DISCRETE MORSE THEORY FOR TOTALLY NON-NEGATIVE
FLAG VARIETIES

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Abstract. In a seminal 1994 paper [20], Lusztig extended the theory of total
positivity by introducing the totally non-negative part \((G/P)\geq 0\) of an arbitrary
(generalized, partial) flag variety \(G/P\). He referred to this space as a “remarkable
polyhedral subspace,” and conjectured a decomposition into cells, which was subse-
quently proven by the first author [27]. In this article we use discrete Morse theory
to show that the cell decomposition of \((G/P)\geq 0\) is polyhedral in the following
sense: closures of cells are collapsible and hence contractible. This answers a ques-
tion posed by Lusztig in 1996 [23], and generalizes a later result of Lusztig’s [21],
that \((G/P)\geq 0\) – the closure of the top-dimensional cell – is contractible. Fur-
thermore, we show that the boundary of each cell – hence in particular the boundary
of \((G/P)\geq 0\) – is homotopy equivalent to a sphere.

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1. Introduction

The classical theory of total positivity studies matrices whose minors are all positive. Lusztig dramatically generalized this theory with a 1994 paper [20] in which

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he introduced the totally positive part of a real reductive group $G$ (totally positive matrices are recovered when $G$ is a general linear group). Lusztig also defined the (totally) positive part $(G/P)_{>0}$ of an arbitrary (generalized, partial) flag variety $G/P$, and its closure, the (totally) non-negative part $(G/P)_{\geq 0}$. Lusztig referred to $(G/P)_{\geq 0}$ as a “remarkable polyhedral subspace” \cite{20}, and conjectured a decomposition into cells, which was subsequently validated by the first author \cite{27}. (The stronger statement that this cell decomposition is a CW complex was proved in \cite{31}.)

A notion of total positivity for toric varieties also exists \cite[Section 4.1]{13}, so it is natural to compare the properties of the spaces $(G/P)_{\geq 0}$ to their toric counterparts. The non-negative part of a toric variety is homeomorphic – via the moment map – to its moment polytope \cite[Section 4.2]{13}, and in this sense is polyhedral. By contrast, the cell decomposition of the non-negative part of $G/P$ cannot be modeled by any polytope. For example, the totally non-negative part of the Grassmannian $Gr_{2,4}(\mathbb{R})$ has one top-dimensional cell of dimension 4 and four 3-dimensional cells; but there is no 4-dimensional polytope with four facets.

Nevertheless, in this article we show that the cell decomposition of $(G/P)_{\geq 0}$ has favorable topological properties. Our main result is that the closure of each cell is collapsible, hence contractible, and the boundary of each cell is homotopy equivalent to a sphere.

In \cite{21} Lusztig showed that the totally nonnegative parts $X_{\geq 0}$ are contractible for $X = G, U^+, U^-$ and $G/B$. His proof also shows that $(G/P)_{\geq 0}$ is contractible. Our contractibility result is a generalization of this and can be viewed as adding the intersections of opposed Bruhat cells $R_{\sigma,\tau}$ to the list of varieties $X$ with $X_{\geq 0}$ contractible.

The proof of our main result relies heavily on:

- Parameterizations of cells \cite{24} in terms of reduced expressions,
- the related construction of attaching maps for cells of $(G/P)_{\geq 0}$ \cite{31}, which proved that this cell complex is actually a CW complex,
- the description of the face poset of $(G/P)_{\geq 0}$ (the partially ordered set of closures of cells) \cite{28} and combinatorial results about it \cite{34}.

In addition, we draw on a variety of topological and combinatorial techniques. The most important tool is Forman’s Discrete Morse Theory \cite{12} for general CW complexes. In Section 7 we explain ideas of Chari \cite{10} and Kozlov \cite{17} which imply that we can think of a discrete Morse function as a certain kind of matching on the Hasse diagram of the face poset in which matched edges are regular. Therefore, we can reduce the problem of proving contractibility of closures of cells to the problem of constructing optimal Morse matchings (in this case with a single critical cell of dimension 0), such that whenever a cell $\sigma$ is matched to $\tau$, $\sigma$ is a regular face of $\tau$.

Proving that $\sigma$ is a regular face of $\tau$ seems quite difficult in general, particularly since our attaching maps are defined in a non-explicit way in terms of the canonical basis. However, for some pairs of cells $\sigma \prec \tau$ which we call ‘good’, we can show
that there is a parameterization for $\tau$ given by positive polynomials in $t_1, \ldots, t_p$ such that when we send $t_p$ to infinity (while keeping the other $t_i$’s finite), we get a parameterization for $\sigma$. In this case, one can show that the polytope $P$ which is the domain of the attaching map $h$ for $\tau$ has a face $P'$ which is the domain of an attaching map $h'$ for $\sigma$. We go on to prove that in this ‘good’ case, $\sigma$ is a regular face of $\tau$.

Knowing that certain edges of the face poset of $(G/P)_{\geq 0}$ are regular, we need to construct optimal Morse matchings on the Hasse diagram of the face poset of the closure of each cell, such that each matched edge is provably regular. Our proof uses Dyer’s reflection orders and his EL-labeling of Bruhat order [11], as well as the observation in Section 7.3 that one can pass explicitly from an EL-labeling of a CW poset to a Morse matching. This latter observation follows from work of Chari [10].

To get Morse matchings in which matched edges are provably regular, we need to make careful choices of reflection orders.

It is possible that something even stronger than our main result is true, namely, that $(G/P)_{\geq 0}$ together with its cell decomposition is a regular CW complex. This was conjectured by the second author in [34] and would imply in particular that $(G/P)_{\geq 0}$ is homeomorphic to a ball. Recall that a CW complex is regular if the closure of each cell is homeomorphic to a closed ball, and the boundary of each cell is homeomorphic to a sphere. Combinatorial evidence for this conjecture was given in [34], where it was shown that the face poset of $(G/P)_{\geq 0}$ is the face poset of some regular CW complex homeomorphic to a ball. The main achievement of the present paper is to prove the homotopy-equivalence version of this conjecture.

One might hope to use a recent result of Hersh [14], which gives a criterion for determining when each attaching map for a CW complex is a homeomorphism on its entire domain. However, this result could not be applied directly to the CW structure constructed in [31]: the attaching maps of [31] do not have that property.

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have the opposite Borel subgroup $B^-$ such that $B^+ \cap B^- = T$, and its unipotent radical $U^-$. Denote the set of simple roots by $\Pi = \{ \alpha_i \mid i \in I \} \subset \Phi^+$. For each $\alpha_i \in \Pi$ there is an associated homomorphism $\phi_i : SL_2 \to G$. Consider the 1-parameter subgroups in $G$ (landing in $U^+, U^-$, and $T$, respectively) defined by

$$x_i(m) = \phi_i \left( \begin{array}{cc} 1 & m \\ 0 & 1 \end{array} \right), \quad y_i(m) = \phi_i \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right), \quad \alpha_i^\vee(t) = \phi_i \left( \begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array} \right),$$

where $m \in \mathbb{R}, t \in \mathbb{R}^*, i \in I$. The datum $(T, B^+, B^-, x_i, y_i; i \in I)$ for $G$ is called a pinning. The standard pinning for $SL_d$ consists of the diagonal, upper-triangular, and lower-triangular matrices, along with the simple root subgroups $x_i(m) = I_d + mE_{i,i+1}$ and $y_i(m) = I_d + mE_{i+1,i}$ where $I_d$ is the identity matrix and $E_{i,j}$ has a 1 in position $(i, j)$ and zeroes elsewhere.

2.2. Folding. If $G$ is not simply laced, then one can construct a simply laced group $\hat{G}$ and an automorphism $\tau$ of $\hat{G}$ defined over $\mathbb{R}$, such that there is an isomorphism, also defined over $\mathbb{R}$, between $G$ and the fixed point subset $\hat{G}^\tau$ of $\hat{G}$. Moreover the groups $G$ and $\hat{G}$ have compatible pinnings. Explicitly we have the following.

Let $\hat{G}$ be simply connected and simply laced. We apply the same notations as in Section 2.1 for $G$, but with a dot, to our simply laced group $\hat{G}$. So we have a pinning $(\hat{T}, \hat{B}^+, \hat{B}^-, x_i, y_i, i \in \hat{I})$ of $\hat{G}$, and $\hat{I}$ may be identified with the vertex set of the Dynkin diagram of $\hat{G}$.

Let $\sigma$ be a permutation of $\hat{I}$ preserving connected components of the Dynkin diagram, such that, if $j$ and $j'$ lie in the same orbit under $\sigma$ then they are not connected by an edge. Then $\sigma$ determines an automorphism $\tau$ of $\hat{G}$ such that

1. $\tau(\hat{T}) = \hat{T},$
2. $\tau(x_i(m)) = x_{\sigma(i)}(m)$ and $\tau(y_i(m)) = y_{\sigma(i)}(m)$ for all $i \in \hat{I}$ and $m \in \mathbb{R}$.

In particular $\tau$ also preserves $\hat{B}^+, \hat{B}^-$. Let $\hat{I}$ denote the set of $\sigma$-orbits in $\hat{I}$, and for $\bar{i} \in \hat{I}$, let

$$x_{\bar{i}}(m) := \prod_{i \in \bar{i}} x_i(m),$$

$$y_{\bar{i}}(m) := \prod_{i \in \bar{i}} y_i(m).$$

The fixed point group $\hat{G}^\tau$ is a simply laced, simply connected algebraic group with pinning $(\hat{T}^\tau, \hat{B}^{\tau+}, \hat{B}^{\tau-}, x_{\bar{i}}, y_{\bar{i}}, \bar{i} \in \hat{I})$. There exists, and we choose, $\hat{G}$ and $\tau$ such that $\hat{G}^\tau$ is isomorphic to our group $G$ via an isomorphism compatible with the pinnings.

2.3. Flag varieties. The Weyl group $W = N_G(T)/T$ acts on $X(T)$ permuting the roots $\Phi$. We set $s_i := \hat{s}_iT$ where $\hat{s}_i := \phi_i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$. Then any $w \in W$ can be expressed as a product $w = s_{i_1} s_{i_2} \ldots s_{i_m}$ with $\ell(w)$ factors. This gives $W$ the structure of a Coxeter group; we will assume some basic knowledge of Coxeter systems and
Bruhat order as in [15]. We set \( \dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \ldots \dot{s}_{i_m} \). It is known that \( \dot{w} \) is independent of the reduced expression chosen.

We can identify the flag variety \( G/B \) with the variety \( B \) of Borel subgroups, via

\[
gB^+ \iff g \cdot B^+ := gB^+ g^{-1}.
\]

We have the Bruhat decompositions

\[
B = \bigsqcup_{w \in W} B^+ \dot{w} \cdot B^+ = \bigsqcup_{w \in W} B^- \dot{w} \cdot B^+
\]

of \( B \) into \( B^+ \)-orbits called Bruhat cells, and \( B^- \)-orbits called opposite Bruhat cells.

**Definition 2.1.** For \( v, w \in W \) define

\[
\mathcal{R}_{v,w} := B^+ \dot{w} \cdot B^+ \cap B^- \dot{v} \cdot B^+.
\]

The intersection \( \mathcal{R}_{v,w} \) is non-empty precisely if \( v \leq w \) in the Bruhat order, and in that case is irreducible of dimension \( \ell(w) - \ell(v) \), see [16].

Let \( J \subset I \). The parabolic subgroup \( W_J \subset W \) corresponds to a parabolic subgroup \( P_J \) in \( G \) containing \( B^+ \). Namely, \( P_J = \bigsqcup_{w \in W_J} B^+ \dot{w} B^+ \). Consider the variety \( \mathcal{P}^J \) of all parabolic subgroups of \( G \) conjugate to \( P_J \). This variety can be identified with the partial flag variety \( G/P_J \) via

\[
gP_J \iff gP_J g^{-1}.
\]

We have the usual projection from the full flag variety to a partial flag variety which takes the form \( \pi = \pi^J : B \rightarrow \mathcal{P}^J \), where \( \pi(B) \) is the unique parabolic subgroup of type \( J \) containing \( B \).

