SPECTRAL THEORY FOR 1-BODY STARK OPERATORS

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ABSTRACT. We investigate spectral theory for a one-body Stark Hamiltonian under minimum regularity and decay conditions on the potential (actually allowing sub-linear growth at infinity). Our results include Rellich’s theorem, the limiting absorption principle, radiation condition bounds and Sommerfeld’s uniqueness, and most of these are stated and proved in sharp form employing Besov-type spaces. For the proofs we adopt a commutator scheme by Ito–Skibsted [IS]. A feature of the paper is a particular choice of an escape function related to parabolic coordinates, which conforms well with classical mechanics for the Stark Hamiltonian. The whole setting of the paper, such as the conjugate operator and the Besov-type spaces, is generated by this single escape function. We apply our results in the sequel paper [AIIS].

CONTENTS

1. Introduction 1
2. Setting and results 2
3. Conjugate operator 7
4. Rellich’s theorem 9
4.1. A priori super-cubic-exponential decay estimate 10
4.2. Absence of super-cubic-exponentially decaying eigenfunctions 12
5. LAP bounds 13
5.1. Key bounds and local compactness 13
5.2. Proof of LAP bounds 15
6. Radiation condition bounds 17
6.1. Key bounds 17
6.2. Proof of radiation condition bounds 19
6.3. Applications 20
References 22

1. Introduction

In this paper we investigate spectral theory for a perturbed Stark Hamiltonian on the Euclidean space of dimension $d \geq 2$. Let us split the space variable of $\mathbb{R}^d$ as $(x, y) = (x, y_2, \ldots, y_d) \in \mathbb{R} \times \mathbb{R}^{d-1}$ and apply the Stark field in the positive $x$-direction. The free Stark Hamiltonian is given by

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) - x = \frac{1}{2}(p_{x_1}^2 + p_{y_2}^2 + \cdots + p_{y_d}^2) - x; \quad p = -i\partial.$$

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We perturb it, and consider

\[ H = H_0 + q, \]

where \( q \) is a multiplication operator by a real-valued function \( q \in L^2_{\text{loc}}(\mathbb{R}^d) \) (with more regularity for \( d \geq 4 \)). We assume that \( q \) has a weaker growth rate at infinity than the Stark field in some appropriate sense.

We are going to present a spectral analysis of \( H \), and our main results are Rellich’s theorem, LAP (Limiting Absorption Principle), radiation condition bounds and Sommerfeld’s uniqueness result. The precise statements will be given in Section 2. These results are known for perturbations of the free Laplacian but seem to a substantial degree to be missing for Stark Hamiltonians even for the 1-body case, definitely in the sharp form as derived here. We refer to [AH, He, Ya1, Ya2, Wh] for directly related spectral results for 1-body Stark Hamiltonians, and to [Si, Ta1, Ta2, Sk, HMS1] for \( N \)-body generalizations.

The stationary scattering theory for Stark Hamiltonians is not fully developed, although asymptotic completeness of the time-dependent wave operators was established long ago, even for \( N \)-body Stark Hamiltonians [AT1, AT2, HMS2]. In our sequel paper [AIIS] we study the stationary scattering theory for the one-body problem for a more restrictive class of potentials than considered here. In particular we shall derive detailed information on the scattering matrix using results from this paper (in particular Sommerfeld’s uniqueness result).

We prove our results using the commutator scheme developed by Ito–Skibsted [IS], and the choice of an escape function \( f \), given by (2.1), is a novelty of the present paper. In the scheme of [IS] an escape function plays an central role, generating the ‘conjugate operator’ \( A \) and the associated Besov spaces \( \mathcal{B}, \mathcal{B}^* \) and \( \mathcal{B}^*_0 \), see (2.11) and (2.4), respectively. Our escape function \( f \) is intimately related to parabolic coordinates, and it has several appealing features from a classical mechanical viewpoint. As far as we know, it seems to be the first time such \( f \) is employed in commutator theory of Stark Hamiltonians, although superficially there is some similarity with a construction of [HMS1] (for example). We refer to [It1, It2] for applications of the scheme of [IS] to repulsive one-body Hamiltonians.

It is well known that Mourre theory [Mo], under conditions on the potential, yields LAP for Stark Hamiltonians. Although we call \( A \) a ‘conjugate operator’ it does not conform with the notion of conjugate operator of [Mo], in fact our \( A \) is bounded relatively to the Hamiltonian (like the \( A \) of [IS]). Nevertheless the commutator \( i[H, A] \) possesses some positivity justifying our terminology (of course this positivity is very weak and spatially non-uniform), see Lemma 3.1 with \( \Theta \equiv 1 \).

This paper is organized as follows. In Section 2 we present all the assumptions and all the main results of the paper. Section 3 is a short preliminary for proofs of the main results, where we implement a commutator computation. Section 4 is devoted to the proof of Rellich’s theorem, and Section 5 to that of LAP bounds. In Section 6 we first prove the radiation condition bounds for complex values of the spectral parameter, and then we prove LAP, the radiation condition bounds for real values of the spectral parameter and Sommerfeld’s uniqueness result.

### 2. Setting and results

In this section we precisely formulate our setting, and then state all the main results of the paper.
Throughout the paper we fix our escape function:
\[
f = f(x, y) = \chi(r + x) + [1 - \chi(r + x)](r + x)^{1/2}; \quad r = (x^2 + y^2)^{1/2},
\]  
(2.1)
where \( \chi \in C^\infty(\mathbb{R}) \) is a real-valued and smooth cut-off function satisfying
\[
\chi(s) = \begin{cases} 
1 & \text{for } s \leq 1, \\
0 & \text{for } s \geq 2,
\end{cases} \quad \frac{d}{ds}\chi = \chi' \leq 0.
\]  
(2.2)
Such choice of \( f \) is stimulated by [HMS1], but ours is completely different from theirs. One particular difference is that the level surfaces of \( f \) are paraboloids, while those of [HMS1] are distorted spheres. Actually \( r + x \) is exactly one of the components of a choice of parabolic coordinates in \( \mathbb{R}^d \). Thus the gradient vector field of \( f \) is tangent to another family of paraboloids of the converse direction, which asymptotically conforms better with the classical orbits of the Stark Hamiltonian. It is well known that in the parabolic coordinates the method of separation of variables works for the free Stark Hamiltonian, see e.g. [Ti], however our motivation is different.

Letting \( F(S) \) be the characteristic function of a general subset \( S \subseteq \mathbb{R}^d \), we set
\[
F_n = F\left( \{(x, y) \in \mathbb{R}^d | 2^n \leq f(x, y) < 2^{n+1}\} \right) \quad \text{for } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]  
(2.3)
Then define the Besov spaces \( \mathcal{B}^s, \mathcal{B}_0^s \) and \( \mathcal{B}_0^s \) associated with \( f \) as
\[
\mathcal{B} = \left\{ \psi \in L^2_{loc}(\mathbb{R}^d) \mid \sum_{n \in \mathbb{N}_0} 2^{n/2} \| F_n \psi \|_{L^2} < \infty \right\},
\]
\[
\mathcal{B}^s = \left\{ \psi \in L^2_{loc}(\mathbb{R}^d) \mid \sup_{n \in \mathbb{N}_0} 2^{-n/2} \| F_n \psi \|_{L^2} < \infty \right\},
\]  
(2.4)
\[
\mathcal{B}_0^s = \left\{ \psi \in \mathcal{B}^s \mid \lim_{n \to \infty} 2^{-n/2} \| F_n \psi \|_{L^2} = 0 \right\}.
\]
Note that these are Banach spaces with respect to the norms
\[
\| \psi \|_{\mathcal{B}} = \sum_{n \in \mathbb{N}_0} 2^{n/2} \| F_n \psi \|_{L^2}, \quad \| \psi \|_{\mathcal{B}^s} = \| \psi \|_{\mathcal{B}_0^s} = \sup_{n \in \mathbb{N}_0} 2^{-n/2} \| F_n \psi \|_{L^2}.
\]
Note also that, if we introduce the \( f \)-weighted \( L^2 \)-spaces of order \( s \in \mathbb{R} \) as
\[
L^2_s = f^{-s}L^2,
\]
then for any \( s > 1/2 \) the following proper inclusions hold:
\[
L^2_s \subset \mathcal{B} \subset L^2_{1/2} \subset L^2 \subset L^2_{-1/2} \subset \mathcal{B}_0^s \subset \mathcal{B}^s \subset L^2_{-s}.
\]  
(2.5)
Our first theorem is Rellich’s theorem, which asserts absence of generalized eigenfunctions in \( \mathcal{B}_0^0 \) for \( H \) under the following conditions on \( q \), which by [RS, Theorems X.29 and X.38] are sufficient for essentially self-adjointness of \( H \) on \( C^\infty_c(\mathbb{R}^d) \). Define a differential operator \( \partial^f \) in direction to grad \( f \) as
\[
\partial^f = (\partial f) \partial = (\partial_x f) \partial_x + (\partial_y f) \partial_y.
\]  
(2.6)
Let \( H^s = H^s(\mathbb{R}^d) \) denote the standard Sobolev space.

**Condition 2.1.** There exists a splitting \( q = q_1 + q_2 + q_3 \) by real-valued measurable functions \( q_j, j = 1, 2, 3 \), such that for some \( \rho, C > 0 \):

1. \( q_1 \) is continuously differentiable, and satisfies for any \( (x, y) \in \mathbb{R}^d \)
\[
|q_1(x, y)| \leq C f^{-\rho}, \quad \partial^f q_1(x, y) \leq C f^{-1-\rho};
\]  
(2.7)
(2) $q_2$ satisfies for any $(x, y) \in \mathbb{R}^d$
\[ |q_2(x, y)| \leq Cf^{-1-r}; \]

(3) $q_3$ is compactly supported, and the associated multiplication operator is compact as $H^2 \to H^0$.

**Remark 2.2.** Note that $f^2 \leq 2r$ holds true outside some compact subset of $\mathbb{R}^d$, but the converse $cr \leq f^2$ is false. Note also that $|\partial f|^2 = \frac{1}{2}r^{-1}$ for $f$ large, cf. (3.4), and that in general $|\partial f|^2 \leq Cr^{-1}$.

**Condition 2.3.** If $\phi \in L^2_{\text{loc}}(\mathbb{R}^d)$ satisfies

(1) $(H - \lambda)\phi = 0$ for some $\lambda \in \mathbb{R}$ in the distributional sense,

(2) $\phi = 0$ on a non-empty open subset of $\mathbb{R}^d$,

then $\phi = 0$ on $\mathbb{R}^d$.

**Remark.** The property required in Condition 2.3 is called the *unique continuation property*. We consider it as a rather independent topic and will not discuss it in this paper, only referring to [Wo] for some criteria. One sufficient condition in our setting is that, quite roughly speaking, ‘singularities’ of $q_3$ do not separate the space $\mathbb{R}^d$ into plural components. In particular, if $q_3 \equiv 0$, Condition 2.3 holds automatically.

Using the function $\chi$ from (2.2), we define smooth cut-off functions $\chi_m, \bar{\chi}_m, \chi_{m,n} \in C^\infty(\mathbb{R}^d)$ for $m, n \in \mathbb{N}_0$ as

\[ \chi_m = \chi(f/2^m), \quad \bar{\chi}_m = 1 - \chi_m, \quad \chi_{m,n} = \bar{\chi}_m \chi_n. \quad (2.8) \]

**Theorem 2.4.** Assume Conditions 2.1 and 2.3. Let $\lambda \in \mathbb{R}$. If $\phi \in L^2_{\text{loc}}(\mathbb{R}^d)$ satisfies

(1) $(H - \lambda)\phi = 0$ in the distributional sense,

(2) $\bar{\chi}_{m_0}\phi \in B^*_{0}$ for some $m_0 \in \mathbb{N}_0$,

then $\phi = 0$ on $\mathbb{R}^d$.

