Infinitesimal 2-braidings and differential crossed modules

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Abstract
We categorify the notion of an infinitesimal braiding in a linear strict symmetric monoidal category, leading to the notion of a (strict) infinitesimal 2-braiding in a linear symmetric strict monoidal 2-category. We describe the associated categorification of the 4-term relation, leading to six categorified relations. We prove that any infinitesimal 2-braiding gives rise to a flat and fake flat 2-connection in the configuration space of \(n\) particles in the complex plane, hence to a categorification of the Knizhnik-Zamolodchikov connection. We discuss infinitesimal 2-braidings in a 2-category naturally assigned to every differential crossed module, leading to the notion of a quasi-invariant tensor in a differential crossed module. Finally we prove that quasi-invariant tensors exist in the differential crossed module associated to the String Lie-2-algebra.

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Introduction
A linear (strict) monoidal category \(\mathcal{C}\) is a strict monoidal symmetric category \(\mathcal{C} = (C_0, C_1, \otimes, I, B)\), with classes of objects and morphisms \((C_0\) and \(C_1\), respectively), functorial strict tensor product \(\otimes\) (with strict identity \(I\)) and with an involutive braiding \(B\). The designation ‘linear’ comes from the fact that we suppose that given two objects \(x, y \in C_0\), the set of morphisms \(\text{hom}(x, y)\) is given a vector space structure, and the composition of morphisms is bilinear.

Given a linear strict monoidal category \(\mathcal{C} = (C_0, C_1, \otimes, I, B)\), consider objects \(x, y, z \in C_0\). Given a 1-morphism \(f: x \otimes z \to x \otimes z\), we define \(f_{13}^{13} = f_{13}^{13}_{x, y, z}: x \otimes y \otimes z \to x \otimes y \otimes z\) as being (we compose from left to right):

\[ (B_{x,y} \otimes \text{id}_z)(\text{id}_y \otimes f)(B_{y,z} \otimes \text{id}_y) = (\text{id}_x \otimes B_{y,z})(f \otimes \text{id}_y)(\text{id}_x \otimes B_{z,y}). \]

Given \(g: x \otimes y \to x \otimes y\) we also put \(g_{12}^{12} = g_{12}^{12}_{x,y,z} = g \otimes \text{id}_z\). Analogously, given \(h: y \otimes z \to y \otimes z\) we define \(\text{id}_x \otimes h = h_{x,y,z}^{12} = h_{x,y,z}^{23} = h_{x,y,z}^{23} \).
An infinitesimal braiding in a linear strict monoidal category \(\mathcal{C}\) (see [18, XX.4] and [10]) is given by a family of functorial (natural) isomorphisms \(r_{x,y} : x \otimes y \to x \otimes y\), one for each pair of objects \(x, y\) of \(\mathcal{C}\), such that:

1. For each \(x, y \in \mathcal{C}_0\) then \(B_{x,y} r_{y,x} = r_{x,y} B_{x,y}\).
2. For each \(x, y, z \in \mathcal{C}_0\) we have \(r_{x,y \otimes z} = r^{12}_{x,y} + r^{13}_{x,z} + r^{13}_{x,y} r^{23}_{y,z}\).

The naturality condition yields in particular that, given objects \(x, y, z \in \mathcal{C}_0\):

\[
r^{23}_{y,z} r_{x,y \otimes z} = r_{x,y \otimes z} r^{23}_{y,z}.
\]

Combining with the second condition, above, this yields:

\[
r^{23}_{y,z} r^{12}_{x,y} + r^{23}_{y,z} r^{13}_{x,y} = r^{12}_{x,y} r^{23}_{y,z} + r^{13}_{x,y} r^{23}_{y,z}.
\]

Therefore, if we apply the linearity condition, we obtain the well known 4-term relations \([9, 20, 18]\), appearing in the theory of universal Vassiliev knot invariants, namely:

\[
[r^{12}_{x,y} + r^{13}_{x,y} r^{23}_{y,z}] = 0.
\]

The axioms of an infinitesimally braided monoidal category are linear analogues of the axioms of a braided monoidal category. By using Drinfeld associators [14] one can prove that any infinitesimal braiding is the linear counterpart of a certain braided monoidal category \([10, 18]\).

Let \(\mathfrak{g}\) be a Lie algebra. Let \(\mathfrak{U}(\mathfrak{g})\) be its universal enveloping algebra. Let \(\Delta = \Delta^\circ : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})^\otimes \) be the diagonal map. There are two naturally defined linear monoidal categories associated to \(\mathfrak{g}\). The first is the category \(\mathcal{C}_\mathfrak{g}\), whose objects are the naturals \(n \in \mathbb{N}\), and the morphisms are given by pairs \((R, \sigma)\) where:

- \(R \in (\mathfrak{U}(\mathfrak{g})^\otimes)_{0} \subset \mathfrak{U}(\mathfrak{g})^\otimes\) is a \(\mathfrak{g}\)-invariant tensor, namely \([R, \Delta(X)] = 0\) for each \(X \in \mathfrak{g}\).
- \(\sigma \in S_n\) is a permutation of \(\{1,\ldots,n\}\).

(Strictly speaking the set of 1-morphisms \(n \to n\) is the direct sum \(\bigoplus_{\sigma \in S_n} \mathfrak{U}(\mathfrak{g})^\otimes\), and \(\text{hom}(m, n) = \{0\}\) if \(m \neq n\).) The composition of 1-morphisms is given by the obvious semi-direct product law, taking into account that the product of \(\mathfrak{g}\)-invariants tensors in \((\mathfrak{U}(\mathfrak{g})^\otimes)_{0}\) is a \(\mathfrak{g}\)-invariant tensor. Let \(V\) be any representation of \(\mathfrak{g}\). By sending invariant tensors \(R \in (\mathfrak{U}(\mathfrak{g})^\otimes)_{0}\) to the obvious intertwiner \(V^\otimes \to V^\otimes\) (action by \(R\)), and permutations in \(S_n\) to the map that permutes factors in the tensor product, we have a functor \(\mathcal{C}_\mathfrak{g} \to M_\mathfrak{g}\). We will however not use this fact in the following.

It is easy to see that \(\mathcal{C}_\mathfrak{g}\) is a strict symmetric monoidal linear category. Consider a symmetric tensor \(r \in \mathfrak{g}^\otimes \mathfrak{g}\), which is furthermore \(\mathfrak{g}\)-invariant. Then it follows immediately that the family \(r_{m,n} = ((\Delta^\circ \otimes \Delta^\circ)(r), id_{\mathfrak{g}^m})\) is an infinitesimal braiding in the category \(\mathcal{C}_\mathfrak{g}\). Therefore, the 4-term relation \([r^{12} + r^{13}, r^{23}] = 0\) is satisfied by \(r\) (a well known fact).

Recall that the configuration space of \(n\) distinct particles in the complex plane \(\mathbb{C}\) is by definition:

\[
\mathbb{C}(n) = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j \}.
\]

This has a left action of the symmetric group \(S_n\), by permutation of coordinates. The configuration space of \(n\) indistinguishable particles in \(\mathbb{C}\) is defined as \(\mathbb{C}(n)/S_n\). We recall that the fundamental group of \(\mathbb{C}(n)/S_n\) is isomorphic to the braid group in \(n\) strands. Let \(V\) be a vector space. Let \(\mathfrak{gl}(V)\) be the Lie algebra of linear maps \(V \to V\). Consider a family of endomorphisms \(f_{ab} : V \to V\), where \(1 \leq a < b \leq n\), such that \([f_{ab}, f_{a'b'}] = 0\), if \([a,b] \cap [a',b'] = \emptyset\). Define closed 1-forms \(\omega_{ij}\) in the configuration space \(\mathbb{C}(n)\), for \(1 \leq i, j \leq n\) and \(i \neq j\), as:

\[
\omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j}.
\]
Then it is well known, and it is proven for example in [9] [11] [12] [18] [21], that the $\mathfrak{gl}(V)$-valued connection 1-form in the configuration space $C(n)$:

$$A = \sum_{1 \leq i < j \leq n} \omega_{ij} f_{ij}$$

(2)

is flat if, and only if, the 4-term relations $[f_{ij} + f_{ik}, f_{kj}] = 0$ hold. In this case the family $\{f_{ab} : V \to V, 1 \leq a < b \leq n\}$ will be called an infinitesimal Yang-Baxter operator [12]. This is the one of the starting points for constructing the Kontsevich integral [9, 20].

Given a semisimple Lie algebra $\mathfrak{g}$, let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the symmetric invariant tensor associated to the Cartan-Killing form. Given a $\mathfrak{g}$-module $V$, the Knizhnik-Zamolodchikov connection [19] [9] [18] is the form (2), with $f_{ab}$ being the obvious action $r_{ab}$ of $r$ on $V^{\otimes n}$. In this case the connection form $A$ is invariant under the natural action of the symmetric group, thus it descends to a flat connection in the quotient vector bundle over $\mathbb{C}(n)/S_n$. It is proven in [14] [21] [22] that the representation of the braid-group derived from the quotient Knizhnik-Zamolodchikov connection is equivalent to the representation of the braid group derived from the R-matrix of the quantized universal enveloping algebra of $\mathfrak{g}$. This beautiful and deep result is known as the Drinfeld-Kohno Theorem.

The notion of a 2-connection was addressed in [6, 7], by using higher dimensional group theory techniques. The differential analogue of a (strict) Lie 2-group [4] is a (strict) Lie 2-algebra [2]. The latter is equivalent to a differential crossed module $2\Theta = (\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$, where $\mathfrak{h}$ and $\mathfrak{g}$ are Lie-algebras, $\partial : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra map, and $\triangleright$ is a left action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations, such that the following relations, called Peiffer relations, are satisfied:

- $\partial(X \triangleright v) = [X, \partial(v)]$, for each $X \in \mathfrak{g}$ and $v \in \mathfrak{h}$;
- $\partial(v) \triangleright u = [v, u]$, for each $u, v \in \mathfrak{h}$.

We note that any semisimple Lie algebra $\mathfrak{g}$ gives rise to a family of differential crossed modules (all having the same weak homotopy type), each geometrically realizing the Lie algebra 3-cocycle $\omega(X, Y, Z) = \langle X, [Y, Z] \rangle$, where $\langle, \rangle$ is the Cartan-Killing form. In the case of $\mathfrak{sl}_2(\mathbb{C})$, each of these differential crossed modules is called a String Lie-2-algebra. For explicit constructions see [8, 25].

Let $\Theta = (\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$ be a differential crossed module. A $\Theta$-valued 2-connection is locally given by a pair $(A, B)$ of forms, where $A$ is a 1-form with values in $\mathfrak{g}$ and $B$ is a 2-form with values in $\mathfrak{h}$ such that

$$\partial(B) = \mathcal{T}A = dA + [A, A],$$

the curvature of $A$. A local 2-connection can be integrated to give a 2-dimensional (surface) holonomy with values in the associated Lie 2-group; see [6, 7, 15, 16]. Moreover, if a 2-connection $(A, B)$ is flat, meaning that the 2-curvature 3-form $g_{(A, B)} = dB + A \wedge B$ of $(A, B)$ vanishes, then this 2-dimensional holonomy is invariant under homotopy.

In [11] [12] we addressed the categorification of the 4-term relations (1) in the language of differential crossed module-valued 2-connections, which led to the definition of an infinitesimal 2-Yang-Baxter operator in a (strict) Lie 2-algebra, satisfying conditions categorifying the 4-term relations. In this paper we address the categorification of an infinitesimal braiding itself (leading to the definition of a strict infinitesimal 2-braiding, in a linear strict symmetric monoidal 2-category $C_{\mathfrak{g}_0}$) and its applications for defining flat 2-connections in the configuration space of particles in the complex plane. We moreover give the definition of a quasi-invariant tensor in a differential crossed module, which categorifies the notion of an invariant tensor in a Lie algebra. Likewise, each quasi-invariant tensor leads to the construction of an infinitesimal 2-braiding in a certain linear monoidal 2-category $C_{\mathfrak{g}_0}$, naturally associated to a differential crossed module $\Theta$. (We will give explicit details on the construction of $C_{\mathfrak{g}_0}$). Finally, we prove that the String Lie-2-algebra (in the version of Wagemann [25]) contains a non-trivial quasi-invariant tensor. This will give another proof of the main result of [12], where we proved that this particular String Lie-2-algebra contains infinitesimal 2-Yang Baxter operators.

The plan of the paper in the following: in Section [1] we give the definition of a linear symmetric strict monoidal 2-category (stemming out from the definition of a 2-vector space [2]), in which we give a very strict version of the axioms of a braided monoidal 2-category. Any object $x$ of a linear monoidal 2-category gives rise to a differential crossed module $\mathfrak{gl}(x)$. In Section [2] we discuss the definition of a strict infinitesimal
2-braiding \((r, T)\), which is a very direct categorification of the notion of an infinitesimal braiding \(r\), obtained by imposing that \((r, T)\) is a pseudo-natural, rather than a natural transformation. In subsection 2.4 we present the categorified 4-term relations, which are derived from the quasi-naturality of a strict infinitesimal 2-braiding; see Theorems 21 and 22.

Given an infinitesimal 2-braiding \((r, T)\), in a linear symmetric strict monoidal 2-category \(\mathcal{C}\), and an object of \(\mathcal{C}\), we construct a local 2-connection \((A, B)\), in the configuration space \(\mathcal{C}(n)\), of the form:

\[
A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab}, \quad B = 2 \sum_{a < b < c} \omega_{bd} \land \omega_{da} P_{bd} - 2 \sum_{a < b < c} \omega_{da} \land \omega_{ab} Q_{bd}
\]

(see [22] for the definition of \(P\) and \(Q\), which measure the failure of the 4-term relations [1] to hold). In particular we prove in Section 3 that the categorified 4-term relations are equivalent to the vanishing of the 2-curvature 3-form of \((A, B)\).

Finally in Section 4 we work in the framework of differential crossed modules. In subsection 4.2 we categorify the category \(\mathcal{C}_G\) associated to a Lie algebra, in order to define a linear symmetric monoidal 2-category \(\mathcal{C}_G\) (already mentioned above) associated to a differential crossed module \(\mathfrak{g}\). Similarly to the Lie algebra case, sketched above, given a (weak) categorical representation of \(\mathfrak{g}\) in a chain-complex of vector spaces, we can construct a 2-functor from \(\mathcal{C}_G\) into the 2-category of weak categorical representations of \(\mathfrak{g}\), in the 2-category of chain-complexes of vector spaces, chain maps and chain-homotopies (up to 2-fold homotopy); see [11, 12].

In subsection 4.3 we give the definition of a quasi-invariant tensor in a differential crossed module \(\mathfrak{g}\), and prove that each of these gives rise to an infinitesimal 2-braiding in the linear strict monoidal 2-category \(\mathcal{C}_G\). Finally we prove in subsection 4.4 that Wagemann’s String differential crossed module [25] contains a non-trivial quasi-invariant tensor, categorifying the invariant tensor of \(\mathfrak{sl}_2(C) \otimes \mathfrak{sl}_2(C)\) coming from the Cartan-Killing form.

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1 Preliminaries on 2-categories

1.1 Linear 2-categories

We present a minimal extension of the concept of 2-vector space due to Baez and Crans, [2].

Definition 1 (Linear 2-category). A linear 2-category $C = (C_0, C_1, C_2)$ is given by:

1. A set $C_0$ of objects.

2. A set $C_1$ of 1-morphisms, together with source and target maps $s, t : C_1 \to C_0$, as well as an inclusion map $i : C_0 \to C_1$, such that $s \circ i = t \circ i = \text{id}_{C_0}$. The elements of $C_1$ will normally be denoted as:
   
   \[ x \xrightarrow{f} y = s(f) \xrightarrow{f} t(f). \]

3. A set $C_2$ of 2-morphisms, together with source and target maps $s', t' : C_2 \to C_1$, as well as an inclusion map $i' : C_1 \to C_2$. These are such that:
   
   \[ s' \circ i' = t' \circ i' = \text{id}_{C_1}, \]
   \[ s \circ s' = s \circ t' \]
   \[ t \circ s' = t \circ t' \]

We define $\tilde{s}, \tilde{t} : C_2 \to C_0$ as $\tilde{s} = s \circ s' = s \circ t'$ and $\tilde{t} = t \circ s' = t \circ t'$. Each element of $C_2$ will normally be denoted as:

\[ \begin{array}{c}
\xrightarrow{\tilde{s}(T)} \\
\xleftarrow{x} \xrightarrow{\tilde{t}(T)} y
\end{array} = \begin{array}{c}
\xrightarrow{s'(T)} \\
x \xleftarrow{f} \xrightarrow{\mathcal{T}} y \xleftarrow{g} \xrightarrow{\mathcal{T}} z
\end{array} = \begin{array}{c}
\xrightarrow{t'(T)} \\
x \xleftarrow{f \circ g} \xrightarrow{\mathcal{T}} z
\end{array}. \]

4. Given $x, y \in C_0$, we have a vector space structure on each non-empty hom set $C_1(x, y)$, where:
   
   \[ C_1(x, y) = \{ f \in C_1 : s(f) = x \text{ and } t(f) = y \}. \]

5. Given $x, y, z \in C_0$, if both $C_1(x, y)$ and $C_1(y, z)$ are non-empty, we have a bilinear map (the 1-composition):
   
   \[ (f, g) \in C_1(x, y) \times C_1(y, z) \mapsto fg \in C_1(x, z) \]

   This is to be associative and to have units, being given by the map $i$:

   \[ x \xrightarrow{i(x)} x. \]

The 1-composition is denoted in the form:

\[ x \xrightarrow{f} y \xrightarrow{g} z = x \xrightarrow{f \circ g} z. \]

Given that $C_1(x, x) \neq \emptyset$, we also have the zero morphism (the null element of the vector space $C_1(x, x)$):

\[ x \xrightarrow{0_x} x. \]
and since the composition is bilinear we have:

\[ x \xrightarrow{0} x \xrightarrow{f} x = x \xrightarrow{f} x \xrightarrow{0} f = x \xrightarrow{0} x, \]

for each \( f \in C_1(x, x) \).

