Background independent quantizations—the scalar field: II

Wojciech Kamiński, Jerzy Lewandowski and Andrzej Okołów

1 Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, 00-681 Warszawa, Poland
2 Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2 L 2Y5, Canada
3 Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA

E-mail: wkaminsk@fuw.edu.pl, lewand@fuw.edu.pl and oko@fuw.edu.pl

Received 23 May 2006
Published 18 August 2006
Online at stacks.iop.org/CQG/23/5547

Abstract

We are concerned with the issue of the quantization of a scalar field in a diffeomorphism invariant manner. We apply the method used in loop quantum gravity. It relies on the specific choice of scalar field variables referred to as the polymer variables. The quantization, in our formulation, amounts to introducing the ‘quantum’ polymer \( \ast \)-algebra and looking for positive linear functionals, called states. As assumed in our paper, homeomorphism invariance allows us to derive the complete class of the states. They are determined by the homeomorphism invariant states defined on the CW-complex \( \ast \)-algebra. The corresponding GNS representations of the polymer \( \ast \)-algebra and their self-adjoint extensions are derived, the equivalence classes are found, and invariant subspaces characterized. In part I we outlined those results. Here, we present the technical details.

PACS numbers: 04.60.Pp, 04.60.Ds, 04.60.Nc, 03.65.Sq

1. Introduction

1.1. Motivation

Einstein’s theory of gravity coupled (or not) to a matter field is a prominent example of the so-called ‘background independent’ theory. The phrase ‘background independent’ means that the theory is defined on a bare manifold endowed with no geometry or affine structure. In this case, it is natural to look for a corresponding quantum theory that is also manifestly background independent (see [1] for a profound discussion of that issue). It is conceivable that a background independent quantum theory can even be derived from a background dependent framework. Another possibility is that the classical limit of a given quantum theory (the classical GR, for
example) has more symmetries (the diffeomorphisms) than the underlying quantum theory. Therefore, we are not in a position to claim that every background dependent approach to a quantization of a background independent classical theory is wrong. Nonetheless, the first thing one should do is to try it without introducing extra structures. Loop quantum gravity is the only known example. That name comes from the idea [2–4] of using field variables labelled by loops, typically non-intersecting but knotted. Later, embedded graphs were found to be a correct tool [5–7] which gave rise to the present form of the theory. Therefore, there were attempts to promote the theory under a new name like ‘quantum geometry’ [8–12] or ‘quantum spin dynamics’ [16–18]. LQG [19] is a canonical theory, it relies on the 3+1 decomposition of spacetime into the ‘space’ $M$ and ‘time’ $R$. It is invariant with respect to the diffeomorphisms (piecewise analytic) of $M$. It concerns the spacetime geometry and its dynamics, as well as coupled matter fields [20]. The matter fields, however, have to be quantized in a new way, consistent with the LQG and background independent quantization. And this is the point we want to focus on in this work. We are concerned here with the background independent, canonical quantization of the scalar field. The first example of a background independent quantization of the scalar field was proposed by Thiemann in his pioneering work [20]. It was improved and analysed by various authors [21]. In the current paper, we systematically derive a broader class of examples. We show that our class is complete if certain simple assumptions, such as the topological invariance, are made. In fact, we consider only the GNS representations (and their self-adjoint extensions) defined by invariant states introduced on the appropriate $\ast$-algebra. Among the constructed representations we identify those which give essentially self-adjoint operators, the equivalence classes, irreducible representations. Each of them can be applied to the scalar field interacting with the quantum geometry in the frame of LQG. A brief discussion of our results is already published in the form of a letter [22].

Here is an outline of the work. In section 2 we introduce the polymer $\ast$-algebra $\mathcal{A}$, the key object studied in this work. It is constructed by the canonical quantization of a classical field $\phi$ and the canonically conjugate momentum $\pi$, both living on a manifold $M$. The first step is choosing basic field variables. Briefly speaking (see section 2.2 for details), they are as follows: the ‘position variables’ $\hat{e}^k\phi(x)$ (denoted by $\hat{h}_{k,x}$) and the ‘momentum variables’ $\int \pi f$ (denoted by $\pi(f)$), where $k \in R$ and $f$ is a smearing function. The vector space of the smearing functions $\mathcal{F}$ is fixed in this section arbitrarily. Its specific choice will play a crucial role later. The basic variables together with the Poisson bracket form a Lie algebra, still classical. The polymer $\ast$-algebra is defined (section 2.3) by ‘putting hats on’, meaning it is the quantum enveloping algebra of the Lie algebra in question. It still depends on the choice of the smearing function space $\mathcal{F}$. Next, we recall the definition of a state on a $\ast$-algebra and elements of the GNS construction. The conditions that will be imposed on the states are formulated in sections 2.5 and 2.6. Briefly, they are as follows: (i) the diffeomorphism (section 3) and the homeomorphism (section 4–9.1) invariance, and (ii) an extra condition ensuring that for every point $x$, the quantum operators $\rho(\hat{h}_{k,x}), k \in R$, form a one-dimensional group of unitary operators.

The first example of the polymer $\ast$-algebra considered in this work is given by choosing for the smearing functions all the $C^0$, $n = 0, 1, \ldots, \infty$, compactly supported functions on the manifold $M$ (section 3). In this case, we show that there is exactly one $C^0$-diffeomorphism (homeomorphism in the $n = 0$ case) invariant state on the polymer $\ast$-algebra $\mathcal{A}$. The proof is a simplified version of the proof used in [23] in the case of the holonomy-flux $\ast$-algebra. This state coincides with that used thus far in LQG [20, 21].

The proof of that first result is very sensitive to the differentiability class of the smearing functions being equal or greater than the differentiability of the diffeomorphisms. The question that arises naturally, is: how much the uniqueness proved in section 3 depends on those
Background independent quantizations

Differentiability assumptions? Another natural choice for the smearing functions—also used in LQG—is the characteristic functions of bounded regions in \( M \). The study of this case is the subject of the main part of this work.

Specifically, the space \( \mathcal{F}_c \) of the smearing functions we consider throughout sections 4–9 is spanned by the characteristic functions of ball-like regions in the manifold \( M \). The manifold is endowed with a piecewise-analytic structure and the regions are assumed to be piecewise-analytic simplexes (see the appendix). First (section 4), we study the Abelian algebra \( \exp(\otimes \mathcal{F}_c) \) freely generated by the characteristic functions, and its complexification called here the CW-complex \( \ast \)-algebra. That algebra, well defined on its own, can be identified with the subalgebra of the \( \ast \)-polymer algebra generated by all the momentum variables. We find all the homeomorphism invariant states on the CW-complex \( \ast \)-algebra. They can be labelled by the states \( \mu : \mathbb{C}[\tau] \to \mathbb{C} \) where \( \mathbb{C}[\tau] \) is the \( \ast \)-algebra of the polynomials of one real variable.

Every homeomorphism invariant state on the CW-complex \( \ast \)-algebra admits a natural extension to the polymer \( \ast \)-algebra \( \mathfrak{A} \) (section 5). The resulting state defined on \( \mathfrak{A} \) is also homeomorphism invariant. In this way a 1–1 correspondence between the states \( \mu : \mathbb{C}[\tau] \to \mathbb{C} \) and all the homeomorphism invariant states on \( \mathfrak{A} \) which satisfy property 2.10 (mentioned above) was established.

In the remaining part of the work (sections 6–9) we are concerned with the properties of the GNS representations corresponding to the homeomorphism invariant states on \( \mathfrak{A} \) derived in section 4 and their self-adjoint extensions. An explicit form of the GNS representation defined by a state found in section 4 is derived in section 6. The issue of self-adjointness of the resulting operators is studied in section 7. In particular, we characterize all those representations which admit a unique self-adjoint extension according to Schmudgen [24]. The equivalence problem for the self-adjoint extensions of our representations is solved in section 8. The reducibility issue is solved in section 9. The results are summarized in section 10.

The problem we studied is also relevant for the issue of the uniqueness of the diffeomorphism invariant state of quantum geometry in LQG. We elaborate on that in section 10.

2. The polymer \( \ast \)-algebra \( \mathfrak{A} \) and GNS construction

2.1. The classical scalar field

The classical scalar field in the canonical approach consists of a pair of fields \((\phi, \pi)\) defined on an \( N \)-real-dimensional manifold \( M \), where \( \phi \in C^{m_0}(M, \mathbb{R}) \), and \( C^{m_0}(M, \mathbb{R}) \) stands for the space of the \( C^{m_0} \) real-valued functions defined on \( M \), \( m_0 \in \mathbb{N} \) or \( m_0 = \infty \), whereas \( \pi \), called the canonical momentum, is a scalar density of the weight 1. The momentum \( \pi \) can be expressed by a function \( \tilde{\pi} : U \to \mathbb{R} \) defined on an arbitrary region \( U \subset M \) equipped with coordinates \((x^1, \ldots, x^N)\). The function, however, depends on the coordinates in the following way: if \((x^1', \ldots, x^N')\) is another coordinate system defined on \( U \), then the corresponding momentum function \( \tilde{\pi}' \) is such that at every \( x \in U \)

\[
\tilde{\pi}(x) \, d^N x = \tilde{\pi}'(x) \, d^N x'.
\] (2.1)

The fields \( \phi \) and \( \pi \) are called canonically conjugate in the sense of the Poisson bracket, usually written as

\[
\{\phi(x), \tilde{\pi}(y)\} = \delta_x (y).
\] (2.2)
The Poisson bracket will be encoded in the Lie bracket of the Lie algebra introduced in the next subsection.

### 2.2. The classical polymer Lie algebra

**Definition 2.1.** A polymer position variable is an (exponentiated evaluation) function \( h_{k,x} : \mathbb{C}^m_0(M, \mathbb{R}) \rightarrow \mathbb{C} \) assigned to a pair \((k, x) \in \mathbb{R} \times M\) such that \( k \neq 0\), and defined as follows:

\[
h_{k,x} : \phi \mapsto e^{ik\phi(x)}.
\] (2.3)

The set of the polymer position variables is closed with respect to the complex conjugation \( \overline{h_{k,x}} = h_{-k,-x} \).

**Definition 2.2.** The polymer position variable space \( \text{Cyl}^\mathbb{R} \) is the set of all the real-valued linear combinations of all the polymer position variables; that is, the set of all the linear combinations \( h \) such that it can be written in the form

\[
h = \sum_{i=1}^{n} (a_i h_{k_i,x_i} + \overline{a_i} h_{-k_i,-x_i})
\] (2.5)

where \( n \in \mathbb{N}, x_1, \ldots, x_n \in M, k_1, \ldots, k_n \in \mathbb{R} \setminus \{0\}, a_1, \ldots, a_n \in \mathbb{C} \) are arbitrary.

We could well be using only real-valued functions \( \text{Re} h_{k,x} \) and \( \text{Im} h_{k,x} \) up to this point. The complexification will come with the quantization.

A momentum variable will be assigned to a function \( f : M \rightarrow \mathbb{R} \). It can be thought of as the integral \( \int_M \pi f \). However, we will identify it with the operator defined by the Poisson brackets \( \{ h_{k,x}, \int_M \pi f \} \):

**Definition 2.3.** A momentum variable is a linear map \( \pi(f) : \text{Cyl}^\mathbb{R} \rightarrow \text{Cyl}^\mathbb{R} \) and defined by the following action on the polymer position variables:

\[
\pi(f) h_{k,x} = if(x) h_{k,x}.
\] (2.6)

This definition is our version of the Poisson bracket (2.2). We fix a vector subspace \( \mathcal{F} \) of the space of all the real-valued functions \( \text{Map}(M, \mathbb{R}) \) defined on \( M \) and refer to it as the space of the smearing functions. We consider only the momenta defined by the smearing functions \( f \in \mathcal{F} \).

**Definition 2.4.** The momentum variable space \( \Pi_\mathcal{F} \) defined by a given space \( \mathcal{F} \subset \text{Map}(M, \mathbb{R}) \) of the smearing functions is the real vector space spanned by the linear maps \( \pi(f) \) (2.6) such that \( f \in \mathcal{F} \). The vector space \( \mathcal{F} \) itself is referred to as the smearing functions space, and its elements as smearing functions.

Actually, the map

\[
\mathcal{F} \ni f \mapsto \pi(f) \in \Pi_\mathcal{F},
\] (2.7)

is an isomorphism of the vector spaces.

Finally, we define
Definition 2.5. The polymer Lie algebra \((\mathfrak{A}_{cl}, \{\cdot, \cdot\})\) corresponding to a momentum variable space \(\Pi_F\) is the direct sum vector space
\[
\mathfrak{A}_{cl} := \text{Cyl}_1^\mathbb{R} \oplus \Pi_F,
\]
equipped with the following Lie bracket \(\{\cdot, \cdot\}:
\[
\{(h, \pi), (h', \pi')\} := (\pi' h - \pi h', 0).
\]

The Lie bracket \(\{\cdot, \cdot\}\) encodes the structure of the Poisson bracket (2.2) and this is the reason why we denote it in this way (rather than the usual ‘[·, ·]’).

Later in this paper, we will consider the following two examples of the smearing functions space \(\mathcal{F}\) and the corresponding momentum variable spaces:

**Example.**

(i) \(\mathcal{F} = C^0_{\text{c.s.}}(M, \mathbb{R})\)

where the second subscript 0 stands for ‘compactly supported’.

(ii) \(\mathcal{F}\) spanned by the characteristic functions of suitably defined family of open subsets of \(M\).

Our main results will concern the second example. But until the end of this section we keep the smearing functions space \(\mathcal{F}\) general.

2.3. The quantum polymer \(*\)-algebra \(\mathfrak{A}\)

In order to turn the polymer position and momentum variables (definitions 2.1 and 2.3) into quantum operators we will construct from them a \(*\)-algebra. Briefly speaking, the commutation relations of the operators will be defined as \(i\) times the Lie bracket of the corresponding elements of the classical Lie algebra. To do it in an exact way, we formulate a definition analogous to that of the universal enveloping algebra.

Let \((\mathfrak{A}_{cl}, \{\cdot, \cdot\})\) be an arbitrary real Lie algebra.

First, consider a complex Lie algebra \((\mathfrak{A}_{cl}^\mathbb{C}, \{\cdot, \cdot\})\), the complexification of \((\mathfrak{A}_{cl}, \{\cdot, \cdot\})\).

There is the natural complex conjugation \(\bar{\cdot}: \mathfrak{A}_{cl}^\mathbb{C} \to \mathfrak{A}_{cl}^\mathbb{C}\).

Next, consider the huge space \(\bigoplus_{n=0}^{\infty} (\mathfrak{A}_{cl}^\mathbb{C})^\otimes n\).

It has the natural complex associative algebra structure defined by the complex vector space structure and the operation \(\otimes\). There is also naturally defined anti-isomorphism involution \(\cdot^*\) in it, such that
\[
(a_1 \otimes \cdots \otimes a_n)^* = \overline{a}_n \otimes \cdots \otimes \overline{a}_1.
\]

Next, introduce the double-sided ideal \(J\) generated by the following subset:
\[
\{ a \otimes b - b \otimes a - i[a, b] : a, b \in \mathfrak{A}_{cl}^\mathbb{C} \}.
\]

Ideal \(J\) is preserved by the involution \(\cdot^*\), hence \(\cdot^*\) passes to the quotient \(\bigoplus_{n=0}^{\infty} (\mathfrak{A}_{cl}^\mathbb{C})^\otimes n / J\) (and is denoted by the same symbol \(\cdot^*\)).

Finally, define
**Definition 2.6.** The quantum enveloping algebra \((\mathfrak{A}, \cdot^*)\) of a real Lie algebra \((\mathfrak{A}_{\text{cl}}, \{\cdot, \cdot\})\) is the associative, unital, \(*\)-algebra
\[
\mathfrak{A} = \bigoplus_{n=0}^{\infty} (\mathfrak{A}_{\text{cl}}^n)^{\otimes n} / J. \tag{2.14}
\]

From now on, until the end of the paper, we will be considering the quantum enveloping algebra of the polymer Lie algebra:

**Definition 2.7.** A polymer \(*\)-algebra \(\mathfrak{A}\) is the quantum enveloping algebra of a polymer Lie algebra.

Elements of \(\mathfrak{A}\) can be heuristically thought of as quantum operators corresponding to polynomials in the polymer position and momentum variables. (They become operators indeed, only given a representation of \(\mathfrak{A}\)—see the next subsection).

We will be using the following notation: an element \([a_1 \otimes \cdots \otimes a_n] \in \mathfrak{A}\) corresponding to \(a_1 \otimes \cdots \otimes a_n \in \mathfrak{A}_{\text{cl}}^n\) will be denoted by \(\hat{a}_1 \cdots \hat{a}_n\),
\[
\hat{a}_1 \cdots \hat{a}_n := [a_1 \otimes \cdots \otimes a_n]. \tag{2.15}
\]

In particular, we will be denoting
\[
\hat{\pi}(f) \in \mathfrak{A}, \quad \hat{h}_{k,x} = \left[ h_{k,x} \right] \in \mathfrak{A},
\]
and will be calling them the quantum momentum and, respectively, quantum position variable.

The basic commutators \([A, B] = AB - BA\) in the polymer \(*\)-algebra \(\mathfrak{A}\) are
\[
[h_{k,x}, \hat{\pi}(f)] = -kf(x)h_{k,x}, \tag{2.17}
\]
\[
[h_{k,x}, \hat{h}_{k',x'}] = 0 = [\hat{\pi}(f), \hat{\pi}(f)] \tag{2.18}
\]
for arbitrary \(x, x' \in M, k, k' \in \mathbb{R}\setminus\{0\}\) and \(\pi(f) \in \Pi_\mathcal{F}\).

