Apparent Viscosity of Active Nematics in Poiseuille Flow

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Abstract. A Leslie-Erickson continuum hydrodynamic for flowing active nematics has been used to characterize active particle systems such as bacterial suspensions. The behavior of such a system under a plane pressure-driven Poiseuille flow is analyzed. When plate anchoring is tangential and normal, we find the apparent viscosity formula indicating a significant difference between tangential anchoring and normal anchoring conditions for both active rodlike and discoid nematics.

1. Introduction
Active nematics are non-equilibrium complex fluids composed of active or self-propelled particles showing a nematic phase and have attracted much attention in both theoretical modeling [1-3, 5-8] and experiments [9, 11] in the past decade. In these systems, active particles exert forces on the surrounding fluid, resulting in local extensile or contractile stresses proportional to the amount of orientational tensor: \( t^a = \delta \hat{n} n \) where \( \hat{n} \) is the orientation director in the nematic phase which is generally described by a unit vector, and \( \delta \) is the amplitude of the dipolar forces exerted by the particle to swim. The sign of \( \delta \) determines whether the dipolar flow field generated by the active particles is extensile \((\delta < 0)\) or contractile \((\delta > 0)\). In the swimmer literature, the former situation describes "pushers", i.e., most bacteria including E. Coli and Bacillus Subtilis, while the latter corresponds to "pullers" including Chlamydomonas.
Although theoretical modeling efforts [1-3, 5-8] have been made in understanding active nematics, almost all these studies are only focused on sheared systems. Active nematics in Poiseuille flow are
scarcely studied [10, 12]. Here, we will address rheology of active nematics in Poiseuille flow and focus on the apparent viscosity.

2. The governing equations for flowing active nematics

We adopt the Erickson-Leslie-Parodi model [1, 2], in which the nematic order parameter is a fixed magnitude unit vector field $\mathbf{n}$ which evolves according to

$$\frac{\partial \mathbf{n}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{n} = \lambda \mathbf{D} \cdot \mathbf{n} - \Omega \cdot \mathbf{n} + \Gamma \mathbf{h} \quad (1)$$

Where $\mathbf{v}$ is the velocity field of the solvent; $\lambda$ is the flow alignment parameter, $\Gamma$ is a rotational viscosity; $\mathbf{h} = K \Delta \mathbf{n}$ is the molecular field and $K$ is the Frank elasticity constant (single constant approximation), $\Omega = \frac{1}{2} (\nabla \mathbf{v} - \nabla \mathbf{v}^T)$ and $\mathbf{D} = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ are the rate-of vorticity and the rate-of-strain tensors, respectively.

The first two terms on the right-hand side of Eq. (1) describe alignment (or tumbling) of the director field by local shear flow. The third term accounts for the tendency of the ordered nematic to resist distortions, and arises ultimately from excluded volume interactions between individual particles. The flow field $\mathbf{v}$ obeys the Navier-Stokes equation

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\frac{\partial p}{\partial x} \mathbf{I} + \nabla \cdot (\tau + 2 \eta \mathbf{D}) \quad (2)$$

with the continuity equation $\nabla \cdot \mathbf{v} = 0$ to guarantee incompressibility, the stress tensor given by the passive and active contributions, $\tau = t^p + t^a$

$$t^p = -\frac{1}{2} [\mathbf{n} \mathbf{h} + (\mathbf{n} \mathbf{h})^T] + \frac{1}{2} [\mathbf{n} \mathbf{h} - (\mathbf{n} \mathbf{h})^T] \quad (3)$$

where $\rho$ is the fluid density and $\eta$ is the viscosity and $\lambda$ is the flow-alignment parameter. The magnitude of $\lambda$ controls how the director field responds to a shear flow. $|\lambda| > 1$ corresponds to flow-aligning regime in which the director tends to align to the flow direction at the Leslie alignment angle $\theta_L = \frac{1}{2} \cos^{-1} \frac{1}{\lambda}$ while $|\lambda| < 1$ corresponds to flow tumbling regime in which the director continuously rotates under shear. The value of $\lambda$ is mainly determined by the shape of the active particles, $\lambda > 0$ corresponds to a rod-shaped particle, $\lambda < 0$ for a disc-shaped particle and $\lambda = 0$ for a spherical particle. The active contribution can be expressed as [2, 5],

$$t^a = \delta \mathbf{n} \mathbf{n} \quad (4)$$

The sign of $\delta$ determines whether the particles are extensile ($\delta < 0$) or contractile ($\delta > 0$).

