CRYSTALS AND SCHUR $P$-POSITIVE EXPANSIONS

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Abstract. We give a new characterization of Littlewood-Richardson-Stembridge tableaux for Schur $P$-functions by using the theory of $q(n)$-crystals. We also give alternate proofs of the Schur $P$-expansion of a skew Schur function due to Ardila and Serrano, and the Schur expansion of a Schur $P$-function due to Stembridge using the associated crystal structures.

1. Introduction

Let $\mathcal{P}^+$ be the set of strict partitions and let $P_\lambda$ be the Schur $P$-function corresponding to $\lambda \in \mathcal{P}^+$ [12]. The set of Schur $P$-functions is an important class of symmetric functions, which is closely related with representation theory and algebraic geometry (see [10] and references therein). For example, the Schur $P$-polynomial $P_\lambda(x_1, \ldots, x_n)$ in $n$ variables is the character of a finite-dimensional irreducible representation $V_n(\lambda)$ of the queer Lie superalgebra $q(n)$ with highest weight $\lambda$ up to a power of 2 when the length $\ell(\lambda)$ of $\lambda$ is no more than $n$ [13].

The set of Schur $P$-functions forms a basis of a subring of the ring of symmetric functions, and the structure constants with respect to this basis are non-negative integers, that is, given $\lambda, \mu, \nu \in \mathcal{P}^+$,

$$P_\mu P_\nu = \sum_\lambda f_{\mu\lambda}^\nu P_\lambda,$$

for some non-negative integers $f_{\mu\lambda}^\nu$. The first and the most well-known result on a combinatorial description of $f_{\mu\lambda}^\nu$, was obtained by Stembridge [16] using shifted Young tableaux, which is a combinatorial model for Schur $P$- or $Q$-functions [11] [17]. It is shown that $f_{\mu\lambda}^\nu$ is equal to the number of semistandard tableaux with entries in a $\mathbb{Z}_2$-graded set $N = \{ 1' < 1 < 2' < 2 < \cdots \}$ of shifted skew shape $\lambda/\mu$ and weight $\nu$ such that (i) for each integer $k \geq 1$ the southwesternmost entry with value $k$ is unprimed or of even degree and (ii) the reading words satisfy the lattice property. Here we say that the value $|x|$ is $k$ when $x$ is either $k$ or $k'$ in a tableau. Let us call these

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Recently, two more descriptions of \( f^\lambda_{\mu \nu} \) were obtained in terms of semistandard decomposition tableaux, which is another combinatorial model for Schur \( P \)-functions introduced by Serrano \([14]\). It is shown by Cho that \( f^\lambda_{\mu \nu} \) is given by the number of semistandard decomposition tableaux of shifted shape \( \mu \) and weight \( w_0(\lambda - \nu) \) whose reading words satisfy the \( \lambda \)-good property (see \([3, \text{Corollary 5.14}]\)). Here we assume that \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq n \), and \( w_0 \) denotes the longest element in the symmetric group \( S_n \). Another description is given by Grantcharov, Jung, Kang, Kashiwara, and Kim \([6]\) based on their crystal base theory for the quantized enveloping algebra of \( q(n) \) \([7]\). They realize the crystal \( B_n(\lambda) \) associated to \( V_n(\lambda) \) as the set of semistandard decomposition tableaux of shape \( \lambda \) with entries in \( \{1 < 2 < \cdots < n\} \), and describe \( f^\lambda_{\mu \nu} \) by characterizing the lowest weight vectors of weight \( w_0\lambda \) in the tensor product \( B_n(\mu) \otimes B_n(\nu) \). We also remark that bijections between the above mentioned combinatorial models for \( f^\lambda_{\mu \nu} \) are studied in \([4]\) using insertion schemes for semistandard decomposition tableaux.

The main result in this paper is to give another description of \( f^\lambda_{\mu \nu} \) using the theory of \( q(n) \)-crystals, and show that it is indeed equivalent to that of Stembridge. More precisely, we show that \( f^\lambda_{\mu \nu} \) is equal to the number of semistandard tableaux with entries in \( \mathbb{N} \) of shifted skew shape \( \lambda/\mu \) and weight \( \nu \) such that (i) for each integer \( k \geq 1 \) the southwesternmost entry with value \( k \) is unprimed or of even degree and (ii) the reading words satisfy the “lattice property” (see Definitions \([3.3, 3.4] \) and Theorem \([3.5]\)). It is obtained by semistandardizing the standard tableaux which parametrize the lowest weight vectors counting \( f^\lambda_{\mu \nu} \) in \([6]\), where the “lattice property” naturally arises from the configuration of entries in semistandard decomposition tableaux. We show that these tableaux for \( f^\lambda_{\mu \nu} \) are equal to LRS tableaux (Theorem \([3.11]\)), and hence obtain a new characterization of LRS tableaux.

We study other Schur \( P \)- or \( Q \)-positive expansions and their combinatorial descriptions from a viewpoint of crystals. First we consider the Schur \( P \)-positive expansion of a skew Schur function

\[
s_{\lambda/\delta_r} = \sum_{\nu \in \mathcal{P}^+} a_{\lambda/\delta_r \nu} P_\nu
\]

for a skew diagram \( \lambda/\delta_r \) contained in a rectangle \( ((r + 1)^{r+1}) \), where \( \delta_r = (r, r - 1, \ldots, 1) \) \([1]\). We give a combinatorial description of \( a_{\lambda/\delta_r \nu} \) (Theorem \([4.4]\)) by considering a \( q(n) \)-crystal structure on the set of usual semistandard tableaux of shape \( \lambda/\delta_r \) and characterizing the lowest weight vectors corresponding to each \( \nu \in \mathcal{P}^+ \).
As a byproduct we also give a simple alternate proof of Ardila-Serrano’s description of $a_{\lambda/\delta, \nu}$ [1] (Theorem 4.7), which can be viewed as a standardization of our description.

We next consider the Schur expansion of a Schur P-function

$$P_{\lambda} = \sum_{\mu} g_{\lambda \mu} s_{\mu}$$

for $\lambda \in \mathcal{P}^+$. It is equivalent to the expansion of a symmetric function $S_{\mu} = S_{\mu}(x, x)$ in terms of Schur Q-functions $Q_{\lambda} = 2^{\ell(\lambda)} P_{\lambda}$, where $S_{\mu}(x, y)$ is a super Schur function in variables $x$ and $y$. We give a simple and alternate proof of Stembridge’s description of $g_{\lambda \mu}$ [16] (Theorem 5.1) by characterizing the type A lowest weight vectors of weight $w_0 \mu$ in the $q(n)$-crystal $B_n(\lambda)$ when $\ell(\lambda), \ell(\mu) \leq n$.

Finally, we introduce the notion of semistandard decomposition tableaux of shifted skew shape. We consider a $q(n)$-crystal structure on the set of such tableaux, and describe its decomposition into $B_n(\lambda)$’s, which implies that the corresponding character has a Schur P-positive expansion though it is not equal to a skew Schur P-function in general.

The paper is organized as follows. In Section 2, we review the notion of $q(n)$-crystals and related results. In Section 3, we describe a combinatorial description of $f_{\mu \lambda}$ and show that it is equivalent to that of Stembridge. In Sections 4 and 5, we discuss the Schur P-positive expansion of a skew Schur function and the Schur expansion of a Schur P-function, respectively. In Section 6, we discuss semistandard decomposition tableaux of shifted skew shape, and the Schur P-positive expansions of their characters.

2. Crystals for queer Lie superalgebras

2.1. Notation and terminology. In this subsection, we introduce necessary notations and terminologies. Let $\mathbb{Z}_+$ be the set of non-negative integers. We fix a positive integer $n \geq 2$ throughout this paper.

Let $\mathcal{P} = \{ \lambda = (\lambda_i)_{i \geq 1} \mid \lambda_i \in \mathbb{Z}_+, \lambda_i \geq \lambda_{i+1} (i \geq 1), \sum_{i \geq 1} \lambda_i < \infty \}$ be the set of partitions, and let $\mathcal{P}^+ = \{ \lambda = (\lambda_i)_{i \geq 1} \mid \lambda \in \mathcal{P}, \lambda_i = \lambda_{i+1} \Rightarrow \lambda_i = 0 (i \geq 1) \}$ be the set of strict partitions. For $\lambda \in \mathcal{P}$, let $\ell(\lambda)$ denote the length of $\lambda$, and $|\lambda| = \sum_{i \geq 1} \lambda_i$. Let $\mathcal{P}_n = \{ \lambda \mid \ell(\lambda) \leq n \} \subseteq \mathcal{P}$ and $\mathcal{P}_n^+ = \mathcal{P}^+ \cap \mathcal{P}_n$.

The (unshifted) diagram of $\lambda \in \mathcal{P}$ is defined to be the set

$$D_{\lambda} = \{ (i, j) \in \mathbb{N}^2 : 1 \leq j \leq \lambda_i, 1 \leq i \leq \ell(\lambda) \},$$

and the shifted diagram of $\lambda \in \mathcal{P}^+$ is defined to be the set

$$D_{\lambda}^+ = \{ (i, j) \in \mathbb{N}^2 : i \leq j \leq \lambda_i + i - 1, 1 \leq i \leq \ell(\lambda) \}.$$
We identify $D_\lambda$ and $D^+_\lambda$ with diagrams where a box is placed at the $i$-th row from the top and the $j$-th column from the left for each $(i,j) \in D_\lambda$ and $D^+_\lambda$, respectively. For instance, if $\lambda = (6, 4, 2, 1)$, then

$$D_\lambda = \begin{array}{cccc}
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{array} \quad \text{and} \quad D^+_\lambda = \begin{array}{c}
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}
\end{array}.$$ 

Let $A$ be a linearly ordered set. We denote by $W_A$ the set of words of finite length with letters in $A$. For $w \in W_A$ and $a \in A$, let $c_a(w)$ be the number of occurrences of $a$ in $w$.

For $\lambda, \mu \in \mathcal{P}$ with $D_\mu \subseteq D_\lambda$, a tableau of shape $\lambda/\mu$ means a filling on the skew diagram $D_\lambda \setminus D_\mu$ with entries in $A$. For $\lambda, \mu \in \mathcal{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$, a tableau of shifted shape $\lambda/\mu$ is defined in a similar way. For a tableau $T$ of (shifted) shape $\lambda/\mu$, let $w(T)$ be the word given by reading the entries of $T$ row by row from top to bottom, and from right to left in each row. We denote by $w_{\text{rev}}(T)$ the reverse word of $w(T)$. Note that $T_{i,j}$ is not the entry of $T$ at the $(i,j)$-position of the (shifted) skew diagram of $\lambda/\mu$, that is, $(i,j) \in D_\lambda \setminus D_\mu$ or $(i,j) \in D^+_\lambda \setminus D^+_\mu$. For $a \in A$, let $c_a(T) = c_a(w(T))$ be the number of occurrences of $a$ in $T$.

Suppose that $A$ is a linearly ordered set with a $\mathbb{Z}_2$-grading $A = A_0 \sqcup A_1$. For $\lambda, \mu \in \mathcal{P}$ with $D_\mu \subseteq D_\lambda$, let $\text{SST}_A(\lambda/\mu)$ be the set of tableaux of shape $\lambda/\mu$ with entries in $A$ which is semistandard, that is, (i) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (ii) the entries in $A_0$ (resp. $A_1$) are strictly increasing in each column (resp. row). Similarly, for $\lambda, \mu \in \mathcal{P}^+$ with $D^+_\mu \subseteq D^+_\lambda$, we define $\text{SST}_A^+(\lambda/\mu)$ to be the set of semistandard tableaux of shifted shape $\lambda/\mu$ with entries in $A$.

