Nonholonomic Deformations of Disk Solutions and Algebroid Symmetries in Einstein and Extra Dimension Gravity

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April 20, 2005

Abstract

In this article we consider nonholonomic deformations of disk solutions in general relativity to generic off–diagonal metrics defining knew classes of exact solutions in 4D and 5D gravity. These solutions possess Lie algebroid symmetries and local anisotropy and define certain generalizations of manifolds with Killing and/or Lie algebra symmetries. For Lie algebroids, there are structures functions depending on variables on a base submanifold and it is possible to work with singular structures defined by the 'anchor' map. This results in a number of new physical implications comparing with the usual manifolds possessing Lie algebra symmetries defined by structure constants. The spacetimes investigated here have two physically distinct properties: First, they can give rise to disk type configurations with angular/time/extra dimension gravitational polarizations and running constants. Second, they define static, stationary or moving disks in nontrivial solitonic backgrounds, with possible warped factors, additional spinor and/or noncommutative symmetries. Such metrics may have nontrivial limits to 4D gravity with vanishing, or nonzero torsion. The work develops the results of Ref. [1] and emphasizes the solutions with Lie algebroid symmetries following similar constructions for solutions with noncommutative symmetries [2].

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1 Introduction

Disk solutions have been studied intensively in the past for certain important reasons: In astrophysics, disk configurations were discussed as models of certain type galaxies and/or for accretion disks, see e.g. [3]. The Newtonian dust disks are known to be unstable and the same holds in the relativistic case. This was used for a certain numerical work when disk type solutions could be taken as exact initial data for numerical collapse calculations [4]. In the relativistic case, the static disks solutions where obtained as consisting of two counter-rotating streams of matter with vanishing total angular momentum [5]. Infinitely extended dust disks with finite mass were studied in the static case and stationary cases [6]. In analytic form, the first exact solution for a finite stationary dust disk was constructed and analyzed in Refs. [7]. The solution was given for the rigidly rotating dust disk when the corresponding boundary value problem for the Einstein equations was solved with the help of a co–rotating coordinate system. Then, the approach was extended to electro–vacuum solutions in Ref. [8]. A comprehensive review of black hole – disk systems and astrophysical disks is presented in Ref. [9].

The main difficulties for constructing disk solutions in arbitrary backgrounds (in general, anisotropically polarized and with extra dimension dependence and different type of symmetries) is related to technical analytical and numerical problems of integrating the gravitational and matter field equations. Our goal in the present paper is to develop a geometric method of generating such solutions depending on two, three or four variables and to investigate the properties of a such new class of spacetimes. We are motivated to do so because any realistic approach to modern quantum gravity, low–energy string limit and extra dimension physics and phenomenology are related to nonlinear gravitational configurations, nonholonomic frames and generalized symmetries. An alternative motivation is also the study of disk
configurations as possible tests of extra dimensions, torsion fields and topologically nontrivial objects and local anisotropies induced by off–diagonal interactions in the Einstein or string/brane gravity and possible implications in modern cosmology and astrophysics.

In this paper, we adopt our previous strategy for constructing exact solutions by using nonholonomic deformations of certain well known exact solutions formally defined by diagonalizable (by coordinate transforms) metric structures [1, 10, 11, 12, 13, 2]. We consider both the geometry of nonholonomic frames (vielbeins) with associated nonlinear connection (N–connection) structure and the manifolds with metrics and connections possessing generalized Lie algebroid symmetries. Although the gravitational algebroid disk symmetries are discussed only in brief in this paper, they emphasize and conclude the idea of generalized symmetries for 'off–diagonal spacetimes; see details in Refs. [14], on Lie algebroid mathematics and recent applications to strings and field theory and geometric mechanics, and Refs. [15] on Clifford algebroids and exact solutions with Lie algebroid symmetry.

We emphasize that the anholonomic frame method applied in this work is a general one allowing to construct new types of exact solutions of the Einstein (vacuum and nonvacuum ones) and matter field equations reduced by generic off–diagonal ansatz to systems of nonlinear partial differential equations. Surprisingly, this rigorous geometric technique allows to integrate the equations in very general form. Such exact solutions depend on classes of integration functions on one, two and three variables which is very different from the usual ansatz with spherical (cylindrical and/or Lie algebra symmetry) reducing the field equations to algebraic systems or to certain types of nonlinear differential equations when their general solutions depend on integration constants.

The gravitational solutions depending on integration constants, parametrized by 'diagonalizable’ (by coordinate transforms) metrics and subjected to the limit conditions to result in asymptotic Minkowski, or constant curvature spacetimes, are largely applied in modern astrophysics and cosmology (for instance, the are related to black hole physics and inflational scenarios). The most important classes of such solutions are summarized in Ref. [16] and have been generalized for a number of modern string/brane gravitational theories.

Nevertheless, some already elaborated geometric, analytic and numerical methods give rise to more ‘sophisticate’ classes of solutions for vacuum and non–vacuum configurations. They possess new symmetries and depend on classes of integration functions and may describe a new ‘non-perturbative’ and ‘nonlinear’ gravitational physics. It is an important and difficult task to investigate the physical implications of such solutions. For instance, in
Ref. [2], we presented a detailed study of such solutions possessing non-commutative symmetries even they are constructed in the framework of the 'commutative' Einstein gravity and its 'commutative' generalizations. Than, it was understood that the generalized off–diagonal solutions can be also described and classified by Lie algebroid symmetries [15]. In this case, one deals with structure functions instead of Lie algebra constants, one may consider singular maps and unify certain concepts of bundle spaces and nonholonomic manifolds, i.e. manifolds provided with nonintegrable distributions defined by nonholonomic frames and their deformations. The general aim of this paper, is to investigate in detail some classes of such solutions generated by nonholonomic deformations of disk solutions.

It is suggested that the reader will study in advance the Appendix containing the main geometric results on the "anholonomic frame method" before he will consider the content of sections 2–6. We consider necessary and useful to give a number of technical details on nonholonomic differential calculus and proofs related to this method still less known for researches working in gravity and string theories and their applications in astrophysics and cosmology.

We organize our presentation as follows:

The section 2 outlines the main results on disk solutions in general relativity and their conformal transforms and nonholonomic deformations to exact off–diagonal solutions in 4D gravity. There are analyzed two vacuum disk configurations with angular anisotropy: disk solutions with induced torsion and anisotropic disks in general relativity. Other class of solutions is constructed in order to investigate possible disk anisotropies induced by a cosmological constant.

The section 3 is devoted to nonholonomic disk configurations in bosonic string gravity. There are considered two types of torsion fields, the nontrivial string torsion ($H$–field) and a nonholonomically induced torsion, when the dilaton vacuum is constant and the 5D $H$–field ansatz is self–consistently related to a nonholonomic gravitational background in such a manner that their contributions can be approximated by an effective cosmological constant acting as a source of local anisotropy. We analyze four classes of disk like solutions with $H$–field contributions and nontrivial gravitational 3D solitonic backgrounds: 1) string–solitonic disks with angular anisotropy; 2) (time) moving string–solitonic disc configurations; 3) disks with anisotropic polarizations on extra dimension; 4) extra dimension solitonically propagating disks.

The section 4 contains a study of the Dirac equation on nonholonomic gravitational disk configurations. There were constructed two classes of such solutions of the Einstein–Dirac equations defining nonlinear superpositions for gravitational disk – 3D packages of Dirac–waves: 1) metrics with angular
anisotropy and 2) metrics with time moving nonholonomic spinor waves.

The section 5 presents the main ideas and formulas for gravitational Lie algebroids provided with N–connection structure and illustrate how metrics with Lie algebroid symmetry can be defined following the anholonomic frame method. It is shown that the constructions can be adapted to nonholonomic configurations.

The section 6 contains a detailed discussion, motivation of the applied geometric methods and conclusion of results.

2 Nonholonomic Deformations of Disk Solutions

In this section we consider nonholonomic deformations of the disk solutions in general relativity to four and five dimensional (in brief, 4D and 5D) off–diagonal metric ansatz\(^1\) defining also exact solutions in gravity theories, in general, with nontrivial torsion. We analyze two classes of such spacetimes and state the conditions when the generic off–diagonal metrics define exact solutions in vacuum Einstein gravity and/or in presence of sources of local anisotropy defined by a nontrivial cosmological constant.

2.1 Disk solutions in general relativity and their conformal and anholonomic transforms

In order to construct 4D disk type solutions one usually uses a metric ansatz describing the exterior (i.e. the vacuum region) of an axisymmetric, stationary rotating body, written in the Weyl–Lewis–Papapetrou form \[^{16}\],

\[
d s^2_{[\alpha \beta]} = -e^{2U}(dt + ad\phi)^2 + e^{2(k-U)}(d\hat{\rho}^2 + d\hat{\zeta}^2) + e^{-2U}\rho^2 d\phi^2,
\]

where \(\partial_t\) and \(\partial_\phi\) are two commuting asymptotically timelike respectively spacelike Killing vectors and the functions \(U(\hat{\rho}, \hat{\zeta})\), \(a(\hat{\rho}, \hat{\zeta})\) and \(k(\hat{\rho}, \hat{\zeta})\) are taken in a form when the metric defines a solution of the Einstein equations. For a such ansatz, the vacuum field equations are equivalent to the Ernst equation for the potential \(q(z, \bar{z}),\)

\[
q_{zz} + \frac{1}{2(z + \bar{z})}(q_z + q_{\bar{z}}) - \frac{2}{q + \bar{q}}q_zq_{\bar{z}} = 0,
\]

\(^1\)Such metrics can not be diagonalized by coordinate transforms but can be written in effective diagonal forms with respect to some nonholonomic local bases, equivalently, frames, or vielbeins.
where the (complex) Ernst potential is defined by $q = e^{2U} + ib$, $\bar{q}$ is the complex conjugated coordinate and the real function $b$ is related to coefficients in the metric via $b_z = -(i/\hat{\rho})e^{4U}a_z$. The complex variable is defined by $z = \hat{\rho} + i\hat{\zeta}$. \footnote{We note that some our denotations defer from, for instance, those in [7].}

Haven chosen a solution of the Ernst equation, the metric function $U$ is obtained directly from the definition of the Ernst potential where $a$ and $k$ can be constructed from $q$ via quadratures. One uses units where Newton’s gravitational constant $G$ as well as the velocity of light $c$ are equal to 1. The metric functions $e^{2U}$, $e^{2k}$, and $a/\rho_0$ depend uniquely on the normalized coordinates $\hat{\rho}/\rho_0$, $\hat{\zeta}/\rho_0$, and the parameter $\mu = 2\Omega^2\rho_0^2e^{-2V_0}$, where $\Omega$, $\rho_0$ and $V_0$ are respectively the angular velocity, the coordinate radius, and the ‘surface potential’ $V_0(\mu) \equiv U(\hat{\rho} = 0, \hat{\zeta} = 0, \mu)$, respectively. The disk is characterized by the values of coordinates $\hat{\zeta} = 0$ and $0 \leq \hat{\rho} \leq \rho_0$.

The equation (2) is completely integrable. In principle, this also allows to get explicit solutions for the boundary value problems. For infinitesimally thin disks, i.e. for two-dimensionally extended matter sources, one generates global solutions defined by ordinary differential equations in the matter region which state boundary data for the vacuum equations (see details in Refs [17]). This way one constructs solutions to the Ernst equation with free functions which have to be determined by the boundary data. A such solution is unique in the case of Cauchy data and, in general, is related to singular solutions of the elliptic field equations, we refer the reader to Refs. [7] for explicit constructions and discussions.

In this paper, we consider that the functions $U(\hat{\rho}, \hat{\zeta}), a(\hat{\rho}, \hat{\zeta})$ and $k(\hat{\rho}, \hat{\zeta})$ for the metric coefficients in (1) are known from a 4D disk solution defined in explicit form for some boundary data. Our purpose is to deform this metric to certain 4D and 5D generic off–diagonal configurations defining other classes of exact solutions. For such constructions, it is convenient to consider two auxiliary metrics (in general, they do not define any solutions of the Einstein equations) which are conformally equivalent to the mentioned disk solution:

\begin{align*}
\text{cd}[1] & \quad ds^2_{[\text{cd}1]} = e^{2(U-k)}ds^2_{[\text{cd}]1} \\
& = \rho_0^2 d\rho^2 + \rho_0^2 d\zeta^2 + \rho_0^2 \rho^2 e^{-2k} d\varphi^2 - e^{2U-k}(d\tau + a d\zeta)^2 \quad (3)
\end{align*}

and

\begin{align*}
\text{cd}[2] & \quad ds^2_{[\text{cd}2]} = \rho_0^{-2} e^{2k} ds^2_{[\text{cd}1]} \\
& = \rho^{-2} e^{2k} d\rho^2 + \rho^{-2} e^{2k} d\zeta^2 + d\varphi^2 - \rho^{-2} e^{4U}(d\tau + a d\zeta)^2 \quad (4)
\end{align*}
where there are introduced the 'normalized' coordinates \( \rho = \hat{\rho}/\rho_0 \) and \( \zeta = \hat{\zeta}/\zeta_0 \) and, for simplicity, we consider that \( \rho \neq 0 \) for the metric (4) (we have to change the system of coordinates in order analyze the second metric in the vicinity of \( \rho = 0 \)). We also note that we introduced a new time like coordinate

\[
t \to \tau = t + \int \omega^{-1}(\zeta, \varphi) \ d\xi(\zeta, \varphi)
\]

for which

\[
d\tau + a \ d\zeta = dt + a \ d\varphi,
\]

i.e. \( d(\tau-t) = \omega^{-1}d\xi \). This is possible, for instance, if we chose the integrating function \( \omega = a^{-1}e^{\xi-\varphi} \) and put \( \xi = -e^{\xi-\varphi} \).

The auxiliary metrics (3) and (4) can be distinguished from an off-diagonal metric form (with a trivial extension to the 5th coordinate),

\[
ds^2_{\text{[cd]}} = \epsilon(dx^1)^2 + g_{\alpha}(x^i) \left(dx^\alpha \right)^2 + h_{\alpha}(x^i) (e^a)^2,
\]

\[
\epsilon^a = dy^a + N^a_i(x^i) dx^i,
\]

with \( \hat{i}, \hat{k}, \ldots = 2, 3; a, b, \ldots = 4, 5 \) and \( \epsilon = \pm 1 \) (the sign depends on chosen signature for the dimension labelled by the coordinate \( x^1 \)). The data for the coordinates and nontrivial values of the metric and local frame (vielbein) coefficients in (4) are parametrized

\[
x^1 = \chi, x^2 = \rho, x^3 = \zeta, y^4 = v = \varphi, y^5 = \tau,
g_1 = \epsilon, g_2 = \rho_0^2, g_3 = \rho_0^2, h_4 = \rho_0^2 e^{-2k}, h_5 = -e^{2(2U-k)}, N^5 = a, \quad (5)
\]

or

\[
x^1 = \chi, x^2 = \rho, x^3 = \zeta, y^4 = v = \tau, y^5 = \varphi,
g_1 = \epsilon, g_2 = \rho_0^2, g_3 = \rho_0^2, h_4 = -e^{2(2U-k)}, h_5 = \rho_0^2 e^{-2k}, N^4 = a, \quad (6)
\]

In a similar form, we can define the trivial 5D extensions of (4) by considering that

\[
x^1 = \varphi, x^2 = \rho, x^3 = \zeta, y^4 = v = \chi, y^5 = \tau,
g_1 = 1, g_2 = \rho^{-2}e^{2k}, g_3 = \rho^{-2}e^{2k}, h_4 = \epsilon, h_5 = -\rho^{-2} e^{4U}, N^5 = a, \quad (7)
\]

or

\[
x^1 = \varphi, x^2 = \rho, x^3 = \zeta, y^4 = v = \tau, y^5 = \chi,
g_1 = 1, g_2 = \rho^{-2}e^{2k}, g_3 = \rho^{-2}e^{2k}, h_4 = -\rho^{-2} e^{4U}, h_5 = \epsilon, N^4 = a. \quad (8)
\]
In the above presented data, the coordinate $\chi$ is considered to be the extra dimensional one.

By introducing $g_i = (g_1, g_\hat{i})$, when indices of type $i, j, k, \ldots$ run values 1, 2, 3, we define a 5D metric ansatz

$$ds^2 = g_k(x^i)(dx^k)^2 + h_a(x^i)(e^a)^2,$$  \hspace{1cm} (9)

$$e^a = dy^a + N^a_i(x^k)dx^i,$$

which a particular case of the metric \[113\] considered in Appendix. We nonholonomically transform (deform) the coefficients of \[9\],

$$g_k(x^i) \rightarrow g_k(x^\hat{i}) = \eta_k(x^\hat{i})g_k(x^\hat{i}),$$  \hspace{1cm} (10)

$$h_a(x^\hat{i}) \rightarrow h_a(x^i, v) = \eta_a(x^i, v)h_a(x^\hat{i}),$$

$$N^a_i(x^\hat{i}) \rightarrow N^a_i(x^k, v),$$

with $\eta_1 = 1$, to a general 5D off–diagonal metric ansatz

$$ds^2_{[5D]} = g_k(x^\hat{i})(dx^k)^2 + h_a(x^i, v)(e^a)^2,$$  \hspace{1cm} (11)

$$e^a = dy^a + N^a_i(x^k, v)dx^i,$$

with the coefficients not depending on variable $y^5$ but with respective dependencies on $x^i$ (the isotropic, or holonomic variables) and on $v$ (the anisotropic, or anholonomic variable). We conventionally introduced the ”polarization” functions $\eta_\alpha = (\eta_k, \eta_a)$ and the N–connection coefficients $N^\alpha_i$ (see formulas \[81\], \[82\] and \[83\]) which may have corresponding limits $\eta_\alpha \rightarrow 1$ and $N^\alpha_i \rightarrow N^\alpha_i$ related to a conformal transform and extra dimension extension of the disk solution \[11\] to an auxiliar metric \[3\], with parametrizations \[5\] or \[6\], or to \[4\], with parametrizations \[7\] or \[8\].

The values $\eta_\alpha$ and $N^\alpha_i$ will be found as certain general classes of functions solving the gravitational field equations for some vacuum or nonvacuum models in 4D or 5D gravity. This way we shall define nonholonomic deformations of the disk solutions in general relativity to new classes of exact solutions in various types of gravity theories (in general, with nontrivial torsion and matter field sources and extra dimensions).

### 2.2 A metric ansatz with angular anisotropy

The idea to deform nonholonomically the Neugebauer–Meinel disk solution \[7\] to a certain generic off–diagonal vacuum or non–vacuum gravitational anisotropic (on variable $y^4 = v = \varphi$) configuration was proposed in Ref. \[1\].
That approach was valid for small deformations on \( v \) of the Ernst potential and corresponding hyperelliptic functions. In this subsection, we analyze two classes of such locally anisotropic solutions in 4D which hold for any type of nonholonomic deformations (not only for small ones), see the related geometric formalism outlined in Appendix E.

We nonholonomically deform the ansatz (3), defined by data (5) reduced to 4D, i.e., without coordinate \( x^1 \), to a 4D generic off–diagonal metric

\[
\delta s^2_{4D,\varphi} = \rho^2_0 \mathcal{O}(\rho, \zeta) d\rho^2 + \rho^2_0 \mathcal{O}(\rho, \zeta) d\zeta^2 + \rho_0^2 \rho^2 e^{-\frac{2}{\tilde{k}(\rho, \zeta, \varphi)}} \delta \varphi^2 - e^{2[\tilde{U}(\rho, \zeta, \varphi) - \tilde{k}(\rho, \zeta, \varphi)]} \delta \tau^2,
\]

\[
\delta \varphi = d\varphi + w_2(\rho, \zeta, \varphi) d\rho + w_3(\rho, \zeta, \varphi) d\zeta,
\]

\[
\delta \tau = d\tau + n_2(\rho, \zeta, \varphi) d\rho + n_3(\rho, \zeta, \varphi) d\zeta,
\]

where it is emphasized the dependence on the nonholonomic (anisotropic) coordinate \( \varphi \). The polarizations functions and N–coefficients (10) are parametrized in this form:

\[
\eta_2(\rho, \zeta) = \mathcal{O}(\rho, \zeta), \quad \eta_3(\rho, \zeta) = \mathcal{O}(\rho, \zeta), \quad \eta_4(\rho, \zeta, \varphi) = e^{-2[\tilde{k}(\rho, \zeta, \varphi) - \tilde{k}(\rho, \zeta)]},
\]

\[
\eta_5(\rho, \zeta, \varphi) = e^{2[\tilde{U}(\rho, \zeta, \varphi) - \tilde{k}(\rho, \zeta, \varphi)] - 2[\tilde{U}(\rho, \zeta) - \tilde{k}(\rho, \zeta)]},
\]

\[
N^4_{2,3}(\rho, \zeta, \varphi) = w_{2,3}(\rho, \zeta, \varphi), \quad N^5_{2,3}(\rho, \zeta, \varphi) = n_{2,3}(\rho, \zeta, \varphi).
\]

For some classes of solutions, we may consider the limits

\[
\eta \to 1, \eta_4 \to 1, \eta_5 \to 1, w_{2,3} \to 0, n_2 \to 0, N^5_3 = n_3(\rho, \zeta, \varphi) \to N^5_3 = a(\rho, \zeta)
\]

for \( \tilde{k} \to k \) and \( \tilde{U} \to U \) which allows us to use the same boundary conditions as for the non–deformed metrics and a very similar physical interpretation like in the usual 'locally isotropic' disk solutions. But it is not possible to satisfy all such limits and to preserve the condition that all metrics related by such nonholonomic deformations are just solutions of the gravitational field equations. The rigorous procedure is to find certain values \( \eta_a \) and \( N^a_i \) for which certain vacuum, or non–vacuum, solutions are defined and then to analyze which limits of type (13) are possible. In general, two exact solutions may be not related by smooth infinitesimal limits to a fixed type one but they could be related by certain nontrivial conformal and nonholonomic frame transforms. Such metrics may also possess different generalized symmetries, for instance, they may possess Lie algebroid symmetries, see a detailed discussion in Section 5.

