Solvability of linear differential systems in the Liouvillian sense

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Abstract

The paper concerns the solvability by quadratures of linear differential systems, which is one of the questions of differential Galois theory. We consider systems with regular singular points as well as those with (non-resonant) irregular ones and propose some criteria of solvability for systems whose (formal) exponents are sufficiently small.

1 Introduction

Consider on the Riemann sphere \( \mathbb{C} \) a linear differential system

\[
\frac{dy}{dz} = B(z)y, \quad y(z) \in \mathbb{C}^p,
\]

(1)
of \( p \) equations with a meromorphic coefficient matrix \( B(z) \) having singularities at points \( a_1, \ldots, a_n \).

A singular point \( z = a_i \) is said to be regular, if any solution of the system has at most polynomial growth in any sector of small radius with vertex at this point and an opening less than \( 2\pi \). Otherwise the point \( z = a_i \) is said to be irregular.

The Picard–Vessiot extension of the field \( \mathbb{C}(z) \) of rational functions corresponding to the system (1) is a differential field \( F \) obtained by adjoining to \( \mathbb{C}(z) \) all entries of a fundamental matrix \( Y(z) \) of the system (1). One says that the system (1) is solvable by quadratures, if the entries of the matrix \( Y(z) \) are expressed in elementary or algebraic functions and their integrals or, more formally, if the field \( F \) is contained in some extension of \( \mathbb{C}(z) \) obtained by adjoining algebraic functions, exponentials or integrals:

\[
\mathbb{C}(z) = F_1 \subseteq \ldots \subseteq F_m, \quad F \subseteq F_m,
\]

where \( F_{i+1} = F_i(x_i) \) (\( i = 1, \ldots, m - 1 \)), and either \( x_i \) is algebraic over \( F_i \), or \( x_i \) is an exponential of an element in \( F_i \), or \( x_i \) is an integral of an element in \( F_i \). Such an extension \( \mathbb{C}(z) \subseteq F_m \) is called Liouvillian, thus solvability by quadratures means that the Picard–Vessiot extension \( F \) is contained in some Liouvillian extension of the field of rational functions.

Solvability or non-solvability of a linear differential system by quadratures is related to properties of its Galois group. The differential Galois group \( G = \text{Gal}(F/\mathbb{C}(z)) \) of the system (1) (of the Picard–Vessiot extension \( \mathbb{C}(z) \subseteq F \)) is the group of differential automorphisms of the field \( F \) (i.e., automorphisms commuting with differentiation) that preserve elements of the field \( \mathbb{C}(z) \):

\[
G = \left\{ \sigma : F \rightarrow F \mid \sigma \circ \frac{d}{dz} = \frac{d}{dz} \circ \sigma, \sigma(f) = f \quad \forall f \in \mathbb{C}(z) \right\}.
\]

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As follows from the definition, the image \( \sigma(Y) \) of the fundamental matrix \( Y(z) \) of the system (1) under any element \( \sigma \) of the Galois group is a fundamental matrix of this system again, that is, \( \sigma(Y) = Y(z)C \), \( C \in \text{GL}(p, \mathbb{C}) \). As every element of the Galois group is determined uniquely by its action on a fundamental matrix of the system, the Galois group \( G \) can be regarded as a subgroup of the matrix group \( \text{GL}(p, \mathbb{C}) \). Moreover this subgroup \( G \subset \text{GL}(p, \mathbb{C}) \) is algebraic, i.e., closed in the Zariski topology of the space \( \text{GL}(p, \mathbb{C}) \) (the topology whose closed sets are those determined by systems of polynomial equations), see [12, Th. 5.5].

The Galois group \( G \) can be represented as a union of finite number of disjoint connected sets that are open and closed simultaneously (in the Zariski topology), and the set containing the identity matrix is called the identity component. The identity component \( G_0 \subset G \) is a normal subgroup of finite index [12, Lemma 4.5]. Due to the Picard–Vessiot theorem, solvability of the system (1) by quadratures is equivalent to solvability of the subgroup \( G_0 \) [12, Th. 5.12], [13, Ch. 3, Th. 5.1]. (Recall that a group \( H \) is said to be solvable, if there exist intermediate normal subgroups \( \{ e \} = H_0 \subset H_1 \subset \ldots \subset H_m = H \) such that each factor group \( H_i/H_{i-1} \) is Abelian, \( i = 1, \ldots, m \).)

Alongside the Galois group one considers the monodromy group \( M \) of the system (1) generated by the monodromy matrices \( M_1, \ldots, M_n \) corresponding to analytic continuation of a fundamental matrix \( Y(z) \) around the singular points \( a_1, \ldots, a_n \). (The matrix \( Y(z) \) considered in a neighbourhood of a non-singular point \( z_0 \) goes to \( Y(z)M_i \) under an analytic continuation along a simple loop \( \gamma_i \) encircled a point \( a_i \).) As the operation of analytic continuation commutes with differentiation and preserves elements of the field \( \mathbb{C}(z) \) (single-valued functions), one has \( M \subseteq G \). Furthermore the Galois group of a system whose singular points are all regular is a closure of its monodromy group (in the Zariski topology, see [13, Ch. 6, Cor. 1.3]), hence such a system is solvable by quadratures, if and only if the identity component of its monodromy group is solvable.

We are interested in the cases when the answer to the question concerning solvability of a linear differential system by quadratures can be given in terms of the coefficient matrix of the system. For example, in the case of a Fuchsian system (a particular case of a system with regular singular points)

\[
\frac{dy}{dz} = \left( \sum_{i=1}^{n} \frac{B_i}{z - a_i} \right) y, \quad B_i \in \text{Mat}(p, \mathbb{C}),
\]

whose coefficients \( B_i \) are sufficiently small, Yu. S. Ilyashenko and A. G. Khovansky [9] have obtained an explicit criterion of solvability. Namely, the following statement holds:

There exists \( \varepsilon = \varepsilon(n, p) > 0 \) such that a condition of solvability by quadratures for the Fuchsian system (2) with \( \| B_i \| < \varepsilon \) takes an explicit form: the system is solvable by quadratures, if and only if all the matrices \( B_i \) are triangular (in some basis).

