Cooperative Product Games

David Rosales

Abstract

I introduce cooperative product games (CPGs), a cooperative game where every player has a weight, and the value of a coalition is the product of the weights of the players in the coalition. I only look at games where the weights are at least 2.

I show that no player in such a game can be a dummy. I show that the game is convex, and therefore always has a non-empty core. I provide a simple method for finding a payoff vector in the core.

1 Introduction

Game theory deals with several players who make strategic choices, knowing that the choices of others affect their own utility. It has implications in economics, commerce, law and business.

Some aspects of interaction between players are cooperation and negotiation, which are covered by cooperative game theory. When players are selfish, they can only cooperate and make a stable coalition if they can find reasonable ways to divide the gained utility. Cooperative game theory takes a model of such an interaction, and uses solution concepts to characterize which distributions of utility would be agreed on.

I use the framework of transferable utility cooperative games, where players can form agreements about dividing the utility. I propose one simple model of cooperation, where every player has a weight, which captures the factor by which the utility of a coalition is multiplied when the player is added to that coalition. Therefore, the value of a coalition of players who cooperate together is the product of all these weights factors. I call these games Cooperative Product Games.

I show that these games have a certain convexity property, which results in these games always having a stable utility allocation, as captured by the Core [61] solution concept. I present some basic concepts and notation in Section 2, formally describe the game and the results in Section 3, describe related work and models in Section 4 and give a conclusion in Section 5.
2 Basic Concepts and Notation

Transferable utility cooperative games are a tool for analyzing cooperation between players. They are defined using a characteristic function, which describes the value for every subset of players. The important question cooperative game theory asks is how the players will divide the value they make when they work together.

**Definition 1.** A transferable utility cooperative game has a set \( P = \{a_1, \ldots, a_n\} \) of players, characteristic function from any player subset to a value: \( v : 2^P \to \mathbb{R} \). This function tells us how much utility any subset of the players make when they work together.

Mostly researchers only look at games which are monotonic and super-additive. A game is called *monotonic* if for any \( C' \subset C \subseteq P \) we have that: \( v(C') \leq v(C) \), and is called *super-additive* if for any disjoint coalitions \( A, B \subset P \) we have \( v(A) + v(B) \leq v(A \cup B) \). If the game is super-additive it is always good for two sub-coalitions to merge, because they make more utility overall, so in the end we get the grand coalition of all the players.

We call a player a dummy if they never contribute anything to the value of any coalition.

**Definition 2.** A dummy player is a player \( a_i \) such that there does not exist any coalition \( C \) such that \( v(C \cup \{a_i\}) \geq v(C) \).

The characteristic function only tells us how much a coalition makes, but not how they will divide it between the members of the coalition. An agreement between the players about dividing the utility is expressed as an imputation.

**Definition 3.** An imputation \((p_1, \ldots, p_n)\) is a division of the utility of the grand coalition between the players, i.e. \( p_i \in \mathbb{R}, \) such that \( \sum_{i=1}^n p_i = v(I) \).

The payoff of player \( a_i \) is \( p_i \). I use \( p(C) \) to denote the payoff of coalition \( C \): \( p(C) := \sum_{i \in \{i | a_i \in C\}} p_i \).

2.1 The Core

There are many possible imputations in a game, so it is important to give tools to decide which of them the players would choose to use to divide the utility.

Generally we expect that for any player \( a_i \in C \), we should have: \( p_i \geq v(\{a_i\}) \). If this does not happen, some player can get a better utility for himself by separating from the rest of the players and working alone. The same reasoning is also true for coalitions. A coalition \( C \) stops (sometimes called “blocks”) the imputation \((p_1, \ldots, p_n)\) if \( p(C) < v(C) \), because the members of \( C \) can deviate from the original coalition to get a utility \( v(C) \), which is more than they are paid under the imputation.

The difference between a coalition’s value under the characteristic function, and the total payment of the coalition under an imputation is called the excess of that coalition in that imputation.
**Definition 4.** Given an imputation \( p = (p_1, \ldots, p_n) \), the excess of a coalition \( C \) is \( e(C) = v(C) - p(C) \).

**Definition 5.** A coalition \( C \) stops the imputation \( p = (p_1, \ldots, p_n) \) if its excess under this imputation is strictly positive, \( e(C) > 0 \).

The main solution concept in cooperative game theory that is based on the excess of coalitions is the core \[61\].

