ADAPTIVE GRADIENT METHODS CAN BE PROVABLY FASTER THAN SGD AFTER FINITE EPOCHS

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ABSTRACT

Adaptive gradient methods have attracted much attention of machine learning communities due to the high efficiency. However their acceleration effect in practice, especially in neural network training, is hard to analyze, theoretically. The huge gap between theoretical convergence results and practical performances prevents further understanding of existing optimizers and the development of more advanced optimization methods. In this paper, we provide adaptive gradient methods a novel analysis with an additional mild assumption, and revise AdaGrad to SHAdaGrad for matching a better provable convergence rate. To find an \( \epsilon \)-approximate first-order stationary point in non-convex objectives, we prove random shuffling SHAdaGrad achieves a \( \tilde{O}(T^{-1/2}) \) convergence rate, which is significantly improved by factors \( \tilde{O}(T^{-1/4}) \) and \( \tilde{O}(T^{-1/6}) \) compared with existing adaptive gradient methods and random shuffling SGD, respectively. To the best of our knowledge, it is the first time to demonstrate that adaptive gradient methods can deterministically be faster than SGD after finite epochs. Furthermore, we conduct comprehensive experiments to validate the additional mild assumption and the acceleration effect benefited from second moments and random shuffling.

1 Introduction

Stochastic optimization is critical for large scale machine learning, formally, which aims to solve the following finite sum minimization problem:

\[
\min_{x \in \mathbb{R}^d} \ f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\]

where each function \( f_i : \mathbb{R}^d \to \mathbb{R} \) is smooth and possibly non-convex. This problem covers a wide range of models in machine learning, including deep neural networks (DNNs). When training DNNs, adaptive gradient methods \([1, 2, 3]\) are usually much faster than stochastic gradient descent (SGD) in practice, while their theoretical convergence is the same as or even worse than SGD in the non-convex setting \([4, 5, 6, 7]\). This inconsistency between practical performances and theoretical convergence prevents further understanding of existing optimizers and the development of more advanced optimization methods. Thus, closing the gap between practical performances and theoretical results is a very important issue.

Various studies attempt to bridge such a gap from different perspectives. Although these previous work offer very promising insights, they hardly explain why adaptive gradient methods can be faster than SGD theoretically. For

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example, [3] corrects errors in regret convergence analysis. [8, 9] provide the convergence analysis for achieving the global optimum in strongly convex optimization, and [4, 5, 6, 7] investigate the convergence rate for achieving first-order stationary points (FSPs) in non-convex setting to match assumptions in practice. All these studies can only provide similar convergence results to SGD, i.e., $O(T^{-1/2})$ for regret, $O(T^{-1})$ for achieving global optimum and $O(T^{-1/4})$ for achieving FSPs.

We argue that previous studies ignore the effects of random shuffling (sampling without replacement) in their analysis, with random shuffling SGD. (See Table 1 for the details comparison)

In neural network training, instances are usually sent to optimizers

Analysis for random shuffling in optimization.

In this paper, we show that, with an additional mild assumption, adaptive gradient methods can obtain an $\tilde{O}(T^{-1/2})$ convergence rate which outperforms previous best-known results. Specifically, it is $O(T^{-1/3})$ for random shuffling SGD ([10, 11], $O(T^{-1/4})$ for vanilla SGD ([13]) and $O(T^{-1/4})$ for Adam-type optimizers ([4, 5, 6]). From a theoretical point of view, we explain improvement from two observations. First, the combination of some full gradient perturbations and the second moment matrices can provide tighter lower bounds for sufficient descents in the random shuffling setting. Second, tighter sufficient descent lower bounds improve the convergence rate by weakening sufficient conditions required for the convergence. In practice, we first revise AdaGrad with full matrices ([1] (AdaGrad_F) to SHAdaGrad for theoretical proving convenience. Then, we conduct comprehensive experiments to convince readers the mild assumption in our proof, and validate the acceleration effect from the introduction of second moments and random shuffling. To the best of our knowledge, it is the first time to explain adaptive gradient methods can be deterministically faster than SGD after finite epochs both in theory and in practice. The main contributions of this paper are as follows:

- We are the first to analyze the convergence rate of adaptive gradient methods for achieving FSPs in non-convex and random shuffling settings, and provide an $\tilde{O}(T^{-1/2})$ convergence rate to SHAdaGrad, a minor revision of AdaGrad_F, with an additional mild assumption.
- We conduct comprehensive experiments to validate our mild assumption, and present the acceleration effect taken from random shuffling and second moments.

2 Related Work

In this section, we only introduce the work highly related to the analysis of adaptive gradient methods and the random shuffling strategy due to the space limitation. We briefly describe the difference between the existing work and ours, and list all of the convergence results for comparison.

Analysis for adaptive gradient methods

Compared with classic optimization methods for non-convex objectives, e.g., SGD [14], SVRG [15, 13] and SPIDER [16, 17], adaptive gradient methods, e.g., Adagrad [11], Adam [2] and AMSGrad [3], are more popular due to their excellent practical performances for neural network training. These adaptive gradient methods are originally proposed to solve online learning problems, and focus on the convergence analysis of their regret for convex objective functions. To further understand online learning optimizers in neural network training, the convergence analysis for non-convex problems are highly desired. Therefore, [4, 5, 6] and [7] analyze the convergence rate for achieving first-order stationary points (FSPs). Besides, they proposed a series of novel methods for faster convergence and better generalization. However, the convergence results of the proposed methods are usually $O(T^{-1/4})$, which is not better than the vanilla SGD in non-convex settings.

Analysis for random shuffling in optimization.

In neural network training, instances are usually sent to optimizers after random shuffling. With such a pre-processing, random shuffling is considered to be an important ingredient to capture the practical performance of optimization methods in theoretical analysis. Furthermore, the convergence of random shuffling SGD and vanilla SGD is quite different. Compared with the uniform sampling for calculated gradient at each iteration in vanilla SGD, [10, 18] and [11] have fully explained advantages of random shuffling utilization in the convergence rate. They improve the convergence rate from $O(T^{-1})$ and $O(T^{-1/4})$ to $O(T^{-2})$ and $\tilde{O}(T^{-1/3})$ for achieving FSPs in strongly convex and non-convex settings, respectively.

From related work, one may notice that the convergence results of adaptive gradient methods in non-convex and random shuffling settings are still understudied. Within an additional mild assumption, we improve the order of the convergence rate by a factor $\tilde{O}(T^{-1/4})$ compared with vanilla SGD and existing Adam-type optimizers, and $\tilde{O}(T^{-1/6})$ compared with random shuffling SGD. (See Table [1] for the details comparison)
Table 1: Convergence rate comparison of SGD, Adam-type optimizers, random shuffling SGD and SHAdaGrad, where $T$ denotes the number of epoch, and $n$, i.e., the number of instances is considered as a constant.

| Algorithm | Assumptions (L-smoothness+) | Convergence Results |
|-----------|-----------------------------|---------------------|
| vanilla SGD | • $\sigma^2$ bounded variance | $\mathbb{E} \|g_i\|^2 = O\left( \frac{1}{\sqrt{T}} \right)$ |
| Adaptive Gradient Methods Analysis | AMSGrad, AdaFom [4] | • bounded gradients | $\min \mathbb{E} \|g_i\|^2 = O\left( \frac{\ln T + d^2}{\sqrt{T}} \right)$ |
| | AMSGrad, Padam [5] | • initial gradient coordinate lower bound | $\mathbb{E} \|g_i\|^2 = O\left( \frac{1}{T \sqrt{d}} \right)$ |
| | RMSProp, Yogi [6] | • gradient sparsity | $\mathbb{E} \|g_i\|^2 = O\left( \frac{1}{T \sigma^2} \right)$ |
| | AdaGrad, NORM [7] | • $\sigma^2$ bounded variance | $\mathbb{E} \|g_i\|^2 = O\left( \frac{\ln T}{T} \right)$ |
| | GGT [8] | • $\sigma^*$ bounded variance | $\mathbb{E} \|g_i\| = O\left( T^{-1/4} \right)$ |
| Shuffling Analysis | Random Shuffling SGD [10] | • strongly convex functions | $\mathbb{E} \|x_T - x^*\|^2 = O\left( \frac{1}{T^2} \right)$ |
| | Random Shuffling SGD [11] | • bounded gradients | $\mathbb{E} \|g_i\|^2 = O\left( \frac{\ln T}{T^{1/4}} \right)$ |
| SHAdaGrad (ours) (Random Shuffling AdaGrad) | | • bounded gradients | $\mathbb{E} \|g_i\| = O\left( \frac{\sqrt{d} \ln T}{\sqrt{T}} \right)$ |

3 Notation and Preliminaries

In this section, we first introduce notation and preliminaries about objective functions, random shuffling and optimization methods including AdaGrad with full matrices [1] (AdaGrad_F) and SHAdaGrad. Then, we list the commonly used assumptions required for the convergence rate analysis, and define the sufficient descent for the convenience of later explanation.

**Notation of objective functions.** The objective function is defined in eq. 1, where $n$ and $\nabla f_i(x)$ denote the number of instances and the stochastic gradient for the $i$-th instance, respectively. Besides, we call $x \in \mathbb{R}^d$ an $\epsilon$-approximate first-order stationary point, or simply an FSP, if the gradient norm at $x$ satisfies $\|\nabla f(x)\| \leq \epsilon$.

**Notation of random shuffling.** Random shuffling is a sampling strategy to choose the stochastic mini-batch gradient at each iteration, which is different from the uniform sampling in vanilla SGD. Specifically, before the $t$-th epoch begins, random shuffling samples a random permutation $\sigma$ of the $n$ function uniformly and independently, and partitions $\sigma$ into several mini-batches $\{B^1_1, B^1_2, \ldots, B^1_m\}$ which satisfies $B^1_1 \cup B^1_2 \cup \ldots \cup B^1_m = I_n$ and $B^1_j \cap B^1_k = \emptyset, \forall j \neq k$. Then, the mini-batch gradient calculated at iteration $i$ in this epoch is denoted as

$$\nabla f_{B^1_i}(x) := \frac{1}{|B^1_i|} \sum_{k \in B^1_i} \nabla f_k(x). \quad (2)$$

Without loss of generality, we set $|B^1_1| = |B^1_2| = \ldots = |B^1_m|$, i.e., all of the mini-batches have the same number of instances, in the following sections.

**Notation of optimization methods.** We denote $x_{j}^i$ as the parameter at the $j$-th iteration of the $i$-th epoch and

$$H_{i,t} := \left[ \nabla f_{B^1_1}(x_1^i), \nabla f_{B^1_2}(x_2^i), \ldots, \nabla f_{B^1_m}(x_m^i) \right] \in \mathbb{R}^{d \times i}, \quad i \leq m, \quad \delta_{i,t}$$

where $m$ and $d$ are the number of iterations in each epoch and the dimension of the parameters, respectively. With the definition of $H_{i,t}$, we define the matrix

$$G_{i,t} := \sum_{\tau=1}^{i-1} H_{m,\tau} H_{m,\tau}^T + H_{i,t} H_{i,t}^T + \frac{\delta_{i,t}}{T} I, \quad (4)$$
where $\delta_{i,t}$ and $\Gamma$ are the perturbation and the scaling hyper-parameter to keep the positive-definite property for $G_{m,t}$. For any real matrix $M$, the maximum, the minimum and the $i$-th non-zero singular value are denoted as $\lambda_{\min}(M), \lambda_{\max}(M), \lambda_i(M)$, respectively.

