Myopic equilibria, the spanning property and subgame bundles

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Abstract: For a set-valued function $F$ on a compact subset $W$ of a manifold, spanning is a topological property that implies that $F(x) \neq \emptyset$ for interior points $x$ of $W$. A myopic equilibrium applies when for each action there is a payoff whose functional value is not necessarily affine in the strategy space. We show that if the payoffs satisfy the spanning property, then there exist a myopic equilibrium (though not necessarily a Nash equilibrium). Furthermore, given a parametrized collection of games and the spanning property to the structure of payoffs in that collection, the resulting myopic equilibria and their payoffs have the spanning property with respect to that parametrization. This is a far reaching extension of the Kohberg-Mertens Structure Theorem. There are at least four useful applications, when payoffs are exogenous to a finite game tree (for example a finitely repeated game followed by an infinitely repeated game), when one wants to understand a game strategically entirely with behaviour strategies, when one wants to extends the subgame concept to subsets of a game tree that are known in common, and for evolutionary game theory. The proofs involve new topological results asserting that spanning is preserved by relevant operations on set-valued functions.
1 Introduction

Conventionally with games payoffs are multilinear functions of the mixed strategies, meaning that with finitely many pure strategies the payoffs can be represented by multidimensional matrices. One can move away from a multilinear relationship between mixed strategies and payoffs and still get a Nash equilibrium. What counts is that the best reply correspondence of each player in mixed strategies is a convex set, allowing one to apply fixed point theory. If however the relationship between the mixed strategies of a player and her payoffs is merely continuous, there may not be a Nash equilibrium. For example, if the payoff function for a player is convex in her mixed strategies, the best reply correspondence can generate disjoint sets, and the result may be no Nash equilibria.

This lack of a Nash equilibrium can be rectified by expanding the concept of what is a payoff for a player. Instead of associating a payoff to a mixed strategy, one can associate a payoff to each pure strategy or action that the player can use. This broadening of what is a payoff allows one to define a new kind of equilibrium, called a myopic equilibrium. A myopic equilibrium is defined, see (Simon, Spieź and Toruńczyk (2020)), to be a strategy profile such that each action used with positive probability by a player gives the maximum payoff possible from the use of any action of that player. In the context of multilinear payoffs, a myopic equilibrium is the same as a Nash equilibrium. But even when the payoff functions are concave, the myopic equilibria may be far from the Nash equilibria, as we will see in Example 1 below.

In the present paper, we consider parametrized relationships between the mixed strategies and the payoffs, especially when that relationship satisfies the spanning property (Simon, Spieź, Toruńczyk 2002). We show that if the payoffs relative to a parametrization have the spanning property, then the resulting parametrized myopic equilibria also have the spanning property.

This theory has multiple new applications.

First, myopic equilibria are equivalent to equilibria used in evolutionary game theory. The move away from a continuous relationship between mixed strategies and payoffs should broaden the scope of evolutionary game theory.

Second, with extensive form games, the relationship between mixed strategies and behaviour strategies can be seen in a new light. Behaviour strategies are difficult to work with because they lack the multilinear relationship between the strategies and payoffs. Mixed strategies are difficult to work with
because the dimension of the strategy space is much higher than with behaviour strategies. The translation between behaviour strategies and mixed strategies, accomplished by Kuhn (1953), allowed one to go between a strategic structure that was easier to comprehend (behaviour strategies) and one for which a Nash equilibrium exists (mixed strategies). But with myopic equilibria and the spanning property, one can prove the existence of Nash equilibrium entirely through the behaviour strategies.

Third, we can understand what is a subgame in a new way. Subgames are linked closely to the concept of common knowledge, that the players know in common that they have landed in the subgame. A problem arises if the subset of starting points is not a single point; a single subgame is not defined, rather a collection of subgame parametrized by the probability distributions on the starting points of this subset. But with the spanning property and myopic equilibria applied to this parametrization, we can understand a plurality of starting points as a kind of subgame.

Fourth, we can prove the existence of equilibria for some new forms of games through induction, using the conservation of the spanning property in the relationship between the strategies, payoffs, and equilibria.

The following example demonstrates some of the uses of our theory.

**Example 1:**

Consider the following game of three players. Player One chooses between two states, $X$ and $Y$ and Players Two and Three are not informed of the outcome of this choice. Then Players Two and Three play a simultaneous move zero-sum game between each other (so that this game is equivalent to a simultaneous move three person game). Player Two has the actions $L$ and $R$ and Player Three has the actions $l$ and $r$. The payoff matrices for Players Two and Three are the following:

\[
\begin{array}{c|cc}
\text{State X} & l & r \\
L & 9 & 1 \\
R & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{State Y} & l & r \\
L & 1 & 1 \\
R & 1 & 9 \\
\end{array}
\]

where the entries in the matrices represent the payoffs that Player Three must pay to Player Two. Player One gets a payoff of 0 for all combinations except for $R$ and $l$ in State $X$ and $L$ and $r$ in State $Y$. For the $Rl$ combination in State $X$ she gets $1+s$ for some very small $s > 0$ and in State $Y$ she gets 1 for the $Lr$ combination. What are the Nash equilibria of this game and how does it relate to myopic equilibria and a parametrized collection of subgames?
We start with the strategies for Player One. Let $p \in [0, 1]$ be the probability for choosing the state $X$. Assuming that Players Two and Three know this probability $p$, (though not assumed mathematically in the definition of an equilibrium, this can be assumed in its application), their choices are informed by the payoff matrix created by the mixture of $p$ times the matrix for State $X$ plus $1-p$ times the matrix for State $Y$:

$$
\begin{pmatrix}
\beta & 1 - \beta \\
\alpha & 8p + 1 & 1 \\
1 - \alpha & 1 & 9 - 8p
\end{pmatrix}
$$

For the sake of analysis we added $\alpha$ and $\beta$ for the probability that Player Three will choose $l$ and $\alpha$ for the probability that Player Two will choose $L$. If $p$ lies in the interior $(0, 1)$ there are no optimal pure strategies for either player. Assuming indifference between the two actions, we get the formulas

$$(1 + 8p)\beta + 1 - \beta = \beta + (1 - \beta)(9 - 8p)$$

and

$$(1 + 8p)\alpha + 1 - \alpha = \alpha + (1 - \alpha)(9 - 8p),$$

which are symmetric in $\beta$ and $\alpha$ and solve to $\beta = \alpha = 1 - p$. Now consider Player One’s payoff. Her choice for $X$ results in a payoff of $(1 + s)(1 - p)$.

