Interpolating Between Gradient Descent and Exponentiated Gradient Using Reparameterized Gradient Descent

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Abstract
Continuous-time mirror descent (CMD) can be seen as the limit case of the discrete-time MD update when the step-size is infinitesimally small. In this paper, we focus on the geometry of the primal and dual CMD updates and introduce a general framework for reparameterizing one CMD update as another. Specifically, the reparameterized update also corresponds to a CMD, but on the composite loss w.r.t. the new variables, and the original variables are obtained via the reparameterization map. We employ these results to introduce a new family of reparameterizations that interpolate between the two commonly used updates, namely the continuous-time gradient descent (GD) and unnormalized exponentiated gradient (EGU), while extending to many other well-known updates. In particular, we show that for the underdetermined linear regression problem, these updates generalize the known behavior of GD and EGU, and provably converge to the minimum $L_{2-\tau}$-norm solution for $\tau \in [0, 1]$. Our new results also have implications for the regularized training of neural networks to induce sparsity.

1 INTRODUCTION

Mirror descent (MD) [Nemirovsky and Yudin, 1983; Kivinen and Warmuth, 1997] refers to a family of updates which transform the parameters $w \in C$ from a convex domain $C \subset \mathbb{R}^d$ via a link function (a.k.a. mirror map) $f : C \to \mathbb{R}^d$ before applying the descent step. The continuous-time mirror descent (CMD) update, which can be seen as the limit case of (discrete-time) MD, corresponds to the solution of the following ordinary differential equation (ODE) [Nemirovsky and Yudin, 1983; Warmuth and Jagota, 1998; Raginsky and Bouvrie, 2012]:

$$f(\dot{w}(t)) = -\eta \nabla_w L(w(t)),$$

where $\dot{f} := \frac{df}{dt}$ is the time derivative of the link function. The main link functions investigated in the past are $f(w) = w$ and $f(w) = \log(w)$ leading to the gradient descent (GD) and the unnormalized exponentiated gradient (EGU) family of updates. These two link functions are associated with the squared Euclidean and the relative entropy divergences, respectively. For example, the classical Perceptron and Winnow algorithms are motivated using the identity and log links, respectively, when the loss is the hinge loss. A number of papers discuss the difference between the two updates [Kivinen and Warmuth, 1997; Kivinen et al., 2006; Nie et al., 2016; Ghai et al., 2019] and their rotational invariance properties have been explored in [Warmuth et al., 2014]. In particular, the Hadamard problem is a paradigmatic linear problem that shows that EGU can converge dramatically faster than GD when the instances are dense and the target weight vector is sparse [Kivinen et al., 1997; Vishwanathan and Warmuth, 2005]. This property is linked to the strong-convexity of the relative entropy w.r.t. the $L_1$-norm [Shalev-Shwartz et al., 2012].

Contributions In this paper, we introduce a family of tempered updates (parameterized by a temperature $\tau \in \mathbb{R}$) that interpolate between GD (with $\tau = 0$) and EGU (with $\tau = 1$) while covering a wider class of updates such as those motivated using the Burg and inverse divergences. Next, we show that all these updates can be unified as GD updates via a simple reparameterization.\footnote{The normalized version is called EG and the two-sided version EGU\textsuperscript{±}. More about this later.}
For a strictly convex, continuously-differentiable function $F : \mathcal{C} \to \mathbb{R}$ with convex domain $\mathcal{C} \subseteq \mathbb{R}^d$, the Bregman divergence between $\tilde{w}, w \in \mathcal{C}$ is defined as

$$\Delta_F(\tilde{w}, w) = F(\tilde{w}) - F(w) - f(w) \cdot (\tilde{w} - w),$$

where $f := \frac{\partial F}{\partial w} = \nabla_w F$ denotes the gradient of $F$, sometimes called the link function. Trading off the divergence to the last parameter $w_s$ with the current loss lets us motivate the iterative mirror descent (MD) updates [Nemirovsky and Yudin 1983; Kivinen and Warmuth 1997]:

$$w_{s+1} = \arg\min_w \frac{1}{\eta} \Delta_F(w, w_s) + L(w),$$

where $\eta > 0$ is often called the learning rate. Solving for $w_{s+1}$ yields the so-called prox or implicit update (Rockafellar 1976):

$$f(w_{s+1}) = f(w_s) - \eta \nabla_w L(w_{s+1}).$$

This update is typically approximated by the following explicit update that uses the gradient at the old parameter $w_s$ instead:

$$f(w_{s+1}) = f(w_s) - \eta \nabla_w L(w_s).$$

We now show that the CMD update (1) can be motivated similarly by replacing the Bregman divergence in the minimization problem (2) with a “Bregman momentum” which quantifies the rate of change in the value of Bregman divergence as $w(t)$ varies over time: For the convex function $F$, we define the Bregman momentum between $w(t), w_s \in \mathcal{C}$ as the time differential of the Bregman divergence induced by $F$:

$$\dot{\Delta}_F(w(t), w_s) = \dot{f}(w(t)) - f(w_s) \cdot \dot{w}(t) = (f(w(t)) - f(w_s)) \cdot \dot{w}(t).$$

**Proposition 1.** The CMD update

$$\dot{f}(w(t)) = -\eta \nabla_w L(w(t)), \text{ with } w(s) = w_s,$$

is the solution of the following functional:

$$\min_{w(t)} \left\{ \frac{1}{\eta} \dot{\Delta}_F(w(t), w_s) + L(w(t)) \right\}.$$  

**Proof.** Setting the derivatives w.r.t. $w(t)$ to zero, we have

$$0 = \frac{\partial}{\partial w(t)} \left( (f(w(t)) - f(w_s)) \cdot \dot{w}(t) + \eta L(w(t)) \right)$$

$$= H_F(w) \dot{w}(t) + \frac{\partial f(w(t))}{\partial w(t)} (f(w(t)) - f(w_s)) + \eta \nabla_w L(w(t))$$

$$= \dot{f}(w(t)) + \eta \nabla_w L(w(t)) = 0,$$

where $\eta$ is the learning rate.
where we use the fact that \( w(t) \) and \( \dot{w}(t) \) are independent variables, therefore \( \frac{\partial \bar{u}(t)}{\partial w(t)} = 0 \).

Note that the implicit update \( (3) \) and the explicit update \( (4) \) can both be realized as the backward and the forward Euler approximations of \( (1) \), respectively.