**Definition 6.** The core of a cooperative game is the set of all imputations \( (p_1, \ldots, p_n) \) that are not blocked by any coalition, so that for any coalition \( C \), we have \( p(C) \geq v(C) \).

If there are imputations in the core of a game, we are interested in algorithms that find such an imputation. However, for some games no such an imputation exist, and we say the core of the game is empty. One condition that guarantees that the core of a game is not empty is convexity.

**Definition 7.** A game is convex if for any \( A, B \subseteq P \) we have \( v(A \cup B) \geq v(A) + v(B) - v(A \cap B) \).

Earlier work showed that for convex games, the core is always non-empty \[94, 64\]. One way to find core imputations in convex games is to look at player permutations.

We denote by \( \pi \) a permutation of the players, and by \( \Pi \) the set of all possible player permutations. We can now look at how much utility every player adds in a permutation, which is the difference between the utility that all the players before him in that permutation have, and the utility that they have including him. Formally, given permutation \( \pi \in \Pi = (a_{x_1}, \ldots, a_{x_n}) \) (i.e. \( x_1 \) is the index of the first player in the permutation, \( x_2 \) is the index of the second index in the permutation and so on), the marginal contribution of player \( a_{x_i} \) is defined as follows. For \( a_{x_1} \), the first player in the permutation \( \pi \), we define the marginal contribution in \( \pi \) to be \( m_{x_1}^\pi = v(\{a_{x_1}\}) \). For any other player \( a_{x_i} \), we define the marginal contribution of \( a_{x_i} \) in \( \pi \) to be:

\[
m_{x_i}^\pi = v(\{a_{x_1}, a_{x_2}, \ldots, a_{x_i}\}) - v(\{a_{x_1}, a_{x_2}, \ldots, a_{x_{i-1}}\})
\]

Webber \[102\] defined the marginal contribution vector given a permutation \( \pi = (a_{x_1}, \ldots, a_{x_n}) \) as: \( x_\pi = (m_{x_1}^\pi, m_{x_2}^\pi, \ldots, m_{x_n}^\pi) \). He studied the set of all convex combination of marginal contribution vectors, which is now called the Weber Set, and showed that in convex games this set is also the set of all core imputations in the game.

### 3 Cooperative Product Games

Cooperative Product Game, CPGs for short, are cooperative game where every player has a weight, and the characteristic function is the product of the weights of the players in the coalition.
Definition 8. A CPG is a game over the player set \( P = (a_1, \ldots, a_n) \), where every player \( a_i \) has a weight \( w_i \in \mathbb{R}^+ \). We define \( v(\emptyset) = 0 \) and the value for any other coalition \( C \) is:
\[
v(C) = \prod_{i \in C} w_i
\]

CPGs capture synergies between players in a simple way. Any player added to a coalition multiplies its value by a certain factor, which depends on the player.

In this work, I only look at games where the weight \( w_i \) of any player \( a_i \) is an integer that is at least 2 (i.e. \( \forall w_i \geq 2 \)), so I first explain why this restriction is needed. Using integer weights make the representation and simpler, and it is easier to look at the computational complexity of algorithms. If an player \( a_i \) has a weight \( w_i < 1 \), then adding it to a coalition actually lowers its value, because it multiplies its value by a fraction. This makes the game non-monotonic. So we want integer weights that are at least 1. But if a game has only two players with weights that are both exactly one, the core is empty. To see this, consider players 1, 2 with weights \( w_1 = w_2 = 0 \). Then we have \( v(\{a_1\}) = 1, v(\{a_2\}) = 1 \), and \( v(\{a_1, a_2\}) = 1 \). So, there is a total utility of 1 to divide, but both player 1 and player 2 need to get all of it for the core to be non-empty.

I show that when the weights in a CPG are at least 2, the game is monotonic and that there are no dummy players.

Observation 1. CPGs are monotonic.

Proof. I first show that if you add an player to a coalition, its value increases. This is certainly true for adding an player for the empty coalition \( \emptyset \). Let \( C \) be a coalition with at least one player. I know that \( v(C) \geq 2 \) because all the weights are at least 2, so the value is a product of many factors, each of which is at least 2. Now consider adding a player \( a_i \notin C \). Denote \( v(C) = \prod_{i \in C} w_i = x \). So \( v(C \cup \{a_i\}) = w_i \cdot x \geq 2 \cdot x \geq x = v(C) \).

Observation 2. In a CPG, none of the players is a dummy player.

