Uniform Estimates for Dirichlet Problems in Perforated Domains

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Abstract
This paper studies the Dirichlet problem for Laplace’s equation in a domain \( \Omega_{\varepsilon, \eta} \) perforated with small holes, where \( \varepsilon \) represents the scale of the minimal distances between holes and \( \eta \) the ratio between the scale of sizes of holes and \( \varepsilon \). We establish \( W^{1,p} \) estimates for solutions with bounding constants depending explicitly on the small parameters \( \varepsilon \) and \( \eta \). We also show that these estimates are either optimal or near optimal.

Keywords Perforated domain · Laplace’s equation · \( W^{1,p} \) estimate · Homogenization

Mathematics Subject Classification (2010) 35J05 · 35J25 · 35B27

1 Introduction

In this paper we consider the Dirichlet problem for Laplace’s equation,
\[
\begin{aligned}
-\Delta u &= F + \text{div}(f) \quad \text{in } \Omega_{\varepsilon, \eta}, \\
 u &= 0 \quad \text{on } \partial \Omega_{\varepsilon, \eta},
\end{aligned}
\]
(1.1)
in a domain \( \Omega_{\varepsilon, \eta} \) perforated with small holes in \( \mathbb{R}^d \), where \( \varepsilon \in (0, 1] \) represents the scale of the minimal distances between holes and \( \eta \in (0, 1] \) the ratio between the scale of sizes of holes and \( \varepsilon \). Let \( F \in L^2(\Omega_{\varepsilon, \eta}) \) and \( f \in L^2(\Omega_{\varepsilon, \eta}; \mathbb{R}^d) \). Under some general conditions on \( \Omega_{\varepsilon, \eta} \), the Dirichlet problem (1.1) possesses a unique solution \( u \) in \( W^{1,2}_0(\Omega_{\varepsilon, \eta}) \). Moreover, the solution satisfies the energy estimate,
\[
e^{-1} \eta^{d-2} \|u\|_{L^2(\Omega_{\varepsilon, \eta})} + \|\nabla u\|_{L^2(\Omega_{\varepsilon, \eta})} \leq C \left\{ \|f\|_{L^2(\Omega_{\varepsilon, \eta})} + \varepsilon \eta^{\frac{2-d}{2}} \|F\|_{L^2(\Omega_{\varepsilon, \eta})} \right\},
\]
(1.2)
for \( d \geq 3 \), where \( C \) is independent of \( \varepsilon \) and \( \eta \). The purpose of this paper is to investigate the
analogous estimates in the $L^p$ setting for $1 < p < \infty$. More precisely, we study the $W^{1,p}$ estimates,

$$
\begin{align*}
&\|\nabla u\|_{L^p(\Omega_{\varepsilon,\eta})} \leq A_p(\varepsilon, \eta) \|f\|_{L^p(\Omega_{\varepsilon,\eta})} + B_p(\varepsilon, \eta) \|F\|_{L^p(\Omega_{\varepsilon,\eta})}, \\
&\|u\|_{L^p(\Omega_{\varepsilon,\eta})} \leq C_p(\varepsilon, \eta) \|f\|_{L^p(\Omega_{\varepsilon,\eta})} + D_p(\varepsilon, \eta) \|F\|_{L^p(\Omega_{\varepsilon,\eta})},
\end{align*}
$$

(1.3)

and are interested in the explicit dependence of bounding constants $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ on the small parameters $\varepsilon$ and $\eta$. Note that by duality, $B_p(\varepsilon, \eta) = C_{p'}(\varepsilon, \eta)$, where $p' = \frac{p}{p-1}$.

Our study is motivated by the homogenization theory of boundary value problems for elliptic equations and systems in perforated domains, which are used to model various processes in porous media and perforated materials. The asymptotic behavior of the solutions $u = u_\varepsilon$, as $\varepsilon \to 0$, has been studied extensively since 1970’s. In particular, if $\Omega_{\varepsilon,\eta}$ is a bounded Lipschitz domain perforated periodically with $\varepsilon \eta$, the solution of the Dirichlet problem:

$$
-\Delta u_\varepsilon = F \quad \text{in } \Omega_{\varepsilon,\eta} \quad \text{and } u_\varepsilon = 0 \quad \text{on } \partial \Omega_{\varepsilon,\eta},
$$

(1.4)

then $u_\varepsilon/\varepsilon^2 \to \gamma_0 F$ weakly in $L^2(\Omega)$ for some constant $\gamma_0 > 0$. If $\eta = \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, the asymptotic behavior of $u_\varepsilon$ is divided into three cases. In the case of large holes, where $\sigma_\varepsilon = \varepsilon^{\frac{d-d}{2}} \to 0$, $u_\varepsilon/\sigma_\varepsilon^2$ converges strongly in $L^2(\Omega)$ to $\gamma_1 F$ for some constant $\gamma_1 > 0$. In the case of small holes, where $\sigma_\varepsilon \to \infty$, $u_\varepsilon$ converges strongly in $W^{1,2}_0(\Omega)$ to $u_0$, where $u_0$ is the solution of the Dirichlet problem:

$$
-\Delta u_0 = F \quad \text{in } \Omega \quad \text{and } u_0 = 0 \quad \text{on } \partial \Omega.
$$

In the critical case, where $\sigma_\varepsilon \to 1$, the solution $u_\varepsilon$ converges weakly in $W^{1,2}_0(\Omega)$ to $u_0$, where $u_0$ is the solution of the Dirichlet problem:

$$
-\Delta u_0 + \mu_\varepsilon u_0 = F \quad \text{in } \Omega \quad \text{and } u_0 = 0 \quad \text{on } \partial \Omega,
$$

and $\mu_\varepsilon$ is a positive constant [4–6, 10, 11, 13]. Similar results hold for the Stokes equations, 

$$
-\Delta u_\varepsilon + \nabla p_\varepsilon = F \quad \text{and } \text{div}(u_\varepsilon) = 0 \quad \text{in } \Omega_{\varepsilon,\eta},
$$

with Dirichlet condition $u_\varepsilon = 0$ on $\partial \Omega_{\varepsilon,\eta}$ [1]. We refer the reader to [1–3, 8, 9, 12] and the references therein for an introduction to the homogenization theory of Dirichlet problems in perforated domains.

Our main interest in this paper is in uniform regularity estimates for the Dirichlet problem (1.1). In the case $\eta = 1$, the $W^{1,p}$ estimates were established by N. Masmoudi [14] for Laplace’s equation in an unbounded perforated domain given by (1.6). Using a compactness method, the results were extended recently in [16] by the present author to the Stokes equations with $\eta = 1$, whose asymptotic behavior is governed by the so-called Darcy’s law. This paper treats Laplace’s equation in the vanishing volume case where $\eta \to 0$. We are able to establish $W^{1,p}$ estimates with optimal or near optimal bounding constants in a general non-periodic setting. To the best of our knowledge, this paper contains the first results on regularity estimates in $L^p$, which are uniform in both $\varepsilon \in (0, 1]$ and $\eta \in (0, 1]$, for Dirichlet problems in perforated domains. Theorems 1.1 and 1.2 will be useful in the study of optimal convergence rates for solutions $u_\varepsilon$ of (1.4) in $L^p$ and $W^{1,p}$ spaces for $1 < p < \infty$, a topic we plan to return in a future study.
To state our main results, let $Y = [-1/2, 1/2]^d$ be a closed unit cube in $\mathbb{R}^d$ and $\{T_k : k \in \mathbb{Z}^d\}$ a family of closed subsets of $Y$. Throughout the paper we shall assume that each $T_k$ is the closure of a bounded Lipschitz domain, $Y \setminus T_k$ is connected, and that

$$B(0, c_0) \subset T_k, \quad \text{dist}(\partial T_k, \partial Y) \geq c_0 > 0$$

for some $c_0 > 0$. Define

$$\omega_{\varepsilon, \eta} = \mathbb{R}^d \setminus \bigcup_{k \in \mathbb{Z}^d} \varepsilon(k + \eta T_k),$$

where $0 < \varepsilon, \eta \leq 1$. Roughly speaking, the unbounded perforated domain $\omega_{\varepsilon, \eta}$ is obtained from $\mathbb{R}^d$ by removing a hole $\varepsilon(k + \eta T_k)$ of size $\varepsilon \eta$ from each cube $\varepsilon(k + Y)$ of size $\varepsilon$. The distances between holes are bounded below by $c_0 \varepsilon$.

**Theorem 1.1** Let $d \geq 3$ and $1 < p < \infty$. Let $\omega_{\varepsilon, \eta}$ be given by (1.6), where $\{T_k\}$ are the closures of bounded domains with uniform $C^1$ boundaries. Then, for any $F \in L^p(\omega_{\varepsilon, \eta})$ and $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$, the Dirichlet problem,

$$- \Delta u = F + \text{div}(f) \quad \text{in } \omega_{\varepsilon, \eta} \quad \text{and} \quad u = 0 \quad \text{on } \partial \omega_{\varepsilon, \eta},$$

has a unique solution in $W^{1,p}_0(\omega_{\varepsilon, \eta})$. Moreover, the solution satisfies the estimates,

$$\|\nabla u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} C |\varepsilon\eta|^{1 - \frac{d}{2}} \|F\|_{L^p(\omega_{\varepsilon, \eta})} + C_\delta \eta^{-\frac{d}{p} - \frac{1}{p} - \frac{1}{\delta}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p \leq 2, \\ C_\delta |\varepsilon\eta|^{1 - \frac{d}{2} + \frac{2}{p} - \frac{1}{\delta}} \|F\|_{L^p(\omega_{\varepsilon, \eta})} + C_\delta \eta^{-\frac{d}{p} - \frac{1}{p} - \frac{1}{\delta}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 < p < \infty, \end{cases}$$

and

$$\|u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} C |\varepsilon\eta|^{2 - d} \|F\|_{L^p(\omega_{\varepsilon, \eta})} + C_\delta |\varepsilon\eta|^{1 - \frac{d}{2} - \frac{1}{\delta}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p < 2, \\ C_\delta |\varepsilon\eta|^{2 - d} \|F\|_{L^p(\omega_{\varepsilon, \eta})} + C_\delta |\varepsilon\eta|^{1 - \frac{d}{2} - \frac{1}{\delta}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 \leq p < \infty, \end{cases}$$

for any $\delta \in (0, 1)$, where $C$ depends on $d$, $p$ and $\{T_k\}$, and $C_\delta$ also depends on $\delta$.

We point out that the powers of $\varepsilon$ in (1.8) and (1.9) are dictated by scaling. In fact, by rescaling, it suffices to prove the estimates for $\varepsilon = 1$. The powers of $\eta$ in (1.8) and (1.9) are either optimal or near optimal in the sense that the estimates fail for any $\delta < 0$. Indeed, by using a $Y$-periodic function $\chi_{\eta}$, which satisfies the equation

$$- \Delta \chi_{\eta} = \eta^{d - 2} \quad \text{in } \omega_{1, \eta} \quad \text{and} \quad \chi_{\eta} = 0 \quad \text{on } \partial \omega_{1, \eta},$$

we show that lower bounds for $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ in a periodically perforated domain $\omega_{\varepsilon, \eta}$, are given by the corresponding bounding constants in (1.8) and (1.9) with $\delta = 0$. More precisely, if the estimates in (1.3) hold for some $1 < p < \infty$ with the best possible constants $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$, then

$$A_p(\varepsilon, \eta) \geq c |\varepsilon\eta|^{-\frac{d}{2} - \frac{1}{p}},$$

$$D_p(\varepsilon, \eta) \geq c \varepsilon^{2} \eta^{-d},$$

and

$$B_p(\varepsilon, \eta) = C_p(\varepsilon, \eta) \geq \begin{cases} c |\varepsilon\eta|^{-\frac{d}{2}} & \text{for } 1 < p \leq 2, \\ c \varepsilon^{2} \eta^{-d - \frac{d}{p}} & \text{for } 2 < p < \infty, \end{cases}$$

where $c > 0$ depends only $d$, $p$ and $\{T_k\}$. See Sections 4 and 5 for details as well as for $W^{1,p}$ estimates in the case $d = 2$. In [17] J. Wallace and the present author are able to establish the
optimal estimates (1.8)–(1.9) with \( \delta = 0 \) for \( d \geq 3 \) in a periodically perforated domain. We also obtain the optimal \( W^{1,p} \) estimates in the case \( d = 2 \). The proof relies on a large-scale Lipschitz estimate in the periodic setting. It would be very interesting to extend the results in this paper and [17] to the Stokes equations.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) and

\[
\Omega_{\varepsilon, \eta} = \Omega \setminus \bigcup_k \varepsilon \{ k + \eta T_k \},
\]

(1.10)

where the union is taken over those \( k \)'s in \( \mathbb{Z}^d \) for which \( \varepsilon (k + Y) \subset \Omega \).

