NEUTRAL DELAY HILFER FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH FRACTIONAL BROWNIAN MOTION

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(Communicated by George Avalos)

Abstract. In this paper, we study the existence and uniqueness of mild solutions for neutral delay Hilfer fractional integrodifferential equations with fractional Brownian motion. Sufficient conditions for controllability of neutral delay Hilfer fractional differential equations with fractional Brownian motion are established. The required results are obtained based on the fixed point theorem combined with the semigroup theory, fractional calculus and stochastic analysis. Finally, an example is given to illustrate the obtained results.

1. Introduction. Recently, fractional differential equations have received great attention due to their applications in many important applied fields such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics and so on (see [5, 18, 28, 33]). Hilfer [17] proposed a generalized Riemann-Liouville fractional derivative for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. Controllability means to steer a dynamical system from an arbitrary initial state to the desired final state in a given finite interval of time by using the admissible controls, controllability results for linear and nonlinear integer order differential systems were studied by several authors (see [2, 3, 12, 19, 22, 29, 32]). In many applications, a close look at the physical or biological background of the modeling system shows that the change rate of the systems current status often depends not only on the current state but also on the history of the system, see, for example, ([21, 25, 26, 27]). This usually leads to so-called delay differential equations. Further, stochastic delay differential equations driven by fractional Brownian motion have been considered greatly by research community in various aspects due to its salient features for real world problems (see [1, 4, 6, 7, 8, 9, 13]). Many authors have been studied delay stochastic differential equations, for example, Luo et al.
[20] established averaging principle for stochastic fractional differential equations with time-delays. Sakthivel, and Yong [30] obtained sufficient conditions for approximate controllability of fractional differential equations with state-dependent delay. Jing, and Yan [10] discussed existence result for fractional neutral stochastic integro-differential equations with infinite delay. Chadha and Pandey [11] discussed existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay. Zhang et al [34] proved existence and uniqueness of solutions for stochastic differential equations of fractional-order $q > 1$ with finite delays. Ferrante and Rovira [14] studied stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Vadivoo et al [31] discussed the controllability analysis of nonlinear neutral-type fractional-order differential systems with state delay and impulsive effects. However, the existence and controllability results for neutral delay stochastic Hilfer fractional integrodifferential equations with fractional Brownian motion have not yet been considered in the literature, and this fact motivates this work. This paper is prepared as follows. In section 2, we present some basic definitions and lemmas which are useful to prove the main results. In section 3, we deduce the existence of mild solutions of neutral delay stochastic Hilfer fractional integrodifferential equations with fractional Brownian motion. In section 4, we investigate the sufficient conditions for controllability of neutral delay stochastic Hilfer fractional integrodifferential equations with fractional Brownian motion. In the final section, we consider an example to verify the theoretical results.

2. Preliminaries. In order to derive the existence of mild solutions and controllability of neutral delay stochastic Hilfer fractional integrodifferential equations with fractional Brownian motion, we need the following basic definitions and Lemmas.

**Definition 2.1** (see [24]). The fractional integral operator of order $\mu > 0$ for a function $f$ can be defined as

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} ds, \quad t > 0$$

where $\Gamma(\cdot)$ is the Gamma function.

**Definition 2.2** (see [16]). The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ is defined as

$$D^{\nu,\mu}_0 f(t) = I^{\nu(1-\mu)}_0 \frac{d}{dt} I^{(1-\nu)(1-\mu)}_0 f(t).$$

Let $-A : D(A) \to X$ be the infinitesimal generator of an analytic compact semigroup of uniformly bounded linear operators $\{(S(t))_{t \geq 0}\}$. This means that there exists a $M \geq 1$ such that $\|S(t)\| \leq M$. We assume without loss of generality that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $A$, then it is possible to define the fractional power $A^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in $X$ and the expression $\|h\|_\alpha = \|A^\alpha h\|$ defines a norm in $D(A^\alpha)$. If $X_\alpha$ represents the space $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known.

**Lemma 2.1** (see [23]).

1. Let $0 < \alpha \leq 1$, then $X_\alpha$ ia a Banach space.
2. If $0 < \beta \leq \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.
3. For every $0 < \alpha \leq 1$ there exists $M_\alpha > 0$ such that $\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha}, \quad t > 0.$
For $x \in X$, we define two families of operators $\{S_{\nu,\mu}(t) : t \geq 0\}$ and $\{P_{\mu}(t) : t \geq 0\}$ by

$$S_{\nu,\mu}(t) = t^{\nu(1-\mu)} P_{\mu}(t), \quad P_{\mu}(t) = t^{\nu-1} T_{\mu}(t), \quad T_{\mu}(t) = \int_0^\infty \mu \theta \Psi_{\mu}(\theta) S(t^\mu \theta) d\theta, \quad (2.1)$$

where

$$\Psi_{\mu}(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)! \Gamma(1-n\mu)}, \quad \theta \in (0, \infty) \quad (2.2)$$

is a function of Wright-type which satisfies the following equality $\int_0^\infty \theta^\Psi \Psi_{\mu}(\theta) d\theta = \frac{\Gamma(1+\Psi)}{\Gamma(1+\Psi)}$ for $\theta \geq 0$. (2.3)

**Lemma 2.2** (see [15]). The operator $S_{\nu,\mu}$ and $P_{\mu}$ have the following properties.

(i) $\{P_{\mu}(t) : t > 0\}$ is continuous in the uniform operator topology.