Let $N = \{1' < 1 < 2' < 2 < \cdots \}$ be a linearly ordered set with a $\mathbb{Z}_2$-grading $N_0 = \mathbb{N}$ and $N_1 = \mathbb{N}' = \{1', 2', \cdots \}$. Put $[n] = \{1, \ldots, n\}$ and $[n]' = \{1', \ldots, n'\}$, where the $\mathbb{Z}_2$-grading and linear ordering are induced from $N$. For $a \in N$, we write $|a| = k$ when $a$ is either $k$ or $k'$.

### 2.2. Semistandard decomposition tableaux and Schur $P$-functions

Let us recall the notion of semistandard decomposition tableaux [6] [14], which is our main combinatorial object.

**Definition 2.1.**
(1) A word $u = u_1 \cdots u_s$ in $\mathcal{W}_N$ is called a hook word if it satisfies $u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_s$ for some $1 \leq k \leq s$. In this case, let $u_\downarrow = u_1 \cdots u_k$ be the weakly decreasing subword of maximal length and $u_\uparrow = u_{k+1} \cdots u_s$ the remaining strictly increasing subword in $u$.

(2) For $\lambda \in \mathcal{P}^+$, let $T$ be a tableau of shifted shape $\lambda$ with entries in $\mathbb{N}$. Then $T$ is called a semistandard decomposition tableau of shape $\lambda$ if

(i) $T^{(i)}$ is a hook word of length $\lambda_i$ for $1 \leq i \leq \ell(\lambda)$,
(ii) $T^{(i)}$ is a hook subword of maximal length in $T^{(i+1)}T^{(i)}$, the concatenation of $T^{(i+1)}$ and $T^{(i)}$, for $1 \leq i < \ell(\lambda)$.

For any hook word $u$, the decreasing part $u_\downarrow$ is always nonempty by definition.

For $\lambda \in \mathcal{P}^+$, let $SSDT(\lambda)$ be the set of semistandard decomposition tableaux of shape $\lambda$. Let $x = \{x_1, x_2, \ldots\}$ be a set of formal commuting variables, and let $P_\lambda = P_\lambda(x)$ be the Schur $P$-function in $x$ corresponding to $\lambda \in \mathcal{P}^+$ (see [10]). It is shown in [14] that $P_\lambda$ is given by the weight generating function of $SSDT(\lambda)$:

$$P_\lambda = \sum_{T \in SST(\lambda)} x^T,$$

where $x^T = \prod_{i \geq 1} x_i^{c_i(T)}$.

**Remark 2.2.** Recall that the Schur $P$-function $P_\lambda$ can be realized as the character of tableaux $T \in SST(\lambda)$ with no primed entry or entry of odd degree on the main diagonal (cf. [10] [11] [17]). The notion of semistandard decomposition tableaux was introduced in [14] to give a plactic monoid model for Schur $P$-functions. In this paper, we follow its modified version (Definition 2.1) introduced in [6], by which it is more easier to describe $q(n)$-crystals [6 Remark 2.6]. We also refer the reader to [6] for more details on relation between the combinatorics of these two models.

The following is a useful criterion for a tableau to be a semistandard decomposition one, which plays an important role in this paper.

**Proposition 2.3.** ([6 Proposition 2.3]) For $\lambda \in \mathcal{P}^+$, let $T$ be a tableau of shifted shape $\lambda$ with entries in $\mathbb{N}$. Then $T \in SST(\lambda)$ if and only if $T^{(k)}$ is a hook word for $1 \leq k \leq \ell(\lambda)$, and none of the following conditions holds for each $1 \leq k < \ell(\lambda)$:

1. $T_{k,1} \leq T_{k+1,i}$ for some $1 \leq i \leq \lambda_{k+1}$,
2. $T_{k+1,i} \geq T_{k+1,j} \geq T_{k,i+1}$ for some $1 \leq i < j \leq \lambda_{k+1}$,
3. $T_{k+1,j} < T_{k,i} < T_{j+1,i}$ for some $1 \leq i \leq j \leq \lambda_{k+1}$.

Equivalently, $T \in SST(\lambda)$ if and only if $T^{(k)}$ is a hook word for $1 \leq k \leq \ell(\lambda)$, and the following conditions hold for $1 \leq k < \ell(\lambda)$:

(a) if $T_{k,i} \leq T_{k+1,j}$ for $1 \leq i \leq j \leq \lambda_{k+1}$, then $i \neq 1$ and $T_{k+1,i-1} \leq T_{k+1,j}$,
(b) if $T_{k,i} > T_{k+1,j}$ for $1 \leq i \leq j \leq \lambda_{k+1}$, then $T_{k,i} \geq T_{k,j+1}$.

For $\lambda \in \mathcal{P}^+_n$, let $SSDT_n(\lambda)$ be the set of tableaux $T \in SSDT(\lambda)$ with entries in $[n]$. By Proposition 2.3, we see that $SSDT_n(\lambda) \neq \emptyset$ if and only if $\lambda \in \mathcal{P}^+_n$. We denote by $P_\lambda(x_1, \ldots, x_n)$ the Schur $P$-polynomial in $x_1, \ldots, x_n$ given by specializing $P_\lambda$ at $x_{n+1} = x_{n+2} = \cdots = 0$. Then we have $P_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSDT_n(\lambda)} x^T$. For $\lambda \in \mathcal{P}^+_n$, let $H_\lambda^\lambda$ be the element in $SSDT_n(\lambda)$ where the subtableau with entry $\ell(\lambda) - i + 1$ is a connected border strip of size $\lambda_{\ell(\lambda) - i + 1}$ starting at $(i, i) \in D_\lambda^+$ for each $i = 1, \ldots, \ell(\lambda)$, and let $L_\lambda^\lambda$ be the one where the subtableau with entry $n - i + 1$ is a connected horizontal strip of size $\lambda_i$ starting at $(i, i) \in D_\lambda^+$ for each $i = 1, \ldots, \ell(\lambda)$. For example, when $n = 4$ and $\lambda = (4, 3, 1)$, we have

\[
H_4^\lambda = \begin{array}{cccc}
3 & 2 & 2 & 1 \\
2 & 1 & 1 & \\
1 & 
\end{array} \quad \quad L_4^\lambda = \begin{array}{cccc}
4 & 4 & 4 & 4 \\
3 & 3 & 3 & \\
2 & 
\end{array}.
\]

Indeed, $H_\lambda^\lambda$ and $L_\lambda^\lambda$ are the unique tableaux in $SSDT_n(\lambda)$ such that

\[
(c_1(H_\lambda^\lambda), \ldots, c_n(H_\lambda^\lambda)) = \lambda, \quad (c_1(L_\lambda^\lambda), \ldots, c_n(L_\lambda^\lambda)) = w_0\lambda.
\]

Here we assume that $\mathcal{P}^+_n \subset \mathbb{Z}_n$ and the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Z}_n^+$ by permutation, where $w_0$ is the longest element in $\mathfrak{S}_n$.

2.3. Crystals. Let us first review the crystals for the general linear Lie algebra $\mathfrak{gl}(n)$ in [3, 4].

Let $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be the dual weight lattice and $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}) = \bigoplus_{i=1}^n \mathbb{Z}e_i$ the weight lattice with $\langle e_i, e_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. Define a symmetric bilinear form $(\cdot | \cdot)$ on $P$ by $(e_i | e_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. Let $\{ \alpha_i = e_i - e_{i+1} \; (i = 1, \ldots, n-1) \}$ be the set of simple roots, and $\{ h_i = e_i - e_{i+1} \; (i = 1, \ldots, n-1) \}$ the set of simple coroots of $\mathfrak{gl}(n)$. Let $P^+ = \{ \lambda | \lambda \in P, \langle \lambda, h_i \rangle \geq 0 \; (i = 1, \ldots, n-1) \}$ be the set of dominant integral weights.

A $\mathfrak{gl}(n)$-crystal is a set $B$ together with the maps $\text{wt} : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{ -\infty \}$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{ 0 \}$ for $i = 1, \ldots, n-1$ satisfying the following conditions: for $b \in B$ and $i = 1, \ldots, n-1$,

1. $\varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b)$,
2. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \; \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$,
3. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \; \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
4. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b' \in B$,
5. $\tilde{e}_i b = \tilde{f}_i b = 0$ when $\varphi_i(b) = -\infty$.

Here 0 is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{ -\infty \}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. For $\mu \in P$, let $B_\mu = \{ b \in B \mid \text{wt}(b) = \mu \}$. When $B_\mu$
is finite for all \( \mu \), we define the character of \( B \) by
\[
\text{ch} B = \sum_{\mu \in P} |B_\mu| e^\mu, \quad \text{where } e^\mu \text{ is a basis element of the group algebra } \mathbb{Q}[P].
\]

Let \( B_1 \) and \( B_2 \) be \( \mathfrak{gl}(n) \)-crystals. A tensor product \( B_1 \otimes B_2 \) is a \( \mathfrak{gl}(n) \)-crystal, which is defined to be \( B_1 \times B_2 \) as a set with elements denoted by \( b_1 \otimes b_2 \), where

\[
\begin{align*}
\text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle \}, \\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2) \},
\end{align*}
\]

(2.2)

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\varepsilon_i(b_1) \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), 
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), 
\end{cases}
\end{align*}
\]

for \( i = 1, \ldots, n - 1 \). Here we assume that \( 0 \otimes b_2 = b_1 \otimes 0 = 0 \).

For \( \lambda \in \mathcal{P}_n \), let \( B_n(\lambda) \) be the crystal associated to an irreducible \( \mathfrak{gl}(n) \)-module with highest weight \( \lambda \), where we regard \( \lambda \) as \( \sum_{i=1}^n \lambda_i e_i \subset P^+ \). We may regard \( [n] \) as \( B_n(\varepsilon_1) \), where \( \text{wt}(k) = \varepsilon_k \) for \( k \in [n] \), and hence \( W_{[n]} \) as a \( \mathfrak{gl}(n) \)-crystal where we identify \( w = w_1 \ldots w_r \) with \( w_1 \otimes \cdots \otimes w_r \in B_n(\varepsilon_1)^{\otimes r} \). The crystal structure on \( W_{[n]} \) is easily described by so-called the signature rule (cf. [9] Section 2.1). For \( \lambda \in \mathcal{P}_n \), the set \( \text{SST}_{[n]}(\lambda) \) becomes a \( \mathfrak{gl}(n) \)-crystal under the identification of \( T \) with \( w(T) \in W_{[n]} \), and it is isomorphic to \( B_n(\lambda/\mu) \) [9]. In general, one can define a \( \mathfrak{gl}(n) \)-crystal structure on \( \text{SST}_{[n]}(\lambda/\mu) \) for a skew diagram \( \lambda/\mu \). By abuse of notation, we set \( B_n(\lambda/\mu) := \text{SST}_{[n]}(\lambda/\mu) \).

Next, let us review the notion of crystals associated to polynomial representations of the queer Lie superalgebra \( \mathfrak{q}(n) \) developed in [6 7].