**Theorem 1.2** Let \( d \geq 3 \) and \( 1 < p < \infty \). Let \( \Omega_{\varepsilon, \eta} \) be given by (1.10), where \( \Omega \) is a bounded \( C^1 \) domain and \( \{ T_k : k \in \mathbb{Z}^d \} \) satisfies the same conditions as in Theorem 1.1. Then, for any \( f \in L^p(\Omega_{\varepsilon, \eta}, \mathbb{R}^d) \) and \( F \in L^p(\Omega_{\varepsilon, \eta}) \), the Dirichlet problem (1.1) has a unique solution in \( W^{2,p}(\Omega_{\varepsilon, \eta}) \). Moreover, the solution satisfies the estimates in (1.3) with

\[
A_p(\varepsilon, \eta) \leq C_\delta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{\frac{1}{2} - \frac{1}{p}} + \frac{\varepsilon}{p},
\]

\[
B_p(\varepsilon, \eta) \leq \left\{ \begin{array}{ll}
C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{\frac{1}{2}} & \text{if } 1 < p \leq 2, \\
C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{1 - \frac{1}{p} + \delta} & \text{if } 2 < p < \infty,
\end{array} \right.
\]

\[
C_p(\varepsilon, \eta) \leq \left\{ \begin{array}{ll}
C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{\frac{1}{2}} & \text{if } 2 \leq p < \infty, \\
C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{\frac{1}{p} + \delta} & \text{if } 1 < p < 2,
\end{array} \right.
\]

and

\[
D_p(\varepsilon, \eta) \leq C(\varepsilon \eta)^2 \min((\varepsilon \eta)^{-2}, \eta^{-d}),
\]

for any \( \delta \in (0, 1) \), where \( C \) depends on \( d, p, \{ T_k \} \) and \( \Omega \), and \( C_\delta \) also depends on \( \delta \).

Theorem 1.2 shows that the estimates in (1.8) and (1.9) continue to hold with \( \Omega_{\varepsilon, \eta} \) in the place of \( \omega_{\varepsilon, \eta} \), and that if \( \sigma_\varepsilon = \varepsilon \eta^{-\frac{d}{2}} \gg 1 \), better estimates hold in the bounded perforated domain. We mention that analogous estimates to those in Theorem 1.2 are also obtained for \( d = 2 \). See Section 6 for details.

We now describe our approach to Theorems 1.1 and 1.2. The first step is to establish the estimate (1.9) in the case \( 2 \leq p < \infty \), by using the test functions,

\[
v_\ell = \min\{|u|^{p-2}, \ell^{p-2}\} u,
\]

where \( \ell \geq 1 \), in the weak formulation of the elliptic equation \( -\Delta u = F + \text{div}(f) \) in \( \omega_{\varepsilon, \eta} \). This is a classical method that goes back to J. Moser [15] and was used to establish \( L^\infty \) estimates for weak solutions of elliptic equations in divergence form with bounded measurable coefficients. The approach was also used in [14] in the case \( \eta = 1 \). By using a Poincaré inequality for functions in \( W^{1,p}(Y) \) that vanish on \( B(0, \eta) \), we are able to deduce the estimate (1.9) for \( \| u \|_{L^p(\omega_{\varepsilon, \eta})} \) in the case \( 2 \leq p < \infty \). This estimate is sharp and requires no smoothness condition on \( \{ T_k : k \in \mathbb{Z}^d \} \). Next, we use a localization argument to show that

\[
\| \nabla u \|_{L^p(\omega_{\varepsilon, \eta})} \leq C \left\{ (\varepsilon \eta)^{-1} \| u \|_{L^p(\omega_{\varepsilon, \eta})} + \| F \|_{L^p(\omega_{\varepsilon, \eta})} + \| f \|_{L^p(\omega_{\varepsilon, \eta})} \right\},
\]

for \( 2 < p < \infty \), under the assumption that the boundaries of holes \( \{ T_k \} \) are uniformly \( C^1 \). As a result, we obtain an estimate of \( \| \nabla u \|_{L^{p_0}(\omega_{\varepsilon, \eta})} \) for any large \( p_0 > 2 \). Finally, we apply the Riesz–Thorin Interpolation Theorem for \( 2 < p < p_0 \), utilizing the \( L^2 \) energy estimate.
in (1.2) and the estimate of $\nabla u$ in $L^{p_0}$. For any $\delta \in (0, 1)$, we obtain the estimate (1.8) for $2 < p < \infty$ by choosing $p_0$ sufficiently large. The case $1 < p < 2$ follows by a duality argument.

We remark that the approach outlined above works equally well for the bounded perforated domain $\Omega_{\varepsilon, \eta}$. The additional application of the Poincaré inequality on the bounded domain $\Omega$ contributes to the appearance of the factor $\min((\varepsilon \eta)^{-2}, \eta^{-d})$ in Theorem 1.2, which takes value $(\varepsilon \eta)^{-2}$ and yields better estimates in the case $\sigma_{\varepsilon} \gg 1$ (the case of small holes).

## 2 Local Estimates

Recall that $Y = [-1/2, 1/2]^d$. We begin with a Poincaré inequality.

**Lemma 2.1** Let $d \geq 2$ and $1 < p < \infty$. Suppose that $u \in W^{1,p}_0(Y)$ and $u = 0$ on $B(0, \eta)$ for some $0 < \eta < 1/4$. Then

$$
\int_Y |u|^p \, dx \leq C \int_Y |\nabla u|^p \, dx \cdot \begin{cases} 
\eta^{p-d} & \text{if } 1 < p < d, \\
|\ln \eta|^{p-1} & \text{if } p = d, \\
1 & \text{if } d < p < \infty,
\end{cases}
$$

(2.1)

where $C$ depends on $d$ and $p$.

**Proof** The case $p = 2$ is more or less well known. The proof for the general case is similar. We provide a proof for the reader’s convenience. Using $u = 0$ on $B(0, \eta)$, we may write

$$
u(x) = u(r\omega) - u(\eta \omega) = \int_\eta^r \omega \cdot \nabla u(t\omega) \, dt$$

for any $x \in Y$, where $r = |x|$ and $\omega = x/|x|$. It follows by Hölder’s inequality that

$$
|u(x)|^p \leq \int_\eta^r |\nabla u(t\omega)|^p t^{d-1} \, dt \left( \int_\eta^r t^{-\frac{d-1}{p-1}} \, dt \right)^{p-1}.
$$

for $1 < p < \infty$. Thus,

$$
\int_Y |u|^p \, dx \leq C \int_Y |\nabla u|^p \, dx \left( \int_\eta^d t^{-\frac{d-1}{p-1}} \, dt \right)^{p-1},
$$

from which the inequalities in (2.1) follow readily. \(\square\)

For $1 < p < \infty$, let $W^{1,p}_0(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm $\| \nabla \psi \|_{L^p(\Omega)}$ and $W^{-1,p}(\Omega)$ the dual of $W^{1,p}_0(\Omega)$.

The following theorem was proved by D. Jerison and C. Kenig in [7].

**Theorem 2.2** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Let $F \in W^{-1,p}(\Omega)$. Then the Dirichlet problem,

$$
\begin{cases}
-\Delta u = F & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

has a unique solution in $W^{1,p}_0(\Omega)$, if

$$
\left| \frac{1}{p} - \frac{1}{2} \right| < \begin{cases} 
\frac{1}{6} + \delta & \text{for } d \geq 3, \\
\frac{1}{4} + \delta & \text{for } d = 2,
\end{cases}
$$

(2.2)
Lemma 2.5

Let $d$ be a weak solution of
\[ \tilde{\psi}(x) = \tilde{\psi}(x') \quad \text{for} \quad x_d > \psi_i(x'), \]
where $\psi_i$ is a Lipschitz function in $\mathbb{R}^{d-1}$ with $\|\nabla \psi_i\|_\infty \leq M$. By a constant $C$ depending on the Lipschitz character of $\Omega$, we mean that $C$ depends on $(N, M)$. If $\{\Omega_k\}$ is a family of bounded Lipschitz domains with the same $N$ and $M$, we shall call $\{\Omega_k\}$ a family of bounded domains with uniform Lipschitz boundaries. Thus the estimates (2.3) for the domain $\Omega_k$ hold uniformly with a constant $C$ independent of $k$. In the case of $C^1$ domains, we also require that the modules of continuity for $\nabla \psi_i$ are bounded uniformly by a continuous and increasing function $\xi(t)$ on $[0, \infty)$ with $\xi(0) = 0$. We call such $\{\Omega_k\}$ a family of bounded domains with uniform $C^1$ boundaries. In this case the estimates (2.3) for $\Omega_k$ hold uniformly with a constant $C$ independent of $k$ for $1 < p < \infty$.

Remark 2.3

Let $\Omega$ be a bounded Lipschitz domain. We may cover $\partial \Omega$ by a finite number of balls $\{B(x_i, r_0) : i = 1, 2, \ldots, N\}$, where $x_i \in \partial \Omega$ and $r_0 > 0$, so that after a translation and rotation of the coordinate system,
\[
B(x_i, 10r_0) \cap \Omega = B(x_i, 10r_0) \cap \{ (x', x_d) \in \mathbb{R}^d : x_d > \psi_i(x') \},
\]
\[
B(x_i, 10r_0) \cap \partial \Omega = B(x_i, 10r_0) \cap \{ (x', x_d) \in \mathbb{R}^d : x_d = \psi_i(x') \},
\]
where $\psi_i$ is a bounded Lipschitz domain, the estimate above holds under the additional conditions that $p$ satisfies (2.2) (with a uniform $\delta > 0$ independent of $k$). In the case of $C^1$ domains, we also require that the modules of continuity for $\nabla \psi_i$ are bounded uniformly by a continuous and increasing function $\xi(t)$ on $[0, \infty)$ with $\xi(0) = 0$. We call such $\{\Omega_k\}$ a family of bounded domains with uniform $C^1$ boundaries. In this case the estimates (2.3) for $\Omega_k$ hold uniformly with a constant $C$ independent of $k$ for $1 < p < \infty$.

Remark 2.4

Let $F \in L^q(\Omega)$ and $f \in L^p(\Omega; \mathbb{R}^d)$, where $\frac{d}{d'} < p < \infty$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{d'}$. It follows from Theorem 2.2 that if $\Omega$ is a bounded $C^1$ domain in $\mathbb{R}^d$, $d \geq 2$, and $u$ is a weak solution of
\[-\Delta u = F + \text{div}(f) \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
then,
\[ \|\nabla u\|_{L^p(\Omega)} \leq C \left\{ \|F\|_{L^q(\Omega)} + \|f\|_{L^p(\Omega)} \right\}. \] (2.4)

If $\Omega$ is a bounded Lipschitz domain, the estimate above holds under the additional conditions that $p \leq 3$ for $d \geq 3$ and $p \leq 4$ for $d = 2$.

