Charged Dilaton, Energy, Momentum and Angular-Momentum in Teleparallel Theory Equivalent to General Relativity

Gamal G.L. Nashed

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt
e-mail:nasshed@asunet.shams.edu.eg

We apply the energy-momentum tensor to calculate energy, momentum and angular-momentum of two different tetrad fields. This tensor is coordinate independent of the gravitational field established in the Hamiltonian structure of the teleparallel equivalent of general relativity (TEGR). The spacetime of these tetrad fields is the charged dilaton. Our results show that the energy associated with one of these tetrad fields is consistent, while the other one does not show this consistency. Therefore, we use the regularized expression of the gravitational energy-momentum tensor of the TEGR. We investigate the energy within the external event horizon using the definition of the gravitational energy-momentum.

1. Introduction

Quantum mechanics and general relativity (GR) are two very successful and well validated theories within their own domains. The main problem is to unify them into a single consistent theory. One of the most promising models of unification is the string theory. The string theory is classified into two classes which are the closed and the open strings. The gravity is described by the first class while the matter is described by the second one. In case of non-perturbative string theory, there are extended objects known as D-branes. These objectives are surfaces where open strings must begin and finish which provide an alternative approach to the Kaluza-Klein [1]. In this last approach the matter penetrates the extra dimensions, leading to strong constraints from collider physics.

Nowadays there is a growing body of literature about the gravitational field of string matter coupled to an electromagnetically charged dilaton field. Black hole solutions in dilaton gravity were first analyzed by Gibbons and Maeda [2]. Garfinkle et al. [3] have obtained a family of solutions representing static, spherically symmetric charged black holes. Kallosh and Peet [4] in the context of supersymmetric theories investigated these solutions. When the dilaton acquires a mass Gregory and Harvey [5] modified the dilaton black holes. A static spherically symmetric metric around a source coupled to a massless dilaton with both electric and magnetic charged is investigated by Agnes and Camera [6].

*PACS numbers: 04.70.Bw, 04.50.+h, 04.20-Jb.
Keywords: Teleparallel equivalent of general relativity, charged dilaton black holes, Gravitational energy-momentum tensor.
Among various attempts to overcome the problems of quantization and the existence of singular solution in Einstein’s GR, gauge theories of gravity are of special interest, as they based on the concept of gauge symmetry which has been very successful in the foundation of other fundamental interactions. The importance of the Poincaré symmetry in particle physics leads one to consider the Poincaré gauge theory (PGT) as a natural framework for description of the gravitational phenomena \[7\sim13\]. Basic gravitational variables in PGT are the tetrad field \(e^a_{\mu}\) and the Lorentz connection \(A^{ab}_{\mu}\). These variables are associated to the translation and Lorentz subgroups of the Poincaré group. The gauge fields are coupled to the energy-momentum and spin of matter fields, and their field strengths are geometrically identified with the torsion and the curvature.

General geometric arena of PGT, the Riemann-Cartan space \(U_4\), may be a priori restricted by imposing certain conditions on the curvature and the torsion. Thus, Einstein’s GR is defined in Riemann space \(V_4\), which is obtained from \(U_4\) by the requirement of vanishing torsion. Another interesting limit of PGT is the teleparallel or Weitzenböck geometry \(T_4\). The vanishing of the curvature means that parallel transport is path independent. The teleparallel geometry is, in sense, complementary to Riemannian: curvature vanishes, and torsion remains to characterize the parallel transport. For the physical interpretation of the teleparallel geometry there is a one-parameter family of teleparallel Lagrangians which is empirically equivalent to GR \[12, 14, 15\]. If the parameter value \(B = 1/2\) the Lagrangian of the theory coincides, modulo a four-divergence, with the Einstein-Hilbert Lagrangian, and defines (TEGR).

The search for a consistent expression for the gravitating energy and angular-momentum of a self-gravitating distribution of matter is undoubtedly a long-standing problem in GR. It is believed that the energy of the gravitational field is not localizable, i.e., defined in a finite region of the space. The gravitational field does not possess the proper definition of an energy momentum tensor. It is usually to define some energy-momentum and angular-momentum \[16, 17\] which are pseudo-tensors and depend on the second derivative of the metric tensor. These quantities can be annullled by an adequate transformation of coordinates. Bergmann \[16\], Landau-Lifschitz \[17\] justify that the energy and angular momentum are consistent with Einstein’s principle of equivalence. According to this principle “any space-time region, infinitesimal or not, is flat if and only if the Riemann-Christoffel tensor vanishes in this region”. In such a flat space-time, energy of the gravitational field is null. Therefore, it is only possible to define the energy of the gravitational field in whole space-time region and not only in a small region. The Einstein’s GR can also be reformulated in the context of teleparallel geometry \[18\sim27\]. In this geometry the dynamical field is corresponding to orthonormal tetrad field \(e^a_{\mu}\) (\(a, \mu\) are SO(3,1) and space-time indices, respectively). The teleparallel geometry is a suitable framework to address the notions of energy, momentum and angular-momentum of any space-time that admits a \(3 + 1\) foliation \[28\]. Therefore, we consider the TEGR in this work.

In order to calculate the energy and angular momentum we use the Hamiltonian that is formulated for an arbitrary teleparallel theories using Schwinger’s time gauge \[29\sim37\]. In this formulation it is shown that the TEGR is the only viable consistent teleparallel theory of gravity. Maluf and Rocha \[38\] established a theory in which Schwinger’s time gauge has not been incorporated in the geometry of absolute parallelism. In this formulation, the definition of the gravitational angular-momentum arises by suitably interpreting the integral

\[\text{Hamiltonian} = \sum_{a=0}^{3}\sum_{\mu=0}^{3}\sum_{i=0}^{3} \int d^3x \left( e^a_{\mu} \right)_i \left( e^a_{\mu} \right)^* \]

\[\text{where } e^a_{\mu} \text{ is the orthonormal tetrad field, } a, \mu \text{ are indices running from 0 to 3, and } i \text{ is a time index.} \]
form of the constraint equation $\Gamma^{ab} = 0$. This definition has been successfully applied to the gravitational field of a thin, slowly rotating mass shell [39] and for the three-dimensional BTZ black hole [40].

Definitions for the gravitational energy in the context of the TEGR have already been proposed in the literatures [25, 30]. An expressions for the gravitational energy arises from the surface term of the total Hamiltonian is given [41, 42]. These expressions are equivalent to the integral form of the total divergences of the Hamiltonian density developed by Maluf et al. [38]. These expressions yield the same value for the total energy of the gravitational field. However, since these expressions contain the lapse function in the integrand, none of them are suitable to the calculation of the irreducible mass of the Kerr black hole. This is because the lapse function vanishes on the external event horizon of the black hole [30]. The energy expressions [41, 42] neither to be applied to a finite surface integration nor they yield the total energy of the space-time [30]. A good energy-momentum expression for gravitating systems should satisfy a variety of requirements: to give the standard values of the total quantities for asymptotically flat space, to reduce to the material energy-momentum in proper limit and to be positive [43, 44]. No entirely expression has yet been identified. For more details of the topic of quasi-local approach a review article is given [45].

To calculate the energy and momentum, the definition of energy-momentum, i.e., $P^a$, is given which is invariant under global $SO(3, 1)$ transformations. It has been argued elsewhere [46] that $P^a$ makes sense to have a dependence on the frame. The energy-momentum in classical theories of particles and fields does not depend on the frame, and it has been asserted that such dependence is a natural property of the gravitational energy-momentum. It is assumed that a set of tetrads fields is adapted to an observer in the space-time determined by the metric tensor $g_{\mu\nu}$.

We investigate the irreducible mass $M_{irr}$ of the dilaton black hole. This $M_{irr}$ is the total mass of the black hole at the final stage of Penrose’s process of energy extraction, considering that the maximum possible energy is extracted. The $M_{irr}$ is also related to the energy contained within the external event horizon $E(r_+)$ of the black hole (the surface of the constant radius $r = r_+$ defines the external event horizon). Every expression for local or quasi-local gravitational energy must necessary yield the value of $E(r_+)$ in close agreement with $2M_{irr}$, since we know beforehand the value of $M_{irr}$ as a function of the initial angular-momentum of the black hole [47]. The evolution of $2M_{irr}$ is a crucial test for any expression of the gravitational energy. $E(r_+)$ has been obtained by means of different energy expressions [48]. The gravitational energy used in this article is the only one that yields a satisfactory value for $E(r_+)$ and that arises in the framework of the Hamiltonian formulation of the gravitational field.

