GUNNING-NARASIMHAN’S THEOREM WITH A GROWTH CONDITION

FRANC FORSTNERIČ AND TAKEO OHSAWA

Abstract. Given a compact Riemann surface $X$ and a point $x_0 \in X$, we construct a holomorphic function without critical points on the punctured Riemann surface $R = X \setminus \{x_0\}$ which is of finite order at $x_0$.

1. The Statement

Let $X$ be a compact Riemann surface, let $x_0$ be an arbitrary point of $X$, and let $R = X \setminus \{x_0\}$. The set of holomorphic functions on $R$ will be denoted by $\mathcal{O}(R)$. Let $U \subset X$ be a coordinate neighborhood of the point $x_0$ and let $z$ be a local coordinate on $U$ with $z(x_0) = 0$. A holomorphic function $f \in \mathcal{O}(R)$ on $R$ is said to be of finite order (at the point $x_0$) if there exist positive numbers $\lambda$ and $\mu$ such that

\[ |f(z)| \leq \lambda \exp |z|^{-\mu} \quad \text{holds on } U \setminus \{x_0\}.\]

We denote by $\mathcal{O}_{f.o.}(R)$ the set of all holomorphic functions of finite order on $R$. For any $f \in \mathcal{O}_{f.o.}(R)$, the order of $f$ is defined as the infimum of all numbers $\mu > 0$ such that (1) holds for some $\lambda > 0$. By using Poisson-Jensen’s formula it is easy to see that, for any nonvanishing holomorphic function $f$ on $U \setminus \{x_0\}$ satisfying (1), there exist a neighborhood $V \ni x_0$ and a number $\chi > 0$ such that

\[ \frac{1}{|f(z)|} \leq \chi \exp |z|^{-\mu} \quad \text{on } V \setminus \{x_0\} \] (Hadamard’s theorem, c.f. [A, Chap. 5]).

In 1967 Gunning and Narasimhan proved that every open Riemann surface admits a holomorphic function without critical points [GN]. Our goal is to prove the following result for punctured Riemann surfaces.

Theorem 1.1. If $X$ is a compact Riemann surface and $x_0 \in X$ then the punctured Riemann surface $R = X \setminus \{x_0\}$ admits a noncritical holomorphic function of finite order; that is, $\{ f \in \mathcal{O}_{f.o.}(R) : df \neq 0 \text{ everywhere} \} \neq \emptyset$.

We show that this result is the best possible one, except when $X = \mathbb{CP}^1$ is the Riemann sphere in which case $R = \mathbb{C}$.
Proposition 1.2. If \( X \) is a compact Riemann surface of genus \( g \geq 1 \) and \( x_0 \in X \) then every algebraic function \( X \setminus \{x_0\} \to \mathbb{C} \) has a critical point.

In the case when \( X \) is a torus, this was shown in [M, §4].

Proof. Assume that \( f : R = X \setminus \{x_0\} \to \mathbb{C} \) is an algebraic function. Then \( f \) extends to a meromorphic map \( X \to \mathbb{C}P^1 \) sending \( x_0 \) to the point \( \infty \). Let \( d \) denote the degree of \( f \) at \( x_0 \), so \( f \) equals the map \( z \mapsto z^d \) in a certain pair of local holomorphic coordinates at the points \( x_0 \) and \( \infty \). Since \( f^{-1}(\infty) = \{x_0\} \), \( d \) is also the global degree of \( f \). By the Riemann-Hurwitz formula (see [Ha]) we then have

\[
\chi(X) = d_X(\mathbb{C}P^1) - b,
\]

where \( \chi(X) \) is the Euler number of \( X \) and \( b \) is the total branching order of \( f \) (the sum of its local branching orders over the points of \( X \)). If we assume that \( f \) has no critical points on \( R \), then it only branches at \( x_0 \), and its branching order at \( x_0 \) is clearly \( b = d - 1 \). Hence the above equation reads \( 2 - 2g = 2d - (d - 1) = d + 1 \geq 1 \) which is clearly impossible if \( g \geq 1 \).

