Faraday resonance in dynamical bar instability of differentially rotating stars

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We investigate the nonlinear behaviour of the dynamically unstable rotating star for the bar mode by three-dimensional hydrodynamics in Newtonian gravity. We find that an oscillation along the rotation axis is induced throughout the growth of the unstable bar mode, and that its characteristic frequency is twice as that of the bar mode, which oscillates mainly along the equatorial plane. A possibility to observe Faraday resonance in gravitational waves is demonstrated and discussed.

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I. INTRODUCTION

Parametric resonance is widely observed in hydrodynamics, nonlinear optics, chemical reaction and classical oscillatory systems. It is also interesting from the bifurcation theory and the pattern formation. The scientific study in the fluid mechanics dates from the experiments by Faraday in 1831, and is therefore named Faraday resonance (e.g., [1]). Nonlinear dynamics exhibits mode interaction of oscillation in different direction, and possibly causes the resonant growth of a particular mode. Recently, experimental studies of Faraday resonance demonstrate that the system of fluid mechanics [2] and that of Bose-Einstein condensate [3] work perfectly. These agreements in different fields also suggest that Faraday resonance may also occur in an astrophysical context. Quasi-periodic oscillation in gravitational waves from dynamical/ secular instabilities is expected to be excited throughout rotating core collapse, and may drive resonant growth.

Dynamical bar instability in a rotating equilibrium star takes place when the ratio $\beta (\equiv T/W)$ between rotational kinetic energy $T$ and the gravitational binding energy $W$ exceeds the critical value $\beta_{\text{dyn}} \approx 0.27$ for an uniformly rotating incompressible body in Newtonian gravity [4]. Determining the onset of the dynamical bar-mode instability, as well as the subsequent evolution of an unstable star, requires a fully nonlinear hydrodynamic simulation. Simulations performed in Newtonian gravity (e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) have shown that $\beta_{\text{dyn}}$ depends very weakly on the stiffness of the equation of state. $\beta_{\text{dyn}}$ becomes small for stars with high degree of differential rotation [15, 16, 17]. Simulations in relativistic gravitation [18, 19, 20] have shown that $\beta_{\text{dyn}}$ decreases when increases the compactness of the star, indicating that relativistic gravitation enhances the bar-mode instability. Recent numerical simulations show that dynamical bar instability can occur at significantly lower $\beta$ than the threshold $\beta_{\text{crit}} \approx 0.27$ [17, 22, 23, 24, 25]. These recent findings can be classified into the category of a low $T/W$ dynamical instability. This instability may be triggered by the corotation resonance [26, 27], which is completely different from the standard dynamical bar-mode instability triggered by a certain magnitude of rotation [4, 28, 29].

Our main concern in this paper is not to determine the onset of the instability, but to study the dynamical features of the bar. For this purpose, we numerically study the growing behaviour of the azimuthal modes in the nonlinear regime for a longer timescale. One interesting issue of nonlinear evolution is the possibility of resonant growth of other azimuthal modes triggered by the dynamical bar-mode instability. One candidate for such resonance is Faraday resonance, which is excited by the external periodic force. According to the linear approximation of the velocity potential by using an incompressible inviscid liquid in a rectangular tank, the time-dependent behaviour of the liquid surface is expressed by the Mathieu's equation (e.g., [30]). The dynamically unstable bar mode may work for other azimuthal oscillation modes as an external periodic force. The oscillation is not exactly periodic, but rather quasi-periodic, and may trigger a parametric resonance.

