Inverse Analogue of Ailamujia Distribution with Statistical Properties and Applications

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Authors' contributions

This work carried out in collaboration among all authors. Author A. Aijaz designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors A. Ahmad and RT managed the analysis of the study. Author A. Aijaz managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The present paper deals with the inverse analogue of Ailamujia distribution (IAD). Several statistical properties of the newly developed distribution has been discussed such as moments, moment generating function, survival measures, order statistics, shanon entropy, mode and median .The behavior of probability density function (p.d.f) and cumulative distribution function (c.d.f) are illustrated through graphs. The parameter of the newly developed distribution has been estimated by the well known method of maximum likelihood estimation. The importance of the established distribution has been shown through two real life data.

Keywords: Ailamujia distribution; inverse approach; moments; survival analysis; maximum likelihood estimation.

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1 Introduction

H.Q.LV et al. [1] has been established new distribution for modeling real life data and named it Ailamujia distribution with following probability density function and cumulative distribution function.

\[ f(x, \alpha) = 4\alpha^2 x e^{-2\alpha x}; x > 0, \alpha > 0 \]  

(1.1)

The corresponding cumulative distribution function is given as

\[ F(x, \alpha) = 1 - (1 + 2\alpha x)e^{-2\alpha x}, \alpha > 0, x > 0 \]  

(1.2)

They have studied its various statistical and mathematical properties such as mean, variance, median and maximum likelihood estimation. Ailamujia distribution is a functional distribution used to model the repair time and provide assurance to distribution delay time. Recently several authors have done lot of work on Ailamujia distribution. Pan et al. [2] has worked on Ailamujia distribution for interval estimation and hypothesis testing based on small sample size. Long [3] has obtained its Bayesian estimation under type II censoring on the basis of conjugate prior, Jeffrey’s prior and no informative prior distribution. Yu et al. [4] proposed a new method by applying Ailamujia distribution to solve the problem in the production and distribution of battle field injury in campaign macrocosm. Uzma et al. [5] has introduced the weighted analogue of Ailamujia distribution and studied its various characteristics. They showed that weighted analogue of Ailamujia distribution performs better than Ailamujia distribution. Ahmad et al. [6] proposed and studied several structural properties of weighted analogue of inverse Maxwell distribution. Rather et al. [7] proposed a size biased Ailamujia distribution and apply it for analyzing data from engineering and medical science. B.Jaya Kumar et al. [8] introduced area biased distribution and have been studied its various properties and applied the obtained distribution to bladder cancer data. Recently Ahmad et al. [9] introduced two parameter distribution named it Hamza distribution and studied its different mathematical properties.

2 The Inverse Ailamujia Distribution

If \( X \) be a random variable follows Ailamujia distribution having p.d.f (1.1), then \( Y = \frac{1}{X} \) is said to follow inverse Ailamujia distribution (IAD) if its cumulative distribution function (c.d.f) is defined as

\[ F(Y) = P(Y \leq y) = P\left(\frac{1}{x} \leq y\right) = P\left(\frac{1}{y} \leq x\right) = 1 - P(X < \frac{1}{y}) = 1 - F\left(\frac{1}{y}\right) \]

\[ F(Y) = \frac{(2\alpha + y)}{y} e^{-\frac{2\alpha}{y}}, y > 0, \alpha > 0 \]  

(2.1)

The corresponding probability distribution function (p.d.f) is

\[ f(y, \alpha) = 4\alpha^2 \frac{1}{y^3} e^{-\frac{2\alpha}{y}}, y > 0, \alpha > 0 \]  

(2.2)
3 Structural Properties of Inverse Ailamujia Distribution (IAD)

3.1 Moments of inverse Ailamujia distribution

Let $Y$ be a random variable follows inverse Ailamujia distribution. Then the $r^{th}$ moment denoted by $\mu'_r$ of the distribution is obtained as
\[ \mu'_r = E(Y^r) = \int_0^\infty y^r f(y, \alpha) \, dy \]
\[ = \int_0^\infty y^r 4a^2 \frac{1}{y^a} e^{-\frac{2a}{y}} \, dy \]
\[ = 4a^2 \int_0^\infty y^{r-3} e^{-\frac{2a}{y}} \, dy \]

Making the substitution \( \frac{2a}{y} = t \) and after solving the integral, we obtain.

\[ \mu'_r = E(Y^r) = (2a)^r \Gamma(2 - \alpha) \quad (2.3) \]

By substituting \( r = 1 \), we obtain the mean of the distribution

Mean = \( \mu'_1 = 2a \)

We observe that equation (2.3), does not exists for \( r \geq 2 \), which implies that higher order moments does not exists.

