Gravity with free initial conditions: a solution to the cosmological constant problem testable by CMB B-mode polarization

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In standard general relativity the universe cannot be started with arbitrary initial conditions, because four of the ten components of the Einstein’s field equations (EFE) are constraints on initial conditions. In the previous work it was proposed to extend the gravity theory to allow free initial conditions, with a motivation to solve the cosmological constant problem. This was done by setting four constraints on metric variations in the action principle, which is reasonable because the gravity’s physical degrees of freedom are at most six. However, there are two problems about this theory; the three constraints in addition to the unimodular condition were introduced without clear physical meanings, and the flat Minkowski spacetime is unstable against perturbations. Here a new set of gravitational field equations is derived by replacing the three constraints with new ones requiring that geodesic paths remain geodesic against metric variations. The instability problem is then naturally solved. Implications for the cosmological constant $\Lambda$ are unchanged; the theory converges into EFE with nonzero $\Lambda$ by inflation, but $\Lambda$ varies on scales much larger than the present Hubble horizon. Then a new set of constraints is introduced, with a motivation to solve the cosmological constant problem. This was done by setting four constraints on initial conditions, with a motivation to solve the cosmological constant problem. This was done by setting four constraints on initial conditions, with a motivation to solve the cosmological constant problem. This was done by setting four constraints on initial conditions, with a motivation to solve the cosmological constant problem. This was done by setting four constraints on initial conditions, with a motivation to solve the cosmological constant problem. 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I. INTRODUCTION

It is a well known fact that not all of the ten components of the Einstein’s field equations (EFE) are dynamical equations. Because of the freedom of general coordinate transformation, time evolution of all the ten metric components $g_{\mu\nu}$ cannot be determined even if $g_{\mu\nu}$ and their time derivatives are given at an initial spacelike hypersurface. The $\partial_\mu$ components of EFE do not include second time derivatives of any metric component, and $\partial_0^2 g_{\mu\nu}$ do not appear at all in EFE. Therefore the $\partial_\mu$ components of EFE just give four constraints on initial conditions, and the contracted Bianchi identities guarantee that they hold at all time if they are satisfied at an initial spacelike hypersurface. Therefore, when a universe (spacetime) is born, only a special set of physical states are allowed among all that are otherwise physically possible. For example, when we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, the evolution of the universe is simply determined by one of the four constraints (i.e., the Friedmann equation), which does not include second time derivative $\ddot{a}$ of the scale factor. The Hubble parameter $H = \dot{a}/a$ is determined once energy density and curvature are given, though we can imagine a universe with various values of $H$.

From the viewpoint of the action principle to derive EFE, the existence of such four constraints originates from considering variations of all the ten metric components independently, while the physical degrees of freedom (DOFs) of gravity are at most six because of the four from coordinate transformation. However, the Einstein-Hilbert action is already invariant under coordinate transformation, and we do not have to request a stationary action condition about metric variations related to coordinate transformation. This implies that there is a sort of redundancy in deriving the full set of 10-component EFE, and it is not unreasonable to expect that nature may disfavor such a redundancy. Rather, it may be natural to derive gravitational field equations by metric variations constrained into six physical DOFs of gravity. Then the four constraint equations would disappear, and such a theory would be able to describe time evolution of spacetime starting from any physically possible initial states.

Such a theory of gravity was proposed in the previous study (hereafter Paper I), adopting a guiding principle that the gravity theory should be able to describe time evolution starting from free initial conditions. The main motivation of this work was the cosmological con-

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1 As usual, Greek indices run from 0 to 3 and Latin indices are spatial from 1 to 3, with the sign convention same as ref. [1]. A partial derivative is denoted as $\partial_\mu$, or a comma, and a covariant derivative as $\nabla_\mu$ or a semicolon. The fundamental constants $c$ and $h$ are set equal to unity.

2 An infinitesimal coordinate transformation of $x^\mu \rightarrow x'^\mu + \zeta'^\mu$ generates metric variations of $\delta g_{\mu\nu} = -\zeta_{\mu,\nu} - \zeta_{\nu,\mu}$, and the stationary action condition about this leads to the contracted Bianchi identities and the energy-momentum conservation law by a partial integration, rather than EFE.

3 One may consider that a similar argument may also apply for...
constant (Λ) problem (see e.g. [3–6] for reviews). The key is how to set four constraints on metric variations δg_{μν} to extract six gravity DOFs. The unimodular condition, δ(√−g) = (1/2) √−g g^{μν} δg_{μν} = 0, was chosen as the first one[3]. This condition has been studied for a long time starting from Einstein [6–10], and it is interesting concerning the Λ problem because a term Λ_g(x) g_{μν} appears in the field equations as a Lagrange multiplier. However, introducing only this constraint does not solve the Λ problem, because the contracted Bianchi identities require Λ_g to be a universal integration constant, and we do not know how to set its initial value. In Paper I, the three more constraints were assumed to be δg_{0i} = 0, simply to make the 0i components of EFE ineffective. This violates general covariance, and for a consistent theory it was assumed that any spacetime is initially created as a spacelike hypersurface, and the conditions δg_{0i} = 0 hold in preferred reference frames, i.e., synchronous coordinate systems starting from this hypersurface.

