Discrete switching of elastic elements as a way of ensuring damping in Duffing oscillator

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Abstract. With the aim of improving the efficiency of damping oscillations in non-autonomous Duffing system, the elastic element is divided into deformable and accumulating parts; between them four times a period discrete switching is enabled, which consists of the intermittent disconnection and connection parts. The concepts of the spasmodic changes in the initial length of the forged part and the displacement of the protected object static equilibrium state are introduced. Mass transfer between the parts of the elastic element, occurring at the moments of discrete switching, leads to dissipation of mechanical vibrations energy. Using the introduced concepts, characteristics of the positional forces are developed, their harmonic linearization is made, their equivalent stiffness damping and relative attenuation coefficients are found, the influence of amplitude of relative oscillations on them and the parts mass and initial tension ratio are analyzed.

Keywords: Duffing equation, discrete switching parts of an elastic element, displacement of the static equilibrium state, relative attenuation coefficient

1. Introduction

The dynamic characteristics of cushioning systems of objects with discrete switching parts of elastic elements (springs, torsion bars, pneumatic elements), the longitudinal axis of which coincides with the displacement vector, are investigated in detail in [1], [2], [3], [4]. In these systems, the division of the elastic elements into deformable accumulating parts and their discrete switching (DS) in the amplitude positions of the object are introduced. Such modification of connections led to a significant increase in the coefficient of relative damping in the resonance area and, to a large extent, to meet the requirements to damping systems [5], [6] without the introduction of special damping devices. The known oscillations damping mechanisms (Coulomb, linear, quadratic, and hysteretic) are considered in [1, p. 13]; [6, p. 128]; [7], [8]. The closest in the physical nature of the damping generated by the DS, are the systems with jump-up [9], [10], [11], [12]. In [10, p. 53]; [11, p. 44] it is shown that at the jump-up moment, high frequency oscillations occur, which, due to internal and structural friction, result in the dissipation of mechanical energy. A similar statement is contained in the work of A. P. Ivanov [12], which considered the collision of elastic springs with absolutely hard blocker.

In contrast to the systems considered in [9], [10], [11], in which there is no division of the elastic element into parts and no switch unit between them, mass transfer accompanying the DS, happens regardless of the amplitude of force or kinematic excitation.

System with the jump-up described by the Duffing equation, are applied in the construction industry as vibrating tables [14], [15], [16]. In these hardening type systems jump-up is understood as the failure of the steady-state oscillation from the upper reach of amplitude-frequency characteristic (AFC) to the lower one and vice versa. Besides, despite the abrupt change in the initial length of the elastic element, the internal friction is not taken into account. In [17], approximate expressions for the
frequency of breakdowns and their corresponding amplitudes for soft and hard types of poorly damped Duffing oscillator (DO) are found.

The Duffing equation [18] has been studied for 100 years already, because it is a typical representative of nonlinear mechanical systems. In [8], [9], [19], [20] hardening and softening amplitude-frequency response of DO with linear damping are considered. In the articles of K. V. Avramov and Yu. V. Mikhlin and some others [21], [22], [23] free and forced vibrations of a system with the Mises girder as a vibration damper with linear friction and without it are considered. The energy issue of the problem of effective depreciation of objects in the listed works is not considered.

2. Statement of the problem

In this paper we consider the Duffing oscillator, which elastic element is divided into two parts: deformable and accumulating, in both the initial tension is applied. In the positions of the protected object determined according to relative displacement, the DS of the parts is performed, namely, the short-term separation of the connection parts, accompanied by mass transfer from high-frequency oscillations and the subsequent dissipation of vibration energy. The paper is aimed at developing the characteristics of positional force, their harmonic linearization, finding the equivalent stiffness, damping and relative damping, the impact of the relative amplitude of the oscillations, the ratio of the mass of parts and initial tension.

3. Theory and results

3.1. The jump-up law of mass change of OD elastic element parts during their DS

In all the systems of Duffing without discrete and other switching of parts [19], [24, p. 12] the initial tension in the $s_0$ elastic elements with the exception of the systems considered in the book [25, p. 75], is not introduced. Meanwhile, it (figure 1) has a significant impact on all the characteristics of free and forced motion [25]. In addition, if the stiffness of the equilibrium state of the system is zero and $s_0 = 0$ then it is impossible to construct a theory of DO with DS in a dimensionless form. For this reason, in DO with parts DS when the switch is open $I$ the initial tension $s_0$ (figure 1) is introduced. Normalized to the original length of a deformable part, it is found according to the expression

$$S_0 = \frac{s_0}{l_{def,0}}.$$

In a condition of static equilibrium of the system regardless of the value of the initial tension $s_0$ (in a line $\psi_0$) and a turned off switch $I$ (figure 1) the dimensionless total mass of the parts of the elastic element (spring) $M_\Sigma = M_{def,0} + M_{acc} = I + \mu$, regardless of the initial tension $S_0$ is distributed in a way like

$$\mu = M_{acc,0}/M_{def,0} \equiv I_{acc}/l_{def,0}.$$
Figure 1. DO with elastic elements DS 1 is the switch; 2 is the deformable element; 3 is the accumulating element; 4 is the absolutely hard element; 5 is the axis $\psi$ is the vertical axis of symmetry of the system in its condition of static equilibrium; 6 is the initial tension $s_0$ distributed over both parts of the spring.

When kinematic excitation $x(t)$ is exposed to the system by harmonic horizontal displacement of the foundation, the absolute displacement of mass $M$ determined as the sum of the movable $x$ and relative $q_{rel}$ motion, i.e. $q_{abs} = x + q_{rel}$. When vanishing the relative displacement it is necessary to provide the DS of the elastic element parts; otherwise, due to the fact that the dimensionless mass of the deformable part is greater than 1 in a condition of static equilibrium, i.e. $M_{def} > 1$, it begins to dissipate the system instead of its damping due to DS. Therefore, in the Duffing system with DS (figure 1) in contrast to the schemes examined in [1], the parts DS of the elastic element must occur not twice, but four times over the period of oscillations.