(ii) For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_{\mu}(t)$ are linear and bounded operators, and

$$\|P_{\mu}(t)x\| \leq \frac{M t^{\nu-1}}{\Gamma(\mu)} \|x\|, \quad \|S_{\nu,\mu}(t)x\| \leq \frac{M t^{\nu(1-\mu)}}{\Gamma(\nu(1-\mu) + \mu)} \|x\|, \quad (2.4)$$

(iii) $\{P_{\mu}(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.

**Lemma 2.3** For any $x \in H$, $\beta \in (0, 1)$ and $\delta \in (0, 1]$, we have $A T_{\mu}(t)x = A^{1-\beta} T_{\mu}(t) A^\beta x$, $0 \leq t \leq b$ and

$$\|A^\delta T_{\mu}(t)x\| \leq \frac{M C \delta}{t^{\beta \mu}} \frac{\Gamma(2-\delta)}{\Gamma(1 + \mu(1-\delta))} \|x\|, \quad 0 < t \leq b.$$

Fix a time interval $[0, T]$ and let $\{\beta^H(t), t \in [0, T]\}$ be the one dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. That is, $\beta^H$ is a centered Gaussian process with covariance function $R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H})$ (see [7]).

Moreover $\beta^H$ has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s)$$

where $\beta = \{\beta(t), t \in [0, T]\}$ is a Wiener process, and $K_H(t, s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} t^{\frac{1}{2}-H} \int_s^t \int_s^u (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $s < t$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H,H-H^2)}}$ and

$$\beta(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1}, \quad p > 0, \quad q > 0.$$ We put $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\zeta$ the reproducing kernel Hilbert space of the fBm. In fact $\zeta$ is the closure of set of indicator functions $\{1_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_\zeta = R_H(t, s)$.

The mapping $1_{[0,t]} \mapsto \beta^H(t)$ can be extended to an isometry from $\zeta$ onto the first Wiener chaos and we will denote by $\beta^H(\vartheta)$ the image of $\vartheta$ under this isometry.

We recall that for $\psi, \vartheta \in \zeta$ their scalar product in $\zeta$ is given by

$$\langle \psi, \vartheta \rangle_\zeta = H(2H-1) \int_0^T \int_0^T \psi(s) \vartheta(t) |t-s|^{2H-2} ds dt.$$
Let us consider the operator $K^*$ from $\zeta$ to $L^2([0,T])$ defined by
\[
(K^*_n \vartheta)(s) = \int_0^T \vartheta(r) \frac{\partial K_n}{\partial r}(r,s)dr.
\]
Moreover for any $\vartheta \in \zeta$, we have
\[
\beta^H(\vartheta) = \int_0^T (K_n^* \vartheta_n) d\beta(t).
\]
Let $X$ and $Y$ be two real, separable Hilbert spaces and let $L(Y, X)$ be the space of bounded linear operators from $Y$ to $X$. For the sake of convenience, we shall use the same notation to denote the norms in $X$, $Y$ and $L(Y, X)$. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0 \ (n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\} \ (n = 1, 2, \ldots)$ is a complete orthonormal basis in $Y$.

We define the infinite dimensional fBm on $Y$ with covariance $Q$ as
\[
B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t)
\]
where $\beta^H_n$ are real, independent fBm's. The $Y$-valued process is Gaussian, starts from 0, has mean zero and covariance:
\[
E\langle B^H(t), x \rangle \langle B^H(s), y \rangle = R(s,t)\langle Q(x), y \rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].
\]
In order to define Wiener integrals with respect to the $Q$-fBm, we introduce the space $L_2^Q := L_2^Q(Y, X)$ of all $Q$-Hilbert Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y, X)$ is called a $Q$-Hilbert-Schmidt operator, if
\[
\|\psi\|_{L_2^Q}^2 := \sum_{n=1}^{\infty} \|\lambda_n e_n\|^2 < \infty
\]
and that the space $L_2^Q$ equipped with the inner product $\langle \vartheta, \psi \rangle_{L_2^Q} = \sum_{n=1}^{\infty} \langle \vartheta e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let $\phi(s) : s \in [0,T]$ be a function with values in $L_2^Q(Y, X)$, the Wiener integral of $\phi$ with respect to $B^H$ is defined by
\[
\int_0^t \phi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n d\beta^H_n = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} K^* (\phi e_n)(s) d\beta_n(s) \ (2.5)
\]
where $\beta_n$ is the standard Brownian motion.