Owing to the commutation relations, every element of the polymer \(*\)-algebra \(\mathfrak{A}\) corresponding to a momentum variable space \(\Pi_\mathcal{F}\) can be written as a complex linear combination of the elements of the form
\[
\prod_{i=1}^{n} \hat{h}_{k_i,x_i} \prod_{j=1}^{m} \hat{\pi}(f_j), \tag{2.19}
\]
where \(x_i \in M, k_i \neq 0\), and \(\pi(f_j) \in \Pi_\mathcal{F}\), and the cases either \(\prod_{i=1}^{n} \hat{h}_{k_i,x_i}\) or \(\prod_{j=1}^{m} \hat{\pi}(f_j)\) are included as corresponding to \(m = 0\) and, respectively, \(n = 0\).

This decomposition is not necessarily unique, though. To begin with the factors \(h_{k_i,x_i}\) commute among themselves as well as the factors \(\pi(f_j)\). The second source of the non-uniqueness is the identities satisfied by the momentum variables definition 2.3,
\[
\pi(a_1 f_1 + \cdots + a_k f_k) = a_1 \pi(f_1) + \cdots + a_k \pi(f_k), \tag{2.20}
\]
for every finite set of functions \(f_1, \ldots, f_k \in \mathcal{F}\) and real numbers \(a_1, \ldots, a_k\). On the other hand, there are no identities in the vector space \(\Phi\), that is,
\[
a_1 h_{k_i,x_i} + \cdots + a_k h_{k_i,x_i} = 0 \quad \Rightarrow \quad a_1 = \cdots a_k = 0,
\]
provided that \((k_i, x_i) \neq (k_j, x_j)\) for every \(i \neq j\).
2.4. States on \( \ast \)-algebras and GNS representations

This short subsection might be skipped by the reader familiar with the notion of states on \( \ast \)-algebras and the GNS construction.

Quantization of the basic position and momentum variables for the scalar field amounts to finding a representation of the quantum algebra \( \mathfrak{A} \) of the basic variables in the space of operators defined in a Hilbert space. In this paper, we will be concerned with the so-called GNS (Gel’fand–Naimark–Segal) representations defined by a non-negative linear functional on \( \ast \)-algebra \( \mathfrak{A} \).

Recall that

**Definition 2.8.** A state on a unital \( \ast \)-algebra \( \mathfrak{A} \) is a non-trivial linear functional\( \omega : \mathfrak{A} \to \mathbb{C} \), such that for every \( \hat{a} \in \mathfrak{A} \)

\[
\omega(\hat{a}^*\hat{a}) \geq 0, \quad \text{and} \quad \omega(\hat{1}) = 1, \tag{2.22}
\]

where \( \hat{1} \) stands for the unity element.

In fact, if we do not assume the equality above, it follows from the inequality (the positivity condition) itself, that

\[
\omega(\hat{a}^*) = \omega(\hat{a}), \quad \text{and} \quad \omega(\hat{1}) > 0. \tag{2.23}
\]

Therefore, given the positivity, by rescaling we can always achieve the equality in definition 2.8.

Given a state on \( \mathfrak{A} \), we construct the corresponding GNS representation

\[
(\mathcal{H}_0^\omega, \langle \cdot, \cdot \rangle_\omega, \rho_\omega, \Omega_\omega),
\]

where \( \mathcal{H}_0^\omega \) is a vector space endowed with a unitary scalar product \( \langle \cdot, \cdot \rangle_\omega \), \( \rho_\omega \) is a representation of \( \mathfrak{A} \) on \( \mathcal{H}_0^\omega \) and \( \Omega_\omega \) is a vector in \( \mathcal{H}_0^\omega \) which, when viewed as a state on \( \mathfrak{A} \), coincides with \( \omega \) (see equation 2.30). A detailed exposition of the GNS construction for algebras of unbounded operators can be found, for example, in [24]. Here we will only need the following elements and properties that are easy to prove:

(i) The vector space \( \mathcal{H}_0^\omega \) is obtained as the quotient of \( \mathfrak{A} \) considered as a vector space

\[
\mathcal{H}_0^\omega := \mathfrak{A}/J_\omega \tag{2.24}
\]

where the vector subspace

\[
J_\omega := \{ \hat{a} \in \mathfrak{A} : \omega(\hat{a}^*\hat{a}) = 0 \} \tag{2.25}
\]

is a left ideal in \( \mathfrak{A} \).

(ii) The unitary scalar product \( \langle \cdot, \cdot \rangle_\omega \) is defined in \( \mathcal{H}_0^\omega \) by the state \( \omega \), namely

\[
\langle \hat{a}, \hat{b} \rangle_\omega := \omega(\hat{a}^*\hat{b}), \tag{2.26}
\]

where for every \( \hat{c} \in \mathfrak{A} \), \( [\hat{c}] \in \mathfrak{A}/J_\omega \) stands for the equivalence class defined by \( C \).

(iii) To every element \( \hat{a} \) of \( \mathfrak{A} \) we assign a linear (but in general unbounded) operator \( \rho_\omega(\hat{a}) \), defined in the entire vector space \( \mathcal{H}_0^\omega \),

\[
\rho_\omega(\hat{a}) : \mathcal{H}_0^\omega \to \mathcal{H}_0^\omega \tag{2.27}
\]

\[
\rho_\omega(\hat{a})[\hat{b}] := [\hat{a}\hat{b}]. \tag{2.28}
\]
(iv) The vector \( \Omega_\omega \) is
\[
\Omega_\omega = [\hat{1}],
\] (2.29)
and its relation with the state \( \omega \) is
\[
\omega(\hat{a}) = (\Omega_\omega, \rho_\omega(\hat{a})\Omega_\omega)_{\omega},
\] (2.30)
for every \( \hat{a} \in \mathfrak{A} \).

(v) The map \( \rho_\omega : \mathfrak{A} \to \text{End}(\mathcal{H}_0^\omega) \) is a homomorphism of the associative algebra \( \mathfrak{A} \) into the associative algebra of the endomorphisms of the vector space \( \mathcal{H}_0^\omega \).

(vi) The map \( \rho_\omega \) is consistent with the *-structure of \( \mathfrak{A} \) in the following way,
\[
(\rho_\omega(\hat{a}^\ast)[\hat{b}], [\hat{c}])_{\omega} = ([\hat{b}], \rho_\omega(\hat{a})[\hat{c}])_{\omega},
\] (2.31)
for every \( \hat{a}, \hat{b}, \hat{c} \in \mathfrak{A} \).

A representation may be equivalent to a GNS representation. For the sake of precision, we spell out the definition of equivalence:

**Definition 2.9.** Given \((V, (\cdot | \cdot), \rho, v_0)\), where (a) \( V \) is a vector space endowed with a unitary form \((\cdot | \cdot)\), (b) \( \rho \) is an associative epimorphism \( \rho : \mathfrak{A} \to \text{End}(V) \) of a *-algebra \( \mathfrak{A} \) into the endomorphism algebra of \( V \), and (c) \( v_0 \in V \), we say that \((V, (\cdot | \cdot), \rho, v_0)\) is equivalent to a given GNS representation \((\mathcal{H}_0^\omega, (\cdot, \cdot)_\omega, \rho_\omega, \Omega_\omega)\) whenever there exists a unitary vector space isomorphism \( \mathcal{I} : \mathcal{H}_0^\omega \to V \), such that
\[
\rho_\omega = \mathcal{I}^{-1} \circ \rho \circ \mathcal{I}, \quad \text{and} \quad \mathcal{I}(\Omega_\omega) = v_0.
\]
Note that the notion of the Hilbert space is not necessary when one deals with the GNS algebras, as long as we are interested in the elements of the algebra \( \mathfrak{A} \) only.

However, as for every unitary space, one can consider the Hilbert space
\[
\mathcal{H}_\omega := \overline{\mathcal{H}_0^\omega}.
\]
From this point of view, a difference between the current *-algebra case and any other case of a \( C^\ast \)-algebra is that since a general operator \( \rho_\omega(\hat{a}) \) is unbounded, an equality
\[
\hat{a}^\ast = \hat{a}
\] (2.32)
if it takes place, does not imply that \( \rho_\omega(\hat{a}) \) acting in the domain \( \mathcal{H}_0^\omega \) is essentially self-adjoint. If the algebra \( \mathfrak{A} \) is commutative then, remarkably, (2.32) implies that the operator \( \rho_\omega(\hat{a}) \) admits a self-adjoint extension. Indeed, the operator commutes with the anti-linear operation
\[ *
\mathcal{H}_\mu \to \mathcal{H}_\mu \]
induced by the *-structure of \( \mathfrak{A} \), in the commutative case. In general, the self-dual extension is not unique. However, due to the algebra of the problem, this unambiguity does not lead to any inconsistencies. Of course, given a *-algebra, it is always interesting to ask which states define the GNS representations such that the real elements of the algebra correspond to the essentially self-adjoint operators.
2.5. Additional assumption about the representations of $\mathfrak{A}$

In this paper, we apply the GNS construction to the polymer $\ast$-algebra $\mathfrak{A}$ (definition 2.7).

Throughout this paper, we will be assuming about a state $\omega$ on $\mathfrak{A}$ that the corresponding GNS representation $\rho_\omega$ satisfies the following property:

**Property 2.10.** For every point $x \in M$, the set of operators

$$U_x = \{ \rho_\omega(\hat{h}_{k,x}) : k \in \mathbb{R} \},$$

(2.33)

where $h_{0,x}$ stands for the algebra unity element $\hat{1}$, is a group (with the operation of the composition), and

$$\rho_\omega(\hat{h}_{k,x})\rho_\omega(\hat{h}_{k',x}) = \rho_\omega(\hat{h}_{k+k',x}).$$

(2.34)

This assumption reflects the property of the exponentiated evaluation functions

$$e^{i\phi(x)} e^{i\hat{a}(x)} = e^{i(k+k')\phi(x)}.$$

(2.35)

It is analogous to the property

$$\rho_\omega(\hat{\pi}(f_1)) + \rho_\omega(\hat{\pi}(f_2)) = \rho_\omega(\hat{\pi}(f_1 + f_2)),$$

(2.36)

already satisfied as an identity due to the definition of the Lie algebra of the basic variables.

It follows from (2.34) that for every polymer position variable $h_{k,x}$, the operator

$$\rho_\omega(\hat{h}_{k,x}) : H_\omega^0 \to H_\omega^0$$

(2.37)

is the unitary form preserving and bijective, therefore it is extendable by the continuity to a unitary operator in the Hilbert space $H_\omega$. Indeed, it follows that

$$\rho_\omega(\hat{h}_{k,x})\rho_\omega(\hat{h}_{k,x}) = \rho(\hat{1}) = \text{id} = \rho_\omega(\hat{h}_{k,x})\rho_\omega(\hat{h}_{k,x}),$$

(2.38)

and every $k,k' \in \mathbb{R}$, $y \in M$.

We will be assuming the condition (2.34) in this paper. This is equivalent to considering states on the $\ast$-algebra

$$\tilde{\mathfrak{A}} = \mathfrak{A}/\tilde{\mathfrak{J}},$$

(2.41)

where $\tilde{\mathfrak{J}}$ is the two-sided ideal generated by the set

$$\{ \hat{h}_{k,y} - \hat{h}_{k+k',y} : y \in M, k, k' \in \mathbb{R} \}.$$

(2.42)

The canonical epimorphism

$$\mathfrak{A} \to \tilde{\mathfrak{A}}$$

is used to introduce the following notation for elements of $\tilde{\mathfrak{A}}$,

$$\tilde{a} \mapsto \tilde{a}, \quad \hat{h}_{k,x} \mapsto \tilde{\hat{h}}_{k,x}, \quad \hat{\pi}(f) \mapsto \tilde{\hat{\pi}}(f),$$

(2.43)

where $\tilde{1}$, in particular, stands for the unity in $\tilde{\mathfrak{A}}$. A technical observation extensively used below is that every element of the quotient algebra $\tilde{\mathfrak{A}}$ can be represented by a finite linear
combination of elements (2.19) such that \( x_i \neq x_j \) for every pair \( i \neq j \) and the hats are replaced by tildes. In other words, for every finitely supported function \( k: \mathcal{M} \rightarrow \mathbb{R} \) denote

\[
\hat{h}_k = \prod_{x \in \mathcal{M}} \tilde{h}_{x,k(x)}.
\]

The algebra \( \mathfrak{A} \) is spanned by elements of the form

\[
\hat{h}_k, \quad \text{and} \quad \hat{h}_k \prod_{j=1}^{m} \hat{\pi}(f_j),
\]

with all possible \( k, m \in \mathbb{N}, \) and \( f_1, \ldots, f_m \in \mathcal{F}. \)

Whereas, upon property 2.10, each of the operators \( \rho_\omega(\hat{h}_{k,x}) \) will be unitary; operators \( \rho_\omega(\hat{\pi}(f)) \) may or may not be essentially self-adjoint, depending on a state \( \omega. \)

2.6. Symmetries

Suppose that \( \mathfrak{A} \) is the polymer \( * \)-algebra corresponding to a momentum variable space \( \Pi_\mathcal{F} \) (definition 2.4) preserved by the group \( \text{Diff}(M) \) of the diffeomorphisms of \( M \) (of a given class). Every diffeomorphism \( \varphi \in \text{Diff}(M) \) acts on the pair representing the classical scalar field \( (\phi, \pi), \)

\[
\phi \mapsto (\varphi^{-1})^* \phi, \quad \pi \mapsto \varphi_* \pi,
\]

(note that \( \pi, \) as a signed measure, is pushed forward by \( \varphi \) rather than pulled back by its inverse). The action passes naturally to the polymer Lie algebra and next to the polymer \( * \)-algebra. \( \varphi \) defines a \( * \)-automorphism of \( \mathfrak{A}, \)

\[
\sigma_\varphi: \mathfrak{A} \rightarrow \mathfrak{A} \quad \text{(2.47)}
\]

\[
\sigma_\varphi(\hat{h}_{k,x}) = \hat{h}_{k,\varphi(x)} \quad \text{(2.48)}
\]

\[
\sigma_\varphi(\hat{\pi}(f)) = \hat{\pi}((\varphi^{-1})^* f) \quad \text{(2.49)}
\]

The automorphism \( \sigma_\varphi, \) as any automorphism of \( \mathfrak{A}, \) acts naturally in the space of the states defined on \( \mathfrak{A}. \) A state \( \omega \) invariant with respect to that action of the diffeomorphism group \( \text{Diff}(M) \) is called \textit{diffeomorphism invariant.}

Moreover, consider an arbitrary \textit{homeomorphism} \( \varphi. \) In none of the cases considered in this work the space of the smearing functions will be preserved by all the homeomorphisms. However, it may be true that a given momentum variable \( \pi(f) \in \Pi_\mathcal{F}, \) the smearing function \( f \) is mapped by \( \varphi^{-1} \) into a function \( (\varphi^{-1})^* f \) such that still \( (\varphi^{-1})^* f \in \mathcal{F}. \) The smearing functions of this property form a vector subspace of \( \mathcal{F}. \) Denote by \( \Pi_\varphi \) the corresponding vector subspace of \( \mathcal{F}. \) Explicitly,

\[
\Pi_\varphi = \{ f \in \mathcal{F} \mid (\varphi^{-1})^* f \}\]

It comes with the polymer \( * \)-algebra, say \( \mathfrak{A}_\varphi, \) a subalgebra of \( \mathfrak{A}. \) It is obvious how to generalize the definition of the invariance of a state \( \omega \) to the homeomorphisms. The formulae (2.48), (2.49) define a homomorphism

\[
\sigma_\varphi: \mathfrak{A}_\varphi \rightarrow \mathfrak{A} \quad \text{(2.50)}
\]

**Definition 2.11.** A state \( \omega \) defined on the algebra \( \mathfrak{A} \) is called \textit{homeomorphism invariant} if for every homeomorphism \( \varphi \)

\[
\omega(\sigma_\varphi(a)) = \omega(a) \quad \text{(2.51)}
\]

for every \( a \in \mathfrak{A}_\varphi. \)
Certainly, a homeomorphism invariant state on the polymer ∗-algebra $\mathfrak{A}$ is also diffeomorphism invariant. For completeness, let us explain the relation between the diffeomorphism invariance of a state and the diffeomorphism covariance of the corresponding representation. A diffeomorphism invariant state $\omega$ on a polymer ∗-algebra $\mathfrak{A}$ defines a unitary representation $U$ of the diffeomorphism group $\text{Diff}(M)$ in the unitary space $H_\omega$, (2.24)

$$U(\phi)\hat{a} = [\phi^*\hat{a}].$$

The representation $\rho_\omega$ is diffeomorphism covariant, in the sense

$$U(\phi)\rho(\hat{a})U(\phi)^{-1} = \rho(U(\phi)\hat{a})$$

for every $\hat{a} \in \mathfrak{A}$ and every $\phi \in \text{Diff}(M)$.

In the next section, in the case of the momentum variable space $\Pi^{\mathbb{C}^m_0(\pi)}$ we will show that a $\mathbb{C}^m_0$-diffeomorphism invariant state on $\mathfrak{A}$ is unique. This is the known one used in [21]. That state is also homeomorphism invariant.

In sections 4–9 on the other hand, we will consider the momentum variable space $\Pi_\mathcal{F}$ spanned by characteristic functions of regions in $M$. In that case, we will additionally assume the homeomorphism invariance in order to derive a class of new states invariant with respect to the piecewise-analytic diffeomorphisms.

