LOCAL LARGE DEVIATIONS: MCMILLIAN THEOREM FOR MULTITYPE GALTON-WATSON PROCESSES

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Abstract. In this article we prove a local large deviation principle (LLDP) for the critical multitype Galton-Watson process from spectral potential point. We define the so-called a spectral potential $U_K(\cdot, \pi)$ for the Galton-Watson process, where $\pi$ is the normalized eigen vector corresponding to the leading Perron-Frobenius eigenvalue $1$ of the transition matrix $A(\cdot, \cdot)$ defined from $K$, the transition kernel. We show that the Kullback action or the deviation function, $J(\pi, \rho)$, with respect to an empirical offspring measure, $\rho$, is the Legendre dual of $U_K(\cdot, \pi)$. From the LLDP we deduce a conditional large deviation principle and a weak variant of the classical McMillian Theorem for the multitype Galton-Watson process. To be specific, given any empirical offspring measure $\pi$, we show that the number of critical multitype Galton-Watson processes on $n$ vertices is approximately $e^{n(H_\pi, \pi)}$, where $H_\pi$ is a suitably defined entropy.

1. Background

1.1 Introduction

We revisit the typed trees model described by the following procedure: The root carries a random type chosen according to some law on a finite alphabet; given the type of a vertex, the number and types of the children (ordered from left to right) are given, independently of everything else, by an offspring law. We shall refer to the tree together with the types on the tree as a multitype Galton-Watson Process.

Presently there exists some Large deviation Principle(LDP) and Basic Information Theory for the multitype Galton-Watson Process. See, example Dembo et al. [5], Doku-Amponsah [4], Doku-Amponsah [3], Doku-Amponsah [2]. In [5], LDPs were proved for empirical measures of the multitype Galton-Watson trees with the exponential moments of offspring transition kernel being finite, in a topology stronger than weak topology. See [5, pp. 6]. [3] proved an Asymptotic Equipatition Property for hierarchical Data Structures, using [5, Theorem 2.2], for bounded offspring transition kernels. Recently, Doku-Amponsah [2] found a Lossy version of the result [2, Theorem 2.1]. The article [4] extended the LDP result of [5, Theorem 2.2] for the multitype Galton-Watson Process to cover offspring laws (e.g. geometric $\frac{1}{2}$), which were not covered by [5], in the weak topology.

In this article we prove that the LLDP for the multitype Galton-Watson process conditioned on the empirical offspring measure of the process. See Bakhtin [1, Theorem 3.1] for similar results for the
empirical measure of iid random variables. The current article, like the paper by Baktain [1], differs from most of the aforementioned articles, by the lack of any topology on the range of random variables and by the use of the weak topology on the set of measures generated by bounded measurable functions. The main technique used in this article is spectral potential theory. See [1]. To be specific about this approach, we define a spectral potential of the multitype Galton-Watson trees and use it to define an extension of the relative entropy, which we also call the Kullback action. The Kullback action has the rate function of the LDPs in [4] and [5] as special cases. By a proper exponential change of measure, using the Kullback action we prove the LLDP, and from the LLDP we obtain the classical McMillan-Breiman Theorem as a particular case. The conditional LDP is also derived from the LLDP.

1.2 The Multitype Galton-Watson Process. Let us adopt some concepts from the paper Dembo et al. [5]. We shall denote by $GW$ the set of all finite rooted planar trees $T$, by $\mathcal{V} = \mathcal{V}(T)$ the set of all vertices and by $\mathcal{E} = \mathcal{E}(T)$ the set of all edges oriented away from the root, which is always denoted by $\eta$. We write $|\mathcal{V}(T)|$ for the number of vertices in the tree $T$. Let $\mathcal{Y}$ be a finite alphabet with a chosen $\sigma-$ field $\mathcal{F}$ of subsets and write
\[ \mathcal{Y}^* = \bigcup_{n=0}^{\infty} \{n\} \times \mathcal{Y}^n. \]

Note that the child of any vertex $v \in \mathcal{V}(T)$ is characterized by an element of $\mathcal{Y}^*$ and that there is an element $(0, \emptyset)$ in $\mathcal{Y}^*$ symbolizing the absence of a child.