Our aim is to find such functions

\[
\tilde{U}(\rho, \zeta, \varphi), \tilde{k}(\rho, \zeta, \varphi), w_{2,3}(\rho, \zeta, \varphi), n_{2,3}(\rho, \zeta, \varphi)
\]
when gravitational polarizations define a class of exact solutions of the 4D Einstein equations for the canonical d-connection with zero or nonzero cosmological constant $\Upsilon_2, \Upsilon_4 = \Upsilon_0 = \text{const}$, see the method presented in Appendix. We shall also investigate the conditions for the metric coefficients when the ansatz defines vacuum solutions for the Levi–Civita connection, i.e. we shall construct generic off-diagonal solutions with anisotropic dependence on angular variable $\varphi$ in general relativity.

2.2.1 4D vacuum angular anisotropic disk configurations

For vacuum configurations, the function $\varpi(\rho, \zeta)$ from can be defined for any 2D coordinate transforms $\bar{x}^{2,3}(\rho, \zeta)$ when

$$\rho_0^2 \varpi(\rho, \zeta) d\rho^2 + \rho_0^2 \varpi(\rho, \zeta) d\zeta^2 = \varpi(\rho, \zeta) \left[ (d\bar{x}^2) + (d\bar{x}^3)^2 \right]$$

and $\psi = \ln |\varpi|$ is any solution of

$$\psi'' + \psi''' = 0$$

for $\psi' = \partial \psi / \partial x^2$ and $\psi' = \partial \psi / \partial x^3$, see formulas (140) and (141) in Appendix. In a particular vacuum case, we can consider $\varpi = 1$ and $\bar{x}^2 = \rho, \bar{x}^3 = \zeta$.

Anisotropic disks with induced torsion: The functions $\eta_4(\rho, \zeta, \varphi)$ and $\eta_5(\rho, \zeta, \varphi)$, i.e. $h_4 = \eta_4 h_4$ and $h_5 = \eta_5 h_5$, for the values $h_4$ and $h_5$, defined by the disk solution and the conformally transformed metric, are related by formulas (142), or (143), solving the equation (136), see Theorem D.1 in Appendix. For $h_5^* = \partial h_5 / \partial \varphi \neq 0$, we can satisfy the condition $\sqrt{|h_4|} = h_4^{(0)}(\sqrt{|h_5|})^*$, by taking

$$h_4 = \rho_0^2 \rho e^{-2k(\rho, \zeta, \varphi)} \quad \text{and} \quad h_5 = -\tilde{k}^2 e^{2(2U - \tilde{k})}. \quad (14)$$

where

$$h_4^{(0)}(\rho, \zeta) = \rho_0 \rho e^{U(\rho, \zeta) - 2k(\rho, \zeta)} \quad (15)$$

for some functions $\tilde{U}(\rho, \zeta, \varphi)$ and $\tilde{k}(\rho, \zeta, \varphi)$ subjected to the relation $\tilde{k} = e^{2(U - \tilde{U})}$.

For a vacuum configuration, we can chose a value of the integration function $h_4^{(0)}(\rho, \zeta)$ when $w_{2,3} = 0$, see the discussion after formula (146), but this would define more cumbersome formulas for $h_4$ and $h_5$. It is convenient to
consider non–vanishing N–connection coefficients $w_{2,3}$ defined by differentiating on coordinates a functional $s$ on $\tilde{k}, U$ and $\rho_0, \rho$,

$$s(\tilde{k}, U, \rho_0, \rho) = \ln |\sqrt{h_4 h_5}/h_5^*| = \ln |\rho_0 \rho e^{\tilde{k} - 2U}/(\tilde{k} - 1) \tilde{k}^*|$$

computed by using data (14). In result, we get

$$w_2 = (s^*)^{-1} \frac{\partial s}{\partial \rho} \quad \text{and} \quad w_3 = (s^*)^{-1} \frac{\partial s}{\partial \zeta}, \quad (16)$$

which solve the conditions (146). We have to impose the limits $w_{2,3} \to 0$ stating certain limits for $s$ and $\tilde{k}$ for large values of $\rho$ and/or $\zeta$, in order to consider that the locally anisotropic solutions will result in the usual disk metric (1) far away from the locally anisotropic disk configuration.

The rest of N–connection coefficients $n_{2,3}$ are computed by introducing (14) into the first formula (147)

$$n_{2,3} = n_{2,3[1]} (\rho, \zeta) + n_{2,3[2]} (\rho, \zeta) \int \tilde{k}^{-3} e^{\tilde{k}} d\varphi \quad (17)$$

where we introduced all possible dependencies on $\rho$ and $\zeta$ from $h_{4,5}$ in $n_{2,3[2]}$. The functions $n_{2,3[1,2]} (\rho, \zeta)$ have to be defined from certain boundary/limit conditions for a class of solutions. We can state, in a more particular case, that $n_{2[1]}, n_{2[2]} = 0$ and write

$$n_3 = a (\rho, \zeta) \left[ 1 + n_{3[2]} (\rho, \zeta) \int \tilde{k}^{-3} e^{\tilde{k}} d\varphi \right] \to a (\rho, \zeta) \quad (18)$$

for large values of $\rho$ and/or $\zeta$ if we are analyzing some small ”anisotropic” corrections being proportional to some small values $n_{3[2]}$.

Introducing the coefficients (14)–(17) into (12), we get the off–diagonal metric

$$\delta s_{[1D, \varphi]}^2 = \rho_0^2 d\rho^2 + \rho_0^2 d\zeta^2 + \rho_0^2 d\varphi^2 - \tilde{k}^2 (\rho, \zeta, \varphi) e^{2[U(\rho, \zeta) - \tilde{k}(\rho, \zeta, \varphi)]} \delta \tau^2,$$

$$\delta \tau = d\tau + a (\rho, \zeta) \left[ 1 + n_{3[2]} (\rho, \zeta) \int \tilde{k}^{-3} e^{\tilde{k}} d\varphi \right] d\zeta,$$

$$\delta \varphi = d\varphi + \frac{1}{s^*} \left( \frac{\partial s}{\partial \rho} d\rho + \frac{\partial s}{\partial \zeta} d\zeta \right), \quad \text{for}$$

$$s = \ln |\rho_0 \rho e^{\tilde{k} - 2U}/(\tilde{k} - 1) \tilde{k}^*|,$$

12
where, for simplicity, there are considered necessary smooth class coefficients. This ansatz defines a class of vacuum 4D solutions for the Einstein equations with the Ricci tensor computed for the canonical d–connection. The metric coefficients depend on an arbitrary function $\tilde{k}(\rho, \zeta, \phi)$, on integration functions $n_{3[1]}(\rho, \zeta)$ and $n_{3[2]}(\rho, \zeta)$ and on function $U(\rho, \zeta)$ defined by the former Neugebauer–Meinel disk solution. We note that the metric also depends on two another locally isotropic disk functions $a(\rho, \zeta)$ and $k(\rho, \zeta)$ as it is stated by the boundary conditions for the integration functions into the formulas. In the next sections, we shall use certain ansatz generalizing in order to investigate certain locally anisotropic disk solutions in string gravity with nontrivial torsion. We note that such metrics may be used for various calculus concerning physical effects with nonholonomic frames and tests of general relativity.

**Anisotropic disks in general relativity:** We can impose certain constraints on the coefficients of the ansatz in order to generate a class of exact locally anisotropic disk solutions just for the vacuum Einstein gravity. In this case we must satisfy the conditions, see Appendix, stating that the nontrivial coefficients of the Ricci tensor for the Levi–Civita connection are equal to the corresponding coefficients of the Ricci tensor for the canonical d–connection. We restrict the ansatz for such coefficients when $h_5^* = 0$ and $w_k^* = 0$ but $n_k^* \neq 0$. This is possible if we put $w_k = 0$, see the discussion of the formula in Appendix, and consider any functions $\tilde{U}$ and $\tilde{k}$ subjected to the condition

$$2\tilde{U}(\rho, \zeta, \phi) - \tilde{k}(\rho, \zeta, \phi) = \tilde{k}_0(\rho, \zeta)$$

which result in $h_5^* = 0$. In this case, any values of type

$$h_4(\rho, \zeta, \phi) = \rho^2 e^{-2\tilde{k}(\rho, \zeta, \phi)}$$

solve the equation. The nontrivial N–coefficients $n_k$ have to be computed by the second formula in

$$n_{2,3}(\rho, \zeta, \phi) = n_{2,3[1]}(\rho, \zeta) + n_{2,3[2]}(\rho, \zeta) \int e^{-2\tilde{k}(\rho, \zeta, \phi)} d\phi,$$

where we have introduced into $n_k[2](\rho, \zeta)$ the factor $\rho^2$. We have to constrain the functions $n_k[1,2](\rho, \zeta)$ and $\tilde{k}(\rho, \zeta, \phi)$ in a such way as to satisfy which, for $w_k = 0$, holds for any $n_2$ and $n_3$ subjected to the condition

$$n_2' - n_3^* = 0.$$
If we take
\[ n_3 = a(\rho, \zeta) \left[ 1 + n_{3[2]}(\rho, \zeta) \int e^{-2\tilde{k}(\rho, \zeta, \varphi)} d\varphi \right] \rightarrow a(\rho, \zeta) \] (23)
for small values \( n_{3[2]} \), we have to consider some small coefficients \( n_{2[1,2]}(\rho, \zeta) \) defined in a such way that \( n'_3 = n^*_3 \). It is also possible to state that for large values of \( \rho \) and/or \( \zeta \) one has

\[ 2\tilde{U}(\rho, \zeta, \varphi) - \tilde{k}(\rho, \zeta, \varphi) \rightarrow 2U(\rho, \zeta) - k(\rho, \zeta) = \tilde{k}_0(\rho, \zeta). \]

Introducing the values (20)–(23) in (12), we get a vacuum 4D Einstein metric

\[
\delta s^2_{4D,\varphi_E} = \rho^2 d\rho^2 + \rho^2 d\zeta^2 + \rho_0^2 e^{-2k(\rho, \zeta, \varphi)} d\varphi^2 - e^{2\tilde{k}_0(\rho, \zeta)} \delta \tau^2,
\]

\[
\delta \tau = d\tau + \left[ n_{2[1]}(\rho, \zeta) + n_{2[2]}(\rho, \zeta) \int e^{-2\tilde{k}(\rho, \zeta, \varphi)} d\varphi \right] d\rho
\]

\[
+ a(\rho, \zeta) \left[ 1 + n_{3[2]}(\rho, \zeta) \int e^{-2\tilde{k}(\rho, \zeta, \varphi)} d\varphi \right] d\zeta,
\]

for any set of integration functions satisfying the limit conditions to the metric (3). We note that even we may consider some small values \( n_{2[1]}(\rho, \zeta) \) and/or \( n_{2[2]}(\rho, \zeta) \) such functions can not be both zero because only (24) is a solution of the vacuum gravitational equations but not the conformally transformed metric (3). With respect to the nonholonomic frames (82) the new locally anisotropic disk metric has the coefficients very similar to the usual disk solution but with a certain dependence on functions polarized on \( \varphi \). For a more restricted class of solutions, we can search for any periodic polarizations of type

\[ \tilde{k}(\rho, \zeta, \varphi) \simeq k(\rho, \zeta) \cos \omega_0 \varphi \]

when \( \omega_0 \) has to be defined experimentally or computed from certain models of gravitational interactions with nonlinear self-polarization, quantum fluctuations or extra dimension corrections. An alternative way is to treat such solutions as usual disk solutions but self–consistently mapped into nontrivial off–diagonal background defined by a solitonic gravitational wave like we shall consider in details in the next sections.

2.2.2 Disk anisotropy induced by cosmological constants

Let us consider how the disk solution (11), by applying a chain of conformal (2) and static nonholonomic transforms, can be generalized to define exact
off–diagonal solutions of the Einstein equations with cosmological constant. We consider nontrivial sources $\Upsilon_2 = \Upsilon_4 = \lambda_0 = \text{const}$ \cite{156} for the 4D ansatz (without dependence on $x^1$) of the Einstein equations \cite{135}–\cite{138}, see Appendix. The existence and a form of solution is stated by the Corollary D.1, which is the 4D version of the metric \cite{148} for $x^2 = \rho, x^3 = \zeta, y^4 = \nu = \varphi$ and $y^5 = \tau$ and $\epsilon_{23} = 1, \epsilon_4 = 1$ and $\epsilon_5 = -1$.

The function $\varpi(\rho, \zeta)$ from \cite{25} can be defined as a solution of
\begin{equation}
(\ln |\varpi|)'' + (\ln |\varpi|)''' = 2\lambda_0, \tag{26}
\end{equation}
see \cite{144}. One computes
\begin{equation}
\varsigma_4 (\rho, \zeta, \varphi) = \varsigma_{4[0]} (\rho, \zeta) - \frac{\lambda_0}{8} h_0^2 (\rho, \zeta) \int f^* (\rho, \zeta, \varphi) \left[ f (\rho, \zeta, \varphi) - f_0 (\rho, \zeta) \right] d\varphi,
\end{equation}
and, in result, the N–connection coefficients $N^4_{2,3} = w_{2,3} (\rho, \zeta, \varphi)$ and $N^5_{2,3} = n_{2,3} (\rho, \zeta, \varphi)$ can be expressed
\begin{equation}
\text{w}_2 = - \left( \frac{\partial \varsigma_4}{\partial \varphi} \right)^{-1} \left( \frac{\partial \varsigma_4}{\partial \rho} \right) \text{ and } \text{w}_3 = - \left( \frac{\partial \varsigma_4}{\partial \varphi} \right)^{-1} \left( \frac{\partial \varsigma_4}{\partial \zeta} \right), \tag{27}
\end{equation}
and
\begin{equation}
n_{2,3} (\rho, \zeta, \varphi) = n_{2,3[1]} (\rho, \zeta) + n_{2,3[2]} (\rho, \zeta) \int \left[ f^* (\rho, \zeta, \varphi) \right]^2 \varsigma_4 (\rho, \zeta, \varphi) \left[ f (\rho, \zeta, \varphi) - f_0 (\rho, \zeta) \right] d\varphi. \tag{28}
\end{equation}

The class of solutions defined by a such ansatz \cite{24} depends on integration functions $h_0 (\rho, \zeta), f_0 (\rho, \zeta), \varsigma_{4[0]} (\rho, \zeta)$ and $n_{2,3[1,2]} (\rho, \zeta)$, on a solution $\varpi (\rho, \zeta)$ of \cite{26} and on a general function $f (\rho, \zeta, \varphi)$ with $f^* \neq 0$ describing a class of backgrounds to which the disk solution can be mapped self–consistently. The relation to the usual disk solution can be established by considering that
\begin{equation}
\begin{aligned}
\underline{h}_4 &= \rho_0^2 e^{2k} = |\varsigma_{4[0]} (\rho, \zeta)|, \text{ and } \underline{h}_5 = -e^{2(2U - k)} = - |f_0 (\rho, \zeta)|^2, \\
\underline{N}^5_5 &= a = n_{3[1]} (\rho, \zeta)
\end{aligned}
\end{equation}
like in \cite{51}. Such ‘$\varphi$–anisotropic’ solutions are with nontrivial polarizations $\eta_{2,3} = \varpi$ induced by the cosmological constant which modify the former values $g_2 = \rho_0^2, g_3 = \rho_0^2$. 

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One can be imposed certain constraints when the Ricci tensor for the Levi–Civita connection has the same coefficients as the Ricci tensor for the canonical d–connection (all such coefficients being computed with respect to N–adapted vierbeins, see (154) and (153)). For instance, for $h^5 = [f(\rho, \zeta, \varphi) - f_0(\rho, \zeta)]^2$, one holds the condition $h^*_5 \neq 0$. In this case, we must solve the equations (160) by any $n_k$ and $w_k$ when $n_k^* = 0$ and

\[ w^{**} + \frac{w^*_k h^*_5}{2 h_5} = 0. \]  

(29)

We can take $n_3 = n_{3[1]}(\rho, \zeta) = a(\rho, \zeta)$ for $n_{2,3[2]}(\rho, \zeta) = 0$ and any $n_2 = n_{2[1]}(\rho, \zeta)$ satisfying the condition $n^*_2 - n^*_3 = 0$. Such conditions can be solved, for instance, for

\[ f = f_0(\rho, \zeta) \Phi(\varphi) \]  

(30)

when we fix some of the integration functions to be related by

\[ |f_0 h_0|^2 \varsigma_{4[0]}(\rho, \zeta) = -4 \]  

(we may consider that this can be obtained in result of some 2D coordinate transforms). In this case, we get

\[ h_5 = -f_0^2(1 - \Phi)^2 \]  

and

\[ \varsigma_4(\rho, \zeta, \varphi) = \varsigma_{4[0]}(\rho, \zeta) \varsigma(\Phi) \]  

for a functional $\varsigma$ on $\Phi$,

\[ \varsigma\{\Phi\} = 1 - \lambda_0(1 - \Phi)^2. \]

Introducing such parametrizations in (27), we find

\[ w_2 = -\Phi \frac{\partial}{\partial \rho} \ln |\varsigma_{4[0]}| \]  

and

\[ w_3 = -\Phi \frac{\partial}{\partial \zeta} \ln |\varsigma_{4[0]}|, \]  

(31)

for $\Phi \doteq 1/(\ln |\varsigma\{\Phi\}|)^*$. This transform the equations (29) into the equation

\[ \tilde{\Phi}^{**} - \frac{1}{2} \tilde{\Phi}^*(1 - \Phi) = 0 \]  

(32)

for a function $\Phi(\varphi)$. We can approximate, for small values of $\lambda_0$, that

\[ \Phi(\varphi) \approx p_0 + p_3 \varphi^3 + .... \]

for some constants $p_0$ and $p_3$ related in a form to solve (32). Because the functions $w_{2,3}$ defined by (31), for $n_k^* = 0$, satisfy the conditions (161) from Appendix, we get the result that the locally anisotropic metric (25) can be
constrained to define a class of off–diagonal solutions, with anisotropy on $\phi$, in Einstein gravity with cosmological constant $\lambda_0$. The anisotropic terms in such metrics are induced by nontrivial values of $\lambda_0$. In a similar way, one can use other types of generation functions $f(\rho, \zeta, \phi)$ than the parametrization with explicit or non–explicit solutions of the constraints for the integration functions.

It should be emphasized that if the limits are satisfied, one transforms some classes of solutions (vacuum, or nonvacuum, ones, defined by nonholonomic configurations) into a metric which is conformally equivalent to the disk solution, see the metric, but the last one is not just a vacuum solution. We can obtain another type of vacuum solution (with Killing symmetry) after a conformal map. Such nonholonomic maps relate certain class of metrics with generalized Lie algebroid symmetry (see Section 5) to metrics with Killing symmetry. The limit case may connected, by certain conformal and nonlinear coordinate transforms, to a well known exact (disk) solution which allows to state the boundary conditions and compare corresponding physical properties and symmetries both for the deformed and undeformed metrics.

Finally, in this section, we conclude that is possible to ”map” a known disk solution, via superpositions of conformal and nonholonomic frame transforms, into a new class of exact solutions, conventionally called disks with locally anisotropic polarizations or disks in nonholonomic backgrounds.

3 Disks in String Gravity with Solitonic Backgrounds

Since the physical effects related to gravitational disk and matter field configurations may be used as tests of different type of theories of gravity, one may look for implication of generalized disk type solutions in string gravity with nontrivial solitonic backgrounds. In this section, we construct and analyze some classes of such solutions.

3.1 String gravity with nonholonomic $H$–fields

Let us show how the data for the ansatz can be nonholonomically deformed, see relations, to 5D generic off–diagonal metrics defining locally anisotropic disk like configurations in string gravity. For simplicity, we consider a bosonic string model with trivial constant dilaton fields and a so–called $H$–field torsion, $H_{\nu\lambda\rho}$; see, for instance, Refs. for the basic results in string gravity.
We start with an ansatz for the $H$–field, see (109) in Appendix,
\[
H_{\nu\lambda\rho} = \tilde{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho} = \lambda_H \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho},
\] (33)
where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_H = \text{const}$, which satisfies the field equations for $H_{\nu\lambda\rho}$, see (111). The ansatz (33) is not just for the $H$–field like in the former approaches, when the contributions of a such field with coefficients were defined with respect to a coordinate basis and ”summarized” into an effective cosmological constant. In our case, we consider a deformation of $\hat{H}_{\nu\lambda\rho}$ by a locally anisotropic background with nonholonomically induced distortion $\tilde{Z}_{\nu\lambda\rho}$ (112) defined by the d–torsions (36) for the canonical d–connection (94)\(^3\) to an effective $H_{\nu\lambda\rho}$, which can be approximated by a cosmological constant but only with respect to N–adapted frames (82) and (83). Re–defining the constructions with respect to coordinate frames, we get not just a cosmological constant but a more general source defined both by $\tilde{Z}_{\nu\lambda\rho}$ and $\hat{H}_{\nu\lambda\rho}$.