3. $\mathfrak{A}$ defined by $\mathbb{C}^{m_0}_0(M, \mathbb{R})$ smearing functions

In this section, we choose the space $\mathcal{F}$ of the smearing functions to be

$$\mathcal{F} = \mathbb{C}^{m_0}_0(M, \mathbb{R}),$$

that is, the space of the real $\mathbb{C}^{m_0}$-functions of compact support and defined on $M$, where the differentiability class is fixed to be either an integer $m_0 \geq 0$, or $m_0 = \infty$, or $\mathbb{C}^{m_0}$ stands for the semi-analytic functions according to [23].

We consider the polymer ∗-star algebra $\mathfrak{A}$ corresponding to the momentum variable space $\Pi_\mathcal{F}$ as in definition 2.4 with the choice 3.1 of $\mathcal{F}$. We also identify $\Pi_\mathcal{F}$ and $\mathcal{F}$ via the vector space isomorphism (2.7).

In this section, the group $\text{Diff}(M)$ considered in section 2.6 is the group $\text{Diff}^{m_0}(M)$ of all the $\mathbb{C}^{m_0}$-diffeomorphisms of $M$. It turns out that the diffeomorphism invariance and property (2.10) determine a state $\omega$ on $\mathfrak{A}$ completely. The result and the proof we present below are analogous to the result of [23], but much simpler.

**Theorem 3.1.** Suppose $\mathfrak{A}$ is the polymer ∗-algebra (definition 2.7) defined by $\mathcal{F} = \mathbb{C}^{m_0}_0(M, \mathbb{R})$. On $\mathfrak{A}$, there exists exactly one $\mathbb{C}^{m_0}$-diffeomorphism invariant state $\omega_0$ such that the condition (2.40) (equivalent to property 2.10) holds. The state is defined by

$$\omega_0(\hat{a}\hat{\pi}(f)) = 0 = \omega_0(h_{k_1,x_1} \cdots h_{k_n,x_n}),$$

(3.2)

for every $\hat{a} \in \mathfrak{A}$, every momentum variable $\pi(f) \in \Pi_\mathcal{F}$, and every set of labels $\{(k_1,x_1), \ldots, (k_n,x_n)\}$ such that $x_i \neq x_j$ for every $i \neq j$, for all $i = 1, \ldots, n$ and every $n \in \mathbb{N}$.

**Proof.** First, we show that if $\omega_0$ is a diffeomorphism invariant state such property (2.10) is true, then it satisfies (3.2). The main step shows that

$$\omega_0(\hat{\pi}(f)^*\hat{\pi}(f)) = 0,$$

(3.3)

for every $f \in \mathbb{C}^{m_0}_0(M, \mathbb{R})$. The trick is to note that
Lemma 3.2. For every $f \in C_0^m(M, \mathbb{R})$ there exists $g \in C_0^m(M, \mathbb{R})$ and $\epsilon > 0$ such that for every $-\epsilon < \lambda < \epsilon$, there is a $C_0^m(M, \mathbb{R})$-diffeomorphism $\phi_\lambda : M \to M$, such that
\[
g + \lambda f = \phi_\lambda^* g.
\] (3.4)

See [23] for the construction of $\phi_\lambda$.

That is to say, $g + \lambda f$ is just a deformation of $g$, diffeomorphically trivial. Then the diffeomorphism invariance implies
\[
\omega_0(\hat{\pi}(g)\hat{\pi}(f)) = \omega_0(\hat{\pi}(g + \lambda f)\hat{\pi}(g + \lambda f))
\]
\[
= \omega_0(\hat{\pi}(g)\hat{\pi}(f)) + 2\lambda\omega_0(\hat{\pi}(f)\hat{\pi}(f)) + \lambda^2\omega_0(\hat{\pi}(f)\hat{\pi}(f)),
\] (3.5)
for every value $\lambda \in ]-\epsilon, \epsilon[$. Hence, the terms proportional to $\lambda$ and $\lambda^2$ vanish. (Recall also that $\hat{\pi}(f)^* = \hat{\pi}(f)$.) This implies the first equality in (3.2). Note that it also implies
\[
\omega_0(\hat{\pi}(f)A) = \omega_0(A^* \hat{\pi}(f)) = 0,
\] (3.6)
for every $A \in \mathfrak{A}$ and every $\pi(f) \in \Pi_F$.

Now, consider an element $\hat{h} := \hat{h}_{k_1,x_1} \ldots \hat{h}_{k_n,x_n} \in \mathfrak{A}$. Certainly, for every $f \in C_0^m(M, \mathbb{R})$
\[
0 \omega_0(\hat{h}\hat{\pi}(f) - \hat{\pi}(f)\hat{h}) = \left( \sum_{i=1}^n k_i f(x_i) \right) \omega_0(h).
\] (3.7)

If $\hat{h}$ satisfies the assumption in the above equation, then there is $f \in C_0^m(M, \mathbb{R})$ such that the coefficient at $\omega_0(\hat{h})$ in the equation above is not zero, hence the second equality in (3.2) is necessarily true. Hence, we have shown that (3.2) is a necessary condition.

Now, we show that if $\omega_0$ satisfies property (2.10) and (3.2), then it is unique. Note that the only elements of $\mathfrak{A}$ that have not been taken into account in (3.2) are the linear combinations of elements $\hat{h}_{k_1,x_1} \ldots \hat{h}_{k_n,x_n}$ such that there is $i, j \in \{1, \ldots, k\}$ such that
\[
i \neq j, \quad \text{and} \quad y_i = y_j = x.
\] (3.8)

However, equations (3.2), (2.40) and the second equation in (2.8) determine $\omega_0$ even on those elements of $\mathfrak{A}$.

Summarizing, we have shown that if $\omega_0$ is a diffeomorphism invariant state such that property (2.10) is satisfied, then it satisfies (3.2) and is unique.

It is easy to construct a state $\omega_0$ by using (2.34), (3.2). The conditions (2.34), (3.2) define a state on the algebra $\mathfrak{A}$ (see 2.41). The state $\omega_0$ is the pullback of that state to $\mathfrak{A}$; hence it exists. Obviously, it is diffeomorphism invariant.

The GNS representation $\rho_{\omega_0}$ defined by the state $\omega_0$ can be derived explicitly. The first step is to find a convenient notation for the elements of the quotient vector space $H_0^{\omega_0}$.

Every non-zero element of $[\hat{a}] \in \mathfrak{A}/J_{\omega_0}$ can be defined by $\hat{a} \in \mathfrak{A}$ of the following form,
\[
\hat{a} = \prod_{i=1}^n \hat{h}_{k_i,x_i},
\]
and can be labelled by a finitely supported function $k : M \to \mathbb{R}$,
\[
k = \sum_{i=1}^n k_i \mathbf{1}_{[x_i]},
\] (3.9)
where given $x \in M$, $\mathbf{1}_{[x]}$ is the characteristic function of $x$. In particular, the identically zero function labels the element $[\hat{1}] \in \mathfrak{A}/J_{\omega_0}$. Therefore we will write
\[
[k] := \prod_{x \in M} \hat{h}_{k(x), x}
\] (3.10)
where $k : M \to \mathbb{R}$ has a finite support.
Therefore, $\mathcal{H}^{0}_{\omega_0}$ is in this case the complex vector space $\text{Cyl}$ of the finite formal linear combinations of elements $|k\rangle$ labelled by all the functions $k : M \to \mathbb{R}$ of finite support,

$$\mathcal{H}^{0}_{\omega_0} = \text{Cyl} = \left\{ \sum_{i=1}^{n} a_i |k_i\rangle : n \in \mathbb{N}, a_i \in \mathbb{C}, k_i : M \to \mathbb{R} \text{ finitely supported} \right\}. \quad (3.11)$$

Now, the GNS unitary scalar product $(\cdot, \cdot)_{\omega_0}$ defined in $\mathcal{H}^{0}_{\omega_0}$ by the state $\omega_0$ is the following unitary scalar product $(\cdot | \cdot)_{\text{Cyl}}$:

$$(|k_1\rangle | |k_2\rangle)_{\text{Cyl}} = \begin{cases} 1, & \text{if } k_1 = k_2 \\ 0, & \text{if } k_1 \neq k_2. \end{cases} \quad (3.12)$$

Finally, the action $\rho_{\omega_0} : \mathfrak{A} \to \text{End(}\text{Cyl})$ defined on the generators of $\mathfrak{A}$ and the basis in $\text{Cyl}$ can be derived,

$$\rho_{\omega_0}(\hat{h}_{k',x})(|k\rangle) = \left[ \hat{h}_{k',x} \prod_{x \in M} \hat{h}_{k(x),x} \right] |k + k'|_{|1_x\rangle} \quad (3.13)$$

$$\rho(\hat{\pi}(f))(|k\rangle) = \left[ \hat{\pi}(f) \prod_{x \in M} \hat{h}_{k(x),x} \right] = \sum_{x \in M} k(x) f(x) |k\rangle. \quad (3.14)$$

where the product (sum) has only finitely many non-unit (non-zero) factors (terms).

Some more remarks are in order. That unique state is used in LQG for the quantum scalar field coupled with the quantum geometry [20, 21]. The symmetry group of the quantum geometry defined by LQG is the group of semi-analytic diffeomorphisms. As it was shown in [23], lemma 3.2 continues to be true if the $C^{m_0}$ differentiability class is replaced by the semi-analyticity assumption defined therein. Therefore the theorem is true in that case as well.

The representation $\rho_{\omega_0}$ is referred to in the literature as the polymer representation. For the same polymer $\ast$-algebra $\mathfrak{A}$ one may also define the standard Fock state of the scalar field. The comparison between the polymer representation and the Fock representation was discussed in [21].

An important sublety of theorem 3.1 is that both, the smearing functions used in the definition of the basic momentum variables, and the diffeomorphisms $M \to M$ are assumed to be of the same class $C^{m_0}$. The reason is that in the proof of lemma 3.2 (see [23]), the suitable diffeomorphism is constructed from a given smearing function. However, if we fix the differentiability class of the diffeomorphism-symmetry group as $C^{m_0}$, but consider the smearing functions of the class $C^{m'}$ with $m' < m_0$, the proof fails, and, in fact, the theorem is not true. (It is easy to construct a counterexample by using the counterexample to the uniqueness concerning the quantum algebra of the holonomy-flux observable found in [23]).

In the current work, our aim is to use that observation in investigation of the issue of the existence of new diffeomorphism invariant states.

A natural alternative to the $C^{m_0}$ condition used in theorem 3.1 is smearing the momentum variables against the characteristic functions of subsets in $M$. (This is also one of the choices typically used in LQG [20].) With this choice of the space $\mathcal{F}$ in definition 2.3, we will be able to construct a family of new states.
4. The CW-complex ∗-algebra

4.1. The CW-complex vector space \( \mathcal{F}_S \)

4.1.1. Definitions, the Euler characteristics. We assume in this section that the manifold \( M \) is endowed with a piecewise-analytic structure, and consider piecewise-analytic \( k \)-simplexes in \( M \); both notions are precisely defined in the appendix. In particular, if \( S \) is a \( k \)-simplex in \( M \), then, depending on \( k, S \):

- \( 0 \): a 1-point subset of \( M \)
- \( 1 \): an ‘open’ interval embedded in \( M \)
- \( N \): an open ball in \( \mathbb{R}^{\dim M} \) embedded in \( M \) with an inverse local chord, if the \( k = \dim M \).

In the last case \( S \) is referred to just as a ball in \( M \). In the general case \( S \) is referred to just as a simplex (we skip ‘piecewise-analytic’).

**Definition 4.1.** Denote by \( S \) the set of all the \( k \)-simplexes in \( M \), where \( k \) ranges from 0 to \( \dim M \).

Given a subset \( U \subseteq M \), by the characteristic function \( 1_U \) we mean the function

\[
1_U : M \to \mathbb{R} \quad (4.1)
\]

\[
1_U(x) = \begin{cases} 1, & \text{if } x \in U \\ 0, & \text{if } x \not\in U. \end{cases} \quad (4.2)
\]

**Definition 4.2.** The CW-complex vector space \( \mathcal{F}_S \) is the real vector space spanned by the linear combinations of the characteristic functions \( 1_S \) of all the \( k \)-simplexes \( S \) in \( M \):

\[
\mathcal{F}_S = \left\{ \sum_{i=1}^{n} a_i 1_{S_i} : n \in \mathbb{N}, a_i \in \mathbb{R}, S_i \in S \right\}.
\]

The key identities satisfied by the characteristic functions valid for arbitrary subsets \( U_1, U_2 \subseteq M \) are

\[
1_{U_1} + 1_{U_2} - 1_{U_1 \cup U_2} = 0 \quad (4.3)
\]

\[
1_{U_1 \setminus U_2} - 1_{U_1} + 1_{U_1 \cap U_2} = 0. \quad (4.4)
\]

Due to them, despite the fact that we have used only the simplexes to define the space \( \mathcal{F}_S \), there are many other subsets \( U \subseteq M \), such that the characteristic function \( 1_U \in \mathcal{F}_S \).

Indeed, this is the case whenever \( U \) is triangulable (see the appendix).

As a warm up, and to introduce the Euler characteristic, consider a linear map

\[
\chi : \mathcal{F}_S \to \mathbb{R} \quad (4.5)
\]

and assume that it is homeomorphism invariant in the sense that the number \( \chi(1_S) \) depends only on the dimension of simplex \( S \). It is easy to see that this map is uniquely determined by the value of \( \chi(S_0) \) where \( S_0 \) is a 0-simplex, meaning a 1-point subset of \( M \). Indeed, if \( S_1 \) is a 1-simplex, then we can always split it into a disjoint union of two 1-simplexes and one 0-simplex, and consider the corresponding identity between the characteristic functions

\[
S_1 = S_1' \cup S_0' \cup S_0''
\]

\[
1_{S_1} = 1_{S_1'} + 1_{S_0'} + 1_{S_0''}. \quad (4.6)
\]
The linearity and the homeomorphism invariance of $\chi$ applied to the identity imply
\[ \chi(1_S) = -\chi(1_S). \]
Repeating exactly the same calculation for 2-simplex up to dim $M$-simplex, one can convince oneself that for every $k$-simplex $S_k$,
\[ \chi(1_S) = \mu_0 (-1)^k, \quad (4.7) \]
where $\mu_0$ is an arbitrary constant. The map
\[ \mathcal{S} \ni S_k \mapsto \chi_E(S_k) = (-1)^k \in \mathbb{R}, \quad (4.8) \]
where $k$ is the dimension of $S_k$, is called the Euler characteristic.

Thus far we have derived a necessary condition determining the possibly existing linear map $\chi : \mathcal{F}_S \to \mathbb{R}$ up to constant $\mu_0$. Another argument is needed to show that the linear map $\chi$ exists at all. The existence relies on two facts. The first one is that for every $n$-tuple of the simplexes $S_1, \ldots, S_n \in \mathcal{S}$ there is a single triangulation, which is a finite subset $\mathcal{S}' \subset \mathcal{S}$ of pairwise disjoint simplexes in $\mathcal{M}$ such that
\[ S_i = \bigcup_{j=1}^{n_i} S'_{ij}, \quad S'_{ij} \in \mathcal{S}' \]
for every $i = 1, \ldots, n$, and $j = 1, \ldots, n_i$. The second fact is that the Euler characteristic is triangulation independent; that is, for every simplex $S$ and its triangulation that is a decomposition
\[ S = \bigcup_{i=1}^{n} S_i \]
into the union of pairwise disjoint simplexes $S_1, \ldots, S_n$, the identity holds
\[ \chi_E(S) = \sum_{i=1}^{n} \chi_E(S_i). \]

We hope it will not lead to any confusion if we denote by
\[ \chi_E : \mathcal{F}_S \to \mathcal{F}_S \]
the linear extension of the Euler characteristic, that is, the map (4.8) normalized by the condition
\[ \mu_0 = 1, \]
for every 0-simplex in $\mathcal{M}$. We keep calling the extension $\chi_E : \mathcal{F}_S \to \mathbb{R}$ the Euler characteristic functional.

4.1.2. *Homeomorphism invariant, symmetric, bilinear forms.* On the CW-complex vector space $\mathcal{F}_S$, consider a symmetric, bilinear form
\[ \langle \cdot | \cdot \rangle : \mathcal{F}_S \times \mathcal{F}_S \to \mathbb{R}, \]
non-negative, such that for every $s \in \mathcal{F}_S$
\[ \langle s | s \rangle \geq 0. \]

We also assume that the form assigns to a pair of simplexes $S_1, S_2 \in \mathcal{S}$ a number $\langle 1_{S_1} | 1_{S_2} \rangle$ depending only on the homeomorphic characteristics of the pair $\{S_1, S_2\}$. Precisely, we assume that $\langle \cdot | \cdot \rangle$ is homeomorphism invariant in the following sense:
Definition 4.3. A bilinear form $(\cdot|\cdot) : \mathcal{F}_S \times \mathcal{F}_S \to \mathbb{R}$ is called homeomorphism invariant if for every pair of simplexes $S_1, S_2 \in S$ and every homeomorphism $\phi : M \to M$ such that $\phi(S_1), \phi(S_2) \in S$, the form satisfies

$$(1_{\phi(S_1)}|1_{\phi(S_2)}) = (1_{S_1}|1_{S_2}).$$

(4.9)

In this section we will find all the homeomorphism invariant, bilinear, symmetric, non-negative forms on the CW-complex vector space $\mathcal{F}_S$.

