ON THE LARGEST LYAPUNOV EXPONENT FOR PRODUCTS OF GAUSSIAN MATRICES

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Abstract
The paper provides a new formula for the largest Lyapunov exponent of Gaussian matrices, which is valid in the real, complex and quaternion-valued cases. This formula is applied to derive asymptotic expressions for the largest Lyapunov exponent when the size of the matrix is large. In addition, the paper gives new exact formulas for the largest Lyapunov exponents of 3-by-3 real and $d$-by-$d$ quaternion Gaussian matrices.

1. INTRODUCTION

Lyapunov exponents are useful tools that measure sensitivity of a dynamical system with respect to initial conditions. Let $f : X \to X$ be a differentiable map of a manifold $X$ to itself. The dependence on small perturbations in the initial conditions can be measured by the growth of matrix products $P_n = A_n A_{n-1} \ldots A_1$, where $A_k := f'(x_k)$ and $x_k = f(x_{k-1})$. For example, the system can be thought of as “chaotic” if at a typical initial position $x_1$, the products $P_n$ grow exponentially in one direction and contract exponentially in another one.

In order to quantify what is meant by a typical initial position, the manifold $X$ is usually endowed with a probability measure. Then, $A_k$ are random matrices and therefore we are led to the study of random matrix products.

In a seminal paper, Furstenberg and Kesten in [9] showed that $\lim_{n \to \infty} n^{-1} \log \| P_n v_0 \|$ exists under some mild conditions on random matrices $A_k$. This limit does not depend on $v_0 \neq 0$, and it is called the largest Lyapunov exponent. In order to define other Lyapunov exponents, assume that $A_i$ is a sequence of independent, identically distributed random $d$-by-$d$ matrices such that the diagonal elements of $A_i^* A_i$ have finite moments. Then the limit of matrices $(P_n^* P_n)^{1/2n}$ is well defined and has $d$ non-negative real eigenvalues (Oseledec [23], Raghunathan [26]). The logarithms of these eigenvalues are called the Lyapunov exponents of matrices $A_i$. We denote them $\mu_1, \ldots, \mu_d$.

A lot of research went into the study of properties of the Lyapunov exponents, such as their analyticity with respect to small matrix perturbations and with respect to changes in the matrix random process ([27], [14], [24]), the multiplicity of the largest Lyapunov exponent ([10], and
Lyapunov exponents have been generalized to infinite-dimensional operators ([28], [12]), and found important applications in the study of hydrodynamic stability ([29]) and random Schroedinger operators ([3]).

Unfortunately, typically it is not easy to calculate the Lyapunov exponents except by directly simulating the matrix products. There is only a limited number of cases for which an explicit formula is available ([5], [16]), and in some cases it is possible to compute the exponents accurately even if one does not have an explicit formula ([25]).

Recently, Forrester found a formula for the Lyapunov exponents of complex Gaussian matrices with general covariance matrix \( \Sigma \) (see [7]). His method is based on the Harish-Chandra-Itzykson-Zuber integration formula and cannot be applied to the case when matrices are real or quaternion-valued.

In this paper we derive a new formula for the largest Lyapunov exponent, which is applicable not only in the complex-valued case but also for real and quaternion-valued Gaussian matrices. The formula provides an easy way to compute the largest Lyapunov exponent for this class of matrices.

As an application of this formula, we derive asymptotic expressions for the largest Lyapunov exponents of high-dimensional matrices. First, we consider the case when the covariance matrix \( \Sigma \) has a “spike”, that is, when all eigenvalues of \( \Sigma \) except one, equal 1, and the exceptional eigenvalue equals \( \theta > 1 \). We study the asymptotic behavior of the largest Lyapunov exponent when the dimension \( d \) is large and \( \theta / d \to x > 0 \). Next, we consider the case when \( d \) is large and \( \Sigma \) does not have spikes. We find that in this case the largest Lyapunov exponent can be approximated well by a formula from free probability theory.

Another consequence of our main formula is an explicit expression for the largest Lyapunov exponent of a 3-by-3 real Gaussian matrix in terms of standard elliptic functions. Previously, a similar explicit expression was known only in the case of 2-by-2 matrices ([15]).

Finally, we investigate the Lyapunov exponents of quaternion Gaussian matrices. Here we give an alternative argument for the basic determinantal formula for the Lyapunov exponents. (The usual argument is deficient since it relies on the calculus of exterior forms which is not available in the quaternion case.) Then, we derive formulas for all Lyapunov exponents in the case when \( \Sigma = I \) and a formula for the largest Lyapunov exponents in the case of the general \( \Sigma \).

The rest of the paper is organized as follows. Section 2 collects necessary background information about Lyapunov exponents. Section 3 formulates and proves the main formula. Section 4 derives an asymptotic expression for the largest Lyapunov exponent when the covariance matrix \( \Sigma \) has a spike. Section 5 considers the spike-free case and compares our results with a result from free probability theory. Section 6 is about explicit formulas for real Gaussian matrices. Section 7 is about quaternion Gaussian matrices. And Section 8 concludes.
2. Background Information about Lyapunov Exponents

The key fact about Lyapunov exponents is that they satisfy the following relation:

\[ \mu_1 + \ldots + \mu_k = \sup_{n \to \infty} \lim_{n} \frac{1}{n} \log \text{Vol}_k (y_1(n), \ldots, y_k(n)) \]  

where \( y_i(n) = P_n y_i(0) \) and the supremum is over all choices of linearly independent vectors \( y_i(0) \). It can be proved that the supremum is in fact not needed in this formula. In words, the sum of \( k \) largest Lyapunov exponents measures the average growth rate in the volume of a \( k \)-dimensional element when we apply linear transformations specified by matrices \( A_i \).

If these matrices are independent and Gaussian, then this formula can be significantly simplified. Namely, let \( G(i) \) be independent random \( d \)-by-\( d \) matrices whose entries are independent (real) standard Gaussian entries, and \( \Sigma^{1/2} \) is a (real) positive-definite \( d \)-by-\( d \) matrix. Let \( A_i = \Sigma^{1/2} G(i) \). We will call these matrices \textit{real Gaussian matrices with covariance matrix} \( \Sigma \).

The crucial observation is that the distribution of \( A_i^* A_i \) is invariant relative to the transformation

\[ A_i^* A_i \rightarrow Q^* A_i^* A_i Q, \]

where \( Q \) is an arbitrary orthogonal matrix. This implies that the changes in the volume of a \( k \)-dimensional element are independent from step to step and that their distribution is the same as if they were applied to the element spanned by the standard basis vectors \( e_i \),

\[ \mu_1 + \ldots + \mu_k = \mathbb{E} \log \text{Vol}_k (A_1 e_1, \ldots, A_1 e_k) = \frac{1}{2} \mathbb{E} \log \det (G_k^* \Sigma G_k), \]  

where \( G_k \) denotes a random \( d \)-by-\( k \) matrix with the identically distributed standard Gaussian entries. (For details of the argument see [18] and [19].) Sometimes it is useful to write this formula as

\[ \mu_1 + \ldots + \mu_k = \frac{1}{2} \frac{d}{d\mu} \mathbb{E} [\det (G_k^* \Sigma G_k)^\mu] \bigg|_{\mu=0} \]  

This argument works for complex Gaussian matrices as well. For quaternion Gaussian matrices, it should be modified because the theory of exterior products and volume elements is not sufficiently developed for modules over the quaternion skew-field. Luckily, the basic formulas (2) and (3) are still valid, as we will see in a later section.