With respect to the N–adapted frames, the source in (112) is effectively diagonalized,
\[
\Upsilon_{\alpha}^{[H\beta]} = \{\frac{\lambda_H^2}{2}, \frac{\lambda_H^2}{4}, \frac{\lambda_H^2}{4}, \frac{\lambda_H^2}{4}, \frac{\lambda_H^2}{4}\},
\]
and the equations (133), (136) transform respectively into
\[
R_2^3 = R_3^3 = \frac{1}{2g_2g_3} \left[ \frac{(g_2^*)^2}{g_2} - \frac{(g_3^*)^2}{g_3} + \frac{g_2' g_3'}{g_2} + \frac{(g_2')^2}{g_2} - g_2'' \right] = -\frac{\lambda_H^2}{4},
\] (34)
\[
S_4^5 = S_5^5 = -\frac{1}{2h_3 h_5} \left[ h_5^* - h_5^* \left( \ln \sqrt{|h_4 h_5|} \right)^* \right] = -\frac{\lambda_H^2}{4},
\] (35)
when the off–diagonal terms $w_i$ and $n_i$ are defined correspondingly by equations (149) and (150). The solution of (34) can be found as for (26), see also (141) in Appendix, when $\psi = \ln |g_2| = \ln |g_3|$ is a solution of
\[
\epsilon_2 \psi'' + \epsilon_3 \psi''' = -\frac{\lambda_H^2}{2},
\] (36)
where, for simplicity we choose the h–variables $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$, with $\psi^* = \partial \psi / \partial x^2$ and $\psi' = \partial \psi / \partial x^3$ and $\epsilon_{2,3} = \pm 1$.

The Corollary D.1 defines a general class of 5D solutions
\[
\delta s^2 = \epsilon_1 (dx^1)^2 + e^{\psi(x^2,x^3)} \left[ \epsilon_2 (dx^2)^2 + \epsilon_3 (dx^3)^2 \right] + \epsilon_4 h_0^2(x^3) \times \left[ f^* (x^3, v) \right]^2 c_4 \left( x^i, v \right) |(\delta v)^2 + \epsilon_5 [f (x^i, v) - f_0(x^i)]^2 (\delta y^5)^2, \n\]
\[
\delta v = dv + w_k (x^i, v) dx^k, \delta y^5 = dy^5 + n_k (x^i, v) dx^k,
\]
\(^3\)see Definition A.2 in Appendix, on definition of d–connections and related discussion on d–tensors.
with \( \psi(x^2, x^3) \) defined from (36) and the N–connection coefficients \( N^4_i = w_i(x^k, v) \) and \( N^5_i = n_i(x^k, v) \) computed in the form

\[
w_i = -\frac{\partial_i \varsigma_4(x^k, v)}{\varsigma_4^*(x^k, v)}
\]

and

\[
n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{[f^*(x^i, v)]^2}{[f(x^i, v) - f_0(x^i)]^3} \varsigma_4(x^i, v) \, dv,
\]

for

\[
\varsigma_4(x^i, v) = \varsigma_{4[0]}(x^i) - \frac{\kappa_4 \lambda^2_H}{16} h_0^2(x^i) \int f^*(x^i, v) \left[ f(x^i, v) - f_0(x^i) \right] \, dv,
\]

defining an exact solution of the system of string gravity equations (34), (35) and (137), (138), with nonholonomic variables and integration functions \( n_{k[1]}(x^i), \varsigma_{4[0]}(x^i), h_0^2(x^i) \) and \( f_0(x^i) \). This solution is similar to the 4D metric (25), but written for 5D ansatz with arbitrary signatures.

We can choose the function \( f(x^2, x^3, v) \) from (37) as it would be a solution of the Kadomtsev–Petviashvili (KdP) equation [19],

\[
f''' + \epsilon (f' + 6ff^* + f^{***})^* = 0, \quad \epsilon = \pm 1,
\]

or, for another locally anisotropic background, to satisfy the \((2+1)\)-dimensional sine–Gordon (SG) equation,

\[
-f''' + f'' + f^* = \sin f,
\]

see Refs. [20] on gravitational solitons and theory of solitons. In this case, we define a class of 5D nonholonomic spacetimes with string corrections by \( H \)–field self–consistently interacting with 3D gravitational solitons. Such solutions generalized those considered in Refs. [21] for 4D and 5D gravity.

### 3.2 5D disk configurations with string \( H \)–field and solitonic backgrounds

The class of solutions (37) has various parametrizations defining nonholonomic deformations of the disk solution (1) to certain type of generic 5D off–diagonal solutions of bosonic string gravity. We analyze four examples of such generic off–diagonal metrics.
3.2.1 String–solitonic disks with angular $\varphi$–anisotropy

We parametrize the ansatz (37) in a such form that it is a metric of type (11) generated from the data (5) by nonholonomic deformations of type (10), when the angular coordinate $\varphi$ is considered to be that one on which the solutions depend anisotropically. For parametrizations of 5D coordinates

\[ x^1 = \chi, x^2 = \rho, x^3 = \zeta, y^4 = \nu = \varphi, y^5 = \tau, \]

one obtains the metric

\[
\delta s^2 = \epsilon_1 (d\chi)^2 + e^{\psi(\rho, \zeta)} [(dp)^2 + (d\zeta)^2] + h^2_0(\rho, \zeta) \times \]

\[
[f^*(\rho, \zeta, \varphi)]^2 |s_1(\rho, \zeta, \varphi)| (d\varphi)^2 - [f(\rho, \zeta, \varphi) - f_0(\rho, \zeta)]^2 (d\tau)^2,
\]

\[
\delta \varphi = d\varphi + w_1(\chi, \rho, \zeta, \varphi) d\chi + w_2(\chi, \rho, \zeta, \varphi) d\rho + w_3(\chi, \rho, \zeta, \varphi) d\zeta,
\]

\[
\delta \tau = d\tau + n_1(\chi, \rho, \zeta, \varphi) d\chi + n_2(\chi, \rho, \zeta, \varphi) d\rho + n_3(\chi, \rho, \zeta, \varphi) d\zeta,
\]

where $\psi(\rho, \zeta)$ is to generated by a solution of (36), which for $\epsilon_{2,3} = 1$ is a 2D Poisson equation. The N–connection coefficients $w_i$ (38) and $n_i$ (39) are defined by

\[
\varsigma_4(\rho, \zeta, \varphi) = \varsigma_4[0](\rho, \zeta) - \frac{\lambda_H^2}{16} h^2_0(\rho, \zeta) \int f^*(\rho, \zeta, \varphi) [f(\rho, \zeta, \varphi) - f_0(\rho, \zeta)] d\varphi.
\]

The boundary conditions can related to the 4D disk solution (11) via data

\[
g_1 = \epsilon_1, g_2 = \rho_0^2, g_3 = \rho_0^2, h_4 = \rho_0^2 \rho^2 e^{-2k}, h_5 = -e^{2(2U-k)}, N_5 = a,
\]

when the gravitational polarizations $\eta_\alpha$ are computed by using the metric coefficients of (43),

\[
g_{\alpha\beta} = diag \left\{ g_i = \eta_i g_i, h_a = \eta_a h_a \right\} = diag \{ 1, e^{\psi(\rho, \zeta)}, e^{\psi(\rho, \zeta)},
\]

\[
h^2_0(\rho, \zeta) [f^*(\rho, \zeta, \varphi)]^2 |s_1(\rho, \zeta, \varphi)|, [f(\rho, \zeta, \varphi) - f_0(\rho, \zeta)]^2 \}
\]

stated with respect to N–adapted dual funfbein $e^\alpha = (d\chi, d\rho, d\zeta, d\varphi, d\tau)$.

The constructed stationary 'ϕ–anisotropic' solutions may be constrained to be a 4D configuration trivially imbedded in 5D if we put $w_1 = n_1 = 0$ and do not consider dependencies on extra dimension coordinate $\chi$. In this case, the class of metrics with $\eta_\alpha \approx 1$ and $N_1 \approx N_3$ are very similar to the conformal transform of the disk solution, i.e. to (2), but contain certain polarizations induced nonholonomically by the $H$–field in string gravity, approximated together with the N–anholonomic torsion to result in an effective cosmological constant $\lambda_H$. These solutions can be considered to define
some locally anisotropically polarized disks imbedded self–consistently in 5D curved background defined by the function $f(\rho, \zeta, \varphi)$. A such background can be of 3D solitonic nature if we chose $f$ to be any solution of (41), redefined for the signatures $\varepsilon_{2,3,4} = 1$. If the string contributions are normalized in a form to have $\lambda_0 = \lambda^2_H/8$, we may reproduce all geometric and physical properties of the solution (25).

We note that the symmetry properties of (43) should be derived as certain nonholonomic deformations of the 4D Killing disk symmetries. For various classes of integration functions and stated polarizations $\eta_a$, the new type of spacetimes can be characterized by noncommutative symmetries and/or Lie algebroid symmetries, see section 3.

3.2.2 Moving string–solitonic disk configurations

It is possible to deform nonholonomically the metric (3) in a form generating a class of disk like solutions in string gravity with polarizations depending anisotropically on the time like coordinate $\tau$. One uses the coordinate parametrizations $x^1 = \chi, x^2 = \rho, x^3 = \zeta, y^4 = v = \tau$ and $y^5 = \varphi$ and the data (3), when the undeformed (nontrivial) metric and N–connection coefficients are $g_1 = \epsilon_1, g_2 = \rho_0^2$, $g_3 = \rho_0^2$, $h_4 = -e^{2(2U - k)}$, $h_5 = \rho_0^2 \rho^2 e^{-2k}$. The new class of solutions of the system of string gravity equations (34), (35) and (137),(138) is parametrized in the form

$$
\delta s^2 = \epsilon_1 (d\chi)^2 + e^{\psi(\rho, \zeta)} [(d\rho)^2 + (d\zeta)^2] + h_0^2(\rho, \zeta) \times [f^*(\rho, \zeta, \tau)]^2 |\varsigma_4(\rho, \zeta, \tau)| (\delta \tau)^2 - [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)]^2 (\delta \varphi)^2,$$

$$
\delta \tau = d\varphi + w_1(\chi, \rho, \zeta, \tau) d\chi + w_2(\chi, \rho, \zeta, \tau) d\rho + w_3(\chi, \rho, \zeta, \tau) d\zeta,$$

$$
\delta \varphi = d\tau + n_1(\chi, \rho, \zeta, \tau) d\chi + n_2(\chi, \rho, \zeta, \tau) d\rho + n_3(\chi, \rho, \zeta, \tau) d\zeta,$$

where $\psi(\rho, \zeta)$ is also generated by a solution of (36), which for $\varepsilon_{2,3} = 1$ is a 2D Poisson equation and the N–connection coefficients $w_i$ (38) and $n_i$ (39) are computed by using

$$
\varsigma_4(\rho, \zeta, \tau) = \varsigma_{4[0]}(\rho, \zeta) - \frac{\lambda^2_H}{16} h_0^2(\rho, \zeta) \int f^*(\rho, \zeta, \tau) [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)] d\tau.$$

We can restrict the class of solutions for the 4D metrics trivially imbedded in 5D when $w_1 = n_1 = 0$ and the dependence on $\chi$ is not considered for the metric coefficients. The gravitational polarizations $\eta_a$ follows from the metric coefficients in (15),

$$
g_{a\beta} = \text{diag} \left\{ g_1 = \eta_1 g_1, h_a = \eta_a h_a \right\} = \text{diag} \left\{ 1, e^{\psi(\rho, \zeta)}, e^{\psi(\rho, \zeta)} \right\},$$

$$
h_0^2(\rho, \zeta) [f^*(\rho, \zeta, \tau)]^2 |\varsigma_4(\rho, \zeta, \tau)|, [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)]^2$$

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In a more explicit form, we can choose \( f(\rho, \zeta, \tau) \) to be a solution of the 3D signe–Gordon equation (42). This way, there are generating metrics describing 4D locally anisotropically polarized disks moved away in time by certain solitonic waves in 5D spacetime. In a similar form, one had been constructed solutions defining solitically moved black holes [11].

3.2.3 Disks with anistoropic polarizations on extra dimension

By nonholonomic deformations of (4) we can generate disk like solutions in string gravity with \( \chi \)–anisotropy. We consider the data (7) when nontrivial (nondeformed) metric coefficients are \( g_1 = 1, g_2 = \rho^{-2}e^{2k}, g_3 = \rho^{-2}e^{2k}, h_{4} = \epsilon, h_5 = -\rho^{-2}e^{4\psi} \) and \( \Delta_5 = a \) for the local coordinate parametrization \( x^1 = \varphi, x^2 = \rho, x^3 = \zeta, y^4 = v = \chi \) and \( y^5 = \tau \). For such data, the solutions follow from (37),

\[
\delta \tilde{s}^2 = d\varphi^2 + e^{\psi(\rho, \zeta)} d\rho^2 + e^{\psi(\rho, \zeta)} d\zeta^2 + \epsilon_4 h_0^2(\rho, \zeta) \times \left[ f^*(\rho, \zeta, \chi) \right]^2 \quad \text{(46)}
\]

\[
\delta \chi = d\chi + w_1(\varphi, \rho, \zeta, \chi) d\varphi + w_2(\varphi, \rho, \zeta, \chi) d\rho + w_3(\varphi, \rho, \zeta, \chi) d\zeta,
\]

\[
\delta \tau = d\tau + n_1(\varphi, \rho, \zeta, \chi) d\varphi + n_2(\varphi, \rho, \zeta, \chi) d\rho + n_3(\varphi, \rho, \zeta, \chi) d\zeta.
\]

The function \( \psi(\rho, \zeta) \) is a solution of the 2D Poisson equation (36). The N–connection coefficients \( w_i \) (38) and \( n_i \) (39) are defined by

\[
\varsigma_4(\rho, \zeta, \chi) = \varsigma_{4[0]}(\rho, \zeta) - \frac{\chi^2}{16} h_0^2(\rho, \zeta) \int f^*(\rho, \zeta, \chi) [f(\rho, \zeta, \chi) - f_0(\rho, \zeta)] d\chi.
\]

The gravitational polarizations \( \eta_\alpha \) are are defined by using the metric coefficients of (46) and 'nondeformed' data,

\[
g_{\alpha\beta} = \text{diag} \left\{ g_\alpha = \eta_\alpha g_\alpha, h_\alpha = \eta_\alpha h_\alpha \right\} = \text{diag} \{ 1, e^{\psi(\rho, \zeta)}, e^{\psi(\rho, \zeta)} \},
\]

\[
h_0^2(\rho, \zeta) \left[ f^*(\rho, \zeta, \chi) \right]^2 |\varsigma_4(\rho, \zeta, \chi)|, [f(\rho, \zeta, \chi) - f_0(\rho, \zeta)]^2 \}.
\]

The metric (46) defines a stationary disk like 5D solution in string gravity with generic dependence on extra dimension coordinate. We can consider a subclass of solutions when \( w_1 = n_1 = 0 \) and the integration functions are chosen in a such form that one does not have dependence on angular coordinate \( \varphi \). Such spacetimes can not be reduced to certain trivial imbedding of 4D configurations into 5D ones but can model a 4D disk like structure for small nonholonomic deformations. The extra dimension background may be defined for various types of generating functions \( f(\rho, \zeta, \chi) \) which can be of solitonic or other nature, with explicit dependence on the extra dimension coordinate.

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3.2.4 Disks solitonically propagating in extra dimensions

There is another class of nonholonomic deformations of (4) being very different from the constructions with \( \chi \)-anisotropy from the previous subsection. One considers the (8) with the local coordinates \( x^1 = \varphi, x^2 = \rho, x^3 = \zeta, y^4 = v = \tau, y^5 = \chi \) when the (nondeformed) nonzero metric coefficients are \( g_2 = \rho^{-2}e^{2k}, g_3 = \rho^{-2}e^{2k}, h_4 = -\rho_0^{-2}\rho^{-2}e^{4U}, h_5 = \epsilon \) and \( N_3^4 = a \).

The solution with anisotropic dependence on \( \tau \) is given by the ansatz (37) stated for the mentioned parametrizations of coordinates,

\[
\begin{align*}
\delta s^2 &= d\varphi^2 + e^{\psi(\rho, \zeta)}d\rho^2 + e^{\psi(\rho, \zeta)}d\zeta^2 + \epsilon_4 h_0^2(\rho, \zeta) \times \\
&\quad \left[ f^*(\rho, \zeta, \tau)^2|\varsigma_4(\rho, \zeta, \tau)|\delta\tau^2 - [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)]^2\delta\chi^2, \\
\delta\tau &= d\tau + w_1(\varphi, \rho, \zeta, \tau)d\varphi + w_2(\varphi, \rho, \zeta, \tau)d\rho + w_3(\varphi, \rho, \zeta, \tau)d\zeta, \\
\delta\chi &= d\chi + n_1(\varphi, \rho, \zeta, \tau)d\varphi + n_2(\varphi, \rho, \zeta, \tau)d\rho + n_3(\varphi, \rho, \zeta, \tau)d\zeta.
\end{align*}
\]

This metric is characterized by gravitational polarizations \( \eta_\alpha \) relating the ‘nondeformed’ data with the deformed ones,

\[
\begin{align*}
\eta_\alpha &= \text{diag}\left\{ g_i = \eta_i g_{ij}, h_a = \eta_a h_{ab} \right\} = \text{diag}\{1, e^{\psi(\rho, \zeta)}, e^{\psi(\rho, \zeta)}, h_0^2(\rho, \zeta) [f^*(\rho, \zeta, \tau)]^2 |\varsigma_4(\rho, \zeta, \tau)|, [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)]^2 \}. \\
\end{align*}
\]

For this class of metrics, the function \( \psi(\rho, \zeta) \) is also a solution of the 2D Poisson equation (36). The N–connection coefficients \( w_i \) and \( n_i \) are computed by similar formulas (38) and, respectively, (39) but for \( \varsigma_4(\rho, \zeta, \tau) = \varsigma_4(\rho, \zeta) - \frac{\lambda^2}{16} h_0^2(\rho, \zeta) \int f^*(\rho, \zeta, \tau) [f(\rho, \zeta, \tau) - f_0(\rho, \zeta)] d\tau. \)

The solution (47) does not depend explicitly on the extra dimension coordinate. Nevertheless, it is a generic five dimensional one and does not have straightforward limits to configurations containing 4D metrics as trivial imbedding. The extra dimension background may be defined for various types of generating functions \( f(\rho, \zeta, \tau) \) which can be of solitonic or other nature. We can choose \( f(\rho, \zeta, \tau) \) to be a solution of the 3D sine–Gordon equation (42), or inversely, to be a solution of Kadomtsev–Petviashvili (KdP) equation (41). This way we construct generic off–diagonal 5D solutions in string gravity with 3D soliton backgrounds.

4 Disks in Extra Dimensions and Spinors

The geometry of spinors on N–anholonomic spaces was elaborated in Refs. [22]. The main problem was to give a rigorous definition of spinors on spaces
provided with nonintegrable distributions defined by N–connections. In this section, we consider exact solutions of the 5D Einstein–Dirac equations defining certain spinor wave packages on locally anisotropic spaces derived by nonholonomic transforms of disk configurations and giving explicit examples of N–anholonomic spacetimes. We analyze two classes of such solutions.

4.1 Dirac equation on nonholonomic gravitational disk configurations

We consider 5D nonholonomic deformations of (3) starting from the data (5), or (6), when the extra dimension coordinate is time–like. For a d–metric (11) with coefficients

\[ g_{\alpha\beta}(u) = (g_{ij}(u), h_{ab}(u)) = (1, g^i_\alpha(u), h_a(u)), \]

where \( i = 1, 2; a = 0, 1, 2; \) defined with respect to the N–adapted basis (83), we can easily define the funfbein (pentad) fields

\[ e^i_\mu = e^i_\mu e^\mu = \{e_\mu^i = e_\mu^i \delta_\mu^i, e_\mu^a = e_\mu^a \partial_a \}, \]

\[ e^a_\mu = e^a_\mu e^\mu = \{e_\mu^a = e_\mu^a \partial_a \delta_a^a \} \]

satisfying the conditions

\[ g_{ij} = \delta^i_j g_{ij} \text{ and } h_{ab} = \delta^a_b h_{ab}, \]

\[ g_{\hat{i}\hat{j}} = \text{diag}[-1, 11] \text{ and } h_{\hat{a}\hat{b}} = \text{diag}[1, -1] \text{ or } = \text{diag}[-1, 1]. \]

We write

\[ e_\mu^i = \sqrt{|g_i|}\delta_i^\mu \text{ and } e_\mu^a = \sqrt{|h_a|}\delta_a^\mu, \]

where \( \delta_i^\mu \) and \( \delta_a^\mu \) are Kronecker’s symbols.

The Dirac spinor fields on such nonholonomically deformed spaces are parametrized

\[ \Psi(u) = [\Psi^\alpha(u)] = [\psi^\hat{i}(u), \chi^\hat{i}(u)], \]

where \( \hat{\hat{i}} = 0, 1, \) are defined with respect to the 4D Euclidean tangent subspace belonging the tangent space to the 5D N–anholonomic manifold \( V. \) The 4 × 4

\[ \text{In the first works from [22], the problem of rigorous definition of nonholonomic spinor structures was solved for Finsler and Lagrange spaces but the question of constructing a theory of nonholonomic spinor and Dirac operators is also important for (pseudo) Riemannian and Riemann–Cartan spaces [23].} \]
dimensional gamma matrices $\gamma^\alpha = [\gamma^2, \gamma^3, \gamma^4, \gamma^5]$ are defined in the usual way, in order to satisfy the relation
\[
\{ \gamma^\alpha, \gamma^\beta \} = 2g^\alpha_\beta,
\tag{49}
\]
where $\{ \gamma^\alpha, \gamma^\beta \}$ is a symmetric commutator, $g^\alpha_\beta = (1, 1, 1, 1)$, which generates a Clifford algebra distinguished on two holonomic and two anholonomic directions. It is possible to extend the relations (49) for unprimed indices $\alpha, \beta$, by a conventional completing the set of primed gamma matrices with a matrix $\gamma^1$, i.e. write $\gamma^\alpha = [\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5]$ when
\[
\{ \gamma^\alpha, \gamma^\beta \} = 2g^\alpha_\beta.
\]

The coefficients of $N$–anholonomic gamma matrices can be computed with respect to anholonomic bases (83) by using respectively the funfbein coefficients
\[
\hat{\gamma}^\beta(u) = e^\beta_\alpha(u)\gamma^\alpha.
\]
A covariant derivation of the Dirac spinor field, $\nabla_\alpha \Psi$, can be defined by using pentad (equivalently, funfbein) decompositions of the d–metric (11),
\[
\nabla_\alpha \Psi = \left[ e_\alpha + \frac{1}{4} S_{\alpha\beta\gamma}(u) e^\beta_\alpha(u) \gamma^\beta_\gamma \right] \Psi,
\tag{50}
\]
where there are introduced $N$–elongated partial derivatives. The coefficients
\[
(S_{\alpha\beta\gamma}(u) = (D_\gamma e^\alpha_\alpha) e^\beta_\beta e^\gamma_\gamma)
\]
are transformed into rotation Ricci d–coefficients $S_{\alpha\beta\gamma}$ which together with the d–covariant derivative $D_\gamma$ are defined by anholonomic pentads and anholonomic deformations of the Christoffel symbols. In the canonical case, we should take the operator of canonical d–connection $\hat{D}_\gamma$ with coefficients (12).

