Closed $N=2$ Strings: Picture-Changing, Hidden Symmetries and SDG Hierarchy

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Abstract

We study the action of picture-changing and spectral flow operators on a ground ring of ghost number zero operators in the chiral BRST cohomology of the closed $N=2$ string and describe an infinite set of symmetry charges acting on physical states. The transformations of physical string states are compared with symmetries of self-dual gravity which is the effective field theory of the closed $N=2$ string. We derive all infinitesimal symmetries of the self-dual gravity equations in 2+2 dimensional spacetime and introduce an infinite hierarchy of commuting flows on the moduli space of self-dual metrics. The dependence on moduli parameters can be recovered by solving the equations of the SDG hierarchy associated with an infinite set of abelian symmetries generated recursively from translations. These non-local abelian symmetries are shown to coincide with the hidden abelian string symmetries responsible for the vanishing of most scattering amplitudes. Therefore, $N=2$ string theory “predicts” not only self-dual gravity but also the SDG hierarchy.
1 Introduction

The pioneering work of Ooguri and Vafa \cite{1} revealed an intimate connection between self-dual field theories and (classical) \( N=2 \) string theories, formulated in four spacetime dimensions. In particular, four-dimensional manifolds with a metric \( g \) of ultrahyperbolic signature \((++--)\) and a self-dual Riemann tensor arise as exact (to all orders in \( \alpha' \)) classical background configurations for the closed \( N=2 \) string. Indeed, the only physical string degree of freedom in this case is a massless scalar field whose (tree-level) dynamics takes the form of Plebanski’s first \cite{1} or second \cite{2} equation, which both describe self-dual gravity (SDG) \cite{3}, albeit in different gauges. Although the absence of an infinite tower of massive excitations indicates a sort of caricature of a string, this quality makes it amenable to exact solutions, a fact quite rare in string theory. Yet \( N=2 \) strings may not only serve as a testing ground for certain issues in string theory in general but, being consistent quantum theories, they can also serve as a guide in the quantization of self-dual gravity.

The self-duality equations for the Riemann tensor can be considered on complex four-manifolds \( M^C \) with holomorphic metric \( g^C \), and in most papers on SDG just the complex case was considered. In order to investigate the symmetries of the SDG equations, one usually fixes some special form of a tetrad, which corresponds to the choice of a gauge. This choice breaks the invariance of the SDG equations under the gauge group SDiff\((M^C)\) of volume-preserving diffeomorphisms of a manifold \( M^C \). Using different gauges, various symmetries of the SDG equations were uncovered (see e.g. \cite{4,5,6} and references therein). It is fair to say, however, that the connections between these symmetries have not yet been clarified. Also missing is a discussion of the symmetry subalgebras compatible with a real structure on \( M^C \), i.e. symmetry algebras of the SDG equations on Riemannian or Kleinian four-dimensional manifolds with signature \((4,0)\) or \((2,2)\), respectively. Again, the first steps in this direction have been made by Ooguri and Vafa \cite{1}.

We notice that the description of self-duality depends on the orientation of the manifold \( M \), and self-duality can be replaced by anti-self-duality upon changing the orientation of \( M \). This paper will be concerned with the SDG equations. We shall describe the SDG equations on Kleinian four-dimensional manifolds \( M \) of signature \((2,2)\) and discuss their integrability, hidden symmetries, and hierarchies. No complete treatment of these problems yet exists. We shall also discuss a connection between the group-theoretic and the geometric twistor approaches to the symmetries of the SDG equations, describe a general solution of the linearized SDG equations, and therefore present all infinitesimal symmetries of these equations.

If \( N=2 \) string theory “predicts” self-dual gravity, its wealth of symmetries should be obtainable from the stringy description. More precisely, we expect the SDG hierarchy related to the abelian symmetries of the SDG equations to be visible in \( N=2 \) closed string quantum mechanics. Indeed, in an earlier paper with Jünemann \cite{7}, the authors have recently identified part of these hidden string symmetries and have demonstrated that they are the cause of the vanishing of almost all scattering amplitudes. Quite surprisingly, the stringy root of such symmetries is technically the somewhat obscure picture phenomenon \cite{8} which is present whenever covariant quantization meets worldsheet supersymmetry. Global symmetries unbroken by the string background under consideration may be classified with the help of BRST cohomology, and the latter unexpectedly displays a picture dependence \cite{8} (see also \cite{10}). This connection hints at a geometrical interpretation of the picture phenomenon of the closed \( N=2 \) string in terms of flows in the moduli space of self-dual metrics.
2 Review of the closed N=2 string

From the worldsheet point of view, critical closed N=2 strings in flat Kleinian space $\mathbb{R}^{2,2}$ are a theory of $N=(2,2)$ supergravity $(h, \chi, A)$ in 1+1 dimensions coupled to two chiral $N=(2,2)$ massless matter multiplets $(y, \psi)$. The latter’s components are complex scalars (the four string coordinates) and $SO(1,1)$ Dirac spinors (their four NSR partners). The N=2 string Lagrangian, as first written down by Brink and Schwarz [11], reads

$$L = \sqrt{h} \left\{ \frac{1}{2} h^{mn} \partial_m y^A \partial_n y^A + \frac{2}{3} \bar{\psi}^{-A} \gamma^m \overset{\leftrightarrow}{D}_m \psi^+ + A_m \bar{\psi}^{-A} \gamma^m \psi^+ + (\partial_m y^A + \bar{\psi}^{-A} \chi_m^+ \chi_n^+ \gamma^n \gamma^m \chi_n^+ (\partial_m y^A + \bar{\chi}_m \psi^+)) \right\} \eta_{AA} \quad (2.1)$$

where $h_{mn}$ and $A_m$, with $m=0,1$, are the (real) worldsheet metric and $U(1)$ gauge connection, respectively. The worldsheet gravitino $\chi_m$ as well as the matter fields $y^A$ and $\psi^A$ are complex valued, so that the spacetime index $A, \bar{A} = 1, 2$ runs over two values only. Complex conjugation reads

$$(y^A)^* = \bar{y}^\bar{A} \quad \text{but} \quad (\psi^{+A})^* = \bar{\psi}^{-\bar{A}} \quad \text{and} \quad (\chi^+_m)^* = \chi^-_m \quad , \quad (2.2)$$

and $\eta_{AA} = \text{diag}(+-)$ is the flat metric in $\mathbb{C}^{1,1}$. As usual, $\{\gamma^m\}$ are a set of $SO(1,1)$ worldsheet gamma matrices, $\bar{\psi} = \psi \gamma^0$, and $D_m$ denotes the worldsheet gravitationally covariant derivative.

This formulation entails the choice of a complex structure on Kleinian space. A given complex structure breaks the global “Lorentz” invariance of $\mathbb{R}^{2,2}$,

$$\text{Spin}(2,2) = SU(1,1) \times SU(1,1) \rightarrow U(1) \times SU(1,1) \simeq U(1,1) \quad . \quad (2.3)$$

The moduli space of complex structures is the two-sheeted hyperboloid $H^2 = H^2_+ \cup H^2_-$ with $H^2_\pm \simeq SU(1,1)/U(1)$. It can be completed to $CP^1$ by sewing the two sheets together along a circle,

$$CP^1 = H^2_+ \cup S^1 \cup H^2_- \quad . \quad (2.4)$$

Instead of using complex coordinates adapted to $SU(1,1)'$, one may alternatively choose a basis appropriate for $SL(2,\mathbb{R})'$, and employ a real notation for the string coordinates,

$$y^1 = x^1 + ix^2 \quad , \quad y^2 = x^3 + ix^4 \quad , \quad (2.5)$$

by expressing the real coordinates $x^\mu, \mu, \nu, \ldots = 1, 2, 3, 4$, in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})'$ spinor notation,

$$x^{a \dot{a}} = \sigma_\mu^{a \dot{a}} x^\mu = \begin{pmatrix} 0 & x^{13} & x^{23} & x^{43} \\ x^{13} & 0 & x^{24} & x^{34} \\ x^{23} & x^{32} & 0 & x^{24} \\ x^{43} & x^{34} & x^{24} & 0 \end{pmatrix} \quad , \quad \alpha \in \{+,−\} \quad , \quad \dot{\alpha} \in \{+,−\} \quad , \quad (2.6)$$

with the help of chiral gamma matrices $\sigma_\mu$ appropriate for the spacetime metric $\eta_{\mu\nu} = \text{diag}(+-−−)$.

In the real formulation, the tangent space at any point of $\mathbb{R}^{2,2}$ can be split to $\mathbb{R}^2 \oplus \mathbb{R}^2$ which defines a real polarization or cotangent structure. Such a polarization is characterized by a pair of null planes $\mathbb{R}^2$, and the latter are determined by a real null two-form modulo scale or, equivalently, by a real $SL(2,\mathbb{R})$ spinor $v$ modulo scale. Indeed, each null vector $(v_\alpha)$ factorizes into two real spinors, $v_{a \dot{a}} = v_a w_\dot{a}$. Choosing coordinates such that $(v_\alpha) = \begin{pmatrix} 1 \atop 0 \end{pmatrix}$, it becomes clear that a given null plane is stable under the action of

$$B_+ \times SL(2,\mathbb{R})' \quad , \quad \text{with} \quad B_+ := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \bigm| a \in \mathbb{R}^*, \quad b \in \mathbb{R} \right\} \quad , \quad (2.7)$$

2
where $B_+$ acts on $v$ and $SL(2, \mathbb{R})'$ on $w$. The moduli space of cotangent structures thus becomes

$$\text{Spin}(2,2)/[B_+ \times SL(2, \mathbb{R})'] \simeq SL(2, \mathbb{R})/B_+ \simeq S^1$$

(2.8)

which in fact is just the $S^1$ in (2.4). However, it turns out [12] that the real spinor $v$ also encodes the two string couplings,

$$\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \kappa^{1/4} \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix}$$

(2.9)

with $\kappa \in \mathbb{R}^+$ being the gravitational coupling and $\theta \in S^1$ the instanton angle. Since $v$ (including scale) is inert only under the parabolic subgroup of $B_+$ obtained by putting $a=1$, the space of string couplings is that of nonzero real $SL(2, \mathbb{R})$ spinors,

$$\mathbb{R}^+ \times S^1 \simeq \mathbb{R}^2 - \{0\} \simeq \mathbb{C} - \{0\} \ni \kappa^{1/4} e^{i\theta/2} \ .$$

(2.10)

Consequently, fixing the values of the string couplings amounts to breaking the global \textquotedblleft Lorentz\textquotedblright{} invariance of $\mathbb{R}^{2,2}$ in a way different from (2.3),

$$\text{Spin}(2,2) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})' \rightarrow \mathbb{R} \times SL(2, \mathbb{R})' \ ,$$

(2.11)

where $\mathbb{R} \simeq B_+(a=1)$ from eq.(2.7).

The $N=2$ supergravity multiplet defines a gravitini and a Maxwell bundle over the worldsheet Riemann surface. The topology of the total space is labeled by the Euler number $\chi$ of the punctured Riemann surface and the first Chern number (instanton number) $M$ of the Maxwell bundle. It is notationally convenient to replace the Euler number by the \textquotedblleft spin\textquotedblright{} $J := -2\chi = 2n - 4 + 4(\#\text{handles}) \in 2\mathbb{Z} \ .

(2.12)

The Lagrangian is to be integrated over the string worldsheet of a given topology. The first-quantized string path integral for the $n$-point function $A^{(n)}$ includes a sum over worldsheet topologies $(J, M)$, weighted with appropriate powers in the string couplings $(\kappa, e^{i\theta})$:

$$A^{(n)} = \sum_{J=2n-4}^{\infty} \kappa^{J/2} A_{J}^{(n)} = \sum_{J=2n-4}^{\infty} \sum_{M=-J}^{+J} \kappa^{J/2} e^{iM\theta} A_{J,M}^{(n)}$$

(2.13)

where the instanton sum has a finite range because bundles with $|M| > J$ do not contribute. The presence of Maxwell instantons breaks the explicit $U(1)$ factor in (2.3) but the $SU(1,1)$ factor (and thus the whole $\text{Spin}(2,2)$) is fully restored if we let $\kappa^{1/4}(e^{i\theta/2}, e^{-i\theta/2})$ transform as an $SU(1,1)$ spinor. The partial amplitudes $A_{J,M}^{(n)}$ are integrals over the metric, gravitini, and Maxwell moduli spaces. The integrands may be obtained as correlation functions of vertex operators in the $N=2,2$ superconformal field theory on the worldsheet surface of fixed shape (moduli) and topology.

The vertex operators generate from the (first-quantized) vacuum state the asymptotic string states in the scattering amplitude under consideration. They uniquely correspond to the physical states of the $N=2$ closed string and carry their quantum numbers. The physical subspace of the $N=2$ string Fock space in a covariant quantization scheme turns out to be surprisingly small [13]: Only the ground state $|k\rangle$ remains, a scalar on the massless level, i.e. for center-of-mass momentum $k^A$ with $k \cdot k := \eta_{AA} k^A k^A = 0$. The dynamics of this string \textquotedblleft excitation\textquotedblright{} is described by a massless scalar field,

$$\Phi(y) = \int d^4k \ e^{-i(k \cdot y + k \cdot y)} \Phi(k) \ ,$$

(2.14)
whose self-interactions are determined on-shell from the (amputated tree-level) string scattering amplitudes,
\[
(\Phi(k_1)\Phi(k_2)\ldots\Phi(k_n))_{\text{amp}}^{\text{amp}} = A_{2n-4}^{(n)}(k_1,\ldots,k_n;\theta) =: \delta_{k_1+\ldots+k_n} A_{2n-4}^{(n)}(k_1,\ldots,k_n;\theta) .
\]

Interestingly, it has been shown \cite{1} that all tree-level \(n\)-point functions vanish on-shell, except for the two- and three-point amplitudes,
\[
\begin{align*}
\bar{A}_0^{(2)}(k_1,k_2;\theta) &= 1 , \\
\bar{A}_2^{(3)}(k_1,k_2,k_3;\theta) &= -\frac{1}{4} \left[ \epsilon_{AB} k_1^A k_2^B e^{i\theta} - \eta_{AB} (k_1^A k_2^B - \bar{k}_1^B \bar{k}_2^A) - \epsilon_{AB} \bar{k}_1^A \bar{k}_2^B e^{-i\theta} \right]^2 \\
&= \left[ \epsilon_{\alpha\beta} \left( k_1^{+\hat{\alpha}} k_2^{+\hat{\beta}} \cos^2 \frac{\theta}{2} + (k_1^{-\hat{\alpha}} k_2^{-\hat{\beta}} + k_1^{-\hat{\alpha}} k_2^{+\hat{\beta}}) \cos \frac{\theta}{2} \sin \frac{\theta}{2} + k_1^{-\hat{\alpha}} k_2^{-\hat{\beta}} \sin^2 \frac{\theta}{2} \right) \right]^2 \\
&=: \left[ \epsilon_{\alpha\beta} k_1^{+\hat{\alpha}} k_2^{+\hat{\beta}} \right]^2 ,
\end{align*}
\]

with \(\bar{k}_i \cdot k_j + \bar{k}_j \cdot k_i = 0\) due to \(\sum_k k_n = 0\). Note that \(\bar{A}_2^{(3)}\) is totally symmetric in all momenta.

Since we argue that the string couplings \((\kappa,e^{i\theta})\) can be changed at will by global “Lorentz” transformations, it is admissible to make a convenient choice of Lorentz frame. First, we may scale \(\kappa \to 1\) (i.e. put the constant dilaton to zero). Second, the instanton angle \(\theta\) is at our disposal. In the real notation, renaming \(\Phi \to \Psi\), one sees that taking \(\theta=0\) reduces the amplitude to its \(M=+J\) contribution \cite{2},
\[
\bar{A}_2^{(3)}(k_1,k_2,k_3;\theta=0) = \left[ \epsilon_{\alpha\beta} k_1^{+\hat{\alpha}} k_2^{+\hat{\beta}} \right]^2 ,
\]

which translates to a cubic interaction \cite{1}
\[
\mathcal{L}_{\text{int}} = \frac{\kappa}{6} e^{\hat{\alpha}\hat{\beta}} \bar{\epsilon}^{\hat{\gamma}\hat{\delta}} \Psi \partial_{+\hat{\alpha}} \partial_{+\hat{\beta}} \Psi \partial_{+\hat{\gamma}} \partial_{+\hat{\delta}} \Psi .
\]

The resulting equation of motion reads
\[
-\Box \Psi + \frac{\kappa}{2} e^{\hat{\alpha}\hat{\beta}} \bar{\epsilon}^{\hat{\gamma}\hat{\delta}} \partial_{+\hat{\alpha}} \partial_{+\hat{\beta}} \Psi \partial_{+\hat{\gamma}} \partial_{+\hat{\delta}} \Psi = 0
\]

and is known as Plebanski’s second equation. It describes the dynamics of the single-helicity \((h=+2)\) graviton in 2+2 self-dual gravity. More precisely, the self-dual Riemann tensor reduces to the \((0,2)\) Weyl tensor \(C_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\); which in light-cone gauge goes back to Plebanski’s prepotential \(\Psi\) \cite{3},
\[
C_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \partial_{+\hat{\alpha}} \partial_{+\hat{\beta}} \partial_{+\hat{\gamma}} \partial_{+\hat{\delta}} \Psi ,
\]

which is subject to the second-order equation \cite{20}.

In the complex notation, renaming \(\Phi \to \phi\), the \(U(1)\) factor in \cite{23} can be restored by averaging over all cotangent structures. In this manner, \(\bar{A}_2^{(3)}\) simplifies to
\[
\int \frac{d\theta}{2\pi} \bar{A}_2^{(3)}(k_1,k_2,k_3;\theta) = -\frac{1}{4} \left[ \eta_{AB} (k_1^A k_2^B - \bar{k}_1^B \bar{k}_2^A) \right]^2 + \frac{1}{2} \epsilon_{AB} k_1^A k_2^B \bar{\epsilon}_{AB} \bar{k}_1^A \bar{k}_2^B
\]

\[
= \frac{\delta}{6} \epsilon_{AB} \epsilon_{AB} k_1^A k_2^B \bar{k}_1^A \bar{k}_2^B
\]

\textsuperscript{1} The \(SO(2,2)\) transformation properties of this interaction become manifest when this term is rewritten as \(\frac{\delta}{6} T_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}^{(+)} \bar{\epsilon}^{\hat{\alpha}\hat{\beta}} \bar{\epsilon}^{\hat{\gamma}\hat{\delta}} \Psi \partial_{+\hat{\alpha}} \partial_{+\hat{\beta}} \Psi \partial_{+\hat{\gamma}} \partial_{+\hat{\delta}} \Psi\), with a self-dual projector \(T^{(+)}\) having nonzero components \(T^{(+)}_{++++} = 1\) only.
which leads to a cubic interaction
\[ \mathcal{L}_{\text{int}} = \frac{\kappa}{6} \epsilon^{AB} \epsilon^{\bar{A}\bar{B}} \phi \partial_A \partial_{\bar{A}} \phi \partial_B \partial_{\bar{B}} \phi \quad (2.23) \]
The corresponding equation of motion is
\[ -\square \phi + \frac{\kappa}{2} \epsilon^{AB} \epsilon^{\bar{A}\bar{B}} \partial_A \partial_{\bar{A}} \phi \partial_B \partial_{\bar{B}} \phi = 0 \quad (2.24) \]
and is called Plebanski’s first equation. It also describes 2+2 self-dual gravity but in a different parametrization. In particular, the metric of a self-dual spacetime is Ricci-flat and Kähler, with a Kähler potential
\[ \Omega(y, \bar{y}) = \eta^A_{\bar{A}} y^A \bar{y}^{\bar{A}} + \phi(y, \bar{y}) \quad (2.25) \]
where \( \phi \) must satisfy the second-order equation (2.24).