The choice for $Y$, given $p$, gives a payoff of $(1-s)(1-p)p$. The choice for $X$, given $p$, gives a payoff of $(1-p)p$. This means that the choice for $p$ yields an expected payoff of $f(p) := (1+s)(1-p)p^2 + (1-p)^2p$. To optimize the payoff we must take the derivative $f'(p) = 1 + (2s - 2)p - 3sp^2$ and set it to zero, which solves to
\[ p = \frac{s - 1 + \sqrt{1 + 4s + s^2}}{3s}. \] For \( s \) very small the solution calls for \( p \) very close to \( \frac{1}{2} \), with an expected payoff very close to \( \frac{1}{4} \). Finding the Nash equilibrium of a one player game is a problem of optimization and as long as the payoff function is continuous there will be at least one optimal strategy, and if the payoff function is strictly concave there will be a unique optimal strategy. Indeed the payoff function of this game is strictly concave, as the second derivative of \( f \) is \( 2s - 2 - 6sp \), which is always negative as long as \( s \) is less than 2. However the concept of Nash equilibrium is not the right one for understanding the three person game broken into two parts, the first part being that of Player One’s choice and the second part collection of games between Players Two and Three. It corresponds to the situation where Player One can commit herself to a mixed strategy and maximizes the payoff accordingly. But what if Player One “takes back control” to choose the action giving the highest payoff? By doing so she will destroy the equilibrium property of the one player game and after it is re-established through the myopic equilibrium concept she will have a worse payoff, though one relevant to the three player game. This has similarity to the myth of Odysseus and the Sirens, where Odysseus optimizes by binding himself.

Conventionally the game played between Players Two and Three is not considered a subgame because it does not involve common knowledge of a single state, even though the players do know in common that their choices pertain to a subset of size two. This “subgame” is really a continuum of subgames, one for each probability distribution on this subset of size 2.

We generalize the concept of subgame to subsets of vertices in a game tree known in common, and call it a subgame bundle. These subsets are closed with respect to actions and the knowledge of the players. Some requirement of common knowledge is needed on the subsets that can define a subgame bundle, because otherwise a player could be forced to play the same in two locations belonging to different subgames.

A problem with generalizing the concept of a subgame to subsets is that the set of equilibria, as a function of the probability distributions on the subset of starting positions, in general cannot be approximated by continuous functions. The above Example 1 used a zero-sum game played between the remaining players, whose equilibria as a function of distributions will be an upper-semi-continuous and non-empty convex valued correspondence. However in the more general context of non-zero-sum games the equilibrium payoff
correspondence will not have this nice property.

We know from the Structure Theorem of Kohlberg and Mertens (1986) that there will be a topological structure to the equilibrium correspondence, a homotopic relation that implies the spanning property. There are two advantages to working with the spanning property over the homotopic property of Kohlberg and Mertens. First the spanning property is more general. With repeated games of incomplete information on one side the equilibrium payoff correspondence as a function of the probability distributions has the spanning property (Simon, Spież, Toruńczyk 2002), but there is no reason to believe that it has the homotopic property. Second, there is no indication yet that the homotopic property is robust with respect to composition, though we prove that the spanning property has such a robustness.

The myopic equilibrium concept is not entirely new. The mathematical structure behind a myopic equilibrium is identical to that of a “Nash equilibria of population densities” from evolutionary game theory. Though the mathematical formalities are the same, the concepts are very different. With evolutionary game theory, there is a continuum of animals belonging to a species. Each species has a variety of types and a distribution of types is equivalent mathematically to a mixed strategy. What we call a myopic equilibrium is in the context of evolutionary game theory a kind of Nash equilibrium because each individual animal seeks to maximize a utility independent of the species as a whole. The distribution of types will influence the payoffs for each type, and that introduces a potentially non-affine structure to those payoffs.

The rest of this paper is organized as follows. In the second section we provide the necessary topological apparatus and prove new results about the spanning property, giving conditions under which it is preserved by operations on correspondences like taking intersections, taking cartesian products or taking weighted sums. In the third section we define “game bushes”, “game bundles”, “subgame bushes”, and “subgame bundles” and use results from §2 to show the robustness of the spanning property with respect to myopic equilibria, that if the payoffs have the spanning property then the equilibrium payoffs of the game bundle has the spanning property. In the fourth section we consider subgames, factor games, and a new concept of subgame perfection. Finally we consider related questions and problems.
2 The spanning property of correspondences

Let $X$ and $Y$ be metrizable spaces. By a correspondence $F : X \to Y$ we mean here any compact subset of $X \times Y$. Hence the same correspondence may also be denoted as $F \subset X \times Y$ and we often switch from one notation to the other; however, the use of $F : X \to Y$ displays the asymmetric role which $X$ and $Y$ play below. Given such an $F$ and a subset $W$ of $X$ we call $F \cap (W \times Y)$ the restriction of $F : X \to Y$ to $W$ and denote it $F|W$. The image of $F|W$ under the projection to $Y$ is denoted by $F(W)$. We also write $F(x) := F(\{x\})$ for $x \in X$ and $\text{dom}(F) := \{x \in X : F(x) \neq \emptyset\}$, the domain of $F$. The usual challenge we encounter is to show that a given point $x \in X$ is in $\text{dom}(F)$, i.e., that $F(x) \neq \emptyset$.

It is convenient to note that if $F : X \to Y$ and $G : Y \to Z$ are correspondences, then the formula $H(x) = G(F(x))$ defines a correspondence, i.e., $\bigcup_{x \in X} \{x\} \times H(x)$ is a compact subset of $X \times Z$. We write $G \circ F$ for $H$.

Until Theorem 1 below we let $M$ be a compact, connected $d$-manifold with boundary $\delta M$ (possibly empty) and $W$ be a compact subset of $M$. By $\delta W$ we denote the boundary of the interior of $W$ with respect to $M$. Important to us is that there exists a well-defined element of the Čech homology group $H_d(W, \delta W; \mathbb{Z}_2)$, denoted here by $[W]$, with the following properties:

**Fact 1.**

a) If $W$ is a compact, connected $d$–manifold then $[W]$ is the $\mathbb{Z}_2$–fundamental class of $W$, the unique non-zero element of the Čech homology group $H_d(W, \delta W; \mathbb{Z}_2)$.

b) With $\partial[W] \in H(\delta W; \mathbb{Z}_2)$ standing for the image of $[W]$ under the boundary operator, one has that the image of $\partial[W]$ in $H(W \setminus \{w\}; \mathbb{Z}_2)$ under the inclusion–induced map is non-zero, for each point $w$ in the interior of $W$.

A proof with $M$ a sphere is given in (Simon, Spież, Toruńczyk 2002); using the orientability of $M$ over $\mathbb{Z}_2$ a generalization causes no difficulty. Of course, $[W]$ depends on the ambient manifold $M$, e.g. it is null when $W$ is a disk embedded in the sphere $S^3$, but non-null if this disk is being considered as a subset of $S^2$.

