DEFORMATION OF LEBRUN’S ALE METRICS WITH NEGATIVE MASS

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ABSTRACT. In this article we investigate deformations of a scalar-flat Kähler metric on the total space of complex line bundles over \( \mathbb{CP}^1 \) constructed by C. LeBrun. In particular, we find that the metric is included in a one-dimensional family of such metrics on the four-manifold, where the complex structure in the deformation is not the standard one.

1. Introduction

In 1988 C. LeBrun [15] explicitly constructed an example of anti-self-dual (ASD) Kähler metric on the total space of the complex line bundle \( \mathcal{O}(-n) \) over \( \mathbb{CP}^1 \), which is asymptotically locally Euclidean (ALE) and whose mass is negative when \( n > 2 \). Significance of the metric is not only in that it provides counter-example to the generalized positive action conjecture, but also in that, it naturally appears in a typical example ([16, Section 5]) of degeneration of compact ASD manifolds as one of the two pieces (see also [3, 12]). In this degeneration, the other piece is an ALE hyper-Kähler manifold constructed by Gibbons-Hawking [4] and Hitchin-Kronheimer [5] [14].

Because the LeBrun metrics can be thought as a natural generalization of the Burns metric and the Eguchi-Hanson metric on \( \mathcal{O}(-1) \) and \( \mathcal{O}(-2) \) respectively, and since these two metrics are rigid as ALE ASD metrics, one might think that the LeBrun’s metrics could not be deformed as an ASD structure. However, in a very recent work, by establishing an index theorem for the deformation complex on compact ASD orbifolds, J. Viaclovsky [27] has shown that the versal family of the LeBrun’s ASD structure on \( \mathcal{O}(-n) \) is non-trivial, and that the moduli space of ASD structures near the LeBrun’s one is at least \( (4n - 12) \)-dimensional. The purpose of the present paper is to answer some questions which naturally arise from that work.

We recall that from the ALE condition, LeBrun’s metric can be conformally compactified by adding one point at infinity, and consequently we obtain an ASD structure on a compact orbifold \( \widetilde{\mathcal{O}}(-n) \). We call this ASD orbifold as the LeBrun orbifold. In Section 2 by making use of the twistor space of the LeBrun orbifold, we reprove that the parameter space of the versal family for the LeBrun orbifold is smooth and real \( (4n - 8) \)-dimensional. Here we are considering versal family of ASD structures on the fixed orbifold \( \widetilde{\mathcal{O}}(-n) \). Because the LeBrun orbifold has an effective U(2)-action [15], the parameter space of the above versal family also has a natural U(2)-action. Then by determining this U(2)-action and classifying all U(2)-orbits whose dimension is less than four, we prove the following result:

**Theorem 1.1.** Let \( n \geq 3 \), \( B \) be an open neighborhood of the origin in \( \mathbb{R}^{4n-8} \), and \( \{[g_t] | t \in B \subset \mathbb{R}^{4n-8} \} \) be the versal family of ASD structures on the orbifold \( \widetilde{\mathcal{O}}(-n) \) for the LeBrun’s

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ASD structure, so that \([g_0]\) is equal to the LeBrun’s ASD structure. If we take \(B\) sufficiently small, the following holds.

(i) If \(n = 3\), for any \(t \in B\) with \(t \neq 0\), we have \(\text{Aut}_0[g_t] \simeq \text{U}(1)\), and the moduli space is 1-dimensional at \([g_t]\).

(ii) If \(n \geq 3\), there exist \(U(2)\)-invariant, mutually disjoint connected subsets \(B_1, \cdots, B_{[n/2]}\) of \(B\) such that \(t \in B_1 \cup \cdots \cup B_{[n/2]}\) implies \(\text{Aut}_0[g_t] \simeq \text{U}(1)\). Moreover, the moduli space of these \(U(1)\)-invariant ASD structures is 1-dimensional at \([g_t]\) if \(t \in B_1\), 3-dimensional at \([g_t]\) if \(t \in B_2 \cup B_3 \cup \cdots \cup B_{[n/2]}\).

(iii) If \(n > 3\) and \(t \not\in B_1 \cup B_2 \cup \cdots \cup B_{[n/2]}\), then \(\text{Aut}_0[g_t] = \{e\}\) and the dimension of the moduli space is \((4n - 12)\)-dimensional at \([g_t]\).

(iv) If \(n = 4\), there exists another \(U(2)\)-invariant connected subset \(B_0\) of \(B\) for which the following holds: \(t \in B_0\) implies \(\text{Aut}_0[g_t] \simeq \text{SU}(2)\), and the moduli space of these \(SU(2)\)-invariant ASD structures is 1-dimensional at \([g_t]\). Further if \(t \not\in B_0 \cup B_1 \cup B_2\), then the moduli space is 4-dimensional at \([g_t]\).

Here, for a conformal structure \([g]\), \(\text{Aut}_0[g]\) denotes the identity component of the conformal automorphism group of \([g]\), and for a number \(k\), \([k]\) means the largest integer not greater than \(k\). Theorem 1.1 classifies all small deformations of the LeBrun’s ASD structure which are equivariant with respect to a subgroup of \(U(2)\) of positive dimension, and gives an answer to a question by Viaclovsky [27, 1.4. Question (3)(4)] regarding deformations of the LeBrun orbifold. For the \(SU(2)\)-equivariant deformation of the metrics on \(4\)-manifold, it would be interesting to find concrete description of them, under the work of Hitchin [6] concerning \(SU(2)\)-invariant ASD metrics in general.

In Section 3 we study deformations of LeBrun’s ASD orbifold which preserves Kählerity on the smooth locus, again by using twistor space. The key for such investigation is a theorem of Pontecorvo [22] which expresses the Kählerity of an ASD structure in terms of certain divisor on the twistor space. Especially we prove the following result:

**Theorem 1.2.** For any \(n \geq 3\), on the 4-manifold \(\mathcal{O}(-n)\), there exists a one-dimensional smooth family \(\{(J_t, g_t)\}\) of complex structures and ALE, ASD Kähler metrics, which satisfies the following properties:

(i) \(g_0\) coincides with the LeBrun metric, and \(J_0\) is the standard complex structure,

(ii) if \(t \neq 0\), \(g_t\) is not conformal to the LeBrun metric. Further the complex surface \((\mathcal{O}(-n), J_t)\) is biholomorphic to an affine surface in \(\mathbb{C}^{n+1}\). Furthermore, the Kähler surface \((\mathcal{O}(-n), J_t, g_t)\) admits a non-trivial \(U(1)\)-action.

In particular if \(t \neq 0\) the complex structure \(J_t\) on \(\mathcal{O}(-n)\) is different from the standard one. More explicitly, the affine surface in \(\mathbb{C}^{n+1}\) in the theorem can be concretely obtained as follows. Let \(F_{n-2} := \mathbb{P}(\mathcal{O}(n - 2) \oplus \mathcal{O})\) be the ruled surface over \(\mathbb{CP}^1\), and \(\Gamma\) and \(h\) the unique negative section and a fiber of the ruling respectively. Then the linear system \(|\Gamma + (n - 1)h|\) induces an embedding \(F_{n-2} \subset \mathbb{CP}^{n+1}\). Thus if we remove a generic hyperplane section (which is a \((+n)\)-rational curve) from the image of \(F_{n-2}\), we get an affine surface in \(\mathbb{C}^{n+1}\). This is nothing but the affine surface in the theorem. Note that if \(p : \mathbb{C}^{n+1} \to \mathbb{CP}^n\) denotes the projection from the origin, the total space \(\mathcal{O}(-n)\) can be realized as the minimal resolution of the cone for the projection \(p\) over the rational normal curve in \(\mathbb{CP}^n\). Then the smooth affine surface discussed above is obtained by varying the defining (quadratic) equations of the cone and taking a simultaneous resolution of the cone singularity in the family (see [21, §8]).
Theorem 1.2 provides a partial answer to a Viaclovsky’s question [27, 1.4. Question (2)] concerning scalar-flat Kähler deformations of the LeBrun metric. From the proof the family of the Kähler metrics in Theorem 1.2 is obtained from the $U(1)$-equivariant family over the subset $B_1$ in Theorem 1.1 by restricting onto any 1-dimensional linear subspace in $B_1$. We also investigate other non-trivial $U(1)$-equivariant deformations of the LeBrun’s orbifold, which are over $B_2, B_3, \cdots, B_{\lfloor n/2 \rfloor}$ in Theorem 1.1 and show that they do not preserve Kählerity of the metric, in contrast with the above one. Further we also observe that for these deformations the corresponding twistor spaces are non-Moishezon. This would be natural in light of similar phenomena in the case of twistor spaces on $n\mathbb{CP}^2$.

After writing this paper, Michael Lock and Jeff Viaclovsky [18] extended the index theorem in [27] to general compact ASD orbifolds with cyclic quotient singularities, and showed for example that the ALE SFK metrics constructed by Calderbank-Singer [2] on the minimal resolution of the quotient $\mathbb{C}^2/\Gamma$, $\Gamma$ being a cyclic group, admit a non-trivial deformation as ALE ASD metrics. But it is not straightforward to see that the method used in this paper can be applied to the twistor spaces of their spaces, since the singularities on the twistor spaces are not Gorenstein any more (i.e. the canonical divisor is not a Cartier divisor), which makes the key divisor $S$ non-Cartier.

Notation. We write $F_n$ for the ruled surface $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O})$ over $\mathbb{CP}^1$. If $X$ is a subset of a twistor space, we denote by $\overline{X}$ for the image of $X$ under the real structure.

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2. The Kuranishi family of the twistor space

2.1. The twistor space of the LeBrun orbifold. First we briefly recall the LeBrun’s ALE metric with negative mass from [15]. For more details, one can also consult a paper by Viaclovsky [26, Sections 2.3, 5.2]. Fix any integer $n > 2$. On $\mathbb{C}^2$, the metric is written as

$$g_{LB} := \frac{dr^2}{(1 - \frac{1}{r^2})(1 + \frac{1}{r^2})} + r^2 \left[ \sigma_1^2 + \sigma_2^2 + \left(1 - \frac{1}{r^2}\right) \left(1 + \frac{n - 1}{r^2}\right) \sigma_3^2 \right],$$

where $r$ is the Euclidean distance form the origin, and $\sigma_1, \sigma_2, \sigma_3$ are left-invariant coframe of $SU(2) = S^3$. Clearly this metric has singularities at the unit sphere. Let $\zeta := e^{2\pi i/n}$ and consider the action of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ generated by

$$(z, w) \mapsto (\zeta z, \zeta w).$$

After dividing $\mathbb{C}^2$ by this action and resolving the resulting singularity at the origin, the metric (2.1) defines a non-singular Riemannian metric on the total space of the holomorphic line bundle $\mathcal{O}(-n) \to \mathbb{CP}^1$, which is locally asymptotically Euclidean (ALE) at infinity. Moreover, the metric is Kähler with respect to the complex structure on $\mathcal{O}(-n)$. In particular, it is anti-self-dual (ASD). Furthermore from the ALE property, after an appropriate conformal change, the metric extends to a one-point compactification $\overline{\mathcal{O}(-n)} = \mathcal{O}(-n) \cup \{\infty\}$ as an orbifold ASD structure. For brevity we call the orbifold $\overline{\mathcal{O}(-n)}$ equipped with this ASD structure as the LeBrun orbifold.
We also recall that the isometry group of the LeBrun orbifold is the unitary group $U(2)$, and the $U(2)$-action on the open subset $\mathcal{O}(-n)$ is realized from the natural $U(2)$-action on $\mathbb{C}^2$ through the quotient by $\mathbb{Z}_n$ and the minimal resolution.

