Decomposition of nonholonomic mechanics models

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Abstract. In the paper, the problem of singularly perturbed differential systems decomposition by the method of integral manifolds is studied and the application of the method to the problems of non-holonomic mechanics is considered.

1. Introduction

When a non-holonomic model is used, the external force field is usually not completely specified. However, the degenerate (limiting) system is defined. In this case, the “experimental material” is used on certain quasi-speeds which are linear forms of generalized velocities with coefficients that depend on generalized coordinates. The original system of equations can be extended using the behavior of the solutions of the adjoint system [1, 2]. In practice, the first integral of the type of the law of conservation of energy always exists and, at the same time, there are also attractors, non-isolated equilibrium points, etc. The point is that a non-holonomic model is the initial approximation to the slow component on the integral (invariant) manifold of the original problem. But in the fast-slow system there is also the fast component of the solution, which, in particular, largely takes into account a dissipation, leaving the non-holonomic model only traces of its presence (sometimes, quite expressive). The classical use of a non-holonomic system of equations for the study of theoretical mechanics problems requires its justified application at an infinite or, at least, an extremely long time interval: the verification of the uniform continuous time dependence on initial conditions, the construction of the first integrals, the verification of the existence of an invariant measure, etc. For this goal the theory of integral manifolds can be applied [3]. In this paper, the model of Chaplygin sleigh [4] is considered as an example of the investigation of the non-holonomic problem by the methods integral manifolds. Some theoretical and applied results along these lines were obtained in[5, 6].

The main object of our consideration is the following system of differential equations:

\[ \dot{x} = f(x, y, t, \varepsilon), \]  
\[ \varepsilon \dot{y} = g(x, y, t, \varepsilon), \]

where \( x \) and \( f \) are vectors in Euclidean spaces \( \mathbb{R}^m \), \( y \) and \( g \) are vectors in \( \mathbb{R}^n \), \( t \in \mathbb{R} \), and \( \varepsilon \) is a small positive parameter.

The goals of the paper are to construct a transformation reducing (1)-(2) to the system

\[ \dot{v} = F(v, t, \varepsilon), \]
\[ \varepsilon \dot{z} = G(v, z, \varepsilon), \]

and to discuss some applications to the problems of non-holonomic mechanics.

2. Slow integral manifold

Suppose that system (1), (2) confirms with the following hypothesis.

I. Equation \( g(x, t, y, 0) = 0 \) has an isolated solution \( y = h_0(x, t) \) for \( t \in R, x \in R^m \).

II. The functions \( f, g, \) and \( h_0 \) are \((k + 2)\) times continuously differentiable \((k \geq 0)\) in
\[ \Omega_0 = \{(x, y, t, \varepsilon) : x \in R^m, \|y - h_0(x, t)\| < \rho, t \in R, 0 \leq \varepsilon \leq \varepsilon_0\} \]

III. The eigenvalues \( \lambda_i(x, t) \) \((i = 1, \ldots, n)\) of the matrix
\[ B(x, t) = \frac{\partial g}{\partial y}(x, h_0(x, t), t, 0) \]
satisfy the inequality
\[ Re \lambda_i(x, t) \leq -2\gamma < 0. \quad (3) \]

Under such assumptions the system (1), (2) has the slow integral manifold \( y = h(x, t, \varepsilon) \). The flow on this manifold is described by the \( m \)-dimensional system
\[ \dot{x} = f(x, h(x, t, \varepsilon), t, \varepsilon). \quad (4) \]

An exact calculations of \( h(x, t, \varepsilon) \) is generally impossible, and various approximations are necessary. One possibility is the asymptotic expansions of \( h(x, t, \varepsilon) \) in integer power of the small parameter \( \varepsilon \):
\[ h(x, t, \varepsilon) = h_0(x, t) + \varepsilon h_1(x, t) + \varepsilon^2 h_2(x, t) + \ldots \]