6. Given \( x, y \in C_0 \), define:

\[ C_2(x, y) = \{ f \in C_2 : s(T) = x \text{ and } t(T) = y \}. \]

In the case when \( C_2(x, y) \) is not empty we impose that we are given a vector space structure on it, such that \( s', t' : C_2(x, y) \to C_1(x, y) \) and also \( i' : C_1(x, y) \to C_2(x, y) \) each are linear. Let \( 0^2_x \) be the null vector of the vector space \( C_2(x, x) \). We in particular have that \( i'(0_x) = 0^2_x \).

7. Let \( x, y \in C_0 \). Given \( f, g \in C_1 \), with \( s(f) = s(g) = x \) and \( t(f) = t(g) = y \) define the 2-hom set:

\[ C_2(f, g) = \{ T \in C_2 : s'(T) = f \text{ and } t'(T) = g \}. \]

Given \( f, g, h \in C_1(x, y) \) we are to have a vertical composition map

\[ (T, S) \in C_2(f, g) \times C_2(g, h) \mapsto \begin{array}{c} \uparrow S \\ \uparrow T \end{array} \in C_2(f, h). \]

This will normally be denoted as:

\[
\begin{array}{c}
\circlearrowleft \uparrow S \\
\circlearrowleft \uparrow T \\
\end{array}
\begin{array}{c}
\downarrow f \\
\downarrow g \\
\end{array}
\begin{array}{c}
x \\xrightarrow{h} \ x \ \\
\uparrow i'(f) \ \\
y \\
\end{array}
\]

This vertical composition is to be associative and to have units, given by \( i' : C_1 \to C_2 \):

\[
\begin{array}{c}
\circlearrowleft \uparrow i'(f) \\
\end{array}
\begin{array}{c}
x \ \\
y \\
\end{array}
\]

Given \( x, y \) with \( C(x, y) \neq \emptyset \), consider the vector space of composable 2-morphisms:

\[ \text{Comp}(x, y) = \{ (T, S) \in C_2(x, y) \times C_2(x, y) : t'(T) = s'(S) \}. \]

We suppose that

\[ (T, S) \in \text{Comp}(x, y) \mapsto \begin{array}{c} \uparrow S \\ \uparrow T \end{array} \]

is a linear map.

8. Let \( x, y, z \in C_0 \). Given \( f, g \in C_1(x, y) \), we have a right whiskering map:

\[ (T, h) \in C_2(f, g) \times C_1(y, z) \mapsto Th \in C_2(fh, gh), \]
which will be visualised as:

\[
\begin{array}{c}
x \xymatrix{\uparrow T \ar[r]^f & y \ar@/^/[rr] & \downarrow g \ar@/^/[rr] & z}
\end{array}
\]

and required to be bilinear as a map \( C_2(x, y) \times C_1(y, z) \to C_2(x, z) \). In particular

\[
T_0 \circ x = 0^2_x \text{ and } 0^2_x \circ f = 0^2_x f, \quad \forall x \in C_0, f \in C_1(x, x), T \in C_2(x, x).
\]

We analogously have a left whiskering map, visualised as:

\[
\begin{array}{c}
z \xymatrix{\downarrow g \ar[r]^h & x \ar@/^/[rr] & \downarrow f \ar@/^/[rr] & \uparrow T \ar[r]^y & \downarrow h \ar@/^/[rr] & \uparrow T y \ar[r]^z & y}
\end{array}
\]

These whiskering maps are to be distributive with respect to the 2-composition of 2-morphisms:

\[
\frac{S}{T} f = \frac{Sf}{Tf} \quad \text{and} \quad f \frac{S}{T} = \frac{fS}{fT}
\]

and also associative:

\[
f(Tg) = (fT)g, \quad (Tf)g = T(fg), \quad (fg)T = f(gT).
\]

(The previous identities are to hold for any triple of composable morphisms). In addition, the interchange law is also to be satisfied: if we are given 1-morphisms:

\[
x \xymatrix{f \ar[r] & y \text{ and } y \xymatrix{g \ar[r] & z}}
\]

and 2-morphisms:

\[
\begin{array}{c}
x \xymatrix{\uparrow T \ar[r]^f & y \text{ and } y \xymatrix{\uparrow T' \ar[r]^f & z}}
\end{array}
\]

then we should have:

\[
\frac{Tg'}{fT'} = \frac{gT'}{Tf'}. \quad (3)
\]

Therefore we can define the horizontal composition of two 2-morphisms, as a particular instance of vertical composition:

\[
\begin{array}{c}
x \xymatrix{\uparrow T \ar[r]^f & y \text{ and } y \xymatrix{\uparrow T' \ar[r]^f & z}}
\end{array}
\]
where
\[ T T' = \frac{Tg'}{fT'} = \frac{gT'}{Tf'} \]  
(4)

Remark 2. Note that we could state conditions 4, 6 and 7 of the definition of a linear 2-category as saying that, given \( x, y \in C_0 \), then either the category having objects \( C_1(x, y) \) and morphisms \( C_2(x, y) \) is empty, or it is a Baez-Crans 2-vector space, [2].

Lemma 3. Let \( x, y, z, w \in C_0 \). Let \( T = x \xrightarrow{f} y, S = z \xrightarrow{g} w \) be 2-morphisms. Given a 1-morphism \( y \xrightarrow{h} z \) we have:

\[ (Th)S = T(hS) \]

Proof. By definition of horizontal composition
\[ T(hS) = \frac{f'(hS)}{T(hg)} \]
whereas
\[ (Th)S = \frac{(f'h)S}{(Th)g} \]
These coincide due to the associativity of the whiskering. \( \square \)

Consider an object \( x \in C_0 \). Let \( V(x, x) = \{ A \in C_2(x, x) : s'(A) = 0_x \} \). One has a map:
\[ (f, A) \in C_1(x, x) \times V(x, x) \xrightarrow{i} i'(f) + A \in C_2(x, x) \]
Since \( s' \) and \( t' \) are linear maps, we can see that \( s'(i'(f) + A) = f \) and \( t'(i(f) + A) = f + t'(A) \). In the opposite direction, given \( T \) in \( C_2(x, x) \) we can consider the pair
\[ \mu(T) = (s'(T), \overrightarrow{T}) \in C_1(x, x) \times V(x, x) \]
where \( \overrightarrow{T} \) is the arrow part of \( T \), namely [2]:
\[ \overrightarrow{T} = T - i'(s'(T)) \]  
(5)
Therefore \( T = i(s'(T)) + \overrightarrow{T} \), i.e. we can decompose a 2-morphism into an arrow part plus (the image via \( i \) of) the source 1-morphism. Note that \( t'(\overrightarrow{T}) = t'(T - i'(s'(T))) = t'(T) - s'(T) \). Pictorially:

Lemma 4. The above correspondence between \( C_1(x, x) \times V(x, x) \) and \( C_2(x, x) \) is one-to-one.
Proof. Taking into account the linearity of \( s' \) and \( t' \), if \((f, A) \in C_1(x, x) \times V(x, x)\) we have:

\[
\begin{align*}
\mu(\eta(f, A)) &= \mu(t'(f) + A) = (s'(t'(f) + A), i'(f) + A - i'(s'(t'(f) + A))) \\
&= (f, i'(f) + A - i'(f)) = (f, A) \in C_1(x, x) \times V(x, x);
\end{align*}
\]

On the other hand, if \( T \in C_2(x, x) \) we have:

\[
\eta(\mu(T)) = \eta(s(T), T - i'(s'(T))) = i'(s(T)) + T - i'(s(T)) = T \in C_2(x, x).
\]

The next lemma shows how vertical composition of 2-morphisms behaves under this correspondence.

**Lemma 5.** If \((A, B)\) is a pair of vertically composable 2-morphisms in \( C_2(x, x) \) then:

\[
\begin{align*}
\frac{B}{A} &= \eta(s'(A), A + B) = i'(s(A)) + A + B.
\end{align*}
\]  \( \tag{6} \)

**Proof.** See \([2, \text{pg. 11}]\). Since \( A \) and \( B \) are vertically composable 2-morphisms, \( t'(A) = s'(B) \). By linearity of vertical composition we can write

\[
\begin{align*}
\frac{B}{A} &= i'(s'(B)) + \overrightarrow{A} \\
&= i'(s'(A)) + \overrightarrow{A} + \overrightarrow{B} + i'(0_\alpha)
\end{align*}
\]

and now it is easy to recognize in the right hand side \( i'(s'(A)) + \overrightarrow{A} \) as the first summand and \( \overrightarrow{B} \) as the second summand. Note that \( t : C_1 \rightarrow C_2 \) is the 2-identity map. \( \square \)

**Remark 6.** Note that in particular we can see that any 2-morphism is invertible, with respect to the vertical composition. For this reason the inverse of \( f \xrightarrow{A} g \) will be denoted by \( g \xleftarrow{A} f \), thus \( -A = -\overrightarrow{A} \).

**Example 7** (Chain complexes). The most natural non-trivial example of a linear 2-category is probably the category Chain of chain complexes of vector spaces, chain maps, and chain homotopies (up to 2-fold homotopy), except that we have a class (rather than a set) of objects. As objects of Chain we take all chain-complexes of vector spaces. Given chain-complexes \( x \) and \( y \), the set of 1-morphisms is the vector space of chain-maps \( x \rightarrow y \), with 1-composition being the composition of chain-maps.

Given chain complexes \( x \) and \( y \), of vector spaces, with boundary \( \partial \), recall that a chain-homotopy \( s \) is a degree 1 map \( x \rightarrow y \) with \( s - t = \alpha \circ \partial - \partial \circ \alpha \). Then given chain maps \( f, g : x \rightarrow y \) the set of two morphisms \( f \Rightarrow g \) is the set of pairs \((f, s)\), where \( s \) is a homotopy (up to 2-fold homotopy) connecting \( f \) and \( g \), therefore \( f - g = s \circ \partial + \partial \circ s \). The vertical composition of 2-morphisms corresponds to the sum of homotopies, whereas the whiskering is obtained from the composition of homotopies with chain-maps. It is essential to consider homotopies up to 2-fold homotopy in order that the interchange law holds. For details see \([11, 12] \).

### 1.2 Differential crossed modules from linear 2-categories

Recall that a differential crossed module \([2, 4]\) is given by a Lie algebra morphism \( \partial : \mathfrak{h} \rightarrow \mathfrak{g} \) together with a left action \( \triangleright \) by derivations of \( \mathfrak{g} \) on \( \mathfrak{h} \), such that:

- \( \partial(X \triangleright v) = [X, \partial(v)] \), for each \( v \in \mathfrak{h} \) and \( X \in \mathfrak{g} \);
- \( \partial(v) \triangleright w = [v, w] \), for each \( v, w \in \mathfrak{h} \).
Let $C = (C_0, C_1, C_2)$ be a linear 2-category. Given an object $x \in C_0$ one has a differential crossed module
\[ \mathfrak{gl}(x) = (\beta: \mathfrak{gl}^1(x) \to \mathfrak{gl}^0(x), \triangleright), \]
especially constructed in [2, 4]. The Lie algebra $\mathfrak{gl}^0(x)$ is given by all morphisms $C_1(x, x)$, with bracket:
\[ \{f, g\} = fg - gf \]
(recall that $C_1(x, x)$ is a vector space). The Lie algebra $\mathfrak{gl}^1(x)$ is given by all 2-morphisms $V(x, x)$, i.e. of the form
\[ \xymatrix{ x \ar@/^/[r]^{f} \ar@/_/[r]_{g} & x, } \]
the bracket being
\[ \{T, S\} = TS - ST. \]
Here $TS$ is the horizontal composition of 2-morphisms. We have an action of $\mathfrak{gl}^0(x)$ on $\mathfrak{gl}^1(x)$ given by
\[ f \triangleright T = fT - Tf. \]
This is an action by derivations due to the associativity of the whiskering. We have a Lie algebra morphism $\beta: \mathfrak{gl}^1(x) \to \mathfrak{gl}^0(x)$, where $\beta = t'$. Clearly
\[ \beta(f \triangleright T) = f \triangleright \beta(T). \]
Let us see that $\beta(T) \triangleright S = \{S, T\}$. Note that $s'(T) = s'(S) = 0_x$. Let $f = t'(T)$ and $g = t'(S)$.
\[ \{T, S\} = TS - ST = \frac{fS}{T0_x} - \frac{Sf}{0_xT} = \frac{fS}{0_x^2} - \frac{Sf}{0_x^2} = \frac{fS}{i'(0_x)} - \frac{Sf}{i'(0_x)} = fS - Sf = \beta(T) \triangleright S. \]
Note that $i: C_1 \to C_2$ is the 2-identity map.

### 1.3 Strict monoidal linear 2-categories

**Definition 8.** A strict monoidal linear 2-category $(C, \otimes, I) = (C_0, C_1, C_2, \otimes, I)$ is given by:

1. A linear 2-category $C = (C_0, C_1, C_2)$.
2. For any $x, y \in C_0$ an object $x \otimes y$ of $C_0$. We are to have that $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, for each objects $x, y, z \in C_0$.
3. An object $I$ of $C$. We suppose that $x \otimes I = I \otimes x = x$, for each object $x$.
4. For any $x \in C_0$ and any $y \xrightarrow{f} z \in C_1$, we have 1-morphisms:
\[ x \otimes y \xrightarrow{x \otimes f} x \otimes y \text{ and } y \otimes x \xrightarrow{f \otimes x} z \otimes y. \]

Tensoring with an object is to be distributive with respect to composition of morphisms, namely
\[ x \otimes (f \otimes y) = (x \otimes f) \otimes y \text{ and } (f \otimes x) \otimes y = f \otimes (x \otimes y). \]

and moreover it is to define a linear map $C_1(x, y) \to C_1(x \otimes z, y \otimes z)$. The same is to hold for left tensoring by $z \in C_0$. Tensoring with $I$ is supposed to not change anything: $1 \otimes f = f = f \otimes 1$ for each 1-morphism $f$. Tensoring morphisms with objects is to be associative, for instance:
\[ x \otimes (f \otimes y) = (x \otimes f) \otimes y \text{ and } (f \otimes x) \otimes y = f \otimes (x \otimes y). \]
5. For any $x \in C_0$ and $y \xrightarrow{f} z \in C_2$ we are to be given 2-morphisms:

\[
x \otimes \begin{array}{c}
g \downarrow \quad \uparrow S \\
\end{array} \quad z = x \otimes y \quad \uparrow x \otimes S \\
\otimes z,
\]

Moreover, tensoring with an object is to be linear and distributive with respect to vertical composition of 2-morphisms. Tensoring with $I$ is supposed to not change anything. Finally, whiskering commutes with tensoring with objects: given $x, y, z \in C_0$ with $C_2(x, y)$ and $C_1(y, z)$ not empty, we suppose that

\[(A, f) \in C_2(x, y) \times C_1(x, y) \mapsto Af \in C_2(x, y)\]

commutes with left and right tensoring. For instance

\[x \otimes (Af) = (x \otimes A)(x \otimes f) \text{ and } (Af) \otimes x = (A \otimes x)(f \otimes x).\]

The same is of course to hold for left whiskering. Tensoring with objects is to be associative.

6. We now impose the usual interchangeability conditions for strict 3-categories.

(i) Given $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$ we impose that:

\[x \otimes x' \xrightarrow{x \otimes f'} x \otimes y' \xrightarrow{f' \otimes y'} y \otimes y' = x \otimes x' \xrightarrow{f \otimes x'} y \otimes x' \xrightarrow{y \otimes f'} x' \otimes y'.\]

This permits us to define the tensor product of morphisms $f \otimes f' : x \otimes x' \to y \otimes y'$ as being one of the previous compositions.

(ii) Given $y \xrightarrow{f} z \in C_2$ and $x \xrightarrow{h} x'$ we have the following equality of 2-morphisms:
We define:
\[
\left( \begin{array}{c}
g \\\ny \\
\downarrow S \\
z \\
\end{array} \right) \otimes (x \rightarrow x') = \left( \begin{array}{c}
y \otimes x \\
\downarrow S \otimes h \\
z \otimes x' \\
\end{array} \right)
\]
as being one of the previous compositions. The same identity is to hold also for left tensoring.

(iii) Given
\[
\left( \begin{array}{c}
x \\
y \\
\downarrow S \\
x' \\
\end{array} \right)
\quad \text{and} \quad \left( \begin{array}{c}
x' \\
y' \\
\downarrow S' \\
x'' \\
\end{array} \right)
\]
we have that:
\[
\left( \begin{array}{c}
x \otimes x' \xrightarrow{g \otimes g'} y \otimes y' \\
\downarrow f \otimes f' \\
\end{array} \right) = \left( \begin{array}{c}
x \otimes x' \xrightarrow{f \otimes f'} y \otimes y' \\
\downarrow g \otimes g' \\
\end{array} \right)
\]
We define
\[
\left( \begin{array}{c}
x \otimes x' \\
\downarrow S \otimes S' \\
y \otimes y' \\
\end{array} \right)
\]
as being one of the previous compositions.

Example 9 (Chain-complexes again). The 2-category Chain of chain complexes is nearly a strict monoidal category, except that the tensor product of objects is not strictly associative; for details, in a different language, see \[11, 12\]. The tensor product of chain-complexes is the usual one; see \[13\], as it is the tensor product of chain maps, and the tensor product of chain-maps and homotopies, up to 2-fold homotopy. Note however that given two 2-morphisms \(f \Rightarrow g\) and \(f' \Rightarrow g'\) then the tensor product is:
\[
\left( f \Rightarrow g \right) \otimes \left( f' \Rightarrow g' \right) = f \otimes f' \xrightarrow{g \otimes g'} g \otimes g' = f \otimes f' \xrightarrow{(g \otimes g' + f \otimes f')} g \otimes g'.
\]
Again we must consider homotopies up to 2-fold homotopy in order that the last equality holds.