For diagonal d–metrics, the funfbein coefficients can be taken in their turn in diagonal form and the corresponding gamma matrix $\hat{\gamma}^\alpha(u)$ for anisotropic curved spaces are proportional to the usual gamma matrix in flat spaces $\gamma^\alpha$. The Dirac equations on nonholonomic manifolds are written in the simplest form with respect to anholonomic frames,
\[
(i\hat{\gamma}^\alpha(u) \nabla_\alpha - M)\Psi = 0,
\tag{51}
\]
where $M$ is the mass constant of the Dirac field. The Dirac equation is the Euler–Lagrange equation for the Lagrangian
\[
\mathcal{L}^{(1/2)}(u) = \sqrt{|g|} \{[\Psi^+(u) \hat{\gamma}^\alpha(u) \nabla_\alpha \Psi(u) - (\nabla_\alpha \Psi^+(u))\hat{\gamma}^\alpha(u) \Psi(u)] - M \Psi^+(u) \Psi(u)\},
\tag{52}
\]
where by $\Psi^+(u)$ we denote the complex conjugation and transposition of the column $\Psi(u)$. Varying $\mathcal{L}^{(1/2)}$ on d–metric (52) we obtain the symmetric energy–momentum d–tensor

$$\Upsilon_{\alpha\beta}(u) = \frac{i}{4}[\Psi^+(u)\tilde{\gamma}_\alpha(u)\nabla_\beta\Psi(u) + \Psi^+(u)\tilde{\gamma}_\beta(u)\nabla_\alpha\Psi(u) - (\nabla_\alpha\Psi^+(u))\tilde{\gamma}_\beta(u)\Psi(u) - (\nabla_\beta\Psi^+(u))\tilde{\gamma}_\alpha(u)\Psi(u)]. \quad (53)$$

We can verify that for a such class of metrics one holds the condition $\hat{D}_\gamma e^\alpha = 0$ resulting in zero Ricci rotation coefficients,

$$S_{\alpha\beta\gamma}(u) = 0 \text{ and } \nabla_\alpha\Psi = e_\alpha\Psi,$$

and the N–anholonomic Dirac equations (51) transform into

$$(i\gamma^\alpha e_\alpha - M)\Psi = 0. \quad (54)$$

Further simplifications are possible for Dirac fields depending only on coordinates $(x^1 = \chi, x^2 = \rho, x^3 = \zeta)$, i. e. $\Psi = \Psi(x^k)$ when the equation (54) transforms into

$$(i\gamma^1\partial_\chi + i\gamma^2\frac{1}{\sqrt{|g_2|}}\partial_2 + i\gamma^3\frac{1}{\sqrt{|g_3|}}\partial_3 - M)\Psi = 0.$$

The equation (54) simplifies substantially in $\tilde{x}$–coordinates

$$(\chi, \tilde{x}^2 = \tilde{x}^2(\rho, \zeta), \tilde{x}^3 = \tilde{x}^3(\rho, \zeta)),$$

defined in a form when are be satisfied the conditions

$$\frac{\partial}{\partial \tilde{x}^2} = \frac{1}{\sqrt{|g_2|}}\partial_2 \text{ and } \frac{\partial}{\partial \tilde{x}^3} = \frac{1}{\sqrt{|g_3|}}\partial_3 \quad (55)$$

We get

$$(i\gamma^1\frac{\partial}{\partial \chi} + i\gamma^2\frac{\partial}{\partial \tilde{x}^2} + i\gamma^3\frac{\partial}{\partial \tilde{x}^3} - M)\Psi(\chi, \tilde{x}^2, \tilde{x}^3) = 0. \quad (56)$$

The equation (56) describes the wave function of a Dirac particle of mass $M$ propagating in a three dimensional Minkowski flat plane which is imbedded as a N–adapted distribution into a 5D spacetime when the extra dimension coordinate may be considered to be time–like.

The solution $\Psi = \Psi(\chi, \tilde{x}^2, \tilde{x}^3)$ of (56) is searched in the form

$$\Psi = \begin{cases} 
\Psi^+(\tilde{x}) = \exp[-i(k_1\chi + k_2\tilde{x}^2 + k_3\tilde{x}^3)]\varphi^1(k) & \text{for positive energy;} \\
\Psi^-(\tilde{x}) = \exp[i(k_1\chi + k_2\tilde{x}^2 + k_3\tilde{x}^3)]\chi^1(k) & \text{for negative energy,}
\end{cases} \quad (26)$$
with the condition that $k_1$ is identified with the positive energy and $\varphi^1(k)$ and $\chi^1(k)$ are constant bispinors. To satisfy the Klein–Gordon equation we must have

$$k^2 = -(k_1)^2 + (k_2)^2 + (k_3)^2 = M^2.$$ 

The Dirac equation implies

$$(\sigma^i k_i - M)\varphi^1(k) \text{ and } (\sigma^i k_i + M)\chi^1(k),$$

where $\sigma^i(i = 1, 2, 3)$ are Pauli matrices corresponding to a realization of gamma matrices as to a form of splitting to usual Pauli equations for the bispinors $\varphi^1(k)$ and $\chi^1(k)$.

In the rest frame for the horizontal plane parametrized by coordinates $\tilde{x} = \{\chi, \tilde{x}^2, \tilde{x}^3\}$ there are four independent solutions of the Dirac equation,

$$\varphi^1_{(1)}(M, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \varphi^1_{(2)}(M, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\chi^1_{(1)}(M, 0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \chi^1_{(2)}(M, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

We consider wave packets of type (for simplicity, we can use only superpositions of positive energy solutions)

$$\Psi^+(\zeta) = \int \frac{d^3p}{2\pi^3} \frac{M}{\sqrt{M^2 + (k^2)^2}} \sum_{[\alpha]} b(p, [\alpha]) \varphi^{[\alpha]}(k) \exp \left[-ik_i\tilde{x}^i\right]$$

when the coefficients $b(p, [\alpha])$ define a current (the group velocity)

$$J^2 \equiv \sum_{[\alpha]} \int \frac{d^3p}{2\pi^3} \frac{M}{\sqrt{M^2 + (k^2)^2}} |b(p, [\alpha])|^2 \frac{p^2}{\sqrt{M^2 + (k^2)^2}}$$

with $|p^2| \sim M$ and the energy–momentum d–tensor (53) has nontrivial coefficients

$$\Upsilon^1_1 = 2\Upsilon(\tilde{x}^2, \tilde{x}^3) = k_1 \Psi^+\gamma_a \Psi, \Upsilon^2_2 = -k_2 \Psi^+\gamma_2 \Psi, \Upsilon^3_3 = -k_3 \Psi^+\gamma_3 \Psi$$

where the holonomic coordinates can be reexpressed $\tilde{x}^i = \tilde{x}^i(x^i)$. We must take two or more waves in the packet and choose such coefficients $b(p, [\alpha])$, 

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satisfying corresponding algebraic equations, in order to get from \(58\) the equalities
\[
\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon(\tilde{x}^2, \tilde{x}^3) = \Upsilon(x^2 = \rho, x^3 = \zeta),
\] (59)
required by the conditions \(53\).

We note that the ansatz for the 5D metric \(11\) and 4D spinor fields depending on 3D h–coordinates \(\tilde{x} = \{\chi, \tilde{x}^2, \tilde{x}^3\}\) reduces the Dirac equation to the usual one projected on a flat 3D spacetime. This configuration is N–adapted, because all coefficients are computed with respect to N–adapted frames. The spinor source \(\Upsilon(x^2, x^3)\) induces an additional nonholonomic deformation.

### 4.2 Anisotropic gravitational disk–spinor fields

In this subsection, we construct two new classes of solutions of the Einstein–Dirac fields generalizing the 5D disk metrics in string gravity \(37\) to contain contributions from the energy momentum d–tensor
\[
\Upsilon_\alpha^\beta = \{2\Upsilon(\rho, \zeta), \Upsilon(\rho, \zeta), \Upsilon(\rho, \zeta), 0, 0\}
\]
for a Dirac wave packet satisfying the conditions \(58\) and \(59\). This preserves the same gravitational field equations \(135\), i.e. \(34\), \(137\) and \(138\) but modifies \(35\) to
\[
S_4^4 = S_5^5 = -\frac{1}{2h_4h_5} \left[ h_2^* - h_5 \left( \ln \sqrt{|h_4h_5|} \right)^* \right] = -\frac{\lambda_H^2}{4} - \Upsilon(\rho, \zeta). \quad (60)
\]
As a particular case we can put \(\lambda_H = 0\). Following the Corollary D.1, see Appendix, the solution of the spinorially modified equations can be written in the form
\[
\begin{align*}
\delta s^2 &= -(d\chi)^2 + e^{\psi(\rho, \zeta)} \left[ (d\rho)^2 + (d\zeta)^2 \right] + \epsilon_4 h_0^2(x^i) \left[ f^*(\rho, \zeta, v) \right]^2 \times |\varsigma_4(\rho, \zeta, v)| \left( \delta v \right)^2 + \epsilon_5 \left[ f(\rho, \zeta, v) - f_0(\rho, \zeta) \right]^2 \left( \delta y^5 \right)^2, \\
\delta v &= dv + w_1(\chi, \rho, \zeta, v) d\chi + w_2(\chi, \rho, \zeta, v) d\rho + w_3(\chi, \rho, \zeta, v) d\zeta, \\
\delta y^5 &= dy^5 + n_1(\chi, \rho, \zeta, v) d\chi + n_2(\chi, \rho, \zeta, v) d\rho + n_3(\chi, \rho, \zeta, v) d\zeta.
\end{align*}
\] (61)
From this ansatz, we can distinguish two classes of solutions: 1) If we take \(y^4 = v = \varphi\) and \(y^5 = \tau\), with \(\epsilon_4 = 1\) and \(\epsilon_5 = -1\), we generate a spinor source generalization of \(43\). 2) For \(y^4 = v = \tau\) and \(y^5 = \varphi\), with \(\epsilon_4 = -1\) and \(\epsilon_5 = 1\), we obtain a spinor generalization of \(15\).

We note that the formulas for the N–connection coefficients, \(N_i^4 = w_i(x^k, v)\) and \(N_i^5 = n_i(x^k, v)\), are derived for a different generation
function (40), than that used in (38) and (39). In our case, we have

\[ \varsigma_4 (\rho, \zeta, v) = \varsigma_{4[0]} (\rho, \zeta) - \frac{\epsilon_1}{4} h_{0}(\rho, \zeta) \times \]

\[ \int \left[ -\frac{\lambda^2 H}{4} - \Upsilon(\rho, \zeta) \right] f^*(\rho, \zeta, v) [f (\rho, \zeta, v) - f_0(\rho, \zeta)] dv. \]  

Substituting this value, respectively, in (149) and (150), we can compute

\[ w_i = -\frac{\partial \varsigma_4 (\rho, \zeta, v)}{\varsigma_4^* (\rho, \zeta, v)} \]  

and

\[ n_k = n_{k[1]} (\chi, \rho, \zeta) + n_{k[2]} (\chi, \rho, \zeta) \int \frac{[f^* (\rho, \zeta, v)]^2}{[f (\rho, \zeta, v) - f_0(\rho, \zeta)]^2} \varsigma_4 (\rho, \zeta, v) dv. \]  

The function \( \psi = \ln |g_2| = \ln |g_3| \) from (61) is just the solution of (34) with signatures \( \epsilon_2 = \epsilon_3 = 1 \), \( \psi \) is a solution of

\[ \psi^{**} + \psi'' = -\frac{\lambda^2 H}{2}, \]

with \( \psi^* = \partial \psi / \partial \rho \) and \( \psi' = \partial \psi / \partial \zeta \).

We can fix a disk–solitonic background on which packages of 3D Dirac wave propagate self–consistently if we take the function \( f (\rho, \zeta, v) \) from the coefficients (64), computed by formulas (63) and (64), to be a solution of the 3D solitonic equation (41) or (42).

5 Lie Algebroid Symmetries and Disk Configurations

The nonholonomically deformed disk solutions do not possess, in general, Killing symmetries. In this section we show that it is possible to distinguish certain configurations with Lie algebroid symmetry. We restrict our considerations to the class of Riemann–Cartan spacetimes defining Lie algebroid configurations [15]. On details on Lie algebroids geometry and applications, we refer to [14].

5.1 Gravitational Lie algebroids and N–connections

Let us consider a vector bundle \( \mathcal{E} = (E, \pi, M) \), with a surjective map \( \pi : E \to M \) of the total spaces \( E \) to the base manifold \( M \), where dimensions
are finite ones, \( \dim E = n + m \) and \( \dim M = n \). The Lie algebroid structure 
\[ \mathcal{A} \cong (E, [\cdot, \cdot], \rho) \]
is defined by an anchor map \( \rho : E \to TM \) (\( TM \) is the 
tangent bundle to \( M \)) and a Lie bracket on the \( C^\infty(M) \)-module of sections 
of \( E \), denoted \( \text{Sec}(E) \), such that 
\[
[X, fY] = f[X, Y] + \rho(X)(f)Y
\]
for any \( X, Y \in \text{Sec}(E) \) and \( f \in C^\infty(M) \).

For applications to gravity and string theories and spaces provided with 
nonholonomic frame (vielbein) structure, one works on a general manifold 
\( V, \dim V = n + m \), which is a (pseudo) Riemannian spacetime, or a 
more general one with nontrivial torsion. A Lie algebroid structure can be 
modelled locally on a nonholonomic spacetime \( V \) defined by a Whitney type 
sum 
\[ TV = hV \oplus vV \]  
(65)

stating a splitting into certain conventional horizontal \( (h) \) and vertical \( (v) \) 
subspaces. This distribution is in general nonintegrable (nonholonomic) and 
defines a \( N \)-connection structure. A such manifold is called \( N \)-anholonomic 
(see Ref. \([15, 23]\)). An anchor structure is defined as a map \( \hat{\rho} : V \to hV \) 
and the Lie bracket structure is considered on the spaces of sections 
\( \text{Sec}(vV) \). For the purposes of this paper, we can consider Riemann–Cartan 
nonholonomic manifolds admitting a locally fibered structure induced by the 
splitting (65) when the Lie algebroid constructions are usual ones but with 
formal substitutions \( E \to V \) and \( M \to hV \).

One defines the Lie algebroid structure in local form by its structure 
functions \( \rho^i_a(x) \) and \( C^c_{ab}(x) \) defining the relations 
\[
\rho(v_a) = \rho^i_a(x) e_i = \rho^i_a(x) \partial_i, \quad (66)
\]
\[
[v_a, v_b] = C^c_{ab}(x) v_c \quad (67)
\]
and subjected to the structure equations 
\[
\rho^i_a \frac{\partial \rho^k_b}{\partial x^j} - \rho^i_b \frac{\partial \rho^k_a}{\partial x^j} = \rho^i_c C^c_{ab} \quad \text{and} \quad \sum_{cyclic(a,b,c)} \left( \rho^i_a \frac{\partial C^d_c}{\partial x^j} + C^d_{af} C^f_{bc} \right) = 0. \quad (68)
\]

We shall omit underlying of coordinate indices if it will not result in ambigui-
gities. Such equations are standard ones for the Lie algebroids but defined 
on \( N \)-anholonomic manifolds. In brief, we call such spaces to be Lie \( N \)-
algebroids. We can consider that spacetimes provided with Lie algebroid 
structure generalize the class of manifolds possess Lie algebra symmetry to a case when the structure constants transform into structure functions.
depending on holonomic coordinates, see (68). On such spaces, we can work with singular structures defined by the so-called ‘anchor’ map.

The Lie algebroid and N–connection structures can be established in a mutually compatible form by introducing the ”N–adapted” anchor

\[ \hat{\rho}_j^a(x, u) \doteq e_j^i(x, u) e_a^x(u, x) \rho_{xi}^j(x) \] (69)

and ”N–adapted” (boldfaced) structure functions

\[ C_{ag}^f(x, u) = e_f^j(x, u) e_a^x(u, x) e_g^y(x, u) C_{by}^f(x), \] (70)

respectively, into formulas (66), (67) and (68). One follows that the Lie algebroids on N–anholonomic manifolds are defined by the corresponding sets of functions \( \hat{\rho}_j^a(x, u) \) and \( C_{ag}^f(x, u) \) with additional dependencies on v–variables \( u^b \) for the N–adapted structure functions. For such generalized Lie N–algebroids, the structure relations became

\[ \hat{\rho}_j^a(x, u) = \hat{\rho}_h^i(x, u) e_i, \] (71)

\[ [v_d, v_b] = C_{db}^f(x, u) v_f \] (72)

and the structure equations of the Lie N–algebroid are written

\[ \sum_{cyclic(a,b,c)} \left( \hat{\rho}_j^a e_j(\hat{\rho}_h^i) + C_{ag}^f C_{be}^c - C_{be}^f C_{ag}^c Q^{\bar{f} i b e}_{\bar{f} a} \right) = 0, \] (73)

for \( Q^{\bar{f} i b e}_{\bar{f} a} = e_{\bar{f} a} e_{\bar{f} e} e_{\bar{f} b} e_{\bar{f} e} e_{\bar{f} i} e_{\bar{f} j} e_{\bar{f} a} e_{\bar{f} e} \) with the values \( e_{\bar{f} a} \) and \( e_{\bar{f} i} \) defined by the N–connection. The Lie N–algebroid structure is characterized by the coefficients \( \hat{\rho}_h^i(x, u) \) and \( C_{db}^f(x, u) \) stated with respect to the N–adapted frames (83) and (82).

### 5.2 Disks with Lie Algebroid symmetry

Let us analyze the conditions when a class of metrics of type (11) models a Lie algebroid provided with N–connection structure. We re–write the metric in dual form,

\[ g = g^{\alpha \beta}(u) e_\alpha \otimes e_\beta = g^{ij}(u) e_i \otimes e_j + h^{ab}(u) v_b \otimes v_b, \]

\[ = (g_i)^{-1} e_i \otimes e_i + (h_a)^{-1} v_a \otimes v_a \]

where \( v_a \doteq e_a^x(x, u) \d \partial / \partial u^a \) satisfy the Lie N–algebroid conditions \( v_a v_b - v_b v_a = C_{ab}^f(x, u)v_d \) of type (72) with \( e_i \) being of type (83). The functions \( g_i \)
and \( h_a \) are any data defining a nonholonomic deformation of a disk solutions considered in the previous sections.

It is possible to introduce an anchor map (71) in the form
\[
\hat{\rho}_{\alpha'}(u) = e^{i}_{\alpha'}(u) e^{\alpha}_{\alpha'}(u) \hat{\rho}_{\alpha}(x),
\]
defined in the form (69) by some matrices of type (84) and (85), and write the canonical relation
\[
g^{ij}(u) = h^{\alpha'\beta'}(u) \hat{\rho}_{\alpha'}(u) \hat{\rho}_{\beta'}(u).
\]
As a result, the \( h \)-component of the \( d \)-metric is represented
\[
g^{ij}(u) e_i \otimes e_j = h^{\alpha'\beta'}(u) e^{i}_{\alpha'}(u) e^{\alpha}_{\alpha'}(x) \hat{\rho}_{\alpha}(x) e_j^{\beta'}(u) e^{\beta}_{\beta'}(x) e_i \otimes e_j = h^{\alpha'\beta'}(u) \hat{\rho}_{\alpha'}(x) \hat{\rho}_{\beta'}(x) \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta},
\]
where \( \hat{\rho}_{\alpha'}(x) = e^{\alpha}_{\alpha'}(x) \hat{\rho}_{\alpha}(x) \). For effective diagonal metrics, with respect to \( N \)-adapted frames, the formulas (74) and (75) simplify substantially. We conclude that the ansatz (11) admits a Lie algebroid type structure with the structure functions \( \rho_{\alpha}(x) \) and \( C_{\alpha\beta}(x) \) if and only if the contravariant \( h \)-component of the corresponding \( d \)-metric, with respect to the local coordinate basis, can be parametrized in the form
\[
g^{ij}(u) = h^{\alpha'\beta'}(u) \rho^{\alpha'}_{\alpha'}(x) \rho^{\beta'}_{\beta'}(x),
\]
i.e.
\[
(g_i)^{-1}(u) = \sum_a (h_a)^{-1}(u) \left[ \rho_{\alpha}(x) \right]^2.
\]
The anchor \( \rho_{\alpha'}(x) \) may be treated as a vielbein transform depending on \( x \)-coordinates lifting the horizontal components of the contravariant metric on the \( v \)-subspace. The Lie type structure functions \( C_{\alpha\beta}(x, u) \) define certain anholonomy relations for the basis \( v_a \). On a general Lie \( N \)-algebroids we can consider any set of coefficients \( \hat{\rho}_{\alpha'}(u) \) and \( C_{\alpha\beta}^d(x, u) \) not obligatory subjected to the data (74).

