MASS DEPENDENCE OF QUANTUM ENERGY INEQUALITY BOUNDS

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ABSTRACT. In a recent paper [J. Math. Phys. 47 082303 (2006)], Quantum Energy Inequalities were used to place simple geometrical bounds on the energy densities of quantum fields in Minkowskian spacetime regions. Here, we refine this analysis for massive fields, obtaining more stringent bounds which decay exponentially in the mass. At the technical level this involves the determination of the asymptotic behaviour of the lowest eigenvalue of a family of polyharmonic differential equations, a result which may be of independent interest. We compare our resulting bounds with the known energy density of the ground state on a cylinder spacetime. In addition, we generalise some of our technical results to general $L^p$-spaces and draw comparisons with a similar result in the literature.

1. INTRODUCTION

Quantum Energy Inequalities (QEIs) quantify the extent to which a quantum field can violate the energy conditions of classical general relativity. For example, the real scalar field in $d$-dimensional Minkowski space admits physically reasonable states\(^1\) for which the expected energy density is negative, thus violating the Weak Energy Condition. Moreover, the energy density at any given point is unbounded from below. However, the theory also satisfies a QEI bound \([10, 12]\) which asserts that averages of the (normal ordered) energy density along an inertial curve $\gamma$, parameterized by proper time and with velocity $u^a$, obey

\[
\int d\tau \langle u^a u^b; T_{ab}; (\gamma(\tau)) \rangle \psi g(\tau)^2 \geq -K_d \int_m^\infty \frac{du}{\pi} u^d |\hat{g}(u)|^2 Q_d(u/m)
\]

for all physically reasonable states $\psi$, where $g$ is any smooth, compactly supported real-valued function, $\hat{g}(u) = \int e^{-iu\tau} g(\tau)\,d\tau$ is its Fourier transform,

\[
Q_d(x) = \frac{d}{x^d} \int_1^x dy y^2(y^2 - 1)^{(d-3)/2}
\]

and the constant $K_d = A_{d-2}/(2d(2\pi)^{d-1})$, with $A_k$ the area of the unit $k$-sphere. Similar QEIs are known for a variety of free field theories in flat and curved spacetimes, and also for positive energy conformal field theories in two-dimensional Minkowski space. We refer the reader to the reviews \([9, 22]\) and \([11]\) for references and applications of QEIs.

For many purposes it is convenient to have a simpler form for the QEIs. One way of doing this was developed recently in \([11]\): estimating $\langle u^a u^b; T_{ab}; (\gamma(\tau)) \rangle \psi$ by its supremum over an open

\(^1\)By ’physically reasonable’, we mean Hadamard states, which have smooth normal-ordered two-point functions.
particularly useful, unless supplemented with a discussion of how the constants $\tau_0$ limit, as is the simpler formula (nearly) exponential decay can be obtained; our main result is that the above bound is satisfied for $m \tau_0 > 0$. For massless scalar fields in even-dimensional Minkowski space, this may be converted to an eigenvalue problem, leading to the bound

$$\sup_{\tau \in I} \langle u^a u^b : T_{ab} : (\gamma(\tau)) \rangle \psi \geq -K_d \sqrt{\frac{2}{\pi}} \Gamma(2n+1) \Gamma(n+1) \Gamma(2n+1/2) \frac{a b}{m^{d/2}}$$

for all real-valued $g$ compactly supported in $I$. As the left-hand side is independent of $g$, we are free to optimize the inequality over the class of permissible $g$. We were not able to optimize over $n$ in (3) either, because this is a rather weak estimate if $m \tau_0 \gg 1$: fixing any $g$, it is easy to show that the right-hand side of (3) decays faster than any inverse polynomial as $m \tau_0 \to \infty$. Thus we know that bounds of the type

$$\sup_{\tau \in I} \langle u^a u^b : T_{ab} : (\gamma(\tau)) \rangle \psi \geq -C_{d,n} m^{d/2}$$

exist for suitable constants $C_{d,n}$ and any integer $n \geq d/2$. This information in itself is not particularly useful, unless supplemented with a discussion of how the constants $C_{d,n}$ grow with $n$. The purpose of the present paper is to investigate this question. In fact we will find that (nearly) exponential decay can be obtained; our main result is that the above bound is satisfied with

$$C_{d,n} = K_d \Gamma(2n+1) \Gamma(n+1) \Gamma(2n+1/2) \frac{a b}{m^{d/2}}$$

for $n \geq d/2$. Here $K'_d = K_d$ except for $d = 2$, where $K'_2 = 6 K'_2/5$.

We are also free to optimize over $n$. This procedure leads to a bound

$$\sup_{\tau \in I} \langle u^a u^b : T_{ab} : (\gamma(\tau)) \rangle \psi \geq -Q(m, \tau_0)$$

where

$$Q(m, \tau_0) \sim K'_d \sqrt{\pi} m^{d/2} (m \tau_0)^{1/2} e^{-m \tau_0/2}$$

for $m \tau_0 \gg 1$.

