A Goldstone “Miracle”:
The Absence of a Higgs Fine Tuning Problem
in the Spontaneously Broken O(4) Linear Sigma Model

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More than four decades ago, B.W. Lee and K. Symanzik proved that, in a generic set of O(4) linear sigma models (LΣM) in the $f_a - m_a^2/\lambda^2$ half-plane, Ward-Takahashi identities, along with tadpole renormalization, i.e. a Higgs Vacuum Stability Condition, force all S-matrix ultra-violet quadratic divergences (UV-QD) to be absorbed into the physical renormalized pseudo-scalar pion mass squared, $m_\pi^2$. We show that all such UV-QD, together with any finite remnants, therefore vanish identically in the Goldstone-mode limit, $m_\pi^2 \rightarrow m_{\pi,\text{NGB},\text{LΣM}}^2 \equiv 0$, where pions are Nambu-Goldstone Bosons (NGB) and axial-vector current conservation is restored, $\partial^\mu \tilde{A}_\mu \rightarrow 0$ (i.e. when Lee and Symanzik’s Goldstone Symmetry Restoration Condition is enforced, as required by Goldstone’s theorem for the spontaneously broken theory). The scalar Higgs mass is therefore not quadratically fine-tuned in the spontaneously broken theory. Hence Goldstone-mode O(4) LΣM symmetries are sufficient to ensure that any finite remnant, after UV-QD cancellation, does not suffer from the Higgs Fine Tuning Problem or Naturalness Problem. This is contrary to the lore that quadratic divergences in the Higgs mass, observable already at 1-loop, lead to such problems in the O(4) LΣM, and are then directly inherited by the Standard Model; but it is consistent with 1-loop results found in canonical introductory quantum field theory textbooks where the cancellation of such 1-loop quadratic divergences is noted but its importance is not.

It was recently shown that, including all-orders perturbative electro-weak and QCD loops, and independent of 1-loop ultraviolet regularization scheme, all finite remnants of UV-QD in the Standard Model (SM) S-Matrix are absorbed into NGB mass-squareds $m_{\pi,\text{NGB},\text{SM}}^2$, and vanish identically when $m_{\pi,\text{NGB},\text{SM}}^2 \rightarrow 0$, i.e. in the spontaneously broken SM. The SM Higgs mass is therefore not fine-tuned. This paper’s results on the O(4) LΣM lead us to a simple and intuitive understanding of that SM result, arising from the embedding of O(4) LΣM into the SM.

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I. INTRODUCTION

The idea that ultra-violet quadratic divergences (UV-QD) in the Standard Model (SM) require extreme fine-tunings in order to result in weak scale physics, has had enormous impact in motivating the possible existence of Beyond the Standard Model physics, such as supersymmetry, technicolor, large extra dimensions, etc. These UV-QD are canonically demonstrated (at one-loop) in the O(4) LΣM [52], which is embedded in the Electroweak sector of the SM. It is then argued that these $O(A^2)$ contributions to, say the Higgs mass-squared, must be fine-tuned against similarly large counter-terms to obtain a small $O(10^4 GeV^2)$ Higgs mass-squared (within the LΣM) and similar magnitude weak gauge boson masses. In this paper, we will show that, contrary to this lore the O(4) LΣM is perfectly well behaved, i.e. there is no fine-tuning necessary to keep the Higgs mass small, in the “Goldstone mode limit” where the O(4) symmetry is spontaneously broken, leaving behind three precisely massless Nambu-Goldstone Bosons (NGB)\textsuperscript{[1–3]}. This is because all UV-QD are contained in the mass squared of the NGB, so enforcing the Goldstone Theorem (i.e. that the NGBs are indeed massless)\textsuperscript{[2, 3]} eliminates all UV-QD including that in the Higgs mass without fine-tuning. It also ensures the Higgs Vacuum Stability Condition (Higgs-VSC), that the physical vacuum is stable against the spontaneous creation or disappearance of Higgs particles, although that is by necessity enforced even off the Goldstone mode limit. Notably, the Goldstone mode limit is precisely the appropriate limit of the O(4) LΣM for the SM, where those three massless NGBs of the O(4) LΣM are eaten by the three gauge bosons of the three broken generators of the SU(2)xU(1) symmetry. For the specific case of weak-scale Goldstone-mode O(4) LΣM, the Naturalness problem and the Higgs Fine-Tuning problem would have been the same problem, which we here call the Higgs Fine-Tuning Problem (HFTP). We show that the O(4) LΣM theory does not suffer from a HFTP in Goldstone mode.

Although the recent trend has been to argue that the SM is only a low-energy effective theory of some more complete theory, our goal is to understand the properties of the actual SM. If the SM lacks fine-tuning problems related to quadratic divergences, then clearly the onus is on those proposing any extension of the SM to demon-
strate that they maintain that problem-free existence; for it would not be a problem of the SM, but of the extension. Because we are interested only in the disposition of the the UV-QD of the SM, we turn our attention to the much simpler O(4) LΣM, which has the same UV-QD structure as the SM in the appropriate zero gauge coupling limit. The taming of logarithmic divergences and the calculation of finite renormalization contributions is not the subject of this paper, and it is important that those are sensitive to the difference between the SM and the O(4) LΣM. Because we are interested in the disposition only of UV-QD arising from quantum loops in the O(4) LΣM, and not in the logarithmic divergences or finite parts, we are able to treat the O(4) LΣM in isolation, as a stand-alone flat-space quantum field theory:

- It is not embedded or integrated into any higher-scale “Beyond the O(4) LΣM” physics.
- Loop integrals are cut off at a short-distance UV scale, $\Lambda$.
- Although the cut-off can be taken to be near the Planck scale $\Lambda \simeq M_{Pl}$, quantum gravitational loops are not included.

For pedagogical simplicity, we will ignore in all calculations, discussions and formulae any logarithmic divergences and finite contributions from loops, as these are unnecessary to (and distracting from) our explanation of the absence of any fine-tuning demanded by quadratic divergences. Thus, while for dimension-2 coefficients of relevant operators we will carefully distinguish between renormalized values of quantities (e.g. $\langle H \rangle^2$, $f_\pi^2$) and the corresponding quadratically divergent bare counter-term coefficient $\delta \langle H \rangle^2$, we will not distinguish between things that are only logarithmically or finitely renormalized, such as between bare fields/dimensionless couplings and their renormalized values. We will also drop all vacuum energy and vacuum bubble contributions as beyond the scope of this paper. Furthermore, this paper concerns stability and symmetry restoration protection against only UV-QD. It does not address any of the other, more usual, stability issues of the SM (cf. the discussion in, for example, [4], and references therein) or the O(4) LΣM (in part because the solutions to those issues are generically different in the SM than in the O(4) LΣM).

In a recent paper [5], it was shown that, including all-orders perturbative electro-weak and QCD loops, and independent of 1-loop UV regularization scheme, all finite remnants of ultra-violet quadratic divergences (UV-QD) in the Standard Model (SM) S-Matrix are absorbed into Nambu Goldstone Boson (NGB) mass-squareds $m_{\pi,NGB,SM}^2$ and vanish identically when $m_{\pi,NGB,SM}^2 \to 0$, i.e. in the spontaneously broken SM. Therefore, the SM Higgs mass is not fine-tuned. But, as a successful theory of nature, the SM is necessarily quite baroque, and sometimes even opaque. In this paper, we clarify that SM result by ignoring most SM parameters, gauge bosons and ghosts. We take the zero-gauge-coupling limit in 1PI 1-loop and multi-loop nested UV-QD. The resulting Goldstone-mode O(4) LΣM, which is embedded in the SM, provides a simple and intuitive understanding of the results in Ref. [5]. Fortunately for most readers, all relevant O(4) LΣM 1-loop Feynman diagrams were calculated and agreed on long ago. We refer the interested reader to that vast literature [6–45] for specifics. Motivated by that literature, we use Euclidean metric and the Feynman-diagram naming convention of Ref. [6].

The structure of the remainder of this paper is:

- Section 2 clarifies the correct renormalization of spontaneously broken O(4) LΣM coupled to Standard Model (SM) quarks and leptons. Most of the calculations (if not the effective Lagrangian presentation) in Section 2A through 2E (and probably 2F) are not new and have been common knowledge for more than four decades. Specifically:
  - Section 2A introduces the (fermion-free) O(4) LΣM in the $f_\pi$ vs. $m_\pi^2/\lambda^2$ half-plane.
  - Section 2B calculates 1-loop UV-QD in 2-point self-energies and 1-point tadpoles and shows that Ward-Takahashi identities, together with tadpole renormalization, i.e. a Higgs Vacuum Stability Condition (Higgs-VSC), force all these to be absorbed into the physical renormalized pseudo-scalar pion (pole) mass-squared $m_\pi^2$.
  - Section 2C defines the HFTP which emerges in 1-loop-corrected $m_\pi^2 \neq 0$ O(4) LΣM and, in particular, the $m_\pi^2 \neq 0$, $f_\pi \to 0$ “Wigner-mode” limit.
  - Section 2D studies the opposite Goldstone-mode $m_\pi^2 = 0$ $f_\pi \neq 0$ limit, and shows that the Goldstone Symmetry Restoration Condition (Goldstone-SRC) embedded there causes all remnants of 1-loop UV-QD to cancel identically as $m_\pi^2 \to m_{\pi,NGB,LΣM}^2 = 0$ (exactly zero!), and the theory to have no HFTP there.
  - Section 2E extends our bosonic Goldstone-mode O(4) LΣM results to all-orders in loop-perturbation theory.
  - Section 2F further extends Goldstone-mode O(4) LΣM results to include virtual SM quarks and leptons.

