Classical and quantum Reissner-Nordström black hole thermodynamics and first order phase transition

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Abstract First we consider classical Reissner-Nordström black hole (CRNBH) metric which is obtained by solving Einstein-Maxwell metric equation for a point electric charge $e$ inside of a spherical static body with mass $M$. It has 2 interior and exterior horizons. Using Bekenstein-Hawking entropy theorem we calculate interior and exterior entropy, temperature, Gibbs free energy and heat capacity at constant electric charge. We calculate first derivative of the Gibbs free energy with respect to temperature which become a singular function having a singularity at critical point $M_c = \frac{2|e|}{\sqrt{3}}$ with corresponding temperature $T_c = \frac{1}{24\pi \sqrt{3}|e|}$. Hence we claim first order phase transition is happened there. Temperature same as Gibbs free energy takes absolutely positive (negative) values on the exterior (interior) horizon. The Gibbs free energy takes two different positive values synchronously for $0 < T < T_c$ but not for negative values which means the system is made from two subsystem. For negative temperatures entropy reaches to zero value at $T \to -\infty$ and so takes Bose-Einstein condensation single state. Entropy increases monotonically in case $0 < T < T_c$. Regarding results of the work presented at Wang and Huang (Phys. Rev. D 63:124014, 2001) we calculate again the mentioned thermodynamical variables for remnant stable final state of evaporating quantum Reissner-Nordström black hole (QRNBH) and obtained results same as one in case of the CRNBH. Finally, we solve mass loss equation of QRNBH against advance Eddington-Finkelstein time coordinate and derive luminosity function. We obtain switching off of QRNBH evaporation before than the mass completely vanishes. It reaches to a could Lukewarm type of RN black hole which its final remnant mass is $m_{\text{final}} = |e|$ in geometrical units. Its temperature and luminosity vanish but not in Schwarzschild case of evaporation. Our calculations can be take some acceptable statements about information loss paradox (ILP).

Keywords Reissner-Nordström black holes · Negative temperatures · Heat capacity · Phase transition · Dark matter · Gibbs free energy · Liquid helium · Bose-Einstein condensation · Quantum fields · Backreaction · Luminosity · Mass loss · Information loss paradox

1 Introduction

Since the seminal work of Hawking (1974) and Bekenstein (1973), we have understood that black holes behave as thermal objects containing characteristics such as temperature, entropy and et cetera. Hawking radiation has not yet been directly observed, of course; a typical stellar mass black hole has a Hawking temperature of well under a micro-Kelvin, far lower than that of the cosmic microwave background temperature $\approx 2.7$ K. However the thermal properties of black holes are studied in the literature and there is well understood that they have temperature

$$k T_{\text{Hawking}} = \frac{\hbar \kappa}{2\pi}$$

and entropy

$$S_{BH} = \frac{A_{\text{horizon}}}{4G \hbar}.$$
Fig. 1 Diagram of the entropy (17), temperature (18) and heat capacity (20) are plotted against $M$ with ‘dot’, ‘dash’ and ‘solid’ lines respectively for particular charges $e = \pm 1$. Temperature values on the vertical axis are re-scaled as $\times 10000$ but not for entropy and heat capacity. A $-$ phase transition is happened at positive critical temperature $T_c(M_c) = \frac{1}{24\pi \sqrt{3}} |e|$, with $M_c = \frac{2|e|}{\sqrt{3}}$.

Surface gravity by applying second law of the black hole thermodynamics:

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega_H \delta J + \Phi_H \delta e$$

(3)

where ‘$\Omega_H$’ is angular velocity and ‘$\Phi_H$’ is electric potential given on the horizon. Also ‘$e$’ and ‘$M$’ is electric charge and mass of the black hole respectively. Surface area of the event horizon of a black hole never decreases such that

$$\delta A \geq 0$$

(4)

In a typical thermodynamic system, temperature is a measure of average energy of microscopic constituents and entropy counts the number of microstates. However one can see that the Bekenstein-Hawking entropy depends on both Plank’s and Newton’s constant and so obtain a good statement: Statistical mechanic description of black hole thermodynamics might be tell us something profound about quantum gravity (Valeri and Novikov 1998; Papantonopoulos 2009). A compendious review about the black hole thermodynamics and its relation with other topics as quantum gravity, generalization of thermodynamic laws, statistical properties and information loss paradox is given in Carlip (2015) (see also references therein).

This encourages us to study some thermodynamical aspects about the CRNBH metric. Here we calculate its entropy $A(M, e)$, temperature $T(M, e)$, Gibbs free energy $G(M, e)$ and heat capacity $C_e(M, e)$ at constant charge $e$ on the interior (Cauchy) and exterior horizons. We obtain corresponding temperature takes negative (positive) values for interior (exterior) horizon of the CRNBH metric (see Figs. 1 and 2). For negative temperatures on the interior horizon the Gibbs free energy takes negative values but for positive temperatures will be have positive values (see Fig. 3). Entropy and so the black hole microstates degrees of freedom decreases by increasing initial mass $M$ (compare dot line at Figs. 1 and 2). This happened for negative temperatures (compare dash line at Figs. 1 and 2). Thus we can result the entropy reaches to zero value when negative tem-
temperatures approach to negative infinity and so the black hole can be take Bose-Einstein condensation state with zero momentum and minimum energy (see Fig. 4). We obtain corresponding heat capacity calculated on exterior horizon which exhibits with discontinuity at critical point $M_c = \frac{2|e|}{\sqrt{3}}$. This means that phase transition is happened there (see solid line at Fig. 1).

