ON STABLE DEGENERATIONS OF COHEN-MACAULAY MODULES OVER SIMPLE SINGULARITIES OF TYPE \((A_n)\)

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Dedicated to Professor Yuji Yoshino on the occasion of his sixtieth birthday.

Abstract. We study the stable degeneration problem for Cohen-Macaulay modules over simple singularities of type \((A_n)\). We prove that the stable hom order is actually a partial order over the ring and are able to show that the stable degenerations can be controlled by the stable hom order.

1. Introduction

The concept of degenerations of modules introduced in representation theory for studying the structure of the module variety over a finite dimensional algebra \[1, 9, 12, 13, 16, 17\]. Classically, Bongartz [1] investigated the degeneration problem of modules over an artinian algebra in relation with the Auslander-Reiten quiver. In [16], Zwarra gave a complete description of degenerations of modules over representation finite algebras by using some order relations for modules known as the hom order, the degeneration order and the extension order. Now a theory of degenerations is considered for not only module categories, but derived categories [7] or stable categories [14], more generally triangulated categories [10].

Let \(R\) be a commutative Gorenstein local \(k\)-algebra which is not necessary finite dimensional. Yoshino [14] introduced a notion of the stable analogue of degenerations of (maximal) Cohen-Macaulay \(R\)-module in the stable category \(\text{CM}(R)\). The notion of the stable degenerations is closely related to the ordinary degenerations. In fact, the author [4] give a complete description of degenerations of Cohen-Macaulay modules over a ring of even dimensional simple singularity of type \((A_n)\) by using the description of stable degenerations over it. Hence it is also important for the study of degeneration problem to investigate the description of stable degenerations.

The purpose of this paper is to describe stable degenerations of Cohen-Macaulay modules over simple singularities of type \((A_n)\).

\[ k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2). \]

More precisely, we discuss the stable degeneration problem over the ring which is of odd dimensional. First we compare Auslander-Reiten theory on \(\text{CM}(R)\) with that on \(\text{CM}(R)\). We look into the relation between AR sequences and AR triangles of Cohen-Macaulay modules (Proposition 2.2). And we consider an order relation on \(\text{CM}(R)\) which is the stable analogue of the hom order (Definition 2.6). We shall show that it is actually a partial order if a ring is of finite representation type with certain assumptions.

In Section 3 we devote to describe stable degenerations of Cohen-Macaulay modules over the ring above. We shall show that all stable degenerations can be controlled by the stable hom order (Theorem 3.6). To show this, we use the stable analogue of the argument over finite dimensional algebras in [16].
2. Stable hom order on Cohen-Macaulay modules

Throughout the paper \( R \) is a Henselian Gorenstein local ring that is \( k \)-algebra where \( k \) is an algebraically closed field of characteristic 0. For a finitely generated \( R \)-module \( M \), we say that \( M \) is a Cohen-Macaulay \( R \)-module if 

\[
\text{Ext}^i_R(M, R) = 0 \quad \text{for any } i > 0.
\]

We denote by \( \text{CM}(R) \) the category of Cohen-Macaulay \( R \)-modules with all \( R \)-homomorphisms. And we also denote by \( \text{CM}^0(R) \) the stable category of \( \text{CM}(R) \). The objects of \( \text{CM}^0(R) \) are the same as those of \( \text{CM}(R) \), and the morphisms of \( \text{CM}^0(R) \) are elements of \( \text{Hom}^0_R(M, N) = \text{Hom}_R(M, N)/\text{P}(M, N) \) for \( M, N \in \text{CM}(R) \), where \( \text{P}(M, N) \) denote the set of morphisms from \( M \) to \( N \) factoring through free \( R \)-modules. For a Cohen-Macaulay module \( M \) we denote it by \( M_0 \) to indicate that it is an object of \( \text{CM}^0(R) \). For a finitely generated \( R \)-module \( M \), take a free resolution 

\[
\cdots \to F_1 \xrightarrow{d} F_0 \to M \to 0.
\]

We denote \( \text{Im}_d \) by \( \Omega M \). We note that this defines the functor giving an auto-equivalence of \( \text{CM}(R) \). Since \( R \) is Gorenstein, it is known that \( \text{CM}^0(R) \) has a structure of a triangulated category with the shift functor defined by the functor \( \Omega^{-1} \). We recommend the reader to [3, Chapter 1], [11, Section 4] for the detail. Since \( R \) is Gorenstein, by the definition of a triangle, \( L \to M \to N \to L[1] \) is a triangle in \( \text{CM}^0(R) \) if and only if there is an exact sequence 

\[
0 \to L \to M' \to N \to 0
\]

in \( \text{CM}(R) \) with \( M' \cong M \) in \( \text{CM}(R) \), that is, \( M' \) is isomorphic to \( M \) up to free summand. Since \( R \) is Henselian, \( \text{CM}(R) \), hence \( \text{CM}^0(R) \), is a Krull-Schmidt category, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

In the paper we use the theory of Auslander-Reiten (abbr. AR) sequences and triangles of Cohen-Macaulay modules. Let us recall the definitions of those notions. See [11] for AR sequences and [3, 8] for AR triangles.

**Definition 2.1.** Let \( X, Y \) and \( Z \) be Cohen-Macaulay \( R \)-modules.

(1) A short exact sequence \( \Sigma : 0 \to Z \to Y \xrightarrow{f} X \to 0 \) is said to be an AR sequence ending in \( X \) (or starting from \( Z \)) if it satisfies

(A1) \( X \) and \( Z \) are indecomposable.

(A2) \( \Sigma \) is not split.

(A3) If \( g : W \to X \) is not a split epimorphism, then there exists \( h : W \to Y \) such that \( g = f \circ h \).

(2) We also say that a triangle \( \Sigma : Z \to Y \xrightarrow{f} X \xrightarrow{w} Z[1] \) is an AR triangle ending in \( X \) (or starting from \( Z \)) if it satisfies

(ART1) \( X \) and \( Z \) are indecomposable.

(ART2) \( w \neq 0 \).

(ART3) If \( g : W \to X \) is not a split epimorphism, then there exists \( h : W \to Y \) such that \( g = f \circ h \).

**Proposition 2.2.** Let \( \Sigma : 0 \to Z \to Y \xrightarrow{f} X \to 0 \) be an AR sequence ending in \( X \). Then \( \Sigma : Z \to Y \xrightarrow{f} X \xrightarrow{w} Z[1] \) is an AR triangle ending in \( X \).