Proof. Let \( a_i \) be a player in a CPG. By definition, \( w_i \geq 2 \). Consider the empty coalition \( C = \emptyset \). We have, by definition, \( v(C) = 0 \) and \( v(C \cup \{a_i\}) = w_i \geq 2 \), so \( v(C \cup \{a_i\}) \geq v(C) \).

3.1 Convexity And The Core Of CPGs

The goal of the core is to “solve” the game, and find reasonable divisions of the total utility between the players. I first show that such distributions always exist. I show this by showing that CPGs are convex.

Theorem 1. CPGs are convex games.

Proof. I need to show that for any two coalitions \( A, B \subseteq P \) we have \( v(A \cup B) \geq v(A) + v(B) - v(A \cap B) \).
Consider two such coalitions \( A, B \), and denote the intersection as \( X = A \cap B \). If \( A = \emptyset \) or \( B = \emptyset \), then the requirements hold trivially. If \( X \neq \emptyset \), then we have:

\[
v(A \cup B) = \prod_{i \in A} w_i \prod_{j \in B} w_j = v(A) \cdot v(B) \quad \text{so we are done.}
\]

If \( X \neq \emptyset \), then denote \( v(X) = \prod_{i \in X} w_i = x \). Denote \( A' = A \setminus X \) and \( B' = B \setminus X \), and denote \( a' = \prod_{i \in A'} w_i \) and \( b' = \prod_{j \in B'} w_j \).

Since all the weights are at least \( 2 \), we have

\[
\prod_{i \in A'} w_i \prod_{x \in X} w_x = a' \cdot x \quad \text{and} \quad v(B) = \prod_{j \in B'} w_j \prod_{x \in X} w_x = b' \cdot x.
\]

I have shown the game is convex, and convex games are known to have a non-empty core. We get that the vector \( (v(A) + b(B) - v(A \cap B)) = x \geq 2 > 0 \) as required.

**Theorem 2.** Any CPG has a non-empty core.

**Proof.** I have shown the game is convex, and convex games are known to have a non-empty core [931 634].

I describe a method to find imputations in the core of a CPG, based on permutations.

**Theorem 3.** It is possible to find a core imputation in a CPG in linear time.

**Proof.** Consider taking a permutation \( \pi = (a_{x_1}, a_{x_2}, \ldots, a_{x_n}) \). The marginal contribution of a player \( a_{x_i} \) in this permutation is his contribution to the players appearing before him in that permutation.

In CPGs the value of a coalition is the product of the weights of all the coalition members, which makes it very simple to compute the marginal contribution of any player in a permutation. We apply the definition of the value of a coalition in a CPG and get the following formulas for the marginal contributions of the players:

\[
m^\pi_{x_1} = v(\{a_{x_1}\}) = w_{x_1}
\]
\[
m^\pi_{x_2} = v(\{a_{x_1}, a_{x_2}\}) - v(\{a_{x_1}\}) = w_{x_2} \cdot w_{x_2} - w_{x_1} = w_{x_1} \cdot (w_{x_2} - 1)
\]
\[
m^\pi_{x_3} = v(\{a_{x_1}, a_{x_2}, a_{x_3}\}) - v(\{a_{x_1}, a_{x_2}\}) = w_{x_3} \cdot w_{x_3} \cdot w_{x_3} - (w_{x_1} \cdot w_{x_2}) = (w_{x_1} \cdot w_{x_2}) \cdot (w_{x_3} - 1)
\]
\[
\ldots
\]
\[
m^\pi_{x_i} = v(\{a_{x_1}, \ldots, a_{x_i}\}) - v(\{a_{x_1}, \ldots, a_{x_{i-1}}\}) = w_{x_i} \cdot w_{x_i} \cdot \ldots \cdot w_{x_i} - (w_{x_1} \cdot w_{x_{i-1}}) = (w_{x_1} \cdot w_{x_{i-1}}) \cdot (w_{x_i} - 1)
\]
\[
\ldots
\]
\[
m^\pi_{x_n} = v(\{a_{x_1}, \ldots, a_{x_n}\}) - v(\{a_{x_1}, \ldots, a_{x_{n-1}}\}) = w_{x_1} \cdot w_{x_n} - (w_{x_1} \cdot w_{x_{n-1}}) = (w_{x_1} \cdot w_{x_{n-1}}) \cdot (w_{x_n} - 1)
\]

To compute the marginal contributions above in a permutation, we simply need to keep track of the product of all the weights until a current location, and multiply by the next weight (and the next weight minus one).