- Section 3 presents some conclusions drawn from the arguments and calculations of the other sections.
- Appendix A shows that the vanishing of 1-loop UV-QD in Goldstone-mode O(4) LΣM does not depend on the choice of n-dimensional regularization or Pauli-Villars UV cut-off regularization.
- Appendices B and E show detailed calculations of 1-loop UV-QD in O(4) LΣM with bosons and with virtual SM quarks and leptons, respectively.
• Appendix C discusses a subtlety which arises in Goldstone-mode O(4) LEM when calculations begin using the Goldstone-mode Lagrangian. This results in the 1-loop UV-QD “Goldstone-mode Renormalization Prescription” (GMRP), which enforces spontaneous symmetry breaking (SSB) and the Goldstone Theorem. The GMRP, when extended to the SM, clarifies its no-HIPTP result [5].

• Appendix D shows details for the extension of our results to all-loop-orders for Goldstone-mode O(4) LEM.

II. O(4) LINEAR SIGMA MODEL, WITH EITHER MASSIVE PIONS $m_\pi^2 \neq 0$, OR NGB PIONS $m_\pi^2 \to m_{\pi,\text{NGB}}^2; \Sigma_{\text{LSM}} \equiv 0$

A. O(4) LSME in the $f_\pi$ vs. $m_\pi^2/\lambda^2$ half-plane

We follow closely the pedagogy (and much of the notation) of Benjamin W. Lee [40] and use the linear representation of $\Phi$ to make manifest the physical content of Higgs Vacuum Stability Condition (Higgs-VSC) tadpole renormalization, as well as the exact not-fine-tuned cancellation of UV-QD, when a Goldstone Symmetry Restoration Condition (Goldstone-SRC) enforces the well-known Goldstone Theorem [2, 3] as $m_\pi^2 \to m_{\pi,\text{NGB}}^2 = 0$. We begin with the pure scalar bare Lagrangian:

$$L_{\text{LSM}}^{\text{Bare}} = -|\partial_\mu \Phi|^2 V_{\text{LSM}}^{\text{Bare}}$$

$$V_{\text{LSM}}^{\text{Bare}} = \lambda^2 \left( \Phi^\dagger \Phi - \frac{1}{2} (H^2 + \delta(H)^2) \right)^2 \epsilon H$$

$$- L_{\text{LSM}}^{\text{CounterTerm}} - L_{\text{LSM}}^{\text{SymmetryBreaking}},$$

where

$$L_{\text{LSM}}^{\text{CounterTerm}} = \lambda^2 \delta(H)^2 \left( \Phi^\dagger \Phi - \frac{1}{2} (H^2)^2 \right),$$

$$L_{\text{LSM}}^{\text{SymmetryBreaking}} = \epsilon H.$$  

Here $\lambda^2 > 0$, while

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} H + i \pi_3 \\ -\pi_2 + i \pi_1 \end{bmatrix}, \quad \pi_{\pm} = \frac{1}{2} (\pi_1 \mp i \pi_2);$$

$$H \equiv h + \langle H \rangle, \quad \langle \Phi \rangle = \begin{bmatrix} \langle H \rangle \\ 0 \end{bmatrix}, \quad \text{and} \quad \langle h \rangle = 0.$$  

Vacuum energy/bubble contributions are ignored. We assume $\lambda^2 = O(1)$ and loosely refer to the real scalar $h$ as the physical “Higgs.”

Apart from $L_{\text{LSM}}^{\text{SymmetryBreaking}}$, the theory has O(4) symmetry with conserved vector currents (CVC) $V_\mu$ and partially conserved axial-vector currents (PCAC) $A_\mu$:

$$V_\mu = \bar{\pi} \times \partial_\mu \pi$$

$$A_\mu = \bar{\pi} \partial_\mu H - H \partial_\mu \pi.$$  

Inclusion of explicit symmetry breaking yields

$$\partial_\mu V_\mu = 0,$$

$$\partial_\mu A_\mu = -\epsilon \pi.$$  

The first of the connected Green’s function Ward-Takahashi identities, connecting the vacuum with the on-shell one-pion state of momentum $q_\mu$, reads [40–51]

$$\langle 0 | A_\mu^i (x) | \pi^j (q) \rangle = -i \delta^{ij} f_\pi q_{\mu} \bar{\pi}^i e^{iq_\mu x_\mu},$$  

where $f_\pi$ is defined as the experimental value of the renormalized pion decay constant [40–51]. Its divergence is

$$\partial_\mu (0 | A_\mu^i (x) | \pi^j (q) \rangle) = \delta^{ij} f_\pi q_\mu \bar{\pi}^i e^{iq_\mu x_\mu}$$

$$= -i \delta^{ij} f_\pi m_\pi^2 \bar{\pi}^i e^{iq_\mu x_\mu}.$$  

The final equality is due to the pion being on-shell, so $m_\pi^2 \geq 0$ is the physical renormalized pseudo-scalar pion (pole) mass-squared.

To all-loop-orders of perturbation theory, equations 6 and 7 receive only logarithmically divergent and finite corrections [40–45], so that, including all-orders UV-QD, but ignoring (as un-interesting for this paper) logarithmic
divergences and finite contributions:
\[ \partial_\mu \tilde{A}_\mu = -f_\pi m_\pi^2 \phi \]
\[ \varepsilon = f_\pi m_\pi^2 \]
\[ L_{\text{Symmetry Breaking}}^{\Sigma M} = f_\pi m_\pi^2 H. \] (8)

B.W. Lee and K. Symanzik then analyze the generic set of quantum field theories in the \( f_\pi \) vs. \( m_\pi^2/\lambda^2 \) half-plane (cf. Figure 1).

**B. Inclusion of 1-loop ultra-violet quadratic divergences**

The UV-QD 1-loop Lagrangian, evaluated at zero momentum, including all 1-loop 2-point self-energy (Figures 3, 4 and 5) and 1-loop 1-point tadpole-function (Figure 6) UV-QDs, is (cf. Appendix B):
\[ L_{\Sigma M}^{1\text{-loop}:\Lambda^2} = C_{\Sigma M}^{\text{Unrenorm}:1\text{-loop}:\Lambda^2} \Lambda^2 \left( \Phi^4 - \frac{\langle H \rangle^2}{2} \right), \] (9)

with \( C_{\Sigma M}^{\text{Unrenorm}:1\text{-loop}:\Lambda^2} \) a dimensionless constant. The form of Eq. 9, explicitly calculated in Appendix B, follows from Lee/Symanziks proof [40–51] that the theory is properly renormalized, throughout the \( f_\pi \) vs. \( m_\pi^2/\lambda^2 \) half-plane, with the same UV-QD graphs and counter-terms as the symmetric Wigner-mode limit: \( \langle H \rangle^2 \to 0 \) holding \( m_\pi^2 \neq 0 \). Noticing that at tree level the Higgs mass \( m_h^2 \approx 2\lambda^2\langle H \rangle^2 \), we suggestively (and recognizably [6] \( 38,39 \)) rewrite Eq. B8 (cf. Appendix B):
\[ C_{\Sigma M}^{\text{Unrenorm}:1\text{-loop}:\Lambda^2} = \left( -6\lambda^2 \right) \left( \frac{\Lambda^2}{16\pi^2} \right) \] (10)
\[ = \langle H \rangle^2 \frac{3m_\pi^2}{16\pi^2} \Lambda^2 \]

(Here \( \langle H \rangle^2 \) indicates that this is currently true only at tree level, although we shall find below that \( m_h^2 \approx 2\lambda^2\langle H \rangle^2 \) holds to all orders in loops.)

The reader is reminded to take the Wigner-mode limit carefully, taking \( \langle H \rangle^2 \to 0 \) but holding \( \lambda^2 \) fixed. Appendix A shows that this result is independent of whether

n-dimensional regularization or Pauli-Villars regularization is used. To fully exacerbate and reveal any HFTP, we imagine \( \Lambda \approx M_P, P \) near the Planck scale.

Using the bare Lagrangian \( L_{\Sigma M}^{\text{Bare}} \), we form a 1-loop-UV-QD-improved effective Lagrangian, which includes all scalar 2-point self-energy and 1-point tadpole 1-loop UV-QD (but ignores, for us un-interesting, 1-loop logarithmically divergent, finite contributions and vacuum energy/bubbles):
\[ L_{\Sigma M}^{\text{Effective}:1\text{-loop}:\Lambda^2} \equiv \frac{1}{2} L_{\Sigma M}^{\text{Bare}:\Lambda^2} + L_{\Sigma M}^{1\text{-loop}:\Lambda^2} \]
\[ = -|\partial_\mu \Phi|^2 - V_{\Sigma M}^{\text{Renorm}:1\text{-loop}:\Lambda^2}, \] (11)

where
\[ V_{\Sigma M}^{\text{Renorm}:1\text{-loop}:\Lambda^2} = \lambda^2 \left[ -f_\pi \langle H \rangle^2 + \langle h \rangle \right] - f_\pi m_\pi^2 \delta \langle H \rangle^2 + C_{\Sigma M}^{\text{Unrenorm}:1\text{-loop}:\Lambda^2} \Lambda^2 \left[ \Phi^4 - \frac{\langle H \rangle^2}{2} \right] \] (12)
\[ = \lambda^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi_+ + \pi_- \right] + \langle H \rangle h - f_\pi m_\pi^2 \]
\[ + m_\pi^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi_+ + \pi_- \right], \]

and
\[ m_h^2 = -\left( \lambda^2 \delta \langle H \rangle^2 + C_{\Sigma M}^{\text{Unrenorm}:1\text{-loop}:\Lambda^2} \Lambda^2 \right). \]

(13)

Because of the way it enters Eq. 12, \( m_h^2 \) is the physical renormalized pseudo-scalar pion (pole) mass-squared of Eqs. 7 and 8.

