Lee-Yang theorems and the complexity of computing averages

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Abstract

We study the complexity of computing average quantities related to spin systems, such as the mean magnetization and susceptibility in the ferromagnetic Ising model, and the average dimer count (or average size of a matching) in the monomer-dimer model. By establishing connections between the complexity of computing these averages and the location of the complex zeros of the partition function, we show that these averages are #P-hard to compute, and hence, under standard assumptions, computationally intractable. In case of the Ising model, our approach requires us to prove an extension of the famous Lee-Yang Theorem from the 1950s.

1 Introduction

1.1 Background

Many natural computational problems in combinatorics, statistics and statistical physics can be cast in the following framework. We are given as input a graph $G = (V, E)$ which implicitly defines a set $\Omega = \Omega(G)$ of combinatorial structures, or configurations (such as matchings in $G$, or $k$-colorings of its vertices). A weight function $w : \Omega \to \mathbb{R}^+$ assigns a positive weight to every element $\sigma \in \Omega$, giving rise to a probability distribution $\pi(\sigma) = \frac{w(\sigma)}{Z}$; here the normalizing factor $Z := \sum_{\sigma \in \Omega} w(\sigma)$ is called the partition function, and is typically not known. (While it is easy to compute the weight $w(\sigma)$ for any given $\sigma$, computing the sum $Z$ is usually hard. Note that the size of $\Omega$ is typically exponential in the size of $G$.)

The final ingredient is a non-negative function, or observable, $f : \Omega \to \mathbb{R}^+ \cup \{0\}$, so that $f(\sigma)$ is also easily computable for each $\sigma \in \Omega$. Our goal is to compute the average of $f$ with respect to $\pi$, i.e.,

$$\langle f \rangle := \sum_{\sigma} \pi(\sigma) f(\sigma) = \frac{\sum_{\sigma} w(\sigma)f(\sigma)}{Z}.$$

As illustration we present two classical examples, both of which we will develop extensively in our later results.

Example 1: The Ferromagnetic Ising Model. Here the configurations are assignments of spin values $\{+, -\}$ to the vertices of $G$, i.e., $\Omega = \{+, -\}^V$. The weight of a configuration $\sigma$ is

$$w_I(\sigma) := \beta^d(\sigma) \lambda^p(\sigma), \quad (1)$$

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where \( d(\sigma) \) is the number of disagreements in \( \sigma \) (i.e., the number of edges \( \{u, v\} \in E \) with \( \sigma(u) \neq \sigma(v) \)), and \( p(\sigma) \) is the number of vertices \( v \in V \) with \( \sigma(v) = + \). The model has two parameters: the edge potential \( \beta \), satisfying \( 0 < \beta \leq 1 \), which governs the strength of the interaction between neighboring spins; and the vertex activity \( \lambda > 0 \), which specifies the tendency for spins to be \( + \). The probability distribution \( \pi(\sigma) = \frac{w_I(\sigma)}{Z_I} \) is the familiar Gibbs distribution, and \( Z_I := Z_I(G, \beta, \lambda) \) is the associated partition function. Note that when \( \beta < 1 \), this distribution favors agreement between neighboring spins. Similarly, the distribution favors \('+\)' (respectively, \('-\)') spins when \( \lambda > 1 \) (respectively, when \( \lambda < 1 \)).

An important observable here is the magnetization \( p(\sigma) \), which is just the number of \(+\)-spins in \( \sigma \). Its average, the mean magnetization, is a fundamental quantity in statistical physics:

\[
\langle p \rangle := \frac{\sum_{\sigma} w_I(\sigma)p(\sigma)}{Z_I}.
\]

Other widely studied averages include the mean energy \( \langle d \rangle \) (the average size of the cut between \(+\)-spins and \(-\)-spins) and the susceptibility \( \chi := \langle p^2 \rangle - \langle p \rangle^2 \) (the variance of the magnetization).

**Example 2: Matchings, or the Monomer-Dimer Model.** The configurations \( \Omega \) are all matchings (independent sets of edges) in \( G \). The weight of a matching \( \sigma \) is

\[
w_M(\sigma) := \lambda^{u(\sigma)} \prod_{e \in \sigma} \gamma_e,
\]

where \( u(\sigma) \) is the number of unmatched vertices (monomers) in \( \sigma \). The parameter \( \lambda > 0 \) is the vertex weight (or monomer activity), while for each edge \( e \in E \), \( \gamma_e \) is an edge weight (or dimer activity). The Gibbs distribution \( \pi(\sigma) = \frac{w_M(\sigma)}{Z_M} \) is a natural weighted distribution on matchings, and the partition function \( Z_M := Z_M(G, \{\gamma_e\}_{e \in E}, \lambda) \) is the weighted matching polynomial of \( G \).

A natural observable here is \( u(\sigma) \), the number of unmatched vertices (or monomers). Note that \( (|V| - \langle u \rangle)/2 \) is just the average size of a (weighted) matching in \( G \) (or equivalently, the average number of dimers).

Observe that the Ising model partition function \( Z_I \) may be written as a polynomial in \( \lambda \) (actually a bivariate polynomial in \( \lambda \) and \( \beta \)):

\[
Z_I = \sum_{k=0}^{\lvert V \rvert} \alpha_k \lambda^k,
\]

where \( \alpha_k = \sum_{\sigma:p(\sigma)=k} \beta^{d(\sigma)} \).

The mean magnetization then becomes

\[
\langle p \rangle = \frac{\sum_k k \alpha_k \lambda^k}{Z_I} = \frac{DZ_I}{Z_I},
\]

where \( D \) denotes the differential operator \( \lambda \frac{\partial}{\partial \lambda} \). Similarly, the mean energy and susceptibility \( \chi \) can be written

\[
\langle d \rangle = \beta \frac{D^2 Z_I}{Z_I}; \quad \chi := \langle p^2 \rangle - \langle p \rangle^2 = \frac{D^2 Z_I}{Z_I} - \left( \frac{DZ_I}{Z_I} \right)^2.
\]

For matchings, the partition function \( Z_M \) is nothing other than the matching polynomial

\[
Z_M = \sum_{k=0}^{\lvert V \rvert} \alpha_k \lambda^k,
\]

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where \( \alpha_k = \sum_{\sigma : u(\sigma) = k} \prod_{e \in \sigma} \gamma_e \) is a weighted sum over matchings with \( k \) unmatched vertices. The average number of monomers is then

\[
\langle u \rangle = \frac{D Z_M}{Z_M},
\]

where \( D \) again denotes the differential operator \( \lambda \frac{\partial}{\partial \lambda} \).

Equations (3)–(5), which express averages as the ratio of some derivative of the partition function to the partition function itself, are in fact no accident; they are a consequence of the fact that the Gibbs distribution takes the form \( w(\sigma) = \exp(-H(\sigma))/Z \), where the Hamiltonian \( H(\sigma) \) is a sum of natural observables.

The subject of this paper is the computational complexity of computing natural averages such as (3)–(5). While the complexity of computing partition functions has been widely studied in the framework of Valiant’s class \( \#P \) of counting problems\(^1\) (see, e.g., [10–13, 19, 22]), we are not aware of any corresponding results for the exact computation of averages. In the approximate setting, by contrast, it is well known that (at least for the wide class of self-reducible problems, which includes all the examples above) approximate computation of the partition function is polynomial time equivalent to sampling (approximately) from the Gibbs distribution \( \pi \) [27]; and sampling from \( \pi \) clearly allows us to approximate averages to any desired accuracy (this is because the observables one is concerned with in these situations admit \textit{a priori} absolute bounds which are polynomial in the input size).

What if we are interested in exact computation? It is tempting to argue that computing an average as in, say, (3) is at least as hard as computing the partition function \( Z_I \), because (3) is a rational function and thus by evaluating it at a small number of points we could recover the numerator and denominator polynomials by rational interpolation. Since the partition function is \( \#P \)-hard\(^2\) in almost all cases of interest (including \( Z_I \) and \( Z_M \) above at all but trivial values of the parameters), we would be done.

The problem with this argument is that, viewed as polynomials in the variable \( \lambda \), \( Z_I \) and its derivative \( D Z_I \) may have common factors (equivalently, viewed as polynomials in the complex variable \( \lambda \), they might have common zeros); and in this case we are clearly not able to recover \( Z_I \) by rational interpolation. Indeed it seems hard \textit{a priori} to rule out the possibility that non-trivial interactions between \( Z_I \) and its derivative could conspire to make the average much easier to compute than \( Z_I \) itself. Thus we are naturally led to the following question:

\textbf{Question: Is it possible for the partition function \( Z \) and its derivative to have common zeros?}\(^3\)

If the answer is no, then we will be able to conclude that computing the average is as hard as computing \( Z \) itself, and thus \( \#P \)-hard in all interesting cases.

The main goal of this paper is to carry through this program using resolutions of the above question in several interesting cases. Before proceeding, we mention a possible alternative approach to deal with the issue of repeated zeros. Since a generic polynomial does not have repeated zeros, one could try to argue that any given graph \( G \) can be perturbed so that its partition function has distinct zeros, and so that the magnetization of the perturbed graph is close to the magnetization of the original graph. One could then perform the interpolation operations with respect to the perturbed graph, and hope that if the perturbations are small enough, then the reduction still goes through. Indeed, this is our intuition for why the magnetization (and other averages) should be hard to compute.

\(^1\)#P is the natural counting analog of the complexity class \( NP \) of decision problems. See Appendix A for details.

\(^2\)#P-hard functions are as hard to compute as any function in \( \#P \), and hence, widely believed to be intractable. See Appendix A for details.

\(^3\)Note that a common zero of \( Z \) and \( D Z \) corresponds to a repeated zero of \( Z \), so this question is equivalent to the question of whether \( Z \) may have repeated zeros.
However, it is not clear how to convert this intuition into a formal proof: in addition to a rather involved error analysis, this would require showing that the partition function of a “perturbed” Ising model behaves like a generic polynomial with respect to the structure of its zeros, which seems no easier than answering the Question above. Our approach sidesteps this issue by tackling the question directly, and in addition establishes a non-trivial property of the zeros of the partition function that may be of independent interest.

1.2 Contributions

The question of common zeros actually turns out to be a deeper issue of wider interest in statistical physics and complex analysis. The study of the zeros of the partition function dates back to the work of Lee and Yang in 1952: the famous Lee-Yang Circle Theorem \cite{30} proves the remarkable fact that the zeros of the ferromagnetic Ising partition function $Z_I$ always lie on the unit circle in the complex plane. This classical theorem, which has since been re-proved many times in different ways \cite{3, 33, 40}, was developed initially as an approach to studying phase transitions, but has since spawned a more global theory connected with the Laguerre-Pólya-Schur theory of linear operators preserving stability of polynomials. (See the Related Work section below.)

Somewhat surprisingly, despite much activity in this area, the question of the location of the zeros of the derivative $DZ_I$ (or equivalently, of repeated zeros of $Z_I$ itself) remains open.

The main technical contribution of this paper is to resolve this question as follows.

**Theorem 1.1.** Let $G = (V, E)$ be a connected graph, and suppose $0 < \beta < 1$. Then the zeros of the polynomial $DZ_I(G, \beta, \lambda)$ (in $\lambda$) satisfy $|\lambda| < 1$.

Since the Lee-Yang Theorem says that all zeros of $Z_I$ satisfy $|\lambda| = 1$, Theorem 1.1 immediately implies that $Z_I$ and $DZ_I$ have no common zeros.

**Remark.** The restriction that $G$ be connected is needed: there exist disconnected graphs for which the conclusion of the theorem does not hold. A simple example is a graph consisting of two isomorphic disconnected subgraphs. For the same reason we require $\beta < 1$. We also note that standard facts from complex analysis (in particular, the Gauss-Lucas theorem) imply that the zeros of $DZ_I$ lie in the convex hull of those of $Z_I$, and hence within the closed unit circle. The content of Theorem 1.1 is that they must lie in the interior of the circle. This refinement is of course crucial for our application.

Before moving on, let us briefly mention our approach to the proof. We actually prove a more general result concerning the zeros of the multivariate partition function $Z_I(G, \beta, \{\lambda_v\}_{v \in V})$; see Theorem 3.2 in Section 3. (The Lee-Yang Theorem itself is also often stated in multivariate form.) Our proof is based on a delightful combinatorial proof of the Lee-Yang Theorem due to Asano \cite{3}, which begins with the empty graph (which trivially satisfies the theorem) and builds the desired graph $G$ by repeatedly adding edges one at a time; by a careful induction one can show that the Lee-Yang property is preserved under each edge addition. Our proof follows a similar induction, but the argument is more delicate because we are working with the more complicated polynomial $DZ_I$ rather than $Z_I$. In particular, in the inductive step we need to evoke a non-trivial correlation inequality due to Newman \cite{33}.

Our first computational complexity result follows as an almost immediate corollary of Theorem 1.1.

