Variational Estimation of the Large-scale Time-dependent Meridional Circulation in the Sun: Proofs of Concept with a Solar Mean Field Dynamo Model

Ching Pui Hung1,2, Allan Sacha Brun2, Alexandre Fournier1, Laurene Jouve2,3, Olivier Talagrand4, and Mustapha Zakari1

1 Institut de Physique du Globe de Paris, Sorbonne Paris Cité, Université Paris Diderot UMR 7154 CNRS, F-75005 Paris, France
2 Laboratoire AIM Paris-Saclay, CEA/IRFU Université Paris-Diderot CNRS/INSU, F-91191 Gif-Sur-Yvette, France
3 Université de Toulouse, UPS-OMP, Institut de Recherche en Astrophysique et Planétologie, F-31028 Toulouse Cedex 4, France
4 Laboratoire de météorologie dynamique, UMR 8539, Ecole Normale Supérieure, Paris Cedex 05, France

Received 2017 February 23; revised 2017 October 1; accepted 2017 October 4; published 2017 November 10

Abstract

We present in this work the development of a solar data assimilation method based on an axisymmetric mean field dynamo model and magnetic surface data. Our midterm goal is to predict quasi-cyclic solar activity. Here we focus on the ability of our algorithm to constrain the deep meridional circulation of the Sun based on solar magnetic observations. To that end, we develop a variational data assimilation technique. Within a given assimilation window, the assimilation procedure minimizes the differences between the data and the forecast from the model by finding an optimal meridional circulation in the convection zone and an optimal initial magnetic field via a quasi-Newton algorithm. We demonstrate the capability of the technique to estimate the meridional flow through a closed-loop experiment involving 40 years of synthetic, solar-like data. By assimilating the synthetic magnetic proxies, we are able to reconstruct a stochastic time-varying meridional circulation that is also slightly equatorially asymmetric. We show that the method is robust in estimating a flow whose level of fluctuation can reach 30% about the average, and that the horizon of predictive capability of the method is of the order of one cycle length.

Key words: dynamo – magnetohydrodynamics (MHD) – methods: data analysis – Sun: activity – Sun: interior – Sun: magnetic fields

1. Introduction

1.1. Solar Activity: Observations and Models

The Sun is an active star. Solar activity includes surface magnetic variability, solar eruption, and coronal activity and its effects on planets through magnetic disturbances. The Sun is a nonlinear system, and it is a real challenge to predict its future activity. Since solar activity impacts space weather, which in turn alters our modern technology-based society significantly, it has become increasingly important to obtain good solar predictions. The most common index to quantify solar activity is the sunspot number (SSN). (For recent discussion of the SSN, see Clette & Lefèvre 2012; Clette et al. 2014; Svalgaard & Schatten 2016; Vaquero et al. 2016). Sunspots are dark areas on the solar disk, where a mostly vertical magnetic field of ∼3 kG peak values is present (Stix 2002). In 1850, Rudolf Wolf introduced the relative SSN \( R_s = k(10^g + s) \), where \( g \) is the number of sunspot groups, \( s \) is the number of individual sunspots, and \( k \) is a constant accounting for the differences in observations from various observers and astronomers (Wolf 1850). The corresponding sunspot series started in 1749. In addition to the SSN, the surface magnetic field of the Sun is also an important observable. Observations of the solar magnetic field can be traced back at least as early as 1908 through the pioneering observations of Hale (1908). Systematic daily observations of the solar magnetic field over the solar disk started in the early 1970s at the Kitt Peak National Observatory, with synoptic maps nearly continuously measured from early 1975 through mid 2003 (Hathaway 2010). Tracing the surface radial magnetic field as a function of time and latitude, averaged over longitude, enables the so-called butterfly diagram to be constructed. It shows the position where sunspots appear during a solar cycle and exhibits their phase relationship with the strength of the polar field. One of the most prominent features of solar activity is the quasi-periodicity of the sunspot cycles of 11 years. Those cycles, however, vary in both their period and their amplitude (for more recent time series, consult Svalgaard et al. 2017).

The long-term (multidecadal) variation shows randomness, but with highs in the SSN every seven or eight cycles (Gleissberg 1939; Usoskin 2013). Furthermore, sometimes the solar activity is broken up; periods of such depression are called grand minima. The significant modulation of solar activity raises questions regarding its predictability. Studies suggest that the predictability also depends on whether or not the source of the variability of the solar dynamo is deterministic (Tobias et al. 1998; Ossendrijver et al. 2002; Brandenburg & Spiegel 2008); even a weak stochastic perturbation can lead to a loss in predictability (Bushby & Tobias 2007).

Dynamo models based on magnetohydrodynamics are a common class of models established to account for the solar activity (Charbonneau 2010). The model used in our assimilation framework (to be discussed below) is a dynamo model based on the mean field induction equation in spherical coordinates with azimuthal symmetry. Its mechanism was proposed by Babcock (1961) and elaborated by Leighton (1969). This model can also account for Joy’s law (Hale et al. 1919). Numerical studies of the so-called Babcock–Leighton dynamo model are widely established (e.g., Dikpati & Charbonneau 1999; Jouve & Brun 2007, and references therein).

In this flux-transport solar dynamo model, the meridional circulation in the convection zone is the key ingredient determining the length of the solar cycle. The effects of the meridional circulation on the magnetic cycle and magnetic field are investigated in detail in Jouve & Brun (2007), Hazra et al. (2014), and Belucz et al. (2015). A meridional circulation with one cell per hemisphere is frequently used as a reference in this model to account for the cycle length, maxima, and phase
relationship in solar activity. Additional cells in the radius and latitude can result in different effects on the advection of the magnetic field and cycle length. A two-cell-in-radius meridional flow implies the presence of a return flow at mid-depth, which slows down the transport of the flux from the surface to the tachocline, resulting in a longer cycle length. The flow becomes poleward at the tachocline, thus introducing a poleward branch in the time–latitude plot of the toroidal flow at the base of the convection zone. The toroidal field at the base is weaker than that in the unicellular case as the polar fields are advected from the bottom at low latitudes rather than being brought from the poles. On the other hand, for a dynamo model with a two-cell-in-latitude (in each hemisphere) meridional flow, the cycle length is shorter than in the unicellular case because of the shorter primary conveyor belt, while a poleward branch in the toroidal field at the tachocline is also present, as in the two-cell-in-radius case. For a larger number of latitudinal cells, the toroidal field at the base is also weaker than in the unicellular case as the dynamo is confined to low latitudes where the differential rotation is smaller (Belucz et al. 2015). It is also found that the influence of having several radial cells on the model is stronger than that of adding cells in latitude (see Jouve & Brun 2007, for details).

Although a flux-transport dynamo model with unicellular meridional circulation is commonly used to account for the 11 year solar activity, a recent estimate of meridional circulation from helioseismology below the solar surface suggests the possibility of more complex flow structures. For example, in Zhao et al. (2013a), a meridional circulation with two cells in the radial direction is reported, though the errors of the estimate below 0.80R⊙ (R⊙ is the solar radius) are considerably higher than those at the surface. In Schad et al. (2013), more complicated structures like two cells in radius and four cells in latitude are suggested, based on the perturbation of solar p-mode eigenfunctions by meridional flow. A submerged meridional cell, which disrupts the orderly poleward flow and equatorial symmetry in those years (Haber et al. 2002b), was discovered using the local helioseismic technique of ring diagram analysis of MDI data from 1998–2001. Time–distance helioseismic measurements using GONG data also suggest a multicellular large-scale meridional flow in the convection zone (Kholikov et al. 2014). In summary, there is no unique in-depth conclusion on the meridional flow structure, which also piques interests in estimating the meridional flow with an independent method resting on a dynamo model.

1.2. Solar Prediction and Data Assimilation Methods

Because of the irregular nature of solar activity discussed above, a wide range of solar prediction methods have been developed, from studies of geomagnetic precursors to extrapolation methods based on time series analysis of past activity and correlation studies (Hathaway et al. 1999), to more sophisticated methods using numerical models that simulate the evolution of the system on the basis of the relevant physical equations. Such numerical models require the definition of adequate initial conditions, which are obtained using the technique of data assimilation (Petrovay 2010; Dikpati & Gilman 2006; Pesnell 2016). Data assimilation is an emerging technique in solar cycle and activity prediction, which is a way to incorporate observations in numerical models (Brun 2007). Suppose some solar observations are available on a time interval. By controlling the initial condition and key control parameters of a numerical model, the task of the data assimilation method is to obtain a model trajectory that can best account for the observations.

Modern data assimilation techniques can be split into two general classes, sequential and variational. The Kalman filter and Ensemble Kalman Filter (EnKF) are common methods for the sequential class and make use of observations on the fly, as soon as they are available. For the variational approach, by controlling selected parameters of the physical model, an optimal fit of the data is obtained over the entire time window, making use of all observations available. A common example is 4D-Var, in which the minimization of the objective function can be implemented by the development of an adjoint model (Fournier et al. 2010; Talagrand 2010). The respective merits and drawbacks of the sequential and variational approaches have been discussed at length (see e.g., Fournier et al. 2010, Section 2.2.3 and references therein). Suffice it to say here that both lead to similar answers (identical in the linear case with Gaussian error statistics) and that a sequential method is generally easier to implement than a variational method (which requires the implementation of the adjoint model). The variational approach is more flexible, and it uses all of the observations available over a given time window to define an optimal initial setup at the beginning of the window. For sequential data assimilation, the use of EnKF assimilation in the analysis or prediction of solar activity, for example, is illustrated by Kitiashvili & Kosovichev (2008) and Dikpati et al. (2014). On the other hand, the use of the variational data assimilation method with solar dynamo models is illustrated by Jouve et al. (2011). In that paper, an αΩ mean field dynamo model defined on a Cartesian coordinate system is adopted. A corresponding adjoint model is developed, followed by a twin experiment that successfully estimates the spatial dependence of the physical ingredients of the model, such as the profile and strength of the α-effect. Similar developments based on a flux-transport dynamo model in axisymmetric spherical coordinates are presented in Hung et al. (2015, hereafter Paper I) to estimate the steady meridional flow of the model with synthetic magnetic observations as a first step toward predicting the solar cycle.

