EQUIVARIANT HOMOTOPY AND DEFORMATIONS OF DIFFEOMORPHISMS

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Abstract. We present a way of constructing and deforming diffeomorphisms of manifolds endowed with a Lie group action. This is applied to the study of exotic diffeomorphisms and involutions of spheres and to the equivariant homotopy of Lie groups.

1. Introduction

In this paper we investigate the geometric and algebraic relation between two areas:

On the one hand, the study of the symmetries of geometric generators of homotopy groups: it has long been a theme in homotopy theory to produce generators and elements of homotopy groups that are “nice” with respect to symmetry and geometric properties (e.g. [Bo], and, for more recent results, [Pu] and the references therein). On the other hand, we have the construction of deformations of diffeomorphisms, or the non-existence of such, i.e., orientation-preserving diffeomorphisms in different isotopy classes, which give rise for example to exotic spheres ([KM, Du]). Let us note that both areas have important applications in physics, [KR] for geometric generators of homotopy groups, and [AR] for exotic phenomena.

We noticed the link between the two in a particular example, the relationship between a distinguished generator of $\pi_6(S^3)$ and an exotic diffeomorphism of $S^6$. In this paper we abstract this principle and reach what we call “equivariant J-process”, since it is analogous to the construction of the J-homomorphism in topology: we want to remark right away that in the present work we are mainly interested in the algebra of the J-process, instead of the topology. Our main result says that equivariantly symmetry-preserving deformations of elements of homotopy groups provide in a canonical way deformations of diffeomorphisms; and we give examples in which the lack of equivariance of the homotopy deformations implies that the deformed maps cease to be diffeomorphisms at some point (or, conversely, that no equivariant deformation exists). This process begins the distillation and abstraction of the phenomena that appears in the authors’ research in exotic maps and involutions ([Du, DMR, ADPR, DPR]), in order to search for a constructive and algebraic theory of exotic phenomena.

The paper is organized as follows: in section 2 we describe the main technique, which is actually quite easy to prove; we believe that the relevant issue here is the abstraction of the principle and the consequences of its application in concrete examples. In section 3 we explain the example that in fact motivated the main result:
the equivariant differential geometry of a Blakers-Massey element (a generator of \( \pi_6(S^3) \)) and its relation to the exotic diffeomorphisms (i.e., not deformable to the identity through diffeomorphisms) of the 6-sphere, and exotic involutions of the 6-sphere and 5-sphere. This setting will provide several applications of the main results, first deforming this exotic diffeomorphism of \( S^6 \) to a rational (still exotic) diffeomorphism, deforming the exotic involutions to rational maps, and then showing the non-existence of equivariant deformations of maps even though it is known that non-equivariant deformations exist. These applications are done in section 4. We finally comment on some future directions in section 5.

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2. The equivariant J-process

Let us describe the equivariant J-process, which has two ingredients:

- A Lie group \( G \) acting differentiably on a manifold \( M \) (from now on we assume that all groups, actions, maps are in the differentiable category). We denote the action of \( G \) on \( M \) by a dot \( g \cdot m, g \in G, m \in M \).

- A map \( \alpha : M \to G \). Note that in the case where \( M \) is a \( n \)-sphere, the homotopy class of the map \( \alpha \) then represents an element in the homotopy group \( \pi_n(G) \).

In this situation, we have

**Definition.** Define \( J_\alpha : M \to M \), the J-process self map of \( M \) associated to \( \alpha \), by

\[
J_\alpha(m) = \alpha(m) \cdot m.
\]

We have then

**Theorem 2.1.** Let \( J_\alpha : M \to M \) be a J-process associated to a map \( \alpha : M \to G \) and an action of \( G \) on \( M \). If \( \alpha \) is \( G \)-equivariant with respect to the conjugation action of \( G \) on itself, then \( J_\alpha \) is a bijection with inverse \( (J_\alpha)^{-1} = J_{\alpha^{-1}} \) and the powers are given by \( J_\alpha^n = J_\alpha \circ \cdots \circ J_\alpha = J_{\alpha^n} \).