We can provide an alternative definition of Bregman momentum in terms of the dual of \( F \). Function if \( F^*(\theta) = \sup_{\omega \in \mathcal{S}} (\theta \cdot \omega - F(\omega)) \) denotes the Fenchel dual of \( F \) and \( \bar{w} = \arg\sup_{\omega \in \mathcal{S}} (\theta \cdot \omega - F(\omega)) \), then the following relation holds between the pair of dual variables \( (w, \theta) \):

\[
\dot{w}(t) = f^*(\theta(t)) = H_F(\theta(t)) \dot{\theta}(t),
\]

\[
\dot{\theta}(t) = f(w(t)) = H_F(w(t)) \dot{w}(t).
\]

This pairing lets us rewrite the Bregman momentum in its dual form:

\[
\Delta_F(w(t), w_0) = \Delta_{F^*}(\theta(t), \theta(0)) = (\theta(t) - \theta(0))^T H_{F^*}(\theta(t)) \dot{\theta}(t).
\]

An expanded derivation is given in Appendix B. Using \( (1) \), we can rewrite the CMD update \( (1) \) as

\[
\dot{w}(t) = -\eta \nabla_w L(w(t)),
\]

which corresponds to a natural gradient update w.r.t. the Riemannian metric \( H_F \). Using \( H_F(w) = H_F^{-1}(\theta) \) and \( \nabla_w L(w) = H_F^{-1}(\theta) \nabla_{\theta} L \circ f^*(\theta) \), the update can also be written equivalently in the dual domain \( \theta \) as a natural gradient update w.r.t. the Riemannian metric \( H_{F^*} \), or as a CMD with the link \( f^* \):

\[
\dot{\theta}(t) = -\eta H_{F^*}^{-1}(\theta(t)) \nabla_{\theta} L \circ f^*(\theta(t)),
\]

\[
\dot{f}^*(\theta(t)) = -\eta H_{F^*}^{-1}(\theta(t)) \nabla_{\theta} L \circ f^*(\theta(t)).
\]

The above equivalence between the primal and dual versions of the natural gradient update can also be seen as a special form of the reparameterization method presented in the next section (Corollary 1).

The CMD update naturally generalizes to the case when there exists a number of constraints on the parameter \( w(t) \). Essentially, the gradient on the r.h.s. is replaced by a projected gradient (Proof given in Appendix B).

**Proposition 2.** The CMD update with the additional constraint \( \psi(w(t)) = 0 \) for some function \( \psi : \mathbb{R}^d \rightarrow \mathbb{R}^m \) s.t. \( \{w \in \mathcal{C} | \psi(w(t)) = 0 \} \) is non-empty, amounts to the projected gradient update

\[
\dot{w}(t) = -\eta P_{\psi}(w(t)) \nabla_w L(w(t)),
\]

where (denoting by \( J_{\psi}(w(t)) := \nabla_w \psi(w(t)) \))

\[
P_{\psi} := I_d - H_F^{-1} J_{\psi}^T (J_{\psi} H_F^{-1} J_{\psi}^T)^{-1} J_{\psi},
\]

is the projection matrix onto the tangent space at \( w(t) \). Equivalently, the update can be written as a projected natural gradient descent update

\[
\dot{w}(t) = -\eta P_{\psi}(w(t)) \nabla_w L(w(t)).
\]

3 REPARAMETERIZATION

We now establish the first main result of the paper.

**Theorem 1.** Let \( w = q(u) \in \mathbb{R}^d \) where \( u \in \mathbb{R}^k \) with \( k \geq d \) is a vector of parameters. The CMD update on \( w \) w.r.t. the convex function \( F \),

\[
\dot{w}(t) = -\eta \nabla_w L(w(t)),
\]

coincides with the CMD update for parameters \( u \) using the convex function \( G \) (and link \( g := \nabla_u G \)) on the composite loss \( L \circ q \),

\[
\dot{u}(t) = -\nabla_u L \circ q(u(t)).
\]

provided that range(\( q \) \( ) \subseteq \text{dom}(G) \) holds and we have \( H_F^{-1}(w) = J_q(u) H_G^{-1}(u) J_q(u)^T \) for all \( w = q(u) \).

**Proof.** Note that \( \dot{w}(t) = \frac{\partial w(t)}{\partial u(t)} \dot{u}(t) = J_q(u(t)) \dot{u}(t) \). Also, \( \nabla_u L \circ q(u(t)) = J_q(u(t))^T \nabla_w L(w(t)) \).

The CMD update on \( u \) with the link function \( g(u) \) can be written as \( \dot{u}(t) = -\eta H_G^{-1}(u(t)) \nabla_u L \circ q(u(t)) \). Thus (dropping \( t \) for simplicity),

\[
\dot{u} = -\eta H_G^{-1}(u) J_q(u)^T \nabla_u L(w).
\]

Multiplying by \( J_q(u) \) from the left yields

\[
\dot{w} = -\eta J_q(u) H_G^{-1}(u) J_q(u)^T \nabla_w L(w).
\]

Comparing the result to \( (9) \) concludes the proof. \( \square \)

Note that Theorem 1 shows, in general, how the local geometry is affected by the reparameterization function \( q \). For instance, the primal \( (1) \) and dual \( (11) \) equivalence is an immediate consequence\(^4\).

**Corollary 1.** The primal \( (1) \) and dual \( (11) \) CMD updates are equivalent via the special case of reparameterization where \( w(t) = f^*(\theta(t)) \) (or \( \theta(t) = f(w(t)) \)).

\(^4\)Note that the equivalence of the primal-dual updates was already shown in Warmuth and Jagota (1998) for the continuous case and in (Raskutti and Mukherjee, 2015) for the discrete case (where it is only one-sided).
In the following, we also provide the constrained reparameterized updates for completeness, for which the proof is given in Appendix D.

**Theorem 2.** The constrained CMD update \eqref{cmd} coincides with the reparameterized projected gradient update on the composite loss,

\[
g^*(u(t)) = -\eta P_{\psi q}(u(t))\nabla_u L \circ q(u(t)) ,
\]

where \((J_{\psi q}(u(t)) := J_q^T (u(t))\nabla_w \psi(q(u(t)))\)

\[
P_{\psi q} := I_k - H_{\psi}^{-1} J_{\psi q}^T (J_{\psi q} H_{\psi}^{-1} J_{\psi q}^T)^{-1} J_{\psi q} ,
\]

is the projection matrix onto the tangent space at \(u(t)\).

An important case is when \(g\) is the identity function, i.e. the reparameterized CMD update is continuous GD on \(u\). We now show that EGU and many other updates can be performed using GD via a simple reparameterization. For this, we first introduce the tempered Bregman divergence.

### 4 TEMPERED BREGMAN UPDATES

The tempered relative entropy divergence \cite{Amid2019} is defined based on the tempered logarithm link function \cite{Naudts2002}:

\[
f_\tau(w) = \log_\tau(w) = \frac{1}{1 - \tau}(w^{1-\tau} - 1),
\]

for \(w \in \mathbb{R}^d_{\geq 0}\) and \(\tau \in \mathbb{R}\). The \(\log_\tau\) function is shown in Figure 2 for different values of \(\tau \geq 0\). Note that \(\tau = 1\) recovers the standard log function as a limit point. The \(\log_\tau(w)\) link function is the gradient of the convex function

\[
F_\tau(w) = \sum_i \left( w_i \log_\tau w_i + \frac{1}{2 - \tau} (1 - w_i^{2-\tau}) \right)
\]

\[
= \sum_i \left( \frac{1}{(1 - \tau)(2 - \tau)} w_i^{2-\tau} - \frac{1}{1 - \tau} w_i + \frac{1}{2 - \tau} \right).
\]

The convex function \(F_\tau\) induces the following tempered Bregman divergence \footnote{The second form is more commonly known as \(\beta\)-divergence \cite{Cichocki2010} with \(\beta = 2 - \tau\).}

\[
\Delta_{F_\tau}(\tilde{w}, w) = \sum_i \left( \tilde{w}_i \log_\tau \tilde{w}_i - \tilde{w}_i^{2-\tau} - w_i^{2-\tau} \right)
\]

\[
= \frac{1}{1 - \tau} \sum_i \left( \frac{w_i^{2-\tau} - w_i^{2-\tau}}{2 - \tau} - (\tilde{w}_i - w_i) w_i^{1-\tau} \right) . \tag{14}
\]

For \(\tau = 0\), we obtain the squared Euclidean divergence \(\Delta_{F_0}(\tilde{w}, w) = \frac{1}{2} \| \tilde{w} - w \|^2\) and for \(\tau = 1\), the relative entropy \(\Delta_{F_1}(\tilde{w}, w) = \sum_i (\tilde{w}_i \log(\tilde{w}_i/w_i) - \tilde{w}_i + w_i)\) (See \cite{Amid2019} for an extensive list of examples).