3. Fast integral manifold

Let us introduce a new variable \( z \) by the formula \( y = z + h(x, t, \varepsilon) \) to consider the behaviour in some neighbourhood of the slow integral manifold \( y = h(x, t, \varepsilon) \). Further, we introduce an additional variable \( v \), which satisfies the equation
\[ \dot{v} = f(v, h(v, t, \varepsilon), t, \varepsilon), \]

describing the flow on the slow integral manifold. On the next step, we introduce else one variable \( w \) by the formula \( x = v + w \). For \( z, v \) and \( w \) we have the differential system
\[ \dot{v} = F(v, t, \varepsilon), \quad (5) \]
\[ \dot{w} = W(v, w, z, t, \varepsilon), \quad (6) \]
\[ \varepsilon \dot{z} = B(v, t)z + Z(v, w, z, t, \varepsilon), \quad (7) \]

where
\[ F(v, t, \varepsilon) = f(v, h(v, t, \varepsilon), t, \varepsilon), \]
\[ B(v, t) = \frac{\partial g}{\partial y}(v, h_0(v, t), t, 0), \]
\[ Z(v, w, z, t, \varepsilon) = g(v + w, z + h(v + w, t, \varepsilon), t, \varepsilon) - B(v, t)z - \varepsilon \frac{\partial h}{\partial t}(v + w, t, \varepsilon) \]
\[ - \varepsilon \frac{\partial h}{\partial x}(v + w, t, \varepsilon)f(v + w, z + h(v + w, t, \varepsilon), t, \varepsilon), \]
\[ W(v, w, z, t, \varepsilon) = f(v + w, z + h(v, w, t, \varepsilon), t, \varepsilon) - F(v, t, \varepsilon). \]
The functions $Z$ and $W$ satisfy the inequalities
\[
|Z(v, w, z, t, \varepsilon)| \leq c\|z\|(\|z\| + \|w\| + \varepsilon),
\]
\[
|W(v, w, z, t, \varepsilon)| \leq c(\|z\| + \|w\|),
\]
\[
Z(v, w, z, t, \varepsilon) - Z(\bar{v}, \bar{w}, \bar{z}, t, \varepsilon)
\leq c[(\varepsilon + \|\dot{z}\| + \|\dot{w}\|)(\|\dot{z}\||\|v - \bar{v}\| + \|z - \bar{z}\|) + \|w - \bar{w}\|),
\]
\[
W(v, w, z, t, \varepsilon) - W(\bar{v}, \bar{w}, \bar{z}, t, \varepsilon)
\leq c[(\|\dot{z}\| + \|\dot{w}\|)|v - \bar{v}| + \|w - \bar{w}\| + \|z - \bar{z}\|],
\]
where
\[
\|\dot{z}\| = \max\{\|z\|, \|\dot{z}\|\}, \quad \|\dot{w}\| = \max\{\|w\|, \|\dot{w}\|\}.
\]

**Theorem 1** If all assumptions I–III hold for $k \geq 1$, then there exists such $\varepsilon_2$, $0 < \varepsilon_2 \leq \varepsilon_1$, than for all $\varepsilon \in (0, \varepsilon_2]$ system (5)–(7) possesses the integral manifold $w = \varepsilon H(v, z, t, \varepsilon)$, the flow on which is described by the differential system
\[
\dot{v} = F(v, t, \varepsilon),
\]
\[
\varepsilon \dot{z} = G(v, z, t, \varepsilon),
\]
\[
G(v, z, t, \varepsilon) = B(v, t)z + Z(v, \varepsilon H(v, z, t, \varepsilon), z, t, \varepsilon).
\]
In many cases $H$ can be found as an asymptotic expansion
\[
H(v, z, t, \varepsilon) = H_0(t, v, z) + \varepsilon H_1(t, v, z) + \varepsilon^2 H_2(t, v, x) + \ldots
\]
from the corresponding invariance equation
\[
\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial v} F(v, t, \varepsilon) + \frac{\partial H}{\partial z} [B(v, t)z + Z(v, \varepsilon H, z, t, \varepsilon)] = W(v, \varepsilon H, z, t, \varepsilon).
\]

**4. Splitting transformation**

Our main goal is the constructing of the transformation
\[
x = v + \varepsilon H(v, z, t, \varepsilon),
\]
\[
y = z + h(x, t, \varepsilon),
\]
which reduces the original system (1), (2) to the form
\[
\dot{v} = F(v, t, \varepsilon),
\]
\[
\varepsilon \dot{z} = G(v, z, t, \varepsilon).
\]
Let $(x(t), y(t))$ be a solution to (1), (2) with an initial condition $x(t_0) = x_0$, $y(t_0) = y_0$. There exists a solution $(v(t), z(t))$ of (14), (15) with the initial condition $v(t_0) = v_0$, $z(t_0) = z_0$, such that
\[
x(t) = v(t) + \varepsilon H(v(t), z(t), t, \varepsilon), \quad y(t) = z(t) + h(x(t), t, \varepsilon).
\]
It is sufficient to show that (16) takes place under $t = t_0$. Setting $t = t_0$ in (16) we obtain
\[
x_0 = v_0 + \varepsilon H(v_0, z_0, t_0, \varepsilon), \quad y_0 = z_0 + h(x_0, t_0, \varepsilon),
\]
and, therefore, \( z_0 = y_0 - h(x_0, t_0, \varepsilon) \).

