A nanoscale window for probing Planck scale phenomena

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Abstract

The noncommutative space provides a framework to understand phenomena in Planck scale physics. However, there is no any direct experimental evidence to demonstrate the existence of noncommutative space. We propose an experimental scheme based on the Aharonov-Bohm effect in the nano-scale quantum mechanics to probe the phenomena in the noncommutative phase space. By the Seiberg-Witten map, the free electrons of the nano-scale ring in noncommutative phase space can be mapped equivalently to the quantum mechanical (Heisenberg’s algebra) phase space with an extra effective magnetic flux. We introduce two variables related to the persistent current in the ring to probe the noncommutative phase space effect. We give a value-independent criterion to detect the existence of the noncommutative phase space. Namely the answer for existence or nonexistence of the noncommutative phase space depends only on the trend of the curves, rather than the values of the observation data. It can be expected to catch the noncommutative phase-space effect by this scheme.

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I. INTRODUCTION

When physicist horizon extended from the macroscopic scale to the Planck scale create a weird world in the Planck scale, noncommutative geometry and algebra. The fundamental difficulties in theoretical physics, such as the singularity in particle physics, nonlocality and dark energy in quantum gravity and quantum cosmology, hint the possibility of existence of the finite length and time (Planck scale), by which physicist can understand consistently all phenomena in physics world.[1] The minimum length and time leads to noncommutative spacetime, namely noncommutative algebra and geometry rules the Planck world. What is the noncommutative spacetime? The noncommutative concepts in physics can be traced back to the angular momentum algebra in classical mechanics. These concepts are upgraded to any pair of conjugate variables called Heisenberg’s algebra in quantum mechanics. It infers that the elementary particles have an intrinsic uncertainty and nonlocal behavior in the microscopic world. These ideas are generalized further to noncommutative space to remove singularity in particle physics and to understand the nonlocality of gravity and quantum cosmology. [1] The figure 1 shows the family of noncommutative space. The Heisenberg’s space is defined by

\[ [x_i, p_j] = i\hbar \delta_{ij} \] 
\[ [x_i, x_j] = [p_i, p_j] = 0 \]  

where \( i, j = 1, 2, 3 \). This is also called as a canonical commutative relation or Heisenberg’s algebra. This noncommutative algebra can be generalized to [2]

\[ [\hat{x}_i, \hat{p}_j] = i\hbar \Delta_{ij} \] 
\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij} \] 
\[ [\hat{p}_i, \hat{p}_j] = i\tilde{\theta}_{ij} \]  

where \( \theta_{ij} \) and \( \tilde{\theta}_{ij} \) are antisymmetric real constant \( 3 \times 3 \) matrices. These noncommutative algebra are equivalent to the Weyl-Moyal correspondence in field theories on noncommutative spaces [1] and can be realized by the Seiberg-Witten map, [2]

\[ \hat{x}_\mu = a_{\mu\nu}x_\nu + b_{\mu\nu}p_\nu \] 
\[ \hat{p}_\mu = c_{\mu\nu}x_\nu + d_{\mu\nu}p_\nu \]
between the canonical commutative relation and noncommutative phase space. For two-dimensional system, the noncommutative matrices can be obtained by the Seiberg-Witten map

$$\Delta_{ij} = \left( \begin{array}{cc} \alpha^2 + \frac{\theta\bar{\theta}}{2\alpha^2h^2} & \frac{\theta\bar{\theta}}{4\alpha^2h^2} \\ \frac{\theta\bar{\theta}}{4\alpha^2h^2} & \alpha^2 + \frac{\theta\bar{\theta}}{2\alpha^2h^2} \end{array} \right)$$

with

$$\theta_{ij} = \frac{1}{\hbar} \left( \begin{array}{cc} 0 & \theta \\ -\theta & 0 \end{array} \right); \quad \bar{\theta}_{ij} = \frac{1}{\hbar} \left( \begin{array}{cc} 0 & \bar{\theta} \\ -\bar{\theta} & 0 \end{array} \right)$$

If we may set the constraint of $\alpha$ and $(\theta, \bar{\theta})$, $\alpha^2 + \frac{\theta\bar{\theta}}{2\alpha^2h^2} = 1$ such that $\Delta_{ij} = \left( \begin{array}{cc} 1 & \frac{1-\alpha^2}{2} \\ \frac{1-\alpha^2}{2} & 1 \end{array} \right)$. When $\alpha = 1$, $\Delta_{ij} = \delta_{ij}$ and $\theta\bar{\theta} = 0$.

