GROUP REPRESENTATION ON REFLEXIVE SPACES

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Abstract. Algebras which admit representations on reflexive Banach spaces seem to be a good generalisation of Arens regular Banach algebras, and behave in a similar way to $C^*$-algebras and Von Neumann algebras. Such algebras are called weakly almost periodic Banach algebras (or in abbreviated form $WAP$-algebras). In this paper, for weighted group convolution measure algebra we construct a representation on reflexive space.

Keywords: $WAP$-algebra, dual Banach algebra, Arens regularity, weak almost periodicity

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1. Introduction and Preliminaries

The dual $A^*$ of a Banach algebra $A$ can be turned into a Banach $A$—module in a natural way, by setting

$$
\langle f \cdot a, b \rangle = \langle f, ab \rangle \quad \text{and} \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle
$$

for all $a, b \in A, f \in A^*$. A dual Banach algebra is a Banach algebra $A$ such that $A = (A_*)^*$, as a Banach space, for some Banach space $A_*$, and such that $A_*$ is a closed $A$—submodule of $A^*$; or equivalently, the multiplication on $A$ is separately weak*-continuous. A functional $f \in A^*$ is said to be weakly almost periodic if $\{ f \cdot a : \|a\| \leq 1 \}$ is relatively weakly compact in $A^*$. We denote by $WAP(A)$ the set of all weakly almost periodic elements of $A^*$. As pointed out by Pym[14], $\lambda \in A^*$ is weakly almost periodic if and only if $\lim_m \lim_n \langle a_m b_n, \lambda \rangle = \lim_n \lim_m \langle a_m b_n, \lambda \rangle$ whenever $(a_m)$ and $(b_n)$ are sequences in unit ball of $A$ and both repeated limits exist. For more about weakly almost periodic functionals, see [6]. It is known that the multiplication of a Banach algebra $A$ has two natural but, in general, different extensions (called Arens products) to the second dual $A^{**}$ each turning $A^{**}$ into a Banach algebra. When these extensions are equal, $A$ is said to be (Arens) regular. It can be verified that $A$ is Arens regular if and only if $WAP(A) = A^*$. Further information for the Arens regularity of Banach algebras can be found in [5, 6]. If $A$ and $B$ are Banach algebras, the linear operator $\phi : A \to B$ is said to be bounded below if $\inf\{\|\phi(a)\| : \|a\| \geq 1\} > 0$. $WAP$-algebras, as a generalization of the Arens algebras, are defined as follows.

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regular algebras, has been introduced and extensively studied in [8]. Indeed, a Banach algebra $A$ for which the natural embedding of $A$ into $\mathcal{WAP}(A)^*$ is bounded below, is called a $\mathcal{WAP}$-algebra. It has also known that $A$ is a $\mathcal{WAP}$-algebra if and only if it admits an isomorphic representation on a reflexive Banach space. If $A$ is a $\mathcal{WAP}$-algebra, then $\mathcal{WAP}(A)$ separates the points of $A$ and so $\mathcal{WAP}(A)$ is $\omega^*$-dense in $A^*$. It can be readily verified that every dual Banach algebra, and every Arens regular Banach algebra, is a $\mathcal{WAP}$-algebra for comparison see [12]. Moreover group algebras are also $\mathcal{WAP}$-algebras, however, they are neither dual Banach algebras, nor Arens regular in the case where the underlying group is not discrete, see [4, Corollary3.7] and [17, 12].

The paper is organized as follows. In section 2, we construct a representation on reflexive space.

2. Group Measure Algebras

Let $G$ be a locally compact group with left Haar measure $\lambda$. A Borel measurable function $\omega \geq 1$ on $G$ is called a weight or weight function if $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. Throughout this section, let $1 \leq p < \infty$ be a real number and $q$ is such that $1/p + 1/q = 1$, then $q$ is called the exponential conjugate of $p$. The functions $f : G \to \mathbb{C}$ such that $f\omega \in L^p(G)$ form a linear space which is denoted by $L^p(G, \omega)$. Then $\|f\|_{p,\omega} = \|f\|_p$ defines a norm on $L^p(G, \omega)$. The dual space of $L^1(G, \omega)$ denoted by $L^\infty(G, 1/\omega)$. It consists of all complex-valued measurable functions $g$ on $G$ such that $g/\omega \in L^\infty(G)$. We equipped $L^\infty(G, 1/\omega)$ with the norm $\|g\|_{\infty,\omega} = \|g/\omega\|_\infty$. Then

$$C_0(G, 1/\omega) = \{f : G \to \mathbb{C} : f/\omega \in C_0(G)\}$$

is a subspace of it. The dual space of $L^p(G, \omega)$ is $L^q(G, 1/\omega)$ consist of all measurable functions $g$ on $G$ such that $g/\omega \in L^q(G)$ by duality

$$\langle f, g \rangle := \int_G f(x)g(x)d\lambda(x)$$

for all $f \in L^p(G, \omega)$ and $g \in L^q(G, 1/\omega)$.