The main aim of the present work is to reformulate the solution given by Garfinkle et al. [3] within the framework of TEGR and then, compute energy, momentum and angular momentum using the energy-momentum tensor. In §2 we briefly review the TEGR theory for gravitational, electromagnetic and dilaton and then we derive the equations of motion. A summary of the derivation of energy and angular-momentum is given in §3. In §4, we study the two tetrad fields and then calculate the energy and angular-momentum. To calculate the energy associated with the second tetrad field we use the regularized expression for the gravitational energy-momentum in §5. The final section is devoted to discussion and conclusion.
2. The TEGR for gravitation, electromagnetic and dilaton

In a space-time with absolute parallelism the parallel vector fields $e^a_\mu$ define the non-symmetric affine connection
\[ \Gamma^\lambda_{\mu\nu} \overset{\text{def.}}{=} e^a_\lambda e^a_{\mu\nu}, \tag{1} \]
where $e_{\mu\nu} = \partial_\nu e_{\mu\ast}$. The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$, given by Eq. (1), is identically vanishing. The metric tensor $g_{\mu\nu}$ is defined by
\[ g_{\mu\nu} \overset{\text{def.}}{=} \eta_{ab} e^a_\mu e^b_\nu, \tag{2} \]
with $\eta_{ab} = (-1, +1, +1, +1)$ is the metric of Minkowski space-time.

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by
\[ L_G = e L_G = - \frac{e}{16\pi} \left( \frac{T^{abc} T_{abc}}{4} + \frac{T^{abc} T_{bac}}{2} - T^a T_a \right) - L_m = - \frac{e}{16\pi} \Sigma^{abc} T_{abc} - L_m, \tag{3} \]
where $e = \text{det}(e^a_\mu)$. The tensor $\Sigma^{abc}$ is defined by
\[ \Sigma^{abc} \overset{\text{def.}}{=} \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^{b} - \eta^{ab} T^{c} \right). \tag{4} \]

$T^{abc}$ and $T^a$ are the torsion tensor and the basic vector field defined by
\[ T^{a}_{\mu\nu} \overset{\text{def.}}{=} e^a_\lambda T^\lambda_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu, \quad T_{bc}^{a} \overset{\text{def.}}{=} e^c_b e^a_\nu T_{a\mu\nu}, \quad T^{b}_{a} \overset{\text{def.}}{=} T^{a}_{b\cdot}. \tag{5} \]
The quadratic combination $\Sigma^{abc} T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term [30]. $L_m$ represents the Lagrangian density for matter fields.

The electromagnetic Lagrangian density $L_{e.m.}$ is [50]
\[ L_{e.m.} = e L_{e.m.} = e \, e^{-2\xi} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{6} \]
with $F_{\mu\nu}$ being the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$ and is defined by $F_{\mu\nu} \overset{\text{def.}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$.

Finally the dilaton Lagrangian density $L_D$ is [3]
\[ L_D = e \, L_D = 2e \, (\nabla \xi)^2, \tag{7} \]

---

*space-time indices $\mu$, $\nu$, $\cdots$ and SO(3,1) indices $a$, $b \cdots$ run from 0 to 3. Time and space indices are indicated to $\mu = 0$, $i$, and $a = (0), (i)$.

1Throughout this paper we use the relativistic units $c = G = 1$ and $\kappa = 8\pi$.

2Heaviside-Lorentz rationalized units will be used throughout this paper.
with $\xi$ being the dilaton.

The gravitational, electromagnetic and dilaton field equations for the system described by $L_G + L_{e.m.} + L_D$ are the following:

$$\begin{align*}
e_{\alpha\lambda}e_{\beta\mu}\partial_\nu\left(e^\Sigma^{\beta\lambda\nu}\right) - e\left(\Sigma^\lambda_{\quad \alpha}T_{\beta\nu\mu} - \frac{1}{4}e_{\alpha\mu}T_{\beta\epsilon\delta}\Sigma^{\beta\epsilon\delta}\right) &= \frac{1}{2}\kappa eT_{\alpha\mu}, \\
\nabla_\mu\left(e^{-2\xi}F^{\mu\nu}\right) &= 0, \\
\nabla^2\xi + \frac{1}{2}e^{-2\xi}F^2 &= 0,
\end{align*}$$

where

$$T_{\mu\nu} = 2\left\{\nabla_\mu\xi\nabla_\nu\xi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\xi\nabla_\sigma\xi + e^{-2\xi}\left(g_{\rho\sigma}F_{\mu\rho}F^{\sigma\rho} - \frac{1}{4}g_{\mu\nu}F^2\right)\right\}.$$ 

It is possible to prove by explicit calculations that the left hand side of the symmetric field equation (8) is exactly given by [30]

$$\frac{e}{2}\left[R_{\alpha\mu}(e) - \frac{1}{2}e_{\alpha\mu}R(e)\right].$$

The axial-vector part of the torsion tensor $a_\mu$ is defined by

$$a_\mu \overset{\text{def}}{=} \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} = \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}, \quad \text{where} \quad \epsilon_{\mu\nu\rho\sigma} \overset{\text{def}}{=} \sqrt{-g}\delta_{\mu\nu\rho\sigma},$$

and $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$.

### 3. Energy, momentum and angular-momentum

In the context of Einstein’s GR, rotational phenomena is certainly not a completely understood issue. The prominent manifestation of a purely relativistic rotation effect is the dragging of inertial frames. If the angular-momentum of the gravitational field of isolated system has a meaningful notion, then it is reasonable to expect the latter to be somehow related to the rotational motion of the physical sources.

The angular-momentum of the gravitational field has been addressed in the literature by means of different approaches. The oldest approach is based on pseudotensors [16, 17], out of which angular-momentum superpotentials are constructed. An alternative approach assumes the existence of certain Killing vector fields that allow the construction of conserved integral quantities [51]. Finally, the gravitational angular-momentum can also be considered in the context of Poincaré gauge theories of gravity [52], either in the Lagrangian or in the Hamiltonian formulation. In the latter case it is required that the generators of spatial rotations at infinity have a well defined functional derivatives. From this requirement a certain surface integral arises, whose value is interpreted as the gravitational angular-momentum.

The Hamiltonian formulation of TEGR is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component
Therefore, this quantity will arise as a Lagrange multiplier [53]. The momentum canonically conjugated to \( e_{ai} \) is given by \( \Pi^{ai} = \frac{\delta L}{\delta \dot{e}_{ai}} \). The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form \( L = p \dot{q} - H \), in terms of \( e_{ai}, \Pi^{ai} \) and the Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form [28]

\[
H = e_{ai} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k,
\]  

(10)

plus a surface term. Here \( \alpha_{ik} \) and \( \beta_k \) are Lagrange multipliers that are identified as

\[
\alpha_{ik} = \frac{1}{2} (T_{i0k} + T_{k0i}) \quad \text{and} \quad \beta_k = T_{00k},
\]  

(11)

and \( C^a, \Gamma^{ik} \) and \( \Gamma^k \) are first class constraints. The Poisson brackets between any two field quantities \( F \) and \( G \) is given by

\[
\{F, G\} = \int d^3 x \left( \frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta \Pi^{ai}(x)} - \frac{\delta F}{\delta \Pi^{ai}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right).
\]  

(12)

We recall that the Poisson brackets \( \{\Gamma^{ij}(x), \Gamma^{kl}(x)\} \) reproduce the angular-momentum algebra [30].

