Timelike \( B_2 \)-slant helices in Minkowski space \( \mathbb{E}^4_1 \)

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Abstract

We consider a unit speed timelike curve \( \alpha \) in Minkowski 4-space \( \mathbb{E}^4_1 \) and denote the Frenet frame of \( \alpha \) by \( \{T, N, B_1, B_2\} \). We say that \( \alpha \) is a generalized helix if one of the unit vector fields of the Frenet frame has constant scalar product with a fixed direction \( U \) of \( \mathbb{E}^4_1 \). In this work we study those helices where the function \( \langle B_2, U \rangle \) is constant and we give different characterizations of such curves.

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1 Introduction and statement of results

A helix in Euclidean 3-space \( \mathbb{E}^3 \) is a curve where the tangent lines make a constant angle with a fixed direction. A helix curve is characterized by the fact that the ratio \( \tau / \kappa \) is constant along the curve, where \( \tau \) and \( \kappa \) denote the torsion and the curvature, respectively. Helices are well known curves in classical differential geometry of space curves [8] and we refer to the reader for recent works on this type of curves [4, 12]. Recently, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed direction [5]. They characterize a slant helix iff the function

\[
\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)'
\]

is constant. The article [5] motivated generalizations in a twofold sense: first, by considering arbitrary dimension of Euclidean space [7, 10]; second, by considering

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analogous problems in other ambient spaces, for example, in Minkowski space $\mathbb{E}_1^n$ [1, 3, 6, 11, 13].

In this work we consider the generalization of the concept of helix in Minkowski 4-space, when the helix is a timelike curve. We denote by $\mathbb{E}_4^4$ the Minkowski 4-space, that is, $\mathbb{E}_4^4$ is the real vector space $\mathbb{R}^4$ endowed with the standard Lorentzian metric

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system of $\mathbb{R}^4$. An arbitrary vector $v \in \mathbb{E}_4^4$ is said spacelike (resp. timelike, lightlike) if $\langle v, v \rangle > 0$ or $v = 0$ (resp. $\langle v, v \rangle < 0$, $\langle v, v \rangle = 0$ and $v \neq 0$). Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_4^4$ be a (differentiable) curve with $\alpha'(t) \neq 0$, where $\alpha'(t) = d\alpha/dt(t)$. The curve $\alpha$ is said timelike if all its velocity vectors $\alpha'(t)$ are timelike. Then it is possible to re-parametrize $\alpha$ by a new parameter $s$, in such way that $\langle \alpha'(s), \alpha'(s) \rangle = -1$, for any $s \in I$. We say then that $\alpha$ is a unit speed timelike curve.

Consider $\alpha = \alpha(s)$ a unit speed timelike curve in $\mathbb{E}_4^4$. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the moving frame along $\alpha$, where $T, N, B_1$ and $B_2$ denote the tangent, the principal normal, the first binormal and second binormal vector fields, respectively. Here $T(s), N(s), B_1(s)$ and $B_2(s)$ are mutually orthogonal vectors satisfying

$$\langle T, T \rangle = -1, \langle N, N \rangle = \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 1.$$

Then the Frenet equations for $\alpha$ are given by

$$\begin{bmatrix}
T' \\
N' \\
B_1' \\
B_2'
\end{bmatrix} = 
\begin{bmatrix}
0 & \kappa_1 & 0 & 0 \\
\kappa_1 & 0 & \kappa_2 & 0 \\
0 & -\kappa_2 & 0 & \kappa_3 \\
0 & 0 & -\kappa_3 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix},$$

(2)

Recall the functions $\kappa_1(s), \kappa_2(s)$ and $\kappa_3(s)$ are called respectively, the first, the second and the third curvatures of $\alpha$. If $\kappa_3(s) = 0$ for any $s \in I$, then $B_2(s)$ is a constant vector $B$ and the curve $\alpha$ lies in a three-dimensional affine subspace orthogonal to $B$, which is isometric to the Minkowski 3-space $\mathbb{E}_3^3$.

