Kelvin transformation and inverse multipoles in electrostatics

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Abstract
The inversion in the sphere or Kelvin transformation, which exchanges the radial coordinate for its inverse, is used as a guide to relate distinct electrostatic problems with dual features. The exact solution of some nontrivial problems are obtained through the mapping from simple highly symmetric systems. In particular, the concept of multipole expansion is revisited from a point of view opposed to the usual one: the sources are distributed in a region far from the origin while the electrostatic potential is described at points close to it.

Keywords: Kelvin transformation, inverse multipoles, conformal transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

Mapping a difficult problem into an easier or previously solved one is a powerful strategy both in mathematics and physics. Electromagnetism is a theory in which this is often possible and rewarding. For example, two arbitrary charge distributions together with their respective electrostatic potentials are related by Green’s reciprocation theorem, allowing known results about a simple arrangement of charges to be translated into information about a more complex configuration [1]. Another useful technique in electrostatics is the Kelvin transformation [2]—also known as inversion in the sphere—which, among other things, maps planes into spheres and vice versa, and by means of which some difficult problems can be solved [3, 4]. Such mappings are often suggested by symmetries of the theory. Here we explore certain
aspects of the application of the Kelvin transformation to electrostatics that we find instructive and, to our knowledge, have not been discussed elsewhere.

Electromagnetism is an example of a successful theory with impressive experimental corroboration both at classical and quantum realms. Lorentz invariance insures its validity at high velocities, while the gauge symmetry establishes a paradigm for the description of other fundamental interactions and is linked to charge conservation and the absence of photon mass. The latter aspect guarantees the scale invariance of the electromagnetic theory. A more subtle property of the theory, intimately related to scale invariance, is its conformal invariance. As has been recently stressed [5], the conformal symmetry of electromagnetism is characteristic to four-dimensional space–time.

Within the conformal symmetry group, the special conformal transformation means a spacetime coordinate inversion followed by a spacetime translation and another inversion,

\[ \frac{x_\mu}{x^2} = \frac{y_\mu}{y^2} + b_\mu, \quad (1.1) \]

and takes the form

\[ x_\mu = \frac{y_\mu + x^2 b_\mu}{1 + 2b \cdot \bar{x} + b^2 \bar{x}^2}, \quad (1.2) \]

where \( a \cdot b = a_\mu b^\mu \) and \( a^2 = a_\mu a^\mu \). Its implications to the dynamics of charges have been discussed in [6] and a more complete investigation of this symmetry is presented in [7], where use is made of the general covariant formalism.

This article deals with conformal transformations analogous to equation (1.2) which affect spatial variables alone, so that the time variable is left untouched. In particular it will be focused on an essential ingredient of the special conformal transformation, namely the spatial inversion

\[ \bar{x}_i = \frac{R^2}{r^2} x_i, \quad (1.3) \]

where \( R \) is a positive constant and \( r = \sqrt{x^\cdot x} \) is the radial variable. This mapping, variously known as inversion in the sphere or Kelvin transformation [2], leaves electrostatics invariant, the focus of our interest. It is worth remarking that, in general, magnetostatics is not left invariant by (1.3). This kind of coordinate change has been explored in the framework of electrical engineering [8], but some of its features have not been appreciated from the physicist’s point of view. For instance, it allows the description of infinitely extended systems starting from localised ones.

An important tool for the study of localised charge distributions is the multipole expansion, which has been widely explored not only in electromagnetism but also in other macroscopic field theories such as gravitation. In the latter case, the study of Newtonian and Einsteinian orbits is an explicit example [9, 10]. Perturbations of the Newtonian gravitational potential imply planetary perihelion advance. In this case, the decisive perturbations, which stem from space–time curvature, where once thought to be due to a solar oblateness that would give rise to a quadrupole contribution to the Sun’s gravitational potential [11, 12]. In the case of electrostatics, the standard textbooks devote great attention to the multipole expansion [13]. Its applications are wide ranging, from the quantum-mechanical study of asymmetric atoms [14] up to electromagnetic radiation and scattering of electromagnetic waves [15]. As a rule, one is interested in describing a field at points far from a localised source distribution, as, for example, in the discussion of the electric field created by a point dipole or by a uniformly polarised spheroidal electret embedded in an infinite dielectric [16].
An unusual point that will be addressed here is the transformation of the multipole expansion of the electrostatic potential under the inversion in the sphere (1.3). This gives rise to an interchange of the roles of points close to and distant from the origin.

The paper is organised as follows. Section 2 deals with the effect on the Poisson equation of inversions in the sphere. Their impact on the multipole expansion of the electrostatic potential is investigated in section 3. The following sections are dedicated to applications. In section 4 the duality of spherical shells leads naturally to the concept of self-dual and anti-self-dual models, and to the role they play in the method of images. In section 5 the consideration of eccentric spheres leads to the discussion of a general conformal transformation. In section 6 the mapping from spheres into planes is discussed stressing the topology change induced by the inversion transformation. In section 7 the relationship between a cylinder and a special torus is studied. Finally, some conclusions are presented and further applications are pointed out.

2. Electrostatics and inversion transformation

Let us start by considering the role of the inversion in the sphere

\[ \mathbf{r} \xrightarrow{S} \hat{\mathbf{r}} = \frac{R^2}{r^2} \mathbf{r} \]  

(2.1)

in electrodynamics\(^4\). Since all information on the electrostatic field is embodied in the potential \( \Phi \), all is needed is a description of its fate under transformation (2.1), which is denoted by \( \Sigma \) and whose inverse is

\[ \mathbf{r} = \frac{R^2}{r^2} \hat{\mathbf{r}}. \]  

(2.2)

In order to determine how solutions of the Poisson equation

\[ \nabla^2 \Phi = -\frac{1}{\varepsilon_0} \rho \]  

(2.3)

are mapped into other solutions by the inversion operation, we start from the Laplacian in spherical coordinates:

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] = \frac{1}{r^2} [D_r + D_\theta^2 + L^2]. \]  

(2.4)

Here \( D_r = r \partial/\partial r \) and \( L^2 \) is a differential operator acting on the angular variables alone [18]. The angular operator is invariant because transformation (2.1) does not change angles:

\[ \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'}{r r'} = \frac{\mathbf{r} \cdot \mathbf{r}'}{r r'}, \]  

(2.5)

Furthermore, taking the modulus of both sides of (2.2) we find

\[ r = \frac{R^2}{\hat{r}}. \]  

(2.6)

\(^4\) The positive parameter \( R \) is required for dimensional consistency and defines the radius of an invariant sphere. A change in \( R \) means a scale transformation.
from which it follows that
\[
D_r = \hat{r} \frac{\partial}{\partial \hat{r}} = \hat{r} \frac{\partial}{\partial \hat{r}} \hat{r} = \hat{r} \left( -\frac{R^2}{\hat{r}^2} \right) \frac{\partial}{\partial \hat{r}} = -\hat{r} \frac{\partial}{\partial \hat{r}} = -D_r.
\] (2.7)

Certainly, the term linear in \(D_r\) spoils the invariance of the Laplacian operator (2.4) under inversions.

