Maximal and other operators in exponential Orlicz and Grand Lebesgue Spaces.

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Abstract

We derive in this preprint the exact up to multiplicative constant non-asymptotical estimates for the norms of some non-linear in general case operators, for example, the so-called maximal functional operators, in two probabilistic rearrangement invariant norm: exponential Orlicz and Grand Lebesgue Spaces.

We will use also the theory of the so-called Grand Lebesgue Spaces (GLS) of measurable functions.

Key words and phrases: Measure and probability, measurable functions, random variable (r.v.), operators, maximal functional operator, tail of distribution, contraction, Lebesgue-Riesz, Orlicz and Grand Lebesgue Spaces (GLS), martingales, Doob’s inequality and theorem, Dunford-Schwartz operator, generating function, Lyapunov’s inequality, Young-Orlicz function, conditional expectation, Young-Fenchel transform, rearrangement invariant (r.i.) space.

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1.1. Definitions. Notations. Previous results. Statement of problem.

Let \((X, B, \mu)\) be a probability space: \(\mu(X) = 1\). We will denote by \(|f|_p = |f|L(p)\) the ordinary Lebesgue - Riesz \(L(p)\) norm of arbitrary measurable numerical valued function \(f : X \to \mathbb{R}\):

\[
|f|_p = |f|L(p) = |f|L_p(X, \mu) := \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad p \in [1, \infty).
\]

**Definition 1.1.** The operator \(Q : L(p) \to L(p), \ p \in (1, \infty)\), not necessary to be linear, acting from any \(L(p)\) to one, is said to be of a type \(\lambda, \nu; \lambda, \nu = \text{const}, \ \lambda \geq \nu \geq 0\), write

\[
Q \in \text{Type}(\lambda, \nu) = \text{Type}[L(p) \to L(p)](\lambda, \nu),
\]

iff for some finite constant \(Z = Z[Q]\) and for certain interval \(p \in [1, b), \ b = \text{const} \in (1, \infty)\)

\[
|Q[f]|_p \leq Z \frac{p^\lambda}{(p - 1)^\nu} |f|_p,
\]

or equivalently

\[
\|Q\|[L(p) \to L(p)] \leq Z[Q] \frac{p^\lambda}{(p - 1)^\nu}.
\]  \(1.1a\)

Note that the function \(f(\cdot)\) in the left-hand side of inequality (1.1) may be vector function; moreover, one can consider the relation of the form

\[
g(x) = Q(f) = Q(f_1, f_2, \ldots, f_N), \ N = 1, 2, \ldots, \infty,
\]

such that

\[
|g|_p \leq Z(f) \cdot \frac{p^\lambda}{(p - 1)^\nu} \cdot \psi(p)
\]  \(1.2\)
for certain positive continuous function $\psi = \psi(p), \ p \in [1, b)$. The concrete form of these function will be clarified below. For instance, one can choose the function $\psi(\cdot)$ as a *natural function* for the family of the r.v. $\{f_i\}$:

$$
\psi(p) := \sup_i |f_i|_p,
$$

if it is finite at last for one value $p = b, \ b \in (1, \infty]$.

This approach may be used for instance in the martingale theory, see [8], where $\lambda = \nu = 1$, and

$$
g = \max_{i=1}^N f_i,
$$

There are many examples for such operators satisfying the estimate (1.2) (or (1.1)): Doob’s inequality for martingales [8], [13]; singular integral operators of Hardy-Littlewood type [30], [31], [24], Fourier integral operators [27], [30], pseudodifferential operators [33], theory of Sobolev spaces [1] etc.

Note that in the last two examples, as well as in [31], [24], $\lambda = \nu = 1$.

Especially many examples are delivered to us the theory of the so-called maximal operators, see [1], [8], [27] etc. For instance, let $\{\phi_k\}, \ k = 0, 1, 2, \ldots$ be ordinary complete orthonormal trigonometric system on the set $[-\pi, \pi]$ equipped with *normalized* Lebesgue measure $d\mu = dx/(2\pi)$ and let $f \in L(p), \ p \in (1, \infty)$. Denote by

$$
s_M(x) = \sum_{k=0}^M (g, \phi_k) \phi_k(x)
$$

the partial Fourier sum for $f(\cdot)$ and

$$
g(x) := \sup_{M \geq 1} |s_M(x)|;
$$

then the inequality (1.2) holds true and herewith $\lambda = 4$; $\nu = 3$; see [27].

Note that there are several examples in which $\lambda > 0$, but $\nu = 0$; see e.g. [32], [28].
Let us turn now our attention on the ergodic theory, see e.g. [3], [20], [21] etc. To be more concrete, suppose \( T = [0, 1] \) with the classical Lebesgue measure \( \mu \). Let \( f : T \to \mathbb{R} \) be certain measurable function. Denote as ordinary
\[
  f^o(t) := \inf\{y, \ y > 0 : \mu\{s : |f(s)| > y\} \leq t\}, \ t \in (0, 1]
\]
and
\[
  f^{oo}(t) := \frac{1}{t} \int_0^t f^o(x) \, dx.
\]

Introduce for arbitrary rearrangement invariant (r.i.) space \( E \) builted over \( (T, \mu) \) by \( H(E) \) another (complete) r.i. space as follows

\[
  H(E) := \{f, \ f \in L_1 : f^{oo} \in E\} \subset E
\]
equipped with the norm

\[
  ||f||_{H(E)} \overset{def}{=} ||f^{oo}||_E. \tag{1.3}
\]

Further, let an operator \( A \) be an \( L_1 - L_\infty \) contraction, for instance,

\[
  A = A_\theta = A_\theta[f](t) = f(\theta(t)),
\]
where \( \theta(\cdot) \) is an invertible ergodic measure preserving transformation of the set \([0, 1]\). Define the following maximal Dunford-Schwartz operator, not necessary to be linear

\[
  B_A[f](t) := \sup_{n=2,3,...} \left[ \frac{1}{n} \sum_{k=0}^{n-1} A^k[f](t) \right]. \tag{1.4}
\]

M.Braverman in [3] proved that

\[
  ||B_A[f]||_{H(E)} \leq ||f||_E. \tag{1.5}
\]
In particular, if \( E = L_p(T), \ 1 < p < \infty \), the estimate (1.5) takes the form

\[
  |B_A[f]|_p \leq \frac{p}{p-1} |f|_p, \ 1 < p \leq \infty. \tag{1.6}
\]
So, the inequality (1.1) is satisfied for the operator $Q = B_A$ again with the parameters $\lambda = \nu = 1$.