3. Plane Poiseuille flow

We consider a plane Poiseuille flow between two parallel plates located at $y = \pm H$ with an imposed small pressure gradient $\frac{\partial p}{\partial x}$, where $\frac{\partial p}{\partial x}$ is constant and negative. We assume strong particle anchoring and the velocity boundary condition no-slip at the plates given by
where \( \mathbf{n}_0 \) is the initial director across the plates in absence of Poiseuille flow. Figure 1 depicts the cross section of the shear flow on the \((x, y)\) plane. We consider the in-plane orientation of an active nematic confined to the plane \((x, y)\) and strong anchoring boundary conditions. The director \( \mathbf{n} \) confined to the plane \((x, y)\) and parameterized by the director angle, \( \mathbf{n} = (\cos \varphi, \sin \varphi, 0) \). Variations in the direction of flow \((x)\) and primary vorticity direction \((z)\) are suppressed.

\[
\mathbf{n}_0 = (\cos \varphi_0, \sin \varphi_0, 0), \quad \nu(\pm H) = 0
\]

\( \varphi_0 \) is the initial director across the plates in absence of Poiseuille flow. Figure 1 depicts the cross section of the shear flow on the \((x, y)\) plane. We consider the in-plane orientation of an active nematic confined to the plane \((x, y)\) and strong anchoring boundary conditions. The director \( \mathbf{n} \) confined to the plane \((x, y)\) and parameterized by the director angle, \( \mathbf{n} = (\cos \varphi, \sin \varphi, 0) \). Variations in the direction of flow \((x)\) and primary vorticity direction \((z)\) are suppressed.

**Figure 1.** The geometry of the plane Poiseuille flow. The boundary anchoring for the director is \( \mathbf{n}_0 = (\cos \varphi_0, \sin \varphi_0, 0) \) and the velocity boundary condition is no-slip \( \nu(\pm H) = 0 \).

We consider a uniform plate anchoring boundary condition, either parallel to the flow direction, called tangential anchoring, i.e. \( \varphi_0 = 0 \), or perpendicular to the plates, called normal or homeotropic anchoring, i.e. \( \varphi_0 = \pi/2 \). We linearize the governing system of equations with respect to \( -\frac{\partial \varphi}{\partial x} \) with the boundary conditions given by (5). The linearized system is

\[
\frac{\partial \varphi}{\partial t} = A \frac{\partial^2 \varphi}{\partial y^2} + B \frac{\partial v_x}{\partial y},
\]

\[
\rho \frac{\partial v_x}{\partial t} = 1 + \frac{\partial \tau_{xy}}{\partial y},
\]

\[
\tau_{xy} = C \frac{\partial^2 \varphi}{\partial y^2} + D \frac{\partial v_x}{\partial y} + \delta \varphi
\]
Where $A = \Gamma K$, $C = -KB$, $D = \eta$, and $B = \begin{cases} \frac{\lambda - 1}{2}, & \text{if } \varphi_0 = 0 \\ \frac{\lambda + 1}{2}, & \text{if } \varphi_0 = \frac{\pi}{2} \end{cases}$

### 3. Steady states and apparent viscosity

In order to establish the apparent viscosity formula, we need the steady state solutions of the system (6) in the confined geometry which are given by

$$v_x = \frac{AHr \coth ry}{B\delta} \left(1 - \frac{\cosh ry}{\cosh rH}\right),$$

$$\varphi = \frac{H}{\delta} \left(\frac{\sinh ry}{\sinh rH} - \frac{y}{H}\right),$$

where $r = \begin{cases} \sqrt{\frac{(\lambda - 1)\delta}{2(AD - BC)}}, & \text{if } \varphi_0 = 0 \\ -\sqrt{\frac{(\lambda + 1)\delta}{2(AD - BC)}}, & \text{if } \varphi_0 = \frac{\pi}{2} \end{cases}$. The apparent viscosity defined by $\eta_{\text{app}} = \frac{2H^3}{3F}$ [4], where

$$F = \int_0^H v_x(y)dy = \frac{AH}{B\delta}(rH \coth rH - 1)$$

is the flow rate per unit length. The resulting apparent viscosity is given by

$$\eta_{\text{app}} = \frac{2H^2B\delta}{3A(rH \coth rH - 1)}$$

Figure 2. The apparent viscosity versus the activity for rodlike and discoid swimmers in flow-aligning (solid line) and tumbling regimes (dashed line) at tangential anchoring conditions. The values of parameters are: $\Gamma = 0.2$, $K = 0.04$, $\eta = 1.27$ and $H = 1$.