### 3. Total positivity for flag varieties

#### 3.1. The totally non-negative part of \( G/P \) and its cell decomposition.

**Definition 3.1.** [20] The totally non-negative part \( U^-_{\geq 0} \) of \( U^- \) is defined to be the semigroup in \( U^- \) generated by the \( y_i(t) \) for \( t \in \mathbb{R}_{\geq 0} \).

The totally non-negative part of \( B \) (denoted by \( B_{\geq 0} \) or by \( (G/B)_{\geq 0} \)) is defined by

\[
B_{\geq 0} := \{ u \cdot B^+ \mid u \in U^-_{\geq 0} \},
\]

where the closure is taken inside \( B \) in its real topology.

The totally non-negative part of a partial flag variety \( \mathcal{P}^J \) (denoted by \( \mathcal{P}^J_{\geq 0} \) or by \( (G/P_J)_{\geq 0} \)) is defined to be \( \pi^J(B_{\geq 0}) \).

Lusztig [20, 22] introduced natural decompositions of \( B_{\geq 0} \) and \( \mathcal{P}^J_{\geq 0} \).

**Definition 3.2.** [20] For \( v, w \in W \) with \( v \leq w \), let

\[
\mathcal{R}_{v,w;>0} := \mathcal{R}_{v,w} \cap B_{\geq 0}.
\]

We write \( W^J \) (respectively \( W^J_{\text{max}} \)) for the set of minimal (respectively maximal) length coset representatives of \( W/W_J \).
Definition 3.3. [22] Let $\mathcal{I}^J \subset W^J_{\text{max}} \times W_J \times W^J$ be the set of triples $(x, u, w)$ with the property that $x \leq wu$. Given $(x, u, w) \in \mathcal{I}^J$, we define $\mathcal{P}^J_{x,u,w;>0} := \pi^J(\mathcal{R}_{x,u,w;>0}) = \pi^J(\mathcal{R}_{x,u-1,w;>0})$.

The first author [27] proved that $\mathcal{R}_{v,w;>0}$ and $\mathcal{P}_{x,u,w;>0}$ are semi-algebraic cells of dimension $\ell(w) - \ell(v)$ and $\ell(wu) - \ell(x)$, respectively.

3.2. Parameterizations of cells. In [24], Marsh and the first author gave parameterizations of the cells $\mathcal{R}_{v,w;>0}$, which we now explain.

Let $v \leq w$ and let $w = (i_1, \ldots, i_m)$ encode a reduced expression $s_{i_1} \ldots s_{i_m}$ for $w$. Then there exists a unique subexpression $s_{i_1} \ldots s_{i_k}$ for $v$ in $w$ with the property that, for $l = 1, \ldots, k$,

$$s_{i_1} \ldots s_{i_l} s_{i_r} > s_{i_1} \ldots s_{i_l}$$

whenever $j_l < r \leq j_{l+1}$, where $j_{k+1} := m$. This is the “rightmost reduced subexpression” for $v$ in $w$, and it is called the “positive subexpression” in [24]. We use the notation

$$v_+ := \{j_1, \ldots, j_k\},$$

$$v_+^c := \{1, \ldots, m\} \setminus \{j_1, j_2, \ldots, j_k\},$$

for this special subexpression for $v$ in $w$. Note that this notation only makes sense in the context of a fixed $w$.

Now we can define the map

$$\phi_{v_+,w} : (\mathbb{C}^*)^{v_+^c} \to \mathcal{R}_{v,w},$$

$$(t_r)_{r \in v_+^c} \mapsto g_1 \ldots g_m \cdot B^+,$$

where

$$g_r = \begin{cases} s_{t_r}, & \text{if } r \in v_+, \\ y_{t_r}, & \text{if } r \in v_+^c. \end{cases}$$

Theorem 3.4. [24, Theorem 11.3] The restriction of $\phi_{v_+,w}$ to $(\mathbb{R}_{>0})^{v_+^c}$ defines an isomorphism of semi-algebraic sets,

$$\phi_{v_+,w}^0 : (\mathbb{R}_{>0})^{v_+^c} \to \mathcal{R}_{v,w;>0}.$$

We note that this parameterization generalizes Lusztig’s parametrization of totally nonnegative cells in $U_{>0}$ from [20]. Namely $U_{>0} = \bigsqcup_{w \in W} U_{>0}(w)$, for

$$U_{>0}(w) := \{y_1(t_1)y_2(t_2) \ldots y_m(t_m) \mid t_i \in \mathbb{R}_{>0}\},$$

where $w = (i_1, \ldots, i_m)$ is a/any reduced expression of $w$. Clearly, $\mathcal{R}_{1,w}^0 = U_{>0}(w) \cdot B^+$.

3.3. Change of coordinates under braid relations. In the simply laced case there is a simple change of coordinates [20, 29] which describes how two parameterizations of the same cell are related when considering two reduced expressions which differ by a commuting relation or a braid relation.

If $s_i s_j = s_j s_i$ then $y_i(a)y_j(b) = y_j(b)y_i(a)$ and $y_i(a)s_j = s_j y_i(a)$.

If $s_i s_j s_i = s_j s_i s_j$ then
(1) $y_i(a)y_j(b)y_i(c) = y_j \left( \frac{bc}{a+c} \right) y_i(a+c)y_j \left( \frac{ab}{a+c} \right)$,
(2) $y_i(a)s_jy_i(b) = y_j \left( \frac{b}{a} \right) y_i(a)s_jx_j \left( \frac{b}{a} \right)$,
(3) $\dot{s}_j\dot{s}_iy_j(a) = y_i(a)\dot{s}_j\dot{s}_i$.

In case (2) Lemma 11.4 from [24] implies that the factor $x_j \left( \frac{b}{a} \right)$ disappears into $B^+$ without affecting the remaining parameters when this braid relation is applied in the parametrization of a totally nonnegative cell.

The changes of coordinates have also been computed for more general braid relations and have been observed to be invertible, subtraction-free, homogeneous rational transformations [1, 29].

3.4. Total positivity and canonical bases for simply laced $G$. Assume that $G$ is simply laced. Let $U$ be the enveloping algebra of the Lie algebra of $G$; this can be defined by generators $e_i, h_i, f_i$ ($i \in I$) and the Serre relations. For any dominant character $\lambda$ there is a finite-dimensional simple $U$-module $V(\lambda)$ with a non-zero vector $\eta$ such that $e_i\eta = 0$ and $h_i\eta = \lambda_i\eta$ for all $i \in I$. The pair $(V(\lambda), \eta)$ is determined up to unique isomorphism.

There is a unique $G$-module structure on $V(\lambda)$ such that for any $i \in I$, $a \in \mathbb{R}$ we have

$$x_i(a) = \exp(a e_i) : V(\lambda) \to V(\lambda), \quad y_i(a) = \exp(a f_i) : V(\lambda) \to V(\lambda).$$

Then $x_i(a)\eta = \eta$ for all $i \in I$, $a \in \mathbb{R}$, and $t\eta = \lambda(t)\eta$ for all $t \in T$. Let $B(\lambda)$ be the canonical basis of $V(\lambda)$ that contains $\eta$ [18]. We now collect some useful facts about the canonical basis.

**Lemma 3.5.** [22, 1.7(a)]. For any $w \in W$, the vector $w\eta$ is the unique element of $B(\lambda)$ which lies in the extremal weight space $V(\lambda)^{w(\lambda)}$. In particular, $w\eta \in B(\lambda)$.

We define $f_i^{(p)}$ to be $\frac{f_i^p}{p!}$.

**Lemma 3.6.** Let $s_{i_1}\ldots s_{i_n}$ be a reduced expression for $w \in W$. Then there exists $a \in \mathbb{N}$ such that $f_{i_1}^{(a)}\dot{s}_{i_2}\ldots\dot{s}_{i_n}\eta = \dot{s}_{i_1}\dot{s}_{i_2}\ldots\dot{s}_{i_n}\eta$. Moreover, $f_{i_1}^{(a+1)}\dot{s}_{i_2}\ldots\dot{s}_{i_n}\eta = 0$.

**Proof.** This follows from Lemma 3.5 and properties of the canonical basis, see e.g. the proof of [19, Proposition 28.1.4].

4. $(G/P)_{\geq 0}$ as a CW complex: Attaching maps using toric varieties

Recall that a $CW$ complex is a union of cells $X$ with additional requirements on how cells are glued: in particular, for each cell $\sigma$, one must define a (continuous) attaching map $h : B \to X$ where $B$ is a closed ball, such that the restriction of $h$ to the interior of $B$ is a homeomorphism with image $\sigma$.

Even given the parameterizations of cells, it is not obvious how to define attaching maps. One needs to extend the domain of each map $\phi_{>0,w}$ from $(\mathbb{R}_{>0})^{V_{\mathbb{R}}}$ (an open ball) to a closed ball. However, a priori it is not clear how to let the parameters approach 0 and infinity. In this section we explain how, following earlier work for
Grassmannians [26], the authors [31] defined attaching maps for the cells and proved that the cell decomposition of \((G/P)_{\geq 0}\) is a CW complex.

Lemma 4.1 is the key to defining attaching maps. It says that one can compactify \((\mathbb{R}_{>0})^r\) inside a toric variety related to the parameterization, obtaining a closed ball (the non-negative part of the toric variety). We refer to [13] and [32] for the basics on toric varieties and their non-negative parts. Let \(S \in \mathbb{Z}^r\) be a finite set whose elements are ordered, \(m_0, \ldots, m_k\), and thought of as corresponding to monomials, \(t^m = t_1^{m_1}t_2^{m_2} \cdots t_r^{m_r}\). We let \(X_S\) denote the toric subvariety of \(\mathbb{P}^k\) associated to \(S\), and \(X_S^{>0}\) and \(X_S^{\geq 0}\) its positive and non-negative parts, respectively. Explicitly, \(X_S\) is the closure of the image of the associated map

\[
\chi = \chi_S : t = (t_1, \ldots, t_r) \mapsto [m_0^t, m_1^t, \ldots, m_k^t]
\]

from \((\mathbb{C}^*)^r\) to \(\mathbb{P}^k\), while \(X_S^{>0}\) and \(X_S^{\geq 0}\) are obtained as the image of \(\mathbb{R}_{>0}^r\) and its closure.

A fact which is crucial here is that \(X_S^{>0}\) is homeomorphic to a closed ball. More specifically, \(X_S\) has a moment map which gives a homeomorphism to the convex hull \(B_S\) of \(S\).

**Lemma 4.1.** [26] Suppose we have a map \(\phi : (\mathbb{R}_{>0})^r \to \mathbb{P}^N\) given by

\[
(t_1, \ldots, t_r) \mapsto [p_1(t_1, \ldots, t_r), \ldots, p_{N+1}(t_1, \ldots, t_r)],
\]

where the \(p_i\)'s are Laurent polynomials with positive coefficients. Let \(S\) be the set of all exponent vectors in \(\mathbb{Z}^r\) which occur among the (Laurent) monomials of the \(p_i\)'s, and let \(B_S\) be the convex hull of the points of \(S\). Then the map \(\phi\) factors through the totally positive part \(X_S^{>0}\), giving a map \(\Phi_{>0} : X_S^{>0} \to \mathbb{P}^N\). Moreover, \(\Phi_{>0}\) extends continuously to the closure to give a well-defined map \(\Phi_{\geq 0} : X_S^{\geq 0} \to \Phi_{>0}(X_S^{>0})\). Note that if we precompose with the isomorphism \(B_S \cong X_S^{>0}\) given by the moment map, we can consider the domain of \(\Phi_{>0}\) to be the polytope \(B_S\), a closed ball.

