Multicomponent Calogero Model of $B_N$-Type Confined in Harmonic Potential

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Abstract

A new one-dimensional model, the multicomponent Calogero model of $B_N$-type confined in the harmonic potential, is introduced. The Lax pair of this model is determined, and then a set of functionally independent conserved operators are constructed. Moreover, the energy spectrums of the above model are obtained by three different methods.

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There has been a recent burst of interest in a family of one-dimensional many body systems with the $1/r^2$-type interaction. For comprehensive reviews of these systems see Refs. [1,2] and references therein. The two-body interacting potential of these systems are supposed to be translationally invariant. Mathematically speaking, this property reflects the invariance under the action of the Weyl group of type $A_{N-1}$, where $N$ is the number of particles. In the case of continuous models without internal degrees of freedom, Olshanetsky and Perelomov have studied corresponding systems associated with other Weyl groups. See the survey paper Ref. [3]. More recently, it has become apparent that the models associated with the Weyl group of type $B_N$ (or $BC_N$) are important to describe the one-dimensional physics with boundaries. For example, the non-relativistic dynamics of quantum sine-Gordon solitons in the presence of a boundary was described by the Sutherland model of $BC_N$-type (with sinh-interaction) [4]. This model is also related to the physics of the quantum electric transport in mesoscopic systems [5,6]. The Haldane-Shastry model, which is the discrete version of the Calogero-Sutherland model, with the free boundary conditions was defined by generalizing the ordinary (i.e. $A_{N-1}$-type) model to the one associated with the Weyl group of type $BC_N$ [7,8].

It is important to study the application of other $1/r^2$ models associated with the Weyl group of type $B(C)_N$ to the one-dimensional physics with boundaries. For the first step of these directions, we will introduce and analyze the multicomponent generalizations of the Calogero model of $B_N$-type.

Before defining the new model, we briefly recall the Weyl group of type $B_N$ and its action on the physical space. In this letter, we will consider the one-dimensional system with internal degrees of freedom. Internal degrees of freedom means that the particle has some spin (color) $\sigma \in \Omega$. Here $\Omega$ is a finite set with $\# \Omega = r$, that is, we consider the $r$-component system. Then $N$-particle system is specified by $(q_1 \sigma_1, q_2 \sigma_2, \ldots, q_N \sigma_N)$, where $(q_1, q_2, \ldots, q_N) \in R^N$ are particle coordinates and $(\sigma_1, \sigma_2, \ldots, \sigma_N) \in \Omega^N$. The Weyl group action can be defined on both the space of particle coordinates $R^N$ and the space of spins $\Omega^N$. Firstly, we define the action on $R^N$. In general, we consider the group $W$ of the
coordinate transformations

\[(q_1, q_2, \ldots, q_N) \mapsto (\epsilon_1 q_{\sigma(1)}, \epsilon_2 q_{\sigma(2)}, \ldots, \epsilon_N q_{\sigma(N)})\]

of \(\mathbb{R}^N\), where \(\sigma\) are the elements of the \(N\)-th symmetric group \(S_N\) and \(\epsilon_j \in \{\pm 1\}\). We call \(W\) the Weyl group\(^1\) of type

\[A_{N-1}, \quad \text{if } \epsilon_j = 1, \text{ (for all } j),\]

\[B_N, (C_N, \text{ or } BC_N), \quad \text{if } \epsilon_j \in \{\pm 1\},\]

\[D_N, \quad \text{if } \epsilon_j \in \{\pm 1\} \text{ and } \prod_{k=1}^{N} \epsilon_k = 1.\]

Of course, the Weyl group of type \(A_{N-1}\) is same as the symmetric group \(S_N\).

Secondly, we consider the \(B_N\)-action on \(\Omega^N\). For this purpose, we introduce the operators, \(P_j, P_{jk}\) and \(\bar{P}_{jk}\) which are defined as follows. Operators, \(P_j, P_{jk}\) and \(\bar{P}_{jk}\), are acting only on the spin variable. The operator \(P_j\) acts on the \(j\)-th particle by \(\sigma_j \mapsto \sigma_j^* \in \Omega\) such that \(P_j^2 = 1\) (i.e. \(\sigma_j^{**} = \sigma_j\)). The operator \(P_{jk}\) acts by the permutations of \(j\)-th and \(k\)-th spins,

\[(\cdots, q_j \sigma_j, \cdots, q_k \sigma_k, \cdots) \mapsto (\cdots, q_j \sigma_k, \cdots, q_k \sigma_j, \cdots).\]

