Ground-state properties of one-dimensional anyon gases

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We investigate the ground state of the one-dimensional interacting anyonic system based on the exact Bethe ansatz solution for arbitrary coupling constant \(0 \leq c \leq \infty\) and statistics parameter \((0 \leq \kappa \leq \pi)\). It is shown that the density of state in quasi-momentum \(k\) space and the ground state energy are determined by the renormalized coupling constant \(c'\). The effect induced by the statistics parameter \(\kappa\) exhibits in the momentum distribution in two aspects: Besides the effect of renormalized coupling, the anyonic statistics results in the nonsymmetric momentum distribution when the statistics parameter \(\kappa\) deviates from 0 (Bose statistics) and \(\pi\) (Fermi statistics) for any coupling constant \(c\). The momentum distribution evolves from a Bose distribution to a Fermi one as \(\kappa\) varies from 0 to \(\pi\). The asymmetric momentum distribution comes from the contribution of the imaginary part of the non-diagonal element of reduced density matrix, which is an odd function of \(\kappa\). The peak at positive momentum will shift to negative momentum if \(\kappa\) is negative.

I. INTRODUCTION

The physical systems with fractional statistics have been a subject of great interest [1, 2, 3]. Although initial studies of fractional statistics have focused on two-dimensional systems [4, 5, 6, 7], anyons (the particles obeying the fractional statistics) have also found applications in various one-dimensional (1D) systems [2, 7, 8, 9]. Recently, many theoretical works have been dedicated to study the 1D anyon gas [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] since the integrable model of anyon gas with a \(\delta\)-function interaction was introduced by Kondo [8] despite no proof of the realization of anyon gas in 1D. Because the statistical properties are related with the topological order, currently anyons also draw intensive attention in topological quantum computation. Particularly, with the rapid progress in the regime of cold atom and with the excellent controllability of the neutral atoms trapped in the optical lattice, cold atoms have become a popular experimental platform in many research regimes. Rotating Bose-Einstein condensates and cold atoms in optical lattice have also been proposed to create, manipulate, and test anyons [20, 21, 22].

On the other hand, the experimental progress of trapped 1D cold atom systems [23, 24, 25, 20] has triggered more and more theorists to study the many body physics of 1D correlated systems beyond the mean-field theory [27, 28, 29, 30, 31, 32]. Further, the ability of tuning the effective 1D interactions by Feshbach resonance leads experiment accessible to the strong correlation regime [25, 26]. In the light of recent experiments, studies of the integrable model have re-attracted much attention today [33, 34, 35, 36, 37] because with its exact solution we can obtain the properties in the full physical regime even in the strong correlated limit. By applying the exact solution of Lieb-Liniger to the 1D interacting Bose gas in a hard-wall trap [38, 39], the evolution from the Bose-Einstein condensate to "fermionized" Tonks-Girardeau (TG) gas was explicitly and exactly displayed in our previous works [35, 36], which has also been confirmed by various numerical methods [40, 41, 42]. Although the density distribution of Bose gas in the limit of infinite repulsion is identical to the free Fermi gas [43], the momentum distributions exhibit quite different properties from the free Fermi gas because of their respective statistics [44, 45]. It is well known that the different permutation symmetry leads to the physics of Bose gases in many aspects differing from the physics of Fermi gases. As a natural generalization, the anyon gas interpolates between the Bose and Fermi gas. No doubt the integrable anyon model provides us the possibility to study the effect of generalized permutation symmetry in an exact manner although it is still a theoretically hypothetical model. So far much attention has focused on its thermodynamic properties and correlation functions [10, 11], however no general result for the momentum distribution has been given except for the anyon gas in the infinite interaction limit (or the anyonic TG gas) [10, 11]. In this paper, we investigate the ground state of the 1D anyonic system with finite number of anyons in the whole regime of interaction parameter, and particularly we shall study how the fractional statistics affect the ground state properties, such as the ground state energy and the momentum distribution. By numerically solving the Bethe ansatz equations, we obtain the ground state wave function and thus the momentum distribution for different coupling constant and statistics parameter. The obvious properties of anyonic statistics are displayed in the

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momentum distribution and with the change of statistics parameter the system exhibits the Bose statistics, Fermi statistics and the fractional statistics in between.

The paper is organized as follows. In Sec. II, we give a brief review of 1D anyonic model and introduce its solution with Bethe ansatz method. In Sec. III, we present the density of ground state in quasi-momentum space, ground state energy and momentum distributions for different coupling constant and statistics parameter. A brief summary is given in Sec. IV.