**Lemma 2.4** (see [7]). If $\psi : [0,T] \to L_2^Q(Y, X)$ satisfies $\int_0^T \|\psi(s)\|^2_{L_2^Q} < \infty$ then the above sum in (2.5) is well defined as $X$-valued random variable and we have
\[
E \left\| \int_0^t \psi(s)dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^T \|\psi(s)\|^2_{L_2^Q} ds.
\]

3. **Existence and uniqueness.** In this section, we study the existence and uniqueness of mild solutions of neutral delay Hilfer fractional integro-differential equations with fractional Brownian motion in the following form
\[
\begin{aligned}
& D_{\alpha^+}^{\gamma \nu}[x(t) + g(t, x(t - \theta(t)))] + Ax(t) = f(t, x(t - \chi(t))) \\
& + \int_0^t \sigma(s, x(s - \varphi(s))) dB^H(s), \quad t \in J = [0, T], \\
& x(t) = \varphi(t), \quad -\tau \leq t < 0, \\
& I_{0+}^{(1-\nu)(1-\mu)}x(0) = \varphi(0)
\end{aligned}
\]
where $D^{\nu,\mu}_{t+}$ is the Hilfer fractional derivative, $0 \leq \nu \leq 1$, $\frac{1}{2} < \mu < 1$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $S(t)_{t \geq 0}$, in a Hilbert space $X$, $B^H_t$ is a fractional Brownian motion on a real separable Hilbert space $Y$ and $\varphi \in C([\tau, 0), L^2(\Omega, X))$ is continuous function. The functions $\theta$, $\chi$, $g : [0, +\infty) \to [0, \tau]$ ($\tau > 0$) are continuous and $f$, $g : [0, +\infty) \times X \to X$, $\sigma : [0, +\infty) \times X \to L^2(Y, X)$ are appropriate Lipschitz type functions.

Let $(\Omega, F, P)$ be a complete probability space furnished with complete family of right continuous increasing sub-$\sigma$-algebras $\{F_t : t \in J\}$ satisfying $F_t \subset F$. Through this paper, $R_T := C([-\tau, T], L^2(\Omega, X))$ be the Banach space of all continuous functions $\xi$ from $[-\tau, T]$ into $L^2(\Omega, X)$, equipped with the supremum norm $\|\xi\|_{R_T} := \sup_{t \in [-\tau, T]} (E \|\xi(t)\|^2)^{1/2}$.

Define $K = \{x : t(1-\nu)(1-\mu)x(t) \in R_T, \ x(t) = \varphi(t) \text{ for } t \in [-\tau, 0)\}$ with norm $\|\cdot\|_K$ defined by $\|\cdot\|_K = (\sup_{t \in J} E \|t(1-\nu)(1-\mu)x(t)\|^2)^{1/2}$. Obviously, $K$ is a Banach space.

Here $L^2_0(Y, X)$ denotes the space of all $Q$-Hilbert-Schmidt operators from $Y$ into $X$.

**Definition 3.1.** A $X$-valued process $\{x(t), \ t \in [-\tau, T]\}$, is called a mild solution of equation (3.1) if

(i) $x(\cdot) \in C([-\tau, T], L^2(\Omega, X))$,

(ii) $x(t) = \varphi(t), \ -\tau \leq t < 0$,

(iii) For arbitrary $t \in J$, we have

\[
\begin{align*}
\{x(t) &= S_{\tau, t}(\varphi(0) + g(0, \varphi(0)) - g(t, x(t - \theta(t)))) - \int_0^t A P_{\nu}(t-s)g(s, x(s - \theta(s)))ds \\
&+ \int_0^t P_{\mu}(t-s)f(s, x(s - \chi(s)))ds + \int_0^t P_{\mu}(s - \xi)\sigma(\xi, x(\xi - \varrho(\xi)))d B^H(\xi)ds .
\end{align*}
\]

In this paper we need the following assumptions.

**H1** There exist finite positive constants $C_i = C_i(T)$, $i = 1$, $2$, such that the function $f : [0, +\infty) \times X \to X$ satisfies the following Lipschitz conditions: for all $t \in J$ and $x, \ y \in X$ the inequalities $\|f(t, y) - f(t, x)\|^2 \leq C_1 \|y - x\|^2$ and $\|f(t, x)\|^2 \leq C_2 (1 + \|x\|^2)$ are valid.

**H2** The function $g$ is $X_\beta$-valued, and there exist constants $C_i = C_i(T)$, $i = 3$, $4$, such that for all $t \in J$ and $x, \ y \in X$ the following inequalities are satisfied:

(i) $\|A^\beta g(t, y) - A^\beta g(t, x)\|^2 \leq C_3 \|y - x\|^2$;

(ii) $\|A^\beta g(t, x)\|^2 \leq C_4 (1 + \|x\|^2)$;

(iii) $\|C_i A^{-\beta}\|^2 < 1$, $i = 3$, $4$.

**H3** There exist finite positive constants $C_i = C_i(T)$, $i = 5$, $6$, such that, for all $t \in J$ and $x, \ y \in X$

$\|\sigma(t, y) - \sigma(t, x)\|_{L^2_0}^2 \leq C_5 \|y - x\|^2$ and $\|\sigma(t, x)\|_{L^2_0}^2 \leq C_6 (1 + \|x\|^2)$.

We can now prove the existence and uniqueness of mild solution for the system (3.1).