The following result constructs attaching maps for cells of \((G/P)_{\geq 0}\) [31].

**Theorem 4.2.** [31] For any \(G/B\) we can construct a positivity preserving embedding \(i : G/B \to \mathbb{P}^N\), for some \(N\), with the following property. For any totally non-negative cell \(\mathcal{R}_{x,w;>0}\) and parameterization \(\phi_{x,w}^{>0}\) as above, the composition

\[
i \circ \phi_{x,w}^{>0} : (\mathbb{R}_{>0})^{\mathbb{X}_+^r} \sim \mathcal{R}_{x,w;>0} \to \mathbb{P}^N
\]

takes the form

\[
i \circ \phi_{x,w}^{>0} : t = (t_r)_{r \in \mathbb{X}_+^r} \mapsto [p_1(t), \ldots, p_{N+1}(t)],
\]

where the \(p_j\)'s are polynomials with positive coefficients. Applying Lemma 4.1 to \(i \circ \phi_{x,w}^{>0}\), we get an attaching map \(\Phi_{x,w}^{>0} : X_{x,w}^{>0} \to \overline{\mathcal{R}_{x,w;>0}}\) where the non-negative toric variety \(X_{x,w}^{>0}\) is homeomorphic to its moment polytope \(B_{x,w}\).

In Theorem 4.2, the map \(i\) is defined as follows. When \(G\) is simply laced, we consider the representation \(V = V(\rho)\) of \(G\) with a fixed highest weight vector \(\eta\) and corresponding canonical basis \(B(\rho)\). We let \(i : B \to \mathbb{P}(V)\) denote the embedding
which takes $g \cdot B_+ \in \mathcal{B}$ to the line $\langle g \cdot \eta \rangle$. This is the unique $g \cdot B_+$-stable line in $V$. We specify points in the projective space $\mathbb{P}(V)$ using homogeneous coordinates corresponding to $\mathcal{B}(\rho)$. The theorem then follows using the positivity properties of the canonical basis in simply laced type.

If $G$ is not simply laced, we use a folding argument to deduce the result from the simply laced case: the map $i$ is given by $i'$ from Lemma 4.3 as we will now explain. Let $\hat{G}$ be the simply laced group with automorphism corresponding to $G$. We identify $G$ with $\hat{G}$ and use all of the notation from Section 2.2. For any $\bar{i} \in \bar{I}$ there is a simple reflection $s_i$ in $W$, which is represented in $\hat{G}$ by $
abla s_{\bar{i}} := \prod_{i \in \bar{i}} \hat{s}_i$. In this way any reduced expression $w = (\bar{i}_1, \ldots, \bar{i}_m)$ in $W$ gives rise to a reduced expression $\hat{w}$ in $\hat{W}$ of length $\sum_{k=1}^m |\bar{i}_k|$, which is determined uniquely up to commuting elements. To a subexpression $v$ of $w$ we can then associate a unique subexpression $\hat{v}$ of $\hat{w}$ in the obvious way.

Lemma 4.3. [31] Let $v, w$ be in $W$ with $v \leq w$.

1. We have $\mathcal{R}_{v, w; \geq 0} = \hat{\mathcal{R}}_{v, w; \geq 0} \cap \mathcal{B}_\tau$.
   In particular the composition $i' : \mathcal{R}_{v, w} \hookrightarrow \hat{\mathcal{R}}_{v, w} \to \mathbb{P}(V(\rho))$ is positivity preserving.

2. Suppose $w = (\bar{i}_1, \ldots, \bar{i}_m)$ is a reduced expression for $w$ in $W$, and $v_+ = (j_1, \ldots, j_k)$ is the positive subexpression for $v$. Then we have a commutative diagram,

$$
\begin{array}{ccc}
\mathcal{R}_{v_+, w+; \geq 0} & \xrightarrow{i} & \hat{\mathcal{R}}_{v_+, w+; \geq 0} \\
\phi_{\bar{\nu}_+, w}^0 & \uparrow & \phi_{\nu_+, w}^0 \\
\mathbb{R}_{\geq 0} & \xrightarrow{\bar{i}} & \mathbb{R}_{\geq 0},
\end{array}
$$

where the top arrow is the usual inclusion, the vertical arrows are both isomorphisms, and the map $\bar{i}$ has the form

$$(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_k),$$

where each $t_i$ is repeated $|\bar{i}_j|$ times on the right hand side.

Remark 4.4. For partial flag varieties we can also use Theorem 4.2 to construct an attaching map for each $\mathcal{P}_{x, u, w; \geq 0}$. The projection $\pi^j : \mathcal{R}_{x, u, w; \geq 0} \to \mathcal{P}_{x, u, w; \geq 0}$ is a homeomorphism so we take the composition $\Phi_{\bar{x}_u, w; \geq 0}^0 \circ \pi^j$ as our attaching map.

Theorem 4.5. [31] $(G/P)_{\geq 0}$ is a CW complex.
5. The poset $Q^I$ of cells of $(G/P_J)_{\geq 0}$ and a regularity criterion

In this section we will review the description of the face poset of $(G/P_J)_{\geq 0}$ which was given by the first author \cite{pos}. We will then prove Theorem 5.3 giving a condition which ensures that a cell $\sigma$ is a regular face of another cell $\tau$ with respect to the attaching map of $\tau$.

Definition 5.1. Let $K$ be a finite CW complex, and let $Q$ denote its set of cells, augmented by a least element $\hat{0}$. The notation $\sigma^{(p)}$ indicates that $\sigma$ is a cell of dimension $p$. We write $\tau > \sigma$ if $\sigma \neq \tau$ and $\sigma \subset \overline{\tau}$, where $\overline{\tau}$ is the closure of $\tau$, and we say $\sigma$ is a face of $\tau$. By convention we also say that $\tau > \hat{0}$ for all $\tau$. This gives $Q$ the structure of a partially ordered set, which we refer to as the face poset of $K$.

A description of the face poset of $(G/P_J)_{\geq 0}$ was given in \cite{pos}.

Theorem 5.2. \cite{pos} Fix $W$ and $W_J$, the Weyl group and its parabolic subgroup corresponding to $G/P_J$. Let $Q^J$ denote the face poset of $(G/P_J)_{\geq 0}$ with its decomposition into totally nonnegative cells. The elements of $Q^J$ are indexed by $I^J \cup \hat{0}$, where $I^J$ is as in Definition 3.3.

The order relations in $Q^J$ are described in terms of Weyl group combinatorics by

$$P_{x,u,w;>0}^J \leq P_{x',u',w';>0}^J$$

if and only if there exist $u_1,u_2 \in W_J$ with $u_1u_2 = u$ and $\ell(u) = \ell(u_1) + \ell(u_2)$, such that $x'u'^{-1} \leq xu_2^{-1} \leq wu_1 \leq w'$. Moreover $\hat{0} < P_{x,u,w;>0}^J$ for all $(x,u,w) \in I^J$.

Remark 5.3. When $G/P_J$ is a (type $A$) Grassmannian, $Q^J$ is the poset of cells of the totally non-negative Grassmannian, first studied by Postnikov \cite{post}.

When $P_{x,u,w;>0}^J < P_{x',u',w';>0}^J$ and $\dim P_{x,u,w;>0}^J = \dim P_{x',u',w';>0}^J + 1$, we will write $P_{x,u,w;>0}^J < P_{x',u',w';>0}^J$.

Suppose a cell $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let $B$ be a closed ball of dimension $p + 1$, and let $h : B \rightarrow K$ be the attaching map for $\tau$, i.e. $h$ is a continuous map that maps $\text{Int}(B)$ homeomorphically onto $\tau$. The following definition is essential to discrete Morse theory for general CW complexes, as collapses of cells must take place along regular edges.

Definition 5.4. \cite{dis} Definition 1.1] We say that $\sigma^{(p)}$ is a regular face of $\tau^{(p+1)}$ (with respect to the attaching map $h$ for $\tau$) and that $(\sigma, \tau)$ is a regular edge, if

1. $h : h^{-1}(\sigma) \rightarrow \sigma$ is a homeomorphism,
2. $h^{-1}(\sigma)$ is a closed $p$-ball.

To use discrete Morse theory in our situation we must find enough regular edges. However, the toric varieties and attaching maps in Theorem 4.2 are constructed using the canonical basis, and hence are not at all explicit. Thus at first glance it might seem hopeless to deduce whether a cell $\sigma$ is a regular face of $\tau$ with respect to an attaching map $h$ for $\tau$. Fortunately, by the following result we do have a situation
of which we can prove regularity of a pair of faces. We will first prove Theorem 5.5 in the case of complete flag varieties, and then generalize it to partial flag varieties.

**Theorem 5.5.** Consider $P^I_{x,u,w;>0} \supset P^J_{x',u',w;>0}$ in $(G/P_J)_{>0}$ and let $w = (i_1, \ldots, i_m)$ be a reduced expression for $w$. We call this pair of cells **good** with respect to $w$ if the positive subexpression $x'x_{+1}^1$ is equal to $xu_{+1}^1 \cup \{k\}$ and moreover $xu_{+1}^1$ contains $\{k+1, \ldots, m\}$. In this case $P^J_{x',u',w;>0}$ is a regular face of $P^I_{x,u,w;>0}$ with respect to the attaching map $\Phi_{xu_{+1}^1,w}^0 \circ \pi^J$.

When the choice of reduced expression $w$ and the attaching map are clear from context, we will sometimes omit the phrase **with respect to** $w$ or **with respect to** the attaching map.

**Proposition 5.6.** Choose a reduced expression $w = (i_1, \ldots, i_m)$ for $w$, and suppose that the pair $R_{v,u;>0} \supset R_{v',u;>0}$ is good with respect to $w$. Suppose that $v_+$ and $v'_+$ are related by $v'_+ = v_+ \cup \{k\}$. Then $X_{v'_+,w}$ can be identified with a sub-toric variety of $X_{v+,w}$, and its moment polytope $B_{v'_+,w}$ is a facet of the moment polytope $B_{v+,w}$ of $X_{v+,w}$. Moreover, the attaching map $\Phi_{v'_+,w}^0 : X_{v'_+,w}^0 \to R_{v,u;>0}$ restricts to $X_{v'_+,w}^0$ to give the attaching map $\Phi_{v'_+,w}^0$ for $\overline{R_{v',u;>0}}$.

**Proof.** Let us first consider the case that $G$ is simply-laced. By our assumptions the parameterizations of the two cells take the form

\[
\phi_{v'_+,w}^0 = g_1 \cdots g_{k-1}y_{i_k}(t_k)\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot B^+,
\]

\[
\phi_{v'_+,w}^0 = g_1 \cdots g_{k-1}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot B^+.
\]

If we compose the parameterization $\phi_{v'_+,w}^0$ with the inclusion $i : R_{v,u} \hookrightarrow \mathbb{P}^N$ from Theorem 4.2 we get a map

\[
t = (t_{h_1}, \ldots, t_{h_v}, t_k) \mapsto [p_1(t), \ldots, p_{N+1}(t)],
\]

where the $p_j$’s are polynomials with positive coefficients. We note that, by the definition of the map $i$,

\[
[p_1(t), \ldots, p_{N+1}(t)] = \langle g_1 \cdots g_{k-1}y_{i_k}(t_k)\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta \rangle,
\]

with notation as in Section 3.4 and where we identify the projective space $\mathbb{P}^N$ with $\mathbb{P}(V(\rho))$ using the canonical basis.