The lower bounds for the inequalities of the form (1.1), (1.2), i.e. the lower bounds for the operator $Q$ with at the same parameters $\lambda, \nu$ may be found, for instance, in [11], [15].

Notice [3], [9] that there are rearrangement invariant spaces $E$ for which the norms $\| \cdot \|_E$ and $\| \cdot \|_{H(E)}$ are not equivalent. For example,

$$H(L \ln^{n-1} L) = L \ln^n L, \ n \geq 1,$$

see [3], proposition 1.2.

**We intend in this preprint to extend the inequality (1.1) (or (1.2)) into the wide class of another rearrangement invariant Banach functional spaces: exponential Orlicz spaces and into Grand Lebesgue Spaces.**

In detail, let $Y_1, Y_2$ be two rearrangement invariant (r.i.) Banach functional spaces over $(X, B, \mu)$, in particular, Orlicz spaces or Grand Lebesgue ones. We set ourselves the goal to estimate of the correspondent operator norms

$$\|Q\|_{Y_1 \to Y_2} = \sup_{0 \neq f \in Y_1} \left[ \frac{\|Qf\|_{Y_2}}{\|f\|_{Y_1}} \right]. \quad (1.7)$$

2. **Grand Lebesgue Spaces (GLS).**

Let $(X, B, \mu)$ be again the source probability space. Let also $\psi = \psi(p), \ p \in [1, b), \ b = \text{const} \in (1, \infty]$ be certain bounded from below: $\inf \psi(p) > 0$ continuous inside the semi-open interval $p \in [1, b)$ numerical valued function. We can and will suppose without loss of generality

$$\inf_{p \in [1, b)} \psi(p) = 1 \quad (2.0)$$

and $b = \sup\{p, \ \psi(p) < \infty\}$, so that $\text{supp} \ \psi = [1, b)$ or $\text{supp} \ \psi = [1, b]$. The set of all such a functions will be denoted by $\Psi(b) = \{\psi(\cdot)\}; \ \Psi := \Psi(\infty)$. 
By definition, the (Banach) Grand Lebesgue Space (GLS) \( G\psi = G\psi(b) \) consists on all the real (or complex) numerical valued measurable functions (random variables, r.v.) \( f : X \to \mathbb{R} \) defined on our probability space and having a finite norm

\[
||f|| = ||f||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1,b)} \left[ \frac{|f|_p}{\psi(p)} \right].
\]  

(2.1)

The function \( \psi = \psi(p) \) is said to be generating function for this space.

Furthermore, let now \( \eta = \eta(z), z \in S \) be arbitrary family of random variables defined on any set \( z \in S \) such that

\[
\exists b = \text{const} \in (1, \infty), \forall p \in [1,b) \Rightarrow \psi_S(p) := \sup_{z \in S} |\eta(z)|_p < \infty.
\]

The function \( p \to \psi_S(p) \) is named as a natural function for the family of random variables \( S \). Obviously,

\[
\sup_{z \in S} ||\eta(z)||_{G\Psi_S} = 1.
\]

The family \( S \) may consists on the unique r.v., say \( \Delta : \psi_\Delta(p) := |\Delta|_p, \)

if of course the last function is finite for some value \( p = p_0 > 1. \)

Note that the last condition is satisfied if for instance the r.v. \( \Delta \) satisfies the so-called Kramer’s condition; the inverse proposition is not true.

The generating \( \psi(\cdot) \) function in (1.2) may be introduced for instance as natural one for some famoly of a functions.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [2], chapters 1, 2; and were investigated in particular in many works, see e.g. [4], [5], [6], [14], [16], [18], [22], chapters 1,2; [23], [24] etc. We refer here some used in the sequel facts about these spaces and supplement more.

The so-called tail function \( T_f(y), y \geq 0 \) for arbitrary (measurable) numerical valued function \( f \) is defined as usually

\[
T_f(y) \overset{\text{def}}{=} \max(\mu(f \geq y), \mu(f \leq -y)), y \geq 0.
\]
It is known that

\[ |f|^p_p = \int_X |f|^p(x) \mu(dx) = p \int_0^\infty y^{p-1} T_f(y) \, dy \]

and if \( f \in G\psi, \ f \neq 0 \), then

\[ T_f(y) \leq \exp \left( -v^*_\psi(\ln(y/||f||G\psi)) \right), \ y \geq e \ ||f||G\psi, \quad (2.2) \]

where

\[ v(p) = v_\psi(p) := p \ln \psi(p). \]

Here and in the sequel the operator (non-linear) \( f \rightarrow f^* \) will denote the famous Young-Fenchel transform

\[ f^*(u) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(f)} (xu - f(x)). \]

Conversely, the last inequality may be reversed in the following version: if

\[ T_\zeta(y) \leq \exp \left( -v^*_\psi(\ln(u/K)) \right), \ u \geq e \ K, \]

and if the auxiliary function \( v(p) = v_\psi(p) \) is positive, finite for all the values \( p \in [1, \infty) \), continuous, convex and such that

\[ \lim_{p \to \infty} \psi(p) = \infty, \]

then \( \zeta \in G(\psi) \) and besides \( ||\zeta|| \leq C(\psi) \cdot K. \)

Let us consider the so-called \textit{exponential} Orlicz space \( L(M) \) builded over source probability space with correspondent Young-Orlicz function

\[ M(y) = M[\psi](y) = \exp \left( v^*_\psi(\ln |y|) \right), \quad |y| \geq e; \ M(y) = Cy^2, \ |y| < e. \]

The exponentiality implies in particular that the Orlicz space \( L(M) \) is not separable as long as the correspondent Young-Orlicz function \( M(y) = M[\psi](y) \) does not satisfy the \( \Delta_2 \) condition.