While formula (2) allows one to compute all Lyapunov exponents, it is essentially a multidimensional integral which can be computationally demanding. For this reason, it is of interest to obtain a more explicit way for the Lyapunov exponent calculation.

For real Gaussian matrices and the simplest situation when \( \Sigma = \sigma^2 I \) and \( I \) is the identity matrix, Newman showed in [19] that

\[ \mu_i = \frac{1}{2} \left[ \log (2\sigma^2) + \Psi \left( \frac{d - i + 1}{2} \right) \right], \]  

where \( \sigma \) is the standard deviation of the Gaussian entries and \( \Psi \) is the digamma function.
where $\Psi(x)$ is the digamma function, $\Psi(x) := (\log \Gamma(x))'$. (At the positive integer points, $\Psi(n) = \sum_{k=1}^{n-1} \frac{1}{k+\gamma}$, where $\gamma = 0.5772\ldots$ is the Euler constant. At half-integers, $\Psi(n + 1/2) = \sum_{k=1}^{n-1/2} -2 \log 2 - \gamma$. The asymptotic behavior of the digamma function is given by the formula $\Psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} (1 + O\left(\frac{1}{z^2}\right))$.

In particular if we normalize $\sigma^2 = 1/d$, then for $d = 1$ the largest Lyapunov exponent $\mu_1 = [-\log 2 - \gamma]/2$ and for $d \to \infty$, $\mu_1 = -\frac{1}{2d} + O\left(\frac{1}{d^2}\right)$.

Another explicit formula is known for the sum of all Lyapunov exponents. Indeed, if $k = d$, then $\det (G_k^* \Sigma G_k) = \det G_d G_d^* \det \Sigma$, and therefore formula (3) becomes

$$\mu_1 + \ldots + \mu_d = \frac{1}{2} \log \det \Sigma + \frac{d}{d\mu} \left. E \left[ \det (G_d G_d^*)^\mu \right] \right|_{\mu=0}$$

In [7], Forrester showed that this implies that

$$\mu_1 + \ldots + \mu_d = \frac{1}{2} \sum_{i=1}^{d} \left( \log \left( \frac{2}{y_i} \right) + \Psi \left( \frac{i}{2} \right) \right), \quad (5)$$

where $y_i$ are eigenvalues of $\Sigma^{-1}$. (See formula 2.25 in [7]).

Forrester has also established analogues of formulas (4) and (5) for the complex-valued Gaussian matrices.

(Recall that in general the density for a Gaussian matrix $A$ with covariance matrix $\Sigma$ is given by $P(A) = c_\beta \det \left( \Sigma^{-k} \right) \exp \left[ -\frac{\beta}{2} \text{Tr} \left( A^* \Sigma^{-1} A \right) \right]$, where $\beta = 1, 2, 4$ for real, complex or quaternion matrices and $c_\beta$ is a normalization constant. Equivalently, $A$ can be obtained as $\Sigma^{1/2} G$, where $\Sigma^{1/2}$ is a real positive definite matrix and $G$ is real, complex or quaternion matrix with independent entries. The entries of $G$ have components that are real Gaussian variables with variance $1/\beta$.)

Namely, Forrester showed that in the case of the complex-valued Gaussian matrices with $\Sigma = \sigma^2 I$,

$$2\mu_i = \log \sigma^2 + \Psi \left( d - i + 1 \right)$$

(see Proposition 1 in [7] and note that the absence of $1/2$ before $\Psi$ is a typo.)

If $\sigma^2 = 1/d$, then for $d = 1$ the largest Lyapunov exponent $\mu_1 = -\gamma/2$ and for $d \to \infty$, $\mu_1 = -\frac{1}{d} + O\left(\frac{1}{d^2}\right)$.

The sum rule in the complex valued case is

$$\mu_1 + \ldots + \mu_d = \frac{1}{2} \sum_{i=1}^{d} \left( \log \left( \frac{1}{y_i} \right) + \Psi (i) \right).$$

A significant advance that Forrester achieved in the complex-valued case is an explicit formula for all Lyapunov exponents valid in the case of general $\Sigma$. Namely, it is shown in [7]
that

\[ \mu_k = \frac{1}{2} \Psi (k) + \frac{1}{2} \prod_{i<j} (y_i - y_j) \det \begin{bmatrix} y_j^{i-1} & \log y_j & y_j^{k-1} \\ (\log y_j) y_j^{k-1} \\ y_j^{i-1} \end{bmatrix} \]  

where \( y_i \) are eigenvalues of \( \Sigma^{-1} \). In particular for \( k = 1 \), one can re-write this as

\[ \mu_1 = \frac{1}{2} \left[ \Psi (1) - \sum_{j=1}^{d} \log y_j \prod_{l \neq j} (1 - y_j/y_l) \right] \]

provided that all \( y_i \) are different.

The proof of formula (7) is based on the Harish-Chandra-Itzykson-Zuber integral and cannot be directly generalized to the case of real or quaternion Gaussian matrices.

In fact, it appears that for the real-valued case with general \( \Sigma \), an explicit formula (due to Mannion [15]) is only known for products of 2-by-2 Gaussian matrices:

\[ \mu_1 = \frac{1}{2} \left[ \Psi (1) + \log \left( \frac{1}{2} \text{Tr} \Sigma + \sqrt{\det \Sigma} \right) \right] . \]  

(Some explicit formulas are also known for 2-by-2 random matrices with non-Gaussian entries, see [16]. In addition, there are methods which sometime allow one to compute Lyapunov exponents efficiently even when explicit formulas are not available, see [25].)

Our goal is to derive an explicit formula for the largest Lyapunov exponent which would be applicable in the real and quaternion-valued case with general \( \Sigma \).

3. AN INTEGRAL FORMULA FOR THE LARGEST LYAPUNOV EXPONENT

Our main formula is as follows.

**Theorem 3.1.** Let \( A_i \) be independent Gaussian matrices with covariance matrix \( \Sigma \). Let the entries be real, complex or quaternion, according to whether \( \beta = 1, 2, \) or 4. Assume that the eigenvalues of \( \Sigma \) are \( \sigma_i^2 = 1/y_i \). Then, the following formula holds for the largest Lyapunov exponent of \( A_i \),

\[ 2\mu_1 = \Psi (1) + \log \left( \frac{2}{\beta} \right) + \int_0^\infty \left[ 1_{[0,1]} (x) - \prod_{i=1}^{d} \left( 1 + \frac{x}{y_i} \right)^{-\beta/2} \right] \frac{dx}{x} . \]  

**Proof:** We start with (2) and note that a \( d \)-by-\( k \) Gaussian matrix \( \Sigma^{1/2} G_k \) has the density function

\[ p_{\Sigma^{1/2} G_k} (X) = c \det (\Sigma^{-k}) \exp \left[ -\frac{\beta}{2} \text{Tr} \left( XX^* \Sigma^{-1} \right) \right] . \]
If one changes variables to non-zero eigenvalues and eigenvectors of $XX^*$ then one obtains the density for eigenvalues of $G_k^* \Sigma G_k$,

$$p(\lambda_1, \ldots, \lambda_k) = c' \det(\Sigma^{-k}) \prod_{1 \leq i < j \leq k} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^k \lambda_i^{\beta(d-k+1)/2-1} \times \int \exp \left( -\frac{\beta}{2} \text{Tr} \left( U A U^* \Sigma^{-1} \right) \right) (U^* dU),$$

where $(U^* dU)$ denote the Haar measure on the unitary group of real/complex/quaternion $d$-by-$d$ matrices, and $A$ is the diagonal $d$-by-$d$ matrix with entries $\lambda_1, \ldots, \lambda_k, 0, \ldots, 0$.