The other interesting issue of nonlinear evolution is the duration of the bar shape, when it forms. This is quite important for gravitational wave detection. We basically believe that once the dynamical bar instability takes place, the system generates quasi-periodic gravitational waves for a period sufficient enough to be detected in the ground-based gravitational wave detectors. The only causes to destruct a bar are dissipative effects such as viscosity and gravitational radiation. The typical timescale of such effects takes place in the secular timescale, which is much longer than the dynamical one of the system. Therefore the standard picture is that the bar can persist in its shape until the secular timescale. However, recent numerical simulation shows that a bar destructs its shape in the dynamical timescale [20]. The authors argue a possible cause of the destruction of bar as

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azimuthal mode coupling. Although there was in the past a debate between the two numerical simulations about the persistence of a bar \[12, 31\], the different outcomes were considered as the different accuracy level of the center of mass condition at that time. Since there is the only one group that claims the destruction of bar structure in the dynamical timescale with a satisfaction of the center of mass condition, it is worth investigating the destruction of a bar employing a different computational code. In order to focus on this topic, it is sufficient to investigate this topic in three-dimensional hydrodynamics in Newtonian gravity.

This paper is organized as follows. In Sec. II we present the basic equations of our hydrodynamic simulation in Newtonian gravity. In Sec. III are discussed the numerical results of our findings of Faraday resonance. In Sec. IV we briefly summarize our findings. Throughout this paper, we use the geometrized units with \( G = c = 1 \) \[32\] and adopt Cartesian coordinates \((x, y, z)\) with the coordinate time \( t \). Note that Latin index takes \( (x, y, z) \).

II. BASIC EQUATIONS

A. Newtonian Hydrodynamics

We construct a three dimensional Newtonian hydrodynamics code assuming an adiabatic \( \Gamma \)-law equation of state

\[ P = (\Gamma - 1)\rho \varepsilon, \tag{2.1} \]

where \( P \) is the pressure, \( \Gamma \) the adiabatic index, \( \rho \) the mass density and \( \varepsilon \) the specific internal energy density. For perfect fluids the Newtonian equations of hydrodynamics consist of the continuity equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v^i)}{\partial x^i} = 0, \tag{2.2} \]

the energy equation

\[ \frac{\partial e}{\partial t} + \frac{\partial (\rho e v^i)}{\partial x^i} = -\frac{1}{\Gamma} \varepsilon^{-1} P_{vis} \frac{\partial v^i}{\partial x^i}, \tag{2.3} \]

and the Euler equation

\[ \frac{\partial (\rho v_i)}{\partial t} + \frac{\partial (\rho v_i v^j)}{\partial x^j} = -\frac{\partial P}{\partial x^i} - \rho \frac{\partial \Phi}{\partial x^i}. \tag{2.4} \]

Here \( v^i \) is the fluid velocity and \( \Phi \) the gravitational potential; and \( e \) is defined according to

\[ e = (\rho \varepsilon)^{1/\Gamma}. \tag{2.5} \]

We compute the artificial viscosity pressure \( P_{vis} \) from \[32\]

\[ P_{vis} = \begin{cases} C_{vis} \rho (\delta v)^2, & \text{for } \delta v \leq 0; \\ 0, & \text{for } \delta v \geq 0, \end{cases} \tag{2.6} \]

where \( \delta v \equiv 2\delta x \partial_t v^i, \delta x(= \Delta x = \Delta y = \Delta z) \) is the local grid spacing and where we choose the dimensionless parameter \( C_{vis} = 2 \). When evolving the above equations we limit the stepsize \( \Delta t \) by an appropriately chosen Courant condition.

The gravitational potential is determined by the Poisson equation

\[ \Delta \Phi = 4\pi \rho, \tag{2.7} \]

with the outer boundary condition

\[ \Phi = -\frac{M}{r} - \frac{d_i x^i}{r^2} + O(r^{-3}). \tag{2.8} \]

Here \( M \) is the total mass

\[ M = \int_V \rho dx^3 \tag{2.9} \]

and \( d_i \) is the dipole moment

\[ d_i = \int_V \rho x_i dx^3. \tag{2.10} \]

B. Initial Data

As initial data, we construct differentially rotating equilibrium models with an algorithm based on Hachisu \[33\] and adopt cylindrical two dimensional coordinate to compute the axisymmetric equilibrium star. Individual models are parameterized by the ratio of the polar to equatorial radius \( R_p/R_{eq} \), and a parameter of dimension length \( d \) that determines the degree of differential rotation through

\[ \Omega = \frac{j_0}{d^2 + \varpi^2}. \tag{2.11} \]