For measurable functions $f$ and $g$ on $G$, the convolution multiplication

$$(f \ast g)(x) = \int_G f(y)\ g(y^{-1}x)\ d\lambda(y)\quad (x \in G),$$

is defined at each point $x \in G$ for which this makes sense; i.e., the function $y \mapsto f(y)\ g(y^{-1}x)$ is $\lambda$-integrable. Then $f \ast g$ is said to exist if $(f \ast g)(x)$ exists for almost all $x \in G$.

Since $\omega \geq 1$ hence $L^p(G, \omega) \subseteq L^p(G)$ and $L^q(G) \subseteq L^q(G, 1/\omega)$.

The map

$$\gamma_p : L^1(G, \omega) \longrightarrow B(L^p(G, \omega))$$

is defined by $\gamma_p(a)(\xi) = a \ast \xi$ for all $a \in L^1(G, \omega)$ and $\xi \in L^p(G, \omega)$. 

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Let \((\gamma_p)_* : L^p(G, \omega) \hat{\otimes} L^q(G, 1/\omega) \to L^\infty(G, 1/\omega)\) be such that

\[
\langle (\gamma_p)_*(\xi \otimes \eta), a \rangle = \langle (\xi \otimes \eta), \gamma_p(a) \rangle = \langle \eta, \gamma_p(a)(\xi) \rangle \\
= \langle \eta, a \ast \xi \rangle = \int_G \int_G \eta(t)a(s)\xi(s^{-1}t)d\lambda(t)d\lambda(s) \\
= \langle \eta \ast \xi, a \rangle
\]

such that \(\eta \ast \xi(s) = \int_G \xi(s^{-1}t)\eta(t)d\lambda(t)\) and \(\xi(s) = \xi(s^{-1})\) for all \(s \in G\). Then \((\gamma_p)_*^* = \gamma_p\).

**Lemma 2.1.** Let \(G\) be a locally compact group, and let \(\omega\) be a weight on \(G\). Let \(1 \leq p, q < \infty\).

(1) Every compactly supported function in \(L^p(G)\) (respectively \(L^q(G)\)) belongs to \(L^p(G, \omega)\) (respectively \(L^q(G, 1/\omega)\)).

(2) \(C_{00}(G)\) is dense in \(L^p(G, \omega)\) (respectively \(L^q(G, 1/\omega)\)).

**Proof.** (1) By [11, Lemma 1.3.3], the weight \(\omega\) is bounded away from both zero and infinity on compact subsets of \(G\). If \(f \in L^p(G)\) with compact support \(K = \text{supp}f\) then for all \(x \in K\), \(\omega(x) < b\) for some \(b > 0\). Then \(\|f\|_{p, \omega} \leq b\|f\|_p\). Hence \(f \in L^p(G, \omega)\).

(2) By (1) \(C_{00}(G) \subseteq L^p(G, \omega)\). To show that \(C_{00}(G)\) is dense in \(L^p(G, \omega)\), let \(f \in L^p(G, \omega)\) and \(\epsilon > 0\) be given. Since \(f\omega \in L^p(G)\), there exists \(h \in C_{00}(G)\) such that \(\|h - f\omega\|_p^p \leq \epsilon\). Let \(S\) denote the compact support of \(h\) and observe that \(\omega(x) \geq \delta\) for some \(\delta > 0\) and all \(x \in S\). Since \(\omega\) is bounded on \(S\), \(\omega|_{S} \in L^p(S)\) and hence there exist a continuous function \(\eta : S \to \mathbb{R}\) such that \(\eta(x) \geq \delta\) for all \(x \in S\) and

\[
\int_S |\eta(x) - \omega(x)|^p d\lambda(x) \leq \frac{\epsilon\delta^p}{\|h\|_\infty^p}.
\]

Now define a function \(g\) on \(G\) by \(g(x) = \frac{h(x)}{\eta(x)}\) for \(x \in S\) and \(g(x) = 0\) for \(x \notin S\). Since \(1/\eta(x) \leq 1/\delta\) for all \(x \in S\), it is easily verified that \(g\) is continuous on \(G\). Thus \(g \in C_{00}(G)\) and

\[
\|g - f\|^p_{p, \omega} = \int_S \omega(x)^p|g(x) - f(x)|^p d\lambda(x) + \int_{G\setminus S} \omega(x)^p|f(x)|^p d\lambda(x),
\]
We estimate the first integral on the right as follows:

\[
\int_S \omega(x)^p |g(x) - f(x)|^p \, d\lambda(x) \leq \int_S \omega(x)^p \frac{\|h(x)\|}{\eta(x)} \frac{\|h(x)\|}{\omega(x)}^p \, d\lambda(x) \\
+ \int_S \omega(x)^p \frac{\|h(x)\|}{\omega(x)} - f(x)|^p \, d\lambda(x) \\
= \int_S \frac{\|h(x)\|^p}{\eta(x)^p} |\omega(x) - \eta(x)|^p \, d\lambda(x) \\
+ \int_S |h(x) - \omega(x)f(x)|^p \, d\lambda(x) \\
\leq \frac{\|h\|_\infty}{\delta p} \int_S |\omega(x) - \eta(x)|^p \, d\lambda(x) \\
+ \int_S |h(x) - \omega(x)f(x)|^p \, d\lambda(x) \\
\leq \epsilon + \int_S |h(x) - \omega(x)f(x)|^p \, d\lambda(x).
\]

It follows that

\[
\|g - f\|_{p,\omega}^p \leq \epsilon + \int_S |h(x) - \omega(x)f(x)|^p \, d\lambda(x) + \int_{G \setminus S} \omega(x)^p |f(x)|^p \, d\lambda(x) \\
= \epsilon + \int_G |h(x) - \omega(x)f(x)|^p \, d\lambda(x) \leq 2\epsilon.
\]

This shows that \(C_{00}(G)\) is dense in \(L^p(G, \omega)\).

