On the blow-up analysis at collapsing poles for solutions of singular Liouville-type equations

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ABSTRACT

We analyze a blow-up sequence of solutions for Liouville-type equations involving Dirac measures with “collapsing” poles. We consider the case where blow-up occurs exactly at a point where the poles coalesce. After proving that a "quantization" property still holds for the "blow-up mass," we obtain precise pointwise estimates when blow-up occurs with the least blow-up mass. Interestingly, such estimates express the exact analogue of those previously obtained for solutions of "regular" Liouville equations where the "collapsing" Dirac measures are neglected. Such information will be used in a forthcoming paper to describe the asymptotic behavior of minimizers of the Donaldson functional introduced by Goncalves and Uhlenbeck in 2007, yielding to mean curvature 1-immersions of surfaces into hyperbolic 3-manifolds.

1. Introduction

In this note, we analyze the behavior of a sequence of solutions for Liouville-type equations involving Dirac measures whose poles coalesce at a point where the solutions blow-up. Namely, we have blow-up at a point of “collapsing” singularities. This situation naturally arises in the study of Toda systems of Liouville type in the analysis of the so called “shadow” system discussed in [1]. It has motivated the work in [2, 3], where (after [4]) the phenomenon of “blow-up without concentration” was recorded and illustrated by various examples.

More conveniently, for the following discussion, we notice that the role of the poles may be replaced by the zeroes of the weight functions appearing in the equations governing the “regular” part of the given solution sequence. In this formulation, the issue is to understand the blow-up behavior of a (regular) sequence at a point of “collapsing” zeroes.

In fact, this is exactly what may occur when studying the asymptotic behavior of constant mean curvature (CMC) c-immersions of a closed-oriented surface S of genus \( g \geq 2 \) into hyperbolic 3-manifolds. Following [5], it was recently established in [6] that for \(|c| < 1\), the moduli space of all such (CMC) c-immersions can be parametrized by elements of the tangent bundle \( T'(T_g(S)) \). Recall that \( T_g(S) \) is the space of all conformal structures on \( S \) identified up to biholomorphisms in the same homotopic class of the identity. Actually, for \(|c| < 1\), the authors in [6] could label any such (CMC) c-immersion with the minimizer...
(and unique critical point) of the Donaldson functional associated to a given element of $T(T_g(S))$, as introduced in [5]. In this respect we refer to [7] for a discussion of a different approach towards a possible parametrization of CMC immersions.

In [8], we initiated our investigation about the existence of analogous (CMC) 1-immersions. As established in [8], those immersions can be detected only as “limits”, as $c \to 1^-$, of the (CMC) c-immersions obtained in [6]. In view of the work of Bryant [9] about (CMC) 1-immersions into the hyperbolic space $\mathbb{H}^3$, we expect that those (CMC) c-immersions develop at the limit: $c \to 1^-$, finitely many “punctures,” corresponding to possible ramification points. We hope to capture such behavior by means of our blow-up analysis, with the “punctures” being realized by the blow-up points. But we face a delicate situation, exactly when, at the limit, the pull-back metrics of the (CMC) c-immersions blow-up at points of” collapsing” zeroes of the holomorphic quadratic differentials identified by the second fundamental form of the immersion. Indeed, recall that any holomorphic quadratic differential admits exactly $4(\beta - 1)$ zeroes in $X$, counted with multiplicity (see [10]).

The pointwise estimates we establish here (see Theorem 5) have helped us to obtain in [8] the first existence result about (regular) (CMC) 1-immersions of surfaces of genus 2 into 3-manifolds of sectional curvature $-1$, see [8] for details.

More precisely, we start by showing that even in the” collapsing” case, the blow-up mass (see (1.12) below) is” quantized” and must take values in $8\pi \mathbb{N}$. This is a somewhat expected property, as it has been worked out already in the context of systems in [1, 11] and in [2] only for two collapsing zeroes. Nevertheless, for the sake of completeness, we have chosen to provide in Section 2 a detailed proof (see Theorem 1), together with other useful extensions of known facts.

More importantly, we obtain sharp pointwise estimates when blow-up occur with the least blow-up mass $8\pi$. Interestingly, such estimates are exactly the analogue of the estimates obtained by Li in [12] for solution-sequences of Liouville equations with non-vanishing weight functions, where none of the” collapsing” issues discussed here arise.

To state our result, (after a translation) we localize our analysis in a small ball $B_r$ around the origin. Therefore, we consider a solution sequence $u_k$ satisfying:

$$\begin{aligned}
-\Delta u_k &= h_k e^{u_k} - 4\pi \sum_{j=1}^{s} x_j \delta_{p_{j,k}} \\
\int_{B_r} h_k e^{u_k} &\leq C
\end{aligned} \quad (1.1)$$

$$\int_{B_r} h_k e^{u_k} \leq C \quad (1.2)$$

with

$$s \geq 2 \quad \text{and distinct points } p_{j,k} \to 0, \quad \text{as } \quad k \to +\infty, \quad j = 1, \ldots, s \quad (1.3)$$

$$0 < a \leq h_k \leq b, \quad |\nabla h_k| \leq A \quad \text{in } \quad B_r. \quad (1.4)$$

In addition, we require the following bounded oscillation property:

$$\max_{\partial B_r} u_k - \min_{\partial B_r} u_k \leq C \quad (1.5)$$

which is always verified when $u_k$ is actually a “localization” of a globally defined function, for example, over a Riemann surface (see e.g. Theorem 4).
We define the "regular" part \( \zeta_k \) of \( u_k \) as given by
\[
    u_k(x) = \zeta_k(x) + \sum_{j=1}^{n} 2x_j \ln |x - p_{j,k}|
\]
which defines a smooth function in \( B_r \) and satisfies
\[
    -\Delta \zeta_k = (\Pi_{j=1}^{s} |x - p_{j,k}|^{2\gamma_j}) h_k e^{\zeta_k} \quad \text{in} \quad B_r \tag{1.7}
\]
\[
    \int_{B_r} W_k e^{\zeta_k} \leq C \quad \text{with} \quad W_k(x) = (\Pi_{j=1}^{s} |x - p_{j,k}|^{2\gamma_j}) h_k(x) \tag{1.8}
\]
\[
    \max_{\partial B_r} \zeta_k - \min_{\partial B_r} \zeta_k \leq C \tag{1.9}
\]
We assume that \( \zeta_k \) admits a blow-up point at the origin, that is
\[
    \zeta_k(0) = \max_{B_r} \zeta_k \to +\infty, \quad \text{as} \quad k \to +\infty. \tag{1.10}
\]
Since by our assumptions we know that \( \zeta_k \) can admit at most finitely many blow-up points (see Remark 2.1), by taking \( r > 0 \) smaller if necessary, we can further assume that the origin is the only blow-up point for \( \zeta_k \) in \( B_r \), namely:
\[
    \forall \ \varepsilon \in (0, r) \ \exists \ C_\varepsilon > 0 : \ \max_{B_r \setminus B_\varepsilon} \zeta_k \leq C_\varepsilon. \tag{1.11}
\]
Finally, we define the "blow-up mass" of \( \zeta_k \) at the origin as follows
\[
    \sigma = \lim_{\delta \to 0} \lim_{k \to +\infty} \int_{B_\delta} W_k e^{\zeta_k}. \tag{1.12}
\]
The main results contained in this note can be summarized as follows:

**Theorem.** Assume that \( \zeta_k \) satisfies (1.7)–(1.11). If \( x_j \in \mathbb{N} \) and (1.3), (1.4) hold then, for \( \sigma \) in (1.12) we have:
\[
    \sigma \in 8\pi\mathbb{N}.
\]
Furthermore, if \( \sigma = 8\pi \) then:
\[
    p_{j,k} \neq 0 \ \forall \ j = 1, \ldots, s, \ W_k(0) = \Pi_{j=1}^{s} |p_{j,k}|^{2\gamma_j} h_k(0) > 0 \quad \text{and} \quad W_k(0) \xrightarrow{k \to +\infty} 0^+ \tag{1.13}
\]
\[
    \zeta_k(x) = \ln \frac{e^{\zeta_k(0)}}{(1 + W_k(0)e^{\zeta_k(0)}|x|^2)^2} + O(1) \quad \text{in} \quad B_r,
\]
\[
    \int_{B_r} |\nabla \zeta_k|^2 = 16\pi (\zeta_k(0) + \ln(W_k(0))) + O(1);
\]
\[
    \zeta_k(0) + \min_{\partial B_r} \zeta_k + 2 \ln W_k(0) = O(1) \tag{1.14}
\]
\[
    u_k(0) + \min_{\partial B_r} u_k \to +\infty, \quad \text{as} \quad k \to +\infty, \quad u_k \text{ in } (1.6).
\]
By (1.14), we see that our original sequence \( u_k \) satisfying (1.1), (1.2) and (1.5) blows-up at the origin if and only if its regular part \( \zeta_k \) (see (1.6)) blows-up there as well. Furthermore, as already observed, it is remarkable that the pointwise estimate (1.13) is the exact analogue of the one established in [12], when \( W_k(0) \) is bounded uniformly
away from zero, and none of the issues (discussed here) about “collapsing of zeros” arise.

Our results are just a first contribution toward the understanding of the blow-up phenomena at “collapsing” zeroes (or poles). We hope they open the way to the description of “multiple” blow-up profiles, already present in the “non-collapsing” case, according to the analysis in [13–15].

### 2. Blow-up at collapsing zeroes: local analysis

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and regular set. We consider the sequence: $\eta_k \in C^2(\Omega) \cap C^0(\Omega)$, satisfying the following Liouville-type problem:

$$
\begin{cases}
-\Delta \eta_k = W_k e^{\eta_k} \text{ in } \Omega \\
\max_{\partial \Omega} \eta_k - \min_{\partial \Omega} \eta_k \leq C \\
\int_{\Omega} W_k e^{\eta_k} \leq C
\end{cases}
$$

with a weight function $W_k \in L^\infty(\Omega)$.

After the work of Brezis–Merle [16], a vast literature is now available, concerning the asymptotic behavior of $\eta_k$ (possibly along a subsequence), as $k \to +\infty$, according to various assumptions on $W_k$ and its vanishing behavior, see for example [4, 17–21].

Motivated by our applications in [8], where we describe the asymptotic behavior of minimizers of the Donaldson functional introduced in [5] (see [6]), here, we shall take $W_k$ to satisfy:

$$W_k \geq 0 \text{ and } \|W_k\|_{L^\infty(\Omega)} + \int_{\Omega} \frac{1}{(W_k)^{\varepsilon_0}} \leq C, \text{ for some } \varepsilon_0 > 0. \tag{2.4}$$

A first important information about the sequence $\eta_k$ is the following well-known result, stemming from [16]:

**Proposition A.** Assume (2.1)–(2.3) and (2.4). If $\lim_{k \to +\infty} \int_{\Omega} W_k e^{\eta_k} < 4\pi$, then $\eta_k^+$ is uniformly bounded in $C^0_{loc}(\Omega)$.

**Proof.** See Proposition 5.3.13 in [21].

Consequently, as in [16], it is natural to define $z_0 \in \Omega$ a blow-up point for the sequence $\eta_k$, if the following holds

$$\exists \ z_k \to z_0 \text{ with } \eta_k(z_k) \to +\infty \text{ as } k \to \infty.$$

Moreover, we call the blow-up mass of $\eta_k$ at $z_0$ the value:

$$\sigma(z_0) = \lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(z_0)} W_k e^{\eta_k}. \tag{2.5}$$

**Remark 2.1.** By virtue of Proposition A we know that $\sigma(z_0) \geq 4\pi$, and so, by the assumption (2.3), we know that $\eta_k$ can admit at most a finite number of blow-up points.

From now on we shall denote by $S$ the finite set (possibly empty) of all blow-up points of $\eta_k$ in $\Omega$, and refer to $S$ as the blow-up set.

The following holds.
Proposition B. Under the assumptions of Proposition A, $\eta_k^+$ is uniformly bounded in $C^0_\text{loc}(\Omega \setminus S)$, along a subsequence.

Proof. See Proposition 5.3.17 in [21].

The information above allows us to provide the following "rough" description about the asymptotic behavior of (a subsequence of) $\eta_k$. It is a general version of analogous statements available in literature (under stronger assumptions on the vanishing properties of $W_k$) starting with [16, 17, 20] and then [2–4]. See also [1, 11, 22–25] for analogous results in the context of Liouville-type systems.

