FINITE SUBGROUPS OF THE HOMEOMORPHISM GROUP OF A COMPACT TOPOLOGICAL MANIFOLD ARE ALMOST NILPOTENT

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Abstract. Around twenty years ago Ghys conjectured that finite subgroups of the diffeomorphism group of a compact smooth manifold $M$ have an abelian normal subgroup of index at most $a(M)$, where $a(M)$ depends only on $M$. First we construct a family of counterexamples to this conjecture including, for example, the product space $T^2 \times S^2$.

Following the first appearance of our counterexample on the arXiv Ghys put forward a revised conjecture, which predicts only the existence of a nilpotent normal subgroup of index at most $n(M)$. Our main result is the proof of the revised Ghys conjecture. More generally, we show that the same result holds for homeomorphism groups of not necessarily compact topological manifolds with finitely generated homology groups.

Our proofs are based on finite group theoretic results which provide a general strategy for proving similar Jordan-type theorems.

1. Introduction

The following is a summary of two classical results due to H. Minkowski [27] and C. Jordan [19]. See [15] and [5] for a modern rendering of Minkowski’s and Jordan’s original arguments.

Theorem 1.1 (Minkowski, Jordan). If $K$ is a number field, then there is a constant $B = B(d, K)$ such that for any finite group $G < \text{GL}(d, K)$, one has $|G| \leq B$. If $K$ is an arbitrary field of characteristic zero, then there is a constant $J = J(n)$ such that for any finite group $G < \text{GL}(d, K)$, there exists an abelian normal subgroup $A \leq G$ of index at most $J$.

Let us say that a group $G$ has the Jordan property if there is a constant $a(G)$ such that any finite subgroup $G$ of $G$ has an abelian normal subgroup of index at most $a(G)$.

#1We would like to dedicate this work to our teacher, Gabriella Thiry (Gabi néni).
abelian. This terminology was introduced a few years ago by Popov \cite{popov2009}. Jordan’s theorem states that the groups $GL(d, \mathbb{C})$ have the Jordan property. Boothby and Wang \cite{boothby1965} proved that connected Lie groups have the Jordan property (see also \cite{wang1964}). Moreover, connected algebraic groups have the Jordan property \cite{springer1970}.

Around twenty years ago Étienne Ghys conjectured in several lectures (see \cite{ghys1993}) that for any compact smooth manifold $M$, there is a constant $a(M)$ such that any finite group acting smoothly and effectively on $M$ has an abelian normal subgroup of index at most $a(M)$, that is, the diffeomorphism group of $M$ has the Jordan property. This conjecture first appeared in print in a 2011 survey of Fisher on the Zimmer program \cite{fisher2011}.

The particular case in which $M$ is a sphere was also independently asked in several talks by Walter Feit and later by Bruno Zimmermann. A positive answer to the Feit-Zimmermann question was recently obtained by Mundet i Riera \cite{mundet2020}. Zimmermann \cite{zimmermann1990} has shown using Thurston’s geometrization conjecture (proved by Perelman) that the Ghys conjecture holds for compact 3-manifolds. Mundet i Riera has confirmed the Ghys conjecture in several other cases: for tori \cite{mundet2020}, acyclic manifolds, homology spheres and manifolds with nonzero Euler characteristic \cite{mundet2020}.

Despite the rising expectations, the authors of the present paper proved in \cite{csikos2022} that the conjecture is false: the diffeomorphism group of $T^2 \times S^2$ does not have the Jordan property. Our counter-example was inspired by an algebro-geometric construction of Yu. G. Zarhin \cite{zarhin2020}, who proved that the birational automorphism group of $E \times \mathbb{P}^1$, where $E$ is an elliptic curve, does not have the Jordan property.

In view of the above results, Mundet i Riera \cite{mundet2020} posed the intriguing problem to characterize compact smooth manifolds for which Ghys’ conjecture is true.

Following the appearance of our counterexample \cite{csikos2022}, Ghys revised his conjecture. In a talk \cite{ghys2022}, he suggested the following.

**Conjecture 1.2** (Ghys). Let $M$ be a compact smooth manifold. Then any finite subgroup of $\text{Diff}(M)$ has a nilpotent normal subgroup of index at most $n(M)$.

Note that Jordan’s theorem can be derived as a corollary of the classical Zassenhaus lemma, which states that there exists a neighborhood $\Omega$ of the identity in $GL(d, \mathbb{R})$ such that if $\Gamma$ is a discrete subgroup of $GL(d, \mathbb{R})$, then $\Gamma \cap \Omega$ generates a nilpotent subgroup. Hence the revised Ghys conjecture is related to earlier work of Ghys aiming at obtaining
analogues of the Zassenhaus lemma for analytic diffeomorphism groups (see \[8\]).

It was suggested by Fisher \[8\] that for the original question of Ghys, it would be more natural to ask about finite groups of homeomorphisms and not assume the differentiability of the maps.

Our main result is a proof of the revised Ghys conjecture. Actually, we prove the following stronger result.

**Theorem 1.3.** Let \(M\) be a compact topological manifold. If a finite group \(G\) acts continuously and effectively on \(M\), then it has a nilpotent normal subgroup \(N \triangleleft G\) of index at most \(n(M)\), where \(n(M)\) depends only on the homotopy type of \(M\).

For our proof to work it is essential to allow non-compact manifolds as well. Therefore we prove the following more general result, which implies Theorem 1.3 immediately.

**Theorem 1.4.** Let \(M\) be a (possibly non-compact) topological manifold whose homology group \(H_*\(M;\mathbb{Z}\))\) is finitely generated (as an abelian group). If a finite group \(G\) acts continuously and effectively on \(M\), then it has a nilpotent normal subgroup \(N \triangleleft G\) of index at most \(n(\dim(M), H_*\(M;\mathbb{Z}\))\), where \(n(\dim(M), H_*\(M;\mathbb{Z}\))\) depends only on \(\dim(M)\) and \(H_*\(M;\mathbb{Z}\)\).

In our counterexamples showing that the diffeomorphism group of a compact smooth manifold \(M\) may not have the Jordan property (see section 2), the finite groups acting on \(M\) are 2-step nilpotent as are the groups in numerous further examples obtained by Mundet i Riera \[29\] and Dávid R. Szabó \[44\] (see also \[45\]) based on our construction. This raises the question whether Theorem 1.3 can be strengthened as follows.

**Question 1.5.** Let \(M\) be a compact smooth manifold. Is there a bound \(n'(M)\) such that any finite subgroup of \(\text{Diff}(M)\) has a 2-step nilpotent normal subgroup of index at most \(n'(M)\)?

In view of the results of the Ph.D. thesis of Dávid R. Szabó \[45\] an affirmative answer would be best possible.

An affirmative answer to the above question was proved by Mundet i Riera and Sáez-Calvo \[31\] in dimension 4. Their proof exploits the fact that in dimension 4, the class of freely acting finite subgroups of \(\text{Diff}(M)\) do have the Jordan property, therefore if a subgroup is far from abelian, then it must have elements with non-empty fixed point set. We remark that this approach is limited to dimension 4, as we shall see in section 2 that the Heisenberg type groups justifying the
failure of the Jordan property of the diffeomorphism group of $T^2 \times S^3$ act freely on the product $T^2 \times S^3$.

The question whether groups related to some geometric structure have some variant of the Jordan property has been asked in other contexts as well. Serre [39] proved that the plane Cremona group $\text{Cr}_2(\mathbb{C})$, the group of birational automorphisms of the projective plane $\mathbb{CP}^2$, has the Jordan property. He asked if this also holds for the higher rank Cremona groups $\text{Cr}_n(\mathbb{C})$.

Serre's question was answered positively for the Cremona group $\text{Cr}_3(\mathbb{C})$ by Prokhorov and Shramov [36]. More generally, they have shown that if the BAB (Borisov-Alexeev-Borisov) conjecture holds, then all Cremona groups have the Jordan property. The BAB conjecture has been confirmed by Birkar [1]. This, in particular, completes the proof of a positive answer to the question of Serre.

Popov [34] extended Serre’s question to birational automorphism groups of general varieties. He proved that the birational automorphism groups of complex algebraic surfaces have the Jordan property, except for surfaces birationally equivalent to one of the surfaces $E \times \mathbb{P}^1$, where $E$ is any elliptic curve. Zarhin [48] has shown, that $\text{Bir}(E \times \mathbb{P}^1)$ does not have the Jordan property. Later Prokhorov and Shramov [35] proved, that for any complex algebraic variety $X$, all finite subgroups of $\text{Bir}(X)$ have a soluble normal subgroup of bounded index. Using their ideas, Guld [14] has shown that actually finite subgroups of $\text{Bir}(X)$ have a nilpotent normal subgroup of bounded index and nilpotency class at most two. We consider this as a further indication that Question 1.5 is the right question to ask.

In a recent paper, Mundet i Riera [29] obtained similar results for the group of symplectomorphisms of a compact symplectic manifold. In yet another recent paper, Prokhorov and Shramov [37] have classified algebraic threefolds for which the group of birational automorphisms does not have the Jordan property.

It turns out (see Theorem 4.10) that the finite groups we encounter have bounded rank in the following sense.

**Definition 1.6.** The rank of a finite group $G$ is the minimal integer $r$ such that every subgroup $H$ of $G$ is $r$-generated.

Various group-theoretic properties, like being abelian-by-bounded or nilpotent-by-bounded can be established for finite groups $G$ of bounded rank by proving that the same property holds for subgroups of $G$ with a very simple structure. This somewhat surprising principle has been found independently by Mundet i Riera and Turull [32] and the authors of the present paper (see a remark concerning this in [32]). The results
of \cite{32} have found applications much earlier in \cite{30}. We waited with the publication of our variant till we could complete the proof of the applications described here.

An essential tool in the course of proving Theorem 1.4 is the following group theoretic reduction theorem, which is our variant of the above principle. This reduction allows us to consider only very special type of groups in the geometrical arguments.

**Theorem 1.7** (Reduction Theorem). Let $G$ be a finite group of rank $r$. Then for all $T > 0$, there is an integer $I(r, T) > 0$ such that one of the following holds.

(a) $G$ has a nilpotent normal subgroup of index at most $I(r, T)$.

(b) For some distinct primes $p$ and $q$ the group $G$ has a subgroup of the form $P \rtimes \mathbb{Z}_m$, where $P$ is a special $p$-group, $m$ is a power of $q$, and the image of $\mathbb{Z}_m$ in Aut($P$) has order at least $T$. In particular, we have $|P| \leq p^{2r}$.

A similar reduction theorem for checking the Jordan property was obtained earlier by the present authors, and independently by Mundet i Riera and Turull \cite{32}.

Theorem 1.7 will be proved in section 3 as Theorem 3.14. Both the proofs of Theorem 4.10 and Theorem 1.7 rely on group theoretic techniques, in particular, on the Classification of Finite Simple Groups (CFSG). It is an interesting question whether the use of CFSG can be avoided. The generality of our statement suggests that, perhaps, group theoretic arguments can be completely eliminated. Moreover, the analogous algebro-geometric results do not use CFSG at all. This leads to the following:

**Question 1.8.** Is there a geometric proof of Theorem 1.3?

**Scheme of the proof of Theorem 1.4.** The starting point is to establish an upper bound $r$ on the rank of the group $G$. For compact manifolds and abelian groups, it is a classical result of Mann-Su \cite{25}, our more general situation is tackled in \cite{6} (see Theorem 4.10 below).

The rank bound and Theorem 1.7 imply, with a little bit of thought, that it is enough to prove the theorem for groups of the form $G \cong P \rtimes \mathbb{Z}_m$, where $P$ is a $p$-group of size $|P| \leq p^{2r}$ for some prime $p$, and $m$ is not divisible by $p$. So from now on $G$ is such a group.

Minkowski’s bound (Theorem 1.11) gives us a subgroup $H \leq G$ of bounded index which acts trivially on $H^*(M; \mathbb{Z})$. By the universal coefficient theorem, $H$ acts trivially on $H_*(M; \mathbb{F}_p)$ as well. We replace $G$ with $H$, so from now on $G$ acts trivially on $H_*(M; \mathbb{F}_p)$. 
Let $U \subset M$ be the largest open subset where the $P$-action is free. By a result of [6], $\dim H_*(U; \mathbb{Z}_p)$ is bounded from above, and it is tempting to replace $M$ with $U$ to reduce our statement to free actions, but [6] gives us no control over the $\mathbb{Z}_m$-action on $H_*(U; \mathbb{Z}_p)$. In section 6 we study this situation. The Borel Fixed Point Formula (see Proposition 4.8) gives us points whose stabilizer subgroups in $P$ are isomorphic to $\mathbb{Z}_p$. The main result of [6] implies that these cyclic subgroups are $\mathbb{Z}_m$-invariant for some $\mathbb{Z}_m \leq \mathbb{Z}_m$ of bounded index. After a careful analysis of the fixed point submanifolds of these cyclic subgroups and the corresponding $\mathbb{Z}_p \times \mathbb{Z}_m$-actions around them, we can safely remove these submanifolds from $M$. Repeating this step inductively, we arrive at $U$, and we deduce that there is a subgroup $\mathbb{Z}_m'' \leq \mathbb{Z}_m$ of bounded index which acts trivially on $H_*(U; \mathbb{Z}_p)$ (see Lemma 6.1). Finally, we replace $M$ with $U$, and reduce the statement to the case of free action of $P$.

Note, that in the above reductions we lost control of the integer homology, we only have a bound on the mod $p$ homology of $M$. Moreover, even if we started with a compact manifold, we lost compactness along the way. So it is essential for our proof to generalize the conjecture of Ghys and include open manifolds.

So we are reduced to the following scenario. $G = P \rtimes \mathbb{Z}_m$ as above, $G$ acts on a topological manifold $M$ such that $\dim H_*(M; \mathbb{Z}_p) = B$ is bounded, and $P$ acts freely. Let $\mathbb{Z}_m'$ be the image of the conjugacy action of $\mathbb{Z}_m$ on $P$. Then $G$ has a nilpotent subgroup of index $m'$, namely $P \rtimes \mathbb{Z}_m/m' = P \times \mathbb{Z}_m/m'$, so it is enough to bound $m'$.

Lemma 5.2 tackles this for the case when $P$ is an elementary abelian $p$-group. The idea is the following. It is more convenient to use Poincaré duality and switch to cohomology. We study the action of $\mathbb{Z}_m'$ on the Borel spectral sequence calculating the cohomology of $M/P$ (see Proposition 4.15). For each $i, j$, one can easily determine the irreducible representations of $\mathbb{Z}_m'$ which show up in $E_{i,j}^2$. On the other hand, $E_{i,j}^\infty = 0$ for $i + j > n$, so a lot of cancellations must occur. But an irreducible representation can be cancelled only by the same representation appearing in certain other $E_{k,l}^2$ terms with $k + l = i + j - 1$. Using general representation theory we find a bound $N$ such that all irreducible representations of $\mathbb{Z}_m'$ must occur in $\bigoplus_{i+j \leq N} E_{i,j}^2$. This gives a bound on $m'$, and proves the theorem for this case.