**Remark.** We show in [AIIS] that under more restrictive conditions on $q$ there are lots of generalized eigenfunctions in $B^*$, see Remark 2.11 below. Thus we can consider Theorem 2.4 to be optimal.

The proof of Theorem 2.4 will be given in Section 4. The following corollary is obvious by Theorem 2.4.

**Corollary 2.5.** There is no pure point spectrum for $H$, that is $\sigma_{\text{pp}}(H) = \emptyset$.

**Remark.** Theorem 2.4 and hence Corollary 2.5 hold true also for an escape function

\[ f_1 = f_1(x,y) = \chi(x) + [1 - \chi(x)]|x|^{1/2} \quad (2.9) \]

instead of (2.1). Obviously Theorem 2.4 is a stronger statement with $f$ rather than $f$ replaced by $f_1$. The setting with $f_1$ is very similar to the one of [Ya1]. We note that [Ya1] does not discuss absence of eigenvalues, and that the assumptions are not completely comparable. For example we allow a growing long-range part in the direction of the Stark field, while in [Ya1] the potential can only grow in the classically forbidden region. In the direction of the field the potential in [Ya1] is assumed to be short-range. On the other hand the singular part in [Ya1] can have unbounded support.
Our second theorem is \textit{LAP bounds} for the resolvent
\[ R(z) = (H - z)^{-1} \text{ for } z \in \mathbb{C} \setminus \mathbb{R}. \]

We shall need an additional condition to treat the classical forbidden region.

\textbf{Condition 2.6.} \textit{Conditions 2.1 and 2.3 hold. In addition, for any } f_0 \geq 1 \text{ \textit{lim}}
\[ \mu \to -\infty \left( \inf \{ -x + q(x,y) \mid x < \mu, \, f(x,y) \leq f_0 \} \right) = \infty. \]

For a compact interval \( I \subseteq \mathbb{R} \) we write
\[ I_\pm = \{ z \in \mathbb{C} \mid \text{Re } z \in I, \, \pm \text{Im } z \in (0,1) \}, \]
respectively. In addition, we introduce a differential operator \( p^f \) and a matrix \( \ell \) as
\[ p^f = -i \partial^f, \quad \ell_{jk} = |\partial^f|^2 \delta_{jk} - (\partial^j f)(\partial^k f), \quad (2.10) \]
cf. (2.6). Note that \( \ell \) represents a projection onto the orthogonal complement of \( \text{grad} f \), scaled by \( |\partial^f|^2 \). In particular, \( \ell \) is non-negative.

The Einstein summation convention is adopted throughout the paper, although tensorial superscripts are avoided. For a general linear operator \( T \) we write \( \langle T \rangle \psi = \langle \psi, T \psi \rangle \).

\textbf{Theorem 2.7.} \textit{Assume Condition 2.6. Let } I \subseteq \mathbb{R} \text{ \textit{be a compact interval. Then there exists } C > 0 \text{ \textit{such that } for any } z \in I_\pm \text{ and } \psi \in \mathcal{B} \text{ the state } \phi = R(z) \psi \text{ satisfies}}
\[ \| \phi \|_{\mathcal{B}^*} + \| (1 - x/r)^{1/2} \phi \|_{L^2_{1/2}} + \| p^f \phi \|_{\mathcal{B}^*} + \langle p_j f^{-1} \ell_{jk} p_k \rangle^{1/2} \phi \leq C \| \psi \|_{\mathcal{B}}. \]

\textbf{Remark.} The finiteness of the second term on the left-hand side means that \( \phi \) has a slightly stronger decay rate in directions not parallel to \( x \), cf. (2.5). The bound actually reproduces a result of [Ad] for the 1-body case. Similarly, the derivatives \( p \phi \) have slightly stronger decay rates in directions orthogonal to \( \text{grad} f \), as expressed by the finiteness of the fourth term, cf. (2.14) below.

The proof of Theorem 2.7 will be given in Section 5. The following corollary follows directly from Theorem 2.7.

\textbf{Corollary 2.8.} \textit{There is no singularly continuous spectrum for } \( H \), that is \( \sigma_{sc}(H) = \emptyset. \)

\textbf{Remark.} Corollaries 2.5 and 2.8 assert that the spectrum \( \sigma(H) \) is purely absolutely continuous. Although Theorems 2.4 and 2.7 are much more detailed results, stability of purely absolute continuous spectrum is of its own interest. See e.g. [BCDSSW, NP, Ki, Sa, CK] for related results, most of which depend on 1-dimensional techniques.

Thirdly, we provide radiation condition bounds for \( R(z) \), which describe the leading oscillation of the resolvent along \( \text{grad} f \). Define a differential operator \( A \) as
\[ A = [H_0, i f] = \text{Re } p^f = p^f - \frac{i}{2}(\Delta f), \quad (2.11) \]
cf. (2.10). Note that \( A \) is essentially self-adjoint on \( C^\infty_0(\mathbb{R}^d) \), and by using Condition 2.1 and Remark 2.2 one easily checks that \( \mathcal{D}(A) \supset \mathcal{D}(H) \). Let \( I \subseteq \mathbb{R} \) be a compact interval, and we choose an asymptotic complex phase \( a = a_z \) as
\[ a = \chi_I \left[ \sqrt{(r - q_1 + z)/r} \pm \frac{1}{2} f r^{-2} \right] \quad \text{for } z \in I_\pm, \quad (2.12) \]
respectively. Here $l \in \mathbb{N}_0$ is chosen dependently on $I$ such that $\text{Re}((r - q_1 + z)/r)$ is uniformly positive for all $z \in I_\pm$ and $(x, y) \in \text{supp } \tilde{\chi}_l$. The branch of square root is fixed such that $\text{Re} \sqrt{w} > 0$ for $w \in \mathbb{C} \setminus (-\infty, 0]$.

Let us further impose an additional condition that slightly strengthens the second bound of (2.7). Let us use shorthand notation

$$\bar{\partial} = |\partial f| \partial, \quad \bar{p} = |\partial f| p.$$ (2.13)

Note that then in particular we have

$$(p^f)^*(p^f) + p_j \ell_{jk} p_k = p|\partial f|^2 p = (\bar{p})^* \bar{p}.$$ (2.14)

**Condition 2.9.** Condition 2.6 holds. In addition, there exist $\tilde{\rho}, C > 0$ such that

$$|\tilde{\partial} q_1| \leq C f^{-1-\tilde{\rho}}.$$ (2.13)

With $\rho, \tilde{\rho} > 0$ from Condition 2.1 and 2.9 we set

$$\beta_c = \min\{1/2, \rho, \tilde{\rho}\}.$$ (2.14)

**Theorem 2.10.** Assume Condition 2.9. Let $I \subseteq \mathbb{R}$ be a compact interval, and choose $l \in \mathbb{N}_0$ as above. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $z \in I_\pm$ and $\psi \in f^{-\beta}B$ the states $\phi = R(z) \psi$ satisfy

$$\|f^\beta (1 - x/r)^{1/2} \phi\|_{L^2_{1/2}} + \|f^\beta (A + a) \phi\|_{B^r} + \langle p_j f^{2\beta-1} \ell_{jk} p_k \rangle^{1/2}_\phi \leq C \|f^\beta \psi\|_{B^r},$$

respectively.

**Remark 2.11.** Our choice of $a$ is partly taken for technical convenience. It is not claimed to be canonical and we do not consider Theorem 2.9 to be optimal for $\rho, \tilde{\rho} > 1/2$. In fact we show in [AIIS] that in some cases, some $\beta > 1/2$ are allowed for a different choice of $a$ (including the particular form of $a_{\pm}^{\text{sim}}$) in that case that in fact any $\beta \in [0, 4]$ can be chosen in that case (note that intuitively $\frac{L^2}{2r} \approx 1$). Note for comparison that Corollary 2.13 in this case implies the bounds

$$\|f^\beta (A \mp a_{\pm}^{\text{sim}}) \phi\|_{B^r} \leq C_{\beta} \|f^\beta \psi\|_{B^r}$$

for $\beta < 1/2$, but the result does not imply this bound if $\beta > 1/2$.

In [AIIS] we construct WKB approximations. For the above simple case these read

$$(f^{d-2}r)^{-1/2} \exp(\pm i \frac{1}{2} f^3) \xi(y/f) \in B^r \setminus B^r_0$$

for a dense set of functions $\xi \in L^2(\mathbb{R}^{d-1})$ in the other parabolic coordinates $g = y/f$. Radiation bounds are related to WKB approximations. Thus manifestly

$$(A \mp a_{\pm}^{\text{sim}})(f^{d-2}r)^{-1/2} \exp(\pm i \frac{1}{2} f^3) \xi(g) = 0 \text{ for } f > \sqrt{2}.$$ (2.14)

Of course this assertion relies on the particular form of $a_{\pm}^{\text{sim}}$ (including the particular imaginary part).

A proof of Theorem 2.10 will be given in Section 6.

Finally we present applications of Theorems 2.4, 2.7 and 2.10. The first application is LAP. (We distinguish between ‘LAP bounds’ and ‘LAP’.)
Corollary 2.12. Assume Condition 2.9. Let \( I \subseteq \mathbb{R} \) be a compact interval. For any \( k = 0, 1, s > 1/2 \) and \( \epsilon \in (0, \min\{\beta_c, s - 1/2\}) \) there exists \( C > 0 \) such that for any \( z, z' \in I_+ \) or \( z, z' \in I_- \)
\[
\|\hat{p}^k R(z) - \hat{p}^k R(z')\|_{L^2(I_+^2, L^2)} \leq C|z - z'|^\epsilon. \tag{2.15}
\]
In particular, \( \hat{p}^k R(z) \) for \( k = 0, 1 \) have uniform limits as \( I_+ \ni z \to \lambda \in I \) in the norm topology of \( \mathcal{L}(L^2_+, L^2_-) \), which one denotes by
\[
\hat{p}^k R(\lambda \pm i0) = \lim_{z \to \lambda \pm i0} \hat{p}^k R(z) \text{ in } \mathcal{L}(L^2_+, L^2_-), \tag{2.16}
\]
respectively. Moreover, these limits \( \hat{p}^k R(\lambda \pm i0) \) belong to \( \mathcal{L}(\mathcal{B}, \mathcal{B}^*) \).

Combining Theorem 2.10 and Corollary 2.12 we obtain radiation condition bounds for real spectral parameters by taking limits. Thus we need respective limits
\[
a_\pm := \lim_{z \to \lambda \pm i0} a_z \text{ for } \lambda \in I.
\]

Corollary 2.13. Assume Condition 2.9. Let \( I \subseteq \mathbb{R} \) be a compact interval, and choose \( l \in \mathbb{N}_0 \) as above. Then for all \( \beta \in (0, \beta_c) \) there exists \( C > 0 \) such that for any \( \lambda \in I \) and \( \psi \in f^{-\beta} \mathcal{B} \) the states \( \phi = R(\lambda \pm i0)\psi \) satisfy
\[
\|f^{\beta}(1 - x/r)^{1/2}\phi\|_{L^2_{1/2}} + \|f^{\beta}(A \mp a_\pm)\phi\|_{\mathcal{B}}^* + \langle p_j f^{2\beta - 1} \ell_{jk} p_k \rangle^{1/2}_{\phi} \leq C\|f^{\beta}\psi\|_{\mathcal{B}},
\]
respectively.

