Conductivity activation energy for bilayer heterostructures at integer filling factors

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Integer filling factors of a hetero-structure 2D electron gas (2DEG) in an external magnetic field are special ones because a huge degeneracy of the ground state is gone here. It justifies the Hartree-Fock approximation with the accuracy limited only by normally a small parameter: \( V_{int}/\hbar \omega_0 \), where \( V_{int} \) is the energy of the Coulomb interaction and \( \omega_0 \) is the frequency of the cyclotron resonance. Such an approach predicts the ground state of a single layer 2DEG at \( \nu = 1 \) to be a ferromagnet with the degenerate total spin orientation. The elementary excitations of 2DEG are electron-hole pairs or excitons and in the close binding limit of vanishing momentum they transform into the elementary excitations of a ferromagnet - spin-waves. The latter are gapless \(^1\) and do not interact with each other \(^2\) if Zeeman energy is neglected - the two consequences of the Goldstone theorem. In the limit of large momentum the electron and the hole of an exciton are well separated and they become the elementary charged excitations. In the Ref.\(^3\) it was shown that a special topological spin texture in a 2D ferromagnet called a skyrmion \(^4\) carries a unit charge while costing only half the exciton’s energy to appear.

The case of a bilayer 2DEG turned out to be a more rich one. The Hartree-Fock approximation does not apply here except for two limiting cases. The first one is the case of well separated layers which is a common setup in the experiment \(^5\) \(^6\) and where, theoretically, one starts from the two single layer ferromagnets and makes the perturbation expansion in powers of interlayer
interactions \[\mathcal{I}\]. And the second one is the symmetric case defined in such a way that one can freely rotates an electron spinor in both layer and spin spaces. The latter requires to approximate the Coulomb interaction by its symmetric part and to neglect all symmetry breaking fields like Zeeman energy. The first attempts in this direction dealt with the case of filling factor \(\nu = 1\) and relied heavily on the assumption of a saturated spin polarization of electrons \[\mathcal{I}, \mathcal{I}\). This symmetric approximation turned out to be useful to determine the exciton energy in bilayer \[\mathcal{I}\]. Recent works Refs.\[\mathcal{I}, \mathcal{I}\) specialize to the bilayer heterostructure case \(\nu = 2\), employ the Hartree-Fock approximation and predict a phase diagram that features three phases: the ferromagnetic, the canted antiferromagnetic and a special spin-singlet phase. In this paper we reproduce the phase diagram of the Refs.\[\mathcal{I}, \mathcal{I}\) isolating the symmetric and the symmetry breaking parts of the Hamiltonian in a consistent way. Our approach reveals the Hartree-Fock phase diagram to be indeed exact in the limit \(V_{\text{anis}}/V_{\text{sym}} \rightarrow 0\), where \(V_{\text{sym}}\) is the \(SU(4)\)-symmetric part of the bilayer Hamiltonian whereas \(V_{\text{anis}}\) is anisotropy interactions that reduce the bilayer Hamiltonian symmetry to \(SU(2) \otimes SU(2)\). We prove the stability of all phases with respect to long-range spatial perturbations.

Our new results concern the energy of topological excitations in bilayers. We find that low-energy excitations over the bilayer ground state is governed by the \(U(4)/U(\nu) \otimes U(4 - \nu)\) coset non-linear Sigma Model. We identify charged excitations in the bilayer with skyrmions or topological excitations of this non-linear Sigma Model \[\mathcal{I}, \mathcal{I}, \mathcal{I}\). We calculate the skyrmion energy gap to vary dramatically over the bilayer parameter space and we find a sharp dip of this gap in the canted antiferromagnetic phase of the bilayer. Our work was motivated by recent measurements of the diagonal conductivity activation energy \[\mathcal{I}\). In this paper we suggest to identify the experimental activation energy with the energy gap of skyrmion, and we have got a qualitative agreements with the results of Ref.\[\mathcal{I}\) along this line.

**Hamiltonian of 2DEG bilayer**

The electronic Hamiltonain of a 2DEG in a confining potential \(V(\vec{p})\) and in an external magnetic field \(H\) consists of a one-particle part as well as a
Figure 1: Schematic view of a confining potential $V(\xi)$ in a typical Bilayer setup

Coulomb interaction part:

$$H = \int \psi_\alpha^+(\vec{\rho}) \left( \frac{1}{2m} \left[ -i\vec{\nabla} + \vec{A}(\vec{\rho}) \right]^2 + V(\vec{\rho}) - |g|\mu_B H \sigma_\alpha^z \right) \psi_\beta(\vec{\rho}) d^3\vec{\rho} + \frac{1}{2} \int \int \frac{e^2}{|\vec{\rho} - \vec{\rho}'|} \psi_\alpha^+(\vec{\rho}) \psi_\beta^+(\vec{\rho}') \psi_\beta(\vec{\rho}') \psi_\alpha(\vec{\rho}) d^3\vec{\rho} d^3\vec{\rho}',$$

(1)

where $\alpha, \beta = \pm$ are spin indices and thereafter a sum over repeated indices is implied. We use such units that $\hbar = 1$, $e = c$ and $H = B = 1$. The latter implies that all distances can be expressed in terms of the so-called magnetic length: $l_H = \sqrt{\hbar/eH} = 1$. We split three coordinates $\vec{\rho}$ into a perpendicular to the layer coordinate $\xi$ and two in-plane coordinates $\vec{r} = (x, y) = (z, \bar{z})$. We assume that the confining potential is uniform over the plane: $V(\vec{\rho}) = V(\xi)$, and represents a double well structure in the transverse direction as shown on the Fig.1, with the two wells being separated by the distance $d$. We use only two eigen functions: the lowest energy symmetric $\chi_S(\xi)$ and antisymmetric $\chi_A(\xi)$, from a set of one-electron eigen functions in the confining potential $V(\xi)$ and we expand an electron second-quantized operator in terms of these two eigen functions:
\[
\psi_\alpha(\vec{r}) = \sum_{\tau,n,p} \chi_\tau(\xi) \phi_{n,p}(\vec{r}) c_{n\alpha \tau p}, \quad (2)
\]
where \(c_{n\alpha \tau p}^+\) and \(c_{n\alpha \tau p}\) are electron creation and annihilation operators, \(\phi_{n,p}(z\bar{z})\) is an electron wave function number \(p\) in the Landau gauge in the \(n\)'s Landau Level, the index \(\tau = 1,2\) being the layer index and the layer wave functions read:
\[
\chi_{1,2}(\xi) = \frac{\chi_S(\xi) \pm \chi_A(\xi)}{\sqrt{2}}. \quad (3)
\]
We restrict our model to the case of a sufficiently strong magnetic field, such that the cyclotron energy \(\hbar \omega_0\) dominates over the Coulomb, Zeeman and the level splitting: \(E_A - E_S\), energies. Thus, we specialize to the lowest Landau Level and retain only the term \(n = 0\) in (2).