In fact, we see that any algebraic function \( f : R = X \setminus \{x_0\} \to \mathbb{C} \) with degree \( d \) at \( x_0 \) must have precisely \( (d + 1) - (2 - 2g) = d + 2g - 1 \) branch points in \( R \) when counted with algebraic multiplicities. \( \Box \)

2. Preliminaries

We assume that \( X \) and \( R = X \setminus \{x_0\} \) are as above.

Proposition 2.1. For any effective divisor \( \delta \) on \( X \) whose support does not contain the point \( x_0 \) there exists \( f \in O_{f.o.}(R) \) whose zero divisor \( f^{-1}(0) \) coincides with \( \delta \).

Proof. Since holomorphic vector bundles over noncompact Riemann surfaces are trivial by Grauert’s Oka principle, there exists a holomorphic function \( f_0 \) on \( R \) whose zero divisor equals \( \delta \). Let \( V \) be a disc neighborhood of the point \( x_0 \) in \( X \), with a holomorphic coordinate \( z \) in which \( z(x_0) = 0 \), such that \( f_0 \) does not vanish on \( V \setminus \{x_0\} \). Let \( m \in \mathbb{Z} \) denote the winding number of \( f \) around the point \( x_0 \). Choose a meromorphic function \( h \) on \( X \) such that \( h(z) = c(z)z^m \) on \( z \in V \) for some nonvanishing holomorphic function \( c \) on \( V \), and such that all remaining zeros and poles of \( h \) lie in \( X \setminus \overline{V} \). Then \( f_0/h \) is a nowhere vanishing holomorphic function with winding number zero in \( V \setminus \{x_0\} \), and hence \( \log(f_0/h) \) has a single valued holomorphic branch on \( V \setminus \{x_0\} \). Choose a smaller disc \( W \subset V \) centered at \( x_0 \). By solving a Cousin-I problem we find a holomorphic functions \( u_1 \) on \( X \setminus \overline{W} \) and \( u_2 \) on \( V \setminus \{x_0\} \) such that \( u_1 - u_2 = \log(f_0/h) \) holds on \( V \setminus \overline{W} \), and such that \( x_0 \) is a pole of the function \( u_2 \). Hence, letting \( f = he^{-u_2} \) on \( V \setminus \{x_0\} \) and \( f = f_0e^{-u_1} \) on \( R \setminus W \), we obtain a function \( f \in O_{f.o.}(R) \) satisfying \( f^{-1}(0) = \delta \). \( \Box \)
Let $L \to X$ be a holomorphic line bundle and let $h$ be a fiber metric of $L$. A holomorphic section $s$ of $L$ over $R$ is said to be of finite order if the length $|s|$ of $s$ with respect to $h$ satisfies on $U \setminus \{x_0\}$, as a function of the local coordinate $z$, that

$$|s|(z) \leq \lambda \exp |z|^{-\mu}$$

for some $\lambda, \mu \in (0, \infty)$. The order of $s$ is defined similarly as in the case of holomorphic functions. Since every holomorphic line bundle over $X$ is associated with a divisor, Proposition 2.1 implies the following:

**Proposition 2.2.** For any holomorphic line bundle $L$ over $X$, there exists a holomorphic section $s$ of the restricted bundle $L|_R$ such that $s$ is of finite order and $s(x) \neq 0$ for all $x \in R$.

**Proof.** Let $v$ be any meromorphic nonzero section of $L$. Let $p_1, \ldots, p_m$ (resp. $q_1, \ldots, q_n$) be the poles (resp. the zeros) of $v$ in $R$. By Proposition 2.1 there exist functions $f, g \in \mathcal{O}_{f.o.}(R)$ such that $p_1 + p_2 + \ldots + p_m$ (resp. $q_1 + \ldots + q_n$) is the zero divisor of $f$ (resp. of $g$). Then the section $s = fv/g$ satisfies the stated properties. □

**Corollary 2.3.** There exists a holomorphic 1-form of finite order on $R$ which does not vanish anywhere.

Let $\omega$ be a nowhere vanishing holomorphic 1-form of finite order on $R$ guaranteed by Corollary 2.3. Then, Theorem 1.1 is equivalent to saying that there exists a function $g \in \mathcal{O}_{f,a}(R)$ such that $g^{-1}(0) = \emptyset$ and $\int_\gamma g \omega = 0$ holds for any 1-cycle $\gamma$ on $R$, for the primitives of $g\omega$ will then be without critical points and clearly of finite order, the converse being obvious.

