Optimal mitigation with endogenous learning and a cumulative constraint: with application to negative emissions of carbon dioxide

Ashwin K Seshadri

Centre for Atmospheric and Oceanic Sciences, and Divecha Centre for Climate Change, Indian Institute of Science, Bangalore, 560012, India (ashwin@fastmail.fm; ashwins@iisc.ac.in)

Abstract

Large-scale extraction of carbon dioxide (CO₂) from Earth’s atmosphere ("negative emissions") is important for stringent climate change mitigation scenarios, and we examine optimal (i.e. least-cost) pathways of negative emissions in the presence of learning by doing ("endogenous learning"). Optimal pathways solve a variational problem involving minimization of discounted costs subject to a constraint on total negative emissions across time. A minimum pathway exists if the marginal cost curve of negative emissions is increasing with annual rate of emissions reduction. In the absence of endogenous learning, the optimal pathway has annual negative emissions increasing with time: with more rapid increase in emissions rate occurring in case of large discount rate and slower increase of the cost curve. Endogenous learning can have contrary effects depending on how it is included in models. This paper identifies a basic distinction, between additive and multiplicative effects on marginal costs of endogenous learning, which governs its qualitative effects in such models. If endogenous learning is best modeled as a negative addition to the cost
function, shifting the cost curve downward, the optimal pathway has higher emissions rate early on when compared to the no-learning case, however with emissions increasing with time. In contrast if endogenous learning is a multiplicative effect, scaling down marginal cost uniformly, then benefits of learning are slowly manifest as marginal cost rises and the optimal pathway begins at lower emissions rates that increase more rapidly as compared to if endogenous learning were absent.

1 Introduction

As the urgency of limiting global warming grows, techniques for removing carbon dioxide (CO$_2$) from Earth’s atmosphere (“negative emissions”) are coming into greater scientific and policy focus (Clarke et al., 2014; Hansen et al., 2016; Minx et al., 2017). Several techniques for negative emissions are in existence, including afforestation and reforestation, bioenergy with carbon capture and sequestration, soil carbon sequestration, enhanced weathering, ocean fertilization, and direct air capture (Hansen et al., 2016; Minx et al., 2017; Smith et al., 2015). Negative emissions techniques present the opportunity to partly decouple radiative forcing from the energy infrastructure (Keith, 2009), and climate modeling analyses have indicated their importance for emissions scenarios limiting global warming to 2 degrees C or below (van Vuuren et al., 2011). Without them, higher rates of global decarbonization would become essential (Boucher et al., 2016; van Vuuren et al., 2018; Kriegler et al., 2018).

Therefore several studies have begun to examine various aspects of negative emissions, such as cost, scale, feasibility and effects of deploying them at large scales for limiting global warming (Caldecott et al., 2015; Boucher et al., 2016; Fuss et al., 2016; Sanchez and S. Callaway, 2016; Field and Mach, 2017; Peters and Geden, 2017; Psarras et al., 2017; Reiner, 2018). Currently, negative emissions are often included in studies of climate change mitigation. A basic question regarding their deployment is the available capacity of negative emissions and overall costs of implementation (Smith et al., 2015; Field and Mach, 2017).
A basic aspect of climate change mitigation is that global warming from CO₂ is approximately proportional to its cumulative emissions across time (Allen et al. (2009); Matthews et al. (2009); Allen and Stocker (2014)), and independent of the precise path that emissions takes, owing to the long atmospheric lifetime of CO₂ (Seshadri (2017)).

In the presence of negative emissions, the relationship between global warming and CO₂ emissions is no longer independent of the trajectory of emissions (Zickfeld et al. (2016)).

In Earth-system models, large-scale removal of CO₂ from the atmosphere leads to some excess CO₂ that had been previously taken up by oceans and biosphere to return to the atmosphere (Cao and Caldeira (2010)), and hence CO₂ removal can be modeled by subtracting from total emissions (Jones et al. (2016)). Cumulative emissions is therefore the most relevant metric for CO₂ even in the presence of substantial removal, making cumulative negative emissions across time an important factor in mitigation policy. Therefore it is essential to study optimal pathways of a negative emissions under a constraint on cumulative CO₂ removal across time.

Climate policy is often analyzed in the context of endogenous learning, i.e. learning by doing (Arrow (1962); Wing (2006); van der Zwaan et al. (2002); Baker and Shittu (2008); Gillingham et al. (2008)). Implementation of a technology and associated investments in research and development has the potential to lower its costs, and this influences suitable pathways of emissions reductions across time (Goulder and Mathai (2000); van der Zwaan et al. (2002); Gillingham et al. (2008)). Even in the presence of endogenous learning, undertaking rapid mitigation too early might impose unnecessary costs, whereas delaying too long would not leave sufficient time for costs to become reduced via learning. How to balance these considerations, in the context of time-discounting? To examine this, we introduce a highly abstracted model of negative emissions costs that takes into consideration increasing marginal costs as well as endogenous learning.

The situation for negative emissions is quite different from that of decarbonization for which, generally, the marginal costs would increase over an extend period of time of several decades as different technologies are brought into action. For negative emissions, where the same techniques (such as bioenergy with carbon capture) can be expected to
be deployed year after year, the average cost in any individual year can be modeled as an increasing function of the level of negative emissions in that year. Furthermore, in this case, learning by doing can be modeled as a function of cumulative negative emissions until that time. This describes effects on average cost of accumulated knowledge as more or less the same basket of techniques is employed each year to expand the total volume of negative emissions across time.

Negative emissions therefore presents a particularly straightforward case study to examine the more general problem of endogenous learning. Several authors have considered the economics of endogenous learning, starting with the work of Arrow (1962), who modeled it as a function of cumulative investment across time. Although endogenous learning is not always present directly in integrated assessment models of climate change, several previous studies have considered its implications for CO$_2$ mitigation across time (e.g. Goulder and Mathai (2000); van der Zwaan et al. (2002); Wing (2006)). One method of modeling endogenous learning is through changes in the cost curve (Baker et al. (2008)), and prior work has included such an approach (Gillingham et al. (2008)). There are contradictory lessons from modeling of endogenous learning (Gillingham et al. (2008)), showing both early (van der Zwaan et al. (2002)) and delayed (Goulder and Mathai (2000)) mitigation as a result.

At the outset, opposite arguments about the effects of endogenous learning can sound plausible. One one hand, a higher emissions rate early on can be beneficial over time for it reduces future costs. On the other hand, it might be beneficial to begin at lower emissions rates that only gradually increase with time as benefits of lowered costs become significant. The previous studies show that details of how endogenous learning is modeled matter for the lessons about optimal mitigation trajectories (Goulder and Mathai (2000); van der Zwaan et al. (2002); Gillingham et al. (2008)).

This issue is reexamined in the present work, by considering optimal pathways for negative emissions in the presence of endogenous learning and a cumulative emissions constraint. Section 2 introduces a model for the average cost of negative emissions, in the presence of endogenous learning, where effects of learning are modeled by making costs
decrease as a function of cumulative emissions. Section 3 uses a variational approach
to examine minimum expenditure pathways of negative emissions in the presence of a
cumulative emissions constraint. Section 4 considers the optimal starting time for neg-
avative emissions, in the presence of a cumulative emissions constraint. The analysis of
Sections 2-4 considers only a model of endogenous learning for which learning enters
through an additive term (of negative sign) in the cost function. Section 5 considers op-
timal pathways when endogenous learning enters through a multiplicative effect. There
is considerable evidence for costs reducing as a multiplicative function of cumulative
production \( (\text{Nagy et al. (2013)}) \). It is shown that additive and multiplicative forms of
modeling endogenous learning have different effects, corresponding to early and late mit-
egation respectively. Section 6 introduces a general cost function that subsumes these
particular cases, and explores the differences further. We thereby identify a basic dis-
tinction in such models of endogenous learning, that between additive and multiplicative
effects on marginal costs, which governs qualitatively the effects on optimizing mitigation
pathways.

2 Average costs of negative emissions and annual
mitigation expenditures

Our basic setup is as follows. Let the cumulative negative emissions necessary for meeting
a global warming goal at some time horizon be \( C_N \), with its value determined exogenously.
For algebraic convenience, the end of this time-horizon is defined as \( t = 0 \). Negative
emissions starts at \( t = -T \), with \( T > 0 \). One goal is to determine a suitable \( T \) that
is reasonable, given our understanding of costs of implementing negative emissions. A
second goal is to understand how, after \( t = -T \), the negative emissions rate should vary
in time. Denoting negative emissions rate in year \( t \) by \( n(t) \), we define cumulative negative
emissions until \( t \) \( (-T \leq t \leq 0) \) as \( N(t) = \int_{-T}^{t} n(s) ds \). We can therefore represent the
negative emissions rate as \( \dot{N}(t) \equiv \frac{dN(t)}{dt} \).
In the absence of endogenous learning, the average cost of negative emissions is an increasing function \( \gamma \left( \dot{N} (t) \right) \) of negative emissions rate in year \( t \), as cheaper techniques are employed first. The effect on the average cost of endogenous learning can be modeled directly through a term in \( N (t) \), describing effects of accumulated knowledge as the volume of negative emissions expands. Previous authors have shown evidence in a number of sectors for cost reductions depending on cumulative production (Nagy et al. (2013)).