**Proof.** We shall show that \( \Sigma \) satisfies (ART1), (ART2) and (ART3).

(ART1) It is obvious.

(ART2) If \( w \) is zero, then \( \Sigma \) is split. Thus there exists \( g : X \to Y \) such that \( f \circ g = 1_X \). Note that \( f \circ g \in \text{radEnd}_R(X) \) since \( \Sigma \) is not split. It yields that \( f \circ g = 1_X \in \text{radEnd}_R(X) \). This is a contradiction and \( w \) must be non zero.
(ART3) Let \( g : W \to X \) be not a split epimorphism. Then \( g : W \to X \) is also not a split epimorphism. By (AR3), we have a morphism \( h : W \to Y \) such that \( f \circ h = g \). We conclude that \( f \circ h = g \). \( \square \)

We say that \( \text{CM}(R) \) (resp. \( \text{CM}(R) \)) admits AR sequences (resp. AR triangles) if there exists an AR sequence (resp. AR triangle) ending in \( X \) (resp. \( X \)) for each indecomposable Cohen-Macaulay \( R \)-module \( X \). We also say that \( (R, m) \) is an isolated singularity if each localization \( R_p \) is regular for each prime ideal \( p \) with \( p \neq m \). If \( R \) is an isolated singularity, \( \text{CM}(R) \) admits AR sequences (cf. \[11, Theorem 3.2\]). As a corollary of Proposition 2.2, \( \text{CM}(R) \) admits AR triangles if \( R \) is an isolated singularity.

**Corollary 2.3.** If \( R \) is an isolated singularity, we have an 1-1 correspondence between the set of isomorphism classes of AR sequences in \( \text{CM}(R) \) and that of AR triangles in \( \text{CM}(R) \).

**Proof.** According to Proposition 2.2 we can define the mapping from the set of AR sequences to the set of AR triangles by taking its triangles. Note that AR triangles (resp. AR sequences) ending in \( X \) (resp. \( X \)) is unique up to isomorphism of triangles (resp. sequences) for a giving indecomposable \( X \) (resp. \( X \)) (see \[3, 8\]). Hence it follows from Proposition 2.2 that the mapping is surjective. The injectivity of the mapping is clear. \( \square \)

**Remark 2.4.** Reiten and Van den Bergh\[8\] show that a Hom-finite \( k \)-linear triangulated category \( T \) admits AR triangles if and only if \( T \) has a Serre functor. We can show that \( \text{CM}(R) \) is a Hom-finite triangulated category which has a Serre functor if \( R \) is an isolated singularity. Actually \( \text{CM}(R) \) has a Serre functor \( \tau(-)[-1] \cong (-)[d-1] \) where \( d = \dim R \). (Cf. \[11, Lemma 3.10\]. See also \[6, Corollary 2.5\].) Here \( \tau \) is an AR translation (Remark 2.12). Therefore we can also show that \( \text{CM}(R) \) admits AR triangles by \[8\].

**Lemma 2.5.** \[11, Lemma 3.9\] Let \( M \) and \( N \) be finitely generated \( R \)-modules. Then we have a functorial isomorphism

\[
\text{Hom}_R(M, N) \cong \text{Tor}^R_1(\text{Tr}M, N).
\]

Here \( \text{Tr}M \) is an Auslander transpose of \( M \).

According to Lemma 2.5, \( \text{Hom}_R(M, N) \) is of finite dimensional as a \( k \)-modules for \( M, N \in \text{CM}(R) \) if \( R \) is an isolated singularity. Thus the following definition makes sense.

**Definition 2.6.** For \( M, N \in \text{CM}(R) \) we define \( M \leq_{\text{hom}} N \) if \( [X, M] \leq [X, N] \) for each \( X \in \text{CM}(R) \). Here \( [X, M] \) is an abbreviation of \( \text{dim}_k \text{Hom}_R(X, M) \).

Now let us consider the full subcategory of the functor category of \( \text{CM}(R) \) which is called the Auslander category. We give a brief review of the Auslander category (see \[11, Chapter 4, 13\] for the detail). The Auslander category \( \text{mod}(\text{CM}(R)) \) is the category whose objects are finitely presented contravariant additive functors from \( \text{CM}(R) \) to the category of Abelian groups and whose morphisms are natural transformations between functors. The following lemma is one of the advantage of considering \( \text{mod}(\text{CM}(R)) \).

**Lemma 2.7.** \[11, Theorem 13.7\] A group homomorphism

\[
\gamma : G(\text{CM}(R)) \to K_0(\text{mod}(\text{CM}(R)));
\]

defined by \( \gamma(M) = [\text{Hom}_R(M, M)] \) for \( M \in \text{CM}(R) \), is injective. Here \( G(\text{CM}(R)) \) is a free Abelian group \( \bigoplus \mathbb{Z} \cdot X \) where \( X \) runs through all isomorphism classes of indecomposable objects in \( \text{CM}(R) \).
We denote by $\text{mod}(\text{CM}(R))$ the full subcategory $\text{mod}(\text{CM}(R))$ consisting of functors $F$ with $F(R) = 0$. Note that every object $F \in \text{mod}(\text{CM}(R))$ is obtained from a short exact sequence in $\text{CM}(R)$. Namely we have the short exact sequence $0 \to L \to M \to N \to 0$ such that
\[
0 \to \text{Hom}_R(, L) \to \text{Hom}_R(, M) \to \text{Hom}_R(, N) \to F \to 0
\]
is exact in $\text{mod}(\text{CM}(R))$. Note that since $F \in \text{mod}(\text{CM}(R))$ is a subfunctor $\text{Ext}^1_R(, L)$ for some $L \in \text{CM}(R)$, $F(X)$ has finite length for each $X \in \text{CM}(R)$ if $R$ is an isolated singularity. Therefore we can define a group homomorphism associated with $X$ in $\text{CM}(R)$
\[
(2.1) \quad \varphi_X : K_0(\text{mod}(\text{CM}(R))) \to \mathbb{Z}; [F] \mapsto \dim_k F(X).
\]
If $0 \to Z \to Y \to X \to 0$ is an AR sequence in $\text{CM}(R)$, then the functor $S_X$ defined by an exact sequence
\[
0 \to \text{Hom}_R(, Z) \to \text{Hom}_R(, Y) \to \text{Hom}_R(, X) \to S_X \to 0
\]
is a simple object in $\text{mod}(\text{CM}(R))$ and all the simple objects in $\text{mod}(\text{CM}(R))$ are obtained in this way from AR sequences. We say that $R$ is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay $R$-modules. We note that if $R$ is of finite representation type, then $R$ is an isolated singularity (cf. [11, Chapter 3.]).

