GENERALIZED COHOMOLOGIES AND THE PHYSICAL SUBSPACE OF THE SU(2) WZNW MODEL

Michel DUBOIS-VIOLETTE
Laboratoire de Physique Théorique et Hautes Energies
Université Paris XI, Bâtiment 211
F-91 405 Orsay Cedex, France
flad@qcd.th.u-psud.fr

and

Ivan T. TODOROV
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia
and
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
BG-1784 Sofia, Bulgaria
todorov@inrne.acad.bg

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Abstract

The zero modes of the monodromy extended $SU(2)$ WZNW model give rise to a gauge theory with a finite dimensional state space. A generalized BRS operator $A$ such that $A^h = 0$ ($h = k + 2 = 3, 4, \ldots$ being the height of the current algebra representation) acts in a $(2h-1)$-dimensional indefinite metric space $\mathcal{H}_I$ of quantum group invariant vectors. The generalized cohomologies $\text{Ker} A^n / \text{Im} A^{h-n}$ ($n = 1, \ldots, h-1$) are 1-dimensional. Their direct sum spans the physical subquotient of $\mathcal{H}_I$.

Introduction

The canonical approach to the quantum Wess-Zumino-Novikov-Witten (WZNW) model (see [1] [5] [10] [6] [7] [8] [9] and references therein) singles out a non-trivial finite dimensional problem involving the (“quantum group” [1]) zero modes, its infinite dimensional counterpart (generated by the chiral vertex operators) being relatively trivial. The discussion in [5] [6] is based on a lattice formulation of the theory whose continuum limit is treated (following the canonical approach of [10] [6]) in [8], [9]. The result is a finite dimensional gauge model: the physical state space $\mathcal{H}$ appears as a quotient, $\mathcal{H} = \mathcal{H}' / \mathcal{H}''$, where $\mathcal{H}'$ is singled out in the original (still finite dimensional) indefinite metric space by a finite set of constraints while $\mathcal{H}''$ is the subspace of “null” (zero norm) vectors of $\mathcal{H}'$.

In this paper we realize the physical space $\mathcal{H}$ as a direct sum of generalized cohomologies (of the type considered in [1] and [8], see also [12]). For a level $k (= 1, 2, \ldots)$ representation of the $A_1^{(1)} (= \hat{\mathfrak{su}}(2))$ Kac-Moody algebra, the “BRS charge” $A$ satisfies $A^h = 0$ where $h = k + 2$ (in general, the height $h$ is the sum of the level $k$ and the dual Coxeter number of the underlying simple Lie algebra). The reduction of the original $h^4$ dimensional tensor product space $\mathcal{F} \otimes \mathcal{F}$ of chiral zero modes is performed in two steps. First, in Section 1, we review the results of refs [8] [8] ending up with a $(2h-1)$ dimensional space $\mathcal{H}_I$ of $U_q(sl_2) \otimes U_q(sl_2)$ invariant vectors (with a deformation parameter $q = e^{i\pi} \hbar$). Secondly, in Section 2, we represent $\mathcal{H}$ as a direct sum of 1-dimensional cohomologies $\text{Ker} A^n / \text{Im} A^{h-n}$, $n = 1, \ldots, h-1$ ($A^n$ being regarded as endomorphisms of $\mathcal{H}_I$).
1 The Fock space of left and right zero modes
and its quantum group invariant subspace

The solution of the WZNW equations of motion is customarily written in a factorized form:

\[ g(x,t) = u(x-t)\bar{u}(x+t) \quad \text{(classically, } g, u, \bar{u} \in SU(2)) \]  

(1.1)

Each of the quantized chiral fields, \( u \) and \( \bar{u} \), is expanded into a sum of a positive and a negative frequency part multiplied by a (\( q \)-deformed) creation and annihilation operator:

\[
u^\alpha_\beta(x) = a^+_{\beta}u^\alpha(x, N) + a^-_{\gamma}u^\alpha(x, \bar{N}), \quad \bar{u}^\beta_\gamma(\bar{x}, \bar{N}) = \bar{a}^+_{\gamma}\bar{u}^\beta_\gamma(\bar{x}, \bar{N}),
\]

\( \alpha, \beta, \gamma = 1, 2 \)  

(1.2)

Here \( N \) and \( \bar{N} \) (defined modulo \( 2h \) for \( h = k + 2 \)) are the (quantum) dimension operators:

\[
a^\pm_{\beta} = q^{\pm N_{\beta}}a_{\beta}^\pm, \quad \bar{a}^\pm_\gamma = q^{-\frac{1}{2}}\bar{a}^\pm_\gamma \]  