The constraint \( C^a \) is written as \( C^a = -\partial_i \Pi^{ai} + h^a \), where \( h^a \) is an intricate expression of the field variables. The integral form of the constraint equation \( C^a = 0 \) motivates the definition of the gravitational energy-momentum \( P^a \) four-vector [30]

\[
P^a = -\int_V d^3 x \partial_i \Pi^{ai},
\]  

(13)

where \( V \) is an arbitrary volume of the three-dimensional space. In the configuration space we have

\[
\Pi^{ai} = -4\kappa \sqrt{-g} \Sigma^{a0i} \quad \text{with} \quad \partial_\nu (\sqrt{-g} \Sigma^{a\lambda\nu}) = \frac{1}{4\kappa} \sqrt{-g} e^a_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \quad \text{where}
\]

\[
t^{\lambda\mu} = \kappa \left( 4\Sigma^{bc\lambda} T_{bc}^{\mu} - g^{\lambda\mu} \Sigma^{bcd} T_{bcd} \right).
\]  

(14)

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. By making \( \lambda = 0 \) in Eq. (14) and identifying \( \Pi^{ai} \) in the left side of the latter, the integral form of Eq. (13) is written as

\[
P^a = \int_V d^3 x \sqrt{-g} e^a_\mu \left( t^{0\mu} + T^{0\mu} \right).
\]  

(15)

Eq. (15) suggests that \( P^a \) is now understood as the gravitational energy-momentum [30]. The spatial component \( P^{(i)} \) form a total three-momentum, while temporal component \( P^{(0)} \) is the total energy [17].
It is possible to rewrite the Hamiltonian density of Eq. (10) in the equivalent form [39]

\[ H = e_{ab} C^a + \frac{1}{2} \lambda_{ab} \Gamma^{ab}, \quad \text{with} \quad \lambda_{ab} = -\lambda_{ba}, \]  

(16)

are the Lagrangian multipliers that are identified as \( \lambda_{ik} = \alpha_{ik} \) and \( \lambda_{0k} = -\lambda_{k0} = \beta_k \). The constraints \( \Gamma^{ab} = -\Gamma^{ba} \) [28] embodies both constraints \( \Gamma^{ik} \) and \( \Gamma^k \) by means of the relation

\[ \Gamma^{ik} = e_a i e_b k \Gamma^{ab}, \quad \text{and} \quad \Gamma^k \equiv \Gamma^{0k} = e_a \epsilon_b k \Gamma^{ab}. \]  

(17)

The constraint \( \Gamma^{ab} \) can be reads as

\[ \Gamma^{ab} = M^{ab} + 4\kappa \sqrt{-g} e_{(0)} c \left( \Sigma^{ab} - \Sigma^{bc} \right). \]  

(18)

In similarity to the definition of \( P^a \), the integral form of the constraint equation \( \Gamma^{ab} = 0 \) motivates the new definition of the space-time angular-momentum. The equation \( \Gamma^{ab} = 0 \) implies

\[ M^{ab} = -4\kappa \sqrt{-g} e_{(0)} c \left( \Sigma^{abc} - \Sigma^{bca} \right), \]  

(19)

Maluf et al. [30, 39] defined

\[ L^{ab} = \int_V d^3 x e_{\mu} a e_{\nu} b M^{\mu \nu}, \]  

(20)

as the 4-angular-momentum of the gravitational field for an arbitrary volume \( V \) of the three-dimensional space. In Einstein-Cartan type theories there also appear constraints that satisfy the Poisson bracket as given by Eq. (12). However, such constraints arise in the form \( \Pi^{[ij]} = 0 \), and so a definition similar to Eq. (20), i.e., interpreting the constraint equation as an equation for the angular-momentum of the field, is not possible. Definition (20) is three-dimensional integral. The quantities \( P^a \) and \( L^{ab} \) are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory. Moreover, these quantities transform covariantly under global \( SO(3, 1) \) transformations [39].

4. Tetrad fields with spherical symmetry

Now we will consider two simple configuration of tetrad fields and discuss their physical interpretation as reference frames. The first one in quasi-orthogonal coordinate system can be written as [54]

\[ e_{(0)} = A, \quad e_{\alpha} a = C x^a, \quad e_{(0)} = D x^a, \]
\[ e_{a} a = \delta_{a} B + F x^a x^a + \epsilon a_{\alpha} S x^\beta, \]  

(21)

where \( A, C, D, B, F, \) and \( S \) are unknown functions of \( r \). It can be shown that the unknown functions \( D \) and \( F \) can be eliminated by coordinate transformations [55, 56], i.e., by making
use of freedom to redefine $t$ and $r$, leaving the tetrad field (21) having four unknown functions in the quasi-orthogonal coordinates. Thus the tetrad field (21) without the unknown functions $D$ and $F$ and also without the two unknown functions $C$ and $S$ will be used in the following discussion for the calculations of energy, momentum and angular-momentum but in the spherical coordinate. Therefore, the tetrad field (21) can be written in the spherical coordinates without the unknown functions $D$, $F$, $C$ and $S$ as

$$\left(e_{1a}^\mu\right) = \begin{pmatrix}
\frac{1}{A} & 0 & 0 & 0 \\
0 & B \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R(r)} & -\frac{\sin \phi}{R(r) \sin \theta} \\
0 & B \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R(r)} & \frac{\cos \phi}{R(r) \sin \theta} \\
0 & B \cos \theta & -\frac{\sin \theta}{R(r)} & 0
\end{pmatrix}. \quad (22)$$

The other configuration of tetrad field that has a simple interpretation as a reference frame can has the form

$$\left(e_{2a}^\mu\right) = \begin{pmatrix}
\frac{1}{A} & 0 & 0 & 0 \\
0 & B & 0 & 0 \\
0 & 0 & \frac{1}{R(r)} & 0 \\
0 & 0 & 0 & \frac{1}{R(r) \sin \theta}
\end{pmatrix}. \quad (23)$$

The two tetrads (22) and (23) are related by a local Lorentz transformation which keeps spherical symmetry, i.e., the tetrad (22) can be written in terms of the tetrad (23) using the following local Lorentz transformation

$$\left(e_{1a}^\mu\right) = \Lambda^\mu_\nu \left(e_{2a}^\nu\right), \text{ where } \Lambda^\mu_\nu = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
0 & \cos \theta & -\sin \theta & 0
\end{pmatrix}. \quad (24)$$

The space-time associated with the two tetrad fields (22) and (23) is the same and has the form

$$ds^2 = -A^2 dt^2 + \frac{1}{B^2} dr^2 + R(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (25)$$

Now we are going to calculate the energy, momentum and angular-momentum associated with the two tetrad fields (22) and (23). For asymptotically flat space-times $P^0$ yields the ADM energy [57]. In the context of tetrad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition

$$e_{\alpha \mu} \cong \eta_{\alpha \mu} + \frac{1}{2} h_{\alpha \mu} (1/r), \quad (26)$$

and by the condition $\partial_\mu e^a_\mu = O(1/r^2)$ in the asymptotic limit $r \to \infty$. An important property of tetrad fields that satisfy Eq. (26) is that in the flat space-time limit one has $e^a_\mu(t, x, y, z) = \delta^a_\mu$, and therefore the torsion tensor $T^a_\mu_\nu = 0$. 


Now we are going to apply Eq. (13) to the tetrad field (22) to calculate the energy content. We perform the calculations in the spherical coordinate. Eqs. (22) and (23) assumed that the reference space is determined by a set of tetrad fields $e^{a}_\mu$ for the flat space-time such that the condition $T^{a}_{\mu\nu} = 0$ is satisfied. Using Eq. (5) in Eq. (22), the non-vanishing components of the torsion tensor are given by

$$T^{(0)}_{01} = \frac{A'}{A}, \quad T^{(2)}_{12} = \frac{(1 - R'(r)B)}{R(r)B} = T^{(3)}_{13},$$  \hspace{1cm} (27)$$

and the non-vanishing component of the tensor $T^{(a)}$ defined by Eq. (5) is given by

$$T^{(1)} = \frac{B(r) \{2A(r) - 2R'(r)A(r)B(r) - A'(r)B(r)R(r)\}}{R(r)A}. \hspace{1cm} (28)$$

The axial vector associated with Eq. (22) is vanishing identically due to the fact that the tetrad field of Eq. (22) has a spherical symmetry [54].