**Theorem 1.2.** For any fixed $0 < \beta < 1$ and any fixed $\lambda \neq 1$, the problem of computing the mean magnetization of the Ising model on connected graphs is #P-hard. Moreover, the problem remains #P-hard even when the input is restricted to graphs of maximum degree at most $\Delta$, for any fixed $\Delta \geq 4$. 

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Note that in the case $\lambda = 1$ the mean magnetization is trivially $|V|/2$ by symmetry. Theorem 1.2 confirms that in all non-trivial cases, the problem of computing the fundamental average quantity associated with the Ising model is as hard as it could possibly be. Furthermore, the result also holds for bounded degree graphs, which are relevant in the statistical physics setting. The result can also be extended to arbitrary ferromagnetic two-spin systems and to planar graphs: the details can be found in Appendix D.

We also prove a similar (but slightly weaker) result for the susceptibility of the Ising model.

**Theorem 1.3.** For any fixed $0 < \beta < 1$, the problem of computing the susceptibility of the Ising model on connected graphs, when $\lambda$ is specified in unary, is $\#P$-hard. Moreover, the problem remains $\#P$-hard even when the input is restricted to graphs of maximum degree at most $\Delta$ for any $\Delta \geq 3$.

**Remark.** The requirement that $\lambda$ be part of the input seems to be an artifact of the rational interpolation operations we use in our proof. In particular, our proof of Theorem 1.2 shows hardness for fixed $\lambda$ by “simulating” different values of $\lambda$ by suitably modifying the graph. To adapt this reduction approach to prove hardness for susceptibility (at fixed values of $\lambda$) seems to require the polynomial time computation of magnetization as a subroutine. However, we conjecture that computing the susceptibility should be hard even for fixed values of $\lambda$ (including $\lambda = 1$).

We then proceed beyond the Ising model, and ask about the hardness of computing averages in the monomer-dimer model (i.e., weighted matchings). A classical result of Heilmann and Lieb [26] establishes an analog of the Lee-Yang Theorem for the zeros of the monomer-dimer partition function $Z_M$; however, Heilmann and Lieb also present examples of (connected) graphs $G$ for which $Z_M$ has repeated zeros, so we cannot hope to prove an analog of Theorem 1.1 in this case. On the other hand, Heilmann and Lieb show that if $G$ contains a Hamiltonian path then all the zeros of $Z_M$ are simple. We are able to capitalize on this fact by adapting existing $\#P$-hardness reductions for $Z_M$ in such a way that the instances of $Z_M$ that appear in the reduction always contain a Hamiltonian path. Specifically, we present a reduction from the problem MONOTONE 2-SAT of counting satisfying assignments of a monotone 2-CNF formula to computing $Z_M$ in Hamiltonian graphs $G$. The reduction is an elaboration of Valiant’s original $\#P$-completeness proof for the permanent [41].

This leads to our third computational complexity result.

**Theorem 1.4.** For any fixed $\lambda > 0$, the problem of computing the average number of dimers (equivalently, average size of a matching) in the monomer-dimer model on connected graphs with edge weights in the set $\{1, 2, 3\}$ is $\#P$-hard. Moreover, the problem remains $\#P$-hard even when the input is restricted to graphs of maximum degree at most $\Delta$, for any $\Delta \geq 5$.

**Remark.** Note that our hardness result requires a small finite number (three) of different values for the edge weights. However, this requirement can be removed if $G$ is allowed to have parallel edges; the theorem then holds for any single fixed non-zero edge weight (including the uniform case in which all edge weights are 1).

1.3 Related Work

The study of the location of zeros of the partition function was initiated by Yang and Lee [44] in connection with the analysis of phase transitions. In the follow-up paper [30], they instantiated this approach for the ferromagnetic Ising model by proving the celebrated Lee-Yang theorem on the location of zeros of the partition function and using it to conclude that the ferromagnetic Ising model can have at most one phase transition. The Lee-Yang approach has since become
a cornerstone of the study of phase transitions, and has been used extensively in the statistical physics literature: see, e.g., [3, 26, 33, 40] for specific examples, and Ruelle’s book [37] for background. In a slightly different line of work, Biskup et al [5, 6] studied a novel approach to Lee-Yang theorems for a general class of spin systems on lattice graphs using asymptotic expansions of the partition function. Zeros of partition functions have also been studied in a purely combinatorial setting without reference to the physical interpretation: see, for example, Choe et al [15] for a collection of such results about zeros of a general class of partition functions. Another important example is the work of Chudnovsky and Seymour [16], who show that the zeros of the independence polynomial of claw-free graphs lie on the real line. There have also been attempts to relate the Lee-Yang program to the Riemann hypothesis [34].

Lee-Yang theorems have also been studied in mathematics in connection with the theory of stability preserving operators. The main problem underlying this area is the characterization of linear operators that preserve the class of polynomials, called \( \Omega \)-stable polynomials, whose zeros lie in some fixed closed set \( \Omega \). This research area has its origins in the work of Laguerre [29] and of Pólya and Schur [36], and also has connections to control theory [17] and to electrical circuit theory [9]. It has also seen considerable recent activity, especially through the breakthrough results of Borcea and Brändén, who completely characterize stability preserving operators for multivariate polynomials in various important settings [7, 8]. Although the study of stability preserving operators is closely related to our problem, there is a crucial difference in that we require our linear differential operator to not only preserve the stability of the partition function, but in fact to improve it, by restricting the possible locus of the zeros of the derivative to the open interior of the locus of the zeros of the partition function itself.

In the statistical physics literature, we are aware of only two works which consider the multiplicity of the zeros of the Ising partition function: Heilmann and Lieb [26] and Biskup et al [5, 6]. In [26], a theorem similar to our Theorem 1.1 is proven in the special case when the underlying graph \( G \) has a Hamiltonian path and \( \beta \) is close enough to 1 (depending upon the graph \( G \)). Similarly, in the special case of the Ising model, the results of [5] imply our result but only when \( \beta \) is close to 0, and only in the special case of lattice graphs [4, 6]. Note that neither of these results appears to be sufficient for the purposes of our hardness result.

The classification of counting problems associated with partition functions (via so-called dichotomy theorems) has also recently been a very active area of research. For several interesting general classes of partition functions, these theorems characterize the partition function as being either computable in polynomial time or \#P-hard [10–13, 19, 22]. However, there appear to be no analogous results on the complexity of averages such as the magnetization.

A related area that we do not deal with in this work is the problem of approximate counting. Recent progress in this area has shown that the complexity of approximating the partition function, as well as that of the related problem of approximate sampling, is closely related to the phase transition phenomenon [38,39,43]. However, it is not clear whether hardness results analogous to [38,39] can be proven for the approximate computation of the magnetization.

2 Preliminaries

2.1 The models

Let \( G = (V, E) \) be an undirected graph. The two models we will be concerned with are the ferromagnetic Ising model and the monomer dimer model, both of which have already been defined in Section 1.1.

Ferromagnetic Ising model. Recall that in the ferromagnetic Ising model, a configuration \( \sigma : V \to \{+, -\} \) is an assignment of +/- spins to the vertices of \( G \). The model is characterized by an
edge potential $0 < \beta \leq 1$, and a vertex activity $\lambda > 0$. The weight function $w_I(\sigma)$ defined in (1) induces a probability distribution over configurations with an associated partition function $Z_I(G, \beta, \lambda) := \sum_{\sigma} w_I(\sigma)$. We shall be concerned with the mean magnetization $M(G, \beta, \lambda) := \langle p \rangle$, which is the average number of +--spins in a configuration, and the susceptibility $\chi := \langle p^2 \rangle - \langle p \rangle^2$, which is the variance of the same quantity. As in (3)-(4), these quantities can be written in terms of the derivatives of $Z_I$ with respect to $\lambda$.

For our discussion of the zeros of $Z_I(G, \beta, \lambda)$, we will also need a generalization of the Ising model in which the vertex activities can vary across vertices of $G$. Suppose that the vertex activity at vertex $v$ is $z_v$. The weight of a configuration $\sigma$ is then defined as

$$w_I(\sigma) := \beta^{d(\sigma)} \prod_{v:s(v) = +} z_v,$$

and the partition function is given by $Z_I(G, \beta, (z_v)_{v \in V}) = \sum_{\sigma} w_I(\sigma)$. Consider the linear differential operator $D_G$ defined as follows:

$$D_G := \sum_{v \in V} z_v \frac{\partial}{\partial z_v}.$$

As in (3)–(4), we can then write the magnetization $M_I(G, \beta, (z_v)_{v \in V})$ as

$$M(G, \beta, (z_v)_{v \in V}) = \frac{D_G Z_I(G, \beta, (z_v)_{v \in V})}{Z_I(G, \beta, (z_v)_{v \in V})}.$$

**Monomer-dimer model.** Recall that in the monomer-dimer model, the configurations are matchings of $G$. The model is characterized by edge weights $\gamma_e > 0$ for every edge $e$ in $E$ and a vertex activity $\lambda > 0$. The weight $w_M(\sigma)$ of a matching $\sigma$ is as described in (2), and the associated partition function is defined by $Z_M(G, (\gamma_e)_{e \in E}, \lambda) := \sum_{\sigma} w_M(\sigma)$.

The average number of monomers $U(G, (\gamma_e)_{e \in E}, \lambda) := \langle u \rangle$ can be written (as in (5)) in terms of the derivative of $Z$: in particular $U(G, (\gamma_e)_{e \in E}, \lambda) = \frac{D_M Z_M(G, (\gamma_e)_{e \in E}, \lambda)}{Z_M(G, (\gamma_e)_{e \in E}, \lambda)}$. The average dimer count $D(G, (\gamma_e)_{e \in E}, \lambda)$ (equivalently, the average size of a matching) can be obtained from $U$ by the simple relation

$$D(G, (\gamma_e)_{e \in E}, \lambda) = \frac{n - U(G, (\gamma_e)_{e \in E}, \lambda)}{2},$$

where $n$ is the number of vertices in $G$.

**Remark.** In our definitions above, vertex activities are restricted to be positive real numbers. Although this is the physically (and computationally) relevant setting, in our proofs and in our discussion of Lee-Yang theorems we will need to work with vertex activities that are arbitrary complex numbers. The expressions for the quantities defined above still remain valid.

### 2.2 Zeros of partition functions

We first consider the location of the complex zeros of the partition function of the ferromagnetic Ising model. In a seminal paper Lee and Yang proved the following striking theorem [30].

**Theorem 2.1** ([30]). Let $G$ be any undirected graph and suppose $0 < \beta \leq 1$. Then the complex zeros of $Z_M(G, \beta, z)$, considered as a polynomial in $z$, satisfy $|z| = 1$.

Actually, Lee and Yang proved the following multivariate version of their theorem, the proof of which was later considerably simplified by Asano [3].
Theorem 2.2 ([3, 30]). Let $G = (V, E)$ be a connected undirected graph, and suppose $0 < \beta < 1$. Suppose $(z_v)_{v \in V}$ is a set of complex valued vertex activities such that $|z_v| \geq 1$ for all $v \in V$, and $|z_u| > 1$ for at least one $u \in V$. Then $Z_I(G, \beta, (z_v)_{v \in V}) \neq 0$.

Theorem 2.2 is readily seen to imply Theorem 2.1 by setting $z_v = z$ for all $v \in V$. We now consider the partition function of the monomer-dimer model. In [25], Heilmann and Lieb stated the following result (see [26] for the complete proof).

Theorem 2.3 ([25, 26]). Let $G = (V, E)$ be any graph, and $(\gamma_e)_{e \in E}$ be a collection of positive real edge weights. The complex zeros of $Z_M(G, (\gamma_e)_{e \in E}, z)$, considered as a polynomial in $z$, satisfy $\Re(z) = 0$. Further, if $G$ contains a Hamiltonian path, all the zeros are simple.

In [26], Heilmann and Lieb also gave examples of connected graphs $G$ in which $Z_M$ has repeated zeros. This is in contrast to the Ising model, where, as we prove in Section 3, connectedness is sufficient to ensure that the zeros are simple.

2.3 Rational interpolation

In our hardness reductions, we will need a few well known facts about interpolation of rational functions. While it is clear that it is not in general possible to determine all coefficients of a rational function given its values at any number of points, this can be done if we impose a few simple conditions, as stated in the following theorem.

Theorem 2.4 ([32]). Suppose $R(x) = \frac{p(x)}{q(x)}$ where $\gcd(p(x), q(x)) = 1$ and both $p(x)$ and $q(x)$ are of degree $n$. Suppose $\tilde{p}(x)$ and $\tilde{q}(x)$ are polynomials of degree at most $n$ satisfying

$$\frac{\tilde{p}(x_i)}{\tilde{q}(x_i)} = R(x_i)$$

for $2n + 2$ distinct values $x_1, x_2, \ldots, x_{2n+2}$. Then there is a constant $c$ such that $p(x) = c\tilde{p}(x)$ and $q(x) = c\tilde{q}(x)$.