1.4 Symmetric strict monoidal linear 2-categories
This is a very restricted case of the definition of a braided monoidal 2-category in \[5, 17\].
Definition 10. A totally symmetric strict monoidal linear 2-category $C = (C, \otimes, I, B)$ is given by:

1. A strict monoidal linear 2-category $(C, \otimes, I)$.

2. For any 2-objects $x$ and $y$ an invertible 1-morphism

$$x \otimes y \xrightarrow{B_{x,y}} y \otimes x.$$

This is to satisfy the following properties:

(a) For any two objects $x$ and $y$ we have: $B_{x,y}B_{y,x} = \text{id}_{x \otimes y}$

(b) For any three objects $x, y, z$:

$$\left(x \otimes y \otimes z \xrightarrow{B_{x,y,z}} y \otimes z \otimes x\right) = \left(x \otimes y \otimes z \xrightarrow{B_{y,z,x}} y \otimes x \otimes z \xrightarrow{\text{id}_{x \otimes y}} y \otimes z \otimes x\right), \quad (7)$$

and analogously for $B_{x \otimes y, z}$. Moreover, $B_{\text{id}, x} = B_{x, \text{id}} = \text{id}_x$.

(c) Given $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$ we have:

$$\left(x \otimes x' \xrightarrow{f \otimes f'} y \otimes y' \xrightarrow{B_{x,y}} y' \otimes y\right) = \left(x \otimes x' \xrightarrow{B_{x',y}} x' \otimes x \xrightarrow{f' \otimes f} y' \otimes y\right), \quad (8)$$

(d) Given $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$ we have:

$$\left(A \otimes A' \xrightarrow{B_{y,y}} B_{(x,x')} \left(A' \otimes A\right)\right). \quad (9)$$

Note that properties (a), (c), (d) are equivalent to saying that $B$ is an involutive natural transformation between the strict 2-functors $\otimes$ and $\otimes^{\text{op}}$ (the opposite tensor product).

Example 11. Forgetting about the associativity morphisms, the basic example of a totally symmetric strict monoidal linear 2-categories is the 2-category of chain complexes, chain-maps, and homotopies up to 2-fold homotopies. The symmetry is a graded version of the flip $[13, 24]$. See $[11, 12]$ for details, in a different language.

The following has an essentially obvious proof, and follows from Mac Lane coherence theorem for braided tensor categories $[23]$, in the particular case of symmetric categories.

Lemma 12. Consider a totally symmetric strict monoidal linear 2-category $C = (C, \otimes, I, B)$. Let $a_1, \ldots, a_n$ be objects of $C$. Given any permutation $\sigma$ of $\{1, \ldots, n\}$ there exists a unique map:

$$M_{\sigma} : a_1 \otimes \ldots \otimes a_n \rightarrow a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)}$$

constructed from compositions of the braidings $B_{x,y}$ (where $x$ and $y$ are tensor products of objects in $\{a_1, \ldots, a_n\}$) possibly tensored by tensor products of objects in $\{a_1, \ldots, a_n\}$.

As an example for $n = 3$ and $\sigma = (13)$ then

$$x \otimes y \otimes z \xrightarrow{M_{\sigma}} z \otimes y \otimes x$$

can be defined as (for instance):

$$B_{x, y \otimes z} (B_{y \otimes z} \otimes x) = (x \otimes B_{y, z})B_{x, y \otimes z} = (B_{x, y} \otimes z)(y \otimes B_{x, z})(B_{y, z} \otimes x).$$

13
We now introduce a notation, which will be used in the following section, to give the definition of a strict infinitesimal 2-braiding. Let \( x_1, \ldots, x_n \) be objects of \( C = (C, \otimes, I, B) \), a totally symmetric strict monoidal linear 2-category. Given distinct indices \( a \) and \( b \) in \( \{1, \ldots, n\} \), consider a morphism \( r: x_a \otimes x_b \to x_b \otimes x_a \) (first case), or \( r: x_b \otimes x_a \to x_b \otimes x_a \) (second case). Put:
\[
 x_1 \otimes x_2 \otimes \ldots \otimes x_n \xrightarrow{\rho^b} x_1 \otimes x_2 \otimes \ldots \otimes x_n
\]
as being the composition (in the first case):
\[
 x_1 \otimes x_2 \otimes \ldots \otimes x_n \xrightarrow{M_r} X \otimes x_a \otimes x_b \otimes X \xrightarrow{\otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otimes \otime
we have:

\[
\begin{align*}
(x \otimes y \otimes z \xrightarrow{r_{12}^z} x \otimes y \otimes z) &= (x \otimes y \otimes z \xrightarrow{r_{x,y \otimes z}} x \otimes y \otimes z) \\
(x \otimes y \otimes z \xrightarrow{r_{13}^x} x \otimes y \otimes z) &= (x \otimes y \otimes z \xrightarrow{x \otimes r_{x,y \otimes z}} x \otimes y \otimes z) \\
(x \otimes y \otimes z \xrightarrow{r_{13}^y} x \otimes y \otimes z) &= (x \otimes y \otimes z \xrightarrow{y \otimes x \otimes z \xrightarrow{y \otimes x \otimes z \xrightarrow{r_{x,y \otimes z}} x \otimes y \otimes z}})
\end{align*}
\]

These type of notation extends immediately: Given \( R: x_i \otimes x_j \otimes x_c \rightarrow x_i \otimes x_j \otimes x_c \) we can define

\[
R^{abc}: x_1 \otimes x_2 \otimes \ldots \otimes x_n \rightarrow x_1 \otimes x_2 \otimes \ldots \otimes x_n
\]

by using the obvious analogue of (10), and this does not depend on the permutation chosen by the same naturality arguments as in the proof of Lemma 13. Analogously, given a 2-morphism:

\[
R^{abc}:
\]

we can unambiguously define a 2-morphism:

\[
R^{abc}:
\]

### 2 Strict infinitesimal 2-braidings

#### 2.1 Definition of a strict infinitesimal 2-braiding

We present a direct categorification of the concept of an infinitesimal braided category; see [18, XX.4] and [10], called there infinitesimal symmetric category.

**Definition 14.** A strict infinitesimal 2-braiding \((r, T)\) in a symmetric strict monoidal linear 2-category \(C = (C, \otimes, I, B)\) is given by the following data.

1. For any pair of objects \(x\) and \(y\), a 1-morphism \(r_{x,y}: x \otimes y \rightarrow x \otimes y\). These are to satisfy the linearity conditions:

\[
\begin{align*}
r_{x,y \otimes c} &= r_{x,y}^{12} + r_{x,z}^{13}, \\
r_{x \otimes y,z} &= r_{x,z}^{13} + r_{y,z}^{23}
\end{align*}
\]

(14)

thus for example:

\[
\begin{align*}
r_{x,y \otimes z \otimes w} &= r_{x,y}^{12} + r_{x,z}^{13} + r_{x,w}^{14}, \\
r_{x \otimes y,z \otimes w} &= r_{x,z}^{13} + r_{x,w}^{14} + r_{y,z}^{23} + r_{y,w}^{24}
\end{align*}
\]
2. For any morphism \( x \xrightarrow{f} x' \) and any object \( y \), a 2-morphism \( T_{(f,y)} \) such that:

\[
\begin{array}{c}
\xymatrix{
 x \otimes y & x' \otimes y \\
 x \ar[r]^{r_{x,y}} & x' \ar@/^/[r]_{T_{(f,y)}} & x' \otimes y
}
\end{array}
\]

(15)

In other words, \( T(f, y) \) measures the failure of functoriality of \( r \) with respect to the morphism \( f \otimes y \). This is to be natural with respect to 2-morphisms. If \( S : f \Rightarrow g \) then \( T_{(f,y)} \) and \( T_{(g,y)} \) have to be related via a vertical composing with \( S \): the request is

\[
(S \otimes y)r_{x',y} = T_{(g,y)} \Rightarrow T_{(f,y)} (S \otimes y)
\]

which is graphically visualized as

\[
\begin{array}{c}
\xymatrix{
 x \otimes y & x' \otimes y & x' \otimes y \\
 x \ar[r]^{r_{x,y}} & x' \ar@/^/[r]_{T_{(f,y)}} & x' \otimes y
}
\end{array}
\]

(16)

Moreover we suppose that \( T \) satisfies the linearity condition

\[
T_{(f+f',y)} = T_{(f,y)} + T_{(f',y)}
\]

(18)

and that, if \( x \xrightarrow{f} x' \xrightarrow{f'} x'' \), we have:

\[
T_{f f',y} = \frac{(f \otimes y)T_{(f',y)}}{T_{(f,y)}(f' \otimes y)}.
\]

(19)

Graphically this means:

\[
\begin{array}{c}
\xymatrix{
 x \otimes y & x' \otimes y & x' \otimes y \\
 x \ar[r]^{r_{x,y}} & x' \ar@/^/[r]_{T_{(f',y)}} & x' \otimes y
}
\end{array}
\]

3. Similarly, for any object \( x \) and morphisms \( y \xrightarrow{f} y' \), a 2-morphism \( T_{(x,f)} \):

\[
\xymatrix{
 x \otimes y & x' \otimes y & x' \otimes y \\
 x \ar[r]^{r_{x,y}} & x' \ar@/^/[r]_{T_{(x,f)}} & x' \otimes y
}
\]

16
As before, this is to be natural with respect to all 2-morphisms \( S \): \( f \Rightarrow g \), in the sense that

\[
\frac{T(x,f)}{r_{x,y}(x \otimes S)} = \frac{(x \otimes S)r_{x,y'}}{T(x,g)} .
\]

(20)

Moreover, the obvious analogues of linearity (18) and composition (19) properties hold.

4. We suppose that that given \( x \xrightarrow{f} x' \) and \( y \xrightarrow{g} y' \), the 2-morphisms \( T(f, g) \) and \( T(x, y) \) satisfy the following interchange law:

\[
\frac{(f \otimes g)T(x', g)}{T(f, g)(x' \otimes g)} = \frac{(x \otimes g)T(f, y')}{T(x, g)(f \otimes y')} .
\]

(21)

Graphically:

The above equality allows to define unambiguously \( T(f, g) \) as one of the two sides of (21).

5. Finally we suppose that given \( f : x \rightarrow x' \) the following linearity conditions hold:

\[
\begin{align*}
T(f, y \otimes z) &= T^{12}_{(f, y)} + T^{13}_{(f, z)} \\
T(f \otimes y, z) &= T^{13}_{(f, z)}
\end{align*}
\]

(22)

\[
\begin{align*}
T(y \otimes f, z) &= T^{13}_{(y, f)} + T^{23}_{(z, f)} \\
T(y, f \otimes z) &= T^{23}_{(z, f)}
\end{align*}
\]

(23)

**Remark 15.** If we do not consider the linearity conditions, the previous definition simply expresses the fact that the pair \((r, T)\) is a pseudo-natural transformation \( \otimes \Rightarrow \otimes \), from the 2-functor \( \otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \) to itself.

**Remark 16.** Equations (14), (22) and (23) could easily be weakened to hold only up to a coherent 2-morphism, which is the type of generality appearing in the definition of a braided-monoidal 2-category; \( \mathfrak{R} \). We will however not need this generality. This is nevertheless the reason for the term “strict” in “strict infinitesimal 2-braiding”.

### 2.2 The failure of the 4-term relations and coherent infinitesimal 2-braidings

Remember that in an infinitesimally symmetric category the natural endomorphism \( r : \otimes \Rightarrow \otimes \) satisfies the so-called 4-term relations or infinitesimal braid relations: for generic objects \( x, y, z \) we have

\[
\begin{align*}
[r_{x,y', x', y}^{12} + r_{x', y}^{13} + r_{y', z}^{23}] = 0 ,

[r_{x,y}^{12} + r_{x,y}^{13} + r_{y,z}^{23}] = 0 .
\end{align*}
\]

These relations express the functoriality of \( r_{x,y \otimes z} \) (resp. \( r_{x \otimes y, z} \)) with respect to the morphism \( x \otimes r_{y,z} \) (resp. \( x \otimes r_{y,z} \)). Consider now a totally symmetric monoidal linear strict 2-category, with an infinitesimal 2-braiding \((r, T)\). Since \((r, T)\) is a pseudo-natural transformation, \( r \) is not functorial and so it does not satisfy the 4-term relations. We can however measure this failure in terms of the 2-morphism \( T \) suitably evaluated. Let \( x, y, z \in \mathcal{C}_0 \). Let us consider the 2-morphism \( P_{x,y,z} := T_{(x,y,z)} \). Explicitly:

\[
\begin{align*}
[r_{x,y \otimes z} + P_{x,y,z}] &= r_{x, y \otimes z} ,

r_{x, y \otimes z} &= T_{(x,y,z)} .
\end{align*}
\]

(24)
From its definition, and using the linearity of \( r \):

\[
x \otimes y \otimes z \xrightarrow{T_{(x,y,z)}} x \otimes y \otimes z = x \otimes y \otimes z \quad (25)
\]

Similarly we consider the 2-morphism \( Q_{x,y,z} := T_{(x,y,z)} \). Explicitly:

\[\begin{align*}
\tau_{x,y}(x \otimes z) &= T_{(x,y,z)} \\
&= (r_{x \otimes y, z})r_{x, y \otimes z}.
\end{align*}\]

As before, by linearity of \( r \) it is:

\[
x \otimes y \otimes z \xrightarrow{T_{(x,y,z)}} x \otimes y \otimes z = x \otimes y \otimes z. \quad (27)
\]

We then see that \( P_{x,y,z} \) and \( Q_{x,y,z} \) are 2-morphisms connecting the left and right sides of the 4-term relations.

Closely related to these, there are three naturally defined 2-morphisms, which connect a combination of 4-term relations sides:

\[
r_{23}^{yz} x + r_{12}^{xy} + r_{13}^{yz} x \xrightarrow{T_{(x,y,z)}} r_{23}^{yz} x + r_{12}^{xy} x + r_{13}^{yz} x.
\]

They are

\[
P_{2,1,3}^{x,y,z} + i'(r_{23}^{yz} x), \quad Q_{1,2,3}^{x,y,z} + i'(r_{23}^{yz} x),
\]

and the composition

\[
r_{23}^{yz} x + r_{12}^{xy} + r_{13}^{yz} x \xrightarrow{Q_{2,3}^{x,y,z}} r_{23}^{yz} x + r_{12}^{xy} x + r_{13}^{yz} x
\]

**Definition 17** (Coherent infinitesimal 2-braiding). An infinitesimal 2-braiding is called coherent if the previous three morphisms all coincide. Passing to the arrow parts of each 2-morphism, this is to say that for each objects \( x, y, z \) we must have:

\[
\overrightarrow{P_{g,x}^{x,y,z}} = - (\overrightarrow{P_{x,y,z}} + \overrightarrow{Q_{x,y,z}}) = \overrightarrow{Q_{x,y,z}}.
\]

**2.3 Totally symmetric infinitesimal 2-braiding**

**Definition 18.** An infinitesimal 2-braiding in a symmetric monoidal linear 2-category \( C = (C, \otimes, I, B) \) is called totally symmetric if for any objects \( x, x', y, y', z \in C_0 \) and morphisms \( x \xrightarrow{f} x' \), \( y \xrightarrow{g} y' \) the following conditions hold:

(i) \( r \) and \( T \) are functorial with respect to the braiding \( B \):

\[
\begin{align*}
&\quad r_{x,y} = B_{x,y} r_{y,x} B_{y,x} \\
&T_{(f,g)} = B_{x,y} T_{(g,f)} B_{y',x'}
\end{align*}\]

(28) (29)
(ii) \( r \) is functorial with respect to \( B \) in the second argument:
\[
    r_{x,y; z}(x \otimes B_{y,z}) = (x \otimes B_{y,z})r_{x,z;y}
\]  
and furthermore:
\[
    T(x,B_{y,z}) = i'(r_{x,y; z}(x \otimes B_{y,z})).
\]  
(in the equation above recall that \( i' : C_1 \to C_2 \) is the 2-identity map.)

(iii) similarly, \( r \) is functorial with respect to \( B \) in the first argument:
\[
    r_{x,y; z}(B_{x,y} \otimes z) = (B_{x,y} \otimes z)r_{y,x; z}
\]  
and furthermore
\[
    T(B_{x,y; z}) = i'(r_{x,y; z}(B_{x,y} \otimes z))r_{x,y; z}.
\]

Some comment on the previous properties: the graphical presentation of (29) is
\[
\begin{align*}
    x \otimes y \xrightarrow{B_{y,z}} y \otimes x \\
    \quad \uparrow T(g,f) \quad \uparrow T(f,g)
\end{align*}
\]
\[y' \otimes x' \xrightarrow{B_{y',z'}} x' \otimes y' = x \otimes y.
\]

A consequence of (31) is that \( s'(T(x,B_{y,z})) = t'(T(x,B_{y,z})) \), where
\[
    s'(T(x,B_{y,z})) = (r_{x,y; yz})(x \otimes B_{y,z}) = (r_{x,y}^{12} + r_{x,z}^{13})B_{y,z}^{23},
\]
\[
    t'(T(x,B_{y,z})) = (x \otimes B_{y,z})(r_{y,z; yz}) = B_{y,z}^{23}(r_{x,y}^{12} + r_{x,z}^{13}),
\]
which is exactly equation (30), while (31) expresses the stronger property that \( T(x,B_{y,z}) \) is in fact the identity 2-morphism. Combining this with (19) we get
\[
    B_{y,z}^{23}T_{(y,z; yz)}B_{y',z'}^{23} = T_{(y,z; yz)}.
\]

**Lemma 19.** Suppose that the infinitesimal 2-braiding is totally symmetric. We have, for each three objects \( x, y, z \):
\[
    P_{x,z,y} = P_{x,y,z}^{132}, \quad P_{x,y,z} = Q_{x,y,z}^{231}, \quad Q_{x,z,y} = Q_{y,z,x}^{231}
\]

**Proof.** The first equality is proved as:
\[
    P_{x,z,y} = T(x,r_{y,z}) = (x \otimes B_{z,y})(x \otimes B_{y,z})T(x,r_{y,z})(x \otimes B_{y,z})(x \otimes B_{y,z}) = (x \otimes B_{z,y})T(x,r_{y,z})(x \otimes B_{y,z}) = (x \otimes B_{y,z})P_{x,y,z}(x \otimes B_{y,z}) = P_{x,y,z}^{132}
\]
where we used (31) in the second step. The third equality follows similarly. The second equality is proved, using (20), as:
\[
    P_{x,y,z} = T(x,r_{y,z}) = T^{231}_{y,z,x} = Q^{231}_{y,z,x} = Q^{231}_{x,y,z}.
\]

Combining with Definition 17, we reach the following result whose easy proof is omitted.