One of the key steps in our argument is to show that the eigenvalue $\lambda_n$ obeys

$$\frac{2 \Gamma(n+1) \Gamma(2n+1/2)}{\Gamma(n+1/2) R_n} < \lambda_n < \frac{2 \Gamma(n+1) \Gamma(2n+1/2)}{\Gamma(n+1/2)}$$

for $n \geq 1$, where $R_n = F_3(1/2, 1, -n; 1 - 2n, n + 1; 1)$ is given in terms of the hypergeometric function $F_3$ [1]; moreover, $R_n \to 1$ as $n \to \infty$, so both bounds are asymptotic to $\lambda_n$ in this limit, as is the simpler formula $\sqrt{2}(2n)!$ (which we do not claim to be a bound). We were not
able to locate this fact, which may be of independent interest, in the literature. Some analogous results for the same operator, but with different boundary conditions, are known; see Sec. 6.

Clearly, the results presented here represent a considerable improvement on (4), and permit many of the results of [11] to be strengthened. The main thrust of [11] is that the Minkowski space results just mentioned also apply in curved spacetimes, provided the segment of the inertial curve $\gamma$ parameterized by $I$ may be contained in a ‘sufficiently large’ region in which the metric is Minkowskian. An example will be given in Sec. 5.

The paper has the following structure: in Sec. 2 we reduce our problem to the eigenvalue problem (5); estimates for the eigenvalues $\lambda_n$ are obtained in Sec. 3 and the optimisation over $n$ mentioned above is performed in Sec. 4. Sec. 5 contains an example in which our bound may be compared against a known value for the expectation value of the energy density. In this particular instance, the ratio of the actual energy density to our bound tends to zero exponentially as the mass increases so we cannot conclude that our bounds are asymptotically sharp. (Equally, we cannot rule out this possibility.) They nonetheless represent a distinct improvement on earlier results. In Sec. 6 we consider the underlying reasons for the success of the strategy employed in the previous sections. A mixture of theoretical and numerical evidence suggests that the sequence of solution operators to (5) may be ‘asymptotically rank 1’; a phenomenon which has been established elsewhere for solution operators to the same differential equation but with different boundary conditions. Finally, in Sec. 7 we show how our analysis may be extended to determine the $L^p$-operator norms on the solution operator to (5); this is of independent interest and allows some comparison between our main results (in an $L^2$-context) and a result obtained in [6] (which is related to the $L^1$ version of our problem).

2. Reduction to an eigenvalue problem

As initial preparation, we notice that the integrand in (2) is bounded from above by $y^{d-1}$ if $d > 2$, and hence $Q_d(x) < 1$ for $x > 1$. In the case $d = 2$, it may be shown [11] that $Q_2(x) < 6/5$ on this domain. Accordingly, (3) implies

$$\sup_{\tau \in I} \langle u a^b : T_{ab} : (\gamma(\tau)) \rangle \psi \geq -K'_d \int m \frac{du}{\pi} u^d |\hat{g}(u)|^2 \int d\tau g(\tau)^2$$

for all real-valued $g \in C_0^\infty(I)$, where $K'_d = K_d$ for $d > 2$, $K'_2 = 6K_2/5$. Since the right-hand side is invariant under translations in $g$, we may assume without loss of generality that $I = (-\tau_0/2, \tau_0/2)$. Writing $g(\tau) = h(2\tau/\tau_0)$, where $h \in C_0^\infty(-1, 1)$, we obtain

$$\sup_{\tau \in I} \langle u a^b : T_{ab} : (\gamma(\tau)) \rangle \psi \geq -\frac{2^d K'_d}{\tau_0^d} \int_1^\infty \frac{du}{\pi} y^d |\hat{h}(y)|^2 \int dt h(t)^2$$

for all real-valued $h \in C_0^\infty(-1, 1)$, where $x = m\tau_0/2$.