Then, the iteration paradigm of both AdaGrad_F and its variant SHAdaGrad can be formulated as 

$$x_{i+1}^t = x_i^t - \eta_{i,t} G_{i,t}^{-\frac{1}{2}} \nabla f_{B_i^t}(x_i^t),$$  

(5)

where $\frac{\delta_{i,t}}{\lambda_{i,t}}$ in $G_{i,t}$ is a constant for AdaGrad_F while adaptive for shuffled AdaGrad (SHAdaGrad).

**Main assumptions.** We list commonly used assumptions [16, 6, 7] in the typical analysis as follows:

**Assumption 1.** We assume the following

1. The $\Delta := f(x_1^1) - f^* < \infty$ where $f^* := \inf_{x \in \mathbb{R}^d} f(x)$ is the global infimum value of $f(x)$.

2. (L-Smooth Assumption) The component function $f_i(x)$ is L-smooth, i.e., for all $x, y \in \mathbb{R}^d$ and $i \in \mathbb{I}_n$,

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|.$$

3. (Variance Bounded Assumption) The stochastic gradient has a bounded variance, i.e., for any $i \in \mathbb{I}_n$,

$$\mathbb{E}_i \left[ \|\nabla f_i(x) - \nabla f(x)\|^2 \right] \geq c_i^2.$$

4. (Gradient Bounded Assumption) The norm of stochastic gradient is upper bounded, i.e., for any $i \in \mathbb{I}_n$,

$$\|\nabla f_i(x)\| \leq G.$$

With L-Smooth Assumption in Assumption [1], we next introduce the definition of sufficient descent, which plays an important role for understanding the core idea of this paper.

**Definition 1.** We denote the sufficient descent as the deterministic negative term in RHS of L-Smooth inequality about the objective function.

For example, if we set the step size of SHAdaGrad to be a fixed constant, we provide the sufficient descent about $\nabla f(x_1^1)$ as follows.

$$f(x_{m+1}^t) \leq f(x_1^1) + \nabla f^\top(x_1^1) (x_{m+1}^t - x_1^1) + \frac{L}{2} \|x_{m+1}^t - x_1^1\|^2$$

$$\leq f(x_1^1) + \nabla f^\top(x_1^1) \left[ \sum_{i=1}^m \eta G_{i,t}^{-\frac{1}{2}} \nabla f_{B_i^t}(x_i^1) \right] + \frac{L}{2} \|x_{m+1}^t - x_1^1\|^2$$

$$= f(x_1^1) + \eta \nabla f^\top(x_1^1) \left[ \sum_{i=1}^m G_{m,t}^{-\frac{1}{2}} \nabla f_{B_i^t}(x_i^1) - \sum_{i=1}^m G_{i,t}^{-\frac{1}{2}} \nabla f_{B_i^t}(x_i^1) \right]$$

a positive upper bound

$$+ \frac{L}{2} \|x_{m+1}^t - x_1^1\|^2 - \frac{m}{2} \eta \nabla f^\top(x_1^1) G_{m,t}^{-\frac{1}{2}} \nabla f(x_1^1),$$

(6)

where $\odot$ follows from L-Smooth Assumption in Assumption [1] and $\odot$ follows from eq. [5]

### 4 Core Idea: Reducible Gradient Perturbation Sequence

In this section, we introduce the underlying ideas behind the convergence rate improvement for achieving first-order stationary points (FSPs) in non-convex optimization. We introduce the concept of Reducible Gradient Perturbation Sequence (RGPS) which is defined as

$$s_t = \frac{1}{m} \sum_{i=1}^m \nabla f_{B_i^t}(x_i^1), \quad t \in \mathbb{I}_T.$$

(7)

In random shuffling setting, RGPS can establish strong connections with both the update paradigm of adaptive gradient methods and full gradients, which provides the sufficient descent a $\Theta(\|\nabla f(x_1^1)\|)$ lower bound rather than the common result $\Theta(\|\nabla f(x_1^1)\|^2)$. Besides, the upper bound $U(T)$ of sufficient descents in our analysis is similar to previous work,
which means the sufficient conditions for convergence only request $U(T) \leq \epsilon$ rather than $U(T) \leq \epsilon^2$. Hence, a better provable convergence rate can be obtained by introducing RGPS. Specifically, we take AdaGrad, FedAdaGrad, and SHAdaGrad as examples to explain how RGPS works in the convergence analysis, and organize the details guided by answering the following two questions

1. How SHAdaGrad obtains a tighter sufficient descent lower bound with RGPS?
2. How the tighter sufficient descent lower bound benefits the convergence in SHAdaGrad?

### 4.1 RGPS is a Coupling of Gradients and Sufficient Descents

In this section, we answer the first question proposed in section 4. First, we introduce the relation between the sufficient descent lower bound and the convergence rate. Then, combining with section 4.1, we provide an explanation of the connection between the sufficient descent lower bound and the convergence rate improvement in SHAdaGrad.

In particular, we provide two lemmas to explain that RGPS is a coupling of the full gradient sequence and the sufficient descent about $s^t$.

**Lemma 4.1.** Suppose Assumption 4 and Assumption 2 hold, we have

$$\|\nabla f(x^t_1) − s^t\| \leq O(\eta/\sqrt{t}),$$

(8)

if the step size is fixed at each iteration.

This lemma illustrates that $s^t$ is close to $\nabla f(x^t_1)$ when the fixed step size $\eta$ is small enough. With triangle inequality, it also denotes that the full gradient norm $\|\nabla f(x^t_1)\|$ can be bounded by $\|s^t\|$.

**Lemma 4.2.** Suppose Assumption 4 and Assumption 2 hold, in SHAdaGrad, if $\delta_{m,t} \leq tmG^2$, $0 \leq \delta_{m,t} - \delta_{m,t-1} \leq m\lambda_{\max}(H_{m,t}H_{m,t}^\top)$ and $\Gamma \geq m$, we have

$$\sqrt{\frac{\delta_{m,t} - \delta_{m,t-1}}{t}} \|s^t\| \leq O\left(s^t_1 G_{m,t}^{-\frac{1}{2}}s^t_1\right).$$

(9)

This lemma reveals the connection between $\|s^t\|$ and the sufficient descent about $s^t$. According to the special structure of $G_{m,t}$, we are able to provide $\|s^t\|$ as the lower bound of the quadratic form $s^t_1 G_{m,t}^{-\frac{1}{2}}s^t_1$. Combining Lemma 4.1 with Lemma 4.2, $s^t_1 G_{m,t}^{-\frac{1}{2}}s^t_1$ can be even lower bounded:

$$\sqrt{\frac{\delta_{m,t} - \delta_{m,t-1}}{t}} \|\nabla f(x^t_1)\| = O\left(s^t_1 G_{m,t}^{-\frac{1}{2}}s^t_1\right) + O\left(\frac{1}{t}\right)$$

(10)

by using RGPS as a bridge.

On the other hand, if we investigate the sufficient descent about $\nabla f(x^t_1)$ directly, we obtain a lower bound of the sufficient descent as

$$\frac{1}{\sqrt{t}} \|\nabla f(x^t_1)\| \leq O(\nabla f^T(x^t_1)G_{m,t}^{-\frac{1}{2}}\nabla f(x^t_1)),$$

(11)

due to the definition of $G_{m,t}$ and Gradient Bounded Assumption in Assumption 1. When the parameter $x^t_1$ is close to an FSP, $\|\nabla f(x^t_1)\|$ is close to 0 due to L-Smooth Assumption in Assumption 1. With a lower order of $\|\nabla f(x^t_1)\|$, LHS of eq. (10) is a undoubtedly better lower bound compared with that in eq. (11) when $\delta_{m,t} - \delta_{m-1,t}$ has a constant lower bound, and the upper bound of RHS in eq. (10) is almost the same as that in eq. (11).

### 4.2 Tight Lower Bounds Weaken Sufficient Conditions for the Convergence

In this section, we answer the second question proposed in section 4. First, we introduce the relation between the sufficient descent lower bound and the convergence rate. Then, combining with section 4.1, we provide an explanation about the convergence rate improvement in SHAdaGrad.

The relation between the sufficient descent lower bound and the convergence rate. If we analyze the convergence through investigating the lower bound of the sufficient descent about $\nabla f(x^t_1)$ in SHAdaGrad, we have following inequalities:

$$m\eta \sum_{t=1}^{T} \frac{C}{\sqrt{t}} \|\nabla f(x^t_1)\|^2 \overset{\Omega}{\leq} m\eta \sum_{t=1}^{T} \nabla f^T(x^t_1) G_{m,t}^{-\frac{1}{2}} \nabla f(x^t_1) \overset{\Theta}{\leq} \nabla f(x^t_1) - f^* + \Theta(T\eta^a),$$

(12)
where \(\oplus\) follows from eq. 11 and \(\otimes\) can be obtained through providing the telescoping sum of eq. 6 and scaling the terms with positive upper bounds to \(\Theta(\eta^\alpha) (\alpha \geq 0)\). If a random variable \(\tau\) follows \(P[\tau = i] = \frac{1}{i^{\frac{3}{2}} - \frac{1}{2}}\), we obtain
\[
\mathbb{E}_\tau \left[ \|\nabla f(x_1^\tau)\|^2 \right] \leq \frac{f(x_1^\tau) - f^*}{\eta m \sqrt{T}} + \Theta(\sqrt{T}) \mathbb{E}_\tau \left[ \|\nabla f(x_1^\tau)\|^2 \right] \leq \Theta \left( T^{\frac{1}{2} - \frac{1}{2}} \right). \tag{13}
\]
As a result, the sufficient condition for achieving FSP, \(\mathbb{E}_\tau [\|\nabla f(x_1^\tau)\|^2] \leq \epsilon^2\), is \(RHS \leq \epsilon^2\) for eq. 13. The convergence rate of SHAdaGrad is at least \(O(T^{-1/4})\) if we lower bound the sufficient descent about \(\nabla f(x_1^\tau)\) like previous work. As a result, we can conclude that the order of \(\|\nabla f(x_1^\tau)\|\) in the lower bound of sufficient descent directly decide the order of \(\epsilon\) in RHS of the sufficient condition for convergence. The order of \(\|\nabla f(x_1^\tau)\|\) higher, the convergence rate worse.

The convergence rate improvement in SHAdaGrad. eq. 10 in section 4.1 shows that the order of \(\|\nabla f(x_1^\tau)\|\) in the lower bound of the sufficient descent about \(s_t\) is significantly smaller than that in eq. 11. Hence, similar to eq. 12 we can approximately provide
\[
\sum_{t=1}^{T} \eta \|\nabla f(x_1^\tau)\| \leq \eta \sum_{t=1}^{T} s_t G_m^{-\frac{1}{2}} s_t + \Theta (\eta^2 \ln T) \sum_{t=1}^{T} f(x_1^\tau) - f^* + \Theta (\eta^\alpha \sum_{t=1}^{T} t^{-\beta} \sqrt{\sum_{t=1}^{T} f(x_1^\tau)}) \sum_{t=1}^{T} t^{-\beta} \sqrt{T}, \tag{14}
\]
where \(\oplus\) follows from eq. 11 \(\otimes\) can be obtained by techniques similar to the inequality \(\oplus\) in eq. 12. The constants satisfy \(\alpha \geq 0, \beta \geq 1\). Notice that \(T_1\) in eq. 14 is corresponding to \(\Theta (T \eta^\alpha)\) in eq. 12 and \(T_2\) in eq. 14 is from the gap between \(s_t\) and \(\nabla f(x_1^\tau)\). Similar to eq. 13 we obtain
\[
\mathbb{E}_\tau \left[ \|\nabla f(x_1^\tau)\| \right] \leq \frac{f(x_1^\tau) - f^*}{\eta \sqrt{T}} + \Theta (\eta^\alpha \sqrt{T}) + \Theta \left( \eta \ln T \right). \tag{15}
\]
As a result, the sufficient condition for the parameters achieving FSP, \(\mathbb{E}_\tau [\|\nabla f(x_1^\tau)\|] \leq \epsilon\), is \(RHS \leq \epsilon\) for eq. 15. That is to say, the convergence rate of SHAdaGrad is near \(O(T^{-1/2})\) which is better than previous best-known results.