**Lemma 20** (Jacobi type identity). Given a coherent totally symmetric infinitesimal 2-braiding \( C = (C, \otimes, I, B) \), for each \( x, y, z \in C_0 \) we have:
\[
    p^{123}_{x,y,z} + p^{312}_{x,y,z} + p^{231}_{x,y,z} = 0.
\]

19
2.4 The categorified 4-term relations

Let us denote for simplicity the left and right sides of the 4-term relations as follows:

\[
\begin{align*}
r_{12}^{13} f_{y, z}^{12} + r_{y, z}^{13} f_{x, y}^{13} &= s'(P_{x, y, z}), \\
r_{12}^{13} f_{x, y}^{12} + r_{x, y}^{12} f_{y, z}^{12} &= t'(P_{x, y, z}), \\
r_{y, z}^{12} f_{x, y}^{23} + r_{x, y}^{23} f_{y, z}^{23} &= s'(Q_{x, y, z}), \\
r_{y, z}^{12} f_{x, y}^{13} + r_{x, y}^{13} f_{y, z}^{13} &= t'(Q_{x, y, z}).
\end{align*}
\]

These 4-term relations are therefore satisfied if \( P_{x, y, z} \) and \( Q_{x, y, z} \) reduce to the vertical identity 2-morphisms between their equal source and target. When \( P_{x, y, z} \) and \( Q_{x, y, z} \) are non-trivial, let us consider the 2-morphisms \( T_{(v(P_{x, y, z}), w)} \) and \( T_{(v'(P_{x, y, z}), w)} \). From the definition of an infinitesimal 2-braiding, these 2-morphisms will satisfy five naturality conditions, which we now describe.

Since \( s'(P_{x, y, z}) \xrightarrow{a} t'(P_{x, y, z}) \), the two \( T \) morphisms are related by the naturality condition (16). We recall its content: every time we have a 2-morphism \( A : s'(A) \xrightarrow{A} t'(A) \) the 2-morphisms \( T_{(v(A), w)} \) and \( T_{(v'(A), w)} \) are to satisfy

\[
\frac{(A \otimes w)r_{X, w}}{T_{(v'(A), w)}} = \frac{T_{(v(A), w)}r_{X, w}(A \otimes w)}{T_{(v'(A), w)}}.
\]

We are now considering the case \( X = x \otimes y \otimes z \) and \( A = P_{x, y, z} \). This leads to the first naturality condition

\[
\frac{(P_{x, y, z} \otimes w)r_{x \otimes y \otimes z, w}}{T_{(s'(P_{x, y, z}), w)}} = \frac{T_{(v(P_{x, y, z}), w)}r_{x \otimes y \otimes z, w}(P_{x, y, z} \otimes w)}{T_{(v'(P_{x, y, z}), w)}}
\]

which can be visualised as:

![Diagram](image)

We now derive the relations between \( r \) and \( P_{x, y, z} \) associated to (37). It is convenient to use the decomposition of 2-morphisms into the source and arrow part, \( A = i'(s'(A)) + \vec{A} \). We then observe that the source parts of the two vertical compositions in (37) agree; hence only relations between the arrow parts of the vertical compositions are left, and they are computed using Lemma 5.

\[
\begin{align*}
\frac{(P_{x, y, z} \otimes w)r_{x \otimes y \otimes z, w}}{T_{(v(P_{x, y, z}), w)}} &= \vec{T}_{(v'(P_{x, y, z}), w)} + (P_{x, y, z} \otimes w)r_{x \otimes y \otimes z, w} \\
\frac{T_{(v'(P_{x, y, z}), w)}}{r_{x \otimes y \otimes z, w}(P_{x, y, z} \otimes w)} &= r_{x \otimes y \otimes z, w}(P_{x, y, z} \otimes w) + \vec{T}_{(v'(P_{x, y, z}), w)}.
\end{align*}
\]
Now we compute each term separately. We use linearity of $r$, and $\overrightarrow{fA} = f \overrightarrow{A}$. So, for example, we have that $\overrightarrow{T_{(f',g)}} = f \overrightarrow{T_{(f,g)}} + \overrightarrow{T_{(f,g)}f'}$. We obtain:

$$T'((p_{x,y,z})_w) = \overrightarrow{T_{(r_{12}^w, r_{13}^w, r_{23}^w, x, y, z, w)}} = \overrightarrow{T_{(r_{12}^w, r_{13}^w, x, y, z, w)}} + \overrightarrow{T_{(r_{23}^w, x, y, z, w)}}$$

and analogously

$$T'((p_{x,y,z})_w) = \overrightarrow{T_{(r_{12}^w, r_{13}^w, r_{23}^w, x, y, z, w)}} = \overrightarrow{T_{(r_{12}^w, r_{13}^w, x, y, z, w)}} + \overrightarrow{T_{(r_{23}^w, x, y, z, w)}}$$

Also

$$(r_{14}^{w,x}, r_{24}^{w,y}, r_{34}^{w,z}) \cdot \overrightarrow{P_{x,y,z}^{123}} - r_{12}^{w,x} \cdot \overrightarrow{Q_{y,z,w}^{234}} + r_{23}^{w,y} \cdot \overrightarrow{Q_{x,z,w}^{124}} = 0 \quad (39)$$

A second naturality condition is provided by the relation between $T_{(w,x', p_{y,z})}$ and $T_{(w,y', p_{x,z})}$. Again derived from (16), it reads

$$(w \otimes P_{x,y,z})_w r_{x,y,z} w_{y,z} = \overrightarrow{T_{(w', p_{y,z})}} - \overrightarrow{T_{(w', p_{x,z})}}$$

and it is graphically visualizes as

$$w \otimes x \otimes y \otimes z.$$
Two analogous relations are derived from naturality conditions of $T_{(w,s'(Q_{x,y,z}))}$ and $T_{(w,s'(Q_{x,y,z}))}$ (resp. $T_{(w,s'(Q_{x,y,z}))}$ and $T_{(w,s'(Q_{x,y,z}))}$) with respect to the 2-morphism $Q_{x,y,z}$. Similarly to (37) we have

\[
\frac{T_{(w,s'(Q_{x,y,z}))}}{r_{w,s(yz)}}(Q_{x,y,z} \otimes w) = T_{(w,s'(Q_{x,y,z}))}(Q_{x,y,z} \otimes w)
\]

which is displayed as

\[
\begin{array}{c}
\text{(43)}
\end{array}
\]

Following the same line of computations as before, equating the arrow parts of both sides we get the relation

\[
(r_{r,y,z}^{14} + r_{r,y,z}^{24} + r_{r,y,z}^{34}) \Rightarrow Q_{r,y,z}^{123} + r_{r,y,z}^{12} \Rightarrow (Q_{r,y,z}^{123} + Q_{r,y,z}^{234}) - (r_{r,y,z}^{13} + r_{r,y,z}^{23}) \Rightarrow Q_{r,y,z}^{124} = 0 .
\]

Similarly to (40) we also know that it holds

\[
\frac{T_{(w,s'(Q_{x,y,z}))}}{r_{w,s(yz)}}(w \otimes Q_{x,y,z}) = T_{(w,s'(Q_{x,y,z}))}(w \otimes Q_{x,y,z})
\]

or, graphically:

\[
\begin{array}{c}
\text{(46)}
\end{array}
\]

or, graphically:

\[
\begin{array}{c}
\text{(47)}
\end{array}
\]
This eventually leads to the relation

\[
(r_{x,y}^{12} + r_{w,z}^{13} + r_{w,z}^{14}) \Rightarrow Q_{x,y,z}^{234} + r_{x,y}^{23} \Rightarrow (p_{w,x,z}^{124} + p_{w,y,z}^{134}) - (r_{x,y}^{24} + r_{y,z}^{34}) \Rightarrow p_{w,x,y}^{123} = 0 .
\] (48)

Finally, a further condition is obtained by considering

\[
r_{x,y} : x \otimes y \to x \otimes y , \quad r_{z,w} : z \otimes w \to z \otimes w
\]

and writing the interchange law (21), that assures \( T(r_{x,y}^{12}, r_{w,z}^{13}) \) is well defined. In our present setting it reads

\[
\frac{(r_{x,y} \otimes z \otimes w) T(x \otimes y, r_{w,z})}{T(r_{x,y} \otimes z \otimes w)} = \frac{(x \otimes y \otimes r_{z,w}) T(r_{x,y} \otimes z \otimes w)}{T(r_{x,y} \otimes z \otimes w)} .
\] (49)

Graphically:

Writing \( (49) \) along the arrow parts (the source parts of both sides are trivially equal) and using \( (22) \) we compute:

\[
\begin{align*}
&= r_{x,y} \Rightarrow T(x \otimes y, r_{w,z}) + T(r_{x,y} \otimes z \otimes w) - r_{x,y} \Rightarrow T(r_{x,y} \otimes z \otimes w) = \\
&= r_{x,y} \Rightarrow T(x \otimes y, r_{w,z}) + T(r_{x,y} \otimes z \otimes w) - r_{x,y} \Rightarrow T(r_{x,y} \otimes z \otimes w) = \\
&= r_{x,y} \Rightarrow p_{w,x,z}^{124} + p_{w,y,z}^{234} - r_{x,y}^{12} \Rightarrow (Q_{x,y,z}^{234} + Q_{x,y,z}^{124}) = 0 .
\end{align*}
\] (51)

We have finished proving our first main theorem:

**Theorem 21.** Consider an infinitesimal 2-braiding \((r, T)\) in a symmetric strict monoidal linear 2-category \( \mathcal{C} = \)

23
(C, ⊗, I, B). Let $P_{x,y,z}$ and $Q_{x,y,z}$ be as defined in (23) and (26). The equations below hold:

$$
(r_{x_{14},z_{y,0}}^{14} + r_{w_{y,0}}^{24} + r_{z_{y,0}}^{34}) \triangleright (p_{y_{x,y,0}}^{123} - (r_{x_{14},z_{y,0}}^{14} + r_{x_{13},z_{y,0}}^{13} \triangleright (Q_{x_{y,y,0}}^{23} + r_{x_{24},z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{12} + Q_{x_{y,y,0}}^{14})) = 0, \tag{52}
$$

$$
(r_{x_{24},z_{y,0}}^{24} + r_{w_{y,0}}^{13} + r_{z_{y,0}}^{34}) \triangleright (p_{y_{x,y,0}}^{123} - r_{x_{24},z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{23} + r_{x_{24},z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{12} + Q_{x_{y,y,0}}^{14})) = 0, \tag{53}
$$

$$
(r_{x_{13},z_{y,0}}^{13} + r_{w_{y,0}}^{24} + r_{z_{y,0}}^{34}) \triangleright (p_{y_{x,y,0}}^{123} + r_{x_{13},z_{y,0}}^{13} \triangleright (Q_{x_{y,y,0}}^{23} + r_{x_{24},z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{12} + Q_{x_{y,y,0}}^{14})) = 0, \tag{54}
$$

$$
(r_{x_{13},z_{y,0}}^{13} + r_{w_{y,0}}^{24} + r_{z_{y,0}}^{34}) \triangleright (p_{y_{x,y,0}}^{123} + r_{x_{13},z_{y,0}}^{13} \triangleright (Q_{x_{y,y,0}}^{23} + r_{x_{24},z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{12} + Q_{x_{y,y,0}}^{14})) = 0. \tag{55}
$$

Continuing the notation of the previous Theorem, suppose that the infinitesimal 2-braiding $(r, T)$ is coherent, Definition (17). Looking at equation (56), and applying $B_{y,z}$ to both sides of it, we reach that for any objects $x, y, z, w$ of $C$:

$$
r_{x_{13},z_{y,0}}^{13} \triangleright (p_{y_{x,y,0}}^{124} + p_{x_{y,y,0}}^{24}) - r_{z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{13} + Q_{x_{y,y,0}}^{14}) = 0. \tag{56}
$$

If $(r, T)$ is coherent then:

$$
r_{x_{13},z_{y,0}}^{13} \triangleright (p_{y_{x,y,0}}^{124} + p_{x_{y,y,0}}^{24}) - r_{z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{13} + Q_{x_{y,y,0}}^{14}) = 0. \tag{57}
$$

Therefore:

**Theorem 22.** In the conditions of Theorem 21 if $(r, T)$ is coherent then furthermore we have:

$$
r_{x_{13},z_{y,0}}^{13} \triangleright (p_{y_{x,y,0}}^{124} + p_{x_{y,y,0}}^{24}) + r_{z_{y,0}}^{24} \triangleright (Q_{x_{y,y,0}}^{13} + Q_{x_{y,y,0}}^{14}) = 0. \tag{57}
$$

The relations appearing in Theorems 21 and 22 are our proposal for a categorification of the 4-term relations.

### 3 A related construction: the categorified Knizhnik-Zamolodchikov connection

We now re-interpret the construction of the categorified Knizhnik-Zamolodchikov (KZ in the following) connection [11][12], emphasizing the close relation with strict infinitesimal 2-braidings. Let us fix a positive integer $n \in \mathbb{N}$. Let $C(n)$ be the configuration space of $n$ distinct particles in the complex plane:

$$
C(n) = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j \}. \tag{58}
$$

This has an obvious left action $S_n \ni \sigma \mapsto L_\sigma \in \text{diff}(C(n))$ of the symmetric group $S_n$. The configuration space of $n$ indistinguishable particles in $C$ is defined as $C(n)/S_n$. Define closed 2-forms $\omega_{ij}$ in the configuration space $C(n)$, for $1 \leq i, j \leq n$ and $i \neq j$:

$$
\omega_{ij} = \frac{dz_i - dz_j}{z_i - z_j}. \tag{59}
$$

Clearly for each $\sigma \in S_n$:

$$
L_\sigma^* (\omega_{ij}) = \omega_{\sigma^{-1}(i) \sigma^{-1}(j)} \tag{59.1}
$$

We recall the well known Arnold’s relation [11], for each distinct indices $i, j, k \in \{1, \ldots, n\}$:

$$
\omega_{ij} \wedge \omega_{jk} + \omega_{ik} \wedge \omega_{ij} + \omega_{ij} \wedge \omega_{ij} = 0. \tag{59.2}
$$

Note that for each distinct indices $i, j \in \{1, \ldots, n\}$, we have $\omega_{ij} = \omega_{ji}$.

Consider an infinitesimal braiding $r$ in a linear monoidal category $\mathcal{C}$ (see [18][10] and the Introduction). Choose an object $x \in \mathcal{C}$. Given a positive integer $n$ and $1 \leq a < b \leq n$ we thus have a morphism $r_{ab} : x^{\otimes a} \to x^{\otimes b}$. 24
Recall the construction of the differential crossed module \( \mathfrak{gl}(x) \) associated to \( \mathfrak{c} \) and \( x \), outlined in Section 1.2

The \( \mathfrak{gl}(x) \)-valued KZ-connection associated to the triple \((\mathfrak{c}, r, x)\) is:

\[
A = \sum_{1 \leq a < b \leq n} \omega_{ab} r^{ab}
\]

Therefore the curvature of \( A \) is:

\[
\mathcal{F}_A = A \wedge A = 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ab} \wedge r^{ac} \right] + 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ad} \wedge r^{bc} \right] + 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ad} \wedge r^{bc} \right] .
\]

The last term is (by using Arnold relation (59)):

\[
-2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ab} \wedge r^{bc} \right] - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ad} \wedge r^{bc} \right]
\]

By using the fact that \( \omega_{ab} = \omega_{ba} \) we conclude:

\[
\mathcal{F}_A = -2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ab} \wedge r^{bc} \right] - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ad} \wedge r^{bc} \right] + 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \left[ r^{ad} \wedge r^{bc} \right].
\]

We can see that the curvature of \( A \) vanishes exactly because \( r \), being an infinitesimal braiding operator, does satisfy the 4-term relations.