Lemma 2.5

Let $d \geq 2$ and $d' = \frac{d}{d-1} < p < \infty$. Let $T$ be the closure of an open subset of $Y$ with $C^1$ boundary. Assume that $0 \in T$ and $Y \setminus T$ is connected. Let $u \in W^{1,p}(\tilde{Y} \setminus \eta T)$ be a weak solution of
\[
\begin{cases}
-\Delta u = F + \text{div}(f) & \text{in} \quad \tilde{Y} \setminus \eta T, \\
u = 0 & \text{on} \quad \partial(\eta T),
\end{cases}
\]
where $\tilde{Y} = (1 + c_0)Y$ and $0 < \eta \leq 1$. Then
\[ \int_{Y \setminus \eta T} |\nabla u|^p \ dx \leq C \left\{ \eta^{-p} \int_{\tilde{Y} \setminus \eta T} |f|^p \ dx + \int_{\tilde{Y} \setminus \eta T} |F|^p \ dx + \int_{\tilde{Y} \setminus \eta T} |f|^p \ dx \right\}, \] (2.5)
where $C$ depends on $d$, $p$, $c_0$ and $T$. If $\partial T$ is Lipschitz, the estimate above holds under the additional conditions on $p$ in (2.2).
Proof We may assume \( \eta \in (0, 1/4) \). The case \( \eta \in [1/4, 1] \) follows from the \( W^{1,p} \) estimate (2.4) for Laplace’s equation in \( C^1 \) and Lipschitz domains, by a localization argument. In fact, the case \( \eta \in (0, 1/4) \) also follows readily from (2.4) by a localization argument. To see this, choose a cut-off function \( \varphi \in C_0^\infty (\bar{Y} \setminus \eta T) \) such that \( 0 \leq \varphi \leq 1 \) in \( \bar{Y} \setminus \eta T \), \( \varphi = 1 \) in \( Y \setminus 2\eta T \), and

\[
|\nabla \varphi| \leq C\eta^{-1}, \quad |\nabla^2 \varphi| \leq C\eta^{-2} \quad \text{in} \ 2\eta T \setminus \eta T,
|\nabla \varphi| + |\nabla^2 \varphi| \leq C \quad \text{in} \ \bar{Y} \setminus Y.
\]

Using

\[-\Delta(u\varphi) = F\varphi + \div(f\varphi) - f \cdot \nabla \varphi - 2\div(u\nabla \varphi) + u \Delta \varphi\]

in \( \mathbb{R}^d \), we obtain

\[
\left\| \nabla (u\varphi) \right\|_{L^p(\mathbb{R}^d)} \leq C \left\{ \left\| f \varphi \right\|_{L^p(\mathbb{R}^d)} + \left\| u \nabla \varphi \right\|_{L^p(\mathbb{R}^d)} + \left\| F \varphi \right\|_{L^q(\mathbb{R}^d)} + \left\| \nabla \varphi \right\|_{L^q(\mathbb{R}^d)} + \left\| u \Delta \varphi \right\|_{L^q(\mathbb{R}^d)} \right\},
\]

where \( d' < p < \infty \) and \( \frac{1}{q} = \frac{1}{p} + \frac{1}{d} \). This gives

\[
\left\| \nabla u \right\|_{L^p(Y \setminus 2\eta T)} \leq C \left\{ \left\| f \right\|_{L^p(\bar{Y} \setminus \eta T)} + \left\| F \right\|_{L^q(\bar{Y} \setminus \eta T)} + \eta^{-1} \left\| u \right\|_{L^p(\bar{Y} \setminus \eta T)} \right\}, \tag{2.6}
\]

where we have used the observation \( \left\| u \right\|_{L^q(2\eta T \setminus \eta T)} \leq C \eta \left\| u \right\|_{L^p(2\eta T \setminus \eta T)} \).

Finally, if \( \partial T \) is \( C^1 \) and \( u = 0 \) on \( \partial T \), it follows from (2.4) by a rescaling argument that

\[
\left\| \nabla u \right\|_{L^p(2\eta T \setminus \eta T)} \leq C \left\{ \eta^{-1} \left\| u \right\|_{L^p(3\eta T \setminus \eta T)} + \left\| f \right\|_{L^p(3\eta T \setminus \eta T)} + \eta \left\| F \right\|_{L^p(3\eta T \setminus \eta T)} \right\} \tag{2.7}
\]

for \( 1 < p < \infty \). If \( \partial T \) is Lipschitz, the same is true under the additional conditions in (2.2). The estimate (2.7), together with (2.6), yields (2.5).

3 Global Estimates

Let \( \{T_k : k \in \mathbb{Z}^d \} \) be a family of closed subsets of \( Y = [-1/2, 1/2]^d \). Assume that each \( T_k \) is the closure of a bounded Lipschitz domain and satisfies the condition (1.5). We also assume that \( Y \setminus T_k \) is connected.

Lemma 3.1 Let \( \omega_{\varepsilon, \eta} \) be given by (1.6). Then, for any \( u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \),

\[
\int_{\omega_{\varepsilon, \eta}} \left| u \right|^2 \, dx \leq C \int_{\omega_{\varepsilon, \eta}} \left| \nabla u \right|^2 \, dx \cdot \begin{cases} \varepsilon^2 \eta^{2-d} & \text{if} \ d \geq 3, \\ \varepsilon^2 \left| \ln(\eta/2) \right| & \text{if} \ d = 2, \end{cases} \tag{3.1}
\]

where \( 0 < \varepsilon, \eta \leq 1 \) and \( C \) depends on \( d \) and \( c_0 \) in (1.5).

Proof By rescaling we may assume \( \varepsilon = 1 \). Assume \( d \geq 3 \). By applying Poincaré’s inequality (2.1), we obtain

\[
\int_{k+(Y \setminus \eta T_k)} \left| u \right|^2 \, dx \leq C \eta^{2-d} \int_{k+(Y \setminus \eta T_k)} \left| \nabla u \right|^2 \, dx
\]

for each \( k \in \mathbb{Z}^d \), where we have used the assumption \( B(0, c_0) \subset T_k \). The inequality (3.1) with \( \varepsilon = 1 \) and \( d \geq 3 \) follows by summing the inequalities above over \( k \in \mathbb{Z}^d \). The proof for the case \( d = 2 \) is the same.
Let $\Omega_{\varepsilon, \eta}$ be given by (1.10), where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$. Then, for $u \in W^{1,2}_0(\Omega_{\varepsilon, \eta})$,

$$
\int_{\Omega_{\varepsilon, \eta}} |u|^2 \, dx \leq C \int_{\Omega_{\varepsilon, \eta}} |\nabla u|^2 \, dx, \tag{3.2}
$$

where $C$ depends on $\Omega$. This follows by extending $u$ by zero to $\Omega$ and applying Poincaré’s inequality for $\Omega$. Moreover,

$$
\int_{\Omega_{\varepsilon, \eta}} |u|^2 \, dx \leq C \int_{\Omega_{\varepsilon, \eta}} |\nabla u|^2 \, dx \cdot \left\{ \begin{array}{ll}
\varepsilon^2 \eta^{2-d} & \text{if } d \geq 3, \\
\varepsilon^2 |\ln(\eta/2)| & \text{if } d = 2,
\end{array} \right. \tag{3.3}
$$

for any $u \in W^{1,2}_0(\Omega_{\varepsilon, \eta})$. To see (3.3), we may assume $\varepsilon$ is sufficiently small. For otherwise, the desired estimate follows from (3.2). We cover $\Omega$ by a family of cubes $\{\varepsilon(k+Y)\}$ of size $\varepsilon$, where $k \in \mathbb{Z}^d$ and $\varepsilon(k+Y) \cap \Omega \neq \emptyset$. The case $\varepsilon(k+Y) \subset \Omega$ is handled in the same manner as in the proof of Lemma 3.1. If $\varepsilon(k_0 + Y) \cap \partial \Omega \neq \emptyset$ for some $k_0 \in \mathbb{Z}^d$, we may find $k_1 \in \mathbb{Z}^d$ such that $\varepsilon(k_1 + Y) \subset \Omega$ and $\varepsilon(k_0 + Y) \subset 5\varepsilon(k_1 + Y)$.

It follows from the proof of Lemma 2.1 that

$$
\int_{\varepsilon(k_0+Y)} |u|^2 \, dx \leq \int_{5\varepsilon(k_1+Y)} |u|^2 \, dx \leq C \varepsilon^2 \eta^{2-d} \int_{5\varepsilon(k_1+Y)} |\nabla u|^2 \, dx
$$

for $d \geq 3$, where we have extended $u$ by zero to $\mathbb{R}^d$. As a result,

$$
\int_{\Omega_{\varepsilon, \eta}} |u|^2 \, dx \leq C \varepsilon^2 \eta^{2-d} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \leq C \varepsilon^2 \eta^{2-d} \int_{\Omega_{\varepsilon, \eta}} |\nabla u|^2 \, dx.
$$

The proof for the case $d = 2$ is the same.

It is not hard to see that

$$
W^{1,2}_0(\omega_{\varepsilon, \eta}) = \left\{ u \in W^{1,2}(\mathbb{R}^d) : u = 0 \text{ on } \mathbb{R}^d \setminus \omega_{\varepsilon, \eta} \right\}.
$$

Using Lemma 3.1 and the Lax–Milgram Theorem, one may show that, given $F \in L^2(\omega_{\varepsilon, \eta})$ and $f \in L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$, the Dirichlet problem

$$
\begin{cases}
-\Delta u = F + \text{div}(f) & \text{in } \omega_{\varepsilon, \eta}, \\
u = 0 & \text{on } \partial \omega_{\varepsilon, \eta},
\end{cases} \tag{3.4}
$$

has a unique solution in $W^{1,2}_0(\omega_{\varepsilon, \eta})$. Moreover, if $d \geq 3$, the solution satisfies

$$
\|\nabla u\|_{L^2(\omega_{\varepsilon, \eta})} + \varepsilon^{-1} \eta^{-d/2} \|u\|_{L^2(\omega_{\varepsilon, \eta})} \leq C \left\{ \varepsilon \eta^{-d/2} \|F\|_{L^2(\omega_{\varepsilon, \eta})} + \|f\|_{L^2(\omega_{\varepsilon, \eta})} \right\}, \tag{3.5}
$$

and if $d = 2$,

$$
\|\nabla u\|_{L^2(\omega_{\varepsilon, \eta})} + \varepsilon^{-1} |\ln(\eta/2)|^{-1/2} \|u\|_{L^2(\omega_{\varepsilon, \eta})} \leq C \left\{ \varepsilon |\ln(\eta/2)|^{1/2} \|F\|_{L^2(\omega_{\varepsilon, \eta})} + \|f\|_{L^2(\omega_{\varepsilon, \eta})} \right\}. \tag{3.6}
$$

The constants $C$ in (3.5) and (3.6) depend only on $d$ and $c_0$ in (1.5).

The following is one of the key estimates in this paper.
Theorem 3.3 Let $\omega_{\epsilon, \eta}$ be given by (1.6) and $2 \leq p < \infty$. Given $F \in L^2(\omega_{\epsilon, \eta}) \cap L^p(\omega_{\epsilon, \eta})$ and $f \in L^2(\omega_{\epsilon, \eta}; \mathbb{R}^d) \cap L^p(\omega_{\epsilon, \eta}; \mathbb{R}^d)$, let $u$ denote the weak solution of (3.4) in $W^{1,2}_0(\omega_{\epsilon, \eta})$. Then $u \in L^p(\omega_{\epsilon, \eta})$ and

$$
\|u\|_{L^p(\omega_{\epsilon, \eta})} \leq C \left\{ \frac{\epsilon^2 \eta^{2-d}}{2} \| F \|_{L^p(\omega_{\epsilon, \eta})} + \epsilon \eta^{2-d} \| f \|_{L^p(\omega_{\epsilon, \eta})} \right\},
$$

(3.7)

where $d \geq 3$ and $C$ depends on $d$, $p$ and $c_0$. If $d = 2$, we have

$$
\|u\|_{L^p(\omega_{\epsilon, \eta})} \leq C \left\{ \epsilon^2 \ln(\eta/2) \| F \|_{L^p(\omega_{\epsilon, \eta})} + \epsilon \ln(\eta/2)^{1/2} \| f \|_{L^p(\omega_{\epsilon, \eta})} \right\}.
$$

(3.8)

Proof We give the proof for the case $d \geq 3$. The proof for the case $d = 2$ is similar. For $\ell \geq 1$, let $v_\ell = \min\{ |u|^{p-2}, \ell^{p-2} u \}$, where $p > 2$ (the case $p = 2$ is given by (3.5) and (3.6). Note that

$$
\nabla v_\ell = (p-2)|u|^{p-2} \chi_{\{|u|<\ell\}} \nabla u + \min\{ |u|^{p-2}, \ell^{p-2} \} \nabla u = (p-1)|u|^{p-2} \chi_{\{|u|<\ell\}} \nabla u + \ell^{p-2} \chi_{\{|u|\geq \ell\}} \nabla u.
$$

It follows that $v_\ell \in W^{1,2}_0(\omega_{\epsilon, \eta})$ and

$$
\int_{\omega_{\epsilon, \eta}} \nabla u \cdot \nabla v_\ell \, dx = \int_{\omega_{\epsilon, \eta}} F v_\ell \, dx - \int_{\omega_{\epsilon, \eta}} f \cdot \nabla v_\ell \, dx.
$$