**Definition 2.4.** A \( \mathfrak{q}(n) \)-crystal is a set \( B \) together with the maps \( \text{wt} : B \to P, \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\} \) and \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \) for \( i \in I := \{1, \ldots, n-1, \bar{1}\} \) satisfying the following conditions:

1. \( B \) is a \( \mathfrak{gl}(n) \)-crystal with respect to \( \text{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i \) for \( i = 1, \ldots, n - 1 \),
2. \( \text{wt}(b) \in \bigoplus_{i \in [n]} \mathbb{Z} \varepsilon_i \) for \( b \in B \),
3. \( \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i, \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i \) for \( b \in B \),
4. \( \tilde{f}_i b = b' \) if and only if \( b = \tilde{e}_i b' \) for all \( b, b' \in B \),
5. for \( 3 \leq i \leq n - 1 \), we have
   (i) the operators \( \tilde{e}_i \) and \( \tilde{f}_i \) commute with \( \tilde{e}_i, \tilde{f}_i \),
   (ii) if \( \tilde{e}_i b \in B \), then \( \varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) \) and \( \varphi_i(\tilde{e}_i b) = \varphi_i(b) \).
Let $B_n$ be a $q(n)$-crystal which is the $\mathfrak{gl}(n)$-crystal $B_n(\epsilon_1)$ together with $\tilde{f}_T[1] = [2]$ (in dashed arrow): 

\[
\begin{array}{cccccc}
1 & \cdots & 2 & \cdots & 3 & \cdots & n-1 & \cdots & n \\
\tilde{f}_T & & & & & & & & \\
\end{array}
\]

Here we write $b \stackrel{i}{\rightarrow} b'$ if $\tilde{f}_ib = b'$ for $b, b' \in B$ and $i \in I \setminus \{T\}$ as usual, and $b \stackrel{T}{\rightarrow} b'$ if $\tilde{f}_Tb = b'$.

For $q(n)$-crystals $B_1$ and $B_2$, the tensor product $B_1 \otimes B_2$ is the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ where the actions of $\tilde{e}_T$ and $\tilde{f}_T$ are given by

\[
\tilde{e}_T(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_Tb_1 \otimes b_2, & \text{if } \langle \epsilon_1, \text{wt}(b_2) \rangle = \langle \epsilon_2, \text{wt}(b_2) \rangle = 0, \\
b_1 \otimes \tilde{e}_Tb_2, & \text{otherwise},
\end{cases}
\]

(2.3)

\[
\tilde{f}_T(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_Tb_1 \otimes b_2, & \text{if } \langle \epsilon_1, \text{wt}(b_2) \rangle = \langle \epsilon_2, \text{wt}(b_2) \rangle = 0, \\
b_1 \otimes \tilde{f}_Tb_2, & \text{otherwise}.
\end{cases}
\]

Then it is easy to see that $B_1 \otimes B_2$ is a $q(n)$-crystal. In particular, $W_{[n]}$ is also a $q(n)$-crystal.

Let $B$ be a $q(n)$-crystal. Suppose that $B$ is a regular $\mathfrak{gl}(n)$-crystal, that is, each connected component in $B$ is isomorphic to $B_n(\lambda)$ for some $\lambda \in \mathcal{P}_n$. Let $W = \mathfrak{S}_n$ be the Weyl group of $\mathfrak{gl}(n)$ which is generated by the simple reflection $r_i$ corresponding to $\alpha_i$ for $i = 1, \ldots, n - 1$. We have a group action of $W$ on $B$ denoted by $S$ such that

\[
S_{r_i}(b) = \begin{cases} 
\tilde{f}_i^{[\text{wt}(b), h_i]}b, & \text{if } \langle \text{wt}(b), h_i \rangle \geq 0, \\
\tilde{e}_i^{[\text{wt}(b), h_i]}b, & \text{if } \langle \text{wt}(b), h_i \rangle \leq 0,
\end{cases}
\]

for $b \in B$ and $i = 1, \ldots, n - 1$. For $2 \leq i \leq n - 1$, let $w_i \in W$ be such that $w_i(\alpha_i) = \alpha_1$, and let

(2.4)

\[
\tilde{e}_T = S_{w_i^{-1}} \tilde{e}_T S_{w_i}, \quad \tilde{f}_T = S_{w_i^{-1}} \tilde{f}_T S_{w_i}.
\]

For $b \in B$, we say that $b$ is a $q(n)$-highest weight vector if $\tilde{e}_ib = \tilde{e}_Tb = 0$ for $1 \leq i \leq n - 1$, and $b$ is a $q(n)$-lowest weight vector if $S_{w_0}b$ is a $q(n)$-highest weight vector.

For $\lambda \in \mathcal{P}^+$, let $B_n(\lambda) = SSDT_n(\lambda)$, and consider an injective map

(2.5)

\[
\begin{array}{ccc}
B_n(\lambda) & \xrightarrow{\iota} & W_{[n]} \\
T & \xrightarrow{w_{\text{rev}}(T)} & w_{\text{rev}}(T).
\end{array}
\]

Then we have the following.

**Theorem 2.5.** ([6] Theorem 2.5]) Let $\lambda \in \mathcal{P}_n^+$ be given.
(a) The image of $B_n(\lambda)$ in (2.3) together with $\{0\}$ is invariant under the action of $\bar{e}_i$ and $\bar{f}_i$ for $i \in I$, and hence $B_n(\lambda)$ is a $q(n)$-crystal.

(b) The $q(n)$-crystal $B_n(\lambda)$ is connected where $H^\lambda_n$ is a unique $q(n)$-highest weight vector and $L^\lambda_n$ is a unique $q(n)$-lowest weight vector.

Remark 2.6. In [7], a semisimple tensor category over the quantum superalgebra $U_q(q(n))$ is introduced, and it is shown that each irreducible highest weight module $V_n(\lambda)$ in this category, parametrized by $\lambda \in \mathcal{P}_n^+$, has a crystal base. Furthermore, it is shown in [6, Theorem 2.5(c)] that the crystal of $V_n(\lambda)$ is isomorphic to $B_n(\lambda)$.

Let $B_1$ and $B_2$ be $q(n)$-crystals. For $b_1 \in B_1$ and $b_2 \in B_2$, let us say that $b_1$ and $b_2$ are equivalent and write $b_1 \equiv b_2$ if there exists an isomorphism of $q(n)$-crystals $\psi : C(b_1) \rightarrow C(b_2)$ such that $\psi(b_1) = b_2$ where $C(b_i)$ denotes the connected component of $b_i \in B_i$ ($i = 1, 2$) as a $q(n)$-crystal.

![Diagram of q(3)-crystal B3(3,1)](image)

Figure 1. The $q(3)$-crystal $B_3(3,1)$
By [7] Theorem 4.6, each connected component in $B_n^\otimes (N \geq 1)$ is isomorphic to $B_n(\lambda)$ for some $\lambda \in \mathcal{P}_n^+$ with $|\lambda| = N$. Indeed, for $b = b_1 \otimes \cdots \otimes b_N \in B_n^\otimes$, there exists a unique $\lambda \in \mathcal{P}_n^+$ and $T \in B_n(\lambda)$ such that $b \equiv T$. In particular, $b$ is a $q(n)$-lowest (resp. $q(n)$-highest) weight vector if and only if $b \equiv L^\lambda_n$ (resp. $H^\lambda_n$).

The following lemma plays a crucial role in characterization of $q(n)$-lowest weight vectors in $B_n^\otimes$ and hence describing the decompositions of $B_n^\otimes$ and $B_n(\mu) \otimes B_n(\nu)$ ($\mu, \nu \in \mathcal{P}_n^+$) into connected components in [6].

**Lemma 2.7.** ([6] Lemma 1.15, Corollary 1.16]) For $b = b_1 \otimes \cdots \otimes b_N \in B_n^\otimes$, the following are equivalent:

1. $b$ is a $q(n)$-lowest weight vector;
2. $b' = b_2 \otimes \cdots \otimes b_N$ is a $q(n)$-lowest weight vector and $\varepsilon_{b_1} + \text{wt}(b') \in w_0\mathcal{P}_n^+$, where $w_0$ is the longest element of $S_n$.
3. $\text{wt}(b_M \otimes \cdots \otimes b_N) \in w_0\mathcal{P}_n^+$ for all $1 \leq M \leq N$.

Hence, we have the following immediately by Lemma 2.7.

**Corollary 2.8.** For $\lambda^{(1)}, \ldots, \lambda^{(s)} \in \mathcal{P}_n^+$ and $T_1 \otimes \cdots \otimes T_s \in B_n(\lambda^{(1)}) \otimes \cdots \otimes B_n(\lambda^{(s)})$, the following are equivalent:

1. $T_1 \otimes \cdots \otimes T_s$ is a $q(n)$-lowest weight vector,
2. $T_r \otimes \cdots \otimes T_s \in B_n(\lambda^{(r)}) \otimes \cdots \otimes B_n(\lambda^{(r)})$ is a $q(n)$-lowest weight vector for all $1 \leq r \leq s$.

Note that we do not have an analogue of Lemma 2.7 for $q(n)$-highest weight vectors.

**Remark 2.9.** Let $m \geq n$ be a positive integer, and put $t = m - n$. For $N \geq 1$, let $\psi_t : B_m^\otimes \rightarrow B_n^\otimes$ be the map given by $\psi_t(u_1 \otimes \cdots \otimes u_N) = (u_1 + t) \otimes \cdots \otimes (u_N + t)$. Then for $\lambda \in \mathcal{P}_n^+$ and $u \in B_m^\otimes$ we have $u \equiv L^\lambda_m$ if and only if $\psi_t(u) \equiv L^\lambda_n$. This implies that the multiplicity of $B_n(\lambda)$ in $B_m^\otimes$ is equal to that of $B_m(\lambda)$ in $B_n^\otimes$ for $\lambda \in \mathcal{P}_n^+$.

3. **Littlewood-Richardson rule for Schur $P$-functions**

For $\lambda, \mu, \nu \in \mathcal{P}_n^+$, the shifted Littlewood-Richardson coefficients $f_{\mu \nu}^\lambda$, are the coefficients given by

$$P_\mu P_\nu = \sum_\lambda f_{\mu \nu}^\lambda P_\lambda.$$  \hfill (3.1)

In this section we give a new combinatorial description of $f_{\mu \nu}^\lambda$ using the theory of $q(n)$-crystals. We also show that our description of $f_{\mu \nu}^\lambda$ is equivalent to the Stembridge’s description [10].
3.1. Shifted Littlewood-Richardson rule.

**Definition 3.1.** Let \( w = w_1 \cdots w_N \) be a word in \( W_N \). Let \( m_k = c_k(w) + c_k'(w) \) for \( k \geq 1 \). We define \( w^* = w_1^* \cdots w_N^* \) to be the word obtained from \( w \) after applying the following steps for each \( k \geq 1 \):

1. Consider the letters \( w_i \)'s with \( |w_i| = k \). Label them with 1, 2, \ldots, \( m_k \) (as subscripts), first enumerating the \( w_p \)'s with \( w_p = k \) from left to right, and then \( w_q \)'s with \( w_q = k' \) from right to left.
2. After the step (1), remove all ‘‘ in each labeled letter \( k_j \), that is, replace any \( k_j \) with \( k_j \) for \( c_k(w) < j \leq m_k \).

