DE RHAM THEORY OF EXPLODED MANIFOLDS

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Abstract. This paper extends de Rham theory of smooth manifolds to exploded manifolds. Included are versions of Stokes’ theorem, De Rham cohomology, Poincare duality, and integration along the fiber. The resulting De Rham cohomology theory of exploded manifolds is used in a separate paper [4] to define Gromov Witten invariants of exploded manifolds.

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1. Introduction

The goal of this paper is to describe a version of de Rham cohomology for exploded manifolds which extends de Rham cohomology for smooth manifolds. At first glance the most natural extension of de Rham cohomology would be to take the complex of smooth or $C^\infty$ differential forms on an exploded manifold with the usual differential $d$. Unfortunately, this naive extension does not have good properties - for example, in a smooth connected family of exploded manifolds, the cohomology defined this way might change. Moreover, the tools of integration and Poincare duality are not available for this naive extension.

Instead, we shall use a sub-complex $\Omega^*(B)$ of $C^{\infty-\frac{1}{2}}$ differential forms on $B$, defined below in definition 1.2. In the case that $B$ is a smooth manifold, $\Omega^*(B)$ is the usual complex of smooth differential forms. We shall show in Corollary 11.2 that the cohomology $H^*(B)$ does not change in connected families of exploded manifolds. (This fact is nontrivial to prove because families of exploded manifolds are not always locally trivial.) As suggested by the names of the sections of this paper, many of the standard tools of de Rham cohomology still apply for $\Omega^*(B)$.

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From now on, some knowledge of the definitions and notation from [2] shall be necessary to understand this paper. Recall that coordinates on \( \mathbb{R}^n \times T_P \) are given by

\[
x_j : \mathbb{R}^n \times T_P^m \rightarrow \mathbb{R} \quad \text{for } 1 \leq j \leq n
\]

and

\[
\tilde{z}_i : T_P^m \rightarrow \mathbb{C}^* t^\mathbb{R} \quad \text{for } 1 \leq i \leq m
\]

Smooth or \( C^{\infty, 1} \) differential one forms on \( \mathbb{R}^n \times T_P^m \) are given by smooth or \( C^{\infty, 1} \) functions times \( dx_j \) and the real and imaginary parts of \( \tilde{z}_i^{-1} d\tilde{z}_i \). These differential forms are not ideal for de Rham cohomology, as even compactly supported forms may not have finite integral.

**Example 1.1** (A compactly supported form with infinite integral).

\( T^1_1 := T^1_{[0, \infty)} \) has a single coordinate

\[
\tilde{z} : T^1_1 \rightarrow \mathbb{C}^* t^{[0, \infty)}
\]

Consider the two form \( \alpha \) given by the wedge product of the real and imaginary parts of \( \tilde{z}^{-1} d\tilde{z} \). Over any tropical point \( t^a \in T^1_1 \) in the tropical part of \( T_1^1 \), there is a \( \mathbb{C}^* \) worth of points corresponding to a choice of coefficient \( c \) of \( \tilde{z} = ct^a \). On the \( \mathbb{C}^* \) worth of points over each tropical point of \( T^1_1 \), \( \alpha \) is a nonzero \( \mathbb{C}^* \) invariant volume form, so by any straightforward definition of integration, \( \alpha \) should have infinite integral. Similarly, if \( \alpha \) is multiplied by any continuous function \( f : T^1_1 \rightarrow \mathbb{R} \) which is nonzero when \( |\tilde{z}| = 0 \), the integral of \( f\alpha \) is again infinite. This is because if \( f\alpha \) is restricted to the \( \mathbb{C}^* \) worth of points over any point in \((0, \infty) \subset T^1_1 \) is a nonzero \( \mathbb{C}^* \) invariant volume form, and hence has infinite integral.

Recall that

\[
[ct^a] = \begin{cases} 
0 & \text{if } a > 0 \\
1 & \text{if } a = 0
\end{cases}
\]

and that the topology on \( T^1_1 \) is a non-Hausdorff topology in which every open subset is the pullback of some open subset in \( \mathbb{C} \) under the map \([\tilde{z}] : T^1_1 \rightarrow \mathbb{C}\). It follows that \( f\alpha \) may be compactly supported and still have infinite integral.

There are several possible fixes to this problem - we shall consider forms which do not contain the real part of \( \tilde{z}_i^{-1} d\tilde{z}_i \), where it is an obstacle to integration. In particular, we shall require that our differential forms vanish on integral vectors, which are the vectors \( v \) so that \( vf \) is an integer times \( f \) for all exploded functions \( f \). (For example, the integral vectors on \( T^1_1 \) are integers times the real part of \( \tilde{z} \frac{\partial}{\partial z} \), wherever \( |\tilde{z}| = 0 \).)

For Stokes' theorem to work out correctly, we shall also require the following condition: Given any map \( f : T^1_{[0, \infty)} \rightarrow B \), we shall assume that our differential forms vanish on all of the vectors in the image of \( df \).

As an example to see that some restriction is necessary for Stokes' theorem to hold, consider a compactly supported form \( \theta \) on \( T^1_1 \) given by the imaginary part of \( \tilde{z}_i^{-1} d\tilde{z}_i \), multiplied by a smooth, compactly supported function \( f \) which is 1 when \( |\tilde{z}| = 0 \). Then the integral of \( d\theta \) is \( 2\pi \) rather than 0.

**Definition 1.2.** Let \( \Omega^k(B) \) be the vector space of \( C^{\infty, 1} \) differential \( k \) forms \( \theta \) on \( B \) so that

1. for all integral vectors \( v \), the differential form \( \theta \) vanishes on \( v \),
2. and for all maps \( f : T^1_{[0, \infty)} \rightarrow B \), the differential form \( \theta \) vanishes on all vectors in the image of \( df \).
Denote by $\Omega^k_B \subset \Omega^k(B)$ the subspace of forms with complete support. (Say that a form has complete support if the set where it is non zero is contained inside a complete subset of $B$ - in other words, a compact subset with tropical part consisting only of complete polytopes.)

Clearly, the usual wedge product, exterior differential, and interior product with a $C^{\infty,1}$ vector field are all defined and obey the usual properties on $\Omega^*(B)$. Moreover, given any $C^{\infty,1}$ map $g : B \to C$, the pullback $g^*$ of differential forms sends forms in $\Omega^*(C)$ to forms in $\Omega^*(B)$. This is because $dg$ always sends integral vectors to integral vectors, and sends any vector in the image of $df : \mathbb{T}(T_{(0,\infty)}B) \to TB$ to a vector in the image of $d(g \circ f)$.

**Definition 1.3.** Denote the homology of $(\Omega^*(B),d)$ by $H^*(B)$, and the homology of $(\Omega^*_c(B),d)$ by $H^*_c(B)$.

We shall show in section 3 that given an assumption about the topology of $B$ akin to the existence of a finite good cover, $H^*_c(B)$ is dual to $H^*(B)$.

### 2. Mayer Vietoris sequence

Below we shall prove that the usual Mayer Vietoris sequence holds. This requires partitions of unity, which are constructed in section 10.

**Lemma 2.1 (Mayer Vietoris sequence).** Given open subsets $U$ and $V$ of an exploded manifold $B$, the Mayer Vietoris sequences

\[
0 \to \Omega^*(U \cup V) \xrightarrow{\partial_0 + \partial_1 - \partial_2} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\partial_3 + \partial_4 - \partial_5} \Omega^*(U \cap V) \to 0
\]

\[
0 \to \Omega^*_c(U \cap V) \xrightarrow{\partial_0 + \partial_1 - \partial_2} \Omega^*_c(U) \oplus \Omega^*_c(V) \xrightarrow{\partial_3 + \partial_4 - \partial_5} \Omega^*_c(U \cup V) \to 0
\]

are exact sequence of chain complexes.

**Proof:**

The proof is identical to the proof in the case of smooth manifolds given in 11. We shall discuss the first exact sequence first.

As usual in the Mayer Vietoris sequence, the first map is the direct sum of the restriction of forms from $U \cup V$ to $U$ and $V$, which is an injective chain map. Then the second map is the restriction of forms on $U$ to $U \cap V$ minus the restriction of forms from $V$ to $U \cap V$. This is a chain map, and its kernel is the forms which agree on $U \cap V$, which obviously agrees with the image of the first map. It remains to verify that this second map is surjective. Choose a partition of unity for $U \cup V$ subordinate to $U$ and $V$, so we have smooth functions $\rho_U$ and $\rho_V$ on $U \cup V$ which sum to 1 and which are supported inside $U$ and $V$ respectively. Then any form $\theta \in \Omega^*(U \cap V)$ is the image of $\rho_U \theta \oplus (\rho_V \theta)$ in $\Omega^*(U) \oplus \Omega^*(V)$.

Now for the second exact sequence. The first map is given by inclusion of completely supported forms in $U \cap V$ to completely supported forms in $U$ and $V$. This is clearly an injective chain map. The second map is given by inclusion of completely supported forms in $U$ to $U \cup V$ plus the inclusion of completely supported forms in $V$ to $U \cup V$. Again, it is clear that this is a chain map. The kernel consists of forms which cancel each other on $U \cap V$, and which are also supported in $U \cup V$. This agrees with the image of the first map. To see that the second map is surjective, suppose that $\theta \in \Omega^*_c(U \cup V)$. Then $\theta$ is the image of $\rho_U \theta \oplus \rho_V \theta \in \Omega^*_c(U) \oplus \Omega^*_c(V)$. □
3. Integration and Stokes’ theorem

We shall show below that if $B$ is oriented and $n$-dimensional, then the integral of compactly supported forms in $\Omega^*(B)$ is well defined.

**Example 3.1 ($\Omega^2(T^1_1)$).**

On $T^1_1$, the integral vectors are integer multiples of the real part of $\bar{z} \frac{\partial}{\partial \bar{z}}$ at points where $|\bar{z}| = 0$. Any two form in $\Omega^2(T^1_1)$ must therefore vanish wherever $|\bar{z}| = 0$. What remains is the subset of $T^1_1$ where $\bar{z} \in \mathbb{C}^*$. Let $[\bar{z}] = e^{r+i\theta}$ and denote the imaginary part of $\bar{z}^{-1} d\bar{z}$ by $d\theta$ and denote the real part by $dr$. If $\alpha \in \Omega^2(T^1_1)$, then

$$\alpha = f(r, \theta) dr \wedge d\theta$$

where $f$ is a smooth function of $r$ and $\theta$, and for any $\delta < 1$, the size of $f$ or any of its derivatives is bounded by $e^{\delta r}$ as $r \to -\infty$. If $\alpha$ is compactly supported on $T^1_1$, that corresponds to $f$ vanishing when $r$ is sufficiently large. (Of course, $\alpha$ being compactly supported in $T^1_1$ does not imply that $\alpha$ is compactly supported in $\mathbb{C}^* \subset T^1_1$.) The integral of $\alpha$ is finite if $\alpha$ is compactly supported in $T^1_1$ and given by

$$\int_{T^1_1} \alpha = \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(r, \theta) dr d\theta$$

Note that in general, a top dimensional form in $\Omega^*(B)$ will vanish on every strata apart from those strata of $B$ which are smooth manifolds (and therefore have no nonzero integral vectors). We can therefore define the integral of a top dimensional form on an oriented exploded manifold $B$ to be the sum of the integrals of this form over these smooth strata. This integral is well defined if the integral over each smooth strata is well defined and the sum of these integrals is well defined.

**Definition 3.2.** If $\alpha$ is a top dimensional form on an oriented exploded manifold $B$, define the integral of $\alpha$ to be the sum of the integral of $\alpha$ over all strata of $B$ which are smooth manifolds.

$$\int_B \alpha = \sum_{[B_i] = \text{point}} \int_{B_i} \alpha$$

**Lemma 3.3.** If a top dimensional form $\alpha \in \Omega^*(B)$ is compactly supported, then the integral of $\alpha$ is finite.

**Proof:**

By using a partition of unity, we may assume that $\alpha$ is compactly supported within a single coordinate chart $\mathbb{R}^n \times T_p^m$. The strata of this coordinate chart which are smooth manifolds are the strata over the (zero dimensional) corners of the polytope $P$. As $P$ will have only a finite number of such corners, we need only verify the finiteness of our integral over one strata of our coordinate chart.

We must deal with the fact that our corner of $P$ may not be standard. Pulling back $\alpha$ to a refinement of $\mathbb{R}^n \times T_p^m$ will not change the integral. We may subdivide our corner of $P$ so that the corresponding corner of each new cell has exactly $m+1$ edges. Therefore, by taking a refinement and again using a partition of unity, we may assume that a neighborhood of the corner of $P$ at our strata is isomorphic to a neighborhood of 0 in the image of some integral affine map applied to the standard quadrant $[0, \infty)^m$. It follows that our strata is contained in the image of a proper map from $\mathbb{R}^n \times T_{(0,\epsilon)^m}$ to our coordinate chart which restricted to our strata is a covering map of some positive degree.
It therefore suffices to prove the lemma for a compactly supported form $\alpha \in \Omega^{n+2m}(\mathbb{R}^n \times T^m_{[0,\epsilon]})$. Use coordinates $[\tilde{z}_k] = e^{x_k+i\theta_k}$ on our strata. Then

$$\alpha = f(x,r,\theta) \prod dx \prod dr_k \wedge d\theta_k$$

where $f$ is smooth and bounded by some constant times $e^{\frac{1}{2} \sum k r_k}$. Furthermore, on the support of $f$, $|x|$ and $r$ are bounded above. The integral of $\alpha$ on our strata is therefore finite and well defined.