If we take the limit as $t_k \to \infty$ we obtain a new map

\[
(5.1) \quad t' = (t_{h_1}, \ldots, t_{h_v}) \mapsto [p'_1(t'), \ldots, p'_{N+1}(t')]
\]

\[
= \lim_{t_k \to \infty} \langle g_1 \cdots g_{k-1}y_{i_k}(t_k)\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta \rangle = \langle g_1 \cdots g_{k-1}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta \rangle.
\]

To justify this last equality, recall from Section 3.4 that $y_{i_k}(t_k) = \exp(t_k f_{i_k})$ on $V(\rho)$; also note that if $a$ is the positive integer such that

\[
\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta = \hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta,
\]

\[
f_{i_k}^{(a)}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta = 0,
\]

\[
f_{i_k}^{(a+1)}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta = 0,
\]

\[
f_{i_k}^{(a+1)}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta = 0,
\]

\[
f_{i_k}^{(a)}\hat{s}_{i_{k+1}} \cdots \hat{s}_{i_m} \cdot \eta = 0.
\]
as in Lemma 3.6 then \( a \) is the highest power of \( t_k \) appearing in any \( p_j \). Moreover, as we are working in projective coordinates we may divide each \( p_j(t) \) by \( t_k^a \) and take

\[
p_j'(t') = \lim_{t_k \to \infty} \frac{1}{t_k^a} p_j(t).
\]

The monomials of \( p_j(t) \) which don’t vanish in this limit are precisely those which are multiples of this maximal power, \( t_k^a \).

It follows that the toric variety \( X_{\mathbf{v}', \mathbf{w}} \) is the sub-toric variety of \( X_{\mathbf{v}, \mathbf{w}} \) which is given precisely by those monomials which are multiples of \( t_k^a \) (and other coordinates set to zero). Its moment polytope can be identified with the face of \( B_{\mathbf{v}, \mathbf{w}} \) cut out by the hyperplane \( x_k = a \). Moreover from (5.1) it follows that the attaching map \( \Phi_{\mathbf{v}', \mathbf{w}}^\geq : X_{\mathbf{v}', \mathbf{w}}^\geq \to \mathcal{R}_{\mathbf{v}', \mathbf{w}}^{\geq 0} \) is the restriction of \( \Phi_{\mathbf{v}, \mathbf{w}}^\geq \). This also implies that \( B_{\mathbf{v}', \mathbf{w}} \), which is isomorphic to \( X_{\mathbf{v}', \mathbf{w}}^\geq \), has codimension 1 in \( B_{\mathbf{v}, \mathbf{w}} \), making it a facet.

In the non simply-laced case the proof is analogous. However now the attaching map for \( \mathcal{R}_{p, \mathbf{w}}^{p, \geq 0} \) is obtained from \( \phi_{\mathbf{v}, \mathbf{w}}^\geq \circ \tilde{\iota} \) as in Lemma 4.3 where \( \phi_{\mathbf{v}, \mathbf{w}}^\geq \) is the corresponding parameterization in the related simply laced group \( \mathcal{G} \). So we are looking at parameterizations of \( \mathcal{R}_{\mathbf{v}, \mathbf{w}}^{p, \geq 0} \) and \( \mathcal{R}_{\mathbf{v}', \mathbf{w}}^{p, \geq 0} \) embedded into \( \mathcal{R}_{\mathbf{v}, \mathbf{w}} \) and \( \mathcal{R}_{\mathbf{v}', \mathbf{w}} \), respectively, which take the form

\[
\begin{align*}
\mathbf{t} & \mapsto \tilde{g}_1 \ldots \tilde{g}_{k-1} y_{ik,1}(t_k) y_{ik,2}(t_k) \ldots y_{ik,l}(t_k) \hat{s}_{ik+1} \ldots \hat{s}_{im} \cdot B^+, \\
\mathbf{t'} & \mapsto \tilde{g}_1 \ldots \tilde{g}_{k-1} \hat{s}_{ik+1} \ldots \hat{s}_{im} \cdot B^+, 
\end{align*}
\]

where \( \hat{s}_{ik} = \hat{s}_{ik,1} \cdot \hat{s}_{ik,2} \ldots \hat{s}_{ik,l} \).

As before, there are unique positive integers \( a_1, \ldots, a_l \) such that

\[
f_{ik,1}^{(a_1)} \ldots f_{ik,l}^{(a_1)} \hat{s}_{ik+1} \ldots \hat{s}_{im} \cdot \eta = \hat{s}_{ik,1} \hat{s}_{ik,2} \ldots \hat{s}_{ik,l} \hat{s}_{ik+1} \ldots \hat{s}_{im} \cdot \eta, 
\]

and for each \( 1 \leq h \leq l \), if we increase the corresponding exponent by 1, we have

\[
f_{ik,h}^{(a_h+1)} \ldots f_{ik,l}^{(a_1)} \hat{s}_{ik+1} \ldots \hat{s}_{im} \cdot \eta = 0.
\]

Now the composition \( \iota' \circ \phi_{\mathbf{v}', \mathbf{w}}^\geq \) for \( \iota' \) as in Lemma 4.3 takes the form

\[
\mathbf{t} \mapsto [p_1(t), \ldots, p_{N+1}(t)],
\]

for polynomials \( p_j \) with positive coefficients. And by the observation about the \( f_{ik,h} \)’s, the maximal power of \( t_k \) in any of the \( p_j \)’s is \( t_k^{a_1 + \ldots + a_l} \).

Finally we look at what happens if \( t_k \) tends to infinity and repeat the arguments from the simply-laced case. In this case \( X_{\mathbf{v}', \mathbf{w}} \) is the sub-toric variety of \( X_{\mathbf{v}, \mathbf{w}} \), which is given by those monomials which are multiples of \( t_k^{a_1 + \ldots + a_l} \) (and other coordinates set to zero), and its moment polytope can be identified with the face of \( B_{\mathbf{v}, \mathbf{w}} \) cut out by the equations \( x_1 = a_1, \ldots, x_l = a_l \). Moreover from the analogue of (5.1) it follows that the attaching map \( \Phi_{\mathbf{v}', \mathbf{w}}^\geq : X_{\mathbf{v}', \mathbf{w}}^\geq \to \mathcal{R}_{\mathbf{v}', \mathbf{w}}^{\geq 0} \) is the restriction of \( \Phi_{\mathbf{v}, \mathbf{w}}^\geq \). This also implies that \( B_{\mathbf{v}', \mathbf{w}} \), which is isomorphic to \( X_{\mathbf{v}', \mathbf{w}}^\geq \), has codimension 1 in \( B_{\mathbf{v}, \mathbf{w}} \), making it a facet. \( \square \)
Remark 5.7. We note that in the situation of Proposition 5.6 we have also shown that for any point \( \chi(t_1, \ldots, t_r) \) in \( X_{G/B}^{t_1, \ldots, t_r} \), with \( \chi \) as in Section 5.1, and any positive integer \( c \), the limit \( \lim_{z \to -\infty} \chi(t_1, \ldots, t_r, z^c t_r) \) lies in \( X_{G/B}^{t_1, \ldots, t_r} \).

Remark 5.8. Proposition 5.6 is a big step towards proving Theorem 5.5 for \( (G/B)_{\geq 0} \). To relate our notation to Definition 5.4, let \( \tau = R_{v,w;>0}, \sigma = R_{v',w;>0}, h = \Phi_{v,w}^{>0}, \) and \( h' = \Phi_{v',w}^{<0} \). By Proposition 5.6, \( h^{-1}(\sigma) \) contains \( X_{v,w}^{>0} \). If we could show that this is an equality, then because \( h|_{X_{v,w}^{>0}} \) is the attaching map \( h' \), the restriction \( h|_{h^{-1}(\sigma)} = h|_{X_{v,w}^{>0}} \) would be a homeomorphism, proving that Definition 5.4 (1) is satisfied. Furthermore, \( h^{-1}(\sigma) = X_{v',w}^{>0} \) is a closed ball of appropriate dimension, verifying (2).

Proposition 5.9. Suppose that \( w > v' > v \) and we have a reduced expression \( w = (i_1, \ldots, i_j, \ldots, i_m) \) such that \( v' = s_{i_j+1} s_{i_{j+2}} \ldots s_{i_m} \), and \( v \) is obtained from \( v' \) by removing a unique factor \( s_k \) for \( j + 1 \leq k \leq m \). Suppose furthermore that we have a sequence \((c_1, c_2, \ldots, c_j, c_k) \in \mathbb{Z}^{j+1} \) such that for some/any fixed \( t_1, t_2, \ldots, t_j, t_k > 0 \) the family of elements

\[
g_z \cdot B^+ := y_{i_1}(z^{c_1} t_1) \ldots y_{i_j}(z^{c_j} t_j) s_{i_{j+1}} \ldots s_{i_k} \ldots y_{i_m}(z^{c_m} t_m) B^+, z > 0
\]

in \( R_{w,>0} \) tends as \( z \to \infty \) to an element of \( R_{v',>0} \). Then we must have \( c_1 = \cdots = c_j = 0 \) and \( c_k > 0 \).

Proof. Recall that since \( w \) ends in \( v' \), that is, \( \ell(wv'^{-1}) = \ell(w) - \ell(v') \), having a continuous map

\[
\pi = \pi_{w,v'^{-1}} : B^+w \cdot B^+ \to B^+w'v'^{-1} \cdot B^+.
\]

See for example Section 4.3 of [24]. In terms of our parameterizations, if \( x < w \) and we consider an element

\[
\phi_{x,w}(t) = g_1 \cdots g_m \cdot B^+ \in R_{x,w}
\]

then \( \pi(g_1 \cdots g_m \cdot B^+) \) is just given by deleting the last \( m - j \) factors spelling out the \( v' \):

\[
\pi(g_1 \cdots g_m \cdot B^+) = g_1 \cdots g_j \cdot B^+.
\]

Note that in particular \( \pi \) preserves total nonnegativity and takes both \( R_{v',>0} \) and \( R_{w,>0} \) to the same cell, namely \( R_{e,uv'^{-1};>0} \).

Now since the limit of \( g_z \cdot B^+ \) is assumed to lie in \( R_{v',>0} \) everything is taking place in \( B^+w \cdot B^+ \), the domain of \( \pi \), and we can apply \( \pi \) to \( g_z \cdot B^+ \) before and after taking the limit \( z \to \infty \):

\[
\lim(\pi(g_z \cdot B^+)) = \pi(\lim(g_z \cdot B^+)) \in R_{e,uv'^{-1};>0}.
\]

So we see that

\[
\pi(g_z \cdot B^+) = y_{i_1}(z^{c_1} t_1) \ldots y_{i_j}(z^{c_j} t_j) \cdot B^+
\]
is a 1-parameter family in $\mathcal{R}_{v,w^{v-1};0}$ whose limit point as $z \to \infty$ again lies in $\mathcal{R}_{v,w^{v-1};0}$. However, suppose that one of the $c_1, \ldots, c_j$ is nonzero. Then taking the limit would certainly give something that left the cell $\mathcal{R}_{v,w^{v-1};0}$ and went to a smaller one. So all of these $c_i$ must be zero.

Given that the $c_i$ are zero for $i \leq j$, it is clear that $c_k$ must be positive for the limit of the original family to lie in $\mathcal{R}_{v,w;0}$.

\begin{proof}
Let us first assume that $w_0 = (j_1, \ldots, j_n)$ ends with a reduced expression for $v'$. Then we are in the situation of Proposition 5.9 and so in the terms of the coordinates of the parameterization $\phi_{v,w}^v$, there is a unique vector $C \in \mathbb{Z}^r$ giving a 1-parameter family $g_z \cdot B^+ = \phi_{v,w}^v(z^C \cdot t)$, whose limit point lies in $\mathcal{R}_{v,w;0}$.