The key step is the following lemma:

Proposition 4.4. Suppose that $\dim M > 1$ and $M$ is connected. Suppose $(\cdot|\cdot) : \mathcal{F}_S \times \mathcal{F}_S \to \mathbb{R}$ is a homeomorphism invariant symmetric, bilinear, non-negative form. Then, for any two balls $B_0, B_1 \in S$

$$(1_{B_1} - 1_{B_0}|1_{B_1} - 1_{B_0}) = 0.$$  \hspace{1cm} (4.10)

Proof. Consider first the special case of the balls, when $B_0 \subset B_1$ and the difference $B_1 \setminus B_0$ is a tube; that is, there is a local coordinate system $(x_1, \ldots, x_N)$ in $M$ such that $B_1 \setminus B_0 = R = \{(x^1, \ldots, x^N) \mid r_0^2 \leq (x^1)^2 + (x^2)^2 < r_1^2, |x_i| < H, 2 < i < N\}$. \hspace{1cm} (4.11)

We have $1_{B_1} - 1_{B_0} = 1_R$, therefore, in particular $1_R \in \mathcal{F}_S$. \hfill $\square$

Lemma 4.5. Suppose $R \subset M$ is a tube defined in a local coordinate system by (4.11). Suppose $M$ and $(\cdot|\cdot)$ satisfy the assumptions of proposition 4.4. Then,

$$(1_R|1_R) = 0.$$ \hspace{1cm} (4.12)

Proof. The trick consists in cutting out the tube $R$ into $n$ disjoint, homeomorphic pieces

$$R = R_1^{(n)} \cup \cdots \cup R_n^{(n)},$$

such that each two pairs of pairs $(R, R_i^{(n)})$ and $(R, R_j^{(n)})$ are homeomorphic to each other and

$$1_{R_i^{(n)}} \in S$$

for every $i = 1, \ldots, n$. Indeed, we can choose some $n \in \mathbb{N}$ and using the same coordinates define

$$R_i^{(n)} = \left\{(x^1, \ldots, x^N) \in R : x^1 = r \cos \phi, x^2 = r \sin \phi, r \in \mathbb{R}, (I - 1) \frac{2\pi}{n} \leq \phi < I \frac{2\pi}{n} \right\}.$$  \hspace{1cm} (4.13)

Then,

$$1_R = 1_{R_1^{(n)}} + \cdots + 1_{R_n^{(n)}}, \quad \text{and} \quad 1_{R_i^{(n)}} \in \mathcal{F}_S$$

for every $i = 1, \ldots, n$. Now, the number in question can be written in the following form,

$$(1_R|1_R) = n(1_R|1_{R_1^{(n)}})$$  \hspace{1cm} (4.12)

where we have used the homeomorphism invariance of the form $(\cdot|\cdot)$. The left-hand side of the equality is independent of $n$. The right-hand side obviously depends on $n$. However, the

$\text{It is useful to have in mind the identification of a ball with a cylinder.}$
factor \((1_R|1_{R'}^n)\), again due to the homeomorphism invariance, is \(n\) independent! The only solution of equation (4.12) valid for arbitrary \(n\) is
\[
(1_R|1_R) = 0. \tag{□}
\]

In the continuation of the proof of proposition 4.4 we will use the known fact that given a symmetric, bilinear and non-negative form in a vector space, the ‘zero norm’ vectors form a vector space, that is, in our case
\[
(s|s) = 0, \quad \text{and} \quad (s'|s') = 0 \quad \Rightarrow \quad (s + as'|s + as') = 0
\]
for every \(a \in \mathbb{R}\).

**Lemma 4.6.** Suppose \(B_0\) and \(B_0'\) are two disjoint balls in \(M\) (that is, \(B_0, B_0' \subseteq \mathcal{S}\) are simplexes of the maximal dimension). Suppose \(M\) and \((\cdot | \cdot)\) satisfy the assumptions of proposition 4.4. Then,
\[
(1_{B_0} - 1_{B_0'}|1_{B_0} - 1_{B_0'}) = 0.
\]

**Proof.** Given the balls \(B_0\) and \(B_0'\), there is a single ball \(B_1\) such that (i) \(B_0 \subseteq B_1\) and \(R = B_1 \setminus B_0\) is a tube, and (ii) \(B_0' \subseteq B_1\) and \(R' = B_1 \setminus B_0'\) is a tube. (The notion of ‘tube’ is defined by (4.11).) We can write
\[
1_{B_0} - 1_{B_0'} = 1_{B_1} - 1_{B_0} + 1_{B_1} - 1_{B_0'} = -1_R + 1_{R'}.
\]
The application of lemma 4.5 concludes the proof of lemma 4.6. \(\tag{□}\)

Finally, suppose \(B_1, B_2 \subseteq M\) are arbitrary two balls in \(M\). Let \(B_3\) be a ball in \(M\) disjoint from \(B_1 \cup B_2\). We have
\[
1_{B_1} - 1_{B_2} = 1_{B_1} + 1_{B_3} - 1_{B_2}.
\]
and lemma 4.6 applies to the first difference as well as to the second one. Hence, proposition 4.4 follows.

The conclusion from proposition 4.4 is that if we fix a ball \(B_0\) in \(M\), then for every other ball \(B\) and every \(s \in \mathcal{F}_S\),
\[
(1_B|s) = (1_{B_0}|s)
\]
provided the assumptions of proposition 4.4 are satisfied.

It follows that the form \((\cdot | \cdot)\) can be determined via proposition 4.4 up a normalization factor. To see it, note first

**Lemma 4.7.** For every simplex \(S\) in \(M\), there exist balls \(B_1, \ldots, B_k\), such that
\[
1_S = \sum_{i=1}^n m_i 1_{B_i}, \tag{4.13}
\]
where \(m_1, \ldots, m_n\) are integers such that
\[
\chi_E(S) = (-1)^{\dim M} \sum_{i=1}^n m_i. \tag{4.14}
\]

**Proof.** The formula (4.14) follows directly from the application of the Euler characteristic functional to (4.13). Given a \(k\)-simplex \(S\) in \(M\) such that \(k < \dim M\), it is easy to see that there
are three $k + 1$-simplexes $S'$, $S''$ and $S'''$ in $M$ such that $1_s = 1_{s'} - 1_{s''} - 1_{s'''}$ and all the terms on the right-hand side are characteristic functions of $k + 1$-simplexes. Repeating this observation sufficiently many times we get (4.13).

Therefore, we can calculate now for arbitrary two elements $s, s' \in F_S$,

$$
(s|s') = \left( \sum_{i=1}^{n} m_i 1_{B_i} \right) \left( \sum_{i=1}^{n'} m_i' 1_{B_i'} \right) = \chi_E(s) \chi_E(s')(1_{B_0}|1_{B_0}),
$$

(4.15)

where we have used that

$$(1_{B_0}|1_{B_0}) = (1_{B_0}|1_{B_0}).$$

Hence, $(\cdot|\cdot)$ is determined indeed up to a constant factor. Due to the independence of the Euler characteristic of triangulation, the formula above does define a bilinear form $(\cdot|\cdot) : F_S \times F_S \to \mathbb{R}$, given any value

$$
\lambda = (1_{B_0}|1_{B_0})
$$

In this way, we have proved that

**Theorem 4.8.** There exists a unique, modulo the rescaling by an arbitrary factor $\mu_2 > 0$, homeomorphism invariant, bilinear, symmetric form $(\cdot|\cdot)$ defined on the CW-complex vector space $F_S$, provided $\dim M > 1$ and $M$ is connected. The form is defined as follows,

$$(\cdot|\cdot) = \mu_2 \chi_E(\cdot) \chi_E(\cdot),
$$

(4.16)

where $\chi_E : F_S \to \mathbb{R}$ is the Euler characteristic functional.

Note, finally, that it follows from theorem 4.8 that the resulting unitary space $F_S/[s \in F_S : (s|s) = 0]$ is just one dimensional. If we denote by $[1_B]$ the element of the space $F_S/[s \in F_S : (s|s) = 0]$ corresponding to a ball $B$ in $M$, then the projection map is

$$
F_S \ni s \mapsto (-1)^{\dim M} \chi_E(s)[1_B].
$$

Note also that the element $[1_B]$ is unique, and it does not depend on which ball $B$ in $M$ we use to define it.

### 4.2. Homeomorphism invariant states on $\exp(\odot F_S^\mathbb{C})$

Consider the complexification $F_S^\mathbb{C}$ of the (real) CW-complex vector space $F_S$. We will denote by $\odot$ the symmetrized tensor product, and use the following notation for an element $s \in F_S^\mathbb{C}$

$$
S^{\odot n} := s \odot \cdots \odot s
$$

($n$ copies of $s$ on the right-hand side) and extend this notation to the space $F_S^\mathbb{C}$ itself

$$
(F_S^\mathbb{C})^{\odot n} := F_S^\mathbb{C} \odot \cdots \odot F_S^\mathbb{C},
$$

(4.17)

that is $(F_S^\mathbb{C})^{\odot n}$ is the space of the symmetric elements of the tensor product $F_S^\mathbb{C} \odot \cdots \odot F_S^\mathbb{C}$. Consider the following vector space:

$$
\exp(\odot F_S^\mathbb{C}) := \bigoplus_{n=0}^{\infty} (F_S^\mathbb{C})^{\odot n}.
$$

(4.19)
The space has the natural commutative \( \ast \)-algebra structure. Indeed, \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \) is by definition a complex vector space. The associative composition operation is the symmetrized tensor product \( \odot \), and the star operation is defined by using the complex conjugation \( \bar{\cdot} \): in \( \mathcal{F}_{\mathcal{S}}^\mathbb{C} \),

\[
(a \odot \cdots \odot b)(c \odot \cdots \odot d) = a \odot \cdots \odot b \odot c \odot \cdots \odot d \\
(a \odot \cdots \odot b)^\ast = \bar{b} \odot \cdots \odot \bar{a}.
\]

(4.20)

**Definition 4.9.** The \( \ast \)-algebra \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \) is called the CW-complex \( \ast \)-algebra.

As we will see, this algebra is naturally isomorphic to subalgebra of \( \mathfrak{A} \) generated by momentum variables \(5.1\).

**Definition 4.10.** A state \( \tilde{\omega} : \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \to \mathbb{C} \) is called homeomorphism invariant if for every \( k \)-tuple \( S_1, \ldots, S_k \in \mathcal{S} \) of simplexes in \( M \) and every homeomorphism \( \varphi : M \to M \) such that still \( \varphi(S_1), \ldots, \varphi(S_k) \in \mathcal{S} \), the following is true:

\[
\tilde{\omega}(1_{\varphi(S_1)} \odot \cdots \odot 1_{\varphi(S_k)}) = \tilde{\omega}(1_{S_1} \odot \cdots \odot 1_{S_k}).
\]

In this section, we derive all the homeomorphism invariant states on \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \).

Suppose that

\[
\tilde{\omega} : \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \to \mathbb{C}
\]

is a homeomorphism invariant state. It is clear that \( \tilde{\omega} \) defines a homeomorphism invariant, symmetric, bilinear form \( (\cdot|\cdot) : \mathcal{F}_{\mathcal{S}} \times \mathcal{F}_{\mathcal{S}} \to \mathbb{R} \), namely

\[
(s|s') = \tilde{\omega}(s \odot s')
\]

(recall that the subspace \( \mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{S}}^\mathbb{C} \) consists of the real elements such that \( \bar{s} = s \)). Given any two balls \( B_1 \) and \( B_2 \) in \( M \), proposition 4.4 guarantees that the difference \( 1_{B_1} - 1_{B_2} \in \mathcal{F}_{\mathcal{S}} \) satisfies

\[
\tilde{\omega}((1_{B_1} - 1_{B_2}) \odot (1_{B_1} - 1_{B_2})) = 0.
\]

Now, it follows from the basic properties of the commutative \( \ast \)-algebras that for every \( A, A' \in \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \)

\[
\tilde{\omega}(A \odot (1_{B_1} - 1_{B_2}) \odot A') = 0.
\]

(4.21)

Using this observation, we conclude that the state \( \tilde{\omega} \) is completely characterized by the following sequence of the numbers,

\[
\mu_n := \tilde{\omega}(1_{B_0}^{\odot n}), \quad n \in \mathbb{N}.
\]

(4.22)

where \( B_0 \in \mathcal{S} \) is an arbitrarily fixed ball in \( M \). Indeed, given any finite set of balls \( B_1, \ldots, B_n \) in \( M \) the identity (4.21) applied \( n \) times implies

\[
\tilde{\omega}(1_{B_1} \odot \cdots \odot 1_{B_n}) = \tilde{\omega}(1_{B_0}^{\odot n}).
\]

Furthermore, due to (4.13), (4.14), the subset of all the elements of \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \) of the form \( 1_{B_1} \odot \cdots \odot 1_{B_n} \), where \( B_i \) are balls in \( M \), spans the vector space \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \).

In other words, every homeomorphism invariant state \( \tilde{\omega} \) on \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \) is unambiguously determined by its restriction to the unital subalgebra \( \mathbb{C}[1_{B_0}] \) of \( \exp(\odot \mathcal{F}_{\mathcal{S}}^\mathbb{C}) \) generated by an element \( 1_{B_0} \), where \( B_0 \) is an arbitrarily fixed ball in \( M \). It is clear that \( \mathbb{C}[1_{B_0}] \) is isomorphic to the \( \ast \)-algebra \( \mathbb{C} [\tau] \) of complex-valued polynomials of one real variable; let us explain the notation:

\[
\tau : \mathbb{R} \to \mathbb{R}, \quad \tau (r) = r
\]
is the identity map,
\[ C[\tau] = \left\{ \sum_{j=1}^{n} a_j \tau^{j-1} : n \in \mathbb{N}, a_j \in C \right\} \]
and the isomorphism \( C[1_{B_0}] \rightarrow C[\tau] \) is defined just by
\[ 1_{B_0} \mapsto \tau. \] (4.23)

Thus we have arrived at

**Lemma 4.11.** Every homeomorphism invariant state \( \check{\omega} \) on the \( \ast \)-algebra \( C[\tau] \) is determined by a state \( \mu \) defined on the \( \ast \)-algebra \( C[\tau] \), according to the following formula,
\[ \check{\omega}(1_{B_1} \cdots 1_{B_k}) = \mu(\tau^k), \] (4.24)
for every \( k \in \mathbb{N} \).

In lemma 4.11 the existence of the state \( \check{\omega} \) defined on \( \exp(\odot F_S C) \) has been assumed. Now, we turn to the existence issue itself. We will show that

**Lemma 4.12.** Every state \( \mu \) defined on the polynomial \( \ast \)-algebra \( C[\tau] \) defines a unique homeomorphism invariant state \( \check{\omega} \) on \( \exp(\odot F_S C) \) such that (4.24).

**Proof.** Let us begin by constructing a \( \ast \)-homomorphism
\[ \text{Eul} : \exp(\odot F_S C) \rightarrow C[\tau] \]
such that for every ball \( B' \) in \( M \),
\[ 1_{B'} \mapsto \tau \in C[\tau]. \]

The linear extension of the equality above to the vector space \( F_S C \), the complexification of \( F_S \), is defined by using the Euler characteristic (4.8) in the following way:
\[ \text{Eul} : F_S C \rightarrow C[\tau], \quad s \mapsto (-1)^{\dim M} \chi_E(s) \tau. \] (4.25)

Because \( F_S C \) generates the commutative algebra \( \exp(\odot F_S C) \) freely modulo the linear relations satisfied by its elements, the above formula can be uniquely extended to the whole algebra \( \exp(\odot F_S C) \).
\[ \text{Eul}(s_1 \odot \cdots \odot s_n) := \text{Eul}(s_1) \cdots \cdot \text{Eul}(s_n) \in C[\tau]. \] (4.26)

Suppose \( \mu \) is a state on \( C[\tau] \). The pullback
\[ \check{\omega} := \mu \circ \text{Eul} \]
is certainly a state on \( C[\tau] \). The pullback
\[ \check{\omega} := \mu \circ \text{Eul} \]
is also homeomorphism invariant. Finally, the state \( \check{\omega} \) satisfies (4.24). □

Let us summarize our observations by the following:

**Theorem 4.13.** There is a 1–1 correspondence between the space of the homeomorphism invariant states on the CW-complex \( \ast \)-algebra \( \exp(\odot F_S C) \) and the space of the states on the polynomial algebra \( C[\tau] \). The correspondence is defined by the pullback \( \text{Eul}^* \) with the \( \ast \)-homomorphism \( \text{Eul} : \exp(\odot F_S C) \rightarrow C[\tau] \) (4.25), (4.26).

The careful reader would notice that the polynomial algebra \( C[\tau] \) has emerged just as isomorphic to the unital subalgebra of the CW-complex algebra, generated by a fixed single ball \( 1_{B_0} \). In fact, using the polynomial algebra representation has many advantages. To begin
with, an example of a state $\mu$ on the polynomial algebra $\mathbb{C}[\tau]$ is defined by a probability (regular, Borel) measure $d\mu$ on $\mathbb{R}$, via

$$\mu(\tau^k) := \int_{\mathbb{R}} \tau^k \, d\mu(\tau),$$

for every $k \in \mathbb{N}$, provided all the functions $\tau^k$ are integrable. Conversely, every state $\mu$ on the polynomial $\ast$-algebra $\mathbb{C}[\tau]$ can be constructed in that way. This non-trivial fact follows from the existence of a self-adjoint extension for the operator $\rho_{\mu}(\tau)$, where $\rho_{\mu}$ is the GNS representation (see section 2.4; the measure is unique if the extension is unique).

Given a homeomorphism invariant state $\tilde{\omega}$ defined on the CW-complex $\ast$-algebra $\exp(\odot \mathcal{F}_S^C)$ and the corresponding state $\mu$ defined on the polynomial algebra $\mathbb{C}[\tau]$ we construct now the corresponding GNS representation.