5 SHAdaGrad achieves an \(\tilde{O}(T^{-1/2})\) Convergence Rate

In this section, we show the convergence rate of adaptive gradient methods for achieving first-order stationary points (FSPs) in non-convex optimization can be \(O(T^{-1/2})\). Note that our theoretical results are based on SHAdaGrad, a variant of AdaGrad with full matrices (AdaGrad_F), and is just proposed for analytic convenience. Besides, we compare the total complexity between SHAdaGrad and random shuffiling SGD to illustrate that adaptive gradient methods can be faster than SGD after finite epochs, theoretically.

**Algorithm 1:** SHAdaGrad with full matrices

**Input:** The step size \(\eta > 0\), the iteration number in one epoch \(m\), the number of instances \(n\);

**Variables:** \(H_{c,t} \in \mathbb{R}^{d \times d}, g_{c,t} \in \mathbb{R}^{d \times d}, \delta_c = 0\);

**Initialization:** \(x_{m+1}^0, \sigma_p = 0\);

for \(t < 1\) to \(T\) do

   Initialize \(x_t^0 = x_{m+1}^0, \sigma_p = 0\);

   while \(\sigma_p < c_{\delta_c m^2}^2\) do

      Random shuffle the instances and get a partition \(\{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_m\}\);

      Calculate \(\sigma_p = \sum_{j=1}^{m} \|\nabla f_{\mathbb{B}_j}(x_t^\tau)\|^2\);

   end

   for \(i < 1\) to \(m\) do

      Receive the mini-batch stochastic gradient \(g_{i,t} = \nabla f_{\mathbb{B}_i}(x_t^\tau)\);

      \(\delta_c = \delta_c + \|g_{i,t}\|^2\);

      Let \(H_{i,t} = [g_{i,1}, g_{i,2}, \ldots, g_{i,t}]\);

      \(x_{t+1}^\tau = x_t^\tau - \eta \cdot (H_{i,t} H_{i,t}^\top + \delta_c I)^{-\frac{1}{2}} g_{i,t}\);

   end
SHAdaGrad, a modified AdaGrad for theoretically analytic convenience. We list the main differences between SHAdaGrad and AdaGrad as follows. First, SHAdaGrad requires a lower bound for the sum of mini-batch gradient norms, and obtains such lower bound with the sampling strategy (Step. 6 to Step. 8). Second, AdaGrad_F only considers the perturbation $\frac{m}{\eta T}$ as a constant, while SHAdaGrad has an adaptive perturbation which is related to the $l_2$ norm of mini-batch gradients (Step. 12). On the other hand, Algorithm which almost have a same update paradigm (Step. 14) as AdaGrad_F. Hence, it preserve benefits from second moments of adaptive gradient methods.

In the following, we provide our additional mild assumptions, the convergence results and the total complexity of SHAdaGrad. Due to space limitations, the details of proof arguments are provided in the supplementary materials.

**Assumption 2.** We assume $d \geq m$, $H_{m,t}$ has full column rank and bounded condition number formulated as

$$
\lambda_{\max}(H_{m,t}^T H_{m,t}) / \lambda_{\min}(H_{m,t}^T H_{m,t}) \leq c_{\kappa},
$$

where $d$ denotes the dimension of the parameters, and $m$ is the number of iterations in each epoch.

In Assumption 2, $d \geq m$ follows the over-parameterized property in most neural network training. Besides, we validate bounded condition numbers with experiments in section 6.

**Theorem 5.1.** Under Assumption 1 and Assumption 2, if $\eta \leq \frac{c^2_{\kappa}}{16nLG}$, $\Gamma \geq m$ and the hyper-parameter $\delta_{j,i}$ satisfy

$$
\delta_{j,i} = \sum_{p=1}^{i-1} \sum_{q=1}^{m} \left\| \nabla f_{B_p}(x_q^p) \right\|^2 + \sum_{q=1}^{m} \left\| \nabla f_{B_q}(x_q^i) \right\|^2, \forall j \in \mathbb{I}_T, i \in \mathbb{I}_m,
$$

we have

$$
\mathbb{E}_t \left[ \left\| \nabla f(x_t^i) \right\| \right] \leq \frac{C_0}{\eta \sqrt{T}} + \frac{C_1}{\sqrt{T}} + \frac{C_2 \eta}{\sqrt{T}} + \frac{C_3 \eta^2}{\sqrt{T}} + \frac{C_4 \ln(T)}{\sqrt{T}} + \frac{C_5 \eta \ln(T)}{\sqrt{T}},
$$

where $C_0, C_1, \ldots, C_5$ are constants which are independent with $T$ and defined in the proof.

Assuming that $L$, $G$ and $c_{\kappa}$ are known. Then, we can choose the following learning rate to obtain a concrete bound.

**Corollary 5.2.** Let $\{x_t^i\}$ be the sequence generated by Algorithm and $x_{out}$ be its output. For given tolerance $\epsilon > 0$, under the same conditions as Theorem 5.1 if we choose $\eta = \frac{c^2_{\kappa}}{16nLG}$, $\Gamma = m$ and $m = n$, then to guarantee

$$
\mathbb{E}_t \left[ \left\| \nabla f(x_t^i) \right\| \right] = \frac{\sum_{i=1}^{T} \frac{1}{\sqrt{T}} \left\| \sum_{j=1}^{m} \frac{1}{m} \nabla f_{B_q}(x_q^i) \right\|}{\sum_{i=1}^{T} \frac{1}{\sqrt{T}}} \leq \epsilon,
$$

it requires nearly $T = \lceil 36C_{\max}^n d \epsilon^2 \rceil$ outer iterations, where $C_{\max}$ is constant independent with $T$, $n$, $d$ and defined in the proof. In expectation, the total number of gradient evaluation is nearly

$$
T = \lceil 36 \left[ 1 - \exp \left( - \frac{c^2_{\kappa}}{32G^4} \right) \right]^{-1} C_{\max}^n d \epsilon^2 \rceil.
$$

To guarantee eq. 19 the total complexity required by random shuffling SGD is $O(\text{C_{sgd}n} \epsilon^{-3})$. That is to say, for a rough comparison, if $\epsilon \leq O(\frac{c_{sgd}}{\text{C_{sgd}n} \epsilon^{-3}})$ then Algorithm 1 seems to have advantages over random shuffling SGD in non-convex settings. From this point of view, it seems that Algorithm 1 seems inefficient when $n$ and $d$ is large. However, our analysis focuses on explaining that the introduction of second moments is beneficial for adaptive gradient methods to reduce the dependence on $T$, and our convergence rate may be loose in that it does not take into account a tight dependence on $n$ and $d$ in our complexity results.

**6 Experiments**

In this section, we conduct comprehensive experiments to validate the additional mild assumption, i.e., Assumption and the acceleration effect from second moments.

The paper then proceeds to introduce the experimental settings for the image classification tasks. We used the CIFAR-10 dataset, and test a highly simplified CNN model, whose architecture can be found in our supplementary materials. To compare convergence rates among SGD, AdaGrad, AdaGrad_F, SHAdaGrad and their random shuffling version, e.g., SGD_s, AdaGrad_s, etc, we ran 200 epochs, and set the learning rate for different optimizers as theoretical suggested in Table 2.

From fig. 1 we validate the condition number of $H_{m,t}$ will not increase with the number of iteration growth as Assumption presented. From fig. 2 we have two observations. First, random shuffling can actually take faster convergence for different optimizers in neural network training except for AdaGrad_F. Second, adaptive gradient methods are usually faster than SGD in both uniform sampling and random shuffling settings.
Table 2: Hyper-parameters selection of different optimizers.

| Optimizers   | Hyper-Parameters | Selection of $\eta$ |
|--------------|------------------|---------------------|
| SGD_u        | $\eta_t = \eta \cdot t^{-1/2}$ | $\{1.0, 0.1, 0.01\}$ |
| SGD_s        | $\eta_t = \eta \cdot t^{-1/3}$ | $\{1.0, 0.1, 0.01\}$ |
| AdaGrad_u    | $\eta_t = \eta \cdot t^{-1/2}$ | $\{0.1, 0.01, 0.001\}$ |
| AdaGrad_s    | $\eta_t = \eta \cdot t^{-1/2}$ | $\{0.1, 0.01, 0.001\}$ |
| SHAdaGrad_u  | $\eta_t = \eta, \Gamma = d$    | $\{1.0, 0.1, 0.01\}$ |
| SHAdaGrad_s  | $\eta_t = \eta, \Gamma = d$    | $\{1.0, 0.1, 0.01\}$ |
| AdaGrad_F_u  | $\eta_t = \eta$               | $\{1.0, 0.1, 0.01\}$ |
| AdaGrad_F_s  | $\eta_t = \eta$               | $\{1.0, 0.1, 0.01\}$ |

Figure 1: The condition number of SHAdaGrad is bounded for different learning rates.

Figure 2: Convergence of optimizers (_u) and their random shuffling version (_s) on CIFAR-10 image classification tasks, which shows acceleration effect taken from the random shuffling and adaptive learning rates.

7 Conclusion

In this paper, we provide a novel perspective to illustrate that Adagrad variants can be faster than SGD after finite epochs in non-convex and random shuffling settings. Under an additional mild assumption, we propose a minor revision of Adagrad, named SHAdaGrad, and obtain a better convergence rate, i.e., $\tilde{O}(T^{-1/2})$, compared with previous best-known results. Besides, we conduct extensive experiments to validate the additional mild assumption and the acceleration effect taken from the introduction of second moments and random shuffling.
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A Notations and Assumptions for the Appendix

In this section, we introduce some notations and assumptions used in this paper.

A.1 Notations

We denote the objective function as follows

\[
\min_{x \in \mathbb{R}^d} \ f(x) := \mathbb{E} \left[ F(x; \zeta) \right],
\]

(20)

where \( F(x, \zeta) \) is the stochastic component indexed by some random variable \( \zeta \). \( F(x, \zeta) \) is smooth, and possibly non-convex. Let \( \nabla F(x, \zeta) \) denote the stochastic gradient of \( f(x) \).

The finite-sum objective is a special case of Eq. (20) where \( f(x) \) with finite sampled stochastic variables \( \zeta \). It can be formulated as

\[
\min_{x \in \mathbb{R}^d} \ f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)
\]

(21)

whose stochastic gradient for the \( i \)-th instance is \( \nabla f_i(x) \).