The Lagrangian may be generalized to curved Kleinian space [15] by introducing a Kählerian metric background (plus a constant dilaton and axion). The one-loop beta function equations then demand the Kähler metric to be Ricci-flat, implying again a self-dual Riemann tensor [16]. The deviation from flat \( R^2 \) may therefore be parametrized either by a nontrivial Weyl tensor expressed in \( \Psi \) or by the perturbation \( \phi \) of the flat Kähler potential, in accordance with the string dynamics. We learn that the arbitrariness of the Lorentz frame for the \( N=2 \) string implies the equivalence of different descriptions of self-dual gravity, in particular the equivalence of Plebanski’s first and second equations as a matter of gauge choice. For the remainder of this paper, we will mainly use Plebanski’s first equation, corresponding to complex string variables, adapted to \( SU(1,1) \) notation.

We conclude that, at tree-level, the \( N=2 \) closed string is indeed identical to self-dual gravity.

3 Self-dual gravity

3.1 Self-dual gravity equations

Let \( M \) be an oriented four-manifold of class \( C^\omega \) with a nondegenerate metric \( g \) of signature \((++--)\) and a volume element \( \epsilon \). Having the metric \( g \), one can introduce the Levi-Civita connection \( \Gamma \), the Riemann tensor \( R \) and the Ricci tensor \( \text{Ric} \). If \( U \) is an open subset of \( M \) and \( x^\mu : U \to \mathbb{R}^4 \) are (local) coordinates on \( U \), \( \mu, \nu, ... = 1, ..., 4 \), then these objects have components \( g = (g_{\mu\nu}), \Gamma = (\Gamma^\rho_{\mu\nu}), \text{R} = (R^\rho_{\mu\sigma\nu}), \text{Ric} = (R_{\mu\nu}) = \left( R^\rho_{\mu\sigma\nu} \right) \), and the volume element \( \epsilon \) has the form
\[ \epsilon = \frac{1}{4!} \sqrt{\det g} \, \varepsilon_{\mu_1 ... \mu_4} dx^{\mu_1} \wedge ... \wedge dx^{\mu_4}, \]
where \( \varepsilon_{\mu_1 ... \mu_4} \) is a skew-symmetric symbol with \( \varepsilon_{1234} = 1 \).

Let us consider the following equations on a metric \( g = (g_{\mu\nu}) \):
\[ * \mathcal{R} = \mathcal{R} \iff \frac{1}{2} \epsilon^{\mu_1 \mu_2 \nu_1 \nu_2} R^\rho_{\mu_1 \nu_1 \nu_2} = R^\rho_{\mu_2 \nu_1}, \quad (3.1) \]
where \( * \) is the Hodge star operator and \( \epsilon^{\mu_1 \mu_2 \nu_1 \nu_2} := \sqrt{\det g} \, g^{\mu_1 \rho} g^{\mu_2 \sigma} \varepsilon_{\rho \sigma \nu_1 \nu_2} \). A metric \( g \) satisfying eqs. (3.1) is called a self-dual (SD) or left-flat metric [17, 18, 19, 20, 21]. It is easy to see that SD metrics are Ricci-flat, i.e. they satisfy Einstein’s vacuum field equations. Notice that the definition of self-duality will be replaced by the definition of anti-self-duality if we change the orientation of the manifold \( M \), e.g. from local coordinates \((x^1, x^2, x^3, x^4)\) on \( U \) we go over to \((x^1, x^2, x^3, -x^4)\).
Equations (3.1) can be rewritten in a simpler form of equations on divergence-free vector fields. Namely, let $\epsilon$ be a volume form on an oriented 4-manifold $M$. Then for a vector field $\varphi$ on $M$ a divergence of $\varphi$ is defined by

$$L_\varphi \epsilon = (\text{div} \varphi) \epsilon,$$

where $L_\varphi$ is the Lie derivative along $\varphi$. Thus, “divergence-free” is a synonym for “volume-preserving”, and we shall consider the algebra $sdiff(M)$ of volume-preserving vector fields on $M$. Now, let us take four pointwise linearly independent vector fields $T_\alpha \in sdiff(M)$ and suppose they satisfy the self-duality equations

$$\frac{1}{2} \epsilon^{\alpha_2 \beta_2} [T_{\alpha_2}, T_{\beta_2}] = [T_{\alpha_1}, T_{\beta_1}],$$

(3.2)

where $\epsilon^{\alpha_2 \beta_2} := \eta^{\alpha_2 \alpha_3} \eta^{\beta_2 \beta_3} \epsilon_{\alpha_3 \beta_3 \alpha_1 \beta_1}$, $\eta^{-1} = (\eta^{-1})^{\alpha \beta} = \text{diag}(+1, +1, -1, -1)$ and $\alpha, \beta, ...$ are tangent (Lorentz) indices. Let $f$ be a scalar function defined by $f^2 = 4 \epsilon (T_1, T_2, T_3, T_4)$, where $\{T_\alpha\}$ is a solution of eqs.(3.3). Then one may define a tetrad $e_\alpha$ and a (contravariant) metric $g^{-1}$ by formulae

$$e_\alpha := f^{-1} T_\alpha \iff e^\alpha = f^{-1} T^\alpha,$$

(3.3)

$$g^{-1} := f^{-2} \eta^{\alpha \beta} T_\alpha T_\beta \iff g_{\mu \nu} = f^{-2} \eta^{\alpha \beta} T^\mu_\alpha T^\nu_\beta,$$

(3.4)

and the metric (3.4) will be SD. Conversely, every SD metric arises in this way, and eqs.(3.2) are equivalent to eqs.(3.1). For detailed proofs and references see e.g. [22].

We call eqs.(3.2) the self-dual gravity (SDG) equations. They are invariant under the transformations

$$T_\alpha \mapsto \delta^\phi_{\theta} T_\alpha = [\varphi, T_\alpha],$$

(3.5)

where $\varphi$ is any divergence-free vector field on $M$, i.e. $\varphi \in sdiff(M)$. For discussion of Lorentz transformations see [1]. Equations (3.2) may be interpreted as the self-dual Yang-Mills equations for Yang-Mills fields with the gauge group $SDiff(M)$ of volume-preserving transformations of $(M, \epsilon)$ (see [22] and references therein).

### 3.2 The first Plebanski equation

Let us introduce complex divergence-free vector fields

$$W_1 := \frac{1}{\sqrt{2}} (T_1 - iT_2), \quad W_2 := \frac{1}{\sqrt{2}} (T_3 - iT_4), \quad W_1 := \frac{1}{\sqrt{2}} (T_1 + iT_2), \quad W_2 := \frac{1}{\sqrt{2}} (T_3 + iT_4),$$

(3.6)

i.e. $W_1, W_2, W_1, W_2 \in sdiff^C(M) = sdiff(M) \otimes \mathbb{C}$. Then eqs.(3.2) may be rewritten in the form

$[W_1, W_2] = 0, \quad (3.7)$

$[W_1, W_1] - [W_2, W_2] = 0, \quad (3.8)$

$[W_1, W_2] = 0. \quad (3.9)$

We see that eqs.(3.7) and (3.8) have the form of “zero commutator” conditions. Notice that $\{W_A, W_B\}$ are null vector fields since

$$\eta_{11} := \eta(W_1, W_1) = 1, \quad \eta_{22} := \eta(W_2, W_2) = -1,$$

$$\eta(W_A, W_B) = \eta(W_A, W_B) = \eta(W_1, W_2) = \eta(W_2, W_1) = 0,$$

$$\eta(W_A, W_B) = \eta(W_A, W_B) = \eta(W_1, W_2) = \eta(W_2, W_1) = 0,$$
where \( A, B, \ldots = 1, 2, \bar{A}, \bar{B}, \ldots = 1, 2 \).

We have defined complex vector fields (3.1) so that \( W_A = \bar{W}_A \). Notice that eqs. (3.7) and (3.9) are invariant w.r.t. transformations from the complexification \( \text{SDiff}^C(M) \) of the group \( \text{SDiff}(M) \). We shall consider transformations from \( \text{SDiff}^C(M) \) such that after their action we shall have \( W_A \neq \bar{W}_A \) but the metric will be real. These complex transformations may be used for the partial fixing of a coordinate system. Namely, one can always introduce complex coordinates \( y^\bar{A}, y^\bar{A} := (y^\bar{A}) = \tilde{y}^\bar{A} \) on \( U \subset M \) so that \( W_1 \) and \( W_2 \) become coordinate derivatives (Frobenius theorem), i.e.

\[
W_A = \partial_A := \frac{\partial}{\partial y^A},
\]

and eq. (3.7) is identically satisfied. Then one can solve eq. (3.8) by choosing a gauge

\[
W_1 = \partial_1 \partial_2 \Omega \partial_2 - \partial_2 \partial_2 \partial_1, \quad W_2 = \partial_1 \partial_1 \Omega \partial_2 - \partial_2 \partial_1 \partial_1,
\]

where \( \Omega(y^\bar{A}, y^\bar{A}) \) is a real-valued scalar function, \( \partial_A := \partial/\partial y^A \). Substituting (3.11) into eq. (3.9), we obtain the first Plebanski equation

\[
\partial_1 \partial_2 \Omega \partial_2 \partial_1 - \partial_1 \partial_1 \Omega \partial_2 \partial_2 \partial_1 = 1.
\]

Thus, the SDG equations (3.7)-(3.9) can be reduced to one equation (3.12) on a scalar function \( \Omega \).

We introduce the antisymmetric \( \varepsilon \)-symbols,

\[
\varepsilon_{12} = -\varepsilon_{21} = \varepsilon_{12} = -\varepsilon_{21} = 1, \quad \varepsilon^{12} = -\varepsilon^{21} = \varepsilon^{12} = -\varepsilon^{21} = -1,
\]

\[
\varepsilon^B_A = \eta^{BC} \varepsilon_{CA}, \quad \varepsilon^B_A = \eta^{BC} \varepsilon_{CA} \quad \Rightarrow \quad \varepsilon^1_1 = \varepsilon^2_2 = \varepsilon^1_1 = \varepsilon^2_2 = 1.
\]

By using (3.13), eqs. (3.10) - (3.12) can be rewritten in the form

\[
W_A = \partial_A, \quad W_A = \varepsilon^B_A \varepsilon^{BC} \partial_B \partial_C \Omega, \quad (3.14)
\]

\[
\varepsilon^{AB} \varepsilon^{\bar{A}\bar{B}} \partial_A \partial_{\bar{A}} \Omega \partial_B \partial_{\bar{B}} \Omega = -2. \quad (3.15)
\]

The vector fields (3.14) are divergence-free with respect to the volume form \( \epsilon = dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4 \) and coincide with a tetrad because \( f = 1 \). Dual basis of \( \{W_A, W_{\bar{A}}\} \) has the form

\[
\theta^A = \partial_A \Omega dy^A, \quad \theta^{\bar{A}} = -\partial_{\bar{A}} \Omega dy^\bar{A}, \quad \theta^A = dy^A.
\]

The above choice of coordinates, a tetrad and a dual tetrad is convenient because the metric \( g \) in this gauge takes the Kähler form

\[
g = \eta_{AB} \theta^A \theta^B = 2 \partial_A \partial_B \Omega dy^A dy^B
\]

with the Kähler potential \( \Omega \).

### 3.3 Linear systems for the SDG equations

In Sect.3.2 it has been shown that the first Plebanski equation (3.12) can be obtained from the ASDG equations (3.7)-(3.9) by fixing a gauge. That is why we shall consider ‘Lax pairs’ for
where \( \lambda \) on \( H \) equations (3.7)-(3.9), and Lax pairs for eq.(3.12) will be obtained by substituting the tetrad (3.14). For the sake of simplicity we shall consider real analytic solutions of eqs.(3.7)-(3.9) and (3.12).

Equations (3.7)-(3.9) can be obtained as the compatibility conditions of the following linear system of equations:

\[
\begin{align*}
\mathcal{L}^+_1 \phi^+_1 &:= (W_1 - \lambda W_2)\phi^+_1 = 0, \\
\mathcal{L}^+_2 \phi^+_2 &:= (W_2 - \lambda W_1)\phi^+_2 = 0, \\
\mathcal{L}^+_3 \phi^+_3 &:= \partial^\lambda \phi^+_3 = 0,
\end{align*}
\]

(3.17)

where \( \lambda \) is the complex ‘spectral parameter’, and \( \phi^+_a(x, \lambda) \) are smooth functions for \( x \in U, \ |\lambda| \leq 1 \) and holomorphic in \( \lambda \) for \( |\lambda| < 1, a, b, \ldots = 1, 2, 3 \). Indeed, the compatibility conditions

\[ [\mathcal{L}^+_a, \mathcal{L}^+_b] = 0 \]

(3.18)
of eqs.(3.17) are identical to the SDG equations (3.7)-(3.9).

For the same equations (3.7)-(3.9) one can write another linear system

\[
\begin{align*}
\mathcal{L}^-_1 \phi^-_1 &:= (\lambda^{-1} W_1 - W_2)\phi^-_1 = 0, \\
\mathcal{L}^-_2 \phi^-_2 &:= (\lambda^{-1} W_2 - W_1)\phi^-_2 = 0, \\
\mathcal{L}^-_3 \phi^-_3 &:= \partial^\lambda \phi^-_3 = 0,
\end{align*}
\]

(3.19)

where \( |\lambda| > 0 \) and functions \( \phi^-_a(x, \lambda) \) are smooth in \( x \in U, \ |\lambda| \geq 1 \) and holomorphic in \( \lambda \) for \( |\lambda| > 1 \). The compatibility conditions

\[ [\mathcal{L}^-_a, \mathcal{L}^-_b] = 0 \]

(3.20)
of eqs.(3.19) are identical to the SDG equations (3.7)-(3.9).

The auxiliary ‘spectral parameter’ \( \lambda \) in the linear systems (3.17) and (3.19) is connected with the group \( SU(1, 1) \). This group acts on the sphere \( S^2 \simeq \mathbb{C}P^1 \), and the Riemann sphere decomposes into the disjoint union

\[ S^2 = H^2_+ \cup S^1 \cup H^2_2 \]

(3.21)
of orbits of \( SU(1, 1) \), where

\[ H^2 := H^2_+ \cup H^2_2 \]

is the two-sheeted hyperboloid, \( H^2_+ \simeq SU(1, 1)/U(1) \) is the upper half of the hyperboloid and \( H^2_2 \simeq SU(1, 1)/U(1) \) is the lower half of the hyperboloid. Using the stereographic projection, one can identify \( \bar{H}^2_+ \) with the open disk \( |\lambda| < 1 \) in \( \mathbb{C} \) and \( \bar{H}^2_- \) with the domain \( |\lambda| > 1 \) in \( \mathbb{C} \cup \infty \).

One may consider the closures

\[ \bar{H}^2_+ = H^2_+ \cup S^1 = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}, \quad \bar{H}^2_2 = H^2_2 \cup S^1 = \{ \lambda \in \mathbb{C} \cup \infty : |\lambda| \geq 1 \}. \]

Then \( \lambda \in \bar{H}^2_+ \) in (3.17), \( \lambda \in \bar{H}^2_- \) in (3.19) and \( S^1 \) is a common boundary of the spaces \( \bar{H}^2_+ \) and \( \bar{H}^2_- \). Notice that the metric

\[ ds^2 = \frac{4d\lambda d\bar{\lambda}}{(1 - \lambda \bar{\lambda})^2} \]

(3.22)
on \( H^2 \) is invariant under the action of the group \( SU(1, 1) \) and singular at \( |\lambda| = 1 \). This is a metric of the Poincaré model of Lobachevskii geometry.
3.4 Twistors, complex structures and the SDG equations

Let us consider a bundle $Z \to M$ of complex structures over a 4-manifold $M$. Following ref. [18], we shall call $Z$ a twistor space of $M$. Since we are interested in local geometry of the manifold $M$, we shall consider the restriction of the twistor bundle $Z \to M$ to an open subset $U \subset M$ and put $\mathcal{P} := Z|_U$.

Recall that on any oriented SD manifold $M$ there exists a family of complex structures covariantly constant w.r.t. the Levi-Civita connection, and they are parametrized by the two-sheeted hyperboloid $H^2$ described in Sect.3.3, where

$$H^2_+ \simeq SO(2,2)/U(1,1) \simeq SU(1,1)/U(1) \simeq SL(2,\mathbb{R})/SO(2),$$

$$H^2_- \simeq SO(2,2)/U(1,1) \simeq SU(1,1)/U(1) \simeq SL(2,\mathbb{R})/SO(2)$$

are two orbits of the group $SO(2,2)$ acting on complex structures $J \in SO(2,2)$. Therefore, in our case the twistor space $\mathcal{P}$ of $U$ coincides, as a smooth manifold, with the direct product

$$\mathcal{P} \simeq U \times H^2 = (U \times H^2_+) \cup (U \times H^2_-) = \mathcal{P}_+ \cup \mathcal{P}_-,$$

i.e. $\mathcal{P}$ is the disjoin union of $\mathcal{P}_+ = U \times H^2_+$ and $\mathcal{P}_- = U \times H^2_-.$

The vector fields $L^+_\alpha$ from the linear system (3.17) span the (0,1) tangent space of $\mathcal{P}_+$, and the vector fields $L^-_\alpha$ from (3.19) span the (0,1) tangent space of $\mathcal{P}_-$. The compatibility conditions of these linear systems are the integrability conditions of an almost complex structure on the twistor space $\mathcal{P}$. From the other hand, these conditions are equivalent to the condition of self-duality of the metric $g$ on $U \subset M$, which is the reformulation to the ultrahyperbolic case of the well-known twistor correspondence between SD geometry of $\mathcal{P}$ and complex geometry of $\mathcal{P}$. Notice that if an almost complex structure on $\mathcal{P}$ is integrable, then $\phi^\alpha_\pm$ may be taken as complex analytic coordinates on $\mathcal{P}_\pm \subset \mathcal{P}$. Moreover, from (3.17) and (3.19) it is clear that one may always take $\phi^\alpha_+ (x, \lambda) = \lambda$, $\phi^\alpha_- (x, \lambda) = \lambda^{-1}$.

The space

$$\mathcal{P}_0 = U \times S^1$$

is the $S^1$ bundle over $U$ of real anti-self-dual null bivectors $L^0_1 \wedge L^0_2$, modulo scale, where

$$L^0_1 = W_1 - \lambda W_2, \quad L^0_2 = \lambda^{-1} W_2 - W_1,$$

(3.23)

$\lambda \in S^1$. Put another way, $\mathcal{P}_0$ is a set of pairs $(x, L^0_1(x) \wedge L^0_2(x))$, where $x \in U$, and

$$L^0_1 \wedge L^0_2 = \lambda^{-1} W_1 \wedge W_2 + (W_1 \wedge W_1 - W_2 \wedge W_2) - \lambda W_1 \wedge W_2$$

is an anti-self-dual null bivector at this point, parametrized by $\lambda \in S^1$. If we introduce the spaces

$$\tilde{\mathcal{P}}_+ := U \times \tilde{H}_+^2, \quad \tilde{\mathcal{P}}_- := U \times \tilde{H}_-^2,$$

then $\mathcal{P}_0 = \tilde{\mathcal{P}}_+ \cap \tilde{\mathcal{P}}_-$ and our linear systems (3.17), (3.19) are defined on the subsets $\tilde{\mathcal{P}}_+, \tilde{\mathcal{P}}_-$ of the space

$$\tilde{\mathcal{P}} := \tilde{\mathcal{P}}_+ \cup \mathcal{P}_0 \cup \tilde{\mathcal{P}}_- \simeq U \times S^2.$$

The space $\mathcal{P}_0 = \tilde{\mathcal{P}}_+ \cap \tilde{\mathcal{P}}_-$ is the common domain of the coordinates $\phi^\alpha_+$ and $\phi^\alpha_-$. Therefore there exist smooth functions $f^\alpha_{+-}$ such that on $\mathcal{P}_0$ we have

$$\phi^\alpha_+ = f^\alpha_{+-}(\phi^-_+).$$

(3.24)
On the space $\tilde{\mathcal{P}}$ one can introduce a map $\tau : \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$ called a real structure. It is an antiholomorphic involution, defined by the formula
\[
\tau(\phi^1(x, \lambda), \phi^2(x, \lambda), \phi^3(x, \lambda)) = (\phi^1(\bar{x}, \bar{\lambda}^{-1}), \phi^2(\bar{x}, \bar{\lambda}^{-1}), \phi^3(\bar{x}, \bar{\lambda}^{-1})),
\]
where $\phi^a(x, \lambda)$ are local coordinates on $\tilde{\mathcal{P}}$, $\bar{x} = x$. This involution takes the complex structure on $\tilde{\mathcal{P}}$ to its conjugate. We may choose the coordinates $\phi^a_{\pm}$ on $\tilde{\mathcal{P}}_{\pm}$ in such a way that
\[
\phi^1_{\pm} = \tau(\phi^2_{\pm}), \quad \phi^2_{\pm} = \tau(\phi^3_{\pm}), \quad \phi^3_{\pm} = \tau(\phi^1_{\pm}).
\]

Considering these conditions on $\mathcal{P}_0$ and substituting them into (3.24), we obtain the reality conditions on functions $\phi^a_{\pm}$.