Defining, for any such $h$,

$$H_{d,h}(x) = \frac{\int_x^\infty \frac{du}{\pi} y^d |\hat{h}(y)|^2}{\int_{-1}^1 dt |h(t)|^2},$$
our problem is now to estimate from above the function
\[ x \mapsto \inf_h H_{d,h}(x) \]
for \( x \gg 1/2 \). Writing \((Df)(t) = i df/\,dt\), observe that for any \( n \geq d/2 \), and \( x > 0 \)
\[ \|h\|^2 H_{d,h}(x) \leq \frac{x^d}{x^{2n}} \int_x^\infty \frac{dy}{\pi} |\hat{D^n h}(y)|^2 \]
\[ \leq x^{d-2n} \int_0^\infty \frac{dy}{\pi} |\hat{D^n h}(y)|^2 \]
\[ = x^{d-2n} \int_\infty^\infty \frac{dy}{2\pi} |\hat{D^n h}(y)|^2 \]
\[ = x^{d-2n} \int_{-1}^1 dt |(D^n h)(t)|^2 \]
using the monotone decrease of \( y^{d-2n} \) on \( \mathbb{R}^+ \), the fact that \(|\hat{D^n h}(u)|^2\) is even because \( h \) is real-valued, Parseval’s identity, and the fact that \( h \) is supported on \((-1, 1)\). Introducing the usual \( L_2 \)-inner product \( \langle \cdot, \cdot \rangle \) on \((-1, 1)\) (by convention, this is linear in the second slot) and its associated norm \( \| \cdot \| \), we can write the last expression in the form
\[ H_{d,h}(x) \leq x^{d-2n} \frac{\int_{-1}^1 dt |(D^n h)(t)|^2 \|h\|^2}{\langle h, h \rangle} = x^{d-2n} \frac{\langle D^n h, D^n h \rangle}{\langle h, h \rangle} \]
(using the symmetry of \( D^n \) on \( C_0^\infty (-1, 1) \)) and minimise the right-hand side over \( h \) (excluding the identically zero function). By Theorem X.23 in [20], the infimum is the lowest element of the spectrum of the Friedrichs extension \( A \) of \( D^{2n} \) on \( C_0^\infty (-1, 1) \), whose domain is the intersection of Sobolev spaces [2] \( D(A) = W_0^{n,2}(-1, 1) \cap W_0^{2n,2}(-1, 1) \) [see, e.g., Sec. II.B in [13]]. Moreover, the operator \( A \) has compact resolvent, by a straightforward modification of the proof of Theorem XIII.73 in [21], so \( A \) has purely discrete spectrum. Using elliptic regularity, the eigenvectors of \( A \) are smooth solutions to (5), and since they belong to \( W_0^{n,2}(-1, 1) \) they obey the boundary conditions \( \psi(\pm 1) = \psi'(\pm 1) = \cdots = \psi^{(n-1)}(\pm 1) = 0 \).

To summarise: we have established that
\[ \inf_h H_{d,h}(x) \leq x^{d-2n} \lambda_n, \]
where \( \lambda_n \) is the minimal eigenvalue of (5) subject to the boundary conditions just mentioned.

3. Estimates for the Minimal Eigenvalue

Let \( G : [-1, 1]^2 \to \mathbb{R} \) denote Green’s function for (5), and \( T \) denote the associated solution operator
\[ (Tf)(t) = \int_{-1}^1 G(t, s)f(s) \, ds. \]
Since this is the inverse to the original problem, we seek the maximal eigenvalue, or spectral radius, of \( T \), which we shall denote \( r(T) \). Numerical investigation suggests that, for large \( n \), the
eigenfunction associated with the maximal eigenvector is well approximated by \( f(t) = (1-t^2)^n \); this observation leads us to rigorous bounds via the following fact.

**Lemma.** If \( a \) and \( b \) are positive constants such that \( af(t) \leq (Tf)(t) \leq bf(t) \) for all \( t \in [-1, 1] \), then \( a \leq r(T) \leq b \).

This is part of the general theory of order-preserving operators (see, for example, [16, Lemmas 9.1, 9.4]), but for the reader’s convenience we include the short proof.

**Proof of Lemma.** Since \( 0 \leq af(t) \leq (Tf)(t) \), we can square and integrate to give \( \|af\| \leq \|Tf\| \) and hence \( a \leq \|T\| \), where \( \|T\| \) denotes the operator norm of \( T \) acting on \( L^2(-1, 1) \). Since \( T \) is self-adjoint, \( \|T\| = r(T) \), so we have \( a \leq r(T) \). Note that the operator norm inequality is true for any \( L^p \) norm, not just for \( p = 2 \); we exploit this fact in Sec. 7.

The image under \( T \) of any \( L^2 \) function is \( 2n - 1 \) times differentiable and has derivatives of order up to \( n - 1 \) equal to zero at both endpoints; in particular, it can be written as \( p(t) = (1-t^2)^n q(t) = f(t)q(t) \), where \( q \in C[-1, 1] \). The Banach space \( X \) of all such functions, with norm

\[
\|p\|_f = \sup_{t \in (-1,1)} \frac{|p(t)|}{f(t)}
\]

is therefore \( T \)-invariant and contains all of the \( L^2 \) eigenfunctions of \( T \). In particular, the spectral radius of \( T \) as an operator on \( X \) is the same as its spectral radius as an operator on \( L^2(-1, 1) \).

For any \( p \in X \), we have by definition

\[
-b\|p\|_f f(t) \leq (Tf)(t) \leq b\|p\|_f f(t)
\]

Since \( G \) is non-negative [4], we can apply \( T \) to this inequality to give

\[
-b\|p\|_f f(t) \leq (Tp)(t) \leq b\|p\|_f f(t)
\]

Since \( (Tf)(t) \leq bf(t) \), we have

\[
-b\|p\|_f f(t) \leq (Tp)(t) \leq b\|p\|_f f(t)
\]

which is to say that \( \|Tp\|_f \leq b\|p\|_f \). This shows that the operator norm of \( T \) on \( X \) is no larger than \( b \), and hence that \( r(T) \leq b \). \( \square \)

Note that the lower bound on \( r(T) \) does not depend on the positivity of the Green function \( G \).