In this study, we are going to extend the framework developed in Paper I by adding the initial conditions to the control vector and estimating a time-dependent meridional circulation.

In Paper I, we included the meridional circulation as the main control parameter of our data assimilation pipeline. We verified that the variational assimilation method is capable of estimating the meridional circulation of the model by minimizing the misfit between synthetic magnetic observations and model trajectory. Again, this study assumed a steady meridional circulation. In reality, the solar cycle is significantly modulated, and the meridional flow is fluctuating (Basu & Antia 2010; Ulrich 2010; Komm et al. 2015; for more information on the observations of the meridional flow, see Haber et al. 2002a, 2003; Zhao et al. 2004, 2012, 2013a; Švanda et al. 2007, 2008; Schad et al. 2013; Upton & Hathaway 2014), so the next step of development is to verify the applicability of the method to capture the variability of the modulated activity.
Similar studies were performed recently, for example, by Dikpati et al. (2014, 2016). In Dikpati et al. (2014), a numerical experiment was used to reconstruct the time-varying amplitude of the flow by applying EnKF to the Babcock–Leighton flux-transport dynamo model. In this work, we apply a variational data assimilation method to reconstruct a time-varying meridional circulation by ingesting synthetic observations produced by a dynamo model with meridional flow modulated in both amplitude and shape.

We present our work as follows. In Section 2, we describe the motivation and methodology of the assimilation framework. In Section 3, we present the results of the numerical experiment. We discuss the results of hindcasting in Section 3.1. Then, we investigate the predictive capability of the assimilation procedure and of the model in Section 3.2. Furthermore, we test the robustness of the procedure by inverting the synthetic observations based on a meridional flow with different levels of fluctuations (Section 3.3). We summarize and discuss our results in Section 4. Along with Paper I, in the Appendices, we describe the Babcock–Leighton mean field dynamo model (Appendix A), and we include some details about the algorithm, which incorporates the initial condition in the assimilation procedure (Appendix B). Finally, we give a brief analysis of the observation of the flow at the surface of the Sun (Appendix C).

2. Methodology

2.1. Generation of Synthetic Data Based on a Dynamo Model with a Time-varying Meridional Circulation

We presented in Paper I a first step toward predicting future solar activity levels using variational data assimilation. As a proof of concept, we performed twin experiments for which the assimilated data were produced by the flux-transport (Babcock–Leighton) model itself. Details of the model and its numerical implementation can be found in Appendix A. The system is axisymmetric and we express the magnetic field \( \mathbf{B}(r, t) \) as a sum of its toroidal and poloidal components, and the latter is further expressed as the curl of a vector potential under the axisymmetric assumption:

\[
\mathbf{B}(r, t) = \mathbf{B}_\psi(r, t)e_\psi + \nabla \times [A_\phi(r, t)e_\phi],
\]

where \( e_\psi \) is the azimuthal unit vector, the first and second terms are the toroidal and poloidal components of the magnetic field, respectively, and \( A_\phi e_\phi \) is the vector potential of the poloidal field. The model equations are partial differential equations describing the time evolution of \( A_\psi \) and \( B_\psi \). This model is very similar to the one in Paper I, except that (i) the meridional flow (defined with \( \psi \)) is steady in Paper I but time dependent in this work, and (ii) the diffusion profile is slightly modified here compared with that in Paper I. The axisymmetric meridional circulation is described using a stream function \( \psi(r, t) \), in which \( r \) and \( t \) denote position in the meridional plane and time, respectively.

Since the flux-transport model we adopted was based on a constant meridional circulation, the regular and periodic synthetic activity it generated lacked some of the salient features of solar activity, namely, its variability in cycle length and amplitude. In fact, the duration of the 23 sunspot cycles since 1749 is distributed broadly about 11 ± 3 years.

![Figure 1. Stream functions of those two components of the meridional circulation used to generate synthetic observations. The left one (\( \psi_1 \)) is the stream function for unicellular flow; the right one (\( \psi_2 \)) is the stream function for the equatorially antisymmetric flow. Note that the equatorial parity of the stream function is opposite to that of the corresponding flow.](image)

To be able to account for these important observational facts, we make the meridional flow of our flux-transport model time dependent, and write the corresponding stream function \( \psi(r, t) \) as the sum of a constant (background) term \( \bar{\psi}(r) \) and a time-dependent term (of zero mean) \( \psi'(r, t) \):

\[
\psi(r, t) = \bar{\psi}(r) + \psi'(r, t). \tag{2}
\]

In this study, \( \bar{\psi} \) corresponds to an equatorially antisymmetric, one-cell-per-hemisphere constant flow, the maximum surface amplitude of which is \( v_0 = 22.3 \text{ m s}^{-1} \). This flow pattern will be denoted \( \psi_1 \) henceforth, and its streamlines are shown in the left panel of Figure 1. The integration of the model with \( \bar{\psi} \) alone leads to a regular activity of period 11.5 years.

The fluctuating part \( \psi'(t) \) comprises two components, the amplitudes of which are time dependent: the first is \( \psi_1 \) and the second (\( \psi_2 \) henceforth) corresponds to an equatorially symmetric, two-cells-per-hemisphere (on the meridional plane, one radial node) flow, shown in the right panel of Figure 1. As indicated in this figure, the total flow is therefore a combination of two components and can be written as

\[
\psi(r, t) = c_1(t) \times \psi_1(r) + c_2(t) \times \psi_2(r). \tag{3}
\]

We specify the explicit expression of \( \psi(r, t) \) in this case in terms of its expansion in a chosen set of basis functions in Appendix A. The coefficients \( c_1 \) and \( c_2 \) are constructed as follows:

\[
c_1(t) = 1 + A_1 F[\delta_1(t)], \tag{4}
\]

\[
c_2(t) = A_2 F[\delta_2(t)], \tag{5}
\]

in which each \( \delta_1(t) \) is a random number (drawn from a uniform distribution) with its amplitude normalized so that \( \delta_1 = 1 \) implies a maximum surface velocity equal to \( v_0 \).

Substituting Equations (4) and (5) into Equation (3), we see that in this study the time-independent part \( \bar{\psi}(r) \) is \( \psi_1(r) \), and the time-dependent part is

\[
\psi'(r, t) = A_1 F[\delta_1(t)] \psi_1(r) + A_2 F[\delta_2(t)] \psi_2(r). \tag{6}
\]

The width \( \tau_1 \) of the interval between two consecutive values of \( \delta_1 \) is chosen based on the available observational evidence. As shown in Appendix C, a spectral decomposition of the solar
surface flow inferred by Ulrich (2010) shows that the equatorially symmetric flows are dominant with respect to their antisymmetric counterparts. The autocorrelation times of the amplitudes vary from $\sim 1$ year for the antisymmetric modes to three years and more for the symmetric modes. In this study, for the sake of simplicity, we shall take that time to be three years for both families, and consequently set $t_{\tau} = 3$ years.

We next interpolate in time between two consecutive values of $d_i$ using a sine function, and this interpolation is symbolized by the $F$ operator in Equations (4) and (5). To explicitly define $F$, for any nonnegative integer $n$, suppose random numbers $\delta_{i,n}, i = 1, 2$ are generated at $t = n\tau$, then

$$ F[\delta_i(t)] = \frac{1}{2}\left[\delta_{i,n} + \delta_{i,n+1} + (\delta_{i,n} - \delta_{i,n+1}) \times \cos[\pi(t/n - n)]\right], $$

for $n\tau \leq t < (n + 1)\tau$.

Figure 2 shows an example of a realization of $(c_1, c_2)$, for which the chosen level of fluctuation amounts to 30% of the mean flow (in other words, $A_1 = A_2 = 0.3$ and the maximum surface velocity that can originate from $\psi_2$ alone is 7 ms$^{-1}$, and the maximum total fluctuation at the surface (from $\psi'(r, t)$) can reach $\sim 14$ ms$^{-1}$).

The level of fluctuation in $\psi'$ controls the amount of variability in the simulation, which can be assessed statistically.

We show the histograms of the cycle duration of the model for different fluctuation levels $A_1 (A_2)$ in Figure 3, namely, 10%, 20%, and 30%. The cycle length is defined by the time between two consecutive minima of the modeled magnetic proxy, which will be defined shortly after (Equations (8) and (9)). Each corresponding model was integrated for a long enough time to enable 100 cycles to be achieved. The statistics shown here are separated into their northern and southern contributions. For a perturbation of 30% of the flow speed, the cycle length varies from $\sim 9$ to $\sim 14$ years, which is in reasonable agreement with observations based on the available records of the 23 cycles at our disposal.

Unless otherwise stated, we will use this fluctuation level of 30% in the remainder of this study. An example of a realization of the meridional flow is shown in Figure 4 for the $\theta$
Figure 4. (a) Latitudinal component of the flow at the surface as a function of time, in the case of a fluctuation of the meridional flow characterized by \( A_1 = A_2 = 0.3 \). (See the text for details.) The assimilation period in the numerical experiment that follows is indicated by the dashed vertical lines. The sign convention is positive for a flow due south. (b) Latitudinal component of the flow in time–radius contour plots at latitude 45° (top) and –45° (bottom); same flow setup as in (a).