We say that a correspondence $F : W \to Y$ has property $\mathcal{S}$ if $[W]$ lies in the image of the projection–induced homomorphism $H(F, F|\delta W; \mathbb{Z}_2) \to H(W, \delta W; \mathbb{Z}_2)$ of the relative Čech homology groups with $\mathbb{Z}_2$–coefficients. If $F$ is given as, say, $F \subset W \times Y_1 \times Y_2$ and we wish to consider it as a correspondence from $W$ to $Y_1 \times Y_2$ (rather than e.g. from $Y_1$ to $W \times Y_2$),
then above we say that $F$ has property $S$ for $W$, and similarly when there are more or two factors. Here, $S$ is an abbreviation for “spanning”, addressing to Fact 2a) below:

**Fact 2.** a) If $F : W \to Y$ has property $S$ then the domain of $F$ contains the closure of the interior of $W$ relative to the ambient manifold $M$.

b) If $W' \subset W$ are compact subsets of $M$ and $F : W \to Y$ has property $S$ then $F|W'$ hast it either.

c) Let $W$ and $W'$ be compact subsets of $M$ and let correspondences $F : W \to Y$ and $F' : W' \to Y$ coincide on $W \cap W'$. If $F'$ takes values in singletons only, then $F \cup F'$ has property $S$ for $W \cup W'$.

d) Suppose a correspondence $F : W \to Y$ is such that for each neighbourhood $U$ of $\delta W$ in $M$ and each neighborhood $V$ of $F$ in $W \times Y$ there exists a compact set $G \subset V$ with property $S$ for a compact set $D$ satisfying $\delta D \subset U$ and $D \subset W \cup U$. Then, $F$ has property $S$ for $W$.

Claim a) above follows from Fact 1b), and for the other ones see (Simon, Spie\'z, Toru\'nczyk 2002) (again, with $M$ a sphere).

**Theorem 1.** Let $W, X, Y$ be compact manifolds and $F, G \subset W \times X \times Y$ be correspondences with property $S$ for $W \times X$ and for $W \times Y$, respectively, such that the projections $F \to Y$ and $G \to X$ are null–homotopic. Then, $F \cap G$ has property $S$ for $W$.

**Proof.** 1). Let’s first assume none of $W, X, Y$ has a boundary. Let $\alpha \in H(F)$ be a homology class mapped to $[W \times X]$ under the homomorphism induced by the projection to $W \times X$. The projection of $F$ to $Y$ being homotopic to a constant (say, $y_o$) one has $\alpha = [W \times X \times \{y_o\}]$ in $H(W \times X \times Y)$ – by which we mean that the images of these two classes under the inclusion-induced homomorphisms coincide.

Similarly, there exists a class $\beta \in H(G)$ mapped to $[W \times Y]$ under the homomorphism induced by the projection of $G$ to $W \times Y$ and such that for some $x_o \in X$ one has $\beta = [W \times \{x_o\} \times Y]$ in $H(W \times X \times Y)$.

Below, we’ll be using the properties of the ”intersection pairing” described in (Dold, 1972, §VIII.13). So, there is a well-defined element $\alpha \bullet \beta$ of $H(F \cap G)$. In $H(W \times X \times Y)$ this element is equal to $[W \times X \times \{y_o\}] \bullet [W \times \{x_o\} \times Y]$, by what said earlier. The last product is however equal to $[W \times \{(x_o, y_o)\}]$, for the manifolds $X \times \{y_o\}$ and $\{x_o\} \times Y$ intersect transversally at $(x_o, y_o)$. And since $[W \times \{(x_o, y_o)\}]$ is mapped to $[W]$ under the homomorphism induced by the projection to $W$, so is $\alpha \bullet \beta$. Thus $F \cap G$ has property $S$ for $W$. 

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2). Let now $\delta W = \emptyset$ and $\delta Y \neq \emptyset$. We then consider $Y$ as a subset of $Y := Y \cup_{\delta Y} Y'$, a union of two copies $Y$ and $Y'$ of $Y$ intersecting along $\delta Y$. Let $h$ be a homeomorphism of $Y$ onto $Y'$ which restricts to identity on $\partial Y$, and let $G' := (id_{W \times X} \times h)(G)$ and $G := G \cup G'$.

We first prove that $\tilde{G}$ has property $S$ for $W \times \tilde{Y}$. To this end we take an $\alpha \in H(G,G|(W \times \delta Y))$ which witnesses property $S$ of $G$ (i.e., is mapped to $[W \times Y]$ by the projection–induced homomorphism), set $\alpha' := h_{*}(\alpha)$ and let $\tilde{\alpha} \in H(\tilde{G},\tilde{G}|(\partial W \times Y);\mathbb{Z}_{2})$ be the image of $\alpha \oplus \alpha'$ in the relative Mayer–Vietoris sequence for $(\tilde{G},\tilde{G}|(\partial W \times Y)), (\tilde{G}',\tilde{G}'|(\partial W \times Y'))$. We readily see that $\partial \tilde{\alpha} = 0$, so $\tilde{\alpha}$ can be considered as an element of $H(\tilde{G})$. Also, the projection–induced homomorphism maps $\tilde{\alpha}$ to the image of $[W \times Y] \oplus [W \times Y']$ under an analogous Mayer–Vietoris sequence, and this image is equal to $[W \times \tilde{Y}]$. This shows that $\tilde{G}$ indeed has property $S$ for $W \times \tilde{Y}$. (We skipped a technical point dealing with why are all the pairs involved excisive.)

Moreover, the projection of $\tilde{G}$ to $X$ factors through the projection $G \to X$ and hence is null-homotopic. The analogous property of the projection $F \to \tilde{Y}$ is even more obvious. Since $\delta \tilde{Y} = \emptyset$, if additionally $\delta X = \emptyset$ then $F \cap \tilde{G}$ has property $S$ for $W$, by 1) above. However, $F \cap \tilde{G} = F \cap G$ and thus the assertion is true if $\delta W = \delta X = \emptyset$ (or if $\delta W = \delta Y = \emptyset$, by symmetry).

3). The assertion also holds true if both $X$ and $Y$ do have a boundary but $W$ doesn’t. This is so because then the reasoning of 2) still applies once the reference to 1) gets replaced by one to the conclusion of 2). (No additional assumption on $X$ is being made this time.)

4). Finally, if $\delta W \neq \emptyset$ then we replace $W$ by $\tilde{W} := W \cup_{\delta W} W$, and $F$ and $G$ by $\tilde{F} := F \circ (q \times id_{X \times Y})$ and $\tilde{G} := G \circ (q \times id_{X \times Y})$ respectively, where $q : \tilde{W} \to W$ is a retraction. As in 2) above we infer that $\tilde{F}$ has property $S$ for $\tilde{W} \times X$ and $\tilde{G}$ has it for $\tilde{W} \times Y$. Hence, the correspondence $\tilde{F} \cap \tilde{G}$ has property $S$ for $\tilde{W}$, by 1) to 3) above. Since on $W$ it restricts to $F \cap G$, the desired conclusion follows from Fact 2b). □

**Addendum to Theorem 1.** The conclusion of Theorem 1 holds also true if $X$ and $Y$ are as before and $W$ is any compact subset of a compact, connected ambient manifold $M$ that is PL or is of dimension neither 4 nor 5.