Insight into the tadpole term \( \langle H \rangle - f_\pi \) \( m_\pi^2 \) in Eq. 12 follows by carefully defining the properties of the vacuum, including all effects of UV-QD, and imposing the Higgs-VSC on the vacuum and excited states:
**Higgs VSC:** The physical Higgs particle must not simply disappear into the exact UV-corrected vacuum.

For the purposes of this paper, we take the UV-corrected vacuum to be the UV-QD-corrected vacuum. It is important to note several things:

1. The UV-QD-corrected vacuum includes all perturbative UV-QD corrections, including
   - 1-loop when referenced in subsections II B, II C and II D,
   - 1PI multi-loop when referenced in subsection II E and Appendix D, and
   - 1-loop O(4) LEM with fermions in subsection II F.

2. By definition, \( H = \langle H \rangle + h \), and \( \langle \Phi^4 \rangle = \frac{1}{2} \langle H \rangle^2 \) is the exact vacuum expectation value (VEV).

3. The physical Higgs particle \( h \) has exactly zero VEV: \( \langle h \rangle \equiv 0 \).
4. Exact tadpole renormalization is to be imposed to all orders in perturbation theory.

It is important to realize that Higgs-VSC tadpole renormalization does not constitute fine-tuning; rather it is a stability condition on the vacuum and excited states of the theory.

Lee/Symanzik impose the Higgs-VSC essentially by hand [40–45]. They ensure that

\((\langle H \rangle - f_\pi) m_\pi^2 h = 0\) \hspace{1cm} (14)

in the entire \( f_\pi - m_\pi^2/\lambda^2 \) half-plane by enforcing

\(\langle H \rangle = f_\pi. \)

After tadpole renormalization, the effective 1-loop Lagrangian (keeping 1-loop UV-QD, but ignoring uninteresting logarithmic divergences, finite parts and vacuum energy/bubbles), can be re-written:

\[
\begin{align*}
L_{\Sigma M}^{\text{Effective}};1\text{-loop};\Lambda^2 & = -|\partial_\mu \Phi|^2 - V_{\Sigma M}^{\text{Renorm};1\text{-loop}};\Lambda^2, \\
V_{\Sigma M}^{\text{Renorm};1\text{-loop}};\Lambda^2 & = \\
\lambda^2 [\Phi^\dagger \Phi - \frac{1}{2} \left( f_\pi^2 - \frac{m_\pi^2}{\lambda^2} \right)]^2 - f_\pi m_\pi^2 H
\end{align*}
\]

with

\[H = h + f_\pi.\] \hspace{1cm} (17)

Here \(m_\pi^2\) is still the physical renormalized pseudo-scalar pion (pole) mass-squared of Eq. 13, and

\[m_\pi^2 \equiv m_\pi^2 + 2\lambda^2 f_\pi^2 \geq m_\pi^2.\] \hspace{1cm} (18)

A cross-section of \(V_{\Sigma M}^{\text{Renorm};1\text{-loop}};\Lambda^2\), with \(m_\pi^2/\lambda^2 - f_\pi^2 < 0\) and \(m_\pi^2 > 0\) is plotted in Figure 2. We observe that, because of the explicit symmetry breaking term \(f_\pi m_\pi^2 H\) in Eq. 16, it is not possible to change the results of subsections II C through II F, by changing to the unitary \(\Phi\) representation [4] or by re-scaling \(\langle H \rangle\) (cf. Appendix C).

The results (except the effective Lagrangian presentation and the inclusion of UV-QD into \(m_\pi^2\)) in this subsection are not new [40–45]. We simply observe here that Ward-Takahashi identities, together with Higgs-VSC tadpole renormalization, are sufficient to force all UV-QD in the S-Matrix of the \(O(4)\) \(\Sigma M\) to be absorbed into the physical renormalized pseudo-scalar pion (pole) mass-squared, \(m_\pi^2\). We also remark that this is not surprising. The quadratic divergences are ultra-violet constants. They therefore retain no “memory” of the low-energy \((\langle H \rangle \ll \Lambda)\) breaking of the \(O(4)\) symmetry. \(H\) and \(h\) are the same to UV-QD, and, by the \(O(4)\) symmetry, indistinguishable from the \(\pi^i\).

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2 This is distinct from the observation that in the Goldstone mode this Higgs-VSC condition (Eq. 14) is also guaranteed by the Goldstone-SRC, i.e. by \(m_\pi^2 = 0\). Whether or not Eq. 15 can therefore be relaxed in Goldstone mode, does not change the conclusions of this paper.
• observable properties of the theory be stable against minute variations of fundamental parameters [52, 53]; and

• electroweak radiative corrections be the same order (or much smaller) than the actually observed values [6].

The problem is widely regarded as exacerbated in O(4) LΣM (and the SM) because even if one fine tunes the parameters at 1-loop, one must retune them from scratch at 2-loop, and again at each order in loops. This is sometimes called the Technical Naturalness Problem For

Weak-scale O(4) LΣM with

3

weak-scale O(4) LΣM with

m2 ≠ 0, these problems are all the same problem, the HFTP.3

O(4) LΣM in Wigner mode [40–45], with unbroken O(4) symmetry and four degenerate massive scalars (i.e. 1 scalar + 3 pseudo-scalars), is defined as the limit

at 2-loop, and again at each order in loops. This is

less pseudo-scalar Nambu-Goldstone Bosons, is defined

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where

HFTP.

3

Weak-scale O(4) LΣM also separately suffers a Weak Hierarchy problem because it is unable to predict or explain the enormous splitting between the weak scale and the next larger scale (e.g. classical gravitational physics Planck scale MF). We make no claim to address that aesthetic problem here.

1. The Goldstone-SRC includes all perturbative quadratically divergent corrections, including

• 1-loop when referenced in subsections II D and II F;

• 1PI Multi-loop when referenced in subsection II E and in Appendix D.
2. In generic theories spanning the $f_\pi$ vs. $m_\pi^2/\lambda^2$ half-plane (Figure 1), it would be necessary to impose this Goldstone-SRC (essentially by hand) in order to force the theory to the Goldstone-mode O(4) LΣM limit, i.e. onto the y-axis, $m_\pi^2 \to m_{\pi;NGB;LΣM}^2 \equiv 0$. (Where $m_\pi^2$ is the loop-induced Nambu-Goldstone Boson (NGB) mass (see Appendix C)).

3. In more modern language, for spontaneously broken O(4) LΣM, it is necessary to explicitly enforce the Goldstone Theorem order-by-order so that the pions remain exactly massless to all orders of loop perturbation theory. That is the purpose of Lee/Symanzik's Goldstone-SRC. One thus has no choice but to operate in the $m_\pi^2 \to m_{\pi;NGB;LΣM}^2 \equiv 0$ to realize the Goldstone Theorem.

4. Since UV-QD (in the generic LΣM theory’s S-Matrix) are all packed into the physical renormalized pseudo-scalar pion (pole) mass-squared (as demanded by O(4) symmetry, and as shown by explicit calculation[40]), the Goldstone Theorem also forces any finite remnant of UV-QD to be exactly zero in the Goldstone-mode (i.e. spontaneously broken) limit of the theory.

5. We have ignored infra-red(IR) subtleties as beyond the scope of this paper.

It is easy to see that $V_{\text{Renorm;1-loop};A^2}$ in Eq. 16 (shown in Figure 2 with $m_\pi^2 \neq 0$) only has NGB when "bottom-of-the-wine-bottle Goldstone symmetry is restored: i.e. in Goldstone mode with

$$m_\pi^2 \to m_{\pi;NGB;LΣM}^2 \equiv 0 \quad (27)$$

is restored exactly and identically. Re-writing Eq. 11 after tadpole renormalization

$$L^{\text{Effective;1-loop};A^2|_{\text{Goldstone;LΣM}}} = -|\partial_\mu \Phi|^2 - V_{\text{Renorm;1-loop};A^2|_{\text{Goldstone;LΣM}}},$$

where

$$V_{\text{Renorm;1-loop};A^2|_{\text{Goldstone;LΣM}}} = \lambda^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi_+ \pi_- + f_\pi h \right]^2 + m_{\pi;NGB;LΣM}^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi_+ \pi_- \right].$$

Here

$$m_{\pi;NGB;LΣM}^2 = \frac{(-\lambda^2 \delta(H))^2 + C_{\text{Unrenorm;1-loop};A^2} \Lambda^2}{16\pi^2}.$$  

The crucial observation here is that proper 1-loop enforcement of the Goldstone Theorem $m_{\pi;NGB;LΣM}^2 \equiv 0$, requires imposition (essentially by hand) of Lee/Symanski's Goldstone-SRC [40-45]

$$0 \equiv m_{\pi;NGB;LΣM}^2 = \left( -\lambda^2 \delta(H)^2 + 6\lambda^2 \frac{\Lambda^2}{16\pi^2} \right). \quad (30)$$

cancel identically, without fine-tuning, thus avoiding any Higgs Fine Tuning Problem.