Figure 5. Top: time–latitude representation of the toroidal field at the tachocline. Bottom: time–latitude evolution of the magnetic field in the line of sight at the surface.
The meridional flow is dominated by a unicellular structure in each hemisphere, with equatorial asymmetric fluctuations. This meridional circulation is chosen for our numerical tests in this work as the unicellular structure is usually observed (Basu & Antia 2010; Ulrich 2010), although helioseismological studies suggest the presence of counter cells in the convection zone (Haber et al. 2002a; Schad et al. 2013; Zhao et al. 2013b). In particular, we present the surface flow in Figure 2, and note again that a similar time variability is also reported in the Sun (e.g., Ulrich & Boyden 2005; Komm et al. 2015).

The plots show the asymmetry of the flow about the equator. The corresponding simulated magnetic field is shown in Figure 5, which shows the advection of the toroidal field toward the equator at the base of the convection zone, and the polar branch at the surface shows that the radial field is advected polewards.

Since the model does not produce sunspots per se, we introduce a proxy for the total SSN in the form of a pseudo-Wolf number \( \tilde{W}^{\theta} \) defined as

\[
\tilde{W}^{\theta}(t) = \left( \int_{\theta = 0}^{\theta = \pi} \int_{r = 0.7r_s}^{r = 0.71r_s} B^{\theta}_{\odot}(r, \theta, t) r^2 \sin \theta \, drd\theta \right)^2,
\]

where the superscript \( ^\theta \) denotes observations and the radial coordinate \( r \) is normalized by the solar radius \( r_s \). We further decompose \( \tilde{W}^{\theta} \) into its north and south components:

\[
\tilde{W}^{\theta}(t) = \tilde{W}_{N}^{\theta}(t) + \tilde{W}_{S}^{\theta}(t),
\]

in which the north (south) component \( \tilde{W}_{N}^{\theta} (\tilde{W}_{S}^{\theta}) \) is computed by restricting the integration in Equation (8) to the northern (southern) hemisphere. In the radius, the integral is restricted to a thin layer (between 0.70\( R_{\odot} \) and 0.71\( R_{\odot} \)) where toroidal flux tubes are thought to originate. The corresponding pseudo-Wolf numbers are shown in Figure 6. As the flow applied is equatorially asymmetric, so are the corresponding magnetic proxies. Furthermore, there is a clear phase difference between \( \tilde{W}_{N}^{\theta}(t) \) and \( \tilde{W}_{S}^{\theta}(t) \), which suggests symmetric and antisymmetric dynamo modes as well (DeRosa et al. 2012). In these figures, note that the y-axis and rightmost dotted lines represent the edges of the 40 year time window over which we will conduct our assimilation experiments.

The synthetic (and noised) time series of \( \tilde{W}_{N}^{\theta}(t) \) and \( \tilde{W}_{S}^{\theta}(t) \) will constitute one kind of synthetic observation used in our assimilation experiments. The other class of data will consist of synthetic (and noised) maps of the line-of-sight component of the magnetic field at the model surface, \( B_{\odot}^{\theta} \), defined as

\[
B_{\odot}^{\theta}(\theta, t) = B_{\odot}^{\theta}(r = 1, \theta, t) \sin \theta = e_r \cdot \nabla \times (A_{\odot} e_\theta)
\]

\[
= (\cos \theta + \sin \theta \partial_\theta)A_{\odot}^{\theta}(r = 1, \theta, t).
\]

The level of noise should be consistent with that of the observations of the Sun. We estimate the noise of the surface magnetic field from the ratio of the coefficient of the monopole component to the coefficient of the dipole component of the observed field (the former, theoretically, should be zero for a noise-free situation). The data are available at WSO, and the ratio is \( \sim 10\% \). For the modeled SSN proxy \( \tilde{W}_{N}, \tilde{W}_{S} \), we refer to real SSN data; the average uncertainty of the data is about 10% of the root mean square of the whole time series (estimated from the sunspot series provided by the Solar Influences Data Analysis Center, SIDC). Therefore, we add 10% noise (with respect to the root mean square of the observables) to the synthetic data \( B_{\odot}^{\theta} \) and \( \tilde{W}_{N}, \tilde{W}_{S} \) for our numerical experiment.

Note that this 10% added noise differs from the stochastic forcing \( A_{\odot} \) of the meridional flow; it is in addition to the fluctuating time series.

2.2. Assimilation Setting

In this section, we describe the data assimilation procedure that we developed to minimize the misfit between synthetic observations and the magnetic trajectories of the dynamo model by estimating the meridional circulation and the initial conditions that give an optimal fit to the data. We also present some technical details in Appendix B. The meridional circulation depends on time, and from a study of the observed
surface flow (Basu & Antia 2010; Ulrich 2010; Komm et al. 2015), the temporal variability is of the order of one year. Therefore, we use an assimilation window of width one year, and we will assimilate data for 40 consecutive years. We should stress at this stage that within this one-year window, the flow is steady. It can vary from one window to the next, if the data demand it; the flow is therefore, mathematically speaking, piecewise constant over intervals of constant width one year. We choose a course of 40 years because in the future we intend to apply our method to invert the magnetic field on the solar surface, using the systematic, daily observations of the field on the solar disk from WSO (which are digitized and available from 1975 onward).

We include the initial magnetic field of the model in the control parameters; this is a new property of our method compared to Paper I. In Paper I, the initial condition was approximated by the magnetic configuration of a dynamo field produced by a model with a steady flow. This approximation gets worse when the field is based on a time-varying flow, to the point where it precludes the success of the assimilation. The assimilation model relies on solving the flux-transport model as an initial value problem, so we need better control of the initial conditions. As a result, we add it to the control vector together with the flow. We then schematically express the control vector \( x \) as

\[
x_n = (x_{n,IC}, x_{n,MC})^T.
\]

Here, the subscript \( 1 \leq n \leq 40 \) denotes the step of the assimilation window. The component \( x_{n,IC} \) represents the initial conditions in the parameter space, and \( x_{n,MC} \) is the meridional circulation, which is represented by \( c_1 \) and \( c_2 \). For our current study, there will be two coefficients representing two different structures of the flow. For the initial condition state vector \( x_{n,IC} \), we will discuss below that we restrict its dimension to \( m = 20 \), and further justify the consistency between this choice and the results in Appendix B.

The initial conditions for the assimilation model, \( A_\phi(r, \theta, t_i) \) and \( B_\phi(r, \theta, t_i) \), where \( t_i \) is the beginning of the assimilation window, are defined on a grid of \( n_r \times n_q = 129 \times 129 \) points. However, if we represent the initial condition pointwise in the parameter space, the number of parameters \( (2n_r n_q \sim 32,000) \) will be too large compared to the number of observations, which will result in overfitting. For the latter, let \( N_p^\ell \) and \( N_p^q \) be the number of sampling in time and latitude respectively, and the total number of observations \( N^o = N_p^\ell N_p^q + 2N_\phi^o \). (The second term on the right-hand side corresponds to \( \hat{W}_p^\ell \) and \( \hat{W}_p^q \) (if they are included as observations).) At the same grid size, an assimilation window of one year (sampled on a monthly basis, i.e., \( N_\phi^o = 12 \)) of the surface magnetic field (spatial sampling in every latitudinal grid point except at the poles \( N^\ell_p = 127 \)) only gives \( N^o \sim 1500 \). To stay realistic, we do not want an artificially fine sampling in time, which, of course, can give a higher \( N^o \). In practice, the sampling frequency of the real magnetic field is, for example, daily in WSO down to 45 s cadence with HMI onboard the SDO satellite (Schou et al. 2012). The latitudinal resolution of the real data also depends on the instrument used.

For instance, the maximum spherical harmonic degree \( \ell_{\max} \) is 60 for WSO maps and about 190 for MDI (note that HMI has 16 times the resolution of MDI; Scherrer et al. 1995). Therefore, we choose to represent the magnetic field on the meridional plane with a truncated set of basis functions rather than pointwise. This comes down to constructing the covariance matrix of the dynamo magnetic field \( P \) to account for the magnetic variability of the Sun. We define and discuss the construction of such a covariance matrix \( P \) in detail in Appendix B. We can then describe our initial magnetic state by retaining only the most prominent eigenvectors of \( P \) as a basis.

Figure 7 shows the eigenvalue spectrum \( \lambda \) and the approximation of the magnetic field driven by a simple uncellular meridional flow with the eigenbasis of its own covariance matrix. We define the error of approximating the field as

\[
dX/X = \sqrt{\int_D (X_{\text{approx}} - X_{\text{true}})^2 \, da / \int_D X_{\text{true}}^2 \, da},
\]

where \( da \sim r \, dr \, d\theta \), \( X \) is \( A_\phi \) or \( B_\phi \), and \( dX/X \) is the error in approximating \( A_{\phi,\text{true}} \) or \( B_{\phi,\text{true}} \) with \( A_{\phi,\text{approx}} \) or \( B_{\phi,\text{approx}} \), respectively. The domain of integration, \( D \), is the meridional plane.

We can see that we get a good approximation with only \( m = 20 \) basis functions (Figure 7 (b)), so we will limit the number of parameters for the initial condition at \( m = 20 \) under this representation. We also update the covariance matrix \( P_\ell \) with the most recent forecast at the end of each year \( n \). The dimension of \( x_{n,IC} \) is 20; together with the two parameters in \( x_{n,MC} \), the dimension of the parameter space is 22, well below \( N^o = 1500 \).

For the first year of the assimilation window, the initial guess for the initial condition \( (x_{1,IC}^g) \) and meridional flow \( (x_1^g_{MC}) \) comes from a dynamo model based on a uncellular flow with a magnetic cycle of 22 years (the superscript \( g \) stands for “guess”). Then, for the subsequent data assimilations, the initial guess will be the forecast magnetic field and velocity at the end of the previous assimilation. The former is obtained by evolving the dynamo model for one year with \( x_{n-1,IC} \) and \( x_{n-1,MC}^g \) (with the superscript \( f \) standing for forecast); the latter is simply \( x_{n-1,MC}^g \). Within each one-year window, we estimate the coefficients of the stream function and initial condition which give the minimum misfit, and consequently, we obtain an estimate of the time variation of the flow profile in Figure 2 by approximating it with a piecewise constant function.