On \( N \)-anholonomic manifolds, it is more convenient to work with the \( N \)-adapted relations (75) than with (76). For effectively diagonal \( d \)-metrics, such anchor map conditions must be satisfied both for \( \eta_a = 1 \) and nontrivial values of \( \eta_a \), i.e.
\[
g^i = h^4 \left( \hat{\rho}_{\alpha} \right)^2 + h^5 \left( \hat{\rho}_{\beta} \right)^2,
\]
\[
g^j = g^4 \left( \hat{\rho}_{\alpha} \right)^2 + g^5 \left( \hat{\rho}_{\beta} \right)^2,
\]
(77)
for $\eta \alpha \to 1$, where $g^i = 1/g_i$, $h^a = 1/h_a$, $\eta^\alpha = 1/\eta_\alpha$ and $g^\alpha = 1/g_\alpha$. The real solutions of (77) are

\[
\left( \hat{\rho}_4^i \right)^2 = \frac{g^i g_4 H_4^i}{g_4}, \quad \left( \hat{\rho}_5^i \right)^2 = -\frac{g^i g_5 H_5^i}{g_5},
\]

(78)

where

\[
H_a^i = \eta_a \frac{1 - \eta_4/\eta_k}{\eta_5 - \eta_4},
\]

$\eta_1 = 1$, for any parametrization of solutions with a set of values of $g^\alpha$ and $\eta^\alpha$ for which $(\hat{\rho}_a^i)^2 \geq 0$. The nontrivial anchor coefficients can be related to a general solution of type (11) via gravitational polarizations $\eta_\alpha$ from (10) when $g_k = \eta_k g_\alpha$ and $h_a = \eta_a h_\alpha$. In a particular case, we may compute the anchor nontrivial components (78) for an explicit solutions, for instance, taking the data (44).

The final step in Lie algebroid classification of such classes of metrics is to define $C_{ab}(x, u)$ from the algebraic relations defined by the first equation in (73) with given values for $\hat{\rho}_a^i$, see (78), and defined $N$–elongated operators $e_i$. In result, the second equation in (73) will be satisfied as a consequence of the first one. This restrict the classes of possible v–frames, $v_b = e_b^i(x, u) \partial/\partial u^i$, where $e_b^i(x, u)$ have to satisfy the algebraic relations (72). We conclude, that the Lie N–algebroid structure imposes certain algebraic constraints on the coefficients of vielbein transforms.

Finally, we note that the Lie algebroid classification of N–anholonomic spaces can be re–written in terms of Clifford algebroids elaborated in Ref. [15]. A such approach is more appropriate in spinor gravity and for Einstein–Dirac fields. In general, the spinor constructions can be included in the left or right minimal ideals of Clifford algebras generalized to Clifford spaces (C–spaces, elaborated in details in Refs. [24]), in our case provided with additional N–connections and/or algebroid structure.

6 Discussion and Conclusions

In this paper we constructed explicit exact solutions describing nonholonomic deformations of four dimensional (in brief, 4D) disk solutions to generic off–diagonal metrics in 4D and 5D gravity. The solutions depend on three and, respectively, four coordinates, posses local anisotropies and generalized symmetries. We identified the circumstances under which they may resemble the usual disk solutions but with certain nonlinear polarizations in the static and stationary cases and/or propagating in time or extra dimension. There were also analyzed the conditions when the solutions may be constrained to define vacuum and nonvacuum configurations in Einstein gravity.
Our results and the specific properties of the applied geometric method, in more detail, are as follows:

Let us firstly consider the main features of the so-called anholonomic frame method elaborated in our works \[1, 10, 11, 12, 13, 2\] and its differences from well known approaches to constructing disk, instanton, black hole or toroidal solutions in gravity theories:

The very complicated structure of the Einstein equations and matter field equations in a general curved spacetime gives none hope to find general solutions of such systems of nonlinear partial differential equations. The bulk of known physically important solutions in gravity (see, for instance, Ref. \[16\]) were constructed by some particular types of metric and (for non–vacuum configurations) matter field ansatz reducing the field equations to any systems of algebraic or nonlinear ordinary differential equations. The solutions of such equations depend on (integration) constants which are physically defined from some symmetry prescriptions \(^5\) and some boundary (asymptotic) conditions like the request to get in the limit the Minkowski flat spacetime and the Newton gravitational low. Such solutions are usually parametrized by diagonal metrics, vielbeins and connections which in corresponding coordinates depends on one time, or one space, like coordinate, or there are considered some stationary (rotating) and/or wave type dependencies.

In our works, we used more general ansatz for the gravitational and matter fields, see metric (114) in Appendix, where (for convenience) there are summarized the main results on the anholonomic frame method. In 4D and 5D, such generic off–diagonal metric ansatz \(^6\) depend respectively on three and four variables (coordinates). The constructions can be also performed by analogy for higher or lower dimensions. In the vacuum case, a such ansatz reduces the Einstein equations to a system of nonlinear partial differential equations, see Theorem C.1 which can be integrated in very general form, see Theorem D.1. To prove such results, we applied new geometric concepts and methods which came from the Finsler and Lagrange geometry \[25\] but further adapted to purposes of gravity theories on nonholonomic (pseudo) Riemannian and Riemann–Cartan–Weyl spacetimes \[2, 21, 26\]. The new classes of solutions depend not only on integration constants but also on certain types of integration functions of one, two and three variables, for 4D, and on functions on four variables, in 5D. This is a general property of solutions of systems of partial differential equations.

By applying the anholonomic frame method, many rigorous and sophis-

\(^5\) for instance, one impose the spherical or cylindrical symmetry of spacetime, or some Lie algebra symmetries like in anisotropic cosmology models

\(^6\) they can not be diagonalized by any coordinate transforms
icate solutions of Einstein’s equations have been constructed, see \[1, 10, 11, 12, 13, 2, 15, 26\] and there presented references. A part of such solutions have been considered for physically relevant situations (like locally anisotropic black hole, wormhole, Taub NUT ... solutions). Nevertheless, there is still a problem of complete understanding the physical meaning of solutions depending not only on arbitrary physical constants but on classes of functions arising by integrating systems of nonlinear partial differential equations.

The first fundamental property of such solutions is that they describe classes of spacetimes characterized by certain types of nonholonomy relations for vielbeins with associated nonlinear (N-connection) structure defined by generic off-diagonal metric terms. In this approach, one deals with induced torsions and the gravitational and matter field interactions are modelled on nonholomnomic manifolds, i.e. on manifolds with prescribed nonholonomic, equivalently, anholonomic, or nonintegrable distributions. As a matter of principle, the constructions can be equivalently redefined with respect to coordinate frames with vanishing of induced torsions but the formulas became very cumbersome and non adapted to the fundamental geometric objects. On such nonholonomic spaces, it is convenient to work with more general classes of linear connections (not only with the Levi–Civita one) which allows us to apply the method in string and gauge gravity models when the torsion fields are not trivial. The (psudo) Riemannian configurations can be emphasized by certain special classes of constraints on the nonholonomic frame structure. But even in such cases, the N–connection may be nontrivial and the constructions depend on some classes of integration functions.

The second fundamental property of the nonholonomic spacetimes (defined by generic off–diagonal metrics and nonholonomic frames) is that they are characterized by more general symmetries than in the case of usual Killing spacetimes with spherical, or cylindrical, or with Lie algebra symmetries. In Ref. [2], we analyzed in details some examples of exact solutions characterized by noncommutative symmetries which (surprisingly) are present even on real 'off–diagonal' spacetimes and can be emphasized if the nonholonomic deformations of metrics are associated to certain types of Siberg–Witten transforms.

Developing such ideas, we proposed to characterize the new classes of generic off–diagonal solutions also by Lie algebroid symmetries, or corresponding generalizations to Clifford algebroids [15] for Einstein–Dirac systems. Roughly speaking, the concept of Lie algebroid generalizes the concept of Lie algebra to a case when the structure constants transform into structure functions depending on a base manifold coordinates and one can work with singular structures defined by the so–called ‘anchor’ map (see the
main concepts and results in Refs. [14]). In a slight different interpretation, a spacetime with Lie algebroid symmetry may be considered as a geometric construction modelling the fibered and/or bundle spaces on nonholonomic manifolds if anholonomic frames and their deformations are introduced into consideration. This allows us to work with classes of solutions characterized by structure functions and nonholonomic frame structures associated to integration functions defined by constructing more general type of solutions of the Einstein equations.

Of course, we may impose further constraints and see what happen when such ansatz depend on one time, or space/extra dimension, like variable and the Einstein equations transform into some systems of algebraic or nonlinear ordinary differential equations. These are very restricted cases when we loose certain quality and fundamental properties of the generic gravitational and matter field equations. The real world and gravity can not be completely analyzed only by ansatz resulting in ordinary differential equations, or algebraic systems. In general, the gravitational and matter field interactions are described, by multi–dimensional nonlinear and nonperturbative effects which can not be derived only from solutions of ordinary differential equations. The anholonomic frame method is a geometric one giving the possibility to construct exact solutions of gravitational and matter field equations reduced to systems of nonlinear partial equations on 2–4 variables. Here, we note that it is possible to consider more particular classes of integration functions when 'far away' from such nonholonomic gravitational–matter configurations the spacetime will have a Minkowski or (anti) de Sitter limit. But for certain finite, or infinite, regions, the nonholonomic spacetimes will be characterized, for instance, by algebroid and/or noncommutative symmetries.

A special interest presents the classes of metrics with algebroid symmetries which can be defined by a chain of conformal and/or nonholonomic deformations of a well known class of exact solutions describing certain physically interesting situations. In this work we investigated in details how a given disk solution (for instance, any Neugebauer–Meinel one [7]) can be nonholonomically extended to another type of solutions, depending correspondingly on classes of functions on 2–4 variables, in nonholonomic 4D and 5D gravity. We considered examples from general gravity, with nontrivial cosmological constant, bosonic string gravity with so–called $H$–field torsion. There were also analyzed various type of nontrivial gravitational 3D solitonic background deformations and nonholonomic Einstein–Dirac solutions with disk like symmetries. For small deformations, such new solutions seem to preserve the properties of usual disk solutions but with additional anisotropic polarizations and new type of symmetries. In general, the disk 'character' of solutions is broken by nonholonomic transforms.
Finally, we note that in our further works we shall be interested in constructing and investigation of certain exact solutions describing nonlinear superpositions of some black ellipsoid/torus/hole, or wormhole, configurations with a polarized disk configuration, or to say that a disk configuration is moving self-consistently in a solitonic background in 4D or extra dimension. This would be a further development of results in [10, 11, 12, 13] in order to generate solutions with nontrivial Lie algebroid and/or noncommutative symmetries and see their applications in modern cosmology and astrophysics.

Acknowledgement: The work is supported by a sabbatical fellowship of the Ministry of Education and Research of Spain.

A Vielbeins and N–Connections

In this section we recall some basic facts on the geometry of nonholonomic frames (equivalently, vielbeins) with associated nonlinear connection (N–connection) structure in Riemann–Cartan spaces, see Ref. [26] for more general constructions in generalized Finsler–affine geometry.

A spacetime is modelled as a manifold $V^{n+m}$ of dimension $n + m$, with $n \geq 2$ and $m \geq 1$. The local coordinates are labelled in the form $u^\alpha = (x^i, y^a)$ when Greek indices $\alpha, \beta, ...$ split into subclasses like $\alpha = (i, a)$, $\beta = (j, b)$ ... where the Latin indices (the so–called horizontal, h, ones) $i, j, k, ...$ run values $1, 2, ... n$ and (the vertical, v, ones) $a, b, c, ...$ run values $n + 1, n + 2, ..., n + m$. We denote by $\pi^\top : TV^{n+m} \to TV^n$ the differential of a map $\pi : V^{n+m} \to V^n$ defined by fiber preserving morphisms of the tangent bundles $TV^{n+m}$ and $TV^n$. The kernel of $\pi^\top$ is just the vertical subspace $vV^{n+m}$ with a related inclusion mapping $i : vV^{n+m} \to TV^{n+m}$.

Definition A.1. A nonlinear connection (N–connection) $N$ on space $V^{n+m}$ is defined by the splitting on the left of an exact sequence

$$0 \to vV^{n+m} \to TV^{n+m} \to TV^{n+m}/vV^{n+m} \to 0,$$

i. e. by a morphism of submanifolds $N : TV^{n+m} \to vV^{n+m}$ such that $N \circ i$ is the unity in $vV^{n+m}$.

Equivalently, a N–connection is defined by a Whitney sum of horizontal (h) subspace $(hV^{n+m})$ and vertical (v) subspaces,

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \quad (79)$$

A spacetime provided with N–connection structure is denoted $V^{n+m}$. This is a nonholonomic manifold because the distribution (79), in general, is not
integrable, i.e. nonholonomic (equivalently, anholonomic). In brief, we call such spaces to be N–anholonomic because their nonholonomy is defined by a N–connection structure. We shall use boldfaced indices for the geometric objects adapted to a N–connection.

Locally, a N–connection is defined by its coefficients $N^a_i (u) = N^a_i (x, y)$, i.e.

$$N = N^a_i (u) d^i \otimes \partial_a,$$

characterized by the N–connection curvature

$$\Omega = \frac{1}{2} \Omega^a_{ij} d^i \wedge d^j \otimes \partial_a,$$

with coefficients

$$\Omega^a_{ij} = \delta_{[j} N^a_{i]} = \partial N^a_i / \partial x^j - \partial N^a_j / \partial x^i + N^b_i \partial N^a_j / \partial y^b - N^b_j \partial N^a_i / \partial y^b. \tag{80}$$

The linear connections are defined as a particular case when the coefficients are linear on $y^a$, i.e. $N^a_i (u) = \Gamma^a_{bj} (x) y^b$.

A general metric structure may be written in the form

$$g = g_{\alpha \beta} (u) e^\alpha \otimes e^\beta = g_{ij} (u) d^i \otimes d^j + h_{ab} (u) \delta^a \otimes \delta^b, \tag{81}$$

where

$$e^\beta = (d^i, \delta^a) \equiv \delta u^\alpha = (\delta x^i = dx^i, \delta y^a = dy^a + N^a_i (u) dx^i) \tag{82}$$

is dual to

$$e_\alpha = (\delta_i, \partial_a) \equiv \frac{\delta}{\delta u^\alpha} = \left( \frac{\delta}{\delta x^i} = \partial_i - N^a_i (u) \partial_a, \frac{\partial}{\partial y^a} \right). \tag{83}$$

For constructions on spaces provided with N–connection structure, we have to consider ‘N–elongated’ operators instead of usual partial derivatives, like $\delta_j$ in (80) and (83).

We emphasize that a set of coefficients $N^a_i$ defines a special class of frame transforms parametrized by matrices of type

$$e^\alpha_\beta = \begin{bmatrix} e^i_\alpha (u) & N^b_i (u) e^\beta_\alpha (u) \\ 0 & e^a_\alpha (u) \end{bmatrix}, \tag{84}$$

$$e^\beta_\alpha = \begin{bmatrix} e^i_\beta (u) & -N^b_k (u) e^k_\alpha (u) \\ 0 & e^a_\beta (u) \end{bmatrix}. \tag{85}$$

---

We also use boldfaced symbols in order to emphasize that some geometric objects are defined with respect to vielbeins with associated N–connection structure.
In a particular case, one put \( e_i^i = \delta_i^i \) and \( e_a^a = \delta_a^a \) with \( \delta_i^i \) and \( \delta_a^a \) being the Kronecker symbols, defining a global splitting of \( V^{n+m} \) into "horizontal" and "vertical" subspaces (respectively, h- and v–subspaces) with the vielbein structure
\[
e_\alpha = e_\alpha^\alpha \partial_\alpha \quad \text{and} \quad e_\beta = e_\beta^\beta du_\beta,
\]
where we underline the indices related to the local coordinate basis \( \partial_\alpha = \partial/\partial u_\alpha \) and \( du_\beta \). We shall omit underlining of indices if this will not result in ambiguities.

The metric (81) can be equivalently written in "off–diagonal" form
\[
g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta,
\]
for the coefficients
\[
g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_i^e h_{ae} \\ N_j^e h_{be} & h_{ab} \end{bmatrix}
\]
computed with respect to a coordinate co–basis \( du^\alpha = (dx^i, dy^a) \).

The N–coframe (82) satisfies the anholonomy relations
\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w_{\alpha\beta\gamma}(u) e_\gamma
\]
with nontrivial anholonomy coefficients \( w_{\alpha\beta\gamma}(u) \) computed
\[
w^a_{\ ji} = -w^a_{\ ij} = \Omega^a_{\ ij}, \quad w^b_{\ ia} = -w^b_{\ ai} = \partial_a N_i^b.
\]

It should be noted that, in general, a metric (86) is generic off–diagonal, i.e. it cannot be diagonalized by any coordinate transforms. In general, the anholonomy coefficients (88), defined by the off–diagonal representation (86), or equivalently by the block representation (81), do not vanish.

The geometric constructions can be adapted to the N–connection structure. For instance, one can be elaborated this approach to the theory of linear connections:

**Definition A.2.** A distinguished connection (d–connection) \( D = \{ \Gamma^\alpha_{\beta\gamma} \} \) on \( V^{n+m} \) is a linear connection conserving under parallelism the Whitney sum (79).

One also uses the term d–tensor for the decompositions of tensor with respect to N–adapted bases and say that (81) is a d–metric structure. The N–adapted components \( \Gamma^\gamma_{\alpha\beta}(u) \) of a d–connection \( D = \{ D_\alpha \} \) (equivalently, a covariant derivative) are defined by the equations
\[
\Gamma^\gamma_{\alpha\beta}(u) = (D_\alpha e_\beta)] e^\gamma.
\]
The operations of h- and v-covariant derivations, \( D^h \) \( \{ L^{jk}, L^{bk} \} \) and \( D^v \) \( \{ C^{jk}, C^{bc} \} \) are introduced by corresponding h- and v–parametrizations of \((88)\),

\[
L^{jk} = (D_k \delta^j) \| d^i, \quad L^{bk} = (D_k \partial_b) \| \delta^a, \quad C^{jk} = (D_c \delta_j) \| d^i, \quad C^{bc} = (D_c \partial_b) \| \delta^a.
\]

The components \( \Gamma_{\gamma \alpha \beta} = (L^{jk}, L^{bk}, C^{jk}, C^{bc}) \) completely define a linear connection \( \mathbf{D} \) in \( N \)-adapted form the global splitting of \( \mathbf{V}^{n+m} \) into h- and v–subspaces. We can consider a corresponding d–connection 1–form

\[
\Gamma^\gamma_\alpha = \Gamma_{\alpha \beta}^\gamma e^\beta
\]

and say that a d–connection \( \mathbf{D}_\alpha \) is compatible with a metric \( g \) if

\[
Dg = 0. \quad (90)
\]

Using the covariant derivative \( D \), we can define the torsion tensor

\[
\mathbf{T}^\alpha = D e^\alpha = de^\alpha + \Gamma^\gamma_\beta \wedge e^\beta \quad (91)
\]

and the curvature tensor

\[
\mathbf{R}^\alpha_\beta = D \Gamma^\alpha_\beta = d \Gamma^\alpha_\beta - \Gamma^\gamma_\beta \wedge \Gamma^\alpha_\gamma \quad (92)
\]

where ”\( \wedge \)” denotes the antisymmetric product of forms.

If a spacetime \( \mathbf{V}^{n+m} \) is provided with both \( N \)-connection \( \mathbf{N} \) and d–metric \( g \) structures, there is a unique linear symmetric and torsionless connection \( \nabla \), called the Levi–Civita connection. This connection is metric compatible, i.e. \( \nabla g_{\alpha \beta} = 0 \) for \( g_{\alpha \beta} = (g_{ij}, h_{ab}) \), see \((81)\), with the coefficients

\[
\nabla \Gamma_{\alpha \beta \gamma} = g(e_\alpha, \nabla_\gamma e_\beta) = g_{\alpha \tau} \nabla \Gamma_{\tau \beta \gamma},
\]

computed as

\[
\nabla \Gamma_{\alpha \beta \gamma} = \frac{1}{2} \left[ e_\beta g_{\alpha \gamma} + e_\gamma g_{\alpha \beta} - e_\alpha g_{\gamma \beta} + g_{\alpha \tau} w_{\gamma \beta}^\tau + g_{\beta \tau} w_{\alpha \gamma}^\tau - g_{\gamma \tau} w_{\beta \alpha}^\tau \right] \quad (93)
\]

with respect to \( N \)-frames \( e_\beta \) \((83)\) and \( N \)-coframes \( e^\alpha \) \((82)\), this formula is proved for any nonholonomic frames, for instance, in Refs. \([27]\). We note that the Levi–Civita connection is not adapted to the \( N \)-connection structure: its h– and v–coefficients can not defined in a form preserved under coordinate and frame transforms.