Notice that given the evaluations at the points $x_i$ one can write down a system of $2n + 2$ homogeneous linear equations for the $2n + 2$ unknown coefficients of $p$ and $q$. The theorem then guarantees that this system has rank exactly $2n + 1$. Thus, since Gaussian elimination can be implemented to run in strongly polynomial time (see, e.g., [21]), a polynomial time algorithm for evaluating $R$ immediately yields a polynomial time algorithm for determining some $\tilde{p}$ and $\tilde{q}$ satisfying the conditions of the above theorem. If we know at least one non-zero coefficient of $p$ or $q$, we can then determine the proportionality constant $c$, and hence $p$ and $q$ also, in time polynomial in $n$.

3 An extended Lee-Yang theorem

In this section we prove Theorem 1.1, our extension of the classical Lee-Yang theorem. Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$, with vertex activity $z_i$ at the $i$th vertex. When clear from the context, we will write $Z(G)$ and $M(G)$ for the partition function $Z_I(G, \beta, (z_v)_{v \in V})$ and the mean magnetization $M(G, \beta, (z_v)_{v \in V})$ of the Ising model on $G$. In terms of the linear operator $D_G$ defined in Section 2.1, we then have $M(G) = D_G Z(G)/Z(G)$.

For convenience, we will use the shorthand $Y' = D_G Y$ (when $G$ is clear from the context) in this section. Notice that this is slightly non-standard, as this shorthand is usually used for the actual derivative. In particular, when all the $z_i$ are equal to $z$, we have $Y' = z \frac{d}{dz}$ with our
notation. Also, observe that the operator $D_G$ obeys the usual product rule: $(Y_1Y_2)' = Y_1'Y_2 + Y_1Y_2'$.

In our proof, we will also need the following generalization of the partition function. We call an assignment of positive integer valued weights $w : V \rightarrow \mathbb{Z}^+$ to the vertices of $G$ legal if $w(v)$ is at least equal to the degree of $v$, for all $v \in V$.

**Definition 3.1.** Let $w$ be a legal collection of weights for $G$. The weighted partition function $Z_w(G)$ is then defined as

$$Z_w(G) := \sum_{\sigma \in \{+,-\}^V} \beta^{d(\sigma)} \prod_{v: \sigma(v) = +} z_v^{w(v)}, \tag{6}$$

where, as before, $d(\sigma)$ is number of disagreeing edges in the configuration $\sigma$.

Notice that the multivariate Lee-Yang theorem (Theorem 2.2) holds also for the weighted partition function, since all the weights are positive integers and we are effectively just changing variables from $z_v$ to $z_v^{w(v)}$.

We will also need the following consequence of a correlation inequality of Newman [33], whose proof can be found in Appendix B.

**Theorem 3.1 ([33, Theorem 3.2]).** Let $G$ be any graph, and let $w$ be a legal collection of weights for $G$. Suppose $0 < \beta < 1$, and $|z_v| \geq 1$ for all $v \in V$ are such that $Z_w(G) \neq 0$. Then

$$\Re(M(G)) = \Re\left(\frac{Z_w'(G)}{Z_w(G)}\right) \geq n/2.$$  

Here, $\Re(z)$ denotes the real part of $z$.

In the special case of real valued activities, the above theorem is equivalent to the well known Griffiths inequality [24], which states the intuitive fact that in a ferromagnetic Ising model where all activities favor the + spin, the magnetization must be at least $n/2$.

For ease of reference in the inductive proof, we give a name to the property we want to establish. Recall that when all the vertex activities are equal to $z$, the classical Gauss-Lucas theorem, together with the Lee-Yang theorem, implies that the zeros of the derivative $D_G(Z(G))$ lie on or inside the unit circle. Our goal is to establish that they actually lie inside the unit circle. Accordingly, we use the following terminology:

**Definition 3.2 (Strict Gauss-Lucas property).** A graph $G = (V,E)$ has the strict Gauss-Lucas property (SGLP) if for every set of activities such that $|z_v| \geq 1$ for all $v \in V$, and every $0 < \beta < 1$, one has $D_G Z(G) \neq 0$. The graph has the weighted strict Gauss-Lucas property (WSGLP) if for all legal weights $w$, $D_G Z_w(G) \neq 0$ necessarily holds under the same conditions.

Note that WSGLP easily implies SGLP: we simply choose $w(v) = \Delta$ for all $v$, where $\Delta$ is the maximum degree of $G$. From WSGLP, we then have that whenever $|z_v| \geq 1$ and $0 < \beta < 1$, $D_G Z_w(G) = \Delta D_G Z(G) = \neq 0$, and hence $D_G Z(G) \neq 0$. Thus Theorem 1.1 is implied by the following more general statement.

**Theorem 3.2.** Every connected graph has the weighted strict Gauss-Lucas property, and hence also the strict Gauss-Lucas property.

We now proceed to prove Theorem 3.2, using induction on the number of edges in the graph $G$. We first consider the base case of a connected graph with a single edge.

---

4Recall that we are using here the slightly non-standard notation $Z_w'(G) = D_G Z_w(G)$, as described at the beginning of this section.
Lemma 3.3 (Base Case). Let \( G \) be the graph consisting of a single edge connecting two vertices. Then \( G \) has the weighted strict Gauss-Lucas property.

Proof. In this case we have \( Z_w(G) = z_1^{w_1} z_2^{w_2} + \beta(z_1^{w_1} + z_2^{w_2}) + 1 \) and therefore \( D_G Z_w(G) = (w_1 + w_2) z_1^{w_1} z_2^{w_2} + \beta(w_1 z_1^{w_1} + w_2 z_2^{w_2}), \) with \( w_1, w_2 \geq 1. \) When \( |z_1|, |z_2| \geq 1, \) the latter vanishes only if \( w_1 + w_2 = \beta \frac{w_2}{z_1^{w_1}} + \frac{w_1}{z_2^{w_2}} \leq \beta \left( \frac{w_2}{|z_1|} + \frac{w_1}{|z_2|^2} \right) \leq \beta (w_1 + w_2), \)

which cannot hold since \( 0 < \beta < 1. \)

For the inductive case, we require two operations: adding a new vertex to the graph, and merging two existing vertices. These operations are formalized in the following lemmas.

Lemma 3.4 (Adding a vertex). Suppose \( G = (V, E) \) is a connected graph satisfying the weighted Gauss-Lucas property. Let \( u \) be a vertex not in \( V. \) Then, the graph \( G_1 \) obtained by attaching the new vertex \( u \) to any vertex (say \( v_1 \)) of \( G \) also has the weighted Gauss-Lucas property.

Lemma 3.5 (Merging vertices). Suppose \( G = (V, E) \) is a connected graph satisfying the weighted strict Gauss-Lucas property. Consider any two vertices, say \( v_1 \) and \( v_2, \) in \( G \) that are not connected by an edge. The graph \( G_1 \) obtained by merging \( v_1 \) and \( v_2 \) into a single vertex \( v \) (while making all the edges incident on \( v_1 \) and \( v_2 \) incident on \( v \)) also has the weighted strict Gauss-Lucas property.

Before proceeding with the proofs of the above lemmas, we show how to use them to prove Theorem 3.2.

Proof of Theorem 3.2. We will prove by induction on \( m \) that any connected graph with at most \( m \) edges satisfies WSGLP. By Lemma 3.3, this statement is true when \( m = 1. \) Now suppose that the statement is true when \( m = k, \) and consider any connected graph \( G \) with \( k + 1 \) edges.

In case \( G \) has a cycle, there exist vertices \( u \) and \( v \) such that the edge \( \{u, v\} \) can be removed from \( G \) to obtain a connected graph \( H. \) Since \( H \) has at most \( k \) edges, \( H \) satisfies WSGLP by the inductive hypothesis. Let \( v_1 \) be a vertex not in \( G. \) By Lemma 3.4, the graph \( H \cup \{\{u, v_1\}\} \) satisfies WSGLP. We can now merge \( v_1 \) and \( v \) to obtain \( G, \) which therefore satisfies WSGLP by Lemma 3.5.

In case \( G \) is a tree, there exists an edge \( \{u, v\} \) such that \( v \) is of degree 1. Again, we obtain a connected graph \( H \) with at most \( k \) edges by removing the edge \( \{u, v\}. \) By the inductive hypothesis, \( H \) satisfies WSGLP, and hence by Lemma 3.4, \( G \) does too. This completes the induction. \( \square \)

Remark. Note that the proof of Theorem 3.2 given above holds also when the graph \( G \) is allowed to have parallel edges and self-loops. This will be useful in our extension to general two-state ferromagnetic spin systems in Appendix D.

We turn now to the proofs of Lemmas 3.4 and 3.5, for which we will need the following additional lemma.

Lemma 3.6. Let \( G \) be a connected graph. Fix any set \( S \) of vertices of \( G, \) and let \( Z_w^+(S) \) denote the partition function restricted to configurations on the subgraph \( G - S, \) with all the vertices in \( S \) fixed to have spin +. Consider any set of vertex activities satisfying \( |z_v| \geq 1 \) for \( v \in G - S. \) Then, for \( 0 < \beta < 1 \) and any set of permissible weights on the vertices of \( G, \) we have \( Z_w^+(S) \neq 0 \) and \( \Re (Z_w^+(S) / Z_w^+(S)) \geq 0. \) In particular, for any positive real \( a, \) we have \( Z_w^+(S)^\prime + a Z_w^+(S) \neq 0. \)
Proof. Observe that $Z_w^+(S)$ is proportional to the product of weighted partition functions on connected components of the graph $G - S$, where the activities on the vertices connected to $S$ in these components (of which there is at least one in each component) have increased in magnitude by a factor of at least $1/\beta > 1$. We can therefore conclude using Theorem 2.2 that $Z_w^+(S) \neq 0$. The second condition $\Re(Z_w^+(S)' / Z_w^+(S)) \geq 0$ then follows from Theorem 3.1 applied to $G - S$. \hfill \Box

We first prove Lemma 3.5, since its proof is somewhat simpler.

Proof of Lemma 3.5. Consider any legal weight assignment on $G_1$. If the weight of $v$ in $G_1$ is $w_v$, we can write $w_v = w_1 + w_2$ such that the weight assignment giving weights $w_1$ and $w_2$ to $v_1$ and $v_2$ respectively is legal for $G$. By partitioning into four cases based on the spins of $v_1$ and $v_2$, we can write the corresponding weighted partition function $Z_w(G)$ and its derivative as

$$Z_w(G) = A z_1^{w_1} z_2^{w_2} + C z_1^{w_1} + D z_2^{w_2} + B;$$
$$Z_w(G)' = (A' + (w_1 + w_2)A) z_1^{w_1} z_2^{w_2} + (C' + w_1C) z_1^{w_1} + (D' + w_2D) z_2^{w_2} + B', \tag{8}$$

for polynomials $A, B, C, D$ in the remaining variables $z_i$. Notice that in the notation of Lemma 3.6, $A = Z_w^+([v_1, v_2])$. Similarly, denoting the activity at the merged vertex by $z$, we have the following expressions for $G_1$:

$$Z_w(G_1) = A z_1^{w_1+w_2} + B; \tag{9}$$
$$Z_w(G_1)' = (A' + (w_1 + w_2)A) z_1^{w_1+w_2} + B'. \tag{10}$$

with $A$ and $B$ as defined above. Now consider any fixing of the activities such that $|z_i| \geq 1$ for $i > 2$. Since $G$ satisfies the weighted strict Gauss-Lucas property, we get by setting $z_1 = z_2$ in eq. (8) that the (univariate) polynomial

$$(A' + (w_1 + w_2)A) z^{w_1+w_2} + (C' + w_1C) z^{w_1} + (D' + w_2D) z^{w_2} + B'$$

in $z$ has no zeros satisfying $|z| \geq 1$. Also, we know from Lemma 3.6 that $A' + (w_1 + w_2)A \neq 0$. Thus, we must have that the product of the zeros, $B'/(A' + (w_1 + w_2)A)$, satisfies

$$\left| \frac{B'}{A' + (w_1 + w_2)A} \right| < 1.$$

However, using eq. (10), this implies that if $|z_i| \geq 1$ for $i > 2$, then $Z_w(G_1)'$ can be zero only if $|z| < 1$, and hence $G_1$ satisfies the weighted strict Gauss-Lucas property. \hfill \Box

Finally, we give the proof of Lemma 3.4.