Let $\beta$ be a probability distribution on $\mathcal{Y}$ and $\mathcal{K} : \mathcal{Y} \times \mathcal{Y}^* \rightarrow [0, 1]$ be an offspring transition kernel. We describe the law $\mathbb{P}$ of a multitype Galton-Watson process $Y$, see Mode [7], by the following procedure:
- The root $\eta$ carries a random type $Y(\eta)$ chosen according to the probability measure $\beta$ on $\mathcal{Y}$.
- For every vertex with type $a \in \mathcal{Y}$ the offspring number and types are given independently of everything else, by the offspring law $\mathcal{K}\{\cdot \mid a\}$ on $\mathcal{Y}^*$. We write
\[ \mathcal{K}\{\cdot \mid a\} = \mathcal{K}\{(N, Y_1, \ldots, Y_N) \in \cdot \mid a\}, \]
i.e. we have a random number $N$ of descendants with types $Y_1, \ldots, Y_N$.

We shall consider $Y = ((Y(v), C(v)), v \in \mathcal{V})$ under the joint law of tree and offspring. Interpret $Y$ as a multitype Galton-Watson process and $Y(v)$ as the type of vertex $v$. For each typed tree $Y$ and each vertex $v$ we denote by $C(v) = (N(v), Y_1(v), \ldots, Y_{N(v)}(v)) \in \mathcal{Y}^*$, the number and types of the offsprings of $v$, ordered from left to right. Notice that the children of the root (denoted by $\eta$) are ordered, but the root itself is not. Denote, for every $c = (n(a), a_1(c), \ldots, a_n(c)) \in \mathcal{Y}^*$ and $a \in \mathcal{Y}$, the multiplicity of the symbol $a$ in $c$ by
\[ m(a, c) = \sum_{i=1}^{n(c)} 1_{\{a_i = a\}}. \]

Define the matrix $A$ with index set $\mathcal{Y} \times \mathcal{Y}$ and nonnegative entries by
\[ A(a, b) = \sum_{c \in \mathcal{Y}^*} \mathcal{K}\{c \mid b\} m(a, c), \text{ for } a, b \in \mathcal{Y}. \]

$A(a, b)$ is the expected number of offspring of type $a$ of a vertex of type $b$. Let $A^*(a, b) = \sum_{k=1}^{\infty} A^k(a, b) \in [0, \infty]$. We say that the matrix $A$ is irreducible if $A^*(a, b) > 0$; for all $a, b \in \mathcal{Y}$.

The multitype Galton-Watson Process is called irreducible if the matrix $A$ is irreducible. It is called critical (subcritical, supercritical) if the largest eigenvalue of the matrix $A$ is 1 (less than 1, greater than 1 resp.). Let $\pi$ be the eigenvector corresponding to the largest Perron-Frobenius eigenvalue $1$.
For every multitype Galton-Watson tree $Y$, the empirical offspring measure $M_Y$ is defined by

$$M_Y(a, c) = \frac{1}{|T|} \sum_{v \in V} \delta_{(Y(v), C(v))}(a, c), \quad \text{for } (a, c) \in \mathcal{Y} \times \mathcal{Y}^*.$$ 

We call $\rho$ (normalized to a probability vector). Then, by the Perron-Frobenius Theorem, $\pi$ is unique, if the Galton-Watson tree is irreducible. See Dembo et al. [6, Theorem 3.1.1].

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We call $\rho$ shift-invariant if $\rho_1(a) = \sum_{(b, c) \in \mathcal{Y} \times \mathcal{Y}^*} m(a, c)\rho(b, c)$, for all $a \in \mathcal{Y} \times \mathcal{Y}^*$.

We denote by $\mathcal{M}(\mathcal{Y} \times \mathcal{Y}^*)$ the space of probability measures $\rho$ on $\mathcal{Y} \times \mathcal{Y}^*$ with $\int n \rho(da, dc) < \infty$, using the convention $c = (n, a_1, \ldots, a_n)$. Denote by $\mathcal{M}_s(\mathcal{Y} \times \mathcal{Y}^*)$ the space of positive measures on $\rho$ on $\mathcal{Y} \times \mathcal{Y}^*$ with $\int n \rho(da, dc) < \infty$, by $\mathcal{B}(\mathcal{Y} \times \mathcal{Y}^*)$ the space of all real-valued bounded measurable functions on $\mathcal{Y} \times \mathcal{Y}^*$, by $\mathcal{B}_+(\mathcal{Y} \times \mathcal{Y}^*)$ the space of continuous linear functionals on $\mathcal{B}(\mathcal{Y} \times \mathcal{Y}^*)$ and by $\mathcal{B}_+(\mathcal{Y} \times \mathcal{Y}^*)$ the collection of all positive linear functionals on $\mathcal{B}(\mathcal{Y} \times \mathcal{Y}^*)$.