For \(q\) is similar to \(p\), but \(g\) must be defined by \(g(x) = \eta(x)h(x)\). \(\square\)

**Remark 2.2.** If \(\alpha \leq \omega \leq \beta\), then \(L^p(G, \omega) = L^p(G)\) and the norms \(||.||_{p,\omega}\) and \(||.||_p\) are equivalent and \(C_{00}(G))\) is dense in \(L^p(G)\), hence dense in \(L^p(G, \omega)\).

For the function \(\xi\) on \(G\) and \(x \in G\) left and right translates \(L_x \xi\) and \(R_x \xi\) defined by \(L_x \xi(y) = \xi(x^{-1}y)\) and \(R_x \xi(y) = \xi(xy)\) for all \(y \in G\).

**Lemma 2.3.** Let \(G\) be a locally compact group, and let \(\omega\) be a weight on \(G\), \(1 < p < \infty\) and \(\xi \in L^p(G, \omega)\).

1. For all \(x \in G\), \(L_x \xi \in L^p(G, \omega)\) and \(||L_x \xi||_{p,\omega} \leq \omega(x)||\xi||_{p,\omega}\).
2. The map \(x \rightarrow L_x \) from \(G\) into \(L^p(G, \omega)\) is continuous.
3. Let \(q\) be the exponential conjugate of \(p\). If \(\omega\) is continuous on \(G\) and \(\eta \in L^q(G, 1/\omega)\). Then \(\eta \ast \xi \in C_0(G, 1/\omega)\).
4. For every relatively compact neighborhood \(V\) of \(e\) in \(G\), let \(u_V \in L^p(G, \omega)\) be such that \(u_V \geq 0\) and \(||u_V||_{p,\omega} = 1\) and \(u_V = 0\) almost everywhere on \(G \setminus V\). Then, given \(\xi \in L^p(G, \omega)\) and \(\epsilon > 0\),

\[
||u_V \ast \xi - \xi||_{p,\omega} < \epsilon
\]

for all sufficiently small \(V\).
Proof. (1) follows simply from submultiplicativity of $\omega$:

$$||L_x\xi||_{p,\omega}^p = \int_G |\xi(x^{-1}t)|^p \omega(t)^p d\lambda(t)$$

$$= \int_G |\xi(x^{-1}t)|^p \omega(x^{-1}t)^p \frac{\omega(t)^p}{\omega(x^{-1}t)^p} d\lambda(t)$$

$$\leq \omega(x)^p \int_G |\xi(x^{-1}t)|^p \omega(x^{-1}t)^p d\lambda(t)$$

$$= \omega(x)^p ||\xi||_{p,\omega}^p$$

(2) Let $\epsilon > 0$. Since $C_{00}(G)$ is dense in $L^p(G,\omega)$. There exists $g \in C_{00}(G)$ such that $||\xi - g||_{p,\omega} < \epsilon/3$. Let $x \in G$ and choose a compact neighbourhood $V$ of $x$ in $G$. Let

$$C = \sup\{\omega(s) : s \in V.suppg\} < \infty$$

Then for $y \in V$,

$$||L_xg - L_yg||_{p,\omega}^p = \int_G |g(x^{-1}t) - g(y^{-1}t)|^p \omega(t)^p d\lambda(t)$$

$$\leq C^p \int_{V.suppg} |g(x^{-1}t) - g(y^{-1}t)|^p d\lambda(t)$$

$$= C^p ||L_xg - L_yg||_{p,\omega}^p$$

when $y \to x$ converges to zero. Since $\omega$ is locally bounded:

$$||L_x\xi - L_y\xi||_{p,\omega} \leq ||L_x(\xi - g)||_{p,\omega} + ||L_xg - L_yg||_{p,\omega} + ||L_y(\xi - g)||_{p,\omega}$$

$$\leq (\omega(x) + \omega(y))||\xi - g||_{p,\omega} + ||L_xg - L_yg||_{p,\omega} < \epsilon$$

(3) For $x \in G$, by Hölder inequality:

$$\int_G |\xi(x^{-1}y)|\eta(y)|d\lambda(y)| = \int_G |L_x\xi(y)||\eta(y)|d\lambda(y) \leq ||L_x\xi||_{p,\omega}||\eta||_{q,\omega}.$$  

So $\eta \ast \hat{\xi}$ is defined everywhere and bounded on $G$ by $\omega(x)||\xi||_{p,\omega}||\eta||_{q,\omega}$. For $x, y \in G$, Hölder inequality gives

$$||\eta \ast \hat{\xi}(x) - \eta \ast \hat{\xi}(y)|| \leq ||L_x\xi - L_y\xi||_{p,\omega}||\eta||_{q,\omega}.$$  

The map $t \to L_t\xi$ from $G$ into $L^p(G,\omega)$ is continuous and therefore we obtain that $\eta \ast \hat{\xi}$ is continuous. To prove $\eta \ast \hat{\xi} \in C_0(G,1/\omega)$, note first that $\xi, \eta \in C_{00}(G)$ whenever $\eta \ast \hat{\xi} \in C_{00}(G)$.