Phase transition phenomena and Bose-Einstein condensation state are well known properties in statistical systems. For instance a $\Lambda$-transition type is happened usually for isotope He$^4$ of liquid helium at low critical temperatures as 2.17 K. The isotope He$^3$ has nuclear spin $\frac{1}{2}$ and so obeys Fermi-Dirac statistics, while the isotope He$^4$ with nuclear spin 0 obeys Bose-Einstein statistics. At very low temperatures where quantum effects become important, He$^3$ and He$^4$ as quantum fluids will have identical chemical properties but with different masses or energies. Two isotopes He$^3$ and He$^4$ exhibit very different behavior due to the difference in their statistics. Liquid He$^4$ which is a boson liquid exhibits at rather straightforward transition to a superfluid state at 2.19 Kelvin. This can be understood as a (Bose-Einstein) condensation of particles into a single quantum state with zero momentum. Liquid He$^3$ also undergoes at transition to a superfluid state, but at a much lower temperature ($2.7 \times 10^{-3}$) Kelvin. The mechanism for a latter isotope is different from the liquid He$^4$ where quasi-particles form bound pairs with spin $s = 1$ and relative angular momentum $l = 1$. One can see experimental diagram of $\Lambda$-transition for liquid He$^4$ system at Fig. 7.5 of Pathria (1972) or Fig. 4.22 of Reichl (1987) (see also p. 153 in Huang 2001).

According to the latter phenomena about the liquid helium we see that RHS of Fig. 3 in this article shows that matter content of the CRNBH namely $M$ takes two different Gibbs free energy at temperatures $0 < T < T_c$. This means the CRNBH will be has two different phases (two subsystem) at particular temperature given by $0 < T < T_c$. These 2 phases reach to a single state at critical temperature $T_c = \frac{|e|}{24\sqrt{3}m_\ast}$. LHS of Fig. 3 shows that the CRNBH takes Bose-Einstein condensation state when temperature approaches to negative infinite values $T \rightarrow -\infty$, where entropy takes zero value and so microstates degrees of freedom vanishes. This happened on the interior horizon of the CRNBH metric.

Gibbs free energy is the chemical potential that is minimized $\Delta G = 0$ when a system reaches equilibrium at constant pressure and temperature. In other words its derivative with respect to the reaction coordinate of the system vanishes at the equilibrium point. A reaction with a negative (positive) Gibbs free energy will (not) proceed spontaneously. In other words when $\Delta G < 0$ ($\Delta G > 0$), the system will be has spontaneous (non-spontaneous) reactions with constant pressure and temperature.

Negative temperatures have physical meaning and is possible for ordinary statistical systems if there exists an upper limit for the energy of the given system. Ordinary systems such as freely moving particle or a harmonic oscillator possess usually kinetic energy of motion which is obviously unbounded and so cannot be suitable candidates for systems with negative temperature. In these ordinary translational and vibrational degrees of freedom (and so entropy) of a body increases without limit as the energy increases. Two-level spin systems for instance magnetic dipoles in presence of external magnetic field are suitable candidates where the system exhibits with negative temperature when population of dipoles with higher energy state is more than that in the lower energy state (see for instance Sect. 3.9, Pathria 1972). For these systems variations of entropy with respect to their energy takes negative sign $\Delta A < 0$ (for instance see dot line in Fig. 2 given in this work).

In summary, our study about thermodynamical aspect of the CRNBH metric predicts that it is formed from two subsystem: (a) Matter content inside of interior horizon $0 < r < r_-$ with negative temperatures, reaching to Bose-Einstein condensation state and (b) matter content inside between interior and exterior horizon $r_- < r < r_+$. with positive temperatures exhibiting to a first order phase transition.

At second part of the paper we extend aim of the work to a QRNBH metric by using results of the work presented by Wang and Huang (2001). We calculate again its entropy, temperature, Gibbs free energy and heat capacity at constant electric charge. Bobo et al. solved time-independent back-reaction metric equation of quantum perturbed RN black hole in the presence of massless, charge-less quantum scalar field. They obtained metric of final state as remnant stable static black hole which its horizon radiiuses are greater than the classical horizon radiiuses $r_{\pm}^{Q} > r_{\pm}$.
Our mathematical calculations for QRNBH in static regime of the backreaction metric solution predict results same one which obtained in cases of CRNBH metric (see Figs. 5, 6, 7, 8 and compare with Figs. 1, 2, 3, 4). Also we calculate luminosity of evaporating QRNBH and obtained a switching off for it before that its mass function disappear completely. Also we give some suitable statement about ILP.

Organization of the paper is given as follows.

In Sect. 2 we call CRNBH metric as static solution of Einstein-Maxwell equation. Also we discuss its stress tensor which treats as an anisotropic hydrostatic fluid with negative barotropic and anisotropy indexes. Also we call its internal and external horizon radiuses. In Sect. 3 we calculate entropy, positive temperatures, Gibbs free energy and heat capacity on its exterior horizon. In Sect. 4 we calculate entropy, negative temperature, Gibbs free energy and heat capacity on its interior horizon. Also we discuss about its Bose Einstein condensation state. In Sect. 5 we call QRNBH interior and exterior horizons by using results of Wang and Huang (2001) and calculate corresponding entropy, temperature, Gibbs free energy and heat capacity at constant electric charge by using results of Wang and Huang (2001). We
obtain time dependent mass function by solving mass loss equation. Also we calculate luminosity of the QRNBH and give some acceptable physical statements about ILP. Section 6 denotes to concluding remark.

2 Classical Reissner-Nordström black hole

CRNBH is metric solution of Einstein-Maxwell equation $G_{\mu\nu} = 8\pi T^{EM}_{\mu\nu}$ where

$$T^{EM}_{\mu\nu} = -F_{\mu\alpha}F^{\alpha}_{\nu} + \frac{1}{4}g_{\mu\nu}(F_{\alpha\beta}F^{\alpha\beta})$$

(5)

is electromagnetic fields stress tensor of antisymmetric electromagnetic tensor field $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The above stress tensor is written with metric signature ($-,+,+,+$) by setting $c = G = h = 1$. It is trace free as $g^{\mu\nu}T^{EM}_{\mu\nu} = 0$ because the electromagnetic fields propagate at the invariant speed $c$ evaluated in all reference frames by all observers. The invariance of speed of light is one of postulates of special (and also general) relativity theory. In various alternative gravity theories ‘variable speed of light’ (VSL) is accepted as hypothesis because velocity of the light rays take slow down when traveling through a medium. VSL should not be confused with faster than light theories. Notable VSL attempts have been done by Einstein itself (Einstein 1907) and other researchers as Dicine (1957) (see also Clayton and Moffat 1999).

