TWO-ORBIT K"AHLER MANIFOLDS AND MORSE THEORY

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Abstract. We deal with compact K"ahler manifolds $M$ acted on by a compact Lie group $K$ of isometries, whose complexification $K^C$ has exactly one open and one closed orbit in $M$. If the $K$-action is Hamiltonian, we obtain results on the cohomology and the $K$-equivariant cohomology of $M$.

1. Introduction

Complete smooth complex algebraic varieties $M$ with an almost homogeneous action of a linear algebraic group $G$ have been studied by several authors (see e.g. [AK], [O], [HS]). When the complement $N$ of the open $G$-orbit is homogeneous, the variety is called a two-orbit variety. Akhiezer ([AK]) gave a classification of two-orbit varieties in the case where $N$ is a complex hypersurface, while Feldmüller ([F]) extended this classification under the hypothesis that $N$ has codimension two and $G$ is reductive. Recently, Cupit-Foutou ([C]) gave the full classification when $G$ is reductive.

We will be dealing with a compact K"ahler manifold $M$, which is acted on by a compact Lie group $K$ of isometries, whose complexification $K^C$ acts on $M$ with two orbits. Under the hypothesis that the $K$-action is Hamiltonian with moment map $\mu$, we study the critical set of the squared moment map $|\mu|^2$, applying standard Morse-theoretic results due to Kirwan ([K]) and obtaining information on the cohomology and $K$-equivariant cohomology of $M$.

Our main result is the following.

Theorem 1. Let $M$ be a compact K"ahler manifold and let $K$ be a compact connected Lie group of isometries acting on $M$ in a Hamiltonian fashion. If the complexification $K$ acts on $M$ with two orbits, then

i) $K$ is semisimple and $M$ is simply connected projective algebraic;

ii) the Hodge numbers $h^{p,q}(M) = 0$ if $p \neq q$;

iii) the function $f : M \to \mathbb{R}$ given by $|\mu|^2$, where $\mu : M \to \mathfrak{k}$ is the moment mapping, is a Bott-Morse function; it has only two critical submanifolds given by the closed $G$-orbit $N$, which realizes the maximum of $f$ and by a $K$-orbit $S$, which realizes the minimum. The Poincaré polynomial $P_M(t)$ of $M$ satisfies

$$P_M(t) = t^k \cdot P_N(t) + P_S(t) - (1 + t)R(t),$$

where $k$ is the real codimension of $N$ in $M$ and $R(t)$ is a polynomial with positive integer coefficients. In particular $\chi(M) = \chi(N) + \chi(S)$;

iv) if $\chi(M) > \chi(N)$, then $f$ is a perfect Bott-Morse function, i.e.

$$P_M(t) = t^k \cdot P_N(t) + P_S(t).$$

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v) The $K$-equivariant Poincaré series of $M$ is given by
\[ P^K_M(t) = t^k \cdot P^K_N(t) + P^K_S(t). \]

We remark here that the statement of the Theorem holds for every $K$-invariant Kähler form on $M$, not only for those which are induced by the Fubini-Study Kähler form of a projective space where the manifold might be $K^C$-equivariantly embedded.

In order to prove Theorem 1, we will need some properties of the critical point set of the squared moment map. In [1], it is proved that, given a compact Lie group acting holomorphically on some $\mathbb{P}(V)$ with moment map $\mu$, then two critical points of $||\mu||^2$ which belong to the same $K^C$-orbit lie on the same $K$-orbit; the proof of this fact is essentially based on the explicit expression of the moment map for the $K$-action on the projective space endowed with its canonical Fubini-Study Kähler form. We slightly generalize such result to a class of compact Kähler manifolds which are acted on by a compact Lie group of isometries with moment map $\mu$ and we put the following

**Definition 2.** Given a compact Kähler manifold $(M, \omega)$ acted on by a compact Lie group $K$ of isometries with moment map $\mu$, we say that $\mu$ has the *Ness property* if two critical points for $||\mu||^2$ which are in the same $G$-orbit belong to the same $K$-orbit.

Note that if $0 \in \mu(M)$, then two points in $\mu^{-1}(0)$ which lie in the same $G$-orbit belong to the same $K$-orbit by a well-known fact due to [1], p.97.

