Integrability of the one dimensional Schroedinger equation

Thierry COMBOT

Abstract

We present a definition of integrability for the one dimensional Schroedinger equation, which encompasses all known integrable systems, i.e. systems for which the spectrum can be explicitly computed. For this, we introduce the class of rigid functions, built as Liouvillian functions, but containing all solutions of rigid differential operators in the sense of Katz, and a notion of natural of boundary conditions. We then make a complete classification of rational integrable potentials. Many new integrable cases are found, some of them physically interesting.

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1. Introduction

In this article, we are interested in the definition of integrability and the search of integrable potentials of the one dimensional Schroedinger equation

\[
\frac{d^2 \psi}{dz^2} + (V(z) + E)\psi(z) = 0
\]

where \( V \) is a rational function, and \( E \) a parameter. The problem is not only to find solutions of equation (1) under a more or less explicit form, but above all to compute the set \( S \) (called the spectrum) parameters values \( E \) such that equation (1) admits a solution with particular properties (called boundary conditions).
conditions). The most typical condition required is the square integrable condition
\[ \int_{-\infty}^{\infty} |\psi(z)|^2 dz < \infty \]

Equation (1) is the quantum equivalent of a one degree of freedom Hamiltonian system. In classical mechanics, this system is always integrable in the sense that we are always able to express the solutions in terms of quadrature. In this quantum equivalent, it is no longer the case.

There has been various ways to define the meaning of \( V \) quantum integrable. Here we focus on complete integrability, i.e. finding all the eigenfunctions, which is more restrictive to partial integrability cases, as in [1] where only finitely many eigenstates are found. The closest notion to the one presented in this article is the following

**Definition 1 (Integrability definition in [2]).** The equation (1) is said to be integrable if for all \( E \) in the spectrum, the solutions of equation (1) are Liouvillian.

This definition seems to contain all “quantum integrable” cases, at least in the case of discrete spectrum. However, there are several inconvenience

- This does not allow (at least a priori) to compute the set \( S \). Indeed, the integrability check has to be done the other way. Assume we know the spectrum \( S \), the system is integrable if and only if equation (1) has Liouvillian solutions. This can however, for a fixed \( E \), be done algorithmically through the Kovacic algorithm [3].

- The notion of integrability is strongly dependant of the boundary conditions. This happens for example in the following equation

\[ \frac{d^2\psi}{dz^2} + (z^{-1} + E)\psi(z) = 0 \]

If the boundary condition is being square integrable on \( \mathbb{R}^+ \), then the system is integrable with a discrete spectrum \( \mathcal{S} \). If we look for square integrable solutions on \( \mathbb{R}^- \) there are none, and if only near 0 then \( \mathcal{S} = \mathbb{C} \). In this continuous case, the system is not integrable in the above sense.
The Liouvillian condition is somewhat arbitrary. In particular, there are other functions which are dubbed “nice” but not Liouvillian, see [4, 5, 6].

So our purpose is to build a definition of integrability which is as most as possible independent of the boundary conditions, which allows to compute in an algebraic manner the spectrum, and which is large enough to contain all cases dubbed to be “quantum integrable”. For this we construct a class of functions in section 2, we name “rigid functions”, the name coming from the notion of rigid operators introduced by Katz [7], to which they are closely linked.

**Definition 2.** A potential $V \in \mathbb{C}(z)$ is said to be quantum integrable if for all $E \in \mathbb{C}$, the solutions of equation (1) are rigid functions.

In section 2 is also introduced a notion of natural boundary conditions. The boundary conditions are natural if they can be expressed in terms of monodromy and Stokes matrices, see Definition 6. We prove in particular that the classical square integrability condition is “almost” equivalent to a natural boundary condition, in the sense that there exists a natural boundary condition which gives an infinite discrete set of energies containing those for which the square integrability condition is satisfied (as long as this one is not always satisfied). We will see moreover how to compute explicitly the spectrum from the expressions of solutions in terms of rigid functions and boundary conditions. The quantum integrable potentials split naturally in two categories. The discrete type, for which there exist natural boundary conditions leading to an infinite countable spectrum, and the continuous one, for which any natural boundary condition leads to finite or continuous spectrum. The latter is related to isomonodromic deformations. In section 3, we prove Theorem 1 the quantum integrable potentials $V$ should have eigenfunctions of 4 possible forms. In section 4, we prove the main theorem of the article, a classification of quantum integrable rational potentials. The families of integrable potentials are generated by Pade interpolation/series. The section 5 and the Appendix is devoted to examples and the explicit generation of the quantum integrable potentials of these families, as their presentation uses Pade interpolation and Pade series which makes their construction not immediate, although straightforward. Among them, two physically interesting new quantum integrable potentials are solved in details

$$V(z) = -z^2 - 2 - \frac{8}{2z^2 + 1} + \frac{16}{(2z^2 + 1)^2}$$
\[ V(z) = \frac{1}{z} - \frac{4}{z^2 + 2z + 2} + \frac{8}{(z^2 + 2z + 2)^2} \]

For the rest of the article, we will note \( W(\mu, \nu, z) \) a non zero solution of the differential equation

\[ y''(z) + \left( -\frac{1}{4} + \frac{\mu}{z} + \frac{1/4 - \nu^2}{z^2} \right) y(z) = 0. \]

Moreover, from now on, the \( ' \) will be the differentiation in \( z \).

**Theorem 1.** If \( V \in \mathbb{C}(z) \) is quantum integrable, then up to affine coordinate change and addition of a constant to \( V \) the Schrödinger equation has solutions of one of the following forms

\[
\psi(z, E) = \frac{z^{3/2} \left( \frac{M(z, E)}{z} \right) W(E/4, \nu, z^2) + W'(E/4, \nu, z^2)}{\sqrt{M(z, E)^2 z^2 + M(z, E) z - M'(z, E) z^2 - z^4 + z^2 E - 4 \nu^2 + 1}}
\]

\[
\psi(z, E) = \frac{z \left( \frac{M(z, E)}{\sqrt{-4E}} \right) W((-4E)^{-1/2}, \nu, z \sqrt{-4E}) + W'((-4E)^{-1/2}, \nu, z \sqrt{-4E})}{\sqrt{4M(z, E)^2 z^2 + 4z^2 E - 4M'(z, E) z^2 - 4 \nu^2 + 4z + 1}}
\]

\[
\psi(z, E) = \frac{z \left( \frac{M(z, E)}{\sqrt{-4E}} \right) W(0, \nu, z \sqrt{-4E}) + W'(0, \nu, z \sqrt{-4E})}{\sqrt{4M(z, E)^2 z^2 + 4z^2 E - 4M'(z, E) z^2 - 4 \nu^2 + 1}}
\]

\[
\psi(z, E) = \frac{z \left( \frac{M(z, E)}{(z + E)^{3/2}} \right) W(0, \frac{1}{3}, \frac{4i}{3}(z + E)^{3/2}) + W'(0, \frac{1}{3}, \frac{4i}{3}(z + E)^{3/2})}{(z + E)^{-1/4} \sqrt{M(z, E)^2 + E - M'(z, E) + z}}
\]

with \( M(z, E) \) rational in \( z, E \).

The case \( M = \infty \) has to be included, and effectively leads to quantum integrable potentials. Remark that given a solution \( \psi \) of the Schrödinger equation, we can recover the potential as \( V + E = -\psi''/\psi \). Thus the function \( M \), and even its restriction to a generic value of \( E \), completely defines the potential \( V \) in the above expressions. In particular, the potential \( V(z) + E \) can be written as a rational function of \( z, E, M \) and its derivatives. Remark that however, all rational \( M \) do not lead to potentials, as \( -\psi''/\psi \) should be of the form \( V(z) + E \). This will be the condition to obtain an integrable potential. We now present the classification results, i.e. a set of \( M \) functions leading to all quantum integrable potentials \( V \).
Theorem 2. A quantum integrable potential \( V \in \mathbb{C}(z) \) comes from a function \( M \) given by

- In case 1 of Theorem 1, the rational interpolation with numerator denominators degrees in \( E \) less than \( n/2, (n-1)/2 \) given by \( M(z, \epsilon_1(4k + 2) + 4\epsilon_2 \nu) = \)

\[
- \frac{\partial}{\partial z} \ln \left( z^{2\epsilon_1 \epsilon_2 \nu + 1} e^{\epsilon_1 z^2/2} F_1(-k, 2\epsilon_1 \epsilon_2 \nu + 1, \epsilon_1 z^2) \right)
\]

(2)

for \( n \) points of the form \( \epsilon_1(4k + 2) + 4\epsilon_2 \nu, k \in \mathbb{N}, \epsilon_1, \epsilon_2 = \pm 1 \)

- In case 2 of Theorem 1, the rational interpolation with numerator denominators degrees in \( E \) less than \( n/2, (n-1)/2 \) given by \( M(z, -(2\epsilon \nu + 2k + 1)^{-2}) = \)

\[
- \frac{\partial}{\partial z} \ln \left( z^{\epsilon \nu + 1/2} e^{-\frac{z^2}{2\epsilon \nu + 2k + 1}} F_1(-k, 2\epsilon \nu + 1, \frac{2z}{2\epsilon \nu + 2k + 1}) \right)
\]

(3)

for \( n \) points \( -(2\epsilon \nu + 2k + 1)^{-2} \) with \( k \in \mathbb{N}, \epsilon = \pm 1 \).

- In case 3 of Theorem 1, the singular \( M = \infty \).

- In case 3 of Theorem 1 with \( \nu = 0 \), the rational function with numerator denominators degrees in \( E \) less than \( n/2, (n-1)/2 \) defined by the series

\[
M(z, E) = -\frac{\partial}{\partial z} \ln \left( \sum_{i=0}^{n-1} D_i F(z) E^{n-i} \right) + O(E^n)
\]

with \( D = -\partial_z^2 - 1/(4z^2), F(z) = P_1(z^2) + \ln z P_2(z^2) \) and \( \deg P_1 = n-1, \deg P_2 \leq n/2 - 1 \).

- In case 3 of Theorem 1 with \( \nu = 1/2 \), the rational function with numerator denominators degrees in \( E \) less than \( n/2, (n-1)/2 \) defined by the series

\[
M(z, E) = -\frac{\partial}{\partial z} \ln \left( \sum_{i=0}^{n-1} (-1)^i \partial_z^{2i} F(z) E^{n-i} \right) + O(E^n)
\]

with \( F \) polynomial, \( \deg F = 2n - 1 \) or \( 2n - 2 \).
• In case 4 of Theorem 1 the singular $M = \infty$.

Remarks
The case $n = 0$ (no interpolation points or series) will conventionally give $M = \infty$ (constant infinite function) and this convention allows to recover the potentials $z^2 + \alpha/z^2, 1/z + \alpha/z^2, z, \alpha/z^2$ which are singular cases in our classification.

The interpolation points could be not distinct: for specific values of $\nu$, two interpolations points given by different $k, \epsilon_1, \epsilon_2$ can be equal. The rational interpolation is then given by a limit process when $\nu$ tends to the specific value.

The $M$ function used to express a quantum integrable potential is not unique. This is due to recurrence relations between Whittaker functions. This induces a homographic transformation on $M$, and so infinitely many $M$ can give the same potential.

2. Quantum integrability of 1D rational potentials

As said before, a quantum problem is given by a potential $V \in \mathbb{C}(z)$ and some additional conditions on the solutions we are searching. We want an integrability definition that is as generic as possible, i.e. not depending on these boundary conditions but only to the potential $V$. Still some boundary conditions seem more natural than others. For example, asking that a solution should vanish on some fixed point seem too arbitrary to be acceptable. Indeed, this condition has the physical sense of an infinite wall at an arbitrary point, and so adding such boundary condition corresponds to the transformation $V(z) \to V(z) + \delta_a(z)$ where $\delta_a$ is a Dirac at $a \in \mathbb{C}$. This can be understood as a modification of the potential (adding a singularity to $V$) more than just a boundary condition for the quantum problem. So we need to restrict ourselves to “admissible” boundary conditions.

2.1. Natural boundary conditions

Definition 3 (singularities). Let us consider a linear differential equation

$$a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t) = 0$$

with $a_i$ polynomials, $a_n \neq 0$ and relatively prime. The roots of $a_n$ are called singularities. If $\alpha$ is not a root of $a_n$, $\alpha$ is called a regular point. At a
singularity $\alpha$, if the system admits a converging Puiseux series (possibly with logs) basis of solutions, the point $\alpha$ is called singular regular, else $\alpha$ is called singular irregular. If moreover these Puiseux series are Laurent series, we call $\alpha$ a meromorphic singularity, and if polynomial series, apparent singularity.

Near a meromorphic singularity, the solutions of the differential equation are univalued. In the even more special case of an apparent singularity, the point $\alpha$ is not a singularity for any solution of the differential equation (so is the origin of “apparent”).

**Definition 4 (monodromy).** Let us consider a linear differential equation

$$a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t) = 0$$

with $a_i$ polynomials, $a_n \neq 0$ and relatively prime. Let $\alpha \in \mathbb{C}$ be a regular point, and $B$ a series basis solution at $\alpha$. Let $\gamma \subset \mathbb{C}$ be a closed oriented curve not containing singular points, with $\alpha \in \gamma$. By analytic continuation, we can extend the basis of solution $B$ at $\alpha$ along $\gamma$. After one loop, we obtain a solution basis $B'$. As $B, B'$ are both solutions basis at $\alpha$, there exists a matrix $M$, such that $B' = BM$, called the *monodromy matrix along* $\gamma$.

Remark that if the monodromy around a point is trivial, then it is either a regular point or at worst a meromorphic singularity. Indeed, the monodromy around a point encode the local multivaluation of the solutions of the differential equation.

**Definition 5 (Stokes).** Let us consider a linear differential equation

$$a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t) = 0$$

with $a_i$ polynomials, $a_n \neq 0$ and relatively prime. Let $\alpha \in \mathbb{C}$ be a singular irregular point. We can construct a basis of solutions at $\alpha$ with formal power series of the form

$$e^{\sum_{i=1}^{\infty} c_i(z-\alpha)^{-i/p}} (z-\alpha)^\gamma \ln(z-\alpha)^k \sum_{i=0}^{\infty} b_i (z-\alpha)^i$$

Such formal series solution can be identified with a Gevrey function solution of the differential equation following a particular direction towards $\alpha$ except
for finitely many directions, called singular directions, and the directions between them called sectors. For each sector, the formal basis is identified to a Gevrey function basis, and going from one sector to the next defines a basis change, the Stokes matrix. The monodromy matrix generated by the truncated formal series solution along a small loop around $\alpha$ is called the \textit{formal monodromy}. The monodromy matrix defined as in definition \[\text{above}\] along a small loop around $\alpha$ is called the \textit{true monodromy}.