In the following, we derive the CMD updates using the time derivative of \eqref{log_tau} as the tempered Bregman momentum. Notice that the link function \(\log_\tau(x)\) is only defined for \(x \geq 0\) when \(0 < \tau\). In order to have a weight \(w \in \mathbb{R}^d\), we use the EGU trick \cite{Kivinen1997} by maintaining two non-negative weights \(w_+\) and \(w_-\) and setting \(w = w_+ - w_-\). We call this the **tempered EGU** update. As our second main result, we show that that continuous tempered EGU updates interpolate between continuous GD and continuous EGU (for \(\tau \in [0, 1)\)). Furthermore, these updates can be simulated by continuous GD on a new set of parameters \(u\) using a simple reparameterization. We show that on the underdetermined linear regression problem, under certain assumptions, these updates converge to the solution with the smallest \(L_{2-\tau}\)-norm.

#### 4.1 TEMPERED EGU±

We first introduce the generalization of the EGU± updates using the tempered Bregman divergence \eqref{log_tau}. Let \(w(t) = w_+(t) - w_-(t)\) with \(w_+(t), w_-(t) \in \mathbb{R}^d_{\geq 0}\) and \(w_+(0) = w_-(0) = w_0\). The tempered EGU± updates are motivated by

\[
\argmin_{w_+, w_- \in \mathbb{R}^d_{\geq 0}} \left\{ \frac{1}{\eta} \dot{B}_\tau + L(w(t)) \right\},
\]

where \(\dot{B}_\tau := \Delta_{F_\tau}(w_+(t), w_0) + \Delta_{F_\tau}(w_-(t), w_0)\).

Enforcing the constraints using the Lagrange multipliers \(\lambda_+(t), \lambda_-(t) \in \mathbb{R}^d_{\geq 0}\), we have

\[
\log_\tau w_+(t) = -\eta \nabla_w L(w(t)) + \lambda_+(t) , \quad \log_\tau w_-(t) = +\eta \nabla w L(w(t)) + \lambda_-(t) . \tag{15}
\]
Using $\log \cdot w(t) = \tilde{w}(t) \odot w(t) \odot$ and applying the KKT conditions $w_+(t) \odot \lambda_+(t) = 0$ and $w_-(t) \odot \lambda_-(t) = 0$ gives

$$
\begin{align*}
\dot{w}_+(t) &= (w_+(t))^{\odot} \odot \left( - \eta \nabla_w L(w(t)) \right), \\
\dot{w}_-(t) &= (w_-(t))^{\odot} \odot \left( + \eta \nabla_w L(w(t)) \right),
\end{align*}
$$

which means we can simply ignore $\lambda_+(t)$ and $\lambda_-(t)$ in (15). Integrating (15) over $t$ and applying the inverse link yields

$$
\begin{align*}
w_+(t) &= \exp_{\tau}(\log \cdot w_0 - \eta \int_0^t \nabla_w L(w(z)) \, dz), \\
w_-(t) &= \exp_{\tau}(\log \cdot w_0 + \eta \int_0^t \nabla_w L(w(z)) \, dz),
\end{align*}
$$

where $\exp_{\tau}(\cdot) := [1 + (1 - \tau) x]^{1/\tau}$. Note that $\tau = 1$ is a limit case which recovers the standard exp function and the updates (17) become the standard EGU$\pm$. Additionally, the GD updates are recovered at $\tau = 0$. As a result, the tempered EGU$\pm$ updates (17) interpolate between GD and EGU$\pm$ for $\tau \in [0, 1]$ and generalize beyond for values of $\tau > 1$ and $\tau < 0$.

Note that the tempered EGU$\pm$ updates make use of two $d$-dimensional non-negative weight vectors and then use a $d$-dimensional difference for inference. The EGU$\pm$ updates (for which the double weight trick was introduced originally) can be reformulated using the $d$-dimensional link function $\arcsinh(w)$ which behaves as a “two-sided logarithm”. Also the associated $\arcsinh(w)$ based Bregman divergence (Warmuth [2007]; Ghai et al. [2019]) can be used to analyze EGU$\pm$. In the appendix, we show that for the special case of $w_+(0) = w_-(0) = 1$, the tempered EGU$\pm$ updates can also be represented using the $d$-dimensional tempered link function $\arcsinh(w)$.

### 4.2 REPARAMETERIZED TEMPERED EGU$\pm$

We now show that the updates (15) can be obtained via a reparameterization on a set of weights $u_+, u_- \in \mathbb{R}^d$.

**Proposition 3.** The updates (15) can be realized by GD updates on $u_+, u_- \in \mathbb{R}^d$ and setting $w_+ = q_\tau(u_+)$ and $w_- = q_\tau(u_-)$ where

$$q_\tau(u) = \left( \frac{2}{2 - \tau} \right) - \frac{2}{\tau - \tau} |u|^{\frac{2}{\tau - \tau}}, \tau \neq 2. \quad (18)$$

**Proof.** Note that the Jacobian is

$$J_{q_\tau}(u_+) = \left( \frac{2}{2 - \tau} \right) \frac{2}{\tau - \tau} \mathrm{diag} \left( \text{sign}(u_+) \odot |u_+|^{\frac{2}{\tau - \tau}} \right).$$

Thus, $J_{q_\tau}(u_+) J_{q_\tau}^{-1}(u_+) = \text{diag} \left( (w_+)^{\odot} \right)$. A similar construction holds for $w_-$ and $u_-$. Applying the results of Theorem 3 yields the same dynamics as in (15). For completeness, we write the reparameterized tempered EGU$\pm$ updates as

$$
\begin{align*}
\dot{u}_+(t) &= -\eta \nabla_{u_+} L \odot w_+(t), \\
\dot{u}_-(t) &= -\eta \nabla_{u_-} L \odot w_-(t),
\end{align*}
$$

where $w(u_+, u_-) := q_\tau(u_+) - q_\tau(u_-)$. Note that in the reparameterized update, the non-negativity of the reparameterized weights is naturally imposed by the absolute value in the reparameterization.

Special cases of the reparameterized tempered EGU$\pm$ updates include GD with $\tau = 0$ and EGU$\pm$ with $\tau = 1$, where the latter corresponds to $w = u_+ \odot u_- - u_- \odot u_-$. This form of updates was recently discussed in (Vaskevicus et al. [2019]) for sparse signal recovery from an underdetermined system of linear measurements. Similar approaches have been applied for mapping the replicator dynamics from the simplex to the unit sphere (Akимов [1979]; Sandholm [2010]). Additionally, values of $0 < \tau < 1$ interpolate between GD and EGU. However, any other value of $\tau \in \mathbb{R} \setminus \{2\}$ also corresponds to a valid update. Note that $\tau = 2$ is a special case, which according to (Amid et al. [2019]) corresponds to the Burg divergence

$$\Delta_{\text{Burg}}(\tilde{w}, w) = \sum_i \left( \frac{w_i}{\tilde{w}_i} - \log \left( \frac{w_i}{\tilde{w}_i} \right) - 1 \right).$$

To find the update for Burg, we point out that the shifted version of reparameterization (18),

$$q_\tau(u) = \left( \frac{2}{2 - \tau} \right) - \frac{2}{\tau - \tau} |u|^{\frac{2}{\tau - \tau}},$$

also satisfies the conditions in Proposition 3. For this choice of reparameterization, we have $q_2(u) = \exp(u)$. For simplicity, we consider the reparameterization in Eq. (18) and adopt $q_2(u) := \exp(u)$ for the special case of $\tau = 2$.