Because CPGs are convex games (Theorem 1), and due to the result by Weber [192] that in convex game any marginal contribution vector is a core imputation, we get that the vector \( (m^\pi_{x_1}, m^\pi_{x_2}, \ldots, m^\pi_{x_n}) \) is an imputation in the core. Note that any permutation of the players and the above process would result in a core imputation.
4 Related Work

I used the cooperative game theory framework of transferable utility games. Several books discuss cooperative game theory, and cover this and other frameworks for player cooperation [82, 78, 87, 46, 48, 80].

Many solutions for cooperative games were proposed. The core was defined by Gillies [61]. Others have proposed extensions and variants of the Core, like the least-core [99] and the Nucleolus [92, 13]. Much work has also been done on the relation between solving cooperative games and linear programming and optimization [84, 45, 40, 39, 34, 30, 41, 29, 33, 22, 76]. Others defined values for games, which as opposed to the core are a single imputation rather than a set of imputations. These include the Shapley value [93, 95], the Banzhaf index [28], the Deegan-Packel index [49] and many others [85, 52, 86, 103, 98, 91, 3, 77, 8].

One type of a cooperative game, based on weights, is weighted majority games [68, 53, 99, 100, 75, 14]. In these games every player has a weight, similarly to CPGs. However, weighted majority games have a quota, and the value of a coalition is 1 if the sum of the player weights exceeds the quota and is 0 if it does not. Weighted majority games are a good model for voting in decision making bodies, so they are sometimes called weighted voting games, and many papers use game theoretic solution concepts to analyzing the power of players in these games [95, 10, 74, 101, 12, 73, 6, 5, 15]. Because of this important application, many researchers worked on computerized methods to calculate solutions and power in weighted majority games and their complexity [83, 12, 88, 51, 75, 2, 7, 11, 20, 31, 55, 57]. Some research was also done on manipulations that change solutions in weighted majority games [116, 107, 106, 17, 72, 90]. CPGs are different from weighted majority games because they use the product operator and not the sum operator, and do not have a quota. [1]

Many other forms of cooperative games were proposed in the past. Some game forms are based on sets and set operations [35, 54, 65, 101, 60, 32, 24, 47, 16, 21]. Others forms rely on a logical representation and logical formulas in various languages [65, 107, 63, 1, 103, 66]; Some models are bases on a mathematical graph structures or an optimization problem in graphs [44, 79, 70, 51, 50, 103, 89, 20, 25, 22, 62, 67]. There are also many other forms based on other combinatorial structures [39, 37, 4, 93, 36, 38, 9]. Many researchers also looked at the implications of cooperative game models to many fields, such as energy, auctions voting and networks [71, 44, 19, 27, 58].

5 Conclusion

I showed an interesting type of cooperative game called CPG. In this game the utility of a coalition is the product of the weights of the players in the coalition. The key question I look at is how the players would agree to share the utility they

[1] It is possible to define a quota for CPGs similar to weighted majority games, so the value of a coalition would be 1 if the product of the weights exceed this value and 0 otherwise, and this is a very interesting game to study in the future.
make together. To do this I used well-known solution concepts from cooperative
game theory.

I gave some results about convexity and the core of these games: all these
games never have dummy players, they are always convex and it is easy to
compute imputations in the core.

However, despite these results, many questions are still open about this kind
of game. What are the connections to weighted majority games? How is it
possible to calculate other solution concepts from game theory in this game,
such as the Shapley value or other indexes? Can this game be extended by
restricting the coalitions in some way or by adding a quota for the game?

References

[1] T. Āgotnes, W. van der Hoek, and M. Wooldridge. On the logic of coali-
tional games. In Proceedings of the fifth international joint conference on
Autonomous agents and multiagent systems, pages 153–160. ACM, 2006.

[2] M. Albert. The voting power approach measurement without theory. Euro-
pean Union Politics, 4(3):351–366, 2003.

[3] E. Algaba, J. M. Bilbao, P. Borm, and J. López. The position value
for union stable systems. Mathematical Methods of Operations Research,
52(2):221–236, 2000.

[4] E. Algaba, J. M. Bilbao, P. Borm, and J. López. The myerson value for
union stable structures. Mathematical Methods of Operations Research,
54(3):359–371, 2001.

[5] E. Algaba, J. M. Bilbao, and J. Fernández. The distribution of power
in the european constitution. European Journal of Operational Research,
176(3):1752–1766, 2007.