This gives

$$
c \int_{\omega_{\epsilon, \eta}} \min\{ |u|^{p-2}, \ell^{p-2} \} |\nabla u|^2 \, dx
\leq \int_{\omega_{\epsilon, \eta}} |F| \min\{ |u|^{p-2}, \ell^{p-2} \} |u| \, dx + \int_{\omega_{\epsilon, \eta}} |f| \min\{ |u|^{p-2}, \ell^{p-2} \} |\nabla u| \, dx,
$$

where $c$ depends on $p$. By using the Cauchy inequality we obtain

$$
\int_{\omega_{\epsilon, \eta}} \min\{ |u|^{p-2}, \ell^{p-2} \} |\nabla u|^2 \, dx
\leq C \left\{ \int_{\omega_{\epsilon, \eta}} |F| \min\{ |u|^{p-2}, \ell^{p-2} \} |u| \, dx + \int_{\omega_{\epsilon, \eta}} |f|^2 \min\{ |u|^{p-2}, \ell^{p-2} \} \, dx \right\}.
$$

(3.9)

Next, let

$$
w_\ell = \min\{ |u|^{\frac{p}{2}-1}, \ell^{\frac{p}{2}-1} \} u.
$$

Note that $w \in W^{1,2}_0(\omega_{\epsilon, \eta})$ and

$$
c \min\{ |u|^{\frac{p}{2}-1}, \ell^{\frac{p}{2}-1} \} |\nabla u| \leq |\nabla w_\ell| \leq C \min\{ |u|^{\frac{p}{2}-1}, \ell^{\frac{p}{2}-1} \} |\nabla u|,
$$
where $C > 0$, $c > 0$ depend only on $d$ and $p$. It follows by Lemma 3.1 and (3.9) that
\[
\int_{\omega_{e,\eta}} |w_\ell|^2 \, dx \leq C \varepsilon^2 \eta^{2-d} \int_{\omega_{e,\eta}} |\nabla w_\ell|^2 \, dx \leq C \varepsilon^2 \eta^{2-d} \left\{ \left\| F \right\|_{L^p(\omega_{e,\eta})} \left( \int_{\omega_{e,\eta}} (\min(|u|^{p-2}, \ell \theta^{-2}) |u|^{p'}) \, dx \right)^{1/p'} \right. \\
+ \left. \left\| f \right\|_{L^p(\omega_{e,\eta})} \left( \int_{\omega_{e,\eta}} (\min(|u|^{p-2}, \ell \theta^{-2}))^{p/p'} \, dx \right)^{1-2/p} \right\},
\]
where we have used Hölder’s inequality for the last step.

Finally, we observe that
\[
|w_\ell|^2 = \min(|u|^{p-2}, \ell \theta^{-2}) |u|^2,
\]
and thus
\[
(\min(|u|^{p-2}, \ell \theta^{-2}) |u|^{p'}) \leq |w_\ell|^2,
\]
where we have used the assumption $p > 1$. As a result, we obtain
\[
\left\| w_\ell \right\|_{L^2(\omega_{e,\eta})}^2 \leq C \varepsilon^2 \eta^{2-d} \left\{ \left\| F \right\|_{L^p(\omega_{e,\eta})} \left| w_\ell \right|_{L^2(\omega_{e,\eta})}^{2/p'} + \left\| f \right\|_{L^p(\omega_{e,\eta})} \left| w_\ell \right|_{L^2(\omega_{e,\eta})}^{2-4/p} \right\} \leq C(\varepsilon^2 \eta^{2-d})^p \left\| F \right\|_{L^p(\omega_{e,\eta})}^p + C \left( \varepsilon^2 \eta^{2-d} \right)^{2/p} \left\| f \right\|_{L^p(\omega_{e,\eta})}^p \left( 1/2 \right) \left| w_\ell \right|_{L^2(\omega_{e,\eta})}^2,
\]
where we have used Young’s inequality for the last step. Thus, we have proved that
\[
\int_{\omega_{e,\eta}} |w_\ell|^2 \, dx \leq C(\varepsilon^2 \eta^{2-d})^p \int_{\omega_{e,\eta}} |F|^p \, dx + C \left( \varepsilon^2 \eta^{2-d} \right)^{p/2} \left( \int_{\omega_{e,\eta}} |f|^p \, dx \right),
\]
from which the estimate (3.7) follows by letting $\ell \to \infty$ and using Fatou’s Lemma.

The estimates in Theorem 3.3 also hold for a bounded perforated domain.

**Theorem 3.4** Let $\Omega_{e,\eta}$ be given by (1.10), where $\Omega$ is a bounded Lipschitz domain. Let $2 \leq p < \infty$. Given $F \in L^p(\Omega_{e,\eta})$ and $f \in L^p(\Omega_{e,\eta}; \mathbb{R}^d)$, let $u$ denote the weak solution of (1.1) in $W^{1,2}_0(\Omega_{e,\eta})$. Then $u \in L^p(\Omega_{e,\eta})$ and
\[
\|u\|_{L^p(\Omega_{e,\eta})} \leq C \left\{ \min(1, \varepsilon^2 \eta^{2-d}) \left\| F \right\|_{L^p(\Omega_{e,\eta})} + \min(1, \varepsilon \eta^{2/d}) \left\| f \right\|_{L^p(\Omega_{e,\eta})} \right\}, \tag{3.10}
\]
where $d \geq 3$ and $C$ depends on $d$, $p$, $c_0$ and $\Omega$. If $d = 2$, we have
\[
\|u\|_{L^p(\Omega_{e,\eta})} \leq C \left\{ \min(1, \varepsilon^2 |\ln(\eta/2)|) \left\| F \right\|_{L^p(\Omega_{e,\eta})} + \min(1, \varepsilon |\ln(\eta/2)|^{1/2}) \left\| f \right\|_{L^p(\Omega_{e,\eta})} \right\}. \tag{3.11}
\]

**Proof** By Remark (3.2), the estimates in (3.1) continue to hold if $\omega_{e,\eta}$ is replaced by $\Omega_{e,\eta}$. Consequently, an inspection of the proof of Theorem 3.3 shows that the estimates (3.7) and (3.8) hold with $\Omega_{e,\eta}$ in the place of $\omega_{e,\eta}$. Moreover, by using the Poincaré inequality (3.2), the same argument as in the proof of Theorem 3.3 also gives
\[
\|u\|_{L^p(\Omega_{e,\eta})} \leq C \left\{ \left\| F \right\|_{L^p(\Omega_{e,\eta})} + \left\| f \right\|_{L^p(\Omega_{e,\eta})} \right\}, \tag{3.12}
\]
for $d \geq 2$ and $2 \leq p < \infty$, where $C$ depends on $d$, $p$ and $\Omega$. By combining (3.12) with (3.7) and (3.8), we obtain (3.10) and (3.11).
4 Estimates of $u$

For $1 < p < \infty$ and $\varepsilon, \eta \in (0, 1]$, we introduce operator norms,

$$A_p(\varepsilon, \eta) = \sup \left\{ \| \nabla u \|_{L^p(\omega_{\varepsilon, \eta})} : u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \text{ is a weak solution of } -\Delta u = \text{div}(f) \right\},$$

in $\omega_{\varepsilon, \eta}$ with $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $\| f \|_{L^p(\omega_{\varepsilon, \eta})} = 1$, \ (4.1)

$$B_p(\varepsilon, \eta) = \sup \left\{ \| u \|_{L^p(\omega_{\varepsilon, \eta})} : u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \text{ is a weak solution of } -\Delta u = F \right\},$$

in $\omega_{\varepsilon, \eta}$ with $F \in L^p(\omega_{\varepsilon, \eta}) \cap L^2(\omega_{\varepsilon, \eta})$ and $\| F \|_{L^p(\omega_{\varepsilon, \eta})} = 1$, \ (4.2)

$$C_p(\varepsilon, \eta) = \sup \left\{ \| u \|_{L^p(\omega_{\varepsilon, \eta})} : u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \text{ is a weak solution of } -\Delta u = f \right\},$$

in $\omega_{\varepsilon, \eta}$ with $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $\| f \|_{L^p(\omega_{\varepsilon, \eta})} = 1$, \ (4.3)

$$D_p(\varepsilon, \eta) = \sup \left\{ \| u \|_{L^p(\omega_{\varepsilon, \eta})} : u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \text{ is a weak solution of } -\Delta u = F \right\},$$

in $\omega_{\varepsilon, \eta}$ with $F \in L^p(\omega_{\varepsilon, \eta}) \cap L^2(\omega_{\varepsilon, \eta})$ and $\| F \|_{L^p(\omega_{\varepsilon, \eta})} = 1$, \ (4.4)

**Lemma 4.1** Let $A_p(\varepsilon, \eta), B_p(\varepsilon, \eta), C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ be defined above. Then

$$A_p(\varepsilon, \eta) = A_p(1, \eta), \quad B_p(\varepsilon, \eta) = \varepsilon B_p(1, \eta), \quad C_p(\varepsilon, \eta) = \varepsilon C_p(1, \eta), \quad D_p(\varepsilon, \eta) = \varepsilon^2 D_p(1, \eta).$$

**Proof** This follows by a simple rescaling, using the observation that if $-\Delta u = F + \text{div}(f)$ in $\omega_{\varepsilon, \eta}$ and $v(x) = u(\varepsilon x)$, then

$$-\Delta v(x) = \varepsilon^2 F(\varepsilon x) + \varepsilon \text{div}\{ f(\varepsilon x) \}$$

for $x \in \omega_{1, \eta}$. \ 

**Lemma 4.2** Let $1 < p < \infty$. Then

$$A_p(\varepsilon, \eta) = A_{p'}(\varepsilon, \eta), \quad B_p(\varepsilon, \eta) = C_{p'}(\varepsilon, \eta), \quad \text{and } D_p(\varepsilon, \eta) = D_{p'}(\varepsilon, \eta),$$

where $p' = \frac{p}{p-1}$.

**Proof** This follows by a duality argument. Indeed, let $u, w \in W^{1,2}_0(\omega_{\varepsilon, \eta})$ be weak solutions of $-\Delta u = F + \text{div}(f)$ in $\omega_{\varepsilon, \eta}$ and $-\Delta w = G + \text{div}(g)$ in $\omega_{\varepsilon, \eta}$, respectively, where $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$, $g \in L^{p'}(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$, $F \in L^p(\omega_{\varepsilon, \eta}) \cap L^2(\omega_{\varepsilon, \eta})$ and $G \in L^{p'}(\omega_{\varepsilon, \eta}) \cap L^2(\omega_{\varepsilon, \eta})$. Then

$$\int_{\omega_{\varepsilon, \eta}} uG \, dx - \int_{\omega_{\varepsilon, \eta}} \nabla u \cdot g \, dx = \int_{\omega_{\varepsilon, \eta}} \nabla u \cdot \nabla w \, dx$$

$$= \int_{\omega_{\varepsilon, \eta}} wF \, dx - \int_{\omega_{\varepsilon, \eta}} \nabla w \cdot f \, dx.$$ 

**Theorem 4.3** Let $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ be defined by (4.3) and (4.4), respectively.
1. For $2 \leq p < \infty$ and $\varepsilon, \eta \in (0, 1]$,

$$C_p(\varepsilon, \eta) \leq \begin{cases} C_\varepsilon \eta^{\frac{2-d}{2}} & \text{for } d \geq 3, \\ C_\varepsilon |\ln(\eta/2)|^{\frac{1}{2}} & \text{for } d = 2. \end{cases} \tag{4.5}$$

2. For $1 < p < \infty$ and $\varepsilon, \eta \in (0, 1]$,

$$D_p(\varepsilon, \eta) \leq \begin{cases} C_\varepsilon^2 \eta^{\frac{2-d}{2}} & \text{for } d \geq 3, \\ C_\varepsilon^2 |\ln(\eta/2)| & \text{for } d = 2. \end{cases} \tag{4.6}$$

The constants $C$ depend only on $d, p$ and $c_0$ in (1.5).

**Proof** The estimates for $2 \leq p < \infty$ in (4.5) and (4.6) are given by Theorem 3.3. The case $1 < p < 2$ for $D_p(\varepsilon, \eta)$ follows from the case $2 < p < \infty$ by Lemma 4.2.