Define an exploded manifold with boundary to be an abstract exploded space $M$ locally isomorphic to $(-\infty,0] \times \mathbb{R}^n \times T^m_P$. As usual, if $M$ is oriented, the boundary $\partial M$ is oriented in a way consistent with giving the boundary of $(-\infty,0] \times \mathbb{R}^n \times T^m_P$ the usual orientation on $\mathbb{R}^n \times T^m_P$, so a positively oriented top dimensional form on $\partial M$ is obtained by inserting an outward pointing normal vector into a top dimensional positively oriented form on $M$. We can now state Stokes’ theorem for exploded manifolds:

**Theorem 3.4** (Stokes’ theorem). If $M$ is an oriented exploded manifold with boundary and $\theta \in \Omega_\ast^\epsilon(M)$, then

$$\int_M d\theta = \int_{\partial M} \theta$$

**Proof:**

We shall use Stokes’ theorem for smooth manifolds. Because of the linearity of the equation we must prove, we may use a partition of unity to reduce to the case when $M$ is covered by a single coordinate chart. Consider the integral of $d\theta$ over a single strata of the coordinate chart, $M'$. We must deal with the following problem: even though $d\theta$ is compactly supported on $M$, it may not have compact support on $M'$.

The tropical part of $M'$ is a 0 dimensional corner of the polytope $M$. Identify $M$ with a polytope $P \subset [0,\infty)^m$ so that $M'$ is 0. Using the corresponding map $M \rightarrow T^m_{[0,\infty)^m}$, we may consider the coordinates $\tilde{z}_i$ on $T^m_{[0,\infty)^m}$ as coordinates on $M$. Consider the hypersurface $N_\epsilon \subset M'$ where $|\tilde{z}_1 \ldots \tilde{z}_m| = \epsilon$, oriented as the boundary of the region $M'_\epsilon$ where $|\tilde{z}_1 \ldots \tilde{z}_m| \geq \epsilon$. Our form $\theta$ is compactly supported when restricted to $M'_\epsilon$, so we can use Stokes’ theorem for manifolds.

$$\int_{M'} d\theta = \lim_{\epsilon \rightarrow 0} \int_{M'_\epsilon} d\theta = \int_{\partial M'} \theta + \lim_{\epsilon \rightarrow 0} \int_{N_\epsilon} \theta$$

Consider the integral $\int_{N_\epsilon} \theta$ as $\epsilon \rightarrow 0$. Because $\theta$ is compactly supported, it vanishes when any smooth monomial is large enough. Because $\theta$ vanishes on integral vectors and is one less than top dimensional, $\theta$ vanishes on all strata of $M$ with tropical dimension at least 2. Therefore, $\theta$ is bounded by a constant times $w_S^\delta$ for any $\delta < 1$ where $S$ consists of all strata of $M$ with dimension at least 2. (Recall from [2] that $w_S$ is a finite sum of absolute values of smooth monomials which vanish on all the strata in $S$.) Note that $w_S$ is a sum of exponentials, so the integral of $\theta$ over the regions where $w_S$ is small will also be small. It follows that in the limit $\epsilon \rightarrow 0$ the integral $\int_{N_\epsilon} \theta$ is concentrated in the directions corresponding to the edges of $M$ attached to our strata, and that

$$\lim_{\epsilon \rightarrow 0} \int_{N_\epsilon} \theta = \sum_{\text{edges } e} \int_{N_e} \theta$$

where $N_e$ is an appropriately oriented hypersurface where $|\tilde{z}_1 \ldots \tilde{z}_m|$ is constant in the strata corresponding to an edge $e$ attached to our corner. (The sum is over
all these edges.) Because \( \theta \) vanishes on integral vectors, and is constant in the direction of integral vectors, the integral of \( \theta \) over \( N_e \) is equal to the integral of \( \theta \) over any appropriately oriented hypersurface in this strata given by \( \{ |\tilde{z}^\alpha| = c \} \), so long as the integral vectors in this strata are transverse to this hypersurface. The fact that the support of \( \theta \) is complete implies that if \( \theta \) does not vanish on \( N_e \), there must be a corner of \( M \) at the other end of the edge \( e \). The same calculation of the integral of \( d\theta \) for this other corner will yield a contribution of the integral of \( \theta \) over \( N_e \) with the opposite orientation, so all the contributions from \( \int_{N_e} \theta \) cancel, and we obtain that

\[
\int_M d\theta = \int_{\partial M} \theta
\]
as required.

\[\square\]

4. Cohomology of a coordinate chart

In this section, we calculate \( H^* \) for all standard coordinate charts and \( H^*_c \) for coordinate charts \( \mathbb{R}^n \times T^m_P \) for which \( P \) is complete.

**Lemma 4.1.** Let \( P \subset \mathbb{R}^m \) be an integral affine polytope. Suppose that the directions of the infinite rays in \( P \) span the last \( k \) coordinate directions in \( \mathbb{R}^m \). Then \( H^*(\mathbb{R}^n \times T^m_P) \) is equal to the free exterior algebra generated by the imaginary parts of \( \tilde{z}^{-1}_i d\tilde{z}_i \) for \( i = 1, \ldots, m-k \).

**Proof:**

Note that if \( M \) is a smooth manifold, \( H^*(M) \) is the usual De Rham cohomology of \( M \). There is an obvious map of a torus \( T^m \rightarrow \mathbb{R}^n \times T^m_P \) which pulls back the above forms to non trivial homology classes in \( H^*(T^m) \), so \( H^*(\mathbb{R}^n \times T^m_P) \) must contain a copy of the free exterior algebra generated by the above differential forms.

We shall prove that each class in \( H^*(\mathbb{R}^n \times T^m_P) \) may be represented by a differential form which is constant in standard coordinates and contains no \( dx_i \) factors. Our proof shall follow the proof of the Poincare lemma in [1].

Consider the map \( K : \Omega^* \rightarrow \Omega^{*-1} \) given by

\[
K(\theta)(x_1, x_2, \ldots) = \int_0^{x_1} \frac{i}{\sqrt{1}} \theta(s, x_2, \ldots) ds
\]

then

\[
dK(\theta) = (1 - dx_1 \wedge i \frac{\partial}{\partial x_1}) \int_0^{x_1} \frac{i}{\sqrt{1}} \theta(s, x_2, \ldots) ds + dx_1 \wedge i \frac{\partial}{\partial x_1} \theta
\]

so

\[
(Kd + dK)(\theta) = \int_0^{x_1} \left( i \frac{\partial}{\partial x_1} d + (1 - dx_1 \wedge i \frac{\partial}{\partial x_1}) dx_1 \wedge i \frac{\partial}{\partial x_1} \right) \theta(s, x_2, \ldots) ds + dx_1 \wedge i \frac{\partial}{\partial x_1} \theta
\]

Suppose that \( i \frac{\partial}{\partial x_1} \theta = 0 \), then

\[
(Kd + dK)(\theta) = \int_0^{x_1} L_{\frac{\partial}{\partial x_1}} \theta(s, x_2, \ldots) ds
\]

Suppose that \( \theta = dx_1 \wedge \alpha \) where \( i \frac{\partial}{\partial x_1} \alpha = 0 \). Then

\[
(Kd + dK)dx_1 \wedge \alpha = \int_0^{x_1} 0 ds + dx_1 \wedge \alpha
\]

Therefore, in general

\[
(Kd + dK)\theta = \theta - (1 - dx_1 \wedge i \frac{\partial}{\partial x_1}) \theta(0, x_2, \ldots)
\]
It follows that we can represent the cohomology class of any closed form \( \theta \) with the closed form \((1 - dx_1 \wedge i \frac{\partial}{\partial x_1}) \theta(0, x_2, \ldots) \) which is independent of \( x_1 \) and \( dx_1 \).

Similarly, we may represent any class in \( H^*(\mathbb{R}^n \times T^p_m \mathbb{P}) \) by a form pulled back from \( T^p_m \mathbb{P} \) under the obvious projection map.

Now we have reduced to the case of differential forms on \( T^p_m \mathbb{P} \). The standard basis for differential forms on \( T^p_m \mathbb{P} \) is given by the exterior algebra generated by the real and imaginary parts of \( \tilde{z}^{-1} d\tilde{z} \), so we can consider forms in \( \Omega^*(T^p_m \mathbb{P}) \) as maps from \( T^p_m \mathbb{P} \) to \( \mathbb{R}^{2m} \). (Of course, not all \( \mathcal{C}^\infty_0 \), maps to \( \mathbb{R}^{2m} \) will correspond to forms in \( \Omega^* \) because of the condition that forms in \( \Omega^* \) vanish on integral vectors.) We wish to show that any cohomology class can be represented by a form which is constant in this basis. In particular, if \( P^o \) is the interior strata of \( P \), then we shall show that a closed form \( \theta \) represents the same cohomology class as \( e_{P^o} \theta \). (Using notation from the definition of \( \mathcal{C}^\infty_0 \) in [2].)

For each strata of \( P \), choose an integral vector \( \alpha \) pointing towards the interior of \( P \). Choose these vectors consistently so that the vectors for adjacent strata differ only by a vector contained within one of the strata.

Then on each strata, consider the vector field

\[
v := \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial r_i}\]

where \( \frac{\partial}{\partial r_i} \) is the real part of \( \tilde{z}_i \frac{\partial}{\partial \tilde{z}_i} \). This vector field \( v \) is not a globally defined vectorfield on \( T^p_m \mathbb{P} \) because it may change by an integral vector from one strata of \( T^p_m \mathbb{P} \) to the next. We can think of \( v \) as a ‘vectorfield defined up to integral vectors’. As differential forms \( \theta \in \Omega^*(T^p_m \mathbb{P}) \) always vanish on integral vectors, \( i_v \theta \) is still a well defined form in \( \Omega^*(T^p_m \mathbb{P}) \) even though \( v \) might jump by an integral vectorfield when changing from strata to strata. Let \( \Phi_{tv} \) be the flow of the vectorfield \( v \) for time \( t \) on each strata. We will not be overly worried by the fact that \( \Phi_{tv} \) does not give a globally defined map from \( T^p_m \mathbb{P} \) to itself. Note that the flow of any integral vectorfield preserves any \( C^\infty_0 \)-differential form, therefore the ambiguity in the definition of \( v \) does not affect how forms are changed by the flow of \( v \). Therefore, if \( \theta \in \Omega^*(T^p_m) \), then \( \Phi_{tv}^* \theta \in \Omega^*(T^p_m) \) for all \( t \).

Given \( \theta \in \Omega^*(T^p_m) \) define

\[
K \theta := \int_{-\infty}^{0} \Phi_{tv}^* i_v \theta dt
\]

Now check that \( K \theta \in \Omega^* \). Note that \( i_v \theta \) vanishes on \( T^p_{P^o} \subset T^p_m \mathbb{P} \), and \( \Phi_{tv} \) travels towards this central strata as \( t \to -\infty \). The fact that \( i_v \theta \) vanishes on the central strata and is \( C^\infty_0 \) implies that restricted to any compact subset, there exists a
constant \( e \) so that
\[ |\Phi_{i_v}^* \epsilon \theta| < ce^{-\frac{1}{2}t} \]
and similar estimates hold for any derivative of \( \theta \). Therefore, on each strata \( K\theta \) is well defined and smooth. It is clear that \( K\theta \) vanishes on all vectors that forms in \( \Omega^* \) should vanish on, so it remains to check that \( K\theta \) is \( C^\infty \) everywhere. At this stage, the reader must be familiar with the section defining \( C^\infty \) in [2].

Note that for any strata \( S \), \( e_S \Phi_{i_v}^* \epsilon \theta = \Phi_{i_v}^* e_S \epsilon \theta \). Therefore for any collection of strata \( S \), \( \Delta_S K\theta = K\Delta_S \theta \). We must prove that \( w_S^{-\delta} \Delta_S K\theta \) is bounded on any compact subset for any \( \delta < 1 \) (and we must prove a similar estimate for any derivative of \( \theta \)). For any \( 0 < \epsilon < 1 - \delta \), \( \Delta_S \theta \) is bounded by a constant times \( w_S^{\epsilon} \) on any compact subset. The weight \( w_S \) is a sum of absolute values of smooth monomials which vanish on \( P^n \), so \( \Phi_{i_v}^* w_S \) is bounded on compact subsets by a constant times \( c' \) for \( t < 0 \). It follows that \( \Phi_{i_v}^* \Delta_S \epsilon \theta \) is bounded by a constant times \( w_S^{\delta} c't \) on compact subsets for \( t < 0 \). Integrating this gives that \( w_S^{-\delta} \Delta_S K\theta \) is bounded on any compact subset. For any constant vectorfield \( X \), note that \( L_X K\theta = KL_X \theta \), so the bounds for the derivatives of \( \theta \) follow from the same argument, and \( K\theta \in \Omega^*(T^n_{\mathbb{P}}) \).