Proposition 2.1. Let $\eta_k$ satisfy (2.1)-(2.3) with $W_k \to W$ uniformly in $C^0_\text{loc}(\Omega)$, and assume that (2.4) holds. Then (along a subsequence), $\eta_k$ satisfies one of the following alternatives, as $k \to +\infty$:

(i) $\eta_k \to -\infty$ uniformly on compact sets of $\Omega$,

(ii) $\eta_k \to \eta_0$ in $C^2_\text{loc}(\Omega)$, with $\eta_0$ satisfying:

\[
\begin{cases}
-\Delta \eta_0 = We^{\eta_0} \text{ in } \Omega; \\
\int_{\Omega} We^{\eta_0} \leq C,
\end{cases}
\]

(iii) (blow-up): There exists a finite set $S \neq \emptyset$ of blow-up points of $\eta_k$ in $\Omega$ such that, as $k \to +\infty$:

(a) either (blow-up with "concentration")

\[
\eta_k \to -\infty \text{ uniformly on compact sets of } \Omega \setminus S
\]

\[
W_k e^{\eta_k} \to \sum_{q \in S} \sigma(q)\delta_q, \text{ weakly in the sense of measures},
\]

(b) or (blow-up without concentration)

\[
\eta_k \to \eta_0 \text{ in } C^2_\text{loc}(\Omega \setminus S);
\]

\[
W_k e^{\eta_k} \to \sum_{q \in S} \sigma(q)\delta_q + We^{\eta_0}, \text{ weakly in the sense of measures},
\]

and

\[
\begin{cases}
-\Delta \eta_0 = We^{\eta_0} + \sum_{q \in S} \sigma(q)\delta_q \text{ in } \Omega; \\
\int_{\Omega} We^{\eta_0} \leq C.
\end{cases}
\]

Moreover, the blow-up mass $\sigma(q) \geq 4\pi$, $\forall q \in S$.

Proof. As already mentioned, the claimed results are available in literature under various assumptions on $W_k$, and for completeness we highlight here the main arguments involved. First, we observe that the sequence:

\[
\phi_k = \eta_k - \min_{\partial \Omega} \eta_k
\]

is uniformly bounded in $\partial \Omega$ and by the Green representation formula we have:

\[
\phi_k(x) = \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{1}{|x-y|} \right) W_k e^{\eta_k} dy + \Psi_k
\]

with suitable $\Psi_k \to \Psi$ uniformly in $C^2(\Omega)$. 

Recalling that $\mathcal{S}$ is the (possibly empty) blow-up set of $\eta_k$ in $\Omega$, by Proposition B we know that $\eta_k^+$ is uniformly bounded on compact sets of $\Omega \setminus \mathcal{S}$. Therefore, we can use well-known potential estimates to conclude that (along a subsequence):

$$\phi_k \to \phi_0 \quad \text{uniformly in} \quad C^{1,2}_\text{loc}(\Omega \setminus \mathcal{S}).$$

(2.8)

Next, letting $s_0 = \frac{\nu_0}{1+\nu_0} \in (0,1)$ with $\nu_0 > 0$ in (2.4), we check:

$$\min_{\partial \Omega} \eta_k = \min_{\Omega} \eta_k \leq \int_{\Omega} \eta_k^+ \leq C \int_{\Omega} \eta_k^+ \leq C \left( \int_{\Omega} W_k e^{\eta_k} \right)^{s_0} \left( \int_{\Omega} \frac{1}{W_k^{s_0}} \right)^{1-s_0} \leq C.$$

So (along a subsequence)

$$\text{either } \min_{\partial \Omega} \eta_k \to -\infty \quad \text{on compact sets of } \Omega \setminus \mathcal{S} \quad \text{or} \quad |\min_{\partial \Omega} \eta_k| \leq C. \quad (2.9)$$

If the first alternative holds in (2.9), then (along a subsequence) we deduce alternative (i) in case $\mathcal{S} = \emptyset$, or in case $\mathcal{S} \neq \emptyset$ then blow-up with concentration in alternative (iii).

On the contrary in case the second alternative holds in (2.9), then by virtue of (2.8), we find (along a subsequence) that

$$\eta_k \to \eta_0 \quad \text{uniformly in} \quad C^0_\text{loc}(\Omega \setminus \mathcal{S}).$$

Hence, we conclude that either (ii) or alternative b) of (iii) hold according to whether $\mathcal{S} = \emptyset$ or $\mathcal{S} \neq \emptyset$, and in view of Remark 2.1 the proof is completed.

As discussed in [2, 3], all the alternatives of Proposition 2.1 can actually occur. When alternative (iii) holds, then to better understand the behavior of $\eta_k$ around a blow-up point $q \in \mathcal{S}$, it is crucial to identify the specific value of the blow-up mass $\sigma(q)$ in (2.5). To this purpose, for the weight function $W_k$, we shall work under the following assumption,

$$|\nabla W_k| \leq A \quad \text{and} \quad W_k \to W \quad \text{in} \quad C^0_\text{loc}(\Omega).$$

(2.10)

According to the results in [17, 20], the value of $\sigma(q)$ depends on whether the limiting function $W$ in (2.10) vanishes or not at $q \in \mathcal{S}$. If locally around $q \in \mathcal{S}$ there holds:

$$W_k(x) = |x - p_k|^{2a} h_k(x) \quad \text{in} \quad B_r(q), \quad a \geq 0, \quad p_k \to q, \quad (2.11)$$

$$0 < a \leq h_k \leq b \quad \text{and} \quad |\nabla h_k| \leq A \quad \text{in} \quad B_r \quad (2.12)$$

then, we know the following:

**Theorem A.** ([17, 20]) If $\eta_k$ in Proposition 2.1 satisfies alternative (iii) and for $q \in \mathcal{S}$ the weight function $W_k$ satisfies (2.11) and (2.12) in $B_r(q)$, then, only alternative a) occurs, namely we have blow-up with ”concentration” as described in (2.6) and

(i) if $a = 0$ in (2.11) then $\sigma(q) = 8\pi$,

(ii) if $a > 0$ in (2.11) then $\sigma(q) = 8\pi(1 + a)$.

In this note, we shall focus to the case where, for $q \in \mathcal{S}$, we have $W(q) = 0$ and $q$ is the accumulation point of different zeroes of $W_k$ (collapsing of zeroes).
In view of the applications we have in mind, we consider the case where the zeroes of $W_k$ have integral multiplicity, and more precisely for $q \in \mathcal{S}$ and $r > 0$ sufficiently small, we assume:

$$W_k(x) = \left( \prod_{j=1}^{s} |x - p_{j,k}|^{2\alpha_j} \right) h_k(x), \quad x \in B_r(q), \quad \alpha_j \in \mathbb{N}, \quad s \geq 2 \quad (2.13)$$

$$p_{j,k} \neq p_{l,k} \quad \text{for} \quad j \neq l \quad \text{and} \quad p_{j,k} \to q, \quad \text{as} \quad k \to +\infty, \quad \forall \ j = 1, \ldots, s. \quad (2.14)$$

In this case, we start by proving a quantization property for the blow-up mass in (2.5), which completes the result in [2] (where only two zeroes of $W_k$ coalesce at $q$) and follows as in [1, 11], where analogous information were deduced in the context of systems.

**Theorem 1.** Suppose that $\eta_k$ in Proposition 2.1 satisfy alternative (iii) and for $q \in \mathcal{S}$ assume (2.12)–(2.14). Then, $\sigma(q) \in 8\pi \mathbb{N}$.

To prove Theorem 1 we need some preliminaries. First of all, we can” localize” our analysis around the blow-up point $q$. Indeed by means of (2.7), we have

$$g_k(x) = \min_{\partial \Omega} g_k + \frac{1}{2\pi} \int_{\Omega} \ln \left( \frac{1}{|x - y|} \right) W_k e^{\varphi_k} dy + \Psi_k$$

with $\Psi_k$ uniformly bounded in $C^2(\Omega)$. Hence, we easily check (as in [26]) that $\eta_k$ satisfies the bounded oscillation property around $q$.

Thus, after a translation, we can take $q = 0$, and for $r > 0$ sufficiently small, we need to analyze a sequence $n_k \in C^{2}(Br) \cap C^{0}(\overline{Br})$ satisfying:

$$\begin{cases}
-\Delta \xi_k = (\prod_{j=1}^{s} |x - p_{j,k}|^{2\alpha_j}) h_k(x) e^{\xi_k} & \text{in} \quad Br \\
\max_{\partial Br} \xi_k - \min_{\partial Br} \xi_k \leq C \\
\max_{Br} \xi_k = \xi_k(x_k) \to +\infty, \quad \text{and} \quad x_k \to 0, \quad \text{as} \quad k \to +\infty,
\end{cases} \quad (2.15)$$

$$p_{j,k} \neq p_{l,k} \quad \text{for} \quad j \neq l, \quad p_{j,k} \to 0 \quad \text{as} \quad k \to +\infty, \quad \forall \ j = 1, \ldots, s, \quad (2.16)$$

where $s \geq 2, \ \alpha_j \in \mathbb{N}, \ h_k$ satisfies (2.12) in $B_r$,

$$\int_{Br} W_k e^{\xi_k} \leq C, \quad \text{where} \quad W_k(x) = (\prod_{j=1}^{s} |x - p_{j,k}|^{2\alpha_j}) h_k(x). \quad (2.17)$$

Finally, without loss of generality, in view of (2.12), we can assume also that,

$$h_k \to h, \quad \text{in} \quad C^{0}_{\text{loc}}(Br), \quad \text{with} \quad h(0) = 1 \quad (2.18)$$

and zero is the only blow-up point of $\xi_k$ in $B_r$, that is:

$$\forall \ 0 < \delta < r \ \exists \ C_{\delta} > 0 : \ \max_{Br \setminus B_{\delta}} \xi_k \leq C_{\delta}. \quad (2.19)$$

Clearly, under the above assumptions, we have that (2.10) holds with

$$W(x) = |x|^{2\alpha} h(x), \quad h(0) = 1 \quad \text{and} \quad x = \sum_{j=1}^{s} \alpha_j \in \mathbb{N}. \quad (2.20)$$
Furthermore, Proposition 2.1 applies to $\zeta_k$ in $B_r$, and by (2.17) and (2.21) we know that (along a subsequence) $\zeta_k$ satisfies alternative (iii) with $S = \{0\}$, and we have

$$m := \lim_{\delta \to 0} \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B_\delta} W_k e^{\zeta_k} \geq 2.$$  

(2.23)

Since, as above, by Green representation formula, we can write:

$$\zeta_k(x) = \min_{B_r} \zeta_k + \frac{1}{2\pi} \int_{B_r} \ln \left( \frac{1}{|x-y|} \right) W_k e^{\zeta_k} dy + \psi_k, \ x \in B_r,$$  

with $\psi_k$ uniformly bounded in $C^2(B_r)$, we can use the information in part (iii) of Proposition 2.1, to deduce that (along a subsequence) the following holds:

$$\nabla \zeta_k \to -m \frac{x}{|x|^2} + \nabla \phi, \ \text{uniformly in} \ C^1_{\text{loc}}(B_r \setminus \{0\}), \ \text{as} \ k \to +\infty,$$  

(2.25)

with a suitable $\phi$ smooth in $B_r$.

At this point, to establish Theorem 1, it suffices to prove the following:

**Theorem 2.** Under the above assumptions, in (2.23) we have that $m \in 4\mathbb{N}$.

**Proof.** We set $x = \sum_{j=1}^{s} x_j \in \mathbb{N}$ and note that $x \geq 2$. Letting

$$\tau_k = \max_{j=1,\ldots,s} |p_{j,k}| \to 0, \ \text{as} \ k \to +\infty,$$  

(2.26)

we define

$$q_{j,k} = \frac{p_{j,k}}{\tau_k}, \ j = 1, \ldots, s,$$

and since $|q_{j,k}| \leq 1$, along a subsequence, we can assume:

$$q_{j,k} \to q_j \ \text{as} \ k \to +\infty, \ j = 1, \ldots, s.$$

Therefore, (recalling the normalization $h(0) = 1$)

$$0 \leq W_{1,k}(x) := (\prod_{j=1}^{s} |x - q_{j,k}|^{2\tau_j}) h_k(\tau_k x) \to \prod_{j=1}^{s} |x - q_{j}|^{2\tau_j} = W_1(x),$$  

(2.27)

as $k \to +\infty$, uniformly on compact sets of $\mathbb{R}^2$. We should keep in mind that $W_1(x)$ vanishes exactly at the set

$$Z_1 := \{q_1, \ldots, q_s\},$$  

(2.28)

and the $q_j$’s may not be distinct. We define

$$\varphi_k(x) = \zeta_k(\tau_k x) + 2(x + 1) \ln \tau_k \ \text{in} \ D_k = \{|x| \leq \frac{r}{\tau_k}\}$$  

(2.29)

satisfying:

$$-\Delta \varphi_k = W_{1,k}(x)e^{\varphi_k} \ \text{in} \ D_k$$

$$\int_{D_k} W_{1,k}e^{\varphi_k} \leq C.$$

By scaling (2.24), for any given $x_1, x_2 \in \mathbb{R}^2$ and $k$ large, we have:
\[ \varphi_k(x_1) - \varphi_k(x_2) = \frac{1}{2\pi} \int_D \ln \left( \frac{|x_2 - y|}{|x_1 - y|} \right) W_{1,k} e^{\varphi_k} + T_k(x_1, x_2) \]  

(2.30)

with \( T_k \) uniformly bounded in \( C^2_{\text{loc}} \). Therefore (as in lemma 2.2 of [26]), we find \( R_0 > 1 \) sufficiently large, such that \( \forall R \geq R_0 \) there holds,

\[
\max_{\partial B_R} \varphi_k - \min_{\partial B_R} \varphi_k \leq C_R,
\]

with a suitable constant \( C_R \), depending on \( R \) only.