It is straightforward, although a little tedious, to show that we can get rid of the term linear in \(D_r\) by the following device:
\[
\nabla^2 \Phi = \nabla^2 (r^{-\frac{3}{2}} \Phi) = r^{-\frac{1}{2}} \left[ -\frac{1}{4} + D_r^2 + L^2 \right] (r^{-\frac{3}{2}} \Phi).
\] (2.8)

The Poisson equation is thus written as
\[
\left[ -\frac{1}{4} + D_r^2 + L^2 \right] r^{-\frac{3}{2}} \Phi(r) = -\frac{1}{\epsilon_0} r^2 \rho(r).
\] (2.9)

This suggests to define
\[
\hat{r}^{1/2} \hat{\Phi}(\hat{r}) = r^{1/2} \Phi(r) \implies \hat{\Phi}(\hat{r}) = \frac{r^2}{\hat{r}^2} \Phi(r) = \frac{R}{\hat{r}} \Phi(r)
\] (2.10)
and, similarly,
\[
\hat{r}^{5/2} \hat{\rho}(\hat{r}) = r^{5/2} \rho(r) \implies \hat{\rho}(\hat{r}) = \frac{r^5}{\hat{r}^5} \rho(r) = \frac{R^5}{\hat{r}^5} \rho(r)
\] (2.11)
with \(r\) given in terms of \(\hat{r}\) by (2.2). With the basic definitions (2.10) and (2.11) the Poisson equation is preserved, giving rise to a pair of dual electrostatics problems related by the space inversion \(S\):
\[
\nabla^2 \Phi(r) = -\frac{1}{\epsilon_0} \rho(r) \implies \nabla^2 \hat{\Phi}(\hat{r}) = -\frac{1}{\epsilon_0} \hat{\rho}(\hat{r}).
\] (2.12)

Thus, we have two electrostatic problems that are derived from each another by means of the inversion transformation. The transformation (2.10) of one harmonic function into another appears in [19] under the name of a Kelvin transformation. It is worthy of note, and it is easy to check, that inversion in the sphere is an involution, that is, \(\hat{r} = r\), \(\hat{r} = \rho\) and \(\hat{\Phi} = \Phi\).

Some subtleties deserve to be stressed. Notice that a gauge transformation \(\Phi'(r) = \Phi(r) + \Phi_0\), which does not change the electric field \(E(r)\) in the original setup, affects the physics described by the dual system through the ‘addition’ of a point particle of charge \(\hat{Q}_0 = 4\pi \epsilon_0 \Phi_0\) at the origin, since \(\hat{\Phi}'(\hat{r}) = \hat{\Phi}(\hat{r}) + \frac{\Phi_0}{\hat{r}}\). Under this perspective, the spacial inversion maps a physical system into a class of systems related by the addition of a monopole at the origin.

Before turning to our main issues of interest, let us digress a little on the mathematical origin of transformations (2.10) and (2.11). With the help of (2.6) it is easy to show that under transformation (2.1) the volume element changes as follows:
\[
\hat{\Omega} = \hat{r}^2 d\hat{r} \hat{\Omega} = \left( \frac{R}{\hat{r}} \right)^6 \hat{r}^2 d\hat{r} d\Omega = \left( \frac{R}{\hat{r}} \right)^6 dV.
\] (2.13)
We also have
\[ | \mathbf{r} - \mathbf{r}' |^2 = \left( \frac{R^2 - \mathbf{r}^2}{r^2} - \frac{R^2 - \mathbf{r}'^2}{r'^2} \right)^2 = R^4 \left( \frac{\mathbf{r} - \mathbf{r}'^2}{r^2 r'^2} \right)^2 \implies | \mathbf{r} - \mathbf{r}' | = \frac{R^2}{rr'} | \mathbf{r} - \mathbf{r}' |. \] (2.14)

These results, combined with (2.10) and (2.11), prompt a rederivation of Kelvin’s inversion theorem by means of a change of integration variables in the Coulomb law equation:
\[ \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{| \mathbf{r} - \mathbf{r}' |} + \Phi_0 \iff \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d\mathbf{r}' \frac{\tilde{\rho}(\mathbf{r}')}{| \mathbf{r} - \mathbf{r}' |} + \Phi_0 R / \mathbf{r}. \] (2.15)

Note that the original system defined by the charge density \( \rho \) is, in fact, associated with a family of dual systems, with volume charge density \( \tilde{\rho} \) and an arbitrary point charge located at the origin.

Note, finally, that the total electric charge is not preserved by a Kelvin transformation:
\[ Q = \int \rho(\mathbf{r}) d\mathbf{V} = \int \left( \frac{R}{r} \right)^{\frac{3}{2}} \left( \frac{R}{r} \right)^{\frac{5}{2}} \rho(\mathbf{r}) dV = \int \frac{R}{r} \rho(\mathbf{r}) dV \neq Q. \] (2.16)

In particular, a finite-charge system may be mapped into an infinite-charge system and vice versa.