Now consider
\[ (Kd + dK)\theta = \int_{-\infty}^0 \Phi_{i_v}(i_v d + di_v)\theta dt = \theta - \lim_{t \to -\infty} \Phi_{i_v}^* \theta - \epsilon P - \theta \]
It follows that if \( \theta \in \Omega^*(T^n_{\mathbb{P}}) \) is any closed differential form, we may represent the same cohomology class by the constant differential form \( \epsilon P - \theta \). The lemma follows as the only constant differential forms in \( \Omega^*(T^n_{\mathbb{P}}) \) are the forms mentioned in the statement of the lemma.

Note that \((T^1)^n_m\) has the same cohomology as \( \mathbb{C}^m \). Using a good cover (constructed in Lemma 10.2), we may use this to prove that the cohomology of the explosion of any compact complex manifold relative to a normal crossing divisor is equal to the cohomology of the original manifold.

**Corollary 4.2.** If \( M \) is a compact complex manifold with a normal crossing divisor, then the smooth part map
\[ \text{Expl} M \xrightarrow{\mathbb{1}} M \]
induces an isomorphism on cohomology
\[ [-]^* : H^*(M) \xrightarrow{\cong} H^*(\text{Expl} M) \]
More generally, if \( B \) has a finite good cover in the sense of lemma 10.3, and all polytopes in the tropical part of \( P \) are quadrants \([0, \infty)^m\) then \([ - ]^* \) is an isomorphism on cohomology.
\[ [-]^* : H^*([B]) \xrightarrow{\cong} H^*(B) \]

**Proof:**

Choose a finite good cover \( \{U_i\} \) of \( B \) in the sense of Lemma 10.2, so the intersection of any number of these \( U_i \) is either empty or isomorphic to \( \mathbb{R}^n \times T^n_{\mathbb{P}} \). By assumption the only possible polytopes \( P \) are quadrants \([0, \infty)^n\), Lemma 4.1 tells us that \( H^*(\mathbb{R}^n \times (T^1)^n_m) \) is generated by the constant functions. The smooth part of \( \mathbb{R}^n \times (T^1)^m \) is \( \mathbb{R}^n \times \mathbb{C}^m \), so the smooth part map \([ - ] \) induces an isomorphism on cohomology:
\[ [-]^* : H^*(\mathbb{R}^n \times \mathbb{C}^m) \xrightarrow{\cong} H^*(\mathbb{R}^n \times (T^1)^m) \]
Therefore, if $B$ has a good cover by a single open set, our lemma holds. We may now proceed by induction over the cardinality of a good cover using the Mayer Vietoris sequence from Lemma 2.1.

Suppose that our lemma holds for all exploded manifolds satisfying our tropical part assumption with a good cover containing at most $k$ sets. Then suppose that $B$ has a good cover $\{U, V_1, \ldots, V_k\}$. Let $V = \bigcup_{i=1}^k V_i$. Then our lemma holds for $U$, $V$ and $U \cap V$. Then the smooth part map gives the following commutative diagram involving Mayer Vietoris sequences

\[
\begin{array}{cccc}
\Omega^* (U \cup V) & \rightarrow & \Omega^* (U) \oplus \Omega^* (V) & \rightarrow & \Omega^* (U \cap V) \\
\uparrow \ [.]^* & & \uparrow \ [.]^* \oplus \ [.]^* & & \uparrow \ [.]^* \\
\Omega^* ([U] \cup [V]) & \rightarrow & \Omega^* ([U]) \oplus \Omega^* ([V]) & \rightarrow & \Omega^* ([U] \cap [V])
\end{array}
\]

Considering the induced maps on the homology long exact sequence and using the five lemma then implies that

\[\ [\cdot]^* : H^*([U] \cup [V]) \rightarrow H^*(U \cup V)\]

is an isomorphism. By induction, our lemma must hold for $B$ so long as $B$ has a finite good cover and the tropical part of $B$ contains only quadrants. The tropical part of the Expl $M$ contains only quadrants, and Lemma 10.2 implies that if $M$ is compact, Expl $M$ has a finite good cover, so our lemma also holds for Expl $M$.

Note that it is not true in general that $[B]$ has the same cohomology as $B$. For example, $T_{[0,1]}$ has the same cohomology as $\mathbb{C}^*$, but the smooth part of $T_{[0,1]}$ is two copies of $\mathbb{C}$ glued at $0$.

Stokes’ theorem implies that $(\alpha, \theta) \mapsto \int_B \alpha \wedge \theta$ gives a bilinear pairing $H^k_b(B) \times H^{n-k}(B) \rightarrow \mathbb{R}$ where the dimension of $B$ is $n$. We shall now start to prove that in many situations, this pairing is non-degenerate, so Poincaré duality holds.

Lemma 4.3 below computes $H^*_c(\mathbb{R}^n \times T^n_P)$ for polytopes $P$ which are complete and contain no entire lines. Note that computation of $H^*_c$ in the case where $P$ is a complete polytope which may contain entire lines follows, because there exists an obvious projection map $\pi : \mathbb{R}^n \times T^n_P \rightarrow \mathbb{R}^n \times T^n_{P'}$ so that $P'$ is complete and contains no lines, and so that $\pi^*$ is a bijection on both $\Omega^*$ and $\Omega^*_c$.

**Lemma 4.3.** If $P$ is a complete polytope which contains no entire lines, then the integration pairing

\[(\alpha, \theta) \mapsto \int_{\mathbb{R}^n \times T^n_P} \alpha \wedge \theta\]

gives an identification of $H^*_c(\mathbb{R}^n \times T^n_P)$ with the dual of $H^*(\mathbb{R}^n \times T^n_P)$.

**Proof:**

Choose a basis $\{ \zeta_v := [e^a \bar{z}^{\bar{a}}] \}$ for the smooth monomials on $T^n_P$. Then consider the differential form

\[d \left( \sum_{i=1}^n \frac{1}{2} |x_i|^2 + \sum_v |\zeta_v|^2 \right)\]

as giving a smooth map

\[f : \mathbb{R}^n \times T^n_P \rightarrow \mathbb{R}^{n+m}\]

by obtaining the first $n$ components of $f$ by inserting $\frac{\partial}{\partial x_i}$ and the last $m$ components by inserting the real part of $\bar{z} \frac{\partial}{\partial z}$. We shall check below that this map $f$ is proper, and the image of a strata $S$ with $k$ dimensional tropical part is a cone $C_S$.
in $\mathbb{R}^{n+m}$ with codimension $k$. (Throughout this proof, we shall use $S$ to refer to both a strata of $\mathbb{R}^n \times T^m_P$, and its tropical part which is a strata of the polytope $P$.)

\[ P \quad \text{image of } f \]

By averaging we may represent any class in $H^*_c$ by a differential form which is preserved by the vector fields given by the imaginary part of $\tilde{z}_i \frac{\partial}{\partial \tilde{z}_i}$. Any such closed differential form breaks up into a sum of closed differential forms in the form of $\alpha \wedge \beta$ where $\alpha$ is closed and vanishes on the imaginary part of $\tilde{z}_i \frac{\partial}{\partial \tilde{z}_i}$ and $\beta$ is some product of the imaginary parts of $\tilde{z}_i^{-1}d\tilde{z}_i$. Our goal below shall be to show that the cohomology class of $\alpha$ may be represented by $f^*\alpha'$. We shall then modify $\alpha$ using knowledge of $H^*_c(\mathbb{R}^{n+m})$.

Let us examine the map $f$.

\[ f = \text{id}_{\mathbb{R}^n} + \sum_v 2|\zeta_v|^2 v \]

In the above, identify $v \in \mathbb{Z}^m$ with the corresponding vector in $0 \times \mathbb{R}^m \subset \mathbb{R}^n \times \mathbb{R}^m$. This formula implies that $f$ is proper and that the image of $f$ restricted to a particular strata is contained inside the cone $C_S$ which is $\mathbb{R}^n$ times the positive span of all $v$ so that $\zeta_v$ is nonzero on our strata. Taking the derivative of $f$ gives

\[ Df = \text{id}_{\mathbb{R}^n} + \sum_v 4|\zeta_v|^2 |v|^2 \pi_v \]

Where

\[ |v|^2 \pi_v = \left( \sum_{i=1}^m v_i dr_i \right) v \]

and $dr_i$ indicates the real part of $\tilde{z}_i^{-1}\tilde{z}_i$. If we regard the vector space with basis $\left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial r_i} \right)$ as $\mathbb{R}^{n+m}$, then $\pi_v$ indicates orthogonal projection onto the subspace spanned by $v$. Regarded this way, $Df$ is a symmetric, positive semidefinite matrix. It follows from this formula that on any particular strata $S$, $Df$ is surjective onto the tangent space to the cone $C_S$. Combined with the properness of $f$, this implies that the image of $f$ restricted to $S$ is the interior of this cone $C_S$.

Note that our differential form $\alpha$ restricted to any strata $S$ must be equal to the pullback under $f$ of some differential form $\alpha'$, which is a smooth differential form on the interior of $C_S$. In general, it will not be true that $\alpha'$ comes from a smooth differential form on $\mathbb{R}^{n+m}$ - for that we shall need to modify $\alpha$.

Consider the operator $K$ defined in equation \[1\] in the proof of \[4.1\]. Choose some compactly supported smooth function $\rho$ on $T^m_P$ which is 1 in a neighborhood of the interior strata $P^0$ of $T^m_P$, and modify $\alpha$ to the form $\alpha + d\rho K\alpha$. This modified form
(which we shall again call $\alpha$) is still compactly supported, but has the property that in a neighborhood of this interior strata, $\alpha$ is the pullback of some smooth form via the composition of the map $f$ with the orthogonal projection to $C_{P^*}$.

Suppose that for all strata $S$ of $P$ with dimension greater than $k$, there exists a neighborhood of $S$ on which $\alpha$ is the pullback of a smooth form under the composition of $f$ with the orthogonal projection to $C_S$. We shall now modify $\alpha$ so that the same holds for the strata $S$ with dimension $k$. Let $F$ indicate the smallest face of $P$ which contains $S$ (in other words $F$ is the closure of $S \subseteq P$). Using the implicit function theorem for exploded manifolds proved in [2], we may identify a neighborhood of our strata $S$ with $C_S$.)

We may therefore modify $\alpha$ so that each strata $S$ has a neighborhood so that $\alpha$ is the pullback of a smooth form under $f$ composed with orthogonal projection to $C_S$. It follows that this modified form $\alpha$ is $f^*\alpha'$ for some smooth closed form $\alpha$ on $R^n$.

Use $d\theta_i$ to indicate the imaginary part of $\bar{z}_i^{-1}dz_i$. We now have that $H^*_c$ is generated by differential forms $f^*\alpha' \wedge \beta$ where $\beta$ is some product of the $d\theta_i$. Choose some standard form $\alpha_0$ with integral $1$ which is compactly supported in the interior of the image of $f$. We shall now show that we may exchange $\alpha'$ for $(f\alpha')\alpha_0$. Assume that the span of the unbounded directions in $P$ is the plane given by the first $k$ coordinate directions. If $\beta$ contains $d\theta_1 \cdots d\theta_k$, then $\alpha'$ must vanish on the boundary of $\bigcup S C_S$, therefore $\alpha'$ is compactly supported inside the interior of $\bigcup S C_S$ (which is diffeomorphic to $R^{n+m}$). Therefore

$$\alpha' - d\gamma = \alpha_0 \int \alpha'$$

where $\gamma$ is compactly supported inside the interior of $\bigcup S C_S$. As $f^*\gamma \wedge \beta \in \Omega^*_c$, the modified form $(f\alpha')f^*\alpha \wedge \beta$ represents the same class in $H^*_c$ as $\alpha \wedge \beta$.

Suppose that some unbounded strata $S$ of $P$ is contained in the first coordinate plane. If $\beta$ does not contain $d\theta_1$, then $\alpha'$ is not required to vanish on $C_S$, which is a codimension one face of the image of $f$. In this case, we may choose a compactly supported form $\gamma$ which vanishes on the cones which $\alpha$ is required to vanish on, and for which $d\gamma = \alpha$ on the image of $f$. As $f^*\gamma \wedge \beta \in \Omega^*_c$, we have that in this case $\alpha \wedge \beta$ represents the zero cohomology class in $H^*_c$. Similarly, if $\beta$ does not contain $d\theta_1 \cdots d\theta_k$, $\beta$ is a sum of forms which vanish in a similar fashion on some unbounded dimension $1$ strata, so $\alpha \wedge \beta = 0 \in H^*_c$.

In conclusion, we have shown that $H^*_c$ is generated by forms

$$f^*\alpha_0 \wedge d\theta_1 \cdots d\theta_k \wedge \beta$$

where $\beta$ is some product of $d\theta_i$ for $k < i \leq m$. Lemma [1.1] showed that $H^*$ is generated as an exterior algebra by $d\theta_i$ for $k < i \leq m$. The integration pairing on our space of differential forms times $H^*$ is therefore nondegenerate, therefore all the above forms represent independent classes in $H^*_c$, and the integration pairing identifies $H^*_c$ with the dual of $H^*$ as required.

□
5. Poincare duality

**Theorem 5.1 (Poincare duality).** If $B$ is a complete oriented exploded manifold so that each map $T \rightarrow B$ is constant, then the integration pairing gives an isomorphism between $H^*(B)$ and its dual.

