DENSITY OF DIAGONALIZABLE MATRICES IN SETS OF STRUCTURED MATRICES DEFINED FROM INDEFINITE SCALAR PRODUCTS

Abstract
For an (indefinite) scalar product $[x, y]_B = x^H By$ for $B = \pm B^H \in \text{GL}_n(\mathbb{C})$ on $\mathbb{C}^n \times \mathbb{C}^n$ we show that the set of diagonalizable matrices is dense in the set of all $B$-selfadjoint, $B$-skewadjoint, $B$-unitary and $B$-normal matrices.

1. Introduction
Whenever $B \in \text{GL}_n(\mathbb{C})$ is some arbitrary (nonsingular) matrix, the function $[\cdot, \cdot]_B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, $(x, y) \mapsto x^H By$ ($x^H := x^T$) defines a nondegenerate sesquilinear form on $\mathbb{C}^n \times \mathbb{C}^n$. In case $B$ is not necessarily Hermitian positive definite, such forms are also referred to as indefinite scalar products [12, Sec. 2]. In this work, we particularly consider such products for the cases $B = B^H$, $B = -B^H$ (with $B^H$ denoting the conjugate transpose of $B$) and $B^H B = I_n$ (i.e. $B$ is unitary). Indefinite scalar products with $B = \pm B^H$ are called orthosymmetric while those with $B^H = B^{-1}$ are called unitary [12, Def. 3.1, 4.1].

Several classes of matrices $A \in \text{M}_n(\mathbb{C})$ are naturally related to $[\cdot, \cdot]_B$:

(a) A matrix $J \in \text{M}_n(\mathbb{C})$ is called $B$-selfadjoint if $[Jx, y]_B = [x, Jy]_B$ holds for all $x, y \in \mathbb{C}^n$. It follows that $x^H J^H By = x^H BJy$ holds for all $x, y \in \mathbb{C}^n$ if $J$ is $B$-selfadjoint. This means we have $J^H B = BJ$, that is, $J = B^{-1}J^H B$. The set of $B$-selfadjoint matrices is denoted by $\mathcal{J}(B)$. 


(b) A matrix $L \in \mathbf{M}_n(\mathbb{C})$ is called $B$-skewadjoint if $[Lx, y]_B = [x, -Ly]_B$ holds for all $x, y \in \mathbb{C}^n$. It follows from this equation as in (a) that $L$ is $B$-skewadjoint if and only if $-L = B^{-1}L^HB$. The set of all $B$-skewadjoint matrices is denoted by $\mathbb{L}(B)$.

(c) A matrix $G \in \mathbf{M}_n(\mathbb{C})$ is called $B$-unitary if $[Gx, Gy]_B = [x, y]_B$ holds for all $x, y \in \mathbb{C}^n$. This means that $x^HB^HBy = x^HBy$ has to hold for all $x, y \in \mathbb{C}^n$ and implies that $G^HBG = B$. The set of all $B$-unitary matrices is denoted by $\mathbb{G}(B)$.

(d) A matrix $N \in \mathbf{M}_n(\mathbb{C})$ is called $B$-normal if $NB^{-1}N^H = B^{-1}N^HBN$ holds. The set of all $B$-normal matrices is denoted by $\mathcal{N}(B)$.

A great many of problems in control systems theory, matrix equations or differential equations involve matrices from the classes of matrices defined above (see, e.g., [10, 12] and the references therein). The monograph [4] contains a comprehensive treatment and applications for the Hermitian case $B = B^H$.

Assume $B = I_n$ is the $n \times n$ identity matrix. Then the sets of $B$-selfadjoint, $B$-skewadjoint, $B$-unitary and $B$-normal matrices $A \in \mathbf{M}_n(\mathbb{C})$ coincide with the sets of Hermitian ($A = A^H$), skew-Hermitian ($A = -A^H$), unitary ($A^HA = I_n$) and normal ($AA^H = A^HA$) matrices. It is a well-known fact that every matrix belonging to any of these four sets of matrices is semisimple, i.e., diagonalizable [6]. In case $B \neq I_n$ the situation is different: for instance, consider the $n \times n$ reverse identity matrix $R_n$ (which is Hermitian but not positive definite) and some basic $n \times n$ Jordan block $J(\lambda)$ for some $\lambda$, i.e.

$$J(\lambda) = \begin{bmatrix}
\lambda & 1 & & \\
& \lambda & 1 & \\
& & \ddots & \ddots \\
& & & \lambda & 1 \\
& & & & \lambda \\
\end{bmatrix}, \quad R_n = \begin{bmatrix}1 & \cdots & \\
& \ddots & \ddd \\
& & 1 \\
\end{bmatrix}. \quad (1)$$

If $\lambda \in \mathbb{R}$, then $J(\lambda) \in \mathcal{J}(R_n)$. However, $J(\lambda)$ is the prototype of a matrix which is not diagonalizable (see also [4] Ex. 4.2.1). Similar examples can be found for the other types of matrices and other $B$. In general, for a (skew)-Hermitian matrix $B$, a matrix $A \in \mathbf{M}_n(\mathbb{C})$ belonging to $\mathcal{J}(B), \mathbb{L}(B), \mathbb{G}(B)$ or $\mathcal{N}(B)$ need not be semisimple. Thus, given some (skew)-Hermitian $B \in \mathbf{Gl}_n(\mathbb{C})$, the following question arises: “How large is the subset of semisimple matrices in the sets $\mathcal{J}(B), \mathbb{L}(B), \mathbb{G}(B)$ and $\mathcal{N}(B)$?” The answer to this question is simply “every matrix is semisimple” in case $B = I_n$ (and whenever $B$ is Hermitian positive definite), but seems to be unknown for general indefinite or skew-Hermitian $B$.

\footnote{Notice that the set $\mathcal{N}(B)$ includes the sets $\mathcal{J}(B), \mathbb{L}(B)$ and $\mathbb{G}(B)$.}
In this work we consider the question raised above from a topological point of view. Recall that $M_n(\mathbb{C})$ can be considered as a topological space with basis $B_R(A) = \{ A' \in M_n(\mathbb{C}) : \|A - A'\| < R \}$ for $A \in M_n(\mathbb{C})$ and $R \in \mathbb{R}$, $R > 0$ (see, e.g. [7, Sec. 11.2]). The sets $J(B), L(B), G(B)$ and $N(B)$ can thus be interpreted as topological spaces on their own equipped with the subspace topology [1, Sec. 1.5]. For instance, a subset $S \subset N(B)$ is open (in this subspace topology) if $S$ is the intersection of $N(B)$ with some open subset of $M_n(\mathbb{C})$. If a property does not hold for all elements in a topological space, it is usually reasonable to ask whether it holds for a dense subset. Thus, a second question, in this context naturally related to the first one, is “As not all matrices in $J(B), L(B), G(B)$ or $N(B)$ are semisimple, is the subset of semisimple matrices in these sets at least dense?”

Since the sets $J(B), L(B), G(B)$ and $N(B)$ are defined by matrix equations, they are (topologically) closed subsets of $M_n(\mathbb{C})$ (for instance, $N(B)$ is $B$-normal if and only if it satisfies the equation $NB^{-1}N^HB = B^{-1}N^HB$). Consequently, although the set of semisimple matrices is dense in $M_n(\mathbb{C})$ [11, Cor. 7.3.3], there is no direct reasoning why this fact should carry over to $J(B), L(B), G(B)$ or $N(B)$. In fact, a closed subset of $M_n(\mathbb{C})$ need not contain any diagonalizable matrix at all (see [7] for an example). This work is devoted to the second question raised above and gives a positive answer. Our main result can be stated as follows:

**Main result.** Let $B = \pm B^H \in GL_n(\mathbb{C})$. Then the set of diagonalizable matrices is dense in $J(B), L(B), G(B)$ and $N(B)$.