Remark that along a small loop around a singular regular point, it is easy to compute the monodromy matrix. The difficulty appear when $\gamma$ encompass several singularities. The path $\gamma$ can be deformed, but still we need to know how a Puiseux series solutions at one singularity reconnects with another at the other singularity. Let us now define the notion of natural boundary conditions.

\textbf{Definition 6 (Natural boundary conditions).} Let us consider equation (1). A natural boundary condition on solutions of (1) can be written under the form

$$\prod_{i=1}^{m} M_i^{w_i} \in J$$

where $M_i$ are Stokes or monodromy matrices, $w_i \in \mathbb{Z}$ and $J$ a set of conjugacy class of matrices.

The product encodes a path with integer turns around singularities and integer many crossing of singular directions. The fact that the condition has to be about a conjugacy class of matrices and not equal to a particular matrix is because of the arbitrary initial basis choice. Indeed, we have fixed a common point and basis arbitrary, and so if we want to get rid of this arbitrary choice, we need to consider that the matrices $M_i$ are defined up a \textbf{common} basis change:

$$(M_1, \ldots, M_m) \rightarrow (P^{-1}M_1P, \ldots, P^{-1}M_mP) \quad P \in \text{GL}_2(\mathbb{C})$$

\textbf{Proposition 1.} The spectrum for equation (1) with natural boundary conditions is the set of roots of

- a holomorphic function $f(E)$ if $\lim_{z=\infty} V(z) = \infty$
- a holomorphic function $f(\ln E), E \in \mathbb{C}^*$ if $\lim_{z=\infty} V(z) = 0$
Remark that if $V$ converges at infinity, we can always assume it converges to 0 as we can always make an energy shift for $E$.

**Proof.** Equation (1) comes with a parameter $E$, which plays a fundamental role. The monodromy and Stokes matrices depend a priori on this parameter. Let us first remark that the singularities of (1) do not move with respect to $E$. The same applies for singular directions, except possibly at infinity for $E = 0$ for which singular direction crossing is possible: indeed, the asymptotic behaviour of solutions change when $E = 0$ if $V$ tends to zero at infinity.

Thus when $\lim_{z \to \infty} V(z) = \infty$, the monodromy and stokes can be globally defined on $\mathbb{C}$ as functions of $E$. And as our equation depends analytically on $E$, all these matrices are holomorphic functions of $E$. The natural boundary conditions are put on this matrices, and so this gives the first case of the corollary.

When $\lim_{z \to \infty} V(z) = 0$, the monodromy and stokes are defined on $\mathbb{C}^*$. However, $\mathbb{C}^*$ is not simply connected, and thus this does not imply that these matrices are globally defined on $\mathbb{C}^*$. We need to consider the universal covering of $\mathbb{C}^*$. Our equation depends analytically on $E$, all these matrices are locally holomorphic functions of $E$. Locally holomorphic functions on the universal covering of $\mathbb{C}^*$ are holomorphic function in $\ln E$, and thus the corollary follows.

Let us now remark that the matrices $M_i$ of Definition 6 depend on $E$, and they are not well defined for $E = 0$ when $\lim_{z \to \infty} V(z) = 0$. Thus in this case the problem to know whether $E = 0$ belongs to the spectrum is not defined through this presentation of natural boundary conditions. This can be explicitly seen on the example $V(z) = 1/z$, for which the solutions of the Schrödinger equation are

$$\psi(z, E) = \mathcal{W}\left(-\frac{i}{2\sqrt{E}}, \frac{1}{2}, 2i\sqrt{E}z\right)$$

The case $E = 0$ is a singularity of this equation, as singular directions at the irregular point $\infty$ are crossing. The Whittaker function simplifies in the Bessel function. So from now on, this problem will be skipped completely by assuming that $E \in \mathbb{C}^*$ when $V(z)$ converges (and then assuming it converges to 0).
**Definition 7.** Let us consider Schrödinger equation (1) with $\lim_{z=\infty} V(z) = 0$ if $V$ converges. We say that $V$ is of continuous type if, up to common basis change, the monodromy and Stokes matrices do not depend on $E \in \mathbb{C}^*$. Else the equation is said of discrete type.

Said otherwise, in the continuous case, the transformation $V(z) \rightarrow V(z) + \epsilon$ is an isomonodromic iso-Stokes deformation. Such kind of deformations are very rare, and have been analysed by Painlevé, leading to the so-called Painlevé equations [8].

**2.2. The square integrability condition**

The most classical boundary condition is square integrability of one solution

$$\int_{\mathbb{R}} |\psi|^2 \, dz < \infty$$

Although this seems to be a global condition (and so the word boundary would be inappropriate), the solutions $\psi$ are always regular outside the singularities of the differential equation. So the condition of square integrability comes down to analysing the behaviour at singularities. For our definition of natural boundary condition to be reasonable, it should include this square integrability condition. This will not always be exactly the case, so let us define a little larger notion

**Definition 8.** Let us consider equation (1) with $\lim_{z=\infty} V(z) = 0$ if convergent, and some boundary conditions. Let us note $\mathcal{S}$ the set of $E \in \mathbb{C}^*$ satisfying these boundary conditions. We say that these boundary conditions are almost natural if there exist natural boundary conditions, defining a set $\mathcal{C}$, and such that

$$\mathcal{S} \subset \mathcal{C} \quad \dim \mathcal{S} = \dim \mathcal{C}$$

In this definition, an almost natural condition is “close” to a natural boundary condition in the sense that if $\mathcal{S}$ is discrete infinite (the case with physical sense), then one can find a set $\mathcal{C}$ containing it which is also discrete infinite. With the set $\mathcal{C}$, the “structure” of the spectrum $\mathcal{S}$ is known, we just have to remove some “errors”.

**Definition 9.** Let us consider $\alpha$ a real singularity of the Schrödinger equation (1). We say that the singularity $\alpha$ is active if the space of formal series solutions at $\alpha$ contains exactly a subspace of dimension 1 of square integrable near $\alpha$ formal series.
Proposition 2. We consider the Schrödinger equation (1) with $\lim_{z \to \infty} V(z) = 0$ if $V$ converges. Let us consider a fixed based point and a series solution basis at this point. For all $\alpha_i \in \mathbb{R}$ active singularities of equation (1), consider the Stokes matrices going from direction $\mathbb{R}^-$ to $\mathbb{R}^+$ and the true monodromy matrices, and denote $G$ the multiplicative group generated. Let us note

$$ S = \{ E \in \mathbb{C}^*, \exists \psi \text{ solution of (1) with } \int_{\mathbb{R}} |\psi|^2 \, dz < \infty \} $$

$$ C_1 = \{ E \in \mathbb{C}^*, G \text{ cotriangularizable} \} $$

$$ C_2 = \{ E \in \mathbb{C}^*, G \text{ codiagonalizable} \} $$

$$ C_3 = \{ E \in \mathbb{C}^*, G \subset I_2 \} $$

Assume $S \neq \mathbb{C}^*, \emptyset$. Then there is one $C_i$ discrete countable such that $S \subset C_i$.

Remark that the $C_i$ come from natural boundary conditions. But we have not a priori $S = C_i$ for some $i$. This is however the case for typical quantum integrable physical systems as the potentials $V(z) = z^2, 1/z$. If $S$ is infinite countable (which is typically the interesting case), then the inclusion $S \subset C_i$ is strong, and so we can say that the square integrable condition is almost equal to a natural boundary condition as one of the $C_i$ contains $S$ and is of same dimension.

Corollary 1. The square integrability condition is almost natural.

Proof. Let us consider an $E \in S$ and $\alpha \in \mathbb{R}$ be a singularity. In the general irregular case with $\alpha \in \mathbb{R}$, we have such kind of behaviour

$$ \sum_{i=1}^{m} c_i (z-\alpha)^{-i/p} (z - \alpha)^{\gamma} \ln(z - \alpha)^k, \quad p \in \mathbb{N}^*, k \in \{0, 1\}, \; c_i, \gamma \in \mathbb{C} $$

A basis of such formal solutions lives in a differential field extension over the field of Laurent series, and thus we can attach to it a differential Galois group. Remark that the Schrödinger equation is unimodular, the Wronskian is constant, and so this group is in $SL_2(\mathbb{C})$. As it is diagonal, it is generated by one matrix, we will note in this proof $M_\alpha$. We can moreover assume that $M_\alpha$ is in $G$, as $M_\alpha$ belong to the local differential Galois group at $\alpha$ (i.e. the Galois group over the base field of meromorphic functions on an open neighbourhood of $\alpha$).
Depending on the parameters, these formal series can be square integrable or not. Let us denote $E_{int}$ the subspace of formal series square integrable. The dimension of this space can be 1, 2. As there is a square integrable solution, there is a non zero element of $E_{int}$ which is sent to $E_{int}$ be the Stokes matrix $S$ going from direction $\mathbb{R}^-$ to $\mathbb{R}^+$. And the same for the true monodromy. Let us remark that this is automatically satisfied if $E_{int}$ is of dimension 2, so we can restrict ourselves to the case of dimension 1, for which $\alpha$ is called an active singularity. This has to be satisfied simultaneously for all real singularities, and so all true monodromy matrices and Stokes matrices from $\mathbb{R}^-$ to $\mathbb{R}^+$ at active singularities have to stabilize a common vector space. In other words, the group $G$ has to be cotriangularizable. Thus $S \subset C_1$.

If $C_1$ is discrete countable, then the proposition is proved. So we can now assume that $C_1 = \mathbb{C}^*$. In other words, the group $G$ is triangular for all $E$. But Proposition 2 has the hypothesis $S \neq \mathbb{C}^*$ and so the vector space $E_{int}$ is not always stabilized by $G$. Let us now remark that the matrix $M_\alpha$ always stabilizes the vector space $E_{int}$. And this matrix at an active singularity is in the group $G$.

Let us first assume that there exist an active singularity such that $M_\alpha$ is not identity. Now two cases:

- either $M_\alpha$ is diagonalizable with distinct eigenvalues. Then after to basis change, we can assume that $M_\alpha$ is diagonal and the group $G$ is triangular (recall that $M_\alpha \in G$). As $E_{int}$ is of dimension 1, is not generated by the vector $(1, 0)$ (else it would be stabilized by $G$ for all $E$) and is stabilized by $M$, then

$$E_{int} = \mathbb{C}.(0, 1).$$

If $E \in S$, we have that this vector space is stabilized by $G$. Thus $G$ stabilize two supplementary 1-dimensional vector spaces, and so is diagonal. This gives $S \subset C_2$.

- either $M_\alpha$ is not diagonalizable. So both eigenvalues of $M_\alpha$ are equal to 1, and $M_\alpha$ is triangular (after basis change). However, $M_\alpha$ stabilize $E_{int}$, which can only be $\mathbb{C}.(1, 0)$. But then $E_{int}$ is stabilized by $G$ for any $E$, and this would implies $S = \mathbb{C}^*$. Impossible.

The last remaining case is when all matrices $M_\alpha$ at active singularities are identity. Then the formal series solutions cannot have nor exponentials,
nor fractional/irrational powers. So all active singularities are meromorphic singularities (formal series solutions are Laurent series), and thus the group \( G \) is reduced to identity, i.e. \( C_2 = \mathbb{C}^* \).

If \( C_2 \) is discrete countable, then the proposition is proved. So we can now assume that \( C_2 = \mathbb{C}^* \). In other words, the group \( G \) is diagonal for all \( E \). But Proposition 2 has the hypothesis \( S \neq \mathbb{C}^* \) and so the vector space \( E_{int} \) is not always stabilized by \( G \). The matrices \( M_\alpha \) at active singularities are also diagonal, and stabilize the vector space \( E_{int} \). However this cannot be \( \mathbb{C}.(1,0), \mathbb{C}.(0,1) \), as else it would be stabilized by \( G \) for all \( E \). So these matrices must have a third stable vector space. And so they are identity. This implies that the group \( G \) is reduced to identity. So any vector space is stabilized by \( G \), and thus in particular \( E_{int} \). Thus \( S \subset C_3 \). Finally, if \( C_3 = \mathbb{C}^* \), the group \( G \) is identity for all \( E \), and thus \( S \) is either \( \emptyset \) or \( \mathbb{C}^* \).

\[ \square \]

For the computation of the spectrum of quantum integrable system under the square integrability condition, we will first compute the sets \( C_i \) which can be found algebraically from the monodromy/Stokes matrices. Then we obtain a countable discrete set of “candidates” and we can look more precisely the behaviour of solutions at singularities to check which energies satisfy the square integrability condition.

**Other examples of almost natural boundary condition**

- Prescribed singular behaviour at one singularity
- Prescribed radius of convergence for series solutions
- Analyticity of solutions of a particular domain
- To belong to the Bargman space (holomorphic functions with \( |f(z)|^2 \exp(-z^2/2) \) integrable)
- Prescribed ramification/coverings of the Riemann sphere

**2.3. Rigid functions**

Our objective now is to compute the spectrum, and more precisely defined a class of Schroedinger equation (1) for which the monodromy and Stokes matrices can be explicitly computed. At first view, this seems to be
intractable, as analytic continuation of formal series is used everywhere to define these matrices. However, the idea of rigid operators \([7]\) is to gather all algebraic information we have at our disposal, and try to compute these matrices.

The main information we have is local monodromy in the singular regular case (so a small loop around a singularity) and the formal monodromy matrix for irregular singular points. Making a whole turn around this irregular singular point gives moreover a multiplicative relation between formal monodromy (known), true monodromy (unknown), and Stokes matrices (unknown). Finally we have a global structure: if we have \(n\) singularities, making a turn around \(n-1\) of them equals to making a turn around the one left (recall that we are on the Riemann sphere). However, these matrices are not known in a common basis.

2.3.1. Regular case

Let us first focus on the regular singular case, i.e. no irregular singularities at all. To summarize, we search \(M_1,\ldots,M_m \in GL_n(\mathbb{C})\) such that

\[M_1 \cdots M_{m-1} = M^{-1}_m\]

just knowing the \(M_i\) up to conjugacy. Of course, one just has to find the \(M_1,\ldots,M_m\) up to common basis change. Is this enough to find the \(M_i\)? Sometimes \([9]\).

**Definition 10.** Let us consider a linear differential operator

\[y(t) \longrightarrow a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t)\]

with \(a_i\) polynomials, \(a_n \neq 0\) relatively prime, and with only regular singularities. The operator is said to be rigid if the monodromy matrices are uniquely defined up to common basis change by their conjugacy class given by local monodromy.