In this article we propose a refinement of the above assertion in which it is sufficient that the eigenvalues of the residue matrices \( B_i \) be small (the estimate is given), and we also propose a generalization to the case of a system with irregular singular points.

2 A local form of solutions of a system near its singular points

A singular point \( a_i \) of the system (1) is said to be Fuchsian, if the coefficient matrix \( B(z) \) has a simple pole at this point.

Due to Sauvage’s theorem, a Fuchsian singular point of a linear differential system is regular (see [3, Th. 11.1]). However, the coefficient matrix of a system at a regular singular point may
in general have a pole of order greater than one. Let us write the Laurent expansion of the
coefficient matrix \( B(z) \) of the system (1) near its singular point \( z = a \) in the form

\[
B(z) = \frac{B_{-r-1}}{(z-a)^{r+1}} + \ldots + \frac{B_{-1}}{z-a} + B_0 + \ldots, \quad B_{-r-1} \neq 0. \tag{3}
\]

The number \( r \) is called the Poincaré rank of the system (1) at this point (or the Poincaré rank of
the singular point \( z = a \)). For example, the Poincaré rank of a Fuchsian singularity is equal
to zero.

The system (1) is said to be Fuchsian, if all its singular points are Fuchsian (then it can be
written in the form (2)). The system whose singular points are all regular will be called regular singular.

According to Levelt’s theorem [15], in a neighbourhood of each regular singular point \( a_i \) of
the system (1) there exists a fundamental matrix of the form

\[
Y_i(z) = U_i(z)(z - a_i)^A_i(z - a_i)^\tilde{E}_i, \tag{4}
\]

where \( U_i(z) \) is a holomorphic matrix at the point \( a_i \), \( A_i = \text{diag}(\varphi_1^i, \ldots, \varphi_p^i) \) is a diagonal integer
matrix whose entries \( \varphi_j^i \) organize in a non-increasing sequence, \( \tilde{E}_i = (1/2\pi i) \ln \tilde{M}_i \) is an upper-
triangular matrix (the normalized logarithm of the corresponding monodromy matrix) whose
eigenvalues \( \rho_j^i \) satisfy the condition

\[
0 \leq \Re \rho_j^i < 1.
\]

Such a fundamental matrix is called a Levelt matrix, and one also says that its columns form a
Levelt basis in the solution space of the system (in a neighborhood of the regular singular point
\( a_i \)). The complex numbers \( \beta_j^i = \varphi_j^i + \rho_j^i \) are called the (Levelt) exponents of the system at the
regular singular point \( a_i \).

If the singular point \( a_i \) is Fuchsian, then the corresponding matrix \( U_i(z) \) in the decomposition
(4) is holomorphically invertible at this point, that is, \( \det U_i(a_i) \neq 0 \). It is not difficult to check
that in this case the exponents of the system at the point \( a_i \) coincide with the eigenvalues of the
residue matrix \( B_i \). And in the general case of a regular singularity \( a_i \) there are estimates for the
order of the function \( \det U_i(z) \) at this point obtained by E. Corel [4] (see also [3]):

\[
r_i \leq \ord_{a_i} \det U_i(z) \leq \frac{p(p-1)}{2} r_i,
\]

where \( r_i \) is the Poincaré rank of the regular singular point \( a_i \). These estimates imply the
inequalities for the sum of exponents of the regular system over all its singular points, which are
called the Fuchs inequalities:

\[
- \frac{p(p-1)}{2} \sum_{i=1}^n r_i \leq \sum_{i=1}^n \sum_{j=1}^p \beta_j^i \leq - \sum_{i=1}^n r_i \tag{5}
\]

(the sum of exponents is an integer).

Let us now describe the structure of solutions of the system (1) near one of its irregular
singular points. We suppose that the irregular singularity \( a_i \) of Poincaré rank \( r_i \) is non-resonant,
that is, the eigenvalues \( b_1^i, \ldots, b_p^i \) of the leading term \( B_{-r_i-1} \) of the matrix \( B(z) \) in the expansion
(3) at this point are pairwise distinct. Let us fix a matrix \( T_i \) reducing the leading term \( B_{-r_i-1} \)
to the diagonal form

\[
T_i^{-1} B_{-r_i-1} T_i = \text{diag}(b_1^i, \ldots, b_p^i).
\]
Then the system possesses a uniquely determined formal fundamental matrix $\Y_i(z)$ of the form

$$\Y_i(z) = \F_i(z)(z - a_i)^{A_i} e^{Q_i(z)},$$

where

a) $\F_i(z)$ is a matrix formal Taylor series in $z - a_i$ and $\F_i(a_i) = T_i$;

b) $A_i$ is a constant diagonal matrix whose diagonal entries are called the \textit{formal exponents} of the system (1) at the irregular singular point $a_i$;

c) $Q_i(z) = \text{diag}(q^1_i(z), \ldots, q^N_i(z))$ is a diagonal matrix whose diagonal entries $q^j_i(z)$ are polynomials in $(z - a_i)^{-1}$ of degree $r_i$ without a constant term,

$$q^j_i(z) = -\frac{b^j_i}{r_i}(z - a_i)^{-r_i} + o((z - a_i)^{-r_i}).$$

Furthermore a punctured neighbourhood of the point $a_i$ can be covered by a set $\{S_i^1, \ldots, S_i^{N_i}\}$ of "good" open sectors with vertex at this point (which we take to be arranged in counterclockwise order starting with $S_i^1$) such that in each $S_i^j$ there exists a \textit{unique} genuine fundamental matrix

$$Y_i^j(z) = F_i^j(z)(z - a_i)^{A_i} e^{Q_i(z)} \quad (6)$$

of the system (1) whose factor $F_i^j(z)$ has the asymptotic expansion $\F_i(z)$ in $S_i^j$ (see [19, Th. 21.13, Prop. 21.17]). In every intersection $S_i^j \cap S_i^{j+1}$ the fundamental matrices $Y_i^j$ and $Y_i^{j+1}$ necessarily differ by a constant invertible matrix:

$$Y_i^{j+1}(z) = Y_i^j(z)C_i^j, \quad C_i^j \in \text{GL}(p, \mathbb{C}),$$

and it is understood that the logarithmic term $(z - a_i)^{A_i}$ is analytically continued from $S_i^1$ to $S_i^2$, from $S_i^2$ to $S_i^3$, ..., from $S_i^{N_i}$ to $S_i^1$, so that

$$Y_i^1(z)e^{2\pi i A_i} = Y_i^{N_i}(z)C_i^{N_i} \quad \text{in} \quad S_i^{N_i} \cap S_i^1. \quad (7)$$