To find suitable constants, we shall find an exact formula for \( Tf \). Although there is an explicit formula for the Green function [4], its use would involve some integrals which are not obviously tractable; we therefore exploit the fact that \( f \) and \( Tf \) are both polynomials of known degree, to reduce the differential equation to a finite system of linear equations.

Since \( f \) is a polynomial of degree \( 2n \), all of its \( (2n) \)th order integrals are polynomials of degree \( 4n \), exactly one of which satisfies the boundary conditions, or equivalently has a factor of \( (1-t^2)^n \). Moreover, the differential operator and boundary conditions commute with reflection in the origin, so the same is true of \( T \); since \( f \) is even, \( Tf \) is also even. In view of the factor \( (1-t^2)^n \), it is convenient to write \( (Tf)(t) = (1-t^2)^n P(1-t^2) \), where \( P \) is a polynomial of degree \( n \), say \( P(z) = \sum_{k=0}^n \alpha_k z^k \). The bounds for the spectral radius are then

\[
\min_{t \in [-1,1]} P(1-t^2) \leq r(T) \leq \max_{t \in [-1,1]} P(1-t^2).
\]
To determine $P$, we must solve the equation

$$(-1)^n \frac{d^{2n}}{dt^{2n}} \sum_{k=0}^{n} \alpha_k (1 - t^2)^{n+k} = (1 - t^2)^n.$$ 

We approach this simply by expanding the powers of $1 - t^2$ using the binomial theorem, differentiating, and equating coefficients.

$$(1 - t^2)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r t^{2r}$$

We simplify this expression following the procedure in [19, §3.3]. Letting $r = k - j$, we have

$$\alpha_j = \sum_{k=j}^{n} (-1)^{j+k} \binom{n+k}{n+j} \frac{(2k)!}{[2(n+k)]!} \binom{n}{k}.$$

We simplify this expression following the procedure in [19, §3.3]. Letting $r = k - j$, we have

$$\alpha_j = \sum_{r=0}^{n-j} (-1)^r \binom{n+r+1}{n+j} \frac{(2j+2r)!}{(2n+2j+2r)!} \binom{n}{j+r}.$$
and the ratio of term $r + 1$ to term $r$ is
\[
\frac{(r + j + 1/2)(r + j-n)}{(r + j + n + 1/2)(r + 1)}.
\]
(note that this formula gives 0 if $r = n - j$). We can therefore identify the sum as a hypergeometric function
\[
\alpha_j = \frac{(2j)!}{[2(n+j)]!} \left( \begin{array}{c} n \\ j \end{array} \right) _2F_1(j + 1/2, j - n; j + n + 1/2; 1).
\]
Gauss’s identity [19, §3.5] states that, provided $\text{Re}(c - a - b) > 0$,
\[
_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.
\]
We apply to this to give
\[
_2F_1(j + 1/2, j - n; j + n + 1/2; 1) = \frac{\Gamma(j + n + 1/2)\Gamma(2n - j)}{\Gamma(n)\Gamma(2n + 1/2)}
\]
and can therefore conclude that
\[
\alpha_j = \frac{n\Gamma(1/2 + n + j)\Gamma(2n - j)\Gamma(1 + 2j)}{\Gamma(1/2 + 2n)\Gamma(2n + 1 + 2j)\Gamma(1 + j)\Gamma(n + 1 - j)}.
\]
Since each $\alpha_j$ is positive, $P(1 - t^2)$ attains its minimum and maximum over $[-1, 1]$ at $±1$ and 0, respectively, and we have
\[
\alpha_0 = P(0) \leq r(T) \leq P(1) = \sum_{j=0}^{n} \alpha_j.
\]
From the point of view of the application, the lower bound is the more important one (and, as already mentioned, does not depend on the positivity of the Green function). Before we consider this, though, we shall show that the ratio of the upper and lower bounds tends to 1 as $n \to \infty$, so both bounds are in fact asymptotically equal to $r(T)$ as $n \to \infty$. The ratio is
\[
R_n = \frac{1}{\alpha_0} \sum_{j=0}^{n} \alpha_j
\]
and we can use the same technique as before to identify this sum as a hypergeometric function. The ratio of two successive terms is given by
\[
\frac{\alpha_{j+1}}{\alpha_j} = \frac{(2j + 1)(n - j)}{2(2n - 1 - j)(n + j + 1)} = \frac{(j + 1/2)(j - n)(j + 1)}{(j + 1 - 2n)(j + n + 1)(j + 1)}
\]
so
\[
\frac{1}{\alpha_0} \sum_{j=0}^{n} \alpha_j = 3F_2(1/2, 1, -n; 1 - 2n, n + 1; 1).
\]
We can now calculate the limit; our strategy is influenced by an unpublished calculation of T.H. Koornwinder, also employed in [18]. We first expand in Pochhammer symbols to give