**Example 3.2.**

\[
\begin{align*}
  w &= 11'1'11' & \rightarrow & 11_1'1_21_3'1_4'1_5 & \rightarrow & w^* = 11_11_21_41_5' \\
  w &= 21'12'2_12_1 & \rightarrow & 2_11_4'2_3'2_21_3 & \rightarrow & w^* = 2_11_41_22_21_3
\end{align*}
\]

**Definition 3.3.** Let \( w = w_1 \cdots w_N \in W_N \) be given. We say that \( w \) satisfies the “lattice property” if the word \( w^* = w_1^* \cdots w_N^* \) associated to \( w \) given in Definition 3.1 satisfies the following for \( k \geq 1 \):

1. \((L1)\) if \( w_i^* = k_1 \), then no \( k + 1 \) \( j \) for \( j \geq 1 \) occurs in \( w_1^* \cdots w_{i-1}^* \).
2. \((L2)\) if \( (w_s^*, w_t^*) = (k + 1, i, k_{i+1}) \) for some \( s < t \) and \( i \geq 1 \), then no \( k + 1 \) \( j \) for \( i < j \) occurs in \( w_s^* \cdots w_t^* \).
3. \((L3)\) if \( (w_s^*, w_t^*) = (k_{j+1}, k + 1) \) for some \( s < t \) and \( j \geq 1 \), then no \( k_i \) for \( i \leq j \) occurs in \( w_s^* \cdots w_t^* \).

**Definition 3.4.** For \( \lambda, \mu, \nu \in \mathcal{P}^+ \), let \( F^\lambda_{\mu\nu} \) be the set of tableaux \( Q \) such that

1. \( Q \in SST^+_N(\lambda/\mu) \) with \( c_k(Q) + c_k'(Q) = \nu_k \) for \( k \geq 1 \),
2. \( k \geq 1 \), if \( x \) is the rightmost letter in \( w(Q) \) with \( |x| = k \), then \( x = k \),
3. \( w(Q) \) satisfies the “lattice property” in Definition 3.3.

Then we have the following characterization of \( f^\lambda_{\mu\nu} \).

**Theorem 3.5.** For \( \lambda, \mu, \nu \in \mathcal{P}^+ \), we have

\[ f^\lambda_{\mu\nu} = \left| F^\lambda_{\mu\nu} \right|, \]

that is, the shifted LR coefficient \( f^\lambda_{\mu\nu} \) is equal to the number of tableaux in \( F^\lambda_{\mu\nu} \).

**Proof.** Choose \( n \) such that \( \lambda, \mu, \nu \in \mathcal{P}^+_n \). Put

\[
(3.2) \quad L^\lambda_{\mu\nu} = \{ T | T \in B_n(\nu), \ T \otimes L^\mu_n \equiv L^\lambda_n \}.
\]

By Corollary 2.8, we have

\[
(3.3) \quad B_n(\nu) \otimes B_n(\mu) \cong \bigsqcup_{\lambda \in \mathcal{P}^+_n} B_n(\lambda)^{\oplus |L^\lambda_{\mu\nu}|}.
\]
Hence we have $|L_{\mu\nu}^\lambda| = f_{\mu\nu}^\lambda = f_{\nu\mu}^\lambda$ from (2.1) and the linear independence of Schur $P$-polynomials $P_\lambda(x_1, \ldots, x_n)$’s.

Let us prove $f_{\mu\nu}^\lambda = |F_{\mu\nu}^\lambda|$ by constructing a bijection

\[(3.4) \quad L_{\mu\nu}^\lambda \xrightarrow{\text{bij}} F_{\mu\nu}^\lambda, \quad T \xrightarrow{\text{bij}} QT.\]

Let $T \in L_{\mu\nu}^\lambda$ be given. Assume that $w_{\text{rev}}(T) = u_1 \cdots u_N$ where $N = |\nu|$. By Lemma 2.7 there exists $\mu^{(m)} \in \mathcal{P}_n^+$ for $1 \leq m \leq N$ such that

(i) $(u_{N-m+1} \cdots u_N) \otimes L_n^\mu \equiv L_n^\mu^{(m)}$ and $\mu^{(N)} = \lambda$,

(ii) $\mu^{(m)}$ is obtained by adding a box in the $(n-u_m+1)$-st row of $\mu^{(m-1)}$.

Here we assume that $\mu^{(0)} = \mu$. Recall that

\[w_{\text{rev}}(T) = T^{(\ell(\nu))} \cdots T^{(1)},\]

where $T^{(k)} = T_{k,1} \cdots T_{k,\lambda_k}$ is a hook word for $1 \leq k \leq \ell(\nu)$. We define $QT$ to be a tableau of shifted shape $\lambda/\mu$ with entries in $\mathbb{N}$, where $\mu^{(m)}/\mu^{(m-1)}$ is filled with

\[(3.5) \quad \begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)\downarrow}, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)\downarrow}, \end{cases}\]

for some $1 \leq k \leq \ell(\nu)$. In other words, the boxes in $QT$ corresponding to $T^{(k)\downarrow}$ are filled with $k'$ from right to left as a vertical strip and then those corresponding to $T^{(k)\uparrow}$ are filled with $k$ from left to right as a horizontal strip.

By construction, it is clear that $QT \in SST_N^{\lambda/\mu}$ with $c_{k'}(QT) + c_k(QT) = \nu_k$ for $1 \leq k \leq \ell(\nu)$. Let $w(Q_T) = w_1 \cdots w_N$. Since $T^{(k)}$ is a hook word for each $k$ and the rightmost letter, say $u_m$, in $T^{(k)\downarrow}$ is strictly smaller than the leftmost letter $u_{m+1}$ in $T^{(k)\downarrow}$, the entry $k$ in $QT$ corresponding to $u_m$ is located to the southeast of all $k'$’s in $QT$. So the conditions Definition 3.3(1) and (2) are satisfied.

It remains to check that $w(Q_T)$ satisfies the “lattice property”. Note that if we label $k$ and $k'$ in (3.5) as $k_j$ and $k'_j$, respectively when $u_m = T_{k,j}$, then it coincides with the labeling on the letters in $w(Q_T)$ given in Definition 3.1(1). Now it is not difficult to see that the conditions Proposition 2.3(1), (2), and (3) on $T$ implies the conditions Definition 3.3(L1), (L2), and (L3), respectively. Therefore, $QT \in F_{\mu\nu}^\lambda$.

Finally the correspondence $T \mapsto QT$ is injective and also reversible. Hence the map (3.4) is a bijection. This completes the proof.

□

**Remark 3.6.** We see from Remark 2.9 that $|L_{\mu\nu}^\lambda|$ does not depend on $n$ for all sufficiently large $n$. Hence (3.3) also implies the Schur $P$-positivity of the product $P_\mu P_\nu$. 
Remark 3.7. For $T \in L^\lambda_{\mu \nu}$, let $\hat{Q}_T$ be the tableau of shifted shape $\lambda/\mu$, which is defined in the same way as $Q_T$ in the proof of Theorem 3.5 except that we fill $\mu^{(m)}/\mu^{(m-1)}$ with $m$ in (3.5) for $1 \leq m \leq N$. Then the set \{ $\hat{Q}_T \mid T \in L^\lambda_{\mu \nu}$ \} is equal to the one given in [6, Theorem 4.13] to describe $f^\lambda_{\mu \nu}$. For example,

$$\begin{align*}
T_1 &= \begin{array}{ccc}
3 & 3 & 1 \\
2 & & \\
\end{array} \in L^{(4,3,1)}_{(3,1)(3,1)} & \quad \hat{Q}_{T_1} = \begin{array}{c}
1 \\
2 & 3 \\
4 \\
\end{array} & \quad Q_{T_1} = \begin{array}{c}
1' \\
2 \\
\end{array} \\
T_2 &= \begin{array}{ccc}
4 & 2 & 3 \\
3 & & \\
\end{array} \in L^{(4,3,1)}_{(3,1)(3,1)} & \quad \hat{Q}_{T_2} = \begin{array}{c}
3 \\
1 & 4 \\
2 \\
\end{array} & \quad Q_{T_2} = \begin{array}{c}
1' \\
1 \\
\end{array}
\end{align*}$$

3.2. Stembridge’s description of $f^\lambda_{\mu \nu}$.

Definition 3.8. Let $w = w_1 \cdots w_N$ be a word in $W_N$ and $w_\text{rev}$ be the reverse word of $w$. Let $\tilde{w}$ be the word obtained from $w$ by replacing $k$ by $(k + 1)'$ and $k'$ by $k$ for each $k \geq 1$. Suppose that $w\tilde{w}_\text{rev} = a_1 \cdots a_{2N}$, and let $m_k(i) = c_k(a_1 \cdots a_i)$ for $k \geq 1$ and $0 \leq i \leq 2N$. Then we say that $w$ satisfies the lattice property if

$$(3.6) \quad m_k+1(i) = m_k(i) \text{ implies } |a_{i+1}| \neq k + 1 \text{ for } k \geq 1 \text{ and } i \geq 0.$$ 

Here we assume that $m_k(0) = 0$.

Definition 3.9. For $\lambda, \mu, \nu \in \mathcal{P}^+$, let $\text{LRS}^\lambda_{\mu \nu}$ be the set of tableaux $Q$ such that

1. $Q \in \text{SST}^+_N(\lambda/\mu)$ with $c_k(Q) + c_{\nu'}(Q) = \nu_k$ for $k \geq 1$,
2. for $k \geq 1$, if $x$ is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,
3. $w(Q)$ satisfies the lattice property in Definition 3.8.

We call $\text{LRS}^\lambda_{\mu \nu}$ the set of Littlewood-Richardson-Stembridge tableaux.

Theorem 3.10. ([16, Theorem 8.3]) For $\lambda, \mu, \nu \in \mathcal{P}^+$, we have

$$f^\lambda_{\mu \nu} = \left| \text{LRS}^\lambda_{\mu \nu} \right|,$$

that is, the shifted LR coefficient $f^\lambda_{\mu \nu}$ is equal to the number of tableaux in $\text{LRS}^\lambda_{\mu \nu}$.

Theorem 3.11. For $\lambda, \mu, \nu \in \mathcal{P}^+$, we have

$$F^\lambda_{\mu \nu} = \text{LRS}^\lambda_{\mu \nu}.$$ 

Proof. Since Definition 3.3(1) and (2) are the same as Definition 3.9(1) and (2), respectively, it suffices to show that for any $Q \in \text{SST}^+_N(\lambda/\mu)$, $w := w(Q)$ satisfies the “lattice property” in Definition 3.8 if and only if $w$ satisfies the lattice property in Definition 3.8. We assume that $N = |\nu|$, $w = w_1 \cdots w_N$, $w^* = w^*_1 \cdots w^*_N$, and $w\tilde{w}_\text{rev} = a_1 \cdots a_{2N}$.
Suppose that \( w \) satisfies the “lattice property” in Definition 3.3. We use induction on \( 1 \leq i \leq 2N \) to show that \( a_1 \cdots a_{2N} \) satisfies (3.6). We first observe from (L1) that \( a_1 = 1 \) or \( 1' \), and \( a_1 \) satisfies (3.6) since \( m_k(0) = 0 \) for all \( k \geq 1 \).

We now assume that \( a_1 \cdots a_i \) for some \( 1 \leq i < 2N \) satisfies (3.6). Suppose for the sake of contradiction that \( m_{k+1}(i) = m_k(i) = m \) and \( |a_{i+1}| = k + 1 \) for some \( k \geq 1 \). Here \( m > 0 \) by (L1). By induction hypothesis, there exist \( s < t \leq i \) such that \( a_s = k \) with \( m_k(s) = m \) and \( a_t = k + 1 \) with \( m_{k+1}(t) = m \). Note that for each \( k \geq 1 \)

\[
(3.7) \quad c_k(w) + c_k(w) > c_{k+1}(w) + c_{(k+1)'}(w),
\]

which implies that the number of \( k \)'s in \( w \wedge \text{rev} \) is greater than the number of \( k + 1 \)'s in \( w \wedge \text{rev} \). So we can choose an integer \( u > i + 1 \) such that \( a_u = k \) and \( m_k(u) = m + 1 \).