The matrices $C_i^1, \ldots, C_i^{N_i}$ are called \textit{Stokes matrices} of the system (1) at the non-resonant irregular singular point $a_i$. They satisfy the relation

$$e^{2\pi i A_i} = M_i^1 C_i^1 \cdots C_i^{N_i}, \quad (8)$$

where $M_i^1$ is the monodromy matrix of $Y_i^1$ at the point $a_i$. Indeed, the fundamental matrix $Y_i^1$ can be continued from $S_i^1$ into $S_i^2$ as $Y_i^2(C_i^1)^{-1}$, since $Y_i^1 = Y_i^2(C_i^1)^{-1}$ in $S_i^1 \cap S_i^2$. Furthermore it is continued from $S_i^2$ into $S_i^3$ as $Y_i^3(C_i^2)^{-1}C_i^1$, etc. Finally, in $S_i^{N_i}$ it becomes equal to $Y_i^{N_i}(C_i^1 \cdots C_i^{N_i-1})^{-1}$. Then it comes back into $S_i^1$ as $Y_i^1 e^{2\pi i A_i}(C_i^1 \cdots C_i^{N_i})^{-1}$ according to (7), whence the relation (8) follows. It is also known that all the eigenvalues of any Stokes matrix are equal to 1, that is, the Stokes matrices are unipotent (see [19, Prop. 21.19] or [19, Th. 15.2]).
3 Linear differential systems and meromorphic connections on holomorphic vector bundles

Let us recall some notions concerning holomorphic vector bundles and meromorphic connections in a context of linear differential equations. Here we mainly follow [11, Ch. 3] or [7] (see also [3]).

In an analytic interpretation, a holomorphic bundle $E$ of rank $p$ over the Riemann sphere is defined by a cocycle $\{g_{\alpha\beta}(z)\}$, that is, a collection of holomorphic matrix functions corresponding to a covering $\{U_\alpha\}$ of the Riemann sphere:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(p, \mathbb{C}), \quad U_\alpha \cap U_\beta \neq \emptyset.$$  

These functions satisfy the conditions

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I \quad (\text{for } U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset).$$

Two holomorphically equivalent cocycles $\{g_{\alpha\beta}(z)\}$, $\{g'_{\alpha\beta}(z)\}$ define the same bundle. Equivalence of cocycles means that there exists a set $\{h_\alpha(z)\}$ of holomorphic matrix functions $h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$ such that

$$h_\alpha(z)g_{\alpha\beta}(z) = g'_{\alpha\beta}(z)h_\beta(z). \quad (9)$$

A section $s$ of the bundle $E$ is determined by a set $\{s_\alpha(z)\}$ of vector functions $s_\alpha : U_\alpha \rightarrow \mathbb{C}^p$ that satisfy the conditions $s_\alpha(z) = g_{\alpha\beta}(z)s_\beta(z)$ in intersections $U_\alpha \cap U_\beta \neq \emptyset$.

A meromorphic connection $\nabla$ on the holomorphic vector bundle $E$ is determined by a set $\{\omega_\alpha\}$ of matrix meromorphic differential 1-forms that are defined in the corresponding neighbourhoods $U_\alpha$ and satisfy gluing conditions

$$\omega_\alpha = (dg_{\alpha\beta})g_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \quad (\text{for } U_\alpha \cap U_\beta \neq \emptyset). \quad (10)$$

Under a transition to an equivalent cocycle $\{g'_{\alpha\beta}\}$ connected with the initial one by the relations $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I$, the 1-forms $\omega_\alpha$ of the connection $\nabla$ are transformed into the corresponding 1-forms

$$\omega'_\alpha = (dh_\alpha)h_\alpha^{-1} + h_\alpha\omega_\alpha h_\alpha^{-1}. \quad (11)$$

Conversely, the existence of holomorphic matrix functions $h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$ such that the matrix 1-forms $\omega_\alpha$ and $\omega'_\alpha$ (satisfying the conditions $10$ for $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ respectively) are connected by the relation $11$ in $U_\alpha$, indicates the equivalence of the cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ (one may assume that the intersections $U_\alpha \cap U_\beta$ do not contain singular points of the connection).

Vector functions $s_\alpha(z)$ satisfying linear differential equations $ds_\alpha = \omega_\alpha s_\alpha$ in the corresponding $U_\alpha$, by virtue of the conditions $10$ can be chosen so that a set $\{s_\alpha(z)\}$ determines a section of the bundle $E$, which is called horizontal with respect to the connection $\nabla$. Thus horizontal sections of a holomorphic vector bundle with a meromorphic connection are determined by solutions of local linear differential systems. The monodromy of a connection (the monodromy group) characterizes ramification of horizontal sections under their analytic continuation along loops in $\mathbb{C}$ not containing singular points of the connection 1-forms and is defined similarly to the monodromy group of the system $1$. A connection may be called Fuchsian (logarithmic), regular or irregular depending on the type of the singular points of its 1-forms (as singular points of linear differential systems).

If a bundle is holomorphically trivial (all matrices of the cocycle can be taken as the identity matrices), then by virtue of the conditions $10$ the matrix 1-forms of a connection coincide
on non-empty intersections $U_\alpha \cap U_\beta$. Hence horizontal sections of such a bundle are solutions of a global linear differential system defined on the whole Riemann sphere. Conversely, the linear system (11) determines a meromorphic connection on the holomorphically trivial vector bundle of rank $p$ over $\mathbb{C}$. It is understood that such a bundle has the standard definition by the cocycle that consists of the identity matrices while the connection is defined by the matrix 1-form $B(z)dz$ of coefficients of the system. But for us it will be more convenient to use the following equivalent coordinate description (a construction already appearing in [3]).