If $\xi \in L^p(G,\omega), \eta \in L^q(G,1/\omega)$, then for $1 \leq r < \infty$, since $C_{00}(G)$ is dense in $L^r(G,\omega)$ and $L^r(G,1/\omega)$, there exist $(\xi_n)$ and $(\eta_n)$ in $C_{00}(G)$ such that $||\xi - \xi_n||_{p,\omega} \to 0$ and $||\eta - \eta_n||_{q,\omega} \to 0$. Then for all $x \in G$,

$$||\eta \ast \hat{\xi}(x) - \eta_n \ast \hat{\xi}_n(x)||/\omega(x) \leq ||\eta \ast (\hat{\xi} - \hat{\xi}_n)(x)||/\omega(x) + ||(\eta - \eta_n) \ast \hat{\xi}(x)||/\omega(x)$$

$$\leq ||\xi - \xi_n||_{p,\omega}||\eta||_{q,\omega} + ||\xi||_{p,\omega}||\eta - \eta_n||_{q,\omega},$$

which tends to 0 as $n \to \infty$. It follows that $(\eta \ast \hat{\xi})/\omega \in C_0(G)$. 


Now, since the map $y \mapsto L_y g$ from $G$ into $L^p(G, \omega)$ is continuous, we find a neighbourhood $W$ of $e$ in $G$ such that, for all $y \in W$,

$$||L_y g - g||_\infty \leq \frac{\epsilon}{3 \lambda(V, \text{suppg})^{1/p}}$$

Together with the above estimate we get for all $V \subseteq W$,

$$||u_V \ast \xi - \xi||_{p,\omega} \leq ||u_V \ast (\xi - g)||_{p,\omega} + ||u_V \ast g - g||_{p,\omega} + ||g - \xi||_{p,\omega}$$

$$\leq (||u_V||_{p,\omega} + 1)||\xi - g||_{p,\omega} + \frac{\epsilon}{3} < \epsilon$$

Now, the Young's construction [17, Theorem 4] for weighted convolution measure algebras.

**Theorem 2.4.** Let $G$ be a locally compact group, and let $\omega \geq 1$ be a continuous weight on $G$. Then weighted convolution measure algebra $M_b(G, \omega)$ is a WAP-algebra.

**Proof.** Let $\{p_n\}$ be some sequence in $(1, \infty)$ such that $p_n \to 1$. Let $E$ be the direct sum, in an $\ell_2$-sense, of the spaces $L^{p_n}(G, \omega)$. To be exact,

$$E = \{\{\xi_n\} : \xi_n \in L^{p_n}(G, \omega), ||\{\xi_n\}||_E < \infty\}$$

with $||\{\xi_n\}||_E = (\sum_{n=1}^{\infty} ||\xi_n||_{p_n,\omega}^2)^{1/2}$, then $E$ is reflexive.
The mapping \( \Theta : M_b(G, \omega) \rightarrow B(E) \) is defined by \( \Theta(\mu)((\xi_n)) = \{ \mu * \xi_n \} \). Consider the adjoint map \( \Theta^* : E \otimes E^* \rightarrow M_b(G, \omega)^* \) given by
\[
\langle \Theta^*(\xi \otimes \eta), \mu \rangle = \langle \eta, \Theta(\mu)(\xi) \rangle = \sum_{n=1}^{\infty} \langle \eta_n, \hat{\Theta}(\mu)(\xi_n) \rangle = \sum_{n=1}^{\infty} \langle \mu, \eta_n * \xi_n \rangle
\]
where \( \xi = \{ \xi_n \} \in E, \eta = \{ \eta_n \} \in E^* \) and \( \mu \in M_b(G, \omega) \).

In particular, \( \Theta^* \) maps into \( C_0(G, 1/\omega) \). So that \( \Theta \) is \( \omega^* - \omega^* \) continuous.

