Gauge Symmetry from Integral Viewpoint

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Abstract

The purpose of this paper is to investigate the gauge symmetry of classical field theories in integral formalism. A gauge invariant theory is defined in terms of the invariance of the physical observables under the coordinate transformations in principal bundle space. Through the detailed study on the properties of non-Abelian parallel transporter under gauge transformations, we show that it is not generally a two-point spinor, i.e. an operator to be affected only by the gauge group elements at the two end points of the parallel transport path, except for the pure gauge situation, and therefore the local gauge symmetry for non-Abelian models is found to be broken in non-perturbative domain. However, an Abelian gauge theory is proved to be strictly invariant under local gauge transformation, as it is illustrated by the invariance of the interference pattern of electrons in Aharonov-Bohm effect. The related issues of the phenomenon are discussed.

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1 Introduction

It has been accepted without doubt that the present-day physical theories of the fundamental interactions in nature largely follow from the principle of gauge symmetry, which postulates that any physically acceptable field model should be invariant under gauge transformations. For a concrete gauge field model it is constructed from the principle as follows: 1) The requirement for the invariance of the field action $S$ under the action of a finitely dimensional Lie group leads to the existence of conserved currents (Noether theorem); 2) The generalization of the invariance of the action under the global gauge transformations to the local ones introduces in the coupling of the currents through a gauge field. In fact it is just the second step, i.e. the replacement of $\partial_\mu$ in action by $D_\mu$, that enables us to have complete theories to describe the matter fields and the interactions coupling matters. The quantized gauge field model (standard model) has been successful to a large extent in explaining and predicting the experimental results in elementary particle physics—for an overview, see e.g. Ref.[1].

The invariance of field action, or more exactly the invariance of Lagrangian density under gauge transformations, constitutes the basis for the differential formalism of a gauge field theory. According to it, two field models $\langle L, A_\mu, \psi \rangle$ and $\langle L', A'_\mu, \psi' \rangle$ are equivalent if they are related to each other by the form invariance of Lagrangian density $L$, and therefore by the invariance of local field equations, under gauge transformations including the global (GT1) and local (GT2a for matter field and GT2b for gauge field) ones. Here we have used the notations in Ref.[2]. It is assumed as a matter of course that other formalisms of gauge field theory, e.g. the integral formalism [3-4], should be the same as the differential one with respect to the inherent gauge symmetry. To meet the requirement, the gauge transformation of a parallel transporter, the counterpart of gauge field $A_\mu(x)$ in the integral formalism, should be a two-point spinor\footnote{It is termed in analogy to two-point tensor in [5-6] for the parallel transporter of a tensor field in spacetime manifold.}, i.e. that to be affected only by the gauge transformation GT2a at the two end points which the parallel transporter connects. Contrary to this assumption, however, we prove that the two-point spinor property for a parallel transporter for a non-Abelian gauge field is conditional if the field is non-trivial and, therefore, the gauge freedom for a non-Abelian theory is much more restricted than what we see from differential
point of view on gauge theories.

In this paper, we first clarify in the framework of fibre bundle some commonly used terminologies such as gauge transformation and gauge invariance, then give a definition of gauge invariant field theory in terms of the invariance of physical observables. After proving the equivalence of gauge transformation GT2b and two-point spinor property of infinitesimal parallel transporter in the form of one-order expansion, we will study in detail the property of finite parallel transporter in non-Abelian and Abelian gauge theories. Finally, some related issues will be discussed.

2 Gauge transformation and gauge invariance

In this section we begin with the clarification of some terminologies regarding a gauge field theory, which are by no means uniform throughout the literature. We here adopt the definition of gauge theory in Ref.[2]:

A gauge field theory is the one that is derived from the gauge principle and represents the geometry of a principal fibre bundle. The gauge group is given by the structure group of the bundle.

Since the geometrical framework of fibre bundle provides a natural mathematical setting for the representation of physical gauge theories, we will largely use the language for the clarification of the terminologies we will use in our study.

A principal fibre bundle is defined as a tuple \( \langle E, M, \pi, G \rangle \) with bundle space \( E \), base space \( M \), projection map \( \pi: E \to M \), and structure group \( G \) which is homeomorphic to fibre space \( F \). The base space \( M \) we study in this paper is Minkowskian or Euclidean spacetime for simplicity. In fibre bundle language a matter field is given as a cross section in the associated vector bundle of principal bundle. It is a smooth map \( \psi: M \to E \), which satisfies \( \pi \circ \psi(x) = x \) for \( \forall x \in M \). Of course there is no continuous cross section unless the principle bundle is with a trivial topology, i.e. the product of base and fibre space. To describe gauge field \( A_\mu(x) \) (throughout the paper \( A_\mu \) is the short handed symbol for \( \sum_{a=1}^{\dim G} A_\mu^a T^a \), where \( T^a \)s are the generators of the Lie algebra of gauge group), we need to identify \( i \lambda A_\mu(x) \), where \( \lambda \)

\footnote{The above-mentioned gauge symmetry principle is close to the definition of gauge postulate in Ref.[2], and should be distinguished from the term gauge principle there.}
is the coupling constant, with the local representation \( B_\mu(x) \) [7, p. 160] of connection form \( \omega \), which is given as [8, p. 255]

\[
\omega = -g^{-1}(x)dg(x) + g^{-1}(x)B_\mu(x)g(x)dx^\mu
\]  

(1)

in the local coordinate \( \varphi: (g(x), x) (g \in G, x \in M) \), and which determines a unique decomposition \( TE = PE \oplus HE \) of the tangent space \( TE \) of the principal bundle \( E \) into a ‘perpendicular’ and a ‘horizontal’ part.

Cross section at each point \( x \in U_\alpha \), an arbitrary neighborhood in base space, can be regarded locally as an element of gauge group. A choice of a cross section is equivalent to the choice of a local coordinate, and the cross section in a local moving frame \( \{e_i\} \) is given as \( \psi(x) = \{\psi^i(x)\} \). Therefore the local gauge transformations GT2a and GT2b can be derived from the transformation of local coordinates

\[
T_{\alpha\beta} = \varphi_{\beta}^{-1} \circ \varphi_{\alpha} : G \times U_{\alpha\beta} \rightarrow G \times U_{\alpha\beta}
\]  

(2)

Under this transformation the cross section \( \psi(x) \) in bundle space \( E_\alpha \) transforms as

\[
\psi'(x) = g(x)\psi(x)
\]  