For simplicity, overall the average cost of negative emissions is represented as a function increasing in \( \dot{N} (t) \) and decreasing with \( N (t) \), by model

\[
\gamma \left( N (t), \dot{N} (t) \right) = \gamma_0 - \gamma_1 N (t) + \gamma_2 \dot{N} (t)^
u
\]  

(1)

where coefficients \( \gamma_0, \gamma_1 \) and \( \gamma_2 \) are positive. This formula can also be represented as \( \gamma(t) = \gamma_0 - \gamma_1 N(t) + \gamma_2 n(t)^
u \). The last term describes effects of increasing marginal cost as emissions rate in year \( t \) increases, whereas the second term describes effects of endogenous learning on lowered average cost as cumulative negative emissions grow.

Then the total expenditure on negative emissions in year \( t \) is

\[
E_N(t) = \gamma(t) n(t) = \left\{ \gamma_0 - \gamma_1 N(t) + \gamma_2 \dot{N} (t)^
u \right\} \dot{N} (t)
\]  

(2)

where we have returned to representing emissions rate \( n(t) \) by \( \dot{N} (t) \) to lay the ground for solving the variational problem in the next section.

3 Minimum-expenditure pathways

Given the previous development for a single year, the total expenditure on negative emissions across our time-horizon between \( t = -T \) and \( t = 0 \), with the time-discounting rate being \( \delta \), is

\[
F_N = \int_{-T}^{0} e^{-\delta(t+T)} E_N \left( N(t), \dot{N} (t) \right) dt
\]  

(3)
and writing this as $e^{-\delta T}\int_{-T}^{0} g(t)\,dt$, with

$$g(t) = e^{-\delta t}E_N\left( N(t), \dot{N}(t) \right) dt$$ (4)

we seek to minimize the functional in Eq. (3) subject to constraint on cumulative negative emissions $\int_{-T}^{0} \dot{N}(t)\,dt = C_N$. Recall that $C_N$ is the specified volume of negative emissions across time, determined exogenously in our setup. For fixed $T$, minimizing the functional $\int_{-T}^{0} g(t)\,dt$ involves finding a solution to the Euler-Lagrange equation for a constrained minimization problem (Appendix 1)

$$\frac{\partial g}{\partial N} + \lambda \frac{\partial n}{\partial N} = \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{N}} + \lambda \frac{\partial n}{\partial \dot{N}} \right)$$ (5)

and, using $\frac{\partial g}{\partial N} = -\gamma_1 e^{-\delta t} \dot{N}(t)$, $\frac{\partial g}{\partial \dot{N}} = e^{-\delta t} \left( \gamma_0 - \gamma_1 N(t) + (\nu + 1) \gamma_2 \dot{N}(t)^\nu \right)$, $\frac{\partial n}{\partial N} = 0$ and $\frac{\partial n}{\partial \dot{N}} = 1$, the Euler-Lagrange equation becomes the 2nd-order differential equation in $N(t)$

$$\nu (\nu + 1) \gamma_2 \dot{N}(t)^{\nu - 1} \ddot{N}(t) = \delta \left\{ \gamma_0 - \gamma_1 N(t) + (\nu + 1) \gamma_2 N^\nu \right\}$$ (6)

with boundary conditions on cumulative emissions $N(-T) = 0$ and $N(0) = C_N$. Let us examine a few special cases before turning to the general case:

### 3.1 Constant marginal costs ($\nu = 0$)

With constant marginal costs, described by $\nu = 0$, the Euler-Lagrange equation becomes $\delta \{ \gamma_0 + \gamma_2 - \gamma_1 N(t) \} = 0$. In the presence of time-discounting where $\delta > 0$, existence of a solution requires that $N(t) = (\gamma_0 + \gamma_2) / \gamma_1$, which is constant in time. That would contradict the boundary conditions on $N(t)$ and such a solution cannot be achieved. A minimizing pathway does not exist for constant marginal costs in case $\delta \neq 0$.

With constant marginal costs, a minimizing pathway occurs only if $\delta = 0$, in which case the Euler-Lagrange equation is trivially solved. All pathways of $N(t)$ satisfying
the boundary conditions would result in the same total discounted expenditure, and the
pathway can be chosen arbitrarily. Given that the state of constant marginal costs is
unrealistic, as well as the special character of the associated solution, which occurs only
for \( \delta = 0 \), we do not dwell on this situation in the following. Instead, we examine only
cases with \( \nu > 0 \) and \( \gamma_2 > 0 \), where marginal costs are non-trivial and increasing.

### 3.2 Increasing marginal costs (\( \nu > 0 \) and \( \gamma_2 > 0 \), zero marginal
cost technologies are available (\( \gamma_0 = 0 \)), and there is no
endogenous learning (\( \gamma_1 = 0 \))

In case \( \gamma_0 = \gamma_1 = 0 \) the Euler-Lagrange equation becomes \( \ddot{N}(t) = \frac{\delta}{\nu} \dot{N}(t) \), which is
integrated between \(-T\) and \( t \) for \( \dot{N}(t) = \dot{N}(-T) e^{\frac{\delta}{\nu}(t+T)} \). Integrating again

\[
N(t) = \int_{-T}^{t} \dot{N}(s) ds = \frac{\nu \dot{N}(-T)}{\delta} \left( e^{\frac{\delta}{\nu}(t+T)} - 1 \right) \tag{7}
\]

Substituting boundary condition \( N(0) = C_N \), we solve for \( \dot{N}(-T) = \frac{\delta}{\nu} \frac{C_N}{e^{\frac{\delta}{\nu}T} - 1} \), and hence

\[
N(t) = \frac{C_N}{e^{\frac{\delta}{\nu}T} - 1} \left( e^{\frac{\delta}{\nu}(t+T)} - 1 \right) \tag{8}
\]

Negative emissions rate \( \dot{N}(t) = \frac{\delta}{\nu} \frac{C_N}{e^{\frac{\delta}{\nu}T} - 1} e^{\frac{\delta}{\nu}(t+T)} \) increases exponentially in this case.

The increase in rate of negative emissions follows from time-discounting. For the limiting
case of zero time-discounting, i.e. where \( \frac{\delta}{\nu} T \to 0 \) we can approximate \( e^{\frac{\delta}{\nu}T} - 1 \cong \frac{\delta}{\nu} T \)
and \( e^{\frac{\delta}{\nu}(t+T)} - 1 \cong \frac{\delta}{\nu} (t + T) \), so that \( N(t) \cong \frac{C_N}{\nu} (t + T) \) and \( \dot{N}(t) = C_N/T \) is constant.

Alternately the Euler-Lagrange equation for \( \delta = 0 \), i.e. \( \ddot{N}(t) = 0 \), requires emissions rate
\( \dot{N}(t) \) to be constant in time.

At the same time, the shape of the average cost function plays a role. If \( \nu \) is large making
average cost increase rapidly, the solution has emissions rate growing slowly so as not to
impose large costs in the future when emissions rate is higher. Balance between time-
discounting of future costs and avoiding large expenditures in the future is governed by
rate constant $\delta/\nu$. This equals the rate of time-discounting only for average cost functions increasing linearly with negative emissions rate.

### 3.3 Zero time-discounting ($\delta = 0$) and there is endogenous earning ($\gamma_1 > 0$)

Returning to the general case of the Euler-Lagrange Eq. (6), but with $\delta = 0$ we obtain $
abla (t) = \nabla (t) = 0$. Since $\nabla (t)$ cannot be zero for all $t$, otherwise there would be no negative emissions, solution requires $\nabla (t) = 0$. This is integrated for $N (t) = at + b$, and boundary conditions are solved for $a = C_N / T$ and $b = C_N$, yielding

$$ N (t) = C_N \left( 1 + \frac{t}{T} \right) $$

and $\dot{N} (t) = C_N / T$. For the average cost function in Eq. (1), zero time-discounting yields a minimizing solution with constant rate of negative emissions, even in the presence of endogenous learning where $\gamma_2 > 0$.