Now we are going to apply Eq. (13) to the tetrad field (22) using Eqs. (27) and (28) to calculate the energy content. We perform the calculations in the spherical coordinate. The only required component of $\Sigma^{\mu\nu\lambda}$ is

$$\Sigma^{(0)01} = -\frac{R(r) \sin \theta \{1 - R'(r)B\}}{4\pi}. \hspace{1cm} (29)$$

Using Eq. (29) in (13) we obtain

$$P^{(0)} = E = -\int_{S \rightarrow \infty} dS_{k} \Pi^{(0)k} = -\frac{1}{4\pi} \int_{S \rightarrow \infty} dS_{k} e^{\Sigma^{(0)0k}} = R(r) \{1 - R'(r)B\}. \hspace{1cm} (30)$$

Now let us apply expression (13) to the evaluation of the irreducible mass by fixing $V$ to be the volume within the $r = r_{+}$ surface where $r_{+}$ is the external horizon, i.e., $B = 0$. Therefore,

$$P^{(0)} = E = -\int_{S_{i}} dS_{i} \Pi^{(0)i} = -\int_{S} d\theta d\phi \Pi^{(0)1}(r, \theta, \phi), \hspace{1cm} (31)$$

where the surface $S$ is determined by the condition $r = r_{+}$. The expression of $\Pi^{(0)1}$ will be obtained by considering Eq. (14) using Eq. (4) and Eq. (5). The expression of $\Pi^{(0)1}(r, \theta, \phi)$ for the tetrad (22) reads

$$\Pi^{(0)1}(r, \theta, \phi) = \frac{\sin \theta R(r_{+}) \{1 - R'(r_{+})B(r_{+})\}}{4\pi}, \hspace{1cm} (32)$$

integrate Eq. (32) on the surface of constant radius $r = r_{+}$ where $r_{+}$ is the external horizon of the black hole. On this surface the second term of Eq. (32) vanishes, i.e., $B(r_{+}) = 0$. Therefore, on the surface $r = r_{+}$ we get

$$P^{(0)} = E = R(r_{+}). \hspace{1cm} (33)$$
Eq. (33) consistent with the results obtained before when \( R(r_+) = r_+ \) [30, 45], otherwise we obtained a different result. It is shown [3] that the unknown functions in the metric given by Eq. (25) may have the value

\[
A = \frac{1}{B} = \sqrt{1 - \frac{2M}{r}}, \quad R(r) = r\sqrt{1 - \frac{Q^2 e^{-2\xi_0}}{rM}}. \tag{34}
\]

For the value of the unknown functions given by Eq. (34) to satisfy the field equation (8) the dilaton, the vector potential, the Maxwell field and the energy momentum tensor must have the form

\[
\xi = \xi_0 - \frac{1}{2} \ln \left( 1 - \frac{Q^2 e^{-2\xi_0}}{rM} \right), \quad A_3 = Q \cos \theta, \quad F_{23} = Q \sin \theta,
\]

\[
T^0_0 = \frac{Q^2 (4rM^2 e^{2\xi_0} - 6M^2 Q^2 + rQ^2)}{4r^3 (rMe^{2\xi_0} - Q^2)^2}, \quad T^1_1 = \frac{Q^2 (4rM^2 e^{2\xi_0} - 2MQ^2 - rQ^2)}{4r^3 (rMe^{2\xi_0} - Q^2)^2}
\]

\[
T^2_2 = T^3_3 = -\frac{Q^2 (4rM^3 e^{4\xi_0} - 6rM^2 Q^2 e^{2\xi_0} + 2MQ^4 - r^2 MQ^2 e^{2\xi_0} + rQ^4)}{4r^3 (rMe^{2\xi_0} - Q^2)^3}, \tag{35}
\]

where \( M, Q, \xi, \xi_0 \) are the mass, charge, dilaton and the asymptotic value of the dilaton respectively. As is clear from Eq. (33) that if the charge \( Q = 0 \) then \( R(r) = r \) and the irreducible mass will coincides with that obtained before [30, 45]. Equation (33) tell us that the energy associated with the solution given by Eq. (34) on the surface \( R(r_+) \) is different from what is well known [30, 45]. To over come such problem let us use a coordinate transformation which makes \( R(r) \) that appear in Eq. (34) to be \( r \).

Now we are going to redefine the radial coordinate to be

\[
r = \sqrt{R^2 + \frac{Q^4 e^{-4\xi_0}}{4M^2} + \frac{Q^2 e^{-2\xi_0}}{2M}}. \tag{36}
\]

Using the coordinate transformation (36) in Eq. (22) we get

\[
(e_{1\mu}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\sqrt{1 - \frac{4M^2 e^{2\xi}}{\lambda(R)}} & \sigma(R)\sqrt{1 - \frac{4M^2 e^{2\xi}}{\lambda(R)}} \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\
0 & \frac{\sigma(R)\sqrt{1 - \frac{4M^2 e^{2\xi}}{\lambda(R)}} \sin \theta \sin \phi}{2MR} & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
0 & 0 & \sigma(R)\sqrt{1 - \frac{4M^2 e^{2\xi}}{\lambda(R)}} \cos \theta & -\frac{\sin \theta}{R} \\
0 & 0 & 0 & \frac{\sigma(R)\sqrt{1 - \frac{4M^2 e^{2\xi}}{\lambda(R)}} \cos \phi}{2MR}
\end{pmatrix}, \tag{37}
\]

where

\[
\lambda(R) \overset{\text{def}}{=} \sqrt{4R^2 M^2 + Q^4 e^{-4\xi_0} e^{2\xi_0} + Q^2}, \quad \sigma(R) \overset{\text{def}}{=} \sqrt{4R^2 M^2 + Q^4 e^{-4\xi_0}}, \tag{38}
\]
and the associated spacetime of Eq. (37) is given by

\[ ds^2 = -\left(1 - \frac{4M^2e^{2\xi}}{\lambda(R)}\right)dt^2 + \frac{4M^2R^2}{\sigma^2(R)\left(1 - \frac{4M^2e^{2\xi}}{\lambda(R)}\right)}dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2). \] (39)

When the dilaton \( \xi \) and the charge \( Q \) are vanishing then, Eq. (37) will be identical with the tetrad field that reproduce Schwarzschild spacetime [58]. When the dilation solution is vanishing then Eq. (37) will behave asymptotically like

\[
\begin{pmatrix}
  \frac{2R^2+2MR+3M^2-Q^2}{2R^2} & 0 & 0 & 0 \\
  0 & \frac{\sin\theta \cos\phi(2R^2-2MR-M^2+Q^2)}{2R^2} & \frac{\cos\theta \cos\phi}{R} & -\frac{\sin\phi}{R}\sin\theta \\
  0 & \frac{\sin\theta \sin\phi(2R^2-2MR-M^2+Q^2)}{2R^2} & \frac{\cos\theta \sin\phi}{R} & \frac{\cos\phi}{R}\sin\theta \\
  0 & \frac{\cos\theta(2R^2-2MR-M^2+Q^2)}{2R^2} & -\frac{\sin\theta}{R} & 0
\end{pmatrix},
\] (40)

and the associated metric of Eq. (40) has the form

\[ ds^2 \simeq -\left(1 - \frac{2MR-Q^2}{R^2}\right)dt^2 + \left(1 + \frac{2MR-Q^2}{R^2}\right)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \] (41)

which is the asymptotic form of Reissner–Nordström metric [58].

Repeat the calculations of energy using the tetrad field given by Eq. (37). Using Eq. (5) in Eq. (37), the non-vanishing components of the torsion tensor are given by

\[
T^{(0)}_{01} = \frac{2\lambda'(R)M^2e^{2\xi_0}}{\lambda(R)[\lambda(R) - 4M^2e^{2\xi_0}]},
\]

\[
T^{(2)}_{12} = \frac{\sigma(R)\sqrt{\lambda(R) - 4M^2e^{2\xi_0}} - 2MR\sqrt{\lambda(R)}}{R\sigma(R)\sqrt{\lambda(R) - 4M^2e^{2\xi_0}}} = T^{(3)}_{13},
\] (42)

and the non-vanishing component of the tensor \( T^{(a)} \) is given by

\[
T^{(1)} = -\frac{\sigma(R)\left\{\sigma(R)M^2e^{2\xi_0}(4\lambda(R) - RX(R)) + 2RM\sqrt{\lambda^4(R) - 4\lambda^3(R)M^2e^{2\xi_0} - \lambda^2(R)\sigma(R)}\right\}}{2R^3M^2\lambda^2(R)},
\] (43)