We will assume throughout this work that all the three curvatures satisfy $\kappa_i(s) \neq 0$ for any $s \in I$, $1 \leq i \leq 3$.

**Definition 1.1.** A unit speed timelike curve $\alpha : I \to \mathbb{E}_4^4$ is said to be a generalized (timelike) helix if there exists a constant vector field $U$ different from zero and a vector field $X \in \{T, N, B_1, B_2\}$ such that the function

$$s \mapsto \langle X(s), U \rangle, \quad s \in I$$

is constant.
In this work we are interested by generalized timelike helices in $E_4^1$ where the function $\langle B_2, U \rangle$ is constant. Motivated by the concept of slant helix in $E_3$ [10], we give the following

**Definition 1.2.** A unit speed timelike curve $\alpha$ is called a $B_2$-slant helix if there exists a constant vector field $U$ such that the function $\langle B_2(s), U \rangle$ is constant.

Our main result in this work is the following characterization of $B_2$-slant helices in the spirit of the one given in equation (1) for a slant helix in $E_3$:

A unit speed timelike curve in $E_4^1$ is a $B_2$-slant helix if and only if the function

$$\frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)^2 - \left( \frac{\kappa_3}{\kappa_2} \right)^2$$

is constant.

When $\alpha$ is a lightlike curve, similar computations are been given by Erdogan and Yilmaz in [2].

## 2 Basic equations of timelike helices

Let $\alpha$ be a unit speed timelike curve in $E_4^1$ and let $U$ be a unit constant vector field in $E_4^1$. For each $s \in I$, the vector $U$ is expressed as linear combination of the orthonormal basis $\{T(s), N(s), B_1(s), B_2(s)\}$. Consider the differentiable functions $a_i$, $1 \leq i \leq 4$,

$$U = a_1(s)T(s) + a_2(s)N(s) + a_3(s)B_1(s) + a_4(s)B_2(s), \quad s \in I, \quad (3)$$

that is,

$$a_1 = -\langle T, U \rangle, \quad a_2 = \langle N, U \rangle, \quad a_3 = \langle B_1, U \rangle, \quad a_4 = \langle B_2, U \rangle.$$

Because the vector field $U$ is constant, a differentiation in (3) together (2) gives the following ordinary differential equation system

$$\begin{align*}
    a_1' + \kappa_1 a_2 &= 0 \\
    a_2' + \kappa_1 a_1 - \kappa_2 a_3 &= 0 \\
    a_3' + \kappa_2 a_2 - \kappa_3 a_4 &= 0 \\
    a_4' + \kappa_3 a_3 &= 0
\end{align*} \quad (4)$$
In the case that \( U \) is spacelike (resp. timelike), we will assume that \( \langle U, U \rangle = 1 \) (resp. \(-1\)). This means that the constant \( M \) defined by

\[
M := \langle U, U \rangle = -a_1^2 + a_2^2 + a_3^2 + a_4^2
\]  

is 1, \(-1\) or 0 depending if \( U \) is spacelike, timelike or lightlike, respectively.

We now suppose that \( \alpha \) is a generalized helix. This means that there exists \( i \), \( 1 \leq i \leq 4 \), such that the function \( a_i = a_i(s) \) is constant. Thus in the system (4) we have four differential equations and three derivatives of functions.

The first case that appears is that the function \( a_1 \) is constant, that is, the function \( \langle \mathbf{T}(s), U \rangle \) is constant. If \( U \) is timelike, that is, the tangent lines of \( \alpha \) make a constant (hyperbolic) angle with a fixed timelike direction, the curve \( \alpha \) is called a timelike cylindrical helix [6]. Then it is known that \( \alpha \) is timelike cylindrical helix iff the function

\[
\frac{1}{\kappa_3^2} \left( \frac{\kappa_1}{\kappa_2} \right)^2 + \left( \frac{\kappa_1}{\kappa_2} \right)^2
\]

is constant [6].