$$L^{\text{Effective;1-loop};A^2|_{\text{Goldstone;LΣM}}}$$

in Eq. 28 and 31 gives the sensible, at worst logarithmically divergent and not-fine-tuned Higgs mass:

$$m_h^2 = 2\lambda^2 f_\pi^2. \quad (33)$$

Before going forward, let us now summarize the general 1-loop and specific 1-loop Goldstone-mode observations so far, and which are extended to all-loop-orders in subsection II E and Appendix D. We emphasize that most of the calculations in Section II are not new; rather, they have been common knowledge for more than four decades. What is new are certain observations that apply specifically to spontaneously broken O(4) LΣM (i.e. in Goldstone mode):

1. Spontaneously broken O(4) LΣM must be viewed as a limiting case of Lee/Symanski's generic set of $m_\pi^2/\lambda^2 \geq 0$ theories (i.e. the $m_\pi^2/\lambda^2 = 0$ line (y-axis) in the $f_\pi$ vs. $m_\pi^2/\lambda^2$ half-plane shown in Figure 1), where two conditions must be explicitly imposed on the O(4) LΣM vacuum and on Goldstone-mode excited states:

- **Higgs VSC**: everywhere in the $f_\pi - m_\pi^2/\lambda^2$ half-plane the Higgs must not simply disappear into the vacuum, so $\langle H \rangle \equiv f_\pi$;

- **Goldstone-SRC**: the masses of Nambu-Goldstone Bosons must be fixed to exactly zero in the Goldstone-mode (y-axis) to enforce the Goldstone Theorem.
2. Ward-Takahashi identities and tadpole renormalization force all UV-QD in the S-Matrix to be absorbed into the physical renormalized pseudo-scalar pion (pole) mass-squared $m_\pi^2$ everywhere in the half-plane.

3. All 1-loop UV-QD divergences can be obtained by first calculating them in the unbroken theory (Wigner mode) and then breaking the symmetry throughout the half-plane, and specifically on the y-axis (Goldstone mode), where O(4) is broken only spontaneously. These 1-loop results do not depend on choice of UV regularization scheme (e.g. n-dimensional or Pauli-Villars cut-off, as shown in Appendix A).

4. All relevant dimension-2 operators (i.e. with 1-loop UV-QD coefficients) form a renormalized Higgs potential (Figure 2) that is well-defined everywhere in the half-plane and can be minimized at the 1-loop-corrected VEV. Although the coefficients of the relevant dimension-2 scalar self-energy-2-point and tadpole-1-point relevant operators take their natural scale $\delta(H)^2 \sim \Lambda^2$, there is still no need to fine-tune away the UV-QD to get a weak-scale Higgs mass-squared $m_H^2 G^{\text{ren}} \simeq O(1)$ if experiment so demands, once $m_\pi^2 \to m_\pi^2 \text{NGB;L}\Sigma M \equiv 0$, is fixed.

5. The Goldstone Theorem insists that the NGBs are exactly massless, $m_\pi^2 \equiv 0$. All UV-QD in $m_\pi^2$, together with all logarithmically divergent and all finite remnants, vanish identically and exactly as $m_\pi^2 = 0$. This forces all UV-QD in the S-Matrix to cancel exactly, with finite remnant exactly zero. They are not absorbed into renormalized parameters, including, but especially $m_\pi^2$.

6. The root cause of this fine-tuning-free disappearance of UV-QD is imposition of a Higgs-VSC and a Goldstone-SRC on the spontaneously broken vacuum and its excited states as demanded by Lee and Symanzik.

7. Our no-fine-tuning-theorem for a weak-scale Higgs mass (in the spontaneously broken O(4) LΣM) is then simply another (albeit un-familiar) consequence of the Goldstone Theorem, an exact property of the spontaneously broken vacuum and spectrum.

8. There is no possibility (or need) to cancel 1-loop UV-QD between virtual bosons and fermions in the O(4) LΣM. (After all, we have so far not allowed for any fermions.)

9. It is un-necessary to impose any new Beyond the O(4) LΣM symmetries or other physics: weak-scale spontaneously broken O(4) LΣM already has sufficient symmetry to force all S-Matrix UV-QD to vanish and to ensure that it does not suffer a HFTP.

10. The fact that all 1-loop UV-QD in the SM Higgs self-energy and mass cancel exactly after tadpole renormalization has been known [30, 39] for more than 3 decades. We have traced that SM result to the Goldstone mode O(4) linear sigma model embedded in the spontaneously broken SM, giving a simple and intuitive understanding of the result of [5], i.e. that the SM Higgs mass is not fine-tuned.

An important subtlety, which gives a clear 1-loop UV-QD “Goldstone-Mode Renormalization Prescription” (GMRP) for both Goldstone-mode O(4) LΣM and the Standard Model, is discussed in Appendix C. There we re-do the calculations and analysis in this section, but beginning with the Goldstone-mode O(4) LΣM bare Lagrangian, Eq. 1 with $\epsilon = 0$:

$$L_{\text{bare;A}} =$$

$$-|\partial_\mu \Phi|^2 - \lambda \frac{1}{2} \left( \Phi \Phi - \frac{1}{2} \right)^2 (\langle H \rangle^2 + \delta(H)^2)^2$$

The subtlety arises after UV-QD cancellation, when enforcing $m_\pi^2 \to m_\pi^2 \text{NGB;L}\Sigma M \equiv 0$ and SSB and avoiding a “fine-tuning discontinuity.”

The same thing happens when SM calculations begin with the spontaneously broken SM bare Lagrangian: one must enforce $m_\pi^2,SM \to m_\pi^2,\text{NGB;SM} \equiv 0$ in that GMRP, which is correct for the SM [5].

E. Goldstone-mode weak-scale spontaneously broken O(4) LΣM has no HFTP at any loop-order

The reader should worry that cancellation of 1-loop UV-QD is insufficient to demonstrate that the theory does not require fine-tuning. UV-QD certainly appear at multi-loop orders and fine-tuning $\delta(H)^2$ might yet be required. If each loop order contributed a factor $\hbar/16\pi^2 \simeq 10^{-2}$, then cancellation of UV-QD to more than 16 1PI loops would be required to defeat a factor of $G^{\text{ren}} \Lambda^2 \lesssim 10^{-32}$ for $\Lambda \gtrsim M_{\text{Pl}}$.

As we remarked above, B.W. Lee, and K. Symanzik renormalized generic O(4) LΣM (i.e. in the full $f_\pi - m_\pi^2/\Lambda^2$ half-plane of Figure 1) to all-loop-orders more than forty years ago. We simply remind the reader of their results [40–45], of important follow-up results [46–48] and of some accessible textbook presentations [49–51] relevant to this paper. In particular:

- The UV divergence and counter-term structure is the same throughout the $f_\pi$ vs. $m_\pi^2/\Lambda^2$ half-plane (Figure 1), giving proper interolation between the symmetric Wigner mode ($f_\pi = 0, m_\pi^2 \neq 0$) and the spontaneously broken Goldstone mode ($f_\pi \neq 0, m_\pi^2 = 0$).

- The Ward-Takahashi Identity, CVC and PCAC (Eqs. 5, 6 and 7) receive only logarithmically divergent and finite corrections. Ignoring those, but including all-orders UV-QD, Eq. 7 is still true and $m_\pi^2$...
is still the all-loop-corrected physical renormalized pion (pole) mass-squared.

Appendix D demonstrates just such exact cancellation and shows that 1PI multi-loop fine-tuning is unnecessary. Appendix D then shows that SSB, careful definition of the vacuum – Goldstone-SRC enforcement of the Goldstone Theorem and Higgs-VSC tadpole renormalization – are sufficient to force all-loop-orders S-matrix UV-QD into \( m^2_\pi \) (in \( L_{\Sigma M}^{{\text{Eff.\;All-loop};\Lambda^2} \)) and to therefore vanish identically and exactly in the \( m^2_\pi \rightarrow m^2_{\pi:\text{NGB}+\Sigma M} \equiv 0 \) limit. They are not absorbed into renormalized parameters, such as the Higgs mass squared. There is no surviving remnant of UV-QD at any order of perturbation theory, nor is there a HFTP in UV-QD-corrected spontaneously broken O(4) \( \Sigma M \).

It is easy to see from Appendix D that the observations and lessons at the end of subsection II D can be extended to all orders of bosonic O(4) \( \Sigma M \), at least in loop-perturbation theory. We now add fermions to the O(4) \( \Sigma M \), and show that the spontaneously broken, O(4) \( \Sigma M \) with Standard Model fermions remains free of any HFTP.

F. \( O(4) \Sigma M \) in the \( f_\pi \) vs. \( m^2_\pi/\lambda^2 \) half-plane with SM quarks and leptons

All-orders renormalization of the generic bosonic O(4) \( \Sigma M \) was long ago extended to include parity-doublet fermions by J.L. Gervais and B.W. Lee [44]. The full reasoning necessary to secure renormalized inclusion of SM quarks and leptons in the \( f_\pi \) vs. \( m^2_\pi/\lambda^2 \) half-plane is beyond the scope of this paper (i.e. involves infra-red and regularization subtleties [40–51]). We merely remind the reader of the key complication, that SM fermion masses vanish in the symmetric Wigner mode (\( f_\pi = 0, \lambda^2_\pi \neq 0 \)), as

\[
m_{\text{quark, lepton}} \propto \langle H \rangle = f_\pi, \quad (35)
\]

If we couple SM quarks and leptons to O(4) \( \Sigma M \) in the usual way, then new UV-QD are proportional to Yukawa couplings squared, and appear only in coefficients of relevant dimension-2 scalar-sector operators. They are assembled and included in the scalar-sector effective Lagrangian as in Section 2B.