The next proposition gives sufficient conditions in order to find invariant Kähler forms with the Ness property.

**Proposition 3.** Let $M$ be a compact complex manifold and let $K$ be a connected, compact semisimple Lie group of holomorphic transformations of $M$ such that $K^C$ has an open orbit in $M$. If $\omega$ is a $K$-invariant Kähler form, then there exists a $K$-invariant Kähler form $\omega'$ which is cohomologous to $\omega$ and whose corresponding moment map has the Ness property.

We will also need the following result, which might have an independent interest (see also [1]).

**Proposition 4.** Let $M$ be a compact Kähler manifold which is acted on by a compact connected Lie group $K$ of isometries in a Hamiltonian fashion with moment map $\mu$. If a point $x \in M$ realizes the maximum of $||\mu||^2$, then the orbit $K \cdot x$ is complex.

We remark here that Cupit-Foutou [1] has deduced from her classification of two-orbit varieties that all such manifolds are spherical, i.e. a Borel subgroup of $G$ has an open orbit. Using the fact that the moment map relative to $K$ separates orbits when $M$ is spherical (see [1], [1]), we see that the critical sets $C_\beta$ of $f = ||\mu||^2$ (see section 2 for basic definitions) consist of single $K$-orbits, making the study of the Ness property unnecessary. We preferred to avoid using the fact that two-orbit varieties are spherical, with the hope of finding a new classification of such manifolds with a symplectic approach.

2. **Basics on moment mappings and Morse theory**

We here briefly recall some basic results due to F.Kirwan [1] about the geometry of the moment map; namely we will first deal with a compact Kähler manifold $M$
which is acted on effectively by a compact connected group of isometries $K$; we recall that $K$ acts automatically by holomorphic transformations and there is a holomorphic action on $M$ of the complex Lie group $G := K^\mathbb{C}$.

A smooth map $\mu : M \to \mathfrak{t}^*$ is said to be a moment map if

$$d\mu_x(X)(\xi) = \omega_x(X, \hat{\xi})$$

for all $\xi \in \mathfrak{t}$ and $X \in T_xM$, where we denote by $\hat{\xi}$ the fundamental Killing vector field on $M$ induced by the one parameter subgroup $\exp(t\xi)$ for $\xi \in \mathfrak{t}$. If $\mu$ is equivariant, then we will call $\mu$ an equivariant moment map; it is known (see [HW]) that an equivariant moment map exists if and only if $K$ acts trivially on the Albanese torus $\text{Alb}(M)$. In the sequel we will identify $\mathfrak{t} \cong \mathfrak{t}^*$ by means of a $Ad(K)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{t}$ and we will consider $\mu$ as a $\mathfrak{t}$-valued map.

The critical points of the function $f : M \to \mathbb{R}$ given by $f(x) := ||\mu||^2$, where $||\cdot||$ is the norm on $\mathfrak{t}$ induced by the scalar product $\langle \cdot, \cdot \rangle$, are given by those points $p \in M$ such that $\mu(p)_p = 0$; the function $f$ is in general not a Bott-Morse function, while for every $\beta \in \mathfrak{t}\setminus\{0\}$ the height function $\mu_\beta := (\mu, \beta)$ is (see [AM]). According to [K], if $\beta \in \mathfrak{t}$, $\beta \neq 0$, we denote by $Z_\beta$ the union of those connected components of the critical set of $\mu_\beta$ on which $\mu_\beta$ assumes the value $||\beta||^2$, while the Morse stratum $Y_\beta$ associated to $Z_\beta$ consists of all points of $M$ whose paths of steepest descent under $\mu_\beta$ converge to points of $Z_\beta$. The subset $Y_\beta$ is a complex submanifold (see [K], p. 89) and the Hessian $H(\mu_\beta)$ restricted to $T_xY_\beta \times T_xY_\beta$ is positive semi-definite, where $x \in Z_\beta$.