In the remaining of this section we shall show that the estimates in Theorem 4.3 are sharp. To this end, we consider a periodically perforated domain,

$$\omega_{\varepsilon, \eta} = \mathbb{R}^d \setminus \bigcup_{k \in \mathbb{Z}^d} \varepsilon(k + \eta T), \tag{4.7}$$

where $T$ is the closure of a bounded Lipschitz subdomain of $Y$. We assume that $Y \setminus T$ is connected, $B(0, c_0) \subset T$, and that $\text{dist}(\partial T, \partial Y) \geq c_0 > 0$. Let $\chi_\eta$ be a $Y$-periodic function in $H^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$-\Delta \chi_\eta = \eta^{d-2} \text{ in } \omega_{1, \eta} \text{ and } \chi_\eta = 0 \text{ on } \mathbb{R}^d \setminus \omega_{1, \eta}, \tag{4.8}$$

where $\eta \in (0, 1]$. The existence and uniqueness of $\chi_\eta$ may be proved by using the Lax–Milgram Theorem on the Hilbert space $\{u \in H^1_{\text{per}}(Y) : u = 0 \text{ in } \eta T\}$, where $H^1_{\text{per}}(Y)$ denotes the closure of the set of smooth $Y$-periodic functions in $H^1(Y)$.

In (4.9) and (4.10) below, by $A \approx B$, we mean that there exist $C > 0$ and $c > 0$ independent of $\eta$ such that $c \leq A/B \leq C$.

**Lemma 4.4** Let $\chi_\eta$ be the $Y$-periodic function defined by (4.8). Then, if $d \geq 3$,

$$\int_Y \chi_\eta \, dx \approx 1 \quad \text{and} \quad \left(\int_Y |\nabla \chi_\eta|^2 \, dx\right)^{1/2} \approx \eta^{d-2}. \tag{4.9}$$

If $d = 2$, we have

$$\int_Y \chi_\eta \, dx \approx |\ln(\eta/2)| \quad \text{and} \quad \left(\int_Y |\nabla \chi_\eta|^2 \, dx\right)^{1/2} \approx |\ln(\eta/2)|^{\frac{1}{2}}. \tag{4.10}$$

**Proof** Suppose $d \geq 3$. Using

$$\int_Y |\nabla \chi_\eta|^2 \, dx = \eta^{d-2} \int_Y \chi_\eta \, dx, \tag{4.11}$$

we obtain

$$\int_Y |\nabla \chi_\eta|^2 \, dx \leq \eta^{d-2} \left(\int_Y |\chi_\eta|^2 \, dx\right)^{1/2} \leq C \eta^{d-2} \left(\int_Y |\nabla \chi_\eta|^2 \, dx\right)^{1/2},$$

where we have used Poincaré inequality (2.1) for $p = 2$. It follows that

$$\left(\int_Y |\nabla \chi_\eta|^2 \, dx\right)^{1/2} \leq C \eta^{d-2} \quad \text{and} \quad \int_Y \chi_\eta \, dx \leq C,$$
where $C$ depends only on $d$ and $c_0$. To prove the reverse inequalities, we construct a function

$$\psi \in H^1_{\text{per}}(Y)$$

such that $0 \leq \psi \leq 1$ in $Y$, $\psi = 1$ in $Y \setminus C \eta T$, $\psi = 0$ on $\eta T$, and $|\nabla \psi| \leq C \eta^{-1}$. Note that

$$\int_Y \psi \, dx \approx 1 \quad \text{and} \quad \left( \int_Y |\nabla \psi|^2 \, dx \right)^{1/2} \leq C \eta^{d-2}.$$ 

This yields

$$\eta^{d-2} \int_Y \psi \, dx = \int_Y \nabla \chi_\eta \cdot \nabla \psi \, dx \leq C \eta^{d-2} \|\nabla \chi_\eta\|_{L^2(Y)}.$$ 

Hence, $\eta^{d-2} \leq C \|\nabla \chi_\eta\|_{L^2(Y)}$, which, together with (4.11), also gives $\int_Y \chi_\eta \, dx \geq c > 0$.

Consider the case $d = 2$. The proof for the upper bounds is the same. However, the same choice of $\psi$ only produces lower bounds for $\int_Y \chi_\eta \, dx$ and $\|\nabla \chi_\eta\|_{L^2(Y)}$ by a positive constant $c$. For optimal lower bounds, we may assume that $\eta$ is sufficiently small and $\eta T \subset B(0, C_0 \eta) \subset B(0, 1/4)$. Let $\psi$ be the function in $H^1_{\text{per}}(Y)$ such that

$$\left\{ \begin{array}{ll}
\psi = 1 - \frac{\ln(1/2)}{\ln(C_0 \eta)} & \text{in } Y \setminus B(0, 1/2), \\
\psi(x) = 1 - \frac{\ln(1/2)}{\ln(C_0 \eta)} & \text{for } x \in B(0, 1/2) \setminus B(0, C_0 \eta), \\
\psi = 0 & \text{in } B(0, C_0 \eta).
\end{array} \right.$$ (4.12)

A direct computation shows that $\int_Y \psi \, dx \geq c > 0$ and

$$\int_Y |\nabla \psi|^2 \, dx = |\ln(C_0 \eta)|^{-2} \int_{B(0,1/2) \setminus B(0,C_0 \eta)} |x|^{-2} \, dx \leq C |\ln \eta|^{-1}.$$ 

It follows that

$$c \leq \int_Y \psi \, dx = \int_Y \nabla \chi_\eta \cdot \nabla \psi \, dx \leq C |\ln \eta|^{-1/2} \|\nabla \chi_\eta\|_{L^2(Y)}.$$ 

As a result, we have proved that $\|\nabla \chi_\eta\|_{L^2(Y)} \geq c |\ln(\eta/2)|^{1/2}$. By (4.11), this yields $\int_Y \chi_\eta \, dx \geq c |\ln(\eta/2)|$. □

The next theorem shows that the estimate (4.6) for $D_p(\varepsilon, \eta)$ is sharp for $d \geq 2$ and $1 < p < \infty$.

**Theorem 4.5** Let $D_p(\varepsilon, \eta)$ be defined by (4.4) for the periodically perforated domain $\omega_{\varepsilon, \eta}$ in (4.7). Then, for $d \geq 3$,

$$c \varepsilon^2 \eta^{2-d} \leq D_p(\varepsilon, \eta) \leq C \varepsilon^2 \eta^{2-d},$$ (4.13)

and for $d = 2$,

$$c \varepsilon^2 |\ln(\eta/2)| \leq D_p(\varepsilon, \eta) \leq C \varepsilon^2 |\ln(\eta/2)|,$$ (4.14)

where $1 < p < \infty$ and $C, c > 0$ depend only on $d$, $p$ and $c_0$.

**Proof** The upper bounds are given by Theorem 4.3. In view of Lemmas 4.1 and 4.2, to prove the lower bounds, we may assume $\varepsilon = 1$ and $1 < p \leq 2$. Fix $R > 10$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^d$, $\varphi = 1$ in $B(0, R)$, $\varphi = 0$ outside of $B(0, 2R)$, and $|\nabla \varphi| \leq CR^{-1}$, $|\nabla^2 \varphi| \leq CR^{-2}$. Using

$$- \Delta(\chi_\eta \varphi) = \eta^{d-2} \varphi - 2 \nabla \chi_\eta \cdot \nabla \varphi - \chi_\eta \Delta \varphi$$ (4.15)
in $\omega_{1,\eta}$ and $\chi_{\eta}\varphi = 0$ on $\partial\omega_{1,\eta}$, we obtain
\[
c R^d \|\chi_{\eta}\|_{L^p(Y)} \leq \|\chi_{\eta}\varphi\|_{L^p(\omega_{1,\eta})}
\leq D_p(1, \eta) \left\{ C \eta^{d-2} R^d + CR^{-1+\frac{d}{p}} \|\nabla\chi_{\eta}\|_{L^p(Y)} + CR^{-2+\frac{d}{p}} \|\chi_{\eta}\|_{L^p(Y)} \right\},
\]
where we have used the periodicity of $\chi_{\eta}$. It follows that
\[
\|\chi_{\eta}\|_{L^1(Y)} \leq D_p(1, \eta) \left\{ C \eta^{d-2} + CR^{-1} \|\nabla\chi_{\eta}\|_{L^p(Y)} + CR^{-2} \|\chi_{\eta}\|_{L^p(Y)} \right\}
\] (4.16)
for any $R > 10$, where we have used the fact that $\|\chi_{\eta}\|_{L^p(Y)} \geq \|\chi_{\eta}\|_{L^1(Y)}$. By letting $R \to \infty$ in (4.16), we see that
\[
D_p(1, \eta) \geq c \eta^{2-d} \|\chi_{\eta}\|_{L^1(Y)}.
\]
In view of Lemma 4.4, this gives the lower bounds in (4.13) and (4.14) for $\varepsilon = 1$. □

The estimate (4.5) for $C_p(\varepsilon, \eta)$ is also sharp.

**Theorem 4.6** Let $C_p(\varepsilon, \eta)$ be defined by (4.3) for the periodically perforated domain $\omega_{\varepsilon, \eta}$ in (4.7). Then, for $d \geq 3$,
\[
c \varepsilon \eta^{\frac{2-d}{2}} \leq C_p(\varepsilon, \eta) \leq C \varepsilon \eta^{\frac{2-d}{4}},
\] (4.17)
and for $d = 2$,
\[
c \varepsilon |\ln(\eta/2)|^\frac{1}{2} \leq C_p(\varepsilon, \eta) \leq C \varepsilon |\ln(\eta/2)|^\frac{1}{2},
\]
where $2 \leq p < \infty$ and $C, c$ depend only on $d, p$ and $T$.

**Proof** The upper bounds are given by Theorem 4.3. To prove the lower bounds, in view of Lemmas 4.1 and 4.2, we assume $\varepsilon = 1$ and consider $B_p(1, \eta)$ for $1 < p \leq 2$.

Let $\varphi$ and $\chi_{\eta}$ be the same as in the proof of Theorem 4.5. Then
\[
\|\nabla(\chi_{\eta}\varphi)\|_{L^p(\omega_{1,\eta})} \leq B_p(1, \eta) \left\{ C \eta^{d-2} R^d + CR^{-1+\frac{d}{p}} \|\nabla\chi_{\eta}\|_{L^p(Y)} \right. \]
\[
\left. +CR^{-2+\frac{d}{p}} \|\chi_{\eta}\|_{L^p(Y)} \right\}.
\] (4.18)

By Sobolev imbedding, $\|\chi_{\eta}\varphi\|_{L^q(\mathbb{R}^d)} \leq C \|\nabla(\chi_{\eta}\varphi)\|_{L^p(\mathbb{R}^d)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$ and $1 < p < d$. It follows that
\[
\|\chi_{\eta}\|_{L^q(Y)} \leq B_p(1, \eta) \left\{ C \eta^{d-2} R + C \|\nabla\chi_{\eta}\|_{L^p(Y)} + CR^{-1} \|\chi_{\eta}\|_{L^p(Y)} \right\}
\] (4.19)
for any $R \geq 10$.

Suppose that $d \geq 3$ and $1 < p \leq 2$. Note that $\|\nabla\chi_{\eta}\|_{L^p(Y)} \leq \|\nabla\chi_{\eta}\|_{L^2(Y)} \leq C \eta^{\frac{d-2}{2}}$ and
\[
\|\chi_{\eta}\|_{L^p(Y)} \leq \|\chi_{\eta}\|_{L^2(Y)} \leq C \|\nabla\chi_{\eta}\|_{L^2(Y)} \eta^{\frac{2-d}{2}} \leq C.
\]
By letting $R = C \eta^{\frac{2-d}{2}}$ in (4.19), we obtain
\[
C_p'(1, \eta) = B_p(1, \eta) \geq c \eta^{\frac{2-d}{2}}
\]
for $1 < p \leq 2$. This gives the lower bound in (4.17). In the case $d = 2$ and $1 < p < 2$, we choose $R = C |\ln(\eta/2)|^{1/2}$ in (4.19). In view of (4.10) we obtain
\[
B_p(1, \eta) \geq c |\ln(\eta/2)|^{1/2}. \tag{4.20}
\]
Finally, note that by (4.18), we have
\[ \|\nabla \chi_\eta\|_{L^p(Y)} \leq B_p(1, \eta) \left\{ C_\eta^{d-2} + CR^{-1}\|\nabla \chi_\eta\|_{L^p(Y)} + CR^{-2}\|\chi_\eta\|_{L^p(Y)} \right\}. \]
By letting \( R \to \infty \), we obtain
\[ B_p(1, \eta) \geq c_\eta^{2-d}\|\nabla \chi_\eta\|_{L^p(Y)} \] (4.21)
for \( d \geq 2 \) and \( 1 < p < \infty \). As a result, we see that (4.20) also holds for \( d = 2 \) and \( p = 2 \).