\[
R_n = \sum_{k=0}^{n} \frac{(1/2)_k(1)_k(-n)_k}{(1-2n)_k(n+1)_k k!}
\]

where the sum terminates at \( n \) because \((-n)_k = 0 \) for \( k > n \). The \((1)_k\) term in the numerator cancels with the \( k! \) term in the denominator, and the other symbols can be expanded to give

\[
R_n = \sum_{k=0}^{n} c_{n,k}
\]

where

\[
c_{n,k} = \frac{\left((-n)(1-n)\ldots(k-1-n)\right)\left((1/2)(3/2)\ldots(k-1/2)\right)}{\left((1-2n)(2-2n)\ldots(k-2n)\right)\left((n+1)(n+2)\ldots(n+k)\right)} \leq \frac{1}{2^k}
\]

The last step is true because each term on the numerator is no larger than the corresponding term on the denominator; specifically,

\[
n - r + 1 \leq 2n - r; \quad 2r - 1 \leq n + r \quad (1 \leq r \leq n).
\]

We also have \( c_{n,0} = 1 \) and, for each \( k > 0 \), \( c_{n,k} \to 0 \) as \( n \to \infty \) (because \( c_{n,k} \) is a rational function of \( n \) whose denominator has degree \( k \) greater than its numerator). We now have \( 0 \leq c_{n,k} \leq d_k \), \( \sum_{k=1}^{\infty} d_k < \infty \) and, for each \( k \), \( c_{n,k} \to \delta_{k0} \) as \( n \to \infty \). It follows from Tannery’s theorem (i.e., the Dominated Convergence Theorem on a measure space of countably many atoms of mass 1) that

\[
R_n = \sum_{k=0}^{n} c_{n,k} \xrightarrow{n \to \infty} \sum_{k=0}^{\infty} \delta_{k0} = 1.
\]

We now know that

\[
\alpha_0 = \frac{\Gamma(n + 1/2)}{2\Gamma(n + 1/2)\Gamma(2n + 1/2)}
\]

is a lower bound for \( r(T) \), asymptotically equal to \( r(T) \) as \( n \to \infty \). It follows from Stirling’s formula that

\[
r(T) \sim \frac{1}{\sqrt{2(2n)!}}
\]

as \( n \to \infty \); but this is greater than \( \alpha_0 \) for large \( n \), so we cannot conclude that this is a lower bound for \( r(T) \).
4. Optimisation over $n$

We know from the calculations in the previous section that
\[
\inf_{\mathcal{D}} H_{d,h}(x) \leq x^{d-2n} \frac{2\Gamma(n+1)\Gamma(2n+1/2)}{\Gamma(n+1/2)} =: \exp(F(n)).
\]
We now optimise over $n$ for fixed $x$, allowing, for the moment, non-integer values of $n$. The logarithmic derivative is
\[
F'(n) = -2 \log(x) + \Psi(n+1) + 2\Psi(2n+1/2) - \Psi(n+1/2)
\]
where $\Psi$ is the digamma function. We seek a critical point of $F$, so wish to solve the equation
\[
\log(x) = \frac{1}{2} \Psi(n+1) + \Psi(2n+1/2) - \frac{1}{2} \Psi(n+1/2).
\]
The right-hand side has an asymptotic expansion
\[
\log(2n) + \frac{1}{4n} + O(1/n^2)
\]
(a straightforward calculation from [1, Equation 6.3.18]) so we have
\[
x = 2n \exp(1/(4n)) \exp(O(1/n^2)) = 2n(1 + 1/(4n) + O(1/n^2))(1 + O(1/n^2)) = 2n + 1/2 + O(1/n).
\]
It is apparent that $n \sim x/2$ as $n \to \infty$, so $O(1/n) = O(1/x)$ and we can solve the equation to give
\[
n = x/2 - 1/4 + O(1/x). \quad (13)
\]
Next, we have (by definition of the polygamma function and using [1, Equation 6.4.12])
\[
F''(n) = \psi_1(n+1) + 4\psi_1(2n+1/2) - \psi_1(n+1/2) = 2/n + O(1/n^2)
\]
At the critical point, (13) is true; multiplying by $4/(nx)$ (and remembering $O(1/n) = O(1/x)$) yields
\[
\frac{2}{n} = \frac{4}{x} + O(1/x^2)
\]
so, at the critical point, $F''(n) = 4/x + O(1/x^2)$. We can therefore expand $F$ about its critical point $n_0$ ($F$ is analytic in the right half-plane) to give
\[
F(n_0 + \delta) = F(n_0) + \delta^2 \left(\frac{2}{x} + O(1/x^2)\right) + \frac{\delta^3}{3!} F'''(n_0 + \zeta)
\]
where $\zeta$ lies between 0 and $\delta$. We also have
\[
F'''(n) = -\frac{2}{n^2} + O(1/n^3)
\]
(using [1, Equation 6.4.13]) so, if we assume $|\delta| < 1$, which implies that $|\zeta| < 1$, we have $F'''(n_0 + \zeta) = O(1/x^3)$ (uniformly in $\delta$ and $\zeta$). This gives
\[
F(n_0 + \delta) = F(n_0) + \frac{2\delta^2}{x} + O(1/x^2). \quad (14)
\]
Now choose $\delta$ such that $|\delta| \leq 1/2$ and $n_0 + \delta \in \mathbb{N}$, so $\exp(F(n_0 + \delta))$ is the bound we seek. Since $2\delta^2/x + O(1/x^2) \to 0$ as $x \to \infty$, we have $\exp(F(n_0 + \delta)) \sim \exp(F(n_0))$ as $x \to \infty$.