3.2 Moment generating function of inverse Ailamujia distribution

Let \( Y \) be a random variable follows inverse Ailamujia distribution then the moment generating function denoted by \( M_Y(t) \) is obtained as.

\[ M_Y(t) = \int_0^\infty e^{ty} f(y, \alpha) \, dy \]

Using Taylor’s theorem, we get

\[ = \int_0^\infty \left( 1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \cdots \right) f(y, \alpha) \, dy \]
\[ = \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} y^r f(y, \alpha) \, dy \]
\[ = \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty y^r f(y, \alpha) \, dy \]
\[ = \sum_{r=0}^\infty \frac{t^r}{r!} \Gamma(2 - r) \]

3.3 Characteristics function of inverse Ailamujia distribution

Let \( Y \) be a random variable follows inverse Ailamujia distribution then the moment generating function denoted by \( \phi_Y(t) \) is obtained as.

\[ \phi_Y(t) = \int_0^\infty e^{ity} f(y, \alpha) \, dy \]
Using Taylor's theorem, we get
\[
\int_0^\infty \left\{ 1 + ity + \frac{(ity)^2}{2!} + \frac{(ity)^3}{3!} + \cdots \right\} f(y, \alpha) dy = \int_0^\infty \sum_{r=0}^{\infty} \frac{(ity)^r}{r!} y^r f(y, \alpha) dy
\]
\[
= \sum_{r=0}^{\infty} \frac{(ity)^r}{r!} \int_0^\infty y^r f(y, \alpha) dy = \sum_{r=0}^{\infty} \frac{(ity)^r}{r!} (2\alpha)^r \Gamma(2 - r)
\]

### 3.4 Harmonic mean of inverse Ailamujia distribution

The harmonic mean (H) is given as.
\[
\frac{1}{\bar{H}} = E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y, \alpha) dy = 4\alpha^2 \int_0^\infty \frac{1}{y^4} e^{-\frac{2\alpha}{y}} dy
\]

Making the substitution \( t = \frac{2\alpha}{y} = \frac{t}{\alpha} \) and after solving the integral, we obtain
\[
\frac{1}{\bar{H}} = \frac{1}{\alpha}
\]

### 3.5 Mode and median of inverse Ailamujia distribution

Taking the log function to the p.d.f of inverse Ailamujia distribution, we have
\[
\log f(y, \alpha) = 2 \log 2\alpha - 3 \log y - \frac{2\alpha}{y}
\]

Differentiate (2.4), w.r.t \( y \), we get
\[
\frac{\partial \log f(y, \alpha)}{\partial y} = -3y^{-1} + \frac{2\alpha}{y^2}
\]

Substituting \( \frac{\partial \log f(y, \alpha)}{\partial y} = 0 \), we get
\[
y = \frac{2\alpha}{3} \Rightarrow M_0 = y_0 = \frac{2\alpha}{3}
\]

Using the empirical formula for median, we get
\[
M_d = \frac{1}{3} M_0 + \frac{2}{3} \mu_1 = \frac{14\alpha}{9}
\]
4 Shannon Entropy of Inverse Ailamujia Distribution

The concept of information entropy was introduced by Shanon in 1948. The entropy can be interpreted as the average rate at which information is produced by a random source of data and is given by

\[ H(y, \alpha) = -E[\log f(y, \alpha)] \]

\[ = -E \left[ \log \left( 4\alpha^2 \frac{1}{y^3} e^{\frac{2\alpha}{y}} \right) \right] \]

\[ = -E \left[ 2 \log 2\alpha - 3 \log y - \frac{2\alpha}{y} \right] \]

\[ = -2 \log 2\alpha + 3E(\log y) + 2\alpha E \left( \frac{1}{y} \right) \quad (4.1) \]

Now

\[ E(\log y) = \int_{0}^{\infty} \log y \, f(y, \alpha) dy \]

\[ = 4\alpha^2 \int_{0}^{\infty} \log y \, \frac{1}{y^3} e^{\frac{2\alpha}{y}} dy \]

Making the substitution \( \frac{2\alpha}{y} = t \) and after solving the integral, we get

\[ E(\log y) = \log 2\alpha - \Gamma(2) \quad (4.2) \]