Plugging the wave function (2) into the Hamiltonian (1) we find a 2DEG Hamiltonian as:
\[
H = \frac{1}{2m} c_{n\alpha \tau p}^+ c_{n\alpha \tau p} - c_{n\alpha \tau p}^+ \left( t \tau^x_{\tau_1 \tau_2} + \mu^z \tau^z_{\tau_1 \tau_2} \right) c_{n\alpha \tau p} - |g| \mu_B H c_{n\alpha \tau p} \sigma^z_{\alpha \beta} c_{n\alpha \tau p} + \\
+ \frac{1}{2} \sum_{p_1 p_4} \int \int d^2 \vec{r} d^2 \vec{r}' \, V^{\tau_1 \tau_4}_{\tau_2 \tau_3}(\vec{r} - \vec{r}') \phi_{p_1}^*(\vec{r}) \phi_{p_2}^*(\vec{r}') \phi_{p_3}(\vec{r}) \phi_{p_4}(\vec{r}) \\
c_{n\alpha \tau p_1} c_{n\alpha \tau p_2} c_{n\alpha \tau p_3} c_{n\alpha \tau p_4}, \quad (4)
\]
where we have defined a hopping constant:
\[
t = \frac{1}{2} \int \int d^2 \vec{r} d\xi \, \phi_p^*(\vec{r}) \chi_{\tau_1}(\xi) \tau^x_{\tau_1 \tau_2} V(\vec{r}) \chi_{\tau_2}(\xi) \phi_p(\vec{r}), \quad (5)
\]
as well as an external electrostatic potential created by an asymmetric gate charge:
\[
\mu^z = \frac{1}{2} \int \int d^2 \vec{r} d\xi \, \phi_p^*(\vec{r}) \chi_{\tau_1}(\xi) \tau^z_{\tau_1 \tau_2} V(\vec{r}) \chi_{\tau_2}(\xi) \phi_p(\vec{r}), \quad (6)
\]
whereas the Coulomb interaction matrix reads:
\[
V^{\tau_1 \tau_4}_{\tau_2 \tau_3}(\vec{r} - \vec{r}') = \int \int \frac{\chi_{\tau_1}(\xi) \chi_{\tau_2}(\xi') \chi_{\tau_3}(\xi') \chi_{\tau_4}(\xi)}{\sqrt{(\xi - \xi')^2 + (\vec{r} - \vec{r}')^2}} \, d\xi d\xi'. \quad (7)
\]
We use notations: \(\tau^x, \tau^y\) and \(\tau^z\), for the Pauli matrices in the layer space whereas we use notations: \(\sigma^x, \sigma^y\) and \(\sigma^z\), for the Pauli matrices in the spin space. The hopping constant can be related to the splitting of the symmetric
and the antisymmetric levels: \( t = E_A - E_S \). The electrostatic potential \( \mu^z \), which can be viewed as a difference between the chemical potentials in the two layers, breaks down the symmetry between the two wells of \( V(\xi) \) potential. This term appears naturally when a single gate is fabricated to control the electron density in the bilayer. In the limit \( d \to 0 \), \( \mu^z \) vanishes too, whereas in the limit of large layer separation: \( d \to \infty \), \( \mu^z \to \infty \) and electrons reside only on the layer adjacent to the gate. We assume that the energy of a capacitor formed by the two layers is much lower than the characteristic Coulomb energy \( e^2/\kappa l_H^3 \), per area, where \( \kappa \) is the dielectric constant. We note the invariance of the Coulomb energy: \( (\xi, \xi') \leftrightarrow (-\xi, -\xi') \) under the following transformations: \( \tau_1 \leftrightarrow \tau_4 \), \( \tau_2 \leftrightarrow \tau_3 \) as well as \( (\tau_1 \tau_4) \leftrightarrow (\tau_2 \tau_3) \). To fully exploit these symmetries we cast the Eq.7 into a more suitable representation:

\[
V^{\tau_1 \tau_4}(\vec{r} - \vec{r}') = V^{\mu\nu}(\vec{r} - \vec{r}')\tau^{\mu\nu}\tau^{\tau_1 \tau_4},
\]

where \( \tau^0 \) is the unit matrix, \( V^{\mu\nu} \) is a \( 3 \times 3 \) symmetric interaction matrix with indices \( \mu, \nu \) running over a set \( (0, z, x) \). If there is a symmetry of the Coulomb interaction under an exchange of layers: \( (\xi, \xi') \leftrightarrow (-\xi, -\xi') \) and \( 1 \leftrightarrow 2 \) then it restricts further values of the interaction matrix: \( V^{0z} = 0 \) and \( V^{zz} = 0 \). But in the presence of a gate asymmetry we shall keep the matrix element: \( V^{0z} \). Therefore, the Coulomb interaction matrix for symmetric bilayer 2DEG depends on four parameters: \( V^{00} > 0, V^{0z}, V^{zz} > 0 \). We note also, that \( V^{0x} \sim \chi, V^{xx} \sim \chi^2 \), whereas \( V^{zz} \sim d^2/|z|^3 \) as \( |z| \to \infty \). In the following we shall neglect \( V^{xx} \) matrix element.

Next, we split the total bilayer Hamiltonian \( (4) \) into two parts: the first one contains a dominant Coulomb energy term:

\[
H^{\text{sym}} = \frac{1}{2m} \sum_{\alpha \tau \alpha' \tau'} c_{\alpha \tau p}^+ c_{\alpha' \tau' q} + \frac{1}{2} \int \frac{d^2 \vec{q}}{(2\pi)^2} V^{00}(\vec{q}) N(\vec{q}) N(-\vec{q}),
\]

where \( V^{\mu\nu}(\vec{q}) \) is the Fourier transform of \( V^{\mu\nu} (\vec{r}) \) multiplied by a factor \( \exp(-\vec{q}^2/4) \) and the electron density operator reads:

\[
N(\vec{q}) = \sum_p c_{\alpha \tau p}^+ c_{\alpha \tau p - q} \exp -iqx(p - q_x/2).
\]