In [K] it is proved that $f$, although not nondegenerate in the sense of Bott, induces nevertheless a smooth stratification $\{S_\beta \mid \beta \in B\}$ of $M$ for some appropriate choice of a $K$-invariant metric on $M$. The stratum to which a point of $M$ belongs is determined by the limit set of its positive trajectory under the flow $-\text{grad} f$; the indexing set $B$ is a finite subset of the positive Weyl chamber $\mathfrak{t}_+$. The strata $S_\beta$ are all locally closed $K$-invariant submanifolds of $M$ of even dimension. In [K] it is also proved that the set of critical points of $f$ is the disjoint union of the closed subset $\{C_\beta = K(Z_\beta \cap \mu^{-1}(\beta)) \mid \beta \in B\}$ of $M$, and the image, under $\mu$, of each connected component of the critical set of $f$ is a single adjoint orbit in $\mathfrak{t} \cong \mathfrak{t}^*$. More precisely, for each $\beta \in B$, $C_\beta$ consists of those critical points of $f$ whose image under $\mu$ lies in the adjoint orbit of $\beta$; thus the function $f$ takes the constant value $||\beta||^2$ on $C_\beta$.

The introduction of the submanifolds $S_\beta$ is given in [K] in the general setting of symplectic manifolds. If we restrict our attention on Kähler ones, one can give a different characterization of the submanifolds $S_\beta$: a point $x$ belongs to $S_\beta$ if and only if $\beta$ is the closest point to the origin of $\mu(Gx) \cap \mathfrak{t}_+$ ([K], p. 90).

3. Proof of the main results

We start proving Proposition 3, which will be useful in proving Theorem 1.

Proof of Proposition 3: First of all we note that the Albanese map $\alpha : M \to \text{Alb}(M) := \text{Alb}(M)$ is $K$-equivariant and it induces a surjective homomorphism $\alpha_* : G \to \text{Aut}(\text{Alb}(M)) \cong \text{Aut}(M)$ by [P], where $G = K^\mathbb{C}$; since $G$ is semisimple, we have that $\text{Aut}(M) = \{0\}$, i.e. the first Betti number $b_1(M) = 0$. It then follows from [O] that $M$, being $G$-almost homogeneous, is simply connected algebraic and $h^{2,0}(M) = 0$. This implies that $H^2(M, \mathbb{C}) = H^1(M, \Omega^1_M)$ and therefore
\[ H^2(M, \mathbb{R}) = H^{(1,1)}(M, \mathbb{R}), \] where \( H^{(1,1)}(M, \mathbb{R}) = H^2(M, \mathbb{R}) \cap H^1(M, \Omega^1_M) \). Since \( H^2(M, \mathbb{Z}) \) is a lattice of maximal rank in \( H^2(M, \mathbb{R}) \), there exists a basis \( c_1, \ldots, c_m \) of \( H^2(M, \mathbb{R}) \) with each \( c_i \in H^2(M, \mathbb{Z}) \cap K \), where \( K \) denotes the open Kähler cone in \( H^2(M, \mathbb{R}) \). So, given a Kähler form \( \omega \), we can find positive real numbers \( a_j \) such that \( |\omega| = \sum a_j c_j \); since each \( c_j \) is the Chern class of a positive line bundle, this means that there exist embeddings \( \phi_j : M \to \mathbb{P}(V_j) \) such that \( \omega \) is cohomologous to

\[ \omega' := \sum_j \lambda_j \cdot (\phi_j)^*(\omega_{FS}(\mathbb{P}(V_j))), \]

where \( \omega_{FS}(\mathbb{P}(V_j)) \) denotes the Kähler form of the Fubini-Study Kähler metric on \( \mathbb{P}(V_j) \) and \( \lambda_j \) are real positive numbers. Note that \( K \) acts trivially on \( H^1(M, \mathcal{O}^*) \) since \( b_1(M) = 0 \) implies that \( H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \); this means that the projective embeddings can be taken to be \( G \)-equivariant.

We will now prove that the moment map \( \mu' \) corresponding to \( \omega' \) has the Ness property, following the lines of the proof in [K], p.1303; we will prove almost all her statements in full generality, while we will use the special form of \( \omega' \) only in Lemma 6. Note that the semisimplicity of \( K \) implies the existence and the uniqueness of \( \mu' \).