(3)

with the local moving frame transformation in the tangent space of the associate vector bundle of principal bundle: \( \{e_i\} \rightarrow \{e'_i\} \). Thus we have got the gauge transformation GT2a. To obtain gauge transformation GT2b, we will make use of connection form Eq.(1). In two different coordinates \( (g_1(x), x) \) and \( (g_2(x), x) \), if a connection form is expressed as [8, p. 256]

\[
\omega_{E_\alpha} = -g_1^{-1}(x)dg_1(x) + g_1^{-1}(x)B^{(1)}_\mu(x)g_1(x)dx^\mu
\]

\[
= -g_2^{-1}(x)dg_2(x) + g_2^{-1}(x)B^{(2)}_\mu(x)g_2(x)dx^\mu,
\]  

(4)

then the relation of the local representations of connection form is therefore found to be

\[
B^{(2)}_\mu(x) = -g(x)\frac{\partial g^{-1}}{\partial x^\mu}(x) + g(x)B^{(1)}_\mu(x)g^{-1}(x),
\]  

(5)

where \( g = g_1g_2^{-1} \). Gauge transformation GT2b has been derived in this way if the local representation \( B_\mu(x) \) is identified with the gauge field \( i\lambda A_\mu \). From Eq. (3) and Eq. (5) we obviously see that the local coordinate transformation Eq. (2) can be realized by any
$C^1$ gauge group element $g(x)$ (with continuous first derivatives with respect to spacetime coordinates $x$), which also preserves the invariance of the field action constructed by $\psi(x)$ and $A_\mu(x)$. In this sense we call a $C^1$ coordinate transformation $T_{\alpha\beta}(x)$ a ‘general local gauge transformation’.

In Eq. (3) the action of gauge group is performed in the same fibre over each point in the base space, so we call it ‘perpendicular action’ of gauge group. With connection form $\omega$ we can also define another type of action of gauge group, that is, parallel transport of fibres from one point in base space to another. It involves different points in base space, so it is termed ‘horizontal action’ of gauge group. In this paper we denote a parallel transporter, the gauge group element which implements such action, as $\Phi_\gamma(B_\mu; x, x_0)$, where the transport path $\gamma$ is an arbitrary piecewise smooth curve which connects two end points, say $x_0$ and $x$, and is parametrized by the path variable $t$. As a point in fibre space, cross section $\psi(x)$ is transported accordingly:

$$\psi(x) = \Phi_\gamma(B_\mu; x, x_0)\psi(x_0)$$

(6)

Parallel transport is determined by the ‘horizontal direction’ of the bundle space [8, p. 253]:

$$\omega = -\Phi^{-1}d\Phi + \Phi^{-1}B_\mu d\Phi^\mu = 0.$$  

(7)

Over a smooth segment of the path on which there is a definite tangent vector field, $dx^\mu(t)/dt$, it is equivalent to the following matrix differential equation:

$$\frac{d\Phi}{dt}(t) = B(t)\Phi(t),$$

(8)

$$\Phi(t_0) = I,$$

(9)

where $\Phi(t) = \Phi(B(t); t, t_0)$, and $B(t) = B_\mu(x(t))dx^\mu(t)/dt$. There are two types of the solution to the equation:

$$\Phi(t) = Pexp\left(\int_{t_0}^{t} dsB(s)\right) = I + \int_{t_0}^{t} dsB(s)$$

$$+ \int_{t_0}^{t} dsB(s)\int_{t_0}^{s} ds_1B(s_1) + \cdots + \int_{t_0}^{t} dsB(s)\cdots\int_{t_0}^{s_{n-1}} ds_nB(s_n) + \cdots,$$

(10)
for the case of non-Abelian gauge group, and
\[ \Phi(t) = \exp \left( \int_{t_0}^{t} ds B(s) \right) \]  
(11)
for the case of Abelian gauge group, since the differential equation in the latter case reduces to an ordinary linear differential equation. A parallel transporter along a piecewise smooth curve or a gauge group element of ‘horizontal action’ is therefore given as the product of these operators defined on the monotonic smooth segments. Naturally there is the accompanying differential equation for parallel transport of matter field:
\[ \frac{d\psi}{dt}(t) - i\lambda A_\mu(x(t)) \frac{dx^\mu}{dt}(t)\psi(t) = D_\mu \psi(x(t)) \frac{dx^\mu}{dt}(t) = 0, \]  
(12)
\[ \psi(t_0) = \psi_0. \]  
(13)
From these results it is concluded that, under a parallel transport, the total wave function for a matter field not only changes its amplitude but also undergoes a ‘rotation’ in the internal space such as isospin space in SU(2) gauge theory.

Parallel transporter plays an important role in the integral formalism of gauge theory. It adequately describes all physics contained in the gauge theory and, moreover, the summation of it is a physically measurable quantity because it defines the transition amplitude of a particle moving along a classical trajectory in the presence of gauge field \( A_\mu \). For overviews of gauge field theories with \( \Phi = P\exp \oint_C A_\mu(x)dx^\mu \) as the dynamical variable (loop space formalism), see e.g. Ref.[9, chap. 7], Ref.[10, chap. 4]

We are now in a position to define gauge invariance, the central concept in gauge field theory. Following the definition of gauge field theory in Ref.[2] cited at the beginning of the section, we give the definition as follows:

**A gauge invariant field theory** is a gauge field theory which is invariant under the bundle coordinate transformation \( T_{\alpha\beta} \), that is, all the observables including the local and non-local ones should be invariant under its induced transformations GT1 and GT2. The classification of global and local invariant theories is according to whether \( T_{\alpha\beta} \) is spacetime dependent or not.

Obviously from this definition we see that the gauge freedom of a gauge field theory is comprised of the actions of gauge group that preserve the invariance of the theory. The
distinction of the definition from those elsewhere lies in the invariance of the non-local operators, such as \( \bar{\psi}(x_2)\Phi_\gamma(A_\mu; x_2, x_1)\psi(x_1) \), the operator for the bound state wave function [10]. Should the operator be invariant under local gauge transformations, the parallel transporter must be a two-point spinor, i.e. it transforms as follows:

\[
\Phi_\gamma(A'_\mu; x_2, x_1) = g(x_2)\Phi_\gamma(A_\mu; x_2, x_1)g^{-1}(x_1),
\]

under the local gauge transformations. This two-point spinor property of \( \Phi_\gamma \) is regarded as the natural consequence of the general covariance in bundle space (see e.g. [12]), since the physics should be independent of the choice of coordinate. In fibre bundle language we can say that the commutation of ‘perpendicular action’ and ‘horizontal action’ of gauge group guarantees a coordinate-free description of physics contained in a gauge theory. In addition to the gauge fixing terms which are introduced in the Lagrangian density for the appropriate physical situations, however, we find that this requirement actually imposes more restriction on the gauge freedom of a gauge theory, because we will prove later that it is conditional in non-Abelian gauge theories.