**Theorem 3.1.** Let the assumptions (H1)-(H3) are satisfied and fix $\frac{1}{2} < \mu \beta < 1$, and let $T > 0$ be any finite real number. Then, the system (3.1) has a unique mild solution on $[-\tau, T]$ provided that

\[
\zeta := \left[ \frac{25 \mu^2 C_3 C_4^2 T^{2 \mu - 1}}{(2 \mu - 1) \Gamma^2(1 + \mu \beta)} + \frac{50 C_5 H M^2 T^{2 \mu + 2 H - 1}}{(2 \mu - 1) (2 \mu + 2 H - 1) \Gamma^2(\mu)} \right] < 1.
\]

**Proof.** Consider the map $\Pi$ on $K$ defined by

$$(\Pi x)(t) = \varphi(t), \ t \in [-\tau, 0),$$
and for all \( t \in J \)

\[
\left\{ \begin{array}{l}
(\Pi x)(t) = S_{\nu,\mu}(t)[\varphi(0) + g(0, \varphi(0))] - g(t, x(t - \theta(t))) - \int_0^t AP_{\nu}(t - s)g(s, x(s - \theta(s)))ds \\
+ \int_0^t P_{\nu}(t - s)f(s, x(s - \chi(s)))ds + \int_0^t \int_0^\infty P_{\nu}(s - \xi)\sigma(\xi, x(\xi - g(\xi)))dB^H(\xi)ds.
\end{array} \right.
\]

(3)

It will be shown that the operator \( \Pi \) from \( K \) into itself has a fixed point. We show that \( \Pi \) maps \( K \) into itself.

From (3.3), for \( t \in J \), we have

\[
\|(\Pi x)(t)\|^2_K = \sup_{t \in J} t^{2(1-\nu)(1-\mu)}E\|(\Pi x)(t)\|^2 \\
\leq 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}E\|S_{\nu,\mu}(t)[\varphi(0) + g(0, \varphi(0))]\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}E\|g(t, x(t - \theta(t)))\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}E\|\int_0^t AP_{\nu}(t - s)g(s, x(s - \theta(s)))ds\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}E\|\int_0^t P_{\nu}(t - s)f(s, x(s - \chi(s)))ds\|^2 \\
+ E\|\int_0^t \int_0^s P_{\nu}(s - \xi)\sigma(\xi, x(\xi - g(\xi)))dB^H(\xi)ds\|^2 \\
\leq \frac{25M^2(1 + 2E\|\varphi(0)\|^2\|\nu(1 - \mu) \mu \|)}{\Gamma^2(\nu(1 - \mu) + \mu)} + \frac{25 T^{2(1-\nu)(1-\mu)}}{2(2\mu - 1)(2\mu + 2H - 1)\Gamma^2(\mu)}(1 + E\|x\|^2) \quad (4)
\]

Therefore \( \Pi \) maps \( K \) into itself.

We show that \( (\Pi x)(t) \) is continuous on \( J \) for any \( x \in K \). Let \( 0 < t \leq T \) and \( \epsilon > 0 \) be sufficiently small, then,

\[
\|(\Pi x)(t + \epsilon) - (\Pi x)(t)\|^2_K = \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|(\Pi x)(t + \epsilon) - (\Pi x)(t)\|^2 \\
\leq 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|(S_{\nu,\mu}(t + \epsilon) - S_{\nu,\mu}(t))[\varphi(0) + g(0, \varphi(0))]\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|g(t + \epsilon, x(t + \epsilon - \theta(t + \epsilon))) - g(t, x(t - \theta(t)))\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|\int_0^t AP_{\nu}(t + \epsilon - s)g(s, x(s - \theta(s)))ds - \int_0^t AP_{\nu}(t - s)g(s, x(s - \theta(s)))ds\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|\int_0^t P_{\nu}(t + \epsilon - s)f(s, x(s - \chi(s)))ds - \int_0^t P_{\nu}(t - s)f(s, x(s - \chi(s)))ds\|^2 \\
+ 25 \sup_{t \in J} t^{2(1-\nu)(1-\mu)}\|\int_0^\infty \int_0^\infty P_{\nu}(s - \xi)\sigma(\xi, x(\xi - g(\xi)))dB^H(\xi)ds \\
- \int_0^t \int_0^\infty P_{\nu}(s - \xi)\sigma(\xi, x(\xi - g(\xi)))dB^H(\xi)ds\|^2. \quad (5)
\]