We now consider \( x \) and \( y \) two critical points that belong to the same \( G \)-orbit. Recall that we can assume that \( \mu'(x), \mu'(y) \neq 0 \). First, note that if \( f' = \|\mu'\|^2 \) then \( f'(y) = f'(x) \). In fact, \( x \in C_\beta \subset S_\beta \), where \( \beta = \mu'(x) \) and, by the \( G \)-invariance of \( S_\beta \), \( y = g \cdot x \in S_\beta \), since \( y \) is critical it must lie in \( C_\beta \) on which \( f' \) is constant. We can also assume that \( \beta \in \mathfrak{t}_+ \). Let now take \( P \) the parabolic subgroup corresponding to \( \beta \); by the Levi decomposition \( P = L \cdot U \), where \( L = K_\beta^C \) is the Levi subgroup of \( P \), and \( U \) its unipotent radical. We have that \( G = KPU \), hence \( y = g \cdot x = klu \cdot x \), for some \( k \in K, l \in L \) and \( u \in U \). Since \( y \) is a critical point of \( f' \) if and only if the \( K \)-orbit of \( K \cdot y \) consists of critical points of \( f' \), it is enough to prove the result for \( y = lu \cdot x \). Using this notation we state

**Lemma 5.** If \( \alpha \in \mathfrak{t} \setminus \mathfrak{k}_e \), then \( \mu'_\alpha(\exp t\mathfrak{i} \alpha \cdot x) \) is a strictly increasing function of \( t \).

**Lemma 6.** Suppose \( x \) and \( y = lu \cdot x \) are critical points of \( f' \) with \( f'(x) = f'(y) \neq 0 \). Then \( u = e_G \), the identity in \( G \).

Assuming the lemmas, we will deduce our claim. By Lemma 5, \( y = l \cdot x \); if \( l \in G_x \) then \( y = x \) and we are done. Otherwise \( y = l \cdot x \) and \( l \notin G_x \). We will show that this implies that \( f'(y) > f'(x) \), a contradiction. Since \( K_\beta^C/K_\beta \) is a symmetric space, there is a geodesic joining the cosets \( [e] \) and \( [l] \). Thus there is a one parameter subgroup \( \exp(t\alpha_1) \), with \( \alpha_1 \in \mathfrak{k}_\beta \), such that \( l = \exp(t\alpha_1)k_1 \) for \( k_1 \in K_\beta \) and \( t_0 \in \mathbb{R}^+ \). It suffices to prove the result for \( y = \exp(t\alpha_1) \cdot x \). Since \( \beta \neq 0 \) we take \( \alpha_0 = -\beta \). We can assume \( \alpha_1 \) to be of length 1 and perpendicular to \( \alpha_0 \), otherwise we substitute \( \alpha_1 \) with its component in \( \mathfrak{k}_\beta \cap (\mathbb{R} \cdot \beta)^\perp \). Complete \( \alpha_0, \alpha_1 \) to an orthonormal basis \( \{\alpha_0, \alpha_1 \ldots \alpha_m\} \) of \( \mathfrak{k} \). We have

\[ f'(z) = \|\mu'(z)\|^2 = \sum_{i=0}^m \mu'^2_{\alpha_i}(z) \]

and

\[ \|\mu'(y)\|^2 = \|\mu'(\exp(t\alpha_1) \cdot x)\|^2 \geq \mu'^2_{\alpha_0}(\exp t\alpha_1 \cdot x) + \mu'^2_{\alpha_1}(\exp t\alpha_1 \cdot x). \]
One can easily prove that $\mu'^2_{\alpha_0}(\exp t\alpha \cdot x)$ is constant and equal to $||\mu'(x)||^2$; on the other hand, $\mu'_\alpha(x) = 0$ and $\mu'_\alpha(\exp t\alpha \cdot x)$ is strictly increasing, by Lemma 6, so that $\mu'^2_{\alpha_1}(\exp t\alpha \cdot x)$ is positive. Hence $f'(y) > f'(x)$. This contradiction proves Proposition 3. □

Now we prove the two lemmas.