As the last application, we provide Sommerfeld’s uniqueness result.

Corollary 2.14. Assume Condition 2.9. Let \( \lambda \in \mathbb{R}, \phi \in f^{\beta} \mathcal{B}^* \) and \( \psi \in f^{-\beta} \mathcal{B} \) with \( \beta \in (0, \beta_c) \). Then \( \phi = R(\lambda \pm i0)\psi \) holds if and only if both of the following hold:

(1) \( (H - \lambda)\phi = \psi \) in the distributional sense;
(2) \( (A \mp a_\pm)\phi \in f^{-\beta} \mathcal{B}^*_0 \).

The proofs of Corollaries 2.12, 2.13 and 2.14 will be given in Section 6.

3. Conjugate operator

This is a short preliminary section for the proofs of our main theorems in the following sections. Here we compute an explicit expression for a weighted commutator
\[
[H, iA]_{\Theta} := i(H\Theta A - A\Theta H).
\]

For various choices of the weight function \( \Theta \in C^\infty(\mathbb{R}^d) \) (see (4.1), (4.9), (5.1) and (6.3) for concrete expressions) this ‘commutator’, with \( A \) given as in (2.11), tends to be positive (for this reason \( A \) is referred to as a conjugate operator). Implementation of a commutator is always haunted by the ‘domain problem’, however, as long as there is a common core for operators involved, in the present case \( C^\infty_c(\mathbb{R}^d) \), an approximation argument works easily. In this paper we do not elaborate further on domains for readability. Actually we have rigorously treated such problems in previous works in more complicated situations (like in cases with boundaries), cf. [IS].

For the moment we only assume that \( \Theta \in C^\infty(\mathbb{R}^d) \) is a function only of \( f \), and that for some \( m \in \mathbb{N}_0 \) and for all \( k \in \mathbb{N}_0 \)
\[
\text{supp } \Theta \subseteq \{ (x, y) \in \mathbb{R}^d \mid f(x, y) \geq 2^m \}, \quad |\Theta^{(k)}| \leq C_k, \tag{3.1}
\]
where $\Theta^{(k)}$ denotes the $k$-th derivative of $\Theta$ in $f$. In the later arguments we may let $m \in \mathbb{N}_0$ be sufficiently large, so that

$$\text{supp } \Theta \cap \text{supp } q_3 = \emptyset.$$ \hfill (3.2)

We note that on $\text{supp } \Theta$ we can write derivatives of $f$ as

$$\partial_x f = \frac{1}{2} f r^{-1},$$
$$\partial_y f = \frac{1}{2} f^{-1} r^{-1} y,$$
$$\partial_x \partial_x f = \frac{1}{2} f^{-1} r^{-1} - \frac{1}{2} f r^{-2} - \frac{1}{2} f^{-1} r^{-3} x^2,$$
$$\partial_y \partial_y f = \frac{1}{2} f^{-1} r^{-1} \delta_{jk} - \frac{1}{2} f^{-3} r^{-2} y_j y_k - \frac{1}{2} f^{-1} r^{-3} y_j y_k,$$
$$\partial_x \partial_y f = \partial_y \partial_x f = -\frac{1}{2} f^{-1} r^{-2} y - \frac{1}{2} f^{-1} r^{-3} xy.$$ \hfill (3.3)

In particular, we also have expressions on $\text{supp } \Theta$:

$$\partial^j f = \frac{1}{2} f^{r^{-1}} \partial_x + \frac{1}{2} f^{-1} r^{-1} y \partial_y,$$
$$(\partial^j r) = \frac{1}{2} f^{-1},$$
$$|\partial f|^2 = \frac{1}{2} r^{-1},$$
$$\Delta f = \frac{1}{2} (d - 2) f^{-1} r^{-1},$$
$$\partial_j \partial_k f = f^{-1} \ell_{jk} - f^{-1} \partial_j |f|^2 (\partial_k r)(\partial_k r),$$ \hfill (3.4)

see (2.6) and (2.10) for $\partial^j$ and $\ell$, respectively.

**Lemma 3.1.** Assume Condition 2.1. Then, as quadratic forms on $C_c^\infty(\mathbb{R}^d)$,

$$[H, iA]_\Theta = A \Theta' A + p_j f^{-1} \Theta' \ell_{jk} \partial_k + p_j f^{-1} |\partial f|^2 (\delta_{jk} - (\partial_j r)(\partial_k r)) \Theta p_k$$
$$+ \frac{1}{2} f^{-1} (1 - x/r) \Theta - \frac{1}{4} |\partial f|^4 \Theta'' - \frac{1}{2} (\partial^j |\partial f|^2) \Theta'' - \frac{1}{2} f^{-1} |\partial f|^4 \Theta''$$
$$+ q_4 \Theta' + q_3 \Theta - 2 \text{Im} (q_2 \Theta p') - 2 \text{Re} (f^{-1} |\partial f|^2 \Theta H) - \text{Re} (|\partial f|^2 \Theta' H)$$

with

$$q_4 = -\frac{1}{4} (\Delta |\partial f|^2) + f^{-1} |\partial f|^2 (\Delta f) - f^{-1} (|\partial f|^2 |\partial f|^2) + f^{-2} |\partial f|^4 + |\partial f|^2 q_2,$$
$$q_5 = -\frac{1}{4} (\Delta^2 f) - \frac{1}{2} f^{-1} (\Delta |\partial f|^2) + \frac{1}{2} f^{-2} |\partial f|^2 (\Delta f) + f^{-2} (|\partial f|^2 |\partial f|^2) - f^{-3} |\partial f|^4$$
$$+ 2 f^{-1} |\partial f|^2 q - (\partial^j q_1) + (\Delta f) q_2.$$ In particular, $$|q_4| \leq C f^{-1 - \min(3, \rho)} r^{-1}, \quad q_5 \geq -C f^{-1 - \min(6, \rho)}.$$ **Proof.** Using the adjoint of the expression (2.11), we can compute

$$[H, iA]_\Theta = \text{Im} ((p')^* \Theta p') + 2 \text{Im} ((p')^* \Theta (-x + q)) + \text{Re} ((\Delta f) \Theta H)$$
$$= (p')^* \Theta' p' + p_j (\partial_j \partial_k p) \Theta p_k - \frac{1}{2} p (\Delta f) \Theta p - \frac{1}{2} p |\partial f|^2 \Theta' p$$
$$+ (x - q_1) (\Delta f) \Theta + (x - q_1) |\partial f|^2 \Theta' + (\partial_x f) \Theta - (\partial^j q_1) \Theta$$
$$- 2 \text{Im} (q_2 \Theta p') + \text{Re} ((\Delta f) \Theta H).$$ \hfill (3.5)

We combine the third, fifth and tenth terms of (3.5) as

$$-\frac{1}{2} p (\Delta f) \Theta p + (x - q_1) (\Delta f) \Theta + \text{Re} ((\Delta f) \Theta H)$$
$$= -\frac{1}{4} (\Delta (\Delta f) \Theta) + (\Delta f) q_2.$$ \hfill (3.6)
and, similarly, the fourth and sixth terms of (3.5) as
\[ -\frac{1}{2}p|\partial f|^2 \Theta' p + (x - q_1)|\partial f|^2 \Theta' \]
\[ = -\frac{1}{2}(\Delta|\partial f|^2 \Theta') + |\partial f|^2 \Theta' - \text{Re}(|\partial f|^2 \Theta'H). \]

In addition, let us add to the right-hand side of (3.5) the following “zero” term:
\[ 0 = pf^{-1}|\partial f|^2 \Theta p - 2xf^{-1}|\partial f|^2 \Theta - \frac{1}{2}(\Delta f^{-1}|\partial f|^2 \Theta) \]
\[ + 2f^{-1}|\partial f|^2 q \Theta - 2 \text{Re}(f^{-1}|\partial f|^2 \Theta'H). \]

Then by (3.5), (3.6), (3.7) and (3.8) we obtain
\[ [H, iA]_\Theta = (p')^\ast \Theta' p' + pf^{-1}|\partial f|^2 \Theta p + p_j(\partial_j \partial_k f) \Theta p_k + (\partial, f) \Theta \]
\[ - 2xf^{-1}|\partial f|^2 \Theta - \frac{1}{2}(\Delta(\Delta f) \Theta) - \frac{1}{4}(\Delta |\partial f|^2 \Theta') - \frac{1}{2}(\Delta f^{-1}|\partial f|^2 \Theta) \]
\[ + |\partial f|^2 q \Theta' + 2f^{-1}|\partial f|^2 q \Theta - (\partial f q_1) \Theta + (\Delta f)f \Theta \]
\[ - 2 \text{Im}(q \Theta' p') - 2 \text{Re}(f^{-1}|\partial f|^2 \Theta'H) - \text{Re}(|\partial f|^2 \Theta'H). \]

Next, we expand the sixth to eighth terms of (3.9) as
\[ -\frac{1}{2}(\Delta(\Delta f) \Theta) - \frac{1}{4}(\Delta |\partial f|^2 \Theta') - \frac{1}{2}(\Delta f^{-1}|\partial f|^2 \Theta) \]
\[ = -\frac{1}{4}|\partial f|^4 \Theta'' - \frac{1}{2}(\Delta f)^2 \Theta'' - \frac{1}{2}(\partial f^2 |\partial f|^2 \Theta'') - \frac{1}{2}f^{-1}|\partial f|^4 \Theta'' \]
\[ - \frac{1}{2}(\partial f^{3} \Theta) - \frac{1}{4}(\Delta |\partial f|^2 \Theta') - \frac{1}{4}(\Delta f)^2 \Theta' - \frac{1}{2}f^{-1}|\partial f|^2 (\Delta f) \Theta' \]
\[ - f^{-1}(\partial f^{2} |\partial f|^2 \Theta' + f^{-2}|\partial f|^4 \Theta' - \frac{1}{4}(\Delta^2 f) \Theta - \frac{1}{2}f^{-1}(\Delta |\partial f|^2 \Theta) \]
\[ + \frac{1}{2}f^{-2}|\partial f|^2 (\Delta f) \Theta + f^{-2}(\partial f^2 |\partial f|^2 \Theta - f^{-3}|\partial f|^4 \Theta'. \]

Then the first term of (3.9) combined with the second, fifth and seventh terms of (3.10) makes the first term of asserted identity. Inserting expressions from (3.3) and (3.4) into the second to fourth terms of (3.9), we obtain the second to third terms of the asserted identity. The rest terms of (3.9) and (3.10) clearly correspond to the rest terms of the asserted identity. Hence we are done.

4. Rellich’s theorem

In this section we prove Theorem 2.4. The proof reduces to the following two propositions. We basically proceed along the lines of [IS], but here we need to discuss cubic exponential decay estimates while in [IS] linear exponential decay estimates suffice. This appears as a unique feature for the Stark Hamiltonian.

Throughout the section we impose Conditions 2.1 and 2.3.

**Proposition 4.1.** If a function \( \phi \in L^2_{\text{loc}}(\mathbb{R}^d) \) satisfies for some \( m_0 \in \mathbb{N}_0 \):

1. \( (H - \lambda)\phi = 0 \) in the distributional sense,
2. \( \chi_{m_0} \phi \in B^*_0 \),

then \( \chi_{m_0} \exp(\alpha f^3) \phi \in B^*_0 \) for any \( \alpha \geq 0 \).