II. FORMULATION OF THE MODEL AND ITS SOLUTION

We consider the one dimensional system of $N$ anyons in length $L$ with second quantized Hamiltonian

$$H_A = \frac{\hbar^2}{2m} \int_0^L dx \frac{\partial \Psi_A^\dagger}{\partial x} \frac{\partial \Psi_A}{\partial x} + \frac{g_{1D}}{2} \int_0^L dx \Psi_A^\dagger (x) \Psi_A^\dagger (x) \Psi_A (x) \Psi_A (x),$$

in which $m$ is the mass of anyon and $g_{1D}$ denotes the effective interacting constant between anyons. The field operator $\Psi_A^\dagger (x)$ and $\Psi_A (x)$ obey the anyonic commutation relations

$$\Psi_A (x_1) \Psi_A^\dagger (x_2) = e^{-i\kappa(x_1-x_2)} \Psi_A^\dagger (x_2) \Psi_A (x_1) + \delta (x_1-x_2),$$

$$\Psi_A^\dagger (x_1) \Psi_A^\dagger (x_2) = e^{i\kappa(x_1-x_2)} \Psi_A^\dagger (x_2) \Psi_A^\dagger (x_1),$$

where the sign function $\epsilon (x)$ gives $-1$, $0$ or $1$ depending on whether $x$ is negative, zero, or positive. The parameter related with statistics and will be restricted in the regime $[0, \pi]$ in the present work. Particularly, $\kappa = 0$ and $\pi$ correspond to Bose statistics and Fermi statistics, respectively. A standard rescaling procedure brings the Hamiltonian into a dimensionless one

$$H_A = \int_0^L dx \left[ \frac{\partial \Psi_A^\dagger}{\partial x} \frac{\partial \Psi_A}{\partial x} + c \Psi_A^\dagger (x) \Psi_A^\dagger (x) \Psi_A (x) \Psi_A (x) \right]$$

with $c$ being the dimensionless coupling constant. The Hamiltonian take the same form as the Lieb-Liniger Bose model except that the field operator satisfy anyonic statistics. Similar to the Bose model the eigen problem for the Hamiltonian can be reduced to solve the problem of $N$-anyons with $\delta$-interaction

$$H \Psi (x_1, \ldots, x_N) = E \Psi (x_1, \ldots, x_N)$$

with

$$H = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2\epsilon \sum_{j<l} \delta(x_j-x_l).$$

The anyonic commutation relations require the wavefunction satisfy the generalized symmetry

$$\Psi (x_1, \ldots, x_N) = e^{-i\omega} \Psi (x_N, x_{N-1}, \ldots, x_1)$$

with the anyonic phase

$$\omega = \kappa \left[ \sum_{k=j+1}^l \epsilon(x_j-x_k) - \sum_{k=j+1}^{l-1} \epsilon(x_l-x_k) \right].$$

Using the coordinate Bethe ansatz we can obtain the wavefunction

$$\Psi (x_1, \ldots, x_N) = \prod_{q} \theta(x_{qN} - x_{qN-1}) \cdots \theta(x_{q2} - x_{q1}) \times \exp \left( -i\frac{\kappa}{2} \Lambda (x_{q1}, x_{q2}, \ldots, x_{qN}) \right) \times \varphi (x_{q1}, x_{q2}, \ldots, x_{qN}),$$

where $Q$ is used to label the region $0 \leq x_{q1} \leq x_{q2} \leq \cdots \leq x_{qN} \leq L$ and $\theta (x-y)$ is the step function. In different region the phase factors of the wavefunction are determined by $\Lambda = \sum_{j<l} \epsilon (x_j-x_l)$, which results in the exchange symmetry dictated by eq. (4). The wave function $\varphi (x_1, \ldots, x_N)$ is taken as the Bethe ansatz type

$$\varphi (x_{q1}, \ldots, x_{qN}) = \prod_{P} \left[ A_{P_{1}} \cdots A_{P_{N}} \exp \left( i \sum_{j} k_{P_{j}} x_{qj} \right) \right].$$

