. INTRODUCTION

Distinguishing between quantum objects is one of the most fundamental tasks in information theory. Because of its particular importance in quantum communication \([1,2]\) and quantum cryptography \([3]\), discriminating among these objects plays a very important role in quantum information protocols.

The quantum state discrimination problem has been the subject of many researches during recent decades \([4]\). The starting point of these efforts date back to 1978 when Helstrom addressed this issue in his famous book \([2]\). He studied the case of two states and obtained the minimum-error probability in the framework of quantum detection and estimation theory \([5]\). Since then, based on the different approaches on the issue (error-free protocols with a possibility of failure and protocols with minimum-error probability) many developments have been achieved \([6–19]\). Finding which one is the most suitable is highly dependent on the purpose of the process.

The extension to the general case of N possible states \( \{\rho_i\}_{i=1}^N \) with associated a priori probabilities \( \{p_i\}_{i=1}^N \) is not a straightforward problem. For these problems, however, there are necessary and sufficient conditions on the optimal measurement operators \( \pi_i \), known as the Helstrom conditions \([2, 20, 21]\).

\[
\Gamma - p_i \rho_i \geq 0 \quad \forall i, \quad (1)
\]

\[
\pi_j (p_j \rho_j - p_i \rho_i) \pi_i = 0 \quad \forall i, j, \quad (2)
\]

where \( \Gamma = \sum_i p_i \rho_i \pi_i \) is a positive Hermitian operator known as Lagrange operator. In fact, these two conditions are not independent and the second condition can be obtained from the first condition \([20, 21]\). So, the main condition which is both necessary and sufficient condition for an optimal measurement is Eq. (1). However, we can still use Eq. (2) in our procedure. It is also of particular importance to note that all minimum-error measurements that give the optimal probability define a unique Lagrange operator \([22]\).

The Helstrom conditions can be used to check the optimality of a candidate measurement, however, they cannot be used to construct the optimal measurement for a general problem. They can be useful for the problem with some symmetries among states which can be a guide for guessing the form of measurement in such a problem \([23–28]\).

Although, for a while, only problems with certain symmetries seemed solvable, in recent years there have been successes in solving problems by employing the conditions \([11, 12]\). Particularly, for the qubit states, the Bloch representation provides a useful tool to solve the problem of ME discrimination of qubit states. For example, Hunter in 2004 presented a complete solution for pure qubit states with equal a priori probabilities \([29]\). Samsonov in 2009 applied the necessary and sufficient conditions for indicating an algorithmic solution for N pure qubit states \([30]\). Deconinck and Terhal in 2010 used the discrimination duality theorem and the Bloch sphere representation for a geometric and analytic representation of the optimal guessing strategies among qubit states \([31]\). Bae in 2013 represented a geometric formulation for a case with equal priori probabilities in a situation where quantum state geometry is clear \([32]\), and in the same year a complete analysis for three mixed states of qubit systems was done by Ha et al. \([33]\). Weir et al. in 2017 used the Helstrom conditions constructively and analytically for solving a problem with any number of qubit states with arbitrary priori probabilities \([34]\) with
giving the central role in their approach to the inverse of Γ instead of Γ itself. Later, they use their method to find optimal strategies for trine states. In this paper, we address the problem of ME discrimination of N mixed qubit states with arbitrary a priori probabilities. The problem were studied for the problem of two states and the problem of equally likely pure qubit states. In 2010, Deconinck and Terhal indicated a general algorithm to find the Lagrange operator Γ by using the geometric properties of the Bloch sphere. Our main purpose in this paper is to extend their work and give an analytical method in detail to find all solutions of a general mixed qubit states problem, as well as studying some properties of these solutions.

Our starting point is the Helstrom conditions. Employing these conditions and the Bloch representation for qubit states, we show that one can find all possible ME POVM answers of a given problem with just knowing the Lagrange operator Γ. Generally, we solve the problem for both equal and arbitrary priori probabilities. For equal a priori probabilities, we introduce the notion of circumsphere. Furthermore, for a general problem with arbitrary a priori probabilities, we establish a general structured instruction to obtain Γ, with some practical tips involving the polytope of states motivating us to introduce two classes of unchanged guessing probability and unchanged measurement operators. Moreover, this instruction will be used to find Γ for a general ME problem with two, three, and four qubits (which the case for four-qubit states is completely new). Equipped with these tools, we are able to find all possible ME discrimination measurements of the problem. By introducing nondecomposable POVM answers of the problem an alternative way to obtain a general optimal answer is achievable by considering a convex combination of all these nondecomposable POVM sets.

We show that for an optimal strategy, there might be some states that are not detectable, so their related POVM elements are null; same as we call them as nonguessable and nearly guessable states. The other states which are detectable are called guessable states. Based on the fact that there is always possible to have an POVM answer with at most four detectable states for a typical ME discrimination problem of qubit states, we investigate these four detectable states in our instruction for finding Γ and in our solved examples of two, three, and four mixed qubit states. We consider the problem of trine states with arbitrary priori probabilities as a specific case of generic three states to compare our results with already known answers.

We introduce the concept of nondecomposable POVM qubit measurements. The mentioned instruction can be used to find all nondecomposable measurements for a problem of qubit state discrimination. These subsets do not include nondetectable states and have at most four elements. Finding all of these nondecomposable subsets enable us to find all possible ME answers of a given problem by establishing convex combination of these sets.

This paper is organized as following: In Sec. we review the problem of ME discrimination in a general way. In Subsec. [II A] we reformulate the problem for qubit states using the Bloch vector representation, and represent a way to find all optimal answers of a problem using the Lagrange operator Γ. In Subsec. [II B] we will first discuss the uniqueness of an optimal measurement, then introducing guessable, unguessable, and nearly guessable states, as well as the concept of decomposable and nondecomposable subsets of a POVM answer. In Subsec. [II C] we consider equal priori probability problems and introduce a minimal sphere, covering all of the Bloch vectors of the states, as circumsphere. In Subsec. [II D] we consider a general problem of arbitrary priori probability of qubits by introducing the hyperbola related to every two states, and then constructing a four-step instruction to find Γ for the most general case of qubits and introduce two classes of equivalent problems: (i) unchanged guessing probability, and (ii) unchanged measurement operators. Next, in Sec. III we solve a general problem of mixed qubit states with arbitrary priori probabilities for two, three, and four qubits cases and for the case of three qubits compare our results for the case of trine states with the known results in [35]. We show the method of constructing all optimal answers with the example of four Symmetric Pure States in [II D]. The last section includes the summary and conclusions.

II. QUANTUM STATE DISCRIMINATION

Suppose that we are given N states \( \rho_i, i = 1, \cdots, N \), with priori probabilities \( p_i \). Our task is to discriminate among these states by performing the optimal measurement with minimum probability of error. Such measurement needs to have \( N \) outcomes corresponding, respectively, to each of the \( N \) states \( \rho_i \). A general measurement can be described by a positive operator-valued measure (POVM) \[ \pi \subseteq \{ \pi_j \} \], which is defined by a set of operators \( \{ \pi_j \} \) satisfying

\[
\pi_j \geq 0, \quad \sum_j \pi_j = 1. \tag{3}
\]

Each of the possible measurement outcomes \( j \) is characterized by the corresponding POVM element \( \pi_j \). The probability of observing outcome \( \pi_j \) when the state of system is \( \pi_i \) is

\[
P(j|i) = \text{Tr}(\pi_j \rho_i). \tag{4}
\]

Then, the probability of making a correct guess when the given set is \( \{ p_i, \rho_i \} \) will be

\[
P_{\text{corr}} = \sum_{i=1}^{N} p_i \text{Tr}(\rho_i \pi_i). \tag{5}
\]
In a minimum-error discrimination approach the aim is to seek the optimal measurement that gives rise to the minimum average probability of error occurred during the process or, equivalently, to the maximum probability of making a correct guess given by $P_{\text{guess}} = \max\{P_{\text{corr}}\}$. This is achievable, as we mentioned previously, if and only if the POVM elements satisfy the Helstrom conditions 11 and 12.

To continue, let us reformulate conditions 11 and 12 by summing the second one over $j$ which results in

$$
\Gamma - \tilde{\rho}_i \geq 0 \quad \forall i, \\
(\Gamma - \tilde{\rho}_i)\pi_i = 0 \quad \forall i,
$$

where we have defined $\tilde{\rho}_i = p_i\rho_i$.

From these equations (6) and (7) (and its Hermitian conjugate $\pi_i(\Gamma - \tilde{\rho}_i) = 0 \quad \forall i$), in the case that $\Gamma - \tilde{\rho}_i$ is not a full rank operator, one can easily see that $\text{supp}\{\Gamma - \tilde{\rho}_i\} \perp \text{supp}\{\pi_i\}$, or equivalently, $\text{supp}\{\Gamma - \tilde{\rho}_i\} \subseteq \ker\{\pi_i\}$, where $\text{supp}\{\cdot\}$ and $\ker\{\cdot\}$ denotes the support and kernel of an operator, respectively, or in other words, equations (6) and (7) imply that both determinants $\text{det}\{\Gamma - \tilde{\rho}_i\}$, $\text{det}\{\pi_i\}$ have to be zero.

So far, everything is quite general and no reference has been made to qubit states. In what follows we consider, however, the qubit cases.