Finally, in the general case, when $P$ is an arbitrary $p$-group of order $p^\rho$ for some $\rho \leq 2r$ (see Lemma 5.3 for the precise statement) we use induction on $\rho$. Let $E$ be the subgroup consisting of the elements of order at most $p$ in the center of $P$. This is a characteristic subgroup
of $P$, hence $\mathbb{Z}_m$ normalizes it. First we apply the already known case to the subgroup $E \rtimes \mathbb{Z}_m \leq G$, which gives us a subgroup $\mathbb{Z}_{m_1} \leq \mathbb{Z}_m$ of bounded index which commutes with $E$. So $E$ is in the center of the subgroup $G_1 = P \rtimes \mathbb{Z}_{m_1} \leq G$. Then we apply the induction hypothesis to the $G_1/E$ action on the manifold $M/E$, and find a subgroup $\mathbb{Z}_{m_2} \leq \mathbb{Z}_{m_1}$ of bounded index which commutes with $P/E$. Since $m_2$ is not divisible by $p$, this $\mathbb{Z}_{m_2}$ must commute with $P$ as well. Therefore $P \rtimes \mathbb{Z}_{m_2} = P \times \mathbb{Z}_{m_2}$ is nilpotent. This completes the induction.

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2. Counterexamples

In this section, we construct a family of compact real analytic manifolds, the real analytic diffeomorphism groups of which do not have the Jordan property. These manifolds will be the total spaces of bundles over the 2-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with compact fibers and structure group $G$, where $G$ is a compact connected Lie group with finite center acting effectively on the fibers. It will be shown that infinitely many of the Heisenberg type groups $G_n = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_n$ with multiplication rule

$$ (a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + ab') $$

have an effective real analytic action on such a manifold, while the index of an abelian subgroup of $G_n$ is at least $n$.

The simplest examples of this type, the total spaces of the oriented $S^2$-bundles over $T^2$, appeared first in the preprint of the authors [7] motivated by the ideas of Zarhin [48]. We remark that in the special case of the trivial $S^3$ bundle with structure group $\text{SU}(2)$, our construction provides a free action of all the groups $G_n$ on the product manifold $T^2 \times S^3$. This shows that the diffeomorphism group of $T^2 \times S^3$ does not even have the weakened Jordan property considering only freely acting finite groups of diffeomorphisms.
2.1. **Principal bundles over the torus.** There is a simple way to classify real analytic principal bundles over the torus $T^2$.

**Lemma 2.1.** Let $G$ be a connected compact Lie group. Then the isomorphism classes of real analytic principal $G$-bundles over the torus $T^2$ are classified by the elements of the fundamental group $\pi_1(G)$.

**Proof.** Real analytic and topological classification of principal $G$-bundles coincide (see [13]), so we focus on the latter one. Principal $G$-bundles over the torus are classified by the homotopy classes of maps $T^2 \to BG$ into a classifying space $BG$ of principal $G$-bundles. Consider the usual CW structure of $T^2$ with a single 2-cell glued to the 1-skeleton $T^2(1)$ equal to a bouquet of two circles. Since $G$ is connected, $BG$ is simply connected, and $[T^2, BG] = [T^2/T^2(1), BG] = \pi_2(BG) \cong \pi_1(G)$.

If $H$ is a Lie group, $\xi = (E \to B)$ is a principal $H$-bundle, then for any smooth manifold $F$ equipped with a left $H$-action $\theta$, there is a fiber bundle $\xi_\phi[F] = (E \times_H F \to B)$ associated to the principal bundle $\xi$, having fibers diffeomorphic to $F$. The correspondence between the principal $G$-bundles over $T^2$ and $\pi_1(G)$ is canonical in the following sense. If $\phi: H \to G$ is a smooth homomorphism from $H$ into the Lie group $G$, then $H$ acts on $G$ by left translations via the homomorphism $\phi$, and the associated bundle $\xi_\phi[G]$ is a principal $G$-bundle. In the case when the base space of $\xi$ is $T^2$, the element of $\pi_1(G)$ corresponding to $\xi_\phi[G]$ is the image of the element of $\pi_1(H)$ corresponding to $\xi$ under the induced homomorphism $\phi_*: \pi_1(H) \to \pi_1(G)$.

As a special case, principal $S^1$-bundles over the torus $T^2$ correspond to elements of $\pi_1(S^1) \cong \mathbb{Z}$. The isomorphism with $\mathbb{Z}$ can be given explicitly by fixing an orientation of the torus and assigning to each circle bundle $\xi$ the integral of its first Chern class $c_1(\xi)$ over the torus. We shall denote by $\xi^n$ the principal $S^1$-bundle over $T^2$ with Chern number $n$.

If $G$ is a compact connected Lie group with finite center, then $\pi_1(G)$ is finite, so there are only a finite number of isomorphism classes of principal $G$-bundles over $T^2$. In particular, if $G$ is simply connected, e.g., $G = SU(n)$, then all the $G$ bundles over $T^2$ are trivial. Since every element of $\pi_1(G)$ can be represented by a power of an injective group homomorphism $\phi: S^1 \to G$, any principal $G$-bundle over $T^2$ is isomorphic to the bundle $\xi_{\phi[k]}^k[G] \cong \xi_{\phi}^k[f][G]$ for a suitably chosen injective homomorphism $\phi$ and integer $k$. If $\phi$ represents an element of order $m$ in $\pi_1(G)$, then $\xi_{\phi}^k[G]$ and $\xi_{\phi}^l[G]$ are isomorphic principal $G$ bundles if and only if $m$ divides $k - l$. 


2.2. Weak automorphisms of bundles over the torus. Recall that by a weak isomorphism between the smooth bundles $\xi = (E \xrightarrow{\pi} B)$ and $\xi' = (E' \xrightarrow{\pi'} B')$ with the same fiber type we mean a pair of diffeomorphisms $\Phi: E \to E'$ and $\bar{\Phi}: B \to B'$ such that $\bar{\Phi} \circ \pi = \pi' \circ \Phi$ and $\Phi$ is a structure preserving isomorphism between the fibers.

A weak isomorphism is uniquely defined by the map $\Phi$. Consequently, the group of weak automorphisms of a bundle can be embedded into the diffeomorphism group of the total space as a subgroup.

**Lemma 2.2.** For any integer $n \geq 2$, the Heisenberg type group $G_n = (\mathbb{Z}^n \times \mathbb{Z}^n) \rtimes \mathbb{Z}^n$ with multiplication rule (1) acts effectively by real analytic weak automorphisms on a real analytic model of the principal $S^1$-bundle $\xi^n$.

**Proof.** We construct a real analytic model of $\xi^n$ for $n \in \mathbb{Z}$ as follows. Let $\Xi = (\mathbb{R}^2 \times S^1 \to \mathbb{R}^2)$ be the trivial principal $S^1$-bundle over the plane. For a given $n \in \mathbb{Z}$, consider the action of $\mathbb{Z}^n$ on $\Xi$ by weak automorphisms given by the action $\Phi_n: \mathbb{Z}^n \times (\mathbb{R}^2 \times S^1) \to \mathbb{R}^2 \times S^1$,

$$\Phi_n((k,l),(x,y,z)) = (x + k, y + l, e^{2\pi i kny}z)$$

on the total space, where $(k,l) \in \mathbb{Z}^2$, $(x,y) \in \mathbb{R}^2$, and $z \in S^1 \subset \mathbb{C}$. As $\Xi$ and the action $\Phi_n$ are real analytic, factoring $\Xi$ with this action, we obtain a real analytic model $E_n \xrightarrow{\pi^n} T^2$ for the principal $S^1$-bundle $\xi^n$. Denote the $\Phi_n(\mathbb{Z}^2)$-orbit of $(x,y,z) \in \mathbb{R}^2 \times S^1$ by $\langle x,y,z \rangle_n \in E_n$.

Take the ring of modulo $n$ residue classes of integers as a model for $\mathbb{Z}^n$, and denote by $[k]_n \in \mathbb{Z}^n$ the residue class of $k \in \mathbb{Z}$. Then the demanded action of $G_n$ on the bundle $\xi^n$ is induced by the action $\Psi_n: G_n \times E_n \to E_n$,

$$\Psi_n(([k]_n,[l]_n,[m]_n),(x,y,z)_n) = \langle x + k/n, y + l/n, e^{2\pi i (ky + mz)/n}z \rangle_n$$

on the total space of $\xi^n$. □

**Lemma 2.3.** If $A < G_n$ is a commutative subgroup, then $|G_n : A| \geq n$.

**Proof.** See [48, Section 3.]. □

**Theorem 2.4.** Let $G$ be a connected compact Lie group with finite center, $F$ be a compact real analytic manifold with an effective real analytic left $G$-action $\theta$. Then for any principal $G$-bundle $\eta$ over the torus $T^2$, the weak automorphism group of the associated bundle $\eta[aF]$ contains infinitely many of the groups $G_n$ as a subgroup. In particular, the diffeomorphism groups of the total spaces of these bundles do not have the Jordan property.
Proof. As it was observed at the end of subsection 2.1 we can find an injective homomorphism $\phi: S^1 \to G$ such that $\eta \cong \xi_0^k[G]$ for some $k \in \mathbb{Z}$, and if $\eta$ corresponds to an element of order $m$ in $\pi_1(G)$ by the statement of Lemma 2.1 then $\eta \cong \xi_0^l[G]$ whenever $k \equiv l \mod m$. By Lemma 2.2 and the naturality of the associated bundle construction, if the integer $l \geq 2$ is congruent to $k$ modulo $m$, then $G_l$ acts effectively on $\xi^l$ by weak automorphisms, and this action induces an action of $G_l$ also on the associated bundles $\eta \cong \xi_0^l[G]$ and $\eta_\theta[F] \cong (\xi_0^l[G])_\theta[F]$. Since $\phi$ is injective, $\theta$ is effective, the resulting $G_l$-actions on $\eta_\theta[F]$ and its total space are effective. Representing these bundles with their real analytic models described above, all these actions will be real analytic. □

3. The group theoretic reduction theorem

In this section, we prove our key group-theoretic tool, Theorem 1.7 and other related results. These results can be used to show that, under certain conditions, a finite group contains an abelian or nilpotent normal subgroup of small index.

We will start with an account of the theory of finite groups of bounded rank. Recall that the rank of a finite group $G$ is the minimal integer $r$ such that every subgroup $H$ of $G$ is $r$-generated.

The following lemma is proved in [16, Corollary 1.8].

Lemma 3.1. If every elementary abelian subgroup of a finite group $G$ has rank at most $d$, then $G$ has rank at most $1/2d^2 + 2d + 1$.

It is an easy observation that a finite abelian subgroup of $GL(n, \mathbb{C})$ has rank at most $n$. Moreover, as shown in [20], the rank of any finite subgroup of $GL(n, \mathbb{C})$ is at most $3n/2$. Groups of bounded rank also occur naturally in the study of finite groups acting on compact topological manifolds. Namely, as mentioned in the introduction, by a result of Mann and Su [25], the ranks of elementary abelian groups acting faithfully by homeomorphisms on a given compact topological manifold $M$ are bounded. This implies that actually there is a bound on the ranks of finite groups acting on $M$ by homeomorphisms.

Shalev [40] has given a description of the structure of finite groups of bounded rank.

Theorem 3.2. Let $G$ be a finite group of rank $r$. Then there exists a chain $1 \triangleleft G_2 \triangleleft G_1 \triangleleft G$ of characteristic subgroups of $G$ such that

(a) $|G/G_1|$ is bounded in terms of $r$;
(b) $|G_1/G_2| \cong S_1 \times \cdots \times S_k$, where $0 \leq k \leq r$, and for $1 \leq i \leq k$, the group $S_i = X_{n_i}(p_i^{e_i})$ is a simple group of Lie type, such that $n_i$ and $e_i$ are bounded in terms of $r$;
(c) $G_2$ is soluble.

Note that the above structure theorem relies on the Classification of Finite Simple Groups (CFSG) in an essential way. It would be most interesting to prove such a result without CFSG. This would be a natural analogue of the CFSG-free description of the structure of finite linear groups of bounded dimension over arbitrary fields. That structure theorem was first obtained by Weisfeiler [47] using CFSG and later Larsen and Pink [21] gave an amazing “elementary” proof.

We will use Theorem 3.2 to reduce the proof of Theorem 1.7, the main result of this section, to the soluble case. We prove the following.

**Proposition 3.3.** Let $G$ be a finite group of rank $r$ and assume that all soluble subgroups $S$ of $G$ contain a nilpotent normal subgroup of index at most $t$. Then $G$ contains a nilpotent normal subgroup of index bounded in terms of $r$ and $t$.

**Proof.** If $Q$ is a section (quotient of a subgroup) of $G$, then it satisfies the same condition on soluble subgroups as $G$. This follows easily using the well-known fact that if $Q$ is isomorphic to $H/N$, where $H$ is a minimal subgroup of $G$ having $Q$ as a quotient, then $N$ must be nilpotent. (In fact $N$ is contained in the Frattini subgroup $\Phi(H)$, which is known to be nilpotent. To see this, we use another well-known fact, namely that $\Phi(H)$ is the intersection of the maximal subgroups of $H$. If such a maximal subgroup $M$ does not contain $N$, then $MN = H$ and hence $H/N$ is isomorphic to $M/(H \cap N)$, a contradiction.)

We claim that if the section $Q$ is isomorphic to $\text{PSL}(2,q)$, then we have $q \leq 2t + 1$. Indeed, assume the contrary. $Q$ is the quotient of $\text{SL}(2,q)$ by its center $Z = \{I, -I\}$, where $I$ denotes the unit matrix. (In characteristic two, $I = -I$ and $Z = \{I\}$.) Hence $\text{SL}(2,q)$ also satisfies the same condition on soluble subgroups. Let $B$ denote the subgroup of upper triangular matrices. $B$ is the semidirect product of the subgroups

$$P = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ x \in \mathbb{F}_q \right\} \quad \text{and} \quad D = \left\{ \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \ y \in \mathbb{F}_q, \ y \neq 0 \right\},$$

where $\mathbb{F}_q$ denotes the base field. $P$ is normal, abelian, and the conjugate of an element $X = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $Y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ is $\begin{pmatrix} 1 & xy^2 \\ 0 & 1 \end{pmatrix}$. In particular, if $x \neq 0$ then $X$ does not commute with $Y$ unless $y = \pm 1$, so the centralizer $C_B(X)$ is $PZ$, and therefore $|C_B(X)| \leq 2q$. On the other hand $B$ is soluble, hence it has a nilpotent normal subgroup $N$ of index at most $t < \frac{q-1}{2}$. But $|B| = q(q-1)$, so $|N| > q - 1 = |D|$, hence $N$ intersects $P$ nontrivially. Now $P$ is an abelian Sylow subgroup of
$B$, hence $P \cap N$ is an abelian Sylow subgroup of the nilpotent normal subgroup $N$, hence $N$ centralizes $N \cap P$. By the above estimate $|N| \leq 2q$, hence $|B| \leq |N|t < 2q^{t-1} = |B|$, a contradiction.

Now let $S_i$ be one of the simple groups in the statement of Theorem 3.2 and assume that $S_i$ is not isomorphic to a Suzuki group. By a result of Liebeck, Nikolov and Shalev [23], apart from the Suzuki groups, every finite simple group of Lie type over a field of $q$ elements can be written as a product of $c d^2$ subgroups isomorphic to $SL(2,q)$ or $PSL(2,q)$, where $l$ is the Lie rank of the group, and $c$ is some absolute constant. By the above claim, for any such subgroup, we have $q \leq 2t + 1$, and $l \leq r$. Hence we have $|SL(2,q)| \leq (2t + 1)^3$. It follows that $|S_i| \leq (2t + 1)^3 c r^2$. This implies that the product of the orders of the simple groups $S_i$ which are not Suzuki groups is at most $(2t + 1)^3 c r^3$. If $S_i$ is a Suzuki group, then it has bounded order (since the size of the field of definition is bounded). Our statement follows. □

Next we will describe the structure of finite soluble groups of bounded rank. For this we need the following standard description of soluble groups with trivial Frattini subgroup (see [17, III-4.2 and III-4.5]).