### 3.4 General case

The general case of the Euler-Lagrange Eq. (6) does not have an explicit solution and must be solved numerically. For small discount rate $\delta$ we may consider a perturbation to the previous solution. Expanding the solution as $N (t) = N_0 (t) + \delta N_1 (t)$, so that $\dot{N} (t) = \dot{N}_0 (t) + \delta \dot{N}_1 (t)$, etc., and substituting into Eq. (6) and collecting like powers of $\delta$, we obtain upto $0^{th}$-order in $\delta$

$$ \dot{N}_0 (t) = 0 $$

which is solved as before for $\dot{N}_0 (t) = 0$, so that $N_0 (t) = at + b$. Boundary conditions are $N_0 (-T) = 0$ and $N_0 (0) = C_N$, yielding $N_0 (t) = C_N \left( 1 + \frac{t}{T} \right)$. 

9
For the 1st-order correction in $\delta$ we obtain the following differential equation

$$
\nu (\nu + 1) \gamma_2 \tilde{N}_0 (t)^{\nu-1} \tilde{N}_1 (t) = \left\{ \gamma_0 - \gamma_1 N_0 (t) + (\nu + 1) \gamma_2 \tilde{N}_0 (t)^\nu \right\}
$$

(11)

Boundary conditions are $N_1 (-T) = 0$ and $N_1 (0) = 0$, for maintaining overall boundary conditions on $N (t)$. This is integrated for

$$
\nu (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^{\nu-1} N_1 (t) = \frac{1}{2} \left\{ \gamma_0 - \gamma_1 C_N + (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^\nu \right\} (t + T) t - \frac{1}{6} \gamma_1 \frac{CN}{T} (t^2 - T^2) t
$$

(12)

Differentiating in time for $\dot{N}_1 (t)$ and substituting $\dot{N} (t) = \dot{N}_0 (t) + \dot{N}_1 (t) \delta$ the emissions rate is

$$
n (t) = \frac{C_N}{T} + \frac{\delta}{2\nu (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^{\nu-1}} \left\{ \gamma_0 - \gamma_1 C_N + (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^\nu \right\} (2t + T) - \frac{1}{3} \gamma_1 \frac{CN}{T} (3t^2 - T^2)
$$

(13)

equaling, at $t = -T$

$$
n (-T) = \frac{C_N}{T} - \frac{\delta T}{2\nu (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^{\nu-1}} \left\{ \gamma_0 - \frac{1}{3} \gamma_1 C_N + (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^\nu \right\}
$$

(14)

and $t = 0$

$$
n (0) = \frac{C_N}{T} + \frac{\delta T}{2\nu (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^{\nu-1}} \left\{ \gamma_0 - \frac{2}{3} \gamma_1 C_N + (\nu + 1) \gamma_2 \left( \frac{CN}{T} \right)^\nu \right\}
$$

(15)

Of course, this calculation is valid only if $\nu > 0$ and $\gamma_2 > 0$, so that average costs are increasing with emissions rate. For $\gamma_0 = \gamma_1 = 0$, we have $n (-T) = \frac{C_N}{T} \left( 1 - \frac{\delta T}{\nu^2} \right)$ and $n (0) = \frac{C_N}{T} \left( 1 + \frac{\delta T}{\nu^2} \right)$. Emissions rate increases with time, proportionally to $\delta/\nu$. The increase of the negative emissions rate with time is a direct result of non-zero $\delta$, but other factors modulate the effect. Larger $\nu$, making average cost more sensitive to emissions
rate, tends to make optimal emissions rate more constant in time. Larger $\gamma_0$, introducing a larger fixed contribution to the average cost and thereby making it less sensitive to emissions rate, makes the optimal emissions rate grow more rapidly in time.

Endogenous learning, with $\gamma_1 > 0$, has a countervailing effect to that of discounting in time. By lowering average costs as cumulative negative emissions grows, endogenous learning as modeled by Eq. (1) mitigates partly the influence of time-discounting by reducing the difference between $n(0)$ and $n(-T)$. This effect is discussed further in the following.

### 3.5 A special case

As discussed previously, the general form of the Euler-Lagrange Eq. (6) cannot be integrated explicitly. However there is a special case when it reduces to a familiar form. If average cost increases proportionally to emissions rate, i.e. $\nu = 1$, the Euler-Lagrange equation becomes for $\gamma_2 > 0$

$$\dot{N}(t) - \delta \dot{N}(t) + \frac{1}{2} \delta \gamma_1 N(t) = \frac{1}{2} \delta \gamma_0$$

which, being a linear equation, is solved explicitly for

$$N(t) = \left( C_N - \frac{\nu}{\gamma_1} \right) e^{-\lambda_2 T} - e^{-\lambda_1 T} e^{\lambda_1 t} + \frac{\left( C_N - \frac{\nu}{\gamma_1} \right) e^{-\lambda_2 T} - \frac{\nu}{\gamma_1} e^{\lambda_1 t} + \gamma_0}{\gamma_1} \tag{16}$$

with $\lambda_1$ and $\lambda_2$ being the roots of quadratic equation $\lambda^2 - \delta \lambda + \frac{1}{2} \delta \gamma_1 = 0$, or $\lambda_{1,2} = \frac{\delta \pm \sqrt{\delta^2 - \frac{2\gamma_1}{\delta}}} {2}$. Furthermore, if $\frac{\gamma_1}{\delta \gamma_2} \ll 1$, these can be approximated as $\lambda_1 = \delta - \frac{\gamma_1}{2 \gamma_2}$ and $\lambda_2 = \frac{\gamma_1}{2 \gamma_2}$. In this case, $\lambda_1$ is slightly smaller than discount rate $\delta$ but approximately equal to it, while $\lambda_2$ by comparison is very small.

Let us carry this logic further, assuming that $\lambda_2$ can be approximated by zero. Then

$$N(t) \cong C_N \frac{e^{\lambda_1 t} - e^{-\lambda_1 T}}{1 - e^{-\lambda_1 T}} = C_N \frac{e^{\lambda_1 (t+T)} - 1}{e^{\lambda_1 T} - 1} \tag{17}$$
which is similar to the solution of Section 3.2 upon replacing $\delta/\nu$ there by $\lambda_1$ in the present analysis. With $\nu = 1$, in the absence of endogenous learning, emissions rate would grow at rate $\delta$ whereas in the present analysis it grows at lower rate $\lambda_1 = \delta - \frac{\gamma_1}{\gamma_2}$ and therefore must be larger initially in order to meet the cumulative emissions constraint.

This summarizes qualitative effects of endogenous learning for the average cost model of Eq. (1). Endogenous learning, represented by $\gamma_1$, decreases emissions growth rate by an amount increasing with ratio $\gamma_1/\gamma_2$, but does not alter the preference for delayed emissions in the optimal trajectory, since $\lambda_1 > 0$. Furthermore, endogenous learning by itself is inconsequential without time-preference, and if $\delta = 0$ the eigenvalues would simply be $\lambda_{1,2} = 0$, leading to constant emissions rate across time, irrespective of any benefits of learning by doing.

4 Optimal starting time for negative emissions

Section 3 examined optimal pathways of negative emissions on the assumption that its starting time $t = -T$ has been specified. Complete characterization of the negative emissions pathway includes specification of $T$. A general answer to the question of the optimal starting time is far from obvious. Starting too early might risk incurring large expenditures before they are necessary, whereas delaying abatement risks higher costs owing to larger average emissions rates and could unduly postpone benefits of endogenous learning. Here we examine how the total discounted expenditure depends on the choice of starting time, while assuming the minimum-expenditure pathways determined above for the corresponding cases:

1. In the simplest case, for $\delta = 0$ (Section 3.3), we obtained constant emissions rate solutions and $N(t) = C_N \left(1 + \frac{t}{T}\right)$. Since time-discounting is absent, discounted

\[^1\text{This approximation is especially valid for large } \gamma_2, \text{ with the resulting solution clearly independent of } \gamma_0. \text{ If the average cost rises rapidly with the emissions rate, the minimum value does not influence the optimal emissions trajectory, but of course will affect the costs as shown in the following section.}\]
mitigation cost

\[ F_N = \int_{-T}^{0} \left( \gamma_0 - \gamma_1 N(t) + \gamma_2 \dot{N}(t)^\nu \right) \dot{N}(t) \, dt \]  

becomes

\[ F_N = C_N \left\{ \gamma_0 - \frac{1}{2} \gamma_1 C_N + \gamma_2 \left( \frac{C_N}{T} \right)^\nu \right\} \]

which is minimized by making \( T \) as long as possible. Since marginal costs increase with the emissions rate, i.e. \( \nu > 0 \), negative emissions should begin as early as possible. This is not influenced by the presence of endogenous learning which, of course, brings the discounted cost down if \( \gamma_1 > 0 \).