This part of the Hamiltonian is invariant under uniform rotations from the \( SU(4) \) Lee group in the combined spin and layer space. Every of its eigen energy is hugely degenerate. Given any eigen state \( |\Psi\rangle_0 \) a set of related
eigen states can be generated by applying rotations: $|\Psi\rangle = U|\Psi\rangle_0$, where $U \in SU(4)$. For Landau level filling factor $\nu = 1$, $\nu = 2$ and $\nu = 3$ we assume that the the bilayer ground state is uniform over $p$-orbitals:

$$\Psi = \prod_{i=1}^{\nu} \prod_p c_{\alpha_i \tau_i p}^+ |\text{empty}\rangle,$$

and we prove in the next section that this state is stable with respect to long-range spatial perturbations. One can easily check by inspection that any such wave-function (11) represents an eigen function of the $H^{\text{sym}} (9)$. The remaining few terms in the Hamiltonian (4) are treated like perturbations:

$$H^{\text{anis}} = -c_{\alpha_1 \tau_1 p}^+ \left( t^{x \tau_1 \tau_2} + \mu^z \tau_{1 \tau_2} \right) c_{\alpha_2 \tau_2 p} - |g| \mu_B H c_{\alpha \tau p}^+ \sigma_{\alpha \beta}^z c_{\beta \tau p} +$$

$$+ \frac{1}{2} \int \frac{d^2 \vec{q}}{(2\pi)^2} V^{\mu \nu}(\vec{q}) T^\mu(\vec{q}) T^\nu(\vec{-q}),$$

where (see e.g. [9])

$$T^\mu(\vec{q}) = \sum_p c_{\alpha \tau_1 p}^+ \tau_{1 \tau_2}^\mu c_{\alpha \tau_2 p - \vec{q}} \exp -i \frac{q_x}{2} \left( \vec{p} - \frac{\vec{q}}{2} \right)$$

with $(\mu \nu) \neq (00)$. The Hamiltonian (12) breaks down the $SU(4)$ symmetry but it is still invariant under separate rotations in the spin and layer space: $SU(2) \otimes SU(2)$. We shall call this part of the Hamiltonian the anisotropy Hamiltonian. It lifts the degeneracy of eigen states of the $SU(4)$-symmetric Hamiltonian (3). An important point to note here is that a splitting of energy levels is determined by matrix elements of weak anisotropy Hamiltonian (12) truncated to a linear space of the symmetric Hamiltonian (3) level degeneracy. There are no Fermi-liquid type renormalizations of the constants of the anisotropy Hamiltonian (12) due to the $SU(4)$-symmetric Hamiltonian (3). In other word the mean-field Hartree-Fock approach is perfect for the $\nu = 1$, $\nu = 2$ and $\nu = 3$ cases.

Our guiding analogy in treating the total bilayer Hamiltonian (3, 12) lies in the theory of magnetism. We will see below that there exists a local order parameter: $Q$, very much like magnetization. And we aim to express the total bilayer Hamiltonian (3, 12) in terms of this order parameter $Q$. The exchange-like Hamiltonian (3) has to be expanded in powers of spatial variations of order parameter $Q(\vec{r})$ with the second power of gradients being
the important contribution, whereas only locally homogeneous \( Q \) has to be retain in the anisotropy Hamiltonian (12). In the next section we carry out the first step whereas in the next-to-next section we transform the anisotropy energy.

**SU(4) Symmetric Case**

In this section we specialize to the \( SU(4) \)-symmetric part of the bilayer 2DEG Hamiltonian (9) which is invariant under the global rotations of a four component electron spinor by the \( 4 \times 4 \) matrix \( U \) from the \( SU(4) \) Lee group. We find it useful the ground state of operators \( c \) to be the reference state. Any non-homogeneous state is then generated by rotation: \( U(t, \vec{r}) \). And the action of bilayer is some functional of it:

\[
S[U(t, \vec{r})] = -i \text{ tr log } \int \mathcal{D}c^+(t)\mathcal{D}c(t) \exp \left( i \int L dt \right)
\]

where the symmetric Lagrangian of bilayer 2DEG reads:

\[
L = \int c^+_{\alpha_1} \left[ i \frac{\partial}{\partial t} - \frac{1}{2m} \left( -i \vec{\nabla} + \vec{A}_0 + \vec{\Omega}_{\alpha_2\gamma_2} \right)^2 \right] c_{\beta_2} \, d^2\vec{r} + \\
+ \int c^+_{\alpha_1} \Omega^t_{\alpha_2\gamma_2} c_{\beta_2} \, d^2\vec{r} - \frac{1}{2} \int \frac{d^2\vec{q}}{(2\pi)^2} V^{00}(\vec{q}) N(\vec{q}) N(-\vec{q}),
\]

where \( \vec{A}_0 = (0, A^y) \) is the vector potential of the external magnetic field and

\[
\Omega^t = i U^+ \frac{\partial}{\partial t} U, \quad \vec{\Omega} = -i U^+ \vec{\nabla} U.
\]

In the limit of slow spatial variations of rotation \( U(t, \vec{r}) \) this functional has an expansion in powers of \( \Omega \). In this context the functional (14) is called an effective low-energy Goldstone Action, and we are going to find it in this section. All calculations follows step in step those done in Ref. [12] for the case of a single layer, and here we emphasized only the points of difference.

Calculations of the Goldstone Action can be carried through for three filling factors of the bilayer 2DEG: \( \nu = 1, \nu = 2 \) and \( \nu = 3 \), at once. We start with choosing the reference state: i) \( n = 1 \), one electron with spin up fills every orbital \( p \) of the lowest Landau Level of the first layer: \( |+\rangle_1 \); ii)
n = 2, two electrons - one with spin up on the first layer and the other with spin down on the second layer - fill every orbital \( p \) the lowest Landau Level: \(|+\rangle_1|->\rangle_2\); iii) three electrons - one with spin down on the first layer and the two other with spin up and down on the second layer - fill every orbital \( p \) the lowest Landau Level: \(|-\rangle_1|+\rangle_2 |-\rangle_2\). The case iii) reduces to the case i) if one makes the electron-hole transformation.

The one-electron Green functions defined for the reference state of the Hamiltonian (15) in the homogeneous limit: \( \vec{\Omega} = 0 \), reads:

\[
G^0_{\alpha\tau_1,\beta\tau_2}(\epsilon) = \frac{1}{2} \left( \Sigma^0_{\alpha\beta,\tau_1\tau_2} + \Sigma^z_{\alpha\beta,\tau_1\tau_2} \right) \frac{1}{\epsilon + E_0 - \mu - i0^+} + \frac{1}{2} \left( \Sigma^0_{\alpha\beta,\tau_1\tau_2} - \Sigma^z_{\alpha\beta,\tau_1\tau_2} \right) \frac{1}{\epsilon - \mu + i0^+},
\]

where \( \mu \) is the chemical potential,

\[
\Sigma^0_{\alpha\beta,\tau_1\tau_2} = \sigma^0_{\alpha\beta} \tau^0_{\tau_1\tau_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

and in the case \( n = 1, 3 \):

\[
\Sigma^z_{\alpha\beta,\tau_1\tau_2} = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

whereas in the case \( n = 2 \):

\[
\Sigma^z_{\alpha\beta,\tau_1\tau_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