Since \( W_{1,k} \) in (2.27) satisfies (2.4) and (2.10) in any open and bounded set of \( \mathbb{R}^2 \) (for large \( k \)), we can, therefore, apply Proposition 2.1 to \( \varphi_k \) in any ball \( B_R \) with \( R \geq R_0 \).

Furthermore, by a diagonalization process, along a subsequence, we can set

\[
\mu = \lim_{R \to +\infty} \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B_R} W_{1,k} e^{\varphi_k} \leq m,
\]

(2.31)

and we can describe the asymptotic behavior of \( \varphi_k \), as \( k \to +\infty \), by one of the following alternatives:

(i) \( \varphi_k \to -\infty \) uniformly on compact sets of \( \mathbb{R}^2 \) and \( \mu = 0 \),

(ii) \( \varphi_k \to \varphi_0 \) in \( C^2_{\text{loc}}(\mathbb{R}^2) \), \( -\Delta \varphi_0 = W_1(x)e^{\varphi_0} \) in \( \mathbb{R}^2 \),

\[
\mu = \frac{1}{2\pi} \int_{\mathbb{R}^2} W_1(x)e^{\varphi_0} \ (W_1 \text{ is given in (2.27))}
\]

(2.32)

(iii) There exists a finite blow-up set

\[ S_\varphi := \{ q : \exists z_k \to q \text{ and } \varphi_k(z_k) \to +\infty \} \]

such that

(a) either \( \varphi_k \to -\infty \) uniformly on compact sets of \( \mathbb{R}^2 \setminus S_\varphi \),

\[
W_{1,k} e^{\varphi_k} \to \sum_{q \in S_\varphi} \sigma(q) \delta_q \text{ weakly in the sense of measures,}
\]

(2.35)

\[
\mu = \frac{1}{2\pi} \sum_{q \in S} \sigma(q) \text{ with } \sigma(q) \geq 4\pi.
\]

(2.36)

In addition we know that,

\[
\sigma(q) = 8\pi \text{ if } q \neq j, \forall j = 1, \ldots, s,
\]

\[
\sigma(q) = 8\pi(1 + \alpha_j) \text{ if } q = q_j \text{ and } q \neq q_k \text{ for } k \neq j;
\]

(2.37)

(b) or \( \varphi_k \to \varphi_0 \) in \( C^2_{\text{loc}}(\mathbb{R}^2 \setminus S_\varphi) \),

\[
W_{1,k} e^{\varphi_k} \to \sum_{q \in S} \sigma(q) \delta_q + W_1 e^{\varphi_0} \text{ weakly in the sense of measures,}
\]

(2.38)

\[-\Delta \varphi_0 = W_1(x)e^{\varphi_0} + \sum_{q \in S_\varphi} \sigma(q) \delta_q \text{ in } \mathbb{R}^2,\]

(2.39)

\[
\mu = \frac{1}{2\pi} \left( \int_{\mathbb{R}^2} W_1(x)e^{\varphi_0} + \sum_{q \in S_\varphi} \sigma(q) \right).
\]

(2.40)
Furthermore, when alternative b) holds in (ii) then, necessarily: $S \varphi \subset Z_1$ (see (2.28)), and blow-up takes place at points in $Z_1$ where different zero points of $W_{1,k}$ collapse together. Namely, for $q \in S \varphi$ there holds:

$$q = q_{j_1} = q_{j_2} = \ldots = q_{j_m} \in Z_1$$

with $m \geq 2$, and $1 \leq j_1 < \ldots < j_m \leq s$.  

(2.41)

Moreover, in this case $\sigma(q)$ must satisfy:

$$4\pi \leq \sigma(q) < 4\pi(\varepsilon(q) + 1) \quad \text{for} \quad \varepsilon(q) := \sum_{k=1}^{m} \varepsilon(q_k). \quad (2.42)$$

Indeed, by (2.39), we have that $\varphi_0$ admits a logarithmic singularity at $q$ of order $\frac{\sigma(q)}{2\pi}$.

At the same time, we must ensure the integrability of $|x - q|^{2\varepsilon(q)}e^\varphi_0$ around $q$ (recall (2.27) and see (2.40)), and this is possible only if the inequality given in (2.42) holds.

When either alternative (ii) or b) in (iii) hold, then we can argue similarly around infinity. Therefore, by (2.33) and (2.34), or by (2.39) and (2.40), respectively, we find that,

$$\varphi_0(x) = -\mu \ln(|x|) + O(1), \quad \text{for} \quad |x| \geq 1; \quad (2.43)$$

(see e.g. [27]), and again, by the given integrability condition, we derive that

$$\mu = 2(\varepsilon + 1 + t) \quad \text{for some} \quad t > 0. \quad (2.44)$$

With the help of the information above we can establish the following:

Claim 1: If $\varphi_k$ satisfies alternative (iii), then $\sigma(q) \in 8\pi\mathbb{N}$, $\forall \ q \in S \varphi$.  

(2.45)

Since $x_j \in \mathbb{N}$, $\forall \ j = 1, \ldots, s$, by virtue of (2.37) to establish the claim we only have to consider the case where $q \in S \varphi$ satisfies (2.41).

To this purpose, we go back to the original sequence $\xi_k$. Actually we replace $\xi_k(x)$ with $\xi_k(x + \frac{p_{1,k} + p_{2,k}}{2})$ and $\frac{p_{1,k} + p_{2,k}}{2} \to 0$, so we can assume, without loss of generality, that,

$$p_{1,k} = -p_{2,k}, \quad \forall \ k \in \mathbb{N}. \quad (2.46)$$

We proceed by induction on $s$, the number of different zeroes of $W_k$ collapsing at the origin, the only blow-up point of $\xi_k$ (see (2.15), (2.17), (2.18), (2.21)).

In case $s = 2$, then, as shown in [2], we easily reach the desired conclusion. Indeed, by (2.26) and (2.46) we have: $\tau_k = |p_{1,k}| = |p_{2,k}|$, so by letting: $q_k = \frac{p_{1,k}}{\tau_k}$ we have $|q_k| = 1$ and $\frac{p_{1,k}}{\tau_k} = -q_k$. Thus, (along a subsequence) we have:

$$-\Delta \varphi_k = |x - q_k|^{2\varepsilon_k} |x + q_k|^{2\varepsilon_k} h_k(\tau_k x) e^{\varphi_k} \quad \text{in} \quad D_k,$$

$$q_k \to q_0 \neq 0, \quad \text{with} \quad q_1 = q_0 \quad \text{and} \quad q_2 = -q_0.$$ 

So, for the scaled sequence $\varphi_k$ we do not have to face the possibility of a further “collapsing” of zeroes, and the desired conclusion follows simply by (2.37).  

Next, let $s \geq 3$, and assume that $q \in S \varphi$ satisfies

$$q = q_{j_1} = q_{j_2} = \ldots = q_{j_m}, \quad j_i \in \{1, \ldots, s\} \quad \text{and} \quad m \geq 2.$$
It suffices to show that \( m \leq s - 1 \), since then by the induction assumption we could conclude that \( \sigma(q) \in 8\pi\mathbb{N} \), as desired. As above, for suitable \( q_0 \in \mathbb{R}^2 \), we have: \( q_1 = q_0 \) and \( q_2 = -q_0 \). In case \( q_0 \neq 0 \), we see that, if \( q \neq \pm q_0 \), then \( q \in \{q_3, ..., q_s\} \) and necessarily \( m \leq s - 2 \). While if \( q = q_0 \) or \( q = -q_0 \), then \( q \neq q_2 \) or \( q \neq q_1 \), respectively, and again \( m \leq s - 1 \). Next, we suppose that \( q_0 = 0 \). In case \( q \neq 0 \), then, we conclude as above that \( m \leq s - 2 \). Hence, suppose that \( q = q_0 = 0 \), and let \( j_0 = \{1, ..., s\} \) so that (along a subsequence): \( \tau_k = |p_{j_0, k}| \). Consequently \( |q_{j_0}| = 1 \) and so \( q \neq q_{j_0} \) and we can still conclude that \( m \leq s - 1 \). So, if \( q \) coincides with \( m \)-collapsing zeros in \( Z_1 \), then we have shown that, \( 2 \leq m \leq s - 1 \), and as already mentioned, the desired conclusion follows by the induction hypothesis.

To proceed further, we recall the following:

**Lemma A** (Lemma 2.1 of [22]). Let \( u \) satisfy

\[
\begin{align*}
\Delta u + e^u &= 4\pi \sum_{j=1}^{N} \beta_j \delta_{q_j} \quad \text{in} \quad \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^u &< +\infty
\end{align*}
\]

with \( \beta_j \in \mathbb{N} \cup \{0\} \) and \( q_j \in \mathbb{R}^2 \), \( j = 1, ..., N \). Then,

\[
\int_{\mathbb{R}^2} e^u = 4\pi \left( \sum_{j=1}^{N} \beta_j + 1 + t \right) \in 8\pi\mathbb{N} \quad \text{with} \quad t > 0.
\]

By combining Claim 1 (see (2.45)) and Lemma A we obtain:

**Claim 2:** \( \mu \in 4\mathbb{N} \cup \{0\} \). (2.47)

To establish (2.47), notice that in case \( \varphi_k \) satisfies (i) then \( \mu = 0 \). While if \( \varphi_k \) satisfies (ii), then we can apply Lemma A to the function,

\[
u := \varphi_0 + \sum_{j=1}^{s} 2x_j \ln |x - q_j|, \quad x_j \in \mathbb{N}
\]

(2.48)

to conclude, that

\[
\mu = \frac{1}{2\pi} \int_{\mathbb{R}^2} \prod_{j=1}^{s} |x - q_j|^{2x_j} e^{\varphi_0} = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^u \in 4\mathbb{N}.
\]

Conversely, if \( \varphi_k \) satisfies alternative a) of (iii), then, by (2.36) and (2.45), still we find: \( \mu \in 4\mathbb{N} \).

Finally, if alternative b) in (iii) holds, then with the notation in (2.41), (2.42) we see that \( u \) in (2.48) satisfies:

\[
\Delta u + e^u = 4\pi \sum_{q_j \in Z_1 \setminus S_0} x_j \delta_{q_j} + 4\pi \sum_{q \in S_0} \left( x(q) - \frac{\sigma(q)}{4\pi} \right) \delta_q
\]

and in view of (2.42), we can check that Lemma A applies to \( u \). As a consequence

\[
\int_{\mathbb{R}^2} W_1(x) e^{\varphi_0} = \int_{\mathbb{R}^2} \prod_{j=1}^{s} |x - q_j|^{2x_j} e^{\varphi_0} = \int_{\mathbb{R}^2} e^u \in 8\pi\mathbb{N}.
\]

Therefore, by (2.40) and (2.45), we derive that \( \mu \in 4\mathbb{N} \) as desired.
Notice in particular that, when \( \varphi_k \) satisfies either (ii) or alternative b) of (iii), then, \( \mu \) must satisfy (2.44) with \( t \in \mathbb{N} \).

Finally to conclude the proof of Theorem 2, we recall the following relation between \( m \) and \( \mu \) established, e.g. in [1]:

\[
m^2 - \mu^2 = 4(1 + \alpha)(m - \mu).
\] (2.49)

For completeness we have included a proof of (2.49) in the Appendix.