The motivation of just allowing free initial conditions may not be strong enough to consider an alternative theory of gravity. However, the proposed theory gives a solution both for the smallness and coincidence problems of Λ. A universe is assumed to start with highly inhomogeneous conditions, and hence Λ_g(x) is not a universal constant. However, once a portion of the universe starts inflation [20–25], anisotropy rapidly disappears and Λ_g(x) converges into a cosmological constant term Λ. Hence the universe can be described by EFE+Λ within the present-day Hubble horizon, but the final total value of Λ ≡ Λ_g + Λ_{vac} can be positive or negative and changes on scales much larger than the Hubble horizon, where Λ_{vac} is the contribution from vacuum energy density of all relevant fields in the particle physics theory. Then the anthropic argument for the cosmological constant [26–27] applies; galaxy formation is possible only in the regions of |ρA| ≲ |ρM|, like concentration of human populations to coastal areas on Earth regarding Λ as altitude, where ρA and ρM are energy densities of Λ and matter, respectively. The probability distribution of Λ is unchanged by these. The term Ξ_{μν} comes from the unimodular condition on δg_{μν}, and the tensor Ξ_{μν} comes from the three conditions of δg_{0i} = 0. In this theory, it is assumed that any spacetime realized in nature is finite toward past. In particular, the initial spacelike hypersurface at the birth of the universe can be defined at least as a classical theory. Then we can define a synchronous coordinate system starting from this hypersurface, in which g_{00} = 1 and g_{0i} = 0 throughout the spacetime. In this reference frame, it is assumed that δg_{0i} do not represent gravity DOFs, and hence the 0i components of EFE simply become ineffective by Ξ_{μν}, which takes the form of Ξ_{00} = Ξ_{ij} = 0 with three nonzero components of Ξ_{μν} = Ξ_{i0}. Since the initial hypersurface of constant time is fixed, the synchronous condition allows only transformations within spatial coordinates [x^μ = f^μ(x^i)], and the form of Ξ_{μν} is unchanged by these. The term Ξ_{μν} can be converted into any coordinate systems if we define the ordinary tensor transformation law, and then eqs. (11) become covariant. The theory is derived from the same Lagrangians as general relativity and hence the stationary action condition does not depend on choice of reference frames. However, the general principle of relativity is violated because the constraints on δg_{μν} to extract the gravity DOFs have a preferred frame. Since the preferred frame can physically be specified by the initial spacelike hypersurface, this is a consistent theory to

II. STABILITY OF THE FIELD EQUATIONS IN PAPER I

The gravitational field equations proposed in Paper I are

\[ G^{μν} - κ T^{μν} = Λ_g(x) g^{μν} + Ξ^{μν}, \]

where \( G^{μν} = R^{μν} - (1/2) R g^{μν} \) is the Einstein tensor, \( κ = 8πG \), \( G \) the Newton’s gravitational constant, and \( T^{μν} \) the energy-momentum tensor of matter. The term including a scalar field \( Λ_g(x) \) comes from the unimodular condition on δg_{μν}, and the tensor Ξ_{μν} comes from the three conditions of δg_{0i} = 0. In this theory, it is assumed that any spacetime realized in nature is finite toward past in timelike directions, and the initial spacelike hypersurface at the birth of the universe can be defined at least as a classical theory. Then we can define a synchronous coordinate system starting from this hypersurface, in which g_{00} = 1 and g_{0i} = 0 throughout the spacetime. In this reference frame, it is assumed that δg_{0i} do not represent gravity DOFs, and hence the 0i components of EFE simply become ineffective by Ξ_{μν}, which takes the form of Ξ_{00} = Ξ_{ij} = 0 with three nonzero components of Ξ_{μν} = Ξ_{i0}. Since the initial hypersurface of constant time is fixed, the synchronous condition allows only transformations within spatial coordinates [x^μ = f^μ(x^i)], and the form of Ξ_{μν} is unchanged by these. The term Ξ_{μν} can be converted into any coordinate systems if we define the ordinary tensor transformation law, and then eqs. (11) become covariant. The theory is derived from the same Lagrangians as general relativity and hence the stationary action condition does not depend on choice of reference frames. However, the general principle of relativity is violated because the constraints on δg_{μν} to extract the gravity DOFs have a preferred frame. Since the preferred frame can physically be specified by the initial spacelike hypersurface, this is a consistent theory to
determine evolution of the gravitational fields and spacetime.

However, there is a problem in these equations, which becomes apparent when we consider linear perturbations from the flat Minkowski spacetime. In the synchronous frame, we consider perturbations of $h_{ij}$, $\Lambda_g(x)$, and $\Xi^{\mu\nu}$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $\eta_{\mu\nu}$ is the metric of the Minkowski spacetime as usual. The contracted Bianchi identities of eqs. (1) lead to $\partial_\mu \Lambda_g g^{\mu
u} + \nabla_\mu \Xi^{\mu\nu} = 0$, and the $\nu = 0$ and $i$ components are:

$$\partial_0 \Lambda_g + \partial_i \Xi^{0i} = 0$$  \hspace{1cm} (2)

$$-\partial_i \Lambda_g + \partial_0 \Xi^{0i} = 0$$  \hspace{1cm} (3)

Therefore we find

$$\delta_0^2 \Lambda_g + \Delta \Lambda_g = 0$$  \hspace{1cm} (4)

where $\Delta$ is the three-dimensional Laplacian. This is clearly unstable, because a mode of wavenumber $k$ will grow exponentially. The field $\Lambda_g(x)$ is a scalar and hence this result does not depend on choice of coordinate systems. When we consider a plane wave propagating into the $x^3$ direction, the mode $(h_{11} + h_{22})$ is prohibited in the ordinary EFE because the $00$ and $33$ components give

$$(h_{11} + h_{22})_{33} = (h_{11} + h_{22})_{00} = 0$$  \hspace{1cm} (5)

However, in this theory because of the freedom about $\Lambda_g(x)$ we find

$$(h_{11} + h_{22})_{00} + (h_{11} + h_{22})_{33} = 0$$  \hspace{1cm} (6)

which also grows exponentially. Therefore, even if the theory predicts that inflation produces a spatially flat universe obeying EFE+$\Lambda$ with negligible $\Xi^{\mu\nu}$ (Paper I), a small perturbation will rapidly grow after inflation and the Minkowski spacetime is not stable. This is the problem that should be solved in the new field equations presented below.