### 5 MINIMUM-NORM SOLUTIONS FOR LINEAR REGRESSION

Gunasekar et al. [2017] showed that on the underdetermined linear regression problem, using the factorization $W = UU^\top$ for the weight matrix $W$ and applying continuous-time GD on $U$ achieves the minimum $L_1$-norm solution. Note that in the vector (i.e. diagonal weight matrix) case, this corresponds to setting $w = u \odot u$ and running continuous-time GD on $u$, which according to Theorem 1 is equivalent to running EGU update on $w$. The fact that the EGU update favors the minimum $L_1$-norm solution is linked to the strong-convexity of the negative entropy function $F_1(w) = \sum_i (w_i \log w_i - w_i)$ (which induces the EGU updates) w.r.t. the $L_1$-norm.
The focus here is to show that, under similar assumptions, the tempered updates converge to the solution with the smallest $L_{2-\tau}$-norm when $\tau \in [0, 1]$. For a start, the following result from Amid et al. [2019] establishes strong-convexity of $F_\tau$ function.

**Remark 1 (Amid et al. [2019]).** The function $F_\tau$, with $0 \leq \tau \leq 1$, is $R^{-\tau}$-strongly convex over the set \( \{ w \in \mathbb{R}^d_+ : \|w\|_{2-\tau} \leq R \} \) w.r.t. the $L_{2-\tau}$-norm.

The strong convexity of the $F_\tau$ function w.r.t. the $L_{2-\tau}$-norm suggests that the updates motivated by the tempered Bregman divergence (14) yield the minimum $L_{2-\tau}$-norm solution in certain settings. We verify this by considering the vector and matrix updates for the linear regression problem.

**5.1 VECTOR CASE**

Let \( \{ x_n, y_n \}_{n=1}^N \) denote the set of input-output pairs and let $X$ be the design matrix for which the $n$-th row is equal to $x_n$. Also, let $y$ denote the vector of targets. Consider the tempered EGU updates (15) on the weights $w(t) = w_+(t) - w_-(t)$ where $w_+(t), w_-(t) \geq 0$ and $w_+(0) = w_-(0) = w_0$. Following (17), we have

\[
\begin{align*}
    w_+(t) &= \exp_\tau \left( \log_\tau w_0 - \eta \int_0^t X^\top \delta(z) \, dz \right), \\
    w_-(t) &= \exp_\tau \left( \log_\tau w_0 + \eta \int_0^t X^\top \delta(z) \, dz \right),
\end{align*}
\]

where $\delta(t) = X (w_+(t) - w_-(t))$.

**Theorem 3.** Consider the underdetermined linear regression problem where $N < d$. Let $E = \{ w \in \mathbb{R}^d : Xw = y \}$ be the set of solutions with zero error. Given $w(\infty) \in E$, then the tempered EGU updates (17) with temperature $0 \leq \tau \leq 1$ and initial solution $w_0 = \alpha 1$ converge to the minimum $L_{2-\tau}$-norm solution in $E$ in the limit $\alpha \to 0$.

**Proof.** We show that the solution of the tempered EGU satisfies the dual feasibility and complementary slackness KKT conditions for the following optimization problem (omitting $t$ for simplicity):

\[
\begin{align*}
    \min_{w_+, w_-} & \quad \| w_+ - w_- \|_{2-\tau}^2, \\
    \text{s.t.} & \quad X (w_+ - w_-) = y \quad \text{and} \quad w_+, w_- \geq 0.
\end{align*}
\]

Imposing the constraints using a set of Lagrange multipliers $\nu_+, \nu_- \geq 0$ and $\lambda \in \mathbb{R}$, we have

\[
\begin{align*}
    \min_{w} \sup_{\nu_+, \nu_- \geq 0, \lambda} \left\{ \| w_+ - w_- \|_{2-\tau}^2 + \lambda^\top \left( X (w_+ - w_-) - y \right) - \nu_+^\top \nu_+ - \nu_-^\top \nu_- \right\}.
\end{align*}
\]

The set of KKT conditions are

\[
\begin{align*}
    w_+, w_- & \geq 0, \\
    Xw & = y, \\
    + \text{sign}(w) \odot |w|^{(1-\tau)} - X^\top \lambda & \not\equiv 0, \\
    - \text{sign}(w) \odot |w|^{(1-\tau)} + X^\top \lambda & \equiv 0, \\
    (\text{sign}(w) \odot |w|^{(1-\tau)} - X^\top \lambda) \odot w_+ & = 0, \\
    (\text{sign}(w) \odot |w|^{(1-\tau)} - X^\top \lambda) \odot w_- & = 0,
\end{align*}
\]

where $w = w_+ - w_-$. The first condition is imposed by the form of the updates (17) and the second condition is satisfied by the assumption at $t \to \infty$. Using $w_0 = \alpha 1$ with $\alpha \to 0$, we have

\[
\begin{align*}
    w_+(t) &= \exp_\tau \left( -\frac{1}{1-\tau} - \eta \int_0^t X^\top \delta(z) \, dz \right) \\
    &= \left[ - (1-\tau) \eta \int_0^t \delta(z) \right]_{\tau=0}^{\infty}, \\
    w_-(t) &= \exp_\tau \left( -\frac{1}{1-\tau} + \eta \int_0^t X^\top \delta(z) \, dz \right) \\
    &= \left[ + (1-\tau) \eta \int_0^t \delta(z) \right]_{\tau=0}^{\infty}.
\end{align*}
\]

Setting $\lambda = -(1-\tau) \eta \int_0^\infty \delta(z)$ satisfies the remaining KKT conditions.

**Corollary 2.** Under the assumptions of Theorem 3, the reparameterized tempered EGU updates (19) also recover the minimum $L_{2-\tau}$-norm solution where $w(t) = q_\tau(u_+(t)) - q_\tau(u_-(t))$.

**5.2 PARTIAL RESULTS ON THE MATRICES**

The linear regression problem can be generalized to the matrix case where the inputs $X_n$ are symmetric matrices and the goal is to learn a symmetric weight matrix $W \in \mathbb{S}^d$ which minimizes the squared error

\[
\min_{W \in \mathbb{S}^d} \frac{1}{2} \| X(W) - y \|_2^2,
\]

in which $X(W) := \text{tr} (X_n W)$.