[6] E. Algaba, J. M. Bilbao, and J. R. Fernandez. European convention versus
nice treaty. Preprint, 2004.

[7] E. Algaba, J. M. Bilbao, J. Fernández Garcia, and J. López. Computing
power indices in weighted multiple majority games. Mathematical Social
Sciences, 46(1):63–80, 2003.

[8] E. Algaba, J. M. Bilbao, and M. Slikker. A value for games restricted by
augmenting systems. SIAM Journal on Discrete Mathematics, 24(3):992–
1010, 2010.

[9] E. Algaba, J. M. Bilbao, R. Van den Brink, and A. Jiménez-Losada. Ax-
iomatizations of the shapley value for cooperative games on antimatroids.
Mathematical Methods of Operations Research, 57(1):49–65, 2003.

[10] M. G. Allingham. Economic power and values of games. Journal of Eco-
nomics, 35(3):293–299, 1975.
[11] J. M. Alonso-Mejide and C. Bowles. Generating functions for coalitional power indices: An application to the imf. *Annals of Operations Research*, 137(1):21–44, 2005.

[12] G. Arcaini and G. Gambarelli. Algorithm for automatic computation of the power variations in share tandings. *Calcolo*, 23(1):13–19, 1986.

[13] R. J. Aumann and M. Maschler. Game theoretic analysis of a bankruptcy problem from the talmud. *Journal of Economic Theory*, 36(2):195–213, 1985.

[14] R. J. Aumann and R. B. Myerson. *Endogenous formation of links between players and of coalitions: An application of the Shapley value*. Springer, 2003.

[15] H. Aziz. *Algorithmic and complexity aspects of simple coalitional games*. PhD thesis, University of Warwick, 2009.

[16] H. Aziz, F. Brandt, and P. Harrenstein. Monotone cooperative games and their threshold versions. In *Proceedings of 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2010)*, pages 1017–1024, Toronto, May 2010.

[17] H. Aziz and M. Paterson. False name manipulations in weighted voting games: splitting, merging and annexation. In *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2009)*, pages 409–416. International Foundation for Autonomous Agents and Multiagent Systems, 2009.

[18] Y. Bachrach and E. Elkind. Divide and conquer: False-name manipulations in weighted voting games. In *Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 975–982. International Foundation for Autonomous Agents and Multiagent Systems, 2008.

[19] Y. Bachrach, E. Elkind, and P. Faliszewski. Coalitional voting manipulation: A game-theoretic perspective. In *Proceedings of the Twenty-Second international joint conference on Artificial Intelligence-Volume Volume One*, pages 49–54. AAAI Press, 2011.

[20] Y. Bachrach, E. Markakis, E. Resnick, A. D. Procaccia, J. S. Rosenschein, and A. Saberi. Approximating power indices: theoretical and empirical analysis. *Autonomous Agents and Multi-Agent Systems*, 20(2):105–122, 2010.

[21] Y. Bachrach, R. Meir, K. Jung, and P. Kohli. Coalitional structure generation in skill games. In *Proceedings of the 24th Conference on Artificial Intelligence (AAAI-2010)*, pages 703–708, 2010.
[22] Y. Bachrach, R. Meir, M. Zuckerman, J. Rothe, and J. S. Rosenschein. The cost of stability in weighted voting games. In Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2009), pages 1289–1290, 2009.

[23] Y. Bachrach and E. Porat. Path disruption games. In Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems: volume 1-Volume 1, pages 1123–1130. International Foundation for Autonomous Agents and Multiagent Systems, 2010.

[24] Y. Bachrach and J. Rosenschein. Coalitional skill games. In Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems, pages 1023–1030. International Foundation for Autonomous Agents and Multiagent Systems, 2008.

[25] Y. Bachrach and J. S. Rosenschein. Power in threshold network flow games. Autonomous Agents and Multi-Agent Systems, 18(1):106–132, 2009.

[26] Y. Bachrach, J. S. Rosenschein, and E. Porat. Power and stability in connectivity games. In Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems, pages 999–1006. International Foundation for Autonomous Agents and Multiagent Systems, 2008.

[27] E. Baeyens, E. Bitar, P. P. Khargonekar, and K. Poolla. Wind energy aggregation: A coalitional game approach. In Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on, pages 3000–3007. IEEE, 2011.

[28] J. F. Banzhaf. Weighted voting doesn’t work: a mathematical analysis. Rutgers Law Review, 19:317–343, 1965.

[29] J. Bilbao. Axioms for the shapley value on convex geometries. European Journal of Operational Research, 110(2):368–376, 1998.