It is proved in [11, (13.7.4)] that for each object $F$ in $\text{mod}(\text{CM}(R))$, there is a filtration by subobjects $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ such that each $F_i/F_{i-1}$ is a simple object in $\text{mod}(\text{CM}(R))$ if $R$ is of finite representation type.

We also remark that, since $\text{CM}(R)$ is a Krull-Schmidt category,
\[
S_X(Y) = \begin{cases} 1 & \text{if } X \cong Y, \\ 0 & \text{if } X \ncong Y. \end{cases}
\]
for an indecomposable module $Y \in \text{CM}(R)$. See [11] (4.11) for instance.

**Lemma 2.8.** If $R$ is of finite representation type, then we have the equality in $K_0(\text{mod}(\text{CM}(R)))$
\[
\text{[Hom}_R(, M)] = \sum_{X_i \in \text{indCM}(R)} [X_i, M] |S_{X_i}|
\]
for each $M \in \text{CM}(R)$.

**Proof.** For $F = \text{Hom}_R(, M)$, $F(R) = 0$, so that $F \in \text{mod}(\text{CM}(R))$. Since $R$ is of finite representation type, $F$ has a filtration by simple objects $S_{X_i}$ (see above). Hence we have the equality in $K_0(\text{mod}(\text{CM}(R)))$:
\[
[F] = \sum_{X_i \in \text{indCM}(R)} c_i [S_{X_i}].
\]
By using homomorphism in (2.1), we see that
\[
[X_i, M] = \dim_k F(X_i) = \dim_k \sum_{X_i \in \text{indCM}(R)} c_i \dim_k S_{X_j}(X_i) = c_j.
\]
Therefore we obtain the equation in the lemma.

**Theorem 2.9.** Let $R$ be of finite representation type and $M$ and $N$ be Cohen-Macaulay $R$-modules. Suppose that $[X, M] = [X, N]$ for each $X \in \text{CM}(R)$. Then $M \oplus \Omega M \cong N \oplus \Omega N$. 
We have a long exact sequence as follows:

\[ 0 \rightarrow \text{Hom}_R(-, \Omega M) \rightarrow \text{Hom}_R(-, P_M) \rightarrow \text{Hom}_R(-, M) \rightarrow \text{Hom}_R(-, M) \rightarrow 0, \]

where \( P_M \) is a free \( R \)-module. Thus we have

\[ [\text{Hom}_R(-, M)] + [\text{Hom}_R(-, \Omega M)] - [\text{Hom}_R(-, P_M)] = [\text{Hom}_R(-, N)] + [\text{Hom}_R(-, \Omega N)] - [\text{Hom}_R(-, P_N)]. \]

Hence,

\[ [\text{Hom}_R(-, M)] + [\text{Hom}_R(-, \Omega M)] + [\text{Hom}_R(-, P_N)] = [\text{Hom}_R(-, N)] + [\text{Hom}_R(-, \Omega N)] + [\text{Hom}_R(-, P_M)]. \]

According to Lemma 2.7, we get

\[ M \oplus \Omega M \oplus P_N \cong N \oplus \Omega N \oplus P_M. \]

Therefore \( M \oplus \Omega M \cong N \oplus \Omega N \). \( \square \)

It immediately follows from the theorem that

**Corollary 2.10.** Let \( R \) be of finite representation type and \( M \) and \( N \) be Cohen-Macaulay \( R \)-modules. Suppose that \( U \cong U[1] \) for each indecomposable Cohen-Macaulay \( R \)-module \( U \). Then \([X, M] = [X, N] \) for each \( X \in \text{CM}(R) \) if and only if \( M \cong N \). Particularly, \( \leq_{\text{hom}} \) is a partial order on \( \text{CM}(R) \).

**Remark 2.11.** The stable hom order \( \leq_{\text{hom}} \) is not always a partial order on \( \text{CM}(R) \) even if the base ring \( R \) is a simple singularity of type \((A_n)\). Let \( R = k[[x, y, z]]/(x^3 + y^2 + z^2) \), that is, \( R \) is a two dimensional simple singularity of type \((A_2)\). And let \( I \) (resp. \( J \)) be an ideal generated by \((x, y)\) (resp. \((x^2, y)\)). Note that the set \([I, J]\) is a complete list of non free indecomposable Cohen-Macaulay \( R \)-modules (see [11] Chapter 10 for instance). Then it is easy to see that \([L, L] = [J, J] = 1\). However \( L \neq J \). Thus \( \leq_{\text{hom}} \) is not a partial order on \( \text{CM}(R) \).

For later reference we state some results on the stable hom order. In what follows, we always assume that \( R \) is an isolated singularity. We denote by \( \mu(M, Z) \) the multiplicity of \( Z \) as a direct summand of \( M \). For an AR triangle \( \bar{Z} \rightarrow \bar{Y} \rightarrow \bar{X} \rightarrow \bar{Z}[1] \), we denote \( \bar{Z} \) (resp. \( \bar{X} \)) by \( \tau \bar{X} \) (resp. \( \tau^{-1} \bar{Z} \)).

**Remark 2.12.** For an AR sequence \( 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \), \( Z \) (resp. \( X \)) is also denoted by \( \tau X \) (resp. \( \tau^{-1} Z \)) (see [11] Definition 2.8).). By virtue of Proposition 2.2, \( \tau \bar{X} \cong \tau \bar{X} \) for each non free indecomposable Cohen-Macaulay \( R \)-module \( X \).