Proof of Lemma 3.4. Note that any legal set of weights for $G_1$ can be obtained by adding one to the weight $w_1$ of $v_1$ in a legal set of weights $w$ of $G$, and then assigning $u$ an arbitrary weight $w_0 \geq 1$. With a slight abuse of notation, we denote these related weight assignments (one on $G$ and the other on $G_1$) by the same letter $w$. We now partition the terms in $Z_w(G)$ based on the spin of $v_1$ to get

$$Z_w(G) = A z_1^{w_1} + B;$$
$$Z_w(G)' = (A' + w_1A) z_1^{w_1} + B',$$
where \( w_1 \) is the weight of \( v_1 \) in \( G \). Here, \( A, B \) are polynomials in the remaining variables \( z_i \), and \( A \) is of the form \( Z_w(\{v_1\}) \) in the notation of Lemma 3.6. We again assume \( 0 < \beta < 1 \) and \( |z_i| \geq 1 \) for \( i > 1 \). We now consider \( G_1 \). Denoting the activity at \( u \) by \( z \), we can write

\[
Z_w(G_1) = A(\beta + z^{w_0}) z_1^{w_1+1} + B(1 + \beta z^{w_0});
\]

\[
Z_w(G_1)' = (A' + w_1 A)(\beta + z^{w_0}) z_1^{w_1+1} + A(\beta + (w_0 + 1) z^{w_0}) z_1^{w_1+1} + B'(1 + \beta z^{w_0}) + w_0 \beta B z^{w_0}.
\]

Now suppose that \( G_1 \) does not satisfy the weighted strict Gauss-Lucas property, and hence \( |z| \) and \( |z_1| \) are both also at least 1, but \( Z_w(G_1)' = 0 \). It follows from Theorem 3.1 that we then also have \( Z_w(G_1) = 0 \). We now proceed to derive a contradiction to the above observations. For convenience, we denote \( z_1^{w_1+1} \) by \( y \) in what follows.

Using Lemma 3.6, we know that \( A \neq 0 \) and that \( A' + w_1 A \neq 0 \) for our setting of activities. By Theorem 2.2 applied to \( Z_w(G) \) and the weighted strict Gauss-Lucas property applied to \( Z_w(G)' \), we get

\[
\frac{|B|}{A} \leq 1, \quad \text{and} \quad \frac{B'}{A' + w_1 A} < 1. \tag{11}
\]

Also, since \( Z_w(G_1) = 0 \), we must have

\[
y = -\frac{B(1 + \beta z^{w_0})}{A'} \frac{\beta + z^{w_0}}{\beta + z^{w_0}}. \tag{12}
\]

Notice that \( y \) is well defined since \( A \neq 0 \), \( |z| \geq 1 \) and \( \beta < 1 \). Further, since \( \beta < 1 \), either one of \( |z| > 1 \), or \( |B| < |A| \) would imply that \( |y| < 1 \), which is a contradiction to our assumption that \( |z_1| \geq 1 \) (since \( y = z_1^{w_1+1} \)). Thus, we must have

\[
|z| = 1, \quad \text{and} \quad \left| \frac{B}{A} \right| = 1. \tag{13}
\]

Now, substituting the value of \( y \) from eq. (12) into \( Z_w(G_1)' = 0 \), we get

\[
B'(1 + \beta z^{w_0}) + \beta w_0 B z^{w_0} = \left( (A' + w_1 A)(\beta + z^{w_0}) + A(\beta + (w_0 + 1) z^{w_0}) \right) \frac{B(1 + \beta z^{w_0})}{A'} \frac{\beta + z^{w_0}}{\beta + z^{w_0}}.
\]

Dividing through by \((A' + w_1 A)(1 + \beta z^{w_0})\), setting \( c = A/(A' + w_1 A) \) and rearranging terms, we get

\[
\frac{B'}{A' + w_1 A} = \frac{B}{A} \left( 1 + c + w_0 c \left( \frac{z^{w_0}}{\beta + z^{w_0}} + \frac{1}{1 + \beta z^{w_0}} - 1 \right) \right)
\]

\[
= \frac{B}{A} \left( 1 + c + w_0 c \left( 2 \Re \left( \frac{z^{w_0}}{\beta + z^{w_0}} \right) - 1 \right) \right), \quad \text{since} \quad |z| = 1. \tag{14}
\]

Notice that these divisions are well defined since \( A' + w_1 A \neq 0 \), and \( \beta < 1 \) and \( |z| = 1 \) implies that \( (1 + \beta z^{w_0}) \neq 0 \) as well. Note also that \( c \) is of the form \( 1/(w_1 + c') \) where \( \Re(c') = \Re(A'/A) \geq 0 \) by Lemma 3.6 and our earlier observations about \( A \): it therefore follows that \( \Re(c) \geq 0 \). However, we then calculate that for \( |z| = 1 \), the factor inside the braces in (14) has real part (and hence absolute value) at least 1. Using \( |B|/|A| = 1 \) from (13), we then see that the right hand side of (14) always has absolute value at least 1, which gives us the required contradiction to (11). This shows that \( G_1 \) satisfies the weighted strict Gauss-Lucas property. \( \Box \)
4 Hardness of computing the mean magnetization

In this section, we use our extended Lee-Yang theorem (Theorem 1.1) to prove Theorems 1.2 and 1.3 via reductions from the problem of computing the partition function of the Ising model, which is known to be #P-hard even for bounded degree graphs [11, 19]. More specifically, we will use the following #P-hardness result.

**Theorem 4.1 ([11, Theorem 1],[19, Theorem 5.1]).** Fix $\beta$ satisfying $0 < \beta < 1$. The problem of computing the partition function $Z_I(G, \beta, 1)$ of the Ising model on connected graphs of fixed maximum degree $\Delta \geq 3$ is #P-hard.

For simplicity, we prove here a version of Theorem 1.2 without the bounded degree constraint. The extension to bounded degree graphs requires some more work and is proved in Appendix C.

**Proof of Theorem 1.2.** We assume $\lambda > 1$, since the case $\lambda < 1$ is symmetrical. For given $0 < \beta < 1$, suppose that we have an algorithm $\mathcal{A}$ which, given a connected graph $G$, outputs the mean magnetization $M(G, \beta, \lambda)$ in polynomial time. Let $G$ be a graph of $n$ vertices. Notice that as a rational function in $z$, $M(G, \beta, z)$ is a ratio of the two polynomials, $DZ_I(G, \beta, z)$ and $Z_I(G, \beta, z)$, which are both of degree $n$. Further, since $G$ is connected, these polynomials are co-prime by Theorem 1.1. Thus, if we could efficiently evaluate $M(G, \beta, z)$ at $2n + 2$ distinct points $z$ using algorithm $\mathcal{A}$, we could uniquely determine the coefficients of $Z_I(G, \beta, z)$ by Theorem 2.4 (since we know that the constant term in $Z_I(G, \beta, z)$ is 1). We could then determine $Z_I(G, \beta, 1)$ in polynomial time. Theorem 4.1 would then imply that computing the mean magnetization for the given values of the parameters $\beta$ and $\lambda$ is #P-hard.

In order to evaluate $M(G, \beta, z)$ at $2n + 2$ distinct values, we consider the graph $G(k)$ obtained by attaching $k$ new neighbors to each vertex of $v$. We then have

$$Z_I(G(k), \beta, \lambda) = (1 + \beta \lambda)^n z^n I(G, \beta, \lambda_k),$$

$$M(G(k), \beta, \lambda) = \frac{k n \beta \lambda}{1 + \beta \lambda} + \left[ 1 + \frac{k \lambda (1 - \beta^2\right]}{(1 + \beta \lambda)(\beta + \lambda)} \right] M(G, \beta, \lambda_k),$$

where $\lambda_k = \lambda \left( \frac{\beta + \lambda}{1 + \beta \lambda} \right)^k$. Notice that when $\beta < 1$, all the $\lambda_k$ are distinct, and further, $M(G, \beta, \lambda_k)$ can be easily determined given $M(G(k), \beta, \lambda)$. Therefore, we can evaluate $M(G(k), \beta, \lambda)$ for $0 \leq k \leq 2n + 1$ using the algorithm $\mathcal{A}$, and then using eqs. (15) and (16) we can determine $M(G, \beta, \lambda_k)$ in polynomial time. Since these evaluations are at distinct points, the reduction is complete.

**Proof of Theorem 1.3.** For a given $\beta$ as specified in the theorem, suppose that there is a polynomial time algorithm $\mathcal{A}$ which, given a graph $G$ of maximum degree $\Delta$, and a value of $\lambda$ in unary, outputs the susceptibility $\chi(G, \beta, \lambda)$. Notice that as a rational function in $z$, $\chi(G, \beta, z)$ is a ratio of the two polynomials $Z_I(G, \beta, z) \cdot D^2 Z_I(G, \beta, z) - (D Z_I(G, \beta, \lambda))^2$ and $Z_I(G, \beta, z)^2$, which are both of degree $2n$. Further, since $G$ is connected, these polynomials are co-prime by Theorem 1.1. To see this, notice that any common complex zero of these two polynomials must be a common zero of $Z_I(G, \beta, \lambda)$ and $D Z_I(G, \beta, \lambda)$, which is prohibited by Theorem 1.1.

To complete the reduction, we notice that we can choose $4n + 2$ distinct values of $\lambda$ in the interval $(0, 1)$ all of which have a unary representation length of at most $5n$. Thus, using $\mathcal{A}$, we can efficiently evaluate $\chi(G, \beta, z)$ at $4n + 2$ distinct values of $z$. By Theorem 2.4 we can then use these evaluations to uniquely determine the coefficients of $Z_I(G, \beta, z)^2$ (since we already know that the constant coefficient is 1), and hence, $Z_I(G, \beta, 1)$, in polynomial time. Because of Theorem 4.1, this implies that the problem of computing the susceptibility at the given value of $\beta$ is #P-hard. 


5 Hardness of computing the average dimer count

In this section we prove Theorem 1.4 by reducing the \#P-hard problem #MONOTONE-2SAT to the problem of computing the average dimer count. The reduction is similar in structure to Valiant’s original proof for the \#P-hardness of the problem of counting perfect matchings. However, since we will need to do rational interpolation, we need the zeros of the partition function to be simple, so by Theorem 2.3 we will need to ensure that the graph appearing as the output of the reduction always has a Hamiltonian path. The formal properties satisfied by our reduction are stated in the following theorem.

**Theorem 5.1.** There exists a polynomial time algorithm \( A \) which, when given as input a Monotone 2-SAT formula \( \phi \), outputs a weighted graph \( G \) with the following properties:

1. The weights in \( G \) are drawn from the set \( \{1, 2, 3\} \).
2. Suppose \( \phi \) has \( \nu \) variables and \( \mu \) clauses. Then, given the total weight \( W \) of perfect matchings in \( G \), the number of satisfying assignments of \( \phi \) can be determined in polynomial time from \( W, \mu, \) and \( \nu \).
3. \( G \) contains a Hamiltonian path.

We observe here that Valiant’s reduction from \#3-SAT [41] can be easily modified so that it satisfies properties 1 and 2. However, it is property 3 that is crucial for our purposes, since it allows the use of Theorem 2.3. We first show how Theorem 5.1 can be used to immediately prove a slightly weaker version of Theorem 1.4, which shows hardness only on general graphs. The proof showing hardness for bounded degree graphs can be found in Appendix C.

**Proof of Theorem 1.4.** Fix any \( \lambda > 0 \), and suppose that there exists a polynomial time algorithm \( B \) which, given a connected graph \( H \), with edge weights in the set \( \{1, 2, 3\} \) outputs \( D(H, (\gamma_e)_{e \in E}, \lambda) \). In the following, we suppress the dependence on edge weights \((\gamma_e)_{e \in E}\) for clarity of notation. Given a Monotone 2-SAT formula \( \phi \), we can then produce the graph \( G = A(\phi) \) in polynomial time. Let \( n \) be the number of vertices in \( G \). Since \( G \) contains a Hamiltonian path, Theorem 2.3 implies that \( Z_M(G, z) \) and \( DZ_M(G, z) \) have no common zeros. Thus, being able to use algorithm \( B \) to evaluate \( D(G, z) \) (and hence \( U(G, z) \)) at \( 2n + 2 \) different values of \( z \) would allow us to uniquely determine the coefficients of \( Z_M(G, z) \) in polynomial time by rational interpolation (Theorem 2.4), since we already know that the coefficient of \( z^n \) is 1. This would allow us to obtain \( W \) (which is the constant term in \( Z_M(G, z) \)), and hence, by property 2, also the number of satisfying assignments of \( \phi \), in polynomial time. This would show that the problem of computing \( D(G, \lambda) \) is \#P-hard (since \#MONOTONE-2SAT is \#P-hard [42]).