**Proposition 4.2.** If a function \( \phi \in L^2_{\text{loc}}(\mathbb{R}^d) \) satisfies for some \( m_0 \in \mathbb{N}_0 \):

1. \( (H - \lambda)\phi = 0 \) in the distributional sense,
2. \( \chi_{m_0} \exp(\alpha f^3) \phi \in B^*_0 \) for any \( \alpha \geq 0 \),

then \( \phi = 0 \) on \( \mathbb{R}^d \).

Propositions 4.1 and 4.2 will be proved in Subsections 4.1 and 4.2, respectively.
4.1. A priori super-cubic-exponential decay estimate. Here we are going to prove Proposition 4.1. Choose a weight function Θ to be of the form
\[ \Theta = \Theta_{m,n,\nu}^{\alpha,\beta,\delta} = \chi_{m,n} e^{\theta}, \]
\[ \theta = \Theta_{\nu}^{\alpha,\beta,\delta} = 2\alpha f^3 + 6\beta \int_0^f s^2(1 + s/2^\nu)^{-3-\delta} ds \]  
with parameters \( \alpha, \beta \geq 0, \delta > 0 \) and \( m,n,\nu \in \mathbb{N} \). Note that Θ actually satisfies (3.1) for large \( m \). We denote the derivatives in \( f \) by primes as before. If we set
\[ \theta_0 = 1 + f/2^\nu \]
for notational simplicity, then
\[ \theta' = 6\alpha f^2 + 6\beta f^2 \theta_0^{-3-\delta}, \quad \theta'' = 12\alpha f + 12\beta f \theta_0^{-3-\delta} - 6\beta(3 + \delta)2^{-\nu} f^2 \theta_0^{-4-\delta}, \ldots \]
Noting that \( 2^{-\nu} \theta_0^{-1} \leq f^{-1} \), we have
\[ \left| (\theta - 2\alpha f^3)^{\delta^k} \right| \leq C_{\delta,k} \beta f^{3-k} \theta_0^{-3-\delta} \] for \( k = 1, 2, \ldots \).

The following form inequality plays an essential role in the proof of Proposition 4.1.

**Lemma 4.3.** Fix any \( \alpha_0 \geq 0 \) and \( \delta \in (0, \min\{2, \rho\}) \) in the definition (4.1) of Θ. Then there exist \( \beta, c, C > 0 \) and \( n_0 \in \mathbb{N} \) such that uniformly in \( \alpha \in [0, \alpha_0], n > m \geq n_0 \) and \( \nu \geq n_0 \), as quadratic forms on \( D(H) \),
\[ \Im(A \Theta(H - \lambda)) \geq cf^{-1} \theta_0^{-\delta} \Theta - C f^{-1} \left( \chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2 \right) e^\theta \]
\[ + \Re(\gamma(H - \lambda)), \]
where \( \gamma = \gamma_{m,n,\nu} \) is a certain function satisfying \( \text{supp} \gamma \subseteq \text{supp} \chi_{m,n} \) and \( |\gamma| \leq C e^\theta \) uniformly in \( n > m \geq n_0 \) and \( \nu \geq n_0 \).

**Proof.** Fix any \( \alpha_0 \geq 0 \) and \( \delta \in (0, \min\{2, \rho\}) \). We will fix small \( \beta \in (0, 1] \) and large \( n_0 \in \mathbb{N} \) in the last step of the proof. For the moment we only assume \( n_0 \in \mathbb{N} \) is large enough that (3.2) holds for all \( m \geq n_0 \). All the estimates below are uniform in \( \alpha \in [0, \alpha_0], \beta \in (0, 1], n > m \geq n_0 > 0 \) and \( \nu \geq n_0 \).

By Lemma 3.1 we can bound
\[ 2 \Im(A \Theta(H - \lambda)) \]
\[ = [H, iA]_\Theta + \lambda|\partial f| e^{\lambda|\partial f|} \]
\[ \geq A\Theta A + p_j f^{-1} \Theta \xi_{jk,p} + \tfrac{1}{2} f^{-1}(1 - x/r)\Theta \]
\[ - \tfrac{3}{2} |\partial f|^4 \theta'' \Theta - \tfrac{1}{4} |\partial f|^4 \theta^3 \Theta - \tfrac{1}{2} (|\partial f|^2 \theta^2) \Theta - \tfrac{1}{2} f^{-1} |\partial f|^4 \theta^2 \Theta \]
\[ - C_1 Q - 2 \Re(f^{-1} |\partial f|^2 \Theta(H - \lambda)) - \Re(|\partial f|^2 \Theta'(H - \lambda)) \]
\[ \geq (A + \tfrac{1}{2} |\partial f|^2 \theta') \Theta(A - \tfrac{1}{2} |\partial f|^2 \theta') + p_j f^{-1} \Theta \xi_{jk,p} + \tfrac{1}{2} f^{-1} r^{-1} (r - x) \theta_0^{-\delta} \Theta \]
\[ + \tfrac{1}{2} |\partial f|^4 \theta'(\theta'' - 2 f^{-1} \theta') \Theta - C_2 Q + \Re(\gamma_1(H - \lambda)). \]

Here and below we gather admissible error terms in \( Q \), which is of the form
\[ Q = f^{-4} |\chi_{m,n}''| e^\theta + f^{-2} |\chi_{m,n}'| e^\theta + |\chi_{m,n}'| e^\theta + f^{-1-\min\{2, \rho\}} \Theta \]
\[ + pr^{-1} |\chi_{m,n}'| e^{\theta p} + pf^{1-\rho} r^{-1} \Theta p. \]
Actually \(-Q\) can be bounded below as
\[ -Q \geq -C_3 f^{-1-\min\{2, \rho\}} \Theta - C_3 f^{-1} \left( \chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2 \right) e^\theta \]
\[ - 2 \Re(r^{-1} |\chi_{m,n}'| e^{\theta(H - \lambda)}) - 2 \Re(f^{-1-\rho} r^{-1} \Theta(H - \lambda)), \]  
(4.4)
and we will see that this will be negligibly absorbed by other terms of (4.3).

Let us further combine and bound the first and second terms of (4.3) in the following manner. Choose \( n_0 = n_{0, \beta} \) large enough depending on \( \beta \in (0, 1] \), so that
\[
\theta' \geq 6\beta(f\theta_0)^{-1}3f^{-1}\theta_0^{-\delta} \geq 6\beta2^{3(n_0-1)}f^{-1}\theta_0^{-\delta} \geq \frac{1}{2}f^{-1}\theta_0^{-\delta} \text{ on supp } \Theta.
\]

Then we have the first and second terms of (4.3) bounded as
\[
\begin{align*}
(A + \frac{1}{2} |\partial f|^2 \theta')\theta' \Theta (A - \frac{1}{2} |\partial f|^2 \theta') & + p_jf^{-1}\Theta\ell_{jkp_k} \\
\geq & \frac{1}{2} (A + \frac{1}{2} |\partial f|^2 \theta')f^{-1}\theta_0^{-\delta}\Theta (A - \frac{1}{2} |\partial f|^2 \theta') + \frac{1}{2} p_jf^{-1}\theta_0^{-\delta}\Theta\ell_{jkp_k} \\
\geq & \frac{1}{2} (p^2)^{\ast}f^{-1}\theta_0^{-\delta}\Theta p^2 - \frac{1}{8} f^{-1} |\partial f|^4 \theta^2 \theta_0^{-\delta}\Theta + \frac{1}{2} p_j f^{-1} \theta_0^{-\delta}\Theta\ell_{jkp_k} - C_4Q \\
\geq & \frac{1}{2} p f^{-1} |\partial f|^2 \theta_0^{-\delta}\Theta p - \frac{1}{8} f^{-1} |\partial f|^4 \theta^2 \theta_0^{-\delta}\Theta - C_4Q \\
\geq & f^{-1} |\partial f|^2 x\theta_0^{-\delta}\Theta + \frac{1}{8} f^{-1} |\partial f|^4 \theta^2 \theta_0^{-\delta}\Theta - C_5Q + \text{Re} (f^{-1} |\partial f|^2 \theta_0^{-\delta}\Theta(H - \lambda)).
\end{align*}
\]

On the other hand, it is clear that the fourth term of (4.3) is bounded as
\[
\begin{align*}
\frac{1}{4} |\partial f|^4 \theta' (\theta'' - 2 f^{-1} \theta') \Theta & \geq -C_6 \beta f^{-1} \theta_0^{-\delta} \Theta.
\end{align*}
\]

Now by (4.3), (4.5), (4.6) and (4.4) we obtain
\[
2 \text{Im} (A \Theta (H - \lambda)) \geq \frac{1}{2} (1 - C_7 \beta) f^{-1} \theta_0^{-\delta} \Theta - C_7 f^{-1} \min (2, \rho) \Theta
\]
\[
- C_7 f^{-1} (x_n^{2} + x_{n-1,n+1}^{2}) e^{\theta} + \text{Re} (\gamma_2 (H - \lambda)).
\]

By choosing \( \beta \in (0, 1] \) small enough, and then \( n_0 \in \mathbb{N} \) large enough we obtain the asserted inequality. \( \square \)

**Proof of Proposition 4.1.** Let \( \phi \in L^2_{\text{loc}}(\mathbb{R}^d) \) and \( m_0 \in \mathbb{N}_0 \) satisfy the assumptions of the assertion, and set
\[
\alpha_0 = \sup \{ \alpha \geq 0 \mid \bar{x}_{m_0} \exp (\alpha f^3) \phi \in B^s_0 \}.
\]

Assume \( \alpha_0 < \infty \), and let us deduce a contradiction. Fix any \( \delta \in (0, \min (2, \rho)) \), and choose \( \beta > 0 \) and \( n_0 \geq 0 \) as in Lemma 4.3. We may let \( n_0 \geq m_0 + 3 \) without loss of generality. If \( \alpha_0 = 0 \), let \( \alpha = 0 \) so that we automatically have \( \alpha + \beta > \alpha_0 \). Otherwise, we choose \( \alpha \in [0, \alpha_0] \) such that \( \alpha + \beta > \alpha_0 \). With such \( \alpha \) and \( \beta \) we evaluate the inequality (4.2) in the state \( \chi_{m-2, n+2} \), and then we obtain for any
\[
n > m \geq n_0 \text{ and } \nu \geq n_0
\]
\[
\| (f^{-1} \theta_0^{-\delta} \Theta)^{1/2} \phi \|^2 \leq C_m \| \chi_{m-1,m+1} \phi \|^2 + C_\nu 2^{-\nu/2} \| \chi_{n-1,n+1} \exp (\alpha f^3) \phi \|^2.
\]

The second term on the right-hand side of (4.7) vanishes in the limit \( n \to \infty \) since \( \bar{x}_{m_0} \exp (\alpha f^3) \phi \in B^s_0 \), and hence by Lebesgue’s monotone convergence theorem
\[
\| (\bar{x}_m f^{-1} \theta_0^{-\delta} e^\theta)^{1/2} \phi \|^2 \leq C_m \| \chi_{m-1,m+1} \phi \|^2.
\]

Next we let \( \nu \to \infty \) in (4.8) invoking again Lebesgue’s monotone convergence theorem, and then it follows that
\[
\chi_m^{1/2} f^{-1/2} \exp ((\alpha + \beta) f^3) \phi \in L^2(\mathbb{R}^d).
\]

This implies \( \chi_m^{1/2} \exp (\kappa f^3) \phi \in B^s_0 \) for any \( \kappa \in (0, \alpha + \beta) \), which contradicts that \( \alpha + \beta > \alpha_0 \). \( \square \)
4.2. Absence of super-cubic-exponentially decaying eigenfunctions. Next we prove Proposition 4.2. In order to prove it we choose $\Theta$ to be

$$\Theta = \Theta_{m,n}^\alpha = \chi_{m,n} e^\theta; \quad \theta = \theta^\alpha = 2\alpha f^3$$

with parameters $\alpha \geq 1$ and $m, n \in \mathbb{N}$. We first prove the following form inequality similar to Lemma 4.3, however, focusing on different parameters. We remark that Lemma 4.4 will be implemented similarly to Lemma 4.3.