We start with the application of the GNS construction to the state $\mu$:

$$\mathcal{H}_\mu^0 = \mathbb{C}[\tau]/J_\mu,$$

equipped with the unitary scalar product

$$\langle [P_1], [P_2] \rangle_\mu := \mu(P_1 P_2).$$

Since the state $\tilde{\omega}$ on the algebra $\exp(\odot \mathcal{F}_S^C)$ is the pullback obtained by the epimorphism $\text{Eul} : \exp(\odot \mathcal{F}_S^C) \to \mathbb{C}[\tau]$, there is the natural identification

$$\langle \hat{\mathcal{H}}^0_\mu, \langle \cdot, \cdot \rangle_\mu \rangle = \langle \hat{\mathcal{H}}^0_\omega, \langle \cdot, \cdot \rangle_\omega \rangle.$$  \hspace{1cm} (4.30)

And the action of the GNS representation $\rho_\omega$ becomes

$$\rho_\omega : \exp(\odot \mathcal{F}_S^C) \to \text{End}(\mathcal{H}_\mu^0), \quad \rho_\omega(A) P := [\text{Eul}(A) P]$$

using the map (4.25), (4.26). Obviously,

$$\Omega_\omega = \Omega_\mu = [1],$$

meaning the (equivalence class of) zero-order polynomial 1.

**Proposition 4.14.** $\langle \hat{\mathcal{H}}^0_\mu, \langle \cdot, \cdot \rangle_\mu, \rho_\mu, \Omega_\mu \rangle$ defined by (4.28), (4.29), (4.31), (4.32) above is (equivalent to) the GNS representation (2.4) corresponding to the state $\tilde{\omega}$.

### 5. DEFINITION BY THE SMearing CHARACTERISTIC FUNCTIONS

In this section, we combine the definitions and results of the previous section 4 with the notion of the polymer $\ast$-algebra of section 2. We consider here the polymer $\ast$-algebra $\mathfrak{A}$ defined by the space of the smearing functions $\mathcal{F}$ taken to be the CW-complex vector space $\mathcal{F}_S$,

$$\mathcal{F} = \mathcal{F}_S.$$  \hspace{1cm} (5.1)

At first sight, it may seem surprising that in our definition we admit also the smearing functions supported on measure zero lower dimensional simplexes. However, we remember from the previous section that the simplex vector space $\mathcal{F}_S$ is spanned by the characteristic functions of the open balls in $M$. Therefore the emergence of the measure zero supports is a result of the linear structure. Still, a state on the corresponding polymer $\ast$-algebra $\mathfrak{A}$ might just ignore (in a suitable sense) the quantum momentum variables corresponding to those measure zero sets. We are not making that assumption, though.
We will construct now all the homeomorphism invariant (in the sense of definition 2.10) states on \( \mathfrak{A} \) which satisfy property (2.10). In the next sections, we will study the corresponding GNS representations, their equivalence classes and the irreducibility.

The momentum variable space \( \Pi_{\mathcal{F}_S} \) (see definition 2.3) corresponding to that space of the smearing functions will be denoted by

\[ \Pi := \Pi_{\mathcal{F}_S}. \]

Since the momentum variables are labelled by the smearing functions, and the smearing functions are labelled in this section by (suitable) subsets in \( M \), we will often use the following notation for the momentum variables assigned to a \( k \)-simplex \( S \) in \( M \) (or more generally, a triangulable subset \( S \subset M \)),

\[ \pi_S := \pi(1_S), \quad \hat{\pi}_S := \hat{\pi}(1_S). \quad (5.2) \]

**Definition 5.1.** The subalgebra \( \mathfrak{P} \) of the polymer \( * \)-algebra \( \mathfrak{A} \) generated by all the quantum momentum variables \( \hat{\pi}(s) \) where \( s \) ranges the whole space \( \mathcal{F}_S \) of the smearing functions is called the quantum momentum algebra.

The quantum momentum algebra \( \mathfrak{P} \) is naturally isomorphic to the CW-complex algebra \( \exp(\bigodot \mathcal{F}_S^C) \),

\[ \exp(\bigodot \mathcal{F}_S^C) \rightarrow \mathfrak{P} \quad (5.3) \]

\[ 1_{S_1} \odot \cdots \odot 1_{S_n} \mapsto \hat{\pi}_{S_1} \cdots \hat{\pi}_{S_n}, \quad (5.4) \]

where the assignment defined for all the \( n \)-tuples of \( k \)-simplexes \( S_i, i = 1, \ldots, n, \in \mathbb{N} \), determines the isomorphism.

Now, every homeomorphism invariant state \( \omega : \mathfrak{A} \rightarrow \mathbb{C} \) defines by the restriction a homeomorphism invariant state on \( \mathfrak{P} \). Therefore, due to proposition 4.4, for every two balls \( B_1, B_2 \) in \( M \),

\[ \omega((\hat{\pi}_{B_1} - \hat{\pi}_{B_2})^2) = 0, \quad (5.5) \]

and since \((\hat{\pi}_B)^* = \hat{\pi}_B \) for every ball \( B \), the equality implies that for every \( \hat{a} \in \mathfrak{A} \),

\[ \omega(\hat{a}(\hat{\pi}_{B_1} - \hat{\pi}_{B_2})) = 0 = \omega((\hat{\pi}_{B_1} - \hat{\pi}_{B_2})\hat{a}). \quad (5.6) \]

This property leads us to the following observation:

**Lemma 5.2.** Every homeomorphism invariant state \( \omega \) on \( \mathfrak{A} \) vanishes on the following elements of \( \mathfrak{A} \),

\[ \omega(\hat{h}_{k_1,x_1}) = \omega \left( \prod_{j=1}^{n} \hat{h}_{k_j,x_j} \right) = \omega \left( \prod_{j=1}^{n} \hat{h}_{k_j,x_j} \prod_{l=1}^{m} \hat{\pi}(f_l) \right) = 0, \quad (5.7) \]

where all \( k_j \neq 0 \), the points \( x_j \in M \) are pairwise different, and \( n > 0 \).

**Proof.** Consider

\[ \hat{a} = \prod_{j=1}^{n} \hat{h}_{k_j,x_j} \prod_{l=1}^{m} \hat{\pi}(f_l) \quad (5.8) \]

such that all \( k_j \neq 0 \), the points \( x_j \notin M \) are pairwise different, and \( n > 0 \). It is possible to find two balls \( B_1 \) and \( B_2 \) in \( M \), such that \( B_2 \) does not contain any of the points \( x_j, j = 1, \ldots, n \), whereas \( B_1 \) contains exactly one of the points, say \( x_1 \). Then,

\[ [\hat{\pi}_{B_1} - \hat{\pi}_{B_2}, \hat{a}] = k_1 \hat{a}. \]
Acting with $\omega$ on both sides of the above equation and taking advantage of the fact that $\hat{\pi}_B - \hat{\pi}_B$ is self-adjoint element of $\mathfrak{A}$, we get
\[ 0 = \omega(\hat{\pi}_B - \hat{\pi}_B)\hat{a} - \omega(\hat{a}(\hat{\pi}_B - \hat{\pi}_B)) = k_1 \omega(\hat{a}). \] (5.9)

But, the action of every state $\omega : \mathfrak{A} \to \mathbb{C}$ which satisfies property (2.10) is determined by the action on the elements used in lemma 5.2 and the restriction of $\omega$ to the quantum momentum variable subalgebra $\mathfrak{P}$. Therefore, the following is true:

**Lemma 5.3.** Every homeomorphism invariant state $\omega$ on $\mathfrak{A}$ which satisfies property 2.10 is determined by its restriction to the quantum momentum subalgebra $\mathfrak{P}$.

The converse is also true:

**Lemma 5.4.** For every state $\tilde{\omega}$ on the quantum momentum subalgebra $\mathfrak{P}$ there exists a unique extension to a homeomorphism invariant state $\omega$ of property 2.10 defined on the polymer $*$-algebra $\mathfrak{A}$.

**Proof.** It is convenient in what follows to use the quotient algebra $\tilde{\mathfrak{A}}$ (2.41), and the canonical epimorphism $\mathfrak{A} \to \tilde{\mathfrak{A}}$.

The quantum momentum subalgebra $\mathfrak{P}$ is naturally isomorphic to its image $\tilde{\mathfrak{P}}$ upon the map $\mathfrak{A} \to \tilde{\mathfrak{A}}$ therefore the subalgebras and their elements will be identified.

Every state $\tilde{\omega}$ on the subalgebra $\mathfrak{P}$ there defines a unique $\mathbb{C}$-linear extension $\tilde{\omega} : \tilde{\mathfrak{A}} \to \mathbb{C}$,

given by the formulae (5.7) (where all $k_j \neq 0$, the points $x_j \in M$ are pairwise different, and $n > 0$) with $\omega$ replaced by $\tilde{\omega}$ and the hats replaced by the tildes.

It remains to show that the extension is non-negative. The extension $\tilde{\omega}$ is just the pullback $\tilde{\omega} = P^* \tilde{\omega}$ (5.14)

is non-negative as well and defines a state on $\mathfrak{A}$ via the pullback $\mathfrak{A} \to \tilde{\mathfrak{A}}$. □
Combining lemmas 5.3 and 5.4 with theorem 4.13 leads to a complete characterization of the homeomorphism invariant states on the polymer $*$-algebra $\mathfrak{A}$ provided property 2.10 is satisfied. The states can be labelled by states on the polynomial algebra $\mathbb{C}[\tau]$ (we recall that the following algebras are naturally identified: $\exp(\circ \mathcal{F}_{\mathfrak{C}}) = \mathfrak{P} = \tilde{\mathfrak{P}}$; the last one being the image of the previous one in $\tilde{\mathfrak{A}}$):

**Theorem 5.5.** Suppose $\mathfrak{A}$ is the polymer $*$-algebra defined by the space of the smearing functions (5.1). There is a natural bijection between the set of the homeomorphism invariant states on $\mathfrak{A}$ which satisfy property 2.10 and the set of all the states on the polynomial algebra $\mathbb{C}[\tau]$. The bijection is the pullback defined by the linear map $\mathfrak{A} \rightarrow \tilde{\mathfrak{A}} \rightarrow \mathbb{C}[\tau]$ where $\mathfrak{A} \rightarrow \tilde{\mathfrak{A}}$ is the canonical projection on the quotient (2.41), whereas the map $\mathcal{E} : \tilde{\mathfrak{A}} \rightarrow \mathbb{C}[\tau]$ is defined as follows:

\[
\mathcal{E}_{|P} = \text{Eul}
\]

\[
\mathcal{E}_{|\tilde{\mathfrak{A}}} = 0.
\]

(where we use the notation of (4.25), (4.26), (2.44)) for every non-zero finitely supported function $k$ on $M$.

6. Explicit form of GNS representations of $\mathfrak{A}$

We characterize, in this section, the GNS representations (see those of the states described by theorem 5.5. We use the notation and definitions introduced in section 2.4.

Let $\omega$ be a homeomorphism invariant state on $\mathfrak{A}$ whose GNS representation $\rho_\omega$ satisfies property 2.10. Recall that according to theorem 5.5, the state $\omega$ is the pullback of a state $\mu$ on the polynomial algebra $\mathbb{C}[\tau]$. We will also use that, as explained in section 2.5, $\omega$ is the pullback of a state $\tilde{\omega}$ on the algebra $\tilde{\mathfrak{A}}$ (see (2.41)).

The first step is a characterization of the unitary space $(H_0^\omega, \langle \cdot, \cdot \rangle_\omega)$ in which the representation is defined; that is, the quotient space $\mathfrak{A}/J_\omega$ endowed with the unitary scalar product $\langle \cdot, \cdot \rangle_\omega$. For this purpose, we will use the unitary spaces: $(\text{Cyl}, (\cdot | \cdot)_{\text{Cyl}})$ (3.11), (3.12) and $(\mathbb{C}[\tau]/J_\mu, \langle \cdot, \cdot \rangle_\mu)$ (4.28), (4.29). We will also use the $*$-algebra homomorphism $\text{Eul}$ (4.26), (4.25) defined also on the quantum momentum algebra $\mathbb{C}[\tau]$.

**Lemma 6.1.** There is a natural vector space isomorphism

\[
\mathcal{I} : \mathfrak{A}/J_\omega \rightarrow \text{Cyl} \otimes \mathbb{C}[\tau]/J_\mu,
\]

unitary with respect to the unitary forms $(\cdot, \cdot)_\omega$ and, respectively, $(\cdot \cdot) \otimes (\cdot, \cdot)_\mu$, and such that, for every element $b$ of the quantum momentum algebra $\mathfrak{P}$ (see definition 5.1) and every $n$-tuple of quantum position variables $\hat{h}_{k_1,x_1}, \ldots, \hat{h}_{k_n,x_n}$, including $\hat{h}_{0,x_i} = \hat{1}$,

\[
\mathcal{I} : \left[ \prod_{i=1}^n \hat{h}_{k_i,x_i} b \right] \mapsto \left[ \sum_{j=1}^n k_j 1_{x_j} \right] \otimes [\text{Eul}(\hat{b})],
\]

**Proof.** We know that the vector space $\mathfrak{A}/J_\omega$ is isomorphic to $\tilde{\mathfrak{A}}/J_\tilde{\omega}$ (see (2.5)). A general element $\tilde{a} \in \tilde{\mathfrak{A}}$ can be written in the form (5.13). Consider the following map,

\[
\tilde{\mathfrak{A}} \rightarrow \text{Cyl} \otimes \mathbb{C}[\tau]/J_\mu \quad \sum_{j=1}^n \hat{h}_k \hat{b}_j \mapsto \sum_{j=1}^n |k_j \rangle \otimes [\text{Eul}(\hat{b}_j)],
\]
(as in the previous section, the subalgebra $\mathfrak{P} \subset \mathfrak{A}$ is identified with its image in $\mathfrak{A}$). The kernel of (6.3) is given by the left-hand side such that

$$\mathbb{C}[\tau]/J_{\mu} \ni [\text{Eul}(\hat{b}_j)] = 0, \quad j = 1, \ldots, n.$$  \hspace{1cm} (6.4)

On the other hand,

$$\hat{\omega} \left( \sum_{j=1}^{n} \hat{h}_k \hat{b}_j \right) = \sum_{j=1}^{n} \mu (\text{Eul}(\hat{b}_j)^* \text{Eul}(\hat{b}_j)).$$  \hspace{1cm} (6.5)

Therefore the kernel of (6.3) coincides with the left ideal $J_{\mu}$. Obviously, the map (6.2) is onto. Therefore, it passes to a vector space isomorphism, the isomorphism $\mathcal{I}$.

The calculation (6.5) also shows that $\mathcal{I}$ is unitary, if $\mathbb{A}/J_{\mu}$ is endowed with the unitary scalar product $\langle \cdot, \cdot \rangle_\omega$ and the space $\text{Cyl} \otimes \mathbb{C}[\tau]/J_{\mu}$ with the tensor product $\langle \cdot | \cdot \rangle_{\text{Cyl}} \otimes \langle \cdot, \cdot \rangle_{\mu}$.

Next, we turn to the GNS representation $\rho_\omega$ of $\mathfrak{A}$ defined by the state $\omega$. Our task amounts to evaluating the action of the operators $\mathcal{I} \circ \rho_\omega(\hat{h}_{k,x}) \circ \mathcal{I}^{-1}$, and $\mathcal{I} \circ \rho_\omega(\hat{s}(s)) \circ \mathcal{I}^{-1}$ in $\text{Cyl} \otimes H_{\mu}$.

