Strong gravitational lensing in a squashed Kaluza-Klein Gödel black hole

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Abstract

We investigate the strong gravitational lensing in a squashed Kaluza-Klein black hole immersed in the Gödel universe with global rotation. Our result show that the strong gravitational lensing in the squashed Kaluza-Klein Gödel black hole spacetime has some distinct behaviors from that in the Kerr case. In the squashed Kaluza-Klein Gödel black hole spacetime, the photon sphere radius, the minimum impact parameter, the coefficient $\bar{a}$, $\bar{b}$ and the deflection angle $\alpha(\theta)$ in the $\phi$ direction are independent of whether the photon goes with or against the global rotation of the Gödel Universe. While in the Kerr black hole, the values of these quantities for the prograde photons are different from those for the retrograde photons. Moreover, the coefficient of $\bar{b}$ increases with $j$ in the squashed Kaluza-Klein Gödel black hole, but decreases with $a$ in the Kerr case. We also probe the influence of the squashed effect on the strong gravitational lensing in this black hole and find that in the extremely squashed case $\rho_0 = 0$, the coefficient $\bar{a}$ is a constant 1 and is independent of the global rotation of the Gödel Universe. Furthermore, we assume that the gravitational field of the supermassive central object of the Galaxy can be described by this metric and estimate the numerical values of the coefficients and the main observables in the strong gravitational lensing.

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I. INTRODUCTION

It is well known that photons would be deviated from their straight paths as they pass close to a compact and massive body. Gravitational lensing is such a phenomena resulting from the deflection of light rays in a gravitational field, which plays an essential role in the astrophysics because it can help us extract the information about the distant stars which are otherwise too dim to be observed. The strong gravitational lensing is caused by a compact object with a photon sphere (such as black hole)\cite{1}. When the photons pass close to the photon sphere, the deflection angles of the light rays diverge and an observer can detect two infinite sets of faint relativistic images on each side of the black hole \cite{2–4}. It is shown that the relativistic images can provide us not only some important signatures about black holes in the Universe, but also profound verification of alternative theories of gravity in their strong field regime. Thus, the strong gravitational lensing by black holes has been studied extensively in various theories of gravity \cite{5–11, 14–17}. Most studies of the strong gravitational lensing are focused on black holes immersed in the rather idealized isotropic homogeneous Universe.

It is more reasonable to consider the Universe background as homogeneous but with global rotation since the rotation is a universal phenomenon in our Universe. An original exact solution for the rotating Universe was obtain by Gödel through solving Einstein equations with pressureless matter and negative cosmological constant \cite{18}, which is a four-dimensional spacetime and exhibits close timelike curves through every point. The generalizations of Gödel Universe in the minimal five-dimensional gauged supergravity has been found in \cite{19, 20}. Like in the original four-dimensional Gödel solution, there also exist the close timelike curves in these five-dimensional supersymmetric Gödel spacetimes. Exact solutions describing various black holes immersed in the five-dimensional rotating Gödel Universe were found in \cite{21–25}, which obey to the usual black hole thermodynamics. The studies indicate that the quasinormal modes \cite{26} and Hawking radiation \cite{27} of these black holes are considerably affected by the rotating cosmological background. Using the squashing transformation, the new squashed Kaluza-Klein black hole solutions in the rotating universe have been obtained in \cite{28, 29}. In \cite{30}, the wave dynamics of the charged squashed Kaluza-Klein Gödel black hole has been investigated, which shows that the squashed effect and the cosmological rotational effect changes behavior of quasinormal modes. The main purpose of this paper is to study the strong gravitational lensing in the neutral squashed Kaluza-Klein Gödel black hole and to see what effects of the cosmological rotation and compactness of the extra dimension on the coefficients and the observables of gravitational lensing in the strong field limit.
The plan of our paper is organized as follows. In Sec.II we introduce briefly the neutral squashed Kaluza-Klein black hole immersed in the five-dimensional rotating Gödel Universe. In Sec.III we adopt to Bozza’s method [10–12] and obtain the deflection angles for light rays propagating in the squashed Kaluza-Klein Gödel black hole. In Sec.IV we suppose that the gravitational field of the supermassive black hole at the center of our Galaxy can be described by this metric and then obtain the numerical results for the main observables in the strong gravitational lensing. At last, we present a summary.

