Hypo-EP operators with reference to indefinite inner product

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Abstract
Let \(H\) be a Hilbert space, and let \(T : H \to H\) be a bounded linear operator with closed range. In this paper, we discuss about the hypo-EP operators with reference to indefinite inner product, which is weaker than the case of EP operators. Moreover, we characterize such operators and some fundamental properties have been established.

Keywords
Indefinite inner product, range symmetric, hypo normal operators, hypo-EP operators.

AMS Subject Classification
15A09, 46C20, 47B20.

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1. Introduction
Throughout this paper \(\langle \cdot, \cdot \rangle\) denotes the indefinite inner product space in \(\mathbb{C}^n\) is a sesquilinear form \([x, y]\) with the regularity condition \([x,y] = 0, \forall y \in \mathbb{C}^n\) only when \(x = 0\). Indefinite inner product in \(\mathbb{C}^n\) is defined by \([x,y] = \langle x, Jy \rangle\), where \(\langle \cdot, \cdot \rangle\) denotes the conventional unitary space inner product in \(\mathbb{C}^n\) with \(J^2 = I\).

A matrix has an inverse only if it is square and even then only if it is non-singular. Usually singular and rectangular matrices do not possess inverse. In numerous areas of applied mathematics for some kind of partial inverse of a matrix that is singular or even rectangular. Such inverse are called generalized inverse. The concept of generalized inverse for the matrices was introduced first by Moore [9] in 1920, who defined a unique inverse (called by him the “general reciprocal”) for every finite matrix. In 1955 Penrose [10] sharpened and extended the Bjerhammar’s results on the linear systems, and showed that the Moore’s inverse of a matrix is unique and reduces to the ordinary case when applied to the nonsingular matrices. This initiated the development of the research area, which is now called the theory of generalized inverses.

Inquiries of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors \((2, 12)\). The indefinite product of matrices and applications to indefinite inner product space and extended some formulae from Euclidean space to an indefinite inner product space were investigated by Ramanathan et al. [12] in 2004. The class of EP matrices was analogously developed by Jayaraman [3] in 2012 in indefinite inner product space, known as the class of range symmetric matrices and established some equivalent conditions for a matrix to be range symmetric. Radojevic [11] in 2014 investigated the new results for EP matrices in indefinite inner product space. In 2014 Meenakshi [6] extended the concept of range symmetric matrices in indefinite inner product space. Commutators in indefinite inner product space were studied by Meenakshi [7] in 2015. Meenakshi [8] in 2016 extended the product range symmetric matrices in indefinite inner product space. In 2018 Krishnaswamy et al. [4] studied the positive semidefinite matrices with reference to indefinite inner product. On sums of range symmetric matrices with reference to indefinite inner product were extended by Krishnaswamy et al. [5] in 2019.

There are two different values for dot product of vectors in indefinite inner product space. To overcome this difficulty, Ramanathan et al. introduced a new matrix product, called indefinite matrix multiplication was introduced and some of
its properties are established in [12].

In this paper \( H_1, H_2 \) and \( H \) are separable Hilbert spaces with inner product. Let \( B(H_1, H_2) \) be the set of all bounded linear operators from \( H_1 \) into \( H_2 \). Let \( B_C(H_1, H_2) \) be the subspace of all \( T \in B(H_1, H_2) \) such that the range of \( T \) is closed in \( H_2 \). If \( H_1 = H_2 = H \), we write \( B(H) = B(H, H) \) and \( B_C(H) = B_C(H, H) \). For \( T \in B(H_1, H_2) \), \( Ra(T), Nu(T) \) and \( \perp \) denotes the range space, null space and orthogonality with reference to indefinite inner product.

### 2. Hypo-J-EP operator

**Definition 2.1.** [11] For \( T \in B_C(H_1, H_2) \) has a Moore Penrose J-inverse \( T^+ \) that is \( T^+ \) is the unique solution for the equations:

(i) \( T \circ T^+ \circ T = T \)
(ii) \( T^+ \circ T \circ T^+ = T^+ \)
(iii) \( (T \circ T^+)[][T^+ = T \circ T^+ \)
(iv) \( (T^+ \circ T)[][T^+ = T \circ T^+ \)

**Definition 2.2.** For an operator \( T \in B_C(H) \), if \( [T^+, T] \geq 0 \) then \( T \) is called hypo-J-EP operator.

From Definition 2.2, [1], [3] and [7] the following Theorem follows immediately.

**Theorem 2.3.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Then the following statements are equivalent:

(i) \( T \) is Hypo-J-EP
(ii) \( Ra(T) \subseteq Ra(T^+) \)
(iii) \( Ra(T) \subseteq Ra(T^+) \)
(iv) \( T^+ \circ T^+[T^+ = T \circ T^+ \)
(v) \( T \circ T^+ \geq ||T \circ T^+|| \) for all \( x \in H \).

**Theorem 2.4.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Then \( T \) is hypo-J-EP if and only if \( ||T^+ \circ x||[2] \leq ||T^+ \circ x||[2] \) for all \( x \in H \).

**Proof.** Suppose that \( T \) is hypo-J-EP operator. Then from Theorem 2.3, \( T \) satisfies the following condition \( ||(T^+ \circ T) \circ x|| \geq ||(T \circ T^+) \circ x|| \) for all \( x \in H \).