However, \( B \) only allows us to evaluate \( U(G, z) \) at \( z = \lambda \). In order to “simulate” other values of \( \lambda \), we consider the graph \( G(k) \) obtained by attaching \( k \) new vertices to each vertex of \( G \) with unit weight edges. We then have

\[
Z_M(G(k), \lambda) = \lambda^{nk}Z_M(G, \lambda_k); \tag{17}
\]
\[
U(G(k), \lambda) = nk + \frac{\lambda^2 - k}{\lambda^2 + k}U(G, \lambda_k), \tag{18}
\]

where \( \lambda_k = \lambda + k/\lambda \). Thus, by choosing \( 2n + 2 \) different values of \( k \), none of which is equal to \( \lambda^2 \), we can determine \( U(G, z) \) at \( 2n + 2 \) different values of \( z \) by running \( B \) on \( G(k) \) and using eq. (18). This completes the proof. \( \square \)

In the rest of this section, we proceed to prove Theorem 5.1 in a sequence of steps. For simplicity, we will describe our reduction in terms of cycle covers in a directed graph rather than perfect matchings in an undirected graph (this also allows us to directly compare our
gadget construction with that of Valiant [41] at various steps). Given a weighted directed graph
\( G = (V, E) \), we define the undirected bipartite graph \( \text{Bip}(G) = (V \times \{0, 1\}, E') \) where the edge
\( \{(x, 0), (y, 1)\} \) is in \( E' \) with weight \( \gamma_e \) if and only if \( (x, y) \) is an edge in \( E \) with the same weight.
Note that a subset \( S \subseteq E \) forms a cycle cover of weight \( w \) in \( G \) if and only if the corresponding subset of edges \( S' = \{(x, 0), (y, 1)\} \mid (x, y) \in S \) forms a perfect matching of weight \( w \) in \( \text{Bip}(G) \). In particular, the total weight of all perfect matchings in \( \text{Bip}(G) \) is the same as the total weight of all cycle covers of \( G \).

Later, while arguing about the existence of Hamiltonian paths in graphs of the form \( \text{Bip}(G) \), we will find it convenient to use the following short-hand notation for simple paths in the graph \( \text{Bip}(G) \) in terms of the edges and vertices of \( G \). Consider any simple path \((x_1, 1), (x_2, 0), (x_3, 1), (x_4, 0), \ldots, (x_l, 1)\) where we have assumed for simplicity that \( l \) is odd. The edges corresponding to this path in \( G \) are \( x_1 \leftarrow x_2, x_2 \rightarrow x_3, x_3 \leftarrow x_4, \ldots, x_{l-1} \rightarrow x_l \). Notice that alternate edges are traversed in reverse in this representation. The path can therefore be represented as \( x_1 \leftarrow x_2 \rightarrow x_3 \leftarrow x_4 \rightarrow \ldots \rightarrow x_l \). Similarly for a path starting on the other side, say \((x_1, 0), (x_2, 1), (x_3, 0), (x_4, 1)\), we have the representation \( x_1 \rightarrow x_2 \leftarrow x_3 \rightarrow x_4 \). Notice that a path \( p_2 \) starting at a vertex \( v \) in this notation can be appended to a path \( p_1 \) ending at \( v \) if and only if the arrows at \( v \) in \( p_1 \) and \( p_2 \) respectively are in opposite directions. We will refer to this notation as the alternating path representation. Further, given an alternating path representation of a path, we will refer to edges going right (such as \( x_1 \rightarrow x_2 \) in the last example) as forward edges, and edges going left (such as \( x_2 \leftarrow x_3 \) in the above example) as backward edges.

5.1 Overview of the reduction

We now look at the basic structure of our reduction, which is an elaboration of Valiant’s reduction [41] as modified by Papadimitriou [35] and presented in [2]. Recall that given a monotone 2-SAT formula \( \phi \), the reduction needs to produce in polynomial time a directed graph \( G \) such that the number of satisfying assignments of \( \phi \) can be easily determined from the total weight of cycle covers of \( G \), and such that \( \text{Bip}(G) \) has a Hamiltonian path. Our first step is to introduce a shared variable in all the clauses of \( \phi \): this shared variable will be useful later in showing the existence of a Hamiltonian path through the gadget.

**Observation 5.2.** Let \( \phi = \bigwedge_{i=1}^{\mu} c_i \) be a monotone 2-SAT formula with \( \mu \) clauses, \( \nu \) variables, and \( s \) satisfying assignments. Let \( \tau \) be a variable not appearing in \( \phi \) and consider the 3-SAT formula
\[
\phi' = \bigwedge_{i=1}^{\mu} (\tau \lor c_i).
\]

The number of satisfying assignments of \( \phi' \) is \( s' := 2^\nu + s \).

Notice that each clause in \( \phi' \) has exactly three variables, and that the number of satisfying assignments of \( \phi \) can be easily determined given the number of satisfying assignments of \( \phi' \).

We start the construction of \( G \) by creating a separate variable gadget (see Figure 1) for each of the variables \( \tau, x_1, x_2, \ldots, x_\nu \) occurring in \( \phi' \). This gadget has an external dotted edge for each appearance of the variable in the formula, and is designed so that any cycle cover must either use all the dotted edges in a particular gadget, or none of them.

As done in Valiant’s reduction, we then introduce a clause gadget (see Figure 2) for each clause in \( \phi' \). Each clause gadget has one external dotted edge for each literal in the clause, and is designed so that no cycle cover can include all the dotted edges; and so that for any other subset of the dotted edges, there is exactly one cycle cover including all the edges in the subset and no others. For each clause gadget, we label each of the three dotted edges in the gadget
with one of the three literals appearing in the clause. However, in this step, we ensure that in each gadget the $b \rightarrow c$ dotted edge is the one labeled with the literal $\tau$, since this is needed to show that the final construction has a Hamiltonian path. We now “pair” each dotted edge appearing in a clause gadget with a dotted edge corresponding to the same literal in a variable gadget, so that each dotted edge appears in exactly one pair.

We first consider cycle covers which obey the constraint that they must choose exactly one edge from each such pair. We claim that the number of cycle covers satisfying this “pairing” constraint equals the number of satisfying assignments of $\phi'$. To see this, we associate a truth assignment with every cycle cover by setting the variable $v$ to true if the cycle cover uses all the dotted edges in the variable gadget for $v$, and to false if it uses none of the dotted edges. Notice that because of the pairing constraint, a cycle cover is uniquely determined by specifying its assignment. Further, given the above properties of the clause gadget, exactly those cycle covers are permitted whose associated assignments are satisfying assignments of $\phi'$.

We now enforce the “pairing” constraint referred to above using a gadget similar to Valiant’s XOR-gadget. The XOR-gadget has two ports (labeled $a$ and $d$), each of which admits one incoming and one outgoing edge (see Figure 3). To ensure the “pairing” constraint for a pair of dotted edges $e_1 \rightarrow f_2$ and $e_2 \rightarrow f_2$, we replace them by the incoming-outgoing pair of a single XOR-gadget (see Figure 3b). The gadget has the property that after the replacement, the weight of every cycle cover which would have included exactly one of the two dotted edges $e_1 \rightarrow f_1$ and $e_2 \rightarrow f_2$ in the original graph gets multiplied by a factor of 2 (for each replacement made), while the weight of any cycle covers not satisfying the pairing constraint becomes 0 (see Appendix E for a proof). The total weight of all cycle covers in the final graph so obtained is therefore $2^{l} s'$, where $s'$ is the number of satisfying assignments and $l$ is the total number of literals in $\phi'$ (since one XOR-gadget is needed to replace the pair of dotted edges for each literal). Further, replacing a pair of edges by a XOR gadget does not change the in-degree or out-degree of any vertex already present.

Note that the XOR-gadget has edges of weight $-1$, which are not permitted in the monomer-dimer model. This can be remedied by replacing the $-1$ weight edges by a large chain of edges...
(of length, say, $m^2$ where $m$ is the number of edges in the original graph) of weight 2, with individual vertices in the chain having self loops (of weight 1). The total weight of cycle covers in the new graph modulo $2^{m^2} + 1$ then gives the total weight of cycle covers in the original graph.

This last step of replacing the $-1$ edge by a long chain presents a challenge since we will need to include all the vertices in the chain in our Hamiltonian path (equivalently, all $-1$ weight edges must appear in the Hamiltonian path). For this reason, we cannot use Valiant’s XOR-gadget directly. Our XOR-gadget, on the other hand, is such that the $-1$ weight edges can always be included in our Hamiltonian path. However, we have to be careful in the orientation of the XOR-gadgets in order to be able to construct a Hamiltonian path later: when replacing a pair of dotted edges one of which belongs to $\tau$’s variable gadget, we orient the XOR-gadget so that the incoming edge at vertex $a$ in the XOR-gadget comes from the variable gadget. At all other pairs, we orient the XOR-gadgets so that the incoming edge at the vertex $a$ comes from a clause gadget.

![Diagram](image)

Figure 3: Replacing dotted edges $e_1 \rightarrow f_1$ and $e_2 \rightarrow f_2$

5.2 Analyzing the reduction

We now proceed to analyze the output of the reduction to complete the proof of Theorem 5.1. The use of XOR-gadgets to enforce the “pairing” constraint as described above introduces a factor of 2 for each literal appearing in the clause, and therefore the total weight of cycle covers after this step is $2^{3\mu s'} \leq 2^{6\mu}$. To get rid of the $-1$ weight edges in the XOR-gadgets, we replace each such edge by a chain of $\kappa = 6\mu - 1$ vertices with self-loops of weight 1 and connecting edges of weight 2. We call this final graph $G$. Since the initial total weight of cycle covers was at most $2^{6\mu}$, the weight of cycle covers in $G$ (and hence the total-weight of all perfect matchings in $\text{Bip}(G)$), modulo $2^{\kappa+1} + 1$, is exactly $2^{3\mu s'}$. Since all steps in the construction of $\text{Bip}(G)$ starting from $\phi$ can be done in time polynomial in the representation size of $\phi$, this proves parts 1 and 2 of Theorem 5.1.

We now proceed to prove part 3, that is, that $\text{Bip}(G)$ has a Hamiltonian path. We will use the alternating path notation described above in order to keep our discussion in terms of the vertices and edges of $G$, and we will call this representation of a Hamiltonian path in $\text{Bip}(G)$ an alternating Hamiltonian path. In an alternating Hamiltonian path, each vertex of $G$ is visited exactly twice, and the length of the alternating path between the two visits is odd. This is equivalent to saying that all vertices must appear exactly twice in an alternating Hamiltonian path, with all vertices except the first vertex in the path having one incoming forward edge, and
one incoming backward edge.

Our gadgets so far are designed to have alternating Hamiltonian paths which can be pieced together to form an alternating Hamiltonian path for $G$, and hence, we only need to list these paths and show how to stitch them together. We begin with alternating Hamiltonian paths in the clause gadget.

**Observation 5.3.** The clause gadget in Figure 2 has the alternating Hamiltonian path

$$c \leftarrow 0 \rightarrow a \leftarrow c \rightarrow b \leftarrow a \rightarrow 0 \leftarrow b$$

which uses all the dotted edges except the $b \rightarrow c$ dotted edge.

Recall that in the construction of the reduction, we ensured that the new variable $\tau$ was associated with the $b \rightarrow c$ dotted edge in each clause gadget. This will be used to connect the above alternating Hamiltonian path in different clause gadgets via connections to the variable gadget for $\tau$ at the $b \rightarrow c$ edge. Also, in the final construction, the dotted edges in the alternating Hamiltonian path will be replaced by detours into the associated XOR-gadget.

We now consider the XOR-gadget in Figure 3a. It turns out that in some cases, we will need to traverse the XOR-gadget partially, so that a path enters at $a$ via a backward edge, uses the $a \rightarrow d$ edge, and then leaves via a backward edge at $d$. In order to cover the rest of the vertices, we will then need to construct an alternating path that enters at $a$ and leaves at $d$ via forward edges, and covers all the vertices except $a$ and $d$ twice. Another complication with the XOR-gadget is the presence of two $-1$ weight edges which need to be replaced with chains of vertices with self loops. However, this will not be a problem if we can ensure that both of the alternating paths described above use both the $-1$ weight edges, since an edge in an alternating path can always be replaced by a chain of vertices with self-loops. We now show that, as we claimed above, our modified XOR-gadget satisfies all of these conditions.

**Observation 5.4.** The XOR-gadget in Figure 3a has the alternating Hamiltonian path

$$a \leftarrow b \rightarrow b \leftarrow a \rightarrow d \leftarrow c \rightarrow c \leftarrow d.$$  

The gadget also has the following alternating path which enters at $a$ and leaves from $d$ using forward edges, but which does not otherwise visit these vertices:

$$a \leftarrow b \rightarrow b \leftarrow c \rightarrow c \leftarrow d.$$  

Moreover, both these paths use the $-1$ weight edges $b \rightarrow a$ and $c \rightarrow c$.

**Remark.** Since the XOR-gadget is connected to variable gadgets (except those for variable $\tau$) in $G$ via an outgoing edge at $a$ and an incoming edge at $d$, it will be possible to replace the $a \rightarrow d$ edge in the alternating Hamiltonian path above by a detour into the connected variable gadget when constructing an alternating Hamiltonian path in $G$. Similarly, it will be possible to use the $a \rightarrow d$ edge as a replacement for the dotted edge in the variable gadget that was replaced by the XOR-gadget, at the cost of visiting the vertices $a$ and $d$ once. The role of the second alternating path is to visit the remaining vertices in a XOR-gadget which has already been partially traversed in this way.