And, the operator \(\bar{P}_{jk}\) is defined by \(\bar{P}_{jk} = P_j P_k P_{jk}\). These operators represent the \(B_N\)-action on \(\Omega^N\). We note that these operators satisfy the relations\(^2\)

\[P_j^2 = P_{jk}^2 = \bar{P}_{jk}^2 = 1, \quad P_{jk} = P_{kj}, \quad \bar{P}_{jk} = \bar{P}_{kj},\]

\[P_j P_k = P_k P_j, \quad P_j P_{jk} = P_{jk} P_j, \quad P_j P_{jk} = P_{jk} P_k = P_{jk} P_j,\]

\[P_{jk} P_{kl} = P_{kl} P_{jk} = P_{jl} P_{jk}, \quad \bar{P}_{jk} \bar{P}_{kl} = \bar{P}_{jl} \bar{P}_{jk} = \bar{P}_{kl} \bar{P}_{jl},\]

where \(j, k, \) and \(l\) are distinct in the last two formulae.

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\(^1\)For the complete definition of the Weyl groups associated with the simple Lie algebras, see\(^3\).

Among others, the groups \(A_{N-1}, B_N, \) and \(D_N\) (also \(C_N\) and \(BC_N\)) are of particular interest because they are related to quantum \('N'\)-body interacting systems.

\(^2\)They also satisfy the so-called reflection equations\(^4\) which are the generalized Yang-Baxter equations associated with the Weyl group of type \(B_N\).
Let us introduce models which are multicomponent generalizations of the Calogero model of $B_N$-type. The Hamiltonians are

$$H_1 = \sum_{j=1}^{N} p_j^2 + V_{B_N}, \quad (5)$$

$$H_2 = H_1 + V_{\text{conf}}, \quad (6)$$

where $p_j = -i \frac{\partial}{\partial q_j}$, and

$$V_{B_N} = \sum_{j,k=1, j \neq k}^{N} \frac{\lambda(\lambda - P_{jk})/(q_j - q_k)^2 + \lambda(\lambda - \bar{P}_{jk})/(q_j + q_k)^2}{(q_j - q_k)^2} + \sum_{j=1}^{N} \frac{\lambda_1(\lambda_1 - P_{j})}{q_j^2}, \quad (7)$$

$$V_{\text{conf}} = \omega^2 \sum_{j=1}^{N} q_j^2. \quad (8)$$

Here $\lambda, \lambda_1 \in \mathbb{R}$ are the coupling constants and $\omega > 0$. We note that the $B_N$-action on $\Omega^N$ induces a $B_N$-action on the operators $P_j, P_{jk}$ and $\bar{P}_{jk}$. Therefore, Hamiltonians $H_1$ and $H_2$ are $B_N$-invariant, in spite that they are not invariant under the $B_N$-action on $\mathbb{R}^N$. Then we refer to the model with the Hamiltonian $H_1$ ($H_2$) as the (confined) spin Calogero model of $B_N$-type ($B_N$-(C)SC model). These models are natural generalizations of the models which have been studied in Refs. [11–17]. We note that the $B_N$-SC model has been analyzed by Cherednik [18], but his treatment is quite different from ours and the Lax pair of this model has not been given. The $B_N$-CSC model is new.

Here we give some comments on the above Hamiltonians. The term $\lambda(\lambda - \bar{P}_{jk})/(q_j + q_k)^2$ in $V_{B_N}$ expresses the two-body interaction between the $j$-th particle and the “mirror-image” (we place a mirror at the origin $q = 0$) of the $k$-th particle. The term $\lambda_1(\lambda_1 - P_{j})/q_j^2$ in $V_{B_N}$ represents the potential of the impurity sitting on the origin. These terms violate the translational invariance. Note that $V_{\text{conf}}$ also violates the translational invariance.

In what follows, we only treat the Hamiltonian $H_2$, because almost all calculations necessary for the Hamiltonian $H_1$ are done in similar manner, or are included in those for Hamiltonian $H_2$.