In case of unmoving point particle with electric charge $e$ and negligible mass $m_e \ll M$, located inside of spherical body with mass $M$, one can obtain nonzero components of the electromagnetic fields as $F_{tr} = -F_{rt} = E(r) = \frac{e}{r}$ in units $\frac{1}{4\pi\epsilon_0} = 1$. In the latter case the electromagnetic field stress tensor (5) takes a simple form as

$$T^{(EM)}_{\mu\nu} = \frac{1}{8\pi} \left( \frac{e}{r^2} \right)^2 \delta(1 1 -1 -1).$$

(6)

If we assume that the above electric field stress tensor treats same as anisotropic hydrostatic perfect fluid, then corresponding effective density will be

$$\rho_{eff} = T^t_t = \frac{e^4}{8\pi r^4},$$

(7)

effective radial pressure become

$$p^t_{eff} = T^r_r = \frac{e^4}{8\pi r^4},$$

(8)

and effective transverse (tangential) pressure is

$$p^t_{eff} = T^\theta_\theta = T^{\phi\phi} = - \frac{e^4}{8\pi r^4}.$$  

(9)

Corresponding hydrostatic pressure $p$ is defined by

$$p = \frac{p^r + 2p^t}{3} = - \frac{e^4}{24\pi r^4}.$$  

(10)

and anisotropic stress tensor is defined by

$$\Pi^\mu_{\nu} = (p^r - p^t) \delta_{\nu\mu} \left\{ \begin{array}{c} \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \end{array} \right\}$$

$$= \frac{e^4}{4\pi r^4} \delta_{\nu\mu} \left\{ \begin{array}{c} \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \end{array} \right\}.  \quad (11)$$

Inserting (7), (8), (9) and (10), one can obtain corresponding barotropic and anisotropy indexes respectively as

$$\gamma = \frac{p_{eff}}{\rho_{eff}} = -\frac{1}{3}$$

(12)

and

$$\Delta = \frac{p^t_{eff} - p^r_{eff}}{\rho} = -2.  \quad (13)$$

The barotropic index (12) has negative value and so negative pressure of electric field of point particle produces anti-gravity (repulsive force), relative to attractive force of the mass $M$ in the CRNBH metric

$$ds^2 = -\left( 1 - \frac{2M}{r} + \frac{e^2}{r^2} \right)dt^2 + \frac{dr^2}{\left( 1 - \frac{2M}{r} + \frac{e^2}{r^2} \right)}$$

$$+ r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  

(14)

The above metric solution is obtained by inserting (6) and solving the Einstein-Maxwell equation $G_{\mu\nu} = 8\pi T^{EM}_{\mu\nu}$ for a spherically symmetric static line element

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  

(15)

The metric solution (14) is asymptotically flat and called as metric of CRNBH. Exterior and interior event horizons is obtained by solving $g_{tt}(r) = 0$ as $r_+ = M + \sqrt{M^2 - e^2}$ and $r_- = M - \sqrt{M^2 - e^2}$ respectively but its apparent horizon is obtained from equation $g^{tt}(\partial_r \delta_{tt}) = 0$ as $r_{\pm} = M \pm \sqrt{M^2 - e^2}$. Its interior (exterior) event horizon coincide with interior (exterior) apparent horizon. Interior horizon is called usually as ‘Cauchy’ horizon and cannot be seen via observers located outside of exterior horizon. All horizons exist for $0 \leq |e|_M \leq 1$ and for $|e|_M > 1$ the horizons are disappeared and so the metric solution (14) leads to Schwarzschild space time where $r_+ = 2M$, $r_- = 0$ (Wald 1984; Ghaffarnejad et al. 2013).

It is well known, negative values for barotropic index of an arbitrary fluid is usually related to dark matter and dark energy which can be support carefully acceleration of our expanding universe. Also a true cosmological constant $\Lambda$ may be responsible for the data as $\gamma_{DE} = -1$ but it is possible that a dynamical mechanism is at work. Dark energy has several candidates as quintessence and non-canonical (negative kinetic energy) $k$-essence scalar fields. Experimental tests determine $-1.38 < \gamma_{DE} < -0.82$ for the barotropic
index of dark energy by using WMAP data and CMB experiments (Melchiorri et al. 2003). On the other hand strong evidence, from a large number of independent observations indicates that dark matter is composed by yet unknown weakly interacting elementary particles. Since these particles are required to have small random velocities at early times, they are called cold dark matter (CDM) with barotrophic index $-1 < \gamma_{CDM} < 0$ (Serra and Romero 2011).

Equation (12) accords the dark matter barotrophic index and so one can obtain a correspondence between electric field stress tensor of point particle (6) and unknown CDM stress tensor. However, physical effects of the electric charge stress tensor (6) cases to break Schwarzschild black hole horizon radius $r_{Sch} = 2M$ to two different horizon radiuses called as interior horizon $r_- = M - \sqrt{M^2 - e^2}$ and exterior horizon $r_+ = M + \sqrt{M^2 - e^2}$ for CRNBH which both are less than the Schwarzschild one.

In the following we will restrict ourselves to a typical situation $e^2 < M^2$ for which the horizons are not destructed and so there is not naked singularity. Physically the condition $e^2 < M^2$ means the black hole has a remnant case and so may have implications about ILP. We will discuss about ILP at last section of the paper. We calculate now entropy temperature, Gibbs free energy and heat capacity of CRNBH metric (14) at constant electric charge for both interior and exterior event horizons.

### 3 Entropy on exterior horizon

The CRNBH is static with no angular momentum $\Omega_H = 0$ and so with no electric current density $J = 0$. Thus it cannot contribute to electromagnetic interactions and its electric charge will be invariant with no electromagnetic radiation but can be contribute with gravitational and electrostatic self-interactions only. With these boundary conditions one can rewrite (3) for the CRNBH as

$$\delta M = T \delta A + \Phi \delta e$$

where $A$, $T$, $M$ and $\Phi$ are entropy, temperature, mass and electric potential respectively defined on horizons hypersurface. First we calculate mentioned thermodynamical functions on the exterior horizon of CRNBH metric as follows.