II. THE SQUASHED KALUZA-KLEIN GÖDEL BLACK HOLE SPACETIME

The static neutral squashed Kaluza-Klein Gödel black hole is described by the metric [28]

\[ ds^2 = -f(r)dt^2 + \frac{k(r)^2}{V(r)}dr^2 - 2g(r)\sigma_3 dt + h(r)\sigma_2^2 + \frac{r^2}{4}[k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2], \] (1)

with

\[ \sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \]
\[ \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \]
\[ \sigma_3 = d\psi + \cos \theta d\phi. \] (2)

Here coordinates \( \theta \in [0, \pi) \), \( \phi \in [0, 2\pi) \) and \( \psi \in [0, 4\pi) \), and \( r \) runs in the range \( (0, r_{\infty}) \). The metric functions are

\[ f(r) = 1 - \frac{2M}{r^2}, \quad g(r) = jr^2, \quad h(r) = -j^2r^2(r^2 + 2M), \]
\[ V(r) = 1 - \frac{2M}{r^2} + \frac{16j^2M^2}{r^2}, \quad k(r) = \frac{V(r_{\infty})r_{\infty}^4}{(r_{\infty}^2 - \sqrt{r_{\infty}^2 - r^2})^2}. \] (3)

The parameter \( M \) is the mass of the black hole, and \( j \) denotes the scale of the Gödel background. The Killing horizon of the black hole is given by the equation \( V(r) = 0 \), so the radius of the black hole horizon is \( r_H^2 = 2M - 16j^2M^2 \). When \( j = 0 \), Eq. (3) reduces to the five-dimensional Kaluza-Klein black hole with squashed horizon [31]. As \( r_{\infty} \to \infty \), one has \( k(r) \to 1 \), which means the squashing effect disappears and the five-dimensional neutral black hole is recovered in the Gödel universe [21].

Using coordinate transformation \( \rho = \rho_0 \frac{r^2}{r_{\infty}^2 - r^2} \) and \( \tau = \sqrt{\frac{\rho_0}{\rho_0 + \rho_M}}t \), we can find that the metric (1) can be rewritten as

\[ ds^2 = -F(\rho)d\tau^2 + \frac{K(\rho)}{G(\rho)}d\rho^2 + \rho^2K(\rho)(d\theta^2 + \sin^2 \theta d\phi^2) - 2H(\rho)\sigma_3 d\tau + D(\rho)\sigma_3^2, \] (4)
where

\begin{align*}
F(\rho) &= 1 - \frac{\rho_M}{\rho}, \quad G(\rho) = 1 - \frac{\rho_H}{\rho}, \quad K(\rho) = 1 + \frac{\rho_0}{\rho}, \\
H(\rho) &= \frac{j r_\infty^2}{K \rho_0} \sqrt{\rho_0 (\rho_0 + \rho_M)} , \quad D(\rho) = \frac{r_\infty^2}{4K} - \frac{j^2 r_\infty^4}{(\rho + \rho_0)^2 (\rho_0 + \rho_M)} \left[ \rho (2\rho_M + \rho_0) + \rho_0 \rho_M \right],
\end{align*}

with

\begin{align*}
\rho_M = \rho_0 \frac{2M}{r_\infty^2 - 2M}, \quad \rho_H = \rho_0 \frac{r_H^2}{r_\infty^2 - r_H^2}.
\end{align*}

Here \( \rho_H \) is the radius of the black hole event horizon. The parameter \( \rho_0 \) is a scale of transition from five-dimensional spacetime to an effective four-dimensional one. The positive parameters \( r_\infty \) and \( \rho_0 \) and \( \rho_H \) are related by \( r_\infty^2 = 4\rho_0 (\rho_0 + \rho_H) \). The Komar mass of the black hole is given by \( M = \pi r_\infty \rho_M / G_5 \) \( ^{32,33} \), where \( G_5 \) is the five-dimensional gravitational constant. Since in the squashed Kaluza-Klein black hole spacetime the relationship between \( G_5 \) and \( G_4 \) (the four-dimensional gravitational constant) can be expressed as \( G_5 = 2\pi r_\infty G_4 \) \( ^{32,33} \), one can obtain that the quantity \( \rho_M = 2G_4 M \). The relation between \( \rho_H \) and \( \rho_M \) is

\begin{align*}
\rho_H = \frac{2(\rho_0 + \rho_M)}{\sqrt{1 + 64j^2 \rho_M^2} + 1} - \rho_0.
\end{align*}

It is clear that the radius of the black hole event horizon \( \rho_H \) decreases with increase of the Gödel parameter \( j \). As \( j \) vanishes, one can find that \( \rho_H \) is coincide with \( \rho_M \).