Now, following Olshanetsky-Perelomov [3] and Ujino-Wadati [13], we construct a set of conserved operators for the $B_N$-CSC model. To start with, we take the $2N \times 2N$ matrices $L, M$ and $Q$ as follows;
\[
L = \begin{pmatrix} L & S \\ -S & -L \end{pmatrix}, \quad M = \begin{pmatrix} M & T \\ T & M \end{pmatrix}, \quad Q = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix},
\]

where \( L, M, S, T, \) and \( Q \) are \( N \times N \) matrices depending on \( p_j, q_j, P_j, P_{jk} \) and \( \bar{P}_{jk} \):

\[
L_{jk} = \delta_{jk} (p_j) + (1 - \delta_{jk}) \left( i\lambda \frac{P_{jk}}{q_j - q_k} \right),
\]

\[
M_{jk} = \delta_{jk} \left( 2\lambda \sum_{l=1, l \neq j}^{N} \left\{ \frac{P_{jl}}{(q_j - q_l)^2} + \frac{\bar{P}_{jl}}{(q_j + q_l)^2} \right\} \right) + \lambda \frac{P_{jk}}{q_j^2},
\]

\[
S_{jk} = \delta_{jk} \left( i\lambda_1 \frac{P_j}{q_j} \right) + (1 - \delta_{jk}) \left( i\lambda \frac{\bar{P}_{jk}}{q_j} \right),
\]

\[
T_{jk} = \delta_{jk} \left( -\lambda_1 \frac{P_j}{q_j^2} \right) + (1 - \delta_{jk}) \left( -2\lambda \frac{\bar{P}_{jk}}{(q_j + q_k)^2} \right),
\]

\[
Q_{jk} = \delta_{jk} (iq_j).
\]

These operators satisfy the so-called sum-to-zero conditions \[19\] and the extended Lax equations which will lead to conserved operators for the \( B_N \)-CSC model. The sum-to-zero conditions are the following conditions on \( M \),

\[
\sum_{\nu=1}^{2N} M_{\mu\nu} = \sum_{j=1}^{N} M_{jk} + \sum_{j=1}^{N} T_{jk} = 0, \text{ for all } \nu,
\]

\[
\sum_{\mu=1}^{2N} M_{\nu\mu} = \sum_{j=1}^{N} M_{kj} + \sum_{j=1}^{N} T_{kj} = 0, \text{ for all } \nu,
\]

where \( k = \nu \) for \( 1 \leq \nu \leq N, k = \nu - N \) for \( N + 1 \leq \nu \leq 2N \). These are obvious from the formulae \[11\] and \[13\]. The extended Lax equations for the \( B_N \)-CSC model are

\[
[H_2, L_{\mu,\nu}^\pm] = \sum_{\sigma=1}^{2N} (L_{\mu,\sigma}^\pm M_{\sigma\nu} - M_{\mu\sigma} L_{\sigma\nu}^\pm) \pm 2\omega L_{\mu,\nu}^\pm, \quad (\mu, \nu = 1, 2, \cdots, 2N),
\]

where \( L^\pm = L \pm 2\omega Q \). These equations are equivalent to the following equations

\[
[H_2, L_{jk}^\pm] = \sum_{m=1}^{N} (L_{jm}^\pm M_{mk} - M_{jm} L_{mk}^\pm) + \sum_{m=1}^{N} (S_{jm} T_{mk} + T_{jm} S_{mk}) \pm 2\omega L_{jk}^\pm,
\]

\[
[H_2, S_{jk}] = \sum_{m=1}^{N} (S_{jm} M_{mk} - M_{jm} S_{mk}) + \sum_{m=1}^{N} (L_{jm}^\pm T_{mk} + T_{jm} L_{mk}^\mp) \pm 2\omega S_{jk},
\]

where, \( j, k = 1, 2, \cdots, N \), and \( L^\pm = L \pm 2\omega Q \). By using the relations \[9\], we can prove \[16\].
Let us consider the operators
\[ L^{(m_1,n_1,m_2,n_2,\cdots)} = (L^+)^{m_1}(L^-)^{n_1}(L^+)^{m_2}(L^-)^{n_2} \cdots, \] (19)
where \( m_j, n_j \in \{0, 1, 2, 3, \cdots\} \).

From the extended Lax eq. (16), we can find that
\[ \left[ \mathcal{H}, L^{(m_1,n_1,m_2,n_2,\cdots)} \right]_{\mu,\nu} = \left[ L^{(m_1,n_1,m_2,n_2,\cdots)}, \mathcal{M} \right]_{\mu,\nu} \] (20)
if \( \sum_j n_j = \sum_k m_k \). Then, due to the sum-to-zero condition (15), we see that the operators
\[ I_n = 2N \sum_{\mu,\nu=1}^{2N} O \left( L^+ L^- \right)_{\mu,\nu}, \quad n = 1, \cdots, N, \] (21)
commute with \( \mathcal{H} \). Here \( \mathcal{O} \) denotes the Weyl ordered product,
\[ \mathcal{O}(X^n Y^n) = \frac{n! n!}{(2n)!} (X^n Y^n + X^{n-1}Y^n X + X^{n-1}Y^{n-1}XY + X^{n-1}Y^{n-2}XY^2 + \cdots + X^n Y^{n-1}Y) \]
\[ + X^{n-2}Y^n X^2 + X^{n-2}Y^{n-1}X^2 Y + X^{n-2}Y^{n-1}XY X + \cdots + X^n Y^{n-1}XYX^{n-2} \]
\[ + \cdots \]
\[ + Y^n X^n \). \] (22)