The effective action can be written as:

\[
S = S_0 + S_2,
\]

where

\[
S_0 = i \text{ tr } \log \frac{G}{G_0}
\]
and $S_2$ is conveniently represented in terms of diagrams in the Fig. 2. As it is explained in the Ref. [12], the first order perturbation correction to the Green function: $\delta G = G - G_0$, contains an electron propagation on the first excited Landau Level. Thus, we also need the bare Green function of an excited electron on this level:

$$G^{1}_{\alpha\tau_1,\beta\tau_2}(\epsilon) = \frac{1}{2} \left( \frac{1}{\epsilon - 1/m + E_1 - \mu + i0^+} \right) \left( \sum_{\alpha\beta,\tau_1\tau_2}^0 + \sum_{\alpha\beta,\tau_1\tau_2}^z \right) + \frac{1}{2} \left( \frac{1}{\epsilon - 1/m - \mu + i0^+} \right) \left( \sum_{\alpha\beta,\tau_1\tau_2}^0 - \sum_{\alpha\beta,\tau_1\tau_2}^z \right),$$  \hspace{1cm} (23)

where

$$E_0 = 2E_1 = \sqrt{\frac{\pi}{2}} \frac{e^2}{\kappa l_H}.$$  \hspace{1cm} (24)

As it is explained in the Ref. [12] the gradient vector field enters the following terms in the Hamiltonian

$$H_1 = \frac{1}{2m} \int c^{\dagger}_{\alpha\tau_1} \left( \Omega^+_{\alpha\beta,\tau_1\tau_2} \tilde{H}_- + \Omega^-_{\alpha\beta,\tau_1\tau_2} \tilde{H}_+ \right) c_{\beta\tau_2} \, d^2\mathbf{r},$$  \hspace{1cm} (25)

$$H_2 = \frac{1}{2m} \int c^{\dagger}_{\alpha\tau_1} \left[ (\tilde{\Omega}_z^2)_{\alpha\beta,\tau_1\tau_2} - i (\partial_{\mu} \tilde{\Omega}_\mu)_{\alpha\beta,\tau_1\tau_2} \right] c_{\beta\tau_2} \, d^2\mathbf{r},$$  \hspace{1cm} (26)

Figure 2: Second order Hartree-Fock diagrams
where the operators $\hat{\Pi}_\pm$ shift an electron between the adjacent Landau Levels:

$$
\hat{\Pi}^-_{n\phi} (\vec{r}) = \sqrt{2n\phi_{n-1p}(\vec{r})}, \quad \hat{\Pi}^+_{n\phi} (\vec{r}) = \sqrt{2n\phi_{n\phi}(\vec{r})},
$$

though only the $n = 0$ and $n = 1$ Landau Level states are relevant for our problem. The two new gradient vector fields in the Hamiltonian (25) are defined as follows:

$$
\Omega_\pm = -i U^+ (\partial_y \mp i \partial_x) U,
$$

These two components are real: $\Omega_- = \Omega_+^\ast$. An expansion of the 2DEG action up to the second power of the Hamiltonian (25) reads:

$$
\delta S_0 = i \text{tr} (H_1 G_0) + \frac{i}{2} \text{tr} (H_1 G_0 H_1 G_0) + i \text{tr} (H_2 G_0).
$$

Combining this expansion with diagrams of the Fig. 2 we find the low-energy Goldstone Hamiltonian as follows:

$$
H_G = \frac{E}{8} \int \frac{d^2\vec{r}}{2\pi} \text{tr} \left( (\Sigma^0 - \Sigma^z)\Omega_- (\vec{r}) (\Sigma^0 + \Sigma^z)\Omega_+ (\vec{r}) \right)
$$

The insertion matrices in (30) are non-negative diagonal ones and they represent the occupation number for the electron states:

$$
N = \frac{1}{2} \left( \Sigma^0 + \Sigma^z \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

and

$$
\Sigma^0 - N = \frac{1}{2} \left( \Sigma^0 - \Sigma^z \right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
$$

where blocks are $1 \times 1$ and $3 \times 3$ in the case $\nu = 1, 3$ and $2 \times 2$ in the case $\nu = 2$. It follows immediately that $H_G \geq 0$.

The matrices $N$ and $1 - N$ can be viewed as projector operators that allow only those rotations into the Hamiltonian (31) that do change the ground state. It is useful to understand what particular sub-set of $SU(4)$ Lee group these physical rotations form. The vector field $\Omega_\mu$ can be expanded in the basis of fifteen generators of $SU(4)$ Lee group: $\{ \Sigma^l \}$, with $l = 1..15$. And we subdivide them into two complementary sets: the first one includes those generators that do commute with occupation number matrices (31, 32), and we shall called it an (even) set, whereas the second one includes the remaining
generators, and we shall call it the (**odd**) set. Generators of the (**even**) set constitute an algebra itself. This algebra has a normal Abelian subalgebra formed by a single generator: $\Sigma^z - \text{tr}\Sigma^z/4$. A Lee group built around the (**even**) set of generators is called a stabilizer sub-group $S$ of $SU(4)$ Lee group. The (**odd**) set always contains an even number of generators. Specifically, eight in the case of $\nu = 2$ and six in the case $\nu = 1, 3$.

The Hamiltonian (30) must be invariant under the time reversal symmetry. The time reversal operator can be chosen as a complex conjugate operator: $U \rightarrow U^*$. It follows that $\Omega \rightarrow -\Omega^T$. Now it is evident that $\Omega_-$ and $\Omega_+$ get interchanged under the time reversal in (30), and this changes the Hamiltonian. But we remember that the time reversal is always accompanied by inverting the magnetic field $B_z$ and, thus, we can restore the time reversal symmetry by multiplying the antisymmetric part of the Hamiltonian $H_G$ by the sign of the magnetic field:

$$H_G = \frac{E_1}{2} \int \frac{d^2\vec{r}}{2\pi} \left[ \text{tr} \left( (\Sigma^0 - N)\Omega_{\mu}N\Omega_{\mu} \right) + i \text{sgn}(B_z)\epsilon_{\mu\nu}\text{tr} (\Omega_{\mu}\Omega_{\nu}N) \right] \quad (33)$$

Now if we define a useful gradient vector field:

$$\Omega^z_\mu(\vec{r}) = \frac{i}{2}\text{tr} \left( U^+(\vec{r})\Sigma^z\partial_\mu U(\vec{r}) \right), \quad (34)$$

then it is straightforward to rewrite the Eq.(33):

$$H_G = \frac{E_1}{2} \int \frac{d^2\vec{r}}{2\pi} \left[ \text{tr} \left( (\Sigma^0 - N)\Omega_{\mu}N\Omega_{\mu} \right) + \text{sgn}(B_z)\text{curl} \Omega^z \right], \quad (35)$$