In general, the integral is difficult to compute for $\beta = 1$ or $\beta = 4$. (For $\beta = 2$, this can be done by using the Harish-Chandra-Itzykson-Zuber formula.) However, in the case of $k = 1$, there is a formula discovered independently in [17], [30], and [22]. (See also [6] for a simple derivation.) It holds for $\beta = 1, 2, 4$. Suppose $A$ and $B$ are diagonal matrices with diagonal elements $(a, 0, \ldots, 0)$ and $(b_1, \ldots, b_d)$, respectively. Let $G^{(\beta)} (d)$ denote the group of unitary $d$-by-$d$ matrices in the real, complex, and quaternion case, respectively. Then

$$\int_{G^{(\beta)} (d)} \exp (\text{Tr} \left( U A U^* B \right)) (U^* dU) = \Gamma \left( \frac{\beta d}{2} \right) a^{1-\beta d/2} \frac{1}{2\pi i} \int_C e^{az} \prod_{i=1}^d (z - b_i)^{-\beta/2} dz,$$

where the contour of integration goes in counterclockwise around the points $b_i$. In the case of $\beta = 1$, the contour starts at $+\infty$ or $-\infty$ depending on the sign of $a$, and the branch of $(z - b_i)^{-1/2}$ should be chosen appropriately.

Specializing to the case $a = -\frac{\beta}{2} \lambda < 0$, and $b_i = y_i$, we get the following formula for the density of $G_1^* \Sigma G_1$,

$$p_{\beta} (\lambda) = c \prod_{i=1}^d y_i^{\beta d/2-1} \left( -\frac{\beta}{2} \lambda \right)^{1-\beta d/2} \frac{1}{2\pi i} \int_C e^{-\beta \lambda z/2} \prod_{i=1}^d (z - y_i)^{-\beta/2} dz = c' \int_C e^{-\beta \lambda z/2} \prod_{i=1}^d (z - y_i)^{-\beta/2} dz,$$

where $c'$ is a normalization constant. Next, we use formula (2) and find that

$$\mu_1 = \frac{1}{2c_{\beta}} \int_0^\infty \log \lambda \left[ \int_C e^{-\beta \lambda z/2} \prod_{i=1}^d (z - y_i)^{-\beta/2} dz \right] d\lambda,$$

where the contour of integration goes in a counterclockwise direction around the points $y_i$, and

$$c_{\beta} = \int_C \int_0^\infty e^{-\beta \lambda z/2} \prod_{i=1}^d (z - y_i)^{-\beta/2} dz d\lambda.$$

(Note that one can write similar although more complicated formulas for other Lyapunov exponents by using a generalization of formula (10), which was recently discovered by Onatski in [21].)
By changing the order of integration in (12) and computing the inner integral, we find that
\[
2 \mu_1 = \Psi(1) + \frac{1}{c_\beta} \left[ \frac{1}{2\pi i} \int_C \log \left( \frac{2}{\beta z} \right) \left( \frac{2}{\beta z} \right)^d \prod_{i=1}^d (z - y_i)^{-\beta/2} \, dz \right],
\]
where
\[
c_\beta = \frac{1}{2\pi i} \int_C \left( \frac{2}{\beta z} \right)^d \prod_{i=1}^d (z - y_i)^{-\beta/2} \, dz
= \frac{2}{\beta} \prod_{i=1}^d (-y_i)^{-\beta/2}.
\]
This implies that
\[
2 \mu_1 = \Psi(1) + \log \left( \frac{2}{\beta} \right) + \frac{1}{2\pi i} \int_C \log (z) \prod_{i=1}^d \left( 1 - \frac{z}{y_i} \right)^{-\beta/2} \, \frac{dz}{z},
\]
where the contour of integration goes in a counterclockwise direction around the points \( y_i \).

Next, we move the contour of integration so that it starts at \(-\infty\), goes along the upper edge of the real axis to \(-r\), then circles around the 0 in the clockwise direction, and then returns to \(-\infty\) along the lower edge of the real axis. Then by computing the integrals over the two rays and the circle, and by taking the radius of the circle to zero, we find that
\[
-\frac{1}{2\pi i} \int_C \log (z) \prod_{i=1}^d \left( 1 - \frac{z}{y_i} \right)^{-\beta/2} \, \frac{dz}{z} = \lim_{r \to 0} \left\{ \int_r^\infty \prod_{i=1}^d \left( 1 + \frac{x}{y_i} \right)^{-\beta/2} \, \frac{dx}{x} + \log r \right\}
= \int_0^1 \left( \prod_{i=1}^d \left( 1 + \frac{x}{y_i} \right)^{-\beta/2} - 1 \right) \frac{dx}{x} + \int_1^\infty \prod_{i=1}^d \left( 1 + \frac{x}{y_i} \right)^{-\beta/2} \, \frac{dx}{x},
\]
which proves formula (9). □

4. ASYMPTOTIC BEHAVIOR FOR A MODEL WITH A SPIKE

Assume that all \( y_i = 1 \), for \( i = 1, \ldots, d - 1 \) and \( y_d = 1/\theta < 1 \). This means that the covariance matrix has a spike \( \theta > 1 \), or informally that one of the rows in matrices \( A_i \) has the size which is \( \sqrt{\theta} \) larger than other rows. We ask the question about the behavior of the largest Lyapunov exponent when \( d \), or \( \theta \), or both, are large. Assume first that \( \beta = 2 \). We can write
\[
2 \mu_1 = \Psi(1) + \int_0^\infty \left[ 1_{[0,1]}(x) - \frac{1}{(1 + x)^{d-1}} \frac{1}{1 + \theta x} \right] \frac{dx}{x}
= \Psi(d) + \int_0^\infty \left[ \frac{1}{(1 + x)^{d-1}} \left( \frac{1}{1 + x} - \frac{1}{1 + \theta x} \right) \right] \frac{dx}{x},
= \Psi(d) + f_d,
\]
where
\[
f_d = (\theta - 1) \int_0^\infty \frac{1}{(1 + x)^d (1 + \theta x)} \, dx \leq \frac{\theta - 1}{d}
\]
Hence, if $\theta = O(d)$ and $d \to \infty$, then

$$2\mu_1 \sim \Psi(d) \sim \log d.$$ 

In other words, in this case the spike $\theta$ in $\Sigma$ cannot influence the leading order asymptotics of the largest Lyapunov exponent.

It is still interesting to find out what is the contribution of the spike $\theta$ to the Lyapunov exponent even though it is of a lower order than the leading asymptotics. (Indeed, the leading term asymptotics can be removed if we rescale all the entries in the matrices $A_i$ by $\sigma = 1/\sqrt{d}$.)