**Proof.** The equivariance hypothesis on \( \alpha \) translates to \( \alpha(g \cdot m) = g\alpha(m)g^{-1} \). Then, we just compute:

\[
J_{\alpha^{-1}}(J_\alpha(m)) = J_{\alpha^{-1}}(\alpha(m) \cdot m) = \alpha^{-1}(\alpha(m) \cdot m) \cdot (\alpha(m) \cdot m)
\]

\[
= ((\alpha(m)\alpha^{-1}(m)\alpha^{-1}(m)) \cdot (\alpha(m) \cdot m) \quad \text{by equivariance}
\]

\[
= \alpha^{-1}(m) \cdot (\alpha(m) \cdot m) = m.
\]

The proof of the power formula is similar.

Consider now a one parameter family \( \alpha_t : M \to G \) of differentiable maps. If this deformation satisfies the equivariance property for all \( t \), then there is a one parameter family \( J_{\alpha_t} \) of J-processes of \( M \); theorem 2.1 then guarantees that this deformation is through diffeomorphisms. We shall see in section 4 that sometimes
Theorem 2.2. Let \( M \) be a \( G \)-manifold, \( \alpha : M \to G \) satisfying the hypothesis of theorem 2.1. If in addition there is an involution \( \delta \) of \( M \) such that \( \alpha(\delta(m)) = \alpha^{-1}(m) \), and \( \delta \) commutes with the \( G \)-action (thus producing a \( G \times \mathbb{Z}_2 \)-action on \( M \)), then the \( J \)-process \( \delta \circ J \) is another involution of \( M \).

Proof. Compute:
\[
\delta J_\alpha(\delta J_\alpha(m)) = \delta J_\alpha(\delta(\alpha(m) \cdot m)) \\
= \delta[\delta(\alpha(m) \cdot m)](\delta(\alpha(m) \cdot m)) \\
= \delta[\alpha^{-1}(m) \cdot \delta(\alpha(m) \cdot m)], \text{ by the equivariance of } \delta \text{ and the inverse map,} \\
= \delta[\alpha^{-1}(m) \cdot (\delta \alpha(m) \cdot m)], \text{ by the } G\text{-equivariance of } \alpha, \\
= \delta \delta[\alpha^{-1}(m) \cdot (\alpha(m) \cdot m)], \text{ since } \alpha \text{ and } \delta \text{ commute,} \\
= m, \text{ by } \delta^2 = 1 \text{ and the group action property.}
\]

\[\blacksquare\]

In the next sections we apply these results to an important special case.

3. Blakers-Massey elements and exotic diffeomorphisms

3.1. An equivariant generator of \( \pi_6(S^3) \). The objective of this section is to study the geometry –in particular, the equivariant geometry– of a distinguished generator of \( \pi_6(S^3) \cong \mathbb{Z}_{12} \). A map \( f : S^6 \to S^3 \) is called by us a Blakers-Massey element if its class \( [f] \in \pi_6(S^3) \) generates \( \pi_6(S^3) \). Recall that \( \pi_6(S^3) \) classifies the principal \( S^3 \)-bundles over \( S^7 \), each element of \( \pi_6(S^3) \) corresponding to an equivalence class of \( S^3 \)-bundles over \( S^7 \). For example, the bundle \( S^3 \cdots Sp(2) \to S^7 \) corresponds to a generator of \( \pi_6(S^3) \); the Gromoll-Meyer exotic sphere (\([GM]\)) of non-negative curvature is a given as a Riemannian quotient of the total space. Also, Grove and Ziller (\([GZ]\)) constructed cohomogeneity one metrics of non-negative curvature on principal \( SO(4) \)-bundles over \( S^4 \), allowing the construction of metrics of non-negative curvature on the exotic 7-spheres that are bundles over \( S^4 \); some of these bundles are covered by \( S^3 \)-bundles over \( S^7 \).

