Some noteworthy alternating trilinear forms

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Abstract. Given an alternating trilinear form \( T \in \text{Alt}(\times^3 V_n) \) on \( V_n = V(n, \mathbb{F}) \) let \( \mathcal{L}_T \) denote the set of \( T \)-singular lines in \( \text{PG}(n-1) = \mathbb{P}V_n \), consisting that is of those lines \( \langle a, b \rangle \) of \( \text{PG}(n-1) \) such that \( T(a, b, x) = 0 \) for all \( x \in V_n \). Amongst the immense profusion of different kinds of \( T \) we single out a few which we deem noteworthy by virtue of the special nature of their set \( \mathcal{L}_T \).

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1. Introduction

We will deal with a finite-dimensional vector space \( V_n = V(n, \mathbb{F}) \) and the associated projective space \( \text{PG}(n-1, \mathbb{F}) = \mathbb{P}V_n \). The \( \binom{n}{2} \)-dimensional space \( \text{Alt}(\times^2 V_n) \) consisting of alternating bilinear forms on \( V_n \) is of course very well understood. If \( n = 2m \), or if \( n = 2m + 1 \), then the nonzero elements \( B \in \text{Alt}(\times^2 V_n) \) fall into \( m \) \( \text{GL}(n, \mathbb{F}) \)-orbits \( \{ \Omega_k \}_{k=1,2,...,m} \), where \( \Omega_k \) consists of those \( B \) which have rank \( 2k \). For a given \( B \in \text{Alt}(\times^2 V_n) \) a point \( \langle a \rangle \in \mathbb{P}V_n \) is said to be \( (B-)\text{singular} \) whenever \( B(a, x) = 0 \) holds for all \( x \in V_n \). Consequently if \( n \) is odd then \( B \)-singular points exist for any \( B \), while if \( n = 2m \) is even then only when \( B \) is on the orbit \( \Omega_m \) do \( B \)-singular points not exist.

In the present paper we consider instead the \( \binom{n}{3} \)-dimensional space \( \text{Alt}(\times^3 V_n) \) consisting of alternating trilinear forms on \( V_n \). In contrast with \( \text{Alt}(\times^2 V_n) \) the mathematics of the space \( \text{Alt}(\times^3 V_n) \) is much more complicated (and interesting!). In particular the orbit structure of \( \text{Alt}(\times^3 V_n) \) is only known in certain low-dimensional cases. Alternating trilinear forms have been classified in dimension \( n \leq 7 \) over an arbitrary field, see [3,11], and also in dimension 8 over \( \mathbb{C} \) and \( \mathbb{R} \), see [4,6]. Over \( \mathbb{C} \) there are 23 orbits in dimension \( n = 8 \), but in dimension \( n = 9 \) the number of orbits is known to be infinite.

Over a finite field \( \text{GF}(q) \) there are of course, in any finite dimension \( n \), “only” a finite number of \( \text{GL}(n, q) \)-orbits. But in fact the number of orbits increases
extremely rapidly with increasing $n$. To demonstrate this it will suffice to use a crude upper bound on the order of the group $GL(n, q)$, namely $|GL(n, q)| \ll q^{n^2}$, which holds on account of the inclusion $GL(V_n) \subset L(V_n, V_n)$. Since $\wedge^3 V_{n,q}$ is of size $q^{n(n-1)(n-2)/6}$ it follows that $|\wedge^3 V_{n,q}|/|GL(n, q)| \gg q^{n(n^2-9n+2)/6}$. In particular for $n = 10$ we have $|\wedge^3 V_{10}|/|GL(10, q)| \gg q^{20}$, and so even on the ridiculous assumption that the stabiliser group of every $T \in \wedge^3 V_{10}$ is trivial the number of $GL(10, q)$-orbits would be substantially more than $q^{20}$. And for $n = 20$ the number of $GL(20, q)$-orbits in $\wedge^3 V_{20}$ is much more than $q^{740}$.

The violence of the combinatorial explosion which takes place for $n > 8$ is really quite startling! This occurs even over the smallest fields. For on setting $N(n, q) = |\wedge^3 V_{n,q}|/|GL(n, q)|$ we find for $q = 2$ the following approximate values

| $n$  | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|------|-----|-----|-----|-----|-----|-----|-----|
| $N(n, 2)$ | 0.00010 | 0.000053 | 0.00021 | 0.0135 | 27.6 | $3.6 \times 10^{6}$ | $6.1 \times 10^{13}$ |

Faced with this great multitude of orbits for alternating trilinear forms one naturally hopes that there are a few orbits which are singled out by having some special property and which thus deserve further attention. In the case of alternating bilinear forms the outstanding $GL(n, \mathbb{F})$-orbit occurs of course in even dimension $n = 2m$ and consists of those $B \in \text{Alt}(\times^2 V_n)$ which have no singular points, the stabiliser groups being $\approx \text{Sp}(2m, \mathbb{F})$. Now in the case of $T \in \text{Alt}(\times^3 V_n)$ one may define a point $\langle a \rangle \in \mathbb{P}V_n$ to be $T$-singular whenever $T(a, x, y) = 0$ holds for all $x, y \in V_n$. Also one may define a subspace $\text{rad} T$ of $V_n$ by

$$\text{rad} T = \{a \in V_n : T(a, x, y) = 0 \text{ for all } x, y \in V_n\}$$ (1)

and call $T$ non-degenerate whenever $\text{rad} T = \{0\}$. But, just as in the bilinear case, there is not much interest in degenerate $T$, since one naturally switches one’s attention to the non-degenerate trilinear form induced in the lower-dimensional quotient space $V_n/\text{rad} T$. However of crucial importance in the case of $T \in \text{Alt}(\times^3 V_n)$ are those projective lines $\langle a, b \rangle$ in $\text{PG}(n-1) = \mathbb{P}V_n$ which are $T$-singular, satisfying that is

$$T(a, b, x) = 0 \text{ for all } x \in V_n.$$ (2)

For a given $T \in \text{Alt}(\times^3 V_n)$ we will denote by $\mathcal{L}_T$ the set consisting of all the $T$-singular lines in $\text{PG}(n-1, \mathbb{F}) = \mathbb{P}V_n$.

**Remark 1.** The space $\text{Alt}(\times^3 V_n)$ of alternating 3-forms is naturally isomorphic to the space $\wedge^3 V_n^*$ of dual trivectors, and sometimes statements concerning an element $T \in \text{Alt}(\times^3 V_n)$ will be phrased in terms of its isomorphic image $t \in \wedge^3 V_n^*$. If $\{f_i\}_{1 \leq i \leq n}$ is the basis for $V_n^*$ dual to the basis $\{e_i\}_{1 \leq i \leq n}$ for $V_n$ then, on writing $f_{ijk} := f_i \wedge f_j \wedge f_k$, we have

$$t = \sum_{1 \leq i < j < k \leq n} c_{ijk} f_{ijk}, \text{ where } c_{ijk} := T(e_i, e_j, e_k).$$ (3)