**Theorem 4.1.** Suppose that $A_i$ are independent $d$-by-$d$ Gaussian matrices with the covariance matrix $\Sigma$, and that the eigenvalues of $\Sigma$ are $\sigma_i^2 = 1$ for $i = 1, \ldots, d-1$, and $\sigma_d^2 = \theta > 1$. Let $\theta = d/t$, where $0 < t < d$. In the complex case ($\beta = 2$), we have the following estimate,

$$2\mu_1 = \log d + e^t \int_1^\infty e^{-tx} \frac{dx}{x} + O_t(1/d),$$

$$= \log d - e^t \text{Ei}(-t) + O_t(1/d),$$

where $\text{Ei}(x)$ is the exponential integral function. In the real case ($\beta = 1$),

$$2\mu_1 = \log d + e^{t/2} \int_1^\infty e^{-tx/2} \frac{dx}{\sqrt{x}(\sqrt{x} + 1)} + O_t(1/d),$$

**Proof:** For the complex case, we have

$$2\mu_1 = \Psi(d) + \int_0^\infty \frac{1}{(1 + x)^{d-1}} \left[ \frac{1}{1 + x} - \frac{1}{1 + \theta x} \right] \frac{dx}{x}$$

$$= \Psi(d) + \int_0^\infty \frac{1}{(1 + u/d)^d} \left[ \frac{1}{1 + u/d} - \frac{1}{1 + u/t} \right] \frac{du}{u}$$

$$= \log d + \int_0^\infty e^{-u} \left[ 1 - \frac{1}{1 + u/t} \right] \frac{du}{u} + O_t(1/d)$$

$$= \log d + \int_0^\infty e^{-xt} \frac{du}{1 + x} + O_t(1/d)$$

For the real case, we assume $d = 2k$ (the other case is similar) and write:

$$2\mu_1 = 2 \log + \Psi(k) + \int_0^\infty \frac{1}{(1 + x)^{k-1/2}} \left[ \frac{1}{\sqrt{1 + x}} - \frac{1}{\sqrt{1 + \theta x}} \right] \frac{dx}{x}$$

$$= \log 2 + \Psi(k) + \int_0^\infty \frac{1}{(1 + u/k)^{k-1/2}} \left[ \frac{1}{\sqrt{1 + u/k}} - \frac{1}{\sqrt{1 + 2u/t}} \right] \frac{du}{u}$$

$$= \log d + \int_0^\infty e^{-u} \left[ 1 - \frac{1}{\sqrt{1 + 2u/t}} \right] \frac{du}{u} + O_t(1/d)$$

$$= \log d + e^{t/2} \int_1^\infty e^{-tx/2} \frac{dx}{\sqrt{x}(\sqrt{x} + 1)} + O_t(1/d).$$

(The last step uses the change of variable $x = 1 + 2u/t$.) □
This theorem explains what happens if both $d$ and $\theta$ approach infinity at the same rate. Alternatively, if $\theta > 1$ is fixed and $d$ approaches infinity, then $t = d/\theta \to \infty$ and the asymptotic expansion for the exponential integral function,

$$\text{Ei} (-t) = -\frac{e^{-t}}{t} \sum_{k=0}^{\infty} \frac{k!}{(-t)^k},$$

suggests that $2\mu_1 = \Psi (d) + 1/t + O(t^{-2}) = \Psi (d) + (\theta/d) + O((\theta/d)^2)$. Since the error term in the formula (17) depends on $t$, this argument is not quite satisfactory. However, we can rigorously obtain the following result.
**Theorem 4.2.** Suppose that $A_i$ are $d$-by-$d$ complex Gaussian matrices with the covariance $\Sigma$, and that the eigenvalues of $\Sigma$ are $\sigma_i^2 = 1$ for $i = 1, \ldots, d - 1$, and $\sigma_d^2 = \theta > 1$. Then, for the largest Lyapunov exponent $\mu_1$, we have

$$\lim_{d \to \infty} d (\mu_1 - \log d) = \theta - \frac{3}{2}.$$ 

**Proof:** One can check that for $d \geq 1$, the additional term $f_d$ defined in (16) satisfies the following recursion:

$$f_d = \left( \frac{\theta - 1}{\theta} \right) \left( \frac{1}{d} + f_{d+1} \right).$$

(18) Then we obtain a convergent series for $f_d$,

$$f_d = s \sum_{k=0}^{\infty} \frac{s^k}{d + k},$$

where $s := (\theta - 1) / \theta$. Hence,

$$df_d = s \sum_{k=0}^{\infty} \frac{s^k}{1 + k/d}.$$ 

By using the monotone convergence theorem, we find that

$$\lim_{d \to \infty} df_d = \frac{s}{1 - s} = \theta - 1.$$ 

Together with the asymptotic expansion for the digamma function, $\Psi(d) = \log d - \frac{1}{2d} + O\left( \frac{1}{d^2} \right)$, this limit implies the statement of the theorem. $\square$

We can also use the recursion in (18) and the initial condition $f_1 = \log \theta$ in order to obtain

$$f_d = \left( \frac{\theta}{\theta - 1} \right)^{d-1} \left( \log \theta - \sum_{k=1}^{d-1} \frac{1}{k} \left( 1 - \frac{1}{\theta} \right)^k \right).$$

(19) From formula (19) we can see that in the remaining case, when $d$ is fixed and $\theta$ goes to infinity, we have $f_d \sim \log \theta - \Psi(d) + \Psi(1)$, hence $2\mu_1 \sim \log \theta - \gamma$.

### 5. A comparison with results from free probability

Let $x_i$ be free, identically distributed non-commutative random variables. (For an introduction to free probability, see [20] or [13].) Assume that $x_i$ are bounded and define $\Pi_n := x_n \ldots x_1$. In [11] it was found that the limit of $n^{-1} \log \|\Pi_n\|$ exists and equals $\frac{1}{2} \log E (x_1^* x_1)$, where $E$ denote the trace operation in the free probability space.

Note that the limit of $n^{-1} \log \|\Pi_n\|$ is the largest Lyapunov exponent for the product of variables $x_i$. How does the free probability result about the largest Lyapunov exponent compares with our formula for Gaussian matrices?

In free probability theory, it is known that as $d$ approach infinity, the $d$-by-$d$ Gaussian matrices $A_i^{(d)}$ with covariance matrix $d^{-1}I_d$ converge in distribution to the so-called circular element, which we will denote $a$. (Technically, the element $a$ is a non-selfadjoint operator with the Brown measure uniformly distributed over the unit circle.) In addition, suppose that matrices $\Sigma_d$ are diagonal with eigenvalues $\theta_i > 0$, $i = 1, \ldots, d$, and assume that the
distribution \( \frac{1}{d} \sum_{i=1}^{d} \delta_{\theta_i} \) weakly converges to a compactly supported measure \( \nu \). Then, matrices \( \Sigma_d \) converge in distribution to a self-adjoint non-commutative random variable \( s \) in the sense of free probability theory. This variable has the spectral probability measure \( \nu \).

Since in this paper we are interested in Lyapunov exponents of matrices \( \Sigma_d^{1/2} A_i^{(d)} \), it is natural to consider products of non-commutative random variables \( s_1^{1/2} \ldots s_{n-1}^{1/2} a_n \) where \( a_i \) are free circular elements. Since \( a_i \) are unitarily invariant and free of \( s \), the norm of the product \( s_1^{1/2} a_n s_1^{1/2} \ldots a_1 \) is the same as the norm of the product \( s_n^{1/2} a_n s_{n-1}^{1/2} a_{n-1} \ldots s_1^{1/2} a_1 \), where \( s_i \) are self-adjoint non-commutative random variables which have the same distribution as \( s \), and which are free from each other and from all of \( a_n \). Hence we can define \( x_i = s_i^{1/2} a_i \) and apply the result from [11]. This result shows that the largest Lyapunov exponent of \( x_i \) equals

\[
\frac{1}{2} \log E(a^* sa) = \frac{1}{2} \left( \log E(aa^*) + \log E(s) \right) = \frac{1}{2} \log E(s)
\]

(20)

(The first equality uses the fact that \( E \) is a trace and that \( s \) is free of \( a \). The second equality uses a property of circular elements. The third equality follows from the connection of the trace \( E \) and the spectral probability measure \( \nu \), and from the assumption on the distribution of \( \theta_i \).)