At first we consider a covering $\{U_\alpha\}$ of the punctured Riemann sphere $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$ by simply connected neighbourhoods. Then on the corresponding non-empty intersections $U_\alpha \cap U_\beta$ one defines the matrix functions of a cocycle, $g'_{\alpha\beta}(z) \equiv \text{const}$, which are expressed in terms of the monodromy matrices $M_1, \ldots, M_n$ of the system (11) via the operations of multiplication and taking the inverse (see [7]). In this case the matrix differential 1-forms $\omega'_\alpha$ defining a connection are equal to zero. Further the covering $\{U_\alpha\}$ is complemented by small neighbourhoods $O_i$ of the singular points $a_i$ of the system, thus we obtain the covering of the Riemann sphere $\mathbb{C}$. To non-empty intersections $O_i \cap U_\alpha$ there correspond matrix functions $g'_{\alpha}(z) = Y_i(z)$ of the cocycle, where $Y_i(z)$ is a germ of a fundamental matrix of the system whose monodromy matrix at the point $a_i$ is equal to $M_i$ (so, for analytic continuations of the chosen germ to non-empty intersections $O_i \cap U_\alpha \cap U_\beta$ the cocycle relations $g_{i\alpha}g_{\alpha\beta} = g_{i\beta}$ hold). The matrix differential 1-forms $\omega'_i$ determining the connection in the neighbourhoods $O_i$ coincide with the 1-form $B(z)dz$ of coefficients of the system. To prove holomorphic equivalence of the cocycle $\{g'_{i\alpha}, g'_{i\alpha}\}$ to the identity cocycle it is sufficient to check existence of holomorphic matrix functions

$$h_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C}), \quad h_i : O_i \rightarrow \text{GL}(p, \mathbb{C}),$$

such that

$$\omega'_\alpha = (dh_\alpha)h^{-1}_\alpha + h_\alpha\omega_\alpha h^{-1}_\alpha, \quad \omega'_i = (dh_i)h^{-1}_i + h_i\omega_i h^{-1}_i. \quad (12)$$

Since we have $\omega_\alpha = B(z)dz$ and $\omega'_i = 0$ for all $\alpha$, the first equation in (12) is rewritten as a linear system

$$d(h^{-1}_\alpha) = (B(z)dz)h^{-1}_\alpha,$$

which has a holomorphic solution $h^{-1}_\alpha : U_\alpha \rightarrow \text{GL}(p, \mathbb{C})$ since the 1-form $B(z)dz$ is holomorphic in a simply connected neighbourhood $U_\alpha$. The second equation in (12) has a holomorphic solution $h_i(z) \equiv I$, as $\omega_i = \omega'_i = B(z)dz$.

One says that a bundle $E$ has a subbundle $E' \subset E$ of rank $k < p$ that is stabilized by a connection $\nabla$, if the pair $(E, \nabla)$ admits a coordinate description $\{g_{i\alpha}\}, \{\omega_\alpha\}$ of the following blocked upper-triangular form:

$$g_{i\alpha} = \begin{pmatrix} g_{i\alpha} & * \\ 0 & g_{i\beta} \end{pmatrix}, \quad \omega_\alpha = \begin{pmatrix} \omega_\alpha^1 \\ 0 \end{pmatrix},$$

where $g_{i\alpha}$ and $\omega_\alpha^1$ are blocks of size $k \times k$ (then the cocycle $\{g_{i\alpha}\}$ defines the subbundle $E'$ and the 1-forms $\omega_\alpha^1$ define the restriction $\nabla'$ of the connection $\nabla$ to the subbundle $E'$).

**Example 1.** Consider a system (11) whose monodromy matrices $M_1, \ldots, M_n$ are of the same blocked upper-triangular form, and the corresponding holomorphically trivial vector bundle $E$ with the meromorphic connection $\nabla$. Suppose that in a neighbourhood of each singular point $a_i$ of the system there exist a fundamental matrix $Y_i(z)$ whose monodromy matrix is $M_i$, and a holomorphically invertible matrix $\Gamma_i(z)$ such that $\Gamma_i Y_i$ is a blocked upper-triangular matrix
(with respect to the blocked upper-triangular form of the matrix $M_i$). Let us show that to the common invariant subspace of the monodromy matrices there corresponds a vector subbundle $E' \subset E$ that is stabilized by the connection $\nabla$.

We use the above coordinate description of the bundle and connection with the cocycle $\{g'_{i\beta}, g'_{i\alpha}\}$ and set $\{\omega'_{i\alpha}, \omega'_{i\beta}\}$ of matrix 1-forms. The matrices $g'_{i\alpha\beta}$ are already blocked upper-triangular since the monodromy matrices $M_1, \ldots, M_n$ are (and $\omega'_{i\alpha} = 0$), while the matrices $g'_{i\alpha} = Y_i$ can be transformed to such a form, $\Gamma_i g'_{i\alpha} = \Gamma_i Y_i$. Thus changing the matrices $g'_{i\alpha}$ onto $\Gamma_i g'_{i\alpha}$ and matrix 1-forms $\omega'_{i\alpha}$ onto

$$\Gamma_i \omega'_{i\alpha} \Gamma_i^{-1} + (d\Gamma_i) \Gamma_i^{-1},$$

we pass to the holomorphically equivalent coordinate description with the cocycle matrices and connection matrix 1-forms having the same blocked upper-triangular form.

The following auxiliary lemma points to a certain block structure of a linear differential system in the case when the corresponding holomorphically trivial vector bundle with the meromorphic connection has a holomorphically trivial subbundle that is stabilized by the connection.