The twistor space of the LeBrun orbifold is implicitly constructed in his different paper [16, Section 3], and we now recall the construction, according to [11]. Let $n \geq 3$ be an integer as above. Over $\mathbb{CP}^1 \times \mathbb{CP}^1$, take a rank-3 vector bundle $E_n := \mathcal{O}(n-1,1) \oplus \mathcal{O}(1,n-1) \oplus \mathcal{O}$, and consider the associated $\mathbb{CP}^2$-bundle $\mathbb{P}(E_n) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. Let $X \subset \mathbb{P}(E_n)$ be a hypersurface defined by

$$\tag{2.3} xy = (u - v)^n t^2,$$

where $(u,v)$ are non-homogeneous coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $(x,y,t)$ are fiber coordinates on the bundle $E_n$. This is an equation which takes values in the line bundle $\mathcal{O}(n,n)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$. $X$ is equipped with an anti-holomorphic involution. See [11] for its concrete form. Next define divisors $E$ and $\overline{E}$, and a curve $L'_{\infty}$ lying on $\mathbb{P}(E_n)$ by

$$\tag{2.4} E = \{x = t = 0\}, \quad \overline{E} = \{y = t = 0\} \quad \text{and} \quad L'_{\infty} = \{x = y = u - v = 0\}.$$

These are included in $X$, and $E$ and $\overline{E}$ are sections of the projection $X \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$. $L'_{\infty}$ is non-singular and isomorphic to $\mathbb{CP}^1$. Moreover we have $E \cap \overline{E} = \emptyset$ and $(E \cup \overline{E}) \cap L'_{\infty} = \emptyset$. The threefold $X$ has $A_{n-1}$-singularities along the curve $L'_{\infty}$, and this is exactly the singular locus of $X$. By looking the normal bundle in $X$, the section $E$ can be blown-down in a unique way to $\mathbb{CP}^1$ along a projection $\mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, and the same for $\overline{E}$. Let $\mu : X \rightarrow Z_{\text{LB}}$ be the blowdown of $E \cup \overline{E}$ obtained this way, and put $C = \mu(E)$ and $\overline{C} = \mu(\overline{E})$ for the image rational curves. We have $N_C/Z_{\text{LB}} \simeq N_{\overline{C}}/Z_{\text{LB}} \simeq \mathcal{O}(1-n)^{\mathbb{P}^2}$ for the normal bundles. These curves play significant role for studying deformations of $Z_{\text{LB}}$. We write the curve $\mu(L'_{\infty})$ by $L_{\infty}$. Then the variety $Z_{\text{LB}}$ is exactly the twistor space of the LeBrun orbifold $\mathcal{O}(-n)$, the curve $L_{\infty} \subset Z_{\text{LB}}$ is the twistor line over the orbifold point $\infty$, and the anti-holomorphic involution on $Z_{\text{LB}}$ induced from that on $X$ is the real structure (see [11, Theorem 3.3]). For a later purpose we further define other two divisors on $X$ as

$$\tag{2.5} D' = \{x = u - v = 0\}, \quad \overline{D'} = \{y = u - v = 0\}.$$

These are over the diagonal $\Delta := \{u = v\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$, and we have $D' \cap \overline{D'} = L'_{\infty}$. The two divisors $D'$ and $\overline{D'}$ are non-singular and biholomorphic to the ruled surface $\mathbb{P}_n$. Note that these are not Cartier divisors. Because $\Delta$ is a $(1,1)$-curve, the blowdown $\mu$ induces a biholomorphic map $D' \rightarrow \mu(D')$ and $\overline{D'} \rightarrow \mu(\overline{D'})$. We write the images by $D = \mu(D')$ and $\overline{D} = \mu(\overline{D'})$. These are again non-Cartier divisors on $Z_{\text{LB}}$. Since $X$ is a hypersurface in a smooth space, the canonical line bundle $K_X$ naturally makes sense by adjunction formula, and we obtain

$$\tag{2.6} K_X \simeq \pi^* \mathcal{O}(-2, -2) - (E + \overline{E}),$$

where $\pi : X \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ denotes the projection. Noting $D' + \overline{D'} \in |\mu^* \mathcal{O}(D + \overline{D}) - E - \overline{E}|$, this implies $D + \overline{D} \in |K_{\text{LB}}^{1/2}|$ on $Z_{\text{LB}}$. By Pontecorvo’s theorem [22], this divisor gives a reason for the LeBrun metric to be Kähler, in terms of twistor space.

The $U(2)$-action on the LeBrun orbifold naturally induces a holomorphic $U(2)$-action on the twistor space $Z_{\text{LB}}$, which clearly preserves the twistor line $L_{\infty}$. Looking at the normal bundles, we readily see that the divisors $D, \overline{D}$ and the curves $C$ and $\overline{C}$ on $Z_{\text{LB}}$ are invariant under this $U(2)$-action.
2.2. **Locally trivial deformations of the twistor space.** Before going to the actual computations for the twistor spaces, we briefly recall well-known facts regarding deformation theory for general compact complex varieties. For a complex variety $Y$ which may have singularities, let $\Omega_Y$ be the sheaf of Kähler differentials on $Y$ as usual, and we define the tangent sheaf of $Y$ as

$$\Theta_Y := \mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y).$$

If $Y$ is a hypersurface in a smooth space $V$ (just as our $X$ in $\mathbb{P}(\mathcal{E}_n)$), in terms of local defining equation $f = 0$ of $Y$ in $V$, this can be concretely written as

$$\Theta_Y = \{v|_Y \mid v \in \Theta_V, v(f) = 0\}.$$  

(2.7)

Then if $Y$ is compact, the Zariski tangent space of the Kuranishi family of *locally trivial* deformations of $Y$ is identified with the cohomology group $H^1(\Theta_Y)$, and the obstruction space is $H^2(\Theta_Y)$. (In this article we do not need to consider general deformations which are not locally trivial, and so we do not need Ext-groups.) In particular, if $H^2(\Theta_Y) = 0$, the parameter space of the Kuranishi family is identified with an open neighborhood of the origin in $H^1(\Theta_Y)$.

For the present twistor space, we have the following

**Proposition 2.1.** For the twistor space $Z_{LB}$ of the LeBrun orbifold $\hat{O}(-n)$, we have

$$H^2(\Theta_{Z_{LB}}) = 0.$$  

Moreover in terms of the $U(2)$-invariant rational curves $C$ and $\overline{C}$ in $Z_{LB}$, we have a $U(2)$-equivariant isomorphism

$$H^1(\Theta_{Z_{LB}}) \simeq H^1(N_{C/Z_{LB}} \oplus N_{\overline{C}/Z_{LB}}).$$  

(2.8)

In particular, $h^1(\Theta_{Z_{LB}}) = 4n - 8$ (since $N_{C/Z_{LB}} \simeq N_{\overline{C}/Z_{LB}} \simeq \mathcal{O}(1 - n)^{\oplus 2}$).

**Proof.** We imitate the calculations given in [10 Section 1.2]. In this proof for simplicity we write $Z$ for $Z_{LB}$. Let $\Theta_{X,E+\overline{E}}$ denote the subsheaf of the tangent sheaf $\Theta_X$ consisting of vector fields which are tangent to the divisor $E \cup \overline{E}$. (Since $X$ is smooth at $E \cup \overline{E}$, this naturally makes sense.) We define the subsheaf $\Theta_{Z,C+\overline{C}}$ of $\Theta_Z$ in a similar way. Then a computation using local coordinates shows that the blowdown $\mu : X \to Z$ induces a $U(2)$-equivariant isomorphism $\Theta_{X,E+\overline{E}} \simeq \mu^*\Theta_{Z,C+\overline{C}}$. This induces an equivariant isomorphism

$$H^i(\Theta_{X,E+\overline{E}}) \simeq H^i(\Theta_{Z,C+\overline{C}}), \quad i \geq 0.$$  

(2.9)

From the normal bundles of $E$ and $\overline{E}$ in $X$, we readily obtain $H^i(\Theta_{X,E+\overline{E}}) \simeq H^i(\Theta_X)$ for any $i \geq 0$. Hence from (2.9) we have an equivariant isomorphism

$$H^i(\Theta_{Z,C+\overline{C}}) \simeq H^i(\Theta_X), \quad i \geq 0.$$  

(2.10)

For computing the RHS, let $\pi : X \to \mathbb{C}P^1 \times \mathbb{C}P^1 =: Q$ be the projection as in (2.6), and consider the natural homomorphism $d\pi : \Theta_X \to \pi^*\Theta_Q$. Since $\pi$ is clearly submersion outside the line $L_{\infty}$, the support of the cokernel sheaf for $d\pi$ is contained in $L_{\infty}$. Further for a point $x \in L_{\infty}$, the image of the differential $(d\pi)_x : T_xX \to T_{\pi(x)}Q$ is readily seen to be the subspace $T_{\pi(x)}\Delta$, where $\Delta$ is the diagonal of $Q$ as before. Therefore the cokernel sheaf of $d\pi$ is exactly the normal sheaf $N_{\Delta/Q}$ under the identification $L_{\infty} \simeq \Delta$ by $\pi$, and if we write $\mathcal{F}$ for the image sheaf of the homomorphism $d\pi$, we obtain an exact sequence

$$0 \to \mathcal{F} \to \pi^*\Theta_Q \to N_{\Delta/Q} \to 0.$$  

(2.11)

(...)
As the map $H^0(\pi^*\Theta_Q) \to H^0(N_{\Delta/Q})$ is clearly surjective, this sequence easily implies
\begin{equation}
H^i(\mathcal{F}) = 0, \quad i > 0.
\end{equation}
On the other hand for the kernel sheaf $\Theta_{X/Q}$ of $d\pi$, which consists of vertical vector fields, noting $\Theta_{X/Q} \simeq \mathcal{O}_X(E + \mathcal{E})$ from an obvious vertical vector field, and taking the direct image $\pi_*$ of the standard exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(E + \mathcal{E}) \to \mathcal{O}_E(E) \oplus \mathcal{O}_E(E) \to 0$, we obtain $H^i(\Theta_{X/Q}) = 0$ for $i > 0$. Hence from the exact sequence
\begin{equation}
0 \to \Theta_{X/Q} \to \Theta_X \to \mathcal{F} \to 0
\end{equation}
we obtain $H^i(\Theta_X) \simeq H^i(\mathcal{F})$ for any $i > 0$. Hence by (2.12) and (2.10) we get $H^i(\Theta_X) = H^i(\Theta_{Z,C+\mathcal{C}}) = 0$ for any $i > 0$. Therefore the standard exact sequence
\begin{equation}
0 \to \Theta_{Z,C+\mathcal{C}} \to \Theta_Z \to N_{C/Z} \oplus N_{C/Z}^\sigma \to 0
\end{equation}
induces the required isomorphism $H^1(\Theta_Z) \simeq H^1(N_{C/Z} \oplus N_{C/Z}^\sigma)$ as well as the vanishing $H^2(\Theta_Z) = 0$. The last isomorphism is clearly $U(2)$-equivariant, since all isomorphisms and the exact sequences we have used are clearly $U(2)$-invariant.