**Lemma 4.4.** There exist $c, C > 0$ and $n_0 \in \mathbb{N}$ such that uniformly in $\alpha \geq 1$ and $n > m \geq n_0$, as quadratic forms on $\mathcal{D}(\mathcal{H})$,

$$\text{Im}(A\Theta(H - \lambda)) \geq c\alpha^2 f^3 r^{-2}\Theta - C\alpha^2 f^{-1} \left(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2\right) e^\theta + \text{Re}(\gamma(H - \lambda)),$$

where $\gamma = \gamma_{m,n}^\alpha$ is a certain function satisfying $\text{supp}\gamma \subseteq \text{supp}\chi_{m,n}$ and $|\gamma| \leq C\alpha e^\theta$ uniformly in $\alpha \geq 1$ and $n > m \geq n_0$.

**Proof.** In this proof all the estimates are uniform in $\alpha \geq 1$ and $n > m \geq n_0$. We will retake $n_0 \in \mathbb{N}$ larger, if necessary, each time it appears below.

By Lemma 3.1 we bound

$$2\text{Im}(A\Theta(H - \lambda)) = [H, \mathcal{A}] + \lambda |\partial f|^2 \Theta'$$

$$\geq A\theta' A + p_j f^{-1} \Theta \ell_{jk} p_k + \frac{1}{2} f^{-1} (1 - x/r) \Theta$$

$$- \frac{3}{4} |\partial f| |\partial f|^2 \Theta' - \frac{1}{4} \Delta |\partial f|^4 \Theta' - \frac{1}{8} \Delta \partial f^2 |\partial f|^2 \Theta - \frac{1}{8} f^{-1} |\partial f|^4 \Theta'$$

$$- C_1 Q - 2 \text{Re}(f^{-1} |\partial f|^2 \Theta(H - \lambda)) - \text{Re}(\text{Re}(f^{-1} |\partial f|^2 \Theta'))$$

$$\geq (A + \frac{1}{2} |\partial f|^2 \Theta') \Theta(A - \frac{1}{2} |\partial f|^2 \Theta') + p_j f^{-1} \Theta \ell_{jk} p_k + \frac{1}{2} f^{-1} (r - x) \Theta$$

$$- C_2 Q + \text{Re}(\gamma_1(H - \lambda)),$$

where $Q$ consists of admissible error terms:

$$Q = f^{-4} |\chi_{m,n}'' e^\theta + \alpha f^{-2} |\chi_{m,n} e^\theta + \alpha^2 |\chi_{m,n}' e^\theta + \alpha f^{-1 - \min(2, \rho)} r^{-1} \Theta$$

$$f^{-1 - \min(6, \rho) \Theta} + p r^{-1} |\chi_{m,n}' e^\theta + \alpha f^{-1 - \rho} r^{-1} \Theta p.$$}

Note that $Q$ satisfies

$$-Q \geq -C_3 \alpha^2 f^{-1 - \min(2, \rho)} r^{-2} \Theta - C_3 f^{-1 - \min(2, \rho)} \Theta$$

$$- C_3 \alpha^2 f^{-1} \left(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2\right) e^\theta$$

$$- 2 \text{Re}(r^{-1} |\chi_{m,n}' e^\theta(H - \lambda)) - 2 \text{Re}(f^{-1 - \rho} r^{-1} \Theta(H - \lambda)).$$

Let us combine and bound the first and second terms of (4.11) as

$$(A + \frac{1}{2} |\partial f|^2 \Theta') \Theta(A - \frac{1}{2} |\partial f|^2 \Theta') + p_j f^{-1} \Theta \ell_{jk} p_k$$

$$\geq \frac{1}{2} (A + \frac{1}{2} |\partial f|^2 \Theta') f^{-1} \Theta(A - \frac{1}{2} |\partial f|^2 \Theta') + \frac{1}{2} p_j f^{-1} \Theta \ell_{jk} p_k$$

$$\geq \frac{1}{2} p_j f^{-1} |\partial f|^2 \Theta p - \frac{1}{2} f^{-1} |\partial f|^4 \Theta' - C_4 Q$$

$$\geq \frac{1}{2} f^{-1} |\partial f|^2 \Theta + f^{-1} |\partial f|^2 x \Theta - C_5 Q + \text{Re}(f^{-1} |\partial f|^2 \Theta(H - \lambda))$$

by (4.11), (4.12) and (4.13) we obtain

$$2\text{Im}(A\Theta(H - \lambda)) \geq (\frac{9}{2} - C_6 f^{-\min(2, \rho)}) \alpha^2 f^3 r^{-2} \Theta + (\frac{1}{2} - C_6 f^{-\min(2, \rho)}) f^{-1} \Theta$$

$$- C_6 \alpha^2 f^{-1} \left(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2\right) e^\theta + \text{Re}(\gamma_2(H - \lambda)).$$
Lemma 5.1. Fix any function $\theta$ and functions in $\mathcal{L}_{\text{loc}}^2(\mathbb{R}^d)$ with large $\chi$ where $\chi > 0$. We evaluate the inequality (4.10) in the state $\chi_{m-2,n+2}^0$, and then it follows that for any $\alpha \geq 1$ and $n > m \geq n_0$

\[
\|f^{3/2}r^{-1}\chi_{m,n}^{1/2} \exp(\alpha f^3)\phi\|^2 \leq C_m \|\chi_{m-1,m+1} \exp(\alpha f^3)\phi\|^2
\]

or

\[
\|f^{3/2}r^{-1}\chi_{m}^{1/2} \exp(\alpha f^3)\phi\|^2 \leq C_m \|\chi_{m-1,m+1} \exp(\alpha f^3)\phi\|^2.
\]

(4.14)

Since $\bar{\chi}_m \exp(\alpha f^3)\phi \in \mathcal{B}_0^\ell$ for any $\alpha > 0$, the second term on the right-hand side of (4.14) vanishes in the limit $n \to \infty$. Hence by the Lebesgue monotone convergence theorem we obtain

\[
\|f^{3/2}r^{-1}\chi_{m}^{1/2} \exp(\alpha f^3)\phi\|^2 \leq C_m \|\chi_{m-1,m+1} \exp(\alpha f^3)\phi\|^2,
\]

or

\[
\|f^{3/2}r^{-1}\chi_{m}^{1/2} \exp(\alpha f^3)\phi\|^2 \leq C_m \|\chi_{m-1,m+1} \exp(\alpha f^3)\phi\|^2.
\]

(4.15)

Now assume $\bar{\chi}_{m+2} \phi \not\equiv 0$. The left-hand side of (4.15) grows exponentially as $\alpha \to \infty$ whereas the right-hand side remains bounded. This is a contradiction. Thus $\bar{\chi}_{m+2} \phi \equiv 0$. Then by Condition 2.3 we obtain $\phi \equiv 0$ on $\mathbb{R}^d$. □

5. LAP bounds

In this section we prove LAP bounds asserted in Theorem 2.7. Technically, we split $\phi = R(z)\psi$ into two parts according to the size of $f$. We bound the part of $\phi$ with large $f$ employing a commutator computation from Lemma 3.1 for a weight $\Theta = \Theta_{m,\nu}^\delta = \bar{\chi}_m \theta$;

\[
\theta = \theta^\delta = \int_0^{f/2^\nu} (1 + s)^{-1-\delta} \, ds = \left[1 - (1 + f/2^\nu)^{-\delta}\right]/\delta
\]

with parameters $\delta > 0$ and $m, \nu \in \mathbb{N}_0$. On the other hand, the part of $\phi$ with small $f$ can be controlled by local compactness for which we make use of Condition 2.6. These preliminary arguments are given in Subsection 5.1, and the proof of Theorem 2.7 in Subsection 5.2.

5.1. Key bounds and local compactness. Let us denote the derivatives of functions in $f$ by primes as in the previous sections. Then we have

\[
\theta' = (1 + f/2^\nu)^{-1-\delta}/2^\nu, \quad \theta'' = -(1 + \delta)(1 + f/2^\nu)^{-2-\delta}/2^{2\nu}.
\]

(5.2)

The function $\theta$ has the following properties.

Lemma 5.1. Fix any $\delta > 0$ in (5.1). Then there exist $c, C, C_k > 0$, $k = 2, 3, \ldots$, such that for any $k = 2, 3, \ldots$ and uniformly in $\nu \in \mathbb{N}_0$

\[
c/2^\nu \leq \theta \leq \min\{C, f/2^\nu\},
\]

\[
c\min\{2^\nu, f\}^\delta f^{-1-\delta}\theta \leq \theta' \leq f^{-1}\theta,
\]

\[
0 \leq (-1)^{k-1}\theta^{(k)} \leq C_k f^{-k}\theta.
\]

We omit the proof of Lemma 5.1, see e.g. [IS, Lemma 4.2]. The following proposition provides key bounds for the proof of Theorem 2.7.
Proposition 5.2. Assume Condition 2.1. Let $I \subseteq \mathbb{R}$ be a compact interval, fix any $\delta \in (0, \min\{2, \rho\})$ in (5.1). Then there exist $C > 0$ and $n \in \mathbb{N}_0$ such that for any $\nu \in \mathbb{N}_0$, $z \in I_\pm$ and $\psi \in \mathcal{B}$ the states $\phi = R(z)\psi$ satisfy

$$
\|\theta^{1/2}\phi\|^2 + \|1 - x/r\|^2 \theta^{1/2}\phi\|^2_{L^2_{\pm}} + \|\theta^{1/2}A\phi\|^2 + \langle p_j f^{-1}\theta\ell_j p_k \rangle_{\phi} \\
\leq C \left( \|\phi\|_{L^2_{\pm}} \|\psi\|_{L^2_{\pm}} + \|A\phi\|_{L^2_{\pm}} \|\psi\|_{L^2_{\pm}} + \|\chi_\alpha \theta^{1/2}\phi\|^2 \right). 
$$

(5.3)

Proof. Fix $I$ and $\delta$ as in the assertion. We choose $m \in \mathbb{N}_0$ in (5.1) large enough that (3.2) holds. It suffices to show that there exist $c_1, C_1 > 0$ and $n \in \mathbb{N}_0$ such that uniformly in $z \in I_\pm$ and $\nu \in \mathbb{N}_0$

$$
\text{Im}(A\Theta(H - z)) \geq c_1' \theta + c_1 f^{-1}(1 - x/r)\theta + c_1 A\theta A + c_1 p_j f^{-1}\theta\ell_j p_k \\
- C_1\chi_\alpha^2 \theta + \text{Re}(\gamma_1(H - z)),
$$

(5.4)

where $\gamma_1 = \gamma_1_{z, \nu}$ is a certain uniformly bounded complex-valued function: $|\gamma_1| \leq C_1$. In fact, deduction of (5.3) from (5.4) is straightforward by taking expectation of (5.4) in the state $\phi = R(z)\psi$. Hence we prove (5.4) in what follows.