Here we will work with the 2-norm of matrices\footnote{Our results hold in the same way for any submultiplicative and unitarily invariant matrix norm.}, i.e.

$$\|A\|_2 := \max_{\|x\|_2 = 1} \|Ax\|_2.$$ 

The density result stated above is thus equivalent to the fact that, for any $A \in J(B), L(B), G(B)$ or $N(B)$ and any $\varepsilon > 0$ there exists some diagonalizable $A'$ from the same class of matrices such that $\|A - A'\|_2 < \varepsilon$. Our motivation for the analysis of density stems from the fact that, for certain $B$ such as $B = R_n$ (with $R_n$ as in (1)) or

$$B = J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in M_{2n}(\mathbb{R}),$$

semisimple matrices in $J(B), L(B), G(B)$ or $N(B)$ can all be transformed to a sparse and nicely structured (canonical) form by a $B$-unitary similarity transformation as shown in [2]. From this point of view our results imply that, even if the transformation established in [2] does not exists for some specific matrix $A \in J(B), L(B), G(B)$ or $N(B)$, it will exist for matrices arbitrarily close to $A$ from the same structure class.
In Section 2 some preliminaries and basic results are given. Section 3 gives the proofs for the main results on the density of semisimple matrices in \( J(B), L(B), G(B) \) and \( N(B) \). We focus on the case \( B = B^H \) in Sections 3.1 to 3.3 and discuss the case \( B = -B^H \) in Section 3.4. Some conclusions are given in Section 4.

2. Preliminaries and Basic Results

The set of all \( j \times k \) matrices over \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) is denoted by \( M_{j \times k}(\mathbb{F}) \). Whenever \( j = k \), we use the short-hand-notation \( M_k(\mathbb{F}) = M_{k \times k}(\mathbb{F}) \). The notation \( \text{GL}_k(\mathbb{F}) \) refers to the general linear group over \( \mathbb{F}^k \) (i.e. the set of \( k \times k \) nonsingular matrices over \( \mathbb{F} \)). The set of all eigenvalues of a matrix \( A \in M_k(\mathbb{C}) \) is called the spectrum of \( A \) and is denoted \( \sigma(A) \). The multiplicity of an eigenvalue \( \lambda \in \mathbb{C} \) as a root of \( \det(A - xI_k) \) is called its algebraic multiplicity and is denoted by \( m(A, \lambda) \). For each \( \lambda \in \sigma(A) \) any vector \( v \in \mathbb{C}^k \) that satisfies \( Av = \lambda v \) is called an eigenvector for \( \lambda \). The set of all those vectors corresponding to \( \lambda \in \sigma(A) \) is called the eigenspace for \( \lambda \). It is a subspace of \( \mathbb{C}^k \) and its dimension is called the geometric multiplicity of \( \lambda \).

The conjugate transpose of a vector/matrix is denoted with the superscript \( ^H \) while \( ^T \) is used for the transposition without complex conjugation. A matrix \( A \in M_n(\mathbb{C}) \) is called semisimple (or diagonalizable), if there exists some \( S \in \text{GL}_n(\mathbb{C}) \) such that \( S^{-1}AS \) is a diagonal matrix.

Let \( B \in \text{GL}_n(\mathbb{C}) \). For any \( A \in M_n(\mathbb{C}) \) let \( A^* := B^{-1}A^H B \). The matrix \( A^* \) is usually referred to as the adjoint for \( A \), cf. [12, Sec.2]. The sets of all \( B \)-selfadjoint, \( B \)-skewadjoint, \( B \)-unitary and \( B \)-normal matrices as introduced in Section 1 can now be characterized by the equations \( A = A^* \), \( A = -A^* \), \( A^* = A^{-1} \) and \( AA^* = A^*A \), respectively. Notice that the mapping \( ^* : A \mapsto A^* \) is \( \mathbb{R} \)-linear, that is, for any \( A, C \in M_n(\mathbb{C}) \) and any \( \alpha, \gamma \in \mathbb{R} \) it holds that

\[
(\alpha A + \gamma C)^* = \alpha A^* + \gamma C^*.
\]

Moreover, \( (AC)^* = C^*A^* \) holds, so \( ^* \) is antihomomorphic.

**Lemma 1.** Let \( B \in \text{GL}_n(\mathbb{C}) \). Then the sets \( J(B) \) and \( L(B) \) are \( \mathbb{R} \)-subspaces of \( M_n(\mathbb{C}) \). Moreover, \( G(B) \) is a subgroup of \( \text{GL}_n(\mathbb{C}) \).

**Proof.** The first statement follows immediately since \( ^* : A \mapsto A^* \) is \( \mathbb{R} \)-linear.

Now let \( G \in G(B) \). Then, as \( G^H BG = B \) holds, \( \det(G^H) \det(G) = 1 \) follows, so \( G \) is nonsingular. Moreover, \( G^{-H} BG^{-1} = B \) follows, so \( G^{-1} \in G(B) \). Finally, for \( G, F \in G(B) \), we obtain

\[
(FG)^H B (FG) = G^H (F^H BF) G = G^H BG = B.
\]

Therefore, \( FG \in G(B) \). Since \( I_n \in G(B) \), the proof is complete. \( \square \)
The sets $J(B)$ and $\mathbb{L}(B)$ are often referred to as the Jordan and Lie algebras (cf. [12, Sec. 2]) for $[\cdot, \cdot]_B$ since $J(B)$ is closed under the operation $A \odot C := \frac{1}{2}(AC + CA)$ whereas $\mathbb{L}(B)$ is closed under the Lie bracket $[A, C] := AC - CA$. The following Corollary 1 shows that $J(B)$ and $\mathbb{L}(B)$ can never be $\mathbb{C}$-subspaces of $\mathbb{M}_n(\mathbb{C})$. In fact, multiplication by $i$ gives a possibility to easily switch between the sets $J(B)$ and $\mathbb{L}(B)$.

**Corollary 1.** Let $B \in \text{Gl}_n(\mathbb{C})$. If $J \in J(B)$, then $iJ \in \mathbb{L}(B)$. On the other hand, if $L \in \mathbb{L}(B)$, then $iL \in J(B)$.

**Proof.** Let $J \in J(B)$. Notice that

$$B^{-1}(iJ)^H B = B^{-1}(-iJ^H)B = -i(B^{-1}J^H B) = -iJ$$

for any $J \in J(B)$. Thus $iJ \in \mathbb{L}(B)$. On the other hand, if $L \in \mathbb{L}(B)$, then $B^{-1}(iL)^H B = -iB^{-1}L^H B = iL$, so $iL \in J(B)$. \(\Box\)

The following Lemma 2 can easily be verified by a straightforward calculation (see also [13, Sec. 1]). Therefore, the proof is omitted.

**Lemma 2.** Let $B \in \text{Gl}_n(\mathbb{C})$ and $A \in \mathbb{M}_n(\mathbb{C})$. Furthermore, let $T \in \text{Gl}_n(\mathbb{C})$ and consider

$$A' := T^{-1}AT \quad \text{and} \quad B' := T^HBT.$$ 

Then $A$ is $B'$-selfadjoint/$B'$-skewadjoint/$B'$-unitary/$B'$-normal if and only if $A'$ is $B'$-selfadjoint/$B'$-skewadjoint/$B'$-unitary/$B'$-normal.