Let us now suppose that what we have is an infinitesimal 2-braiding \((r, T)\) in a linear monoidal symmetric 2-category \( \mathfrak{c} \). The previous calculation of the curvature of \( A = \sum_{1 \leq a < b \leq n} \omega_{ab} r^{ab} \) remains the same. Recalling (24) and (26) we set

\[
P^{abc} = P^{abc}_{x,x,x} \quad \text{and} \quad Q^{abc} = Q^{abc}_{x,x,x}.
\]

Since the differential \( \partial \) in \( \mathfrak{gl}(x) \) is the target map, we have

\[
\partial(P^{abc}) = [r^{bc}, r^{ab} + r^{ac}] \quad \text{and} \quad \partial(Q^{abc}) = [r^{ab}, r^{ac} + r^{bc}].
\]

Therefore we can write

\[
\mathcal{F}_A = 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \partial(P^{abc}) - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} \partial(Q^{abc})
\]

and so we set

\[
B = 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} P^{abc} - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} Q^{abc}.
\]

Going back to the previous basis:

\[
B = -2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} P^{abc} - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} P^{abc} - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} Q^{abc}
\]

\[
= 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} (P^{abc} + Q^{abc}) - 2 \sum_{a < b < c} \omega_{bc} \wedge \omega_{ca} P^{abc}.
\]

This shows that with respect with the generators in [11, 12] we have

\[
P^{abc} + Q^{abc} = K^{abc}, \quad P^{abc} = -K^{abc}
\]

Now we condense our notation: we denote \( a \wedge \beta = a \beta \) and we put

\[
[ab] = \omega_{ab}, \quad [abc] = \omega_{ab} \wedge \omega_{bc} = [ab][bc], \quad W_{ab, cde} = r^{bc} \circ p^{cde}, \quad Z_{ab, cde} = -r^{bc} \circ Q^{cde}.
\]
In this notation we simply write

\[ B = 2 \sum_{a < b < c} [bca] P_{abc} - \sum_{a < b < c} [cab] Q_{abc} \]

and we compute the 2-curvature \( S_{(A,B)} = A \wedge B \) to be:

\[
S_{(A,B)} = \sum_{a < b < c < d} [cd][bca]W_{cdabc} + [cd][cab]Z_{cdabc} + [bd][bca]W_{bdabc} + [bd][cab]Z_{bdabc} + [ad][bca]W_{adabc} + [ad][cab]Z_{adabc} \\
\quad + \sum_{a < b < c < d} [cd][bda]W_{cdabd} + [cd][db]Z_{cdabd} + [cb][bda]W_{cbabd} + [cb][db]Z_{cbabd} + [ca][bda]W_{caabd} + [ca][db]Z_{caabd} \\
\quad + \sum_{a < b < c < d} [bd][cda]W_{bdacd} + [bd][dac]Z_{bdacd} + [bc][cda]W_{bcacd} + [bc][dac]Z_{bcacd} + [ab][cda]W_{abacd} + [ab][dac]Z_{abacd} \\
\quad + \sum_{a < b < c < d} [ab][cdb]W_{abbcd} + [ab][dbc]Z_{abbcd} + [ac][cdb]W_{acbcd} + [ac][dbc]Z_{acbcd} + [ad][cdb]W_{adbcd} + [ad][dbc]Z_{adbcd}
\]

Consider now the following closed differential forms:

\[
\{ \omega_{i,j,k} \wedge \omega_{j,k,l} \wedge \omega_{i,j,k} \text{ s.t. } i_k < j_k \text{ and } j_k < j_{k'} \text{ for } k < k' \}
\]

with all indices running in \( \{1, \ldots, n\} \). These are linearly independent, which follows from the main result of \([1]\) and can easily be proven directly. In fact these forms are known to span a subspace of the space \( \Omega^3(\mathbb{C}(n)) \) of differential 3-forms in the configuration space \( \mathbb{C}(n) \), generating the Rham cohomology in degree three. Using our notation, these six linearly independent closed forms in \( \Omega^3(\mathbb{C}(n)) \) are:

\[
\{ [ab][ac][ad], [ab][bc][ad], [ab][ac][bd], [ab][bc][bd], [ab][ac][cd], [ab][bc][cd] \} \quad (1 \leq a < b < c < d \leq n).
\]

In terms of this basis, each of the 3-forms appearing in the 24 terms expressing \( S_{(A,B)} \) is written as a vector with six components. We gather these in the matrix \( M \) below, in order of appearance:

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

This is convenient to describe the flatness of the KZ 2-connection in terms of linear algebra. Indeed consider the vector \( V \) whose 24 components are the coefficients along the 3-forms in the previous expression of the 2-curvature \( S_{(A,B)} \):

\[
V = (W_{cdabc}, Z_{cdabc}, W_{bdabc}, Z_{bdabc}, W_{adabc}, Z_{adabc}, W_{cdabd}, Z_{cdabd}, W_{bdabd}, Z_{bdabd}, W_{adabd}, Z_{adabd}, W_{cdacd}, Z_{cdacd}, W_{bdacd}, Z_{bdacd}, W_{adacd}, Z_{adacd}, W_{abacd}, Z_{abacd}, W_{abbcd}, Z_{abbcd}, W_{acbcd}, Z_{acbcd}, W_{adbcd}, Z_{adbcd})
\]

Then the vanishing of the 2-curvature \( S_{(A,B)} \) is equivalent to the matrix equation \( MV = 0 \). Given that the rank of \( M \) is six, this leads to six independent equations in the variables \( Z \) and \( W \). These 2-flatness conditions have previously appeared in \([11,12]\), only with a different choice of generators.

What we want to point out here is that these six 2-flatness conditions are equivalent to the ones appearing in Theorems \([21,22]\). To see this we left-multiply \( M \) by the matrix \( N \) (with rank six):

\[
N = \begin{pmatrix}
0 & -1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Therefore:

\[
NM = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\

\end{pmatrix}
\]

Then the six equations coming from \(NMV = 0\) are the equations appearing in Theorems 21 and 22. We have proved the following result, which provides a new interpretation of the flatness conditions of the KZ 2-connection.

**Theorem 23.** Consider a coherent infinitesimal 2-braiding \((\epsilon, T)\) in a linear symmetric strict monoidal 2-category \(\mathcal{C}\). Choose and object \(x\) of \(\mathcal{C}\). Then the pair \((A, B)\), where:

\[
A = \sum_{1 \leq a < b \leq n} \omega_{ab} t^{ab}, \quad B = 2 \sum_{a < b < c < d} \omega_{bd} \wedge \omega_{ad} p^{bd} - 2 \sum_{a < b < c < d} \omega_{cd} \wedge \omega_{ab} q^{ab}.
\]

\((\omega_{ij}, P^{ijk}, Q^{ijk}\text{ defined as before})\) is a flat (and fake flat) 2-connection in the configuration space \(\mathcal{C}(n)\), with values in the differential crossed module \(\mathfrak{g}(x)\) associated to \((\mathcal{C}, x)\).

By using exactly the same techniques as in \([11, 12]\) we can prove that:

**Theorem 24.** In the conditions of the previous theorem, if the infinitesimal 2-braiding is totally symmetric, then the 2-connection \((A, B)\) is invariant under the obvious action of the symmetric group.

## 4 Infinitesimal 2-braidings for a differential crossed module

### 4.1 Preliminaries

#### 4.1.1 Summary of notation and algebra actions

Given a Lie algebra \(\mathfrak{g}\), let \(\mathcal{U}(\mathfrak{g})\) denote its universal enveloping algebra. Let \(n\) be a positive integer. Recall that we have an algebra (in fact Hopf algebra) isomorphism:

\[
\mathcal{U}(\mathfrak{g}^{	ext{ten}}) \rightarrow \mathcal{U}(\mathfrak{g})^{	ext{ten}}
\]

being (for \(X_i \in \mathfrak{g}\), where \(i \in \{1, \ldots, n\}\))

\[
(X_1, X_2, \ldots, X_n) \mapsto X_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes X_2 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes X_n.
\]  

(64)

Let \(\mathfrak{g} = (\partial : \mathfrak{h} \rightarrow \mathfrak{g}, \mathfrak{v})\) be a differential crossed module, which will be fixed throughout this subsection. Consider the semidirect product \(\mathfrak{e} = \mathfrak{h} \rtimes \mathfrak{g}\). This is a Lie algebra with Lie bracket:

\[
[(u, X), (v, Y)] = \left([u, v] + X \cdot v - Y \cdot u, [X, Y]\right), \quad \forall X, Y \in \mathfrak{g} \text{ and } u, v \in \mathfrak{h}.
\]

Note that \(\mathfrak{h}\) and \(\mathfrak{g}\) (more precisely, their images under the maps \(\mathfrak{h} \ni v \mapsto (v, 0) \in \mathfrak{e}\) and \(\mathfrak{g} \ni X \mapsto (0, X) \in \mathfrak{e}\)) are Lie subalgebras of \(\mathfrak{e}\). We have a Lie algebra map:

\[
\beta : \mathfrak{e} \ni (v, X) \mapsto X + \partial(v) \in \mathfrak{g}.
\]

By restriction of the adjoint action of \(\mathfrak{e}\) on itself we recover the action of \(\mathfrak{g}\) on \(\mathfrak{h}\) and actions (by Lie algebra derivations) of \(\mathfrak{g}\) and of \(\mathfrak{h}\) on \(\mathfrak{e}\). All these actions canonically lift to the corresponding enveloping algebras, and to tensor products of them. As an example, since there is an action \(\triangleright\) of \(\mathfrak{g}\) on \(\mathfrak{h}\) by Lie algebra derivations, then \(\mathfrak{g}\) acts on \(\mathcal{U}(\mathfrak{h})\) by algebra derivations, explicitly as:

\[
X \triangleright (v_1 v_2 \ldots v_n) = ((X \triangleright v_1) v_2 \ldots v_n) + (v_1 (X \triangleright v_2) \ldots v_n) + \cdots + (v_1 v_2 \ldots (X \triangleright v_n)).
\]
The Lie algebra maps $\partial: \mathfrak{h} \to \mathfrak{g}$ and $\beta: \mathfrak{e} \to \mathfrak{g}$ intertwine the $\mathfrak{g}$-actions. We then have algebra maps $\partial: \mathfrak{u}(\mathfrak{h}) \to \mathfrak{u}(\mathfrak{g})$ and $\beta: \mathfrak{u}(\mathfrak{e}) \to \mathfrak{u}(\mathfrak{g})$, which also intertwine the $\mathfrak{g}$-actions.

By considering the product action (which is an action by derivations) of $\mathfrak{g}^{\text{fin}}$ on $\mathfrak{h}^{\text{fin}}$, namely

$$(X_1, \ldots, X_n) \triangleright (v_1, \ldots, v_n) = (X_1 \triangleright v_1, \ldots, X_n \triangleright v_n),$$

and the product map $\partial: \mathfrak{h}^{\text{fin}} \to \mathfrak{g}^{\text{fin}}$, so that $\partial(v_1, \ldots, v_n) = (\partial(v_1), \ldots, \partial(v_n))$, we define a differential crossed module $(\mathfrak{h}, \mathfrak{g})$. Therefore the previous description applies to it, and it will be used without much comment. In particular we have maps $\partial: \mathfrak{u}(\mathfrak{h})^{\text{fin}} \to \mathfrak{u}(\mathfrak{g})^{\text{fin}}$ and $\beta: \mathfrak{u}(\mathfrak{e})^{\text{fin}} \to \mathfrak{u}(\mathfrak{g})^{\text{fin}}$, and we have a Lie algebra action by algebra derivations of $\mathfrak{g}^{\text{fin}}$ on $\mathfrak{u}(\mathfrak{e})^{\text{fin}} \cong \mathfrak{u}(\mathfrak{e})^{\text{fin}}$.

Recall that we have a Lie algebra morphism $\Delta: \mathfrak{g} \to \mathfrak{g}^{\text{fin}}$, being $\Delta(x) = (x, \ldots, x)$. This diagonal map, together with similar ones induced on enveloping algebras, permit us to define actions, by algebra derivations, of $\mathfrak{g}$ and of $\mathfrak{h}$ on $\mathfrak{u}(\mathfrak{e})^{\text{fin}} \cong \mathfrak{u}(\mathfrak{e})^{\text{fin}}$, which will have a prime importance later.

### 4.1.2 An auxiliary vector space

From now on we will make no distinction between $\mathfrak{u}(\mathfrak{e})^{\text{fin}}$ and $\mathfrak{u}(\mathfrak{a})^{\text{fin}}$, where $\mathfrak{a}$ is a Lie algebra. Let $\mathfrak{g}^{\text{fin}} = (\partial: \mathfrak{h} \to \mathfrak{g}, \star)$ be a differential crossed module. As before we put $\mathfrak{e} = \mathfrak{h} \triangleright \mathfrak{g}$. By the Poincaré-Birkhoff-Witt Theorem both $\mathfrak{u}(\mathfrak{g})$ and $\mathfrak{u}(\mathfrak{h})$ are subalgebras of $\mathfrak{u}(\mathfrak{e})$. Inside the algebra $\mathfrak{u}(\mathfrak{e})^{\text{fin}}$, we let $A_n$ be the smallest vector space containing all elements of the form

$$x \triangleright y, \text{ where } x, y \in \mathfrak{u}(\mathfrak{g})^{\text{fin}} \text{ and } v \in \mathfrak{h}^{\text{fin}}.$$ 

By restriction of the action of $\mathfrak{g}^{\text{fin}}$ on $\mathfrak{u}(\mathfrak{e})^{\text{fin}}$ by derivations, $A_n$ is a $\mathfrak{g}^{\text{fin}}$-module.

**Definition 25.** Let $n$ be a positive integer. The vector space $\mathfrak{U}^{(n)}$ is given by the subspace $A_n$, modulo the relations:

$$x \partial(u) y \triangleright z = x u y \partial(v) z,$$

where $u, v \in \mathfrak{h}^{\text{fin}} \subset \mathfrak{u}(\mathfrak{e})^{\text{fin}}$ and $x, y, z \in \mathfrak{u}(\mathfrak{g})^{\text{fin}}$.

We provide some examples since the notation may be misleading. If $n = 1$ we have, for each $u, v \in \mathfrak{h}$,

$$\partial(u) v = u \partial(v).$$

With $n = 2$ we have, for each $u, v \in \mathfrak{h}$,

$$(\partial(u), 0) (0, v) = (u, 0) (0, \partial(v))$$

or, by passing from $\mathfrak{u}(\mathfrak{e} \oplus \mathfrak{e})$ to the isomorphic $\mathfrak{u}(\mathfrak{e}) \otimes \mathfrak{u}(\mathfrak{e})$ (see the morphism $\mathfrak{b}$)

$$(\partial(u) \otimes 1) (1 \otimes v) = (u \otimes 1) (1 \otimes \partial(v))$$

which is simply written as

$$\partial(u) \otimes v = u \otimes \partial(v).$$

This adds of course to the relation obtained from

$$\partial(u, 0) (v, 0) = (u, 0) \partial(v, 0),$$

which is

$$\partial(u) v \otimes 1 = (u \partial(v)) \otimes 1.$$ 

Similarly when $n = 3$ we have, for each $u, v \in \mathfrak{h}$:

$$\partial(u, 0, 0) (0, v) = (u, 0, 0) \partial(0, 0, v).$$
from which we obtain
\[ \partial(u) \otimes 1 \otimes v = u \otimes 1 \otimes \partial(v). \]
Therefore if \( a \in \mathcal{U}(g) \) and \( u, v \in \mathfrak{h} \):
\[ \partial(u) \otimes a \otimes v = u \otimes a \otimes \partial(v). \]
Of course different relative positions of \( u \) and \( v \) will give analogous relations.

Clearly the algebra map \( \beta: \mathcal{U}(\mathfrak{g}^{\text{en}}) \to \mathcal{U}(g^{\text{en}}) \) descends to a vector space map:
\[ \beta: \mathcal{U}(n) \to \mathcal{U}(g^{\text{en}}). \]
As mentioned before, we have an action of \( g \) on \( \mathcal{U}(\mathfrak{g}^{\text{en}}) \) by algebra derivations. This is defined from the obvious action of \( g^{\text{en}} \) on \( \mathcal{U}(\mathfrak{g}^{\text{en}}) \) by algebra derivations, and the diagonal map \( \Delta: g \to g^{\text{en}}. \)

**Lemma 26.** The action of \( g^{\text{en}} \) on \( \mathcal{U}(\mathfrak{g}^{\text{en}}) \) descends to an action of \( g \) on the vector space \( \mathcal{U}(n) \).

**Proof.** Given \( X \in g^{\text{en}}, u, v \in \mathcal{U}(n) \) and \( x, y, z \in \mathcal{U}(g^{\text{en}}) \) we have:
\[
X \mapsto (x u y \partial(v) z) = (X \mapsto x) u y \partial(v) z + x (X \mapsto u) y \partial(v) z + x u y (X \mapsto \partial(v)) z + x u y \partial(v) (X \mapsto z)
= (X \mapsto x) u y \partial(v) z + x (X \mapsto u) y \partial(v) z + x u y (\partial(X \mapsto v)) z + x u y \partial(v) (X \mapsto z).
\]
Clearly the difference between the previous two expressions is in the kernel of the projection \( A_n \to \mathcal{U}(n) \). \( \square \)

As a consequence of the previous Lemma, \( g \) acts on \( \mathcal{U}(n) \).

Note that the maps:
\[ (x, a) \in \mathcal{U}(g^{\text{en}}) \times A_n \mapsto xa \in A_n, \quad (x, a) \in \mathcal{U}(g^{\text{en}}) \times A_n \mapsto ax \in A_n \]
clearly directly descend to maps
\[ \mathcal{U}(g^{\text{en}}) \times \mathcal{U}(n) \to \mathcal{U}(n). \]
These will be frequently used. Now the crucial property of all the construction.

**Lemma 27** (The defining relations of \( \mathcal{U}(n) \)). For each \( a, b \in \mathcal{U}(n) \) we have:
\[ \beta(a) b = a \beta(b). \tag{65} \]

**Proof.** By bilinearity, it suffices proving this result on vector space generators. Given \( x, y, x', y' \in \mathcal{U}(g^{\text{en}}) \) and \( u, u' \in \mathfrak{h} \) we have:
\[ \beta(x u y) (x' u' y') = x \partial(u) y x' \partial(v') y' = x u y x' \partial(v') y' = (x u y) \beta(x' u' y'). \] \( \square \)

Let \( m \) and \( n \) be positive integers. There exist obvious inclusions \( i_m \) and \( i_n \) of \( g^{\text{en}} \) and \( g^{\text{en}} \) inside \( \mathfrak{g}^{\text{en}(m+n)} \), being, for example \( i_m(X_1, \ldots, X_m) = (X_1, \ldots, X_m, 0, \ldots, 0) \) and \( i_n(X_1, \ldots, X_n) = (0, \ldots, 0, X_1, \ldots, X_n) \), where \( X_i \in g \). These extend to the associated enveloping algebras. Clearly \( i_m(A_n) \subset A_{m+n} \), and also this defines an inclusion map \( i_m: \mathcal{U}(m) \to \mathcal{U}(m+n) \), commuting with \( \beta \). Given \( a \in \mathcal{U}(m) \) and \( l \in \mathcal{U}(g) \) we define
\[ a \otimes l = i_m(a) i_m(l) = i_m(l) i_m(a). \]
As a consequence of the defining relations \( (65) \) we have, for \( a \in \mathcal{U}(m) \) and \( b \in \mathcal{U}(n) \):
\[ a \otimes \beta(b) = \beta(a) \otimes b. \tag{66} \]

We finish by introducing a map \( \mathfrak{h} \otimes \mathcal{U}(g) \to \mathcal{U}(n) \). By definition:
\[ (v, r) \mapsto v \triangleright r = \Delta(v) r - r \Delta(v). \tag{67} \]
Note that given \( v \in \mathfrak{h} \) and \( r, r' \in \mathcal{U}(g) \):
\[ v \triangleright (r r') = \Delta(v) r r' - r r' \Delta(v) = \Delta(v) r r' - r \Delta(v) r' + r \Delta(v) r' - r r' \Delta(v) = (v \triangleright r) r' + r (v \triangleright r'). \]
4.2 A strict 2-category $\mathcal{C}_{\delta}$ derived from a differential crossed module $\delta$

In [11, 12] we lifted the flatness conditions of the KZ 2-connection from the differential crossed module where the KZ connection takes value to an auxiliary short complex of vector spaces $\tilde{\mathcal{O}}: \mathcal{U}^{(n)} \to g^{\otimes n}$ (see e.g [11] Section 4.3). This auxiliary complex is where the universal relations expressing the flatness conditions live, prior to being represented on $\mathfrak{gl}(V)$ for some complex of vector spaces $V$. This construction mimics what happens with the 4-term relations, whose universal form lives in $g^{\otimes n}$ and is then represented as KZ flatness conditions in $\mathfrak{gl}(V^{\otimes n})$ for some $g$-module $V$.

