We present preliminary results on the $\rho$ meson decay width estimated from the scattering phase shift of the $I=1$ two-pion system. The phase shift is calculated by the finite size formula for non-zero total momentum frame (the moving frame) derived by Rummukainen and Gottlieb, using the $N_f=2$ improved Wilson fermion action at $m_\pi/m_\rho = 0.41$ and $L = 2.53$ fm.
1. Introduction

Lattice study of the $\rho$ meson decay is an important step for understanding of the dynamical aspect of hadron reactions induced by strong interactions. There are already three studies [1, 2, 3]. The earlier two studies employed the quenched approximation ignores the decay into two ghost pions. The most recent work, while using the $N_f = 2$ dynamical configurations, concentrated on the $\rho \to \pi\pi$ transition amplitude rather than the full matrix including the $\rho \to \rho$ and $\pi\pi \to \pi\pi$ amplitudes. All three studies were carried out at an unphysical kinematics $m_\pi/m_\rho > 1/2$.

In this work we attempt to carry out a more rigorous approach. We estimate the decay width from the scattering phase shift for the $l = 1$ two-pion system. The finite size formula presented by Rummukainen and Gottlieb [4] is employed for an estimation of the phase shift. Calculations are carried out with $N_f = 2$ full QCD configuration previously generated for a study of light hadron spectrum with a renormalization group improved gauge action and a clover fermion action at $\beta = 1.8, \kappa = 0.14705$ on a $12^3 \times 24$ lattice [5]. The parameters determined from the spectrum analysis are $1/a = 0.92$ GeV, $m_\pi/m_\rho = 0.41$, and $L = 2.53$ fm. All calculations of this work are carried out on VPP5000/80 at the Academic Computing and Communications Center of University of Tsukuba.

2. Method

In order to realize a kinematics such that the energy of the two-pion state is close to the resonance energy $m_\rho$, we consider the non-zero total momentum frame (the moving frame) [3] with the total momentum $p = 2\pi/L \cdot e_3$. The initial $\rho$ meson is assigned a polarization vector parallel to $p$. One of the final two pions carry the momentum $p$, while the other pion is at rest. The energies ignoring hadron interactions are then given by $E_1^0 = \sqrt{m_\rho^2 + p^2 + m_\rho}$ for the two-pion state and $E_2^0 = \sqrt{m_\rho^2 + p^2}$ for the $\rho$ meson. We neglect higher energy states whose energies are much higher than $E_1^0$ and $E_2^0$. On our full QCD configurations, the invariant mass for the two-pion state takes $\sqrt{s} = 0.97 \times m_\rho$, while $1.47 \times m_\rho$ is expected for the zero total momentum. The $\rho$ meson at zero momentum cannot decay energetically, so that it can be used to extract $m_\rho$.

The hadron interactions shift the energy from $E_1^0$ to $E_n (n = 1, 2)$. These energies $E_n$ are related to the two-pion scattering phase shift $\delta(\sqrt{s})$ through the Rummukainen-Gottlieb formula [3], which is an extension of the Lüscher formula [2] to the moving frame. The formula for the total momentum $p = p e_3$ and the $A_{\Sigma}$ representation of the rotation group on the lattice reads

$$\frac{1}{\tan \delta(\sqrt{s})} = \frac{1}{2\pi^2 q^2} \sum_{r \in \Gamma} \frac{1 + (3r_3^2 - r^2)/q^2}{r^2 - q^2} , \tag{2.1}$$

where $\sqrt{s} = \sqrt{E^2 - p^2}$ is the invariant mass, $k$ is the scattering momentum ($\sqrt{s} = 2\sqrt{m_\rho^2 + k^2}$), $\gamma$ is the Lorentz boost factor ($\gamma = E/\sqrt{s}$), and $q = kL/(2\pi)$. The summation for $r$ in (2.1) runs over the set

$$\Gamma = \{r | r_1 = n_1, r_2 = n_2, r_3 = (n_3 + \frac{pL}{22\pi})/\gamma, \; n \in Z^3 \} . \tag{2.2}$$