The abrupt change in the mass of deformable elastic element $M_{def}$, occurring at the moments of its switching with the accumulating element, regardless of the period quarter of relative fluctuations, $k = I, II, III, IV$ is described as follows

$$
M_{def} = \left( \frac{M_{def}^I + M_{def}^{unl}}{2} + \eta \frac{M_{def}^I - M_{def}^{unl}}{2} \right), \tag{1}
$$

where $M_{def}^I$ is the mass of the deformable part when it is loaded in the first and in the third part of the period, $M_{def}^{unl}$ at its unloading in the II and IV parts of the period, $\eta$ is the Heaviside function that takes in (1) value $+1$ on the segments of the relative oscillations phase, $\psi_I = \left[ 0, \frac{\pi}{2} \right]$, $\psi_{III} = \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right]$ and $-1$ on the segments $\psi_{II} = \left[ \frac{\pi}{2}, \pi \right]$, $\psi_{IV} = \left[ \frac{3\pi}{2}, 2\pi \right]$.

Expression (1) may be written in the following form

$$
M_{def} = M_{def}^{av} + \eta M_{def}^{unl}, \tag{2}
$$
where $M_{\text{def}}^{\text{ampl}}$ is the mean mass of a deformable part, which is superimposed on the amplitude of mass $M_{\text{def}}$.

Equivalent restoring $F_{\text{rel}}(q_{\text{rel}}, \mu, S_0)$ and dissipative $F_{\text{diss}}(q_{\text{rel}}, \mu)$ forces are generated by a relative movement $q_{\text{rel}} = q - x$. Taking into account the filtering properties of the system approximate periodic solution for the generalized coordinate $q_{\text{rel}}$ is advisable to be written in the form [26], [27]

$$q_{\text{rel}} = m_{q,\text{rel}} + a_{q,\text{rel}} \cos \omega t,$$

(3)

where $m_{q,\text{rel}}$ is the displacement of the center of oscillation against the state of static equilibrium; $a_{q,\text{rel}}$ is the amplitude of the forced relative oscillations, $\omega$ is the frequency of excitation.

Dimensionless mass of the deformable part after DS in the amplitude positions of the system in the first and third quarters of the period is found by the expression

$$M_{\text{def}}' = \left(\mu + 1\right) \left[1 - \frac{\mu}{\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}}\right],$$

(4)

where $A_{q,\text{rel}} = a_{q,\text{rel}}/l_{\text{def},0}$, and minimum mass of this part of the second and fourth quarters of the period –

$$M_{\text{def}}'' = \left(\mu + 1\right) \left(\frac{l}{1 + \mu}\right) = l.$$  

(5)

In accordance with (1) and taking into account (4) and (5), the expression for the mass of the deformable part (figure 2) is written in the form

$$M_{\text{def}} = \frac{(\mu + 1)}{2} \left[1 - \frac{\mu}{\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}}\right] + \frac{l}{2} + \eta \left[\frac{(\mu + 1)}{2} \left(1 - \frac{\mu}{\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}}\right) - \frac{l}{2}\right].$$

(6)

Differentiating (6) with respect to mass parts $\mu$, we find that $\mu_{\text{exp}}|_{A_{q,\text{rel}}=2} \approx 1.95$.

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**Figure 2.** The mass of the deformable part of the elastic element:

- $M_{\text{def}}'$ – when its loading at the I or III part of the period,
- $M_{\text{def}}''$ – when its unloading at the II or IV part of the oscillation period: $l$ is the vertical plane $\mu \approx 1.95$, going through the extremum of the cross section surface $M_{\text{def}}'$ by vertical plane $A_{q,\text{rel}} = 2$. 

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At \( A_{q,rel} = 0 \) \( \mu_{exp} \rvert_{A_{q,rel}=0} = 1 \). This means that the extrema in the surface sections \( M'_{def} \) in figure 2 by planes \( A_{q,rel} = \text{const} \) are located on a spatial curve.

In accordance with the expression (2), the summands in(6) are written as

\[
M^\text{av}_{def} = \frac{\mu + 1}{2} \left( 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right) + \frac{l}{2}
\]

is the expression for the mean mass, and

\[
M^\text{ampl}_{def} = \frac{\mu + 1}{2} \left( 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right) - \frac{l}{2}
\]

is the expression for its amplitude values.

If the elastic element is not divided into two parts, i.e. \( \mu = 0 \), regardless of the symmetry of the system, this element according to (7), (8) has a constant mass equal to \( l \).

Subtracting (7) from (8) a relationship between the mean amplitude and the masses of deformable part is established

\[
M^\text{av}_{def} = l + M^\text{ampl}_{def}.
\]

3.2. The displacement of the equilibrium state of an object \( \Delta Q_{rel} \)

3.2.1 Changing the initial length of the elastic deformable element \( \Delta L_{def} \) while its DS with the accumulating element

The abrupt change in the mass of the deformable part (1), occurring in moments of DS using (2), (7), (8) leads to the dimensionless changes in the initial length of the deformable part \( \Delta L_{def} \) (Figure 3), which can be found as

\[
\Delta L_{def} = \Delta M_{def} = M_{def} - M_{def}^\text{av} + \eta M^\text{ampl}_{def} - l = \Delta L^\text{av}_{def} + \eta \Delta L^\text{ampl}_{def}
\]

Taking into account (7), (8) the expression (10) can be rewritten in the following form

\[
\Delta L_{def} = \frac{\mu + 1}{2} \left[ 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right] - l \eta \left[ 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right] - \frac{l}{2} + \eta \left[ 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right] - \frac{l}{2}.
\]

In the first and third quarters, when \( \eta = +1 \) (11) takes the form

\[
\Delta L^l_{def} = \left( \mu + 1 \right) \left[ 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right] - l,
\]

while in II and IV quarters take the following form

\[
\Delta L^u_{def} = 0.
\]