Here, we describe the random shuffling setting in optimization procedure. Before each epoch, e.g., \( i \)-th epoch, beginning, we sample some permutation \( \sigma_i \) of the set \( \{1, 2, \ldots, n\} \), and partitions \( \sigma_i \) into mini-batch of equal size \( \{\mathcal{B}^i_1, \mathcal{B}^i_2, \ldots, \mathcal{B}^i_{n}\} \), where we require \( \mathcal{B}^i_1 \cup \mathcal{B}^i_2 \cup \ldots \cup \mathcal{B}^i_{n} = \sigma_i \) and \( \mathcal{B}^i_j \cap \mathcal{B}^i_k = \emptyset, \forall j \neq k \). Then, the mini-batch gradient calculated at iteration \( j \) in this epoch corresponds to \( \nabla f_{\mathcal{B}^i_j}(x) \) denoted as

\[
\nabla f_{\mathcal{B}^i_j}(x) := \frac{1}{|\mathcal{B}^i_j|} \sum_{k \in \mathcal{B}^i_j} f_k(x).
\]

Moreover, We denote \( x^i_j \) as the parameter at the \( j \)-th iteration of the \( i \)-th epoch and

\[
H_{i,t} := \left[ \nabla f_{\mathcal{B}^i_1}(x^i_1), \nabla f_{\mathcal{B}^i_2}(x^i_2), \ldots, \nabla f_{\mathcal{B}^i_{n}}(x^i_{n}) \right] \in \mathbb{R}^{d \times i},
\]

(23)

where \( m \) and \( d \) are presented as the number of iterations in each epoch and the dimension of the parameters, respectively. With the definition of \( H_{m,t} \), we define the matrix

\[
G_{i,t} := \sum_{\tau=1}^{t-1} H_{m,\tau} H_{m,\tau}^T + H_{i,t} H_{i,t}^T + \delta_{i,t} \Gamma I,
\]

(24)

where \( \delta_{i,t} \) and \( \Gamma \) are the perturbation and the scaling hyper-parameter to keep the positive-definite property for \( G_{m,t} \). Besides, for any real matrix \( M \), we denote the maximum, the minimum and the \( i \)-th non-zero singular value as \( \lambda_{\max}(M), \lambda_{\min}(M), \lambda_i(M) \), respectively.

A.2 Assumptions

In this subsection, we list our assumptions where Assumption 3 introduces some common assumptions used in various previous work \([16, 6, 7]\), and Assumption 4 is required in our proof additionally. To illustrate the rationality, we validate Assumption 4 with various experiments.

Assumption 3. We assume the following

1. The \( \Delta := f(x^i) - f^* < \infty \) where \( f^* = \inf_{x \in \mathbb{R}^d} f(x) \) is the global infimum value of \( f(x) \).

2. The component function \( f_i(x) \) is \( L \)-smooth, i.e., for all \( x, y \in \mathbb{R}^d \) and \( i \in \mathbb{I}_n \), \( \|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\| \).

3. The stochastic gradient has a bounded variance, i.e., for any \( i \in \mathbb{I}_n \), \( \mathbb{E} \left[ \|\nabla f_i(x) - \nabla f(x)\|^2 \right] \geq \sigma^2 \).

4. The norm of stochastic gradient is upper bounded, i.e., for any \( i \in \mathbb{I}_n \), \( \|\nabla f_i(x)\| \leq \max \{c_\sigma, G'\} := G \).

Assumption 4. Without loss of generality, we assume \( d \gg m \), \( H_{m,t} \) has full column rank and bounded condition number formulated as

\[
\frac{\lambda_{\max}(H_{m,t}^T H_{m,t})}{\lambda_{\min}(H_{m,t}^T H_{m,t})} \leq c_{\kappa}.
\]


B Existing Lemmas

Lemma B.1 (Conjugate Rule in [20]). Suppose that $M \succeq 0$. For every $A$, the matrix $A^* MA \succeq 0$ where $A^*$ means the conjugate transpose matrix of $A$. In particular,

$$M \preceq N \implies A^* MA \preceq A^* NA.$$  \hfill (25)

Lemma B.2 (Hoeffding’s inequality). Let $z_1, z_2, \ldots, z_n$ be independent bounded random variables with $z_i \in [a, b]$ for all $i$, where $-\infty < a < b < \infty$. Then

$$P\left[\frac{1}{n} \sum_{i=1}^{n} (z_i - \mathbb{E}[z_i]) \geq t\right] \leq \exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$  \hfill (26)

and

$$P\left[\frac{1}{n} \sum_{i=1}^{n} (z_i - \mathbb{E}[z_i]) \leq -t\right] \leq \exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$  \hfill (27)

for all $t \geq 0$.

Lemma B.3 (Lemma 13 in [1]). Let $N \succeq M \succeq 0$ be symmetric $d \times d$ matrices. Then $N^{\frac{1}{2}} \succeq M^{\frac{1}{2}}$.

Proof. This lemma had been proved in [1], we include a proof for the convenience of readers. Let $\lambda$ be a eigenvalue of $N^{\frac{1}{2}} - M^{\frac{1}{2}}$, corresponding to some eigenvector $x$. Hence, we have $(N^{\frac{1}{2}} - \lambda I)x = M^{\frac{1}{2}}x$. Taking the inner product of both size with $x^T N^{\frac{1}{2}} x$, we have

$$x^T N^{\frac{1}{2}} x - \lambda x^T N^{\frac{1}{2}} x = x^T \left(N^{\frac{1}{2}} - \lambda I\right) x$$

$$= x^T N^{\frac{1}{2}} M^{\frac{1}{2}} x \leq \|N^{\frac{1}{2}} x\| \cdot \|M^{\frac{1}{2}} x\| = \sqrt{x^T N x \cdot x^T M x} \leq \frac{x^T N x}{T_2}. $$  \hfill (28)

Thus, with $T_1 \leq T_2$ and $x^T N^{\frac{1}{2}} x \geq 0$, we obtain $\lambda \geq 0$ to complete the proof. \qed

Lemma B.4. Let $N \succeq M \succeq 0$ be symmetric $d \times d$ matrices. Then $N^{-\frac{1}{2}} \succeq M^{-\frac{1}{2}}$.

Proof. Since $N \succeq M$, we have $M^{-\frac{1}{2}} N M^{-\frac{1}{2}} = \left(M^{-\frac{1}{2}} N^{\frac{1}{2}}\right) \left(N^{\frac{1}{2}} M^{-\frac{1}{2}}\right) \succeq I$ because of Lemma B.1. Commuting the product of two matrix does not change the eigenvalues, hence all eigenvalues of $N^{\frac{1}{2}} M^{-1} N^{\frac{1}{2}}$ are larger than 1. Utilizing Lemma B.1 again, then we obtain $M^{-1} \succeq N^{-1}$ to complete the proof. \qed

Lemma B.5 (Sherman-Morrison formula). Suppose $M \in \mathbb{R}^{d \times d}$ is an invertible square matrix and $u, v \in \mathbb{R}^d$ are column vectors. Then $M + uv^T$ is invertible if and only if $1 + v^T Mu \neq 0$. In this case,

$$\left(M + uv^T\right)^{-1} = M^{-1} - \frac{M^{-1} uv^T M^{-1}}{1 + v^T M^{-1} u}$$  \hfill (29)
C Important Lemmas

Lemma C.1. In SHAdaGrad, suppose that the Assumption 3 hold, and the perturbation satisfies $0 \leq \delta_{i+1,t} - \delta_{i,t} \leq G^2$. For any $1 \leq i \leq j \leq m$ and $\Gamma \geq 1$, we have

$$G_{j,t}^{\frac{3}{2}} \leq G_{i,t}^{\frac{3}{2}} + \sqrt{2mG} \cdot I$$

Proof. It can be easily checked when $i = j$ holds. Therefore, we only need to prove $i < j$. To simplify notations in the following proof, we set $\Delta := \sum_{k=i+1}^{j} \nabla f_{B_k}^\top(x_k^i) \nabla f_{B_k}^\top(x_k^i)$. Then, according to the definition, we have

$$G_{j,t} = G_{i,t} + \Delta G + \frac{\delta_{j,t} - \delta_{i,t}}{\Gamma} \cdot I.$$

With the following fact,

$$\lambda_{max} (\Delta G) \leq tr (\Delta G) \leq (j-i)G^2 \leq mG^2$$

and $\delta_{j,t} - \delta_{i,t} = \sum_{k=i}^{j-1} (\delta_{k+1,t} - \delta_{k,t}) \leq (j-i)G^2 \leq mG^2$, (31) where (1) follows from the gradient bounded condition, the forth item in Assumption 3 we have

$$\Delta G + \frac{\delta_{j,t} - \delta_{i,t}}{\Gamma} \cdot I \leq m \left(1 + \frac{1}{\Gamma}\right) G^2 \cdot I \leq 2mG^2 \cdot I,$$

where (1) follows from the fact $\Gamma \geq 1$. Then, we obtain

$$G_{j,t}^{\frac{3}{2}} \leq G_{i,t} + 2mG^2 \cdot I$$

and $\frac{3}{2}$ follows from Eq. 30 and Eq. 32. (2) follows from Lemma B.3. Thus, we complete the proof.

Lemma C.2. In SHAdaGrad, suppose that the Assumption 3 hold, and the perturbation $\delta_{i,t}$ is no decreasing. For any $1 \leq i \leq m$, we have

$$\|x_{i+1}^t - x_i^t\|^2 \leq \min \{ \eta_{i,t}, \eta_{i,t}^2 G^2 \lambda_{min}^{-1} (G_1, t) \}$$

Proof. According to iterations in SHAdaGrad, if we set $\overline{G}_{i,t} := G_{i,t} - \nabla f_{B_i}^\top(x_i^t) \nabla f_{B_i}^\top(x_i^t)$, we have

$$\|x_{i+1}^t - x_i^t\|^2 = \|\eta_{i,t} G_{i,t}^{\frac{3}{2}} \nabla f_{B_i}^\top(x_i^t)\|^2 = \eta_{i,t}^2 \nabla f_{B_i}^\top(x_i^t) G_{i,t}^{-1} \nabla f_{B_i}^\top(x_i^t)$$

$$\leq \eta_{i,t}^2 \nabla f_{B_i}^\top(x_i^t) \left( \overline{G}_{i,t} + \nabla f_{B_i}^\top(x_i^t) \nabla f_{B_i}^\top(x_i^t) \right)^{-1} \nabla f_{B_i}^\top(x_i^t)$$

$$\leq \eta_{i,t}^2 \left( \nabla f_{B_i}^\top(x_i^t) \overline{G}_{i,t}^{-1} \nabla f_{B_i}^\top(x_i^t) - \frac{\|\nabla f_{B_i}^\top(x_i^t) \overline{G}_{i,t}^{-1} \nabla f_{B_i}^\top(x_i^t)\|^2}{1 + \nabla f_{B_i}^\top(x_i^t) \overline{G}_{i,t}^{-1} \nabla f_{B_i}^\top(x_i^t)} \right) \leq \eta_{i,t}^2,$$ (35)

where (1) follows from Lemma B.3. For $T_1$ in Eq. 35, we also have

$$T_1 \leq \eta_{i,t}^2 \nabla f_{B_i}^\top(x_i^t) \overline{G}_{i,t}^{-1} \nabla f_{B_i}^\top(x_i^t) \leq \frac{\eta_{i,t}^2}{\lambda_{min} (G_1, t)} \|\nabla f_{B_i}^\top(x_i^t)\|^2 \leq \eta_{i,t}^2 G^2 \lambda_{min}^{-1} (G_1, t),$$ (36)

where (1) follows from the fact $G_{i,t} \geq G_1, t$ and Lemma B.4. (2) follows from the gradient bounded condition, i.e., the forth item in Assumption 3. Combining Eq. 35 with Eq. 36, we obtain

$$\|x_{i+1}^t - x_i^t\|^2 \leq \min \{ \eta_{i,t}^2, \eta_{i,t}^2 G^2 \lambda_{min}^{-1} (G_1, t) \}$$ (37)

to complete the proof.
Corollary C.3. In SHAdaGrad, suppose that the Assumption 3 hold, and the perturbation $\delta_{i,t}$ is no decreasing. For any $1 \leq i \leq m$, we have
\[
\|x_{i}^t - x_{i}^{t}\|^2 \leq \min \left\{ \eta_{i,t}^{-2} \lambda_{\min}^{-1} (G_{1,t}) (i-1)^2 G^2, \eta_{i,t}^2 (i-1)^2 \right\},
\] (38)
when step size satisfies $\eta_{1,t} = \eta_{2,t} = \ldots = \eta_{m,t} = \eta_{.,t}$.