**Remark.** As pointed out above, it does not seem possible to include the two $-1$ weight edges in both the above alternating paths in Valiant’s original construction. This necessitated the construction of our new XOR-gadget in which the $-1$ weight edges are part of both the paths. The edges in our construction are the same as those in Valiant’s construction, but the weights have been chosen differently.
We now consider the variable gadget shown in Figure 1. We first work as if the dotted edges are present. In this case, for any vertex \( v \) in the gadget except the leftmost vertex, we can construct an alternating Hamiltonian path which covers all the vertices in the gadget, uses all the dotted edges except the one between \( v \) and its predecessor and can be appended to an alternating path that enters via a forward edge at \( v \) and leaves via a forward edge at \( v \)’s predecessor. When the dotted edges are replaced by a XOR-gadget, this alternating path can still be traversed as described in the remarks following Observation 5.4, by instead following a forward edge to the \( d \) vertex of the XOR-gadget, following the \( a \rightarrow d \) edge in reverse, and then entering the variable gadget at the successor of \( v \) via the outgoing edge at the \( a \) vertex of the XOR-gadget. It is for this reason that we enforced above the condition that when a XOR-gadget is connected to a variable gadget for a variable other that \( \tau \), it is oriented so that the incoming external edge at its \( d \) vertex comes from the variable gadget.

We now start constructing the alternating Hamiltonian path in \( G \) starting at the left-most vertex in the variable gadget for \( \tau \). If all the dotted edges were present, this gadget is just a chain of vertices, and hence there is an alternating Hamiltonian path that covers all its vertices. However, each dotted edge has been replaced by an outgoing edge to the \( a \) vertex and an incoming edge from the \( d \) vertex of a XOR-gadget. Thus, instead of following the dotted edges, our alternating path will take a detour into the corresponding XOR-gadget, and after traversing several other vertices, return via its \( d \) vertex to visit the other vertices in the variable gadget for \( \tau \). Thus, we need to show that these detours into the XOR-gadgets can be used to make the alternating path go through all the other vertices in \( G \) twice while respecting the required parity constraints.

We consider one such detour. We suppose that the XOR-gadget in question connects to a clause gadget \( C \) for the clause \( \tau \lor v_1 \lor v_2 \). While following the alternating path for the XOR-gadget, we bypass the \( a \rightarrow d \) edge of the XOR-gadget and instead take a detour into the \( c \) vertex of \( C \). We then start following the alternating path in Observation 5.3. If the dotted edges were present, we would be able to complete an alternating path covering all vertices in \( C \) and then return via a forward edge from the \( b \) vertex of \( C \) into the \( d \) vertex of the XOR-gadget. We could then complete the alternating Hamiltonian path in the XOR-gadget, and return via a forward edge into the variable gadget for \( \tau \). However, since the dotted edges have been replaced by XOR-gadgets, we would need to take detours into the XOR-gadgets replacing them. Suppose we are trying to replace the dotted edge \( c \rightarrow a \), corresponding to the literal \( v_1 \). At this point there can be two cases:

### Case 1.
Suppose that the vertices of the variable gadget for \( v_1 \) have still not been covered by our growing alternating Hamiltonian path. Consider the XOR-gadget \( X \) replacing the \( c \rightarrow a \) dotted edge of \( C \). We consider the alternating Hamiltonian path in Observation 5.4 starting at the \( a \) vertex of \( X \). We follow this path until we need to use the \( a \rightarrow d \) edge. At this point, we take a detour into the variable gadget for \( v_1 \) via a forward edge at vertex \( a \) of \( X \). The vertex \( u \) we connect to in the variable gadget cannot be a leftmost vertex, since its predecessor \( u' \) is connected to vertex \( d \) of \( X \) via a \( u' \rightarrow d \) edge. As discussed above, we will therefore get an alternating Hamiltonian path for the vertex gadget which will leave the gadget through the \( u' \rightarrow d \) edge (though this will end up using the \( a \rightarrow d \) edges in all other XOR-gadgets corresponding to occurrences of \( v_1 \)). We can then complete the alternating Hamiltonian path for \( X \), and this gives us an alternating path starting with a backward edge at vertex \( a \) of \( X \), ending with a backward edge at vertex \( d \) of \( X \), and covering all vertices in \( X \) and the variable gadget of \( v_1 \), while also using the \( a \rightarrow d \) edge in XOR-gadgets corresponding to all other occurrences of \( v_1 \). We then use the forward edge from vertex \( c \) of \( C \) to vertex \( a \) of \( X \) and the forward edge from vertex \( d \) of \( X \) to vertex \( a \) of \( C \) to replace the dotted \( c \rightarrow a \) edge by the above
alternating path.

**Case 2.** Suppose that the variable gadget for the vertex \( v_1 \) has already been covered by our growing Hamiltonian path. Then, as seen in Case 1, in the XOR-gadget \( X \) corresponding to the \( c \to a \) edge, the vertex \( a \) has already been visited using a backward edge, while \( d \) has already been visited via a forward edge. Consider the second alternating path in Observation 5.4. Traversing this alternating path from \( a \) to \( d \) will satisfy the remaining covering requirements for all the vertices in \( X \). Thus, as in Case 1 above, we can replace the \( c \to a \) dotted edge in the alternating Hamiltonian path for \( C \) by an edge from vertex \( c \) of \( C \) to vertex \( a \) of \( X \) and an edge from vertex \( d \) of \( X \) to vertex \( a \) of \( X \). As before, this ensures that the vertices of this XOR-gadget are covered while traversing the alternating Hamiltonian path for \( C \).

Observe that since each clause gadget is connected to the variable gadget for \( \tau \), and since all other variable and XOR-gadgets are connected to at least one of the clause gadgets, the above alternating path eventually covers all of the individual gadgets. This completes the proof for the existence of the alternating Hamiltonian path in \( G \), and hence the proof of Theorem 5.1.

6 Future Work

This work leaves open the complexity of computing several other average quantities; the most pertinent of which is \( \langle d \rangle \), the average size of cuts under the Ising measure. The obvious approach of attempting rational interpolation over \( \beta \) via an analog of our Theorem 1.1 does not directly work, since Lee-Yang theorems do not hold in the same generality for the \( \beta \) parameter. A related problem is the complexity of computing the susceptibility \( \chi \) at fixed values of \( \lambda \) (in particular, \( \lambda = 1 \)), where, again, analyzing the partition function and its derivatives as polynomials in \( \beta \) may prove helpful.

Extensions of our results to the antiferromagnetic Ising model and the hard-core (weighted independent sets) model also remain open: again, our current approach would need to be modified, since Lee-Yang theorems do not in general hold for antiferromagnetic systems. In particular, the best known analog of the Lee-Yang theorem for the hard-core model, due to Chudnovsky and Seymour [16], works only for claw-free graphs. Similarly, the complexity of computing averages in spin systems with more than two spin values (such as the Potts model or proper colorings) remains open.

Finally, we mention potential connections with the large literature on stability preserving operators. As indicated in Section 1.3, this field is usually concerned with operators that preserve the region in which the zeros of a polynomial lie. Our extended Lee-Yang Theorem (Theorem 1.1) is an apparently rare example in which the operator actually makes this region strictly smaller. We conjecture that there may be more applications of this phenomenon.

**Acknowledgements.** We thank Aris Anagnostopoulos for several fruitful discussions which led to the formulation of the problems considered in this paper. We also thank Marek Biskup, Christian Borgs, Petter Brändén, Charles Newman, David Ruelle, Alan Sokal and Marc Thurley for helpful comments.

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A  An overview of complexity theoretic terminology

For the benefit of readers who may not be familiar with terminology from complexity theory, we provide here an overview of the relevant definitions. For more details, we refer to the textbook of Arora and Barak [2].

The conventional notion of efficiency in complexity theory is computability in time polynomial in the size of the input. For example, the problem of finding the shortest path between two specified vertices in a graph (presented as an adjacency matrix) can be solved in “polynomial time”, e.g., using Dijkstra’s algorithm. This notion of efficiency is captured by the complexity class P. For technical reasons we consider decision problems, where the answer is either ‘YES’ or ‘NO’. For example, the decision version of the shortest path problem would be to determine, given an integer $\ell$, if there is a path of length at most $\ell$ between the two specified vertices. A decision problem is said to be in the class P if there is an algorithm for it that runs in time polynomial in the size of the input. Examples of problems known to be in P include deciding if a graph has an Eulerian tour, deciding whether an integer (presented in binary) is prime [1], etc.

The class NP is the class of decision problems for which a polynomial time algorithm can verify the correctness of a ‘YES’ answer given a certificate of correctness. For example, we consider SAT, the problem of deciding whether a given Boolean formula has a satisfying assignment. While this problem is conjectured to not belong in the class P, it is clearly in NP: the certificate for a formula being satisfiable is just a satisfying assignment, if any, of the formula. Formally, a problem is said to be in NP if there is an algorithm $A$ and a polynomial $p$ such that

1. If $I$ is a ‘YES’ instance, then there exists a certificate $C$ such that on input $(I, C)$, $A$ runs in time at most $p(|I|)$ and outputs ‘YES’.

2. If $I$ is a ‘NO’ instance, then for any certificate $C$, on input $(I, C)$, $A$ runs in time at most $p(|I|)$ and outputs ‘NO’.

Another example of a problem in NP is deciding whether a graph $G$ has a Hamiltonian cycle (where the certificate is a Hamiltonian cycle, if any, in the graph). Both of the above examples are also NP-hard, that is, if there is a polynomial time algorithm for either of them, then there are polynomial time algorithms for all problems in NP. It is clear that $P \subseteq NP$, but it is a long standing conjecture in complexity theory that $P \subsetneq NP$. NP-hardness has long been the standard notion of intractability for decision problems, since the pioneering work of Cook [18], Levin [31] and Karp [28].

The class #P, introduced by Valiant [41], is a counting analog of NP: any problem in #P involves counting the number of certificates (possibly zero) that would make an algorithm for an
NP-problem accept. An example of a problem in $\#P$ is counting the number of satisfying assignments of a given SAT formula. As in the case of NP-hardness, a counting problem is called $\#P$-hard if a polynomial time algorithm for the problem implies the existence of polynomial time algorithms for all problems in $\#P$.

Counting versions of most NP-complete problems (such as counting the number of satisfying assignments of a Boolean formula) can be shown to be $\#P$-hard. However, there are decision problems in $P$ whose counting analogs are still $\#P$-hard, and it is this fact that makes the theory of $\#P$-hardness a non-trivial extension of the theory of NP-hardness. A famous example is counting the number of dimer coverings (perfect matchings) of a given graph: Edmonds [20] showed that there is a polynomial time algorithm for checking whether a graph has a dimer covering,\(^\text{5}\) however, counting the number of such coverings in general graphs is $\#P$-hard [41]. $\#P$-hardness has since become the standard notion of intractability for counting problems and for the computation of partition functions in statistical physics: indeed, almost all interesting classes of partition functions are known to be $\#P$-hard to compute (see, e.g., [11]).

B Proof of Theorem 3.1

As indicated earlier, Theorem 3.1 is an easy corollary of the following result of Newman [33] (restated in our notation).

**Theorem B.1 ([33, Theorem 3.2 and eq. 3.5]).** Consider the ferromagnetic Ising model on a graph $G = (V, E)$ on $n$ vertices, where we allow the edge potentials also to be variable, with the condition that the edge potential $\beta_{uv}$ on every edge $uv$ satisfies $0 < \beta_{uv} < 1$. Let $(z_v)_{v \in V}$ be a collection of complex vertex activities such that $|z_v| > 1$ for all $v \in V$. Then,

$$\Re\left(M(G, (\beta_{uv})_{uv \in E}, (z)_{v \in V})\right) > \frac{n}{2}.$$ 

We now proceed with the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Let $w$ be any weight assignment (not necessarily legal) of positive integral weights to the vertices of $G$. Consider the graph $H$ obtained from $G$ by appending to each vertex $v$ of $G$ a chain $C_v$ of $w(v) - 1$ vertices. Further, we let the edge potential be $\beta$ on all edges of $H$ which were present in $G$, and $0 < \gamma < 1$ on all the edges which are either part of some $C_v$, or connect a vertex $v$ to its associated chain $C_v$. Let $(y_v)_{v \in V}$ be a set of vertex activities on $V$. Henceforth, we will drop the subscript and refer to this set of activities as $(y_v)$. With a slight abuse of notation, we also denote by $(y_v)$ the collection of activities on $H$ such that for any vertex $x$ in $H$ such that $x \in C_v$ for some $v \in V$, we have $y_x = y_v$.