3 Joint of differential and integral formalism

In practice a parallel transporter \( \Phi_\gamma(x, x_0) \) can be expressed as infinite product of local operator too. It is obtained through discretization of Eq. (8):

\[
\frac{\Phi(t + \delta) - \Phi(t)}{\delta} = B(t)\Phi(t)
\]

The solution of the difference equation is

\[
\Phi(t + n\delta) = \lim_{n \to \infty} \prod_{i=0}^{n-1} (I + \delta B(i\delta)) \Phi(t_0).
\]

If there is a definite tangent vector at each point of a segment of the parallel transport path, we get the alternative form for a parallel transporter over the segment:

\[
Pexp \int_{t_0}^{t} dsA(s) = \lim_{\Delta t \to 0} (I + A(t_{n-1})\Delta t) \cdots (I + A(t_0)\Delta t),
\]

where \( A(t_i) = i\lambda A_\mu(x(t_i))dx^\mu(t_i)/dt, i = 1, 2, \cdots, n - 1. \)

Infinitesimal parallel transporter \( I + i\lambda A_\mu dx^\mu \) bridges over the connection between differential and integral formalism of gauge theory. When \( \Delta x^\mu \to 0 \), one-order approximation

\[
g(x + \Delta x) \approx g(x) + \frac{\partial g(x)}{\partial x^\mu} \Delta x^\mu
\]
can be regarded to be an exact equality, and therefore we find that infinitesimal parallel transporter, $I + i \lambda A_\mu dx^\mu$, transforms as a two-point spinor under gauge transformation GT2b:

$$I + i \lambda A'_\mu dx^\mu = \left( g(x) + \frac{\partial g(x)}{\partial x^\mu} dx^\mu \right) \left( I + i \lambda A_\mu dx^\mu \right) g(x)$$

$$= g(x + dx)(I + i \lambda A_\mu dx^\mu)g(x).$$

(19)

Obviously there is the equivalence of gauge transformation GT2b (Eq. (5)) and two-point spinor property of infinitesimal parallel transporter.

Two-point spinor property for an infinitesimal parallel transporter is crucial for the invariance of parallel transport equation of $\psi$ (Eq. (12)). In terms of infinitesimal parallel transporter the covariant change of $\psi(x)$, which means the difference of the field at $x$ and that parallely transported from $x + dx$, can be expressed as

$$\delta^{\text{cov}} \psi(x) = (I - i \lambda A_\mu(x)dx^\mu)\psi(x + dx) - \psi(x) = D_\mu \psi(x)dx^\mu$$

(20)

Under local gauge transformation it is transformed according to Eq. (3) and Eq. (19) to

$$\delta^{\text{cov}} \psi'(x) = g(x) \left( \frac{\partial}{\partial x^\mu} - i \lambda A_\mu(x) \right) \psi(x)dx^\mu.$$  

(21)

Thus the differential equation for the parallel transport of $\psi(x)$, which can also be express as

$$\lim_{\delta t \to 0} \frac{\delta^{\text{cov}} \psi}{\delta t} = 0,$$

(22)

is transformed covariantly under gauge transformation, as long as infinitesimal parallel transporter is a two-point spinor. This is consistent with the invariance of the ‘horizontal direction’ in bundle space under the action of structure group.

4 Properties of parallel transporter in non-Abelian gauge theory

From Eq. (17) there are obviously the following three properties of parallel transporter

$$\Phi_{\gamma_2 \gamma_1} (A_\mu) = \Phi_{\gamma_2} (A_\mu) \Phi_{\gamma_1} (A_\mu);$$
\[ \Phi_{\bar{\gamma}}(A_\mu) = (\Phi_{\gamma}(A_\mu))^{-1}; \]
\[ \Phi_{\gamma}(CA_\mu C^{-1}) = C\Phi_{\gamma}(A_\mu)C^{-1}. \]

Here \( \bar{\gamma} \) denotes the inverse path of \( \gamma \) and \( C \) is a global gauge transformation.

In this section we will primarily study the property of non-Abelian parallel transporter under local gauge transformation. It was always taken for granted that finite parallel transporter should be a two-point spinor too, since it can be pieced together with infinite number of infinitesimal parallel transporter, which have been proved to be two-point spinors. Along with the analysis on the causes for the false statement, we will give a detailed study on parallel transport equation, finite and infinitesimal parallel transporter and their connection under gauge transformations.

4.1 Investigation into differential equation Eq. (8) and Eq. (12)

The differential equation that determines the parallel transport of matter field \( \psi(x) \) (Eqs. (8) and (12)) is of the type

\[ \frac{dx}{dt}(t) = W(t)x(t). \]  

(23)

Let us study the behavior of the equation under the transformation

\[ x(t) = L(t)y(t). \]  

(24)

Substituting Eq. (24) into Eq. (23), we have

\[ \frac{d}{dt}(L(t)y(t)) = \frac{dL}{dt}(t)y(t) + L(t)\frac{dy}{dt}(t) = W(t)L(t)y(t). \]  

(25)

Then Eq. (23) is transformed to

\[ \frac{dy}{dt}(t) = W'(t)y(t), \]  

(26)

with

\[ W'(t) = -L^{-1}(t)\frac{d}{dt}L(t) + L^{-1}(t)W(t)L(t). \]  

(27)

The relation between \( W \) and \( W' \) is just that of gauge transformation GT2b if \( L^{-1} \) is identified with gauge group element \( g \) and, therefore, it is concluded that

\[ \psi'(t) = Pexp\left(\int_{t_0}^t dsA'(s)\right) \psi'(t_0) = g(t)Pexp\left(\int_{t_0}^t dsA(s)\right) g^{-1}(t_0)\psi'(t_0). \]  

(28)
Seemingly the procedure will lead to the conclusion Eq. (14), but definitely it has only proved the two-point spinor property of the parallel transporter when the field $\psi(t)$ is parallelly transported over a one-dimensional range $[t_0, t]$, a trivial field theory, because all the operations in the procedure are stuck to the one-dimensional range $[t_0, t]$ and the integral in Eq. (28) should be interpreted as a one-variable integral. In any one-dimensional situation we can always find a local gauge transformation GT2b that makes gauge field $A(t)$ vanish identically because the differential equation

$$\frac{dg}{dt}(t)g^{-1}(t) + g(t)A(t)g^{-1}(t) = 0$$  \hspace{1cm} (29)$$

has definite solutions, then the two-point spinor property holds absolutely (see Appendix A).

In fact, a parallel transport path should be regarded as the map $[t_0, t] \rightarrow M$ rather than $[t_0, t]$ itself. The point will be clearly seen through the discussion in the following subsections.