There are fixed points of the action of $\tau$ on $\tilde{\mathcal{P}}$, and they form a three-dimensional real manifold called a real twistor space $\mathcal{T}(U)$ of $U$. In the flat case this manifold coincides with the projective space $\mathbb{R}P^3$ [23, 15]. The space $\mathcal{P}_0$ is fibred over $\mathcal{T}(U)$ by real 2-manifolds called $\beta$-surfaces, and the vector fields (3.23) span their tangent spaces. Coordinates on the real twistor space $\mathcal{T}(U)$ may be introduced in the following way. Consider three functions $\phi^0_a(x, \lambda)$ on $\mathcal{P}_0$ which are real analytic for $x \in U$, $\lambda = \exp(i\gamma) \in S^1$. So they extend holomorphically in $\lambda$ to a neighbourhood $\mathcal{U}_0$ of $U \times S^1$ in $\tilde{\mathcal{P}} = U \times S^2$. Then consider linear differential equations
\[
\mathcal{L}^0_1 \phi^0_a = (W_1 - \lambda W_2) \phi^0_a = 0,
\]
\[
\mathcal{L}^0_2 \phi^0_a = (\lambda^{-1} W_2 - W_1) \phi^0_a = 0,
\]
where $\mathcal{L}^0_A$ are vector fields on $\mathcal{P}_0$ defined by formulae (3.23). The compatibility conditions of the linear system (3.27), (3.28) are identical to the SDG equations. Therefore eqs. (3.27), (3.28) always have solutions if $\{W_A, W_A\}$ satisfy eqs. (3.7)-(3.9). Moreover, one may always take $\phi^3_0 = \lambda$ and impose on $\phi^1_0, \phi^2_0$ the reality condition such that
\[
\overline{\phi^1_0}(x, \lambda) = \phi^2_0(x, \lambda).
\]
Such functions $(\phi^1_0, \phi^2_0, \phi^3_0) = (\phi^1_0, \phi^2_0, \exp(i\gamma))$ may be considered as coordinates on real twistor space $\mathcal{T}(U)$ of $U$. Notice that extensions of these real analytic functions $\phi^0_a$ to a neighbourhood $\mathcal{U}_0$ of $U \times S^1$ are holomorphic coordinates on $\mathcal{U}_0$ satisfying the reality conditions
\[
\overline{\phi^0_a}(x, \lambda^{-1}) = \phi^0_a(x, \lambda),
\]
which are reduced to the condition (3.29) on $\mathcal{P}_0$.

To sum up, by virtue of the twistor correspondence all the information about a SD metric on $U$ is encoded in a complex structure of the twistor space $\mathcal{P}$ of $U$, which is defined by the linear systems (3.17), (3.19) and complex coordinates $\phi^a_{\pm}$. From the other hand, according to the Kodaira-Spencer deformation theory, all the information about the complex structure of $\mathcal{P}$ can be extracted from the transition functions $f^a_{\pm}$ on $\mathcal{P}_0$. But for preserving readability we shall not develop this correspondence further.
3.5 The second Plebanski equation

To describe the second Plebanski equation, it is necessary to go to a real null tetrad \( \{X_A, X_A^\dagger\} \) on \( U \subset M \), \( A, B, ... = 1, 2, \bar{A}, \bar{B}, ... = 1, 2 \). For this we set

\[
\lambda = \frac{i - \zeta}{i + \zeta} \tag{3.30}
\]

in eqs.\((3.17)\) and \((3.19)\). The transformation \((3.30)\) is a linear-fractional transformation of the complex plane that carries the unit disk \( |\lambda| < 1 \) to the upper half-plane \( \text{Im} \zeta > 0 \), the domain \( |\lambda| > 1 \) to the lower half-plane \( \text{Im} \zeta < 0 \) and the circle \( |\lambda| = 1 \) to the real axis \( \text{Im} \zeta = 0 \). The metric on \( H^2 \) in the coordinates \( \zeta, \bar{\zeta} \) takes the form

\[
ds^2 = -\frac{4d\zeta d\bar{\zeta}}{(\zeta - \bar{\zeta})^2}. \tag{3.31}
\]

This is a metric of Klein’s model of Lobachevskii geometry. It is invariant under the action of the group \( SL(2, \mathbb{R}) \) and singular when \( \text{Im} \zeta = 0 \) (real axis).

It is easy to show that after the above transformation \( \lambda \mapsto \zeta \), eqs.\((3.17)\) may be rewritten as

\[
\frac{1}{(i + \zeta)}(X_1 - \zeta X_2)\chi^a_+ = 0, \quad \frac{1}{(i + \zeta)}(X_2 - \zeta X_1)\chi^a_+ = 0, \quad \partial_\zeta \chi^a_+ = 0, \tag{3.32}
\]

where \( \text{Im} \zeta \geq 0 \) and

\[
X_1 = \frac{1}{\sqrt{2}}(T_2 + T_4), \quad X_2 = \frac{1}{\sqrt{2}}(T_3 - T_1), \quad X_1 = \frac{1}{\sqrt{2}}(T_2 - T_4), \quad X_2 = \frac{1}{\sqrt{2}}(T_3 + T_1)
\]

are null vector fields since

\[
\eta(X_1, X_1) = -\eta(X_2, X_2) = 1, \quad \eta(X_A, X_B) = \eta(X_A^\dagger, X_B^\dagger) = \eta(X_1, X_2) = \eta(X_2, X_1) = 0. \tag{3.33}
\]

Analogously, eqs.\((3.19)\) may be rewritten in the form

\[
\frac{1}{(i - \zeta)}(X_1 - \zeta X_2)\chi^a_- = 0, \quad \frac{1}{(i - \zeta)}(X_2 - \zeta X_1)\chi^a_- = 0, \quad \partial_\zeta \chi^a_- = 0, \tag{3.34}
\]

where \( \text{Im} \zeta \leq 0 \). Functions \( \chi^a_+(x, \zeta) \) and \( \chi^a_-(x, \zeta) \) are real analytic for \( x \in U, \text{Im} \zeta \geq 0 \) and \( x \in U, \text{Im} \zeta \leq 0 \), respectively. The functions \( \chi^a_+ \) are holomorphic in \( \zeta \) for \( \text{Im} \zeta > 0 \) and \( \chi^a_- \) are holomorphic in \( \zeta \) for \( \text{Im} \zeta < 0 \). Moreover, one may always take \( \chi^3_+ = \chi^3_- = \zeta \). Notice that the reality condition \((3.26)\) takes the form

\[
(\chi^1_+(x, \zeta), \chi^2_+(x, \zeta), \chi^3_+(x, \zeta), \chi^1_-(x, \zeta), \chi^2_-(x, \zeta), \chi^3_-(x, \zeta)) = (\chi^2_-(x, \zeta), \chi^1_-(x, \zeta), \chi^3_-(x, \zeta)).
\]

The compatibility conditions of each of the linear systems \((3.32)\) and \((3.34)\) are reduced to the equations

\[
[X_1, X_2] = 0, \tag{3.35}
\]

\[
[X_1, X_1] - [X_2, X_2] = 0, \tag{3.36}
\]

\[
[X_1, X_2] = 0, \tag{3.37}
\]

which are the SDG equations \((3.2)\) rewritten in terms of real divergence-free vector fields \( \{X_A, X_A^\dagger\} \). Again, as in the case of complex divergence-free vector fields, two equations from eqs.\((3.35)-(3.37)\)
has the form of “zero commutator” conditions and one can choose real coordinates such that \( \{X_A\} \) or \( \{X_{\tilde{A}}\} \) coincide with coordinate derivatives. Then for two remaining vector fields one can choose a parametrization such that eq. (3.36) will be identically satisfied, and the SDG equations will be reduced to one equation on a scalar function.

Let us choose real coordinates \((z^A, z^{\tilde{A}})\) so that

\[ X_A = \partial_A := \frac{\partial}{\partial z^A}, \quad \text{(3.38)} \]

and take vector fields \(\{X_{\tilde{A}}\}\) in the form

\[ X_{\tilde{1}} = \partial_2 \partial_1 \Omega \partial_2 - \partial_2 \partial_2 \Omega \partial_1, \quad X_{\tilde{2}} = \partial_1 \partial_1 \Omega \partial_2 - \partial_1 \partial_2 \Omega \partial_1, \quad \text{(3.39)} \]

where \(\Omega = \Omega(z^A, z^{\tilde{A}})\) is a real-valued scalar function and \(\partial_{\tilde{A}} := \partial/\partial z^{\tilde{A}}\). Then eq. (3.36) is identically satisfied, and eq. (3.37) is reduced to the first Plebanski equation

\[ \partial_1 \partial_2 \Omega \partial_2 \Omega - \partial_1 \partial_1 \Omega \partial_2 \partial_2 \Omega = 1 \]

in the real null coordinates \((z^A, z^{\tilde{A}})\).

Using the same coordinates \((z^A, z^{\tilde{A}})\), instead of (3.38), (3.39) one can choose the following tetrad:

\[ X_A = \partial_A, \quad X_{\tilde{1}} = \partial_1 + \partial_1 \partial_2 \Psi \partial_2 - \partial_2^2 \Psi \partial_1, \quad X_{\tilde{2}} = \partial_2 + \partial_1^2 \Psi \partial_2 - \partial_2 \partial_2 \Psi \partial_1. \quad \text{(3.40)} \]

Then eqs. (3.35), (3.36) are identically satisfied, and eq. (3.37) is reduced to the second Plebanski equation

\[ \partial_1 \partial_1 \Psi - \partial_2 \partial_2 \Psi + \partial_1 \partial_2 \Psi \partial_1 \partial_2 \Psi - \partial_1^2 \Psi \partial_2^2 \Psi = 0 \quad \text{(3.41)} \]

on a real-valued scalar function \(\Psi(z^A, z^{\tilde{A}})\). This equation coincides with eq. (2.20) after introducing the coupling constant \(\kappa\) by changing \(\Psi \rightarrow \kappa \Psi\).

The vector fields (3.38), (3.39), (3.40) are divergence-free with respect to the volume form \(\epsilon = dz^1 \wedge dz^2 \wedge dz^{\tilde{1}} \wedge dz^{\tilde{2}}\) and dual basis of the tetrad (3.40) has the form

\[ \sigma^1 = dz^1 + \partial_1 \partial_2 \Psi dz^2 + \partial_2^2 \Psi dz^{\tilde{1}}, \quad \sigma^2 = dz^2 - \partial_1 \partial_2 \Psi dz^{\tilde{1}} - \partial_1^2 \Psi dz^{\tilde{2}}, \quad \sigma^{\tilde{A}} = dz^{\tilde{A}}. \]

In terms of \(\Psi\) the metric \(g\) has the form

\[ g = \eta_{AB} \sigma^A \sigma^B = 2(dz^1 dz^{\tilde{1}} - dz^2 dz^{\tilde{2}} + \partial_2^2 \Psi dz^1 dz^{\tilde{1}} + 2\partial_1 \partial_2 \Psi dz^{\tilde{1}} dz^{\tilde{2}} + \partial_1^2 \Psi dz^{\tilde{2}} dz^{\tilde{2}}). \quad \text{(3.42)} \]

**Remark.** The Plebanski equations are special cases of the SDG equations appeared after the gauge fixing and solving two equations from eqs. (3.2). The first Plebanski equation is an analogue of the SD Yang-Mills equations in the so-called Yang gauge, and the second Plebanski equation is an analogue of the SD Yang-Mills equations in the so-called Leznov gauge.
4 Symmetries of self-dual gravity

4.1 Symmetries of the SDG equations from the twistor viewpoint

We consider the SDG equations (3.7)-(3.9). The linearization of these equations has the form

\[ [\delta W_1, W_2] + [W_1, \delta W_2] = 0, \quad [\delta W_1, W_2] + [W_1, \delta W_2] = 0, \]

\[ [\delta W_1, W_1] + [W_1, \delta W_1] - [\delta W_2, W_2] - [W_2, \delta W_2] = 0. \]  

(4.1)

To find (infinitesimal) symmetries of eqs. (3.7)-(3.9) means to find solutions \( \delta W_A, \delta W_A \) of eqs. (4.1) for all given solutions \( W_A, W_A \) of eqs. (3.7)-(3.9).

As discussed in Sect. 3.3, eqs. (3.17)-(3.19) can be rewritten as eqs. (3.18) or eqs. (3.20) depending on \( \lambda \in H^2_+ \) or \( \lambda \in H^2_\). Since the vector fields \( L^3_3 \) trivially commute with the vector fields \( L^\pm_A \), one may consider only the following equations:

\[ \mathcal{P}_+ : \quad [L^+_1, L^+_2] = 0, \]  

\[ \mathcal{P}_- : \quad [L^-_1, L^-_2] = 0, \]  

(4.2)

(4.3)

defined on the subsets \( \mathcal{P}_+ = U \times H^2_+ \) and \( \mathcal{P}_- = U \times H^2_{-} \) of the space \( \mathcal{P} \) (see Sect. 3.4). Accordingly, equations (4.1) can be rewritten in the form of the following equations on \( \mathcal{P}_\pm \):

\[ [\delta L^+_1, L^+_2] + [L^+_1, \delta L^+_2] = 0, \]  

\[ [\delta L^-_1, L^-_2] + [L^-_1, \delta L^-_2] = 0, \]  

(4.4)

(4.5)

where

\[ \delta L^+_1 := \delta W_1 - \lambda \delta W_2, \quad \delta L^+_2 := \delta W_2 - \lambda \delta W_1, \]  

\[ \delta L^-_1 := \frac{1}{\lambda} \delta W_1 - \delta W_2, \quad \delta L^-_2 := \frac{1}{\lambda} \delta W_2 - \delta W_1. \]  

(4.6)

(4.7)

Solutions of eqs. (4.4) on \( \mathcal{P}_+ \) are related to solutions of eqs. (4.5) on \( \mathcal{P}_- \) by

\[ \delta L^+_A = \lambda \delta L^-_A \]  

(4.8)

on the overlap \( \mathcal{P}_0 = \mathcal{P}_+ \cap \mathcal{P}_- \), since eqs. (4.4) and (4.5) on the subsets of \( \mathcal{P} \) encode the same equations (4.1) on \( U \subset M \).

General solutions of eqs. (4.4) and (4.5) have the form

\[ \delta \psi_+ L^+_A = [\psi_+, L^+_A], \]

\[ \delta \psi_- L^-_A = [\psi_-, L^-_A], \]  

(4.9)

where \( \psi_+ = \psi_+^u(x, \lambda) \partial_{\mu} \) is a vector field on \( U \times H^2_+ \), the components of which are smooth for \( x \in U \), \( \lambda \in H^2_+ \) and holomorphic in \( \lambda \) for \( |\lambda| < 1 \), and \( \psi_- = \psi_-^u(x, \lambda) \partial_{\mu} \) is a smooth vector field on \( U \times H^2_{-} \), the components of which are holomorphic in \( \lambda \) for \( |\lambda| > 1 \). These vector fields should preserve a volume form \( \epsilon \) on \( U \).

By substituting the solutions (4.3) into eqs. (4.8), we obtain the equations

\[ [\psi_+ - \psi_-, L^+_A] = 0 \]  

on the intersection \( \mathcal{P}_0 \) of \( \mathcal{P}_+ \) and \( \mathcal{P}_- \). These equations mean that the vector fields \( \psi_+ \) and \( \psi_- \) are not completely arbitrary.
Symmetries of eqs. (4.2), (4.3) and therefore of the SDG equations (3.7)-(3.9) may be described as follows. Consider the space $\mathcal{P} \simeq U \times \mathbb{C}P^1$, the subsets $\mathcal{P}_\pm \subset \mathcal{P}$ and the nonempty intersection $\mathcal{P}_0 = \mathcal{P}_+ \cap \mathcal{P}_-$ (see Sect.3.4). On $\mathcal{P}_0 = U \times S^1$ we introduce a vector field $\psi = \psi^\mu \partial_\mu$ satisfying the equations

$$[\psi, \mathcal{L}_A^n] = 0$$

and preserving a volume form $\epsilon$ on $U$. Then by expanding $\psi$ in a Fourier series in $\lambda = \exp(i\gamma) \in S^1$ we have

$$\psi = \sum_{n=-\infty}^{\infty} \lambda^n \psi^n,$$

$$\psi = \psi_+ - \psi_-,$$

$$\psi_+ := \psi_0^0 + \sum_{n=1}^{\infty} \lambda^n \psi^n_+,$$

$$\psi_- := \psi_0^0 - \sum_{n=1}^{\infty} \lambda^{-n} \psi^{-n},$$

$$\psi_+^0 - \psi_-^0 = \psi_0^0,$$

where $\psi^n = \psi^n \mu \partial_\mu$ are vector fields on $U$. As a function of $\lambda$, each component $\psi^n_\mu$ of $\psi_+$ is the limit of a holomorphic function on the open disk $|\lambda| < 1$ and $\psi^n_\mu$ is the limit of a holomorphic function on the exterior $|\lambda| > 1$, including the point $\lambda = \infty$. Put another way, $\psi_+$ extends continuously to a vector field on $\mathcal{P}_+$ with components holomorphic in $\lambda \in H^2_+$, and $\psi_-$ extends continuously to a vector field on $\mathcal{P}_-$ with components holomorphic in $\lambda \in H^2_-$. The splitting (4.11)-(4.13) of $\psi$ is unique up to

$$\psi_0^0 \mapsto \psi_0^0 + \varphi, \quad \psi_0^0 \mapsto \psi_0^0 + \varphi,$$

for some vector field $\varphi = \varphi^\mu(x) \partial_\mu \in \operatorname{sdiff}(M)$. Using the transformations (4.14), one can always choose a gauge in which $\psi_0^0$ or $\psi_0^0$ is zero. Finally, using $\psi_\pm$ from (4.11)-(4.13) we define the transformations (4.9) of the vector fields $\mathcal{L}_A^\pm$. By construction, these $\delta_\psi \mathcal{L}_A^\pm$ satisfy eqs. (4.4), (4.5) and (4.8).