### III. Deflection Angle in the Squashed Kaluza-Klein Gödel Black Hole Spacetime

In this section, we will study deflection angles of the light rays when they pass close to a squashed Kaluza-Klein Gödel black hole, and then probe the effects of the Gödel parameter \( j \) and the scale of extra dimension \( \rho_0 \) on the deflection angle and the coefficients in the strong field limit. Here we consider only the case the light ray is limited in the equatorial plane \( \theta = \frac{\pi}{2} \). With this condition, we get the reduced metric for the squashed Kaluza-Klein Gödel black hole

\begin{align*}
ds^2 &= -A(\rho) dt^2 + B(\rho) d\rho^2 + C(\rho) d\phi^2 + D(\rho) d\psi^2 - 2H(\rho) dt d\psi,
\end{align*}

where

\begin{align*}
A(\rho) &= F(\rho), \quad B(\rho) = \frac{K(\rho)}{G(\rho)}, \quad C(\rho) = \rho^2 K(\rho),
\end{align*}

and

\begin{align*}
F(\rho) &= 1 - \frac{\rho_M}{\rho}, \quad G(\rho) = 1 - \frac{\rho_H}{\rho}, \quad K(\rho) = 1 + \frac{\rho_0}{\rho}, \\
H(\rho) &= \frac{j r_\infty^2}{K \rho_0} \sqrt{\rho_0 (\rho_0 + \rho_M)} , \quad D(\rho) = \frac{r_\infty^2}{4K} - \frac{j^2 r_\infty^4}{(\rho + \rho_0)^2 (\rho_0 + \rho_M)} \left[ \rho (2\rho_M + \rho_0) + \rho_0 \rho_M \right],
\end{align*}

with

\begin{align*}
\rho_M = \rho_0 \frac{2M}{r_\infty^2 - 2M}, \quad \rho_H = \rho_0 \frac{r_H^2}{r_\infty^2 - r_H^2}.
\end{align*}
For the null geodesics, we can obtain three constants of motion

\[ E = - g_{\mu\mu} \ddot{x}^\mu = A(\rho) \dot{t} + H(\rho) \dot{\psi}, \]

\[ L_\phi = g_{3\mu} \ddot{x}^\mu = C(\rho) \dot{\phi}, \]

\[ L_\psi = g_{4\mu} \ddot{x}^\mu = D(\rho) \dot{\psi} - H(\rho) \dot{t}, \]  

(10)

where a dot represents a derivative with respect to affine parameter \( \lambda \) along the geodesics. With these three constants, one can find that the equations of motion of the photon can be simplified as

\[ \frac{dt}{d\lambda} = \frac{D(\rho) E - H(\rho) L_\psi}{H^2(\rho) + A(\rho) D(\rho)}, \]

\[ \frac{d\phi}{d\lambda} = \frac{L_\phi}{C(\rho)}, \]

\[ \frac{d\psi}{d\lambda} = \frac{H(\rho) E + A(\rho) L_\psi}{H^2(\rho) + A(\rho) D(\rho)}. \]

(11)

\[ \left( \frac{d\rho}{d\lambda} \right)^2 = \frac{1}{B(\rho)} \left[ \frac{D(\rho) E - 2H(\rho) E L_\psi - A(\rho) L_\psi^2}{H^2(\rho) + A(\rho) D(\rho)} - \frac{L_\psi^2}{C(\rho)} \right]. \]

(12)

From the \( \theta \)-component of the null geodesics in the equatorial plane, we can obtain

\[ \frac{d\phi}{d\lambda} \left[ D(\rho) \frac{d\psi}{d\lambda} - H(\rho) \frac{dt}{d\lambda} \right] = 0, \]

(13)

which implies that either \( \frac{d\phi}{d\lambda} = 0 \) or \( L_\psi = D(\rho) \frac{d\psi}{d\lambda} - H(\rho) \frac{dt}{d\lambda} = 0 \). As done in the usual squashed Kaluza-Klein black hole spacetime \[17\], here we set \( L_\psi = 0 \), which implies that the total angular momentum \( J \) of the photon is equal to the constant \( L_\phi \). In doing so, one can get the effective potential for the photon passing close to the black hole

\[ V(\rho) = \frac{1}{B(\rho)} \left[ \frac{D(\rho) E}{H^2(\rho) + A(\rho) D(\rho)} - \frac{L_\psi^2}{C(\rho)} \right]. \]

(14)

Making use of this effective potential, one can obtain that the impact parameter and the photon sphere equation are

\[ u = J = \sqrt[4]{\frac{C(\rho) D(\rho)}{H(\rho)^2 + A(\rho) D(\rho)}}, \]

(15)

and

\[ D(\rho) \left[ H(\rho)^2 + A(\rho) D(\rho) \right] C'(\rho) - C(\rho) \left[ D(\rho)^2 A'(\rho) + 2D(\rho) H(\rho) H'(\rho) - H(\rho)^2 D'(\rho) \right] = 0, \]