However the hypothesis that \( U \) is timelike can be dropped and we can assume that \( U \) has any causal character, as for example, spacelike or lightlike. We explain this situation. In Euclidean space one speaks on the angle that makes a fixed direction with the tangent lines (cylindrical helices) or the normal lines (slant helices). In Minkowski space, one can only speak about the angle between two vectors \( \{u, v\} \) if both are spacelike (Euclidean angle) or both are timelike and are in the same timecone (hyperboilc angle). See [9, page 144]. This is the reason to avoid any reference about 'angles' in Definition 1.1.

Suppose now that the function \( \langle \mathbf{T}(s), U \rangle \) is constant, independent on the causal character of \( U \). From the expression of \( U \) in (3), we know that \( a'_1 = 0 \) and by using (4), we obtain \( a_2 = 0 \) and

\[
a_3 = \frac{\kappa_1}{\kappa_2} a_1, \ a_3' = \kappa_3 a_4, \ a'_4 = \kappa_3 a_3 = 0.
\]

Consider the change of variable \( t(s) = \int_0^s \kappa_3(x)dx \). Then \( \frac{dt}{ds}(s) = \kappa_3(s) \) and the last two above equations write as \( a_3''(t) + a_3(t) = a_4''(t) + a_4(t) = 0 \). Then one obtains that there exist constants \( A \) and \( B \) such that

\[
a_3(s) = A \cos \int_0^s \kappa_3(s)ds + B \sin \int_0^s \kappa_3(s)ds
\]
\[
a_4(s) = -A \sin \int_0^s \kappa_3(s) \, ds + B \cos \int_0^s \kappa_3(s) \, ds.
\]
Since \(a_2^2 + a_3^2 = \langle U, U \rangle + a_1^2\) is constant, and
\[
a_4 = \frac{1}{\kappa_3} \kappa' = \frac{1}{\kappa_3} \left( \frac{\kappa_1}{\kappa_2} \right)' a_1,
\]
it follows that
\[
\frac{1}{\kappa_3^2} \left( \frac{\kappa_1}{\kappa_2} \right)^2 + \left( \frac{\kappa_1}{\kappa_2} \right)^2 = \text{constant}.
\]
Then one can prove the following

**Theorem 2.1.** Let \(\alpha\) be a unit speed timelike curve in \(E^4_1\). Then the function \(\langle T(s), U \rangle\) is constant for a fixed constant vector field \(U\) if and only if the function
\[
\frac{1}{\kappa_3^2} \left( \frac{\kappa_1}{\kappa_2} \right)^2 + \left( \frac{\kappa_1}{\kappa_2} \right)^2
\]
is constant.

When \(U\) is a timelike constant vector field, we re-discover the result given in [6].

3 Timelike \(B_2\)-slant helices

Let \(\alpha\) be a \(B_2\)-slant helix, that is, a unit speed timelike curve in \(E^4_1\) such that the function \(\langle B_2(s), U \rangle\), \(s \in I\), is constant for a fixed constant vector field \(U\). We point out that \(U\) can be of any causal character. In the particular case that \(U\) is spacelike, and since \(B_2\) is too, we can say that a \(B_2\)-slant helix is a timelike curve whose second binormal lines make a constant angle with a fixed (spacelike) direction.

Using the system (3), the fact that \(\alpha\) is a \(B_2\)-slant helix means that the function \(a_4\) is constant. Then (4) gives \(a_3 = 0\) and (3) writes as
\[
U = a_1(s)T(s) + a_2(s)N(s) + a_4B_2(s), \quad a_4 \in \mathbb{R}
\]
where
\[
a_2 = \frac{\kappa_3}{\kappa_2} a_4 = -\frac{1}{\kappa_1} a'_1, \quad a'_2 + \kappa_1 a_1 = 0.
\]
We remark that \(a_4 \neq 0\): on the contrary, and from (4), we conclude \(a_i = 0, 1 \leq i \leq 4\), that is, \(U = 0\): contradiction.
It follows from (7) that the function \( a_1 \) satisfies the following second order differential equation:

\[
\frac{1}{\kappa_1} \frac{d}{ds} \left( \frac{1}{\kappa_1} a_1' \right) - a_1 = 0.
\]