The right hand side of (2.1) can be evaluated by the method described in Ref. [2].
In order to calculate $E_1$ and $E_2$ we construct a $2 \times 2$ matrix time correlation function,

$$G(t) = \begin{pmatrix}
(0|\left(\pi\pi\right)^\dagger(t)\left(\pi\pi\right)(t_s)|0)
& (0|\left(\pi\pi\right)^\dagger(t)\rho_3(t_s)|0)

(0|\rho_3^\dagger(t)(\pi\pi)(t_s)|0)
& (0|\rho_3^\dagger(t)\rho_3(t_s)|0)
\end{pmatrix}. \tag{2.3}
$$

Here, $\rho_3(t)$ is an interpolating operator for the neutral $\rho$ meson with the momentum $p = 2\pi/L \cdot e_3$ and the polarization vector parallel to $p$; $(\pi\pi)(t)$ is an interpolating operator for the two pions given by

$$(\pi\pi)(t) = \frac{1}{\sqrt{2}} \left( \pi^- (p,t) \pi^+(0,t) - \pi^+(p,t) \pi^-(0,t) \right), \tag{2.4}$$

which belongs to the $A_2^-$ and iso-spin representation with $I = 1, I_3 = 0$.

We can extract the two energy eigenvalues by a single exponential fitting of the two eigenvalues $\lambda_1(t, t_R)$ and $\lambda_2(t, t_R)$ of the normalized matrix $M(t, t_R) = G(t)G^{-1}(t_R)$ with some reference time $t_R$ assuming that the lower two states dominate the correlation function.

In order to construct the meson state with non-zero momentum we introduce a $U(1)$ noise $\xi_j(x)$ in three-dimensional space whose property is

$$\frac{1}{N_R} \sum_{j=1}^{N_R} \xi_j^\dagger(x)\xi_j(y) = \delta^3(x-y) \quad \text{for} \quad N_R \to \infty. \tag{2.5}$$

We calculate the quark propagator

$$Q(x,t|q,t_s,\xi_j) = \sum_y \left( D^{-1} \right)(x,t;y,t_s) \cdot \left[ e^{i q \cdot y} \xi_j(y) \right], \tag{2.6}$$

regarding the term in the square bracket as the source. The two point function of the meson with the spin content $\Gamma$ and the momentum $p$ can be constructed from $Q$ as

$$\frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-i p \cdot x} \cdot \left\langle \gamma_5 Q^\dagger(x,t|0,t_s,\xi_j) \gamma_5 \Gamma^\dagger Q(x,t|p,t_s,\xi_j) \Gamma \right\rangle, \tag{2.7}$$

where the bracket refers to the trace for color and spin indeces.

The quark contraction for the $\pi\pi \to \pi\pi$ and the $\pi\pi \to \rho$ components of $G(t)$ are given by

$$G_{\pi\pi \to \pi\pi}(t) = \begin{pmatrix}
\begin{array}{cc}
\text{p} & 0 \\
0 & \text{p}
\end{array}
& - \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

\end{pmatrix} + \begin{pmatrix}
\begin{array}{cc}
\text{p} & 0 \\
0 & \text{p}
\end{array}
& - \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

\end{pmatrix}$$

$$G_{\pi\pi \to \rho}(t) = \begin{pmatrix}
\begin{array}{cc}
\text{p} & 0 \\
0 & \text{p}
\end{array}
& - \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

& \begin{array}{c}
\text{+}
\end{array}

\end{pmatrix}$$

where the four verteces for the $\pi\pi \to \pi\pi$ and three verteces for the $\pi\pi \to \rho$ components refer to the pion or the $\rho$ meson with definite momentum. The time direction is upward in the diagrams, and the $\rho \to \pi\pi$ component is given by changing the time direction.
The first term of the $\pi\pi \rightarrow \pi\pi$ component in (2.8) can be calculated by introducing another $U(1)$ noise $\eta_j(x)$ having the same property as $\xi_j(x)$ in (2.5):

$$\frac{1}{N_R} \sum_{j=1}^{N_R} \sum_{x,y} e^{-ip \cdot x} \left( \langle Q^+(x,t|0,t_s,\xi_j) Q(x,t|p,t_s,\xi_j) \rangle \langle Q^+(y,t|0,t_s,\eta_j) Q(y,t|0,t_s,\eta_j) \rangle \right).$$

(2.9)

The second term of (2.8) is obtained by exchanging the momentum of the sink in (2.9).