Proof. It can be easily checked that $i = 1$ holds in Eq. (38). Therefore, we only need to prove $i \geq 2$. We have
\[
\|x_{i}^t - x_{i}^{t}\|^2 = \sum_{j=1}^{i-1} (x_{j+1}^t - x_{j}^t)^2 \leq \sum_{j=1}^{i-1} \|x_{j+1}^t - x_{j}^t\|^2 \leq (i-1) \sum_{j=1}^{i-1} \|x_{j+1}^t - x_{j}^t\|^2,
\] (39)
where \(\oplus\) follows from the triangle inequality and \(\odot\) follows from the Cauchy-Schwarz inequality. According to the iteration of SHAdaGrad, for any $j \in I_{i-1}$ we have
\[
\|x_{j+1}^t - x_{j}^t\|^2 \leq \min \left\{ \eta_{i,t}^{-2} \lambda_{\min}^{-1} (G_{1,t}) \right\},
\] (40)
where \(\oplus\) follows from Lemma C.2. Combining Eq. (39) with Eq. (40) we obtain
\[
\|x_{i}^t - x_{i}^{t}\|^2 \leq \min \left\{ \eta_{i,t}^{-2} \lambda_{\min}^{-1} (G_{1,t}) (i-1)^2 G^2, \eta_{i,t}^2 (i-1)^2 \right\}.
\] (41)
Thus, we complete the proof.

Lemma C.4. In SHAdaGrad, suppose that the Assumption 3 hold, and the perturbation $\delta_{i,t}$ is no decreasing. If the step size in the $i$-th epoch satisfies $\eta_{i,t} \leq \frac{c^2}{16nL \delta^2}$, we have
\[
\sum_{j=1}^{m} \|\nabla f_{B_j}(x_i^t)\|^2 \geq \frac{c^2 m^2}{16n}
\]
with probability at least $1 - \exp \left(-\frac{m^2 c^4}{32 n^2 G^4} \right)$.

Proof. To simplify notations in the following proof, we set $s_i = \sum_{j=1}^{m} \|\nabla f_{B_j}(x_i^t)\|^2$. We have
\[
\mathbb{E}_{\sigma_i} [s_i] = \sum_{j=1}^{m} \left( \mathbb{E}_{B_j} \left[ \|\nabla f_{B_j}(x_i^t)\|^2 \right] \right).
\] (42)
With the symmetry of the permutation $\sigma_i$, there is $P [B_1^t = \tilde{\sigma}] = P [B_2^t = \tilde{\sigma}] = \ldots = P [B_m^t = \tilde{\sigma}]$ for any specific subset $\tilde{\sigma} \subseteq \mathbb{I}_m$ where $|\tilde{\sigma}| = |B_j^t|$, $\forall j \in \mathbb{I}_m$. Thus, when the sample size of mini-batch $|\tilde{\sigma}| = n/m \leq (n+1)/2$, the expectation $\mathbb{E}_{\sigma_i} [s_i]$ can be reformulated as
\[
\mathbb{E}_{\sigma_i} [s_i] = m \mathbb{E}_{\sigma} \left[ \|\nabla f_{\tilde{\sigma}}(x_i^t)\|^2 \right] = m \mathbb{E}_{\sigma} \left[ \|\nabla f_{\tilde{\sigma}}(x_i^t) - \mathbb{E}_{\tilde{\sigma}} \left[ \|\nabla f_{\tilde{\sigma}}(x_i^t)\|^2 \right] \right]
\]
\[
\geq m \mathbb{E}_{\tilde{\sigma}} \left[ \|\nabla f_{\tilde{\sigma}}(x_i^t) - \mathbb{E}_{\tilde{\sigma}} \left[ \|\nabla f_{\tilde{\sigma}}(x_i^t)\|^2 \right] \right] \geq m \left( 1 - \frac{n/m - 1}{n} \right) \geq \frac{c^2 m^2}{2n},
\] (43)
where \(\oplus\) is established because of the property of sampling without replacement variance. Besides, for any $i \in \mathbb{I}_m$, we have $s_i \in [0, mG^2]$. According to Lemma B.2, we have
\[
P \left[ s_i - \frac{c^2 m^2}{4n} \leq \frac{c^2 m^2}{8n} \right] \leq P \left[ s_i - \mathbb{E} [s_i] \leq \frac{c^2 m^2}{8n} \right] \leq \exp \left( -\frac{m^2 c^4}{32 n^2 G^4} \right).
\] (44)
That is to say, \( s_i \geq \frac{c^2 m^2}{8n} \) establishes with probability at least \( 1 - \exp\left( -\frac{m^2 c^4}{32n G^4} \right) \). When the step size in the \( i \)-th epoch is small enough, i.e., \( \eta_{.,i} \leq \frac{c^2 m^2}{16n G} \), we have

\[
\sum_{j=1}^{m} \| \nabla f_{B_j}(x_1^i) \| \cdot \| \nabla f_{B_j}(x_j^i) - \nabla f_{B_j}(x_1^i) \| \\
\leq G \sum_{j=1}^{m} \| \nabla f_{B_j}(x_j^i) - \nabla f_{B_j}(x_1^i) \| \leq LG \sum_{j=1}^{m} \| x_j^i - x_1^i \| \\
\leq LG \sum_{j=1}^{m} \sum_{k=1}^{j-1} \| x_{k+1}^i - x_k^i \| \leq LG \sum_{j=1}^{m} \sum_{k=1}^{j-1} \eta_{.,i} = LG \sum_{j=1}^{m} \eta_{.,i} (j-1) \\
\leq \frac{LG \sum_{j=1}^{m} \eta_{.,i} m^2}{2} \leq \frac{c^2 m^2}{32n},
\]

(45)

where \( \odot \) follows from Lemma C.2. Thus, we have

\[
\sum_{j=1}^{m} \| \nabla f_{B_j}(x_j^i) - \nabla f_{B_j}(x_1^i) + \nabla f_{B_j}(x_1^i) \|^2 \geq \sum_{j=1}^{m} \left[ \| \nabla f_{B_j}(x_1^i) \| - \| \nabla f_{B_j}(x_j^i) - \nabla f_{B_j}(x_1^i) \| \right]^2 \\
\geq \sum_{j=1}^{m} \| \nabla f_{B_j}(x_1^i) \|^2 - 2 \sum_{j=1}^{m} \| \nabla f_{B_j}(x_1^i) \| \cdot \| \nabla f_{B_j}(x_j^i) - \nabla f_{B_j}(x_1^i) \| \overset{\odot}{\geq} \frac{c^2 m^2}{16n}
\]

(46)

with probability at least \( 1 - \exp\left( -\frac{m^2 c^4}{32n G^4} \right) \), where we have \( \odot \) due to Eq. 44 and Eq. 45 Then, the proof is completed. \( \square \)
D Convergence Rate of SHAdaGrad on Non-Convex and Shuffling Settings

Lemma D.1. In SHAdaGrad, suppose that the Assumption 3 and Assumption 4 hold. If the hyper-parameter $\delta_{j,i}$, $\Gamma$ and the step size $\eta$ satisfy

$$\delta_{j,i} = \sum_{p=q=1}^{i-1} \left\| \nabla f_{B_{i}^{t}}(x_{j}^{p}) \right\|^2 + \sum_{q=1}^{j} \left\| \nabla f_{B_{i}^{t}}(x_{i}^{q}) \right\|^2, \quad \forall \ i \in \mathbb{I}, \ j \in \mathbb{I}, \ 1 \leq \Gamma \leq n \quad \text{and} \quad \eta \leq \frac{c^2}{16nLG}.$$

(47)

Then, we have

$$\frac{\eta_{t}}{4m} \left( \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{m}^{t}) \right) \neq \frac{m}{2} \left( \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \leq f(x_{1}^{t}) - f(x_{m+1}^{t}) + \min \left\{ \frac{2\eta_{i}^{2}L^2G^2m^3}{3\lambda_{min}(G_{1,t})}, \frac{2\eta_{i}^{3}L^2m^3}{3\sqrt{\lambda_{min}(G_{m,t})}} \right\} + \min \left\{ \frac{3\eta_{i}m^{1,5}G^3}{\sqrt{2} \lambda_{min}(G_{1,t})}, \frac{3\eta_{i}m^{1,5}G}{\sqrt{2}} \right\},$$

(48)

for the $t$-th epoch in SHAdaGrad.

Proof. According to the iteration of SHAdaGrad, when $1 \leq i \leq j \leq m$, the outer product matrix $G_{i,t}$ have the following properties

$$G_{j,i} \geq G_{i,t} \geq G_{i,t} \geq G_{i,t} \geq G_{i,t},$$

(49)

where $\odot$ follows from Lemma B.3 and $\oplus$ follows from Lemma B.4.