Recall that any two reduced expressions for $w_0$ can be related by braid and commuting relations, and suppose now that $w_0$ is any reduced expression for $w_0$. It suffices to prove that if the statement of the Proposition holds for $w_0$ then it also holds for any $w'_0$ obtained from $w_0$ by a braid relation or commuting relation.

This is obvious in the case of a commuting relation. Now suppose $w_0$ and $w'_0$ are related by a more general braid relation, and $C$ is the vector associated (up to positive scalar multiple) to $w_0$. The braid relation gives us a change of coordinates $\kappa(t) = t'$ which is rational, homogeneous and subtraction-free, see Section 3.3. We let $t'_z := \kappa(z^C \cdot t)$. For example, if $w_0$ is the longest element of the symmetric group $S_3$, then applying Formula (1) of Section 3.3 to $y_1(z^{c_1} t_1) y_2(z^{c_2} t_3) y_3(z^{c_3} t_3) \cdot B^+$, gives

$$y_2 \left( \frac{z^{c_2 + c_3 t_3} t_3}{z^{c_1} t_1 + z^{c_3} t_3} \right) y_1 \left( z^{c_1} t_1 + z^{c_3} t_3 \right) y_2 \left( \frac{z^{c_1 + c_3} t_1 t_3}{z^{c_1} t_1 + z^{c_3} t_3} \right) \cdot B^+,$$

and the entries are the components of $t'_z$. Because the components of $t'_z$ are subtraction-free, the maximal power of $z$ in each one dominates the limit as $z \to \infty$. In this example, we have therefore

$$\lim_{z \to \infty} \left( \phi_{v_0, w'_0}^{v_0}(t'_z) \right) = \lim_{z \to \infty} \left( \phi_{v_0, w'_0}^{v_0}(z^{c_2 + c_3 - \max(c_1, c_3)} q_1(t), z^{\max(c_1, c_3)} q_2(t), z^{c_1 + c_2 - \max(c_1, c_3)} q_3(t)) \right)$$

where the $q_i(t)$ are new rational, subtraction-free functions in the $t_j$'s, and we define $C' := (c_2 + c_3 - \max(c_1, c_3), \max(c_1, c_3), c_1 + c_2 - \max(c_1, c_3))$. In general, the same procedure can be applied to define a $C'$ out of the original $C$ as well as new rational,
transformation, which one may compare to the zones 9.1 and 9.2 of [20].

C unique (up to positive scalar multiple) choice of vector C to positive scalar multiple.

However this implies that

\( R \in \mathbb{Z}^r \) with the required property for our new reduced expression \( w_0' \).

To prove uniqueness, suppose \( D' \in \mathbb{Z}^r \) is a different element such that

\[
\lim_{z \to \infty} (\phi_{v^+, w_0'}(z^{C'} \cdot t')) \in R \in \mathbb{Z}^r
\]

for some \( t' \in R > 0 \). Then we may apply the coordinate transformation back from the reduced expression \( w_0' \) to \( w_0 \). Thus \( D' \) is transformed by a piece-wise-linear transformation to a \( D \in \mathbb{Z}^r \) such that

\[
\lim_{z \to \infty} (\phi_{v^+, w_0}(z^{D} \cdot t)) \in R \in \mathbb{Z}^r.
\]

However this implies that \( D \) is a positive multiple of \( C \), and by applying the original transformation again, that \( D' \) was a positive multiple of \( C' \).

Proposition 5.11. Choose \( w > v' > v \) and a reduced expression \( w = (i_1, \ldots, i_m) \). Let \( v^+_c = \{h_1, \ldots, h_r\} \) for \( h_1 < \cdots < h_r \) and suppose that \( v^+_c \) is equal to \( v^+_c \cup \{h_r\} \).

Write \( g_1 \cdots g_m \cdot B^+ \) for an element of \( R_{v^+, w_0} > 0 \) given in the parameterization corresponding to \( (v^+, w) \). As in Proposition 5.10, for \( (c_{i_1}, \ldots, c_{i_r}) \in \mathbb{Z}^r \) and \( z > 0 \), we let \( g_z \cdot B^+ \) be \( g_1 \cdots g_m \cdot B^+ \) with each \( y_{h_i}(t_{h_i}) \) replaced by \( y_{h_i}(z^{c_{i_r}} t_{h_i}) \). Then if the family of elements \( g_z \cdot B^+ \) in \( R_{v^+, w_0} > 0 \) tends as \( z \to \infty \) to an element of \( R_{v^+, w_0} > 0 \), we must have \( c_{i_1} = \cdots = c_{i_r} = 0 \) and \( c_{i_r} > 0 \).

Proof. Suppose \( (c_{i_1}, \ldots, c_{i_r}) \in \mathbb{Z}^r \) has the property,

\[
g_z \cdot B^+ \text{ has limit in } R_{v^+, w_0} > 0 \text{ as } z \to \infty,
\]

(5.2)

for the corresponding one-parameter family \( g_z \cdot B^+ \). Choose a reduced expression \( w_0 = (j_1, \ldots, j_{n-m}, i_1, \ldots, i_m) \) for \( w_0 \) ending with \( w \), and let us fix \( u_1, \ldots, u_{n-m} \in \mathbb{R}_{>0} \). Then we obtain a new one-parameter family,

\[
y_{j_1}(u_1) \cdots y_{j_{n-m}}(u_{n-m}) g_z \cdot B^+,
\]

which lies in \( R_{v^+, w_0} > 0 \) for \( z > 0 \) and tends to an element in \( R_{v^+, w_0} > 0 \) as \( z \to \infty \). Now Proposition 5.10 is applicable and we have that \( C = (0, \ldots, 0, c_{i_1}, c_{i_2}, \ldots, c_{i_r}) \) is the unique (up to positive scalar multiple) choice of vector \( C \in \mathbb{Z}^{n-m+r} \) such that the corresponding 1-parameter family in \( R_{v^+, w_0} > 0 \) tends to a point in \( R_{v^+, w_0} > 0 \). It follows that the original \( r \)-tuple \( (c_{i_1}, \ldots, c_{i_r}) \) satisfying (5.2) is also uniquely determined up to positive scalar multiple.
Now it only remains to prove that (5.2) holds for \((c_{h_1}, \ldots, c_{h_{r-1}}, c_{h_r}) = (0, \ldots, 0, 1)\). But this is clear, by the same argument we used for (5.1) in the proof of Proposition 5.6.

We now turn to the proof of Theorem 5.5.

**Proof of Theorem 5.5.** We begin with the full flag variety case. Recall the natural inclusion \(X_{v'}^0 \subseteq X_{v'}^0, w_0\), given by Proposition 5.6, for which \(\Phi_{v', w_0}^0(X_{v'}^0, w_0) = \mathcal{R}_{v', w_0}^0\). By Remark 5.8, it suffices to prove the claim that

\[
(\Phi_{v', w}^0)^{-1}(\mathcal{R}_{v', w}; 0) = X_{v', w}^0.
\]

Suppose we have \(x' \in X_{v'}^0, w_0\) such that \(\Phi_{v', w_0}^0(x') \in \mathcal{R}_{v', w_0}^0\). We can approach \(x'\) from a point in the interior, \(X_{v'}^0, w_0\), by a 1-parameter family. Namely,

\[
x' = \lim_{z \to \infty} \chi(z^C \cdot t) = \lim_{z \to \infty} \chi(z^{c_1} t_1, \ldots, z^{c_r} t_r),
\]

for some \(t \in \mathbb{R}_{>0}^r, C \in \mathbb{Z}^r\) and \(\chi\) the map from (4.1) associated to \(X_{v'}^0, w_0\). Therefore

\[
\Phi_{v', w_0}^0(x') = \lim_{z \to \infty} \chi(z^C \cdot t) = \lim_{z \to \infty} g_z \cdot B^+,\n\]

where \(g_z \cdot B^+\) is as in Proposition 5.11. Since we assumed \(\Phi_{v', w_0}^0(x') \in \mathcal{R}_{v', w_0}^0\), and are in the ‘good’ situation, Proposition 5.11 implies that \(C = (0, \ldots, 0, c)\), for positive \(c\). But this implies that

\[
x' = \lim_{z \to \infty} \chi(z^C \cdot t) \in X_{v', w_0}^0,
\]

see Remark 5.7. Therefore the claim (5.3) holds and the Theorem is true for the full flag variety.

Now consider the case of \(G/P_j\). We have that \(\pi^J\) gives an isomorphism from \(\mathcal{R}_{xu^{-1}, w_0}^0\) to \(P_{x, u, w_0}^J\) and from \(\mathcal{R}_{x'u^{-1}, w_0}^0\) to \(P_{x'u', u, w_0}^J\). We’ve already proved that \(\mathcal{R}_{x'u^{-1}, w_0}^0\) is a regular face of \(\mathcal{R}_{xu^{-1}, w_0}^0\) with respect to the attaching map \(\Phi_{xu_{1}^{-1}, w}^0\).

Recall that the attaching map for \(P_{x, u, w_0}^J\) is simply \(\pi^J \circ \Phi_{xu_{1}^{-1}, w}^0\).

When we restrict \(\pi^J\) to \(\overline{\mathcal{R}_{xu^{-1}, w_0}^0}\), it is straightforward to check that the preimage of \(\mathcal{P}_{x'u', w_0}^J\) is \(\mathcal{R}_{x'u^{-1}, w_0}^0\). By Theorem 5.5 in the full flag variety case, we know that \(\Phi_{xu_{1}^{-1}, w}^0(\mathcal{R}_{x'u^{-1}, w_0}^0) = X_{x'u_{1}^{-1}, w}^0\) and \(\Phi_{xu_{1}^{-1}, w}^0\) is a homeomorphism from \(X_{x'u_{1}^{-1}, w}^0\) to \(\mathcal{R}_{x'u^{-1}, w_0}^0\). Therefore \((\pi^J \circ \Phi_{xu_{1}^{-1}, w}^0)^{-1}(\mathcal{R}_{x'u^{-1}, w_0}^0) = X_{x'u_{1}^{-1}, w}^0\) and \(\pi^J \circ \Phi_{xu_{1}^{-1}, w}^0\) is a homeomorphism from \(X_{x'u_{1}^{-1}, w}^0\) to \(\mathcal{R}_{x'u^{-1}, w_0}^0\). It follows that \(\mathcal{P}_{x'u', u, w_0}^J\) is a regular face of \(P_{x, u, w_0}^J\) with respect to the attaching map \(\pi^J \circ \Phi_{xu_{1}^{-1}, w}^0\).

\(\square\)

6. Preliminaries on poset topology

6.1. Preliminaries. Poset topology is the study of combinatorial properties of a partially ordered set, or poset, which reflect the topology of an associated simplicial
or cell complex. In this section we will review some of the basic definitions and results of poset topology.

Let \( P \) be a poset with order relation \(<\). We will use the symbol \(<\) to denote a covering relation in the poset: \( x < y \) means that \( x < y \) and there is no \( z \) such that \( x < z < y \). Additionally, if \( x < y \) then \([x, y]\) denotes the closed interval from \( x \) to \( y \); that is, the set \( \{z \in P \mid x \leq z \leq y \} \).

We will often identify a poset \( P \) with its Hasse diagram, which is the graph whose vertices represent elements of \( P \) and whose edges depict covering relations.

The natural geometric object associated to a poset \( P \) is the realization of its order complex (or nerve). The order complex \( \Delta(P) \) is the simplicial complex whose vertices are the elements of \( P \) and whose simplices are the chains \( x_0 < x_1 < \cdots < x_k \) in \( P \).

A poset is called bounded if it has a least element \( \hat{0} \) and a greatest element \( \hat{1} \). The atoms of a bounded poset are the elements which cover \( \hat{0} \). Dually, the coatoms are the elements which are covered by \( \hat{1} \). A finite poset is said to be pure if all maximal chains have the same length, and graded, if in addition, it is bounded. An element \( x \) of a graded poset \( P \) has a well-defined rank \( \rho(x) \) equal to the length of an unrefinable chain from \( \hat{0} \) to \( x \) in \( P \).