Here \( \Omega \) is the angular velocity, \( j_0 \) a constant parameter with units of specific angular momentum, and \( \varpi \) the cylindrical radius. The parameter \( d \) determines the length scale over which \( \Omega \) changes; uniform rotation is achieved in the limit \( d \to \infty \). For the construction of initial data we also assume a polytropic equation of state

\[ P = \kappa \rho^{1+1/n}, \tag{2.12} \]

where \( n = 1/(\Gamma - 1) \) is the polytropic index and \( \kappa \) a constant. In the absence of shocks, the polytropic form of the equation of state is conserved by the \( \Gamma \)-law equation of state (Eq. \[2.1\]).

We also compute the virial identity, which is identically zero in the equilibrium star, to show the accuracy level as

\[ V_{Nwt} = \frac{|2T_{tot} - W + 3\Pi|}{W}, \tag{2.13} \]
where

\[ T_{\text{tot}} = \frac{1}{2} \int \rho v^i v_i d^3x, \quad (2.14) \]
\[ W = -\frac{1}{2} \int \rho \Phi d^3x, \quad (2.15) \]
\[ \Pi = \int P d^3x. \quad (2.16) \]

Note that we have divided the value by a gravitational binding energy \( W \) so that the value \( V_{\text{Net}} \) is regarded as a relative error of the system. We summarize our four different rotating equilibrium stars in Table II.

C. Gravitational Waveforms

We compute approximate gravitational waveforms by evaluating the quadrupole formula. In the radiation zone, gravitational waves can be described by a transverse-traceless, perturbed metric \( h_{ij}^{TT} \) with respect to a flat spacetime. In the quadrupole formula, \( h_{ij}^{TT} \) is found from [34]

\[ h_{ij}^{TT} = \frac{2}{r} \frac{d^2}{dt^2} f_{ij}^{TT}, \quad (2.17) \]

where \( r \) is the distance to the source, \( I_{ij} \) the quadrupole moment of the mass distribution (see Eq. 36.42b in Ref. [34]), and where \( TT \) denotes the transverse-traceless projection. Choosing the direction of the wave propagation to be along the \( z \)-axis (rotational axis of the equilibrium star), we determine the two polarization modes of gravitational waves from

\[ h_+^{(z)} = \frac{1}{2} (h_{xx}^{TT} - h_{yy}^{TT}) \quad \text{and} \quad h_\times^{(z)} = h_{xy}^{TT}. \quad (2.18) \]

For observers along the \( z \)-axis, we thus have

\[ \frac{r h_+^{(z)}}{M} = \frac{1}{2M} \frac{d}{dt} (I_{xx} - I_{yy}), \quad (2.19) \]
\[ \frac{r h_\times^{(z)}}{M} = \frac{1}{M} \frac{d}{dt} I_{xy}. \quad (2.20) \]

Using the same procedure, the observers along the \( x \)-axis detect the wave propagates as

\[ \frac{r h_+^{(x)}}{M} = \frac{1}{2M} \frac{d}{dt} (I_{yy} - I_{zz}), \quad (2.21) \]
\[ \frac{r h_\times^{(x)}}{M} = \frac{1}{M} \frac{d}{dt} I_{yz}. \quad (2.22) \]

The number of time derivatives \( I_{ij} \) that have to be taken out can be reduced by using the continuity equation (Eq. 2.2)

\[ \dot{I}_{ij} = \int (\rho v^i x^j + \rho x^i v^j) d^3x, \quad (2.23) \]

in equations (2.19) – (2.22) (see Ref. [35]).