Search for rigid operators is still on going, and is known as the Deligne-Simpson problem \([10]\):

**Problem 1.** Let \(n, m\) be two positive integers. Find all \(m\)-uplet \((G_1,\ldots,G_m)\) of conjugacy classes of \(GL_n(\mathbb{C})\) such that the equation

\[M_1 \cdots M_m = I_n \quad M_i \in G_i, \quad i = 1\ldots m\]

admits a unique solution up to common conjugacy.
For small dimensions, the problem is solved, and in particular for $n = 2$, the only possible solution leads to a famous operator, Gauss hypergeometric differential equation.

2.3.2. Irregular case

The definition of rigid operators in the irregular case can be done through a limiting process, the confluence. Using the parameters in a family of regular rigid operator, we make two singularities fuse with simultaneous rescaling. This leads to an equation with one less singularity, but irregular. Moreover, we have

- The limit direction when the fusion occurs becomes a singular direction
- The Stokes matrices are the limits of the monodromy matrices around each singularity of the fusion
- Multiple singularities can fuse simultaneously, leading to several singular direction and Stokes matrices

So the irregular generalisation of rigid operators is straightforward

**Definition 11.** Let us consider a linear differential operator

$$y(t) \rightarrow a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t)$$

with $a_i$ polynomials, $a_n \neq 0$ relatively prime. The operator is said to be rigid if either it is regular and rigid according to Definition 10, or is produced by a limit confluence process of a family of regular and rigid operators according to Definition 10.

As the order is conserved by the limiting process, we only have to look at confluence processes for the Gauss hypergeometric equation. Outside elementary functions, this produces the Whittaker differential equation (and Bessel differential equation as a specialization).

2.3.3. Galois rigidity

There an additional global algebraic structure we have not used yet to compute our monodromy/Stokes matrices, which is the differential Galois group. In particular, due to Ramis theorem, we know that monodromy/Stokes matrices belong to the differential Galois group. This group can be computed algebraically and automatically thanks to the Kovacic algorithm. It is an algebraic Lie subgroup of $GL_n(\mathbb{C})$. So this suggests the following generalization of the Deligne Simpson problem (already raised in [10]).
Problem 2. Let $n, m$ be two positive integers and $G$ an algebraic Lie subgroup of $GL_n(\mathbb{C})$. Find all $m$-uptet $(G_1, \ldots, G_m)$ of conjugacy classes of $G$ such that the equation

$$M_1 \cdots M_m = I_n \quad M_i \in G_i, \; i = 1 \ldots m$$

admits finitely many solutions in $G$ up to common conjugacy.

Definition 12 (Galois rigid operators). Let us consider a linear differential operator

$$y(t) \rightarrow a_n(t)y^{(n)}(t) + \cdots + a_0(t)y(t)$$

with $a_i$ polynomials, $a_n \neq 0$ relatively prime. The operator is said to be rigid if it is

- either regular and if the monodromy matrices are defined up to a finite choice up to common basis change by their conjugacy class given by local monodromy and their inclusion to the differential Galois group.

- a limit confluence process of a family of regular and rigid operators of the above case.

Theorem 3. The Galois rigid operators of order 2 without meromorphic singularities are, up to hyperexponential multiplication and Moebius transformation

- The hypergeometric equation.

- The Whittaker equation.

- The logarithm equation $zy'' + y'$.

- Any operator with dihedral Galois group over $\mathbb{C}(z)$ and diagonal Galois group over $\mathbb{C}(\sqrt{z})$.

- Any operator with diagonal Galois group or finite Galois group.

Proof. Let us treat each possible differential Galois group.

If $\text{Gal} = GL_2(\mathbb{C})$ or $SL_2(\mathbb{C})$, this has already been done in [9, 10] for regular operators. The only possible case is the Gauss hypergeometric equation. Its confluence gives the Whittaker equation.
A triangular group. We are first looking for Fuchsian equations. For each matrix, we know its conjugacy class, and so in particular its eigenvalues. By multiplying the solutions by a hyperexponential function, we can fix one eigenvalue to 1 for each matrix $M_i$, and we choose the upper left one. As all the matrices should be upper triangular, only one coefficient is still unknown $\beta_i$

$$M_i = \begin{pmatrix} 1 & \beta_i \\ 0 & \lambda_i \end{pmatrix}$$

If the two eigenvalues are equal, then $\lambda_i = 1$ and we moreover know if $\beta_i$ is 0 or not. The diagonalizable case with double eigenvalue case leads only to the identity matrix, and so a meromorphic singularity, which is forbidden. The only information known on the $M_i$ is the multiplicative relation $M_1 \ldots M_m = I_2$. So the relation becomes

$$\prod_{i=1}^m M_i = I_2$$

The upper right coefficient of this product is

$$\sum_{i=1}^m \left( \prod_{j=i+1}^m \lambda_j \right) \beta_i$$

The other coefficients of the product do not give us additional information. This upper right coefficient is a linear form in the $\beta_i$, the unknowns. As we have removed the case for which $\beta_i$ is known to be zero, this linear form is the only relation we have on the $\beta_i$. Remark now that we are searching the matrices $M_i$ up to common basis change. Here we have to keep the triangular form, so we can make a triangular basis change. Such basis change multiply by some constant the upper right coefficient of the $M_i$. So the rigidity problem comes down to find when equation (4) has finitely many solutions up to transformation

$$(\beta_i)_{i=1 \ldots m} \rightarrow (\alpha \beta_i)_{i=1 \ldots m} \quad \alpha \in \mathbb{C}^*$$

So this encodes a projective plane of dimension $m - 2$. This has finitely many points if and only if $m = 2$.

Thus we have at most 2 singularities. If there is only one, then it should be meromorphic, as a multivalued function has at least two ramification points on the Riemann sphere. So it has exactly 2 singularities, and using Moebius
transformation, we can fix one of them at 0, and the other one at infinity.
As the differential equation is Fuchsian, the differential Galois group is the
Zariski closure of the monodromy matrices group. So in particular this group
has a common eigenvector of eigenvalue 1. And thus a rational solution. So
up to multiplication by a rational function, this solution can be sent to 1,
and we obtain that our equation writes down
\[
\frac{d^2 y}{dz^2} - a(z) \frac{dy}{dz} = 0
\]
with \(a \in \mathbb{C}(z)\). The rational function can only have one pole at zero, and 0
should be singular regular. So \(a(z) = cz^{-1}\). The solutions of this equation
can be written
\[
y(z) = \int z^c dz
\]
If \(c \neq -1\), then we obtain a hyperexponential solution, and thus the Galois
group is diagonal, which is included in another case. The case \(c = -1\)
gives the differential equation of the theorem. Remark to conclude that this
equation cannot have an irregular confluence.

The dihedral case. The Galois group is not connected. If the proje ctive
Galois group is not finite (next case of the theorem), then there ar e exactly
two components. These can be written
\[
\begin{pmatrix}
\lambda & 0 \\
0 & \mu
\end{pmatrix}
\begin{pmatrix}
0 & \lambda \\
\mu & 0
\end{pmatrix}
\]
In the first case, knowing the eigenvalues allows to determine the matrix.
In the second case, this only gives us the product \(\lambda \mu\). By multiplying the
solutions by a hyperexponential, we can assume the Galois group being uni-
modular. And so that the determinant of these matrices should be 1. This
implies that \(\lambda \mu = -1\) in the second case. Then for each monodromy ma-
trix in the second component, we have one unknown. In the other hand,
monodromy matrices in the identity component are fully known.

The multiplicative relation can be written
\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_1^{-1}
\end{pmatrix}
\begin{pmatrix}
0 & \mu_1 \\
-\mu_1^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_2 & 0 \\
0 & \lambda_2^{-1}
\end{pmatrix}
\cdots
\begin{pmatrix}
\lambda_p & 0 \\
0 & \lambda_p^{-1}
\end{pmatrix}
\begin{pmatrix}
0 & \mu_p \\
-\mu_p^{-1} & 0
\end{pmatrix} = I_2
\]
We simplified directly the relation by multiplying successive diagonal mat ri-
ces. In the equation, the \(\lambda\)'s are known, the \(\mu\)'s are unknown. Remark that
we need an even number of non diagonal matrices, i.e. \( p \) even. Writing down
the product, we obtain just one equation from identification

\[
\prod_{i=1, i \text{ odd}}^{p} \lambda_i \mu_i = \prod_{i=1, i \text{ even}}^{p} \lambda_i \mu_i
\]

As the monodromy matrices are searched up to common basis change, using
a diagonal basis change (as we need to keep the diagonal structure of
the identity component), we can multiply the \( \mu_i \)'s by an arbitrary complex
number. Still, even up to a multiplication of all the \( \mu_i \)'s, this equation has
finitely many solutions only if \( p = 2 \).

We also know that the differential Galois group is dihedral, and so the
solutions can be written under the form

\[
y(z) = A e^{\int f(z) + \sqrt{g(z)} dz} + B e^{\int f(z) - \sqrt{g(z)} dz} \quad f, g \in \mathbb{C}(z)
\]

There are exactly two monodromy matrices outside the identity component
of the Galois group. The corresponding singularities are root-poles of odd
order of \( g \). Using a Moebius transformation, we can put them at 0, \( \infty \), and
thus

\[
y(z) = A e^{\int f(z) + \sqrt[2]{g(z)} dz} + B e^{\int f(z) - \sqrt[2]{g(z)} dz} \quad f, g \in \mathbb{C}(z)
\]

These are exactly the solutions of operators of order 2 with rational coefficients
whose differential Galois group is diagonal over \( \mathbb{C}(\sqrt{z}) \).

For the diagonal Galois group case, the eigenvalues of the monodromy
matrices are known, they are all codiagonalizable, and thus we know all the
monodromy matrices. In the finite projective case, after multiplication by
a hyperexponential function, the Galois group becomes finite, and thus the
monodromy matrices are known up to a finite choice, as they belong to the
Galois group which is finite.

\[\Box\]

2.3.4. The class of rigid functions

We can now define the class of functions we are interested in. These are
built in a same way as Liouvillian functions, except that “basic” functions
are now solutions of Galois rigid operators.
Definition 13 (Rigid functions). The field of rigid functions \( \mathcal{R} \) is the smallest differential field with the following properties

- \( \mathcal{R} \) contains the solutions of all Galois rigid operators

- if \( f \in \mathcal{R} \), then for any algebraic function \( g \), \( f \circ g \in \mathcal{R} \) (algebraic pullbacks)

We moreover define field of rigid functions of order \( n \), \( \mathcal{R}_n \), as the smallest differential field containing the solutions of all Galois rigid operators of order \( \leq n \) and all their algebraic pullbacks.

Examples

A Liouvillian function can be written in finite terms using integrals, exponentials, and algebraic functions. In our rigid function field, integrals are not allowed. For a Liouvillian function to be rigid, we have to somehow compute these integrals more explicitly. However, this does not imply that these integrals should be elementary functions. Indeed, Liouvillian elementary functions are rigid functions, but Liouvillian rigid functions are not always elementary. This kind of construction has already been suggested by Mark van Hoeij in [11], where he suggest \((F, O)\) differential fields. The field of base functions \( F \) is exactly the same, coming from rigid differential operators, but he adds the integrals in the set of allowed operators \( O \), which is excluded here.

The error function

\[
\int e^{z^2} \, dz = z^{-1/2} e^{z^2/2} W\left(\frac{1}{4}, \frac{1}{4}, z^2\right)
\]

is Liouvillian, is not elementary, but still is rigid. This is because this integral admits a representation in terms of the Whittaker function \( W \), which is a rigid function (as a confluence of the Gauss hypergeometric equation).

The elliptic integrals of the first kind

\[
\int \frac{1}{\sqrt{(z-a)(z-b)(z-c)}} \, dz
\]

These can be expressed as Heun function [12]. The cross ratio (see [13]) is not a constant with respect to \( a, b, c \). So if this integral could be expressed using a
hypergeometric function (Whittaker functions are excluded due to the irregular singularity), then it would lead to a parametric algebraic transformation between Heun and hypergeometric function. Such parametric transformations have been classified in [14], and a transformation for general elliptic integral of the first kind is not included. This integral cannot be expressed in terms of elementary functions in general, and thus is not rigid for generic $a, b, c$. Remark that some exceptional values of $a, b, c$ for which this function is rigid are known

\[
\int \frac{1}{\sqrt{z^3 - 1}} \, dz = -\frac{2}{\sqrt{z}} \, _2F_1 \left( \frac{1}{2}, 1/6, 7/6, \frac{1}{z^3} \right)
\]

\[
\int \frac{1}{\sqrt{z^3 - z}} \, dz = -\frac{2}{\sqrt{z}} \, _2F_1 \left( \frac{1}{2}, 1/4, 5/4, \frac{1}{z^2} \right)
\]

\[
\int \frac{1}{\sqrt{z^3 - az^2}} \, dz = \frac{2}{a} \arctan \left( \frac{\sqrt{z - a^2}}{a} \right)
\]

As the monodromy group in these cases can be explicitly computed, this implies that the complex lattice of corresponding elliptic functions can be explicitly computed. And indeed, those have exceptional properties, being triangular, square and collapsed respectively.

In general, it is difficult to prove that some function is not rigid. In the case of a differential equation of order 2 with generic exponents at singularities, we cannot find an algebraic pullback mapping these singularities to only 3 points, because it would imply a rational relation between the exponents (see pullbacks of hypergeometric equation to themselves in [15]). The cases with parameters are thus much easier, and it is mostly done for Heun functions in [14], as exponents typically depend on the parameter. This is exactly our situation as the energy level $E$ is a parameter in our Schroedinger equation, and appears in the asymptotic behaviour of the solutions.

3. Rigid eigenfunctions

We now proceed to the proof of Theorem [1].

**Notation:** In the following of the article, we will use both the notations $f(z, E)$ and $f(z)$, which will be two different functions, and we will always precise the variables in case of ambiguity.
3.1. Asymptotic analysis

The possible asymptotic behaviours of eigenfunctions are of the following form
\[ z^{\gamma} e^{\max(0,n) + 2} \sum_{k=1}^{\max(0,n)+2} a_k z^{k/2} \]
with \( n \) the asymptotic exponent of \( V \) at infinity (if \( V \) converges at infinity, we always assume that it converges to 0). The exponent \( \gamma \) encodes the formal monodromy at infinity. Let us look now at the dependence in function of \( E \).

