CORRIGENDUM: ON SUBADDITIONITY OF THE LOGARITHMIC KODAIRA DIMENSION

OSAMU FUJINO

Abstract. John Lesieutre constructed an example satisfying $\kappa_\sigma \neq \kappa_\nu$. This says that the proof of the inequalities in Theorems 1.3, 1.9, and Remark 3.8 in [O. Fujino, On subadditivity of the logarithmic Kodaira dimension, J. Math. Soc. Japan 69 (2017), no. 4, 1565–1581] is insufficient. We claim that some weaker inequalities still hold true and they are sufficient for various applications.

Contents

1. Introduction
2. On [F2, Theorems 1.3 and 1.9]
3. On [F2, Remark 3.8], [N, Chapter V, 4.1. Theorem (1)], and so on
4. References

1. Introduction

In [Les], John Lesieutre constructs a smooth projective threefold $X$ and a pseudo-effective $\mathbb{R}$-divisor $D$ on $X$ such that $\kappa_\sigma(D) = 1$ and $\kappa_\nu(D) = 2$. This means that the equality $\kappa_\sigma = \kappa_\nu$ does not always hold true. In the proof of [F2, Theorem 1.3], we used the following lemma (see [F2, Lemma 2.8]), which is a special case of [Leh, Theorem 6.7 (7)].

**Lemma 1.1.** Let $D$ be a pseudo-effective Cartier divisor on a smooth projective variety $X$. We fix some sufficiently ample Cartier divisor $A$ on $X$. Then there exist positive constants $C_1$ and $C_2$ such that

$$C_1 m^{\kappa_\sigma(X,D)} \leq \dim H^0(X, O_X(mD + A)) \leq C_2 m^{\kappa_\sigma(X,D)}$$

for every sufficiently large $m$.

Unfortunately, the proof of [Leh, Theorem 6.7 (7)] in [Leh] (see also [E]) depends on the wrong fact that $\kappa_\sigma = \kappa_\nu$ always holds. Moreover, Lesieutre’s example says that [Leh, Theorem 6.7 (7)] is not true when $D$ is an $\mathbb{R}$-divisor. Therefore, this trouble damages [F2, Theorems 1.3, 1.9, and Remark 3.8]. This means that the proof of the inequalities in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)] is incomplete.

In this paper, we explain that slightly weaker inequalities than the original ones in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)] still hold true. Fortunately, these weaker inequalities are sufficient for [F2, Corollaries 1.5 and 1.6] and some other applications. Note that one of the main purposes of [F2] is to reduce Iitaka’s subadditivity conjecture on the logarithmic Kodaira dimension $\kappa$ (see [F2, Conjecture 1.1]) to a special case of the generalized abundance conjecture (see [F2, Conjecture 1.4]). For that purpose, one of the weaker inequalities in Theorem 2.1 below is sufficient.

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Remark 1.2. Kenta Hashizume reduces Iitaka’s subadditivity conjecture on the logarithmic Kodaira dimension $\pi$ to the generalized abundance conjecture for sufficiently general fibers (see [H, Theorem 1.2]). In some sense, his result is better than the one in [F2]. We note that his proof uses [GL, Theorem 4.3] (see Theorem 3.2 below) and that the proof of [GL, Theorem 4.3] uses [N, Chapter V, 4.2. Corollary] which follows from [N, Chapter V, 4.1. Theorem (1)]. Fortunately, the inequality (3.3) below, which is weaker than the one in [N, Chapter V, 4.1. Theorem (1)], is sufficient for our purpose. So there are no troubles in [H].

Remark 1.3. We note that [F1, Lemma 2.4.9] is nothing but [Leh, Theorem 6.7 (7)]. Fortunately, however, we do not use it directly in [F1].

It is highly desirable to solve the following conjecture.

Conjecture 1.4. Let $X$ be a smooth projective variety and let $D$ be a pseudo-effective $\mathbb{R}$-divisor on $X$. Then there exist a positive integer $m_0$, a positive rational number $C$, and an ample Cartier divisor $A$ on $X$ such that

\begin{equation}
Cm^{\kappa_s(X,D)} \leq \dim H^0(X, \mathcal{O}_X([mm_0D]+A))
\end{equation}

holds for every large positive integer $m$.

