From a stochastic maximal inequality to infinite-dimensional martingales

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Abstract

As an alternative to the well-known methods of “chaining” and “bracketing” that have been developed in the study of random fields, a new method, which is based on a stochastic maximal inequality derived by using the Taylor expansion, is presented. The inequality dealing with finite-dimensional discrete-time martingales is pulled up to infinite-dimensional ones by using the monotone convergence arguments. The main results are some weak convergence theorems for sequences of separable random fields of discrete-time martingales under the uniform topology with the help also of entropy methods. As special cases, some new results for i.i.d. random sequences, including a new Donsker theorem and a moment bound for suprema of empirical processes indexed by classes of sets or functions, are obtained.

1 Introduction: outline of new approach

The starting point of our approach in this paper will be to derive an inequality, which may be called a stochastic maximal inequality, described as follows; the proof will be given in Section 2.

Theorem 1 Let \( \{\xi^i; i \in I_F\} \) be a finite-dimensional martingale difference sequence on a discrete-time stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_k)_{k=0,1,...,\mathbb{P}}\), where \( I_F \) is a finite set and each \( \xi^i = (\xi^i_k)_{k=1,2,...} \) is a 1-dimensional martingale difference sequence such that \( \mathbb{E}[(\xi^i_k)^2] < \infty \) for all \( i,k \). Then, there exists a (1-dimensional) martingale \( M = (M_n)_{n=0,1,...} \) such that \( M_0 = 0 \) and that for any \( n = 1,2,..., \)

\[
\max_{i \in I_F} \left( \sum_{k=1}^{n} \xi^i_k \right)^2 \leq \sum_{k=1}^{n} \mathbb{E} \left[ \max_{i \in I_F} (\xi^i_k)^2 \bigg| \mathcal{F}_{k-1} \right] + M_n.
\]
The above result is a generalization of the most important special case of the Doob-Meyer decomposition theorem in the 1-dimensional case: for any \( n = 1, 2, \ldots \),

\[
\left( \sum_{k=1}^{n} \xi_k \right)^2 = \sum_{k=1}^{n} E[(\xi_k)^2|\mathcal{F}_{k-1}] + M_n,
\]

where \((M_n)_{n=0,1,\ldots}\) is a martingale such that \(M_0 = 0\). The “stochastic maximal inequality” plays the role of an alternative to some known, maximal inequalities based on the naive inequalities “\( \max_i |X_i|^p \leq \sum_i |X_i|^p \)” for given \( p \geq 1 \) and “\( \max_i \psi(X_i) \leq \sum_i \psi(X_i) \)” for given, non-decreasing, convex function \( \psi \) such that \( \psi(0) = 0 \); c.f. pages 96–97 of van der Vaart and Wellner [42].

Having the “stochastic maximal inequality” in hands, we aim to establish some central limit theorems (CLTs) in \( \ell^\infty \)-spaces via new tightness criteria (more precisely, some criteria for asymptotic equicontinuity in probability) for sequences of separable random fields of discrete-time martingales. Along the way of our study, we will obtain some supremal inequalities in such settings.

The rest part of this introductory section is intended to be a self-contained exposition of this study, like an independent, short review note. For illustrations, we shall start with explaining our motivation through a historical review for one of the main subjects in Subsection 1.1 and finish with presenting some special cases of the main results applied to i.i.d. random sequences in Subsection 1.4; the proofs of the theorems announced there will be given in Subsection 4. In Subsection 1.2 we present an outline of the new approach based on the monotone convergence theorem in order to make our discussion in Subsection 1.4 go smoothly. Some preliminaries and notations will be provided in Subsection 1.3 and the organization of the rest part of the paper will be stated in the last subsection of the current section.

### 1.1 Motivation

Let a probability space \((\mathcal{X}, \mathcal{A}, P)\) be given, and denote the \( L^p(P) \)-seminorm on \( L^p(P) = L^p(\mathcal{X}, \mathcal{A}, P) \) by \( \| \cdot \|_{P,p} \) for every \( p \geq 1 \). Let us given a non-empty subset \( \mathcal{H} \) of \( L^2(P) \) with an envelope function \( H \in L^2(P) \), that is, a measurable function \( H \) on \((\mathcal{X}, \mathcal{A})\) such that \( |h| \leq H \) holds for every \( h \in \mathcal{H} \). Equip \( \mathcal{H} \) with the pseudometric \( \rho_{P,2} \) defined by \( \rho_{P,2}(g,h) = \|g - h\|_{P,2} \). For given i.i.d. sample \( X_1, \ldots, X_n \) from the law \( P \), we define the empirical process
\( \mathbb{G}_n = \{ \mathbb{G}_{n,t} h; (t,h) \in [0,1] \times \mathcal{H} \} \) by

\[
\mathbb{G}_{n,t} h = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (h(X_k) - Ph), \quad \forall (t,h) \in [0,1] \times \mathcal{H}.
\]

(1)

The problem to seek some sufficient conditions under which the sequence of random fields \( \{ \mathbb{G}_{n,t} h; h \in \mathcal{H} \} \) or \( \{ \mathbb{G}_{n,t} h; (t,h) \in [0,1] \times \mathcal{H} \} \) converges weakly to a Gaussian random field, in other words, some sufficient conditions for the class \( \mathcal{H} \) to have the Donsker property, was initiated by the landmark paper of Dudley \[11\], and was lively studied in the 80’s. Two types of sufficient conditions for the Donsker property have been studied, namely, the uniform entropy condition (Kolčinskii \[18\] and Pollard \[31\]) and the \( L^2 \)-bracketing entropy condition (Ossiander \[30\]; c.f., Andersen \textit{et al.} \[1\]); see Dudley \[12\], van der Vaart and Wellner \[42\] and references therein for refinements and generalizations up to the cases of row-independent arrays. Some attempts to remove the assumption of independence were started around 1990 by some authors including Doukhan \textit{et al.} \[10\] and Dedecker and Louhichi \[8\] who considered some stationary sequences based on mixing conditions, and Levental \[22\], Bae \[3\], Bae and Levental \[4\], \[5\], as well as \[23\], \[24\], \[25\], \[26\], \[27\], who considered some martingale cases. All the above results are based on some integrability conditions for certain entropies of the class \( \mathcal{H} \) described in terms of \( L^2 \)-type pseudometrics.

In contrast, in this paper we will show that, in the situation where the random field under consideration is separable with respect to a suitable pseudometric on \( \mathcal{H} \), it is not necessary to assume any “integrability” conditions for entropies of the class \( \mathcal{H} \). This phenomenon may look surprising at first sight. However, the results in the current paper might be able to be imagined earlier, because an innovative result of van der Vaart and van Zanten \[43\] concerning the Donsker property for empirical processes of 1-dimensional, regular, ergodic diffusion processes already gave us an important message on this issue. They proved that a necessary and sufficient condition for the property is that there exists a bounded and continuous version of the Gaussian random fields in the limit, which has been characterized by the deep paper of Talagrand \[35\] in terms of majorizing measures introduced by Xavier Fernique. The important thing for us is that no entropy type condition was assumed in their main theorem, which suggested that some new Donsker theorems, with no or weaker entropy conditions, could be obtained if we start our argument with assuming, e.g., the separability of random fields.
1.2 Outline of new approach

A family \( X = \{X(t); t \in \mathbb{T}\} \) of real-valued, Borel measurable random variables \( X(t) \) indexed by elements \( t \) of a non-empty set \( \mathbb{T} \) defined on a common probability space is said to be a random field. Let us first “recall” the definition of the separability of a random field when a pseudometric \( \rho \) is equipped with the parameter set \( \mathbb{T} \). Here, the meaning of the quotation marks for the word “recall” is that since Joseph L. Doob introduced the definition of the separability for random fields in 1953 (see Doob [9]), there have appeared several variations of the definition which are slightly different from Doob’s original one, depending on different purposes. The following form of the definition is taken from the authoritative book of Ledoux and Talagrand [20].