5 Estimates of \( \nabla u \)

In this section we establish upper and lower bounds for \( A_p(\varepsilon, \eta) \) and \( B_p(\varepsilon, \eta) \), defined by (4.1) and (4.2), respectively. Throughout this section we assume \( \{T_k\} \) are the closures of open subsets of \( Y \) with uniform \( C^1 \) boundaries and satisfy (1.5).

**Theorem 5.1** Let \( \omega_{\varepsilon, \eta} \) be given by (1.6). Let \( u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \) be a weak solution to the Dirichlet problem:
\[ -\Delta u = \text{div}(f) \quad \text{in} \quad \omega_{\varepsilon, \eta} \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial \omega_{\varepsilon, \eta}, \]
where \( f \in L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \) for some \( 1 < p < \infty \). Then
\[ \|\nabla u\|_{L^p(\omega_{\varepsilon, \eta})} \leq C_\delta \eta^{-\frac{d-1}{2}}|\ln(\eta/2)|^{1/2}\|f\|_{L^p(\omega_{\varepsilon, \eta})} \] (5.1)
for any \( \delta \in (0, 1) \), where \( C_\delta \) depends on \( d, p, \delta \) and \( \{T_k\} \).

**Proof** By rescaling and duality we may assume \( \varepsilon = 1 \) and \( p > 2 \). It follows by Lemma 2.5 that for each \( k \in \mathbb{Z}^d \),
\[ \int_{k+(Y\setminus T_k)} |\nabla u|^p \, dx \leq C \left\{ \eta^{-p} \int_{k+(\overline{Y}\setminus T_k)} |u|^p \, dx + \int_{k+(\overline{Y}\setminus T_k)} |f|^p \, dx \right\}, \]
where \( C \) depends on \( d, c_0, \) and the uniform \( C^1 \) characters of \( \{T_k\} \). By summing the inequalities above over \( k \in \mathbb{Z}^d \), we obtain
\[ \int_{\omega_{1, \eta}} |\nabla u|^p \, dx \leq C \left\{ \eta^{-p} \int_{\omega_{1, \eta}} |u|^p \, dx + \int_{\omega_{1, \eta}} |f|^p \, dx \right\}. \] (5.2)
This, together with (3.7), gives
\[ \|\nabla u\|_{L^p(\omega_{1, \eta})} \leq C_p \eta^{-\frac{d}{2}}\|f\|_{L^p(\omega_{1, \eta})} \]
for any \( p > 2 \) and \( d \geq 3 \). In the case \( d = 2 \), we may use (5.2) and (3.8) to obtain
\[ \|\nabla u\|_{L^p(\omega_{1, \eta})} \leq C_\eta^{-1} |\ln(\eta/2)|^{1/2}\|f\|_{L^p(\omega_{1, \eta})} \] (5.3)
for \( p > 2 \). Note that for \( p = 2 \), we have the energy estimate,
\[ \|\nabla u\|_{L^2(\omega_{1, \eta})} \leq \|f\|_{L^2(\omega_{1, \eta})}. \]
Thus, by Riesz–Thorin Interpolation Theorem, if \( d \geq 3 \) and \( 2 < p < p_0 \),
\[ \|\nabla u\|_{L^p(\omega_{1, \eta})} \leq C_\eta^\alpha\|f\|_{L^p(\omega_{1, \eta})}, \]
where
\[ \alpha = \frac{d}{2} \cdot \frac{1}{2} - \frac{1}{p} \cdot \frac{1}{2} - \frac{1}{p_0}. \]
Since

\[ \alpha \to -d \left( \frac{1}{2} - \frac{1}{p} \right) \quad \text{as} \quad p_0 \to \infty, \]

for any \( \delta \in (0, 1) \), there exists \( p_0 > p \) such that

\[ \alpha > -d \left( \frac{1}{2} - \frac{1}{p} \right) - \delta. \]

As a result, we have proved that if \( d \geq 3 \),

\[ \| \nabla u \|_{L^p(\omega_{\delta, \eta})} \leq C_\delta \eta^{-d} \left( \frac{1}{2} - \frac{1}{p} \right)^{-\delta} \| f \|_{L^p(\omega_{\delta, \eta})} \]

for any \( p > 2 \) and \( \delta \in (0, 1) \). Using (5.3), the same argument also yields (5.1) for \( p > 2 \) in the case \( d = 2 \). The logarithmic factor can be absorbed into \( \eta^{-\delta} \).

\[ \square \]

**Theorem 5.2** Let \( \omega_{\varepsilon, \eta} \) be the same perforated domain as in Theorem 5.1. Let \( u \in W^{1,2}_0(\omega_{\varepsilon, \eta}) \) be a weak solution to the Dirichlet problem: \( -\Delta u = F \) in \( \omega_{\varepsilon, \eta} \) and \( u = 0 \) on \( \partial \omega_{\varepsilon, \eta} \), where \( F \in L^2(\omega_{\varepsilon, \eta}) \cap L^p(\omega_{\varepsilon, \eta}) \) for some \( 1 < p < \infty \). Then, if \( d \geq 3 \),

\[ \| \nabla u \|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} C \varepsilon \eta^{1-\frac{d}{p}} \| F \|_{L^p(\omega_{\varepsilon, \eta})} & \text{if} \ 1 < p \leq 2, \\ C \varepsilon \eta^{1-d+\frac{d}{p}-\delta} \| F \|_{L^p(\omega_{\varepsilon, \eta})} & \text{if} \ 2 < p < \infty, \end{cases} \]

(5.4)

for any \( \delta \in (0, 1) \), where \( C_\delta \) depends on \( d, p, \delta \) and \( \{T_k\} \). If \( d = 2 \), one has

\[ \| \nabla u \|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} C \varepsilon \| \ln(\eta/2) \|^{\frac{1}{2}} \| F \|_{L^p(\omega_{\varepsilon, \eta})} & \text{if} \ 1 < p \leq 2, \\ C \varepsilon \eta^{1-d+\frac{d}{p}-\delta} \| F \|_{L^p(\omega_{\varepsilon, \eta})} & \text{if} \ 2 < p < \infty, \end{cases} \]

(5.5)

for any \( \delta \in (0, 1) \).

**Proof** Since \( B_p(\varepsilon, \eta) = C_p(\varepsilon, \eta) \) for \( 1 < p < \infty \), the estimates (5.4) and (5.5) for the case \( 1 < p \leq 2 \) are given by Theorem 4.3.

To treat the case \( 2 < p < \infty \), we may assume \( \varepsilon = 1 \) by rescaling. Suppose \( d \geq 3 \). Let \( u \in W^{1,2}_0(\omega_{1, \eta}) \) be a weak solution of \( -\Delta u = F \) in \( \omega_{1, \eta} \). Note that by Lemma 2.5, for each \( k \in \mathbb{Z}^d \),

\[ \int_{k+(Y \setminus \eta T_k)} |\nabla u|^p \ dx \leq C \left\{ \eta^{-p} \int_{k+(Y \setminus \eta T_k)} |u|^p \ dx + \int_{k+(Y \setminus \eta T_k)} |F|^p \ dx \right\}, \]

(5.6)

where \( C \) depends on \( d, p, c_0 \), and the uniform \( C^1 \) characters of \( \{T_k\} \). It follows by summation that

\[ \| \nabla u \|_{L^p(\omega_{1, \eta})} \leq C \left\{ \eta^{-1} \| u \|_{L^p(\omega_{1, \eta})} + \| F \|_{L^p(\omega_{1, \eta})} \right\} \]

\[ \leq C \eta^{1-d} \| F \|_{L^p(\omega_{1, \eta})} \]

for any \( p > 2 \), where we have used (3.7) for the last inequality. By Riesz–Thorin Interpolation Theorem, this, together with the \( L^2 \) estimate

\[ \| \nabla u \|_{L^2(\omega_{1, \eta})} \leq C \eta^{1-d} \| F \|_{L^2(\omega_{1, \eta})}, \]

yields

\[ \| \nabla u \|_{L^p(\omega_{1, \eta})} \leq C \eta^{d} \| F \|_{L^p(\omega_{1, \eta})}, \]



\[ \square \]

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where \( 2 < p < p_0 \) and
\[
\alpha = 1 - d + \frac{d}{2} \cdot \frac{\frac{1}{p} - \frac{1}{p_0}}{2 - \frac{1}{p_0}}.
\]

Since \( \alpha \to 1 - d + \frac{d}{p} \) as \( p_0 \to \infty \), by choosing \( p_0 \) sufficiently large, we obtain
\[
\| \nabla u \|_{L^p(\omega_{1,\eta})} \leq C_\delta \eta^{1-d+\frac{d}{p} - \delta} \| F \|_{L^p(\omega_{1,\eta})}
\]
for any \( \delta \in (0, 1) \).

The proof for the case \( d = 2 \) and \( 2 < p < \infty \) is similar. It follows by (5.6) and (3.8) that
\[
\| \nabla u \|_{L^p(\omega_{1,\eta})} \leq C \eta^{-1} \| \ln(\eta/2) \| \| F \|_{L^p(\omega_{1,\eta})},
\]
for any \( p > 2 \). This, together with the \( L^2 \) estimate
\[
\| \nabla u \|_{L^2(\omega_{1,\eta})} \leq C \| \ln(\eta/2) \|^{1/2} \| F \|_{L^2(\omega_{1,\eta})},
\]
yields the desired estimate by interpolation. \(\square\)

Next we show that the estimates in Theorems 5.1 and 5.2 are near optimal.

**Lemma 5.3** Let \( \chi_\eta \) be the \( Y \)-periodic function defined by (4.8). Let \( 2 < p < \infty \). Then, if \( d \geq 3 \),
\[
\left( \int_Y |\chi_\eta|^p \, dx \right)^{1/p} \leq C \quad \text{and} \quad \left( \int_Y |\nabla \chi_\eta|^p \, dx \right)^{1/p} \geq c \eta^{\frac{d}{p} - 1}.
\]
If \( d = 2 \), we have
\[
\left( \int_Y |\chi_\eta|^p \, dx \right)^{1/p} \leq C |\ln(\eta/2)| \quad \text{and} \quad \left( \int_Y |\nabla \chi_\eta|^p \, dx \right)^{1/p} \geq c \eta^{\frac{2}{p} - 1}.
\]

**Proof** Let \( \varphi \) be a cut-off function in \( C_0^\infty(\mathbb{R}^d) \) such that \( 0 \leq \varphi \leq 1 \) in \( \mathbb{R}^d \), \( \varphi = 1 \) in \( B(0, R) \), \( \varphi = 0 \) outside of \( B(0, 2R) \), \( |\nabla \varphi| \leq CR^{-1} \) and \( |\nabla^2 \varphi| \leq CR^{-2} \). Since \( \chi_\eta \varphi = 0 \) on \( \partial \omega_{1,\eta} \) and
\[
-\Delta(\chi_\eta \varphi) = \eta^{d-2} \varphi - 2\nabla \chi_\eta \cdot \nabla \varphi - \chi_\eta \Delta \varphi
\]
in \( \omega_{1,\eta} \), it follows from (3.7) that if \( d \geq 3 \),
\[
\| \chi_\eta \varphi \|_{L^p(\omega_{1,\eta})} \leq C_p \eta^{2-d} \left\{ \eta^{d-2} R^d \varphi + R^{-1+\frac{d}{p}} \| \nabla \chi_\eta \|_{L^p(Y)} + R^{-2+\frac{d}{p}} \| \chi_\eta \|_{L^p(Y)} \right\}.
\]
This yields
\[
\| \chi_\eta \|_{L^p(Y)} \leq C_p \eta^{2-d} \left\{ \eta^{d-2} + R^{-1} \| \nabla \chi_\eta \|_{L^p(Y)} + R^{-2} \| \chi_\eta \|_{L^p(Y)} \right\}.
\]
By letting \( R \to \infty \) we obtain \( \| \chi_\eta \|_{L^p(Y)} \leq C \) for \( d \geq 3 \) and \( 2 < p < \infty \). If \( d = 2 \), we may use (3.8) in the place of (3.7) to obtain \( \| \chi_\eta \|_{L^p(Y)} \leq C |\ln(\eta/2)| \) for \( 2 < p < \infty \).