III. A NEW APPROACH TO DERIVE THE GRAVITATIONAL FIELD EQUATIONS

A. Variations keeping geodesics

The constraints of $\delta g_{0i} = 0$ in Paper I were introduced in a rather ad hoc way simply to make the $0i$ components of EFE ineffective, and this may be the cause of the difficulty described in [4]. In this work the unimodular condition is kept as one of the four constraints because it has a connection to the cosmological constant term, and see also Paper I for some theoretical motivations to introduce this condition in the action principle. Then we should seek for new three constraints on $\delta g_{\mu\nu}$ on a more solid physical basis. Here geodesics may be relevant, because the effective number of components of the geodesic equations is indeed three. The principle of equivalence tells us that gravity can be erased in any local inertial frames, and hence it is reasonable to expect that a geodesic should remain geodesic against metric variations related to the physical DOFs of gravity. It is difficult to find covariant constraints on $\delta g_{\mu\nu}$ keeping any geodesics, and here the same assumption as Paper I is adopted that any spacetime realized in nature starts from a well-defined spacelike hypersurface. The paths of fixed spatial coordinates in the synchronous reference frame starting from the initial hypersurface are geodesics, and they are natural “backbone” of spacetime when we consider evolution of spacetime to the timelike direction. Therefore it is assumed that metric variations should keep these geodesics in the action principle to derive the gravitational field equations.

Let $u^\mu = dx^\mu/d\tau$ be the four-velocity field of these backbone geodesics $x^\mu(\tau)$, where $\tau$ is the proper time along the geodesics. Therefore $u^\alpha \nabla_\alpha u^\mu = 0$, and in the synchronous frame $\tau = x^0$ and $w^\mu = (1,0,0,0,0)$. The condition of keeping geodesics is given by $\delta (u^\alpha \nabla_\alpha u^\mu) = 0$ against $\delta g_{\mu\nu}$. Here, it should be noted that the backbone paths $x^\mu(x^0)$ are kept unchanged by $\delta g_{\mu\nu}$, and accordingly $u^\mu$ changes by $\delta g_{\mu\nu}$ because a four-velocity includes proper time that depends on metric, as $dt^2 = g_{\mu\nu} dx^\mu dx^\nu$. Metric and four velocity fields after variations are denoted as $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + \delta g_{\mu\nu}$ and $\tilde{u}^\mu \equiv u^\mu + \delta u^\mu$, respectively, and then variations of $u^\mu$ become

$$\delta u^\mu = \delta \left( \frac{dx^\mu}{d\tau} \right) = \frac{\delta (dx^\mu)}{d\tau} u^\mu = -\frac{1}{2} \delta g_{\rho\sigma} u^\rho u^\sigma u^\mu \propto u^\mu$$  \hspace{1cm} (7)

Because $\tilde{u}^\mu \tilde{u}_\mu = 1$ holds after variation, we find

$$\tilde{u}^\alpha \nabla_\alpha (\tilde{u}_\mu \tilde{u}^\mu) = 2 \tilde{u}_\mu \tilde{u}^\alpha \nabla_\alpha \tilde{u}^\mu = 0$$  \hspace{1cm} (8)

where $\nabla_\alpha$ is a covariant derivative with $\tilde{g}_{\mu\nu}$. Note that $\tilde{u}^\alpha \nabla_\alpha \tilde{u}^\mu$ is not necessarily zero, if the path is no longer geodesic after variation. Therefore,

$$\tilde{u}_\mu \tilde{u}^\alpha \nabla_\alpha \tilde{u}^\mu = \delta (u_\mu u^\alpha \nabla_\alpha u^\mu) = u_\mu \delta (u^\alpha \nabla_\alpha u^\mu) = 0$$  \hspace{1cm} (10)

i.e., variations of the geodesic equations are perpendicular to $u^\mu$. This means that the condition of keeping geodesics consists of three constraints rather than four, which is not surprising because there are only three independent components in the geodesic equations.

Variations of the geodesic equations can be written with $\delta u^\mu$ and $\delta \Gamma^\mu_{\rho\sigma}$ as

$$\delta (u^\alpha \nabla_\alpha u^\mu) = \delta u^\alpha \nabla_\alpha u^\mu + u^\alpha \delta \nabla_\alpha (\delta u^\mu) + u^\gamma u^\delta \delta \Gamma^\mu_{\alpha\beta}$$  \hspace{1cm} (11)
where $\Gamma^\mu_{\rho\sigma}$ are the Christoffel symbols. Using eq. (5) and the geodesic equations, the first term on the right hand side vanishes and we find

$$\delta(u^\alpha \nabla_{\alpha} u^\mu) = \frac{1}{2} u^\alpha u^\rho \nabla_{\alpha} \delta g_{\rho\sigma} + u^\alpha u^\beta \delta \Gamma^\mu_{\alpha\beta} \quad (12)$$

Because this is perpendicular to $u^\mu$ and the first term on the right hand side is proportional to $u^\mu$, in the synchronous frame these constraints are equivalent to requesting that the spatial components of the second term are zero, i.e.,

$$2u^\alpha u^\beta \delta \Gamma^i_{\alpha\beta} = g^{ij}(2\partial_0 \delta g_{ij} - \partial_j \delta g_{00}) = 0 \quad (13)$$

Therefore the three constraints can simply be expressed as

$$2\partial_0 \delta g_{0i} - \partial_i \delta g_{00} = 0 \quad (14)$$

B. The new field equations

Here a new set of field equations is derived from the action principle by adopting the constraints (eq. 14) in the synchronous coordinate system, instead of $\delta g_{0i} = 0$ in Paper I. An important feature of the constraints (eq. 14) is that they include derivatives of metric, in contrast to another constraint of the unimodular condition. In derivation of local differential field equations generally, variation of the action $\delta S$ is expressed with $\delta g_{ij}$ (but without their derivatives using partial integrations), and then $\delta S = 0$ is requested for a local metric variation that is nonzero only at an infinitesimal region. A constraint without derivatives can easily be incorporated in such a local metric variation, but it is not simple when a constraint includes derivatives, leading to nonlocal field equations.