**Proposition 2.13.** Let \( M \) be a Cohen-Macaulay \( R \)-module. And let \( U \) be an indecomposable non free Cohen-Macaulay \( R \)-module and \( \Sigma : U \xrightarrow{\varphi} E_U \xrightarrow{f} \tau^{-1} U \rightarrow U[1] \) be an AR triangle starting from \( U \). Suppose that \( U[1] \) is isomorphic to \( U \) in \( \text{CM}(R) \). Then we have

\[ \mu(M, U) = \frac{1}{2}([L, M] + [\tau^{-1} L, M] - [E_U, M]). \]

**Proof.** Apply \( \text{Hom}_R(-, M) \) to the triangle \( \Sigma \), we have a long exact sequence as follows:

\[ \text{Hom}_R(\tau^{-1} U[-1], M) \xrightarrow{\text{Hom}_R(\tau^{-1} U, M)} \text{Hom}_R(\bar{E}_U[-1], M) \xrightarrow{\text{Hom}_R(\bar{E}_U, M)} \text{Hom}_R(\bar{U}[-1], M) \rightarrow. \]
Since $\text{End}_R(X)/\text{radEnd}_R(X) \cong k$ for each non free indecomposable Cohen-Macaulay module $X$ and by the property of an AR triangle, we have

$$\text{Ker} \, \text{Hom}_R(f[i], M) \cong \text{Coker} \, \text{Hom}_R(g[i-1], M) \cong k^{\mu(M,U)^i)}$$

for all $i \in \mathbb{Z}$. In particular

$$0 \to k^{\mu(M,U[-1])}) \to \text{Hom}_R(\tau^{-1}U, M) \to \text{Hom}_R(E_U, M) \to \text{Hom}_R(U, M) \to k^{\mu(M,U)} \to 0$$

By the assumption, $U \cong U[-1]$, so that

$$2\mu(M,U) = [U, M] + [\tau^{-1}U, M] - [E_U, M].$$

We obtain the required equation. \qed

Remark 2.14. By using Proposition 2.13 one can show that $\leq_{\text{hom}}$ is a partial order on $\text{CM}(R)$ without the assumption that $R$ is of finite representation type.

Lemma 2.15. Let $M$ and $N$ be Cohen-Macaulay $R$-modules with $M \leq_{\text{hom}} N$. And let $U$ be a non free and indecomposable direct summand of $N$. Suppose that $[U, M] = [U, N]$ and $U \cong U[1]$. Then $U$ is also a direct summand of $M$.

Proof. We assume that $U$ is not a direct summand of $M$. Take the AR triangle:

$$U \to E_U \to \tau^{-1}U \to U[1].$$

Applying $\text{Hom}_R(-, M)$ to the triangle, we have an exact sequence

$$0 \to \text{Hom}_R(U, M) \to \text{Hom}_R(E_U, M) \to \text{Hom}_R(U, M) \to 0. $$

Thus the equality $[\tau^{-1}U, M] + [U, M] = [E_U, M]$ holds. On the other hand, since $U$ is a direct summand of $N$, we also have

$$\text{Hom}_R(\tau^{-1}U, N) \to \text{Hom}_R(E_U, N) \to \text{Hom}_R(U, N).$$

Particularly the strict inequality $[\tau^{-1}U, N] + [U, N] > [E_U, N]$ holds. Since $U \cong U[1]$, it yields that $U \cong \tau^{-1}U$ (Remark 2.14). And $[U, M] = [U, N]$ by the assumption, we conclude that $[E_U, M] > [E_U, N]$. However this makes a contradiction since $M \leq_{\text{hom}} N$. Therefore $U$ is a direct summand of $M$. \qed

Proposition 2.16. Let $M$ and $N$ be Cohen-Macaulay $R$-modules with $M \leq_{\text{hom}} N$. Suppose that $[U, M] = [U, N]$ and $U \cong U[1]$ for each non free and indecomposable direct summand $U$ of $N$. Then $M \cong N$.

Proof. It follows from Lemma 2.15. \qed

3. Stable degenerations of Cohen-Macaulay modules

In this section, we shall describe the stable degenerations of Cohen-Macaulay modules over simple singularities over type $(A_n)$, namely

$$k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2).$$

First let us recall the definition of stable degenerations of Cohen-Macaulay modules.

Definition 3.1. [13] Definition 4.1 Let $M, N \in \text{CM}(R)$. We say that $M$ stably degenerates to $N$ if there exists a Cohen-Macaulay module $Q \in \text{CM}(R \otimes_k V)$ such that $Q[1/t] \cong M \otimes_k K$ in $\text{CM}(R \otimes_k K)$ and $Q \otimes_V V/tV \cong N$ in $\text{CM}(R)$.

It is known that the ring $k[[x_0, x_1, x_2, \cdots, x_d]]$ is of finite representation type, so that it is an isolated singularity. If a ring is an isolated singularity, there is a nice characterization of stable degenerations. See also [10] Theorem 1.
Theorem 3.2. [14] Theorem 5.1, 6.1] Consider the following three conditions for Cohen-Macaulay $R$-modules $M$ and $N$:

1. $M \oplus P$ degenerates to $N \oplus Q$ for some free $R$-modules $P$, $Q$.
2. $M$ stably degenerates to $N$.
3. There is a triangle

$$\mathbb{Z} \longrightarrow M \oplus \mathbb{Z} \longrightarrow N \longrightarrow \mathbb{Z}[1]$$

in $\text{CM}(R)$.

If $R$ is an isolated singularity, then (2) and (3) are equivalent. Moreover, if $R$ is artinian, the conditions (1), (2) and (3) are equivalent.

Remark 3.3. In general, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. And it is required that the endomorphism of $\mathbb{Z}$ in the triangle in (3) is nilpotent. However, if $R$ is an isolated singularity, we do not need the nilpotency assumption. The theorem also says that $M$ and $N$ give the same class in the Grothendieck group of $\text{CM}(R)$ if $M$ stably degenerates to $N$. See for [3] Chapter 3 for the definition of Grothendieck groups of triangulated categories.

We state order relations with respect to stable degenerations and triangles.

Definition 3.4. [4] Definition 3.2., 3.3.] Let $M$ and $N$ be Cohen-Macaulay $R$-modules.

1. We denote by $M \leq_{st} N$ if $N$ is obtained from $M$ by iterative stable degenerations, i.e. there is a sequence of Cohen-Macaulay $R$-modules $L_0, L_1, \ldots, L_r$ such that $M \cong L_0$, $N \cong L_r$ and each $L_i$ stably degenerates to $L_{i+1}$ for $0 \leq i < r$.
2. We say that $M$ stably degenerates by a triangle to $N$, if there is a triangle of the form $U \to M \to V \to U[1]$ in $\text{CM}(R)$ such that $U \oplus V \cong N$. We also denote by $M \leq_{\text{tri}} N$ if $N$ is obtain from $M$ by iterative stable degenerations by a triangle.

Remark 3.5. It has shown in [14] that the stable degeneration order is a partial order. Moreover, if there is a triangle $U \to M \to V \to U[1]$, then we can show that $M$ stably degenerates to $U \oplus V$ (cf. [4] Remark 3.4. (2))). Hence $M \leq_{\text{tri}} N$ induces $M \leq_{st} N$. It also follows from Theorem 3.3 that $M \leq_{st} N$ induces that $M \leq_{\text{hom}} N$.