We now consider the following four cases:

**Case 1.** Let \( 1 \leq s < t < i + 1 \leq N \). In this case \( w^*_{s} = k_m, w^*_{t} = k + 1_m \) and \( w^*_{s+1} = k + 1_M \) for some \( M \geq m + 1 \). (i) If \( u \leq N \), then we have \( (w^*_{s}, w^*_{s+1}, w^*_{u}) = (k + 1_m, k + 1_M, k_{m+1}) \), which contradicts (L2). (ii) If \( N < u < 2N - s + 1 \), then \( a_{2N-u+1} = w_{2N-u+1} = k' \) \((s < 2N - u + 1 \leq N)\) but no \( k \) occurs in \( w_{s+1} \cdots w_N \) which contradicts Definition 3.4(2). (iii) If \( 2N - s + 1 < u \leq 2N \), then we have \( (w^*_{2N-u+1}, w^*_{u}, w^*_{t}) = (k_{m+1}, k_m, k + 1_m) \), which contradicts (L3).

**Case 2.** Let \( 1 \leq s < t \leq N \leq i + 1 \leq 2N \). In this case \( w^*_{s} = k_m \) and \( w^*_{t} = k + 1_m \). Since \( w_{2N-u+1} = k' \) \((2N - u + 1 < N)\), we have \( s \neq 2N - u + 1 \). (i) If \( s < 2N - u + 1 \), then we have \( w_{2N-u+1} = k' \) but no \( k \) in \( w_{s+1} \cdots w_n \) since \( m_k(i) = m \), which contradicts Definition 3.4(2). (ii) If \( 2N - u + 1 < s \), then we have \( (w^*_{2N-u+1}, w^*_{s}, w^*_{t}) = (k_{m+1}, k_m, k + 1_m) \), which contradicts (L3).

**Case 3.** Let \( 1 \leq s \leq N < t < i + 1 \leq 2N \). In this case \( w^*_{s} = k_m, w^*_{2N-t+1} = k + 1_M \). If \( a_{i+1} = (k+1) \), then \( a_{2N-i} = k \) but it is impossible from the assumption \( m_k(i) = m \). So \( a_{i+1} = k + 1 \) and \( w^*_{2N-i} = k + 1_m \). (i) If \( s < 2N - t + 1 \), then we have \( w_{2N-t+1} = (k+1)' \) \((2N - t + 1 \leq N)\) but no \( k \) in \( w_{2N-t+2} \cdots w_N \) since \( m_{k+1}(i) = m \), which contradicts Definition 3.3(2). (ii) If \( 2N - t + 1 < s \), then by (3.7) there is an integer \( v > u \) such that \( a_v = k \) and \( m_k(v) = m + 2 \). So we have \( (w^*_{2N-v+1}, w^*_{2N-u+1}, w^*_{2N-i}) = (k_{m+2}, k_{m+1}, k + 1_m) \), which contradicts (L3).

**Case 4.** Let \( N < s < t < i + 1 \leq 2N \). In this case \( w^*_{2N-s+1} = k_m \) and \( w^*_{2N-t+1} = k + 1_m \). (i) If \( a_{i+1} = k + 1 \), then \( w_{2N-i} = (k+1)' \) and \( w^*_{2N-i} = k + 1_m \). By (3.7) there is an integer \( v > u \) such that \( a_v = k \) and \( m_k(v) = m + 2 \). So we have \( (w^*_{2N-v+1}, w^*_{2N-u+1}, w^*_{2N-i}) = (k_{m+2}, k_{m+1}, k + 1_m) \), which contradicts (L3). (ii) If \( a_{i+1} = (k+1)' \), then \( w_{2N-i} = k \) and \( w^*_{2N-i} = k_M \) for some \( M < m \). So we have \( (w^*_{2N-u+1}, w^*_{2N-i}, w^*_{2N-t+1}) = (k_{m+1}, k_M, k + 1_m) \), which contradicts (L3).

Conversely, we assume that \( w \) satisfies the lattice property in Definition 3.8. We first claim that \( w \) satisfies (L1). Given \( k \geq 1 \), let \( w^*_i = k_1 \) for some \( 1 \leq i \leq N \). If
$w* = k + 1$ for some $1 \leq j < i$, then it follows that $m_k(j - 1) = m_{k+1}(j-1) = 0$ and $a_j = k + 1$, which contradicts (3.6). Hence $w$ satisfies (L1).

Next, we claim that $w$ satisfies (L2). Suppose that there is a triple $(w^*_s, w^*_u, w^*_t) = (k+1, k, k+1, k_i+1)$ for some $k \geq 1$, $i < j$, and $1 \leq s < u < t \leq N$. We may assume that $j = i + 1$. Since $w^*_s = k + 1$ is placed to the left of $w^*_u = k + 1_{i+1}$, it follows from Definition 3.3 that $a_s = k + 1$, and from Definition 3.9(2) that $a_u = k + 1$. Since $w$ satisfies the lattice property, there is a positive integer $v < s$ such that $w^*_u = k_i$, i.e., $a_v = k$ and $m_k(v) = i$ for some $v < s$. We have $m_{k+1}(u - 1) = m_k(u - 1) = i$ and $a_u = k + 1$, a contradiction. So $w$ satisfies (L2).

Finally, we claim that $w$ satisfies (L3). Suppose for the sake of contradiction that $(w^*_s, w^*_u, w^*_t) = (k_{j+1}, k_i, k + 1, j)$ for some $k \geq 1$, $i \leq j$, and $1 \leq s < u < t \leq N$. We may assume that $i = j$. Since $w^*_u = k_{j+1}$ is placed to the left of $w^*_k = k_j$, it follows that $a_s = k'$. We consider four cases depending on the primedness of $a_u$ and $a_t$ as follows:

Case 1. Let $a_u = k'$ and $a_t = (k + 1)'$. It follows that $a_{2N-u+1} = k (m_k(2N - u + 1) = j)$ and $a_{2N-t+1} = k + 1 (m_k(2N - u + 1) = j)$. So we have $m_{k+1}(2N - t) = m_k(2N - t) = j - 1$ and $a_{2N-t+1} = k + 1$, as desired.

Case 2. Let $a_u = k'$ and $a_t = k + 1$. It follows that $a_{2N-u+1} = k, m_k(2N - u + 1) = j$ and $m_{k+1}(t) = j$. Since $t < 2N - u + 1$, we have $m_k(t) < m_{k+1}(t)$. So there is an integer $0 \leq \hat{t} < t$ such that $m_k(\hat{t}) = m_{k+1}(\hat{t}) < j$ and $a_{t+1} = k + 1$, as desired.

Case 3. Let $a_u = k$ and $a_t = (k + 1)'$. It follows that $a_{2N-t+1} = k + 1$ and $m_{k+1}(2N - t + 1) = j$. From $m_k(2N - s + 1) = j + 1$ and $2N - t + 1 < 2N - s + 1$ we have $m_k(2N - t + 1) = m_{k+1}(2N - t + 1) = j$. If there is another $k + 1$ between $w^*_s$ and $w^*_u$, then we obtain the desired contradiction. Otherwise, $m_k(2N - u) = m_{k+1}(2N - u)$ and $a_{2N-u+1} = (k + 1)'$, as desired.

Case 4. Let $a_u = k$ and $a_t = k + 1$. From $a_s = k'$ ($w^*_s = k_{j+1}$) it follows that $m_k(2N - u) = j$. If $m_{k+1}(2N - u) = j$, from $a_{2N-u+1} = (k + 1)'$ we get a contradiction. If $m_{k+1}(2N - u) > j$, by choosing the smallest integer $\hat{t} > t$ such that $m_{k+1}(\hat{t}) = j + 1$ this leads to a contradiction.

Indeed, we have shown in the proof of Theorem 3.11 that

**Corollary 3.12.** Let $w \in \mathcal{W}_N$ be such that

1. $(c_k(Q) + c_k^e(Q))_{k \geq 1} \in \mathcal{P}^+$,

2. for $k \geq 1$, if $x$ is the rightmost letter in $w$ with $|x| = k$, then $x = k$.

Then $w$ satisfies the “lattice property” in Definition 3.3 if and only if $w$ satisfies the lattice property in Definition 7.8.

**Remark 3.13.** A bijection from $\text{LRS}^\lambda_{\mu\nu}$ to $\text{L}_m^\lambda_{\mu\nu}$ is also given in [4, Theorem 4.7], which coincides with the inverse of the map $T \mapsto Q_T$ in (3.4) (see also the remarks
in [4, p.82]). The proof of [4, Theorem 4.7] use insertion schemes for two versions of semistandard decomposition tableaux and another combinatorial model for $f^\lambda_{\mu\nu}$ by Cho [3] as an intermediate object between LRS$^\lambda_{\mu\nu}$ and $L^\lambda_{\mu\nu}$.

On the other hand, we prove more directly that the map $T \mapsto Q_T$ in (3.4) is a bijection from $L^\lambda_{\mu\nu}$ to LRS$^\lambda_{\mu\nu}$ by using a new characterization of the lattice property in Theorem 3.11.

4. Schur $P$-expansions of skew Schur functions

4.1. The Schur $P$-expansion of $s_{\lambda/\delta_r}$. For $r \geq 0$, let us denote by $\delta_r$ the partition $(r, r - 1, \ldots, 1)$ if $r \geq 1$, and $(0)$ if $r = 0$. We fix a non-negative integer $r$.

Let $\lambda \in \mathcal{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{(r+1)(r+1)}$. Here $((r + 1)^{r+1})$ means the rectangular partition $(r + 1, \ldots, r + 1)$ with length $r + 1$. For instance, the diagram

\[ D_{(5,4,4,4,2)/\delta_4} = \]

is contained in $D_{(5^7)}$.

It is shown in [1, 5] that the skew Schur function $s_{\lambda/\delta_r}$ has a non-negative integral expansion in terms of Schur $P$-functions

\[ s_{\lambda/\delta_r} = \sum_{\nu \in \mathcal{P}^+} a_{\lambda/\delta_r \nu} P_\nu, \]

together with a combinatorial description of $a_{\lambda/\delta_r \nu}$. Moreover it is shown that these skew Schur functions are the only ones (up to rotation of shape by $180^\circ$), which have Schur $P$-positivity. In this section, we give a new simple description of $a_{\lambda/\delta_r \nu}$ using $\mathfrak{q}(n)$-crystals.

First we consider a $\mathfrak{q}(n)$-crystal structure on $B_n(\lambda/\delta_r)$, which is a slight generalization of [7 Example 2.10(d)].

**Proposition 4.1.** Let $\lambda \in \mathcal{P}_n$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{(r+1)^{r+1}}$. Then the $\mathfrak{gl}(n)$-crystal $B_n(\lambda/\delta_r)$ as a subset of $W_{[n]}$ together with $0$ is invariant under $\tilde{e}_T$ and $\tilde{f}_T$. Hence $B_n(\lambda/\delta_r)$ is a $\mathfrak{q}(n)$-crystal.