For every element $\langle k \rangle \otimes [P]$, we will denote

$$\mathcal{I}^{-1}[k] \otimes [P] =: \left[ \hat{h}_k b_P \right].$$

Given any quantum position variable $\hat{h}_{k,x}$, we have

$$\mathcal{I} \circ \rho_\omega(\hat{h}_{k,x}) \circ \mathcal{I}^{-1}[k] \otimes [P] = \mathcal{I} \circ [\hat{h}_k b_P] = \mathcal{I} \left[ \hat{h}_k \hat{h}_k b_P = |k + k 1_{(1)}\rangle \otimes [P]. \right. \hspace{1cm} (6.6)$$

And given any embedded CW-complex $s \in \mathcal{F}_S$,

$$\mathcal{I} \circ \rho_\omega(\hat{s}(s)) \circ \mathcal{I}^{-1}[k] \otimes [P] = \mathcal{I} \circ [\hat{s}(s) b_P] = \mathcal{I} \left[ \hat{h}_k \left( b_{\text{Eul}(s)} b_P + \sum_{x \in M} k(x)s(x) b_P \right) \right]$$

$$= \left| k \right\rangle \otimes \left( \sum_{x \in M} k(x)s(x) [P] + [\text{Eul}(s) P] \right). \hspace{1cm} (6.7)$$

In conclusion,

**Theorem 6.2.** Suppose $\mathfrak{A}$ is the polymer $*$-algebra defined by the space of the smearing functions (5.1). Suppose $\omega$ is a homeomorphism invariant state on $\mathfrak{A}$ such that property 2.10 is satisfied. Let $\mu : \mathbb{C}[\tau] \rightarrow \mathbb{C}$ be the state corresponding to $\omega$ by theorem 5.5. The GNS representation $(\mathcal{H}_\mu^0, \langle \cdot, \cdot \rangle_{\mu}, \rho_\mu, \Omega_\mu)$ corresponding to $\omega$ (see section 2.4) is equivalent to $(D, \langle \cdot, \cdot \rangle, \rho, \Omega)$ where (see (3.11), (3.12) and (4.28, 4.29))

$$D = \text{Cyl} \otimes H_{\mu}^0,$$

equipped with the unitary scalar product,

$$\langle \cdot, \cdot \rangle = \langle \cdot | \cdot \rangle_{\text{Cyl}} \otimes \langle \cdot, \cdot \rangle_{\mu},$$

and with the distinguished vector,

$$\Omega = |0\rangle \otimes [1],$$

and the representation $\rho$ is defined as follows,

$$\rho(\hat{h}_{k,x}) |k\rangle \otimes [P] = |k + k 1_{(1)}\rangle \otimes [P].$$

$$\rho(\hat{s}(s)) |k\rangle \otimes [P] = |k\rangle \otimes \left( \sum_{x \in M} k(x)s(x) [P] + (-1)^{\dim P} \chi_{E}(s) \rho_\mu(\tau) [P] \right) \hspace{1cm} (6.11)$$

where GNS representation $(\mathcal{H}_\mu^0, \langle \cdot, \cdot \rangle_{\mu}, \rho_\mu, \Omega_\mu)$ corresponds to the state $\mu : \mathbb{C}[\tau] \rightarrow \mathbb{C}$. 
7. Self-adjoint extensions

7.1. Momentum self-adjoint representations

We turn now to the issue of self-adjoint extension of the GNS representations of the states on the polymer $*$-algebra $\mathfrak{A}$ considered in sections 5 and 6. Suppose $\omega : \mathfrak{A} \rightarrow \mathbb{C}$ is a homeomorphism invariant state considered in theorem 5.5 and determined by a state $\mu : \mathbb{C}[\tau] \rightarrow \mathbb{C}$ on the polynomial algebra. We will study the corresponding representation $(D, \langle \cdot, \cdot \rangle, \rho, \Omega)$ of (6.8)–(6.11) equivalent to the GNS representation corresponding to the state $\omega$. We will also use the Hilbert space $H$ defined by the completion $H = \overline{D}$, (7.1) as well as the GNS representation $(H^0_\mu, \langle \cdot, \cdot \rangle_\mu, \rho_\mu, \Omega_\mu)$ corresponding to the state $\mu : \mathbb{C}[\tau] \rightarrow \mathbb{C}$ and the Hilbert space $H_\mu = H^0_\mu$. (7.2)

For every $\hat{a} \in \mathfrak{A}$ the operator $\rho(\hat{a})$ is defined in the domain $D \subset H$. We know that for every (real-valued) smearing function $s \in \mathcal{F}_S$, by the GNS construction, the operator $\rho(\hat{a}(s))$ is symmetric. That is, for every pair of vectors $|\hat{a}\rangle \otimes [P], |\hat{b}\rangle \otimes [Q]$(7.2).

It is easy to note that every operator $\rho(\hat{a}(s))$ is essentially self-adjoint if

$$\chi_E(s) = 0.$$  

Indeed, given $s \in \mathcal{F}_S$ such that (7.3), we have

$$\rho(\hat{a}(s))|k\rangle \otimes [P] = \sum_{x \in M} k(x)s(x)|k\rangle \otimes [P],$$

for every finitely supported function $k$ and every polynomial $P \in \mathbb{C}[\tau]$. The eigenvectors $|k\rangle \otimes [P]$ span the domain $D$ of the operator, which shows the essential self-adjointness.

Now we will consider the case $\chi_E(s) \neq 0$.

The following theorem shows that the issue of the self-adjointness of the quantum momentum operators $\rho(\hat{a}(s))$ comes down to the self-adjointness of the corresponding GNS operators $\rho_\mu(\tau)$ defined in the Hilbert space $H_\mu$ spanned by polynomials.

**Theorem 7.1.** Let $(D, \langle \cdot, \cdot \rangle, \rho, \Omega)$ be any of the representations (6.8)–(6.11) of the polymer $*$-algebra $\mathfrak{A}$. The following two conditions are equivalent:

(i) For every smearing function $s \in \mathcal{F}_S$, the operator $\rho(\hat{a}(s))$ is essentially self-adjoint in the Hilbert space $H$ (7.1).

(ii) The operator $\rho_\mu(\tau)$ is essentially self-adjoint in the Hilbert space $H_\mu$ (7.2).

**Proof.** We will show that for every $s \in \mathcal{F}_S$, the images of the operators $\rho(\hat{a}(s)) \pm i$ are dense in the Hilbert space $H$, that is, the equalities

$$\rho(\hat{a}(s)) \pm iD = H,$$

(7.4)
are true, if and only if
\[ (\rho_\mu(\tau) \pm i) \mathcal{H}_\mu^0 = \mathcal{H}_\mu. \] (7.5)
The last condition (7.5) is necessary and sufficient for the essential self-adjointness of the operator \( \rho_\mu(\tau) \).

The first step of the proof is to fix an arbitrary vector \( |k\rangle \in \text{Cyl} \), and apply the operator in question to the subspace
\[ \{ |k\rangle \} \otimes \mathcal{H}_\mu^0 \subset \text{Cyl} \otimes \mathcal{H}_\mu^0 = D. \] (7.6)
The image is
\[ (\rho(\hat{\pi}(s)) \pm i) \{ |k\rangle \} \otimes \mathcal{H}_\mu^0 = \{ |k\rangle \} \otimes \left( (-1)^M \chi_E(s) \rho_\mu(\tau) + \sum_{x \in M} k(x) s(x) \pm i \right) \mathcal{H}_\mu^0. \] (7.7)
Focus attention on the second factor of the tensor product on the right-hand side. Suppose \( \chi_E(s) \neq 0 \). (7.8)

The factor is dense in the Hilbert space \( \mathcal{H}_\mu \), if and only if the operator \( \rho_\mu(\tau) \) is essentially self-adjoint. (On the other hand, if we assume that \( \chi_E(s) = 0 \), then the second factor is unconditionally dense in \( \mathcal{H}_\mu \).)

Now,
\[ (\rho(\hat{\pi}(s)) \pm i) D = \bigoplus_{|k\rangle \in \text{Cyl}} \{ |k\rangle \} \otimes \left( (-1)^M \chi_E(s) \rho_\mu(\tau) + \sum_{x \in M} k(x) s(x) \pm i \right) \mathcal{H}_\mu^0. \] (7.9)
Clearly, in the case (7.8) the right-hand side is dense in \( \mathcal{H} \) if and only if the operator \( \rho_\mu(\tau) \) is essentially self-adjoint. (If \( \chi_E(s) = 0 \), the right-hand side equals \( D \) for every \( \omega \) and \( \mu \).)

\[ \square \]

Definition 7.2. Every representation \( (D, \langle \cdot, \cdot \rangle, \rho, \Omega) \) of the polymer \( \ast \)-algebra defined in (6.8)–(6.11) which satisfies conditions (1) and (2) will be called momentum self-adjoint.

Theorem 7.1 provides a unique self-adjoint extension for each of the quantum momentum operators, provided certain conditions are satisfied. Talking about representations, however, we need a suitable extension of the common domain of all the operators of a given GNS representation. The appropriate framework can be found in [24]. We use some elements of that framework in the next subsection and combine with our very results.

7.2. Unique self-adjointed extension

Definition 7.3. A Hermitian representation of a \( \ast \)-algebra \( \mathfrak{A} \) is a triple \( (D, \langle \cdot, \cdot \rangle, \rho) \) which consists of

- a vector space \( D \) equipped with a unitary scalar product \( \langle \cdot, \cdot \rangle \),
- an algebra homomorphism \( \rho : \mathfrak{A} \rightarrow \text{End}(D) \) into the algebra of linear operators defined in \( D \),

such that
\[ \langle \phi, \rho(a)\psi \rangle = \langle \rho(a^*)\phi, \psi \rangle, \quad \forall \phi, \psi \in D. \]

Given a Hermitian representation \( (D, \langle \cdot, \cdot \rangle, \rho) \) we also consider the natural Hilbert space completion
\[ \mathcal{H} = \overline{D}. \]
polymer representation of A this is not sufficient to generate all of ρ(a) of the adjoint operator ⟨·⟩. For every operator ρ(a) : D → D defined by a Hermitian representation, the domain D^∗ a of the adjoint operator ρ(a)^∗ is an extension of D.

D ⊂ D^∗ a ⊂ H.

The operator ρ(a) has the natural extension to the operator adjoint to ρ(a)^∗, ρ(a)^∗ : D^∗ a → H.

Whereas D^∗ a may not be preserved by the operator; it can be easily shown that the following intersection is preserved:

D^∗ := \bigcap_{b ∈ A} D^∗ b.

ρ(a)^∗ (D^∗) ⊂ D^∗, ∀a ∈ A.

\textbf{Definition 7.4 [24].} Given a Hermitian representation (D, ⟨·, ·⟩, ρ) of a ∗-algebra A, the triple (D^∗, ⟨·, ·⟩, ρ^∗) is called the adjoint extension of (D, ⟨·, ·⟩, ρ), if

• D^∗ is defined by (7.11), and
• for every a ∈ A, ρ^∗ (a) : D^∗ → D^∗ is the following linear operator:

ρ^∗ (a) := ρ(a)^∗|D^∗.

(7.13)

It is not hard to check that the adjoint extension of a Hermitian representation of a ∗-algebra A is a representation of A in the sense that

ρ^∗ (ab) = ρ^∗ (a)ρ^∗ (b).

In general though, we do not know if (D^∗, ⟨·, ·⟩, ρ^∗) is Hermitian. That property can be ensured by some stronger conditions.

\textbf{Lemma 7.5 [24].} Let (D, ⟨·, ·⟩, ρ) be a Hermitian representation of a ∗-algebra A, and (D^∗, ⟨·, ·⟩, ρ^∗) be its adjoint extension. Let A ⊂ A be the set of elements a ∈ A such that

⟨v, ρ^∗ (a) w⟩ = ⟨ρ(a)^∗ v, w⟩, ∀v, w ∈ D^∗.

(7.14)

Suppose A generates the algebra A. Then (D^∗, ⟨·, ·⟩, ρ^∗) is a Hermitian representation of the algebra A.

We apply now the definitions formulated above and lemma 7.5 to the representations of the polymer ∗-algebra. Obviously, each of the representations (D, ⟨·, ·⟩, ρ, Ω) defined by (6.8)–(6.11) of theorem 6.2 defines just a Hermitian representation (D, ⟨·, ·⟩, ρ) of the polymer ∗-algebra (by skipping the cyclic vector Ω). Consider the adjoint extension (D^∗, ⟨·, ·⟩, ρ^∗). In that case the set A ⊂ A introduced in lemma 7.5 contains every quantum position variable ˆh_{ks} and all those quantum momentum variables ˆπ(s) which satisfy χ_E(s) = 0. Indeed, each operator ρ(ˆh_{ks}) is unitary in D, whereas every operator ρ(ˆπ(s)) is essentially self-adjoint. But this is not sufficient to generate all of A. Nonetheless, in the case of a momentum self-adjoint representation of A (according to definition 7.2), we also have ˆπ(s) ∈ A for every quantum
momentum variable. Then, the set $A$ does generate all the algebra $\mathfrak{A}$. Hence, the adjoint extension is a Hermitian representation itself. We can also somewhat simplify the definition of the domain $D^*$ in this case. Indeed, each of the domains $D^*_h, x$ is just the full Hilbert space $\bar{D}$.

It follows that in (7.11) the elements $b = \hat{\pi}(s_1) \cdots \hat{\pi}(s_k)$ are sufficient to achieve the set $D^*$ (note that $D^*_\rho(a) \cap D^*_\rho(b) \subset D^*_\rho(a + b)$ and $D^*_\rho(\hat{\pi}(s_1) \cdots \hat{\pi}(s_k)) = D^*_\rho(\hat{\pi}(s_1) \cdots \hat{\pi}(s_k) \rho(\hat{h}_x))$, and we arrive at the following conclusion:

**Corollary 7.6.** Let $(D, \langle \cdot, \cdot \rangle, \rho, \Omega)$ be one of the representations (6.8)–(6.11). Suppose $(D, \langle \cdot, \cdot \rangle, \rho, \Omega)$ is momentum self-adjoint according to definition 7.2. Then, the adjoint extension $(D^*, \langle \cdot, \cdot \rangle, \rho^*)$ of the Hermitian representation $(D, \langle \cdot, \cdot \rangle, \rho)$ of $\mathfrak{A}$ is a Hermitian representation. Moreover,

$$D^* = \bigcap_{s_1, \ldots, s_n \in \mathcal{F}_S} D^*_{\hat{\pi}(s_1) \cdots \hat{\pi}(s_n)}$$

(7.15)

We do not need a complete characterization of the spaces $D^*_{\hat{\pi}(s_1) \cdots \hat{\pi}(s_n)}$ used in corollary 7.6. However, it is easy to note that all of them contain a useful common subspace (the one in the middle below),

$$D \subset \text{Cyl} \otimes D_\tau \subset D^*,$$

(7.16)

where the subspace $D_\tau \subset \mathcal{H}_\mu$ (see (7.2)) defined in the next paragraph, is the domain of an essentially self-adjoint extension of the operator $\rho_\mu(\tau)$.

Now, the Hilbert space $\mathcal{H}_\mu$ is unitary isomorphic with the space $L^2(\mathbb{R}, d\mu)$, where $d\mu$ is a measure (i.e. regular, Borel measure) on $\mathbb{R}$ such that

(a) $\mu(\tau^n) = \int_\mathbb{R} \tau^n \, d\mu(\tau), n = 0, 1, \ldots, n, \ldots$, and

(b) the subspace of polynomials is dense in $L^2(\mathbb{R}, d\mu)$.

This measure $d\mu$ is uniquely determined by the state $\mu$ provided $\mu$ satisfies condition (2) of theorem 7.1. The unitary isomorphism maps $\tau \in \mathbb{C}[\mathbb{R}]$ into the identity function $\tau : \mathbb{R} \rightarrow \mathbb{R}$. Henceforth, we will be identifying the Hilbert spaces $\mathcal{H}_\mu$ and $L^2(\mathbb{R}, d\mu)$. Via that equivalence, the subspace $D_\tau$ is

$$D_\tau = \left\{ f \in L^2(\mathbb{R}, d\mu) : \int_\mathbb{R} |\tau^{2n} f(\tau')|^2 \, d\mu(\tau') < \infty, \forall n \in \mathbb{N} \right\}.$$  

(7.17)

Finally,

**Definition 7.7.** A Hermitian representation of a $\ast$-algebra $\mathfrak{A}$ is called self-adjoint if it coincides with its own adjoint extension.

It can be shown (see [24], section 8.1) that the adjoint extensions $(D^*, \langle \cdot, \cdot \rangle, \rho^*)$ defined in corollary 7.6 are all self-adjoint Hermitian representations.

### 7.3. Non-unique self-adjoint extensions

We will generalize here the unique extension of a momentum self-adjoint representation introduced in the previous subsection to a general case. Consider now an arbitrary state $\mu : \mathbb{C}[\mathbb{R}] \rightarrow \mathbb{C}$, that is, do not assume property (2) of theorem 7.1. Still, there exists a measure $d\mu^*$ on $\mathbb{R}$ such that the conditions (a) and (b) hold. The measure defines an essentially self-adjoint extension $\rho_{d\mu^*}(\tau)$ of the operator $\rho_\mu(\tau)$. The domain of the extension is the subspace
Identifying again the Hilbert spaces $\mathcal{H}_\mu$ and $L^2(\mathbb{R}, d\mu^*)$, the extension is defined in $D_\tau$ (7.17) by

$$
(\rho^{cl}_{d\mu^*}(\tau) f)(\tau') = \tau' f(\tau').
$$

Therefore we will denote it just by $\hat{\tau}$,

$$
\hat{\tau} := \rho^{cl}_{d\mu^*}(\tau). \tag{7.19}
$$

This extension defines naturally the extension of the GNS representation (6.8)–(6.11) corresponding to $\mu$, to a Hermitian representation $(D_{d\mu^*}, \langle \cdot, \cdot \rangle, \rho_{d\mu^*})$, where $D_{d\mu^*} \subset \mathcal{H}$ such that (we skip the subscript $d\mu^*$ at the scalar product because it coincides with the scalar product in $\mathcal{H}$)

- $D_{d\mu^*} = \text{Cyl} \otimes D_\tau$ equipped with the scalar product of $\text{Cyl} \otimes L^2(\mathbb{R}, d\mu^*)$, and
- the representation is defined as follows,

$$
\rho_{d\mu^*}(\hat{\h}_k,s)|\langle k \rangle \otimes f = |k + k\mathbf{1}|1(s) \otimes f, \tag{7.20}
$$

$$
\rho_{d\mu^*}(\hat{\pi}(s))|\langle k \rangle \otimes f = |k \rangle \otimes \left(\sum_{x \in M} k(x)s(x)f + (-1)^{\text{dim}M} \chi_E(s)\hat{\tau} \cdot f\right). \tag{7.21}
$$

In the same way as it was explained in the previous subsection, it can be shown that the adjoint extension of the Hermitian representation $(D_{d\mu^*}, \langle \cdot, \cdot \rangle, \rho_{d\mu^*})$ is a Hermitian representation and a self-adjoint representation of $\mathfrak{A}$ in the sense of definition 7.7.

Finally, in the case of a momentum self-adjoint representation (see definition 7.2), the extension introduced in the current subsection coincides with the unique extension introduced in section 7.2.

8. Equivalence of representations

**Definition 8.1.** Two representations $(D_1, \langle \cdot, \cdot \rangle_1, \rho_1)$ and $(D_2, \langle \cdot, \cdot \rangle_2, \rho_2)$ are called equivalent if there exists a unitary space isomorphism $I : D_1 \to D_2$ satisfying

$$
I^{-1} \circ \rho_2 \circ I = \rho_1. \tag{8.1}
$$

Here $I$ is called an intertwining map between the representations.