Let $B \in \text{Gl}_n(\mathbb{C})$. Then any $A \in \mathbb{M}_n(\mathbb{C})$ can always be expressed as $A = S + K$ with

$$S = \frac{1}{2}(A + A^*) \in \mathbb{M}_n(\mathbb{C}), \quad K := \frac{1}{2}(A - A^*) \in \mathbb{M}_n(\mathbb{C}). \quad (2)$$

Assuming that $B = \pm B^H$, it is easy to see that $S \in J(B)$ and $K \in \mathbb{L}(B)$. In this case, $A = S + K$ can be interpreted as a $B$-analogue of the Toeplitz decomposition stated in [8, Thm. 4.1.2]. The next Lemma 3 shows that, in case $B^H B = I_n$ additionally holds, $S$ and $K$ in (2) are the best approximations to $A$ from $J(B)$ and $\mathbb{L}(B)$ with respect to any unitarily invariant matrix norm. The proof of Lemma 3 is similar to [3, Thm. 2] (see also [15, Lem. 8.3]).

**Lemma 3.** Let $B = \pm B^H \in \text{Gl}_n(\mathbb{C})$ with $B^H B = I_n$ and let $\| \cdot \|$ be any unitarily invariant matrix norm.

(a) Let $A \in \mathbb{M}_n(\mathbb{C})$ and $S := \frac{1}{2}(A + A^*)$. Then $S \in J(B)$ and it holds that $\|A - S\| \leq \|A - C\|$ for any other matrix $C \in J(B)$.

(b) Let $A \in \mathbb{M}_n(\mathbb{C})$ and $K := \frac{1}{2}(A - A^*)$. Then $K \in \mathbb{L}(B)$ and it holds that $\|A - K\| \leq \|A - C\|$ for any other matrix $C \in \mathbb{L}(B)$. 

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Proof. (a) Let $\| \cdot \|$ be some unitarily invariant matrix norm and let $A \in M_n(\mathbb{C})$ be given. It follows from $B = \pm B^H$ that $(A^*)^* = A$. Therefore, we have $S^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + A) = S$, so $S \in \mathcal{J}(B)$.

Now let $C \in \mathcal{J}(B)$ be arbitrary. Then
\[
\| A - S \| = (1/2)\| A - A^* \| = \| (A - C) - (A^* - C) \|
\]
\[
= (1/2)\| (A - C) - (A - C)^* \| \leq (1/2)(\| A - C \| + \| (A - C)^* \|)
\]
\[
= (1/2)(\| A - C \| + \| (A - C)^H \|) = \| A - C \|.
\]

In conclusion we have $\| A - S \| \leq \| A - C \|$. The proof for (b) proceeds along the same lines. \qed

Let
\[
J_k(\lambda) := \begin{bmatrix}
\lambda & 1 \\
\lambda & 1 \\
\vdots & \ddots \\
\lambda & 1 \\
\end{bmatrix} \in M_k(\mathbb{C})
\]
denote a basic Jordan block for $\lambda$ if $\lambda \in \mathbb{R}$ and define $J_k(\lambda) = J_k(\lambda) \oplus J_k(\bar{\lambda})$ with $2k = k$ in case $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here and in the following, the notation $\oplus$ is used to denote the direct sum of two matrices, i.e. $X \oplus Y = \text{diag}(X, Y)$. The next two theorems for the case when $B = B^H$ are taken from [4] and will be useful for some proofs in Section 3. For convenience, we omit the subscript $k$ for the Jordan blocks in Theorem 1.

**Theorem 1** ([4] Thm. 5.1.1). Let $B = B^H \in \text{Gl}_n(\mathbb{C})$ and $A \in \mathcal{J}(B)$. Then there is some $T \in \text{Gl}_n(\mathbb{C})$ such that $T^{-1}AT = J$ and $T^HBT = \tilde{B}$ where
\[
J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus J(\lambda_{\alpha+1}) \oplus \cdots \oplus J(\lambda_\beta)
\]
is a Jordan normal form for $A$ where $\lambda_1, \ldots, \lambda_\alpha$ are the real eigenvalues of $A$ and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ are the nonreal eigenvalues of $A$ from the upper half-plane. Moreover,
\[
\tilde{B} := T^HBT = \eta_1 P_1 \oplus \cdots \eta_\alpha P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta
\]
where, for $k = 1, \ldots, \beta$, $P_k = R_p$ is the $p \times p$ reverse identity matrix (see also [1]) with $p \times p$ being the size of $J(\lambda_k)$ and $\eta = \{\eta_1, \ldots, \eta_\alpha\}$ is an ordered set of signs $\pm 1$.

The transformation defined in Theorem 2 can be interpreted as a $B$-analogue of the well-known Cayley transform [5].

**Theorem 2** ([4] Prop. 4.3.4). Let $B = B^H \in \text{Gl}_n(\mathbb{C})$ and $A \in \mathcal{J}(B)$. Let $w \in \mathbb{C}$ be a nonreal number with $w \notin \sigma(A)$ and let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Then
\[
U = \alpha(A - wI_n)(A - wI_n)^{-1} \in \mathcal{G}(B)
\]
and \( \alpha \notin \sigma(U) \). Conversely, if \( U \in \mathbb{G}(B) \), \( |\alpha| = 1 \) and \( \alpha \notin \sigma(U) \), then for any \( w \neq \overline{w} \) we have

\[
A = (wU - \overline{w}\alpha I_n)(U - \alpha I_n)^{-1} \in \mathbb{J}(B)
\]

and \( w \notin \sigma(A) \). The formulas (3) and (4) are inverse to one other.

2.1 Symmetric Polynomials and 1-Regularity

A matrix \( A \in M_n(\mathbb{C}) \) is called 1-regular\(^3\), if the eigenspace for any \( \lambda \in \sigma(A) \) is one-dimensional. In particular, \( A \) is 1-regular if and only if there is only one basic Jordan block for each eigenvalue \( \lambda \) in the Jordan decomposition of \( A \). This implies that a diagonalizable matrix is 1-regular if and only if all its eigenvalues are (pairwise) distinct. Moreover, it follows directly that 1-regularity is preserved under similarity transformations.

The following Theorem\(^3\) characterizes 1-regular matrices and is of central importance in the further discussion.

**Theorem 3** ([14 Prop. 1.1.2]). Let \( A \in M_n(\mathbb{C}) \). Then each of the following statements is equivalent to \( A \) being 1-regular.

(a) The centralizer of \( A \) in \( M_n(\mathbb{C}) \) coincides with \( \mathbb{C}[A] \). That is, for all \( A' \in M_n(\mathbb{C}) \) commuting with \( A \), there is some polynomial \( p(x) \in \mathbb{C}[x] \) such that \( A' = p(A) \).

(b) The dimension of \( \mathbb{C}[A] \) equals \( n \), i.e. the matrices \( I_n, A, A^2, \ldots, A^{n-1} \) are linearly independent over \( \mathbb{C} \).

The next Proposition\(^1\) shows that, for any given \( A \in \mathbb{J}(B) \) we can always find a 1-regular matrix from \( \mathbb{J}(B) \) that commutes with \( A \). From now one we confine ourselves to the case \( B = B^H \) and postpone the discussion of the situation for \( B = -B^H \) to Subsection 3.4.

**Proposition 1.** Let \( B = B^H \in \text{Gl}_n(\mathbb{C}) \) and \( A \in \mathbb{J}(B) \). Then there exists a 1-regular matrix \( C \in \mathbb{J}(B) \) such that \( A \) and \( C \) commute.