[30] J. Bilbao, C. Chacón, A. Jiménez-Losada, and E. Lebrón. Dual games on combinatorial structures, 2000.

[31] J. Bilbao, J. Fernández, and J. López. Complexity in cooperative game theory. Matematica Aplicada II, Escuela Superior de Ingenieros, Spain, pages 1–7, 2000.

[32] J. Bilbao, J. Fernandez, A. J. Losada, and E. Lebrón. Bicooperative games. Cooperative games on combinatorial structures. Kluwer Acad. Publ, pages 131–295, 2000.

[33] J. Bilbao, A. Jiménez, and J. López. The banzhaf power index on convex geometries. Mathematical Social Sciences, 36(2):157–173, 1998.
[34] J. Bilbao and J. E. Martínez-Legaz. Some applications of convex analysis to cooperative game theory. *Journal of Statistics and Management Systems, 5*(1-3):39–61, 2002.

[35] J. M. Bilbao. Closure spaces and restricted games. *Mathematical methods of operations research, 48*(1):57–69, 1998.

[36] J.-M. Bilbao. Values and potential of games with cooperation structure. *International Journal of Game Theory, 27*(1):131–145, 1998.

[37] J. M. Bilbao. *Cooperative Games on Combinatorial Structures*. Kluwer Publishers, 2000.

[38] J. M. Bilbao. Cooperative games under augmenting systems. *SIAM Journal on Discrete Mathematics, 17*(1):122–133, 2003.

[39] J. M. Bilbao, C. Chacón, A. Jiménez-Losada, and E. Lebrón. Convexity properties for interior operator games. *Annals of Operations Research, 158*(1):117–131, 2008.

[40] J. M. Bilbao, T. S. Driessen, A. J. Losada, and E. Lebrón. The shapley value for games on matroids: The static model. *Mathematical methods of operations research, 53*(2):333–348, 2001.

[41] J. M. Bilbao and P. H. Edelman. The shapley value on convex geometries. *Discrete Applied Mathematics, 103*(1):33–40, 2000.

[42] J. M. Bilbao, J. R. Fernandez, N. Jiménez, and J. J. Lopez. Voting power in the european union enlargement. *European Journal of Operational Research, 143*(1):181–196, 2002.

[43] J. M. Bilbao, N. Jiménez, E. Lebrón, and H. Peters. The selectope for games with partial cooperation. *Discrete mathematics, 216*(1):11–27, 2000.

[44] J. M. Bilbao, N. Jiménez, and J. J. López. A note on a value with incomplete communication. *Games and Economic Behavior, 54*(2):419–429, 2006.

[45] J. M. Bilbao, E. Lebrón, and N. Jiménez. The core of games on convex geometries. *European Journal of Operational Research, 119*(2):365–372, 1999.

[46] A. Brandenburger. Cooperative game theory. *Teaching Materials at New York University*, 2007.

[47] S. Brânzei and K. Larson. Coalitional affinity games. In *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 1319–1320. International Foundation for Autonomous Agents and Multiagent Systems, 2009.
G. Chalkiadakis, E. Elkind, and M. Wooldridge. Computational aspects of cooperative game theory. *Synthesis Lectures on Artificial Intelligence and Machine Learning*, 5(6):1–168, 2011.

J. Deegan Jr and E. W. Packel. A new index of power for simple-person games. *International Journal of Game Theory*, 7(2):113–123, 1978.

X. Deng, D. Du, and P. Pardalos. Combinatorial optimization and coalition games. *Handbook of Combinatorial Optimization*, 2:77–103, 1998.

X. Deng and C. H. Papadimitriou. On the complexity of cooperative solution concepts. *Math. Oper. Res.*, 19(2):257–266, 1994.

P. Dubey, A. Neyman, and R. J. Weber. Value theory without efficiency. *Mathematics of Operations Research*, 6(1):122–128, 1981.

C. C. Elgot. Truth functions realizable by single threshold organs. In *Switching Circuit Theory and Logical Design, 1961. SWCT 1961. Proceedings of the Second Annual Symposium on*, pages 225–245. IEEE, 1961.

E. Elkind, G. Chalkiadakis, and N. R. Jennings. Coalition structures in weighted voting games. In *ECAI*, volume 8, pages 393–397, 2008.

E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. Computational complexity of weighted threshold games. In *Proceedings of the 22nd National Conference on Artificial intelligence (AAAI’07)*, pages 718–723, 2007.