For \( v_0 \) we have the equation

\[
v_0 = x_0 - H(v_0, z_0, t_0, \varepsilon) = V(v_0),
\]

which has the unique solution for any \( x_0 \in \mathbb{R}^m \) and fixed \( z_0 \) and \( t_0 \), where

\[
\|z_0\| = \|y_0 - h(x_0, t_0, \varepsilon)\| \leq \rho_2
\]

for some \( \rho_2 \).

The following statement is true.

**Theorem 2** Let all assumption of Theorem 1 are hold. Then there exist such numbers \( \varepsilon_2 \) and \( \rho_2 \) that for all \( \varepsilon \in (0, \varepsilon_2] \) any solution \( x = x(t, \varepsilon) \), \( y = y(t, \varepsilon) \) of system (1), (2) with the initial condition \( x(t_0, \varepsilon) = x_0 \), \( y(t_0, \varepsilon) = y_0 \), where \( \|y_0 - h(x_0, t_0, \varepsilon)\| \leq \rho_2 \), can be represented in form of (16).

This statement means that in the \( \rho_2 \)-neighbourhood of the slow integral manifold \( y = h(x, t, \varepsilon) \) of system (1), (2) can be reduced to the form (14), (15) by the splitting transformation (12), (13). Thus, system (1), (2) was split into two subsystems, the first of which is independent and contains a small parameter in a regular manner. Note that the initial value \( v_0 \) can be calculated from (17) in a form of an asymptotic expansion:

\[
v_0 = v_{00} + \varepsilon v_{01} + \varepsilon^2 v_{02} + \ldots
\]

For example, \( v_{00} = z_0 \), \( v_{01} = -H(x_0, z_{00}, t_0, 0) \), where \( z_{00} = y_0 - h(x_0, t_0) \).

Consider two simple examples with the splitting transformation.

**Example 1** The system

\[
\dot{x} = x, \quad \varepsilon \dot{y} = -y - x^2
\]

by the transformation

\[
x = v, \quad y = z - v^2/(1 + 2\varepsilon)
\]

is reduced to the form

\[
\dot{v} = v, \quad \varepsilon \dot{z} = -z.
\]

**Example 2** The system

\[
\dot{x} = y, \quad \varepsilon \dot{y} = -y - y^2
\]

by the transformation

\[
x = v - \varepsilon \ln(1 + z), \quad y = z
\]

is reduced to the form

\[
\dot{v} = 0, \quad \varepsilon \dot{z} = -z - z^2.
\]

5. Systems that are linear with respect to fast variables

5.1. General case

Consider the differential system

\[
\dot{x} = f_0(x, t, \varepsilon) + F_1(x, t, \varepsilon)y,
\]

\[
\varepsilon \dot{y} = g_0(x, t, \varepsilon) + G_1(x, t, \varepsilon)y,
\]

where \( x \in \mathbb{R}^m \), \( y \in \mathbb{R}^n \), \( t \in \mathbb{R} \).
Suppose that the following representations take place

\[ F_1(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j F_{1,j}(x, t), \quad G_1(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j G_{1,j}(x, t), \]

\[ f_0(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j f_{0,j}(x, t), \quad g_0(x, t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j g_{0,j}(x, t). \]

Here, \( G_{1,0} = G_{1,0}(x, t) \) plays the role of matrix \( B(x, t) \). The formulae for the coefficients of asymptotic expansions of slow integral manifold \( h = h(x, t, \varepsilon) \) take the form

\[ h_0 = G_{1,0}^{-1} g_{0,0}, \]

\[ h_k = G_{1,0}^{-1} \left[ \frac{\partial h_{k-1}}{\partial t} + \frac{k-1}{k} \frac{\partial h_{p}}{\partial x} (f_{0,k-1-p} + \sum_{j=0}^{k-1-p} F_{1,j} h_{k-p-1-j}) \right] - g_{0,k} - \sum_{j=1}^{k} G_{1,j} h_{k-j}, \quad k \geq 1. \]  