In this section we aim at giving a categorical interpretation of the auxiliary complex involved in the KZ 2-flatness conditions. We associate to a differential crossed module $\delta$ a linear 2-category $\mathcal{C}_{\delta}$ where the universal structure responsible for the KZ 2-flatness, namely a quasi-invariant tensor in $\delta$, defines an infinitesimal 2-braiding.

4.2.1 Objects and 1-morphisms

Let $\delta = (\partial: h \to g, \tau)$ be a differential crossed module. Given a positive integer $n$, we have a natural action of the permutation group $S_n$ on $\mathcal{U}(\delta^{\otimes n})$. This clearly descends to an action on the vector space $\mathcal{U}^{(n)}$.

We now define a linear 2-category $\mathcal{C}_{\delta}$ associated to $\delta$. The objects are given by non-negative integers $n \in \mathbb{N}$. The set of 1-morphisms is only non-empty in the case $n \to n$. In this case a 1-morphism is given by:

(a) an element $R \in \mathcal{U}(\delta)^{\otimes n}$;
(b) a linear map $\zeta: g \to \mathcal{U}^{(n)}$;
(c) a permutation $\sigma \in S_n$;
(d) an integer $k$ (all of the construction would make sense considering $k \in \mathbb{C}$);

It is depicted as $n \xrightarrow{(R, \zeta, \sigma, k)} n$. This data is subject to the following conditions:

(i) $X \circ R = \beta(\zeta(X))$, $\forall X \in g$;
(ii) $u \circ R = \zeta(\partial(u))$, $\forall u \in h$ (here $\circ$ is the map in $\delta$);
(iii) $\zeta([X, Y]) = X \circ \zeta(Y) - Y \circ \zeta(X)$, $\forall X, Y, Z \in g$.

Remark 28 (Decorated permutations). To simplify the notation we will sometimes abbreviate $S_n \times \mathbb{Z} \ni (\sigma, k) = \tau$, and call such a pair a 'decorated permutation'. Given any representation $i \mapsto \sigma(i)$ of $S_n$, $i = (1, \ldots, n)$, we write $\tau(i) = \sigma(i)$ if $\tau = (\sigma, k)$.

Note that if $(R, \zeta, \sigma, k)$ and $(R', \zeta', \sigma', k)$ are morphisms $n \to n$ then so is $(R + R', \zeta + \zeta', \sigma, k)$. Therefore the set of morphisms $n \to n$ decomposes as a direct sum

$$\text{hom}(n, n) = \bigoplus_{(\sigma, k) \in S_n \times \mathbb{Z}} \text{hom}_{\delta, k}(n, n)$$

where $\text{hom}_{\delta, k}(n, n)$ is the vector space of morphisms $n \to n$ whose underlying decorated permutation is $(\sigma, k)$. The composition of 1-morphisms is

$$n \xrightarrow{(R, \zeta, \sigma, k)} n \xrightarrow{(R', \zeta', \sigma', k')} n = n \xrightarrow{(\tau_0(R'), \tau_0(\zeta') + \zeta_0(\tau(R'), \sigma', k + k'))} n.$$ 

This is associative and has units. We now check that conditions (i)-(iii) of the definition of a 1-morphism are satisfied by the composition of two 1-morphisms. For (i) and (ii) we use the fact that on $\mathcal{U}(\delta)^{\otimes n}$ the $g$ and
\( \mathfrak{b} \) actions commute with the \( S_n \) action: then for every \( X \in \mathfrak{g}, u \in \mathfrak{b} \) we have

\[
X \ast (R \sigma(R')) = (X \ast R) \sigma(R') + R(X \ast \sigma(R')) = (X \ast R) \sigma(R') + R(\sigma(X) \ast R') \\
= \beta(\zeta(X)) \sigma(R') + R(\beta(\zeta'(X))) = \beta(\zeta(X)) \sigma(R') + R(\beta(\zeta'(X))) \\
= \beta(\zeta(X)) \sigma(R') + R(\zeta'(X)) \\
\]

\( u \ast (R \sigma(R')) = (u \ast R) \sigma(R') + R(u \ast R') = \zeta(\partial(u)) \sigma(R') + R(\zeta'(\partial(u))). \)

As for condition (iii), evaluating the lhs on \( X, Y \in \mathfrak{g} \):

\[
R \sigma(\zeta'([X, Y])) + \zeta([X, Y]) \sigma(R') = R \sigma(X) \ast \zeta'(Y) - Y \ast \zeta'(X) + (X \ast \zeta(Y) - Y \ast \zeta(X)) \sigma(R') \\
while, by using the fact that \( \mathfrak{g} \) acts by derivations on \( \mathfrak{u}(v) \) and \( X \ast R = \beta(\zeta(X)) \), the rhs reads
\]

\[
X \ast (R \sigma(\zeta'(Y)) + X \ast (\zeta(Y)) \sigma(R')) - Y \ast (R \sigma(\zeta'(X)) + Y \ast (\zeta(X)) \sigma(R')) = \\
= \beta(\zeta(X)) \sigma(\zeta'(Y)) + R(X \ast \sigma(\zeta'(Y)) + (X \ast \zeta(Y)) \sigma(R')) + \zeta(Y) \sigma(\beta(\zeta'(X))) + \\
- \beta(\zeta(Y)) \sigma(\zeta'(X)) - R(Y \ast \sigma(\zeta'(X)) - (Y \ast \zeta(X)) \sigma(\zeta'(Y))) - \zeta(X) \sigma(\beta(\zeta'(Y)))
\]

Subtracting the two sides yields:

\[
\beta(\zeta(X)) \sigma(\zeta'(Y)) + \zeta(Y) \sigma(\beta(\zeta'(X))) - \beta(\zeta(Y)) \sigma(\zeta'(X)) - \zeta(X) \sigma(\beta(\zeta'(Y)))
\]

which vanishes in \( \mathfrak{u}(v) \) by \( \mathfrak{g} \). Finally, note that given a positive integer \( n \) and \( \tau, \tau' \in S_n \times \mathbb{Z} \), the composition map:

\[
\text{hom}_1(n, n) \times \text{hom}_v(n, n) \rightarrow \text{hom}_{v \tau}(n, n)
\]

is bilinear.

4.2.2 2-morphisms and whiskering

The set of 2-morphisms \( n \xrightarrow{(R, \zeta, \tau)} n' \xrightarrow{(R', \zeta', \tau')} n' \) is non-empty only if \( n = n' \) and \( \tau = \tau' \). In this case a 2-morphism \( (R, \zeta, \tau) \xrightarrow{T} (R', \zeta', \tau) \) is given by an element \( T \in \mathfrak{u}(v) \), such that:

(i) \( R' = R + \beta(T) \);

(ii) \( \zeta'(X) = \zeta(X) + X \ast T \), \( \forall X \in \mathfrak{g} \).

We denote a 2-morphism as:

\[
\begin{array}{c}
\xymatrix{ n \ar@<-0.5ex>[r]_{(R, \zeta, \tau)} & n', \ar@<0.5ex>[l]^{(R', \zeta', \tau')}
\end{array}
\]

Note that if \( n \xrightarrow{(R, \zeta, \tau)} n \) then also \( n \xrightarrow{(R + \beta(T), \zeta, \tau)} n, \) for any \( T \in \mathfrak{u}(v) \). This is because, for \( X \in \mathfrak{g} \) and \( v \in \mathfrak{b} \):

\[
X \ast (R + \beta(T)) = \beta(\zeta(X)) + \beta(X \ast T) = \beta(\zeta(X)) + \beta(\zeta'(X) - \zeta(X)) = \beta(\zeta'(X)),
\]

\[
v \ast (R + \beta(T)) = \zeta(\partial(v)) + \partial(v) \ast T = \zeta(\partial(v)) + \zeta'(\partial(v)) - \zeta(\partial(v)) = \zeta'(\partial(v)),
\]

where we used \( \mathfrak{g} \). Also given \( X, Y \in \mathfrak{g} \):

\[
\zeta'([X, Y]) = \zeta([X, Y]) - [X, Y] \ast T \\
= X \ast \zeta(Y) - Y \ast \zeta(X) - X \ast (Y \ast T) + Y \ast (X \ast T) \\
= X \ast \zeta'(Y) - Y \ast \zeta'(X).
\]
The vertical composition of 2-morphisms is

\[(R, \zeta, \tau) \xrightarrow{T} (R', \zeta', \tau) \xrightarrow{T'} (R'', \zeta'', \tau) = (R, \zeta, \tau) \xrightarrow{T+T'} (R'', \zeta'', \tau).\]

It is trivial to check that \(T + T'\) does connect \((R, \zeta, \tau)\) and \((R'', \zeta'', \tau)\).

The set of 2-morphisms connecting 1-morphisms whose underlying decorated permutation is \(\tau\) is denoted by \(\text{hom}^2(n, n)\). It is in one-to-one correspondence with \(\text{hom}_2(n, n) \times \mathcal{U}^{(n)}\) and it is therefore a vector space. With this identification, the source and target maps \(s', t'\): \(\text{hom}^2(n, n) \to \text{hom}_2(n, n)\) have the form:

\[s'((R, \zeta, \tau) \times T) = (R, \zeta, \tau), \quad \text{and} \quad t'((R, \zeta, \tau) \times T) = (R + \beta(T), \zeta', \tau),\]

where \(\zeta'(X) = \zeta(X) - X \triangleright T\). Therefore both \(s'\) and \(t'\) are linear. For each object \(n \in \mathbb{N}\) the vector space of 2-morphisms is the direct sum:

\[\text{hom}^2(n, n) = \bigoplus_{\tau \in S_3 \times \mathcal{Z}} \text{hom}^2_\tau(n, n).\]

The whiskering are naturally defined as:

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
(R, \zeta, \tau) \\
\xrightarrow{T}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
(R', \zeta', \tau') \\
\xrightarrow{T'}
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
(R'', \zeta'', \tau'') \\
\xrightarrow{T''}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

That these 2-morphisms have the correct source and target requires using the relations (55) for \(\mathcal{U}^{(n)}\). For example, that \(X \triangleright T = \zeta'(X) - \zeta(X)\) for the left whiskering is proved as

\[
X \triangleright (R'' \tau''(T)) = (X \triangleright R'') \tau''(T) + R''(X \triangleright \tau''(T)) = \beta(\zeta''(X))\tau''(T) + R'' \tau''(\zeta(X)) - R'' \tau''(\zeta(X))
\]

\[
= \zeta''(X)\beta(\tau''(T)) + R'' \tau''(\zeta'(X)) - R'' \tau''(\zeta(X))
\]

\[
= \zeta''(X)\tau''(R') - \zeta''(X)\tau''(R) + R'' \tau''(\zeta'(X)) - R'' \tau''(\zeta(X))
\]

\[
= R'' \tau''(\zeta'(X)) + \zeta''(X)\tau''(R') - R'' \tau''(\zeta(X)) - \zeta''(X)\tau''(R).
\]

The relations (56) are also essential to prove the interchange condition, which essentially follows from

\[S \tau(\beta(T)) = \beta(S) \tau(T), \quad \forall S, T \in \mathcal{U}^{(n)}.\]

Finally, note that the left and right whiskering defines bilinear maps:

\[\text{hom}_2(n, n) \times \text{hom}^2_\tau(n, n) \to \text{hom}^2_\tau(n, n), \quad \text{hom}^2_\tau(n, n) \times \text{hom}_2(n, n) \to \text{hom}^2_\tau(n, n).\]

4.2.3 The monoidal structure

We write \(1_n\) for the identity of \(\mathcal{U}^{(n)} \cong \mathcal{U}^{(n^\text{op})}\). Given \(\sigma' \in S_n\) we put \(n \otimes \sigma'\) as being the permutation of \(\{1, \ldots, n+n'\}\) which leaves \(\{1, \ldots, n\}\) static, whereas \((n \otimes \sigma')(i) = \sigma'(i-n) + n\) if \(i > n\). We analogously define \(\sigma \otimes n'\). We write \(n \otimes (\sigma', k') = (n \otimes \sigma', k')\), for a decorated permutation \((\sigma', k') \in S_n \times \mathbb{Z}^\text{op}\).

**Definition 29.** The monoidal structure \(\otimes : \mathcal{C}_\text{es} \times \mathcal{C}_\text{es} \to \mathcal{C}_\text{es}\) of \(\mathcal{C}_\text{es}\) is defined as follows:

1. given two objects \(n, n'\in \mathcal{C}_\text{es}\), it is \(n \otimes n' = n + n'\);
2. Given an object \( n' \) and a 1-morphism \( n \xrightarrow{(R,\zeta,\eta)} n \), it is
\[
n' \otimes n \xrightarrow{(R,\zeta,\eta)} (n + n')
\]
similarly, it is \( n' \otimes n \xrightarrow{(R,\zeta,\eta)} (n' + n) \).

3. Given an object \( n' \) and a 2-morphism \( T : (R,\zeta,\tau) \Rightarrow (R',\zeta',\tau) \) it is
\[
\left( \begin{array}{c}
(n' \otimes n) \\
(R,\zeta,\tau)
\end{array} \right) \xrightarrow{T} \left( \begin{array}{c}
(n' \otimes 1_n) \\
(R,\zeta,\tau)
\end{array} \right)
\]
\[
\left( \begin{array}{c}
(1_n \otimes n) \\
(R,\zeta,\tau)
\end{array} \right) \xrightarrow{T \otimes 1_n} \left( \begin{array}{c}
(1_n \otimes n') \\
(R,\zeta,\tau)
\end{array} \right)
\]

This definition is clearly distributive with respect to compositions of 1- and 2-morphisms, as well as the whiskering. We comment the interchange laws, see Definition \([8]\) condition 6. The condition 6.(i) between 1-morphisms and tensor product holds, being
\[
(R,\zeta,\tau) \otimes (R',\zeta',\tau') = (R \otimes R', R \otimes \zeta' + \zeta \otimes R', \tau \otimes \tau').
\]
The condition 6.(ii) between 2-morphisms and 1-morphisms also holds, being:
\[
\left( \begin{array}{c}
(n \otimes m) \\
(R,\zeta,\tau)
\end{array} \right) \xrightarrow{(R \otimes R', R \otimes \eta \otimes R', \tau \otimes \tau')} \left( \begin{array}{c}
(n \otimes m) \\
(R,\zeta,\tau)
\end{array} \right)
\]
(to prove that the latter 2-morphisms has the correct source and target requires using \([66]\)). Finally, condition 6.(iii) requires that given 2-morphisms
\[
\left( \begin{array}{c}
(R + \beta(T),\zeta',\tau) \\
(R,\zeta,\tau)
\end{array} \right) \xrightarrow{T \otimes (P,\eta,\nu)} \left( \begin{array}{c}
(R + \beta(S),\eta',\nu) \\
(P,\eta,\nu)
\end{array} \right)
\]
it holds
\[
(R + \beta(T),\zeta',\tau) \otimes S = T \otimes (P + \beta(S),\eta',\nu)
\]
The previous equality follows from
\[
T \otimes R' + (R + \beta(T)) \otimes T' = R \otimes T' + T \otimes (R + \beta(T'))
\]
(both sides of which can be taken as the definition of \( T \otimes S \) which in turn follows from relations \([66]\).)

Given objects \( n \) and \( m \) there is a natural symmetric braiding \( B_{n,m} : n + m \to m + n \), essentially given by the permutation \( \sigma_{n,m} \) that exchanges \( n \) and \( m \). Namely:
\[
\sigma_{n,m}(i) = \begin{cases} 
i + m & \text{if } i \leq n, \\
i - n & \text{if } i > n \end{cases}, \quad i = 1, \ldots, n, n + 1, \ldots, n + m
\]

It is a nice exercise to check the remaining conditions of Definition \([10]\) finishing to prove that \( \mathcal{C} \) is a symmetric strict monoidal linear 2-category.
4.3 Infinitesimal braidings in a differential crossed module: quasi-invariant tensors

Let \( \mathcal{G} = (\mathfrak{g} : h \rightarrow g, \triangleright) \) be a differential crossed module. The diagonal map \( \mathfrak{u}(g) \rightarrow \mathfrak{u}(g)^{\otimes n}, x \mapsto x \otimes 1 \ldots \otimes 1 + c.p. \) (c.p. stands for cyclic permutations), will be denoted by \( \Delta^n \), and similarly for \( c \) and \( h \), where \( c = h \triangleright g \). The fact that, if \( \sigma \in S_n \), then \( \sigma \circ \Delta^n = \Delta^n \) will be used several times below.