It is a classic result \([La]\) \([Tg]\) that \( \pi_6(S^3) \cong \mathbb{Z}_{12} \); the traditional generator is obtained from the commutator of quaternions as follows: consider the map \( \hat{h} : S^3 \times S^3 \to S^3 \) given by \( \hat{h}(x,y) = xyx^{-1}y^{-1} \), where we consider \( S^3 \) as the group of unit quaternions. Since \( \hat{h}(1, y) = \hat{h}(x, 1) = 1 \), \( h \) factors through a map \( \hat{h} : S^3 \wedge S^3 \cong S^6 \to S^3 \) which generates \( \pi_6(S^3) \) (\([Tg]\)). In [DMR], by means of studying the geometry of geodesics of certain metrics on bundles over \( S^7 \), there is a construction
of a differentiable generator $b$ of $\pi^6(S^3)$ that is directly defined on $S^6$: consider the map $b : S^6 \to S^3$ defined as follows: the sphere $S^6$ is expressed as the set

$$S^6 = \{ (p, w) \in \mathbb{H} \times \mathbb{H} / \mathbb{R}(p) = 0, |p|^2 + |w|^2 = 1 \},$$

where $\mathbb{H}$ denotes the quaternions. Define $b : S^6 \to S^3$ by

$$b(p, w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{w}{|w|}, & w \neq 0 \\ -1, & w = 0 \end{cases}$$

where $e^x = \cos(|x|) + \sin(|x|) \frac{x}{|x|}$ denotes the exponential map of the Lie group $S^3$ of unit quaternions. The map $b$, which is a priori not even continuous at points of the form $(p, 0)$, is in fact analytic, and it generates $\pi^6(S^3)$ (see [DMR] and section 4.1).

Having a differentiable map as a representative, obtained by geometrical methods, invites the study of its geometry, in particular with respect to group actions; we shall presently see that this distinguished generator has many symmetry properties and is a cohomogeneity one map. Write $SO(4) = S^3 \times S^3 / \cong$, where $(q, r) \cong (-q, -r)$, and $SO(3) = S^3 / \pm 1$.

We represent $SO(4)$ and $SO(3)$ as follows Define $(q, r) \cdot (p, w) = (qp\bar{q}, rw\bar{q})$. This representation embeds $SO(4)$ in the exceptional Lie group $G_2$: the group $SO(3)$ is represented by the standard quaternionic conjugation $C_q(x) = qx\bar{q}$. The map $(q, r) \mapsto \pm q$ provides an epimorphism $\phi : SO(4) \to SO(3)$. Then, a simple computation shows that $b((q, r) \cdot (p, w)) = qb(p, w)\bar{q}$. Thus, we have the following commutative diagram,

$$\begin{array}{ccc}
SO(4) \times S^6 & \to & S^6 \\
(\phi, b) \downarrow & & \downarrow b \\
SO(3) \times S^3 & \to & S^3
\end{array}$$

And thus the Blakers-Massey element $b$ is equivariant. Note that both actions are cohomogeneity one: the conjugation action on $S^3$ has as quotient the interval $[0, 1]$, realized by the map $R : S^3 \to [-1, 1]$, $R(\theta) = Re(\theta)$; the regular orbits are diffeomorphic to $S^2$ and $\pm 1$ are the two singular orbits. The $SO(4)$ action on $S^6$ has as the invariant function $S : S^6 \to [-1, 1]$, $S(p, w) = |p|^2 - |w|^2 = 2|p|^2 - 1 = 1 - 2|w|^2$; the singular orbits are $S^2$ $(w = 0)$ and $S^3$ $(p = 0)$. The regular orbits are diffeomorphic to $S^2 \times S^3$, that is, the set $\{(p, w) / |p| = r_1, |w| = r_2 \}$ when $r_1$ and $r_2$ are both non-zero.