#### 3.1 Positive temperatures

Inserting exterior horizon radius $r_+ = M + \sqrt{M^2 - e^2}$, surface area defined by $A_+ = 4\pi r_+^2$, become

$$A_+(M, e) = 4\pi \left( M + \sqrt{M^2 - e^2} \right)^2.$$  

(17)

According to the Bekenstein-Hawking entropy theorem Eq. (17) describes entropy of exterior horizon counter part of the CRNBH. Using (16) and varying (17) with respect to $M$ and $e$, one can obtain corresponding temperature $T_+$ and electric potential $\Phi_+$ respectively as

$$T_+(M, e) = \frac{\sqrt{M^2 - e^2}}{8\pi (M + \sqrt{M^2 - e^2})^2}$$

(18)

and

$$\Phi_+(M, e) = \frac{e}{M + \sqrt{M^2 - e^2}}$$

(19)

which become real functions only for $M^2 \geq e^2$. So the CRNBH should be has remnant case with mass lower limit as $M_{min} = |e|$. Heat capacity of exterior horizon at constant electric charge $e$ is obtained as

$$C_e^+(M, e) = 8\pi \left[ \frac{(M + \sqrt{M^2 - e^2})^2 \sqrt{M^2 - e^2}}{M - 2\sqrt{M^2 - e^2}} \right]$$

(20)

by using the definition

$$C_e^+(M, e) = T_+(M, e) \frac{\partial A_+(M, e)}{\partial T_+(M, e)} e$$

$$= T_+(M, e) \frac{\partial A_+(M, e)}{\partial M} e.$$  

(21)

Diagram of the solutions (17), (18) and (20) are plotted against $M$ for particular electric charges $e = \pm 1$ in Fig. 1. Diagram of the heat capacity (20) exhibits with a singularity (see solid line in Fig. 1) at critical mass $M_c = \frac{2|e|}{\sqrt{3}} \simeq 1.1547|e|$ and its sign is changed from positive to negative values (Ghaffarnejad 2013). Temperature and entropy do not exhibit with singularity and take positive values always for all $M$ (see dash and dot lines respectively in Fig. 1). Entropy diagram satisfies (4).

#### 3.2 Phase transition

First (second) order phase transitions is happened in ordinary statistical systems when first (second) derivative of its Gibbs free energy with respect to the temperature has singularity at critical temperature $T_c$. Gibbs free energy of our CRNBH is defined by

$$G_+ = M - T_+ A_+ - \Phi_+ e$$

(22)

where $M$ is total energy (mass), $A_+$ entropy and $\Phi_+$ is electric potential on the exterior horizon $r_+$. Furthermore electric charge $e$ is a constant. Inserting (17), (18) and (19) Eq. (22) become

$$G_+(M, e) = \frac{\sqrt{M^2 - e^2}}{2}$$

(23)

which with $e = 0$ reduces to the Helmholtz free energy of a Schwarzschild black hole as $M_c$. In case of Lukewarm type of CRNBH where $M = |e|$ the Gibbs free energy vanishes and so the Lukewarm black hole will be has an equilibrium
state thermodynamically. Applying (18) and (23) one can obtain

\[ \frac{\partial G_+}{\partial T} = \frac{8\pi M (M + \sqrt{M^2 - e^2})^2}{M - 2\sqrt{M^2 - e^2}} \]  

(24)

which has singularity at critical point \( \frac{M}{|e|} = \frac{2}{\sqrt{3}} \). Its singular point is same as one which is obtained for heat capacity (20). Thus CRNBH metric exhibits with first order phase transition. Second order derivative of the Gibbs energy (23) is obtained with respect to temperature \( T \) as

\[ \frac{\partial^2 G_+}{\partial T^2} = 128\pi^2 (M + \sqrt{M^2 - e^2})^4 \times \frac{(M^2 + e^2 - 2\sqrt{M^2 - e^2})}{(M - 2\sqrt{M^2 - e^2})^3} \]  

(25)

which is not well behaved for the critical point \( M_c = \frac{2|e|}{\sqrt{3}} \). This is necessary and sufficient conditions for the system to perform a first order phase transition (see p. 83 in Reichl 1987). In other words the entropy and temperature are single valued at the critical point \( M_c = \frac{2|e|}{\sqrt{3}} \) but not the heat capacity or first order derivative of Gibbs energy with respect to temperature. Our results agree with the work presented by Davis (1977). In case \( 1 < \frac{M}{|e|} < \frac{2}{\sqrt{3}} \), the heat capacity (20) takes positive values which means the RN black hole is in equilibrium with its surrounding heat bath, but in case \( \frac{M}{|e|} > \frac{2}{\sqrt{3}} = 1.1547 \) it become disequilibrium because the heat capacity (20) takes negative values and so a phase transition is happened there (see solid line in Fig. 1). In the latter case the CRNBH will be radiate its energy and is shrunk quantum mechanically by interacting the thermal Hawking radiation (see Figs. 9 and 10). So we must be consider backreaction corrections of the thermal Hawking radiation on event horizons of evaporating RN black hole to obtain corresponding thermodynamical quantities. We will do this in Sects. 5 and 6. We complete now the present subsection by recalling other claim for phase transition of CRNBH presented by Meitei et al. (2010).

The electric potential (19) has not singular point and so dose not exhibit discontinuity (see also Eq. (2.3) in Lousto 1997), but Meitei et al. are shown in Meitei et al. (2010), that the CRNBH electric potential (19) can be rewritten as

\[ \Phi = \left( \frac{\partial M}{\partial e} \right)_T = \frac{er_+}{2e^2 - Mr_+} \]  

(26)

This form of electric potential exhibits with discontinuity at the critical point \( \frac{M}{|e|} = \frac{2}{\sqrt{3}} \). Hence they claimed that the discontinuity is not physical and so the phase transition is apparent. The latter statement cannot be correct in my opinion because of singularity which is happened on the first derivative of the corresponding Gibbs free energy. Eliminating \( M \) between (17), (18) and (23) one can obtain temperature dependent form of the Gibbs free energy and the entropy respectively as follows

\[ T_+(G_+) = \frac{G_+}{4\pi(\sqrt{4G_+^2 + e^2} + 2G_+)} \]  

(27)

and

\[ A_+(G_+) = 4\pi \left[ (4G_+^2 + e^2)^{\frac{1}{2}} + 2G_+ \right]^2 \]  

(28)

where \( G_+ > 0 \) for all \( M > |e| \) (see (23)). Their diagram are given in RHS of Figs. 3 and 4 respectively.