**Definition 1 (Separable random field)** A random field \( X = \{X(t); t \in \mathbb{T}\} \) with parameter \( t \) of a pseudometric space \((\mathbb{T}, \rho)\) is said to be separable if there exists a \( \mathbb{P} \)-null set \( N \in \mathcal{F} \) and a countable subset \( \mathbb{T}^* \) of \( \mathbb{T} \) such that it holds for every \( \omega \in N^c \), \( t \in \mathbb{T} \), and any \( \varepsilon > 0 \) that

\[
X(t)(\omega) \in \overline{\{X(s)(\omega); s \in \mathbb{T}^*, \rho(s, t) < \varepsilon\}},
\]

where the closure is taken in \((-\infty, \infty]\).

If \( \mathbb{T} \) itself is a countable set, then \( X \) is separable with respect to any pseudometric \( \rho \) on \( \mathbb{T} \). If \((\mathbb{T}, \rho)\) is a separable pseudometric space and almost all sample paths \( t \sim X(t) \) are \( \rho \)-continuous, then \( X \) is separable with respect to \( \rho \). If \( \mathbb{T} = (-\infty, \infty) \), or \([a, \infty)\), or \([a, b]\) and so on, and if almost all sample paths of \( t \sim X(t) \) is right-continuous with respect to a pseudometric \( \rho \) on \( \mathbb{T} \), then \( X \) is separable with respect to \( \rho \).

Now, let us present an overview of our approach. In order to establish the asymptotic tightness for sequences of random fields \( X_n = \{X_n(t); t \in \mathbb{T}\} \), if we assume that the random fields are separable with respect to a pseudometric \( \rho \) on \( \mathbb{T} \), our main task is to obtain some appropriate bounds for

\[
P \left( \sup_{s, t \in \mathbb{T}^* \atop \rho(s, t) < \delta} |X_n(s) - X_n(t)| > \varepsilon \right) \quad (2)
\]

or

\[
E \left[ \sup_{s, t \in \mathbb{T}^* \atop \rho(s, t) < \delta} |X_n(s) - X_n(t)| \wedge 1 \right], \quad (3)
\]
where $\mathbb{T}^*$ is a countable, dense subset of $\mathbb{T}$, for all (sufficiently large) $n \in \mathbb{N}$. Our plan to accomplish this purpose consists of the following two steps based on a given, increasing sequence $(\mathbb{T}_m)_{m=1,2,\ldots}$ of finite subsets of $\mathbb{T}^*$ such that $\mathbb{T}_m \uparrow \mathbb{T}^*$ as $m \to \infty$.

**Step 1** We should first obtain some “bounds”, which are not depending on $m$, for either of (2) or (3) replacing $\mathbb{T}^*$ by $\mathbb{T}_m$.

**Step 2** Then, let $m \to \infty$ to obtain the desired inequalities for $\mathbb{T}^*$ by the property of a probability measure concerning a monotone sequence of events for (2) or by the dominated/monotone convergence theorem for (3).

Our new inequality given in Theorem 1 is useful for Step 1 above with the help of Lenglart’s inequality or the optional sampling theorem for 1-dimensional stochastic processes. A key point is that the inequality holds for any finite subset $\mathbb{T}_F$ of $\mathbb{T}^*$, not changing the form of the right-hand side.

### 1.3 Some more preliminaries and notations

To illustrate our results for martingales, the rest part of this section is devoted to announcing some results in the most important special cases of i.i.d. random sequences. Let us prepare some more notations and conventions here. When a non-empty set $\mathbb{T}$ is given, we denote by $\ell^\infty(\mathbb{T})$ the space of real-valued, bounded functions on $\mathbb{T}$, and equip it with the uniform metric. We discuss the weak convergence issues in the framework of the theory developed by J. Hoffmann-Jørgensen and R.M. Dudley; see Part I of van der Vaart and Wellner [42]. Given $\epsilon > 0$, the $\epsilon$-covering number $N(\epsilon, \mathbb{T}, \rho)$ of a pseudometric space $(\mathbb{T}, \rho)$ is defined as the smallest number of balls with $\rho$-radius $\epsilon$ needed to cover $\mathbb{T}$. Given $\epsilon > 0$, a probability space $(\mathcal{X}, \mathcal{A}, P)$ and $p \geq 1$, the $\epsilon$-bracketing number $N[\| \cdot \|](\epsilon, \mathcal{H}, \rho_{P,p})$ of a class $\mathcal{H} \subset L^p(P)$ is defined as the smallest number $N$ such that: there exists $N$-pairs $(u^m, l^m)$, $m = 1, \ldots, N$, of elements of $L^p(P)$, satisfying $\rho_{P,p}(u^m, l^m) < \epsilon$ for all $m$’s, such that for every $h \in \mathcal{H}$ there exists some $m$ such that $l^m \leq h \leq u^m$. The notations “$\overset{P}{\longrightarrow}$” and “$\overset{P}{\Rightarrow}$” mean the convergence in probability and in law, respectively. We refer to van der Vaart and Wellner [42] for the weak convergence theory in $\ell^\infty$-spaces, and to Jacod and Shiryaev [17] for the standard definitions and notations in the theory of semimartingales; hereafter, they are abbreviated to vdV-W [42] and J-S [17], respectively.
1.4 Announcements of results for i.i.d. random sequences

All results announced in this subsection will be proved in Section 4 by applying more general results for discrete-time martingales.

1.4.1 A bracketing CLT for i.i.d. random fields

The first weak convergence theorem in this paper is a bracketing CLT for i.i.d. sequences of separable random fields. Some related works can be found in Chapter 2.11 of vdV-W [42], but our entropy assumption (4) is weaker than those for the known results.

Theorem 2 Let \((X_k)_{k=1,2,\ldots}\) be i.i.d. copies of a random field \(X = \{X(\theta); \theta \in \Theta\}\) indexed by a set \(\Theta\) such that \(E[X(\theta)] = 0\) and \(E[(X(\theta))^2] < \infty\) for every \(\theta \in \Theta\). Suppose that \(\Theta\) is totally bounded with respect to the pseudometric \(\rho\) defined by 
\[
\rho(\theta, \theta') = \sqrt{E[(X(\theta) - X(\theta'))^2]},
\]
that \(X\) is separable with respect to \(\rho\), and that \(\sup_{\theta \in \Theta} |X(\theta)| < \infty\) almost surely. Suppose also that for any \(\epsilon \in (0,1]\) there exist \(N(\epsilon)\)-pairs \((L^{\epsilon,m}, U^{\epsilon,m})\), \(m = 1,\ldots,N(\epsilon)\), of random variables such that 
\[
\sqrt{E[(U^{\epsilon,m} - L^{\epsilon,m})^2]} < \epsilon
\]
and that for every \(\theta \in \Theta\) there exists some \(m \in \{1,\ldots,N(\epsilon)\}\) such that 
\[
L^{\epsilon,m} \leq X(\theta) \leq U^{\epsilon,m}.
\]
If the condition 
\[
\lim_{\epsilon \to 0} \epsilon^2 \log N(\epsilon) = 0
\]
is satisfied, then the sequence \((n^{-1/2} \sum_{k=1}^{n} X_k)_{n=1,2,\ldots}\) converges weakly in \(\ell^\infty(\Theta)\) to \(G\), where \(G = \{G(\theta); \theta \in \Theta\}\) is a centered Gaussian random field with covariances \(E[G(\theta)G(\theta')] = E[X(\theta)X(\theta')]\). Moreover, almost all sample paths of \(G\) are uniformly continuous with respect to \(\rho\).

1.4.2 A Donsker theorem for empirical processes

Let us consider empirical processes indexed by classes of functions as in Subsection 1.1. The result stated below may be viewed as an extension of a “special case” of some Donsker theorems for empirical processes that have been proved under some integrability conditions for entropies, such as 
\[
\int_0^1 \sup_Q \sqrt{\log N(\epsilon)(\|H\|_{Q,2}, \mathcal{H}, \rho_{Q,2})} d\epsilon < \infty,
\]
where “\(\sup_Q\)” is taken over all probability measures \(Q\) on \((X, \mathcal{A})\), or 
\[
\int_0^1 \sqrt{\log N(\epsilon)(\mathcal{H}, \rho_{P,2})} d\epsilon < \infty;
\]
see, e.g., Theorems 2.5.2 and 2.5.6 of vdV-W [42].

The quotation marks for the above phrase “special case” mean that our result below does not completely include the Donsker theorems stated above, because we have to assume the separability of random fields.

