Numerical solution of the Harry Dym equation using Chebyshev spectral method via Lie group method

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Abstract: This study represents the solution of the Harry Dym equation (HDE) that was calculated using the Lie symmetry group method. This method transforms the controlled partial differential equation with boundary and initial conditions to the problem that containing boundary value of ordinary differential equation. The Chebyshev spectral approximation method is used to solve this problem numerically. This approach is performing by using the highest order derivatives of Chebyshev approximations, as starting steps, followed by getting approximations to the derivatives of lower order. Our obtained numerical results were compared with other works is done. It is seen that the proposed method for the Harry Dym equation gives more accurate results comparing with mentioned works.

Introduction

The nonlinear and linear equations of fractional partial differential are applied widely in many sciences, such as signal processing, analytical chemistry, biology, electromagnetism, fluid mechanics, acoustic, and other physical processes (for instance see [1], [2]).

The Harry Dym equation which is fractional and nonlinear is defined as:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u^3(x,t) \frac{\partial^3 u(x,t)}{\partial x^3}
\]  

(1)

The fractional derivative's order is defined by \( \alpha \), which is a positive parameter and the function of the space \( x \) and the time \( t \) is presented by \( u(x,t) \).

In this study, we take \( \alpha = 1 \) in equation (1), which becomes:

\[
\frac{\partial u(x,t)}{\partial t} = u^3(x,t) \frac{\partial^3 u(x,t)}{\partial x^3}
\]  

(2)

And with respect to the initial condition

\[
u(x,0) = (a - \frac{3\sqrt{b}}{2} x)^{2/3} = p(x)
\]  

(3)

In addition to the boundary conditions

\[
u(0,t) = q_0(t),
\]

\[
u(1,t) = q_1(t),
\]
\[
\frac{\partial u(0,t)}{\partial x} = r(t) \quad (4)
\]

Where \(a\) and \(b\) are appropriate constants and \(p, q_0, q_1, r\) are produced from the exact solution of equation (2), which was reported by Mokhtari [3] as

\[
u(x,t) = \left[ a - \frac{3\sqrt{b}}{2}(x + bt) \right]^{2/3} \quad (5)
\]

Equation (2) is called classical nonlinear Harry Dym equation or the time fractional Harry Dym equation.

There are many authors who applied analytical and numerical methods to solve the classical nonlinear Harry Dym equation. Al-Khaled and Alquran used Adomian decomposition method (ADM) to acquire approximate solution of time fractional generalized Harry Dym equation [4]. The Harry Dym nonlinear equation solved by Fonseca, applying methods of Lattice-Boltzmann and a solitary wave [5]. To achieve the solution of the nonlinear Harry-Dym equation with approximate analysis, Ghiasi and Saleh employed the homotopy analysis method (HAM) [6]. A q-homotopy analysis method (q-HAM) is used to obtain analytical solutions of the time- fractional Harry Dym equation [7]. Kumar, Singh, and Kiliman solved the nonlinear fractional Harry Dym equation using the Adomian decomposition method (ADM) and the homotopy perturbation Sumudu transform method (HPSTM) [8]. In [9] Kumar, The homotopy perturbation procedure (HPM) was used by Tripathi and Singh to obtain an approximate solution to the time fractional Harry Dym equation. Maitama and Abdullahi solved both linear and nonlinear fractional partial differential equations. They used the natural homotopy perturbation method (NHPM), which is a mixture of the natural transform method (NTM) and the homotopy perturbation method (HPM) [10]. The Adomian decomposition method was used by Mokhtari. He applied power series, variational iteration, and direct integration to create exact solutions for travelling wave of the Harry Dym equation [3]. Rawashdeh used the fractional reduced differential transform procedure (FRDTM) to calculate the solutions to the nonlinear Harry Dym equation [11]. In [12], Shunmugarajan applied the homotopy analysis method (HAM) to get a solution that could be approximate for time fractional generalized Harry Dym equation. The reconstruction of the variational iteration method (RVIM) along with the homotopy perturbation method (HPM) were used by Soltani and Khorshidi. They obtained the Harry Dym equation analytical solution of [13]. Recently, Assabaai and Mukherji used Lie group method to find the appropriate and precise solutions of the Harry-Dym equation [14].