For \( \xi, \eta \in C_0(G) \), we have that
\[
\lim_{p \rightarrow 1} || \xi ||_{p, \omega} = || \xi ||_{1, \omega}, \quad \lim_{q \rightarrow \infty} || \eta ||_{q, \omega} = || \eta ||_{\infty, \omega}.
\]
By lemma 2.3 for any \( \eta \in C_0(G) \) and \( \varepsilon > 0 \) we can find some \( u_V \in C_0(G) \) with \( ||u_V||_{1, \omega} = 1 \) and \( ||\eta * \hat{u}_V - \eta ||_{\infty, \omega} < \varepsilon \). As \( p_n \rightarrow 1 \), we can find \( n \in \mathbb{N} \) with \( ||u_V||_{p_n, \omega} < 1 + \varepsilon \) and \( ||\eta||_{q_n, \omega} < (1 + \varepsilon)^2 ||\eta||_{\infty, \omega} \). It follows that
\[
||\eta * \hat{u}_V||_{\infty, \omega} \leq ||u_V||_{p_n, \omega} ||\eta||_{q_n, \omega} < (1 + \varepsilon)^2 ||\eta||_{\infty, \omega}
\]
and that
\[
\langle \eta, \Theta(\mu)(u_V) \rangle = \langle \mu, \eta * \hat{u}_V \rangle \geq ||\mu|| - \varepsilon ||\mu||
\]
By taking suitable supremums, it now follows:
\[
(1 - \varepsilon)||\mu|| \leq \sup\{ ||\langle \mu, \eta \rangle \| : \eta \in C_0(G), ||\eta|| \leq 1 \} - \varepsilon ||\mu|| \\
\leq \sup\{ ||\langle \eta, \Theta(\mu)(u_V) \rangle \| : \eta \in C_0(G), ||\eta|| \leq 1 \} \\
= ||\Theta(\mu)(u_V)|| \leq ||\Theta|| ||\mu||
\]
Since \( \varepsilon \) is arbitrary \( \Theta \) is isometric. \( \square \)

Let \( G \) be a locally compact group, and let \( \omega \) be a Borel-measurable weight function on it. Then the Fourier-Stieltjes algebra \( B(G) \) and the weighted measure algebra \( M_b(G, \omega) \) are dual Banach algebras(see [12]), also the Fourier algebra \( A(G) \) and \( L^1(G, \omega) \) are closed ideals in them respectively. Hence all are \( \mathcal{WAP} \)-algebras.

The next example shows that Young’s construction can not work for semigroups.

**Examples 2.1.** Let \( S = (\mathbb{N}, \text{min}) \). Then \( \ell_1(S) \) is \( \mathcal{WAP} \)-algebras. But we can’t apply Young’s construction for this semigroup. Let \( f(n) = \frac{1}{n^\alpha} \) and \( g(n) = \frac{1}{\sqrt{n}} \) defined for all \( n \in \mathbb{N} \).

\[
||f||_1 = \sum_{n \in \mathbb{N}} \frac{1}{n^{\alpha/2}} < \infty, \quad ||g||_4 = \sum_{n \in \mathbb{N}} \frac{1}{n^{3/4}} < \infty
\]
Then \( f \in \ell_1(S) \) and \( g \in \ell_4(S) \) but \( f \ast g \not\in \ell_4(S) \) since
\[
f \ast g(k) = \sum_{n,m=k} f(n)g(m) = \sum_{m=k} \frac{1}{k^2} \frac{1}{\sqrt{m}} + \sum_{n=k} \frac{1}{n^2} \frac{1}{\sqrt{k}} = \infty
\]
for all \( k \in \mathbb{N} \).

3. Relation between \( C_0(S) \) and \( \text{WAP}-\text{algebra} \)

Let \( S = \mathbb{N} \). Then for \( S \) equipped with min multiplication, the semigroup algebra \( \ell_1(S) \) is a WAP-algebra but is not neither Arens regular nor a dual Banach algebra. While, if we replace the min multiplication with max then \( \ell_1(S) \) is a dual Banach algebra (so a WAP-algebra) which is not Arens regular. If we change the multiplication of \( S \) to the zero multiplication then the resulted semigroup algebra is Arens regular (so a WAP-algebra) which is not a dual Banach algebra. This describes the interrelation between the concepts of being Arens regular algebra, dual Banach algebra and WAP-algebra.

**Definition 3.1.** Let \( X, Y \) be sets and \( f \) be a complex-valued function on \( X \times Y \).

1. We say that \( f \) is a cluster on \( X \times Y \) if for each pair of sequences \( (x_n), (y_m) \) of distinct elements of \( X, Y \), respectively
\[
\lim_n \lim_m f(x_n, y_m) = \lim_m \lim_n f(x_n, y_m)
\]
whenver both sides of (3.1) exist.

2. If \( f \) is cluster and both sides of (3.1) are zero (respectively positive) in all cases, we say that \( f \) is 0-cluster (respectively positive cluster).

In general \( \{ f_\omega : f \in \text{wap}(S) \} \neq \text{wap}(S,1/\omega) \). By using [2, Lemma 1.4] the following is immediate.

**Lemma 3.2.** Let \( \Omega(x, y) = \frac{\omega(x)}{\omega(x, \omega(y))} \), for \( x, y \in S \). Then

1. If \( \Omega \) is cluster, then \( \{ f_\omega : f \in \text{wap}(S) \} \subseteq \text{wap}(S,1/\omega) \);
2. If \( \Omega \) is positive cluster, then \( \text{wap}(S,1/\omega) = \{ f_\omega : f \in \text{wap}(S) \} \).