**Theorem 3** Suppose that the class \( \mathcal{H} \subset L^2(P) \) has an envelope function \( H \in L^2(P) \); it is assumed that the class \( \mathcal{H} \) does not depend on \( n \). Suppose also that
\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \log N(\epsilon, \mathcal{H}, \rho_{P,2}) = 0. \tag{5}
\]
Given i.i.d. sample \( X_1, \ldots, X_n \) from the law \( P \), define \( G_n = \{ G_{n,h} : (t,h) \in [0,1] \times \mathcal{H} \} \) by (1).

(i) If the random field \( h \sim G_{n,1}(h) \) is separable with respect to \( \rho_{P,2} \) for every \( n = 1, 2, \ldots \), then the sequence \( (G_{n,1})_{n=1,2,\ldots} \), where \( G_{n,1} = \{ G_{n,1,h} : h \in \mathcal{H} \} \), converges weakly in \( \ell^\infty(\mathcal{H}) \) to \( G_{P,1} \), where \( G_{P,1} = \{ G_{P,1,h} : h \in \mathcal{H} \} \) is a centered Gaussian random field with covariances \( E[G_{P,1}gG_{P,1}h] = Pg_h - PgPh \). Moreover, almost all sample paths \( h \sim G_{P,1}h \) are uniformly continuous with respect to \( \rho_{P,2} \).

(ii) If the random field \( h \sim G_{n,1}(h) \) is separable with respect to \( \rho_{P,2} \) for every \( t \in [0,1] \) and \( n = 1, 2, \ldots \), then the sequence \( (G_n)_{n=1,2,\ldots} \), where \( G_n = \{ G_n,h : (t,h) \in [0,1] \times \mathcal{H} \} \), converges weakly in \( \ell^\infty([0,1] \times \mathcal{H}) \) to \( G_P \), where \( G_P = \{ G_P,h : (t,h) \in [0,1] \times \mathcal{H} \} \) is a centered Gaussian process with covariances \( E[G_{P,s}gG_{P,t}h] = (s \wedge t)(Pg_h - PgPh) \). Moreover, almost all sample paths \( t \sim G_P(t,h) \) are uniformly continuous with respect to the pseudometric \( \bar{\rho} \) defined by \( \bar{\rho}((s,g),(t,h)) = \sqrt{|s-t|} \vee \rho_{P,2}(g,h) \).

A key point of the above result is that the condition (5) implies that \( (\mathcal{H}, \rho_{P,2}) \) is totally bounded, and thus it is possible to choose a countable, dense subset \( \mathcal{H}^* \subset \mathcal{H} \) for the separability of the random fields \( h \sim G_{n,t}(h) \) with respect to \( \rho_{P,2} \).

### 1.4.3 Moment bounds for suprema of empirical processes

In the modern theory of empirical processes, another issue, which is closely related to Donsker theorems and is of interest by themselves, is the supremal inequalities. They are some bounds for \( (E[\sup_{h \in \mathcal{H}} |G_{n,1,h}|^p])^{1/p} \) for given \( p \geq 1 \), which have been known to be important, e.g., for deriving the rate of convergence of various \( M \)-estimators; see the sophisticated treatments presented in the book of vdV-W [42], as well as the original works of van de Geer [37], [38], [39] and Birgé and Massart [6], among others, based on some supremal inequalities not for \( L^p \)-norms but for large deviation probabilities.
See van de Geer [40], Kosorok [19] and Talagrand [36] for comprehensive expositions on this topic.

Although some useful supremal inequalities including Theorems 2.14.2 and 2.14.5 of vdV-W [42] as well as recent results of van de Geer and Lederer [41] have been established, up to our knowledge no result without entropy type condition is known until the present time. We will obtain a new supremal inequality without any entropy type assumption, by assuming some mild restrictions to the class $\mathcal{H}$ instead.

**Theorem 4** Let $\mathcal{H}$ be a class of elements in $\mathcal{L}^2(P)$ with an envelope function $H \in \mathcal{L}^2(P)$; it is assumed that the class $\mathcal{H}$ does not depend on $n$. Suppose also that either of the following (a) or (b) is satisfied:

(a) $\mathcal{H}$ is a countable set;

(b) the random field $\{G_{n,h}; h \in \mathcal{H}\}$ is separable with respect to $\rho_{P,2}$, where $G_{n,t}$ is defined by (1).

Then, it holds for any $n = 1, 2, \ldots$ that:

$$\sqrt{\mathbb{E}\left[\sup_{h \in \mathcal{H}} (G_{n,1}h)^2\right]} \leq ||H||_{P,2} + \sup_{h \in \mathcal{H}} |Ph|;$$

$$\mathbb{E}\left[\sup_{h \in \mathcal{H}} \left|\frac{1}{n} \sum_{k=1}^{n} (h(X_k) - Ph)\right|\right] \leq \frac{1}{\sqrt{n}} \left\{ ||H||_{P,2} + \sup_{h \in \mathcal{H}} |Ph| \right\}.$$

The following example, which may look contradicting with Theorem 4 at first sight, is due to Taiji Suzuki.

**Example 1 (T. Suzuki)** Let $P$ be the uniform distribution on $[0, 1]$, and put $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}_n$, where

$$\mathcal{H}_n = \left\{ \sum_{i=1}^{2n} v_i 1_{[(i-1)/2n, i/2n)}; \ v_i \in \{0, 1\}, \ i = 1, \ldots, 2n \right\}.$$

Then, for any realization of $X_1, \ldots, X_n$, it is possible to find an element $h \in \mathcal{H}_n$ such that $\mathbb{P}_n h = \frac{1}{n} \sum_{k=1}^{n} h(X_k) = 1$ and $Ph = 1/2$. Thus, it holds that

$$\sup_{h \in \mathcal{H}} |\mathbb{P}_n h - Ph| \geq \sup_{h \in \mathcal{H}_n} |\mathbb{P}_n h - Ph| = 1/2.$$

On the other hand, if the second displayed inequality in Theorem 4 were true for this class $\mathcal{H}$, then it should imply that $\mathcal{H}$ is uniform Glivenko-Cantelli, which should contradict with the above fact.
Notice, however, that the class \( \mathcal{H} \) contains some non-measurable functions, and that the class \( \mathcal{H}_n \) does depend on \( n \). Thus there is no contradiction here.

A more general remark is the following.

**Remark 1** In Theorem [1] the martingale difference sequences \((\xi_i^k)_{k=1,2,\ldots}, i \in I_F\), should not depend on the index "\( n \)" that is contained in the martingale \( M = (M_n)_{n=0,1,\ldots} \). For example, one should not "easily" multiply \( \xi_i^k \)'s by positive constants \( c_n \), depending on the index "\( n \)"

\[
\max_{i \in I_F} \left( \sum_{k=1}^{n} c_n \xi_i^k \right)^2 \leq c_n^2 \sum_{k=1}^{n} \mathbb{E} \left[ \max_{i \in I_F} (\xi_k^i)^2 \middle| \mathcal{F}_{k-1} \right] + M'_n,
\]

because \( n \sim M'_n = c_n^2 M_n \) is not a martingale any more. It should be warned that some careful consideration is necessary for such operations when one applies the stochastic maximal inequality or its consequences.

To close this subsection, let us observe a consequence of the above result. In general, it is known that a class of sets is uniform Glivenko-Cantelli if and only if its VC-dimension is finite; see, e.g., Theorem 6.4.5 of Dudley [12]. As a consequence from the second inequality of Theorem 4, we obtain the following fact.