From (11) it follows that the mean and peak value of changes in the initial length of the deformable part (figure 3) can be found from the expression

\[
\Delta L^\text{av}_{def} = \Delta L^\text{ampl}_{def} = \frac{\mu + 1}{2} \left[ 1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}} \right].
\]
Figure 3. a) 3D-graphs of the initial length of a deformable part: 1 – when its loading at the first and third quarters $\Delta l_{def}^1$ according to (12); 2 – when its unloading in the second and fourth parts of the period $\Delta l_{def}^2$ according to (13); 3 are the surfaces of the mean and peak values of the displacement $\Delta l_{def}^{av} = \Delta l_{def}^{ampl}$ according to (14); b) an abrupt change in the initial length of the deformable part of DO, occurring in the moments of DS

3.2.2 Piecewise-constant stiffness of DO with DS as a function of the oscillation phase

Substituting into the expression for the dimensionless stiffness of the deformable and the accumulating parts of the spring [1, p. 103]

\[ c_{def} = \frac{I}{1 + \Delta l_{def}}; \quad c_{acc} = \frac{I}{\mu - \Delta l_{def}} \]

the expression for the dimensionless change in the length of the deformable part (11) we get

\[ c_{def} = \frac{I}{\left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) + \frac{I}{2} + \eta \left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) - \frac{I}{2}} \]

\[ \left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) + \frac{I}{2} - \eta \left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) - \frac{I}{2} \right) \]  \hspace{1cm} (15)

\[ c_{acc} = \frac{1}{\mu - \left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) + \frac{I}{2} - \eta \left(\mu + 1\right) \left(1 - \frac{\mu}{\sqrt{A_{q,rel}^2 + (\mu + 1)^2}}\right) - \frac{I}{2}} \]  \hspace{1cm} (16)

At the moments of switching, the deformable and accumulating elements are connected in series, therefore their overall stiffness can be found as

\[ c_{bind} = \frac{c_{def} c_{acc}}{c_{def} + c_{acc}} = \frac{I}{\mu + 1} \]  \hspace{1cm} (17)

For a given mass ratio of the elastic elements $\mu$ of the parts stiffness in the intervals between the DS moments (15) and (16) depend only on the amplitude of the vibration $A_{q,rel}$, while the stiffness $c_{bind}$ (17) does not depend on it (figure 4).
Figure 4. The stiffness curves of: a) a deformable part, b) an accumulating part;
1..5 – in the intervals between the switching according to (15) and (16); 6 is the mean value of stiffness of the parts when the amplitude \( A_{\text{rel},1} = 2 \) and 7 is the stiffness of series-connected elements at the moments of switching according to (17).

This ratio of the mass \( \mu = 2 \) is chosen because it is close to the value at which there occurs the maximum mass transfer between the parts (vertical plane in figure 2).

The highest stiffness \( c_{\text{def}} = c_{\text{def}}^\text{unl} = 1 \) in the system is achieved at the II and IV quarters of the period of relative fluctuations (figure 4) where the Heaviside function \( \eta = -1 \) and the initial length change of the deformable part \( \Delta L_{\text{def}} = 0 \).

### 3.2.3. The distribution of the total initial tension \( S_0 \) on both parts of the elastic element in DS moments

In the DS moments the proportion of primary tension \( S_0 \) applied to the deformable part at the II and IV quarters are found according to the formula

\[
S_{0,\text{def}}^{\text{unl}} = k_{\text{def}}^{\text{unl}} S_0 \quad \text{in which}
\]

\[
k_{\text{def}}^{\text{unl}} = \frac{1}{1 + 1/\mu} = \frac{1}{\mu + 1} \quad \text{(18)}
\]

while at the first and third quarters – according to the formula

\[
S_{0,\text{def}}^l = \frac{c_{\text{acc}}^l}{c_{\text{def}}^l + c_{\text{acc}}^l} S_0 = k_{\text{def}}^l S_0. \quad \text{(19)}
\]

Substituting expressions (15) and (16) into (19) for stiffness of deformable and accumulating parts at the first and third quarters, respectively, we receive for coefficient \( k_{\text{def}}^l \) expression

\[
k_{\text{def}}^l = 1 - \frac{\mu}{\sqrt{A_{\text{rel}}^2 + (\mu + 1)^2}}. \quad \text{(20)}
\]

The general expression for the coefficient \( k_{\text{def}} \), taking into account the proportion of initial tension \( S_0 \) applied to the deformable element and depending on the part of the oscillation period can be written as

\[
k_{\text{def}} = \frac{k_{\text{def}}^l + k_{\text{def}}^{\text{unl}}}{2} + \eta \frac{k_{\text{def}}^l - k_{\text{def}}^{\text{unl}}}{2}. \quad \text{(21)}
\]
After substituting in the expression (21) formulas (18) and (20) it will have the following form

\[
k_{\text{def}} = \frac{(2 + \mu)\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2} - \mu(\mu + 1)}{2(\mu + 1)\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}} + \eta\mu \frac{\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}}{(\mu + 1)\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2}}.
\]  
(22)

3.2.4. The influence of DO with DS parameters: the amplitude oscillations of the \( A_{q,\text{rel}} \) mass ratio \( \mu \) and the initial tension \( S_0 \) on the object displacement \( \Delta Q_{\text{rel}} \)

Occurring during the oscillations of non-autonomous DO with DS Duffing type periodic variation of the initial length of the deformable part \( \Delta L_{\text{def}} \) (figure 3) leads to a periodic displacement of the equilibrium state \( \Delta Q_{\text{rel}} \) of the object mass \( M \) (Figure 1) in which the positional force is equal to zero (figure 5).

