Manifold Embeddings by Heat Kernels of Connection Laplacian

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Abstract
We show that any closed $n$-dimensional manifold $(M, g)$ can be embedded by a map constructed using the heat kernels of the connection Laplacian as well as maps constructed using truncated heat kernel at a certain time $t$ from a $\delta$-net $\{q_i\}_{i=1}^{N_0}$ via a rescaling technique. Both the time $t$ and $N_0$ are bounded in terms of the dimension, bounds on the Ricci curvature and its derivative, the injectivity radius, and the volume. Moreover, both maps can be made arbitrarily close to an isometry.

Keywords  Diffusion maps · Vector diffusion maps · Embedding dimensions · Almost isometric embedding · Dimension reduction · Heat kernels

1 Introduction

1.1 Motivation
This work is motivated by the manifold learning problem in machine learning. Data collected for the purpose of machine learning are often in a high-dimensional space, but are believed to satisfy certain low-dimensional structure, that is, the collected dataset can be well approximated by a low-dimensional manifold sitting inside a high-dimensional Euclidean space. For example, gray-scale $n \times n$ images of a fixed object in space taken from all different angles yield data points in $\mathbb{R}^{n^2}$, while the intrinsic dimensionality of the space of all images of the objection is two, that is, the dimensionality of the surface of the object. One example of such a dataset is cryo-electron microscopy (EM) images for the single-particle reconstruction problem, where one uses a large number of cryo-EM projection images from various viewing angles to reconstruct the 3D structure of the macromolecular complex. For interested
readers, see [1–4] for the single-particle reconstruction problem for visualization of biological molecules.

How to analyze a dataset under this manifold assumption is generally called the Manifold Learning problem. One particular goal is to represent the possibly nonlinear low-dimensional manifold and to reduce the dimensionality of the space where the dataset lies inside.

1.2 The Finite-Dimensional Embedding Problem

Mathematically, the Manifold Learning problem can be formulated as asking if it is possible to embed a manifold (hence the dataset) into a finite-dimensional Euclidean space that is bi-Lipschitz, even isometric. Although the embedding problem was first positively answered by Whitney [5], and the isometrically embedding problem was first solved by Nash [6], the approaches are not canonical and are not essentially feasible for data analysis. In Berard–Besson–Gallot’s breakthrough paper [7], the spectral embedding idea is explored. They show that a class of closed manifolds can be embedded into the sequence space $\ell^2$ by their heat kernels. Given an $L^2(M)$ orthonormal basis of eigenfunctions $\{\phi_i\}_{i=0}^\infty$ of an $n$-dimensional closed manifold $M$ such that $\Delta \phi_i = -\lambda \phi_i$ with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, they consider the family of maps $\psi_t : M \rightarrow \ell^2$,

$$\psi_t(p) = \sqrt{2}(4\pi)^{n/4}t^{(n+2)/4}\{e^{-\lambda_i t/2}\phi_i(p)\}_{i \geq 1} \text{ for } t > 0 \quad (1.1)$$

and construct embeddings using the fact that orthonormal eigenfunctions separate points.

Note that, in practice, one is only able to work with finitely many eigenfunctions and eigenvalues. Thus, the next natural question is whether one can embed the manifold by finite eigenfunctions and eigenvalues.

Consider the Laplace operator on graph approximations of the underlying manifold. In 2003, Belkin and Niyogi [8] developed an algorithm, Eigenmaps, defined by

$$p \mapsto (\phi_1(p), \cdots, \phi_N(p)) \quad (1.2)$$

using the first $N$ eigenfunctions of the Laplace operator. In 2006, Coifman and Lafon [9] developed an algorithm, Diffusion Maps (DM),

$$p \mapsto (e^{-\lambda_1 t/2}\phi_1(p), \cdots, e^{-\lambda_N t/2}\phi_N(p)) \text{ for } t > 0 \quad (1.3)$$

using the first $N$ eigenfunctions and eigenvalues of the Laplace operator. Empirically, both algorithms give useful representation of “embedding” of the manifold and have been applied successfully to problems in machine learning.

In 2008, Jones, Maggioni, and Schul [10, 11] showed that any smooth $n$-dimensional manifold admitting charts in which the metric is $C^\alpha$ can be locally embedded in $\mathbb{R}^n$ by eigenfunctions of the Laplace operator. They also observed a “heat coordinate” system on manifolds, called heat triangulation, is more stable than eigenfunction coordinates. In 2014, the question, whether a manifold can be embedded with finitely many
eigenfunctions, is positively answered separately by Bates [12] and Portegies [13]. In Portegies [13], it is further shown that the embedding is almost isometric with a prescribed error bound.

### 1.3 Spectral Embeddings Based on the Connection Laplacian for Vector Fields

The spectral embedding mentioned above depends on the Laplace–Beltrami operator. In 2012, Singer and Wu [14, 15] developed the algorithm, vector diffusion maps (VDM), which makes use of the eigen-vector fields and eigenvalues of the connection Laplacian. In brief, the VDM with the diffusion time $t > 0$ is defined by the eigen-vector fields of the connection Laplacian

$$V_t : M \to \ell^2$$

$$p \mapsto \left( e^{-(\lambda_i + \lambda_j)t/2} \langle X_i, X_j \rangle(p) \right)_{i,j=1}^{\infty},$$

where $p \in M$ and $X_i$ is the $i$-th eigen-vector field of the connection Laplacian associated with the eigenvalue.

The VDM is originally motivated by studying the cryo-electron microscope problem, in particular the class averaging algorithm [16–18]. The VDM method helps aligning images and find similar images (nearest neighbors) more effectively and outperforms the DM. Although the VDM method provides promising results in several applications, unlike the DM method, it still lacks theoretical understandings.

### 1.4 Challenges in Studying of the VDM

For the DM, one studies a parabolic equation and elliptic equations with respect to the Laplace–Beltrami operator. By working on a harmonic coordinate chart $\{x_a\}$, there is only the principal part of the equations, no lower order terms. That is,

$$u_t = g^{ab} \partial_x a \partial_x b u$$

(1.5)

For the VDM, one needs to study a parabolic system and elliptic systems with respect to the connection Laplacian. Let $\{x^a\}$ be a harmonic coordinate system on $M$ and $X = X^a \frac{\partial}{\partial x^a}$ be a vector field. One has

$$\Delta_{TM} \left( X_c^c \partial_c \right) = -g^{ab} \left( \partial_a \partial_b X^c + 2\Gamma^c_{bd} \partial_a X^d + \Gamma^c_{ae} \Gamma^e_{bd} X^d + \partial_a \Gamma^c_{bd} X^d \right) \partial_c$$

(1.6)

It requires more delicate analysis to take care of the lower order terms to obtain desired estimates. In [19], Lin and Wu investigated estimates of eigen-vector fields and truncations of heat kernels and its derivatives. In the end, we showed that one is able to construct an embedding of a manifold with finite eigen-vector fields of the connection Laplacian that is bi-Lipschitz. Note that we did not know if the map can be made arbitrarily close to isometry.
1.5 Main Contribution

The main contribution in this paper is to show that one can construct an embedding of a manifold with heat kernels of the connection Laplacian that can be made arbitrarily close to an isometry. Our approach is inspired by Portegies’ rescaling technique [13]. We build a fundamental solution from scratch using the parametrix method, with careful estimates.