In order to construct the other terms of (2.8) we calculate a quark propagator of another type by the source method,

$$W(x,t|k,t_1|q,t_s,\xi_j) = \sum_z (D^{-1})(x,t;z,t_1) \cdot \left[ e^{ik \cdot x} Q(z,t_1|q,t_s,\xi_j) \right],$$

(2.10)

where the term in the square bracket is regarded as the source in solving the propagator. Using $W$ we can construct the third to sixth terms in the $\pi\pi \rightarrow \pi\pi$ component of (2.8) by

$$3rd = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle W^+(x,t|0,t_s|p,t_s,\xi_j) W(x,t|0,t_s,\xi_j) \rangle \right),$$

$$4th = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle W(x,t|0,t_s|p,t_s,\xi_j) W^+(x,t|0,t_s,\xi_j) \rangle \right),$$

$$5th = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle W(x,t|p,t_s|0,t_s,\xi_j) W^+(x,t|0,t_s,\xi_j) \rangle \right),$$

$$6th = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle W^+(x,t|p,t_s|0,t_s,\xi_j) W(x,t|0,t_s,\xi_j) \rangle \right).$$

(2.11)

The two terms of $\pi\pi \rightarrow \rho$ of (2.8) can be similarly constructed by

$$1st = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle W^+(x,t|0,t_s|\xi_j) (\gamma_5 \gamma_3) Q(x,t|0,t_s,\xi_j) \rangle \right),$$

$$2nd = \frac{1}{N_R} \sum_{j=1}^{N_R} \sum_x e^{-ip \cdot x} \left( \langle Q^+(x,t|0,t_s,\xi_j) (\gamma_5 \gamma_3) W(x,t|p,t_s|0,t_s,\xi_j) \rangle \right).$$

(2.12)

In this work we set the source at $t_s = 4$ and impose the Dirichlet boundary condition in the time direction. We calculate the $Q$-type propagators for four sets of $q$ and the $U(1)$ noise in (2.6): ($q$, noise) = \{(0,\xi), (0,\eta), (p,\xi), (-p,\xi)\}. The $W$-type propagators are calculated for 22 sets of $k$, $t_1$ and $q$ in (2.10): ($k,t_1|q$) = \{(p,t_s|0), (-p,t_s|0), (0,t_s|p), (0,t_s|0), (0,t_s|p), (0,t_s|0), (0,t_s|0)\}, with the same $U(1)$ noise $\xi$. All diagrams for the time correlation function can be calculated with combinations of these propagators. We choose $N_R = 10$ for the number of $U(1)$ noise. We carry out additional measurements to reduce statistical errors using the source operator is located at $t_s + T/2$ and the Dirichlet boundary condition is imposed at $T/2$. We average over the two measurements for the analysis. Thus we calculate 520 quark propagators for each configuration. The total number of configurations analyzed are 800 separated by 5 trajectories [5].

3. Results

In Fig. 1 we plot the real part of the diagonal components ($\pi\pi \rightarrow \pi\pi$ and $\rho \rightarrow \rho$) and the imaginary part of the off-diagonal components ($\pi\pi \rightarrow \rho$, $\rho \rightarrow \pi\pi$) of $G(t)$. Our construction of
$\rho$ meson decay from the lattice

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Figure 2: Normalized eigenvalues $\lambda_1(t, t_R)$ and $\lambda_2(t, t_R)$.

$G(t)$ is such that the sink and source operators are identical for a sufficiently large number of the $U(1)$ noise. In this case we can prove that $G(t)$ is an Hermitian matrix and the off-diagonal parts are pure imaginary from $P$ and $CP$ symmetry. We find that this is valid within statistics, but the statistical errors of the $\rho \rightarrow \pi\pi$ component is larger than those of $\pi\pi \rightarrow \rho$ in Fig. 1. In the following analysis we substitute $\rho \rightarrow \pi\pi$ by $\pi\pi \rightarrow \rho$ to reduce the statistical error.