Now consider any collection of activities such that $|y_v| > 1$ for all $v \in V$. From Theorem B.1, we get that for any $\gamma \in (0, 1)$,

$$\Re\left(M(H, \{\beta, \gamma\}, (y_v))\right) > \frac{n}{2}. \quad (19)$$

Since $|y_v| > 1$ for all $v$, the Lee-Yang theorem\(^\text{6}\) implies that both $Z_I(H, \{\beta, \gamma\}, (y_v))$ and $Z_w(G, \beta, (y_v))$ are non-zero (when $0 < \gamma < 1$). We can therefore take the limit $\gamma \to 0$ in eq. (19) to get

$$\frac{n}{2} \leq \lim_{\gamma \to 0} \Re\left(M(H, \{\beta, \gamma\}, (y_v))\right) = \Re\left(\left\frac{Z_w(G, \beta, (y_v))}{Z_w(G, \beta, (y_v))}\right\right) = \Re\left(M(G, \beta, (y_v))\right). \quad (20)$$

\(^5\)Edmond’s result was, in fact, a crucial step towards the definition of the class P.

\(^6\)Although we stated Theorem 2.2 only for uniform edge potentials, Asano’s proof [3] in fact supports our current conclusion with variable edge potentials and a possibly disconnected graph.
We now take a sequence \( ((y^\ell_v))_{\ell=1}^{\infty} \) of activity assignments such that \( |y^\ell_v| > 1 \) for all \( \ell \) and \( v \) and such that \( \lim y^\ell_v = z_v \). Since we assume in the hypothesis of the theorem that \( Z_w(G, \beta, (z_v)_{v \in V}) \neq 0 \), we can take the limit \( \ell \to \infty \) in eq. (20) to get

\[
\frac{n}{2} \leq \lim_{\ell \to \infty} \Re \left( M \left( G, \beta, (y^\ell_v) \right) \right) = \Re \left( M(G) \right),
\]

which completes the proof. \( \square \)

## C Hardness with a smaller blowup in degree

Recall that, in our proofs of Theorems 1.2 and 1.4 in Sections 4 and 5, we realized different values of \( \lambda \) required for the interpolation by attaching \( k \) extra vertices to each vertex of \( G \). This necessarily entails a large increase in the degree of \( G \). In this section, we give an alternative way of realizing different values of \( \lambda \) which entails an increase in degree of exactly one, and which therefore allows us to complete the proofs of the stronger, degree-bounded version of Theorems 1.2 and 1.4.

We denote by \( P_k \) a path of \( k \) vertices. Let \( p_k^+ \) (respectively, \( p_k^- \)) be the partition function \( Z_I(P_k, \beta, \lambda) \) restricted to configurations in which the leftmost vertex of \( P_k \) is fixed to spin ‘+’ (respectively, ‘−’). We also set \( r_k := \frac{p_k^+}{p_k^-} \). Similarly, we denote by \( y_k \) the partition function \( Z_M(P_k, \lambda) \), where we assume that all edges in \( P_k \) have weight one, and suppress the dependence on edge weights for clarity of notation. Notice that \( p_1^+ = 1 \) and \( p_1^- = r_1 = y_1 = \lambda \). We further define \( y_0 = 1 \). The following recurrence relations show that, for fixed \( \beta \) and \( \lambda \), \( p_k^+, p_k^-, r_k \) and \( y_k \) can be computed in time polynomial in \( k \):

\[
\begin{align*}
p_k^+ &= \lambda (\beta p_{k-1}^- + p_{k-1}^+); \\
p_k^- &= \beta p_{k-1}^+ + p_{k-1}^-; \\
r_k &= \lambda \frac{\beta + r_{k-1}}{1 + \beta r_{k-1}}; \\
y_k &= \lambda y_{k-1} + y_{k-2}.
\end{align*}
\]

Notice that \( p_k^+, p_k^-, r_k \) and \( y_k \) are all functions of \( \lambda \). We note that values of their derivatives with respect to \( \lambda \) can also be computed in time polynomial in \( k \) via the following recurrence relations: \( \dot{p}_1^+ = \dot{y}_0 = 0, \dot{p}_1^- = \dot{r}_1 = \dot{y}_1 = 1 \), and

\[
\begin{align*}
\dot{p}_k^+ &= \frac{p_k^+}{\lambda} + \lambda (\beta \dot{p}_{k-1}^- + \dot{p}_{k-1}^+); \\
\dot{p}_k^- &= \beta \dot{p}_{k-1}^+ + \dot{p}_{k-1}^-; \\
\dot{r}_k &= \frac{\dot{p}_k^+ p_{k-1}^- - \dot{p}_k^- p_{k-1}^+}{(p_k^-)^2}; \\
\dot{y}_k &= y_{k-1} + \lambda y_{k-1} + y_{k-2}.
\end{align*}
\]

Here, we use the dot notation for the derivative with respect to \( \lambda \). Using a simple induction, one can also show that when \( \beta < 1, \dot{r}_k > 0 \) for all \( k \).

Now consider a connected graph \( G \). For \( k \geq 1 \), we define \( G(k) \) as the graph obtained by attaching to each vertex \( v \) of \( G \) a different instance of the path \( P_k \), such that \( v \) is connected to the “leftmost” vertex of \( P_k \) via an edge. Notice that the maximum degree of \( G(k) \) is one more than the maximum degree of \( G \). We first consider the Ising model on the graphs \( G(k) \). We have

\[
Z_I(G(k), \beta, \lambda) = (p_{k+1}^-)^n Z_I(G, \beta, \lambda_k),
\]

(29)
where $n$ is the number of vertices in $G$ and $\lambda_k = r_{k+1}$. Notice that when $0 < \beta < 1$, the sequence $\lambda_k$ is strictly increasing and greater than 1 (respectively, strictly decreasing and less than 1) when $\lambda > 1$ (respectively, when $\lambda < 1$): this follows from the observation that the right hand side of the recurrence (23) is a strictly increasing function of $r_{k-1}$, and that $r_2 > r_1$ (respectively, $r_2 < r_1$) when $\lambda > 1$ (respectively, when $\lambda < 1$). We also have

$$M(G(k), \beta, \lambda) = \frac{n\lambda p_{k+1}}{p_{k+1}} + \frac{\lambda r_{k+1}}{r_{k+1}} M(G, \beta, \lambda_k). \quad (30)$$

We now complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** As in the partial proof in Section 4, we assume $\lambda > 1$ (since the case $\lambda < 1$ is symmetrical) and suppose that we have a polynomial time algorithm $A$ which, given a connected graph $G$ of maximum degree at most $\Delta \geq 4$, outputs the mean magnetization $M(G, \beta, \lambda)$ in polynomial time.

Now consider any connected graph $G$ of maximum degree at most $\Delta - 1 \geq 3$. As shown in the partial proof in Section 4, Theorem 1.1 implies that if we can efficiently evaluate $M(G, \beta, z)$ at $2n + 2$ distinct values of $z$ using our hypothetical algorithm $A$, we can uniquely determine the coefficients of $Z_f(G, \beta, z)$, and hence also $Z_f(G, \beta, 1)$, in polynomial time. In view of Theorem 4.1, this would imply that the problem of computing the mean magnetization in graphs of maximum degree at most $\Delta$ for parameter values $\beta$ and $\lambda$ is #P-hard.

In order to evaluate $M(G, \beta, z)$ at $2n + 2$ distinct values, we evaluate $M(G(k), \beta, \lambda)$ for $1 \leq k \leq 2n + 2$ using our hypothetical algorithm $A$. Notice that this can be done since the construction of the $G(k)$ (as given in this section) implies that they have maximum degrees which are at most one larger than the maximum degree of $G$. Using eqs. (21) to (27) and (30), and the fact that $r_k > 0$ for all $k$, we can then determine $M(G, \beta, \lambda_k)$ in polynomial time. Since $\lambda_k$ is a strictly increasing sequence, these evaluations are at distinct points, and hence the reduction is complete.

We now consider the monomer-dimer model on the graphs $G(k)$. We have

$$Z_M(G(k), \lambda) = y_k^n Z_M(G, \lambda_k), \quad (31)$$

where $n$ is the number of vertices in $G$ and $\lambda_k = y_{k+1}/y_k$. We also have

$$U(G(k), \lambda) = n\lambda t_k + \lambda(t_{k+1} - t_k)U(G, \lambda_k), \quad (32)$$

where $t_k = y_k/y_{k+1}$. It turns out that the sequence $(\lambda_{2k})_{k \geq 0}$ is strictly increasing and hence consists of distinct values, and further that $t_{2k+1} - t_{2k} > 0$ for all $k$. This follows easily from the following explicit solutions for the $y_k$ and the $\lambda_k$:

$$y_k = \frac{\xi^{k+1} - \eta^{k+1}}{\xi - \eta}; \quad \lambda_k = \frac{\xi^{k+2} - \eta^{k+2}}{\xi^{k+1} - \eta^{k+1}},$$

where

$$\xi = \frac{1}{2} \left( \lambda + \sqrt{\lambda^2 + 4} \right) > 0; \quad \eta = \frac{1}{2} \left( \lambda - \sqrt{\lambda^2 + 4} \right) < 0.$$  

Notice that $t_{k+1} - t_k > 0$ for even $k$ implies that for such $k$, we can determine $U(G, \lambda_k)$ given $U(G(k), \lambda)$, using equations (24), (28) and (32). We can now complete the proof of Theorem 1.4 for the bounded degree case.
Proof of Theorem 1.4. As in the partial proof in Section 5, we fix any $\lambda > 0$, and suppose that there exists a polynomial time algorithm $B$ which, given a connected graph $H$ with edge weights in the set $\{1, 2, 3\}$, and of maximum degree at most $\Delta \geq 5$, outputs $D(H, \lambda)$ (recall that we are suppressing explicit dependence on the edge weights for clarity of notation). Given a Monotone 2-SAT formula $\phi$, we then produce the graph $G = A(\phi)$ in polynomial time. Notice that in the construction of $G$ as given in Section 5.1, each vertex has degree at most $4 \leq \Delta - 1$: this corresponds to the maximum of the in-degrees and the out-degrees over all vertices in the directed version of the reduction.

As argued in the partial proof in Section 5, Theorem 2.3 and the existence of a Hamiltonian path in $G$ together imply that if we could use algorithm $B$ to evaluate $D(G, z)$ (and hence $U(G, z)$) at $2n + 2$ different values of $z$, then we can determine the number of satisfying assignments of $\phi$ in polynomial time. This would in turn imply that computing $D(G, \lambda)$ for graphs of maximum degree at most $\Delta$ is #P-hard.

As before, in order to realize other values of $\lambda$, we consider the graphs $G(k)$ (as described in this section), for $k = 0, 2, 4, \ldots, 4n + 4$. Notice that the maximum degree of $G(k)$ is one more than that of $G$, and hence is at most $\Delta$. Further, as argued in the remarks following eqs. (31) and (32), these choices of $k$ ensure that the values $\lambda_k$ are distinct, and that $U(G, \lambda_k)$ can be easily determined from $U(G(k), \lambda)$. We can therefore determine $U(G, z)$ at $2n + 2$ different values of $z$ by running $B$ on the $G(k)$, as required.

\[\Box\]

D Hardness for general two state ferromagnetic spin systems and planar graphs

We now show how to extend our results to general two-state ferromagnetic spin systems. Recall that a general two-state spin system [23] is parametrized by a $(+, +)$ edge potential $\alpha_1$, a $(-, -)$ edge potential $\alpha_2$, and a vertex activity $\lambda$. As before, given a graph $G = (V, E)$, we define a probability distribution over the set of configurations $\sigma : V \to \{+,-\}$ via the weights $w_S(\sigma)$ given by

$$w_S(\sigma) = \lambda^{p(\sigma)} \alpha_1^{e_+(\sigma)} \alpha_2^{e_-(\sigma)},$$

where $e_+(\sigma)$ (respectively, $e_-(\sigma)$) denotes the number of edges with ‘+’ (respectively, ‘-’) spin on both end-points, while $p(\sigma)$ denotes the number of vertices with + spin. The partition function $Z_S(\sigma)$ and the magnetization $M_S(\sigma)$ are given by

$$Z_S(G, \alpha_1, \alpha_2, \lambda) := \sum_{\sigma \in \{+,-\}^V} w_S(\sigma);$$

$$M_S(G, \alpha_1, \alpha_2, \lambda) := \langle p \rangle = \frac{\sum_{\sigma} p(\sigma) w_S(\sigma)}{Z_S(G, \alpha_1, \alpha_2, \lambda)}.$$

Remark. The Ising model corresponds to the special case $\alpha_1 = \alpha_2 = \beta$.

It is well known that general two-state spin systems can be represented in terms of an Ising model in which the activity at each vertex depends upon the degree of the vertex [23]. In particular, if $G$ is a $\Delta$-regular graph then all vertex activities in the equivalent Ising model are the same, and one has

$$w_S(\sigma) = \alpha_2^{\lfloor E \rfloor} w_I(\sigma)$$

where the Ising model has an edge potential $\beta = 1/\sqrt{\alpha_1 \alpha_2}$ and a vertex activity $\lambda' = \lambda(\alpha_1/\alpha_2)^{\Delta/2}$. A two-spin system is called ferromagnetic if the above translation produces a ferromagnetic Ising model, that is, when $\alpha_1 \alpha_2 \geq 1$. 
However, the above translation does not allow us to directly translate our hardness result for the ferromagnetic Ising model, since our results were not derived for regular graphs. We will instead do a reduction similar to the ones done in our earlier proof, but starting from the following somewhat stronger hardness result for the partition function.