The organization of the paper is as follows. In Sect. 2, we introduce background on the heat kernels and state the main results. In Sect. 3, we review the $C^{k,\alpha}$-harmonic radius and introduce the rescaled heat kernel $\bar{K}_{TM}$ defined in (3.5) that satisfies the rescaled parabolic system (3.6). Following [20, Sect. 4 in Chap. 9], we construct a fundamental solution $\Gamma$ for (3.6) and show that $\Gamma$ is close to the standard Euclidean heat kernel on a local harmonic chart. We also show that any solution with sufficient decay is close to this fundamental solution $\Gamma$. Thus, the rescaled heat kernel $\bar{K}_{TM}$ is close to the standard Euclidean heat kernel. In Sect. 4, we prove our main Theorem and Corollary.

Remark 1 Note that due to the lack of canonical isometry from $L^2(TM)$ to $\ell^2$, we are not able to extend our main results, embeddings with heat kernels, to a similar almost isometry result for embeddings with eigen-vector fields.

Remark 2 Although we do not have numerical example or applications, it seems to be a natural generalization of its counterpart with the Laplace-Beltrami operators, where one embeds a manifold using its heat kernels. In [10], such a “heat coordinate” system on manifolds is called heat triangulation and is claimed much more stable than eigenfunction coordinates because heat kernels are local objects and good statistical control on smoothness of heat kernels.

Remark 3 The main result in the work is an existing Theorem. How to select parameters such diffusion time $t$ is so-called the parameter selection problem. The common practice is to choose parameters empirically. The parameter selection problem is important. However, it is outside of the scope of the work here.

2 Heat Kernel and Main Result

In this section, we review some background that is required for the main result, as well as the Gaussian bound for the heat kernel that is used in the proofs of the main results.

2.1 Heat Kernel of Connection Laplacian

Let $M$ be an $n$-dimensional smooth closed manifold. We denote by $K_{TM}(p, t; q)$ the heat kernel of the connection Laplacian $\Delta_{TM}$ on the manifold $M$. The heat kernel can be expressed as follows:

$$K_{TM}(p, t; q) = \sum_{i=1}^{\infty} e^{-\lambda_i t} X_i(p) \otimes X_i(q),$$
where \( t > 0, p, q \in M, \lambda_i \) are eigenvalues so that \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \{ \lambda_i \}_{i=1}^\infty \) is an \( L^2(TM) \)-orthonormal basis formed by eigen-vector fields of \( \Delta TM \). We define the \( m \)-th order truncated heat kernel \( K^{(m)}_{TM} \) by

\[
K^{(m)}_{TM}(p, t; q) := \sum_{i=1}^{m} e^{-\lambda_i t} X_i(p) \otimes X_i(q).
\]

Let \( \| \cdot \|_{HS} \) denote the Hilbert-Schmidt norm. We have

\[
\| K_{TM}(p, t; q) \|_{HS}^2 = \text{Tr} \left( K_{TM}(p, t; q)^* K_{TM}(p, t; q) \right)
= \sum_{i,j=1}^{\infty} e^{-(\lambda_i + \lambda_j)t} \langle X_i(p), X_j(p) \rangle \langle X_i(q), X_j(q) \rangle;
\]

and

\[
\| K^{(m)}_{TM}(p, t; q) \|_{HS}^2 = \sum_{i,j=1}^{m} e^{-(\lambda_i + \lambda_j)t} \langle X_i(p), X_j(p) \rangle \langle X_i(q), X_j(q) \rangle.
\]

### 2.2 Main Result

For given \( \kappa, i_0, V > 0 \), consider the set of (isometry classes of) closed smooth manifolds of dimension \( n \)

\[
\mathcal{M}_{n, \kappa, i_0, V} := \{ (M^n, g) : |Rc|, |\nabla Rc| \leq \kappa, \ \text{inj}(M) \geq i_0, \ \text{Vol}(M) \leq V \},
\]

where \( \text{inj}(M) \) denotes the injectivity radius of \( M \) and \( Rc \) and \( \nabla Rc \) denote the Ricci curvature and its covariant derivative, respectively.

We prove the following Theorem and Corollary:

**Theorem 4** Given \( \epsilon > 0 \), there exists \( t_0 = t_0(n, \kappa, i_0, \epsilon) \) so that for all \( t < t_0 \) there exists \( \delta = \delta(n, \kappa, i_0, \epsilon, t) \) such that for all \( (M, g) \in \mathcal{M}_{n, \kappa, i_0, V} \) and every \( \delta \)-net \( \{ q_1, \cdots, q_N \} \) on \( M \), the map

\[
H : (M^n, g) \to (\mathbb{R}^N_0, g_{\text{can}})
\]

\[
p \mapsto \frac{(2t)^{(3n+2)/4}}{V_e} \left( |A_1|^{1/2} \| K_{TM}(p, t; q_1) \|_{HS}^2 
+ \cdots 
+ |A_N|^{1/2} \| K_{TM}(p, t; q_N) \|_{HS}^2 \right)
\]

(2.1)

is an embedding and the dilatation, which is the operator norm of the differential, is controlled by a given \( \epsilon \),

\[
1 - \epsilon < |(dH)_p|^2 < 1 + \epsilon \quad \text{for all } p \in M.
\]

(2.2)
Here \( \{A_i\}_{i=1}^{N_0} \) is a partition of \( M \) so that \( A_i \subset B_{\delta}(q_i) \) for all \( i \) and

\[
V_e := \left( \int_{\mathbb{R}^n} (\partial x_1 \|\Gamma_E(0, \frac{1}{2}; y)\|^2_{HS}) \, dy \right)^{1/2},
\]

(2.3)

where \( \Gamma_E \) is the standard Euclidean heat kernel on \( \mathbb{R}^n \times \mathbb{R} \) whose definition is reviewed in (3.7).

Moreover, there exists an \( N = N(n, \kappa, i_0, V, \epsilon, t) \) so that for all \( m \geq N \), the same statements hold for the map \( H^{(m)} \) defined by truncated heat kernels

\[
H^{(m)}(p) := \frac{(2t)^{(3n+2)/4}}{V_e} \left( \frac{A_1}{2} \| K^{(m)}_{TM}(p, t; q_1) \|^2_{HS}, \ldots, \frac{1}{2} \| K^{(m)}_{TM}(p, t; q_{N_0}) \|^2_{HS} \right)
\]

(2.4)

We comment that the \( N_0 \) is not necessary universal. It is for every choice of covering sets that gives a \( \delta \)-net, where \( \delta \) depends on the geometry, \( \epsilon \) and \( t \).