There is a type of d–connections which are similar to the Levi–Civita connection and satisfy certain metricity conditions, such metrics being adapted
to the N–connection. One considers the so–called canonical d–connection
\[ \hat{\Gamma}_\alpha = \hat{\Gamma}_\alpha^{\beta}\epsilon^\beta, \]
which ”minimally” extends the Levi–Civita connection in order to be N–adapted, metric compatible and defined only by the coefficients of metric (86) (equivalently by the block representation (81)), see a general proof in [26]. It is defined by the components
\[ \hat{\Gamma}_\gamma^{\alpha\beta} = (\hat{L}_{jk}, \hat{L}_{bk}, \hat{C}_{jc}, \hat{C}_e), \]
where
\[ \hat{L}_{jk} = \frac{1}{2} g^{ir} \left( \frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right), \]
\[ \hat{L}_{bk} = \frac{\partial N_k^a}{\partial y^b} + \frac{1}{2} h^{ac} \left( \frac{\delta h_{bc}}{\delta x^k} - \frac{\partial N_k^d}{\partial y^a} h_{dc} - \frac{\partial N_k^d}{\partial y^c} h_{db} \right), \]
\[ \hat{C}_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c}, \]
\[ \hat{C}_e = \frac{1}{2} h^{ad} \left( \frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right). \]
This connection satisfies the torsionless conditions for the h– and v–subspaces, when, respectively, \( \hat{T}_{jk} = \hat{T}_{bk} = 0 \). On such subspaces, it has the usual properties of the Levi–Civita connection but, in general, it contains certain additional coefficients induced by the nonholonomic frame structure and N–connection coefficients (see below the formulas (96) but re–defined for \( \hat{L}_{jk}, \hat{C}_{jc}, \) and \( \hat{C}_e \)).

The coefficients of the Levi–Civita connection \( \nabla \Gamma_\beta \gamma \) and of the canonical d–connection \( \hat{\Gamma}_\beta \gamma \) are related by the formulas
\[ \nabla \Gamma_\beta \gamma = \left( \hat{L}_{jk}, \hat{L}_{bk}, \hat{C}_{jc} + \frac{1}{2} g_{jk} \Omega_{jk}^{a} h_{ca}, \hat{C}_e \right), \]
where \( \Omega_{jk}^{a} \) is the N–connection curvature (80). The proof of this results consists from a straightforward computation of the coefficients of \( \nabla \Gamma_\beta \gamma \) with respect to the nonholonomic bases (82) and (83) and a re–definition of coefficients following (94).

The torsion \( T_{\alpha \beta \gamma} = (T_{ijk}, T_{ja}, T_{aij}, T_{bi}, T_{bc}) \) of a d–connection \( \Gamma_\gamma^{\alpha\beta} \) is defined by corresponding d–torsions
\[ T_{ijk} = \hat{L}_{jk} - \hat{L}_{kj}, T_{ja} = C_{ja}, T_{aij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i} = \Omega_{ji}^{a}, \]
\[ T_{bi} = P_{bi}^a \frac{\delta N^a_i}{\delta y^b} - L_{bj}^a, T_{bc} = S_{bc}^a = C_{bc}^a - C_{eb} \]
computed in explicit form by distinguishing the formulas (91) with respect to the N–adapted vielbeins (83) and (82). We note that on (pseudo) Riemannian
In spacetimes the \( d \)-torsions can be induced by the \( N \)-connection coefficients and reflect the anholonomic character of the \( N \)-adapted vielbein structure. Such objects vanish when we transfer our considerations with respect to holonomic bases for a trivial \( N \)-connection and zero "vertical" dimension.

The curvature \( \mathbf{R}^a_{\beta\gamma\tau} = (R^i_{hjk}, R^a_{bijk}, P^c_{bka}, S^i_{jbc}, S^a_{bcd}) \) of a \( d \)-connection \( \Gamma^\gamma_{\alpha\beta} \) is defined by corresponding \( d \)-curvatures

\[
R^i_{hjk} = \left( \frac{\delta L^i_{hj}}{\delta x^k} - \frac{\delta L^i_{hk}}{\delta x^j} + L^m_{hj}L^i_{mk} - L^m_{hk}L^i_{mj} - C^i_{,ka}Q^a_{jk}, \right) \\
R^a_{bijk} = \left( \frac{\delta L^a_{bijk}}{\delta x^j} - \frac{\delta L^a_{bk}}{\delta x^i} + L^c_{bijk}L^a_{ck} - L^c_{bck}L^a_{ej} - C^a_{,bc}Q^e_{jk}, \right)
\]

\[
P^i_{jka} = \left( \frac{\partial L^i_{jk}}{\partial y^k} - \left( \frac{\partial C^a_{ja}}{\partial x^k} + L^i_{,jk}C^a_{ja} - L^i_{ja}C^a_{,jk} \right) \right) + C^a_{jb}P^b_{,ka}, \\
P^c_{bka} = \left( \frac{\partial L^c_{bk}}{\partial y^a} - \left( \frac{\partial C^d_{ba}}{\partial x^k} + L^c_{dk}C^d_{ba} - L^d_{dk}C^c_{da} \right) \right) + C^c_{bd}P^d_{,ka},
\]

\[
S^i_{jbc} = \left( \frac{\partial C^i_{jb}}{\partial y^c} - \frac{\partial C^i_{jc}}{\partial y^b} + C^d_{jb}C^i_{,hc} - C^d_{jc}C^i_{,hb} \right), \\
S^a_{bcd} = \left( \frac{\partial C^a_{bc}}{\partial y^d} - \frac{\partial C^a_{bd}}{\partial y^c} + C^e_{bc}C^a_{,ed} - C^e_{bd}C^a_{,ec} \right).
\]

Such formulas follow from an explicit coefficient calculus of (92) with respect to the \( N \)-adapted vielbeins (83) and (82). They are equivalent to the formulas given in [27, 16] but rewritten in a form adapted to vielbein transforms (84) and (85).

The Ricci tensor

\[ \mathbf{R}^a_{\alpha\beta} = \mathbf{R}^T_{\alpha\beta\tau} \]

is characterized by four \( d \)-tensor components \( \mathbf{R}^a_{\alpha\beta} = (R_{ij}, R_{ia}, R_{ai}, S_{ab}) \), where

\[
R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -2P^k_{,ia} = -P^k_{ika}, \\
R_{ai} \doteq 1P^i_{aib}, \quad S_{ab} \doteq S^c_{abc}.
\]

It should be emphasized that because, in general, \( 1P^i_{ai} \neq 2P^i_{ia} \) the Ricci \( d \)-tensors are non symmetric (this a nonholonomic frame effect). A such tensor became symmetric with respect to holonomic vielbeins and for the Levi–Civita connection.
Contracting with the inverse to a d–metric of type \([81]\) in \(V^{n+m}\), we can introduce the scalar curvature of a d–connection \(D\),
\[
\hat{R} = g^{\alpha \beta} R_{\alpha \beta} \div R + S,
\]
where \(R \div g^{ij} R_{ij}\) and \(S \div h^{ab} S_{ab}\) and compute the Einstein tensor
\[
G_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} \hat{R}.
\]
In the vacuum case, \(G_{\alpha \beta} = 0\), that the Ricci d–tensors \([98]\) vanish.

\section*{B \ N–Anholonomic Frames and String Gravity}

The Einstein equations for the canonical d–connection \(\hat{\Gamma}^\gamma_{\alpha \beta}\) \([94]\),
\[
\hat{R}_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} \hat{R} = \kappa \Upsilon_{\alpha \beta},
\]
are defined for a general source of matter fields and possible string corrections, \(\Upsilon_{\alpha \beta}\). The model contains a nontrivial torsion \(\hat{T}^\gamma_{\alpha \beta}\) with d–torsions computed by introducing the components of \([94]\) into formulas \([96]\).

We can express the 1–form of the canonical d–connection \(\hat{\Gamma}^\gamma_{\alpha}\) via the deformation of the Levi–Civita connection \(\nabla \Gamma^\gamma_{\alpha}\),
\[
\hat{\Gamma}^\gamma_{\alpha} = \nabla \Gamma^\gamma_{\alpha} + \hat{Z}^\gamma_{\alpha}
\]
where
\[
\hat{Z}^\alpha_{\beta} = e_{\beta} \| \hat{T}_\alpha - e_{\alpha} \| \hat{T}_\beta + \frac{1}{2} \left( e_{\alpha} \| e_{\beta} \| \hat{T}_\gamma \right) e^\gamma.
\]
This mean that it is possible to split all geometric objects into (pseudo) Riemannian and post–Riemannian pieces, for instance,
\[
\hat{R}^\alpha_{\beta} = R^\alpha_{\beta} + \nabla Z^\alpha_{\beta},
\]
\[
\nabla Z^\alpha_{\beta} = \hat{\nabla} Z^\alpha_{\beta} + \hat{Z}^\alpha_{\gamma} \land \hat{Z}^\gamma_{\beta}.
\]
We conclude that the Einstein equations \([101]\) for the canonical d–connection \(\hat{\Gamma}^\gamma_{\alpha}\) constructed for a d–metric \(g_{\alpha \beta} = [g_{ij}, h_{ab}]\) \([81]\) and N–connection \(N^a_i\) are equivalent to the gravitational field equations for the Einstein–Cartan theory with torsion \(\hat{T}^\gamma_{\alpha}\) defined by the N–connection.
The Einstein gravity theory is defined by the condition that the deformation tensor $\nabla Z_{\alpha \beta}^\gamma$ (103) is algebraically constrained that its coefficients satisfy the equations

$$\nabla Z_{\alpha \beta}^\gamma = 0$$

which, as a matter of principle, for vacuum configurations, can be solved for nonzero values of $\hat{Z}_{\alpha}$. In the presence of general matter sources, we may have to impose the condition $\hat{Z}_{\alpha} = 0$ because distortions of the covariant derivatives may be contained in the field equations and energy–momentum of matter.

The Einstein equations (101) can be decomposed into h– and v–components following from (98) and (99),

$$\hat{R}_{ij} - \frac{1}{2} g_{ij} (\hat{R} + \hat{S}) = \Upsilon_{ij},$$

$$\hat{S}_{ab} - \frac{1}{2} h_{ab} (\hat{R} + \hat{S}) = \Upsilon_{ab},$$

$$\hat{P}_{ai} = \Upsilon_{ai},$$

$$-2 \hat{P}_{ia} = \Upsilon_{ia}.$$ (108)

The sources of such equations have to be defined as certain matter field contributions or corrections, for instance, from string gravity. For instance, in the sigma model for bosonic string (see, [18]), the background connection is taken not the Levi–Civita one but a certain deformation by the strength (torsion) tensor

$$H_{\mu \nu \rho} \doteq e_\mu B_{\nu \rho} + e_\rho B_{\mu \nu} + e_\nu B_{\rho \mu}$$

of an antisymmetric field $B_{\nu \rho}$. The connection is of type

$$D_\mu = \nabla_\mu + \frac{1}{2} H_{\mu \nu}^\rho.$$ (109)

For trivial dilaton configurations, we may write

$$R_{\mu \nu} = -\frac{1}{4} H_\mu^\lambda H_\nu^\rho, H_{\mu \nu}^\lambda = 0.$$ (110)

The equations may be re–defined with respect to N–adapted frames [82] and [83] for "boldfaced" values.

Here we consider, for simplicity, a model with zero dilaton field but with nontrivial $H$–field related to the d–torsions induced by the N–connection and canonical d–connection when a class of generic off–diagonal metrics can be
derived from the bosonic string theory if $H_{\nu\lambda\rho}$ and $B_{\nu\rho}$ are related to the d–torsions components $\hat{T}_{\alpha\beta}$. We can take a special ansatz for $B$–field,

$$B_{\nu\rho} = [B_{ij}, B_{ia}, B_{ab}] ,$$

and consider that

$$H_{\nu\lambda\rho} = \hat{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho}$$

(109)

where $\hat{Z}_{\nu\lambda\rho}$ is the distortion of the Levi–Civita connection induced by $\hat{T}_{\alpha\beta}$, see (102), and $\hat{H}_{\nu\lambda\rho}$ is generated by nonholonomic deformations of $H_{\nu\lambda\rho}$.

In this case, the induced by N–connection torsion structure is related to the antisymmetric $H$–field and correspondingly to the $B$–field from string theory. The equations

$$\nabla^\nu H_{\nu\lambda\rho} = \nabla^\nu (\hat{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho}) = 0$$

(110)

impose certain dynamical restrictions to the N–connection coefficients $N_i^a$ and d–metric $g_{\alpha\beta} = [g_{ij}, h_{ab}]$ contained in $\hat{T}_{\alpha\beta}$. If it is prescribed the canonical d–connection $\hat{D}$ on the background space, we can state a model with the equations (110) written in the form

$$\hat{D}^\nu H_{\nu\lambda\rho} = \hat{D}^\nu (\hat{Z}_{\nu\lambda\rho} + \hat{H}_{\nu\lambda\rho}) = 0 ,$$

(111)

where $\hat{H}_{\nu\lambda\rho}$ are computed for stated values of $\hat{T}_{\alpha\beta}$. For trivial N–connections, when $\hat{Z}_{\nu\lambda\rho} \rightarrow 0$ and $\hat{D}^\nu \rightarrow \nabla^\nu$, the $\hat{H}_{\nu\lambda\rho}$ transform into usual $H$–fields.

The dynamics of gravity of nonholonomic string corrections by the field $H_{\nu\lambda\rho}$ is defined by the system of field equations

$$\hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R} = -\frac{1}{4} H_{\alpha\nu\rho} H_{\nu\beta\rho}$$

(112)

and the $H$–field equations (111) In this case, the sources for the equations (105)–(108) are defined by the corresponding $h$– and $v$–projections of

$$\Upsilon^H_{\alpha\beta} = -\frac{1}{4} H_{\alpha\nu\rho} H_{\nu\beta\rho} .$$

It was possible to generate exact solutions for (112) for such ansats for $H_{\nu\lambda\rho}$ when $\Upsilon^H_{\alpha\beta}$ is diagonal with respect to N–adapted frames [2, 26].

C Off–Diagonal Ansatz

In a series of papers [1, 11, 12, 13, 2, 15, 26] we elaborated and developed the anholonomic frame method of constructing exact solutions with generic
off–diagonal metrics (depending on 2–4 variables) in general relativity, gauge gravity and various generalizations to string, brane and generalized Finsler–affine gravity models. In this section, we outline the necessary results and present some details on proofs of such results.

C.1 A Theorem for the 5D Ricci d–tensors

Let us consider a 5D manifold of necessary smooth class provided with N–
connection structure \( N = [N_i^4 = w_i(u^\alpha), N_i^5 = n_i(u^\alpha)] \) and a d–metric of type (81). We compute the components of the Ricci and Einstein tensors for a particular ansatz for d–metric

\[
\delta s^2 = g_1(dx^1)^2 + g_2(x^2,x^3)(dx^2)^2 + g_3(x^4)(dx^3)^2 + h_4(x^k,v)(\delta v)^2 + h_5(x^k,v)(\delta y)^2,
\]

\[
\delta v = dv + w_i(x^k,v)dx^i, \quad \delta y = dy + n_i(x^k,v)dx^i
\]

with \( g_1 = \text{const}, N_i^4 = w_i(x^k,v) \) and \( N_i^5 = n_i(x^k,v) \). The local coordinates are labelled in the form \( u^\alpha = (x^i,y^4,v) \), for \( i = 1,2,3 \). Every coordinate from a set \( u^\alpha \) can may be time like, 3D space like, or extra dimensional. We note that the metric (113) does not depend on variable \( y^5 = y \), but emphasize the dependence on the so–caled ”anisotropic” variable \( y^4 = v \). For simplicity, the partial derivatives will be written in the form \( a^x = \partial a/\partial x^1, a^* = \partial a/\partial x^2, a' = \partial a/\partial x^3, a^* = \partial a/\partial v \).

In equivalent form, we can re–write (113) in off–diagonal form (86) when

\[
\delta s^2 = g_{\alpha\beta}(x^i,v)du^\alpha du^\beta
\]

has the metric coefficients \( g_{\alpha\beta} \) are parametrized by the matrix

\[
\begin{bmatrix}
g_1 + w_{11}h_4 + n_{11}h_5 & w_{12}h_4 + n_{12}h_5 & w_{13}h_4 + n_{13}h_5 & w_1h_4 & n_1h_5 \\
w_{21}h_4 + n_{21}h_5 & g_2 + w_{22}h_4 + n_{22}h_5 & w_{23}h_4 + n_{23}h_5 & w_2h_4 & n_2h_5 \\
w_{31}h_4 + n_{31}h_5 & w_{32}h_4 + n_{32}h_5 & g_3 + w_{33}h_4 + n_{33}h_5 & w_3h_4 & n_3h_5 \\
w_1h_4 & w_2h_4 & w_3h_4 & h_4 & 0 \\
n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5
\end{bmatrix}
\]

where \( w_{ij} = w_iw_j \) and \( n_{ij} = n_in_j \), with the coefficients defined by some necessary smoothly class functions of type

\[
g_1 = \pm 1, g_{2,3} = g_{2,3}(x^2,x^3), h_{4,5} = h_{4,5}(x^i,v),
\]

\[
w_i = w_i(x^i,v), n_i = n_i(x^i,v).
\]
Theorem C.1. The nontrivial components of the 5D Ricci d–tensors $\hat{R}_{\alpha\beta}$, $\hat{R}_{\gamma\delta}(g_{ij}, \hat{R}^i_{\alpha}, \hat{S}_{ab})$, for the d–metric (113) and the canonical d–connection $\hat{\Gamma}^\gamma_{\alpha\beta}(g_{ij})$, all components being computed with respect to the corresponding $N$–anholonomic frames (82) and (83) are stated by formulas

$$
R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_{3}^{**} - \frac{g_2g_3}{2} - \frac{(g_3^{*})^2}{2g_3} - \frac{g_2'g_3'}{2g_3} - \frac{(g_2')^2}{2g_2}],
$$

$$
S_4^4 = S_5^5 = -\frac{1}{2h_4h_5} \left[ h_5^{**} - h_5^* \left( \ln \sqrt{|h_4h_5|} \right)^* \right],
$$

$$
R_{4i} = -w_i^\beta \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5},
$$

$$
R_{5i} = -\frac{h_5}{2h_4} [n_i^{**} + \gamma n_i^*],
$$

where

$$
\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4h_5|}, \beta = h_5^{**} - h_5^* \left[ \ln \sqrt{|h_4h_5|} \right]^*,
$$

$$
\gamma = 3h_5^*/2h_5 - h_4^*/h_4, h_4^* \neq 0, h_5^* \neq 0.
$$

We note that d–metrics with vanishing $h_4^*$ or $h_5^*$ should be analyzed as special cases. In general form, a such theorem is proved for Finsler–affine species in Ref. [26], see also the Appendix to [2] where the results are summarized for Riemann–Cartan and Einstein spaces, in general, with noncommutative symmetries.

C.2 Proof of Theorem C.1

In this appendix, we give the details of the proof for the (pseudo) Riemann–Cartan nonholonomic manifolds. It contains a number of examples of N–adapted differential calculus not presented in physical literature on gravity and strings. We note that such results can be not obtained with standard analytic programs (Mathematica 5 and/or Maple) because in our case we work with the canonical d–connection and N–adapted frames.

It is a cumbersome task to perform tensor calculations (for instance, of the curvature and Ricci tensors) for the off–diagonal ansatz (114) but the formulas simplify substantially for the effectively diagonalized metric (113). In this case, the N–adapted frames of type (82) and (83) and are defined respectively by the pentads (frames, funfbeins)

$$
e^i = dx^i, e^4 = \delta v = dv + w_i (x^k, v) dx^i, e^5 = \delta y = dy + n_i (x^k, v) dx^i
$$

(120)
and the N–elongated partial derivative operators,

\[ e_i = \delta_i = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^i} - w_i \frac{\partial}{\partial v} - n_i \frac{\partial}{\partial y}, \]

\[ e_4 = \frac{\partial}{\partial y^4} = \frac{\partial}{\partial v}, \quad e_5 = \frac{\partial}{\partial y^5} = \frac{\partial}{\partial y}. \]  

(121)

The N–elongated partial derivatives of a function \( f(u^\alpha) = f(x^i, y^a) \) are computed in the form

\[ \delta^2 f = \delta f \frac{\partial}{\partial x^2} = \delta f \frac{\partial}{\partial x} = \frac{\partial f}{\partial x} - N_2^a \frac{\partial f}{\partial y^a} = \frac{\partial f}{\partial x^2} - w_2 \frac{\partial f}{\partial v} - n_2 \frac{\partial f}{\partial y} = f^* - w_2 f' - n_2 f^* \]

where

\[ f^* = \frac{\partial f}{\partial x^2}, \quad f' = \frac{\partial f}{\partial x^3}, \quad f^* = \frac{\partial f}{\partial y^3} = \frac{\partial f}{\partial v}. \]

The N–elongated differential is expressed

\[ \delta f = \frac{\partial f}{\partial u^\alpha} e^\alpha. \]

One perform a N–adapted differential calculus if we work with respect to N–elongated frames and partial derivatives and differentials.