Then \( I_n \)'s are conserved operators. We shall give another expression for the conserved operators. By using the operator \( \mathcal{A}^\epsilon \mathcal{A}^{\epsilon'} = (L - S + \epsilon \omega Q)(L + S + \epsilon' \omega Q) \), where \( \epsilon, \epsilon' \in \{\pm\} \),
we can rewrite (21) in the form,
\[ I_n = \frac{n! n!}{(2n)!} \sum_{\sum_{j=1}^n (\epsilon_j + \epsilon_j') = 0} \sum_{\epsilon, \epsilon'} \left( \mathcal{A}^{\epsilon_1 \epsilon_1'} \cdots \mathcal{A}^{\epsilon_n \epsilon_n'} \right)_{j k}. \] (23)

For example, \( I_1 = \frac{1}{2} \sum_{\epsilon + \epsilon' = 0} \sum_{\epsilon \epsilon' = 1} (\mathcal{A}^{\epsilon \epsilon'})_{j k} = \sum_{j,k=1}^N (L^2 - S^2 + [L,S] - \omega^2 Q^2)_{j k} = \mathcal{H}^2 \). In the case of the \( A_{N-1} \)-type model, the corresponding \( I_1 \) represents the total momentum. On the other hand, in this case, the total momentum is not a conserved operator because the translational invariance is absent in the Hamiltonian \( \mathcal{H}^2 \).

From the explicit form of \( \mathcal{A}^\epsilon \mathcal{A}^{\epsilon'} \), we can find that \( I_n \) is divided into two parts,
\[ I_n(\{q_j\}, \{p_j\}) = I_n^{\text{poly}}(\{q_j\}) + I_n^{\text{homo}}(\{q_j\}, \{p_j\}), \] (24)
where $\mathcal{I}_{n}^{\text{poly}}(\{q_j\})$ is the polynomial of $q_j$'s with the highest degree term $\omega^{2n}\sum_j q_j^{2n}$, and $\mathcal{I}_{n}^{\text{homo}}$ has the form,

$$\mathcal{I}_{n}^{\text{homo}}(\{q_j\},\{p_j\}) = \sum_{j=1}^{N} p_j^{2n} + (q_j\text{'s dependent part}), \quad (25)$$

which satisfies the homogeneity condition $\mathcal{I}_{n}^{\text{homo}}(\{\alpha q_j\},\{\alpha^{-1}p_j\}) = \alpha^{-2n}\mathcal{I}_{n}^{\text{homo}}(\{q_j\},\{p_j\})$.

Then, due to the formula (25), we can conclude that the conserved operators $\mathcal{I}_1, \ldots, \mathcal{I}_N$ are functionally independent. In this way we obtained the conserved operators of the $B_N$-CSC model.

For later use, we define the operators,

$$a_j = \sum_{\mu=1}^{2N} L_{j\mu}^{-} = \sum_{k=1}^{N}(L_{jk}^{-} + S_{jk}) \quad (26)$$

$$= p_j - i\omega q_j + i\lambda \sum_{l=1, l\neq j}^{N} \left( \frac{P_{jl}}{q_j - q_l} + \frac{\bar{P}_{jl}}{q_j + q_l} \right) + i\lambda_1 \frac{P_j}{q_j}, \quad (27)$$

$$a_j^\dagger = \sum_{\mu=1}^{2N} L_{\mu j}^{+} = \sum_{k=1}^{N}(L_{kj}^{+} - S_{kj}) \quad (28)$$

$$= p_j + i\omega q_j - i\lambda \sum_{l=1, l\neq j}^{N} \left( \frac{P_{jl}}{q_j - q_l} + \frac{\bar{P}_{jl}}{q_j + q_l} \right) - i\lambda_1 \frac{P_j}{q_j}, \quad (29)$$

where $j = 1, 2, \ldots, N$. These satisfy the following commutation relations,

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \quad (30)$$

$$[a_j, a_k^\dagger] = \delta_{jk}(2\omega + M_{jj} - T_{jj}). \quad (31)$$

Then it is easy to see that the Hamiltonian $\mathcal{H}_2$ is factorized as

$$\mathcal{H}_2 = \sum_{j=1}^{N} a_j^\dagger a_j + \mathcal{E}, \quad (32)$$

where $\mathcal{E} = \omega(N + 2\lambda \sum_{j<k}(P_{jk} + \bar{P}_{jk}) + 2\lambda_1 \sum_j P_j)$. The operator $\mathcal{E}$ is related to the second Casimir operator of the Lie algebra $so_{2N+1}$.