**Remark 2.2.** For the LeBrun twistor space on $n\mathbb{CP}^2$ constructed in [16], there exist similar curves $C_9$ and $C_0$ and the cohomology group $H^1(\Theta_{Z})$ is a direct sum of $H^1(N_{C_9/Z} \oplus N_{C_9/Z}^\sigma)$ with another cohomology group (see the exact sequence (1.14) in [8]). The latter cohomology group precisely corresponds to deformations as LeBrun twistor spaces. In the present case this cohomology vanishes as in the above proof, and all non-trivial deformations yield non-LeBrun twistor spaces.

Proposition 2.1 means that the parameter space of the Kuranishi family of locally trivial deformations of $Z_{LB}$ may be identified with a neighborhood of the origin in the cohomology group $H^1(N_{C/Z_{LB}} \oplus N_{C/Z_{LB}}^\sigma)$, which is $(4n - 8)$-dimensional over $\mathbb{C}$. Deformations as twistor spaces can be obtained by restricting the Kuranishi family to the real locus $H^1(N_{C/Z_{LB}} \oplus N_{C/Z_{LB}}^\sigma)^\sigma$ in the neighborhood, where $\sigma$ denotes the real structure of $Z_{LB}$. We call this restricted family as the *versal family of twistor spaces* for $Z_{LB}$, and the corresponding family of ASD structures on $\mathcal{O}(-n)$ as the *versal family of ASD structures* for the LeBrun orbifold. We note that there is a natural $U(2)$-equivariant isomorphism
\begin{equation}
H^1(N_{C/Z_{LB}}) \simeq H^1(N_{C/Z_{LB}} \oplus N_{C/Z_{LB}}^\sigma)^\sigma,
\end{equation}
which sends an element $\eta \in H^1(N_{C/Z_{LB}})$ to the pair $(\eta, \sigma^*\eta)$ in the real diagonal. Therefore as far as we are concerned with the versal family of twistor spaces or ASD structures, the $U(2)$-action on one half
\[ H^1(N_{C/Z_{LB}}) \simeq \mathbb{C}^{2n-4} \simeq \mathbb{R}^{4n-8} \]
is fundamental, which we next discuss.

### 2.3. Explicit form of the $U(2)$-action on $H^1$

For expressing the result we consider the natural representation space $\mathbb{C}^2$ of $U(2)$ (acted by the multiplication of matrices), and for each non-negative integer $m$ we write $S^m\mathbb{C}^2$ for the $m$-th symmetric product, where $S^0\mathbb{C}^2$ means the trivial representation on $\mathbb{C}$. For convenience we promise $S^m\mathbb{C}^2 = 0$ if $m < 0$. Let $\mathbb{C}$ be a 1-dimensional representation of $U(2)$ obtained by multiplying the $l$-th power of the determinant, and we write
\[ S^l_l\mathbb{C}^2 := S^m\mathbb{C}^2 \otimes \mathbb{C}^l. \]
Of course we have \( \dim \mathbb{C} S^m \mathbb{C}^2 = m + 1 \) for any \( m \geq 0 \). This is an irreducible representation of \( U(2) \) for any \( m, l \geq 0 \). Under these notations we have

**Proposition 2.3.** Suppose \( n \geq 3 \). Then under the above notation, the \( U(2) \)-action on \( H^1(N_{C/Z_{LB}}) \) is equivalent to the direct sum

\[
S^{n-2} \mathbb{C}^2 \oplus S^{n-4} \mathbb{C}^2.
\]

(Note that the second direct summand vanishes when \( n = 3 \).)

For the proof of Proposition 2.3 we first recall that the \( U(2) \)-action on the open subset \( \mathcal{O}(-n) \) of the LeBrun orbifold is induced from the natural \( U(2) \)-action on \( \mathbb{C}^2 \) via the quotient and minimal resolution. From the \( \mathbb{Z}_n \)-action on \( \mathbb{C}^2 \) in (2.2), we can use the power \( z^n := \xi \) as a fiber coordinate of the line bundle \( \mathcal{O}(-n) \) on an affine open subset of \( \mathbb{C}P^1 \). If we put \( u := w/z \), which is a coordinate on the affine subset, the pair \( (\xi, u) \) can be used as coordinates on an open subset of \( \mathcal{O}(-n) \). Under these coordinates the \( U(2) \)-action on \( \mathcal{O}(-n) \) is explicitly given as

\[
(\xi, u) \mapsto A \left( (\alpha + \beta u)^n \xi, \frac{\gamma + \delta u}{\alpha + \beta u} \right), \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2).
\]

**Proof of Proposition 2.3.** We again write \( Z \) for \( Z_{LB} \). We first note that via the twistor fibration, the divisor \( D \) minus the line \( L_{\infty} \) can be \( U(2) \)-equivariantly identified with the open subset \( \mathcal{O}(-n) \), while the curves \( C \) and \( \overline{C} \) are identified with the zero-section of the line bundle \( \mathcal{O}(-n) \). From the inclusions \( C \subset D \subset Z \), we have the standard exact sequence

\[
0 \to N_{C/D} \to N_{C/Z} \to N_{D/Z}|_C \to 0
\]

for the normal bundles, which is \( U(2) \)-equivariant. Further since \( D + \overline{D} \in |K_{Z_{LB}}^{-1/2}| \) we have

\[
K_D \simeq K_Z + D|_D \simeq (-2D - 2\overline{D} + D)|_D \simeq (-D - 2\overline{D})|_D
\]

and hence, since \( \overline{D} \cap C = \emptyset \), by restricting this to \( C \), \( [D]|_C \simeq K_{D/C}^{-1} \). Therefore

\[
N_{D/Z}|_C \simeq [D]|_C \simeq K_{D/C}^{-1}|_C \simeq N_{C/D} \otimes K_{C}^{-1},
\]

where the last isomorphism is from adjunction formula. All these isomorphisms are clearly \( U(2) \)-equivariant. Therefore since \( N_{C/D} \simeq \mathcal{O}(-n) \) and \( N_{C/D} \otimes K_{C}^{-1} \simeq \mathcal{O}(2 - n) \), as \( n > 2 \), from (2.18) we get an equivariant exact sequence

\[
0 \to H^1(N_{C/D}) \to H^1(N_{C/Z}) \to H^1(N_{C/D} \otimes K_{C}^{-1}) \to 0.
\]

(This is not true if \( n = 2 \).) From this we obtain a \( U(2) \)-equivariant isomorphism

\[
H^1(N_{C/Z}) \simeq H^1(N_{C/D}) \oplus H^1(N_{C/D} \otimes K_{C}^{-1}).
\]

Since the standard open covering of \( C = \mathbb{C}P^1 \) is not \( U(2) \)-invariant, it seems difficult to compute the action on \( H^1 \) by using Cech cohomology (as we did in [3]). So we convert it to that on \( H^0 \) by Serre duality. Namely from (2.21), the \( U(2) \)-action on \( H^1(N_{C/Z}) \) can be identified with the dual of the \( U(2) \)-action on \( H^0(N_{C/D} \otimes K_{C}) \oplus H^0(N_{C/D} \otimes K_{C}^2) \). As \( U(2) \subset O(4) \), the dual action is equivalent to the original one. So we compute the \( U(2) \)-action on the two direct summands. For these we use the above coordinates \( (\xi, u) \).
We first compute the U(2)-action on the former space $H^0(N_{C/D}^{-1} \otimes K_C)$. We put $U := \mathbb{CP}^1 \setminus \{(0 : 1)\}$, where the coordinate $u$ is valid. We use the 1-form $d\xi$ as a frame of the co-normal bundle $N_{C/D}^{-1}$ over $U$. For $A \in U(2)$, by (2.17), we have
\begin{equation}
A^* d\xi = d((\alpha + \beta u)\xi) = n(\alpha + \beta u)^{n-1}\beta du + (\alpha + \beta u)^n d\xi.
\end{equation}
So over the zero-section $\{\xi = 0\}$, we have
\begin{equation}
A^* d\xi = (\alpha + \beta u)^n d\xi.
\end{equation}
On the other hand, for the frame of the canonical bundle $K_C$ on $U$, we use the 1-form $du$. For the pull-back of $du$ under $A$, we have
\begin{equation}
A^* du = d\left(\frac{\gamma + \delta u}{\alpha + \beta u}\right) = \frac{\alpha\delta - \beta\gamma}{(\alpha + \beta u)^2} du.
\end{equation}
By (2.23) and (2.24), the U(2)-action on the line bundle $N_{C/D}^{-1} \otimes K_C$ is given by
\begin{equation}
d\xi \otimes du \xrightarrow{A^*} (\alpha\delta - \beta\gamma)(\alpha + \beta u)^{n-2} d\xi \otimes du.
\end{equation}
Since $\deg(N_{C/D}^{-1} \otimes K_C) = n - 2$, any global section of $N_{C/D}^{-1} \otimes K_C$ can be written as $P(u) d\xi \otimes du$ for some polynomial $P(u)$ with $\deg P(u) \leq n - 2$. For this section, by (2.25), we obtain
\begin{equation}
P(u) d\xi \otimes du \xrightarrow{A^*} (\alpha\delta - \beta\gamma) \left\{(\alpha + \beta u)^{n-2} P\left(\frac{\gamma + \delta u}{\alpha + \beta u}\right)\right\} d\xi \otimes du.
\end{equation}
The ingredient of the brace is a polynomial whose degree is at most $(n - 2)$, and the assignment $P(u) \mapsto (\alpha + \beta u)^{n-2} P((\gamma + \delta u)/(\alpha + \beta u))$ is exactly the $(n - 2)$-th symmetric product of the natural representation of U(2). Therefore noting the determinant in (2.26), the U(2)-action on $H^0(N_{C/D}^{-1} \otimes K_C)$ is equivalent to $S^{n-2}C^2 \otimes C_1 = S_1^{n-2}C^2$. Thus we obtain the first direct summand in (2.16).

The U(2)-action on the latter space $H^0(N_{C/D}^{-1} \otimes K_C^2)$ can be readily obtained from the above computations if we notice that $d\xi \otimes (du)^2$ can be used as a frame over $U$, instead of $d\xi \otimes du$. Namely, by taking the tensor product of (2.23) with the square of (2.24), the U(2)-action on $H^0(N_{C/D}^{-1} \otimes K_C^2)$ is exactly $S_2^{n-4}C^2$. This gives the second direct summand of (2.16), and we have finished a proof of Proposition 2.3.