To prove the lower bounds for \( \| \nabla \chi_\eta \|_{L^p(Y)} \), we construct a function \( \psi \in H^1_{per}(Y) \) as in the proof of Lemma 4.4. Note that
\[
\int_Y \psi \, dx \approx 1 \quad \text{and} \quad \left( \int_Y |\nabla \psi|^p \, dx \right)^{1/p'} \leq C \eta^{-1+\frac{d}{p'}}.
\]
It follows that
\[ C \eta^{d-2} \leq \| \nabla \chi_\eta \|_{L^p(Y)} \| \nabla \psi \|_{L^{p'}(Y)}. \]

This yields
\[ \| \nabla \chi_\eta \|_{L^p(Y)} \geq C \eta^{\frac{d}{p} - 1} \]
for \( d \geq 2. \)

The following theorem provides lower bounds for \( A_p(\epsilon, \eta). \)

**Theorem 5.4** Let \( A_p(\epsilon, \eta) \) be defined by (4.1) for the periodically perforated domain \( \omega_{\epsilon, \eta} \) in (4.7). Then, for \( d \geq 3, \)
\[ A_p(\epsilon, \eta) \geq c \eta^{\frac{d}{p} - 1}, \quad (5.9) \]
and for \( d = 2, \)
\[ A_p(\epsilon, \eta) \geq c |\ln(\eta/2)|^{\frac{1}{2}} \eta^{-2\frac{1}{2} - \frac{1}{p}}, \quad (5.10) \]
where \( 1 < p < \infty \) and \( c > 0 \) depends only on \( d, p \) and \( T. \)

**Proof** In view of Lemmas 4.1 and 4.2 we may assume that \( \epsilon = 1 \) and \( 2 \leq p < \infty. \) We first consider the case \( d \geq 3. \) Let \( u \) be a weak solution of \(-\Delta u = \text{div}(f)\) in \( \omega_{1, \eta} \) with \( u = 0 \) on \( \partial \omega_{1, \eta}. \) By Sobolev imbedding, we have
\[ \| u \|_{L^{p'}(\omega_{1, \eta})} \leq C \| \nabla u \|_{L^{p'}(\omega_{1, \eta})} \leq C A_p(1, \eta) \| f \|_{L^{p'}(\omega_{1, \eta})}, \]
where \( \frac{1}{q} = \frac{1}{p'} - \frac{1}{d} \) and \( 1 < p' < d. \) By duality this implies that if \(-\Delta v = G \) in \( \omega_{1, \eta} \) and \( v = 0 \) on \( \partial \omega_{1, \eta}, \) then
\[ \| \nabla v \|_{L^p(\omega_{1, \eta})} \leq C A_p(1, \eta) \| G \|_{L^q(\omega_{1, \eta})}. \]
As a result, we have proved that if \(-\Delta u = F + \text{div}(f)\) in \( \omega_{1, \eta} \) and \( u = 0 \) on \( \partial \omega_{1, \eta}, \) then
\[ \| \nabla u \|_{L^p(\omega_{1, \eta})} \leq C A_p(1, \eta) \left\{ \| F \|_{L^q(\omega_{1, \eta})} + \| f \|_{L^p(\omega_{1, \eta})} \right\}, \quad (5.11) \]
where \( d' < p < \infty \) and \( \frac{1}{q} = \frac{1}{p'} + \frac{1}{d}. \)

Let \( \varphi \in C_0^\infty((B(0, R))) \) be a cut-off function such that \( \varphi = 1 \) in \( B(0, R), \) \( 0 \leq \varphi \leq 1, \)
\[ |\nabla \varphi| \leq C/R, \] and \( |\nabla^2 \varphi| \leq C/R^2, \) where \( R \geq d. \) Let \( \chi_\eta \) be the 1-periodic function in Lemma 5.3. Since \(-\Delta (\chi_\eta \varphi) = \eta^{d-2} \varphi - 2\text{div}(\chi_\eta \nabla \varphi) + \chi_\eta \Delta \varphi\) in \( \omega_{1, \eta} \) and \( \chi_\eta \varphi = 0 \) on \( \partial \omega_{1, \eta}, \) we deduce from (5.11) that
\[ \| \nabla (\chi_\eta \varphi) \|_{L^p(\omega_{1, \eta})} \leq C A_p(1, \eta) \left\{ \eta^{d-2} R^{\frac{d}{q} + 1} + R^{-1} \eta^{\frac{d}{p} + 1} \right\}, \]
where we have used (5.7). Since \( \chi_\eta \) is \( Y\)-periodic, it is not hard to see that
\[ \| \nabla (\chi_\eta \varphi) \|_{L^p(\omega_{1, \eta})} \geq \| \nabla \chi_\eta \|_{L^p(B(0, R))} \approx R^{\frac{d}{q}} \| \nabla \chi_\eta \|_{L^p(Y)}. \]

It follows that
\[ \| \nabla \chi_\eta \|_{L^p(Y)} \leq C A_p(1, \eta) \left\{ \eta^{d-2} R + R^{-1} \right\} \leq C A_p(1, \eta) \eta^{\frac{d}{q} - 1}. \]
where we have chosen $R = C \eta^{-\frac{d+2}{2}}$. This, together with (5.7), yields (5.9) for $2 \leq p < \infty$ and $\varepsilon = 1$.

The proof for the case $d = 2$ and $2 < p < \infty$ is similar. Indeed, using (5.8), we may show that

$$
\|\nabla (\chi \eta \varphi)\|_{L^p(\omega_{\varepsilon, \eta})} \leq C A_p(1, \eta) \left\{ R^{\frac{2}{p}+1} + |\ln(\eta/2)| R^{\frac{2}{p}-1} \right\}.
$$

It follows that

$$
\|\nabla \chi \eta\|_{L^p(Y)} \leq C A_p(1, \eta) \left\{ R + |\ln(\eta/2)| R^{-1} \right\}
$$

where we have let $R = C |\ln(\eta/2)|^{1/2}$. This, together with (5.8), gives (5.10) for $2 < p < \infty$ and $\varepsilon = 1$.

Finally, in the case $d = 2$ and $p = 2$, we note that by Lemma 3.1,

$$
C_2(1, \eta) \leq C A_2(1, \eta) |\ln(\eta/2)|^{1/2}.
$$

By Theorem 4.6, $C_2(1, \eta) \geq c |\ln(\eta/2)|^{1/2}$. It follows that $A_2(1, \eta) \geq c > 0$. \hfill \Box

The next theorem gives lower bounds for $B_p(\varepsilon, \eta)$.

**Theorem 5.5** Let $B_p(\varepsilon, \eta)$ be defined by (4.2) for the periodically perforated domain $\omega_{\varepsilon, \eta}$ in (4.7). Then, for $d \geq 3$,

$$
B_p(\varepsilon, \eta) \geq \begin{cases} 
  c \varepsilon \eta^{1-d} & \text{for } 1 < p \leq 2, \\
  c \varepsilon \eta^{1-d+\frac{d}{p}} & \text{for } 2 < p < \infty,
\end{cases}
$$

where $c > 0$ depends only on $d$, $p$ and $T$. If $d = 2$, we have

$$
B_p(\varepsilon, \eta) \geq \begin{cases} 
  c \varepsilon |\ln(\eta/2)|^{\frac{1}{2}} & \text{for } 1 < p \leq 2, \\
  c \varepsilon \eta^{1+\frac{d}{p}} & \text{for } 2 < p < \infty.
\end{cases}
$$

**Proof** Since $B_p(\varepsilon, \eta) = C_p(\varepsilon, \eta)$, the case $1 < p \leq 2$ is contained in Theorem 4.6. To treat the case $2 < p < \infty$, we assume $\varepsilon = 1$. As in the proof of Theorem 4.5, we may use the equation (4.15) to deduce that

$$
\|\nabla \chi \eta\|_{L^p(Y)} \leq C \eta^{d-2} B_p(1, \eta).
$$

By Lemma 5.3 this leads to

$$
B_p(1, \eta) \geq c \eta^{-d} \|\nabla \chi \eta\|_{L^p(Y)} \geq c \eta^{1-d+\frac{d}{p}}
$$

for $d \geq 2$. \hfill \Box

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1** Let $d \geq 3$ and $1 < p < \infty$. It follows from Theorems 4.3, 5.1 and 5.2 that if $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d) \cap L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $F \in L^p(\omega_{\varepsilon, \eta}) \cap L^2(\omega_{\varepsilon, \eta})$, the weak solution of (1.7) in $W_0^{1,2}(\omega_{\varepsilon, \eta})$ satisfies the estimates (1.8) and (1.9). By a density argument this implies that for any $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $F \in L^p(\omega_{\varepsilon, \eta})$, the Dirichlet problem (1.7) has a solution $u$ in $W_0^{1,p}(\omega_{\varepsilon, \eta})$, which satisfies the estimates (1.8) and (1.9).
To prove the uniqueness, let us assume that \( \Delta u = 0 \) in \( \omega_{1,\eta} \) and \( u = 0 \) on \( \partial \omega_{1,\eta} \) for some \( u \in W^{1,p}(\omega_{1,\eta}) \) and \( p > 1 \). By \( L^\infty \) estimates for harmonic functions in Lipschitz domains,

\[
\max_{k+(Y \setminus T_k)} |u| \leq C_\eta \int_{k+(Y \setminus T_k)} |u| \, dx,
\]

where \( C_\eta \) depends on \( \eta \), but not on \( k \). Since \( u \in L^p(\omega_{1,\eta}) \), we see that \( u \in L^\infty(\omega_{1,\eta}) \) and that \( u(x) \to 0 \) as \( |x| \to \infty \). By applying the maximum principle in \( B(0, R) \cap \omega_{\eta,\eta} \) and then letting \( R \to \infty \), we conclude that \( u \equiv 0 \) in \( \omega_{1,\eta} \). This completes the proof of Theorem 1.1. \( \square \)

Theorem 5.1 treats the case \( d \geq 3 \). The estimates for \( d = 2 \) are summarized below.

**Theorem 5.6** Suppose \( d = 2 \) and \( 1 < p < \infty \). Let \( \omega_{\varepsilon,\eta} \) be the same as in Theorem 1.1. Then, for any \( F \in L^p(\omega_{\varepsilon,\eta}) \) and \( f \in L^p(\omega_{\varepsilon,\eta}; \mathbb{R}^d) \), the Dirichlet problem (1.7) has a unique solution in \( W^{1,p}_0(\Omega_{\varepsilon,\eta}) \). Moreover, the solution satisfies the estimates,

\[
\|\nabla u\|_{L^p(\omega_{\varepsilon,\eta})} \leq \begin{cases} 
C\varepsilon |\ln(\eta/2)|^{1/2} \|F\|_{L^p(\omega_{\varepsilon,\eta})} + C\delta \eta^{-2\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} \|f\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 1 < p \leq 2,

C\delta \varepsilon^{-1 + \frac{2}{p} - \frac{1}{2}} \|F\|_{L^p(\omega_{\varepsilon,\eta})} + C\delta \eta^{-2\frac{1}{2} - \frac{1}{p} - \frac{1}{2}} \|f\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 2 < p < \infty,
\end{cases}
\]

and

\[
\|u\|_{L^p(\omega_{\varepsilon,\eta})} \leq \begin{cases} 
C\varepsilon^2 |\ln(\eta/2)| \|F\|_{L^p(\omega_{\varepsilon,\eta})} + C\delta \varepsilon^{-\frac{1}{2} - \delta} \|f\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 1 < p < 2,

C\varepsilon^2 |\ln(\eta/2)| \|F\|_{L^p(\omega_{\varepsilon,\eta})} + C\varepsilon |\ln(\eta/2)| \|f\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 2 \leq p < \infty,
\end{cases}
\]

for any \( \delta \in (0, 1) \), where \( C \) depends on \( p \) and \( \{T_k\} \), and \( C_\delta \) also depends on \( \delta \).

**Proof** See (5.1) and (5.5) for (5.12), and (4.5)–(4.6) for (5.13). \( \square \)

### 6 Estimates in a Bounded Perforated Domain

In this section we consider the Dirichlet problem (1.1) in a bounded perforated domain \( \Omega_{\varepsilon,\eta} \) given by (1.10). Throughout this section, unless indicated otherwise, we assume that \( \Omega \) is a bounded \( C^1 \) domain and that \( \{T_k : k \in \mathbb{Z}^d\} \) are the closures of bounded sub-domains of \( Y \) with uniform \( C^1 \) boundaries.