A special case involving the solution of Eq. (8) that needs to be clarified is a periodic $A(s)$. Without the loss of generality we suppose $A(s + 2\pi) = A(s)$. Then for a parallel transport path, $[0, 4\pi] \rightarrow M$, we have

$$\Phi(4\pi, 2\pi)\Phi(2\pi, 0) = Pexp \left( \int_{2\pi}^{4\pi} dsA(s) \right) Pexp \left( \int_{0}^{2\pi} dsA(s) \right).$$

Its term of the order $(i\lambda)^2$ is

$$\int_{0}^{2\pi} dsA(s) \int_{0}^{s} ds'A(s') + \int_{2\pi}^{4\pi} dsA(s) \int_{2\pi}^{s} ds'A(s') + \int_{2\pi}^{4\pi} dsA(s) \int_{0}^{2\pi} dsA(s)$$

$$= \int_{0}^{4\pi} dsA(s) \int_{0}^{s} ds'A(s') + (\int_{0}^{2\pi} dsA(s))^2,$$  \hspace{1cm} (30)$$

with the presence of the periodic function $A(s)$. If the $(i\lambda)^2$ term of $\Phi(4\pi, 0) = Pexp \int_{0}^{4\pi} dsA(s)$ should also be formally given as $\int_{0}^{4\pi} dsA(s) \int_{0}^{s} ds'A(s')$ according to Eq. (10), the integral factor $\int_{0}^{s} ds'A(s')$ in it is indefinite due to the period $2\pi$ of $A(s)$. To guarantee a definite group composition law:

$$\Phi(4\pi, 2\pi)\Phi(2\pi, 0) = \Phi(4\pi, 0),$$

we must specify that the parallel transporter constructed by a periodic $A(s)$ should be expressed as the product of those along the monotonic one, i.e. $\Phi(2\pi n, 0) = \Phi^n(2\pi, 0)$ in the example.
4.2 Investigation into infinite product Eq. (17)

To see if the two-point spinor property of finite parallel transporter, a non-local operator, can be obtained through piecing together two-point spinor property of local infinitesimal parallel transporter, we need to study the infinite product form of finite parallel transporter Eq. (17), since we have proved that an infinitesimal parallel transporter will be a two-point spinor only when it is in the form of one-order expansion. The transport path, \([t_0, t] \to M\), is divided into countably infinite small range \([t_i, t_{i+1}]\), for \(i = 0, 1, \cdots, n\), then a finite parallel transporter after gauge transformation GT2 becomes the following path-ordered infinite product:

\[
\Phi(A_\mu'(x); t, t_0) = (I + A'(t_n-1)\Delta t) \cdots (I + A'(t_1)\Delta t)(I + A'(t_0)\Delta t), \tag{31}
\]

where \(A'(t_i) = i\lambda A_\mu'(x(t_i)) \frac{dx^\mu}{dt}(t_i)\), and \(\Delta t \to 0\). With the two-point spinor property of an infinitesimal parallel transporter in the form of the one-order expansion along the transport path, it is equal to

\[
(U(t_{n-1}) + \Delta U(t_{n-1}))(I + A(t_{n-1})\Delta t)U^{-1}(t_{n-1}) (U(t_{n-2}) + \Delta U(t_{n-2}))(I + A(t_{n-2})\Delta t) \cdots \nonumber \\
\cdots (U(t_0) + \Delta U(t_0))(I + A(t_0)\Delta t)U^{-1}(t_0),
\]

where \(U(t) = g(x(t))\). If we take

\[
U(t_{i+1}) = U(t_i) + \Delta U(t_i) = U(t_i) + \frac{dU}{dt}(t_i)\Delta t, \tag{32}
\]

for \(i = 1, 2, \cdots, n - 1\), then all the group elements in the middle will be canceled in couples and a two-point spinor \(\Phi(A_\mu(x); t, t_0)\) will be obtained.

To check the validity of the argument, we can study the relation of the gauge group elements at the two end points of the transport path. The simplest case is a smooth group element function \(U(t)\) on the transport path, i.e. there are infinite-th partial derivatives with respect to spacetime variables at each point of the path, \([t_0, t] \to M\). According to Eq. (32), the gauge group elements are related by iterative one-order expansion along the transport path, so we obtain the following relation (see Appendix B) of the gauge group elements at the end points of a smooth transport path:

\[
U(t) = U(t_{n-1}) + \frac{dU}{dt}(t_{n-1})\Delta t
\]
\[
U(t_{n-2}) + 2 \frac{dU}{dt}(t_{n-2}) \Delta t + \frac{dU^2}{dt^2}(t_{n-2})(\Delta t)^2 = \cdots \\
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{d}{dt} \right)^k U(t_0)(\Delta t)^k,
\]

where \(\Delta t = (t - t_0)/n\). When \(n \to \infty\), it reduces to the Taylor expansion in variable \(t\):

\[
U(t) = U(t_0) + (t - t_0)U'(t_0) + \frac{1}{2!}(t - t_0)^2U''(t_0) + \cdots. \quad (34)
\]

Changing the variable of differentials \(t\) to spacetime variables \(x\), we have

\[
g(x) = g(x_0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( (x^\mu - x_0^\mu) \frac{\partial}{\partial x^\mu} \right)^k g(x_0).
\]

This is not definitely consistent with the Taylor expansion of \(g(x)\) directly with respect to spacetime variables \(x\), if \(g(x)\) is only a \(C^k(k < \infty)\) function, a general local gauge transformation, on the subset of \(M\), into which the parallel transport path is embedded. The relation of the Taylor expansion in path parameter \(t\) and spacetime variables \(x\) is given in Appendix C. From the above discussion it can be seen that it is improper to treat the parallel transport of matter field on \(M\) as a one-dimensional problem in the parameter space \([t_0, t]\).