**Lemma 1.** If a holomorphically trivial vector bundle $E$ of rank $p$ over $\mathbb{C}$ endowed with a meromorphic connection $\nabla$ has a holomorphically trivial subbundle $E' \subset E$ of rank $k$ that is stabilized by the connection, then the corresponding linear system $[\mathbb{L}]$ is reduced to a blocked upper-triangular form via a constant gauge transformation $\tilde{y}(z) = Cy(z), C \in \text{GL}(p, \mathbb{C})$. That is,

$$CB(z)C^{-1} = \begin{pmatrix} B'(z) & \ast \\ 0 & \ast \end{pmatrix},$$

where $B'(z)$ is a block of size $k \times k$.

**Proof.** Let $\{s_1, \ldots, s_p\}$ be a basis of global holomorphic sections of the bundle $E$ (which are linear independent at each point $z \in \mathbb{C}$) such that the 1-form of the connection $\nabla$ in this basis is the 1-form $B(z)dz$ of coefficients of the linear system. Consider also a basis $\{s'_1, \ldots, s'_p\}$ of global holomorphic sections of the bundle $E'$ such that $s'_1, \ldots, s'_k$ are sections of the subbundle $E'$, $(s'_1, \ldots, s'_p) = (s_1, \ldots, s_p)C^{-1}, C \in \text{GL}(p, \mathbb{C})$.

Now choose a basis $\{h_1, \ldots, h_p\}$ of sections of the bundle $E$ such that these are horizontal with respect to the connection $\nabla$ and $h_1, \ldots, h_k$ are sections of the subbundle $E'$ (this is possible since $E'$ is stabilized by the connection). Let $Y(z)$ be a fundamental matrix of the system whose columns are the coordinates of the sections $h_1, \ldots, h_p$ in the basis $\{s_1, \ldots, s_p\}$. Then

$$\tilde{Y}(z) = CY(z) = \begin{pmatrix} k \times k & * \\ 0 & * \end{pmatrix}$$

is a blocked upper-triangular matrix, since its columns are the coordinates of the sections $h_1, \ldots, h_p$ in the basis $\{s'_1, \ldots, s'_p\}$. Consequently, the transformation $\tilde{y}(z) = Cy(z)$ reduces the initial system to a blocked upper-triangular form. \hfill \square

The degree $\deg E$ (which is an integer) of a holomorphic vector bundle $E$ endowed with a meromorphic connection $\nabla$ may be defined as the sum

$$\deg E = \sum_{i=1}^{n} \text{res}_{a_i} \text{tr} \omega_i$$

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of the residues of local differential 1-forms \( \text{tr} \omega_i \) over all singular points of the connection (the notation "\( \text{tr} \)" is for the trace), where \( \omega_i \) is the local matrix differential 1-form of the connection \( \nabla \) in a neighbourhood of its singular point \( a_i \). Later when calculating the degree of a bundle we will apply the Liouville formula \( \text{tr} \omega_i = d \ln \det Y_i \), where \( Y_i \) is a fundamental matrix of the local linear differential system \( dy = \omega_i y \).

4 Solvability of regular singular systems with small exponents

Consider a system \( \text{(I)} \) with regular singular points \( a_1, \ldots, a_n \) of Poincaré rank \( r_1, \ldots, r_n \) respectively. If the real part of the exponents of this system is sufficiently small, then the following necessary condition, of solvability by quadratures, holds.

**Theorem 1.** Let for some \( k \in \{1, \ldots, p-1\} \) the exponents \( \beta^j_i \) of the regular singular system \( \text{(I)} \) satisfy the condition

\[
\text{Re} \beta^j_i > -1/nk, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,
\]

and, for each pair \( \beta^j_i \not\equiv \beta^l_i \pmod{\mathbb{Z}} \), \( i = 1, \ldots, n \), one of the conditions

\[
\text{Re} \beta^j_i - \text{Re} \beta^l_i \notin \mathbb{Q} \quad \text{or} \quad \text{Im} \beta^j_i \neq \text{Im} \beta^l_i.
\]

Then the solvability of the system \( \text{(I)} \) by quadratures implies the existence of a constant matrix \( C \in \text{GL}(p, \mathbb{C}) \) such that the matrix \( CB(z)C^{-1} \) has the form as in Theorem 1:

\[
CB(z)C^{-1} = \begin{pmatrix} B'(z) & * \\ 0 & * \end{pmatrix},
\]

where \( B'(z) \) is an upper-triangular matrix of size \( k \times k \).

**Remark 1.** Though the inequalities \( \text{(13)} \) restrict the real parts of the exponents from below, together with the estimates \( \text{(5)} \) they provide boundedness from above.

**Remark 2.** The sum of the Poincaré ranks of a regular singular system whose exponents satisfy the condition \( \text{(13)} \) is indeed restricted because of the Fuchs inequalities \( \text{(5)} \), namely, \( \sum_{i=1}^n r_i < p/k \).

The proof of the theorem is based on two auxiliary lemmas.

**Lemma 2.** Let the exponents \( \beta^j_i \) of the regular singular system \( \text{(I)} \) satisfy the condition \( \text{(13)} \). If the monodromy matrices of this system are upper-triangular, then there is a constant matrix \( C \in \text{GL}(p, \mathbb{C}) \) such that the matrix \( CB(z)C^{-1} \) has the form as in Theorem 1.

**Proof.** We use a geometric interpretation (exposed in the previous section) according to which to the regular singular system \( \text{(I)} \) there corresponds the holomorphically trivial vector bundle \( E \) of rank \( p \) over the Riemann sphere endowed with the meromorphic connection \( \nabla \) with the regular singular points \( a_1, \ldots, a_n \).

Since the monodromy matrices \( M_1, \ldots, M_n \) of the system are upper-triangular there exists, as shown in Example 1, a flag \( E^1 \subset E^2 \subset \ldots \subset E^p = E \) of subbundles of rank \( 1, 2, \ldots, p \) respectively that are stabilized by the connection \( \nabla \). Indeed, a fundamental matrix \( Y \) determining the monodromy matrices \( M_1, \ldots, M_n \) of the system can be represented near each singular point \( a_i \) in the form

\[
Y(z) = T_i(z)(z - a_i)^{E_i}, \quad E_i = (1/2\pi i) \ln M_i,
\]
where $T_i$ is a meromorphic matrix at the point $a_i$. This matrix can be factored as $T_i = V_i P_i$, with a holomorphically invertible matrix $V_i$ at $a_i$ and an upper-triangular matrix $P_i$ which is a polynomial in $(z - a_i)^{\pm 1}$ (see, for example, [11, Lemma 1]). Thus the matrix $V_i^{-1} Y$ is upper-triangular.