4 Entropy on interior horizon

Interior horizon \( r_- = M - \sqrt{M^2 - e^2} \) cannot be observed by an observer located outside of exterior horizon but its
thermodynamics properties can be take some physical statements as follows. One can calculate entropy of interior horizon of the CRNBH as
\[ A_-(M, e) = 4\pi r_-^2 = 4\pi (M - \sqrt{M^2 - e^2})^2. \] (29)

### 4.1 Negative temperatures

Varying the entropy equation (29) with respect to \( M \) and \( e \) and comparing with (16) one can obtain corresponding temperature and electric potential on the interior horizon as follows
\[ T_-(M, e) = -\frac{1}{8\pi} \frac{\sqrt{M^2 - e^2}}{(M - \sqrt{M^2 - e^2})^2} \] (30)
and
\[ \Phi_-(M, e) = \frac{e}{M - \sqrt{M^2 - e^2}}. \] (31)

Regarding (21) we can obtain heat capacity equation for the interior horizon as
\[ C_e^-(M, e) = -8\pi \frac{(M - \sqrt{M^2 - e^2})^2 \sqrt{M^2 - e^2}}{M + 2\sqrt{M^2 - e^2}}. \] (32)

which does not exhibit with a singularity. Temperature (30) takes negative values always and we give diagram of the solutions (29), (30) and (32) in Fig. 2. Temperature, entropy and heat capacity are plotted against \( M \) with dash, dot and solid lines respectively for particular electric charges \( e = \pm 1 \). Gibbs free energy on the interior horizon is given by
\[ G_-(M, e) = M - T_\ast A_\ast - \Phi_\ast e. \] (33)

Inserting (29), (30) and (31) explicit form of the Gibbs free energy (33) become
\[ G_-(M, e) = -\frac{1}{2} \sqrt{M^2 - e^2}. \] (34)
Applying (30) and (34) one obtain
\[ \partial G_- = \frac{4\pi M(M - \sqrt{M^2 - e^2})^2}{M + 2\sqrt{M^2 - e^2}} \] (35)
which has not a singular point. Eliminating mass parameter \( M \) between (30) and (34), temperature dependent function of Gibbs free energy become
\[ T_-(G_-) = \frac{G_-}{4\pi (4G_-^2 + e^2 + 2G_-^2)} \] (36)
where \( G_- < 0 \) (see (34)). RHS and LHS of Fig. 3 describes the temperature equations (27) and (36) against Gibbs free energy respectively for particular charges \( e = \pm 1 \). Negative temperature \( T < 0 \) appears in ordinary statistical systems having a finite energy maximum \( E_{\text{max}} \) where entropy is continuously increasing (decreasing) then energy and temperature decrease (increase).

In the present work initial mass of the CRNBH is its finite energy maximum \( E_{\text{max}} = M \) which decays in the presence of Hawking radiation (see Figs. 9, 10 and 11) and we will consider its effects on the thermodynamical properties of the CRNBH in Sects. 5 and 6. Abatement of the CRNBH entropy is happened on the interior horizon by increasing its mass (see dot and dash lines at Fig. 2). Inserting (34) the entropy equation (29) can be rewritten as
\[ A_-(G_-) = 4\pi \left[ \frac{(4G_-^2 + e^2)^{\frac{1}{2}}}{2G_-} \right]^2 + 2G_-^2. \] (37)

Diagram of Eqs. (28) and (37) are plotted against Gibbs free energy in RHS and LHS of Fig. 4 respectively. RHS of the figure shows rise of the entropy monotonously by increasing positive values of Gibbs free energy on the exterior horizon where the temperature takes positive values. LHS of the figure shows decrease of entropy to a zero value on the interior horizon by decreasing negative values of the Gibbs free energy to negative infinity. Namely, when entropy takes a zero value on the interior horizon then the matter counter part inside of the interior horizon reaches to a Bose-Einstein condensation state with zero momentum and minimum energy (see Sect. 4.2 for more discussion). Exterior horizon exhibits with a first order phase transition for particular mass \( M_0 = \frac{2|m|}{\sqrt{3}} \) where thermodynamical functions take the following critical values.
\[ A_-(M_0) = \frac{4\pi e^2}{3}, \quad A_+(M_0) = 4\pi e^2, \] (38)
\[ G_{\pm}(M_0) = \pm \frac{|e|}{2\sqrt{3}}, \] (39)
\[ C_e^-(M_0) = -\frac{2\pi e^2}{3}, \quad C_e^+(M_0) \to \pm \infty, \] (40)
\[ T_+(M_0) = -\frac{9}{24\pi \sqrt{3}|e|}, \quad T_-(M_0) = -\frac{9}{24\pi \sqrt{3}|e|}. \] (41)
and
\[
\Phi_-(M_c) = \frac{3}{\sqrt{3}} \frac{e}{|e|}, \quad \Phi_+(M_c) = \frac{1}{\sqrt{3}} \frac{e}{|e|}. \tag{42}
\]

In summary after than the phase transition is happened then the system will be proceed to take a Bose-Einstein condensation single state as follows.

### 4.2 Bose-Einstein condensation

Photons (bosons with spin \(s = 1\)) transport point particle electric field to other places in space time, from point of quantum electrodynamic (QCD) view. Thus we can be assume inside of the CRNBH is accumulated with more photons with total relativistic energy \(M\) and total zero spin \(s = 0\). Total number of boson particles is conserved in a bosonic gas and it is suitable statistical system which can be reach to a phase transition. In the latter system when temperature decreases to its critical value \(T \rightarrow T_c\) all of boson particles take zero momentum and so minimum energy. Corresponding entropy (degrees of freedom) decreases and phase transition is happened at critical temperature \(T_c\) and for temperatures \(T < T_c\) the bosonic gas reaches from gas state to a Bose-Einstein condensation state. Figure 2 (dot line) shows decrease of entropy at negative temperatures by increasing \(M\) and so for CRNBH the Bose-Einstein condensation state may be happened at negative temperatures.