We will denote by $\overline{I}$ the unique extension of the map $I$ to the completions $\overline{D_1}$ and $\overline{D_2}$ of the domains.

We will study now the issue of the equivalence in the context of the momentum self-adjoint representations considered in section 7.2, as well as in the more general context of the representations defined in section 7.3. To make our results as general as possible, we consider the (self-)adjoint extensions of the representations. We show that the equivalence issue comes down to the absolute continuity of the measures on $\mathbb{R}$ labelling the representations.

**Theorem 8.2.** Let $(D_1, \langle \cdot, \cdot \rangle_1, \rho_1)$ and $(D_2, \langle \cdot, \cdot \rangle_2, \rho_2)$ be the representations of the algebra $\mathfrak{A}$ defined by (7.20)–(7.21), and the measures $d\mu^* = d\mu_1, d\mu_2$.

- Suppose their adjoint extensions $(D_1^*, \langle \cdot, \cdot \rangle_1, \rho_1^*)$ and $(D_2^*, \langle \cdot, \cdot \rangle_2, \rho_2^*)$ are equivalent. Then the following holds:

  (a) The measures $d\mu_1$ and $d\mu_2$ are absolutely continuous with respect to each other.
(b) For every unitary isomorphism \( I : D_1^* \to D_2^* \) such that
\[
I^{-1} \circ \rho_2^* \circ I = \rho_1^*
\]
there is \( h \in L^2(\mathbb{R}, d\mu_2) \) such that \( I \) is of the following form for every \( |k\rangle \in \text{Cyl} \), and every \( f \in L^2(\mathbb{R}, d\mu_1) \):
\[
|k\rangle \otimes f \mapsto |k\rangle \otimes hf.
\]
Moreover,
\[
|h|^2 = \frac{d\mu_1}{d\mu_2}.
\]
Conversely, if the measures \( d\mu_1 \) and \( d\mu_2 \) are absolutely continuous, the map (8.3) with
\[
h := \sqrt{\frac{d\mu_1}{d\mu_2}}
\]
twines the representations \((D_1, \langle \cdot, \cdot \rangle, \rho_1)\) and \((D_2, \langle \cdot, \cdot \rangle, \rho_2)\) (not only the adjoint extensions).

**Proof.** Suppose that the representations \((D_1^*, \langle \cdot, \cdot \rangle_1, \rho_1^*)\) and \((D_2^*, \langle \cdot, \cdot \rangle_2, \rho_2^*)\) are equivalent and \( I : D_1 \to D_2 \) is an intertwining map. The sketch of the proof of (8.3) is simple. Every intertwining map \( I \) is determined by its action on the cyclic vector \(|0\rangle \otimes 1\). But, as we argue below, each intertwining map \( I \) has to satisfy
\[
I : |0\rangle \otimes 1 \mapsto |0\rangle \otimes h,
\]
where \( h \in L^2(\mathbb{R}, d\mu_2) \). Next, we find that the map \( I \) determined by (8.5) is exactly the one defined in (8.3). Below we give the details. The only technical subtlety we have to be careful about is that we are dealing with the adjoint extension of the representation \( \rho_2 \).

The vector \(|0\rangle \otimes 1\) is annihilated by the following subclass of the momentum operators:
\[
\rho_1(\hat{\tau}(s))|0\rangle \otimes 1 = 0, \quad \text{whenever} \quad \chi_\mathcal{E}(s) = 0.
\]
This property has to be preserved by every intertwiner, therefore
\[
\rho_2^*(\hat{\tau}(s))I(|0\rangle \otimes 1) = 0, \quad \text{whenever} \quad \chi_\mathcal{E}(s) = 0.
\]
But every vector \( \psi \in D_2^* \) such that
\[
\rho_2^*(\hat{\tau}(s))\psi = 0, \quad \forall s \in \mathcal{F}_S : \chi_\mathcal{E}(s) = 0,
\]
is necessarily of the form \(|0\rangle \otimes h\) (that statement is more obvious in the case of \( \psi \in D_2 \) and \( \rho_2 \), and it generalizes to the adjoint extensions), hence
\[
I(|0\rangle \otimes 1) = |0\rangle \otimes h, \quad h \in L^2(\mathbb{R}, d\mu_2).
\]
Next we determine the action of \( I \) on the vectors \(|0\rangle \otimes \tau^n, n = 1, \ldots \). We have
\[
I(|0\rangle \otimes \tau^n) = \rho_2^*(\hat{\tau}(s_1)) \cdots \rho_2^*(\hat{\tau}(s_n))|0\rangle \otimes h,
\]
with any \( s_1, \ldots, s_n \in \mathcal{F}_S \) such that \( \chi_\mathcal{E}(s_i) = 1, \ i = 1, \ldots, n \). Again we observe that the right-hand side of (8.10), if denoted by \( \psi \), satisfies (8.8). Hence, for every \( s \in \mathcal{F}_S \), the operator \( \rho_2^*(\hat{\tau}(s)) \) preserves the subspace \(|0\rangle \otimes L^2(\mathbb{R}, d\mu_2) \cap D_2^* \). Fix \( s \) such that \( \chi_\mathcal{E}(s) = 1 \) and consider the restriction
\[
\rho_2^*(\hat{\tau}(s)) : |0\rangle \otimes L^2(\mathbb{R}, d\mu_2) \cap D_2^* \to |0\rangle \otimes L^2(\mathbb{R}, d\mu_2) \cap D_2^*.
\]
It follows from (7.21) and (7.19) that the operator (8.11) coincides with the operator \( \hat{\tau} \) (as expected). Therefore, we have derived
\[
I(|0\rangle \otimes \tau^n) = |0\rangle \otimes \hat{\tau}^n \cdot h = |0\rangle \otimes h \tau^n.
\]
Finally, the action of the intertwiner $I$ on elements $|k\rangle \otimes \tau^n$ is determined in an obvious way:

$$I(|k\rangle \otimes \tau^n) = \left( \rho_1 \left( \prod_{x \in M} \hat{h}_{k(x), x} \right) \right) |0\rangle \otimes \tau^n = \left( \rho_2 \left( \prod_{x \in M} \hat{h}_{k(x), x} \right) \right) |0\rangle \otimes h \tau^n = |k\rangle \otimes h \tau^n.$$  

By the continuity the map extends to (8.3).

The absolute continuity of the measures $d\mu_1$ and $d\mu_2$ is obvious as well as the Nikodym–Radon derivative (8.4).

Conversely, given two measures $d\mu_1$ and $d\mu_2$ on $\mathbb{R}$ which are absolutely continuous, one can check by inspection that the map (8.3) is unitary, maps $D_1 \to D_2,$ and intertwines the representations $\rho_1$ and $\rho_2.$  

\[\Box\]

9. Invariant subspaces

In this section, we solve the issue of the irreducibility of the representations of the polymer $*$-algebra $\mathcal{A}$ introduced in (6.8), (6.9), (6.11) and in sections 7.2 and 7.3.

Consider a representation $(D, \langle \cdot, \cdot \rangle, \rho)$ of the algebra $\mathcal{A}.$ As before, we will also use the corresponding Hilbert space $\mathcal{H}$ (7.1). It is defined by the completed tensor product (we denote it by $\otimes$) of the following spaces, $C_{\text{yl}},$ $H_\mu = C[\tau]/J_\mu,$ (9.1)

the completions with respect to the unitary scalar products $\langle \cdot | \cdot \rangle_{\text{yl}},$ and, respectively $\langle \cdot , \cdot \rangle_\mu,$ that is

$$\mathcal{H} = C_{\text{yl}} \otimes H_\mu.$$  

(9.2)

We present a quite strong, general result which shows that the issue boils down to the irreducibility of the representations of the polynomial algebra $C[\tau]$ corresponding to the state $\mu: \mathbb{C} \to \mathbb{C}.$

Recall that each quantum position variable $\hat{h}_{k,x}$ and every quantum momentum $\hat{\pi}(s)$ such that (7.3).

Each of those operators with the domain $D$ (6.8) is essentially self-adjoint. Therefore, the operator $\exp(i\rho(\hat{\pi}(s)))$ can be defined uniquely as an unitary operator in $\mathcal{H}.$ We will assume below that the unitary operators mentioned above preserve a common subspace in $\mathcal{H},$ and we will come to a quite strong conclusion.

**Theorem 9.1.** Let $(D, \langle \cdot , \cdot \rangle, \rho)$ be a representation of the polymer $*$-algebra defined by (6.8), (6.9), (6.11), corresponding to a state $\omega: \mathfrak{A} \to \mathbb{C}$ determined by a state $\mu: \mathbb{C} \to \mathbb{C}.$ Suppose $\mathcal{H}$ is a Hilbert subspace of the Hilbert space $\mathcal{H}$ (9.2) such that

$$\rho(\hat{h}_{k,x})(\mathcal{H}) \subset \mathcal{H},$$  

(9.4)

$$\exp(i\rho(\hat{\pi}(s)))(\mathcal{H}) \subset \mathcal{H},$$  

(9.5)

for every quantum position variable $\hat{h}_{k,x}$ and every quantum momentum $\hat{\pi}(s)$ such that (7.3). Then, there is a Hilbert subspace $\mathcal{H}_\mu$ of $H_\mu$ such that

$$\mathcal{H} = C_{\text{yl}} \otimes \mathcal{H}_\mu.$$  

(9.6)
Proof. Given $s \in \mathcal{F}_S$, every eigenvalue of the operator $\exp(i\rho_{\omega}(\hat{\pi}(s)))$ can be written as

$$
\lambda_{k,s}^s = \exp \left( i \sum_{x \in M} k(x)s(x) \right),
$$

(9.7)

where $k$ is a finitely supported function on $M$, and for every $k$ the number $\lambda_{k,s}^s$ is an eigenvalue of the operator $\exp(i\rho_{\omega}(\hat{\pi}(s)))$. To each of the eigenvalues $\lambda_{k,s}^s$, there is assigned the corresponding subspace $\mathcal{H}_{\lambda_{k,s}^s}$ of the eigenvectors. We have

$$
|k\rangle \otimes \mathcal{H}_\mu \subset \mathcal{H}_{\lambda_{k,s}^s},
$$

(9.8)

for every finitely supported function $k'$ such that

$$
\sum_{x \in M} k'(x)s(x) = \sum_{x \in M} k(x)s(x).
$$

However, if we fix a finitely supported function $k$ and consider the common part of all the sets $\mathcal{H}_{\lambda_{k,s}^s}$, then it is easy to show that

$$
\bigcap_{s \in \mathcal{F}_S: \chi_E(s)=0} \mathcal{H}_{\lambda_{k,s}^s} = |k\rangle \otimes \mathcal{H}_\mu.
$$

(9.9)

Let us turn now to the preserved subspace $\tilde{\mathcal{H}}$. Denote by

$$
P_{\tilde{\mathcal{H}}} : \mathcal{H} \to \tilde{\mathcal{H}}
$$

the orthogonal projection onto $\tilde{\mathcal{H}}$. For every fixed $s \in \mathcal{F}_S$ such that (7.3), and a finitely supported function $k$, due to the assumption that $\tilde{\mathcal{H}}$ is preserved by the unitary operator $\exp(i\rho_{\omega}(\hat{\pi}(s)))$ and by the general properties of the unitary operators, it follows that

$$
P_{\tilde{\mathcal{H}}} (\mathcal{H}_{\lambda_{k,s}^s}) \subset \mathcal{H}_{\lambda_{k,s}^s}.
$$

(9.10)

Therefore

$$
P_{\tilde{\mathcal{H}}} (|k\rangle \otimes \mathcal{H}_\mu) = \bigcap_s P_{\tilde{\mathcal{H}}} (\mathcal{H}_{\lambda_{k,s}^s}) \subset |k\rangle \otimes \mathcal{H}_\mu,
$$

(9.11)

where the range of $s$ on the right-hand side is the same as in (9.9). Hence, there is a Hilbert subspace $\tilde{\mathcal{H}}_{k,\mu} \subset \mathcal{H}_\mu$, such that

$$
P_{\tilde{\mathcal{H}}} (|k\rangle \otimes \mathcal{H}_\mu) = |k\rangle \otimes \tilde{\mathcal{H}}_{k,\mu}.
$$

(9.12)

The relevance of this space consists in the following orthogonal decomposition of the invariant space $\tilde{\mathcal{H}}$,

$$
\tilde{\mathcal{H}} = P_{\tilde{\mathcal{H}}} \left( \bigoplus_k |k\rangle \otimes \tilde{\mathcal{H}}_{k,\mu} \right) = \bigoplus_k |k\rangle \otimes \tilde{\mathcal{H}}_{k,\mu},
$$

(9.13)

where $k : M \to \mathbb{R}$ ranges the set of all the finitely supported functions.

Next we will show that in fact the subspaces $\tilde{\mathcal{H}}_{k,\mu}$ are necessarily all the same, independent of $k$. To see that, we introduce for every finitely supported function $k : M \to \mathbb{R}$ the following unitary operator:

$$
U_k = \prod_{x \in M} \rho(\hat{h}_{k(s),x}).
$$

(9.14)
By the definition of the operators $U_k$, and $\rho(\hat{h}_{k,z})$, 
\[ U_k(|k'\rangle \otimes \mathcal{H}_\mu') = |k+k'\rangle \otimes \mathcal{H}_\mu', \]  
(9.15) 
for every two finitely supported functions $k$ and $k'$ defined on $M$, and every Hilbert subspace $\mathcal{H}_\mu' \subset \mathcal{H}_\mu$. On the other hand, the invariant space $\tilde{\mathcal{H}}$ is in particular both $U_k$ and $(U_k)^{-1} = U_{-k}$ invariant, hence 
\[ U_k(\tilde{\mathcal{H}}) = \tilde{\mathcal{H}}. \]  
(9.16) 
In conclusion, 
\[ |k\rangle \otimes \tilde{\mathcal{H}}_{k,\mu} = U_k(|0\rangle \otimes \tilde{\mathcal{H}}_{0,\mu}) \]  
(9.17)  
\[ = U_k(|0\rangle \otimes \mathcal{H}_\mu \cap \tilde{\mathcal{H}}) = U_k(|0\rangle \otimes \tilde{\mathcal{H}}_{0,\mu}) \]  
(9.18)  
\[ = |k\rangle \otimes \tilde{\mathcal{H}}_{0,\mu}. \]  
(9.19) 
Therefore, the subspace pointed out in the conclusion of this theorem is 
\[ \tilde{\mathcal{H}}_{\mu} = \tilde{\mathcal{H}}_{0,k}. \]  
□

To complete the conclusion of theorem 9.1, consider a Hermitian representation $(D, \langle \cdot, \cdot \rangle, \rho)$ of $\mathfrak{A}$. Suppose it is either defined by a momentum self-adjoint GNS representation and by formulae (6.8), (6.9), (6.11), or it is one of the representations (7.20), (7.21). It any case, each of the operators $\rho(\hat{h}(s)), s \in FS$ is essentially self-adjoint, hence it defines a unitary operator $\exp(\i \rho(\hat{h}(s)))$. Suppose the subspace $\tilde{\mathcal{H}} = \mathbb{C}y1 \otimes \tilde{\mathcal{H}}_{\mu}$ of theorem 9.1 is preserved by each of the operators $\exp(\i \hat{h}(s)))$. Then the subspace 
\[ \tilde{\mathcal{H}}_{\mu} \subset L^2(\mathbb{R}, d\mu^*) \]  
is necessarily preserved by the operator $\exp(\i \lambda \hat{h}), \lambda \in \mathbb{R}$ (see 7.19). On the other hand, generically, it is not hard to find such a subspace $\tilde{\mathcal{H}}_{\mu}$. To see an example, suppose there is a $d\mu^*$-measurable, proper subset $V \subset \mathbb{R}$. Define a Hilbert subspace $\tilde{\mathcal{H}}_{\mu} \subset L^2(\mathbb{R}, d\mu^*)$ to be spanned by (classes of) all the square integrable functions of supports contained in $V$. The Hilbert space $\tilde{\mathcal{H}} = \mathbb{C}y1 \otimes \tilde{\mathcal{H}}_{\mu}$ has the non-trivial (dense in $\tilde{\mathcal{H}}$) intersection with the domain $D^*$ of the adjoint extension 
\[ \tilde{\mathcal{D}} = D^* \cap \tilde{\mathcal{H}}. \]  
(9.20) 
The subspace $\tilde{\mathcal{D}}$ is invariant with respect to representation $\rho^*$. Hence, the restriction of $\rho^*$ to $\tilde{\mathcal{D}}$ is a new Hermitian representation of the algebra $\mathfrak{A}$.

That representation is equivalent to the adjoint extension $(D_{\mu'}^*, \langle \cdot, \cdot \rangle', \rho_{\mu'}^*)$ of the representation $(D_{\mu}; \langle \cdot, \cdot \rangle, \rho_{\mu})$ defined by (7.20), (7.21) and a measure 
\[ d\mu' = 1_V \ d\mu^*. \]  
The only case in which there is no invariant subspace $\tilde{\mathcal{H}}$ in theorem 9.1, is the state $\mu : \mathbb{C} \rightarrow \mathbb{C}$ defined by a measure $d\mu = \delta_{r_0}$ supported at a single point $r_0 \in \mathbb{R}$.
10. Discussion

10.1. Summary of the results

The polymer $*$-algebra $\mathcal{A}$ (definition 2.7) contains the momentum subalgebra generated by the smeared momentum variables (definition 2.3). The study of diffeomorphism invariant states on the momentum subalgebra is crucial for characterization of possible diffeomorphism invariant states on $\mathcal{A}$.