The following remark will be important in the sequel:

**Remark 3.1.** Consider the composition $\beta = P \circ b : S^6 \to SO(3)$, where $P$ is the canonical double cover projection $S^3 \to SO(3)$. Then taking the $\pm$ equivalence classes in the formula $b((q, r) \cdot (p, w)) = qb(p, w)\bar{q}$ we also get a commutative diagram

$$\begin{array}{ccc}
SO(4) \times S^6 & \to & S^6 \\
(\phi, \beta) \downarrow & & \downarrow \beta \\
SO(3) \times SO(3) & \to & SO(3)
\end{array}$$
Let us also study at the powers of $b$:

$$b^n(p, w) = \begin{cases} \frac{w^n e^{\pi p}}{\overline{w}}, & w \neq 0 \\ (-1)^n, & w = 0 \end{cases}$$

Then all the powers of the Blakers-Massey element have the same equivariance properties.

### 3.2. The Blakers-Massey element and exotic maps.

This concrete map $b$ is a fundamental building block of exotic maps (degree one diffeomorphisms of spheres not isotopic to the identity, and free involutions not conjugate to the antipodal map); see [Du, DMR, ADPR]. Define

$$\sigma(p, w) = (b(p, w) p \bar{b}(p, w))^{-1}, \quad b(p, w) p \bar{b}(p, w)^{-1}$$

then $\sigma : S^6 \to S^6$ is a degree one diffeomorphism not isotopic to the identity ([Du, DMR]), and is a generator of the groups $\Gamma_7$ of isotopy classes of diffeomorphisms of $S^6$; ([DMR]); it is unknown ([KM]) that $\Gamma_7 \cong \mathbb{Z}_{28}$. The map $\sigma^k$ represents $k \in \mathbb{Z}_{28}$, and thus in particular any diffeomorphism of $S^6$ can be deformed to $\sigma^k$ for infinitely many $k$. Two maps $\sigma^k$ and $\sigma^\ell$ are isotopic to each other if and only if $k \equiv \ell \mod 28$, however let us remark that no explicit isotopy is known between $\sigma^k$ and $\sigma^{k+28r}$, $r \neq 0$. Note that the map $\sigma$ is $SO(3)$ (and not $SO(4)$) equivariant.

It is not obvious that $\sigma$ is a diffeomorphism; in [Du] this fact is established by indirect geometric methods, and in [DMR] it is observed that the inverse of $\sigma$ is given by $\sigma(p, w) = (b(p, w)^{-1} p \bar{b}(p, w), b(p, w)^{-1} p \bar{b}(p, w))$. Note that if $b$ were constant, this inverse is immediate; but since $b$ depends on $(p, w)$, in general such a result would not be true. However, we shall see that the equivariance is the structural reason for $\sigma$ being a diffeomorphism by applying the main theorem (compare 3.2 of [DMR]); in order to do this, let us make precise how the J-process works in this case: let $B : S^6 \to SO(7)$ be given by $B(x) = \Delta \circ P \circ b(x) = \Delta \circ \beta$, where $\Delta : SO(3) \to SO(7)$ is the diagonal embedding of $SO(3) \times SO(3)$ in $SO(7)$ (with a 1 in the middle), i.e.

$$\Delta(T) = \begin{pmatrix} T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T \end{pmatrix}$$

Note that the image of $B$ falls into a subgroup of $SO(7)$ that is isomorphic to $SO(3)$. Then we are in the setup of theorem [2.1] $B : S^6 \to SO(3)$ is equivariant with respect to the conjugation action of $SO(3)$ in itself, and $\sigma$ can be simply written as the $J$-process $\sigma(x) = J_{B(x)}(x) = B(x)x$, since the projection of $S^3$ to $SO(3)$ is realized by the standard quaternionic conjugation $q \mapsto T_q$, $T_q(x) = qxq$. Thus we recover the results of [DMR] in a more structural way: $\sigma$ is a diffeomorphism, and the powers of $\sigma$ are given by $\sigma^k(x) = B^k(x)x$.

Also, $\sigma$ allows the construction of exotic involutions: $(p, w) \mapsto -\sigma(p, w)$ is a free involution of $S^6$ that is not conjugate to the antipodal involution; since $\sigma$ acts on $(p, w)$ by a quaternionic conjugation, it preserves the real part of $w$ and thus this involution restricts to an (also exotic) involution of the $S^5$ defined by $\text{Re}(w) = 0$, where it has a simple pictorial description ([ADPR, ADPR-1]).
the fact that $-\sigma$ is an involution follows immediately now by theorem 2.2, taking
$\delta$ to be the antipodal involution of $S^n$ and $\alpha$ the Blakers-Massey element, since
$b(-p, -w) = b(p, w)$.