At this point, we see that, if \( \varphi_k \) satisfies (i) then, \( \mu = 0 \) and from (2.49) we find \( m = 4(\alpha + 1) \in 4\mathbb{N} \), as \( \alpha = \sum_{j=1}^{\ell_2} \alpha_j \in \mathbb{N} \). In case \( \varphi_k \) satisfies (ii) then by (2.44) and (2.47), \( \mu \in 4\mathbb{N} \) and \( \mu > 2(\alpha + 1) \). Therefore, from (2.49) we must have that necessarily \( m = \mu \in 4\mathbb{N} \). Finally, if \( \varphi_k \) satisfies alternative (iii), then, again from (2.49) we have that, either \( m = \mu \in 4\mathbb{N} \) or \( m = 4(\alpha + 1) - \mu \), with \( \alpha \in \mathbb{N} \) and \( \mu \in 4\mathbb{N} \). Thus, in any case, \( m \in 4\mathbb{N} \), and the proof is completed. \( \square \)

The “local” results above can be used to describe the asymptotic behavior of solutions for Liouville-type equations on a compact Riemann surface \((X, g)\). Denote by \( d_g(\cdot, \cdot) \) the distance in \((X, g)\). We consider the sequence \( v_k \in C^{2,2}(X) \) to satisfy:

\[
-\Delta v_k = R_k e^{v_k} - f_k \quad \text{in} \quad X, \quad R_k(z) = \left( \Pi_{j=1}^{\ell_2}(d_g(z, z_j, k)^{2\alpha_j}) \right) g_k(z), \quad z \in X; \tag{2.50}
\]

with

\[
g_k \in C^1(X) : \quad a \leq g_k \leq b, \quad |\nabla g_k| \leq A \quad \text{and} \quad g_k \to g_0 \quad \text{in} \quad C^0(X); \tag{2.51}
\]

\[
z_{j, k} \in X : \quad z_{j, k} \neq z_{l, k}, \quad j \neq l \quad \text{and} \quad z_{j, k} \to z_j, \quad j = 1, \ldots, N; \tag{2.52}
\]

\[
f_k \in C^{0,2}(X), \quad f_k \to f_0 \quad \text{in} \quad L^p(X) \quad \text{for some} \quad p > 1, \quad \int_X f_0 \, dA \neq 0; \tag{2.53}
\]

and so,

\[
R_k(z) \to R_0(z) := \left( \Pi_{j=1}^{\ell_2}(d_g(z, z_j)^{2\alpha_j}) \right) g_0(z) \quad \text{in} \quad C^0(X).
\]

As before we assume that,

\[
x_j \in \mathbb{N}, \quad j = 1, \ldots, N. \tag{2.54}
\]

We denote by \( Z = \{z \in X : R_0(z) = 0\} = \{z_1, \ldots, z_N\} \) the zero set of \( R_0 \), formed by the points point \( z_j \) given in (2.53), for \( j = 1, \ldots, N \). Again we must keep in mind that such points may not be distinct, since, at the limit, different zeroes of \( R_k \) could coalesce at the same zero of \( R_0 \). Therefore, we let \( Z_0 \subset Z \) be the subset (possibly empty) of \( Z \), given by such “collapsing” zeroes, namely:

\[
Z_0 = \{z \in Z : \exists \ s \geq 2, \ 1 \leq j_1 < \ldots < j_s \leq N \quad \text{such that} \quad z = z_{j_1} = \ldots = z_{j_s} \quad \text{and} \quad z \notin Z \setminus \{z_{j_1}, \ldots, z_{j_s}\}\}.
\]

By combining the “local” results obtained above, we can establish the following:

**Theorem 3.** Let \( v_k \) satisfy (2.50) and assume (2.51)-(2.55). Then, along a subsequence, one of the following alternatives holds:
(i) \textbf{(compactness):} $v_k \to v_0$ in $C^2(X)$ with
\[
-\Delta v_0 = R_0 e^{v_0} - f_0, \text{ in } X
\] (2.56)

(ii) \textbf{(blow-up):} There exists a finite blow-up set
\[
S = \{ q \in X : \exists \ q_k \to q \text{ and } v_k(q_k) \to +\infty, \text{ as } k \to +\infty \}
\]
such that, $v_k^+$ is uniformly bounded in $C^0_{\text{loc}}(X \setminus S)$ and, as $k \to +\infty$,

(a) \text{ either (blow-up with concentration):}
\[
v_k \to -\infty \text{ uniformly on compact sets of } X \setminus S,
R_k e^{v_k} \to \sum_{q \in S} \sigma(q) \delta_q \text{ weakly in the sense of measures, } \sigma(q) \in 8\pi\mathbb{N}.
\]

In particular, $\int_X f_0 \, dA \in 8\pi\mathbb{N}$ in this case.

(b) \text{ or (blow-up without concentration):}
\[
v_k \to v_0 \text{ in } C^2_{\text{loc}}(X \setminus S),
R_k e^{v_k} \to R_0 e^{v_0} + \sum_{q \in S} \sigma(q) \delta_q \text{ weakly in the sense of measures},
\]
\[
-\Delta v_0 = R_0 e^{v_0} + \sum_{q \in S} \sigma(q) \delta_q - f_0 \text{ in } X, \sigma(q) \in 8\pi\mathbb{N}.
\]

Furthermore, if alternative b) of (ii) holds then $S \subset Z_0$ and so, in this case, blow-up occurs only at points where different zeroes of $R_k$ coalesce at the limit.

\textbf{Proof.} If $\max_X v_k \leq C$, then the right-hand side of (2.50) is uniformly bounded in $L^p(X), \ p > 1$. Hence, by setting $v_k = w_k + d_k$ with $d_k = \int_X v_k$, then, by elliptic estimates (see [28]), we see that $w_k$ is uniformly bounded in $C^{1,2}(X)$, and so along a subsequence, we obtain that, $w_k \to w_0$ in $C^1(X)$, as $k \to +\infty$. After integration of (2.50), we have:
\[
\int_X e^{v_k} R_k = \int_X f_k \to \int_X f_0, \text{ as } k \to +\infty.
\] (2.57)
Since by assumption: $\int_X f_0 \neq 0$, then from (2.57) we see that necessarily, $\int f_0 > 0$, and so $d_k \to \log \left( \frac{\int_X f_0}{\int_X R_0 e^{v_0}} \right) = d_0$. Consequently, $v_k \to v_0 = w_0 + d_0$ in $C^1(X)$ and $v_0$ satisfies (2.56). Thus alternative (i) holds in this case.

Next, assume that, (along a subsequence)
\[
\max_X v_k \to +\infty, \text{ as } k \to +\infty.
\]
This implies that the blow-up set of (a subsequence of) $v_k$ is not empty, that is:
\[
S = \{ q \in X : \exists \ q_k \in X : q_k \to q \text{ and } v_k(q_k) \to +\infty \} \neq \emptyset.
\]

At this point, around any $q \in X$, we introduce local conformal coordinates centered at the origin, and (with abuse of notation) we shall denote in the same way the local expressions of $v_k$ and $f_k$ in such coordinates, as defined in a small ball $B_r$. In this way, in $B_r$ we may consider the sequence $\theta_k$ satisfying:
\[-\Delta \theta_k = f_k \quad \text{in} \quad B_r \quad \text{and} \quad \theta_k = 0 \quad \text{in} \quad \partial B_r.\]

Hence, \(\theta_k\) is uniformly bounded in \(C^{1,2}(X)\) and (along a subsequence) we may assume that, \(\theta_k \to \varphi_0\) in \(C^1(X)\), as \(k \to +\infty\), with \(\Delta \theta_0 = f_0\) in \(B_r\) and \(\theta_0 = 0\) in \(\partial B_r\). Therefore, the new sequence \(\xi_k = v_k - \theta_k\) satisfies:

\[-\Delta \xi_k = W_k e^{\xi_k} \quad \text{in} \quad B_r\]

with suitable \(W_k \geq 0\) such that, in \(B_r\) the following holds:

\[|\nabla W_k| \leq A \quad \text{and} \quad W_k \to W_0 \quad \text{uniformly}.\]

From (2.57) also we have:

\[\int_{B_r} W_k e^{\xi_k} \leq C,\]

with suitable constant \(C > 0\). Furthermore,

- if \(W_0(0) = 0\) then \(W_k(z) = \prod_{j=1}^s |z - p_{j,k}|^{2s_j} h_k(z)\) for \(s \geq 2\), and \(h_k\) satisfies (2.12) in \(B_r\), \(x_j \in \mathbb{N}\) and \(p_{j,k} \to 0\), as \(k \to \infty\).

In any case, \(W_k\) satisfies (2.4) in \(B_r\) (with suitable \(\varepsilon_0 > 0\)), and therefore, by Remark 2.1 and (2.57) we conclude that the blow-up set \(S\) is finite. Also, by using the Green representation formula for \(v_k\) in \(X\) then, in the usual way (see e.g. [26] or [21]), for every compact set \(K \subset X \setminus S\) we can check that,

\[\max_K v_k - \min_K v_k \leq C, \quad \text{(2.58)}\]

with suitable \(C > 0\) depending only on \(K\).

Thus, by (2.58), we can apply alternative (iii) of Proposition 2.1 and Theorem 1 to obtain (in conformal coordinates) the local blow-up description of \(v_k\) around any blow-up point \(q \in S\). At the same time the property (2.58) also allows us to patch together such local information and arrive at the desired statement in (ii).

Furthermore, as a direct consequence of Theorem 3, we can recover a compactness result, well-known in the non-collapsing case:

**Corollary 2.1.** Under the assumptions of Theorem 3, if

\[
\limsup_{k \to +\infty} \int_X R_k e^{\xi_k} < 8\pi
\]

then, along a subsequence, \(\xi_k \to \xi_0\) in \(C^{1,2}(X)\) with \(\xi_0\) satisfying (2.56).

### 3. Local estimates in case of least blow-up mass

This section is devoted to provide a more detailed description about a sequence \(\xi_k\) which satisfies the local problem (2.15) and admits a blow-up point at the origin and (2.18) holds. Namely, \(\xi_k\) blows-up at a point where different zeroes of the weight function \(W_k\) “collapse” together.
We focus to the case of least blow-up mass $8\pi$, namely when (2.23) holds as follows:

$$m = \lim_{\delta \to 0} \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B_1} W_k e^{\zeta_k} = 4. \quad (3.1)$$

This is a first important step toward the more involved description of “multiple” blow-up profiles, and we refer to [2] for some progress in this direction, $s = 2$ in (2.18).

With the notation of the previous section, a first consequence of (3.1) is the following:

**Proposition 3.1.** Let $\zeta_k$ satisfy (2.15)–(2.21) and suppose that (3.1) holds. Then, $m = \mu = 4$ and the sequence $\varphi_k$ in (2.29) must satisfy alternative a) of (iii) in Proposition 2.1, with a unique blow-up point $q_0$.

Furthermore if $W_1(q_0) = 0$ (i.e. $q_0 \in Z_1 = \{q_1, \ldots, q_s\}$), then, $q_0$ must satisfy (2.41), namely different zeroes of $W_{1,k}$ collapse at $q_0$.

**Proof.** First of all we can exclude that $\varphi_k$ satisfies alternative (i). Indeed, in this case $\mu = 0$, and so by (2.49) we would have $m = 4(1 + x) > 4$, in contradiction with (3.1). Therefore, $0 < \mu \leq m$, and since by (2.47) we have $\mu \in 4\mathbb{N}$, we see that necessarily $m = \mu = 4$. This fact allows us to conclude also that $\varphi_k$ cannot satisfy alternative (ii) or alternative b) of (iii). In fact, in this situation, by (2.44), we would have: $\mu = 2(x + 1 + t)$, with $x$ and $t$ positive integers, a contradiction to (3.1). So $\varphi_k$ can only satisfy alternative a) of (iii) with a single blow-up point $q_0$. Furthermore, if $q_0 \in Z_1 = \{q_1, \ldots, q_s\}$, then (2.37) together with (3.1) allow us to conclude that, $s \geq 2$ and different zeroes of $W_{1,k}$ must converge to $q_0$. \hfill \Box

To proceed further, we use a different more convenient normalization for $\zeta_k$ from the previous section. In fact, we observe that, the translated function: $\tilde{\zeta}_k(x + x_k)$ (with the point $x_k$ defined in (2.17)) satisfies an analogous problem, possibly in a smaller ball around the origin, where properties (2.15)–(2.21) continue to hold, simply with the point $p_{j,k}$ replaced by $(p_{j,k} - x_k) \to 0$, as $k \to +\infty$. More importantly, since $x_k \to 0$ for $k \to +\infty$, then both sequences $\zeta_k(x)$ and its translated $\tilde{\zeta}_k(x + x_k)$ admit the same value of $m$ in (2.23). Thus, without loss of generality, we can assume that,

$$\tilde{\zeta}_k(0) = \max_{B_{r}} \tilde{\zeta}_k \to +\infty, \quad \text{as} \quad k \to +\infty. \quad (3.2)$$

Also, after relabeling the indices, (along a subsequence) we may take:

$$0 \leq |p_{1,k}| \leq |p_{2,k}| \leq \ldots \leq |p_{s,k}|, \quad \text{with} \quad s \geq 2; \quad (3.3)$$

so that,

$$\tau_k = |p_{s,k}| > 0, \quad (3.4)$$

$$q_{j,k} = \frac{p_{j,k}}{\tau_k} \to q_j, \quad \text{as} \quad k \to +\infty, \quad \forall \ j = 1, \ldots, s, \quad (3.5)$$

$$0 \leq |q_1| \leq |q_2| \leq \ldots \leq |q_s| = 1. \quad (3.6)$$

It is important to notice that, with this new normalization, we no longer expect (2.46) to hold. We have:
Theorem 4. Let $\xi_k$ satisfy (2.15), (2.16), (2.18)–(2.21) and assume (3.1)–(3.6). Then, the points $p_{j,k} \neq 0, 1 \leq j \leq s$ and there exists $s_1 \in \{2, \ldots, s\}$ such that (along a subsequence) the following holds, as $k \to +\infty$:

$$z_{j,k} := \frac{p_{j,k}}{|p_{s,k}|} \to z_j \neq 0, \; \forall \; j = 1, \ldots, s_1,$$  \hspace{1cm} (3.7)

and if $s_1 < s$ then,

$$\frac{p_{j,k}}{|p_{s,k}|} \to q_j \neq 0 \; \text{ and } \; \frac{|p_{j,k}|}{|p_{s,k}|} \to +\infty, \; \forall \; j = s_1 + 1, \ldots, s.$$  \hspace{1cm} (3.8)