The remaining part of the article is organized in the following manner: Section 2 contains the main results of the article: Theorem 2.1, Corollary 2.2 and Theorem 2.3. In Section 3 these results are proved.

## 2. Statement of main results

Write $\langle f, \sigma \rangle := \sum_{y \in \mathcal{Y}} \sigma(y)f(y)$ and define the spectral potential $U_K(g, \pi)$ of the multitype Galton-Watson process $Y$ by

$$U_K(g, \pi) = \log \langle e^g, \pi \otimes K \rangle.$$ 

(2.1)

Observe that (2.1) possesses all the remarkable properties mentioned in [1, pp.536-538], i.e. (i) It is finite on $\mathcal{D}(\pi) := \{g : \mathcal{Y} \times \mathcal{Y}_s \to \mathbb{R} \mid (e^g, \pi \otimes K) < \infty\}$ (ii) It is monotone (iii) it is additively homogeneous and it is convex in $g$. For $\rho \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}_s)$ we call a nonlinear functional

$$J(\pi, \rho) := \begin{cases} H(\rho \parallel \pi \otimes K) & \text{if } \rho \text{ is shift-invariant and } \rho_1 = \pi, \\ +\infty & \text{otherwise}. \end{cases}$$ 

(2.2)

Kullback action.

We denote by $P_n(y) = \mathbb{P}_n\{Y = y\} = \mathbb{P}\{Y = y \mid |\mathcal{V}| = n\}$ the distribution of the multitype Galton-Watson tree $x$ conditioned to have $n$ vertices. In Theorem 2.1 below we state our main result, the LLD for the multitype Galton-Watson tree.

**Theorem 2.1 (LLD).** Suppose $y = (y(v) : v \in \mathcal{V})$ is an irreducible, critical multitype Galton-Watson tree on $n$ vertices, with finite type space $\mathcal{Y}$ and an offspring kernel $K$. Then,

(i) for any functional $\rho \in \mathcal{M}_s(\mathcal{Y} \times \mathcal{Y}_s)$ and a number $\varepsilon > 0$, there exists a weak neighborhood $B_{\rho}$ such that

$$P_n\{y \in \text{GW} \mid M_y \in B_{\rho}\} \leq e^{-nJ(\pi, \rho) - n\varepsilon + o(n)},$$

(ii) for any $\rho \in \mathcal{M}_s(\mathcal{Y} \times \mathcal{Y}_s)$, a number $\varepsilon > 0$ and a fine neighborhood $B_{\rho}$ we have the asymptotic estimate:

$$P_n\{y \in \text{GW} \mid M_y \in B_{\rho}\} \geq e^{-nJ(\pi, \rho) + n\varepsilon - o(n)},$$
Next we state the McMillian-Breiman Theorem for the multitype Galton-Watson tree. For any \( a \in \mathcal{Y} \) and the conditional probability measure \( \pi(c \mid a) \) on \( \mathcal{Y}_0^* \) we define an entropy by

\[
H_{\pi}(a) := - \sum_{c \in \mathcal{Y}_0^*} \pi(c \mid a) \log \pi(c \mid a).
\]

We write \( \mathcal{Y}_0^* = N_0 \bigcup \bigcup_{n=0}^{\infty} \{n\} \times \mathcal{Y}^n \) and state a corollary of Theorem 2.1 below:

**Corollary 2.2 (McMillian Theorem).** Suppose \( \mathcal{Y}_0 \) is the space of all irreducible, critical multitype Galton-Watson trees with finite type space \( \mathcal{Y} \) and an offspring kernel supported on \( \mathcal{Y}_0^* \).