The spectrum of gravitational waveform can be computed as

\[ S^{(x,z)} = |\tilde{h}_+^{(x,z)}|^2 + |\tilde{h}_\times^{(x,z)}|^2, \quad (2.24) \]

where

\[ \tilde{h}_+^{(x,z)} = \int dt h_+^{(x,z)} e^{i\omega t}. \quad (2.25) \]

D. Diagnostics

We monitor the conservation of mass \( M \) (Eq. 2.9), angular momentum \( J \)

\[ J = \int \rho (x v^y - y v^x) d^3x, \quad (2.26) \]

and the location of the center of mass \( x_{CM}^i \)

\[ x_{CM}^i = \frac{\int \rho x^i d^3x}{\int \rho |d^3x|}. \quad (2.27) \]

Due to our flux-conserving difference scheme the mass \( M \) is also conserved up to a round-off error, except if matter leaves the computational grid.

To monitor the development of the azimuthal modes \((m = 1, 2, 3, 4)\) and the one in the \( z \)-direction, we compute the following five diagnostics

\[ D = \langle e^{i m \phi} \rangle_{m=1} = \frac{1}{M} \int \rho \frac{x + i y}{\sqrt{x^2 + y^2}} d^3x, \quad (2.28) \]
\[ Q = \langle e^{i m \phi} \rangle_{m=2} = \frac{1}{M} \int \rho \frac{x^2 - y^2 + i(2xy)}{x^2 + y^2} d^3x, \quad (2.29) \]
\[ O = \langle e^{i m \phi} \rangle_{m=3} = \frac{1}{M} \int \rho \frac{x^2 - 3y^2 + iy(3x^2 - y^2)}{(x^2 + y^2)^{3/2}} d^3x, \quad (2.30) \]
\[ M_4 = \langle e^{i m \phi} \rangle_{m=4} = \frac{1}{M} \int \rho \frac{x^4 - 6x^2 y^2 + y^4 + i(4x^2 y^2(x^2 - y^2))}{(x^2 + y^2)^2} d^3x, \quad (2.31) \]
\[ D_z = \frac{1}{MR_p} \int \rho |z| d^3x, \quad (2.32) \]

where a bracket denotes the density weighted average. When we compute the four diagnostics in the equatorial plane \((D^{(eq)}, Q^{(eq)}, O^{(eq)}, M_4^{(eq)})\), we change the integral volume from \(d^3x\) to \(dxdy\) and \(M\) to \(M_{\text{eq}}\). Note that \(M_{\text{eq}}\) is the rest-mass density integrated only in the equatorial plane.
The page contains a detailed discussion on the evolution of differentially rotating stars, focusing on the nonlinear behavior of non-axisymmetric dynamical bar instabilities. The authors study four different models, each detailed in Table I, to investigate the non-axisymmetric dynamical bar instabilities. They choose the axis of rotation to align with the z axis and assume planar symmetry across the equator. The computations are performed using a grid of 401 x 401 x 101 covering the equatorial diameter of the equilibrium star as 121 grid points.

III. NUMERICAL RESULTS

Here we show our evolution of the differentially rotating stars. We terminate the integration when the relative error of the rest mass exceeds $\sim 10^{-4}$, since the only violation of the rest mass conservation is caused by the matter outflow at the outer boundary of computation. We also terminate the integration when the time exceeds 20 ~ 40 central rotation periods, which are sufficient to enhance all $m$ modes. Note that our code never crashes throughout the evolution.

To enhance any dynamically unstable mode, we disturb the initial equilibrium density $\rho_{eq}$ by a non-axisymmetric perturbation according to

$$\rho = \rho_{eq} \left[ 1 + \delta^{(2)} \frac{x^2 + 2xy - y^2}{R_{eq}^2} + \delta^{(4)} \frac{x^4 - 6x^2y^2 + y^4 + 4xy(x^2 - y^2)}{R_{eq}^4} \right], \quad (3.1)$$

where we set $\delta^{(2)} = \delta^{(4)} = 10^{-2}$.