![Diagram](image)

**Figure 5.** To calculate the displacement of the equilibrium state of DO with DC of elastic elements \( \Delta Q_{\text{rel}} \) in the first and third quarters: pos. 1 – the switch is open, pos. 2 – the switch is closed; the share of initial tension \( S_{0,\text{def}} \), with which the stretched deformable part in the first and third quarters is determined by the expression (19), while the change in the length of the deformable part of the elastic element \( \Delta L_{\text{def}} \) along (11) when \( \eta = +1 \)

The displacement \( \Delta Q_{\text{rel}} \) (figure 5) can be found as the relation

\[
\sqrt{A_{q,\text{rel}}^2 + (\mu + 1)^2} - (\mu + 1) - \Delta L_{\text{def}} + k_{\text{def}} S_0 = 0.
\]  
(23)

Solving (23) with respect to \( \Delta Q_{\text{rel}} \), we obtain an expression for the displacement of the equilibrium state of the object in the I quarter.

In the third quarter of the period \( \Delta Q_{\text{rel}}^1 = -\Delta Q_{\text{rel}}^3 \). A single expression for displacement of the static equilibrium state of the object is written in the form

\[
\Delta Q_{\text{rel}}^1 = \Delta Q_{\text{rel}}^3 = \pm \sqrt{2\mu \Delta L_{\text{def}} - 2\mu k_{\text{def}} S_0 + 2\Delta L_{\text{def}} S_0 + \Delta L_{\text{def}}^2 - 2\Delta L_{\text{def}} k_{\text{def}} S_0 + k_{\text{def}}^2 S_0^2}.
\]  
(24)

Substituting a periodic change in the length of the deformable part of the elastic element \( \Delta L_{\text{def}} \) (11), coefficient \( k_{\text{def}} \) (22) and \( \eta = +1 \) into (24), we obtain the expression for displacement of the static equilibrium state of DO with parts DS \( \Delta Q_{\text{rel}}^{1,3} \) in the first and third quarters of the oscillation period (figure 6).
Figure 6. a) periodic displacement surfaces of the equilibrium state of the system $\Delta Q_{rel}$ at $S_0 = 0$:

1 – $\Delta Q_{rel}^1$ and 2 – $\Delta Q_{rel}^2$, respectively for the first and third quarters of the oscillation period;

3 – for the second and fourth quarters $\Delta Q_{rel}^3 = 0$;

b) are the same surfaces depending on the mass ratio $\mu$ and the initial tension $S_0$ at the amplitude $A_{q,rel} = 2$

Negative values of the curves in the cross sections of the surfaces in figure 6a by vertical planes $\mu = \text{const}$ (figure 7 a), and $A_{q,rel} = \text{const}$ (figure 7 b), marked with dashes, represent the motion of the system with increasing negative coordinates $Q_{rel}$, i.e. from left to right.

Figure 7. Dependences of the equilibrium state periodic displacement:

a) – on the relative oscillations amplitude $A_{q,rel}$ at $1, 1' - \mu = 0, 2, 2' - \mu = 0.1; 3, 3' - \mu = 1;

4, 4' - \mu = 2; 5, 5' - \mu = 4; 6, 6' - \mu = 1000;

b) – on the elastic elements mass ratio $\mu$ at $1, 1' - A_{q,rel} = 0; 2, 2' - A_{q,rel} = 0.1; 3, 3' - A_{q,rel} = 0.5;

4, 4' - A_{q,rel} = 1; 5, 5' - A_{q,rel} = 1.5; 6, 6' - A_{q,rel} = 2$

In the second and fourth quarters of the period of oscillations at $\eta = -1$ (mass $M$ moves from the equilibrium state (figure 1) the equation (23) has the form

$$\sqrt{\Delta Q_{rel}^2 + (\mu + 1)^2} - (\mu + 1) - \Delta L_{def} + k_{um}S_0 = 0,$$

and its solution regarding the displacement $\Delta Q_{rel}$ gives an imaginary value.
\[ \Delta Q_{rel} = i \frac{\sqrt{S_0}}{\mu + 1} \frac{1}{2} (\mu + 1)^2 - S_0 , \]

\( i = \sqrt{-1} \) that only at \( S_0 = 0 \) vanishes in equilibrium.

3.3. Accurate characteristics of the positioning force of DO with parts DS

3.3.1 The potential energy of the deformable part of DO with DS

Regardless of the quarter of the period, the general expression for the potential energy of a deformable part of the system in the intervals between the moments of commutation is written in the form

\[ P(Q_{rel}, \mu) = \frac{1}{2} c_{def} \left[ \sqrt{Q_{rel}^2 + (\mu + 1)^2} - (\mu + 1) - \Delta L_{def} + k_{def} S_0 \right]^2 , \]  

(25)

where the stiffness of the deformable part is found \( c_{def} \) according to expression (15), \( \Delta L_{def} \) is then abrupt change in the initial length of the deformable part, occurring at the moments of DS, – by (11), \( k_{def} \) is the ratio of the initial tension attributable to the deformable part, – by (22).

Assuming that in expression (25) the value of the Heaviside function is consistently \( \eta = +1 \) and \( \eta = -1 \), we obtain two different expressions for potential energy in the first and third quarters and the second and fourth parts of the oscillation period, respectively (figure 8)

– in the first and third quarters

\[ P_{def} = \frac{1}{2} \left[ \sqrt{Q_{rel}^2 + (\mu + 1)^2} - 2 \mu - L + S_0 \right] \left[ \sqrt{Q_{rel}^2 + (\mu + 1)^2} - \mu (\mu - L + S_0) \right]^2 , \]  

(26)

– in the second and fourth quarters

\[ P_{def}^{(2)} = \frac{1}{2} \left( \sqrt{Q_{rel}^2 + (\mu + 1)^2} - (\mu + 1) + \frac{S_0}{\mu + 1} \right)^2 . \]  

(27)

**Figure 8.** The dependence of the potential energy of DO elastic elements DS on its parameters \( S_0, \mu \) and \( A_q,rel \):

a) \( S_0 = 0 \); b) \( S_0 = 0.5 \); I, III – in the first and third quarters of the period along (26); II, IV – in the second and fourth quarters along (27); I, III switching verticals in amplitude, and 2, 4 – in the static equilibrium states; 5, 6 – is the potential energy of one of the deformable parts and both parts at the moments of switching, respectively.
3.3.2 Mixed piecewise-nonlinear characteristics of the DO with DS elements of the elastic elements

Differentiating (25) with respect to the relative coordinate $Q_{\text{rel}}$, we get a general expression to characterize positional force

$$F_{\text{pos}}(A_{\text{rel}}, \mu, S_0) = c_{\text{def}}Q_{\text{rel}} \left( I - \frac{(\mu + 1) + \Delta L_{\text{def}} - k_{\text{def}}S_0}{\sqrt{Q_{\text{rel}}^2 + (\mu + 1)^2}} \right).$$

(28)

The Heaviside function $\eta$ is a part of (28) via the expression for the stiffness of the deformable part $c_{\text{def}}$ (15), changes the initial length of the deformable part $\Delta L_{\text{def}}$ (11) and ratio $k_{\text{def}}$ (22).