Proof. The interior of $W$ is the union of an increasing sequence $(W_{n})$ of compact manifolds with boundaries; see (Kirby and Siebenmann (1977), p. 108). By Theorem 1 and Fact 2b), for each $n$ the appropriate restriction of $F \cap G$ has property $S$ for $W_{n}$, whence Fact 2d) applies. □
Theorem 2. Let $W,X,Y$ be as in Theorem 1 or in the Addendum to it. Furthermore, let $\Phi : W \to X$ and $\Psi : X \to Y$ be correspondences with property $S$ such that the projections $\Phi \to X$ and $\Psi \to Y$ are null-homotopic. Then, $\Psi \circ \Phi : W \to Y$ has property $S$.

Proof. Let $F := \Phi \times Y$ and $G := W \times \Psi$. Then $F,G \subset W \times X \times Y$ are correspondences with property $S$ for $W \times Y$ and for $W \times X$, respectively, and such that the projections $F \to X$ and $G \to Y$ are null-homotopic.

We have $F \cap G = \{(w,x,y) \mid w \in W, x \in \Phi(w), y \in \Psi(x)\}$ and the projection of $F \cap G$ to $W$ is equal to the projection of $F \cap G$ along $X$ to $\Psi \cap \Phi$, followed by the projection of $\Psi \cap \Phi$ to $W$. Since $F \cap G$ has property $S$ for $W$ (by Theorem 1 and the Addendum), so does $\Psi \cap \Phi$. □

Remark. The assumption in Theorems 1 and 2 that the corresponding maps be null-homotopic (i.e., homotopic to constant ones) is automatically satisfied if $X$ and $Y$ are simplices, as in the applications in §3 and §4 below.

Theorem 3. Let $W$ be a compact top-dimensional subset of a PL-manifold and $F_i : W \to Y_i$ $(i = 1, \ldots, l)$ be correspondences with property $S$. Then, this property is also enjoyed by the correspondence $F : W \to \prod_i Y_i$, defined by the formula $F(x) = \prod_i F_i(x)$.

Proof. We may assume all the $Y_i$'s to be normed linear spaces (for they can be embedded into such). For each $i$ let $V_i$ be an open neighborhood of $F_i$ in $X \times Y_i$. We assume this neighbourhood is triangulated consistently with the PL-structure of $W \times \prod_i Y_i$. By the spanning property of $F_i$, for $d = \dim(W)$ there exists a $d$-chain $z_i$ in $V_i$, with boundary in $\partial W \times Y_i$ and such that its class is mapped to $[W]$ by the projection-induced homomorphism.

By general position we may assume that with respect to some triangulation $\mathcal{T}$ of $W$ each $d$-simplex of $z_i$ projects injectively onto some simplex of $\mathcal{T}$.

For each tuple $(\sigma_1, \ldots, \sigma_l)$ of $d$-simplexes $\sigma_i \in z_i$ which project onto the same simplex of $\mathcal{T}$ we define a "diagonal" $k$-simplex in $X \times \prod_i Y_i$ by :

$$\Delta(\sigma_1, \ldots, \sigma_l) := \{(x,y_1, \ldots, y_l) \mid (x,y_i) \in \sigma_i \text{ for each } i \}.$$ 

Then, we let $z$ denote the chain in $X \times \prod_i V_i$ which is the sum of $\Delta(\sigma_1, \ldots, \sigma_l)$’s over all tuples $(\sigma_i)_{i=1}^l$ as above. One can observe that $z$ is a $d$-chain in $(W,\partial W) \times \prod_i V_i$ and its class is mapped to $[W]$ by the projection-induced homomorphism. Thus, $F$ has the spanning property by Fact 2d). □
Corollary 1. In Theorem 3 assume that \( Y_i = \mathbb{R}^A \) for each \( i \), where \( A \) is a finite set. Then, \( \sum_i F_i : W \to \mathbb{R}^A \) defined by the formula \((\sum_i F_i)(x) = \{y_1 + \cdots + y_i \mid y_i \in F_i(x) \text{ for all } i\}\) is a correspondence with property \( S \).

Proof. Let \( Y \subset (\mathbb{R}^A)^l \) be a closed ball containing \( F(W) \), where \( F(x) = \prod_i F_i(x) \subset (\mathbb{R}^A)^l \) for \( x \in W \). Obviously, \( \sum_i F_i = f \circ F \), where \( f(y) = \sum_i y_i \) for \( y = (y_i)_{i=1}^l \in Y \). Hence, it remains to apply Theorems 2 and 3. \( \square \)

Given \( F \subset X \times \mathbb{R}^A \) and a function \( \lambda : X \to \mathbb{R} \) we define \( \lambda F \subset X \times \mathbb{R}^A \) by the formula:

\[(\lambda F)(x) = \{\lambda(x)y : y \in F(x)\} \text{ for } x \in X.\]

Corollary 2. Let \( W \) be a compact PL–manifold and for \( i = 1, 2, \ldots, l \) let it be given a continuous function \( \lambda_i : W \to \mathbb{R} \), a function \( f_i : W \to Y_i \) which is continuous off \( \lambda_i^{-1}(0) \), and a correspondence \( G_i : Y_i \to \mathbb{R}^A \) with property \( S \). Then the correspondence \( \sum_i \lambda_i(G_i \circ f_i) \) has property \( S \).

Proof. By Corollary 1 it suffices to consider the case where \( l = 1 \); we hence write \( \lambda, f \) and \( G \) for \( \lambda_1, f_1 \) and \( G_1 \), respectively. We may assume that \( \lambda \neq 0 \), say \( \sup |\lambda|(W) > 1 \). With \( K_n := \{x \in X \mid |\lambda(x)| > 1/n\} \) it follows from Theorem 2 and Fact 2b) that \( G \circ f|K_n \) has property \( S \), whence \( H_n := \lambda G \circ f|K_n \) has it either. Let \( K \) and \( H \) denote the closures of \( \bigcup_n K_n \) and of \( \bigcup_n H_n \), respectively. Using boundedness of \( G(W) \) we note that \( H \) is compact and \( H = (\lambda G \circ f)|K \); also, from Fact 2d) we infer easily that \( H \) has property \( S \). Moreover \( (\lambda G \circ f)(x) = \{0\} \) for \( x \) off the interior of \( K \) and so the property \( S \) of \( \lambda G \) follows from Fact 2c). \( \square \)

For the last results of this section we introduce notation which is relevant also to the further sections. That is, if \( A \) is a finite set then we let \( \Delta(A) \) denote the \((|A| - 1)\)–simplex spanned by \( A \):

\[\Delta(A) := \{f : A \to [0, 1] \mid \sum_{a \in A} f(a) = 1\}\]

and we consider it being embedded in the \(|A|\)–space \( \mathbb{R}^A \) of all functions from \( A \) to \( \mathbb{R} \), equipped with the norm \( \|f\| := (\sum_a |f(a)|^2)^{1/2} \).

Now, \((A_i)_{i \in N}\) be a partition of \( A \) into non-empty subsets. Corresponding to it we let \( \Delta := \prod_{i \in N} \Delta(A_i) \), considered as a subset of \( \prod_{i \in N} \mathbb{R}^{A_i} = \mathbb{R}^A \). For
a ∈ A and an element v of \( \mathbb{R}^A \) (and hence also for \( v ∈ \Delta \)) we denote by \( v^a \) the a-th coordinate of v.