2. Another case, examined in Section 3.2, had zero marginal cost technologies (\( \gamma_0 = 0 \)) and no endogenous learning (\( \gamma_1 = 0 \)). Substituting for \( \dot{N}(t) = \frac{\delta}{\nu} e^{\frac{\delta}{\nu} T} (e^{\frac{\delta}{\nu} T} - 1) \) as obtained there, the discounted cost can be written as

\[ F_N = \gamma_2 \left\{ \frac{\delta}{\nu} \frac{C_N}{e^{\frac{\delta}{\nu} T} - 1} \right\}^{\nu+1} \int_{-T}^{0} e^{-\delta(t+T)} e^{\frac{\delta}{\nu} T} e^{(\nu+1)(t+T)} \, dt \]

integrating to

\[ F_N = \gamma_2 C_N^{\nu+1} \left( \frac{\delta}{\nu} \right)^\nu \frac{1}{\left( \frac{\delta}{\nu} + 1 \right)^\nu} \]

Since \( \frac{\delta T}{\nu} > 0 \), the value of \( F_N \) is minimized by making \( T > 0 \) as large as possible, or equivalently starting negative emissions as early as possible.

3. Let us turn to the case of Section 3.5 where, since \( \nu = 1 \), the Euler-Lagrange equation could be integrated explicitly. If the second eigenvalue is close to zero, the cumulative emissions rate is approximately \( N(t) \approx C_N \frac{e^{\lambda_1 (t+T)} - 1}{e^{\lambda_1 T} - 1} \), and emissions rate \( \dot{N}(t) \approx \lambda_1 C_N \frac{e^{\lambda_1 (t+T)} - 1}{e^{\lambda_1 T} - 1} \). While appearing like the one above, it originates in a different set of assumptions, and must be considered separately despite the similarities. The value of \( \gamma_0 \) cannot influence the choice of optimal timing under a cumulative
emissions constraint, and we therefore set \( \gamma_0 = 0 \) for clarity, so that discounted cost

\[
F_N = \int_{-T}^{0} e^{-\delta(t+T)} \left\{ -\gamma_1 N(t) \dot{N}(t) + \gamma_2 \dot{N}(t)^2 \right\} dt \tag{23}
\]

integrates to

\[
F_N = \lambda_1 \frac{\gamma_2 \lambda_1 - \gamma_1}{2 \lambda_1 - \delta} \left( \frac{C_N}{e^{\lambda_1 T} - 1} \right)^2 \left\{ e^{(2\lambda_1 - \delta)T} - 1 \right\} + \frac{\gamma_1 \lambda_1}{\delta - \lambda_1} \left( \frac{C_N}{e^{\lambda_1 T} - 1} \right)^2 \left\{ 1 - e^{-(\delta - \lambda_1)T} \right\} \tag{24}
\]

Recall that \( \lambda_1 \cong \delta - \frac{\gamma_1}{2\gamma_2} > 0 \) and this requires that endogenous learning obeys constraint \( \frac{\gamma_1}{\gamma_2} < 2 \). This approximation was based on assuming \( \frac{\gamma_1}{\gamma_2} \ll 1 \). This furthermore ensures that \( 2\lambda_1 - \delta \cong \delta - \frac{\gamma_1}{\gamma_2} > 0 \), so that \( e^{(2\lambda_1 - \delta)T} - 1 \) becomes large for long \( T \). Additionally \( \delta - \lambda_1 > 0 \) so that \( 1 - e^{-(\delta - \lambda_1)T} \rightarrow 1 \) as \( T \) becomes large.

Hence the first term involving \( \gamma_2 \lambda_1 - \gamma_1 \) in Eq. (24) is the dominant one and we further require \( \gamma_2 \lambda_1 - \gamma_1 = \delta \gamma_2 - \frac{3}{2} \gamma_1 > 0 \), entailing that \( \frac{\gamma_1}{\gamma_2} < \frac{2}{3} \). This is, of course, valid given our starting assumption that \( \frac{\gamma_1}{\gamma_2} \ll 1 \) for this approximation.

If endogenous learning is absent, i.e. \( \gamma_1 = 0 \), then \( \lambda_1 = \delta \) and \( F_N = \gamma_2 \left( \frac{\lambda_1}{e^{\lambda_1 T} - 1} \right) C_N^2 \), which corresponds to the previous case of this section. In that case we noted that negative emissions should begin early.

Endogenous learning cannot be expected to undermine this, but we shall complete the analysis to see why. Substituting for \( \lambda_1 \) in Eq. (24), in the first term the terms dependent on \( T \) are of the form \( \left\{ e^{\left(\delta - \frac{\gamma_1}{\gamma_2} \right)T} - 1 \right\} / \left\{ e^{\left(\delta - \frac{\gamma_1}{\gamma_2} \right)T} - 1 \right\}^2 \). The denominator, with larger exponential coefficient and squared term, grows faster with \( T \), so that minimizing discounted expenditure requires \( T \) as large as possible. The second term has term depending on \( T \) of the form \( \left\{ 1 - e^{-\frac{\gamma_1}{\gamma_2} T} \right\} / \left\{ e^{\left(\delta - \frac{\gamma_1}{\gamma_2} \right)T} - 1 \right\}^2 \), which can also be made smaller by increasing \( T \). Therefore even with the model of endogenous learning in Eq. (2) it is optimal to commence negative emissions as early as possible.

In summary, we have shown that for a variety of cases of the average cost formula in

\[\text{In case we had employed the full solution, involving } \lambda_2 \text{ as well, this constraint would not have been needed.} \]
Eq. (1), and in the presence of a cumulative emissions constraint, discounted cost of negative emissions is minimized by starting as early as possible, even in the presence of endogenous learning.

5 Effects of multiplicative model of endogenous learning

We repeat the previous analysis for a cost curve wherein the effect of endogenous learning is modeled differently, as a multiplicative term in cumulative emissions, so that average cost is

\[ \gamma(t) = \left( \gamma_0 + \gamma_2 \dot{N}(t) \right) e^{-\gamma_1 N(t)} \]  

(25)

As before, expenditures on negative emissions in year \( t \) are \( E_N(t) = \gamma(t) \dot{N}(t) \), and discounted expenditures follow Eq. (3). For given \( T \), we seek to minimize \( \int_{-T}^{0} g(t) \, dt \) with \( g(t) = e^{-\delta t} \left( \gamma_0 + \gamma_2 \dot{N}(t) \right) e^{-\gamma_1 N(t)} \dot{N}(t) \), subject to the constraint on cumulative negative emissions between \( t = -T \) and \( t = 0 \). Using \( \partial g / \partial N = -\gamma_0 \gamma_1 e^{-\delta t} \dot{N}(t) e^{-\gamma_1 N(t)} - \gamma_1 \gamma_2 e^{-\delta t} \dot{N}(t)^{\nu+1} e^{-\gamma_1 N(t)} \) and \( \partial g / \partial \dot{N} = \gamma_0 e^{-\delta t} e^{-\gamma_1 N(t)} + (\nu + 1) \gamma_2 e^{-\delta t} \dot{N}(t)^{\nu} e^{-\gamma_1 N(t)} \), the Euler-Lagrange Eq. (5) becomes

\[ \nu (\nu + 1) \gamma_2 \dot{N}(t)^{\nu-1} \dot{N}(t) = \delta \left( \gamma_0 + (\nu + 1) \gamma_2 \dot{N}(t)^{\nu} \right) + \nu \gamma_1 \gamma_2 \dot{N}(t)^{\nu+1} \]  

(26)

Let us examine different cases corresponding to the discussion of Section 3:

1. Constant marginal costs, with \( \nu = 0 \): The Euler-Lagrange equation becomes \( \delta (\gamma_0 + \gamma_2) \). If \( \delta > 0 \) this requires \( \gamma_0 + \gamma_2 = 0 \) which cannot be satisfied, except for zero marginal costs. If \( \delta = 0 \), the equation is trivially satisfied and all pathways of \( N(t) \) satisfying the two boundary conditions yield identical discounted expenditure.

2. Increasing marginal costs (\( \nu > 0 \) and \( \gamma_2 > 0 \)), zero marginal cost technologies are available (\( \gamma_0 = 0 \)), and there is no endogenous learning (\( \gamma_1 = 0 \)): With \( \gamma_0 = \gamma_1 = 0 \),
the Euler Lagrange equation becomes \( \ddot{N} (t) = \frac{\delta}{\nu} \dot{N} (t) \), identical to Section 3.2, yielding the same result that annual negative emissions increases exponentially at rate \( \delta/\nu \). If \( \delta = 0 \), the annual negative emissions rate is constant.