**Proof.** Let $N = |\lambda| - |\delta_r|$. For $T \in B_n(\lambda/\delta_r)$, let $w(T) = w_1 \cdots w_N$. Recall that $T$ is identified with $w(T)$ in $W_{[n]}$. Here we call the box in $D_{\lambda/\delta_r}$ containing $w_i$ the $w_i$-box, and call the set of boxes $(x, r - x + 2) \in D_{\lambda/\delta_r}$ for $1 \leq x \leq r + 1$ the main anti-diagonal of $D_{\lambda/\delta_r}$.
Suppose that \( \tilde{f}_T w(T) \neq 0 \). There exists \( 1 \leq i \leq N - 1 \) such that \( w_i = 1 \) and \( w_j \neq 1, 2 \) for all \( i < j \leq N \), and

\[
\tilde{f}_T^p(w_1 \cdots w_{i-1} 1 w_{i+1} \cdots w_N) = w_1 \cdots w_{i-1} 2 w_{i+1} \cdots w_N,
\]

by the tensor product rule \((2.3)\). We first observe that the entry 1 in \( T \) can be placed only on the main anti-diagonal in \( D_{\lambda/\delta_r} \). If there is a box in \( D_{\lambda/\delta_r} \) below the \( w_i \)-box, then it corresponds to \( w_j \) for some \( j > i \), and hence its entry is greater than 2. Moreover, if there is a box in \( D_{\lambda/\delta_r} \) to the right of the \( w_i \)-box, then its entry is greater than 1 since it is not on the main anti-diagonal. So we conclude that there exists \( T' \in SST_{\nu}(\lambda/\delta_r) \) such that \( w(T') = \tilde{f}_T w(T) \).

Suppose that \( \tilde{e}_T w(T) \neq 0 \). There exists \( 1 \leq i \leq N - 1 \) such that \( w_i = 2 \) and \( w_j \neq 1, 2 \) for all \( i < j \leq N \), and

\[
\tilde{e}_T^p(w_1 \cdots w_{i-1} 2 w_{i+1} \cdots w_N) = w_1 \cdots w_{i-1} 1 w_{i+1} \cdots w_N,
\]

by the tensor product rule \((2.3)\). If the \( w_i \)-box is not on the main anti-diagonal, then the \( w_{i+1} \)-box is placed to the left of the \( w_i \)-box. Then the \( w_{i+1} \)-box is filled with 1 or 2, which contradicts \((4.2)\). So the \( w_i \)-box is on the main anti-diagonal, and thus \( \tilde{f}_T w(T) = w(T') \) for some \( T' \in B_n(\lambda/\delta_r) \). This completes the proof. \( \square \)

**Corollary 4.2.** Under the above hypothesis, the skew Schur function \( s_{\lambda/\delta_r} \) is Schur \( P \)-positive.

**Proof.** Since \( B_n(\lambda/\delta_r) \) is a \( q(n) \)-crystal, the skew Schur polynomial \( s_{\lambda/\delta_r}(x_1, \ldots, x_n) \) is a non-negative integral linear combination of \( P_{\nu}(x_1, \ldots, x_n) \). Then we apply Remark 2.9. \( \square \)

**Definition 4.3.** Let \( \lambda \in \mathcal{P} \) be such that \( D_{\delta_r} \subseteq D_{\lambda} \subseteq D_{(r+1)^r+1} \) and \( \nu \in \mathcal{P}^+ \). Let \( A_{\lambda/\delta_r, \nu} \) be the set of tableaux \( Q \) such that

1. \( Q \in SST_{r+1}(\nu) \) with \( c_k(Q) = \lambda_r - k + 2 - k \) for \( 1 \leq k \leq r + 1 \),
2. \( 1 \leq k \leq r \) and \( 1 \leq i \leq N \),

\[
m_k(i) \leq m_{k+1}(i) + 1,
\]

where \( w_{\text{rev}}(Q) = w_1 \cdots w_N \) and \( m_k(i) = c_k(w_1 \cdots w_i) \).

Then we have the following combinatorial description of \( a_{\lambda/\delta_r, \nu} \).

**Theorem 4.4.** For \( \lambda \in \mathcal{P} \) with \( D_{\delta_r} \subseteq D_{\lambda} \subseteq D_{(r+1)^r+1} \) and \( \nu \in \mathcal{P}^+ \), we have

\[
a_{\lambda/\delta_r, \nu} = |A_{\lambda/\delta_r, \nu}|.
\]

**Proof.** Choose \( n \) such that \( \lambda, \nu \in \mathcal{P}_n^+ \). We may assume that \( \lambda_1 = \ell(\lambda) = r + 1 \). Let

\[
L_{\lambda/\delta_r, \nu} = \{ T \in B_n(\lambda/\delta_r) \mid T \equiv L' \}.
\]
By Proposition 4.1, we have
\[
B_n(\lambda/\delta_r) \cong \bigcup_{\nu \in \mathcal{P}_n^+} B_n(\nu)^{\oplus |L_{\lambda/\delta_r,\nu}|}.
\]

By linear independence of $P_\nu(x_1, \ldots, x_n)$'s for $\nu \in \mathcal{P}_n^+$, we have $a_{\lambda/\delta_r,\nu} = |L_{\lambda/\delta_r,\nu}|$.

Let us construct a bijection
\[
L_{\lambda/\delta_r,\nu} \quad \longrightarrow \quad \Lambda_{\lambda/\delta_r,\nu} \quad \quad \quad T \quad \longmapsto \quad Q_T
\]
as follows. Let $T \in \Lambda_{\lambda/\delta_r,\nu}$ be given. Suppose that $w(T) = u_1 \cdots u_N$, where $N = |\nu|$. By Lemma 2.7, there exists $\nu^{(m)} \in \mathcal{P}_n^+$ for $1 \leq m \leq N$ such that $u_{N-m+1} \cdots u_N \equiv L^{\nu^{(m)}}$, where $\nu^{(1)} = (1)$, $\nu^{(N)} = \nu$, and $\nu^{(m)}$ is obtained by adding a box in the $(n-u_m+1)$-st row of $\nu^{(m-1)}$ for $1 \leq m \leq N$ with $\nu^{(0)} = \emptyset$.

Note that $w_{\text{rev}}(T) = T^{(r+1)} \cdots T^{(1)}$, where $T^{(1)} = T_{n,1} \cdots T_{n,\lambda_1-r_1+1}$ is a weakly increasing word corresponding to the $l$-th row of $T$ for $1 \leq l \leq r + 1$. Let $Q_T$ be a tableau of shifted shape $\nu$ with entries in $\mathbb{N}$, where $\nu^{(m)}/\nu^{(m-1)}$ is filled with $r+2-l$ if $u_m$ occurs in $T^{(l)}$, for some $1 \leq l \leq r+1$. Note that the boxes in $Q_T$ corresponding to $T^{(l)}$ are filled with $r+2-l$ as a horizontal strip. So $Q_T$ satisfies the condition Definition 4.3(1).

For each $k \geq 1$, let us enumerate the letter $k$'s in $Q_T$ from southwest to northeast like $k_1, k_2, \ldots$. Since $T \in \text{SST}_n(\lambda/\delta_r)$, we see that the entry $k_i$ in $Q_T$ corresponds to $T_{l,i}$ for $i \geq 1$, where $l = r+2-k$, and moreover $(k+1,i)$ is located in the southwest of $k_{i+1}$ for $i \geq 2$. This implies the condition Definition 4.3(2), and hence $Q_T \in \Lambda_{\lambda/\delta_r,\nu}$.

Finally, one can check that correspondence $T \mapsto Q_T$ is a bijection. \square

**Example 4.5.** Let $\lambda = (5, 5, 4, 3, 1)$ with $D_\lambda \subseteq D_{(5^5)}$ and $n = 7$. For $\nu = (4, 3, 1)$, we have $L_{\lambda/\delta_4,\nu} = \{T_1, T_2\}$ and $\Lambda_{\lambda/\delta_4,\nu} = \{Q_{T_1}, Q_{T_2}\}$ as follows.

\[
T_1 = \begin{array}{ccc}
6 & \quad & 5 \\
5 & 7 & \quad \\
6 & 6 & 7 \\
7 & 7 & \quad \\
\end{array} \quad T_2 = \begin{array}{ccc}
6 & \quad & 5 \\
5 & 7 & \quad \\
6 & 6 & 7 \\
7 & 7 & \quad \\
\end{array} \quad Q_{T_1} = \begin{array}{ccccc}
1 & 2 & 2 & 4 & \\
3 & 3 & 5 & \quad \\
4 & \quad & \quad & \quad & \\
\end{array} \quad Q_{T_2} = \begin{array}{ccccc}
1 & 2 & 2 & 3 & \\
3 & 4 & 4 & \quad \\
5 & \quad & \quad & \quad & \\
\end{array}
\]

Moreover, we have
\[
h_{(5,5,4,3,1)/\delta_4} = 2P_{(4,3,1)} + P_{(5,2,1)} + P_{(5,3)}.
\]

**4.2. Ardila-Serrano’s expansion of $s_{\delta_r+1}/\mu$**. We fix a non-negative integer $r$. For $\mu \in \mathcal{P}$ with $D_\mu \subseteq D_{\delta_r+1}$, let us recall the result on the Schur $P$-expansion of he skew Schur function $s_{\delta_r+1}/\mu$ by Ardila and Serrano [1].
Let \( N = |\delta_{r+1}| - |\mu| \), and let \( T_{\delta_{r+1}/\mu} \) be the tableau obtained by filling \( \delta_{r+1}/\mu \) with \( 1, 2, \ldots, N \) subsequently, starting from the bottom row to top, and from left to right in each row. For instance,

\[
T_{\delta_{5}/(4,1,1)} = \begin{array}{ccc}
6 & 7 & 8 \\
4 & 5 \\
2 & 3 \\
1
\end{array}
\]

For \( \nu \in \mathcal{P}^+ \) with \( |
u| = N \), let \( B_{\delta_{r+1}/\mu \nu} \) be the set of tableaux \( Q \) such that

1. \( Q \in \text{SST}_{[N]}^{\mu}(\nu) \) where each entry \( i \in [N] \) occurs exactly once,
2. if \( j \) is directly above \( i \) in \( T_{\delta_{r+1}/\mu} \), then \( j \) is placed strictly to the right of \( i \) in \( Q \),
3. if \( i + 1 \) is placed to the right of \( i \) in \( T_{\delta_{r+1}/\mu} \), then \( i + 1 \) is strictly below \( i \) in \( Q \).

**Theorem 4.6.** (Theorem 4.10) For \( \mu \in \mathcal{P} \) with \( D_{\mu} \subseteq D_{\delta_{r+1}} \), the skew Schur function \( s_{\delta_{r+1}/\mu} \) is given by a non-negative integral linear combination of Schur \( P \)-functions

\[
s_{\delta_{r+1}/\mu} = \sum_{\nu \in \mathcal{P}^+} b_{\delta_{r+1}/\mu \nu} P_{\nu},
\]

where \( b_{\delta_{r+1}/\mu \nu} = |B_{\delta_{r+1}/\mu \nu}|. \)

Now we show that Theorem 4.6 (after a little modification of its proof) implies Theorem 4.4. Let \( \lambda \in \mathcal{P} \) be such that \( D_\lambda \subseteq D_{\delta_{r}} \subseteq D_{((r+1)r+1)}. \)

Let \( \nu \in \mathcal{P}^+ \) with \( |
u| = N = |\lambda| - |\delta_{r}| \), and let \( L_{\lambda/\delta_{r} \nu} \) be as in (4.4). Then \( |L_{\lambda/\delta_{r} \nu}| = a_{\lambda/\delta_{r} \nu} \) by (4.4). Let \( T \in L_{\lambda/\delta_{r} \nu} \) be given with \( w(T) = u_1 \cdots u_N \). Recall by Lemma 2.7 that there exists a sequence of strict partitions \( \nu^{(m)} \in \mathcal{P}^+ \) for \( 1 \leq m \leq N \) such that \( u_{N-m+1} \cdots u_N \equiv L^{\nu^{(m)}} \), where \( \nu^{(1)} = (1), \nu^{(N)} = \nu, \) and \( \nu^{(m)} \) is obtained by adding a box in the \((n - u_m + 1)\)-st row of \( \nu^{(m-1)} \) with \( \nu^{(0)} = \emptyset \).