The generalization of Theorem 1 to symmetric matrices $W \in \mathbb{S}^d$ requires a machinery to handle the matrix-matrix derivatives for the Jacobians. Instead, we first consider a natural extension of the reparameterized tempered EGU to the symmetric PSD matrices: given $U_+, U_- \in \mathbb{R}^{d \times d}$, we define

\[
\begin{align*}
    W_+ &= \left( \frac{1}{2-\tau} \right)^{-\frac{1}{2-\tau}} (U_+ U_+^\top)^{\frac{1}{2-\tau}}, \\
    W_- &= \left( \frac{1}{2-\tau} \right)^{-\frac{1}{2-\tau}} (U_- U_-^\top)^{\frac{1}{2-\tau}},
\end{align*}
\]

(21)
and set $W = W_+ - W_-$. The solution for the linear regression problem using the reparameterization \((21)\) can be obtained by solving

$$
\begin{align*}
\dot{U}_+ &= -\eta X^*(\delta) (U_+ U_+)^{-\frac{1}{2+\tau}} U_+ , \\
W_+^{2-\tau} &= \dot{U}_+ U_+^T + U_+ \dot{U}_+^T \\
&= -\eta \left( X^*(\delta) W_+ + W_+ X^*(\delta) \right),
\end{align*}
$$

(22)

where $\delta(t) = X(W(t)) - y$ and $X^*(\nu) = \sum_n X_n \nu_n$. Also, the constant factors are absorbed into the learning rate. A similar set of equations hold for $W_-$ with a sign flip. Solving for $W_+$ and $W_-$ yields

$$
\begin{align*}
W_+(t) &= \left[ W_0^{2-\tau} - \eta \int_s^t S(\delta(z), W_+(z)) \, dz \right]^{\frac{1}{2-\tau}}, \\
W_-(t) &= \left[ W_0^{2-\tau} + \eta \int_s^t S(\delta(z), W_-(z)) \, dz \right]^{\frac{1}{2-\tau}}.
\end{align*}
$$

(23)

where $S(\delta, W_{\pm}) = X^*(\delta)W_{\pm} + W_{\pm} X^*(\delta)$. Update \((23)\) is a generalization of the update given in [Gunasekar et al. 2017], which is obtained by setting $\tau = 1$.

We generalize the results of [Gunasekar et al. 2017] as follows: In the special case where the inputs $X_n$ are symmetric and commutative, i.e., $X_n X_n' = X_n' X_n$ for all $n, n'$, the following result holds.

**Proposition 4.** Given that the inputs $X_n$ to the matrix linear regression problem are symmetric and commutative, if $X(W(\infty)) = y$ holds, the reparameterized update \((23)\) converges to the minimum $L_{2-\tau}$ matrix norm solution when $W_0 = \lim_{\alpha \to 0} \alpha I$.

**Proof.** Note that the optimization problem corresponds to

$$
\min_{W_+, W_-} \|W_+ - W_-\|^{2-\tau}, \quad \text{s.t.} \quad X(W_+ - W_-) = y.
$$

To prove the proposition, it suffices to show that the solution satisfies the following KKT conditions:

$$
\begin{cases}
W_+, W_- \succeq 0 \\
X(W) = y \\
X^*(\nu) \preceq \text{sign}(W) |W|^{1-\tau} \\
\text{sign}(W) |W|^{1-\tau} \preceq X^*(\nu) \\
\left( \text{sign}(W) |W|^{1-\tau} - X^*(\nu) \right) W_+ = 0 \\
\left( \text{sign}(W) |W|^{1-\tau} - X^*(\nu) \right) W_- = 0
\end{cases}
$$

for some $\nu \in \mathbb{R}^n$. Also, $W = W_+ - W_-$ and $\text{sign}(A) = \text{sign}(VDV^\top) = V \text{sign}(D)V^\top$ is the sign function applied to the eigenvalues. The first condition is satisfied by the form of the updates. The second condition follows from the assumption. To show the remaining conditions, first note that the symmetric and commutative inputs $X_n$ are diagonalizable, thus the eigenvectors of $X^*(\nu)$ are fixed and the eigenvalues can be written as $\lambda_i(\nu)$, for $i \in [d]$.

In the limit $W_0 = \lim_{\alpha \to 0} \alpha I$, the reparameterization updates \((23)\) can be written as

$$
\begin{align*}
W_+(t) &= \left[ -\eta \int_s^t S(\delta(z), W_+(z)) \, dz \right]^{\frac{1}{2-\tau}}, \\
W_-(t) &= \left[ +\eta \int_s^t S(\delta(z), W_-(z)) \, dz \right]^{\frac{1}{2-\tau}},
\end{align*}
$$

where $[A]_+ = [VDV^\top]_+ = V[D]_+ V^\top$ is applied to the eigenvalues. Since both $W_+(t)$ and $W_-(t)$ share the same eigenvectors as $X^*(\nu)$ for all $\nu$, it suffices to show that the eigenvalues satisfy the complementary slackness and dual feasibility KKT conditions. Let $\omega_{+,i}(t)$ and $\lambda_i(t)$ denote the $i$-th eigenvector of $W_+(t)$ and $X^*(\nu)$, respectively. For fixed eigenvectors, Equation \((22)\) imposes the following differential equation on the eigenvalues

$$
\frac{d}{dt} \omega_{+,i}^{2-\tau} = 2 - \tau \omega_{+,i}^{2-\tau} \omega_{+,i} \\
-\eta (\lambda_i \omega_{+,i} + \omega_{+,i} \lambda_i) = -2 \eta \omega_{+,i} \lambda_i.
$$

A similar ODE holds for $\omega_{-,i}(t)$ with a sign flip. Imposing the constraint $\omega_{+,i}(t) \geq 0$ with the boundary condition $\omega_{+,i}(0) = 0$ yields

$$
\begin{align*}
\omega_{+,i}(t) &= \left[ -\frac{2 \eta}{2 - \tau} \int_0^t \lambda_i(z) \, dz \right]^{\frac{1}{2-\tau}}, \\
\omega_{-,i}(t) &= \left[ +\frac{2 \eta}{2 - \tau} \int_0^t \lambda_i(z) \, dz \right]^{\frac{1}{2-\tau}}.
\end{align*}
$$

Thus, the updates can be written

$$
\begin{align*}
W_+(t)^{1-\tau} &= \left[ -c \int_0^t X^*(\delta(z)) \, dz \right]_+, \\
W_-(t)^{1-\tau} &= \left[ +c \int_0^t X^*(\delta(z)) \, dz \right]_+,
\end{align*}
$$

(24)

for some constant $c > 0$. This proves the existence of $\nu$ so that the KKT conditions are satisfied. The proof for the case $\tau = 1$ follows similarly to [Gunasekar et al. 2017] using a limit argument. \hfill \square

Next, we provide an alternative dynamic based on generalizing the tempered Bregman divergence \((14)\) to the positive semi-definite (PSD) matrices. Let us first introduce the matrix $\log_\tau$ operator as follows:

$$
\log_\tau A = V \log_\tau(D) V^\top,
$$

where $[D]_+ = [VDV^\top]_+ = V[D]_+ V^\top$.\hfill \square
where \( \text{log}_\tau(D) \) is the diagonal matrix for which the \( \log \tau \) function is applied to the diagonal elements. The matrix tempered Bregman divergence can be defined as

\[
\Delta_F(W, \tilde{W}) = \text{tr} \left( \tilde{W} \log \tau(W) - W \log \tau(W) - \frac{1}{2} (W^{2-\tau} - \tilde{W}^{2-\tau}) \right).
\]

(25)

The continuous-time learning problem for a general symmetric matrix \( W(t) \) can again be achieved via introducing a pair of PSD matrices \( W_+(t), W_-(t) \geq 0 \) such that \( W_+(0) = W_-(0) = W_0 \) and \( W(t) = W_+(t) - W_-(t) \). Following a similar derivation as in the vector case \( \text{(15)} \) and imposing the positive semidefiniteness constraints in the form of Lagrange multipliers \( \lambda_+, \lambda_- \geq 0 \) applied to the eigenvalues of \( W_+, W_-, \) we have

\[
\text{log}_\tau W_+ = -\eta \nabla_W L(W) + V_+ \text{diag}(\lambda_+)^T,
\]

\[
\text{log}_\tau W_- = +\eta \nabla_W L(W) + V_- \text{diag}(\lambda_-)^T,
\]

(26)

where \( V_\pm \) denotes the matrix of eigenvectors of \( W_\pm \). Integrating the r.h.s. yields the following update equations

\[
W_+(t) = \exp_\tau \left( \text{log}_\tau W_0 - \eta \int_0^t \nabla_W L(W(z)) \; dz \right),
\]

\[
W_-(t) = \exp_\tau \left( \text{log}_\tau W_0 + \eta \int_0^t \nabla_W L(W(z)) \; dz \right),
\]

(27)

where matrix \( \exp_\tau \) is defined similarly in terms of the eigen-decomposition of the input.