The authors show the amplitudes of their four diagnostics for all four models in Fig. 1. At the first stage of evolution, the $m = 2$ diagnostic grows exponentially in models I, II, and III, while it stays around the amplitude of $t = 0$ in model IV. Therefore the star of models I, II, and III is determined as dynamically unstable against bar mode, while that of model IV is stable. For the dynamically bar
unstable stars (models I, II, and III), the $m = 2$ diagnostic grows exponentially but the other remaining $m$ modes do not grow at the first evolution stage when imposing a small perturbation (Fig. 4). This result is consistent to the linear perturbation analysis of the dynamically bar unstable stars, which shows that the only dynamically unstable $m$ mode is $m = 2$. Also the result confirms us that the amplitude of perturbation at $t = 0$ is adequate to treat the system linearly ($\delta^{(2)} \approx 10^{-2}$). After that stage the $m = 4$ diagnostic grows exponentially because of the secondary harmonic of $m = 2$ mode, and then the odd $m$ modes are also enhanced. The odd $m$ modes are excited even if we do not impose the perturbation of their corresponding modes in the equilibrium star, since the finite differencing scheme always generates a small amount of all $m$ modes (Fig. 4). However a small fluctuation at the wavefront should occur in nature so that the existence of all $m$ modes, when the bar forms, are quite natural in
the three-dimensional computation (Fig. 1), the equa-

tо reproduce all characteristics of the one obtained from

in every snapshots. Since the equatorial diagnostic (Fig.

coordinate where the center of mass is adjusted to zero

as in the simulation, while the other is the one with the

ing two types of diagnostics in the equatorial plane (e.g.

affects the diagnostics, we have also computed the follow-

check whether the center of mass condition significantly

velocity at the equilibrium equatorial surface. In order to

total value constructed by the total mass and the ve-

also checked the linear momentum conservation in Fig. 3,

We have monitored the center of mass and the linear

momentum throughout the evolution to guarantee that

we do not impose any additional physics in the system.

We have confirmed that the numerical error only allows the star to change the cen-

ter of mass within the one computational grid. We have also checked the linear momentum conservation in Fig. 5 which shows that the relative error is less than 1% of the total value constructed by the total mass and the velocity at the equilibrium equatorial surface. In order to check whether the center of mass condition significantly affects the diagnostics, we have also computed the following two types of diagnostics in the equatorial plane (e.g. 25). One is the diagnostic with the same coordinate as in the simulation, while the other is the one with the coordinate where the center of mass is adjusted to zero in every snapshots. Since the equatorial diagnostic (Fig. 4) reproduces all characteristics of the one obtained from the three-dimensional computation (Fig. 1), the equatorial diagnostic may represent the three-dimensional one.

We compare Figs. 4 and 5 to focus on the effect of the center of mass condition on the diagnostics. For models I and II, the adjustment of the center of mass reduces the amplitude of $D^{(eq)}$ and $O^{(eq)}$ for $t \lesssim 5P_c$. However the condition does not change the exponential growth of $D^{(eq)}$ and $O^{(eq)}$ after $t \gtrsim 5P_c$. For models III and IV the amplitude of $D^{(eq)}$ has been reduced so that the system is stable to $m = 1$. Therefore the linear growth of $D^{(eq)}$ in models III and IV is the outcome of the violation of the center of mass condition.

We also show our equatorial and the meridional density snapshots throughout our integration in Figs. 6 and 7. The symmetry breaking of the bar structure occurs clearly at the time when the spiral arm forms in the equatorial snapshot and in the meridional plane. This becomes clear when we focus on the final snapshots of models I, II, and III.

We also in Fig. 8 show the diagnostics which contain both amplitude and phase. In order to make the picture clear, we first concentrate on the model III, the weakest dynamically unstable bar system of three models.