### C.2.1 Calculation of the N–connection curvature

We compute the coefficients (80), for the d–metric (113) (equivalently, the ansatz (114)), defining the curvature of N–connection \( N_i^a \), by substituting \( N_4^i = w_i(x^k, v) \) and \( N_5^i = n_i(x^k, v) \), where \( i = 2, 3 \) and \( a = 4, 5 \) (for our ansatz, we do not have dependence on \( x^1 \) and \( y^5 \)). The results for nontrivial values are

\[ \Omega_{23}^4 = -\Omega_{23}^4 = w_2' - w_3^* - w_3 w_2^* + w_2 w_3^*, \]

\[ \Omega_{23}^5 = -\Omega_{23}^5 = n_2' - n_3^* - w_3 n_2^* + w_2 n_3^*. \]  

(122)

Such values must be zero if we wont to generate a solution for the Levi–Civita connection. This is a necessary but not enough condition, see (95). Even in this case, the nonholonomy coefficients (88) can be nonzero if there are certain values \( \partial_a N_i^b \neq 0 \).
C.2.2 Calculation of the canonical d–connection

We compute the coefficients $\hat{\Gamma}^{\gamma}_{\alpha \beta} = (\hat{L}^{i}_{jk}, \hat{L}^{a}_{bk}, \hat{C}^{a}_{jc}, \hat{C}^{a}_{bc})$ (94) for the d–metric (113) (equivalently, the ansatz (114)) when $g_{jk} = \{g_{j}\}$ and $h_{bc} = \{h_{b}\}$ are diagonal and $g_{jk}$ depend only on $x^2$ and $x^3$ but not on $y$. We have

$$\begin{align*}
\delta_k g_{ij} &= \partial_k g_{ij} - w_k g_{ij}^* = \partial_k g_{ij}, \quad \delta_k h_{b} = \partial_k h_{b} - w_k h_{b}^* \\
\delta_k w_{i} &= \partial_k w_{i} - w_k w_{i}^*, \quad \delta_k n_{i} = \partial_k n_{i} - w_k n_{i}^* \\
\end{align*}$$

resulting in formulas

$$\hat{L}^{i}_{jk} = \frac{1}{2} g^{ir} \left( \frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right) = \frac{1}{2} g^{ir} \left( \frac{\partial g_{jk}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^r} \right)$$

The nontrivial values of $\hat{L}^{i}_{jk}$ are

$$\hat{L}^{2}_{22} = \frac{g_{2}^*}{2g_{2}}, \quad \hat{L}^{2}_{23} = \frac{g_{2}}{2g_{2}} = \alpha_{2}, \quad \hat{L}^{2}_{33} = -\frac{g_{2}^*}{2g_{2}}$$

$$\hat{L}^{3}_{22} = -\frac{g_{3}^*}{2g_{3}}, \quad \hat{L}^{3}_{23} = \frac{g_{3}^*}{2g_{3}} = \alpha_{3}, \quad \hat{L}^{3}_{33} = \frac{g_{3}}{2g_{3}} = \alpha_{3}.$$ 

In a similar form we compute the components

$$\hat{L}^{a}_{bk} = \partial_b N^{a}_{k} + \frac{1}{2} h^{ac} \left( \partial_k h_{bc} - N^{d}_{k} \frac{\partial h_{bc}}{\partial y^d} - h_{dc} \partial_b N^{d}_{k} - h_{db} \partial_c N^{d}_{k} \right)$$

having nontrivial values

$$\hat{L}^{4}_{42} = \frac{1}{2h_{4}} \left( h_{4}^* - w_{2} h_{4}^* \right) = \delta_{2} \ln \sqrt{|h_{4}|} \div \delta_{2} \beta_{4},$$

$$\hat{L}^{4}_{43} = \frac{1}{2h_{4}} \left( h_{4}^* - w_{3} h_{4}^* \right) = \delta_{3} \ln \sqrt{|h_{4}|} \div \delta_{3} \beta_{4}$$

$$\hat{L}^{5}_{5k} = -\frac{h_{5}}{2h_{4}} n_{k}^*, \quad \hat{L}^{5}_{bk} = \partial_b n_{k} + \frac{1}{2h_{5}} \left( \partial_k h_{b5} - w_{k} h_{b5}^* - h_{5} \partial_b n_{k} \right),$$

$$\hat{L}^{5}_{4k} = n_{k}^* + \frac{1}{2h_{5}} \left( -h_{5} n_{k}^* \right) = \frac{1}{2} n_{k}^*,$$

$$\hat{L}^{5}_{5k} = \frac{1}{2h_{5}} \left( \partial_k h_{5} - w_{k} h_{5}^* \right) = \delta_{k} \ln \sqrt{|h_{4}|} = \delta_{k} \beta_{4}.$$ 

We note that

$$\hat{C}^{a}_{jc} = \frac{1}{2} g^{ak} \frac{\partial g_{jk}}{\partial y^{c}} \div 0$$
because \( g_{jk} = g_{jk}(x^i) \) for the considered ansatz.

The values

\[
\hat{C}_{bc}^a = \frac{1}{2} h^{ad} \left( \frac{\partial h_{bd}}{\partial y^e} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right)
\]

for \( h_{bd} = \text{diag}[h_4, h_5] \) have nontrivial components

\[
\hat{C}_{44}^4 = \frac{h_4^*}{2h_4} \beta_4^*, \quad \hat{C}_{55}^5 = -\frac{h_5^*}{2h_4}, \quad \hat{C}_{45}^5 = \frac{h_5^*}{2h_5} \beta_5^*.
\] (129)

The set of formulas (124)–(129) define the nontrivial coefficients of the canonical d–connection

\[
\widehat{\Gamma}_{\alpha\beta}^\gamma = \left( \hat{L}_{ij}^a, \hat{L}_{bk}^a, \hat{C}_{jc}^a, \hat{C}_{bc}^a \right)
\]

(94) for the 5D ansatz (113).

C.2.3 Calculation of d–torsions

We should put the nontrivial values (124)–(129) into the formulas for d–torsion (96). One holds \( \hat{T}_{ij} = 0 \) and \( \hat{T}_{ab} = 0 \), because the coefficients \( \hat{L}_{ij} \) and \( \hat{C}_{bc}^a \) are symmetric for the chosen ansatz.

We have computed the nontrivial values of \( \Omega_{ji}^a \), see (122) resulting in

\[
\hat{T}_{23}^4 = \Omega_{23}^4 = -\Omega_{23}^4 = w_2^* - w_3 w_2^* + w_2 w_3^*,
\]

\[
\hat{T}_{23}^5 = \Omega_{23}^5 = -\Omega_{23}^5 = n_2^* - n_3 w_3^* + w_2 n_3^*.
\] (130)

One follows

\[
\hat{T}_{jc}^i = -\hat{T}_{cj}^i = \hat{C}_{jc}^i = \frac{1}{2} g_{ik} \frac{\partial g_{jk}}{\partial y^c} \div 0,
\]

see (128). For the components

\[
\hat{T}_{bi}^a = -\hat{T}_{ib}^a = \hat{P}_{bi}^a = \frac{\partial N_{ia}^a}{\partial y^b} - \hat{L}_{bj}^a,
\]

i. e. for

\[
\hat{P}_{bi}^4 = \frac{\partial N_{ia}^4}{\partial y^b} - \hat{L}_{bj}^4 = \partial_b w_i - \hat{L}_{bj}^4 \quad \text{and} \quad \hat{P}_{bi}^5 = \frac{\partial N_{ia}^5}{\partial y^b} - \hat{L}_{bj}^5 = \partial_b n_i - \hat{L}_{bj}^5,
\]

we have the nontrivial values

\[
\hat{P}_{4i}^4 = w_i^* - \frac{1}{2h_4} (\partial_i h_4 - w_i h_4^*) = w_i^* - \delta_i \beta_4, \quad \hat{P}_{5i}^5 = \frac{h_5}{2h_4} n_i^*,
\]

\[
\hat{P}_{4i}^5 = \frac{1}{2} n_i^*, \quad \hat{P}_{5i}^5 = -\frac{1}{2h_5} (\partial_i h_5 - w_i h_5^*) = -\delta_i \beta_5.
\] (131)

The formulas (130) and (131) state the nontrivial coefficients of the canonical d–connection for the chosen ansatz (113).
C.2.4 Calculation of the Ricci d-tensors

Let us compute the value $\tilde{R}_{ij} = \tilde{R}_{ij}^k$ for

$$\tilde{R}_{ij}^k = \frac{\delta \tilde{L}_{ij}^k}{\delta x^k} - \frac{\delta \tilde{L}_{ik}^j}{\delta x^j} + \tilde{L}_{mj}^i \tilde{L}_{mk}^j - \tilde{L}_{hk}^i \tilde{L}_{mk}^j - \tilde{C}_{ha}^i \Omega_{jk}^a,$$

from (124). It should be noted that $\tilde{C}_{ha}^i = 0$ for the ansatz under consideration, see (125). We compute

$$\frac{\delta \tilde{L}_{ij}^k}{\delta x^k} = \partial_k \tilde{L}_{ij}^k + N_k^a \partial_a \tilde{L}_{ij}^k = \partial_k \tilde{L}_{ij}^k + w_k \left( \tilde{L}_{ij}^k \right)^* = \partial_k \tilde{L}_{ij}^k$$

because $\tilde{L}_{ij}^k$ do not depend on variable $y^4 = v$.

Taking the derivative of (124), we obtain

$$\partial_2 \tilde{L}_{22}^2 = \frac{g_2^{\bullet \bullet}}{2g_2} - \frac{(g_2)^2}{2(g_2)^2}, \partial_2 \tilde{L}_{23}^2 = \frac{g_2^{\bullet \prime}}{2g_2} - \frac{g_2 g_2^{\prime \prime}}{2(g_2)^2}, \partial_2 \tilde{L}_{33}^2 = -\frac{g_3^{\bullet \bullet}}{2g_3} + \frac{g_2 g_3^{\prime \prime}}{2(g_3)^2},$$

$$\partial_3 \tilde{L}_{22}^3 = -\frac{g_2^{\bullet \prime}}{2g_2} + \frac{g_2 g_3^{\prime \prime}}{2(g_3)^2}, \partial_3 \tilde{L}_{23}^3 = \frac{g_3^{\bullet \prime \prime}}{2g_3} - \frac{(g_3)^2}{2(g_3)^2}, \partial_3 \tilde{L}_{33}^3 = -\frac{g_3^{\bullet \prime}}{2g_3} + \frac{g_3 g_3^{\prime \prime}}{2(g_3)^2}.$$
Let us denote where \( \hat{C}_{ba|k} \) is the covariant h–derivative. Contracting indices, we have

\[
\hat{R}_{bk} = \hat{P}_{bka} = \frac{\partial \hat{L}^a_{bk}}{\partial y^a} - \hat{C}_{ba|k}^a + \hat{C}_{bd}^a \hat{P}_{ka}^d
\]

from \( \hat{g} \), where \( \hat{C}_{ba|k} \) is the covariant h–derivative. Contracting indices, we have

\[
\hat{R}_{bk} = [1] R_{bk} + [2] R_{bk} + [3] R_{bk}
\]

where

\[
[1] R_{bk} = \left( \hat{L}^4_{bk} \right)^* = \beta_4^*,
\]

\[
[2] R_{bk} = -\partial_k \hat{C}_4 + w_k \hat{C}_4^* + \hat{L}^4_{bk} \hat{C}_4 - \hat{L}^4_{bk} \hat{C}_4 = \delta_k \beta_4 + \beta_4^* + \beta_5^*,
\]

\[
[3] R_{bk} = \hat{C}_{4d}^a \hat{P}_{ka}^d = \hat{C}_{4d}^* \hat{P}_{ka}^d + \hat{C}_{5d}^* \hat{P}_{ka}^d = \beta_4^* (w_k - \delta_k \beta_4) - \beta_5^* \delta_k \beta_5,
\]

and

\[
\hat{C}_4 = \hat{C}_{44}^4 + \hat{C}_{45}^5 = \frac{h_4^*}{2h_4} + \frac{h_5^*}{2h_5} = \beta_4^* + \beta_5^*,
\]

\[
\hat{C}_5 = \hat{C}_{54}^4 + \hat{C}_{55}^5 = 0,
\]

see \( \hat{g} \).

We compute

\[
R_{4k} = [1] R_{4k} + [2] R_{4k} + [3] R_{4k}
\]

with

\[
[1] R_{4k} = \left( \hat{L}^4_{4k} \right)^* = (\delta_k \beta_4)^*,
\]

\[
[2] R_{4k} = -\partial_k \hat{C}_4 + w_k \hat{C}_4^* + \hat{L}^4_{4k} \hat{C}_4 - \hat{L}^4_{4k} \hat{C}_4 = \delta_k \beta_4 + \beta_4^* + \beta_5^* + \beta_4^* (w_k - \delta_k \beta_4) + \beta_5^* \delta_k \beta_5,
\]

\[
[3] R_{4k} = \hat{C}_{44}^4 \hat{P}_{4k}^4 + \hat{C}_{45}^4 \hat{P}_{5k}^4 + \hat{C}_{45}^5 \hat{P}_{5k}^4 = \beta_4^* (w_k - \delta_k \beta_4) - \beta_5^* \delta_k \beta_5,
\]

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Summarizing, we get
\[ \hat{R}_{4k} = w_k \left[ \beta_5^{**} + \left( \beta_4^* \right)^2 - \beta_4^* \beta_5^* \right] + \beta_5^* \partial_k (\beta_4 + \beta_5) - \partial_k \beta_5^*, \]
or, for
\[ \beta_4^* = \frac{h_4^*}{2h_4}, \partial_k \beta_4 = \frac{\partial_k h_4}{2h_4}, \beta_5^* = \frac{h_5^*}{2h_5}, \beta_5^{**} = \frac{h_5^{**} h_5 - (h_5^*)^2}{2(h_5)^3}, \]
we can write
\[ 2h_5 \hat{R}_{4k} = w_k \left[ \frac{(h_5^*)^2}{2h_5} - \frac{h_5^* h_4^*}{2h_4} \right] + \frac{h_5^*}{2} \left( \frac{\partial_k h_4}{h_4} + \frac{\partial_k h_5}{h_5} \right) - \partial_k h_5^* \]
which is equivalent to \[117\].
In a similar way, we compute
\[ \hat{R}_{5k} = [1] R_{5k} + [2] R_{5k} + [3] R_{5k} \]
with
\[ [1] R_{5k} = \left( \hat{L}_{45k}^4 \right)^*, \quad [2] R_{5k} = -\partial_k \hat{C}_5 + w_k \hat{C}_5^* + \hat{L}_{5k}^4 \hat{C}_4, \]
\[ [3] R_{5k} = \hat{C}_{54}^4 \hat{P}_{j4k}^4 + \hat{C}_{55}^5 \hat{P}_{j5k}^5 + \hat{C}_{54}^5 \hat{P}_{j5k}^4 + \hat{C}_{55}^5 \hat{P}_{j5k}^5. \]
We have
\[ \hat{R}_{5k} = \left( \hat{L}_{5k}^4 \right)^* + \hat{L}_{5k}^4 \hat{C}_4 + C_{54}^4 \hat{P}_{j4k}^4 + C_{55}^5 \hat{P}_{j5k}^5 \]
\[ = \left( \frac{h_5^*}{h_4}, \frac{h_5^*}{h_4} \right)^* - \frac{h_5^* h_4^*}{h_4} \left( \frac{h_5^*}{2h_4} + \frac{h_5^*}{2h_5} \right) + \frac{h_5^* h_5}{2h_5 2h_4} n_k^* - \frac{h_5^*}{2h_4 2n_k^*}, \]
or equivalently,
\[ 2h_4 \hat{R}_{5k} = h_5 n_k^{**} + \left( \frac{h_5^*}{h_4} - \frac{3}{2} h_5^* \right) n_k^* \]
which is just the formula \[118\].
For the values
\[ \hat{P}_{jka} = \frac{\partial \hat{L}_{j4k}^i}{\partial y^k} = \left( \frac{\partial \hat{C}_{j4a}}{\partial x^k} + \hat{L}_{j4k}^i \hat{C}_{j4a} - \hat{L}_{j4k}^i \hat{C}^i_{4a} - \hat{L}_{j4k}^i \hat{C}^i_{ja} \right) + \hat{C}_{j4b} \hat{P}_{bka} \]
from \[97\], we obtain zeros because \( \hat{C}_{j4b} = 0 \) and \( \hat{L}_{j4k}^i \) do not depend on \( y^k \).
So,
\[ \hat{R}_{ja} = \hat{P}_{j4a}^i = 0. \]
Taking
\[ \hat{S}_{bcd} = \frac{\partial \hat{C}_{bc}^a}{\partial y^d} - \frac{\partial \hat{C}_{bd}^a}{\partial y^c} + \hat{C}_{bc} \hat{C}_{ed} - \hat{C}_{bd} \hat{C}_{ec}. \]
from (97) and contracting the indices in order to define the Ricci coefficients,
\[ \hat{R}_{bc} = \frac{\partial \hat{C}_{bc}^d}{\partial y^d} - \frac{\partial \hat{C}_{bd}^c}{\partial y^c} + \hat{C}_{bc} \hat{C}_{ed} - \hat{C}_{bd} \hat{C}_{ec} \]
with \( \hat{C}_{bd} = \hat{C}_b \) already computed, see (132), we obtain
\[ \hat{R}_{bc} = \left( \hat{C}_{bc}^4 \right)^* - \partial_c \hat{C}_b + \hat{C}_{bc} \hat{C}_4 - \hat{C}_{bd} \hat{C}_{4c} - \hat{C}_{bd} \hat{C}_{4c} - \hat{C}_{bd} \hat{C}_{5c} - \hat{C}_{bd} \hat{C}_{5c}. \]
There are nontrivial values,
\[ \hat{R}_{44} = \left( \hat{C}_{44}^4 \right)^* - \hat{C}_4 + \hat{C}_{44} \hat{C}_4 - \hat{C}_{44} \hat{C}_4 - \hat{C}_{45}^5 \hat{C}_5 \]
\[ = \beta_4^* - \left( \beta_4^2 + \beta_5^2 \right)^* + \beta_4^2 \left( \beta_5^2 + \beta_4^2 - \beta_5^2 \right) \]
\[ \hat{R}_{55} = \left( \hat{C}_{55}^4 \right)^* - \hat{C}_{55} \hat{C}_4 + 2 \hat{C}_{45} \hat{C}_{45} \]
\[ = - \left( \frac{h_4^*}{2h_4} \right)^* + \hat{h}_5 \left( 2 \beta_5^2 + \beta_4^2 - \beta_5^2 \right) \]
Introducing
\[ \beta_4 = \frac{h_4^*}{2h_4}, \beta_5 = \frac{h_5^*}{2h_5} \]
we get
\[ \hat{R}_4 = \hat{R}_5 = \frac{1}{2h_4h_5} \left[ -h_4^{**} + \left( \frac{h_4^*}{2h_4} \right)^2 + \frac{h_4^* h_5^*}{2h_4} \right] \]
which is just (116).
Theorem C.1 is proven.

C.3 Some consequences of the Theorem C.1

There are certain geometrical properties of the ansatz (114) and its canonical d–connection (94).

Corollary C.1. The non–trivial components of the Einstein tensor \( \hat{G}^\alpha_{\beta} = \hat{R}^\alpha_{\beta} - \frac{1}{2} \hat{R} \delta^\alpha_{\beta} \) for the d–metric (114) given with respect to the N–adapted (co) frames (82) and (83) are
\[ \hat{G}_1^1 = - \left( \hat{R}_2^2 + \hat{S}_4^4 \right), \hat{G}_2^2 = \hat{G}_3^3 = - \hat{S}_4^4, \hat{G}_4^4 = \hat{G}_5^5 = - \hat{R}_2^2. \]
The relations (133) can be derived following the formulas for the Ricci tensor (115)–(118). They impose the condition that the nonholonomically constrained dynamics of such gravitational fields is defined by two independent components $\hat{R}_2^2$ and $\hat{S}_4^i$ and result in:

**Corollary C.2.** The system of 5D Einstein equations is compatible for the generic off–diagonal ansatz (113) if the energy–momentum tensor $\Upsilon_{\alpha\beta}$, given with respect to $N$–adapted frames (82) and (83), is diagonal and satisfies the conditions

$$\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v), \quad \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^2, x^3), \quad \text{and} \quad \Upsilon_1 = \Upsilon_2 + \Upsilon_4. \quad (134)$$

Following the Corollaries C.1 and C.2 and formulas (115)–(118), we can write the Einstein equations for the ansatz (113), equivalently for (114), in the form

$$\hat{R}_2^2 = \hat{R}_3^3 = \frac{1}{2g_2g_3} [g_2^2g_3^2 + (g_3^2)^2 - g_3^6 + (g_2^2)^2 + (g_2')^2 - g_2''] = -\Upsilon_4(x^2, x^3), \quad (135)$$

$$\hat{S}_4^4 = \hat{S}_5^5 = \frac{1}{2h_4h_5} [h_5^2 \left(\ln \sqrt{|h_4h_5|}\right)^* - h_5^*] = -\Upsilon_2(x^2, x^3, v), \quad (136)$$

$$\hat{R}_4^i = -w_i^\beta \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \quad (137)$$

$$\hat{R}_5^i = -h_5^2 \left[n_i^* + \gamma n_i^*\right] = 0. \quad (138)$$

This system of equations can be solved in general forms for different types of vacuum and non–vacuum configurations.

**D General Off–Diagonal Solutions**

One holds:

**Theorem D.1.** The system of second order nonlinear partial differential equations (133)–(138) can be solved in general form if there are given certain values of functions $g_2(x^2, x^3)$ (or, inversely, $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or, inversely, $h_5(x^i, v)$), $\omega(x^i, v)$ and of sources $\Upsilon_2(x^2, x^3, v)$ and $\Upsilon_4(x^2, x^3)$.