A. Discrimination of qubit states

Solving a general problem for minimum-error (ME) discrimination is not an easy task and there is no analytical solution in general. However, there are some approaches for qubit states 24–34. In this work, we consider the problem of the most general case of qubit states using a geometric approach. We use the Bloch vector representation to write $\rho_i$ and $\Gamma$ as

$$
\rho_i = \frac{1}{2}(\mathbb{1} + v_i \cdot \sigma), \\
\Gamma = \frac{1}{2}(\gamma \mathbb{1} + \gamma \cdot \sigma).
$$

Where $v_i = (v_{xi}, v_{yi}, v_{zi})^T$ is the Bloch vector corresponding to the state $\rho_i$, $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$ is a vector constructed by Pauli matrices, and $t$ denotes transposition. Moreover, $\gamma_0 = \text{Tr}(\Gamma) = P_{\text{guess}}$ and $\gamma = (\gamma_x, \gamma_y, \gamma_z)^T$. Nonnegativity of $\rho_i$ requires $0 \leq |v_i| \leq 1$. The lower and upper bounds are achieved for the extreme cases of maximally mixed and pure states, respectively, so $|v_i|$ can be regarded as a kind of purity.

Using Eqs. (8) and (9), one can write

$$
\Gamma - \tilde{\rho}_i = \frac{1}{2}|(\gamma_0 - p_i)\mathbb{1} + (\gamma - \tilde{v}_i) \cdot \sigma|,
$$

where we have defined the subnormalized Bloch vector $\tilde{v}_i = p_i v_i$ associated with the state $\rho_i$ and its priori probability $p_i$. But in this paper, for simplicity, we call both vectors $v_i$ and $\tilde{v}_i$ Bloch vectors. It follows from inequality (10) that

$$
\gamma_0 - p_i \geq |\tilde{v}_i - \gamma|.
$$

So, this equation is equal to the main Helstrom condition, Eq. (4), and from now on, we will check optimality of our answers by Eq. (11).

To continue, if the inequality strictly holds, i.e. positive definiteness of the determinant of $\Gamma - \tilde{\rho}_i$, inevitably corresponding measurement operators $\pi_i$ have to be zero. But if equality holds, i.e. zero determinant of $\Gamma - \tilde{\rho}_i$, then $\pi_i$ does not have to be zero. So, in the case of nontrivial answer we obtain the following useful formula

$$
\gamma_0 - p_i = |\tilde{v}_i - \gamma|.
$$

Regarding $\gamma_0 = P_{\text{guess}}$, this relation implies that the distance between two vectors $\gamma$ and $\tilde{v}_i$ is equal to the difference between two probabilities, i.e. the guessing probability and the priori probability $p_i$ associated with state $\rho_i$. The greater $p_i$, the closer are two vectors $\tilde{v}_i$ and $\gamma$.

Keeping in mind that Eq. (12) does not generally hold for all states. For the sake of simplicity, without loss of generality, we consider that the set of $\{\rho_i\}_{i=1}^N$ and corresponding priori probabilities $\{p_i\}_{i=1}^N$ are arranged in such a way that the first $\{\rho_i\}_{i=M+1}^N$ are those states that the rank of $(\Gamma - \tilde{\rho}_i)$ is less than 2, so it means that these states are detectable in optimal case and the rest states $\{\rho_i\}_{i=M+1}^N$ are states that for them $(\Gamma - \tilde{\rho}_i)$ is a full rank operator, so there is no way to detect them in an optimal way, i.e. $\pi_i = 0$ for $i = M + 1, \cdots, N$. With this terminology and using the fact that $\text{det}(\pi_i) = 0$ when $\Gamma - \tilde{\rho}_i \neq 0$ or full rank, it turns out that the measurement operators $\pi_i$’s for $i = 1, \cdots, M$ must be rank-one and therefore proportional to projectors. As a qubit projector is characterized by its unit vector $\hat{n}_i$ on the Bloch sphere, we have

$$
\pi_i = \frac{\alpha_i}{2}(\mathbb{1} + \hat{n}_i \cdot \sigma),
$$

for $i = 1, \cdots, M$, where $\alpha_i$’s are real and range from zero to one, $\alpha_i = \text{Tr}(\pi_i)$. The completeness relation 15 requires the following conditions on the parameters $\alpha_i$’s

$$
\sum_{i=1}^M \alpha_i = 2,
\sum_{i=1}^M \alpha_i \hat{n}_i = 0,
$$

where $0 \leq \alpha_i \leq 1$. On the other hand, inserting Eqs. (10) and (13) in the Helstrom condition (7), and by using Eq. (12) we find the unit vector $\hat{n}_i$ as

$$
\hat{n}_i = \frac{\tilde{v}_i - \gamma}{|\tilde{v}_i - \gamma|}.
$$

Obviously, knowing vector $\gamma$ is equivalent to know all the unit vectors $\hat{n}_i$ corresponding to the nonzero measurements $\pi_i$.

Corollary 1. As a result of Eqs. (14), $\gamma$ has to be confined in the convex polytope of the points $\tilde{v}_i$’s.
Corollary 2. It can be concluded from Eqs. (12) to (14) that by translation of all vectors \( \tilde{v}_i \)'s, with unchanged \( p_i \)'s, by a fixed vector inside the Bloch sphere nothing will be changed, and the only thing that matters is relative locations of \( \tilde{v}_i \)'s. So, if all of \( \tilde{v}_i \)'s move with the same displacement vector \( a (\tilde{v}_i \to \tilde{v}_i + a) \), the answers will be unchanged. In other words, a set of \( \{p_i, \rho_i'\} \) has the same answer of measurement operators and guessing probability as the initial set \( \{p_i, \rho_i\} \), where

\[
\rho_i' = \frac{1}{2} \left[ I + \left( \frac{v_i + a}{p_i} \right) \sigma \right] \quad \text{with} \quad \left| v_i + \frac{a}{p_i} \right| \leq 1 \tag{16}
\]

This class is characterized by an unchanged guessing probability.

From the above discussion it follows that in order to characterize the optimal measurements, we have to know both the vector \( \gamma \) and the real parameters \( \alpha_i \)'s. If \( \gamma \) is known, then by using Eq. (14) it is an easy task to find all possible sets of parameters \( \{\alpha_i\} \). Each set of \( \{\alpha_i\} \) provides a different answer for the optimal POVM. So, finding \( \gamma \) is our priority that exposes all possible answers. As \( \Gamma \) is determined with a scalar \( \gamma_0 \) and a vector \( \gamma \), we need at most four states to determine \( \Gamma \) satisfying Eq. (12).

On the other hand, if we are given a set of \( \{\pi_i\} \), we are able to find the Lagrange operator \( \Gamma \) (or equivalently \( \gamma \)) with the help of relation \( \Gamma = \sum_i \rho_i \pi_i \). Then by using Eq. (14) we can obtain all \( N - 1 \) possible optimal answers.

B. Types of states and measurements in an optimal strategy

Similar to [31] we distinguish the states in a general qubit state discrimination problem. We showed that \( \gamma \) has to be confined in the convex polytope of the points \( \tilde{v}_i \)'s. So, based on different representation of \( \gamma \) we can redefine the definition of \( \tilde{v}_i \)'s for states in a discrimination problem as unguessable, nearly guessable, and guessable states. Based on their definition, the states for which \( \Gamma - \tilde{\rho}_i \) is a full rank operator (inequality part of Eq. (14)) are called unguessable since their related POVM elements are always null. The nearly guessable states are the states that although for them the operator \( \Gamma - \tilde{\rho}_i \) is not full rank, they do not appear in any optimal measurement and therefor their related measurement operators are null, and finally guessable states are those states that satisfy Eq. (12) and their POVM elements are nonzero for some optimal measurements.

It is important to study under what conditions the optimal measurement of a discrimination problem is unique. First, based on the preceding discussion, we have the following proposition for a discrimination problem with \( N \leq 4 \).

Proposition 3. In case of \( N \leq 4 \), the set of parameters \( \{\alpha_i\} \) satisfying Eq. (14) is unique if the states form a \( N - 1 \) simplex inside Bloch sphere. However, it is neither necessarily the case for \( N > 4 \), nor for \( N \leq 4 \), where the states do not form a \( N - 1 \) simplex, in a sense that each possible set of \( \{\alpha_i\} \) leads to a complete solution for the optimal measurement operators.

Proof. The proof simply comes from the theorem 3.5.6 in [31]. Based on this theorem, a point in an \( k \)-simplex with vertices \( x_0, x_1, \ldots, x_k \) can be written in a unique way of the vertices. So, in the problem of \( N \leq 4 \) qubit state discrimination if these states form a simplex the convex combination for \( \gamma \) is unique.

Consequently, in the cases with non-unique solutions we may face with some problems with different POVM measurements that give us the same optimal guessing probability. Thus, one can think about the possibility of constructing different measurements by combining these measurements in a convex way. This fact motivates us to divide optimal measurements in two different families of measurements which we call them decomposable and nondecomposable measurements.

Definition Consider the POVM \( M = \{\pi_1, \pi_2, \ldots, \pi_m\} \) is an optimal measurement for some discrimination problems. According to (13) we are able to define the set of Bloch unit vectors \( E = \{\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_m\} \) such that their convex polytope contains the Origin (Eq (13)). If one can find a subset \( E' \subset E \) in such a way that its related convex polytope still contains the Origin, then \( M \) is decomposable, otherwise, it is nondecomposable. Based on the three dimensional direction of \( \hat{n}_i \)'s, nondecomposable sets have at most four elements.