If Conjecture 1.4 is true, then there are no troubles in [F2, Theorems 1.3 and 1.9] and [N, Chapter V, 4.1. Theorem (1)].

The following observation may help the reader understand this paper, the trouble in [N, Chapter V, 4.1. Theorem (1)], and Conjecture 1.4.

1.5 (Observation). Let us consider

\[f, g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}\]

such that

\begin{equation}
\limsup_{m \to \infty} f(m) > 0 \quad \text{and} \quad \limsup_{m \to \infty} g(m) > 0.
\end{equation}

We want to prove

\begin{equation}
\limsup_{m \to \infty} (f(m)g(m)) > 0.
\end{equation}

In general, (1.3) does not follow from (1.2). It may happen that $f(m)g(m) = 0$ holds for every $m$. If there exists a positive constant $C$ such that $f(m) \geq C$ for every large positive integer $m$, then we have

\[\limsup_{m \to \infty} (f(m)g(m)) > 0.
\]

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We will freely use the notation in [F2] and work over $\mathbb{C}$, the complex number field, throughout this paper.

2. On [F2, Theorems 1.3 and 1.9]

The proof of the main theorem of [F2], that is, [F2, Theorem 1.3], is incomplete. Here we will prove slightly weaker inequalities.
**Theorem 2.1** (see [F2, Theorem 1.3]). Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let $D_X$ (resp. $D_Y$) be a simple normal crossing divisor on $X$ (resp. $Y$). Assume that $\text{Supp} f^* D_Y \subset \text{Supp} D_X$. Then we have

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)$$

and

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y),$$

where $F$ is a sufficiently general fiber of $f : X \to Y$.

The inequalities in Theorem 2.1 follow from the proof of [F2, Theorem 1.3] without any difficulties.

Before we prove Theorem 2.1, we give a small remark on $\kappa_\sigma$.

**Remark 2.2.** Let $D$ be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective variety $X$. Then $\kappa_\sigma(X, D) = \kappa_\sigma(X, lD)$ holds for every positive integer $l$. We note that $\kappa_\sigma(X, lD) \geq \kappa_\sigma(X, D)$ holds by [N, Chapter V, 2.7. Proposition (1)] since $lD - D = (l - 1)D$ is pseudo-effective. By definition, $\kappa_\sigma(X, lD) \leq \kappa_\sigma(X, D)$ always holds.

Let us prove Theorem 2.1.

**Proof of Theorem 2.1.** In this proof, we will freely use the notation in the proof of [F2, Theorem 1.3]. In the proof of [F2, Theorem 1.3], we have

$$\dim H^0(X, \mathcal{O}_X(mk(K_X + D_X) + A + 2f^* H)) \geq r(mD; A) \cdot \dim H^0(Y, \mathcal{O}_Y(mk(K_Y + D_Y) + H))$$

for every positive integer $m$, where

$$D = k(K_{X/Y} + D_X - f^* D_Y)$$

and

$$r(mD; A) = \text{rank} f_* \mathcal{O}_X(mD + A).$$

We can take a positive integer $m_0$ and a positive real number $C_0$ such that

$$C_0 m^{\kappa(F,D|_F)} \leq r(mm_0 D; A)$$

for every large positive integer $m$. Since $\kappa(F, D|_F) = \kappa(F, K_F + D_X|_F)$, we have

$$\dim H^0(X, \mathcal{O}_X(mm_0 k(K_X + D_X) + A + 2f^* H)) \geq C_0 m^{\kappa(F,K_F+D_X|_F)} \cdot \dim H^0(Y, \mathcal{O}_Y(mm_0 k(K_Y + D_Y) + H))$$

for every positive integer $m$ by (2.1) and (2.4). We may assume that $H$ is sufficiently ample. Then we get

$$\limsup_{m \to \infty} \frac{\dim H^0(X, \mathcal{O}_X(mm_0 k(K_X + D_X) + A + 2f^* H))}{m^{\kappa(F,K_F+D_X|_F)+\kappa_\sigma(Y,K_Y+D_Y)}} > 0$$

by (2.5) and the definition of $\kappa_\sigma(Y, K_Y + D_Y)$. This means that the following inequality

$$\kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y)$$

holds.