Next we will consider the ground state of the $B_N$-CSC model. We shall consider two cases which are characterized by the value of $\lambda$ and $\lambda_1$. First, we consider the case $\lambda, \lambda_1 > 0$. In this case, we can easily find the ground state because in an appropriate representation of
spins the operator $M_{jj} - T_{jj}$ in the r.h.s. of eq. (31) is always positive. Then, the function $\Phi^{(0)}(\{q_k\}, \{\sigma_k\})$ which is a solution to the equations

$$a_j \Phi^{(0)}(\{q_k\}, \{\sigma_k\}) = 0, \quad (j = 1, \cdots, N),$$

(33)

is the ground state of the $B_N$-CSC model, since the term $\sum_j a_j^\dagger a_j$ in r.h.s. of the formula (32) is semi-positive definite. Note that eqs. (33) are equivalent to the so-called $B_N$-type Knizhnik-Zamolodchikov equation [20],

$$\left( \frac{\partial}{\partial q_j} - \lambda \sum_{l=1, l \neq j}^N \left( \frac{P_{jl}}{q_j - q_l} + \frac{\bar{P}_{jl}}{q_j + q_l} \right) - \lambda_1 \frac{P_j}{q_j} \right) \Phi^{KZ}(\{q_k\}, \{\sigma_k\}) = 0,$$

(34)

where we put $\Phi^{(0)}(\{q_k\}, \{\sigma_k\}) = \Phi^{KZ}(\{q_k\}, \{\sigma_k\}) \prod_{l=1}^N e^{-\frac{1}{2} \omega q_l^2}$. We can construct $\Phi^{(0)}(\{q_k\}, \{\sigma_k\})$ in the form,

$$\Phi^{(0)}(\{q_m\}, \{\sigma_m\}) = \phi_{\lambda, \lambda_1}^{(0)}(\{q_m\}) \chi(\{\sigma_m\}),$$

(35)

where

$$\phi_{\lambda, \lambda_1}^{(0)}(\{q_m\}) = \prod_{1 \leq j < k \leq N} |q_j - q_k|^\lambda |q_j + q_k|^\lambda \prod_{l=1}^N |q_l|^\lambda_1 \prod_{n=1}^N e^{-\frac{1}{2} \omega q_n^2}$$

(36)

is the Jastrow-type ground state of the corresponding one-component model, and $\chi(\{\sigma_m\})$ is some function which is invariant under the action of $P_j$ and $P_{jk}$. Note that the above ground state is $B_N$-invariant.

Next we consider the case $\lambda < 0$ or $\lambda_1 < 0$. In this case, the ground state does not need to annihilate by $a_j$’s. However, following [14,15], we conjecture that the (spin-singlet) state

$$\check{\Phi}^{(0)}(\{q_m\}, \{\sigma_m\}) = \phi_{-\lambda, -\lambda_1}^{(0)}(\{q_m\}) \times \prod_{1 \leq j < k \leq N} (q_j - q_k)^{\delta_{j,k}} (q_j + q_k)^{\delta_{j,k}} \exp \left\{ i \frac{1}{2} \text{sgn}(\sigma_j - \sigma_k) \right\} \prod_{l=1}^N q_l$$

(37)

is the ground state of the $B_N$-CSC model with $P_j = 1$ and $\lambda, \lambda_1 < 0$. Note that in the condition $P_j = 1$ the Hamiltonian $H_2$ possesses the $SU(r)$-symmetry. At least, we can show that the state $\check{\Phi}^{(0)}(\{q_m\}, \{\sigma_m\})$ is the eigen state of the Hamiltonian $H_2$ with the eigen value,
\[ \hat{E}^{(0)}_N = \omega \left\{ N - 2\lambda N(N - 1) - 2\lambda_1 N + 2 \sum_{\alpha=1}^r N^2_{\alpha} \right\}, \tag{38} \]

where \( N_{\alpha} \) is the number of particles with spin \( \alpha \) (\( \sum_{\alpha=1}^r N_{\alpha} = N \)).