An exception for this definition is the existence of a measurement operator \( \pi_j \) proportional to unit matrix \( 1 \), then, there is an one-element nondecomposable set \( M' = \{\pi_i = \delta_{ij} 1\} \).

Proposition 4. Consider a POVM \( M = \{\pi_i\}_{i=1}^m \) with its corresponding set \( E = \{\hat{n}_1, \hat{n}_2, \ldots, \hat{n}_m\} \) which is optimal for some minimum-error discrimination problem \( \{p_i, \rho_i\}_{i=1}^N \). If \( M \) is decomposable then there is at least one subset \( E' \subset E \) which its corresponding POVM, \( M' \), is optimal for the same minimum-error discrimination problem \( \{p_i, \rho_i\}_{i=1}^N \).

In other words, any subset \( E' \subset E \) with at most four elements of \( \hat{n}_i \)'s that contains origin in its convex polytope of its elements, and if we cannot discard any of its elements in a way that origin still be in the convex polytope of the remaining elements, then it is a nondecomposable subset.

The proof for this proposition is straightforward according to the definition of decomposable and nondecomposable measurements.

C. Qubit states: Equal priori probabilities

So far, our discussion on discrimination among qubit states was quite general. But now, let us consider the
Suppose that there are special case of $N$ qubit states $\rho_i$ with equal priori probabilities, i.e. $p_i = 1/N$ for $i = 1, \cdots, N$. The Problem for the case of pure qubit states was studied by Hunter [29] and in this section we generalize it to the case of general qubit states including mixed states and then in next section to the case of arbitrary a priori probabilities. In this case, Eqs. (12) and (11) reduce to

$$|v_i - O| = N\gamma_0 - 1, \quad (17)$$

for $i = 1, \cdots, M$ and

$$|v_i - O| < N\gamma_0 - 1, \quad (18)$$

for $i = M + 1, \cdots, N$, respectively, where $O = N\gamma$.

These equations admit geometrical interpretation: Equation (17) defines a sphere with radius $R = N\gamma_0 - 1$ centered at the point $O$. All Bloch vectors, $v_i$’s, are embedded in this sphere, in a sense that $M$ Bloch vectors $v_i$’s with $i = 1, \cdots, M$ lie on the sphere and the remaining $N - M$ Bloch vectors $v_i$’s with $i = M + 1, \cdots, N$ are located somewhere inside the sphere (see Fig 1). As we mentioned previously, the measurement operators associated with the former are given by Eq. (13), however, for latter Bloch vectors we have $\pi_i = 0$, i.e. their corresponding states do not appear to any output of the discrimination process.

Lemma 5. In the optimal measurement, the number of states satisfying Eq. (17) is maximum.

Proof. Suppose that at most $M$ states can be met by Eq. (17). Without loss of generality we label these states from 1 to $M$. Then, the guessing probability can be written as

$$P_{\text{guess}}^M = \max_{\pi_1, \cdots, \pi_M} \sum_{i=1}^{M} \text{Tr}(\hat{\rho}_i \pi_i), \quad \sum_{i=1}^{M} \pi_i = 1. \quad (19)$$

Now, consider a second strategy for which at most $M - 1$ states satisfy Eq. (17). Denoting the corresponding measurement operators by $\pi_i' (i = 1, \cdots, M - 1)$, the guessing probability reads

$$P_{\text{guess}}^{M-1} = \max_{\pi_1', \cdots, \pi_{M-1}'} \sum_{i=1}^{M-1} \text{Tr}(\hat{\rho}_i \pi_i'), \quad \sum_{i=1}^{M-1} \pi_i' = 1. \quad (20)$$

Clearly, $P_{\text{guess}}^{M-1} \leq P_{\text{guess}}^{M}$, since $P_{\text{guess}}^{M-1}$ can be obtained from $P_{\text{guess}}^{M}$ with an extra constraint $\pi_M = 0$.

The lemma implies that in order to reach $P_{\text{guess}}$, we have to choose a sphere with maximum number of states lying on. It is simply the minimal sphere covering all the $v_i$’s (the same result was obtained in [40] with a different approach). We call it circumsphere, defined by $\{R, O\}$, where $R = |v_i - O|$ is its radius, and $O$ is its circumcenter. In view of this and the fact that $0 < R \leq 1$ and $P_{\text{guess}} = \gamma_0$, from Eq. (17), we get

$$\gamma_0 = \frac{1}{N} (1 + R), \quad (21)$$

and

$$\frac{1}{N} < P_{\text{guess}} \leq \frac{2}{N}. \quad (22)$$

Moreover, Eq. (14) reduces to

$$\hat{n}_i = \frac{v_i - O}{|v_i - O|}. \quad (23)$$

Note that, as long as the circumsphere remains fixed, one can modify the angles between $N$ qubit states while the guessing probability remains unchanged. The following corollaries are immediately obtained from the above results.

Corollary 6. Suppose that there are $N$ qubit states with equal priori probabilities, $\{\frac{1}{N}, \rho_i\}_{i=1}^{N}$, covering the circumsphere $\{R, O\}$. Therefore, Eq. (14) is valid with some known $\hat{n}_i$’s from Eq. (23) (note that, the only states which lie on the circumsphere are defined by $\hat{n}_i$). Then

(i) From the above discussions, the circumsphere remains invariant if we add $K$ different qubit states to the circumsphere in which all qubit states have the same priori probabilities $\frac{1}{N+K}$. (ii) It follows from Eq. (13) that all the optimal measurement operators of the first set $\{\frac{1}{N}, \rho_i\}_{i=1}^{N}$ are still optimal answers for the new set $\{\frac{1}{N+K}, \rho_i\}_{i=1}^{N+K}$ because defined $\hat{n}_i$’s from the first problem with $N$ states are still unchanged for the new problem, Eq. (23). However, this is not the only answer of the new problem. (iii) The guessing probability of the new problem is given in terms of the guessing probability $P_{\text{guess}} = \frac{1}{N} (1 + R)$ of the original one as $P_{\text{guess}}^{\text{New}} = \frac{N+K}{N+R} P_{\text{guess}}$, Eq. (21).

Corollary 7. Consider the case of $N$ equiprobable qubit states with the same purity $|v_i|$ located on the circumsphere $\{R, O\}$. If any of the following statements be true then the other ones will also be true. In other words, these statements can be interchangeably used in this particular case.
(i) The circumcenter is at Origin, i.e. $O = 0$.

(ii) The radius of the circumsphere is given by $R = |v_i|$.

(iii) The guessing probability is $P_{\text{guess}} = \frac{1}{N} (1 + |v_i|)$.

(iv) The optimal measurement operators are given by Eq. (13) with $\hat{n}_i = \frac{v_i}{|v_i|}$.

All implications are trivial and can be inferred by the results given above. A particular case is when there are $N$ equiprobable pure states with $O$ at Origin is an example which achieves its maximum value $P_{\text{guess}} = 2/N$. This is because the radius of this circumsphere can not be greater than one.

D. Qubit states: Arbitrary priori probabilities

Now, we are taking a step further by considering a general problem of a set of qubit states with arbitrary priori probabilities. Before we provide an instruction to find the answers of a general problem of a set of qubit states with arbitrary priori probabilities, consider that one is able to find the Lagrange operator $\Gamma$ by using Eq. (12). Based on the previous discussion, we know that one of the answers of an ME problem is determined with $N$ equiprobable pure states, $|v_i| = 1$. It follows from the corollary that the guessing probability of $N$ equiprobable pure qubit states with $O$ at Origin is an example which achieves its maximum value $P_{\text{guess}} = 2/N$. This is because the radius of this circumsphere can not be greater than one.

From point $\hat{v}_l$ on the connecting line between $\hat{v}_i$ and $\hat{v}_m$. As $R_{lm}$'s are distances, they are definitely positive. So, possible candidates for $\gamma_0$ and $\gamma$ are

\[
\gamma_0 = \frac{1}{2} \left( (p_l + p_m) + d_{lm} \right),
\]

and

\[
b = \sqrt{c^2 - a^2},
\]

It follows from Eq. (11) that only the left branch of the hyperbola could provide a candidate answer for $\gamma$.

In the case of two-state case the candidate $\gamma$ is located at the distance of

\[
R_{lm} = c - a = \frac{1}{2} \left[ d_{lm} - p_{lm} \right],
\]

from point $\hat{v}_l$ on the connecting line between $\hat{v}_i$ and $\hat{v}_m$. As $R_{lm}$'s are distances, they are definitely positive. So, possible candidates for $\gamma_0$ and $\gamma$ are

\[
\gamma_0 = \frac{1}{2} \left( (p_l + p_m) + d_{lm} \right),
\]

and

\[
\gamma = \frac{1}{2} \left[ (v_l + v_m) + \frac{p_{lm}}{d_{lm}} (\hat{v}_l - \hat{v}_m) \right] = \frac{1}{2} \left[ (1 + \frac{p_{lm}}{d_{lm}}) \hat{v}_l + (1 - \frac{p_{lm}}{d_{lm}}) \hat{v}_m \right],
\]

where the primes indicate that they are still candidate answers. Now, the question is: which two states to pick? In this case we have the following proposition

**Proposition 8.** In a case where a two element optimal POVM exists the answer can be obtained by considering two states which maximize $\gamma_{0,lm}$.