The Orlicz \( ||\cdot||L(M) = ||\cdot||L(M[\psi](\cdot)) \) and \( ||\cdot||G\psi \) norms are quite equivalent:
\[ ||f||G\psi \leq C_1||f||L(M) \leq C_2||f||G\psi. \]

\[ 0 < C_1 = C_1(\psi) < C_2 = C_2(\psi) < \infty. \quad (2.3) \]

Furthermore, let now \( \eta = \eta(z), \ z \in W \) be arbitrary family of measurable functions (random variables) defined on any set \( W \) such that

\[ \exists b = \text{const} \in (1, \infty], \ \forall p \in [1, b) \ \Rightarrow \psi_W(p) := \sup_{z \in W} |\eta(z)|_p < \infty. \quad (2.4) \]

The function \( p \to \psi_W(p) \) is named as a \textit{natural} function for the family of random variables \( W \). Obviously,

\[ \sup_{z \in W} ||\eta(z)||G\Psi_W = 1. \]

The family \( W \) may consists on the unique r.v., say \( \Delta \) :

\[ \psi_{\Delta}(p) := |\Delta|_p, \quad (2.5) \]

if of course the last function is finite for some value \( p = p_0 > 1 \).

Note that the last condition is satisfied if for instance the r.v. \( \zeta \) satisfies the so-called Kramer’s condition; the inverse proposition is not true.

**Example 2.0.** Let us consider also the so-called \textit{degenerate} \( \Psi \) — function \( \psi_{(r)}(p) \), where \( r = \text{const} \in [1, \infty) \):

\[ \psi_{(r)}(p) \overset{\text{def}}{=} 1, \ p \in [1, r]; \]

so that the correspondent value \( b = b(r) \) is equal to \( r \). One can extrapolate formally this function onto the whole semi-axis \( R^1_+ \):

\[ \psi_{(r)}(p) := \infty, \ p > r. \]

The classical Lebesgue-Riesz \( L_r \) norm for the r.v. \( \eta \) is quite equal to the GLS norm \( ||\eta||G\psi_{(r)} \) :
Thus, the ordinary Lebesgue-Riesz spaces are particular, more precisely, extremal cases of the Grand-Lebesgue ones.

**Example 2.1.** For instance, let \( \psi \) function has a form

\[
\psi(p) = \psi_m(p) = p^{1/m}, \quad m = \text{const} > 0. \tag{2.6}
\]

The function \( f : X \to \mathbb{R} \) belongs to the space \( G_{\psi_m} \) :

\[
||f||_{G_{\psi_m}} = \sup_{p \geq 1} \left\{ \frac{|f|_p}{p^{1/m}} \right\} < \infty
\]

if and only if the correspondent tail estimate is follow:

\[
\exists V = V(m) > 0 \Rightarrow T_f(y) \leq \exp \{- (y/V(m))^m \}, \quad y \geq 0.
\]

The correspondent Young-Orlicz function for the space \( G_{\psi_m} \) has a form

\[
M_{\psi_m}(y) = \exp(|y|^m), \quad |y| > 1; \quad M_m(y) = e^{y^2}, \quad |y| \leq 1.
\]

There holds for arbitrary function \( f \)

\[
||f||_{G_{\psi_m}} \asymp ||f||_{L(M_m)} \asymp V(m),
\]

if of course as a capacity of the value \( V = V(m) \) we understand its minimal positive value from the relation (2.7).

The case \( m = 2 \) correspondent to the so-called subgaussian case, i.e. when

\[
T_f(y) \leq \exp \{- (y/V(2))^2 \}, \quad y > 0. \tag{2.7}
\]

It is presumes as a rule in addition that the function \( f(\cdot) \) has a mean zero:

\[
\int_X f(x) \mu(dx) = 0. \quad \text{More examples may be found in [4], [16], [22].}
\]
We bring a more general example, see [17]. Let \( m = \text{const} > 1 \) and define \( q = m' = m/(m-1) \). Let also \( L = L(y), y > 0 \) be positive continuous differentiable \textit{slowly varying} at infinity function such that

\[
\lim_{\lambda \to \infty} \frac{L(y/L(y))}{L(y)} = 1.
\]  

(2.8)

Introduce a following \( \psi \) - function

\[
\psi_{m,L}(p) \overset{def}{=} p^{1/m} L^{-1/(m-1)} \left( p^{(m-1)^2/m} \right), \quad p \geq 1,
\]  

(2.9a)

and a correspondent exponential tail function

\[
T^{(m,L)}(y) \overset{def}{=} \exp \left\{ -q^{-1} y^q L^{-/(q-1)} \left( y^{q-1} \right) \right\}, \quad y > 0.
\]  

(2.9b)

The following implication holds true:

\[
0 \neq f \in G\psi_{m,L} \iff \exists C = \text{const} \in (0, \infty), \ T_f(y) \leq T^{(m,L)}(y/C).
\]  

(2.10)

A particular cases: \( L(y) = \ln^r(y + e), \ r = \text{const}, \ y \geq 0 \); then the correspondent generating functions have a form

\[
\psi_{m,r}(p) = m^{-1} p^m \ln^{-r/(m-1)}(p + 1),
\]  

(2.11a)

and correspondingly the tail function

\[
T^{m,r}(y) = \exp \left\{ -y^q (\ln y)^{-(q-1)r} \right\}, \quad y \geq e.
\]  

(2.11b)

**Example 2.2.** Bounded support of generating function.

Introduce the following tail function

\[
T^{<b,\gamma,L>}(x) \overset{def}{=} x^{-b} (\ln x)^{\gamma} L(\ln x), \quad x \geq e,
\]  