Negative temperatures have several physical features and we address results of some suitable works as: (1) The Bose-Einstein condensation phase transition can be happened at negative temperatures (Mosk 2005). (2) Dark matter perfect fluid with negative pressure corresponds boson fields with negative temperatures (Mosk 2005). (3) Negative temperature can be create an attractively interacting ensemble of ultra-cold bosons which are stable and could not collapsed for arbitrary atom numbers (Braun et al. 2013) (see also Carr 2013).

As an extension of our work it is useful to consider Hawking radiation temperature of the CRNBH as dynamical backreaction effects on the CRNBH itself and obtain corrected counter part for entropy, temperature Gibbs free energy and heat capacity at constant electric charge \(e\) for final state of evaporating QRNBH metric in the next section. In the context of alternative gravity theories in absence of torsion \(f(R)\) and in presence of torsion \(f(T)\) effects black hole thermodynamics are studied also in the literature by more authors which can be addressed to for instance Cembranos et al. (2011) and Gamal and Nashed (2015) respectively (see also references therein).

### 5 Quantum Reissner-Nordström black hole

Using perturbation series expansion method presented by York (1985), backreaction corrections of conformal anomaly invariant massless, charge-less quantum scalar fields is used to solve time independent backreaction metric equation of an evaporating QRNBH metric by Wang and Huang (2001). They obtained final state of the evaporating metric as remnant stable fat black hole metric where exterior and interior quantum event horizon radiiues \(r_Q^{EH}\) are obtained approximately as (see Eqs. (68) and (69) in Wang and Huang 2001)
\[
r_+^{EH} \approx r_+ \left[ 1 + \epsilon \left( \frac{r_+ - r_-}{r_+ + r_-} \right) \frac{D}{M} \right], \tag{43}
\]
and
\[
r_-^{EH} \approx r_- \left[ 1 - \epsilon \left( \frac{r_+ + r_-}{r_+ - r_-} \right) \frac{D}{M} \right], \tag{44}
\]
where \(0 < \epsilon < 1\) is order parameter of perturbation series expansion of the well known metric backreaction equation \(G_{\mu\nu} = 8\pi (T_{\mu\nu}^{\text{class}} + < T_{\mu\nu}^{\text{ren}}>)\) and the cutoff length \(D\) is related to radiation energy of interacting quantum fields as \(D = E_{\text{rad}}\) (see Eq. (77) in Wang and Huang 2001). In absence of quantum field backreaction corrections \(D = 0\) the perturbed event horizons radius lead to the classical values \(r_\pm\). Inserting \(r_\pm = M \pm \sqrt{M^2 - e^2}\), the relations (43) and (44) become respectively
\[
r_+^{EH} \approx (M + \sqrt{M^2 - e^2}) \left( 1 + \frac{\epsilon D}{\sqrt{M^2 - e^2}} \right), \tag{45}
\]
and
\[
r_-^{EH} \approx (M - \sqrt{M^2 - e^2}) \left( 1 - \frac{\epsilon D}{\sqrt{M^2 - e^2}} \right). \tag{46}
\]
Regarding Bekenstein-Hawking entropy theorem (2), one can obtain exterior and interior quantum perturbed entropy function by calculating exterior and interior horizon surface area \(A_{\pm}^Q(M, e) = 4\pi (r_\pm^{EH})^2\) as
\[
A_{\pm}^Q(M, e) \approx 4\pi \left( M \pm \sqrt{M^2 - e^2} \right)^2 \times \left( 1 \pm \frac{\epsilon D}{\sqrt{M^2 - e^2}} \right)^2. \tag{47}
\]
Varying the above entropy functions with respect to \(M\) one can obtain temperature of exterior and interior horizons of QRNBH metric respectively as follows
\[
T_{\pm}^Q(M, e) \approx \pm \frac{\sqrt{M^2 - e^2}}{8\pi (M \pm \sqrt{M^2 - e^2})^2} \times \left( 1 \pm \frac{\epsilon D(M - 2\sqrt{M^2 - e^2})}{M^2 - e^2} \right). \tag{48}
\]
Using (16) one can write a suitable equation for the electric potential at constant entropy as
\[
\Phi = \left( \frac{\partial M}{\partial e} \right)_A \tag{49}
\]
where subscript \(A\) denotes to a constant entropy condition. Using (47) at constant entropy \(A_{\pm}^Q = \text{constant} ,\) Eq. (49) up to second order terms \(O(\epsilon^2)\) reduces to the following form.
\[ \Phi_\pm^Q(M, e, D) = \left( \frac{\partial M}{\partial e} \right)_{(A,D)} = \frac{e}{M \pm \sqrt{M^2 - e^2}} \times \left[ 1 \pm \frac{\epsilon D(\sqrt{M^2 - e^2} \pm M)}{(M^2 - e^2)(\mp M^2 \pm e^2 - M\sqrt{M^2 - e^2})} \right] \]  

(50)

where constant Hawking radiation energy of quantum massless, charge-less scalar field \( D = E_{\text{rad}} \) is independent of \( e, M \) and so its variation with respect to \( e \) and \( M \) vanishes. One can calculate heat capacity for exterior and interior horizons by applying (21) respectively as

\[ C_e^{Q+}(M) \simeq \frac{8\pi \sqrt{M^2 - e^2}(M + \sqrt{M^2 - e^2})^2}{M - 2\sqrt{M^2 - e^2}} \times \left[ 1 - \epsilon D \left[ \frac{2(M^2 - 3e^2 + 2M\sqrt{M^2 - e^2})}{(M^2 - e^2)(M - 2\sqrt{M^2 - e^2})} \right] \right] \]  

(51)

and

\[ C_e^{Q-}(M) \simeq -\frac{8\pi \sqrt{M^2 - e^2}(M - \sqrt{M^2 - e^2})^2}{M + 2\sqrt{M^2 - e^2}} \times \left[ 1 + \epsilon D \left[ \frac{3e^2 - 6M\sqrt{M^2 - e^2}}{(M^2 - e^2)(M + 2\sqrt{M^2 - e^2})} \right] \right]. \]  