It is not difficult to verify that if vector fields $\psi$ and $\eta$ on $\mathcal{P}_0$ satisfy eqs. (4.10), then the vector field $[\psi, \eta]$ will also satisfy eqs. (4.10) by virtue of the Jacobi identities. Therefore, the space of vector fields satisfying eqs. (4.10) forms an algebra with the standard commutator of vector fields. These vector fields can be considered as representatives of ‘free’ vector fields on the real twistor space $T(U)$ of $U$. Notice that to impose reality conditions on symmetries of equations (3.7)-(3.9) and (4.2), (4.3) it is necessary to know reality conditions for vectors from a tetrad $\{W_A, W_A\}$. For example, if $W_A = W_A$, then on vector fields $\psi_\pm$ from formulae (4.11)-(4.13) one can impose the following reality conditions:

$$\psi_+(x, \lambda) = \overline{\psi_-(x, \lambda^{-1})} \quad \iff \quad \psi^n_+ = \overline{\psi^n_-},$$

where $n = 0, 1, \ldots$. Then we shall have $\delta_\psi W_A = \delta_\psi W_A$.

Suppose that not only the vector field $\psi$, but also the vector fields $\psi_\pm$ from (4.11)-(4.13) satisfy eqs. (4.10), i.e.

$$[\psi_\pm, \mathcal{L}_A^n] = 0,$$

Then from formulae (4.6), (4.7) and (4.9) we see that for such $\psi_\pm$ the symmetry transformations are trivial, $\delta_\psi W_A = \delta_\psi W_A = 0$. Vector fields $\psi_+$ and $\psi_-$ satisfying eqs. (4.15) are in a one-to-one correspondence with holomorphic vector fields on $\mathcal{P}_+$ and $\mathcal{P}_-$, respectively. Indeed, $\mathcal{L}_A^+$ are vector fields of type (0,1) on $\mathcal{P}_+$ (see Sect.3.4) and therefore have the form

$$\mathcal{L}_a^+ = \mathcal{L}_a^+ b \frac{\partial}{\partial \phi^b_+}.$$
From eqs. (4.13) for components of a vector field

\[ \psi_+ = \psi_+^a \frac{\partial}{\partial \theta_+^a} + \psi_+^b \frac{\partial}{\partial \theta_+^b} \]

we obtain

\[ \frac{\partial}{\partial \theta_+^b} \psi_+^a = 0, \quad (4.16) \]

\[ \psi_+^b \frac{\partial}{\partial \theta_+^b} \mathcal{L}_+^{+a} - \mathcal{L}_+^{+b} \frac{\partial}{\partial \theta_+^b} \psi_+^a + \psi_+^b \frac{\partial}{\partial \theta_+^b} \mathcal{L}_+^{+a} = 0. \quad (4.17) \]

Equations (4.16) mean that the vector field \( \psi_+ \) has arbitrary holomorphic components \( \psi_+^a \) in a complex holonomic basis \( \{ \partial/\partial \theta_+^a, \partial/\partial \theta_+^b \} \) of vector fields on \( P_+ \) and fixed \((0,1)\) components \( \psi_+^b \) determined by eqs. (4.17). Analogously, solutions \( \psi_- \) of eqs. (4.13) are in a one-to-one correspondence with holomorphic vector fields on \( P_- \).

**Remark.** The quotient space of the space of solutions \( \psi \) to eqs. (4.10) by the subspace of solutions \( \psi \pm \) to eqs. (4.13) can be described in terms of sheaf cohomology groups and Kodaira-Spencer deformation theory. For discussion in Euclidean signature and references see [24].

### 4.2 Transformations of tetrads on self-dual manifolds

The explicit form of the transformations of a (conformal) SD tetrad \( \{ W_A, W_A \} \) on \( U \subset M \) can easily be obtained from formulae (4.4)-(4.13). Namely, let us choose any solution \( \psi \) of eqs. (4.10) and split it by formulae (4.11)-(4.13). Then we have

\[ \delta_\psi W_1 = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_+, W_1 - \lambda W_2] = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_-, W_1 - \lambda W_2] = [\psi_0^+, W_1] = [\psi_0^-, W_1] - [\psi_1^-, W_2], \quad (4.18) \]

\[ \delta_\psi W_2 = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_+, W_2 - \lambda W_1] = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_-, W_2 - \lambda W_1] = [\psi_0^+, W_2] = [\psi_0^-, W_2] - [\psi_1^-, W_1], \quad (4.19) \]

\[ \delta_\psi W_1 = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_-, W_1 - \frac{1}{\lambda} W_2] = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_+, W_1 - \frac{1}{\lambda} W_2] = [\psi_0^+, W_1] = [\psi_0^-, W_1] - [\psi_1^+, W_2], \quad (4.20) \]

\[ \delta_\psi W_2 = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_-, W_2 - \frac{1}{\lambda} W_1] = \oint_{S^1} \frac{d\lambda}{2\pi i} [\psi_+, W_2 - \frac{1}{\lambda} W_1] = [\psi_0^+, W_2] = [\psi_0^-, W_2] - [\psi_1^-, W_1], \quad (4.21) \]

where the contour \( S^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) circles once around \( \lambda = 0 \) and \( \psi_0^\pm \) are coefficients in the Fourier series (4.13).

Let us consider solutions \( \psi_\pm \) from (4.9)-(4.13) such that \( \psi = \psi_+ - \psi_- = 0 \) on \( P_0 \). The condition \( \psi_+ = \psi_- \) on \( P_0 \) means that \( \psi_\pm \) are the restrictions \( \psi_+ = \tilde{\varphi}|_{P_+}, \psi_- = \tilde{\varphi}|_{P_-} \) of a globally defined vector field \( \tilde{\varphi} \) on \( \tilde{P} \simeq U \times \mathbb{C}P^1 \). Such vector fields \( \tilde{\varphi} \) are the pull-back of vector fields
\[ \varphi = \varphi^\mu(x) \partial_\mu \in \text{sdiff}(M) \] to \( \tilde{P} \). They have the form \( \tilde{\varphi} = \varphi + \varphi^\lambda(x, \lambda) \partial_\lambda \), where \( \varphi^\lambda(x, \lambda) \) are quadratic in \( \lambda \). For the vector fields \( \tilde{\varphi} \) we have \( \varphi(x) \equiv \psi^0_+ = \psi_0^- \) and from formula (4.18)-(4.21) we obtain

\[ \delta_\varphi W_A = [\varphi, W_A], \quad \delta_{\tilde{\varphi}} W_A = [\tilde{\varphi}, W_A]. \quad (4.22) \]

The transformations (4.22) coincide with the transformations (3.5) generated by the group \( \text{SDiff}(M) \) of volume-preserving diffeomorphisms of \((M, \epsilon)\).

To see a connection with the previous “recursion” descriptions of symmetries of the SDG equations on complex 4-manifolds we consider a vector field \( \rho \in \text{sdiff}(M) \) such that

\[ \delta_0^\rho W_A := [\rho, W_A] = 0, \quad \delta_0^\rho W_\bar{A} := [\rho, W_\bar{A}] \neq 0. \quad (4.23) \]

Then we choose a solution \( \psi_\rho \) of eqs. (4.10) on \( P_0 \) such that

\[ \psi^0_\rho = -\rho, \quad (4.24) \]

where \( \psi^0_\rho \) is zero component in the Fourier series (4.11). By substituting the expansion (4.11) of \( \psi_\rho \) in a Fourier series into formula (4.10), we obtain the recursion relations

\[ [W_1, \psi^{n+1}_\rho] = [W_2, \psi^n_\rho], \quad [W_2, \psi^{n+1}_\rho] = [W_1, \psi^n_\rho]. \quad (4.25) \]

It is easy to see that if we put

\[ \delta^n_\rho W_A := 0, \quad \delta^n_\rho W_\bar{A} := [W_A, \psi^n_\rho], \quad (4.26) \]

then eqs. (4.25) and formulae (4.26) exactly reproduce the symmetries of the SDG equations known before (see [6] and references therein).

The transformations (4.26) can be obtained from formulae (4.18)-(4.21). Namely, notice that if \( \psi_\rho \) satisfies eqs. (4.10), then for any \( n \in \mathbb{Z} \) the vector field

\[ \lambda^{-n} \psi_\rho \]

also satisfies eqs. (4.10). Then split the vector field (4.27) according to formulae (4.11)-(4.13),

\[ \lambda^{-n} \psi_\rho = (\lambda^{-n} \psi_\rho)_+ - (\lambda^{-n} \psi_\rho)_-, \quad (4.28) \]

and using (4.14) choose the gauge \( (\lambda^{-n} \psi_\rho)_+^0 = 0 \), i.e.

\[ \lambda^{-n} \psi_\rho = (\lambda^{-n} \psi_\rho)_+ - (\lambda^{-n} \psi_\rho)_- = (\lambda \psi^{n+1}_\rho + \lambda^2 \psi^{n+2}_\rho + ...) - (-\psi^n_\rho - \lambda^{-1} \psi^{n-1}_\rho - ...). \quad (4.29) \]

Then we have

\[ (\lambda^{-n} \psi_\rho)_+^0 = 0, \quad (\lambda^{-n} \psi_\rho)_-^0 = -\psi^n_\rho, \quad (4.30) \]

and formulae (4.18)-(4.21) are reduced to formulae (4.26). Thus the known symmetries (4.27) of the SDG equations are generated by the vector fields (4.27) on the space \( P_0 \approx U \times S^1 \) by formulae (4.18)-(4.21). We shall return to a correspondence between the ‘twistor’ and ‘recursion relations’ approaches to the description of symmetries when considering the Plebanski equations.
4.3 Symmetries of the Plebanski equations

Symmetries of the first Plebanski equation (3.12) can be obtained by substituting the explicit form (3.10), (3.11) of a tetrad \{W_A, W_A\} parametrized by a real function \( \Omega \) into formulae (4.11), (4.11), (4.18)-(4.21) or (4.23), (4.23), (4.27)-(4.30). If we want to preserve the gauge \( W_A = \partial_A \), then we should impose the conditions \( \delta \psi W_A = 0 \). From the formulae of Sect.4.2 it follows that for this it is sufficient to choose always \( \psi^0_+ = 0 \) in the splitting (4.11)-(4.13). In particular, let us consider divergence-free vector fields \( \rho = \rho A \delta A \) satisfying the equations \([\partial_A, \rho] = 0\) and generating the algebra that we denote by \( \text{sdiff}_C \). Then if we put \( (\psi_\rho)_+ = -\psi^0_\rho = \rho \), then by formulae (4.24)-(4.30) we obtain the well-known Lie algebra of symmetries \( \text{sdiff}_C \) or \( \text{sdiff}_C \otimes \mathbb{C} [\lambda, \lambda^{-1}] \) (cf. [3]).

Notice that symmetries \( \delta \psi W_A \) are connected by formulae (4.18)-(4.21) with Lie derivatives along local vector fields \( \psi \_\pm \) on \( \mathcal{P}_\pm \). Having the explicit form of diffeomorphism-type transformations \( \delta \psi : W_A \to \delta \psi W_A \), one can set up a problem of finding functional transformations \( \delta \psi : \Omega \to \hat{\delta} \psi \Omega \) of the Kähler potential \( \Omega \) of a SD metric inducing the same change of metric (3.16) and do not use a tetrad \( \{W_A, W_A\} \). Here we used formulae (3.14) and (4.18)-(4.21). In general, if we use only the symmetric \( \Omega \) that the equations (3.12), (3.14) as under the action by \( \delta \psi \) on \( W_A \). In other words, we want to find such \( \hat{\delta} \psi \Omega \) that the equations

\[
\varepsilon^{AB} \varepsilon^{CB} \partial_B \partial_B (\hat{\delta} \psi) \partial_C = \delta \psi W_A = [\psi^0_-, W_A] = [\psi^0_-, \varepsilon^{AB} \varepsilon^{CB} \partial_B \partial_B \Omega \partial_C]
\]

are satisfied. Here we used formulae (3.14) and (4.18)-(4.21). In general, if we use only the metric (3.14) and do not use a tetrad \( \{W_A, W_A\} \), then we can consider functional transformations \( \hat{\delta} : \Omega \to \hat{\delta} \Omega \) by themselves without considering the transformations of a tetrad.

Consider the transformation \( \hat{\delta} : \Omega \to \hat{\delta} \Omega \) and substitute \( \Omega + \hat{\delta} \Omega \) into the Plebanski equation (3.12). Leaving terms with \( \hat{\delta} \Omega \) of the power not higher than one (linearization), we see that \( \hat{\delta} \Omega \) satisfy the ultrahyperbolic wave equation

\[
\mathbf{g}^{\mu \nu} \nabla_\mu \nabla_\nu \hat{\delta} \Omega = \frac{1}{\sqrt{\text{det} g}} \partial_\mu (\sqrt{\text{det} g \mathbf{g}^{\mu \nu} \partial_\nu \hat{\delta} \Omega}) = 0
\]

(4.32)

where we used formulae (3.14) and (3.16). Thus, finding all infinitesimal symmetries of the first Plebanski equation is reduced to finding all solutions to the wave equation in a self-dual background.

The general solution of the wave equation on Riemannian manifolds with SD conformal structures was described by Hitchin [25] in twistor terms and used by Park [3] in discussion of symmetries to the first Plebanski equation on complex 4-manifolds. In the case of ultrahyperbolic signature (2,2), the general solution of eq.(4.32) can also be described in twistor terms.

Consider real analytic functions \( \phi^1_0(x, \lambda), \phi^2_0(x, \lambda) = \phi^3_0(x, \lambda), \phi^4_0(x, \lambda) = \lambda = \exp(i\gamma) \) satisfying eqs. (3.27), (3.28) on \( \mathcal{P}_0 = U \times S^1 \). Let us act on eqs. (3.27) by the vector field \( W_1 \), on eqs. (3.28) by the vector field \( -\lambda W_2 \) and sum them. Taking into account that the vector fields \( \{W_A, W_A\} \) have the form (3.10), (3.11) and the function \( \Omega \) satisfies the Plebanski equation (3.12), we obtain that functions \( \phi^0_0(x, \lambda) \) satisfy the wave equation (4.32) on \( U \) for any \( \lambda = \phi^0_0 \in S^1 \). Therefore an arbitrary complex-valued smooth function \( F(\phi^1_0, \phi^2_0, \phi^3_0) \) of \( \phi^0_0 \) also satisfies eq.(4.32). As a function of \( \phi^0_0 \) it represents a free function on the 3-dimensional real twistor space \( \mathcal{T}(U) \) of \( U \) introduced in Sect.3.4.

Expanding a smooth function \( F(\phi^1_0, \phi^2_0, \phi^3_0, \lambda) \) in a Fourier series in \( \lambda \), we have

\[
F(\phi^1_0(x, \lambda), \phi^2_0(x, \lambda), \lambda) = F(x, \lambda) = \sum_{n=-\infty}^{\infty} \lambda^n F^n(x).
\]
Then general smooth complex solution of eq. (4.32) has the form
\[ \hat{\delta} F \Omega = F^0(y^A, y^\bar{A}) = \oint_{S^1} \frac{d\lambda}{2\pi i\lambda} F(\phi^1_0, \phi^2_0, \lambda), \] (4.34)
where \( S^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). Substituting the expansion (4.33) into the equations \( \mathcal{L}^0_A F = 0 \) which follow from eqs. (3.27), (3.28) we have the recursion relations
\[ W_1 F^{n+1} = W_2 F^n, \quad W_2 F^{n+1} = W_1 F^n, \] (4.35)
where \( n \in \mathbb{Z} \). Using eqs. (4.35), one can find \( F^n \) for any \( n \) by choosing \( F^0 \) as a ‘seed solution’.

We call the map
\[ \mathcal{R} : F^n \mapsto F^{n+1} \] (4.36)
the recursion operator. This operator is not unique because of the ambiguity in the inversion of \( W_1 \) and \( W_2 \). Moreover, in the absence of boundary conditions, \( F^n \) is determined only up to the addition of a function \( f(y^A) + g(y^\bar{A}) \), where \( f \) and \( g \) are arbitrary functions of \( y^A \) and \( y^\bar{A} \), respectively. In terms of the recursion operator \( \mathcal{R} \) we have
\[ F^{n+1} = \mathcal{R} F^n \] (4.37)
and by iterating we obtain \( F^n = \mathcal{R}^n F^0 \).

To see a correspondence with the symmetries known before and described e.g. in \cite{7}, let us consider a complex divergence-free vector field \( \rho = \rho^A \partial_A \) satisfying the equations \( [\partial_{\bar{A}}, \rho] = 0 \). Substitute \( \delta^0_\rho W_A = [\rho, W_A] \) into eqs. (4.31), (4.32) and take into account that \( \partial_A \rho^A = 0 \). After short calculations we see that
\[ \hat{\delta}^0_\rho \Omega = \rho^A \partial_A \Omega \] (4.38)
satisfy eqs. (4.31) and define a trivial complex solution of eq. (4.32). To generate nontrivial symmetries of the Plebanski eq. (3.12), let us take the complex solution (4.34) of eq. (4.32) and choose a function \( F = F_\rho = \sum_n \lambda^n F^n_\rho \) such that \( F^0_\rho = \rho^A \partial_A \Omega = \hat{\delta}^0_\rho \Omega \). Functions \( F^n_\rho \) for \( n \neq 0 \) can be found by using the recursion relations (4.33)-(4.34). Then
\[ \hat{\delta}^n_\rho \Omega := F^n_\rho \] (4.39)
satisfy eq. (4.32) and therefore are symmetries of the first Plebanski equation. Formula
\[ \rho^A_{n+1} = \varepsilon^{BA} \partial_B F^n_\rho \]
connects functions \( F^n_\rho \) with functions \( \rho^A_{n+1} \) used for description of symmetries to the first Plebanski equation in \cite{3}. The above symmetries are in a one-to-one correspondence with coefficients \( F^n_\rho \) in the expansion of the function \( F_\rho(\phi^1_0, \phi^2_0, \lambda) \) in \( \lambda \) and generate the affine Lie algebra \( \text{diff}_2^C \otimes \mathbb{C}[\lambda, \lambda^{-1}] \).

Notice that the wave equation (4.32) is invariant under the change \( y^A \leftrightarrow y^\bar{A} \). If \( \eta = \eta^\bar{A} \partial_{\bar{A}} \) is any divergence-free vector field satisfying \( [\partial_A, \eta] = 0 \), then
\[ \hat{\delta}^0_\eta \Omega := \eta^\bar{A} \partial_{\bar{A}} \Omega \] (4.40)
is also a solution of eq. (4.32). Notice that vector fields \( \rho \) from (4.38), (4.39) and \( \eta \) from (4.40) are independent, commute and generate the Lie algebra \( \text{diff}_2^C \oplus \text{diff}_2^C \simeq w_\infty \oplus w_\infty \). Taking
\[ F^0_\eta := \eta^A \partial_A \Omega \] as a starting point of the recursion procedure (4.35)-(4.37), we introduce \( F^n_\eta \) and \( F_\eta = \sum_n \lambda^n F^n_\eta \). Then we obtain an infinite number of new symmetries by putting \[ \hat{\delta}^n_\eta \Omega := F^n_\eta. \] These symmetries form the algebra \( \text{diff} \mathbb{C}_2 \otimes \mathbb{C}[\lambda, \lambda^{-1}] \).

To introduce transformations preserving the reality of the Kähler potential, let us consider a function \( F(\phi^1_0, \phi^2_0, \lambda) \) that is real analytic and therefore extendable to a holomorphic function \( F(\phi^1_0, \phi^2_0, \lambda) \) on a neighbourhood \( U_0 \) of \( T(U) \) in \( \mathcal{P} \), where it satisfies
\[
F(\phi^1_0(x, \lambda^{-1}), \phi^2_0(x, \lambda^{-1}), \lambda^{-1}) = F(\phi^1_0(x, \lambda), \phi^2_0(x, \lambda), \lambda).
\] (4.42)

On \( \mathcal{P}_0 \) such a function is real, i.e.
\[ F(\phi^1_0, \phi^2_0, \lambda) = F(\phi^1_0, \phi^2_0, \lambda), \]
and by formula (4.34) it gives the general real analytic solution \( \hat{\delta}_F \Omega \) of eq.(4.32) since \( F^0 = F^0 \).