\textbf{Remark 2.4.} From the above proof, elements of the representation spaces $S_1^{n-2}C^2$ and $S_2^{n-4}C^2$ are polynomials in $u$. As we have put $u = w/z$ where $(z, w)$ is the coordinates on $\mathbb{C}^2$, this is equivalent to saying that the representation spaces are homogeneous polynomials of the two variables $z$ and $w$ (of degree $(n - 2)$ and $(n - 4)$ respectively.) This will be useful later when identifying subgroups of U(2).

\subsection*{2.4. Dimension of the moduli spaces.} Next we would like to compute, by utilizing Proposition 2.3, dimension of the moduli space of ASD structures on the orbifold $\mathcal{O}(-n)$ obtained as small deformations of the LeBrun metric. For this, we need to compute dimension of orbits for the U(2)-action obtained in Proposition 2.3.

The case $n = 3$ is very simple, because the U(2)-action (2.16) is just a 2-dimensional representation $S_1^1C^2$. If we restrict this to the subgroup SU(2), we get a natural representation of SU(2), and any orbit is diffeomorphic to a 3-sphere, except the origin. Moreover
U(2)-orbits and SU(2)-orbits evidently coincide, and the stabilizer subgroup at any point (except the origin) is isomorphic to U(1).

For investigating the case $n > 3$, we next compute dimension of U(2)-orbits in the space $S^m_{l} \mathbb{C}^2$ for any $m \geq 2$ and $l \geq 0$. We identify $S^m_{l} \mathbb{C}^2$ with the space of homogeneous polynomials of $z$ and $w$ of degree $m$; in particular a natural basis is provided by

$z^m, z^{m-1}w, z^{m-2}w^2, \ldots, w^m.$

Note that by Remark 2.24, the variables $z, w$ are identical to the ones in the coordinates $(z, w)$ we used in Section 2.3. We can classify all lower-dimensional orbits as follows:

**Proposition 2.5.** Suppose $m \geq 2$ and $l \geq 0$, and for each integer $j$ with $0 \leq j \leq m$, let $O_j \subset S^m_{l} \mathbb{C}^2$ be the U(2)-orbit going through the monomial $z^{m-j} w^j$. Then we have the following: (i) the orbit $O_j$ is 3-dimensional for any $j$, (ii) the orbits $O_0, O_1, \ldots, O_m$ are all U(2)-orbits in $S^m_{l} \mathbb{C}^2$ which are not 4-dimensional, except the origin, (iii) the coincidence $O_j = O_k$ occurs iff $j = k$ or $j + k = m$ holds.

**Proof.** By thinking $(z, w)$ as homogeneous coordinates on $\mathbb{CP}^1$ we identify the space of homogeneous polynomials of degree $m$ with $H^0(\mathbb{CP}^1, \mathcal{O}(m))$. There is a natural U(2)-action on this space, under which it is identified with $S^m_{l} \mathbb{C}^2$ as a U(2)-module. Hence the U(2)-module $S^m_{l} \mathbb{C}^2$ is identified with $\mathbb{C}_l \otimes H^0(\mathcal{O}(m))$. In order to classify lower-dimensional orbits, it is enough to classify all polynomials $P(z, w) \in \mathbb{C}_l \otimes H^0(\mathcal{O}(m))$ whose stabilizer subgroup is of positive dimension. We assert that this is the case exactly when the set of roots $Z_P := \{(z, w) \in \mathbb{CP}^1 | P(z, w) = 0\}$ satisfies one of the following conditions: (1) $Z_P$ consists of a single point, (2) $Z_P$ consists of two points and moreover they are invariant under the involution $(z, w) \mapsto (\overline{z}, -\overline{w})$ on $\mathbb{CP}^1$.

For this suppose first that the polynomial $P(z, w)$ satisfies the condition (1). Then since the U(2)-action on $\mathbb{CP}^1$ is transitive, we can suppose that $P(z, w) = az^m$ for some $a \in \mathbb{C}^*$. It is elementary to see that the identity component of the stabilizer subgroup for this monomial (viewed as an element of $S^m_{l} \mathbb{C}^2$) is a U(1)-subgroup of U(2). Therefore the stabilizer subgroup is of positive dimension. Second suppose that $P(z, w)$ satisfies the condition (2). Then under an identification $\mathbb{CP}^1 \simeq S^2$ the two roots of $P(z, w) = 0$ in $\mathbb{CP}^1$ form an anti-podal pair. Recalling that the natural U(2)-action on $\mathbb{CP}^1 \simeq S^2$ is isometric with respect to the standard metric on $S^2$, this means that the natural U(2)-action on the space of anti-podal pairs of points is also transitive. Therefore we can suppose that $P(z, w) = z^{m-j} w^j$ for some $0 < j < m$. Then again it is elementary to see that the stabilizer subgroup at $P(z, w) \in S^m_{l} \mathbb{C}^2$ is 1-dimensional. Thus if $P(z, w)$ satisfies (1) or (2), then the stabilizer subgroup at this $P(z, w) \in S^m_{l} \mathbb{C}^2$ is of 1-dimensional. This means the assertion (i) of the proposition.

Conversely for (ii) suppose that the roots of $P(z, w) \in S^m_{l} \mathbb{C}^2$ do not satisfy (1) nor (2). If there are more than two roots, then elements of U(2) preserving the set of the roots constitute a finite subgroup at most. This implies that the stabilizer subgroup at the point $P(z, w) \in S^m_{l} \mathbb{C}^2$ is also a finite subgroup. If there are exactly two roots but the roots are not an anti-podal pair, then because of the isometricity of the U(2)-action on $S^2$, elements of U(2) which preserve the two roots constitute a finite subgroup at most, since such an element has to preserve four points. Therefore we again obtain that the stabilizer subgroup at $P(z, w) \in S^m_{l} \mathbb{C}^2$ is a finite subgroup. Therefore the stabilizer subgroup is zero-dimensional if $P(z, w)$ does not satisfy (1) nor (2). Hence from the transitivity of the
natural U(2)-action on $S^2$, we obtain that if the U(2)-orbit through $P(z, w) \in S_1^m \mathbb{C}^2$ is not four-dimensional, then $P(z, w) \in O_j$ for some $0 \leq j \leq m$. This proves the assertion (ii).

For the final assertion (iii), $O_j = O_{m-j}$ is clear because there actually exists an element of $U(2)$ which interchanges $z^{m-j}w^j$ and $z^jw^{m-j}$ as elements of $S_1^m \mathbb{C}^2$. Moreover, if $O_j = O_k$, the set of multiplicities of the two roots must equal. This implies $k \in \{j, m-j\}$.

The stabilizer subgroup at the monomials in the space $(2.16)$ is concretely given as follows:

**Lemma 2.6.** For a pair $(m_1, m_2)$ of integers define a subgroup $G(m_1, m_2) \subset T^2 \subset U(2)$ by

$$G(m_1, m_2) := \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} ; \alpha, \beta \in \mathbb{R}, m_1\alpha + m_2\beta = 0 \right\}.$$  

Then the identity component of the stabilizer subgroup at the monomial $z^{m-k}w^k \in S_1^m \mathbb{C}^2$ is $G(m + l - k, l + k)$. In particular the identity component coincides for the two monomials

$$z^{n-2-k}w^k \in S_1^{n-2} \mathbb{C}^2 \quad \text{and} \quad z^{n-3-k}w^{k-1} \in S_2^{n-4} \mathbb{C}^2$$

in the representation $(2.16)$.

**Proof.** This is elementary and we omit a proof. The last coincidence is a direct consequence of the concrete form of the stabilizer subgroup.

From Lemma 2.6 we put all monomials in $S_1^{n-2} \mathbb{C}^2$ and $S_2^{n-4} \mathbb{C}^2$ as in the following table:

| $S_1^{n-2} \mathbb{C}^2$ | $z^{n-2}$ | $z^{n-3}w$ | $z^{n-4}w^2$ | ... | $z^2w^{n-4}$ | $zw^{n-3}$ | $w^{n-2}$ |
|-------------------------|-----------|------------|------------|-----|-------------|----------|--------|
| $S_2^{n-4} \mathbb{C}^2$ | $z^{n-4}$  | $z^{n-5}w$ | $z^{n-6}w^2$ | ... | $zw^{n-3}$ | $w^{n-4}$ |        |
| stabilizer              | $G(n-1, 1)$ | $G(n-2, 2)$ | $G(n-3, 3)$ | ... | $G(3, n-3)$ | $G(2, n-2)$ | $G(1, n-1)$ |

This reads, for example, that the two monomials $z^{n-3}w$ and $z^{n-4}$ have the same stabilizer group $G(n-2, 2)$ as the identity component.

With these preliminary results, we investigate automorphisms and the moduli space of the ASD structures on $\mathcal{O}(-n)$ which appear from the versal family of the twistor space $Z_{\text{LB}}$. Let $p : \mathcal{F} \to B$ be the versal family for $Z_{\text{LB}}$, where $p^{-1}(0) = Z_{\text{LB}}$. As before, the parameter space $B$ can be $U(2)$-equivariantly identified with an invariant neighborhood of the origin in the vector space $S_1^{n-2} \mathbb{C}^2 \oplus S_2^{n-4} \mathbb{C}^2$. For each subgroup $G \subset U(2)$ let $B^G$ the subspace of $G$-invariant elements, which is an intersection of $B$ with the linear subspace of $S_1^{n-2} \mathbb{C}^2 \oplus S_2^{n-4} \mathbb{C}^2$ consisting of $G$-invariant elements. By restricting $p$ over $B^G$, we obtain a versal family of $G$-equivariant deformations for $Z_{\text{LB}}$. In particular $Z_{\text{LB}}$ admits a non-trivial $G$-equivariant deformation if $B^G \neq 0$. These considerations readily mean the following

**Proposition 2.7.** Suppose $n \geq 3$ and let $G$ be a closed connected subgroup of $U(2)$ which satisfies $\dim G \geq 1$. Assume that the LeBrun’s ASD structure on $\mathcal{O}(-n)$ admits a non-trivial $G$-equivariant deformation. Then (i) if $n \neq 4$, we have $G = G(k, n-k)$ for some $k$ satisfying $1 \leq k < n$. So $G$ is isomorphic to $U(1)$. (ii) If $n = 4$ and $G \neq SU(2)$, the same conclusion holds. Moreover there exists an $SU(2)$-equivariant deformation.