We inject this expression in equation (1), giving

- if \( n \geq 3 \), then \( \gamma \) and all the \( a_i \) are constant
- if \( n = 2 \), then \( \gamma \) is affine in \( E \) and all the \( a_i \) are constant
- if \( n = 1 \), then \( \gamma = -1/4 \), \( a_3 \) constant, \( a_2 = 0 \) and \( a_1 \) linear in \( E \).
- if \( n = -1 \), then \( a_2 = \sqrt{-E} \), \( \gamma \sqrt{-E} \) constant.
- if \( n \leq -2 \), then \( a_2 = \sqrt{-E} \), \( \gamma = 0 \).

To find rigid eigenfunctions, we need to search two type of rigid solutions: the ones coming from hypergeometric or Whittaker functions, and the Liouvillian ones. Let us first remark that equation (1) has an irregular singularity at infinity. If a solution is rigid and non Liouvillian, then its expression has to involve a non solvable hypergeometric or Whittaker function, which are solutions of equations of order 2. Thus they should have an expression of the form

\[ \psi(z, E) = h(z, E)(M(z, E)F(f(z, E)) + F'(f(z, E))) \]

with \( h \) hyperexponential, \( f, M \) algebraic and \( F \) hypergeometric or Whittaker, with parameters possibly depending on \( E \). The quotient of the two possible asymptotic behaviours of solutions of equation (1) has always an essential singularity at infinity. If the function \( F \) was hypergeometric, then the quotient of two solutions of the form (3) would have a Puiseux/log series at infinity. Thus the function \( F \) should be of Whittaker type.

So non Liouvillian eigenfunctions should be of the form
\[ e^{\int g(z, E)dz} (M(z, E)W(\mu(E), \nu(E), f(z, E)) + W'(\mu(E), \nu(E), f(z, E))) \]
where \( g, M, f \) are algebraic in \( z, E \) (\( E \) being the parameter). Computing the second or differential equation whose solution is this function, the \( g \) can be expressed in function of \( f, M \) through the condition that no term in \( \psi' \) appears in this equation. We then obtain for \( V + E \) a large rational expression depending on \( M, f \), their derivatives in \( z \) and \( E, \nu(E), \mu(E) \).

### 3.2. Whittaker pullbacks

A rigid solution related to (a non solvable) Whittaker function of a second order differential equation is of the form

\[
e^{\int g(z)dz} (M(z)W(\mu, \nu, f(z)) + W'(\mu, \nu, f(z)))
\]

with \( f, M, g \) algebraic. Still if we restrict ourselves to differential equation with rational coefficients, the \( f, M, g \) cannot be arbitrary algebraic functions (see [10] for Bessel functions).

**Proposition 3.** If the function

\[
e^{\int g(z)dz} (M(z)W(\mu, \nu, f(z)) + W'(\mu, \nu, f(z)))
\]

with \( f, M, g \) algebraic is solution of a second order unimodular differential equation with \( W \) non solvable, then \( f \) is either rational, or the square root of a rational function with \( \mu = 0 \).

**Proof.** We are searching a pullback transformation and gauge transformation which sends an unimodular differential equation with rational coefficients (the Whittaker equation) to an unimodular differential equation with rational coefficients. Both these transformations are algebraic, as the function \( \exp(\int g(z)dz) \) can be expressed algebraically in \( M, f, z \) and their derivatives. So the functions we are looking are of the form

\[
M_1(z)W(\mu, \nu, f(z)) + M_2(z)W'(\mu, \nu, f(z))
\]

with \( M_1, M_2, f \) algebraic. Thus in particular the pullback function \( f \) cannot be an arbitrary algebraic function as this function has to satisfy a rational linear differential equation.

We now consider the function \( \sigma \) which sends one value of \( f \) to the others (recall \( f \) is algebraic and so multivalued). This function is the Galois action on the branches of \( f \). And thus the action of \( \sigma \) on (6) produces another
solution of the differential equation. As we assumed the differential Galois group of the Whittaker function being $SL_2(\mathbb{C})$, this implies a relation of the form

$$W_1(\mu, \nu, \sigma(z)) = S_1(z)W_1(\mu, \nu, z) + S_2(z)W_2(\mu, \nu, z)$$

with $S_1, S_2$ algebraic, $W_1, W_2$ a basis of solutions of the Whittaker equation. Let us make some precisions about the function $\sigma$. The differential equation has rational coefficients, the singularities of (6) should not depend on which branch of $f$ we choose. Thus if $f(z) = 0, \infty$ for some branch, this should be the same for the other branches (and 0, $\infty$ cannot be exchanged as they lead to a different type of singularity, one regular, the other irregular). This implies in particular that $\sigma$ is univalued at 0, $\infty$, and their value are 0, $\infty$. Let us now act the Galois group of the Whittaker equation on the above relation. We consider a path in $\mathbb{C}^*$ and a corresponding monodromy/Stokes matrix $A$, assumed to be diagonal (possible as the Galois group is $SL_2(\mathbb{C})$) with eigenvalues $\alpha, 1/\alpha, \alpha$ not root of unity, acting on the basis $W_1, W_2$. As $\sigma$ is algebraic, the path is not always closed on its associated Riemann surface. Still, if we take a suitable power of $A$, noted $B$, this will correspond to a closed path of the Riemann surface associated to $\sigma$ ($\sigma$ being algebraic, its monodromy group is finite). When applying $B$ infinitely many times, this gives after taking a limit

$$W_1(\mu, \nu, \sigma(z)) = S_1(z)W_1(\mu, \nu, z) \text{ or }$$

$$W_1(\mu, \nu, \sigma(z)) = S_2(z)W_2(\mu, \nu, z)$$

Using the unimodular property, we get that $S_{1,2}(z) = c\sqrt{\sigma(z)}$, $c \in \mathbb{C}$.

Let us now consider a ramification point $\alpha$ of $\sigma$, outside 0, $\infty$. We have the relation

$$\frac{W_1(\mu, \nu, \sigma(z))}{\sqrt{\sigma(z)}} = cW_1(\mu, \nu, z)$$

Near $\alpha$, the righthandside is analytic. The function $W_1$ can be chosen arbitrary (solution of the Whittaker equation), and is analytic at $\alpha$. Thus the left hand side (for a generic choice of $W_1$) is not analytic. Thus such ramification point $\alpha$ does not exist.

This implies that $\sigma$ has at most two ramification points 0, $\infty$, and moreover knowing that $\sigma(0) = 0, \sigma(\infty) = \infty$, this implies that

$$\sigma(z) = az^r, \quad a \in \mathbb{C}^*, \quad r \in \mathbb{Q}_+^*$$
Now looking at asymptotics near infinity, we find that the only possible exponent is $r = 1$. Now we need to express $W_1(\mu, \nu, az)$ in function of $W_1, W_2$. The only possibility is $a = 1$ or $a = -1$ with $\mu = 0$. This implies that $f$ is either rational, or the square root of a rational function with $\mu = 0$.

\[ \square \]

We now need to find $M, f$ leading to a function $V$ depending only on $z$ and not on the parameter $E$.

**Definition 14.** We consider a linear differential equation with a parameter $E$. We say that this equation has no mobile singularity if the position of the singularities does not depend on $E$. Similarly, we say that a rational function has no mobile singularity (respectively root) if its poles (respectively root) do not depend on the parameter $E$.

The Schrödinger equation has no mobile singularities. So the solutions \[ (5) \] should have no mobile singularities. In particular, the Whittaker functions have ramification points at 0, $\infty$, and this will restrict the possible pullback transformations $f$.

**Proposition 4.** The pullback function $f(z, E)$ has to be of the form

\[ w(E)f(z) \text{ or } \]

\[(w_1(E)f(z) + w_2(E))^k \text{ with } \nu = \pm \frac{1}{2k}, \; k \in \mathbb{N} \setminus \{0, 1\} \text{ or} \]

\[(w_1(E)f(z) + w_2(E))^k \text{ with } \nu = \pm \frac{1}{2k}, \; \mu = 0, \; k \in \frac{1}{2} \mathbb{N} \setminus \{0, 1/2, 1\} \]

**Proof.** Let us first consider the case $\mu \neq 0$. Then we have $f(z, E)$ rational in $z$ according to Proposition 3. The Schrödinger equation has no mobile singularities. The values $f = \infty$ always lead to singularities of \[ (6) \], and so cannot depend on $E$. The point 0 is a regular singularity of the Whittaker equation with exponents $1/2 + \nu, 1/2 - \nu$. Let us consider a root $\alpha$ of $f$, with multiplicity $k \in \mathbb{N}^*$. The function $W(\mu(E), \nu(E), f(z, E))$ admits a Puiseux series in $z$ near $z = \alpha$, with first term exponent $(1/2 + \nu)k$ or $(1/2 - \nu)k$. Now if $\alpha$ depends on $E$, it cannot be a singularity of the Schrödinger equation (not even an apparent one). Now taking into account the gauge transformation,
the function $\psi$ admits a Puiseux series with first term exponent $(1/2 + \nu)k + \gamma$ or $(1/2 - \nu)k + \gamma$, with $\gamma$ depending on the gauge transformation. If we want $\alpha$ not being a singularity of the Schrödinger equation, we need these exponents to be 0, 1. And thus we need

$$(1/2 + \nu)k - (1/2 - \nu)k = \pm 1$$

and thus $\nu = \pm 1/(2k)$. Remark moreover that $\nu = \pm 1/2$ leads to a logarithmic singularity for the Whittaker function, and thus $\alpha$ would always be a singularity.

Thus if $\nu \notin \{\pm 1/(2k), k \in \mathbb{N} \setminus \{0, 1\}\}$, then all the roots of $f$ do not depend on $E$. And so is of the form $w(E)f(z)$. Let us now assume $\nu = \pm 1/(2k), k \in \mathbb{N} \setminus \{0, 1\}$. The pullback function is of the form $F(z)Q(z, E)^k$, with $F$ rational, $Q$ polynomial with simple roots in $z$. Let us now look at critical points of $f$. If such a critical point $\alpha$ is not on the level $f = 0$, then it will give a singularity. We have

$$f'(z, E) = Q(z, E)^{k-1}(F'(z)Q(z, E) + F(z)Q'(z, E))$$

and so the right factor $F'(z)Q(z, E) + F(z)Q'(z, E)$ cannot have roots depending on $E$ (as else it would lead to a mobile singularity). And so

$$F'(z)Q(z, E) + F(z)Q'(z, E) = w_1(E)S(z)$$

This is a non homogeneous linear differential equation in $Q$, and the solutions are of the form

$$Q(z, E) = w_2(E)F(z)^{-1/k} + w_1(E)P_1(z)$$

Let us remark that we can assume that $w_1, w_2$ are not $\mathbb{C}$-dependant, as this would lead again to a pullback function of the form $w(E)f(z)$. And so both functions $F(z)^{-1/k}, P_1(z)$ have to be polynomials. Let us note $F(z) = 1/P_2(z)^k$, giving

$$f(z, E) = \left(w_2(E) + w_1(E)\frac{P_1(z)}{P_2(z)}\right)^k$$

which gives the second case of the Proposition.

Let us now consider the case $\mu = 0$. Then $f(z, E)$ is a square root of a function rational in $z$. The same arguments as before still apply, except
that the root multiplicity $k$ can be half-integer. So if $\nu \notin \{\pm 1/(2k), k \in 1/2\mathbb{N} \setminus \{0, 1/2, 1\}\}$, then the pullback is of the form $w(E)f(z)$, and else is of the form

$$f(z, E) = \left(w_2(E) + w_1(E)\frac{P_1(z)}{P_2(z)}\right)^k$$

giving the third case of the Proposition.

□

**Proposition 5.** The possible pullback functions $f(z, E)$ for non-Liouvilian eigenfunctions are up to affine transformation

- $z^2$ with $\mu = E/4$, $\nu$ constant.
- $2iz\sqrt{E}$ with $\mu = 1/(2i\sqrt{E})$, $\nu$ constant.
- $2iz\sqrt{E}$ with $\mu = 0$, $\nu$ constant.
- $\frac{4i}{3}(z + E)^{3/2}$ with $\mu = 0, \nu = \pm 1/3$.

**Proof.** Let us note $n$ the asymptotic exponent of $V$ at infinity. Recall that we can always assume that when $V$ converges at infinity, it converges to 0. And thus that $n \in \mathbb{Z}^\ast$. The proof is made by disjunction of cases of possible $n \in \mathbb{Z}^\ast$.

We first remark that if $n \geq 2$, then the pullback function $f$ has to be of the form $w(E)f(z)$ according to the asymptotic expansion, and moreover, $w(E)$ is constant. In this more special case $n \geq 3$, we have moreover that the formal monodromy at infinity is constant. The formal monodromy at infinity of the Whittaker function is encoded by $\mu$, and thus $\mu$ has to be constant. So $f, \mu, \nu$ do not depend on $E$, only $M$ can depend on $E$ (and $g$ as a consequence)

$$\psi(z, E) = e^{\int g(z, E)dz} (M(z, E)W(\mu, \nu, f(z)) + W'(\mu, \nu, f(z)))$$

After computation, we find that $g$ is algebraic in $f, E$ and their derivatives. So we make a series expansion of $\psi$ in $E$ near $E = \infty$. After multiplying by a suitable power of $E$, this produces a limit function $s(z)$, smooth almost everywhere. And thus $\psi''(z, E)/\psi(z, E)$ has a limit when $E$ tends to infinity. Impossible as $\psi''(z, E)/\psi(z, E) = V(z) + E$. 

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Let us now study the case $n = 2$. Recall that $\mu$ is affine (non constant) in $E$, so we can make a parameter change and consider $\mu$ as the parameter instead of $E$. So $\psi''(z, E)/\psi(z, E)$ should be affine in $\mu$. Making a series expansion of this at $\mu = \infty$, we find
\[
\frac{\mu f''(z)^2}{f(z)} + o(\mu)
\]
Thus after possibly affine variable change, we can assume $f(z) = z^2$. This gives by the way a relation between $\mu$ and $E$, $E = 4\mu + C$, and we can assume the constant $C$ is zero by putting it into $V$. To conclude, remark that $\nu$ is the exponent at the singularity 0 of the Whittaker function. In the Schrödinger equation, the exponents do not depend on $E$, and thus so does $\nu$.

For the case $n = 1$, the asymptotic should be of the form
\[
z^{-1/4}e^{a_1 z^{3/2} + a_1(E)z^{1/2}}
\]
with $a_1$ affine in $E$. Looking at the possible pullbacks in Proposition 4, the only possible one is
\[
w_1(E)(f(z) + w_2(E))^{3/2}, \quad \mu = 0, \quad \nu = \pm 1/3
\]
and thus $w_1$ is constant, $w_2$ affine in $E$. After possibly adding a constant to $V$, we can assume $w_2(E) = E$. We find that $g$ is algebraic in $f, E$ and their derivatives and we express $\psi''(z, E)/\psi(z, E)$ in function of $f, M, E$ algebraically. We then make a series expansion at $E = \infty$, giving
\[
-\frac{9}{16}w_1^2 f'(z)^2 E + o(E)
\]
So after possibly affine variable change, we can assume $f(z) = z$ and thus $w_1 = 4i/3$.