Indeed, in terms of components \( F^n \) of the expansion of the function \( F \) in \( \lambda \), the reality conditions (4.42) have the form
\[ \overline{F^n} = F^{-n}. \]

Hence \( F^0 = F^0 \) and formula (4.34) gives a real solution of eq.(4.32).

Having in mind all the above, let us take as \( \eta \) in formulae (4.10) a vector field complex conjugate to the vector field \( \rho \) from formulae (1.38), (1.39), i.e. choose \( \eta = \overline{\rho} \). Then starting with \( F^0_{\overline{\rho}} = \overline{\rho}^A \partial_A \Omega \) and using eqs.(4.35)-(4.37), one can introduce a function \( F_{\overline{\rho}} \) conjugate to \( F_{\rho} \) in a sense that
\[ F_{\rho}(x, \lambda^{-1}) = F_{\overline{\rho}}(x, \lambda) \iff \overline{F^n_{\rho}} = F^{-n}_{\overline{\rho}}. \]

After this we can introduce symmetries
\[ \Delta^n_{F_{\rho} + F_{\overline{\rho}}} \Omega = (\hat{\delta}^n_{F_{\rho}} + \hat{\delta}^{-n}_{F_{\overline{\rho}}}) \Omega = F^n_{\rho} + F^{-n}_{\overline{\rho}} = F^n_{\rho} + \overline{F^n_{\overline{\rho}}}, \]
\[ \Delta^n_{i(\rho - \overline{\rho})} \Omega = i(\hat{\delta}^n_{\rho} - \hat{\delta}^{-n}_{\overline{\rho}}) \Omega = i(F^n_{\rho} + F^{-n}_{\overline{\rho}}) = i(F^n_{\rho} - \overline{F^n_{\overline{\rho}}}), \]
(4.43)
(4.44)

preserving the reality of \( \Omega \).

Symmetries of the second Plebanski equation (3.41) can be described analogously. Namely, linearizing eq.(3.41), we obtain the wave equation
\[ (\partial_1 \partial_1 - \partial_2 \partial_2 + 2 \partial_1 \partial_2 \Psi \partial_2 \partial_2 - \partial^2_1 \Psi \partial_1^2 - \partial^2_2 \Psi \partial_2^2) \hat{\delta} \Psi = 0, \]
(4.45)

where we use the metric inverse to the metric (3.42). Using further the linear system (3.32) or (3.34), we can introduce real analytic functions \( \lambda^1_0(x, \zeta), \lambda^2_0(x, \zeta), \lambda^3_0(x, \zeta) = \zeta \) on \( \mathcal{P}_0, \zeta \in \mathbb{R} \cup \infty \).

From formula (3.30) for \( \lambda = \exp(i\gamma) \) we obtain
\[ \zeta = \tan(\frac{\gamma}{2}). \]

By introducing a smooth function \( F(\lambda^1_0(x, \gamma), \lambda^2_0(x, \gamma), \gamma) \), one can represent a general solution of eq.(4.43) in the form
\[ \hat{\delta}_F \Psi = \int_0^{2\pi} \frac{d\gamma}{2\pi} F(\lambda^1_0, \lambda^2_0, \gamma). \]
For discussion of this formula in the case of the flat space $\mathbb{R}^{2,2}$ see e.g. [23, 15]. One can introduce symmetries generated by vector fields on $U$ by formulae analogous to (4.38), (4.39), (4.40), (4.41) and (4.43), (4.44), that we shall not do.

4.4 Relationships between SDG symmetries

We shall show here that all symmetries of the SDG equations described in Sections 4.1-4.3 are connected with each other.

In Sect.3.4 we mentioned that all information about a SD metric $g$ on $U$ is encoded in transition functions $f_{\pm}^a$ on the space $P_0$, $a = 1, 2, 3$. Accordingly, infinitesimal variations of the SD metric are encoded in variations $\delta f_{\pm}^a$ of these transition functions. In its turn, $\delta f_{\pm}^a$ are nothing but components $\psi^a$ in the expansion

$$\psi = \psi^a \frac{\partial}{\partial \phi_+^a} + \psi^\bar{a} \frac{\partial}{\partial \phi_-^\bar{a}}$$

in the basis $\{\partial/\partial \phi_+^a, \partial/\partial \phi_-^\bar{a}\}$ of a vector field $\psi$ satisfying eqs.(4.10), i.e.

$$\delta f_{\pm}^a = \psi^a.$$

Analogously, the holomorphic components $\psi_{\pm}^a$ of vector fields $\psi_{\pm}$ from formulae (4.11)-(4.13) define infinitesimal variations

$$\delta \phi_{\pm}^a = \psi_{\pm}^a$$

of the coordinates $\phi_{\pm}^a$ on $\tilde{P}_{\pm}$.

Recall that $\phi_+^3 = \lambda, \phi_-^3 = \lambda^{-1}$ and in Sections 4.1-4.3 we considered transformations such that $\delta \phi_{\pm}^3 = \psi_{\pm}^3 = 0 \Rightarrow \psi^3 = 0, \delta \lambda = 0$. The imposing of these restrictions is connected with the fact that under deformations of the space $\tilde{P}$ the holomorphic projection

$$\pi : \tilde{P} \to \mathbb{C}P^1$$

must be preserved. The choice of the gauge $\psi_{\pm}^3 = \psi^3 = \psi^\lambda = 0$ satisfies this condition and is always possible, which is proved in the twistor approach (see e.g. [17, 13]).

Notice that on fibres of the bundle (4.46) one can define a closed 2-form $\omega$, holomorphic on the space $\tilde{P}$ [17, 19]. In fact, the equations $d\omega = 0$ are equivalent to the SDG equations [24]. In the local coordinates $\phi_{\pm}^A$ on $\tilde{P}$ this 2-form is

$$\omega|_{\tilde{P}^+} = d\phi_+^1 \wedge d\phi_+^2 \quad \omega|_{\tilde{P}^-} = d\phi_-^1 \wedge d\phi_-^2.$$

On $P_0$ we have

$$d\phi_+^1 \wedge d\phi_+^2 = \lambda^2 d\phi_0^1 \wedge d\phi_0^2 = \lambda^2 d\phi_-^1 \wedge d\phi_-^2,$$

which means the quadratic dependence of $\omega|_{\tilde{P}^+}$ on $\lambda$. In other words, $\omega$ is a 2-form on $\tilde{P}$ with values in sections of the holomorphic line bundle $\pi^*O(2) \to \tilde{P}$ (see e.g. [21]). By deforming $\tilde{P}$ one should preserve $\omega$ (cf. [17, 19, 21]) and therefore vector fields $\psi$ satisfying (4.10) have to preserve not only a volume 4-form $\epsilon$ on $M$, but also the 2-form $\omega$ on $\tilde{P}$. Consider a neighbourhood $U_0$ of $P_0$, a basis $\{\partial/\partial \phi_0^a, \partial/\partial \phi_0^\bar{a}\}$ on $U_0$ and expand $\psi = \psi^A \partial/\partial \phi_0^A + \psi^\bar{A} \partial/\partial \phi_0^\bar{A}$, where it is taken into account that $\psi^3 = \psi^\lambda = 0$. Then we obtain

$$\mathcal{L}_{\psi} \omega = 0 \iff \psi^A \frac{\partial}{\partial \phi_0^A} = \varepsilon^{BA} \frac{\partial F^A}{\partial \phi_0^B} \frac{\partial}{\partial \phi_0^A}.$$

(4.47)
This means that the vector field $\psi$ acts on functions $G(\phi, \lambda)$ as a Hamiltonian vector field defined on fibres of the bundle (4.46). In (4.47) the Hamiltonian $F(\phi_{0}, \lambda) = F(\phi_{0}, \overline{\phi}_{0}, \lambda)$ is a real analytic function on $\mathcal{P}_0$ that represents a free function on the real twistor space $T(U)$ of $U$. Thus, formula (4.47) defines the connection between vector fields $\psi$ from (4.10), (4.11)-(4.13) and functions $F$ from (4.33), (4.34).

Notice that from formulae (4.47) it is easy to see the algebraic properties of the transformations $\hat{\delta}_F \Omega$ defined by eq. (4.34). Namely,

$$\{\hat{\delta}_F, \hat{\delta}_G\} = \hat{\delta}_{\{F,G\}} \Omega,$$

where

$$\{F,G\} := \varepsilon^{AB} \frac{\partial F}{\partial \phi^A_{0}} \frac{\partial G}{\partial \phi^B_{0}}$$

is the Poisson bracket on fibres of the bundle (4.46).

5 Self-dual gravity hierarchy

5.1 The solution space of the SDG equations

In what follows we will briefly discuss symmetries from the viewpoint of geometry of infinite-dimensional solution spaces. From now on, eqs. (3.7)-(3.9) will be called the SDG equations if a tetrad $\{W_{\bar{A}}, W_A\}$ is chosen in the form (3.10), (3.11). These equations are equivalent to the first Plebanski equation. So, the self-dual tetrads (3.10), (3.11) and self-dual metrics (3.16) are parametrized by a function $\Omega$ satisfying eqs. (3.12). Consider the space of all solutions to the SDG equations and denote it by $\mathcal{N}$. Then formula (3.16) defines the projection

$$p : \mathcal{N} \to \mathcal{M} \quad (5.1)$$

doing the space $\mathcal{N}$ on the space $\mathcal{M}$ of the self-dual metrics describing physical degrees of freedom of the SDG gravity.

Let us consider transformations of the space $\mathcal{N}$ into itself defined by the formula

$$\Omega \mapsto \Omega' = \Omega + f(y^A) + g(y^{\bar{A}}), \quad (5.2)$$

where $f$ and $g$ are arbitrary functions of $y^A$ and $y^{\bar{A}}$, respectively. It is easy to see that $\Omega$ and $\Omega'$ determine the same tetrad (3.10), (3.11) and metric (3.16) since functions $f(y^A) + g(y^{\bar{A}})$ belong to the kernel of the projection (5.1). It should be stressed that we consider the space $\mathcal{N}$ of local solutions to the SDG equations.

In Sections 4.1-4.4 we have described solutions $\delta \Omega$ of the linearized (around $\Omega$) SDG equations. From the geometric point of view, solutions of the linearized SDG equations are vector fields on the solution space $\mathcal{N}$, i.e. sections of the tangent bundle $T\mathcal{N}$. Therefore infinitesimal symmetries of the SDG equations form the algebra $\text{Vect}(\mathcal{N})$ of vector fields on $\mathcal{N}$. As it has been shown above, this algebra is isomorphic to the algebra of vector fields on the real twistor space $T(U)$.

Notice that describing symmetry algebras of the SDG equations generated by algebras of vector fields on $U \subset M$ is equivalent to defining homomorphisms of these algebras into the algebra $\text{Vect}(\mathcal{N})$. For instance, in Sections 4.2 and 4.3 we have actually described the homomorphism of the algebra $\text{sdiff}_2 \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ into the algebra $\text{Vect}(\mathcal{N})$. In other words, by considering some
given group $\mathcal{G}$ (e.g. the loop group $\text{LSDiff}_{2}^{\mathbb{C}} = \mathcal{C}^{\infty}(S^{1}, \text{SDiff}_{2}^{\mathbb{C}})$) as a candidate for the symmetry group of the SDG equations, one should say about defining its action on the solution space $\mathcal{N}$. Accordingly, one can consider the infinitesimal action of the group $\mathcal{G}$ on $\mathcal{N}$ and a homomorphism of its Lie algebra into the algebra $\text{Vect}(\mathcal{N})$ of vector fields on $\mathcal{N}$.

5.2 Abelian symmetries and flows on the solution space

We consider eqs. (5.7)-(5.9) with a tetrad of the form (3.10), (3.11), parametrized by a function $\Omega$. In Sections 4.2-4.4 we discussed symmetries of these equations generated by vector fields $\hat{\delta}$ and find all $\hat{\delta}$ with (5.4) and (5.5), (5.6) define the action of the algebra $\text{sdiff}_{\mathbb{C}}$ where $\hat{\psi}$ and $\hat{\psi}$ we shall find all $\hat{\psi}$ and substitute them into formulae (4.26), (4.27), (4.28), (4.29), (4.30) we set $\hat{\psi}(0)$, (4.31), (4.32), (4.33), (4.34) define the action of the algebra $\mathbb{C}^{2} \otimes \mathbb{C}[\lambda]$ on $\{W_{B}, W_{B}\}$ and $\Omega$.

In Sect.5.1 we discussed infinitesimal symmetries of the SDG equations as vector fields on the solution space $\mathcal{N}$. In particular, an infinite number of commuting symmetries (5.3), (5.4) corresponds to an infinite number of commuting vector fields $\hat{\delta}$ on $\mathcal{N}$ with the components $F_{A}^{n} = \hat{\delta}_{n}^{A}\Omega$ at the point $\Omega \in \mathcal{N}$, $n = 0, 1, \ldots$. To these vector fields we can correspond the system of differential equations

$$\frac{\partial}{\partial y_{A}} \Omega = F_{A}^{n}$$

(5.7)

on $\mathcal{N}$, where $t_{n}^{A} \in \mathbb{C}$ are complex parameters, $A = 1, 2$, $n = 0, 1, \ldots$. Let us also introduce the parameters $t_{1}^{1} \in \mathbb{C}$ by the formula

$$\frac{\partial}{\partial t_{1}^{1}} \Omega = \varepsilon_{A}^{B} \frac{\partial}{\partial y_{B}} \Omega,$$

(5.8)

where $\varepsilon_{B}^{A}$ are given by formulae (3.12). The dependence of $\Omega$ on $t = (t_{n}^{A})$ can be recovered by solving successively eqs. (5.7), (5.8). Solutions $\Omega(t) = \Omega(y^{1} + \varepsilon_{B}^{A} t_{n-1}^{B}, y^{1} + t_{0}^{A}, t_{1}^{1}, \ldots)$ of the dynamical systems (5.7), (5.8) are called integral curves or flows of vector fields $\hat{\delta}_{n}^{A}$. For small $t_{n}^{A}$’s we have

$$\Omega(t) = \Omega(t = 0) + \sum_{A,n} t_{n}^{A} F_{A}^{n} + O(t).$$

(5.9)

We can represent $\Omega$ as a function of the parameters $t_{n}^{A}$ with $n \geq -1$ alone, with dependence on the coordinates $y^{1}, y^{A}$ recovered by substituting $y^{1} + t_{1}^{1}$ for $t_{2}^{1}$, and so on. Recall that
solutions of the linearized SDG equations are vector fields tangent to the flows and eqs. (5.4), (5.8) are mutually consistent because the flows commute. We denote by $\Gamma_+$ the group corresponding to the algebra $\mathbb{C}^2 \otimes \mathbb{C}[\lambda]$ with the generators $\{\partial_A^n\}$. The parameters $t^A_n$ can be considered as coordinates on a subspace $\mathbb{T}$ of the solution space $\mathcal{N}$. Notice that the space $\mathcal{N}$ of local solutions to the SDG equations can be described as an orbit of the loop group $\text{LSDiff}_2^\mathbb{C}$, and the coordinates $t = (t^A_n) = (t^A_{-1}, t^A_0, t^A_1, \ldots)$ parametrize an orbit $\mathbb{T}$ of the abelian subgroup $\Gamma_+$ of the group $\text{LSDiff}_2^\mathbb{C}$. The action of this group on a seed solution $\Omega(t = 0)$ embeds it in a family of new solutions $\Omega(t)$, labelled by the parameters $t = (t^A_n)$. To sum up, the flows are generated by the translations along the vector fields $\partial^n_A = \partial/\partial t^n_A$ on the orbit $\mathbb{T}$ of the group $\Gamma_+$ acting on the solution space $\mathcal{N}$.

5.3 Higher flows and SDG hierarchy

Let us move on to more concrete description of flows on the solution space $\mathcal{N}$ of the SDG equations. First of all, using equations (4.25)–(4.30), we show how $\delta^n_A$ are connected with $\hat{\delta}^n_A$. We have

\[
\delta^n_A W_1 = \partial_1 \partial_2 \delta^n_A \Omega \partial_2 - \partial_2 \partial_2 \hat{\delta}^n_A \Omega \partial_1 = \partial_1 \partial_2 \delta^n_A \Omega \partial_2 - \partial_2 \partial_2 \hat{\delta}^n_A \Omega \partial_1 = \partial^n_A W_1,
\]

\[
\delta^n_A W_2 = \partial_1 \partial_1 \delta^n_A \Omega \partial_2 - \partial_2 \partial_1 \delta^n_A \Omega \partial_1 = \partial_1 \partial_1 \delta^n_A \Omega \partial_2 - \partial_2 \partial_1 \delta^n_A \Omega \partial_1 = \partial^n_A W_2.
\]

We see that $\delta^n_A W_B = \partial^n_A W_B$, $\delta^n_A \Omega = \partial^n_A \Omega$ and therefore we can identify $\hat{\delta}^n_A$ with $\hat{\delta}^n_A$ and do not write a hat over $\hat{\delta}$. On the other hand, from eqs. (4.25) we have

\[
\partial^n_A W_1 = \delta^n_A W_1 = [W_1, \psi^n_A] = \partial_2 \psi^n_A + 1,
\]

\[
\partial^n_A W_2 = \delta^n_A W_2 = [W_2, \psi^n_A] = \partial_1 \psi^n_A + 1,
\]

and substituting (5.10), (5.11) into (5.12), (5.13), one can show that

\[
\psi^{n+1}_A = \partial^n_A \psi,
\]

where

\[
\psi := \partial_1 \phi \partial_2 - \partial_2 \phi \partial_1,
\]

\[
\phi := \Omega - \Omega_0 = \Omega - (y^1 y^1 - y^2 y^2).
\]

The function $\phi$ is often used instead of $\Omega$ by considering the first Plebanski equation.

Now we should take into account that by definition $\partial_A \Omega = \partial^n_A \Omega$, $\partial_A \Omega = \varepsilon B \partial^{-1}_A \Omega$. Then we have

\[
W_1 = \partial_1 = \partial_2^{-1}, \quad W_2 = \partial_2 = \partial_1^{-1},
\]

\[
W_A = \partial^n_A + \psi^0_A, \quad \psi^0_A = \partial^n_A \psi,
\]

when acting on $\phi^n_A$. Let us introduce the operators

\[
\mathcal{L}^n_A := \partial^n_A - \lambda (\hat{\partial}^{n+1}_A + \psi^{n+1}_A),
\]

with $n = -1, 0, 1, \ldots$, which for $n = -1$ coincide with the operators $\mathcal{L}^+_A$ from eqs. (3.17). Then we consider the linear equations

\[
\mathcal{L}^m_A \phi^n_A = 0,
\]

where $\phi^n_A$ are complex coordinates on $\mathcal{P}_+ \subset \mathcal{P}$. The compatibility conditions of eqs. (5.20) have the form

\[
[\mathcal{L}^m_A, \mathcal{L}^n_B] = 0 \iff [\partial^m_A - \lambda (\hat{\partial}^{m+1}_A + \psi^{m+1}_A), \partial^n_B - \lambda (\hat{\partial}^{n+1}_B + \psi^{n+1}_B)] = 0.
\]
These equations can be rewritten as follows:

\[ \partial^m_A \psi^n_B + \partial^n_B \psi^{-m}_A = 0, \quad (5.22) \]
\[ \partial^n_A \psi^n_B + \partial^n_B \psi^{-n}_A + [\psi_A^{m+1}, \psi_B^{n+1}] = 0, \quad (5.23) \]
where \( m, n = -1, 0, 1, \ldots \).