(16)

respectively. Here we set \( E = 1 \). The equations \[15\] and \[16\] are more complex than those in the usual spherical symmetric black hole spacetime. As the Gödel parameter \( j \to 0 \), we find that the function \( H(\rho) \to 0 \),
which yields that the impact parameter \cite{15} and the photon sphere equation \cite{16} reduce to those in the usual neutral squashed Kaluza-Klein black hole spacetime \cite{17}. The radius of the photon sphere is the largest real root of Eq. \cite{16}, which can be expressed as

\[ \rho_{ps} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \]  

with

\[ A = 2\rho_M^2 [1 - 32j^2\rho_0(\rho_0 + 2\rho_M) + \sqrt{1 + 64j^2\rho_M^2}], \]

\[ B = -(3\rho_0^3 + 9\rho_0^2\rho_M + 8\rho_0\rho_M^2 + 6\rho_M^3) - 32j^2\rho_0^2\rho_M^2(3\rho_0 + 7\rho_M) + \rho_0(3\rho_0^3 + 9\rho_0\rho_M + 10\rho_M^2)\sqrt{1 + 64j^2\rho_M^2}, \]

\[ C = -2\rho_0\rho_M[2\rho_M^2 + \rho_0^2 + 2\rho_0\rho_M + 32j^2\rho_0^2\rho_M - \rho_0(\rho_0 + 2\rho_M)\sqrt{1 + 64j^2\rho_M^2}]. \]  

\( (17) \)

\( (18) \)

Obviously, it depends on both the Gödel parameter \( j \) and the scale of transition \( \rho_0 \). As \( j \to 0 \), one can get \( \rho_{ps} = \frac{3\rho_M - \rho_0 + \sqrt{\rho_0^2 + 9\rho_0\rho_M + 9\rho_M^2}}{4} \), which is consistent with that in the usual squashed Kaluza-Klein black hole spacetime \cite{17}. As \( \rho_0 \) approaches zero, the radius of the photon sphere becomes \( \rho_{ps} = \frac{3\rho_M}{1 + \sqrt{1 + 64j^2\rho_M^2}} \), which decreases with the Gödel parameter \( j \). In Fig.\((1)\), we set \( \rho_M = 1 \) and plot the variety of the radius of the photon sphere \( \rho_{ps} \) with the parameters \( j \) and \( \rho_0 \). It shows that with the increase of \( \rho_0 \), \( \rho_{ps} \) increases for the smaller \( j \) and decreases for the larger \( j \). For fixed \( \rho_0 \), it is easy to obtain that \( \rho_{ps} \) decreases monotonically with the increase of the Gödel parameter \( j \). Moreover, it is well known that in the Kerr black hole, the photon sphere radiuses are different for the photons winding in the same direction (i.e., \( a > 0 \)) or in the converse direction (i.e., \( a < 0 \)) of the black hole rotation \cite{11,13}. However, one can obtain that in the squashed Kaluza-Klein Gödel black hole spacetime the photon sphere radius \( \rho_{ps} \) is independent of the sign of the Gödel parameter \( j \) since all of quantities \( A, B \) and \( C \) in Eqs.\textsuperscript{17} and \textsuperscript{18} are the function of \( j^2 \). Thus, the photon sphere radius in the black hole spacetime keeps the same whether the photon moves in the same or converse direction of the global rotation of the Gödel Universe. This means that the effects of the global rotation parameter \( j \) of the Gödel Universe background on the gravitational lensing is different from that of the rotation parameter \( a \) of the black hole itself.

In the squashed Kaluza-Klein Gödel black hole spacetime, the deflection angles of \( \phi \) and \( \psi \) for the photon coming from infinite are

\[ \alpha_\phi(\rho_s) = I_\phi(\rho_s) - \pi, \]  

\[ \alpha_\psi(\rho_s) = I_\psi(\rho_s) - \pi, \]

\( (19) \)

\( (20) \)
FIG. 1: Variety of the quantity $\rho_{ps}/\rho_M$ with $\rho_0/\rho_M$ and $j$ in the squashed Kaluza-Klein Gödel black hole spacetime.

respectively. Here $\rho_s$ is the closest approach distance, $I_\phi(\rho_s)$ and $I_\psi(\rho_s)$ are

\[
I_\phi(\rho_s) = 2 \int_{\rho_s}^{\infty} \frac{\sqrt{B(\rho)F(\rho)C(\rho_s)}}{C(\rho)} \frac{1}{\sqrt{\mathcal{F}(\rho) - \frac{F(\rho)C(\rho_s)}{C(\rho)}}} d\rho, \tag{21}
\]

\[
I_\psi(\rho_s) = 2 \int_{\rho_s}^{\infty} \frac{H(\rho)}{D(\rho)} \sqrt{\frac{B(\rho)F(\rho_s)}{F(\rho)}} \frac{1}{\sqrt{\mathcal{F}(\rho) - \frac{F(\rho)C(\rho_s)}{C(\rho)}}} d\rho, \tag{22}
\]

with

\[
\mathcal{F}(\rho) = \frac{H^2(\rho) + A(\rho)D(\rho)}{D(\rho)}. \tag{23}
\]