Moreover

$$\xi_k(0) + 2 \ln |p_{s,k}| + 2 \sum_{j=1}^{s} z_j \ln |p_{j,k}| \to +\infty.$$  \hspace{1cm} (3.9)

Proof. If

$$q_{j,k} = \frac{p_{j,k}}{|p_{s,k}|} \to q_j \neq 0, \; \forall \; j = 1, \ldots, s$$  \hspace{1cm} (3.10)

then, it suffices to choose $s = s_1$. Indeed, we only need to check (3.9). To this purpose, by the normalization (3.2), we see that $q_0 = 0$ in Proposition 3.1. So for the sequence $\varphi_k$ in (2.29) we have

$$\varphi_k(0) = \sup_{D_k} \varphi_k \to +\infty.$$  \hspace{1cm} (3.11)

Conversely, by (2.29), (3.4), and (3.10), we find:

$$\varphi_k(0) = \xi_k(0) + 2 \ln \tau_k + 2 \ln |p_{s,k}|$$

$$= \xi_k(0) + 2 \ln |p_{s,k}| + 2 \sum_{j=1}^{s} z_j \ln |p_{j,k}| + O(1),$$

and so, (3.9) follows with $s_1 = s$ in this case. Since $|q_s| = 1$, let us now assume that there exists $\bar{s} \in \{1, \ldots, s - 1\}$ such that,

$$q_1 = q_2 = \ldots = q_{\bar{s}} = 0, \; \text{ and } \; q_j \neq 0 \; \text{ for } \; j = \bar{s} + 1, \ldots, s.$$  \hspace{1cm} (3.12)

Since the origin is a blow-up point for $\varphi_k$, by Proposition 3.1 we know that (3.12) must hold with $\bar{s} \geq 2$. Clearly, Theorem 1 applies to $\varphi_k$ around the origin, and it implies that,

$$m_\varphi = \lim_{\delta \to 0, k \to +\infty} \lim 2\pi \int_{B_\delta} W_{1,k} e^{\varphi_k} \in 4\mathbb{N}.$$  \hspace{1cm}

Since $m_\varphi \leq m = 4$, we see that necessarily $m_\varphi = 4$. This information allows us to apply Proposition 3.1 to $\varphi_k$ (around the origin), with $0 \leq |q_{1,k}| \leq |q_{2,k}| \leq \ldots \leq |q_{s,k}| \to 0$, as $k \to +\infty$. Therefore, by setting:

$$\tau_{1,k} = |q_{s,k}| \; \text{ and } \; \bar{x} = \sum_{j=1}^{\bar{s}} x_j \in \mathbb{N}, \; \bar{x} \geq 2,$$  \hspace{1cm} (3.13)

for the new” scaled” function:
\( \varphi_{1,k}(x) := \varphi_k(\tau_{1,k}x) + 2(\bar{z} + 1) \ln \tau_{1,k} \)

we know that (along a subsequence), \( \varphi_{1,k}(0) \to \infty \), as \( k \to +\infty \), and actually the origin is the only blow-up point of \( \varphi_{1,k} \), where “concentration” occurs, as described by alternative a) of (iii) in Proposition 2.1.

Furthermore (along a subsequence), as \( k \to +\infty \), we have:

\[ \frac{q_{j,k}}{\tau_{1,k}} \to q_j^{(1)} \land \, \forall \, j = 1, \ldots, \bar{s}; \]

with \( 0 \leq |q_1^{(1)}| \leq |q_2^{(1)}| \leq \ldots \leq |q_{\bar{s}}^{(1)}| = 1 \). If

\( q_j^{(1)} \neq 0, \land \, j = 1, \ldots, \bar{s}; \) (3.14)

then, from (3.12) we deduce that, as \( k \to +\infty \),

\[ \frac{|p_{j,k}|}{|p_{s,k}|} = \frac{|q_{j,k}|}{\tau_{1,k}} \to |q_j^{(1)}| \neq 0, \land \, j = 1, \ldots, \bar{s}, \]

\[ \frac{|p_{j,k}|}{|p_{s,k}|} \to |q_j| \neq 0, \land \frac{|p_{j,k}|}{|p_{s,k}|} = \frac{|q_{j,k}|}{\tau_{1,k}} \to +\infty, \land \, j = \bar{s} + 1, \ldots, s. \] (3.15)

So, if (3.14) holds then (3.7) and (3.8) are verified with \( s_1 = \bar{s} \).

Moreover, to show that also (3.9) holds with \( s_1 = \bar{s} \), simply note that \( \varphi_{1,k}(0) \to +\infty \), as \( k \to +\infty \), and in view of (2.29), (3.13), (3.15) we have

\[ \varphi_{1,k}(0) = \varphi_k(0) + 2(\bar{z} + 1) \ln \tau_{1,k} \]

\[ = \bar{z}(0) + 2(\bar{z} + 1) \ln |p_{k,\bar{s}}| + 2(\bar{z} + 1) \ln \frac{|p_{k,\bar{s}}|}{|p_{k,\bar{s}}|} \]

\[ = \bar{z}(0) + 2(\bar{z} + 1) \ln |p_{k,\bar{s}}| + 2(\bar{z} - \bar{z}) \ln |p_{k,\bar{s}}| \]

\[ = \bar{z}(0) + 2 \ln |p_{k,\bar{s}}| + \sum_{j=1}^{\bar{s}} 2\bar{z}_j \ln |p_{k,j}| + o(1). \]

Hence, the desired conclusion follows in this case.

On the contrary, if \( q_j^{(1)} \) vanishes for some \( j \in \{1, \ldots, \bar{s}\} \), then, we can repeat the argument above for the sequence \( \varphi_{1,k} \), and eventually continue in this way, by taking further scalings. However, at each step the number of “collapsing” zeroes decreases, and moreover Proposition 3.1 applies to each of the new scaled sequences. Therefore, this procedure must stop after finitely many steps.

We also emphasize that, in case we assume “by contradiction” that \( p_{1,k} = 0 \) identically, then we would end up with a sequence blowing-up at zero and satisfying a Liouville-type problem with a weight function vanishing at the origin with the same order of \( |x|^{2\bar{z}_j} \). Thus, to such a sequence, we can apply (2.42) (see [17]) and obtain a blow-up mass \( m = \frac{\sigma(0)}{2\bar{z}} = 4(1 + \bar{z}_j) > 4 \), a contradiction. Hence \( p_{1,k} \neq 0 \), and so by (2.18) and (3.3) we conclude that \( p_{j,k} \neq 0 \) for all \( 1 \leq j \leq s \), as claimed. Consequently, after finitely many steps, we must arrive at sequence which blows-up at zero and satisfying a Liouville-type problem with a weight function never vanishing around the origin.
In other words we have found $s_1 \in \{2, \ldots, s\}$ for which (3.15) holds with $\bar{s}$ replaced by $s_1$. And as before, we check that (3.9) is also satisfied.

Theorem 4 identifies the appropriate scale to use to gain good control on $\xi_k$ in a tiny neighborhood around the origin. Indeed, according to Theorem 4, we define:

$$\varepsilon_k = |p_{e_{i},k}| \to 0, \text{ as } k \to +\infty$$

(3.16)

and let

$$z_{j,k} = \frac{p_{j,k}}{\varepsilon_k} \to z_j \neq 0 \text{ for } j = 1, \ldots, s_1.$$

If $s_1 < s$, we set:

$$\varepsilon_{j,k} := \frac{\varepsilon_k}{|p_{j,k}|} \to 0, \quad \hat{p}_{j,k} := \frac{p_{j,k}}{|p_{j,k}|} \to \hat{p}_j, \quad |\hat{p}_j| = 1 \; \forall \; j = s_1 + 1, \ldots, s.$$

We define

$$v_k(x) = \zeta_k(\varepsilon_k x) + 2 \ln \varepsilon_k + \sum_{j=1}^{s} 2z_j \ln |p_{j,k}|,$$

satisfying:

$$-\Delta v_k(x) = \prod_{j=1}^{s} |x - z_{j,k}|^{2z_j} V_k(x)e^{v_k(x)} \text{ in } D_k = B_{\frac{1}{\varepsilon_k}}$$

(3.17)

with

$$V_k(x) = \prod_{j=s_1+1}^{s} |\varepsilon_{j,k} x - \hat{p}_{j,k}|^{2z_j} h_k(\varepsilon_k x) A_k$$

(3.18)

Clearly, the zero set of $R_0$ in (3.18) is given by:

$$Z = \{z_1, \ldots, z_s\},$$

where the $z_j$'s may not be necessarily distinct, but they satisfy:

$$0 < \delta_0 < |z_1| \leq |z_2| \leq \ldots \leq |z_s| < \frac{L_0}{4}$$

(3.19)

with suitable $\delta_0 > 0$ and $L_0 \geq 1$. Also we have:

$$\int_{D_k} R_k(x)e^{v_k} \leq C.$$ 

More importantly, in view of (3.7), (3.8), (3.9), we can check that,

$$v_k(0) = \max_{D_k} v_k \to +\infty, \quad \text{as } k \to +\infty.$$
By (3.1), we see that the origin is the only blow-up point of $v_k$, where the well-known blow-up analysis of [16, 17, 20] applies. As a consequence we find:

$$R_k e^{v_k} \to 8\pi\delta_0 \quad \text{weakly in the sense of measures, as } k \to +\infty,$$

locally on compact sets. Furthermore (see corollary 5.6.57 of [21]), for every $R > 0$ the following well-known estimate holds:

$$|v_k(x) - \ln \frac{e^{v_k(0)}}{(1 + \varepsilon_k^2 + R_k(0)|x|^2)^{\frac{3}{2}}}| \leq C_R \quad \text{in } \bar{B}_R,$$

with a suitable constant $C_R > 0$ depending on $R$ only. In particular, from (3.21) we derive:

$$|v_k(x) + v_k(0)| \leq C_R, \quad \forall \ x \in \partial B_R.$$

(3.22)

$$\max_{\partial B_R} v_k - \min_{\partial B_R} v_k \leq C_R.$$

(3.23)

**Lemma 3.1.** Let $L_0 > 1$ fixed to satisfy (3.19). Then, for every $R > L_0$ there exists $C_R$ such that,

$$\max_{\frac{1}{2} \leq |y| \leq R} \{v_k(y) + 2(\bar{z} + 1) \ln |y|\} \leq C_R$$

(3.24)

with $\bar{z} = \sum_{j=1}^{s_i} z_j \in \mathbb{N}$.

**Proof.** We start by observing the following:

**Claim:** For all $\varepsilon > 0$ there exist $k_\varepsilon \in \mathbb{N}$ and $r_\varepsilon > 0$ such that, for $\delta \in (0, r_\varepsilon)$ and $R \geq \frac{L_0}{4}$ we have:

$$\int_{B_\delta \setminus B_{r_\varepsilon}} W_k e^{v_k} = \int_{B_\delta \setminus B_{r_\varepsilon}} R_k e^{v_k} < \varepsilon, \forall \ k \geq k_\varepsilon;$$

(3.25)

with $\varepsilon_k$ given in (3.16).

To establish (3.25), recall that by (3.1) we find $k_\varepsilon \in \mathbb{N}$ and $r_\varepsilon > 0$ such that

$$\int_{B_\delta} W_k e^{v_k} \leq 8\pi + \frac{\varepsilon}{2}, \quad \forall \ k \geq k_\varepsilon \quad \text{and} \quad \delta \in (0, r_\varepsilon).$$

Conversely, by taking $k_\varepsilon$ larger if necessary, from (3.20), also we have that

$$\int_{B_{r_\varepsilon} \setminus B_{\frac{L_0}{4}}} W_k e^{v_k} = \int_{B_{r_\varepsilon} \setminus B_{\frac{L_0}{4}}} R_k e^{v_k} \geq 8\pi - \frac{\varepsilon}{2}, \quad \forall \ k \geq k_\varepsilon$$

and we immediately derive (3.25).