Variation of the action by $\delta g_{ij}$ in general relativity is

$$\delta S = \frac{1}{2\kappa} \int (G^{ij} - \kappa T^{ij}) \sqrt{-g} \delta g_{ij} \, d^4x \quad (15)$$

and define $\xi^i(x)$ in a synchronous coordinate system as integrations of the 0i components over time ($t = x^0$), as

$$\xi^i(x) \equiv \int_{t_0}^t \left(G^{0i} - \kappa T^{0i}\right) \sqrt{-g} \, dt' + C^i \quad (16)$$

where integration is over a path of constant spatial coordinates (i.e., a backbone geodesic) starting from the initial hypersurface on which time is $t_0$, and $C^i(x^i) = \xi^i(t_0, x^j) \sqrt{-g}$ are integration constants. Then the 0i and $i0$ metric variations in eq. (15) can be converted into a variation about $\delta g_{00}$ using the constraint (14) as

$$2 \int (G^{0i} - \kappa T^{0i}) \sqrt{-g} \, d^4x \quad (17)$$

$$= 2 \int \partial_0(\xi^i \sqrt{-g}) \delta g_{0i} \, d^4x \quad (18)$$

$$= -2 \int \xi^i \sqrt{-g} \partial_0 \delta g_{0i} \, d^4x \quad (19)$$

$$= - \int \xi^i \sqrt{-g} \partial_i \delta g_{00} \, d^4x \quad (20)$$

$$= \int \partial_i(\xi^i \sqrt{-g}) \sqrt{-g} \delta g_{00} \, d^4x \quad (21)$$

where the boundary terms are assumed to vanish as usual. Now the stationary action condition about $\delta g_{00}$ and $\delta g_{ij}$, and the definition of $\xi^i$ (eq. 16) can be combined into ten-component new field equations as follows:

$$G^{ij} - \kappa T^{ij} = \Lambda g^{ij} + \Xi^{ij} \quad (22)$$

Here, the unimodular condition is also adopted to generate the term $\Lambda g^{ij} g^{ij}$, and $\xi^i \equiv \xi^i \sqrt{-g}$. Note that now $\Xi^{ij}$ is different from that in Paper I; it is similar but $\Xi^{00}$ is nonzero and related to $\Xi^{ij}$ through $\xi^i$ in this new version. Since $\xi^i$ is a time integration of the 0i component of EFE, these are integro-differential equations about time, i.e., nonlocal.

If we define a tensor $\Xi^{ij}$ by the ordinary transformation law, the field equations (22) would become a covariant form holding in any coordinate systems. However, the above expression of $\Xi^{ij}$ by $\xi^i$ (eqs. 23–25) is valid in synchronous coordinate systems, and there is a freedom of coordinate transformation within the spatial coordinates, $x^i = f^i(x')$, even if the initial spacelike hypersurface is fixed. Therefore if $\Xi^{ij}$ can be defined as a tensor, eqs. (23–25) should be common in all these synchronous systems under an appropriate transformation law of $\xi^i$, which can be shown as follows. The transformation law of $\Xi^{ij}$ by the spatial coordinate transformation should be

$$\Xi^{00} = \Xi^{00} \quad (26)$$

$$\Xi^{0i} = \partial_{x^j} \xi^i \Xi^{0j} \quad (27)$$

$$\Xi^{ij} = \partial_{x^k} \partial_{x^j} \xi^i \Xi^{kl} = 0 \quad (28)$$

In a synchronous frame, the Christoffel symbol in eq. (23) is $\Gamma^i_{00} = (1/2) g^{ij} g_{ij0}$, which is invariant under spatial coordinate transformations. Then $\Xi^{00}$ obeys the tensor transformation law if $\xi^i$ are transformed as spatial
components of a four-vector $\xi^\mu$ whose 0-th component is $\xi^0 = \xi^0 = 0$. Using this four vector, eq. \eqref{eq:constraint_00} becomes $\Xi^{00} = -\nabla_0 \xi^\mu$, which is a scalar and hence consistent with the transformation law of $\Xi^{00}$ (eq. \eqref{eq:transform_00}). Therefore $\Xi^{\mu\nu}$ is consistently expressed by eqs. \eqref{eq:constraint_00}–\eqref{eq:constraint_00} in all the synchronous coordinate systems starting from the initial hypersurface. Finally, if we define $\xi^{00}$ in non-synchronous coordinate systems as those transformed from the synchronous systems by the ordinary tensor law, a tensor $\Xi^{\mu\nu}$ is defined consistently in any coordinate systems.

### C. Ambiguity in spacetime determination from initial conditions and its removal

The role of a gravitational theory should be to determine time evolution of spacetime, i.e., $g_{ij}$ in a synchronous frame, when $g_{ij}$, $\partial_0 g_{ij}$, and $T^{\mu\nu}$ on the initial hypersurface are given. However, the equations \eqref{eq:constraint_00} are not yet sufficient. The second time derivatives $\partial_0^2 g_{ij}$ appear only in the $ij$ components, but these also include $\Lambda_g (x)$. Though $\Lambda_g$ can be erased using the 00 component, $\partial_0 \xi^i$ then appears in the $ij$ components, which must be determined independently. One may consider that the three constraints from the 0i components determine $\partial_0 \xi^i$, but the 0i components determine only $\partial_0 \xi^0$, with no constraints on $\xi^i$. If we consider $\xi^i$ as new fields of physical DOFs, such an ambiguity may not be a problem. However, our motivation is just to remove the constraints on initial conditions in EFE with increased DOFs of metric dynamics, rather than to complicate the gravity theory by introducing new physical fields. If there is another constraint on the initial conditions of $\Lambda_g$ and/or $\partial_0 \xi^i$, the spacetime evolution would be unambiguously determined without new physical fields.