**Lemma 3.4.** Let $G$ be a finite soluble group and suppose that $\Phi(G) = 1$. Then the Fitting subgroup $F = F(G)$ of $G$ is a direct product of minimal elementary abelian normal subgroups of $G$, say $F = E_1 \times \cdots \times E_r$. The group $G$ acts upon $F$ by conjugation; the kernel of this action is $F$. This action defines an embedding of $G/F$ into $\text{Aut}(E_1) \times \cdots \times \text{Aut}(E_r)$ in such a way that $G/F$ induces an irreducible linear group $G_i = G/C_G(E_i)$ on each of the vector spaces $E_i$.

We will use some results concerning the structure of primitive linear groups, which are essentially due to Suprunenko [41], see [46, Lemma 2.2] for a concise description. Recall that a subgroup of $G \leq \text{GL}(n,q)$ is an *imprimitive linear* group if the space $V = V(k,q)$ decomposes as a direct sum $V_1 \times \cdots \times V_k$ of subspaces ($k \geq 2$) and $G$ permutes the $V_i$’s. It is a *primitive linear* group if no such $G$-invariant decomposition exists.

**Lemma 3.5 ([41], [46]).** Let $G < \text{GL}(n,q)$ be a maximal soluble primitive linear group. Then the following statements hold.

(a) There exists a unique maximal normal abelian subgroup $A$ of $G$.
(b) $A$ is isomorphic to the multiplicative group of non-zero elements of an extension $F_{q^k} \otimes F_q$, where $k$ divides $n$. In particular, $A$ is a cyclic group of order $q^k - 1$. 
(c) Setting $C = C_G(A)$, the quotient group $G/C$ is isomorphic to a subgroup of the Galois group $\text{Gal}(\mathbb{F}_{q^k} : \mathbb{F}_q)$. In particular, $|G/C| \leq k$.

(d) Denoting by $B$ a maximal subgroup of $C$ such that $B/A$ is a maximal normal abelian subgroup of $G/A$, the index $|B/A|$ equals $(n/k)^2$.

(e) $B/A = F(C/A)$. In particular, $C/B$ embeds into $\text{Aut}(B/A)$.

**Corollary 3.6.** If $G < \text{GL}(n, q)$ is a soluble primitive linear group, then it has a cyclic normal subgroup $A$ of order at most $q^n$ such that $q$ and $|A|$ are relatively prime and the index of $A$ in $G$ is bounded in terms of $n$.

**Proof.** We may clearly assume that $G$ is actually a maximal primitive soluble group. The index of the cyclic subgroup $A$ is at most $k(n/k)^2 |\text{Aut}(B/A)|$. But the order of $\text{Aut}(B/A)$ is bounded in terms of $n$ since $|B/A|$ is bounded in terms of $n$. Our statement follows. □

This, in turn, implies the following.

**Proposition 3.7.** If $G < \text{GL}(n, q)$ is a soluble completely reducible linear group, then it has an abelian normal subgroup $A$ of order at most $q^n$ such that $q$ and $|A|$ are relatively prime and the index of $A$ in $G$ is bounded in terms of $n$.

**Proof.** It is enough to show that $G$ has a (not necessarily normal) abelian subgroup $B$ with the required properties, since then we can take $A$ to be the normal core of $B$ in $G$. Since $G$ is a subdirect product of irreducible linear groups of degrees say $d_1, \ldots, d_m$ with $\sum d_i = n$, we may clearly assume that in fact $G$ is irreducible. There exists a non-refinable $G$-invariant decomposition $V = V_1 \times \cdots \times V_k$ into $l$-dimensional subspaces with $n = kl$. Now $G$ permutes the subspaces $V_i$ and the kernel $N$ of this action has index at most $n!$ in $G$. If $N_i$ is the normalizer and $C_i$ is the centralizer of $V_i$ in $G$, then $G_i = N_i/C_i$ acts primitively on $V_i$. Hence $G_i$ has a cyclic normal subgroup of order at most $q^l$ and coprime to $q$ such that its index in $G_i$ is bounded in terms of $n$. It is clear that $N$ is embeddable into $G_1 \times \cdots \times G_k$ and it follows that $N$ has an abelian subgroup $B$ of order at most $q^n$, such that $q$ and $|B|$ are relatively prime and the index of $B$ in $N$ is bounded in terms of $n$ (since the same holds for the direct product $G_1 \times \cdots \times G_k$ itself). This proves the proposition. □

We call a finite group of the form $E \rtimes C$, where $E$ is an elementary abelian $p$-group and $C$ is a cyclic group of prime power order acting
faithfully on $E$ such that $p$ does not divide $|C|$, a group of affine cyclic type. We denote such an extension group by $\text{Aff}(E, C)$.

We will use the following (see [17, III-3.4 and III-4.5]).

**Lemma 3.8.** Let $G$ be a finite group. Then we have

(a) $\Phi(G/\Phi(G)) = 1$ and 
(b) $F(G/\Phi(G)) = F(G)/\Phi(G)$.

Next we prove the promised structure theorem for soluble groups of bounded rank.

**Theorem 3.9.** Let $G$ be a soluble group of rank $r$. Then $G/F(G)$ has an abelian normal subgroup $A$ of order at most $|F(G)/\Phi(G)|$ and of index bounded in terms of $r$. Moreover, if $G$ has no section $\text{Aff}(E, C)$ of affine cyclic type with $|C| > T$, then we have $|A| \leq (T!)^r$.

**Proof.** By Lemma 3.8, it is sufficient to consider soluble groups with trivial Frattini subgroup. We use Lemma 3.4 and the notation of that lemma. By Proposition 3.7, each of the quotient groups $G_i$ has an abelian normal subgroup $A_i$ of size at most $E_i$ and of index bounded in terms of $r$, say at most $f(r)$. Moreover, $|E_i|$ and $|A_i|$ are relatively prime. Hence, by our conditions, the orders of the elements of $A_i$ are not divisible by prime powers larger than $T$ and therefore the exponent of $A_i$ divides $T!$. Consider the inverse images $\hat{A}_i$ of the subgroups $A_i$ in $G/F(G)$. Let $A$ be the intersection of the subgroups $\hat{A}_i$. Since $G/F(G)$ is embeddable into the direct product of the $G_i$, and $A$ corresponds to a subgroup of the direct product of the $A_i$, we see that $A$ is an abelian normal subgroup of size at most $|E_1| \cdots |E_k| = |F(G)/\Phi(G)|$. Moreover, the exponent of $A$ divides $T!$.

Since $G$ itself can be generated by $r$ elements, the intersection of all of its subgroups of index at most $f(r)$ has bounded index in terms of $r$ (see [24, Cor. 1.1.2]), hence the same is true for $G/A$.

Now $A$ is abelian of rank at most $r$, hence it is the product of at most $r$ cyclic subgroups of order at most $T!$, and therefore $|A| \leq (T!)^r$. The proof is complete. \qed

Combining the previous facts we prove our first result which characterizes nilpotent by bounded groups among groups of bounded rank.

**Theorem 3.10.** Let $G$ be a finite group of rank $r$. Assume that $G$ has no section $\text{Aff}(E, C)$ of affine cyclic type with $|C| > T$. Then $G$ has a nilpotent normal subgroup of index bounded in terms of $r$ and $T$. 

Proof. By Theorem \ref{thm:subgroupconditions}, if $H$ is a soluble group satisfying the above conditions, then $|H/F(H)|$ is bounded in terms of $r$ and $T.$ In particular, this holds for all soluble subgroups of a group $G$ satisfying these conditions. Using Proposition \ref{prop:bound} we obtain our statement. \hfill \qed

Note that if a finite group has a nilpotent normal subgroup of index $T,$ then it can not have a section $\text{Aff}(E,C)$ of affine cyclic type with $|C| > T.$

We will also give another characterization in terms of certain excluded subgroups.

We need the following.

Lemma \ref{lem:characterization}. Let $A \cong \text{Aff}(E,C)$ be a group of affine cyclic type. Assume that for a finite group $G$, we have $G/N \cong A$ and that no proper subgroup of $G$ has a quotient isomorphic to $A.$ Then $G$ is a group of the form $P \rtimes \mathbb{Z}_m,$ where $P$ is a $p$-group, $m$ is a power of a prime $q \neq p,$ and the image of $\mathbb{Z}_m$ in $\text{Aut}(P)$ has order at least $|C|.$

Proof. By our condition on the minimality of $G,$ the normal subgroup $N$ is contained in the Frattini subgroup $\Phi(G)$ of $G$ (see the proof of Proposition \ref{prop:bound}), hence it is nilpotent. Moreover, the pre-image $\hat{E}$ of $E$ in $G$ is a normal subgroup of $G$ which satisfies $\hat{E}' \subseteq N \subseteq \Phi(G).$ Hence $\hat{E}$ is nilpotent by Wielandt’s theorem ([17, III-3.11]).

If $S$ is a Sylow subgroup of $\hat{E}$ of order coprime to $|E|$ and $|C|,$ then it is normal in $G$ (since it is a characteristic subgroup of $\hat{E}$) and by the Schur-Zassenhaus theorem (see [42, Ch. 2, (8.10)]) it has a complement. But then this complement has a quotient isomorphic to $A,$ a contradiction.

The Sylow $p$-subgroup $S_p$ is also normal in $G,$ and it follows that the quotient $G/S_p$ is a $q$-group for some prime $q \neq p.$ This quotient has a cyclic subgroup which projects onto $C.$ By our minimality condition, $G$ is actually the split extension of $S_p$ by this cyclic subgroup. Our statement follows. \hfill \qed

Recall that a $p$-group $P$ is said to be a special $p$-group if either $P$ is an elementary abelian $p$-group or we have $\Phi(G) = Z(G) = G'$ and $G'$ is elementary abelian. In particular, such a group has class at most 2 and exponent at most $p^2.$

We need a well-known result of Hall-Higman (see [43, Ch. 4, (4.19)]).

Theorem \ref{thm:Hall-Higman} (Hall-Higman). Assume that the group $A$ acts on a group $H,$ where $A$ and $H$ have coprime order. Suppose that a subgroup $R$ of $A$ acts on $H$ non-trivially. Let $P$ be an $A$-invariant subgroup of $H$ that is minimal among the $A$-invariant subgroups of $H$ on which $R$ acts

### \textbf{References}

[17], [42], [43]...
nontrivially. Then $P$ is a special $p$-group for some prime $p$. The group $R$ acts on $\Phi(P)$ trivially, and the group $A$ acts on $P/\Phi(P)$ irreducibly.

Remark 3.13. This theorem implies immediately that in the conclusion of Lemma 3.11 we may take $P$ to be a special $p$-group (we apply Theorem 3.12 by setting $A$ to be $\mathbb{Z}_m$, $H$ to be $P$ and $R$ to be the smallest subgroup of $\mathbb{Z}_m$ which acts on $H$ nontrivially).

We arrived at the central result of this section.

Theorem 3.14 (same as Theorem 1.7). Let $G$ be a finite group of rank $r$. Then for all $T > 0$, there is an integer $I(r, T) > 0$ such that one of the following holds.

(a) $G$ has a nilpotent normal subgroup of index at most $I(r, T)$.

(b) For some distinct primes $p$ and $q$ the group $G$ has a subgroup of the form $P \rtimes \mathbb{Z}_m$, where $P$ is a special $p$-group, $m$ is a power of $q$, and the image of $\mathbb{Z}_m$ in $\text{Aut}(P)$ has order at least $T$. In particular, we have $|P| \leq p^{2r}$.

Proof. If $G$ has a section of affine cyclic type with $|C| > T$, then (b) holds by Lemma 3.11 and Remark 3.13. Otherwise, by Theorem 3.10, $G$ has a nilpotent normal subgroup of index bounded in terms of $r$ and $T$. □

As a corollary of Theorem 3.14 we will also show that if a finite group $G$ of bounded rank does not contain certain abelian-by-cyclic subgroups, then $G$ is abelian-by-bounded. While we do not use this corollary in our paper, it should be useful for proving analogues of Jordan’s Theorem in various situations. The corollary was explicitly stated in a talk by the second named author [38]. A somewhat weaker result has independently (see the last sentence on page 824 of [32] concerning this issue) been obtained by Mundet i Riera and Turull as the main theorem of [32]. Our arguments are very much different from those in [32].

We need to prove two lemmas on finite $p$-groups.

The proof of the first lemma is based on the following useful result of Chermak-Delgado (see [18, 1.41]).

Proposition 3.15. Let $G$ be a finite group. Then $G$ has a characteristic abelian subgroup $N$ such that $|G : N| \leq |G : A|^2$ for every abelian subgroup $A$ of $G$.

Lemma 3.16. Let $P$ be a finite $p$-group. Assume that all metabelian normal subgroups of $P$ have an abelian subgroup of index at most $t$. Then $P$ has an abelian normal subgroup of index at most $t^{4 \log t}$. 
Proof. Let $A$ be an abelian normal subgroup of maximal order in $P$. Let $\tilde{B}$ be an abelian normal subgroup of maximal order in $\tilde{P} = P/A$ and set $b = |\tilde{B}|$. By a classical result of Burnside-Miller (see [12, Cor. 2 of Ch. 2, Thm. 1.17]), we have $|\tilde{P}| \leq p^{(\log p)b(\log p + 1)/2} \leq b^{\log b}$. The inverse image $B$ of $\tilde{B}$ in $P$ is a metabelian normal subgroup of $P$, hence by assumption it has an abelian subgroup of index at most $t$. By Proposition 3.15, $B$ has an abelian characteristic subgroup $C$ of index at most $t^2$. But $C$ (as a characteristic subgroup of a normal subgroup) is a normal subgroup of $P$. Hence by assumption we have $|C| \leq |A|$, and therefore $t^2 \geq b$. We obtain $|\tilde{P}| \leq (t^2)^{2\log t} = t^{4\log t}$, as required.

Lemma 3.17. Let $P$ be a finite metabelian $p$-group of rank $r$. Assume that all abelian-by-cyclic subgroups of $P$ have an abelian subgroup of index at most $t$. Then $P$ has an abelian normal subgroup of index at most $t^{2r}$.

Proof. By a result of Gillam [11], a metabelian group $P$ has an abelian normal subgroup $A$ such that the order of any abelian subgroup of $P$ is at most $|A|$. It is easy to see that a metabelian group of rank $r$ is a product of $2r$ cyclic subgroups. Hence the quotient $\tilde{P} = P/A$ is a product of $2r$ cyclic groups $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_{2r}$. The inverse image $C_i$ of $\tilde{C}_i$ in $P$ is an abelian-by-cyclic group, and by assumption, $A$ is an abelian subgroup of $C_i$ of largest order. By our condition, we have $|C_i| \leq |C_i : A| \leq t$ (for $i = 1, \ldots, 2r$). Hence $|P : A| \leq t^{2r}$, as required.

Corollary 3.18. Let $G$ be a finite group of rank $r$. Then for all $T > 0$, there is an integer $J(r, T) > 0$ such that one of the following holds:

(a) $G$ has an abelian normal subgroup of index at most $J(r, T)$.

(b) For some distinct primes $p$ and $q$ the group $G$ contains the split extension of an elementary abelian $p$-group $E$ by a cyclic group $C$ of order $q^t$, such that the image of $C$ in $\text{Aut}(E)$ has order at least $T$.