More generally, if $B$ has a finite good cover in the sense of Lemma 10.2 and each polytope in the tropical part of $B$ is complete and contains no entire lines, then the integration pairing identifies $H^*(B)$ with the dual of $H^*_c(B)$.

**Proof:**

We shall use Lemma 10.2 which states that any complete exploded manifold $B$ must have a finite good cover by open sets so that any intersection is isomorphic to a standard coordinate chart $\mathbb{R}^n \times T^n_B$. The condition that each map $T \rightarrow B$ is constant implies that the polytope $P$ contains no entire lines, so we may apply the result of Lemma 5.3 to know that the integration pairing identifies the dual of $H^*(\mathbb{R}^n \times T^n_B)$ with $H^*_c(\mathbb{R}^n \times T^n_B)$. The proof may now proceed as in the case of smooth manifolds by induction over the size of our open cover using the Mayer Vietoris sequences from Lemma 2.1.

In particular, suppose that the dimension of $B$ is $n$. Define the differential $d' = (-1)^{n+1-k}d$ on $\Omega^k_c$. The Mayer Vietoris sequence obviously is still exact for this new differential, and the homology of $(\Omega^*_c, d')$ is obviously the same as the homology of $(\Omega^*_c, d)$. This sign modification allows the following formula:

$$\int_B (da) \wedge \beta = \int_B \alpha \wedge d' \beta$$

Let $C_\ast$ denote the dual chain complex to $(\Omega^*_c, d')$. The above formula implies that the integration pairing on any oriented $n$ dimensional manifold gives a chain map $\Omega^* \rightarrow C_\ast$. We shall now verify that the corresponding map between Mayer Vietoris sequences is commutative:

$$\begin{array}{cccc}
0 & \rightarrow & \Omega^*(U \cup V) & \rightarrow & \Omega^*(U) \oplus \Omega^*(V) & \rightarrow & \Omega^*(U \cap V) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & C_\ast(U \cup V) & \rightarrow & C_\ast(U) \oplus C_\ast(V) & \rightarrow & C_\ast(U \cap V) & \rightarrow & 0 \\
\end{array}$$

Following $\alpha \in \Omega^*(U \cup V)$ across and down, then evaluating on $\beta_1 \oplus \beta_2 \in \Omega^*_c(U) \oplus \Omega^*_c(V)$ gives

$$\int_U \alpha \wedge \beta_1 + \int_V \alpha \wedge \beta_2$$

Following $\alpha$ down and across, then evaluating on $\beta_1 \oplus \beta_2$ gives

$$\int_{U \cup V} \alpha \wedge (\beta_1 + \beta_2)$$

Therefore the first square commutes. The commutativity of the second square amounts to the equation

$$\int_{U \cap V} (\alpha_1 - \alpha_2) \wedge \beta = \int_U \alpha_1 \wedge \beta + \int_V \alpha_2 \wedge (-\beta)$$

where $\beta$ is compactly supported in $U \cap V$.

Therefore, taking homology gives a commutative diagram

$$\begin{array}{cccc}
H^{k-1}(U \coprod V) & \rightarrow & H^{k-1}(U \cap V) & \rightarrow & H^k(U \cup V) & \rightarrow & H^k(U \cap V) & \rightarrow & H^k(U \cap V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{n+1-k}(U \coprod V) & \rightarrow & H^{n+1-k}(U \cap V) & \rightarrow & H^{n-k}(U \cup V) & \rightarrow & H^{n-k}(U \cap V) & \rightarrow & H^{n-k}(U \cap V) \\
\end{array}$$

The downward arrows above are given by the integration pairing. Say that Poincare duality holds if this integration pairing map is an isomorphism. The Five Lemma
implies that if Poincare duality holds on \( U, V \) and \( U \cap V \), then Poincare duality holds on \( U \cup V \).

Suppose that Poincare duality holds on all oriented exploded manifolds satisfying our assumptions on their tropical part and having a good cover containing at most \( k \) members. Suppose that \( B \) satisfies the tropical part assumptions and has a good cover \( \{U_1, \ldots, U_{k+1}\} \). Then Poincare duality must hold for \( U_{k+1}, \bigcup_{i=1}^{k} U_i \) and \( U_{k+1} \cap \bigcup_{i=1}^{k} U_i \). The above argument then gives that Poincare duality must hold for \( B \). By induction starting with Lemma 4.3, Poincare duality must hold for all oriented exploded manifolds that admit a finite good cover and which have a tropical part containing only complete polytopes that contain no lines. Lemma 10.2 states that complete exploded manifolds have a finite good cover, and our theorem follows.

□

Note that Poincare duality as stated in Theorem 5.1 does not imply the usual relationship between intersections of submanifolds and wedge products of Poincare duals. We will explore this relationship more when we return to Poincare duality in section 8.

6. Integration along the fiber

In this section, we define integration along the fibers \( f_i \) for suitable maps \( f \).

Given a \( C^{\infty,1} \) map \( f : A \rightarrow B \), and a compactly supported differential form \( \theta \) on \( A \), we may regard \( \theta \) as a current (something dual to the space of differential forms), then push forward this current to obtain a current on \( B \). If this current on \( B \) is also represented by a differential form, we call this differential form \( f_!(\theta) \). In particular, when it exists, \( f_i \theta \) has the property that for all differential forms \( \alpha \) on \( B \),

\[
\int_B \alpha \wedge f_! \theta = \int_A (f^* \alpha) \wedge \theta
\]

In the case of smooth manifolds, \( f_i \) exists if \( f \) is a submersion. In our case, we must be careful that \( df \) restricted to the subspace spanned by integral vectors is also surjective.

**Theorem 6.1.** Let \( A \) and \( B \) be oriented exploded manifolds, and suppose that \( f : A \rightarrow B \) is a \( C^{\infty,1} \) map which satisfies

(1) \( f \) is a submersion in the sense that

\[
df : T_x A \rightarrow T_{f(x)} B
\]

is surjective

(2) \( df : \mathbb{Z} T_x A \rightarrow \mathbb{Z} T_{f(x)} B \)

is a surjective map on integral vectors

Then if the fiber of \( f \) is \( n \) dimensional, there exists a linear chain map \( f_i : \Omega^*_c(A) \rightarrow \Omega^*_{c-i}(B) \), with the property that

\[
\int_B \alpha \wedge f_i \theta = \int_A f^* \alpha \wedge \theta
\]

for all \( \alpha \in \Omega^*_c(B) \).

If all polytopes \( P \) in the tropical part of \( B \) are complete and contain no entire lines, then \( f_i(\theta) \) is uniquely determined by the above property.
Proof:
The discussion on fiber products in [2] implies that each fiber of \( f \) is a \( C^{\infty, 1}_\mathbb{A} \) exploded manifold. The top wedge of the cotangent space of the fibers is a \( C^{\infty, 1}_\mathbb{A} \) vector bundle over \( \mathbb{A} \), and the fact that \( f \) is a submersion implies that the pullback of \( \bigwedge T^*\mathbb{B} \) is a vector bundle over \( \mathbb{A} \), therefore the tensor of these two bundles is a \( C^{\infty, 1}_\mathbb{A} \) vector bundle \( E \) over \( \mathbb{A} \). From \( \theta \in \Omega^*_\mathbb{A}(\mathbb{A}) \), we can associate a \( C^{\infty, 1}_\mathbb{A} \) section \( \theta' \) of \( E \) as follows: inserting a top dimensional polyvector \( v \) tangent to the fiber of \( f \) into the righthand places of \( \theta \) gives a form \( \theta \wedge v \) which vanishes on the kernel of \( df \). Therefore \( \theta \wedge v \) must be a section of the pullback of \( \bigwedge T^*\mathbb{B} \). The we may define \( \theta' \) as the unique section of \( E \) so that \( \theta'(v) = \theta \wedge v \). It is obvious that this definition of \( \theta' \) does not actually depend on the choice of top dimensional polyvector \( v \). As \( \theta \) has complete support and vanishes on all the relevant vectors, \( \theta' \) restricted to the fiber \( f^{-1}(p) \) is in \( \Omega^*_\mathbb{B}(f^{-1}(p)) \otimes \bigwedge T^*\mathbb{B} \). Orient the fibers of \( f \) so that if \( \alpha \) is a volume form on \( \mathbb{B} \) and \( \beta \) is a volume form on the fibers of \( f \), then \( \ast \alpha \wedge \beta \) is a volume form on \( \mathbb{A} \).

We may therefore integrate \( \theta' \) along the fiber \( f^{-1}(p) \) to obtain a form \( f_\ast(\theta)(p) \in \bigwedge T^*\mathbb{B} \). We must now verify that \( f_\ast(\theta) \) defined this way is in \( \Omega^*_\mathbb{B}(\mathbb{B}) \), and verify that it satisfies our defining property.

As \( f \) is a submersion, any \( C^{\infty, 1}_\mathbb{A} \) vector field \( v \) on \( \mathbb{B} \) may be lifted to a \( C^{\infty, 1}_\mathbb{A} \) vectorfield \( \tilde{v} \) on \( \mathbb{A} \) so that \( df(\tilde{v}) = v \). Let \( \Phi_{\tilde{v}} \) indicate the flow of the vectorfield \( \tilde{v} \) on \( \mathbb{A} \) and \( \Phi_v \) indicate the flow of the vectorfield \( v \) on \( \mathbb{B} \). We have that \( f \circ \Phi_{\tilde{v}} = \Phi_v \circ f \) and \( f_\ast \Phi_{\tilde{v}} = \Phi_v \circ f_\ast \). As the map \( f_\ast \) is linear, and \( \Phi_v \circ f_\ast(\theta) \) is differentiable in \( t \), \( f_\ast \Phi_{\tilde{v}} \circ \theta \) is also differentiable in \( t \) and

\[
f_\ast L_{\tilde{v}}(\theta) = L_v f_\ast(\theta)
\]

Given a vector field \( v \) on \( \mathbb{B} \), note that that the section \( (i_v \theta)' \) of our bundle \( E \) does not depend on the choice of lift \( \tilde{v} \). If at \( p \), \( v \) is an integral vectorfield, then the second assumption on \( f \) implies that given any point \( q \in f^{-1}(p) \), we may choose our lift \( \tilde{v} \) so that \( \tilde{v} \) is a linear combination of integral vectorfields at \( q \), therefore \( (i_v \theta)' \) vanishes around \( q \). Therefore \( (i_v \theta)' \) vanishes on \( f^{-1}(p) \), so \( f_\ast(\theta) \) vanishes on integral vectors. Similarly, given any map of \( T^1_{(0, \infty)} \mathbb{A} \) passing through \( p = f(q) \), the second assumption above on \( f \) implies that this map may be covered by a map \( g \) of \( T^1_{(0, \infty)} \mathbb{A} \) composed with \( f \) so that the image of \( g \) contains \( q \). It follows that if \( v \) is in the image of such a map, \( (i_v \theta)' \) must vanish. Therefore \( f_\ast(\theta) \) must vanish on the tangent space of the image of any map from \( T^1_{(0, \infty)} \mathbb{A} \). Therefore, \( f_\ast(\theta) \) vanishes on all the vectors which it should vanish on.

As the image of any complete set is complete, \( f_\ast(\theta) \) has complete support. Therefore, to check that \( f_\ast(\theta) \in \Omega^*_\mathbb{B}(\mathbb{B}) \), it remains to check that \( f_\ast(\theta) \) is \( C^{\infty, 1}_\mathbb{A} \). To do this, we work locally in a single coordinate chart \( U' \) on \( \mathbb{A} \) and \( U \) on \( \mathbb{B} \). Our assumptions on \( f \) imply that the image of every strata of \( U' \) under \( f \) is a strata of \( U \). By modifying our chart \( U' \) using the implicit function if necessary, we may assume that the pullback of monomial functions from \( U \) are monomial functions on \( U' \). It follows that given any strata \( S \) of \( U' \) and \( C^{\infty, 1}_\mathbb{A} \) function \( g \) on \( \mathbb{B} \), \( e_\ast f_\ast g = e_\ast L_S g \). Note that \( f_\ast(\theta) \) depends only on position in \( [U] \) - if \( p \) and \( p' \) have the same image in \([U]\), then the integrals used to compute \( f_\ast(\theta) \) are the same on the fiber over \( p \) and \( p' \). It follows that \( e_\ast f_\ast(\theta) \) makes sense.

Given any set \( S \) of strata of \( U' \), let \( S' \) be the set of strata of \( U' \) sent to \( S \) by \( f \). Let \( S' \) be a single strata of \( U' \). Then

\[
e_{S'} f_\ast(\theta) = \sum_{T \in S'} f_\ast(e_T \theta)
\]
If \( T_1 \) and \( T_2 \) are two distinct strata in \( S' \), then \( e_{T_1} e_{T_2} \theta' = 0 \), so
\[
\Delta_S f \theta = f_l (\Delta_S \theta)
\]
It follows that for any set \( S \) of strata of \( U \),
\[
\Delta_S f \theta = f_l (\Delta_S \theta)
\]
As \( df \) is surjective on integer vectors, every smooth monomial which vanishes on all strata in \( S' \) is divisible by the pullback of a smooth monomial vanishing on all strata in \( S \). Therefore, we may choose \( w_{S'} = f^* w_{S} \). It follows that
\[
\Delta_{S'} f \theta = f_l (\Delta_{S'} \theta)
\]
As we already have that \( L_{v_l f} \theta = f_l L_{v_{f^* \theta}} \), it follows that \( f \theta \in C^\infty (B) \) if \( \theta \in \Omega^*_c (a) \).