Proof of Lemma 4. We will show that $\frac{d}{dt}\mu'_\alpha(\exp(it\alpha) \cdot x) > 0$. Using the definition of $\mu'_\alpha$ we get
\[
\frac{d}{dt}\mu'_\alpha(\exp(it\alpha) \cdot x) = \frac{d}{dt} < \mu'(\exp(it\alpha) \cdot x), \alpha >= < \mu'_{\exp t\alpha \cdot x}(J\alpha), \alpha >
\]
then, using the definition of $\mu'$, we have that the last term is equal to $\omega'_{\exp t\alpha \cdot x}(J\alpha, \alpha)$ and then equal to $g_{\exp t\alpha \cdot x}(\alpha, \alpha)$ which is strictly positive. □

Proof of Lemma 5. Let $y = lu \cdot x$. Using the uniqueness of $\mu'$ we have that
\[
\mu'(x) = \sum_i \lambda_i \mu'_i
\]
where $\lambda_i$ are real positive numbers, that have been already introduced, and $\mu'_i$ are the moment maps corresponding to the forms $(\phi_i)^*(\omega_{F_S}(P(V_i)))$. Using the same arguments as in [5], p.1305, we have that, if $u \neq e$, for each $\mu'_i$ the following strict inequality holds
\[
(3.1) \quad \mu'_{\alpha_0}(lu \cdot x) > \mu'_{\alpha_0}(x).
\]
We compute $\mu'_{\alpha_0}(lu \cdot x)$ and we prove that it is positive: indeed
\[
\mu'_{\alpha_0}(lu \cdot x) = \sum_i \lambda_i < \mu'_i(lu \cdot x), \alpha_0 > = \sum_i \lambda_i \mu'_{\alpha_0}(lu \cdot x)
\]
and, applying the strict inequality (3.1), we have that the last term is strictly greater than $\sum_i \lambda_i \mu'_{\alpha_0}(x) = \mu'_{\alpha_0}(x) = ||\mu'(x)|| > 0$. Hence
\[
f'(y) = ||\mu'(lu \cdot x)||^2 > \mu'^2_{\alpha_0}(lu \cdot x) > \mu'^2_{\alpha_0}(x) = ||\mu'(x)||^2 = f'(x)
\]
so $u$ must be $e$ and Lemma 5 is proved. □

We next prove Proposition 4.

Proof of Proposition 4. Let $\beta = \mu(x)$, which we can suppose to lie in $t_+$. Then $x \in C_\beta = K \cdot (Z_\beta \cap \mu^{-1}(\beta))$. We now claim that $Z_\beta = \mu^{-1}(\beta)$. Indeed, if $p \in Z_\beta$, then $\mu_\beta(p) = ||\beta||^2$ and
\[
||\beta||^2 \leq \langle \mu(p), \beta \rangle \leq ||\mu(p)|| \cdot ||\beta|| \leq ||\beta||^2,
\]
and therefore $\mu(p) = \beta$, i.e. $p \in \mu^{-1}(\beta)$). Viceversa, if $p \in \mu^{-1}(\beta)$, then $||\mu(p)||^2$ is the maximum value of $f := ||\mu||^2$ and therefore $\beta p = 0$; moreover $\mu_\beta(p) = ||\beta||^2$ and therefore $p \in Z_\beta$. This implies that $\mu^{-1}(\beta)$ is a complex submanifold and that $C_\beta = K \cdot \mu^{-1}(\beta)$. We now claim that $S_\beta = C_\beta$. Indeed, if $\gamma_t(q)$ denotes the flow of $-\text{grad}(f)$ through a point $q$ belonging to the stratum $S_\beta$, then $\gamma_t(q)$ has a limit point in the critical subset $C_\beta$: since $f(\gamma_t(q))$ is non-increasing for $t \geq 0$ and $f(C_\beta)$ is the maximum value of $f$, we see that $f(\gamma_t(q)) = ||\beta||^2$ for all $t \geq 0$, that is $S_\beta \subseteq C_\beta$ and therefore $S_\beta = C_\beta$. 5
This implies that $C_\beta = S_\beta$ is a smooth complex submanifold of $M$ (see [K], p. 89) and for every $y \in \mu^{-1}(\beta)$, we have
\[ T_y S_\beta = T_y(K \cdot y) + T_y(\mu^{-1}(\beta)). \]
Now, if $v \in T_y(\mu^{-1}(\beta))$, then $v = Jw$ for some $w \in T_y(\mu^{-1}(\beta))$ and for every $X \in \mathfrak{g}$ we have
\[ 0 = \langle d\mu_y(w), X \rangle = \omega_y(w, \dot{X}_y) = \omega_y(Jv, \dot{X}_y) = g_y(v, \dot{X}_y), \]
meaning that $T_y(\mu^{-1}(\beta))$ is $g$-orthogonal to $T_y(K \cdot y)$. Since both $S_\beta$ and $\mu^{-1}(\beta)$ are complex, this implies that $K \cdot y$ is a complex orbit. \hfill \Box