(1.3)

The Fock space vacuum \( |\text{vac}\rangle \) is assumed to satisfy

\[
a^-|\text{vac}\rangle = 0 = \bar{a}_-|\text{vac}\rangle, \quad q^{-N_{\beta}}|\text{vac}\rangle = |\text{vac}\rangle = q^{\bar{N}^{-1}}|\text{vac}\rangle. \]  

(1.4)

Introducing the left and right monodromies \( M \) and \( \bar{M} \) by

\[
\quad u(x + 2\pi) = u(x)M, \quad \bar{u}(\bar{x} + 2\pi) = \bar{M}^{-1}\bar{u}(\bar{x})
\]

(1.5)

(and viewing \( M \) and \( \bar{M} \) as new dynamical variables) we demand that the expansion (1.2) diagonalizes the monodromy matrices:

\[
a^\pm M = a^\pm q^{\mp(N + \frac{1}{2})}, \quad \bar{M}\bar{a}^\pm = q^{\mp(N + \frac{1}{2})}\bar{a}^\pm.
\]

(1.6)

Furthermore, \( M \) and \( \bar{M} \) commute with the chiral vertex operators \( u_\pm \) and \( \bar{u}_\pm \). The quantum exchange relations for \( u \) and \( \bar{u} \) (derived within the canonical approach in \([4],[7]\)) imply the following quadratic relations among the “\( q \)-oscillators”:

\[
a^\varepsilon a^\varepsilon\prod_A = a^\varepsilon_{\rho}a_{\varepsilon}(\prod_A a^\rho_{\alpha}) = 0 \quad (\varepsilon = \pm), \quad a^\mp a^\pm \prod_A = \pm \frac{\langle N \rangle}{\frac{[2]}{2}}\mathcal{E},
\]

\[
(a^-a^+ - a^+a^-) \prod_A = a^-a^+ - a^+a^- = [N]\mathcal{E}, \quad [N] = \frac{q^N-q^{-N}}{q-q^{-1}}.
\]

(1.7)
and identical relations for the bar sector \((a^\pm \rightarrow \bar{a}_\mp, N \rightarrow \bar{N})\).

Here \(\mathcal{E}(= (\mathcal{E}_{\alpha\beta}))\) is the \(U_q(sl_2)\) invariant \((q\)-skewsymmetric\) tensor and \(\Pi_A\) is the projector on the \(q\)-skewsymmetric part of the tensor product of two \(U_q(sl_2)\) spinors:

\[
(\mathcal{E}_{\alpha\beta}) = \begin{pmatrix}
0 & -q^{1/2} \\
-q^{-1/2} & 0
\end{pmatrix} = (\mathcal{E}^{\alpha\beta}),
\]

\[
(\Pi_A)_{\alpha'\beta'}^{\alpha\beta} = \frac{1}{(2)}\mathcal{E}^{\alpha\beta} \mathcal{E}_{\alpha'\beta'} \quad (\Pi_A^2 = \Pi_A, \mathcal{E} \Pi_A = \mathcal{E}).
\]

On the other hand the left and right sectors completely decouple, so that

\[
[a_\epsilon^\alpha, \bar{a}_{\epsilon'}^{\beta'}] = 0 \quad (= [M_\alpha^\beta, \bar{M}_{\beta'}^{\alpha'}]).
\]

The quantum generators are read off the components \(M_{\pm}\) of the Gauss decomposition of the monodromy matrix:

\[
q^{3/2} M = M_+ M_-^{-1}, \quad M_+ = \begin{pmatrix}
qu^{-\frac{H}{2}} & (q^{-1} - q)F q^{\frac{H}{2}} \\
0 & q^{H/2}
\end{pmatrix},
\]

\[
M_-^{-1} = \begin{pmatrix}
qu^{-\frac{H}{2}} & 0 \\
(q^{-1} - q)E q^{\frac{H}{2}} & q^{H/2}
\end{pmatrix}
\]