Figure 2 shows how the apparent viscosity depends on the activity for rodlike and discoid swimmers at tangential anchoring boundary conditions. We summarize the noticeable features below.

- The apparent viscosity of a flow-aligning rodlike puller system or a tumbling rodlike pusher system is thickened by the activity while the apparent viscosity of a flow-aligning rodlike
pusher or a tumbling rodlike puller system is thinned by the activity.

- The apparent viscosity of a discoid system is always thinned by the activity.
- The apparent viscosity could be negative for all rodlike systems but only for discoid fuller system in a high activity regime.

**Figure 3.** The apparent viscosity versus the activity for rodlike and discoid swimmers in flow-aligning (solid line) and tumbling regimes (dashed line) at normal anchoring conditions. The values of parameters are: $\Gamma = 0.2$, $K = 0.04$, $\eta = 1.27$ and $H = 1$.

Figure 3 shows how the apparent viscosity depends on the activity for rodlike and discoid swimmers at normal boundary conditions. We summarize the noticeable features below.

- The apparent viscosity of a rodlike system is always thinned by the activity.
- The apparent viscosity of a flow-aligning discoid system is thickened by the activity while the apparent viscosity of a tumbling discoid system is thinned by the activity.
- The apparent viscosity could be negative for all rodlike and discoid system in a high activity regime.

We notice that the apparent viscosity of rodlike systems is enhanced by one order of magnitude for normal relative to parallel anchoring; however the apparent viscosity of discoid systems is diminished one order of magnitude for normal relative to parallel anchoring.

Common swimming bacteria, such as Bacillus Subtilis, E. Coli, and many others, rodlike swimmers (length about $\sim 5\mu m$, diameter of the order of $\sim 1\mu m$) are flow-aligning rodlike pushers. The apparent viscosity of these systems is thinned by the activity for tangential anchoring. In contrast, Chlamydomonas Reinhardtii are tumbling pullers. The apparent viscosity of such systems is thinned for a tangential anchoring.
4. Conclusions
We have used the Leslie-Ericken–Parodi continuum theory to model flowing active nematics. We establish the apparent viscosity formula active nematics subject to a plane Poiseuille flow. The results are consistent with the previous results on rodlike swimmers and discoid swimmers, experimentally and theoretically. We look forward to tests of other predictions in experiments on active nematics in future.

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References
[1] Cui Z Zeng X and Su J 2014 Steady States of Sheared Active Nematics Advances in Applied Mathematics and Mechanics 6 p 75-86
[2] Cui Zhenlu and Zeng X 2012 Rheology of Sheared Bacterial Suspensions IMA 155 p 217
[3] Cui Zhenlu 2011 Weakly Sheared Active Suspensions: Hydrodynamics, Stability and Rheology Phys. Rev. E 83 p 031911
[4] Cui Z Calderer M C and Wang Q 2006 Mesoscale structure in flows of weakly sheared cholesteric liquid crystalline polymers Discrete and Continuous Dynamical Systems-Series B 6 p 291
[5] Fielding SM Marenduzzo D Cates ME 2011 Nonlinear dynamics and rheology of active fluids: Simulations in two dimensions Phys. Rev. E 83 p 041910
[6] Giomi L et al 2010 Sheared active fluids: Thickening, thinning, and vanishing viscosity Phys. Rev. E 81 p 051908
[8] Hatwalne Y et al 2004 Rheology of active-particle suspensions Phys. Rev. Lett. 92 p 118101
[9] Rafaï S Jibuti L and Peyla P 2010 Effective Viscosity of Microswimmer Suspensions Phys. Rev. Lett. 104 p 098102
[10] Ravnik M and Yeomans J M 2013 Confined Active Nematic Flow in Cylindrical Capillaries Phys. Rev. Lett. 110 p 026001
[11] Sokolov A et al 2009 Reduction of Viscosity in Suspension of Swimming Bacteria Phys. Rev. Lett. 103 p 148101
[12] Zöttl A and Stark H 2012 Nonlinear dynamics of a microswimmer in Poiseuille flow Phys. Rev. Lett. 108 p 218104