This ambiguity arises because the field equations include both $\xi^i$ and their time derivatives. This is a consequence of the nonlocal nature of the constraints to keep geodesics, which lead to the field equations including integrations of the 0i components of EFE over the past ($\xi^i$). The nonlocality is somewhat concordant with the basic assumption of the theory that there are preferred reference frames defined by the initial spacelike hypersurface, which is also a global property of the spacetime. Then it would be natural to relate the initial conditions about $\xi^i$ and $\Lambda_g$ to the initial spacelike hypersurface. A natural initial condition is that dynamical equations about $\partial_0^2 g_{ij}$ are not affected by $\xi^i$ or $\Lambda_g$ on the initial hypersurface. This is achieved by requesting a covariant condition $\Lambda_g = 0$ on the initial hypersurface, because the effect of $\xi^i$ is propagated to the $ij$ components through $\Lambda_g$ in the 00 component. Therefore, in this theory it is assumed that any spacetime realized in nature is born as a spacelike hypersurface on which $\Lambda_g = 0$, by physics beyond the level of a classical theory.

Now the theory can determine spacetime evolution without ambiguity from given initial conditions about $g_{ij}$, $\partial_0 g_{ij}$, and $T^{\mu\nu}$. The principle of free initial conditions is satisfied because of the freedom about $\partial_0 \xi^i$ and $\partial_0 \xi^0$ in the 00 and 0i components, respectively. The initial value of $\partial_0 \xi^i$ is determined by the 00 component with $\Lambda_g = 0$, and evolution of $\xi^i$ is determined by the time derivative of the 00 component. Therefore the $ij$ components to determine $\partial_0^2 g_{ij}$ are the same as EFE on the initial hypersurface, and deviation from EFE emerges by evolution of $\Lambda_g$ from zero, which is related to nonlocal integration of physical quantities over the past backbone paths starting from the initial hypersurface. Note that $\xi^i$ affect the metric evolution only through the form of $\partial_0 \xi^i$, and hence any $\xi^i$ on the initial hypersurface give the same spacetime solution if $\partial_0 \xi^i$ is the same.

### D. Covariant derivation by Lagrange multipliers

The fact that $\Xi^{\mu\nu}$ becomes a tensor using the four-vector field $\xi^\mu$ implies that a more covariant derivation of the field equations \eqref{eq:constraint_00} in general coordinate systems may be possible, and indeed it is, using the concept of Lagrange multipliers as shown below. An important difference from the case of the unimodular condition is that the constraints to keep geodesics include metric derivatives, and hence derivatives of multipliers appear in the field equations by partial integrations, which is related to the nonlocal nature of the constraints as already discussed. If the gravity plus matter action $S$ is stationary against $\delta g_{\mu\nu}$ under the constraints of keeping geodesics, $\delta (u^\alpha \nabla_\alpha u^\mu) = 0$, a modified action $S \equiv S + S_C$ should be stationary with a Lagrange multiplier four-vector $\xi^\mu$, where

$$S_C = \frac{1}{\kappa} \int g \sqrt{-g} \left( \xi^\mu u^\nu \nabla_\alpha u^\mu \right) d^4x \tag{29}$$

and a factor of $1/\kappa$ is introduced to make the resultant field equations equivalent to eqs. \eqref{eq:constraint_00}. Because of the orthogonality between $u^\mu$ and $\delta (u^\alpha \nabla_\alpha u^\mu)$, a condition of $\xi^\mu u^\mu = 0$ can be set. Variation of $S_C$ by $\delta g_{\mu\nu}$ is then

$$\delta S_C = \frac{1}{\kappa} \int g \sqrt{-g} \left( \xi^\mu \delta (u^\alpha \nabla_\alpha u^\mu) \right) d^4x = \frac{1}{\kappa} \int g \sqrt{-g} \left( \xi^\mu u^\nu \partial_\nu S_C \right) d^4x \tag{30}$$

Another possible constraint to remove the ambiguity is setting $\Xi^{00} = -\nabla_0 \xi^\mu = 0$, rather than $\Lambda_g = 0$ on the initial hypersurface. However, in this option fluctuation of $\Lambda_g$ produced by quantum effect during inflation would be too large to be consistent with the presently observed universe (see \cite{Weyl}).
where the contribution from the first term on the right hand side of eq. 13 vanishes by \( \xi_\nu u^\nu = 0 \), and we do not have to consider variation of \( \sqrt{-g} \) because of the geodesic equations for unvaried quantities. Since variations of the Christoffel symbols \( \delta \Gamma^\mu_{\alpha \beta} \) are a tensor, \( \delta S_C \) should be a scalar. We find

\[
\delta S_C = \frac{1}{2} \int \left\{ \xi^\mu u^\alpha u^\beta \left[ g^{\mu \rho} (\delta g_{\rho \beta, \alpha} + \delta g_{\alpha \rho, \beta} - \delta g_{\alpha \beta, \rho}) + \delta g^{\mu \rho} (g_{\rho \beta, \alpha} + \alpha g_{\rho \beta, \alpha} - g_{\alpha \beta, \rho}) \right] \right\} \sqrt{-g} \, d^4x
\]

\[
= -\frac{1}{2} \int \left[ \nabla_\alpha (\xi^\rho u^\alpha u^\beta) \delta g_{\rho \beta} + \nabla_\beta (\xi^\rho u^\alpha u^\beta) \delta g_{\alpha \rho}
- \nabla_\mu (\xi^\rho u^\alpha u^\beta) \delta g_{\alpha \beta}
+ (\text{terms including } \Gamma^\mu_{\rho \nu}) \right] \sqrt{-g} \, d^4x
\]

where partial integrations with vanishing boundaries have been used to get the second line, and the terms including Christoffel symbols appear by converting partial derivatives into covariant ones and \( \partial_\alpha (\sqrt{-g}) \equiv \Gamma^\mu_{\rho \nu} \sqrt{-g} \). Since \( \delta S_C \) is a scalar, the terms including Christoffel symbols should vanish, which of course can be checked by a direct calculation. Therefore,

\[
\delta S_C = \frac{1}{2} \int \Xi^{\mu \nu} \delta g_{\mu \nu} \sqrt{-g} \, d^4x
\]

where

\[
\Xi^{\mu \nu} \equiv \nabla_\mu (\xi^\rho u^\nu u^\rho) + \nabla_\nu (\xi^\rho u^\mu u^\rho) - \nabla_\rho (\xi^\rho u^\mu u^\nu)
\]

Then the field equations become eq. 22 after adding the unimodular condition. It is straightforward by component calculations that this tensor \( \Xi^{\mu \nu} \) is exactly the same as eqs. 23, 24 in a synchronous coordinate system.