**Proof.** We apply Theorem\(^1\) to \( A \) and \( B \) (the subscript indicating the size of the Jordan blocks is omitted). So, there exists some \( T \in \text{Gl}_n(\mathbb{C}) \) such that

\[
\tilde{A} := T^{-1}AT = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus J(\lambda_{\alpha+1}) \oplus \cdots \oplus J(\lambda_\beta)
\]

where \( \lambda_1, \ldots, \lambda_\alpha \in \mathbb{R} \) are the real eigenvalues of \( A \) and \( \lambda_{\alpha+1}, \ldots, \lambda_\beta \in \mathbb{C} \) are the nonreal eigenvalues of \( A \) from the upper half-plane. Moreover,

\[
\tilde{B} := T^HBT = \eta_1 P_1 \oplus \cdots \eta_\alpha P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta
\]

\(^3\)A 1-regular matrix is sometimes also called nonderogatory.
where, for \( k = 1, \ldots, \beta \), \( P_k = R_p \) (the \( p \times p \) reverse identity matrix, see (1)) with \( p \times p \) being the size of \( J(\lambda_k) \), and each \( \eta_j \) is either +1 or −1. According to Lemma 2 we have \( \tilde{A} \in \mathbb{J} (\tilde{B}) \) since \( A \in \mathbb{J} (B) \). Now let \( a_1, \ldots, a_\alpha \in \mathbb{R} \) and \( a_{\alpha+1}, \ldots, a_\beta \in \mathbb{C} \setminus \mathbb{R} \) be arbitrary and pairwise distinct values and consider

\[
\tilde{C} := J(a_1) \oplus \cdots \oplus J(a_\alpha) \oplus J(a_{\alpha+1}) \oplus \cdots \oplus J(a_\beta)
\]

where each \( J(a_k) \) has the same size as \( J(\lambda_k) \), \( k = 1, \ldots, \beta \). Observe that \( \tilde{C} \in \mathbb{J} (\tilde{B}) \) and that \( \tilde{A} \tilde{C} = \tilde{C} \tilde{A} \) holds. Moreover, \( \tilde{C} \) is 1-regular since the values \( a_k \) are all distinct. We now apply the reverse transformation to obtain

\[
A = T \tilde{A} T^{-1}, \quad C := T \tilde{C} T^{-1}, \quad B = T^{-H} \tilde{B} T^{-1}.
\]

Note that now \( AC = CA \) holds and that \( C \in \mathbb{J} (B) \) (according to Proposition 2 since we had \( \tilde{C} \in \mathbb{J} (\tilde{B}) \)). Finally, as \( \tilde{C} \) was 1-regular so is \( C \).

Using Corollary 1 the result from Proposition 1 can easily be extended to \( \mathbb{L}(B) \).

**Corollary 2.** Let \( B = B^H \in \textbf{GL}_n(\mathbb{C}) \) and \( A \in \mathbb{L}(B) \). Then there exists a 1-regular matrix \( C \in \mathbb{L}(B) \) such that \( A \) and \( C \) commute.

**Proof.** This follows immediately from Corollary 1 and Proposition 1. Additionally, notice that \( C \in \textbf{M}_n(\mathbb{C}) \) is 1-regular if and only if \( iC \) is 1-regular.

A polynomial \( p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) in \( n \geq 1 \) unknowns is called symmetric if

\[
p(x_1, \ldots, x_n) = p(x_{\tau(1)}, \ldots, x_{\tau(n)})
\]

holds for all permutations \( \tau \) of \( 1, 2, \ldots, n \). The following Theorem 4 will be central in the next section.

**Theorem 4** ([14] Prop. 7.1.10). Let \( p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) be a symmetric polynomial and \( f : \textbf{M}_n(\mathbb{C}) \to \mathbb{C} \) be a function given by

\[
f(A) = p(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)) =: p(A)
\]

where \( \lambda_k(A), k = 1, \ldots, n \), denote the eigenvalues of \( A \). Then there is a polynomial \( q(x_{11}, x_{12}, \ldots, x_{nn}) \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}] \) in \( n^2 \) unknowns such that

\[
f(A) = q(a_{11}, a_{12}, \ldots, a_{nn}) \quad \forall \ A = [a_{i,j}]_{i,j} \in \textbf{M}_n(\mathbb{C}).
\]

**Proof.** The proof follows the one from [14]. Let \( A = [a_{i,j}]_{i,j} \in \textbf{M}_n(\mathbb{C}) \) and let \( \chi_A(x) = \det(xI_n - A) \) be the characteristic polynomial of \( A \). Assume that

\[
\chi_A(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n.
\]

for some \( c_k \in \mathbb{C}, k = 1, \ldots, n \). It is well known that \( c_1, \ldots, c_n \) are polynomials in the entries \( a_{ij}, 1 \leq i, j \leq n \), of \( A \).
Furthermore, let \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) denote the eigenvalues of \( A \). Then \( \chi_A(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) \) and its expansion gives

\[
\chi_A(x) = x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)^n s_n
\]  

(6)

for the coefficients \( s_1, \ldots, s_n \in \mathbb{C} \). A closer inspection reveals that \( s_1, \ldots, s_n \) are given by the \( n \) elementary symmetric polynomials in \( \lambda_1, \ldots, \lambda_n \) (see also [9, Sec. 2]), that is

\[
s_1 = \sum_i \lambda_i, \quad s_2 = \sum_{i < j} \lambda_i \lambda_j, \quad s_3 = \sum_{i < j < k} \lambda_i \lambda_j \lambda_k, \ldots, \quad s_n = \prod_i \lambda_i.
\]  

(7)

A comparison of coefficients in (5) and (6) yields: each elementary symmetric polynomial \( s_k = s_k(\lambda_1, \ldots, \lambda_n) \) in (7) in the eigenvalues of \( A \) agrees with a certain polynomial \( c_k = p_k(a_{11}, a_{12}, \ldots, a_{nn}) \) in the entries \( a_{ij} \) of \( A \). In consequence, this is true for any symmetric polynomial \( q(\lambda_1, \ldots, \lambda_n) \) since \( q \) can always be expressed as a polynomial in \( s_1, \ldots, s_n \) (cf. [9, Thm. 2.20]).

\[
\mathbf{Example \, 1.} \quad \text{We give three applications of Theorem 4 that will be important in the further discussion. Each of these examples can be found in [14, Sec. 7].}
\]

First, let \( A = [a_{ij}]_{ij} \in \mathbf{M}_n(\mathbb{C}) \) with eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \).

(i) The function \( p : \mathbf{M}_n(\mathbb{C}) \to \mathbb{C} \) given by

\[
p(A) := p(\lambda_1(A), \ldots, \lambda_n(A)) = \prod_k \lambda_k(A)
\]

determines whether \( A \) is invertible. That is, \( p(A) = 0 \) if \( A \) is singular and \( p(A) \neq 0 \) otherwise. As \( p \) is a symmetric polynomial in \( \lambda_1(A), \ldots, \lambda_n(A) \), there is some \( q(x_{11}, \ldots, x_{nn}) \) is \( n^2 \) unknowns with \( q(A) := q(a_{11}, a_{12}, \ldots, a_{nn}) = p(A) \) for all \( A \in \mathbf{M}_n(\mathbb{C}) \). The polynomial \( q \) in \( a_{11}, a_{12}, \ldots, a_{nn} \) is given by the determinant.

(ii) The function \( p : \mathbf{M}_n(\mathbb{C}) \to \mathbb{C} \) given by

\[
p(A) := p(\lambda_1(A), \ldots, \lambda_n(A)) = \prod_{k \neq j} (\lambda_k(A) - \lambda_j(A)),
\]

determines whether \( A \) has a multiple eigenvalue. That is, \( p(A) \neq 0 \) if all eigenvalues of \( A \) are (pairwise) distinct and \( p(A) = 0 \) otherwise. Notice that \( p(\lambda_1(A), \ldots, \lambda_n(A)) \) is a symmetric polynomial in \( \lambda_1(A), \ldots, \lambda_n(A) \). According to Theorem 4 there exists a polynomial \( q \in \mathbb{C}[x_{11}, \ldots, x_{nn}] \) in \( n^2 \) unknowns such that

\[
q(A) = q(a_{11}, a_{12}, \ldots, a_{nn}) = p(\lambda_1(A), \ldots, \lambda_n(A)) = p(A)
\]

for all \( A = [a_{ij}]_{ij} \in \mathbf{M}_n(\mathbb{C}) \).
Now assume \[ A = [a_{ij}]_{ij} \in \mathbf{M}_{m \times n}(\mathbb{C}). \] Recall that a \( k \times k \) minor of \( A \) is the determinant of a submatrix \( A' \in \mathbf{M}_{k \times k}(\mathbb{C}) \) obtained from \( A \) by deleting \( m-k \) rows and \( n-k \) columns. Thus (the value of) each minor is expressible as a polynomial in some entries of \( A \) according to \( (i) \).