Similarly, we can take a positive integer $m_1$ and a positive real number $C_1$ such that

$$C_1 m^{\kappa(Y,K_Y+D_Y)} \leq \dim H^0(Y, \mathcal{O}_Y(mm_1 k(K_Y + D_Y))),$$

$$\leq \dim H^0(Y, \mathcal{O}_Y(mm_1 k(K_Y + D_Y) + H))$$

by [F2, Theorem 1.3].
for every large positive integer \( m \) by the definition of \( \kappa(Y, K_Y + D_Y) \) if \( H \) is a sufficiently ample Cartier divisor. Then, by (2.1) and (2.8), we have
\[
\dim H^0(X, O_X(mm_1 k(K_X + D_X) + A + 2f^*H)) \\
\geq C_1 m^\kappa(Y, K_Y + D_Y) \cdot r(mm_1 D; A)
\]
for every large positive integer \( m \). Therefore, we get
\[
\limsup_{m \to \infty} \frac{\dim H^0(X, O_X(mm_1 k(K_X + D_X) + A + 2f^*H))}{m^{\kappa(Y, K_Y + D_Y) + \kappa(Y, K_Y + D_Y)}} > 0
\]
when \( A \) is sufficiently ample. Note that
\[
\sigma(m_1 D|_F; A|_F) = \max \left\{ k \in \mathbb{Z}_{\geq 0} \cup \{-\infty\} \left| \limsup_{m \to \infty} \frac{r(mm_1 D; A)}{m^k} > 0 \right. \right\}
\]
for a sufficiently general fiber \( F \) of \( f : X \to Y \) and that
\[
\kappa_\sigma(F, K_F + D_X|_F) = \kappa_\sigma(F, D|_F)
\]
\[
= \kappa_\sigma(F, m_1 D|_F)
\]
\[
= \max \{ \sigma(m_1 D|_F; A|_F) \mid A \text{ is very ample} \}.
\]
Hence we have the inequality
\[
\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)
\]
by (2.10). \( \square \)

Of course, we have to weaken inequalities in [F1, Theorem 4.12.1 and Corollary 4.12.2] following Theorem 2.1.

Theorem 1.9 in [F2] has the same trouble as [F2, Theorem 1.3]. Of course, we can prove slightly weaker inequalities.

**Theorem 2.3** (see [F2, Theorem 1.9]). Let \( f : X \to Y \) be a proper surjective morphism from a normal variety \( X \) onto a smooth complete variety \( Y \) with connected fibers. Let \( D_X \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( (X, D_X) \) is lc and let \( D_Y \) be a simple normal crossing divisor on \( Y \). Assume that \( \text{Supp} f^*D_Y \subset [D_X] \), where \([D_X]\) is the round-down of \( D_X \). Then we have
\[
\kappa_\sigma(X, K_X + D_X) \geq \kappa_\sigma(F, K_F + D_X|_F) + \kappa(Y, K_Y + D_Y)
\]
and
\[
\kappa_\sigma(X, K_X + D_X) \geq \kappa(F, K_F + D_X|_F) + \kappa_\sigma(Y, K_Y + D_Y),
\]
where \( F \) is a sufficiently general fiber of \( f : X \to Y \).

It is obvious how to modify the proof of [F2, Theorem 1.9] in order to get Theorem 2.3. For the details, see the proof of Theorem 2.1 above.

As we said above, the inequalities in Theorem 2.1 are sufficient for [F2, Corollaries 1.5 and 1.6]. The reader can check it without any difficulties.