(i) For any empirical offspring measure \( \pi \) on \( \mathcal{Y} \times \mathcal{Y}_0^* \) and \( \varepsilon > 0 \), there exists a neighborhood \( B_{\pi} \) such that

\[
\text{Card} \left( \left\{ y \in \mathcal{Y}_0 \mid M_y \in B_{\pi} \right\} \right) \geq e^{n(H_{\pi}, \pi) + \varepsilon}.
\]

(ii) for any neighborhood \( B_{\pi} \) and \( \varepsilon > 0 \), we have

\[
\text{Card} \left( \left\{ y \in \mathcal{Y}_0 \mid M_y \in B_{\pi} \right\} \right) \leq e^{n(H_{\pi}, \pi) - \varepsilon},
\]

where \( \text{Card}(A) \) means the cardinality of \( A \).

**Remark 1** Note that if the transition kernel of the multitype Galton-Watson process is bounded. Thus, there exists \( N_0 \in \mathbb{N} \) such that \( K\{N > N_0 \mid a\} = 0 \), for all \( a \in \mathcal{Y} \), then, we could recover the results of [3, Theorem 2.1] by setting \( \pi(c \mid a) = K\{c \mid a\} \), for \( a \in \mathcal{X} \). Thus, we have

\[
\text{Card} \left( \left\{ y \in \mathcal{Y}_0 \right\} \right) \approx e^{n(H_{K}, \pi)}.
\]

Finally, we state in Theorem 2.3 the full LDP for the multitype Galton-Watson tree.

**Theorem 2.3 (LDP).** Suppose \( y = (y(v) : v \in \mathcal{Y}) \) is an irreducible, critical multitype Galton-Watson tree with finite type space \( \mathcal{Y} \) and offspring kernel \( K \).

(i) Let \( F \) be an open subset of \( \mathcal{M}(\mathcal{Y} \times \mathcal{Y}^*) \). Then we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n \left\{ y \in \mathcal{G} \mathcal{M} \mid M_y \in F \right\} \geq - \inf_{\rho \in F} J(\pi, \rho).
\]

(ii) Let \( \Gamma \) be a closed subset of \( \mathcal{M}(\mathcal{Y} \times \mathcal{Y}^*) \). The we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n \left\{ y \in \mathcal{G} \mathcal{M} \mid M_y \in \Gamma \right\} \leq - \inf_{\rho \in \Gamma} J(\pi, \rho).
\]

3. **Proof Main of Results**

3.1 **Properties of the Kullback action.** Next, we state a key ingredient (Lemma 3.1) in the proof of our main result, Theorem 2.1. This Lemma gives remarkable properties of 2.2 above, which will help us circumvent the topological problems faced in [1] and [3]. To state the lemma, we denote by \( C \) the space of continuous functions \( g : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R} \) and notice that, the proof below follows similar ideas as the proof of [1, Lemma 2.2] for the empirical measures on measurable spaces.
Lemma 3.1. The following holds for the Kullback action or divergence function $J(\pi, \rho)$.

(i) $J(\pi, \rho) = \sup_{g \in C} \{ \langle g, \rho \rangle - U_K(g, \pi) \}$.

(ii) The function $J(\pi, \cdot)$ is convex and lower semi-continuous on the space $\mathcal{B}_*(\mathcal{Y} \times \mathcal{Y}_*)$.

(iii) For any real $c$, the set $\{ \rho \in \mathcal{B}_*(\mathcal{Y} \times \mathcal{Y}_*) : J(\pi, \rho) \leq c \}$ is weakly compact.

Proof. (i) Let $g \in C$ be such that $\langle g, \rho \rangle$ approximates the functional $\langle \phi, \rho \rangle$ and $U_K(g, \pi)$ approximates $U_K(\phi, \pi)$ where $\phi \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}_*)$. Suppose $\rho$ is absolutely continuous with respect to $\pi \otimes K$. Thus, there exists $g$ such that $\rho = g\pi \otimes K$. For $t > 0$, we define the approximating function $g_t \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}_*)$ as follows

$$g_t(a, c) := \begin{cases} \log g(a, c), & \text{if } e^{-t} < g(a, c) < e^t, \\ t, & \text{if } g(a, c) > e^t \\ -t, & \text{if } g(a, c) < e^{-t} \end{cases} \quad (3.1)$$

for all $(a, c) \in \mathcal{Y} \times \mathcal{Y}_*$. Now, for $t \to \infty$ we have

$$\langle e^{g_t}, \pi \otimes K \rangle = \int g_t \mathbb{1}_{\{e^{-t} < g < e^t\}} \pi \otimes K(da, dc) + \int e^t \mathbb{1}_{\{g > e^t\}} \pi \otimes K(da, dc)$$