We focus on explicit calculation of 1-loop UV-QD contributions, due to SM quarks and leptons, in the generic \( f_\pi \) vs. \( m^2_\pi/\lambda^2 \) half-plane. The total UV-QD 1-loop Lagrangian, calculated in Appendix E (Eq. E7) is

\[
L_{\Sigma M+SM}\text{qukl}^{1-\text{loop};\Lambda^2} = \frac{-6\lambda^2 f^2_\pi + 4 \sum_{\mu} m_\mu^2 + 4 \sum_{\nu} m_\nu^2}{16\pi^2 \langle H \rangle^2} \Lambda^2 (\Phi^\dagger \Phi - \frac{1}{2} \langle H \rangle^2)
\]

with

\[
C_\text{Unrenorm;1-\text{loop};\Lambda^2} = \frac{6\lambda^2 f^2_\pi + 4 \sum_{\mu} m_\mu^2 + 4 \sum_{\nu} m_\nu^2}{16\pi^2 \langle H \rangle^2}
\]

recognizable as the zero-gauge-coupling limit of 1-loop UV-QD in the SM [6, 38, 39], where to this order \(-6\lambda^2 f^2_\pi = -3m^2_\pi \).

After Higgs-VSC tadpole renormalization (cf. subsection II B), the effective scalar-sector Lagrangian, keeping 1-loop UV-QD (but ignoring un-interesting logarithmic divergences, finite parts and vacuum energy/bubbles) can again be written:

\[
L_{\Sigma M+SM}\text{qukl}^{\text{Effective};1-\text{loop};\Lambda^2} = -|\partial_\mu \Phi|^2 - V_{\Sigma M+SM}\text{qukl}^{\text{Renorm};1-\text{loop};\Lambda^2},
\]

\[
V_{\text{Renorm};1-\text{loop};\Lambda^2} = -\lambda^2 \left( \Phi^\dagger \Phi - \frac{1}{2} \langle f^2_\pi - \frac{m^2_\pi}{\lambda^2} \rangle \right) - f_\pi m^2_\pi H
\]

with \( H = h + f_\pi \).

Here

\[
m^2_\pi \equiv \lambda^2 \delta \langle H \rangle^2 + C_\text{Unrenorm;1-\text{loop};\Lambda^2}(\langle H \rangle = f_\pi)
\]

is still the physical renormalized pseudo-scalar pion (pole) mass-squared, and

\[
m^2_\pi = m^2_\pi + 2\lambda^2 f^2_\pi \geq m^2_\pi.
\]

Once again, the physical renormalized pseudo-scalar pion pole mass-squared \( m^2_\pi \) has absorbed all 1-loop UV-QD, which now includes those from virtual SM quarks and leptons. Therefore, the results in each of the sections above are generalized to include all UV-QD traceable to virtual SM quarks and leptons:

- Section II C (1-loop): At 1-loop the generic O(4) \( \Sigma M \) in the \( f_\pi \) vs. \( m^2_\pi/\lambda^2 \) half-plane of Figure 1 (i.e. away from the y-axis, \( m^2_\pi = 0 \)), and including Wigner mode (x-axis), has (additional) surviving finite remnants of UV-QD due to virtual SM quarks and leptons, and continues to have a HFTP.

- Section II D (1-loop): It is obvious that

\[
V_{\text{Renorm;1-\text{loop};\Lambda^2}}^{\Sigma M+SM}\text{qukl} \text{ in Eq. } 38 \text{ and Figure } 2 \text{ has Nambu-Goldstone Bosons only when "bottom-of-the-wine-bottle" Goldstone symmetry is restored, i.e. in Goldstone mode on the } m^2_\pi = 0 \text{ line. After Goldstone-SRC, the Goldstone Theorem forces all 1-loop UV-QD and their finite remnants to vanish exactly and identically:}
\]

\[
m^2_\pi = -\lambda^2 \delta \langle H \rangle^2 - \left[ \frac{-6\lambda^2 f^2_\pi + 4 \sum_{\mu} m_\mu^2 + 4 \sum_{\nu} m_\nu^2}{16\pi^2 \langle H \rangle^2} \right] \Lambda^2
\]

\[
\rightarrow m^2_\pi;\text{NGB};\Sigma M+SM\text{qukl} \equiv 0
\]

\[
m^2_\pi = 2\lambda^2 f^2_\pi.
\]
where the sum is over all flavors, and for quarks over all colors.

• Eq. 42 fixes the numerical value of \( \delta(H)^2 \) so that there is exactly zero finite remnant after UV-QD cancellation. \( m^2_{\pi} \) is still the physical renormalized scalar Higgs pole mass-squared: it is still not fine-tuned. Therefore, including virtual SM quarks, leptons and scalars, spontaneously broken O(4) LΣM has no surviving finite remnants of cancelled 1-loop UV-QD and therefore has no HFTP.

• \( L^{Effective;\ 1-loop;\ A^2}_{Goldstone;\ LΣM+SMqℓ;\ Φ} \) (1-loop): The 1-loop Goldstone-mode scalar-sector effective Lagrangian, including all 1-loop UV-QD effects due to virtual scalars and SM quarks and leptons, is as given in Eq. 38 with \( m^2_{\pi} = 0 \). This is the same as Eq. 31. Therefore

\[
L^{Effective;\ 1-loop;\ A^2}_{Goldstone;\ LΣM+SMqℓ;\ Φ} = L^{Effective;\ 1-loop;\ A^2}_{Goldstone;\ LΣM;\ Φ}. \tag{43}
\]

• Section II.E (1PI multi-loops): Calculation of 1PI naive degrees of divergence reveals [40–51] that UV-QD can only appear in the scalar sector effective Lagrangian, where they are exactly cancelled by scalar sector counterterms. Therefore, when including UV-QD traceable to virtual SM quarks and leptons, the 1-loop observations and lesson in Section II.D are generalized in to all-perturbative-loop-orders.

• That all 1-loop UV-QD due to virtual SM fermions cancel exactly in the SM Higgs self-energy and mass cancel after tadpole renormalization has been known [30, 39] for more than 3 decades.

### III. CONCLUSION

Belief in huge non-vanishing remnants of cancelled ultra-violet quadratic divergences, and consequent fine-tuning problems in field theories with fundamental scalars, fruitfully formed much of the original motivation for certain proposed Beyond the Standard Model (BSM) physics. A partial list would include: Technicolor [52], low-energy supersymmetry [55], little Higgs theories [56] and Lee-Wick theories [57].

Susskind [52] was probably the first to motivate BSM physics out of concern for fine-tunings in the scalar sector. He argued that fundamental scalars are bad because they lead to quadratic divergences that must be fine tuned away. The motivation for technicolor was to replace the fundamental scalar Higgs doublet with a composite condensate of new fermions, called techniquarks, with QCD-like confining interactions. Fundamental scalars do result in quadratic divergences that must be fine tuned away under many circumstances (e.g. in Section II.C), and so most renormalizable field theories with fundamental scalars do have a scalar fine-tuning problem. However, as we have showed, the Goldstone Theorem can protect a limited number of scalars (at least those in the same multiplet as the Nambu-Goldstone Bosons) from such fine-tuning problems. The O(4) Linear Sigma Model (LΣM) with one Higgs doublet, the same scalar structure as the standard model (SM), is one example of such a theory that is protected in Goldstone mode, i.e. when spontaneously broken. This is an important loophole in the canonical objection to fundamental scalars. More particularly, it is a loophole that the stand-alone Standard Model, unadorned by BSM physics, makes use of [5] so that the SM Higgs mass is not-fine-tuned.

Low energy supersymmetry (SUSY) aims to solve the Higgs fine-tuning problem by the miraculous cancellation of the quadratically divergent portion of the loop contributions of particles against those of their superpartners. The key observation is that if the particle is a boson it’s superpartner is a fermion, and vice versa – the \(-1\) that fermions pick up in loops relative to bosons is key to understanding this cancellation. (But not sufficient – the numerical values of the coefficients must be equal, and are as a consequence of the supersymmetry.) Low energy SUSY might be used to solve the scalar fine-tuning problem of theories that have one. However, since the spontaneously broken O(4) LΣM (and the SM [5]) does not have this problem, low energy SUSY can only solve the scalar fine-tuning problems of BSM physics. For example, Grand Unified Theories will generically have a Higgs fine-tuning problem, as finite contributions to the Higgs mass-squared will be of order the GUT scale, and will need to be fine tuned down to the weak scale. Thus SUSY GUTs can make use of low energy SUSY to avoid a Higgs Fine-Tuning Problem.

The role of the Goldstone Theorem in taming the Higgs fine-tuning problem in spontaneously broken O(4) LΣM (and the Standard Model [5]) appears to have gone unnoticed. As we have shown, because quadratic divergences are sensitive only to ultraviolet physics, the O(4) symmetry forces the quadratically divergent contributions to the Higgs mass-squared to be identical to those of the other three bosons in its multiplet. This is true whether the O(4) symmetry is spontaneously broken, explicitly broken or unbroken. In the unbroken (Wigner-mode) and explicitly broken phases of the theory, this still leaves a fine-tuning if the boson masses are meant to be small compared to whatever scale the cutoff is taken to be. However, in the spontaneously broken phase the Goldstone Theorem guarantees that the companions of the Higgs are exactly massless Nambu-Goldstone Bosons, and thus automatically eliminates any common UV-QD contributions to the masses of the NGBs and the Higgs.