**Corollary 5** The VC-dimension of any countable subset of a Borel \( \sigma \)-field on a metric space is finite.

### 1.4.4 An extension of Jain-Marcus’ theorem

If a given pseudometric space \((\Theta, \rho)\) is totally bounded, then any random field \(\{X(\theta); \theta \in \Theta\}\) whose almost all sample paths are \(\rho\)-continuous is \(\rho\)-separable. We shall present an extension of Jain and Marcus’ [16] theorem (see also Araujo and Giné [2] who considered the cases of row independent arrays), where it was assumed that

\[
\int_0^1 \sqrt{\log N(\epsilon, \Theta, \rho)} d\epsilon < \infty,
\]

to the case where

\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \log N(\epsilon, \Theta, \rho) = 0. \tag{6}
\]

We denote the space of bounded, \(\rho\)-continuous functions on \(\Theta\) by \(C_b(\Theta, \rho)\), and equip it with the uniform metric.
Theorem 6 Let \((X_k)_{k=1,2,...}\) be i.i.d. copies of a random field \(X = \{X(\theta); \theta \in \Theta\}\) indexed by a totally bounded pseudometric space \((\Theta, \rho)\) such that \(E[X(\theta)] = 0\) and \(E[X(\theta)^2] < \infty\) for every \(\theta \in \Theta\). Suppose that the condition (6) is satisfied and that there exists a random variable \(L\) such that

\[ |X(\theta) - X(\theta')| \leq L \rho(\theta, \theta'), \quad \forall \theta, \theta' \in \Theta, \quad \text{and} \quad E[L^2] < \infty. \]

Then, the sequence \((n^{-1/2} \sum_{k=1}^{n} X_k)_{n=1,2,...}\) converges weakly in \(C_b(\Theta, \rho)\) to \(G\), where \(G = \{G(\theta); \theta \in \Theta\}\) is a centered Gaussian random field with covariances \(E[G(\theta)G(\theta')] = E[X(\theta)X(\theta')]\). Moreover, almost all sample paths of \(G\) are uniformly continuous with respect to \(\rho\).

1.4.5 An infinite-dimensional version of the classical CLT

Let us present an infinite-dimensional version of the classical CLT for an i.i.d. random sequence \(X_1, X_2, ...,\) where the dimension of each random vector is “infinite” in the sense that each \(X_k = \{X_i^k; i \in I\}\) has infinitely many coordinates \(i \in I\). The result below could be proved also as a special case of some known CLTs in Banach spaces. However, the claim in the following form does not seem to be written explicitly in the literature; see, e.g., Ledoux and Talagrand [20] for this issue, as well as Section 2.1.4 of vdV-W [42] for a discussion on the relationship between the central limit theory in Banach spaces and that for empirical processes.

Theorem 7 Let \(I\) be a subset of \(\mathbb{Z}^d\) for some \(d \in \mathbb{N}\). Let \((X_k)_{k=1,2,...}\) be i.i.d. copies of \(\ell_\infty(I)\)-valued random variable \(X = \{X^i; i \in I\}\) such that \(E[X^i] = 0\) and \(E[(X^i)^2] < \infty\) for every \(i \in I\) and that

\[ \lim_{N \to \infty} \sup_{\|i\| \geq N} E \left[ \sup_{\|j\| \geq \|i\|} (X^j - X^i)^2 \right] \log(1 + N) = 0, \]

where \(\|\cdot\|\) denotes the \(d\)-dimensional Euclidean norm. Then, the sequence \((n^{-1/2} \sum_{k=1}^{n} X_k)_{n=1,2,...}\) converges weakly in \(\ell_\infty(I)\) to \(G\), where \(G = \{G^i; i \in I\}\), is a tight, Borel measurable random variable such that the distribution of every finite-dimensional marginal is Gaussian with mean zero and covariances \(E[G^iG^j] = E[X^iX^j]\). Moreover, \(i \sim G^i\) is uniformly continuous with respect to the pseudometric \(\rho\) on \(I\) defined by \(\rho(i, j) = \sqrt{E[(X^i - X^j)^2]}\) almost surely.
1.5 Organization of the paper

The organization of the rest part of this paper is the following. The stochastic maximal inequality (Theorem 1) and its consequence (Corollary 8), which will be the core of our study, are prepared in Section 2. By using the corollary, in Subsection 3.1 we will first provide two kinds of inequalities, namely, (i) some bounds for (2nd order) expectations for suprema, and (ii) infinite-dimensional Lenglart’s inequalities, both for discrete-time martingales. The main results concerning weak convergence in $\ell^\infty$-spaces are established in Subsection 3.2 by using infinite-dimensional Lenglart’s inequality. The results announced in Subsection 1.4 will be proved in Section 4. The paper finishes with stating some concluding remarks in Section 5.

The results given in this paper would be useful for the development of high-dimensional statistics (see, e.g., Bühlmann and van de Geer [7]). For example, the results concerning the rate of convergence of some Dantzig selectors by Fujimori and Nishiyama [14], [15] could be improved by using the approach presented in this paper. Some other statistical applications will be presented in the forthcoming book [29].

2 Proof of stochastic maximal inequality

Let us start our discussion with proving the stochastic maximal inequality announced at the beginning of Section 1. Throughout this section, let a discrete-time stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,1,2,\ldots}, \mathbb{P})$ be given.

Proof of Theorem 1. We may set $1_F = \mathbb{I}_m := \{1, \ldots, m\}$ without loss of generality. For every $i \in \mathbb{I}_m$ and $t \in [0, \infty)$, define $X^i_t := \left(\sum_{k \leq t} \xi^i_k\right)^2$, $Y^i_t := 1 \{i = \min\{j; j \in \mathbb{I}_m, t\}\}$ with $\mathbb{I}_{m,t} := \{i \in \mathbb{I}_m; X^i_t = \max_{j \in \mathbb{I}_m} X^j_t\}$, where we make a convention that $X^i_t = 0$ for $t \in [0,1)$. Then $Y^i_t$’s take values in $\{0, 1\}$, and it holds that $\sum_{i \in \mathbb{I}_m} Y^i_t = 1$. 

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By the Taylor expansion, it holds for any \( \nu > 0 \) and any \( n = 1, 2, \ldots \) that

\[
\max_{i \in I_m} X_n^i = \sum_{i \in I_m} X_n^i Y_n^i
\]

\[
\leq \sum_{i \in I_m} X_n^i e^{-\nu(Y_n^i-1)^2/2}
\]

\[
= \sum_{i \in I_m} \sum_{k=1}^{n} \{X_k^i e^{-\nu(Y_k^i-1)^2/2} - X_{k-1}^i e^{-\nu(Y_{k-1}^i-1)^2/2}\}
\]

\[
= \sum_{i \in I_m} \sum_{k=1}^{n} e^{-\nu(Y_k^i-1)^2/2}(X_k^i - X_{k-1}^i)
\]

\[+ \sum_{i \in I_m} \sum_{k=1}^{n} X_{k-1}^i \cdot (-\nu(Y_{k-1}^i-1))e^{-\nu(Y_{k-1}^i-1)^2/2}(Y_k^i - Y_{k-1}^i)
\]

\[+ \sum_{i \in I_m} \sum_{k=1}^{n} (-\nu(Y_k^i-1))e^{-\nu(Y_k^i-1)^2/2}(X_k^i - X_{k-1}^i)(Y_k^i - Y_{k-1}^i)
\]

\[+ \frac{1}{2} \sum_{i \in I_m} \sum_{k=1}^{n} \tilde{X}_k^i \cdot (-\nu + \nu^2(Y_k^i-1))e^{-\nu(Y_k^i-1)^2/2}(Y_k^i - Y_{k-1}^i)^2
\]

\[=: R_n(\nu),
\]

where \( (\tilde{X}_{k-1}^i, \tilde{Y}_k^i) \) is a point on the segment connecting the points \( (X_{k-1}^i, Y_{k-1}^i) \) and \( (X_k^i, Y_k^i) \). Here, note that one of the components of the fourth term of \( R_n(\nu) \), namely,

\[\tilde{X}_k^i \cdot (-\nu)e^{-\nu(Y_k^i-1)^2/2}(Y_k^i - Y_{k-1}^i)^2,
\]

is non-positive. Thus we have that \( R_n(\nu) \leq R'_n(\nu) \), where

\[
R'_n(\nu) = \sum_{i \in I_m} \sum_{k=1}^{n} e^{-\nu(Y_k^i-1)^2/2}(X_k^i - X_{k-1}^i)
\]

\[+ \sum_{i \in I_m} \sum_{k=1}^{n} X_{k-1}^i \cdot (-\nu(Y_{k-1}^i-1))e^{-\nu(Y_{k-1}^i-1)^2/2}(Y_k^i - Y_{k-1}^i)
\]

\[+ \sum_{i \in I_m} \sum_{k=1}^{n} (-\nu(Y_k^i-1))e^{-\nu(Y_k^i-1)^2/2}(X_k^i - X_{k-1}^i)(Y_k^i - Y_{k-1}^i)
\]

\[+ \frac{1}{2} \sum_{i \in I_m} \sum_{k=1}^{n} \tilde{X}_k^i \cdot \nu^2(Y_k^i-1)^2e^{-\nu(Y_k^i-1)^2/2}(Y_k^i - Y_{k-1}^i)^2.
\]
As $\nu \to \infty$, the coefficient $e^{-\nu(Y_{k-1}^i - 1)^2/2}$ for the first term converges to $Y_{k-1}^i$, and the second, third, and fourth terms vanish; thus we obtain that

$$
\lim_{\nu \to \infty} R'_n(\nu) = \sum_{i \in I_m} \sum_{k=1}^n Y_{k-1}^i (X_k^i - X_{k-1}^i),
$$

which implies that

$$
\max_{i \in I_m} X_{k-1}^i \leq \sum_{k=1}^n Y_{k-1}^i (X_k^i - X_{k-1}^i).
$$