4. Applications

4.1. Deformations of diffeomorphisms and involutions. As mentioned in section 2, the equivariant J-process technique provides a “canonical” way of deforming diffeomorphisms $J_\alpha(m)$ through equivariant deformations of the corresponding maps $\alpha$. We will use this technique to deform the diffeomorphisms $\sigma_k : S^6 \to S^6$, which represent any isotopy class of diffeomorphisms of $S^n$, to rational maps; this could pave the way to the algebro-geometric study of such exotic diffeomorphisms.

The steps in the deformation are as follows: we will construct a homotopy $H(s, p, w) : [0, 1] \times S^6 \to S^6$ between the Blakers-Massey element and a rational map; this homotopy will be equivariant for all values of the deformation parameter $s$.

By theorem 2.1, the maps $J_{H_s} : S^6 \to S^6$ will be diffeomorphisms; and therefore this procedure furnishes a deformation between the exotic diffeomorphism $\sigma$ and a rational map $R$, in the same isotopy class of $\sigma$ and therefore also a generator of the group $\Gamma_7$. Then powers of $\sigma$ representing all other isotopy classes are also taken care of by theorem 2.1, since $J_{H_k} = J_{H_s} = R^k$.

In order to construct this deformation $H_s$, all we need to do is to reconsider proposition 1 of [DMR] carefully and equivariantly. Spelling out the exponential in the Blakers-Massey element, we have

$$b(p, w) = \cos(\pi|p|) + \frac{\sin(\pi|p|)}{|p|(1 - |p|^2)}wp\bar{w}.$$ 

The functions $x \mapsto \sin(\pi x)$ and $x \mapsto x(1 - x^2)$ are both odd, positive on $(0, 1)$ and have a zero of order 1 at $x = 0$ and $x = 1$; therefore $g(x) = \sin(\pi x)/(x(1 - x^2))$ is an even, positive, differentiable function on $[0, 1]$, (in particular this explains the analyticity of $b$). We will homotop $g$ affinely to the constant function 1; in order to deal with $\cos(\pi|p|)$, we use the function $c(x) = 1 - 4x^2$. The function $c$ is the simplest even function satisfying the property that it has the same sign as $\cos(\pi x)$ on $[0, 1]$.

Now consider the $r(p, w) = (1 - 4|p|^2 + wp\bar{w})$ and the affine homotopy

$$\hat{H}(s, p, w) = \hat{H}_s(p, w) = (1 - s)b(p, w) + sr(p, w).$$

For any $s$, the map $\hat{H}_s$ is equivariant with respect to conjugation since $b$ is equivariant and $r$ is a polynomial in $|p|, p$ and $w$. Also, the expressions in $|p|$ are all even, and therefore these maps can be written in terms of $|p|^2, p, w$, but since $p$ is purely imaginary, $|p|^2 = -p^2$ and all expressions involved are analytic expressions in the non-commuting quaternionic variables $p, w$ and $\bar{w}$.

Rewriting $\hat{H}$, we have

$$\hat{H}_s(p, w) = [(1 - s)\cos(\pi|p|) + sc(|p|)] + [(1 - s)g(|p|) + s]wp\bar{w}.$$
Then the sign properties of $c(x)$ and $g(x)$ imply that $\hat{H}(s,p,w)$ is never zero. Then $H(s,p,w) = \hat{H}(s,p,w)/|\hat{H}(s,p,w)|$ furnishes an equivariant homotopy between the Blakers-Massey element and the map $Q : S^6 \to S^3$.