In order to get a physical result, the twisted boundary condition is used here. Under the twisted boundary condition, we have the Bethe ansatz equations

$$\exp (ik_j L) = \prod_{l=1(l\neq j)}^N \frac{ik_l - ik_j + \epsilon'}{ik_l - ik_j - \epsilon'},$$

with $j = 1, 2, \ldots, N$, where

$$\epsilon' = \frac{c'}{\cos(\kappa/2)}$$

is the renormalized coupling constant and $\kappa$ the statistics parameter. It is deserved to notice that the system is reduced to the Lieb-Liniger Boson model when $\kappa = 0$ and the system is reduced to the free Fermi one when $\kappa = \pi$. The coefficients $A_{P}$ take the form of

$$A_{P_{1}P_{2} \cdots P_{N}} = \epsilon_{P} \prod_{j<l}^N (ik_{P_{j}} - ik_{P_{j}} + \epsilon'),$$

in which $\epsilon_{P}$ denotes a $+(-)$ sign factor associated with even (odd) permutations. With the set of quantum number $\{k_{j}\}$ known as quasimomentum the energy eigenvalue
and the total momentum can be formulated as

$$ E = \sum_{j=1}^{N} k_j^2 $$

and $P = \sum_{j=1}^{N} k_j$. Taking the logarithm of Bethe ansatz equations, we have

$$ k_j L = n_j \pi - \sum_{l=1(l \neq j)}^{N} \left( \arctan \frac{k_j - k_l}{c'} \right). $$

In the following only the case of $c > 0$ will be investigated. The set of integer $\{n_j\}$ determine an eigenstate and for the ground state $n_j = j - (N + 1)/2$ ($1 \leq j \leq N$).

### III. PROPERTIES OF THE GROUND STATE

By numerically solving the set of transcendental equations eq. (9), the quasimomentum $\{k_j\}$ and thus the wave function can be decided exactly. For $c > 0$, the set of $k_i$ is unique and real, therefore the density of state in $k$ space $\rho(k)$ can be formulated as

$$ L \rho \left( \frac{k_j + k_{j+1}}{2} \right) = \frac{1}{k_{j+1} - k_j}. $$

In Fig. 1, we plot the density of ground state in quasimomentum $k$ space with the interaction parameter $c$ fixed to $c = 0.1$ for several $\kappa$. It is shown that the density of state of anyons with coupling constant $c$ and statistical parameter $\kappa$ is identical to the density of state of Lieb-Liniger Bose model with effective coupling constant $c' = c/cos(\kappa/2)$. This is clear by comparing the Bethe ansatz equation (10) with that of Lieb-Liniger model [38]. When $\kappa = 0$, the model is reduced to Bose model and $k$ mainly distribute in the regime around zero. While for the case of $\kappa = \pi$, we have $k_j = (j - (N + 1)/2)2\pi/L$ with $j = 1, \cdots, N$ even for the weak coupling $c = 0.1$, which is the same as the situation of $c' = \infty$ of the Lieb-Liniger Bose model. Here the statistics parameter $\kappa$ affects the density of state in quasi-momentum space only by renormalizing the coupling constant $c$ into $c'$. For the case of $c = 0.1$, we note that when $\kappa$ approaches $\pi$ minor change of $\kappa$ will result in the great change of the density of ground state in $k$ space. For instance, there are obvious difference among the densities corresponding to $\kappa = 0.9\pi, 0.99\pi$, and $\pi$ due to the dramatic changes of the effective coupling constant $c' = 0.64, 6.37$ and $\infty$ correspondingly. In Fig. 2, we display the ground-state energy versus $\kappa$ for different coupling constant $c$. The ground-state energy increases with the increase of $\kappa$ and the coupling constant $c$. For the weak coupling the energy of the system is small for most $\kappa$ but it abruptly increases and arrives at the maximum when $\kappa$ approach $\pi$. In the strong coupling limit ($c \rightarrow \infty$) the ground-state energy is almost a constant independent of the statistics parameter. Similarly the ground-state energy of the anyon gas with coupling constant $c$ and statistical parameter $\kappa$ is identical to that of the corresponding Bose gas with effective coupling constant $c' = c/cos(\kappa/2)$.