(c) $G$ contains an abelian-by-cyclic $p$-subgroup $P$ which does not have an abelian normal subgroup of index at most $T$.

Proof. Assume that (c) does not hold. We observe that for $p > T$ this implies that any $p$-subgroup of $G$ is abelian. Assume also that (b) does not hold.

We first show using Theorem 3.14 that in this case $G$ has a nilpotent normal subgroup of index bounded in terms of $r$ and $T$. Let $H$ be a subgroup of $G$ which is an extension of a special $p$-group $P$ by $\mathbb{Z}_m$ ($m$ a power of some prime $q$ distinct from $p$). As noted above, either $P$ is
elementary abelian or it has order at most $T^{2r}$. If a cyclic group $\mathbb{Z}_m$ acts on $P$ then in the first case the image of $\mathbb{Z}_m$ in $\text{Aut}(P)$ has order at most $T$ by our assumption (that (b) does not hold). In the second case the order of the image of $\mathbb{Z}_m$ in $\text{Aut}(P)$ is at most $|P| \leq T^{2r}$. Now Theorem 3.14 implies that $G$ has a nilpotent normal subgroup of index at most $I(r, T^{2r})$. In particular this is a bound for the index of $F(G)$ in $G$.

Now let $G$ be a $p$-group for some prime $p \leq T$ for which (c) does not hold, i.e., assume that all abelian-by-cyclic subgroups of $G$ contain an abelian normal subgroup of index at most $T$. If $H$ is any metabelian subgroup of $G$, then by Lemma 3.17, $H$ has an abelian normal subgroup of index at most $T^{2r}$. Using Lemma 3.16 (with $t = T^{2r}$), we see that $G$ itself has an abelian normal subgroup of index at most $(T^{2r})^{4 \log T^{2r}} = T^{16r^{4} \log T}$.

Consider now the general case. The Fitting subgroup $F(G)$ is the direct product of its Sylow $p$-subgroups. For $p > T$ these are abelian and for $p \leq T$ we have just proved that they have an abelian normal subgroup of index bounded in terms of $r$ and $T$. Hence the same holds for $F(G)$ itself. But we have already proved that the index of $F(G)$ is bounded in terms of $r$ and $T$. The proof of the corollary is complete. □

4. Preliminaries in topology

In this section, we collect a number of basic results from topology. Though the main theorems in the introduction give bounds on the complexity of finite group actions on a topological manifold in terms of the dimension and the integer homology group of the manifold, we also want to use homology with coefficients in the $p$-element field $\mathbb{F}_p$, and cohomology.

The singular homology of a topological space $X$ with integer coefficients is denoted by $H_*(X; \mathbb{Z})$, homology of $X$ with coefficients in the field $\mathbb{F}_p$ is denoted by $H_*(X; \mathbb{F}_p)$. For a sheaf $\mathcal{A}$ on $X$, we denote by $H^*_c(X; \mathcal{A})$ the sheaf cohomology of $\mathcal{A}$ with compact support in the sense of Grothendieck [12], see also [4, Chapter II].

For a finite group $G$, the group cohomology of a $G$-module $M$ is denoted by $H^*(G; M)$, see [3, Section IV-2].

Let $M$ be a $d$-dimensional topological manifold. For every prime $p$, $M$ has a mod $p$ orientation sheaf $\mathcal{O}_{p,M}$, or shortly $\mathcal{O}_p$, which is the sheafification of the presheaf which assigns to an open subset $U \subseteq M$ the relative singular homology group $H_d(M, M \setminus U; \mathbb{F}_p)$, see [4, Example I-1.11].
Although our main results deal with topological manifolds, in section 6 we need to work with more general spaces, cohomology manifolds over the field $\mathbb{F}_p$. Cohomology manifolds are defined e.g. in [4, Definition V-16.7], they are homology manifolds by [4, Definition V-16.8]. The orientation sheaf of a homology manifold over $\mathbb{F}_p$ is the sheafification of the presheaf which assigns to an open subset $U$ the Borel-Moore homology of $U$, see [4, Definition V-9.1] and [4, page 293]. Topological manifolds are cohomology manifolds over any field, and the above definitions of the orientation sheaf are compatible.

We need some general notation for group actions.

**Definition 4.1.** Let $G$ be a group acting on a set $X$. Denote by $G_x$ the stabilizer subgroup of an element $x \in X$. For subgroups $H \leq G$, we denote by $X^H$ the fixed point subset of $H$. We will study the set of all stabilizer subgroups

$$\text{Stab}(G, X) = \{ G_x \mid x \in X \}.$$ 

**Definition 4.2.** Let $G$ be a group. A $G$-manifold is a topological manifold equipped with a continuous action of $G$. The $G$-manifold is **effective** if the $G$-action is effective.

Many of our constructions involving the orientation sheaf will be functorial with respect to open embeddings due to the following simple proposition, which follows from the fact that the definition of the orientation sheaf is local, and does not involve any choices.

**Proposition 4.3.** For any open embedding $f : N \to M$ of cohomology manifolds over $\mathbb{F}_p$, there is a natural isomorphism

$$\mathcal{O}_{p,N} \cong f^* \mathcal{O}_{p,M}.$$ 

**Proposition 4.4.** If a finite group $G$ acts continuously on a topological manifold $M$, then the action can be lifted canonically to a $G$-action on the sheaf $\mathcal{O}_{p,M}$, and on its cohomology $H^*_c(M; \mathcal{O}_{p,M})$. Moreover, if the $G$-action on $M$ is free and $f : M \to M/G$ denotes the quotient map, then $\mathcal{O}_{p,M} \cong f^* \mathcal{O}_{p,M/G}$ canonically.

**Proof.** Follows immediately from Proposition 4.3. \qed

**Proposition 4.5** (Poincaré duality). Let $M$ be a $d$-dimensional topological manifold. For all $0 \leq i \leq d$, there are canonical isomorphisms

$$\Delta_M : H^*_c(M; \mathcal{O}_p) \cong H_{d-i}(M; \mathbb{F}_p).$$

Moreover, every open embedding $f : N \hookrightarrow M$ induces a commutative diagram

**Proof.** Follows immediately from Proposition 4.3. \qed
\[
\begin{array}{ccc}
H_c^i(N; \mathcal{O}_{p,N}) & \longrightarrow & H_c^i(M; \mathcal{O}_{p,M}) \\
\Delta_N & & \Delta_M \\
H_{d-i}(N; \mathbb{F}_p) & \longrightarrow & H_{d-i}(M; \mathbb{F}_p)
\end{array}
\]

**Proof.** Let \( \mathcal{A} \) be the constant sheaf on \( M \) with \( \mathbb{F}_p \) coefficients, then \( \mathcal{O}_{p,M} \otimes \mathcal{A} \cong \mathcal{O}_{p,M} \), and our \( \Delta_M \) is constructed in \cite{4} Theorem V-9.2. Moreover, let \( \mathcal{A}' \leq \mathcal{A} \) be the subsheaf which is equal to \( \mathcal{A} \) on the image of \( f \) and 0 outside. Proposition 4.3 implies that \( H_c^i(N; \mathcal{O}_{p,N}) \cong H_c^i(M; \mathcal{O}_{p,M} \otimes \mathcal{A}') \), hence the diagram (2) is also constructed in \cite{4} Theorem V-9.2. We remark that Borel-Moore homology with compact supports, which appears in the cited theorem of \cite{4}, coincides with singular homology in the case of topological manifolds. \( \square \)

**Proposition 4.6.** Let \( M \) be a \( d \)-dimensional cohomology manifold. Then \( H_c^d(M; \mathcal{O}_p) \) is isomorphic to the free \( \mathbb{F}_p \)-module on the connected components of \( M \), and this isomorphism is a natural transformation for open embeddings. In particular, \( \dim H_c^d(M; \mathcal{O}_p) \) is the cardinality of the connected components of \( M \).

**Proof.** By Poincaré duality \cite{4} V-9.2, \( H_c^d(M, \mathcal{O}_p) \) is isomorphic to the Borel-Moore homology \( H_0^c(M, \mathbb{F}_p) \) with compact support. Then our statement is a special case of \cite{4} V-5.14. \( \square \)

Smith theory describes the fixed point set of finite \( p \)-groups acting continuously on topological manifolds. In particular, these are disjoint unions of cohomology manifolds.

**Proposition 4.7.** Let \( G \) be a finite \( p \)-group acting continuously and effectively on a topological manifold \( M \). Then each connected component \( F \) of \( M^G \) is a cohomology manifold over \( \mathbb{F}_p \), and we have an isomorphism

\[
\mathcal{O}_{p,M} \big|_F \cong \mathcal{O}_{p,F}
\]

which is a natural transformation with respect to \( G \)-equivariant open embeddings. If \( M \) is connected and \( |G| > 2 \) then \( M \setminus M^G \) is connected. Furthermore, we have the inequalities

\[
\dim H_c^*(M^G; \mathcal{O}_p) \leq \dim H_c^*(M; \mathcal{O}_p), \quad \dim H_c^*(M \setminus M^G; \mathcal{O}_p) \leq 2 \dim H_c^*(M; \mathcal{O}_p).
\]

**Proof.** Most of this is contained in \cite{6} Proposition 2.5. If \( M \) is connected and \( |G| > 2 \) then \( \dim Z_M \leq \dim(M) - 2 \) by \cite{3} Theorem V/2.6 and \( M \setminus M^G \) is connected by \cite{3} Corollary I/4.7. \( \square \)
For an elementary abelian $p$-group acting on a manifold, Borel’s Fixed Point Formula [3, Theorem XIII-4.3] helps us to understand the local behavior of stabilizer subgroups and their fixed point submanifolds.

**Proposition 4.8** (Borel). Let $M$ be a topological $d$-manifold, and let $G$ be an elementary abelian $p$-group acting continuously and effectively on $M$. Let $x \in M^G$ be a fixed point of $G$. For a subgroup $H \leq G$, denote by $d(H)$ the dimension of the connected component of $x$ in $M^H$. Then

$$d - d(G) = \sum_{H \leq G} (d(H) - d(G)),$$

where $H$ runs through the subgroups of $G$ of index $p$.

For a $p$-group acting on a manifold, Ignasi Mundet i Riera and the authors investigated the global behavior of stabilizer subgroups and their fixed point submanifolds. Their main result is the following.

**Proposition 4.9** ([6, Theorem 5.1]). For all integers $d, B \geq 0$, there is a bound $\tilde{C} = \tilde{C}(d, B)$ with the following property. Let $M$ be a topological $d$-manifold such that $\dim H^*(M; \mathcal{O}_p) \leq B$. Then each finite $p$-group $G$ acting continuously on $M$ has a characteristic subgroup $H \leq G$ of index at most $\tilde{C}$ such that $|\text{Stab}(H, M)| \leq \tilde{C}$.

An old result of Mann-Su [25] gives an upper bound on the rank of an elementary abelian $p$-group $G$ acting on a compact topological manifold $M$. IGNASI MUNDET I RIERA AND THE AUTHORS extended this to not necessarily compact manifolds and to arbitrary finite groups $G$.

**Theorem 4.10** ([6, Theorem 1.8]). Let $G$ be a finite group acting continuously and effectively on a topological manifold $M$ such that $H^*(M; \mathbb{Z})$ is finitely generated. Then the rank of $G$ is bounded in terms of $\dim(M)$ and $H^*(M; \mathbb{Z})$.

4.1. **Equivariant cohomology.** The most important topological tools in our paper, equivariant cohomology and a spectral sequence converging to it, were essentially introduced by Borel (see [3, section IV-3.1]). For discrete group actions, a different but equivalent approach to these notions is due to Grothendieck [12]. We prefer to use the latter approach here.

**Definition 4.11.** If $G$ is a discrete group, $X$ is a topological space equipped with a continuous action of $G$, or shortly a $G$-space, and $R$ is a commutative unital ring, then a sheaf of $R$-modules $\pi : A \to X$ on $X$ is a $G$-sheaf if we are given a continuous action $\tau$ of $G$ on the sheaf
space $\mathcal{A}$ for which $\tau_g(\mathcal{A}_x) = \mathcal{A}_{gx}$ and the restriction of $\tau_g|_{\mathcal{A}_x} : \mathcal{A}_x \to \mathcal{A}_{gx}$ is a module isomorphism for every $g \in G$ and $x \in X$.

The most important examples of $G$-sheaves in this paper are the orientation sheaves of topological manifolds with a group action (see Proposition 4.4).

Suppose now that $G$ is a discrete group. If we are given a $G$-sheaf $\mathcal{A}$ of $R$-modules on the $G$-space $X$, then we can consider the $R$-module $\Gamma^G_c(\mathcal{A})$ of $G$-equivariant sections of $\mathcal{A}$ with compact support. The functor $\Gamma^G_c(-)$ is left exact, so it gives rise to the right derived functors $R^i\Gamma^G_c(-)$.

**Definition 4.12** (Grothendieck [12, Sec. 5.7]). The $i$-dimensional equivariant sheaf cohomology of the $G$-sheaf $\mathcal{A}$ of $R$-modules on the $G$-space $X$ with compact support is the $R$-module

$$H^i_{G,c}(X; \mathcal{A}) = R^i\Gamma^G_c(\mathcal{A}).$$

Group cohomology is an important special case of equivariant cohomology.

**Definition 4.13.** Let $G$ be a discrete group, $V$ an $RG$-module. Let $\mathcal{V}$ be the $G$-sheaf on a one point space $\{p\}$, with stalk $V$ at $p$, where $G$ acts on the stalk via the $RG$-module structure. Then the group cohomology of $G$ with coefficients in $V$ is

$$H^*(G; V) = H^*_G(\{p\}; V).$$

In the following cases the equivariant cohomology can be computed easily.

**Proposition 4.14.** Let $M$ be a cohomology manifold over $\mathbb{F}_p$ and let $G$ be a finite group acting continuously on $M$.

(a) If the action is free, then

$$H^*_{G,c}(M; \mathcal{O}_p) \cong H^*_c(M/G; \mathcal{O}_p).$$

(b) If the action is trivial then

$$H^*_{G,c}(M; \mathcal{O}_p) \cong H^*_c(M; \mathcal{O}_p) \otimes H^*(G; \mathbb{F}_p).$$

Both isomorphisms are natural transformations for $G$-equivariant open embeddings.

**Proof.** For (a) we refer to [4, IV (32)] or [6, Prop. 2.7 (a)]. (b) is a special case of the Künneth formula, we refer to [6, Proposition 2.7] for a complete proof. \(\square\)

\footnote{Grothendieck used the notation $H^*_c(X; G, \mathcal{A})$ for this group.}
In the general case, the following spectral sequence is a basic tool to compute equivariant cohomology.

**Proposition 4.15** ([12, Sec. 5.7]). Let $G$ be a finite group acting continuously on a topological manifold $M$. Then there exists a spectral sequence

$$E_t^{i,j} = H^{i+j}_c(M; O_p), \quad E_2^{i,j} = H^i(G; H^j_c(M; O_p)),$$

which is functorial in $M$ with respect to $G$-equivariant open embeddings.

The following long exact sequence is another useful tool for calculations.

**Proposition 4.16.** Let $G$ be a finite $p$-group acting continuously on a topological manifold $M$. With the notation $F = M^G$ and $U = M \setminus F$, we have a long exact sequence

$$\cdots H^n_{G,c}(U; O_{p,U}) \to H^n_{G,c}(M; O_{p,M}) \to H^n_{G,c}(F; O_{p,F}) \to H^{n+1}_{G,c}(U; O_{p,U}) \cdots$$

which is functorial for $G$-equivariant open embeddings.