We shall show that such $g$ can be found in a subset of $\mathcal{O}_{f,a}(R)$ consisting of functions of the form $\exp \int_{x_1}^x \eta$ where $\eta$ are meromorphic 1-forms on $X$ which are holomorphic on $R$ and $x_1 \in R$ is an arbitrary fixed point in $R$.

Let us denote by $\Omega^1_{alg}(R)$ (resp. $\mathcal{O}_{alg}(R)$) the set of meromorphic 1-forms (resp. meromorphic functions) on $X$ which are holomorphic on $R$. The general theory of coherent algebraic sheaves on affine algebraic varieties implies the following (c.f. [S] or [Ha]).

**Proposition 2.4.** Every element of $H^1(R; \mathbb{C})$ is represented by an element of $\Omega^1_{alg}(R)$ as a de Rham cohomology class.

Let $K$ be a compact set in $R$ and let $\mathcal{O}(K)$ denote the set of all continuous functions on $K$ which are holomorphically extendible to some open neighborhoods of $K$ in $X$. Then the Runge approximation theorem says the following in our situation.

**Proposition 2.5.** For any compact set $K \subset R$ such that $R \setminus K$ is connected, the image of the restriction map $\mathcal{O}_{alg}(R) \to \mathcal{O}(K)$ is dense with respect to the topology of uniform convergence.
The proof of Theorem 1.1 to be given below is basically a combination of Corollary 2.3, Proposition 2.4 and Proposition 2.5. In order to make a short cut argument, we shall apply a refined version of Proposition 2.5 (Mergelyan’s theorem) below.

3. Proof of Theorem 1.1

For any $C^1$ curve $\alpha: [0,1] \to X$ we denote by $|\alpha|$ its trace, i.e., $|\alpha| = \{\alpha(t): 0 \leq t \leq 1\}$. If $\alpha$ is closed ($\alpha(0) = \alpha(1)$), we denote by $[\alpha]$ its homology class in $H_1(X; \mathbb{Z})$.

Let $g$ denote the genus of $X$. There exist simple closed real-analytic curves $\alpha_1, \ldots, \alpha_{2g}$ in $R$ satisfying

\begin{equation}
H_1(X; \mathbb{Z}) = \sum_{i=1}^{2g} \mathbb{Z}[\alpha_i]
\end{equation}

such that $\cap_{i=1}^{2g} |\alpha_i| = \{p\}$ holds for some point $p \in R$ and such that, putting $\Gamma = \cup_{i=1}^{2g} |\alpha_i|$, the complement $R \setminus \Gamma$ is connected.

Let $\omega$ nowhere vanishing holomorphic 1-form of finite order on $R$ furnished by Corollary 2.3. For each curve $\alpha_i$ there is a neighborhood $U_i \supset |\alpha_i|$ in $R$ and a biholomorphic map $\varphi_i$ from an annulus $A_r = \{w \in \mathbb{C}: 1 - r < |w| < 1 + r\}$ onto $U_i$ for a sufficiently small $r > 0$ such that the positively oriented unit circle $\{|w| = 1\}$ is mapped by $\varphi_i$ onto the curve $\alpha_i$, with $\varphi_i(1) = p$. For $i = 1, \ldots, 2g$ put

\[ \varphi_i^* \omega = H_i(w) \, dw, \quad n_i = \frac{1}{2\pi \sqrt{-1}} \int_{|w|=1} d \log H_i \in \mathbb{Z}. \]

By Proposition 2.4 there exists $\xi \in \Omega^1_{alg}(R)$ such that

\begin{equation}
(3) \quad n_i = \frac{1}{2\pi \sqrt{-1}} \int_{\alpha_i} \xi, \quad i = 1, \ldots, 2g.
\end{equation}