Now it is clear that the density of state in $k$ space and the ground-state energy are determined completely by the renormalized coupling constant $c'$. Furthermore, from the eqs. [38] and [40], we see that the energy level structure of the anyon gas is exactly the same as the corresponding bosonic model with renormalized coupling constant $c'$. Therefore the thermodynamic properties of the anyon gas are the same as the well known thermodynamic properties of the 1D boson gas [38, 40] with the
effective coupling $c' = c/cos(\kappa/2)$. This implies that the intrinsic difference between the anyon gas and the corresponding Bose gas is hard to be distinguished just by the thermodynamic properties. However, due to the different exchange symmetry of the wave functions, the observables associated with the wave functions rather than the square of wave functions, such as the off-diagonal reduced density matrix and the momentum distributions, should display quite different behaviors. In terms of the ground state wave function $\Psi(x_1, \cdots, x_N)$, the one-body reduced density matrix is given by

$$\rho(x, x') = \frac{N \int_{0}^{L} dx_2 \cdots dx_N \Psi^*(x, x_2, \cdots, x_N) \Psi(x', x_2, \cdots, x_N)}{\int_{0}^{L} dx_1 \cdots dx_N |\Psi(x_1, x_2, \cdots, x_N)|^2},$$

while the momentum distribution

$$n(p) = \frac{1}{2\pi} \int_{0}^{L} dx \int_{0}^{L} dx' \rho(x, x') e^{-ip(x-x')}$$  \hspace{1cm} (11)$$

is the Fourier transformation of $\rho(x, x')$. Below we rescale the momentum distribution $n(p)$ and momentum $p$ into dimensionless forms in the units of $L/2\pi$ and $2\pi/L$ respectively and the original notations are reserved.

To give a concrete example, we display the momentum distribution of the anyon gas with coupling constant $c = 0.1$ and $N = 4$ in Fig. 3 for various statistics parameters. At $\kappa = 0$, the momentum distribution shows a typical Boson-type distribution with an obvious peak around the zero momentum point, whereas the momentum distribution is identical to that of free spinless Fermions at $\kappa = \pi$, exhibiting shell structure. In these two situations the momentum distributions are symmetric. When $\kappa$ deviates from these two extrema points, the momentum distribution becomes asymmetric: the anyons distribute with more probability in the regime of positive momentum whereas with relatively small probability in the other side. With the increase of $\kappa$, the distribution redistributes between the regime of positive and negative $p$, and gets wider with more and more peaks appearing. Finally, $N$ peaks appear at $\kappa = \pi$ and the momentum distribution shows the characteristic shell structure of
Fermions.

As a comparison, in Fig. 4 we display the momentum distribution of the Lieb-Liniger boson gas for $N = 4$ with the renormalized coupling constants $c = 0.1/cos(\kappa/2) = 0.1, 0.12, 0.18, 0.51, 2.04$ and $\infty$ corresponding to those for anyons in Fig. 3, where we have $c = 0.1$ and $\kappa = 0, 3\pi/8, 5\pi/8, 7\pi/8, 31\pi/32$, and $\pi$ respectively. The momentum distributions for $c = 0.1, 0.12, 0.18, 0.51, 2.04$ are not obviously different in Fig. 4 as they all fall in the weakly interacting regime. We also display the momentum distribution for $c = 50$ in Fig. 4. The height of the peak shrinks with the increase of the repulsive interaction and stronger repulsive interaction between the bosonic atoms tends to spread out the distribution widely. Even in the TG limit, the momentum distribution of bosons does not show shell structure like free fermions. For all coupling constant, bosons always congregate symmetrically in the regime around $p = 0$ with the most probability and with the increase of momentum $p$ the probability diminishes rapidly. Obviously, the asymmetry in the momentum distribution of anyons can not be attributed to the special permutation symmetry of the anyon wavefunction characterized by the statistical parameter $\kappa$ because the renormalized coupling constant is an even function of $\kappa$. The momentum distribution is not so because of the existence of the imaginary part of the non-diagonal part of the one body density matrix, which is an odd function of $\kappa$. If $\kappa$ is negative the momentum distribution is equal to the distribution for positive $\kappa$ with the mapping between the positive momentum and negative momentum. Our results interpolate between the known results of Bose and Fermi gas, respectively, with both of momentum distributions being symmetric. When the statistics parameter deviates from these two points, the momentum distribution is asymmetric and evolves from the Bose distribution to Fermi distribution with the increase of $\kappa$. Although the thermodynamic properties of anyon system are even functions of the statistics parameter $\kappa$ because the renormalized coupling constant is even function of it, the momentum distribution is not so because of the existence of the imaginary part of the non-diagonal part of the one body density matrix, which is an odd function of $\kappa$. If $\kappa$ is negative the momentum distribution is equal to the distribution for positive $\kappa$ with the mapping between the positive momentum and negative momentum. Our results interpolate between the known results of Bose and Fermi gas and clarify the effect of generalized permutation symmetry of the anyon gas in an exact manner.