3. Zero time-discounting \( (\delta = 0) \) with endogenous earning \( (\gamma_1 > 0) \): Then the Euler-Lagrange equation is

\[
\ddot{N} (t) = \frac{\gamma_1}{\nu + 1} \dot{N} (t)^2
\]

which is integrated for

\[
N (t) = -\frac{\nu + 1}{\gamma_1} \ln \left\{ 1 - \left( \frac{t + T}{T} \right) \left( 1 - e^{-\frac{\gamma_1 C_N}{\nu + 1}} \right) \right\}
\]

Emissions is

\[
\dot{N} (t) = \frac{\nu + 1}{\gamma_1 T} \left( 1 - e^{-\frac{\gamma_1 C_N}{\nu + 1}} \right) \left( 1 - \left( \frac{t + T}{T} \right) \left( 1 - e^{-\frac{2\gamma_1 C_N}{\nu + 1}} \right) \right)
\]

which takes boundary values \( \dot{N} (-T) = \frac{\nu + 1}{\gamma_1 T} \left( 1 - e^{-\frac{\gamma_1 C_N}{\nu + 1}} \right) \) and \( \dot{N} (0) = \frac{\nu + 1}{\gamma_1 T} \left( 1 - e^{-\frac{2\gamma_1 C_N}{\nu + 1}} \right) e^{\frac{2\gamma_1 C_N}{\nu + 1}} \equiv e^{\frac{2\gamma_1 C_N}{\nu + 1}} \dot{N} (-T) \). Emissions increases with time if \( \gamma_1 > 0 \), owing to endogenous learning, even in the absence of time-discounting. This is in contrast to the result of Section 3.3, where endogenous learning had no effect in case \( \delta = 0 \), for the model with endogenous learning’s effect represented as an additive contribution to average cost.

4. For the case with zero marginal cost technologies, i.e. \( \gamma_0 = 0 \), we obtain

\[
\nu (\nu + 1) \gamma_2 \ddot{N} (t) = \delta (\nu + 1) \gamma_2 \dot{N} (t) + \nu \gamma_1 \gamma_2 \dot{N} (t)^2
\]

which, being independent of \( N (t) \), can be written as the 1-st-order differential equation

\[
\nu (\nu + 1) \gamma_2 \dot{n} (t) = \delta (\nu + 1) \gamma_2 n (t) + \nu \gamma_1 \gamma_2 n (t)^2
\]

Substituting \( n (t) = 1/x (t) \), and differentiating we obtain

\[
\nu (\nu + 1) \gamma_2 \dot{x} (t) = -\delta (\nu + 1) \gamma_2 x (t) - \nu \gamma_1 \gamma_2. \]

This is
integrated easily, and we obtain finally

$$\dot{N}(t) = \dot{N}(-T) \frac{e^{\delta(t+T)}}{1 - \frac{\nu_{b+1}}{\nu} \dot{N}(-T) \left(e^{\delta(t+T)} - 1\right)}$$

(31)

Endogenous learning $\gamma_1 > 0$ makes emissions rate start slowly and subsequently grow more rapidly than the effect of time-discounting alone (recall that if $\gamma_1 = 0$, $\dot{N}(t) = \dot{N}(-T) e^{\delta(t+T)}$), because the denominator becomes smaller with time. This is contrary to the result in Section 3 for the additive learning effect, where endogenous learning makes emissions grow more slowly than owing to time-discounting alone.

6 Differences between additive and multiplicative endogenous learning models

While there are of course similarities in the influence of the two average costs functions described above, the differences are quite substantial, especially in relation to effects of endogenous learning. In the first cost function, endogenous learning has no influence in the absence of time-discounting, while in the second endogenous learning favors increasing emissions rates even in the absence of time-discounting. In the presence of time-discounting, the effects of endogenous learning are opposite for the two cost functions.

Prior studies of mitigation in the presence of endogenous learning have suggested contrary effects, leading to both early (van der Zwaan et al. 2002) and delayed emissions reductions (Goulder and Mathai 2000). Each of these effects is present in the two average cost functions examined in this paper. We now introduce a more generic cost function that subsumes both of these models, making the differences between them clearer.
6.1 A generic model for additive and multiplicative endogenous learning effects

We consider average cost

\[ \gamma(t) = \left( \gamma_0 + f(N) + \gamma_2 \dot{N}(t)^\nu \right) h(N) \]  

(32)

where \( \gamma_0 + \gamma_2 \dot{N}(t)^\nu \) describes the cost curve, increasing in emissions rate, in the absence of endogenous learning. Functions \( f(N) \) and \( h(N) \) describe effects of endogenous learning.

In Section 2, \( f(N) = -\gamma_1 N(t) \) and \( h(N) = 1 \), whereas in Section 5, \( f(N) = 0 \) and \( h(N) = e^{-\gamma_1 N(t)} \). More generally, \( f(N) \leq 0 \) with \( f(0) = 0 \) is increasing in magnitude with \( N \), whereas \( h(N) \) is decreasing in \( N \) and obeys \( 0 < h(N) \leq 1 \) and \( h(0) = 1 \). These additive and multiplicative contributions are not exclusive, and both terms can be simultaneously present in the average cost formula.

Using \( g(t) = e^{-\delta t} \gamma(t) \dot{N}(t) \), the Euler-Lagrange Eq. (5) becomes, after some algebra

\[ \nu (\nu + 1) \gamma_2 \dot{N}(t)^{\nu-1} \dot{\tilde{N}}(t) = -\nu \gamma_2 \dot{N}(t)^{\nu+1} \frac{dh}{dN}(N) \]

\[ + \delta \left\{ \gamma_0 + (\nu + 1) \gamma_2 \dot{N}(t)^\nu + f(N) \right\} \]

(33)

since \( h(N) > 0 \). Let us examine the two basic models separately:

1. In the additive model \( h(N) = 1 \) and \( \frac{dh}{dN}(N) = 0 \), and therefore the Euler-Lagrange Eq. (33) reduces to

\[ \nu (\nu + 1) \gamma_2 \dot{N}(t)^{\nu-1} \dot{\tilde{N}}(t) = \delta \left\{ \gamma_0 + (\nu + 1) \gamma_2 \dot{N}(t)^\nu + f(N) \right\} \]

(34)

extending Eq. (3) to a general additive endogenous learning contribution \( f(N) \) to average cost. In the absence of time-discounting we have \( \ddot{\tilde{N}}(t) = 0 \), leading to constant emissions rate \( \dot{N}(t) \).

In the presence of discounting, since \( f(N) < 0 \) for \( N > 0 \), endogenous learning
decreases the pace at which emissions rate $\dot{N}(t)$ increases with time.

2. For the multiplicative model with $f(N) = 0$ and $h(N)$ decreasing with $N$, endogenous learning has an influence even with $\delta = 0$. In this case, since $\frac{dh}{dN} < 0$ the net contribution is positive, as in Eq. (30), with its effect being to increase emissions rate with time.

If both effects $f(N)$ and $h(N)$ are active, the net result of $f(N)$ tending to decrease abatement rates with time and $h(N)$ doing the opposite would depend on their relative magnitudes.

### 6.2 Effects on marginal cost curve

Given the importance of these differences, let us consider the influence of these two models of endogenous learning on the marginal cost curve. Average cost $\gamma(N, \dot{N})$ is related to marginal cost $\beta(N, \dot{N})$ as $\gamma(N, \dot{N}) = \frac{1}{N} \int_0^{\dot{N}} \beta(N, \dot{s}) \, d\dot{s}$ where $\dot{s}$ is integrated from 0 to $\dot{N}$. Substituting for the form of the average cost in Eq. (32), marginal cost is related as

$$\int_0^{\dot{N}} \beta(N, \dot{s}) \, d\dot{s} = \left( \gamma_0 + f(N) + \gamma_2 \dot{N}(t)^\nu \right) h(N) \dot{N}$$

and differentiating with respect to $\dot{N}$

$$\beta(N, \dot{N}) = \left\{ \gamma_0 + (\nu + 1) \gamma_2 \dot{N}^\nu + f(N) \right\} h(N)$$

makes it apparent that $f(N)$ is an additive contribution to the marginal cost curve whereas $h(N)$ multiplies it throughout.

In any given year, $N$ is approximately determined by previous emissions, and marginal cost is determined by emissions rate $\dot{N}$. Additionally, the marginal cost curve is influenced by the history of past emissions, depending on the combination of $f(N)$ and $h(N)$ at work in the model. The additive term $f(N)$, being negative, reduces marginal cost by the same amount for all values of emissions rate, thus effecting a downward shift in the
curve. In contrast multiplication by \( h(N) \) pivots down marginal cost, reducing it by the same factor for all values of \( \dot{N} \).

### 6.3 Interpreting the effects of the two models

The effect of the multiplicative model through varying \( h(N) \) can be interpreted through analogy with effects of discounting in time, which also induces increasing emissions rate. The multiplicative term scales-down future costs as \( N \) grows, and this can be interpreted as amplifying the discounting of future costs.