3. Inverse multipoles

Since the Kelvin transformation takes points near the origin into points far from the origin and vice versa, the transformation of the multipole expansion seems worth studying. Suppose all the sources of a system are contained inside the sphere of inversion, that is, \( \rho(\mathbf{r}) = 0 \) for \( r \geq R \). Then, for exterior points the potential \( \Phi(\mathbf{r}) \) can be expressed in terms of a multipole expansion, obtained, for instance, from a series expansion of equation (2.15) in inverse powers of \( r \) in the form [13]
\[ \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0} \sum_{l,m=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}(\theta, \phi) q_{lm} \frac{1}{r^{l+1}}, \] (3.1)
where \( Y_{lm}(\theta, \phi) \) are spherical harmonics and the spherical multipole moments \( q_{lm} \) are given by
\[ q_{lm} = \int_{r<R} d\mathbf{V} Y_{lm}(\theta, \phi) \rho(\mathbf{r}) r^l, \] (3.2)
where the bar denotes complex conjugate. The corresponding system obtained by inversion (2.1) is, contrastingly, free of charges inside the sphere of inversion, that is, \( \tilde{\rho}(\mathbf{r}) = 0 \) for \( \tilde{r} < R \). With the help (2.6) and (2.10) the exterior expansion (3.1) is transformed into the interior expansion
\[ \tilde{\Phi}(\mathbf{r}) = \frac{1}{\varepsilon_0} \sum_{l,m=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_{lm}(\theta, \phi) \tilde{q}_{lm} \tilde{r}^l, \] (3.3)
with the inverse multipole moments \( \tilde{q}_{lm} \) defined as
\[ \tilde{q}_{lm} = \int_{r>R} d\mathbf{V} \tilde{Y}_{lm}(\theta, \phi) \tilde{\rho}(\mathbf{r}) \frac{1}{\tilde{r}^{l+1}}. \] (3.4)
In terms of the multipole moments \( q_{lm} \), the inverse multipole moments \( s_{lm} \) are given by
\[
\tilde{s}_{lm} = \frac{q_{lm}}{R^{2l+1}}.
\]

Arguably, both the Kelvin transformation and the inverse multipole expansion (3.3) might be given a more attentive consideration by textbooks.

In terms of Cartesian coordinates the multipole expansion (3.1) takes the form
\[
\Phi(r) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{Q}{r} + P \cdot r + \frac{1}{2} \sum_{ij} Q_{ij} x_i x_j + \ldots \right],
\]
with
\[
Q = \int \rho dV, \quad P = \int \rho r dV, \quad Q_{ij} = \int \rho (3x_i x_j - r^2 \delta_{ij}) dV, \quad \ldots
\]

The dual potential turns out to be given by the Maclaurin expansion
\[
\tilde{\Phi}(\tilde{r}) = \frac{1}{4\pi\varepsilon_0} \left[ \tilde{S}_0 + \tilde{S} \cdot \tilde{r} + \frac{1}{2} \sum_{ij} \tilde{S}_{ij} x_i x_j + \ldots \right]
\]
with coefficients
\[
\tilde{S}_0 = \int \frac{\tilde{\rho}}{\tilde{r}} dV, \quad \tilde{S} = \int \frac{\tilde{\rho} \tilde{r}}{\tilde{r}^3} dV, \quad \tilde{S}_{ij} = \int \frac{\tilde{\rho}}{\tilde{r}^5} (3\tilde{x}_i \tilde{x}_j - \tilde{r}^2 \delta_{ij}) dV, \quad \ldots
\]

The coefficients \( \tilde{S}_{ij;\ldots;k} \) in the inverse multipole expansion (3.8) are essentially the \( n \)th partial derivatives of the transformed potential \( \tilde{\Phi}(\tilde{r}) \) computed at the transformed origin (\( \tilde{r} = 0 \)), which are related to the corresponding derivatives of the original potential \( \Phi(r) \) at infinity (\( r = \infty \)). Of course, the cartesian inverse multipole moments \( \tilde{s}_{ij;\ldots;k} \) can be expressed in terms of the spherical inverse multipole moments (3.4) of the transformed charge density.

In the case of non-overlapping charge distributions there is an expression for their electrostatic interaction energy in terms of direct and inverse multipole moments that may be of some interest. Let systems \( A \) and \( B \) have disjoint charge distributions, so that \( \rho_A(r) \) vanishes outside a sphere of radius \( R \) whereas \( \rho_B(r) \) vanishes inside the same sphere. The electrostatic interaction energy of the two systems is
\[
U_{AB} = \int \rho_A(r) \Phi_B(r) dV = \int_{r < R} \rho_A(r) \Phi_B(r) dV.
\]

Inserting in the above equation the inverse multipole expansion (3.3) for \( \Phi_B(r) \) without the tildes one finds
\[
U_{AB} = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} q_{lm}^{(A)} s_{lm}^{(B)},
\]
where \( q_{lm}^{(A)} \) and \( s_{lm}^{(B)} \) are the direct and inverse spherical multipole moments associated with systems \( A \) and \( B \), respectively. In terms of cartesian multipole moments we have
\[
U_{AB} = \frac{1}{4\pi\varepsilon_0} \left[ \mathbf{Q}^{(A)} \mathbf{S}_0^{(B)} + \mathbf{P}^{(A)} \cdot \mathbf{S}^{(B)} + \sum_{ij} Q_{ij}^{(A)} S_{ij}^{(B)} + \ldots \right].
\]

In words, the interaction energy is a sum of the interaction energies between the direct and inverse multipole moments of the non-overlapping charge distributions. For an application of the concept of inverse multipoles in chemical physics, see [17].
## 4. Spherical shell and self-duality

As one of the simplest examples, let us take a spherical shell of radius $R_1$ with centre at the origin and a surface charge density $\sigma(\theta, \phi)$. The corresponding volume charge density is

$$\rho(r) = \sigma(\theta, \phi) \delta(r - R_1). \tag{4.1}$$

Now we consider the system obtained by inversion with respect to radius $R$. From (2.11) and well-known properties of the Dirac delta function such as $\delta(ax) = |a|^{-1} \delta(x)$, $\delta(-x) = \delta(x)$ and $f(x)\delta(x-a) = f(a)\delta(x-a)$, we find

$$\tilde{\rho}(\mathbf{r}) = \sigma(\theta, \phi) \delta(r - \tilde{R}_1), \tag{4.2}$$

where $\tilde{\sigma}(\theta, \phi) = (R_1/R)^2 \sigma(\theta, \phi)$ and $\tilde{R}_1 = R_1^2/R_0$. The dual system is a concentric spherical shell with radius $\tilde{R}_1$ and surface charge density $\tilde{\sigma}$. The original and transformed total charges are related through $\tilde{Q} = (R/R_1)Q$, with similar relations for the higher multipole moments. Expressing the surface charge density as $\sigma(\theta, \phi) = \sum_{l,m} \sigma_{lm}(\theta, \phi)$, the potential is given by

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell + 1} Y_{\ell m}(\theta, \phi) \sigma_{\ell m} R_1 \left[ \left( \frac{R_1}{r} \right)^{\ell+1} \Theta(r - R_1) + \left( \frac{r}{R_1} \right)^{\ell} \Theta(R_1 - r) \right], \tag{4.3}$$

where $\Theta$ is the Heaviside step function: $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. This leads to an expression for $\Phi$ exactly analogous to equation (4.3) with the replacements $(r, R_1) \rightarrow (\tilde{r}, \tilde{R}_1)$. In this process, the direct multipole moments of the original spherical shell charge distribution are mapped into the inverse multipole moments of the transformed system, and vice-versa.