The \(U_q\) covariance of \(a^\pm\) is expressed by the canonical exchange relations

\[
M_+ a_\epsilon^\alpha = a_\epsilon^\alpha R^+_\pm M_\pm, \quad M_-^{\pm1} a_\epsilon^\alpha = a_\epsilon^{\pm1} M^{\pm1}
\]

or

\[
[E, a_1^\pm] = 0 = Fa_2^\pm - qa_2^\pm F, \quad [E, a_2^\pm] = a_1^\pm q^H,
\]

\[
F a_1^\pm - q^{-1} a_1^\pm F = a_2^\pm, \quad q^H a_1^\pm = a_1^{\pm1} q^{H+1}, \quad q^H a_2^\pm = a_2^{\pm1} q^{H-1}.
\]

The \(U_q(sl_2)\) raising and lowering operators \(E\) and \(F\) can be expressed as bilinear combinations of \(a^+\) and \(a^-\) with coefficients in the Cartan subalgebra (and similarly for \(\bar{E}\) and \(\bar{F}\)):

\[
E = -q^{-1/2} a_1^+ a_1^-, \quad F = a_2^+ a_2^- q^{3/2-H}, \quad \bar{E} = -\bar{a}_2^+ \bar{a}_2^- q^{\frac{1}{2} + H}, \quad \bar{F} = q^{\frac{1}{2}} \bar{a}_2^+ \bar{a}_2^-.
\]
(Note that the role of the indices 1 and 2 for the bar sector are reversed; we have, in particular, $q^H \bar{a}_\varepsilon^1 = \bar{a}_\varepsilon^1 q^{H-1}$, etc. instead of (1.12).)

We also note the relations

$$\left( q^{3/2} - q^{-1/2} \right) a_\pm^1 a_\mp^\dagger = q^H - q^{1+\mathbb{N}}, \quad \left( q^{1/2} - q^{-3/2} \right) a_\pm^2 a_\mp^\dagger = q^H - q^{\pm\mathbb{N}-1}$$

(1.14)

$$\left( q^{3/2} - q^{-1/2} \right) \bar{a}_\pm^1 \bar{a}_\mp^2 = q^{-H} - q^{1+\mathbb{N}}, \quad \left( q^{1/2} - q^{-3/2} \right) \bar{a}_\pm^2 \bar{a}_\mp^1 = q^{-H} - q^{\pm\mathbb{N}-1}.$$  

(1.15)

The diagonal action of the left and right $U_q(sl_2)$ is generated by the coproduct

$$L_\pm = \bar{M}_\pm^{-1} M_\pm \text{ satisfying } [L_\pm, a_\varepsilon^\beta \bar{a}_\varepsilon^\beta] = 0.$$  

(1.16)

In other words, we have (assuming $[X, \bar{Y}] = 0$ for $X, Y = E, F, H$ and using the same notation for the $U_q$ generators in the arguments of $\Delta$ as for the operators in the first term of the tensor product in the right-hand side)

$$\Delta(E) = q^H \bar{E} + E, \quad \Delta(F) = F q^{-\bar{H}} + \bar{F}, \quad \Delta(q^H) = q^{H+\bar{H}}.$$  

(1.17)

The algebra $\mathcal{B} \otimes \bar{\mathcal{B}}$ generated by $a^\varepsilon, \bar{a}_\varepsilon$ (and the Cartan’s) has, for $q$ satisfying (1.3), a huge ideal which will be represented by the zero operator in the vacuum Fock space $\mathcal{F} \otimes \bar{\mathcal{F}}$ (see [11]) :

$$(a^\varepsilon^\beta)^h \mathcal{F} \otimes \mathcal{F} = 0 = (\bar{a}_\varepsilon^\beta)^h \mathcal{F} \otimes \bar{\mathcal{F}} = (q^{2h^\varepsilon^{H-1}} - 1)\mathcal{F} \otimes \mathcal{F}.$$  

(1.18)

Each of the factors $\mathcal{F}$ and $\bar{\mathcal{F}}$ is then an $h^2$ dimensional space with basis ($|n_1, n_2\rangle$) and ($|\bar{n}_1, \bar{n}_2\rangle$) where

$$|n_1 n_2\rangle = (a_+^{n_1} a_+^{n_2})|\text{vac}\rangle \in \mathcal{F}, \quad |\bar{n}_1 \bar{n}_2\rangle = (\bar{a}_+^{\bar{n}_1} \bar{a}_+^{\bar{n}_2})|\text{vac}\rangle \in \bar{\mathcal{F}}$$

(1.19)

0 ≤ $n_1, n_2, \bar{n}_1, \bar{n}_2$ ≤ $h - 1$.