The right-hand side can be written as

$$
\sum_{i \in I_m} \sum_{k=1}^n Y_{k-1}^i \left( \mathbb{E}[(\xi_k^i)^2 | \mathcal{F}_{k-1}] + M_n \right)
= \sum_{k=1}^n \mathbb{E} \left[ \sum_{i \in I_m} Y_{k-1}^i (\xi_k^i)^2 | \mathcal{F}_{k-1} \right] + M_n
\leq \sum_{k=1}^n \mathbb{E} \left[ \max_{i \in I_m} (\xi_k^i)^2 | \mathcal{F}_{k-1} \right] + M_n,
$$

where $M = (M_n)_{n=0,1,...}$ is a martingale given by $M_0 = 0$ and

$$
M_n = \sum_{i \in I_m} \sum_{k=1}^n Y_{k-1}^i (X_k^i - X_{k-1}^i - \mathbb{E}[(\xi_k^i)^2 | \mathcal{F}_{k-1}]), \quad n = 1, 2, ....
$$

The proof is finished.

As a consequence, we can easily deduce the following corollary by using the optional sampling theorem.

**Corollary 8** Under the same situation for $\{\xi_i; i \in I_F\}$ as in Theorem 7, it holds for any bounded stopping time $T$ that

$$
\mathbb{E} \left[ \max_{i \in I_F} \left( \sum_{k=1}^T \xi_i^k \right)^2 \right] \leq \mathbb{E} \left[ \sum_{k=1}^T \mathbb{E} \left[ \max_{i \in I_F} (\xi_k^i)^2 | \mathcal{F}_{k-1} \right] \right].
$$

### 3 Infinite-dimensional martingales

This section is devoted to pulling some devices for finite-dimensional discrete-time martingales up to infinite-dimensional ones by using the monotone convergence theorem and the elementary properties of monotone sequence of events for probability measures.
3.1 Supremum inequalities

Corollary together with the monotone convergence arguments yields the following theorem.

**Theorem 9** Let $\mathbb{I}$ be a countable set. For every $i \in \mathbb{I}$, let $(\xi^i_k)_{k=1,2,...}$ be a martingale difference sequence defined on a discrete-time stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,1,...}, P)$, which is common for all $i$'s, such that $E[\sup_{i \in \mathbb{I}} (\xi^i_k)^2] < \infty$ for all $k$.

(i) It holds for any bounded stopping time $T$ that

$$E\left[\sup_{i \in \mathbb{I}} \left(\sum_{k=1}^{T} \xi^i_k\right)^2\right] \leq E\left[\sum_{k=1}^{T} E\left[\sup_{i \in \mathbb{I}} (\xi^i_k)^2 \mid \mathcal{F}_{k-1}\right]\right].$$

(ii) It holds for any stopping time $T$ and any constants $\varepsilon > 0$ and $\delta > 0$ that

$$P\left(\sup_{n \leq T} \sup_{i \in \mathbb{I}} \left|\sum_{k=1}^{n} \xi^i_k\right| \geq \varepsilon\right) \leq \frac{\delta}{\varepsilon^2} + P\left(\sum_{k=1}^{T} E\left[\sup_{i \in \mathbb{I}} (\xi^i_k)^2 \mid \mathcal{F}_{k-1}\right] \geq \delta\right).$$

(ii') For any constant $c > 0$, it holds for any stopping time $T$ and any constants $\varepsilon > 0$ and $\delta > 0$ that

$$P\left(\sup_{n \leq T} \sup_{i \in \mathbb{I}} \left|c \sum_{k=1}^{n} \xi^i_k\right| \geq \varepsilon\right) \leq \frac{\delta}{\varepsilon^2} + P\left(c^2 \sum_{k=1}^{T} E\left[\sup_{i \in \mathbb{I}} (\xi^i_k)^2 \mid \mathcal{F}_{k-1}\right] \geq \delta\right).$$

**Proof.** (i) Observe the displayed inequality in Corollary where $\mathbb{I}_F$ is any finite subset of $\mathbb{I}$. First replace “$\max_{i \in \mathbb{I}_F}$” by “$\sup_{i \in \mathbb{I}}$” on the right-hand side. Next enumerate the elements of the class $\mathbb{I}$ with $\mathbb{I}_m = \{i_m; m \in \mathbb{N}\}$, and then apply the monotone convergence theorem to the left-hand side with $\mathbb{I}_F = \mathbb{I}_m = \{i_1, \ldots, i_m\}$ as $m \to \infty$.

(ii) The result for the case of $\mathbb{I}$ is a finite set, say $\mathbb{I}_F$, easily follows from Lenglart’s inequality, because Corollary says that the non-negative, adapted process $(X_n)_{n=0,1,2,...}$ given by $X_0 = 0$ and

$$X_n = \max_{i \in \mathbb{I}_F} \left(\sum_{k=1}^{n} \xi^i_k\right)^2, \quad n = 1, 2, ...,$$

is $L$-dominated by the increasing predictable process $(A_n)_{n=0,1,...}$ given by $A_0 = 0$ and

$$A_n = \sum_{k=1}^{n} E\left[\max_{i \in \mathbb{I}_F} (\xi^i_k)^2 \mid \mathcal{F}_{k-1}\right], \quad n = 1, 2, ...;$$
see Lenglart [21] or Definition I.3.29 of J-S [17] for the $L$-domination property, as well as Theorem 4 in Section VII.3 of Shiryaev [33] for Lenglart’s inequality for discrete-time stochastic processes.

To prove the result for $I$ rather than $I_F$, repeat the same argument as in the proof of (i), where the monotone convergence theorem for the left-hand side is replaced by the elementary property of monotone sequence of events for probability measures.

(ii’) The result easily follows from the inequality obtained in (ii). $\square$

3.2 Asymptotic equicontinuity in probability

In this subsection, we will establish some results concerning weak convergence in the space $\ell^\infty(\Theta)$ with the uniform metric, in the case where a pseudometric space $\rho$ is equipped with the set $\Theta$.

3.2.1 Preliminaries

Let us start with giving a brief review for the modern version of Prohorov’s [32] theory developed by J. Hoffmann-Jørgensen and R.M. Dudley.