We now turn to the spectrum of the \( B_N \)-CSC model. We will derive the spectrum by using three different methods. The first is the explicit construction of the excited states (we call the direct method), more precisely the triangulation of the Hamiltonian \( H_2 \). The second is the renormalized-harmonic oscillator (RHO) method \[21\] which is a variant of the asymptotic Bethe-ansatz method. The third is the operator method. In what follows, we fix the spin configuration \( \{ N_{\alpha} \} \).

To begin with, we consider the direct method. We introduce the \( B_N \)-invariant free boson bases (Notice that the model is confined in the harmonic potential),

\[
\psi(\{n_m\}) = \sum_{\epsilon_1, \ldots, \epsilon_N \in \{\pm 1\}} \sum_{\sigma \in S_n} \prod_{a=1}^r H_{n_{\sigma(a)}}(\sqrt{\omega} \epsilon_a q_a). \tag{39}
\]

Here \( H_n \) is the Hermite polynomial, and \( n_1, n_2, \ldots, n_N \) are even non-negative integers such that \( n_1 \geq n_2 \geq \cdots \geq n_N \). We will fix the ordering \( \succ \) of the above basis. For the two sets of even non-negative integers, \( n_1 \geq n_2 \geq \cdots \geq n_N \) and \( m_1 \geq m_2 \geq \cdots \geq m_N \), we write \( \psi(\{n_j\}) \succ \psi(\{m_j\}) \) if the first nonvanishing difference \( n_k - m_k \) is positive. Then we can show that with respect to the above ordering the Hamiltonian \( H_2 \) is triangular in the basis \( \{ \hat{\Psi}(\{n_j\}) = \hat{\Phi}^{(0)}(\{q_m\}, \{\sigma_m\})\psi(\{n_j\}) \} \) where \( \hat{\Phi}^{(0)}(\{q_m\}, \{\sigma_m\}) \) is the Jastrow-type ground state with the ground state energy \( \hat{E}^{(0)}_N \). Therefore, the energy spectrum is obtained by reading the diagonal elements labeled in terms of the quantum numbers \( \{n_j\} \). The result is

\[
\hat{E}_N = \hat{E}^{(0)}_N + 2\omega \sum_{j=1}^N n_j. \tag{40}
\]

We note that \( \sum_{j=1}^N n_j \) is always the even non-negative integer. This is the essential difference from the \( A_{N-1} \)-case and corresponds to the existence of mirror particles.

\[3\] Due to the formula \( H_n(-x) = (-1)^n H_n(x) \) for the Hermite polynomial, summands in the r.h.s. of (39) with odd \( n_j \) are vanished. This reflects the \( B_N \)-invariance.
Next, we construct the RHO solution which is due to Kawakami [21]. The essence of the RHO method is that all the interaction effects are incorporated in terms of the renormalized quantum numbers of oscillators. This method is supported by the following observation. From the formula (24), in the asymptotic region \(0 \ll q_1 \ll q_2 \ll \cdots \ll q_N\), the operators \(\{ \mathcal{I}_n \}\) have the form \(\{ \sum_j (p_j^{2n} + \omega^{2n} q_j^{2n}) \}\) which are the conserved operators of the \(N\)-independent harmonic oscillators.

We shall consider the case which corresponds to the ground state \(\tilde{\Phi}^{(0)}\) given by the formula (37). In the RHO method, the energy spectrum \(\tilde{E}_N\) is given by

\[
\tilde{E}_N = 2\omega \sum_{j=1}^{N} (m_j^{(1)} + \frac{1}{2}).
\]

The renormalized quantum number \(m_j^{(1)}\) together with \(m_j^{(\alpha)} (\alpha = 2, \cdots, r)\) are to be determined by the nested equations (cf. Ref. [22]),

\[
m_j^{(1)} = I_j^{(1)} - \sum_{k=1}^{M_2} \left\{ \text{sgn}(m_k^{(2)} - m_j^{(1)}) + \text{sgn}(m_k^{(2)} + m_j^{(1)}) \right\}
\]

\[
+ (-\lambda + 1) \sum_{l=1, l \neq j}^{M_1} \left\{ \text{sgn}(m_j^{(1)} - m_l^{(1)}) + \text{sgn}(m_j^{(1)} + m_l^{(1)}) \right\} + (-\lambda_1 + 1)\text{sgn}(m_j^{(1)}),
\]