Using Eqs. (42) and (43) to calculate the energy content. The only required component of \( \Sigma^{\mu\nu\lambda} \) is

\[
\Sigma^{(0)01} = -\frac{\sin\theta\left\{2MR\sqrt{\lambda(R)} - \sigma(R)\sqrt{\lambda(R) - 4M^2e^{2\xi_0}}\right\}}{8M\pi\sqrt{\lambda(R)}},
\] (44)

Substituting Eq. (44) in (13) we obtain
\[ P^{(0)} = E = -\int_{S \to \infty} dS_k \Pi^{(0)k} = -\frac{1}{4\pi} \int_{S \to \infty} dS_k \Sigma^{(0)0k} = \frac{2MR\sqrt{\lambda(R) - \sigma(R)\sqrt{\lambda(R) - 4M^2e^{2\xi_0}}}}{2M\sqrt{\lambda(R)}} \]

\[ = R - e^{-2\xi_0} \left( \sqrt{4R^2M^2e^{4\xi_0} + Q^4\sqrt{4M^2R^2e^{4\xi_0} + Q^4 + Q^2 - 4M^2e^{2\xi_0}}} \right) \]

\[ \approx M - \frac{4Q^2M^2e^{-2\xi_0} - 4M^4 + Q^4e^{-4\xi_0}}{8M^2R} + O\left(\frac{1}{R^2}\right), \quad (45) \]

where we have used the definitions of \( \lambda(R) \) and \( \sigma(R) \) given by Eq. (38). For large \( R \), i.e., \( \lim_{R \to \infty} \), Eq. (45) will give the ADM [59]. If the asymptotic dilaton \( \xi_0 \) is vanishing then the asymptotic form of the energy can be obtain from Eq. (45) to have the value

\[ E \approx M - \frac{4Q^2M^2 - 4M^4 + Q^4}{8M^2R}, \quad (46) \]

which is the energy of Reissner-Nordström space-time when \( Q^4 = 0 \) and \( M^2 = 0 \) [58].

Apply expression (13) to the evaluation of the irreducible mass by fixing \( V \) to be the volume within the \( R = R_+ \) surface where \( R_+ \) is the external horizon, i.e., \( \left(1 - \frac{4M^2e^{2\xi}}{\lambda(R_+)}\right)\sigma^2(R_+)=0 \). Therefore,

\[ P^{(0)} = E = -\int_S dS_1 \Pi^{(0)1} = -\int_S d\theta d\phi \Pi^{(0)1}(R, \theta, \phi) \quad (47) \]

where the surface \( S \) is determined by the condition \( R = R_+ \). The expression of \( \Pi^{(0)1} \) will be obtained by considering Eq. (14) using Eqs. (4) and (5). The expression of \( \Pi^{(0)1}(R, \theta, \phi) \) for the tetrad (37) reads

\[ \Pi^{(0)1}(R, \theta, \phi) = \frac{2MR_+\sqrt{\lambda(R_+)} - \sigma(R_+)}{8M\pi\sqrt{\lambda(R_+)}} \]

\[ \sqrt{\lambda(R_+) - 4M^2e^{2\xi_0}} \] \( \sigma(R_+) \), \( \sigma(R_+) \). Therefore, on the surface \( R = R_+ \) integration of Eq. (48) will give

\[ P^{(0)} = E = R_+, \quad (49) \]

which is a satisfactory result that is obtained before [30, 45].

Using Eq. (14) in Eq. (37) to calculate the momentum and angular-momentum associated with the first tetrad field given by Eq. (37). In this case we get

\[ \Pi^{(1)1}(R, \theta, \phi) = 0. \quad (50) \]

Substitute Eq. (50) in Eq. (13) we get

\[ P^{(1)} = \int_V dV \partial_1 \Pi^{(1)1}(R, \theta, \phi) = \int_S dS_1 \Pi^{(1)1}(R, \theta, \phi) = 0. \quad (51) \]
By the same method we obtain

\[ \Pi^{(2)}(R, \theta, \phi) = 0, \quad P^{(2)} = 0, \quad \Pi^{(3)}(R, \theta, \phi) = 0, \quad P^{(3)} = 0. \]  

(52)

The results of Eqs. (51) and (52) are expected results since the space-time given by Eq. (37) is a spherically symmetric static space-time. Therefore, the spatial momentum associated with any static solution is identically vanishing [59].

We use Eqs. (19) and Eq. (4) in Eq. (20) to calculate the components of the angular-momentum. Finally we get

\[ M^{(0)(1)}(R, \theta, \phi) = -R \sin^2 \theta \cos \phi \left\{ 2MR \sqrt{\lambda^2(R) - 4M^2 \lambda(R)e^{2\xi_0}} + \sigma(R)\lambda(R) - 4\sigma(R)M^2e^{2\xi_0} \right\} \frac{1}{4\pi \sigma(R)\lambda(R)}, \]

\[ M^{(0)(2)}(R, \theta, \phi) = -R \sin^2 \theta \sin \phi \left\{ 2MR \sqrt{\lambda^2(R) - 4M^2 \lambda(R)e^{2\xi_0}} + \sigma(R)\lambda(R) - 4\sigma(R)M^2e^{2\xi_0} \right\} \frac{1}{4\pi \sigma(R)\lambda(R)}, \]

\[ M^{(0)(3)}(R, \theta, \phi) = -R \sin \theta \cos \theta \left\{ 2MR \sqrt{\lambda^2(R) - 4M^2 \lambda(R)e^{2\xi_0}} + \sigma(R)\lambda(R) - 4\sigma(R)M^2e^{2\xi_0} \right\} \frac{1}{4\pi \sigma(R)\lambda(R)}, \]

\[ M^{(1)(2)}(R, \theta, \phi) = M^{(1)(3)}(R, \theta, \phi) = M^{(2)(3)}(R, \theta, \phi) = 0. \]  

(53)

Using Eq. (53) in Eq. (20) we get

\[ L^{(0)(1)} = \int_0^\pi \int_0^{2\pi} \int_0^\infty d\theta d\phi dR M^{(0)(1)}(R, \theta, \phi) = 0, \]

by the same method we can get

\[ L^{(0)(2)} = L^{(0)(3)} = L^{(1)(2)} = L^{(1)(3)} = L^{(2)(3)} = 0. \]  

(54)

It is of interest to note that the vanishing of \( L^{(0)(1)} \), \( L^{(0)(2)} \) is due to the appearance of terms like \( \sin \phi \) and \( \cos \phi \) while the vanishing of \( L^{(0)(3)} \) is due to the appearance of term like \( \sin \theta \cos \theta \).

To repeat the same computation for the tetrad (23) it is sufficient to use the rules of transformation of the conserved quantities [60]. Therefore, the required component of \( \Sigma^{\mu\nu\lambda} \) needed to calculate the energy of the tetrad field (23) has the form

\[ \Sigma^{(0)01} = -\frac{B(r)R(r)R'(r)\sin \theta}{4\pi}. \]  

(55)

Substituting (55) in (13) we obtain

\[ P^{(0)} = E = -\oint_{S\rightarrow \infty} dS_k \Pi^{(0)k}(r, \theta, \phi) = -\frac{1}{4\pi} \oint_{S\rightarrow \infty} dS_k e^{\Sigma^{(0)0k}} = -\frac{B(r)R(r)R'(r)}{2M}. \]  

(56)

When the asymptotic dilaton \( \xi_0 = 0 \) and the charge \( Q = 0 \) then the asymptotic form of the above form of energy is given by

\[ E \approx M - r, \]  

(57)
which is different from the ADM form [59]. This is due to the fact that the components of the torsion when $M = 0$, $Q = 0$ and $\xi = 0$ do not vanishing identically contradict the flatness condition given by Eq. (26). Therefore, in this case we are going to use the regularized expression for the gravitational energy-momentum [30].