### 4.3 Condition for the preservation of parallel transport under gauge transformation

First we start with the following equation in the parameter space \([t_0, t]\):

\[
\psi(t) - \psi(t_0) = P \exp \left( \int_{t_0}^{t} ds A(s) \right) \psi(t_0) - \psi(t_0) \\
= \int_{t_0}^{t} ds A(s) \left( I + \int_{t_0}^{s} ds' A(s') + \cdots \right) \psi(t_0) \\
= \int_{t_0}^{t} ds A(s) P \exp \left( \int_{t_0}^{s} ds' A(s') \right) \psi(t_0) = \int_{t_0}^{t} ds A(s) \psi(s), \quad (36)
\]

where \(A(s) = i \lambda A_\mu(x(s))dx^\mu(s)/ds\). It is the integral equation of the parallel transport of fermions and is equivalent to Eq. (12). If the concerned parallel transporter is a two-point spinor, this equation should transform covariantly under gauge transformation. On the spacetime manifold \(M\), the transformed Eq. (36) after a coordinate transformation in principal bundle space (Eq. (2)) is

\[
\psi'(x) = \psi'(x_0) + i \lambda \int_{x_0}^{x} dz A'_\mu(z) \psi'(z)
\]
Figure 1: A spinor $\psi_0$ at $x_1$ is parallelly transported to $x_2$ along a family of curves connecting the two points.

$$= g(x_0)\psi(x_0) + \int_{x_0}^{x} dz^\mu (i\lambda g(z)A_\mu(z) + \partial_\mu g(z))\psi(z),$$  \hspace{1cm} (37)

if $\psi(z)$ produced by parallel transport, a $C^\infty$ extension of $\psi(s)$ to $M$ (the transport curves under consideration are embedding ones), transforms covariantly. To treat the integral in the equation as a one-parameter integral in $[t_0, t]$, we have

$$\psi'(t) = U(t_0)\psi(t_0) + \int_{t_0}^{t} ds \left( U(s)A(s)\psi(s) + \frac{d}{ds}(U(s)\psi(s)) \right) - \int_{t_0}^{t} ds \ U(s)\frac{d}{ds}\psi(s)$$

$$= U(t)\psi(t) = U(t)(\psi(t_0) + \int_{t_0}^{t} dsA(s)\psi(s)), \hspace{1cm} (38)$$

where $U(t) = g(x(t))$ and Eq. (12) has been considered. So the integral equation for the parallel transport of fermions appears to transform covariantly under gauge transformation.

However, as we will clarify as follows, such a treatment actually leads to the contradiction with facts. Let’s study a situation described by the figure 1. The corresponding integral equations on $\gamma_1$ and $\gamma_2$ are given as follows:

$$Pexp \left( \int_{\gamma_1} dsA(s) \right) \psi(t_0) - \psi(t_0) = \int_{\gamma_1} dsA(s)\psi(s), \hspace{1cm} (39)$$

$$Pexp \left( \int_{\gamma_2} dsA(s) \right) \psi(t_0) - \psi(t_0) = \int_{\gamma_2} dsA(s)\psi(s). \hspace{1cm} (40)$$
We have the following relation after subtracting Eq. (40) from Eq. (39)

\[
P_{\text{exp}} \left( \int_{\gamma_1} ds A(s) \right) \psi(t_0) - P_{\text{exp}} \left( \int_{\gamma_2} ds A(s) \right) \psi(t_0) = \int_{\gamma_1} ds A(s) \psi(s) - \int_{\gamma_2} ds A(s) \psi(s)
\]

\[
= \int_{\gamma_2 \circ \gamma_1} ds A(s) \psi(s) \tag{41}
\]

The left hand side of it can be rewritten as

\[
P_{\text{exp}} \left( \int_{\gamma_1} ds A(s) \right) (I - P_{\text{exp}} \left( \int_{\gamma_1} ds A(s) \right) \times P_{\text{exp}} \left( \int_{\gamma_2} ds A(s) \right) ) \psi(t_0)
\]

\[
= P_{\text{exp}} \left( \int_{\gamma_1} ds A(s) \right) ( - \int_{\gamma_1 \circ \gamma_2} ds A(s) \psi(s) ), \tag{42}
\]

if Eq. (36) is considered. The final equation is therefore given as

\[
(P_{\text{exp}} \left( \int_{\gamma_1} ds A(s) \right) - I) \int_{\gamma_2 \circ \gamma_1} ds A(s) \psi(s) = 0. \tag{43}
\]

It indicates a trivial result for the parallel transport of \( \psi_0 \) in the parameter space \([t_0, t]\), which goes against facts. Thus we have shown again that the differential form \( A(t) dt \) in the integrals of all the concerned equation have to be interpreted not only as the one-form in the parameter space \([t_0, t]\) but also as the one-form \( i\lambda A_\mu(x) dx^\mu \) on spacetime manifold M itself.

Next we will reveal the actual condition for the two-point spinor property of parallel transporter. We consider the situation in the above figure again and set up a coordinate of a homotopic curve family, \( x^\mu(t, s), \) on the surface surrounded by \( \gamma_1 = x^\mu(t, 0) \) and \( \gamma_2 = x^\mu(t, 1). \) At the two end points there is \( \frac{\partial}{\partial s} x^\mu(0, s) = \frac{\partial}{\partial s} x^\mu(1, s) = 0. \) Due to the parallel transport of \( \psi_0 \) along the family of curves, there is a fermion field \( \psi(t, s) \) on the surface. If the parallel transporters \( \Phi_\gamma \) are two-point spinors, the difference between the transported fermions along \( \gamma_1 \) and \( \gamma_2 \) respectively to \( x_2 \) will transform covariantly under gauge transformation:

\[
\Delta \psi'(x_2) = g(x_2) \Delta \psi(x_2) = U(1, s)(\Phi_{\gamma_2} - \Phi_{\gamma_1}) \psi_0 \tag{44}
\]

According to the Stokes theorem in the surface parameter space \( \{t, s\}, \)

\[
\Delta \psi(x_2) = ig \int_0^1 A_\mu(x(t, 1)) \psi(t, 1) \frac{\partial x^\mu}{\partial t} dt - ig \int_0^1 A_\mu(x(t, 0)) \psi(t, 1) \frac{\partial x^\mu}{\partial t} dt
\]

\[
= ig \int_{\gamma_1 \circ \gamma_2} A_\mu(x(t, s)) \psi(t, s) \frac{\partial x^\mu}{\partial t} dt
\]
\[
= \int_0^1 ds \int_0^1 dt \, ig\left( \frac{\partial}{\partial t}(A_\mu(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial s} - \frac{\partial}{\partial s}(A_\mu(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial t} \right) \}. \tag{45}
\]

In the transformed coordinate in principal bundle space (after the action in Eq. (2)), \(\Delta \psi'(x_2)\) is expressed similarly as
\[
\Delta \Psi'(x_2) = ig \oint_{\bar{\gamma}_{12}} A'_\mu(x(t,s))\psi'(t,s) \frac{\partial x^\mu}{\partial t} dt
\]
\[
= \int_0^1 ds \int_0^1 dt \, ig\left( \frac{\partial}{\partial t}(g(x(t,s))A_\mu(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial s} \right)
- \frac{\partial}{\partial s}(g(x(t,s))A_\mu(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial t} \right) \}
+ \int_0^1 ds \int_0^1 dt \left( \frac{\partial}{\partial t}(\partial_\mu g(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial s} \right)
- \frac{\partial}{\partial s}(\partial_\mu g(x(t,s))\psi(t,s)) \frac{\partial x^\mu}{\partial t} \right) \}, \tag{46}
\]

if \(\psi(t,s)\) is supposed to transform covariantly. Only when the \(\Delta \psi'(x_2)\) calculated in the transformed bundle coordinate (Eq. (46)) and that transformed from \(\Delta \psi(x_2)\) in Eq.(45) by \(g(x_2)\) are equal, will the parallel transporters under consideration be truly two-point spinors. In other words it is the necessary condition for the two-point spinor property of parallel transporter. From explicit calculation, however, their equivalence is true only in the following two situations:

1) \(g(x) = const\), a global gauge transformation GT1;
2) a pure gauge situation with \(F_{\mu\nu}(x) = 0\) identically.