It appears that the requirement that the Higgs vacuum is stable (against spontaneous appearance or disappearance of physical Higgs) fixes the natural scale of the remaining contributions to \( m^2_{\pi} \) to be \( f_\pi^2 \) by requiring that the Higgs vacuum expectation value \( \langle H \rangle \) be exactly equal to \( f_\pi = (\frac{G^{Exp}_{μ}}{\sqrt{2}})^{-1/2} \). This leaves \( m^2_{\pi} \) to pick up only finite and logarithmically divergent contributions.
We have shown that S-Matrix UV-QD sum exactly to zero in the spontaneously broken O(4) \( LΣM \). This is traced to the UV-QD Goldstone-Mode Renormalization Prescription (see Appendix C): after Higgs-Vacuum Stability Condition tadpole renormalization, the UV-QD-corrected theory requires explicit enforcement of the Goldstone Theorem by imposition of Lee/Symanzik’s Goldstone-Symmetry Restoration Condition. All S-Matrix UV-QD, and their finite remnants, therefore vanish identically. Our no-fine-tuning-theorem for a weak-scale Higgs mass is therefore simply another (albeit unfamiliar) consequence of the Goldstone Theorem, an exact property of the O(4) \( LΣM \) vacuum and excited states. It is un-necessary to impose any new symmetries or new physics: Goldstone-mode O(4) \( LΣM \) already has sufficient symmetry to force all S-Matrix UV-QD remnants to vanish identically and to ensure that there is no Higgs Fine Tuning Problem.

Although the stand-alone Goldstone-mode O(4) \( LΣM \), with loop integrals cut off at some much higher energy UV scale, is insensitive to that higher scale (up to terms proportional to \( \sim \ln \Lambda \)), that may be spoiled if it is embedded/integrated into some higher scale “Beyond the O(4) \( LΣM \)” theory. But that is not the problem of Goldstone-mode O(4) \( LΣM \)s. It is up to those introducing the new high-scale physics to maintain the Goldstone “miracle,” or, at worst, augment it with a natural mechanism of their own.

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but
\[(2 - n)A(m^2) \propto \Gamma \left(2 - \frac{n}{2}\right) \quad (A3)\]
\[\xrightarrow{n \to 4} \sim m^2 \ln \left(1 + \frac{\Lambda^2}{m^2}\right) + \text{finite}.\]

Meanwhile
\[nB_{22}(q^2, m_1^2, m_2^2) - A(m^2) \quad (A4)\]
\[\xrightarrow{n \to 4} \sim m^2 \ln \left(1 + \frac{\Lambda^2}{m^2}\right) + \text{finite}.\]

The vanishing of UV-QD in Goldstone-mode O(4) LΣM does not therefore depend on whether dimensional regularization or Pauli-Villars UV cut-off regularization is used.

**Appendix B: 1-loop UV-QD Lagrangian, \(L^{1\text{-}\text{loop}, \Lambda^2}_{\Sigma M, \text{hh}}\) in O(4) LΣM (without fermions), in the \(f_\pi - m_\pi^2/\lambda^2\) half-plane**

We are only interested in UV-QD contributions to 1-loop effective Lagrangians: these can be evaluated at zero momentum.

### 1. 1-loop 2-point functions

The UV-QD contributions of 1-loop 2-point \(hh\) Higgs self-energy diagrams \([6, 38, 39]\) (Figure 3) to the 1-loop UV-QD 2-point Lagrangian are (with the Feynman diagram naming convention of Ref. \([6]\)): \(P_{9A}\) with virtual \(\pi_3, P_{9B}\) with virtual \(\pi_+\), and \(P_{10}\) with virtual \(h\):

\[
L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, hh} = \Lambda^2 (p_{9A} + p_{9B} + p_{10}) \frac{h^2}{2} \quad (B1)
\]

with
\[p_{9A} = -\frac{\Lambda^2}{16\pi^2}; \quad p_{9B} = -2\frac{\Lambda^2}{16\pi^2}; \quad p_{10} = -\frac{3\Lambda^2}{16\pi^2}. \quad (B2)\]

The UV-QD contribution of each of the 1-loop 2-point \(\pi_3 \pi_3\) neutral-pseudoscalar self-energy diagrams (Figure 4) in \(L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, \pi_3 \pi_3}\), and the 1-loop 2-point charged-pseudoscalar \(\pi_\pm \pi_\mp\) self-energy diagrams (Figure 5) in \(L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, \pi_\pm \pi_\mp}\) are related (by explicit calculation and by O(4) symmetry) to their associated 1-loop 2-point \(hh\) Higgs self-energy diagrams (Figure 3) in \(L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, hh}\) by Clebsch-Gordon coefficients and combinatorics.

Adding all these into the UV-QD 1-loop 2-point Lagrangian gives:

\[
L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, \Phi^4 \Phi} = L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, hh} + L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, \pi_3 \pi_3} + L^{1\text{-}\text{loop}, 2\text{-}\text{point}; \Lambda^2}_{\Sigma M, \pi_\pm \pi_\mp} \quad (B3)
\]

\[= \Lambda^2 \left(p_{9A} + p_{9B} + p_{10}\right) \left(\frac{h^2}{2} + \frac{\pi^2_+}{2} + \pi_+ \pi_-ight).\]
2. 1-loop 1-point functions

With \( \langle H \rangle ^2 > 0 \) the theory also receives UV-QD contributions from 1-loop 1-point \( \langle H \rangle h \) tadpole diagrams [6, 38, 39] (Figure 6): \( T_{1A} \) with a virtual \( \pi_3 \) loop, \( T_{1B} \) with a virtual \( \pi_\pm \) loop, \( T_4 \) with a virtual \( h \) loop.

The contributions of these tadpole diagrams to S-matrix elements (e.g. boson propagator tadpole insertions) can be incorporated via a 1-loop 1-point tadpole Lagrangian

\[
L_{1\text{-loop};1\text{-point};\Lambda^2}^{1\text{-loop}} = (t_{1A} + t_{1B} + t_4) \Lambda^2 h. \tag{B4}
\]

But explicit calculation (or O(4) symmetry [6, 38–45, 50] and visual comparison of Figures 3 and 6) shows that the UV-QD in each of the 1-loop 1-point tadpole diagrams is proportional to the UV-QD in its associated 1-loop 2-point Higgs self-energy diagram:

\[
t_{1A} = p_{9A} \langle H \rangle, \quad t_{1B} = p_{9B} \langle H \rangle, \quad t_4 = p_{10} \langle H \rangle, \tag{B5}
\]

so that

\[
L_{1\text{-loop};2\text{-point};\Lambda^2}^{1\text{-loop}} = (p_{9A} + p_{9B} + p_{10}) \Lambda^2 \langle H \rangle h. \tag{B6}
\]

We can therefore form the total UV-QD 1-loop O(4) LΣM Lagrangian

\[
L_{1\text{-loop};1\text{-point};\Lambda^2}^{1\text{-loop}} = L_{1\text{-loop};2\text{-point};\Lambda^2}^{1\text{-loop}} + L_{1\text{-loop};1\text{-point};\Lambda^2}^{1\text{-loop}}
\]

\[
= (p_{9A} + p_{9B} + p_{10}) \Lambda^2 \left( \frac{h^2}{2} + \frac{\pi_3^2}{2} + \pi_+ \pi_- + \langle H \rangle h \right)
\]

\[
= C_{\text{renorm};1\text{-loop};\Lambda^2} \Lambda^2 \left( \Phi^\dagger \Phi - \frac{1}{2} \langle H \rangle^2 \right)
\]

where

\[
C_{\text{renorm};1\text{-loop};\Lambda^2} = (p_{9A} + p_{9B} + p_{10}) \tag{B8}
\]

\[
= -\frac{6\lambda}{16\pi^2}.
\]

Appendix C: Goldstone-Mode Renormalization Prescription (GMRP) and the fine-tuning discontinuity

In order to better understand a correct self-consistent renormalization prescription for the SM, we re-do the calculations and analysis of Section II D, but beginning with the Goldstone-mode O(4) LΣM bare Lagrangian, i.e. Eq. 1 with \( \epsilon = 0 \).
with
\[ V_{\text{Goldstone; } LΣM}^{\text{Bare; } Λ^2} = \lambda^2 \left[ \Phi^1 \Phi - \frac{1}{2} \left( \langle H \rangle^2 + \delta \langle H \rangle^2 \right) \right]^2 \] (C2)

\[ = \lambda^2 \left[ \Phi^1 \Phi - \frac{1}{2} \langle H \rangle^2 \right]^2 - V_{\text{CounterTerm; } LΣM}^{\text{C}} \]

where
\[ V_{\text{CounterTerm; } LΣM}^{\text{C}} = \lambda^2 \delta \langle H \rangle^2 \left( \Phi^1 \Phi - \frac{1}{2} \langle H \rangle^2 \right) \]

\[ H \equiv h + \langle H \rangle, \text{and}\langle h \rangle = 0. \] (C3)