It should be noted that if \( M_b(S) \) is Arens regular (resp. dual Banach algebra) then \( M_b(S,\omega) \) is so. We don’t know that if \( M_b(S) \) is WAP-algebra, then \( M_b(S,\omega) \) is so. The following Lemma give a partial answer to this question.

**Corollary 3.3.** Let \( S \) be a locally compact topological semigroup with a Borel measurable weight function \( \omega \) such that \( \Omega \) is cluster on \( S \times S \).

1. If \( M_b(S) \) is a WAP-algebra, then so is \( M_b(S,\omega) \);
2. If \( \ell_1(S) \) is a WAP-algebra, then so is \( \ell_1(S,\omega) \).

**Proof.** (1) Suppose that \( M_b(S) \) is a WAP-algebra so \( \text{wap}(S) \) separates the points of \( S \). By lemma 3.2 for every \( f \in \text{wap}(S) \), \( f_\omega \in \text{wap}(S,1/\omega) \). Thus the evaluation map \( \epsilon : S \to \tilde{X} \) is one to one.

(2) follows from (1).
Corollary 3.4. For a locally compact semi-topological semigroup $S$,

1. If $C_0(S) \subseteq \text{wap}(S)$, then the measure algebra $M_b(S)$ is a WAP-algebra.
2. If $S$ is discrete and $c_0(S) \subseteq \text{wap}(S)$, then $\ell_1(S)$ is a WAP-algebra.

Proof. (1) By [3, Corollary 4.2.13] the map $\epsilon : S \to S^{\text{wap}}$ is one to one, thus $M_b(S)$ is a WAP-algebra.

(2) follows from (1). □

Dales, Lau and Strauss [7, Theorem 4.6, Proposition 8.3] showed that for a semigroup $S$, $\ell_1(S)$ is a dual Banach algebra with respect to $c_0(S)$ if and only if $S$ is weakly cancellative. If $S$ is left or right weakly cancellative semigroup, then $\ell_1(S)$ is a WAP-algebra. The next example shows that the converse is not true, in general.

Example 3.5. Let $S = (\mathbb{N}, \min)$ then $\text{wap}(S) = c_0(S) \oplus \mathbb{C}$. So $\ell_1(S)$ is a WAP-algebra but $S$ is neither left nor right weakly cancellative. In fact, for $f \in \text{wap}(S)$ and all sequences $\{a_n\}, \{b_m\}$ with distinct elements in $S$, we have $\lim_m f(b_m) = \lim_m \lim_n f(a_nb_m) = \lambda = \lim_n \lim_m f(a_nb_m) = \lim_n f(a_n)$, for some $\lambda \in \mathbb{C}$. This means $f - \lambda \in c_0(S)$ and $\text{wap}(S) \subseteq c_0(S) \oplus \mathbb{C}$. The other inclusion is clear.

If $\{x_n\}$ and $\{y_m\}$ are sequences in $S$ we obtain an infinite matrix $\{x_ny_m\}$ which has $x_ny_m$ as its entry in the $m$th row and $n$th column. As in [2], a matrix is said to be of row type $C$ (resp. column type $C$) if the rows (resp. columns) of the matrix are all constant and distinct. A matrix is of type $C$ if it is constant or of row or column type $C$.

J.W. Baker and A. Rejali in [2, Theorem 2.7(v)] showed that $\ell_1(S)$ is Arens regular if and only if for each pair of sequences $\{x_n\}, \{y_m\}$ with distinct elements in $S$ there is a submatrix of $\{x_ny_m\}$ of type $C$.

A matrix $\{x_ny_m\}$ is said to be upper triangular constant if $x_ny_m = s$ if and only if $m \geq n$ and it is lower triangular constant if $x_ny_m = s$ if and only if $m \leq n$. A matrix $\{x_ny_m\}$ is said to be $W$-type if every submatrix of $\{x_ny_m\}$ is neither upper triangular constant nor lower triangular constant.

Theorem 3.6. Let $S$ be a semigroup. The following statements are equivalent:

1. $c_0(S) \subseteq \text{wap}(S)$.
2. For each $s \in S$ and each pair $\{x_n\}, \{y_m\}$ of sequences in $S$,
   \[ \{\chi_s(x_ny_m) : n < m\} \cap \{\chi_s(x_ny_m) : n > m\} \neq \emptyset; \]
3. For each pair $\{x_n\}, \{y_m\}$ of sequences in $S$ with distinct elements, $\{x_ny_m\}$ is a $W$-type matrix;
4. For every $s \in S$, every infinite set $B \subseteq S$ contains a finite subset $F$ such that $\cap \{sb^{-1} : b \in F\} \setminus (\cap \{sb^{-1} : b \in B \setminus F\})$ and $\cap \{b^{-1}s : b \in F\} \setminus (\cap \{b^{-1}s : b \in B \setminus F\})$ are finite.
Proof. (1)⇔ (2). For all \( s \in S \), \( \chi_s \in wap(S) \) if and only if
\[
\{ \chi_s(x_ny_m) : n < m \} \cap \{ \chi_s(x_ny_m) : n > m \} \neq \emptyset.
\]
(3)⇒ (1) Let \( c_0(S) \not\subseteq wap(S) \) then there are sequences \( \{x_n\}, \{y_m\} \) in \( S \) with distinct elements such that for some \( s \in S \),
\[
1 = \lim_{m \to \infty} \lim_{n \to \infty} \chi_s(x_ny_m) \neq \lim_{n \to \infty} \lim_{m \to \infty} \chi_s(x_ny_m) = 0.
\]
Since \( \lim_n \lim_m \chi_s(x_ny_m) = 0 \), for \( 1 > \varepsilon > 0 \) there is a \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \lim_m \chi_s(x_ny_m) < \varepsilon \). This implies for all \( n \geq N \), \( \lim_m \chi_s(x_ny_m) = 0 \). Then for \( n \geq N \), \( 1 > \varepsilon > 0 \) there is a \( M_n \in \mathbb{N} \) such that for all \( m \geq M_n \) we have \( \chi_s(x_ny_m) < \varepsilon \). So if we omit finitely many terms, for all \( n \in \mathbb{N} \) there is \( M_n \in \mathbb{N} \) such that for all \( m \geq M_n \) we have \( x_ny_m \neq s \). As a similar argument, for all \( m \in \mathbb{N} \) there is \( N_m \in \mathbb{N} \) such that for all \( n \geq N_m \), \( x_ny_m = s \).

Let \( a_1 = x_1, b_1 \) be the first \( y_n \) such that \( a_1y_n = s \). Suppose \( a_m, b_n \) have been chosen for \( 1 \leq m, n < r \), so that \( a_nb_m = s \) if and only if \( n \geq m \). Pick \( a_r \) to be the first \( x_m \) not belonging to the finite set \( \cup_{1 \leq n \leq r} \{ x_m : x_my_n = s \} \). Then \( a_rb_n \neq s \) for \( n < r \). Pick \( b_r \) to be the first \( y_n \) belonging to the cofinite set \( \cap_{1 \leq n \leq r} \{ y_n : x_my_n = s \} \). Then \( a_nb_m = s \) if and only if \( n \geq m \). The sequences \( \{a_m\}, \{b_n\} \) so constructed satisfy \( a_nb_m = s \) if and only if \( n \geq m \). That is, \( \{a_nb_m\} \) is not of \( W \)-type and this is a contradiction.

(1)⇒ (3). Let there are sequences \( \{x_n\}, \{y_m\} \) in \( S \) such that \( \{x_ny_m\} \) is not a \( W \)-type matrix, (say) \( x_ny_m = s \) if and only if \( m \leq n \). Then
\[
1 = \lim_{m \to \infty} \lim_{n \to \infty} \chi_s(x_ny_m) \neq \lim_{n \to \infty} \lim_{m \to \infty} \chi_s(x_ny_m) = 0.
\]
So \( \chi_s \not\in wap(S) \). Thus \( c_0(S) \not\subseteq wap(S) \).

(4)⇔ (1) This is Ruppert criterion for \( \chi_s \in wap(S) \), see [16, Theorem 4].

**Example 3.7.**

(i) Let \( S \) be the interval \( [\frac{1}{2}, 1] \) with multiplication \( x \cdot y = \max\{\frac{1}{2}, xy\} \), where \( xy \) is the ordinary multiplication on \( \mathbb{R} \). Then for all \( s \in S \setminus \{\frac{1}{2}\}, x \in S \), \( x^\alpha s \) is finite. But \( x^{-\frac{1}{2}} = \left[\frac{1}{2}, \frac{1}{x}\right] \). Let \( B = \left[\frac{1}{2}, \frac{3}{4}\right] \). Then for all finite subset \( F \) of \( B \),
\[
\bigcap_{x \in F} x^{-\frac{1}{2}} \setminus \bigcap_{x \in B \setminus F} x^{-\frac{1}{2}} = \left[\frac{2}{3}, \frac{1}{2x_F}\right]
\]
where \( x_F = \max F \). By [16, Theorem 4] \( \chi_{\frac{1}{2}} \not\in wap(S) \). So \( c_0(S \setminus \{\frac{1}{2}\}) \oplus \mathbb{C} \not\subseteq wap(S) \). It can be readily verified that \( \epsilon : S \to S^{wap} \) is one to one, so \( \ell_1(S) \) is a WAP-algebra but \( c_0(S) \not\subseteq wap(S) \). This is a counter example for the converse of Corollary 3.4.