Clearly, the right hand side of (3.4) tends to zero as \( \epsilon \to 0 \). Hence, \( (\Pi x)(t) \) is continuous on \( J \).
We are going to show that \((\Pi x)(t)\) is a contraction on \(K\). Let \(x, y \in K\), for any \(t \in (0, T)\) be fixed, then
\[
E\|\Pi y(t) - (\Pi x)(t)\|^2 \leq 25E\|y(t, y(t - \theta(t))) - g(t, x(t - \theta(t)))\|^2 \\
+ 25E\|\int_0^t AP_\mu(t-s)y(s, x(s - \theta(s))) - g(s, x(s - \theta(s)))ds\|^2 \\
+ 25E\|\int_0^t P_\mu(t-s)[f(s, x(s - \chi(s))) - f(s, x(s - \chi(s)))ds]\|^2 \\
+ 25E\|\int_0^t \int_0^s P_\mu(s-\xi)[\sigma(\xi, y(x(\xi))) - \sigma(\xi, x(\xi))]dB^H(\xi)ds\|^2 \\
\leq \frac{25\mu^2 C_2 C_2^2 (1 + \beta)2\mu\beta - 1}{(2\mu - 1)^2(1 + \mu\beta)} + \frac{50C_2HM2^\mu 2\mu + 2H - 1}{(2\mu - 1)(2\mu + 2H - 1)}E\|y(t) - x(t)\|^2.
\]
Therefore,
\[
\sup_{t \in J} t^2(1 - \nu)(1 - \mu) E\|\Pi y(t) - (\Pi x)(t)\|^2 \leq \zeta \sup_{t \in J} t^2(1 - \nu)(1 - \mu) E\|y(t) - x(t)\|^2.
\]
This implies that
\[
\|\Pi y - \Pi x\|^2_K \leq \zeta \|y - x\|^2_K.
\]
Hence, \(\Pi\) is a contraction on \(K\). From the Banach fixed point theorem, \(\Pi\) has a unique fixed point \(x(t)\) which satisfies \(x(t) = \varphi(t)\) on \([-\tau, 0)\). Therefore the system (3.1) has a unique mild solution on \([-\tau, T]\), and the proof is completed.

4. Controllability result. In this section, we will establish a set of sufficient conditions for controllability of neutral delay Hilfer fractional integrodifferential equations with fractional Brownian motion in the following form
\[
\begin{align*}
\begin{cases}
D_0^\nu \mu [x(t) + g(t, x(t - \theta(t)))] + Ax(t) = f(t, x(t - \chi(t))) + Bu(t) + \\
\int_0^t \sigma(s, x(s - \theta(s)))dB^H(s), & t \in J = (0, T], \\
x(t) = \varphi(t), & -\tau \leq t < 0, \\
I_0^{(1 - \nu)(1 - \mu)}x(0) = \varphi(0)
\end{cases}
\end{align*}
\] (4.1)

where the control function \(u(\cdot)\) is given in \(L^2(J, U)\), the Hilbert space of admissible control functions with \(U\) a Hilbert space. The symbol \(B\) stands for a bounded linear from \(U\) into \(X\).

**Definition 4.1.** A \(X\)-valued process \(\{x(t), t \in [-\tau, T]\}\), is called a mild solution of equation (4.1) if
(i) \(x(\cdot) \in C([-\tau, T], L^2(\Omega, X))\),
(ii) \(x(t) = \varphi(t), -\tau \leq t < 0\),
(iii) For arbitrary \(t \in J\), we have
\[
\begin{align*}
\begin{cases}
x(t) = S_{0, \nu}(t)[\varphi(0) + g(0, \varphi(0))] - g(t, x(t - \theta(t))) - \int_0^t AP_\mu(t-s)g(s, x(s - \theta(s)))ds \\
+ \int_0^t P_\mu(t-s)[f(s, x(s - \chi(s))) + Bu(s)]ds + \int_0^t \int_0^s P_\mu(s-\xi)[\sigma(\xi, x(\xi))]dB^H(\xi)ds.
\end{cases}
\end{align*}
\] (4.2)

**Definition 4.2.** The system (4.1) is said to be controllable on the interval \([-\tau, T]\), if for every initial stochastic process \(x(t) = \varphi(t)\) defined on \([-\tau, 0)\), there exists a stochastic control \(u \in L^2(J, U)\) such that the mild solution \(x(t)\) of the system (4.1) satisfies \(x(T) = x_1\), where \(x_1\) and \(T\) are the preassigned terminal state and time respectively.

Fix \(\frac{1}{2} < \mu\beta < 1\) and let \(T > 0\) be any finite real number.

To establish the result, we need the following additional hypothesis
(H4) The linear operator \(W\) from \(U\) into \(X\) defined by
\[
Wu = \int_0^T P_\mu(T-s)Bu(s)ds
\]
has a fixed point. This fixed point is then a solution of equation (4.1).

**Proof.** Using the assumption (H4), define the control

\[ u(t) = W^{-1}\{x_1 - S_{v,\mu}(T)[\varphi(0) + g(0, \varphi(0))] + g(T, x(T - \theta(T))) + \int_0^T A\rho(t-s)g(s, x(s - \theta(s)))ds - \int_0^T P_\mu(T-s) f(s, x(s - \chi(s)))ds - \int_0^T \int_0^s P_\mu(s - \xi) \sigma(\xi, x(\xi - \varrho(\xi))dB^H(\xi))d\xi(t) \} \]

It shall now be shown that when using this control, the operator \( \Pi^* \) defined by \( \Pi^*(x)(t) = \varphi(t) \) for \( t \in (-\tau, 0) \),

\[
\langle \Pi^* x)(t) \rangle = S_{v,\mu}(t)[\varphi(0) + g(0, \varphi(0))] - g(t, x(t - \theta(t))) - \int_0^t A\rho(t-s)g(s, x(s - \theta(s)))ds
+ \int_0^t P_\mu(t-s) f(s, x(s - \chi(s)))ds + \int_0^t \int_0^s P_\mu(s - \xi) \sigma(\xi, x(\xi - \varrho(\xi))dB^H(\xi))d\xi(t)
+ \int_0^t P_\mu(T-s) g(T, x(\theta(T))) - \int_0^t \int_0^s P_\mu(s - \xi) \sigma(\xi, x(\xi - \varrho(\xi))dB^H(\xi))d\xi(s)ds \eta d\bar{\eta} \in J \]

has a fixed point. This fixed point is then a solution of equation (4.1).