**E. Linear perturbation analysis**

To examine the stability of flat spacetime in this theory, consider linear perturbations about \( h_{ij} \), \( \Lambda_g \) and \( \xi^i \) from the Minkowski spacetime without matter in a synchronous coordinate. The contracted Bianchi identities \( \nabla_\mu (\Lambda_g g^{\mu \nu} + \Xi^{\mu \nu}) = 0 \) for the 0 and i components are:

\[
\partial_0 \Lambda_g + \partial_0 (\partial_0 \xi^i) + \partial_i (\partial_0 \xi^i) = \partial_0 \Lambda_g = 0 , \quad (35)
- \partial_0 \Lambda_g + \partial_0^2 \xi^i = 0 . \quad (36)
\]

Therefore at the level of linear perturbation, the scalar field \( \Lambda_g(x) \) is always a constant along the time coordinate. This is in sharp contrast to the theory proposed in Paper I in which \( \Lambda_g \) exponentially grows, and hence the instability problem has disappeared.

Next we consider plane wave perturbations along the \( x^3 \) direction, and hence derivatives about \( x^1 \) and \( x^2 \) disappear. The 10 components of the field equations (multiplied by a factor of 2) become

\[
00 : (h_{11} + h_{22}),_{33} = 2 \Lambda_g - 2 \xi^3,_{3} \quad (37)
11 : \Box h_{22} + h_{33},_{00} = -2 \Lambda_g \quad (38)
22 : \Box h_{11} + h_{33},_{00} = -2 \Lambda_g \quad (39)
33 : (h_{11} + h_{22}),_{00} = -2 \Lambda_g \quad (40)
12 : -\Box h_{12} = 0 \quad (41)
13 : -h_{13},_{00} = 0 \quad (42)
23 : -h_{23},_{00} = 0 \quad (43)
01 : -h_{13},_{03} = -2 \xi^3,_{0} \quad (44)
02 : -h_{23},_{03} = -2 \xi^3,_{0} \quad (45)
03 : (h_{11} + h_{22}),_{03} = -2 \xi^3,_{0} \quad (46)
\]

where \( \Box \equiv \eta^{\mu \nu} \partial_\mu \partial_\nu \) and the superscripts in eqs. 22 have been converted into subscripts by the ordinary tensor laws, and note that \( \Xi_{00} = \Xi^{00} \) and \( \Xi_{0i} = g_{ij} \Xi^{ij} \sim -\Xi_{0i} \) up to the first order perturbations. It can be seen that the two modes of gravitational waves \( (h_{12} \text{ and } h_{11} - h_{22}) \) in EFE are unchanged.

Since free initial conditions are allowed in this theory, we consider plane wave perturbations \( h_{ij} = A_{ij} \exp(ikx^3) \) for all the \( ij \) components on the initial hypersurface. For simplicity, here particular solutions with initial conditions of \( \Lambda_g = 0 \) and \( h_{ij},_{00} = 0 \) are considered.

Then because \( \partial_0 \Lambda_g = 0, \Lambda_g \) is zero throughout the spacetime, making the \( ij \) components the same as EFE. The difference from EFE appears as \( \xi^i \) and \( \xi^3 \) only in the \( 0\mu \) components. There is no acceleration for \( h_{13} \) and \( h_{23} \), but the difference from EFE is that their first time derivatives can be nonzero. However, we have assumed \( h_{ij},_{00} = 0 \) initially for the particular solutions considered here, and hence they are kept constant with time. Then these two modes are not different from those possible in EFE, and they can be erased by a linear infinitesimal coordinate transformation \( x^\mu = x^\mu + \zeta^\mu \) and \( h^i_{(i)} = h_{ij} - \zeta_{j,i} - \zeta_{j,i} \), where \( \zeta_\mu \equiv \eta_{\mu \nu} \zeta^\nu \). For example, \( h_{13} \) disappears by a spatial coordinate transformation keeping the synchronous condition, \( \zeta = A_{13} (ik)^{-1} \exp(ikx^3) \).

Now give a look at the mode \( h_{11} + h_{22} \), which is also free and hence \( A_{11} + A_{22} \) becomes constant by the assumed initial condition of \( h_{ij},_{00} = 0 \). It should be noted that this mode is prohibited in EFE because of the 00 component, \( (h_{11} + h_{22}),_{33} = 0 \), in contrast to \( h_{13} \) and \( h_{23} \). But in this theory \( A_{11} + A_{22} \) can be nonzero by the freedom of \( \xi^3 \). This mode cannot be erased by coordinate transformations, and it may have observational consequences that are different from EFE. Finally, the mode \( h_{33} \) feels a constant acceleration as

\[
2h_{33},_{00} = -2\xi^3,_{3} = (h_{11} + h_{22}),_{33} . \quad (47)
\]

\[\text{[7]}\] Though this theory assumes \( \Lambda_g = 0 \) on the initial hypersurface when a spacetime was born, nonzero fluctuations of \( \Lambda_g \) may be produced in later nonlinear cosmological evolution. However, as argued in [10] these conditions are expected to hold for fluctuations generated by inflation.
Though it seems to evolve in time as \( h_{33} = (Bt^2 + A_{33}) \exp(ikx^3) \) where \( A_{33} \) is constant and \( B = -(A_{11} + A_{22})k^2/4 \), this depends on the choice of a coordinate system. In fact, this mode is transformed into

\[
\begin{align*}
    h'_{00} &= -2B \frac{1}{k^2} \exp(ikx^3) \tag{48} \\
    h'_{03} &= h'_{33} = 0 \tag{49}
\end{align*}
\]

by a coordinate transformation of

\[
\begin{align*}
    \zeta_0 &= 2Bt \frac{1}{k^2} \exp(ikx^3) \tag{50} \\
    \zeta_3 &= (Bt^2 + A_{33}) \frac{1}{ik} \exp(ikx^3). \tag{51}
\end{align*}
\]