With the $L$–Lipschitz continuous gradient assumption, the second item in Assumption 3 we have

$$f(x_{m+1}^{t}) - f(x_{1}^{t}) \leq \nabla f^{T}(x_{1}^{t}) \left( x_{m+1}^{t} - x_{1}^{t} \right) + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$= \nabla f^{T}(x_{1}^{t}) \left[ \sum_{i=1}^{m} -\eta_{i} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right] + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$= \nabla f^{T}(x_{1}^{t}) \left[ \sum_{i=1}^{m} \left( -\eta_{i} \nabla f_{B_{i}^{t}}(x_{1}^{t}) + \eta_{i} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \right] + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$- \sum_{i=1}^{m} \eta_{i} \nabla f^{T}(x_{1}^{t}) G_{m,t}^{-\frac{1}{2}} \nabla f_{B_{i}^{t}}(x_{1}^{t}).$$

(50)

We set the step sizes of different iterations to be the same in one epoch, i.e., $\eta_{1,t} = \eta_{2,t} = \ldots = \eta_{m,t} = \eta_{t}$. Then, we obtain

$$f(x_{m+1}^{t}) - f(x_{1}^{t}) \leq \eta_{t} \nabla f^{T}(x_{1}^{t}) \left[ \sum_{i=1}^{m} \left( G_{m,t}^{-\frac{1}{2}} \nabla f_{B_{i}^{t}}(x_{1}^{t}) - G_{i,t}^{-\frac{1}{2}} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \right] + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$- \eta_{t} \left( \nabla f(x_{1}^{t}) + \frac{1}{m} \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{1}^{t}) + \frac{1}{m} \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right)$$

$$= \eta_{t} \nabla f^{T}(x_{1}^{t}) \left[ \sum_{i=1}^{m} \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right] + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$+ \eta_{t} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_{i}^{t}}(x_{1}^{t}) - \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \right] \nabla f_{B_{i}^{t}}(x_{1}^{t}) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$- \eta_{t} \left( \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right)$$

$$= \eta_{t} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_{i}^{t}}(x_{1}^{t}) - \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \right] \nabla f_{B_{i}^{t}}(x_{1}^{t}) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) + \frac{L}{2} \left\| x_{m+1}^{t} - x_{1}^{t} \right\|^2$$

$$- \eta_{t} \left( \sum_{i=1}^{m} \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right) \nabla f_{B_{i}^{t}}(x_{1}^{t}) \right)$$

$$\leq \eta_{t} \min \left\{ \frac{3\eta_{i}m^{1,5}G^{3}}{\sqrt{2} \lambda_{min}(G_{1,t})}, \frac{3\eta_{i}m^{1,5}G}{\sqrt{2}} \right\},$$

(51)
We next bound $T_1$ and $T_2$ separately. First for $T_1$ in Eq. \ref{eq:T1} we have

$$T_1 = \eta_{t,t} \left( \nabla f(x_1^t) - \frac{1}{m} \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) + \frac{1}{m} \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]$$

$$= \eta_{t,t} \left( \nabla f(x_1^t) - \frac{1}{m} \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]$$

$$+ \frac{\eta_{t,t}}{m} \left( \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]. \tag{52}$$

For $T_{1,1}$ we have

$$T_{1,1} \equiv \eta_{t,t} \left( \nabla f(x_1^t) - \frac{1}{m} \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \nabla f_{B_t^i}(x_i^t)$$

$$\leq \frac{\eta_{t,t}}{m} \left[ \sum_{i=1}^m \left( \nabla f_{B_t^i}(x_i^t) - \nabla f_{B_t^i}(x_1^t) \right) \right] ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \left( \nabla f_{B_t^i}(x_i^t) - \nabla f_{B_t^i}(x_1^t) \right) \right]$$

$$+ \frac{\eta_{t,t}}{4m} \left( \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]. \tag{53}$$

where $\odot$ follows from the invertibility of $G_{m,t}$ and $\ominus$ follows from the Cauchy-Schwarz inequality.

Similarly, for $T_{1,2}$ we have

$$T_{1,2} = \frac{\eta_{t,t}}{m} \left( \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \nabla f_{B_t^i}(x_i^t)$$

$$\leq \eta_{t,t} \left[ \frac{1}{4m} \left( \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) \right] ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right]$$

$$+ \frac{1}{m} \left( \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]. \tag{54}$$

Submitting Eq. \ref{eq:T11} and Eq. \ref{eq:T12} back into Eq. \ref{eq:T1} we obtain

$$T_1 \leq \frac{\eta_{t,t}}{m} \left[ \sum_{i=1}^m \left( \nabla f_{B_t^i}(x_i^t) - \nabla f_{B_t^i}(x_1^t) \right) \right] ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \left( \nabla f_{B_t^i}(x_i^t) - \nabla f_{B_t^i}(x_1^t) \right) \right]$$

$$+ \frac{5\eta_{t,t}}{4m} \left( \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \left[ \sum_{i=1}^m \left( G_{m,t}^{-\frac{1}{2}} - G_{i,t}^{-\frac{1}{2}} \right) \nabla f_{B_t^i}(x_i^t) \right]$$

$$+ \frac{\eta_{t,t}}{4m} \left( \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right) ^\top G_{m,t}^{-\frac{1}{2}} \left( \sum_{i=1}^m \nabla f_{B_t^i}(x_i^t) \right). \tag{55}$$
Similarly, $T_2$ in Eq. 51 satisfies

$$T_2 = \frac{\eta_{t,t}}{4m} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \right]^{\top} \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right).$$

As a result, we plug Eq. 55 and Eq. 56 into Eq. 51 and obtain that

$$f(x_{m+1}^t) - f(x_i^t) \leq 2\eta_{t,t} \frac{m}{m} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \right]^{\top} \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) + \frac{L}{2} \left\| x_{m+1}^t - x_i^t \right\|^2. \tag{57}$$

For $T_1$ in Eq. 57, we have

$$T_1 = \frac{2\eta_{t,t}}{m} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \right]^{\top} \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{\sqrt{\lambda_{\min}}}.$$  

$$\leq 2\eta_{t,t} \frac{m}{m} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \right]^{\top} \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{\sqrt{\lambda_{\min}}}.$$  

$$\leq \frac{2\eta_{t,t} L^2}{\sqrt{\lambda_{\min}}} \sum_{i=1}^{m} \left\| x_i^t - x_i^t \right\|^2 \leq \frac{2\eta_{t,t} L^2}{\sqrt{\lambda_{\min}}} \sum_{i=1}^{m} \left\{ \lambda_{\min}^{-1} (G_{1,t}) G^2, 1 \right\} \sum_{i=1}^{m} (i - 1)^2 \leq \frac{2\eta_{t,t} L^2 G^2 m^3}{3 \lambda_{\min} G_{1,t}} \cdot \frac{2\eta_{t,t} L^2 m^3}{3 \lambda_{\min} G_{m,t}} \tag{58}$$

where (1) follows from the L-Lipschitz continuous gradient assumption, the second item in Assumption 3 and (2) follows from Corollary C.3.

With similar techniques, we relax $T_2$ in Eq. 57 as follows

$$T_2 = \frac{5\eta_{t,t}}{4m} \left[ \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \right]^{\top} \sum_{i=1}^{m} \left( \nabla f_{B_t}(x_i^t) - \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right).$$  

$$= \frac{5\eta_{t,t}}{4m} \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right) \frac{1}{G_{m,t}^\frac{1}{2}} \left( \sum_{i=1}^{m} \nabla f_{B_t}(x_i^t) \right). \tag{59}$$
where follows from the Cauchy-Schwarz inequality. For each in the last equation of Eq. we have

\[
\nabla f_i^T(x_i^t) \left( G_{m,t} - G_{i,t} \right) G_{m,t}^{-1} \left( G_{m,t} - G_{i,t} \right) \nabla f_i^T(x_i^t)
\]

\[
= \nabla f_i^T(x_i^t) \left( G_{m,t}^2 - 2G_{i,t}^2 + G_{i,t} G_{m,t}^2 \right) \nabla f_i^T(x_i^t)
\]

\[
\leq \nabla f_i^T(x_i^t) \left( G_{m,t}^2 - G_{i,t}^2 \right) \nabla f_i^T(x_i^t) = \nabla f_i^T(x_i^t) G_{m,t}^{-1} \left( G_{m,t}^2 - G_{i,t}^2 \right) \nabla f_i^T(x_i^t)
\]

where follows from \(G_{m,t}^2 - G_{i,t}^2 \leq 0\) stated in Eq. 49. follows from Lemma C.1 and follows from the gradient upper bound assumption, the forth point in Assumption 3 and Lemma B.5. After submitting Eq. back into Eq. we have

\[
T_2 \leq \min \left\{ \frac{5\eta_{,t}m\lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2\lambda_{min}(G_{1,t})}}, \frac{5\eta_{,t}m\lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2}} \right\}.
\]

For \(T_3\) in Eq. we have

\[
T_3 = \frac{L}{2} \left[ \sum_{i=1}^{m} -\eta_{,t} G_{i,t}^2 \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right]^{}\|G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right]^{\top} \| \leq L\eta_{,t}^2 \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right]^{\top} \leq \frac{\lambda_{min}(G_{m,t}^2)}{4mL}
\]

Hence, if the step size is small enough, we then obtain

\[
\eta_{,t} \leq \frac{c_{\sigma}}{16nLG} \quad \frac{c_{\sigma}}{16nL} \leq \frac{c_{\sigma} m}{4\sqrt{n}L} \leq \frac{c_{\sigma}}{4mL} \leq \frac{\lambda_{min}(G_{m,t}^2)}{4mL}
\]

\[
\Rightarrow \eta_{,t} I \leq \frac{4mL}{G_{m,t} G_{m,t}} \leq \frac{4mL}{G_{m,t}^2} \leq \frac{\lambda_{min}(G_{m,t}^2)}{4mL}
\]

\[
\Rightarrow \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \nabla f_i^T(x_i^t) \leq \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \nabla f_i^T(x_i^t) \leq \frac{\lambda_{min}(G_{m,t}^2)}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right)
\]

where follows from the definition of constant \(G\). follows from Lemma C.4 and follows from Lemma B.1. Besides, with the same step size upper bound, we have

\[
L\eta_{,t}^2 \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right]^{\top} \|G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \| \leq \frac{\eta_{,t}}{4mL} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right) \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right) \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right) \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right) \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right) \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( G_{m,t}^{-1} \left[ \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right] \left( G_{m,t}^{-1} - G_{m,t}^{-1} \right) \nabla f_i^T(x_i^t) \right)
\]

where follows from Eq. and follows from similar techniques with Eq. Hence, combining Eq. with Eq. we obtain

\[
T_3 \leq \frac{\eta_{,t}}{4mL} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right)^{\top} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) + \min \left\{ \frac{\eta_{,t} m \lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2}}, \frac{\eta_{,t} m \lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2}} \right\}. \]

Submitting Eq. and Eq. back into Eq. we obtain

\[
\eta_{,t} \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right)^{\top} \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \left( \sum_{i=1}^{m} \nabla f_i^T(x_i^t) \right) + \min \left\{ \frac{\eta_{,t} m \lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2}}, \frac{\eta_{,t} m \lambda_{min}(G_{1,t})^{1.5}}{2\sqrt{2}} \right\}
\]

\[
to complete the proof.
Lemma D.2. In SHAdaGrad, suppose that the Assumption 3 and Assumption 4 hold, $\Gamma \geq 1$, the perturbation $\delta_{i,i}$ is no decreasing and $\delta_{m,i} \leq imG^2$, we have

$$
\begin{align*}
\sum_{j=1}^{m} \nabla f_{B_j}(x^*_j)^\top G_{m,i}^{-\frac{1}{2}} \sum_{j=1}^{m} \nabla f_{B_j}(x^*_j) & \geq \sqrt{\frac{\delta_{m,i} - \delta_{m,i-1}}{4G^2imd\Gamma}} \sum_{j=1}^{m} \nabla f_{B_j}(x^*_j) \top (G_{m,i} - G_{m,i-1})^{-\frac{1}{2}} \sum_{j=1}^{m} \nabla f_{B_j}(x^*_j) \\
\text{(67)}
\end{align*}
$$

Proof. To simplify notations in the following analysis, we set $H_{m,i} := \sum_{\tau=1}^{\ell} H_{m,i} \Gamma_{m,i,k}^{\top}$. Then, we have

$$
\begin{align*}
\left[tr\left(H_{m,i}^{\frac{1}{2}}\right)\right]^2 & = \left[\sqrt{\lambda_1\left(H_{m,i}\right)} + \sqrt{\lambda_2\left(H_{m,i}\right)} + \ldots + \sqrt{\lambda_d\left(H_{m,i}\right)}\right]^2 \\
& \leq \left[\lambda_1\left(H_{m,i}\right) + \lambda_2\left(H_{m,i}\right) + \ldots + \lambda_d\left(H_{m,i}\right)\right] \cdot d = \text{tr}\left(H_{m,i}\right) \cdot d \leq imG^2d \\
\text{(68)}
\end{align*}
$$

where $\odot$ establishes because of the definition of matrix $H_{m,i}$:

$$
\text{tr}\left(H_{m,i}\right) = \sum_{j=1}^{m} \sum_{k=1}^{m} \left\| \nabla f_{B_j}(x_k) \right\|^2 \leq imG^2. \\
\text{(69)}
$$