By Lemmas 3.1, 5.1 and the Cauchy–Schwarz inequality we can bound uniformly in $z = \lambda \pm i\mu \in I_\pm$ and $\nu \in \mathbb{N}_0$

$$
2\text{Im}(\Theta(H - z)) \\
\geq A\Theta' A + p_j f^{-1}\Theta\ell_j p_k + p_j f^{-1}|\partial f|^2(\delta_j - (\partial_j r)(\partial_k r))\Theta p_k \\
+ \frac{1}{2} f^{-1}(1 - x/r)\Theta - \frac{1}{4}(f'\Theta')\partial f^2(\delta_j - (\partial_j r)(\partial_k r))\Theta p_k + q_\Theta \Theta' - 2\text{Im}(q_\Theta p') - 2 \text{Re}(f^{-1}\partial f^2 \Theta H - \text{Re}(\partial f^2 \Theta' H) - 2\lambda \text{Im}(A\Theta) + 2\Gamma \text{Re}(A\Theta)
$$

(5.5)

$$
\geq \frac{1}{2} A\Theta' A + \frac{1}{2} p_j f^{-1}\Theta\ell_j p_k + \frac{1}{2} p_j f^{-1}|\partial f|^2 \Theta' p + \frac{1}{2} f^{-1}(1 - x/r)\Theta - C_2 Q \\
- 2\text{Re}(f^{-1}\partial f^2 \Theta(H - z)) - \text{Re}(\partial f^2 \Theta'(H - z)) + 2\Gamma \text{Re}(\Theta p'),
$$

(5.5)

where $Q$ is an admissible error of the form

$$
Q = f^{-1}\min(2, \rho)\theta + pr^{-1} f^{-1}\rho \theta p.
$$

We rewrite and bound the third term on the right-hand side of (5.5) as

$$
\frac{1}{2} p_j |\partial f|^2 \Theta' p \geq \frac{1}{2} |\partial f|^2 \Theta' p
$$

(5.6)

$$
= \frac{1}{2} \text{Re}(|\partial f|^2 \Theta'(H - z)) + \frac{1}{2}(x - q + \lambda)|\partial f|^2 \Theta' + \frac{1}{2}(\Delta |\partial f|^2 \Theta')
$$

(5.6)

$$
\geq \frac{1}{2} \text{Re}(|\partial f|^2 \Theta'(H - z)) + \frac{1}{2} r^{-1} x \Theta' - C_3 Q.
$$

As for the eighth term of (5.5), we use the Cauchy–Schwarz inequality and Lemma 5.1, and then

$$
\mp 2\Gamma \text{Re}(\Theta p') \geq -C_4 \Gamma pr^{-1} \Theta p - C_5 \Gamma
$$

(5.7)

$$
\geq -2C_4 \Gamma \text{Re}(r^{-1} \Theta(H - z)) - C_6 \Gamma
$$

(5.7)

$$
\geq -2C_4 \Gamma \text{Re}(r^{-1} \Theta(H - z)) \pm C_6 \text{Im}(H - z).
$$

By (5.5), (5.6) and (5.7) we obtain

$$
2\text{Im}(A\Theta(H - z)) \geq \frac{1}{2} A\theta A + \frac{1}{2} p_j f^{-1}\theta\ell_j p_k + \frac{1}{4} f^{-1}(1 - x/r)\theta + \frac{1}{4} \theta' \\
- C_7 Q + \text{Re}(\gamma_2(H - z)).
$$

(5.8)
Finally we combine and bound the fourth and fifth terms of (5.8) as, for large \( n \in \mathbb{N}_0 \),
\[
\frac{1}{4}\theta' - C_7 Q \geq \frac{1}{8}\theta' - C_8 \chi^2 \theta - C_9 \text{Re}(r^{-1} f^{-1} \rho(H - z)). \tag{5.9}
\]
Hence by (5.8) and (5.9) the assertion follows. \( \square \)

For the proof of Theorem 2.7 we also use local compactness of the following form.

**Proposition 5.3.** Assume Condition 2.6. Then for any \( l \in \mathbb{N}_0 \) and compact interval \( I \) the mapping
\[
\chi_l P_H(I) : L^2 \to L^2
\]
is compact, where \( P_H(I) \) denotes the spectral projection for \( H \) onto \( I \).

**Proof.** Fix any \( l \in \mathbb{N}_0 \) and any compact interval \( I \). We let \( \{ \psi_k \}_{k \in \mathbb{N}_0} \subseteq L^2 \) be a bounded sequence, and set \( \phi_k = \chi_l P_H(I) \psi_k \). First, using Condition 2.6 we have
\[
\| \phi_k \|^2 + \| p \phi_k \|^2 \leq \| \phi_k \|^2 + 2 \langle H \phi_k - 2 \langle -x + q \rangle \phi_k \leq C_1 \| \phi_k \|^2 + C_2 \langle H \rangle \phi_k \leq C_3 \| \psi_k \|^2.
\]
Hence the sequence \( \{ \phi_k \}_{k \in \mathbb{N}_0} \) is bounded in the standard Sobolev space \( H^l(\mathbb{R}^d) \). Then by Rellich’s compact embedding theorem and the diagonal argument it suffices to show that
\[
\lim_{\nu \to \infty} \sup_k \| \eta_{\nu} \phi_k \| = 0; \quad \eta_{\nu}(x, y) = 1 - \chi(-x/2^\nu), \tag{5.10}
\]
see (2.2) for \( \chi \). Let \( \epsilon > 0 \). Using again Condition 2.6 we deduce that for any large \( \nu \in \mathbb{N}_0 \), independent of \( k \in \mathbb{N}_0 \),
\[
\| \eta_{\nu} \phi_k \|^2 \leq \epsilon \langle -x + q \rangle \eta_{\nu} \phi_k \leq \epsilon \langle H \rangle \eta_{\nu} \phi_k \leq C_4 \epsilon \| \psi_k \|^2;
\]
where \( C_4 > 0 \) does not depend on \( \epsilon > 0 \) or \( k \in \mathbb{N}_0 \). This verifies (5.10), and hence we are done. \( \square \)

5.2. **Proof of LAP bounds.** Now we prove Theorem 2.7 employing Propositions 5.2, 5.3 and a contradiction argument.

**Proof.** Let \( I \) be a compact interval.

**Step 1.** First we reduce the proof of Theorem 2.7 to the single bound
\[
\| \phi \|_{B^*} \leq C_1 \| \psi \|_{B}; \quad \phi = R(z) \psi. \tag{5.11}
\]
Assume (5.11) holds true. Fix any \( \delta \in (0, \min\{2, \rho\}) \). Then by Proposition 5.2 and (5.11) there exists \( C_2 > 0 \) such that uniformly in \( \epsilon_1 \in (0, 1) \) and \( \nu \in \mathbb{N}_0 \)
\[
\| (1 - x/r)^{1/2} \theta^{1/2} \phi \|_{L^2}^2 + \| \theta^{1/2} A \phi \|^2 + \| p_j f^{-1} \theta \ell_{jk} \phi \|_{B^*}^2 \leq \epsilon_1 \| \phi \|_{B^*}^2 + \epsilon_1^{-1} C_2 \| \psi \|_{B}^2. \tag{5.12}
\]
For each \( \nu \geq 0 \), restricting the integral region to \( \{ 2^\nu \leq f < 2^{\nu+1} \} \), we can bound the second term on the left-hand side of (5.12) as
\[
\| \theta^{1/2} A \phi \|^2 \geq 3^{-(1+\delta)} 2^{-\nu} \| F_\nu A \phi \|^2, \tag{5.13}
\]
where $F_\nu$ is from (2.3). As for the first and third terms on the same side, letting $\nu = 0$ and using Lemma 5.1, we have
\[
\|(1 - x/r)^{1/2} \theta^{1/2} \phi\|^2_{L^2_{-1/2}} + \langle p_j f^{-1} \theta \ell_j \phi \rangle \phi \geq c_1 \|(1 - x/r)^{1/2} \phi\|^2_{L^2_{-1/2}} + c_1 \langle p_j f^{-1} \ell_j \phi \rangle \phi.
\]
(5.14)

We use (5.13) and (5.14) separately in (5.12). The bound with the right-hand side of (5.14) is independent of $\nu$, and for the bound with the right-hand side of (5.13) we take the supremum in $\nu \in \mathbb{N}_0$. Then we obtain uniformly in $\epsilon_1 \in (0, 1)$
\[
c_2 \|(1 - x/r)^{1/2} \phi\|^2_{L^2_{-1/2}} + c_2 \|A \phi\|^2_{L^2} + c_2 \langle p_j f^{-1} \ell_j \phi \rangle \phi \leq \epsilon_1 \|A \phi\|^2_{L^2} + c_1 \|\psi\|^2_{L^2}.
\]

Therefore by letting $\epsilon_1 \in (0, c_2/2)$ it follows that
\[
\|(1 - x/r)^{1/2} \phi\|^2_{L^2_{-1/2}} + \|p_j f\|^2_{L^2} + \langle p_j f^{-1} \ell_j \phi \rangle \phi \leq C_3 \|\psi\|^2_{L^2}.
\]

Hence Theorem 2.7 reduces to the single bound (5.11).

Step 2. Next we prove (5.11) arguing by contradiction. So assume there exist $z_k \in I_\pm$ and $\psi_k \in B$ such that
\[
\lim_{k \to \infty} \|\psi_k\|^2_{L^2} = 0, \quad \|\phi_k\|^2_{L^2} = 1; \quad \phi_k = R(z_k) \psi_k.
\]
(5.15)

By the time-reversal property we may assume that $z_k \in I_\pm$. In addition, by choosing a subsequence we may assume that $z_k$ converges to some $z \in \mathbb{T}$. If $\text{Im} z > 0$, then (5.15) contradicts the bounds
\[
\|\phi_k\|_{L^2} \leq \|R(z_k) \psi_k\| \leq \|R(z_k)\|_{L(L^2)} \|\psi_k\| \leq \|R(z_k)\|_{L(L^2)} \|\psi\|_{L^2}
\]
as $k \to \infty$. Hence we have a real limit
\[
\lim_{k \to \infty} z_k = z = \lambda \in I.
\]
(5.16)

Let $s > 1/2$. By choosing a further subsequence we may assume $\phi_k$ converges weakly to some $\phi \in L^2_{-s}$. Then, in fact, $\phi_k$ converges strongly to $\phi \in L^2_{-s}$. To verify this let us fix $s' \in (1/2, s)$ and $h \in C_c^\infty(\mathbb{R})$ with $h = 1$ on a neighborhood of $I$, and decompose for any $l \in \mathbb{N}_0$
\[
f^{-s} \phi_k = (f^{-s} h(\cdot)) \chi_k^s(f^s \phi_k) + (f^{-s} h(\cdot)) f^s(\chi_k^s f^{s'}) (f^{-s'} \phi_k) + f^{-s} (1 - h(\cdot)) R(z_k) \psi_k.
\]
(5.17)

By (5.15) we see that the last term on the right-hand side of (5.17) converges to 0 in $L^2$. Since $f^{-s} h(\cdot) f^s$ is a bounded operator, by choosing $l \in \mathbb{N}_0$ sufficiently large the second term of (5.17) can be arbitrarily small in $L^2$. Lastly, since $f^{-s} h(\cdot) f^s$ is compact by Proposition 5.3, the first term of (5.17) converges strongly in $L^2$. Therefore $\phi_k$ converges to $\phi$ in $L^2_{-s}$:
\[
\lim_{k \to \infty} \phi_k = \phi, \quad \text{in } L^2_{-s}.
\]
(5.18)

By (5.15), (5.16) and (5.18) it follows that
\[
(H - \lambda) \phi = 0 \quad \text{in the distributional sense}.
\]
(5.19)

In addition, we can verify $\phi \in B_0^s$. In fact, let us fix $\delta = 2s - 1 > 0$ and apply Proposition 5.2 to $\phi_k$. Then, letting $k \to \infty$ and using (5.15), (5.18) and Lemma 5.1, we obtain for all $\nu \in \mathbb{N}_0$
\[
\|\theta^{1/2} \phi\| \leq \|\chi_n \theta^{1/2} \phi\| \leq C \alpha^2 \|\chi_n f^{1/2} \phi\|.
\]
(5.20)
Then by letting $\nu \to \infty$ in (5.20) we obtain $\phi \in B_0^*$.
Therefore by (5.19) and Theorem 2.4, we have $\phi = 0$, but this is a contradiction. In fact, we can prove, as in Step 1,
\[
1 = \|\phi_k\|_{B_0}^2 \leq C_5 (\|\psi_k\|_{B}^2 + \|\chi_n \phi_k\|_{L}^2).
\]
But the right-hand side can be made arbitrarily small (in particular smaller than 1) by taking $k$ big enough.

\[\square\]

6. Radiation condition bounds

Here we prove Theorem 2.10 and Corollaries 2.12, 2.13, and 2.14. For simplicity of arguments we prove the assertions only for the upper sign. For the proof of Theorem 2.10 the form inequality (6.4) below is a key ingredient, cf. (4.2), (4.10) and (5.4) in the former sections.