**Proof.** The assertion except the final one in (ii) follows immediately from Propositions 2.5 and the above table. When $n = 4$, the second direct summand in $(2.16)$ becomes the 1-dimensional space $\mathbb{C}_2$, and the identity component of the stabilizer subgroup at any point on $\mathbb{C}_2$ is clearly the subgroup $SU(2)$, except the origin. Therefore if we restrict the
versal family of ASD structures on $\mathcal{O}(-4)$ to the real 2-dimensional subspace $\{0\} \oplus \mathbb{C}_2 \subset S_1^2 \mathbb{C}^2 \oplus \mathbb{C}_2 \simeq H^1(\Theta_{Z_{\text{LB}}})^G$, we obtain the required SU(2)-equivariant deformation. \hfill \square

For the moduli space of the invariant ASD structures in Proposition 2.7 if $t \in B^G$, the fiber $p^{-1}(t)$ has a (holomorphic) $G$-action of course. However, there can exist a subgroup of $U(2)$ which acts non-trivially on $B^G$, and it gives an identification between different fibers of $p$ over $B^G$. Thus the subspace $B^G$ itself cannot be considered as a moduli space of $G$-invariant ASD structures in general, and instead the actual moduli space is considered to be the quotient space of the subspace $B^G$ under the action of the subgroup of $U(2)$ consisting of elements which preserve $B^G$.

For example, if $G = G(n - 2, 2)$, the subspace $B^G$ is $B \cap \langle z^{n-3}w, z^{n-4} \rangle \subset \mathbb{C}$, which is 4-dimensional over $\mathbb{R}$. It is easy to see that the subgroup of $U(2)$ consisting of elements which preserve this subspace is the maximal torus $T^2$ consisting of diagonal matrices. Further orbits of the $T^2$-action on $\langle z^{n-3}w, z^{n-4} \rangle \subset \mathbb{C}$ is 1-dimensional, except the origin. Consequently we obtain that the moduli space of these $U(1)$-invariant ASD structures on $\mathcal{O}(-n)$ is 3-dimensional. By the same argument, we get the following

**Proposition 2.8.** Let $n \geq 3$ and $k$ satisfy $1 \leq k < n$, and consider ASD structures on $\mathcal{O}(-n)$ obtained as the $G(k, n - k)$-equivariant small deformation as in Proposition 2.7. Then the moduli space of $U(1)$-invariant ASD structures on $\mathcal{O}(-n)$ obtained by this equivariant deformation is 1-dimensional if $k \in \{1, n - 1\}$, and 3-dimensional if $k \notin \{1, n - 1\}$.

For the SU(2)-equivariant deformation the case $n = 4$, the moduli space is 1-dimensional.

It is already immediate to give a proof of Theorem 1.1 in the introduction.

**Proof of Theorem 1.1.** This is an immediate consequence of Propositions 2.3, 2.5, 2.8 and Lemma 2.6. More concretely for the $U(2)$-invariant subset $B_i$ in $B$ in the theorem, it is enough to take the union of all $U(2)$-orbits which go through:

- the real 2-dimensional subspace $\langle z^{n-2} \rangle \setminus \{0\}$ for the case $i = 1$,
- the real 4-dimensional subspace $\langle z^{n-i-1}w^{i-1}, z^{n-i-2}w^{i-2} \rangle \setminus \{0\}$ for the case $1 < i \leq \lfloor n/2 \rfloor$,
- the real 2-dimensional subspace $\{0 \oplus \mathbb{C}_2\} \setminus \{0\}$ in the case of $(n, i) = (4, 0)$.

For the dimension of the above $U(2)$-invariant subsets, we readily have $\dim B_0 = 2$, $\dim B_1 = 4$ and $\dim B_i = 6$ if $i \notin \{0, 1\}$.

## 3. Deformations preserving Kählerian property

The investigation in the last section concerns versal and equivariant deformations of the LeBrun’s ASD structure on $\mathcal{O}(-n)$ as an ASD orbifold. Since the LeBrun metric is Kähler, from differential geometric point of view, it would be desirable to obtain deformations of the metric preserving not only anti-self-duality but also Kählerian property. In this section again by using twistor spaces, we find, for any $n \geq 3$, a deformation of the LeBrun metric on $\mathcal{O}(-n)$ which keeps anti-self-duality as well as Kählerity. The deformation is realized as one of the $U(1)$-equivariant deformation we found in the last section. Meanwhile we also show that the corresponding twistor spaces are Moishezon for these deformations. We also show that for other $U(1)$-equivariant deformations, the deformed twistor spaces are not Moishezon.
The key tool for finding such a deformation is of course Pontecorvo’s theorem \[22\] Theorem 2.1, which means that an anti-self-dual conformal structure on a 4-manifold \(M\) carries a Kähler representative for a complex structure if and only if the twistor space possesses a divisor \(D\) which is mapped diffeomorphically to \(M\) by the twistor fibration, and which satisfies \(D + \overline{D} \in |K^{-1/2}|\); then the conformal class has a Kähler representative with respect to the complex structure of \(D\), and then derive information about existence of a reducible member of \(|K^{-1/2}|\).

As we already mentioned, for the twistor space \(Z_{\text{LB}}\) of the LeBrun orbifold \(\mathcal{O}(-n)\), the divisor \(D\) in Section \[2.1\] gives a reason for the LeBrun metric to be Kähler with respect to the standard complex structure on \(\mathbb{C}P^1\). One would naturally think that we should investigate deformations of the pair \((Z_{\text{LB}}, D + \overline{D})\), which might actually work. However, the divisor \(D\) itself is not a Cartier divisor on \(Z_{\text{LB}}\), and deformation theory of such a pair might be subtle. Therefore here we take a real irreducible divisor \(S \in |K^{-1/2}|\) and consider deformations of the pair \((Z_{\text{LB}}, S)\).

3.1. Deformation of the pair \((Z_{\text{LB}}, S)\). In general if \(X\) is a complex variety and \(Y\) is a reduced Cartier divisor on \(X\), the subsheaf \(\Theta_{X,Y}\) of the tangent sheaf \(\Theta_X\) is naturally defined as

\[
\Theta_{X,Y} = \{v \in \Theta_X : v(g)/g \in \mathcal{O}_X\},
\]

where \(g \in \mathcal{O}_X\) is a local equation of \(Y\). (When \(X\) and \(Y\) are non-singular, this is exactly the sheaf of vector fields on \(X\) which are tangent to \(Y\).) This sheaf plays the same role for deformations of the pair \((X, Y)\) as the sheaf \(\Theta_X\) plays for deformations of \(X\) itself. Namely, if \(X\) is compact, the cohomology group \(H^1(\Theta_{X,Y})\) is the Zariski tangent space of the Kuranishi family of locally trivial deformations of the pair \((X, Y)\), and \(H^2(\Theta_{X,Y})\) is the obstruction space. In particular, if \(H^2(\Theta_{X,Y}) = 0\), the parameter space of the Kuranishi family is naturally identified with an open neighborhood of the origin in \(H^1(\Theta_{X,Y})\).

As a real irreducible divisor \(S \in |K^{-1/2}|\) we first take any real non-singular \((1,1)\)-curve \(\mathcal{C}\) on \(Q = \mathbb{C}P^1 \times \mathbb{C}P^1\) which is different from the diagonal \(\Delta\), and put \(S := \mu(\pi^{-1}(\mathcal{C}))\). (Recall that \(\pi : X \to Q\) is a projection and \(\mu : X \to Z_{\text{LB}}\) is the blowdown of the divisor \(E \cup \overline{E}\).) From the formula \[2.6\] this actually belongs to \(|K^{-1/2}|\) on \(Z_{\text{LB}}\). Then the maximal subgroup of the automorphism group \(U(2)\) of \(Z_{\text{LB}}\) which preserves \(S\) is isomorphic to a torus \(T^2\), and under this action \(S\) has a structure of a toric surface. We note that the complex structure of \(S\) is independent of the choice of the \((1,1)\)-curve \(\mathcal{C}\). Obviously the above subgroup \(T^2 \subset U(2)\) preserves not only \(S\) but also the two divisors \(D\) and \(\overline{D}\) too. It is easy to see that the surface \(S\) satisfies the following properties:

- the intersection \(S \cap L_{\infty}\) consists of two points, which are mutually conjugate,
- \(S\) has \(A_{-1}\)-singularities at these two points, and is non-singular except these points.

For investigating deformation of the pair \((Z_{\text{LB}}, S)\), we first show the following

**Proposition 3.1.** We have the following vanishing:

\[
H^2(\Theta_S) = H^2(\Theta_{Z_{\text{LB}}(-S)}) = 0.
\]

**Proof.** In this proof we again write \(Z\) for \(Z_{\text{LB}}\). Our proof for \(H^2(\Theta_S) = 0\) is quite analogous to \(H^2(\Theta_Z) = 0\) in the proof of Proposition \[2.1\]. Put \(S \cap L_{\infty} = \{p, \overline{p}\}\). Since \(S\) is biholomorphic to \(\pi^{-1}(\mathcal{C})(\subset X)\), we have two exact sequences

\[
0 \to \Theta_{S/E} \to \Theta_S \to \mathcal{G} \to 0 \text{ and } 0 \to \mathcal{G} \to \pi^*\Theta_E \to \mathbb{C}_p \oplus \mathbb{C}_{\overline{p}} \to 0,
\]
where $\mathcal{G}$ is the image sheaf of the canonical homomorphism $\Theta_S \to \pi^*\Theta_E$. Since $\mathcal{G} \simeq \mathbb{CP}^1$, the induced map $H^0(\pi^*\Theta_E) \to H^0(\mathbb{C}_p \oplus \mathbb{C}_r)$ is easily seen to be surjective. Further we have $H^i(\pi^*\Theta_E) \simeq H^i(\Theta_E) = 0$ for any $i \geq 0$. Hence the second sequence of (3.3) implies $H^i(\mathcal{G}) = 0$ for $i \geq 1$. Hence the first one in (3.3) means $H^2(\Theta_{S/\mathcal{G}}) \simeq H^2(\Theta_S)$. Further, from an obvious vector field which vanishes on $C \cup \mathcal{C}$, we have $\Theta_{S/\mathcal{G}} \simeq \mathcal{O}_S(C + \mathcal{C})$, and we readily have $H^2(\mathcal{O}_S(C + \mathcal{C})) = 0$. Therefore $H^2(\Theta_S) = 0$ follows.