This is no longer a synchronous coordinate system, but the mode becomes a static gravitational field. The initial fluctuation \( A_{33} \) disappears in \( h'_{00} \), and \( B \) is related to the mode \( (h_{11} + h_{22}) \), and hence the mode \( h_{33} \) is essentially the same as \( (h_{11} + h_{22}) \). This implies that the growing mode \( h_{33} \propto t^2 \) is an artifact by the constrained choice of synchronous coordinate systems, which becomes a static perturbed Minkowski spacetime in non-synchronous frames.

\[\text{IV. IMPLICATIONS FOR COSMOLOGY}\]

\[\text{A. Solving the cosmological constant problem}\]

Implications for the cosmological constant problem are not much changed from those obtained in Paper I. It is reasonable to assume that the universe started from a highly inhomogeneous state without the four constraints of EFE on initial conditions. In the theory \( \Lambda_g = 0 \) is assumed on the initial hypersurface, but the linear perturbation theory cannot be adopted in such a state and hence \( \Lambda_g(x) \) would soon evolve to nonzero values. Then in some regions of the spacetime inflation begins by the vacuum energy density of an inflaton field with a Hubble parameter value of \( H \). The inflating regions become isotropic and homogeneous, and hence anisotropic quantities such as \( \xi^i \) should become unimportant, and the contracted Bianchi identities ensure that \( \Lambda_g(x) \) converges to a universal constant. This should be examined by numerical studies in future work, but can be verified at the linear theory level in the flat FLRW metric as follows. Consider linear perturbations about \( h_{ij}, \delta \Lambda_g, \) and \( \xi^i \) from the FLRW metric in a synchronous gauge, ignoring matter perturbations for simplicity. Here \( h_{ij} \) are defined by \( g_{ij} = -a^2(\delta_{ij} - h_{ij}) \) so that they become the same as those in (III) and (III) when \( a = 1 \). The fluctuation \( \delta \Lambda_g(x) \) is discriminated from the nonzero constant \( \Lambda_g \) of the background metric. In this case

\[
\begin{align*}
    \Xi^{00} &= -\partial_t \xi^i \tag{52} \\
    \Xi^{0i} &= \partial_0 \xi^i + 3H \xi^i \tag{53}
\end{align*}
\]

and the contracted Bianchi identities \( \nabla_{\mu}(\delta \Lambda_g g^{\mu\nu} + \Xi^{\mu\nu}) = 0 \) become

\[
\begin{align*}
    \nu = 0 : & \quad \partial_0 \delta \Lambda_g = 0 \tag{54} \\
    \nu = i : & \quad -\frac{1}{a^2} \partial_0 \delta \Lambda_g + \partial_i \Xi^{0i} + 5H \Xi^{0i} = 0. \tag{55}
\end{align*}
\]

Therefore \( \delta \Lambda_g \) becomes constant about time in the linear theory also in the FLRW metric. From the second equation, \( \Xi^{0i} \) should exponentially decay if \( \Delta H \Xi^{0i} \geq |\partial_0 \delta \Lambda_g/a^2| \), and hence \( \Delta H \Xi^{0i} \) is limited below \( |\partial_0 \delta \Lambda_g/a^2| \). Consider a fluctuation whose comoving wavenumber is \( k_i \sim H \) at the beginning of inflation, at which \( a \) is normalized to unity. Then we find \( |\Xi^{0i}| \leq |\delta \Lambda_g|/a^2 \) and \( |\Xi^{00}| \sim |\Xi^{0i}| \), meaning that \( \Xi^{\mu\nu} \) is unimportant compared with \( \delta \Lambda_g g^{\mu\nu} \) (i.e., \( |\Xi^{00}| \ll |\delta \Lambda_g| \) and \( |\Xi^{0i}| \ll |\delta \Lambda_g|/a \) when \( a \gg 1 \). Therefore fluctuations at the beginning of inflation rapidly disappear and the field equations converge into EFE with a nonzero cosmological constant \( \Lambda \) on scales much smaller than the comoving scale of initial inhomogeneity before inflation \( (k_i^{-1}) \). The symmetry of general covariance is thus spontaneously restored. However, the fluctuations of \( \Lambda_g \) on the comoving scale \( k_i^{-1} \) remain after inflation, with the final value of \( \Lambda = \Lambda_g + \Lambda_{\text{vac}} \) in the presently observed universe. Then the cosmological constant problem is solved by the anthropic argument, as explained in (II) (see Paper I for more discussions).

\[\text{B. Primordial metric anisotropy generated by inflation}\]

In (IV.A) it was shown that the observable universe should be described by EFE+\( \Lambda \) after inflation at the level of a classical theory. It is the standard paradigm that the large scale structures observed in the present universe are generated by quantum fluctuations of the inflaton field, and the two modes of primordial gravitational waves \( (h_{12} \) and \( h_{11} - h_{22} ) \) are also generated by quantum fluctuations of metric, which may be observed as B-mode polarization of CMB (see (351) for a review). These quantum fluctuations are assumed to become classical fluctuations when the wavelength of a mode becomes larger than the Hubble horizon during inflation, though there are some fundamental theoretical issues about such a process.

In the proposed theory, the four constraints on initial conditions in EFE have been removed, and DOFs of metric dynamics are increased. As shown in (III) the two gravitational wave modes are not changed in the new theory. The newly increased modes do not obey the wave equation, but quantum fluctuations should exist for any dynamical DOF. Therefore if the standard paradigm of quantum generation of density and metric fluctuations is correct, all the six modes of \( h_{ij} \) fluctuations are expected to appear. The amplitude of primordial gravitational waves is predicted to be \( \Delta^2_h \sim k^4(|h_k|^2) \sim GH^2 \), where \( \langle |h_k|^2 \rangle \) is the power spectrum of metric fluctuations of a comoving wavenumber \( k \). This can be derived
by the uncertainty principle about time \((H^{-1})\) and kinetic energy within the Hubble volume \((\dot{h}^2/G \times H^{-3})\) using \(\dot{h} \sim H h\). The same argument can be applied to the new modes, and hence they are expected to have similar amplitudes to those of the two gravitational wave modes. An important difference is, however, that the new modes are not oscillatory. Matter fluctuations and gravitational waves are predicted to be Gaussian, because the ground state wave function of the harmonic oscillator is Gaussian. This implies that the non-oscillatory new modes would have strong deviation from Gaussian.