If we change variables in the above equation as \( \frac{1}{\kappa_1} \frac{d}{ds} = \frac{d}{dt} \), that is, \( t = \int_0^s \kappa_1(s) ds \), then we get

\[
\frac{d^2 a_1}{dt^2} - a_1 = 0.
\]

The general solution of this equation is

\[
a_1(s) = A \cosh \int_0^s \kappa_1(s) ds + B \sinh \int_0^s \kappa_1(s) ds,
\]

where \( A \) and \( B \) are arbitrary constants. From (7) and (8) we have

\[
a_2(s) = -A \sinh \int_0^s \kappa_1(s) ds - B \cosh \int_0^s \kappa_1(s) ds.
\]

The above expressions of \( a_1 \) and \( a_2 \) give

\[
A = -\left[ \frac{\kappa_3}{\kappa_2} \sinh \int_0^s \kappa_1(s) ds + \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)' \cosh \int_0^s \kappa_1(s) ds \right] a_4,
\]

\[
B = -\left[ \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)' \sinh \int_0^s \kappa_1(s) ds + \frac{\kappa_3}{\kappa_2} \cosh \int_0^s \kappa_1(s) ds \right] a_4.
\]

From (10),

\[
A^2 - B^2 = \left[ \frac{1}{\kappa_1^2} \left( \frac{\kappa_3}{\kappa_2} \right)' \right] a_4^2.
\]

Therefore

\[
\frac{1}{\kappa_1^2} \left( \frac{\kappa_3}{\kappa_2} \right)' - \frac{\kappa_3^2}{\kappa_2^2} = \text{constant} := m.
\]

Conversely, if the condition (11) is satisfied for a timelike curve, then we can always find a constant vector field \( U \) such that the function \( \langle B_2(s), U \rangle \) is constant: it is sufficient if we define

\[
U = \left[ -\frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)' \mathbf{T} + \frac{\kappa_3}{\kappa_2} \mathbf{N} + \mathbf{B}_2 \right].
\]

By taking account of the differentiation of (11) and the Frenet equations (2), we have that \( \frac{dU}{ds} = 0 \) and this means that \( U \) is a constant vector. On the other hand, \( \langle B_2(s), U \rangle = 1 \). The above computations can be summarized as follows:
Theorem 3.1. Let \( \alpha \) be a unit speed timelike curve in \( E_4^1 \). Then \( \alpha \) is a \( B_2 \)-slant helix if and only if the function

\[
\frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)^2 - \left( \frac{\kappa_3}{\kappa_2} \right)^2
\]

is constant.

From (5), (8) and (9) we get

\[ A^2 - B^2 = a_4^2 - M = a_4^2 m. \]

Thus, the sign of the constant \( m \) agrees with the one \( A^2 - B^2 \). So, if \( U \) is timelike or lightlike, \( m \) is positive. If \( U \) is spacelike, then the sign of \( m \) depends on \( a_4^2 - 1 \). For example, \( m = 0 \) iff \( a_4^2 = 1 \). With similar computations as above, we have

Corollary 3.2. Let \( \alpha \) be a unit speed timelike curve in \( E_4^1 \) and let \( U \) be a unit spacelike constant vector field. Then \( \langle B_2(s), U \rangle^2 = 1 \) for any \( s \in I \) if and only if there exists a constant \( A \) such that

\[
\frac{\kappa_3}{\kappa_2}(s) = A \exp \left( \int_0^s \kappa_1(t)dt \right)
\]

As a consequence of Theorem 3.1, we obtain other characterization of \( B_2 \)-slant helices. The first one is the following

Corollary 3.3. Let \( \alpha \) be a unit speed timelike curve in \( E_4^1 \). Then \( \alpha \) is a \( B_2 \)-slant helix if and only if there exists real numbers \( C \) and \( D \) such that

\[
\frac{\kappa_3}{\kappa_2}(s) = C \sinh \int_0^s \kappa_1(s)ds + D \cosh \int_0^s \kappa_1(s)ds, \tag{12}
\]

Proof. Assume that \( \alpha \) is a \( B_2 \)-slant helix. From (7) and (9), the choice \( C = -A/a_4 \) and \( D = -B/a_4 \) yields (12).