The additive model requires more discussion for placing it in proper perspective. Consider emissions rate \( \dot{N}(t) = \frac{CN}{T} + \epsilon x(t) \), with \( \epsilon \ll 1 \) so that \( \epsilon x(t) \) is a small perturbation to a trajectory with constant emissions rate. Cumulative negative emissions is \( N(t) = \int_{-T}^{t} \dot{N}(s) ds = CN \left(1 + \frac{t}{T}\right) + \epsilon X(t) \), where \( X(t) = \int_{-T}^{t} x(s) ds \). For satisfying boundary conditions on \( N(t) \) we require \( X(-T) = X(0) = 0 \). If only the additive model in \( f(N) \) is active (i.e. \( h(N) = 1 \)), the expenditure on negative emissions in year \( t \) is \( E_N(t) = \gamma_0 \dot{N}(t) + f(N) \dot{N}(t) + \gamma_2 \dot{N}(t)^{\nu+1} \), and substituting the models of \( N(t) \) and \( \dot{N}(t) \) above

\[
E_N(t) = \gamma_0 \left\{ \frac{CN}{T} + \epsilon x(t) \right\} + \left\{ \frac{CN}{T} + \epsilon x(t) \right\} f \left( \frac{CN}{T} \right) + \gamma_2 \left( \frac{CN}{T} + \epsilon x(t) \right)^{\nu+1} \tag{37}
\]

After approximating \( f \) above by its 1st-order Taylor series and some algebra, the discounted expenditure in case \( \delta = 0 \) simplifies to

\[
F_N = \int_{-T}^{0} E_N(t) dt = \gamma_0 CN + \frac{CN}{T} \int_{-T}^{0} f \left( \frac{CN}{T} \right) dt + \gamma_2 \int_{-T}^{0} n(t)^{\nu+1} dt \tag{38}
\]

as described in Appendix 2. There it is shown that in the presence of additive endogenous learning, the first-order effect on total costs of small perturbations, described by \( \epsilon x(t) \), to the constant emissions trajectory is zero, in the absence of discounting. That is, if \( \epsilon \ll 1 \) and \( \delta = 0 \), endogenous learning has the same effect on the total costs as it would for a
constant emissions rate trajectory, as measured by the term $\frac{C_N}{T} \int_{-T}^{0} f \left( C_N \left( 1 + \frac{t}{T} \right) \right) dt$.

The effect of non-constant trajectories enters through the last term in Eq. (38). Since $\nu > 0$, $p (n) \equiv n^{\nu+1}$ is a convex function of $n$. Emissions rate $n (t) = \frac{C_N}{T} + \epsilon x (t)$, and its expectation in time is given by $\mathbb{E}n = \frac{1}{T} \int_{-T}^{0} n (t) dt$. Using boundary conditions on $X (t)$, $\mathbb{E}n = \frac{C_N}{T}$. Additionally, $\mathbb{E}p (n) = \frac{1}{T} \int_{-T}^{0} p (n (t)) dt = \frac{1}{T} \int_{-T}^{0} n (t)^{\nu+1} dt$.

From Jensen’s inequality, for a convex function $p (n)$, $\mathbb{E}p (n) \geq p (\mathbb{E}n)$. But $p (\mathbb{E}n) = \left( \frac{C_N}{T} \right)^{\nu+1}$. Hence $\frac{1}{T} \int_{-T}^{0} n (t)^{\nu+1} dt \geq \left( \frac{C_N}{T} \right)^{\nu+1}$, or equivalently $\gamma_2 \int_{-T}^{0} \left( \frac{C_N}{T} + \epsilon x (t) \right)^{\nu+1} dt \geq \gamma_2 \int_{-T}^{0} \left( \frac{C_N}{T} \right)^{\nu+1} dt$, with equality occurring only for $\epsilon = 0$. Minimizing total cost across time requires $\epsilon = 0$, and an additive endogenous learning effect does not modify the optimal trajectory in case $\delta = 0$, entailing a constant emissions rate.

Perhaps simpler intuition can be found, but such would not be a substitute for mathematical argument. What is clear, however, is that the above effect results from convexity of the function $n (t)^{\nu+1}$, describing the contribution of increasing marginal cost to the expenditure in year $t$. This convexity arises from $\nu > 0$; and if $\nu = 0$, Eq. (34) would be satisfied for arbitrary emissions pathways if time-discounting were also absent.

The present section also shows that the qualitative results of Sections 3 and 5 do not depend on the particular models of $f (N)$ and $h (N)$ employed there. If endogenous learning shifts downward the marginal cost curve through an additive effect, constant emissions rate is optimal for $\delta = 0$ regardless of the particular model of $f (N) < 0$. For $\delta > 0$ higher emissions are favored early on (compared to the no-learning case) because the term contributes negatively to $\ddot{N} (t)$ in Eq. (33). However it does not alter the preference for increasing emissions rate over time, as seen in the following. Substituting from Eq. (36), the Euler-Lagrange Eq. (33) can be represented in terms of marginal cost $\beta (N, \dot{N})$ as $\nu (\nu + 1) \gamma_2 \ddot{N} (t)^{\nu-1} \dot{N} (t) = -\nu \gamma_2 \ddot{N} (t)^{\nu+1} \frac{\ddot{N}}{h (N)} + \delta \beta (N, \dot{N}) / h (N)$. Since marginal cost is always positive, even in the presence of endogenous learning, the term $\delta \beta (N, \dot{N}) / h (N)$ that includes the additive learning effect causes a positive contribution to $\ddot{N} (t)$, causing emissions rate $\dot{N} (t)$ to increase over time. Hence, additive learning slows the growth rate of negative emissions without making it zero.

---

3We recall that $\dot{N} (t)$, describing rate of negative emissions, is non-negative.
In contrast, if endogenous learning scales down the marginal cost curve through a multiplicative term, this favors increasing emissions over time since, if \( \frac{dh}{dN} < 0 \), this term contributes positively to \( \dot{N}(t) \) in Eq. (33). Multiplicative learning therefore increases the growth rate of negative emissions.

7 Conclusions

Global warming from anthropogenic CO\(_2\) is approximately proportional to its cumulative emissions across time (Allen et al. (2009); Matthews et al. (2009)). Although path independence between global warming and cumulative CO\(_2\) emissions is less accurate in scenarios with substantial negative emissions (Zickfeld et al. (2016)), cumulative negative emissions remains a useful metric for mitigation policy studies. Therefore it is useful to consider the discounted costs of alternate negative emissions pathways, while keeping fixed the total negative emissions across time.

This paper introduces idealized models to examine optimal pathways of negative emissions that minimize discounted costs, in the presence of endogenous learning (i.e. "learning by doing"). While several studies have included negative emissions in scenarios for meeting global warming goals (van Vuuren et al. (2011); Friedlingstein et al. (2014); Fuss et al. (2016); Hansen et al. (2016); Jones et al. (2016); Field and Mach (2017); Kriegler et al. (2018); van Vuuren et al. (2018)), we are not aware of explicit analyses of optimal pathways of negative emissions in the presence of a cumulative emissions constraint, let alone in the presence of endogenous learning. Such optimal pathways can be posed naturally as solutions to a suitably chosen variational problem, which seeks to minimize discounted costs of negative emissions subject to a constraint on total negative emissions across time.

Our model for negative emissions costs assumes that the same types of activities are repeated each year, in increasing order of cost. Therefore the cost of negative emissions in any given year is determined by integrating over the marginal cost curve. In the absence of endogenous learning, the marginal cost is, from Eq. (30), \( \beta(\dot{N}) = \gamma_0 + (\nu + 1) \gamma_2 \dot{N}^\nu \), with \( \dot{N} \) being the negative emissions rate, i.e. emissions in a given year. Then, in case the
marginal cost curve is non-constant with $\nu > 0$, as it generally is, the cost of mitigation
in year $t$ is $E_N(t) = \left\{ \gamma_0 + \gamma_2 \dot{N}(t)^{\nu} \right\} \dot{N}(t)$, being a convex function of the emissions rate.

The stipulation that $\nu > 0$ ensures that the Euler-Lagrange equation in Eq. (6) is a
2-point boundary value problem, so that in general a minimizing pathway of negative
emissions can be found satisfying the two boundary conditions on cumulative negative
emissions. Only in case $\nu = 0$ does the Euler-Lagrange equation become an algebraic
equation in $N(t)$ (i.e. ”degenerate”), with a minimum solution existing only if time-
discount rate $\delta = 0$: in which case all pathways have the same discounted cost and there
is no unique minimum.

Given the unrealistic state of constant marginal costs and the special character of the
associated minimum solution, existing only in the absence of time-discounting, we have
not considered this setting in much of the present paper, which mostly examines situations
with $\nu > 0$. In the latter circumstance, in the absence of time-discounting and endogenous
learning the least-cost pathway involves a constant rate of negative emissions across time.
With non-zero discount rate $\delta$, the emissions rate in the least-cost pathway grows at rate
$\delta/\nu$. Larger discount rate causes the optimal solution to have an emissions rate that is
small at first but growing more rapidly, whereas a larger exponent $\nu$ of the marginal cost
curve induces a higher emissions rate in the beginning that grows more slowly thereby
making it more equal across time.