Substituting in (28) the values of the Heaviside function $\eta = +1$ and $\eta = -1$ we obtain the expressions for the positional forces in $I$ and $III$ and in $II$ and $IV$ quarters, respectively (Figure 9),

$$F_{\text{pos}}^I(A_{\text{rel}}, \mu, S_0) = \left( \sqrt{Q_{\text{rel}}^2 + (\mu + 1)^2} - 2\mu - I + S_0 \right) \sqrt{A_{\text{rel}}^2 + (\mu + 1)^2} - \mu (\mu - I + S_0) \right)$$

(29)

$$F_{\text{pos}}^{\text{unl}}(\mu, S_0) = \left( 1 - \frac{\mu + I - S_0}{\mu + 1} \right) Q_{\text{rel}}.$$

(30)

Figure 9. The influence of the common initial tension $S_0$ on the characteristics of the positioning force of DO with the DS of elastic element parts: $a) - S_0 = 0; b) - S_0 = 0.5; curves 5, 6 – characteristics of the restoring force of one deformable part when the mass ratio $\mu = 0$ and of both parts at the moments of switching respectively;

$\Delta Q_{\text{rel}}$ is the displacement from the static equilibrium state of DO with DS

Substituting into the expression for the positional force $I$ and $III$ (29) and (30) the dimensionless relative displacement

$$Q_{\text{rel}} = q_{\text{rel}}/l_{\text{def}} = A_{q_{\text{rel}}} \cos \psi,$$

(31)

we obtain the expressions for positional force in the quarters during the period which are necessary for its harmonic linearization:
3.4. Harmonic linearization of a mixed piecewise-nonlinear characteristics of DO with DS of elastic elements parts

3.4.1 Ratios of harmonically linearized approximation of positional force
Substituting expressions (32), (33) for position force \( F_{\text{pos}}(\psi) \) and \( F_{\text{unl}}(\psi) \) into the expressions for the coefficients of its harmonic linearized approximation [26], [27]

\[
A_{\text{pos}}(\psi) = A_{\text{pos},\psi} \cos \psi + A_{\text{pos},\sin \psi} \sin \psi,
\]

integrating by quarters and summing the results, we obtain for the amplitude of the potential component of the position force (figure 10) the following expression

\[
A_{\text{pos}}(\psi) = \frac{1}{\pi} \left[ F_{\text{pos}}(\psi) \cos \psi \right] \mu, A_{\text{pos},\psi}, S_0, K \left( \frac{A_{\text{pos},\psi}}{(\mu + 1)^2} \right), E \left( \frac{A_{\text{pos},\psi}^2}{(\mu + 1)^2} \right),
\]

where, by \( K \left( \frac{A_{\text{pos},\psi}}{(\mu + 1)^2} \right) \) and \( E \left( \frac{A_{\text{pos},\psi}^2}{(\mu + 1)^2} \right) \) complete elliptic integrals of the first and second kind [28], respectively, are denoted.

**Figure 10.** The surfaces of the amplitudes of the potential and restoring components of DO positional force with DS (figure 1): 1 are the DO systems with DS; 2 of one deformable part \( \mu = 0 \); 3 in the DS moments with a total length of the elastic element parts \( \mu + 1 \), \( \mu = 2 \); 4, 5 is the vertical plane \( \mu = 0 \) and horizontal plane \( A_{\text{def}} = 0 \) respectively.

The amplitude of the dissipative components of the positioning force is given by
In expressions (34), (35) positional force of the fourth period (figure 11 a) is found according to equations (32) and (33).

The maximum amplitude of the dissipative component $A_{def}$ (figure 11 b) is achieved when the value of the ratio of the mass $\mu$ is about 2.5 times smaller than the maximum cross-section of the surface of the mass of the deformable part in the first and third quarters of the oscillation period by plane $A_{q,rel} = 2$ (figure 2).

Figure 11. a) the effect of initial tension $S_0$ on accurate positional force according to (32) and (33), vertical lines indicate the DS of DO parts; b) 1 is the surface of the amplitude of the dissipative component of the positional force of the DO deformable part with DS (figure 1); 2 – is the vertical plane passing through the maximum section of the surface 1 by plane $A_{q,rel} = 2$

3.4.2 The equivalent coefficients of stiffness and damping

Dividing the amplitude of the potential components of the positioning force (34) on the amplitude of relative displacement $A_{q,rel}$, we obtain an expression for the equivalent stiffness of OD with DS

$$c_{eq}(A_{q,rel}, \mu, S_0) = \frac{A'_{def}}{A_{q,rel}}. \tag{36}$$

In the limit when $A_{q,rel} \to 0$ this formula gives the expression for the stiffness of the linearized system in a small

$$c_{eq}^{st} = \lim_{A_{q,rel} \to 0} \frac{A'_{def}}{A_{q,rel}} = \frac{S_0}{(\mu + 1)^2}. \tag{37}$$