**Theorem 4.** Let \( F : W × \Delta → \mathbb{R}^A \) be a correspondence with property \( S \), where \( W \) is compact manifold or is as in the Addendum to Theorem 1. Let further \( D ⊂ \Delta × \mathbb{R}^A \) denote the set of all pairs \((σ,v)\) such that

\[
\text{for all } i ∈ N \text{ and } a ∈ A_i, \text{ if } σ^a > 0 \text{ then } v^a = \max\{v^b : b ∈ A_i\}. \quad (1)
\]

Then, \( F ∩ (W × D) \) has property \( S \) for \( W \).

**Corollary 3.** With \( W \) and \( F \) as above define \( E : W → \mathbb{R}^N \) by \((w,r) ∈ E \) if and only if for some \((w,σ,v) ∈ F \) one has both (1) above and \( r^i = \max_{a ∈ A_i} v^a \) for each \( i ∈ N \). Then, \( E \) has property \( S \).

**Proof.** Theorems 2 and 4 do the job, because \( E = f ◦ (F ∩ (W × D)) \) where \( f : \mathbb{R}^A → \mathbb{R}^N \) is defined by the formula \( f(v) = (\max\{v^a : a ∈ A_i\})_{i ∈ N} \).

**Proof of Theorem 4.** Let \( G \) denote the closure of \( D \) in \( \Delta × S \), where \( S \) is the sphere \( \mathbb{R}^A \cup \{∞\} \). By Theorem 1 and its Addendum, combined with the contractibility of \( \Delta \) and of \( \mathbb{R}^A \), it suffices to prove that \( G \) has property \( S \) for \( S \).

To this end let us recall that there exists a retraction \( r : \mathbb{R}^A → \Delta \) such that, for \( σ ∈ \Delta \) and \( v ∈ \mathbb{R}^A \), condition (1) holds true if and only if \( r(σ + v) = σ \). (See (Simon, Spież, Toruńczyk, 2020), Lemma 1). This \( r \) is the product of nearest–point retractions \( r_i : \mathbb{R}^{A_i} → \Delta(A_i), \ i ∈ N. \) The formulas \( φ(σ,v) = σ + v \) and \( ψ(v) = (r(v), v - r(v)) \) hence define mutually inverse mappings \( ψ : \mathbb{R}^A → D \) and \( φ : D → \mathbb{R}^A \). It follows that \( φ : D → \mathbb{R}^A \) is a homeomorphism which is at finite distance from the projection \( (σ,v) → v \). The straight-line homotopy between these two mappings is thus proper, and accordingly the projection \( p : D ∪ \{∞\} → S \) is homotopic to the homeomorphism \( φ ◦ id_{\{∞\}} \).

However, the projection of \( G \) to \( S \) is equal to the composition \( p ◦ q \), where \( q : G → D ∪ \{∞\} \) is a map that squeezes the set \( G_∞ := G ∩ (\Delta × \{∞\}) \) to \( \{∞\} \) and is the identity elsewhere. It is readily seen that \( G_∞ = \Delta × \{∞\} \), whence \( q \) induces a homology–isomorphism by Vietoris’ theorem. Since so did \( p \), also \( p ◦ q \) induces such an isomorphism. □

### 3 Game bushes

We have to modify the concept of a finite game tree so that the terminal points of the game contain variable payoffs and also there could be a variety
of locations where the game starts. Without first stipulating the payoffs, we call this modification a *game bush*. Some may prefer the term “game forest”. But our structure involves an informational interaction between the different starting points (roots) and terminal points (leaves); it is not merely an independent collection of games. After we add a payoff structure, it becomes a “game bundle”. In Simon, Spież, and Toruńczyk (2020) the concept of a “truncated game tree” was introduced, a game tree with a unique starting point and an informational structure to the terminal points. However we prefer a “game bush” to a “truncated game bundle” and prefer a “subgame bush” to a “truncated subgame bundle”. Except for the terminology, the concepts follow in parallel those of that article.

Let there be a finite set $N$ of players. There is a finite directed graph $(V, V)$ (arrows between vertices) such that the directed graph is acyclic without the orientation of the edges (meaning that for every vertex there is only one path leading to that vertex from the roots). The subset $R$ of initial vertices is called the roots. $T$ is the set of terminal points and every path of arrows starts at a root and ends at a terminal point, with each terminal point determining a unique such path of arrows. We allow $T$ and $R$ to have a non-empty intersection, meaning that in part of the game bush the game could start and end simultaneously. The set $D$ of nodes is the subset $V \setminus T$ and these are the vertices (except for the roots) to which comes exactly one arrow and from which, without loss of generality, come at least two arrows. Toward a root, there is no arrow, and from a root comes at least two arrows.

For each player $n \in N$ there is a subset $D_n \subseteq D$ such that $\forall i \neq n D_i \cap D_n = \emptyset$. Define $D_0$ to be the set $D \setminus (\bigcup_{n \in N} D_n)$. Also for every $n \in N$ there is a partition $P_n$ of $D_n$. For every $W \in P_n$ there is a set of actions $A^n_W$ such that there is a bijective relationship between $A^n_W$ and the arrows leaving every $v \in W$. For every $v \in D_0$ there is a probability distribution $p_v$ on the arrows leaving the node $v$, and therefore also on the nodes following directly after $v$ in the tree. We assume without loss of generality that $p_v$ gives positive probability to every arrow leaving $v$. The subset $D_0$ is where Nature makes a choice. Nature is not a player because it has no payoffs and its actions are involuntary and randomized rather than choices in the usual sense.

At any node $v \in W \in P_n$ only the player $n$ is making any decision, and this decision determines completely which vertex follows $v$. At the nodes $v$ in $V_0$ nature is making a random choice, according to $p_v$, concerning which
vertex follows $v$. If the game is at the node $v \in D_n$ and $v \in W \in P_n$, then Player $n$ is informed that the node is in the set $W$ and that player has no additional information, so that inside $W$ player $n$ cannot distinguish between nodes within $W$. A set $W \in P_n$ is called an *information set* of Player $n$.

The inspiration behind its definition is that a game bush could be part of a larger game, either at the start, the end, or in the middle. With conventional game trees, there is only one root. The introduction of multiple roots creates new problems. As a game bush could represent a kind of subgame, information may be inherited from previous play. Therefore for each player $n \in N$ we require that there is a partition $R_n$ of the roots $R$ representing what each player knows at the start of the game bush. And then to define the payoffs and potentially to connect the game bush to further play in subsequent game bushes, we require for each player $n \in N$ that there is a partition $Q_n$ of the terminal points.

Now we extend the definition of a game bush to that of a game bundle.