**Proof.** For simplicity, consider that $\gamma_{0,12}$ is the maximum. Assume that there are states $i$ and $j$ with $\gamma_{0,ij} < \gamma_{0,12}$ that satisfy Helstrom conditions. It means that

\[
\gamma_{0,ij} = |\gamma_{ij} - \hat{v}_l|, \quad \text{for} \quad l = i, j \quad (31)
\]

\[
\gamma_{0,ij} = |\gamma_{ij} - \hat{v}_l|, \quad \text{for} \quad l \neq i, j \quad (32)
\]

then, writing inequality for $l = 1, 2$ and summing them gives

\[
2\gamma_{0,ij} > |\gamma_{ij} - \hat{v}_1| + |\gamma_{ij} - \hat{v}_2| + (p_1 + p_2)
\]

\[
\geq |\hat{v}_1 - \hat{v}_2| + (p_1 + p_2) = 2\gamma_{0,12},
\]

and

\[
\gamma_{0,ij} = \frac{1}{2} \left( (p_l + p_m) + d_{lm} \right),
\]

\[
b = \sqrt{c^2 - a^2},
\]

It follows from Eq. (11) that only the left branch of the hyperbola could provide a candidate answer for $\gamma$.

In this case of two-state case the candidate $\gamma$ is located at the distance of

\[
R_{lm} = c - a = \frac{1}{2} \left[ d_{lm} - p_{lm} \right],
\]

from point $\hat{v}_l$ on the connecting line between $\hat{v}_i$ and $\hat{v}_m$. As $R_{lm}$'s are distances, they are definitely positive. So, possible candidates for $\gamma_0$ and $\gamma$ are

\[
\gamma_0 = \frac{1}{2} \left( (p_l + p_m) + d_{lm} \right),
\]

and

\[
\gamma = \frac{1}{2} \left[ (v_l + v_m) + \frac{p_{lm}}{d_{lm}} (\hat{v}_l - \hat{v}_m) \right] = \frac{1}{2} \left[ (1 + \frac{p_{lm}}{d_{lm}}) \hat{v}_l + (1 - \frac{p_{lm}}{d_{lm}}) \hat{v}_m \right],
\]

where the primes indicate that they are still candidate answers. Now, the question is: which two states to pick? In this case we have the following proposition

**Proposition 8.** In a case where a two element optimal POVM exists the answer can be obtained by considering two states which maximize $\gamma_{0,lm}$.

**Proof.** For simplicity, consider that $\gamma_{0,12}$ is the maximum. Assume that there are states $i$ and $j$ with $\gamma_{0,ij} < \gamma_{0,12}$ that satisfy Helstrom conditions. It means that

\[
\gamma_{0,ij} = |\gamma_{ij} - \hat{v}_l|, \quad \text{for} \quad l = i, j \quad (31)
\]

\[
\gamma_{0,ij} = |\gamma_{ij} - \hat{v}_l|, \quad \text{for} \quad l \neq i, j \quad (32)
\]

then, writing inequality for $l = 1, 2$ and summing them gives

\[
2\gamma_{0,ij} > |\gamma_{ij} - \hat{v}_1| + |\gamma_{ij} - \hat{v}_2| + (p_1 + p_2)
\]

\[
\geq |\hat{v}_1 - \hat{v}_2| + (p_1 + p_2) = 2\gamma_{0,12},
\]
the second inequality comes from the triangle inequality. So, we end up in $\gamma_{0i} > \gamma_{0j}$ which is in contradiction with our first assumption $\gamma_{0i} < \gamma_{0j}$.

To continue, we should find two states which maximizes Eq. (24). Then test these two states using the following condition

$$p_i + |\tilde{v}_i - \gamma| \geq p_i + |\tilde{v}_i - \gamma|, \quad \forall i \quad (34)$$

which comes from the Helstrom condition, Eq. (11). The cases $i = l, m$ are trivial. If the Eq. (24) is maximized for more than a pair of states, we must test Helstrom condition for each of them. Note that ME discrimination might have some answers and what we need is just one of them to find $\gamma$. If the condition (33) is not met we proceed to the next step.

**Step (iii).**— In this step we consider three-state case. Let us label them by $l, m, n$. $\gamma'$ is constructed from these three states by drawing three hyperbolas arising from each pair of states $l, m, n$ (lm, ln, mn). The convex polytope in this case is a triangle and it is enough to just calculate the point that two hyperbolas meet ($\gamma'$). This point must be inside the triangle. The related probability can be obtained from Eq. (45). As before, for three states that maximizes this probability we must test Helstrom condition. The cases $i = l, m, n$ are trivial. Likewise the second step, we might have more than one three-state case to maximize Eq. (45). In this case we must test the Helstrom condition for each of them.

**Step (iv).**— If we still do not have the answer, we have to consider four-state case. This step must reveal the answer since we know that there is always a nondecomposable answer with maximum four number of states. In this case, although there are $\binom{4}{2} = 6$ hyperbolas, it is enough that three hyperbolas, with a common focus point $\tilde{v}_m$, meet at one point inside the polytope to obtain $\gamma'$. The related probability can be obtained from Eq. (55). A four-state case that maximizes this probability must be states we are searching for to find $\gamma$. As before, this four-state case that maximizes Eq. (55) might not be unique.

Deriving of equation (15) and Eq. (55) for steps (iii) and (iv) respectively will be discussed in next section.

By looking at the Bloch representation of $\Gamma$, Eq. (9), we see that there are just four unknowns to be determined, a scalar $\gamma_0$ and a three dimensional vector $\gamma$. To this purpose, one can easily find them by considering four qubit states with their corresponding Bloch vectors (See Eq. (12)). So, we do not expect to have more than four states. Furthermore, as a typical ME discrimination problem has, at least, a nondecomposable answer, one can see that these four steps are enough to find $\gamma$. Although we introduced a way to find required states which lead to the Lagrange operator, the fourth step seems less probable, because we often expect that to have at least one nondecomposable POVM answer with less number of elements for a given probable.

At this point, it is useful to explicitly express some remarkable results:

- According to the above instruction, equation (26) has a key role to find $\gamma$, which is unaffected as long as the relative distances between states $\tilde{d}_m$’s, and also priori probabilities $p_i$’s are consistent. It suggests that by translation or rotation of the polytope, or equivalently all $\tilde{v}_i$’s as a whole, with a fixed vector inside the Bloch sphere, $P_{\text{guess}}$ is consistent. This class is characterized by an *unchanged guessing probability*.

- Furthermore, by translation and rescaling of the polytope inside the Bloch sphere, with unaffected $p_i$’s, measurement operators which are defined by Eq. (15) will also be unchanged. This class is also characterized by an *unchanged measurement operators*.

- Based on the mentioned instruction, to have a complete solution of a general qubit state discrimination we need to find all nondecomposable sets, $E' = \{\tilde{n}_1, \tilde{n}_2, \cdots\}$, of the problem (It can be simply done, after finding $\gamma$, by looking at the geometry of the $\tilde{n}_i$’s; see the explanation after proposition (4)). Then, we are able to obtain all of the possible optimal POVM’s using convex combinations of all nondecomposable sets, i.e. $\sum_{i=1}^{k} \beta_i M_i'$. Where $k$ is the number of all nondecomposable sets, $0 \leq \beta_i \leq 1$, and $\sum_{i=1}^{k} \beta_i = 1$ to satisfy completeness relation of a POVM measurement.

For this purpose, in the next section, we will solve the problem for these cases analytically. Some of these results like the problem of two states discrimination have been known for a long time.

### III. SOLUTION IN QUBIT STATE DISCRIMINATION FOR $N \leq 4$

#### A. Two States Case

As the first example, consider the case of two arbitrary qubit states $\rho_1$ and $\rho_2$ with priori probabilities $p_1 \geq p_2$. We proceed following the steps described above. First, if $\Gamma = \hat{\rho}_0$ then $\pi_1 = 1, \pi_2 = 0$ and $P_{\text{guess}} = \gamma_0 = p_1$. But, if this condition is not met, from Eq. (11) $\alpha_1 = \alpha_2 = 1$, and POVM’s are projectors given by Eq. (13) with $\hat{n}_1 = -\tilde{n}_2 = \frac{\tilde{d}_1 - \tilde{d}_2}{\tilde{d}_1 - \tilde{d}_2}$ and $\hat{d}_{12} = |\tilde{v}_1 - \tilde{v}_2|$. The associated guessing probability is then given by Eq. (29)

$$P_{\text{guess}} = \frac{1}{2}(1 + \tilde{d}_{12}), \quad (35)$$

which is the well-known Helstrom relation for two states \cite{2}.
B. Three States

We now consider a general case of three arbitrary qubit states with priori probabilities $p_1 \geq p_2 \geq p_3$. To proceed further, note that discrimination of an arbitrary set of three qubit states can be reduced to the discrimination of three qubit states, all embedded in $x-z$ plane, defined by

$$
\rho_1 = \frac{1}{2} \begin{pmatrix} 1 + a & 0 \\ 0 & 1 - a \end{pmatrix},
$$

$$
\rho_2 = \frac{1}{2} \begin{pmatrix} 1 + b \cos \theta & b \sin \theta \\ b \sin \theta & 1 - b \cos \theta \end{pmatrix},
$$

$$
\rho_3 = \frac{1}{2} \begin{pmatrix} 1 + c \cos \phi & -c \sin \phi \\ -c \sin \phi & 1 - c \cos \phi \end{pmatrix},
$$

where $\theta$ and $\phi$ are defined in FIG. 3. For a proof see Appendix A. The corresponding Bloch vectors $\vec{v}_i$’s are given by

$$
\vec{v}_1 = (0, ap_1)^t, \\
\vec{v}_2 = (bp_2 \sin \theta, bp_2 \cos \theta)^t, \\
\vec{v}_3 = (-cp_3 \sin \phi, cp_3 \cos \phi)^t,
$$

where $(x, z)^t$ is a simplified representation for $(x, 0, z)^t$.