We continue to ignore vacuum energy. Using this bare Lagrangian and the 1-loop UV-QD result in Eq. 10, we form the 1-loop UV-QD-improved effective Goldstone-mode O(4) LΣM Lagrangian, which includes all scalar 2-point 1-loop UV-QD self-energies and 1-point 1-loop UV-QD tadpoles (but ignores un-interesting 1-loop logarithmically divergent, finite contributions and vacuum energy/bubbles):

\[ L_{\text{Effective; } 1\text{-loop; } LΣM} = L_{\text{Goldstone; } LΣM}^{\text{Bare; } Λ^2} + L_{\text{1\-loop; } LΣM}^{1\text{-loop; } Λ^2} \] (C4)

\[ = -\partial^2 \Phi^2 - V_{\text{Goldstone; } LΣM}^{\text{counter; } 1\text{-loop; } Λ^2} \]

where
\[ V_{\text{Goldstone; } LΣM}^{\text{counter; } 1\text{-loop; } Λ^2} = \lambda^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi^+ \pi^- + \langle H \rangle h \right]^2 \] (C5)

\[ + m_{\pi; NGB; LΣM}^2 \left[ \frac{h^2}{2} + \frac{\pi^2}{2} + \pi^+ \pi^- + \langle H \rangle h \right] \]

and
\[ m_{\pi; NGB; LΣM}^2 = - \left( \lambda^2 \delta \langle H \rangle^2 + \frac{6\lambda^2}{16\pi^2} Λ^2 \right). \] (C6)

A 1-loop-induced finite (after cancellation of UV-QD) mass-squared \( m_{\pi; NGB; LΣM}^2 \) for the three Nambu-Goldstone Bosons has appeared! But it is only an artifact of not having yet properly enforced the Goldstone Theorem – a contribution to the physical renormalized pion mass-squared appearing in Lee/Symanzik’s generic O(4) LΣM theories before taking the Goldstone-mode limit. In other words, quantum loops have moved the theory off the Goldstone-mode limit. In other words, quantum loops have moved the theory off the Goldstone-mode limit. Proper renormalization must move it back onto the \( m_\pi^2 = 0 \) line [40–45] in order to enforce the Goldstone Theorem. We note that the Higgs Vacuum Stability Condition is automatically satisfied when \( m_\pi^2 = 0 \) (but not elsewhere in the \( f_\pi \) vs. \( m_\pi^2 / \lambda^2 \) half-plane) (Figure 1). Proper renormalization must move it back onto the \( m_\pi^2 = 0 \) line [40–45] in order to enforce the Goldstone Theorem. We note that the Higgs Vacuum Stability Condition is automatically satisfied when \( m_\pi^2 = 0 \) (but not elsewhere in the \( f_\pi \) vs. \( m_\pi^2 / \lambda^2 \) half-plane). This “miracle” [50] is none other than the O(4) invariance.

Section II D proves that the self-consistent UV-QD GMRP for O(4) LΣM is to impose Lee/Symanzik’s two conditions:

1. Goldstone SRC: \( \lambda^2 \delta \langle H \rangle^2 \) is used to set \( m_{\pi; NGB; LΣM}^2 \) to zero identically, with exactly zero finite remnant, and restore axial current conservation.

2. Higgs VSC: Set \( \langle H \rangle = f_\pi \) NoFineTuning to the experimental value, its name emphasizing that...
fine-tuning is un-necessary for a weak scale \((f_{\pi}^{\text{NoFineTuning}})^2 G^\mu_\nu = O(1)\). (Note that that is necessary, even though it would appear that the Goldstone-SRC eliminates all tadpoles, because the theory must be well-behaved off as well as on the Goldstone line.)

We now recover Eq. 31:

\[
H = h + f_{\pi}^{\text{FineTuned}}, \quad \langle h \rangle = 0.
\]

This prescription clearly generalizes to include SM quarks and leptons (Section II F) and all-loop-orders (Section II E), in Goldstone-mode O(4) LΣM.

**Fine-tuning discontinuity:** Ignoring vacuum energy, re-write the Goldstone-mode O(4) LΣM 1-loop Eq. C5 as:

\[
\begin{align*}
V_{\text{Renorm;1-loop}} & = |f_{\pi}^{\text{FineTuned}}|^2 + \frac{\pi_3^2}{2} + \pi_+ + \pi_- \quad \text{(C8)} \\
\end{align*}
\]

and re-scale the Higgs VEV

\[
H = h + f_{\pi}^{\text{FineTuned}}, \quad \langle h \rangle = 0.
\]

The temptation is to regard the Higgs VEV in Eq. C10 as fine-tuned. If it were, the two different renormalization approaches, Eqs. (31 and C8) would generate a “fine-tuning discontinuity” in the 1-loop UV-QD renormalization of Goldstone-mode O(4) LΣM:

\[
\Delta f_{\pi}^2 = (f_{\pi}^{\text{FineTuned}})^2 - (f_{\pi}^{\text{NoFineTuning}})^2 \quad \text{(C11)}
\]

\[
= \delta(H)^2 + (-6) \frac{\Lambda^2}{16\pi^2}
\]

\[
\text{with}\]

\[
\frac{m_{\pi}^2}{\Lambda^2;\text{NGB};\text{LΣM}}\]

\[
\text{even after cancellation of UV-QD. But Lee/Symonzik provide the self-consistent way out of this nasty discontinuity: i.e. they insist that Goldstone mode is simply the m_{\pi}^2 = 0 line (y-axis) in the f_{\pi} vs. m_{\pi}/\Lambda^2 half-plane (Figure 1) [40–45]. This is essential – to realize that the O(4) LΣM must be defined in the half-plane because the quantum loops contribute to m_{\pi}^2. Everywhere in the plane, the UV-QD are cancelled by counter-terms, but only on the Goldstone line does the Goldstone Theorem force m_{\pi}^2 to have a specific value \ll \Lambda^2, namely m_{\pi}^2 = 0. The Goldstone-Mode UV-QD Renormalization Prescription, m_{\pi}^2 \to m_{\pi}^2;\text{NGB};\text{LΣM} \equiv 0, then forces}

\[
\Delta f_{\pi}^2 = 0 \quad \text{(C13)}
\]

identically, with exactly zero finite remnant.

**Appendix D:** Finite remnants of cancelled UV-QD vanish identically in Goldstone mode O(4) LΣM to all perturbative loop-orders. The Higgs mass is *not* fine-tuned.

B.W.Lee and K.Symonzik renormalized generic O(4) LΣM to all-loop-orders throughout the \(f_{\pi} vs. m_{\pi}^2/\Lambda^2\) half-plane [40–45]. Aside from vacuum bubbles (ignored here), they proved that 1PI multi-loop UV-QD can only appear in the coefficients of:

- Scalar-sector 2-point operators proportional to \(\Phi^\dagger \Phi\);
- Scalar-sector 1-point tadpole operators, whose 1PI loops contain at least one 3-boson vertex.

It was then shown that the counter-term Lagrangian

\[
L^\text{CounterTerm};\Lambda^2 = \lambda^2 \langle H \rangle^2 \left(\frac{h^2 + \pi_3^2}{2} + \pi_+ + \pi_- + \langle H \rangle h\right)
\]

\[
\text{is guaranteed to remove all 1PI UV-QD in the S-Matrix [40–51]. It follows that the multi-loop UV-QD contribution can be written}
\]

\[
L^\text{All-loops};\Lambda^2
\]

\[
\text{with}\]

\[
C^\text{Unrenorm;All-loops};\Lambda^2 \left(\frac{h^2 + \pi_3^2}{2} + \pi_+ + \pi_- + \langle H \rangle h\right) \Lambda^2
\]

\[
\text{with}\]

\[
\text{Now form the all-loop-UV-QD-improved effective Lagrangian, including all-orders scalar 2-point self-energy and 1-point tadpole UV-QD (but ignoring un-interesting logarhythmically divergent and finite contributions and vacuum energy/bubbles):}
\]

\[
L^\text{Effective;All-loop};\Lambda^2 = L^\text{Bar};\Lambda^2 + L^\text{All-loop};\Lambda^2
\]

\[
= -|\partial_\mu \Phi|^2 + V^\text{Renorm;All-loop};\Lambda^2, \quad \text{(D3)}
\]

where

\[
V^\text{Renorm;A-loop};\Lambda^2
\]

\[
= \lambda^2 \left[\frac{h^2}{2} + \frac{\pi_3^2}{2} + \pi_+ + \pi_- + \langle H \rangle h\right]^2 \quad \text{(D4)}
\]

\[
+ m_{\pi}^2 \left[\frac{h^2}{2} + \frac{\pi_3^2}{2} + \pi_+ + \pi_-\right] + (\langle H \rangle - f_{\pi}) m_{\pi}^2 h.
\]
Here
\[ m_{\pi}^2 = -\lambda^2 \delta(H)^2 + C^{Unrenorm;All-loops;\Lambda^2}_{LSM} \]  
(D5)
and
\[ m_h^2 = m_{\pi}^2 + 2\lambda^2 \langle H \rangle^2 \geq m_{\pi}^2, \]  
(D6)
Once again, \( m_{\pi}^2 \), the all-loop-corrected physical renormalized pion (pole) mass-squared (and the solution to a highly non-linear equation), has absorbed all UV-QD, this time to all perturbative loop-orders [40–45].