$$+ \int e^{-t} \mathbb{1}_{\{e^{-t} > g\}} \pi \otimes K(da, dc) \to \langle g, \pi \otimes K \rangle = (1, \rho) = 1$$

$$\langle g_t, \pi \otimes K \rangle = \int g \log g \mathbb{1}_{\{e^{-t} < g < e^t\}} \pi \otimes K(da, dc)$$

$$+ \int t \mathbb{1}_{\{g > e^t\}} \pi \otimes K(da, dc) + \int -t \mathbb{1}_{\{e^{-t} > g\}} \pi \otimes K(da, dc) \to \langle g \log g, \pi \otimes K \rangle = (\log g, \rho).$$

Therefore we have $\lim_{t \to \infty} (\langle g_t, \pi \otimes K \rangle - \langle e^{g_t}, \pi \otimes K \rangle) \to J(\pi, \rho)$ which proves Lemma 3.1(i).

Suppose $\rho$ is not absolutely continuous with respect to $\pi \otimes K$. Thus, there exists an $\varepsilon > 0$ such that for any $\eta > 0$ there exists $B_\delta \subset \mathcal{Y} \times \mathcal{Y}_*$ with $\pi \otimes K(B_\delta) \leq \delta$, and at the same time we have $\rho(B_\delta) > \varepsilon$. For this $\delta$ we define the function

$$g_\delta(a, c) := \begin{cases} -\log \delta, & \text{if } (a, c) \in B_\delta, \\ 0, & \text{if } (a, c) \notin B_\delta. \end{cases} \quad (3.2)$$

Then we have $\lim_{t \to \infty} (\langle g_\delta, \pi \otimes K \rangle - \langle e^{g_\delta}, \pi \otimes K \rangle) \geq -\varepsilon \log \delta - \langle e^{g_\delta}, \pi \otimes K \rangle \geq -\varepsilon \log \delta - \log(2)$.

Taking limit as $\delta \downarrow 0$ we have $J(\pi, \rho) = +\infty$, which ends the proof of Lemma 3.1(i).

(ii)& (iii). Observe from the variational formulation of the relative entropy, see Dembo et al. [6], and Lemma 3.1(i) that $J(\pi, \rho)$ reduces to equation (2.2) above. Now the relative entropy is convex and lower semi-continuous, and all its level sets are compact. Hence we have $J(\pi, \rho)$ convex and lower semi-continuous, and all its level sets are weakly compact in the weak topology, which ends the proof of the Lemma.

Note that Lemma 3.1(i) above implies the so-called variational principle. See, example Komkov [8].
3.2 Proof of Theorem 2.1. By Lemma 3.1 for any $\varepsilon > 0$ there exists a function $g \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}_*)$ such that

$$J(\pi, \rho) - \frac{\varepsilon}{2} < \langle g, \rho \rangle - U_K(g, \pi).$$

Let us define the probability distribution $\tilde{P}$ by

$$\tilde{P}_n(y) = \frac{\tilde{P}\{y, |\mathcal{V}| = n\}}{\tilde{P}\{|\mathcal{V}| = n\}} = \pi(y) \prod_{v \in \mathcal{V}, |\mathcal{V}| = n} e^{g(y(v), c(v)) - U_K(g, \pi) K\{c(v)|y(v)\}/\tilde{P}\{|\mathcal{V}| = n\}}.$$

Then, it is not too hard to observe that we have

$$\frac{dP_n(y)}{d\tilde{P}_n(y)} = \prod_{v \in \mathcal{V}, |\mathcal{V}| = n} e^{-g(y(v), c(v)) + U_K(g, \pi)} = e^{-\sum_{v \in \mathcal{V}, |\mathcal{V}| = n} g(y(v), c(v)) + nU_K(g, \pi) \tilde{P}\{|\mathcal{V}| = n\}/\tilde{P}\{|\mathcal{V}| = n\}}.$$

Now we define a neighbourhood of the functional $\rho$ as follows:

$$B_\rho = \left\{ \omega \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}_*) : \langle g, \omega \rangle > \langle g, \rho \rangle - \frac{\varepsilon}{2} \right\}.$$  

Using [5, Lemma 3.1], under the condition $M_y \in B_\rho$ we have that

$$\frac{dP_n(y)}{d\tilde{P}_n(y)} < e^{nU_K(g, \pi) - n(\rho) + n\varepsilon/2 + o(n)} < e^{-nJ(\pi, \rho) + n\varepsilon + o(n)}.$$