[Diagram of Bloch representation of three states $\rho_1$, $\rho_2$, and $\rho_3$ with corresponding probabilities $p_1 \geq p_2 \geq p_3$.]

With the assumption that $p_1$ is the greatest priori probability, first we have to check whether $\hat{p}_1$ is equal to $\Gamma$ or not. So, with the help of Eq. (24), we need to check the following conditions

$$
|(-bp_2 \sin \theta, ap_1 - bp_2 \cos \theta)^t| \leq p_{12},
$$

$$
|(cp_3 \sin \phi, ap_1 - cp_3 \cos \phi)^t| \leq p_{13}.
$$

(38)

(39)

Where $|(x, z)^t| = \sqrt{x^2 + z^2}$. Obviously, if the above conditions are not met, we must calculate $P_{\text{guess}}$ from Eq. (29) for every two states and then check the Helstrom condition using Eq. (34). The explicit form of this conditions to detect one of the pairs of $(\rho_1, \rho_2)$, $(\rho_1, \rho_3)$ or $(\rho_2, \rho_3)$, respectively, are

$$
|\langle x'_{12} + cp_3 \sin \phi, z'_{12} - cp_3 \cos \phi \rangle^t| \leq \tilde{R}_{12} + p_{13},
$$

(40)

$$
|\langle x'_{13} - bp_2 \sin \theta, z'_{13} - bp_2 \cos \theta \rangle^t| \leq \tilde{R}_{13} + p_{12},
$$

(41)

$$
|\langle x'_{23}, z'_{23} - ap_1 \rangle^t| \leq \tilde{R}_{23} - p_{12}.
$$

(42)

with the corresponding $\gamma_{ij} = (x'_{ij}, z'_{ij})^t$ for these states from Eq. (30)

$$
\begin{align*}
x'_{12} &= \frac{bp_2 \tilde{R}_{12} \sin \theta}{2R_{12} + p_{12}}, \\
z'_{12} &= ap_1 - \frac{(ap_1 - bp_2 \cos \theta)\tilde{R}_{12}}{2R_{12} + p_{12}}, \\
x'_{23} &= \frac{bp_2 \sin \theta - (bp_2 \sin \theta + cp_3 \sin \phi)\tilde{R}_{23}}{2R_{12} + p_{23}}, \\
z'_{23} &= \frac{bp_2 \cos \theta - (bp_2 \cos \theta - cp_3 \cos \phi)\tilde{R}_{23}}{2R_{12} + p_{23}}, \\
x'_{13} &= \frac{-cp_3 \tilde{R}_{13} \sin \phi}{2\tilde{R}_{13} + (p_{13})}, \\
z'_{13} &= ap_1 - \frac{(ap_1 - cp_3 \cos \phi)\tilde{R}_{13}}{2\tilde{R}_{13} + (p_{13})},
\end{align*}
$$

where $p_{ij} := p_i - p_j$ and $R_{lm}$ is defined in Eq. (28).

For any specific problem with known $p_i$, $\theta$ and $\phi$, each condition of Eqs. (40) to (42) that is satisfied will be the only solution of the problem. Accordingly, the optimal POVM elements are given by $\pi_i = \frac{1}{2}(\mathbb{1} + \hat{n}_i \cdot \sigma)$, (see Eq. (13)), where the corresponding unit vectors of the nonzero measurement elements can be written as

$$
\hat{n}_l = -\hat{n}_m = \frac{\vec{v}_l - \vec{v}_m}{d_{lm}}
$$

(44)

Finally, if none of the above conditions were met, the solution for $\gamma$ must be found while all three states are detectable. It means that three hyperbolas should met at one point. Note that since three points are embedded in a plane it is enough to find the intersection between two hyperbolas in this plane. So, by using the properties of hyperbolas the guessing probability can be obtained as

$$
P_{\text{guess}} = \gamma_0 = p_1 + |\gamma - \hat{v}_1|
$$

$$
= p_1 + \frac{p_{12}^2 - p_{13}^2}{2(d_{12} \cos(\alpha) + p_{12})},
$$

(45)
So, the answers of the optimal operators are \( \pi \) with a rotation matrix, Appendix B. In case of \( \rho \) and a three elements nondecomposable POVM is optimal.

\[ \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{i\frac{4\pi}{3}} \end{pmatrix} \]

FIG. 4. Representation of three qubit states problem when a three elements nondecomposable POVM is optimal.

\[ \alpha = \arctan \left( \frac{\hat{d}_{13}(\hat{d}_{12}^2 - p_{12}^2) \sin(\beta)}{\hat{d}_{13}(\hat{d}_{12}^2 - p_{12}^2) \cos(\beta) - \hat{d}_{12}(\hat{d}_{12}^2 - p_{12}^2)} \right) \]

\[ + \arccos \left( \frac{(\hat{d}_{13}^2 - p_{13}^2)p_{12} - (\hat{d}_{12}^2 - p_{12}^2)p_{13}}{\sqrt{\hat{d}_{13}^2(\hat{d}_{12}^2 - p_{12}^2)^2 + \hat{d}_{12}^2(\hat{d}_{13}^2 - p_{13}^2)^2 - 2\hat{d}_{12}\hat{d}_{13}(\hat{d}_{12}^2 - p_{12}^2)(\hat{d}_{13}^2 - p_{13}^2) \cos(\beta)}} \right), \]

and

\[ \beta = \arccos \left( \frac{\hat{d}_{12}^2 + \hat{d}_{13}^2 - \hat{d}_{21}^2}{2\hat{d}_{12}\hat{d}_{13}} \right). \] (46)

To find the vector \( \gamma \), we can use the geometric of triangle in FIG. 4. For this purpose, by using the Gram-Schmidt method, we can use two edges of the triangle \( \hat{v}_2 - \hat{v}_1 \) and \( \hat{v}_3 - \hat{v}_1 \) to construct an orthonormal basis from them for the plain which the triangle lies in. If we show the orthonormal basis by \( \hat{k}_1 \) and \( \hat{k}_2 \), then

\[ \hat{k}_1 = \frac{\hat{v}_2 - \hat{v}_1}{d_{12}}, \quad \hat{k}_2 = \frac{\hat{f}}{|\hat{f}|} \] (47)

where

\[ \hat{f} = (\hat{v}_3 - \hat{v}_1) - \hat{k}_1 \left[ \frac{(|\hat{v}_2 - \hat{v}_1|)}{d_{12}} \right] \left[(\hat{v}_3 - \hat{v}_1)] \right]. \]

Hence, \( \gamma \) can be written as

\[ \gamma = \frac{\hat{d}_{12}^2 - p_{12}^2}{2(d_{12}\cos(\alpha) + p_{12})} \left( \cos(\alpha)\hat{k}_1 + \sin(\alpha)\hat{k}_2 + \hat{v}_1 \right). \] (48)

So, POVM elements can be obtained using Eqs. (13) and (15).

It is of the notice that we first altered the Bloch vectors with a rotation matrix, Appendix A and a displacement. In case of \( \rho_i \) and \( \rho_j \) to be two detectable states, optimal operators are \( \pi_i = \rho_n \) and \( \pi_j = \rho - \rho_n \) with \( \rho_n = \hat{v}_3 - \hat{v}_1 \).

So, the answers of the optimal operators \( \{\pi_i\} \) are easily obtained with the vectors of the original problem. But, for the case in which three states are detectable, after obtaining \( \rho_n \)’s we must rotate them back, with inverse of rotation matrix, to the original problem and then use the formula \( \pi_i = \rho_n \) to get the optimal operators for the original problem.

Here, for more illustration of this case, let us consider discrimination among trine states, i.e. the qubit states associated with equidistant points on the surface of the Bloch sphere. They are defined by

\[ |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \]

\[ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{2\pi}{3}}|1\rangle), \]

\[ |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{4\pi}{3}}|1\rangle). \] (49)

Then, based on the previous discussion, it is possible to rotate these states on the Bloch sphere to align them on \( x - z \) plane

\[ |\psi_1\rangle = |0\rangle, \]

\[ |\psi_2\rangle = \frac{1}{2}(|0\rangle + \sqrt{3}|1\rangle), \]

\[ |\psi_3\rangle = \frac{1}{2}(|0\rangle - \sqrt{3}|1\rangle). \] (50)

By writing the corresponding density matrices and comparing them with Eq. (36), we get \( a = b = c = 1 \) and \( \theta = \phi = 2\pi/3 \).