The rest of the paper, we devote to describe stable degenerations over the one dimensional simple singularity of type $(A_n)$. Namely we consider the ring of the form:

$$R = k[[x, y]]/(x^{n+1} + y^2).$$

As stated in [11] Proposition 5.11, if $n$ is an even integer, the set of ideals of $R$

$$\{ I_i = (x^i, y) \mid 1 \leq i \leq n/2 \}$$

is a complete list of isomorphic indecomposable non free Cohen-Macaulay $R$-modules. On the other hand, if $n$ is an odd integer, then

$$\{ I_i = (x^i, y) \mid 1 \leq i \leq (n-1)/2 \} \cup \{ N_+ = R/(x^{(n+1)/2} + \sqrt{-1} y), N_- = R/(x^{(n+1)/2} - \sqrt{-1} y) \}$$

is a complete list of the ones (cf. [11] Paragraph 9.9)).

In this section, we shall show

Theorem 3.6. Let $R = k[[x, y]]/(x^{n+1} + y^2)$ and $I_i$, $N_+$ and $N_-$ as above.

1. If $n$ is an even integer, the stable hom order coincides with the stable degeneration order. Particularly we have

$$I_1 \leq_{st} I_2 \leq_{st} \cdots \leq_{st} I_{n/2}.$$
(2) If $n$ is an odd integer, then we have
\[ I_1 \leq_{st} I_2 \leq_{st} \cdots \leq_{st} I_{(n-1)/2} \leq_{st} N_+ \oplus N_-. \]
and
\[ N_+ \leq_{st} N_+ \oplus I_1 \leq_{st} \cdots \leq_{st} N_+ \oplus I_{(n-1)/2} \leq_{st} N_+ \oplus N_+ \oplus N_- \quad \text{(double sign corresponds).} \]

To show this, we use the stable analogue of the argument in [16].

The lemma below is well known for the case in an Abelian category (cf. [16, Lemma 2.6]). The same statement follows in an arbitrary triangulated category, not necessary CM$(\mathbb{R})$. (The author thanks Yuji Yoshino for telling me this argument.)

**Lemma 3.7.** Let
\[ \Sigma_1 : N_1 \xrightarrow{(f_1, v)} L_1 \oplus N_2 \xrightarrow{(u, g_1)} L_2 \longrightarrow N_1[1] \]
and
\[ \Sigma_2 : M_1 \xrightarrow{(f_2, w)} N_1 \oplus M_2 \xrightarrow{(v, g_2)} N_2 \longrightarrow M_1[1] \]
be triangles in an arbitrary triangulated category. Then we also have the following triangle.
\[ M_1 \rightarrow L_1 \oplus M_2 \rightarrow L_2 \rightarrow M_1[1]. \]

**Proof.** We consider the following triangle associated with $\Sigma_2$:
\[ M_1 \xrightarrow{(-f_1 \circ f_2, w)} L_1 \oplus N_1 \oplus M_2 \xrightarrow{(1 \quad f_1 \quad 0 \quad 0 \quad 1 \quad f_2 \quad w)} N_2 \longrightarrow M_1[1]. \]
We remark that the left morphism is given by
\[
\begin{pmatrix}
-f_1 \circ f_2 \\
-f_1 \\
-w
\end{pmatrix}
= \begin{pmatrix} 1 & -f_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} f_1 \\ v \\ 0 \end{pmatrix},
\]
to make the diagram below:
\[
\begin{array}{ccc}
N_1 & \xrightarrow{f_1} & L_1 \oplus N_2 \xrightarrow{(u, g_1)} L_2 \longrightarrow N_1[1] \\
\| & & \| \\
N_1 & \xrightarrow{(0, 1)} & L_1 \oplus N_1 \oplus M_2 \longrightarrow L_1 \oplus M_2 \longrightarrow N_1[1] \\
\| & & \| \\
M_1 & \xrightarrow{(-f_1 \circ f_2, w)} & M_1.
\end{array}
\]
By the octahedral axiom, we obtain the required triangle. \qed

Combining (the Abelian version of) Lemma 3.7 with Lemma 2.8, the stable hom order can be calculate easily from datum of AR triangles. For instance, let $R = \mathbb{k}[[x, y]]/(x^{n+1} + y^2)$ where $n$ is even. The AR sequences are
\[ 0 \rightarrow I_i \rightarrow I_{i-1} \oplus I_{i+1} \rightarrow I_i \rightarrow 0 \]
for \( i = 1, \cdots, n/2 \) where \( I_0 = R \) and \( I_{n/2+1} = I_{n/2} \). Then we have
\[
I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{n/2} \rightarrow I_{n/2} \rightarrow \cdots \rightarrow I_1 \rightarrow R
\]
The diagram shows that
\[
\text{Hom}_R(-, I_j) = \sum 2[S_i]
\]
in \( K_0(\text{mod}(\text{CM}(R))) \). Thus \([I_i, I_j] = 2 \) for \( i, j \in 1, \cdots, n/2 \). Moreover since \([- , I_j]\) is an additive function on \( \text{CM}(R) \), one can show that
\[
[I_i, I_j] \text{ is an even number for each } i, j.
\]

Taking Proposition \([2, 13]\) and \((3.2)\) into consideration, we make the following definition.

**Definition 3.8.** Let \( M \) and \( N \) be Cohen-Macaulay \( R \)-modules. We define a function \( \delta_{M \rightarrow N}(\mathord{-}) \) on \( \text{CM}(R) \) by
\[
\delta_{M \rightarrow N}(\mathord{-}) = \frac{1}{2}([-N] - [-M]).
\]

For a triangle \( \Sigma : \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{L}[1] \), we also define a function \( \delta_{\Sigma}(\mathord{-}) \) on \( \text{CM}(R) \) by
\[
\delta_{\Sigma}(\mathord{-}) = \frac{1}{2}([-\mathcal{L}] + [-\mathcal{N}] - [-\mathcal{M}]).
\]

We say that a triangle \( \mathcal{Z} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{X} \xrightarrow{\mathcal{Z}[1]} \mathcal{Y}[1] \) is without isomorphisms if \( f \in \text{rad}(\mathcal{Y}, \mathcal{X}) \) and \( g \in \text{rad}(\mathcal{Z}, \mathcal{Y}) \). Let \( \mathcal{Z} \rightarrow \mathcal{Y} \rightarrow \mathcal{X} \rightarrow \mathcal{Z}[1] \) be any triangle. As in the same case of a sequence, there is a triangle without isomorphisms \( \mathcal{Z}' \rightarrow \mathcal{Y}' \rightarrow \mathcal{X}' \rightarrow \mathcal{Z}'[1] \) for such that \( \mathcal{Z} \cong \mathcal{Z}' \oplus \mathcal{U}, \mathcal{Y} \cong \mathcal{Y}' \oplus \mathcal{U} \oplus \mathcal{V} \) and \( \mathcal{X} \cong \mathcal{X}' \oplus \mathcal{V} \) for some \( \mathcal{U}, \mathcal{V} \in \text{CM}(R) \). (Cf. \([10]\) Paragraph 2.7.)