A schematic view of the procedure is shown in Figure 8. We start from a guess state representing a dynamo model based on a uncellular meridional circulation \( x_{\ell,IC}^g \), \( x_{\ell,MC}^g \) and with the input of the magnetic observations of the first year, we get the forecast state \( x_{\ell,IC}^g, x_{\ell,MC}^g \) from the assimilation procedure. Based on the forecast state, we can evaluate the initial guess state \( x_{\ell,IC}^g, x_{\ell,MC}^g \) of the second year, and we repeat the assimilation procedure when new observations are available.

Note that the covariance matrix is evaluated and therefore modified after each year, so that the projection on the corresponding truncated eigenbasis gives a good approximation of the initial condition in each assimilation.

2.3. Data and Objective Function

We aim to minimize an objective function defined in terms of the differences between the observations and the model trajectory,

\[
\mathcal{J} = \sum_{\alpha} \sum_{i=1}^{N_{\phi,\alpha}} \sum_{j=1}^{N_{\phi,\alpha}} \frac{[y_\alpha(\theta_j, t_i) - y_\alpha^c(\theta_j, t_i)]^2}{\sigma^2_\alpha(\theta_j, t_i)},
\]

where \( y_\alpha(\theta_j, t_i) \) are the observational data in latitude \( \theta_j \) and time \( t_i \), and \( y_\alpha^c(\theta_j, t_i) \) are the corresponding model values. The summation is performed for all \( \alpha \) parameters and all observational time points. The term \( \sigma^2_\alpha(\theta_j, t_i) \) represents the uncertainty in the observations, which is typically estimated from the model itself. The objective function \( \mathcal{J} \) essentially measures the squared difference between the model and the observations weighted by their uncertainties.
\(\alpha\) denotes the type of magnetic proxy \(y\) to be compared. The proxies with the superscript \(^o\) stand for observations, and those without a superscript for the forecast values; \(\sigma_\alpha\) stands for the uncertainty of the measurement. For each type \(\alpha\), we sum the observations over the observation times and latitudes, \(N_{\alpha,t}\) and \(N_{\alpha,\theta}\), respectively. Recall that \(J\) is defined over an interval of total duration one year.

As mentioned above, the synthetic observations used for the experiment are the magnetic sunspot proxy (Equation (8)) and the surface line-of-sight magnetic field \(B^t_{\text{los}}\) (Equation (10)). Historically, sunspot series given in Wolf numbers started from 1749, and daily, continuous, and digitized observations of the surface magnetic field of the Sun became available later. Therefore, we first look for the possibility of estimating the (synthetic) time-varying flow with the assimilation procedure by ingesting the modeled synthetic sunspot proxy (Equation (8)) as the only observable. This is to investigate the feasibility of estimating the meridional circulation of the Sun since 1749. However, this would be more difficult as SSN is only one value (two for hemispheric SSN) at a particular observation time, instead of a latitude map provided by \(B^t_{\text{los}}\).

We first make this relatively more challenging by attempting to assimilate the synthetic hemispheric sunspot proxy only, with an assimilation window of one year, with various sampling frequencies, from monthly to every six days. We find that in these attempts, the estimate of flow in the first-year assimilation is not physical, as the surface flow is found to be 20 times higher than the truth. This gives an unstable dynamo model for further assimilation after the first year, making the algorithm unstable. This is because the information contained in the data is not rich enough to estimate the meridional circulation as well as the initial magnetic field within the assimilation window concerned. Moreover, with the SSN alone, there is a sign ambiguity for the magnetic field. Furthermore, we showed in Paper I that compared with temporal dependence in the observations, latitudinal dependence is more important for the estimation of internal dynamics.

To proceed, we can add more information to the pipeline. For example, we can add constraints to the optimization

Figure 7. (a) Eigenvalue spectrum of the covariance matrix of the dynamo field for a steady unicellular flow. The eigenvalues \(\lambda\) are normalized with the greatest eigenvalue \(\lambda_{\text{max}}\) in the plot. (b) Error in the approximation of the magnetic field of the same dynamo field at a particular time \(t_0 = 2.91(R^2/\eta)\) as a function of the size of a truncated eigenbasis. Black: error in the poloidal field. Red: error in the toroidal field.
procedure based on physical knowledge as a background term in the objective function, which does not need more observations. Or, we can add more observations within the assimilation window. In this study, we are going to include the characteristics of the butterfly diagram in the observations. As a result, we introduce more observations with spatial data distribution in order to help the minimization of the objective function. A more effective objective function to be minimized can then be

\[
\mathcal{J} = \sum_{i=1}^{N^o} \left\{ \sum_{j=1}^{N^o} \frac{[B_{los}(\theta_j, t_i) - B_{los}(\theta_j, t_i)]^2}{\sigma_{B_{los}}^2(\theta_j)} + \frac{[W_N(t_i) - \tilde{W}_{N}(t_i)]^2}{\sigma_{W_N}^2} + \frac{[W_S(t_i) - \tilde{W}_{S}(t_i)]^2}{\sigma_{W_S}^2} \right\},
\]

where \(\sigma_{B_{los}}, \sigma_{W_N}, \) and \(\sigma_{W_S}\) in this study are the fractions of the root mean squares of the line-of-sight surface field, and the synthetic SSN proxy in the northern and southern hemispheres, respectively, in order to model the uncertainties of the data, i.e.,

\(\sigma_{\alpha}(\theta_j) = \epsilon_{\alpha} \sqrt{\langle \gamma_{\alpha}^N \rangle^2}, \) where \(\epsilon_{\alpha}\) is the level of noise of the species \(\alpha\) and \(\langle \cdot \rangle\) is averaging over time. As stated in Section 2.1, the noise levels added to \(B_{los}^o\) and \(\tilde{W}_{N}, \tilde{W}_{S}\) for our numerical experiment are of order 10%.

The total number of observations is \(N^o = (N_{\theta}^o + 2)N_t^o\). In our case, within an assimilation window of one year, sampling monthly \(N_t^o = 12\) and uniformly in latitude \(N_{\theta}^o = 127\), we have \(N^o = 1548\). (Here we also tested that for a coarse sampling in latitude, say \(N_{\theta}^o = 63\), we can get similar results and performance. For a systematic study of the effect of latitude sampling on the assimilation procedure, see Paper I).

The normalized misfit is defined as

\[
\mathcal{J}_{\text{norm}} = \sqrt{\frac{\sum_{\alpha} \mathcal{J}_{\alpha}}{N_{\alpha}^o}}.
\]

An optimal fit gives \(\mathcal{J}_{\text{norm}} \sim 1\), while \(\mathcal{J}_{\text{norm}} \gg 1\) indicates that the misfit is too large considering the noise added to the synthetic observations, and \(\mathcal{J}_{\text{norm}} \ll 1\) implies statistical overfitting.

Figure 8. Schematic diagram illustrating the data assimilation procedure used in this study. Integer subscripts refer to discrete time indices.
3. Results of Assimilation Pipeline

In this section, we demonstrate that by assimilating, in a sequence of windows of width one year, the synthetic observations displayed in Figures 5 and 6, we are able to estimate the meridional flow shown in Figure 2. We illustrate the data of 40 years under study and the first year of data for assimilation in Figure 5 with broken dashed lines. We start the assimilation with a unicellular flow as an initial guess for the meridional circulation. For the initial condition on the magnetic field components $A_\phi$ and $B_\phi$ for the first year of the assimilation, we conduct two trials with two different guesses. The first guess is a dynamo field based on a unicellular flow ($c_1 = 1$, $c_2 = 0$ in Equation (3)), where the fields have a definite parity about the equator, i.e., symmetric for $A_\phi$ and antisymmetric for $B_\phi$. The second guess is a dynamo field based on a unicellular flow but slightly modified with an antisymmetric flow which contributes to 1% of the $v_\theta$ (of the background flow at the surface). The flow is then slightly asymmetric and so does the corresponding dynamo field. The motivation behind the second trial is an attempt to account for the equatorially asymmetric nature of the synthetic observations in Figures 5 and 6. We discuss separately the hindcast of the data assimilation for 40 years and the ability of the model to forecast beyond the 40th year. For the latter, we estimate the magnetic field 25 years after the latest assimilation, making a total study of 65 years. For clarity and convenience in the discussion, in the following, in our figures where a time evolution is shown, $t = 0$ corresponds to the time at which we start to ingest observations, i.e., $t = 0$ at the left broken vertical line in Figure 5, at (model time) year 1144 of the synthetic observations.

In the following, the term dynamical trajectory refers to the time series of the magnetic field in the computational domain, as predicted by the numerical dynamo model. The true, or reference, trajectory is the one obtained using the combination of the control parameters, initial condition, and time-dependent meridional flow used to generate the synthetic data. This reference trajectory serves as a gauge to evaluate the quality of the assimilated trajectory. The assimilated trajectory has an initial magnetic field vector and initial meridional flow that are not those of the reference trajectory, and the goal of the assimilation is precisely to have this trajectory get closer to the true trajectory. In contrast, the term free run refers to the trajectory obtained, starting from this wrong initial setup, without assimilating any data.

3.1. Hindcast by Assimilation of the Synthetic Data

In this section, we discuss the results of the reconstructed meridional circulation, and the misfit of the data and the estimate of the magnetic field when data are available. This is possible, as under the basis of the numerical experiment the flow driving the dynamo and the resulting magnetic field on the meridional plane are known.