The two eigenvalues $\tilde{\lambda}_1(t, t_R)$ and $\tilde{\lambda}_2(t, t_R)$ for the matrix $M(t, t_R) = G(t)G^{-1}(t_R)$ are shown in Fig. 2. We set the reference time $t_R = 9$ and normalize the eigenvalues by the correlation function for the free two-pion system, $\langle 0|\pi(-p,t)\pi(p,t_s)|0\rangle \langle 0|\pi(0,t)\pi(0,t_s)|0\rangle$. Thus the slope of the figure corresponds to the energy difference $\Delta E_n = E_n - E_1^0$. We observe that the energy difference for $\tilde{\lambda}_1$ is negative and that for $\tilde{\lambda}_2$ is positive. This means that the two-pion scattering phase shift is positive for the lowest state and negative for the next higher state.

We extract the energy difference $\Delta E_n$ for both states by a single exponential fitting of the normalized eigenvalues $\lambda_1$ and $\lambda_2$ for the time range $t = 10 - 16$. Then we reconstruct the energy $E_n$ in the moving frame by adding the energy of the two free pions, i.e., $E_n = \Delta E_n + E_1^0$, and convert it
to the invariant mass $\sqrt{s}$. Substituting $\sqrt{s}$ into the Rummukainen-Gottlieb formula (2.1) we obtain the scattering phase shift:

$$\tan\left(\sqrt{s}\right) = 0.7880 \pm 0.0082 \quad 0.0773 \pm 0.0033$$

$$0.962 \pm 0.024 \quad -0.43 \pm 0.12$$

(3.1)

The $\rho$ meson mass obtained at zero momentum is $m_\rho = 0.858 \pm 0.012$. Hence the sign of the scattering phase shifts at $\sqrt{s} < m_\rho$ is positive (attractive interaction) and that at $\sqrt{s} > m_\rho$ is negative (repulsive interaction) as expected. The corresponding results for $\sin^2\delta(\sqrt{s})$, which is proportional to the scattering cross section of the two-pion system, are plotted in Fig. 3 together with the position of $m_\rho$.

In order to estimate the $\rho$ meson decay width at the physical quark mass we parameterize the scattering phase shift by the effective $\rho \to \pi\pi$ coupling constant $g_{\rho\pi\pi}$,

$$\tan\left(\sqrt{s}\right) = \frac{g_{\rho\pi\pi}^2}{6\pi} \cdot \frac{k^3}{\sqrt{s}(M_R^2 - s)}$$

(3.2)

with $g_{\rho\pi\pi}$ defined by the effective Lagrangian,

$$L_{\text{eff.}} = g_{\rho\pi\pi} \cdot \epsilon_{abc} (k_1 - k_2) \mu \rho_{\mu}^a(p) \pi^b(k_1) \pi^c(k_2)$$

(3.3)

where $k$ is the scattering momentum and $M_R$ is the resonance mass. The coupling $g_{\rho\pi\pi}$ generally depends on the quark mass and the energy, but our present data at a single quark mass do not provide this information. Here we assume that these dependences are small and try to estimate $g_{\rho\pi\pi}$ and $M_R$ from our results in (3.1). We also estimate the $\rho$ meson decay width at the physical
The resonance mass $M_R$ obtained from the scattering phase shift is consistent with $aM_R = 0.858 \pm 0.012$ obtained from the $\rho$ meson with zero momentum. The $\rho$ meson decay width $\Gamma_\rho$ at the physical quark mass is consistent with experiment (150 MeV). In Fig. 3 we indicate the position of $M_R$ and draw the line given by (3.2) with $g_{\rho \pi \pi}$ and $M_R$ in (3.5).

4. Summary

We have shown that a direct calculation of the $\rho$ meson decay width from the scattering phase shift for the $I = 1$ two-pion system is possible with present computing resources. However, several issues remain which should be investigated in future work. The most important issue is a proper evaluation of the quark mass and energy dependence of the effective $\rho \to \pi \pi$ coupling constant $g_{\rho \pi \pi}$. This constant is used to obtain the physical decay width at $m_\pi/m_\rho = 0.18$ from our results at $m_\pi/m_\rho = 0.41$ by a long chiral extrapolation. In principle we can estimate the decay width from the scattering phase shift without such a parameterization, if we have data for several energy values at or near the physical quark mass. This will be our goal toward the lattice determination of the $\rho$ meson decay.

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