An asymptotic expansion for a ratio of products of gamma functions

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Abstract

An asymptotic expansion of a ratio of products of gamma functions is derived. It generalizes a formula which was stated by Dingle, first proved by Paris, and recently reconsidered by Olver.

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1 Introduction

Our starting point is the Gaussian hypergeometric function $F(a, b; c; z)$ and its series representation

$$\frac{1}{\Gamma(c)} F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c + n)_n} \frac{1}{n!} z^n, \quad |z| < 1,$$

which here is written in terms of Pochhammer symbols

$$(x)_n = x(x+1) \ldots (x+n-1) = \Gamma(x+n)/\Gamma(x).$$

The hypergeometric series appears as one solution of the Gaussian (or hypergeometric) differential equation, which is characterized by its three regular
singular points at \( z = 0, 1, \infty \). The local series solutions at 0 and 1 of this differential equation are connected by the continuation formula [4]

\[
\frac{1}{\Gamma(c)} F(a, b; c; z) = \frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; 1 + a + b - c; 1 - z) \\
+ \frac{\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} (1 - z)^{c-a-b} F(c - a, c - b; 1 + c - a - b; 1 - z), \quad (|\arg(1 - z)| < \pi).
\]

Here we want to show that Eq. (1) implies an interesting asymptotic expansion for a ratio of products of gamma functions, of which only a special case was known before.

By applying the method of Darboux [4, 8] to (1), we derive in Sec. 2 the formula in question. The behaviour of this and a related formula is discussed in Sec. 3 and illustrated by a few numerical examples.

2 Derivation of an asymptotic expansion for a ratio of products of gamma functions

It is well-known that the late coefficients of a Taylor series expansion contain information about the nearest singular point of the expanded function [3]. In this respect we want to analyze the continuation formula (1), in which then only the second, at \( z = 1 \) singular term \( R \) is relevant, which may be written as

\[
R = \frac{\Gamma(a + b - c)\Gamma(1 + c - a - b)}{\Gamma(a)\Gamma(b)} \sum_{m=0}^{\infty} \frac{(c-a)_m(c-b)_m}{\Gamma(1 + c - a - b + m)!} (1 - z)^{c-a-b+m}.
\]

By means of the binomial theorem in its hypergeometric-series-form, we may expand the power factor

\[
(1 - z)^{c-a-b+m} = \sum_{n=0}^{\infty} \frac{\Gamma(a + b - c - m + n)}{\Gamma(a + b - c - m)!} z^n.
\]

Interchanging then the order of the summations and simplifying by means of the reflection formula of the gamma function, we arrive at

\[
R = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(c-a)_m(c-b)_m}{m!} \frac{\Gamma(a + b - c - m + n)}{n!} z^n.
\]
This is to be compared with the left-hand side $L$ of (4), which is

$$L = \frac{1}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n.$$  

Comparison of the coefficients of these two power series, which according to Darboux [4] and Schäfke and Schmidt [8] should agree asymptotically as $n \to \infty$, then yields

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} = \sum_{m=0}^{M} (-1)^m (c-a)_m (c-b)_m m! \Gamma(a+b-c-m+n)$$

$$+ O(\Gamma(a+b-c-M-1+n)).$$

By means of

$$O(\Gamma(a+b-c-M-1+n)) = \Gamma(a+b-c+n)O(n^{-M-1})$$

and the reflection formula of the gamma function, the relevant formula (2) may also be written as

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^{M} \frac{(c-a)_m (c-b)_m}{m!(1+c-a-b-n)_m} + O(n^{-M-1}).$$

The asymptotic expansion for a ratio of products of gamma functions in this form (3) or the other (2) seems to be new. It is only the special case when $c = 1$ which is known. This special case was stated by Dingle [2], first proved by Paris [7], and reconsidered recently by Olver [5], who has found a simple direct proof. His proof, as well as the proof of Paris, can be adapted easily to the more general case when $c$ is different from 1. Still another proof is available [6] which includes an integral representation of the remainder term. Our derivation of Eq. (2) or (3) is significantly different from all the earlier proofs of the case when $c = 1$.

3 Discussion and numerical examples

We now want to discuss our result in the form (3). First we observe that the substitution $c \to a+b-c$ leads to the related formula

$$\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(a+b-c+n)} = 1 + \sum_{m=1}^{M} \frac{a-c)_m (b-c)_m}{m!(1-c-a-b-n)_m} + O(n^{-M-1}).$$
Which of (3) or (4) is more advantageous numerically depends on the values of the parameters, and in this respect the two formulas complement each other. Table 1 shows an example with a set of parameters for which (3) gives more accurate values than (4), while Table 2 contains an example for which (4) is superior to (3).

For finite \(n\) and \(M \to \infty\) the series on the right-hand side of (3) converges if \(\text{Re}(1 - c - n) > 0\). The same is true for (4) if \(\text{Re}(1 + c - a - b - n) > 0\). Then, in both cases, the Gaussian summation formula yields

\[
\frac{\Gamma(1 - c - n)\Gamma(1 + c - a - b - n)}{\Gamma(1 - a - n)\Gamma(1 - b - n)},
\]

which, by means of the reflection formula of the gamma function, is seen to be equal to

\[
\frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)\Gamma(a + b - c + n)\sin(\pi[a + n])\sin(\pi[b + n])}\sin(\pi[a + b - c + n])
\]

Otherwise (2) – (4) are divergent asymptotic expansions as \(n \to \infty\).