In this section we always assume Condition 2.9. Furthermore, we throughout the section fix a compact interval $I$ and $l \in \mathbb{N}_0$ as in (2.12), so that the phase $a$ is always a fixed function. We may let $l \in \mathbb{N}_0$ be large without loss of generality, so that the formulas from (3.3) and (3.4) are available on $\text{supp} \, a$, and also that $\text{supp} \, a \cap \text{supp} \, q_3 = \emptyset$.

6.1. Key bounds. We first present basic properties of $a$.

Lemma 6.1.  
(1) There exists $C > 0$ such that for any $z \in I_+$ and $(x, y) \in \mathbb{R}^d$
\[
|a| \leq C, \quad \text{Im} \, a \geq \frac{1}{3} \chi \ell \, br^{-2}, \quad |\tilde{\partial}a| \leq C \, f^{-1-\min\{2,\rho, \tilde{\rho}\}} \, r^{-1},
\]
where $\tilde{\partial}$ is from (2.13).

(2) Let $m \in \mathbb{N}_0$ with $m \geq l + 2$. Then for any $z \in I_+$ one can write
\[
\tilde{\chi}_m(H - z) = \tilde{\chi}_m[(A + a)r(A - a) + p_j r \ell_{jk} p_k + r - x + q_6]
\]
with
\[
q_6 = (p'ar) + ra^2 - r + q - z + \frac{1}{4} r (\Delta f)^2 + \frac{1}{2} (\partial^l r \Delta f).
\]
The function $q_6$ in particular satisfies for some $C' > 0$
\[
\tilde{\chi}_m |q_6| \leq C' \tilde{\chi}_m f^{-1-\min\{2,\rho, \tilde{\rho}\}}.
\]

Proof. The bounds in (1) follow from straightforward computations, and here we only do (2). Using the formulas from (3.3) and (3.4), we can rewrite
\[
\tilde{\chi}_m(H - z) = \tilde{\chi}_m[(p')^* r p' + \rho_j r \ell_{jk} p_k - x + q - z]
\]
\[
= \tilde{\chi}_m[A r A + \rho_j r \ell_{jk} p_k - x + q - z + \frac{1}{4} r (\Delta f)^2 + \frac{1}{2} (\partial^l r \Delta f)]
\]
\[
= \tilde{\chi}_m[(A + a)r(A - a) + \rho_j r \ell_{jk} p_k + r - x + q_6]
\]
with $q_6$ given as (6.1). The last two terms of (6.1) obviously satisfy (6.2). In addition we can compute on $\sup \tilde{\chi}_m$, using the formulas from (3.3) and (3.4),
\[
(p'ar) + a^2 r - r + q - z = \frac{1}{4} (z r^{-2} + 2(\partial^l q_1) - f r^{-2} q_1) \sqrt{(z + r - q_1) / r}
\]
\[
+ \frac{1}{8} r^{-2} - \frac{3}{16} f^2 r^{-3} + q_2.
\]
Hence we can verify (6.2). \[\square\]
We will employ the following weight functions:
\[
\Theta = \Theta_{\mu, \nu}^\beta = \tilde{x}_m \theta^{2\beta},
\]
\[
\theta = \theta_\nu^\beta = \int_0^{1/2\nu} (1 + s)^{-1-\delta} \, ds = \left[ 1 - (1 + f/2\nu)^{-\delta} \right]/\delta \tag{6.3}
\]
with parameters $\beta, \delta > 0$ and $\mu, \nu \in \mathbb{N}_0$. Note that $\theta$ is the same as that in Section 5, and hence Lemma 5.1 is available. We denote derivatives in $f$ by primes as in (5.2).

**Lemma 6.2.** Let $\beta \in (0, 1/2)$. Fix any $m \in \mathbb{N}_0$ with $m \geq l + 2$, and fix any $\delta > 0$ in (6.3). Then there exist $c, C > 0$ such that uniformly in $z \in I_+$ and $\nu \in \mathbb{N}_0$, as quadratic forms on $\mathcal{D}(H)$,
\[
\text{Im}((A - a)^* \Theta(H - z)) \\
\geq \frac{c}{f} (1 - \frac{x}{r}) \Theta + c(A - a)^* \tilde{x}_m \theta^{2\beta - 1}(A - a) + c p_j f^{-1} \ell jkp_k \\
- C f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta} \theta^{2\beta} + \text{Re}(\gamma \theta^{2\beta}(H - z)),
\]
where $\gamma = \gamma_{z, \nu}$ is a certain function satisfying $|\gamma| \leq C f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta}$.

**Proof.** In this proof we repeatedly use the formulas from (3.3) and (3.4) without mentioning. Fix $\beta \in (0, 1/2)$, $m \in \mathbb{N}_0$ and $\delta > 0$ as in the assertion. By Lemmas 6.1 we write
\[
\begin{align*}
2 \text{Im}((A - a)^* \Theta(H - z)) \\
&= 2 \text{Im}((A - a)^* \Theta(A + a)r(A - a)) + 2 \text{Im}((A - a)^* \Theta p_j r \ell jkp_k) \\
&\quad + 2 \text{Im}((A - a)^* \Theta(r - x)) + 2 \text{Im}((A - a)^* \Theta q_0),
\end{align*}
\]
and we further compute each term on the right-hand side of (6.5). All the estimates below are uniformly in $z \in I_+$ and $\nu \in \mathbb{N}_0$.

By Lemma 6.1 the first term of (6.5) can be computed and bounded as
\[
\begin{align*}
2 \text{Im}((A - a)^* \Theta(A + a)r(A - a)) \\
&= (A - a)^*(\partial^j \tilde{x}_m) \theta^{2\beta} r(A - a) + 2\beta(A - a)^* \tilde{x}_m r|\partial f|^2 \theta^{2\beta - 1}(A - a) \\
&\quad - (A - a)^*(\partial^{tr}\Theta)(A - a) + 2(A - a)^*(\text{Im} a) r \Theta(A - a) \\
&\geq \beta(A - a)^* \tilde{x}_m \theta^{2\beta - 1}(A - a).
\end{align*}
\]
For the second term of (6.5) we use Lemma 6.1, the Cauchy–Schwarz inequality and Lemma 5.1. Omitting some computations, we finally obtain
\[
\begin{align*}
2 \text{Im}((A - a)^* \Theta p_j r \ell jkp_k) \\
&= 2 \text{Im}(p_j Ar \Theta \ell jkp_k) + 2 \text{Im}(\{A, p_j\} r \Theta \ell jkp_k) - 2 \text{Im}(p_j a^* r \Theta \ell jkp_k) \\
&\quad - 2 \text{Im}([a, p_j] r \Theta \ell jkp_k) \\
&\geq p_j f^{-1} \Theta \ell jkp_k - \beta p_j \tilde{x}_m \theta^{2\beta - 1} \ell jkp_k + \frac{4}{7} p f^{-2} \Theta p - \frac{1}{2} (p^\prime)^* f^{-1} r^{-1} \Theta p^r \\
&\quad - \text{Im}((\partial j \Delta f) r \Theta \ell jkp_k) - 2 \text{Re}((\partial j a^*) r \Theta \ell jkp_k) - C_1 Q \\
&\geq (1 - \beta - \epsilon_1) p_j f^{-1} \Theta \ell jkp_k - \frac{4}{7} p f^{-2} (r - x) \Theta p - C_2 \epsilon_1 Q,
\end{align*}
\]
where $\epsilon_1 \in (0, 1)$ is a small constant fixed below, $C_2 > 0$ is independent of $\epsilon_1$, and $Q$ is an admissible error of the form
\[
Q = f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta} \theta^{2\beta} + p f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta} r^{-1} \theta^{2\beta} p.
\]
As for the third term of (6.5), we simply compute and bound it by Lemma 5.1 as
\[
2 \text{Im}((A - a)^* \Theta(r - x)) \\
\geq -\beta \bar{x}_m r^{-1}(r - x) \theta' \theta^{2\beta - 1} + \frac{1}{2} f r^{-2}(r - x) \Theta - C_3 Q \geq (\frac{1}{2} - \beta) f^{-1}(1 - x/r) \Theta + \frac{1}{2} x f^{-1} r^{-2}(r - x) \Theta - C_3 Q. \tag{6.8}
\]

The last term of (6.5) is bounded by using the Cauchy–Schwarz inequality and Lemmas 6.1 as
\[
2 \text{Im}((A - a)^* \bar{x}_m \theta^{2\beta} q_0) \geq -\epsilon_1 (A - a)^* \bar{x}_m f^{-1 - \delta} \theta^{2\beta} (A - a) - C_4 \epsilon_1^{-1} Q. \tag{6.9}
\]

By (6.5), (6.6), (6.7), (6.8) and (6.9) we have
\[
2 \text{Im}((A - a)^* \Theta(H - z)) \\
\geq (\frac{1}{2} - \beta) f^{-1}(1 - x/r) \Theta + (A - a)^* (\beta \theta' - \epsilon_1 f^{-1 - \delta} \theta) \bar{x}_m \theta^{2\beta - 1}(A - a) \\
+ (1 - \beta - \epsilon_1) p_j f^{-1} \Theta \ell_{jkq} - \frac{1}{2} p f^{-1} r^{-2}(r - x) \Theta p \\
+ \frac{1}{2} x f^{-1} r^{-2}(r - x) \Theta - C_5 \epsilon_1^{-1} Q. \tag{6.10}
\]

The first term on the right-hand side of (6.10) conform with the assertion, and so do the second and third by using Lemma 5.1 and choosing small \( \epsilon_1 \in (0, 1) \). Let us combine the fourth and fifth terms of (6.10) as
\[
- \frac{1}{2} p f^{-1} r^{-2}(r - x) \Theta p + \frac{1}{2} x f^{-1} r^{-2}(r - x) \Theta \\
\geq -C_6 f^{-1} r^{-1}(1 - x/r) \Theta - \frac{1}{2} \text{Re}(f^{-1} r^{-2}(r - x) \Theta(H - z)) - C_6 Q. \tag{6.11}
\]

Finally we bound the remainder term \( Q \) as
\[
-Q \geq -C_7 f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta} \theta^{2\beta} - 2 \text{Re}(f^{-1 - \min\{4, 2\rho, 2\bar{\rho}\} + \delta} r^{-1} \theta^{2\beta}(H - z)). \tag{6.12}
\]

Hence by (6.10), (6.11) and (6.12) the assertion follows. \( \square \)