It is curious that some systems are invariant under the Kelvin transformation. Consider two concentric spherical shells with radii $R_1$ and $R_2$ whose respective surface charge densities are $\sigma_1$ and $\sigma_2$. This system is the same as its dual as long as we choose $\sigma_2 = (R_1/R_2)^{3/2} \sigma_1$ for transformations with respect to the radius $R = \sqrt{R_1 R_2}$. The concept of self-dual models can be generalised to include systems that lack spherical symmetry. Suppose the charge density does not change upon an inversion with respect to radius $R$, that is, $\tilde{\rho} = \rho$. This means that $\rho$ and $\tilde{\rho}$ are the same function: $\tilde{\rho}(\mathbf{r}) = \rho(\mathbf{r})$. Making use of (2.11), this self-duality condition can be written in spherical coordinates as

$$\rho(r, \theta, \phi) = \left( \frac{R}{r} \right)^3 \rho \left( \frac{R^2}{r}, \theta, \phi \right). \tag{4.4}$$

Anti-self-duality, $\tilde{\rho} = -\rho$, that is,

$$\rho(r, \theta, \phi) = -\left( \frac{R}{r} \right)^3 \rho \left( \frac{R^2}{r}, \theta, \phi \right), \tag{4.5}$$

also plays a role in electrostatics. The paradigmatic example is the method of images for finding the potential of the system composed of a point charge $Q$ near a grounded spherical conductor of radius $R$, which we take as the radius of inversion. Choosing coordinates such that the charge $Q$ lies on the $z$-axis, the exterior problem corresponds to $\rho(\mathbf{r}) = Q \delta^3(\mathbf{r} - R_1 \hat{z})$ with $R_1 > R$. The image charge inside the sphere is described by
$$\rho_{\text{int}}(r) = -(R/R_i)Q_3(r - (R^2/R_i)\hat{z}) = -\rho(r),$$

as can be verified by using the properties of the three-dimensional Dirac delta function. In other words, the image charge arises from inversion with respect to the grounded sphere in such a way that the total charge density of the system composed by both interior and exterior charges is anti-self-dual: $\rho_T = \rho - \rho_i$. Changing the spherical shell potential to a nonvanishing constant value amounts to performing a gauge transformation on the potential and impacts the inverted system by the addition of an extra point charge at the origin, breaking the anti-self-duality condition.

5. Eccentric spheres

Now we study the effect of an inversion in the sphere on asymmetric systems. Consider the mutually conjugate systems, $\tilde{S} \iff S$, related by a Kelvin transformation with respect to a sphere of radius $R$ centred at the origin. Let $\tilde{S}$ consist of a uniformly charged spherical shell of radius $R_1$ with its centre $c_i$ displaced upwards from the origin along the $z$-axis by the distance $\bar{d} < R_1$, as depicted in figure 1. This breaks the spherical symmetry with respect to the origin. Now, the associated system, $S$, is a little bit less obvious than the ones previously considered. The spherical surface on which the charges of system $\tilde{S}$ are located is described parametrically by

$$\begin{align*}
(x, y, \bar{z}) = (\bar{R}_i \sin \alpha \cos \phi, \bar{R}_i \sin \alpha \sin \phi, \bar{R}_1 \cos \alpha + \bar{d}),
\end{align*}$$

(5.1)

where $\alpha$ is the polar angle measured from its centre $c_i$. By means of the inverse Kelvin transformation (2.2) this sphere is be mapped to

$$\begin{align*}
(x, y, z) = \frac{R^2}{r^2}(\bar{x}, \bar{y}, \bar{z}).
\end{align*}$$

(5.2)

A straightforward but lengthy algebra shows that the system $S$ consists of another spherical shell of radius...
Although for system $S$ the electric charge is still spread on a spherical shell, it is no longer uniformly distributed, since distances of charge elements to the origin are not constant. Interestingly, however, the potentials for both systems are remarkably simple. For system $\tilde{S}$ let us define the regions $A$, with $\sqrt{x^2+y^2+(z-d)^2} < R_1$, and $B$, with $\sqrt{x^2+y^2+(z-d)^2} > R_1$, corresponding respectively to interior and exterior solutions

$$\overline{\Phi}_A(r) = \frac{Q_A}{4\pi\varepsilon_0 \tilde{R}_1} \quad \text{and} \quad \overline{\Phi}_B(r) = \frac{Q_B}{4\pi\varepsilon_0 \sqrt{x^2+y^2+(z-d)^2}}$$

$$= \frac{\tilde{Q}}{4\pi\varepsilon_0 \sqrt{d^2+r^2-2rd \cos \theta}}.$$  

Using equation (2.10), the $S$-system potential turns out to be given in the exterior region $A$, with $\sqrt{x^2+y^2+(z+d)^2} > R_1$, and interior region $B$, with $\sqrt{x^2+y^2+(z+d)^2} < R_1$, respectively, by

$$\Phi_A(r) = \frac{Q_A}{4\pi\varepsilon_0 r} \quad \text{and} \quad \Phi_B(r) = \frac{Q_B}{4\pi\varepsilon_0 \sqrt{x^2+y^2+2rs \cos \theta}}.$$  

Here we introduced $s = (R_1^2 - d^2)/d$ and the parameters $Q_A = (R/\tilde{R}_1)\tilde{Q}$ and $Q_B = (R/d)\tilde{Q}$ with dimension of charge.

The result (5.5) can be described in terms of associated virtual point charges. The charge $Q_A$, located within region $B$, at the origin, describes the potential in the exterior region $A$, while the charge $Q_B$, located in region $A$ at $r = s\tilde{R}_1$, dictates the field in region $B$, corresponding to the interior solution. The system $S$ presents thus a nonuniform charge distribution.
on a spherical shell with a sort of lensing effect. The interior and exterior potentials coincide with the ones produced by specific point charges located outside each region, neither of them being localised at the sphere geometric centre, see figure 1.