5. Regularized expression for the gravitational energy-momentum and localization of energy

An important property of the tetrad fields that satisfy the condition of Eq. (26) is that in the flat space-time limit $e^a_\mu (t, x, y, z) = \delta^a_\mu$, and therefore the torsion $T^\lambda_{\mu \nu} = 0$. Hence for the flat space-time it is normally to consider a set of tetrad fields such that $T^\lambda_{\mu \nu} = 0$ in any coordinate system. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (26). Moreover for such tetrad fields the torsion $T^\lambda_{\mu \nu} \neq 0$ for the flat space-time [61]. It might be argued, therefore, that the expression for the gravitational energy-momentum (13) is restricted to particular class of tetrad fields, namely, to the class of frames such that $T^\lambda_{\mu \nu} = 0$ if $E_a^\mu$ represents the flat space-time tetrad field [61]. To explain this, let us calculate the flat space-time of the tetrad field of Eq. (23) using (34) which is given by

$$
(E_{2a}^\mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r \sin \theta}
\end{pmatrix}.
$$

Expression (58) yields the following non-vanishing torsion components:

$$
T^{(2)}_{12} = -\frac{1}{r} = T^{(3)}_{13}, \quad T^{(3)}_{23} = -\cot \theta.
$$

The tetrad field (58) when written in the Cartesian coordinate will have the form

$$
(E_{2a}^\mu(t, x, y, z)) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\
0 & -\frac{x z}{r \sqrt{x^2+y^2}} & \frac{y z}{r \sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r} \\
0 & -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0
\end{pmatrix}.
$$

In view of the geometric structure of Eq. (60), we see that, Eq. (23) does not display the asymptotic behavior required by Eq. (26). Moreover, in general the tetrad field (60) is
adapted to accelerated observers [28, 30, 61]. To explain this, let us consider a boost in the $x$-direction of Eq. (60). We find

$$\left( E_{2a}^{\mu}(t, x, y, z) \right) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ \frac{v\gamma x}{r} & \frac{\gamma x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{v\gamma y}{r\sqrt{x^2+y^2}} & \frac{\gamma y}{r\sqrt{x^2+y^2}} & \frac{y}{r\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r} \\ -\frac{v\gamma y}{r\sqrt{x^2+y^2}} & -\frac{\gamma y}{r\sqrt{x^2+y^2}} & \frac{x}{r\sqrt{x^2+y^2}} & 0 \end{pmatrix},$$

(61)

where $v$ is the speed of the observer and $\gamma = \frac{1}{\sqrt{1-v^2}}$. For a static object in a space-time whose four-velocity is given by $u^\mu = (1, 0, 0, 0)$ we may compute its frame components $u^a = e^a_\mu u^\mu = (\gamma, \frac{v\gamma x}{r\sqrt{x^2+y^2}}, \frac{v\gamma xz}{\sqrt{x^2+y^2}}, \frac{-v\gamma y}{\sqrt{x^2+y^2}})$. It can be shown that along an observer’s trajectory whose velocity is determined by $u^a$ the quantities

$$\phi^{(k)}_{(j)} = u^i \left( E_{2}^{(k)}_{(m)} \partial_k E_{2(j)}^{(m)} \right),$$

(62)

constructed out from Eq. (61) are non vanishing. This fact indicates that along the observer’s path the spatial axis $E_{2(a)}^{\mu}$ rotate [28, 61]. In spite of the above problems discussed for the tetrad field of Eq. (23) it yields a satisfactory value for the total gravitational energy-momentum, as we will discussed.

In Eq. (13) it is implicitly assumed that the reference space is determined by a set of tetrad fields $e^a_\mu$ for flat space-time such that the condition $T^a_{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time tetrad fields for which $T^a_{\mu\nu} \neq 0$. In this case Eq. (13) may be generalized [28, 61] by adding a suitable reference space subtraction term, exactly like in the Brown-York formalism [62, 63].

We will denote $T^a_{\mu\nu}(E) = \partial_\mu E^a_{\nu} - \partial_\nu E^a_{\mu}$ and $\Pi^{\alpha\beta}(E)$ as the expression of $\Pi^{\alpha\beta}$ constructed out of the flat tetrad $E^a_\mu$. The regularized form of the gravitational energy-momentum $P^a$ is defined by [28, 61]

$$P^a = -\int_V d^3x \partial_k \left[ \Pi^{ak}(e) - \Pi^{ak}(E) \right].$$

(63)

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by tetrad fields $E^a_\mu$, obtained from $e^a_\mu$ by requiring the vanishing of the physical parameters like mass, angular-momentum, etc. Assuming that the space-time is asymptotically flat then Eq. (63) can have the form [28, 61]

$$P^a = -\oint_{S_{\rightarrow \infty}} dS_k \left[ \Pi^{ak}(e) - \Pi^{ak}(E) \right],$$

(64)

where the surface $S$ is established at spacelike infinity. Eq. (64) transforms as a vector under the global SO(3,1) group [30].

We may likewise establish the regularized expression for the gravitational 4-angular momentum. It reads

$$L_{ab} = \int d^3x \left[ M_{ab}^{\mu\nu}(e) - M_{ab}^{\mu\nu}(E) \right].$$

(65)
Now we are in a position to prove that the tetrad field (23) yields a satisfactory value for the total gravitational energy-momentum. We will integrate Eq. (64) over a surface of constant radius \( x^1 = r \) and require \( r \to \infty \). Therefore, the index \( k \) in (64) takes the value \( k = 1 \). We need to calculate the quantity

\[
\Sigma^{(0)01} = e^{(0)}_0 \Sigma^{001} = \frac{1}{2} e^{(0)}_0 (T^{001} - g^{00} T^1).
\]

Evaluate the above equation we find

\[
\Pi^{(0)1}(e) = -\frac{1}{4\pi} e \Sigma^{(0)01} = -\frac{\sin \theta (2rM - Q^2 e^{-2\xi_0}) \sqrt{1 - \frac{2M}{r}}}{8\pi M},
\]

and the expression of \( \Pi^{(0)1}(E) \) is obtained by just making \( M = 0, Q = 0 \) and \( \xi_0 = 0 \) in Eq. (66), it is given by

\[
\Pi^{(0)1}(E) = -\frac{1}{4\pi} r \sin \theta.
\]

Thus the gravitational energy of the tetrad field of Eq. (23) is given by

\[
P^{(0)} = \int d\theta d\phi \frac{1}{4\pi} \sin \theta \left( r - \frac{(2rM - Q^2 e^{-2\xi_0}) \sqrt{1 - \frac{2M}{r}}}{2M} \right),
\]

\[
r - \frac{(2rM - Q^2 e^{-2\xi_0}) \sqrt{1 - \frac{2M}{r}}}{2M} \approx M + \frac{Q^2 e^{-2\xi_0}}{2M} + O \left( \frac{1}{r} \right).
\]

which is exactly the ADM when \( Q^2 = 0 \) up to \( O(1/r) \). Eq. (68) tells us that when \( (2rM - Q^2 e^{-2\xi_0}) = 0 \) the form of the energy given by Eq. (68) will effect and this is one of the defect of the solution given by Eq. (34) [3]. Therefore, we use the coordinate transformation given by Eq. (36). The tetrad (23) after using transformation (36) will have the form

\[
(e_{2a}^\mu) = \begin{pmatrix}
\frac{1}{\sqrt{1 - \frac{4M^2 e^{2\xi_0}}{\lambda(R)}}} & 0 & 0 & 0 \\
0 & \sigma(R) \sqrt{1 - \frac{4M^2 e^{2\xi_0}}{\lambda(R)}} & 0 & 0 \\
0 & 0 & \frac{1}{R} & 0 \\
0 & 0 & 0 & \frac{1}{R \sin \theta}
\end{pmatrix},
\]

Repeat the calculations done above the non-vanishing components of the torsion tensor and the vector field \( T^{(a)} \) of the tetrad field given by Eq. (69) have the form

\[
T^{(0)}_{01} = -\frac{2M^2 \lambda'(R)e^{2\xi_0}}{\lambda(R)[\lambda(R) - 4M^2 e^{2\xi_0}]}, \quad T^{(2)}_{12} = T^{(3)}_{13} = -\frac{1}{R}, \quad T^{(3)}_{23} = -\cot \theta,
\]

\[
T^{(1)} = -\frac{\sigma(R) \{\lambda'(R) - 4\lambda(R)M^2 e^{2\xi_0} - RM^2 \lambda'(R)e^{2\xi_0}\}}{R^3 M^2 \lambda^2(R)}, \quad T^{(2)} = -\frac{\cot \theta}{R^2}.
\]
The only required component of $\Sigma^{\mu\nu\lambda}$ needed to calculate the energy using the regularized expression given by Eq. (64) is

$$\Sigma^{(0)01}(e) = \frac{\sigma(R)\sqrt{R(R) - 4M^2 e^{2\xi_0}}}{2M\sqrt{\lambda(R)}} \sin \theta, \quad \Sigma^{(0)01}(E) = \frac{R \sin \theta}{4\pi}. \quad (71)$$