Interestingly, Connes and Rieffel construct a mathematical formulation on noncommutative geometry and algebra, which provide a mathematical language of noncommutative space physics.[3] However, so far there has not been direct experimental evidences to demonstrate the existence of noncommutative geometry or space even though a lot of phenomena in particle physics and gravity can be described by the noncommutative space or phase-space language. The difficulty to directly find out experimental evidence of the existence of noncommutative space is because the noncommutative space phenomena are predicted occurring in the Planck scale.
In this paper, we apply the Seiberg-Witten map to study the Aharonov-Bohm effect in noncommutative phase space. The noncommutative effect induces an effective magnetic flux that produces the persistent current in a mesoscopic ring. We propose a scheme to probe the effective magnetic flux coming from the noncommutative phase space even the effect is very weak. Since the persistent current and magnetic flux in mesoscopic rings can be implemented by nanotechnology, it can be expected to detect the noncommutative phase space effect by this scheme.

II. QUANTUM RING IN NONCOMMUTATIVE PHASE SPACE

The Hamiltonian of the two-dimensional (2D) free electron in noncommutative phase space can be written as

\[ H_{nc} = \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) \]  

which can be mapped to the Heisenberg’s (canonical) commutative space

\[ H_{nc} = \frac{1}{2m^*} [(p_x + eA_x)^2 + (p_y + eA_y)^2] \]  

where \( m^* = m/\alpha \) is the effective mass and the effective vector potential is

\[ A_x = \frac{\tilde{\theta}}{2e\alpha^2 \hbar} y; \quad A_y = -\frac{\tilde{\theta}}{2e\alpha^2 \hbar} x \]  

and the effective magnetic field

\[ B_z = \frac{\tilde{\theta}}{e\alpha^2 \hbar} \]  

It can be seen that the free electron in the 2D noncommutative phase space is equivalent to electron in an effective magnetic field induced by the noncommutative phase space effect, which provides a way to detect the existence of noncommutative phase space by probing the effect of the effective magnetic field or vector potential coming from the noncommutative phase space.

We consider a one-dimensional ring in an external magnetic field along the ring axis. The magnetic field is constant inside the ring such that the electron states depend only on the total magnetic flux in the ring. In the polar coordinate system, \( x = R \cos \varphi, y = R \sin \varphi \), by using the Seiberg-Witten map, the Hamiltonian in noncommutative phase space is written as
FIG. 2: (Color online): The nano scale ring in the noncommutative phase space on the left side, which is mapped (Seiberg-Witten) to Heisenberg’s (canonical) space on the right side.

\[ H_{nc} = -\frac{\hbar^2}{2m^*R^2} \left[ \frac{\partial}{\partial \phi} + i \left( \frac{\phi}{\phi_0} - \frac{\phi_{nc}}{\phi_0} \right) \right]^2 - \frac{3\hbar^2}{8m^*R^2} \frac{\phi_{nc}^2}{\phi_0^2} \] (14)

where \( \phi_{nc} = \frac{2\pi R^2 \tilde{\theta}}{e\hbar a^2} \) is the effective magnetic flux coming from the noncommutative phase space and \( \phi_0 = \frac{\hbar}{e} \) is the flux quanta \( (e < 0) \). \( \phi \) is the external magnetic flux in the ring. (see Fig. 2) For convenience, we introduce the dimensionless magnetic flux, \( f_{nc} \equiv \frac{\phi_{nc}}{\phi_0} \) and \( f = \frac{\phi}{\phi_0} \), the Hamiltonian of quantum ring can be rewritten as