It will be shown that the operator \( \Pi^* \) from \( K \) into itself has a fixed point.

We show that \( \Pi^* \) maps \( K \) into itself.

From (4.3) for \( t \in J \), we have

\[
\|(\Pi^* x)(t)\|_K^2 \leq 36 \sup_{t \in J} t^{2(1-\nu)(1-\mu)} E \||\Pi^* x)(t)\|_K^2
+ 36 \sup_{t \in J} t^{2(1-\nu)(1-\mu)} E \|S_{v,\mu}(t)[\varphi(0) + g(0, \varphi(0))]\|^2
+ 36 \sup_{t \in J} t^{2(1-\nu)(1-\mu)} E \|g(t, x(t - \theta(t)))\|^2
+ 36 \sup_{t \in J} t^{2(1-\nu)(1-\mu)} E \|\int_0^t A\rho(t-s)g(s, x(s - \theta(s)))ds\|^2
+ 36 \sup_{t \in J} t^{2(1-\nu)(1-\mu)} [E \|\int_0^t P_\mu(t-s) f(s, x(s - \chi(s)))ds\|^2]
+ E \|\int_0^t \int_0^s P_\mu(s - \xi) \sigma(\xi, x(\xi - \varrho(\xi))dB^H(\xi))ds\|^2]
\]
Clearly, the right hand side of (4.4) tends to zero as \(\epsilon \to 0\). Hence, \((\Pi^* x)(t)\) is continuous on \(J\). We are going to show that \((\Pi^* x)(t)\) is a contraction on \(K\).
Let $x, y \in K$, for any $t \in (0, T]$ be fixed, then

$$
E\|((\Pi^*y)(t) - (\Pi^*x)(t))\|^2 \\
\leq 36E\|g(t, y(t - \theta(t))) - g(t, x(t - \theta(t)))\|^2 \\
+ 36E\|\int_0^t AP_{\mu}(t-s)[g(s, y(s - \theta(s))) - g(s, x(s - \theta(s)))]ds\|^2 \\
+ 36E\|\int_0^t P_{\mu}(t-s)[f(s, y(s - \chi(s))) - f(s, x(s - \chi(s)))]ds\|^2 \\
+ 36E\|\int_0^t \int_0^s P_{\mu}(s-\xi)[\sigma(\xi, y(\xi - \varrho(\xi))) - \sigma(\xi, x(\xi - \varrho(\xi)))]dDB^H(\xi)ds\|^2 \\
+ 36E\|W^{-1}\|\|B\|^2 \int_0^t \|P_{\mu}(t-\eta)\|^2 (E\|g(T, y(T - \theta(T))) - g(T, x(T - \theta(T)))\|^2 \\
+ E\|\int_0^T AP_{\mu}(t-s)[(s, y(s - \theta(s))) - g(s, x(s - \theta(s)))]ds\|^2 \\
+ E\|\int_0^T P_{\mu}(T-s)[f(s, y(s - \chi(s))) - f(s, x(s - \chi(s)))]ds\|^2 \\
+ E\|\int_0^T \int_0^s P_{\mu}(s-\xi)[\sigma(\xi, y(\xi - \varrho(\xi))) - \sigma(\xi, x(\xi - \varrho(\xi)))]dDB^H(\xi)ds\|^2(\eta)d\eta
\leq \frac{36\mu^2 C_3 C_4^2 (1 + \beta) T^{2\mu H}}{2(\mu - 1)\Gamma^2(1 + \mu H)} + \frac{72 C_5 H M^2 T^{2\mu H - 1}}{(2\mu - 1)(2\mu + 2\mu H - 1)\Gamma^2(\mu)} \|E\|y - x\|^2 + \tilde{\zeta}E\|y - x\|^2.
$$

Therefore,

$$
\sup_{t \in J} t^{2(1-\nu)(1-\mu)} E\|((\Pi^*y)(t) - (\Pi^*x)(t))\|^2 \leq \tilde{\zeta} \sup_{t \in J} t^{2(1-\nu)(1-\mu)} E\|y(t) - x(t)\|^2.
$$

This implies that

$$
\|\Pi^*y - \Pi^*x\|^2 \leq \tilde{\zeta}\|y - x\|^2.
$$

Hence, $\Pi^*$ is a contraction on $K$. From the Banach fixed point theorem, $\Pi^*$ has a unique fixed point $x(t)$ which satisfies $x(t) = \varphi(t)$ on $[-\tau, 0)$. Therefore the system (4.1) has a mild solution satisfying $x(T) = x_1$. Thus, system (4.1) is controllable on $[-\tau, T]$.