The previous section (about spikes in \( \Sigma_d \)) shows that this formula does not always provide a correct limit value for Gaussian matrices. Indeed, we have seen that if all \( \theta_i \) except one equal \( 1/d \) and the exceptional \( \theta \) equals \( 1/t \), then the large-\( d \) limit of the largest Lyapunov exponent is not \( \frac{1}{2} \log(1 + 1/t) \), which would be suggested by formula (20). However, under some mild conditions this formula does give the correct limit.

**Theorem 5.1.** Suppose that \( A_i^{(d)} \) are \( d \times d \) Gaussian matrices with covariance matrix \( \Sigma_d \), and let the eigenvalues of \( \Sigma_d \) be \( \theta_{i,d} \) where \( i = 1, \ldots, d \). Assume that \( \theta_{i,d} \) are bounded, \( 1 \leq \theta_{i,d} \leq L \), and that

\[
\lim_{d \to \infty} \text{Tr} \left( \Sigma_d \right) := \lim_{d \to \infty} \frac{1}{d} \sum_{i=1}^{d} \theta_{i,d} = \lambda.
\]

Then, for the largest Lyapunov exponents of \( A_i^{(d)} \), we have

\[
\lim_{d \to \infty} \mu_1^{(d)} = \frac{1}{2} \log \lambda.
\]

**Proof:** First, a straightforward calculation shows that

\[
\Psi(1) + \log \left( \frac{2}{\beta} \right) + \int_{0}^{\infty} \left[ 1_{[0,1]}(x) - e^{-(\beta/2) \lambda x} \right] \frac{dx}{x} = \log \lambda.
\]

If we compare this with the formula (9), then we find that it is sufficient to show that

\[
\int_{0}^{\infty} \left[ \prod_{i=1}^{d} \left( 1 + \frac{\theta_i}{d} \right)^{-\beta/2} e^{-(\beta/2) \lambda x} \right] \frac{dx}{x} \to 0
\]

(21)
The figure shows the dependence of the error of the free probability approximation on the dimension. The solid blue lines are for $\beta = 2$ and the dashed red lines are for $\beta = 1$. For circle markers, one half of the eigenvalues of the covariance matrix equals $1/d$ and the other half equals to $3/d$. For the squares, the eigenvalues are placed uniformly at equal distance from each other between $1/d$ and $3/d$.

as $d \to \infty$. First, consider

$$I_1 = \int_0^\infty \left[ \prod_{i=1}^d \left( 1 + \frac{\theta_i}{d} x \right)^{-\beta/2} - \prod_{i=1}^d \exp \left( -\frac{\beta \theta_i}{2 d} x \right) \right] \frac{dx}{x}.$$

We split $I_1$ in two integrals, $I'_1 + I''_1$, the first one is over the interval from 0 to $M$, and the other is over the interval from $M$ to infinity. For the second integral, we have

$$\int_M^\infty \exp \left( -\frac{\beta}{2} \frac{\sum \theta_i}{d} x \right) \frac{dx}{x} = \int_M^\infty \frac{e^{-t} dt}{t} \leq e^{-M(\beta/2)} \to 0.$$
as $M \to \infty$ and the convergence is uniform in $d$. In addition,

\[
\int_{M}^{\infty} \prod_{i=1}^{d} \left(1 + \frac{\theta_i}{x} \right)^{-\beta/2} \frac{dx}{x} \leq \int_{M/d}^{\infty} (1 + t)^{-d\beta/2} \frac{dt}{t} \\
\leq \frac{d}{M} \int_{M/d}^{\infty} (1 + t)^{-d\beta/2} \frac{dt}{t} \\
= \frac{d}{M \cdot d^{(\beta/2) - 1}} \left(1 + \frac{M}{d}\right)^{-d\beta/2} \to 0
\]

as $M \to \infty$, again uniformly in $d > 1$. We conclude that for every $\varepsilon > 0$, we can find $M_0$ such that $|I_1''| \leq \varepsilon$ for all $M \geq M_0$ and all $d > 1$. In words, we can make $I_1''$ arbitrarily small uniformly in $d$ by taking $M$ sufficiently large.

For the integral $I_1'$, we estimate the integrand by using the fact that if $|z_i| \leq 1$ and $|w_i| \leq 1$, then

\[
\left| \prod_{i=1}^{d} z_i - \prod_{i=1}^{d} w_i \right| \leq \sum_{i=1}^{d} |z_i - w_i|,
\]

(see Lemma 1 of Section 27 in Billingsley [2]). Since

\[
\left| \left(1 + \frac{\theta_i}{x}\right)^{-\beta/2} - \exp \left(-\frac{\beta \theta_i}{2} x \right) \right| \leq C \left(\frac{\theta_i}{d}\right)^2
\]

for all $x \leq d/L$, therefore (for $d \geq LM$), we estimate

\[
I_1' \leq \int_{0}^{M} C \frac{L^2}{d} x dx = \frac{C L^2 M^2}{2d}.
\]

For a fixed $M$, this can be made arbitrarily small by choosing $d$ sufficiently large.

Hence, $I_1 \to 0$ as $d \to \infty$.

Similarly,

\[
I_2'' := \int_{M}^{\infty} \left(e^{-\langle\beta/2\rangle x} - e^{-\langle\beta/2\rangle d^{-1} \langle\sum \theta_i\rangle x}\right) \frac{dx}{x} \to 0
\]

as $M \to \infty$ uniformly in $d$, and for

\[
I_2' := \int_{0}^{M} \left(e^{-\langle\beta/2\rangle x} - e^{-\langle\beta/2\rangle d^{-1} \langle\sum \theta_i\rangle x}\right) \frac{dx}{x},
\]

we estimate

\[
|I_2'| \leq \int_{0}^{M} C \left| \lambda - d^{-1} \langle\sum \theta_i\rangle \right| dx \\
= CM \left| \lambda - d^{-1} \langle\sum \theta_i\rangle \right| \to 0
\]

as $d \to \infty$ for a fixed $M$. Altogether, the convergence of $I_1$ and $I_2 := I_2' + I_2''$ to zero proves (21) and completes the proof. □
6. The Largest Lyapunov Exponent for Real Gaussian Matrices

It is easy to check that in the case when $\Sigma = I_d$ and $\beta = 1$, the integral in (9) leads to Newman’s formula (4).

For example, if $d$ is even, $d = 2k$, then

$$\frac{1}{(1 + x)^k} = \frac{1}{1 + x} - \frac{1}{1 + x} - \cdots - \frac{1}{(1 + x)^k},$$

and

$$\int_0^\infty \left[ 1_{[0,1]}(x) - (1 + x)^{-k} \right] \frac{dx}{x} = 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} = \Psi(k) - \Psi(1),$$

in agreement with formula (4).