Put \( m = n = -1 \) in eqs. (5.21)-(5.23) and use (5.14)-(5.16) and (5.17), (5.18). Then eqs. (5.21)-(5.23) coincide with the SDG equations which are reduced to the first Plebanski equation in the coordinates \( t_{-1}, t_{0}^A \).

\[ \partial^0_1 \partial^{-1}_1 \phi - \partial^0_1 \partial^{-1}_1 \phi + \partial^0_1 \partial^{-1}_1 \phi \partial^0_1 \partial^{-1}_1 \phi - \partial^0_1 \partial^{-1}_2 \phi \partial^0_1 \partial^{-1}_2 \phi = 0. \quad (5.24) \]

It is easy to see that for \( m = -1, n \geq 0 \) eqs. (5.21)-(5.23) coincide with eqs. (5.12), (5.13) defining symmetries of the SDG equations. Using (5.14)-(5.16), these equations can be rewritten as equations on the function \( \phi \),

\[ \partial^0_1 \partial^{-1}_A \phi - \partial^0_1 \partial^{-1}_B \phi + \partial^{-1}_1 \partial^0_1 \phi \partial^{-1}_2 \partial^0_1 \phi - \partial^{-1}_2 \partial^0_1 \phi \partial^{-1}_2 \partial^0_1 \phi = 0, \quad (5.25) \]
\[ \partial^0_2 \partial^{-1}_A \phi - \partial^0_2 \partial^{-1}_B \phi + \partial^{-1}_2 \partial^0_2 \phi \partial^{-1}_2 \partial^0_2 \phi - \partial^{-1}_2 \partial^0_2 \phi \partial^{-1}_2 \partial^0_2 \phi = 0. \quad (5.26) \]

If we differentiate eqs. (5.25) w.r.t. \( t^2_{-1} \) and eqs. (5.26) w.r.t. \( t^1_{-1} \) and subtract the second equation from the first equation, then we obtain the linearized first Plebanski equation with \( \partial^0_1 \phi = \partial^0_1 \phi \). Thus, it follows from eqs. (5.25), (5.26) that \( \partial^0_1 \phi \) satisfies to the linearized (around \( \phi \)) SDG equations. Finally, for \( m \geq 0, n \geq 0 \) eqs. (5.21)-(5.23) coincide with commutativity conditions for flows generated by symmetries \( \delta^0_A \phi = \partial^0_A \phi \) on the solution space \( N \) of the SDG equations.

To rewrite eqs. (5.21)-(5.23) in terms of the function \( \phi \) for all \( m, n \geq -1 \), let us substitute the expression (5.14) for \( \psi^{n+1}_A \) into eqs. (5.21)-(5.23). Then we have

\[ \partial^{m+1}_A \partial^n_B \phi - \partial^0_A \partial^{n+1}_B \phi + \partial^0_A \partial^{m}_B \partial^n_B \phi - \partial^0_A \partial^{m}_B \partial^n_B \phi = 0, \quad (5.27) \]
where \( m, n = -1, 0, 1, \ldots \). The main conclusion is that the function \( \phi \) depends on an infinite number of variables \( t = (t^A_n) \), four of which can be identified with the coordinates on \( U: t^A_{-1} = \epsilon^B_A y^B, t^A_0 = y^A \). Variables \( t^A_n, n \geq -1 \), parametrize an orbit \( T \) of the abelian symmetry group \( \Gamma \subset \text{LSDiff}_2 \) in the solution space \( N \) of the SDG equations. This interpretation of abelian symmetries and higher flows, according to which integrable equations are described as dynamical systems on an infinite-dimensional Grassmannian manifolds associated with loop groups, is standard in the theory of integrable systems (see e.g. [27, 28, 29, 30]). So solutions of any integrable equation depend on an infinite number of “times” and obey an infinite set of differential equations - hierarchy - generated by flows along the higher “times” (higher flows).

We shall call eqs. (5.27) the SDG hierarchy generated by recursion from the translations along \( \partial_1 = \partial^0_1 \) and \( \partial_2 = \partial^0_2 \). The vector fields \( \partial^0_A \) labelled by consecutive values of \( n \) are related by the recursion operator \( R: \partial^{n+1}_A = R \partial^{m}_A \Rightarrow \partial^0_A = R \partial^0_A \) (see Sect. 4.3 for the definition of \( R \)). One can also introduce an SDG hierarchy generated by inverse recursion from the translations \( \partial_1 \) and \( \partial_2 \). To obtain these equations we introduce an infinite number of complex parameters \( t^{-n}_{-1}, n = -1, 0, 1, \ldots \), so that \( \partial^0_1 \Omega = \epsilon^0_A \partial_B \Omega, \partial^0_2 \Omega = \partial_A \Omega \). We also introduce an infinite number of linear differential operators

\[ L^{-n}_A := \partial^{-n}_A - \frac{1}{\lambda} (\partial^{-n-1}_A + \psi^{-n-1}_A), \quad (5.28) \]
where $\psi^m_{\bar{A}}$ are some vector fields on $U$, and consider linear differential equations

$$\mathcal{L}^m_{\bar{A}} \phi^n_{\bar{A}} = 0,$$

(5.29)

where $\phi^n_{\bar{A}}$ are complex coordinates on $\bar{\mathcal{P}}_u \subset \mathcal{P}$, $a = 1, 2, 3$, $n = -1, 0, 1, 2, \ldots$. Notice that eqs. (5.29) with $n = -1$ can be obtained from eqs. (3.17) by the action of the operator $\tau$ of real structure (see Sect. 3.4), i.e. by the combination of map $\lambda \mapsto \bar{\lambda}^{-1}$ and complex conjugation.

The compatibility conditions of eqs. (5.29) are the equations

$$[\mathcal{L}^m_{\bar{A}}, \mathcal{L}^n_{\bar{B}}] = 0 \Leftrightarrow [\partial^m_{\bar{A}} - \frac{1}{\chi}(\partial^m_{\bar{A}} - \psi^m_{\bar{A}})][\partial^n_{\bar{B}} - \frac{1}{\chi}(\partial^n_{\bar{B}} - \psi^n_{\bar{B}})] = 0,$$

(5.30)

which are equivalent to the equations

$$\partial^m_{\bar{A}} \psi^n_{\bar{B}} - \partial^n_{\bar{B}} \psi^m_{\bar{A}} = 0,$$

(5.31)

$$\partial^{m-1}_{\bar{A}} \psi^n_{\bar{B}} - \partial^{n-1}_{\bar{B}} \psi^m_{\bar{A}} = 0,$$

(5.32)

It is not difficult to verify that for $m = n = -1$ eqs. (5.30)-(5.32) are equivalent to the first Plebanski equation, since $\psi^0_{\bar{A}} = \bar{\phi}_{\bar{A}}, \tilde{\psi} = \partial^0_{\bar{A}} \phi - \partial^0_{\bar{A}} \phi^n_{\bar{B}}$. For $m = -1, n \geq 0$ eqs. (5.30)-(5.32) are reduced to the equations

$$\partial^m_{\bar{A}} \bar{\psi}^{-n} = \delta^n_{\bar{B}} \bar{\psi}^{-n} = \varepsilon^n_{\bar{B}} \delta^n_{\bar{A}} \bar{\psi}^{-n},$$

(5.33)

defining symmetries of the SDG equations. These symmetries form the algebra $\mathbb{C}^2 \otimes \mathbb{C}[\lambda^{-1}]$ with the generators $\delta^n_{\bar{A}} = \delta^n_{\bar{A}}$, $n \geq -1$. In (5.33) $\bar{\psi}^{-n}$ is complex conjugate to $W_B$.

One can easily obtain from eqs. (5.33) and definitions (3.10), (3.11), that

$$\psi^{-n}_{\bar{A}} = \partial^{n-1}_{\bar{A}} \tilde{\psi},$$

(5.34)

where $\tilde{\psi}$ is the vector field complex conjugate to the vector field $\tilde{\psi}_{\bar{A}}$. By substituting (5.34) into (5.30)-(5.32), we obtain the equations

$$\partial^{m-1}_{\bar{A}} \bar{\phi} - \partial^{n-1}_{\bar{B}} \bar{\phi} + \partial^m_{\bar{A}} \partial^0_{\bar{A}} \phi - \partial^m_{\bar{A}} \partial^0_{\bar{B}} \phi = 0,$$

(5.35)

where $m, n \geq -1$. Now recall that imposing the reality conditions on symmetry transformations has led us in Sect. 4.3 to the conditions $\delta^n_{\bar{A}} \Omega = \bar{\partial}^n_{\bar{A}} \Omega$, where $\Omega$ is connected with $\phi$ by formula (5.10). But $\bar{\partial}^{-n}_{\bar{A}} \Omega = \partial^{-n}_{\bar{A}} \Omega = \delta^{-n}_{\bar{B}} \Omega$ and therefore we obtain that

$$\partial^{-n}_{\bar{A}} \Omega = \bar{\partial}^n_{\bar{A}}.$$

(5.36)

So eqs. (5.35) of the SDG hierarchy generated by $\partial_{\bar{A}}$ are complex conjugation of eqs. (5.27) and therefore do not contain new degrees of freedom and do not give any additional information in comparison with eqs. (5.27). As such, we shall consider only eqs. (5.27) and call them the SDG hierarchy.

### 5.4 Twistor geometry and truncated SDG hierarchy

The flows generated by $\partial_{\bar{A}}$ and $\partial^0_{\bar{A}}$ are space-time translations

$$\phi(t^{-1}_{\bar{A}}, t^0_{\bar{A}}, \ldots) \mapsto \phi(t^{-1}_{\bar{A}} + \alpha^{-1}_{\bar{A}}, t^0_{\bar{A}} + \alpha^0_{\bar{A}}, \ldots)$$
and therefore give rather trivial new solutions. But the higher flows generated by $\partial^p_A$ with $n \geq 1$ give less trivial new solutions. We have shown above that the SDG equations are embedded in an infinite system of partial differential equations, and every solution of the SDG equations can be extended to a solution of the infinite system of equations (5.27) of the SDG hierarchy.

To understand the geometric meaning of these equations, let us truncate the SDG hierarchy, i.e. suppose that

$$\partial^p_A \phi = 0 \text{ for } n > r,$$

where $r \geq 0$ is an integer. Then we have a finite number of $2(r + 1)$ linear equations (5.20) with $-1 \leq n \leq r - 1$ on complex functions $\phi^n_\alpha$ on $\hat{\mathcal{P}}_+ \subset \hat{\mathcal{P}}$ and a finite system of eqs. (5.27) with $m, n = -1, 0, ..., r - 1$ on a function $\phi$ of $2(r + 2)$ variables $t = (t^4_{-1}, t^4_0, ..., t^4_r)$. This finite system of equations on $\phi$ we shall call the SDG hierarchy up to level $r$, and denote it by SDG($r$). In particular, SDG(0) equations coincide with the first Plebanski equation, and the SDG hierarchy equations are obtained when $r \to \infty$ and will be denoted by SDG($\infty$).

The parameters $t^4_n$ with $n = -1, 0, ..., r$ are local coordinates on some complex manifold of dimension $2(r + 2)$ which we denote by $M_r$. In the special case $r = 0$, the space $M_0$ is a self-dual 4-manifold described locally by the Plebanski equations. Manifolds $M_r$ and their geometry have been described by Alekseevsky and Graev [31] in the differential geometry setting. We shall briefly describe their approach and compare it with our approach based on ideas of symmetry, integrability etc.

Recall that any holomorphic line bundle over the projective line $\mathbb{CP}^1$ is isomorphic to the holomorphic line bundle $\mathcal{O}(r + 1) \to \mathbb{CP}^1$ of Chern class $(r + 1)$ for some integer $r$. For $r \geq -1$ the space of holomorphic sections of $\mathcal{O}(r + 1)$ is $H^0(\mathbb{CP}^1, \mathcal{O}(r + 1)) \simeq S^{r+1} \mathbb{C}^2 = \mathbb{C}^{r+2}$. The space $\mathbb{C}^{r+2}$ is the space of irreducible representation of type $(\frac{1}{2}(r + 1), 0)$ of the group $SL(2, \mathbb{C})$ having noncompact real forms $SU(1, 1) \simeq SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R})$. A holomorphic vector bundle $\mathcal{N}$ of rank $l$ over the Riemann sphere $\mathbb{CP}^1$ is called $(r + 1)$-isotonic if it is decomposed into a direct sum of $l$ copies of the line bundle $\mathcal{O}(r + 1) : N \simeq l\mathcal{O}(r + 1) = \mathcal{O}(r + 1) \oplus ... \oplus \mathcal{O}(r + 1)$ [31]. We shall consider the case $l = 2$, i.e. the bundle $\mathcal{N} = \mathcal{O}(r + 1) \oplus \mathcal{O}(r + 1)$. The space of holomorphic sections $H^0(\mathbb{CP}^1, \mathcal{N})$ of this bundle $\mathcal{N}$ carries a tensor product structure (Grassmann structure), since

$$H^0(\mathbb{CP}^1, \mathcal{N}) = H^0(\mathbb{CP}^1, \mathcal{N} \otimes \mathcal{O}(-r - 1)) \otimes H^0(\mathbb{CP}^1, \mathcal{O}(r + 1)) \simeq \mathbb{C}^2 \otimes \mathbb{C}^{r+2}.$$  

On the space $\mathbb{C}^2$ we always define the antisymmetric bilinear form with components $\varepsilon_{AB}$ (a volume form). Therefore the automorphism group of the bundle $\mathcal{N} = \mathcal{O}(r + 1) \oplus \mathcal{O}(r + 1)$ is the group $SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C})$, where the left group $SL(2, \mathbb{C})$ acts irreducibly on $\mathbb{C}^2$ and the right group $SL(2, \mathbb{C})$ acts irreducibly on $\mathbb{C}^{r+2}$. Consider a complex 3-dimensional manifold $\mathcal{Z}$. Let $\pi : \mathcal{Z} \to \mathbb{CP}^1$ be a holomorphic fibration over the Riemann sphere $\mathbb{CP}^1$. A rational curve $P$ in $\mathcal{Z}$ is called $(r + 1)$-isotonic if its normal bundle $N_P = T\mathcal{Z}/TP$ is isomorphic to the bundle $\mathcal{O}(r + 1) \oplus \mathcal{O}(r + 1)$ over $\mathbb{CP}^1 \to \mathcal{Z}$ [31]. For such curves the space $H^0(P, N_P)$ of holomorphic sections of $N_P$ is isomorphic to the space $\mathcal{E}_P \otimes H_P \simeq \mathbb{C}^2 \otimes \mathbb{C}^{r+2}$. A section $\sigma : \mathbb{CP}^1 \to \mathcal{Z}$ is called $(r + 1)$-isotonic if the curve $\sigma(\mathbb{CP}^1)$ is $(r + 1)$-isotonic [31]. At last, let us consider the bundle $L = \mathcal{O}(2r + 2)$ over $\mathbb{CP}^1 \to \mathcal{Z}$ and its pull-back $\pi^* L$ to $\mathcal{Z}$. Denote by $\lambda^2_\pi(L)$ the sheaf over $\mathcal{Z}$ of holomorphic 2-forms on fibres of $\pi$ and put $\lambda^2_\pi(L) = \lambda^2_\pi \otimes \pi^* L$. A section $\omega \in H^0(\mathcal{Z}, \lambda^2_\pi(L))$ of the sheaf $\lambda^2_\pi(L)$ is called a relative symplectic structure of type $L$ on $\mathcal{Z}$ if it is closed and nondegenerate on fibres [31]. The integer $\deg L = c_1(L) = 2(r + 1)$ is called the weight of the structure $\omega$. 26
Then there exists a family of (structure.

(ii) the bundle admits a (r+1)-isotonic section σ of π,

(iii) there exists a holomorphic relative symplectic 2-form ω of weight 2(r + 1) on Z.

Then there exists a family of (r+1)-isotonic sections containing σ and the parameter space of holomorphic sections is a 2(r + 2)-dimensional complex manifold Mr with a natural $SL(2, \mathbb{C}) \otimes 1_{r+2}$-structure.

For proof see [31]. Notice only that the proof is based on a theorem of Kodaira [32] which asserts that if the sheaf cohomology group $H^1(\sigma(\mathbb{C}P^1), N_\sigma)$ vanishes then a sufficiently small deformation of the (r+1)-isotonic section σ remains a (r+1)-isotonic section and may be integrated to a deformation of the projective line $\sigma(\mathbb{C}P^1)$, which makes the parameter space of all holomorphic sections of Z a complex manifold Mr with the tangent space $T_{\sigma'}M_r$ at the point $\sigma' \in M_r$ isomorphic to

$$H^0(\sigma'(\mathbb{C}P^1), N_{\sigma'}) \simeq E_{\sigma'} \otimes H_{\sigma'} \simeq \mathbb{C}^2 \otimes \mathbb{C}^{r+2}.$$ (5.39)

Therefore the tangent bundle $TM_r$ is isomorphic to the tensor product $E \otimes H$ of a holomorphic vector bundle $E \to M_r$ of rank 2 with fibres $E_{\sigma'} \simeq \mathbb{C}^2$ at $\sigma' \in M_r$ and the trivial vector bundle $H = M_r \times \mathbb{C}^{r+2}$. The holonomy group $SL(2, \mathbb{C}) \simeq SL(2, \mathbb{C}) \otimes 1_{r+2}$ of the manifold $M_r$ is very “small”, which impose severe restrictions on geometry of $M_r$. In particular, a metric on $M_r$ is parametrized by the function $\phi(t_1^A, t_0^A, ..., t_r^A)$, and any metric connection always has a torsion except the case $r = 0$, when one can introduce the standard Levi-Civita connection. Notice that (r+1)-isotonic sections, that are real (i.e. preserved by $\tau$), are parametrized by a real 2(r+2)-dimensional submanifold of $M_r$.

How are the results by Alekseevsky and Graev connected with ours? What is the geometric meaning of linear equations (5.20) and the SDG(r) equations? Answers are the following. As a manifold $\mathcal{Z}$ we considered the 3-dimensional complex manifold $\bar{\mathcal{P}} = \mathcal{P}_+ \cup \mathcal{P}_0 \cup \mathcal{P}_-$ that is diffeomorphic to the manifold $U \times S^2$ as a real manifold (see Sect.4.4). This manifold $\bar{\mathcal{P}}$ is a holomorphic fibre bundle

$$\pi : \bar{\mathcal{P}} \to \mathbb{C}P^1$$ (5.40)

over the projective line $\mathbb{C}P^1$ (see Sect.4.4). Sections $\sigma$ of the bundle (5.40) over $\bar{\mathcal{P}}_+ \subset \mathbb{C}P^1$ are defined by the functions $\phi^A, \sigma(\bar{\mathcal{P}}_+) = (\lambda, \phi^A)$, and over $\bar{\mathcal{P}}_- \subset \mathbb{C}P^1$ by the functions $\phi^A, \sigma(\bar{\mathcal{P}}_-) = (\lambda^{-1}, \phi^A)$, where $\phi^A_{\pm}$ were introduced in Sect.3.3. The relative symplectic 2-form $\omega$ in the local coordinates $\phi^A_{\pm}$ on $\bar{\mathcal{P}}_\pm$ has the form

$$\omega|_{\bar{\mathcal{P}}_+} = d\phi^1_+ \wedge d\phi^2_+,$$

$$\omega|_{\bar{\mathcal{P}}_-} = d\phi^1_- \wedge d\phi^2_-,$$ (5.41)

and was introduced in Sect.4.4.