Let us estimate the degree of each subbundle $E^i$, $j \leq k$, noting that in a neighbourhood of each singular point $a_i$ the initial system is transformed via a holomorphically invertible gauge transformation in a system with a fundamental matrix of the form

$$Y_i(z) = \left( \begin{array}{cc} U'_i(z) & 0 \\ 0 & * \end{array} \right) (z - a_i) \left( \begin{array}{cc} A'_i & 0 \\ 0 & * \end{array} \right) (z - a_i) \left( \begin{array}{cc} E'_i & 0 \\ 0 & * \end{array} \right)$$

such that the matrix $Y_i'(z) = U'_i(z)(z - a_i)^{A'_i}(z - a_i)^{E'_i}$ is a Levelt fundamental matrix for a linear system of $j$ equations with the regular singular point $a_i$. The matrix 1-form of coefficients of this system in a neighbourhood of $a_i$ is a local 1-form of the restriction $\nabla^j$ of the connection $\nabla$ to the subbundle $E^i$, and the exponents $\beta'_i, \ldots, \beta''_i$ of this system (the eigenvalues of the matrix $A'_i + E'_i$) form a subset of the exponents of the initial system at $a_i$. Therefore,

$$\text{Re } \beta'_i > -1/nk, \quad l = 1, \ldots, j.$$

The degree of the holomorphically trivial vector bundle $E^p$ is equal to zero, and for $j \leq k$ one has:

$$\deg E^j = \sum_{i=1}^{n} \text{res}_{a_i} d \ln \det Y'_i(z) = \sum_{i=1}^{n} \text{ord}_{a_i} \det U'_i(z) + \sum_{i=1}^{n} \text{tr}(A'_i + E'_i) =$$

$$= \sum_{i=1}^{n} \text{ord}_{a_i} \det U'_i(z) + \sum_{i=1}^{n} \sum_{l=1}^{j} \text{Re } \beta'_i > -j/k \geq -1.$$

As the degree of a subbundle of a holomorphically trivial vector bundle is non-positive, one has $\deg E^j = 0$, hence all the subbundles $E^1 \subset \ldots \subset E^k$ are holomorphically trivial (a subbundle of a holomorphically trivial vector bundle is holomorphically trivial, if its degree is equal to zero, see [11, Lemma 19.16]). Now the assertion of the lemma follows from Lemma 1. \hfill \Box

A matrix $A$ will be called $N$-resonant, if there are two eigenvalues $\lambda_1 \neq \lambda_2$ such that $\lambda_1^N = \lambda_2^N$, that is,

$$|\lambda_1| = |\lambda_2|, \quad \text{arg } \lambda_1 - \text{arg } \lambda_2 = \frac{2\pi}{N} j, \quad j \in \{1, 2, \ldots, N - 1\}.$$

Let a group $M \subset \text{GL}(p, \mathbb{C})$ be generated by matrices $M_1, \ldots, M_n$. If these matrices are sufficiently close to the identity (in the Euclidean topology) then the existence of a solvable normal subgroup of finite index in $M$ implies their triangularity, see Theorem 2.7 [13, Ch. 6]. According to the remark following this theorem, the requirement of proximity of the matrices $M_i$ to the identity can be weakened as follows.

**Lemma 3.** There is a number $N = N(p)$ such that if the matrices $M_1, \ldots, M_n$ are not $N$-resonant, then the existence of a solvable normal subgroup of finite index in $M$ implies their triangularity.

**Proof of Theorem 1.** From the theorem assumptions it follows that the identity component $G^0$ of the differential Galois group $G$ of the system (11) is solvable, hence $G^0$ is a solvable normal subgroup of $G$ of finite index. Then the monodromy group $M$ of this system also has a solvable normal subgroup of finite index, namely $M \cap G^0$. 


As follows from the definition of the exponents $\beta_{ji}$ of a linear differential system at a regular singular point $a_i$, these are connected with the eigenvalues $\mu_{ij}$ of the monodromy matrix $M_i$ by the relation
\[ \mu_{ij} = \exp(2\pi i \beta_{ji}). \]
Therefore,\[ \mu_{ij} = \exp(2\pi i (\Re \beta_{ji} + i \Im \beta_{ji})) = e^{-2\pi i \Im \beta_{ji}}(\cos(2\pi \Re \beta_{ji}) + i \sin(2\pi \Re \beta_{ji})) \]
and for any $N$ the matrices $M_i$ are non $N$-resonant by the conditions (14) on the numbers $\beta_{ji}$.

As a consequence of Theorem 1 we obtain the following refinement of the Ilyashenko–Khovansky theorem on solvability by quadratures of a Fuchsian system with small residue matrices.

**Corollary 1.** Let the eigenvalues $\beta_{ji}$ of the residue matrices $B_i$ of the Fuchsian system (2) satisfy the condition
\[ \Re \beta_{ji} > -\frac{1}{n(p-1)}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p, \] (15)
and, for each pair $\beta_{ji} \not\equiv \beta_{il} (\mod \mathbb{Z})$, $i = 1, \ldots, n$, one of the conditions (14). Then the solvability of the Fuchsian system (2) by quadratures is equivalent to the simultaneous triangularity of the matrices $B_i$.

**Proof.** The necessity of simultaneous triangularity is a direct consequence of Theorem 1, since the exponents of the Fuchsian system (2) at $a_i$ coincide with the eigenvalues of the residue matrix $B_i$. Sufficiency follows from a general fact that any linear differential system with an (upper-) triangular coefficient matrix is solvable by quadratures (one should begin with the last equation). \( \square \)

**Remark 3.** The inequalities (15) restricting the real parts of the exponents of the Fuchsian system from below also provide their boundedness from above because of the Fuchs relation $\sum_{i=1}^n \sum_{j=1}^p \beta_{ij} = 0$ (see (5)). Namely,
\[ -\frac{1}{n(p-1)} < \Re \beta_{ji} < \frac{np-1}{n(p-1)}. \]
In particular, the integer parts $\varphi_{ji}$ of the numbers $\Re \beta_{ji}$ for such a system have to belong to the set $\{-1, 0, 1\}$.