**Lemma 3.9.** ([10] Lemma 3.1.) Let \( R \) be of finite representation type and let \( M \) and \( N \) be Cohen-Macaulay \( R \)-modules with \( M \leq \text{hom}(N) \) and let \( \Sigma : \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{L}[1] \) be a triangle without isomorphisms such that \( \delta_{\Sigma} \leq \delta_{M \rightarrow N} \). Suppose that \( \mathcal{X} \cong \mathcal{X}[1] \) and \( \delta_{\Sigma}(\mathcal{X}) \) is even for each indecomposable \( \mathcal{X}, \mathcal{X}' \in \text{CM}(R) \). Then there exists a triangle \( \Phi : \mathcal{Z} \rightarrow \mathcal{Y} \rightarrow \mathcal{V} \rightarrow \mathcal{Z}[1] \) without isomorphisms such that \( \delta_{\Phi}(\mathcal{Y}) = \delta_{M \rightarrow N}(\mathcal{Y}) \).

**Proof.** If \( \delta_{\Sigma}(\mathcal{W}) = \delta_{M \rightarrow N}(\mathcal{W}) \), we have nothing to prove. Otherwise, we assume that there exists an indecomposable direct summand \( W_1 \) of \( W = W_1 \oplus W_2 \) such that \( \delta_{\Sigma}(W_1) < \delta_{M \rightarrow N}(W_1) \).

Let \( f \) be the morphism \( \mathcal{W} \rightarrow \mathcal{W}_1 \) in the triangle \( \Sigma \). Take the AR triangle of \( \mathcal{W}_1 \) and construct a pullback diagram:
\[
\begin{array}{ccc}
\tau W_1 & \rightarrow & E_{W_1} \rightarrow W_1 \rightarrow \tau W_1[1] \\
\bigg\| & & \bigg\| \\
\tau W_1 & \rightarrow & E \rightarrow \mathcal{L} \rightarrow \tau W_1[1].
\end{array}
\]

Since \( \Sigma \) is without isomorphisms, \( f \) is not isomorphism. By the property of an AR triangle (ART3), the bottom triangle splits, so that \( E \cong \mathcal{L} \oplus \tau W_1 \). Then we have a new triangle:
\[
\Psi : \mathcal{L} \oplus \tau W_1 \rightarrow E_{W_1} \oplus \mathcal{U} \rightarrow W_1 \rightarrow (\mathcal{U} \oplus \tau W_1)[1].
\]

Applying Lemma \([3.7]\) to the triangles \( \Sigma \) and \( \Psi \), we get
\[
\Theta : \mathcal{L} \oplus \tau W_1 \rightarrow W_2 \oplus E_{W_1} \rightarrow \mathcal{V} \rightarrow (\mathcal{U} \oplus \tau W_1)[1].
\]

It is easy to see that we have the following equality
\[
\delta_{\Theta}(X) = \delta_{\Sigma}(X) + \delta_{\Psi}(X) = \delta_{\Sigma}(X) + \mu(X, \mathcal{W}_1)
\]
for each $X \in \text{CM}(R)$. Therefore $\delta_\Sigma \leq \delta_\Delta \leq \delta_{M,N}$. Repeating this procedure, we obtain the required triangle noting that this process stops since $R$ is of finite representation type. 

\begin{theorem}
Let $R$ be of finite representation type. Suppose that $X \cong X[1]$ and $[X,X']$ is even for each indecomposable $X, X' \in \text{CM}(R)$. Then $M \leq_{\text{hom}} N$ if and only if $M \leq_{st} N$ for Cohen-Macaulay $R$-modules $M$ and $N$.

\begin{proof}
As mentioned in Remark 3.6, $M \leq_{st} N$ implies that $M \leq_{\text{hom}} N$.

To show the converse, we assume that $M \leq_{\text{hom}} N$. For each indecomposable Cohen-Macaulay module $X$, we set $r(X) = \min\{\delta_{M,N}(X), \mu(N,X)\}$. We denote by $F$ the set of all the isomorphism classes of non free indecomposable Cohen-Macaulay modules with $r(X) > 0$:

$$F = \{X \mid r(X) > 0\} / \cong.$$ 

Let $N_1 = \bigoplus_{X \in F} X^{r(X)}$ and $N_2 = \bigoplus_{X \in \text{CM}(R)} X^{\mu(N,X) - r(X)}$. Then $N_1 \oplus N_2 \cong N$ in $\text{CM}(R)$.

Since $R$ is an isolated singularity, we can take an Auslander-Reiten triangle:

$$\Sigma_X : \tau X \to E_X \to X \to \tau X[1].$$

Consider a triangle $\Sigma$ which is a direct sum of $r(X)$ copies of triangles $\Sigma_X$ where $X$ runs through all modules in $F$:

$$\Sigma : \bigoplus_{X \in F} (\tau X)^{r(X)} \to \bigoplus_{X \in F} E_X^{r(X)} \to \bigoplus_{X \in F} X^{r(X)} \to \bigoplus_{X \in F} (\tau X[1])^{r(X)}.$$ 

Here we note that $\bigoplus_{X \in F} X^{r(X)} \cong N_1$. Then we have an inequality $\delta_{\Sigma}(X) = r(X) \leq \delta_{M,N}(X)$ for each $X \in \text{CM}(R)$. By virtue of Lemma 3.3, we have a triangle $\Sigma : Z \to Y \to N_1 \to Z[1]$ such that $\delta_{\Sigma} \leq \delta_\Sigma \leq \delta_{M,N}$ and $\delta_Y = \delta_{M,N}(W)$. Since $0 \leq \delta_{M,N} - \delta_\Sigma = \delta_{M,N} - \delta_{M,N}(W)$, we see that $M \oplus Z \cong N_2 \oplus Y$. Moreover $\delta_{M \oplus Z} = \delta_{\Sigma}(N_1) = \delta_{M,N}(N_1) - \delta_Y = 0$.