We have defined our map \( f_l \) in the same way as the integration over fibers map for smooth manifolds with the sign convention chosen so that
\[
\int_A f^* \alpha \wedge \theta = \int_B \alpha \wedge f_l \theta
\]
The above formula uniquely characterizes \( f_l \) in the case that \( A \) and \( B \) are smooth manifolds. A quick calculation using Stokes theorem gives that
\[
\int_A f^* \alpha \wedge d \theta = \int_B \alpha \wedge df_l \theta
\]
therefore, in the case of smooth manifolds \( f_l \) is a chain map. As our \( f_l \) is simply obtained by a sum of \( f_l \) for smooth manifold components, our \( f_l \) is also a chain map.

7. Fiber products and integration along the fiber

In Lemma 7.3 below, we shall show that \( f_l \) transforms well under fiber products. In order to do this, we need to specify the orientation convention we shall use for fiber products. Fiber products of exploded manifolds are defined in [2]. It is also shown in [2] that if \( f \) and \( g \) are transverse, then the derivatives of the maps in the following commutative diagram
\[
\begin{array}{ccc}
A & \times_g B & \overset{f'}{\longrightarrow} & B \\
\downarrow g' & & \downarrow g \\
A & \overset{f}{\longrightarrow} & C
\end{array}
\]
give a short exact sequence
\[
0 \longrightarrow T_{(p_1, p_2)} (A \times_g B) \overset{(df', dg')}{\longrightarrow} T_{p_1} A \times T_{p_2} B \overset{d(f-g)}{\longrightarrow} T_{f(p_1)} C \longrightarrow 0
\]
In other words, the same relationship between tangent spaces as in the case of manifolds holds, so we may orient fiber products of exploded manifolds as we orient fiber products of manifolds. In particular, the above exact sequence and commutative diagram imply that
- \( df' \) gives an isomorphism between \( \ker dg' \) and \( \ker dg \)
- \( df \) gives an isomorphism between \( \coker dg' \) and \( \coker dg \)
- \( dg' \) gives an isomorphism between \( \ker df' \) and \( \ker df \)
- \( dg \) gives an isomorphism between \( \coker df' \) and \( \coker df \)
- \( \ker d(g \circ f') = \ker df' \oplus \ker dg' \cong \ker df \oplus \ker dg \)
\[ \text{coker } d(g \circ f') \cong \text{coker } df \oplus \text{coker } dg. \]

**Definition 7.1.** Given a map of oriented vector spaces \( A : X \rightarrow Y \) we shall use the following shorthand for an orientation convention for \( \ker A \) relative \( \text{coker } A \). By saying the identification

\[ \text{coker } A \oplus X = \ker A \oplus Y \]

is an oriented isomorphism, we mean that given any metric on \( X \) and \( Y \), the natural map

\[ A' : \text{coker } A \oplus X \rightarrow \ker A \oplus Y \]

is an oriented isomorphism. This map \( A' \) restricts to \( \text{coker } A \) to be the identification of \( \text{coker } A \) with the orthogonal complement of \( A(X) \subset Y \), and restricts to \( X \) to be the orthogonal projection onto \( \ker A \) and the map \( A \).

Of course, the relative orientation of \( \ker A \) and \( \text{coker } A \) given by the isomorphism \( A' \) does not depend on the choice of metrics on \( X \) and \( Y \).

We shall find the following way of arranging kernels and cokernels convenient.

\[ \text{coker } df \oplus TA \ f \times_g \ B \oplus \text{coker } dg = \ker df \oplus TC \oplus \ker dg \]

**Definition 7.2** (Orientation convention for fiber products). Let \( A, B \) and \( C \) be oriented exploded manifolds, and let \( f : A \rightarrow C \) and \( g : B \rightarrow C \) be transverse maps. Orient \( \ker df \) relative to \( \text{coker } df \) so that the identification

\[ \text{coker } df \oplus TA = \ker df \oplus TC \]

is an oriented isomorphism. On the other hand, orient \( \ker dg \) relative to \( \text{coker } dg \) so that the following identification gives an oriented isomorphism:

\[ TB \oplus \text{coker } dg = TC \oplus \ker dg \]

Then orient \( TA \ f \times_g \ B \) so that the following identification is an oriented isomorphism:

\[ \text{coker } df \oplus TA \ f \times_g \ B \oplus \text{coker } dg = \ker df \oplus TC \oplus \ker dg \]

The reader unfamiliar with this orientation convention should verify the following observations:

1. The above convention agrees with the usual convention for orienting products.
2. Given two transverse submanifolds \( A \) and \( B \) of a manifold \( M \), with normal bundles \( N_A \) and \( N_B \) oriented by the convention

\[ TA \oplus N_A = TM \] and \( TB \oplus N_B = TM \]

then \( A \cap B \) considered as a fiber product is oriented so that

\[ T(A \cap B) \oplus N_B \oplus N_A = TM \]

Be warned that some readers may consider this the usual convention for orienting \( B \cap A \)! See also example 8.2

3. The orientation of \( B \ g \times_f \ A \) differs from the orientation of \( A \ f \times_g \ B \) by

\[ (-1)^{\text{dim } A - \text{dim } C \cdot \text{dim } B - \text{dim } C} \]

4. The above convention for orienting the tangent space at a point of \( A \ f \times_g \ B \) does give a well defined orientation on \( A \ f \times_g \ B \). (You must check that deforming \( df \) and \( dg \) continuously doesn’t lead to any discontinuous change in orientation convention.)
(5) If the normal bundle of \( A \times g \times B \subset A \times B \) is identified with the pullback of \( TC \) using \( df - dg \), then the identification
\[
T(A \times g \times B) \oplus TC = T(A \times B)
\]
changes orientation by the sign
\[
(-1)^{\dim B \dim C}
\]
Of course, if we used \( dg - df \) to identify our normal bundle with the pullback of \( TC \), then the sign would be \((-1)^{\dim C(\dim C + \dim B)}\), which agrees with the convention found on page 114 of [5].

(6) The above convention makes the fiber product associative in the sense that where defined,
\[
(A \times g \times B) \circ k' \times (B \times \times C) = (A \times g \times B) \times \times C = (A \times g \times B) \times \times C
\]
The proof of associativity is not entirely trivial - a sketch is below. It helps to consider the following commutative diagram:
\[
\begin{array}{ccc}
(A \times g \times B) & \xrightarrow{f'} & (B \times \times k) \\
\downarrow & & \downarrow h' \\
A & \xrightarrow{f} & M_1
\end{array}
\quad
\begin{array}{ccc}
B \times \times k & \xrightarrow{h} & M_2 \\
\downarrow g & & \downarrow g' \\
C & \xrightarrow{g'} & C
\end{array}
\]
Note that \( df, df' \) and \( df'' \) have the same kernel and cokernel. Our orientation convention is equivalent to requiring that the relative orientations of these kernels and cokernels are the same, and that the orientations of \( A \times g \times B \) is such that the following identifications are oriented isomorphisms
\[
coker df' \oplus TA \times g \times B = \ker df' \oplus TB \\
coker df \oplus TA = \ker df \oplus TM_1
\]
It follows that
\[
A \times g \times \times k' \times (B \times \times k) = (A \times g \times B) \times \times k' \times (B \times \times k)
\]
Similarly, our orientation convention can be described only considering the downward pointing maps. All is as above, except that the kernels and cokernels now go on the right in the above identifications. It follows that
\[
(A \times g \times B) \circ f' \times \times k C = (A \times g \times B) \times \times k' \times (B \times \times k)
\]

**Lemma 7.3.** Suppose that \( A, B \) and \( C \) are oriented exploded manifolds, \( f : A \to B \) satisfies the conditions enumerated in Theorem 6.1 for \( f_1 \) to exist, and \( g : C \to B \) is a complete \( C^\infty L \) map.

Consider the following commutative diagram involving the fiber product of \( g \) and \( f \):
\[
\begin{array}{ccc}
C \times g \times f A & \xrightarrow{f'} & C \\
\downarrow g' & & \downarrow g \\
A & \xrightarrow{f} & B
\end{array}
\]
Then \( f'_1 \) also exists, and the following diagram is commutative
\[
\begin{array}{ccc}
\Omega^*_c(C \times g \times f A) & \xrightarrow{f'_1} & \Omega^*_c(C) \\
\uparrow (g')^* & & \uparrow g^* \\
\Omega^*_c(A) & \xrightarrow{f'_1} & \Omega^*_c(B)
\end{array}
\]
Proof:
As \( g \) is complete, \( g' \) is also complete, therefore
\[
(g')^* : \Omega^*(A) \rightarrow \Omega^*(C \times_f A)
\]
is well defined.

As noted in the section on fiber products in [2], if \( f \) is a submersion and \( df \) is also surjective on integral vectors, \( f' \) is a submersion which is also surjective on integral vectors. Therefore \( f' \) satisfies the conditions of Theorem 6.1 and \( f' \) exists.

It remains to verify that \( g^* \circ f = f'_* \circ g'^* \).

Note that \( g' \) gives an isomorphism between the fibers of \( f' \) and the fibers of \( f \). When we consider \(((g')^*(\theta))'\) on the fibers with values in \((f')^* \wedge T^*C\), this form can be obtained from the corresponding form \( \theta' \) by applying \((g'^*) \otimes (f'^* \circ g'^*)\). So
\[
((g')^*(\theta))' = (g')^* \otimes (f'^* \circ g'^*)(\theta') \in \Omega^{\text{top}}(\wedge (f'^*)^{-1}(p)) \otimes (f')^* \wedge P_C
\]
As we obtain \( f'_* (g')^* \theta \) by integrating \(((g')^*(\theta))'\) along the fibers of \( f' \), if the fibers of \( f' \) are oriented the same as the fibers of \( f \),
\[
g^* \circ f_*(\theta) = f'_* \circ (g')^* (\theta)
\]
Recall that to define integration along fibers, we orient so that
\[
TB \oplus \ker df = TA
\]
and
\[
TC \oplus \ker df' = TC \times_f A
\]
On the other hand, to orient \( C \times_f A \), we make the oriented identifications
\[
\ker dg \oplus TC = \ker dg \oplus TB
\]
\[
TA = TB \oplus \ker df
\]
\[
\ker dg \oplus TC \times_f A = \ker dg \oplus TB \oplus \ker df
\]
Inserting the first of the above three equations into the last equation then gives that
\[
TC \times_f A = TC \oplus \ker df
\]
is an oriented isomorphism. Therefore, with our orientation convention, the fibers of \( f \) and \( f' \) have the same orientation. It follows that
\[
g^* \circ f_*(\theta) = f'_* \circ (g')^* (\theta)
\]
as required. \( \square \)

8. Poincare duality and fiber products

In this section, we consider the relationship between Poincare duality and fiber products. In particular, we consider the relationship between Poincare duality and refinements, and the relationship between Poincare duality and intersection products.

Suppose that \( B' \rightarrow B \) is a refinement map. The corresponding inclusion \( H^*(B) \rightarrow H^*(B') \) need not be an isomorphism. For example, suppose that \( B \) is a refinement of \( \mathbb{T}^n \) corresponding to subdividing \( \mathbb{R}^n \) into a toric fan. Then \( H^*(B) \) is isomorphic to the cohomology of corresponding toric manifold \([B]\). Further subdividing this toric fan will produce a toric manifold with higher dimensional cohomology.
Suppose now that we have a map \( f : C \to B \) where \( C \) is a complete oriented exploded manifold and Poincare duality holds for \( B \). Then there exists some closed form \( \theta \in \Omega^*_c(B) \) so that for all \( \alpha \in \Omega^*(B) \),

\[
\int_C f^* \alpha = \int_B \alpha \wedge \theta
\]

This form \( \theta \) may be unsatisfactory for the following reason: Given any refinement \( B' \to B \), the fiber product gives a refined map \( f' : C' \to B' \). Ideally, the pullback of \( \theta \) to \( B' \) will then be the Poincare dual to \( f' \), but this may not be the case because there may be classes in \( H^*(B') \) which are not pulled back from classes in \( H^*(B) \).

**Lemma 8.1.** Suppose that \( C \) is a complete exploded manifold and \( f : C \to B \) is a \( C^\infty,1 \) map so that

\[
 df : \mathbb{Z}T_x C \to \mathbb{Z}T_{f(x)} B
\]

is surjective. Then given any neighborhood \( N \) of \( f(C) \subset B \), there exists a closed form \( \theta \in \Omega^*_c(B) \) supported in \( N \) which is Poincare dual to \( f \) in the sense that

\[
\int_B \alpha \wedge \theta = \int_C f^* \alpha
\]

for all closed \( \alpha \in \Omega^*(B) \).