The behaviours in the diagnostics are clearly understood once we compute the spectra of the diagnostics (Fig. 9). From the spectra we find the following two remarkable issues. One is that the spectra $|F_2|^2$, $|F_3|^2$, $|F_4|^2$ take a peak around $\omega_{\text{bar}} \approx 5 \sim 6P_c^{-1}$ for models I, II, III, and the other is that $|F_3|^2$, $|F_4|^2$, $|F_5|^2$ take a peak around $\omega_{\text{quad}} \approx 2\omega_{\text{bar}} \approx 10 \sim 12P_c^{-1}$ for bar unstable stars. Combining the present feature with the behaviour of the five diagnostics explained before (Fig. 1), the dynamically unstable bar acts as follows.

Firstly the $m = 2$ mode grows and acts as a dominant mode of all because of the dynamical bar instabil-

TABLE I: Four different rotating equilibrium stars in Newtonian gravity of $\Gamma = 2$, $d/R_{\text{eq}} = 1$.

| Model | $R_0/R_{\text{eq}}$ | $T/W$ | $V_{\text{tw}}$ |
|-------|-----------------|-------|-------------|
| I     | 0.225           | 0.281 | 8.29 x 10^{-5} |
| II    | 0.250           | 0.277 | 8.79 x 10^{-5} |
| III   | 0.275           | 0.268 | 7.95 x 10^{-5} |
| IV    | 0.300           | 0.256 | 9.47 x 10^{-5} |

FIG. 4: Same as Fig. 1 but the diagnostics are only computed in the equatorial plane.
Next the $m = 4$ mode grows because of the secondary harmonic of the $m = 2$ mode. In fact the saturation amplitude of the $m = 4$ is approximately $\approx 0.3$ for model I, 0.2 for model II, and 0.04 for model III, all of which are the order of the square of the saturation amplitude of the $m = 2$ ($\approx 0.6^2$ for model I, 0.5$^2$ for model II, 0.2$^2$ for model III). After that Faraday resonance occurs, which is clearly found in both $D_z$ and $|F_z|^2$ from the fact $\omega_{\text{quad}} \approx 2\omega_{\text{bar}}$.

Note that Faraday resonance occurs in the fluid mechanics when the oscillation of the vertical direction is twice ($2\omega$) as much as the one in the horizontal direction ($\omega$) in the weakly nonlinear interaction [1, 30]. The reason why the resonance does not clearly appear in model I is either the strongly nonlinear effect or the insufficient duration time of quasi-periodic oscillation for computing the spectrum. Then, there is a resonance between $m = 1$ and $m = 2$, $m = 3$ and $m = 4$. The possibility of such resonances is three wave interaction: either $m = 1$ ($\omega_{\text{bar}}$) and $m = 2$ ($\omega_{\text{bar}}$) generates $m = 3$ ($\omega_{\text{bar}} + \omega_{\text{bar}}$) or $m = 3$ ($2\omega_{\text{bar}}$) and $m = 2$ ($\omega_{\text{bar}}$) generates $m = 1$ ($2\omega_{\text{bar}} - \omega_{\text{bar}}$) in the dominant part. It is the fact found in the nonlinear behaviour of the dynamically unstable bar system.

The gravitational waveform and its spectrum have been computed by the quadrupole formula observed along the rotational axis and in the equatorial plane (Figs. 10 – 13). There are two remarkable features in gravitational waves from the viewpoint of nonlinear behaviour. One is that the quasi-periodic oscillation does not last until the radiation reaction timescale but decays because of the symmetry breaking of the dynamical bar. The duration period is related to the degree of nonlinearity of the bar mode instability, which is estimated from the inclination angle of the amplitude of the $m = 1$ ($\Re[D]$) and $m = 3$ ($\Re[O]$) diagnostics. In the present case, the duration period of the bar structure is estimated as $\sim 10P_c$ for model I, $\sim 15P_c$ for model II, and $\sim 35P_c$ for model III. The other is that Faraday resonance has clearly appeared in the spectrum of gravitational waveform observed at least in the equatorial plane. Since we adopt quadrupole formula to compute gravitational waves, the higher order harmonics of the unstable bar mode such as $m = 4$ mode cannot be seen in this spectrum. Therefore a peak around $\omega \approx 12P_c^{-1}$ in Fig. 11 indicates the fact of an oscillation along the $z$-axis, which is the evidence of Faraday resonance. We have also computed the gravitational waveform and its spectrum observed along the $z$-axis and found that there is no peak around $\omega \approx 12P_c^{-1}$ in model III (Fig. 13). The fact also supports that a peak around $\omega \approx 12P_c^{-1}$ in Fig. 11 is the outcome of Faraday resonance, since an oscillation along $z$-direction can be clearly observed by gravitational waves in the equatorial plane, not in the rotation axis. When we increase the degree of nonlinearity, the above feature of the Faraday resonance in gravitational waves can be also seen in the equatorial plane.