Define $Q$ to be the meet partition $\bigwedge_{n \in N} Q_n$, the unique finest partition such that for every $n \in N$ every member of $Q_n$ is contained in some member of $Q$. For any $C \in Q$ we assume there is a correspondence $F_C \subseteq \Delta(C) \times R^{C \times N}$ of *continuation* payoffs. The correspondences $F_C$ for all $C \in Q$ together with the game bush define a game bundle. The partition $Q$ represents common knowledge in the end points. We believe that the payoffs must be defined by such a partition. A non-empty overlap of different subsets could create an unresolvable ambiguity concerning what are the payoffs. And an overlap between an information set and a set defining the subgame would create problems for the subgame concept.

A game bundle is not a single game, rather a collection of games parametrized by the distributions on the set $R$ of roots. Furthermore, because the payoffs are defined by correspondences, there could be different payoff consequences for the same set of actions and ending at the same terminal point (or indeed none where the correspondence may be empty).

A game tree, the conventional game in extensive form, is a game bundle such that the set $R$ (of roots) is a singleton, $Q$ is the discrete partition on $T$ (meaning the collection $\{\{t\} \mid t \in T\}$) and the correspondence $F_t$ for every $t \in T$ is defined by a singleton $\{(\delta_t, r)\}$ for some payoff $r \in \mathbb{R}^N$ (where $\delta_r$ is the Kronecker delta).
For every player \( n \in N \) let \( S_n \) be the finite set of pure decision strategies of the players in the game bush, by which we mean a function that decides, at every set \( W \) in \( \mathcal{P}_n \) deterministically which member of \( A^n_W \) should be chosen. If each such \( A^n_W \) has cardinality \( l \) and there are \( k \) such sets in \( \mathcal{P}_n \) then the cardinality of \( S_n \) is \( l^k \).

For our purposes, the set \( \Delta := \prod_{n \in N} \Delta(S_n) \) will be the set of strategies, what are called the mixed strategies. A behaviour strategy is a choice of a point in \( \Delta(A^n_W) \) for each choice of \( n \in N \) and \( W \in \mathcal{P}_n \). Later we will establish new relations between these different types of strategies through subgame bundles and myopic equilibria.

For every \( q \in \Delta(R) \), \( \sigma \in \Delta := \prod_{n \in N} \Delta(S_n) \), and \( C \in \mathcal{Q} \) let \( p_{q,\sigma}(C) \) be the probability of reaching \( C \) through \( \sigma \) and \( q \in \Delta(R) \). If this probability is positive, define \( P_{q,\sigma}(\cdot|C) \) to be the conditional probability on \( C \) induced by \( q \) and \( \sigma \).

Given a game bundle with \( q \in \Delta(R) \), \( \sigma \in \Delta := \prod_{n \in N} \Delta(S_n) \), and \( C \in \mathcal{Q} \) let \( p_{q,\sigma}(C) \) be the probability of reaching \( C \) through \( \sigma \) and \( q \in \Delta(R) \). If this probability is positive, define \( P_{q,\sigma}(\cdot|C) \) to be the conditional probability on \( C \) induced by \( q \) and \( \sigma \).

Given a game bundle with \( q \in \Delta(R) \), and \( \sigma \in \Delta := \prod_{n \in N} \Delta(S_n) \) define \( f^n_\phi(s) \) to be the expected value of \( y^{n,\phi} \) as determined by the \( q \in \Delta(R) \) and transitions determined by the \( s \) and the \((\sigma^j \mid j \neq n)\), whereby the conditional probabilities \( p_{q,\sigma} \) are still determined by the entire \( \sigma = (\sigma^j \mid j \in N) \). A plan \( \phi \) for \( (q,\sigma) \) is an \( m \)-equilibrium if for all \( s \in S_n \) with \( \sigma^n(s) > 0 \) it follows that \( f^n_\phi(s) = \max_{t \in S_n} f^n_\phi(t) \). The payoff of a plan \( \phi \) is a vector in \( \mathbb{R}^N \) such that the \( n \) coordinate is the expected value of of the \( f^n_\phi(s) \), which for an \( m \)-equilibrium is the common value for the \( f^n_\phi(s) \) for all those \( s \) with positive probability.

Next we show that the correspondence of \( m \)-equilibria has a certain special property as long as the correspondences defining the continuation payoffs have that same special property.

The next theorem applies to the context of a game bundle defined by a partition \( \mathcal{Q} \) and correspondences \( F_C \) for every \( C \in \mathcal{Q} \).

**Theorem 5.** If for every \( C \in \mathcal{Q} \) the correspondence \( F_C \subseteq \Delta(C) \times \mathbb{R}^{C \times N} \) has the spanning property then the correspondence defined by the payoffs of \( m \)-equilibria has the spanning property with respect to \( \Delta(R) \).
Remark: We follow approximately a previous argument of Simon, Spieź, and Toruńczyk (2020).

Proof: Let $\epsilon > 0$ be given and let $B$ be a positive quantity larger than any payoff from the correspondences $F_C$. If $p_{q,\sigma}(C)$, the probability of reaching $C$ with $q$ and $\sigma$, is at least $2\epsilon$ then define $\lambda_{q,\sigma}(C) = 1$. If $p_{q,\sigma}(C) \leq \epsilon$ then $\lambda_{q,\sigma}(C) = 0$. And if $\epsilon < p_{q,\sigma}(C) < 2\epsilon$ then let $\lambda_{q,\sigma}(C) = \frac{p_{q,\sigma}(C) - \epsilon}{2\epsilon}$.

For every choice of $\epsilon > 0$ and $C \in Q$ define the correspondence $F_\epsilon \subseteq \Delta(R) \times \Delta \times \mathbb{R}^{N \times T}$ by $F_\epsilon(q, \sigma)^N = \lambda_{q,\sigma} F_C(q, p_{q,\sigma}(\cdot|C) + (1 - \lambda_{q,\sigma}(C))) \{r_B\}$, where $r_B$ is the point whose entry is $B$ for all choices. Due to Corollary 2 the correspondence $F_\epsilon$ has the spanning property. Due to Corollary 3 (with $\Delta(R)$ taking the role of $W$ in Corollary 3), the correspondence $G_\epsilon$ defined by the myopic equilibria from $F_\epsilon$ plans has the spanning property. And because the spanning property is defined using Čech homology, the cluster limit $G$ of the $G_\epsilon$ for any sequence of $\epsilon$ going to zero also has the spanning property.

Now consider any payoff obtained from $G$; we need to show that it comes from an $m$-equilibrium. Suppose for a given $(q, \sigma)$ that $y^N$ is from a sequence of the $F_\epsilon$. As there are fixed values for the minimal non-zero probability for reaching any terminal point $e \in T$, and the sequence of $\epsilon$ went to zero, it follows for any $C \in Q$ given positive probability by $q$ and $\sigma$ that $\lambda < 1$ was not used in the definition of the payoffs in $G_\epsilon$ for some sufficiently small $\epsilon$. And if $C \in Q$ is given zero probability by the $q$ and $\sigma$, due to the large positive $B$, any cluster of vectors from the correspondence $F_C$ can be used for the event that some terminal point in $C$ is reached (and if there is no such cluster, meaning that eventually only $B$ was used in the definition of the $G_\epsilon$, then any vector from $F_C$ would do). Given the inequalities defining the myopic equilibria of $G$, after removing the use of $B$ those inequalities remain to show that $G$ is a subset of the myopic equilibria of the game bundle. □

4 Subgames and factor games

We extend the definition of a subgame from that defined on a single state to one defined on a subset of possible states.