As in the usual black hole spacetime, both of the deflection angles increase when parameter $\rho_s$ decreases. If $\rho_s$ is equal to the radius of the photon sphere $\rho_{ps}$, one can find that both of the deflection angles diverge, which means that the photon is captured by the black hole in this case. Moreover, from Eq. (21) and (22), one can find that in the squashed Kaluza-Klein Gödel black hole spacetime both of the deflection angles depend on the parameters $j$ and $\rho_0$, which implies that we could in theory detect the global rotation of the universe and the extra dimension through the gravitational lens. It is interesting to note that the integral $I_\phi(\rho_s)$ is function of $j^2$, which yields that the deflection angle $\alpha_\phi(\rho_s)$ is independent of whether the photon goes with or against the global rotation of the Gödel Universe. However, from Eq. (22), we find that the integral $I_\psi(\rho_s)$ contains the factor $j$, and then the deflection angle $\alpha_\psi(\rho_s)$ for the photon traveling in the same direction as the global rotation of the Gödel Universe is different from that traveling in converse direction, which is similar to that of the deflection angle in the $\phi$ coordinate in the four-dimensional Kerr black hole spacetime [11, 12]. The main reason is that in the equatorial plane $\theta = \pi/2$ the global rotation of the spacetime is in the $\psi$ direction rather than in the $\phi$ direction. As $j$ vanishes, one can find that the deflection angle of $\psi$ tends to zero since
\( H(\rho) = 0 \), which reduces to that of in the usual squashed Kaluza-Klein black hole spacetime \[17\].

Here we will limit our attention to investigate the deflection angle in the \( \phi \) direction for the light rays passing close to the black hole in the equatorial plane since it can be observed by astronomical experiments. Moreover, it is very convenient for us to compare with the results obtained in the usual black hole spacetime.

Defining a variable \( z = 1 - \frac{\rho_s}{\rho} \), one can rewrite Eq. (21) as \[10, 11\]

\[
I_{\phi}(\rho_s) = \int_0^1 R(z, \rho_s) f(z, \rho_s) dz,
\]  

(24)

with

\[
R(z, \rho_s) = 2\frac{\rho_s^2}{\rho M^2} \sqrt{B(\rho) F(\rho) C(\rho_s)}
\]

\[
= 2\left\{ \frac{\rho_s^2 (\rho_s + \rho_0) (1 + \sqrt{1 + 64 j^2 \rho_s^2})}{2 \rho_M^2 [\rho_s + \rho_0 (1 - z)] - \rho_0 (\sqrt{1 + 64 j^2 \rho_M^2} - 1) [(1 - z) \rho_0 M + \rho_s (2 \rho_M + \rho_0)]} \right\}^{\frac{1}{2}},
\]

(25)

and

\[
f(z, \rho_s) = \frac{1}{\sqrt{F(\rho_s) - F(\rho) C(\rho_s)/C(\rho)}}.
\]

(26)

Here we consider only the small \( j \) case since the small rotation of the Gödel cosmological background seems

![Variety of the coefficient \( a \) with \( \rho_0/\rho_M \) and \( j \) in the squashed Kaluza-Klein Gödel black hole spacetime.](image)

the most reasonable in phenomenology. Moreover, the function \( R(z, \rho_s) \) in the small \( j \) case is regular for all values of \( z \) and \( \rho_s \). From Eq. (26), one can find that the function \( f(z, \rho_s) \) diverges as \( z \) tends to zero. Thus, the integral (24) can be split into the divergent part \( I_D(\rho_s) \) and the regular one \( I_R(\rho_s) \)

\[
I_D(\rho_s) = \int_0^1 R(0, \rho_s) f_0(z, \rho_s) dz,
\]

\[
I_R(\rho_s) = \int_0^1 [R(z, \rho_s) f(z, \rho_s) - R(0, \rho_s) f_0(z, \rho_s)] dz.
\]

(27)
Expanding the argument of the square root in \( f(z, \rho_s) \) to the second order in \( z \), we have

\[
f_s(z, \rho_s) = \frac{1}{\sqrt{p(\rho_s)z + q(\rho_s)z^2}},
\]

where

\[
p(\rho_s) = \frac{\rho_s}{C(\rho_s)} \left[ C'(\rho_s)F(\rho_s) - C(\rho_s)F'(\rho_s) \right],
\]

\[
q(\rho_s) = \frac{\rho_s^2}{2C(\rho_s)} \left[ 2C'(\rho_s)C(\rho_s)F'(\rho_s) - 2C'(\rho_s)^2F(\rho_s) + F(\rho_s)C(\rho_s)C''(\rho_s) - C^2(\rho_s)F''(\rho_s) \right].
\]

(28)

(29)

Obviously, as \( \rho_s \) tends to \( \rho_{ps} \), one can obtain easily that the leading term of the divergence in \( f_s(z, \rho_s) \) is \( z^{-1} \) since the coefficient \( p(\rho_s) \) approaches zero, which implies that the integral \( (24) \) diverges logarithmically.