In the $C^{(n)}$ compactly supported smearing functions case (section 3), there is no non-trivial $C^{(n)}$-diffeomorphism invariant state on the momentum subalgebra. This is the meaning of the identity (3.3) derived from the invariance. This fact and the assumption property 2.10 determine the unique state on $\mathcal{A}$ (3.2). The corresponding representation (3.13), (3.14) of $\mathcal{A}$ is the one used in LQG in the quantization of the scalar field. The uniqueness was not a surprise, and the argument used in the proof was similar to that of [23].

The remaining part of the work was focused on the case of the momentum variables $\pi(s)$ defined by smearing against the characteristic functions of regions in $\mathcal{M}$. The result is the construction of new states. Now, the momentum subalgebra—identified with the CW-complex algebra $\exp(\odot F_{C}S)$ in this case (section 4.2)—does admit non-trivial, diffeomorphism invariant states. In order to construct explicit examples, we assumed a greater symmetry, namely the homeomorphism invariance (see section 2.6). We succeeded in deriving all the states. The complete class is characterized in theorem 4.13.

Again, each of the states defined on the momentum subalgebra determines a state on the polymer $*$-algebra $\mathcal{A}$ upon the assumption property 2.10. This leads to the derivation of all the homeomorphism invariant states defined on the polymer $*$-algebra $\mathcal{A}$ which satisfy property 2.10 (theorem 5.5). The states are labelled by all the states $\mu : \mathbb{C}[\tau] \to \mathbb{C}$ in a 1–1 manner, where $\mathbb{C}[\tau]$ is the $*$-algebra of the polynomial functions on $\mathbb{R}$, and $\tau = \text{id} : \mathbb{R} \to \mathbb{R}$.

The resulting GNS representations of $\mathcal{A}$ corresponding to the derived states are provided in an explicit form in theorem 6.2. The properties of the representations are analysed in sections 7–9.

Given any of the representations $\rho$ of $\mathcal{A}$, the momentum operators $\rho(\hat{\pi}(s))$ may or may not be essentially self-adjoint. The necessary and sufficient conditions for that momentum self-adjointness are formulated in terms of the labelling state $\mu : \mathbb{C}[\tau] \to \mathbb{C}$ in theorem 7.1.

Whereas the self-adjoint extension of each of the essentially self-adjoint momentum operators is quite well understood, the corresponding extension of the entire representation $\rho$ requires special care. The suitable framework was derived by Schmudgen [24]. We show that for each of our momentum self-adjoint representations $\rho$ of $\mathcal{A}$, the adjoint extension $\rho^*$ (definition 7.4) is a self-adjoint (definition 7.7) Hermitian representation of $\mathcal{A}$. The remaining representations we found are also extended to homeomorphism invariant, self-adjoint according to [24]. Hermitian representations of $\mathcal{A}$. Each resulting self-adjoint representation $\rho_{\mu}^{*}$ of $\mathcal{A}$ is determined by a measure $d\mu$ on $\mathbb{R}$, provided the algebra of the polynomials is dense in $L^2(\mathbb{R}, d\mu)$. This is the outcome of section 7.

Two self-adjoint representations $\rho_{\mu_1}^{*}$ and $\rho_{\mu_2}^{*}$, constructed in section 7 are equivalent if and only if measures $d\mu_1$ and $d\mu_2$ are absolutely continuous with respect to each other (theorem 8.2). The intertwining Hilbert space isomorphism is found as well.

The issue of reducibility of the representations considered above is solved in section 9. A general characterization of invariant subspaces is given (theorem 9.1). The final conclusion is

**Corollary 10.1.** Suppose a representation $(\rho, D, \langle \cdot, \cdot \rangle)$ of the polymer $*$-algebra $\mathcal{A}$ is either

- the representation defined by (6.8, 6.9, 6.11) and a state $\mu : \mathbb{C}[\tau] \to \mathbb{C}$, and assume it is momentum self-adjoint;
Then, there is no proper subspace $\tilde{H} \subset \bar{D}$ preserved by the unitary extensions of all the operators $\exp(i\rho(\hat{\pi}(s)))$ and $\rho(\hat{h}_{k,x})$, unless the state $\mu$ (the measure $d\mu^*$) is the delta measure $\delta_{\tau_0}$ supported at $\tau_0 \in \mathbb{R}$. The definition of the representation reads

$$(D, \langle \cdot, \cdot \rangle) = (\text{Cyl}, \langle \cdot|\cdot \rangle_{\text{Cyl}}),$$

$$(10.1)$$

$$\rho(\hat{h}_{k,x})|k\rangle = |k + k1_{[\xi]}\rangle,$$

$$\rho(\hat{\pi}(s))|k\rangle = \left(\sum_{x \in M} k(x)s(x) + \tau_0(-1)^{\dim M} \chi_{E}(s)\right)|k\rangle.$$  

If $\tau'_0 \neq \tau_0$, then the representation $\rho'$ corresponding to $\tau'_0$ is inequivalent to $\rho$.

10.2. Relevance for the holonomy-flux algebra

The polymer $*$-algebra considered in this work is used in loop quantum gravity for the quantum scalar field coupled with the quantum geometry. The quantum geometry itself is described in terms of the holonomy-flux $*$-algebra [23]. The holonomy-flux $*$-algebra has a similar structure to the polymer $*$-algebra, and one could say that the latter is a simplified version of the former one. In particular, the quantum flux variable $\hat{P}(f)$—the counterpart of the quantum momentum operators—is defined by a smearing function $f : M \rightarrow su(2)$ of the support contained in an oriented 2-simplex $s$ in a three-dimensional piecewise-analytic manifold $M$, and taking values in the Lie algebra $su(2)$. In particular, $f$ is often assumed to be of the form

$$f = 1_s\xi, \quad \xi \in su(2).$$

The corresponding momentum can be denoted by $\hat{P}_{s,\xi}$. Naturally, the question arises, if the class of the topological states defined on the simplex algebra in section 4 of the current paper can be used to define a class of new diffeomorphism invariant states on the holonomy-flux algebra. Indeed, by exactly the same argument as in section 4 we prove that for every topologically invariant state $\omega$

$$\omega((\hat{P}_{s,\xi})^*(\hat{P}_{s,\xi} - \hat{P}_{s',\xi})) = 0,$$ 

for two arbitrary piecewise-analytic 2-simplexes $s$ and $s'$ in $M$. However, in this case a diffeomorphism which flips the orientation of $s$ into the opposite one is, on the one hand, a symmetry of $\omega$ but on the other hand maps

$$\hat{P}(s, \xi) \mapsto -\hat{P}_{s,\xi}.$$  

(10.4)

It follows from (10.3), (10.4) that

$$\omega(\hat{P}_{s,\xi}^* \hat{P}_{s,\xi}) = 0.$$  

Eventually, the only state is the one used in LQG. However, if we relaxed the topological invariance assumption, in favour of proper diffeomorphism invariance, perhaps new states could be found on the holonomy-flux algebra defined by the characteristic functions. On the other hand, we have the uniqueness result [23] valid in the case of the compactly supported, $C^{(0)}$-smearing functions $f$ (analogous to those in section 3).

Acknowledgments

We have benefited from discussions with Abhay Ashtekar, John Baez, Marcin Bobieński, Witold Marciszewski, Tadeusz Mostowski, Hanno Sahlmann, Lee Smolin, Thomas Thiemann and Andrzej Trautman. The work was partially supported by the Polish Ministerstwo Nauki i Informatyzacji grant no 1 P03A 015 29.
Appendix. Piecewise-analytic simplexes and manifolds

In the main part of this work we are using piecewise-analytic simplexes embedded in a manifold $M$ endowed with an (appropriately defined) piecewise-analytic structure. Those notions are defined below. The property of the piecewise-analytic simplexes crucial in this paper is property A.7 below.

**Definition A.1** [26]. A subset $X \subset \mathbb{R}^N$ is called semi-analytic if it has the following property: for every $x \in \overline{X}$ (the completion of $X$) there is a neighbourhood $U \ni x$ in $\mathbb{R}^N$, and analytic functions $f_{ij}: U \to \mathbb{R}, i = 1, \ldots, n, j = 1, \ldots, n_i$ such that

$$U \cap X = \bigcup_{i=1}^{n} X_i, \quad X_i = \{ x \in U : f_{ij}(x) \leq 0, j = 1, \ldots, n_i \}.$$  \hspace{1cm} (A.1)

Obviously, every analytic diffeomorphism maps semi-analytic sets into semi-analytic sets. However, the class of the diffeomorphisms of that property is larger. Therefore, we define

**Definition A.2.** A $C^{(n)}$-diffeomorphism $\phi: U \to U'$, where $U$ and $U'$ are open subsets in $\mathbb{R}^N, 0 \leq n < \infty$, is called piecewise-analytic if for every semi-analytic subset $X$ of $\mathbb{R}^N$ the subsets $\phi(X \cap U)$ and $\phi^{-1}(X \cap U')$ are semi-analytic.

A subfamily of piecewise-analytic diffeomorphisms was introduced in [23] and called semi-analytic diffeomorphisms. Many examples were constructed. In particular, it was shown that for every point $x \in \mathbb{R}^N$, and every neighbourhood $U$ of $x$, there is a semi-analytic diffeomorphism which moves $x$ but is the identity map on $\mathbb{R}^N \setminus U$. In this sense, the structure we are fixing in $\mathbb{R}^N$ to consider the semi-analytic sets admits local degrees of freedom.

Given a manifold $M$, a structure compatible with the semi-analytic sets can be defined as follows. Let $M$ be a differentiable (or topological) manifold of the differentiability class $C^{(n)}, 0 < n < \infty$ (respectively, $n = 0$). Suppose the maximal atlas admits a subatlas $((U_I, \chi_I))_{I \in I}$ labelled by some set $I$, such that for every two chard $\chi_I$ and $\chi_J$, the map $\chi_J \circ \chi_I^{-1} : \chi_I(U_I \cap U_J) \to \chi_J(U_I \cap U_J)$ is a piecewise-analytic $C^{(n)}$-diffeomorphism. We call the family $((\chi_I, U_I))_{I \in I}$ a piecewise-analytic atlas on $M$ and extend to a maximal piecewise-analytic atlas. A chard $(\chi, U)$ belonging to the maximal piecewise-analytic atlas is called a piecewise-analytic chard.

**Definition A.3.** A manifold endowed with a piecewise-analytic atlas is called piecewise analytic.

Obviously, every analytic manifold is piecewise analytic. A piecewise-analytic diffeomorphism $M \to M$ of a piecewise-analytic manifold $M$ is a diffeomorphism such that it itself and its inverse preserve the maximal piecewise-analytic atlas.

Now we turn to semi-analytic simplexes. As before, we begin with $\mathbb{R}^N$:

**Definition A.4.** A 0-simplex in $\mathbb{R}^N$ is a point. If $k > 0$, a piecewise-analytic $k$-simplex in $\mathbb{R}^N$ is a semi-analytic set $S$ such that there is a homeomorphism $\phi : \mathbb{R}^N \to \mathbb{R}^N$ which maps $S$ and the completion $\overline{S}$ onto the following sets:

$$\phi(S) = \left\{ (x^1, \ldots, x^N) \in \mathbb{R}^N : \sum_{i=1}^{k} x^k < 1, \text{ and } x^{k+1}, \ldots, x^N = 0 \right\}.$$  \hspace{1cm} (A.2)
\[\phi(\tilde{S}) = \left\{(x^1, \ldots, x^N) \in \mathbb{R}^N : \sum_{i=1}^{k} x^i \leq 1, \text{and } x^{k+1}, \ldots, x^N = 0\right\}. \quad (A.3)\]

And next we proceed with a manifold:

**Definition A.5.** A k-simplex in a piecewise-analytic manifold is a subset \(S \subset M\) such that there is a piecewise-analytic chord \((\chi, U)\) which maps \(S\) onto a semi-analytic k-simplex in \(\mathbb{R}^N\).

The simplexes are used for partitions called triangulations:

**Definition A.6.** A piecewise-analytic triangulation of subsets \(X_1, \ldots, X_n \subset M\), where \(M\) is a piecewise-analytic manifold, is a family of pairwise disjoint subsets \(S_1, \ldots, S_m \subset M\), such that \(S_i\) is a \(k_i\)-simplex in \(M\) for \(i = 1, \ldots, n\), and

\[X_i = \bigcup_{k=1}^{n_i} S_{j_k}, \quad i = 1, \ldots, n, \quad 1 \leq j_k \leq m. \quad (A.4)\]

The property of the semi-analytic sets and piecewise-analytic triangulations crucial for our work is [26] (see Lojasiewicz, page 463, theorems 1 and 2):

**Property A.7.** Every finite family of piecewise-analytic simplexes in a piecewise-analytic manifold \(M\) admits a piecewise-analytic triangulation.

Let us remark on the status of the framework we are using. The theory of the semi-analytic sets is well established in the mathematical literature [26]. We apply the deep results of that theory. There seems to be no unique generalization of the notion of semi-analyticity to a category of ‘semi-analytic’ manifolds, though. (Probably each generalization has its drawbacks and limitations.) In the current paper, we introduce the weakest, from the point of view of our aims, definition of a manifold structure compatible with the definition of the semi-analytic sets. It relies on a somewhat implicit (but not empty) definition of the piecewise-analytic diffeomorphisms in \(\mathbb{R}^N\). In [23] we introduced a family of explicitly defined ‘semi-analytic’ diffeomorphisms, and used them to define ‘semi-analytic’ manifolds. They form a subclass in the class of the piecewise-analytic manifolds defined above. Finally, our definitions should also be compared with similar ideas of [27].

**References**

[1] Smolin L 2005 The case for background independence Preprint hep-th/0507235
[2] Rovelli C and Smolin L 1988 Knot theory and quantum gravity Phys. Rev. Lett. 61 1155–8
[3] Rovelli C and Smolin L 1990 Loop representation for quantum general relativity Nucl. Phys. B 331 80–152
[4] Ashtekar A 1991 Lectures on Non-Perturbative Canonical Gravity (Notes Prepared in Collaboration with R S Tate) (Singapore: World Scientific)
[5] Ashtekar A and Lewandowski J 1994 Representation theory of analytic holonomy algebras Knots and Quantum Gravity ed J C Baez (Oxford: Oxford University Press)
[6] Baez J C 1994 Generalized measures in gauge theory Lett. Math. Phys. 31 213–23
[7] Lewandowski J 1994 Topological measure and graph-differential geometry on the quotient space of connections Int. J. Mod. Phys. D 3 207–10
[8] Lewandowski J 1997 Volume and quantizations Class. Quantum Grav. 14 71–6
[9] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry: I. Area operators Class. Quantum Grav. 14 A55–A81
[10] Ashtekar A and Lewandowski J 1997 Quantum theory of geometry: II. Volume operators Adv. Theor. Math. Phys. 1 388–429
[11] De Pietri R 1997 Spin networks and recoupling in loop quantum gravity *Nucl. Phys. Suppl.* **57** 251–4

[12] De Pietri R 1997 On the relation between the connection and the loop representation of quantum gravity *Class. Quantum Grav.* **14** 53–70

[13] Thiemann T 1998 Closed formula for the matrix elements of the volume operator in canonical quantum gravity *J. Math. Phys.* **39** 3347–71

[14] Thiemann T 1998 A length operator for canonical quantum gravity *J. Math. Phys.* **39** 3372–92

[15] Major S A 1999 Operators for quantized directions *Class. Quantum Grav.* **16** 3859–77

[16] Thiemann T 1996 Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity *Phys. Lett. B* **380** 257–64

[17] Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839–73

[18] Thiemann T 1998 QSD: III. Quantum constraint algebra and physical scalar product in quantum general relativity *Class. Quantum Grav.* **15** 1207–47

[19] Thiemann T 2001 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press) (Preprint gr-qc/0110034) (at press)

Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* **21** R53 (Preprint gr-qc/0404018)

Smolin L 2004 An invitation to loop quantum gravity Preprint hep-th/0408048

Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)

[20] Thiemann T 1998 QSD: V. Quantum gravity as the natural regulator of matter quantum field theories *Class. Quantum Grav.* **15** 1281–314

Thiemann T 1998 Kinematical Hilbert spaces for Fermionic and Higgs quantum field theories *Class. Quantum Grav.* **15** 1487–512 (Preprint gr-qc/9705021)

[21] Ashtekar A, Lewandowski J and Sahlmann H 2003 Polymer and Fock representations for a scalar field *Class. Quantum Grav.* **20** L11–21 (Preprint gr-qc/0211012)

[22] Kamiński W, Lewandowski J and Bobrowski B 2006 Background independent quantizations—the scalar field: I *Class. Quantum Grav.* **23** 2761–70

[23] Lewandowski J, Okołow A, Sahlmann H and Thiemann T 2005 Uniqueness of diffeomorphism invariant states on holonomy-flux algebras Preprint gr-qc/0504147

[24] Schmüdgen K 1990 *Unbounded Operator Algebras and Representation Theory* (Basel: Birkhauser)

[25] Simon B 1999 The classical moment problem as a self-adjoint finite difference operator Preprint math-ph/9906008

[26] Łojasiewicz S 1964 Triangulation of semi-analytic sets *Ann. Scuola. Norm. Sup. Pisa* **18** 449–74

Bierstone E and Milman P D 1988 Semianalytic and subanalytic sets *Publ. Math. IHES* **67** 5–42

[27] Fleischhack C 2004 Representations of the Weyl algebra in quantum geometry Preprint math-ph/0407006