**Remark**

The nilpotency relation (1.18) allows to define a finite dimensional counterpart of the Bernard-Felder cohomology [2]. To this end we define a “BRS charge” $Q_n,$
with domain the (homogeneous) subspace \( F_{h+n} \) of \( F \) spanned by vectors of the form (1.19) with \( n_1 + n_2 + 1 = h + n \), setting
\[
Q_n = (a_1^- a_2^-)^n : F_{h+n} \to F_{h-n}
\]
while extending its action to \( F_{h-n} \) as
\[
(a_1^- a_2^-)^{h-n} : F_{h-n} \to 0.
\]

Each of the associative algebras \( B \) and \( \tilde{B} \) (and hence, their tensor product) admits a linear anti-involution (“transposition”) \( X \to \tilde{t}X \) and an associated \( U_q(\mathfrak{sl}_2) \) invariant bilinear form \( \langle \bullet | \bullet \rangle \) such that
\[
\tilde{t}a_+^\alpha = -\mathcal{E}_+^\alpha a_-^\beta, \quad \tilde{t}a_-^\alpha = \mathcal{E}_-^\alpha a_+^\beta, \quad \tilde{t}\tilde{a}_+^\alpha = \tilde{a}_-^\alpha \mathcal{E}_-^\alpha, \quad \tilde{t}\tilde{a}_-^\alpha = -\tilde{a}_+^\alpha \mathcal{E}_+^\alpha
\]
and
\[
\langle m_1 m_2 | n_1 n_2 \rangle = q^{-n_1 n_2} [n_1]! [n_2]! \delta_{m_1} \delta_{m_2} \delta_{n_1} \delta_{n_2}.
\]

The following statement is a consequence of Proposition 4.1 of [9].

**PROPOSITION 1** The operators \( \Delta(E), \Delta(F) \) (1.17) and
\[
B = a_+^\alpha \tilde{a}_-^\alpha, \quad \tilde{t}B = -a_-^\alpha \tilde{a}_+^\alpha
\]
give rise to two commuting copies of \( U_q(\mathfrak{sl}_2) \) : \([t^{t}B, \Delta(X)] = 0 \) for \( X = E, F \) and
\[
[\Delta(E), \Delta(F)] = [H + H] = \Delta([H]), \quad q^{H+H} \left( \begin{array}{c} \Delta(E) \\ \Delta(F) \end{array} \right) = \left( \begin{array}{c} \Delta(E) \\ \Delta(F) \end{array} \right) q^{H+H \pm 2},
\]
\[
[B, \tilde{t}B] = [N - \tilde{N}], \quad q^{N-\tilde{N}} \left( \begin{array}{c} B \\ \tilde{t}B \end{array} \right) = \left( \begin{array}{c} B \\ \tilde{t}B \end{array} \right) q^{N-\tilde{N} \pm 2}.
\]

The subspace \( \mathcal{H}_I \) of \( F \otimes \tilde{F} \) which consist of \( U_q(\mathfrak{sl}_2) \Delta \otimes U_q(\mathfrak{sl}_2)B, \tilde{t}B \) invariant vectors is spanned by
\[
\{ A^{+[n]} \vert \text{vac} \rangle, n = 0, \ldots, 2h - 2 \}, \quad A^\pm = a_+^\alpha \tilde{a}_-^\beta = A^+_1 + A^+_2,
\]
\[
A^{[n]} = \frac{1}{[n]!} A^n = \sum_{k=m}^{n-m} q^{k(n-k)} A_1^{[k]} A_2^{[n-k]}, m = \max(0, n - h + 1).
\]
A central result of [8] is the monodromy invariance of \( g(x, t) \) when restricted to the physical subspace. It is reflected (within the finite-dimensional zero-mode problem under consideration) in the property

\[
a^\varepsilon M \bar{M}^{-1} \bar{a}_\varepsilon \mathcal{H}_I = a^\varepsilon \bar{a}_\varepsilon \mathcal{H}_I
\]

which follows from (1.5) and from

\[
\left( q^{\pm(N-\bar{N})} - 1 \right) \mathcal{H}_I = 0.
\]

### 2 Generalized cohomologies in \( \mathcal{H}_I \)

The quadratic relations (1.7) for \( a^\pm_{\alpha} \) (and their counterparts for \( \bar{a}^\pm_{\beta} \)) together with (1.3) and (1.9) imply

\[
[A, A^+] = [N + \bar{N}] =: [2\hat{N}], \quad q^{\pm\hat{N}} A^\pm = A^\pm q^{\pm(N\pm1)} \quad \text{for} \quad A \equiv A^-.
\]

(Note that condition (1.26) implies that the operator \( \hat{N} \) introduced in (2.1) has an integer spectrum on \( \mathcal{H}_I \).)