1.1 Chebyshev Expansion Method

The Chebyshev polynomial of the first kind of degree \(n\), \(T_n(x)\), has \((n+1)\) extreme located in the interval \([-1,1]\) at the points

\[
x_i = \cos\left(\frac{i\pi}{N}\right), \quad i = 0, 1, \ldots, N \quad (6)
\]

Let us assume that the function \(f(x)\) is described and behaves well in the interval \([-1,1]\). Then we have the approximation

\[
f(x) = \sum_{r=0}^{N} a_r T_r(x) \quad (7)
\]

where the Chebyshev coefficients \(a_r\) are defined as

\[
a_r = \frac{2}{N} \sum_{i=0}^{N-1} f(x_i) T_r(x_i) + \frac{2}{N} \left( \frac{f(x_0) T_r(x_0) + f(x_N) T_r(x_N)}{2} \right) \quad (8)
\]
Here \( T_r(x) \) is the \( r^{th} \) polynomial of Chebyshev. The approximate formula (7) is exact at \( x = x_i \) given by equation (6).

If we use equation (8) in equation (7), then after certain manipulations we have the following approximation

\[
\int_{-1}^{1} f(x) \, dx = B[f]
\]  

(9)

where \( B \) is an \((N + 1) \times (N + 1)\) matrix (see [15]), the column elements \([f]\) are given by \( f(x_i) \) where \( x_i \) are the Chebyshev points

\[
x_i = -\cos \left( \frac{i\pi}{N} \right), \quad i = 0, 1, \ldots, N
\]  

(10)

The approximations of the integral is given by the right-hand side of equation (9) at the points (6).

In other meaning, the integral at the points (10) is evaluate rather than evaluating the Chebyshev coefficients of the integral. The two approximations are equal, i.e. if the integral at the points (10) is known, its Chebyshev coefficients can be directly obtained from equation (8). We note that the computation of the elements of \( B \) does not depend on the integral.

Note: For the interval \([0,1]\), the elements of matrix \( B \) are to be halved, and the Chebyshev points will be

\[
x_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{i\pi}{N} \right) \right], \quad i = 0, 1, \ldots, N,
\]

such that the Chebyshev polynomials are given by

\[
T_r^* (x) = T_r(2x - 1), \quad 0 \leq x \leq 1
\]  

(11)

where all the \( T_r^* (x) \) properties can be obtained from those of \( T_r(2x - 1) \).

2. Method Description

Firstly, we apply Lie symmetry group analysis for the nonlinear Harry-Dym equation (2) to derive symmetry generators of it. We took one-parameter \( \varepsilon \)-Lie group point of transformations [16], [17], which makes equation (2) invariant. These transformations are given by

\[
\begin{align*}
\tilde{x} &= x(x,t,u;\varepsilon) = x + \varepsilon \xi_1(x,t,u) + O(\varepsilon^2), \\
\tilde{t} &= t(x,t,u;\varepsilon) = t + \varepsilon \xi_2(x,t,u) + O(\varepsilon^2), \\
\tilde{u} &= u(x,t,u;\varepsilon) = u + \varepsilon \phi(x,t,u) + O(\varepsilon^2)
\end{align*}
\]  

(12)

The infinitesimal generator is associated with equation (12)

\[
X = \xi_1(x,t,u) \frac{\partial}{\partial x} + \xi_2(x,t,u) \frac{\partial}{\partial t} + \phi(x,t,u) \frac{\partial}{\partial u}
\]  

(13)