**Remark.** In this section, we allow graphs to have parallel edges and self-loops. In computing the degree of a vertex, each self-loop is counted separately. As observed in the remark following the proof of Theorem 3.2 in Section 3, our extended Lee-Yang theorem (Theorem 1.1) holds also in this setting.

**Theorem D.1** ([14, Theorem 1]). Fix \( \alpha_1, \alpha_2 > 0 \) with \( \alpha_1 \alpha_2 > 1 \) and \( \Delta \geq 3 \). The problem of computing the partition function \( Z_S(G, \alpha_1, \alpha_2, 1) \) on \( \Delta \)-regular graphs is \#P-hard.

We will then prove the following theorem. We will also show later in this section that the \#P-hardness holds even when the input is restricted to planar graphs.

**Theorem D.2.** Fix \( \alpha_1, \alpha_2, \lambda > 0 \) and \( \Delta \geq 4 \) such that \( \alpha_1 \alpha_2 > 1 \). The problem of computing the magnetization \( M_S(G, \alpha_1, \alpha_2, \lambda) \) on connected graphs of degree at most \( \Delta \) is \#P-hard, except when \( \alpha_1 = \alpha_2 \) and \( \lambda = 1 \), in which case it can be solved in polynomial time.

**Remark.** Notice that when \( \alpha_1 \alpha_2 = 1 \), the problem reduces to the case of a graph consisting of isolated vertices, and hence can be solved in polynomial time. Similarly, in the case \( \alpha_1 = \alpha_2 \) and \( \lambda = 1 \), the two spins are symmetric, and the magnetization is therefore \( n/2 \), where \( n \) is the number of vertices in \( G \).

Before proceeding with the proof of Theorem D.2, we will need to analyze the model on graphs \( G(k) \) defined in Appendix C. As before, we begin by analyzing the model on the path \( P(k) \). We denote by \( p^+_k \) (respectively, \( p^-_k \)) the partition function \( Z_S(P_k, \alpha_1, \alpha_1, \lambda) \) restricted to configurations in which the leftmost vertex is fixed to ‘+’ (respectively, ‘−’). We also define the ratio \( r_k = p^+_k / p^-_k \). Similarly, we denote by \( m^+_k \) (respectively, \( m^-_k \)) the average magnetization of the path \( P_k \) conditioned on the leftmost vertex being fixed to ‘+’ (respectively, ‘−’). We have \( p^+_1 = r_1 = \lambda, p^-_1 = 1 \) and \( m^+_1 = 1, m^-_1 = 0 \), and the following recurrences for \( k \geq 1 \):

\[
p^+_k = \lambda(\alpha_1 p^+_{k-1} + p^-_{k-1}) \\
p^-_k = \alpha_2 p^-_{k-1} + p^+_{k-1} \\
r_k = \frac{\alpha_1 r_{k-1} + 1}{\alpha_2 + r_{k-1}} \\
m^+_k = 1 + \frac{\alpha_1 m^+_{k-1} p^+_{k-1} + m^-_{k-1} p^-_{k-1}}{\alpha_1 p^+_{k-1} + p^-_{k-1}} \\
m^-_k = \frac{\alpha_2 m^-_{k-1} p^-_{k-1} + m^+_{k-1} p^+_{k-1}}{\alpha_2 p^-_{k-1} + p^+_{k-1}}
\]

Under the condition \( \alpha_1 \alpha_2 > 1 \), one can prove using a simple induction that for all \( k \geq 1 \), \( m^+_k - m^-_k > 0 \), and that when \( (\alpha_1 - 1) \lambda - (\alpha_2 - 1) > 0 \) (respectively, when \( (\alpha_1 - 1) \lambda - (\alpha_2 - 1) < 0 \)), the \( r_k \) form a strictly increasing (respectively, strictly decreasing) sequence, and hence are all distinct.

For reasons that will become clear shortly, we need the \( r_k \) to be distinct, and hence we will need to handle the remaining case \( (\alpha_1 - 1) \lambda - (\alpha_2 - 1) = 0 \) specially. We observe that unless \( \alpha_1 = \alpha_2 = 1 \), or \( \alpha_1 = \alpha_2 \) and \( \lambda = 1 \), both of which are excluded in the hypothesis of the theorem,
we cannot have both \((\alpha_1 - 1)\lambda - (\alpha_2 - 1) = 0\) and \((\alpha_1^2 - 1)\lambda - (\alpha_2^2 - 1) = 0\). To take advantage of this, we will modify \(P(k)\) by replacing each edge in \(P(k)\) by two parallel edges. We call the resulting graph \(P(k)\)', and again define the quantities \(p_k^+, p_k^-, r_k, m_k^+\) and \(m_k^-\) by recursion on \(P(k)'\). Notice that for \(k = 1\), these quantities are the same as those for \(P(k)\); however for \(k \geq 2\), we now need to modify the recurrences above by replacing \(\alpha_1\) and \(\alpha_2\) by \(\alpha_1^2\) and \(\alpha_2^2\) respectively. As before, we have \(m_k^+ - m_k^- \geq 0\) for all \(k \geq 1\). Further, by our observation, the \(r_k\) form a strictly monotone sequence. Thus, in the case \((\alpha_1 - 1)\lambda - (\alpha_2 - 1) = 0\), we redefine \(G(k)\) to use the paths \(P(k)'\) in place of \(P(k)\). In what follows, we will assume that \(G(k)\) are appropriately defined taking into account the values of \(\lambda, \alpha_1\) and \(\alpha_2\), and will not explicitly keep track of the above modification. Notice that the maximum degree of \(G(k)\) is still at most \(\max(\Delta + 1, 3)\), where \(\Delta\) is the maximum degree of \(G\).

Given the above definition of \(G(k)\), we have the relations

\[ Z_S(G(k), \alpha_1, \alpha_2, \lambda) = (\alpha_2 p_k^- + p_k^+)^n Z_S(G(\alpha_1, \alpha_2, \lambda_k)) \]  
\[ M_S(G(K), \alpha_1, \alpha_2, \lambda) = nm_k^- + (m_k^+ - m_k^-) M_S(G(k), \alpha_1, \alpha_2, \lambda_k), \]

where \(\lambda_k = (\frac{\alpha_1 r_k + 1}{\alpha_2 + r_k})\). Since the \(r_k\) form a strictly monotone sequence, it follows that (since \(\alpha_1 \alpha_2 > 1\)) so do the \(\lambda_k\). In particular, all the \(\lambda_k\) are distinct.

Proof of Theorem D.2. Proceeding as in the proof of Theorem 4.1, we fix any \(\lambda > 0\) and \(\alpha_1, \alpha_2 > 0\) satisfying \(\alpha_1 \alpha_1 > 1\), and suppose that there exists a polynomial time algorithm \(B\) which, given a connected graph \(G\) of maximum degree at most \(\Delta \geq 4\), outputs \(M_S(G, \alpha_1, \alpha_2, \lambda)\).

Now consider any connected regular graph \(G = (V, E)\) of degree \(d := \Delta - 1 \geq 3\) on \(n\) vertices. From the translation in eq. (33), we see that for any \(\lambda > 0\),

\[ Z_S(G, \alpha_1, \alpha_2, \lambda) = \alpha_2^{\frac{1}{2}} Z_I \left( G, \beta, \lambda \left( \frac{\alpha_1}{\alpha_2} \right)^{d/2} \right), \]  
\[ M_S(G, \alpha_1, \alpha_2, \lambda) = M_I \left( G, \beta, \lambda \left( \frac{\alpha_1}{\alpha_2} \right)^{d/2} \right), \]

where \(\beta = 1/\sqrt{\alpha_1 \alpha_2} < 1\). Theorem 2.4 along with our main Theorem 1.1 then implies that if we can efficiently evaluate \(M_S(G, \alpha_1, \alpha_2, z)\) at \(2n + 2\) distinct values of \(z\) using our hypothetical algorithm \(B\), we can uniquely determine the coefficients of \(Z_S(G, \alpha_1, \alpha_2, z)\), and hence also the value of \(Z_S(G, \alpha_1, \alpha_2, 1)\), in polynomial time. In view of Theorem D.1, this would imply that the problem of computing the mean magnetization in graphs of maximum degree at most \(\Delta\) for parameter values \(\alpha_1, \alpha_2\) and \(\lambda\) is \#P-hard.

In order to evaluate \(M_S(G, \alpha_1, \alpha_2, z)\) at \(2n + 2\) distinct values of \(z\), we instead compute \(M_S(G(k), \alpha_1, \alpha_2, \lambda)\), for \(1 \leq k \leq 2n + 2\), using our hypothetical algorithm \(B\). Notice that this can be done since the construction of the \(G(k)\) implies that they have maximum degrees which are at most one larger than the maximum degree of \(G\). Using eqs. (34) to (38) and (40), and the fact that \(m_k^+ - m_k^- > 0\) for all \(k\), we can then determine \(M_S(G, \alpha_1, \alpha_2, \lambda_k)\) in polynomial time. Since \(\lambda_k\) is a strictly monotone sequence as shown in the discussion above, these evaluations are at distinct values of \(z\), and hence the reduction is complete.

Extension to planar graphs

Cai and Kowalczyk [14] also proved the following planar graph version of Theorem D.1.

Theorem D.3 ([14, Theorem 1]). Fix \(\alpha_1, \alpha_2 > 0\) with \(\alpha_1 \alpha_2 > 1\), \(\alpha \neq \alpha_2\) and \(\Delta \geq 3\). The problem of computing the partition function \(Z_S(G, \alpha_1, \alpha_2, 1)\) on planar \(\Delta\)-regular graphs is \#P-hard.
In order to extend Theorem D.2 to planar $\Delta$-regular graphs, we consider the cases $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \alpha_2 = \alpha$ separately. In case $\alpha_1 \neq \alpha_2$, we proceed exactly as in the proof of Theorem D.2 given above, except that we start with a planar $d$-regular graph $G$ in the reduction, and use Theorem D.3 instead of Theorem D.1 as our starting hardness result. Since $G$ is planar, so are the $G(k)$, and hence we see that computing $M_S(H, \alpha_1, \alpha_2, \lambda)$ on planar graphs $H$, for $\alpha_1, \alpha_2$ and $\lambda$ satisfying the condition $\alpha_1 \neq \alpha_2$ in addition to the conditions of Theorem D.2 is #P-hard.

We now turn to the case $\alpha_1 = \alpha_2 = \alpha > 1$ (with $\lambda \neq 1$). In this case, we start with the fact that computing $Z_S(G, 2\alpha, \frac{\alpha}{2}, 1)$ on planar $\Delta$-regular graphs is #P hard (this is a direct corollary of Theorem D.3). We again proceed exactly as in proof of Theorem D.2, starting with an arbitrary planar $d$-regular graph $G$, and noting that the $G(k)$ are planar too. Notice that the proof then shows that assuming the existence of a polynomial time algorithm to compute the magnetization in planar graphs of degree at most $d + 1$, we can evaluate the coefficients of the polynomial $Z_s(G, \alpha, \alpha, z)$, and hence also the quantity $Z_S(G, \alpha, \alpha, 2^d)$. However we then use the translation to the Ising model given above to see that

$$Z_S(G, \alpha, \alpha, 2^d) = \left(\frac{\alpha}{2}\right)^{|E|} Z_S(G, 2\alpha, \frac{\alpha}{2}, 1),$$

which shows that we can also evaluate $Z_S(G, 2\alpha, \frac{\alpha}{2}, 1)$ in polynomial time. This establishes the #P-hardness in the remaining case $\alpha_1 = \alpha_2$ (with $\lambda \neq 1$).

We thus see that in Theorem D.2, the input graphs can be restricted to be planar, and the same hardness result still holds.

E XOR-gadget

Claim E.1. The total weight of cycle covers of the XOR-gadget in Figure 3a is 2 when either

- $a$ is connected to an external incoming edge and $d$ is connected to an external outgoing edge;

or

- $a$ is connected to an external outgoing edge and $d$ is connected to an external incoming edge.

For all other external connections of $a$ and $d$, the total weight of cycle covers of the gadget is 0.

Proof. When the total number of incoming external edges at $a$ and $d$ is not equal to the total number of outgoing external edges, the XOR-gadget cannot admit a cycle cover due to parity constraints, and thus, the total weight of all cycle covers in these cases is trivially zero. A simple way to see this is that a cycle cover corresponds to a perfect matching in the natural undirected bipartite representation of the gadget discussed above. When the numbers of external incoming and outgoing edges are not equal, the bipartite graph does not remain balanced and hence cannot have a perfect matching. For all other configurations, in which the number of external incoming and outgoing edges are equal, the weights of all cycle covers can be shown to have the claimed value by exhaustive enumeration. □