By expanding the potential (5.4) in powers of $\tilde{d}/\tilde{r}$ a multipole expansion is obtained for the exterior solution in which all multipole terms appear. At the same time, only the first term occurs for the interior solution inverse multipole expansion, since the exact potential in this region is itself the lowest order term in powers of $\tilde{r}/\tilde{d}$. On the other hand, for the dual potential (5.5), the opposite occurs. All inverse multipole terms appear for the interior solution. For the exterior solution, in spite of the $S$-system charge distribution not being uniform, only the monopole contribution appears outside the spherical shell. This is an example of the interplay between direct and inverse multipole terms discussed in section 3.

For completeness let us quote the results for the case in which the origin lies outside the $\tilde{S}$-system sphere, which occurs for $\tilde{d} > \tilde{R}$, figure 2. The transformed sphere is still another sphere of radius $\tilde{R}_1 = R^2 \tilde{R}_1/ (\tilde{d}^2 - \tilde{R}_1^2)$ with centre $c_\tilde{f}$ displaced upwards to $z = \tilde{d} = R^2 \tilde{d}/ (\tilde{d}^2 - \tilde{R}_1^2)$. Now, differently from the previous case, the $\tilde{S}$-system interior region is mapped into the $S$-system interior region, and the same occurs to their respective exterior regions. The $S$-system interior potential $\phi_A$ is due to a virtual charge $Q_A$ at the origin, which belongs to the exterior region $B$, whereas the exterior field $\phi_B$ is determined by the virtual charge $Q_B$ located within the interior region $A$ at $z = (d^2 - R_1^2)/d$.

### 5.1. Special conformal transformation

Let us now discuss the full special conformal transformation on the space variables and provide an illustrative example. Consider the sequence of transformations applied to a system labelled $S'$: (i) $S' \Rightarrow \tilde{S}'$, a Kelvin transformation $\mathbf{r}' \rightarrow \tilde{\mathbf{r}}'$ with radius $R$, which according to (2.10) relates $\phi(\mathbf{r}')$ to $\tilde{\phi}(\tilde{\mathbf{r}}')$; (ii) $\tilde{S}' \Rightarrow \tilde{S}$, a translation along the $\tilde{z}$-axis by $\tilde{d}$, $\tilde{\mathbf{r}}' = \tilde{\mathbf{r}}'' - \tilde{d}$, which establishes the relation $\tilde{\phi}(\tilde{\mathbf{r}}') = \tilde{\phi}(\tilde{\mathbf{r}} - \tilde{d})$; (iii) $\tilde{S} \Rightarrow S$, another Kelvin transformation $\tilde{\mathbf{r}}'' \rightarrow \mathbf{r}$ with respect to the same sphere of radius $R$, which relates $\tilde{\phi}(\tilde{\mathbf{r}}')$ to $\phi(\mathbf{r})$. A straightforward computation leads to the mapping

$$\mathbf{r}' = \frac{\mathbf{r} - \mathbf{D} r^2}{1 - 2\mathbf{D} \cdot \mathbf{r} + D^2 r^2}, \quad (5.6)$$

where $\mathbf{D} = (\tilde{d}/R^2)\tilde{\mathbf{z}}$, and to the following relation between the potentials for systems $S$ and $S'$:

$$\phi(\mathbf{r}) = (1 + 2\mathbf{D} \cdot \mathbf{r}') + D^2 r^2 \tilde{\phi}(\tilde{\mathbf{r}}'). \quad (5.7)$$

Equation (5.6) defines the spatial version of the special conformal transformation (1.2), while (5.7) is the expected behaviour for a scalar field under such a transformation [5, 6].

Considering for $S'$ a spherical shell of radius $R_1'$ uniformly charged with charge $Q'$, the system $\tilde{S}'$ is a shell of radius $\tilde{R}_1' = R^2/R_1'$ and total charge $\tilde{Q}' = (R/\tilde{R}_1')Q'$. On the other hand, the system $S$ represents the spherical shell with translated centre, the same radius $\tilde{R}_1 = \tilde{R}_1'$ and the same uniformly distributed charge $\tilde{Q} = \tilde{Q}'$. The system $S$, in its turn, has been described at the beginning fo this section, see figure 1. Summarising, we witness that the special conformal symmetry maps the uniformly charged spherical shell of total charge $Q'$ centred at the origin with radius $R_1'$ into the previously described non-uniformly charged spherical shell of radius $R_1 = R_1'/|1 - D^2 R_1'^2|$, with the same total charge and centred at $z = -D^2 R_1'^2/ (1 - D^2 R_1'^2)$. The potential produced by this configuration has been described in (5.5). By increasing $D$ from zero, the targeted sphere is continuously enlarged and has its
centre displaced downwards. When \( DR_1' \) approaches 1, the spherical shell tends to be infinitely large, with its centre infinitely displaced downwards. Then, beyond the critical value \( D = 1/R_1' \), the centre of the spherical shell emerges on the upper half line \( z > 0 \) and its radius progressively diminishes while its centre approaches the origin.

6. From sphere to plane

The critical case of spherical shells that pass through the origin deserves a separate study. It corresponds to the intermediate situation between those with the origin inside or outside the spherical shell \( S \) we discussed in the previous section. Consider, therefore, a system composed of two spheres of radius \( L \) that touch each other at the origin and are defined by

\[
s^2 + (z + L)^2 = L^2, \tag{6.1}
\]

where \( s = \sqrt{x^2 + y^2} \), as shown in figure 3. On the surface of each sphere let us attach the charge density

\[
\sigma_s(r) = \pm \sigma_0 R_3^3, \tag{6.2}
\]

where \( \sigma_+ \) and \( \sigma_- \) correspond respectively to the spheres above and below the \( xy \)-plane.

The idea is to obtain the electrostatic potential for this system by applying the Kelvin transformation \( (2.1) \). Since equations (6.1) are equivalent to \( r^2 = \pm 2zL \), it follows at once that the spherical surfaces are mapped to the planes \( \bar{z} = \pm \bar{L} = \pm R^2/2L \).