(52)

Also one can obtain corresponding Gibbs free energy of QRNBH metric up to second order terms \( O(\epsilon^2) \) by using (22) and (33) for the relations (47), (48) and (50) as follows

\[ G_{\pm Q}(M, e, D) \simeq \pm \frac{\sqrt{M^2 - e^2}}{2} - \epsilon D \times \left[ M(M^2 - e^2)(M \pm \sqrt{M^2 - e^2}) \pm 2e^2 \right] \times \left[ 2(M^2 - e^2)^2 [M \pm \sqrt{M^2 - e^2}] \right]. \]  

(53)

Inserting \( \epsilon = 0 \) the above quantum corrected solutions lead to the classical counterparts given in Sect. 4. In perturbative approach we must be set \( \epsilon D \ll M \) in Eqs. (43) and (44). For instance it is evaluated for unstable circular photon orbits \( r_{ph} = 3M \) of Schwarzschild metric solution in York (1985) as

\[ \epsilon \left( \frac{D}{M} \right) \simeq 3.1 \times 10^{-4}. \]  

(54)

Hence we will consider the above sample together with \( e = \pm 1 \) to plot diagrams of Eqs. (47), (48), (51), (52) and (53).

Their diagrams are given at Figs. 5, 6, 7, and 8. Comparing these figures one can result that a QRNBH metric will be take a Bose Einstein condensation single state at negative temperature where corresponding entropy reaches to a zero value on the interior horizon but phase transition is happened on the exterior horizon. According to York’s idea (York 1985) and considering time independent regime of perturbation of quantum matter fields, our calculations predict same thermodynamical behavior for CRNBH and QRNBH metric to exhibit phase transition at critical temperature and Bose-Einstein condensation state.

5.1 Mass loss and the switching off effect

According to all our calculations given in Sects. 4 and 5 for CRNBH and QRNBH metrics, the corresponding thermodynamical variables become real quantities for

\[ M \geq |e| \]  

(55)

which in geometrical units is a minimum mass for evaporating RN black hole (see all figures) because for \( M < |e| \) horizons of the CRNBH and QRNBH metrics disappear and leads to naked singularity. One can set the above minimum mass to be equal with the Plank mass \( M_p = G^{-\frac{1}{2}} \) where semiclassical approach of quantum gravity (quantum fields in curved space) become invalid. In the semiclassical approach of quantum gravity Einstein tensor (geometry) treats as classical field but matter stress tensor treats as quantum and so the modified Einstein metric (backreaction) equation is written as \( G_{\mu\nu} = 8\pi (\langle T_{\text{quant}}^{\mu\nu} \rangle_{\text{ren}} - \langle T_{\text{quant}}^{\mu\nu} \rangle_{\text{ren}} \) is renormalized (ten) expectation value of quantum matter stress tensor operator. Several solutions of the backreaction equation is given in the literature where the authors seek final state of evaporating black holes. They solved time independent and time dependent version of the backreaction equation and obtained remnant stable black holes with no naked singularities (see for instance Ghaffarnejad 2007 and references therein). However we calculate now QRNBH mass loss and show that it reaches to a remnant finally.

Line element of evaporating RN black hole (14) can be written near the horizon as Vaidya form as

\[ ds^2 \simeq -\left( 1 - \frac{r_+(v)}{r} \right) dv^2 + 2dr dv + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \]  

(56)

with the associated stress energy tensor

\[ \langle T_{\mu\nu}^{\text{quant}} \rangle_{\text{ren}} = -\frac{1}{8\pi r^2} \frac{dr_+(v)}{dv} \delta_{\mu\nu}, \]  

(57)

where \((v, r)\) is advance Eddington-Finkelstein coordinates system. The black hole luminosity at distance \( r \) is defined by

\[ L(r, v) = 4\pi r^2 \langle \dot{r}^r_{\text{ren}} \rangle \]  

(58)

which by inserting (57) can be rewritten as

\[ L = -\frac{1}{2} \frac{dr_+(v)}{dv} \]  

(59)

where negative sign describes inward flux of negative energy across the horizon which causes the black hole to shrink.
The Stefan-Boltzmann law for luminosity of black body radiation is defined by

\[ L_{SB} = AT^4 \]  

(60)

where \( A \) is surface area of black body and \( T \) is its temperature. If (59) satisfies (60) on the apparent horizon of the QRNBH metric then we can write mass loss equation of the black hole as follows

\[ \frac{dr_+(v)}{dv} = -2\xi A_+(v) T_+^2(v) \]  

(61)

where the normalization constant \( \xi \) depends linearity on the number of massless fields and will control the rate of evaporation. The functions \( r_+(v) \), \( A_+(v) \) and \( T_+(v) \) are apparent horizon radius, its surface area and Hawking radiation temperature respectively. They are defined explicitly by replacing initial mass \( M \) with mass function \( m(v) \) for Eqs. (17) and (18) such as follows

\[ A_+(v) = 4\pi r_+^2(v) \]  

(62)

and

\[ T_+(v) = \frac{\sqrt{m(v)^2 - e^2}}{8\pi (m(v) + \sqrt{m(v)^2 - e^2})^2} \]  

(63)

where we defined

\[ r_+(v) = m(v) + \sqrt{m(v)^2 - e^2}. \]  

(64)

Inserting (62), (63) and (64) the mass loss equation (61) reduces to the following form

\[ \frac{dm(v)}{dv} = -\frac{\xi}{2^{16}\pi^3} \left[ \frac{m^2(v) - e^2}{[m(v) + \sqrt{m(v)^2 - e^2}]^2} \right]^2 \]  

(65)

which has solution as

\[ -v^*(m^*) = (m^{*2} - 1)^2 \left[ \frac{64}{13} m^{*6} - \frac{1072}{143} m^{*4} - \frac{1240}{429} m^{*2} - \frac{523}{3003} \right] - \frac{64}{13} m^{*13} + \frac{272}{11} m^{*11} - \frac{152}{3} m^{*9} + \frac{377}{7} m^{*7} - 31 m^{*5} + 9 m^{*3} - m^* + C \]  