(20)

The invariance equation (11) for the fast integral manifold \( H = H(v, z, t, \varepsilon) \) in this case takes the form

\[ \varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial v} [f_0(v, t, \varepsilon) + F_1(v, t, \varepsilon) h(v, t, \varepsilon)] + \frac{\partial H}{\partial z} G_1(v + \varepsilon H, t, \varepsilon) \]

\[ - \varepsilon \frac{\partial h}{\partial x} (v + \varepsilon H, t, \varepsilon) F_1(v + \varepsilon H, t, \varepsilon) z = f_0(v + \varepsilon H, t, \varepsilon) - f_0(v, t, \varepsilon) \]

\[ + F_1(v + \varepsilon H, t, \varepsilon) (z + h(v + \varepsilon H, t, \varepsilon)) - F_1(v, t, \varepsilon) h(v, t, \varepsilon). \]

Setting \( \varepsilon = 0 \), we obtain

\[ \frac{\partial H_0}{\partial z} G_{1,0}(v, t) z = F_{1,0}(v, t) z. \]

It is possible to represent \( H_0(v, t, z) \) in the form \( H_0(v, t, z) = D_0(v, t) z \), where matrix \( D_0(v, t) \) satisfies the equation

\[ D_0(v, t) G_{1,0}(v, t) = F_{1,0}(v, t), \]

and, therefore,

\[ H_0(v, t, z) = F_{1,0}(v, t) G_{1,0}^{-1}(v, t) z. \]

Neglecting terms of order \( o(\varepsilon) \), we use the transformation

\[ x = v + \varepsilon H_0(v, z, t), \quad y = z + h_0(x, t) + \varepsilon h_1(x, t) \]

(21)

to reduce system (18) to a nonlinear block triangular form:

\[ \dot{v} = f_{0,0}(v, t) + F_{1,0}(v, t) h_0(v, t) + \varepsilon [f_{0,1}(v, t) + F_{1,0}(v, t) h_1(x, t) + F_{1,1}(v, t) h_0(v, t)] + O(\varepsilon^2), \]

\[ \varepsilon \dot{z} = [G_{1,0}(v, t) + \varepsilon (G_{1,1}(v, t) + \frac{\partial G_{1,0}}{\partial x}(v, t) H_0(v, z, t)) \]

\[ - \frac{\partial h_0}{\partial x}(v, t) F_{1,0}(v, t)] z + O(\varepsilon^2). \]  

(22)
5.2. Partial cases

5.2.1. Autonomous systems. If the r.h.s. of (18) do not contain the variable \( t \) then the formulae (20) take forms

\[
\begin{align*}
\up{h}_0 &= G_{1,0}^{-1} g_{0,0}, \\
\up{h}_k &= G_{1,0}^{-1} \sum_{p=0}^{k-1} \frac{\partial \up{h}_{p}}{\partial \up{x}} (f_{0,k-1-p} + \sum_{j=0}^{k-1-p} F_{1,j} h_{k-p-1-j}) \\
&- g_{0,k} - \sum_{j=1}^{k} G_{1,j} h_{k-j}], \quad k \geq 1.
\end{align*}
\]

The invariance equation (11) for \( H = H(v,z,\varepsilon) \) in this case takes the form

\[
\begin{align*}
\varepsilon \frac{\partial H}{\partial v} [f_0(v,\varepsilon) + F_1(v,\varepsilon) h(v,\varepsilon)] &+ \frac{\partial H}{\partial z} [G_1(v + \varepsilon H,\varepsilon) \\
- \varepsilon \frac{\partial h}{\partial x} (v + \varepsilon H,\varepsilon) F_1(v + \varepsilon H,\varepsilon)] z = f_0(v + \varepsilon H,\varepsilon) - f_0(v,\varepsilon) \\
+ &F_1(v + \varepsilon H,\varepsilon) (z + h(v + \varepsilon H,\varepsilon)) - F_1(v,\varepsilon) h(v,\varepsilon).
\end{align*}
\]

Setting \( \varepsilon = 0 \), we obtain

\[
\frac{\partial H_0}{\partial z} G_{1,0}(v) z = F_{1,0}(v) z.
\]

Thus,

\[
H_0(v,z) = F_{1,0}(v) G_{1,0}^{-1}(v) z.
\]