**Proof of Theorem 1.** We will first prove that $K$ is semisimple. Let $\mathfrak{g}$ be the Lie algebra of the center of $\mathfrak{k}$, the Lie algebra of $K$, and let $X \in \mathfrak{g}$ be a vector such that $\exp(X)$ generates the torus $Z^0(K)$. We will show that $X$ acts trivially on $M$; let $C$ denote the zero set of $X$.

If $\mu : M \to \mathfrak{g}$ denotes a moment mapping for the $K$-action, then $\phi : \pi \circ \mu : M \to \mathbb{R} \cdot X$, where $\pi : \mathfrak{g} \to \mathbb{R} \cdot X$ denotes the orthogonal projection w.r.t. an $\text{Ad}(K)$-invariant scalar product on $\mathfrak{g}$, is a function such that $d\phi_X(Y) = \omega(Y, X)$ for all $Y \in T_x M$ and all $x \in M$, where $\omega$ is the Kähler form of $M$. Since critical points of $\phi$ are exactly the zeros of $X$ in $M$, we have that there are at least two connected components of the zero set of $X$; on the other hand if $x \in \Omega$ is a point where $X$ vanishes and $\Omega = G/H$ is the open $G$-orbit, then $\mathfrak{g} \cap \mathfrak{h}$ and therefore $Z^0(K)$ acts trivially on $\Omega$, hence on the whole $M$. But then the zero set of $X$ lies in the closed orbit $N$, which is connected, so that $C = N$, a contradiction.

Arguing as in the proof of Proposition 3, we see that the first Betti number $b_1(M) = 0$. It then follows from [Q] that $M$, being $G$-almost homogeneous, is simply connected and projective algebraic.

We now consider the function $f := ||\mu||^2$ and its critical set $C$. We first note that $C$ does not coincide with $M$, i.e. the function $f$ is not constant; indeed otherwise, by Proposition 4, every $K$-orbit would be complex, contrary to the uniqueness of the closed $G$-orbit.

By Proposition 4, we see that the closed orbit $N$ consists exactly of those points at which the function $f$ realizes the maximum value; we denote by $\beta_{\text{max}}$ the point $\mu(N) \cap t_+$. We now denote by $\beta_{\text{min}}$ the point in the convex polytope $\mu(M) \cap t_+$ which is the closest point to the origin; then $||\beta_{\text{min}}||^2$ is the minimum value of $f$. Moreover we know ([K]) that a point $x$ belongs to the stratum $S_\beta$ for some $\beta \in t_+$ if and only if $\beta$ is the closest point to the origin of $\mu(G \cdot x) \cap t_+$; it then follows that the whole orbit $G \cdot x$ belongs to $S_{\beta_{\text{min}}}$ and therefore there are exactly two critical values of $f$, that is $||\beta_{\text{max}}||^2$ and $||\beta_{\text{min}}||^2$; moreover $S_{\beta_{\text{min}}} = \Omega$, where $\Omega$ is the open $G$-orbit.

Since $M = S_{\beta_{\text{max}}} \cup S_{\beta_{\text{min}}}$, we see that a critical point of $f$ must lie either in the closed orbit $N$ or in $C_{\beta_{\text{min}}}$. We here prove that $C_{\beta_{\text{min}}}$ consists of a single $K$-orbit. Indeed we have seen in Proposition [K] that there exists a Kähler form $\omega'$ on $M$ which is cohomologous to $\omega$ and whose corresponding moment map has the Ness property. Since $\omega_t = tw' + (1 - t)\omega$, $t \in [0, 1]$, is a homotopy by symplectic forms connecting $\omega$ with $\omega'$, by the equivariant Moser homotopy Theorem ([MS], p. 91) there exists a $K$-equivariant symplectomorphism
\[ F : (M, \omega) \to (M, \omega'). \]
Now note that the (unique) moment map corresponding to $\omega$ is $\mu = \mu' \circ F$, hence the critical points of $f$ are taken by $F$ to critical points of $f'$. Using the same arguments as above, we see that $\Omega = S_{\beta_{\min}}'$, where $S'$ denotes the stratum w.r.t. $\mu'$; hence the Ness property implies that the minimal critical set for $f'$ consists of a single $K$-orbit and, using the $K$-equivariance of $F$, we conclude that also $C_{\beta_{\min}}$ consists of a single $K$-orbit. We have thus proved the following