Similarly, in the case of $d = 2$, the integrand in (9) has only irrationalities of the type $1/\sqrt{P(x)}$ where $P(x)$ are quadratic polynomials. Hence they can be computed in terms of elementary functions, and the result is equivalent to Mannion’s formula (8).

If $d = 3$ or $d = 4$, then we have elliptic integrals in formula (9). These integrals can be brought to one of available standard forms if desired. In particular, one can reduce the integrals to the standard Legendre integrals. We will do it here for the case $d = 3$. Let the inverses of eigenvalues of $\Sigma$ be $y_1 > y_2 \geq y_3$ and define the following quantities:

$$k^2 := \frac{y_1 - y_3}{y_1 - y_2}, \quad n := \frac{y_1}{y_1 - y_2}, \quad \alpha := 2\sqrt{\frac{y_2 y_3}{y_1 (y_1 - y_2)}},$$

and

$$\varphi(x) := i \sinh^{-1} \left( \sqrt{\frac{y_2 - y_1}{y_1 + x}} \right).$$

Recall that Legendre’s incomplete elliptic integral of the first kind is defined by the formula

$$F(\varphi, k^2) = \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

and Legendre’s incomplete elliptic integral of the third kind is

$$\Pi(\varphi, n, k^2) = \int_0^{\sin \varphi} \frac{dt}{(1 - nt^2) \sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

**Proposition 6.1.** For 3-by-3 real Gaussian matrices $A_i$, the largest Lyapunov exponent is given by the formula

$$\Psi(1) + \log(2) - \alpha F(\varphi(0), k^2) + \lim_{x \to 0} \left[ \alpha \Pi(\varphi(x), n, k^2) - \log x \right].$$

For dimensions larger than 4, it is difficult to give explicit formulas without integral expressions. However, for matrices of even dimension, if all eigenvalues of $\Sigma$ have multiplicity two, then there are no irrationalities in the integrand, and we have the following result.
Proposition 6.2. Suppose the dimension $d$ is even, $d = 2k$, and the eigenvalues of $\Sigma^{-1}$ come in pairs, $y_i = y_{i+k}$ for all $i = 1, \ldots, k$. Assume that $y_i$ are distinct for $i = 1, \ldots, k$. In this case, we have the following formula for the largest Lyapunov exponent

$$
\mu_1 = \frac{1}{2} \left[ \Psi (1) + \log 2 - \frac{1}{\prod_{1 \leq l_1 < l_2 \leq k} (y_{l_2} - y_{l_1})} \det \left( \begin{bmatrix} \log y_j & j=1, \ldots, k \\ y_j^i-1 & j=1, \ldots, k \end{bmatrix} \right) \right]
$$

$$
= \frac{1}{2} \left[ \Psi (1) + \log 2 - \sum_{j=1}^k \frac{1}{\prod_{l \neq j} (1 - y_j/y_i)} \right].
$$

Proof: From (11), we have the following formula for the density of $G^* \Sigma G$,

$$
p_1 (\lambda) = c' \frac{1}{2\pi i} \int C e^{-\lambda z/2} \prod_{i=1}^k (z - y_i)^{-1} \, dz.
$$

If in addition all $y_i$ are distinct, then we get by computing residues

$$
p_1 (\lambda) = c' \sum_{i=1}^k e^{-\lambda y_i/2} \prod_{j \neq i} (y_i - y_j)^{-1}.
$$

Hence, we can compute

$$
E \lambda^\mu = c' \Gamma (1 + \mu) \sum_{i=1}^k 2^{1+\mu} y_i^{-1-\mu} \prod_{j \neq i} (y_i - y_j)^{-1},
$$

which shows that

$$
c' = \left( \sum_{i=1}^k \frac{2}{y_i} \prod_{j \neq i} (y_i - y_j)^{-1} \right)^{-1},
$$

and therefore, from (3) we get

$$
\mu_1 = \frac{1}{2} \left[ \Psi (1) + \log 2 - \frac{\sum_{i=1}^k (\log y_i) y_i^{-1} \prod_{j \neq i} (y_i - y_j)^{-1}}{\sum_{i=1}^k y_i^{-1} \prod_{j \neq i} (y_i - y_j)^{-1}} \right]
$$

$$
= \frac{1}{2} \left[ \Psi (1) + \log 2 - \frac{F(\log y, y)}{F(1, y)} \right],
$$

where $F(l, y)$ is defined by the formula

$$
F(l, y) := \sum_{i=1}^k l_i \prod_{j \neq i} y_j / \prod_{j \neq i} (y_j - y_i).
$$

By using the Vandermonde determinant formula, one can obtain the identity

$$
F(l, y) = \det \left( \begin{bmatrix} l_i & i=1, \ldots, k \\ y_j & j=1, \ldots, k \end{bmatrix} \right) / \prod_{1 \leq l_1 < l_2 \leq k} (y_{l_2} - y_{l_1}).
$$
Hence \( F(1, y) = 1 \) and we obtain that

\[
\mu_1 = \frac{1}{2} \left[ \Psi(1) + \log 2 - \frac{1}{\prod_{1 \leq l_1 < l_2 \leq k} (y_{l_2} - y_{l_1})} \det \left( \begin{array}{c} \log y_j \\ y_j^{-1} \end{array} \right) \right] + \frac{1}{2} \left[ \Psi(1) + \log 2 - \sum_{j=1}^k \frac{\log y_j}{\prod_{l \neq j} (1 - y_j/y_l)} \right].
\]

(The equality in the second line follows by an easy transformation of \( F(\log y, y) \).) □

7. LYAPUNOV EXPONENTS FOR QUATERNION GAUSSIAN MATRICES

Recall that the algebra of real quaternions \( \mathbb{Q} \) is a non-commutative division algebra isomorphic to a subalgebra of all 2-by-2 complex matrices \( M_2(\mathbb{C}) \) generated over \( \mathbb{R} \) by the identity matrix and matrices

\[
i = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]

which are called quaternion units. The quaternions are typically written as \( q = s + x i + y j + z k \), where \( s, x, y, z \) are real. The conjugate of \( q \) is \( q^* = s - x i - y j - z k \), and the norm of \( q \) is \( |q| := (q^* q)^{1/2} = (s^2 + x^2 + y^2 + z^2)^{1/2} \).

Quaternion matrices are matrices whose entries are quaternions. They obey the usual rules of matrix addition and multiplication. The dual of a quaternion matrix \( X \) is defined as a matrix \( X^* \), for which \( (X^*)_{lk} = (X_{kl})^* \). Self-dual quaternion matrices are defined by the property that \( X^* X = X \). Unitary quaternion matrices are matrices with the property that \( X^* X = X X^* = I \), where \( I \) is the identity matrix.

If we represent each quaternion by a 2-by-2 complex matrix, then every quaternion matrix \( X \) is represented by a \( 2n \)-by-\( 2n \) complex matrix which we denote \( \varphi(X) \). Let \( J \) be a \( 2n \)-by-\( 2n \) block-diagonal matrix with the blocks \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) on the main diagonal. Complex matrices \( A \) that satisfy the condition \( A^T = -J A J \) are called skew-Hamiltonian, and complex matrices \( V \) that satisfy the condition \( V^T J V = J \) are called symplectic. A quaternion matrix \( X \) is self-dual if and only if \( \varphi(X) \) is skew-Hamiltonian, and \( X \) is unitary if and only if \( \varphi(X) \) is symplectic.