To force the theory to Goldstone mode, Lee/Symaznik’s two conditions are imposed:

**Higgs VSC Tadpole renormalization:** The physical Higgs particle must not simply disappear into the exact UV-QD-corrected vacuum, defined in Section II A:
\[ \langle H \rangle = f_\pi; H = h + f_\pi; \langle h \rangle = 0 \]  
(D7)
Thus
\[ V^{\text{Renorm};All-loops;\Lambda^2}_{LSM} = \lambda^2 \left[ \Phi^I \Phi - \frac{1}{2} \left( f_\pi^2 - \frac{m_{\pi}^2}{\lambda^2} \right) \right]^2 - f_\pi m_{\pi}^2 h \]  
(D8)
with
\[ m_{\pi}^2 = -\lambda^2 \delta(H)^2 - C^{Unrenorm;All-loops;\Lambda^2}_{LSM} \]  
(D9)
\[ m_h^2 = m_{\pi}^2 + 2\lambda^2 f_\pi^2 \geq m_{\pi}^2, \]  
(D10)
A cross-section of \( V^{\text{Renorm};All-loops;\Lambda^2}_{LSM} \) (through its absolute minimum and its local maximum) is plotted in Figure 2.

**Goldstone-SRC:** NGB masses must be identically and exactly zero: \( m_{\pi}^2 \rightarrow m_{\pi;NGB;LSM}^2 = 0 \). Thus
\[ 0 = m_{\pi;NGB;LSM}^2 \]  
(D11)
\[ = -\lambda^2 \delta(H)^2 - C^{Unrenorm;All-loops;\Lambda^2}_{LSM} \]  
(D9)
\[ = m_{\pi}^2 + 2\lambda^2 f_\pi^2 \]  
(D10)
But \( m_{\pi}^2 \) is not only the coefficient of \( \frac{1}{2} \pi_3^2 \) and of \( \pi_+ \pi_- \), the pion mass terms in \( V^{\text{Renorm};All-loops;\Lambda^2}_{LSM} \) (cf. Eq. D4), and hence in the effective Lagrangian \( L^{Effective;All-loops;\Lambda^2}_{LSM;\Phi} \) (cf. Eq. D3). \( m_{\pi}^2 \) is also, by naive power counting, the quadratically divergent coefficient of \( \frac{1}{2} h^2 \), the Higgs mass term. Therefore \( m_{\pi}^2 \) not only is not quadratically divergent, but there are no finite remnants of the cancellation of the quadratic divergences. This is achieved without fine-tuning because it is a direct consequence of the Goldstone Theorem for spontaneously broken O(4) LSM, not a choice. Furthermore
\[ L^{Effective;All-loops;\Lambda^2}_{Goldstone;LSM} \]  
(D12)
where
\[ V^{\text{Renorm};A-loop;\Lambda^2}_{LSM} \]  
(D13)
\[ = \lambda^2 \left[ \Phi^I \Phi - \frac{1}{2} f_\pi^2 \right]^2 \]  
(D14)
gives the sensible, at worst logarithmically divergent and not-fine-tuned Higgs mass:
\[ m_h^2 = 2\lambda^2 f_\pi^2. \]  
(D15)

**Appendix E: Virtual SM quark and lepton contributions to the 1-loop UV-QD Lagrangian in the \( f_\pi - m_{\pi}^2/\lambda^2 \) half-plane.** 1-loop-UV-QD-improved Goldstone-mode scalar-sector effective Lagrangian

UV-QD contributions to the 1-loop effective Lagrangian can be calculated at zero momentum. Including virtual SM quarks and leptons (Reference [6]: graphs \( P_{11;1} \) where \( i \) runs over these fermions), 1-loop 2-point function \( hh \) Higgs self-energy diagrams (shown for the representative third-generation quarks and leptons in Figure 7):

![Graph of third-generation quarks and leptons contributing to \( L^{1-loop;2-point;\Lambda^2}_{LSM+SMeson;hh} \). There are identical graphs for the other two generations.](image)

\[ L^{1-loop;2-point;\Lambda^2}_{LSM+SMeson;hh} = \Lambda^2 \left( p_{9A} + p_{9B} + p_{10} + p_{11} + \frac{1}{2} h^2 \right) \]  
(E1)
with
\[ p_{11} = \sum_{i=fermions} p_{11;i}. \]  
(E2)
Here
\[ 8\pi^2 p_{11;i} = y_i^2 \]  
(E3)
is the square of the Yukawa coupling to the ith fermion \([6, 38, 39]\).

With \((H)^2 > 0\), the theory also receives 1-loop 1-point function \((H) h\) tadpole diagrams \([6, 38, 39]\) (Ref. \([6]\) naming conventions: graphs \(T_{6i}\), where \(i\) runs over all virtual SM quarks and leptons). (These are shown for the representative third-generation quarks and leptons in Figure 8.) Their contribution to S-matrix elements (e.g. boson and fermion propagator tadpole insertions) can be incorporated via a 1-loop 1-point tadpole Lagrangian

\[
L_{LSM + SMqkl}^{\text{1-loop;1-point}; A^2} = (t_{1A} + t_{1B} + t_{1} + t_{0}) A^2 h. \tag{E4}
\]

But explicit calculation shows \([6, 38, 39]\) that the UV-QD in each of the 1-loop 1-point tadpole diagrams in Figure 8 is proportional to the UV-QD in its associated 1-loop 2-point Higgs self-energy diagram in Figure 7, so that

\[
L_{LSM + SMqkl}^{\text{1-loop;1-point}; A^2} = (p_{0A} + p_{0B} + p_{l0} + p_{11}) A^2 \langle H \rangle h. \tag{E5}
\]

Furthermore, the UV-QD contribution of each of the 1-loop 2-point \(\pi_3\pi_3\) neutral pseudoscalar self-energy diagrams (Figure 9) in \(L_{LSM + SMqkl}; \pi_3\pi_3\) and of each of the 1-loop 2-point \(\pi_+\pi_-\) charged pseudoscalar self-energy diagrams (Figure 10) in \(L_{LSM + SMqkl}; \pi_+\pi_-\) are related (by explicit calculation and O(4) symmetry) to their associated 1-loop 2-point \(hh\) Higgs self-energy diagrams (Figure 7) in \(L_{LSM + SMqkl}; hh\) by Clebsch-Gordon coefficients and combinatorics. The total UV-QD 1-loop Lagrangian is

\[
L_{LSM + SMqkl}^{\text{1-loop;1-point}; A^2} = (p_{0A} + p_{0B} + p_{l0} + p_{11}) \times \tag{E6}
\]

\[
× \Lambda^2 \left( \frac{1}{2} h^2 + \frac{1}{2} \pi_3^2 + \pi_+ \pi_- + \langle H \rangle h \right) = C^{\text{Unrenorm;1-loop}; A^2} \Lambda^2 \left( \Phi \Phi - \frac{1}{2} \langle H \rangle^2 \right)
\]

with

\[
C^{\text{Unrenorm;1-loop}; A^2} = (p_{0A} + p_{0B} + p_{l0} + p_{11}).
\]

\(C^{\text{Unrenorm;1-loop}; A^2}\) is a constant. With quark and lepton masses-squared \(m_i^2 = \frac{1}{2} \langle H \rangle^2\), the coefficient in Eq. E6 can famously \([6]\) be written

\[
C^{\text{Unrenorm;1-loop}; A^2} \Lambda^2 = \left( \frac{3m_1^2 + 4 \sum_{\text{quarks}} m_i^2 + 4 \sum_{\text{leptons}} m_i^2}{16\pi^2 \langle H \rangle^4} \right) \Lambda^2. \tag{E7}
\]

(Where the sums are over flavor and, for the quarks, color.)

---

**FIG. 8.** Graphs of third-generation quarks and leptons contributing to \(L_{LSM + SMqkl}^{\text{1-loop;1-point}; A^2}\). There are identical graphs for the other two generations.

**FIG. 9.** Graphs of third-generation quarks and leptons contributing to \(L_{LSM + SMqkl}^{\text{1-loop;2-point}; A^2}\). There are identical graphs for the other two generations.
FIG. 10. Graphs of third-generation quarks and leptons contributing to \( L_{1\text{-loop};2\text{-point};\Lambda^2} \). There are identical graphs for the other two generations.

Now write the bare scalar-sector Lagrangian

\[
L_{\text{Bare};\Lambda^2;\Phi} = -|\partial_\mu \Phi|^2 - V_{\text{Bare};\Lambda^2;\Phi} - V_{\text{Bare};\Lambda^2;\Sigma} \equiv 0
\]

and form the scalar sector 1-loop-UV-QD-improved effective Lagrangian, including 1-loop 2-point self-energies and 1-loop 1-point tadpole UV-QD (but ignoring uninteresting logarithmically divergent and finite contributions and vacuum energy/bubbles):

\[
L_{\text{Effective};\Lambda^2;\Phi} = L_{\text{Bare};\Lambda^2;\Phi} + L_{1\text{-loop};\Lambda^2;\Phi} + L_{1\text{-loop};\Sigma;\Phi} \equiv 0
\]

with

\[
V_{\text{Renorm};\Lambda^2} = \lambda^2 \left[ \frac{1}{2} h^2 + \frac{1}{2} \langle H \rangle \right]^2
\]

and

\[
m_{\pi;\Sigma;\Phi}^2 = 2\lambda^2 f_\pi^2
\]

Comparison with Eq. 25 reveals that the scalar sector effective Goldstone-mode \( O(4) \) \( L_{\Sigma M} \) Lagrangians and Higgs mass are unchanged by UV-QD quantum loops with virtual SM quarks and leptons:

\[
L_{\text{Effective};1\text{-loop};\Lambda^2;\Phi} = L_{\text{Goldstone};\Lambda^2;\Phi} \equiv 0
\]

These results generalize to all-loop-orders.