**Corollary 5** Given \( \epsilon > 0 \), there exists \( t_0 = t_0(n, \kappa, i_0, \epsilon) \) so that for all \( t < t_0 \) there exists \( N_0 = N_0(n, \kappa, i_0, \epsilon, t, V) \) such that for all \( (M, g) \in \mathcal{M}_{n, \kappa, i_0, V} \), there exist points \( \{q_1, \ldots, q_{N_0}\} \) on \( M \) such that the map

\[
\mathcal{H} : (M^n, g) \to (\mathbb{R}^{N_0}, g_{can})
\]

\[ p \mapsto \frac{(2t)^{(3n+2)/4}}{V_e} A \left( \| K_{TM}(p, t; q_1) \|^2_{HS}, \ldots, \| K_{TM}(p, t; q_{N_0}) \|^2_{HS} \right)
\]

(2.5)

is an embedding, where \( A = A(n, \kappa, i_0, V, \epsilon, t) > 0 \) is a small constant. In addition, the local dilatation satisfies

\[
1 - \epsilon < |d\mathcal{H}_p| < 1 + \epsilon \quad \text{for all } p \in M.
\]

(2.6)

Moreover, there exists an \( N = N(n, \kappa, i_0, V, \epsilon, t) \) so that for all \( m \geq N \), the same statements hold with every heat kernel \( K_{TM} \) replaced by the truncated heat kernel \( K^{(m)}_{TM} \):

\[
\mathcal{H}^{(m)} : (M^n, g) \to (\mathbb{R}^{N_0}, g_{can})
\]

\[ p \mapsto \frac{(2t)^{(3n+2)/4}}{V_e} A \left( \| K^{(m)}_{TM}(p, t; q_1) \|^2_{HS}, \ldots, \| K^{(m)}_{TM}(p, t; q_{N_0}) \|^2_{HS} \right)
\]

(2.7)

Note that this \( N_0 \) is universal as we choose a partition that depends on the geometry, \( t \) and \( \epsilon \).

In order to prove the main result, we heavily rely on the Gaussian bound of the heat kernel of the connection Laplacian, which is stated below.
2.3 Gaussian Bound of the Heat Kernel

Let $K_M(p, t; q)$ denote the heat kernel of the Laplace–Beltrami operator $\Delta_M$ on $(M, g)$. By the Faber–Krahn inequality [21, Sect. 14.2] and the exponential decay [21, Theorem 15.14] of the heat kernel $K_M(p, t; q)$, we have for $p, q$ in a ball of radius less than the $C^{2,\alpha}$-harmonic radius $r_h$, of which definition is reviewed in Sect. 3.1,

$$K_M(p, t; q) \leq C(n) \left(1 + \frac{d(p, q)^2}{t}\right)^{n/2} \exp \left(-\frac{d(p, q)^2}{4t}\right)$$  \hspace{1cm} (2.8)

where $d(p, q)$ denotes the geodesic distance between $p, q \in M$ for constants $a(n) > 0$ and $C(n) > 0$ depending only on $n$.

In addition, the semigroup domination theorem by Hess–Schrader–Uhlenbrock [22, Theorem 3.1] states the following in our setting

$$\text{Tr} \left(e^{t\Delta_T M}\right) \leq n \text{Tr} \left(e^{t\Delta_M}\right),$$

which implies that

$$\|K_T M(p, t; q)\|_{HS} \leq nK_M(p, t; q) \leq C(n) \left(1 + \frac{d(p, q)^2}{t}\right)^{n/2} \exp \left(-\frac{d(p, q)^2}{4t}\right).$$  \hspace{1cm} (2.9)

3 Fundamental Solutions on Charts

The goal in this section is to show that the rescaled heat kernel, $\tilde{K}_T M$ defined in (3.5), is close to the standard heat kernel $\Gamma_E$. We start with reviewing the harmonic radius, and there is a universal lower bound for the harmonic radius that is independent of the point on the manifold. This allows us to work on a harmonic chart of certain size. In Lemma 8, we construct a fundamental solution $\Gamma$ of the rescaled parabolic system (3.6) that is close to the Euclidean heat kernel using the parametrix method as presented in Friedman’s book [20, Sect. 4 Chap. 9]. In Lemma 9, we show that the rescaled heat kernel $\tilde{K}_T M$ and the fundamental solution $\Gamma$ are close by applying the Schauder estimates [23] and the semigroup domination theory. Throughout the paper, the constant $C$ varies line by line.

3.1 The $C^{k,\alpha}$-Harmonic Radius

Recall that (see [24]) given $Q > 1$ and $\alpha \in (0, 1)$, the $C^{k,\alpha}$-harmonic radius at $p$ in $M$ is defined as the largest number $r_h = r_h(Q, k, \alpha)(p)$ such that on the geodesic ball $B_{r_h}(p)$ of center $p$ and radius $r_h$, there is a harmonic coordinate chart $u : B_{r_h}(p) \subset M \to U \subset \mathbb{R}^n$ such that $u(p) = 0$, $g_{ab}(p) = \delta_{ab}$ and the metric tensor is $C^{k,\alpha}$ controlled in these coordinates. Namely, if $g_{ab}, a, b = 1, \cdots, n$ are the components of $g$ in these coordinates, then
\[ Q^{-1} \delta_{ab} \leq g_{ab} \leq Q \delta_{ab} \] as bilinear forms \( (3.1) \)

\[
\sum_{1 \leq |\beta| \leq k} r^{|\beta|}_h \sup_p |\partial^\beta g(p)| + \sum_{|\beta|=k} r^{2+\alpha}_h \sup_{p \neq q} \frac{|\partial^\beta g_{ij}(p) - \partial^\beta g_{ij}(q)|}{d(p, q)\alpha} \leq Q - 1 , \quad (3.2)
\]

where \( d(p, q) \) denotes the geodesic distance on \((M, g)\). The harmonic radius of \((M, g)\), denoted as \( r_h(Q, k, \alpha)(M) \), is defined by

\[
r_h(Q, k, \alpha)(M) := \inf_{p \in M} r_h(Q, k, \alpha)(p). \quad (3.3)
\]

For any manifold \((M, g) \in M_{n, \kappa, i_0, V}\), it is proved by Hebey–Herzlich [24, Corollary of Theorem 6] that there exist \( C^{2, \alpha} \)-harmonic coordinate charts whose size is independent of the choice of points. Here, we restate the Corollary for our case.

**Corollary 6** Let \( \alpha \in (0, 1) \) and \( Q > 1 \). Let \((M, g)\) be a smooth-closed \( n \)-dimensional Riemannian manifold. Suppose for some \( \kappa > 0, i_0 > 0 \)

\[
|Rc|, |\nabla Rc| \leq \kappa \quad \text{and} \quad \text{inj}(M) \geq i_0.
\]

Then, there exists a constant \( C = C(n, Q, \alpha, \kappa, i_0) \) such that the \( C^{2, \alpha} \)-harmonic radius

\[
r_h = r_h(Q, 2, \alpha)(M) \geq C.
\]

That is, for any \( M \in M_{n, \kappa, i_0, V}\), there is a universal \( C^{2, \alpha} \)-harmonic radius \( r_h \) that is independent of the choice of point. Let \( p \in M \) and let \( u : B_{r_h}(p) \to \mathbb{R}^n \) be a harmonic coordinate chart with \( u(p) = 0 \) satisfying (3.1) and (3.2). For a vector field \( X = X^a \partial_a \), by a direct calculation, we have

\[
\Delta M \left( X^c \partial_c \right) = -g^{ab} \left( \partial_a \partial_b X^c + 2 \Gamma^c_{bd} \partial_d X^b + \Gamma^e_{ac} \Gamma^d_{bd} X^d + \partial_a \Gamma^c_{bd} X^d \right) \partial_c.
\]

\( (3.4) \)

Note that the coefficients on the right-hand side of the equation (3.4) are controlled in \( C^{2, \alpha} \)-harmonic coordinates by (3.2).