6.2. Proof of radiation condition bounds. Here we prove the radiation condition bounds, Theorem 2.10.

Proof of Theorem 2.10. For \( \beta = 0 \) the assertion is obvious by Theorem 2.7. Hence we may let \( \beta \in (0, \beta_\epsilon) \). Take any \( m \geq l + 2 \) and \( \delta \in (0, \min\{4, 2\rho, 2\bar{\rho}\} - 2\beta) \), and apply Lemma 6.2 to the state \( \phi = R(z)\psi \) with \( \psi \in f^{-\bar{\rho}} B \) and \( z \in I_+ \). Then by the Cauchy–Schwarz inequality, Theorem 2.7 and Lemma 5.1
\[
\| (1 - x/r)^{1/2} \Theta^{1/2} \phi \|_{L^2_{1/2}}^2 + \| \bar{x}^{1/2} \theta^{1/2} \theta^{2\beta - 1/2} (A - a) \phi \| |^2 + \langle p_j f^{-1} \Theta \ell_{jkq} \rangle_{\phi} \\
\leq C_1 \left[ \| \Theta^{1/2} (A - a) \phi \|_{B^*} \| \theta^{2\beta} \psi \|_{B} + \| f^{-(1 + \min\{4, 2\rho, 2\bar{\rho}\} - \delta)/2} \theta^{2\beta} \phi \| |^2 \right. \\
\left. + \| f^{-(1 + \min\{4, 2\rho, 2\bar{\rho}\} - \delta)/2} \theta^{2\beta} \psi \| |^2 \right] \tag{6.13}
\]
\[
\leq C_2 2^{-2\beta \nu \bar{\nu}} \left[ \| \bar{x}^{1/2} f^{\beta} (A - a) \phi \|_{B^*} \| f^{\beta} \psi \|_{B} + \| f^{\beta} \psi \|_{B}^2 \right].
\]

Here we have \( f^{\beta} (A - a) \phi = f^{\beta} (A - a) R(z) \psi \in B^* \) for each \( z \in I_+ \) (seen by commuting \( R(z) \) and powers of \( f \)) and hence the right-hand side of (6.13) is finite. Then it
follows by (6.13) that
\[ 2^{2\beta}\| (1 - x/r)^{1/2}\Theta^{1/2}\phi \|_{L^2_{1/2}}^2 + 2^{2\beta}\| \bar{\chi}_m^{1/2}\theta^{1/2}\bar{\theta}^{1/2}(A - a)\phi \|_{L^2_{1/2}}^2 \]
\[ + 2^{2\beta} \langle p_j f^{-1} \Theta \ell_{jk} p_k \rangle \phi \leq C_2 \left[ \| \bar{\chi}_m^{1/2} f^\beta (A - a)\phi \|_B \| f^\beta \psi \|_B + \| f^\beta \psi \|_B^2 \right]. \] (6.14)

In the second term on the left-hand side of (6.14) restrict the integral region to \( \{ 2^\nu \leq f < 2^{\nu+1} \} \) and take supremum in \( \nu \in \mathbb{N}_0 \), and then we obtain
\[ c_1 \| \bar{\chi}_m^{1/2} f^\beta (A - a)\phi \|_B^2 \leq C_2 \left[ \| \bar{\chi}_m^{1/2} f^\beta (A - a)\phi \|_B \| f^\beta \psi \|_B + \| f^\beta \psi \|_B^2 \right]. \]

By the Cauchy–Schwarz inequality this implies
\[ \| \bar{\chi}_m^{1/2} f^\beta (A - a)\phi \|_B \leq C_3 \| f^\beta \psi \|_B. \] (6.15)

As for the first and third terms on the left-hand side of (6.14) we first bound \( \theta^{2\beta} \geq (f\theta)^{2\beta} \), then take the limit \( \nu \to \infty \) using the Lebesgue monotone convergence theorem, and use (6.15) to estimate the right-hand side, yielding
\[ \left\| \bar{\chi}_m^{1/2} f^\beta (1 - x/r)^{1/2}\phi \right\|_{L^2_{1/2}}^2 + \langle p_j \bar{\chi}_m^{1/2} f^{2\beta-1} \ell_{jk} p_k \rangle \phi \leq C_4 \| f^\beta \psi \|_B^2. \] (6.16)

From (6.15) and (6.16) we can remove the cut-off \( \bar{\chi}_m^{1/2} \) by using Theorem 2.7. Hence we are done. \( \square \)

6.3. Applications. Finally we prove Corollaries 2.12, 2.13 and 2.14 as applications of Theorems 2.4, 2.7 and 2.10.

6.3.1. LAP.

Proof of Corollary 2.12. Let \( s > 1/2 \) and \( \epsilon \in (0, \min\{\beta_c, s - 1/2\}) \) as in the assertion. Let \( s' = s - \epsilon \). Decompose for \( n \in \mathbb{N}_0 \) and \( z, z' \in I_+ \) as
\[ R(z) - R(z') = \chi_n R(z) \chi_n - \chi_n R(z') \chi_n \]
\[ + (R(z) - \chi_n R(z) \chi_n) - (R(z') - \chi_n R(z') \chi_n). \] (6.17)

We estimate terms on the right-hand side of (6.17) as follows. By Theorem 2.7 we can estimate uniformly in \( n \in \mathbb{N}_0 \) and \( z, z' \in I_+ \) as
\[ \| R(z) - \chi_n R(z) \chi_n \|_{L^2_{1/2}, L^2_{1/2}} \]
\[ \leq \| f^{-s} R(z) \bar{\chi}_n f^{-s} \|_{L^2} + \| f^{-s} \bar{\chi}_n R(z) \chi_n f^{-s} \|_{L^2} \]
\[ \leq C_1 2^{-(s-s')n} = C_1 2^{-\epsilon n}, \] (6.18)

and, similarly,
\[ \| R(z') - \chi_n R(z') \chi_n \|_{L^2_{1/2}, L^2_{1/2}} \leq C_2 2^{-(s-s')n} = C_2 2^{-\epsilon n}. \] (6.19)

As for the first and second terms of (6.17), noting \( \bar{a}_z = a_z \) and
\[ i[H, \chi_{n+1}] = \text{Re}(\chi'_{n+1} p') = \text{Re}(\chi'_{n+1} A), \] (6.20)

we can rewrite them as
\[ \chi_n R(z) \chi_n - \chi_n R(z') \chi_n \]
\[ = \chi_n R(z) \{ \chi_{n+1} (H - z') - (H - z) \chi_{n+1} \} R(z') \chi_n \]
\[ = \frac{1}{2} \chi_n R(z) \chi_{n+1} (A - a_{z'}) R(z') \chi_n + \frac{1}{2} \chi_n R(z) (A + a_z) \chi_{n+1} R(z') \chi_n \]

\[
- \frac{1}{2} \chi_n R(z)(a_z - a_{z'}) \chi_{n+1}' R(z') \chi_{m} + (z - z') \chi_n R(z) \chi_{m} R(z') \chi_{n} \\
- (z - z') \chi_m R(z) \chi_{n+1}(a_z + a_{z'})^{-1} (A - a_{z'}) R(z') \chi_n \\
+ (z - z') \chi_n R(z) (A + a_z)^* \chi_{n+1}(a_z + a_{z'})^{-1} R(z') \chi_n \\
- (z - z') \chi_n R(z) [\lambda, \chi_{m+1}(a_z + a_{z'})^{-1}] R(z') \chi_n.
\]

Here \( m \in \mathbb{N}_0 \) is chosen so that \((a_z + a_{z'})^{-1}\) is non-singular on \( \text{supp} \chi_m \). Then by Theorems 2.7 and 2.10 we have uniformly in \( n \in \mathbb{N}_0 \) and \( z, z' \in I_+ \)

\[
\| \chi_n R(z) \chi_{n} - \chi_n R(z') \chi_{n}\|_{L(L^2, L^2_{-1})} \leq C_3 2^{-e_n} + C_4 2^{(1-e)n} |z - z'|. \quad (6.21)
\]

By (6.17), (6.18), (6.19) and (6.21), we obtain uniformly in \( n \in \mathbb{N}_0 \) and \( z, z' \in I_+ \)

\[
\| R(z) - R(z') \|_{L(L^2, L^2_{-1})} \leq C_4 2^{-e_n} + C_3 2^{(1-e)n} |z - z'|.
\]

Now, if \( |z - z'| \leq 1 \), we choose \( n \in \mathbb{N}_0 \) with \( 2^n \leq |z - z'|^{-1} \leq 2^{n+1} \), and then we obtain

\[
\| R(z) - R(z') \|_{L(L^2, L^2_{-1})} \leq C_5 |z - z'|^{e'}. \]

The same bound is trivial for \( |z - z'| > 1 \), and hence the Hölder continuity (2.15) for \( R(z) \) is obtained. The Hölder continuity (2.15) for \( \tilde{R}(z) \) follows by that for \( R(z) \) and the first resolvent equation.

The existence of the limits (2.16) follows immediately from (2.15). By Theorem 2.7 the limits \( R(\lambda \pm i0) \) and \( \tilde{R}(\lambda \pm i0) \) map into \( B^* \), and moreover by density argument these limits extended continuously to maps \( B \to B^* \). Hence the assertions are verified. \( \Box \)

6.3.2. Radiation condition bounds for real spectral parameters.

Proof of Corollary 2.13. The assertion is from Theorem 2.10, Corollary 2.12 and approximation arguments. Here we only note the elementary property

\[
\| \psi \|_{B^*} = \sup_{m \in \mathbb{N}_0} \| \chi_m \psi \|_{B^*} \text{ for } \psi \in B^*.
\]

Hence we are done. \( \Box \)

6.3.3. Sommerfeld’s uniqueness result.

Proof of Corollary 2.14. Let \( \lambda \in \mathbb{R}, \phi \in f^\beta B^* \) and \( \psi \in f^{-\beta} B \) with \( \beta \in [0, \beta_c) \). First we let \( \phi = \tilde{R}(\lambda + i0) \psi \). Then by Corollaries 2.12 and 2.13 we see that (1) and (2) of the corollary hold. Conversely assuming (1) and (2) of the corollary we let

\[
\phi' = \phi - \tilde{R}(\lambda + i0) \psi.
\]

Then by Corollaries 2.12 and 2.13 it follows that

1. \((H - \lambda) \phi' = 0 \) in the distributional sense,
2. \( \phi' \in f^\beta B^* \) and \((A - a_+ \phi' \in f^{-\beta} B_0^* \).

In addition we have \( \phi' \in B_0^* \). To see this we use functions \( \chi_{\nu} \) as is (2.8), but considering now arbitrary \( \nu \in [0, \infty) \). Noting the identity

\[
2 \text{Im}(\chi_{\nu}(H - \lambda)) = (\text{Re} a_+) \chi_{\nu}' + \text{Re}(\chi_{\nu}(A - a_+)),
\]

cf. (2.11), we have the bound

\[
0 \leq \langle (\text{Re} a_+) \chi_{\nu}' \rangle_{\phi'} = \text{Re}(\chi_{\nu}'(A - a_+)_{\phi'}. \quad (6.22)
\]

Using (2') above we deduce that the right-hand side is bounded as a function of \( \nu \geq 0 \), leading to the conclusion that \( \phi' \in B^* \). Next, taking the limit \( \nu \to \infty \).
in (6.22) using again (2′), we indeed obtain \( \phi' \in B_0^\ast \). Then by (1′) above and Theorem 2.4 it follows that \( \phi' = 0 \), i.e. \( \phi = R(\lambda + i0)\psi \). Hence we are done. \( \square \)

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