$$Q(p,w) = \frac{1 + 4p^2 + wp\bar{w}}{\sqrt{(1 + 4p^2)^2 - |w|^4p^2}}$$

Now the map $R(p,w) = (Q(p,w)p\bar{Q}(p,w), Q(p,w)p\bar{Q}(p,w))$ is a rational diffeomorphism of $S^6$ that is not isotopic to the identity; its powers are rational diffeomorphisms representing all isotopy classes of diffeomorphisms of $S^6$. Writing $R$ explicitly, we have

$$R(p,w) = \left( \frac{(1 + 4p^2 + wp\bar{w})p(1 + 4p^2 - wp\bar{w})}{(1 + 4p^2)^2 - |w|^4p^2}, \frac{(1 + 4p^2 + wp\bar{w})w(1 + 4p^2 - wp\bar{w})}{(1 + 4p^2)^2 - |w|^4p^2} \right).$$

For each value $s$ of the deformation parameter, the map $H_s$ satisfies the hypothesis of Theorem 2.2 with respect to the antipodal involution of $S^n$. Therefore $-J_{H_s}$ is a deformation of the exotic involution $\sigma$ of $S^6$, through involutions that also restrict to $S^5$, and, since close enough involutions are easily seen to be conjugate, these involutions are all exotic. At the end of the deformation we reach the involution $-R(p,w)$, which is a rational involution of $S^6$. Note that when restricted to $S^5$, defined by $Re(w) = 0$, $\bar{w} = -w$ and the map

$$R(p,w) = -\left( \frac{(1 + 4p^2 - wpw)(1 + 4p^2 + wpw)}{(1 + 4p^2)^2 - w^4p^2}, \frac{(1 + 4p^2 - wpw)w(1 + 4p^2 + wpw)}{(1 + 4p^2)^2 - w^4p^2} \right)$$

is an exotic involution of $S^5$ defined by a rational map in the non-commuting quaternionic variables $p$ and $w$.

Let us remark that deforming $\alpha$ through plain (not necessarily equivariant) homotopies produces a deformation of $J_0$ that is not necessarily through diffeomorphisms. We shall take advantage of this in the next section.

**4.2. The equivariant Serre problem.** It is know $\pi_6(S^3) \cong \mathbb{Z}_{12}$. However, no explicit deformation of twelve times a generator is known. The authors call this the Serre problem: to find an explicit homotopy between the 12th power of the Blakers-Massey element and the identity, or, in other terms, to understand how the quaternions are homotopy commutative in the 12 power. This problem is still open, although significant advances have been made recently ([Pü]); a solution to the Serre problem has far-reaching consequences, for example, the writing of explicit non-cancellation phenomena and new models for exotic spheres (see, e.g., [Rö97]). We show that, although the generator and all its powers are represented by equivariant maps, there is no equivariant solution to the Serre problem (cf. Theorem 4.1).

We believe that, in addition to the statement of the theorem (which can probably be proven using standard methods of equivariant homotopy theory), the method of proof by using the relationship between equivariant homotopy and isotopy through explicit formulas is of independent interest: an equivariant homotopy would imply that the order of the group of homotopy 7-spheres divides 12 and we know it is isomorphic to $\mathbb{Z}_{28}$ ([KM, DK]); therefore such a homotopy is not possible. We also extend this result to homotopies of the Blakers-Massey element inside other groups.
Theorem 4.1. There exists no differentiable homotopy \( \phi : [0,1] \times S^6 \to S^3 \) between \( b^{12} \) and a constant map such that for each \( t \in [0,1] \), \( \phi(t,\cdot) : S^6 \to S^3 \) is \( SO(3) \)-equivariant.

Proof. We first adapt the Blakers-Massey element to the proposition above, by considering lifting to \( S^3 \) and considering \( b : S^6 \to S^3 \) as an \( S^3 \)-equivariant map. An \( SO(3) \)-equivariant homotopy between \( b^{12} \) and the constant map then lifts to an \( S^3 \)-equivariant homotopy \( b_t(p,w) \) such that \( b_0(p,w) = 1 \) and \( b_1(p,w) = b(p,w) \). If such a homotopy exits, the maps

\[
\sigma_t^{12}(p,w) = (b_t(p,w))^{12}pb_t(p,w)^{-12}, b_t(p,w)^{12}pb_t(p,w)^{-12},
\]

furnish an isotopy between \( \sigma^{12} \) and the identity diffeomorphism. But by standard differential topology methods (see, for example, [Ko]), the map \( \phi : \pi_0(\text{Diff}(S^6)) \to \Gamma_7, \phi(f) = D^7 \cup_f D^7 \), is an isomorphism ([Mi]), where \( \Gamma_7 \) is the group of differentiable structures on \( S^7 \) under the connected sum operation. Thus \( \sigma^{12} \) represents 12 in the group \( \Gamma_7 \equiv Z_{28} ([KM]) \), and we get a contradiction. \( \square \)