Thus, the deflection angle in the \( \phi \) direction in the strong field region can be approximated very well as

\[
\alpha(\theta) = -\bar{a} \log \left( \frac{\theta D_{OL}}{u_{ps}} - 1 \right) + \tilde{b} + O(u - u_{ps}),
\]

(30)

with

\[
\bar{a} = \frac{R(0, \rho_{ps})}{2\sqrt{q(\rho_{ps})}},
\]

\[
\tilde{b} = -\pi + b_R + \bar{a} \log \frac{\rho_s^2}{2u_{ps}} \left[ C''(\rho_{ps})F(\rho_{ps}) - C(\rho_{ps})F''(\rho_{ps}) \right],
\]

\[
b_R = I_R(\rho_{ps}), \quad u_{ps} = \frac{C(\rho_{ps})}{F'(\rho_{ps})}.
\]

(31)

The quantity \( D_{OL} \) is the distance between observer and gravitational lens. Repeating the operations above, one can obtain a similar strong gravitational lensing formula for the deflection angle in the \( \psi \) direction \( (\alpha_\psi(\theta)) \) in which the forms of coefficients are different from that of \( \bar{a} \) and \( \tilde{b} \) in Eq. (31). As \( \rho_s \) tends to \( \rho_{ps} \), we find...
that the deflection angle $\alpha_\psi(\theta)$ also diverges logarithmically. However, $\alpha_\psi(\theta)$ cannot actually be observed by astronomical experiments. So we do not consider it in the following discussion.

Combining with Eqs. (17), (30) and (31), we can probe the properties of strong gravitational lensing in the squashed Kaluza-Klein Gödel black hole spacetime and explore the effects of the rotation parameter of cosmological background $j$ on the deflection angle in the strong field limit. In the Figs. (2)-(3), we plot the variations of the coefficients $\bar{a}$ and $\bar{b}$ in the deflection angle (30) with the parameters $j$ and $\rho_0$. As $j$ tends to zero, these quantities reduce to those in the squashed Kaluza-Klein black hole in the cosmological background without rotation [17]. Moreover, we can see that for fixed $j$, both of the coefficients $\bar{a}$ and $\bar{b}$ increase with the size of the extra dimension $\rho_0$, which are similar to those in the usual squashed Kaluza-Klein black hole spacetime. But these two coefficients increase more quickly than in the case $j = 0$. For fixed $\rho_0$, the coefficient $\bar{a}$ in general increases with the increase of $j$. However, we also find that in the extremely squashed case $\rho_0 = 0$, $\bar{a}$ is a constant 1 and is independent of the Gödel parameter $j$. Fig.(3) tells us that the coefficient $\bar{b}$ increases with $j$ for arbitrary $\rho_0$, which is different from the change of the coefficient $\bar{b}$ with the rotation parameter $a$ in Kerr black hole spacetime. In Fig.(4), We present the deflection angle $\alpha(\theta)$ evaluated at $u = u_{ps} + 0.003$, which shows that the larger parameters $j$ and $\rho_0$ lead to the bigger deflection angle $\alpha(\theta)$ for the light propagated in the squashed Kaluza-Klein Gödel black hole spacetime [17]. Comparing with those in the Schwarzschild black hole spacetime, we could extract information about both the rotation of the cosmological background and the size of the extra dimension by using strong field gravitational lensing.
IV. OBSERVATIONAL GRAVITATIONAL LENSING PARAMETERS

Let us now see how the Gödel parameter \( j \) and the scale parameter \( \rho_0 \) affect the observables in the strong gravitational lensing. Assuming that the spacetime of the supermassive black hole at the Galactic center of Milky Way can be described by the squashed Kaluza-Klein Gödel black hole metric, we can estimate the numerical values for the coefficients and observables of gravitational lensing in the strong field limit.

We consider the simplest geometric disposition when the source, the lens and the observer are highly aligned so that the lens equation in strong gravitational lensing can be simplified as

\[
\beta = \theta - \frac{D_{LS}}{D_{OS}} \Delta \alpha_n, \tag{32}
\]

where the quantity \( D_{LS} \) denote the distance between the lens and the source. \( D_{OS} \) is the distance between the observer and the source, which is related to \( D_{LS} \) and \( D_{OS} \) by \( D_{OS} = D_{LS} + D_{OL} \) for this simplest geometry. \( \beta \) and \( \theta \) are the angular separations between the source and the lens, and between the imagine and the lens, respectively. \( \Delta \alpha_n = \alpha - 2n\pi \) is the offset of deflection angle, and \( n \) is an integer. Since \( u_{ps} \ll D_{OL} \), one can find that the \( n \)-th image position \( \theta_n \) and the \( n \)-th image magnification \( \mu_n \) can be approximated as