For $n = -1$ the pullback function $f$ is of the form $w(E)f(z)$, and combining this with the asymptotic expansion, we have that $f(z, E)$ is of the form $f(z)\sqrt{-E}$.

The asymptotic data also give us that $\gamma\sqrt{-E}$ is constant, where $\gamma$ encodes the formal monodromy exponent at infinity of $\psi(z, E)$. The $\mu$ parameter in
\( \mathcal{W} \) encodes the formal monodromy of \( \mathcal{W} \) at infinity, and thus we obtain that \( \gamma = \mu \) up to a (integer) constant. Thus we have a relation of the form (knowing that \( \gamma \neq 0 \))

\[
E = \frac{\alpha}{\mu^2} + \beta, \quad \alpha \neq 0
\]

Remark that we can assume \( \beta = 0 \) as a constant can be put in the potential \( V \). We can also assume \( \alpha = -1/4 \) by making a dilatation of the coordinate system (which multiplies \( E \) by a constant), giving \( \mu = 1/(2i\sqrt{E}) \). The parameter \( \nu \) also cannot depend on \( E \) as the exponents of the Schrödinger equations do not depend on \( E \). We obtain a large expression for the potential \( V \) depending on \( M, f, E \), and we make a series expansion at \( E = \infty \), giving

\[
-\frac{1}{4} f'(z)^2 E + O(1)
\]

Thus we have that \( f(z) = 2iz \) (up to affine variable change), and thus the pullback function is \( 2iz\sqrt{E} \).

For \( n \leq -2 \), according to asymptotics, we need to have \( \mu = 0 \). Using Proposition 4, the possible pullbacks are of the form \( w(E)f(z) \) or

\[
w_1(E)(f(z) + w_2(E))^k, \quad k \in \frac{1}{2}\mathbb{Z} \setminus \{0, 1/2, 1\}, \quad \nu = \frac{1}{4k}
\]

Now using the asymptotics in \( z \), we have

\[
w_1(E)(f(z) + w_2(E))^k = z\sqrt{-E} + O(1)
\]

For \( k \geq 3/2 \), such series expansion is impossible, and so the only possible pullbacks are of the form \( w(E)f(z) \).

With the asymptotics, we obtain moreover \( w(E) = \sqrt{-E} \). Computing the corresponding potential and making a series expansion at \( E = \infty \), we obtain

\[
-\frac{1}{4} f'(z)^2 E + o(E)
\]

And thus up to affine variable change the pullback is of the form \( 2iz\sqrt{E} \).

\[\square\]
3.3. Liouvillian pullbacks

Let us now look at Liouvillian eigenfunctions. The finite projective case and the log case of Theorem 3 are impossible due to the asymptotic behaviour at infinity. For the diagonal case, we need to search all Schroedinger equation with diagonal Galois group (for all $E$). This implies that there exists a hyperexponential solution for all $E$.

**Lemma 4.** If the Schroedinger equation (1) has one hyperexponential solution

$$
\psi(z, E) = e^{\int F(z, E) dz}
$$

then the space of solutions of equation (1) is of the form

$$
e^{\int g(z, \sqrt{-E}) dz} \left( A M(z, E) \sqrt{-E} \frac{ch(z \sqrt{-E})}{1} + B sh(z \sqrt{-E}) \right) \quad A, B \in \mathbb{C} \quad (7)
$$

with $g, M$ rational in both variables.

**Proof.** We first write $F$ under partial fraction decomposition. After integration, we obtain a logarithmic part and a rational part. We know there are no mobile singularities, that the residues are constant, and that in the finite irregular singularity case the exponential part does not depend on $E$, we deduce that

$$
e^{\int F(z, E) dz} = \prod_{i=1}^{p} (z - z_i(E))^{\alpha_i} e^{P(z, E) + H(z)}
$$

with $P$ polynomial in $z, E$ and $H$ rational in $z$ of negative degree. We now look at possible asymptotic expansions at $z = \infty$. Let us look at the monodromy at infinity. Recall that for $n = 2, -1$, the true monodromy should depend on $E$. However, all the $\alpha_i$ are constant in $E$. The Stokes phenomenon here is trivial, and thus the true monodromy at infinity is the sum of the $\alpha_i$. So it cannot depend on $E$. For $n = 1$, non rational terms are required, and so is also impossible.

For $n \geq 3$, we obtain that $P$ is constant in $E$ when $n \geq 1$, and true monodromy at infinity is constant. This implies

$$
\lim_{E \to \infty} e^{\int F(z, E) dz} E^\beta = s(z) \neq 0
$$

for a suitable $\beta$. Thus $\psi(z, E)$ would converge after rescaling to $s(z)$. Impossible as $\psi''(z, E)/\psi(z, E) = V(z) + E$. Thus $n \leq -2$.  

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We conclude that \( P(z, E) = z\sqrt{-E} \). We can now act the Galois group element \( \sqrt{-E} \rightarrow -\sqrt{-E} \) to obtain for free a new hyperexponential solution.

Let us now remark that if an \( \alpha_i \) is not a positive integer, then \( z_i(E) \) is a singularity of the Schrodinger equation. And thus \( z_i \) cannot depend on \( E \). Such a term can be put in factor for both hyperexponential solutions. This gives a solution space of the form

\[
e^{\int g(z, \sqrt{-E}) dz} \left( Ae^{z\sqrt{-E}} T(z, \sqrt{-E}) + Be^{-z\sqrt{-E}} T(z, -\sqrt{-E}) \right) \quad A, B \in \mathbb{C}
\]

with \( T \) polynomial, \( g \) rational. Rewriting the terms using ch, sh instead of exponentials, and then putting in the coefficient in front of sh and then inside the \( e^{\int g(z, \sqrt{-E}) dz} \) by changing \( g \), we obtain the expression (7). And the expression of \( M \) in function of \( T \) proves that it is indeed rational in \( E \).

This Lemma implies in particular that if we have one hyperexponential solution, then we have two and the solution space of equation (1) is the solution space of a differential equation which is a rational gauge transformation of the equation \( y'' + Ey = 0 \).

Let us now look at the dihedral case, for which the space of solutions is of the form

\[
Ae^{\int R_1(z) dz} + Be^{\int R_2(z) dz}
\]

with \( R_1, R_2 \) belonging to an extension of degree 2 over \( \mathbb{C}(z) \). Remark all rigid functions in the dihedral case are not algebraic pull-backs of Galois rigid operators with dihedral Galois group. Indeed, we need to take into account the field operations, as in the following example

\[
(\sqrt{z^2 + 1} + 1)^\sqrt{2} e^{z\sqrt{z^2 + 1}}
\]

This is not a gauge transformation/algebraic pullback of a solution of a dihedral Galois rigid equation. However, the term in the exponential can be written \( \sqrt{z^2(z^2 + 1)} \) and so is a pullback of \( \exp \sqrt{z} \). So is \( (\sqrt{z^2 + 1} + 1)^\sqrt{2} \) with \( \sqrt{z^2 + 1} \). Still the expression in the exponential has to be elementary, and thus the integrals \( \int R_i(z) dz \) have to be elementary.

**Proposition 6.** A Schrodinger equation (1) cannot have a rigid solution space with dihedral Galois group.
PROOF. The space of solutions is of the form
\[ Ae^{\int R_1(z,E)dz} + Be^{\int R_2(z,E)dz} \]
where \( R_1, R_2 \) are solutions of the Ricatti equation associated to \( (1) \). This equation has base field coefficients \( \mathbb{C}(E) \). We now use Theorem 5.4 of [17], saying that \( R_1, R_2 \) are in an extension of degree 2 over \( \mathbb{C}(z,E) \). And thus \( R_1, R_2 \in \mathbb{C}(z,E, \sqrt{f(z,E)}) \) for some polynomial \( f \). We can now rewrite the solution space under the form
\[ e^{\int g(z,E)dz} \left( Ae^{\int \sqrt{f(z,E)}F(z,E)dz} + Be^{-\int \sqrt{f(z,E)}F(z,E)dz} \right) \]
with \( g, f, F \) rational functions in \( z, E \). We also know that the Schroedinger equation is unimodular, giving a condition on \( g \) allowing it to be expressed in function of \( f, F \). We find in particular that the solution space should be of the form
\[ \frac{1}{f(z,E)^{1/4}} \left( Ae^{\int \sqrt{f(z,E)}F(z,E)dz} + Be^{-\int \sqrt{f(z,E)}F(z,E)dz} \right) \]
with \( f, F \) rational in \( z, w \) rational in \( E \). We can moreover assume \( f(z) \) polynomial with only simple roots.

Let us now look at singularities. The solutions should not have mobile singularities, and thus \( f(z,E), F(z,E) \) cannot have mobile roots/poles. And thus we can write our solution space under the form
\[ \frac{1}{f(z)^{1/4}} \left( Ae^{\int w(z,E)f(z)dz} + Be^{-\int w(z,E)f(z)dz} \right) \]
with \( f, F \) rational in \( z, w \) rational in \( E \). We can moreover assume \( f(z) \) polynomial with only simple roots.

Now recall that our solution has to be rigid as well. And thus the integral has to be an elementary function. So we have
\[ \int \sqrt{w(E)f(z)}F(z)dz = \sqrt{w(E)f(z)}Q(z) + \sum \alpha_i \sqrt{w(E)} \ln u_i(z) \]
with \( Q \in \mathbb{C}(z), u_i \in \mathbb{C}(z, \sqrt{f(z)}) \).

Let us remark that if \( w \) is constant, then we would obtain a solution space not depending in \( E \), which is impossible. On the other hand, we know that in the Schroedinger equation \( (1) \), there are no mobile singularities, the residues
are constant, and in the finite irregular singularity case the exponential part does not depend on $E$. So we deduce that $Q$ is a polynomial.

We now look at the asymptotic behaviour in $z$. The cases $n \geq 2$ are impossible, as $E$ has to appear in the exponential part. The case $n = 1$ is also impossible as $\sqrt{w(E)}$ has to appear as a factor of all terms in the exponential. Remain the cases $n \leq -1$, for which we should have

$$\sqrt{w(E)}f(z)Q(z) \sim \sqrt{-E}z$$

As $f$ is a polynomial and cannot be constant (as else the Galois group would be diagonal instead of dihedral), we conclude that $f$ is of degree 2, and $Q$ is constant. And so after possibly an affine coordinate change, we obtain

$$w(E) = -E, \quad f(z) = z^2 + 1, \quad Q(z) = 1$$

Thus the Schroedinger equation should have a solution of the form

$$\psi(z, E) = \frac{S(z)\sqrt{-E}e^{\sqrt{-E}\sqrt{z^2+1}}}{(z^2 + 1)^{1/4} \sqrt{\frac{S'(z)^2}{S(z)} + \frac{S(z)}{\sqrt{z^2 + 1}}} + \frac{z^2}{\sqrt{z^2 + 1}} + o(E)}$$

with

$$S(z) = \prod_i u_i(z)^{\alpha_i}$$

We then compute a series expansion of $-\psi/\psi$ at $E = \infty$, giving

$$-\frac{\psi''(z, E)}{\psi(z, E)} = E \left( \frac{S'(z)^2}{S(z)^2} + \frac{2zS'(z)}{\sqrt{z^2 + 1}S(z)} + \frac{z^2}{z^2 + 1} \right) + o(E)$$

The integrability condition is

$$\frac{S'(z)^2}{S(z)^2} + \frac{2zS'(z)}{\sqrt{z^2 + 1}S(z)} + \frac{z^2}{z^2 + 1} = 1$$

and this cannot be satisfied for a function $S(z)$ of the required form.

\[ \square \]
3.4. Proof of Theorem 1

The hyperexponential factor \( \exp \int g(z, \sqrt{-E}) \, dz \) can be computed explicitly by just requiring that the differential equation whose solution is the eigenfunction does not have a \( \psi' \) term. This gives a first order differential equation, and this equation admits surprisingly very simple solutions, giving the prefactors in the expressions of Theorem 1. The Liouvillian case of Lemma 4 is included as a special case of the third case of Theorem 1, with \( \nu = 1/2 \). Indeed, for \( \mu = 0, \nu = 1/2 \), the Whittaker function simplifies and simply becomes an exponential. Finally we have to prove that \( M(z, E) \) is not only algebraic but rational. In the Liouvillian case, this is already proved by Lemma 1. We remark that \( M \) as written in Theorem 1 corresponds to a gauge transformation of a differential equation with rational coefficients in \( z, E \). In the non solvable case, if \( M \) was algebraic and not rational, the Galois action would generate more solutions. Impossible, as we already have a vector space of dimension 2 of solutions (a single solution of a non solvable equation generates a basis of solutions under the action of the differential Galois group).

Remark that the expressions of the gauge transformations in Theorem 1 are not chosen as simple as possible, but these choices will simplify proofs of Theorem 2.

4. Classification of integrable potentials

We will now describe all the possible \( M \) of Theorem 1 leading to a potential \( V \).

4.1. Mobile singularities

We first prove the following Proposition, valid for the 4 families of eigenfunctions of Theorem 1.

**Proposition 7.** Let \( \psi \) be a function of the form given by Theorem 1 and \( H \) the denominator under the square root. The function \( \psi \) is a solution of a Schroedinger equation if and only if

\[
H(z, E) = \frac{w(E)P(z)}{Q(z, E)}
\]

with \( w, P, Q \) polynomials and \( \deg_E w \geq \deg_E \text{numer}(M) + \deg_E \text{denom}(M) + 1 \).
Proof. Let us first remark that as the Schroedinger equation is linear, the singularities of the solutions are always singularities of the potential. And thus the denominators in the expression of Theorem I are singularities of the equation, and thus poles of $V$. But then these poles should not depend on $E$. Then we can write

$$H(z, E) = \frac{w(E)P(z)}{Q(z, E)}$$

with $w, P, Q$ polynomials.

We have that $\deg_E \text{num}(H) = \deg_E w$ and so we just have to prove that $\deg_E \text{num}(H) \geq \deg_E \text{num}(M) + \deg_E \text{denom}(M) + 1$. Just noting $M = R/S$, replacing and taking the numerator, we find $\deg_E w = \max(2 \deg_E R, 2 \deg_E S + 1)$. This however skips the possible problem of simplifications of the fraction.