### 6.2. Shellability and edge-labelings

A pure finite simplicial complex \( \Delta \) is said to be shellable if its maximal faces can be ordered \( F_1, F_2, \ldots, F_n \) in such a way that \( F_k \cap (\bigcup_{i=1}^{k-1} F_i) \) is a nonempty union of maximal proper faces of \( F_k \) for \( k = 2, 3, \ldots, n \).

Certain edge-labelings of posets can be used to prove that the corresponding order complexes are shellable. These techniques were pioneered by Bjorner [2], and Bjorner and Wachs [3].

One technique that can be used to prove that an order complex \( \Delta(P) \) is shellable is the notion of lexicographic shellability, or EL-shellability, which was first introduced by Bjorner [2]. Let \( P \) be a graded poset, and let \( \mathcal{E}(P) \) be the set of edges of the Hasse diagram of \( P \), i.e. \( \mathcal{E}(P) = \{(x, y) \in P \times P \mid x \geq y\} \). An edge labeling of \( P \) is a map \( \lambda : \mathcal{E}(P) \to \Lambda \) where \( \Lambda \) is some poset (usually the integers). Given an edge labeling \( \lambda \), each maximal chain \( c = (x_0 \geq x_1 \geq \cdots \geq x_k) \) of length \( k \) can be associated with a \( k \)-tuple \( \sigma(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)) \). We say that \( c \) is an increasing chain if the \( k \)-tuple \( \sigma(c) \) is increasing; that is, if \( \lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k) \). The edge labeling allows us to order the maximal chains of any interval of \( P \) by ordering the corresponding \( k \)-tuples lexicographically. If \( \sigma(c_1) \) lexicographically precedes \( \sigma(c_2) \) then we say that \( c_1 \) lexicographically precedes \( c_2 \) and we denote this by \( c_1 <_L c_2 \).

**Definition 6.1.** An edge labeling is called an EL-labeling (edge lexicographical labeling) if for every interval \([x, y]\) in \( P \),

1. there is a unique increasing maximal chain \( c \) in \([x, y]\), and
2. \( c <_L c' \) for all other maximal chains \( c' \) in \([x, y]\).
If one has an EL-labeling of $P$, it is not hard to see that the corresponding order on maximal chains gives a shelling of the order complex \([2]\). Therefore a graded poset that admits an EL-labeling is said to be EL-shellable.

Given an EL-labeling $\lambda$ of $P$ and $x \in P$, we define $\text{Last}_\lambda(x)$ to be the set of elements $z \preceq x$ such that $\lambda(z \preceq x)$ is maximal among the set $\{\lambda(y \preceq x) \mid y \preceq x\}$.

### 6.3. Face posets of cell complexes

When analyzing a CW complex $K$, it is sometimes useful to study its face poset $\mathcal{F}(K)$, as in Definition \([5,1]\). The face poset is a natural poset to study particularly if the CW complex has the subcomplex property, i.e. if the closure of a cell is a union of cells.

The class of regular CW complexes is particularly nice. Recall that a CW complex is regular if the closure of each cell is homeomorphic to a closed ball and if additionally the closure minus the interior of a cell is homeomorphic to a sphere. In general, the order complex $\|\mathcal{F}(K) - \{\hat{0}\}\|$ does not reveal the topology of $K$. However, the following result shows that regular CW complexes are combinatorial objects in the sense that the incidence relation of cells determines their topology.

**Proposition 6.2.** [4, Proposition 4.7.8] Let $K$ be a regular CW complex. Then $K$ is homeomorphic to $\|\mathcal{F}(K) - \{\hat{0}\}\|$.

We will call a poset $P$ a CW poset if it is the face poset of a regular CW complex.

There is a notion of shelling for regular cell complexes (which is distinct from the notion of shelling of the order complex), due to Bjorner and Wachs. Such a shelling is a certain ordering on the coatoms of the face poset. We don’t need the precise definition, only the following result that an EL-labeling of the face poset of a regular cell complex gives rise to a shelling.

**Theorem 6.3.** [6, Theorem 5.11] If $P$ is the face poset of a regular CW complex $K$, then any EL-labeling of $P$ gives rise to a shelling of $K$. To go from the EL-labeling to the shelling one chooses the ordering on coatoms which is specified by the order on edges between the unique greatest element and the co-atoms.

### 7. Discrete Morse theory for general CW complexes

In this section we review Forman’s powerful discrete Morse theory \([12]\). The theory comes in three “flavors”: for simplicial complexes, regular CW complexes, and general CW complexes. In each setting, one needs to find a certain discrete Morse function, and then the main theorem says that the space in question is homotopy equivalent to another simpler space obtained by collapsing non-critical cells.

The first two flavors of the theory are the simplest and most widely used, because in these two settings, a result of Chari \([10]\) implies that a discrete Morse function is equivalent to a matching on the face poset of the CW complex. To work with the third flavor of the theory, one must check some additional technical conditions:

\(^1\)If there is not a unique maximal cell then we will add a greatest element $\hat{1}$ to ensure that the poset is bounded. But all cell complexes in this paper will have a unique maximal cell.
the \textit{discrete Morse hypothesis}, as well as an extra topological condition included in the definition of discrete Morse function. However, as we will see in Theorem 7.6, it is enough to find a matching on the face poset of a CW complex with the \textit{subcomplex property} such that matched edges are regular. Although this result will not be surprising to the experts, we could not find it in the literature and so we give a careful exposition here. The proof follows from an argument of Kozlov \cite[Proof of Theorem 3.2]{17}.

7.1. Forman’s discrete Morse theorem for general CW complexes. Let $K$ be a finite CW complex and let $Q$ be its poset of cells.\footnote{Although Theorem 3.2 of \cite{17} was in the more restricted setting of regular CW complexes, the proof still holds in our situation.} Recall the definition of regular face from Definition 5.4.

Definition 7.1. \cite[p. 102]{12} A function $f : Q \to \mathbb{R}$ is a discrete Morse function if for every $\sigma^{(p)}$ of dimension $p$, the following conditions hold:

1. $\#\{\tau^{(p+1)} | \tau^{(p+1)} > \sigma \text{ and } f(\tau) \leq f(\sigma)\} \leq 1$.
2. $\#\{v^{(p-1)} | v^{(p-1)} < \sigma \text{ and } f(v) \geq f(\sigma)\} \leq 1$.
3. If $\sigma$ is an irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$.
4. If $v^{(p-1)}$ is an irregular face of $\sigma$ then $f(v) < f(\sigma)$.

Note that a Morse function is a function which is “almost increasing.” Indeed, one should think of a Morse function as a function which specifies the order in which to attach the cells of a homotopy-equivalent CW complex \cite{17}.

Definition 7.2. We say that a cell $\sigma^{(p)}$ is critical if

1. $\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} = 0$, and
2. $\#\{v^{(p-1)} < \sigma \mid f(v) \geq f(\sigma)\} = 0$.

Let $m_p(f)$ denote the number of critical cells of dimension $p$.

For each cell $\sigma$ of a CW complex $K$, let $\text{Carrier}(\sigma)$ denote the smallest subcomplex of $K$ containing $\sigma$. If $K$ has the subcomplex property (see Section 6.3), then for any $\sigma$, $\text{Carrier}(\sigma)$ is its closure, and hence condition (1) of Definition 7.3 below is satisfied.

Definition 7.3. \cite[p. 136]{12} Give a CW complex $K$ and a discrete Morse function $f$, we say that $(K, f)$ satisfies the Discrete Morse Hypothesis if:

1. For every pair of cells $\sigma$ and $\tau$, if $\tau \subset \text{Carrier}(\sigma)$ and $\tau$ is not a face of $\sigma$, then $f(\tau) \leq f(\sigma)$.
2. Whenever there is a $\tau > \sigma^{(p)}$ with $f(\tau) < f(\sigma)$ then there is a $\tilde{\tau}^{(p+1)}$ with $\tilde{\tau} > \sigma$ and $f(\tilde{\tau}) \leq f(\tau)$.

The following is Forman’s main theorem for general CW complexes.

Theorem 7.4. \cite[Theorem 10.2]{12} Let $K$ be a CW complex satisfying the Discrete Morse Hypothesis, and $f$ a discrete Morse function. Then $K$ is homotopy equivalent to a CW complex with $m_p(f)$ cells of dimension $p$.
7.2. Discrete Morse functions as matchings. Chari [10] pointed out that when the CW complex is regular, one can depict a Morse function $f$ as a certain kind of matching on the Hasse diagram of the poset of cells. Given such an $f$, we define a matching $M(f)$ on the Hasse diagram of $Q$ whose edges correspond to the pairs of cells in which we get equality in (1) or (2) of Definition 7.1.

We define a Morse matching $M$ on a poset $Q$ to be a matching on the Hasse diagram such that if edges in $M$ are directed from lower to higher-dimension elements and all other edges are directed from higher to lower-dimension elements, then the resulting directed graph $G(M)$ is acyclic. We refer to any elements of $Q$ which are not matched by $M$ as critical elements (or critical cells).

In the situation of arbitrary CW complexes, a Morse matching such that matched edges are regular gives rise to a discrete Morse function satisfying property (2) of the Discrete Morse Hypothesis, as the following lemma shows. The proof of this lemma is a simple translation of an argument of Kozlov [17, Proof of Theorem 3.2].

**Lemma 7.5.** Let $M$ be a Morse matching on the face poset $Q$ of a CW complex, such that each edge in $M$ corresponds to a regular pair of faces in the CW complex. Then there exists a discrete Morse function $f_M$, satisfying property (2) of the Discrete Morse Hypothesis, whose critical cells are exactly the critical cells of $M$.

**Proof.** We will inductively assign positive integer labels to each of the elements of $Q$, producing a function $f_M$. At each step, let $x$ be one of the elements of $Q$ of minimal rank (dimension) among those not yet labeled, and let $i$ be the smallest positive integer not yet appearing as a label in $Q$. If $x$ is not in $M$ and hence critical, label $x$ with $i$. If $x$ is not critical, then we must have $(x, y) \in M$, where $x < y$. If each $z < y$ in $Q$ is labeled, then label both $x$ and $y$ with $i$. Otherwise, there exists $x_1 < y$ in $Q$ where $x_1$ is not labeled; repeat the argument with $x_1$ taking the place of $x$. Either we will label $x_1$ or a pair $(x_1, y_1) \in M$, or, since $G(M)$ is acyclic, we will find $x_2 \neq x$, $x_2 \neq x_1$, $y_1 > x_2$, etc.

Since there are finitely many elements of $Q$, the process will terminate. Since we never label an element $y \in Q$ until we have labeled each $x < y$, $f_M$ has the property that for $x < y$, $f_M(x) \leq f_M(y)$. Therefore condition (2) of the Discrete Morse Hypothesis is satisfied. The only case in which $f_M(x) = f_M(y)$ is when $(x, y) \in M$, i.e. $(x < y)$ is a regular pair of faces – and so conditions (3) and (4) of Definition 7.1 are satisfied. Conditions (1) and (2) of Definition 7.1 are satisfied because $M$ is a matching. Finally, it is clear that the cells which are critical with respect to $M$ are exactly those which are critical with respect to Definition 7.2. \hfill \Box

We now re-state Forman’s Morse Theorem for general CW complexes in terms of Morse matchings.

**Theorem 7.6.** Let $K$ be a CW complex with the subcomplex property. Suppose its face poset $Q$ has a Morse matching $M$, such that whenever $(\sigma^{(p)}, \tau^{(p+1)}) \in M$, $\sigma$ is a regular face of $\tau$. Let $m_p(M)$ denote the number of critical cells of dimension $p$. Then $K$ is homotopy equivalent to a CW complex with $m_p(M)$ cells of dimension $p$. 