However, the dynamic induced by the tempered Bregman divergence in \( \text{(26)} \) is different than the one obtained from the reparameterization \( \text{(21)} \). As an example, consider the matrix EGU\( \pm \) case for \( \tau = 1 \). The GD update on \( U_+ \) yields

\[
\dot{U}_+ = -\nabla_W L(W) U_+,
\]

\[
\dot{W}_+ = U_+^T U_+ + U_+ \dot{U}_+^T,
\]

\[
= - \left( W_+ \nabla_W L(W) + \nabla_W L(W) W_+ \right),
\]

which is different than the dynamic given in \( \text{(25)} \) for \( \tau = 1 \). (The dynamic for \( W_- \) follows similarly with a sign flip.) Nevertheless, we show that for the underdetermined linear regression problem with commutative inputs, the tempered Bregman updates \( \text{(27)} \) yield the same minimum-norm solution in the limit when the initial solution goes to zero.

Subsituting for the gradients in \( \text{(27)} \) yields

\[
W_+(t) = \exp_{\tau} \left( \text{log}_\tau W_0 - \eta \int_0^t X^* (\delta(z)) \; dz \right),
\]

\[
W_-(t) = \exp_{\tau} \left( \text{log}_\tau W_0 + \eta \int_0^t X^* (\delta(z)) \; dz \right),
\]

(28)

where \( \delta(t) = X(W(t)) - y \) and \( X^*(v) = \sum_n X_n v_n \).

**Proposition 5.** Given that the inputs \( X_n \) to the matrix linear regression problem are symmetric and commutative, if \( X(W(\infty)) = y \) holds, the tempered matrix EGU\( \pm \) update \( \text{(28)} \) converges to the minimum \( L_{2-\tau} \) matrix norm solution when \( W_0 = \lim_{\alpha \to 0_+} \alpha I \).

The proof is given in Appendix [F].

### 6 EXPERIMENTS

In this section, we first consider the GD, EGU\( \pm \), and tempered EGU\( \pm \) updates and their reparameterizations for an underdetermined linear regression problem. Next, we provide preliminary results on reparameterizing deep neural networks.

#### 6.1 MINIMUM-NORM SOLUTIONS FOR LINEAR REGRESSION

We apply the GD, EGU\( \pm \), and the tempered EGU\( \pm \) \((\tau = 0.6)\) updates on a toy underdetermined linear regression problem. For the experiment, we consider \( N = 100 \) samples of \( d = 200 \) dimensional zero-mean unit-variance Gaussian inputs and a linear target obtained with a random Gaussian weight. We apply the updates as well as their reparameterized forms using the weights \( u_+, u_- \in \mathbb{R}^d \). We consider a small learning rate of \( \eta = 0.05 \) and set the initial solutions \( w_+(0) = w_-(0) = 1e-5 \times 1 \). This corresponds to a zero initial solution for the actual weights \( w(0) = w_+(0) - w_-(0) = 0 \). We apply the updates on the full batch for \( 1e+5 \) iterations. The results are shown in Figure [3a]-[c]. As expected, EGU\( \pm \) and GD converge to the minimum \( L_1 \) and \( L_2 \)-norm solutions, respectively. This is also verified by solving for the minimum \( L_1 \) and \( L_2 \)-norm solutions numerically. Additionally, tempered EGU\( \pm \) achieves the minimum \( L_{2-\tau} \)-norm. Finally, note that the reparameterized versions of the algorithms track the original algorithms very closely.

#### 6.2 REPARAMETERIZING WEIGHTS OF NEURAL NETWORKS

We consider a convolutional neural network for classifying the MNIST handwritten digits dataset [LeCun et al. 2010]. The network consists of two convolutional layers of size 32 and 64, followed by two fully connected layers of size 1024 and 10. We train the first network normally for 100 epochs using SGD. The top-1 accuracy obtained on the test set is 99.13%. Next, we train a second network similarly, except we apply the EGU\( \pm \) reparameterization to the weights of the final layer (size 1024×10) where we expect a high amount of redundancy due to...
Figure 3: Experimental results: (a)-(c) norms of the solutions obtained using GD, EGU±, and tempered EGU± (τ = 0.6) along with their reparameterized forms on an underdetermined linear regression problem. (d) absolute values of (a slice of) the weights of the last layer for the vanilla GD and the reparameterized EGU± networks. The L₁-norm of the GD weights is 571.1 while for reparameterized EGU±, the L₁-norm is 176.7.

the large size of the network. As a result, the reparameterized weights naturally exhibit a high amount of sparsity compared to the weights of the vanilla network, as shown in Figure 3(d). As can be seen, most components of the reparameterized weights are concentrated around zero and only a small fraction of the components have large values. In fact, the L₁-norm of the GD weights is 571.1 while for reparameterized EGU±, the L₁-norm is 176.7. To test the significance of the small weights, we clamp the weights of the final layer for which the absolute values are below a certain threshold, to zero during the inference. The top-1 test accuracy results are shown in Table 1 for different values of the threshold and the levels of sparsity achieved. Note that even with 98.90% sparsity, the reparameterized EGU± network achieves 97.48% test accuracy.

7 CONCLUSION AND FUTURE WORK

In this paper, we discussed the continuous-time mirror descent updates and provided a general framework for reparameterizing these updates. Additionally, we introduced the tempered EGU± updates and their reparameterized forms. The tempered EGU± updates include the two commonly used updates, namely, gradient descent and exponentiated gradient updates, and interpolate between the two updates. We showed that under certain conditions for the underdetermined linear regression problem, the tempered EGU± updates converge to the minimum L₂−τ-norm solution. Finally, we expanded the reparameterized updates to the matrix case, generalizing the results of (Gunasekar et al., 2017).

The current work leads to many interesting future directions:

• The reparameterization equivalence theorem holds only in the continuous-time. Clearly, the equivalence relation breaks down after discretization, however, the discretized reparameterized updates seem to track the original updates in many important cases (Amid and Warmuth, 2020). A key research direction is to find general conditions under which the discretized updates closely track the original continuous counterparts.

• A more general treatment of the underdetermined linear regression requires analyzing the results for arbitrary start vectors. Also, extending the matrix results to non-commutative matrices (as discussed in (Gunasekar et al., 2017)) is still an open problem. Furthermore, developing a matrix form of the reparameterization theorem is left for future work.