Now note that for \( A \) the fact \( \text{rank}(A) \leq r < \min\{m,n\} \) is equivalent to the vanishing of all \((r+1) \times (r+1)\) minors of \( A \). As there are \( s := mn(r+1) \) such minors of \( A \), according to Theorem 4, there are \( s \) polynomials \( q_k(x_{11}, \ldots, x_{mn}) \), \( 1 \leq k \leq s \), in \( mn \) unknowns \( x_{11}, x_{12}, \ldots, x_{mn} \) such that \( q_k(A) := q_k(a_{11}, a_{12}, \ldots, a_{mn}) = 0 \) holds for all \( k = 1, \ldots, s \) if and only if \( \text{rank}(A) \leq r \). In particular, if \( \text{rank}(A) > r \), there is at least one \( q_k \) such that \( q_k(A) = q_k(a_{11}, \ldots, a_{mn}) \neq 0 \).

Do not overlook that, in each case considered in Example 1, the polynomials that have been determined do not depend on a special matrix.

Recall that, according to Theorem 3, \( A \in \mathbf{M}_n(\mathbb{C}) \) is 1-regular if and only if \( I_n, A, A^2, \ldots, A^{n-1} \) are linearly independent. In turn this is the case if and only if the matrix

\[
M := \begin{bmatrix}
I_n & A & A^2 & \cdots & A^{n-1}
\end{bmatrix} \in \mathbf{M}_{n^2 \times n}(\mathbb{C}),
\]

whose columns are \( I_n, A, A^2, \ldots \) written in vectorized fashion as \( n^2 \times 1 \) column vectors, has full rank (i.e. \( \text{rank}(M) = n \)). Certainly, the entries of \( M \) are polynomials in the entries of \( A \). The matrix \( M \) has rank \( \leq n-1 \) if and only if all \( n \times n \) minors of \( M \) vanish simultaneously. Taking Example 1 (iii) into account we have the following result:

**Corollary 3.** There exists a collection of \( s \geq 1 \) (nonzero) polynomials \( w_k(x_{11}, \ldots, x_{nn}) \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}] \) in \( n^2 \) unknowns such that \( w_k(A) = w_k(a_{11}, \ldots, a_{nn}) = 0 \) holds for \( A = [a_{ij}]_{ij} \in \mathbf{M}_n(\mathbb{C}) \) and all \( w_k, k = 1, \ldots, s \) if and only if \( A \) is not 1-regular.

In other words, Corollary 3 states that \( w_\ell(A) \neq 0 \) for at least one \( \ell, 1 \leq \ell \leq s \), is sufficient for \( A \in \mathbf{M}_n(\mathbb{C}) \) to be 1-regular. This fact will be very useful for the proofs in the upcoming section.

### 3. Density of Diagonalizable Matrices

In this section we prove our main theorems on the density of diagonalizable matrices in the sets \( \mathcal{J}(B), \mathcal{L}(B), \mathcal{G}(B) \) and \( \mathcal{N}(B) \). For \( B = B^H \) the Lie and Jordan algebras are treated in Subsection 3.1 whereas the set of \( B \)-unitary matrices and the set of \( B \)-normal matrices are considered in Subsections 3.2 and 3.3 respectively. The case \( B = -B^H \) is considered in Subsection 3.4.
3.1 The Lie and Jordan Algebras

We begin by considering the density of diagonalizable matrices in the sets \( \mathbb{J}(B) \) and \( \mathbb{L}(B) \) of \( B \)-selfadjoint and \( B \)-skewadjoint matrices. The result of Proposition 2 will be central for the proof of Theorem 5.

**Proposition 2.** Let \( B = B^H \in M_n(\mathbb{C}) \) be nonsingular. Then \( \mathbb{J}(B) \) and \( \mathbb{L}(B) \) contain a matrix with pairwise distinct eigenvalues.

**Proof.** Let \((m_-, m_+)\) be the inertia\(^4\) of \( B \), that is, denote the number of positive real eigenvalues of \( B \) by \( m_+ \) and the number of negative real eigenvalues of \( B \) by \( m_- \). According to a Theorem of Sylvester (cf. [8, Thm. 4.5.7]) there exists some \( Q \in \text{Gl}_n(\mathbb{C}) \) such that 

\[
B' := Q^H B Q = \begin{bmatrix} -I_{m_-} & \ast \\ \ast & I_{m_+} \end{bmatrix}.
\]

Now for any \( n \) distinct values \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) the matrix 

\[
D = \text{diag}(\alpha_1, \ldots, \alpha_n) \in M_n(\mathbb{C})
\]

is in \( \mathbb{J}(B') \). Thus \( A := Q D Q^{-1} \) is in \( \mathbb{J}(Q^{-H}(Q^H B Q)Q^{-1}) = \mathbb{J}(B) \) according to Lemma 2 and has pairwise distinct eigenvalues. The same statement for \( \mathbb{L}(B) \) follows by taking the diagonal entries \( i\alpha_1, \ldots, i\alpha_n \) for \( D \).

We now prove the main theorem of this section. Its proof makes use of the fact observed in Example 1 (ii). With the use of Proposition 2 it would also follow from [7, Cor. 1].

**Theorem 5.** Let \( B = B^H \in \text{Gl}_n(\mathbb{C}) \).

(a) For any \( J \in \mathbb{J}(B) \) and any \( \varepsilon > 0 \) there exists some diagonalizable \( J' \in \mathbb{J}(B) \) such that \( \|J - J'\|_2 < \varepsilon \). Moreover, \( J' \) can be chosen to have pairwise distinct eigenvalues.

(b) For any \( L \in \mathbb{L}(B) \) and any \( \varepsilon > 0 \) there exists some diagonalizable \( L' \in \mathbb{L}(B) \) such that \( \|L - L'\|_2 < \varepsilon \). Moreover, \( L' \) can be chosen to have pairwise distinct eigenvalues.

**Proof.** Let \( A = [a_{ij}]_{ij} \in M_n(\mathbb{C}) \). Recall that, according to Example 1 (ii), there is some polynomial

\[
q(x_{11}, \ldots, x_{nn}) \in \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}]
\]

in \( n^2 \) unknowns \( x_{ij}, 1 \leq i, j \leq n \), such that \( q(A) = q(a_{11}, \ldots, a_{nn}) = 0 \) if and only if \( A \) has a (i.e. at least one) multiple eigenvalue. Otherwise, that is if all eigenvalues of \( A \) are (pairwise) distinct, \( q(A) \neq 0 \).

\(^4\)As \( B \in \text{Gl}_n(\mathbb{C}) \), we have \( 0 \notin \sigma(B) \). Moreover, as \( B = B^H \), all eigenvalues of \( B \) are real. In conclusion \( B \) has only positive and negative real eigenvalues.
(a) Now assume \( J \in \mathbb{J}(B) \). According to Proposition 2 there is some \( E \in \mathbb{J}(B) \) with distinct eigenvalues. Now consider the family of matrices \( M(z) := zJ + E, z \in \mathbb{C}, \) and the polynomial \( \tilde{q}(z) := q(M(z)) = q(zJ + E) \in \mathbb{C}[z] \) that only depends on the single variable \( z \). Certainly \( \tilde{q}(0) = q(E) \neq 0 \) since the eigenvalues of \( E \) are distinct. Therefore, \( \tilde{q} \neq 0 \) and \( \tilde{q} \) is not the zero-polynomial. As \( \tilde{q}(z) \) has only a finite number of roots, almost all matrices \( M(z_0), z_0 \in \mathbb{C}, \) have distinct eigenvalues. Consequently, the same holds for all \( J + cE = c(c^{-1}J + E), c \in \mathbb{C} \). To guarantee that \( J + cE \in \mathbb{J}(B) \), we confine ourselves to the case \( c \in \mathbb{R} \) (see Lemma 1).