The first term in $H_G$ (33) is the gradient energy whereas the second term is proportional to the topological index of an excited state:

$$Q = \int \text{curl} \Omega^z \frac{d^2\vec{r}}{2\pi} = Z, \quad (36)$$

where $Z$ is the set of integer numbers. The case $Q = \pm 1$ corresponds to the simplest spin skyrmion in the first layer being rotated by a $SU(4)$ matrix to become a general bilayer skyrmion. The energy constant in $H_G$ (33) coincide identically with that of the one-layer case [12], which means that the bilayer skyrmion energy is the same as found for one layer. But there is an important difference between a bilayer skyrmion and a spin skyrmion
in a single ferromagnetic layer. Namely, as was shown in the Ref. [12] any skyrmion carries a charge density in the core:

\[ n(\vec{r}) = \frac{\text{curl} \Omega^z(\vec{r})}{2\pi} = \frac{R^2}{\pi (R^2 + \vec{r}^2)^2}, \quad (37) \]

where \( R \) is the radius of the skyrmion’s core, contrary to that in the bilayer a charge of the skyrmion’s core can be delocalized over the two layers with a much longer tails:

\[ n_{1,2}(\vec{r}) \sim \pm \frac{1}{(R^2 + \vec{r}^2)}. \quad (38) \]

The total charge of two layers does converge according to (37) at large distances from the skyrmion’s core.

The Goldstone Hamiltonian (35) can be cast in a special order parameter representation. To do this we define a non-homogeneous order parameter matrix \( Q \) as follows:

\[ Q(\vec{r}) = U(\vec{r})NU^+(\vec{r}). \quad (39) \]

This electronic order parameter has an important property:

\[ \langle A \rangle = \text{tr}(AQ), \quad (40) \]

where \( A \) is any operator. Inspecting the particular definition of \( N \) it becomes evident that rotations from the denominator sub-group \( S \) leaves the order parameter intact. Thus, rotations in (39) can be restricted to a coset or, in other word, a physical space of the bilayer 2DEG:

\[ \frac{U(4)}{U(\nu) \otimes U(4 - \nu)}. \quad (41) \]

Now it a straightforward calculation to rewrite \( H_G \) in terms of the order parameter matrix:

\[ H = \frac{E_1}{4} \int \text{tr} \left( \vec{\nabla}Q \vec{\nabla}Q \right) \frac{d^2 \vec{r}}{2\pi} + \text{sgn}(B^z) \frac{E_1}{2} \int \epsilon_{\mu\nu} \text{tr} (Q \partial_\mu Q \partial_\nu Q) \frac{d^2 \vec{r}}{2\pi}. \quad (42) \]

In this representation the topological index appears as an index of a map of the order parameter coset space into a 2D plane. The index selection rule is a consequence of a well known homotopy group identity:

\[ Q = \pi_2 \left( \frac{U(4)}{U(\nu) \otimes U(4 - \nu)} \right) = \mathbb{Z}. \quad (43) \]
In the end we have to include the Coulomb energy of charge distribution inside the skyrmion core:

\[
\delta H_G = \frac{1}{2} \int d^2 \vec{r} d^2 \vec{r}' \frac{\text{curl} \Omega^z(\vec{r})}{2\pi} \frac{\text{curl} \Omega^z(\vec{r}')}{2\pi} \frac{e^2}{|\vec{r} - \vec{r}'|}.
\]  

(44)

**Anisotropic Part of Coulomb Energy. Phase Diagram.**

In this section we cast the anisotropic part of the bilayer Hamiltonian (12) in terms of the order parameter matrix \( Q \). It can be conveniently done by the following Hartree-Fock average of \( c \)-operator product in (12):

\[
\tau_\mu \tau_\nu \langle c_\alpha \tau_1 p_1 c_\beta \tau_2 p_2 c_\gamma \tau_3 p_3 c_\delta \tau_4 p_4 > = \delta_{p_1 p_4} \delta_{p_2 p_3} \text{tr}(Q \tau_\mu) \text{tr}(Q \tau_\nu) - \delta_{p_1 p_3} \delta_{p_2 p_4} \text{tr}(Q \tau_\mu Q \tau_\nu),
\]

(45)

where \( \tau_\mu \) acts on four-spinor as \( \tau_\mu \otimes \sigma^0 \). Next, we define the following Coulomb anisotropy constants:

\[
E^{ab} = \int \frac{d z d \bar{z}}{2\pi l_H^2} V^{ab}(|z|) \exp \frac{-|z|^2}{2l_H^2} \approx \int \frac{d z d \bar{z}}{2\pi l_H^2} V^{ab}(|z|),
\]

(46)

where the last approximation holds for \((ab) \neq (00)\) in the limit \( d \ll l_H \).

And, finally, we rewrite the anisotropy Hamiltonian (12) in terms of order parameter matrix \( Q \):

\[
H_{\text{anis}}/N = - \left( t + (\nu - 1)E^{0z} \right) \text{tr}(Q \tau^x) - \left( \mu^z + (\nu - 1)E^{0z} \right) \text{tr}(Q \tau^z) - |g|\mu_B H \text{tr}(Q \sigma^z) + \frac{1}{2} E^{zz} \left[ \text{tr}(Q \tau^z) \text{tr}(Q \tau^z) - \text{tr}(Q \tau^z Q \tau^z) \right],
\]

(47)

where \( N \) is the number of degeneracy of the Landau Level. The Eqs. (12, 47) defines the effective long-range Hamiltonian of a bilayer at integer filling factors. At non-zero temperatures thermal fluctuations of the order parameter soften the anisotropy constants in the Hamiltonian (47).
calculation can be found in eg. [13] and the result reads:

\[
\begin{align*}
(t + (\nu - 1)E^0)_{R} &= (t + (\nu - 1)E^0) \left( \frac{l_h}{R^*} \right)^{sT/E_1} \\
(t + (\nu - 1)E^0)_{R} &= (t + (\nu - 1)E^0) \left( \frac{l_h}{R^*} \right)^{sT/E_1} \\
(\mid g\mid \mu_B H)_{R} &= \mid g\mid \mu_B H \left( \frac{l_h}{R^*} \right)^{sT/E_1} \\
E_{zz} &= E_{zz} \left( \frac{l_h}{R^*} \right)^{sT/E_1}
\end{align*}
\]

(48)

where the spatial scale \( R^2 = l^2 H E_1 / \max(t, \mu^2, \mid g\mid \mu_B H, E_{zz}) \) indicates the excitation wavelength where its anisotropy energy starts to compete with its exchange energy. Note that the three first constants renormalize as an external field whereas \( E_{zz} \) constant renormalizes as an easy-axis anisotropy. Although the Coulomb energy \( E_1 \sim 100K \gg T \sim 1K \) in most experiments, the specific number: \( 24 = 3 \times 8 \), which is related to the order of anisotropy and to the eight degree of freedom for thermal fluctuations in the case of \( SU(4) \) symmetry, makes the renormalization of the constant \( E_{zz} \) noticeable.