**Proof.** This is a special case of the long exact sequence of cohomology with compact support. See [6, Proposition 2.6] for a complete proof. $\square$

We recall some useful facts about the cohomology groups of elementary abelian $p$-groups, which will be used to compute the second page of the spectral sequence in Proposition 4.15. In this paper we do not use the ring structure of $H^*(G; \mathbb{F}_p)$, but in the proposition below, it is easier to describe $H^*(G; \mathbb{F}_p)$ as a graded ring.

**Proposition 4.17.** Let $G$ be an elementary abelian $p$-group of rank $r$.

(a) $H^1(G; \mathbb{F}_p)$ is naturally isomorphic to $\text{Hom}(G, \mathbb{F}_p)$.

(b) For $p = 2$, $H^*(G; \mathbb{F}_2)$ is the symmetric algebra $S^*(H^1(G; \mathbb{F}_p))$.

(c) If $p \geq 3$, then the Bockstein homomorphism $\beta^1 : H^1(G; \mathbb{F}_p) \to H^2(G; \mathbb{F}_p)$ is injective, and

$$H^*(G; \mathbb{F}_p) = \Lambda^*(H^1(G; \mathbb{F}_p)) \otimes S^*(\beta^1(H^1(G; \mathbb{F}_p))),$$

where, for a vector space $V$ over $\mathbb{F}_p$, $\Lambda^*(V)$ and $S^*(V)$ denote the Grassmannian algebra and the symmetric algebra of $V$. In particular, for all $d$, we have a natural isomorphism

$$H^d(G; \mathbb{F}_p) = \bigoplus_{l+2s=d} \Lambda^l(\text{Hom}(G; \mathbb{F}_p)) \otimes S^s(\text{Hom}(G; \mathbb{F}_p)).$$

(d) For all $p$, we have

$$\dim H^d(G; \mathbb{F}_p) = \binom{d + r - 1}{d} = \binom{d + r - 1}{r - 1}.$$
(e) If $V$ is a trivial finite dimensional $\mathbb{F}_pG$-module, then we have
\[ H^d(G; V) \cong H^d(G; \mathbb{F}_p) \otimes V. \]

Proof. The cohomology ring of $\mathbb{Z}_p$ is described in [3, IV-2.1(3)]. Statements (a), (b), (c), (d) for $G = \mathbb{Z}_r^\alpha$ and isomorphism (e) follow from the K"unneth formula. \hfill \square

5. Free actions

In this section we study groups of the form $G = P \rtimes H$ acting effectively on a topological manifold $M$, where $P$ is a $p$-group acting freely on $M$, and $p$ does not divide $|H|$. First we consider the base case $P \cong \mathbb{Z}_r^\alpha$ which will serve as the starting point for our inductive arguments later on.

Our proof needs the extra technical condition that the action of $P$ on $H^*_c(M; \mathcal{O}_p)$ should be trivial. However, when the manifold $M$ will be modified during the proof, we shall loose control over the $P$-action on the cohomology of the new manifold. The following lemma allows us to go around this problem using the $H$-action instead, which will be easier to control.

Lemma 5.1. Let $G$ be a finite group of the form $G = A \rtimes_\alpha H$, where $A \cong \mathbb{Z}_r^\alpha$, $\alpha: H \to \text{Aut}(A)$ is an $H$-action, and $|H|$ is not divisible by $p$. Let $A' \leq A$ denote the subgroup of $H$-invariant elements. Then $A'$ has an $H$-invariant complement $A'' \leq A$. Whenever $G$ acts on a set $X$ so that the subgroup $H$ acts on $X$ trivially, the action of $A''$ on $X$ is trivial as well.

Proof. Since $|H|$ is not divisible by $p$, $A$ decomposes into a direct sum of irreducible $H$-modules by Maschke’s theorem. The sum of the non-trivial summands in this decomposition is a good choice for $A''$.

We have to prove that the kernel $N \triangleleft G$ of the $G$-action on $X$ contains $A''$. By assumption $H \leq N$.

Let $0 \neq B \leq A''$ be any irreducible $H$-submodule of $A''$. Then $B \cap N$ is normal in $G$, so it is an $H$-submodule of $B$. If it were 0, then $B$ would commute with $N$, hence commute with $H$, contrary to the definition of $A''$.

Therefore $B \cap N \neq 0$, hence $B \leq N$ by the irreducibility of $B$. This holds for all irreducible submodules $B \leq A''$, hence $A'' \leq N$. \hfill \square

Lemma 5.2. For all integers $r, d \geq 0$, there is an integer $n_1(r, d)$ with the following property.

Let $G$ be a finite group of the form $G = A \rtimes_\alpha H$, where $A \cong \mathbb{Z}_r^\alpha$, $\alpha: H \to \text{Aut}(A)$ is an $H$-action, and $|H|$ is not divisible by $p$. Let
Let $M$ be a $d$-dimensional topological manifold with an effective $G$-action. Suppose that the subgroup $A$ acts freely on $M$, and the induced action of the subgroup $H$ on $H^*_c(M;\mathcal{O}_p)$ is trivial. Then $H$ has a subgroup $\tilde{H}$ of index at most $n_1(r, d)$ commuting with $A$.

Proof. As in Lemma 5.1, let $A' \leq A$ denote the subgroup of $H$-invariant elements and $A''$ denote its $H$-invariant complement. It is enough to find a subgroup $\tilde{H} \leq H$ of bounded index which commute with $A''$, since this $\tilde{H}$ will commute with $A = A' \oplus A''$ as well.

The $H$-action on $A''$ is a homomorphism $\alpha'': H \to \text{Aut}(A'')$. The subgroup $\tilde{H} = \ker(\alpha'')$ commutes with $A''$, so we need to bound its index, that is the order of the group $K = \text{im}(\alpha'') \cong H/\tilde{H}$. We shall find a bound on $|K|$ using the fact that if $\chi_1, \ldots, \chi_c$ are representatives of the isomorphism classes of the irreducible representations of $K$ over the algebraic closure $\overline{F}_p$ of $F_p$, and $\chi_i$ acts on a vector space of dimension $d_i$, then

$$|K| = \sum_{i=1}^c d_i^2 \leq \left( \sum_{i=1}^c d_i \right)^2. $$

As $p$ does not divide $|K|$, any finite dimensional representation $\tilde{\chi}$ of $K$ over $\overline{F}_p$ can be decomposed into the direct sum of irreducible representations. We shall say that the irreducible representation $\chi_i$ is contained in $\tilde{\chi}$, if the multiplicity $m_i$ of $\chi_i$ in the decomposition $\tilde{\chi} \cong \bigoplus_{i=1}^c m_i \chi_i$ is positive.

By Lemma 5.1, the natural $A''$-action on $H^*_c(M;\mathcal{O}_p)$ is trivial. Applying Proposition 4.15 to the free $A''$-action on $M$, we obtain a spectral sequence

$$E_{i,j}^2 \Rightarrow H^i(M/A'';\mathcal{O}_p),$$

$$E_{i,j}^2 = H^i(A''; H^j_c(M;\mathcal{O}_p)) \cong H^i(A''; F_p \otimes_{\overline{F}_p} H^j_c(M;\mathcal{O}_p)).$$

The group $H$ acts canonically on the entire spectral sequence, and the subgroup $\tilde{H}$ acts trivially on $E_{i,j}^2$. Therefore the $\tilde{H}$-action is trivial on each $E_{i,j}^2$, so the $H$ action induces a $K$-action on the entire spectral sequence. Denote by $\chi^i_{t,j}$ the representation of $K$ induced on $F_p \otimes E_{i,j}^2$ and by $R^i_{t,j}$ the set of irreducible representations contained in $\chi^i_{t,j}$. We collect some facts on the sets $R^i_{t,j}$.

1. The $K$-module $E_{i,j+1}^2$ is a factor of a submodule of $E_{i,j}^2$, hence $R^i_{2,j} \supseteq R^i_{3,j} \supseteq R^i_{4,j} \supseteq \ldots$ is a weakly decreasing sequence.

2. The inclusion map $\chi: K \hookrightarrow \text{Aut}(A'') \hookrightarrow \text{GL}(r, \overline{F}_p)$ is a faithful linear representation of $K$ over $\overline{F}_p$. Denote by

$$\chi^*: K \to \text{GL}(\text{Hom}(\overline{F}_p \otimes A'', \overline{F}_p))$$
the dual representation of \( \chi \), and let \( R^i \) be the set of irreducible representations contained in \( S^i(\chi^*) \) if \( p = 2 \) and in \( \bigoplus_{k+2l=i} \Lambda_k(\chi^*) \otimes S^l(\chi^*) \) if \( p \) is odd. Proposition \( \text{4.17} \) implies that if \( H^i_t(M;\mathcal{O}_p) \neq 0 \), then \( R^i_j = R^i \), otherwise \( R^i_j = \emptyset \).

(3) Introduce the sets \( \tilde{R}^i = \bigcup_{j=0}^i R^j \). We claim that all the irreducible representations \( \chi_1, \ldots, \chi_c \) are contained in the representation \( \bigoplus_{i=0}^{n-1} S^i(\chi^*) \), in particular, they are all contained in \( \tilde{R}^2 \). Indeed, as \( \chi \) is a faithful representation of \( K \), the fixed point set of the action of an element \( g \in K \setminus \{e\} \) on \( \mathbb{F}_p \otimes A'' \) is a proper linear subspace of \( \mathbb{F}_p \otimes A'' \). Since \( \mathbb{F}_p \) is an infinite field, \( \mathbb{F}_p \otimes A'' \) cannot be covered by a finite number of proper linear subspaces, so there exists a vector \( v \in \mathbb{F}_p \otimes A'' \) such that the \( K \)-orbit of \( v \) consists of \( |K| \) distinct points, and we can also choose a linear function \( \ell \in \text{Hom}(\mathbb{F}_p \otimes A'', \mathbb{F}_p) \) which takes different values on the points of the orbit \( Kv \). Then the Lagrange-type polynomial function \( P \in \bigoplus_{i=0}^{n-1} S^i(\text{Hom}(\mathbb{F}_p \otimes A'', \mathbb{F}_p)) \) defined by

\[
P(x) = \prod_{g \in K \setminus \{e\}} \frac{\ell(x) - \ell(gv)}{\ell(v) - \ell(gv)}
\]

takes the value 1 at \( v \) and vanishes at all other points of the orbit \( K v \). Consequently, for any \( g \in K \), the polynomial \( gP \) takes the value 1 at \( gv \) and vanishes at all other points of \( K v \). This implies that the representation of \( K \) induced on the \( K \)-submodule of \( \bigoplus_{i=0}^{n-1} S^i(\text{Hom}(\mathbb{F}_p \otimes A'', \mathbb{F}_p)) \) generated by \( P \) is isomorphic to the regular representation of \( K \) over \( \mathbb{F}_p \). This completes the proof of our claim since all of the irreducible representations \( \chi_1, \ldots, \chi_c \) are contained in the regular representation of \( K \) over \( \mathbb{F}_p \).

(4) By the Poincaré duality, there is a smallest index \( \nu \) such that \( H^i_t(M;\mathcal{O}_p) \neq 0 \). By Proposition \( \text{4.17 (d)} \) and \( \text{(e)} \), \( E^i_j \neq 0 \) for all \( i \geq 0 \), and \( E^i_\nu = 0 \) for all \( j < \nu \). Thus, we have \( E^i_\nu \cong \mathbb{E}^i_\nu / d_\nu \end{eqnarray} \) for all \( t \geq 2 \). This implies that if an irreducible representation of \( K \) is contained in the module \( \mathbb{F}_p \otimes \mathbb{E}^i_\nu \), then it is either contained also in \( \mathbb{F}_p \otimes \mathbb{E}^i_{\nu+1} \) or it must be contained in \( \mathbb{F}_p \otimes \mathbb{E}^{i-\nu+t} \).

If \( i > d-\nu \), then \( H^i_t(M/A'';\mathcal{O}_p) = 0 \) implies \( E^i_\infty = 0 \), so \( \bigcap_{t=2}^\infty R^i_\nu = \emptyset \). By the previous observation, this means that

\[
R^i \subseteq R^i_\nu \subseteq \bigcup_{t=2}^i R^{i-t,\nu+t-1} \subseteq \tilde{R}^{i-2},
\]
therefore $\tilde{R}^i = \tilde{R}^{i-1}$. Iterating this equality, we obtain that for any $i > d - \nu$, we have $\tilde{R}^i = \tilde{R}^{d-\nu}$. Thus, (3) implies that the set $\tilde{R}^{d-\nu} \supseteq \tilde{R}^{2(|K|^{-1})}$ contains all irreducible representations of $K$.

(5) Since all the irreducible representations of $K$ are contained in the representation of $K$ on the module $\otimes_{i=0}^{d-\nu} \mathbb{F}_p \otimes H^i(A'', \mathbb{F}_p)$,

$$\sum_{i=1}^{c} d_i \leq \sum_{i=1}^{d-\nu} \dim H^i(A'', \mathbb{F}_p) \leq \sum_{i=1}^{d-\nu} \left( i + r - 1 \right) \left( d - \nu + r - 1 \right) \frac{r}{r - 1},$$

hence $|K| \leq \left( \frac{d + r}{r} \right)^2$. \hfill \Box

**Lemma 5.3.** For all integers $r, d, B \geq 0$, there is an integer $n_2(r, d, B)$ with the following property.

Let $G = P \rtimes H$ be the semidirect product of a $p$-group $P$ of order at most $p^r$ and a finite group $H$, the order of which is not divisible by $p$. Let $M$ be a connected $d$-dimensional topological manifold with an effective $G$-action. Suppose that $\dim H_c^*(M; \mathcal{O}_p) \leq B$, the $P$-action on $M$ is free, and the induced $H$-action on $H_c^*(M; \mathcal{O}_p)$ is trivial. Then $H$ has a subgroup $\tilde{H}$ of index at most $n_2(r, d, B)$ commuting with $P$.

**Proof.** We prove the lemma via induction on $r$. For $r = 0$, we have $P = \{1\}$, so the lemma holds in this case with $n_2(0, d, B) = 1$. Assume that $r \geq 1$, and $P \neq \{1\}$.

Let $A$ denote the socle of $Z(P)$. It is a characteristic subgroup of $G$ acting freely on $M$, and $A \cong \mathbb{Z}_p^\rho$ for some positive $\rho \leq r$. Lemma 5.2 gives us a subgroup $H_1 \leq H$ of index at most $\max \{ n_1(\rho, d) \mid 0 \leq \rho \leq r \}$ commuting with $A$.