(2.12)

where as before \( L = L(x), \ x \geq 1 \) is positive continuous slowly varying function as \( x \to \infty \), and

\[
b = \text{const} \in (1, \infty), \ \gamma = \text{const} > -1.
\]
Introduce also the following (correspondent!) $\Psi(b)$ function

$$
\psi^{<b,\gamma,L>}(p) \overset{\text{def}}{=} C_1(b, \gamma, L) \left( b - p \right)^{-(\gamma + 1)/b} \left( \frac{1}{b - p} \right), \ 1 \leq p < b. \tag{2.13}
$$

Let the measurable function $f(\cdot)$ be such that

$$
T_f(y) \leq T^{<b,\gamma,L>}(y), \ y \geq e,
$$

then

$$
|f|_p \leq C_2(b, \gamma, L) \psi^{<b,\gamma,L>}(p), \ p \in [1, b) \tag{2.14}
$$
or equivalently

$$
||f|| \in G^{\psi^{<b,\gamma,L>}} \iff ||f||_{G^{\psi^{<b,\gamma,L>}}} < \infty. \tag{2.15}
$$

Conversely, if the estimate (2.14) holds true, then

$$
T_f(y) \leq C_3(b, \gamma, L) y^{-b} (\ln y)^{\gamma + 1} L(\ln y), \ y \geq e \tag{2.16}
$$
or equally

$$
T_f(y) \leq T^{<b,\gamma+1,L>}(y/C_4), \ y \geq C_4 e. \tag{2.16a}
$$

Notice that there is a logarithmic “gap” as $y \to \infty$ between the estimations (2.15) and (2.16). Wherein all the estimates (2.14) and (2.16) are non-improvable, see [17], [18], [24].

**Remark 2.1.** These GLS spaces are used for obtaining of an *exponential estimates* for sums of independent random variables and fields, estimations for non-linear functionals from random fields, theory of Fourier series and transform, theory of operators etc., see e.g. [4], [14], [18], [22], sections 1.6, 2.1 - 2.5.
3. Main result. The case of equal powers.

We consider in this section the case when in the relations (1.1) - (1.2) \( \nu = \lambda = \text{const} > 0 \), i.e.

\[
|g|_p \leq Z_\lambda(\tilde{f}) \cdot \frac{p^\lambda}{(p-1)^\lambda} \cdot \psi(p) = Z \cdot \frac{p^\lambda}{(p-1)^\lambda} \cdot \psi(p), \quad (3.1)
\]

This relation holds true if for example in the relation (1.1) \( f \in G\psi \); one can assume without loss of generality for simplicity \( ||f||G\psi = 1, Z = 1 \), so that \( |f|_p \leq \psi(p), 1 \leq p < b \).

Let us introduce some auxiliary constructions. Let the function \( \psi = \psi(p), \psi \in \Psi(b), b = \text{const} \in (1, \infty) \) be a given. Let also \( q \) be some fixed number inside the set \( (1, b) : 1 < q < b \). Suppose the (measurable) function \( g = g(x) \) satisfies the inequality (3.1). We apply the Lyapunov’s inequality: \( p \in [1, q] \Rightarrow |g|_p \leq |g|_q \), hence

\[
p \in [1, q] \Rightarrow |g|_p \leq |g|_q \leq \left[ \frac{q}{q-1} \right]^\lambda \cdot \psi(q). \quad (3.2a)
\]

We retain the value of the function \( \psi(\cdot) \) on the additional set:

\[
p \in (q, b) \Rightarrow |g|_p \leq \left[ \frac{p}{p-1} \right]^\lambda \cdot \psi(p). \quad (3.2b)
\]

Let us introduce the following \( \psi - \) function \( \tilde{\psi}(p) = \)

\[
\tilde{\psi}_{q,\lambda}(p) := \left[ \frac{q}{q-1} \right]^\lambda \cdot \psi(q) I(p \in [1, q]) + \left[ \frac{p}{p-1} \right]^\lambda \cdot \psi(p) I(p \in (q, b)),
\]

so that

\[
|g|_p \leq Z \cdot \tilde{\psi}_{q,\lambda}(p), 1 \leq p < b. \quad (3.3)
\]

Here and further \( I(p \in A) \) denotes the indicator function of the set \( A \).

So, we have eliminated the possible singularity at the point \( p \to 1 + 0 \).

Let us prove now that

\[
Z^{-1} Q(p, q) := \sup_{\nu \in [1, b]} \frac{\tilde{\psi}_q(p)}{\psi(p)} =: C(b, q, \lambda) < \infty.
\]
We conclude taking into account the restriction $\psi(p) \geq 1$

$$Z^{-1} Q(p, q) = \frac{\tilde{\psi}_q(p)}{\psi(p)} \leq \left[ \frac{q}{q-1} \right]^\lambda \cdot \psi(q) I(p \in [1, q]) + \left[ \frac{p}{p-1} \right]^\lambda I(p \in (q, b));$$

$$\sup_{p \in [1,b]} Z^{-1} Q(p, q) \leq \max \left( \frac{q^\lambda}{(q-1)^\lambda} \psi(q), \left[ \frac{q}{q-1} \right]^\lambda \right) = \left[ \frac{q}{q-1} \right]^\lambda \psi(q). \quad (3.5)$$

Further, let us denote

$$K_{\lambda}[\psi, b] := \inf_{q \in (1,b)} \left\{ \frac{q^\lambda}{(q-1)^\lambda} \psi(q) \right\}, \quad (3.6)$$

then $K_{\lambda}[\psi, b] \in [1, \infty)$; and we derive the following estimate

$$1 \leq \inf_q \sup_p \left\{ \frac{\tilde{\psi}_q(p)}{\psi(p)} \right\} \leq K_{\lambda}[\psi, b] < \infty. \quad (3.7)$$

We get due to proper choice of the parameter $q$:

**Proposition 3.1.** We propose under formulated above notations and conditions, in particular, condition (3.1)

$$\|g\|_G \psi \leq Z \cdot K_{\lambda}[\psi, b] < \infty. \quad (3.8)$$

One can give a very simple upper estimate for the value $K_{\lambda}[\psi, b]$; indeed, we choose in (3.6) $q = 2$ in the case when $b > 2$ and $q = (b+1)/2$ if $b \in (1, 2]$; we get

$$K_{\lambda}[\psi, b] \leq \left[ \frac{b+1}{b-1} \right]^\lambda \psi \left( \frac{b+1}{2} \right) I(b \in (1, 2]) + 2^\lambda \psi(2) I(b > 2).$$

As a slight consequence: if $b < \infty$, then

$$K_{\lambda}[\psi, b] \leq \frac{C(b, \lambda, \psi)}{(b-1)^\lambda}.$$  

where $C(b, \lambda, \psi)$ is continuous bounded function relative the variable $b$ in arbitrary finite segment $1 < b \leq V, V = \text{const} < \infty$. 