As we have seen in the previous section the order parameter can be parameterized by six or eight angles in the case \( \nu = 1, 3 \) or \( \nu = 2 \). Actually, not every of those rotations corresponds to a physically distinct eigen state. We restrict the calculation of the total bilayer energy up to a first order in powers of the anisotropy Hamiltonian, which means that we shall need only its diagonal matrix elements. But, these are real matrix elements of course, despite the fact that in an external magnetic field there is no time reversal symmetry. Hence, if the Hamiltonian is real one so real has to be its ground state. One generates all real eigen states from a reference state by rotations from the \( SO(4) \) sub-group of the \( SU(4) \) group. This group has 6 parameters with two of them falling into the denominator sub-group. Thus, the ground state differs from the reference state by just four rotations. One can view locally the 8D manifold of order parameter as a composition of four unit vectors: magnetization of the first and the second layers and the two hopping-tau vectors which represent the distribution of spin-up and spin-down electron density over the two layers. Now the first two term in the Hamiltonian (12) are external fields acting on these four vectors. On
the other hand the Coulomb energy couples pairs of tau vectors via an exchange interaction. This instructive picture allows us to identify only three special global rotations that do change the total bilayer energy. We start with the case \( \nu = 2 \) and we use a set of trial many electron wave functions parameterized by the three angles of relevant in our case rotation \( s: \)

\[
\prod_p U(\vartheta, -\vartheta) R(\theta_+, \theta_-) c^+_1 c^+_2 |\text{empty}\rangle,
\]

where spins in the layer 1,2 are first rotated by \( \pm \vartheta: \)

\[
U_{\beta\tau_2}^\alpha(\vartheta, -\vartheta) = \left( \frac{\tau^0 + \tau^z}{2} \right)^{\tau_1 \tau_2}_{\alpha\beta} \exp(i\vartheta - \frac{\sigma_y}{2})_{\alpha\beta} + \left( \frac{\tau^0 - \tau^z}{2} \right)^{\tau_1 \tau_2}_{\alpha\beta} \exp(-i\vartheta - \frac{\sigma_y}{2})_{\alpha\beta},
\]

and then wave functions of electrons with spin \( \pm \) spill over the two layers, the process described by two distinct angles: \( \theta_\pm : \)

\[
R_{\beta\tau_2}^\alpha(\theta_+, \theta_-) = \left( \frac{\sigma^0 + \sigma^z}{2} \right)^{\tau_1 \tau_2}_{\alpha\beta} \exp(i\theta_+ - \frac{\tau y}{2})_{\alpha\beta} + \left( \frac{\sigma^0 - \sigma^z}{2} \right)^{\tau_1 \tau_2}_{\alpha\beta} \exp(-i\theta_+ - \frac{\tau y}{2})_{\alpha\beta}.
\]

This set includes the singlet-liquid state at \( \theta_\pm = \pi/2 \) and \( \vartheta = 0 \) and the canted antiferromagnetic state at \( \theta_\pm = 0 \). The order parameter reads:

\[
Q = U R N R^+ U^+,
\]

with \( N \) being the electron density calculated with the reference state of the previous section (see (31)). Now we substitute (52) into the anisotropic Hamiltonian (47) to find the the total anisotropy bilayer energy as:

\[
E_{\text{anis}} = -E^{zz} \cos \theta_+ \cos \theta_- - (t + E^{0x}) \cos \vartheta (\sin \theta_+ + \sin \theta_-) - (\mu^z + E^{0z})(\cos \theta_+ - \cos \theta_-) - |g| \mu_B H \sin \vartheta (\cos \theta_+ + \cos \theta_-)
\]

The minimum of this energy corresponds to three phases: a) ferromagnetic \( \vartheta = \pi/2, \theta_+ = \theta_- = 0 \); b) spin singlet \( \vartheta = 0, \theta_+ = \pi - \theta_- = \theta \); and c) canted antiferromagnetic state otherwise, as it is shown on the Fig.3. It is identical to that found in the Ref.[11]. A line of continuous phase transitions between the ferromagnetic phase and the canted antiferromagnetic phase is given by the following equation:

\[
\left[ (E^{zz} + |g| \mu_B H)^2 - (\mu^z + E^{0z})^2 \right] |g| \mu_B H = \left( t + E^{0x} \right)^2 (E^{zz} + |g| \mu_B H)
\]
In the spin singlet phase the interlayer mixing phase: $\theta$, is determined by the equation:

$$
(E^{zz} \sin \theta + t + E^{0z}) \cos \theta = (\mu^z + E^{0z}) \sin \theta.
$$

(55)

A phase transition line that separate the spin singlet phase from the canted antiferromagnetic phase is given parametrically by the equation:

$$
((t + E^{0x}) \sin \theta - E^{zz} + (\mu^z + E^{0z}) \cos \theta) (t + E^{0z}) = (|g| \mu_B H)^2 \sin \theta,
$$

(56)

where $\theta$ is determined from (55). This phase transition is a continuous one also.

In the case $\nu = 1$ there is no Coulomb interaction energy and the total bilayer energy reads:

$$
E^{\text{anis}} = -t \sin \theta - \mu^z \cos \theta - |g| \mu_B H \cos \vartheta,
$$

(57)

The minimum of this energy is given by electron spin being directed along the magnetic field: $\vartheta = 0$, whereas $\theta = \tan^{-1} t/\mu^z$. There is no phase transition in the case $\nu = 1$ and the only phase can be characterized as ferromagnetic in both the spin and the layer spaces.
The case $\nu = 3$ formally reduces to the case $\nu = 1$ although here the Coulomb interaction energy does not vanish identically. We find renormalizations to the one-particle electron Hamiltonian whereas the total energy being similar to the case $\nu = 1$:

$$E_{\text{anis}} = -(t + 2E_{0x})\sin \theta - (\mu^z + 2E_{0z})\cos \theta - |g|\mu_B H \cos \vartheta, \quad (58)$$

There is no phase transition in this case either.