A sequence $(X_n)_{n=1,2,...}$ of $\ell^\infty(\Theta)$-valued (possibly, non-measurable) maps $X_n = \{X_n(\theta); \theta \in \Theta\}$ defined on probability spaces $(\Omega_n, F_n, P_n)$ is asymptotically $\rho$-equicontinuous in probability if for any constants $\varepsilon, \eta > 0$ there exists a constant $\delta > 0$ such that

$$\limsup_{n \to \infty} P_n^* \left( \sup_{\rho(\theta, \theta') < \delta} |X_n(\theta) - X_n(\theta')| > \varepsilon \right) < \eta,$$

where $P_n^*$ denotes the outer probability measure of $P_n$. Since we will always consider some separable random fields $X_n = \{X_n(\theta); \theta \in \Theta\}$ indexed by a totally bounded pseudometric space $(\Theta, \rho)$, the above condition is equivalent to that there exists a countable, $\rho$-dense subset $\Theta^*$ of $\Theta$ such that for any constants $\varepsilon, \eta > 0$ there exists a constant $\delta > 0$ such that

$$\limsup_{n \to \infty} P_n \left( \sup_{\theta, \theta' \in \Theta^*_{\rho(\theta, \theta') < \delta}} |X_n(\theta) - X_n(\theta')| > \varepsilon \right) < \eta.$$

A sequence $(X_n)_{n=1,2,...}$ of $\ell^\infty(\Theta)$-valued random maps converges weakly to a tight Borel law if and only if there exists a pseudometric $\rho$ with respect to which $\Theta$ is totally bounded, the sequence $(X_n)_{n=1,2,...}$ is asymptotically $\rho$-equicontinuous in probability, and every finite-dimensional marginals
converge weakly to a (tight) Borel law; if, moreover, the tight Borel law on $\ell^\infty(\Theta)$ appearing as the limit is that of a random field $X = \{X(\theta); \theta \in \Theta\}$, then almost all sample paths $\theta \sim X(\theta)$ are uniformly $\rho$-continuous (see Theorems 1.5.4 and 1.5.7 of vdV-W [42]).

A tight Borel law in $\ell^\infty(\Theta)$ is characterized by all of the (tight) Borel laws of finite-dimensional marginals (see Lemma 1.5.3 of vdV-W [42]).

3.2.2 Main results

Our main results may be viewed as some extensions of “special cases” of Donsker theorems for empirical processes of i.i.d. random sequence index by a class of functions to the case of discrete-time martingales under some milder conditions than the integrability condition for bracketing entropies. The phrase “special cases” above means that the random fields are assumed to be separable in our arguments, and the results presented here do not completely include the preceding ones although the restriction additionally imposed here is mild.

Theorem 10 Let $(\Theta, \rho)$ be a totally bounded pseudometric space. Let $\{\xi^\theta; \theta \in \Theta\}$ be a class of martingale difference sequences $(\xi^\theta_k)_{k=1,2,...}$ indexed by $\theta \in \Theta$ defined on a discrete-time stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,1,...}, P)$.

For every $n \in \mathbb{N}$, let a finite stopping time $T_n$ and a constant $c_n > 0$ be given, and suppose that the random field $X_n = \{X_n(\theta); \theta \in \Theta\}$ defined by $X_n(\theta) = \sum_{k=1}^{T_n} \xi^\theta_k$, where $c_n^\theta = c_n \xi^\theta_k$, is separable with respect to $\rho$. Suppose also that the following conditions (i) and (ii) are satisfied for any countable, $\rho$-dense subset $\Theta^*$ of $\Theta$.

(i) It holds that

$$\sum_{k=1}^{T_n} E \left[ |\xi^\theta_k| 1 \left( |\xi^\theta_k| > \varepsilon \right) \right] \to 0, \quad \forall \varepsilon > 0, \quad \forall \theta \in \Theta^*.$$ 

(ii) For any $\varepsilon \in (0, 1]$ there exists a finite partition $\Theta^* = \bigcup_{m=1}^{N(\varepsilon)} \Theta^*_m(\varepsilon)$ such that the diameter (with respect to $\rho$) of $\Theta^*_m(\varepsilon)$ is smaller than $\varepsilon$ for every $m = 1,...,N(\varepsilon)$, and that the following conditions (ii-a) and (ii-b) are satisfied:

(ii-a) $\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log N(\varepsilon) = 0$;

(ii-b) For any $\eta > 0$ there exists a constant $K = K_\eta > 0$ such that for every $\varepsilon \in (0, 1]$,

$$\limsup_{n \to \infty} P \left( \sum_{k=1}^{T_n} E \sup_{\theta \in \Theta^*_m(\varepsilon)} (\xi^\theta_k - \xi^\theta_k')^2 \left| \mathcal{F}_{k-1} \right) \geq K \varepsilon^2 \right) < \eta.$$
for every $\theta \in \Theta_m(\epsilon)$ and $m = 1, 2, ..., N_1(\epsilon)$, and that for every $\epsilon \in (0, 1]$,

$$\lim_{n \to \infty} P \left( \sum_{k=1}^{T_n} E \left[ (\zeta_n^{n,\theta} - \zeta_n^{n,\theta'})^2 | \mathcal{F}_{k-1} \right] \geq K \epsilon^2 \right) = 0$$

for every $\theta, \theta' \in \Theta^*$ such that $\rho(\theta, \theta') < \epsilon$.

Then, the sequence $(X_n)_{n=1,2,...}$ is asymptotically $\rho$-equicontinuous in probability.

Before giving a proof of the above theorem, let us state two corollaries here. The first one is obtained by applying the martingale CLTs in the finite-dimensional case to establish the weak convergence of marginals; the proof is omitted.

**Corollary 11** Suppose that all conditions in Theorem 10 are satisfied. Suppose also that $\sup_{\theta \in \Theta^*} |X_n(\theta)| < \infty$, $P$-almost surely, for every $n \in \mathbb{N}$, where $\Theta^*$ is any countable, $\rho$-dense subset of $\Theta$.

(i) If the sequence $(X_n(\theta))_{n=1,2,...}$ is asymptotically tight in $\mathbb{R}$ for every $\theta \in \Theta^*$, then the sequence $(X_n)_{n=1,2,...}$ is asymptotically tight in $\ell^\infty(\Theta)$, and it converges weakly to a tight, Borel limit in $\ell^\infty(\Theta)$ that has a support in the space of bounded and uniformly $\rho$-continuous functions on $\Theta$.

(ii) Suppose that either of the following (ii-a) or (ii-b) is satisfied:

(ii-a) $\sum_{k=1}^{T_n} \zeta_k^{n,\theta} \zeta_k^{n,\theta'} \xrightarrow{P} C(\theta, \theta')$ for every $\theta, \theta' \in \Theta$, where the limit is a constant;

(ii-b) $\sum_{k=1}^{T_n} E[\zeta_k^{n,\theta} \zeta_k^{n,\theta'} | \mathcal{F}_{k-1}] \xrightarrow{P} C(\theta, \theta')$ for every $\theta, \theta' \in \Theta$, where the limit is a constant, and Lindeberg’s condition

$$\sum_{k=1}^{T_n} E[(\zeta_k^{n,\theta})^2 1\{|\zeta_k^{n,\theta}| > \varepsilon\} | \mathcal{F}_{k-1}] \xrightarrow{P} 0, \quad \forall \varepsilon > 0$$

is satisfied for every $\theta \in \Theta$.

Then, there exists a centered Gaussian random field $G = \{G(\theta); \theta \in \Theta\}$ with covariance $E[G(\theta)G(\theta')] = C(\theta, \theta')$ such that the sequence $(X_n)_{n=1,2,...}$ converges weakly in $\ell^\infty(\Theta)$ to $G$. Moreover, almost all sample paths $\theta \sim G(\theta)$ are bounded and uniformly continuous with respect to $\rho$ and also to the pseudometric $\rho_C$ given by

$$\rho_C(\theta, \theta') = \sqrt{C(\theta, \theta) + C(\theta', \theta') - 2C(\theta, \theta')}.$$
Remark 2 It follows from Sudakov’s \cite{34} minoration that $\Theta$ is totally bounded with respect also to the standard deviation pseudometric $\rho$ of the Gaussian limit $G$ and that $\lim_{\epsilon \downarrow 0} \epsilon^2 \log N(\epsilon, \Theta, \rho_C) = 0$; see, e.g., Theorem 3.18 of Ledoux and Talagrand \cite{20}. See Example 1.5.10 of vdV-W \cite{42} for the last claim concerning the continuity of sample paths with respect to $\rho_C$.

The second corollary says that the condition (ii) in Theorem 10 is easily checked if the random fields are Lipschitz continuous, and the result may be viewed as an extension of Jain-Marcus’ theorem for martingale difference array; cf., Jain and Marcus \cite{16}, Theorem 3.7.16 of Araujo and Giné \cite{2}, Example 2.11.13 of vdV-W \cite{42} and Proposition 4.5 in \cite{25}. The proof is omitted.