**Remark 4.** If each residue matrix $B_i$ of the Fuchsian system (2) only has one eigenvalue $\beta_i$, then the solvability of this system by quadratures is also equivalent to the simultaneous triangularity of the matrices $B_i$ (not depending on values of $\Re \beta_i$). Indeed, in this case each monodromy matrix $M_i$ only has one eigenvalue $\mu_i = e^{2\pi i \beta_i}$, hence is not $N$-resonant. Then the solvability implies the simultaneous triangularity of the monodromy matrices and existence of a flag $E^1 \subset E^2 \subset \ldots \subset E^p = E$ of subbundles of the holomorphically trivial vector bundle $E$ that are stabilized by the logarithmic connection $\nabla$ (corresponding to the Fuchsian system). Since $\deg E = \sum_{i=1}^n p\beta_i = 0$, the degree $\sum_{i=1}^n j\beta_i$ of each subbundle $E^j$ is zero and all these subbundles are holomorphically trivial.
It is natural to expect that for a general Fuchsian system (with no restrictions on the exponents) solvability by quadratures not necessarily implies simultaneous triangularity of the residue matrices. This is indeed illustrated by the following example of A. Bolibrukh.

**Example 2** (A. Bolibrukh [2, Prop. 5.1.1]). There exist points \(a_1, a_2, a_3, a_4\) on the Riemann sphere and a Fuchsian system with singularities at these points, whose monodromy matrices are

\[
M_1 = \begin{pmatrix}
1 & 1 & 1 & -1 & 0 & 0 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix},
M_2 = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & 1 & 1 & -1 & 1 & -1 \\
0 & 0 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix},
M_3 = \begin{pmatrix}
1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & -1 & -1 & 1 & -1 & 1 & 2 \\
0 & 0 & 1 & 1 & -1 & 1 & 2 \\
0 & 0 & 0 & -1 & 1 & -1 & -2 \\
0 & 0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix},
M_4 = \begin{pmatrix}
1 & 0 & 1 & -1 & 1 & 1 & 0 \\
0 & -1 & 1 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix},
\]

whereas the coefficient matrix of this system is not upper-triangular. Moreover, this system cannot be transformed in an upper-triangular form neither via a constant gauge transformation nor even a meromorphic (rational) one preserving the singular points \(a_1, a_2, a_3, a_4\) and the orders of their poles. Thus the residue matrices of this Fuchsian system are not triangular in any basis though the system is solvable by quadratures, since its monodromy group generated by the triangular matrices is solvable.

We notice that Corollary 1 does not apply to this example as its exponents cannot satisfy the conditions (15). Indeed, for any exponent \(\beta^j_i = \varphi^j_i + \rho^j_i\) one has

\[
\rho^j_i = \frac{1}{2\pi i} \ln \mu^j_i, \quad \mu^j_i \in \{-1, 1\},
\]

hence \(\Re \rho^j_i\) is equal to 0 or 1/2. The inequalities (15) imply (for \(n = 4, p = 7\))

\[
\Re \beta^j_i > -\frac{1}{24},
\]

hence \(\varphi^j_i\) is equal to 0 or 1 (see Remark 3). Therefore, the sum of the exponents over all singular points is positive, which contradicts the Fuchs relation.

### 5 A criterion of solvability for a non-resonant irregular system with small formal exponents

Consider the system (1) with non-resonant irregular singular points \(a_1, \ldots, a_n\) of Poincaré rank \(r_1, \ldots, r_n\) respectively. If the real part of the formal exponents of this system is sufficiently small, then the following criterion of solvability by quadratures holds.
Theorem 2. Let at each singular point \( a_i \) the formal exponents \( \lambda^j_i \) of the irregular system (1) be pairwise distinct and satisfy the condition

\[
\text{Re} \lambda^j_i > -\frac{1}{n(p-1)},
\]

and, for every pair \( (\lambda^j_i, \lambda^j_i) \), one of the conditions (14) follows from Ramis’ theorem [18]. Then this system is solvable by quadratures if and only if there is a constant matrix \( C \in GL(p, \mathbb{C}) \) such that \( CB(z)C^{-1} \) is upper-triangular.

Proof. As in Corollary 1, sufficiency does not require a special proof. Let us prove necessity.

Consider a fundamental matrix \( Y \) of the system (1) and the representation of the differential Galois group \( G \) with respect to this matrix. The connection matrices between \( Y \) and the fundamental matrices \( Y^1_i, \ldots, Y^1_n \) (the latter were defined at the end of Section 2) denote by \( P_1, \ldots, P_n \) respectively:

\[
Y(z) = Y^1_i(z)P_i, \quad i = 1, \ldots, n.
\]

Then, as we noted earlier, the monodromy matrices \( M_i = P_i^{-1}M_1^iP_i \) \( (i = 1, \ldots, n) \) with respect to \( Y \) belong to the group \( G \). Moreover, as follows from Ramis’ theorem [18, 1], the corresponding Stokes matrices

\[
\tilde{C}_i^1 = P_i^{-1}C_i^1P_i, \quad \ldots, \quad \tilde{C}_i^{N_i} = P_i^{-1}C_i^{N_i}P_i \quad (i = 1, \ldots, n)
\]

also belong to \( G \). Therefore, by the relation

\[
e^{2 \pi i \tilde{\lambda}_i} = M_i \tilde{C}_i^1 \cdots \tilde{C}_i^{N_i}, \quad \tilde{\lambda}_i = P_i^{-1}\Lambda_iP_i,
\]

(see (5)), the group \( G \) also contains the matrices \( e^{2 \pi i \tilde{\lambda}_i} \) of formal monodromy.