Example 3.a. Let

\[ \psi(p) = \psi_m(p) = p^{1/m}, \quad m = \text{const} > 0, \quad p \in [1, \infty). \]

We obtain after simple calculations

\[ K_m(\lambda) := \inf_{q \in (1, \infty)} \left[ \frac{q^\lambda \psi_m(q)}{(q - 1)^\lambda} \right] = \inf_{q \in (1, \infty)} \left[ \frac{q^{\lambda+1/m}}{(q - 1)^\lambda} \right] = m^{1/m} \cdot (\lambda + 1/m)^{\lambda+1/m} \cdot \lambda^{-\lambda}. \]

(3.9)

In particular,

\[ K_m(1) = m^{-1} (m + 1)^{1+1/m}, \quad m > 0. \]

Note by the way \( \forall \lambda > 0 \Rightarrow \lim_{m \to \infty} K_m(\lambda) = 1. \)

Example 3.b. Let \( b = \text{const} > 1; \quad \beta = \text{const} > 0. \) Define the following tail function

\[ T[b, \beta](y) := C \cdot y^{-b} (\ln y)^{\beta b - 1}, \quad y \geq e, \]

and the following \( \Psi(b) \) function with bounded support

\[ \psi[b, \beta](p) = \left[ \frac{b - p}{b - 1} \right]^{-\beta}, \quad p \in [1, b); \quad \psi[b, \beta](p) = \infty, \quad p \geq b. \]

The tail inequality of the form

\[ T_\eta(y) \leq T[b, \beta](y), \quad y \geq e \]

entails the inclusion \( \eta \in G\psi[b, \beta]. \) The inverse conclusion is not true.

We find after some computations

\[ K_{b, \beta}(\lambda) := \inf_{q \in (1, b)} \left[ \frac{q^\lambda}{(q - 1)^\lambda} \cdot (b - q)^{-\beta} \right] \leq \frac{(\lambda b + \beta)^\lambda \cdot (\lambda + \beta)^{\beta}}{\lambda^\beta \beta^\beta (b - 1)^{\lambda+\beta}}. \]

(3.10)

Example 3.c. Let now \( \psi(p) = \psi_r(p), \quad r = \text{const} > 1. \) It is easily to calculate
\[ K(r) := \inf_{q > 1} \left[ \frac{q^\lambda \psi(q)}{(q-1)\lambda} \right] = \left[ \frac{r}{r-1} \right]^\lambda. \]

Let us return to the theory of operators, see (1.1), (1.2). Namely, assume the operator \( Q \) satisfies the inequality (1.1) or more generally (1.2). It follows immediately from proposition (3.1) the following statement.

**Theorem 3.1.** Suppose the function \( f(\cdot) \) belongs to the space \( G_\psi \) for some generating function \( \psi \) from the set \( \Psi(b) \), \( 1 < b \leq \infty \). Our statement: the function \( g = Q[f] \) from the relations (1.1) (or (1.2)) belongs to the same Grand Lebesgue Space \( G_\psi \), or equivalently to the correspondent exponential Orlicz space \( L(M_\psi) \):

\[
\|g\|_{G_\psi} \leq K_\lambda[\psi,b] Z(Q) \|f\|_{G_\psi},
\]

or equally in the terms of exponential Orlicz spaces

\[
\|g\|_{L(M_\psi)} \leq C(\psi) K_\lambda[\psi,b] Z(Q) \|f\|_{L(M_\psi)}.
\]

**Remark 3.1.** The statement of theorem (3.1) may be reformulated as follows. Under at the same conditions: \( f \in G_\psi \) etc.

\[
\|Q[f]\|_{G_\psi} \leq Z K_\lambda[\psi,b] \|f\|_{G_\psi}
\]

or equally

\[
\|Q(\cdot)\|_{[G_\psi \to G_\psi]} \leq Z K_\lambda[\psi,b].
\]

**Remark 3.2.** Note that the considered here Young-Orlicz function \( M_\psi(y) \) does not satisfy the \( \Delta_2 \) condition, in contradiction to the considered ones in the book [19], section 12.

**Example 3.1.** Suppose the function \( f(\cdot) \) from the estimate (1.1) belongs to the space \( G_\psi_m, m = \text{const} > 0 \):
\[
\sup_{p \geq 1} \left[ \frac{|f|_p}{p^{1/m}} \right] < \infty
\]

or equivalently

\[
\exists C_1 > 0 \Rightarrow T_f(y) \leq \exp(-C_1 y^m), \ y \geq 0.
\]

Then there exists a positive finite constant \( C_3 = C_3(m, \lambda) \) for which

\[
T_g(y) \leq \exp(-C_3(m, \lambda) y^m), \ y \geq 0,
\]

or equivalently

\[
\sup_{p \geq 1} \left[ \frac{|g|_p}{p^{1/m}} \right] < \infty.
\]

More generally, let \( L = L(y), \ y > 0 \) be the positive continuous differentiable \textit{slowly varying} at infinity function such that

\[
\lim_{\lambda \to \infty} \frac{L(y/L(y))}{L(y)} = 1,
\]

i.e. as in the example 2.1. Recall the following notation for \( \psi - \) function

\[
\psi_{m,L}(p) \overset{\text{def}}{=} p^{1/m} L^{-1/(m-1)} \left( p^{(m-1)^2/m} \right), \ p \geq 1, \ m > 1,
\]

and the correspondent exponential tail function

\[
T^{(m,L)}(y) \overset{\text{def}}{=} \exp \left\{ -q^{-1} y^q L^{-(q-1)} \left( y^{q-1} \right) \right\}, \ y > 0,
\]

where \( m = \text{const} > 1, \ q = m/(m - 1) \).