Also

\[ E \left( \frac{1}{y} \right) = 4\alpha^2 \int_{0}^{\infty} \frac{1}{y} \, f(y, \alpha) dy \]

\[ = 4\alpha^2 \int_{0}^{\infty} \frac{1}{y^3} e^{\frac{2\alpha}{y}} dy \]

Making the substitution \( \frac{2\alpha}{y} = t \) and after manipulating integral, we get

\[ E \left( \frac{1}{y} \right) = \frac{1}{\alpha} \quad (4.3) \]

Substituting equations (4.2), (4.3) in (4.1), we get

\[ H(y, \alpha) = -2 \log 2\alpha + 3\{\log 2\alpha - \Gamma(2)\} + 2 \]

5 Survival Measures

The reliability function of a random variable \( y \) is denoted as \( R(y, \alpha) \), can be obtained as

\[ R(y, \alpha) = 1 - F(y, \alpha) \]
Using (2.1) in the above equation, we get

\[ R(y, \alpha) = 1 - \frac{(2\alpha + y)}{y} e^{-\frac{2\alpha}{y}} \]  

(5.1)

The hazard rate function denoted as \( h(y, \alpha) \) of a random variable \( y \) can be obtained as

\[ h(y, \alpha) = \frac{f(y, \alpha)}{R(y, \alpha)} \]  

(5.2)

using equation (2.1) and (5.1) in (5.2), we get

\[ h(y, \alpha) = \frac{4\alpha^2 e^{-\frac{2\alpha}{y}}}{y^2 \left\{ y - (2\alpha + y) e^{-\frac{2\alpha}{y}} \right\}} \]

Also, the reverse hazard rate function denoted as \( h_r(y, \alpha) \), can be obtained as

\[ h_r(y, \alpha) = \frac{f(y, \alpha)}{F(y, \alpha)} \]  

(5.3)

Using equation (2.1) and (2.2) in equation (5.3), we get

\[ h_r(y, \alpha) = \frac{4\alpha^2}{y^2(2\alpha + y)} \]

**Figs. 3.1 and 3.2.** Illustrates different shapes of survival function of inverse Ailamujia distribution

### 6 Order Statistics of Inverse Ailamujia Distribution

Let us suppose \( Y_1, Y_2, Y_3, ..., Y_n \) be random samples of size \( n \) from inverse Ailamujia distribution with p.d.f \( f(y) \) and c.d.f \( F(y) \). Then the probability density function of \( k^{th} \) order statistics is given as.
Now substituting the equation (2.1) and (2.2) in equation (6.1), we obtain the probability density function of $k^{th}$ order statistics is given as

$$f_{Y(k)}(y) = \frac{n!}{(k-1)!(n-k)!} [F(y)]^{k-1}[1-F(y)]^{n-k} f(y), \quad k = 1, 2, 3, \ldots, n$$  \hspace{1cm} (6.1)

The p.d.f of first order $Y_1$ is given as

$$f_{Y(1)}(y) = \frac{4n\alpha^2e^{-\frac{2\alpha}{y}}}{y^{n+2}} \left[ y - (2\alpha + y)e^{-\frac{2\alpha}{y}} \right]^{n-1}$$  \hspace{1cm} (6.2)

And the $n^{th}$ order probability density function $Y_n$ is given as

$$f_{Y(1)}(y) = \frac{4n\alpha^2e^{-\frac{2\alpha}{y}}}{y^{n+2}} \left[ (2\alpha + y)e^{-\frac{2\alpha}{y}} \right]^{n-1}$$

Figs. 4.1 and 4.2. Illustrates different shapes of Hazard function of inverse Ailamujia

7 Estimation of Parameters Inverse Ailamujia Distribution

7.1 Methods of moments

The moment estimator for $\alpha$ denoted as $\hat{\alpha}$ can be obtained by equating population moments with sample moments given as

$$\mu_r = \frac{1}{n} \sum_{i=1}^{n} y_i^r$$
Therefore we have
\[ \frac{1}{n} \sum_{i=1}^{n} y_i = 2\alpha \]
\[ \bar{y} = 2\alpha \Rightarrow \hat{\alpha} = \frac{\bar{y}}{2} \]