\[ H_{nc} = -\varepsilon_0 \left[ \frac{\partial}{\partial \phi} + i (f - f_{nc}) \right]^2 - \frac{3\varepsilon_0}{4} f_{nc}^2 \] (15)

where \( \varepsilon_0 \equiv \frac{\hbar^2}{2m^*R^2} \). The eigenenergy can be obtained by [4]

\[ E_n = \varepsilon_0 (n + f - f_{nc})^2 - \frac{3\varepsilon_0}{4} f_{nc}^2 \] (16)

where \( n = 0, \pm 1, \pm 2, \ldots \). It can be seen that the effective magnetic flux induced by the noncommutative space modifies the eigenenergy levels. Since \( n = 0, \pm 1, \pm 2, \ldots \), and notice that the eigenenergies in Eq. (16) are invariant under \( f - f_{nc} \rightarrow f - f_{nc} + 1 \), we can consider only the domain of \( f - f_{nc} \) within \([ -\frac{1}{2}, \frac{1}{2} ] \) (the first Brillouin flux zone).[6, 7] Suppose there are \( N \) electrons in the ring, and they occupy the energy levels in zero temperature, notice that for the odd-electron ring, \( N = 2k + 1 \), the ground-state energy is \( E_g = \sum_{n=0,\pm 1,\pm 2, \ldots}^{\pm k} E_n \),
and for the even-electron ring, $N = 2k$, $E_g = \left( \sum_{n=0,1,2,\ldots}^{k-1} + \sum_{n=-k,-k+1,-k+2,\ldots}^{-k} \right) E_n$, the ground-state energy of the ring is obtained [4]

$$E_g = \varepsilon_0 \begin{cases} \frac{N^3 - N}{12} + N \left[ (f - f_{nc})^2 - \frac{3}{4} f_{nc}^2 \right] & \text{for } N = 2k + 1 \\ \frac{N^3 + 2N}{12} - N(f - f_{nc}) + N \left[ (f - f_{nc})^2 - \frac{3}{4} f_{nc}^2 \right] & \text{for } N = 2k \end{cases}$$

(17)

Moreover, the ground-state energy is symmetric for $f - f_{nc} = 0$, we can restrict our attention to a half of the first Brillouin flux zone, $[0, \frac{1}{2}]$. The persistent current in the ground state is defined by $J = -\frac{\partial E_g}{\partial \phi}$ and it can be obtained [4]

$$J = J_0 \begin{cases} -2N \frac{\phi}{\phi_0} \left( 1 - \frac{\phi}{\phi_0} \right) & \text{for } N = 2k + 1 \\ N - 2N \frac{\phi}{\phi_0} \left( 1 - \frac{\phi}{\phi_0} \right) & \text{for } N = 2k \end{cases}$$

(18)

where $J_0 = \frac{e}{\hbar} \varepsilon_0$. The persistent current depends on both of the external magnetic flux and the effective magnetic flux induced by the noncommutative phase space. This relationship between the persistent current and the magnetic flux in Eq. (18) provides a way to reveal the noncommutative phase space effect.

### III. SIGNATURES OF NONCOMMUTATIVE-PHASE-SPACE EFFECT

In order to detect the noncommutative-phase-space effect experimentally, we introduce two variables defined by [4]

$$\lambda \equiv \frac{\partial}{\partial \phi} \left( \frac{J}{\phi} \right)$$

(19)

$$\sigma \equiv \frac{\partial}{\partial \phi} \left( \frac{J - NJ_0}{\phi} \right)$$

(20)

as two signatures to detect the noncommutative-phase-space effect experimentally. Thus, we get

$$\lambda = \begin{cases} -2NJ_0 \frac{f_{nc}}{\phi^2} & \text{for } N = 2k + 1 \\ -NJ_0 \left( 1 + 2f_{nc} \right) \frac{1}{\phi^2} & \text{for } N = 2k \end{cases}$$