(66)

where we defined dimensionless time parameter \( v^* \) and mass function \( m^*(v^*) \) by

\[ v^* = \frac{\xi v(m)}{2^{16}\pi^3 e} \]  

(67)

and

\[ m^*(v^*) = \frac{m(v)}{e}. \]  

(68)

\[ C \] is a suitable integral constant which should be set via initial conditions of the evaporation. The mass loss solution (66) shows that the time coordinate \( v^* \) is a real parameter for \( m^* \geq 1 \) and so we choose \( v^* = 0 \) as the moment when evaporation is complete so that

\[ m^*(0) = 1. \]  

(69)

Using (66) and (69) we must be set

\[ C = \frac{16}{3003}. \]  

(70)

Regarding (69) and (70) diagram of mass loss function (66) is plotted against collapsing time \( v^* \) in Fig. 9 for massive QRNBH and in Fig. 10 for low mass QRNBH. Applying (64) and (65) the QRNBH luminosity (59) can be rewritten as

\[ L^*(m^*) = \frac{(m^{*2} - 1)^2}{(m^* + \sqrt{m^{*2} - 1})^6}. \]  

(71)

Diagram of luminosity (71) is plotted against dimensionless mass function \( m^*(v^*) \) in Fig. 11. Diagrams of Figs. 9 and 10 show instability of the evaporating QRNBH which exhibits finally to a remnant stable spherical object with non vanishing minimum mass \( m_{final}(0) = |e| \) (Lukewarm type of RN black hole) and zero temperature (see Eq. (63)) \( T_+(m_{final}) = 0 \) where luminosity switches off to a zero value also (see Fig. 11). It is evident final state of QRNBH evaporation is conflict with final state of the Schwarzschild black hole evaporation reaching to a zero mass with infinite value for luminosity \( L \) and temperature \( T_+ \). This is seen easily by inserting \( e = 0 \) into Eqs. (63) and (65) such that

\[ T_+(v) = \frac{1}{32\pi m(v)}, \quad e = 0, \]  

(72)

\[ m(v) = \left[ M^3 - \frac{3\xi v}{2^{23}\pi^3} \right]^\frac{1}{3} \]  

(73)

and

\[ L = \frac{\xi}{2^{23}\pi^3 m^2(v)} \]  

(74)

which at the finite time

\[ v_{final} = \frac{2^{23}\pi^3 M^3}{3\xi} \]  

(75)

the Schwarzschild black hole evaporation is complete with zero final mass as

\[ m(v_{final}) = 0 \]  

(76)

with

\[ (T_+, L) \to (\infty, \infty). \]  

(77)

### 5.2 Information loss paradox

Information loss paradox (ILP) is one of problems which is related to black holes Hawking radiation. This is described as follows:

Usually one consider that a black hole is formed by the collapse of matter in a pure quantum state, which then evaporates completely into Hawking radiation. This thermal radiation represents a transition from a pure state to a mixed state and so violates unitarity of evolution and is forbidden...
in ordinary quantum mechanics. If the Hawking radiation is assumed to be a pure state, then would appear to require correlations between ‘early’ and ‘late’ Hawking particles that have never been in causal contact. Responding to this paradox Almheiri et al. (2013) argue that we must regard the following statements: (a) the equivalence principle, (b) low energy effective field theory or (c) the non-existence of high-entropy ‘remnants’ at the end of black hole evaporation.

Really, one should obtain a categorical answer for ILP via an unknown pure quantum gravity theory but our calculations in this work predict an acceptable answer for the ILP at a semiclassical approach of quantum gravity (quantum field theory in curve space). Because we show that final state of evaporating QRNBH reaches to a non-vanishing mass remnant stable mini black hole with zero temperature and zero luminosity. The latter remnant will be a lukewarm black hole which may be set its mass with the Plank mass $m_p = G^{-1/2}$ where the semiclassical quantum gravity become invalid. In the latter case there is still a tiny connection with the macroscopic internal region. In either case, the quantum matter field configuration in the internal region are still correlated to external configurations. The loss of information in the exterior part of the space time is analogous (Wald 1992) to the loss of the quantum correlations which occur in any subsystem upon tracing over the quantum state belonging to its complementary subsystem. This loss of information is now inevitable to an outside observer because the dynamics of the evaporation force quantum mechanics to operate in the realm with a varying topology (see Parentani and Piran 1994). But in case of quantum evaporating Schwarzschild metric we see that the black hole mass disappear completely and so one finds two disconnected macroscopic regions: a quite big ‘baby universe’ (Hawking and Laflamme 1988) and a Minkowskian exterior. In summary Hawking radiation for QRNBH can be present some suitable physical statement for ILP but not for the evaporating quantum Schwarzschild black hole.

6 Summary and discussion

In this work we studied thermodynamics of the CRNBH (QRNBH) metric in absence (presence) of backreaction effects of Hawking particles created by interacting of massless, charge-less quantum scalar fields. We obtained that it is unstable thermodynamically and exhibits with a first order phase transition. Evaporation reaches to a remnant stable mini black hole called as Lukewarm black hole where remnant mass is equal to its invariant electric charge and temperature together with luminosity vanishes. Matter content of evaporating QRNBH takes two different values for Gibbs free energy against a single positive temperature on the exterior horizon but not on the interior horizon where the corresponding temperature takes absolutely negative values and entropy reaches to a zero value. Thus we claim that matter content of the QRNBH contain two different phases on the exterior horizon with raising monotonically entropy. But matter content located inside of interior horizon reaches to the Bose Einstein condensation state because its entropy reaches to a zero value at negative infinite temperature. The first order phases transition is happened on the exterior horizon at critical point $\frac{\rho}{m} = \frac{3}{\pi}$. Non-vanishing remnant mass of evaporating QRNBH can be give some suitable physical statements for resolve the information loss paradox. As a future work we extend aim of this work for ensemble of CRNBH and QRNBH to seek phase transition and Bose Einstein condensation state by regarding idea given in Chevalier et al. (2007).

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