Finally, since moving by a distance of up to $1/2$ from the critical point $n_0$ has no effect apart from a multiplicative factor converging to 1, moving from the exact critical point $n_0 = x/2 - 1/4 + O(1/x)$ to the approximate critical point $x/2$ will have no more of an effect. We thus have

$$\exp(F(n_0 + \delta)) \sim \exp(F(x/2))$$

as $x \to \infty$ and we can use Stirling’s formula on $\exp(F)$ to give an asymptotic formula for the bound:

$$\inf_h H_{d,h}(x) \leq \exp(F(n_0 + \delta)) \sim 2\sqrt{\pi x^{d+1/2}}e^{-x}$$

as $x \to \infty$.

5. Example: Cylinder spacetimes

Consider the cylinder spacetime $(N, \eta)$ formed by periodically identifying four-dimensional Minkowski space under a translation in the $z$-direction. The ground state energy density in this spacetime, for the quantized minimally coupled scalar field of mass $m$ is [23, 17]

$$\langle T_{tt} \rangle = -\sum_{n=1}^{\infty} \frac{m^2}{2\pi^2 (nL)^2} K_2(mnL)$$

(14)

where $L$ is the periodicity length. For $mL \gg 1$, the series is dominated by its first term, and so

$$\langle T_{tt} \rangle \sim -\frac{m^4 e^{-mL}}{(2\pi)^{3/2} (mL)^{5/2}}$$

(15)

Now consider a timelike curve $\gamma : (0, \tau_0) \to N$ given by $\gamma(\tau) = (\tau, 0, 0, 0)$, which has total proper duration $\tau_0$. If $\tau_0 \leq L$ we may enclose the curve in an open globally hyperbolic subset

$$D = I^+(\gamma(0)) \cap I^-(\gamma(\tau_0))$$

of the spacetime. Quantum field theory in $D$ is indistinguishable from quantum field theory in its isometric image in the covering Minkowski space. (See [5, 11] for a full presentation of this idea.) Accordingly, we may use Minkowski space QEIs to constrain the energy density in $D$. Applying our results, we obtain an a priori bound $\langle T_{tt} \rangle \geq -Q(m, \tau_0)$. The best constraint is obtained for $\tau_0 = L$; for $mL \gg 1$, this is asymptotically

$$Q(m,L) \sim \frac{m^4 (mL)^{1/2} e^{-mL/2}}{16\pi^2}$$

(16)

Note that although our a priori bound is consistent with the known value of the energy density, it does not adhere to the same exponential law. The same would be true for the ground state on other toroidal quotients of Minkowski space. There are three possible explanations for this. The first of these is that the weaker bound might be needed to accommodate states other than the ground state. Second, it may be that the estimate made in Sec. 2 are too weak (after that point, all our estimates are asymptotically sharp). Thirdly, it may also indicate that our starting form for the quantum energy inequality, which is known not to be optimal, becomes progressively less sharp at large mass.

These regions were called c.e.g.h.s. regions in [11]; here, we use the nomenclature of Sec. 6.6 of [14].
6. Remarks on the role of $(1 - t^2)^n$.

Let $T_n$ denote the solution operator to (5), as defined at the beginning of Sec. 3, $f_n(t) = (1 - t^2)^n$, $\lambda_n$ be the greatest eigenvalue of $T_n$, $E_n$ be the corresponding eigenspace (necessarily one-dimensional and spanned by a non-negative function $u_n$ of unit norm, because of the positivity of the Green function; see, for example, Theorem 11.1(b) in [16]), and $P_n$ be the corresponding spectral projection $P_n f = \langle u_n, f \rangle u_n$.