Equations (5.20) with $n = -1, 0, ..., r-1$ define (r+1)-isotonic sections $\sigma(\bar{\mathcal{P}}_+) = (\lambda, \phi^A(t_1^B, ..., t_r^B, \lambda))$ of the bundle (5.40). These sections exist if the compatibility conditions (5.21)-(5.23) are satisfied. Equations (5.21)-(5.23) are equivalent to eqs. (5.27) with $-1 \leq m, n \leq r-1$ (SDG(r) equations). Notice that if $\phi^A_{\pm}$ define a (r+1)-isotonic section of the bundle (5.40), the relative symplectic form $\omega$ takes values in the bundle $\pi^*\mathcal{O}(2r + 2)$ and therefore on $\mathcal{P}_0 = \mathcal{P}_+ \cap \mathcal{P}_-$ we have

$$d\phi^1_+ \wedge d\phi^2_+ = \lambda^{2r+2} d\phi^1_- \wedge d\phi^2_-.$$ (5.42)

This means that $\omega|_{\mathcal{P}_+}$ may be written as a polynomial of degree $2(r+1)$ in $\lambda \in \bar{\mathcal{P}}_+$. 27
In the “flat” limit, when $\phi = 0$ and the SDG($r$) equations are satisfied identically, solutions of equations (5.20) and (5.29) have the form

$$
\phi^A_+ = \sum_{n=-1}^{r} \lambda^{n+1} t_{n-r-n-1}^A, \quad \phi^A_- = \sum_{n=-1}^{r} \lambda^{n-r} t_{r-n-1}^A.
$$

(5.43)

This means that in the “flat” limit the space $\hat{\mathcal{P}}$ is biholomorphic to the bundle $\mathcal{O}(r + 1)\oplus\mathcal{O}(r + 1) \to \mathbb{C}P^1$, and the functions (5.43) define global sections of this bundle. To clarify the connection with representations of the groups $SL(2, \mathbb{C})$ and $SU(1, 1)$, we introduce on $\mathbb{C}P^1$ homogeneous coordinates $(\lambda^A) = (\lambda^1, \lambda^2)$ so that $\lambda = \lambda^2/\lambda^1$. We can then write the coordinates $t^A_n$ as $t^A_{A_1...A_{r+1}}$, symmetric over indices $A_1...A_{r+1}$, related to the coordinates (5.43) by

$$
\sum_{A_1...A_{r+1}} t^A_{A_1...A_{r+1}} \lambda^{A_1}...\lambda^{A_{r+1}} = (\lambda^1)^{r+1} \sum_{n=-1}^{r} \lambda^{n+1} t^A_{n,r-n} = (\lambda^1)^{r+1} \sum_{n=-1}^{r} \lambda^{n+1} t^A_{r-n-1}.
$$

(5.44)

Here we have introduced $t^A_{n,r-n} := t^A_{r-n-1}$.

The SDG($r$) equations always have a family of solutions containing the trivial solution $\phi = 0$ since due to the Kodaira theorem \[32\] there exist a family of $(r + 1)$-isotonic sections of the bundle (5.40) containing the sections (5.43), (5.44). In particular, eqs. (3.17) define 1-isotonic sections of the bundle (5.40), and such sections exist if $\phi$ satisfies the first Plebanski equation. The existence of solutions $\phi$ of eqs. (5.27) of the SDG hierarchy is equivalent to the existence of $(r + 1)$-isotonic sections of the bundle (5.40) for any integer $r \geq 0$.

6 BRST quantization of the closed N=2 string

6.1 Constraints, ghosts, and pictures

In Sections 4 and 5 we have described hidden symmetries of the self-dual gravity equations and the SDG hierarchy generated by the abelian symmetries. We now come back to the N=2 string to present hidden ‘string’ symmetries, which will be seen to originate from so-called picture-changing and spectral flow operators naturally occurring in the BRST approach. Later on, we will compare those string symmetries with SDG symmetries and find agreement.

The BRST procedure is most convenient for a manifestly $SU(1, 1)$ covariant quantization of the N=2 string \[33\]. In the N=2 superconformal gauge, the N=2 supergravity multiplet is reduced to its moduli plus the super Virasoro constraints

$$
T(z) = -\frac{i}{2} \partial \bar{y} \cdot \partial y - \frac{i}{4} \partial \psi^- \cdot \psi^+ - \frac{i}{4} \partial \psi^+ \cdot \psi^-,
$$

$$
G^+(z) = \partial \bar{y} \cdot \psi^+ , \quad G^-(z) = \partial y \cdot \psi^- ,
$$

$$
J(z) = \frac{i}{2} \psi^- \cdot \psi^+ ,
$$

(6.1)

and their right-moving copy ($\bar{T}, \bar{G}^\pm, \bar{J}$). We have switched to worldsheet light-cone coordinates $(z, \bar{z})$, so that $\partial \equiv \partial_z$ and $\psi$ is one-component Weyl. The dot product denotes the $SU(1, 1)$ invariant scalar product.

A corresponding set of ghost/antighost pairs is introduced,

$$
c, b \quad \gamma^\pm, \beta^\mp \quad c', b' \quad \tilde{c}, \tilde{b} \quad \tilde{\gamma}^\pm, \tilde{\beta}^\mp \quad c', b' ,
$$

(6.2)
in terms of which the BRST operator \( Q = \frac{d}{dz} j(z) + \bar{j}(\bar{z}) \) reads

\[
j = cT + \gamma^+ G^- + \gamma^- G^+ + c'J + c\partial cb + c\partial c' b' - 4\gamma^+ \gamma^- b + 2(\partial \gamma^- \gamma^- - \partial \gamma^+ \gamma^-) b' + c'(\gamma^+ \beta^- - \gamma^- \beta^+)
\]

\[
+ \frac{3}{4} \partial c(\gamma^+ \beta^- + \gamma^- \beta^+) - \frac{3}{4} c(\partial \gamma^+ \beta^- + \partial \gamma^- \beta^+) + \frac{1}{4} c(\gamma^+ \partial \beta^- + \gamma^- \partial \beta^+) \ . \quad (6.3)
\]

The canonical commutation resp. anticommutation relations for the matter and ghost fields are realized on a free field Fock space, which factorizes into a left- and a right-moving part. The left-moving (NS) vacuum is defined by

\[
\pi \equiv \partial \left( \alpha^A_n |0 \rangle = 0 = \bar{\alpha}^A_n |0 \rangle \right.
\]

\[
\alpha^A_n |0 \rangle = 0 = \bar{\alpha}^A_n |0 \rangle \quad \text{for} \quad n \geq 0 \ , \quad \psi^+_r |0 \rangle = 0 = \psi^-_r |0 \rangle \quad \text{for} \quad r \geq \frac{1}{2} \ ,
\]

\[
c_n |0 \rangle = 0 \quad \text{for} \quad n \geq 2 \ , \quad \gamma^+_r |0 \rangle = 0 \quad \text{for} \quad r \geq \frac{1}{2} \ , \quad c'_n |0 \rangle = 0 \quad \text{for} \quad n \geq 1 \ ,
\]

\[
b_n |0 \rangle = 0 \quad \text{for} \quad n \geq 1 \ , \quad \beta^+_r |0 \rangle = 0 \quad \text{for} \quad r \geq -\frac{1}{2} \ , \quad b'_n |0 \rangle = 0 \quad \text{for} \quad n \geq 0 \ . \quad (6.4)
\]

Its turns out, however, that the computation of string amplitudes requires a \( \mathbb{Z}^2 \)-fold copy of our (left-moving NS) Fock space, labelled by so-called pictures charges \((\pi_+, \pi_-)\). A second set \((\tilde{\pi}_+, \tilde{\pi}_-)\) of picture charges occurs for the right-moving Fock space. The vacuum \(|\pi_+, \pi_-\rangle\) of the left-moving \((\pi_+, \pi_-)\) Fock space differs from the above \(|0 \rangle \equiv |\pi_+, \pi_-\rangle\) by the relations

\[
\gamma^+_r |\pi_+, \pi_-\rangle = 0 \quad \text{for} \quad r \geq \frac{1}{2} - \pi_\pm \ , \quad \beta^+_r |\pi_+, \pi_-\rangle = 0 \quad \text{for} \quad r \geq \frac{1}{2} + \pi_\pm \ . \quad (6.5)
\]

Due to the commuting nature of the \((\gamma, \beta)\) ghosts, different vacua \(|\pi_+, \pi_-\rangle\) (and the Fock spaces built over them) are formally connected only through the action of distributions like \(\delta(\gamma^\pm)\) in ghost/antighost modes or by extending the Fock space via bosonization \([3, 34]\). Hence, different pictures are isomorphic but not identical. Unless stated otherwise, normal ordering is understood with respect to \(|0 \rangle = |\pi_+, \pi_-\rangle\). It will turn out that the picture degeneracy is redundant for the enumeration of physical string states but crucial for the understanding of the global symmetry structure.

It is useful to introduce picture operators \(\Pi_\pm\) which measure the picture charges,

\[
\Pi_+ |\pi_+, \pi_-\rangle = \pi_+ |\pi_+, \pi_-\rangle \ , \quad \Pi_- |\pi_+, \pi_-\rangle = \pi_- |\pi_+, \pi_-\rangle \ , \quad (6.6)
\]

and commute with \(Q\). Another important quantum number is the ghost number \(g = g_L + g_R\). We define the left-moving (chiral) ghost number \(g_L\) as the eigenvalue of

\[
G_L = -\oint [bc + b'c' + \beta^+ \gamma^- + \beta^- \gamma^+] - \Pi_+ - \Pi_- \quad (6.7)
\]

so that \(G_L |\pi_+, \pi_-\rangle = 0\), and likewise for \(G_R\). As usual, \(Q\) carries \(g=1\), and \(G_L , G_R\) commute with \(\Pi_\pm\).

An additional feature of \(N=2\) strings is the presence of spectral flow, which may be interpreted as an isometry of the Maxwell moduli space. Its effect on an asymptotic string state is a continuous shift in the monodromies (around the closed string) of all worldsheet spinors according to their Maxwell charge, thereby interpolating between NS and R sectors. Spectral flow acts independently on left- and right-movers. As it leaves the sum \(\pi \equiv \pi_+ + \pi_-\) invariant but continuously changes the difference \(\Delta \equiv \pi_+ - \pi_-\), it is not compact but (for \(\Delta=2\)) relates different (NS) pictures,

\[
S : \quad |\pi_+, \pi_-\rangle \longrightarrow |\pi_+ + 1, \pi_- - 1\rangle \ . \quad (6.8)
\]

29
Actually, this action corresponds to a singular gauge transformation creating a Maxwell instanton at the puncture and may be used to relate string amplitude contributions from different instanton sectors [35, 36]. For a given worldsheet topology \((J, M)\), however, the spectral flow isometry does not alter the amplitude \(A_{J,M}^{(n)}\) since in the sum over all external string legs the picture shifts and the instanton charges must cancel out, according to the selection rule
\[
\sum_{i=1}^{n} (\pi_{\pm})_{i} = \sum_{i=1}^{n} (\tilde{\pi}_{\pm})_{i} = \frac{1}{2} (J \pm M) - n = 2(\#\text{handles}) - 2 \pm \frac{1}{2} M .
\] (6.9)
Hence, it suffices to restrict oneself to NS sectors (i.e. integral pictures) only, as we do in this paper.

6.2 Cohomology for physical states

It is well known that physical states in gauge theories can be identified as BRST cohomology classes of a certain ghost number. For the closed \(N=2\) string, the relevant cohomology is subject to the subsidiary conditions
\[
(b_0 - \tilde{b}_0) |\psi\rangle = 0 \quad \text{and} \quad b'_0 |\psi\rangle = 0 = \tilde{b}'_0 |\psi\rangle
\] (6.10)
and may be termed \(b\)-semirelative and \(b'\)-relative. The BRST cohomology is graded by
\begin{itemize}
  \item ghost number \(g \in \mathbb{Z}\)
  \item picture numbers \((\pi_+, \pi_-) \in \mathbb{Z}^2\)
  \item spacetime momentum \((k^A, \bar{k}^A)\) or \(k^{\alpha\dot{\alpha}} \in \mathbb{R}^{2,2}\)
\end{itemize}
so that one may restrict the analysis to Fock states of a given ghost number, built on picture vacua \(|\pi_+, \pi_-; k\rangle\) dressed with momentum. It is important to distinguish the exceptional case of \(k = 0\) from the generic case \((k \neq 0)\) which contains the propagating modes.

It has been shown [9] that the generic BRST cohomology is non-empty only for
\begin{itemize}
  \item \(g = 2\) and \(g = 3\)
  \item any value for \((\pi_+, \pi_-)\)
  \item any lightlike momentum, \(\bar{k} \cdot k = 0\)
\end{itemize}
and is one-dimensional in each such case. Since the \(g=3\) class is merely conjugate to the \(g=2\) class under the Fock space scalar product, it does not represent a second physical state. Alternatively, we may eliminate it by passing to the \(b\)-relative cohomology, which imposes
\[
(b_0 + \tilde{b}_0) |\psi\rangle = 0
\] (6.11)
on top of the conditions (6.10). Since the relative Fock space factorizes into left- and right-movers, the Künneth theorem tells us that the \((b\text{- and } b'\text{-})\)-relative BRST cohomology also factorizes in the same way. For this reason, it suffices to analyze the relative chiral BRST cohomology which is graded by \((g_{ch}, \pi_+, \pi_-, k)\) and (for lightlike \(k \neq 0\)) nonzero only at \(g_{ch}=1\). In fact, the physical states (at \(g=2\)) are just products of left-moving and right-moving parts, with chiral ghost numbers \(g_L=g_R=1\).

\[\text{Here, } g_{ch} \text{ denotes } g_L \text{ or } g_R, \text{ as usual.}\]
Moreover, for \( k \neq 0 \) one may construct picture-raising and picture-lowering operators which commute with \( Q \) and do not carry ghost number or momentum \([10]\). Together with spectral flow, these operators may therefore be used to define an equivalence relation among all pictures. This projection of the BRST cohomology leaves us with a single, massless, physical mode, in accordance with the result of Ooguri and Vafa \([1]\). It is the string excitation corresponding to the self-dual deformation \( \phi \) of the metric background or to the prepotential \( \Psi \) for a self-dual Riemann tensor. Yet, the discussion of the global symmetries will provide a justification for distinguishing different pictures, since different picture-representatives of the physical mode will be connected with shifts \( \delta \phi \) resp. \( \delta \Psi \). For a related interpretation of the picture phenomenon, see \([37, 38]\).

6.3 Cohomology for global symmetries

It is perhaps less well known that the exceptional (zero-momentum) BRST cohomology at ghost number one harbors all unbroken global symmetry charges of the theory. The argument goes as follows \([39]\):

A current with components \( J_z \) and \( J_{\bar{z}} \) in a two dimensional theory (complex coordinates \( z \) and \( \bar{z} \)) is conserved, i.e. satisfies \( \partial J_z + \partial J_{\bar{z}} = 0 \), when the one-form

\[
\Omega^{(1)} = J_z dz - J_{\bar{z}} d\bar{z}
\]

is closed. The corresponding charge

\[
A = \oint_C \Omega^{(1)}
\]

is conserved if it has the same value for contours \( C \) and \( C' \) that are homologous, i.e. are the boundaries of some surface \( \Sigma \), \( \partial \Sigma = C - C' \). Current conservation implies charge conservation by Stokes' Theorem. In BRST quantization these relations are required to hold only up to BRST commutators. Current conservation then reads

\[
d\Omega^{(1)} = [Q, \Omega^{(2)}]
\]

for some two form \( \Omega^{(2)} = \Omega^{(2)}_z dz \wedge d\bar{z} \). BRST invariance of the charge requires \( [Q, \Omega^{(1)}] = d\Omega^{(0)} \). Applying \( Q \) to this relation implies that \( [Q, \Omega^{(0)}] \) is a constant. Furthermore, this constant must vanish, since otherwise the unit operator were BRST trivial. Summarizing, we have the descent equations

\[
\begin{align*}
[Q, \Omega^{(0)}] & = 0 , \\
[Q, \Omega^{(1)}] & = d\Omega^{(0)} , \\
[Q, \Omega^{(2)}] & = d\Omega^{(1)} .
\end{align*}
\]

Moreover, the whole formalism is unaffected by the replacements

\[
\begin{align*}
\Omega^{(0)} & \rightarrow \Omega^{(0)} + [Q, \alpha] , \\
\Omega^{(1)} & \rightarrow \Omega^{(1)} + d\alpha
\end{align*}
\]

where \( \alpha \) is some form of degree zero. The equations for \( \Omega^{(0)} \) are precisely the defining relations for the BRST cohomology (on operators instead of Fock states). Taking as \( \Omega^{(0)} \) some cohomology class of ghost number \( g \) one can then construct the forms \( \Omega^{(1)} \) and \( \Omega^{(2)} \) of ghost numbers \( g - 1 \) and
It is most natural to choose $\Omega^{(0)}$ to have ghost number one. This results in a charge of ghost number zero that can map physical states to physical states.

As a simple example let us consider target space translation $s$ in bosonic string theory. The only ghost number zero chiral cohomology class is the unit operator, which we may take as the right-moving piece of the closed string cohomology. As left-moving piece we must take a ghost number one state that also has vanishing momentum – the only candidate is $c \partial x^\mu$. The forms $\Omega^{(i)}$ take the form (suppressing the right-moving unit operator)

$$
\Omega^{(0)} = c \partial x^\mu, \quad \Omega^{(1)} = \partial x^\mu dz, \quad \Omega^{(2)} = 0.
$$

The charge is just the center-of-mass momentum operator $p^\mu = \oint dz 2\pi \partial x^\mu$.

To ensure that the charges $A$ generate symmetries on physical states, one must demand not only that they commute with $Q$ but also that their action is compatible with our subsidiary conditions, i.e.

$$
[A, b_0] = 0 = [A, b'_0] \quad \text{and} \quad [A, \tilde{b}_0] = 0 = [A, \tilde{b}'_0].
$$

Since the $(b$- and $b'$-)relative cohomology factorizes into chiral parts, we need only compute the exceptional relative chiral cohomology $H_{rel}^{ch}$. Poincaré duality identifies the cohomologies at

$$
(g_{ch}, \pi) \leftrightarrow (2-g_{ch}, -4-\pi),
$$

and spectral flow equates cohomologies differing only in the value of $\Delta$. Thus, we may restrict ourselves to $\pi \geq -2$ and ignore $\Delta$ and display the current knowledge in the following table

| $g_{ch}$ \(\pi\) | \(\dim H_{rel}^{ch}\) | \(\pi + 1\) | \(\pi + 2\) | \(\pi + 4\) |
|-----------------|----------------|-----------------|----------------|-----------------|
| $-2$            | 0              | 0               | 1              | 0               |
| $-1$            | 0              | 0               | 2              | 2               |
| 0               | 0              | 1               | 4              | $\geq 4$        |
| $\geq 1$        | $\geq \pi + 1$| $\geq 4(\pi + 1)$|

Since the multiplication of cohomology classes preserves the ghost number and picture gradings, the ghost-number zero chiral cohomology by itself forms a ring, the so-called ground ring $H_{rel}^{ch}(g=0)$.[10] This ground ring is rather important since in some sense it generalizes the unit operator, and it reveals much about the global symmetries. It is therefore quite remarkable that the ground ring of the $N=2$ string contains an increasing number of elements in higher pictures.