**IV. CONCLUSION**

We investigate the nonlinear effects of dynamically bar unstable stars by means of three dimensional hydrodynamic simulations in Newtonian gravity. In order to follow the bar shape as long as possible, the initial amplitudes for odd azimuthal perturbations are significantly suppressed in our models.

We find interesting mode coupling in the dynamically unstable system in the nonlinear regime, and that only
before the destruction of the bar. The quasi-periodic oscillation mainly along the rotational axis is induced. The characteristic frequency is twice as big as that of the dynamically unstable bar mode. This feature is quite analogous to the Faraday resonance. Although our finding is only supported by the weakly nonlinear theory of fluid mechanics, we have also found the same feature of parametric resonance even in the strongly nonlinear regime. There is one qualitative difference between Faraday resonance and our numerical result. Faraday resonance has lower frequency than that of the forced oscillation, while our result has higher frequency than that of the bar unstable mode. The fact can be understood by the different regime of the media. Since the media of the rotating star is a perfect fluid, which is only contained inside the star, there should be a cutoff frequency to be amplified. In fact, introducing a cutoff frequency with a polar radius of the star and the sound speed computed by the
mean density, the cutoff frequency $\omega_{\text{cut}}$ is estimated as $\omega_{\text{cut}} \approx \omega_{\text{bar}}$. Therefore a higher frequency than the bar is amplified. The fact also indicates that our finding is interpreted as a parametric resonance.

\[ f_{\text{bar}} \sim 2 \left( \frac{10\text{km}}{R} \right) \left( \frac{T}{W} \right)^{1/2} \left( \frac{M}{R} \right)^{1/2} \text{[kHz]}, \]  
\[ (4.1) \]
FIG. 9: Spectra $|F_m|^2$ and $|F_z|^2$ as a function of $\omega_P c$ for four different rotating stars (see Table I). Solid, dashed, dotted, and dash-dotted line of $|F_m|^2$ denote the values of $m = 1, 2, 3,$ and $4$, respectively.

FIG. 10: Gravitational waveform (+ mode) observed at the $x$ axes for four different rotating stars (See Table I). Note that $\times$ mode is identically zero because we adopt the equatorial symmetry.
FIG. 11: Spectra of gravitational waveform (+ mode) observed at the $x$ axes for four different rotating stars (See Table I).

FIG. 12: Gravitational waveform observed at the $z$ axes for four different rotating stars (See Table I). Solid and dashed line denotes $+$ mode and $\times$ mode, respectively.

$$h_{\text{bar}} \sim 2 \times 10^{-23} \left( \frac{M}{1.4M_\odot} \right) \left( \frac{20\text{Mpc}}{d_{\text{obs}}} \right) \left( \frac{M/R}{0.15} \right) \left( \frac{T/W}{0.25} \right),$$  

(4.2)
of the bar unstable mode, and the amplitude of the parametric resonance is roughly two orders lower (≈ 1%) than that of bar unstable mode. The detection of gravitational waves from parametric resonance may explore the nonlinear phase of the dynamically bar unstable stars such as determining the saturation amplitude of gravitational waveform from the bar unstable system, parametric resonance, and the duration period of the bar structure.

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FIG. 13: Spectra of gravitational waveform observed at the z axes for four different rotating stars (See Table I).
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