3.1.1. Reconstruction of the Time-varying Flow and Minimization of Data Misfit

We show the estimated coefficients of the stream function in Figure 9. By inverting the data, the estimated profiles capture the temporal variation of the stream function reasonably, except at the beginning.

With a unicellular prior for the flow, the difference between the estimate and the reality is obvious for the first few years. The synthetic observations are asymmetric about the equator as they are based on an equatorially asymmetric true flow. The prior for the flow in the first year is symmetric and so are the corresponding dynamo field, the covariance matrix, and eigenbasis of the guess dynamo model. Therefore, such a prior cannot take the asymmetry of the observations into account at the onset of the data assimilation. As the model in the assimilation technique involves solving an initial value problem where the initial conditions are important, the estimation of the meridional flow is inaccurate. However, in the data assimilation of subsequent years, the estimated flow starts capturing the asymmetry, and so does the forecast dynamo field. The corresponding updated covariance matrix gives an eigenbasis that can account for an asymmetric configuration. This shows the ability of the method to adjust the model to give a better approximation of reality. As a result, the estimation of the flow improves starting as soon as the second year. In Figure 10, we plot the stream functions corresponding to the estimated coefficients (Figure 9) and (10 times) the differences between the estimate and the truth in some selected years. It clearly shows that the error in the estimate of the flow decreases at the beginning.

In the trial with a prior based on a slightly equatorially asymmetric flow, as early as the first year, the covariance matrix is able to account partially for the asymmetry of the observations. Therefore, the estimation of the flow in the first year is better than that obtained using a prior based on a purely unicellular flow. We can also identify such a behavior when evaluating the misfit of the synthetic observations in Figure 11. Depending on the assumed prior in the first year, the normalized misfit is considerably higher than unity for the first 5–10 years. It converges toward unity after ~10 years and remains in very good agreement afterwards. Also, irrespective of the first-year assumed prior, the reconstructed flow and the misfit converge to the same value respectively after about 5 ~ 10 years of warm-up time.

The implication here is that the outcome of the assimilation in the first few years depends highly on the initial guess of the initial conditions in the first data assimilation window in this implementation.

Next, we show in Figure 12 the distribution of the misfit of the surface line-of-sight magnetic field as a function of latitude. As the artificial noise added to generate the synthetic observations is normally distributed, theoretically, for optimal fitting, 68% and 95% of the sampled misfit should fall within one times and twice the noise level, respectively. The plot shows such a consistency.

The statistics of the initial guess of the dynamo field determine the basis of representation of the initial conditions in the control parameter space. An initial guess closer to reality gives a more complete representation and vice versa. This can be improved after assimilation in subsequent years. Therefore, there is a spin-up time for the assimilation procedure to adapt to reality, but it is reasonably short compared to the interval over which data are available.

3.1.2. Estimation of the Magnetic Field and Proxies

With the estimate of the parameters $\{x_i\}_{i=0}^{m}$ (flow, initial condition) from the assimilation procedure, we can reconstruct the magnetic field and the magnetic proxy (Equation (8)) within...
the 40 consecutive years. We compute the estimated magnetic field, $A_i$ and $B_i$, and compare it with the true magnetic field on the meridional plane (Figure 13). We measure the (relative) error with Equation (12). We show the difference in Figure 14. The initial guess is based on a unicellular flow. After the first five years of assimilation to capture a dynamo model closer to the truth, the relative errors in the estimated field (inside the 40 consecutive years of the assimilation windows) stay within 10% of the reality. This is about the same as or slightly more than that of the representation of the magnetic configuration introduced in Section 2.2 (Figure 7(b)).

Since the initial magnetic field is in the control parameter space in this assimilation procedure, we also show the estimate of the magnetic field on the meridional plane at the first and 10th years, the reality, and (10 times) their differences in Figure 13. This also shows that the procedure cannot pick up the asymmetry of the field on the first year when the prior is based on a unicellular flow, but the asymmetry can be recovered as the assimilation time evolves (as shown in year 10).

It is believed that the SSN is closely related to the toroidal field in the tachocline (Parker 1993; Charbonneau & MacGregor 1997; Choudhuri et al. 1995; Dikpati & Charbonneau 1999), so it is important to study the effect of data assimilation with the modeled Wolf numbers $W_N^o$ and $W_S^o$ on the reconstruction of the magnetic field. We compare our reference case with the case where only the surface magnetic field is used as the observations in Figure 15. We also present a free run of a 22 year dynamo model, based on a unicellular meridional flow, and evaluate the difference from our reference model in the same figure for comparison. The free run is the situation when there is no data assimilation. In the presence of the synthetic sunspot-like proxy as observations, there is only a tiny improvement in the estimated toroidal field, while the estimated poloidal field is more or less the same. This is consistent to the case we showed earlier, that $W_N^o$ and $W_S^o$ alone do not give enough information for a reasonable estimate of the state vector. The spatial dependence of the observation is important, and such a dependence of the proxy is lost for $W_N^o$ and $W_S^o$ as it is defined as an integration over latitudes. Of course, as discussed earlier there are ways to improve the data assimilation algorithm based on SSN data only. Beyond the 40 year interval of analysis, the error increases when no data are available.

Notice that there are two subtle features of the error in the estimate. (i) The tiny and discontinuous rises in error at the beginning of the yearly assimilation windows shown in Figure 14 are due to an update of the truncated eigenvector of the covariance matrix for each year of assimilation, which are also within a few percent of the true field.

(ii) The errors in Figure 14 show a nearly periodic rise and fall for every sunspot cycle. As this is an evaluation of the relative error of the dynamo field, and the dynamo field possesses a modulation of cycle $\sim 11$ years (or a magnetic cycle of $\sim 22$ years), the relative error can be large if the dynamo field is small. We show the absolute error of the estimate in Figure 15, in which there is no such periodicity in the error.
Figure 10. True and estimated stream functions at different epochs during the assimilation experiment. Also shown at various epochs are 10 times their differences (estimate – reality).
(However, Figure 14 illustrates the size of the error compared with the value of the field, which is not illustrated in Figure 15). In Figure 15, we compare the error with the difference between the true trajectory and that of a free dynamo run with a simple unicellular flow, without assimilation. Compared with the free run, the error decreases in the first five years, and then the estimated field stays close to the true field until the end of the 40 year series. We clearly see the advantage of assimilating data.

Furthermore, we show the fitting of the surface magnetic field at latitudes \( \pm 20^\circ \) in Figure 16. We also show the free run trajectory based on a unicellular flow as reference. The synthetic observations are based on an equatorially asymmetric flow, so the observations are asymmetric about the equator. As a result, we clearly see that the free run quickly goes off track. We also note that the free run trajectory based on the symmetric unicellular flow only fits the observations reasonably in one hemisphere but not in the other (in this case, it gets close to the data in the northern hemisphere.) With the assimilation procedure, taking into account the monthly observations each year, the estimated surface magnetic field reconstructed from the forecast flow gives a smaller misfit in both hemispheres, and clearly the asymmetry is accounted for. During the first few years, the misfit is slightly higher than in later years; as the prior is unicellular flow, it takes time for the procedure to adapt to the asymmetry. Similar results are also observed for the reconstruction of the modeled hemispheric sunspot proxy \( W_N^d \) and \( W_S^d \), defined from the estimated toroidal field at the tachocline, in Figure 17.

Thus, we can conclude that our method is robust and able to reconstruct complex, possibly asymmetric, internal flows from observations of the surface magnetic field, and it yields good agreement with the activity in both hemispheres. We can now test how well it performs for forecasting.

### 3.2. Forecast of the Magnetic Field and Proxies Beyond the Assimilation Window

In this section, we discuss the predictive capability of the procedure based on this flux-transport model. We estimate the magnetic field beyond the 40 years of assimilation, i.e., without assimilation, by evolving the dynamo model in time, based on the forecast magnetic field and the flow at the end of the 40th year.

We show in Figure 15 the difference between the true field and the field obtained from the model beyond 40 years of assimilation. The error starts to grow for \( 10 \sim 20 \) years but remains smaller than that of the free run. After that, it saturates and the magnitude of the error is of the same order as or slightly lower than that of the free run trajectory.

Therefore, if we try to predict the magnetic observations by extrapolating the model based on the magnetic field and the flow at the end of the hindcast, the prediction is reliable within 10 years if we are conservative, and up to 20 years with a low confidence level. After 20 years, there is essentially no predictive capability in this experiment. This is longer than the timescale of the fluctuations, \( \tau = 3 \) years, added to the reference flow to produce the synthetic observations. The reasons are (i) the modeled flow contains a non-fluctuating part \( \bar{\psi}(r) \), which is also captured during the assimilation process, and (ii) the long-term average of the fluctuations is zero, so that the assimilation procedure results in recovering the long-term averaged flow up to a certain extent.

In Figure 16 and in the modeled sunspot-like proxy in Figure 17, we also show the model trajectory after 40 years when no assimilation is performed. In particular, for the modeled sunspot proxy in Figure 17, the trajectory still reasonably fits the observations after 40 years, for \( 1 \sim 2 \) cycles (10 \sim 20 years). Then, the trajectory diverges from the observations after 20 years, but is still closer in phase compared with the free run. We are then confident that our data assimilation model can provide improved predictions in each hemisphere for up to 15 \sim 20 \) years.

### 3.3. Numerical Experiments with Synthetic Data Based on Different Levels of Stochastic Fluctuation on the Flow

We showed the estimation of the profile of the flow through the data assimilation technique using synthetic observations from the flux-transport dynamo model with 30% fluctuations on the meridional circulation. In this section, we study the performance of the assimilation algorithm with respect to the
magnitude of the fluctuations on the meridional flow when generating the synthetic observations. We test the assimilation method with synthetic observations with 10% and 20% fluctuations, together with the 30% case illustrated above.