Let

\[ u(x) = \exp \int_p^x \xi, \quad x \in R. \]

By (2) and (3) the integral is independent of the path in $R$. (Note that the cycle around the deleted point $x_0$ is homologous to zero in $R$.) Hence the function $u$ is well defined, single-valued and nonvanishing on $R$, and $u \in \mathcal{O}_{f,0}(R)$ because $\xi \in \Omega^1_{alg}(R)$. Replacing $\omega$ by $\omega/u$ we obtain a nowhere vanishing 1-form of finite order on $R$, still denoted $\omega$, for which the winding numbers $n_i$ in (3) equal zero. It follows that for every $i = 1, \ldots, 2g$ we have

\[ \varphi_i^* \omega = e^{h_i(w) + c_i} \, dw \]

for some constants $c_i \in \mathbb{C}$ and holomorphic function $h_i$ on the annulus $A_r \subset \mathbb{C}$ with $h_i(1) = 0$. Note that the functions $h_i \circ \varphi_i^{-1}: |\alpha_i| \to \mathbb{C}$ agree at
the unique intersection point $p$ of the curves $|\alpha_i|$, and hence they define a continuous function $H$ on $\Gamma = \cup_i |\alpha_i|$. For every $h \in \mathcal{O}_{alg}(R)$ we have
\[
\int_{\alpha_i} e^{-h} \omega = e^{c_i} \int_{|w|=1} e^{h_i - h \circ \varphi_i} \, dw.
\]
These numbers can be made arbitrarily small by choosing $h$ to approximate $H$ uniformly on $\Gamma$ (which is equivalent to asking that $h_i - h \circ \varphi_i$ is small on $\{|w| = 1\}$ for every $i = 1, \ldots, 2g$). Such $h$ exist by Mergelyan’s theorem: Since $R \setminus \Gamma$ is connected, every continuous function on $\Gamma$ is a uniform limit of functions in $\mathcal{O}_{alg}(R)$ (c.f. [G, Chap. 3]).

We assert that there exist functions $f_i \in \mathcal{O}_{alg}(R)$ for $i = 1, \ldots, 2g$ and a number $\epsilon > 0$ such that, for any $h \in \mathcal{O}_{alg}(R)$ satisfying
\[
\sup_{|\alpha_i|} |h_i \circ \varphi_i^{-1} - h| < \epsilon, \quad i = 1, \ldots, 2g,
\]
there exist numbers $\zeta_i \in \mathbb{C}$ ($i = 1, \ldots, 2g$) such that
\[
\int_{\alpha_j} \left( \sum_{i=1}^{2g} \zeta_i f_i - h \right) \omega = 0, \quad j = 1, \ldots, 2g.
\]

To prove this assertion, which clearly implies Theorem 1.1 (the potential of the 1-form under the integral in (5) is a holomorphic function of finite order and without critical points on $R$), choose functions $f_i \in \mathcal{O}_{alg}(R)$ for $i = 1, \ldots, 2g$ satisfying
\[
e^{c_j} \int_{|w|=1} f_i \circ \varphi_j(w) \, dw = \delta_{ij},
\]
where $\delta_{ij}$ denotes the Kronecker’s delta. Such $f_i$ exist by Proposition 2.5 applied with $K = \Gamma$. After fixing the $f_i$’s, let us choose numbers $0 < \epsilon_0 < 1$ and $C_0 > 1$ in such a way that
\[
\sup_{\Gamma} \left| \exp \left( \sum_{i=1}^{2g} \tau_i f_i \right) - 1 - \sum_{i=1}^{2g} \tau_i f_i \right| \leq C_0 \max_i |\tau_i|^2
\]
holds if $\tau_i \in \mathbb{C}$ and $\max_i |\tau_i| \leq \epsilon_0$.