**Proposition 7.** The inverse images $\mu^{-1}(K; \beta_{\max})$ and $\mu^{-1}(K; \beta_{\min})$ consist exactly of two $K$-orbits and the first one is the closed $G$-orbit.

We will call $S$ the $K$-orbit such that $\mu(S) = K; \beta_{\min}$. We now consider a maximal torus $T$ inside $K$ and prove the following easy Lemma.

**Lemma 8.** The fixed point set $M^T$ is contained in the critical set of $f$.

**Proof.** Indeed if $x \in M^T$, then $T \subseteq K_x$ and by the $K$-equivariance of $\mu$, we have that $\text{Ad}(T)\mu(x) = \mu(x)$. Since $T$ is maximal, we have that $\mu(x) \in t$, where $t$ is the Lie algebra of $T$, and therefore $\mu(x) \in \xi_x$.

From this we see that the set $M^T$ is finite, because a maximal torus of $K$ fixes at most a finite number of points in a $K$-homogeneous space. Using [CL], we deduce that the Hodge numbers $h^{p,q} = 0$ for $p \neq q$.

We have now to prove that the function $f$ is a Bott-Morse function and the rest of the statement (iii) will follow from standard Morse theory.

Let $Z_{\beta_{\min}}$ be the subset of $Z_{\beta}$ consisting of those points $x \in Z_{\beta}$ such that the limit points of the path of steepest descent from $x$ for the function $\|\mu - \beta\|^2$ on $Z_{\beta}$ lie in $Z_{\beta} \cap \mu^{-1}(\beta)$, and let $Y_{\beta_{min}}$ be the inverse image of $Z_{\beta_{min}}$ under the retraction $p_{\beta}: Y_{\beta} \to Z_{\beta}$. Then $Y_{\beta_{min}}$ is an open subset of $Y_{\beta}$ and retracts to $Z_{\beta_{min}}$.

With this notation we recall (see [K]) that the $G$-open orbit $\Omega$ has the structure of a fiber bundle $G \times_{P_{\beta}} Y_{\beta_{min}}$, where $P_{\beta}$ is the parabolic subgroup of $G$ such that $G/P_{\beta} = K/K_{\beta}$ and we recall also that $T_pY_{\beta} \cap T_p(K \cdot p) = T_p(K_{\beta} \cdot p)$ (see Lemma 4.10, p. 48 in [K]).

We start showing that the Hessian of $f$ at a critical point $p$ belonging to the minimum orbit $S$ is non degenerate; we can suppose that $\mu(p) = \beta$, where we have put $\beta = \beta_{\min}$. We have

$$\dim K/K_{p} + \dim Y_{\beta} - \dim (T_p Y_{\beta} \cap T_p(K \cdot p)) =$$

$$= \dim K/K_{p} + \dim Y_{\beta} - \dim (K_{\beta}/K_{p}) = \dim K/K_{\beta} + \dim Y_{\beta} = \dim M,$$

hence

$$T_p(K \cdot p) + T_pY_{\beta} = T_pM.$$

If we consider $W_p := (T_p(K_{\beta} \cdot p))^\perp \cap T_pY_{\beta}$, then $W_p$ is a complement of $T_p(K \cdot p)$ in $T_pM$; we will compute the Hessian of $f$ on vectors $X \in W_p$, namely we will show that $H(f)_p(X, X) > 0$ if $X \in W_p$, $X \neq 0$.