A number \( \lambda \) is called an eigenvalue of a quaternion matrix \( X \) if for some non-zero quaternion vector \( v \), we have \( X v = v \lambda \). (These are the right eigenvalues of \( X \), which are the most convenient in applications.) It is easy to see that if \( \lambda \) is an eigenvalue, then \( q^{-1} \lambda q \) is also an eigenvalue for any quaternion \( q \). However, for self-dual quaternion matrices, all eigenvalues are real scalars (that is, they have zero \( i, j \) and \( k \) parts) and it is possible to show that any \( n \)-by-\( n \) matrix \( X \) of this type has exactly \( n \) eigenvalues (counting with multiplicities); see Zhang [31].
It is possible and useful to generalize the concept of determinant to quaternion matrices. There are several sensible ways to do this ([1]). We will define the determinant of a quaternion matrix as follows:

\[
\det (X) := (\det (\varphi (X)))^{1/2}.
\]

We say a quaternion matrix is Gaussian, if the components of every entry are independent Gaussian random variables with expectation zero and variance 1/4.

The existence of Lyapunov exponents for products of Gaussian quaternion matrices \(A_i\) follows from the corresponding existence result for products of complex representations \(\varphi (A_i)\). Indeed, the eigenvalues of the complex matrix \(\varphi (P_N^* P_N)\) coincide with eigenvalues of the quaternion matrix \(P_N^* P_N\) taken with multiplicity two. Hence, the existence of the limit \((\lambda_i)^{1/N}\) for eigenvalues of \(\varphi (P_N^* P_N)\) implies the existence of the corresponding limit for eigenvalues of \(P_N^* P_N\).

On the other hand, the basic formula ([2]) needs a different justification, since the theory of exterior forms is not available for the quaternion case. This can be done as follows.

We start with the formula ([1]), which in the quaternion case should be formulated as follows:

\[
\mu_1 + \ldots + \mu_k = \lim_{N \to \infty} \frac{1}{2N} \log \det (Q_0^* P_N^* P_N Q_0),
\]

where \(Q_0\) is the \(d\)-by-\(k\) matrix with the columns equal to basis vectors \(e_1, \ldots, e_k\). This formula follows from the corresponding formula for the complex representations of \(P_N\).

In order to analyze the determinant, we define \(Q_i\) as the orthogonal projection on the linear subspace spanned by \(P_i e_1, \ldots, P_i e_k\). Hence \(Q_i\) is a \(k\)-by-\(d\) matrix, and

\[
Q_1^* Q_1 P_i Q_0 = P_i Q_0.
\]

Then,

\[
\det (Q_0^* P_N^* P_N Q_0) = \det (Q_0^* P_{N-1}^* Q_{N-1}^* Q_{N-1} Q_N A_N Q_N^* Q_{N-1}^* P_{N-1} Q_0)
= \det (Q_{N-1}^* P_{N-1} Q_0 Q_0^* P_{N-1}^* Q_{N-1}^* Q_{N-1} A_N Q_N^* Q_{N-1}^* P_{N-1} Q_0)
= \det (Q_{N-1}^* P_{N-1} Q_0 Q_0^* P_{N-1}^* Q_{N-1}^* Q_{N-1} A_N Q_N^* Q_{N-1}^* P_{N-1} Q_0)
\]

In the first line we inserted projectors \(Q_N^* Q_N\) and \(Q_{N-1}^* Q_{N-1}\) by using ([23]). In the second line we moved \(Q_{N-1}^* P_{N-1} Q_0\) to the left, using the properties that the eigenvalues of the product are not changed by this operation. In the third line we used the multiplicativity of the determinant. In the fourth line we moved \(Q_{N-1}^* P_{N-1} Q_0\) to the right in the first determinant and removed \(Q_N^* Q_N\) from the second by using ([23]). In the fifth line we removed \(Q_{N-1}^* Q_{N-1}\) from the first determinant by using ([23]).
The key observation is that the random matrix $A_N^* A_N$ is independent from $P_i$ and $Q_i$ for $i < N$, and that $A_N^* A_N$ has the same distribution as $U^* A_N^* A_N U$ for every unitary transformation $U$ independent of $A_N$. It follows that the random variable $\det (Q_0^* P_N^* P_N Q_0)$ has the same distribution as $\det (Q_0^* P_{N-1}^* P_{N-1} Q_0) \det (Q_0^* A_N^* A_N Q_0^*)$. By applying induction we find that it has the same distribution as the product of $N$ independent copies of $\det (Q_0^* A_1^* A_1 Q_0^*)$.

By the law of large numbers we find that formula (22) becomes

$$\mu_1 + \ldots + \mu_k = \frac{1}{2} \log \det (Q_0^* A_1^* A_1 Q_0)$$

which is the desired formula.

Next, consider the simplest situation when $\Sigma$ is the identity matrix.

**Theorem 7.1.** Let $A_i$ be $d$-by-$d$ Gaussian quaternion matrices with the covariance $\Sigma = \sigma^2 I_d$, then

$$\mu_i = \frac{1}{2} \left[ \log \sigma^2 - \log 2 + \Psi (2 (d - i + 1)) \right],$$

where $\Psi (x) := (\log \Gamma (x))'$ is the digamma function.

**Proof:** We can assume $\sigma^2 = 1$ by a scaling argument. Let $W := G_k^* G_k$. This is a $k$-by-$k$ quaternion Wishart matrix. After changing the variables to eigenvalues and eigenvectors of $W$ as in Proposition 3.2.2 in [8], we find that formula (3) implies

$$\mu_1 + \ldots + \mu_k = \frac{1}{2} \int_0^\infty \ldots \int_0^\infty \prod_{i < j} (\lambda_i - \lambda_j)^4 \prod_{i=1}^k \lambda_i^{\mu+2(d-k)+1} e^{-2\lambda_i} d\lambda_i \left|_{\mu=0} \right.,$$

where $C_d$ is the normalization constant. (It can be obtained by setting $\mu = 0$ in the integral.) This integral can be computed as a Selberg’s integral (Prop. 4.7.3 in [8]) and we find that

$$\mu_1 + \ldots + \mu_k = \frac{1}{2} \left[ k \log 2 + \sum_{j=0}^{k-1} \Psi (2 (d - k + 1 + j)) \right].$$

Another simple formula can be obtained for the sum of all Lyapunov exponents. We note that if $k = d$, then $\det (G_k^* \Sigma G_k) = \det G G^* \det \Sigma$, and therefore formula (3) becomes

$$\mu_1 + \ldots + \mu_d = \frac{1}{2} \log \det \Sigma + \frac{d}{d\mu} \left[ \det (G G^*)^\mu \right] \left|_{\mu=0} \right.$$
We can compute \( \det (GG^*) \) by using the method used in the proof of Theorem 7.1. This computation gives the following result.

**Proposition 7.2.** Let \( A_i \) be \( d \)-by-\( d \) Gaussian quaternion matrices with the covariance matrix \( \Sigma \), which is a positive definite \( d \)-by-\( d \) matrix. Suppose that the eigenvalues of \( \Sigma^{-1} \) are \( y_1, \ldots, y_d \). Then we have,

\[
\mu_1 + \ldots + \mu_d = \frac{1}{2} \left( \sum_{i=1}^{d} \log \left( \frac{1}{2y_i} \right) + \sum_{i=1}^{d} \Psi (2i) \right).
\]

Finally, let us compute the largest Lyapunov exponent in the case of general covariance matrix \( \Sigma \).