The theory assumes \(\Lambda_g = 0\) on the initial spacelike hypersurface when the universe was born as a classical spacetime. It is an interesting question how to treat this for fluctuations classicalized when they become superhorizon during inflation. Ultimately the theory of quantum gravity would be required to answer this question, but it is reasonable to assume \(\delta \Lambda_g = 0\) as the initial condition at classicalization also for such metric perturbations. It should be noted that the initial hypersurface to determine the field equations may also be perturbed from the background isotropic FLRW metric, which may be different for different wavelength modes. Then metric perturbations in various wavelengths cannot be treated in a single synchronous coordinate system, but the synchronous coordinates in which \(\Xi^{\mu \nu}\) takes the simple form of eqs. \((23)–(25)\) may be different for different wavelength modes. Systematic formulations to treat such various modes in a single coordinate system are interesting for future work, but here evolution of a single wavelength mode is considered for simplicity.

Consider a mode of metric perturbation in its synchronous coordinate system. Because \(\delta \Lambda_g = 0\) throughout the spacetime within the linear theory for the FLRW metric, evolution of \(h_{ij}\) is determined by the \(ij\) components of linearized EFE. Classicalization occurs when the mode becomes superhorizon, and hence we can ignore spatial derivative terms in later evolution. In this case the equations for the metric perturbations become \(\dot{h}_{ij} = -3Hh_{ij}\), and hence all components of \(h_{ij}\) are quickly frozen and become constant with time. After inflation, such frozen fluctuations gradually enter the Hubble horizon from shorter wavelength modes. Then the new modes of metric anisotropy should affect the observational signal of CMB B-mode polarization with amplitudes similar to the ordinary two gravitational wave modes, on scales larger than the horizon at the recombination. Especially, the mode \((h_{11} + h_{22})\) cannot be erased by gauge transformations as discussed in III.E. The gravitational wave modes start to oscillate after they become subhorizon, and are damped on the time scale of \(H^{-1}\) by the frictional term in the equations of motion. Though the new modes are not oscillatory, their evolution would also change after entering the horizon. Quantitative evolution must be calculated by a careful treatment of perturbation equations in an expanding universe with coupling to matter fluctuations, which is beyond the scope of this paper.

V. CONCLUDING REMARKS

In this work a new set of gravitational field equations was derived from the action principle, in which the Lagrangians are the same as standard general relativity but four constraints are imposed on \(\delta g_{\mu \nu}\) to extract six DOFs of gravity. Then the four constraints on initial conditions in EFE are removed. This is motivated by a principle that the gravity theory should be able to describe spacetime evolution starting from free initial conditions on the initial spacelike hypersurface. Such field equations were originally proposed in Paper I, but here a new version was derived by replacing the three of the four constraints with more physically motivated ones, requesting that geodesics remain geodesic against variations. Another constraint of the unimodular condition is unchanged. As a result, the theory becomes nonlocal with integro-differential field equations. The constraints of keeping geodesics were introduced independently of the motivation of free initial conditions, but they naturally lead to the field equations similar to those of Paper I and the principle of free initial conditions is satisfied. Furthermore, the instability problem in the version of Paper I naturally disappears. These results lend some credence to the proposed theory as a candidate of the true gravity theory.

The field equations \((22)\) are given in a covariant form, which are derived from the covariant condition of the least action, but the general principle of relativity is violated because the tensor \(\Xi^{\mu \nu}\) takes a simple form in preferred coordinate systems defined by the initial spacelike hypersurface. At the cost of this, the theory allows free initial conditions about the metric, their first time derivatives, and matter distribution. Moreover, the cosmological constant problem is solved in this theory. One may think that the theory is similar to the Einstein-Aether (EA) theory in which a timelike vector field is introduced and hence there is a preferred reference frame \([38, 39]\). However, in the present theory the preferred frame is defined by a synchronous coordinate system with respect to the initial hypersurface, and \(u^\mu\) is not a physical dynamical field. The vector field \(\xi^\mu\) is Lagrange multipliers rather than a dynamical field including kinetic terms, unlike the vector field in the EA theory. The concept of the proposed theory is to increase DOFs of metric dynamics by removing the four constraints on initial conditions in EFE, rather than introducing new physical fields. The proposed theory does not include any new adjustable parameters, while the EA theory includes several.

The theory is indistinguishable from EFE+\(\Lambda\) after inflation as a classical theory, and hence this theory passes all observational/experimental tests supporting general relativity and the standard cosmological model with \(\Lambda\) and cold dark matter. However, it has a particular prediction about the primordial metric anisotropy generated by quantum fluctuations during inflation. Because of the increased DOFs of metric dynamics, new modes of fluc-
tuation would be generated including $h_{11} + h_{22}$ in plane wave solutions to the $x^3$ direction, which is prohibited in standard general relativity. The two gravitational wave modes predicted by EFE are unchanged, and amplitudes of the new modes are similar to those. However, the new modes are non-oscillatory and highly non-Gaussian, unlike the ordinary gravitational waves. These predictions may be tested by future observations seeking for CMB B-mode polarization. A fundamental assumption here is that gravitational fields should be quantized in the same way as other fields, which may not be true. Another possibility is then that primordial metric anisotropy (including ordinary gravitational waves) is not generated at all by quantum metric dynamics during inflation. The solution to the cosmological constant problem by this theory is still valid even in such a case, though the theory would become indistinguishable from EFE+$\Lambda$ within the present Hubble horizon.

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