To establish (3.24), we argue by contradiction and assume there exists $R_1 > L_0$, such that

$$\exists \ y_k \in B_{R_1} \setminus B_{\frac{L_0}{4}}: v_k(y_k) + 2(\bar{z} + 1) \ln |y_k| \to \infty, \quad \text{as } k \to +\infty.$$

Define:

$$\psi_k(x) = v_k(|y_k|x) + 2(\bar{z} + 1) \ln |y_k|, \quad \text{for } x \in \Omega := \left\{ \frac{1}{2} < |x| < 2 \right\}$$
satisfying:

\[- \Delta \psi_k(x) = \left( \prod_{j=1}^{s_k} \left| x - \frac{z_{j,k}}{|y_k|} \right|^{2s_j} \right) V_k(|y_k|x)e^{\psi_k} \quad \text{in} \quad \Omega, \]

\[\psi_k\left( \frac{y_k}{|y_k|} \right) \to +\infty, \quad \text{as} \quad k \to +\infty.\]

Setting:

\[R_{1,k}(x) = \prod_{j=1}^{s_k} \left| x - \frac{z_{j,k}}{|y_k|} \right|^{2s_j} V_k(|y_k|x),\]

in view of (3.19), we check easily that there exist \(0 < a_1 \leq b_1\) and \(A > 0\) such that

\[0 < a_1 \leq R_{1,k}(x) \leq b_1 \quad \text{and} \quad |\nabla R_{1,k}(x)| \leq A \quad x \in \Omega, \quad \int_{\Omega} R_{1,k} e^{\psi_k} \leq C.\]

Therefore, if along a subsequence, we assume that,

\[\frac{y_k}{|y_k|} \to y_0, \quad \text{as} \quad k \to \infty,\]

then \(|y_0| = 1\), and so \(y_0\) is a blow-up point of \(\psi_k\) in \(\Omega\). As above, from [16, 17, 20] we have (along a subsequence)

\[R_{1,k} e^{\psi_k} \rightharpoonup 8\pi \delta_{y_0}, \quad \text{weakly in the sense of measures in} \quad \Omega\]

and therefore, for \(\delta > 0\) sufficiently small, there holds (along a subsequence):

\[\int_{\{ |z - y_k| < \delta |y_k| \}} R_k(z)e^{\psi_k(z)} \, dz = \int_{B_\delta(y_0)} R_{1,k}e^{\psi_k} \geq 4\pi, \quad \text{as} \quad k \to +\infty.\]

Consequently,

\[\int_{\{ |z - y_k| \leq \delta |y_k| \}} R_k(z)e^{\psi_k(z)} \, dz \geq 4\pi, \quad \text{as} \quad k \to +\infty,\]

a contradiction to (3.25).

We can reformulate (3.24) in terms of \(\xi_k\) as follows:

\[\xi_k(x) + 2(\bar{a} + 1) \ln |x| + \sum_{j=s_k+1}^{s} 2x_j \ln |p_{j,k}| \leq C_k, \quad \frac{L_0}{2} \varepsilon_k \leq |x| \leq R\varepsilon_k, \quad (3.26)\]

where the summation term in (3.26) should be dropped in case \(s_1 = s\).

But since for large \(k\) we have:

\[0 \leq W_k(x) \leq C_R |x|^{2s} \prod_{j=s_k+1}^{s} |p_{j,k}|^{2s_j} \quad \text{for} \quad \frac{L_0}{2} \varepsilon_k \leq |x| \leq R\varepsilon_k,\]

then from (3.26) we find:

\[0 \leq W_k(x)e^{\xi_k} \leq \frac{C_R}{|x|^2} \quad \text{for} \quad \frac{L_0}{2} \varepsilon_k \leq |x| \leq R\varepsilon_k \quad (3.27)\]

with suitable \(C_R > 0\) and \(k\) large.
Lemma 3.2. For $\varepsilon > 0$ sufficiently small there exist $k_{\varepsilon} \in \mathbb{N}$ and $C_{\varepsilon} > 0$ such that,

$$\tilde{\zeta}(x) \leq \min_{\partial B_{r}} \tilde{\zeta} + (4 + \varepsilon) \ln \frac{1}{|x|} + C_{\varepsilon}, \quad \text{for} \quad x \in B_{r} \setminus B_{L_0 k} \quad \text{and} \quad k \geq k_{\varepsilon}. \quad (3.28)$$

Proof. First, let us fix $\delta \in (0, r)$ sufficiently small, so that for large $k$ there holds:

$$M_k(\delta) := \frac{1}{2\pi} \int_{B_{\delta}(0)} W_k e^{\hat{\zeta}_k} < 4 + \frac{\varepsilon}{2} \quad \text{and} \quad \int_{B_{\delta} \setminus B_{L_0 k}} W_k e^{\hat{\zeta}_k} < 4\pi\varepsilon. \quad (3.29)$$

Since (3.28) clearly holds in $B_{r} \setminus B_{\delta}$, we are left to establish it in the set

$$\Omega_{k, \delta} = \{x : L_0 e_k \leq |x| \leq \delta\}.$$

In view of (3.29) we can establish the following:

Claim: The inequality (3.28) holds for $x : |x| = L_0 e_k$.

To obtain (3.30) we use (2.24) to write:

$$\tilde{\zeta}(x) = \min_{\partial B_{r}} \tilde{\zeta} + M_k(\delta) \ln \frac{1}{|x|} + \frac{1}{2\pi} \int_{|y| \leq \delta} \ln \left( \frac{|x|}{|x - y|} \right) W_k(y) e^{\hat{\zeta}(y)}$$

$$+ \int_{|y| \leq \delta} \ln \left( \frac{1}{|x - y|} \right) W_k(y) e^{\hat{\zeta}(y)} + O(1).$$

But, for $\delta \leq |y| \leq r$ and $|x| = L_0 e_k$, we see that,

$$|\ln |x - y|| \leq |\ln |y|| + C, \quad \text{and} \quad W_k(y) e^{\hat{\zeta}(y)} \leq C (\text{by (3.27)}),$$

and we deduce:

$$\tilde{\zeta}(x) \leq \min_{\partial B_{r}} \tilde{\zeta} + M_k(\delta) \ln \frac{1}{|x|} + \frac{1}{2\pi} \int_{|y| \leq \delta} \ln \left( \frac{|x|}{|x - y|} \right) W_k(y) e^{\hat{\zeta}(y)} + C. \quad (3.31)$$

To estimate the integral term in (3.31), we let

$$D(x) = \left\{ |y| \leq \frac{|x|}{2} \right\} \cup \left\{ |x - y| \geq \frac{|x|}{2} \right\} \text{and} \quad |y| \leq 2|x|$$

and observe that, if $y \in D(x)$, then $|\ln \frac{|y|}{|x - y|}| \leq 4$ and consequently

$$\left| \int_{D(x)} \ln \left( \frac{|x|}{|x - y|} \right) W_k(y) e^{\hat{\zeta}(y)} \right| \leq C.$$

Moreover, if $2|x| \leq |y| \leq \delta$, then $\frac{|x|}{|x - y|} \leq \frac{|x|}{|y|} \leq 1/|y|$ and therefore,

$$\int_{\{2|x| \leq |y| \leq \delta\}} \ln \left( \frac{|x|}{|x - y|} \right) W_k(y) e^{\hat{\zeta}(y)} \leq 0.$$

So we are left to estimate from above the given integral term on $B_{\frac{r}{2}}(x)$.

Actually, we can easily check (as above) that, for any $t \in (0, \frac{1}{2})$, we can find a suitable constant $C_t > 0$ such that
Therefore, by (3.32), we can use elliptic estimates to conclude that

\[
\int_{B_{\delta}(x)} \ln \left( \frac{|x|}{|x-y|} \right) W_k(y) e^{\tilde{\xi}_k}(y) dy \leq C.
\]

Next notice that, if \( y \in B_{\delta}(x) \) then \( (1-t)L_0\tilde{\eta}_k \leq |y| \leq (1+t)L_0\tilde{\eta}_k \), and so we can use (3.27) to estimate

\[
\int_{B_{\delta}(x)} \ln \left( \frac{|x|}{|x-y|} \right) W_k(y) e^{\tilde{\xi}_k}(y) dy \leq C \int_{(|x-z| \leq t)} \ln \left( \frac{1}{|x|} \right) \frac{1}{|y|^2} dy \\
= C \int_{(|x-z| \leq t)} \frac{1}{|y|^2} dz \leq C.
\]

This information together with (3.29), implies (3.30).

To proceed further, we define:

\[
\phi_k(x) = \tilde{\xi}_k(x) - \min_{\partial B_{\delta}} \tilde{\xi}_k - (4 + \alpha) \ln \frac{1}{|x|}, \quad x \in \Omega_{k,\delta},
\]

and in view of (3.30) we know that, \( \phi_k \) is uniformly bounded from above on \( \partial \Omega_{k,\delta} \), and consequently it satisfies:

\[
\begin{aligned}
-\Delta \phi_k &= W_k e^{\tilde{\xi}_k} \quad \text{in} \quad \Omega_{k,\delta} \\
\phi_k &\leq C \quad \text{in} \quad \partial \Omega_{k,\delta}.
\end{aligned}
\]

We are going to apply a well-known lemma from [16] (see e.g. lemma 5.2.1 of [21]) to the function \( \tilde{\phi}_k \) satisfying:

\[
\begin{aligned}
-\Delta \tilde{\phi}_k &= \tilde{f}_k \quad \text{in} \quad B_{\delta} \\
\tilde{\phi}_k &= C \quad \text{in} \quad \partial B_{\delta},
\end{aligned}
\]  

with \( \tilde{f}_k = \begin{cases} W_k e^{\tilde{\xi}_k} & \text{in} \quad \Omega_{k,\delta} \\
0 & \text{otherwise} \end{cases} \) (3.32)

Thus, as a consequence of the second inequality in (3.29) and lemma 5.2.1 of [21], for any \( 1 \leq q < \frac{1}{\tilde{\alpha}} \) we find a constant \( c_\varepsilon = c_\varepsilon(q) > 0 \) such that, \( \|e^{\tilde{\phi}_k}\|_{L^q(B_{\delta})} \leq c_\varepsilon \). Moreover, by the maximum principle, we know that

\[
\phi_k \leq \tilde{\phi}_k \quad \text{in} \quad \Omega_{k,\delta} \quad \text{and so} \quad \|e^{\phi_k}\|_{L^q(\Omega_{k,\delta})} \leq c_\varepsilon \] (3.33)

for \( 1 \leq q \leq \frac{1}{\tilde{\alpha}} \). Since \( 0 \leq W_k(x) \leq C|x|^{2\tilde{\alpha}} \) in \( \Omega_{k,\delta} \) with \( \tilde{\alpha} \geq 2 \), we find that, \( 0 \leq W_k e^{\tilde{\xi}_k} \leq C|x|^{2(\tilde{\alpha}-2)-\varepsilon e^{\tilde{\xi}_k}} \) and, by virtue of (3.33), we conclude that,

\[
\|\tilde{f}_k\|_{L^p(B_{\delta})} = \|W_k e^{\tilde{\xi}_k}\|_{L^p(\Omega_{k,\delta})} \leq C_\varepsilon \quad \text{for} \quad 1 \leq p < \frac{2}{3\tilde{\alpha}}.
\]

Therefore, by (3.32), we can use elliptic estimates to conclude that \( \tilde{\phi}_k \) is uniformly bounded in \( B_{\delta} \). In turn, from (3.33), we deduce:

\[
\phi_k \leq C \quad \text{in} \quad \Omega_{k,\delta}
\]

and (3.28) is established. \( \square \)

The estimate (3.28) implies in particular that, for any \( \varepsilon > 0 \) sufficiently small, we have:
0 \leq W_k(x)e^{\hat{\xi}_k(x)} \leq C|x|^{2(\bar{\alpha}-2)-\varepsilon}, \text{ for } L_0\varepsilon_k \leq |x| \leq r \text{ and } k \geq k_\varepsilon

with \bar{\alpha} \geq 2. As a consequence, we have:

\int_{L_0\varepsilon_k \leq |x| \leq r} | \ln |x| \cdot W_k(x)e^{\hat{\xi}_k(x)}dx \leq C, \quad (3.34)

and we shall take advantage of (3.34) to refine the estimate (3.28) as follows.