Now let \( \varepsilon > 0 \) be given and choose some \( c_0 \in \mathbb{R} \) with \( |c_0| < \varepsilon/\|E\|_2 \) such that \( \tilde{q}(c_0^{-1}) \neq 0 \). Then \( J' := J + c_0E \in \mathbb{J}(B) \), \( J' \) has \( n \) distinct eigenvalues and
\[
\|J - J'\|_2 = |c_0| \cdot \|E\|_2 < \frac{\varepsilon}{\|E\|_2} \cdot \|E\|_2 = \varepsilon.
\]
Moreover, as \( J' \) has pairwise distinct eigenvalues it is diagonalizable.

(b) The proof for \( L \in \mathbb{L}(B) \) proceeds along the same lines.

**Corollary 4.** Let \( B = B^H \in \mathbb{G}_n(\mathbb{C}) \). The set of matrices with pairwise distinct eigenvalues is dense in \( \mathbb{J}(B) \) and \( \mathbb{L}(B) \).

The proof of Theorem 5 also reveals that any matrix from \( \mathbb{J}(B) \) or \( \mathbb{L}(B) \) can always be expressed as a sum of matrices with pairwise distinct eigenvalues from the same class.

**Corollary 5.** Let \( B = B^H \in \mathbb{G}_n(\mathbb{C}) \). Then every matrix in \( \mathbb{J}(B) \) can be expressed as a sum of two diagonalizable matrices from \( \mathbb{J}(B) \) with pairwise distinct eigenvalues. The same holds for \( \mathbb{L}(B) \).

**Proof.** Using the notation from the proof of Theorem 5 (a), \( E \in \mathbb{J}(B) \) and \( c_0 \in \mathbb{R} \) can be chosen such that \( E \) and \( J + c_0 E \) both have \( n \) distinct eigenvalues. Then \( J = (J + c_0E) - c_0E \) is a sum of two matrices \( J + c_0E, -c_0E \in \mathbb{J}(B) \) which both have pairwise distinct eigenvalues. The proof follows analogously for \( \mathbb{L}(B) \).

Let \( A \in \mathbb{M}_n(\mathbb{C}) \). Using the decomposition \( A = S + K \) with \( S = (1/2)(A + A^*) \) and \( K = (1/2)(A - A^*) \), see 2, accompanied by Corollary 5 we end this section with the following observation related to the set \( \mathcal{N}(B) \) of \( B \)-normal matrices.

**Proposition 3.** Let \( B = B^H \in \mathbb{G}_n(\mathbb{C}) \). Any matrix \( A \in \mathbb{M}_n(\mathbb{C}) \) can be expressed as a sum of four matrices from \( \mathcal{N}(B) \) with each having pairwise distinct eigenvalues.

**Proof.** As any matrix from \( \mathbb{J}(B) \) and \( \mathbb{L}(B) \) can be expressed as a sum of two matrices with \( n \) distinct eigenvalues from the same class, Corollary 5 can be applied to \( A = S + K \) with \( S = (1/2)(A + A^*) \in \mathbb{J}(B) \) and \( K = (1/2)(A - A^*) \in \mathbb{L}(B) \).

\[ \square \]
3.2 The set $\mathbb{G}(B)$ of $B$-unitary matrices

In this section we analyze the set $\mathbb{G}(B)$ and show that it contains a dense subset of diagonalizable matrices. The proof relies on the Cayley transformation (cf. Theorem 2) and uses the result from Theorem 5(a).

**Theorem 6.** Let $B = B^H \in \mathfrak{gl}_n(\mathbb{C})$. For any $G \in \mathbb{G}(B)$ and any $\varepsilon > 0$ there exists some diagonalizable $G' \in \mathbb{G}(B)$ such that $\|G - G'\|_2 \leq \varepsilon$.

**Proof.** Assume $G \in \mathbb{G}(B)$ is given. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$, be chosen such that $\alpha \notin \sigma(G)$ and let $w \neq \overline{w}$ be some fixed number. Then, according to Theorem 2

$$G' := (wG - \overline{w}I_n)(G - \alpha I_n)^{-1} \in \mathbb{J}(B)$$

with $w \notin \sigma(G')$. By Theorem 5 we may construct a sequence $(F'_k)_k \in \mathbb{J}(B)$ of diagonalizable matrices with $F'_k \to G'$ for $k \to \infty$. Again according to Theorem 2, $F_k : = \alpha(F'_k - \overline{w}I_n)(F'_k - wI_n)^{-1} \to G$ for $k \to \infty$ since the transformation is continuous (and both transformations in Theorem 2 are inverse to each other). Certainly, $w \notin \sigma(F'_k)$ has to hold for $F_k$ to be defined. However, $w \notin \sigma(G')$ implies that $w \notin \sigma(F'_k)$ will hold if $F'_k$ is close enough to $G'$ (this can be interpreted as a consequence of the Bauer-Fike Theorem, cf. [5] Thm. 7.2.2], since all $F'_k$ are diagonalizable). Formally, there is some $\eta > 0$ such that $w \notin \sigma(F'_k)$ for all $F'_k \in \mathfrak{M}_n(\mathbb{C})$ with $\|G' - F'_k\|_2 < \eta$. From now on, it suffices to consider only those $F'_k$ from the sequence which are close enough to $G'$ such that $F_k$ is defined.

Now let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$\|F_k - G\|_2 < \varepsilon \quad \text{for all } F_k \text{ such that } \|F'_k - G'\|_2 < \delta$$

due to the continuity of the transformation. Now choose some $F'_j$ from the sequence with $\|F'_j - G'\|_2 < \min\{\delta, \eta\}$ (so, in particular, $w \notin \sigma(F'_j)$). Then $F_j \in \mathbb{G}(B)$ is defined and $\|F_j - G\|_2 < \varepsilon$. As $F'_j$ is diagonalizable, assume $S^{-1}F'_jS = D$ for some diagonal $D \in \mathfrak{M}_n(\mathbb{C})$. Then it follows from a direct calculation that

$$S^{-1}F_jS = \alpha(D - \overline{w}I_n)(D - wI_n)^{-1} \quad (8)$$

is a diagonalization of $F_j$ and the proof is complete. \hfill \qed

**Corollary 6.** Let $B = B^H \in \mathfrak{gl}_n(\mathbb{C})$. The set of matrices with pairwise distinct eigenvalues is dense in $\mathbb{G}(B)$.

**Proof.** This follows from the proof of Theorem 6 and Corollary 4 since the sequence of matrices $(F'_k)_k \in \mathbb{J}(B)$ constructed in the proof of Theorem 6 can be chosen such that all matrices have pairwise distinct eigenvalues. Then, if $S^{-1}F'_jS = D$ for some diagonal $D \in \mathfrak{M}_n(\mathbb{C})$ with $n$ distinct eigenvalues, the matrix $S^{-1}F_jS$ in (8) has distinct eigenvalues, too. \hfill \qed
3.3 The set $\mathcal{N}(B)$ of $B$-normal matrices

We now consider the density of diagonalizable matrices in the set $\mathcal{N}(B)$ of $B$-normal matrices. The following Lemma 4 will be helpful to prove Theorem 7. It shows how to construct $B$-normal matrices from pairs of commuting $B$-selfadjoint matrices.