We first considered a model of endogenous learning where marginal costs became smaller
through subtracting a term proportional to cumulative negative emissions, i.e. $\beta(N, \dot{N}) = \gamma_0 + (\nu + 1) \gamma_2 \dot{N}^{\nu} - \gamma_1 N(t)$. In this model, endogenous learning has no influence on the
optimal pathway if the discount rate is zero. Section 6 showed that this effect arose
from convexity of the total expenditure function, and followed from Jensen’s inequality.
In the presence of time-discounting an additive endogenous learning effect, occurring in
this example as $-\gamma_1 N(t)$, decreases the rate at which emissions grows in the presence of
time-discounting alone.

We generalized this result in Section 6 to additive endogenous learning effects where
$\gamma_1 N(t)$ is replaced by a function increasing in $N(t)$, with the same qualitative results.
Additive endogenous learning effects, shifting downwards the marginal cost curve by a uniform amount increasing with cumulative emissions \( N(t) \), favor early emissions. However in general they do not alter the preference for increasing emissions rate over time, because we still have \( \gamma_0 + (\nu + 1) \gamma_2 \dot{N}(t)^\nu + f(N) > 0 \) in Eq. (34), as marginal costs are positive.

There are alternative ways of modeling the endogenous learning effect. For example, there could be a multiplicative term decreasing in cumulative negative emissions \( N(t) \). Section 5 considered a model where marginal costs are \( \beta(N, \dot{N}) = \{ \gamma_0 + (\nu + 1) \gamma_2 \dot{N}^\nu \} e^{-\gamma_1 N} \), decreasing as an exponential function of cumulative emissions, and Section 6 generalized this to multiplication by arbitrary functions decreasing in \( N(t) \). Such a multiplicative endogenous learning effect, scaling the marginal cost curve down by a factor depending on \( N(t) \), causes the optimal solution to have higher emissions rates later on thereby increasing growth rate of negative emissions. This occurs even in the absence of time-discounting.

At the outset, intuition suggests two possible effects of endogenous learning on the least-cost pathway of emissions. In the presence of endogenous learning, increasing the emissions rate early on can be beneficial for subsequent cost reductions. On the contrary, since endogenous learning is a cumulative effect it is likely to happen anyway and hence it could be beneficial to start at lower emissions rates that increase with time as benefits of learning accumulate. Prior studies have exhibited these contrary effects (e.g. van der Zwaan et al. (2002); Goulder and Mathai (2000); Gillingham et al. (2008)), and this paper shows that each of these effects is present in the two different types of endogenous learning models, additive and multiplicative.

With an additive effect, shifting the marginal cost curve downward, there are benefits to increasing the emissions rate early on as compared to the no-learning case, although the net effect continues to be that of increasing emissions over time. With a multiplicative endogenous learning effect, scaling down the cost curve, benefits of learning make themselves only gradually felt so that it is beneficial to start at lower emissions rates that increase more rapidly than in the absence of endogenous learning. In general both effects might
be present, and which effect dominates is beyond our present scope. Previous authors have shown evidence for cost being proportional to a function of cumulative emissions (Nagy et al. (2013)), suggesting the predominance of multiplicative effects. However the present work merely emphasizes the basic distinction between additive and multiplicative endogenous learning, which is relevant for qualitative understanding of the effects in our idealized setting.

Another aspect examined in this paper is the optimal starting time for negative emissions. It was shown that, whenever marginal costs increase with emissions rate with $\nu > 0$, costs are minimized by starting negative emissions as early as possible and following pathways dictated by the solution to the corresponding Euler-Lagrange equation. Under a cumulative negative emissions constraint, starting early reduces the average negative emissions rate, limiting overall costs. Of course our setting is idealized, and the results serve primarily as a thought experiment, and it is important to develop considerably more realistic treatments that are much more disaggregated (Farmer et al. (2015)), include endogenous and exogenous learning (Nagy et al. (2013); Magee et al. (2016)), treat uncertainty in learning estimates (Nordhaus (2009)), and consider alternatives to exponential discounting (Gollier and Weitzman (2010)).

Appendix 1: Derivation of the Euler-Lagrange equation

For fixed $T$, in order to minimize functional $\int_{-T}^{0} g(t) \, dt$ subject to constraint $\int_{-T}^{0} \dot{N}(t) \, dt = C_N$, we first define $I(N, \dot{N}) = \int_{-T}^{0} g(t) \, dt + \lambda \left( \int_{-T}^{0} \dot{N}(t) \, dt - C_N \right)$. Stationarity requires the first-variation $\delta I$ in the integral due to small changes $\delta N$ and $\delta \dot{N}$ to vanish. The variation $\delta I = I(N + \delta N, \dot{N} + \delta \dot{N}) - I(N, \dot{N})$ is

$$\delta I = \int_{-T}^{0} \left\{ \frac{\partial g}{\partial N} \delta N + \frac{\partial g}{\partial \dot{N}} \delta \dot{N} + \lambda \left( \frac{\partial \dot{N}}{\partial N} \delta N + \frac{\partial \dot{N}}{\partial \dot{N}} \delta \dot{N} \right) \right\} \, dt = 0 \quad (39)$$
and, integrating by parts,  
\[ \int_{-T}^{0} \frac{\partial g}{\partial N} \delta \dot{N} dt = - \int_{-T}^{0} \frac{d}{dt} \left( \frac{\partial g}{\partial N} \right) \delta N dt \]
  after applying the condition that \( \delta N (-T) = \delta N (0) = 0 \) because the variations must preserve boundary conditions of the problem. There is a corresponding equation involving the integral constraint on cumulative negative emissions. Combining these equations yields

\[ \delta I = \int_{-T}^{0} \left\{ \frac{\partial g}{\partial N} \frac{d}{dt} \left( \frac{\partial g}{\partial N} \right) + \lambda \left( \frac{\partial \dot{N}}{\partial N} - \frac{d}{dt} \left( \frac{\partial \dot{N}}{\partial N} \right) \right) \right\} \delta N dt = 0 \] (40)

for arbitrary changes \( \delta N \), yielding the Euler-Lagrange (E-L) Eq. (5).

**Appendix 2: Cumulative expenditures with additive endogenous learning and zero discount rate**

The expenditure on negative emissions in year \( t \) in Eq. (37) is re-written below

\[ E_N (t) = \gamma_0 \left\{ \frac{C_N}{T} + \epsilon x (t) \right\} + \left\{ \frac{C_N}{T} + \epsilon x (t) \right\} f \left( C_N \left( 1 + \frac{t}{T} \right) + \epsilon X (t) \right) \]

\[ + \gamma_2 \left( \frac{C_N}{T} + \epsilon x (t) \right)^{\nu+1} \] (41)

Upon substituting the 1st-order Taylor series expansion in small parameter \( \epsilon \)

\[ f \left( C_N \left( 1 + \frac{t}{T} \right) + \epsilon X (t) \right) \approx f \left( C_N \left( 1 + \frac{t}{T} \right) \right) + \epsilon X (t) f' \left( C_N \left( 1 + \frac{t}{T} \right) \right) \] (42)

into Eq. (41), we obtain

\[ E_N (t) = \frac{C_N}{T} \left\{ \gamma_0 + f \left( C_N \left( 1 + \frac{t}{T} \right) \right) \right\} + \epsilon \frac{d}{dt} \left\{ \gamma_0 X (t) + f \left( C_N \left( 1 + \frac{t}{T} \right) \right) X (t) \right\} \]

\[ + \gamma_2 \left( \frac{C_N}{T} + \epsilon x (t) \right)^{\nu+1} \] (43)
where, for consistency, we have retained only 1st order terms in $\epsilon$ for the series-derived contributions. Total expenditure becomes, for $\delta = 0$

\[
F_N = \int_{-T}^{0} \frac{C_N}{T} \left\{ \gamma_0 + f \left( C_N \left( 1 + \frac{t}{T} \right) \right) \right\} dt + \epsilon \int_{-T}^{0} \frac{d}{dt} \left\{ \gamma_0 X(t) + f \left( C_N \left( 1 + \frac{t}{T} \right) \right) X(t) \right\} dt + \gamma_2 \int_{-T}^{0} \left( \frac{C_N}{T} + \epsilon x(t) \right)^{\nu+1} dt \quad (44)
\]

and, using

\[
\int_{-T}^{0} \frac{d}{dt} \left\{ \gamma_0 X(t) + f \left( C_N \left( 1 + \frac{t}{T} \right) \right) X(t) \right\} dt = \gamma_0 \left( X(0) - X(-T) \right)
\]

\[
+ f \left( C_N \right) X(0) - f \left( 0 \right) X(-T) = 0 \quad (45)
\]

with the last equality following from boundary conditions on $X(t)$, we obtain Eq. (38). Eq. (45) shows that for an additive endogenous learning model the first-order effect on costs of small perturbations, described by $\epsilon x(t)$, to the constant emissions trajectory is zero. With $\epsilon \ll 1$ and $\delta = 0$, endogenous learning has the same effect on the total costs as it would for a constant emissions rate trajectory. The effect of endogenous learning on costs, given by the term $\frac{C_N}{T} \int_{-T}^{0} f \left( C_N \left( 1 + \frac{t}{T} \right) \right) dt$, clearly does not depend on the departure $\epsilon x(t)$.