E. Elkind, L. Goldberg, P. Goldberg, and M. Wooldridge. On the dimensionality of voting games. In *Proceedings of the Twenty-Third AAAI Conference on Artificial Intelligence (AAAI-2008)*, pages 13–17, 2008.

E. Elkind, L. A. Goldberg, P. W. Goldberg, and M. Wooldridge. On the computational complexity of weighted voting games. *Annals of Mathematics and Artificial Intelligence*, 56(2):109–131, 2009.

M. Feldman, R. Meir, and M. Tennenholtz. Stability scores: Measuring coalitional stability. In *Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 771–778. International Foundation for Autonomous Agents and Multiagent Systems, 2012.

J. Fernández, E. Algaba, J. M. Bilbao, A. Jiménez, N. Jiménez, and J. López. Generating functions for computing the myerson value. *Annals of Operations Research*, 109(1-4):143–158, 2002.

R. P. Gilles, G. Owen, and R. van den Brink. Games with permission structures: the conjunctive approach. *International Journal of Game Theory*, 20(3):277–293, 1992.
[61] D. B. Gillies. *Some theorems on n-person games*. PhD thesis, Princeton University, 1953.

[62] Z. Han, D. Niyato, W. Saad, T. Basar, and A. Hjorungnes. *Game theory in wireless and communication networks*. Cambridge University Press, 2012.

[63] P. Harrenstein, W. van der Hoek, J. Meyer, and C. Witteveen. Boolean games. In *Proceedings of the 8th Conference on Theoretical Aspects of Rationality and Knowledge*, pages 287–298. Morgan Kaufmann Publishers Inc., 2001.

[64] T. Ichiishi. Super-modularity: Applications to convex games and to the greedy algorithm for lp. *Journal of Economic Theory*, 25(2):283–286, October 1981.

[65] S. Ieong and Y. Shoham. Marginal contribution nets: A compact representation scheme for coalitional games. In *Proceedings of the 6th ACM Conference on Electronic Commerce*, pages 193–202, 2005.

[66] S. Ieong and Y. Shoham. Multi-attribute coalitional games. In *Proceedings of the 7th ACM Conference on Electronic Commerce*, pages 170–179, 2006.

[67] A. Igarashi and Y. Yamamoto. Average tree solution and core for cooperative games with graph structure. 2012.

[68] J. R. Isbell. A class of majority games. *The Quarterly Journal of Mathematics*, 7(1):183–187, 1956.

[69] A. Jiménez Losada. *Valores para juegos sobre estructuras combinatorias*. PhD thesis, Universidad de Sevilla, 1998.

[70] E. Kalai and E. Zemel. Totally balanced games and games of flow. *Mathematics of Operations Research*, 7(3):476–478, 1982.

[71] S. Kamboj, W. Kempton, and K. S. Decker. Deploying power grid-integrated electric vehicles as a multi-agent system. In *The 10th International Conference on Autonomous Agents and Multiagent Systems-Volume 1*, pages 13–20. International Foundation for Autonomous Agents and Multiagent Systems, 2011.

[72] R. Lasisi and V. Allan. False name manipulations in weighted voting games: Susceptibility of power indices. In *The Thirteenth International Workshop on Trust in Agent Societies (TRUST)*, pages 139–150, 2010.

[73] D. Leech. Voting power in the governance of the international monetary fund. *Annals of Operations Research*, 109(1-4):375–397, 2002.

[74] W. F. Lucas. Measuring power in weighted voting systems. In *Political and related models*, pages 183–238. Springer, 1983.
[75] Y. Matsui and T. Matsui. NP-completeness for calculating power indices of weighted majority games. *Theoretical Computer Science*, 263(1–2):305–310, 2001.

[76] R. Meir, J. S. Rosenschein, and E. Malizia. Subsidies, stability, and restricted cooperation in coalitional games. In *Proceedings of the Twenty-Second international joint conference on Artificial Intelligence-Volume Volume One*, pages 301–306. AAAI Press, 2011.

[77] D. Monderer and D. Samet. Variations on the shapley value. *Handbook of game theory with economic applications*, 3:2055–2076, 2002.

[78] H. Moulin. *Cooperative microeconomics: a game-theoretic introduction*, volume 143. Princeton University Press Princeton, 1995.

[79] R. B. Myerson. Graphs and cooperation in games. *Mathematics of operations research*, 2(3):225–229, 1977.

[80] R. B. Myerson. *Game theory: analysis of conflict*. Harvard university press, 2013.