\[
\theta_n = \theta_0^n + \frac{u_{ps}(\beta - \theta_0^n) e^{\frac{\Delta \alpha_n}{\bar{a}}}}{\bar{a} D_{LS} D_{OL}}, \tag{33}
\]

\[
\mu_n = \frac{u_{ps}^2 (1 + e^{\frac{\Delta \alpha_n}{\bar{b}}}) e^{\frac{\Delta \alpha_n}{\bar{b}}}}{\bar{a} \beta D_{LS} D_{OL}^2}, \tag{34}
\]

respectively. Here \( \theta_0^n \) is the image positions corresponding to \( \alpha = 2n\pi \). In the limit \( n \to \infty \), the minimum impact parameter \( u_{ps} \) is related to the asymptotic position of a set of images \( \theta_\infty \) by a simple form

\[
u_{ps} = D_{OL} \theta_\infty. \tag{35}\]

In order to obtain the coefficients \( \bar{a} \) and \( \bar{b} \), one needs to separate the outermost image from all the others. As in Refs.\[10, 11\], we consider here the simplest situation in which only the outermost image \( \theta_1 \) is resolved as a single image and all the remaining ones are packed together at \( \theta_\infty \). With these simplifications, one can find that the angular separation between the first image and other ones \( s \) and the ratio of the flux from the first image and those from the all other images \( R \) can be expressed as

\[
\begin{align*}
\frac{\theta_1 - \theta_\infty}{\theta_\infty} &= e^{\frac{\Delta \alpha_1}{\bar{a}}}, \\
\frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} &= e^{\frac{\Delta \alpha_1}{\bar{a}}}. \end{align*} \tag{36}
\]
Through measuring these simple observations $s$, $R$, and $\theta_\infty$, one can obtain the strong deflection limit coefficients $\bar{a}$, $\bar{b}$ and the minimum impact parameter $u_{ps}$. Comparing their values with those predicted by the theoretical models, we can obtain the characteristics information about of the lens object stored in the strong gravitational lensing.

FIG. 5: Gravitational lensing by the Galactic center black hole. Variation of the values of the angular position $\theta_\infty$ with parameters $\rho_0/\rho_M$ and $j$ in the squashed Kaluza-Klein black hole spacetime.

FIG. 6: Gravitational lensing by the Galactic center black hole. Variation of the values of the angular separation $s$ with parameters $\rho_0/\rho_M$ and $j$ in the squashed Kaluza-Klein black hole spacetime.

The mass of the central object of our Galaxy is estimated to be $4.4 \times 10^6 M_\odot$ and its distance is around 8.5$kpc$ recently, so the ratio of the mass to the distance $G_4 M/D_{OL} \approx 2.4734 \times 10^{-11}$ 34. Here $D_{OL}$ is the distance between the lens and the observer in the $\rho$ coordination rather than that in $r$ coordination because that in the five-dimensional spacetime the dimension of the black hole mass $M$ is the square of that in the polar coordination $r$. With help of Eqs. (31) and (36), we can estimate the values of the coefficients and observables for gravitational lensing in the strong field limit. The numerical values of $\theta_\infty$, $s$ and $r_m$ (which is related to $R$ by $r_m = 2.5 \log R$) are listed in Table I for the different values of $j$ and $\rho_0/\rho_M$. The dependence of these observables on the parameters $j$ and $\rho_0/\rho_M$ are also shown in Figs.(5)-(7). From Table I and Fig.
r_m = 2.5 \log R.

(5)-(7), we find that for fixed $j$ with the increase of $\rho_0$, the angular position of the relativistic images $\theta_\infty$ and the angular separation $s$ increase, while the relative magnitudes $r_m$ decrease, which is similar to those in usual squashed Kaluza-Klein black hole spacetime \cite{17}. For fixed $\rho_0/\rho_M$, one can obtain that with the increase of $j$, both $\theta_\infty$ and $r_m$ decrease, but the quantity $s$ increases. These information could help us to detect the rotation of the cosmological background in the future.

We now make a comparison between the strong gravitational lensing in the squashed Kaluza-Klein Gödel and four-dimensional Kerr black hole spacetimes. It is well known that for the photon moving along the equatorial plane in the four-dimensional Kerr black hole spacetime \cite{11,12} the photon sphere radius, the minimum impact parameter, the angular position of the relativistic images $\theta_\infty$ and the relative magnitudes $r_m$ in strong gravitational lensing decreases with the rotation parameter $a$ of the black hole, while the coefficient $\bar{a}$, the deflection angle $\alpha(\theta)$ and the observable $s$ in strong gravitational lensing decrease with $a$. These effects of $a$ on the strong gravitational lensing in the Kerr black hole are similar to those of the rotation parameter $j$ of cosmological background in the squashed Kaluza-Klein Gödel case, which can be understandable since both of