Proof. By Lemma 7.5, we have a discrete Morse function $f$ for $K$ satisfying condition (2) of Definition 7.3. Since $K$ has the subcomplex property, condition (1) of Definition 7.3 is satisfied. The result now follows from Theorem 7.4. □

7.3. From edge-labelings to Morse matchings. Both lexicographic shellability and discrete Morse theory are combinatorial tools which can be used to investigate the topology of a CW complex. In this section we will recall a result of Chari [10], which proves the existence of a certain Morse matching given a shelling of a regular CW complex. We will translate this into a statement about constructing a Morse matching from an EL-labeling, and note that one can gain some fairly explicit information about the Morse matching from the EL-labeling.

Recall the notion of pseudomanifold, e.g. from [10]. Note that a shellable pseudomanifold is in particular a regular CW complex which is either a ball or a sphere.

Proposition 7.7. [10, Proposition 4.1] Let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be a shelling of a $d$-pseudomanifold $\Sigma$ and let $v$ be any vertex in $\overline{\sigma_1}$. Then $\Sigma$ admits a Morse matching $M$ such that:

1. If $\Sigma$ is the $d$-sphere then $v$ and $\sigma_m$ are the only critical cells, while if $\Sigma$ is a $d$-ball, then $v$ is the only critical cell.
2. When restricted to $\bigcup_{k=1}^{j} \overline{\sigma_k}$ for $1 \leq j < m$, the only critical cell of $M$ is $v$.

By Theorem 6.3, an EL-labeling of the face poset of a pseudomanifold gives rise to a shelling. Chari used induction to construct the Morse matching of Proposition 7.7. Chari’s proof of Proposition 7.7, applied to a shelling which comes from an EL-labeling, implies the following.

Corollary 7.8. Fix an EL-labeling $\lambda$ of the face poset $Q$ of a pseudomanifold $\Sigma$. Let $M_{\lambda}$ be the Morse matching given by Proposition 7.7. Every $(\sigma \prec \tau) \in M$ has the following property: $\sigma \in \text{Last}_\lambda(\tau)$.

In fact, when the edge labels in $\lambda$ come from a totally ordered set, Chari’s proof of Proposition 7.7 gives the following algorithm for obtaining the Morse matching.

Corollary 7.9. Fix an EL-labeling $\lambda$ of the face poset $Q$ of a pseudomanifold $\Sigma$. Then the Morse matching $M_{\lambda}$ given by Proposition 7.7 can be constructed as follows. Let $\tau$ be the unique greatest element of $Q$ and set $M = \emptyset$.

1. Add $\text{Last}_\lambda(\tau) \prec \tau$ to $M$.
2. Consider each $\sigma \prec \tau$ in the order specified by $\lambda(\sigma \prec \tau)$. For each such $\sigma$ which is not already incident to an edge in $M$, set $\tau := \sigma$. Go to step 1.

Remark 7.10. Chari also gives the extension of Proposition 7.7 to regular CW complexes [10, Theorem 4.2]. Corollaries 7.8 and 7.9 also hold in this situation.

Remark 7.11. Proposition 7.7 and Corollary 7.8 can be useful even when $K$ is a CW complex not known to be regular. In particular, if the face poset $Q$ of $K$ is a CW poset, then there exists a regular CW complex $K_{\text{reg}}$ whose face poset is $Q$. Therefore one can still use these results to construct a Morse matching of $K$. 
8. The Bruhat order, shellability, and reduced expressions

Fix a Coxeter system \((W, I)\) and let \(T\) be the set of reflections. In this section we will review some properties of the Bruhat order \(\leq\) and prove a result (Proposition 8.7) about reduced expressions which will be needed for the proof of Proposition 9.5. The first part of Theorem 8.1 below is due to Bjorner and Wachs [5]. The second part follows from the first together with Bjorner’s result [3] characterizing CW posets.

Theorem 8.1. [5] [3] The Bruhat order of a Coxeter group is thin and (CL)-shellable. Furthermore, an interval with at least two elements is the face poset of a regular CW complex homeomorphic to a ball.\(^4\)

Dyer [11] subsequently strengthened the Bjorner-Wachs result by giving an EL-labeling of Bruhat order. Dyer’s primary tool was his notion of “reflection orders,” certain total orderings of \(T\) which can be characterized as follows.

Definition 8.2. [11, Proposition 2.13] Let \((W, I)\) be a finite Coxeter system with longest element \(w_0\), and let \(T = \{t_1, \ldots, t_n\} \ (n = \ell(w_0))\). Then the total order \(t_1 < t_2 < \cdots < t_n\) on \(T\) is a reflection order if and only if there is a reduced expression \(w_0 = s_{i_1} \cdots s_{i_n}\) such that \(t_j = s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \cdots s_{i_1}\), for \(1 \leq j \leq n\).

Remark 8.3. [11, Remark 2.4] The reverse of a reflection order is a reflection order.

Proposition 8.4. [11] Fix a reflection order \(\preceq\) on \(T\). Label each edge \(x \succ y\) of the Bruhat order by the reflection \(x^{-1} y\). Then this edge labeling together with \(\preceq\) is an EL-labeling; therefore the Bruhat order is EL-shellable.

Definition 8.5. [14] Consider a Coxeter system \((W, I)\). Define a deletion pair in an expression \(s_{i_1} \cdots s_{i_u} \cdots s_{i_t}\) to be a pair \(s_{i_r}, s_{i_t}\) (where \(r < t\)) such that the subexpression \(s_{i_r} \cdots s_{i_t}\) is not reduced but \(s_{i_r} \cdots s_{i_t}\) and \(s_{i_r} s_{i_r} \cdots s_{i_t}\) are each reduced.

E.g. in type A the first \(s_1\) and the last \(s_2\) in \(s_1 s_2 s_1 s_2\) comprise a deletion pair.

Lemma 8.6. [14] Lemma 3.31] If \(s_{i_r} \cdots s_{i_u} \cdots s_{i_t}\) is reduced but \(s_{i_r} \cdots s_{i_t}\) is not, then \(s_{i_u}\) belongs to a deletion pair within \(s_{i_r} \cdots s_{i_t}\).

Proposition 8.7. Consider \(x \leq w\) in a Coxeter group \(W\), and fix a reduced expression \(w = (i_1, \ldots, i_t)\) for \(w\). Let \(x_+ = \{j_1, \ldots, j_k\}\). For any \(p \leq t\), consider the product \(\gamma_1 \cdots \gamma_t\), where
\[
\gamma_r = \begin{cases} 
  s_{i_r}, & \text{if } r \in x_+ \text{ or } r \geq p, \\
  1, & \text{otherwise}.
\end{cases}
\]

Then \(\gamma_1 \cdots \gamma_t\) is reduced.

\(^4\)In fact [8] defined the face poset of a cell complex to be the poset of cells augmented by a least element \(\hat{0}\) and a greatest element \(\hat{1}\); with this convention, intervals in Bruhat order are face posets of CW complexes homeomorphic to spheres. The discrepancy in Theorem 8.1 comes from the fact that we have defined a face poset to be the poset of cells augmented by \(\hat{0}\) only.
Proof. We will prove this by induction. First consider \( p = t \). If \( i_t \in x_+ \) there is nothing to prove, since \( s_{i_1} \cdots s_{i_k} \) is reduced. If \( i_t \notin x_+ \) then assume that \( \gamma_1 \cdots \gamma_t \) is not reduced. This means that \( x s_{i_t} < x \), which contradicts the fact that \( x_+ \) is the positive subexpression for \( x \).

Now by induction assume the proposition holds for any \( p \) between some \( p' \) and \( t \), where \( p' \leq t \). We want to prove it for \( p := p' - 1 \). First suppose that \( p' - 1 \in x_+ \). In this case the product \( \gamma_1 \cdots \gamma_t \) is the same for both \( p = p' \) and \( p = p' - 1 \): in both cases, \( \gamma_{p' - 1} = s_{i_{p' - 1}} \). Therefore by induction it follows that \( \gamma_1 \cdots \gamma_t \) is reduced.

Now suppose that \( p' - 1 \notin x_+ \). In this case the induction hypothesis tells us only that \( \gamma_1 \cdots \gamma_{p' - 2} \gamma_{p' - 1} \cdots \gamma_t \) is reduced; we need to prove that \( \gamma_1 \cdots \gamma_{p' - 2} \gamma_{p' - 1} \gamma_{p' - 2} \cdots \gamma_t \) is reduced. Assume it is not: then by Lemma 8.6, \( s_{i_{p' - 1}} \) belongs to a deletion pair within \( \gamma_1 \cdots \gamma_{p' - 2} s_{i_{p' - 1}} s_{i_{p' - 2}} \cdots s_{i_t} \). Note that \( s_{i_{p' - 1}} \) comprises a consecutive string of generators in a reduced expression and so must be reduced. Also note that by our argument in the first paragraph, \( \gamma_1 \cdots \gamma_{p' - 2} s_{i_{p' - 1}} \) must be reduced: otherwise \( \gamma_1 \cdots \gamma_{p' - 2} s_{i_{p' - 1}} < \gamma_1 \cdots \gamma_{p' - 2} \), which contradicts the fact that \( x_+ \) is a positive subexpression and does not contain \( s_{i_{p' - 1}} \). But we’ve now shown that \( s_{i_{p' - 1}} \) cannot belong to a deletion pair within \( \gamma_1 \cdots \gamma_{p' - 2} s_{i_{p' - 1}} s_{i_{p' - 2}} \cdots s_{i_t} \), a contradiction. \( \square \)

9. Morse matchings and the proof of contractibility

In this section we will construct a Morse matching on the face poset of the closure of an arbitrary cell of \((G/P)_{\geq 0}\), such that matched edges are provably regular. We will then use this to prove our main result: that the closure of each cell is contractible, and the boundary of each cell is homotopy equivalent to a sphere.

Recall the definition of the face poset \( Q^J \) of \((G/P)_J \) from Section 5. Besides having a unique least element \( \emptyset \), \( Q^J \) also has a unique greatest element: this is \( 1 := P_{u_0, u_0, w_0 > 0} \), where \( u_0 \) and \( w_0 \) are the longest elements in \( W_J \) and \( W^J \), respectively.

The following was proved in [34].

**Theorem 9.1.** [34] \( Q^J \) is graded, thin, and EL-shellable. It follows that \( Q^J \) is the face poset of a regular CW complex homeomorphic to a ball.

It will be useful for us to classify the cover relations in \( Q^J \). The following classification is analogous to the one used in [34], with the roles of \( x \) and \( w \) reversed.

**Lemma 9.2.** The cover relations in \( Q^J \) fall into the following three categories.

- **Type 1:** \( P^J_{x', x, w; > 0} < P^J_{x, u, w; > 0} \) such that \( x < x' \). It follows that \( x u^{-1} < x' v^{-1} \).
- **Type 2:** \( P^J_{x, u, w; > 0} < P^J_{x, v, w; > 0} \) such that \( w' \leq w \). It follows that \( w' v < w u \).
- **Type 3:** \( \emptyset < P^J_{x, u, w; > 0} \) where \( P^J_{x, u, w; > 0} \) is a 0-cell. It follows that \( x = w \).

**Remark 9.3.** If \( Q \) is a poset, then the interval poset \( \text{Int}(Q) \) is defined to be the poset of intervals \([x, y]\) of \( Q \), ordered by containment. When \( G/P_J \) is the complete flag variety, i.e. when \( J = \emptyset \), \( Q^J \) is simply the interval poset of the Bruhat order.
Proposition 9.5. \( S \) is isomorphic to the (dual of the) Bruhat interval between \( x \) and \( w \). If \( x < w \), then \( S \) has a Morse matching \( M_x(w) \) in which all matched edges are good. If \( x = w \), then \( M_x(w) \) has no critical cells.