7.2 Method of maximum likelihood estimation

Let \( Y_1, Y_2, ..., Y_n \) be random samples from the inverse Ailamujia distribution. Then the likelihood function of inverse Ailamujia distribution is given as

\[ l = \prod_{i=1}^{n} f(y_i, \alpha) \]
\[ = \prod_{i=1}^{n} \frac{4\alpha^2}{y_i} e^{-2\alpha y_i} = (4\alpha^2)^n \prod_{i=1}^{n} \frac{1}{y_i} e^{-2\alpha \sum_{i=1}^{n} \frac{1}{y_i}} \]

Taking log we get log likelihood function as

\[ \log l = 2n \log 2\alpha - 3 \sum_{i=1}^{n} \log y_i - 2\alpha \sum_{i=1}^{n} \frac{1}{y_i} \]

Differentiating w.r.t \( \alpha \), we get

\[ \frac{\partial \log l}{\partial \alpha} = 2n \frac{1}{2\alpha} - 2 \sum_{i=1}^{n} \frac{1}{y_i} \]

Now equating \( \frac{\partial \log l}{\partial \alpha} = 0 \), we get

\[ \hat{\alpha} = \frac{n}{2S} \]

Where \( S = \sum_{i=1}^{n} y_i^{-1} \)

8 Data Analysis

Data Set 1: In this section we provide an applications which explains the performance of the newly distribution. The data set has been taken from Gross and Clark [10], which signifies the relief times of 20 patients getting an analgesic. We use previous data to associate the fit of the newly developed model with Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distribution. The data are follows.

\[ 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 1.4, 1.1, 8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. \]

In order to compare the two distribution models, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, AICC and BIC values.

\[ AIC = -2lnL + 2k, AICC = AIC + \frac{2k(k+1)}{(n-k-1)} BIC = -2lnL + klnn \]
Table 8.1. ML estimates and criteria for comparison

| Distribution          | Estimates | -2logL | AIC    | AICC   | BIC    |
|-----------------------|-----------|--------|--------|--------|--------|
| Inverse Ailamujia     | 1.7247    | 51.6526| 53.6526| 53.9859| 54.6483|
| Ailamujia Distribution| 0.5263    | 52.3262| 54.3262| 54.6595| 55.3219|
| Exponential Distribution| 0.5263 | 65.674  | 67.674 | 68.0073| 68.6697|
| Inverse Exponential Distribution| 1.724  | 65.3372| 67.3372| 67.6705| 68.3329|
| Lindley Distribution  | 0.8161    | 60.499 | 62.499 | 62.8323| 63.499 |
| Inverse Lindley       | 2.2546    | 63.5142| 65.5142| 65.8475| 66.5099|

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller et al. [11].

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.80, 26.69, 26.77, 26.78
27.05, 27.67, 29.90, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08,
37.09, 39.58, 44.045, 45.29, 45.381

Table 8.2. ML estimates and criteria for comparison

| Distribution          | Estimates | -2logL | AIC    | AICC   | BIC    |
|-----------------------|-----------|--------|--------|--------|--------|
| Inverse Ailamujia     | 29.2153   | 252.2188| 254.2188| 254.4256| 255.6527|
| Ailamujia Distribution| 0.03245   | 252.2306| 254.2306| 254.4374| 255.6645|
| Exponential Distribution| 0.0324 | 274.53288| 276.53288| 276.73977| 277.96678|
| Inverse Exponential Distribution| 29.2152 | 274.523 | 276.523 | 276.72989 | 277.9569 |
| Lindley Distribution  | 0.06298   | 253.9884| 255.9884| 256.1952| 257.4223|
| Inverse Lindley       | 30.1531   | 700.416 | 702.416 | 702.6228 | 703.8499|

From Table 8.1 and 8.2, it has been observed that the inverse Ailamujia distribution (AID) have the lesser AIC, AICC, -2logL and BIC values as compared to Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distribution distributions. Hence we can conclude that inverse Ailamujia distribution leads to a better fit as compared to Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distributions.

9 Conclusion

In this paper the inverse analogue of Ailamujia distribution has been established. Many times inverse of the distributions provide better results for fitting different data taken from various fields. Some statistical properties including moments, moment generating function, characteristics function, mode, median, shanon
entropy, hazard rate function, reverse hazard function has been discussed. The estimation of the parameters of the established distribution has been estimated by the method of moments and maximum likelihood estimator. Finally two real life data sets have been presented to check the performance of the established model.

**Competing Interests**

Authors have declared that no competing interests exist.

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