Let $c = \max_i |c_i|$. By decreasing the number $\epsilon_0 > 0$ if necessary we can assume that
\[
8\pi C_0 e^{1+c} \epsilon_0 < 1.
\]
Choose a constant $C_1 > 0$ such that
\[
|e^t - 1| < C_1 |t| \quad \text{if } |t| < \epsilon_0.
\]
Then, by (6) and (7), it is easy to see that, for any positive number $\epsilon > 0$ satisfying
\[
8\pi C_1 \left( 1 + \sup_{\Gamma} \sum_{i=1}^{2g} |f_i| \right) \epsilon < \epsilon_0
\]
and for any $h \in O_{\text{alg}}(R)$ satisfying (4), the inequality

$$\left| \tau_j - \int_{\alpha_j} \exp \left( \sum \tau_i f_i - h \right) \omega \right| \leq \frac{\epsilon_0}{2}$$

holds for every $j = 1, \ldots, 2g$ whenever $\max_i |\tau_i| \leq \epsilon_0$. Hence, for such a choice of $h$, the map

$$C^{2g} \ni \tau = (\tau_1, \ldots, \tau_{2g}) \xrightarrow{\Phi} (\Phi_1(\tau), \ldots, \Phi_{2g}(\tau)) \in C^{2g},$$

whose $j$-th component is defined by

$$\Phi_j(\tau) = \int_{\alpha_j} \exp \left( \sum_{i=1}^{2g} \tau_i f_i - h \right) \omega,$$

maps the polydisc $P = \{ \tau \in C^{2g} : \max |\tau_i| < \epsilon_0 \}$ onto a neighborhood of the origin in $C^{2g}$. In particular, we have $\Phi(\zeta) = 0$ for some point $\zeta = (\zeta_1, \ldots, \zeta_{2g}) \in P$, and for this $\zeta$ the equations (5) hold. This concludes the proof of Theorem 1.1.

4. Concluding remarks

By a minor adjustment of the proof of Theorem 1.1 one can construct a nowhere vanishing holomorphic 1-form of finite order, $\omega$, on $R$ whose periods $\int_{\alpha_j} \omega$ over the basis curves $[\alpha_j]$ of $H_1(R;\mathbb{Z})$ are arbitrary given complex numbers. In other words, one can prove the following result. (See Kusunoki and Sainouchi [KS] and Majcen [M] for the corresponding result on open Riemann surface and without the finite order condition.)

**Theorem 4.1.** Let $X$ be a compact Riemann surface and $x_0 \in X$. Every element of the de Rham cohomology group $H^1(X;\mathbb{C})$ is represented by a nowhere vanishing holomorphic 1-form of finite order on $R = X \setminus \{x_0\}$.

Since every affine algebraic curve $A \subset \mathbb{C}^N$ is obtained by deleting finitely many points from a compact Riemann surface, Theorem 1.1 implies that every affine algebraic curve admits a noncritical holomorphic function of finite order. One may ask whether the same result also holds on higher dimensional algebraic manifolds:

**Problem 4.2.** Does every affine algebraic manifold $A \subset \mathbb{C}^N$ of dimension $\dim A > 1$ admit a noncritical holomorphic function $f : A \to \mathbb{C}$ of finite order?

Here we say that $f$ is of finite order if $|f(z)| \leq \lambda \exp |z|^\mu$ holds for all $z \in A$ and for some pair of constants $\lambda, \mu > 0$.

Since such $A$ is a Stein manifold, it admits a noncritical holomorphic function according to [1]. The construction in that paper is quite different from the one presented here even for Riemann surfaces, and it does not necessarily give a function of finite order when $A$ is algebraic. The main
difficulty is that the closedness equation $d\omega = 0$ for a holomorphic 1-form, which is automatically satisfied on a Riemann surface, becomes a nontrivial condition when $\dim A > 1$. In particular, this condition is not preserved under multiplication by holomorphic functions, and hence one can not hope to adjust the periods in the same way as was done above.

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Faculty of Mathematics and Physics, University of Ljubljana, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: franc.forstneric@fmf.uni-lj.si

Graduate School of Mathematics, Nagoya University, Chikusaku Furocho, 464-8602 Nagoya, Japan

E-mail address: ohsawa@math.nagoya-u.ac.jp