\[
2 \sum_{l=1, l \neq k}^{M_1} \left\{ \text{sgn}(m_k^{(\alpha)} - m_l^{(\alpha)}) + \text{sgn}(m_k^{(\alpha)} + m_l^{(\alpha)}) \right\} + 2\text{sgn}(m_k^{(\alpha)}) + I_k^{(\alpha)}
\]

\[
= \sum_{j=1}^{M_{\alpha-1}} \left\{ \text{sgn}(m_k^{(\alpha)} - m_j^{(\alpha-1)}) + \text{sgn}(m_k^{(\alpha)} + m_j^{(\alpha-1)}) \right\}
\]

\[
+ \sum_{n=1}^{M_{\alpha+1}} \left\{ \text{sgn}(m_k^{(\alpha)} - m_n^{(\alpha+1)}) + \text{sgn}(m_k^{(\alpha)} + m_n^{(\alpha+1)}) \right\}, \ \alpha = 2, \cdots, r.
\]

Here \(I_j^{(\alpha)} \in \{0, 2, 4, \cdots\}\) is the bare quantum number, and the sign function defined by \(\text{sgn}(x) = 1\) for \(x > 0\) and \(= 0\) otherwise. In the above equation, the quantity \(M_\alpha = \sum_{\beta=\alpha}^{r} N_{\beta}\) was introduced \((M_1 = N, \ M_{r+1} = 0)\). By substituting the nested equations (42) to the formula (41), we obtain

\[
\tilde{E}_N = \tilde{E}_N^{(0)} + 2\omega \sum_{\alpha=1}^{r} \sum_{j=1}^{M_\alpha} I_j^{(\alpha)},
\]

where \(\tilde{E}_N^{(0)}\) was given by the formula (38). The quantity \(\sum_{\alpha=1}^{r} \sum_{j=1}^{M_\alpha} I_j^{(\alpha)}\) which labels the excited states takes values in the even non-negative integers.
Finally, we touch upon the operator method. We define operators

\[ B_n = \frac{1}{2} \sum_{\mu,\nu=1}^{2N} \left( (L^-)^n \right)_{\mu\nu} = \sum_{j,k=1}^{N} \left( (A^-)^n \right)_{jk}, \]  
\[ B_n^\dagger = \frac{1}{2} \sum_{\mu,\nu=1}^{2N} \left( (L^+)^n \right)_{\mu\nu} = \sum_{j,k=1}^{N} \left( (A^+)^n \right)_{jk}, \]  

where \( n \in \{2, 4, 6, \cdots \} \). Note that \( \sum_{\mu,\nu} (L^-)^n_{\mu\nu} = 0 \) for odd \( n \). For example, \( B_2^\dagger = H_2 - 2\omega^2 \sum_{j=1}^{N} q_j^2 + i\omega \sum_{j=1}^{N} (q_j p_j + p_j q_j) \). These satisfy the commutation relations,

\[ \[H_2, B^\dagger_n\] = 2n\omega B^\dagger_n, \quad [H_2, B_n] = -2n\omega B_n, \]  
i.e., these change the energy eigen value by \( 2n\omega \). Let \( |0\rangle \) be the ground state of the \( B_N \)-CSC model with the ground state energy \( \bar{E}^{(0)}_N \). Then we can see that the excited state \( H_2 |\Psi(\{n_j\})\rangle = \bar{E}_N |\Psi(\{n_j\})\rangle \) where \( \bar{E}_N = \bar{E}^{(0)}_N + 2\omega \sum_j n_j \) is constructed by \( |\Psi(\{n_j\})\rangle = \prod_j B^\dagger_{n_j} |0\rangle \). Again, the number \( \sum_j n_j \) is the even non-negative integer.

We can conclude that the above three methods are consistent and give the same results. The correlations via the \( 1/r^2 \)-type interaction appear only in the ground state energy, and excitations with the fixed number of electrons do not include any effects of interactions. Full details will be given in the separate publication.

Finally, we comment on the Yangian symmetry of our model. We introduce the Dunkl operators

\[ D_j = \frac{\partial}{\partial q_j} - \lambda \sum_{l=1,l\neq j}^{N} \left( \frac{1}{q_j - q_l} K_{jl} + \frac{1}{q_j + q_l} \bar{K}_{jl} \right) - \lambda_1 \frac{1}{q_j} K_j, \quad (j = 1, 2, \cdots, N). \]  