In the second case, the transported fermion satisfies the following partial differential equation.
\[
\partial_\mu \psi(x) = igA_\mu(x)\psi(x), \tag{47}
\]

then the general covariance of the integral equation for the parallel transport of fermions can be restored by means of integral by parts with respect to \(\partial_\mu\) in Eq. (37). It is in contrast to the general covariance of the differential equation (Eqs. (12) and (22)) for the parallel transport of fermions that the general covariance of its equivalent integral formalism (Eq. (36)) under gauge transformation is conditional. As has been given by a similar result of the parallel transport of vector field in tangent bundle space based on a Riemannian spacetime [13, p167], the condition \(F_{\mu\nu}(x) = 0\) identically is the sufficient and necessary condition for the equivalence of the ordinary differential equation Eq. (12) and the partial differential
equation Eq. (47). i.e. that for the integrability of parallel transporter \( \Phi_\gamma \), and, through the previous discussion, it is also the sufficient and necessary condition for the preservation of parallel transport of fermions under a general \((C^1)\) local gauge transformation.

### 4.4 Direct verification

The restriction on preservation of parallel transport of fermions in a non-trivial gauge field can be shown directly, if we treat the integrals in the following supposed equation as line integral in spacetime manifold \( M \) rather than the one-parameter integral in the parameter space \([t_0, t]\) itself. It is done by comparing the \((i\lambda)^n\) order terms on the both sides of the supposed equation:

\[
Pexp \left( i\lambda \int_{x_0}^x dz^\mu A_\mu'(z) \right) = g(x)Pexp \left( i\lambda \int_{x_0}^x dz^\mu A_\mu(z) \right) g^{-1}(x_0),
\]

where

\[
g(x) = \exp \left( i\lambda \sum_{a=1}^{\text{dim}G} \omega^a(x)T^a \right) = \exp \left( i\lambda M(x) \right) = I + i\lambda M(x) + \cdots,
\]

and

\[
g^{-1}(x_0) = \exp \left( -i\lambda \sum_{a=1}^{\text{dim}G} \omega^a(x_0)T^a \right) = \exp \left( -i\lambda M(x_0) \right).
\]

After the gauge transformation of \( A_\mu(z) \) is substituted into the left hand side of Eq. (48), we compare the terms of the order \((i\lambda)^2\). The terms of the order with the permutation \( MA_\mu(z) \) on the left hand side of Eq. (48) are given as follows:

\[
(i\lambda)^2 \int_{x_0}^x dz^\mu M(z)A_\mu(z) + (i\lambda)^2 \int_{x_0}^x dM(z) \int_{x_0}^z (dz^\mu)'A_\mu(z') = (i\lambda)^2 \int_{x_0}^x dz^\mu M(z)A_\mu(z)
\]

\[
- (i\lambda)^2 \int_{x_0}^x M(z)d \left( \int_{x_0}^z (dz^\mu)'A_\mu(z') \right) + (i\lambda)^2 M(x) \int_{x_0}^x dz^\mu A_\mu(z).
\]

If the integrals here are interpreted as one-variable integrals in \([t_0, t]\), i.e. \( i\lambda \int_{x_0}^z (dz^\mu)'A_\mu(z') = \int_{t_0}^s ds' A(s') \) and \( A(s)ds = i\lambda A_\mu(z)dz^\mu \), there is

\[
d \left( \int_{t_0}^s ds' A(s') \right) = A(s)ds
\]

as an identity, and the two sides of Eq. (48) will agree up to all orders, as it is always true for the one-dimensional situation. However, as a matter of fact, Eq. (50) implies

\[
d \left( i\lambda \int_{x_0}^z (dz^\mu)'A_\mu(z') \right) = i\lambda A_\mu(z)dz^\mu,
\]

\( ^3 \text{M means the part containing factors such as } dM(z) \text{ and } M(z), \text{ and } A \text{ means those with } A(z) \text{ and } \int_{x_0}^z dz^\mu A_\mu(z). \)
which is true on $M$ only when $A_{\mu}(z)dz^\mu$ is an exact form, i.e. \( \partial_{\mu}A_{\nu}(z) - \partial_{\nu}A_{\mu}(z) = 0 \) identically (see e.g. Ref.[14, p. 10]). Therefore, the consistency for the validity of Eq. (48) in $[t_0, t]$ with that on $M$ doesn’t always hold, if the gauge field under consideration is a non-trivial one.

5 Abelian gauge theory and Aharonov-Bohm effect

The gauge field theory proper with gauge group $U(1)$ is a special type in our discussion. The quantized $U(1)$ gauge field (Quantum Electrodynamics) well describes electromagnetic interaction in nature. The variation of its Lagrangian density

\[
L(x) = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \bar{\psi}(x)(i\gamma^{\mu}D_{\mu} - m)\psi(x)
\]

with respect to field $\bar{\psi}(x)$ leads to Dirac equation

\[
(i\gamma^{\mu}D_{\mu} - m)\psi(x) = 0,
\]

where $D_{\mu} = \partial_{\mu} - ieA_{\mu}$. In the region of spacetime where field strength $F_{\mu\nu}(x)$ identically vanishes, it can be reduced to the equation of free field

\[
(i\gamma^{\mu}\partial_{\mu} - m)\psi_f(x) = 0
\]

through a phase factor called ‘Schwinger String’ [15]:

\[
\psi(x) = \exp\left(ie\int_{x_0}^{x} A_{\mu}(z)dz^\mu\right)\psi_f(x).
\]