In the sequel we put $F := K^{-1/2}$ for simplicity, and show $H^2(\Theta_Z \otimes F^{-1}) = 0$. By taking tensor product with $F^{-1}$ to the exact sequence (2.14), we obtain an exact sequence

$$0 \to \Theta_{Z,C+\mathcal{C}} \otimes F^{-1} \to \Theta_Z \otimes F^{-1} \to (N_{C/Z} \otimes N_{\mathcal{C}/Z}) \otimes F^{-1}|_C \to 0.$$  

Further since $F_C \simeq K_{\mathcal{S}}^{-1}|_C \simeq \mathcal{O}_C(2 - n)$ and $N_{C/Z} \simeq \mathcal{O}_C(1 - n)\mathcal{O}_Z$, the last non-trivial sheaf of this sequence, by taking tensor product with $\mathcal{O}_Z$, we have

$$H^2(\Theta_Z \otimes F^{-1}) \simeq H^2(\Theta_{Z,C+\mathcal{C}} \otimes F^{-1}).$$

For computing the RHS, from the isomorphism $\Theta_{X,E+E} \simeq \mu^*\Theta_{Z,C+\mathcal{C}}$, recalling $\mu^*F \simeq \pi^*\mathcal{O}_Q(1,1) \otimes \mathcal{O}_E(\mathcal{E} + \mathcal{E})$, we have $\mu^*(\Theta_{Z,C+\mathcal{C}} \otimes F^{-1}) \simeq \Theta_{X,E+E} \otimes \pi^*\mathcal{O}_Q(-1,1) \otimes \mathcal{O}_X(\mathcal{E} - \mathcal{E})$. For simplicity we write $\mathcal{L}$ for the sheaf on RHS. From the last isomorphism we have

$$H^2(\Theta_{Z,C+\mathcal{C}} \otimes F^{-1}) \simeq H^2(X, \mathcal{L}).$$

For the RHS of this, from the inclusion $0 \to \Theta_{X,E+E} \to \Theta_X$ we have the exact sequence

$$0 \to \mathcal{L} \to \Theta_X \otimes \pi^*\mathcal{O}_Q(-1,1) \otimes \mathcal{O}_X(\mathcal{E} - \mathcal{E}) \to 0.$$  

The last non-trivial term of (3.7) is clearly isomorphic to $\pi^*\mathcal{O}_Q(-1,1)|_{E_{\mathcal{E},\mathcal{E}}}$, whose all cohomologies vanish. Therefore, writing the middle sheaf as $\mathcal{L}'$, we get

$$H^2(\mathcal{L}) \simeq H^2(\mathcal{L}').$$

For the RHS of this, by taking tensor product with $\pi^*\mathcal{O}_Q(-1,1) \otimes \mathcal{O}_X(\mathcal{E} - \mathcal{E})$ to the exact sequence (2.14), we obtain

$$0 \to \pi^*\mathcal{O}_Q(-1,1) \to \mathcal{L}' \to \mathcal{F} \otimes \pi^*\mathcal{O}_Q(-1,1) \otimes \mathcal{O}_X(\mathcal{E} - \mathcal{E}) \to 0.$$  

Writing $\mathcal{F}'$ for the last non-trivial sheaf of this sequence, by taking a tensor product with $\pi^*\mathcal{O}_Q(-1,1) \otimes \mathcal{O}_X(\mathcal{E} - \mathcal{E})$ to the exact sequence (2.11), we get

$$0 \to \mathcal{F}' \to \pi^*(\mathcal{O}(1,1) \oplus \mathcal{O}(-1,1)) \to \mathcal{O}_{\Delta} \to 0.$$  

From this we get $H^2(\mathcal{F}') = 0$. Therefore from (3.9) we obtain $H^2(\mathcal{L}') = 0$. Hence by (3.8), (3.6) and (3.5) we obtain $H^2(\Theta_Z \otimes F^{-1}) = 0$. 

For investigating deformations of the pair $(Z_{LB}, S)$ we also need the following

**Proposition 3.2.** Let $S$ be a real irreducible member of $|K^{-1/2}|$ as above. Then we have the following exact sequence

$$0 \to \Theta_{Z_{LB}}(-S) \to \Theta_{Z_{LB}, S} \to \Theta_S \to 0.$$  

Hence by Proposition 3.1 we have

$$H^2(\Theta_{Z_{LB}, S}) = 0.$$
Proof. We again write \( Z \) for \( Z_{\text{LB}} \). Since \( \text{Sing} \ S \subset \text{Sing} \ Z \), (3.1) is obvious outside the two singular points of \( S \). Also, in a neighborhood of the singular points, defining equation of \( Z \) and \( S \) in the ambient space \( \mathbb{P}(\mathcal{E}_n) \) can be taken as \( xy-(u-v)^n = 0 \) and \( xy-(u-v)^n = u = 0 \) respectively, and by using these and (2.7) it is easy to obtain concrete form of sections of the sheaves \( \Theta_{Z,S} \) and \( \Theta_S \) in a neighborhood of the singular point. From this the exact sequence (3.1) is known to be available on the two singular points too. \( \square \)

From Propositions 3.1 and 3.2 we readily obtain the following co-stability:

**Proposition 3.3.** The irreducible Cartier divisor \( S \) is co-stable in \( Z_{\text{LB}} \) with respect to locally trivial deformations of \( S \). Namely for any such deformation there exists a locally trivial deformation of the pair \((Z_{\text{LB}}, S)\) which gives the prescribed deformation of \( S \) by restriction.

Proof. We again write \( Z \) for \( Z_{\text{LB}} \). Let \( p : \mathcal{Z} \to B \) and \( \mathcal{S} \subset \mathcal{Z} \) be the Kuranishi family of locally trivial deformations of the pair \((Z, S)\), where \( p^{-1}(0) = Z \) and \( p^{-1}(0) \cap \mathcal{S} = S \). By Proposition 3.2, \( B \) can be identified with an open neighborhood of 0 in \( H^1(\Theta_{Z,S}) \). Let \( \mathcal{S}' \to B' \) be the Kuranishi family of locally trivial deformation of \( S \). As \( H^1(\Theta_S) = 0 \) by Proposition 3.1, the parameter space \( B' \) may be identified with an open neighborhood of the origin in \( H^1(\Theta_S) \). By versality of the Kuranishi family, the family \( \mathcal{S} \to B' \) induces a holomorphic map \( f : B \to B' \) with \( f(0) = 0 \), and the differential \( df \) at 0 is identified with the natural linear map \( H^1(\Theta_{Z,S}) \to H^1(\Theta_S) \). The last map is locally submersion by Proposition 3.2. Therefore \( f \) is locally surjective at 0. By the property \( f^* \mathcal{S}' \simeq \mathcal{S} \) over \( B \), this means the required co-stability. \( \square \)

### 3.2. Concrete deformations of the surface \( S \), and deformations of the pair

Next we concretely construct locally trivial deformations of the singular toric surface \( S \) which preserve \( U(1) \)-action, for some explicit subgroups \( U(1) \) in \( U(2) \). Applying Proposition 3.3 to any one of these deformations, we will obtain non-trivial deformations of the LeBrun metric. It will turn out that some of these deformations preserve Kählerian property.

Fix any \( n \geq 3 \) as before. We first realize our singular toric surface \( S \) in \( Z_{\text{LB}} \) as an explicit birational transform of the product surface \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). Writing \( 0 := (1 : 0) \in \mathbb{CP}^1 \) and \( \infty := (0 : 1) \in \mathbb{CP}^1 \), we take four points and four curves on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) as

\[
q_1 = (0, 0), \quad q_2 = (\infty, 0), \quad q_3 = (0, \infty), \quad q_4 = (\infty, \infty),
\]

\[
C_1 = \mathbb{CP}^1 \times 0, \quad C_2 = \infty \times \mathbb{CP}^1, \quad C_3 = \mathbb{CP}^1 \times \infty, \quad C_4 = 0 \times \mathbb{CP}^1.
\]

We regard \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) as a toric surface by considering the \( T^2 \)-action which preserves the curve \( C_1 + C_2 + C_3 + C_4 \).

For any integer \( k \) satisfying \( 0 < k < n \), we assign a weight \( k \) on the two points \( q_1 \) and \( q_3 \), and a weight \((n-k) \) on the other points \( q_2 \) and \( q_4 \). Under this setting let \( \tilde{S} \to \mathbb{CP}^1 \times \mathbb{CP}^1 \) be the surface obtained by blowing-up the point \( q_i \) for \( m_i \) times for any \( 1 \leq i \leq 4 \), where \( m_i \) is the above weight at \( q_i \). Here, if \( m_i \geq 2 \), the blowup is always done at a \( T^2 \)-fixed point on the strict transforms of the curve \( C_1 \) or \( C_3 \). \( \tilde{S} \) is also a toric surface. The inverse image of the curve \( C_1 + C_2 + C_3 + C_4 \) is a \( T^2 \)-invariant anticanonical curve on \( \tilde{S} \), and it consists of \( 4 + 2n \) components. The self-intersection numbers of the components are given by, up to cyclic permutations,

\[
(3.13) \quad -n, -1, -2, \cdots, -2, -1, -n, -1, -2, \cdots, -2, -1.
\]
In particular, these are independent of $k$, and therefore so is the structure of the toric surface $\tilde{S}$. (Dependence on $k$ will appear later.) Let $C$ and $\overline{C}$ be the two $(-n)$-curves among $\mathcal{E}$. These are strict transforms of the curves $C_1$ and $C_3$. Then in $\tilde{S}$ we can contract the two chains of the $(-2)$-curves to obtain a toric surface with two $A_{n-1}$-singularities. By looking structure as a toric surface, it is easy to see that the last surface is biholomorphic to the surface $S$ in $Z_{\text{LB}}$ we have given in the previous subsection. The contraction map $\tilde{S} \to S$ is nothing but the minimal resolution of the singularities of $S$, and $C$ and $\overline{C}$ are exactly the curves $\mu(E)$ and $\mu(\overline{E})$ under the identification. It is also easy to verify that if we introduce a real structure on the initial surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ by the product which is (anti-podal) $\times$ (complex conjugation), then it naturally lifts to be a real structure on the surface $\tilde{S}$ as well as that on the contracted surface, and the last real structure is exactly the one on the real divisor $S$ in $Z_{\text{LB}}$.

Now we shall give $U(1)$-equivariant, locally trivial deformations of the surface $S$ preserving the real structure, by using the above realization of $S$. The deformations we construct are uniquely and explicitly determined from the value $k$ above. In order to construct locally trivial deformation of $S$, it is enough to give a deformation (in the usual sense) of the minimal resolution $\tilde{S}$ which preserves the two chains of $(-2)$-curves. For fixed integer $k$ with $0 < k < n$ as above, we think the surface $\tilde{S}$ as obtained by blowing up $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the way indicated by the weights $k$ and $n-k$ as above. Then by moving the weighted blowup points $q_1$ and $q_3$ along the curves $C_4$ and $C_2$ freely respectively, we obtain a 2-dimensional family of smooth rational surfaces which can naturally be regarded as deformation of the surface $\tilde{S}$. If $m_i$ is the weight at the point $q_i$ as above, even after the deformation, the iterated blowups at $q_i$ yield $(m_i - 1)$ number of $(-2)$-curves as exceptional curves. Also, the strict transforms of the two curves $C_2$ and $C_4$ are still $(-2)$-curves in the deformed new surface. The union of all these $(-2)$-curves still form two chains of $(-2)$-curves, and each chain yet consists of $(n - 1)$ components. Hence by contracting these two chains simultaneously, we obtain a family of rational surfaces which have two $A_{n-1}$-singularities.

In this way, for each $1 \leq k < n$ we have obtained a locally trivial deformation of the toric surface $S$. From the construction the parameter space of this deformation is naturally identified with the product $C_2 \times C_4$. The real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$ attached above naturally acts on this product, and by restricting the deformation to the real locus, we obtain a deformation of $S$ preserving the real structure. The parameter space of this family is clearly real 2-dimensional. These deformations actually deform the complex structure of $S$ since the deformed new surface is not a toric surface anymore.