(21)

and

$$\sigma = \begin{cases} NJ_0 \left( 1 - 2f_{nc} \right) \frac{1}{\phi^2} & \text{for } N = 2k + 1 \\ -2NJ_0 \frac{f_{nc}}{\phi^2} & \text{for } N = 2k \end{cases}$$

(22)
FIG. 3: (Color online): The signatures of noncommutative phase space for even electron-number rings versus the external magnetic flux. (a) \( \log \lambda \sim \phi \) and \( \log \sigma \sim \phi \) with different electron numbers. (b) \( \log \lambda \sim \log \phi \) and \( \log \sigma \sim \log \phi \) a standard of \( 1/\phi \) behavior. The unit of the magnetic flux is the magnetic flux quantum \( \phi_0 \).

It can be seen that when there exists the noncommutative phase space, both \( \lambda \) and \( \sigma \) are proportional to \( \frac{1}{\phi^2} \), which diverge in the small external magnetic fluxes.

For given parameter \( \tilde{\theta} \leq 1.76 \times 10^{-61}[kg]^2[m]^2[s]^{-2} \) [5] and a mesoscopic ring with radius \( R = 1\mu m \), \( f_{nc} \leq \frac{2\pi R^2 \tilde{\theta}}{\epsilon \hbar^2 \hbar/e} = \frac{R^2 \tilde{\theta}}{\hbar^2 c^2} = 1.5828 \times 10^{-5} \). Suppose that the effective electron number in the ring is about \( 10^4 \sim 10^5 \), we show some theoretical predictions in Figures 3 and 4 based on this idea and parameters. The figure 3 (a) shows the theoretical prediction of the \( \lambda \sim \phi \) behavior in nano-ring with radius \( R = 1\mu m \) with odd-electron number. It can be seen that the \( \lambda \) is divergent in the small \( \phi \) range. Similarly in Fig. 3 (b), we shows the behavior of \( \sigma \sim \phi \). It can be seen that both \( \lambda \) and \( \sigma \) for odd electron number rings have a similar behavior, but the sign is different.

For the even-electron number rings, the behaviors of \( \lambda \) and \( \sigma \) are similar and have only a two-order difference, which are shown in Fig. 4.
FIG. 4: (Color online): The signatures of noncommutative phase space for even-electron number rings versus the external magnetic flux. log \( \lambda \sim \phi \) and log \( \sigma \sim \phi \) with different electron numbers. The unit of the magnetic flux is the magnetic flux quantum \( \phi_0 \).

IV. CRITERION AND SCHEME FOR DETECTING NONCOMMUTATIVE-PHASE-SPACE EFFECT

We can give a criterion to directly detect the existence of the noncommutative phase space.

**Criterion:** if one of the following two cases occurs it infers the existence of the noncommutative phase space:

1. \( \lambda \sim \phi \) is \(-1/\phi^2\) divergent but \( \sigma \sim \phi \) is \(1/\phi^2\) divergent (see Fig. 3);
2. both \( \lambda \sim \phi \) and \( \sigma \sim \phi \) are \(-1/\phi^2\) divergent and \( \lambda < \sigma \) (see Fig. 3).

If (1) occurs it infers that there is odd number of electrons in the ring and (2) means the ring having even number of electrons.

Based on this criterion, we propose an experimental scheme to demonstrate explicitly the existence of the noncommutative phase space.

**Experimental Scheme:** The basic steps include

1. setting up a mesoscopic ring system with an external magnetic field;
2. measuring the persistent current \( J \) versus the external magnetic flux \( \phi \);
3. calculating \( \lambda \) and \( \sigma \) by using the numerical interpolation and derivative techniques.
for estimating the electron number $N$;

(4) plotting $\lambda$ and $\sigma$ versus $\phi$, if we can obtain a qualitative the behaviors of $\lambda$ versus $\phi$ or $\sigma$ versus $\phi$ in Figs 3 and 4, they demonstrate the existence of the noncommutative phase space.