**PROPOSITION 2** Let \( A^\pm_{\alpha} = a^\pm_{\alpha} \bar{a}^\alpha_{\pm} \), \( \alpha = 1, 2 \) (no summation!); then (1.7), (1.9) and (1.18) imply

\[
A^+_2 A^+_1 = q^2 A^+_1 A^+_2, \quad (A^\pm_{\alpha})^h = 0 \implies (A^\pm)^h = 0
\]

\[
A|n\rangle = [n]|n-1\rangle, \quad n = 1, \ldots, 2h - 1 \quad ((\hat{N} - n)|n\rangle = 0, |0\rangle = 0)
\]

where \( |n\rangle \) is defined by \( |n\rangle := (A^+)^{[n-1]}|\text{vac}\rangle \).

**Proof.** The first equation (2.2) follows from (1.9) and the \( q \)-Bose commutation relations

\[
a^\pm_2 a^\pm_1 = qa^\pm_1 a^\pm_2, \quad \bar{a}^\alpha_+ \bar{a}^\alpha_- = q\bar{a}^1_+ \bar{a}^2_-
\]
derived from (1.7). The second one is a direct consequence of (1.18). Finally, for $A^\pm = A_1^\pm + A_2^\pm$ we find

$$A^n = \sum_{k=0}^{n} \binom{n}{k} A_1^k A_2^{n-k},$$

where $\binom{n}{k} = \frac{(n)_+}{(k)_+(n-k)_+}$,

$$\text{with } (n)_+ = \frac{q^{n-1}}{q^2-1},$$

(2.5)

which implies for $n = h$ the last equation (2.2) since

$$\langle h \rangle_+ = q^{h-1}[h] = 0 \text{ for } q^h = -1. \quad (2.6)$$

For $n < h$, (2.3) follows directly from (1.24) and (2.1). For $n \geq h$, it follows from the relations

$$A(A_1^+)^{[n]}|1\rangle = q^{-1}[n](A_1^+)^{[n-1]}|1\rangle, \quad A(A_2^+)^{[n]}|1\rangle = q[n](A_2^+)^{[n-1]}|1\rangle$$

(|1\rangle \equiv |vac \rangle) which allows to set in computing (2.3)

$$(A_\alpha^+)^{[h]}|0\rangle = 0, \quad \alpha = 1, 2 \text{ (in } \mathcal{H}_I). \quad (2.7)$$

The relation

$$^t(A^+) = A \quad (^tA = A^+) \quad (2.8)$$

for the conjugation (1.20) yields, as a corollary of Proposition 2.1, the following Gram matrix (of inner products of the basis vectors)

$$\langle m|n \rangle = \langle 1|A^{[n-1]}(A^+)^{[n-1]}|1\rangle = [n] \delta_{mn}. \quad (2.9)$$

Noting that $[n] = -[h + n] (= [h - n])$ we deduce that the trace of this Gram matrix is zero.

The space $\mathcal{H}_I$ can be viewed as the complexification of the real span $\mathcal{H}_R^I$ of the vectors (1.24). The fact that it admits (unlike $\mathcal{F} \otimes \bar{\mathcal{F}}$) such a real basis (for which the Gram matrix of inner products is real) allows to define a second, antilinear anti-involution on the operator algebra $\mathfrak{A}$ in $\mathcal{H}_I$, defined by

$$(A^\pm)^{[n]} = (A^\mp)^{[n]} \quad (2.10)$$
Clearly, it coincides with the transposition $^t$ for $A^\pm$, but differs when applied to $q^\hat{N}$:

$$(q^\hat{N})^+ = q^{-\hat{N}} \quad (q^+ = q^{-1}).$$

(2.11)

(Note that the relation $q^\hat{N}A = Aq^{-\hat{N}}$ goes into $q^{\hat{N}-1}A^+ = A^+q^{-\hat{N}}$ under transposition and into $q^{1-\hat{N}}A^+ = A^+q^{-\hat{N}}$ under the hermitian conjugation defined by (2.10) and (2.11); the full set of relations (21) remains invariant in both cases.)