Next we show that all these deformations (determined by $k$) of the surface $S$ are equivariant with respect to a $U(1)$-subgroup of $T^2$, and the subgroup depends on the value $k$. For this we consider the $U(1)$-action on $\mathbb{CP}^1 \times \mathbb{CP}^1$ which fixes points on $C_2 \cup C_4$. This $U(1)$-action clearly fixes the four points $q_1, \ldots, q_4$, even after moving $q_1$ and $q_3$. Moreover, it is immediate to see that the $U(1)$-action lifts on the blowup even after moving, and that on the final surface the chains of $(-2)$-curves are invariant under the induced $U(1)$-action. Therefore the $U(1)$-action descends on the contraction of the two chains. Thus our deformation of the surface $S$ is $U(1)$-equivariant.

This $U(1)$ can be naturally regarded as a subgroup of the torus $T^2$, where the last $T^2$ is thought as an automorphism group of the toric surface $\tilde{S}$ preserving the real structure. Then this subgroup has to depend on the number $k$, since the component fixed by the subgroup depends on $k$. (More concretely, there are exactly $k$ components between the curve $C$ and the fixed component.) We write $G(k)$ for this $U(1)$-subgroup of $T^2$. While
$G(k)$ is a subgroup of $T^2$, it is naturally regarded as a subgroup of the automorphism group $U(2)$ of the LeBrun twistor space $Z_{LB}$, since the torus $T^2$ is originally the maximal subgroup of $U(2)$ which preserves the divisor $S$.

On the other hand in Proposition 2.7 for each $1 \leq k < n$ we have obtained the U(1)-subgroup $G(k, n - k)$ for which $Z_{LB}$ admits a non-trivial equivariant deformation. These subgroups coincide:

**Proposition 3.4.** For any $1 \leq k < n$, we have $G(k, n - k) = G(k)$ in $U(2)$.

**Proof.** For the divisors $S$ and $D + \overline{D}$ of $|K^{-1/2}|$ on $Z_{LB}$, the intersection $S \cap (D \cup \overline{D})$ consists of a cycle of six smooth rational curves, two of which are $C$ and $\overline{C}$, while the remaining four components are the inverse image of the two intersection points $\Delta \cap \mathcal{C} \subset Q$ under the projection $\pi$. This cycle is naturally divided into halves by the twistor line $L_\infty$. As $S$ and $D \cup \overline{D}$ are invariant under the $T^2$-action, this cycle is also $T^2$-invariant. Moreover the $T^2$-action on the cycle is effective. Therefore we can identify any U(1)-subgroup of $T^2$ from the action on each component of the cycle.

For the subgroup $G(k, n - k)$, we can readily obtain these actions in a concrete form, by recalling that the torus $T^2$ in which $G(k, n - k)$ is included is exactly the maximal torus of $U(2)$ which consists of diagonal matrices, and that the $U(2)$-action on $\mathcal{C}(-n)$ can be obtained naturally via $Z_n$-quotient and the minimal resolution. (Recall also that the divisor $D \setminus L_\infty$ in $Z_{LB} \setminus L_\infty$ is $U(2)$-equivariantly identified with the open subset $\mathcal{C}(-n)$ by the twistor fibration map.) On the other hand, the $T^2$-action on $S$ was also explicitly given through the above construction, and therefore we can easily recognize the action of the subgroup $G(k)$ on the cycle in concrete forms. The coincidence $G(k, n - k) = G(k)$ follows from these explicit computations. We omit the detail. \qed

From the proposition, by making use of the co-stability obtained in Proposition 3.3 we now have the following

**Proposition 3.5.** For each integer $k$ with $1 \leq k < n$, let $G(k, n - k) \subset U(2)$ be the U(1)-subgroup given as in Proposition 2.8 and $S$ the irreducible member of $|K^{-1/2}|$ as taken in Section 3.7. Then the pair $(Z_{LB}, S)$ admits a $G(k, n - k)$-equivariant, locally trivial deformation. Moreover, the twistor space $Z_{LB}$ itself actually deforms in this deformation.

**Proof.** As constructed above, the surface $S$ admits a $G(k)$-equivariant deformation for which the complex structure actually deforms. By Proposition 3.4, this deformation is also $G(k, n - k)$-equivariant. Applying Proposition 3.3 to this deformation of $S$, there exists a locally trivial deformation of the pair $(Z, S)$ which induces the last deformation of $S$ by restriction, where $Z = Z_{LB}$ as before. This deformation of the pair can be taken $G(k, n - k)$-equivariantly, since the exact sequence (3.11) is $T^2$-equivariant, so that the induced map $H^1(\Theta_{Z,S}) \rightarrow H^1(\Theta_S)$ is also $T^2$-equivariant. Thus we obtain the existence of the $G(k, n - k)$-equivariant deformation of the pair $(Z, S)$. The complex structure of $Z$ actually varies in this deformation, since the complex structure of the divisor $S$ actually deforms, while when we move $S$ inside $Z$, $S$ remains to be a toric surface, so that the complex structure does not vary. \qed

From the proposition, the $G(k, n - k)$-equivariant deformation found in Proposition 2.8 of the LeBrun orbifold can be realized by a deformation of the twistor space for which the divisor $S$ survives. However, since we are taking an irreducible $S \in |K^{-1/2}|$ and not taking the reducible divisor $D + \overline{D}$, we do not know at this stage if the metric is accordingly
deformed in a way that the Kählerian property with respect to some complex structure on $\mathcal{O}(-n)$ is preserved. In the next subsection we answer this affirmatively for $k \in \{1, n - 1\}$.

### 3.3. Deformation preserving Kählerian property

For that purpose we first investigate pluri-anticanonical systems of the singular rational surfaces obtained by the $G(k)$-equivariant deformation in Section 3.2. As in the previous subsection let $n \geq 3$ and $S$ be the real irreducible member of $|K^{-1/2}|$ on $Z_{\text{LB}}$ for the LeBrun metric on $\mathcal{O}(-n)$. We write $S_t$ for the singular rational surface obtained by the $G(k)$-equivariant deformation of $S$ constructed in the last subsection. Since $S_t$ also has only $A_{n-1}$-singularities, the canonical divisor $K$ of $S$ is a Cartier divisor, so the system $|mK^{-1}|$ and the anti-Kodaira dimension (i.e. the Kodaira dimension of $K^{-1}$) makes sense. We denote the latter by $\kappa^{-1}(S_t)$ as usual.

**Proposition 3.6.** The rational surface $S_t$ satisfies the following properties: (i) if $k \in \{1, n - 1\}$, we have $\kappa^{-1}(S_t) = 2$, (ii) if $n = 4$ and $k = 2$, we have $\kappa^{-1}(S_t) = 1$, (iii) if $n > 4$ and $k \notin \{1, n - 1\}$, we have $|mK^{-1}| = \emptyset$ for any $m > 0$.

**Proof.** For (i), from the construction of the $G(k)$-equivariant deformation, the surface $S_t$ is obtained from a non-singular surface by contracting two chains of $(-2)$-curves. We denote the last non-singular surface by $\tilde{S}_t$. (So the contraction $\tilde{S}_t \to S_t$ is the minimal resolution.) If $k \in \{1, n - 1\}$, the surface $\tilde{S}_t$ is exactly the divisor in $|K^{-1/2}|$ on the twistor space over $n\mathbb{CP}^2$ that we have investigated in [9, 10]. In particular, the system $|(n - 2)K^{-1}|$ on $\tilde{S}_t$ induces a surjective degree-two morphism $\tilde{S}_t \to \mathbb{CP}^2$. Hence, since $S_t$ has only $A_{n-1}$-singularities, the degree-two morphism $\tilde{S}_t \to \mathbb{CP}^2$ factors as $\tilde{S}_t \to S_t \to \mathbb{CP}^2$, where $\tilde{S}_t \to S_t$ is the contraction and $S_t \to \mathbb{CP}^2$ is the map associated to $|(n - 2)K^{-1}|$ on $S_t$. This implies $\kappa^{-1}(S_t) = 2$.

For (ii) let $\tilde{S}_t \to S_t$ have the same meaning as above. Then if $n = 4$ and $k = 2$, the surface $\tilde{S}_t$ is the same as the divisor in $|K^{-1/2}|$ on the twistor spaces on $4\mathbb{CP}^2$ of algebraic dimension two which was investigated in [7]. In particular $|K^{-1}|$ on $\tilde{S}_t$ is base point free and induces an elliptic fibration $f : \tilde{S}_t \to \mathbb{CP}^1$. Hence we have $f^*\mathcal{O}(1) \simeq K^{-1}$, which means $\kappa^{-1}(\tilde{S}_t) = 1$.

For (iii) we first consider the surface $\tilde{S}_t$ in the case $n = 5$ and $k \in \{2, 3\}$. Obviously this surface is obtained from the elliptic surface in the last case of $(n, k) = (4, 2)$ by blowing-up two points. Further the two points belong to mutually different fibers of the elliptic fibration, and moreover the two points are smooth point of the fibers. From these we readily deduce that $h^0(mK^{-1}) = 0$ on $\tilde{S}_t$ for any $m > 0$. Therefore since $h^0(mK^{-1})$ for fixed $m$ cannot increase after blowup, we obtain that $h^0(mK^{-1}) = 0$ for $\tilde{S}_t$ in the case $n > 4$ and $k \notin \{1, n - 1\}$. Hence $h^0(mK^{-1}) = 0$ also for the surface $S_t$.

We recall that in Proposition 2.8 we have obtained $G(k, n - k)$-equivariant deformation of the LeBrun orbifold. Correspondingly we have $G(k, n - k)$-equivariant deformation of the twistor space $Z_{\text{LB}}$. For algebraic dimension of these twistor spaces, by using Propositions 3.5 and 3.6 we obtain the following

**Proposition 3.7.** (i) If $k \in \{1, n - 1\}$, all the deformed twistor spaces are Moishezon. at least for small deformations. (ii) If $n = 4$ and $k = 2$, there exists a small deformations whose algebraic dimension is two. (iii) If $n > 4$ and $k \notin \{1, n - 1\}$, there exists a small deformation whose algebraic dimension is zero.

**Proof.** For (i), from the concrete construction of the equivariant deformations of the surface $S$, we readily see that if $k \in \{1, n - 1\}$ the deformed surface $S_t$ has a unique $U(1)$-invariant
real anticanonical curve. Further the curve is a cycle of smooth rational curves consisting of four irreducible components, regardless of the value of $n$. Let $Z_t$ be the twistor space on which the surface $S$ is contained. The exact sequence $0 \to \mathcal{O} \to K^{-1/2} \to K_{S_t}^{-1} \to 0$ on $Z_t$ means that the system $|K^{-1/2}|$ contains a $U(1)$-invariant pencil whose base curve is exactly the above cycle. (When $n > 3$ the pencil is exactly $|K^{-1/2}|$ itself.) Further as any element of this pencil contains the cycle, general members of the pencil also satisfy $\kappa^{-1} = 2$. Thus the twistor space $Z_t$ has a pencil whose general members satisfy $\kappa^{-1} = 2$. This directly implies that $Z_t$ is Moishezon [25]. Further, the twistor spaces obtained as the $G(k, n-k)$-equivariant deformation of the pair $(Z_{LB}, S)$ exhausts the 1-dimensional family obtained in Proposition 2.8 at least for small deformations, since the deformation of the pair actually deforms the complex structure of $Z_{LB}$ by Proposition 3.5. Hence we obtain the assertion (i). The assertion (ii) can be obtained in a similar way. (iii) is much easier. □

**Remark 3.8.** In (iii) of Proposition 3.7 it is very likely that all the twistor spaces in the 3-dimensional family have algebraic dimension zero, at least for small deformations.