Corollary 12 Suppose that all conditions except for (ii) in Theorem 10 are satisfied. Suppose also that the following condition (ii'-a) and (ii'-b) are satisfied, instead of the condition (ii) in Theorem 10:

(ii'-a) $\lim_{\epsilon \downarrow 0} \epsilon^2 \log N(\epsilon, \Theta, \rho) = 0$;

(ii'-b) For every $n \in \mathbb{N}$, there exists an adapted process $(L_n^n)^{k=1,2,...}$ such that $|\zeta^{n,\theta} - \zeta^{n,\theta'}| \leq L^n_k \rho(\theta, \theta')$, $\forall \theta, \theta' \in \Theta$

and that

$$\sum_{k=1}^{T_n} E[(L^n_k)^2|\mathcal{F}_{k-1}] = O_P(1).$$

Then, the sequence $(X_n)_{n=1,2,...}$ is asymptotically $\rho$-equicontinuous in probability.

Proof of Theorem 10 Consider the decomposition $X_n(\theta) = X_n^\alpha(\theta) + \tilde{X}_n^\alpha(\theta)$, $\theta \in \Theta^*$, for any $\alpha > 0$, where

$$X_n^\alpha(\theta) = \sum_{k=1}^{T_n} \left( \zeta^{n,\theta}_{k} 1\{|\zeta^{n,\theta}_{k}| \leq \alpha \} - E \left[ \zeta^{n,\theta}_{k} 1\{|\zeta^{n,\theta}_{k}| \leq \alpha \} | \mathcal{F}_{k-1} \right] \right)$$

and

$$\tilde{X}_n^\alpha(\theta) = \sum_{k=1}^{T_n} \left( \zeta^{n,\theta}_{k} 1\{|\zeta^{n,\theta}_{k}| > \alpha \} - E \left[ \zeta^{n,\theta}_{k} 1\{|\zeta^{n,\theta}_{k}| > \alpha \} | \mathcal{F}_{k-1} \right] \right).$$

First, let us prove that the condition (i) implies that

$$|\tilde{X}_n^\alpha(\theta)| \overset{P}{\longrightarrow} 0, \ \forall \alpha > 0, \ \forall \theta \in \Theta^*. \quad (7)$$
Observe that

\[
\sum_{k=1}^{T_n} E \left[ |\zeta_{n,k}^{n,\theta} - \zeta_{k}^{n,\theta}| > a \right] \xrightarrow{P} 0.
\]

On the other hand, since the non-negative adapted process

\[
\sum_{k=1}^{n} E \left[ |\zeta_{n,k}^{n,\theta}| > a \right] \leq \sum_{k=1}^{T_n} E \left[ |\zeta_{n,k}^{n,\theta}| > a \right] \xrightarrow{P} 0.
\]

Thus (7) has been proved.

Now, fix any \( \varepsilon > 0 \) and \( \eta > 0 \), and choose \( K = K_\eta > 0 \) that meets the conditions in (ii-b). Choose sufficiently small constant \( \delta > 0 \) such that

\[
\frac{K\delta^2}{\varepsilon^2} < \eta,
\]

and then find a finite partition \( \Theta^* = \bigcup_{m=1}^{N(\delta)} \Theta_m(\delta) \) as in the condition (ii) and choose any point \( \theta_m^\delta \) from each of \( \Theta_m(\delta) \)'s. Apply Theorem 9 (ii') to obtain that

\[
P \left( \sup_{\theta \in \Theta_m^*(\delta)} |X_n(\theta) - X_n(\theta_m^\delta)| > \varepsilon \right) \leq \eta + P \left( \sum_{k=1}^{T_n} E \left[ \sup_{\theta \in \Theta_m^*(\delta)} (\zeta_{n,k}^{n,\theta} - \zeta_{k}^{n,\theta_m^\delta})^2 \right] \xrightarrow{P} \geq K\delta^2 \right).
\]

Taking \( \max_{1 \leq m \leq N(\delta)} \) and then \( \limsup_{n \to \infty} \), the first condition in (ii-b) implies that

\[
\limsup_{n \to \infty} \max_{1 \leq m \leq N(\delta)} P \left( \sup_{\theta \in \Theta_m^*(\delta)} |X_n(\theta) - X_n(\theta_m^\delta)| > \varepsilon \right) < 2\eta.
\]

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Here, observe that for any \( \theta, \theta' \in \Theta^* \) such that \( \rho(\theta, \theta') < \delta \) we can find \( m, m' \) such that \( \rho^{\theta_m^\delta, \theta_{m'}^\delta} < 3\delta \) and that for any \( a > 0 \)
\[
|X_n(\theta) - X_n(\theta')| \\
\leq |X_n(\theta) - X_n(\theta_m^\delta)| + |X_n(\theta') - X_n(\theta_{m'}^\delta)| \\
+ |X_n(\theta_m^\delta) - X_n(\theta_{m'}^\delta)| + |X_n(\theta_m^\delta) - X_n(\theta_{m'}^\delta)|.
\]
Thus, recalling (7), once we have proved that
\[
\limsup_{n \to \infty} P \left( \max_{\rho(\theta_m^\delta, \theta_{m'}^\delta) < 3\delta} |X_n^\delta(\theta_m^\delta) - X_n^\delta(\theta_{m'}^\delta)| > \varepsilon' \right) < \eta', \tag{8}
\]
we can conclude that
\[
\limsup_{n \to \infty} P \left( \sup_{\rho(\theta, \theta') < \delta} |X_n(\theta) - X_n(\theta')| > 2\varepsilon + \varepsilon' \right) < 4\eta + \eta'.
\]
Observe that the left-hand side of (8) is bounded by the \( \limsup_{n \to \infty} \) of
\[
\frac{1}{\varepsilon} E \left[ \max_{(m, m') \in M(\delta)} |X_n^\delta(\theta_m^\delta) - X_n^\delta(\theta_{m'}^\delta)| 1_{A_n(\delta)} \right] + P(A_n(\delta)^c),
\]
where \( M(\delta) = \{(m, m') \in \{1, ..., N(|\delta)\}^2; \rho^{\theta_m^\delta, \theta_{m'}^\delta} < 3\delta\} \) and
\[
A_n(\delta) = \left\{ \max_{(m, m') \in M(\delta)} \sum_{k=1}^{T_n} E \left[ \left( \zeta_k^{n, \theta_m^\delta} - \zeta_k^{n, \theta_{m'}^\delta} \right)^2 \bigg| \mathcal{F}_{k-1} \right] < 9K^2 \right\}.
\]
As a preliminary to evaluate the first term, notice that Bernstein’s inequality for discrete-time martingales (Freedman [13], or see, e.g., Corollary 3.3 in [23]) implies that
\[
P(|X_n^\delta(\theta_m^\delta) - X_n^\delta(\theta_{m'}^\delta)| > x, A_n(\delta)) \leq 2 \exp \left( -\frac{x^2}{4ax + 9K\delta^2} \right), \quad \forall x \geq 0,
\]
for every \( (m, m') \in M(\delta) \). It thus follows from Lemma 2.2.10 of vdV-W [42] that
\[
E \left[ \max_{(m, m') \in M(\delta)} |X_n^\delta(\theta_m^\delta) - X_n^\delta(\theta_{m'}^\delta)| 1_{A_n(\delta)} \right] \\
\lesssim 4a \log(1 + N(|\delta)^2) + \sqrt{9K^2 \log(1 + N(|\delta)^2)}, \tag{9}
\]
where the notation “≲” means that the left-hand side is not bigger than the right multiplied by a universal constant.

Let us now finish up the proof of (8). First, by using the condition (ii-a), choose a sufficiently small $\delta > 0$ so that the second term on the line (9) divided by $\varepsilon'$ is small, and next choose a sufficiently small $a > 0$ so that the first term on the line (9) divided by $\varepsilon'$ is small; then let $n \to \infty$ to have not only the convergence (7) but also that

$$\lim_{n \to \infty} \mathbb{P}(A_n(\delta)^c) \leq \lim_{n \to \infty} \sum_{(m,m') \in M(\delta)} \mathbb{P} \left( \sum_{k=1}^{T_n} \mathbb{E} \left[ (\zeta_n^{m} \theta_m^k - \zeta_n^{m'} \theta_m^k)^2 \right] \mathcal{F}_{k-1} \right) \geq K(3\delta)^2 = 0$$

by using the second condition in (ii-b). Thus (8) has been established.