**Remark 7** In [19] where we have similar results but without discussing the Lipschitz control, we write the term \( \partial_a \Gamma^c_{bd} X^d \) in terms of the Ricci curvature and hence having \( |Rc| < \kappa \) is enough to have \( C^{1, \alpha} \)-harmonic radius and bounded coefficients. However, in this paper we will start with a rougher guess in the parametrix process to build a solution that is close to the standard Euclidean heat kernel and need subtler controls of the coefficients. Therefore, the class of manifold \( M_{n, \kappa, i_0, V} \) requires the extra condition \( |\nabla Rc| \leq \kappa \) to guarantee a universal \( C^{2, \alpha} \)-harmonic radius bound.
3.2 Rescaled Heat Kernel

Let \( u : B_{r_0}(p) \subset M \to U \subset \mathbb{R}^n \) be a harmonic coordinate chart with \( u(p) = 0 \). Define a rescaled heat kernel on \(\mathbb{R}^n \times \mathbb{R}_+ \) associated with the heat kernel \( K_{TM} \) of the connection Laplacian as:

\[
\tilde{K}_{TM}(x, s; y) := r^n K_{TM}(u^{-1}(x), sr^2; u^{-1}(y)),
\]

where \( s > 0, r > 0 \) is a scale factor to be chosen later, and \( x, y \) in the proper domain, \( U/r := \{ x \in \mathbb{R}^n \mid r x \in U \} \subset \mathbb{R}^n \). Since \( u \) is a diffeomorphism, denote \((u^{-1})^*g\) to be the induced metric on \( U \), and denote the rescaled metric as \( \bar{g}(x) = (u^{-1})^*g(rx) \). Denote the Christoffel symbol of \((u^{-1})^*g\) as \( \Gamma^c_{ab} \), we have the Christoffel symbol for the rescaled metric satisfying \( \tilde{\Gamma}^c_{ab}(x) = \Gamma^c_{ab}(rx) \). On \((U, (u^{-1})^*g)\), denote \( L = \Delta_{(u^{-1})^*g} - \partial_t \) to be the parabolic system, and \( \tilde{L} \) to be the rescaled version. By a direct calculation, it satisfies the following expansion:

\[
\tilde{L}X^c := \bar{g}^{ab} \left( \partial_a \partial_b X^c + 2r \tilde{\Gamma}^c_{bd} \partial_d X^d + r^2 \tilde{\Gamma}^c_{ae} \tilde{\Gamma}^e_{bd} X^d + r^2 \partial_a \tilde{\Gamma}^c_{bd} X^d \right) - \partial_s X^c = 0,
\]

where \( c = 1 \cdots n \) and \( X^c \) is a function defined on \( U/r \).

We consider the standard Euclidean heat kernel on \(\mathbb{R}^n \times \mathbb{R} \) given by

\[
\Gamma_E(x, t; y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} I_n,
\]

where \( t > 0, x, y \in \mathbb{R}^n \), and \( I_n \) is the \( n \times n \) identity matrix. Note that \( \Gamma_E \) is the heat kernel of the connection Laplacian associated with the trivial tangent bundle of the canonical Euclidean space \( \mathbb{R}^n \). Below, we show that \( \tilde{K}_{TM} \) is close to the standard Euclidean heat kernel \( \Gamma_E \).

3.3 Fundamental Solutions on Charts

Denote \( P_{R, T}(x) \) to be the parabolic cylinder on \(\mathbb{R}^n \times \mathbb{R} \); that is,

\[
P_{R, T}(x) := \{(y, s) \in \mathbb{R}^n \times \mathbb{R}_+ : 0 < s < T, |y - x| < R\},
\]

where \( x \in \mathbb{R}^n \), \( R, T > 0 \). Define \( Z(x, s; y) \) on \(\mathbb{R}^n \) as

\[
Z(x, s; y) := \sqrt{\left| \tilde{g}(y) \right|} \exp \left( -\frac{\tilde{g}_{ab}(y)(x^a - y^a)(x^b - y^b)}{4s} \right) I_n,
\]

where \( \left| \tilde{g}(y) \right| := \det(\tilde{g}_{ab}(y)) \). Note that \( Z(x, s; y) \) is a solution of the principal part of the rescaled system \( \partial_s X^c = \tilde{g}^{ab}(y) \partial_a \partial_b X^c \).

\(\mathcal{C} \) Springer
Lemma 8 Let $\epsilon > 0$, $0 < \alpha < 1$, $1 < Q < \sqrt{2}$ and $R > 0$ be given. Assume that $r$ is small so that $r R < r_h/\sqrt{2}$, where $r_h$ is the $C^{2, \alpha}$-harmonic radius. Consider the rescaled parabolic system (3.6) on a domain $\Omega \times I \subset \mathbb{R}^n \times \mathbb{R}_+$ that contains the rescaled domain $P_{R,4}(0)$. There exists a fundamental solution $\Gamma$ for (3.6) on the domain $\Omega \times I$ with the following pointwise estimate:

$$|\Gamma(x, s; y)| \leq \frac{C(n)}{s^{n/2}} e^{-\frac{|x-y|^2}{8s}} \quad \text{for } 0 < s < 2. \quad (3.9)$$

In particular, for all $y \in B_R(0) \subset \Omega$ and $(x, s) \in P_{R,4}(0)$,

$$|\Gamma(x, s; y) - Z(x, s; y)| \leq \frac{(Q - 1)C(n, \alpha)}{s^{(n-\alpha)/2}} e^{-\frac{|x-y|^2}{8s}}; \quad (3.10)$$

$$|\nabla \Gamma(x, s; y) - \nabla Z(x, s; y)| \leq \frac{(Q - 1)C(n, \alpha)}{s^{(n+1-\alpha)/2}} e^{-\frac{|x-y|^2}{8s}}. \quad (3.11)$$

Moreover, for $(x, s) \notin P_{\frac{R}{2}, \frac{4}{3}}(y)$,

$$|\Gamma(x, s; y) - \Gamma_E(x, s; y)| \leq (Q - 1)C(n, \alpha); \quad (3.12)$$

$$|\nabla \Gamma(x, s; y) - \nabla \Gamma_E(x, s; y)| \leq (Q - 1)C(n, \alpha). \quad (3.13)$$

**Proof** Following the parametrix method in Friedman [20, Sect. 4 in Chap. 9 and Sect. 4 in Chap. 1], we construct $\Gamma$ in the form:

$$\Gamma(x, s; y) = Z(x, s; y) + \int_0^s \int_{\Omega} Z(x, s - \tau; \xi) \Phi(\xi, \tau; y) d\xi d\tau$$

for some $\Phi$ which is an $n \times n$ matrix-valued function defined on $\Omega \times I$. If $\Phi$ is Hölder continuous, then $\Gamma$ satisfies the system (3.6) as a function of $(x, s)$ if and only if

$$\Phi(x, s; y) := LZ(x, s; y) + \int_0^s \int_{\Omega} LZ(x, s - \tau; \xi) \Phi(\xi, \tau; y) d\xi d\tau. \quad (3.14)$$