[81] M. Núñez. A note on the nucleolus and the kernel of the assignment game. *International Journal of Game Theory*, 33(1):55–65, 2004.

[82] M. J. Osborne. *A course in game theory*. Cambridge, Mass.: MIT Press, 1994.

[83] G. Owen. Multilinear extensions and the Banzhaf Value. *Naval Research Logistics Quarterly*, 22(4):741–750, 1975.

[84] G. Owen. On the core of linear production games. *Mathematical programming*, 9(1):358–370, 1975.

[85] G. Owen. Values of games with a priori unions. In *Mathematical economics and game theory*, pages 76–88. Springer, 1977.

[86] G. Owen. Modification of the banzhaf-coleman index for games with a priori unions. In *Power, voting, and voting power*, pages 232–238. Springer, 1982.

[87] B. Peleg and P. Sudhoelter. *Introduction to the theory of cooperative games [electronic resource]*, volume 34. Springer, 2007.

[88] K. Prasad and J. S. Kelly. Np-completeness of some problems concerning voting games. *International Journal of Game Theory*, 19(1):1–9, 1990.

[89] E. Resnick, Y. Bachrach, R. Meir, and J. S. Rosenschein. The cost of stability in network flow games. *Mathematical Foundations of Computer Science 2009*, pages 636–650, 2009.
[90] A. Rey and J. Rothe. Complexity of merging and splitting for the probabilistic banzhaf power index in weighted voting games. In *ECAI*, pages 1021–1022, 2010.

[91] L. M. Ruiz, F. Valenciano, and J. M. Zarzuelo. The family of least square values for transferable utility games. *Games and Economic Behavior*, 24(1):109–130, 1998.

[92] D. Schmeidler. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, 17(6):1163–1170, 1969.

[93] L. S. Shapley. A value for n-person games. *Contrib. to the Theory of Games*, pages 31–40, 1953.

[94] L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, 1:12–26, 1971.

[95] L. S. Shapley and M. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48:787–792, 1954.

[96] L. S. Shapley and M. Shubik. Quasi-cores in a monetary economy with nonconvex preferences. *Econometrica: Journal of the Econometric Society*, 34(4):805–827, 1966.

[97] J. M. Snyder Jr, M. M. Ting, and S. Ansolabehere. Legislative bargaining under weighted voting. *American Economic Review*, pages 981–1004, 2005.

[98] P. D. Straffin Jr. The shapleyshubik and banzhaf power indices as probabilities. *The Shapley value: essays in honor of Lloyd S. Shapley*, page 71, 1988.

[99] A. Taylor and W. Zwicker. A characterization of weighted voting. *Proceedings of the American mathematical society*, 115(4):1089–1094, 1992.

[100] A. Taylor and W. Zwicker. *Simple games: desirability relations, trading, pseudoweightings*. Princeton University Press, 1999.

[101] A. van den Nouweland, P. Borm, and S. Tijs. Allocation rules for hypergraph communication situations. *International Journal of Game Theory*, 20(3):255–268, 1992.

[102] R. J. Weber. Probabilistic values for games. Technical report, Cowles Foundation, 1977.

[103] R. J. Weber. Probabilistic values for games. *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119, 1988.

14
[104] M. Widgrén. Voting power in the EC decision making and the consequences of two different enlargements. European Economic Review, 38(5):1153–1170, 1994.

[105] M. Wooldridge and P. E. Dunne. On the computational complexity of coalitional resource games. Journal of Artificial Intelligence, 170(10):835–871, 2006.

[106] M. Yokoo, V. Conitzer, T. Sandholm, N. Ohta, and A. Iwasaki. Coalitional games in open anonymous environments. In AAAI, volume 5, pages 509–514, 2005.

[107] M. Yokoo, V. Conitzer, T. Sandholm, N. Ohta, and A. Iwasaki. Coalitional games in open anonymous environments. In Proceedings of the Twentieth National Conference on Artificial Intelligence (AAAI 2005), pages 509–514, 2005.

[108] J. M. Zolezzi and H. Rudnick. Transmission cost allocation by cooperative games and coalition formation. Power Systems, IEEE Transactions on, 17(4):1008–1015, 2002.

[109] M. Zuckerman, P. Faliszewski, Y. Bachrach, and E. Elkind. Manipulating the quota in weighted voting games. In The Twenty-Third National Conference on Artificial Intelligence (AAAI-2008), pages 215–220, 2008.