We fix an orthonormal basis $\{X_1, \ldots, X_l\}$ of the Lie algebra $\xi$ w.r.t. the invariant scalar product $\langle \cdot, \cdot \rangle$; then we can write for all $x \in M$, $\mu(x) = \sum_i \mu_i(x) X_i$. Moreover we denote by $M$ the vector field on $M$ given by $M_x = \mu(x)$ for $x \in M$, where for any $X \in \xi$ we denote by $\tilde{X}$ the induced Killing vector field on $M$. Then we have

$$df_x(X) = 2\langle d\mu(x)(X), \mu(x) \rangle = 2\omega_x(X, M_x).$$

It follows that

$$H(f)_x(X, X) = \omega_x(X, \nabla_X M),$$
where $\nabla$ denotes the Levi Civita connection of the Kähler metric $g$. We then have that

$$H(f)_p(X, X) = 2 \sum_i \omega_p(X, X(\mu_i)\hat{X}_i + \mu_i(p)\nabla_X \hat{X}_i).$$

Now we observe that

$$\omega_p(X, \hat{X}_i) = \langle d\mu_X(X), X_i \rangle = X(\mu_i)$$

and therefore

$$H(f)_p(X, X) = 2 \sum_i \omega_p(X, \hat{X}_i)^2 + \omega_p(X, \nabla_X \hat{\beta}).$$

If $\mu_\beta$ denotes as usual the height function $(\mu, \beta)$, we see that $H(\mu_\beta)_p(X, X) = 2\omega_p(X, \nabla_X \hat{\beta})$, so that

$$H(f)_p(X, X) = 2 \sum_i \omega_p(X, \hat{X}_i)^2 + 2H(\mu_\beta)_p(X, X).$$

We now distinguish two cases, whether $\beta = 0$ or $\beta \neq 0$.

If $\beta = 0$, then $H(\beta)_p(X, X) = 2 \sum_i \omega_p(X, \hat{X}_i)^2$ for all $X \in T_p(K \cdot p)^\perp$ and therefore $X$ belongs to the nullity of $H(\beta)_p$ if and only if $JX$ is orthogonal to the tangent space $T_p(K \cdot p)$. But $\beta = 0$ implies that $J(T_p(K \cdot p))$ is contained in the normal space $T_p(K \cdot p)^\perp$; this together with the fact that $p$ belongs to the open $K$-orbit implies that $J(T_p(K \cdot p)) = T_p(K \cdot p)^\perp$ and therefore $X = 0$. This shows that $H(f)_p$ is non degenerate.

If $\beta \neq 0$, then we choose $X \in W_p$. Denote by $V$ the tangent space $T_p(K \cdot p)$. Note that $\omega_p(X, \hat{X}_i) = 0$ if and only if $d\mu_p(X) = 0$ i.e. if and only if $X \in (J\beta)^\perp$ and that $H(\mu_\beta)_p(X, X) = 0$ if and only if $X \in T_pZ_\beta$. We shall prove that, if $H(f)_p(X, X) = 0$, then $X$ must be orthogonal to $V$; hence, since $V + JV = T_pM$, $X$ must be 0. In fact, recall that $p$ is a critical point for $\mu_\beta$ and we have $\hat{\beta}_p = 0$. We consider the skew operator $\nabla \hat{\beta} : T_pM \rightarrow T_pM$; since $e^{t\nabla \hat{\beta}} = d(exp(t\hat{\beta}))_p$, we have that $\nabla \hat{\beta}$ maps $V$ into itself and $T_pZ_\beta = Ker\nabla \hat{\beta}$. The tangent space $V$ then splits as the direct sum of $Ker(\nabla \hat{\beta})_{|V}$ and its orthogonal complement $V'$. Note that $\nabla \hat{\beta}_{|V'}$ is surjective on $V'$, hence, if $Y \in V'$ we have $Y = \nabla \hat{\beta}(Y')$ for some $Y' \in V'$. Using that $\nabla \hat{\beta}$ is skew, we argue that $X \perp V'$. Finally note that $Ker(\nabla \hat{\beta})_{|V'} = T_p(K \cdot p)$, hence, since $X \in W_p$, it is orthogonal to $T_p(K \cdot p)$ and we get our claim. 

The statements in (iii) and (v) follow from the general theory in [K]. As for (iv), we note that if $\chi(M) > \chi(N)$, then $S$ is a compact homogeneous space of positive Euler characteristic; therefore all odd Betti numbers of $S$ vanish (see [BH]) and therefore $f$ is perfect.

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