NOTE ON BOLTHAUSEN-DEUSCHEL-ZEITOUNI'S PAPER ON THE
ABSENCE OF A WETTING TRANSITION FOR A PINNED HARMONIC CRYSTAL
IN DIMENSIONS THREE AND LARGER

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ABSTRACT. The article [I] provides a proof of the absence of a wetting transition for the discrete Gaussian free field conditioned to stay positive, and undergoing a weak delta-pinning at height 0. The proof is generalized to the case of a square pinning-potential replacing the delta-pinning, but it relies on a lower bound on the probability for the field to stay above the support of the potential, the proof of which appears to be incorrect. We provide a modified proof of the absence of a wetting transition in the square-potential case, which does not require the aforementioned lower bound. An alternative approach is given in a recent paper by Giacomin and Lacoin [2].

1. DEFINITIONS AND NOTATIONS

We keep the notations of [1] except for the field which we call \( \phi \) instead of \( X \). Let \( A \) be a finite subset of \( \mathbb{Z}^d \), let \( \phi = (\phi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \) and the Hamiltonian defined as

\[
H_A(\phi) = \frac{1}{8d} \sum_{x,y \in \partial A: |x-y|=1} (\phi_x - \phi_y)^2
\]

where \( \partial A \) is the outer boundary of \( A \). The following probability measure on \( \mathbb{R}^A \) defines the discrete Gaussian free field on \( A \) (with zero boundary condition):

\[
P_A(d\phi) = \frac{1}{Z_A} e^{-H_A(\phi)}d\phi_A \delta_0(d\phi_N)
\]

where \( d\phi_A = \prod_{x \in A} d\phi_x \) and \( \delta_0 \) is the Dirac mass at 0. The partition function \( Z_A \) is the normalization \( Z_A = \int_{\mathbb{R}^A} \exp(-H(\phi_A))d\phi_A \). We will also need the following definition of a set \( A \) being \( \Delta \)-sparse (morally meaning that it has only one pinned point per cell of side-length \( \Delta \)), which we reproduce from [I] page 1215:

Definition 1. Let \( N \in \mathbb{Z}, \Delta > 0, \Lambda_N = \{-[N]/2, \ldots, [N]/2\}^d \) and let \( l_N^{\Delta} = \{z_i\}_{i=1}^{[\sqrt{\Delta}]} \) denote a finite collection of points \( z_i \in \Lambda_N \) such that for each \( y \in \Lambda_N \cap \Delta\mathbb{Z}^d \) there is exactly one \( z \in l_N^{\Delta} \) such that \( |z-y| < \Delta/10 \). Let \( \Lambda_N^{\Delta} \equiv \Lambda_N \setminus \Lambda_N^{\Delta} \).

2. LOWER BOUND ON THE PROBABILITY OF THE HARD WALL CONDITION

The proof of [I] Theorem 6] relies on [I] Proposition 3. Unfortunately, the proof provided in the paper, when applied with \( t > 0 \) provides a lower bound which is a little bit weaker than what is claimed, namely

Proposition 2. Correction of [I] Proposition 3 :
Assume \( d \geq 3 \) and let \( t > 0 \). Then there exist three constants \( c_1, c_2, c_3 > 0 \) depending on \( t \), and \( c_4 > 0 \) independent of \( t \), such that, for all \( \Delta \) integer large enough

\[
\liminf_{N \to \infty} \inf_{l_N^{\Delta}} \frac{1}{(2N+1)^d} \log P_{A_N}(X_i \geq t, i \in A_N) \geq \frac{-d \log \Delta}{\Delta^d} + c_1 \frac{\log \Delta}{\Delta^d} - c_2 e^{-c_4 \sqrt{\log \Delta}}
\]

The statement of [I] Proposition 3] only contains the first two terms. The dependence in \( t \) vanishes between equations (2.4) and (2.5) in [I]. Note that for \( t = 0 \) the third term is irrelevant and the bound coincides with the one stated in the paper.
3. Proof of the absence of a wetting transition in the square-potential case