Proof. We will construct \( M_x(w) \) by using Dyer’s EL-labeling of the Bruhat interval (Proposition 8.4) and Chari’s observation that one can go from a shelling to a Morse matching (Proposition 7.7). To deduce that matched edges are good, we will choose our reflection order carefully and use Corollary 7.8.

Fix a reduced expression \( w = (i_1, \ldots, i_m) \) for \( w \), and choose a reduced expression for \( w_0 \) which begins with \( w^{-1} \). By Definition 8.2, this gives a reflection order. Let \( \prec \) be the reverse of this order; by Remark 8.3, \( \prec \) is also a reflection order.

Label the edge \( R_{v',w;>0} < R_{v,w;>0} \) (where \( v' \succ v \)) in \( S_x(w) \) with the reflection \( \tau \) such that \( v = v'\tau \). By Proposition 8.4, this gives an EL-labeling of \( S_x(w) \). By Theorem 8.1 and Proposition 7.7, if \( S_x(w) \) has at least two elements then this gives a Morse matching on \( S_x(w) \) with a unique critical cell of minimal dimension. Adding back \( R_{w,w;>0} \), we can match this critical cell to \( S_x(w) \), obtaining a Morse matching \( M_x(w) \) on \( S_x(w) \) with no critical cells. Otherwise, if \( S_x(w) \) has one element, i.e. if \( x = w \), then we take \( M_x(w) \) to be the empty matching with one critical cell.

We now need to show that all edges in \( M_x(w) \) are good. By Corollary 7.8, if \( \tau \) labels the edge \( R_{v,w;>0} > R_{v',w;>0} \) (for \( v' \succ v \)) and this edge is in \( M_x(w) \), then among all edge labels going from \( R_{v,w;>0} \) to lower-dimensional cells, \( \tau \) is maximal in \( \prec \). So we need to analyze cover relations corresponding to maximal labels.

Let \( v_+ = \{j_1, \ldots, j_r\} \). Let \( k \) be maximal (1 \( \leq k \leq m \)) such that \( k \notin \{j_1, \ldots, j_r\} \).

We first claim that \( u = \{j_1, \ldots, j_r\} \cup \{k\} \) is a reduced subexpression of \( w \), hence \( R_{u,w;>0} < R_{v,w;>0} \), and that \( u \) is positive. Second, we claim that the label on the edge from \( R_{v,w;>0} \) to \( R_{u,w;>0} \) is maximal among all edge labels from \( R_{v,w;>0} \) down to a lower-dimensional cell.

Proposition 8.7 implies the first claim that \( \{j_1, \ldots, j_r\} \cup \{k\} \) is a reduced subexpression of \( w \). Knowing that it is reduced, it is clear that it is positive.

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5 We remove \( R_{w,w;>0} \) here because it plays the role of \( 0 \), which is not considered to be an element of the face poset when constructing a Morse matching.
To see that the second claim is true, note that by the choice of \( k \), the label on the edge \( R_{v,w} \supset R_{u,w} \) is \( u^{-1}v = s_{i_m} s_{i_{m-1}} \ldots s_{i_k} \ldots s_{i_1} s_{i_m} \). Furthermore, in our reflection order, \( s_{i_m} \triangleright s_{i_m} s_{i_{m-1}} s_{i_m} \triangleright \ldots \triangleright s_{i_{m-1}} s_{i_{m-2}} s_{i_{m-1}} \ldots s_{i_1} s_{i_m} \triangleright \ldots \). Since \( k \) is maximal such that \( k \not\in \{j_1, \ldots, j_r\} \), if we define \( v'' = vs_{i_{m-1}} s_{i_{m-2}} \ldots s_{i_{k-1}} s_{i_{k+1}} s_{i_{k+2}} \ldots s_{i_{i_1}} s_{i_{m}} \) for \( \ell > k \), then \( \ell \in \{j_1, \ldots, j_r\} \) so an expression for \( v'' \) is given by \( \{j_1, \ldots, j_r\} \setminus \{\ell\} \). In particular, \( v'' < v \) and so \( R_{v,w} \supset R_{u,w} \) does not cover \( R_{v'',w} \supset R_{u,w} \). On the other hand, we know that \( R_{v,w} \supset R_{u,w} \) and that the label on this edge is \( s_{i_m} s_{i_{m-1}} \ldots s_{i_k} \ldots s_{i_1} s_{i_m} \). By our choice of reflection order, this label is maximal among all edges from \( R_{v,w} \) down to lower-dimensional cells.

Finally, by the choice of \( k \), and since \( \{j_1, \ldots, j_r\} \cup \{k\} = u \) is positive, this cover relation is good. Therefore every matched edge in \( M_x(w) \) is good. \( \square \)

**Remark 9.6.** If \( x \) is the identity element in \( W \), then the Morse matching constructed in Proposition 9.5 will actually be a multiplication matching by a Coxeter generator. This is a so-called special matching, and is relevant to Kazhdan-Lusztig theory [9]. Anders Bjorner suggested using special matchings to construct acyclic matchings, and realized that one could use them to obtain an acyclic matching for the face poset of the entire space \((G/B)_{\geq 0} \). [8] We are grateful for his insights.

We now turn to the proof of Theorem 9.4.

**Proof.** We partition the elements of the face poset of the closure of \( \mathcal{P}^J_{x,u,w} \) into subsets \( S^J_{x,u} = \{\mathcal{P}^J_{x',w',y} \mid xu^{-1} \leq x'u^{-1} \leq y\} \), where \( y \in W^J \) ranges between \( xu^{-1} \) and \( w \). By Lemma 9.2, the restriction of the face poset \( Q^J \) to \( S^J_{x,u} \) is isomorphic to the (dual of the) Bruhat interval between \( xu^{-1} \) and \( y \), so \( S^J_{x,u} \) and \( S^J_{x,u} \) are isomorphic as posets: we simply identify \( \mathcal{P}^J_{a,b,y} \) with \( R_{a,b} \).

We can now apply Proposition 9.5, which gives us a Morse matching \( M^J_{x,u} \) on \( S^J_{x,u} \) such that all matched edges are good. This matching has either zero or one critical cell, based on whether \( xu^{-1} < y \) or \( xu^{-1} = y \).

We now define

\[
M^J_{x,u,w} = \bigcup_{y \in W^J, xu^{-1} \leq y \leq w} M^J_{x,u}.
\]

Since each \( M^J_{x,u} \) is a matching, and any two matched elements \( \mathcal{P}^J_{a,b,y} \) and \( \mathcal{P}^J_{a',b',y'} \) in \( M^J_{x,u,w} \) have the same third factor \( y \), \( M^J_{x,u,w} \) is also a matching.

Let us assume for the sake of contradiction that there is a cycle in \( G(M^J_{x,u,w}) \). Since each \( G(M^J_{x,u}) \) is acyclic, there must be some edges in the cycle which pass between two different \( S^J_{x,u} \)'s. Each such edge must be directed from the higher-dimensional cell \( \mathcal{P}^J_{a,b,y} \) to the lower-dimensional cell \( \mathcal{P}^J_{a',b',y'} \) for \( y \neq y' \), so by Lemma 9.2, \( y' < y \). So if we traverse the cycle and look at the sequence of poset elements \( \mathcal{P}^J_{a,b,y} \) that we encounter, the third factor can only decrease. Therefore it is impossible to return to the element of the cycle at which we started, which is a contradiction.
As $y$ varies over elements of $W^J$ between $xu^{-1}$ and $w$, we have that $M^J_{xu^{-1}}(y)$ has no critical cells for $xu^{-1} < y$ and it has one critical cell $P^J_{x,u,xu^{-1};>0}$ for $xu^{-1} = y$. Therefore $M^J_{x,u,w}$ has a unique critical cell, the 0-dimensional cell $P^J_{x,u,xu^{-1};>0}$.

Since the face poset of $P^J_{x,u,w;>0}$ has a unique cell of top dimension $\ell(w) - \ell(xu^{-1})$, which is matched in $M^J_{x,u,w}$, when we restrict $M^J_{x,u,w}$ to $\text{bd}\,(P^J_{x,u,w;>0})$, we will get a Morse matching with one additional critical cell of top dimension $\ell(w) - \ell(xu^{-1}) - 1$. This completes the proof of the theorem. □

**Corollary 9.7.** Choose any cell $P^J_{x,u,w;>0}$ of $(G/P^J)_{\geq 0}$. Then there is a Morse matching on the face poset of $\overline{P^J_{x,u,w;>0}}$ with a single critical cell of dimension 0 in which all matched edges are regular; it restricts to a Morse matching on the face poset of $\text{bd}\,P^J_{x,u,w;>0}$ with one additional critical cell of top dimension.

**Proof.** This follows from Theorem 5.5 and Theorem 9.4. □

We now prove our main result.

**Theorem 9.8.** The closure of each cell of $(G/P)_{\geq 0}$ is contractible, and the boundary of each cell of $(G/P)_{\geq 0}$ is homotopy equivalent to a sphere.

**Proof.** Choose an arbitrary cell of $(G/P)_{\geq 0}$ and let $K$ be its closure. Note that $K$ is a CW complex with the subcomplex property because Theorems 4.5 and 5.2 imply that $(G/P)_{\geq 0}$ is. Let $Q$ be the face poset of $K$. By Corollary 9.7, $Q$ has a Morse matching with a unique critical cell of dimension 0, in which all matched edges are regular. Therefore Theorem 7.6 implies $K$ is contractible.

Now let $K'$ be the boundary of an arbitrary cell and let $Q'$ be its face poset. By Corollary 9.7, $Q'$ has a Morse matching with two critical cells, one of dimension 0 and one of top dimension, say $p$, in which all matched edges are regular. Therefore Theorem 7.6 implies that $K'$ is homotopy equivalent to a CW complex with one 0-dimensional cell $\sigma$ and one $p$-dimensional cell whose boundary is glued to $\sigma$. This is precisely a $p$-sphere. □

**Remark 9.9.** Since a Morse function actually gives rise to a concrete collapsing of a CW complex, in fact we have shown that the closure of a cell is collapsible.

**Remark 9.10.** One can give a simpler proof that the closure of each cell in the totally non-negative part of the type A Grassmannian $(Gr_{kn})_{\geq 0}$ is contractible. In that case, one can prove directly that whenever a cell $\sigma$ has codimension 1 in the closure of $\tau$, then $\sigma$ is a regular face of $\tau$. This follows from the technology of [25]; in particular, Theorem 18.3, Lemma 18.9, and Corollary 18.10. Then by Theorem 9.1, the poset of cells of $(Gr_{kn})_{\geq 0}$ is a CW poset with an EL-labeling (hence a shelling), so by Proposition 7.7 we have the requisite Morse matching.

Using Corollary 7.9 we see that there is a more concrete way to describe the matchings $M_x(w)$ of $S_x(w)$. 
Remark 9.11. Fix a reduced expression $w = (i_1, \ldots, i_m)$ for $w$. Start with the maximal element $R_{x, w; > 0}$. Let $k$ be maximal such that $1 \leq k \leq m$ and $k \notin x_+$. Then $x_+ \cup \{k\}$ is the positive subexpression for an element $v > x$. We match $R_{x, w; > 0}$ to $R_{v, w; > 0}$. We next look at each $R_{x', w; > 0}$ of codimension 1 in the order specified by the reflection order. For each $R_{x', w; > 0}$ which has not been matched yet we repeat the procedure, matching $R_{x', w; > 0}$ to the cell $R_{v', w; > 0}$, where $v'_+$ is obtained from $x'_+$ by adding the rightmost generator not yet in $x'_+$. We continue in this fashion, from higher to lower-dimensional elements.

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