The third prolongation of the infinitesimal generator (13) is given by

\[
\text{Pr}^{(3)} X = X + \phi^{'} \frac{\partial}{\partial t} + \phi^{''} \frac{\partial}{\partial u} + \phi^{'''} \frac{\partial}{\partial u^{xxx}}
\]  

(14)

we consider

\[
\Delta = \frac{\partial u(x,t)}{\partial t} - u^3(x,t) \frac{\partial^3 u(x,t)}{\partial x^3}
\]  

(15)

To calculate the infinitesimals \( \xi_1, \xi_2 \) and \( \phi \), applying equation (14) to equation (15), that is,
Pr\(^{(3)}\) \(X(\Delta) = 0\), \textit{when} \(\Delta = 0\) \(\quad (16)\)

Equation (16) is called condition of the invariant, and leads to
\[
\phi' + \alpha \phi^3 - \varepsilon \phi^{xx} = 0
\]
\(\quad (17)\)

Where
\[
\phi' = D_{\xi_i} \phi' - (D_{\xi_i} \xi_j) u j_{\xi_i} - (D_{\xi_j} \xi_j) u j_{\xi_j}, \quad i \neq j
\]
\(\quad (18)\)

where \(\xi_i\) indicates the independent variables. The infinitesimals \(\xi_1, \xi_2\) and \(\phi\) are determined from the coefficients of derivatives \(u_x, u_t\) and derivatives of higher order, that is equating the coefficients of monomials in derivatives by zero. This leads to a partial differential equations system. Using this system we can determine \(\xi_1, \xi_2\) and \(\phi\). These equations are solved to get the infinitesimals solutions \(\xi_1, \xi_2\) and \(\phi\) in the following forms:
\[
\begin{align*}
\xi_1(x,t,u) &= \frac{1}{2} c_3 x^2 + c_4 x + c_5, \\
\xi_2(x,t,u) &= c_1 t + c_2, \\
\phi(x,t,u) &= -\frac{1}{3} (c_1 - 3c_3 x - 3c_4) u
\end{align*}
\]
\(\quad (19)\)

where \(c_i, i = 1,2,\ldots,5\) are arbitrary constants. Equation (19) shows that the nonlinear Harry-Dym equation (2) has the following generators symmetry group:

\[
\begin{align*}
X_1 &= t \partial_t - \frac{1}{3} u \partial_u, \\
X_2 &= \partial_x, \\
X_3 &= \frac{1}{2} x^2 \partial_x + xu \partial_u, \\
X_4 &= xu \partial_x + u \partial_u, \\
X_5 &= \partial_x
\end{align*}
\]
\(\quad (20)\)

Now, We consider the linear combination of generators \(X_2\) and \(X_5\) as \(X_2 - c X_5\), then we get the Similarity transformations
\[
\eta = x + ct, \quad u(x,t) = g(\eta)
\]
\(\quad (21)\)

Substituting equation (21) in to equation (2), we obtain the following ordinary differential equation:
\[
g^3(\eta) g''(\eta) - c g'(\eta) = 0, \quad \eta \in [ct, 1 + ct]
\]
\(\quad (22)\)

with the boundary conditions
\[
\begin{align*}
g &= q_0(t) \text{ and } g' = r(t) \text{ at } \eta \to ct, \\
g &= q_1(t) \text{ at } \eta \to 1 + ct
\end{align*}
\]
\(\quad (23)\)
Using the mapping of algebraic

\[ z = \eta - ct, \]

the boundary region is reduced to the finite domain \([0,1]\). Besides the boundary value problem of equation (22) with equation (23) is transformed to the following boundary value problem of ordinary differential equation:

\[
g^3(z) g^\nu(z) - c g'(z) = 0, \quad z \in [0,1] \tag{24}
g(0) = q_0(t),
g(1) = q_1(t),
g'(0) = r(t) \tag{25}
\]

Now, we use a Chebyshev method, that is by starting with the derivative of the highest order \(g^\nu\) of Chebyshev approximations and generating approximations to derivatives of lower orders \(g^\nu, g'\) and \(g\) as follows:

Let \(\Psi(z) = g^\nu(z)\), then

\[
g^\nu(z) = \int_0^z \Psi(z) \, dz + c_1 \tag{26}
g'(z) = \int_0^z \int_0^z \Psi(z) \, dz \, dz + z \, c_1 + c_2 \tag{27}
g(z) = \int_0^z \int_0^z \int_0^z \Psi(z) \, dz \, dz \, dz + \frac{1}{2} z^2 c_1 + z c_2 + c_3 \tag{28}
\]

From equation (27) and equation (28), we get

\[
c_1 = 2 \left( q_1(t) - q_0(t) - r(t) \right) - 2 \int_0^z \int_0^z \Psi(z) \, dz \, dz,
\]

\[
c_2 = r(t),
\]

\[
c_3 = q_0(t)
\]

We give Chebyshev approximations for equations (26), (27) and (28) as follows:

\[
g^\nu(z_i) = \sum_{j=0}^N \ell^{(2)}_{ij} \Psi(z_j) + d_i \tag{29}
g'(z_i) = \sum_{j=0}^N \ell^{(1)}_{ij} \Psi(z_j) + d_i \tag{30}
g(z_i) = \sum_{j=0}^N \ell^{(3)}_{ij} \Psi(z_j) + d_i \tag{31}
\]

for all \(i, j = 0, 1, ..., N\) and \(z_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{i\pi}{N} \right) \right] \),

where

\[
\ell^{(3)}_{ij} = b_{ij} z_i^3 b_{Nj},
\]

\[
\ell^{(1)}_{ij} = b_{ij} - 2 z_i b_{Nj},
\]

\[
\ell^{(2)}_{ij} = b_{ij} z_i^2 b_{Nj}.
\]
(2) $\ell_{ij} = b_{ij} - 2b_{Nj}$,

(3) $d_i = z_i^2 \left( q_1(t) - q_0(t) - r(t) \right) + z_i \left( r(t) + q_0(t) \right)$,

(1) $d_i = 2z_i \left( q_1(t) - q_0(t) - r(t) \right) + r(t)$,

(2) $d_i = 2 \left( q_1(t) - q_0(t) - r(t) \right)$,

and $b_{ij} = \frac{1}{2!} (z_i - z_j)^2 b_{ij}$, $i, j = 0, 1, ..., N$.

where $b_{ij}$ are the elements of half B matrix as defined in [15].

By using equations (29), (30) and (31) then equation (24) is transformed to the following nonlinear system:

$$
\left( \sum_{j=0}^{N} \ell_{ij} \Psi(z_j) + d_i \right)^3 \Psi_j - c \left( \sum_{j=0}^{N} \ell_{ij}^{(1)} \Psi(z_j) + d_i^{(1)} \right) = 0
$$

(32)

Newton's iteration method is used to solve the system (32), and the approximation to the solution of the original system is obtained at the end.

3. Numerical Examples

Here, some numerical results are given to illustrate the presented method (PM) for the solution of the nonlinear fractional Harry Dym equation. Our numerical results are comparing with other works presented by [4] and [11] (See table 1 and table 2).

Numerical solutions for various values of $x$ and $t$ at $N = 12$ are shown in table 3. Figure 1 and Figure 2 indicate the exact and numerical solution for $t = 0.5, 1$ at $N = 14$ and $N = 12$ respectively.