Suppose that \( g : A \to B \) is any complete \( C^\infty,1 \) map transverse to \( f \). Then \( g^* \theta \) is Poincare dual to the map \( f' \) below

\[
\begin{array}{ccc}
A & g \times f & C \\
\downarrow f' & \downarrow f \\
A & g' & B
\end{array}
\]

in the sense that

\[
\int_A \alpha \wedge g^* \theta = \int_A (f')^* \alpha
\]

for all closed \( \alpha \in \Omega^*(A) \).

**Proof:** We can extend \( f \) to a submersion \( h : C \times \mathbb{R}^n \to B \) satisfying the conditions of Theorem 6.1. (Here \( h \) extends \( f \) in the sense that \( h(p,0) = f(p) \).) Choose a compactly supported form \( \theta_0 \) on \( \mathbb{R}^n \) which integrates to 1, consider this form \( \theta_0 \) as a form on \( C \times \mathbb{R}^n \), then integrate along the fibers of \( h \) to obtain

\[
\theta := h \circ h_0 \in \Omega^*_c(B)
\]

This form \( \theta \) represents the Poincare dual of \( f \). In particular, suppose that \( \alpha \in \Omega^*(B) \) is closed. Then our adaptation of Stokes’ theorem, Theorem 3.4 implies that

\[
\int_{C \times \mathbb{R}^n} h^* \alpha = \int_{C \times 0} h^* \alpha = \int_C f^* \alpha
\]

Therefore,

\[
\int_B \alpha \wedge h_0 \theta_0 = \int_{C \times \mathbb{R}^n} h^* \alpha \wedge \theta_0 = \int_C f^* \alpha
\]

Therefore, \( \theta = h \circ h_0 \) is Poincare dual to \( f \). By choosing \( \theta_0 \) supported close to 0 \( \in \mathbb{R}^n \) we may arrange that \( \theta \) is supported close to the image of \( f \).

Given our complete map \( g \) transverse to \( f \), we may now consider the fiber product

\[
\begin{array}{ccc}
A & g \times h \ (C \times \mathbb{R}^n) & g' \times h \\
\downarrow h' & \downarrow h \\
A & g' & B
\end{array}
\]
Applying Lemma 7.3 gives
\[ g^* \theta = g^* (h_1(\theta_0)) = h'_1 ((g')^*(\theta_0)) \]
so
\[ \int_A \alpha \wedge g^* \theta = \int_{A \times_{(C \times \mathbb{R}^n)}} (h')^* \alpha \wedge (g')^*(\theta_0) \]
Define the map \( r : [0, 1] \times C \times \mathbb{R}^n \rightarrow B \) by
\[ r(t, p, x) = h(p, xt) \]
As \( f(p) = h(p, 0) \) and \( f \) and \( h \) are transverse to \( g \), our new map \( r \) is also transverse to \( g \), so we may take the fiber product
\[ A \times \{0\} \times_{(C \times \mathbb{R}^n)} \times \mathbb{R}^n \rightarrow [0, 1] \times \{0\} \times \mathbb{R}^n \]
As \( g \) is complete, \( \hat{g}' \) is also complete, so \((\hat{g}')^* \theta_0\) is completely supported. We may now apply Stokes theorem. Our map \( r \) restricted to \( t = 1 \) is \( h \), and restricted to \( t = 0 \) is \( \hat{r}(0, p, x) = f(p) \)
Associativity of fiber products then implies that the corresponding boundary of \( A \times \{0\} \times_{(C \times \mathbb{R}^n)} \times \mathbb{R}^n \) is equal to \((A \times f \times C) \times \mathbb{R}^n \). Then
\[ \int_A \alpha \wedge g^* \theta = \int_{(A \times f \times C) \times \mathbb{R}^n} (f')^* (\alpha) \wedge \theta_0 = \int_{A \times f \times C} (f')^* \alpha \]
as required.

Example 8.2 (Intersection of submanifolds and Poincare duality).

suppose that \( A \) and \( C \) are complete exploded manifolds which are submanifolds of the exploded manifold \( B \) in the sense that they can be locally described as the inverse image of a regular value of some \( C^\infty \) valued function. Then we may use the construction of Lemma 8.1 to construct Poincare duals \( \theta_A \) and \( \theta_C \) to \( A \) and \( C \). If \( A \) and \( C \) are transverse, then Lemma 8.1 implies that the \( \theta_C \) restricted to \( A \) is Poincare dual to \( A \cap C \).
Therefore
\[ \int_{A \cap C} \alpha = \int_A \alpha \wedge \theta_C = \int_B \alpha \wedge \theta_C \wedge \theta_A \]
So the Poincare dual to \( A \cap C \) is \( \theta_C \wedge \theta_A \). So with our sign convention intersection products correspond under Poincare duality to wedge products with the order reversed.

Be warned that if neither \( A \) or \( C \) are submanifolds in the above sense, the above formula may not hold. For example, let \( B \) be a refinement of \( T^2 \) corresponding to
dividing \( \mathbb{R}^2 \) into the standard quadrants, and consider \( A := \{ \tilde{z}_1 = \tilde{z}_2 \} \subset B \) and \( C : \{(\tilde{z}_1 + \tilde{z}_2 + 1^3) \in 0^2 \} \subset B \). Note that for any \( \theta \in \Omega^*(B) \)

\[
\int_C \theta = \int_{C'} \theta
\]

where \( C' := \{ \tilde{z}_1 = -\tilde{z}_2 \} \). This is because \( \theta \) must vanish out where \( C \) and \( C' \) differ. Therefore, the Poincare duals of \( C \) and \( C' \) are the same, so if the usual relationship between intersections and wedge products held, the Poincare dual of \( A \cap C \) should be equal to the Poincare dual of \( A \cap C' \). But \( A \cap C \) is a single point and \( A \cap C' \) is empty.

The solution to this problem is to allow a more flexible class of differential forms called refined forms.

9. Refined cohomology

**Definition 9.1.** A refined form \( \theta \in \Gamma^*\Omega^*(B) \) is choice \( \theta_p \in \bigwedge T^*_p(B) \) for all \( p \in B \) so that given any point \( p \in B \), there exists an open neighborhood \( U \) of \( p \) and a complete, surjective, equidimensional submersion

\[
r : U' \rightarrow U
\]

so that there is a form \( \theta' \in \Omega^*(U') \) which is the pullback of \( \theta \) in the sense that if \( v \) is any vector on \( U' \) so that \( dr(v) \) is a vector based at \( p \), then

\[
\theta'(v) = \theta_p(dr(v))
\]

A refined form \( \theta \in \Gamma^*\Omega^*(B) \) is completely supported if there exists some complete subset \( V \) of an exploded manifold \( C \) with a map \( C \rightarrow B \) so that \( \theta_p = 0 \) for all \( p \) outside the image of \( V \). Use the notation \( \Gamma^*\Omega^*_c \) for completely supported refined forms.

Denote the homology of \( \Gamma^*\Omega^*(B), d \) by \( \Gamma H^*(B) \) and \( \Gamma^*\Omega^*_c(B), d \) by \( \Gamma H^*_c(B) \).

For defining the refined cohomology above it should be obvious that

\[
d : \Gamma^k\Omega^*(B) \rightarrow \Gamma^{k+1}\Omega^*(B)
\]

is well defined and \( d^2 = 0 \). Less immediate is the fact that \( \Gamma^*\Omega^*(B) \) is closed under addition and wedge products. If \( \theta_1 \) and \( \theta_2 \) are refined forms, then any point \( p \) has a neighborhood \( U \) with complete surjective, equidimensional submersions

\[
r_1 : U'_1 \rightarrow U
\]

so that \( r_1^*\theta_1 \in \Omega^*(U'_1) \). Then taking the fiber product of \( r_1 \) with \( r_2 \) gives a complete, surjective, equidimensional submersion

\[
r' : U_{1_r} \times_{r_2} U_2 \rightarrow U
\]

so that \( r'^*\theta_1 \in \Omega^*(U_{1_r} \times_{r_2} U_2) \). Therefore, \( \theta_1 + \theta_2 \) and \( \theta_1 \wedge \theta_2 \) are in \( \Gamma^*\Omega^*(B) \).

The existence of partitions of unity combined with Lemma 3.3 implies that the integral of \( \theta \in \Gamma^*\Omega^*_c(B) \) over \( B \) is finite and well defined. In particular if \( \rho \theta \) is supported in \( U \) and the map \( r : U' \rightarrow U \) has degree \( m \), then

\[
\int_U \rho \theta := \frac{1}{m} \int_{U'} r'^* \rho \theta
\]

Note also that given any \( C^\infty \) map \( f : A \rightarrow B \), there is a linear chain map

\[
f^* : \Gamma^*\Omega^*(B) \rightarrow \Gamma^*\Omega^*(A)
\]

defined as usual so that

\[
(f^*\theta)_p(v) := \theta_{f(p)}(df(v))
\]
To see that $f^* \theta$ is actually in $\mathcal{H}^*(\mathbf{A})$, let $r: U' \longrightarrow U$ be a complete, equidimensional submersion onto a neighborhood of $f(p)$ so that $r^* \theta \in \mathcal{H}^*(\mathbf{B})$. Then taking the fiber product of $r: U' \longrightarrow \mathbf{B}$ with $f$ gives a complete equidimensional submersion onto a neighborhood of $p$ so that the pullback of $f^* \theta$ is in $\mathcal{H}^*$, so $f^* \theta \in \mathcal{H}^*(\mathbf{A})$.

Our version of Stokes’ theorem also extends trivially to refined forms in $\mathcal{H}^*_c(\mathbf{B})$. If $\mathbf{B}$ is a complete exploded manifold the integration pairing on $\mathcal{H}^*_c(\mathbf{B})$ is non-degenerate, but as $\mathcal{H}^*_c(\mathbf{B})$ is in general infinite dimensional, this does not imply Poincare duality.

**Theorem 9.2.** Given any submersion $f: \mathbf{B} \longrightarrow \mathbf{C}$, there exists a linear chain map

$$f_! : \mathcal{H}^*_c(\mathbf{B}) \longrightarrow \mathcal{H}^*_c(\mathbf{C})$$

uniquely determined by the property that

$$\int_{\mathbf{C}} \alpha \wedge f_! \beta = \int_{\mathbf{B}} (f^* \alpha) \wedge \beta$$

for all $\beta \in \mathcal{H}^*_c(\mathbf{B})$ and $\alpha \in \mathcal{H}^*(\mathbf{C})$.

**Proof:**

Given any point $p \in \mathbf{C}$, we may take a refinement of a neighborhood of $p$ so that the image of $p$ is contained in a strata which is a smooth manifold. As a smooth form on a manifold is determined by its integral against compactly supported forms, $f_! \beta$ around $p$ is uniquely determined by the property $\int_{\mathbf{C}} \alpha \wedge f_! \beta = \int_{\mathbf{B}} (f^* \alpha) \wedge \beta$. As the right hand side of this equation is linear in $\beta$, it follows that $f_!$ is linear if it exists. Stoke’s theorem implies that if $\alpha \in \mathcal{H}^k(\mathbf{C})$,

$$\int_{\mathbf{C}} \alpha \wedge df_! \beta = (-1)^{k+1} \int_{\mathbf{C}} (d \alpha) \wedge f_! \beta = (-1)^{k+1} \int_{\mathbf{B}} (df^* \alpha) \wedge \beta = \int_{\mathbf{B}} (f^* \alpha) \wedge d \beta$$

so if $f_!$ exists, it is a linear chain map. Using a partition of unity, we may restrict to the case that $f$ is a map between coordinate charts $U$ and $\tilde{V}$ and $\beta$ pulls back to a form in $\mathcal{H}^*_c(U')$. Then, using a partition of unity on $U'$, we may restrict to the case that $\beta$ is supported in a single coordinate chart of $U'$. By relabeling we do not lose generality by assuming that $\beta$ is supported in a single coordinate chart $U$.

The tropical part of $U$ and $\tilde{V}$ are polytopes $\tilde{U}$ and $\tilde{V}$. There exists a coordinate chart $V'$ with a complete equidimensional submersion $\tilde{V}' \longrightarrow V$ so that the image of integral vectors from $U$ in $\tilde{V}$ is always a full sublattice of the image of integral vectors from $V'$. (The tropical part of $V'$ is $\tilde{V}$ with a different integral affine structure.) Then we may choose a refinement $V'' \longrightarrow V$ corresponding to a subdivision of $V$ so that $f(U)$ is a polytope in this subdivision. Suppose that $\alpha \in \mathcal{H}^*V$ pulls back to a $C^{\infty,1}$ form on some $V''' \longrightarrow V$. Then let $\tilde{V}$ be the fiber product of $V'$ with $V''$ and $V'''$, and let $r: \tilde{U} \longrightarrow U$ be the fiber product of $\tilde{V} \longrightarrow V$ with $f: U \longrightarrow V$.