| $\rho_0/\rho_M$ | $\theta_\infty$ ($\mu$ arcsec) | $s$ ($\mu$ arcsec) | $r_m$ (magnitudes) |
|----------------|-------------------------------|--------------------|-------------------|
|                | $j = 0$ $j = 0.03$ $j = 0.06$ $j = 0.09$ | $j = 0$ $j = 0.03$ $j = 0.06$ $j = 0.09$ | $j = 0$ $j = 0.03$ $j = 0.06$ $j = 0.09$ |
| 0              | 26.510 25.955 24.477 22.482 | 0.0338 0.0346 0.0395 0.0493 | 6.8219 6.8219 6.8219 6.8219 |
| 0.1            | 27.374 26.738 25.045 22.765 | 0.0369 0.0387 0.0450 0.0582 | 6.7497 6.7485 6.7451 6.7398 |
| 0.2            | 28.205 27.483 25.565 22.990 | 0.0407 0.0430 0.0509 0.0685 | 6.6838 6.6812 6.6738 6.6620 |
| 0.3            | 29.005 28.194 26.042 23.162 | 0.0445 0.0474 0.0574 0.0803 | 6.6234 6.6193 6.6070 6.5873 |
| 0.4            | 29.779 28.875 26.481 23.285 | 0.0485 0.0520 0.0644 0.0941 | 6.5768 6.5617 6.5440 6.5149 |

TABLE I: Numerical estimation for main observables and the strong field limit coefficients for the black hole at the center of our Galaxy, which is hypothesized to be described by the squashed Kaluza-Klein Gödel black hole spacetime.

FIG. 7: Gravitational lensing by the Galactic center black hole. Variation of the values of the relative magnitudes $r_m$ with parameters $\rho_0/\rho_M$ and $j$ in the squashed Kaluza-Klein black hole spacetime.
\( j \) and \( a \) are the rotation parameters. However, the strong gravitational lensing in the squashed Kaluza-Klein Gödel black hole spacetime has some distinct behaviors from that in the Kerr case. In the squashed Kaluza-Klein Gödel black hole spacetime, the photon sphere radius, the minimum impact parameter, the coefficient \( \bar{a}, \bar{b} \) and the deflection angle \( \alpha(\theta) \) in the \( \phi \) direction are independent of whether the photon goes with or against the global rotation of the Gödel Universe. While in the Kerr black hole, the values of these quantities for the prograde photons are different from those for the retrograde photons. Moreover, the coefficient of \( \bar{b} \) increases with \( j \) in the squashed Kaluza-Klein Gödel black hole, but decreases with \( a \) in the Kerr case. In the extremely squashed case \( \rho_0 = 0 \), we also find that the coefficient \( \bar{a} \) is a constant 1 and is independent of the global rotation of the Gödel Universe. These information could help us to understand further the difference between the global rotation of the cosmological background and the rotation of the black hole itself.

V. SUMMARY

We have investigated the strong gravitational lensing in the neutral squashed Kaluza-Klein black holes immersed in a rotating cosmological background. Besides the influence due to the compactness of the extra dimension, we have disclosed the cosmological rotational effects in the radius of the photon sphere and the deflection angle in the \( \phi \) direction. For fixed \( \rho_0 \), the radius of the photon sphere \( \rho_{ps} \) decreases monotonically with the increase of the Gödel parameter \( j \). As the extra dimension scale \( \rho_0 \) increases, \( \rho_{ps} \) increases for the smaller \( j \) and decreases for the larger \( j \). Moreover, we also find that the larger values of the parameters \( j \) and \( \rho_0 \) lead to the bigger deflection angle in the \( \phi \) direction in the strong gravitational lensing for the light ray propagating in the squashed Kaluza-Klein Gödel black hole spacetime. Our result also show that in the squashed Kaluza-Klein Gödel black hole, the photon sphere radius, the minimum impact parameter, the coefficients \( \bar{a}, \bar{b} \) and the deflection angle \( \alpha(\theta) \) in the \( \phi \) direction in the strong gravitational lensing and the corresponding observables are independent of whether the photon goes with or against the global rotation of the Gödel Universe, which is different from those in the Kerr black hole spacetime.

Assuming that the gravitational field of the supermassive black hole in the Galactic center can be described by this metric, we estimated the numerical values of the coefficients and observables in the strong gravitational lensing. Our results show that the angular position of the relativistic images \( \theta_{\infty} \) and the relative magnitudes \( r_m \) decrease with the increase of the parameter \( j \). The change of the angular separation \( s \) with \( j \) is converse to those of \( \theta_{\infty} \) and \( r_m \). Comparing those with the data from the astronomical observations in the future, we
could detect whether our universe is rotating or not.

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