We now suppose that (12) is satisfied. A straightforward computation gives

\[
\frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)^2 - \left( \frac{\kappa_3}{\kappa_2} \right)^2 = C^2 - D^2.
\]

We now use Theorem 3.1. \( \square \)
We end this section with a new characterization for $B_2$-slant helices. Let now assume that $\alpha$ is a $B_2$-slant helix in $\mathbb{E}^4$. By differentiation (11) with respect to $s$ we get
\[
\frac{1}{\kappa_1^2} \left( \frac{\kappa_3}{\kappa_2} \right)' \left[ \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right) ' \right] - \left( \frac{\kappa_3}{\kappa_2} \right)' = 0, \tag{13}
\]
and hence
\[
\frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)' = \frac{\kappa_3^2}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)' \left[ \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right) ' \right] ,
\]
If we define a function $f(s)$ as
\[
f(s) = \frac{\left( \frac{\kappa_3}{\kappa_2} \right)'}{\left[ \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right) ' \right]'},
\]
then
\[
f(s) \kappa_1(s) = \left( \frac{\kappa_3}{\kappa_2} \right)' . \tag{14}
\]
By using (13) and (14), we have
\[
f'(s) = \frac{\kappa_1 \kappa_3}{\kappa_2} .
\]
Conversely, consider the function $f(s) = \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)'$ and assume that $f''(s) = \frac{\kappa_1 \kappa_3}{\kappa_2}$. We compute
\[
\frac{d}{ds} \left[ \frac{1}{\kappa_1^2} \left( \frac{\kappa_3}{\kappa_2} \right)'^2 - \frac{\kappa_3^2}{\kappa_2} \right] = \frac{d}{ds} \left[ f(s)^2 - \frac{f'(s)^2}{\kappa_1^2} \right] := \varphi(s) . \tag{15}
\]
As $f(s)f'(s) = \left( \frac{\kappa_3}{\kappa_2} \right)' \left( \frac{\kappa_3}{\kappa_2} \right)$ and $f''(s) = \kappa_1' \left( \frac{\kappa_3}{\kappa_2} \right) + \kappa_1 \left( \frac{\kappa_3}{\kappa_2} \right)'$ we obtain
\[
f'(s)f''(s) = \kappa_1 \kappa_1' \left( \frac{\kappa_3}{\kappa_2} \right)^2 + \kappa_1^2 \left( \frac{\kappa_3}{\kappa_2} \right) \left( \frac{\kappa_3}{\kappa_2} \right)'.
\]
As consequence of above computations
\[
\varphi(s) = 2 \left( f(s)f'(s) - \frac{f'(s)f''(s)}{\kappa_1^2} + \frac{\kappa_1' f'(s)^2}{\kappa_1^3} \right) = 0,
\]
that is, the function $\frac{1}{\kappa_1^2} \left( \frac{\kappa_3}{\kappa_2} \right)'^2 - \left( \frac{\kappa_3}{\kappa_2} \right)^2$ is constant. Therefore we have proved the following
Theorem 3.4. Let $\alpha$ be a unit speed timelike curve in $E^4_{1}$. Then $\alpha$ is a $B_2$-slant helix if and only if the function $f(s) = \frac{1}{\kappa_1} \left( \frac{\kappa_3}{\kappa_2} \right)'$ satisfies $f'(s) = \frac{\kappa_1 \kappa_3}{\kappa_2}$.

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