**Anisotropic energy gap of one Skyrmion**

In this section we find an anisotropic part of the total skyrmion gap energy. The non-homogeneous order parameter that represents one skyrmion is given by the Belavin-Polyakov (BP) skyrmion solution for $|Q| = 1$:

$$Q_{BP}(z\bar{z}) = \frac{R^2}{R^2 + |z|^2} \begin{pmatrix} |z|^2 & zR \\ zR & R^2 \end{pmatrix} \quad (59)$$

where only the shown above four matrix elements differs from those in the electron density matrix $N$. The skyrmion order parameter has to be rotated by a homogeneous matrix $RU$ calculated in the previous section in such a way that the order parameter far away from the skyrmion center minimizes the anisotropy energy. In addition to this rotation we have to allow all homogeneous rotations from the denominator sub-group $S$ that actually transform the BP skyrmion order parameter (59): $W$. Thus, a general skyrmion order parameter reads:

$$Q(\vec{r}) = RUWQ_{BP}(z\bar{z})W^\dagger U^\dagger R^\dagger. \quad (60)$$

First, we consider the case $\nu = 2$. Here the matrix $W$ is parameterized by seven angles:

$$\begin{pmatrix} \cos \beta f e^{i(\gamma_f + \alpha_f)} & \sin \beta f e^{i(\gamma_f - \alpha_f)} & 0 & 0 \\ -\sin \beta f e^{i(-\gamma_f + \alpha_f)} & \cos \beta f e^{i(-\gamma_f - \alpha_f)} & 0 & 0 \\ 0 & 0 & \cos \beta e e^{i(\gamma_e + \alpha_e)} & \sin \beta e e^{i(\gamma_e - \alpha_e)} \\ 0 & 0 & -\sin \beta e e^{i(-\gamma_e + \alpha_e)} & \cos \beta e e^{i(-\gamma_e - \alpha_e)} \end{pmatrix} \quad (61)$$
The additional seventh parameter angle of the denominator group just rotates the coordinates: \(z \rightarrow e^{i\gamma_7}z\). We find by explicit calculation that the skyrmion anisotropic energy does not depend on parameters \(\gamma_e, \gamma_f\) and \(\gamma_7\) whereas \(\alpha_e = 0\) and \(\alpha_f = \pi\) correspond to the skyrmion energy minimum. Thus we shall express the anisotropic skyrmion energy in terms of the two relevant parameters \(\beta_e\) and \(\beta_f\) that rotates the core of skyrmion in the empty-electron space and filled-electron space correspondingly. Besides these angles the skyrmion energy depends on free parameters entering the BP solution. The calculation here has been performed only for skyrmion topological index: \(|\mathcal{Q}| = 1\) (59) and in this case there is only one such parameter, namely, the radius \(R\) of the core of skyrmion. There are few different spatial integrals that we encounter in calculation. And only one of them is logarithmically divergent. We calculate the skyrmion anisotropy energy with the logarithmic accuracy. It means that parts of the anisotropy energy coming from different anisotropy sources are all multiplied by the same spatial integral, which we denote by a constant \(A\):

\[
A = \left(\frac{R}{l_H}\right)^2 \log \frac{R^*}{R}. \tag{62}
\]

where \(R^*\) is the inverse mass of Goldstone excitations in the model (35,53). Also, the Zeeman energy of skyrmion reads:

\[
E_{Z}^{skyr} = A|\mathcal{Q}|\mu_BH \left( -\frac{1}{2} \sin \vartheta (\cos \beta_f - \cos \beta_e)(\cos \theta_+ - \cos \theta_-) + \sin \vartheta (\cos \theta_+ + \cos \theta_-) - \cos \vartheta (\sin \beta_f + \sin \beta_e) \sin \frac{\theta_+ + \theta_-}{2} \right)
\tag{63}
\]

The hopping energy of skyrmion reads:

\[
E_{H}^{skyr} = A|\mathcal{Q}| \left( -\frac{1}{2} \cos \vartheta (\cos \beta_f - \cos \beta_e)(\sin \theta_+ - \sin \theta_-) + \cos \vartheta (\sin \theta_+ + \sin \theta_-) - \sin \vartheta (\sin \beta_f + \sin \beta_e) \cos \frac{\theta_+ + \theta_-}{2} \right)
\tag{64}
\]

The gate asymmetry energy of skyrmion reads:

\[
E_{G}^{skyr} = A|\mathcal{Q}| \left( 1/2 (\cos \beta_f - \cos \beta_e)(\cos \theta_+ + \cos \theta_-) - \frac{1}{2} (\cos \beta_f - \cos \beta_e)(\cos \theta_+ - \cos \theta_-) \right)
\tag{65}
\]
And finally the anisotropic Coulomb energy of skyrmion reads:

\[ E_{zz}^{skyr} = A E_{zz} = A \left( -\frac{1}{2} (1 + \cos \beta_f \cos \beta_e) (1 + \cos \theta_+ \cos \theta_-) + \right. \]
\[ \left. + 2 \cos \theta_+ \cos \theta_- - \frac{1}{2} \sin \beta_f \sin \beta_e \sin \theta_+ \sin \theta_- \right). \]

(66)

There is also a contribution to the skyrmion energy coming from a non-homogeneous BP electric charge distribution inside the skyrmion core (44):

\[ E_{C}^{skyr} = \frac{1}{2} \int \int d^2 r d^2 r' \frac{R^2}{\pi (r^2 + R^2)^2} \frac{e^2}{\kappa |r-r'|} \frac{R^2}{\pi (r'^2 + R^2)^2} = \frac{3 \pi^2 e^2}{64 \kappa R}. \]

(67)

The minimum of the total anisotropic skyrmion energy (63, 64, 65, 66): \( \mathcal{E}^{sky} = \mathcal{E}_Z + \mathcal{E}_H + \mathcal{E}_G + \mathcal{E}_{zz} \), over the two parameters \( \beta_e \) and \( \beta_f \) was found numerically and is denoted as: \( \mathcal{E}_{min}^{skyr} \). Next, we find a minimum the total skyrmion energy including (67): \( E_{skyr}^{sky} = A \mathcal{E}_{min}^{skyr} + E_{C}^{skyr} \), with respect to the skyrmion radius \( R \):

\[ \Delta = \frac{Q + |Q|}{2} E_1 + 3 \left( \mathcal{E}_{min}^{skyr} \left( \frac{3 \pi^2 e^2}{128 \kappa l_H} \right)^2 \log \frac{e^2}{\kappa l_H \mathcal{E}_{min}^{skyr}} \right)^{1/3}. \]

(68)

This formula holds in the limit \( \mathcal{E}^{skyr} \ll E_1 \) and, thus, the second term is much smaller than the first term in the Eq. (63) as they are calculated. But, it is important in the case of antiskyrmion \( Q = -|Q| \), the gap is only a relatively small anisotropic energy. The resulting anisotropic part of a skyrmion gap is shown on the Fig. 4 in the case \( \mu^z + E_{0z} = 0 \). Note the two prominent cusp-like lines in the skyrmion gap sheet coincide with the two phase transition lines from the Fig. 3. A view of another cross-section of the skyrmion gap: \( |g| \mu_B H = 0 \), is shown on the Fig. 5. A minima here also coincide with the canted-antiferromagnetic phase. A skyrmion in the ferromagnetic state is a spin-skyrmion with the spin rotations being localized inside one of the two layers, whereas a skyrmion in the spin-singlet state is a layer-skyrmion with the electron density being distributed over the two layers.