**References**

Allen, M. R., and T. F. Stocker (2014), Impact of delay in reducing carbon dioxide emissions, *Nature Climate Change*, 4, 23–26, doi:http://dx.doi.org/10.1038/nclimate2077.

Allen, M. R., D. J. Frame, C. Huntingford, C. D. Jones, J. A. Lowe, M. Meinshausen, and N. Meinshausen (2009), Warming caused by cumulative carbon emissions towards the trillionth tonne, *Nature*, 458, 1163–1166, doi:http://dx.doi.org/10.1038/nature08019.

Arrow, K. J. (1962), The economic implications of learning by doing, *The Review of Economic Studies*, 29, 155–173, doi:10.2307/2295952.
Baker, E., and E. Shittu (2008), Uncertainty and endogenous technical change in climate policy models, *Energy Economics*, 30, 2817–2828, doi:10.1016/j.eneco.2007.10.001.

Baker, E., L. Clarke, and E. Shittu (2008), Technical change and the marginal cost of abatement, *Energy Economics*, 30, 2799–2816, doi:10.1016/j.eneco.2008.01.004.

Boucher, O., V. Belllassen, H. Benveniste, P. Ciais, and P. Criqui (2016), Opinion: In the wake of Paris Agreement, scientists must embrace new directions for climate change research, *Proceedings of the National Academy of Sciences of the United States of America*, 113, 7287–7290, doi:https://doi.org/10.1073/pnas.1607739113.

Caldecott, B., G. Lomax, and M. Workman (2015), Stranded carbon assets and negative emissions technologies, working Paper, Smith School of Enterprise and the Environment, Oxford University.

Cao, L., and K. Caldeira (2010), Atmospheric carbon dioxide removal: long-term consequences and commitment, *Environmental Research Letters*, 5, 1–6, doi:http://dx.doi.org/10.1088/1748-9326/5/2/024011.

Clarke, L., K. Jiang, K. Akimoto, M. Babiker, and G. Blanford (2014), *Climate Change 2014: Mitigation of Climate Change. Contribution of Working Group III to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change*, chap. Assessing Transformation Pathways, pp. 413–510, Cambridge University Press.

Farmer, J. D., C. Hepburn, P. Mealy, and A. Teytelboym (2015), A third wave in the economics of climate change, *Environmental and Resource Economics*, 62, 329–357, doi:https://doi.org/10.1007/s10640-015-9965-2.

Field, C. B., and K. J. Mach (2017), Rightsizing carbon dioxide removal, *Science*, 356, 706–707, doi:10.1126/science.aam9726.

Friedlingstein, P., R. M. Andrew, J. Rogelj, G. P. Peters, and J. G. Canadell (2014), Persistent growth of CO2 emissions and implications for reaching climate targets, *Nature Geoscience*, 7, 709–715, doi:10.1038/ngeo2248.
Fuss, S., C. D. Jones, F. Kraxner, G. P. Peters, P. Smith, and M. Tavoni (2016), Research priorities for negative emissions, *Environmental Research Letters*, 11, 1–11, doi:http://dx.doi.org/10.1088/1748-9326/11/11/115007.

Gillingham, K., R. G. Newell, and W. A. Pizer (2008), Modeling endogenous technological change for climate policy analysis, *Energy Economics*, 30, 2734–2753, doi:10.1016/j.eneeco.2008.03.001.

Gollier, C., and M. L. Weitzman (2010), How should the distant future be discounted when discount rates are uncertain?, *Economics Letters*, 107, 350–353, doi:https://doi.org/10.1016/j.econlet.2010.03.001.

Goulder, L. H., and K. Mathai (2000), Optimal CO2 Abatement in the Presence of Induced Technological Change, *Journal of Environmental Economics and Management*, 39, 1–38, doi:10.1006/jeem.1999.1089.

Hansen, J., M. Sato, P. Kharecha, K. von Schuckmann, D. J. Beerling, and J. Cao (2016), Young People’s Burden: Requirement of Negative CO2 Emissions, doi:https://arxiv.org/abs/1609.05878.

Jones, C. D., P. Ciais, S. J. Davis, P. Friedlingstein, T. Gasser, and G. P. Peters (2016), Simulating the Earth system response to negative emissions, *Environmental Research Letters*, 11, 1–11, doi:10.1088/1748-9326/11/9/095012.

Keith, D. W. (2009), Why capture CO2 from the atmosphere, *Science*, 325, 1654–1655, doi:10.1126/science.1175680.

Kriegler, E., G. Luderer, N. Bauer, L. Baumstark, S. Fujimori, A. Popp, and J. Rogelj (2018), Pathways limiting warming to 1.5 c: a tale of turning around in no time?, *Philosophical Transactions A*, 376, 1–17, doi:0.1098/rsta.2016.0457.

Magee, C. L., S.Basnet, J.L.Funk, and C.L.Benson (2016), Quantitative empirical trends in technical performance, *Technological Forecasting and Social Change*, 104, 237–246, doi:https://doi.org/10.1016/j.techfore.2015.12.011.
Matthews, H. D., N. P. Gillett, P. A. Stott, and K. Zickfeld (2009), The proportionality of global warming to cumulative carbon emissions, *Nature*, 459, 829–832, doi:http://dx.doi.org/10.1038/nature08047.

Minx, J. C., W. F. Lamb, M. W. Callaghan, L. Bornmann, and S. Fuss (2017), Fast growing research on negative emissions, *Environmental Research Letters*, 12, 1–10, doi: https://doi.org/10.1088/1748-9326/aa5ee5.

Nagy, B., J. D. Farmer, Q. M. Bui, and J. E. Trancik (2013), Statistical basis for predicting technological progress, *PLOS One*, 8, 1–7, doi:https://doi.org/10.1371/journal.pone.0052669.

Nordhaus, W. D. (2009), The perils of the learning model for modeling endogenous technical change, national Bureau of Economic Research Working Paper.

Peters, G. P., and O. Geden (2017), Catalysing a political shift from low to negative carbon, *Nature Climate Change*, 7, 619–621, doi:10.1038/nclimate3369.

Psarras, P., H. Krutka, M. Fajardy, Z. Zhang, S. Liguori, N. M. Dowell, and J. Wilcox (2017), Slicing the pie: how big could carbon dioxide removal be?, *WIREs Energy and Environment*, 6, 1–21, doi:10.1002/wene.253.

Reiner, M. H. (2018), The political economy of negative emissions technologies: consequences for international policy design, *Climate Policy*, 18, 306–321, doi:10.1080/14693062.2017.1413322.

Sanchez, D. L., and D. S. Callaway (2016), Optimal scale of carbon-negative energy facilities, *Applied Energy*, 170, 437–444, doi:10.1016/j.apenergy.2016.02.134.

Seshadri, A. K. (2017), Origin of path independence between cumulative CO2 emissions and global warming, *Climate Dynamics*, pp. 1–19, doi:10.1007/s00382-016-3519-3.

Smith, P., S. J. Davis, F. Creutzig, S. Fuss, J. Minx, B. Gabrielle, E. Kato, and R. B. Jackson (2015), Biophysical and economic limits to negative CO2 emissions, *Nature Climate Change*, 6, 42–50, doi:http://dx.doi.org/10.1038/nclimate2870.
van der Zwaan, B., R. Gerlagh, G. Klaassen, and L. Schrattenholzer (2002), Endogenous
technological change in climate change modelling, *Energy Economics*, 24, 1–19, doi:
10.1016/S0140-9883(01)00073-1.

van Vuuren, D. P., E. Stehfest, M. den Elzen, T. Kram, and J. van Vliet (2011), RCP2.6:
exploring the possibility to keep global mean temperature increase below 2 C, *Climatic
Change*, 109, 95–116.

van Vuuren, D. P., E. Stehfest, D. E. H. J. Gernaat, M. van den Berg, and D. L. Bijl
(2018), Alternative pathways to the 1.5 C target reduce the need for negative emission
technologies, *Nature Climate Change*, 8, 391–397, doi:10.1038/s41558-018-0119-8.

Wing, I. S. (2006), Representing induced technological change in models for climate policy
analysis, *Energy Economics*, 28, 539–562, doi:10.1016/j.eneco.2006.05.009.

Zickfeld, K., A. H. MacDougall, and H. D. Matthews (2016), On the proportionality
between global temperature change and cumulative CO2 emissions during periods of
net negative CO2 emissions, *Environmental Research Letters*, 11, 1–9, doi:http://dx.
doi.org/10.1088/1748-9326/11/5/055006.