(See [20, Lemma 5 in Chap. 9, Sect. 4, p. 250] for details.) We show that there is a $\Phi$ of the form:

$$\Phi(x, s; y) = \sum_{i=1}^{\infty} (LZ)_i(x, s; y),$$

where $(LZ)_1 = LZ$ and

$$(LZ)_i(x, s; y) = \int_0^s \int_{\Omega} LZ(x, s - \tau; \xi)(LZ)_{i-1}(\xi, \tau; y) d\xi d\tau,$$

so that $\Phi$ is a formal solution of (3.14). To find such $\Phi$, we first show that $(LZ)_i$ is integrable for all $i \in \mathbb{N}$ and then show the convergence of the series. Since $Z$ is diagonal and identical along the diagonal, it suffices to look at the scalar function:
\[
z(x, s - \tau; y) := \frac{\sqrt{|g(y)|}}{(4\pi(s - \tau))^{n/2}} \exp \left( -\frac{\bar{g}_{ab}(y)(x^a - y^a)(x^b - y^b)}{4(s - \tau)} \right).
\]

For simplicity, denote \( A := \bar{g}_{ab}(y)(x^a - y^a)(x^b - y^b) \). By rewriting

\[
\exp \left( -\frac{A}{4(s - \tau)} \right) = \exp \left( -\epsilon \frac{A}{4(s - \tau)} \right) \exp \left( -(1 - \epsilon) \frac{A}{4(s - \tau)} \right),
\]

and using the inequality \( \sigma^{n/2-\mu} e^{-\epsilon \sigma} \leq \text{constant for fixed constants } \mu, \epsilon, \text{ and } 0 \leq \sigma < \infty \), we obtain a bound for \( z(x, s - \tau; y) \):

\[
|z(x, s - \tau; y)| = \frac{\sqrt{|g(y)|}}{\pi^{n/2}(4(s - \tau))^{\mu}} A^{n-\mu/2} \left( \frac{A}{4(s - \tau)} \right)^{n/2-\mu} \exp \left( -\frac{A}{4(s - \tau)} \right)
\]

\[
= \frac{\sqrt{|g(y)|}}{\pi^{n/2}(4(s - \tau))^{\mu}} A^{n-\mu/2} \left( \frac{A}{4(s - \tau)} \right)^{n/2-\mu} \exp \left( -\epsilon \frac{A}{4(s - \tau)} \right) \exp \left( -(1 - \epsilon) \frac{A}{4(s - \tau)} \right)
\]

\[
\leq C \frac{\sqrt{|g(y)|}}{(s - \tau)^{\mu}|x - y|^{n-2\mu}} \exp \left( -\frac{\lambda_0|x - y|^2}{4(s - \tau)} \right),
\]

for any \( \lambda_0 < Q^{-1}, 0 \leq \mu \leq n/2, \) and some constant \( C = C(n, Q) \). Similarly, it can be proved that

\[
\sum_{c=1}^{n} \left| \frac{\partial z(x, s - \tau; y)}{\partial x^c} \right| \leq \frac{C}{(s - \tau)^{\mu}|x - y|^{n+1-2\mu}} \exp \left( -\frac{\lambda_0|x - y|^2}{4(s - \tau)} \right)
\]

for any \( \lambda_0 < Q^{-1} \) and \( 0 \leq \mu \leq (n + 1)/2 \). Since that \( Z \) is a solution to the principal part of (3.6) and that the coefficients of the rescaled parabolic system (3.6) is controlled under the \( C^{2,\alpha} \) harmonic coordinates, we further have

\[
\left| \frac{\partial z(x, s - \tau; y)}{\partial s} \right| + \sum_{c,d=1}^{n} \left| \frac{\partial z(x, s - \tau; y)}{\partial x^c \partial x^d} \right| \leq \frac{(Q - 1)C(n, \alpha)}{(s - \tau)^{\mu}|x - y|^{n+2-2\mu}} \exp \left( -\frac{\lambda_0|x - y|^2}{4(s - \tau)} \right)
\]

for any \( \lambda_0 < Q^{-1} \) and \( 0 \leq \mu \leq (n + 2)/2 \). Therefore, under the rescaled \( C^{2,\alpha} \) coordinate chart, we have

\[
|LZ(x, s - \tau; y)| \leq \frac{(Q - 1)C(n, \alpha)}{(s - \tau)^{\mu}|x - y|^{n+2-2\mu-\alpha}} \exp \left( -\frac{\lambda_0|x - y|^2}{4(s - \tau)} \right)
\]

(3.15)
for any $\lambda_0 < Q^{-1}$ and $0 \leq \mu \leq (n + 2 - \alpha)/2$. Hence, the singularity $s = \tau, x = y$
is integrable when $1 - \alpha/2 \leq \mu \leq (n + 2 - \alpha)/2$. Recall the integral formula in [20, Lemma 3], that is, for fixed constants $\alpha$ and $\beta$,

$$\int_0^s \int_{\Omega} (s - \tau)^{-a} e^{-\frac{\lambda_0|\xi - \eta|^2}{4(s - \tau)}} \tau^{-b} e^{-\frac{\lambda_0|\eta|^2}{4\tau}} d\xi d\tau \leq \left(\frac{4\pi}{\lambda_0}\right)^{n/2} \frac{\Gamma(n/2 - a + 1)\Gamma(n/2 - \beta + 1)}{\Gamma(n - a - b + 2)} s^{n/2 + 1 - a - \beta} e^{-\frac{\lambda_0|\xi - \eta|^2}{4s}},$$

(3.16)

where $\Gamma(\cdot)$ denotes the gamma function.

Choose $\mu = (n + 2 - \alpha)/2$. We rewrite (3.15) as

$$|LZ(x, s - \tau; y)| \leq \frac{(Q - 1)C(n, \alpha)}{\Gamma(\frac{n}{2})} (s - \tau)^{-(n + 2 - \alpha)/2} \exp\left(-\frac{\lambda_0|x - y|^2}{4(s - \tau)}\right)$$

(3.17)

Using (3.17) and applying (3.16), we have

$$|(LZ)_2(x, s - \tau; y)|$$

$$\leq \int_0^{s - \tau} \int_{\Omega} |LZ(x, s - \tau - \sigma; \xi)||LZ(\xi, \sigma; y)| d\xi d\sigma$$

$$\leq \frac{(Q - 1)^2C(n, \alpha)^2}{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2})} \int_0^{s - \tau}$$

$$\int_{\Omega} (s - \tau - \sigma)^{-\left(1 + \frac{n}{2} - \frac{\alpha}{2}\right)} e^{-\frac{\lambda_0|\xi - \eta|^2}{4(s - \tau) - \sigma}} e^{-\frac{\lambda_0|\eta|^2}{4\sigma}} d\xi d\sigma$$

$$\leq \frac{(Q - 1)^2C(n, \alpha)^2}{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2})} \left(\frac{4\pi}{\lambda_0}\right)^{n/2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2})}{\Gamma(\alpha)} (s - \tau)^{-(1 + \frac{n}{2} - \alpha)} e^{-\frac{\lambda_0|x - y|^2}{4(s - \tau)}}$$

$$= \frac{(Q - 1)^2C(n, \alpha)^2}{\Gamma(\alpha)} \left(\frac{4\pi}{\lambda_0}\right)^{n/2} (s - \tau)^{-(1 + \frac{n}{2} - \alpha)} e^{-\frac{\lambda_0|x - y|^2}{4(s - \tau)}}$$