Let us introduce the following notations

\[ \tilde{\xi}_N = \sum_{x \in \Lambda_N} \mathbb{1}_{[\phi_x \leq a]}, \quad \tilde{\xi}_N' = \sum_{x \in \Lambda_N} \mathbb{1}_{[\phi_x \in (0,a)]}, \]

\[ \Omega_N^+ = \{ \phi_x \geq 0, \ \forall x \in \Lambda \}, \quad \Omega_N^{+} = \{ \phi_x \geq 0, \ \forall x \in \Lambda_N \} \]

\[ \mathcal{A} = \{ x \in \Lambda_N : \phi_x \in [0,a] \} \]

and the following probability measure with square-potential pinning:

\[ \tilde{\mathcal{P}}_{N,a,b} (d\phi) = \frac{1}{Z_{N,a,b}} \exp \left\{ -H(\phi) + \sum_{x \in \Lambda_N} b \mathbb{1}_{[\phi_x \in [0,a]]} \right\} d\phi_{\Lambda_N} \delta_0 (d\phi_N^{+}) \]

in contrast with the measure used in [1]:

\[ \mathcal{P}_{N,a,b} (d\phi) = \frac{1}{Z_{N,a,b}} \exp \left\{ -H(\phi) + \sum_{x \in \Lambda_N} b \mathbb{1}_{[\phi_x \in [-a,a]]} \right\} d\phi_{\Lambda_N} \delta_0 (d\phi_N^{+}). \]

Observe that

\[ \mathcal{P}_{N,a,b} (\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) = \tilde{\mathcal{P}}_{N,a,b} (\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) \]

**Theorem 3.** (Absence of wetting transition, [1] Theorem 6)
Assume \( d \geq 3 \) and let \( a, b > 0 \) be arbitrary. Then there exists \( \epsilon_{a,b}, \eta_{a,b} > 0 \) such that

\[ \tilde{\mathcal{P}}_{N,a,b} (\tilde{\xi}_N > \epsilon_{a,b} N^d | \Omega_N^+) \geq 1 - \exp (-\eta_{a,b} N^d). \]  

(4)

provided \( N \) is large enough.

**Proof.** Let us compute the probability of the complementary event and provide bounds on the numerator and the denominator corresponding to the conditional probability:

\[ \tilde{\mathcal{P}}_{N,a,b} (\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) = \frac{\tilde{\mathcal{P}}_{N,a,b} (\tilde{\xi}_N < \epsilon N^d \cap \Omega_N^+)}{\tilde{\mathcal{P}}_{N,a,b} (\Omega_N^+)} \]

(5)

3.1. Lower bound on the denominator. Writing

\[ \exp \left( \sum_{x \in \Lambda_N} b \mathbb{1}_{[\phi_x \in [0,a]]} \right) = \prod_{x \in \Lambda_N} ((e^b - 1) \mathbb{1}_{[\phi_x \in [0,a]]} + 1) \]

(6)

and using the FKG inequality, we get

\[ \tilde{\mathcal{P}}_{N,a,b} (\Omega_N^+) \geq Z_{N,a,b} \sum_{A \subset \Lambda_N} (e^b - 1)^{|A|} P_N (\mathcal{A} \supset A) P_N (\Omega_N^+ | \mathcal{A} \supset A) P_N (\Omega_N^+ | A \supset \mathcal{A}). \]

(7)

Let us first bound the term \((**)\):

\[ (**) = P_N (\phi \geq 0 \text{ on } \mathcal{A}^c | \phi \in [0,a] \text{ on } A) = \int_{[0,a]^A} P_N (\phi \geq 0 \text{ on } A^c \phi = \psi \text{ on } A) g(\psi) d\psi \]

(8)

for some density function \( g \). Let \( \tilde{\psi} \) be the harmonic extension of \( \psi \) to \( \Lambda_N \setminus A \). Since \( \tilde{\psi} \geq 0 \), we have

\[ (**) = \int_{[0,a]^A} P_N (\phi + \tilde{\psi} \geq 0 \text{ on } A^c | \phi = 0 \text{ on } A) g(\psi) d\psi \]

(9)

\[ = \int_{[0,a]^A} P_{A^c} (\phi + \tilde{\psi} \geq 0 \text{ on } A^c) g(\psi) d\psi \]

(10)

\[ \geq P_{A^c} (\Omega_N^{+}) \]