In the non-relativistic limit, this mathematical transformation corresponds to a physical phenomena. It is the famous Aharonov-Bohm effect that directly demonstrates the effect of gauge potential $A_{\mu}$ on electron field [16,17]. The experiment is arranged to allow two beams of electron from a single source to pass either side of a round coil and impinge on a screen behind. The magnetic field is restricted to within the coil so that field tensor $F_{\mu\nu}$ in where electrons pass is identically zero. If the origin of the space coordinate is chosen at the center of the coil, the one-form of the gauge field will be $A_{\mu}(x)dx^\mu = -y/(x^2+y^2)dx + x/(x^2+y^2)dy$, which cannot be transformed to zero by a $C^\infty$ gauge transformation because it is not the differential of a function over $M/(0, 0)$ [14, p. 6]. Hence we have a double-connected region on $M$. The interference pattern at each point $x$ on the screen is determined by the amplitude,

\[
|\psi_{f_1}(x)\exp\left(ie\int_{\gamma_1} dz^\mu A_{\mu}(z)\right) + \psi_{f_2}(x)\exp\left(ie\int_{\gamma_2} dz^\mu A_{\mu}(z)\right)|^2
\]
\[ |\psi^f_1(x)\exp\left(ie/2 \int_{\Sigma} d\sigma^{\mu\nu}(z)(\partial_{\mu}A_{\nu}(z) - \partial_{\nu}A_{\mu}(z)) \right) + \psi^f_2(x)|^2, \] (56)

where the integral domain is over the cross section of the coil, \(\Sigma\), since the field strength vanishes identically outside. The amplitude is invariant under the local gauge transformations:

\[ \psi'(x) = \exp(i\alpha(x))\psi(x), \] (57)

\[ A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x), \] (58)

because the field strength \(F_{\mu\nu}(x)\) is gauge invariant.

If a double-connected region can be realized in a non-Abelian situation, i.e. the magnetic field in the experiment would be taken place by some non-Abelian gauge field and the electron field by some fermion field coupling to the non-Abelian gauge field, the corresponding amplitude of Eq. (56) will be

\[ |\psi^f_1(x)|^2 + |\psi^f_2(x)|^2 + |\psi^\dagger_f_2(x)\exp\left(i\lambda \int_{\gamma_1} dz^{\mu}A_\mu(z)\right)\psi^f_1(x) + h.c.|^2, \] (59)

where \(\Gamma = \gamma_1 \cup \bar{\gamma}_2\) with the point \(x_0\) as the initial and final point. The phase factor here can be calculated with the help of ‘non-Abelian Stokes theorem’ [18,19]:

\[ P\exp\left(i\lambda \int_{\gamma_1} dz^{\mu}A_\mu(z)\right) = P\exp\left(i\lambda \int_{\gamma_2} A_\mu(z)dz^{\mu}\right) \times P\exp\left(i\lambda \int_{x_0}^{y} A_\mu(z)dz^{\mu}\right), \] (60)

where \(y\) is an arbitrary reference point on \(\Sigma\), and

\[ F_{\mu\nu}(y, z) = P\exp\left(i\lambda \int_{z}^{y} A_\mu dx^{\mu}\right)F_{\mu\nu}(z)P\exp\left(i\lambda \int_{y}^{z} A_\mu dx^{\mu}\right). \]

With regard to Eq. (59), the contribution to the amplitude involving the phase factor around the coil becomes

\[ |\psi^\dagger_f_2(x)g^{-1}(x)\exp\left(i\lambda \int_{\Gamma'} A'_\mu(z)dz^{\mu}\right)g(x)\psi^f_1(x) + h.c.|^2, \]
after the local gauge transformations GT2a and GT2b induced by $T_{\alpha\beta}(x)$. Here $\Gamma' = C \cup S^1 \cup \bar{C}$, with $S^1$ the boundary of the coil and $C$ arbitrary path connecting $x_0$ and $S^1$. It is not equal to the corresponding part in Eq. (59) because $x \not\in \Gamma'$ and the local gauge transformations of the phase factor only involve the gauge group elements on the path $\Gamma'$, so the interference pattern is therefore invariant only under global gauge transformation GT1. Without the consideration of the confinement of fermions in non-Abelian gauge field theories, this imaginary experiment demonstrates that only Abelian gauge field theory is a perfect locally invariant gauge theory in the sense of the definition in Sect.(2), i.e. a coordinate-free description of physics can be realized only in $U(1)$ bundle space.

6 Discussions

Both in model construction [20-22] and lattice simulation [23], non-perturbative approaches to QCD widely involves the application of parallel transporter. The closed parallel transporter (Wilson loop) is an important tool for the study of the non-perturbative phenomena such as the confinement of quarks (see, e.g. Ref.[9, chap. 5]). However, the explicit broken of the local gauge symmetry of the quantities constructed with this non-local operator, parallel transporter, was not understood before. In the perturbative domain, an infinitesimal parallel transporter can be regarded as a good approximation to the real one, because of the asymptotic freedom property for non-Abelian gauge theories. Its two-point spinor property preserves the local gauge invariance GT2 in the perturbative domain. Whether the loss of two-point spinor property of parallel transporter in non-perturbative domain implies some physics requires our further study.

Another issue closely related to gauge symmetry is general covariance in the geometrized theories of gravitation, e.g. general relativity. In fact the concept of gauge invariance originated from H. Weyl’s attempt [24] to unify gravitation and electromagnetism in the framework of Riemannian geometry. General relativity was first treated as a gauge theory of orthonormal frame bundle with the homogeneous Lorentz group as the structure group in Ref.[25]. For a conceptual development of gauge concept and geometrization of fundamental interactions, see Ref.[26]. The largest symmetry in spacetime theories is the invariance of the local field equations under local coordinate transformation, and people used to believe that
this group of differmorphism comprises the gauge freedom of any theory formulated in terms of tensor fields on a spacetime manifold $M$, and therefore all differmorphic models of any spacetime theory represent one and the same physical situation. Hence the term **general covariance** for a physical theory usually refers to the invariance of local field equations under coordinate transformation in tangent bundle space. For a genuine equivalence of physics, however, a tensor field produced by parallel transport should also be covariant under the related transformation law. For example, if a physical process involves the parallel transport of a vector field $n^\mu(x)$ between two points, say $x_1$ and $x_2$, on spacetime manifold, the parallel transporter, $P exp \left( - \int_{x_1}^{x_2} \Gamma^{\sigma}_{\mu\nu}(x) dx^\nu \right)$, of the vector field must be a two-point tensor [5,6]. From the argument in the previous discussion it is true only when the curvature tensor of the spacetime manifold vanishes identically. Generally speaking, the parallel transport of tensor fields should be an ‘absolute element’ [27] in any spacetime theory, i.e. the concerned spacetime transformation should map the parallel transporter as a geometrical object in one model to the corresponding ones in all its equivalent models, so the models should be geometrically rather than topologically equivalent. This requirement imposes stronger condition for the general covariance of a theory than that for the form invariance of local Lagrangian density. Following the argument in this paper we can show the loss of the property in the general situation through the analysis of parallel transporters for tensor fields. When studied from integral point of view, the two most fundamental symmetries, the general covariance in principal and tangent bundle space, are shown to be much more restricted than what we see from differential point of view.