According to the definition of $G_{m,i} \in \mathbb{R}^{d \times d}$ in Eq. (24), we have

$$
\lambda_{max}\left(G_{m,i}^{\frac{1}{2}}\right) = \lambda_{max}\left(H_{m,i}^{\frac{1}{2}} + \frac{\delta_{m,i}}{\Gamma} I\right) \substack{\odot \geq \lambda_{max}\left(H_{m,i}^{\frac{1}{2}} + \frac{\delta_{m,i}}{\Gamma} I\right) \\
\text{where } \odot \text{ follows from the fact } (M + c \cdot I)^{\frac{1}{2}} \leq M^{\frac{1}{2}} + \sqrt{c} \cdot I, \forall A \succeq 0.} \\
\text{and } \delta_{m,i} \leq imG^2. \\
\text{(70)}
$$

Hence, if we set

$$
\beta_i = \sqrt{\frac{\delta_{m,i} - \delta_{m,i-1}}{4G^2imd\Gamma}}, \\
\text{(72)}
$$

then, we obtain

$$
\beta_i\lambda_{max}\left(G_{m,i}^{\frac{1}{2}}\right)^\odot \sqrt{\frac{\delta_{m,i} - \delta_{m,i-1}}{4G^2imd\Gamma}} \substack{\odot \lambda_{min}\left(G_{m,i} - G_{m,i-1}\right)^\frac{1}{2} \\
\text{where } \odot \text{ follows from Eq. (70) and } \odot \text{ follows from Eq. (71).} \\
\text{With the fact } G_{m,i}^{\frac{1}{2}}, (G_{m,i} - G_{m,i-1})^{\frac{1}{2}} \text{ are positive-definite matrices, we have}} \\
\beta_iG_{m,i}^{\frac{1}{2}} \odot (G_{m,i} - G_{m,i-1})^{\frac{1}{2}} \odot \beta_i(G_{m,i} - G_{m,i-1})^{-\frac{1}{2}} \leq G_{m,i}^{\frac{1}{2}} \\
\Rightarrow \sum_{j=1}^{m} \nabla f_{B_j}(x_j)^\top \left[\beta_i\left(G_{m,i} - G_{m,i-1}\right)^{-\frac{1}{2}} \sum_{j=1}^{m} \nabla f_{B_j}(x_j)\right] \\
\leq \sum_{j=1}^{m} \nabla f_{B_j}(x_j)^\top \left[\beta_i\left(G_{m,i} - G_{m,i-1}\right)^{-\frac{1}{2}} \sum_{j=1}^{m} \nabla f_{B_j}(x_j)\right] \\
\text{(74)}
$$

where $\odot$ follows from Lemma B.5. Hence, we complete the proof. \qed
Lemma D.3. In SHAdaGrad, suppose that the Assumption 3 and Assumption 4 hold, and the perturbation \( \delta_{i,t} \) is no decreasing. If \( 0 \leq \delta_{m,i} - \delta_{m,i-1} \leq m \lambda_{\max} \) \( \left( H_{m,i} H_{m,i}^T \right) \) and \( \Gamma \geq m \) then we have

\[
\sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right)^T \left( G_{m,i} - G_{m,i-1} \right) - \frac{1}{2} \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \geq \sqrt{\frac{m}{2e^{2\lambda}}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right\| \tag{75}
\]

Proof. To simplify notations, we abbreviate \( H_{m,i} \) as \( H \) whose SVD can be formulated as

\[
H = U \Sigma V^T, \quad U \in \mathbb{R}^{d \times d}, \Sigma \in \mathbb{R}^{d \times m}, V \in \mathbb{R}^{m \times m}, \tag{76}
\]

where \( U \) and \( V \) are unitary matrices. Specifically, with Assumption 4 \( \Sigma \) and \( V \) can be written as

\[
\Sigma = \begin{bmatrix} \Sigma_0 \end{bmatrix}, \quad \Sigma = \text{diag} \{ \lambda_1, \lambda_2, \ldots, \lambda_m \}, \quad V = [v_1 \ v_2 \ \ldots \ v_m], \quad v_i \in \mathbb{R}^{m \times 1}, \quad \forall i \in \mathbb{I}_m. \tag{77}
\]

Hence, we can reformulate \( \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right\| \) as

\[
\left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right\|^2 = H_{m,i} \left[ \sum_{j=1}^{m} e_j \right]^2 = H^T H \sum_{j=1}^{m} e_j = \left[ \sum_{j=1}^{m} e_j \right] V \Sigma^2 V^T \left[ \sum_{j=1}^{m} e_j \right] = \left( \sum_{j=1}^{m} \lambda_j^2 \right) \left[ \sum_{j=1}^{m} e_j \right]^2 \tag{78}
\]

where \( e_i \) denotes the 0-1 vector whose \( i \)-th coordinate is 1 while other coordinates are 0s. Besides, \( \odot \) in Eq. 78 is established because we set

\[
\sum_{k=1}^{m} e_k = \gamma_1 v_1 + \gamma_2 v_2 + \ldots + \gamma_m v_m, \quad \left\| \sum_{k=1}^{m} e_k \right\|^2 = \sum_{k=1}^{m} \gamma_k^2 = m. \tag{79}
\]

with the full-rank property of matrix \( V \). In addition, we have

\[
\left[ \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right]^T \left( G_{m,i} - G_{m,i-1} \right) - \frac{1}{2} \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \geq \sqrt{\frac{m}{2e^{2\lambda}}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right\|
\]

\[
\geq \frac{\lambda_m^2}{\lambda_1^2 + \delta_{m,i-1}} \sum_{j=1}^{m} \gamma_j^2 = \frac{\lambda_m^2 m}{\lambda_1^2 + \delta_{m,i-1}}. \tag{80}
\]
As a result, if \( \Gamma \geq m \) is established then we have \( \delta_{m,i} - \delta_{m,i-1} \leq \Gamma \tilde{\lambda}_1^2 \) (with the definition of \( \tilde{\lambda}_1 \)) and

\[
\sum_{j=1}^{m} \left\| \nabla f_{B_j}(x_j^i) \right\| \geq \sqrt{\frac{\tilde{\lambda}_1^2 m}{\lambda_1 + \frac{\delta_{m,i} - \delta_{m,i-1}}{1}}} \sum_{j=1}^{m} \frac{\tilde{\lambda}_1^2 \gamma_j^2}{\lambda_1} \geq \frac{\tilde{\lambda}_1^2 m}{\lambda_1 + \frac{\delta_{m,i} - \delta_{m,i-1}}{1}} \sum_{j=1}^{m} \tilde{\lambda}_1^2 \gamma_j^2, \tag{81}
\]

with Assumption 3 to complete the proof.

**Theorem D.4.** In SHAdaGrad, suppose that the Assumption 3 and Assumption 4 hold. If hyper-parameters \( \delta_{j,i}, \Gamma \) and the step size \( \eta \) satisfy

\[
\delta_{j,i} = \frac{i-1}{p=1} \sum_{q=1}^{m} \left\| \nabla f_{B_q}(x_q^i) \right\|^2 + \sum_{q=1}^{j} \left\| \nabla f_{B_q}(x_q^i) \right\|^2, \forall i \in I_T, j \in \mathbb{I}_m, \quad m \leq \Gamma \leq n, \quad \text{and} \quad \eta \leq \frac{c_2^2}{16nLG}, \tag{83}
\]

we have

\[
\mathbb{E}_t \left[ \left\| \nabla f(x_t^i) \right\| \right] \leq \frac{C_0}{\eta \sqrt{T}} + \frac{C_1}{\sqrt{T}} + \frac{C_2 \eta^2}{\sqrt{T}} + \frac{C_3 \eta^2}{\sqrt{T} (1 + T)} + \frac{C_4 \eta \ln(T)}{\sqrt{T}}. \tag{84}
\]

where \( C_0, C_1, \ldots, C_5 \) are constants and defined in the proof.

**Proof.** According to the definition of \( \delta_{j,i} \), we have \( \delta_{m,i} \leq i m G^2 \) due to the gradient norm upper bound assumption, i.e., the forth item in Assumption 3. Besides, we have

\[
\frac{c_2^2 m^2}{16n} \leq \sum_{j=1}^{m} \left\| \nabla f_{B_j}(x_j^i) \right\|^2 = \delta_{m,i} - \delta_{m,i-1} = \text{tr}(H_{m,i} H_{m,i}^\top) \leq m \cdot \lambda_{\max} \left( H_{m,i} H_{m,i}^\top \right), \tag{85}
\]

where \( \circ \) follows from Lemma C.1. Then, we have

\[
\frac{c_2 m}{8 \sqrt{2Gc_\kappa}} \cdot \frac{1}{\sqrt{1 + \frac{1}{n}}} \cdot \frac{1}{\sqrt{i}} \cdot \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) = \frac{c_2 m}{4 \sqrt{n}} \cdot \frac{1}{\sqrt{m}} \cdot \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \leq \frac{\sqrt{\delta_{m,i} - \delta_{m,i-1}}}{2G \sqrt{imdT}} \cdot \sqrt{\frac{m}{2c_\kappa}} \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \tag{86}
\]

\[
\leq \left( \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right) \frac{\delta_{m,i} - \delta_{m,i-1}}{4G^2 \sqrt{imdT}} \left( G_{m,i} - G_{m,i-1} \right)^{\frac{1}{2}} \left( \sum_{j=1}^{m} \nabla f_{B_j}(x_j^i) \right),
\]

where \( \circ \) follows from Lemma C.2. Then, we have
where $\circlearrowleft$ is established due to Eq. 85, $\circlearrowright$ follows from Lemma D.3 and $\circlearrowdown$ follows from Lemma D.2. Hence, we obtain

$$\frac{c_\sigma}{48Gc_\sigma\sqrt{nd}\Gamma} \cdot \frac{\eta_{i}}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| \leq \frac{\eta_{i}}{4m} \left( \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right)^T G_{m,i}^{-1} \left( \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right)$$

$$\circlearrowleft f(x_i^1) - f(x_{m+1}^i) + \frac{2\eta_{i}^3L^2G^2m^3}{3\lambda_{\min}^1(G_{1,i})} + \frac{3\eta_{i}m^{1.5}G^3}{\sqrt{2}\lambda_{\min}(G_{1,i})}$$

$$\circlearrowright f(x_i^1) - f(x_{m+1}^i) + \frac{128\eta_{i}^3L^2G^2m^{1.5}\Gamma^{1.5}}{3c_\sigma^2(i-1)^{1.5}} + \frac{24\sqrt{2}\eta_{i}mG^3}{c_\sigma^2(i-1)m^{0.5}}$$