The transformation (21) can be represented as

\[
x = v + \varepsilon H_0(v,z), \quad y = z + h_0(x) + \varepsilon h_1(x).
\]

In this way, we obtain system (22) in the form

\[
\begin{align*}
\dot{v} &= f_{0,0}(v) + F_{1,0}(v) h_0(v) + \varepsilon [f_{0,1}(v) \\
&+ F_{1,0}(v) h_1(v) + F_{1,1}(v) h_0(v)] + O(\varepsilon^2), \\
\varepsilon \dot{z} &= [G_{1,0}(v) + \varepsilon G_{1,1}(v) + \frac{\partial G_{1,0}}{\partial x}(v) H_0(v,z) \\
&- \frac{\partial h_0}{\partial x}(v) F_{1,0}(v)] z + O(\varepsilon^2).
\end{align*}
\]

5.2.2. Autonomous weakly perturbed systems. Let in (18) matrices \( F_1, G_{1,0} \) are constant and

\[
G_1 = G_{1,0} + \varepsilon G_{1,1}(x),
\]

then formulae (20) take the form

\[
\begin{align*}
\up{h}_0 &= G_{1,0}^{-1} g_{0,0}, \\
\up{h}_k &= G_{1,0}^{-1} \sum_{p=0}^{k-1} \frac{\partial \up{h}_{p}}{\partial \up{x}} (f_{0,k-1-p} + F_1 h_{k-p-1}) \\
&- g_{0,k} - G_{1,1} h_{k-1}], \quad k \geq 1.
\end{align*}
\]
For $H_0(z)$ we obtain

$$H_0(z) = F_1G_{1,0}^{-1}z.$$  

The transformation (24) has the form

$$x = v + \varepsilon H_0(z) + O(\varepsilon^2), \quad y = z + h_0(x) + \varepsilon h_1(x) + O(\varepsilon^2).$$  

As the result, (25) is converted to the following system:

$$\dot{v} = f_0,0(v) + F_1 h_0(v) + \varepsilon [f_0,1(v) + F_1 h_1(v)] + O(\varepsilon^2),$$

$$\varepsilon \dot{z} = [G_1,0 + \varepsilon (G_1,1(v) - \frac{\partial h_0}{\partial x}(v)F_1)]z + O(\varepsilon^2).$$  

(28)

6. Equations of non-holonomic systems

Consider the differential system

$$\dot{x} = f_0(x) + \varepsilon f_1(x) + (F_0(x) + \varepsilon F_1(x))y,$$

$$\varepsilon \dot{y} = g_0(x) + \varepsilon g_1(x) + (G_0(x) + \varepsilon G_1(x))y,$$

(29)

where $x \in R^m$, $y \in R^n$, $t \in R$, which appears under the modelling of some non-holonomic mechanic systems.

This system may be reduced to the form

$$\dot{v} = F(v, \varepsilon),$$

$$\varepsilon \dot{z} = G(v, z, \varepsilon),$$

(30)

(31)

by the transformntion

$$x = v + \varepsilon H(v, z, \varepsilon),$$

$$y = z + h(x, \varepsilon).$$

(32)

(33)

Here,

$$h(x, \varepsilon) = h_0(x) + \varepsilon h_1(x) + O(\varepsilon^2),$$

$$H(v, z, \varepsilon) = H_0(v, z) + O(\varepsilon),$$

$$F(v, \varepsilon) = f_0(x) + F_0(x)h_0(x) + \varepsilon F_0(x)h_1(x) + \varepsilon F_1(x)h_0(x) + \varepsilon f_1(x) + O(\varepsilon^2),$$

$$G(v, z, \varepsilon) = [G_0(v) + \varepsilon G_1(v) + \varepsilon G_0z(v)H_0(v, z) - \varepsilon \frac{\partial h_0}{\partial x}(v) + O(\varepsilon^2)]z,$$

where

$$h_0(x) = -G_0^{-1}(x)g_0(x), \quad h_1(x) = G_0^{-1}(x)[\frac{\partial h_0}{\partial x}(f_0(x) + F_0(x)h_0(x)) - g_0(x) - G_0(x)h_0(x)],$$

$$H_0(v, z) = F_0(v)G_0^{-1}(v)z.$$
7. Chaplygin sleigh

As an example of the investigation of the non-holonomic problem by use of the integral manifolds methods, we can consider the problem of S.A. Chaplygin [4].