**Lemma 4.** Let $B = B^H \in \text{GL}_n(\mathbb{C})$. If $F, G \in \mathcal{J}(B)$ and $F$ and $G$ commute, then $A = F \pm iG \in \mathcal{N}(B)$.

**Proof.** Note that

$$A^* = (F \pm iG)^* = B^{-1}(F \pm iG)^H B = B^{-1}F^H B \pm B^{-1}(iG)^H B = F \mp iG$$

so $AA^* = (F \pm iG)(F \mp iG)$ and $A^*A = (F \mp iG)(F \pm iG)$. Since $FG = GF$ we see that $AA^* = A^*A$ holds. \hfill $\square$

Theorem 7 states that the density result obtained for $\mathcal{J}(B), \mathcal{L}(B)$ and $\mathcal{G}(B)$ before is true for $\mathcal{N}(B)$ under the additional assumption that $B$ is a unitary matrix.

**Theorem 7.** Let $B \in \text{GL}_n(\mathbb{C})$ with $B = B^H$ and $B^H B = I_n$. For any $N \in \mathcal{N}(B)$ and any $\varepsilon > 0$ there exists some diagonalizable $N' \in \mathcal{N}(B)$ such that $\|N - N'\|_2 \leq \varepsilon$.

**Proof.** Let $N \in \mathcal{N}(B)$ be arbitrary. We define $S := \frac{1}{2}(N + N^*) \in \mathcal{J}(B)$, $K := \frac{1}{2}(N - N^*) \in \mathcal{L}(B)$ as in (2) and express $N$ as

$$N = S + K = S - i^2 K = S - iK_H$$  \hspace{1cm} (9)

with $K_H := iK$. Notice that $K_H \in \mathcal{J}(B)$ according to Corollary 1. It follows straightforward that $SK_H = K_HS$ holds, that is

$$SK_H = (1/2)(A + A^*) \cdot (i/2)(A - A^*) = (i/4)(A + A^*)(A - A^*)$$

$$= (i/4)(A - A^*)(A + A^*) = (i/2)(A - A^*) \cdot (1/2)(A + A^*)$$

$$= K_HS,$$

so $S$ and $K_H$ commute.

According to Proposition 1 there exists some 1-regular $E = [e_{ij}]_{ij} \in \mathcal{J}(B)$ such that $K_H$ and $E$ commute, that is, $K_H E = E K_H$ holds. Now, we consider the family of all matrices $M = M(z) = zS + E \in \text{M}_n(\mathbb{C})$ for $z \in \mathbb{C}$. As both $S$ and $E$ commute with $K_H$, so does each $M(z)$. In particular, note that $M(0) = E$ is 1-regular.

According to Corollary 3 there is some $w_\ell(x_{11}, \ldots, x_{nn})$ (from the set of polynomials $w_k$, $k = 1, \ldots, s$, that vanish simultaneously for matrices that are not 1-regular) such that $w_\ell(E) = w_\ell(e_{11}, e_{12}, \ldots, e_{nn}) \neq 0$. Now, as $S$ and $E$ are fixed, consider $\tilde{w}(z) := w_\ell(M(z)) = w_\ell(zS + E) \in \mathbb{C}[z]$ as a single-variable-polynomial and notice that $\tilde{w}(0) = w_\ell(E) \neq 0$. Thus $\tilde{w} \neq 0$.
is not the zero-polynomial. Recall that \( \tilde{w}(z_0) \neq 0 \) is a sufficient condition for \( M(z_0) = z_0 S + E \) to be 1-regular. Consequently, as \( \tilde{w}(z) \) does only have a finite number of roots, \( M(z_0) = z_0 S + E \) will be 1-regular for almost all \( z_0 \in \mathbb{C} \). Therefore \( S + cE = c(c^{-1} S + E) \) is also 1-regular for all but a finite number of nonzero \( c \in \mathbb{C} \).

Now let \( \varepsilon = 2\varepsilon > 0 \) be given. Choose some \( c \in \mathbb{R} \) with \( |c| \leq \varepsilon/(2\|E\|_2) \) such that \( S_c := S + cE \in \mathcal{J}(B) \) is 1-regular. Then

\[
\|S - S_c\|_2 = \|cE\|_2 = |c|\|E\|_2 \leq \frac{\varepsilon}{2\|E\|_2} \|E\|_2 = \frac{\varepsilon}{2}.
\]

As \( S_c \) and \( K_H \) commute (recall that \( S \) and \( E \) both commute with \( K_H \)) and \( S_c \) is 1-regular, there exists some polynomial \( p(x) \in \mathbb{C}[x] \) with \( p(S_c) = K_H \) according to Theorem 3 (a). Moreover, from Theorem 5 (see also Corollary 4), there exists a sequence \( (F_k)_k \in \mathcal{J}(B), k \in \mathbb{N}, \) of diagonalizable matrices with \( F_k \to S_c \). Thus, \( p(F_k) \to K_H \) for \( k \to \infty \) since \( p(x) \) is continuous. Now, for \( \varepsilon/2 \) there exists some \( \delta > 0 \) such that

\[
\|p(S_c) - p(F_k)\|_2 = \|K_H - p(F_k)\|_2 < \frac{\varepsilon}{2} \quad \text{if} \quad \|S_c - F_k\|_2 < \delta \quad (10)
\]

due to the continuity of \( p(x) \). Next, choose some \( F_{\ell} \in \mathcal{J}(B) \) from the sequence \( (F_k)_k \) with \( \|S_c - F_{\ell}\| < \min\{\varepsilon/2, \delta\} \). Then we obtain that

\[
\|S - F_{\ell}\|_2 = \|S - S_c + S_c - F_{\ell}\|_2 \\
\leq \|S - S_c\|_2 + \|S_c - F_{\ell}\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad (11)
\]

As \( p(x) \in \mathbb{C}[x] \) may have complex coefficients, it might be the case that \( p(F_{\ell}) \notin \mathcal{J}(B) \). However, \( G := (1/2)(p(F_{\ell}) + p(F_{\ell})^*) \in \mathcal{J}(B) \) (recall (2)). Assume \( p(x) \) is given by \( \sum_{k=0}^{t} a_k x^k \) for complex coefficients \( a_k \in \mathbb{C}, k = 0, \ldots, t \). Then

\[
p(F_{\ell})^* = B^{-1} \left( \sum_{k=0}^{t} a_k F_{\ell}^k \right)^H B = B^{-1} \left( \sum_{k=0}^{t} \overline{a_k} \left( F_{\ell}^k \right)^H \right) B \\
= B^{-1} \left( \sum_{k=0}^{t} \overline{a_k} \left( F_{\ell}^k \right)^H \right) B = \sum_{k=0}^{t} \overline{a_k} \left( B^{-1} F_{\ell}^k \right)^k = \sum_{k=0}^{t} \overline{a_k} F_{\ell}^k,
\]

so \( p(F_{\ell})^* = q(F_{\ell}) \) with \( q(x) = \sum_{k=0}^{t} \overline{a_k} x^k \). In particular, \( (1/2)(p(F_{\ell}) + q(F_{\ell})) = r(F_{\ell}) = \sum_{k=0}^{t} 2\Re(a_k) F_{\ell}^k \) is a (real) polynomial in \( F_{\ell} \) (where \( \Re(a_k) \) denotes the real part of the complex number \( a_k \)). As

\[
\|p(F_{\ell}) - \frac{1}{2}(p(F_{\ell}) + p(F_{\ell})^*)\|_2 = \|p(F_{\ell}) - r(F_{\ell})\|_2 \leq \|p(F_{\ell}) - X\|_2
\]

holds for any \( X \in \mathcal{J}(B) \) according to Lemma 8 and \( \|p(F_{\ell}) - K_H\|_2 < \varepsilon/2 \) (recall (10) and the choice of \( F_{\ell} \)) we conclude \( \|p(F_{\ell}) - r(F_{\ell})\|_2 \leq \varepsilon/2 \). In
analogy to (11) we obtain

\[ \| K_H - G \|_2 = \| K_H - p(F_\ell) + p(F_\ell) - G \|_2 \]
\[ \leq \| K_H - p(F_\ell) \|_2 + \| p(F_\ell) - G \|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