Consider the subgroup $\tilde{G} = P \rtimes H_1$. $A$ is central (hence normal) in $\tilde{G}$. Therefore the quotient space $N = M/A$ is a $d$-dimensional topological manifold with an induced action of $\tilde{G}/A$. $H_c^*(N; \mathcal{O}_p)$ can be calculated via the Borel spectral sequence (see Proposition 4.13). Since $A$ is central in $\tilde{G}$, the $H_1$-action on the $E_2$ page of the spectral sequence is trivial. Therefore the $H_1$-action on $H_c^*(N; \mathcal{O}_p)$ is trivial. Moreover, we obtain the bound

$$\dim H_c^*(N; \mathcal{O}_p) = \dim H_{A,c}^*(M; \mathcal{O}_p) \leq \sum_{i+j \leq d} \dim E_{i,j}^2 \leq \left( \frac{d + r}{r} \right) B.$$ 

The group $\tilde{G}/A = (P/A) \rtimes H_1$ acts on $N$. If $h \in \tilde{G}$ acts trivially on $N$, then there is a unique continuous map $a: M \to A$ such that $h(x) = a(x)a$. As $A$ is discrete, $M$ is connected, $a(x) \equiv a \in A$ is constant. Then $a^{-1}h$ acts trivially on $M$, so $h = a \in A$. This implies that $\tilde{G}/A$ acts effectively on $N$, so we can apply the induction hypothesis. We
obtain a subgroup $\tilde{H} \leq H_1$ of index at most $n_2(r - 1, d, (\frac{d+r}{r})B)$ acting trivially on $P/A$. Recall that $\tilde{H}$ acts trivially on $A$ as well.

The conjugation action of $\tilde{H}$ lies in the kernel of the restriction homomorphism $\text{Aut}(P) \to \text{Aut}(A) \times \text{Aut}(P/A)$. This kernel is a $p$-group, and $|\tilde{H}|$ is not divisible by $p$. Hence $\tilde{H}$ acts trivially on $P$. Moreover, $|H : \tilde{H}| = |H_1 : H_1| \cdot |H_1 : \tilde{H}|$ is bounded in terms of $r, d, B$. The induction step is complete. □

6. Reduction to free action

The main goal of this section is to prove the following.

**Lemma 6.1.** For all integers $d, B \geq 0$, there is an integer $n_3(d, B)$ with the following property.

Let $p$ be a prime larger than $\tilde{C}(d, B)$ (defined in Proposition 4.9), and let $G = P \rtimes H$ be the semidirect product of a $p$-group $P$ and a finite group $H$ of order not divisible by $p$. Suppose that $G$ acts effectively on a connected topological $d$-manifold $M$ such that $\dim H^i_c(M; O_p) \leq B$, and the induced $H$-action on $H^i_c(M; O_p)$ is trivial. Let $U \subseteq M$ be the largest open subset where the $P$-action is free. Then $U$ is $G$-invariant, connected,

\[
\dim H^i_c(U; O_p) \leq 2 |\text{Stab}(P,M)| B \leq 2 \tilde{C}(d,B) B,
\]

and $H$ has a subgroup $\tilde{H}$ of index at most $n_3(d, B)$ which acts trivially on $H^i_c(U; O_p)$.

**Remark 6.2.** A variant of the above inequality is proved in [6]. However, we prove the rest of this statement by induction, and we need this slightly stronger inequality to make the induction work.

For differentiable action of a finite group on a manifold, each fixed point has invariant neighborhoods homeomorphic to a ball. For continuous actions we use the following replacement.

**Definition 6.3** ([3 Definition I-4.4]). Let $M$ be a topological $d$-manifold and $U \subseteq M$ a connected orientable open subset. A connected open subset $V \subseteq U$ is adapted to $U$ if the induced homomorphism $H^i(V; O_p) \to H^i(U; O_p)$ is an isomorphism for $i = d$, and zero for $i \neq d$.

**Lemma 6.4.** Let $G$ be a finite group, $M$ a $d$-dimensional $G$-manifold, and $x \in M^G$ a fixed point. Then each orientable open neighborhood $U$ of $x$ contains a $G$-invariant orientable connected open neighborhood of $x$ adapted to $U$. 

Proof. Let $W \subseteq U$ be an open neighborhood of $x$ homeomorphic to $\mathbb{R}^d$. Then every connected open subset in $W$ is adapted to $U$. Let $V$ be the intersection of all $G$-translates of $W$, and $V^0 \subseteq V$ the connected component containing $x$. It is $G$-invariant, orientable, connected, and it is adapted to $U$. □

Borel in [3, Lemma V-2.1] identified a direct summand of the equivariant cohomology of a connected oriented $d$-manifold $M$ which is originated from $H^d_c(M; \mathbb{F}_p)$. We need a slightly more precise information than he stated, and we need to generalize this to non-oriented manifolds.

**Proposition 6.5.** Let $G$ be a finite group and $M$ a $d$-dimensional $G$-manifold. The spectral sequence in Proposition 4.15 gives us an edge homomorphism

$$E_M: H^*_{G,c}(M; O_p) \rightarrow H^{*+d}(G; H^d_c(M; O_p)),$$

which is a natural transformation with respect to $G$-equivariant open embeddings. If $M$ is connected and the $G$-action has a fixed point, then

$$(4) \quad H^*(G; H^d_c(M; O_p)) \cong H^*(G; \mathbb{F}_p) \otimes H^d_c(M; O_p),$$

and $E_M$ is surjective.

Proof. By Proposition 4.15 we have $E_2^{*,d} = H^*(G; H^d_c(M; O_p))$, and $E_2^{*,j} = 0$ for all $j > d$. This implies that $E_M$ can be defined as the composition

$$H^*_{G,c}(M; O_p) \rightarrow E^{*-d,d}_\infty \rightarrow E^{*-d,d}_2 = H^{*-d}(G; H^d_c(M; O_p)).$$

Naturality of $E_M$ follows from the naturality of the spectral sequence.

If $M$ is connected, then the $G$-action on $H^d_c(M; O_p)$ is trivial by Proposition 4.15, and therefore (4) holds.

Now assume that $M$ is connected and $M^G \neq \emptyset$. Let $x \in M^G$ be a fixed point and $U \subseteq M$ a connected orientable open neighborhood of $x$. By repeated use of Lemma 6.4, construct connected orientable open $G$-invariant neighborhoods $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_{2d}$ of $x$ such that each $U_i$ is adapted to $U_{i-1}$, and apply [3, Lemma V-2.1] to this sequence. The lemma states that there is a subspace $W \leq H^*_{G,c}(U; O_p)$ isomorphic to $H^{*-d}(G; \mathbb{F}_p) \otimes H^d_c(U; O_p)$ and it follows from its proof that $E_U$ maps $W$ isomorphically onto $H^{*-d}(G; \mathbb{F}_p) \otimes H^d_c(U; O_p)$. In
particular, \( \mathcal{E}_U \) is surjective. Consider now the diagram

\[
H_{G,c}^*(M; \mathcal{O}_p) \xrightarrow{\mathcal{E}_M} H^{* - d}(G; F_p) \otimes H^d_c(M; \mathcal{O}_p)
\]

where \( \phi \) and \( \psi \) are the natural homomorphisms induced by the inclusion \( U \subseteq M \). The map \( \psi \) is an isomorphism by Proposition 4.6, hence \( \mathcal{E}_M \circ \phi = \psi \circ \mathcal{E}_U \) is surjective. This implies that \( \mathcal{E}_M \) is surjective. \( \square \)

We consider the actions of the group \( G \cong \mathbb{Z}_p \). Let \( M \) be a \( G \)-manifold. For differentiable manifolds and smooth actions, the \( G \)-action on the normal bundle of \( M^G \) would give us useful information (see e.g. \([30, \text{Lemma 3.2}]\)). In our situation the normal bundle is not available. Instead, we use the homomorphisms \( \Omega^k_M \) defined below.

**Lemma 6.6.** Consider the group \( G \cong \mathbb{Z}_p \). Let \( M \) be a \( d \)-dimensional \( G \)-manifold, and suppose that \( F = M^G \) is non-empty. For each \( n \geq 1 \) and each \( k \leq d \), we have:

(a) The restriction homomorphism \( H^{d+n}_{G,c}(M; \mathcal{O}_p) \xrightarrow{\text{res}_M} H^{d+n}_{G,c}(F; \mathcal{O}_p) \)

is an isomorphism.

(b) In the following diagram \( K^k_F \) is the natural inclusion induced by the K"unneth isomorphism (see Proposition 4.14(b)), \( \mathcal{E}_M \) is the edge homomorphism defined in Proposition 6.5, and \( \Omega^k_M = \mathcal{E}_M \circ \text{res}_M^{-1} \circ K^k_F \).

\[
\begin{array}{ccc}
H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(F; \mathcal{O}_p) & \xrightarrow{\text{res}_M} & H^{n+d-k}(G; H^k_c(M; \mathcal{O}_p)) \\
\downarrow \mathcal{K}^k_F & & \downarrow \mathcal{E}_M \\
H^{n+d}(F; \mathcal{O}_p) & \xrightarrow{\Omega^k_M} & H^{n+d}_{G,c}(M; \mathcal{O}_p)
\end{array}
\]

The diagram depends on \( M \) functorially with respect to \( G \)-equivariant open embeddings.

(c) Suppose that \( F_0 \) is a \( k \)-dimensional non-empty connected component of \( F \), and let \( M_0 \) denote the connected component of \( M \) containing \( F_0 \). Then \( \Omega^k_M \) maps the direct summand

\[
H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(F_0; \mathcal{O}_p) \leq H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(F; \mathcal{O}_p)
\]

isomorphically onto the direct summand

\[
H^n(G; \mathbb{F}_p) \otimes H^d_c(M_0; \mathcal{O}_p) \leq H^n(G; H^d_c(M; \mathcal{O}_p)).
\]

**Remark 6.7.** We show that (c) is meaningful. \( G \) maps \( M_0 \) into itself, hence \( H^n(G; H^d_c(M_0; \mathcal{O}_p)) \) is a direct summand of \( H^n(G; H^d_c(M; \mathcal{O}_p)) \).
Moreover, by Proposition 4.6 the \( G \)-action on \( H^d_c(M_0; \mathcal{O}_p) \) is trivial, so \( H^n(G; H^d_c(M_0; \mathcal{O}_p)) \cong H^n(G; \mathbb{F}_p) \otimes H^d_c(M_0; \mathcal{O}_p) \). This shows that \( H^n(G; \mathbb{F}_p) \otimes H^d_c(M_0; \mathcal{O}_p) \) is indeed a direct summand of \( H^n(G; H^d_c(M; \mathcal{O}_p)) \).

The same argument shows that \( H^n(G; \mathbb{F}_p) \otimes H^d_c(F_0; \mathcal{O}_p) \) is a direct summand of \( H^n(G; H^d_c(F; \mathcal{O}_p)) \).

**Proof of Lemma 6.6.** Let \( M_* \subseteq M \) be the \( G \)-orbit of a connected component of \( M \). If \( M_* \cap F = \emptyset \), then \( H^{n+d}_{G,c}(M_*; \mathcal{O}_p) = 0 \) by Proposition 4.14(a), and so \( \text{res}_{M_*} = 0 \). Otherwise \( M_* \) is connected and \( M_* \setminus F \) is open, hence \( H^{n+d+1}_{G,c}(M_* \setminus F; \mathcal{O}_p) = 0 \) by Proposition 4.14(a), so Proposition 4.16 implies that \( \text{res}_{M_*} \) is an isomorphism.

Since \( \text{res}_M \) is the direct sum of the isomorphisms \( \text{res}_{M_*} \) for various \( M_* \), (a) follows.

Propositions 4.14(b), 4.16, and 6.5 state that \( K^k_F, \text{res}_M, \) and \( E_M \) are natural transformations. This proves (b).

Finally we prove (c). By Proposition 4.7 \( F_0 \) is a cohomology manifold. Let \( V \subseteq F_0 \) be a connected orientable open subset, and let \( V' \subset V \) be an open subset such that the induced homomorphism

\[
H^j_c(V'; \mathcal{O}_p) \rightarrow H^j_c(V; \mathcal{O}_p)
\]

is \( \begin{cases} 
\text{an isomorphism for } j = k, \\
0 \text{ otherwise.} 
\end{cases} \)

Let \( W \subseteq M \) be an open subset such that \( W \cap F = V \), and let \( U \) be the connected component of \( \bigcap_{g \in G} gW \) containing \( V \). Similarly, let \( W' \subseteq W \) be an open subset such that \( W' \cap F = V' \), and let \( U' \) be the connected component of \( \bigcap_{g \in G} gW' \) containing \( V' \). Then \( U' \subseteq U \) are \( G \)-invariant connected open submanifolds, hence the inclusion map induces an isomorphism

\[
H^d_c(U'; \mathcal{O}_p) \xrightarrow{\cong} H^d_c(U; \mathcal{O}_p) \cong \mathbb{F}_p.
\]
Applying (a), (b), and Remark 6.7 to $U$ and $U'$, we obtain the following diagram.

\[
\begin{array}{c}
H^{n+d}(V'; \mathcal{O}_p) \xrightarrow{\kappa^k_{V'}} H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(V'; \mathcal{O}_p) \\
\downarrow \alpha \hspace{2cm} \downarrow \cong \\
H^{n+d}(V; \mathcal{O}_p) \xrightarrow{\kappa^k_V} H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(V; \mathcal{O}_p) \\
\cong \xrightarrow{\text{res}^{-1}_{U'}} \cong \xrightarrow{\text{res}^{-1}_U} \\
H^{n+d}(U; \mathcal{O}_p) \xrightarrow{\varepsilon_U} H^n(G; \mathbb{F}_p) \otimes H^d_c(U; \mathcal{O}_p) \\
\downarrow \beta \hspace{2cm} \gamma \hspace{2cm} \cong \\
H^{n+d}(U'; \mathcal{O}_p) \xrightarrow{\varepsilon_{U'}} H^n(G; \mathbb{F}_p) \otimes H^d_c(U'; \mathcal{O}_p)
\end{array}
\]

Proposition 4.14(b) and (5) imply that
\[\text{im}(\kappa^k_V) = \text{im}(\alpha),\]
hence
\[\text{im}(\text{res}^{-1}_U \circ \kappa^k_V) = \text{im}(\text{res}^{-1}_U \circ \alpha) = \text{im}(\beta),\]
and therefore
\[\text{im}(\Omega^k_U) = \text{im}(\varepsilon_U \circ \text{res}^{-1}_U \circ \kappa^k_V) = \text{im}(\varepsilon_U \circ \beta) = \text{im}(\gamma \circ \varepsilon_{U'}).\]

But $\varepsilon_{U'}$ is surjective by Proposition 6.5, hence $\Omega^k_{U'}$ is surjective as well. Now $\Omega^k_{U'}$ is a surjective homomorphism between one-dimensional spaces, so it is an isomorphism.

In the following diagram, the vertical arrows are induced by the inclusions $V \subseteq F_0 \subseteq F$ and $U \subseteq M_0 \subseteq M$.

\[
\begin{array}{ccc}
H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(F; \mathcal{O}_p) \xrightarrow{\Omega^k_M} H^n(G; H^d_c(M; \mathcal{O}_p)) \cong H^n(G; \mathbb{F}_p) \otimes H^d_c(M; \mathcal{O}_p) \\
\downarrow \phi \hspace{2cm} \downarrow \psi \\
H^{n+d-k}(G; \mathbb{F}_p) \otimes H^k_c(F_0; \mathcal{O}_p) \hspace{2cm} H^n(G; \mathbb{F}_p) \otimes H^d_c(M_0; \mathcal{O}_p) \\
\end{array}
\]

By Proposition 4.6, $\phi$ and $\psi$ are isomorphisms. This proves (c). □

**Definition 6.8.** Let $W$ and $I$ be finite dimensional $FH$-modules for some field $\mathbb{F}$ and a finite group $H$, and assume that $I$ is irreducible. We shall denote by $\#_I W$ the number of occurrences of $I$ among the composition factors of $W$. 


Smith theory can be used to study the topology of the fixed point set of a $p$-group acting on a manifold (see e.g. Proposition 4.7). We need the following generalization when the action of another group $H$ is taken into account.