Substituting (71) in (64) we obtain

$$P^{(0)} = E = -\oint_{S \to \infty} dS_k \Pi^{(0)k}(R, \theta, \phi) = -\frac{1}{4\pi} \oint_{S \to \infty} dS_k e \Sigma^{(0)0k},$$

$$= R - \frac{\sigma(R)\sqrt{\lambda(R) - 4M^2 e^{2\xi_0}}}{2M\sqrt{\lambda(R)}}$$

$$= R - \frac{\sqrt{4M^2 R^2 + Q^4 e^{-4\xi_0}}}{2M\sqrt{4M^2 R^2 + Q^4 e^{-4\xi_0} e^{2\xi_0} + Q^2 - 4M^2 e^{2\xi_0}}}$$

$$\approx M + O \left( \frac{1}{R} \right), \text{ which is the ADM up to } O \left( \frac{1}{R} \right),$$

$$\approx M - \frac{4Q^2 M^2 e^{-2\xi_0} + 4M^4 - Q^4 e^{-4\xi_0}}{8M^2 R} + O \left( \frac{1}{R^2} \right), \text{ which is the energy of Reissner - Nordström space - time when the asymptotic dilaton } \xi_0 = 0, \quad Q^4 = 0 \text{ and } M^2 = 0 \text{ up to } O \left( \frac{1}{R^2} \right) \quad [58]. \quad (72)$$

By the same method used for the first tetrad given by Eq. (37) we find that the momentum and angular momentum associated with the second tetrad field given by Eq. (69) are

$$\Pi^{(1)1}(R, \theta, \phi) = 0, \quad P^{(1)} = \oint_V dV \partial_1 (\Pi^{(1)1}(R, \theta, \phi)) = \oint_S dS_1 \Pi^{(1)1}(R, \theta, \phi) = 0,$$

$$\Pi^{(2)1}(R, \theta, \phi) = 0, \quad P^{(2)} = 0, \quad \Pi^{(3)1}(R, \theta, \phi) = 0, \quad P^{(3)} = 0. \quad (73)$$

The non vanishing components of the angular-momentum are given by

$$M^{(0)(1)}(e) = \frac{R \sin \theta (\lambda(R) - 4M^2 e^{2\xi_0})}{4\pi \lambda(R)} \approx \sin \theta (R - M) \frac{1}{4\pi} + O \left( \frac{1}{R} \right),$$

$$M^{(0)(1)}(E) \approx \frac{R \sin \theta}{4\pi} + O \left( \frac{1}{R} \right),$$

$$M^{(0)(2)}(R, \theta, \phi) = \frac{M R^2 \cos \theta \sqrt{R(R) - 4M^2 e^{2\xi_0}}}{4\pi \sigma(R) \sqrt{\lambda(R)}},$$

$$M^{(0)(3)}(R, \theta, \phi) = M^{(1)(2)}(R, \theta, \phi) = M^{(1)(3)}(R, \theta, \phi) = M^{(2)(3)}(R, \theta, \phi) = 0. \quad (74)$$

Using Eq. (74) in (65) we get

$$L^{(0)(1)} = \int_0^\pi \int_0^{2\pi} \int_0^\infty d\theta d\phi dR \left[ M^{(0)(1)}(e) - M^{(0)(1)}(E) \right] = M \int_0^\infty dR,$$

which give an infinite result! By the same method we can obtain

$$L^{(0)(2)} = L^{(0)(3)} = L^{(1)(2)} = L^{(1)(3)} = L^{(2)(3)} = 0. \quad (76)$$
It is of interest to note that the non-vanishing of $L^{(0)}(1)$ is due to the appearance of terms like $\sin \theta$ while, the vanishing of $L^{(0)}(2)$ is due to the appearance of terms like $\cos \theta$.

We show by explicit calculation that the energy-momentum tensor which is a coordinate independent does not give a consistent result of the angular momentum when applied to the tetrad field given by Eq. (23) which does not satisfy the boundary condition given by Eq. (26).

6. Main results and Discussion

The main results of this paper are the following:

- Tow different tetrad fields are used. The space-time associated with these tetrad fields is given by Eq. (25).

- The energy of these tetrad fields are calculated using the gravitational energy-momentum tensor, which is a coordinate independent [30]. One of this tetrad field given by Eq. (22) gives a satisfactory results for the energy after using the coordinate transformation given by Eq. (36). The other tetrad field that is given by Eq. (23) its associated energy depends on the radial coordinate.

- Calculations of the torsion components associated with the two tetrad fields are given. From these calculations we show that the torsion components of each tetrad field are different. This may gave an indication why the energy of the two tetrad fields is different.

- We use the regularized expression of the gravitational energy-momentum tensor to calculate the energy associated with the second tetrad field given by Eq. (23).

- We have shown that the energy associated with the second tetrad field did not give the consistent result even after using the regularized expression of the gravitational energy-momentum tensor. Therefore, we use the coordinate transformation given by Eq. (36). Applying this coordinate transformation to the tetrad field (23) we have got a satisfactory value of energy coincides with the value of energy of the first tetrad field.

- Using the definition of the energy and the angular momentum given by Eqs. (13) and (20) we show by explicit calculations that the angular momentum depends on the choice of the frame used.

- The calculation of the irreducible mass is given within the external horizons using the Hamiltonian formulation. From this calculation we show that the external horizons of each model does not play any role on the energy.

- We have shown by explicit calculations that the diagonal tetrad field which given by Eq. (23) suffers from some problems:
  i) It does not satisfy the condition given by Eq. (26) which guarantees the flatness of space-time, consequently the components of the torsion tensor did not vanish when the physical quantities are set equal zero.
  ii) The use of the energy-momentum tensor given by Eq. (13) did not give the consistent results! Therefore, we have used the regularized expression of the energy momentum tensor
and got the consistent result for the energy. Also we have shown that both expressions given by Eqs. (20) and (65) gave an infinite results when we have calculated the angular momentum [59]!

- The construction of the tetrad given by Eq. (23) is the square root of the metric given by Eq. (25) meanwhile, the construction of the tetrad given by Eq. (22) is not the square root of Eq. (25). Possible interpretations of the results given by the second tetrad (which is the square root of the metric) is that it may not be a physical one. Same problem has been appeared [61] for the Kerr solution. We need more studies to confirm this conclusion.

**Acknowledgment**

The author would like to thank the Referee for careful reading, careful checking the mathematics, putting the paper in a more readable form and the comments given for the second tetrad.
References

[1] T. Kaluza, *Sitzungsber. Preuss. Akad. Wiss Berlin Phys. Math.* 33 (1921), P. 966; O. Klein, *Z. Phys.* 37 (1926), 895.

[2] G. Gibbons and K. Maeda, *Nucl. Phys.* B298, (1988), 741.

[3] D. Garfinkle, G.T. Horowitz and A. Strominger, *Phys. Rev.* D43 (1991), 3140.

[4] R. Kallosh and A. Peet, *Phys. Rev.* D46 (1992), 5223; R. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, *Phys. Rev.* D46 (1992), 5278.

[5] R. Gregory and J.A. Harvey, *Phys. Rev.* D47 (1993), 2411.

[6] A.G. Agnese and M. La Camera, *Phys. Rev.* D49 (1994), 2126.

[7] T.W.B. Kibble, *J. Math. Phys.* 2 (1961) 212.

[8] F.W. Hehl, P. Von der Heyde, D. Kerlick and J. Nester, *Rev. Mod. Phys.* 48 (1976), 393.

[9] F.W. Hehl, in: *General Relativity and Gravitation - One Hundred Years after the birth of Albert Einstein*, ed. A. Held (Plenum, New York, 1980) Vol 1.

[10] K. Hayashi, and T. Shirafuji, *Prog. Theor. Phys.* 64 (1980), 866, 883, 1435, 2222; 65, 525.

[11] M. Blagojević and I.A. Nikolić *Phys. Rev.* D62 (2000), 024021.

[12] F.W. Hehl, J. Nitsch and P. von der Heyde, in *General Relativity and Gravitation*, A. Held, ed. (Plenum Press, New York) (1980).