**Proposition 3.2.** We have:

\[ \hat{\xi}_k(x) = \min_{\partial B_r} \hat{\xi}_k + 4 \ln \frac{1}{|x|} + O(1), \quad \text{for } L_0\varepsilon_k \leq |x| \leq r. \quad (3.35) \]

**Proof.** As before, we first establish (3.35) for |x| = L_0\varepsilon_k. To this purpose set,

\[ \mu_k = \frac{1}{2\pi} \int_{|y| \leq 2L_0\varepsilon_k} W_k(y)e^{\hat{\xi}_k(y)}dy = \frac{1}{2\pi} \int_{|z| \leq 2L_0} R_k(z)e^{\nu(z)}dz \to 4, \quad (3.36) \]

as \( k \to +\infty \). Well-known estimates established in [27] (see also [21]) allow us to conclude that,

\[ |\mu_k - 4\nu_k(0)| = O(1). \quad (3.37) \]

Let \( x = \varepsilon_kx' \) with |x'| = L_0 and write

\[ \hat{\xi}_k(x) = \min_{\partial B_r} \hat{\xi}_k + \mu_k \ln \frac{1}{|x|} + \frac{1}{2\pi} \int_{|y| \leq 2L_0\varepsilon_k} \ln \left( \frac{|x|}{|x-y|} \right) W_k(y)e^{\hat{\xi}_k(y)}dy \]

\[ + \frac{1}{2\pi} \int_{|y| \leq 2L_0\varepsilon_k} \ln \left( \frac{1}{|x-y|} \right) W_k(y)e^{\hat{\xi}_k(y)}dy + O(1) \]

\[ = \min_{\partial B_r} \hat{\xi}_k + \mu_k \ln \frac{1}{\varepsilon_k} + \frac{1}{2\pi} \int_{|y| \leq 2L_0} \ln \left( \frac{|x'|}{|x'-y|} \right) R_k(y)e^{\nu(y)}dy \]

\[ + \frac{1}{2\pi} \int_{|y| \leq 2L_0\varepsilon_k} \ln \left( \frac{1}{|y|} \right) W_k(y)e^{\hat{\xi}_k(y)}dy + O(1). \quad (3.38) \]

By virtue of (3.20) and (3.21) we find: \( \int_{|y| \leq L_0} \ln \left( \frac{|x'|}{|x'-y|} \right) R_k(y)e^{\nu(y)}dy \to 0 \), as \( k \to +\infty \), while (3.34) implies that the last integral in (3.38) is uniformly bounded. In conclusion, we have obtained:

\[ \hat{\xi}_k(x) = \min_{\partial B_r} \hat{\xi}_k + \mu_k \ln \frac{1}{\varepsilon_k} + O(1), \quad \text{for } |x| = \varepsilon_kL_0. \quad (3.39) \]

As a consequence, for |y| = L_0, we have:

\[ \nu_k(y) = \hat{\xi}_k(\varepsilon_ky) + 2 \ln \varepsilon_k + \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| \]

\[ = \min_{\partial B_r} \hat{\xi}_k + (2\bar{\alpha} - \mu_k) \ln \varepsilon_k + \sum_{j=k_\varepsilon + 1}^s 2\varepsilon_j \ln |p_{j,k}| + O(1) \]

At this point, we can use (3.22) to deduce the following crucial information:
\[ v_k(0) = -\min_{\partial B_r} \xi_k + (2(\bar{z} + 1) - \mu_k) \ln \frac{1}{\varepsilon_k} + \sum_{j=1}^s 2\varepsilon_j \ln \frac{1}{|p_{j,k}|} + O(1). \]  

(3.40)

Since \(2(\bar{z} + 1) - \mu_k \to 2(\bar{z} - 1) \geq 2\), as \(k \to +\infty\), from (3.37) and (3.40) we obtain that,

\[ 0 < \ln \frac{1}{\varepsilon_k} \leq C v_k(0) \quad \text{and} \quad |\mu_k - 4| \ln \frac{1}{\varepsilon_k} \leq C. \]

(3.41)

for suitable \(C > 0\). Consequently, by (3.39),

\[ |\xi_k(x) - 4 \ln \frac{1}{\varepsilon_k} - \min_{\partial B_r} \xi_k| \leq C, \quad \text{for} \quad |x| = L_0 \varepsilon_k \]

and (3.35) is established for \(|x| = L_0 \varepsilon_k\).

Clearly, by (2.16), (3.35) also holds for \(|x| = r\), and we can argue exactly as above for the function \(\phi_k = \xi_k(x) - \min_{\partial B_r} \xi_k + 4 \ln |x|\), to show that \(\|\phi_k\|_{L^\infty\{R_0 \leq |x| < r\}} \leq C\), and conclude that (3.35) holds.

The estimate (3.35) allows us to show the following additional estimates.

**Proposition 3.3.** Under the above assumptions there holds

\[ v_k(0) = -\min_{\partial B_r} \xi_k + 2 \ln \varepsilon_k - \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| + O(1) \]

(3.42)

\[ |v_k(y) + v_k(0) + 4 \ln |y|| \leq C \quad \text{for} \quad L_0 \leq |y| \leq \frac{r}{\varepsilon_k} \]

(3.43)

\[ \xi_k(0) = -\min_{\partial B_r} \xi_k - 2 \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| + O(1) \]

(3.44)

\[ \int_{B_r} |\nabla \xi_k|^2 = 16\pi \left( v_k(0) + 2 \ln \frac{1}{\varepsilon_k} \right) + O(1) \]

\[ = -16\pi \left( \min_{\partial B_r} \xi_k + \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| \right) + O(1) \]

(3.45)

\[ = 16\pi \left( \xi_k(0) + 2 \sum_{j=1}^s \varepsilon_j \ln |p_{j,k}| \right) + O(1) \]

Proof. From (3.35) we derive that,

\[ v_k(y) = \min_{\partial B_r} \xi_k - 2 \ln \varepsilon_k \]

\[ + \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| + 4 \ln \frac{1}{|y|} + O(1), \quad \text{for} \quad L_0 \leq |y| \leq \frac{r}{\varepsilon_k}. \]

(3.46)

Hence, by using (3.46) with \(|y| = L_0 \varepsilon_k\) together with (3.22), we find:

\[ -v_k(0) = \min_{\partial B_r} \xi_k - 2 \ln \varepsilon_k + \sum_{j=1}^s 2\varepsilon_j \ln |p_{j,k}| + O(1) \]

(3.47)
and (3.42) is established. At this point, by inserting (3.47) into (3.46), we readily obtain (3.43). Also we obtain (3.44) from (3.42), once we recall that,
\[
\begin{aligned}
\varphi_k(0) &= \frac{n_k}{\varphi_k'} + 2\ln e_k + \sum_{j=1}^{s} 2x_j \ln |p_{j,k}|
\end{aligned}
\]
Finally, to obtain (3.45), we multiply both sides of the Eq. (2.15) by:
\[
\begin{aligned}
\frac{n_k}{C_0} &/ \partial_{Br} n_k/C_{21}
\end{aligned}
\]
and then, integrate over \(Br\). We find
\[
\begin{aligned}
\int_{\{x \leq L_0 \}} W_ke^{\varphi_k}(\varphi_k - \min_{\partial B_r} \varphi_k) dx
\end{aligned}
\]
Therefore, by means of (3.35), we obtain:
\[
\begin{aligned}
\int_{\{x \leq L_0 \}} W_ke^{\varphi_k}(\varphi_k - \min_{\partial B_r} \varphi_k) \leq C \int_{B_r} |x|^{2(z-2)} \ln \frac{1}{|x|} dx \leq C
\end{aligned}
\]
(recall that \(\bar{z} \geq 2\)). Since \(\varphi_k\) admits the origin as its only blow-up point on \(B_r\), then, it is uniformly bounded in \(C^1\)-norm on \(\partial B_r\), and in view (2.16) we deduce: \(\left| \int_{\partial B_r} \frac{\partial \varphi_k}{\partial v} (\varphi_k - \min_{\partial B_r} \varphi_k) \right| \leq C\). Finally, we compute:
\[
\begin{aligned}
\int_{\{x \leq L_0 \}} W_k(x)e^{\varphi_k(x)}(\varphi_k(x) - \min_{\partial B_r} \varphi_k(x)) dx
\end{aligned}
\]
(3.48)
Since by (3.36) and (3.41) we have:
\[
\begin{aligned}
\left( v_k(0) + 2 \ln \frac{1}{\varphi_k} \right) \int_{\{|y| \leq L_0 \}} R_k(y)e^{\alpha_k(y)} dy - 8\pi \leq C,
\end{aligned}
\]
then, by (3.42), we can estimate the last term in (3.48) as follows:
\[-2 \left( \min_{\partial B_r} \xi_k + \sum_{j=1}^{s} 2x_j \ln |p_{j,k}| \right) \int_{|y| \leq L_0} R_k(y) e^{\nu_k(y)} \, dy \]
\[= 2 \left( v_k(0) + 2 \ln \frac{1}{\varepsilon_k} \right) \int_{|y| \leq L_0} R_k(y) e^{\nu_k(y)} \, dy \]
\[= \left( v_k(0) + 2 \ln \frac{1}{\varepsilon_k} \right) 16\pi \]
\[+ 2 \left( v_k(0) + 2 \ln \frac{1}{\varepsilon_k} \right) \left( \int_{|y| \leq L_0} R_k(y) e^{\nu_k(y)} \, dy - 8\pi \right) \]
\[= \left( v_k(0) + 2 \ln \frac{1}{\varepsilon_k} \right) 16\pi + O(1). \]

Finally, we use (3.21) to estimate
\[\left| \int_{|x| \leq L_0} R_k(x) e^{\nu_k(x)} (v_k(x) - v_k(0)) \, dx \right| \]
\[\leq C \left| \int_{|x| \leq L_0} \frac{e^{\nu_k(0)}}{1 + e^{\nu_k(0)} h_k(0) |x|^2} \ln \left( 1 + \frac{e^{\nu_k(0)} h_k(0) |x|^2}{8 R_k(0) |x|^2} \right) \, dx \right| \]
\[\leq C \int_{\mathbb{R}^2} \frac{dy}{1 + |y|^2} \ln \left( 1 + |y|^2 \right) \leq C \]
and so (3.45) follows. \qed

Since \( R_k(0) = h_k(0) \), we can combine (3.21) and (3.45) to find
\[|v_k(y) - \ln \frac{e^{\nu_k(0)} h_k(0) |y|^2}{(1 + e^{\nu_k(0)} h_k(0) |y|^2)^2}| \leq C, \quad \text{for } |y| \leq \frac{r}{\varepsilon_k}. \quad (3.49)\]
and obtain in particular that:
\[\int_{|y| \leq \frac{r}{\varepsilon_k}} e^{\nu_k(y)} \, dy \leq C. \]

In view of Theorem 4, we know that \( p_{j,k} \neq 0 \ \forall \ k \in \mathbb{N} \), and so \( W_k(0) = \Pi_{j=1}^{s} |p_{j,k}|^{2\beta} h_k(0) > 0 \). Hence we can formulate (3.49) in terms of \( \xi_k \) as follows:

**Corollary 3.1.** Under the above assumptions we have, \( 0 < W_k(0) \rightarrow 0 \), as \( k \rightarrow +\infty \), and
\[\xi_k(x) = \ln \left( \frac{e^{\xi_k(0)}}{1 + \frac{e^{\xi_k(0)} W_k(0) |x|^2}{8 W_k(0) |x|^2}} \right) + O(1) \quad \text{in } B_r. \quad (3.50)\]

It is interesting to compare (3.50) with the analogous one” bubble” estimate established in [12, 27] (see e.g. theorem 0.3 in [12]) concerning the profile of blow-up solutions of (2.15)–(2.17), when \( W_k \uparrow +\infty \) \( W \) uniformly in \( B_r \) and \( W(0) > 0 \), in which case (3.1) is automatically satisfied (see (i) of Theorem A) and no collapsing issues arise. Indeed, our pointwise estimate in (3.50) is the striking exact analogue of the one provided in theorem 0.3 of [12], carried over to the case where \( W \) satisfies (2.22) and in particular \( W(0) = 0 \).
Furthermore, for the sequence:

\[ u_k(x) = \zeta_k(x) + \sum_{j=1}^{s} 2\pi_j \log |x - p_{j,k}|, \]

satisfying:

\[ -\Delta u_k = h_k e^{u_k} - 4\pi \sum_{j=1}^{s} \pi_j \delta_{p_{j,k}} \quad \text{in} \quad B_r, \tag{3.51} \]

we realize that, the estimate (3.44) stated for \( \zeta_k \) in Proposition 3.3 reduces, in terms of \( u_k \), to the following:

\[ u_k(0) + \min_{\partial B_r} u_k = -\ln W_k(0) \to +\infty, \quad \text{as} \quad k \to +\infty, \]

as stated in (1.14), and implying (by (1.11)) that blow-up for solutions of the singular problem (3.51) is equivalent to blow-up for its "regular" part.

We shall use the estimates established here to describe the asymptotic behavior of minimizers of the Donaldson functional considered in [5, 6], and to obtain in particular that for Riemann surfaces of genus 2, it is always bounded from below, although not coercive. Furthermore, we shall provide rather precise information on when the infimum is attained. In this way, we obtain the first existence result about (CMC) 1-immersions of a closed-orientable surface of genus 2 into hyperbolic 3-manifolds.

For this purpose, we conveniently summarize the results established above for a sequence \( \zeta_k \) satisfying:

\[
\begin{aligned}
-\Delta \zeta_k(x) &= (\Pi_{j=1}^{s} |x - p_{j,k}|^{2\pi_j}) h_k(x) + g_k(x) \quad \text{in} \quad B_r, \\
& \quad s \geq 2 \quad \text{and} \quad \pi_j \in \mathbb{N} \quad \text{for} \quad j = 1, \ldots, s \\
\zeta_k(0) &= \max_{B_r} \zeta_k \to +\infty, \quad \text{as} \quad k \to +\infty, \\
\forall \quad 0 < \delta < r \quad &\exists \quad C_0 > 0 : \max_{B_r \setminus B_{\delta}} \zeta_k \leq C_0 \\
\max_{\partial B_r} \zeta_k - \min_{\partial B_r} \zeta_k &\leq C \\
\int_{B_r} W_k e^{\zeta_k} \leq C, \quad W_k(x) := \left( \Pi_{j=1}^{s} |x - p_{j,k}|^{2\pi_j} \right) h_k(x)
\end{aligned}
\]  

**Theorem 5.** Suppose \( \zeta_k \) satisfies (3.52) with the points \( p_{j,k} \) satisfying (2.18) and (3.3), \( h_k \) satisfying (2.12) and (2.20), and \( g_k \) a convergent sequence in \( L^p(B_r) \), \( p > 1 \).