**Lemma 6.9.** Let $G = P \times H$ be the direct product of a finite $p$-group $P$ for some prime $p$ and a finite group $H$. Let $M$ be a $G$-manifold. Then $M^P$ is an $H$-invariant union of disjoint cohomology manifolds over $\mathbb{F}_p$, and for any irreducible $H$-module $I$, we have

\[
\#_1 H^*_c(M^P; \mathcal{O}_p) \leq \#_1 H^*_c(M; \mathcal{O}_p),
\]

\[
\#_1 H^*_c(M \setminus M^P; \mathcal{O}_p) \leq 2 \#_1 H^*_c(M; \mathcal{O}_p).
\]

**Proof.** We prove the first inequality by induction on $|P|$. Assume first, that $P = \mathbb{Z}_p$. We recall the proof of [4, Theorem II-19.7], and adjust it to our needs.

Let $\pi: M \to M/P$ denote the orbit map, $F = M^P$, and $\iota: F \hookrightarrow M/P$ denote the restriction of $\pi$ to $F$, which is a homeomorphic embedding. Then $H$ acts naturally on $M$, $F$, $M/P$, and the maps $\pi$, $\iota$ are $H$-equivariant. Therefore, by Proposition 4.7, $M^P$ is an $H$-invariant disjoint union of cohomology manifolds over $\mathbb{F}_p$. Let $\text{res}: \pi_*/\mathcal{O}_{p,M} \to \iota_*\tilde{\mathcal{O}}_{p,F}$ denote the restriction homomorphism.

We choose a generator $g \in P$, and set $\tau = 1 - g$, $\sigma = 1 + g + g^2 + \cdots + g^{p-1}$ in the group ring $\mathbb{F}_pP$. For sheaves of $\mathbb{F}_pP$-modules, we denote by $\tilde{\sigma}$ and $\tilde{\tau}$ the automorphisms given by the multiplications with $\sigma$ and $\tau$. Then [4, Theorem II-19.7] gives us the following two exact sequences of sheaves on $M/P$:

\[
0 \to \tilde{\sigma}(\pi_*/\mathcal{O}_{p,M}) \to \pi_*/\mathcal{O}_{p,M} \xrightarrow{\tilde{\tau} \oplus \text{res}} \tilde{\tau}(\pi_*/\mathcal{O}_{p,M}) \oplus \iota_*\tilde{\mathcal{O}}_{p,F} \to 0
\]

\[
0 \to \tilde{\tau}(\pi_*/\mathcal{O}_{p,M}) \to \pi_*/\mathcal{O}_{p,M} \xrightarrow{\tilde{\sigma} \oplus \text{res}} \tilde{\sigma}(\pi_*/\mathcal{O}_{p,M}) \oplus \iota_*\tilde{\mathcal{O}}_{p,F} \to 0
\]

Since $H$ and $g$ commute, $H$ acts canonically on these sheaves, and the homomorphisms are $H$-equivariant. Therefore, the corresponding long exact sequences are sequences of $\mathbb{F}_pH$-modules and module homomorphisms. The parts

\[
H^*_c(M/P; \pi_*/\mathcal{O}_{p,M}) \to H^*_c(M/P; \tilde{\tau}(\pi_*/\mathcal{O}_{p,M})) \oplus H^*_c(F; \mathcal{O}_{p,F}) \to H^*_{c+1}(M/P; \tilde{\sigma}(\pi_*/\mathcal{O}_{p,M})) \to \cdots
\]

and

\[
H^*_{c+1}(M/P; \pi_*/\mathcal{O}_{p,M}) \to H^*_{c+1}(M/P; \tilde{\sigma}(\pi_*/\mathcal{O}_{p,M})) \oplus H^*_{c+1}(F; \mathcal{O}_{p,F}) \to H^*_{c+2}(M/P; \tilde{\tau}(\pi_*/\mathcal{O}_{p,M})) \to \cdots
\]
of these long exact sequences imply the inequalities
\[
\#_1H_c^{2k}(M/P; \tilde{\tau}(\pi_*\mathcal{O}_{p,M})) + \#_1H_c^{2k}(F; \mathcal{O}_{p,F}) \\
\leq \#_1H_c^{2k}(M/P; \pi_*\mathcal{O}_{p,M}) + \#_1H_c^{2k+1}(M/P; \tilde{\sigma}(\pi_*\mathcal{O}_{p,M}))
\]
and
\[
\#_1H_c^{2k+1}(M/P; \tilde{\sigma}(\pi_*\mathcal{O}_{p,M})) + \#_1H_c^{2k+1}(F; \mathcal{O}_{p,F}) \\
\leq \#_1H_c^{2k+1}(M/P; \pi_*\mathcal{O}_{p,M}) + \#_1H_c^{2k+2}(M/P; \tilde{\tau}(\pi_*\mathcal{O}_{p,M}))
\]
for all \(k \geq 0\). Adding these inequalities up for all \(k \geq 0\), and cancelling the repeated terms, we obtain
\[
\#_1H_c^*(F; \mathcal{O}_{p,F}) \leq \#_1H_c^*(M/P; \pi_*\mathcal{O}_{p,M}).
\]
The fibers of \(\pi\) are finite, hence the Leray spectral sequence of \(\pi\) (see [4, Theorem IV-6.1]) degenerates, and so \(H_c^*(M/P; \pi_*\mathcal{O}_{p,M}) \cong H_c^*(M; \mathcal{O}_{p,M})\). This proves the first inequality of the lemma.

The second inequality follows from the first one by Proposition 4.7 and the long exact sequence for cohomology with compact support (see [4, II-10.3]).

Next we do the induction step. We choose a subgroup \(A \cong \mathbb{Z}_p\) in the center of \(G\). Then \(N = M^A\) is a \(G\)-invariant union of disjoint cohomology manifolds over \(\mathbb{F}_p\), and \(G/A\) acts on \(N\) canonically. Applying the induction hypothesis first to the \(A\)-action on \(M\), and then to the \(G/A\)-action on \(N\), we see that our statements hold in this case. This completes the induction step. \(\square\)

Now we are ready to prove the following special case of Lemma 6.1.

**Lemma 6.10.** For all integers \(d, B \geq 0\), there is an integer \(n_4(d, B)\) with the following property. Let \(p\) be a prime, and \(G\) be a finite group of the form \(G = A \times H\), where \(A\) is isomorphic to \(\mathbb{Z}_p\) equipped with an \(H\)-action, and \(|H|\) is not divisible by \(p\). Let \(M\) be a \(d\)-dimensional effective \(G\)-manifold over \(\mathbb{F}_p\) such that \(\dim H_c^*(M; \mathcal{O}_p) \leq B\). Suppose that \(F = M^A\) is nonempty, and the induced \(H\)-action on \(H_c^*(M; \mathcal{O}_p)\) is trivial. Then \(H\) has a subgroup \(\tilde{H}\) of index at most \(n_4(d, B)\) such that

(a) \(\tilde{H}\) commutes with \(A\), and

(b) \(\tilde{H}\) acts trivially on \(H_c^*(F; \mathcal{O}_p)\) and on \(H_c^*(M \setminus F; \mathcal{O}_p)\).

**Proof.** If \(p = 2\), then \(A \cong \mathbb{Z}_2\) has no nontrivial automorphism, so (a) holds with \(\tilde{H} = H\).

Assume now that \(p > 2\). We identify \(\text{Aut}(A)\) with the multiplicative group \(\mathbb{F}_p^*\), then \(H\) acts on \(A\) via a character \(\lambda: H \to \mathbb{F}_p^*\). Proposition 4.7 implies that \(H\) acts on \(H^n(A; \mathbb{F}_p) \cong \mathbb{F}_p^*\) via the character \(\lambda^{−1}/2\) for all \(n \geq 0\).
$H$ permutes the connected components of $F$ and $M$. Let $F_0$ be a connected component of $F$ of dimension say $k$, and let $M_0$ be the connected component of $M$ containing $F_0$. Proposition 4.7 and Proposition 4.6 imply that $F$ has at most $B$ connected components, so $H$ has a subgroup $H_0$ of index at most $B$ that maps $F_0$ into itself, and therefore it maps $M_0$ also into itself. By Proposition 4.6, the $H_0$-action on $H^*_c(F_0; O_p)$ and $H^*_c(M_0; O_p)$ is trivial.

Choose an even integer $n > 0$. Lemma 6.6 gives us an $H_0$-equivariant isomorphism

$$H^{n+d-k}(A; \mathbb{F}_p) \otimes H^k_c(F_0; O_p) \cong H^n(A; \mathbb{F}_p) \otimes H^d_c(M_0; O_p).$$

Let $\lambda_0$ denote the restriction of $\lambda$ to $H_0$. Since $H_0$ acts on the two sides via the characters $\lambda_0^{-\lceil \frac{n+d-k}{2} \rceil}$ and $\lambda_0^{-\frac{n}{2}}$ respectively, we obtain that $\lambda_0^{-\lceil \frac{n+d-k}{2} \rceil} = 1$. Since $k < d$, the subgroup $\tilde{H} = \ker(\lambda_0)$ has index at most $\lceil \frac{d-k}{2} \rceil$ in $H_0$, so it has index at most $B \lceil \frac{d-k}{2} \rceil$ in $H$. Moreover, $\ker(\lambda_0)$ commutes with $A$. This implies (2) for odd primes as well.

In particular, $A \times \tilde{H}$ is a subgroup of $G$. We consider $M$ as an $(A \times \tilde{H})$-manifold over $\mathbb{F}_p$. Then $H^*_c(M; O_p)$ is an $\tilde{H}$-module with no non-trivial composition factors. Lemma 6.9 implies that $H^*_c(F; O_p)$ and $H^*_c(M \setminus F; O_p)$ are also $\tilde{H}$-modules with no non-trivial composition factors. This proves (1). \hfill $\square$

**Lemma 6.11.** For all integers $d, B \geq 0$, there is an integer $n_5(d, B)$ with the following property.

Let $p$ be a prime larger than the bound $\tilde{C}(d, B)$ defined in Proposition 4.9, and let $G = P \rtimes H$ be the semidirect product of a $p$-group $P$ and a finite group $H$ of order not divisible by $p$. Let $M$ be a connected topological $d$-manifold with an effective $G$-action. Suppose that $\dim H^*_c(M; O_p) \leq B$, and the induced $H$-action on $H^*_c(M; O_p)$ is trivial. Let $K \in \text{Stab}(P, M)$ be a stabilizer subgroup different from $\{1\}$. Then $H$ has a subgroup $H_1$ of index at most $n_5(d, B)$ and $K$ has a subgroup $L \neq \{1\}$ such that $M \setminus M^L$ is $H_1$-invariant and the induced $H_1$-action on $H^*_c(M \setminus M^L; O_p)$ is trivial.

**Proof.** Proper subgroups of $P$ have index at least $p$, so by Proposition 4.9

$$|\text{Stab}(P, M)| \leq \tilde{C}(d, B).$$

Conjugates of $K$ are also stabilizer subgroups, so $K$ has at most $\tilde{C}(d, B)$ conjugates. Therefore the normalizer subgroup $N_H(K)$ has index at most $\tilde{C}(d, B)$ in $H$.

Let $E$ be the socle of $K$. This is a non-trivial elementary abelian $p$-subgroup. If $E$ acts freely on $M$, then $H_1 = N_H(K)$ and $L = E$
are a good choice. Otherwise there is a point \( x \in M \) with a non-trivial stabilizer \( E_x \) in \( E \). If \( |E_x| > p \), then Borel’s fixed point formula (Proposition 4.8) implies that there is a nearby point \( y \in M \) whose stabilizer \( E_y \) has index \( p \) in \( E_x \). Iterating this argument we obtain a stabilizer subgroup \( L \in \text{Stab}(E, M) \) of order \( p \).

Since \( E \) is characteristic in \( K \), it is normalized by \( \mathcal{N}_H(K) \). As above, Proposition 4.9 implies that
\[
|\text{Stab}(E, M)| \leq \tilde{C}(d, B),
\]
and \( \mathcal{N}_H(K) \) acts on \( \text{Stab}(E, M) \) via conjugation. Hence, \( \mathcal{N}_H(K) \) has a subgroup \( \tilde{H} \leq \mathcal{N}_H(K) \) of index at most \( \tilde{C}(d, B) \) which normalizes \( L \), and therefore \( M^L \) and \( M \setminus M^L \) are \( \tilde{H} \)-invariant.

Now we apply Lemma 6.10 to the subgroup \( L \times \tilde{H} \leq G \) with its given action on \( M \). We obtain a subgroup \( H_1 \leq \tilde{H} \) with the required properties.

**Proof of Lemma 6.1.** By construction, \( U \) is \( P \)-invariant, and it is also \( H \)-invariant since \( H \) normalizes \( P \). Therefore \( U \) is \( G \)-invariant. Since \( p > \tilde{C}(d, B) \), Proposition 4.9 implies the second part of inequality (3).

We prove the rest of the statement by induction on the size of \( \text{Stab}(P, M) \). If \( |\text{Stab}(P, M)| = 1 \), then \( \{1\} \) is the only stabilizer subgroup, hence the \( P \)-action is free, \( U = M \), and the statement holds. For the induction step we assume that \( |\text{Stab}(P, M)| > 1 \), and the statement holds in all cases with a smaller number of stabilizer subgroups.

Let \( K \in \text{Stab}(P, M) \) be a stabilizer different from \( \{1\} \). Lemma 6.11 gives us a subgroup \( H_1 \leq H \) of index at most \( n_5(d, B) \) and a non-trivial subgroup \( L \leq K \) such that the open submanifold \( V = M \setminus M^L \) is \( H_1 \)-invariant and the \( H_1 \)-action on \( H^*_c(V; O_p) \) is trivial. Moreover, Proposition 4.7 implies that \( V \) is connected and
\[
\dim H^*_c(V; O_p) \leq 2B.
\]
Since \( M^L \) contains all points whose stabilizer is \( K \), we have
\[
\text{Stab}(P, V) \subseteq \text{Stab}(P, M) \setminus \{K\}.
\]
We may apply the induction hypothesis to the subgroup \( P \times H_1 \leq G \) with its given action on \( V \). This gives us a subgroup \( \tilde{H} \leq H_1 \leq H \) with all the required properties. The induction step is complete. \( \square \)

7. **Proof of the Main Theorem**

**Lemma 7.1.** For all integers \( B, \tau \geq 0 \), there is an integer \( I(B, \tau) \) with the following property.

Let \( G \) be a finite group acting continuously on a topological manifold.
such that \( M \) has rank \( B \) and has at most \( \tau \) torsion elements. Then \( \dim H^*_c(M; \mathbb{Q}_p) \leq B + 2\tau \) for all primes \( p \). Moreover, \( G \) has a normal subgroup of index at most \( I(B, \tau) \) whose canonical action on \( H^*_c(M; \mathbb{Q}_p) \) is trivial for all \( p \).

Proof. By the Poincaré duality (Proposition 4.5), it is enough to prove that \( \dim H^*_c(M; \mathbb{Z}_p) \leq B + 2\tau \), and to find a subgroup of bounded index which acts trivially on the homology groups \( H^*_c(M; \mathbb{Z}_p) \) for all primes \( p \).