The dependence of the dynamic stiffness on the amplitude represents the projection of the cross sections of the surface $c_{eq}(A_{q,rel}, \mu, S_0)$ by vertical planes $\mu = const$ onto plane $\mu = 0$. For two values of initial tension, this dependence is shown in figures 12 a) and 12 b).
The expression for the dimensionless resistance coefficient has the following form [1, p. 92]

\[
\beta_{eq}(A_{q,rel}, \mu) = \frac{A'_{F,def}}{\sqrt{c_{eq}(A_{q,rel}, \mu, S_0) \eta A_{q,rel}}},
\]

(38)

where \( A'_{F,def} \) is the amplitude of the dissipative components of the positioning force (35);

\[
\sqrt{c_{eq}(A_{q,rel}, \mu, S_0) \equiv \nu(A_{q,rel}, \mu, S_0)}
\]

is the dimensionless frequency of free DO oscillations with DS;

\[
\eta = \frac{\omega}{\omega_{fr}(A_{q,rel})} = \frac{\omega}{\omega_{nat} \sqrt{c_{eq}(A_{q,rel}, \mu, S_0)}} = \frac{\omega}{\omega_{nat} \nu(A_{q,rel}, \mu, S_0)}
\]

is the dimensionless frequency of the excitation;

\[
\omega_{nat} = \sqrt{\frac{c_{def,0} \cdot M}{\mu}}
\]

is the natural frequency of DO oscillation with DS of parts in figure 1; \( c_{def,0} \) – is the dimensional stiffness of the deformable part.

**Figure 12.** The dependence of the dynamic stiffness of the relative oscillation amplitude \( A_{q,rel} \) and the ratio of the parts mass \( \mu \) with different values of initial tension: a) \( S_0 = 0 \); b) \( S_0 = 0.5 \);

1 – \( \mu = 0 \); 2 – \( \mu = 0.1 \); 3 – \( \mu = 0.2 \); 4 – \( \mu = 1 \); 5 – \( \mu = 0 \); 2 – \( \mu = 0.1 \); 3 – \( \mu = 0.2 \); 4 – \( \mu = 0.6 \);

5 – \( \mu = 1 \); 6 – \( \mu = 2 \); 7 – \( \mu = 4 \); 8 – \( \mu = 6 \); 9 – \( \mu = \infty \)

The sections of the surface of the equivalent resistance coefficient \( \beta_{eq}(A_{q,rel}, \mu) \) (figure 13 a) by vertical planes \( \mu = const \) (figure 13 b) show that regardless of the amplitude \( A_{q,rel} \) of the maximum value, this ratio achieves, when the ratio of the mass \( \mu \approx 1.95 \).
Figure 13. a) 3D dependence of the dimensionless resistance coefficient on the amplitude of the relative oscillations and the mass ratio (surface 1), 2 – vertical plane \( \mu \cong 1.95 \)

3 – the plane of zero damping (\( \mu = 0 \)); b) cross-section of the surface \( \beta_{eq}(A_{q,rel}, \mu) \) by planes

1. \( \mu = 0.1 \); 2. \( \mu = 0.4 \); 3. \( \mu = 1 \); 4. \( \mu = 2 \); 5. \( \mu = 4 \); 6. \( \mu = 10 \); 7. \( \mu = 20 \); 8. \( \mu = 100 \); 9. \( \mu = 1000 \)

The introduction of the initial tension \( S_0 \) increases the stiffness \( c_{eq}(A_{q,rel}, \mu, S_0) \) (figure 12 b) and thereby reduces the equivalent resistance coefficient \( \beta_{eq}(A_{q,rel}, \mu) \) (figure 14).

Figure 14. a) 3D dependence of the dimensionless resistance coefficient of the amplitude of the relative oscillation and the mass ratio (surface 1) at initial tension \( S_0 = 0.5 \), \( \eta = 1 \);

2 – vertical plane \( \mu \cong 1.95 \), 3 – plane of zero damping (\( \mu = 0 \));

b) – sections of the surface \( \beta_{eq}(A_{q,rel}, \mu) \) by planes \( \mu = \text{const} \)
3.5. Equivalent linearization of the positional dissipative component of the force of the elastic element deformable part

3.5.1 The amount of energy dissipated in a period

Substituting in the formula for the number of positional $\beta_{q,rel} (A_{q,rel}, \mu) Q_{rel}$ force of energy $\Delta W = \pi A_{\omega,def} A_{q,rel}$ scattered by dissipative component in the period [1], p. 92] the expression for the amplitude components $A_{\omega,rel}$ (35), we obtain

$$\Delta W = 4\mu \left[ \left( -\mu - \mu^2 - \frac{1}{2} A_{q,rel}^2 \right) \sqrt{A_{q,rel}^2 + (\mu + 1)^2} + \mu^2 + \mu \left( \frac{3}{4} A_{q,rel}^2 + 1 \right) + \frac{A_{q,rel}^2}{4} \right]$$

(39)

As it follows from expression (35) for the amplitude of the dissipative component $A_{\omega,def}$, the amount of energy $\Delta W$ dissipated in the period does not depend on the initial tension $S_0$. Therefore, the area of the hysteresis loops in figure 9 does not depend on this tension either. In the limit when $\mu \to \infty \Delta W \bigg|_{\mu \to \infty} = \lim_{\mu \to \infty} \Delta W = 0$.

The amount of energy dissipated in the period can be found as the area of the hysteresis loop in figure 9:

$$\Delta W' = 2 \left( \int_0^{A_{\omega,rel}} F_{pos} dQ_{rel} - \int_0^{A_{\omega,rel}} F_{pos} dQ_{rel} \right).$$

(40)

Substituting in (40) expressions (29) and (30) for the positioning force of the deformable part $F_{pos}^{\omega}$ and $F_{pos}^{\omega}$ at its loading and unloading, respectively, we obtain expression (39), i.e. $\Delta W' = \Delta W$ (figure 15 a).