For an indecomposable direct summand $U$ of $N_2$, $r(N_1) < \mu(N_1,U)$. Thus $\delta_{\Sigma}(U) = r(U) = \delta_{M,N}(U)$, so that $\delta_{M \oplus Z \oplus N_2 \oplus Y}(N_2) = 0$. Hence $\delta_{M \oplus Z \oplus N_2 \oplus Y}(N_2 \oplus Y) = 0$. By Proposition 3.16, we have $M \oplus Z \cong N_2 \oplus Y$. Therefore, the triangle $\Sigma : Z \to Y \to N_1 \to Z[1]$ induces a triangle $Z \to Y \oplus N_2 \cong M \oplus Z \to N_1 \oplus N_2 \cong N \to Z[1]$. This makes the stable degeneration $M \leq_{st} N$.

\end{proof}

\begin{proof}[Proof of Theorem 3.10]
(1) If $n$ is an even integer, $I_i$ is isomorphic to $\Omega_i$ up to free summand. Thus $I_i \cong I_{i-1}[-1]$. See [11] Proposition 5.11. As shown in [3.2], $[I_i,I_j]$ is an even number. Hence it follows from Theorem 3.10 that the stable degeneration order is equivalent to the stable hom order. By direct calculations of the stable hom order, we get the assertion.

(2) As mentioned in [11] Chapter 9, we see that $I_i \cong \Omega_i$ up to free summand for $i = 1, \ldots, (n-1)/2$. Now let us consider regarding $N_+ \oplus N_-$ as one object $X$. Take the direct sum of triangles $\Sigma X = \Sigma N_+ \oplus \Sigma N_-$ as an analogue of the AR triangle ending in $X$, where $\Sigma N_+ \oplus \Sigma N_-$ is the AR triangle ending in $N_+$ (resp. $N_-$). Then the module which corresponds to $\tau X$ is $\tau N_+ \oplus \tau N_-$, which is isomorphic to $N_+ \oplus N_+ \cong X$. Moreover $X[1] \cong X$. In the same way as [3.2], one can also show that the dimension of $\text{Hom}$ between them is even. Hence the modules composed of $I_1, \ldots, I_{(n-1)/2}$ and $X = N_+ \oplus N_-$ satisfy the assumptions of Theorem 3.10 so that $\leq_{st}$ coincides with $\leq_{\text{hom}}$ on such modules. Therefore we have $I_1 \leq_{st} I_2 \leq_{st} \cdots \leq_{st} I_{(n-1)/2} \leq_{st} X \cong N_+ \oplus N_-$.

To show the remaining inequalities of the clan, it is enough to show that $N_+ \leq_{st} N_+ \oplus I_1$. We shall construct a triangle which gives the degeneration. According to Corollary 2.8 the AR triangles in $\text{CM}(R)$ are the following:

$$I_{i+1} \to I_i \oplus I_{i+2} \to I_{i+1} \to I_{i+1}[1]$$
for $1 \leq i \leq (n-1)/2$ where $I_{(n+1)/2} = N_+ \oplus N_-,$

$$N_+ \rightarrow I_{(n-1)/2} \rightarrow N_+ \rightarrow N_+ [1]$$

and

$$N_+ \rightarrow I_{(n-1)/2} \rightarrow N_+ \rightarrow N_+ [1].$$

By using Lemma 3.7 repeatedly, we have the triangle $N_+ \rightarrow N_+ \rightarrow I_1 \rightarrow N_+ [1].$ Actually,

$$\begin{array}{c}
N_+ \rightarrow N_+ \rightarrow I_{(n-1)/2} \rightarrow \cdots \rightarrow I_2 \rightarrow I_1 \rightarrow 0 \\
\text{and} \quad N_+ \oplus N_- \rightarrow I_{(n-1)/2} \rightarrow \cdots \rightarrow I_2 \rightarrow I_2 \rightarrow I_1,
\end{array}$$

where the last square comes from a triangle which is a direct sum of the AR triangle ending in $N_+$ and $0 \rightarrow N_+ \rightarrow N_+ \rightarrow 0 [1].$ Hence $N_+ \leq_{tri} N_+ \oplus I_1,$ so that $N_+ \leq_{st} N_+ \oplus I_1.$}

\[ \square \]

\[ \text{Remark 3.11. One can show that the inequalities of degenerations in Theorem 3.10 describe minimal degenerations. For the inequalities in (1) and the upper ones in (2), it follows from the facts that they come from a relation with respect to the stable hom order and, for (1), } \]

$$[I_i, I_i] = 2 \quad \text{for } 1 \leq i \leq n/2 \text{ and, for (2), } [I_i, I_i] = 2 \quad \text{for } 1 \leq i \leq (n-1)/2 \text{ and } [I_i, N_+ + N_-] = 2. \]

\[ \text{For the rest inequalities in (2), we show the minimality of } N_+ \leq_{st} N_+ \oplus I_1. \text{ Assume that we have a degeneration between them, that is, } N_+ \leq_{st} U \leq_{st} \tilde{N}_+ \oplus \tilde{I}_1 \text{ for some } U \in \text{CM}(R). \]

\[ \text{Then we claim that } U \cong N_+ \oplus I_1. \text{ Set } U = N_+ \oplus N_- \oplus \bigoplus_{i=1}^{n-1} I_i^{m_i}. \]

\[ \text{Since } N_+ \text{ and } U \text{ give the same class in the Grothendieck group, we have } \]

$$(n_+ - 1)[N_+] + n_-[N_-] + \sum_{i=1}^{n-1} m_i[I_i] = 0 \quad \text{in } K_0(\text{CM}(R)).$$