(3.18)

That is,

$$|(LZ)_2(x, s - \tau; y)| \leq \frac{(Q - 1)^2C(n, \alpha)^2}{\Gamma(\alpha)} \left(\frac{4\pi}{\lambda_0}\right)^{n/2} (s - \tau)^{-(1 + \frac{n}{2} - \alpha)} e^{-\frac{\lambda_0|x - y|^2}{4(s - \tau)}}$$

Similarly,

$$|(LZ)_3(x, s - \tau; y)| \leq \int_0^{s - \tau} \int_{\Omega} |LZ(x, s - \tau - \sigma; \xi)||LZ_2(\xi, \sigma; y)| d\xi d\sigma$$

$$\leq \frac{(Q - 1)^3C(n, \alpha)^3}{\Gamma(\frac{n}{2})\Gamma(\alpha)} \left(\frac{4\pi}{\lambda_0}\right)^n (s - \tau)^{-(1 + \frac{n}{2} - \frac{3\alpha}{2})} e^{-\frac{\lambda_0|x - y|^2}{4(s - \tau)}}$$

(3.19)
Proceeding by induction, we further obtain

\[
| (LZ)_i (x, s - \tau; y) | 
\leq \frac{(Q - 1)^i C(n, \alpha)^i}{\Gamma(\frac{in}{2})} \left( \frac{4\pi}{\lambda_0} \right)^{\frac{in}{2}} (s - \tau)^{- \left( 1 + \frac{1}{2} - \frac{in}{2} \right)} \exp \left( - \frac{\lambda_0 |x - y|^2}{4(s - \tau)} \right),
\]

where \( \lambda_0 < Q^{-1} \). Note that the coefficient \( C(n, \alpha) \) is independent of \( i \) and the Gamma function grows faster than power functions. It follows that the series expansion of \( \Phi \) is convergent and satisfies (3.14). Furthermore,

\[
| \Phi(x, s; y) | \leq \frac{(Q - 1) C(n, \alpha)}{s^{(n+2-\alpha)/2}} e^{- \frac{|x-y|^2}{8s}},
\]

by setting \( \lambda_0 = 1/\sqrt{2} \).

The Hölder continuity of \( \Phi(x, s; y) \) in \( x \) is the same as the proof in [20, Sect. 4 in Chap. 9 and Sect. 4 in Chap. 1] (see (1.4.17) and (9.4.17)). Therefore, we omit the details and have \( \Gamma \) the fundamental solution of (3.6).

By (3.16) and a trivial bound of \( |Z(x, s - \tau; \xi)| \), it follows that

\[
\int_0^s \int_\Omega |Z(x, s - \tau; \xi)\Phi(\xi, \tau; y)| |d\xi| d\tau \leq \frac{(Q - 1) C(n, \alpha)}{s^{(n+\alpha)/2}} e^{- \frac{|x-y|^2}{8s}},
\]

and hence (3.10). In particular, we have (3.9), for \( s \leq 4 \),

\[
|\Gamma(x, s; y)| \leq \frac{D(n, \alpha)}{s^{n/2}} e^{- \frac{|x-y|^2}{8s}},
\]

for some constant \( D(n, \alpha) \). Note that

\[
\nabla \Gamma(x, s; y) = \nabla Z(x, s; y) + \int_0^s \int_\Omega \nabla Z(x, s - \tau; \xi) \Phi(\xi, \tau; y) d\xi d\tau.
\]

Applying (3.16) once again, it follows that

\[
\int_0^s \int_\Omega |\nabla Z(x, s - \tau; \xi)||\Phi(\xi, \tau; y)| d\xi d\tau \leq \frac{(Q - 1) C(n, \alpha)}{s^{(n+1-\alpha)/2}} e^{- \frac{|x-y|^2}{8s}}.
\]

and hence (8) holds.

Inequalities, (3.12) and (3.13), follow from straightforward computations that

\[
\frac{\partial}{\partial x^a} Z(x, s; y) = \frac{1}{2s} \tilde{g}_{ab}(y)(x^b - y^b) Z(x, s; y)
\]

and

\[
\frac{\partial}{\partial x^a} \Gamma_E(x, s; y) = \frac{1}{2s} \delta_{ab}(x^b - y^b) \Gamma_E(x, s; y).
\]
It follows that for \((x, s) \notin P_{\frac{1}{2}, \frac{3}{4}}(y)\),

\[
|\Gamma_E(x, s; y) - Z(x, s; y)| \leq (Q - 1)C(n, \alpha)
\]

and

\[
|\nabla\Gamma_E(x, s; y) - \nabla Z(x, s; y)| \leq (Q - 1)C(n, \alpha)
\]

and hence (3.12) and (3.13). □

With the interior Schauder estimates, the semigroup domination, and the maximum principle on the heat equation \(\partial_t u = \Delta_M u\), we have Lemma 9 which shows that every solution to (3.6) that decays exponentially is close to \(\Gamma\) that we construct in Lemma 8, and hence is close to the standard Euclidean heat kernel \(\Gamma_E\).

**Lemma 9** Let \(\epsilon > 0, C(n) > 0\) and \(0 < \alpha < 1, 1 < Q < \sqrt{2}\), and \(r\) be given.

Suppose \(\Gamma^1_{TM}\) and \(\Gamma^2_{TM}\) are fundamental solutions of (3.6) on \(\mathbb{R}^n \times \mathbb{R}_+\) satisfying the decay rate

\[
|\Gamma^i_{TM}(x, s; y)| \leq \frac{C(n)}{s^{n/2}} e^{-\frac{|x-y|^2}{8s}}, \quad \text{for } i = 1, 2 \text{ and } 0 < s \leq 4.
\]  

(3.20)

There exists \(\tilde{R} = \tilde{R}(n, \alpha, Q, C(n), \epsilon)\) so that for \(R > \tilde{R}\),

\[
|\Gamma^1_{TM}(\cdot, \cdot; y) - \Gamma^2_{TM}(\cdot, \cdot; y)|_{C^{2,0}(P_{2,4}(0))} \leq \epsilon
\]  

(3.21)

for all \(y \in B_R(0)\).

**Proof** By the Schauder type interior estimates [23, Theorem 3], there is a constant \(C = C(n, \alpha, Q)\) so that for \(1 \leq a, b \leq n\),

\[
|(\Gamma^i_{TM})_{ab}(\cdot, \cdot; y)|_{C^{2,0}(P_{2,4}(0))} \leq C \sum_{b=1}^{n} (\Gamma^i_{TM})_{ab}(\cdot, \cdot; y)|_{C^{0}(P_{3,4}(0))}.
\]  

(3.22)

By the semigroup domination theory [22, Theorem 3.1]

\[
\text{Tr} \left( e^{t\Delta_{TM}} \right) \leq n \text{Tr} \left( e^{t\Delta_M} \right),
\]  

(3.23)

and the fact that the associated solutions \(\Gamma^i_M(x, s; y)\), \(i = 1, 2\), of \(\partial_s u = \Delta_M u\) decay exponential \(\frac{C(n)}{s^{n/2}} \exp \left( -\frac{|x-y|^2}{8s} \right)\), we have the bound (3.20).