To continue, we first assume the case with equal prior probabilities. As the three states are pure and located on the surface of the Bloch sphere, finding the answer is straightforward. According to the discussion in Sec. [B] the guessing probability is \( P_{guess} = \frac{1}{3} \) and the corresponding POVM operators for the original problem are \( \pi_i = \frac{2}{3}\langle \psi_i | \psi_i \rangle \), known as the trine measurement [28].
In this region of two-state case the guessing probability can be obtained as
\[
\gamma_x = \frac{2\sqrt{3}(1-2p)(p-\delta)(p+\delta)^2(\delta-3p+1)}{9p^4-4p^3+6p^2\delta^2-12p\delta^2+4\delta^4+\delta^4}, \quad \gamma_y = 0,
\gamma_z = \frac{2(1-2p)(p^2-\delta^2)(-3p^2+(6\delta+1)p+\delta^2-3\delta)}{9p^4-4p^3+6p^2\delta^2-12p\delta^2+4\delta^2+\delta^4}.
\]

Using \( P_{\text{guess}} = \gamma_0 = p_1 + |\tilde{v}_1 - \gamma| \) or Eq. (53), the guessing probability can be calculated
\[
P_{\text{guess}} = \gamma_0 = p + \delta + \sqrt{\gamma_x^2 + (\gamma_z - p - \delta)^2} \leq \frac{2(1-2p)(p^2-\delta^2)(3p^2+\delta^2-2p)}{9p^4-4p^3+6p^2\delta^2-12p\delta^2+4\delta^2+\delta^4}.
\]

The equations (52) and (53), and this \( P_{\text{guess}} \) are in complete agreement with [35] that was obtained with a different approach.

C. Four States

In this section we consider a general problem of four qubit states. According to the previous discussion, we can divide this problem into two cases:

(i) When the four states form a two dimensional convex polytope, in this case according to Caratheodory theorem the vector \( \gamma \) can be written as a convex combination of at most three points, therefore the optimal measurement has at most three nonzero elements. In this case, we only need to solve the problem either by using the three steps instruction in the Sec. C.1 or Eq. (29) is maximum for states \( p_1 \) and \( p_2 \). In this case Eq. (40) gives
\[
\frac{1}{2} \sqrt{\frac{2\delta^2}{d_{12}} + 2 - 3p)^2 + 3(\delta + \frac{2p\delta}{d_{12}})^2} \leq \frac{\tilde{d}_{12}}{2} - 1 + 3p \quad (51)
\]
where \( \tilde{d}_{12} = \sqrt{3p^2 + \delta^2} \). Solving Eq. (51) for \( \delta \), gives us four roots, which the only valid values for \( \frac{1}{3} \leq p \leq \frac{1}{2} \) and \( 0 \leq \delta \) are
\[
\delta \leq \left( 2 - 6p + 5p^2 - 2(1-2p)\sqrt{4p^2-2p+1} \right)^\frac{1}{2}. \quad (52)
\]
In this region of two-state case the guessing probability is
\[
p_{\text{guess}} = \frac{1}{2} \sqrt{3p^2 + \delta^2} + p. \quad (53)
\]

(ii)When four states form a three dimensional polytope, i.e. a tetrahedron. Based on the type of states a four element optimal POVM may exist. If so, to find the guessing probability and optimal measurement we use the properties of tetrahedrons. Each tetrahedron is composed of four triangle faces, six edges, and four vertices. Based on the location of \( \gamma \), the number of detectable states in an optimal way can be specified. If \( \gamma \) be on one vertex then an optimal answer is a no measurement strategy and the other three states are unguessable. If it lies on one edge then an optimal measurement will be a POVM with two non-zero elements, i.e. two guessable states. Furthermore, lying \( \gamma \) on one of the faces means that three states can be detected through an optimal measurement and one state will remain unguessable. Finally, if \( \gamma \) be an interior point of tetrahedron then all four states are guessable, i.e. six hyperbolas meet at a single point. To find the guessing probability in this situation, let us show the vertex \( \gamma \) as an interior point in the tetrahedron \( \tilde{v}_1 \tilde{v}_2 \tilde{v}_3 \tilde{v}_4 \), therefore the guessing probability can be written as
we show how one can find the vector \( \gamma \). To find the vector \( \gamma \), we consider the problem of finding this angle, together with the mathematical background we used to find Eq. 564 for four qubit states problem.

To find the optimal POVM in this case, we first must find the vector \( \gamma \), and then use Eqs. 13 and 15 to find its elements. To find the vector \( \gamma \), first using the vectors \( \vec{v}_1 \), \( \vec{v}_2 \), \( \vec{v}_3 \) and \( \vec{v}_4 \), we construct an orthonormal \( \{ \vec{k}_1, \vec{k}_2, \vec{k}_3 \} \) basis using Gram-Schmidt process

\[
\vec{k}_1 = \frac{\vec{f}_1}{|\vec{f}_1|}, \quad \vec{f}_1 = \frac{\vec{v}_2 - \vec{v}_1}{d_{12}},
\]

\[
\vec{k}_2 = \frac{\vec{f}_2}{|\vec{f}_2|}, \quad \vec{f}_2 = (\vec{v}_1 - \vec{v}_2) - (\vec{v}_1 - \vec{v}_4) \cdot \vec{k}_1 \vec{k}_1
\]

\[
\vec{k}_3 = \frac{\vec{f}_3}{|\vec{f}_3|}, \quad \vec{f}_3 = (\vec{v}_1 - \vec{v}_4) - (\vec{v}_1 - \vec{v}_4) \cdot \vec{k}_1 \vec{k}_1 - (\vec{v}_1 - \vec{v}_4) \cdot \vec{k}_2 \vec{k}_2
\]

Then

\[
\gamma = (\gamma \vec{k}_1) \vec{k}_1 + (\gamma \vec{k}_2) \vec{k}_2 + (\gamma \vec{k}_3) \vec{k}_3,
\]

and all we need is to find coefficients \( \gamma \). For \( \gamma \vec{k}_1 \)

\[
\gamma \vec{k}_1 = \gamma, \quad \frac{\vec{f}_1}{|\vec{f}_1|} = \gamma, \quad \vec{f}_1 = \gamma (\vec{v}_2 - \vec{v}_1)
\]

\[
= \frac{1}{|\vec{f}_1|} \gamma (\vec{v}_2 - \vec{v}_1)
\]

\[
= \frac{1}{|\vec{f}_1|} (\gamma - \vec{v}_1)(\vec{v}_1 - \vec{v}_1) + \vec{v}_1(\vec{v}_2 - \vec{v}_1),
\]

Since \( |\gamma - \vec{v}_1| \) is known 560 then

\[
(\gamma - \vec{v}_1)(\vec{v}_2 - \vec{v}_1) = |\gamma - \vec{v}_1| \hat{d}_{12} \cos(\delta_{\vec{v}_1 \vec{v}_2}),
\]

where \( \delta_{\vec{v}_1 \vec{v}_2} \) is the angle between \( \gamma - \vec{v}_1 \) and \( \vec{v}_2 - \vec{v}_1 \) in triangle \( \gamma \vec{v}_1 \vec{v}_2 \) that can be obtained easily by using the rules of sines in triangle. Then we will have \( \gamma \vec{k}_1 \). The same procedure can be used for finding \( \gamma \vec{k}_2 \) and \( \gamma \vec{k}_3 \).

Here as an example let us consider the problem of four symmetric pure qubit states with equal apriori probabilities which form a polytope having a volume in Bloch sphere i.e. a regular tetrahedron. In this case, the states expressed symmetrically as four points on the Bloch sphere with the related \( \vec{v}_i \) as

\[
\vec{v}_1 = \frac{1}{4} (\sqrt{\frac{8}{9}}, 0, -\frac{1}{3}),
\]

\[
\vec{v}_2 = \frac{1}{4} (-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\frac{1}{3}),
\]

\[
\vec{v}_3 = \frac{1}{4} (-\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, -\frac{1}{3}),
\]

\[
\vec{v}_4 = \frac{1}{4} (0, 0, 1),
\]

The states are equidistant in this example \( \hat{d}_{ij} = \frac{1}{\sqrt{6}} \) then

\[
l_0 = \hat{d}_{12}^4, \quad l_1 = 0, \quad D_{123} = -3 \hat{d}_{12}^4.
\]

After some simple calculation using the method presented in Appendix B angle \( \theta_{12} \) can be obtained as \( \theta = \arccos(\frac{1}{\sqrt{3}}) \). Putting these values in equation 560 we end up with \( p_{\text{guess}} = \frac{1}{2} \) which is consistent with our results for equal apriori probabilities (Corollary 7) as well as the result obtained in 32.
D. Four Symmetric Pure States Case: decomposable and nondecomposable answers

As our last example, to illustrate decomposable sets, let us consider a symmetric set of four pure states with equal priori probabilities, \( p_i = 1/4 \), as

\[
\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta & \sin \theta \\ \sin \theta & 1 - \cos \theta \end{pmatrix},
\]

\[
\rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_4 = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta & -\sin \theta \\ -\sin \theta & 1 - \cos \theta \end{pmatrix},
\]

where \( 0 \leq \theta < \frac{\pi}{2} \) and the corresponding \( v_i \)'s are

\[
v_1 = +\hat{z}, \quad v_2 = \sin \theta \hat{x} + \cos \theta \hat{z},
\]

\[
v_3 = -\hat{z}, \quad v_4 = -\sin \theta \hat{x} + \cos \theta \hat{z}.
\]

Figure 7 shows the place of these vectors on the circum-

sphere (which is the Bloch sphere). Based on the dis-
cussion for the equal priori probabilities, in this case \( O \) is at

Origin.