(87)

where $\circlearrowleft$ follows from Lemma D.1 and $\circlearrowright$ is established when $i \geq 2$ due to the fact

$$\lambda_{\min}(G_{1,i}) \geq \lambda_{\min}(G_{m,i-1}) \geq \frac{\delta_{m,i-1}}{\Gamma} \geq \frac{\sum_{j=1}^{i-1} \sum_{k=1}^{m} \left\| \nabla f_{B_k}(x_i^k) \right\|^2}{\Gamma} \geq \frac{(i-1)c_\sigma^2m^2}{16n\Gamma}$$

(88)

Inequality $\circlearrowleft$ in Eq. 88 follows from Lemma C.4. It should be noticed that when $i = 1$, there is

$$\frac{c_\sigma}{48Gc_\sigma\sqrt{nd}\Gamma} \cdot \eta_{i} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| \circlearrowright f(x_i^1) - f(x_{m+1}^i) + \frac{2\eta_{i}^3L^2m^3}{3\sqrt{\lambda_{\min}(G_{m,t})}} + \frac{3\eta_{i}m^{1.5}G}{\sqrt{2}}$$

$$\leq f(x_i^1) - f(x_{m+1}^i) + \frac{8\eta_{i}^3L^2m^{2.5}\Gamma^{0.5}}{3c_\sigma} + \frac{3\eta_{i}m^{1.5}G}{\sqrt{2}}$$

(89)

where $\circlearrowleft$ follows from Lemma D.1. To achieve some stationary point through SHAdaGrad, for each epoch, we have

$$\frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| \leq \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \left( \nabla f_{B_j}(x_i^j) - \nabla f_{B_j}(x_i^1) \right) \right\|$$

$$\leq \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \left( \nabla f_{B_j}(x_i^j) - \nabla f_{B_j}(x_i^1) \right) \right\|$$

$$\leq \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + \frac{L}{\sqrt{i}} \sum_{j=1}^{m} \left( \min \left\{ \frac{\eta_{i}(j-1)G}{\sqrt{\lambda_{\min}(G_{1,i})}}, \eta_{i}(j-1) \right\} \right)$$

$$\leq \frac{1}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + \frac{2L\eta_{i}m^{0.5}\Gamma^{0.5}}{c_\sigma(i-1)}$$

(90)

where $\circlearrowleft$ follows from Lemma C.3 and $\circlearrowright$ follows from Eq. 88 when $i \geq 2$. Notice that if $i = 1$, we have

$$\left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| \leq \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + L\eta_{i}m^2$$

(91)

Thus, combining Eq. 90 with Eq. 87 when $i \geq 2$, we obtain

$$\frac{\eta_{i}}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| \leq \frac{\eta_{i}}{\sqrt{i}} \left\| \sum_{j=1}^{m} \nabla f_{B_j}(x_i^j) \right\| + \frac{2L\eta_{i}^2m^{0.5}\Gamma^{0.5}}{c_\sigma(i-1)}$$

$$\leq \frac{128L^2G^2\eta_{i}^3m^{1.5}\Gamma^{1.5}}{3c_\sigma^2(i-1)^{1.5}} + \frac{24\sqrt{2}G^3\eta_{i}m\Gamma}{c_\sigma^2(i-1)m^{0.5}}$$

$$\leq \frac{2L\eta_{i}^2m^{0.5}\Gamma^{0.5}}{c_\sigma(i-1)}$$

(92)
where \( \circ \) follows from Eq. 87. Then, we set \( \eta_{\cdot, i} = \eta \) for all \( 1 \leq i \leq T \). Summing up Eq. 92 for \( 1 \leq i \leq T \) and dividing both sides by \( m \eta \), we obtain

\[
\sum_{i=1}^{T} \frac{1}{m \sqrt{T}} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right| = \frac{1}{m} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right| + \sum_{i=2}^{T} \frac{1}{m \sqrt{i}} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right|
\]

\[
\leq \frac{1}{m} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right| + L \eta m + \sum_{i=2}^{T} \frac{1}{m \sqrt{i}} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right|
\]

\[
\ Dems \leq \frac{48Gc_k \sqrt{n d T}}{c \sigma m \eta} \left[ f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right] + L \eta m + \sum_{i=2}^{T} \frac{1}{m \sqrt{i}} \left| \sum_{j=1}^{m} \nabla f_{B_j}(x_1^i) \right|
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

\[
\leq 48Gc_k \sqrt{n d T} \left( f(x_1^1) - f(x_1^{m+1}) + \frac{8n^3L^2m^2n^0.5T^0.5}{3c \sigma} + \frac{3m^3G}{2\sqrt{2}} \right) + L \eta m + \sum_{i=2}^{T} \left( \frac{2LG \eta ^{0.5} T^{0.5}}{c \sigma (i-1)} \right)
\]

where \( \circ \) follows from Eq. 91 \( \circ \) follows from Eq. 89 and \( \circ \) follows from Eq. 92. With the following constants

\[
C_0 = \frac{48Gc_k}{c \sigma} \left[ f(x_1^1) - f(x_1^{m+1}) \right] \cdot n^{0.5} m^{-1.5} d^{0.5} \Gamma^{-0.5}, \quad C_1 = \frac{72 \sqrt{2} G^2 c_k}{c \sigma} \cdot n^{0.5} m^{0.5} d^{0.5} \Gamma^{-0.5},
\]

\[
C_2 = L m, \quad C_3 = \frac{128Gc_k L^2 c_k}{c^2 \sigma} \cdot n^{0.5} m^{-1.5} d^{0.5} \Gamma^{-2}, \quad C_4 = \frac{3^2 \cdot 2^8 \sqrt{2} G^4 c_k}{c^3 \sigma} \cdot n^{1.5} m^{-1.5} d^{0.5} \Gamma^{-1.5}, \quad C_5 = \frac{4L G}{c \sigma} \cdot n^{0.5} m^{0.5} d^{0.5},
\]

we have

\[
\mathbb{E}_\mathcal{T} \left[ \| \nabla f(x_1^i) \| \right] = \frac{\sum_{i=1}^{T} \frac{1}{\sqrt{T}} \left| \sum_{j=1}^{m} \frac{1}{m} \nabla f_{B_j}(x_1^i) \right|}{\sum_{i=1}^{T} \frac{1}{\sqrt{T}}} \leq \frac{C_0}{\eta \sqrt{T}} + \frac{C_1}{\sqrt{\Gamma T}} + \frac{C_2 \eta}{\sqrt{T}} + \frac{C_3 \eta^2}{\sqrt{T}} + \frac{C_4 \ln(T)}{\sqrt{\Gamma T}} + \frac{C_5 \eta \ln(T)}{\sqrt{T}},
\]

if we sample from \( \mathcal{I} \) with probability \( \mathbb{P}[x = i] = \frac{1}{\sqrt{T}} \). It means we achieve some stationary points \( \| \nabla f(x) \| \leq \epsilon \) within \( \tilde{O}(T^{-0.5}) \) in expectation when we set the step size as \( \eta = \Theta(1) \).

**Corollary D.5.** Let \( \{x_1^i\} \) be the sequence generated by SHAAdaGrad and \( x_{out} \) be its output. For given tolerance \( \epsilon > 0 \), under the same conditions as Theorem D.4, if we choose the constant learning rate \( \eta = \frac{c^2}{16mLG} \), \( \Gamma = m \) and the number of iteration in each epoch \( m = n \), then to guarantee

\[
\mathbb{E}_\mathcal{T} \left[ \| \nabla f(x_1^i) \| \right] = \frac{\sum_{i=1}^{T} \frac{1}{\sqrt{T}} \left| \sum_{j=1}^{m} \frac{1}{m} \nabla f_{B_j}(x_1^i) \right|}{\sum_{i=1}^{T} \frac{1}{\sqrt{T}}} \leq \epsilon,
\]

it requires nearly \( T = [36C_{max} n^3 d c^{-2}] \) outer iterations, where \( C_{max} \) is set as

\[
C_{max} = \max \left\{ \frac{384LG^2 c_k (f(x_1^1) - f^*)}{c \sigma}, \frac{72 \sqrt{2} G^2 c_k}{c \sigma}, \frac{c^2}{2G}, \frac{24Gc_k}{c \sigma}, \frac{3^2 \cdot 2^8 \sqrt{2} G^4 c_k}{c^3 \sigma}, \frac{c \sigma}{4} \right\}
\]

(95)
In expectation, the total number of gradient evaluation is nearly \( T = \left\lfloor 36 \left[ 1 - \exp \left( -\frac{c^4}{32G^4} \right) \right]^{-1} C_{\max} n^4 \delta^{-2} \right\rfloor \).

**Proof.** According to Theorem D.4, if we set \( \eta = \frac{c^2}{16nLG} \), \( \Gamma = m \) and \( m = n \), we obtain that

\[
E_t [\| \nabla f (x_t^i) \|] \leq \frac{384L^2 c_c (f(x_t^i) - f^*)}{c^3} \cdot \sqrt{\frac{n^2d}{T}} + \frac{72\sqrt{2G^2 c_c}}{c_c} \cdot \sqrt{\frac{n^3d}{T}} + \frac{c^2}{16G} \cdot \frac{1}{\sqrt{T}} + \left( \frac{c^2}{2G} + \frac{24Gc_c}{L^2} \right) \cdot \sqrt{\frac{n^2d}{T}} + \frac{3^2 \cdot 2^8 \sqrt{2G^4 c_c}}{c^3} \cdot \sqrt{\frac{n^3d}{T}} \ln(T) + \frac{c_c}{4} \cdot \sqrt{\frac{d}{T}} \ln(T)
\]

\[
\leq 6C_{\max} \cdot n^{1.5} \sqrt{d} \cdot \frac{\ln(T)}{\sqrt{T}}.
\]

(96)

Hence, a sufficient condition for achieving FSPs \( E_t [\| \nabla f (x_t^i) \|] \leq \epsilon \) for the objective can be presented as

\[
6C_{\max} \cdot n^{1.5} \sqrt{d} \cdot \frac{\ln(T)}{\sqrt{T}} \leq \epsilon \Leftrightarrow T \geq 36C_{\max} n^4 \delta^{-2},
\]

(97)

where \( \Leftrightarrow \) is established when we ignore the \( \ln \) term.

Besides, for the inner loops, we utilize the rejection sampling to provide a lower bound of \( \delta_{m_t} \). According to Lemma C.4, we can notice that probability of success is at least \( p := 1 - \exp \left( -\frac{m^2 c^4}{32n^2 G^4} \right) \) in every trial (a Bernoulli distribution). Then, let \( r \) be a random variable that indicates number of trials until success. The expectation of \( r \) is

\[
E[r] = \sum_{j=1}^{\infty} jp(1-p)^{j-1} = 1/p, \quad \text{when } p \in (0, 1).
\]

(98)

As a result, it requires \( 1 - \exp \left( -\frac{c^4}{32G^4} \right) \) gradient evaluation for each epoch, and the total number of gradient evaluation is nearly \( T = \left\lfloor 36 \left[ 1 - \exp \left( -\frac{c^4}{32G^4} \right) \right]^{-1} C_{\max} n^4 \delta^{-2} \right\rfloor \) in expectation. \( \Box \)
E The CNN Architecture of the Experiments

Our model architecture is illustrated in Figure 3. The first convolution layer consumes the input image and produces 6-channel output with a $5 \times 5$ convolution kernel. Then a $2 \times 2$ max-pooling layer is utilized, followed by another $5 \times 5$ convolution layer which produces 10-channel output. After two feed-forward layers with 10 units, we predict the classification result using softmax.