The substance of Sec. 3 is that $(T_n f_n)(t)/(\lambda_n f_n(t))$ tends to 1 uniformly in $t$ as $n \to \infty$. It is striking that there is such a simple formula which, in this asymptotic sense, behaves like $c/n$, where $c$ is a constant in the region of $1/4$. The numerical method starts by calculating Green’s function for Equation (5) explicitly for any particular $n$; numerical quadrature schemes can then be employed to calculate the eigenvalues to any required degree of precision. In the table below, $\lambda_1$ and $\lambda_2$ are the first two eigenvalues of $T_n$, the solution operator to (5). The fourth column is converging to 1, illustrating the asymptotic expressions for the dominant eigenvalue derived in Sec. 3. The final column appears to be converging to a limit in the region of $1/4$, as mentioned above.

| $n$ | $\lambda_1$ | $\lambda_2$ | $\sqrt{2}(2n)!\lambda_1$ | $n\lambda_2/\lambda_1$ |
|-----|--------------|--------------|--------------------------|------------------------|
| 5   | 2.01975 × 10^{-7} | 1.04991 × 10^{-8} | 1.03652                  | 0.259911               |
| 10  | 2.96037 × 10^{-19} | 7.56909 × 10^{-21} | 1.01856                  | 0.255861               |
| 15  | 2.69890 × 10^{-33} | 4.56884 × 10^{-35} | 1.01242                  | 0.253928               |
| 19  | 1.36524 × 10^{-45} | 1.81896 × 10^{-47} | 1.00982                  | 0.253144               |
| 20  | 8.74731 × 10^{-49} | 1.10651 × 10^{-50} | 1.00933                  | 0.252995               |

This conjectural asymptotic rank 1 behaviour is known to hold for a closely related problem, in which the boundary conditions in (5) are changed to $\psi^{(j)}(-1) = 0 (0 \leq j \leq n - 1)$, $\psi^{(j)}(+1) = 0 (n \leq j \leq 2n - 1)$. The solution operator for this problem is related to the Riemann-Liouville fractional integration operator, and this yields an asymptotically correct upper bound of $(n!)^2/2^{2n-2}$ for the minimal eigenvalue of the differential operator [24]. Similar bounds were found independently in [15]; for tighter bounds, see [3]. The asymptotic rank 1 property of the solution operators is an immediate consequence of results in [7] and [8] on iterated Volterra convolution operators.

The same property can be seen to hold in another example. If the boundary conditions for Equation (5) are changed to $\psi^{(j)}(-1) = 0 (0 \leq j \leq 2n - 2, j \text{ even})$ and $\psi^{(j)}(+1) = 0 (1 \leq j \leq 2n - 1, j \text{ odd})$ and $T_n$ represent the solution operator, then it is easy to see that
\( T_n = T^n_1 \). The leading eigenvalue of \( T_1 \) is simple, and it follows from the spectral theorem that \( T^n_1 / \| T^n_1 \| \) is asymptotically equal to the associated spectral projection.

7. Results for Other \( L^p \) Spaces

The solution operator to (5)

\[
(T_n f)(t) = \int_{-1}^{1} G_n(t, s) f(s) \, ds
\]

can be thought of as an operator on any of the spaces \( L^p(-1, 1) \) (\( 1 \leq p \leq \infty \)). Denote by \( \| T_n \|_{p,p} \) the operator norm of \( T_n \) acting on \( L^p(-1, 1) \). In Sec. 3, we found asymptotically correct upper and lower bounds for \( \| T_n \|_{2,2} \); denote these by

\[
\frac{a_n}{\sqrt{2(2n)!}} \leq \| T_n \|_{2,2} \leq \frac{b_n}{\sqrt{2(2n)!}}
\]

where \( a_n \) and \( b_n \) both tend to 1 as \( n \to \infty \). Moreover, as remarked in the proof of the Lemma in Sec. 3, the lower bound is valid for all \( L^p \) norms, so in fact we can write

\[
\frac{a_n}{\sqrt{2(2n)!}} \leq \| T_n \|_{p,p}
\]

for all \( p \in [1, \infty] \). It is easy to calculate exactly \( \| T_n \|_{\infty,\infty} \), as follows. By definition, \( T_n 1 = g_n \), where \( g_n^{(2n)} = 1 \) and \( g_n^{(j)}(\pm1) = 0 \) (\( 0 \leq j \leq n - 1 \)); it follows that \( g_n(t) = (1 - t^2)^n / (2n)! \). Since \( \| g_n \|_{\infty} = 1/(2n)! \) and \( \| 1 \|_{\infty} = 1 \), we have a lower bound \( \| T_n \|_{\infty,\infty} \geq 1/(2n)! \). Moreover, for any \( f \in L^\infty(-1, 1) \) and \( t \in [-1, 1] \),

\[
\| (T_n f)(t) \| \leq \| f \|_{\infty} \int_{-1}^{1} G_n(t, s) \, ds = \| f \|_{\infty} (T_n 1)(t) = \frac{1}{(2n)!} (1 - t^2)^n \| f \|_{\infty}
\]

(using the non-negativity of \( G_n \)). Taking a supremum over \( t \in [-1, 1] \) gives \( \| T_n f \|_{\infty} \leq \| f \|_{\infty} / (2n)! \), so \( \| T_n \|_{\infty,\infty} \geq 1/(2n)! \). In combination with the previous inequality, this shows that \( \| T_n \|_{\infty,\infty} = 1/(2n)! \).