Assume there is a simplification. It would mean that a pole $\alpha(E)$ (root of $S$) is not a pole of $H$. Looking at the expression of $\psi$, this implies that $\alpha(E)$ would be a singularity of $\psi$ (as no simplification can occur with the denominator $\sqrt{H}$). And thus $\alpha(E)$ does not depend on $E$, as mobile singularities are forbidden. So factors depending on $z$ only could simplify, but this does not change the degrees in $E$. And thus we have always

$$\deg_E w \geq \max(2 \deg_E R, 2 \deg_E S + 1) \geq \deg_E \text{num}(M) + \deg_E \text{denom}(M) + 1$$

Now we prove the other way of the Proposition. We express $V(z)$ as a rational expression in $E, M$ and its derivatives. We then replace the derivatives of $M$ using $H$. We find that this expression has no singularities at the roots of $w$ (which are roots of $H$ and its derivatives). Knowing that $\deg_E w \geq \deg_E \text{num}(M) + \deg_E \text{denom}(M) + 1$, we obtain that the degree of $V$ is 0. And thus $V$ only depend on $z$, and so is a potential.

Using Proposition 7, we have that $M$ is completely determined using Hermite rational interpolation by its evaluations at roots of $w$, and possibly derivatives in $E$ for multiple roots of $w$. Moreover, such a $M$ will always lead to a quantum integrable potential. Let us now look at what happen at a root $E_0$ of $w$. The denominator $H$ should vanish at $E_0$. We now pose for the following of the proof

$$M(z, E) = -\frac{Y'(z, E)}{Y(z, E)}$$

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The equation $H = 0$ becomes respectively in each of the 4 cases of Theorem 1:

$$z^2 Y''(z, E_0) - zY'(z, E_0) - (z^4 - E_0z^2 - 4\nu^2 + 1)Y(z, E_0) = 0$$

$$4z^2 Y''(z, E_0) + (4E_0z^2 + 4z - 4\nu^2 + 1)Y(z, E_0) = 0$$

$$4z^2 Y''(z, E_0) + (4E_0z^2 - 4\nu^2 + 1)Y(z, E_0) = 0$$

$$Y''(z, E_0) + (z + E_0)Y'(z, E_0) = 0$$

(8)

Now a simple necessary condition for getting a quantum integrable potential is that for all roots of $w$, these equations admit a hyperexponential solution (and this is sufficient when $w$ has no multiple root). The following of the proof of Theorem 2 will be split in 4 parts, each corresponding to one possible eigenfunction class given by Theorem 1.

4.2. Case 1

**Lemma 5 (Galois groups of Whittaker equation in [18]).** The hyperexponential solutions of the first equation (8) are

$$z^{2\epsilon_1\epsilon_2\nu + 1}e^{-\epsilon_1 z^2/2}F_1(-k, 2\epsilon_1\epsilon_2\nu + 1, \epsilon_1 z^2)$$

with $E_0 = \epsilon_1(4k + 2) + 4\epsilon_2\nu$, $k \in \mathbb{N}$, $\epsilon_1, \epsilon_2 = \pm 1$.

Remark that it is possible that the equation admits 2 hyperexponential solutions for some specific $E_0$. This case corresponds to when $E_0$ can be written $\epsilon_1(4k + 2) + 4\epsilon_2\nu$ in two different ways. Now this Lemma gives a condition on the roots of $w$, and gives a formula for $M$ at the roots of $w$. If $w$ has no multiple roots, $M$ can be recovered through Pade interpolation, giving Theorem 2 in the case 1.

We now focus on multiple roots of $w$. As $w$ vanishes at some $E_0$ at order $p \geq 2$, we can differentiate equation (8) with respect to $E_0$, giving us additional equations. Now the condition for getting a quantum integrable potential is that the logarithmic derivative in $z$ of the series solution $Y$ in $E$ at order $p$ has rational coefficients in $z$ (the function $M$ can then be recovered by Pade Hermite interpolation).

**Lemma 6.** Assume $w$ has a double root at $E_0 = \epsilon_1(4k + 2) + 4\epsilon_2\nu$ with $Y(z, E_0)$ the hyperexponential function given by Lemma 5. The function $Y$ leads to a quantum integrable potential if and only if $2\epsilon_1\epsilon_2\nu + k \in \mathbb{N}$. 36
PROOF. We have the equation

\[ M(z, E)^2 z^2 + M(z, E)z - M'(z, E)z^2 - z^4 + z^2 E - 4\nu^2 + 1 = O(E^2) \]

Using Lemma 5, we know that

\[ M(z, E_0) = -\frac{Y'(z)}{Y(z)} \]

with

\[ Y(z) = z^{2\epsilon_1\epsilon_2\nu+1} e^{-\epsilon_1 z^2/2} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu+1)} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu)} F_1(-k, 2\epsilon_1\epsilon_2\nu+1, \epsilon_1 z^2) \]

Noting

\[ M(z, E) = -\frac{Y'(z)}{Y(z)} + (E - E_0)M_1(z) + O((E - E_0)^2) \]

and injecting it in the equation of \( M \) above, we find the solutions for \( M_1 \)

\[ M_1(z) = \frac{z}{Y(z)^2} \int \frac{Y(z)^2}{z} dz \]

The condition for leading to a quantum integrable potential is that \( M(z, E) \) should be rational in \( z, E \), and thus that \( M_1(z) \) should be rational in \( z \). Looking at the integral above, this condition becomes

\[ \int z^{4\epsilon_1\epsilon_2\nu+1} e^{-\epsilon_1 z^2} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu+1)} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu)} F_1(-k, 2\epsilon_1\epsilon_2\nu+1, \epsilon_1 z^2)^2 dz \in e^{-\epsilon_1 z^2} z^{4\epsilon_1\epsilon_2\nu} C(z) \]

We perform a variable change transforming the antiderivative computation in

\[ \int z^{2\epsilon_1\epsilon_2\nu} e^{-\epsilon_1 z} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu+1)} \frac{1}{\Gamma(2\epsilon_1\epsilon_2\nu)} F_1(-k, 2\epsilon_1\epsilon_2\nu+1, \epsilon_1 z)^2 dz \]

This has to be an element of \( e^{-\epsilon_1 z^2} z^{2\epsilon_1\epsilon_2\nu} C(z) \).

Let us first look at when \( \nu \in 1/2\mathbb{Z} \), we are integrating a rational function times exponential. The only possible pole is at \( z = 0 \) (the \( F_1 \) is a polynomial). However, for such \( \nu \), the function \( F_1 \) can become singular. This can be solved using a regularization process, multiplying by some function in \( \nu \), giving

\[ \frac{\Gamma(2\epsilon_1\epsilon_2\nu+1+k)}{\Gamma(2\epsilon_1\epsilon_2\nu+1)} F_1(-k, 2\epsilon_1\epsilon_2\nu+1, \epsilon_1 z) \]
instead of the $1F_1$. Now the valuation at $z = 0$ of

$$z^{2\epsilon_1\epsilon_2\nu}\frac{\Gamma(2\epsilon_1\epsilon_2\nu + 1 + k)^2}{\Gamma(2\epsilon_1\epsilon_2\nu + 1)^2} 1F_1(-k; 2\epsilon_1\epsilon_2\nu + 1, \epsilon_1z)^2$$

is $2\epsilon_1\epsilon_2\nu + k$, and thus if this quantity is non-negative, we are integrating a polynomial times exponential. The condition is then fulfilled.

Let us now prove that for $2\nu \notin \mathbb{Z}$, the condition cannot be satisfied. Let us note $v_{n,k}$ the coefficients in $z^n$ of the polynomial $1F_1(-k; 2\epsilon_1\epsilon_2\nu + 1, \epsilon_1z)$.

Trying to express the antiderivative as an element of $e^{-\epsilon_1z}z^{2\epsilon_1\epsilon_2\nu}C(z)$, we obtain a big linear system, and when $2\nu \epsilon_1 \epsilon_2 \notin \mathbb{Z}$, we can solve it under the condition

$$\sum_{n=0}^{2k} v_{n,k} \epsilon^n \frac{\Gamma(2\epsilon_1\epsilon_2\nu + n + 1)}{\Gamma(2\epsilon_1\epsilon_2\nu)} = 0$$

The coefficients $v_{n,k}$ satisfy holonomic system (see [19, 20] for basic properties) with shifts in $n, k$ as a convolution of $P$-finite sequences, the coefficients of the $1F_1$. So is $\epsilon^n \Gamma(2\epsilon_1\epsilon_2\nu + n + 1)$. As the holonomic property is stable by definite summation, this above sum as a sequence in $k$ also satisfy a recurrence equation. This can be found thanks to the holonomic package [21], giving the relation

$$\sum_{n=0}^{2k} v_{n,k} \epsilon^n \frac{\Gamma(2\epsilon_1\epsilon_2\nu + n + 1)}{\Gamma(2\epsilon_1\epsilon_2\nu)} = \epsilon_1\epsilon_2 \frac{2\Gamma(2\epsilon_1\epsilon_2\nu + 1)\Gamma(k + 1)\nu}{\Gamma(k + 1 + 2\epsilon_1\epsilon_2\nu)}$$

The only admissible root of the righthandside is $\nu = 0$, which is excluded. Remark that the poles of the righthandside are when $2\nu \in \mathbb{Z}$, and thus are excluded (these are exactly the singularities of the $1F_1$ function in $\nu$).

$$\square$$

Double roots are only possible for $2\epsilon_1\epsilon_2\nu + k \in \mathbb{N}$. Looking now at the case with simple roots (with generic $\nu$), we can produce a double root by taking a particular $\nu$: indeed, if there are two roots of the form $4\nu + (4k_1 + 2)$, $-4\nu + (4k_2 + 2)$, they fuse together when $\nu = (k_2 - k_1)/2$. So the double root case at $\epsilon_2\nu_0 + \epsilon_1(k_0 + 1/2)$ can be obtained from the simple root case with two roots

$$\epsilon_2\nu + \epsilon_1(k_0 + 1/2), -\epsilon_2\nu + \epsilon_1(k_0 + 2\epsilon_2\epsilon_1\nu_0 + 1/2)$$

if $\epsilon_1\epsilon_2 = 1$
\[ \epsilon_2\nu + \epsilon_1(k_0 - 2\epsilon_2\nu_0 + 1/2), -\epsilon_2\nu + \epsilon_1(k_0 + 1/2) \text{ if } \epsilon_1\epsilon_2 = -1 \]

So the double root case is included in the simple root case as a limit for a specific \( \nu \).

To conclude, let us prove that triple root or more are not possible

**Lemma 7.** If \( w \) has a triple or more root \( E_0 \), then there is no \( Y \) leading to a quantum integrable potential

**Proof.** We have the equation

\[ M(z, E)^2 z^2 + M(z, E)z - M'(z, E)z^2 - z^4 + z^2 E - 4\nu^2 + 1 = O(E^3) \]

Using Lemma 6, we can assume \( E_0 = \epsilon_1(4k + 2) + 2\epsilon_1(p - k) \) and \( \nu = \epsilon_1\epsilon_2(p - k)/2 \) with \( k, p \in \mathbb{N} \). We also know that

\[ M(z, E_0) = -\frac{Y'(z)}{Y(z)} \]

with

\[ Y(z) = z^{p-k+1}e^{-\epsilon_1z^2/2} _1F_1(-k, p - k, 1, \epsilon_1z^2) \]

Noting

\[ M(z, E) = -\frac{Y'(z)}{Y(z)} + (E - E_0)M_1(z) + (E - E_0)^2M_2(z) + O((E - E_0)^3) \]

and injecting it in the equation of \( M \) above, we find the solutions for \( M_2 \)

\[ M_2(z) = \frac{z}{Y(z)^2} \int \frac{z}{Y(z)^2} \left( \int \frac{Y(z)^2}{z} dz \right)^2 dz \quad (9) \]

As the integrability condition of Lemma 6 is satisfied, we know that

\[ \frac{z}{Y(z)^2} \int \frac{Y(z)^2}{z} dz \in \mathbb{C}(z) \]

Let us now prove that \( M_2 \) has monodromy around 0. More precisely, we will prove that the series expansion at 0 of

\[ \frac{z}{Y(z)^2} \left( \int \frac{Y(z)^2}{z} dz \right)^2 \quad (10) \]
has a non zero residue. However, the residue of this expression does not appear to be holonomic (recall that dividing by \( Y(z) \) is a priori forbidden). Let us first remark that

\[
\tilde{Y}(z) = Y(z) \int \frac{z}{Y(z)^2} dz
\]

is in fact the second independent solution of the first equation (8), and thus is holonomic. We now rewrite the residue expression using integration by parts

\[
\frac{1}{2i\pi} \oint_0 \frac{z}{Y(z)^2} \left( \int \frac{Y(z)^2}{z} dz \right)^2 dz = \\
\frac{1}{2i\pi} \oint_0 \frac{2}{z} Y(z)^2 \int \frac{z}{Y(z)^2} dz \int \frac{Y(z)^2}{z} dz dz dz = \\
\frac{1}{2i\pi} \oint_0 \frac{2}{z} Y(z) \tilde{Y}(z) \int \frac{Y(z)^2}{z} dz dz
\]

which is now clearly holonomic. Thus we can find an holonomic system in \( p, k \) for the monodromy of this expression around 0. We now use the holonomic package [21], and we find the simple formula

\[
\frac{1}{2i\pi} \oint_0 \frac{z}{Y(z)^2} \left( \int \frac{Y(z)^2}{z} dz \right)^2 dz = \frac{\Gamma(k + 1)\Gamma(p + 1 - k)^2}{4\Gamma(p + 1)}
\]

Remark that the formula degenerates for \( p < k \). This is due to the singular definition of function \( Y \), as it has a pole of order one in such case. Noting that the expression is homogeneous of degree 2 in \( Y(z) \), we expect a pole of order 2 of the righthandside. We then regularize the formula by a limit process

\[
\lim_{\alpha \to 0} \alpha^2 \frac{\Gamma(k + 1)\Gamma(p + \alpha + 1 - k)^2}{4\Gamma(p + \alpha + 1)} = \frac{\Gamma(k + 1)}{\Gamma(k - p)^2\Gamma(p + 1)}
\]

We now see that these expressions never vanish for \( p, k \in \mathbb{N} \), implying that \( M_2 \) can never be rational.
4.3. Case 2

The most important remark here is that the second equation of (8) is the same as the first equation of (8) after the variable change

\[ Y(z, E) = k \left( \frac{1}{4} z^2 E, -\frac{4}{E^2} \right) \]

Thus the previous results of last section apply the same, carrying the variable change in \( z \) and \( E \). The change of parameter \( E \) change accordingly the values of \( E_0 \) for which the equation admits a hyperexponential solution, and the multiple root cases are also the same after this variable change. We thus obtain the same following results

**Lemma 8.** The hyperexponential solutions of the second equation (8) are

\[ z^{\nu + 1/2} e^{-\frac{z}{4(2\nu + 2k + 1)}} F_1 \left( -k, 2\nu + 1, \frac{2z}{2\nu + 2k + 1} \right) \]

with \( E_0 = -1/(2\nu + 2k + 1)^2 \), \( k \in \mathbb{N} \), \( \epsilon = \pm 1 \).