(11)
For the term (*), we write \( A = \{x_1, \ldots, x_{|A|}\} \), and \( A_i = \{x_{i+1}, \ldots, x_{|A|}\} \),

\[
(*) = P_N(\phi \in [0, a] \text{ on } A)
\]

\[
= \prod_{i=1}^{|A|} P_N(\phi_{x_i} \in [0, a] | \phi_{x_{i+1}}, \ldots, \phi_{x_{|A|}} \in [0, a])
\]

\[
= \prod_{i=1}^{|A|} \int_{[0,a]^A} P_N(\phi_{x_i} \in [0, a] | \phi = \psi \text{ on } A_i) g_i(\psi) d\psi
\]

for some density \( g_i \). Let \( \tilde{\psi} \) be the harmonic extension of \( \psi \) to \( \Lambda_N \setminus A_i \), we have

\[
(*) = \prod_{i=1}^{|A|} \int_{[0,a]^A} P_N(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a] | \phi = 0 \text{ on } A_i) g_i(\psi) d\psi
\]

\[
= \prod_{i=1}^{|A|} \int_{[0,a]^A} P_N(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a] | \phi = 0) g_i(\psi) d\psi
\]

\[
\geq \prod_{i=1}^{|A|} P_N(\phi_{x_i} \in [0, a])
\]

\[
\geq [c(1/2 \wedge a)]^{|A|}
\]

for some \( c = c(d) > 0 \), since the variance of the free field is bounded in \( d \geq 3 \). The inequality (17) comes from the fact that \( P_N(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a]) \geq P_N(\phi_{x_i} \in [0, a]) \) since \( \tilde{\psi}_{x_i} \in [0, a] \) and \( \phi_{x_i} \) is a centered Gaussian variable. Hence,

\[
\hat{P}_{N,a,b}(\Omega_N^+ < e N^d | \Omega_N^+)
\]

\[
\frac{Z_N}{Z_{N,a,b}} \sum_{A \subset A_N} \exp(J'|A|) P_{A'}(\Omega_{A'}^+)
\]

3.2. Upper bound on the numerator.

\[
\hat{P}_{N,a,b}(\xi_N < e N^d | \Omega_N^+)
\]

\[
\frac{Z_N}{Z_{N,a,b}} \sum_{A : |A| < e N^d} (e^b - 1)^{|A|} P_N(\mathcal{A} \supset A) P_N(\Omega_N^+ | \mathcal{A} \supset A) \leq 1
\]

\[
\leq \frac{Z_N}{Z_{N,a,b}} \|A : |A| < e N^d\| \exp(J e N^d)
\]

with \( J = \log(e^b - 1) + \log(e + \log(1/2 \wedge a)) \), where \( \|X\| \) denotes the cardinality of the set \( X \).

3.3. Upper bound on (5).

\[
\hat{P}_{N,a,b}(\xi_N < e N^d | \Omega_N^+)
\]

\[
\leq \exp(J e N^d) \|A : |A| < e N^d\| \sum_{A \subset A_N} \exp(J'|A|) P_{A'}(\Omega_{A'}^+)
\]

And now we proceed similarly as for the proof with \( \delta \)-pinning potential:

\[
\frac{1}{N^d} \log \hat{P}_{N,a,b}(\xi_N < e N^d | \Omega_N^+) \leq \frac{1}{N^d} \log \left( \exp(J e N^d) \|A : |A| < e N^d\| \right)
\]

\[
- \frac{1}{N^d} \log \sum_{A \subset A_N} \exp(J'|A|) P_{A'}(\Omega_{A'}^+)
\]

The right hand side of (23) can be bounded by \( \epsilon(J + 1 - \log \epsilon) \) as \( N \) tends to infinity (by a rough approximation and the Stirling formula), which in turn can be made as close to 0 as we want by choosing \( \epsilon = \epsilon(J) \) sufficiently small. See [1].
To bound (24) we use [1] Proposition 3 with \( t = 0 \) which matches to our Proposition 2:

\[
(24) \leq -\frac{1}{N^d} \log \sum_{A \subseteq \Lambda_N : A \text{ is } \Delta\text{-sparse}} \exp(J'|A|) P_N(\Omega_A^t)
\]

\[
\leq -\frac{1}{N^d} \left( \left( \frac{N}{\Delta} \right)^d \left[ (d \log \Delta + c_0) + J' - d \log \Delta + c_1 \log \log \Delta \right] \right)
\]

\[
= -\frac{J' + c_0 + c_1 \log \log \Delta}{\Delta^d} < 0 \text{ for } \Delta = \Delta(J') \text{ large enough.}
\]

where \( \Delta\)-sparseness corresponds to Definition 1: a set \( A \subset \Lambda_N \) is \( \Delta\)-sparse if it equals \( A_{\Delta N}^L \), for some set \( l_{nN}^A \)

\[\Box\]

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