We define \( Q_T^{\prime} \) to be the tableau of shifted shape \( \nu \) such that \( \nu^{(m)}/\nu^{(m-1)} \) is filled with \( m \) for \( 1 \leq m \leq N \). Then we have the following.

**Theorem 4.7.** Let \( \lambda \in \mathcal{P} \) be such that \( D_\lambda \subseteq D_{\delta_{r}} \subseteq D_{((r+1)r+1)} \) and \( \nu \in \mathcal{P}^+ \). Then we have a bijection

\[
L_{\lambda/\delta_{r} \nu} \rightarrow B_{\delta_{r+1}/(\lambda^{c})^{\nu}}
\]

\[
T \rightarrow Q_T^{\prime}
\]

where \( \lambda^{c} := (r + 1 - \lambda_{r+1}, r + 1 - \lambda_{r}, \ldots, r + 1 - \lambda_{1}) \) is the complement of \( \lambda \) in \(((r+1)r+1)\).
**Proof.** Let $T'_{\lambda/\delta r}$ be the tableau obtained by filling $\lambda/\delta r$ with $1, 2, \ldots, N$ subsequently, starting from the leftmost column to rightmost, and from bottom to top in each column. For instance, when $\lambda = (5, 4, 4, 4, 2)$ and $r = 4$, we have

$$T'_{\lambda/\delta r} = \begin{array}{ccccc}
9 & 8 & 7 & 5 & 6 \\
4 & 3 & 2 & 1 & 7 \\
3 & 4 & 6 & 1 & 2 \\
\end{array}.$$  

By definition of $Q'_T$, we can check that

1. $Q'_T \in SST_{[N]}^+(\nu)$ where each entry $i \in [N]$ occurs exactly once,
2. if $j$ is directly above $i$ in $T'_{\lambda/\delta r}$, then then $j$ is strictly below $i$ in $Q'_T$,
3. if $i + 1$ is placed to the right of $i$ in $T'_{\lambda/\delta r}$, then $i + 1$ is placed strictly to the right of $i$ in $Q'_T$.

We see that $T_{\delta r+1/(\lambda\nu)'}$ is obtained from $T'_{\lambda/\delta r}$ by flipping with respect to the main anti-diagonal. This implies that $Q'_T \in B_{\delta r+1/(\lambda\nu)'}$. Since the correspondence $T \mapsto Q'_T$ is reversible, it is a bijection.  

**Corollary 4.8.** Under the above hypothesis, we have a bijection

$$A_{\lambda/\delta r} \nu \longrightarrow B_{\delta r+1/(\lambda\nu)'}$$

for $T \in L_{\lambda/\delta r}$.  

Recall that for a skew shape $\eta/\zeta$, we have $s_{\eta/\zeta} = s_{(\eta/\zeta)^{\pi}}$, where $(\eta/\zeta)^{\pi}$ is the (skew) diagram obtained from $\eta/\zeta$ by rotating 180 degree (which can be seen for example by reversing the linear ordering on $\mathbb{N}$ in [2]). Also if $s_{\eta/\zeta}$ has a Schur $P$-expansion, then we have $s_{\eta/\zeta} = s_{\eta '/\zeta'}$ by applying the involution $\omega$ on the ring symmetric function sending $s_{\eta}$ to $s_{\eta'}$ since $\omega(P_{\nu}) = P_{\nu}$ for $\nu \in \mathcal{P}^+$ (see [10] p. 259, Exercise 3.(a)).

Hence we have

$$s_{\lambda/\delta r} = s_{\delta r+1/(\lambda\nu)'} = s_{\delta r+1/\mu \nu},$$

for $\lambda \in \mathcal{P}$ such that $D_{\delta r} \subseteq D_{\lambda} \subseteq D_{(\delta r+1)/\mu \nu+1}$. This implies that

$$a_{\lambda/\delta r} \nu = b_{\delta r+1/\lambda\nu} = b_{\delta r+1/\mu \nu},$$

for $\nu \in \mathcal{P}^+$, where $a_{\lambda/\delta r} \nu$ are given in (4.1). Equivalently, we have

$$a_{\mu \nu}/\delta r \nu = b_{\delta r+1/\mu \nu} = b_{\delta r+1/\mu \nu},$$

for $\mu \in \mathcal{P}$ with $D_{\mu} \subseteq D_{\delta r+1}$. Therefore Theorem 4.6 follows from Theorem 4.4, Corollary 4.8 and (4.6) (or (4.7)).
5. Schur Expansion of Schur $P$-function

For $\lambda \in \mathcal{P}^+$ and $\mu \in \mathcal{P}$, let $g_{\lambda \mu}$ be the coefficient of $s_\mu$ in the Schur expansion of $P_\lambda$, that is,

\[(5.1) \quad P_\lambda = \sum_\mu g_{\lambda \mu} s_\mu.\]

The purpose of this section is to give an alternate proof of the following combinatorial description of $g_{\lambda \mu}$ due to Stembridge.

**Theorem 5.1.** ([16, Theorem 9.3]) For $\lambda \in \mathcal{P}^+$ and $\mu \in \mathcal{P}$, we have

\[g_{\lambda \mu} = |G_{\lambda \mu}|,\]

where $G_{\lambda \mu}$ is the set of tableaux $Q$ such that

1. $Q \in SST_N(\mu)$ with $c_k(Q) + c'_k(Q) = \lambda_k$ for $k \geq 1$,
2. for $k \geq 1$, if $x$ is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,
3. $w(Q)$ satisfies the lattice property.

**Proof.** The proof is similar to that of Theorem 3.5. Choose $n$ such that $\lambda \in \mathcal{P}_n^+$ and $\mu \in \mathcal{P}_n$. Let

\[L_{\lambda \mu} = \{ T | T \in B_n(\lambda), \bar{f}_iT = 0 \ (1 \leq i \leq n-1), \ wt(T) = w_0\mu \}.\]

Then we have as a $\mathfrak{gl}(n)$-crystal

\[(5.2) \quad B_n(\lambda) \cong \bigsqcup_\mu B_n(\mu)^{\oplus|L_{\lambda \mu}|},\]

and hence $g_{\lambda \mu} = |L_{\lambda \mu}|$ by linear independence of Schur polynomials. Let us define a map

\[L_{\lambda \mu} \longrightarrow G_{\lambda \mu}, \quad T \longmapsto Q_T\]

as follows. Let $T \in L_{\lambda \mu}$ be given. Assume that $w_{\text{ev}}(T) = u_1 \cdots u_N$ where $N = |\lambda|$. Since $T$ is a $\mathfrak{gl}(n)$-lowest weight vector, we have by (2.2) that $u_{N-m+1} \otimes \cdots \otimes u_N \in B_n^{\otimes m}$ is a $\mathfrak{gl}(n)$-lowest weight element for $1 \leq m \leq N$. This implies that there exists $\mu^{(m)} \in \mathcal{P}_n$ for $1 \leq m \leq N$ such that $u_{N-m+1} \cdots u_N$ is equivalent as an element of $\mathfrak{gl}(n)$-crystal to a $\mathfrak{gl}(n)$-lowest weight element in $B_n(\mu^{(m)})$, where $\mu^{(N)} = \mu$ and $\mu^{(m)}$ is obtained by adding a box in the $(n-u_m+1)$-st row of $\mu^{(m-1)}$ with $\mu^{(0)} = \emptyset$.

We define $Q_T$ to be a tableau of shape $\mu$ with entries in $\mathbb{N}$, where $\mu^{(m)}/\mu^{(m-1)}$ is filled with

\[
\begin{cases} 
  k', & \text{if } u_m \text{ belongs to } T^{(k)_\uparrow}, \\
  k, & \text{if } u_m \text{ belongs to } T^{(k)_\downarrow},
\end{cases}
\]
for some $1 \leq k \leq \ell(\lambda)$. By almost the same arguments as in the proof of Theorem 3.5 we see that $Q_T$ satisfies the conditions (1) and (2) for $\mathcal{G}_\lambda$, and $w(Q_T)$ satisfies the “lattice property”, which implies that it satisfies the lattice property by Corollary 3.12. (We leave the details to the reader.) Finally the correspondence $T \mapsto Q_T$ is a well-defined bijection. □

Example 5.2. Let $\lambda = (3, 1)$. From Figure 1 we get three $\mathfrak{gl}(3)$-lowest weight vectors in $B_3(\lambda)$

$$
\begin{array}{ccc}
3 & 3 & 3 \\
2 & & \\
\end{array} \quad
\begin{array}{ccc}
3 & 2 & 3 \\
2 & & \\
\end{array} \quad
\begin{array}{ccc}
3 & 2 & 3 \\
1 & & \\
\end{array}.
\end{array}
$$

By applying the mapping $T \mapsto Q_T$ in the proof of Theorem 5.1 to these tableaux we have

$$
\begin{array}{ccc}
1 & 1 & 1 \\
2 & & \\
\end{array} \quad
\begin{array}{ccc}
1 & 1' \\
1 & 2 & \\
\end{array} \quad
\begin{array}{ccc}
1 & 1' \\
1 & & \\
\end{array}.
\end{array}
$$

Thus $P_{(3, 1)} = s_{(3, 1)} + s_{(2, 2)} + s_{(2, 1, 1)}$.

Remark 5.3. Let $\lambda \in \mathcal{P}^+$ be such that $D_{\lambda}^+ \subseteq D_{\delta_r+1}^+$ for some $r \geq 0$. Let $\lambda^{\circ+}$ be a strict partition obtained by counting complementary boxes $D_{\delta_r+1}^+ \setminus D_{\lambda}^+$ in each column from right to left. It is shown in [5] that

$$
s_{\delta_r+1/\lambda} = \sum_{\nu \in \mathcal{P}^{\circ+}} |\nu| = |\lambda| g_{\nu \lambda} P_{\nu^{\circ+}}.
$$

By (4.6) or (4.7), we have $g_{\nu \lambda} = a_{\lambda^{\circ}/\delta_r (\nu^{\circ+})'}$. One may expect that there is a natural bijection between $\mathcal{G}_{\nu \lambda}$ and $A_{\lambda^{\circ}/\delta_r (\nu^{\circ+})}$, but we do not know the answer yet.

6. Semistandard decomposition tableaux of skew shapes

Let $\lambda/\mu$ be a shifted skew diagram for $\lambda, \mu \in \mathcal{P}^+$ with $D_{\mu}^+ \subseteq D_{\lambda}^+$. Without loss of generality, we assume in this section that $\lambda_1 > \mu_1$ and $\ell(\lambda) > \ell(\mu)$.

Let $T$ be a tableau of shifted skew shape $\lambda/\mu$. For $p, q \geq 1$, let $T(p, q)$ denote the entry of $T$ at the $p$-th row and the $q$-th diagonal from the main diagonal in $D_{\lambda}^+$ (that is, $\{(i, i) \mid i \geq 1\} \cap D_{\lambda}^+$) whenever it is defined. Note that $T(p, q)$ is not necessarily equal to $T_{p, q}$ if $\mu$ is nonempty.