The fluctuations in the cycle length of the synthetic observations increase with the level of the fluctuation introduced in the meridional circulation. As illustrated in Figure 3, the spread of the cycle length increases with the level of fluctuation of the flow. For the 30% case, the range of the distribution is comparable to the sunspot cycles, but the lag between the northern and southern hemisphere is perhaps too large compared to that of the real Sun, so the lower level of antisymmetric fluctuations in the flow is useful to assess.

To compare the fitting of the synthetic observations of the flow among three tests (10%, 20%, and 30% fluctuations), we show the corresponding integrated difference between the estimated and true toroidal fields in Figure 18. As the reference case, both the synthetic sunspot proxy and the surface line-of-sight magnetic field are taken as observations. During the 40 years of assimilation, the absolute difference between the truth and estimate increases with the magnitude of the fluctuation of the flow from which the synthetic data are produced. This is because the approximation of the true flow with step functions becomes less accurate as the fluctuation level (effectively the slope of the profile with respect to time) increases. As a result, the difference from the true magnetic configuration is higher. For the hindcast and forecast processes discussed above, i.e., the corresponding results shown in Figures 15–17, the error in the estimate of the field and the predictive capability are of the same order, but is better for lower fluctuation level $A_n$, as could be expected (Figure 18).

Note that in this study, the fluctuation level added to the meridional flow has little effect on spin-up time. The time for the integrated difference to decrease and get flattened about unity is $\sim$6 years for the 10% case, and $\sim$10 years for the 20% and 30% cases. Therefore, the spin-up time for the assimilation procedures to adapt to reality depends mostly on the initial guess of the magnetic configuration in the first year of the assimilation pipeline. Only a guess closer to reality can shorten the spin-up time of the procedure.

We conclude from this study that the assimilation procedure is robust with respect to the fluctuation level of the time-varying flow to be estimated. This is important as the latter affects the time variability of the cycle length (see Figure 3).

4. Discussion and Summary

Our numerical experiment shows the capability of data assimilation to estimate the deep meridional circulation of the Sun using magnetic proxies. As a preparation for analyzing real magnetic observations and for predicting the solar activity in the future, (in particular cycle 25), we adjust the flux-transport model to have solar-like properties such as an 11 year cycle period but modulated in both amplitude and frequency, and a time-varying meridional flow that may be asymmetric with respect to the equator. A stochastic time-varying meridional circulation produces fluctuations in the cycle period and amplitude, which make the simulation more solar-like compared with a dynamo model with a constant meridional flow in terms of the irregularities. We construct synthetic magnetic proxies, like the surface line-of-sight magnetic field and the SSN, by relating them to the surface poloidal field and the toroidal field in the tachocline computed with the flux-transport dynamo model. We also add noise to the data, and the level of noise is consistent with the observations from the real Sun ($\sim$10%; recall Section 2.1).

For the data assimilation method, we now include the initial conditions of the dynamo model as extra control parameters. The representation of the initial conditions is based on the statistical covariance of the dynamo model. We implement this extension within the corresponding adjoint model, such that the resulting framework is capable of estimating the meridional flow as well as the magnetic field within the convection zone throughout the assimilation window. We find that the spectrum of the covariance matrix peaks sharply; this enables a good approximation of the magnetic configuration on the meridional plane by projecting it on a truncated eigenbasis (in our test, 20 eigenmodes are taken) with the dominant eigenvalues, which facilitates the calculations.

We then show that by ingesting the synthetic (monthly) observations on a yearly basis, and within each year applying
the 4D-Var assimilation method, we are able to reconstruct the time-varying flow over the 40 years of the test period very well. The normalized misfit of the data, close to unity, indicates an optimal fit in a statistical sense. We also show that the method is robust for synthetic observations based on stochastic variations of the flow up to at least 30% in terms of the reconstruction of the flow and normalized misfit of data. By studying the time evolution of the differences between the true magnetic field from the data and the forecast magnetic field, and by further comparing it with a free dynamo run where no data assimilation is done, we conclude that in this experiment, the predictive capability of the method is about 15–20 years for the 30% fluctuation in the flow akin to the Sun (exceeding two sunspot cycles for lower fluctuation levels). Starting from a simple equatorial symmetric dynamo field and unicellular meridional flow, the method can give an asymmetric forecast field as well as asymmetric meridional flow, hence it is not impaired by symmetry of any sort. This is a strength as solar poles are known to reverse with a lag of up to two years (Shiota et al. 2012; DeRosa et al. 2012). Although there is a spin-up lasting the first 5–10 years of the assimilation, it is short compared to the period over which data are available; its duration is barely affected by the level of stochastic variation.

Though we prove the performance of the assimilation procedure with synthetic observations produced by the same flux-transport dynamo model, this is not exactly a twin experiment since we use step functions to approximate the flow in our assimilation model instead of trying to reconstruct the exact time-dependent flow that generates the data in Figure 2. In generating the synthetic observations, the choice of the fluctuation level of the antisymmetric component \( A_2 = 0.3 \) may seem excessive, given the resulting phase difference between both hemispheres (up to four years as opposed to one to two years for the Sun). Regardless, we show that our pipeline is capable of reconstructing such an asymmetric configuration, while disentangling the contribution of both symmetric and antisymmetric flow components to the simulated solar activity. In summary, we are confident that our data assimilation pipeline is robust and a promising tool for studying past and future solar activity.

There are, however, several limitations regarding the model and method used in our study. As the flow is perturbed in a stochastic manner, the predictability is limited by the timescale of the stochasticity, three years in our case. An alternative is to introduce fluctuations in the flow in a non-stochastic manner. For example, including the flow as a dynamical variable of the

---

**Figure 13.** Top: meridional plots of the magnetic field at year 1 of the assimilation experiment. From left to right: true poloidal field, estimated poloidal field after assimilation, and ten times the differences of these two; true toroidal field, estimated toroidal field after assimilation, and ten times the differences of these two. Bottom: same, for year 10 of the assimilation.
model, which is coupled nonlinearly with the magnetic field, requires a different formulation and closed equation set. This more deterministic behavior could actually be more easily captured than a purely random variability. The long-term amplitude modulation, such as the Gleissberg cycle, is also absent in the present model, and both can be implemented in future work. Regarding the assimilation, we approximate the time-varying profile of the flow with a linear combination of step functions. However, we can see that with a higher fluctuation on the flow, the effect of the slope of the profile becomes important. The approximation with piecewise constant values will probably give a slightly higher misfit. Therefore, a better approximation of the flow in the assimilation routine is necessary. Thus, one next important step of the improvement is to add the slope of the flow, i.e., the acceleration, to the control vector of the assimilation framework. This will double the number of parameters to represent the stream function. We show that the method is robust in this relatively hard version of the non-constrained numerical experiment. On the other hand, it is possible to extend the applicability of the pipeline by introducing physical constraints to the framework. For example, in the case where we made an attempt to hindcast with $W_N^p$ and $W_S^p$ alone, including more physical information in the form of a background term is a possible improvement.

At this stage, it may be worthwhile to compare our approach and results with those obtained recently by Dikpati and colleagues (Dikpati et al. 2014, 2016, D16 henceforth for the latter). D16 carried out a set of numerical experiments using a sequential assimilation method (the EnKF) applied to a mean field dynamo model that closely resembles the one we use in this study. The purpose of their proof-of-concept experiments (which rest on synthetic data) is to assess the capability of their method to capture the time-dependent behavior of the meridional circulation. To that end, they generate a set of synthetic observations based on a reference trajectory obtained by prescribing a time-dependent meridional circulation. Their meridional circulation has a fixed, one-cell-per-hemisphere configuration, and its time dependence is restricted to its amplitude. The amplitude has a steady and time-varying part.

The time-varying part is deterministic and controlled by a few modes of oscillations with periods of a few years to a decade (see their Figure 1). These deterministic oscillations yield fluctuations of about 40% about the mean (a figure similar to the 30% fluctuations that we generate, but in a stochastic fashion, in this study). Their synthetic observations consist of values of the poloidal (at the top of the convection zone) or toroidal fields (at the bottom of the convection zone). They vary the location and density of observations in their experiments. The true, reference, values are affected by an uncertainty corresponding to a noise level of 4%. This has to be contrasted with observations of the pseudo-number of sunspots and the radial induction in the line of sight used here (affected by relative errors of 10% throughout our study). Dikpati et al. convincingly show that by carrying out an analysis every two weeks (over the course of their 35 year long experiments) using the EnKF, they can recover the time-dependent amplitude of the meridional flow using an ensemble size of 192 members, with each analysis being applied to 10 observations consisting of near-surface poloidal fields from low latitudes and tachocline toroidal fields from mid-latitudes. Success in accurately retrieving the time-dependent amplitude depends on the locations of the available observations (with those at high latitude being less valuable). They also find that a much shorter or longer interval between each update is detrimental to the success of the assimilation. Too short of an interval (e.g., five days) does not allow the system to respond dynamically to a change in the flow amplitude, whereas a too large interval between two updates causes the trajectory of the assimilated system to depart excessively from the “true” trajectory. They do not discuss the predictive capability of their system in the study (recall that we find in our synthetic setup a practical horizon of predictability of about 15 years). Our findings are overall in line with those of D16, in the sense that partial and noised observations of a kinematic dynamo with time-dependent flow features can be used to rather accurately estimate the time-dependent flow in the bulk of the system (not only where observations are available) by using an interpolation based on a physical model (this is essentially what data assimilation is about). The differences between their study and

![Figure 14. Relative difference between the magnetic field estimated from data assimilation and the true magnetic field vs. time, shown in black (red) for the poloidal (toroidal) field.](image-url)
Figure 15. Top: absolute difference between various estimates of the toroidal magnetic field and the true magnetic field vs. time. Blue: free run of the dynamo model (unconstrained by data). Black: data assimilation estimate, with data consisting of magnetic fields in the line of sight and the pseudo-sunspot number. Red: data assimilation estimate, with data restricted to magnetic fields in the line of sight. Bottom: same, but for the poloidal magnetic field.