In the experimental setup [6] \( E^{skyr} \approx E_1 \) and our formulas can be compared with the experimental results only qualitatively. But they found a profound disappearence of the thermal activation gap maximum at some interval on the \( \nu = 2 \) line. In our theory this would correspond to a minimum in the skyrmion gap and we suggest that this indicate the canted antiferromagnetic phase.
Figure 4: Anisotropic part of Skyrmion Energy for a vanishing gate asymmetry.
Figure 5: Anisotropic part of Skyrmion Energy in the case of vanishing Zeeman energy. Spin-singlet phase is on the right side and the canted anti-ferromagnetic phase is on the left side.
In the case $\nu = 1$ we parameterize general rotations from the denominator sub-group by four angles in such a way that the electron density matrix becomes:

$$ WQ_{BP}(\bar{z}z)W^+ = \frac{1}{R^2 + |z|^2} \left( \begin{array}{c} |z|^2 Rz|f\rangle \\ \bar{R}z\langle f| \\ R^2 |f\rangle \langle f| \end{array} \right) $$

(69)

where

$$ |f\rangle = \left( \cos\frac{\beta}{2}, \sin\frac{\beta}{2} \cos\alpha e^{i\lambda_1}, \sin\frac{\beta}{2} \sin\alpha e^{i\lambda_2} \right) $$

(70)

Once again we find that the skyrmion energy does not depend on the parameters $\lambda_1$ and $\lambda_2$. A straightforward calculation shows then that the Zeeman skyrmion energy is

$$ \Delta = 2A|g|\mu_B H \left( 1 - \sin^2\frac{\beta}{2} \cos^2\alpha \right), \quad (71) $$

the Hopping skyrmion energy is

$$ E_{skyr}^H = At \left( -\sin\beta \sin\alpha + \sin\theta \left[ 1 + \cos^2\alpha \sin^2\beta \right] \right) $$

(72)

whereas the gate asymmetry skyrmion energy is:

$$ E_{skyr}^G = A\mu_z \left( -\cos\beta \left[ 1 - \sin^2\frac{\beta}{2} \cos^2\alpha \right] - \cos^2\alpha \sin^2\frac{\theta}{2} + \cos\theta \right) $$

(73)

Note that there is no Coulomb energy in the case $\nu = 1$. Searching for minima of Eq.(71,72,73) varying parameters $\beta$ and $\alpha$ and fixing $\theta = \tan^{-1} t/\mu^2$ we find the minimal anisotropy energy of skyrmion to be:

$$ \Delta = \frac{Q + |Q|}{2} E_1 + \min \left( 2\sqrt{t^2 + \mu^2} , 2|g|\mu_B H \right). $$

(74)

The skyrmion gap is given by the same formula as in the case $\nu = 2$: $|Q|$. Generally, there is no prominent minima in the skyrmion gap here. Nevertheless in the limit $t \ll |g|\mu_B H$ such a minimum occurs.

The case $\nu = 3$ reduces to the case $\nu = 1$ if one simply to renormalize the constant in (74) as it was explained in the previous section.
Conductivity Activation Energy

The conductivity activation energy was measured in experiments for bilayer system \([6]\) and in one layer system \([14]\). It was known for quite some time that this energy is considerably less than a typical exchange constant \(e^2/\kappa l_H\). This fact was the main motivation for the experimental search of topological excitations. If one considers an act of creation of skyrmion and anti-skyrmion pair at large separation then one easily gets the pair energy: \(E_1 = 1/2E_0\). This is definitely less than the creation of electron-hole pair at large separation: \(E_0\), but still of the order of \(e^2/\kappa l_H\), for one layer. In a bilayer system the prominent minimum in the minimal activation energy for skyrmion anti-skyrmion pair is still of the order of \(E_1\), in spite of the fact that experimental conductivity activation energy goes to zero very sharply \([6]\).

This controversy can be overcome by considering the creation of neutral antiskyrmion and electron pair at large separation. First, we consider a one layer case. The energy of the additional electron with reversed spin does not contain exchange Coulomb energy and consist from anisotropy energy only. The energy of anti-skyrmion also has only anisotropic energy \([68]\). The anisotropic energy of electron can be neglected in the limit of large ratio: \(E_1/|g|\mu_B H\), which holds in most experiments. The total anti-skyrmion energy is:

\[
|g|\mu_B B \int (1 - \cos \beta(r)) \frac{d^2 \vec{r}}{2\pi l_H^2} + \frac{1}{2} \int \frac{e^2}{\kappa |\vec{r} - \vec{r}'|} \text{curl} \Omega^z(\vec{r}) \text{curl} \Omega^z(\vec{r}') d^2 \vec{r} d^2 \vec{r}'.
\]

(75)

Plugging the BP solution we calculate this energy to be:

\[
2|g|\mu_B BR^2 \log \frac{R^*}{R} + \frac{3\pi^2 e^2}{64 \kappa R}.
\]

(76)

Minimizing it further with respect to radius of the anti-skyrmion core \(R\) we find the core radius:

\[
R = \frac{1}{2} l_H \left( \frac{3\pi^2 e^2}{32\kappa l_H |g|\mu_B \log(R^*/l_H)} \right)^{1/3}
\]

(77)

and the activation energy:

\[
\Delta = \frac{3}{2} \left( |g|\mu_B H \log(R^*/l_H) \left( \frac{3\pi^2 e^2}{128\kappa l_H} \right)^2 \right)^{1/3}.
\]

(78)
The upper limit under the logarithm $R^*$ is defined by the validity of BP solution. Essentially, it is a distance where gradient energy becomes of the order of Zeeman energy:

$$R^* = l_H \sqrt{\frac{E_1}{|g|\mu_B H}}.$$  

Therefore the logarithmic factor is of the order of unity. If it were rather large one would need to compare the energy of $Q = -1$ and $Q = -2$ antiskyrmions, with the latter being logarithmic free. In any case an important point is that the activation energy is magnetic field dependent:

$$\Delta = KH^{2/3},$$  

which in agreement with the various experimental results [14]. The absolute value of the constant $K$ (78) in this relation also conforms experiment [14]. Note that the Zeeman energy of electron is much less that $\Delta$ (80). In the case of double layer system the situation is more complicated due to large number of parameters defining the $H^{anis}$. Nevertheless, assuming anisotropy energy is small and neglecting for the same reason the anisotropy energy of an electron we get the deep minimum in the canted antiferromagnetic phase in accordance with the experimental result [14]. It should be noted that in the process of creation of anti-skyrmion electron pair the total topological charge of the system is not conserved as opposed to the skyrmion antiskyrmion pair. Therefore, this process goes tentatively on the sample boundary. Also the existence of magnetic field makes it possible to violate electron-hole symmetry usually assumed in theories used the basis of projected on the lowest Landau Level wave functions. The violation of the electron-hole symmetry is related to the topological charge and gives rise to the skyrmion vs antiskyrmion energy difference due to the topological term in the action (35).

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