For example, when $\lambda/\mu = (5, 4, 2)/(3, 1)$, we have

$$
\begin{array}{ccc}
T(1, 1) & T(1, 2) & T(1, 3) \\
T(2, 1) & T(2, 2) & T(2, 3) \\
T(3, 1) & T(3, 2) & T(3, 3)
\end{array} = \begin{array}{ccc}
T_{1, 1} & T_{1, 2} & T_{1, 3} \\
T_{2, 1} & T_{2, 2} & T_{2, 3} \\
T_{3, 1} & T_{3, 2} & T_{3, 3}
\end{array}.
$$
Definition 6.1. For $\lambda, \mu \in \mathcal{P}^+$ with $D^\mu_\lambda \subseteq D^\lambda_\mu$, a skew semistandard decomposition tableau $T$ of shape $\lambda/\mu$ is a tableau of shifted shape $\lambda/\mu$ with entries in $\mathbb{N}$ such that $T^{(k)}$ is a hook word for $1 \leq k \leq \ell(\lambda)$ and the following holds for $1 \leq k < \ell(\lambda)$ and $1 \leq i \leq j \leq \lambda_{k+1}$:

(S1) if $T(k, i) \leq T(k + 1, j)$, then $i \neq 1$ and $T(k + 1, i - 1) < T(k + 1, j)$,
(S2) if $T(k, i) > T(k + 1, j)$, then $T(k, i) \geq T(k, j + 1),$

where we assume that $T(p, q)$ for $p, q \geq 1$ is empty if it is not defined.

Let $SSDT(\lambda/\mu)$ be the set consisting of skew semistandard decomposition tableaux of shape $\lambda/\mu$. Note that when $\mu$ is empty, the set $SSDT(\lambda/\mu)$ is equal to $SSDT(\lambda)$ by Proposition 2.3.

Suppose that $\ell(\lambda) \leq n$. Let $B_n(\lambda/\mu)$ be the set of $T \in SSDT(\lambda/\mu)$ with entries in $[n]$. As in (2.5), consider the injective map

\begin{equation}
B_n(\lambda/\mu) \rightarrow W_n \quad T \rightarrow w_{rev}(T).
\end{equation}

Proposition 6.2. Under the above hypothesis, the image of $B_n(\lambda/\mu)$ in (6.1) together with $\{0\}$ is invariant under the action of $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$, and hence $B_n(\lambda/\mu)$ is a $q(n)$-crystal.

Proof. Choose a sufficiently large $M$ such that all the entries in $L^\mu_M$ are greater than $n$. For a tableau $T$ of shifted shape $\lambda/\mu$ with entries in $[n]$, let $\tilde{T} := L^\mu_M \ast T$ be the tableau of shifted shape $\lambda$, that is, the subtableau of shape $\mu$ in $\tilde{T}$ is $L^\mu_M$ and its complement in $\tilde{T}$ is $T$. By definition of $SSDT(\lambda/\mu)$ and Proposition 2.3 we have

\begin{equation}
T \in B_n(\lambda/\mu) \quad \text{if and only if} \quad \tilde{T} \in B_M(\lambda).
\end{equation}

Let $T \in B_n(\lambda/\mu)$ and $i \in I$ be given. If $\tilde{x}_i \tilde{T} \neq 0 (x = e, f)$, then we have by (6.2) that $\tilde{x}_i \tilde{T} = L^\mu_M \ast T'$ for some $T' \in B_n(\lambda/\mu)$. This implies that $\tilde{x}_i w_{rev}(T) = w_{rev}(T')$. Therefore, the image of $B_n(\lambda/\mu)$ in (6.1) together with $\{0\}$ is invariant under the action of $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$. \hfill \Box

Since $B_n(\lambda/\mu)$ is a subcrystal of $\oplus^{N} B_n(\nu)$ with $N = |\lambda| - |\mu|$, we have

\begin{equation}
B_n(\lambda/\mu) \cong \bigoplus_{\nu \in \mathcal{P}^+_{|\nu|=N}} B_n(\nu) \otimes f^\lambda/\mu(n)
\end{equation}

for some $f^\lambda/\mu(n) \in \mathbb{Z}_+$. Moreover by Remark 2.9 we have

\begin{equation}
f^\lambda/\mu := f^\lambda/\mu(m) = f^\lambda/\mu(n) \quad (m \geq n).
\end{equation}
Proof.
The proof is similar to that of Theorem 3.5. Choose where
\[ P^{\alpha}_{\lambda/\mu} = \sum_{T \in SST(\lambda/\mu)} x^T, \]
then we have from (6.3) and (6.4)
\[ (6.5) \quad P^{\alpha}_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} f^{\lambda/\mu}_{\nu} P_{\nu}. \]

**Example 6.3.** For \( \eta \in \mathcal{P}^+_n \) with \( \ell(\eta) = \ell \), let \( \lambda = \eta + L\delta \ell \) and \( \mu = L\delta \ell \in \mathcal{P}^+_n \), where \( L \geq \eta_1 \). Since each column in \( \lambda/\mu \) has at most one box, we have
\[ B_n(\lambda/\mu) \cong B_n(\eta_1) \otimes \cdots \otimes B_n(\eta_k). \]

By applying Theorem 3.5 repeatedly, we see that \( f^{\lambda/\mu}_{\nu} \) for \( \nu \in \mathcal{P}^+_n \) in this case is equal to the number of tableaux \( Q \) such that
(1) \( Q \in SST^+_N(\nu) \) with \( c_k(Q) + c_{k'}(Q) = \eta_k \) for \( k \geq 1 \),
(2) for each \( k \geq 1 \), if \( x \) is the rightmost in \( w(Q) \) with \( \mid x \mid = k \), then \( x = k \).

One can generalize the notion of “lattice property” in Definition 3.3 to describe the coefficient \( f^{\lambda/\mu}_{\nu} \).

**Definition 6.4.** Let \( w = w_1 \cdots w_N \in \mathcal{W}_N \) be given and let \( w^* = \overline{w}^1 \cdots \overline{w}^s \) be the word associated to \( w \) given in Definition 3.1. For \( \mu \in \mathcal{P}^+ \), we say that \( w \) satisfies the “\( \mu \)-lattice property” if \( w^* \) satisfies the following for each \( k \geq 1 \):

(L1) if \( k > \ell(\mu) \) and \( w^*_i = k_1 \) for \( j \geq 1 \), then no \( k + 1 \) for \( j \geq 1 \) occurs in \( w^*_1 \cdots w^*_{i-1} \),
(L2) if \( (w^*_s, w^*_t) = (k + 1, k_{i+1} - \alpha_k) \) for some \( s < t \) and \( \alpha_k < i \), then no \( k + 1 \) for \( i < j \) occurs in \( w^*_s \cdots w^*_t \),
(L3) if \( (w^*_s, w^*_t) = (k_{j+1} - \alpha_k, k + 1_j) \) for some \( s < t \) and \( \alpha_k < j \), then no \( k_i \) for \( i \leq j - \alpha_k \) occurs in \( w^*_s \cdots w^*_t \),

where \( \alpha_k = \mu_k - \mu_{k+1} \).

**Theorem 6.5.** For \( \lambda, \mu, \nu \in \mathcal{P}^+ \), we have
\[ f^{\lambda/\mu}_{\nu} = \left| F^{\lambda/\mu}_{\nu} \right|, \]
where \( F^{\lambda/\mu}_{\nu} \) is the set of tableaux \( Q \) such that
(1) \( Q \in SST^+_N(\nu) \) with \( c_k(Q) + c_{k'}(Q) = \lambda_k - \mu_k \) for \( k \geq 1 \),
(2) for \( k \geq 1 \), if \( x \) is the rightmost letter in \( w(Q) \) with \( \mid x \mid = k \), then \( x = k \),
(3) \( w(Q) \) satisfies the \( \mu \)-lattice property.

**Proof.** The proof is similar to that of Theorem 3.5. Choose \( n \) such that \( \lambda, \mu, \nu \in \mathcal{P}^+_n \). Put
\[ L^{\lambda/\mu}_{\nu} = \{ T \mid T \in B_n(\lambda/\mu), T \equiv L' \}. \]
From (6.3) and (6.4), we have $|L^\lambda/\mu_\nu| = f^\lambda/\mu_\nu$. Let us define a map

$$L^\lambda/\mu_\nu \mapsto F^\lambda/\mu_\nu$$

as follows. Let $N = |\lambda| - |\mu|$. Suppose that $T \in L^\lambda/\mu_\nu$ is given with $w_{\text{rev}}(T) = u_1 \cdots u_N$. By Lemma 2.7 there exists $\nu^{(m)} \in \mathcal{R}_n^+$ for $1 \leq m \leq N$ such that $u_{N-m+1} \cdots u_N \equiv L^{\nu^{(m)}}$ where $\nu^{(N)} = \nu$ and $\nu^{(m)}$ is obtained by adding a box in the $(n-u_m+1)$-st row of $\nu^{(m-1)}$ with $\nu^{(0)} = \emptyset$.

Note that $w_{\text{rev}}(T) = T^{(\ell(\lambda))} \cdots T^{(1)}$, where $T^{(k)}$ is a hook word for $1 \leq k \leq \ell(\lambda)$. Then we define $Q_T$ to be a tableau of shifted shape $\nu$ with entries in $N$, where $\nu^{(m)}/\nu^{(m-1)}$ is filled with

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)} \uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)} \downarrow, \end{cases}$$

for some $1 \leq k \leq \ell(\lambda)$.

First, by the same argument as in the proof of Theorem 3.5, we see that $Q_T$ satisfies the condition (2) for $F^\lambda/\mu_\nu$ by the same argument as in the proof of Theorem 3.5.

Let us check that $w(Q_T)$ satisfies the $\mu$-lattice property. If we label $k$ and $k'$ in (6.6) as $k_j$ and $k'_j$, respectively, when $u_m = T_{k,j}$, then it coincides with the labeling on the letters in $w(Q_T)$ given in Definition 3.1(1).

Choose a sufficiently large $M$ such that all the entries in $L^\mu_M$ are greater than $n$. Let $S = L^\mu_M \ast T$ (see the proof of Proposition 6.2). Since $S \in B_M(\lambda)$, the conditions Proposition 2.3(1), (2), and (3) on $S$ and hence on $T$ (cf. (6.2)) imply the conditions Definition 6.4(L1), (L2), and (L3), respectively. Therefore, $Q_T \in F^\lambda/\mu_\nu$.

Finally the correspondence $T \mapsto Q_T$ is injective and also reversible. Hence it is a bijection. \[\square\]

**Remark 6.6.** In general, $P^\circ_{\lambda/\mu}$ is not equal to the usual skew Schur $P$-function $P_{\lambda/\mu}$, or $f^\lambda/\mu_\nu$ is not necessarily equal to $f^\lambda/\mu_{\mu'\nu}$. It would be interesting to have a representation-theoretic interpretation of the Schur $P$-expansion of $P^\circ_{\lambda/\mu}$ (6.5).

**Example 6.7.** Let $\lambda = (6, 5, 2, 1)$ and $\mu = (4, 2)$. Then

$$P^\circ_{\lambda/\mu} = 2P_{(6,2)} + 3P_{(5,3)} + 5P_{(5,2,1)} + 4P_{(4,3,1)}$$

since $\bigcup_\nu F^\lambda/\mu_\nu$ consists of

\[
\begin{array}{cccc}
1 & 1 & 2' & 3 & 3 \\
2 & 4 &
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 2 & 2 & 3 & 3 \\
2 & 4 &
\end{array}
\quad
\text{ (when } \nu = (6, 2))
\]
On the other hand, \( P_{\lambda/\mu} = 2P_{(6,2)} + 6P_{(5,3)} + 6P_{(5,2,1)} + 8P_{(4,3,1)} \).

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