Letting \( \xi' = \sqrt{(\xi' + L)^2 + s^2} \) be the radial variables from the centre of each sphere, the volume density associated to the system composed by the two spheres is
\[ \rho(r) = \frac{\sigma_0 R^3}{r^3} [\delta(\zeta_+ - L) - \delta(\zeta_- - L)]. \] (6.3)

As shown in appendix A, it follows that
\[ \bar{\rho}(r) = \sigma_0 [\delta(z - \bar{L}) - \delta(z + \bar{L})]. \] (6.4)

Thus the planes \( \xi = \pm \bar{L} \) are uniformly charged with surface charge densities \( \bar{\sigma}_\pm = \pm \sigma_0 \), respectively—a parallel-plate capacitor. For the electric field of the transformed system we have \( \bar{\mathbf{E}} = -(\sigma_0/\varepsilon_0) \mathbf{\hat{z}} \) in the region between the planes \( \{\xi| < \bar{L}\} \) and \( \bar{\mathbf{E}} = 0 \) in the exterior region \( \{\xi| > \bar{L}\} \). As a consequence, the electrostatic potential of the transformed system can be concisely written as
\[ \bar{\Phi}(\mathbf{r}) = \frac{\sigma_0 \bar{L}}{\varepsilon_0} \Theta(L^2 - z^2) + \frac{\sigma_0 \bar{L}}{\varepsilon_0 |\xi|} \Theta(z^2 - L^2), \] (6.5)

with the understanding that \( \bar{\Phi} = \sigma_0 \bar{L}/\varepsilon_0 \) at \( \xi = \bar{L} \) and \( \bar{\Phi} = -\sigma_0 \bar{L}/\varepsilon_0 \) at \( \xi = -\bar{L} \) inasmuch as \( \Theta(x) \) is not defined at \( x = 0 \).

From this, the potential for the original system (as long as \( r \neq 0 \)) is found to be
\[ \Phi(r) = \frac{\sigma_0}{\varepsilon_0} \left[ \frac{3\xi}{r^3} \Theta \left( L^2 - \frac{z^2 R^4}{r^2} \right) + \frac{\xi}{r\bar{L}\varepsilon_0} \Theta \left( \frac{z^2 R^2}{r^2} - L^2 \right) \right]. \] (6.6)

The first term on the right-hand side of the above equation describes the potential outside both spheres; the second term refers to the interior of either sphere. The interior potential is the same that would be produced by monopoles (point charges) placed at the origin with opposite signs in order to give the potential inside either sphere, while the exterior one is simply a dipole potential. The singularity at the origin is expected, and the opposite signs of the point particles is due to the opposite uniform potentials in the two exterior regions of the parallel-plane capacitor.

For the sake of completeness, let us present the associated electrical fields:
\[ E_r = \frac{\sigma_0 R^3}{\varepsilon_0 r^3} \left[ \frac{3\xi}{r^3} \Theta \left( L^2 - \frac{z^2 R^4}{r^2} \right) + \frac{\xi}{r\bar{L}} \Theta \left( \frac{z^2 R^2}{r^2} - L^2 \right) \right], \] (6.7)

\[ E_z = \frac{\sigma_0 R^3}{\varepsilon_0 r^3} \left[ \left( \frac{3\xi^2}{r^2} - 1 \right) \Theta \left( L^2 - \frac{z^2 R^4}{r^2} \right) + \frac{|\xi|}{L \Theta} \Theta \left( \frac{z^2 R^2}{r^2} - L^2 \right) \right]. \] (6.8)

From the discontinuity of the normal component of the electric field the surface charge density can be recovered.

For the above reasoning we devised the singular charge distribution given by equation (6.2) in order to obtain a uniform density on the associated planes. But, by essentially promoting \( \sigma_0 \) to a function of position on the spheres, new intriguing mappings can be easily constructed, which relate the problems of surface distributions on the two spheres to the associated problem of surface charges on the corresponding planes. Let us illustrate this point by changing the charge density on each sphere, so that, instead of (6.2), for the upper sphere we take
\[ \sigma_\pm(r) = \sigma_{1} + \sigma_{2}\cos\alpha = \sigma_{1} + \sigma_{2}\frac{\xi}{L}, \] (6.9)

while the lower sphere is uncharged: \( \sigma_{r}(r) = 0 \). Here \( \alpha \) is the polar angle measured from the centre of the upper sphere, \( \sigma_{1} \) and \( \sigma_{2} \) being constants. This configuration corresponds to an exterior potential that has only a monopole term, associated to \( \sigma_{1} \), and a dipole term,
controlled by \( \sigma_2 \), for its multipole expansion around the centre of the upper sphere. The charged sphere will be mapped into the single plane \( \bar{z} = \bar{L} \). The charge distribution on the plane, obtained by essentially making the replacement \( \sigma_0 \rightarrow (\sigma_1 - \sigma_2 + 2L^2 \bar{z})/R^2 \) in our previous result, turns out to be nonuniform but axially symmetric:

\[
\bar{\sigma} = (\sigma_1 - \sigma_2) \frac{R^3}{(\bar{s}^2 + \bar{L}^2)^2} + \sigma_1 \frac{R^5}{(\bar{s}^2 + \bar{L}^2)^2}.
\]  

(6.10)

The potential for the exterior of the sphere, determined easily by computing its total charge and dipole moment, is given by

\[
\phi_{\text{out}}(r) = \frac{\sigma_2 L^3}{3\epsilon_0 \hat{z}_+}(r - \bar{L}) + \frac{\sigma_1 L^2}{\epsilon_0 \hat{z}_+},
\]

(6.11)

while for the interior region an analogous argument in terms of inverse multipoles yields

\[
\phi_{\text{in}}(r) = \frac{\sigma_2}{3\epsilon_0}(r - \bar{L}) + \frac{\sigma_1 L}{\epsilon_0}.
\]

(6.12)

The potential \( \phi_{\text{in}} \) gives rise to potential \( \tilde{\phi}_{\text{sup}} \) above the \( \bar{z} = \bar{L} \) plane for the associated system, while \( \phi_{\text{out}} \) is related to the potential \( \tilde{\phi}_{\text{inf}} \) below the said plane. The result is

\[
\tilde{\phi}_{\text{sup}} = \frac{R}{\bar{u}} \left[ \frac{3\sigma_1 - \sigma_2}{3\epsilon_0} \bar{L} + \frac{\sigma_2 R^2}{3\epsilon_0 \bar{u}^2} \bar{z} \right], \quad \tilde{\phi}_{\text{inf}} = \frac{R}{\bar{u}} \left[ \frac{3\sigma_1 - \sigma_2}{3\epsilon_0} \bar{L} + \frac{\sigma_2 R^2}{3\epsilon_0 \bar{u}^2} (2\bar{L} - \bar{z}) \right],
\]

(6.13)

where for the inferior potential we introduced the auxiliary variable \( \bar{u} = \sqrt{\bar{s}^2 + (\bar{z} - 2\bar{L})^2} \).