Suppose the function \( f(\cdot) \) from the estimate (1.1) belongs to the space \( G\psi_{m,L}, \ m = \text{const} > 0 \) :

\[
\sup_{p \geq 1} \left[ \frac{|f|_p}{\psi_{m,L}(p)} \right] < \infty
\]

or equivalently

\[
\exists C_1 = C_1(m, L) > 0 \Rightarrow T_f(y) \leq T^{(m,L)}(y/C_1).
\]

Then there exists a positive constant \( C_3 = C_3(m, L, \lambda) \) for which
\[ T_g(y) \leq T^{(m,L)}(y/C_3) \quad (3.15a) \]

or equivalently

\[ \sup_{p \geq 1} \left[ \frac{|g|_p}{\psi_{m,L}(p)} \right] < \infty. \quad (3.15b) \]

**Example 3.2.** The case of bounded support.

This case is more complicated. Recall the following notation for tail function

\[ T^{<b,\gamma,L>}(x) \overset{\text{def}}{=} x^{-b}(\ln x)^\gamma L(\ln x), \ x \geq e, \]

where as before \( L = L(x), \ x \geq 1 \) is the positive continuous slowly varying function as \( x \to \infty \), and let as before

\[ b = \text{const} \in (1, \infty), \ \gamma = \text{const} > -1. \]

and recall also notation for the following correspondent \( \Psi(b) \) function

\[ \psi^{<b,\gamma,L>}(p) \overset{\text{def}}{=} C_1(b, \gamma, L) (b - p)^{-(\gamma + 1)/b} L^{1/b} \left( \frac{1}{b - p} \right), \ 1 \leq p < b. \]

Let the source (measurable) function \( f(\cdot) \) be such that

\[ ||f|| \in G\psi^{<b,\gamma,L>} \iff ||f||G\psi^{<b,\gamma,L>} < \infty, \quad (3.16) \]

then also \( ||g|| \in G\psi^{<b,\gamma,L>} \) and moreover

\[ ||g||G\psi^{<b,\gamma,L>} \leq K^{b,\gamma}(\lambda) ||f||G\psi^{<b,\gamma,L>}. \quad (3.17) \]

But if we assume the following tail restriction on the function \( f \)

\[ T_f(y) \leq T^{<b,\gamma,L>}(y), \ y \geq e, \quad (3.18) \]

then we conclude only

\[ T_g(y) \leq C_5(b, \gamma, L) y^{-b} (\ln y)^{\gamma + 1} L(\ln y), \ y \geq e \quad (3.19a) \]
or equally

\[ T_g(y) \leq T^{b,\gamma+1,L>}(y/C_6), \ y \geq C_6 e. \quad (3.19b) \]

*Open question:* what is the ultimate value instead “\( \gamma + 1 \)” in the last estimate?

## 4. Main result. The case of different powers.

Let as before some function \( \psi = \psi(p), \ p \in [1, b), \ b = \text{const} \in (1, \infty) \) from the set \( \Psi(b) \) be a given. Suppose in this section that in the inequalities (1.1) or (1.2) \( f \in G_\psi, \ ||f||_{G_\psi} < \infty, \ \lambda > \nu \geq 0, \) and denote \( \Delta = \lambda - \nu; \ (\Delta > 0), \)

\[ \zeta(p) = \zeta[\psi, \Delta](p) := p^\Delta \psi(p). \quad (4.0) \]

Obviously, \( \zeta(\cdot) \in \Psi(b). \)

Let for beginning \( ||f||_{G_\psi} = 1, \) then \( |f|_p \leq \psi(p), \ p \in [1, b). \) We deduce from the inequality (1.1) taking into account the estimate \( |f|_p \leq \psi(p), \ p \in [1, b) \) alike the foregoing section denoting \( g = Q[f] \)

\[ |g|_p \leq Z \left[ \frac{p}{p-1} \right]^\nu p^\Delta \psi(p) = Z \left[ \frac{p}{p-1} \right]^\nu \zeta[\psi, \Delta](p). \quad (4.1) \]

It follows immediately from proposition (3.1) or theorem 3.1

\[ ||g||_{G_\zeta[\psi, \Delta]} \leq Z K_\nu[\zeta[\psi, \Delta]](\zeta, b). \quad (4.2) \]

We proved in fact the following result.

**Theorem 4.1.** Suppose as above that the function \( f(\cdot) \) belongs to the space \( G_\psi \) for some generating function \( \psi \) from the set \( \Psi(b), \ 1 < b \leq \infty. \) Let in (1.1) \( \lambda > \nu \geq 0. \) Our statement: the function \( g = Q[f] \) from the relations (1.1) belongs to the *other* certain Grand Lebesgue Space \( G_\zeta, \) or equivalently to the correspondent exponential Orlicz space \( L(M_\zeta) : \)

\[ ||g||_{G_\zeta[\psi, \Delta]} \leq Z K_\nu(\zeta, b) ||f||_{G_\psi}, \quad (4.3) \]
or equally in the terms of exponential Orlicz spaces

\[ ||g||_{L(M_\zeta)} \leq C(\psi) K_\nu[\zeta(\psi, \Delta)](\nu, b) Z(Q) ||f||_{L(M_\psi)}. \]  

\( (4.3a) \)

**Remark 4.1.** In the case \( b < \infty \) the estimate (4.3) may be simplified as follows.

As long as in this case \( p^\Delta \psi(p) \leq b^\Delta \psi(p) \), we conclude that the operator \( Q \) acts from the space \( G\psi \) into at the same space:

\[ ||Q[f]||_{G\psi} \leq b^\Delta Z K_\nu [b^\Delta \psi, b] ||f||_{G\psi}. \]  