Definition: A subset $S$ of the vertices $V$ of a game bush defines a subgame bush on the vertices $S$ if and only if

(1) one cannot leave $S$ through any choice of action of a player or a choice
of nature (meaning closure by arrows),
(2) for every player $n \in \mathbb{N}$ and every $W \in \mathcal{P}_n$ or $W \in \mathcal{Q}_n$ either $W$ is contained in $S$ or is disjoint from $S$.

At first the condition with $\mathcal{Q}_n$ may seem unnecessary. But if we dropped this condition then every subset of $T$ would define a subgame bush. That would create a problem if the game bush is followed by further play.

If the vertices $S$ define a subgame bush, let $R'$ be the roots of the subgame bush (the vertices in $S$ that are in $R$ or whose predecessor is not in $S$) and $T'$ is the subset of terminal vertices $T' = T \cap S$. The partitions $\mathcal{Q}'_n$ of $T'$ are defined by $\mathcal{Q}'_n = \{F \cap S \mid F \in \mathcal{Q}_n\}$ which by the definition of a subgame bush are equal to $\{F \mid F \in \mathcal{Q}_n, F \subseteq T'\}$. Likewise define $\mathcal{Q}'$ to be the meet partition $\wedge_{n \in \mathbb{N}} \mathcal{Q}'_n$. The partition $\mathcal{R}'_n$ of $R'$ is defined by $u, v$ belonging to the same member of $\mathcal{R}'_n$ if and only if $u$ and $v$ share the same last member of $\mathcal{P}_n$ in the paths to $u$ and $v$, and otherwise, if no such member of $\mathcal{P}_n$ exists, they share the same member of $\mathcal{R}_n$ in the paths to $u$ and $v$.

Given that additionally a game bundle $\Gamma$ is defined, meaning with correspondences $F_C$ for every $C \in \mathcal{Q}$, another game bundle is defined by the subgame bush defined by $S$ and the same correspondences $F_C$ for the sets $C \in \mathcal{Q}$, since $\mathcal{Q}'$ is just a subset of $\mathcal{Q}$. The resulting subgame game bundle we call $\Gamma|_S$, and we see that it is also a game bundle.

Notice that it is possible for a set $S$ defining a subgame bush to have a non-empty intersection with the roots $R$. When this happens, the roots $R'$ of the subgame bush have a non-empty intersection with $R$. Also it is possible for $R'$ to have a non-empty intersection with $T$ (and therefore also with $T'$).

For every $S$ that defines a subgame bush, we define also a factor game bush. Its roots $R''$ are the same as $R$ and its partitions $\mathcal{R}''_n$ are the same as $\mathcal{R}_n$. Its terminal points $T''$ are defined to be $T'' := (T \setminus S) \cup R'$, where $R'$ are the roots of the subgame bush defined by $S$. The partition $\mathcal{Q}''_n$ of $T''$ are defined by $\{A \mid A \in \mathcal{Q}_n, A \cap S = \emptyset\} \cup \mathcal{R}'_n$, (where $\mathcal{R}'_n$ is the partition on the roots of the subgame bush). Because $S$ is containing or disjoint from every member of $\mathcal{Q}_n$, the definition of $\mathcal{Q}''_n$ is straightforward. We define $\mathcal{Q}''$ to be $\wedge_{n \in \mathbb{N}} \mathcal{Q}''_n$.

**Definition:** A game bush has **perfect recall** for every player $n$ if for every $v \in W \in \mathcal{P}_n$ the member of $\mathcal{R}_n$ followed by the information sets in $\mathcal{P}_n$ and actions taken by player $n$ leading to $v \in W$ has no repetitions and this sequence is the same for every $v \in W$. The game bush has perfect recall if it
is has perfect recall for every player.

Next we would like to define a factor game bundle. If $C \in Q'' \cap Q$ is disjoint from $S$, we don’t change the payoff correspondence from the previous definition of $F_C$. But we need to know what should be correspondence $F_C$ when $C \in Q''$ is a subset of $R'$, the roots of the subgame bundle. We would like to say that $C$ also defines a subgame bush, but unfortunately we cannot do that without the assumption of perfect recall. It makes sense to define $F_C$ to be some subset of the m-equilibria of the subgame bundle restricted to $\Delta(C)$ as a subset of $\Delta(R')$.

If a set $S$ of vertices defines a subgame bundle then an m-equilibrium for some $q \in \Delta(R)$ is $S$-perfect if restricted to the subgame bundle defined by $S$ it defines an m-equilibrium for some distribution on the roots $R'$ of $S$ such that when there is positive probability of reaching $S$ that distribution is the conditional probability as implied by the strategies and the initial distribution $q$.

An m-equilibrium is subgame bundle perfect if it is $S$-perfect for all subsets $S$ of vertices that define subgame bundles.

Unless otherwise stated, for the factor game bundle $\Gamma/S$ we define the correspondence $F_C$ for every $C \in Q''$ with $C \subseteq R'$ to be the subgame bundle perfect m-equilibria of the subgame $\Gamma|_S$ as restricted to $\Delta(C)$. If there is some ambiguity concerning what should be $F_C$, we can denote it by $\Gamma^F/S$, where $F$ stands for alternative correspondences.

**Lemma 1.** If the sets $S$ and $T$ define subgame bushes then their union $S \cup T$ and their intersection $S \cap T$ also define subgame bushes.

**Proof:** It follows directly from the definition of subgame bushes as sets that are closed with respect to information sets, partitions, and consequences of actions. \qed

In general, it may be impossible for a player to combine her strategies from $\Gamma/S$ followed by $\Gamma|_S$ in the way desired, as this player may not possess the required memory to do this. If we assumed that these were two distinct players, one playing in the factor game and a different one playing in the subgame bundle, such a combination would not be problematic. But a player performing in both the factor game and the subgame may fail to be capable of performing the necessary mixed strategy in the subgame due to a lack of
memory of what happened in the factor game. To go further, we need the property of perfect recall.

**Lemma 2:** Let $\Gamma$ be a game bundle, $S$ a set defining the subgame bundle $\Gamma|_S$ of $G$, with $R'$ the roots of the subgame defined by $S$. Assume that the game bush of $\Gamma$ has perfect recall. Let $\phi$ be an $m$-equilibrium of the factor game $\Gamma/S$ and $q$ be its induced conditional probability on $R'$, the roots of $S$. The plan $\phi$ combined with any $m$-equilibria of $\Gamma|_S$ corresponding to the conditional probability $q$ of $R'$ (with $q$ equal to anything if the probability of reaching $S$ is zero) defines an $m$-equilibrium of $\Gamma$.