**Theorem 7.3.** Let \( A_i \) be quaternion Gaussian matrices with covariance \( \Sigma \), and assume that \( \Sigma \) is invertible and that the inverses of its eigenvalues, \( y_1, \ldots, y_d \), are all distinct. Then we
have the following formula for the largest Lyapunov exponent,

\[ 2\mu_1 = \Psi (1) - \log (2) - \left( \prod_{i=1}^{d} y_i \right)^2 \sum_{i=1}^{d} \left\{ \frac{1}{y_i^2 \prod_{j \neq i} (y_i - y_j)^2} \left[ 1 - \log y_i \left( 1 + \sum_{j \neq i} \frac{2y_i}{y_i - y_j} \right) \right] \right\}. \] (24)

**Proof:** By formula (15),

\[ 2\mu_1 = \Psi (1) - \log (2) - \frac{1}{2\pi i} \int_C \frac{dz \log z}{z (1 - z/y_1)^2 \ldots (1 - z/y_d)^2}, \]

and the formula (24) is obtained by computing the residues at poles \( z = y_i \). \( \square \)

8. **Conclusion**

We derived new formula (9) for the largest Lyapunov exponent of Gaussian matrices. By using this formula, we found the asymptotic expressions for this exponent in the cases when the covariance matrix has a single spike (Section 4) and when it has no spikes (Section 5). In the latter case, we found that the asymptotic expression agrees with a prediction given by free probability theory.

In the case of real Gaussian matrices, our formula implies that the largest Lyapunov exponent can be written in terms of elliptic functions if the size of matrices is 3 or 4. We derived an explicit formula of this type for the case \( d = 3 \). We have also provided an explicit formula for the case when \( \beta = 1 \) and all eigenvalues of the covariance matrix have multiplicity 2.

For the quaternion case (\( \beta = 4 \)), we derived formulas for all Lyapunov exponents if the covariance matrix \( \Sigma = \sigma^2 I \), and a formula for the largest Lyapunov exponent if \( \Sigma \) is arbitrary.

Our methods depend on a formula discovered by Mo and others. Since this formula has been recently generalized by Onatski, our results can perhaps be extended to give formulas for other Lyapunov exponents. However, the expressions seem to become more complicated.

An interesting open question is whether the asymptotic expressions derived in Sections 4 and 5 remain valid for non-Gaussian random matrices with independent entries.

**References**

[1] Helmer Aslaksen. Quaternionic determinants. *Mathematical Intelligencer*, 18:57, 1996. available at [www.math.nus.edu.sg/aslaksen/](http://www.math.nus.edu.sg/aslaksen/)

[2] Patrick Billingsley. *Probability and Measure*. John Wiley and Sons, third edition, 1995.

[3] P. Bougerol and J. Lacroix. *Products of random matrices with applications to Schrodinger operators*, volume 8 of *Progress in probability and statistics*. Birkhauser: Boston, 1985.

[4] Joel E. Cohen, Harry Kesten, and Charles M. Newman, editors. *Random Matrices and Their Applications*, volume 50 of *Contemporary Mathematics*. American Mathematical Society, 1986.

[5] Joel E. Cohen and Charles M. Newman. The stability of large random matrices and their products. *Annals of Probability*, 12:283–310, 1984.
[6] P. Forrester. Probability densities and distributions for spiked Wishart beta-ensembles. arxiv:1101.2261, 2011.

[7] P. J. Forrester. Lyapunov exponents for products of complex Gaussian random matrices. Journal of Statistical Physics.

[8] P. J. Forrester. Log-gases and random matrices. Princeton University Press, 2010.

[9] H. Furstenberg and H. Kesten. Products of random matrices. Annals of Mathematical Statistics, 31:457–469, 1960.

[10] Y. Guivarch and A. Raugi. Frontiere de Furstenberg, proprietes de contraction et theoremes de convergence. Zeit. Fur Wahrscheinlichkeitstheorie und Verw. Gebiete, 67:265–278, 1985.

[11] V. Kargin. The norm of products of free random variables. Probability Theory and Related Fields, 139:397–413, 2007. arxiv:math/0611593

[12] V. Kargin. Lyapunov exponents of free operators. Journal of Functional Analysis, 255:1874–1888, 2008. arxiv:0712.1378

[13] Vladislav Kargin. Lectures on free probability. arxiv:1305.2611, 2013.

[14] Emile le Page. Regularite de plus grand exposant caracteristique de produit de matrices aléatoires indépendantes et applications. Annales de l’Institut Henri Poincar. Probabilites et Statistiques, 25:109–142, 1989.

[15] David Mannion. Products of 2x2 random matrices. Annals of Applied Probability, 3:1189–1218, 1993.

[16] Jens Marklof, Yves Tourigny, and Lech Wołowski. Explicit invariant measures for products of random matrices. Transactions of the American Mathematical Society, 360:3391–3427, 2008.

[17] M. Y. Mo. The rank 1 real Wishart spiked model I: Finite n analysis. arxiv:1011.5404, 2010.

[18] C. M. Newman. Lyapunov exponents for some products of random matrices: Exact expressions and asymptotic distributions. In Joel E. Cohen, Harry Kesten, and Charles M. Newman, editors, Random Matrices and Their Applications, volume 50 of Contemporary Mathematics, pages 183–195. American Mathematical Society, 1986.

[19] Charles M. Newman. The distribution of Lyapunov exponents: Exact results for random matrices. Communications in Mathematical Physics, 103:121–126, 1986.

[20] Alexandru Nica and Roland Speicher. Lectures on the combinatorics of free probability. volume 335 of London Mathematical Society Lecture Note Series. Cambridge University Press, 2006.

[21] Alexei Onatski. Detection of weak signals in high-dimensional complex-valued data. available at http://www.econ.cam.ac.uk/faculty/onatski/research.html, 2012.

[22] Alexei Onatski, Marcelo J. Moreira, and Mark Hallin. Asymptotic power of sphericity tests in high dimensional data. available at http://www.econ.cam.ac.uk/faculty/onatski/research.html, 2011.

[23] V. I. Oseledec. A multiplicative ergodic theorem. Ljapunov characteristic numbers for dynamical systems. Transactions of the Moscow Mathematical Society, 19:197–231, 1968.

[24] Yuval Peres. Domains of analytic ergodic theorem. Ann. I. H. Poincare, 28:131–148, 1992.

[25] Mark Pollicott. Maximal Lyapunov exponents for random matrix products. Inventiones Mathematicae, 181:209–226, 2010.

[26] M. S. Raghunathan. A proof of Oseledec’s multiplicative ergodic theorem. Israel Journal of Mathematics, 32:356–362, 1979.

[27] David Ruelle. Analyticity properties of the characteristic exponents of random matrix products. The Advances of Mathematics, 32:68–80, 1979.

[28] David Ruelle. Characteristic exponents and invariant manifolds in Hilbert space. The Annals of Mathematics, 115:243–290, 1982.

[29] David Ruelle. Characteristic exponents for a viscous fluid subjected to time dependent forces. Communications in Mathematical Physics, 93:285–300, 1984.
[30] Dong Wang. The largest eigenvalue of real symmetric, Hermitian and Hermitian self-dual random matrix ensembles with rank one external source, part I. to appear in J. of Stat. Physics, arxiv:1012.4144, 2010.

[31] Fuzhen Zhang. Quaternions and matrices of quaternions. Linear Algebra and Its Applications, 251:21–57, 1997.