Note that the image of \( B^k \) is contained in the chain of inclusions \( SO(3) \subset SU(3) \subset G_2 \subset SO(7) \). Theorem 4.1 states that, even though \( B^{12} \) is homotopic (inside of \( SO(3) \)) to the constant map, no equivariant homotopy can exist. Now the “Serre problem” for all the other groups has been solved: there exist explicit generators \( \gamma \) of \( \pi_6(SU(3)) \cong Z_2 \), \( \delta \) of \( \pi_6(G_2) \cong Z_3 \) and explicit homotopies between \( \gamma^6 \) and the constant map [PR] [Pii] and \( \delta^3 \) and the constant map [R92]. Also \( \pi_6(SO(7)) = 0 \). Explicit homotopies between \( b \) and \( \gamma, \delta \) and the constant map inside of the respective groups can be constructed using the geometry of the chain \( SO(3) \subset SU(3) \subset G_2 \subset SO(7) \) [Pii]. Thus, not only we have that \( b^6, b^3 \) and \( b \) are homotopic to the constant map inside of \( SU(3), G_2 \) and \( SO(7) \); explicit homotopies can be written.

Theorem 4.2. The maps \( b^6, b^3, b \) are homotopic to the constant map in \( SU(3), G_2, SO(7) \), respectively, through explicit homotopies. However, no \( SO(3) \)-equivariant homotopy exists.

Proof. All the groups in the chain \( SO(3) \subset SU(3) \subset G_2 \subset SO(7) \) act on \( S^6 \) through the canonical action of \( SO(7) \). Mimic the proof of Theorem 1 with \( \sigma^6, \sigma^3 \) and \( \sigma \) in place of \( \sigma^{12} \). \( \square \)

The symmetry-breaking mechanism of these homotopies is beautifully illustrated in the structure of the group \( \pi_6(SU(3)) \) [PR]: one way of determining the structure of the homotopy of topological groups is by finding a generator \( A \) such that \( A^k = e \), the identity of the group; this is the way that this was done for \( \pi_6(G_2) \cong Z_3 \), by finding a generator such that \( A^3 = e \) ([R92]). The power map \( A \mapsto A^k \) of matrices is of course equivariant under conjugation and this process provides equivariant deformations. However, in the case of \( SU(3) \), the homotopy uses a deformation of the product of matrices through the Cartan subalgebra of \( SU(3) \) and the symmetry is broken; see [PR] for details.
5. Concluding remarks

First we want to note that all the constructions in [DMR, ADPR] related to the Blakers-Massey element and the exotic diffeomorphisms can be generalized by substituting all the quaternions involved by Cayley numbers and modifying the relevant dimensions. However, this passage involves non-trivial modifications of the techniques in the proofs. The unit quaternions are a group, and thus the equivariance properties make sense; the unit Cayley numbers are not a group and what is the right extension of theorem 2.1 and its applications remains to be seen.

The construction in the main theorem, the applications given and the computations in section 3 of [DMR] suggests that there exists an “algebra of exoticity”, which is yet to be described.

It would also be interesting to follow the known, non-equivariant homotopies of the respective powers of the Blakers-Massey element and the identity in $SU(3)$, $G_2$ and $SO(7)$ and study the associated J-process self-maps of $M$, which at some point must cease to be diffeomorphisms and determine the structure of the singularities that appear, which could shed some light in the general question of what makes a diffeomorphism exotic, or, better, how does one detect the exoticity of a degree one diffeomorphism that is given by a formula.

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