\( (4.4) \)

5. **Convergence in the Grand Lebesgue and non-separable Orlicz spaces.**

Let us consider here the sequence of the form

\[ g_n(x) = Q_n(\vec{f}_n) = Q(f_1, f_2, \ldots, f_n), \quad n = 1, 2, \ldots, \infty, \]  

\( (5.0) \)

such that for some non-negative constants \( \lambda, \nu; \lambda \geq \nu \)

\[ \sup_n |g_n|_p \leq Z(\vec{f}) \cdot \frac{p^\lambda}{(p - 1)^\nu} \cdot \psi(p) \]  

\( (5.1) \)

for certain positive continuous function \( \psi = \psi(p) \), \( p \in [1, b) \), \( b = \text{const} \in (1, \infty] \) from the set \( \Psi(b) \).

It follows from theorem 4.1 that

\[ \sup_n ||g_n||_{G\zeta} < \infty, \]  

\( (5.2) \)

where in the case \( \lambda = \nu \Rightarrow \zeta(p) = \psi(p) \).

We suppose in addition to (5.1) (or following (5.2)) that the sequence \( \{g_n(\cdot)\} \) converges in all the norms \( L_p(X, \mu) \), \( p \in [1, b) \);

\[ \exists g_\infty(x) \overset{\text{def}}{=} \lim_{n \to \infty} g_n(x) \]  

\( (5.3a) \)

such that
∀p ∈ [1, b) ⇒ \lim_{n \to \infty} |g_n - g_{\infty}|_p = 0. \quad (5.3b)

Our claim in this section is investigation under formulated before condition the problem of convergence \( g_n \to g_{\infty} \) in more strong norms, concrete: in the CLS sense or correspondingly in Orlicz spaces norms.

The simplest example of (5.2)-(5.3a), (5.3b) give us the theory of martingales. It makes sense to dwell on this in more detail.

This approach may be used for instance in the martingale theory, see [8], where \( \lambda = \nu = 1 \), and

\[ g_n = \max_{i=1}^{n} f_i. \]

where \( \{f_i\} \) is a centered martingale (or semi-martingale) sequence relative certain filtration \( \{F_i\} \):

\[ |g_n|_p \leq \frac{p}{p - 1} |f_n|_p, \quad p \in (1, b), \]

if of course the right-hand side is finite.

J. Neveu proved in [19], pp. 209-220 that if the Orlicz space \( L(M) \) built over our probability space with correspondent Young-Orlicz function \( M(\cdot) \) satisfying the \( \Delta_2 \) condition, or equivalently if the space \( L(M) \) is separable,

\[ \lim_{t \to 0^+} \frac{M(t)}{t} = 0 \]

and

\[ \sup_{n=1,2,...} ||f_n||_{L(M)} < \infty, \]

then there exists almost everywhere a limit

\[ \lim_{n \to \infty} f_n =: f_{\infty} \]

and the convergence in (4.2) take place also in the \( L(M) \) norm:
\[
\lim_{n \to \infty} \| f_n - f_\infty \|_{L(M)} = 0. \quad (5.4)
\]

We must first of all recall some definitions and facts about comparison of GLS from an article [24]. Let \( \psi, \nu \) be two functions from the set \( G\psi(b), \, b \in (1, \infty] \). We will write \( \psi << \nu \), or equally \( \nu >> \psi \), iff

\[
\lim_{p \to b^- 0} \frac{\psi(p)}{\nu(p)} = 0, \quad b < \infty; \quad (5.5a)
\]

\[
\lim_{p \to \infty} \frac{\psi(p)}{\nu(p)} = 0, \quad b = \infty. \quad (5.5b)
\]

There exists an equivalent version (and notion) for Young-Orlicz function, see [26], chapters 2,3.

**Theorem 4.1.** Assume the formulated above notations, conditions (5.3a), (5.3b) remains true. Our statement: for arbitrary \( \Psi(b) \) function \( \tau = \tau(p), \, 1 \leq p < b \) such that \( \tau << \zeta \)

\[
\lim_{n \to \infty} \| f_n - f_\infty \|_{G\tau} = 0, \quad (5.6)
\]

i.e. the sequence \( f_n \) converges not only almost surely but also in arbitrary \( G\tau \) norm for which \( \tau << \zeta \).

**Proof** is very simple. The needed convergence \( f_n, \, n \to \infty \) in the \( G\tau \) norm follows immediately from one of the main results of the article [24], p. 238.

6. 6. Concluding remarks.

**A.** One can consider a more general case as in (1.1), (1.2):

\[
|Q[f]|_p \leq W(p) \| f \|_p, \quad p \in [1, b), \quad f \in G\psi, \quad (6.0)
\]

or equally \( g(x) := Q[f](x) \),

\[
|g|_p \leq W(p) \psi(p), \quad p \in [1, b), \quad (6.1)
\]
where \(W = W(p), p \in (1, b)\) is any measurable function, not necessary to be continuous or bounded.

Indeed, let as above \(q\) be arbitrary number from the open interval \((1, b) : q \in (1, b)\). We have using again the Lyapunov’s inequality

\[
p \in [1, q] \Rightarrow |g|_p \leq I(p \in [1, q]) W(q) \psi(q),
\]

following

\[
|g|_p \leq I(p \in [1, q]) W(q) \psi(q) + I(p \in (q, b)) W(p) \psi(p).
\]

Thus, if we denote \(v(p) = v[W, \psi](p) := \)

\[
\inf_{q \in (1, b)} \{I(p \in [1, q]) W(q) \psi(q) + I(p \in (q, b)) W(p) \psi(p)\} : (6.2)
\]

**Proposition 6.1.**

\[
||Q[f]||_Gv \leq ||f||_G\psi. (6.3)
\]

**B.** In the case of martingales the condition of almost surely convergence follows from the boundedness of its moment: \(\sup_n |f_n|_p < \infty, \exists p \geq 1, \) by virtue of the famous theorem of J.Doob.