**Proof:** Let $S^n_1$ and $S^n_2$ be the decision functions for $\Gamma/S$ and $\Gamma_S$ respectively. Let $s_1$ be a member of $S^n_1$ and $P$ an information set of Player $n$ in the subgame bundle $\Gamma_S$ that is reached with positive probability. If $r$ is the expected payoff for Player $n$ from $s_2$ conditioned on reaching $P$ in the subgame bundle from the strategies used in the factor game $\Gamma/S$, we need to know that $r$ is also the expected payoff from the combination of $s_1$ with $s_2$ conditioned on reaching $P$. This follows from the property of perfect recall, because the distribution on $P$ from using $s_1$ is not any different from the distribution on $P$ used to calculate the expected payoff $r$ in the subgame bundle.

**Theorem 6:** If there is perfect recall, the correspondence of subgame bundle perfect $m$-equilibria has the spanning property, meaning that for every initial probability distribution on the roots there is a subgame bundle perfect $m$-equilibrium.

**Proof:** The proof is by induction on the partially ordered structure of subgame bundles. If there are no subgame bundles, it is true by Theorem 1. Otherwise let $S$ be any subset of vertices defining a subgame. By induction we assume that the correspondence of subgame bundle perfect $m$-equilibria of both $G|_S$ and $G/S$ have the spanning property. Let $U$ be a set of vertices defining a subgame. If $U$ is contained in $S$, we have already assumed the conclusion. If $U$ is not contained in $S$, then by Lemma 1 $G|U$ is a combination of $G|_{U/S}$ and $G|_{U\cap S}$. By Lemma 2 and Theorem 2 the subgame perfect $m$-equilibria of $G|_U$ has the spanning property. \(\square\)
5 Further problems

The main advantage of mixed strategies over behaviour strategies is that one can define the relationship between strategies and payoffs in a straightforward way, through multi-dimensional matrices. However the dimensions of the strategy spaces can become very large, and we would like to reduce these dimensions.

Definition: An extensive form game is solvable if there is a sequence $\emptyset = S_0 \subseteq \cdots \subseteq S_n = V$ such that for each game $G|_{S_i}/S_{i-1}$ no player is called upon to make a choice of action twice (meaning that there are no two $P_1, P_2 \in P_n$ such that $P_2$ follows from $P_1$ in some path).

In a solvable game, in each factor games $G|_{S_i}/S_{i-1}$ there is no distinction between behaviour strategies and pure strategies. Therefore one can understand equilibria of the game entirely through behaviour strategies using the above theorems. The question however is whether there is any computational advantage won through this perspective. So far, the only clear advantage pertains to standard games of incomplete information on one side where a finite game is followed by an infinitely repeated one, as is done in Simon, Spież, and Toruńczyk (2020). There we proved that there exists an equilibrium and if the payoffs from both games come from the same matrices and the payoff is a combination of the discounted and undiscounted, we showed that $\varepsilon$-equilibria exist for every positive $\varepsilon$. With the above Theorem 5 one could hope to improve this result to the existence of an equilibrium.

The following examples demonstrate some of the complexity of what is a subgame and what constitutes subgame perfection.

Example 2:

There are two players. Player One has the choice between $A$ for aggression and $P$ for passivity. If $P$ is chosen the game moves to State $X$ and the payoffs are 1 for Player One and 2 for Player Two. If $A$ was chosen the game moves to State $Y$ and Player Two has the choice between $a$ for aggression and $p$ for passivity. If $a$ is chosen the payoff is 0 for both players and if $p$ is chosen the payoff is 2 for Player One and 1 for Player Two.

There is an equilibrium where Player Two chooses $a$ with certainty and Player One choose $P$ with certainty. Player One is afraid of getting 0, hence willing to choose $P$. And because Player One chooses $P$ with certainty, the “foolish-
ness” of Player Two choosing $a$ never upsets the equilibrium property. This is the model of “mutually assured destruction”, that a player promises to harm all players in the game and that threat works to keep that player from ever confronting the situation where that threat must be carried out. On the other hand the state $Y$ defines a subgame whose only equilibrium involves Player Two choosing $p$, hence also Player One choosing $A$, defining a second and very different equilibrium. We recognize the first equilibrium does not induce an equilibrium of the subgame defined by the state $Y$, but the second equilibrium does. The first equilibrium is subgame perfect, the second is not.

But before we discarded the second equilibrium as irrelevant, notice that it gives a superior payoff to Player One, the player who is performing the interior action in the subgame.

**Example 3:** Now we alter this game slightly so that Player Two cannot distinguish between State $X$ and State $Y$ and has the actions $a$ and $p$ also in State $X$ and the payoff consequences of both $a$ and $p$ from State $X$ are the same as before, a payoff of 2 to Player One and 1 to Player Two. Conventionally, this game has no subgames, so every equilibrium is subgame perfect. In some way it is strategically equivalent to the previous game, as it can be represented in normal form by the same matrix

$$
\begin{array}{c|cc}
& p & a \\
\hline
P & (1,2) & (1,2) \\
A & (2,1) & (0,0) \\
\end{array}
$$

We perceive $X$ and $Y$ together as a subset defining a subgame bundle, with a certainty that the play will reach this subgame bundle. Both equilibria discovered in the previous example, $A$ with $p$ and $P$ with $a$, induce equilibria of the subgame bundle defined by the set $\{X,Y\}$. We introduced a new collection of subgames, and yet the $A$ and $p$ combination cannot be eliminated as a subgame bundle perfect equilibrium. This shows that the subgame bundle perfect concept is not determined exclusively by the normal, or matrix, form of the game.

One would like a concept of subgame bundle perfection that incorporates the most likely probability distribution on a subset $C \in Q$ given that the probability for $C$ is zero. With probability theory, this question motivates the concept of a random variable representing conditional probability with the conditioned set has zero probability. However with games, especially if
the game tree is finite, this question cannot be divorced from the related question of who deviated to bring the play to this forbidden subset and why. Unfortunately hitherto there is no good general answer to this question.

6 References

Dold, A. (1972), *Lectures on Algebraic Topology*, Springer Verlag.

Kohlberg, E. and Mertens, J.-F. (1986), On the Strategic Stability of Equilibria, *Econometrica*, 54 (5), pp. 1003 –1037.

Kirby, R. C. and Siebenmann, L. C. (1977), *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Mathematics Studies 88 (Princeton University Press)

Kuhn, H. (1953), Extensive Games and the Problem of Information, in *Contributions to the Theory of Games I*, Princeton University Press, eds. Kuhn and Tucker, pp. 193-216.

Simon, R.S., Spież, S., Toruńczyk, H. (2002), Equilibrium Existence and Topology in Games of Incomplete Information on One Side, *Transactions of the American Mathematical Society*, Vol. 354, No. 12, pp. 5005-5026.

Simon, R.S., Spież, S., Toruńczyk, H. (2020), Games of Incomplete Information and Myopic Equilibria, *Israel Journal of Mathematics*, to appear.