This allows to define an (indefinite) hermitian inner product $\langle \bullet | \bullet \rangle_I$ in $H_I$ which coincides with the real bilinear form $\langle \bullet | \bullet \rangle$ (inherited from $\mathcal{F} \otimes \bar{\mathcal{F}}$) on $H^R_I$. In what follows we shall only use this hermitian form and shall therefore drop the subscript $I$.

Define the subspace $H'_{\mathcal{H}}$ of $H_I$, on which the hermitian form $\langle \bullet | \bullet \rangle$ is positive semidefinite, by the set of $h - 1$ constraints

$$A^{h-1}(A^+)^n H' = 0 \quad \text{for} \quad n = 0, 1, \ldots, h - 2$$

(2.12)

**PROPOSITION 3** The complement $\text{Coim} (A^{h-1}(A^+)^n)$ of the kernel of the operator $A^{h-1}(A^+)^n$ in $H_I$ is 1-dimensional and given by

$$\text{Coim} (A^{h-1}(A^+)^n) = \{ \mathbb{C}|2h - 1 - n \}$$

(2.13)

The proof is a straightforward consequence of (2.1) and (2.3).

We note that $A^\pm$ weakly commute with the constraints (2.12). For $A^+$ this follows from the easily verifiable relations

$$[A^{h-1}, A^+] = [2\hat{N} + h - 2] A^{h-2}$$

and

$$[2\hat{N} + h - 2] A^{h-2} H' = 0.$$
We also note that the operator $A^{h-1}(A^+)^{h-1}$ vanishes identically in $H_I$ and so does $A^+A^{h-1}$; both assertions follow from the identity $A^+|h\rangle = 0$.

Thus the space $H'$ is $h$-dimensional (spanned by $|n\rangle$ for $1 \leq n \leq h$). Its subspace $H''$ of 0-norm vectors is 1-dimensional: it consists of multiples of the vector $|h\rangle$.

The main result of this section (and of the paper) is the following realization of the physical subquotient

$$H = H'/H''$$

in terms of generalized cohomologies of the type studied in [4] and [3].

**Proposition 4** Each of the generalized cohomologies of the nilpotent operator $A$ is 1-dimensional and can be realized as

$$H^{(n)} = \text{Ker} \ A^n/\text{Im} \ A^{h-n} \approx \{\mathbb{C}|h - n\rangle\}, \ n = 1, \ldots, h - 1.$$  \hfill (2.15)

With this choice the representative subspace is orthogonal to $\text{Im} \ A^{h-n}$. The physical subquotient (2.14) is isomorphic to the direct sum of $H^{(n)}$:

$$H \simeq \bigoplus_{n=1}^{h-1} H^{(n)}.$$  \hfill (2.16)

The null space $H''$ in (2.14) is isomorphic to the intersection of images of $A^n$:

$$H'' = \bigcap_{n=1}^{h-1} \text{Im} \ A^n = \text{Im} \ A^{h-1}(= \text{Im} \ (A^+)^{h-1}).$$  \hfill (2.17)

**Proof.** Ker$A^n$ is $2n$-dimensional and is spanned by the vectors $\{|1\rangle, \ldots, |n\rangle; |h\rangle, \ldots, |h+n-1\rangle\}$. Im $A^{h-n}$ is $(2n-1)$-dimensional and is spanned by the subset of the above, orthogonal to $|h - n\rangle$. This proves (2.15). The rest follows from the explicit knowledge of $H^{(n)}$ and of Im $A^{h-n}$.
3 Discussion : Open problems

The present cohomological treatment of a diagonal $SU(2)$ WZNW model leaves open a number of related questions:

1. The subspace $\mathcal{H}'$ of $\mathcal{F} \otimes \bar{\mathcal{F}}$ (and hence the corresponding subquotient $\mathcal{H}$) was constructed in two steps. We first singled out the subspace $\mathcal{H}_I$ of $\mathcal{F} \otimes \bar{\mathcal{F}}$ of $U_q(sl_2) \otimes U_q(sl_2)$ invariant vectors which admits a hermitian (indefinite) inner product and then introduced the constraints that determine $\mathcal{H}'$. The question whether one can introduce a (generalized) BRS charge in $\mathcal{F} \otimes \bar{\mathcal{F}}$ that takes into account all constraints at the same time is left open.

2. Can non-diagonal $\hat{su}(2)$ models be treated in a similar fashion ?

3. Find the BRS cohomology of diagonal $\hat{su}(n)$ models for $n > 2$; their gauge theory treatment was initiated in [9].

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