Figure 16. Time series of the surface magnetic field in line of sight at latitude 20°. Red: free run of the dynamo model (unconstrained by data). Circles: monthly data extracted from the reference time series. Blue: reference time series. Light blue: data assimilation estimate. Green: time series of the forecast. (b) Same for the field at latitude −20°.
To conclude, we presented here an assimilation method to estimate a time-varying meridional circulation with synthetic magnetic proxies. The method is robust with an optimized data fit and gives a predictive capability of 1 ~ 2 sunspot cycles, depending on the amplitude of the fluctuating part of the sought flow. Future developments include (i) analyzing the magnetic proxies of the real Sun with the data assimilation method, (ii) improving the representation of the meridional circulation in the assimilation framework (e.g., by taking into account the acceleration of the fluid), and (iii) including physical constraints in the objective function.

We acknowledge the financial support of the UnivEarthS Labex program at Sorbonne-Paris-Cité (ANR-10-LABX-0023 and ANR-11-IDEX-0005-02) through project SolarGeoMag. We also acknowledge the support from the ERC PoC SolarPredict project, CNES Solar Orbiter and INSU/PNST grants, and Idex SPC through the DAMSE project. We are grateful to Roger Ulrich for giving us digital access to his surface meridional circulation measurements. We also thank SIDC for access to their sunspot series observations. Wilcox Solar Observatory data used in this study were obtained via the Web site http://wso.stanford.edu on 2017 September 29 03:15:25 PDT, courtesy of J.T. Hoeksema. A.S.B. thanks M. DeRosa and A. Title for useful discussions. Numerical computations were performed on the S-CAPAD platform.
Appendix A

The Babcock–Leighton Flux-transport Mean Field Dynamo Model

This appendix gives a brief description of the flux-transport mean field dynamo model, i.e., the Babcock–Leighton model, with axisymmetry. This is the model used for the assimilation procedure and to generate synthetic observations for our numerical experiment to verify the data assimilation technique. The model equations are (Dikpati & Charbonneau 1999; Jouve & Brun 2007; Jouve et al. 2008; Hung et al. 2015)

\[
\begin{align*}
\partial_t A_\phi &= \eta \left( \nabla^2 - \frac{1}{\omega^2} \right) A_\phi - Re \frac{v_p}{\omega} \cdot \nabla (\omega A_\phi) \\
&\quad + C_s(r, \theta, B_\theta), \tag{16}
\end{align*}
\]

\[
\begin{align*}
\partial_t B_\phi &= \eta \left( \nabla^2 - \frac{1}{\omega^2} \right) B_\phi + \frac{1}{\omega} \frac{\partial (\omega B_\phi)}{\partial r} \frac{\partial (\eta/\eta_t)}{\partial r} \\
&\quad - Re \frac{v_p}{\omega} \cdot \nabla \left( \frac{B_\phi}{\omega} \right) \\
&\quad - Re B_\phi \nabla \cdot v_p + C_\Omega \omega \left[ \nabla \times (A_\phi v_p) \right] \cdot \nabla \Omega, \tag{17}
\end{align*}
\]

where \( A_\phi(r, t) \) and \( B_\phi(r, t) \) are the poloidal potential field and the toroidal field, respectively, \( \omega = r \sin \theta \), \( v_p \) is the poloidal velocity, i.e., the meridional circulation, \( \Omega \) is the profile of the differential rotation, and \( S \) is the source of the poloidal field at the solar surface. The domain is \( (r, \theta) \in [0.6, 1] \times [0, \pi] \). The toroidal field \( B_\phi = 0 \) at the boundary of the domain, and for \( A_\phi \), we impose the pure radial field approximation at the surface, i.e., \( \partial_t (r A_\phi) = 0 \) at \( r = 1 \), and \( A_\phi = 0 \) on all other boundaries. The length is normalized with the solar radius \( R_s \), and the time is normalized with the diffusive timescale \( R_s^2/\eta_t \), where \( \eta_t \) is the envelope diffusivity. We introduce three dimensionless parameters, namely, the Reynolds number based on the meridional flow speed \( Re = R_s v_p/\eta_t \), the strength of the Babcock–Leighton source \( C_s = R_s v_s/\eta_t \), and the strength of the \( \Omega \) effect \( C_\Omega = \Omega_s^2/\eta_t \), with \( \Omega_s = 2\pi \times 456 \text{ nHz} \).

We use the same dynamo model as we did in Paper I here, except that we have some modifications. First, we use a slightly more complex resistivity profile, a two-step profile in the radial direction,

\[
\begin{align*}
\frac{\eta}{\eta_t} &= \frac{\eta_c}{\eta_t} + \frac{\eta_m}{2\eta_t} \left[ 1 + \tanh \left( \frac{r - r_{bm}}{d_1} \right) \right] \\
&\quad + \frac{1}{2} \left[ 1 + \tanh \left( \frac{r - r_2}{d_1} \right) \right], \tag{18}
\end{align*}
\]

where \( \eta_c = 10^9 \text{ cm}^2 \text{ s}^{-1}, \eta_m = 10^{11} \text{ cm}^2 \text{ s}^{-1}, \eta_t = 5 \times 10^{11} \text{ cm}^2 \text{ s}^{-1}, r_{bm} = 0.72, r_2 = 0.95, \) and \( d_1 = 0.016 \). In this

Figure 18. Top: absolute difference between the estimate of the toroidal magnetic field and the true magnetic field vs. time, where the true magnetic field is driven by meridional flow with 10% fluctuation. Blue: free run of the dynamo model (unconstrained by data). Black: data assimilation estimate, with data consisting of magnetic fields in the line of sight and the pseudo-sunspot number. Middle and bottom: same, but for 20% and 30% fluctuations in meridional flow, respectively.
resistivity profile, the high diffusion at the surface produces a lower ratio of the radial magnetic field at the pole to that near the equator (Hotta & Yokoyama 2010). Second, the meridional circulation is also modified. The meridional flow is crucial in this model—it advects the magnetic field poleward at the surface, and equatorward deeper in the convection zone when it is unicellular per hemisphere.

To obtain a dynamo-generated magnetic field with fluctuations in the period and amplitude instead of a constant 22 years and peak amplitude, we use a time-varying meridional circulation for the model. We express the flow in the convection zone as the curl of a stream function,

\[ \psi = \nabla \times (\psi \mathbf{e}_o), \]  

and we expand the stream function as

\[ \psi(r, \theta, t) = -\frac{2}{\pi} \left( \frac{r - r_{mc}}{1 - r_{mc}} \right)^{2.5} (1 - r_{mc}) \]

\[ \times \left[ \sum_{k=1}^{m} \sum_{l=1}^{n} d_{k,l}(t) \sin \left( \frac{k\pi(r - r_{mc})}{2} \right) \right] \]

\[ P_l^k(-\cos \theta) \quad \text{if } r_{mc} \leq r \leq 1 \]

\[ 0 \quad \text{if } r_{bot} \leq r < r_{mc}, \]

where the \( P_l^k \) are the associated Legendre polynomials of order 1. The meridional flow is allowed to penetrate to a radius \( r_{mc} = 0.65 \), i.e., slightly below the base of the convection zone located at \( r_c = 0.7 \). Notice that the radial dependence of the stream function is raised to \( (r - r_{mc})^{2.5} \), compared with \( (r - r_{mc})^{2} \) in Joule et al. (2008) and Paper I. This can give a higher ratio of the maximum flow \( \psi \) at the surface with respect to that of the base of the convection zone, which in turn results in a 22 year magnetic cycle dynamo model with a surface flow \( \sim 20 \text{ ms}^{-1} \) (Yeates et al. 2008), consistent with the observed solar surface flow (Basu & Antia 2010; Ulrich 2010; Komm et al. 2015). The expansion coefficients \( d_{k,l}(t) \) are modulated in time so that the flow is time dependent. Other parameters used in the model include the Reynolds number \( Re = 310 \), \( C_r = 20 \), \( C_\Omega = 2.78 \times 10^4 \), i.e., \( \nu_o = 22.3 \text{ ms}^{-1} \), and \( s_o = 1.44 \text{ ms}^{-1} \).

The grid size is \( n_r \times n_\theta = 129 \times 129 \), and the time step is \( 10^{-6} \), equivalent to 0.112 days. In an illustrative example of the numerical experiment starting from Section 2.1, we chose a model flow (Equation (3)) characterized by \( d_{1,2}(t) = 1/3c_1(t) \), \( d_{2,1}(t) = 0.0865c_2(t) \), and \( d_{2,2}(t) = 0.130c_2(t) \) \( \left( d_{k,l} = 0 \right. \) for other \( (k, \ell) \)). Of course, a model based on stream functions defined by different combination of \( d_{k,l} \) can be investigated.

Appendix B
Assimilation Procedure and Representation of Initial Conditions in the Parameter Space

We present here the technical details of incorporating the initial magnetic field of the dynamo model to the control parameter space as a reference.

The initial conditions for the assimilation model are the magnetic potential of the poloidal field and the toroidal magnetic field on the meridional plane at the beginning of an assimilation window, i.e., \( A_\phi(r, \theta, t_0) \) and \( B_\phi(r, \theta, t_0) \), respectively. To extend the parameter space in the present 4D-Var framework, the initial conditions become part of the implicit dependences of the objective function.