5. Example. Consider the following control neutral delay stochastic Hilfer fractional integro-partial differential equation with fractional Brownian motion in the form

$$
\begin{align*}
D_{0+}^{\alpha} & \left[ x(t, \varepsilon) - G(t, x(t - \theta(t), \varepsilon)) \right] + \frac{\partial^2 x(t, \varepsilon)}{\partial \varepsilon^2} = F(t, x(t - \chi(t), \varepsilon)) + \Theta(t, \varepsilon) \\
& + \int_0^t \sigma(s, x(s - \theta(s), \varepsilon))dDB^H(s), \ 0 \leq \varepsilon \leq \pi, \ t \in J, \\
x(t, 0) = x(t, \pi) = 0, \ t \in J, \\
x(t, 0) = x_0(t), \ 0 \leq \varepsilon \leq \pi, \\
x(t, \varepsilon) = \varphi(t, \varepsilon), \ t \in [-\tau, 0), \ 0 \leq \varepsilon \leq \pi
\end{align*}
$$

(5.1)
Acknowledgments. We would like to thank the referees and the editor for their important comments and suggestions, which have significantly improved the paper.

In order to define the operator $Q : D(A) \subset X \to X$ given by $Ay = -y''$ with $D(A) = \{ y \in X : y'' \in X, y(0) = y(\pi) = 0 \}$. It is well known that $-A$ has discrete spectrum with eigenvalues $-n^2, \ n \in N$ and the corresponding normalized eigenfunctions given by $e_n(\varepsilon) = \sqrt{\frac{2}{\pi}} \sin n \varepsilon, \ n = 1, 2, 3, \ldots$

In addition $(e_n)_{n \in N}$ is a complete orthonormal basis in $X$. Then

$$-Ay = \sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, \ y \in D(A).$$

Furthermore, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operator, $\{S(t)\}_{t \geq 0}$ on $X$ and is given by

$$S(t)y = \sum_{n=1}^{\infty} e^{-t^2 \langle y, e_n \rangle} e_n, \ y \in X, \ t \geq 0.$$

with $\|S(t)\| \leq e^{-t} \leq 1$.

Moreover, the two operators $S_{\frac{\pi}{4}}(t)$ and $P_{\frac{\pi}{4}}(t)$ can be defined by

$$S_{\frac{\pi}{4}}(t)x = \frac{3}{4t^{\frac{1}{4}}} \int_0^t \int_0^\infty \theta(t-s) \frac{\pi}{2} s^{\frac{1}{2}} \Psi\left(\frac{3}{4}\right) S(s^{\frac{1}{2}} \theta) x d\theta ds,$$

$$P_{\frac{\pi}{4}}(t)x = \frac{3}{4} \int_0^\infty \theta t^{\frac{1}{2}} \frac{\pi}{2} \Psi\left(\frac{3}{4}\right) S(s^{\frac{1}{2}} \theta) x d\theta.$$

Clearly,

$$\|P_{\frac{\pi}{4}}(t)\| \leq \frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{4}\right)}, \ \|S_{\frac{\pi}{4}}(t)\| \leq \frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}.$$

In order to define the operator $Q : Y \to R$, we choose a sequence $\{\lambda_n\}_{n \in N} \subset R^+$, set $Qe_n = \lambda_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$. Define the fractional Brownian motion in $Y$ by $B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \delta^H(t)e_n$, where $H \in (\frac{1}{4}, 1)$ and $\{\beta_n^H\}_{n \in N}$ is a sequence of one-dimensional fractional Brownian motions mutually independent.

We define the bounded operator $B : U \to X$ by $B = I$.

Also, We define the following:

$$x(t)(\cdot) = x(t, \cdot), \ f(t, x)(\cdot) = F(t, x(\cdot)), \ g(t, x)(\cdot) = G(t, x(\cdot)), \ \sigma(t, x)(\cdot) = \sigma(t, x(\cdot)).$$

Therefore, with the above choice, the system (5.1) can be written in the abstract form of (4.1). Therefore, all conditions of Theorem 4.1 are satisfied and

$$\tilde{\zeta} = \frac{36 \mu^2 C_5 C_7 \beta^{-1} \Gamma^2(1 + \beta) T^{2H + 1} + 72 C_6 H M^2 \Gamma^2(1 + \mu) T^{2H + 1}}{(2\mu - 1) \Gamma^2(1 + \beta) T^{2H + 1}} \leq 1.$$

Thus, we can conclude that the neutral delay stochastic Hilfer fractional integro-partial differential equation with fractional Brownian motion (5.1) is exactly controllable on the interval $[-\tau, T]$.

**Acknowledgments.** We would like to thank the referees and the editor for their important comments and suggestions, which have significantly improved the paper.
REFERENCES

[1] G. Arthi, J. H. Park and H. Y. Jung, Existence and exponential stability for neutral stochastic integrodifferential equations with impulses driven by a fractional Brownian motion, *Communications in Nonlinear Science and Numerical Simulation*, 32 (2016), 145–157.

[2] G. Arthi and J. H. Park, On controllability of second-order impulsive neutral integrodifferential systems with infinite delay, *IMA J. Math. Control Inf.*, 32 (2015), 639–657.

[3] K. Aissani and M. Benchohra, Controllability of fractional integrodifferential equations with state-dependent delay, *J. Integral Equations Applications*, 28 (2016), 149–167.