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Appendix A

In this appendix we prove the two-point spinor property for parallel transporter when the gauge field $A_\mu(x)$ can be transformed to zero identically. It is in fact to prove
\[
\text{Pexp} \left( i \lambda \int_{x_0}^x dz^\mu A_\mu(z) \right) = \text{Pexp} \left( \int_{x_0}^x dz^\mu \partial_\mu g(z) g^{-1}(z) \right) = g(x)g^{-1}(x_0), \tag{A.1}
\]
where $g(z) = \exp(i \lambda \omega^a(z)T^a) = \exp i \lambda M(z)$. We have
\[
\partial_\mu g(z) g^{-1}(z) = \partial_\mu \left( \sum_{k=0}^{\infty} \frac{1}{k!} (i \lambda M(z))^k \right) \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (i \lambda M(z))^k \right). \tag{A.2}
\]
Substituting it into (A.1), we obtain infinitely many terms containing $M(z)$ as the integral variable, such as $\int d(M^l/l!)(-1)^n/n! M^n)$. All these terms can be grouped into the $(n+1)$-element sets of the order $(i\lambda)^{l+n+x}$ in the form:
\[
\int \cdots \int_{t_0}^{s_{i-1}} d(M^l/l!) \frac{(-1)^k}{k!} M^k \int_{t_0}^{s_i} d \left( \frac{1}{(n-k)!} M^{n-k} \right) \int \cdots,
\]
for $k = 0, 1, \ldots, n$, where the common factors $d \left( M^l/l! \right)$ contributes the order $(i\lambda)^l$, and the common part $\int \cdots$ the order $(i\lambda)^x$. We find that the terms in such a group cancel all together, because $\sum_{k=0}^{n} (-1)^k(1/k!)(1/(n-k)! = 0$ from the identity
\[
(1 - 1)^n = n! \left( \sum_{k=0}^{n} (-1)^k \frac{1}{k! (n-k)!} \right) = 0. \tag{A.3}
\]
In this way only the terms on the right hand side of B.(1) will be left.

Appendix B

Here Eq. (33) is proved by the induction on $n$: If $k = 1$, then for any differentiable function $g(t)$ there is
\[
g(t + \Delta t) = g(t) + dg/dt(t)\Delta t \tag{C.1}
\]
under one-order approximation. Suppose Eq. (33) holds for $k = n - 1$. Then we have
\[
g(t + (n-1)\Delta t) = \sum_{k=0}^{n-1} \binom{n}{k} (d/dt)^k g(t_0)(\Delta t)^k, \tag{C.2}
\]
and
\[
dg/dt(t + (n-1)\Delta t) = \sum_{k=0}^{n-1} \binom{n}{k} (d/dt)^k dg/dt(t_0)(\Delta t)^k. \tag{C.3}
\]
When \( k = n \), it immediately follows that

\[
g(t + n\Delta t) = g(t + (n - 1)\Delta t) + dg/dt (t + (n - 1)\Delta t) \Delta t.
\]  
(C.4)

Substituting B.2 and B.3 into B.4 and considering the relation

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},
\]  
(C.5)

we obtain Eq. (33) for \( k = n \) after the rearrangement of the terms.

Appendix C

First we derive Eq. (35) from Eq. (34). Remember that we suppose \( g(x) \) has infinite-th partial derivatives at each point on the path \( \Gamma, [t_0, t] \rightarrow M \). After the variable \( t \) of the differentials is changed to \( x \), there are

\[
\frac{dU}{dt}(t) = \frac{dx^\mu}{dt}(t)\partial_\mu g(x),
\]

\[
\frac{d^2U}{dt^2}(t) = \frac{d^2x^\mu}{dt^2}(t)\partial_\mu g(x) + \frac{dx^\mu}{dt}(t)\frac{dx^\nu}{dt}(t)\partial_\mu \partial_\nu g(x),
\]

etc. Substituting these results into Eq. (34), we obtain the coefficient of \( \partial_\mu g(x_0) \) as follows:

\[
(t - t_0)\frac{dx^\mu}{dt}(t_0) + \frac{1}{2!}(t - t_0)^2\frac{d^2x^\mu}{dt^2}(t_0) + \cdots = x^\mu - x_0^\mu,
\]
since the path is a smooth one. All the terms of the higher derivatives in Eq. (35) are obtained in the same way.

On the other hand, if there are continuous n-th partial derivatives of \( g(x) \) at each point of \( W = \prod_{\mu=1}^{\dim G} (x^\mu - x_0^\mu) \subset M \), then it can be expanded to the n-th order of \( (x^\mu - x_0^\mu) \) with respect to the spacetime variables. However, with some points \( z \notin \Gamma \) but \( z \in W \), at which the continuous partial derivatives exist only up to the \( m(< n - 1) \)-th, \( g(x) \) cannot be certainly expanded in the form:

\[
g(x) = g(x_0) + \sum_{k=1}^{n-1} \frac{1}{k!} \left( (x^\mu - x_0^\mu) \frac{\partial}{\partial x^\mu} \right)^k g(x_0) + \frac{1}{n!} \left( (x^\mu - x_0^\mu) \frac{\partial}{\partial x^\mu} \right)^n g(\xi),
\]

where \( \xi \) is some point in \( W \).

As an example, we perform the local gauge transformation \( g(x, y) = \exp \left( \sum_{a=1}^{\dim G} (\alpha^a(x, y)) T^a \right) \), where \( \alpha^a(x, y) \) are \( C^\infty \) functions, on two-dimensional plane X-Y. \( g(x, y) \) is a \( C^1 \) function over the domain \( W = [0, 1] \times [0, 1] \) if there are \( \gamma^a \cap W \neq \emptyset \) for some curves \( \gamma^a \) determined by the
equations, $\alpha^{a}(x, y) = 0$. Consider a smooth path $\Gamma$ connecting the point $(0, 0)$ to $(1, 1)$, with $\Gamma \cap \gamma^{a} = \emptyset$ for all $\gamma^{a}$s. Then $U(t)$ is $C^{\infty}$ on $\Gamma$. From the Taylor expansion of $U(t)$ we have

$$g(1, 1) = g(0, 0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^{k} g(0, 0),$$

whereas the Taylor expansion of a $C^{1}$ function $g(x, y)$ on $W$ only definitely gives

$$g(1, 1) = g(0, 0) + (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})g(\xi_{1}, \xi_{2}),$$

where $(\xi_{1}, \xi_{2})$ is some point in $W$. Therefore, it is concluded that the coincidence of the two Taylor expansions Eq. (34) and Eq. (35) requires the same behavior of $g(x)$ on $\Gamma$ and $W$. 
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