Hence, we have

$$P_n\left\{ y \in \mathcal{T} | M_y \in B_\rho \right\} \leq \int_{\{M_y \in B_\rho\}} d\tilde{P}_n(y) \leq \int_{\{M_y \in B_\rho\}} e^{-nJ(\pi, \rho) - n\varepsilon + o(n)} d\tilde{P}_n(y) \leq e^{-nJ(\pi, \rho) - n\varepsilon + o(n)}.$$

Note that $J(\pi, \rho) = \infty$ implies Theorem 2.1(ii), so it is sufficient for us to prove it for a probability measure of the form $\rho = g\pi \otimes K$ and for $J(\pi, \rho) = \langle log g, \rho \rangle < \infty$. Fix any number $\varepsilon > 0$ and any neighbourhood $B_\rho \subset \mathcal{M}(\mathcal{Y} \times \mathcal{Y}_*)$. We define the sequence of sets

$$\mathcal{T}_n := \left\{ y \in \mathcal{T} : M_y \in B_\rho, \left| \int \log gM_y - \int \log g \rho \right| \leq \frac{\varepsilon}{2} \right\}.$$  

Using [5, Lemma 3.1], we observe that for all $x \in \mathcal{T}_n$, we have

$$\frac{dP_n(y)}{d\tilde{P}_n(y)} = \prod_{v \in \mathcal{V}} \frac{1}{g(y(v), c(v))} \frac{\tilde{P}\{|\mathcal{V}| = n\}}{\tilde{P}\{|\mathcal{V}| = n\}} = e^{-n(\log g, M_y) + o(n)} > e^{-n(\log g, \rho) + n\varepsilon/2 + o(n)}.$$

This gives us

$$P_n(\mathcal{T}_n) = \int_{\mathcal{T}_n} dP_n(x) \geq \int_{\mathcal{T}_n} e^{-n(\log g, \rho) + \varepsilon/2 - o(n)} d\tilde{P}_n(x) = e^{-nJ(\pi, \rho) + n\varepsilon - o(n)} \tilde{P}_n(\mathcal{T}_n).$$

Using the law of large numbers we have $\lim_{n \to \infty} \tilde{P}_n(\mathcal{T}_n) = 1$ which completes the proof of the Theorem.

3.3 Proof of Corollary 2.2. The proof of Corollary 2.2 follows from the definition of the Kullback action and Theorem 2.1 if we set $\rho = \omega$ and $K\{c|a\} = 1$ for all $c = (n, a_1, a_2, a_3, ..., a_n) \in \mathcal{Y}_0^n$. 
The proof of Theorem 2.3 below follows from Theorem 2.1 above using similar arguments as in [1, p. 544].

3.4 Proof of Theorem 2.3

Proof. Note that the empirical offspring measure is a probability measure and so belongs to the unit ball in $B_*(\mathcal{Y} \times \mathcal{Y}^*)$. Hence, without loss of generality we may assume that the set $G$ in Theorem 2.3(ii) is relatively compact. See Lemma 3.1 (iii). Choose any $\varepsilon > 0$. Then for every functional $\rho \in \Gamma$ we can find a weak neighbourhood such that the estimate of Theorem 2.1(i) holds. We choose from all these neighbourhood a finite cover of $GM$ and sum up over the estimate in Theorem 2.1(i) to obtain

$$ \lim_{n \to \infty} \frac{1}{n} \log P_n \left\{ y \in GW \mid M_y \in \Gamma \right\} \leq - \inf_{\rho \in \Gamma} J(\pi, \rho) + \varepsilon. $$

As $\varepsilon$ was arbitrarily chosen and the lower bound in Theorem 2.1(ii) implies the lower bound in Theorem 2.3(i), we have the desired results, which ends the proof of the Theorem.

\[ \square \]

4. Conclusion

In this article we have found an LLDP for the multitype Galton-Watson process without any topological restrictions on the space of probability measures on $\mathcal{Y} \times \mathcal{Y}^*$, from a spectral potential point. From the LLDP we deduce other results such as the classical McMillian Theorem and the full conditional large deviation principle for this process. The main technique used is exponential change of measure. These results have thrown more insight on the results of [3] and [4].

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