[13] F.W. Hehl, J.D. MacCrea, E.W. Mielke and Y. Ne’eman, *Phys. Rep.* 258 (1995), 1.

[14] K. Hayashi and T. Shirafuji, *Phys. Rev.* D19 (1979), 3524.

[15] J. Nitsch, in: *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*, eds. Bergman and V. de Sabbata (Plenum, New York, 1980).

[16] P.G. Bergmann and R. Thomson, *Phys. Rev.* 89 (1953), 401.

[17] L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford, 1980).

[18] C. Pellegrini and J. Plebanski, *Mat. Fys. Scr. Dan. Vid. Selsk.* 2 (1963), no.3.

[19] C. Møller, (1961) *Ann. of Phys.* 12 (1961), 118.

[20] C. Møller, “Tetrad fields and conservation laws in general relativity” in Proc. International School of Physics “Enrico Fermi” ed. C. Møller, (Academic Press, London, 1962).

[21] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* 1 (1961), 10.

[22] C. Møller, *Nucl. Phys.* 57 (1964), 330.
[23] K. Hayashi and T. Nakano, *Prog. Theor. Phys.* **38** (1967), 491.

[24] W. Kopzyński, *J. Phys.* **A15** (1982), 493.

[25] C.C. Chang, J.M. Nester and C.M. Chen, *Phys. Rev. Lett.* **83** (1999), 1897; R.S. Tung and J.M. Nester, *Phys. Rev.* **D60** (1999) 021501; J.M. Nester and H.J. Yo Chin, *J. Phys.* **37** (1999) 113; J.M. Nester, F. H. Ho and C. M.Chen, *Quasiloccal Center-of-Mass for Teleparallel Gravity, Proceeding of the 10th. Marcel Grossman Meeting (Rio de Janeiro, 2003)* gr-qc/0403101; J. M. Nester, *Phys. Lett.* **A139** (1989) 112; *J. Math. Phys.* **33** (1992), 910; *Class. Quantum Grav.* **5** (1988), 1003.

[26] N. Toma, *Prog. Theor. Phys.* **86** (1991), 659; T. Kawai and N. Toma, *Prog. Theor. Phys.* **87** (1992), 583 .

[27] V.C. de Andrade and J.G. Pereira, *Phys. Rev.* **D56** (1997), 4689; V.C. de Andrade, L.C.T Guillen and J.G. Pereira, *Phys. Rev. Lett.* **84** (2000), 4533; *Phys. Rev.* **D64** (2001), 027502.

[28] J.W. Maluf and J.F. da Rocha-Neto, *Phys. Rev.* **D64** (2001), 084014.

[29] J. Schwinger, *Phys. Rev.* **130** (1963), 1253.

[30] J. W. Maluf, *J. Math. Phys.* **35** (1994), 335; J. W. Maluf and A. Kneip, *J. Math. Phys.* **38** (1997), 458; J. W. Maluf and J. F. da Rocha-Neto, *J. Math. Phys.* **40** (1999), 1490; J. W. Maluf and A. Goya, *Class. Quant. Grav.* **18** (2001), 5143; J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toribio and K. H. Castello-Branco, *Phys. Rev.* **D65** (2002), 124001; A. A. Sousa and J. W. Maluf, *Prog. Theor. Phys.* **108** (2002), 457.

[31] J.F. da Rocha-Neto and K. H. Castello-Branco, *JHEP* **0311** (2003), 002.

[32] T. Regge and C. Teitelboim, *Ann. Phys. (New York)* **88** (1974), 286.

[33] J. W. York, Jr., *Energy and Momentum of the Gravitational field, in “Essays in General Relativity”* edited by F.J. Tipler (Academic Press, New York, 1980).

[34] R. Beig and N. Ó. Murchadha, *Ann. Phys. (New York)* **174** (1987), 463.

[35] L.B. Szabados, *Class. Quant. Grav.* **20** (2003) 2627.

[36] J. W. Maluf and A.A. Sousa, *Hamiltonian formulation of teleparallel theories of gravity in the time gauge* gr-qc/0002060, (2000).

[37] A. A. Sousa and J. W. Maluf, *Prog. Theor. Phys.* **104** (2000), 531.

[38] J. W. Maluf and J. F. de Rocha-Neto, *Phys. Rev.* **D64** (2001), 084014.

[39] J.W. Maluf, S.C. Ulhoa, F.F. Faria and J.F. da Rocha-Neto, *Class. Quant. Grav.* **23** (2006), 6245.

[40] A. A. Sousa, R.B. Pereira and J.F. da Rocha-Neto, *Prog. Theor. Phys.* **114** (2005), 1179; A. A. Sousa, J.S. Moura and R.B. Pereira gr-qc/0702109.

[41] J. M. Nester, *Int. J. Mod. Phys.* **A4** (1989), 1755.

[42] M. Blagojević and M. Vasilić *Phys. Rev.* **D64** (2001), 044010.
[43] C.C. Change, J.M. Nester, *Grav. & Cosmol.* **6** (2000) 257.

[44] L.L. So, J.M. Nester and H. Chen, *The 7th. conference on Gravitation and Astrophysics;* 
gr-qc/0605150.

[45] L.B. “Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article”, Living 
Rev. Relativity **7**, (2004), 4. http://www.livingreviews.org/lrr-2004-4.

[46] J. W. Maluf, *Annalen Phys.* **14** (2005), 723; *Gravitational and Cosmology* **11** (2005), 
284.

[47] D. Christodoulou, *Phys. Rev. Lett.* **25** (1970), 1596.

[48] G. Bergqvist, *Class. Quant. Grav.* **9** (1992), 1753.

[49] C. Møller, *Mat. Fys. Medd. Dan. Vid. Selsk.* **39** (1978), 13.

[50] T. Kawai and N. Toma, *Prog. Theor. Phys.* **87** (1992), 583.

[51] A. Komar, *Phys. Rev.* **113** (1959), 934; J. Winicour, “Angular Momentum in General 
Relativity”, in General Relativity and Gravitation edited by A. Held (Plenum, New 
York, 1980); A. Ashtekar, “Angular Momentum of Isolated Systems in General Rela-
tivity”, in Cosmology and Gravitation, edited by P.G. Bergmann and V. de Sabbata 
(Plenum, New York 1980).

[52] K. Hayashi and T. Shirafuji, *Prog. Theor. Phys.* **73** (1985), 54; M. Blagojević and M. 
Vasilić *Class. Quant. Grav.* **5** (1988), 1241; T. Kawai, *Phys. Rev.* **D62** (2000), 104014, 
T. Kawai, K. Shibata and I. Tanaka, *Prog. Theor. Phys.* **104** (2000) 505.

[53] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science 
(Monographs Series No. 2) Yeshiva University, New York, 1964).

[54] H.P. Robertson, *Ann. of Math. (Princeton)* **33** (1932), 496.

[55] K. Hayashi and T. Shirafuji, *Phys. Rev.* **D19** (1979), 3524.

[56] T. Shirafuji, G.G.L. Nashed, and K. Hayashi, *Prog. Theor. Phys.* **95** (1996), 665.

[57] R Arnowitt, S. Deser and C.W. Misner, *Gravitation: An Introduction to Current Re-
search* edited by L. Witten (Wiley, N.Y. (1962)).

[58] G.G.L. Nashed and T. Shirafuji, *Int. J. Mod. Phys.* **D16** No. 1, (2007) 65.

[59] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation (Freeman, San Francisco, 
1973)*, P. 435.

[60] Y.N. Obukhov and G.F. Rubilar, *Phys. Rev.* **D 73** (2006), 124017.

[61] J. W. Maluf, M.V.O. Veiga and J. F. da Rocha-neto, *Gen. Rel. Grav.* **39** (2007), 227.

[62] P. Bae(c)kler, R. Hecht, F.W. Hehl and T. Shirafuji, *Prog. Theor. Phys.* **78** (1987), 16.

[63] J.D. Brown and J.W. York, Jr., *Quasi-local energy in general relativity*, Proceedings of the Joint Summer Research conference on Mathematical Aspects of Classical Field 
Theory, edited by M.J. Gotay, J.E. Marsden and V. Moncrief (American Mathematical 
Society, 1991); *Phys. Rev.* **D47** (1993), 1407.