The proof of the theorem is finished. 

4 Proofs of results for i.i.d. random sequences

This section consists of the proofs of the theorems announced in Subsection 1.4. As we will see from now on, all of them are straightforward from the corresponding results in the discrete-time martingale case, because the partial sum process of an i.i.d. random sequence may be viewed as a special case of discrete-time martingales.

Proof of Theorem 2. Define the filtration $(\mathcal{F}_k)_{k=0,1,...}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(X_{k'} : k' \leq k)$. Let us apply Theorem 10 with the help of Corollary 11 (ii) to $\xi^\theta_k = X_k(\theta)$ for every $\theta \in \Theta$, $T_n = n$ and $c_n = 1/\sqrt{n}$.

The condition (i) in Theorem 10 is checked to be satisfied as follows:

$$\sum_{k=1}^{n} \mathbb{E} \left[ |n^{-1/2}X_k(\theta)|1\{|n^{-1/2}X_k(\theta)| > \varepsilon\} \right] \leq n \cdot \mathbb{E} \left[ \frac{n^{-1}(X(\theta))^2}{\varepsilon}1\{|n^{-1/2}X(\theta)| > \varepsilon\} \right] = \frac{1}{\varepsilon} \mathbb{E} \left[ (X(\theta))^21\{|X(\theta)| > n^{1/2}\varepsilon\} \right] \to 0.$$

For every $\varepsilon \in (0,1]$, the finite partition $\Theta = \bigcup_{m=1}^{N[1](\varepsilon)} \Theta^*_{m}(\varepsilon)$ is constructed by firstly setting $\Theta^*_{m}(\varepsilon) = \{\theta \in \Theta; L^{\varepsilon,m} \leq X(\theta) \leq U^{\varepsilon,m}\}$, $m = 1,...,N[1](\varepsilon)$, and then making them disjoint; then the diameter of $\Theta^*_{m}(\Theta)$ with respect to $\rho$ is not larger than $\max_{1 \leq m \leq N[1](\varepsilon)} \sqrt{\mathbb{E}[(U^{\varepsilon,m} - L^{\varepsilon,m})^2]} < \varepsilon$. The condition (ii-a)
has been assumed. The first displayed condition in (ii-b) is satisfied with $K = 1$, because
\[
E \left[ \sup_{\theta' \in \Theta_n(\epsilon)} (X(\theta') - X(\theta))^2 \right] = E \left[ \sup_{\theta' \in \Theta_n(\epsilon)} ((X(\theta') \vee X(\theta)) - (X(\theta') \wedge X(\theta)))^2 \right] 
\leq E \left[ (U_{\epsilon,m} - L_{\epsilon,m})^2 \right] < \epsilon^2
\]
for every $\theta \in \Theta$. The second displayed condition in (ii-b) is also satisfied with $K = 1$ due to the definition of $\rho$.

On the other hand, the condition (ii-b) in Corollary 11 is clearly satisfied. The proof is finished. □

**Proof of Theorem 4.** The claim (i) of this theorem is a special case of Theorem 2 by setting $H = \Theta$ and regarding $h(X_k)$ (in the current theorem) as $X_k(h)$ (in Theorem 2).

It is not difficult to prove the claims in (ii) concerning the random field $(t, h) \sim \mathbb{G}_{n,t} h$ because the infinite-dimensional Lenglart’s inequality (Theorem 9 (ii)) is a bound for the probability of the event $\{\max_{k \leq n} \sup_{i \in I} |\sum_{k'=1}^{k} \xi_{i}^{k'}| \geq \varepsilon\}$, rather than $\{\sup_{i \in I} |\sum_{k'=1}^{n} \xi_{i}^{k'}| \geq \varepsilon\}$.

**Proof of Theorem 3.** Define the filtration $(\mathcal{F}_k)_{k=0,1,...}$ by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(X_{k': k' \leq k})$.

Let us first consider the case (a). The first inequality is proved by applying Theorem 9 (i) to $I = \mathcal{H}$, $\xi_k^h = h(X_k) - Ph$ for every $h \in \mathcal{H}$ and $T = n$, and then multiplying the both sides by $1/n$ before we finally take the square of the both sides. The upper bound is evaluated as
\[
\sqrt{E \left[ \sup_{h \in \mathcal{H}} (h(X_1) - Ph)^2 \right]} 
\leq \sqrt{E \left[ \sup_{h \in \mathcal{H}} (h(X_1))^2 \right]} + \sqrt{\sup_{h \in \mathcal{H}} (Ph)^2} 
\leq ||H||_{P,2} + \sup_{h \in \mathcal{H}} |Ph|.
\]
To prove the second inequality, multiply the left-hand and right-hand sides of the first inequality by $1/\sqrt{n}$.

Finally, the results for the case (b) are proved by reducing the problems to those for the case (a). □
Proof of Theorem 6. The claim is a special case of Corollary 12 by setting 
\[ \zeta_n^k, \theta = n^{-1/2}X_k(\theta). \]

Proof of Theorem 7. Define the filtration \((F_k)_{k=0,1,\ldots}\) by \(F_0 = \{\emptyset, \Omega\}\) and 
\[ F_k = \sigma(X_k^i : k^i \leq k, i \in I) \]
to apply Theorem 10 with the help of Corollary 11 (ii) to \(\Theta = I, \xi_{n,i}^k = X_k^i\) for every 
\(i \in I, T_n = n, \) and \(c_n = 1/\sqrt{n},\) by setting 
\[ \rho(i,j) = \sqrt{E[(X_i^j - X_i^i)^2]}. \]
The finite partition \(\Theta = \bigcup_{m=1}^{N[1](\epsilon)} \Theta_m^*(\epsilon)\) for \(\epsilon \in (0,1]\) is constructed as follows. First choose \(N = N_\epsilon\) which is the minimum positive integer satisfying 
\[ \sup_{i \in I, ||i|| \geq N_\epsilon} E \left[ \sup_{j \in I, ||j|| \geq ||i||} (X_j^i - X_i^i)^2 \right] < (\epsilon/2)^2. \]
Then, construct the finite partition 
\[ \Theta = I = I_\epsilon \cup \left( \bigcup_{i \in I \setminus I_\epsilon} \{i\} \right), \]
where \(I_\epsilon = \{i \in I : ||i|| \geq N_\epsilon\};\) this can be done with \(N[1](\epsilon) \leq (2N_\epsilon)^d + 1.\) If \(N_\epsilon \to \infty\) as \(\epsilon \downarrow 0,\) then the displayed assumption of the theorem implies that 
\[ \lim_{\epsilon \downarrow 0} \epsilon^2 \log(1 + (N_\epsilon - 1)) \]
\[ \leq \lim_{\epsilon \downarrow 0} \left( 4 \sup_{i \in I, ||i|| \geq N_\epsilon - 1} E \left[ \sup_{j \in I, ||j|| \geq ||i||} (X_j^i - X_i^i)^2 \right] \right) \log(1 + (N_\epsilon - 1)) = 0, \]
and thus \(\lim_{\epsilon \downarrow 0} \epsilon^2 \log N[1](\epsilon) = 0\) is satisfied; otherwise, it holds that 
\[ \sup_{\epsilon \in (0,1]} N[1](\epsilon) < \infty, \]
and thus \(\lim_{\epsilon \downarrow 0} \epsilon^2 \log N[1](\epsilon) = 0\) is clearly satisfied.
Checking the other conditions are easy. The proof is finished. \(\square\)

5 Concluding remarks

Aad W. van der Vaart and Jon A. Wellner stated in page 269 of their book [42] that: “The use of entropy in maximal inequalities will probably persist because of its simplicity. However, it is known that entropy inequalities
are not sharp in all cases, while a stronger tool, majorizing measures, gives sharp results, at least for Gaussian processes.”

The message from this paper is the following. In the case of separable random fields indexed by a countable set or a totally bounded set, the new method starting from the stochastic maximal inequality has given some sharp bounds. However, the new method works probably only for clean random fields such as the ones of locally square-integrable martingales where the Itô integrals work well, while the entropy inequalities work for everything. It would therefore be wise to use both methods for different purposes.

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