Although in our derivation \(n\) is a sufficiently large positive integer, the asymptotic expansions (2) – (4) are expected to be valid in a certain sector of the complex \(n\)-plane, and in fact, the proofs of Paris [7] and of Olver [6] apply to complex values of \(n\).

If the series in (3) or (4) converge, their sums are equal to (5), which generally (if neither \(c - a\) nor \(c - b\) is equal to an integer) is different from the left-hand side of (3) or (4). Therefore (3) and (4) can be valid only in the half-planes in which the series do not converge. This means that (3) is an asymptotic expansion as \(n \to \infty\) in the half-plane \(\text{Re}(c-1+n) \geq 0\), and (4) is an asymptotic expansion as \(n \to \infty\) in the half-plane \(\text{Re}(a+b-c-1+n) \geq 0\). Otherwise the series on the right-hand sides represent a different function, namely (3).

A few numerical examples may serve for demonstration of these facts. In Table 3, the series converge to (3) for \(n = 10\), and therefore (3) and (4) are not valid. For \(n = 20\), on the other hand, the series diverge and so (3) and (4) hold. The transition between the two regions is at the line \(\text{Re}(n) = 12.4\) in case of (3) or \(\text{Re}(n) = 12.5\) in case of (4). In Table 4, we see convergence for \(n = -15\) and divergence for \(n = -5\), the transition between the two regions being at the line \(\text{Re}(n) = -10.4\) in case of (3) or \(\text{Re}(n) = -10.5\) in case of (4).
References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[2] R. B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic Press, London, 1973).

[3] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. I.

[4] F. W. J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).

[5] F. W. J. Olver, *Asymptotic expansions of the coefficients in asymptotic series solutions of linear differential equations*, Methods and Applications of Analysis 1, 1–13 (1994).

[6] F. W. J. Olver, *On an asymptotic expansion of a ratio of gamma functions*, Proc. Royal Irish Acad. A95, 5–9 (1995).

[7] R. B. Paris, *Smoothing of the Stokes phenomenon using Mellin-Barnes integrals*, J. Comput. Appl. Math. 41, 117–133 (1992).

[8] R. Schäfke and D. Schmidt, *The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions*, SIAM J. Math. Anal. 11, 848–862 (1980).
Table 1: Values of the right-hand sides of (3) and (4) for the parameters $a = 0.7$, $b = 1.2$, $c = 0.4$.

| $n$ | Right-hand side of (3) | Right-hand side of (4) |
|-----|------------------------|------------------------|
| 1   | 0.9771429              | 0.9744681              |
| 2   | 0.9773113              | 0.9780243              |
| 3   | 0.9772978              | 0.9769927              |
| 4   | 0.9773005              | 0.9774980              |
| 5   | 0.9772995 ←           | 0.9771117 ←           |
| 6   | 0.9773001 ←           | 0.9775615              |
| 7   | 0.9772995              | 0.9767519              |
| 8   | 0.9773003              | 0.9791530              |
| 9   | 0.9772983              | 0.9652341              |
| 10  | 0.9773079              | 1.2823765              |

Exact value of (3) or (4): 0.97729983

Table 2: Values of the right-hand sides of (3) and (4) for the parameters $a = -0.7$, $b = -1.2$, $c = -0.4$.

| $n$ | Right-hand side of (3) | Right-hand side of (4) |
|-----|------------------------|------------------------|
| 1   | 0.968000               | 0.972093               |
| 2   | 0.973760               | 0.972350               |
| 3   | 0.971512 ←            | 0.972324               |
| 4   | 0.973078 ←            | 0.972331               |
| 5   | 0.971231 ←            | 0.972327 ←            |
| 6   | 0.975016 ←            | 0.972330 ←            |
| 7   | 0.959571               | 0.972325               |
| 8   | 1.179434               | 0.972342               |
| 9   | 4.748048               | 0.972163               |
| 10  | 26.430946              | 0.968966               |

Exact value of (3) or (4): 0.97232850
Table 3: Values of the right-hand sides of (3) and (4) for the parameters $a = -11.7$, $b = -11.2$, $c = -11.4$.
| $n = -15$ | $M$ | right-hand side of (3) | right-hand side of (4) |
|-----------|-----|-----------------------|-----------------------|
| 1         | 0.986667 | 0.986957              |
| 2         | 0.985648 | 0.985745              |
| 3         | 0.985453 | 0.985492              |
| 4         | 0.985397 | 0.985415              |
| 5         | 0.985376 | 0.985386              |
| 6         | 0.985368 | 0.985373              |
| 7         | 0.985363 | 0.985367              |
| 8         | 0.985361 | 0.985363              |
| 9         | 0.985360 | 0.985361              |
| 10        | 0.985359 | 0.985360              |

exact value of (3) or (4): 1.97071532

exact value of (5): 0.98535766

| $n = -5$  | $M$ | right-hand side of (3) | right-hand side of (4) |
|-----------|-----|-----------------------|-----------------------|
| 1         | 1.010909 | 1.011111              |
| 2         | 1.009891 | 1.009798              |
| 3         | 1.010254 | ← 1.010331  ←        |
| 4         | 1.009940 | ← 1.009818  ←        |
| 5         | 1.010589 | 1.011015              |
| 6         | 1.005300 | 0.998322              |
| 7         | 0.951894 | 0.887892              |
| 8         | 0.737202 | 0.459630              |
| 9         | 0.134729 | ← 0.725230            |
| 10        | -1.243041 | ← -3.418810         |

exact value of (3) or (4): 1.01011438

exact value of (5): 0.50505719

Table 4: Values of the right-hand sides of (3) or (4) for the parameters $a = 11.7$, $b = 11.2$, $c = 11.4$. 