• Perhaps the most important application of the current work is reparameterizing the weights of deep neural networks for achieving sparse solutions or obtaining an implicit form of regularization that mimics a trade-off between the ridge and lasso methods (e.g. elastic net regularization (Zou and Hastie, 2005)).

Table 1: Different levels of sparsity achieved by thresholding the weights of the reparameterized last layer and the corresponding top-1 test set accuracy. Even with 98.90% sparsity the network achieves 97.48% test accuracy.

| Threshold       | 0   | 1e−3 | 5e−3 | 1e−2 | 5e−2 | 1e−1 |
|----------------|-----|------|-----|------|------|------|
| Sparsity (%)    | 0.00| 25.58| 62.02| 79.08| 97.55| 98.90|
| Accuracy (%)    | 99.02| 99.02| 99.00| 98.96| 98.20| 97.48|

References

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**A  DUAL FORM OF BREGMAN MOMENTUM**

The dual form of Bregman momentum given in [5] can be obtained by first forming the dual Bregman divergence in terms of the dual variables $\theta(t)$ and $\theta_s$ and taking the time derivative, that is,

$$\Delta_F^*(w(t), w_s) = \Delta_F^*(\theta_s, \theta(t))$$

$$= \frac{\partial}{\partial \theta} \left( F^*(\theta_s) - F^*(\theta(t)) - f^*(\theta(t)) \cdot (\theta_s - \theta(t)) \right)$$

$$= -F^*(\theta(t)) + f^*(\theta(t)) \cdot \dot{\theta}(t)$$

$$+ (\theta(t) - \theta_s) \top H^*_F(\theta(t)) \dot{\theta}(t)$$

where we use the fact that $F^*(\theta(t)) = f^*(\theta(t)) \cdot \dot{\theta}(t)$.

**B  CONSTRAINED UPDATES**

We first provide a proof for Proposition 2. Then, we prove Theorem 2.

*Proof of Proposition 2.* We use a Lagrange multiplier $\lambda(t) \in \mathbb{R}^m$ in (5) to enforce the constraint $\psi(w(t)) = 0$ for all $t \geq 0$.

$$\min_{w(t)} \left\{ \frac{1}{\eta} \Delta_F^*(w(t), w_s) + \lambda(t) \cdot \psi(w(t)) \right\}. \quad (29)$$

Setting the derivative w.r.t. $w(t)$ to zero, we have

$$\dot{f}(w(t)) + \eta \nabla_w L(w(t)) + \dot{J}^\top \psi(w(t)) \lambda(t) = 0, \quad (30)$$

where $J^\psi(w(t)) := \nabla_w \psi(w(t))$. In order to solve for $\lambda(t)$, first note that $\dot{\psi}(w(t)) = J^\psi(w(t)) \dot{w}(t) = 0$. Using the equality $\dot{f}(w(t)) = H^*_F(w(t)) \dot{w}(t)$ and multiplying both sides by $J^\psi(w(t)) H^{-1}_F(w(t)) \lambda(t)$ (ignoring $t$)

$$J^\psi(w(t)) \dot{w}(t) + \eta J^\psi(w(t)) H^{-1}_F(w(t)) \nabla L(w(t))$$

$$+ J^\psi(w(t)) H^{-1}_F(w(t)) J^\psi(w(t)) \lambda(t) = 0$$

Assuming that the inverse exists, we can write

$$\lambda(t) = -\eta \left( J^\psi(w(t)) H^{-1}_F(w(t)) J^\psi(w(t)) \right)^{-1} \cdot J^\psi(w(t)) H^{-1}_F(w(t)) \nabla L(w(t)).$$

Plugging in for $\lambda(t)$ yields (13). Multiplying both sides by $H_F(w)$ and using $\dot{f}(w) = H_F(w) \dot{w}$ yields (12).

**C  DISCRETIZED (CONSTRAINED) UPDATES**

The most straight-forward discretization of the unconstrained CMD update (1) is the forward Euler (i.e. explicit) discretization, given in (4). Note that this corresponds to a (approximate) minimizer of the discretized form of (5), that is,

$$\arg\min_{w} \left\{ \frac{1}{\eta} \left( \Delta_F^*(w, w_s) - \Delta_F^*(w, \tilde{w}_s, \tilde{w}_s) \right) + L(w) \right\} = 0.$$ 

An alternative way of discretizing is to apply the approximation on the equivalent natural gradient form (9), which yields

$$w_{s+1} - w_s = -\eta H^{-1}_F(w_s) \nabla_w L(w_s).$$

Despite being equivalent in continuous-time, the two approximations may correspond to different updates after discretization. As an example, for the EG update motivated by $f(w) = \log w$ link, the latter approximation yields

$$w_{s+1} = w_s \odot (1 - \eta \nabla_w L(w_s)), $$

which corresponds to the unnormalized prod update, introduced by [Cesa-Bianchi et al., 2007] as a Taylor approximation of the original EG update.

The situation becomes more involved for discretizing the constrained updates. As the first approach, it is possible to directly discretize the projected CMD update (12).

$$f(\tilde{w}_{s+1}) - f(w_s) = -\eta P_\psi(\tilde{w}_s) \nabla_w L(w_s).$$

However, note that the new parameter $\tilde{w}_{s+1}$ may fall outside the constraint set $C_\psi := \{ w \in \mathcal{C} | \psi(w) = 0 \}$. As a result, a Bregman projection [Shalev-Shwartz et al., 2012] into $C_\psi$ may need to be applied after the update, that is

$$w_{s+1} = \arg\min_{w \in C_\psi} \Delta_F^*(w, \tilde{w}_{s+1}). \quad (31)$$

As an example, for the normalized EG updates with the additional constraint that $w \cdot 1 = 1$, we have $P_\psi(w) = I_d - 1 w^\top$ and the approximation yields

$$\log (\tilde{w}_{s+1}) - \log (w_s) = -\eta (\nabla_w L(w_s) - 1 \mathbb{E}_{w_s} [\nabla_w L(w_s)])$$

where $\mathbb{E}_{w_s} [\nabla_w L(w_s)] = w_s \nabla_w L(w_s)$. Clearly, $\tilde{w}_{s+1}$ may not necessarily satisfy $\tilde{w}_{s+1} \cdot 1 = 1$. Therefore, we apply

$$w_{s+1} = \frac{\tilde{w}_{s+1}}{\| \tilde{w}_{s+1} \|_1}.$$
which corresponds to the Bregman projection onto the unit simplex using the relative entropy divergence [Kivinen and Warmuth [1997]].

An alternative approach for discretizing the constrained update would be to first discretize the functional objective with the Lagrange multiplier (27) and then (approximately) solve for the update. That is,

\[ w_{s+1} = \arg\min_w \left\{ \frac{1}{\eta} \left( \Delta_F(w, w_s) - \Delta_F(w_s, w_s) \right) \right\} + L(w) + \lambda \cdot \psi(w) . \]

Note that in this case, the update satisfies the constraint \( \psi(w_{s+1}) = 0 \) because of (directly) using the Lagrange multiplier. For the normalized EG update, this corresponds to the original normalized EG update in [Littlestone and Warmuth [1994]],

\[ w_{s+1} = \frac{w_s \circ \exp \left( -\eta \nabla_w L(w_s) \right)}{\left\| w_s \circ \exp \left( -\eta \nabla_w L(w_s) \right) \right\|_1} . \]

Finally, it is also possible to discretize the projected natural gradient update [13]. Again, a Bregman projection into \( C_\psi \) may need to be used after the update, that is,

\[ \tilde{w}_{s+1} - w_s = -\eta P_{\psi}(w_s) \tilde{H}_E^{-1}(w_s) \nabla_w L(w(t)) , \]

followed by (31). For the normalized EG update, the first step corresponds to

\[ w_{s+1} = w_s \circ \left( 1 - \eta \nabla_w L(w_s) - 1 E_{w_s}[\nabla_w L(w_s)] \right) , \]

which recovers to the approximated EG update of [Kivinen and Warmuth [1997]]. Note that \( w_{s+1} \cdot 1 = 1 \) and therefore, no projection step is required in this case.