By the maximum principle, there exists \(\tilde{R} = (n, \alpha, Q, C, \epsilon)\) so that for \(y \in B_R(0), R > \tilde{R}\),

\[
|\Gamma^1_M(\cdot, \cdot; y) - \Gamma^2_M(\cdot, \cdot; y)|_{C^0(P_{3,4}(0))} \\
\leq |\Gamma^1_M(\cdot, \cdot; y) - \Gamma^2_M(\cdot, \cdot; y)|_{C^0(\partial B_R(0) \times [0,4])} < \epsilon.
\]  

(3.24)
Combine this with (3.20), we have (3.21).

By Kato’s inequality and Gaussian bounds on $K_M$, we have (2.9), that is, $\tilde{K}_M$ has a Gaussian upper bound. Thus, by Lemmas 8 and 9, $\tilde{K}_M$ is close to $\Gamma_E$.

## 4 Embedding with Heat Kernels

In this section, we prove Theorem 4 and Corollary 5 that manifolds can be embedded with heat kernels as well as how the local dilatation can be controlled by taking the fast decay of heat kernels and the rescaling technique into account.

**Proof** Let $\epsilon$ be given. To determine a scale $r > 0$, we need the following three steps.

**Step 1.** By the Schauder interior estimates [23, Theorem 3] for parabolic systems,

$$|K_M(u^{-1}(\cdot, \cdot; q))|_{C^{2+\alpha}(P_{r_0}^n, \tau(0))} \leq C(n)|K_M(u^{-1}(\cdot, \cdot; q))|_{C^0(P_{r_0}^n, \tau(0))}$$

on a local domain in $\mathbb{R}^n \times \mathbb{R}_+$. By the Gaussian bound of the heat kernel (2.9), we can select $0 < r_0 = r_0(n, \kappa, \iota_0, \epsilon) < \frac{h}{2}$ so that for $0 < t < 2r_0^2$,

$$\int_{M \setminus B_{r_0}^n} |\nabla_q|K_M(p, t; q)|_{HS}^2|dq < \epsilon. \quad (4.1)$$

**Step 2.** Let $R_1 = R_1(n, \epsilon)$ be a radius so that for every $\Gamma(x, s; y)$ satisfying the gradient decay

$$|\nabla_y||\Gamma(x, s; y)||_{HS}^2 \leq D(n) \frac{s^{n+1/2}}{8s} \exp\left(-\frac{|x - y|^2}{8s}\right) \text{ on } (\mathbb{R}^n \setminus B_{r_1}^n(0)) \times \mathbb{R}, \quad (4.2)$$

the integral

$$\int_{\mathbb{R}^n \setminus B_{r_1}^n(0)} |\nabla_y||\Gamma(0, s; y)||_{HS}^2|dy < \epsilon \quad \text{for } \frac{1}{2} \leq s \leq 2. \quad (4.3)$$

**Step 3.** Set $\alpha = \frac{1}{2}$. Choose $1 < Q < \sqrt{2}$ so that $Q - 1 < \epsilon$. Select $R_2 = R_2(n, \alpha = \frac{1}{2}, Q, \epsilon)$ as in Lemma 9 such that the rescaled heat kernel $\tilde{K}_M(0, \frac{1}{2}; y)$ and the solution $\Gamma(0, \frac{1}{2}; y)$ obtained in Lemma 8 are close for all $y \in B_{R_2}(0)$.

We now set

$$r_1 := \min\left(\frac{r_0}{R_1}, \frac{r_h}{2R_2}\right) \quad (4.4)$$

and choose a scale $r < r_1$ and $t = \frac{r^2}{2}$.
Let \( \omega \) denote the modulus of continuity of \( \nabla \parallel \Gamma(E(0, s; \cdot)) \parallel^{2}_{H^2} \). Let \( \delta \) be smaller than \( r \omega^{-1}(\epsilon)/2 \) and \( \{q_{i}\}_{i=1}^{N_{0}} \) be a \( \delta \)-net. Let \( \{A_{i}\}_{i=1}^{N_{0}} \) be a partition of the manifold \( M \) so that \( A_{i} \subset B_{\delta}(q_{i}) \).

Let \( p \in M \). With the chosen \( \alpha = \frac{1}{2} \) and \( Q \), there exists a \( C^{2, \alpha} \)-harmonic coordinate chart \( u : B_{r_{0}}(p) \rightarrow \mathbb{R}^{n} \) with \( u(p) = 0 \). Let \( v = \sum_{a=1}^{n} v_{a} \partial_{a} \in T_{p}M \), \( |v| = 1 \) so that \( \hat{v} = (v^{1}, \cdots, v^{n}) \) is the coordinate of \( v \) under the harmonic coordinate chart. We next show that for any \( t < 2r_{0}^{2} \), where \( r_{0} = r_{0}(n, \kappa, i_{0}, \epsilon, \tau) \), there exists a constant \( \delta = \delta(n, \kappa, i_{0}, \epsilon, t) \) so that for every \( \delta \)-net, \( |dH_{p}(v)|^{2} \) is close to 1.

Let \( I_{\rho}(p) \) denote the subset of \( \{1, \cdots, N_{0}\} \) such that \( A_{j} \cap B_{\rho}(p) \neq \emptyset \) and \( y_{i} := r^{-1}u(q_{i}) \). Then, since \( \bar{K}_{TM}(x, s; y) := r^{n}K_{TM}(u^{-1}(x), sr^{2}; u^{-1}(yr)) \),

\[
|dH_{p}(v)|^{2} = \frac{(2t)^{\frac{3n+2}{2}}}{V_{\epsilon}^{2}} \sum_{i=1}^{N_{0}} |A_{i}|(\nabla \parallel K_{TM}(p, t; q_{i}) \parallel^{2}_{H^{2}} \cdot v)^{2} \]

\[
= \frac{1}{V_{\epsilon}^{2}} \sum_{i \in I_{1}} |A_{i}|(\nabla \parallel \tilde{K}_{TM}(0, \frac{1}{2}; y_{i}) \parallel^{2}_{H^{2}} \cdot \hat{v})^{2} \]

\[
+ \frac{1}{V_{\epsilon}^{2}} \sum_{i \in I_{2}} |A_{i}|(\nabla \parallel \tilde{K}_{TM}(0, \frac{1}{2}; y_{i}) \parallel^{2}_{H^{2}} \cdot \hat{v})^{2} \]

\[
+ \frac{(2t)^{\frac{3n+2}{2}}}{V_{\epsilon}^{2}} \sum_{i \in I_{3}} |A_{i}|(\nabla \parallel K_{TM}(p, t; q_{i}) \parallel^{2}_{H^{2}} \cdot v)^{2} \]

\[
= (I) + (II) + (III) \]

where \( I_{1} = I_{R_{1}}(p) \), \( I_{2} = I_{R_{2}}(p) \setminus I_{1} \), and \( I_{3} = \{1, \cdots, N_{0}\} \setminus I_{1} \cup I_{2} \). Here, we write the first two terms in terms of the rescaled coordinates.