Note that $[I_i] = 0$ and $[N_+] + [N_-] = 0$ in the Grothendieck group of $\text{CM}(R).$ Moreover $[N_+] \neq [N_-] \text{ (cf. } \text{Remark 2.1]), so that they are non zero. If } n_+ > n_-, \text{ then } (n_+ - 1 - n_-)[N_-] = 0. \text{ Thus } n_+ = n_+ - 1. \text{ On the other hand, if } n_+ \leq n_-, \text{ then } (n_+ - n_+ + 1)[N_-] = 0. \text{ We also have } n_+ = n_+ - 1. \text{ This contradicts } n_+ \leq n_- \text{ and } n_+ = n_+ - 1. \text{ Hence } U = N_+ n_-^{-1} \oplus N_- n_-^{-1} \oplus \bigoplus_{i=1}^{(n-1)/2} I_i^{m_i}. \text{ And then } U = N_+ n_-^{-1} \oplus N_- n_-^{-1} \oplus \bigoplus_{i=1}^{(n-1)/2} I_i^{m_i} \leq_{hom} N_+ \oplus I_1. \text{ This induces that } N_+ n_-^{-1} \oplus N_- n_-^{-1} \oplus I_1^{m_1-1} \oplus \bigoplus_{i=2}^{(n-1)/2} I_i^{m_i} \leq_{hom} 0. \text{ Therefore } U \cong N_+ \oplus I_1.$$

In [3], the author also investigate the case when the dimension is odd. The essential part is the following.

\[ \text{Proposition 3.12. [4 Corollary 2.12, Proposition 3.10.] Let } R = k[[x]]/(x^{n+1}). \text{ Then the stable degeneration order coincides with the triangle order on } \text{CM}(R). \]

\[ \text{Proof. We give an outline of a proof for the convenience of the reader.} \]

\[ \text{Since } R \text{ is artinian, } M \text{ stable degenerates to } N \text{ if and only if } M \oplus P \text{ degenerates to } N \oplus Q \text{ for some free } R\text{-modules } P, Q. \text{ As shown in [4 Corollary 2.12], the degeneration order and the extension order are equivalent on } \text{CM}(R). \text{ That is, } N \oplus Q \text{ is obtained from } M \oplus P \text{ by iterative degenerations by extensions. Therefore } N \text{ is obtained from } M \text{ by iterative degenerations by triangles, so that } M \leq_{tri} N. \]

\[ \square \]

\[ \text{Proposition 3.13. [14 Corollary 6.6] Let } R_1 \text{ and } R_2 \text{ be isolated singularities. Suppose there is a } k\text{-linear equivalence } F: \text{CM}(R_1) \rightarrow \text{CM}(R_2) \text{ of triangulated categories. Then, for } M, N \in \text{CM}(R_1), M \text{ stably degenerates to } N \text{ if and only if } F(M) \text{ stably degenerates to } F(N). \]
The following lemma is known as the Kn"orrer’s periodicity (cf. [11 Theorem 12.10]).

**Lemma 3.14.** Let $k$ be an algebraically closed field of characteristic 0 and let $S = k[[x_0, x_1, \cdots, x_n]]$ be a formal power series ring. For a non-zero element $f \in (x_0, x_1, \cdots, x_n)S$, we consider the two rings $R = S/(f)$ and $R^2 = S[[y, z]]/(f + y^2 + z^2)$. Then the stable categories $\text{CM}(R)$ and $\text{CM}(R^2)$ are equivalent as triangulated categories.

Summing up these results, we have the main result of the paper.

**Theorem 3.15.** Let $R$ be a simple singularity of type $(A_n)$. Then the following hold on $\text{CM}(R)$.

1. If $R$ is of odd dimension and $n$ is an even integer, then the stable degeneration order coincides with the stable hom order. Particularly,

   \[
   I_1 \leq_{st} I_2 \leq_{st} \cdots \leq_{st} I_{n/2}.
   \]

   If $n$ is an odd integer, we have

   \[
   I_1 \leq_{st} I_2 \leq_{st} \cdots \leq_{st} I_{(n-1)/2} \leq_{st} N_+ \oplus N_-.
   \]

   and

   \[
   N_\pm \leq_{st} N_\pm + I_1 \leq_{st} \cdots \leq_{st} N_\pm + I_{(n-1)/2} \leq_{st} N_\pm + N_+ \oplus N_- \quad (\text{double sign corresponds}).
   \]

2. If $R$ is of even dimension, then the stable degeneration order coincides with the triangle order.

**Proof.** Combining Kn"orrer’s periodicity (Lemma 3.14) with Proposition 3.13, we have only to deal with the case $\dim R = 1$ to show (1) and the case $\dim R = 0$ to show (2). Hence, by Theorem 3.6 and Theorem 3.12 we obtain the assertion. \qed

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**References**

[1] K. Bongartz, *On degenerations and extensions of finite-dimensional modules.* Adv. Math. 121 (1996), 245–287.

[2] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings.* Preprint (1986), [http://hdl.handle.net/1807/16682](http://hdl.handle.net/1807/16682)

[3] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras,* London Mathematical Society Lecture Note Series 119. Cambridge University Press, Cambridge, 1988. x+208 pp.

[4] N. Hiramatsu and Y. Yoshino, *Examples of degenerations of Cohen-Macaulay modules,* Proc. Amer. Math. Soc. 141 (2013), no. 7, 2275–2288.

[5] N. Hiramatsu, *Degenerations of graded Cohen-Macaulay modules.* To appear in J. Commut. Algebra.

[6] O. Iyama and R. Takahashi, *Tilting and cluster tilting for quotient singularities,* Math. Ann. 356 (2013), no. 3, 1065–1105.

[7] B. Jensen, X. Su and A. Zimmermann, *Degenerations for derived categories.* J. Pure Appl. Algebra 198 (2005), no. 1-3, 281–295.

[8] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality.* J. Amer. Math. Soc. 15 (2002), no. 2, 295–366.

[9] C. Riedtmann, *Degenerations for representations of quivers with relations.* Ann. Scient. École Norm. Sup. 4e série 19 (1986), 275–301.

[10] M. Saorin and A. Zimmermann, *An axiomatic approach for degenerations in triangulated categories.* Preprint.

[11] Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings,* London Mathematical Society Lecture Note Series 146. Cambridge University Press, Cambridge, 1990. viii+177 pp.

[12] Y. Yoshino, *On degenerations of Cohen-Macaulay modules.* J. Algebra 248 (2002), 272–290.

[13] Y. Yoshino, *On degenerations of modules.* J. Algebra 278 (2004), 217–226.
[14] Y. Yoshino, Stable degenerations of Cohen-Macaulay modules, J. Algebra 332 (2011), 500–521.
[15] G. Zwara, A degeneration-like order for modules. Arch. Math. 71 (1998), 437–444.
[16] G. Zwara, Degenerations for modules over representation-finite algebras. Proc. Amer. Math. Soc. 127 (1999), 1313–1322.
[17] G. Zwara, Degenerations of finite-dimensional modules are given by extensions. Compositio Math. 121 (2000), 205–218.

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