To obtain the optimal POVM, we have to find the values

of four unknowns \( \alpha_i \), but from Eq. (35) we have only

three equations. So, we can write \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) in terms

of \( \alpha_4 \) as

\[
\alpha_{1,3} = 1 - (1 \pm \cos \theta) \alpha_4, \quad \alpha_2 = \alpha_4.
\]

Because all the coefficients \( \alpha_i \)'s must be limited between

zero and one \((\alpha_3 \neq 0)\), we find the following bound for

\( \alpha_4 \)

\[
0 \leq \alpha_4 \leq \frac{1}{1 + \cos \theta}.
\]

Obiously, each value of \( \alpha_4 \) in Eq. (66) gives a set \{\( \alpha_i \)\}

that leads to a different answer of optimal measurements

\{\( \pi_i \)\}. As \( \alpha_4 \) has a continuous value, there are infinite

number of answers! But this is not surprising because

we already know that this problem is composed of two

nondecomposable POVM answers:

\[
M'_1 = \{ \pi_1 = \rho_1, \pi_2 = 0, \pi_3 = \rho_3, \pi_4 = 0 \}
\]

\[
M'_2 = \{ \pi_1 = 0, \pi_2 = \frac{1}{1 + \cos \theta} \rho_2, \pi_3 = \frac{2 \cos \theta}{1 + \cos \theta} \rho_3, \pi_4 = \frac{1}{1 + \cos \theta} \rho_4 \}
\]

One can see it clearly from the geometry of the problem

(Fig. 7) that by either omitting the state \( \rho_1 \) or states \( \rho_2 \)

and \( \rho_4 \), the convex polytope of the remaining states still

contains the origin, \( O \). (If either \( \alpha_1 = 0 \) or \( \alpha_2 = \alpha_4 = 0 \),

there are still possible values for set \{\( \alpha_i \)\} to fulfill

Eq. (13).

So, we are able to get a general answer of the problem

from convex combination of these two sets \( \beta_1 M'_1 + \beta_2 M'_2 \),

where \( 0 \leq \beta_1, \beta_2 \leq 1 \) and \( \beta_1 + \beta_2 = 1 \). This general

answer is equivalent to the result of Eqs. (55) and (60).

IV. CONCLUSION

In this paper, we have revisited the problem of minimum-error discrimination for qubit states. For this
aim, we employed the necessary and sufficient Helstrom
condition in a constructive way to obtain the discrimi-
nation parameters for a typical problem of ME qubit
state discrimination. Our tools in this way are the rep-

eresentation of qubit states in terms of the Bloch vectors.

The method is applicable for both equal priori probabili-
ties and the case with arbitrary priori probabilities. For

equal priori probabilities, we show that the maximization

of \( P_{corr} \) can be done by maximizing the number of states that satisfy Eq. (17). It means that, we have to choose

a sphere with maximum number of states on it. For the

case of arbitrary priori probabilities each two qubit states

construct a hyperbola and the desired \( \gamma \) will lie on one of

its parts which is next to the more probable state. Using

these tools, we introduce an instruction to find the La-

grange operator \( \Gamma \). Then, with this Lagrange operator,

we can find all optimal POVM measurements.

we also discuss some properties of the POVM answers

involving the geometric of the polytope of qubit states

inside the Bloch sphere, and introduce some classes of

answers like the classes of unchanged guessing probability

and unchanged measurement operators.

We show that for an optimal strategy, there might be

some states that are not detectable i.e. their related

POVM elements are zero. So, in the problem of ME
discrimination of \( N \) qubits \( \{ \rho_i \}_{i=1}^{N} \) some states might be

undetectable. We indicate them as unguessable, nearly

guessable, and guessable states, assuming that there are

\( M \) number of guessable states in a typical optimal

problem \((1 \leq M \leq N)\).

We show that every POVM set \( M \), can be divided

into a limit number of nondecomposable POVM subsets
Finding all of these subsets is an alternative way of constructing a general ME answer of the given problem. They might be also practical in a case that detecting some states $\rho_j$’s are not interested, so we might be able to use those nondecomposable subsets that do not include $\hat{\Phi}_j$’s. It is also more reasonable when preparing measurement operators are expensive.

To illustrate the proposed instruction, we consider some examples. For the case of three qubit states, by applying some rotation and translation we show that the problem can be reduced to the problem of three qubit states in $x-z$ plane and therefore we obtain a full analysis of the problem using our approach. Then, as a specific case, the problem of trine states with arbitrary priori probabilities is considered. We show that the results are in complete agreement with previous findings. Moreover, We also solve the case of four qubit states for the first time, using the geometry of tetrahedron and the intersection of $\binom{4}{2} = 6$ hyperbolas deriving from each two states. As a final example, we consider a four symmetric pure states case to illustrate the definition of decomposable measurements which were discussed before.

ACKNOWLEDGMENTS

We would like to thank S. J. Akhtarshenas for many fruitful discussions. This work was supported by projects APVV-18-0518 (OPTIQUTE), VEGA 2/0161/19 (HOQIP).

Appendix A: The Rotation Matrix for three qubit states Discrimination

Any qubit state $\rho_i$ can be identified with a point inside the Bloch sphere using its Bloch vector $\mathbf{v}_i$. The same is true for multiplication of the state by its priori probability i.e. $\hat{\rho}_i = p_i \rho_i$ and $\hat{\mathbf{v}}_i = p_i \mathbf{v}_i$. Three states together with their priori probabilities represent three points inside the Bloch sphere. From geometry we know that three non-collinear points determine a plane and each plane can be described by its normal vector which is a vector orthogonal to the plane (i.e. orthogonal to every directional vector of the plane). Having the normal vector in hand, in the next step we can find the rotation matrix which rotates this vector to align it in the $y$-direction. This rotation matrix then rotates each $\hat{\mathbf{v}}_i$ in such a way that the $y$-component of all $\mathbf{v}_i$ become equal. With a translation along $y$-axis, one can eliminate the $y$-components of these rotated vectors. Finally, with an additional rotation in $x-z$ plane, we can rotate $\mathbf{v}_i$ in a way that the state with largest priori probability be aligned in the $z$-direction. Since these rotations and the translation does not affect the relative distances and angles between states, the guessing probability to this new set of states is equal to the original one. The POVM’s can be easily related as well, with a rotation, as was explained in the Sec. [III.B]

To obtain corresponding rotation matrix consider three points $P_1, P_2$ and $P_3$ in the Bloch sphere

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, P_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, P_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}. \tag{A1}$$

The plane containing these three points is defined by

$$ax + by + cz + d = 0, \tag{A2}$$

where the coefficients $a, b$ and $c$ determine the components of normal vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{A3}$$

Since the normal vector $\mathbf{n}$ is the unit vector orthogonal to every direction vector of the plane, it can be obtained by the following equation

$$\hat{\mathbf{n}} = \frac{(P_2 - P_1) \times (P_3 - P_2)}{|(P_2 - P_1) \times (P_3 - P_2)|}. \tag{A4}$$

Our purpose is to rotate this plane such that $\hat{\mathbf{n}}$ aligns to vector $\hat{\mathbf{j}}$

$$\hat{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{A5}$$

The desired rotation can be obtained by the following instruction:

Firstly, let us define the vector $\mathbf{V}$ and matrix $A$ as

$$\mathbf{V} = \hat{\mathbf{n}} \times \hat{\mathbf{j}}, \tag{A6}$$

and

$$A = \begin{pmatrix} 0 & -V_z & 0 \\ V_z & 0 & -V_x \\ 0 & V_x & 0 \end{pmatrix}. \tag{A7}$$

Then the rotation matrix can be written as

$$R = 1 + A + \frac{1-c}{s^2} A^2, \tag{A8}$$

where the coefficients $c$ and $s$ are

$$s = ||\mathbf{V}|| = \sqrt{V_x^2 + V_y^2 + V_z^2},$$

$$c = \hat{\mathbf{n}} \cdot \hat{\mathbf{j}}.$$

The more straightforward way is using the following formulation $R$ for our specific problem

$$R = \begin{pmatrix} 1 - \frac{n_x^2}{1+n_y} & -n_x & -\frac{n_x n_y}{1+n_y} \\ n_x & n_y & n_z \\ -\frac{n_x n_y}{1+n_y} & -n_z & 1 - \frac{n_z^2}{1+n_x} \end{pmatrix}. \tag{A9}$$

where $\mathbf{n} = (n_x, n_y, n_z)$ is defined in Eq. [A4].
Appendix B: Tetrahedron geometry for the problem of Four qubit states discrimination

In this section, we briefly review the material needed for solving the problem of four qubit states discrimination in Sec. [III C]. As we mentioned before, Four qubits that are not on a plane form a tetrahedron in Bloch sphere. We know that a tetrahedron is composed of four triangular faces, six straight edges, and four vertices. There are three types of angles for a tetrahedron: 12 face angles which are regular angles of each triangles; 6 dihedral angles associated to each edge of the tetrahedron which are the angles between each two faces connected together with edges; and 4 solid angles for each vertex of tetrahedron. For the problem of four qubit states, to make these quantities more clear, let us define the following quantities for the lengths of edges and dihedral angles: $d_{ij}$: the length of the edge connecting two qubits $\vec{v}_i$ and $\vec{v}_j$ in the Bloch sphere. $\theta_{ij}$: the dihedral angle between two faces adjoint to the edge $ij$. Using these quantities, there is a relation for a dihedral angle $\theta_{ij}$

$$\cos(\theta_{ij}) = \frac{D_{ij}}{\sqrt{D_{ijk}D_{ijl}}} \tag{B1}$$

where

$$D_{ij} = -\tilde{d}_{ij} + (\tilde{d}_{ik} + \tilde{d}_{jl} + \tilde{d}_{jk} - 2\tilde{d}_{kl})\tilde{d}_{ij} + (\tilde{d}_{ik} - \tilde{d}_{jk})^2(\tilde{d}_{jl} - \tilde{d}_{il}),$$

$$D_{ijk} = -(\tilde{d}_{ij} + \tilde{d}_{ik} + \tilde{d}_{jk})(\tilde{d}_{ij} + \tilde{d}_{ik} - \tilde{d}_{jk})(\tilde{d}_{jk} - \tilde{d}_{ij}). \tag{B2}$$