As mentioned in Section 2.2, we need a representation of \( A_\phi(r, \theta, t) \) and \( B_\phi(r, \theta, t) \) in the parameter space such that the associated dimension is small compared with \( N^2 \sim 1500 \). To address this problem, we represent the magnetic field on the meridional plane with a truncated set of the eigenbasis of the covariance matrix of a dynamo field trajectory. We find that for a magnetic trajectory from the flux-transport dynamo model \( A_\phi(r, \theta, t) \) and \( B_\phi(r, \theta, t) \), if we calculate the covariance matrix over a long time (which covers the 22 year period of the magnetic cycle), the magnetic field at any time in the trajectory \( A_\phi(r, \theta, t_0) \) can be approximated effectively with a linear combination of only the first few eigenvectors of the covariance matrix with leading eigenvalues. We define the field column vector

\[ y(t) = [A_{1,1}(t), ..., A_{n_1,1}(t), A_{1,2}(t), ..., A_{n_1,n_2}(t), B_{1,1}(t), ..., B_{n_1,1}(t), B_{1,2}(t), ..., B_{n_1,n_2}(t)]^T, \]

where \( X_{ij}(t) = X(r_i, \theta_j, t) \) with \( X \) being \( A_\phi \) or \( B_\phi \). \( r_i, \theta_j \) are the spatial grid points of the magnetic field, so the size of the vector is \( 2n_1n_2 \), with \( n_a, n_b \) being the grid size in the radial and polar directions in the coordinate space, respectively. The covariance matrix \( P \) about a particular time \( t_0 \) is defined as

\[ P_{k,l}(t_0) = [\mathbf{y} - \mathbf{y}(t_0)]^T [\mathbf{y} - \mathbf{y}(t_0)]^T, \]

where the over-bar denotes averaging over time, in our case, two magnetic cycles. Notice that the indices \( k, l \) are the indices of the field vector and the covariance matrix, with \( 1 \leq k, l \leq 2n_1n_2 \). The diagonal entries of \( P \) are the variances of \( A_\phi \) and \( B_\phi \) at each grid point, respectively. The off-diagonal entries, depending on the indices, are the covariances of \( A_\phi (B_\phi) \) between any two different grid points or the covariances between \( A_\phi \) and \( B_\phi \) at any two grid points. It measures the auto- and cross-covariances of \( A_\phi \) and \( B_\phi \), and also the correlation between \( A_\phi \) and \( B_\phi \). We diagonalize the matrix, project \( y(t_0) \) onto the eigenbasis, and approximate \( y(t_0) \) in a truncated linear combination of the eigenvectors:

\[ y(t_0) \sim \sum_{i=1}^{m} [w_i^T y(t_0)] w_i, \]
The spectrum of the covariance matrix and the error in the approximation of a dynamo field by a truncated basis of eigenvectors drop rapidly with the level of truncation. The error in Figure 7(b) flattens to a few percent, a consequence of the every other point approximation discussed above; again, this approximation does not impact the overall accuracy of the scheme, which is controlled by the observational noise.

The forward model is initialized with such a representation, and the corresponding adjoint operator is developed similarly. (Recall that the derivative of the objective function with respect to the initial field is the corresponding adjoint field at the beginning of the assimilation window; Paper I.) The covariance matrix in the $n$th step is evaluated from the dynamo model forecast in the $(n - 1)$th assimilation window. For $n = 1$, the dynamo model is a simple one based on unicellular flow. Updating the covariance matrix after each year can ensure that we can capture the change in the dynamics and statistics of the dynamo action, and as a consequence, the initial conditions can be reasonably approximated.

In Section 2.2, we truncate the expansion of the initial condition to $m = 20$ leading eigenvectors, as the spectrum of $P$ and that of the error in expanding a simple dynamo field drop rapidly when the mode number increases (Figure 7). To justify this approximation, we perform the assimilation experiment of our reference case at various $m$ using a simple unicellular flow as the prior for the first year of assimilation. We show the misfit in Figure 20. We can see that at $m = 20$, we have an optimal misfit of $\sim 1$, and for more aggressive truncation of the eigenbasis representation, there will be underfitting. The size of the truncated basis required is related to the spectrum of the covariance matrix. In Figure 21(a), we can see that the covariance matrices for the models forecast during the 40 years of assimilation give broader spectra compared with a unicellular prior. This means that higher eigenmodes are more important for more complicated magnetic configurations as the

Figure 19. First three eigenfunctions (with leading eigenvalues) of the covariance matrix of a dynamo model based on a unicellular flow, expressed on the meridional plane. $P_A$ ($P_B$) is the poloidal (toroidal) component of the eigenvector. The higher modes with lower eigenvalues display more structures on the meridional plane.

Figure 20. Normalized misfit vs. time over the course of the assimilation for different parameterizations of the magnetic component of the control vector. Black (red/blue): five (10/20) eigenmodes are retained after the diagonalization of the covariance matrix constructed from the knowledge of the magnetic field at every other grid point. Green: 20 eigenmodes are retained after the diagonalization of the magnetic covariance matrix constructed from the knowledge of the magnetic field at every grid point.

The Astrophysical Journal, 849:160 (24pp), 2017 November 10
Hung et al.
The assimilation procedure proceeds. To illustrate this, we show the error in expanding the forecast magnetic field at the end of the 40 years of assimilation with the eigenbasis of the final forecast model in Figure 21.(b). Compared with Figure 7(b), the error converges at higher $m$, but is still soundly contained in our chosen size $m = 20$. Therefore, we justify the truncation of the eigenbasis in the representation of the initial condition at $m = 20$ in our tests.

To summarize, the procedures of the data assimilation for our course of 40 years of analysis of synthetic observations are as follows:

1. For $n = 1$, calculate the covariance matrix $P(t_{e,1})$ of the initial guess of the initial conditions. (Here, $t_{e,n}$ and $t_{e,n}$ are, respectively, the starting time and ending time of the assimilation window at the $n$th step, and we have $t_{e,n} = t_{e,n-1}$.) Usually, the guess is the dynamo model based on a unicellular flow with magnetic cycle of 22 years. Diagonalize the covariance matrix and project the guess of the initial magnetic field on the eigenbasis to obtain $x_{1,IC}^g$; the superscript $g$ stands for guess. Combined with the guess of the meridional flow $x_{1,MC}^f$, we have $x_{1}^f$ for the assimilation of the observations of the first year.

2. Based on the synthetic observations of the first year, with an appropriate guess $x_{1}^f$, the data assimilation procedure gives a forecast of the magnetic field and an analyzed meridional flow $x_{1}^f$; the superscript $f$ stands for forecast.

3. For $n > 1$, construct the covariance matrix $P(t_{e,n-1})$ of the dynamo model based on the analyzed flow at the $n - 1$ assimilation $x_{n-1,MC}^f$. Evaluate the eigenbasis of $P(t_{e,n-1})$ and project the analyzed magnetic field from the assimilation window $n - 1$ at $t_{e,n-1}$ and obtain $x_{n,MC}^f$. The initial guess of the flow in step $n$ will be the analyzed flow in step $n - 1$, i.e., $x_{n,MC}^f = x_{n-1,MC}^f$. So, we obtain $x_{n}^g$.

4. Based on the synthetic observations at the $n$th year, with the guess $x_{n}^f$, the data assimilation procedure gives the analysis $x_{n}^f$. The analyzed magnetic field and the estimated flow will give the initial guess $x_{n+1}^g$ and so on until $t_{e,40}$ is reached.
Appendix C

Brief Analysis of the Temporal Variability of the Meridional Flow

In this section, we present an analysis of the surface flow of the Sun which shows temporal variability using data from Ulrich (2010).

In Section 2.1, we mentioned that the correlation time of the spectrum of the surface meridional flow is of order one year. The observed flow on the solar surface can be found, for example, in Ulrich & Boyden (2005) and Ulrich (2010). The flow is dominantly poleward at the surface. We project the flow on the associated Legendre polynomials of order 1 \( P_1^\ell \), and plot the mean square (in time) of the spectrum in Figure 22. The modes, which are odd about the equator, i.e., with even \( \ell \), are dominant over their even parity counterparts, and the spectrum in general decreases with increasing \( \ell \).

To study the temporal variability of the flow, we evaluate the autocorrelation of the expansion coefficients on \( P_1^\ell \) s, for \( \ell = 1, 2, 4, 6 \), and show it in Figure 23. The first three equatorially odd modes display correlation times of at least five years, and the first equatorially even counterpart \( \ell = 1 \) is of correlation times \( \sim 1 \) year. We thus decided to use a modulation of three years for the flow as illustrated in Figures 2 and 4, which results in the time-dependent modulation of the flow, in good agreement with observations.

References

Babcock, H. W. 1961, ApJ, 133, 572
Basu, S., & Antia, H. M. 2010, ApJ, 717, 488
Belczynski, B., Dikpati, M., & Forgacs-Dajka, E. 2015, ApJ, 806, 169
Brandenburg, A., & Spiegel, E. A. 2008, AN, 329, 351
Brun, A. S. 2007, AN, 328, 329
Bushby, P. J., & Tobias, S. M. 2007, ApJ, 661, 1289
Charbonneau, P. 2010, LRSP, 7, 3
Charbonneau, P., & MacGregor, K. B. 1997, ApJ, 486, 502
Choudhuri, A. R., Schussler, M., & Dikpati, M. 1995, A&A, 303, L29
Clette, F., & Lefèvre, L. 2012, JSWSC, 2, A06
Clette, F., Svalgaard, L., Vaquero, J. M., & Cliver, E. W. 2014, SSRv, 186, 35
De Rosa, M. L., Brun, A. S., & Hoeksema, J. T. 2012, ApJ, 757, 96
Dikpati, M., Anderson, J. L., & Mitra, D. 2014, GeoRL, 41, 5361