[4] H. M. Ahmed, Controllability of impulsive neutral stochastic differential equations with fractional Brownian motion, *IMA Journal of Mathematical Control and Information*, 32 (2015), 781–794.

[5] H. M. Ahmed and M. M. El-Borai, Hilfer fractional stochastic integro-differential equations, *Appl. Math. Comput.*, 331 (2018), 182–189.

[6] A. Boufoussi, T. Caraballo and A. Ouahab, Impulsive neutral functional differential equations driven by a fractional Brownian motion with unbounded delay, *Applicable Analysis*, 95 (2016), 2039–2062.

[7] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statistics and Probability Letters*, 82 (2012), 1549–1558.

[8] B. Boufoussi and S. Hajji, Stochastic delay differential equations in a Hilbert space driven by fractional Brownian motion, *Statistics and Probability Letters*, 129 (2017), 222–229.

[9] T. Caraballo, M. J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Analysis: Theory, Methods and Applications*, 74 (2011), 3671–3684.

[10] J. Cui and Y. Litan, Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *Journal of Physics A: Mathematical and Theoretical*, 44 (2011), 335201, 16pp.

[11] A. Chadha and N. Pandey Dwijendra, Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay, *Nonlinear Analysis*, 128 (2015), 149–175.

[12] A. Chadha and V. Antonov, Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces, *Chaos, Solitons & Fractals*, 102 (2017), 140–148.

[13] M. Ferrante and C. Rovira, Convergence of delay differential equations driven by fractional Brownian motion, *J. Evol. Equ.*, 10 (2010), 761–783.

[14] M. Ferrante and C. Rovira, Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, *Bernoulli*, 12 (2006), 85–100.

[15] H. Gu and H. J. J. Trujillo, Existence of mild solution for evolution equation with Hilfer fractional derivative, *Applied Mathematics and Computation*, 257 (2015), 344–354.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific: Singapore, 2000.

[17] R. Hilfer, Experimental evidence for fractional time evolution in glass forming materials, *Chem. Phys.*, 284 (2002), 399–408.

[18] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[19] J. Klamka, Stochastic controllability of linear systems with delay in control, *Bulletin of the Polish Academy of Sciences, Technical Sciences*, 55 (2007), 23–29.

[20] D. Luo, Q. Zhu and Z. Luo, An averaging principle for stochastic fractional differential equations with time-delays, *Applied Mathematics Letters*, 105 (2020), 106290, 8 pp.

[21] J. M. Mahaffy and C. V. Pao, Models of genetic control by repression with time delays and spatial effects, *J. Math. Biol.*, 20 (1984), 39–57.

[22] R. Mabel Lizzy, K. Balachandran and M. Suvithra, Controllability of nonlinear stochastic fractional systems with distributed delays in control, *Journal of Control and Decision*, 4 (2017), 153–168.

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[24] I. Podlubny, *Fractional Differential Equations*, Academic press, San Diego, 1999.

[25] C. V. Pao, Systems of parabolic equations with continuous and discrete delays, *J. Math. Anal. Appl.*, 205 (1997), 157–185.
[26] D. H. Abdel Rahman, S. Lakshmanan and A. S. Alkhajeh, A time delay model of tumour-immune system interactions: Global dynamics, parameter estimation, sensitivity analysis, *Applied Mathematics and Computation*, 232 (2014), 606–623.

[27] F. A. Rihan, C. Tunc, S. H. Saker, S. Lakshmanan and R. Rakkiyappan, Applications of delay differential equations in biological systems, *Complexity*, 2018 (2018), Article ID 4584389, 3 pages.

[28] F. A. Rihan, C. Rajivganthi, P. Muthukumar, Fractional stochastic differential equations with Hilfer fractional derivative: Poisson Jumps and optimal control, *Discrete Dyn. Nat. Soc.*, 2017(2017), Article ID 5394528, 11 pages.

[29] R. Sakthivel, R. Ganesh, Y. Ren and S. M. Anthoni, Approximate controllability of nonlinear fractional dynamical systems, *Communications in Nonlinear Science and Numerical Simulation*, 18 (2013), 3498–3508.

[30] R. Sakthivel and R. Yong, Approximate controllability of fractional differential equations with state-dependent delay, *Results in Mathematics*, 63 (2013), 949–963.

[31] B. Sundara Vadivoo, R. Ramachandran, J. Cao, H. Zhang and X. Li, Controllability analysis of nonlinear neutral-type fractional-order differential systems with state delay and impulsive effects, *International Journal of Control, Automation and Systems*, 16 (2018), 659–669.

[32] J. Wang and H. M. Ahmed, Null controllability of nonlocal Hilfer fractional stochastic differential equations, *Miskolc Math. Notes*, 18 (2017), 1073–1083.

[33] J. R. Wang, M. Feckan and Y. Zhou, A survey on impulsive fractional differential equations, *Fractional Calculus and Applied Analysis*, 19 (2016), 806–831.

[34] X. Zhang, P. Agarwal, Z. Liu, H. Peng, F. You and Y. Zhu, Existence and uniqueness of solutions for stochastic differential equations of fractional-order $q > 1$ with finite delays, *Advances in Difference Equations*, 2017 (2017), 1–18.

Received June 2020; revised May 2021.

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