3. On [F2, Remark 3.8], [N, Chapter V, 4.1. Theorem (1)], and so on

In [F2, Remark 3.8], the author pointed out a gap in the proof of [N, Chapter V, 4.1. Theorem (1)] and filled it by using [Leh, Theorem 6.7 (7)] (see also [F1, Remark 4.11.6]). Therefore, the proof of the inequalities in [N, Chapter V, 4.1. Theorem (1)] is still incomplete.
3.1 (Nakayama’s inequality for $\kappa_{\sigma}$). Here, we will freely use the notation in [N, Chapter V, 4.1. Theorem]. In [N, Chapter V, 4.1. Theorem (1)], Nakayama claims that the inequality
\begin{equation}
\kappa_{\sigma}(D + f^*Q) \geq \kappa_{\sigma}(D; X/Y) + \kappa_{\sigma}(Q)
\end{equation}
holds. Unfortunately, this inequality (3.1) does not follow directly from the inequality
\begin{equation}
h^0(X, [m(D + f^*Q)] + A + 2f^*H) \geq r(mD; A) \cdot h^0(Y, [mQ] + H)
\end{equation}
established in the proof of [N, Chapter V, 4.1. Theorem (1)]. By the same argument as in the proof of Theorem 2.1 above, by using [N, Chapter II, 3.7. Theorem], we can prove
\begin{equation}
\kappa_{\sigma}(D + f^*Q) \geq \kappa_{\sigma}(D; X/Y) + \kappa(Q)
\end{equation}
and
\begin{equation}
\kappa_{\sigma}(D + f^*Q) \geq \kappa(D; X/Y) + \kappa_{\sigma}(Q).
\end{equation}
If we put $D = K_{X/Y} + \Delta$ and $Q = K_Y$, then we have
\begin{equation}
\kappa_{\sigma}(K_X + \Delta) \geq \kappa_{\sigma}(K_{X_y} + \Delta|_{X_y}) + \kappa(K_Y)
\end{equation}
and
\begin{equation}
\kappa_{\sigma}(K_X + \Delta) \geq \kappa(K_{X_y} + \Delta|_{X_y}) + \kappa_{\sigma}(K_Y).
\end{equation}
Lesieutre’s example does not affect [N, Chapter V, 4.1. Theorem (2)] because we do not use $\kappa_{\sigma}$ for the proof of [N, Chapter V, 4.1. Theorem (2)].

We note that the inequalities (3.3) and (3.4) are sufficient for the proof of [F1, Theorem 4.12.8].

The inequality (3.1) has already played an important role in the theory of minimal models. The following result is very well known and has already been used in various papers.

**Theorem 3.2** ([DHP, Remark 2.6] and [GL, Theorem 4.3]). Let $(X, \Delta)$ be a projective klt pair such that $\Delta$ is a $\mathbb{Q}$-divisor. Then $(X, \Delta)$ has a good minimal model if and only if $\kappa_{\sigma}(X, K_X + \Delta) = \kappa(X, K_X + \Delta)$.

In the proof of Theorem 3.2, the inequality (3.1) plays an important role in [DHP, Remark 2.6]. Fortunately, in [DHP, Remark 2.6], the inequality (3.3) is sufficient because we need the inequality (3.1) in the case where $Q$ is a big divisor. In [GL, Theorem 4.3], Gongyo and Lehmann need [N, Chapter V, 4.2. Corollary] in the proof of [GL, Theorem 4.3]. We note that Nakayama uses the inequality (3.1) in the proof of [N, Chapter V, 4.2. Corollary]. Fortunately, we can easily see that [N, Chapter V, 4.2. Corollary] holds true because the inequality (3.3) is sufficient for that proof. We strongly recommend the reader to see [HH, Subsection 2.2] for some related topics. We can find a generalization of Theorem 3.2 (see [HH, Lemma 2.13]).

**Remark 3.3.** In a recent preprint [F3], we introduce the notion of mixed-\(\omega\)-sheaves and discuss some topics related to Nakayama’s inequality for $\kappa_{\sigma}$. For the details, see [F3, Section 11].

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: fujino@math.sci.osaka-u.ac.jp