It is still possible that the equation admits 2 hyperexponential solutions for some specific \( E_0 \). This case corresponds to when \( E_0 \) can be written \(-1/(2\nu + 2k + 1)^2\) in two different ways. Now this Lemma gives a condition on the roots of \( w \), and gives a formula for \( M \) at the roots of \( w \). If \( w \) has no multiple roots, \( M \) can be recovered through Pade interpolation, giving Theorem 2 in the case 2.

We now focus on multiple roots of \( w \). As \( w \) vanishes at some \( E_0 \) at order \( p \geq 2 \), we can differentiate equation (8) with respect to \( E_0 \), giving us additional equations. Now the condition for getting a quantum integrable potential is that the logarithmic derivative in \( z \) of the series solution \( Y \) in \( E \) at order \( p \) has rational coefficients in \( z \) (the function \( M \) can then be recovered by Pade Hermite interpolation).

**Lemma 9.** Assume \( w \) has a double root at \( E_0 = -1/(2\nu + 2k + 1)^2 \) with \( Y(z, E_0) \) the hyperexponential function given by Lemma 8. If \( Y \) leads to a quantum integrable potential, then \( 2\nu + k \in \mathbb{N} \).

**Lemma 10.** If \( w \) has a triple or more root, then there is no \( Y \) leading to a quantum integrable potential.
Again, double roots are only possible for $2\epsilon \nu + k \in \mathbb{N}$. Looking now at the case with simple roots (with generic $\nu$), we can produce a double root by taking a particular $\nu$: indeed, if there are two roots of the form $-(2\nu+2k_1+1)^2$, $-(-2\nu+2k_2+1)^2$, they fuse together when $\nu = (k_2-k_1)/2$. So as in previous section, the double root case can be obtained from the simple root case by fusing two roots through a limiting process for a specific $\nu$.

4.4. Case 3

The third equation is a Bessel equation. It has hyperexponential solutions if and only if $E_0 = 0$. Thus we have $w(E) = E^k$. The equation for $M$ is then

$$4M(z, E)^2z^2 + 4z^2E - 4M'(z, E)z^2 - 4\nu^2 + 1 = O(E^k) \quad (11)$$

**Case $\nu \notin \mathbb{1}/2\mathbb{Z}$**

At $E = 0$, we find only two possible rational solutions

$$M(z, 0) = \frac{2\epsilon \nu - 1}{2z}, \quad \epsilon = \pm 1$$

Let us write

$$M(z, E) = \sum_{i=0}^{k-1} M_i(z)E^i, \quad M_0(z) = \epsilon \nu/z$$

Injecting this in the differential equation, we obtain a list of differential equations in the $M_i$ of the form

$$\frac{2\epsilon \nu - 1}{z}M_i(z) - M_i'(z) = \text{Polynomial}(M_j(z)_{j<i})$$

defining the $M_i$ recursively. The important point is that these differential equations are linear, and that the homogeneous part

$$\frac{2\epsilon \nu - 1}{z}M_i(z) - M_i'(z) = 0$$

has no non-zero rational solutions. Thus the equation admits at most two rational solutions (one for each $\epsilon$). In particular, the $M_i$ are uniquely determined by $M_0$. 

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We now solve equation (11) with zero righthandside. We find two interesting solutions ($\epsilon = \pm 1$)

$$M(z, E) = -\frac{\partial}{\partial z} \ln \left( W(0, \epsilon \nu, \sqrt{-4Ez}) \right)$$

(12)

These solutions satisfy $M(z, 0) = (2 \epsilon \nu - 1)/(2z)$, and thus their series expansion at order $k$ is the unique series solution we are searching. We now note $M_\epsilon(z, E)$ the rational function obtained by Pade series from the series expansion of (12) at order $k$.

The Whittaker function with $\mu = 0$ rewrites in terms of the Bessel function. Now recall there is a recurrence relation between the Bessel function. In particular a linear combination of $W(0, \nu + j, \sqrt{-Ez}), W(0, \nu + 1 + j, \sqrt{-Ez})$ can be rewritten as a linear combination of $W(0, \nu, \sqrt{-Ez}), W(0, \nu + 1, \sqrt{-Ez})$. And this relation gives a homographic transformation on $M$. Using degree considerations in $E$, we find that the two solutions $M_\epsilon(z, E)$ are such that

$$z \left( \frac{M_\epsilon(z, E)}{\sqrt{-4E}} \right) W(0, \nu, z \sqrt{-4E}) + W'(0, \nu, z \sqrt{-4E}) = W(0, \nu + \epsilon k, z \sqrt{-4E})$$

So these possible $M$ give in fact the same eigenfunction as $M = \infty$, proving third case of Theorem 2.

**Case $\nu \in \mathbb{Z}$**

Let us first remark that we can assume $\nu \in [0, 1]$ (as we can always shift $\nu$ by an integer). And so we can assume $\nu = 0$. So the third equation (8) becomes

$$4z^2Y''(z, E_0) + (4E_0z^2 + 1)Y(z, E_0) = 0$$

We can differentiate this equation in $E_0$ up to order $k$, the multiplicity of the root 0 in $w$. Noting

$$Y(z, E) = \sum_{i=0}^{k-1} Y_i(z)E^i + O(E^k), \quad D = -\partial_z^2 - 1/(4z^2)$$

we obtain

$$DY_0 = 0, \quad DY_{i+1} = Y_i, \quad i = 1 \ldots k - 2$$

So the solutions can be obtained by applying the (pseudo) inverse of $D$ iteratively. Let us remark that the equation

$$Df = g, \quad g \in \sqrt{z} \mathbb{C}[z^2] + \ln(z)\sqrt{z} \mathbb{C}[z^2]$$

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has solutions in the vector space \( K = \sqrt{z} \mathbb{C}[z^2] + \ln(z) \sqrt{z} \mathbb{C}[z^2] \). So this vector space is stable by these iterations of taking the inverse of \( D \). At each step, the degrees of the polynomials in \( z^2 \) grows by one.

So, possible functions \( M \) are given by the series expansion

\[
M(z, E) = -\frac{\partial}{\partial z} \ln \left( \sum_{i=0}^{k-1} D^i F(z) E^{k-1-i} \right) + O(E^k)
\]

\[F \in K, \text{ with degrees } k - 1\]

However, we need to check that this series has rational coefficients in \( z \). This is not automatic as the function \( F \) can contain logs. Knowing that \( Y(z, 0) \) should have a rational logarithmic derivative, the only possible solutions are \( Y(z, 0) = a \sqrt{z} \). The constant \( a \) can be assumed to be non zero, as we can multiply by a power of \( E \) without changing \( M \). This condition rewrites in terms of \( F(z) \) by the constraint \( F(z) = \sqrt{z} P_1(z^2) + \ln z \sqrt{z} P_2(z^2) \) with \( \deg P_1 = k - 1 \). The constant \( a \) can further be assumed equal to 1, after multiplication of \( Y \) by a constant, allowing to apply the following Lemma to conclude.

Lemma 11. A series

\[
Y(z, E) = \sum_{i=0}^{k-1} D^i F(z) E^{k-1-i} + O(E^k)
\]

with \( Y(z, 0) = \sqrt{z} \) has a logarithmic derivative which is a series in \( E \) with rational coefficients in \( z \) if and only if

\[
F(z) = \sqrt{z} P_1(z^2) + \ln z \sqrt{z} P_2(z^2), \quad \deg P_2 \leq k/2 - 1
\]

Let us first assume that the series \( Y(z, E) \) can be written under the form

\[
Y(z, E) = e^{-\int M(z, E) dz}
\]

where \( M(z, E) \) is a series in \( E \) with rational coefficients. So after integration, we can obtain logs. Putting \( E = 0 \) in the above expression, and knowing that the first term of \( Y(z, E) \) is \( \sqrt{z} \), we deduce that \( M(z, E) = -1/(2z) + O(E) \). The next terms of the series cannot have singularities outside 0, as \( Y \) does
not. And the possible singularity at 0 is of order 1 at most, due to the form of $Y$ (the singular behaviour at 0 is in ln). Thus we have

$$Y(z, E) = e^{\sum_{i=1}^{k-1} (Q_i(z) + a_i \ln z)E^i + O(E^k)}$$

with $Q_i$ polynomials and $a_i$ constants. We now make a series expansion of the righthandside in $E = 0$, and we see that powers of ln can appear. These are impossible as $Y \in K[[E]]$. A necessary condition to avoid powers of logs is that $a_i = 0, \forall i = 1 \ldots (k - 1)/2$. This implies in particular that the series expansion of $Y$ in $E$ has only polynomials in $z$ as coefficients up to $E^{(k-1)/2}$ included. As the coefficients have the form $D_i F(z)$, we obtain that

$$F(z) = \sqrt{z}P_1(z^2) + \ln z\sqrt{z}P_2(z^2), \quad \deg P_2 \leq k/2 - 1$$

Now let us prove the opposite way. Assume

$$Y(z) = P_1(z^2) + \ln zP_2(z^2), \quad \deg P_2 \leq k/2 - 1,$$

and let us prove that $Y$ has a logarithmic derivative which is a series in $E$ with rational coefficients. For $k = 1, 2$, this can be directly verified. So assume $k \geq 3$. Recall that minus this logarithmic derivative is in fact a solution of the non linear equation

$$4z^2 M(z, E)^2 + 4Ez^2 + 1 - 4z^2 M'(z, E) = O(E^k)$$

We know that $M$ has a series expansion in $E$ with rational coefficients up to $E^{(k-1)/2}$ included. These are moreover odd functions in $z$. Let us prove that this above equation implies that the next terms of the series are also rational. Noting $M(z, E) = -1/(2z) + \sum_{i=1}^{k-1} M_i(z)E^i$, we obtain the relations from the above equation in $M$

$$\sum_{j=1}^{i-1} M_j(z)M_{i-j}(z) - M'_i(z) - M_i(z)/z = 0$$

So these relations give a system of linear differential equations in $M_i, i > (k - 1)/2$. The solutions are

$$M_i(z) = \frac{1}{z} \int z \sum_{j=1}^{i-1} M_j(z)M_{i-j}(z)dz$$

(14)
We know that the \( M_j \) with \( j \leq (k - 1)/2 \) are odd rational in \( z \). Moreover, for \( 1 \leq j \leq (k - 1)/2 \), they have no singularity at \( z = 0 \), as else the coefficients \( a_i \) in equation (13) would be non zero, and so \( Y \) could not satisfy the hypothesis. We also know a priori that \( M \in \mathbb{C}(z)[\ln z][[E]] \). So we just have to prove that logs do not appear when making the integration in equation (14). Let us keep track of the valuation at \( z = 0 \) of the \( M_j \). We have \( \text{val} M_j \geq 1, \ 1 \leq j \leq (k - 1)/2 \). Let us prove by recurrence that the \( M_i \) have no logs and valuation \( \geq -1 \).

For \( 1 \leq i \leq (k - 1)/2 \), it is already done. For larger \( i \), we look the integrand of equation (14), and we see in each product, \( M_j \) or \( M_{i-j} \) has index \( \leq (k - 1)/2 \). Thus the valuation of the sum is at least \( 1 - 1 = 0 \) (using here the recurrence hypothesis \( \text{val} M_j \geq -1, \ \forall j < i \)). Thus the valuation of the integrand is at least 1, and so no logs appear in the integration. Moreover, we then divide by \( z \), dropping the valuation by 1, and thus \( \text{val} M_i \geq -1 \). This gives the Lemma, proving fourth case of Theorem 2.

**Case** \( \nu \in 1/2 + \mathbb{Z} \)

Let us first remark that we can assume \( \nu \in [0, 1] \) (as we can always shift \( \nu \) by an integer). And so we can assume \( \nu = 1/2 \). So the third equation (8) becomes

\[
Y''(z, E_0) + E_0 Y(z, E_0) = 0
\]

We can differentiate this equation in \( E_0 \) up to order \( k \), the multiplicity of the root 0 in \( w \). Noting

\[
Y(z, E) = \sum_{i=0}^{k-1} Y_i(z) E^i O(E^k), \quad D = -\partial_z^2
\]

we obtain

\[
D Y_0 = 0, \quad D Y_{i+1} = Y_i i = 1 \ldots k - 2
\]

So the solutions can be obtained by applying the (pseudo) inverse of \( D \) iteratively. Let us remark that the equation

\[
D f = g, \quad g \in \mathbb{C}[z]
\]

has polynomial solutions. So the vector space of polynomials \( \mathbb{C}[z] \) is stable by these iterations of taking the inverse of \( D \). At each step, the degrees of the polynomial grows by two.
So, possible functions $M$ are given by the series expansion

$$M(z, E) = -\frac{\partial}{\partial z} \ln \left( \sum_{i=0}^{k-1} D^i F(z) E^{k-1-i} \right) + O(E^k)$$

$$F \in \mathbb{C}[z], \deg F \leq 2k - 1$$

Here the series has always coefficients rational in $z$. We finally need to ensure that the precision of the series does not drop by taking the logarithmic derivative, i.e. $Y(z, 0) \neq 0$. This implies $\deg F = 2k - 1$ or $2k - 2$. We then always obtain a rational $M$ through Pade series, proving fifth case of Theorem 2.

4.5. Case 4

The fourth equation (8) is an Airy equation, and never has a hyperexponential solution. Thus only the singular $M = \infty$ remains in this case, leading to an affine potential. This proves the sixth case of Theorem 2.

5. Examples

Outside of the special cases $V(z) = z^2 + \alpha/z^2, 1/z + \alpha/z^2, z, \alpha/z^2$, all the other cases are generated by constructing a gauge transformation function $M$ which is a Pade interpolation or Pade series. In Theorem 2 these non trivial gauge transformations split in four families, corresponding respectively to eigenfunctions of Theorem 1 in case 1, case 2, case 3 with $\nu = 0$, case 3 with $\nu = 1/2$.