Then $\tilde{f} : \tilde{U} \longrightarrow \tilde{V}$ is a submersion which also is surjective on integral vectors. Therefore, Theorem 6.1 implies that there is a linear chain map $\tilde{f}_! : \Omega^*_c(\tilde{U}) \longrightarrow \Omega^*_c(\tilde{V})$ so that

$$\int_{\tilde{V}} \alpha \wedge \tilde{f}_!(r^* \beta) = \int_{\tilde{U}} (\tilde{f}^* \alpha) \wedge r^* \beta$$

for all $\alpha \in \Omega^*(\tilde{V})$ and $\beta \in \Omega^*_c(U)$. Considering $f_! \beta$ as refined form in $\mathcal{H}^*_c(\mathbf{B})$, we have our map $f_!$. As $\tilde{U} \longrightarrow U$ has the same degree as $\tilde{V} \longrightarrow V$, the above formula implies that

$$\int_{\tilde{V}} \alpha \wedge \tilde{f}_! \beta = \int_{\tilde{U}} (f^* \alpha) \wedge \beta$$
Lemma 7.3 implies that this map $f_i$ is independent of further refinement of $\hat{V}$ and $\hat{U}$, so $f_i\beta$ depends only on $\beta$ as an element of $\Omega^*(U)$, not on $\beta$ as an element of $\Omega^*_c(U)$.

□

Lemma 9.3. Suppose that $A$, $B$ and $C$ are oriented exploded manifolds, $f : A \rightarrow B$ is a submersion, and $g : C \rightarrow B$ is a complete $C^{\infty,1}$ map.

Consider the following commutative diagram involving the fiber product of $g$ and $f$:

$$
\begin{array}{c}
\text{C} \\
g \times_f A
\end{array} \xrightarrow{f_i'} \begin{array}{c}
\text{C} \\
g
\end{array}
\begin{array}{c}
\text{A} \\
f
\end{array}
$$

Then $f_i'$ also exists, and the following diagram is commutative

$$
\begin{array}{c}
\Omega^*_{c}(\text{C}) \\
\Omega^*_{c}(g \times_f A)
\end{array} \xrightarrow{(g^*)^*} \begin{array}{c}
\Omega^*_{c}(\text{B}) \\
\Omega^*_{c}(\text{A})
\end{array}
$$

Proof: This lemma has the same proof as Lemma 7.3 except Theorem 9.2 is used instead of Theorem 6.1.

□

Even though Poincaré duality does not hold for $\eta H^*(B)$, the following lemma gives an analogue of the Poincaré dual of a map from a complete manifold.

Lemma 9.4. Suppose that $C$ is a complete exploded manifold and $f : C \rightarrow B$ is a $C^{\infty,1}$ map. Then given any metric on $B$ and distance $r$, there exists a closed form $\theta \in \Omega^*_{c}(B)$ supported within a radius $r$ of $f(C)$ which is Poincaré dual to $f$ in the sense that

$$
\int_B \alpha \wedge \theta = \int_C f^* \alpha 
$$

for all closed $\alpha \in \Omega^*(B)$

Suppose that $g : A \rightarrow B$ is any complete $C^{\infty,1}$ map transverse to $f$. Then $g^* \theta$ is Poincaré dual to the map $f'$ below

$$
\begin{array}{c}
\text{A} \\
g \times_f C
\end{array} \xrightarrow{g'} \begin{array}{c}
\text{C} \\
f
\end{array}
\begin{array}{c}
\text{A} \\
f
\end{array}
$$

in the sense that

$$
\int_A \alpha \wedge (f')^* \theta = \int_A (f')^* \alpha 
$$

for all closed $\alpha \in \Omega^*(A)$

Proof:

The proof of this lemma is identical to the proof of Lemma 8.1 except Theorem 9.2 is used in the place of Theorem 6.1 and Lemma 9.4 is used instead of Lemma 8.1.

□

10. Partitions of unity and good covers

Throughout, this paper, we are assuming that our exploded manifolds considered as topological spaces are second countable. The following lemma constructs a partition of unity subordinate to a given open cover of an exploded manifold.
Lemma 10.1. Given any open cover \( \{U_\alpha\} \) of an exploded manifold \( B \), there exists a partition of unity subordinate to \( \{U_\alpha\} \).

Proof:
Any (second countable) exploded manifold has an exhaustion by compact subsets \( K_i \) so that \( K_{i-1} \) is contained in the interior of \( K_i \). This follows from the observation that this holds for \( \mathbb{R}^n \times T \), and any (second countable) exploded manifold has a countable cover by open subsets isomorphic to \( \mathbb{R}^n \times T \).

A second ingredient needed for construction of partitions of unity is the existence of bump functions. There exists a smooth function with compact support which is positive on any given compact subset of \( \mathbb{R}^n \times T \). Given any point \( p \) in an open subset \( U \) of an exploded manifold, it was proved in [2] that there exists an open neighborhood of \( p \) contained inside \( U \) which is isomorphic to \( \mathbb{R}^n \times T \). Therefore, there exists a smooth non negative function which is positive at \( p \) and which has support compactly contained inside \( U \).

We may now construct partitions of unity as usual. Let \( \{U_\alpha\} \) be any open cover of \( B \). For each point \( p \) in \( K_i \setminus K_{i-1} \), choose a non negative bump function \( \rho_p \) which is positive at \( p \) and which has compact support contained inside \( K_{i+1} \setminus K_{i-1} \) and some \( U_\alpha \). The sets \( \{\rho_p > 0\} \) form an open cover of \( B \) which have a locally finite subcover \( \{\rho_p > 0\} \) for \( i = 1, \ldots \). Then \( \sum \rho_p \) is smooth and positive, so we may divide our functions \( \rho_p \) by this sum to obtain the required partition of unity.

\[ \blacksquare \]

Lemma 10.2. Any compact exploded manifold \( B \) has a finite good cover \( \{U_i\} \) in the sense that the intersection of any number of these \( U_i \) is either empty or isomorphic to \( \mathbb{R}^n \times T \).

Proof:
It was shown in [3] that any exploded manifold has a cover by equivariant coordinate charts isomorphic to open subsets of \( \mathbb{R}^n \times T \). This means that each transition map or its inverse is in the form of a map

\[(x, \bar{z}) \mapsto (f(x), g_1(x)\bar{z}^{\alpha_1}, \ldots, g_n(x)\bar{z}^{\alpha_n})\]

In particular, transition maps of the above type send the lattice of vectorfields \( N \) generated by the real and imaginary parts of \( \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \) to a sublattice of the corresponding lattice in the target. Note that the coordinate charts with more structure are those with higher dimensional tropical part, so these equivariant transition maps never decrease the dimension of the tropical part. We shall assume that if two coordinate charts intersect, the tropical part of one of the coordinate charts is a face of the other coordinate chart (recall that the closure of any strata is a face - this statement does not assume that it is a codimension 1 face!)

By using a partition of unity and reducing the size of coordinate charts where necessary, we can choose a connection \( \nabla \) so that in our coordinate charts, for any vector \( w \) in the lattice of vectorfields \( N \) generated by the real and imaginary parts of \( \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \),

\[ \nabla_w = L_w \text{ and } \nabla w = 0 \]

To achieve the above, proceed as follows: in each coordinate chart, the standard flat connection obeys the above conditions. Choose a finite partition of unity consisting of bump functions compactly supported inside our equivariant coordinate charts, and average these flat connections using this partition of unity. Consider the resulting connection on a coordinate chart \( \mathbb{R}^n \times T \). On the open subset of this coordinate chart which is the complement of the support of all bump functions supported inside charts with lower tropical dimension, the averaged connection obeys
Also, the fact that \( \leq \) is positive definite at \( v \) it has nonnegative second derivative. In particular, if a geodesic has velocity \( v \), then the second derivative of \( |x|^2 \) restricted to the geodesic is
\[
\nabla_v(d|x|^2)(v) = 2 \sum_i (dx_i(v))^2 + 2x_i(\nabla_v dx_i)(v)
\]
Restricted to vectors in the subspace generated by \( \frac{\partial}{\partial x_i} \), the above quadratic form is positive definite at \( x = 0 \), and therefore positive definite on this subspace for \( |x| \) small enough. Let \( w \) be any vector field given by a sum of constants times the real and imaginary parts of \( \tilde{z}_i \frac{\partial}{\partial z_i} \), then
\[
\nabla_{v+w}(d|x|^2)(w) = \nabla_{v+w}(d|x|^2)(w) = 0
\]
Also, the fact that \( \nabla_w = L_w \) implies that
\[
\nabla_w(d|x|^2) = 0
\]
Therefore,
\[
\nabla_{v+w}(d|x|^2)(v+w) = \nabla_v(d|x|^2)(v)
\]
As our quadratic form is positive definite on one subspace and independent of a complimentary subspace, it is positive semidefinite, as required.

Now construct a proper convex function on (a subset of ) our coordinate chart \( \mathbb{R}^n \times T^m_P \). Choose some basis \( \{ \zeta_k : [v^n \hat{z}^m] \} \) of smooth monomials on \( T^m_P \), and consider the function \( f := \sum_k |\zeta_k|^2 \). Restricted to the subspace spanned by the lattice \( N \), this function is convex, as verified by calculation: If \( w_i \) indicates the real part of \( \tilde{z}_i^{-1}d\tilde{z}_i(w) \), then
\[
\nabla_w(df)(w) = \sum_{i,j} 4 |\zeta_{ij}|^2 \alpha_i^2 w_i^2
\]
For \( x \) small, \( |x|^2 \) is strictly convex on the complementary subspace to \( N \) spanned by \( \frac{\partial}{\partial z_i} \). Therefore, if we choose \( \lambda \) large enough, then \( f + \lambda |x|^2 \) will be convex when it is \( \leq 1 \).

Our connection \( \nabla \) on (an open subset of ) \( \mathbb{R}^n \times T^m_P \) defines a connection \( \nabla' \) on a subset of \( \mathbb{R}^n \) follows: Let \( x \) denote the projection to \( \mathbb{R}^n \), and let \( \tilde{v} \) indicate any lift of a vectorfield \( v \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \times T^m_P \) so that \( dx(\tilde{v}) = v \). Then \( \nabla' v_1 v_2 := dx(\nabla_{\tilde{v}_1} \tilde{v}_2) \). Note that every smooth vectorfield on \( \mathbb{R}^n \times T^m_P \) is independent of \( T^m_P \), as \( P^o \) is an open polytope so \( dx(\nabla_{\tilde{v}_1} \tilde{v}_2) \) is indeed a vectorfield on \( \mathbb{R}^n \). We’ll check below that our conditions on \( \nabla \) ensure that this projected connection is well defined:

If \( w \) is in the kernel of \( dx \), then \( \nabla_w \tilde{v} \) is also in the kernel of \( dx \) whenever \( \tilde{v} \) is lifted. This implies that \( dx(\nabla_{\tilde{v}_1} \tilde{v}_2) \) does not depend on the choice of lift of \( v_1 \). The fact that \( \nabla_w = 0 \) for any \( w \in N \) implies that if we instead choose \( w \) in the kernel of \( dx \), then \( \nabla_w \) will also be in the kernel of \( dx \). It follows that \( dx(\nabla_{\tilde{v}_1} \tilde{v}_2) \) does not depend on the choice of lift of \( v_2 \), and the connection \( \nabla' \) is well defined.

Consider the subset \( U \) of our coordinate chart given by
\[
U := \left\{ f + \lambda |x|^2 < \epsilon < \frac{1}{2} \right\} \subset \mathbb{R}^n \times T^m_P
\]
where \( \epsilon \) is chosen small enough that \( f + \lambda |x|^2 \) is still proper and convex on the set where \( f + \lambda x^2 < 2\epsilon \). Denote this set \( \{ f + \lambda x^2 < 2\epsilon \} \) by \( 2U \). Choosing \( \epsilon \) small enough will also ensure that the following two convexity conditions hold for \( U \):
The projection $U'$ of $U \cap \mathbb{R}^n \times T_p^m$ to $\mathbb{R}^n$ is geodesically convex using the connection $\nabla'$. (Note that $f + \lambda |x|^2$ restricted to $U'$ is the convex function $\lambda |x|^2$. Choosing epsilon small enough ensures that this $U'$ will be geodesically convex.)

(2) $U$ is defined by some finite number of inequalities $g_i < 1$ where each $g_i$ is a finite sum of positive functions on $U'$ times the square absolute value of smooth monomial functions, and $g_i$ is proper restricted to each $T_p^m$ fiber of $U' \times T_p^m$.

By using $|x - p|^2$ in place of $|x|^2$, we may cover a neighborhood of our coordinate chart intersected with $\mathbb{R}^n \times T_p^m$ with sets $U$ coming from functions $f$ satisfying the above convexity conditions.

We shall construct our good cover starting by covering the strata with largest tropical dimension and then covering strata in reverse order of tropical dimension. Suppose that we have a collection of equivariant coordinate charts and a finite good cover of all strata of tropical dimension greater than $k$ by sets $U_i$ defined by functions $f_i$ on our coordinate charts satisfying the above convexity conditions. We shall show that we can extend this good cover to strata of tropical dimension $k$.

Choose a cover of the strata of tropical dimension $k$ using coordinate charts with tropical dimension $k$ so that each of these coordinate charts includes in an old coordinate chart via an equivariant map. We may now cover the strata of dimension $k$ by open sets $U$ coming from functions $f$ defined in these new coordinate charts and satisfying the above convexity conditions, and satisfying the extra condition that if $U$ intersects a member $U_i$ of our previously constructed finite good cover, then $U$ is contained entirely inside $2U_i$. This new collection of open sets together with our old good cover is an open cover of the set of strata of dimension at least $k$, which is compact, so we can choose a finite sub cover. It remains to prove that this subcover is a good cover.