Denote by \( \tilde{M} \) the group \( \{e^{2 \pi i \tilde{\lambda}_i}, \tilde{C}_i^1, \ldots, \tilde{C}_i^{N_i}\}_{i=1}^n \) generated by the matrices of formal monodromy and Stokes matrices over all singular points. As follows from the condition of the theorem, the group \( G \) possesses the solvable normal subgroup \( G^0 \) of finite index. Hence the subgroup \( \tilde{M} \) \( \subset G \) possesses the solvable normal subgroup of finite index, \( \tilde{M} \cap G^0 \). Since for any \( N \) the matrices generating the group \( \tilde{M} \) are non-\( N \)-resonant, according to Lemma 3 they are simultaneously reduced to an upper-triangular form by conjugating to some non-degenerated matrix \( \tilde{C} \) (non-resonance of the formal monodromy matrices \( e^{2 \pi i \tilde{\lambda}_i} \) follows from the conditions on the formal exponents, the eigenvalues of the matrices \( \tilde{\Lambda}_i \), and is proved as in Theorem 1; for the Stokes matrices it follows from their unipotence). We may assume that they are already upper-triangular (otherwise we would consider the fundamental matrix \( Y\tilde{C} \) instead of \( Y \)). As follows from [5] Ch. VIII, §1, the relation

\[
\Lambda_i = P_i\tilde{\Lambda}_iP_i^{-1},
\]

where \( \tilde{\Lambda}_i \) is an upper-triangular matrix and \( \Lambda_i \) is a diagonal matrix whose diagonal elements are pairwise distinct, implies that the matrix \( P_i \) writes \( P_i = D_iR_i \), where \( R_i \) is an upper-triangular matrix (the conjugation \( R_i \tilde{\Lambda}_i R_i^{-1} \) annihilates all the off-diagonal elements of the matrix \( \tilde{\Lambda}_i \)) and \( D_i \) is a permutation matrix for \( \Lambda_i \) (that is, the conjugation \( D_i^{-1}\Lambda_iD_i \) permutes the diagonal elements of the matrix \( \Lambda_i \)).

\(^1\text{It is difficult to find the original proof of it, but there exist various variants of this theorem and comments to it proposed by the other authors (see [10] Ths. 1, 6], [14], [16] Th. 2.3.11+[17] Prop. 1.3]).\)
In a neighbourhood of each \( a_i \) we pass from the set of fundamental matrices \( Y_i^1, \ldots, Y_i^{N_i} \), which correspond to the sectors \( S_i^1, \ldots, S_i^{N_i} \), to the fundamental matrices

\[
\tilde{Y}_i^j(z) = Y_i^j(z)P_i \quad \text{(in particular, } \tilde{Y}_i^1 = Y)\]

connected to each other in the intersections \( S_i^j \cap S_i^{j+1} \) by the relations

\[
\tilde{Y}_i^{j+1}(z) = \tilde{Y}_i^j(z)\tilde{C}_i^j.
\]

From the decomposition of \( P_i \) above and from (5) it follows that the matrices \( \tilde{Y}_i^j \) can be written

\[
\tilde{Y}_i^j(z) = F_i^j(z)(z - a_i)^{\lambda_i}e^{Q_i(z)}D_i R_i = F_i^j(z)D_i(z - a_i)^{\lambda_i}e^{Q_i(z)}R_i,
\]

where

\[
\lambda_i' = D_i^{-1} \lambda_i D_i, \quad Q_i'(z) = D_i^{-1} Q_i(z) D_i
\]

are diagonal matrices obtained from the corresponding matrices \( \lambda_i, Q_i(z) \) by a permutation of the diagonal elements. Therefore, in the intersections \( S_i^j \cap S_i^{j+1} \) we have the relations

\[
F_i^{j+1}(z)D_i(z - a_i)^{\lambda_i'}e^{Q_i'(z)}R_i = F_i^j(z)D_i(z - a_i)^{\lambda_i'}e^{Q_i'(z)}R_i\tilde{C}_i^j.
\]

Thus in the sectors \( S_i^1, \ldots, S_i^{N_i} \), which form a covering of a punctured neighbourhood of \( a_i \), there are holomorphically invertible matrices \( F_i^1(z)D_i, \ldots, F_i^{N_i}(z)D_i \) respectively such that in the intersections \( S_i^j \cap S_i^{j+1} \) their quotients

\[
\left( F_i^j(z)D_i \right)^{-1} F_i^{j+1}(z)D_i = (z - a_i)^{\lambda_i'}e^{Q_i(z)}R_i\tilde{C}_i^j R_i^{-1}e^{-Q_i'(z)}(z - a_i)^{-\lambda_i'}
\]

are upper-triangular matrices. As each matrix \( F_i^j(z)D_i \) has the same asymptotic expansion \( \tilde{F}_i(z)D_i \) in the corresponding \( S_i^j \), there exists a matrix \( \Gamma(z) \) holomorphically invertible at the point \( a_i \) such that all the matrices

\[
\tilde{F}_i^j(z) = \Gamma(z)F_i^j(z)D_i, \quad j = 1, \ldots, N_i,
\]

are upper-triangular (according to [1] Prop. 3). In particular,

\[
\Gamma(z)Y(z) = \Gamma(z)\tilde{Y}_1^1(z) = \Gamma(z)F_i^1(z)D_i(z - a_i)^{\lambda_i}e^{Q(z)}R_i = \tilde{F}_1^i(z)(z - a_i)^{\lambda_i}e^{Q_i(z)}R_i
\]

is an upper-triangular matrix. Hence (see Example 1) one has a flag \( E^1 \subset E^2 \subset \ldots \subset E^p = E \) of subbundles of rank \( 1, 2, \ldots, p \) respectively that are stabilized by the connection \( \nabla \). The bounds on the formal exponents imply that these subbundles are holomorphically trivial, whence the assertion of the theorem follows. The proof proceeds as for Lemma 2, there are only two small differences: the first one is that now we deal with formal fundamental matrices of subsystems and their formal exponents, which form a subset of the formal exponents of the initial system at each irregular singular point \( a_i \); the second one is the appearence of an exponential factor in a formal fundamental matrix of a subsystem, but the logarithmic differential of such a factor has a zero residue at \( a_i \).
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