Let \( \mathcal{T} \) be the torsion part of \( H^*_c(M; \mathbb{Z}) \), and \( \mathcal{F} = H^*_c(M; \mathbb{Z}) / \mathcal{T} \cong \mathbb{Z}^B \) be the free part. The \( G \)-action on \( M \) induces a canonical \( G \)-action on the split short exact sequence \( 0 \to \mathcal{T} \to H^*_c(M; \mathbb{Z}) \to \mathcal{F} \to 0 \). This action gives a homomorphism of \( G \) into the automorphism group of this short exact sequence, which is isomorphic to the semidirect product \( \text{Hom}(\mathcal{F}, \mathcal{T}) \rtimes (\text{Aut}(\mathcal{T}) \times \text{Aut}(\mathcal{F})) \). The kernel \( G_0 \) of this homomorphism acts trivially on \( H^*_c(M; \mathbb{Z}) \). A theorem of Minkowski [27] gives an upper bound \( I_0(B) \) on the size of a finite subgroup of \( \text{Aut}(\mathbb{Z}^B) \cong \text{Aut}(\mathcal{F}) \), and \( |\text{Aut}(\mathcal{T})| \leq |\mathcal{T}|^{\log |\mathcal{T}|} \), hence the index of \( G_0 \) in \( G \) is at most \( \tau^{\log \tau} \cdot I_0(B) \cdot \tau^B \).

By the universal coefficient theorem, there is a split exact sequence

\[
0 \to H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p \to H^*_c(M; \mathbb{Z}_p) \to \text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p) \to 0
\]

for all primes \( p \), where

\[
\text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p) = \{ h \in H^*_c(M; \mathbb{Z}) : ph = 0 \},
\]

which has at most \( \tau \) elements. Moreover, \( H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p \) is an elementary abelian \( p \)-group of rank at most \( \tau + B \). This implies that \( \dim H^*_c(M; \mathbb{Z}_p) \leq B + 2\tau \).

The action of \( G \) on \( M \) induces a homomorphism of \( G \) into the automorphism group of the split exact sequence (6), which is isomorphic to the semidirect product of \( \text{Aut}(H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p) \rtimes (\text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p)) \) and the normal subgroup \( \text{Hom}(\text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p), H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p) \). Since \( G_0 \) acts trivially on \( H^*_c(M; \mathbb{Z}) \), the restriction of this homomorphism onto \( G_0 \) yields a homomorphism

\[
G_0 \to \text{Hom}(\text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p), H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p),
\]

the kernel \( G_p \) of which acts trivially on \( H^*_c(M; \mathbb{Z}_p) \). By (7) the torsion group \( \text{Tor}(H^*_c(M; \mathbb{Z}), \mathbb{Z}_p) \) is trivial if \( p > \tau \) and has at most \( \tau \) elements if \( p \leq \tau \). The tensor product \( H^*_c(M; \mathbb{Z}) \otimes \mathbb{Z}_p \) is an elementary abelian \( p \)-group of rank at most \( \tau + B \). Consequently, we have \( G_p = G_0 \) if \( p > \tau \), and \( |G_p| \leq p^{(\tau + B)\tau} \) if \( p \leq \tau \).

The normal subgroup of those elements of \( G \) that act trivially on \( H^*_c(M; \mathbb{Z}_p) \) for every prime \( p \) contains the subgroup \( G_0 \cap \bigcap_{p \leq \tau} G_p \),
therefore, its index is bounded from above by the index of the latter subgroup, which is at most
\[ |G : G_0| \cdot \prod_{p \leq \tau} |G_0 : G_p| \leq \tau^{\log \tau} \cdot I_0(B) \cdot \tau^{B + (r + B)\tau^2}. \]

Now we are ready to present the

Proof of Theorem 1.4. Let \( M_1, \ldots, M_k \) be the connected components of \( M \), \( B \) be the rank of \( H_*(M; \mathbb{Z}) \), \( \tau \) the size of the torsion part of \( H_*(M; \mathbb{Z}) \), and \( d = \dim(M) \). By Proposition 4.6 and Lemma 7.1, we may assume that the canonical \( G \)-action on \( H_*(M; \mathcal{O}_p) \) is trivial, and we have
\[ k = \dim H_c^d(M, \mathcal{O}_p) \leq \dim H_c^*(M; \mathcal{O}_p) \leq B + 2\tau \]
for all primes \( p \). Then \( G \) acts trivially on \( H_0(M, \mathbb{F}_p) \), hence the elements of \( G \) map each component of \( M \) into itself, and we obtain homomorphisms \( \phi_i : G \to \text{Homeo}(M_i) \) corresponding to the action of \( G \) on \( M_i \). Let \( G_i \) denote the image \( \phi_i(G) \) of \( G \) under \( \phi_i \).

Take any of the indices \( 1 \leq i \leq k \). Theorem 4.10 gives us a bound \( \text{rk}(G_i) \leq r = r(d, H_*(M; \mathbb{Z})) \). Using the bounds in Proposition 4.9, Lemma 6.1, and Lemma 5.3 we define the constants
\[ \tilde{C} = \tilde{C}(d, B), \quad n_3 = n_3(d, B), \quad n_2 = n_2(2r, d, 2\tilde{C}B). \]
Apply Theorem 1.7 to each of the groups \( G_i \) with the parameter
\[ T = \max \left( (\tilde{C}2r)! , n_2n_3 \right) + 1. \]
The theorem has two possible outcomes. Assume first that \( 1.7 \) (b) holds, i.e. \( G_i \) has a subgroup of the form \( P \rtimes_H \), where \( P \) is a \( p \)-group of order at most \( p^{2r} \), \( H \) is an abelian group, and \( \alpha : H \to \text{Aut}(P) \) is the action of \( H \) on \( P \), such that
\[ p \nmid |H|, \quad |\alpha(H)| \geq T. \]
This implies that \( (p^{2r})! > |\text{Aut}(P)| \geq T > (\tilde{C}2r)! \), hence \( p > \tilde{C} \). We apply Lemma 6.1 to the subgroup \( P \rtimes_{n_1} H \leq G_i \) acting on \( M_i \). We obtain a \( (P \rtimes_{\alpha} H) \)-invariant open connected submanifold \( U \subseteq M_i \) on which \( P \) acts freely such that
\[ \dim H_c^*(U; \mathcal{O}_p) \leq 2\tilde{C}B, \]
and a subgroup \( H_1 \leq H \) of index at most \( n_3 \) acting trivially on \( H_c^*(U; \mathcal{O}_p) \). We apply Lemma 5.3 to the \( P \rtimes H_1 \) action on \( U \), and obtain a subgroup \( \tilde{H} \leq H_1 \) of index at most \( n_2 \) commuting with \( P \). But then \( \tilde{H} \leq \ker(\alpha) \), hence \( |\text{im}(\alpha)| \leq |H : H_1| \cdot |H_1 : \tilde{H}| \leq n_2n_3 < T \), a contradiction.
Thus $[1.7]$ must hold, i.e., for each $i$, there is a nilpotent normal subgroup $N_i$ of index at most $I(r, T)$ in $G_i$. The map $\phi: G \to G_1 \times \cdots \times G_k$, $\phi(g) = (\phi_1(g), \ldots, \phi_k(g))$ is an injective homomorphism, so $\phi^{-1}(\phi(G) \cap (N_1 \times \cdots \times N_k))$ is a nilpotent normal subgroup of index at most $I(r, T)^{B+2r}$ in $G$. This proves the theorem. □

References

1. Caucher Birkar, Singularities of linear systems and boundedness of Fano varieties, arXiv:1609.05543[math.AG] (2016), 1–33.
2. William M Boothby and Hsien-Chung Wang, On the finite subgroups of connected Lie groups, Commentarii Mathematici Helvetici 39 (1964), no. 1, 281–294.
3. Armand Borel, Seminar on transformation groups, no. 46, Princeton University Press, 1960.
4. Glen E. Bredon, Sheaf theory, vol. 170, Springer Science & Business Media, 2012.
5. Emmanuel Breuillard, An exposition of Jordan’s original proof of his theorem on finite subgroups of $\text{GL}_n(\mathbb{C})$, Laboratoire de Mathématiques, University Paris Sud (2011), 1–8, preprint, http://www.math.u-psud.fr/~breuilla/Jordan.pdf.
6. Balázs Csikós, Ignasi Mundet i Riera, László Pyber, and Endre Szabó, Number of stabilizers in a finite group acting on a manifold, arXiv:2111.14450 (2021).
7. Balázs Csikós, László Pyber, and Endre Szabó, Diffeomorphism groups of compact 4-manifolds are not always Jordan, arXiv:1411.7524.
8. David Fisher, Groups acting on manifolds: around the Zimmer program, Geometry, rigidity, and group actions, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 72–157. MR 2807830 (2012i:22037)
9. Étienne Ghys, The following talks: Groups of diffeomorphisms, Colóquio brasileiro de matemáticas, Rio de Janeiro (Brasil), July 1997; The structure of groups acting on manifolds, Annual meeting of the Royal Mathematical Society, Southampton (UK), March 1999; Some open problems concerning group actions, Groups acting on low dimensional manifolds, Les Diablerets (Switzerland), March 2002; Some Open problems in foliation theory, Foliations 2006, Tokyo (Japan), September 2006.
10. __________, Lecture at IMPA, Rio de Janeiro (Brasil), 2015.
11. J. D. Gillam, A note on finite metabelian $p$-groups, Proc. Amer. Math. Soc. 25 (1970), 189–190. MR 254132
12. Alexandre Grothendieck, Sur quelques points d’algèbre homologique, Tohoku Mathematical Journal, Second Series 9 (1957), no. 2, 119–183.
13. Francesco Guaraldo, On the classification of real analytic fibre bundles, Math. Z. 237 (2001), no. 3, 621–637. MR 1845342
14. Attila Gultz, Finite subgroups of the birational automorphism group are ‘almost’ nilpotent of class at most two, arXiv:2004.11715 [math.AG] (2020), 1–30.
15. Robert M. Guralnick and Martin Lorenz, Orders of finite groups of matrices, Groups, rings and algebras, Contemp. Math., vol. 420, Amer. Math. Soc., Providence, RI, 2006, pp. 141–161. MR 2279238
16. Zoltán Halasi, Károly Podoski, László Pyber, and Endre Szabó, Bounds on the rank and order of abelian subgroups in finite $p$-groups, preprint.
17. B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967. MR 0224703
18. I. Martin Isaacs, *Finite group theory*, Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, 2008. MR 2426855
19. Camille Jordan, *Mémoire sur les équations différentielles linéaires à intégrale algébriques.*, Journal für die reine und angewandte Mathematik 84 (1877), 89–215.
20. L. G. Kovács and Geoffrey R. Robinson, *Generating finite completely reducible linear groups*, Proc. Amer. Math. Soc. 112 (1991), no. 2, 357–364. MR 1047004
21. Michael J. Larsen and Richard Pink, *Finite subgroups of algebraic groups*, J. Amer. Math. Soc. 24 (2011), no. 4, 1105–1158. MR 2813339 (2012f:20148)
22. Dong Hoon Lee, *On torsion subgroups of Lie groups*, Proceedings of the American Mathematical Society 55 (1976), no. 2, 424–426.
23. Martin W. Liebeck, Nikolay Nikolov, and Aner Shalev, *Groups of Lie type as products of $\text{SL}_2$ subgroups*, J. Algebra 326 (2011), 201–207. MR 2746060
24. Alexander Lubotzky and Dan Segal, *Subgroup growth*, Progress in Mathematics, vol. 212, Birkhäuser Verlag, Basel, 2003. MR 1978431
25. L. N. Mann and J. C. Su, *Actions of elementary $p$-groups on manifolds*, Trans. Amer. Math. Soc. 106 (1963), 115–126. MR 0143840 (26 #1390)
26. Sheng Meng and De-Qi Zhang, *Jordan property for non-linear algebraic groups and projective varieties*, Amer. J. Math. 140 (2018), no. 4, 1133–1145. MR 3828043
27. Hermann Minkowski, *Zur Theorie der positiven quadratischen Formen*, J. Crelle 101 (1887), 196–202.
28. Ignasi Mundet i Riera, *Jordan’s theorem for the diffeomorphism group of some manifolds*, Proc. Amer. Math. Soc. 138 (2010), no. 6, 2253–2262. MR 2596066
29. ________, *Finite subgroups of Ham and Symp*, Math. Ann. 370 (2018), no. 1-2, 331–380. MR 3747490
30. ________, *Finite group actions on homology spheres and manifolds with nonzero Euler characteristic*, Journal of Topology 12 (2019), no. 3, 744–758.
31. Ignasi Mundet i Riera and Carles Sáez-Calvo, *Which finite groups act smoothly on a given 4-manifold?*, arXiv:1901.04223 [math.DG] (2019), 1–52.
32. Ignasi Mundet i Riera and Alexandre Turull, *Boosting an analogue of Jordan’s theorem for finite groups*, Adv. Math. 272 (2015), 820–836. MR 3303249
33. Vladimir L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, Affine Algebraic Geometry: The Russell Festschrift, CRM Proceedings and Lecture Notes 54 (2010), 289–311.
34. Vladimir L Popov, *Jordan groups and automorphism groups of algebraic varieties*, Automorphisms in Birational and Affine Geometry, Springer, 2014, pp. 185–213.
35. Yuri Prokhorov and Constantin Shramov, *Jordan property for groups of birational selfmaps*, Compos. Math. 150 (2014), no. 12, 2054–2072. MR 3292293
36. ________, *Jordan property for Cremona groups*, American Journal of Mathematics 138 (2016), no. 2, 403–418.
37. ________, *Finite groups of birational selfmaps of threefolds*, Math. Res. Lett. 25 (2018), no. 3, 957–972. MR 3847342
38. László Pyber, *Around Jordan*, Lecture at the conference ”Simple groups, representations and related topics”, Cambridge, 2015.
39. Jean-Pierre Serre, Le groupe de Cremona et ses sous-groupes finis, Séminaire Bourbaki 1000 (2008), 2008–2009.
40. Aner Shalev, On the fixity of linear groups, Proc. London Math. Soc. (3) 68 (1994), no. 2, 265–293. MR 1253505
41. D. A. Suprunenko, Matrix groups, American Mathematical Society, Providence, R.I., 1976, Translated from the Russian, Translation edited by K. A. Hirsch, Translations of Mathematical Monographs, Vol. 45. MR 0390025
42. Michio Suzuki, Group theory. I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 247, Springer-Verlag, Berlin-New York, 1982, Translated from the Japanese by the author. MR 648772
43. ________, Group theory. II, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 248, Springer-Verlag, New York, 1986, Translated from the Japanese. MR 815926
44. Dávid R. Szabó, Special p-groups acting on compact manifolds, arXiv:1901.07319v2 [math.DG] (2019), 1–11.
45. ________, Jordan type problems via class 2 nilpotent and twisted heisenberg groups, Ph.D. thesis, Central European University, 2021, http://www.etd.ceu.edu/2022/szabo_david.pdf
46. E. P. Vdovin, Regular orbits of solvable linear $p'$-groups, Sib. Élektron. Mat. Izv. 4 (2007), 345–360. MR 2465432
47. Boris Weisfeiler, Post-classification version of Jordan’s theorem on finite linear groups, Proceedings of the National Academy of Sciences 81 (1984), no. 16, 5278–5279.
48. Yuri G. Zarhin, Theta groups and products of abelian and rational varieties, Proc. Edinb. Math. Soc. (2) 57 (2014), no. 1, 299–304. MR 3165026
49. Bruno P. Zimmermann, On Jordan type bounds for finite groups acting on compact 3-manifolds, Arch. Math. (Basel) 103 (2014), no. 2, 195–200. MR 3254363

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