The intersection of a finite number of these new sets $U$ satisfying the above two convexity conditions above clearly still satisfies these convexity conditions, because all transition maps and their inverses are equivariant. Intersection with some of the previously constructed $U_i$ then corresponds to restricting to a subset where the functions $f_i + \lambda_i |x|^2$ are less than some $\epsilon_i$. Restricting a geodesically convex set to a set where a convex function is less than $\epsilon_i$ gives a geodesically convex set, so this intersection obeys the first convexity condition above. The condition that the transition map between the coordinate chart $U$ and the coordinate chart where $f_i = \sum |\zeta_j|^2$ is equivariant, and the fact that the tropical part of $U$ is some face of this coordinate chart implies that $f_i + \lambda_i |x|^2$ in the coordinate chart $\mathbb{R}^n \times T_p^m$ containing $U$ is some sum of positive functions of $\mathbb{R}^n$ times the square absolute value of smooth monomial functions. Therefore the second convexity condition also holds.

It remains to show that a subset $U \subset \mathbb{R}^n \times T_p^m$ satisfying the two convexity conditions above is isomorphic to $\mathbb{R}^n \times T_p^m$. Then our new finite cover will be a good cover of strata of dimension at least $k$, and we can continue extending until we have a finite good cover of the entire exploded manifold. The set $U'$ defined by $U \cap \mathbb{R}^n \times T_p^m = U' \times T_p^m$ is geodesically convex and open, and therefore diffeomorphic to $\mathbb{R}^n$. We can therefore reduce to the case that $U' = \mathbb{R}^n$.

Recall that $U$ is equal to the set where $g_i < 1$ for some finite number of functions $g_i$ which are sums of positive functions on $\mathbb{R}^n$ times square absolute values of monomial functions. Choose a diffeomorphism $\rho : [0,1] \rightarrow [0,\infty)$ so that close to
0, \rho(x) = x. The function
\[ G := \sum_i \rho(g_i) \]
is smooth and proper on each \( T^m_P \) fiber of \( U \). We may assume that these \( g_i \) are a sum of positive functions times the square absolute value of nonconstant monomial functions. Note that at each point of \( T^m_P \), there exists a vector \( v \) so that for all smooth monomials \( \zeta \), \( v |\zeta|^2 \) is positive if the smooth monomial \( \zeta \) is nonzero. Therefore, \( vG > 0 \) if \( G \neq 0 \). If \( \nabla G \) indicates the gradient of \( G \) in the \( T^m_P \) direction using the standard flat metric, then \( \nabla G \) is nonzero whenever \( G \) is nonzero. Let \( v \) be a smooth vectorfield so that \( dx(v) = 0, 0 \leq vG \leq 1 \) and \( vG > 0 \) when \( G > 0 \). This vectorfield \( v \) is complete on \( U \) and for any point \( p \in U \) and \( \epsilon > 0 \), there exists some time \( T \) so that \( G(\Phi_{-tv}(p)) < \epsilon \) for all \( t > T \). Let
\[ G' := \sum_i g_i \]
When \( G \) is small, \( G = G' = \sum g_i \). Note that \( G' \) is proper restricted to \( T^m_P \) fibers of \( \mathbb{R}^n \times T^m_P \) and \( \nabla G' > 0 \) whenever \( G' > 0 \). We can therefore choose some smooth vectorfield \( v' \) so that \( v' = v \) on a neighborhood of \( T^m_P \), \( vG' > 0 \) wherever \( G' > 0 \), and \( vG' \leq 1 \).
Consider the map \( U \rightarrow \mathbb{R}^n \times T^m_P \) given by the limit as \( t \rightarrow \infty \) of
\[ \Phi_{tv'} \circ \Phi_{-tv} \]
the flow for time \(-t\) of \( v \) followed by the flow for time \( t \) of \( v' \). Note that \( v' \) and \( v \) agree in a neighborhood of \( \mathbb{R}^n \times T^m_P \), and \( \Phi_{-tv} \) eventually brings any point into this neighborhood. Therefore, around any point, this limit is simply given by \( \Phi_{tv'} \circ \Phi_{-tv} \) for some large \( t \). It follows that this map is smooth. It is also obviously invertible, as \( \Phi_{-tv} \) also eventually brings each point into a neighborhood of \( T^m_P \). It follows that \( U \) is isomorphic to \( \mathbb{R}^n \times T^m_P \), and we have completed the proof of our lemma.

**11. Cohomology does not change in connected families**

**Lemma 11.1.** Let \( \pi: A \rightarrow \mathbb{R}^n \times T^m_P \) be a family of exploded manifolds, where \( P \) is a bounded polytope. Then given any \( p \in U \), the following is an exact sequence:
\[ 0 \rightarrow H^*(\mathbb{R}^n \times T^m_P) \rightarrow H^*(A) \rightarrow H^*(\pi^{-1}(p)) \rightarrow 0 \]

**Proof:**
Consider a map \( f: \mathbb{R}^n \times T^m_{Q'} \rightarrow \mathbb{R}^n \times T^m_P \) in the form
\[ (x, z) \mapsto (g(x), z^{a_1}, \ldots, z^{a_m}) \]
where \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) is a surjective linear projection, \( a_1 \land \ldots \land a_m \) is nonzero and not a nontrivial integer multiple of any other vector in \( \Lambda^m \mathbb{Z}^{m+m'} \), and \( f: Q \rightarrow P \) is a complete, surjective map which sends strata of \( Q \) to strata of \( P \). The fiber of such a map over a point \( p \) in \( T^m_P \) is
\[ f^{-1}(p) = \mathbb{R}^n \times \mathbb{R}^{m+n-m'} \]
where \( S = f^{-1}(p) \subset Q \). As \( P \) is bounded and \( f \) is a complete map which sends strata of \( Q \) to strata of \( P \), the vector space spanned by unbounded directions in \( S \) is the same as the vector space spanned by the unbounded directions in \( Q \). Let \( l \) indicate the dimension of this vector space spanned by these unbounded directions. Lemma 11.1 allows us to compute \( H^* \) of \( \mathbb{R}^n \times T^m_P \), \( \mathbb{R}^n \times T^m_{Q'} \), and \( f^{-1}(p) \).
\[ \dim(H^1(\mathbb{R}^n \times T^m_P)) = m \]
\[ \dim(H^1(\mathbb{R}^n \times T^m_{Q'})) = m + m' - l \]
\[ \dim(H^1(f^{-1}(p))) = m' - l \]

Any nonzero element of \( H^1(\mathbb{R}^m \times T_{Q}^{m+n'}/f^*(\mathbb{R}^n \times H^1(T_P^n)) \) has nonzero integral on some loop contained inside \( f^{-1}(p) \), therefore the above dimension count above implies that the following is an exact sequence:

\[
0 \longrightarrow H^1(\mathbb{R}^n \times T_P^n) \longrightarrow H^1(\mathbb{R}^m \times T_{Q}^m) \longrightarrow H^1(f^{-1}(p)) \longrightarrow 0
\]

This in turn implies that the following is an exact sequence

\[
0 \longrightarrow H^*(\mathbb{R}^n \times T_P^n) \longrightarrow H^*(\mathbb{R}^m \times T_{Q}^m) \longrightarrow H^*(f^{-1}(p)) \longrightarrow 0
\]

because in each case \( H^* \) is the free exterior algebra on \( H^1 \). The idea of the rest of this proof is to use this local result with Mayer Vietoris and some kind of ‘good’ cover.

Now reexamine our construction of a good cover in the proof of Lemma 10.2. The first step of this construction was to choose equivariant coordinate charts. It is proved in [3] that these equivariant coordinate charts on \( A \) can be chosen so that the projection \( \pi \) is an equivariant map in local coordinates. Using the implicit function theorem and the fact that \( \pi \) is a family, we may also assume that in any of our coordinate charts which project to contain the interior strata of \( \mathbb{R}^n \times T_P^n \), the map \( \pi \) is a restriction of some map in the form of \( f \) above. The connection \( \nabla \) used to construct a good cover on \( A \) may be chosen so that there is some connection \( \pi_*\nabla \) on \( \mathbb{R}^n \times T_P^n \) so that geodesics of \( \nabla \) composed with \( \pi \) are geodesics of \( \pi_*\nabla \).

Chose a point \( p \) in the interior strata of \( \mathbb{R}^n \times T_P^n \). Next, construct a finite good cover of \( \pi^{-1}(p) \subset A \) following the construction in the proof of Lemma 10.2. Note that the open subsets in this cover are constructed as the sets where some convex functions are \( < \epsilon \). We have some freedom to change this \( \epsilon \) a little to construct a finite good cover \( \{U_i\} \) of \( \pi^{-1}(p) \) so that there exists a neighborhood \( N \) of \( p \) so that if \( U \) is any intersection of sets in our open cover, either \( \pi(U) \) contains \( N \), or \( \pi(U) \) is disjoint from \( N \).

By construction, \( \pi \) restricted to \( U \) is equal to the restriction of some map in the same form as the map \( f : \mathbb{R}^n \times T_P^n \longrightarrow \mathbb{R}^m \times T_{Q}^m \) considered above. The argument of Lemma 10.2 applied fiberwise gives a fiber preserving isomorphism of \( U \cap \pi^{-1}(N) \) with \( f^{-1}(N) \).

We may choose our neighborhood \( N \) so that there exists a smooth vectorfield \( v \) who’s time 1 flow \( \Phi_v \) takes \( \mathbb{R}^n \times T_P^n \) to \( N \). We may lift \( v \) to a vectorfield \( v' \) on \( \mathbb{R}^n \times T_{Q}^{m+n'} \) who’s time 1 flow \( \Phi_{v'} \) exists and has image \( f^{-1}(N) \). We therefore have a fiber preserving isomorphism between \( f^{-1}(N) \) and the entire domain of \( f \). Therefore, for any point \( q \in N \), the following is an exact sequence:

\[
0 \longrightarrow H^*(N) \longrightarrow H^*(U \cap N) \longrightarrow H^*(U \cap \pi^{-1}(q)) \longrightarrow 0
\]

This is true for any intersection \( U \) of members of our finite good cover of \( \pi^{-1}(N) \). Therefore using the Mayer Vietoris sequence, the five lemma and induction over the size of a good cover, we get that the following is an exact sequence:

\[
0 \longrightarrow H^*(N) \longrightarrow H^*(\pi^{-1}N) \longrightarrow H^*(\pi^{-1}(q)) \longrightarrow 0
\]

Lifting the vectorfield \( v \) to a smooth vectorfield on \( A \) gives a fiber preserving map which takes \( A \) into \( \pi^{-1}(N) \). Therefore given any point \( q \) in \( \mathbb{R}^n \times T_P^n \),

\[
0 \longrightarrow H^*(\mathbb{R}^n \times T_P^n) \longrightarrow H^*(A) \longrightarrow H^*(\pi^{-1}(q)) \longrightarrow 0
\]

is an exact sequence, as required. □
Corollary 11.2. Cohomology does not change in connected families of exploded manifolds. Given any connected family $\pi : A \to B$ of exploded manifolds and two points $p, q \in B$, there exists an isomorphism $\psi : H^*(\pi^{-1}(p)) \to H^*(\pi^{-1}(q))$ which preserves wedge products and integration.

Proof:
First suppose that $p$ and $q$ are contained inside a single coordinate chart $\mathbb{R}^n \times T^m_P$ on $B$. We may not be able to use Lemma 11.1, because $P$ may not be bounded. On the other hand, we may simply chose a bounded polytope $P' \subset P$ which contains both $p$ and $q$. Then we may restrict our family to $\mathbb{R}^n \times T^m_{P'}$ and apply Lemma 11.1 to see that

$$H^*(\pi^{-1}(p)) = H^*(\pi^{-1}(\mathbb{R}^n \times T^m_{P'}))/\pi^*H^*(\mathbb{R}^n \times T^m_{P'}) = H^*(\pi^{-1}(q))$$

As pullback maps preserve wedge products, this isomorphism preserves the product structure. To see that it preserves integration, suppose that the dimension $\pi^{-1}(p)$ is $k$. Represent a class $\theta \in H^k(\pi^{-1}(p))$ as a closed differential form $\theta \in \Omega^k(\pi^{-1}(\mathbb{R}^n \times T^m_{P'}))$. Restricting $\theta$ to $\pi^{-1}(q)$ is how our isomorphism is constructed. Integrate this class along the fiber to get $\pi_!(\theta)$, which is a closed 0-form on $\mathbb{R}^n \times T^m_{P'}$. In other words $\pi_!(\theta)$ is a constant function - so the integral of $\theta$ along $\pi^{-1}(p)$ is equal to the integral of $\theta$ along $\pi^{-1}(q)$. Therefore, the above isomorphism respects both integration and wedge products.

In general, as $B$ is connected, we may choose a finite sequence of points $p = p_0, p_1, \ldots, p_n = q$ so that $p_i$ and $p_{i+1}$ are in the same coordinate chart. Then applying the above argument to each of the above coordinate charts gives that $H^*(\pi^{-1}(p)) = H^*(\pi^{-1}(q))$ where the isomorphism preserves wedge products and integration.

Note that the cohomology of different fibers in a family is not necessarily canonically isomorphic. If the base of the family is not simply connected, then different homotopy classes of paths in the base may correspond to different isomorphisms.

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