Note that the potential is symmetric under reflexion on the \( \bar{z} = \bar{L} \) plane, corresponding to \( \bar{z} \rightarrow 2\bar{L} - \bar{z} \), as it should be. This solution could be obtained by the method of images for a conducting plane with both a charge and a point dipole placed at the origin. Higher multipole distributions on the sphere will give rise to corresponding higher terms in the associated plane problem. The interested reader is invited to verify that the image problem for the conducting sphere is mapped into the planar image problem.
7. From cylinder to torus

We now illustrate further how the Kelvin transformation can give the electrostatic potential for certain nontrivial systems with localised charge distribution in terms of the potential of an associated charge distribution that extends to infinity.

Let us start by considering an inhomogeneous charge distribution on the surface of a special torus which is constructed by rotating a circle of radius \(S_0\) on the plane \(y = 0\) about the \(z\)-axis. The circle equation is \((x - S_0)^2 + z^2 = S_0^2\), figure 4, so that it is tangent to the \(z\)-axis at the origin. In terms of the cylindrical radial coordinate \(s = \sqrt{x^2 + y^2}\), the equation for the torus surface is \((s - S_0)^2 + z^2 = S_0^2\) or, more simply, \(r^2 = 2S_0s\). Let us assume that its surface charge density is\(^5\)

\[
\sigma(r) = \frac{\sigma_0 R^3}{r^3}.
\]

(7.1)

In order to describe the potential produced by this charge configuration, let us perform an inversion with respect to a sphere of radius \(R\), and solve the associated problem. From \(r = (R^2/r^2)\mathbf{r}\) it follows that \(s = (R^2/r^2)\tilde{s}\), and it is a simple matter to show that the torus is mapped to an infinite cylinder:

\[
r^2 = 2S_0\tilde{s} \leftrightarrow \tilde{s} = \tilde{S}_0,
\]

(7.2)

where

\[
\tilde{S}_0 = \frac{R^2}{2S_0}.
\]

(7.3)

Let us translate the surface charge density (7.1) into a volume density. For this, consider the transformation from the cylindrical to the new orthogonal coordinates according to (see figure 4)

\[
(s, z, \varphi) \longrightarrow (\xi, \alpha, \varphi) = \left(\sqrt{(s - S_0)^2 + z^2}, \tan^{-1}\left(\frac{s - S_0}{z}\right), \varphi\right).
\]

(7.4)

Noting that \(\xi^2 = (s - S_0)^2 + z^2\), the torus surface is given by \(\xi = S_0\), so that

\[
\rho(r) = \frac{\sigma_0 R^3}{r^3} \delta(\xi - S_0)
\]

(7.5)

with \(\xi = \sqrt{(s - S_0)^2 + z^2}\).

From (2.11) we find, as shown in appendix B, that the transformed system charge density is simply

\[
\tilde{\rho}(\tilde{r}) = \sigma_0 \delta(\tilde{s} - \tilde{S}_0),
\]

(7.6)

which represents a uniform surface charge density \(\sigma_0\) glued over the surface of an infinite cylinder of radius \(\tilde{S}_0\) whose symmetry axis coincides with the \(\tilde{z}\)-axis. The potential for this problem can be found by elementary means and is well known: it vanishes inside the cylinder (for a specific gauge choice) and in the exterior region is given by

\[
\tilde{\Phi}(\tilde{r}) = -\frac{\sigma_0 \tilde{S}_0}{\epsilon_0} \ln \frac{\tilde{s}}{\tilde{S}_0}.
\]

(7.7)

From (2.10) we readily obtain the potential produced by the charged torus, namely, it vanishes outside the torus while, inside, it takes the form

\(^5\) This choice is made aiming at the simplicity of the dual system.
The vanishing of the potential outside the torus is ascribable to a point charge at the origin, so that the total charge is zero. A different gauge choice for the potential of the cylinder will change the charge of the point particle at the origin. As a consequence, its exterior potential will no longer vanish.

The singularity at the origin in the torus case is indeed a consequence of the mapping between densities, equation (2.11). Intuitively the inversion maps an infinite extension of the cylinder to a point, giving rise to this singularity. It can be avoided by cutting the charged cylinder, for instance by restricting the polar angle \( \theta \) to the interval \( \theta_\pm \leq \theta \leq \theta_+ \). The cylinder becomes finite. Since the polar angle is preserved by the inversion on the sphere, the torus charge distribution will be restricted to the same angular interval, so that the origin is excluded and the charge density is regular. Note that the vanishing of the potential outside the torus is an artifact of the gauge choice\(^6\). The interested reader is invited to break the azimuthal symmetry by taking, for example, \( \sigma_0 \cos \varphi \) instead of \( \sigma_0 \) as the surface charge density of the cylinder. The torus distribution, associated to this well-known cylinder problem, will present a pure dipole potential for its exterior solution.

8. Concluding remarks

We have discussed some applications of conformal transformations to electrostatics. The inversion in the sphere, an essential ingredient of the special conformal transformations, allows one to relate some elementary problems of electrostatics, usually discussed in undergraduate courses, to other ones, providing intriguing links among them. It also underlies the method of images applied to a spherical conductor. Through its use as a conceptual tool, the notion of inverse multipole expansion emerges naturally by considering the fate of the direct multipole expansion under inversion.

The singularity of the mapping at \( r = 0 \) and \( r = \infty \) has the attractive consequence that it may change the topology of surfaces: an infinite cylinder has been mapped into a (special) torus whereas an infinite plane becomes a spherical shell. By attaching a uniform surface charge density to the cylindrical or plane surfaces there appear singularities in the surface charge distribution of their localised counterparts. On the other hand, nonsingular charge distributions on the torus or sphere lead to exactly soluble charge distributions on the cylinder or plane that vanish at infinity.

By dealing with electrostatics, we hope to have called the reader’s attention to the value of the conformal transformations as a tool to economically introduce concepts that usually are marginally outside the content of an undergraduate intermediate course of electromagnetism.