We outline the main steps of constructing exact solutions which consists the proof of this Theorem.
• The general solution of equation (135) can be written in the form
\[ \varpi = g_0 \exp\left[ a_2 \bar{x}^2(x^2, x^3) + a_3 \bar{x}^3(x^2, x^3) \right], \tag{139} \]
where \( g_0, a_2 \) and \( a_3 \) are some constants and the functions \( \bar{x}^{2,3}(x^2, x^3) \) define any coordinate transforms \( x^{2,3} \rightarrow \bar{x}^{2,3} \) for which the 2D line element becomes conformally flat, i.e.
\[ g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \varpi(x^2, x^3) \left[ (d\bar{x}^2)^2 + \epsilon (d\bar{x}^3)^2 \right], \tag{140} \]
where \( \epsilon = \pm 1 \) for a corresponding signature. In terms of coordinates \( \bar{x}^{2,3} \), the equation (135) transform into
\[ \varpi (\ddot{\varpi} + \varpi'' - \dot{\varpi} - \varpi') = 2\varpi^2 \Upsilon_4(\bar{x}^2, \bar{x}^3) \]
or
\[ \ddot{\psi} + \psi'' = 2\Upsilon_4(\bar{x}^2, \bar{x}^3), \tag{141} \]
for \( \psi = \ln |\varpi| \). The integrals of (141) depends on the source \( \Upsilon_4 \). As a particular case we can consider that \( \Upsilon_4 = 0 \). There are three alternative possibilities to generate solutions of (135). For instance, we can prescribe that \( g_2 = g_3 \) and get the equation (141) for \( \psi = \ln |g_2| = \ln |g_3| \). If we suppose that \( \dot{g}_2 = 0 \), for a given \( g_2(x^2) \), we obtain from (135)
\[ \ddot{g}_3 - \frac{g_2 \ddot{g}_3}{2g_2} = 2g_2 g_3 \Upsilon_4(x^2, x^3) \]
which can be integrated explicitly for given values of \( \Upsilon_4 \). Similarly, we can generate solutions for a prescribed \( g_3(x^3) \) in the equation
\[ \ddot{g}_2 - \frac{g_2' \ddot{g}_3}{2g_2} = 2g_2 g_3 \Upsilon_4(x^2, x^3). \]
We note that a transform (140) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature \( \epsilon = \pm 1 \). In the simplest case with \( \Upsilon_4 = 0 \) the equation (135) is solved by arbitrary two functions \( g_2(x^3) \) and \( g_3(x^2) \).

• For \( \Upsilon_2 = 0 \), the equation (136) relates two functions \( h_4(x^i, v) \) and \( h_5(x^i, v) \) following two possibilities:
  a) to compute
\[ \sqrt{|h_5|} = h_{5[1]}(x^i) + h_{5[2]}(x^i) \int \sqrt{|h_4(x^i, v)|} dv \]
\[ = h_{5[1]}(x^i) + h_{5[2]}(x^i) v, \quad h_4^* (x^i, v) = 0, \tag{142} \]
for $h_4^*(x^i,v) \neq 0$ and some functions $h_{5[1,2]}(x^i)$ stated by boundary conditions;

b) or, inversely, to compute $h_4$ for a given $h_5(x^i,v), h_5^* \neq 0$,

$$\sqrt{|h_4|} = h_{[0]}(x^i) (\sqrt{|h_5(x^i,v)|})^*, \quad (143)$$

with $h_{[0]}(x^i)$ given by boundary conditions. We note that the sourceless equation (136) is satisfied by arbitrary pairs of coefficients $h_4(x^i,v)$ and $h_{5[0]}(x^i)$.

Solutions with $\Upsilon_2 \neq 0$ can be found for an ansatz of type

$$h_5[\Upsilon_2] = h_5, h_4[\Upsilon_2] = \varsigma_4(x^i,v) h_4, \quad (144)$$

where $h_4$ and $h_5$ are related by formula (142), or (143). Substituting (144), we obtain

$$\varsigma_4(x^i,v) = \varsigma_{4[0]}(x^i) - \int \Upsilon_2(x^2,x^3,v) \frac{h_4 h_5}{4 h_5^2} dv, \quad (145)$$

where $\varsigma_{4[0]}(x^i)$ are arbitrary integration functions.

- The exact solutions of (137) for $\beta \neq 0$ are defined from an algebraic equation, $w_i \beta + \alpha_i = 0$, where the coefficients $\beta$ and $\alpha_i$ are computed as in formulas (119) by using the solutions for (135) and (136). The general solution is

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \quad (146)$$

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients $w_k$ could be any functions on $(x^i,v)$. For the vacuum Einstein equations this is a degenerated case imposing the the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the $h_4$ and $h_5$ are related as in the formula (143) but with $h_{[0]}(x^i) = \text{const.}$

- Having defined $h_4$ and $h_5$ and computed $\gamma$ from (119), we can solve the equation (138) by integrating on variable ”$v$” the equation $n_i^* + \gamma n_i^* = 0$. The exact solution is

$$n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4(\sqrt{|h_5|})^{-3} dv, \quad h_5^* \neq 0;$$

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \quad h_5^* = 0; \quad (147)$$

$$= n_{k[1]}(x^i) + n_{k[2]}(x^i) \int (\sqrt{|h_5|})^{-3} dv, \quad h_4^* = 0,$$
for some functions \( n_{k[1,2]}(x^i) \) stated by boundary conditions.

The Theorem \[D.1] is proven.

Summarizing the results for the nondegenerated cases when \( h_4^* \neq 0 \) and \( h_5^* \neq 0 \), we derive an important result for 5D exact solutions parametrized by ansatz of type \[13\], with local coordinates \( u^\alpha = (x^i, y^a) \) when \( x^i = (x^1, x^3), x^3 = (x^2, x^3), y^a = (y^4, v, y^a) \), and for arbitrary signatures \( \epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \) (where \( \epsilon_\alpha = \pm 1 \)):

**Corollary D.1.** Any off–diagonal metric

\[
\delta s^2 = \epsilon_1(dx^1)^2 + \epsilon_2 g_k^k(x^3)(dx^5)^2 + \epsilon_4 h_0^2(x^i) \times \\
[f^*(x^i, v)][s_4(x^i, v)(\delta v)^2 + \epsilon_5 [f(x^i, v) - f_0(x^i)]^2 (\delta y^5)^2, \\
\delta v = d\nu + w_k(x^i, v) dx^k, \delta y^5 = dy^5 + n_k(x^i, v) dx^k, \quad (148)
\]

with coefficients of necessary smooth class, where \( g_k^k(x^i) \) is a solution of the 2D equation \[135\] for a given source \( \Upsilon_4(x^3) \),

\( s_4(x^i, v) = s_{4[0]}(x^i) - \frac{\epsilon_4}{8} h_0^2(x^i) \int \Upsilon_2(x^i, v) f^*(x^i, v) [f(x^i, v) - f_0(x^i)] d\nu, \)

and the \( N–\)connection coefficients \( N^4_i = w_i(x^k, v) \) and \( N^5_i = n_i(x^k, v) \) are

\[
w_i = -\frac{\partial_i s_4(x^k, v)}{s_4^*(x^k, v)} \quad (149)
\]

and

\[
n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \frac{[f^*(x^i, v)]^2}{[f(x^i, v) - f_0(x^i)]^2} s_4(x^i, v) d\nu, \quad (150)
\]

define an exact solution of the system of Einstein equations \[136\]–\[139\] for arbitrary nontrivial functions \( f(x^i, v) \) (with \( f^* \neq 0 \), \( f_0(x^i), h_0^2(x^i), s_{4[0]}(x^i) \), \( n_{k[1]}(x^i) \) and \( n_{k[2]}(x^i) \), and sources \( \Upsilon_2(x^i, v), \Upsilon_4(x^3) \)) and any integration constants and signatures \( \epsilon_\alpha = \pm 1 \) to be defined by certain boundary conditions and physical considerations.

Any metric \[138\] with \( h_4^* \neq 0 \) and \( h_5^* \neq 0 \) has the property to be generated by a function of four variables \( f(x^i, v) \) with emphasized dependence on the anisotropic coordinate \( v \), because \( f^* \div \partial_v f \neq 0 \) and by arbitrary sources.
\( \Upsilon_2(x^k, v), \Upsilon_4 \left( x^\hat{i} \right) \). The rest of arbitrary functions which do not depend on \( v \) have been obtained in result of integration of partial differential equations. This fix a specific class of metrics generated by using the relation (143) and the first formula in (144). We can generate also a different class of solutions with \( h^*_4 = 0 \) by considering the second formula in (142) and respective formulas in (147). The "degenerated" cases with \( h^*_4 = 0 \) but \( h^*_5 \neq 0 \) and inversely, \( h^*_4 \neq 0 \) but \( h^*_5 = 0 \) are more special and request a proper explicit construction of solutions. Nevertheless, such type of solutions are also generic off–diagonal and they could be of substantial interest.

The sourceless case with vanishing \( \Upsilon_2 \) and \( \Upsilon_4 \) is defined by the following:

**Remark D.1.** Any off–diagonal metric (148) with \( \varsigma = 1 \), \( h^2_0(x^i) = h^2 = \text{const}, w_i = 0 \) and \( n_k \) computed as in (150) but for \( \varsigma = 1 \), defines a vacuum solution of 5D Einstein equations for the canonical d–connection (74).

By imposing additional constraints on arbitrary functions from \( N^5_i = n_i \) and \( N^5_i = w_i \), we can select off–diagonal gravitational configurations with such distorsions of the Levi–Civita connection to the canonical d–connections when both classes of linear connections result in the same solutions of the vacuum Einstein equations, see next Appendix E.

### E 4D Nonholonomic Manifolds and Einstein Gravity

The method of constructing 5D solutions with nontrivial torsion can be restricted to generate 4D nonholonomic configurations and generic off–diagonal solutions in general relativity.

#### E.1 Reductions from 5D to 4D

To construct a \( 5D \to 4D \) reduction for the ansatz (113) and (114) is to eliminate from formulas the variable \( x^1 \) and to consider a 4D space (parametrized by local coordinates \( x^2, x^3, v, y^5 \)) being trivially embedded into 5D space (parametrized by local coordinates \( x^1, x^2, x^3, v, y^5 \)) with \( g_{11} = \pm 1, g_{1\alpha} = 0, \alpha = 2, 3, 4, 5 \) with possible 4D conformal and anholonomic transforms depending only on variables \( x^2, x^3, v \). We suppose that the 4D metric \( g_{\hat{\alpha}\hat{\beta}} \) could be of arbitrary signature. In order to emphasize that some coordinates are stated just for a such 4D space we put ”hats” on the Greek indices, \( \hat{\alpha}, \hat{\beta}, \ldots \) and on the Latin indices from the middle of the alphabet, \( \hat{i}, \hat{j}, \ldots = 2, 3 \), where \( u^{\hat{\alpha}} = \left( x^{\hat{i}}, y^\alpha \right) = \left( x^2, x^3, y^4 = v, y^5 \right) \).
In result, the Theorem C.1 and Corollaries C.1 and C.2 and Theorem D.1 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of such a reduction.

- The metric (81) (equivalently, (86)), or, in a more restricted case, of metric (113) (equivalently, with (114)) can be transformed into a 4D one (trivially embedded into a 5D spacetime),

\[ g = g_{\hat{\alpha} \hat{\beta}}(\hat{x}^i, v) \, du^{\hat{\alpha}} \otimes dv^{\hat{\beta}} \]  

with the metric coefficients \( g_{\hat{\alpha} \hat{\beta}} \) parametrized in the form

\[
\begin{bmatrix}
g_2 + n_2^2 h_5 & w_2 n_2 h_5 & w_2 h_4 & n_2 h_5 \\
w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
w_2 h_4 & w_3 h_4 & h_4 & 0 \\
2 h_4 & n_3 h_5 & 0 & n_5 h_5
\end{bmatrix},
\]

where the coefficients are some necessary smoothly class functions of type:

\[
g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^\hat{k}, v),
\]

\[
w_i = w_i(x^\hat{k}, v), n_i = n_i(x^\hat{k}, v); \, \hat{i}, \hat{k} = 2, 3.
\]

- We obtain a quadratic line element

\[
\delta s^2 = g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2,
\]

written with respect to the anholonomic co-frame \((dx^\hat{i}, \delta v, \delta y^5)\), where

\[
\delta v = dv + w_i dx^\hat{i} \quad \text{and} \quad \delta y^5 = dy^5 + n_i dx^\hat{i}
\]

is the dual of \((\delta_i, \partial_4, \partial_5)\), where

\[
\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5.
\]

- If the conditions of the 4D variant of the Theorem C.1 are satisfied, we have the same equations (135) – (138) were we substitute

\[
h_4 = h_4(x^\hat{k}, v) \quad \text{and} \quad h_5 = h_5(x^\hat{k}, v).
\]

As a consequence, we have \(\alpha_i(x^k, v) \rightarrow \alpha_i(x^\hat{k}, v), \beta = \beta(x^\hat{k}, v)\) and \(\gamma = \gamma(x^\hat{k}, v)\) resulting in \(w_i = w_i(x^\hat{k}, v)\) and \(n_i = n_i(x^\hat{k}, v)\).
• One holds the same formulas (142)-(147) from the Theorem D.1 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, \( \hat{i} \rightarrow i \), and holonomic coordinates, \( x^i \rightarrow x^i \), i.e. in the 4D solutions there is not contained the variable \( x^1 \).

• The formulae (133) for the nontrivial coefficients of the Einstein tensor in 4D, stated by the Corollary C.1, are written

\[
\hat{G}_2^2 = \hat{G}_3^3 = -\hat{S}_4^4, \quad \hat{G}_4^4 = \hat{G}_5^5 = -\hat{R}_2^2. \tag{155}
\]

• For symmetries of the Einstein tensor (155), we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary C.2 by the conditions (134), which in 4D are transformed into

\[
\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v), \quad \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^2, x^3). \tag{156}
\]

E.2 Off–diagonal metrics in general relativity

For 4D configurations, we can generate a class of off–diagonal metrics in general relativity if we impose the conditions that the nonholonomic transforms and correspondingly generated distortions \( \hat{Z}_{\alpha\beta} \) (102) of the Levi–Civita connection to the canonical d–connection induce such distortions of the curvature tensor \( \nabla Z^\alpha_\beta \) (103) that there are satisfied the conditions \( \nabla Z^\alpha_\beta\gamma\alpha = 0 \) (104). In this case, the Ricci tensor has the same nontrivial values for the Levi–Civita connection and the canonical d–connection, both computed with respect to N–adapted bases (153) and (154). In this subsection, we demonstrate how the conditions (104) can be solved for a 4D d–metric (152), equivalently, for (151).

The \( h \)– and \( v \)–coefficients of distortion \( \hat{Z}^\tau_{\beta\gamma} \) are defined from (95),

\[
\hat{Z}^\tau_{\beta\gamma} = \left( \hat{Z}^i_{jk} = 0, \hat{Z}^a_{bk} = -\frac{\partial N^a_k}{\partial y^b}, \hat{Z}^i_{jc} = \frac{1}{2} g^{ik} \Omega^a_{jk} h^c_a, \hat{Z}^a_{bc} = 0 \right). \tag{157}
\]

Introducing these values in (103), for

\[
\hat{P}^a_{bk} = \frac{\partial N^a_k}{\partial y^b} - \hat{L}^a_{bk} = \nabla \hat{P}^a_{bk} - \hat{Z}^a_{bk},
\]

we compute the distortions of the d–curvatures (97) which are used for definition of the Ricci d–tensors:

\[
\hat{R}^i_{hjk} = \nabla \hat{R}^i_{hjk} - \hat{Z}_{hka} \Omega^a_{jk},
\]

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\[ \hat{P}^i_{jka} = \nabla p^i_{jka} - \frac{\partial \hat{Z}^i_{ja}}{\partial x^k} + \hat{L}^i_{lk} \hat{Z}^l_{ja} - \hat{L}^i_{jk} \hat{Z}^l_{ia} - \hat{Z}^c_{ak} \hat{C}^i_{jc} - \hat{Z}^c_{ak} \hat{Z}^i_{jc} \]
\[ - \hat{L}^c_{ak} \hat{Z}^i_{jc} - \hat{C}^i_{jb} \hat{Z}^j_{ka} - \hat{C}^i_{jb} \hat{Z}^j_{ka}, \]
\[ \hat{P}^c_{bka} = \nabla p^c_{bka} - \left( \frac{\partial \hat{Z}^c_{bk}}{\partial y^a} + \hat{Z}^c_{dk} \hat{C}^d_{ca} - \hat{Z}^d_{ak} \hat{C}^c_{da} - \hat{Z}^d_{ak} \hat{C}^c_{bd} \right) - \hat{C}^c_{bd} \hat{Z}^d_{ak}, \]
\[ \hat{S}^a_{bcd} = \nabla s^a_{bcd}. \]

Contracting the indices and re-grouping the terms, we get the distortions for the Ricci d–tensors (98)
\[ \hat{R}^i_{ji} = \nabla \hat{R}^i_{ji} - \hat{Z}^i_{ha} \Omega^a_{ji}, \]
\[ \hat{R}^i_{ja} = \nabla \hat{R}^i_{ja} + \frac{\partial \hat{Z}^i_{ja}}{\partial x^i} + \hat{L}^i_{ji} \hat{Z}^l_{ja} - \hat{L}^i_{ji} \hat{Z}^l_{ia} - \hat{L}^c_{ak} \hat{Z}^i_{jc}, \]
\[ \hat{R}^i_{bk} = \nabla \hat{R}^i_{bk} + \frac{\partial \hat{Z}^c_{bk}}{\partial y^c} + \hat{Z}^d_{bk} \hat{C}^c_{dc} - \hat{C}^a_{bd} \hat{Z}^d_{ak}, \]
\[ \hat{R}^i_{ba} = \nabla \hat{R}^i_{ba}. \]

If we consider a subclass of solutions with
\[ \Omega^a_{ji} = 0, \] (158)
when \( \hat{Z}^i_{ja} = 0 \), see (157), we obtain that
\[ \hat{R}^i_{ji} = \nabla \hat{R}^i_{ji}, \quad \hat{R}^i_{ja} = \nabla \hat{R}^i_{ja}, \quad \hat{R}^i_{ba} = \nabla \hat{R}^i_{ba}. \]

One follows that
\[ \hat{R}^\alpha_{\beta} = \nabla \hat{R}^\alpha_{\beta} \]
if \( \hat{R}^i_{bk} = \nabla \hat{R}^i_{bk} \) which hold when
\[ \frac{\partial \hat{Z}^c_{bk}}{\partial y^c} + \hat{Z}^d_{bk} \hat{C}^c_{dc} - \hat{C}^a_{bd} \hat{Z}^d_{ak} = 0. \] (159)

Let us analyze the equations (159) for the ansatz (152) (they hold also for the 5D d–metric (113)). Such metrics do not depend on \( y^5 \). We have
\[ \frac{\partial \hat{Z}^4_{bk}}{\partial v} + \hat{Z}^d_{bk} \hat{C}^c_{dc} - \hat{C}^a_{bd} \hat{Z}^d_{ak} = 0, \]
or, for \( \hat{Z}^4_{bk} \) \( = \frac{\partial \hat{Z}^4_{bk}}{\partial v}, \)
\[ \left( \hat{Z}^4_{ik} \right)^* + \hat{Z}^d_{ik} \hat{C}^c_{bc} - \hat{C}^a_{4d} \hat{Z}^d_{ak} = 0 \] and \( \left( \hat{Z}^4_{5k} \right)^* + \hat{Z}^d_{5k} \hat{C}^c_{dc} - \hat{C}^a_{5d} \hat{Z}^d_{ak} = 0, \]
where \( \hat{Z}_{5k}^d = \partial N_k^d / \partial y^5 = 0 \), see (157). We get

\[
\left( \hat{Z}_{4k}^4 \right)^* + \hat{Z}_{4k}^c \hat{C}_{4c}^a - \hat{C}_{44}^a \hat{Z}_{4k}^c = 0 \text{ and } \hat{Z}_{4k}^c \hat{C}_{5c}^4 = 0.
\]

These formulas, for \( N_k^4 = w_k \) and \( N_k^5 = n_k \), see also formulas (129) and (132), result in

\[
w_k^{**} + \frac{w_k^* h_k^*}{2 h_5} = 0, \tag{160}
\]
\[n_k^* h_k^* h_5 = 0.
\]

There are two possibilities to satisfy this system: 1) to take

\[h_5^* = 0 \text{ which impose } w_k^{**} = 0\]

or 2) to consider that \( h_5^* \neq 0 \) imposing that

\[n_k^* = 0 \text{ for any } w_k^{**} + \frac{w_k^* h_k^*}{2 h_5} = 0.\]

The next step is to see how the constraints (158) may be solved. As a matter of principle, they are satisfied for any \( w_k \) and \( n_k \) for which

\[
w_2' - w_3^* + w_3 w_2^* - w_2 w_3^* = 0, \tag{161}
\]
\[n_2' - n_3^* + w_3 n_2^* - w_2 n_3^* = 0\]

where we put \( n_1 = 0 \) and \( w_1 = 0 \) in order tho have a limit to the 4D configurations, see (122) and (130). For instance, if \( w_1 = 0 \), one reduces (161) to

\[n_2' - n_3^* = 0\]

which can be solved for any \( n_2 = 0 \) and \( n_3 = n_3(x^3, v) \), or, inversely, for any \( n_2 = n_2(x^2, v) \) and \( n_3 = 0 \).

In general, we conclude that any set of functions \( h_{4,5}(x^2, x^3, v) \) and \( w_{2,3}(x^2, x^3, v) \) and \( n_{2,3}(x^2, x^3, v) \) for which the system of equations (160) and (161) is compatible and has nontrivial solutions, defined also as solutions of (135)–(138), can be used for generation of exact solutions of the vacuum 4D (or 5D) solutions of the vacuum Einstein equations. In this case, the Ricci tensors for the Levi–Civita and the canonical d–connections have the same coefficients for the N–adapted bases (153) and (154) (with extensions to 5D (82) and (83)). Such solutions can be extended to certain matter and string like corrections if the effective sources do not depend on covariant derivatives.
Finally, one should be emphasized that even we can nonholonomically constrain the solutions in order to have zero distortions of the Ricci tensor (induced by deforming the Levi Civita to the canonical d–connection one) the distortions of the d–curvature tensors do not vanish. For instance, the components $\hat{R}^a_{bijk}$ distinguished in (97) may be not zero, but they are not used for the definition of the Ricci and Einstein tensor. We can have allowed nontrivial values of the nonholonomically induced canonical torsion (96) because $\partial N^a_j/\partial y^b$ may be not zero even $\Omega_{jk}^a = 0$. This is an anholonomic frame effect which states that a generic off–diagonal vacuum metric may be characterized by certain effective torsion coefficient induced by the off–diagonal terms and related distortions of curvature tensor but with zero distortions of the Ricci tensor.

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