It should be emphasized that the resolution of the numerical estimations in Figs 3 and 4 are based on the parameters of the Planck scale and nano size of the ring. The comparison between the experimental data and the theoretical prediction in Figs 3 and 4 are meaningful and valid for all qualitative level because actually we do not know the exact values of these parameters. In other words, the divergent behavior $\pm 1/\phi^2$ from the experimental data provides a way to predict the values of the noncommutative parameter $\tilde{\theta}$. In practice, we can plot the $\log \lambda \sim \log \phi$ such that we can verify the divergent behavior $\pm 1/\phi^2$ as a linear relation.

In fact, the persistent current in mesoscopic ring have been studied both theoretically and experimentally in the past two decades. [8, 9, 11] Buttiker first predicted that the persistent current occurs in mesoscopic ring and oscillates with an AB flux.[8] The amplitude of the persistent current reaches $(10^{-2} \sim 2) e v_F / 2\pi R$ where $v_F$ is the electronic velocity at Fermi level in the $Cu$ multi-ring system, an isolate $Au$ ring and $GaAs/Al_xGa_{1-x}As$ in the diffusive region at low temperature.[9–11], which agrees with the theoretical prediction. This nanotechnology provides an efficient way to probe the noncommutative phase space. We strongly suggest to rerun the persistent-current experiment to turn out the persistent versus the external magnetic flux data, by which we can investigate the relationship between $(\lambda, \sigma)$ and $\phi$ for detecting the existence of the noncommutative phase space.

On the other hand, Carroll et. al. studied the noncommutative field theory and Lorentz violation. They gave an upper bound of the noncommutative parameter, $\tilde{\theta} \leq (10 TeV)^{-2}$. [12] Falomir et. al. also proposed a scheme to explore the spatial noncommutativity of the scattering differential cross section by the AB effect.[13] It relies on the particle physics experiment involving energies between 200 and 300 GeV for $\tilde{\theta} \leq (10 TeV)^{-2}$ and estimating the typical order of the cross section for neutrino events $10^{-3}$. [13] Obviously, that experimental schemes are much more difficult to be implemented than this scheme because this experimental scheme involves $eV$ energy scale and nanoscale physics.

In fact, there has not been any direct experimental evidences to demonstrate the existence of noncommutative space and phase space, it should be worth studying and exploring from
different aspects and different energy scales even though the concept of noncommutativity originates from the Planck scale physics. Actually, many phenomena in condensed matter physics show the characteristics of noncommutative space in nonrelativistic quantum mechanics, such as an analogy between the Landau’s levels of two-dimensional electron gas in the presence of magnetic field and the free electron in noncommutative phase space,[14, 15], as well as quantum Hall effect.[16], it should be expected some possibilities to catch the noncommutativity especially in the nanoscale condensed matter physics.

V. CONCLUSIONS

In summary, we propose an experimental scheme to probe the existence of noncommutative phase space by the nano-scale quantum ring based on the Seiberg-Witten map. We introduce two variables $\lambda$ and $\sigma$ as two signatures to detect the effect of the noncommutative phase space based on the relationship between the persistent current and the external magnetic flux in the ring. The divergent behaviors of $(\lambda, \sigma)$ versus $\phi$ provide a value-independent scheme for the experimental measurement to catch the effect of the noncommutative phase space. It can be expected to give the answer whether exists the noncommutative phase space.

The noncommutative Heisenberg’s algebra plays a crucial role in quantum mechanics. It
provides an efficient way to understand many novel phenomena beyond classical physics, such as the wave-particle duality of elementary particles, uncertainty of a pair conjugated variables, quantization of physical variables, and nonlocal quantum entanglement. These novel properties in microscopic world imply that the noncommutative Heisenberg’s algebra could be extended to the noncommutative spacetime or phase space to avoid the singularity in particle physics, gravity and early universe in the Planck scale world. It should be a meaningful direction to explore the noncommutativity of space or phase space in the mesoscopic scale world because it is too difficult to directly implement physical experiments in Planck scale. This scheme gives a nano scale window to explore noncommutativity in the Planck’s scale world.

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