Thus we have \( ||T^+ \circ x|| \leq ||T^+ \circ x||[2] \) for all \( x \in H \).

Conversely, we suppose that \( ||T^+ \circ x|| \leq ||T^+ \circ x||[2] \) for all \( x \in H \). Then \( T \circ x = 0 \Rightarrow T^+ \circ x = 0 \). That is \( Nu(T) \subseteq Nu(T^+) \). Hence we have \( Nu(T^+) \subseteq (Nu(T^+))[][1] \). Now, we notice that \( (Nu(T^+))[1] \subseteq (Nu(T^+))[][1] \) which implies \( Ra(T^+) \subseteq Ra(T) \). Therefore \( T \) is Hypo-J-EP.

**Theorem 2.5.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Then \( T \) is hypo-J-EP if and only if one of the following statements are equivalent:

(i) \( T^+ \circ T^+ = T \)
(ii) \( T^+ \circ T^+ \circ T = T^+ \)

**Proof.** (i) From Theorem 2.3, we have that \( T \) is hypo-J-EP \( \iff Ra(T) \subseteq Ra(T^+) \). Thus by Definition 2.1, we have \( T \circ T^+[T^+ = T \circ T^+ \) and \( T \circ T^+ \). The converse is clear from \( T^+ \circ T = Ra(T^+) \).

(ii) It is clear from Definition 2.1, and [1] that \( (T^+ \circ T^+)[x] = (T^+ \circ T^+)[x] \) which implies \( Ra(T^+) \subseteq Ra(T) \).

**Theorem 2.6.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). If \( [T^+, T] = 0 \) then \( T \) is hypo-J-EP.

**Proof.** Since \( [T^+, T] = 0 \), \( T \circ T^+ \circ T = T \) and \( T \circ T^+ \circ T = T \) hold. Then the following statements are equivalent:

(i) \( T^+ \circ T^+ \circ T = T^+ \)
(ii) \( T^+ \circ T^+ \circ T = T^+ \)

**Theorem 2.7.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Suppose that \( T \) is hypo-J-EP. If \( [T \circ T^+, T^+] = 0 \) then \( T \) is J-EP.

**Proof.** Since \( [T \circ T^+, T] = 0 \), \( T \circ T^+ \circ T = T \) holds. Then the following statements are equivalent:

(i) \( T \circ T^+ \circ T = T \)
(ii) \( T \circ T^+ \circ T = T \)

**Theorem 2.8.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Suppose that \( T \) is hypo-J-EP. If \( [T \circ T^+, T^+] = 0 \) then \( T \) is J-EP.

**Proof.** Since \( [T \circ T^+, T] = 0 \), \( T \circ T^+ \circ T = T \) holds. Then the following statements are equivalent:

(i) \( T \circ T^+ \circ T = T \)
(ii) \( T \circ T^+ \circ T = T \)

**Corollary 2.9.** Let \( T \in B_C(H) \) with a bounded Moore Penrose J-inverse \( T^+ \). Suppose that \( T \) is hypo-J-EP. If \( [T \circ T^+, T] = 0 \) then \( T \) is J-EP.
Proof. Since \([T, T^\dagger T] = 0\) \(\Rightarrow T \circ T^\dagger T - T^\dagger T \circ T = 0\). Pre multiply by \(T^\dagger\)
\(\Rightarrow (T^\dagger T) \circ (T \circ T^\dagger) = T^\dagger \circ T\). Which means \(T \circ T^\dagger \geq T^\dagger \circ T\). Thus \(T\) is J-EP by assumption.

Theorem 2.10. Let \(T \in B_0(C(H))\) and let \(\{T_n\}\) be a sequence of hypo-J-EP operators in \(B_0(C(H))\). Let \(T_n^\dagger\) be the Moore Penrose J-inverse of \(T_n\) for every \(n\). Suppose that \(T_n \rightarrow T\) [with respect to the norm \(|| \cdot ||\) on \(B_0(C(H))\)]. Then \(T\) is a hypo-J-EP operator.

Proof. It is clear from [13] that if \(T_n \rightarrow T\) then \(T_n \circ T_n^\dagger \rightarrow T \circ T^\dagger\) and following inequality holds,
\[
||(T_n^\dagger T_n) \circ x - (T^\dagger T) \circ x|| \geq ||(T_n^\dagger T_n) \circ x|| - ||(T^\dagger T) \circ x||.
\]
Hence we have
\[
||(T_n^\dagger T_n) \circ x|| \rightarrow ||(T^\dagger T) \circ x||\text{ for all }x \in H.
\]
Similarly, we obtain
\[
||(T_n \circ T_n^\dagger) \circ x|| \rightarrow ||(T \circ T^\dagger) \circ x||\text{ for all }x \in H.
\]
Therefore by Theorem 2.3, we have
\[
||(T^\dagger T) \circ x|| = \lim_{n \rightarrow \infty} \|| (T_n^\dagger T_n) \circ x|| \geq \lim_{n \rightarrow \infty} \|| (T_n \circ T_n^\dagger) \circ x|| \geq ||(T \circ T^\dagger) \circ x||
\]
That is \(T\) is hypo-J-EP.

3. Conclusion

In this paper we have studied bounded linear operator with closed range. We have established hypo-EP operators with reference to indefinite inner product. Further characterization of hypo-EP operators have been determined.

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