Let $x, y$ are the coordinates of the center of mass, which lies on the axis located in the plane of the wheel; $\varphi$ is the angle of rotation of the body; $\omega = \dot{\varphi}$; $h$ is the distance from the point of touch of the pointed wheel to the center of mass. The kinetic energy of the body is described by the equation

$$T = \frac{1}{2}m(x^2 + y^2) + \frac{1}{2}I_c\dot{\varphi}^2.$$

Here, $m$ is the mass and $I_c$ is the moment of inertia. Taking into account the not completely known reaction $F$ of the underlying surface, which is orthogonal to the wheel plane, the motion equations have the form

$$m\ddot{x} = -F\sin(\varphi),$$

$$m\ddot{y} = F\cos(\varphi),$$

$$I_c\dot{\omega} = -Fh.$$

The variables transformation

$$u = x\cos(\varphi) + y\sin(\varphi),$$

$$v = -x\sin(\varphi) + y\cos(\varphi) - h\varphi,$$

$$\omega = \dot{\varphi}$$

gives

$$m\dot{u} = m\omega(v + h\omega),$$

$$m\dot{v} = \frac{I_c + mh^2}{I_c}F - m\omega u,$$

$$I_c\dot{\omega} = -Fh.$$

Here, $u$ and $v$ are the longitudinal and transverse quasi-speeds at the point of contact of the wheel with the surface. In accordance with the usual condition, we set $v = \varepsilon V$, $|V| \leq C$, $\varepsilon << 1$ and obtain

$$\dot{u} = \omega(\varepsilon V + h\omega),$$

$$\varepsilon V = \frac{I_c + mh^2}{mI_c}F - \omega u,$$

$$I_c\dot{\omega} = -Fh.$$

Setting $V = -\frac{F}{m}$, we have

$$\dot{u} = \omega(\varepsilon V + h\omega),$$

$$I_c\dot{\omega} = -Fh,$$

$$\varepsilon \dot{F} = -\frac{I_c + mh^2}{mI_c}F + m\omega u.$$

This 3D-system is linear with respect to the fast variable $F$ and we can calculate

$$h_0(u, \omega) = \frac{I_c m\omega u}{I_c + mh^2},$$

$$h_1(u, \omega) = \frac{I_c^2}{(I_c + mh^2)^2}hm\omega \left[-\omega^2 + \frac{mu^2}{I_c + mh^2}\right].$$
\[ H_0(z) = \text{col} \left( 0, \frac{h}{I_c + mh^2} \right) z, \]

and to reduce this 3D-system by the transformation

\[ F = z + h_0(u, \omega) + \varepsilon h_1(u, \omega), \]

\[ u = v_1, \quad \omega = v_2 + \varepsilon \frac{h}{I_c + mh^2} z, \]

to the following two subsystems: the slow subsystem

\[ \dot{v}_1 = mh v_1^2 - \varepsilon v_2 \frac{I_c m v_1 v_2}{I_c + mh^2} + O(\varepsilon^2), \]

\[ \dot{v}_2 = -\frac{h}{I_c} \left[ \frac{I_c m v_1 v_2}{I_c + mh^2} + \varepsilon h_1(v_1, v_2) \right] + O(\varepsilon^2), \]

and the fast subsystem

\[ \varepsilon \dot{z} = -\left[ \frac{I_c + mh^2}{I_c} - \varepsilon \frac{mh}{I_c + mh^2} v_1 + O(\varepsilon^2) \right] z. \]

The corresponding initial conditions are

\[ z(0) = F(0) - h_0(u(0), \omega(0)) - \varepsilon h_1(u(0), \omega(0)) + O(\varepsilon^2), \]

\[ v_1(0) = u(0), \quad v_2(0) = \omega(0) - \varepsilon \frac{h}{I_c + mh^2} z(0). \]

Thus, the problem of decomposition of the Chaplygin sleigh model is solved. Note that the developed mathematical apparatus can also be used for solving some problems of computer optics [7-10].

8. References

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Acknowledgments

A. Kobrin was funded by the Russian Foundation for Basic Research (grant 16-01-00429). V. Sobolev was supported by the Russian Foundation for Basic Research and the Government of the Samara Region (grant 16-41-630524) and the Ministry of Education and Science of the Russian Federation under the Competitiveness Enhancement Program of Samara University (2013-2020).