**Definition 30 (Quasi-invariant tensor).** A quasi-invariant tensor in \( \mathcal{G} \) is a triple \((r, c, \zeta)\) where:

(a) \( r = \sum q_a q_t \in g \otimes g \) is a symmetric tensor (thus \( r = \sum q_t q_s \));

(b) \( \xi : g \rightarrow g \otimes h \otimes h \otimes g \subset \mathcal{U}^{(2)} \) is a linear map, whose image is symmetric. We write it as:

\[
\xi(X) = \sum_a \xi_a(X) \mu_a \otimes v_a = \sum_a \xi_a(X) \mu'_a \otimes \mu''_a + \sum_a \xi_a(X) \mu''_a \otimes \mu'_a .
\]

where \( \xi_a : g \rightarrow \mathbb{C} \) is a linear map, and in the second expression we use the notation \( \mu'_a \in g, \mu''_a \in h \) to distinguish the part in \( g \otimes h \) from the one in \( h \otimes g \).

(c) \( c \) is an \( g \)-invariant element in \( \ker(\partial) \subset h \).

The conditions that should be verified are:

(i) \( X \triangleright r = \beta(\xi(X)), \quad \forall X \in g \);

(ii) \( u \triangleright r = \xi(\partial(u)), \quad \forall u \in h \);

(iii) \( \xi([X,Y]) = X \triangleright \xi(Y) - Y \triangleright \xi(X), \quad \forall X,Y \in g \).

Given a quasi-invariant tensor in \( \mathcal{G} \), we can construct a totally symmetric infinitesimal 2-braiding in \( \mathcal{G} \). This may be seen as the categorification of the infinitesimal braiding (in the category of \( g \)-modules) associated to a symmetric and \( g \)-invariant tensor \( r \in g \otimes g \); see the Introduction.

**Theorem 31.** Let \( \mathcal{G} \) be a differential crossed module. Let \((r, \xi, c)\) be a quasi-invariant tensor in \( \mathcal{G} \). Then following definition of \((r, T)\) provides the linear totally symmetric monoidal 2-category \( \mathcal{C}_\mathcal{G} \) with a strict totally symmetric infinitesimal 2-braiding. Given objects \( n, m \) of \( \mathcal{C}_\mathcal{G} \), the morphism \( r_{n,m} : n \otimes m \rightarrow n \otimes m \) is defined as:

\[
r_{n,m} = \left\{ \sum_{1 \leq i \leq m+n} r_{ij}, \sum_{1 \leq i \leq m+n} \xi_{ij}, \text{id}_{S_{m+n}}, 1 \right\} .
\]  

(68)

Here \( 1 \in \mathbb{Z} \) and \( \text{id}_{S_{m+n}} \) denotes the identity of the group \( S_{m+n} \). Given a 1-morphism \((R, \zeta, \sigma, k) : n \rightarrow n \), we set:

\[
T_{(\beta, \zeta, \sigma, k), (m, n)} = - \sum_{n+1 \leq m+n} \sum_{1 \leq i \leq m+n} \xi_{ij} \otimes t_q + kR \otimes \Delta^n(c) = - \sum_{n+1 \leq m+n} \sum_{1 \leq i \leq m+n} \zeta_{ij} \otimes t_q + kR \otimes \Delta^n(c).
\]  

(69)

Analogously we define:

\[
T_{(\beta, \zeta, \sigma, k), (m, n)} = - \sum_{1 \leq i \leq m+n} \sum_{1 \leq j \leq m+n} \zeta_{ij} \otimes t_q + kR \otimes \Delta^n(c) = - \sum_{1 \leq i \leq m+n} \sum_{1 \leq j \leq m+n} \zeta_{ij} \otimes t_q + kR \otimes \Delta^n(c).
\]  

(70)

**Proof.** We need to prove that \( r \) and \( T \) are well defined 1- and 2-morphisms in \( \mathcal{C}_\mathcal{G} \), and that the properties of an infinitesimal 2-braiding, see Definition[14] are satisfied. For the computations it is convenient to write (note \( r = \sum q_t q_s \)):

\[
\sum_{1 \leq i \leq m+n} \sum_{1 \leq j \leq m+n} r_{ij} = \sum_{1 \leq i \leq m+n} \sum_{1 \leq j \leq m+n} s_{ij} \otimes t_q = \sum_{1 \leq i \leq m+n} \sum_{1 \leq j \leq m+n} \zeta_{ij} \otimes \Delta^n(t_q).
\]

(71)
We easily see that conditions (i)-(iii) of Section 4.2.1 are satisfied, so that \( r_{n,m} \) is indeed a 1-morphism in \( \mathcal{C}_g \). Recalling Definition 13 we then need to prove that \( T_{(R,\zeta,\sigma,k),m} \) is indeed a 2-morphism \( (R,\zeta,\sigma) \mapsto r_{n,m} \). To start, we have:

\[
\begin{aligned}
    r_{n,m} \ ( (R,\zeta,\sigma,k) \otimes m ) &= \left\{ \sum_{1 \leq i \leq j \leq m+n} s_{ij} R \otimes t_{i}^j + \sum_{1 \leq i \leq j \leq m+n} s_{ij} \zeta \otimes t_{i}^j, \ \sum_{1 \leq i \leq j \leq m+n} \xi_i (t \otimes v_i^j), \ \sigma \otimes m, k + 1 \right\} \\
    ((R,\zeta,\sigma,k) \otimes m) r_{n,m} &= \left\{ \sum_{1 \leq i \leq j \leq m+n} R s_{ij} \otimes t_{i}^j + \sum_{1 \leq i \leq j \leq m+n} \xi_i R \mu_i^j \otimes v_i^j + \sum_{1 \leq i \leq j \leq m+n} \zeta s_{ij} \otimes t_{i}^j, \ \sigma \otimes m, k + 1 \right\} .
\end{aligned}
\]

We check conditions (i) and (ii) of Section 4.2.2. The first condition follows directly from condition (i) of Section 4.2.1. For the second one, we look at the maps \( g \rightarrow \tau^{(m+n)} \) involved in (71). Their difference is, for each \( X \in g \):

\[
\begin{aligned}
    &\sum_{1 \leq i \leq j \leq m+n} \xi_i (X) R \mu_i^j \otimes v_i^j + \sum_{1 \leq i \leq j \leq m+n} \zeta (X) s_{ij} \otimes t_{i}^j - \sum_{1 \leq i \leq j \leq m+n} s_{ij} \zeta (X) \otimes t_{i}^j - \sum_{1 \leq i \leq j \leq m+n} \xi_i (X) \mu_i^j R \otimes v_i^j \\
    &= - \sum_{1 \leq i \leq j \leq m+n} \xi_i (X) \mu_i^j \triangleright R \otimes v_i^j - \sum_{1 \leq i \leq j \leq m+n} (s_{ij} \triangleright \zeta (X)) \otimes t_{i}^j \\
    &= - \sum_{1 \leq i \leq j \leq m+n} \xi_i (X) \mu_i^j \triangleright R \otimes \Delta^m (v_i) - \sum_{1 \leq i \leq j \leq m+n} s_{ij} \triangleright \zeta (X) \otimes \Delta^m (t_q) .
\end{aligned}
\]

This has to agree with \( X \triangleright T_{(R,\zeta,\sigma,k),m} \). The latter is computed to be:

\[
\begin{aligned}
    X \triangleright T_{(R,\zeta,\sigma,k),m} &= X \triangleright \left\{ - \sum_{1 \leq i \leq j \leq m+n} \zeta (s_{ij}) \otimes \Delta^m (t_q) + k R \otimes \Delta^m (c) \right\} \\
    &= - \sum_{1 \leq i \leq j \leq m+n} X \triangleright (\zeta (s_{ij})) \otimes \Delta^m (t_q) - \sum_{1 \leq i \leq j \leq m+n} \zeta (s_{ij}) \otimes X \triangleright t_{i}^j \\
    &= - \sum_{1 \leq i \leq j \leq m+n} s_{ij} \triangleright \zeta (X) \otimes \Delta^m (t_q) - \sum_{1 \leq i \leq j \leq m+n} \zeta (X,s_{ij}) \otimes \Delta^m (t_q) - \sum_{1 \leq i \leq j \leq m+n} \zeta (s_{ij}) \otimes X \triangleright t_{i}^j .
\end{aligned}
\]

Here passing from the first to the second line we used that \( X \triangleright \Delta^m (c) = 0 \) since \( c \) is \( g \)-invariant and that \( \beta (\zeta (X)) \otimes \Delta^m (c) = \zeta (X) \otimes \Delta^m (\partial c) \), by (66), and this vanishes since \( c \in \ker (\partial) \). Passing from the second to the third line we used \( \zeta ([X,Y]) = X \triangleright \zeta (Y) - Y \triangleright \zeta (X) \). The last two terms now can be rewritten as \( - (\zeta \otimes \Delta^m) (X \triangleright r) \), which then agrees with \( - (\zeta \otimes \Delta^m) \beta (\xi (X)) \). Making this last expression explicit:

\[
\begin{aligned}
    - (\zeta \otimes \Delta^m) \beta (\xi (X)) &= - \sum_{a} \xi_a (X) \zeta (\mu_a^j) \otimes \Delta^m (\partial (\mu_a^j)) + \xi_a (X) \zeta (\partial (\mu_a^j)) \otimes \Delta^m (\mu_a^j) \\
    &= - \sum_{a} \xi_a (X) \beta (\zeta (\mu_a^j)) \otimes \Delta^m (\mu_a^j) + \xi_a (X) \mu_a^j \triangleright R \otimes \Delta^m (\mu_a^j) \\
    &= - \sum_{a} \xi_a (X) \mu_a^j \triangleright R \otimes \Delta^m (\mu_a^j) + \xi_a (X) \mu_a^j \triangleright R \otimes \Delta^m (\mu_a^j) \\
    &= - \sum_{a} \xi_a (X) \mu_a^j \triangleright R \otimes \Delta^m (\nu_a).
\end{aligned}
\]

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so that indeed $X = T_{(R,\zeta,\sigma,k),m}$ agrees with $T_{2}$. The naturality condition (16) is proven similarly. As for condition (19) of the definition of an infinitesimal 2-braiding, note that:

$$T_{((R,\zeta,\sigma,k)(R',\zeta',\sigma',k'),m)} = T_{((R,\zeta,\sigma,k)(R',\zeta',\sigma',k)+R\sigma(\zeta'),\sigma+k',m)} \overset{\Delta^m(t_q)}{\longrightarrow} \left(\sum_q (\zeta(s_q)R + R\sigma(\zeta(s_q)))\otimes \Delta^m(t_q) + (k+k')R\sigma(R') \otimes \Delta^m(c), \right)$$

which clearly is the right hand side of (19). Note that this is the point where the decorations of the permutations become essential.

The interchangeability condition (21) reads as follows: given $(R,\zeta,\sigma,k) : n \to n$ and $(R',\zeta',\sigma',k') : m \to m$ 1-morphisms, then it must hold

$$\frac{T_{((R,\zeta,\sigma,k)(R',\zeta',\sigma',k'),m)}}{T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m)}} = \frac{T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m)}}{T_{(n,((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m))}}$$

This means:

$$(R \otimes 1_n)(\sigma \otimes m)(T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m))}) + T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m))}(\sigma \otimes m)(1_n \otimes R) +$$

$$- (1_n \otimes R')(n \otimes \sigma')(T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m))}) - T_{((R,\zeta,\sigma,k)(n \otimes (R',\zeta',\sigma',k'),m))}(n \otimes \sigma')(R \otimes 1_m) = 0$$

Since $c$ is $g$-invariant the lhs of the previous expression reduces to

$$\sum_q (R\Delta^m(s_q) \otimes \zeta'(t_q) + \zeta(s_q) \otimes \Delta^m(t_q) - \zeta(s_q) \otimes R'\Delta^m(t_q) - \Delta^m(s_q)R \otimes \zeta'(t_q)) =$$

$$= \sum_q (s_q \otimes R \otimes \zeta'(t_q) + \zeta(s_q) \otimes t_q \otimes R)$$

$$= \sum_q (-\beta(\zeta(s_q)) \otimes \zeta'(t_q) + \zeta(s_q) \otimes \beta(\zeta'(t_q))) = 0$$

The remaining properties of Definition 14 follow from simple calculations. \hfill \Box

**Definition 32** (Coherent quasi-invariant tensor). *Given a quasi-invariant tensor $(r,\zeta,c)$, let

$$P = \sum_s s_i \otimes \zeta(t_i) + c \otimes r.$$

Then $(r,\zeta,c)$ is said to be coherent if $P_{123} + P_{231} + P_{312} = 0$.

We omit the following proof, since it only involves lengthy but straightforward computations.

**Theorem 33.** Let $(r,\zeta,c)$ be a coherent quasi-invariant tensor in a differential crossed module $G$. Then the associated infinitesimal 2-braiding in the linear monoidal 2-category $C_{\partial}$ is coherent.

### 4.4 A quasi-invariant tensor in the String Lie-2-algebra

We exhibit an explicit example of a quasi-invariant tensor, in the differential crossed module $G_{\text{String}}$ (defined by Wagemann [25]) representing the String Lie 2-algebra. This can be seen as a re-interpretation of the infinitesimal 2-Yang-Baxter operator in the String Lie 2-algebra that we constructed in [12] Section 4, now in the more powerful framework of quasi-invariant tensors and infinitesimal 2-braidings.

From now on we abbreviate $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$. 

4.4.1 The String differential crossed module

We review the construction of the differential crossed module \( \mathcal{S} \text{tring} \). The original construction is due to Wagemann [25]; see also Sections 4.1 and 4.2 of [12] for omitted proofs. We note also [8], where another (weakly equivalent) differential crossed module was defined, thereby geometrically realizing the Lie algebra 3-cocycle \((X, Y, Z) \in \mathfrak{s}l_2 \times \mathfrak{s}l_2 \times \mathfrak{s}l_2 \mapsto (X, [Y, Z])\). This latter construction is however less algebraic than Wagemann’s.

Let \( \mathfrak{g} \) be a Lie algebra and \( V \) a \( \mathfrak{g} \)-module. An anti-symmetric map \( \omega : \wedge^2(\mathfrak{g}) \to V \) is called a 2-cocycle if its coboundary \( \delta^V(\omega) : \Lambda^3(\mathfrak{g}) \to V \) defined as
\[
\delta^V(\omega)(X, Y, Z) = X \cdot \omega(Y, Z) + Y \cdot \omega(Z, X) + Z \cdot \omega(X, Y) + \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y)
\]
vanishes for all \( X, Y, Z \in \mathfrak{g} \).

Let \( W_1 \) be the Lie algebra of polynomial vector fields in one variable \( x \), with Lie bracket given by the commutator of vector fields:
\[
\left[ f \frac{d}{dx}, g(x) \frac{d}{dx} \right] = \left( f \frac{dg}{dx} - f'g \right)(x) \frac{d}{dx}, \quad \forall f(x) \frac{d}{dx}, g(x) \frac{d}{dx} \in W_1,
\]
where \( f' \) denotes the derivative of \( f \). We identify \( \mathfrak{sl}_2 \subset W_1 \) as the Lie subalgebra generated by:
\[
f = \frac{d}{dx}, \quad k = x \frac{d}{dx}, \quad e = x^2 \frac{d}{dx},
\]
so that the commutation relations read:
\[
[f, e] = 2k; \quad [k, e] = c; \quad [k, f] = -f.
\]
The Cartan-killing form \( \langle \cdot, \cdot \rangle : \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \to \mathbb{C} \) is taken with the normalisation: \( \langle x, y \rangle = \text{Tr}(\text{ad}_x \circ \text{ad}_y) \).

Let \( \mathbb{F}_0 \) be the space of complex polynomials in the variable \( x \), and \( \mathbb{F}_1 \) the space of formal 1-forms \( f(x)dx \), over the space of real numbers, where \( f(x) \) is a complex polynomial. We consider \( \mathbb{F}_0 \) and \( \mathbb{F}_1 \) to be abelian Lie algebras, i.e. with the trivial Lie bracket. They are both \( W_1 \)-modules via the Lie derivative:
\[
\left( f(x) \frac{d}{dx} \right) \cdot g(x) = (fg')(x), \quad \left( f(x) \frac{d}{dx} \right) \cdot dx = (f + f')g(x)dx, \quad \forall f(x) \frac{d}{dx}, g(x) \in \mathbb{F}_1, \quad g(x) \in \mathbb{F}_0, \quad g(x)dx \in \mathbb{F}_1.
\]
hence they are \( \mathfrak{sl}_2 \)-modules as well, by restriction of the module structure. Clearly the formal de Rham differential \( d : \mathbb{F}_0 \to \mathbb{F}_1 \) is a map of \( \mathfrak{sl}_2 \)-modules. We define a map \( \alpha : \wedge^2(\mathfrak{sl}_2) \to \mathbb{F}_1 \) as
\[
\alpha(k, e) = -\alpha(e, k) = dx, \quad \text{and zero otherwise}.
\]
By a direct computation \( \delta^\mathbb{F}_1 \alpha = 0 \), so that \( \alpha \) is a 2-cocycle. Denote \( \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \) the abelian extension of \( \mathfrak{sl}_2 \) by \( \mathbb{F}_1 \) associated to \( \alpha \). We recall that \( \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \) is the vector space \( \mathbb{F}_1 \oplus \mathfrak{sl}_2 \), endowed with the Lie bracket:
\[
\left[ (a, b), (c, d) \right] := \left( y \cdot b - z \cdot a + \alpha(y, z), [y, z] \right), \quad \forall a, b \in \mathbb{F}_1, \quad y, z \in \mathfrak{sl}_2.
\]
Note that every \( \mathfrak{sl}_2 \)-module \( F \) extends trivially to a \( \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \)-module by setting:
\[
(a, y) \cdot f := y \cdot f, \quad (a, y) \in \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2, f \in F.
\]
In particular, we will be interested in the case \( F = \mathbb{F}_0 \). The differential crossed module
\[
\mathcal{S} \text{tring} = (\partial : \mathbb{F}_0 \to \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2)
\]
representing the String Lie 2-algebra fits inside the four-terms exact sequence:
\[
0 \to \mathbb{C} \xrightarrow{i} \mathbb{F}_0 \xrightarrow{\partial} \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \xrightarrow{\pi} \mathfrak{sl}_2 \to 0.
\]
We repeat the relevant structures in \( \mathcal{S} \text{tring} = (\partial : \mathbb{F}_0 \to \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2) \): the Lie bracket in \( \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \) is given in (78), while in \( \mathbb{F}_0 \) is trivial. The \((\mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2)\)-action on \( \mathbb{F}_0 \) is of the type (79), trivially extending the one of \( \mathfrak{sl}_2 \). The maps \( \partial \) and \( \pi \) are explicitly \( \partial(f) = (df, 0) \) and \( \pi(\omega, y) = y \), for \( f \in \mathbb{F}_0 \), \( (\omega, y) \in \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \); hence \( \ker(\partial) = \mathbb{C} \) and \( \text{coker}(\partial) = \mathfrak{sl}_2 \). Therefore, since \( d : \mathbb{F}_0 \to \mathbb{F}_1 \) is the exterior derivative, the kernel of \( \partial : \mathbb{F}_0 \to \mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2 \) is generated by the constant function \( 1_{\mathbb{F}_0} \) in \( \mathbb{C} \).