Here \( K_j, K_{jk} \) and \( \bar{K}_{jk} \) are defined by \( K_j q_j = -q_j K_j, \quad K_{jk} q_k = q_j K_{jk}, \) and \( \bar{K}_{jk} = K_j K_k K_{jk} \). These satisfy the same relations among \( P_j, P_{jk} \) and \( \bar{P}_{jk} \). We find that the Dunkl operators satisfy the relations, \( K_j D_j = -D_j K_j, \quad K_{jk} D_k = D_j K_{jk}, \quad \bar{K}_{jk} D_k = -D_j \bar{K}_{jk}, \) and

\[ [D_j, D_k] = 0, \]  
\[ [q_j, D_k] = \delta_{jk} \left( 1 + \lambda \sum_{l=1,l\neq j}^{N} (K_{jl} + \bar{K}_{jl}) + 2\lambda_1 K_j \right) - (1 - \delta_{jk}) \lambda (K_{jk} - \bar{K}_{jk}). \]  

Then, following Refs. \[8,24-26\], we can construct a monodromy matrix which represents the Yangian symmetry of the \( B_N \)-CSC model. Details will be published elsewhere.
In this letter, we have studied the multicomponent Calogero model of $B_N$-type confined in the harmonic potential. We found that the model have conserved operators as in other one-dimensional models with the $1/r^2$-type interaction. We have constructed the 'Jastrow'-type ground state of our model. Moreover, we have obtained the exact spectrums. For the model associated with the Weyl group of type $D_N$, corresponding quantities are obtained from the $B_N$ case by setting $\lambda_1 = 0$.

The parallel constructions hold for the Sutherland model of $BC_N$-type (the spin Sutherland model of $BC_N$-type ($BC_N$-SS model)), i.e., the periodic model with Hamiltonian

$$\mathcal{H}_3 = \sum_{j=1}^{N} p_j^2 + 2 \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \left\{ \frac{\lambda(\lambda - P_{jk})}{\sin^2 \frac{\pi}{L}(q_j - q_k)} + \frac{\lambda(\lambda - \overline{P}_{jk})}{\sin^2 \frac{\pi}{L}(q_j + q_k)} \right\} + \left( \frac{\pi}{L} \right)^2 \sum_{j=1}^{N} \left\{ \frac{\lambda_1(\lambda_1 - P_j)}{\sin^2 \frac{\pi}{L}q_j} + \frac{\lambda'_1(\lambda'_1 - P_j)}{\cos^2 \frac{\pi}{L}q_j} \right\},$$

(51)

where $L$ is the linear size of the system. For example, $\mathcal{H}_3$ is factorized as

$$\mathcal{H}_3 = \sum_{j=1}^{N} b_j^* b_j + \mathcal{F},$$

(52)

where

$$b_j = -i \frac{\partial}{\partial q_j} + i\lambda \frac{\pi}{L} \sum_{l=1, l \neq j}^{N} \left\{ \cot \frac{\pi}{L}(q_j - q_l)P_{jl} + \cot \frac{\pi}{L}(q_j + q_l)\overline{P}_{jl} \right\} + i\lambda_1 \frac{\pi}{L} \cot \frac{\pi}{L}q_j P_j - i\lambda'_1 \frac{\pi}{L} \tan \frac{\pi}{L}q_j P_j,$$

(53)

$$\mathcal{F} = \left( \frac{2\pi}{L} \right)^2 \left[ \frac{\lambda^2}{2} N(N - 1) + \frac{\lambda'^2}{12} \sum_{j,k,l=1,j,k,l: \text{distinct}}^{N} (P_{jk}P_{kl} + \overline{P}_{jk}\overline{P}_{kl} + P_{jk}\overline{P}_{kl} + \overline{P}_{jk}P_{kl}) \right. + \left. \frac{(\lambda_1 + \lambda'_1)^2}{4} N + \frac{\lambda(\lambda_1 + \lambda'_1)}{4} \sum_{j,k=1,j \neq k}^{N} (P_j + P_k)P_j \right].$$

(54)

Also, the discritization \[7,8,24,30\] of the $B_N$-CSC model and the $BC_N$-SS model can be constructed.

\[4\]The integrability of this model (with $\lambda'_1 = 0$) has been studied by Etingof and Styrkas \[27\] in terms of the representation theory of the Lie algebras. See also \[18,28\].

\[5\]Using the identity $\sin 2x = 2 \sin x \cos x$, we rewrote the term $1/\sin^2(\pi/L)2q_j$. 

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The models which were introduced in this letter may become useful for the one-dimensional physics in the presence of boundaries, in particular, with internal degrees of freedom. For example, those models will be helpful in explaining the internal structure of the FQHE state \cite{31,32}. We hope to turn to this issue in the near future.

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