For \( \varepsilon, \eta \in (0, 1] \), let \( A_p(\varepsilon, \eta) \), \( B_p(\varepsilon, \eta) \), \( C_p(\varepsilon, \eta) \) and \( D_p(\varepsilon, \eta) \) be defined by (4.1), (4.2), (4.3) and (4.4), but with \( \Omega_{\varepsilon,\eta} \) in the place of \( \omega_{\varepsilon,\eta} \).

**Lemma 6.1** Let \( \Omega_{\varepsilon,\eta} \) be given by (1.10) and \( 1 < p < \infty \). For any \( f \in L^p(\Omega_{\varepsilon,\eta}; \mathbb{R}^d) \) and \( F \in L^p(\Omega_{\varepsilon,\eta}) \), the Dirichlet problem (1.1) has a unique solution in \( W^{1,p}_0(\Omega) \). Moreover, the solution satisfies the estimates,

\[
\|\nabla u\|_{L^p(\Omega_{\varepsilon,\eta})} \leq A_p(\varepsilon, \eta) \|f\|_{L^p(\Omega_{\varepsilon,\eta})} + B_p(\varepsilon, \eta) \|F\|_{L^p(\Omega_{\varepsilon,\eta})},
\]

and

\[
\|u\|_{L^p(\Omega_{\varepsilon,\eta})} \leq C_p(\varepsilon, \eta) \|f\|_{L^p(\Omega_{\varepsilon,\eta})} + D_p(\varepsilon, \eta) \|F\|_{L^p(\Omega_{\varepsilon,\eta})}.
\]

**Proof** Since \( \Omega_{\varepsilon,\eta} \) is a bounded \( C^1 \) domain, the existence and uniqueness of solutions for (1.1) are given by [7]. The estimates (6.1) and (6.2) follow by linearity and a density argument. \( \square \)

**Lemma 6.2** The three equations in Lemma 4.2 continue to hold for the domain \( \Omega_{\varepsilon,\eta} \).
Proof The same duality argument for \( \omega_{\epsilon, \eta} \) works equally well for \( \Omega_{\epsilon, \eta} \).

Theorem 6.3 Let \( C_p(\epsilon, \eta) \) and \( D_p(\epsilon, \eta) \) be defined by (4.3) and (4.4), respectively, but with \( \Omega_{\epsilon, \eta} \) in the place of \( \omega_{\epsilon, \eta} \). Then

1. For \( 2 \leq p < \infty \),

\[
C_p(\epsilon, \eta) \leq \begin{cases} 
C \min(1, \epsilon \eta^{2-d}) & \text{if } d \geq 3, \\
C \min(1, \epsilon |\ln(\eta/2)|^{1/2}) & \text{if } d = 2.
\end{cases} 
\] (6.3)

2. For \( 1 < p < \infty \),

\[
D_p(\epsilon, \eta) \leq \begin{cases} 
C \min(1, \epsilon^2 \eta^{2-d}) & \text{if } d \geq 3, \\
C \min(1, \epsilon^2 |\ln(\eta/2)|) & \text{if } d = 2.
\end{cases} 
\] (6.4)

The constants \( C \) depend on \( d, p, c_0 \) and \( \Omega \).

Proof The case \( 2 \leq p < \infty \) is given by Theorem 3.4, while the case \( 1 < p < 2 \) for \( D_p(\epsilon, \eta) \) follows from the fact \( D_p(\epsilon, \eta) = D_p'(\epsilon, \eta) \). Note that the \( C^1 \) conditions for \( \Omega \) and \( \{T_k\} \) are not needed for (6.3) and (6.4).

Lemma 6.4 Let \( u \in W^{1,2}_0(\Omega_{\epsilon, \eta}) \) be a weak solution of the Dirichlet problem (1.1), where \( F \in L^p(\Omega_{\epsilon, \eta}) \) and \( f \in L^p(\Omega_{\epsilon, \eta}; \mathbb{R}^d) \) for some \( 2 < p < \infty \). Then

\[
\|\nabla u\|_{L^p(\Omega_{\epsilon, \eta})} \leq C \left\{ \epsilon^{-1} \|u\|_{L^p(\Omega_{\epsilon, \eta})} + \|F\|_{L^p(\Omega_{\epsilon, \eta})} + \|f\|_{L^p(\Omega_{\epsilon, \eta})} \right\},
\] (6.5)

where \( C \) depends on \( d, p, \Omega \) and \( \{T_k\} \).

Proof We claim that for any \( x_0 \in \Omega_{\epsilon, \eta} \),

\[
\int_{B(x_0, c_1 \epsilon \eta)} |\nabla u|^p \, dx \leq C \int_{B(x_0, 8c_1 \epsilon \eta) \cap \Omega_{\epsilon, \eta}} ((\epsilon \eta)^{-p}|u|^p + |F|^p + |f|^p) \, dx,
\] (6.6)

where \( c_1 = (c_0/100) \). To see this, we consider two cases. In the first case we assume \( B(x_0, 2c_1 \epsilon \eta) \subset \Omega_{\epsilon, \eta} \). The estimate (6.6) then follows by the standard interior estimates for Laplace’s equation. In the second case we assume \( B(x_0, 2c_1 \epsilon \eta) \cap \partial \Omega_{\epsilon, \eta} \neq \emptyset \). Choose \( y_0 \in B(x_0, 2c_1 \epsilon \eta) \cap \partial \Omega_{\epsilon, \eta} \). Then \( B(x_0, c_1 \epsilon \eta) \subset B(y_0, 3c_1 \epsilon \eta) \). Since \( \partial \Omega \) is a \( C^1 \) domain and \( \{T_k\} \) are the closures of bounded domains with uniform \( C^1 \) boundaries, it follows from (2.4) by a localization argument that

\[
\int_{B(y_0, 3c_1 \epsilon \eta) \cap \Omega_{\epsilon, \eta}} |\nabla u|^p \, dx \leq C \int_{B(y_0, 6c_1 \epsilon \eta) \cap \Omega_{\epsilon, \eta}} ((\epsilon \eta)^{-p}|u|^p + |F|^p + |f|^p) \, dx,
\] (6.7)

where we have used the fact \( u = 0 \) on \( \partial \Omega_{\epsilon, \eta} \). Note that \( B(y_0, 6c_1 \epsilon \eta) \subset B(x_0, 8c_1 \epsilon \eta) \). As a result, (6.6) follows from (6.7). Finally, we obtain (6.5) by integrating both sides of (6.6) in \( x_0 \) over \( \Omega_{\epsilon, \eta} \) and using Fubini’s Theorem.

Theorem 6.5 Let \( B_p(\epsilon, \eta) \) be defined by (4.2), but with \( \Omega_{\epsilon, \eta} \) in the place of \( \omega_{\epsilon, \eta} \). Then

1. For \( 1 < p \leq 2 \),

\[
B_p(\epsilon, \eta) \leq \begin{cases} 
C \min(1, \epsilon \eta^{2-d}) & \text{if } d \geq 3, \\
C \min(1, \epsilon |\ln(\eta/2)|^{1/2}) & \text{if } d = 2.
\end{cases} 
\] (6.8)

where \( C \) depends on \( d, p, c_0 \) and \( \Omega \).
2. For $2 < p < \infty$ and $d \geq 2$, 
\[ B_p(\varepsilon, \eta) \leq C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-1}, \eta^{-\frac{d}{2}}) \right\}^{2-\frac{2}{p}+\delta} \quad (6.9) \]
for any $\delta \in (0, 1)$, where $C_\delta$ depends on $d$, $p$, $\delta$, $\{T_k\}$ and $\Omega$.

**Proof** Since $B_p(\varepsilon, \eta) = C_p(\varepsilon, \eta)$ by Lemma 6.2, the estimate (6.8) follows from (6.3). To prove (6.9), we first consider the case $d \geq 3$. Let $u \in W^{1,2}_0(\Omega_{\varepsilon, \eta})$ be a weak solution of $-\Delta u = f$ in $\Omega_{\varepsilon, \eta}$, where $f \in L^p(\Omega_{\varepsilon, \eta})$. By (6.3) and (6.5) we obtain
\[ \| \nabla u \|_{L^p(\Omega_{\varepsilon, \eta})} \leq C(\varepsilon \eta)^{-1} \left\{ \min(1, \varepsilon \eta^{-\frac{d}{2}}) \right\}^2 \| f \|_{L^p(\Omega_{\varepsilon, \eta})} \]
for any $p > 2$. By Riesz–Thorin Theorem, this, together with the $L^2$ estimate,
\[ \| \nabla u \|_{L^2(\Omega_{\varepsilon, \eta})} \leq C \min(1, \varepsilon \eta^{-\frac{d}{2}}) \| f \|_{L^2(\Omega_{\varepsilon, \eta})}, \]
yields
\[ \| \nabla u \|_{L^p(\Omega_{\varepsilon, \eta})} \leq C \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-1}, \eta^{-\frac{d}{2}}) \right\}^{1+t} \| f \|_{L^p(\Omega_{\varepsilon, \eta})}, \]
where
\[ t = \frac{1}{2} - \frac{1}{p} \]
and $2 < p < p_0$. Since $t \to 1 - \frac{2}{p}$ as $p_0 \to \infty$, by choosing $p_0$ sufficiently large, we obtain
\[ \| \nabla u \|_{L^p(\Omega_{\varepsilon, \eta})} \leq C_\delta \varepsilon \eta \left\{ \min((\varepsilon \eta)^{-1}, \eta^{-\frac{d}{2}}) \right\}^{2-\frac{2}{p}+\delta} \| f \|_{L^p(\Omega_{\varepsilon, \eta})}, \]
for any $\delta \in (0, 1)$. The proof for the case $d = 2$ is similar (the factor $\ln(\eta/2)$ is absorbed by $\eta^{-\delta}$). \hfill \square

**Theorem 6.6** Let $A_p(\varepsilon, \eta)$ be defined by (4.1), but with $\Omega_{\varepsilon, \eta}$ in the place of $\omega_{\varepsilon, \eta}$. Then
\[ A_p(\varepsilon, \eta) \leq C_\delta \left\{ \min((\varepsilon \eta)^{-2}, \eta^{-d}) \right\}^{\frac{1}{2} - \frac{1}{p} + \delta} \quad (6.10) \]
for $d \geq 2$ and $1 < p < \infty$, where $C_\delta$ depends on $d$, $p$, $\delta$, $\Omega$ and $\{T_k\}$.

**Proof** Consider the case $d \geq 3$ and $2 < p < \infty$. Let $u \in W^{1,2}_0(\Omega_{\varepsilon, \eta})$ be a weak solution of $-\Delta u = \text{div}(f)$ in $\Omega_{\varepsilon, \eta}$, where $f \in L^p(\Omega_{\varepsilon, \eta}; \mathbb{R}^d)$. It follows from (6.5) and (6.3) that
\[ \| \nabla u \|_{L^p(\Omega_{\varepsilon, \eta})} \leq C(\varepsilon \eta)^{-1} \min(1, \varepsilon \eta^{-\frac{d}{2}}) \| f \|_{L^p(\Omega_{\varepsilon, \eta})}. \]
By interpolation, this, together with the $L^2$ estimate $\| \nabla u \|_{L^2(\Omega_{\varepsilon, \eta})} \leq \| f \|_{L^2(\Omega_{\varepsilon, \eta})}$, yields
\[ \| \nabla u \|_{L^p(\Omega_{\varepsilon, \eta})} \leq C \left\{ \min((\varepsilon \eta)^{-1}, \eta^{-\frac{d}{2}}) \right\}^t \| f \|_{L^p(\Omega_{\varepsilon, \eta})}, \]
where $t$ is the same as in the proof of Theorem 6.5 and $2 < p < p_0$. Since $t \to 1 - \frac{2}{p}$ as $p_0 \to \infty$, we obtain (6.10) for $d \geq 3$. The proof for the case $d = 2$ is similar. \hfill \square

**Proof of Theorem 1.2** This follows from Lemma 6.1 with estimates for $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, and $D_p(\varepsilon, \eta)$ given by Theorems 6.6, 6.5 and 6.3, respectively. The estimate for $C_p(\varepsilon, \eta)$ follows from Theorem 6.5 and the fact $C_p(\varepsilon, \eta) = B_p(\varepsilon, \eta)$. \hfill \square
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