From (76), we can see that \( \ker(\partial) \) is \( \mathfrak{sl}_2 \)-, thus \((\mathbb{F}_1 \rtimes_\alpha \mathfrak{sl}_2)\)-, invariant. This fact will play a prime role later.
4.4.2 The construction of the quasi-invariant tensor

Consider the String differential crossed module $\mathcal{Z} = (\partial : \mathbb{F}_0 \to \mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2, \triangleright)$. Given $X \in \mathfrak{sl}_2$, we put $\overline{X} = (0, X) \in \mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2$. We will also write $\overline{h} = (h, 0)$ for $h \in \mathbb{F}_1$. Let $Q : \mathbb{F}_1 \to \mathbb{F}_0$ be a linear primitive of the exterior differentiation: $d(Q(w)) = w$, for each 1-form $w \in \mathbb{F}_1$. We will choose it so that $Q(dx) = x$. We use $Q$ to introduce the linear map:

$$\omega : \Lambda^2(\mathfrak{sl}_2) \to \mathbb{F}_0$$ such that $\omega(X, Y) = Q(\alpha(X, Y)).$

Recall (77). As a consequence of (60), we can see that:

$$1_{\mathbb{F}_0} \otimes [\overline{X}, \overline{Y}] = 1_{\mathbb{F}_0} \otimes (\alpha(X, Y)_0 [X, Y]) = 1_{\mathbb{F}_0} \otimes \alpha(X, Y) + 1_{\mathbb{F}_0} \otimes [X, Y]$$

$$= 1_{\mathbb{F}_0} \otimes d(\alpha(X, Y)) + 1_{\mathbb{F}_0} \otimes [X, Y] = 1_{\mathbb{F}_0} \otimes \partial(\alpha(X, Y)) + 1_{\mathbb{F}_0} \otimes [X, Y]$$

$$= \partial(1_{\mathbb{F}_0}) \otimes \omega(X, Y) + 1_{\mathbb{F}_0} \otimes [X, Y] = 1_{\mathbb{F}_0} \otimes [X, Y]$$

We therefore have the following absolutely crucial relation in $\mathcal{U}^{(2)}$, valid for all $X, Y \in \mathfrak{sl}_2$:

$$1_{\mathbb{F}_0} \otimes [\overline{X}, \overline{Y}] = 1_{\mathbb{F}_0} \otimes [X, Y]$$ (81)

Let $r = \sum_i s_i \otimes t_i \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ be the infinitesimal braiding of $\mathfrak{sl}_2$ associated to the Cartan-Killing form. Explicitly:

$$r = f \otimes e + e \otimes f - 2k \otimes k = \sum_i s_i \otimes t_i.$$

The following identity (the $\mathfrak{sl}_2$-invariance of $r$) will be used time after time below:

$$\sum_i [s_i, X] \otimes t_i + s_i \otimes [t_i, X] = 0, \quad \text{for all } X \in \mathfrak{sl}_2.$$ (82)

Let $\Phi := \delta^\mathbb{F}_0(\omega) : \Lambda^3 \mathfrak{sl}_2 \to \mathbb{F}_0$ be the coboundary of $\omega$ (an explicit expression is as in (73)).

**Lemma 34.** Given $X, Y \in \mathfrak{sl}_2$ we have:

$$\sum_i \Phi(s_i, X, Y) \otimes t_i = 1_{\mathbb{F}_0} \otimes [X, Y], \quad \sum_i s_i \otimes \Phi(t_i, X, Y) = [X, Y] \otimes 1_{\mathbb{F}_0}$$ (83)

**Proof.** Clearly the map

$$(X, Y) \mapsto \sum_i \Phi(s_i, X, Y) \otimes t_i$$

is bilinear and antisymmetric, so we only need to check (83) for the pairs $(f, e)$, $(f, k)$ and $(k, e)$. This follows from an easy calculation. $\square$

We are now ready to construct the quasi-invariant tensor in $\mathcal{Z}$. Recalling Definition 30, we consider the following objects:

(a) $\overline{r} = \sum_i \overline{s}_i \otimes \overline{t}_i \in (\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2) \otimes (\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2)$ (the obvious lifting of $r$ to $\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2$);

(b) the map $\xi : (\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2) \to ((\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2) \otimes \mathbb{F}_0) \oplus (\mathbb{F}_0 \otimes (\mathbb{F}_1 \rtimes_{\alpha} \mathfrak{sl}_2)) \subset \mathcal{U}^{(2)}$ defined as $\xi = -\xi_0 + C$, where

$$\xi_0(\overline{X}) = \sum_i \omega(s_i, X) \otimes \overline{t}_i + \overline{s}_i \otimes \omega(t_i, X),$$ (84)

$$\xi_0(\overline{h}) = \sum_i s_i \triangleright Q(h) \otimes \overline{t}_i + \overline{s}_i \otimes t_i \triangleright Q(h),$$ (85)

$$C((h, X)) = 1_{\mathbb{F}_0} \otimes \overline{X} + \overline{X} \otimes 1_{\mathbb{F}_0};$$ (86)
We note that since $\ker(\partial)$ is sl₂-invariant, $\xi_0(\bar{h})$ does not depend on the chosen primitive $Q: \mathbb{F}_1 \to \mathbb{F}_0$ of the exterior derivation.

**Theorem 35.** The triple $(\mathfrak{T}, \xi, c)$ is a quasi-invariant tensor in the String differential crossed module $\mathfrak{String}$.

**Proof.** We need to check conditions (i)-(iii) of Definition 30. As for condition (i):

\[
\mathfrak{T} \triangleright \bar{X} = \sum_i \left( (\alpha(X, s_i), [X, s_i]) \otimes \bar{t}_i + \bar{s}_i \otimes (\alpha(X, t_i), [X, t_i]) \right)
\]

\[
= \sum_i \left( \alpha(X, s_i) \otimes \bar{t}_i + \bar{s}_i \otimes \alpha(X, t_i) \right) = -\beta(\xi_0(\bar{X})) = \beta(\xi(\bar{X}))
\]

\[
\bar{h} \triangleright \mathfrak{T} = \sum_i \left( s_i \triangleright h \otimes \bar{t}_i + \bar{s}_i \otimes t_i \triangleright h \right)
\]

\[
= \sum_i -\beta \left( s_i \triangleright Q(h) \otimes \bar{t}_i + \bar{s}_i \otimes t_i \triangleright Q(h) \right) = -\beta(\xi_0(\bar{h})) = \beta(\xi(\bar{h}))
\]

Now condition (ii); if $a \in \mathbb{F}_0$ (remember that $Q: \mathbb{F}_1 \to \mathbb{F}_0$ is a primitive of $d$):

\[
a \triangleright \mathfrak{T} - \xi(\partial(a)) = \sum_i \left( s_i \triangleright (Qd(a) - a) \otimes \bar{t}_i + \bar{s}_i \otimes t_i \triangleright (Qd(a) - a) \right)
\]

and this vanishes since $Qd(a) - a$ is in $\ker(\partial) = \mathbb{C}$ which is sl₂-invariant. Finally, condition (iii); we start by computing:

\[
\xi_0([X, Y]) = \xi_0([X, Y]) - \xi_0(\alpha(X, Y))
\]

\[
= \sum_i \left( \alpha(X, s_i) \otimes \bar{t}_i + \bar{s}_i \otimes \alpha(X, t_i) \right) + s_i \triangleright (Qa(X, Y)) \otimes \bar{t}_i + \bar{s}_i \otimes t_i \triangleright (Qa(X, Y))
\]

\[
\Xi \triangleright \xi_0([X, Y]) = \sum_i \left( X \triangleright \alpha(s_i, Y) \otimes \bar{t}_i + \alpha(s_i, Y) \otimes [X, \bar{t}_i] + [X, \bar{s}_i] \otimes \omega(t_i, Y) - \bar{s}_i \otimes X \triangleright \alpha(t_i, Y) \right)
\]

\[
\Xi \triangleright \xi_0([X, Y]) = \sum_i \left( Y \triangleright \alpha(s_i, X) \otimes \bar{t}_i + \alpha(s_i, X) \otimes [Y, \bar{t}_i] + [Y, \bar{s}_i] \otimes \omega(t_i, X) + \bar{s}_i \otimes Y \triangleright \alpha(t_i, X) \right)
\]

We now claim that

\[
\xi_0([X, Y]) - \Xi \triangleright \xi_0([X, Y]) + \Xi \triangleright \xi_0([X, Y]) = \sum_i \left( \Phi(s_i, X, Y) \otimes \bar{t}_i + \bar{s}_i \otimes \Phi(t_i, X, Y) \right)
\]

\[
= [X, Y] \otimes 1_{\mathbb{F}_0} + 1_{\mathbb{F}_0} \otimes [X, Y]
\]

where the left hand side has been rewritten using (the lift of) (33). To prove the claim one uses the explicit expression of $\Phi$, see (73), and checks that:

\[
\xi_0([X, Y]) - \Xi \triangleright \xi_0([X, Y]) + \Xi \triangleright \xi_0([X, Y]) - \sum_i \left( \Phi(s_i, X, Y) \otimes \bar{t}_i + \bar{s}_i \otimes \Phi(t_i, X, Y) \right) =
\]

\[
\sum_i \left( -\omega(s_i, Y) \otimes [X, \bar{t}_i] - [X, \bar{s}_i] \otimes \omega(t_i, Y) + \omega(s_i, X) \otimes [Y, \bar{t}_i] + [Y, \bar{s}_i] \otimes \omega(t_i, X) + \omega(X, [s_i, Y]) \otimes \bar{t}_i - \omega(Y, [s_i, X]) \otimes \bar{t}_i - \bar{s}_i \otimes \omega(X, [Y, t_i]) - \bar{s}_i \otimes \omega(Y, [t_i, X]) \right).
\]

We know expand each term of the form $[X, \bar{t}_i]$ into $\alpha(X, t_i) + [X, t_i]$. The terms involving the cocycle $\alpha$ are:

\[
\sum_i \left( -\omega(s_i, Y) \otimes \alpha(X, t_i) - \alpha(X, s_i) \otimes \omega(s_i, X) + \omega(s_i, X) \otimes \alpha(Y, t_i) + \alpha(Y, s_i) \otimes \omega(t_i, X) \right)
\]
and they cancel pairwise by a suitable use of \(\ref{66}\) and from the definition of \(\omega;\) for example:
\[
\omega(s_i, Y) \otimes \alpha(X, t_i) = \omega(s_i, Y) \otimes \partial Q \alpha(X, t_i) = \partial \omega(s_i, Y) \otimes \alpha(X, t_i) = \alpha(s_i, Y) \otimes \omega(X, t_i)
\]
The remaining eight terms not containing \(\alpha\) cancel pairwise thanks to the \(\mathfrak{sl}_2\)-invariance of \(r;\) for example:
\[
- \omega(s_i, Y) \otimes \big[ X, t_i \big] - \omega(Y, [s_i, X]) \otimes t_i = 0
\]
and similarly for the other pairs. This proves the claim. We go on with:
\[
C([X, Y]) - X \triangleright C(Y) + Y \triangleright C(X) = C(\alpha(X, Y), [X, Y]) - X \triangleright C(Y) + Y \triangleright C(X)
\]
\[
= 1_{F_l} \otimes [X, Y] + [X, Y] \otimes 1_{F_l},
\]
where we used \(\ref{81}\), so that summing up all the contributions we have eventually proved condition (iii) for \(\xi([X, Y])\). Now the same property for \(\xi([X, h])\). We start by computing:
\[
\xi_0([X, h]) = \xi_0(X \triangleright h) = s_i \triangleright Q(X \triangleright h) \otimes t_i + \bar{s}_i \otimes t_i \triangleright Q(X \triangleright h)
\]
\[
\bar{X} \triangleright \xi_0(h) = \bar{X} \triangleright (s_i \triangleright Q(h)) \otimes \bar{t}_i + s_i \triangleright Q(h) \otimes [\bar{X}, \bar{t}_i] + [\bar{X}, \bar{s}_i] \otimes t_i \triangleright Q(h) + \bar{s}_i \otimes X \triangleright (t_i \triangleright Q(h))
\]
\[
\bar{h} \triangleright \xi_0(X) = - \omega(s_i, X) \otimes \bar{t}_i \triangleright \bar{h} - s_i \otimes \bar{h} \otimes \omega(t_i, X)
\]
and noting that \(X \triangleright (Y \triangleright Q(h)) = X \triangleright (Q(Y \triangleright h)), \) i.e. that \(Q\) is \(\mathfrak{sl}_2\)-equivariant up to \(\mathfrak{sl}_2\)-action. This follows from
\[
\partial (Y \triangleright Q(h) - Q(Y \triangleright h)) = Y \triangleright (\partial Q(h)) - Y \triangleright h = 0
\]
together with the fact that \(\text{ker}(\partial)\) is \(\mathfrak{sl}_2\)-invariant. All this is to write
\[
\xi_0([X, h]) - \bar{X} \triangleright \xi_0(h) + \bar{h} \triangleright \xi_0(X) =
\]
\[
= - \big[ [X, s_i] \triangleright Q(h) \otimes \bar{t}_i - \bar{s}_i \otimes [X, t_i] \triangleright Q(h) - s_i \triangleright Q(h) \otimes [X, \bar{t}_i] - [X, \bar{s}_i] \otimes t_i \triangleright Q(h) + \omega(s_i, X) \otimes \bar{t}_i \triangleright \bar{h} - s_i \otimes \bar{h} \otimes \omega(t_i, X) \big]
\]
\[
= - \left( [X, s_i] \triangleright Q(h) \otimes \bar{t}_i + s_i \triangleright Q(h) \otimes [X, \bar{t}_i] \right) - [\bar{s}_i \otimes [X, t_i] \triangleright Q(h) + [X, \bar{s}_i] \otimes t_i \triangleright Q(h)] + [s_i \triangleright Q(h) \otimes \alpha(X, t_i) - \omega(s_i, X) \otimes t_i \triangleright \bar{h} - s_i \otimes \bar{h} \otimes \omega(t_i, X)].
\]
In the two parenthesis we read the \(\mathfrak{sl}_2\)-invariance of \(r, \) so they vanish. The four terms in the last line cancel pairwise, again by a suitable use of \(\ref{66}\) and from the definition of \(\omega;\) for example:
\[
- s_i \triangleright Q(h) \otimes \alpha(X, t_i) = s_i \triangleright Q(h) \otimes \partial Q \alpha(X, t_i) = \partial (s_i \triangleright Q(h)) \otimes \omega(X, t_i) = s_i \otimes \bar{h} \otimes \omega(X, t_i).
\]
This completes the proof. \(\Box\)

We now address the coherence of the quasi-invariant tensor \((\tau, \xi, c).\) Recalling Definition \(\ref{32}\) this amounts to
\[
P_{123} + P_{231} + P_{312} = 0,
\]
where now \(P\) reads
\[
P = \sum_i \left( [\bar{s}_i \otimes \xi_0(t_i)] + c \otimes \bar{t} \right) + \sum_i \left( - [\bar{s}_i \otimes \xi_0(t_i)] + [\bar{s}_i \otimes C(t_i)] \right) + c \otimes \bar{t}
\]
\[
= \sum_i \left( [\bar{s}_i \otimes \omega(s_i, t_i) \otimes \bar{t}_i - \bar{s}_i \otimes \bar{t}_i] \otimes \omega(t_i, t_i) \right) + \sum_i \bar{s}_i \otimes C(t_i) + c \otimes \bar{t}
\]

**Theorem 36.** The quasi-invariant tensor \((\tau, \xi, c)\) on \(\text{String}\) is coherent for \(c = -2 \cdot 1_{F_l}.\)
Proof. By direct computation. We explicit

\[ P_{123} = -2 \overline{k} \odot x \otimes \overline{f} + 2 \overline{f} \odot x \otimes \overline{k} + 2 \overline{f} \odot \overline{k} \otimes x - 2 \overline{k} \odot \overline{f} \otimes x + \sum_i \left( t_i \otimes 1_{F_0} \otimes t_i + s_i \otimes t_i \otimes 1_{F_0} + c \otimes s_i \otimes t_i \right) \]

\[ P_{231} = -2 \overline{f} \odot \overline{k} \otimes x + 2 \overline{k} \odot \overline{f} \otimes x + 2 x \odot \overline{f} \otimes \overline{k} - 2 x \odot \overline{k} \otimes \overline{f} + \sum_i \left( t_i \otimes s_i \otimes 1_{F_0} + 1_{F_0} \otimes s_i \otimes t_i + t_i \otimes c \otimes s_i \right) \]

\[ P_{312} = -2 x \odot \overline{f} \otimes \overline{k} + 2 x \odot \overline{k} \otimes \overline{f} + 2 \overline{k} \odot x \otimes \overline{f} - 2 \overline{f} \odot x \otimes \overline{k} + \sum_i \left( 1_{F_0} \otimes t_i \otimes s_i + t_i \otimes 1_{F_0} \otimes s_i + s_i \otimes t_i \otimes c \right) \]

and note that in \( P_{123} + P_{231} + P_{312} \) the terms without summation on \( i \) cancel directly, while the sum over \( i \) vanishes for \( c = -2 \cdot 1_{F_0} \).

\[ \square \]

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