To find the guessing probability for qubit states in a case that all states are detectable in an optimal strategy, $\gamma$ has to be an interior point in the tetrahedron. If we connect it with straight lines to each vertex we will end up with 5 tetrahedrons. Considering two tetrahedrons $T_1 = \gamma \vec{v}_1 \vec{v}_2 \vec{v}_3$ and $T_2 = \gamma \vec{v}_1 \vec{v}_2 \vec{v}_4$, we have dihedral angles $\theta^T_{12}$ and $\theta^T_{22}$ for the common edge $\vec{v}_1 \vec{v}_2$ in two tetrahedrons. Therefore, there is a simple relation between these angles and the dihedral angle for the edge $\vec{v}_1 \vec{v}_2$ of the main tetrahedron $\vec{v}_1 \vec{v}_2 \vec{v}_3 \vec{v}_4$ which we show it by $\alpha$.

$$\alpha = \theta^T_{12} + \theta^T_{22}. \tag{B3}$$

Using (B1) for tetrahedron $T_1$ we have

$$\cos^2(\theta^T_{12}) = \frac{D_{12}^T}{\sqrt{D_{123}^T D_{124}^T}},$$

where $D_{12}^T = l_0 + l_1 |\gamma - \vec{v}_1|$. And $l_0$ and $l_1$ were defined earlier. After some algebraic calculations we end up with the following equation

\[ |\gamma - \vec{v}_1| = \]

\[-2D_{123}p_{12}(d_{12}^2 - p_{12}^2)\cos^2(\theta^T_{12}) - l_0 l_1 - \cos(\theta^T_{12})\sqrt{4\cos^2(\theta^T_{12})D_{123}(d_{12}^2 - p_{12}^2)d_{12}^2 + l_0^2(d_{12}^2 - p_{12}^2) + 4p_{12}l_1 l_1 - 4l_0^2}D_{123}(d_{12}^2 - p_{12}^2) \]

\[ = \]

\[4D_{123}(d_{12}^2 - p_{12}^2)\cos^2(\theta^T_{12}) + l_0^2. \]

Similarly, for tetrahedron $T_2$ one can write

\[ |\gamma - \vec{v}_1| = \]

\[-2D_{124}p_{12}(d_{12}^2 - p_{12}^2)\cos^2(\theta^T_{12}) - l'_0 l'_1 - \cos(\theta^T_{12})\sqrt{4\cos^2(\theta^T_{12})D_{124}(d_{12}^2 - p_{12}^2)d_{12}^2 + l'_0^2(d_{12}^2 - p_{12}^2) + 4p_{12}l'_1 l'_1 - 4l'_0^2}D_{124}(d_{12}^2 - p_{12}^2) \]

\[ = \]

\[4D_{124}(d_{12}^2 - p_{12}^2)\cos^2(\theta^T_{12}) + l'_0^2, \]

where $l'_0$ and $l'_1$ can be obtained from $l_0$ and $l_1$ by replacing 3 $\leftrightarrow$ 4. Therefore, by equating these two equations and using the fact that $\theta^T_{12} = \alpha - \theta^T_{12}$, the angle $\theta^T_{12}$ can be obtained. For example for the problem of four symmetric qubit states we [III C] we can easily obtain that $\theta^T_{12} = \theta^T_{12} = \frac{\alpha}{2}$.

\[1\] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North-Holland, 1982).

\[2\] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic, 1976).
[3] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden.:Quantum cryptography. Rev. Mod. Phys. 74, 145–195 (2002).
[4] J. Bae and L. C. Kwek.: Quantum state discrimination and its applications. J. Phys. A: Math. Theor. 48 083001 (2015).
[5] J. A. Bergou, U. Herzog, and M. Hillery.: Discrimination of quantum states. Lect. Notes Phys. 649, 417–465 (2004).
[6] I. D.Ivanovic.:How to differentiate between non-orthogonal states. Phys. Lett. A 123, 257–259 (1987).
[7] A. Peres.:How to differentiate between non-orthogonal states. Phys. Lett. A 128, 19–19 (1988).
[8] D. Dieks.:Overlap and distinguishability of quantum states. Phys. Lett. A 126, 303–306 (1988).
[9] G. Jaeger and A. Shimony.:Optimal distinction between two non-orthogonal quantum states. Phys. Lett. A 197, 83–87 (1995).
[10] A. Chefles.:Unambiguous discrimination between linearly independent quantum states. Phys. Lett. A 239, 339–347 (1998).
[11] A. Chefles and S. M. Barnett.:Optimum unambiguous discrimination between linearly independent symmetric states. Phys. Lett. A 250, 223–229 (1998).
[12] Y. C. Eldar, M. Stojnic, and B. Hassibi.:Optimal quantum detectors for unambiguous detection of mixed states. Phys. Rev. A 69, 062318 (2004).
[13] S. Croke, E. Anderson, S. M. Barnett, C. R. Gilson and J. Jeffers.: Maximum confidence quantum measurements. Phys. Rev. Lett 96, 070401, 2006.
[14] A. Hayashi, T. Hashimoto and M. Horibe, Phys. Rev. A 78, 012333 (2008).
[15] H. Sugimoto, T. Hashimoto, M. Horibe and A. Hayashi, Phys. Rev. A 80, 052322 (2009).
[16] S. A. Ghoreishi, S. J. Akhtarshenas and M. Sarbishaei.:Parametrization of quantum states based on the quantum state discrimination problem. Quantum Inf. Process. 18 , 150 (2019).
[17] S. A. Ghoreishi and M. Ziman.:Minimum-error discrimination of thermal states. Phys. Rev. A 104, 062402 (2021).
[18] D. Ha, J. S. Kim, and Y. Kwon.: Qubit state discrimination using post-measurement information. Quantum Inf. Process. 21 , 2 (2022).
[19] E. R. Loubenets.:General lower and upper bounds under minimum-error quantum state discrimination.” Phy. Rev. A 105, 032410 (2022).
[20] S. M. Barnett and S. Croke.:Quantum state discrimination. Advances in Optics and Photonics, 1, 238-278 (2009).
[21] S. M. Barnett and S. Croke.:On the conditions for discrimination between quantum states with minimum error. J. Phys. A: Math. Theor. 42, 062001 (2009).
[22] S. M. Barnett. Quantum Information (Oxford University Press, 2009).
[23] M. Ban, K. Kurokawa, R. Momose, and O. Hirota.:Optimum measurements for discrimination among symmetric quantum states and parameter estimation. Int. J. Theor. Phys. 36, 1260 (1997).
[24] S. M. Barnett.:Minimum-error discrimination between multiply symmetric states. Phys. Rev. A 64, 030303 (2001).
[25] C. L. Chou.: Minimum-error discrimination between symmetric mixed quantum states. Phys. Rev. A 68, 042305 (2003).
[26] E. Andersson, S. M. Barnett, C. R. Gilson, and K. Hunter.:Minimum-error discrimination between three mirror-symmetric states. Phys. Rev. A 65, 052308 (2002).
[27] C. L. Chou.:Minimum-error discrimination among mirror-symmetric mixed quantum states. Phys. Rev. A 70, 062316 (2004).
[28] C. Mochon.: Family of generalized pretty good measurements and the minimal-error pure-state discrimination problems for which they are optimal. Phys. Rev. A 73, 032328 (2006).
[29] K. Hunter.: Results in Optimal Discrimination. AIP Conference Proceedings 734, 83 (2004);
[30] B. F. Samsonov.:Minimum error discrimination problem for pure qubit states. Phys. Rev. A 80, 052305 (2009).
[31] M. E. Deconinck, B. Terhal.:Qubit state discrimination. Phys. Rev. A 81 , 062304 (2010).
[32] J. Bae.:Structure of minimum-error quantum state discrimination. New J. Phys. 15, 073037 (2013).
[33] D. Ha and Y. Kwon.:Complete analysis for three-qubit mixed-state discrimination. Phys. Rev. A 87, 062302 (2013).
[34] G. Weir, S. M. Barnett, and S. Croke.:Optimal discrimination of single-qubit mixed states. Phys. Rev. A 96, 022312 (2017).
[35] G. Weir, C. Hughes, S. M. Barnett, and S. Croke.:Optimal measurement strategies for the trine states with arbitrary priori probabilities Quantum Sci. Technol. 3 035003 (2018).
[36] A. Peres, Quantum Theory: Concepts and Methods (Kluwer, 1993).
[37] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2010).
[38] T. Heinosaari and M. Ziman The mathematical language of quantum theory: from uncertainty to entanglement (Cambridge: Cambridge University Press, 2011).
[39] I.E. Leonard, J.E. Lewis Geometry of Convex Sets (Wiley, Hoboken, 2016).
[40] J. Bae.:Minimum-error discrimination of qubit states: Methods, solutions, and properties. Phys. Rev. A 87 , 012334 (2013).
[41] K. Hunter.:Measurement does not always aid state discrimination. Phys. Rev. A 68 , 012306 (2003).
[42] K. Wirth and A. Dreiding.:Relations between edge lengths, dihedral and solid angles in tetrahedra. J. Math. Chem. 52 1624 –1638 (2014).