Now we can give a proof of Theorem 1.2 in the introduction.

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1 by restricting the versal family for the twistor space $Z_{LB}$ on $\mathcal{O}(-n)$ to the $U(2)$-invariant subset $B_1$ in $S_1^{n-2} \mathbb{C}^2 \oplus S_2^{n-4} \mathbb{C}^2$, we obtain the versal family of $G(n-1, 1)$-equivariant deformations of $Z_{LB}$. Take any one-dimensional subspace of $\langle z^{n-2} \rangle_\mathbb{C}$ over $\mathbb{R}$ and consider the restriction of the versal family to this subspace. We show that on the open subset $\mathcal{O}(-n)$ the corresponding family of ASD structures provides the required family of ASD ALE Kähler metrics.

Let $Z_t$ be any one of the twistor spaces in this real one-dimensional family, and $L_\infty \subset Z_t$ the twistor line over the orbifold point. We need to show that the linear system $|K^{-1/2}|$ on $Z_t$ carries a real reducible member which contains $L_\infty$. Recall from the proof of Proposition 3.7 (i) that the linear system $|K^{-1/2}|$ on $Z_t$ has a $U(1)$-invariant pencil, and the base locus of the pencil is a cycle of four rational curves. Take a uniformization of the orbifold point $\infty \in \mathcal{O}(-n)$, and let $\infty$ be the point over $\infty$, and $\Gamma \simeq \mathbb{Z}_n$ the group at $\infty$. Take a $\Gamma$-invariant open neighborhood $V$ of the twistor line $L_\infty$, and let $u : \tilde{V} \to V$ be the uniformization corresponding to the above uniformization on the base. Let $L_\infty = u^{-1}(L_\infty)$ be the $\Gamma$-invariant twistor line over $\infty$. Then in the present situation the group $\Gamma$ acts on the normal bundle $N = N_{L_\infty/\tilde{V}}$ as

$$\begin{align*}
(x, y, z) \mapsto (\zeta x, \zeta^{-1} y, z)
\end{align*}$$

where $(x, y)$ is holomorphic fiber coordinates on $N$, $z$ is a coordinate on $L_\infty$, and $\zeta = e^{2\pi i / n}$ is the generator of $\Gamma$ as before. We consider the pullback of the $U(1)$-invariant pencil in $|K^{-1/2}|$ on $Z_t$ to $\tilde{V}$ by the uniformization map $u$. Since $u^* K_{\tilde{V}}^{-1/2} \simeq K_{\tilde{V}}^{-1/2}$, the pullback is a pencil whose members belong to $|K^{-1/2}|$ of the open twistor space $\tilde{V}$.

Now adapting the argument of Kreussler given in the proof of [13, Proposition 3.7], we consider an element $S_0$ of the last pencil on $\tilde{V}$, which is uniquely specified by the property that it goes through a generic point of $L_\infty$. Here, genericity means that the point does not belong to the base curve of the pencil. Then by that argument, the unique divisor $S_0$ is of the form $D_0 + \bar{D}_0$, where $D_0$ and $\bar{D}_0$ are irreducible non-singular, and intersect along $L_\infty$ transversally. (Note that Pedersen-Poon’s result about reducibility of certain divisor used in the Kreussler’s argument does not require for the twistor space to be compact.)
Taking the image of \( S_0 \) under the uniformization map \( u \), it follows that a member of the \( U(1) \)-invariant pencil on the original twistor space \( Z \) which goes through the generic point of \( L_\infty \) is unique, and it contains the whole of \( L_\infty \). Moreover, the last member is clearly real, and at least on the neighborhood \( \tilde{V} \), it consists of two irreducible components \( u(D_0) \) and \( u(D_0) \) whose intersection is precisely \( L_\infty \).

We now show that this divisor is reducible on the whole of \( Z_t \). The \( \Gamma \)-action on the uniformization \( \tilde{V} \) preserves each of the two irreducible components of the divisor \( S_0 \), and from (3.14), it acts on each of the components as merely as a multiplication by \( \zeta \) in the normal direction. This means that the images \( u(D_0) \) and \( u(D_0) \) are non-singular, and the self-intersection numbers of \( L_\infty \) in the components are both \((+n)\). Therefore, if the divisor is irreducible, its normalization would have two disjoint curves whose self-intersection numbers are both \((+n)\). This contradicts Hodge index theorem. Therefore the divisor is reducible on \( Z_t \). Hence by the theorem of Pontecorvo, we obtain that on the smooth locus \( \theta(-n) \), the ASD structure associated to the twistor space \( Z_t \) is represented by a Kähler metric. Also the presence of the above divisor \( D_0 + \overline{D}_0 \), as well as the above degree of the normal bundle of \( L_\infty \) in \( D_0 \) and \( \overline{D}_0 \), mean that the Kähler metric is ALE at infinity (see [17, Proposition 6]).

Finally we detect the complex structure on the regular locus \( \theta(-n) \). For this let \( D_t + \overline{D}_t \) be the reducible member of \(|K^{-1/2}| \) on \( Z_t \) obtained above. Then as both \( D + \overline{D} \) and \( D_t + \overline{D}_t \) are unique reducible member containing the twistor line over the orbifold point \( \infty \), the divisor \( D_t + \overline{D}_t \) is naturally regarded as a deformation of the divisor \( D + \overline{D} \) on \( Z_{LB} \).

We determine the complex structure of \( D_t \). For this, since \( D_t + \overline{D}_t \in \vert K^{-1/2} \vert \), we have, by adjunction formula, \( K_Dt = K_{Z_t} + D_t \vert D_t = K^{1/2} - \overline{D}_t \vert D_t \). Hence, since \( \overline{D}_t \vert D_t \simeq \theta(D_\infty) \), we have

\[
(3.15) \quad K_{D_t}^{-1} \simeq K^{-1/2} \vert D_t + L_\infty.
\]

Let \( S_t \in \vert K^{-1/2} \vert \) be any member of the \( U(1) \)-invariant pencil which is different from \( D_t + \overline{D}_t \).

Then we can write \( S_t \cap D_t = C_1 + C_2 \) for two components \( C_1 \) and \( C_2 \) of the base locus of the pencil. By (3.15) the curve \( C_1 + C_2 + L_\infty \) is an anticanonical curve on \( D_t \), and it is a triangle. On the other hand, noting that on \( \mathbb{F}_n \), the \((-n)\)-section is a base curve of the anticanonical system (as \( n > 2 \)) and it is disjoint from any \((+n)\)-section, \( \mathbb{F}_n \) does not have such a triangle anticanonical curve. Therefore \( D_t \) is not biholomorphic to \( \mathbb{F}_n \).

Now we show from these that \( D_t \) \((t \neq 0)\) is biholomorphic to \( \mathbb{F}_{n-2} \). For this we recall that any small deformation of rational ruled surface \( \mathbb{F}_n \) must be of the form \( \mathbb{F}_{n-2k} \) where \( k \geq 0 \) and \( n - 2k \geq 0 \) (see [24] for the Kuranishi family of \( \mathbb{F}_n \).) Also, in these deformations the ruling \((i.e. \)the projection to \( \mathbb{CP}^1 \)) is preserved. Then as the pair \((D_t, L_\infty)\) is obtained as a small deformation of the pair \((D_0, L_\infty)\) which satisfies \( L_\infty^2 = n \), we still have \( L_\infty^2 = n \) on \( D_t \). This readily means that on \( D_t \simeq \mathbb{F}_{n-2k} \) we have

\[
(3.16) \quad L_\infty \sim \Gamma + (n - k)h \quad \text{(linear equivalence)}
\]

where \( h \) denotes the fiber class of the ruling, and \( \Gamma \) denotes a section of the ruling that satisfies \( \Gamma^2 = -(n - 2k) \). (Of course such a section is unique as long as \( n - 2k > 0 \).) On the other hand, on \( \mathbb{F}_{n-2k} \) we have \( K^{-1} \sim 2\Gamma + (2 + n - 2k)h \). As \( C_1 + C_2 + L_\infty \sim K^{-1} \) as above, it follows that we may suppose that

\[
(3.17) \quad C_1 \sim \Gamma + (1 - k)h, \quad C_2 \sim h.
\]
But since $C_1$ is an irreducible curve, we have $1 - k \geq 0$. Hence as $k > 0$ (since $D_k \not\sim F_n$ as above), we obtain $k = 1$. Thus we have $D_k \simeq F_{n-2}$ and $L_\infty \in |\Gamma + (n - 1)h|$. It is not difficult to show that this linear system induces an embedding $F_{n-2} \subset \mathbb{CP}^{n+1}$ whose image is a non-singular surface of degree $n$. Therefore the complement $F_{n-2} \setminus L_\infty$ is an algebraic surface in $\mathbb{C}^{n+1}$.

Theorem 1.2 gives a partial answer to a question by Viaclovsky \[27, 1.4 \text{ Question (2)}\] concerning scalar-flat Kähler deformations of the LeBrun’s ALE metric. Moreover, in relation with SU(2)-invariant scalar-flat Kähler metrics obtained by Pedersen-Poon \[19\], Theorem 1.2 shows that their metrics are incomplete as long as they are obtained as small deformations of the LeBrun’s ALE metric. On the other hand, when $k \not\in \{1, n - 1\}$, the $G(k, n - k)$-equivariant deformation obtained in Proposition 2.8 does not preserve Kählerity in general, since under the $G(k, n - k)$-equivariant deformation of the pair $(Z_{LB}, S)$ obtained in Proposition 3.5, the divisor $D + D$ can be shown to disappear from the structure of the deformed surface.

Because the twistor spaces obtained in Theorem 1.2 possess the irreducible singular member in $\mathbf{K}^{-1/2}$ whose minimal resolution is exactly the one appeared in \[8, 10\], it is very natural to expect that these twistor spaces on $\mathcal{O}(n)$ has a structure of a double cover of $\mathbb{CP}^3$ (if $n = 3$) or a scroll of planes in $\mathbb{CP}^n$ (if $n > 3$), whose branch divisor is a quartic surface (if $n = 3$) or a cut of the scroll by a quartic hypersurface (if $n > 3$). In other words, it is quite likely that the ASD Kähler metrics on $\mathcal{O}(n)$ in Theorem 1.2 could be obtained as a limit of the ASD structures on $n\mathbb{CP}^2$ which correspond to the twistor spaces investigated in \[9, 10\].

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