**D CONstrained REPARAMETERIZED UPDATE**

We provide the proof for Theorem 2.

**Proof of Theorem 2** Similar to the proof of Proposition 3, we use a Lagrange multiplier \( \lambda(t) \in \mathbb{R}_m \) to enforce the constraint \( \psi \circ q(u(t)) = 0 \) for all \( t \geq 0 \),

\[ \min_{u(t)} \left\{ \frac{1}{\eta} \Delta_C(u(t), u_s) \right\} + L \circ q(u(t)) + \lambda(t) \cdot \psi \circ q(u(t)) . \]

Setting the derivative w.r.t. \( u(t) \) to zero, we have

\[ \dot{g}(w(t)) + \eta \nabla_u L \circ q(w(t)) + J_{\psi \circ q}(u(t)) \lambda(t) = 0 , \]

where \( J_{\psi \circ q}(u(t)) := J_q^T(u) \nabla_u \psi(w(t)) \). In order to solve for \( \lambda(t) \), we use the fact that \( \psi \circ q(u(t)) = J_{\psi \circ q}(u(t)) \dot{u}(t) = 0 \). Using the equality \( \dot{g}(u(t)) = H_G(u(t)) \dot{u}(t) \) and multiplying both sides by \( J_{\psi \circ q}(u(t)) H_G^{-1}(u(t)) \) yields (ignoring \( t \))

\[ J_{\psi \circ q}(u) \dot{u} + \eta J_{\psi \circ q}(u) H_G^{-1}(u) \nabla L \circ q(u) \]

\[ + J_{\psi \circ q}(u) H_G^{-1}(u) J_{\psi \circ q}(u) \lambda(t) = 0 . \]

The rest of the proof follows similarly by solving for \( \lambda(t) \) and rearranging the terms.

Finally, applying the results of Theorem 2 concludes the proof.

**E TEMPERED EGU± AS ARCSINH\(\tau\)**

We first define the following two (element-wise) operators on the non-negative vectors \( a, b \in \mathbb{R}^d_{\geq 0} \),

\[ a \odot_\tau b = \exp_\tau(\log_\tau a + \log_\tau b) \quad (\tau\text{-product}) , \]

\[ a \oslash_\tau b = \exp_\tau(\log_\tau a - \log_\tau b) \quad (\tau\text{-division}) . \]

Note that for \( \tau = 0 \), \( a \odot_0 b = a + b \) and \( a \oslash_0 b = a - b \). Additionally, for \( \tau = 1 \), we have \( a \odot_1 b = a \odot b \) and \( a \oslash_1 b = a \oslash b \). Thus, the \( \tau\text{-product} \) (\( \tau\text{-division} \)) can be seen as an operator that interpolates between the addition (subtraction) and multiplication (division) on the non-negative real numbers for values of \( 0 \leq \tau \leq 1 \). Additionally, we define \( \sinh_\tau : \mathbb{R} \to \mathbb{R} \) for \( \tau \in \mathbb{R} \) as

\[ \sinh_\tau(x) = \frac{1}{2}(\exp_\tau(x) - \exp_\tau(-x)) . \]

**Lemma 1.** For any \( w_+(t) w_-(t) \in \mathbb{R}^d_{\geq 0} \), we have (ignoring \( t \))

\[ \log_\tau(w_+) + \log_\tau(w_-) = 0 \iff w_+ \odot_\tau w_- = c , \]

for all \( t \geq 0 \) and for some \( c \in \mathbb{R}^d_{\geq 0} \).

**Proof.** Note that

\[ w_+ \odot_\tau w_- = c \iff \log_\tau(w_+) + \log_\tau(w_-) = \log_\tau(c) . \]

Taking the derivative of both sides w.r.t. \( t \) completes the proof.

**Proposition 6.** The tempered EGU± updates (16) with \( w_+(0) = w_- (0) = 1 \) can be written as the CMD update on \( w(t) \) using the tempered arcsinh\(\tau\) link function,

\[ \text{arcsinh}_\tau(w(t)) = -\eta \nabla_w L(w(t)) . \]
Proof. Note that from
\[
\log_{\tau}(w_+(t)) = -\eta \nabla_w L(w(t))
\]
\[
\log_{\tau}(w_-(t)) = +\eta \nabla_w L(w(t))
\]
we have \(w_+(t) \odot_{\tau} w_-(t) = c\) for all \(t \geq 0\) where \(c = w_0 \odot_{\tau} w_0\). Using the definition of \(\odot_{\tau}\), we can write (dropping \(t\))
\[
\log_{\tau}(w_-) = \log_{\tau}(c) - \log_{\tau}(w_+) = -\log_{\tau}(w_+ \odot_{\tau} c).
\]

We have
\[
sinh_{\tau} \log_{\tau}(w_+ \odot_{\tau} c) = 
\]
\[
= \frac{1}{2} \left( \exp_{\tau}(\log_{\tau}(w_+ \odot_{\tau} c)) - \exp_{\tau}(-\log_{\tau}(w_+ \odot_{\tau} c)) \right)
\]
\[
= \frac{1}{2} \left( (w_+ \odot_{\tau} c) - (w_- \odot_{\tau} c) \right)
\]

Thus, in general, if we choose the last line to be equal to \(w_+ \odot_{\tau} c\), we have
\[
\text{arcsinh}_{\tau}(w_+ \odot_{\tau} c) = \log_{\tau}(w_+ \odot_{\tau} c)
\]
Taking the derivative of both sides yields
\[
\frac{\partial}{\partial t} \left( \log_{\tau}(w_+ (t)) - \log_{\tau}(w_+ (c)) \right) = \log_{\tau}(w_+(t))
\]
\[
= -\eta \nabla_w L(w(t)).
\]

Setting \(w_0 = 1\) results in the tempered EGU updates (where the constant factor \(1/2\) can be absorbed into the learning rate \(\eta\)).

\[\square\]

F PROOF OF PROPOSITION 5

Proof of Proposition 5. The first KKT condition is satisfied by the form of the updates. The second condition follows from the assumption. To show the remaining conditions, note that the matrix tempered Bregman updates (28) in the limit \(W_0 = \lim_{\alpha \to 0} \alpha I\) can be written as
\[
W_+(t) = \left[ - (1 - \tau) \eta \int_0^t X^*(\delta(z)) \, dz \right] \odot_{\tau}
\]
\[
W_-(t) = \left[ + (1 - \tau) \eta \int_0^t X^*(\delta(z)) \, dz \right] \odot_{\tau}. \quad (32)
\]

Thus, \(W_+(t), W_-(t)\) share the same eigenvectors as \(X^*\). Setting \(\nu = -(1 - \tau) \eta \int_0^\infty \delta(z) \, dz\) satisfies the remaining conditions. The case \(\tau = 1\) also follows by a limit argument, similar to the one given in Gunasekar et al. (2017). \[\square\]