Finally, we arrived at \( \| S - F_\ell \|_2 < \epsilon \) and \( \| K_H - r(F_\ell) \|_2 < \epsilon \). As \( F_\ell, r(F_\ell) \in \mathbb{J}(B) \), note that \( N' := F_\ell - ir(F_\ell) \in \mathcal{N}(B) \) according to Lemma 4. Moreover, we have

\[ \| N - N' \|_2 = \| (S - iK_H) - (F_\ell - ir(F_\ell)) \|_2 = \| (S - F_\ell) - i(K_H - r(F_\ell)) \|_2 \]
\[ \leq \| S - F_\ell \|_2 + \| K_H - r(F_\ell) \|_2 \leq \epsilon + \epsilon = 2\epsilon = \epsilon. \]

As \( F_\ell \) is diagonalizable, so is \( r(F_\ell) \). In addition, as \( F_\ell \) and \( r(F_\ell) \) commute, they are simultaneously diagonalizable [8, Thm. 1.3.21]. This certainly implies \( N' = F_\ell - ir(F_\ell) \) to be diagonalizable (in particular, \( N' \) is a polynomial in \( F_\ell \)). Thus we have found a diagonalizable matrix \( N' \in \mathcal{N}(B) \) with distance at most \( \epsilon \) from \( N \) and the proof is complete. \( \square \)

Applying Lemma 2 and using the result of Theorem 7 we may now prove the density of diagonalizable matrices in \( \mathcal{N}(B) \) without the assumption of \( B \) being unitary.

**Theorem 8.** Let \( B \in \text{Gl}_n(\mathbb{C}) \) with \( B = B^H \). For any \( N \in \mathcal{N}(B) \) and any \( \epsilon > 0 \) there exists some diagonalizable \( N' \in \mathcal{N}(B) \) such that \( \| N - N' \|_2 \leq \epsilon \).

**Proof.** According to Theorem 7 the statement is true if \( B \) is unitary, so assume \( B \in \text{Gl}_n(\mathbb{C}) \) is not unitary. According to [8, Thm. 4.5.7] there exists some \( Q \in \text{Gl}_n(\mathbb{C}) \) such that

\[ B' := Q^H B Q = \begin{bmatrix} -I_{m_-} & \ast \\ \ast & I_{m_+} \end{bmatrix} \]

where \( m_- \) (\( m_+ \)) is the number of negative (positive) eigenvalues of \( B \). According to Lemma 2 we have \( Q^{-1} \mathcal{N}(B) Q = \mathcal{N}(B') \). Since \( B' \) is unitary, Theorem 7 applies and the set of diagonalizable matrices is dense in \( \mathcal{N}(B') \). As any matrix \( A \in \mathcal{N}(B') \) is diagonalizable if and only if \( QAQ^{-1} \in \mathcal{N}(B) \) is diagonalizable the density result follows for \( \mathcal{N}(B) \). \( \square \)

In Sections 3.1, 3.2 and 3.3 we showed the density results of semisimple matrices for \( B = B^H \). Hereby, \( B \) was an arbitrary (indefinite) Hermitian matrix. Before we pass on to the case \( B = -B^H \), do not overlook the special case of \( B \) being positive definite (and \([\cdot, \cdot]_B \) defining a scalar product). Due to Lemma 2 the same situation as for \( B = I_n \) takes place and all matrices in \( \mathbb{J}(B), \mathbb{L}(B), \mathbb{G}(B) \) and \( \mathcal{N}(B) \) are semisimple. In fact, for a Hermitian positive definite matrix \( B \in \text{Gl}_n(\mathbb{C}) \) there exists some \( Q \in \text{Gl}_n(\mathbb{C}) \) such that \( Q^H B Q = I_n \). Whenever \( A \in \mathbb{J}(B) \), then \( A' := Q^{-1} AQ \in \mathbb{J}(I_n) \) is
Hermitian and, in consequence, semisimple. As semisimplicity is preserved under similarity transformation, \( A \) must have been semisimple, too. The same reasoning holds in an analogous way for \( \mathbb{L}(B), \mathbb{G}(B) \) and \( \mathcal{N}(B) \) using Lemma 2.

### 3.4 Skew-Hermitian Sesquilinear Forms

We now consider the case where \( B = -B^H \in \text{Gl}_n(\mathbb{C}) \) is a skew-Hermitian matrix. Fortunately, this situation can be completely traced back to the analysis from Sections 3.1, 3.2 and 3.3.

First note that, if \( B = -B^H \) holds, then \( iB \) is Hermitian (i.e. \((iB)^H = -iB^H = iB\)). Moreover we have that

\[
(iB)^{-1} A^H (iB) = -iB^{-1} A^H (iB) = -i^2 B^{-1} A^H B = B^{-1} A^H B.
\]

This shows that \( A \in \mathbb{J}(B) \) (\( A \in \mathbb{L}(B) \), resp.) if and only if \( A \in \mathbb{J}(iB) \) (\( A \in \mathbb{L}(iB) \), resp.). Moreover

\[
A^H (iB) A = iB \Leftrightarrow i(A^H BA) = iB \Leftrightarrow A^H BA = B,
\]

so \( A \in \mathbb{G}(B) \) if and only if \( A \in \mathbb{G}(iB) \). Finally, the same reasoning reveals that \( \mathcal{N}(B) = \mathcal{N}(iB) \). Therefore, some matrix in any of these sets corresponding to \( B \) can always be interpreted as a matrix from the same set corresponding to \( iB \). The main theorems obtained in the previous sections thus apply directly when \( B = -B^H \).

**Theorem 9.** Let \( B \in \text{Gl}_n(\mathbb{C}) \) with \( B = -B^H \). For any \( A \in \text{M}_n(\mathbb{C}) \) in any of the sets \( \mathbb{J}(B), \mathbb{L}(B), \mathbb{G}(B), \mathcal{N}(B) \) and any \( \varepsilon > 0 \) there exists some diagonalizable \( A' \in \text{M}_n(\mathbb{C}) \) belonging to the same set such that \( \|A - A'\|_2 \leq \varepsilon \). In addition, the set of matrices with pairwise distinct eigenvalues is dense in \( \mathbb{J}(B), \mathbb{L}(B) \) and \( \mathbb{G}(B) \).

**Proof.** Any matrix \( A \in \text{M}_n(\mathbb{C}) \) from \( \mathbb{J}(B), \mathbb{L}(B), \mathbb{G}(B) \) or \( \mathcal{N}(B) \) can be interpreted as a matrix from \( \mathbb{J}(iB), \mathbb{L}(iB), \mathbb{G}(iB) \) or \( \mathcal{N}(iB) \), respectively, and Theorems 5, 6, and 8 apply.

Certainly, there are analogous results for the case \( B = -B^H \) as stated in Corollary 5 and Proposition 3.

### 4. Conclusions

In this work we considered the structure classes of \( B \)-selfadjoint, \( B \)-skewadjoint, \( B \)-unitary and \( B \)-normal matrices defined by an (indefinite) scalar product \([x, y] = x^H By\) on \( \mathbb{C}^n \times \mathbb{C}^n \) for some \( B \in \text{Gl}_n(\mathbb{C}) \). We showed that, if \( B = \pm B^H \), the set of semisimple (i.e. diagonalizable) matrices is dense in the set of all \( B \)-selfadjoint, \( B \)-skewadjoint, \( B \)-unitary and \( B \)-normal matrices.

\[\text{Notice that } n \text{ needs to be even for } B = -B^H \text{ to be nonsingular.}\]
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