All results are taken for constants $a = 4$ and $b = 1$.

| $x$  | $t$  | FRDTM [11] | PM    | Exact solution |
|------|------|------------|-------|----------------|
| 0.0  | 0.002| 2.51858    | 2.51858209 | 2.51858209   |
| 0.004| 2.51732| 2.51732170 | 2.51732170 | 2.51606099   |
| 0.006| 2.51606| 2.51606099 | 2.51606099 | 2.51606099   |
| 0.008| 2.5148 | 2.51479996 | 2.51479996 | 2.51479996   |
| 1    |      |            | 1.8420157829 | 1.8420157829 |
| 0.25 | 1    |            | 1.6528722841 | 1.6528722794 |
| 0.50 |      |            | 1.4521964566 | 1.4521964495 |
| 0.75 |      |            | 1.2365218747 | 1.2365218686 |
| 1.00 | 0.002| 1.84054    | 1.84054188  | 1.84054188   |
| 0.004| 1.83907| 1.83906738 | 1.83906738 | 1.83906738   |
| 0.006| 1.83759| 1.83759229 | 1.83759229 | 1.83759229   |
| 0.008| 1.83612| 1.83611661 | 1.83611661 | 1.83611661   |
| 1    |      |            | 1.0000000000 | 1.0000000000 |
Table 2. Numerical solutions for various values of $x$ and $t = 0.25$ at $N = 16$

| $x$ | ADM [4] | PM | Exact solution |
|-----|---------|----|---------------|
| 0.0 | 2.35999 | 2.3597827 | 2.3597827 |
| 0.5 | 2.02224 | 2.0218949 | 2.0218949 |
| 1.0 | 1.65352 | 1.6528723 | 1.6528723 |

Table 3. Numerical solutions for various values of $x$ and $t$ at $N = 12$

| $t$ | $x$ | Num. solution | Exact solution | Abs. error |
|-----|-----|---------------|---------------|------------|
| 0.005 | 0.00 | 2.5166913813 | 2.5166913813 | 0.000000E+00 |
| 0.25 | 0.25 | 2.3565267420 | 2.3565267336 | 8.38496E-09 |
| 0.50 | 0.50 | 2.1907189698 | 2.1907189586 | 1.17078E-08 |
| 0.75 | 0.75 | 2.0183770310 | 2.0183770226 | 8.38086E-09 |
| 1.00 | 1.00 | 1.8383299068 | 1.8383299068 | 0.000000E+00 |
| 0.05 | 0.00 | 2.4882448800 | 2.4882448800 | 0.000000E+00 |
| 0.25 | 0.25 | 2.3271207277 | 2.3271207193 | 8.37223E-09 |
| 0.50 | 0.50 | 2.1602093021 | 2.1602092909 | 1.15418E-08 |
| 0.75 | 0.75 | 1.9865768252 | 1.9865768169 | 8.36922E-09 |
| 1.00 | 1.00 | 1.8049887641 | 1.8049887641 | 0.000000E+00 |
| 0.5 | 0.00 | 2.1940957902 | 2.1940957902 | 0.000000E+00 |
| 0.25 | 0.25 | 2.0218949004 | 2.0218948922 | 8.19066E-09 |
| 0.50 | 0.50 | 1.8420157938 | 1.8420157829 | 1.09244E-08 |
| 0.75 | 0.75 | 1.6528722876 | 1.6528722794 | 8.21557E-09 |
| 1.00 | 1.00 | 1.4521964495 | 1.4521964495 | 0.000000E+00 |

**Figure 1.** The numerical and exact solutions for $t = 0.5, N = 14$.

**Figure 2.** The numerical and exact solutions for $t = 1.0, N = 12$.

4. Conclusion
In this article, we show how the Lie group is applied to the equation of Harry Dym (HDE). We transform the problem to the ordinary differential equation (ODE) with boundary conditions. As a sequence, the Chebyshev approximation method is applied to solve the problem. In comparison to the exact solutions and the results cited in[4] and [11], the numerical values achieved by the present
process have a better agreement and are more precise.

Acknowledgements
The author, Dr. Mobark Assabaai appreciates Dr. Salim F. Bamsaoud of Hadhramout University's Physics Department for his supportive comments and positive feedback. Also, the author would like to express his gratitude and appreciation to Benevolent Fund for Outstanding Student (BFOS) for the support.

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