For convenience we change the picture labels from $(\pi_+, \pi_-)$ to “spin” labels $(j, m)$ via

$$
\pi_+ = j + m \quad \text{and} \quad \pi_- = j - m \quad \text{on operators}
$$

$$
\pi_+ + 1 = j + m \quad \text{and} \quad \pi_- + 1 = j - m \quad \text{on states}
$$

and likewise for the right movers. The picture offset of the canonical ground state $| -1, -1\rangle$ is responsible for the distinction. The BRST cohomology possesses a natural multiplication rule [10] which we denote by a dot. The (left-moving) ground ring is spanned by the basis elements

$$
O_{j,m,n} = (X_+)^{j+m} \cdot (X_-)^{j-m} \cdot H^n
$$

\[3\] Note that there is one error in the corresponding table of [9].
where the labels range over
\[ j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad \text{and} \quad m, n-m = -j, -j+1, \ldots, +j \quad . \]  

(6.22)

We have introduced the picture-raising operators \[ \Xi \]
\[X_\pm = c \partial \Theta (\beta^\pm) + \delta (\beta^\pm) (G^\mp - 4 \gamma^\pm b + 4 \partial \gamma^\pm b' + 2 \gamma^\pm \partial b') \]  

(6.23)
as well as picture-neutral formal operator \( H \) via
\[ X_+ \cdot H = X_- \cdot S \quad \text{and} \quad X_- \cdot H^{-1} = X_+ \cdot S^{-1} \quad , \]

(6.24)

composed from the spectral flow operator
\[ S = \frac{1}{4} \delta (\gamma^+) \delta (\beta^+) \epsilon_{AB} \psi^+ A \psi^+ B (1+c b') \]  

(6.25)

and its inverse
\[ S^{-1} = - \frac{1}{4} \delta (\beta^-) \delta (\gamma^-) \epsilon_{AB} \psi^- A \psi^- B (1-c b') \quad . \]

(6.26)
The \( \Delta = 0 \) representatives obtain for \( m=0 \) and \( j=\pi/2 \). It is important to note that, while \( S \) can be inverted, \( X_\pm \) do not have inverses \[ [\Xi] \]. This fact limits the range of \( n \) because each factor of \( H \) \((H^{-1})\) must be matched by a factor of \( X_+ \) \((X_-)\) to make sense.

Let us define yet another combination of picture-raising and spectral flow operators, namely
\[ Y_- = X_+ \cdot S^{-1} = X_- \cdot H^{-1} \quad , \]

(6.27)

which allows us to pull out a non-negative power \( \ell = j - m + n \) of \( H \) in
\[ O_{j,m,n} = (X_+)^{j+m} \cdot (Y_-)^{j-m} \cdot H^\ell =: O_{j,m}^\ell \quad \text{with} \quad \ell = 0, 1, \ldots, 2j \quad . \]

(6.28)
The chiral ground ring operators \( O_{j,m}^\ell \) form an infinite abelian algebra under the cohomology product,
\[ O_{j,m}^\ell \cdot O_{j',m'}^{\ell'} = O_{j+j',m+m'}^{\ell+\ell'} \quad . \]

(6.29)
The picture operators \( \Pi_{\pm} \) act as derivations on the ground ring,
\[ [\Pi_{\pm}, O_{j,m}^\ell] = (j \pm m) O_{j,m}^\ell \quad , \]

(6.30)
and represent only the tip of an iceberg, namely an infinite-dimensional algebra of derivations spanned by
\[ D_{j,m}^{\pm,\ell} = O_{j,m}^\ell \cdot (\Pi_{\pm}+1) \quad \text{and} \quad D_{j,m}^{-\ell} = O_{j,m}^\ell \cdot (\Pi_-+1) \quad . \]

(6.31)
The remaining commutators are
\[ [D_{j',m'}^{\pm,\ell'}, O_{j,m}^\ell] = (j\pm m) O_{j+j',m+m'}^{\ell+\ell'} \quad , \]
\[ [D_{j',m'}^{\pm,\ell'}, D_{j,m}^{\pm,\ell}] = (j\pm m) D_{j+j',m+m'}^{\pm,\ell+\ell'} - (j'+m') D_{j+j',m+m'}^{\pm,\ell+\ell'} \quad , \]
\[ [D_{j',m'}^{\pm,\ell'}, D_{j,m}^{-\ell}] = (j\pm m) D_{j+j',m+m'}^{-\ell+\ell'} - (j'-m') D_{j+j',m+m'}^{\pm,\ell+\ell'} \quad . \]

(6.32)

\[ \text{[footnote] Here,} \ \Theta \text{denotes the Heaviside step function. Note that} \ X_\pm \text{differ from the picture-raisers mentioned at the end of the previous subsection.} \]
The algebra of the ground ring and its derivations can be written more concisely in terms of polynomials in two variables \((x, y)\) and vector fields on the \((x, y)\) plane. We first remark that the neutral operator \(H\) can be expressed in terms of momentum operators \(P^A\) and \(\tilde{P}^A\), so that on momentum eigenstates it may be replaced by its eigenvalue, a function \(h(k)\). We also like to point out that the picture-raisers \(X_\pm\) do not carry any spacetime quantum numbers whereas the spectral flow \(S\) (and therefore \(H\)) transform non-trivially under “Lorentz” transformations \([36]\). To keep the spacetime properties manifest it is thus convenient to define

\[
x := X_+ = O^0_\frac{1}{2} + \frac{1}{2} \quad \text{and} \quad y := X_- = Y_+ = O^0_{-\frac{1}{2}} - \frac{1}{2} \cdot H \tag{6.33}
\]
as generators of a ring of polynomials in two variables and to allow for multiplication with arbitrary powers of \(h\). In these variables,

\[
O^\ell_{j,m} = (X_+)^{j+m} \cdot (X_-)^{j-m} \cdot H^{-(j-m)+\ell} = x^{j+m} y^{j-m} h^{-(j-m)+\ell} \tag{6.34}
\]
and the elementary derivations become

\[
\Pi_+ + 1 = x \partial_x \quad \text{and} \quad \Pi_- + 1 = y \partial_y \tag{6.35}
\]
representing the abelian part of the huge algebra of derivations \(D^\pm_\ell\) by vector fields on the \((x, y)\) plane, with elements

\[
p_x(x, y, h) x \partial_x + p_y(x, y, h) y \partial_y \tag{6.36}
\]
containing functions \(p_x\) and \(p_y\) polynomial in \(x\) and \(y\). Note that the translation generators \(\partial_x\) and \(\partial_y\) are missing.

For the complete story, one must combine left- and right-movers. As stated above, symmetry charges carry total ghost number one. Consulting our table, the only way to build a \(g=1\) representative is to combine a \(g_R=0\) class with a \(g_L=1\) class, or vice versa. Only counting the first possibility, we arrive at least at \(4(\pi+1)(\bar{\pi}+1)\) cohomology classes for non-negative values of \(\pi\) and \(\bar{\pi}\), while the \(\pi=1, -1, -2\) sectors are empty.

Since the canonical \(g_L=1\) representatives are just the products \(P^A \cdot O^\ell_{j,m}\) and \(\tilde{P}^A \cdot \tilde{O}^\ell_{j,m}\), with

\[
-iP^A = c \partial y^A - 2\gamma^- \psi^A \quad \text{and} \quad -i\tilde{P}^A = c \partial y^\tilde{A} - 2\gamma^+ \psi^\tilde{A} \tag{6.37}
\]
we obtain the \(4(2j+1)(2\bar{j}+1)\) operators

\[
\Omega^{(0)}_{j,m;\tilde{j},\tilde{m}} = P^A \cdot O^\ell_{j,m} \tilde{O}^\ell_{\tilde{j},\tilde{m}} \quad \text{and} \quad \Omega^{(1)}_{j,m;\tilde{j},\tilde{m}} = \tilde{P}^A \cdot \tilde{O}^\ell_{j,m} \tilde{O}^\ell_{\tilde{j},\tilde{m}} \tag{6.38}
\]
in the \((j, m; \tilde{j}, \tilde{m})\) sector with \(g=1\).

In order to find the symmetry charges \(A\), we have to insert our \(g=1\) zero-forms \(\Omega^{(0)}_{j,m;\tilde{j},\tilde{m}}\) and \(\Omega^{(0)}_{j,m;\tilde{j},\tilde{m}}\) into the descent equations \((6.15)\) and work out the corresponding one-forms \(\Omega^{(1)}_{j,m;\tilde{j},\tilde{m}}\) and \(\Omega^{(1)}_{j,m;\tilde{j},\tilde{m}}\), to be integrated around some contour. This computation has been performed in ref. \([7]\), with the result

\[
A^{A;\tilde{A}}_{j,m;\tilde{j},\tilde{m}} = \oint \frac{dz}{2\pi i} \left[ \int \frac{dw}{2\pi i} b(w) P^A(z) \cdot O^\ell_{j,m}(z) \tilde{O}^\ell_{\tilde{j},\tilde{m}}(z) \right] - \oint \frac{dz}{2\pi i} \left[ P^A(z) \cdot O^\ell_{j,m}(z) \int \frac{dw}{2\pi i} b(w) \tilde{O}^\ell_{\tilde{j},\tilde{m}}(z) \right] \tag{6.39}
\]
and likewise for $A_{j,m;j,m}^{A;\ell;\ell}$. Combining left- and right-movers, one arrives also at more general operators

$$B_{j,m;j,m}^{\pm,\ell} = O_{j,m}^{\ell} \cdot \tilde{O}_{j,m}^{\ell} \cdot (\Pi_{\pm} + 1) \quad \text{and} \quad B_{j,m;j,m}^{\pm,\ell} = O_{j,m}^{\ell} \cdot \tilde{O}_{j,m}^{\ell} \cdot (\tilde{\Pi}_{\pm} + 1)$$

which, together with the charges $A_{j,m;j,m}^{A;\ell;\ell}$ and $A_{j,m;j,m}^{A;\ell;\ell}$, form an enormous non-abelian algebra.

7 Symmetry transformations of physical states

For the discussion of string symmetries, it is revealing not to identify the physical state representatives in the various picture sectors but to distinguish the (boosted) picture-vacua $|\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle$ as different states. Clearly, those states are obtained by applying the ground ring operators $O_{j,m}^{\ell}$ and $\tilde{O}_{j,m}^{\ell}$ to the canonical representative, $|-1, -1; -1, -1; k\rangle$. Focusing on the left-moving part, one obtains

$$O_{j,m}^{\ell}|-1, -1; k\rangle = h(k)^{-(j-m)+\ell} |j+m-1, j-m-1; k\rangle$$

where the phase

$$h(k) = k^1/\tilde{k}^2 = k^2/\tilde{k}^1 = (h(k)^{-1})^*$$

is just the eigenvalue of our formal operator $H$. Hence, there is an equivalence

$$|j+m-1, j-m-1; k\rangle \sim x^{j+m} y^{j-m}$$

modulo powers of $h(k)$.

By virtue of this equivalence, the operators

$$D_{j,m}^{+,\ell} = h^{-(j-m)+\ell} x^{j+m+1} y^{-j-m} \partial_x \quad \text{and} \quad D_{j,m}^{-,\ell} = h^{-(j-m)+\ell} x^{j+m} y^{-j-m+1} \partial_y$$

have a natural action on states as well: they not only shift the quantum numbers $(\pi_+, \pi_-)$ of a state but also pull out an additional factor of $\pi_{\pm} + 1$ from it, since we have added unity to $\Pi_{\pm}$ to account for the picture offset of the $j=0$ ground state. Thus, we may alternatively use $D_{j,m}^{+,\ell}$ (and their right-moving images) to generate all physical states from $|0, -1; k\rangle$ and $|-1, 0; k\rangle$ (times their right-moving partners).

In order to find the transformation laws of the full physical states, we have to apply $A_{j,m;j,m}^{A;\ell;\ell}$ and $B_{j,m;j,m}^{\pm,\ell}$ to the canonical representative, $|\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle$. The action of these charges on the physical states factorizes.

$$A_{j,m;j,m}^{A;\ell;\ell} |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle = O_{j,m}^{\ell} \cdot \tilde{O}_{j,m}^{\ell} \cdot p^{A} |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle$$

$$= h(k)^{-(j-m)+\ell} x^{j+m} y^{-j-m+1} \partial_y (e^{\lambda_0} \cdot p^{A} |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle), \quad (7.5)$$

$$B_{j,m;j,m}^{\pm,\ell} |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle = O_{j,m}^{\ell} \cdot \tilde{O}_{j,m}^{\ell} \cdot (\Pi_{\pm} + 1) |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle$$

$$= h(k)^{-(j-m)+\ell} x^{j+m} y^{-j-m+1} \partial_y (e^{\lambda_0} \cdot (\Pi_{\pm} + 1) |\pi_+, \pi_-, \tilde{\pi}_+, \tilde{\pi}_-; k\rangle), \quad (7.6)$$

\footnote{In these equations, picture and spin labels are unrelated, in order to distinguish between the quantum numbers of the operator and those of the state.}
and similarly for \( A \to \bar{A} \) or \( B \to \bar{B} \). These transformations constitute an infinity of global symmetries which are unbroken in the flat Kleinian background. Their Ward identities constrain the tree-level scattering amplitudes so severely \([3]\) that all but the three-point function must vanish, consistent with the direct computations alluded to earlier.

To make contact with the abelian symmetries of self-dual gravity (see Sect.5), it suffices to consider the sub-algebra of symmetry charges \( \mathcal{A}^{j,00}_{j,m_1,m_1} \), i.e. putting \( \ell=0=\bar{\ell} \). It has the important property that only non-positive powers of the phase \( h(k) \) appear when acting by these charges on physical states. Moreover, since the factor \( h^{-(j+j-m-m)} \) in \((7.5)\) feels neither of the differences \( j-j \) and \( m-\bar{m} \), one can introduce states

\[
||j,m;k\rangle := \sum \sum \sum \langle j_1+m_1-1,j_1-m_1-1;j_2+m_2-1,j_2-m_2-1;k \rangle (7.7)
\]

which are symmetric tensor products of left- and right-moving states. These states form a \((2j+1)\)plet for a fixed value of \( j \) since \( m = -j, -j+1, \ldots, j \). We also introduce combinations of charges,

\[
P^j_m := \eta_{AA} \sum \sum \sum \mathcal{A}^{j,00}_{j_1+m_1,j_2,m_2} (7.8)
\]
corresponding to the symmetric tensor product states \((7.7)\). The action of these charges on the ground state

\[
||0,0;k\rangle = | -1, -1; -1, -1; k \rangle \tag{7.9}
\]

has the form

\[
\delta^j_m ||0,0;k\rangle := P^j_m ||0,0;k\rangle = k_A h^{-(j-m)} ||j,m;k\rangle . \tag{7.10}
\]

Note that for \( j=0 \) we find the translations \( \delta^A_0 ||0,0;k\rangle = k_A ||0,0,k\rangle \) as it should be.

Recall that, for the SDG equations, abelian symmetries generated by recursion from translations \( \partial/\partial y^A = \partial/\partial t^A_0 \) (\( = k_A \) in the momentum representation) read

\[
\delta^j_m \phi(y^B,\bar{y}^B,t_1^B,\ldots) = \frac{\partial}{\partial t^A_n} \phi(y^B,\bar{y}^B,t_1^B,\ldots) , \quad n = 0, 1, \ldots \tag{7.11}
\]

where \( t_1^A, \ldots, t_n^A, \ldots \) parametrize the moduli space of solutions \( \phi \) to the SDG equations (see Sect.5). If we impose on \( \phi \) the condition \((5.37)\), i.e. consider the finite multiplet of symmetries

\[
\delta^j_m \phi = \frac{\partial}{\partial t^A_{j-m}} \phi \tag{7.12}
\]

with \( j \) fixed and \( m = -j, \ldots, j \), then in the momentum representation we will have

\[
\delta^j_m \tilde{\phi}(k) = P^j_{j-m} \tilde{\phi}(k) \tag{7.13}
\]

where \( P^j_{j-m} \) are the momenta corresponding to \( \partial/\partial t^A_{j-m} \). Notice that \( P^j_{j-m} \) implicitly depends on \( \tilde{\phi}(k) \) which renders the transformations \((7.13)\) non-linear in \( \tilde{\phi}(k) \).

As was argued in \([4]\), only the linear part of the symmetries \((7.13)\) can be seen in first-quantized string theory, since in this framework all spacetime symmetries are connected to BRST cohomology. Moreover, in \([7]\) it was shown that the transformations \((7.13)\) can be rewritten in the form

\[
\delta^j_m \tilde{\phi}(k) = k_A h(k)^{-j-m} \tilde{\phi}(k) + O(\tilde{\phi}^2) \tag{7.14}
\]

These are just the non-local transformations derived recursively from the global translations \( y^A \rightarrow y^A + t^A_0 \). The similarity of \((7.10)\) to \((7.14)\) is evident. Thus, the transformations \((7.10)\) of the string ground state \( ||0,0;k\rangle \) corresponding to the field \( \tilde{\phi}(k) \) precisely reproduce the linear part of the symmetries \((7.13)\) of self-dual gravity.
8 Conclusion

This work deepens the identity of closed $N=2$ strings (at tree-level) with self-dual gravity (SDG) on $4D$ manifolds of signature $(++--)$, by the following results. After outlining how the SDG equations emerge in different gauges (Plebanski I and II) from the string dynamics, we have analyzed in detail the non-local hidden symmetries of the SDG equations and introduced an infinite hierarchy of equations (the SDG hierarchy) associated with an infinite number of abelian symmetries. We have then demonstrated how exactly the same symmetries (in linearized form) emerge in first-quantized $N=2$ closed string theory.

Interestingly, the stringy source of those symmetries is a non-trivial ground ring of ghost number zero operators in the chiral BRST cohomology. Such a phenomenon is familiar from the non-critical $2D$ string, where the ground ring was exploited to investigate the global symmetries of the theory, with the result that there are more discrete states and associated symmetries in $2D$ string theory than had been recognized previously [31, 33]. The authors of ref. [33] have wondered if their findings "could be relevant in a realistic string theory with a macroscopic four-dimensional target space". The outcome of this paper answers their question in the affirmative by providing the first four-dimensional if not yet realistic string theory with a rich symmetry structure based on an infinite ground ring.

More concretely, the symmetry charges are constructed from zero momentum operators of picture-raising $X_\pm$, picture charge $\Pi_\pm$, spectral flow $S$, and momentum operators $p^A$, $\bar{p}^\dot{A}$. The abelian subalgebra generated by $X_\pm$, $S$, and $\eta_{AA} \bar{p}^\dot{A}$ was found to coincide with the abelian symmetries of the SDG equations produced by the operators $\partial_A$ and the recursion operator $R$.

Our results ascertain that the non-trivial picture structure of the BRST cohomology is not just an irrelevant technical detail of the BRST approach but indispensable for a deeper understanding of the theory. It is, of course, not a simple task to discover the full symmetry group of a string model. Doing so would roughly correspond to having found a useful non-perturbative definition of the theory. In this paper we have worked in the standard first-quantized formalism which is background-dependent and limits our access to unbroken linear global symmetries.

There remain a number of interesting unresolved issues which should be addressed. Prominent among them is the detection of the non-abelian symmetries of the SDG equations in the $N=2$ string context. Such symmetries constitute an affinization of the $W_\infty$ algebra of volume-preserving $2D$ vector fields. In view of our findings here, it is tempting to relate them to the non-abelian string symmetries generated by the operators $B^{\pm, \ell}_{j, m; \dot{j}, \dot{m}}$ and $\tilde{B}^{\pm, \ell}_{j, m; \dot{j}, \dot{m}}$ introduced in Sect.6.3. Another task is the transfer of our results to the open $N=2$ string pertaining to the self-dual Yang-Mills equations, in Leznov or in Yang gauge. Finally, it would be interesting to find further examples in which the picture structure yields non-trivial information about a theory, like it happens for the relative zero-momentum cohomology of the Ramond sector of the $N=1$ string in flat $9+1$ dimensional spacetime [4]. We hope to return to these problems soon.

Acknowledgment

A.D.P. thanks the Institut für Theoretische Physik der Universität Hannover for its hospitality. The work of A.D.P. was partially supported by the grant RFBR-99-01-01076 and "Freundeskreis der Universität Hannover e.V.". O.L. acknowledges support by DFG under the grant LE-838/5.
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