![Diagram](image)

**Figure 15.** a) 1 – dissipated energy surface; 2 – vertical plane passing through the maximum section of the surface 1 by plane $A_{q,rel} = 2$, achieved at $\mu_{\text{crit}} \approx 0.81$; 3 – the plane of zero damping, at $\mu = 0$ according to (39);

b) surface section in figure a) by planes $A_{q,rel} = \text{const}$:

- $1 - A_{q,rel} = 0$;
- $2 - A_{q,rel} = 0.5$;
- $3 - A_{q,rel} = 1$;
- $4 - A_{q,rel} = 1.5$;
- $5 - A_{q,rel} = 1.75$;
- $6 - A_{q,rel} = 2$

3.5.2 Equivalent relative attenuation coefficient

Equating in accordance with the principle of energy balance [26], [27], dimensionless number of energy dissipated over the period in a linear system, the expression of which is converted to the form [1, p. 95],
to the amount of energy dissipated in a linear DO with parts DS (39), we find an expression for the equivalent relative attenuation coefficient

\[
\Delta W^\text{in} = 2\pi \psi\left(A_{q,\text{rel}},\mu,S_0\right)\end{equation}

(41)

Otherwise, the expression for the relative attenuation coefficient can be obtained by the formula [1, p. 95]

\[
\psi\left(A_{q,\text{rel}},\mu,S_0\right) = \frac{\Delta W}{2\pi A_{q,\text{rel}}^2 c_{eq}\left(A_{q,\text{rel}},\mu,S_0\right)} \end{equation}

(42)

The behavior of surfaces in figure 16, and their sections by planes \(\mu = \text{const}\) in figure 17 indicates a fundamentally different effect of initial tension \(S_0\) on the coefficient \(\psi\left(A_{q,\text{rel}},\mu,S_0\right)\) (41) or (42).

**Figure 16.** The dependence of the surface relative attenuation coefficient \(\psi\) (41), (42) on the amplitude \(A_{q,\text{rel}}\) and mass ratio \(\mu\) at some values of the initial tension \(S_0\): \(1 - S_0 = 0\) and \(2 - S_0 = 0.5\);

3 is the plane with zero damping at \(\mu = 0\);

4 – the level limit plane \(\psi^\text{lim} = \lim_{\mu \to \infty} \psi = 2/k\pi \eta \approx 0.6366\) at \(S_0 = 0\) and \(\eta = 1\).

At zero tension \(S_0\) coefficient \(\psi\) has finite values at amplitude \(A_{q,\text{rel}} = 0\) in the range of the ratios of mass \(\Delta \mu = 0.\infty\), varying in the interval \(\Delta \psi = \left[0, \frac{2}{\pi \eta}\right]\) (figure 17 a); at \(S_0 > 0\) and \(A_{q,\text{rel}} = 0\) coefficient \(\psi = 0\) and its change is monotonically increasing (figure 17 b).
Figure 17. The amplitude dependence of the relative attenuation coefficient:
a) for the initial tension $S_0 = 0$; b) if $S_0 = 0.5$; in both figures a) and b) the mass ratio of parts:
\[ 1 - \mu = 0; \quad 2 - \mu = 0.6; \quad 3 - \mu = 1; \quad 4 - \mu = 2; \quad 5 - \mu = 4; \quad 6 - \mu = 10; \quad 7 - \mu = 1E + 20 \]

If at $S_0 = 0$ (figures 16, 17 a) cascading amplitude dependence $\psi$ in the range of the ratios of mass $\Delta \mu = 0..20$ can be neglected and for it expression (42) can be taken, the analytical limit of which $\lim_{A_{q,rel}} \psi$ cannot be obtained, although the dependence graphs $\psi\left(A_{q,rel}\right)$ (figure 16, the surface 1; figure 17 a), it is are possible to be built, avoiding values $A_{q,rel} = 0$.

To obtain a simple expression $\psi(\mu, \eta)$ for the dependence of the relative attenuation coefficient of the mass ratio $\mu$ and perturbations frequency $\eta$ at the initial tension $S_0 = 0$ let us calculate the number of values $\psi$ at the amplitude $A_{q,rel} = 1E - 10$ (table 1).

Table 1. Numerically obtained values of the relative attenuation coefficient as a function of the mass ratio on the frequency $\eta = 1$ (diamonds in figure 18).

| $\mu$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| $\psi$ | 0  | 0.15 | 0.255 | 0.318 | 0.364 | 0.398 | 0.424 | 0.446 | 0.463 | 0.477 | 0.490 |
| $\mu$ | 11 | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | $\infty$ |
| $\psi$ | 0.500 | 0.509 | 0.517 | 0.524 | 0.531 | 0.536 | 0.541 | 0.546 | 0.550 | 0.554 | 0.6366 |

Further, approximating data from table 1 of nonlinear function
\[
\psi = \frac{2}{\pi \eta} \frac{a \mu}{a \mu + 1},
\]
we obtain the following expression for the relative attenuation coefficient
\[
\psi_{\text{lim}}\bigg|_{A_{q,rel} = 0} \approx \frac{2}{\pi \eta} \frac{0.333 \mu}{0.333 \mu + 1},
\]
(curve 2 in figure 18).
Figure 18. An approximate dependence of the relative attenuation coefficient \( \psi \) of the mass ratio \( \mu \) of DO with parts DS at the initial tension \( S_0 = 0 \): 1 – the numerical values \( \psi \) obtained at an amplitude of relative displacement \( A_{q,rel} = 1 \times 10^1 \); 2 – approximation of these curve values (43); 3 – the limit analytical dependence \( \psi \) for DO with parts DS of the pneumo-element and deformable bodies [1, p. 263]; 4 – the limit value \( \psi^{lim}_{\mu=\infty} = \frac{2}{\pi \eta} \approx 0.6366 \) at the frequency \( \eta = 1 \).