First, note that \( \cup_{i \in I_{3}} A_{i} \) lies inside \( M \setminus B_{2r}(p) \). Because of equation (4.1), (III) is controlled by \( \epsilon \), that is,

\[
(2t)^{\frac{3n+2}{2}} \sum_{i \in I_{3}} |A_{i}|(\nabla \parallel K_{TM}(p, t; q_{i}) \parallel^{2}_{H^{2}} \cdot v)^{2} < C(n)\epsilon. \tag{4.5} \]

Second, since \( \text{diam}(A_{i}) < r \omega(\epsilon) \),

\[
\left| \sum_{i \in I_{2}} |A_{i}|(\nabla \parallel \tilde{K}_{TM}(0, \frac{1}{2}; y_{i}) \parallel^{2}_{H^{2}} \cdot \hat{v})^{2} - \sum_{i \in I_{2}} \int_{\frac{u(A_{i})}{r}} (\nabla \parallel \tilde{K}_{TM}(0, \frac{1}{2}; y) \parallel^{2}_{H^{2}} \cdot \hat{v})^{2} d\mu \right| < C(n)\epsilon \]

where \( d\mu \) denotes the push-forward of the standard volume measure under \( r^{-1}u \).
Note that \( \bigcup_{i \in I_2} A_i \) lies outside of \( B_{rR_1}(p) \) and inside \( B_{rh}(p) \). Thus, \( \bigcup_{i \in I_2} r^{-1}u(A_i) \) is a subset of \( \mathbb{R}^n \setminus B_{R_1}(0) \). By Lemmas 8 and 9, \( \tilde{K}_T(x, \frac{1}{2}; y) \) satisfies the gradient decay (4.2). By the choice of \( R_1 \) in (4.3), we have that

\[
\sum_{i \in I_2} \int_{u(A_i)} (\nabla \| \tilde{K}_T(0, \frac{1}{2}; y) \|_{HS}^2 \cdot \hat{v})^2 d\mu < \epsilon
\]

and that (II) is controlled by \( \epsilon \) as well,

\[
\sum_{i \in I_2} \frac{|A_i|}{r^n} (\nabla \| \tilde{K}_T(0, \frac{1}{2}; y_i) \|_{HS}^2 \cdot \hat{v})^2 < C(n)\epsilon. \tag{4.6}
\]

Last, we show that (I) is close to 1. Let

\[
R := \max(R_1, R_2). \tag{4.7}
\]

Note that \( B_R(0) \subset u(B_{r_1}(p))/r \). By Lemma 9 and our choice of \( Q \) and \( R_2 \), for all \( y \in B_R(0) \), \( \tilde{K}_T(0, \frac{1}{2}; y) \) is close to the solution \( \Gamma(0, \frac{1}{2}; y) \) constructed in Lemma 8, which is close to the Euclidean kernel \( \Gamma_E(0, \frac{1}{2}; y) \). Specifically,

\[
\left| \nabla \| \tilde{K}_T(0, \frac{1}{2}; y) \|_{HS}^2 - \nabla \| \Gamma_E(0, \frac{1}{2}; y) \|_{HS}^2 \right| < C(n)\epsilon.
\]

This implies

\[
\sum_{i \in I_1} \left( (\hat{v} \cdot \nabla \| \tilde{K}_T(0, \frac{1}{2}; y_i) \|_{HS}^2)^2 - (\hat{v} \cdot \nabla \| \Gamma_E(0, \frac{1}{2}; y_i) \|_{HS}^2)^2 \right) \frac{|A_i|}{r^n} < C(n)\epsilon \sum_{i \in I_1} \frac{|A_i|}{r^n} < C(n)\epsilon |B_{R_1}(0)|. \tag{4.8}
\]

Since \( \text{diam}(A_i) < r\omega(\epsilon) \), we have

\[
\left| \sum_{i \in I_1} (\nabla \tilde{v} \| \Gamma_E(0, \frac{1}{2}; y) \|_{HS}^2)^2 \frac{|A_i|}{r^n} - \int_{u(A_i)} (\nabla \tilde{v} \| \Gamma_E(0, \frac{1}{2}; y) \|_{HS}^2)^2 d\mu \right| < C(n)\epsilon, \tag{4.9}
\]
where $d\mu$ denotes the push-forward of the standard volume form under $r^{-1}u$. Due to that $Q^{-1} < g < Q$ and symmetry of $\Gamma_E(x, t; y) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x-y|^2}{4t}}I_n$, we have

$$
|\sum_{i \in I_1} \left( \int_{u(A_i)} (\nabla \|\Gamma_E(0, \frac{1}{2}; y)\|_{HS}^2) d\mu - \int_{u(A_i)} (\nabla \|\Gamma_E(0, \frac{1}{2}; y)\|_{HS}^2) dy \right) | < C(n)(Q - 1)\epsilon < C(n)\epsilon.
$$

(4.10)

Furthermore, by the choice of $R_1$ and (4.3),

$$
|\sum_{i \in I_1} \int_{u(A_i)} (\partial_{x_1} \|\Gamma_E(0, \frac{1}{2}; y)\|_{HS}^2) dy - V^2 | < \epsilon
$$

(4.11)

Combining inequalities (4.8), (4.9), (4.10), and (4.11), we have that (I) is close to 1. Together with (4.5) and (4.6), we conclude that

$$
|dH_p(v)|^2 - 1| \leq C(n)\epsilon
$$

that is, (2.2) in Theorem 4.

As for the second part, (2.4) of Theorem 4, by Lemma 10 in [19], it follows that there exists an $N = N(n, \kappa, i_0, V, \epsilon, t_0)$ so that for any $m \geq N$ and $0 < t \leq t_0$,

$$
\|K^{(m)}_{TM}(\cdot, t; q)\|_{HS} - \|K_{TM}(\cdot, t; q)\|_{HS} \|_\infty < \epsilon \quad (4.12)
$$

$$
\|\nabla K^{(m)}_{TM}(\cdot, t; q)\|_{HS} - \|\nabla K_{TM}(\cdot, t; q)\|_{HS} \|_\infty < \epsilon \quad (4.13)
$$

Therefore,

$$
1 - \epsilon < |dH_p^{(m)}(v)|^2 < 1 + \epsilon.
$$

Next, we prove Corollary 5.

**Proof** For any $A > 0$, we simply replace for each fixed $i$ the point $q_i$ in Theorem 4 by points $q^j_i$, $j = 1, \cdots, N_i$, where $N_i = \lceil |A_i|/A \rceil$. Next, rename all the points $q^j_i$ to $p_k$. It follows that when $A = A(n, \kappa, i_0, \epsilon, t, V)$ small enough,

$$
1 - \epsilon < |dH_p(v)| < 1 + \epsilon.
$$

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Declarations

Conflict of interest  I, Chen-Y un Lin, certify that I have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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