If (3.1) holds, then, the points \( p_{j,k} \neq 0, \ j = 1, \ldots, s \); and there exists \( s_1 \in \{2, \ldots, s\} \) such that (along a subsequence)

\[ z_{j,k} := \frac{p_{j,k}}{|p_{s_1,k}|} \to z_j \neq 0, \quad \forall \quad j = 1, \ldots, s_1, \]

if \( s_1 < s \) then \( \frac{p_{j,k}}{|p_{s,k}|} \to q_j \neq 0 \) and \( \frac{|p_{j,k}|}{|p_{s_1,k}|} \to +\infty, \quad \forall \quad j = s_1 + 1, \ldots, s. \)

Moreover,
\[ \xi_k(0) + 2 \ln |p_{3,k}| + \ln (W_k(0)) \rightarrow +\infty, \quad \text{as} \quad k \rightarrow +\infty, \]
\[ \xi_k(0) + \left( \min_{\partial B_r} \xi_k + 2 \ln (W_k(0)) \right) = O(1) \]
\[ \xi_k(x) = \ln \left( \frac{e^{\xi_k(0)}}{1 + \frac{W_k(0)}{8} e^{\xi_k(0)} |x|^{2^*}} \right) + O(1), \quad x \in B_r \]
\[ \int_{B_r} |\nabla \xi_k|^2 \, dx = 16\pi (\xi_k(0) + \ln (W_k(0))) + O(1). \]

**Proof.** It suffices to observe that the results established above apply to the sequence: \( \hat{\xi}_k := \xi_k + \chi_k \), where \( \chi_k \) is the unique solution of the Dirichlet problem:
\[
\begin{cases}
\Delta \chi_k = g_k & \text{in } B_r \\
\chi_k = 0 & \text{in } \partial B_r.
\end{cases}
\]

Indeed, by elliptic estimates, \( \chi_k \) converges strongly in \( C^{1,\alpha}(B_r) \) and moreover, \( \ln W_k(0) = \sum_{j=1}^s 2\zeta_j \ln |p_j| + O(1) \). Then, it is easy to check that in terms of the original sequence \( \xi_k \), we get exactly the claimed estimates. \( \square \)

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Appendix A: The proof of (2.50)

By virtue of the properties pointed out in Section 2 for the sequence \( \varphi_k \) defined in (2.29), to establish (2.50) we can follow word by word the arguments used in [11] to show the same identity (i.e. (4.9) in [11]), see also [2]. To this purpose we start with the following:

**Lemma A.1.** Let \( \mu \) be defined in (2.31). Then, there exists \( R_0 > 0 \) sufficiently large, such that, as \( k \to +\infty \),

\[
\nabla \varphi_k (x) \to - \mu \frac{x}{|x|^2} + O\left( \frac{1}{|x|^2} \right)
\]

(A.1)

uniformly on compact sets of \( \mathbb{R}^2 \setminus B_{R_0} \).

**Proof.** From (2.30) we have

\[
\nabla \varphi_k (x) = - \frac{\tau_k}{2\pi} \int_{B_k} \left( \frac{\tau_k x - y}{|\tau_k x - y|^2} \right) W_k(y)e^{i\xi_k(y)}dy
\]

\[
= - \frac{\tau_k}{2\pi} \int_{\{|y| \leq \tau_k |x|\}} \left( \frac{\tau_k x - y}{|\tau_k x - y|^2} \right) W_k(y)e^{i\xi_k(y)}dy
\]

\[
- \frac{\tau_k}{2\pi} \int_{\{|y| \leq \tau_k |x|\} \setminus \{|y| \leq \tau_k |x|\}} \left( \frac{\tau_k x - y}{|\tau_k x - y|^2} \right) W_k(y)e^{i\xi_k(y)}dy
\]

\[
= : I_{1,k}(x) + I_{2,k}(x).
\]

Now, for \( R_0 > 1 \) (to be fixed later) let \( |x| \geq R_0 \) and \( \tau_k |x| \leq |y| \leq r \), then, we have:

\[
\left( 1 - \frac{1}{R_0} \right) |y| \leq |\tau_k x - y| \leq \left( 1 + \frac{1}{R_0} \right) |y|
\]

and therefore,

\[
|I_{2,k}| \leq C \int_{\{|y| \leq \tau_k |x|\}} \frac{\tau_k}{|y|} e^{i\xi_k(y)} W_k(y)dy \leq C \frac{1}{|x|^2}, \quad |x| \geq R_0 > 1.
\]

By recalling (2.27) and (2.29), we find:

\[
I_{1,k}(x) = - \frac{1}{2\pi} \int_{\{|y| \leq \tau_k |x|\}} \frac{(x - z)}{|x - z|^2} W_{1,k}(z)e^{i\eta(z)}dz.
\]

Therefore, in case \( \varphi_k \) satisfies alternative (2.32), then (A.1) readily follows. Indeed, \( I_{1,k} \to 0 \), as \( k \to +\infty \), uniformly on compact sets of \( \mathbb{R}^2 \setminus B_{R_0} \) and \( \mu = 0 \) in this case. While, in case \( \varphi_k \) satisfies alternative (2.35), then, for \( R_0 > 1 \) sufficiently large so that, \( S_0 \subset \subset B_{R_0} \), we have, as \( k \to +\infty \):

\[
I_{1,k}(x) \to - \frac{1}{2\pi} \sum_{q \in S_0} \sigma(q) \frac{(x - q)}{|x - q|^2}
\]

uniformly on compact sets of \( \mathbb{R}^2 \setminus B_{R_0} \),

and, in this case (see (2.36)), we have:

\[
- \frac{1}{2\pi} \sum_{q \in S_0} \sigma(q) \frac{(x - q)}{|x - q|^2} = - \mu \frac{x}{|x|^2} + \frac{1}{2\pi} \sum_{q \in S_0} \left( \frac{x}{|x|^2} - \frac{(x - q)}{|x - q|^2} \right)
\]

\[
= - \mu \frac{x}{|x|^2} + O\left( \frac{1}{|x|^2} \right), \quad \text{for } |x| \geq R_0.
\]

So also in this case (A.1) is established. Finally, in case \( \varphi_k \) satisfies alternative (2.33) or (2.38), then by virtue of (2.43) and (2.44) we find that,
\[ W_1(x)e^{\phi_k} \leq \frac{C}{|x|^{2(\rho+1)}} \quad \text{for} \quad |x| \geq R_0 \]  
(A.2)

with \( R_0 \gg 1 \) sufficiently large and suitable constants \( C \geq 0 \) and \( \rho \geq 1 \). In this case, as \( k \to +\infty \), we find:

\[
I_{1,k}(x) \to -\mu \frac{x}{|x|^2} + R_1(x), \quad \text{uniformly on compact sets of } \mathbb{R}^2 \setminus B_{R_0}
\]

with

\[
R_1(x) = \frac{1}{2\pi} \int_{|y| \geq |x|^2} W_1(x)e^{\phi_k} dy + \frac{1}{2\pi} \int_{|y| < |x|^2} \left( \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right) W_1(y)e^{\phi_k(y)} dy + R_2(x),
\]

where \( R_2(x) = 0 \) in case alternative (2.33) holds (i.e. \( S_\varphi = \emptyset \)) or

\[
R_2(x) = \frac{1}{2\pi} \sum_{q \in S_\varphi} \sigma(q) \left( \frac{x}{|x|^2} - \frac{x-q}{|x-q|^2} \right) = O\left( \frac{1}{|x|^2} \right)
\]
in case alternative (2.38) holds (i.e. \( S_\varphi \neq \emptyset \)). Clearly, in view of (A.2), we easily check that,

\[
\int_{|y| \geq |x|^2} W_1(y)e^{\phi_k(y)} \leq \frac{C}{|x|^2}, \quad \forall \ |x| \geq R_0.
\]

By recalling the following easy facts:

if \( |x-y| \geq \frac{|x|}{2} \) then
\[
\left| \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right| \leq 4 \frac{|y|}{|x|^2}
\]

if \( |x-y| \leq \frac{|x|}{2} \) then
\[
\left| \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right| \leq \frac{4}{|x-y|},
\]

and by using (A.2), for \( |x| \geq R_0 \) we derive:

\[
\left| \int_{|y| \leq |x|^2} \left( \frac{x}{|x|^2} - \frac{x-y}{|x-y|^2} \right) W_{1,k}(y)e^{\phi_k} dy \right|
\]

\[
\leq \frac{C}{|x|^2} \int_{\mathbb{R}^2} |y||W_1(y)e^{\phi_k} dy + C \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|} W_1(y)e^{\phi_k} dy
\]

\[
\leq \frac{C}{|x|^2} + C \int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|} \frac{1}{|y|^{2(\rho+1)}} dy
\]

\[
= \frac{C}{|x|^2} + \frac{C}{|x|^{2\rho+1}} \int_{|z| \leq \frac{|x|}{2}} \frac{1}{|x|} \frac{dz}{|z|^{2(\rho+1)}} \leq \frac{C}{|x|^2} \quad \text{for} \quad |x| \geq R_0 \geq 1.
\]

By combining the estimates above, we obtain (A.1) also when \( \phi_k \) satisfies either alternative (2.33) or alternative (2.38), and the proof is completed. \( \Box \)

By using (2.25) and (A.1), we can easily show that, as \( k \to +\infty \):

\[
r \int_{\partial B_r} \left( |\partial_v \xi_k| - \frac{1}{2} |\nabla \xi_k|^2 \right) d\sigma \to 2\pi \frac{m^2}{2} + o_r(1)
\]

with \( m \) in (2.23) and \( o_r(1) \to 0 \), as \( r \to 0 \), and
\[ R \int_{\partial B_R} \left( |\partial_{\nu} \xi_k| - \frac{1}{2} |\nabla \xi_k|^2 \right) d\sigma \to 2\pi \frac{\mu^2}{2} + o_R(1) \]

with \( o_R(1) \to 0 \), as \( R \to +\infty \).

**Lemma A.2.** There holds:

\[ m^2 - \mu^2 = 4(x + 1)(m - \mu). \]

**Proof.** As in [2, 11], we use the Pohozaev identity (see e.g. (5.2.14) in [21]) for \( \xi_k \) (satisfying (2.15)) in \( B_r \setminus B_{R_1} \) and obtain:

\[
\begin{align*}
\int_{\partial B_R} \left( |\partial_{\nu} \xi_k|^2 - \frac{1}{2} |\nabla \xi_k|^2 + W_k e^{\xi_k} \right) d\sigma \\
- R \int_{\partial B_R} \left( |\partial_{\nu} \varphi_k|^2 - \frac{1}{2} |\nabla \varphi_k|^2 + W_{1,k} e^{\varphi_k} \right) d\sigma
\end{align*}
\]

\[= \int_{\partial B_R} \left( 1 + \sum_{j=1}^{s} 2 \frac{y - q_{j,k}}{|y - q_{j,k}|^2} \right) W_{1,k}(y)e^{\varphi_k} dy 
+ \int_{\partial B_R} \left( \tau_k y \cdot \frac{\nabla h_k(\tau_k y)}{h_k(\tau_k y)} \right) W_{1,k}(y)e^{\varphi_k(y)} dy. \tag{A.3}
\]

By using (2.21) and (2.25), and the convergence (A.1) for \( \varphi_k \), we easily check that, for the left-hand side \( (\text{LHS})_k \) of (A.3), we find

\[ (\text{LHS})_k \to 2\pi \left( \frac{m^2}{2} - \frac{\mu^2}{2} \right) + o_r(1) + o_R(1), \text{ as } k \to +\infty. \]

While for the right-hand side \( (\text{RHS})_k \) of (A.3), we have:

\[ (\text{RHS})_k = 2(1 + \alpha) \int_{\partial B_R \setminus B_R} W_{1,k}(y)e^{\varphi_k} 
+ \sum_{j=1}^{s} 2 \frac{q_{k,j}}{|y - q_{k,j}|^2} \int_{\partial B_R \setminus B_R} W_{1,k} e^{\varphi_k} + o_r(1) \]

and we conclude that, as \( k \to +\infty \),

\[ (\text{RHS})_k \to 4\pi (1 + \alpha)(m - \mu) + o_r(1) + o_R(1). \]

Then, we obtain the desired conclusion, by letting \( r \to 0^+ \) and \( R \to +\infty \). \qed