It now follows from the symmetry of \( G_n \) that \( \| T_n \|_{1,1} = 1/(2n)! \) (because \( T_n \) acting on \( L^\infty(-1, 1) \) is the adjoint of \( T_n \) acting on \( L^1(-1, 1) \)).

Information about \( \| T_n \|_{p,p} \) for other values of \( p \) can now be obtained from the Riesz-Thorin interpolation theorem (Theorem IX.17 in [20]). A special case of this, informally stated, is that if \( T \) is bounded on \( L^{p_0} \) and \( L^{p_1} \) and \( p_u^{-1} = (1 - u)p_0^{-1} + up_1^{-1} \) (\( 0 \leq u \leq 1 \)), then \( T \) is bounded on \( L^{p_u} \) and \( \| T \|_{p_u,p_u} \leq \| T \|_{p_0,p_0}^{1-u} \| T \|_{p_1,p_1}^u \). With \( p_0 = 1 \) and \( p_1 = 2 \), we have \( p_u^{-1} = 1 - u/2 \) and

\[
\| T_n \|_{p_u,p_u} \leq \frac{b_n^u}{(2n)!} \frac{1}{2u/2}
\]

or, in terms of some \( p \in [1, 2] \),

\[
\| T \|_{p,p} \leq \frac{b_n^{2/q}}{2(2n)!} \frac{1}{2^{1/q}}
\]
where $p^{-1} + q^{-1} = 1$. Similarly, with $p_0 = 2$ and $p_1 = \infty$, we have $p_u^{-1} = (1 - u)/2$ and
\[
\|T_n\|_{p_u,p_u} \leq b_n^{1-u} \frac{1}{(2n)!} \frac{1}{2^{(1-u)/2}}
\]
or, in terms of some $p \in [2, \infty]$,
\[
\|T_n\|_{p,p} \leq b_n^{2/p} \frac{1}{(2n)!} \frac{1}{2^{1/p}}
\]
In general, for any $p \in [1, \infty]$, we have
\[
\|T_n\|_{p,p} \leq b_n^{2/r} \frac{1}{(2n)!} \frac{1}{2^{1/r}}
\]
where $r = \max(p, q)$. As mentioned above, we also have, for any $p$, the lower bound
\[
\frac{a_n}{\sqrt{2(2n)!}} \leq \|T_n\|_{p,p}
\]
so the decay rate of $\|T^n\|_{p,p}$ is, up to a constant, independent of $p$: $\|T_n\|_{p,p} \asymp 1/(2n)!$.

Finally, we note that the identity
\[
\inf_{n \in \mathbb{D}(-1,1)} \sup_{|h|=1} \|h^{(r)}\|_1 = 2^{r-1} r! \quad (r \in \mathbb{N})
\]
is obtained in Appendix C of [6]. Although there are similarities to the $p = 1$ case above, there are also significant differences: it is an extremum over a hyperplane, as opposed to a ball, and there are no explicit boundary conditions. Our $L^1$ result permits us to prove the related bound
\[
\inf_{n \in \mathbb{D}(-1,1)} \sup_{|h|=1} \|h^{(2n)}\|_1 \geq \|T_n\|^{-1}_{1,1} = (2n)! \quad (n \in \mathbb{N})
\]
which is weaker than the result of [6] (for $r = 2n$) by the geometric factor of $2^{2n-1}$. The origin of this is likely to be the absence of boundary conditions in that result, as it seems that one may approach the bound by nonnegative $h$ (so the difference between the integral and $L^1$ norm is inessential). Boundary conditions enter because our result could equally be stated as the infimum as $h$ varies over the range of $T_n$ in $L^1$, all elements of which obey the specific boundary conditions we have imposed. In fact, (17) was used to obtain a result quite similar in spirit to our main result: namely, that
\[
\inf_{n \in \mathbb{D}(-1,1)} \sup_{h \in \mathbb{D}(-1,1)} \left| y^k \hat{h}(y) \right| \leq \frac{1}{2} \sqrt{\pi} e^{1/4} x^{k+1/2} e^{-x/2}
\]
for $k \in \mathbb{N}_0$ and $x \geq \max\{2k, 2\}$. (We have adapted the formula given in [6] to our own conventions.) Apart from the difference in the extremisation domain, this could be thought of as an $L^1 \rightarrow L^\infty$ version of our $L^2 \rightarrow L^2$ result that
\[
\inf_{n \in \mathbb{D}(-1,1)} \left( \int_x^\infty dy \frac{2\pi}{|y|^2} |y^k \hat{h}(y)|^2 \right)^{1/2} = \inf_{n \in \mathbb{D}(-1,1)} \sqrt{H_{2k, h}(x)/2} \leq \pi^{1/4} x^{k+1/4} e^{-x/2}. 
\]
While the strategy employed in [6] overlaps in part with ours, the key portions of the two arguments (leading to (17) in [6], or our bounds on $\lambda_n$) are quite different.

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