(ii) Take \( T = (\mathbb{N} \cup \{0\},.) \) with 0 as zero of \( T \) and the multiplication defined by
\[
n \cdot m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}
\]
Then $S = T \times T$ is a semigroup with coordinate wise multiplication. Now let $X = \{(k,0) : k \in T\}$, $Y = \{(0,k) : k \in T\}$ and $Z = X \cup Y$. We use the Ruppert criterion [16] to show that $\chi_z \notin \text{wap}(S)$, for each $z \in Z$. Let $B = \{(k,n) : k,n \in T\}$, then $(k, n)^{-1}(k,0) = \{(k,m) : m \neq n\} = B \setminus \{(k,n)\}$. Thus for all finite subsets $F$ of $B$,

$$\left(\bigcap \{(k,n)^{-1}(k,0) \mid (k,n) \in F\}\right) \setminus \left(\bigcap \{(k,0)(k,n)^{-1} : (k,n) \in F\}\right) = \left(\bigcap \{(k,n)^{-1}(k,0) : (k,n) \in B \setminus F\}\right)$$

and the last set is infinite. This means $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0}^{\infty} f(0,n)\chi_{(0,n)} + \sum_{m=0}^{\infty} f(m,0)\chi_{(m,0)}$ be in $\text{wap}(S)$. For arbitrary fixed $n$ and sequence $\{(n,k)\}$ in $S$, we have $\lim_k f(n,k) = \lim_k \lim_m f(n,k) = \lim_m \lim_k f(n,k) = f(n,0)$ implies $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(N \times N)$. Since $\text{wap}(S)$ can not separate the points of $S$ so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n, m) = 2^n 3^m$ for $(n, m) \in S$. Then $\omega$ is a weight on $S$ such that $\omega \in \text{wap}(S,1/\omega)$, so the evaluation map $\epsilon : S \to X$ is one to one. This means $\ell_1(S, \omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counter example for the converse of Corollary 3.3.

(iii) Let $S$ be the interval $[\frac{1}{2}, 1]$ with multiplication $x.y = \max\{\frac{1}{2}, xy\}$, where $xy$ is the ordinary multiplication on $\mathbb{R}$. Then for all $s \in S\\{\frac{1}{2}\}$, $x \in S$, $x^{-1}s$ is finite. But $x^{-1}\frac{1}{2} = [\frac{1}{2}, \frac{1}{2s}]$. Let $B = [\frac{1}{2}, \frac{3}{2}]$. Then for all finite subset $F$ of $B$,

$$\bigcap_{x \in F} x^{-1}\frac{1}{2} \setminus \bigcap_{x \in B \setminus F} x^{-1}\frac{1}{2} = \left[\frac{2}{3}, \frac{1}{2x_F}\right]$$

where $x_F = \max F$. By [16, Theorem 4] $\chi_S \notin \text{wap}(S)$. So $\chi_0(S \setminus \{\frac{1}{2}\}) \subset \subset \text{wap}(S)$. It can be readily verified that $\epsilon : S \to S^{\text{wap}}$ is one to one, so $\ell_1(S)$ is a WAP-algebra but $\chi_0(S) \not\subset \text{wap}(S)$. This is a counter example for the converse of Corollary 3.4.

(iv) Take $T = (\mathbb{N} \cup \{0\},.)$ with 0 as zero of $T$ and the multiplication defined by

$$n.m = \begin{cases} n & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Then $S = T \times T$ is a semigroup with coordinate wise multiplication. Now let $X = \{(k,0) : k \in T\}$, $Y = \{(0,k) : k \in T\}$ and $Z = X \cup Y$. We use the Ruppert criterion [16] to show that $\chi_z \notin \text{wap}(S)$, for each $z \in Z$. Let $B = \{(k,n) : k,n \in T\}$, then $(k, n)^{-1}(k,0) = \{(k,m) : m \neq n\} = B \setminus \{(k,n)\}$. Thus for all finite
subsets $F$ of $B$,
\[
(\cap \{(k,n)^{-1}(k,0) : (k,n) \in F\}) \setminus (\cap \{(k,n)^{-1}(k,0) : (k,n) \in B \setminus F\}) = (\cap \{(k,0)(k,n)^{-1} : (k,n) \in F\}) \\
\setminus (\cap \{(k,n)^{-1}(k,0) : (k,n) \in B \setminus F\}) = (B \setminus F) \setminus F = B \setminus F
\]
and the last set is infinite. This means $\chi_{(k,0)} \notin \text{wap}(S)$. Similarly $\chi_{(0,k)} \notin \text{wap}(S)$. Let $f = \sum_{n=0}^{\infty} f(0,n) \chi_{(0,n)} + \sum_{m=1}^{\infty} f(m,0) \chi_{(m,0)}$ be in $\text{wap}(S)$. For arbitrary fixed $n$ and sequence $\{(n,k)\}$ in $S$, we have $\lim_k f(n,k) = \lim_k \lim_l f(n,l,k) = \lim_l \lim_k f(n,l,k) = f(n,0)$ implies $f(n,0) = 0$. Similarly $f(0,n) = 0$ and $f(0,0) = 0$. Thus $f = 0$. In fact $\text{wap}(S) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Since $\text{wap}(S)$ can not separate the points of $S$ so $\ell_1(S)$ is not a WAP-algebra. Let $\omega(n,m) = 2^n 3^m$ for $(n,m) \in S$. Then $\omega$ is a weight on $S$ such that $\omega \in \text{wap}(S,1/\omega)$, so the evaluation map $\epsilon : S \rightarrow \tilde{X}$ is one to one. This means $\ell_1(S,\omega)$ is a WAP-algebra but $\ell_1(S)$ is not a WAP-algebra. This is a counterexample for the converse of Corollary 3.3.

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