IV. CONCLUSIONS

In summary, we have investigated the ground-state properties of 1D anyon gas based on the exact Bethe ansatz solution for arbitrary coupling constant ($0 \leq c \leq \infty$) and statistical parameter ($0 \leq \kappa \leq \pi$). The density of state and energy are determined by the renormalized coupling constant $c'$. The anyonic system with coupling constant $c$ and statistics parameter $\kappa$ has the same density of state and energy as those of the Lieb-Liniger Bose model with coupling $c' = c/cos(\kappa/2)$. Besides the renormalization of the effective coupling constant, the additional effect induced by the statistics exhibits in the momentum distribution. While in the limit of $\kappa = 0$ and $\kappa = \pi$, the anyon gas is reduced to the Bose gas and free spinless Fermi gas, respectively, with both of momentum distributions being symmetric. When the statistics parameter deviates from these two points, the momentum distribution is asymmetric and evolves from the Bose distribution to Fermi distribution with the increase of $\kappa$. Although the thermodynamic properties of anyon systems are even functions of $\kappa$ because the renormalized coupling constant is even function of it, the momentum distribution is not so because of the existence of the imaginary part of the non-diagonal part of the one body density matrix, which is an odd function of $\kappa$. If $\kappa$ is negative the momentum distribution is equal to the distribution for positive $\kappa$ with the mapping between the positive momentum and negative momentum. Our results interpolate between the known results of Bose and Fermi gas and clarify the effect of generalized permutation symmetry of the anyon gas in an exact manner.

APPENDIX A

The asymmetry of the momentum distribution can be attributed to the special permutation symmetry of the anyon wavefunction characterized by the statistical parameter $\kappa$. To see it clearly, we display the case of two anyons in detail in this appendix. For the two anyon systems, the Bethe ansatz wavefunction takes the form of

$$\Psi(x_1, x_2) = \theta (x_1 < x_2) \exp (i\kappa/2) \varphi(x_1, x_2) + \theta (x_2 < x_1) \exp (-i\kappa/2) \varphi(x_2, x_1),$$

where $\varphi(x_1, x_2) = A_{12} \exp (ik_1 x_1 + ik_2 x_2) + A_{21} \exp (ik_2 x_1 + ik_1 x_2)$. The reduced one body density matrix $\rho(x, x') = \int_0^L dx_2 \Psi \left( x, x_2 \right) \overline{\Psi \left( x', x_2 \right)}$ can be
evaluated with wavefunction, i.e.,
\[
\rho(x, x') = \int_0^L dx_2 \left[ \theta(x < x_2) \theta(x' < x_2) \varphi^*(x, x_2) \varphi(x', x_2) + \theta(x < x_2) \varphi^*(x, x_2) \varphi(x', x_2) \theta(x' < x) \exp(-i\kappa) \varphi(x, x_2) \varphi(x', x_2) + \theta(x < x_2) \varphi^*(x, x_2) \varphi(x', x_2) \right].
\]

When the statistical parameter \(\kappa\) deviates from 0 and \(\pi\), the density matrix is always complex for \(x \neq x'\). From the definition of reduced density matrix, we have \(\rho(x, x') = \rho^*(x', x)\), which implies \(\text{Re} [\rho(x, x')] = -\text{Im} [\rho(x, x')]\) and \(\text{Im} [\rho(x, x')] = -\text{Re} [\rho(x, x')]\). Therefore the momentum distribution can be represented as
\[
n(p) = \frac{1}{2\pi} \int_0^L dx \int_0^L dx' \rho(x, x') e^{-ip(x-x')} = \frac{1}{2\pi} \int_0^L dx \int_0^L dx' \left\{ \text{Re} [\rho(x, x')] \cos (p(x-x')) + \text{Im} [\rho(x, x')] \sin (p(x-x')) \right\}.
\]

The second term is the odd function of momentum \(p\) such that the momentum distribution becomes asymmetric about \(p\). Furthermore, the imaginary part \(\text{Im} [\rho(x, x')]\) is an odd function of \(\kappa\). Therefore, if we take \(\kappa\) as negative (\(\kappa \rightarrow -\kappa\)), the peak at positive momentum as shown in Fig.3 and Fig.5 will shift to negative momentum.

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