**C.** Lower bounds for the norm of considered operators.

Assume in addition to the estimates (1.1), or (1.2), (3.1), that

\[
|f|_p \leq |Q[f]|_p, p \in [1, b), b = \text{const} \in (1, \infty). \quad (6.4)
\]

The inequality (6.3) is true for example for every maximal operators, in the J.Doob’s inequality for martingales etc.

Suppose that the function \(\psi(\cdot)\) is a natural function for appropriate function \(f_0 : X \to R : \psi(p) = |f_0|_p,\) such that \(\forall p < b \Rightarrow \psi(p) < \infty.\) Then the relation (6.4) takes the form
\[ \psi(p) \leq |Q[f_0]|_p, \quad p \in [1, b), \quad \iff \quad |Q[f_0]|_p \geq \psi(p), \]

hence

\[ ||Q[:]|(G\psi \to G\psi) \geq 1. \quad (6.5) \]

References.

1. R.A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.
2. Bennet C., Sharpley R. *Interpolation of operators*. Orlando, Academic Press Inc., (1988).
3. M.Braverman, Ben-Zion Rubstein, A.Veksler. *Domintated ergodic theorems in rearrangement invariant spaces*. Studia Mathematica, **128**, (2), 145-157, (1998).
4. Buldygin V.V., Kozachenko Yu.V. *Metric Characterization of Random Variables and Random Processes*. 1998, Translations of Mathematics Monograph, AMS, v.188.
5. A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces*. Collect. Math. 51, (2000), 131-148.
6. A. Fiorenza and G.E. Karadzhov. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnico 272/03, (2005).
7. Y.Derriennic. *On integrability of the supremum of ergodic ratios*. Ann. Probab., 1, (1973), 338 - 340.
8. J.L.Doob. *Stochastic Processes*. Wiley, New York, 1953.
9. D.Gilat. *The best bound in the L log L inequality of Hardy and Littlewood and its martingale counterpart*. Proc. Amer. Math. Soc., 1992, **97**, 429 - 436.
10. D.Gilat. *On the ratio of the expected maximum of a martingale and the Lp – norm of its last term*. Israel Journal of Mathematics, October 1988, Volume 63, Issue 3, pp 270280.
11. L. Grafakos, A. Montgomery-Smith. *Best constants for uncentered maximal functions*. Bulletin of the London Mathematical Society, 29, (1997), no. 1, 6064.
12. I.Grama, E.Haeusler. *Large deviations for martingales*. Stoch. Pr. Appl.,
13. P. Hall, C. C. Heyde. *Martingale Limit Theory and Its Applications.* Academic Press. 1980, New York, London, Toronto, Sydney.

14. T. Iwaniec and C. Sbordone. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat. Mech. Anal., 119, (1992), 129-143.

15. Paata Ivanisvili, Benjamin Jave, and Fedor Nazaro. *Lower bounds for uncentered maximal functions in any dimension.* arXiv:1602.05895v1 [math.AP] 18 Feb 2016.

16. Kozachenko Yu. V., Ostrovsky E. I. (1985). *The Banach Spaces of random Variables of subgaussian Type.* Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

17. Kozachenko Yu. V., Ostrovsky E., Sirota L. *Relations between exponential tails, moments and moment generating functions for random variables and vectors.* arXiv:1701.01901v1 [math.FA] 8 Jan 2017

18. E. Liflyand, E. Ostrovsky and L. Sirota. *Structural properties of Bilateral Grand Lebesgue Spaces.* Turk. Journal of Math., 34, (2010), 207-219. TUBITAK, doi:10.3906/mat-0812-8

19. Neveu J. *Discrete - parameter Martingales.* North - Holland, Amsterdam, 1975.

20. D. S. Ornstein. *A remark on the Birkhoff ergodic theorem.* Illinois J. Math., 15 (1971), 77 - 79.

21. Adam Osekowski. *Sharp logarithmic inequalities for two Hardy-type operators.* Zeitschrift für Analysis und ihre Anwendungen. European Mathematical Society, Journal for Analysis and its Applications, Volume 20, (2012), Monografie Matematyczne 72 (2012), Birkhuser, 462 pp. DOI: 10.4171/ZAA/XXX.

22. Ostrovsky E. I. (1999). *Exponential estimations for Random Fields and its applications,* (in Russian). Moscow-Obninsk, OINPE.

23. Ostrovsky E. and Sirota L. *Sharp moment estimates for polynomial martingales.* arXiv:1410.0739v1 [math.PR] 3 Oct 2014

24. Ostrovsky E. and Sirota L. *Moment Banach spaces: theory and applications.* HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233-262.

25. De La Pena V. H. *A general class of exponential inequalities for martingales*
and ratios. 1999, Ann. Probab., 36, 1902-1938.

26. Rao M.M., Ren Z.D. *Theory of Orlicz spaces.* (1991), Pure and Applied Mathematics, Marcel Dekker, ISBN 0-8247-8478-2.

27. Juan Arias de Reyna. *Pointwise Convergence of Fourier Series.* Springer, 2002.

28. Derek W. Robinson. *Hardy and Rellich inequalities on the complement of convex sets.* arXiv:1704.03625v1 [math.AP] 12 Apr 2017

29. Secchi, S., Smets, D., and Willem, M. *Remarks on a Hardy-Sobolev inequality.* C. R. Acad. Sci Paris, Serie I, 336, (2003), 811815.

30. E M. Stein and G. Weiss.. *Introduction to Fourier Analysis on Euclidean Spaces.,* (1971), Princeton Univ. Press, Princeton, N.J.

31. Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions.* (PMS-30), (1978), Oxford.

32. Elias M. Stein. *Maximal functions: spherical means.* Proceedings of the National Academy of Sciences, 73 (7), (1976), 21742175.

33. Michael E. Taylor. *Pseudodifferential Operators And Nonlinear PDE.* Springer, 1970, Published by: Princeton University Press.