The limit of the expression (43) is the following

\[
\psi^{lim}_{A_{q,rel}=0} = \lim_{\mu \to \infty} \frac{A_{s,rel} \cdot 0.333 \mu}{2} \cdot \frac{0.333 \mu + 1}{\pi \eta} = \frac{2}{\pi \eta}
\]

From the similar expressions for SA [1] with elastic elements parts DS

\[
\psi^{lim}_{A_{q,rel}=0} \approx \psi^{lim}_{A_{q,rel}=\infty} = \frac{2}{\pi \eta} \cdot \frac{\mu}{\mu + 1},
\]

investigated in [1, p. 263], [2], [3], [4], formula (43) for DO with parts DS is characterized by the presence of the multiplier \( a = 0.333 \), shifting the curve 2 in figure 18 down, however, the limit values of the relative attenuation coefficient \( \psi \) for \( \mu = 0 \) and \( \mu = \infty \) coincide

\[
\psi^{lim}_{\mu=0} = 0 \quad \text{and} \quad \psi^{lim}_{\mu=\infty} = \frac{2}{\pi \eta}.
\]

The increase in initial tension \( S_0 \) at constant amplitude \( A_{q,rel} \) leads to a decrease of the relative attenuation coefficient \( \psi \) (figure 19 a, 19 b).

With increasing the amplitude to value \( A_{q,rel} = 2 \) in the range of the initial tension [0.1..0.5] a monotonous increase in the relative attenuation coefficient \( \psi \) occurs (figure 19 b).
Figure 19. a) the surfaces of the relative attenuation coefficient $\psi$ depending on the mass ratio $\mu$ and the initial tension $S_0$: $1 - A_{q,rel} = 1$; $2 - A_{q,rel} = 2$; $3$ is the limit value $\psi^{lim}_{\mu = \infty} = \frac{2}{(\pi \eta)}$ at the frequency $\eta = 1$;  

b) is the dependence of the coefficient of relative attenuation on the amplitude of relative displacement: $1 - S_0 = 1E - 10; 2 - S_0 = 1E - 5; 3 - S_0 = 1E - 4; 4 - S_0 = 0.1; 5 - S_0 = 0.3; 6 - S_0 = 0.3; 7 - S_0 = 0.5; 8 - S_0 = 0 u \mu = 1E + 20$

3.6. The linearized motion equation of DO with parts of elastic elements DS

The equation of mass motion $M$ (figure 1) taking into account expression (28) for the dimensionless non-conservative positional force $F_{pos}(A_{rel}, \mu, S_0)$ can be written as

$$a\ddot{Q} + c_{rel,0}D_{def}Q_{rel} \left(1 - \frac{(\mu + 1) + \Delta L_{def} - k_{def}S_0}{\sqrt{Q_{rel}^2 + (\mu + 1)^2}} \right) = 0, \quad (44)$$

where $a = M$ is the coefficient of inertia, $c_{def,0}$ is the dimensional stiffness of the deformable part in a state of static equilibrium.

In equation (44), replacing force $F_{pos}(A_{rel}, \mu, S_0)$ by its harmonically linearized approximation [1], [26], [27], we obtain

$$a\ddot{Q}_{rel} + c_{rel,0}D_{def}(A_{F,def}^{c} - A_{F,def}^{r}) = -\ddot{X}.$$

This equation can be transformed to the form [1, p. 97]

$$a\ddot{Q}_{rel} + c_{rel,0}D_{def} \left[\beta_{eq}(A_{q,rel}, \mu, S_0)\ddot{Q}_{rel} + c_{eq}(A_{q,rel}, \mu, S)\dot{Q}_{rel} \right] = -a\ddot{X}. \quad (45)$$

Using its own system time $\tau = \omega_{rel}^{-1}(A_{q,rel}, \mu, S_0)t$ derivatives, included in equation (45) are written in the following form

$$\dot{Q}_{rel} = \omega_{rel}^{-1}v\left(A_{q,rel}, \mu, S_0\right)Q_{rel}^{*}; \ddot{Q}_{rel} = \omega_{rel}^{-2}v^2\left(A_{q,rel}, \mu, S_0\right)Q_{rel}^{*}; \dddot{Q}_{rel} = \omega_{rel}^{-3}v^3\left(A_{q,rel}, \mu, S_0\right)Q_{rel}^{*};$$

$$\dot{X}_{rel} = \omega_{rel}^{-1}v\left(A_{q,rel}, \mu, S_0\right)X_{rel}^{*}; \ddot{X}_{rel} = \omega_{rel}^{-2}v^2\left(A_{q,rel}, \mu, S_0\right)X_{rel}^{*}.$$

Using them, equation (45) can be rewritten in the form

$$\dddot{Q}_{rel} + 2\psi(A_{q,rel}, \mu, S_0)\dot{Q}_{rel}^{*} + Q_{rel}^{*} = -X^{*}, \quad (46)$$
in which the relative attenuation coefficient $\psi(A_{q,rel}, \mu, S_0)$ is determined by expression (41) or (42).
Having solved the equation for the frequency response obtained from (46) relative to the amplitude $A_{q, rel}$ and substituting it in expression (42), we obtain the exact frequency response of the relative attenuation coefficient (solid curves in figure 20). Approximate characteristics of this ratio (dashed curves in figure 20) were built according to expression (43).

**Figure 20.** The frequency response of the relative attenuation coefficient of DO with parts DS:

1. $\mu = 0.4$
2. $\mu = 1$
3. $\mu = 2$
4. $\mu = 4$
5. $\mu = 10$
6. $\mu = \infty$

### 4. Conclusion

Discrete switching of the deformable and accumulating parts of the elastic element in the Duffing oscillator creates a periodic change in the initial length of this element and, as a consequence, the displacement of the equilibrium state and thus adds dissipative properties to the system. The independence of the amount of energy scattered at discrete switching moments makes the frequency characteristics of the relative attenuation coefficient hyperbolic, thereby aligning it with the behavior of the frequency response. Zero initial tension of the elastic element allows to obtain the expression independent on the amplitude of the tangent of mechanical losses and thus build a simple methodology for the selection of basic parameters. Regardless of the mass ratio of the parts, positive initial tension, due to the increase in the equivalent stiffness, reduces the relative attenuation coefficient and makes it almost linearly dependent on the amplitude.

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