Applications of inversion in the sphere to magnetostatics are also of interest. Let the vector potential and the current density be transformed as follows:

\[
\mathbf{A}(\mathbf{r}) \Rightarrow \mathbf{A}(\mathbf{r}) = \frac{R}{R} \mathbf{A}(\mathbf{r}), \quad \mathbf{J}(\mathbf{r}) \Rightarrow \mathbf{J}(\mathbf{r}) = \left(\frac{R}{r}\right)^5 \mathbf{J}(\mathbf{r}).
\]

It can be shown that this transformation maps a magnetostatics problem into another one as long as the gauge condition \( \mathbf{\nabla} \cdot \mathbf{A} = 0 \) is imposed and the current density is transverse, that is, \( \mathbf{J} \cdot \mathbf{r} = 0 \). The reason lies in that the Cartesian components of the vector potential satisfy

\[^6\text{Distinct gauge choices would imply the addition of distinct point charges at the torus origin, that is, (x, y, z) = (0, 0, 0). The choice we made implicitly leads to a zero total charge for the torus.}\]
the Poisson equation with the corresponding components of the current as a source. As discussed at the final paragraphs of section 2, these Poisson equations will be mapped to associated ones. Besides, the conservation of the transformed current is consistently achieved, keeping the transformed system in the realm of magnetostatics. Despite the restriction on \( J \), physically interesting systems such as rotating charged cylinders, spheres or tori are amenable to the mapping. As a relevant example, the Helmholtz coil turns out to be a self-dual system. Since deviations from uniformity of the magnetic field near the centre of symmetry do not present linear, quadratic or cubic terms, the inverse multipole terms for \( l = 2, 3 \) and 4 are not present. Thus, for the exterior potential, the corresponding direct multipole terms are also absent. This is an alternative systematic way of accounting for the absence of these direct multipole terms highlighted in [20]. Exploring the torus to cylinder relationship, it turns out that an infinite uniform solenoid is mapped into a torus whose azimuthal currents create a pure dipole field outside, and zero magnetic field inside the torus. In a more theoretical vein, one might ask if there is any connection between these mappings and the conformal invariance of Maxwell’s electrodynamics in three-dimensional spacetime (which admits a scalar potential formulation) pointed out in [5].

Appendix A

Let us find the charge density on the planes associated with the two charged spherical shells considered in section 6. Our starting point is

\[
\rho(r) = \left( \frac{R}{r} \right)^3 \sigma_0 [\delta(\xi_+ - L) - \delta(\xi_- - L)].
\]  

(A.1)

From a well-known identity for delta functions, we have

\[
\delta(\xi^2 - L^2) = \frac{1}{2L} [\delta(\xi - L) + \delta(\xi + L)] = \frac{1}{2L} \delta(\xi - L),
\]  

(A.2)

where we have used the fact that both \( \xi \) and \( L \) are positive. Taking into account that \( r^2 = \pm 2L \) on the spheres, the volume charge density turns out to be

\[
\rho(r) = \frac{2LR^3}{r^3} \sigma_0 [\delta(\xi_+^2 - L^2) - \delta(\xi_-^2 - L^2)]
\]

\[
= \frac{2LR^3}{r^3} \sigma_0 [\delta((z - L)^2 + s^2 - L^2) - \delta((z + L)^2 + s^2 - L^2)]
\]

\[
= \frac{2LR^3}{r^3} \sigma_0 [\delta(r^2 - 2Lz) - \delta(r^2 + 2Lz)]
\]

\[
= \frac{2LR^3}{r^3} \sigma_0 \left[ R^2 \left( \frac{R^2}{2L} - \frac{R^2}{r^2} \right) - \delta \left( \frac{R^2}{2L} + \frac{R^2}{r^2} \right) \right]
\]

\[
= \left( \frac{R}{r} \right)^5 \sigma_0 \left[ \delta(\bar{z} - \bar{L}) - \delta(\bar{z} + \bar{L}) \right],
\]  

(A.3)

where we used \( \delta(x) = |a|\delta(ax) \) and assumed that \( \tilde{r} \neq 0 \). A comparison with (2.11) yields (6.4).
Appendix B

Let us obtain the cylinder charge density (7.6) from the charge distribution on the torus discussed in section 7, which is given by

\[ \rho(\mathbf{r}) = \frac{\sigma_0 R^3}{r^3} \delta(\xi - S_0). \]  

(B.1)

Noting that \( \xi = \sqrt{(s - S_0)^2 + z^2} \geq 0 \) and \( S_0 > 0 \), so that

\[ \delta(\xi^2 - S_0^2) = \frac{1}{2S_0} \left[ \delta(\xi - S_0) + \delta(\xi + S_0) \right] = \frac{1}{2S_0} \delta(\xi - S_0), \]  

(B.2)

we have

\[ \rho(\mathbf{r}) = \frac{2\sigma_0 R^3 S_0}{r^3} \delta(\xi^2 - S_0^2) = \frac{2\sigma_0 R S_0}{r} \delta \left( \frac{1}{R^2} ((s - S_0)^2 + z^2 - S_0^2) \right). \]  

(B.3)

From this, using equation (2.11), the mapping \( s = \frac{R^2}{r^2} \tilde{s} \) and similarly \( z = \frac{R^2}{r^2} \tilde{z} \), together with the parameter redefinition of \( S_0 = \frac{r^2}{2S_0} \), we find that the transformed charge density is

\[
\tilde{\rho}(\mathbf{r}) = R^3 \frac{\sigma_0 R}{r^3} \delta \left( \frac{1}{R^2} \left( \frac{\tilde{s} R^2}{r^2} - \frac{R^2}{2S_0} \right)^2 + \left( \frac{\tilde{z} R^2}{r^2} - \frac{R^4}{4S_0^2} \right) \right)
\]

\[
= \frac{\sigma_0}{r^2 S_0} \delta \left( \frac{1}{r^2} \left( \tilde{s} \frac{R^2}{r^2} - \frac{R^2}{2S_0} \right)^2 + \tilde{z}^2 - \frac{R^4}{4S_0^2} \right)
\]

\[
= \frac{\sigma_0}{r^2 S_0} \delta \left( \frac{1}{S_0} \tilde{s}^2 - \frac{\tilde{s}^2}{S_0} + \tilde{z}^2 \right) = \frac{\sigma_0}{r^2 S_0} \delta \left( \tilde{s} \frac{R^2}{r^2} - S_0 \right)
\]

as we wished to show.

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