These four families are described completely explicitly: given a set of points or a polynomial (or log-polynomial), we perform a Pade interpolation or Pade series to produce a function $M$, and then a potential $V$. The 4 families can be generated by algorithms given in the Appendix. The Maple code can be directly copied and is able to generate the integrable potentials of Theorem 2. The programs take in input a list of elements (for case 1, 2) or a function (for case 3 with $\nu = 0, 1/2$).

Here we will make explicit computation of the spectrum for one example of each of the 4 families. These examples were chosen as they seem to exhibit interesting properties for physical applications.
An anharmonic potential

Let us consider the potential of the first family

\[ V(z) = -z^2 - 2 - \frac{8}{2z^2 + 1} + \frac{16}{(2z^2 + 1)^2} \]

The potential is analytic on \( \mathbb{R} \), comes from the first case of Theorem 2 with the list \([4\nu + 6]\), giving the gauge (of degree 0 in \( E \))

\[ M(z, E) = -\frac{z^4 - 4\nu z^2 + 4\nu^2 - 4z^2 + 4\nu + 1}{z(-z^2 + 2\nu + 1)} \]

and then taking \( \nu = -3/4 \). The denominator of the expression of the eigenfunction (case 1 Theorem 1) is \( H = (E - 3)z^2 \), and thus the expression becomes singular for \( E = 3 \).

The potential is analytic on \( \mathbb{R} \), and thus so are the eigenfunctions. So the square integrability condition only put a condition near infinity. It is not trivial as the function \( W \) can be exponentially diverging at infinity. So let us first look at the co triangular condition of Proposition 2. The Stokes matrices at infinity of \( W \) are cotriangularizable if and only if \( E \in 2\mathbb{Z} + 1 \). So we already know that the spectrum is a subset of this. It happens that this set leads to Liouvillian functions. We now compute the first eigenfunctions of this potential

\[
\begin{align*}
E = -1 & \quad \frac{e^{-z^2/2}}{2z^2 + 1} \\
E = 5 & \quad \frac{e^{-z^2/2}z(2z^2 + 3)}{2z^2 + 1} \\
E = 7 & \quad \frac{e^{-z^2/2}(4z^4 + 4z^2 - 1)}{2z^2 + 1} \\
E = 9 & \quad \frac{e^{-z^2/2}z(4z^4 - 5)}{2z^2 + 1} \\
E = 11 & \quad \frac{e^{-z^2/2}(8z^6 - 12z^4 - 18z^2 + 3)}{2z^2 + 1} \\
E = 13 & \quad \frac{e^{-z^2/2}z(8z^6 - 28z^4 - 14z^2 + 21)}{2z^2 + 1}
\end{align*}
\]

The polynomial appearing are in fact a linear combination with coefficients in \( \mathbb{C}(z, E) \) of Hermite polynomials. Let us remark that the spectrum is similar to \(-z^2\), except for few “accidents”, \( E = -1, 1, 3 \). The accident \( E = 3 \) is related to the singularity of the Gauge transformation \( M \). In particular, as they are built, the gauge tranformation functions \( M \) always have a particular behaviour at some specific points. However, if we evaluate the eigenfunction in \( E \), the formula breaks downs for these particular \( E \)’s.
A fusion potential

Let us consider a potential of the second family

\[ V(z) = \frac{1}{z} - \frac{4}{z^2 + 2z + 2} + \frac{8}{(z^2 + 2z + 2)^2} \]

The potential is analytic on \( \mathbb{R}^* \), comes from the second case of Theorem 2 with the list \([-1/(2\nu - 1)^2, -1/(2\nu + 2)^2]\), giving the gauge (of degree 1 in \( E \))

\[ M(z, E) = \frac{- (2\nu + 3)(2\nu - 1)(8\nu^4 + 20\nu^3 - 8\nu^2 z + 6\nu^2 - 12\nu z + 2z^2 - 9\nu)}{4z(4\nu^2 + 8\nu - 2z + 3)} E + \]

\[ \frac{- 8\nu^4 - 20\nu^3 + 8\nu^2 z - 30\nu^2 + 12\nu z - 2z^2 - 31\nu + 16z - 6}{4z(4\nu^2 + 8\nu - 2z + 3)} \]

and then taking \( \nu = -1/2 \). The denominator of the expression of the eigenfunction (case 2 Theorem [1]) is

\[ H = \frac{(z^2 + 2z + 2)^2(4E + 1)^2}{4z^2}, \]

and thus the expression becomes singular for \( E = -1/4 \). Remark this is a case of \( w \) with a double root, the fusion occurring when taking \( \nu = -1/2 \).

The square integrability condition puts a condition near infinity and near 0. There is problem on the interval definition of the solutions: the spectrum on \( \mathbb{R} \), on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) are not the same. The square integrability condition
implies that at least 2 of the three matrices involved (monodromy matrix at 0 and 2 Stokes matrices at infinity) should be cotriangularizable. As there is a multiplicative relation between these matrices, this implies that the differential Galois group is triangularizable. In other words, the condition $C_1$ from Proposition 2 are all the same. The condition $C_1$ is then given by $E = -1/(4k^2)$, $k \in \mathbb{N}^*$. The case $k = 1$ leads to a square integrable solution on $\mathbb{R}^-$, the other ones on $\mathbb{R}^+$ (and none on $\mathbb{R}$).

\[
\begin{align*}
E = -1/4 & \quad \frac{e^{z/2}z}{z^2 + 2z + 2} \\
E = -1/16 & \quad \frac{e^{-z/4}z(z^3 + 6z^2 + 18z + 24)}{z^2 + 2z + 2} \\
E = -1/36 & \quad \frac{e^{-z/6}z(z^4 - 4z^3 - 40z^2 - 144z - 216)}{z^2 + 2z + 2} \\
E = -1/64 & \quad \frac{e^{-z/8}z(z^5 - 30z^4 + 50z^3 + 800z^2 + 3200z + 5120)}{z^2 + 2z + 2}
\end{align*}
\]

**Continuous spectrum potentials**

The third case of Theorem 1 gives two types of eigenfunctions, those with the Bessel function, and the Liouvillian ones. The potentials have continuous spectrum as their solutions are isomonodromic with respect to $E$. We give
here two examples for low degree gauge functions $M$.

The fourth case of Theorem 1 with $\nu = 0$, $F(z) = \sqrt{z(a + z^2 + b \ln z)}$, gives for the gauge

$$M(z, E) = \frac{2Ez^2 + Eb - 2}{4z}$$

The denominator of the expression of the eigenfunction (case 3 Theorem 1) is

$$H = \frac{1}{4}E^2(2z^2 + b)^2,$$

giving here an example of a double root of $w$ at $E = 0$. The corresponding potential is

$$V(z) = \frac{1}{4z^2} - \frac{8}{2z^2 + b} + \frac{16b}{(2z^2 + b)^2}$$

The eigenfunctions are given by case 3 of Theorem 1 with the $M$ given above.

The fifth case of Theorem 1 with $\nu = 1/2$, $F(z) = z^4 + az^3 + bz^2 + cz + d$, gives for the gauge

$$M(z, E) = -\frac{3(4z + a)^2 E}{(3a^2z + 12az^2 + 16z^3 + ab - 2c)E - 12a - 48z}$$

The denominator of the expression of the eigenfunction (case 3 Theorem 1) is

$$H = \frac{4z^2(3a^2z + 12az^2 + 16z^3 + ab - 2c)E^3}{(3Ea^2z + 12Eaz^2 + 16Ez^3 + Eab - 2E^2c - 12a - 48z)^2},$$

giving here an example of a triple root of $w$ at $E = 0$. The corresponding potential is $V(z) =

\frac{-96z - 24a}{3a^2z + 12az^2 + 16z^3 + ab - 2c} + \frac{18a^4 + 72a^3z - 72a^2b - 288abz + 144ac + 576cz}{(3a^2z + 12az^2 + 16z^3 + ab - 2c)^2}$

and the (Liouvillian) eigenfunction

$$\frac{(3a^2z + 12az^2 + 16z^3 + ab - 2c)E + (3a^2 + 24az + 48z^2)\sqrt{-E} - 12a - 48z}{3a^2z + 12az^2 + 16z^3 + ab - 2c}e^{z\sqrt{-E}}$$

In these two cases, the monodromy and Stokes do not depend on $E$, and so any natural boundary condition are trivial (leading to a full $C^*$ spectrum or empty spectrum). This implies by the way it is also the case for almost natural boundary conditions.
6. Conclusion

We defined an explicit notion of quantum integrability for 1 dimensional quantum system by building a differential field over \( \mathbb{C}(z) \) with nice properties with respect to the monodromy/Stokes computations. All come down more or less to compute Gauge transformations of hypergeometric or confluent hypergeometric functions and hyperexponential functions. With these notions, we were able to completely classify integrable 1 dimensional quantum problems in this sense. Remark that our classification effectively generates the integrable potentials, but does not answer the opposite question, i.e. given a potential, is it integrable? This can however be done using Theorem 1. Indeed, we need to search for rational gauge transformations of three particular differential equations. There is an algorithm in the Bessel case in [16], and probably soon for Whittaker functions. The quantum integrability would then be decidable in dimension 1.

In the case of discrete spectrum, we always need one time or another to compute monodromy/Stokes matrices, as these appear in the boundary conditions and “produce” the spectrum. However, we do not have a complete understanding of the relation between the spectrum, in particular the gaps appearing in the examples and the singularities of the function \( w \). Moreover, as many possible gauge functions \( M \) are possible for one potential \( V \), and we do not have a canonic gauge for a potential \( V \), the roots of \( w \) can depend on the choice of the gauge \( M \).

In the continuous case, the monodromy/Stokes matrices do not play any role outside of being constant with respect to \( E \). An important point is that the potentials obtained have always poles of order 2, and this seems related to this continuous spectrum property. A natural question would be to ask if there are other systems for which the monodromy/Stokes matrices do not depend on \( E \). Said otherwise, to find all rational functions \( V \in \mathbb{C}(z) \) such that \( \psi''(z) + (V(z) + E)\psi(z) = 0 \) is isomonodromic with respect to the parameter \( E \). This problem is probably related to isomonodromic deformations and Painleve functions [8].

Another possibly way of generalization would be the higher dimensional case. Even in dimension 2, the full classification is probably out of reach as we do not even know all “classical” integrable potentials. But still producing a definition of the same flavour would be interesting. In higher dimension, the notion of commutative observables become important: this is the analogue of the Liouville integrability of Hamiltonian system. So eigenfunctions are
not anymore solution of a single PDE, but several. The natural condition to
put on such a PDE system is holonomicity. Considering the characteristic
variety associated to the corresponding differential ideal, we see that the
holonomicity condition is in fact a dimension condition on this variety, and
so similar to the independence conditions on first integrals in Hamiltonian
systems. So what are the rigid functions solutions of a holonomic PDE
system? The notions of differential Galois group can be generalized as we are
still on some finite dimensional space. The notion of monodromy and Stokes
are possibly more difficult. Hypergeometric functions have been generalized
in many ways in higher dimensions, in particular $A$-hypergeometric functions.
However, the possibility to carry an explicit computation of the monodromy
is here of fundamental importance. Such result have not been yet obtained
for $A$-hypergeometric functions.

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Appendix

These are the Maple code used to generate each of the 4 non trivial families of quantum integrable potentials of Theorem 2. These codes are standalone, i.e. they can be copy/pasted directly into working programs.

```maple
genpot1:=proc(L);
   seq(factor(-diff(ln(DETools[kovacicsols](
      -z*diff(Y(z),z)+z^2*diff(Y(z),z)*Y(z)+(-z^2+L[i]*z^2-4*nu^2+1),Y(z))[1]),z)),i=1..nops(L)));
   CurveFitting[RationalInterpolation]([seq([L[i],1/%[i]],i=1..nops(L))],E);
   simplify(eval(subs(M(z)=z^3/2/sqrt(M(z)),1/(1/4/E^(4*nu,z^2)/32)*W(0,0,E/4,E,nu,z^2)),diff(%,%)));
   simplify(subs((D[3,3])(W))(E/4,nu,z^2)=-(-1/4+E/4/z^2+1/4*(nu^2)/z^2+1/E),diff(%,%)));
   convert(-factor(simplify(subs((D[3,3])(W))(E/4,nu,z^2)=-(-1/4+E/4/z^2+1/4*(nu^2)/z^2+1/E),diff(%,%)/%)),parfrac,z);
end:

genpot2:=proc(L);
   seq(factor(-diff(ln(DETools[kovacicsols](
      4*z^2*diff(Y(z),z)+4*L[i]*z^2-4*nu^2+1),Y(z))[1]),z)),i=1..nops(L)));
   CurveFitting[RationalInterpolation]([seq([L[i],1/%[i]],i=1..nops(L))],E));
   simplify(eval(subs(M(z)=z^3/2/sqrt(M(z)),1/(1/4/E^(4*nu,z^2)/32)*W(0,0,E/4,E,nu,z^2)),diff(%,%)));
   simplify(subs((D[3,3])(W))(E/4,nu,z^2)=-(-1/4+E/4/z^2+1/4*(nu^2)/z^2+1/E),diff(%,%)));
   convert(-factor(simplify(subs((D[3,3])(W))(E/4,nu,z^2)=-(-1/4+E/4/z^2+1/4*(nu^2)/z^2+1/E),diff(%,%)/%)),parfrac,z);
end:

genpot3:=proc(F) local n,i,S;
   n:=degree(coeff(expand(F/sqrt(z)),ln(z),0),z)/2+1:
   S:=E^(-n-1)*F: for i from 1 to n do S:=-diff(S/sqrt(z),z)+E^(-n-1)*F: od:
   numapprox[pade](series(-E^(-n)*diff(ln(collect(S,E,factor)),z),E=0,n),E,[floor(n/2),floor((n-1)/2)]);
   convert(-subs(gamma=sqrt(-E),factor(simplify(subs(E=-gamma^2,subs((D[3,3])(W))(0,0,z(sqrt(-E)),nu,sqrt(-E)),diff(%,%)/%))),parfrac,z));
end:
```

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The input for the two first ones are lists of the form required by Theorem 2, and for the last two are functions, the function $F$ of the form required by Theorem 2. Remark moreover that these last two can handle functions $F$ with parameters. Some implementation tricks have been used:

- The $_1F_1$ functions of Theorem 2 are generated on the fly by the Kovacic algorithm.
- The rational interpolation is made on the function $1/M$, as the default degrees of the interpolation algorithm then meet with the requirements of Theorem 2.
- In the third one, the series defining $M$ is generated recursively through iterated application of differential operator $D$.
- The substitution of $E$ by $-\gamma^2$ is used to force simplifications of the square roots of $E$ (and choose the same valuation for all of them), so that the resulting potential can be put under partial decomposition form.