Spending Is Not Easier Than Trading:
On the Computational Equivalence of
Fisher and Arrow-Debreu Equilibria

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Abstract. It is a common belief that computing a market equilibrium in Fisher’s spending model is easier than computing a market equilibrium in Arrow-Debreu’s exchange model. This belief is built on the fact that we have more algorithmic success in Fisher equilibria than Arrow-Debreu equilibria. For example, a Fisher equilibrium in a Leontief market can be found in polynomial time, while it is PPAD-hard to compute an approximate Arrow-Debreu equilibrium in a Leontief market.

In this paper, we show that even when all the utilities are additively separable, piecewise-linear and concave, computing an approximate equilibrium in Fisher’s model is PPAD-hard. Our result solves a long-term open question on the complexity of market equilibria. To the best of our knowledge, this is the first PPAD-hardness result for Fisher’s model.

1 Introduction

1.1 Market Equilibria: Fisher’s Model vs. Arrow-Debreu’s Model

In 1891, Irving Fisher introduced one of the most fundamental exchange market models in his Ph.D. thesis \cite{fisher}. It considers a market in which there are $m$ buyers and $n$ divisible goods. We denote the amount of good $j$, $j \in [n]$, in the market by $c_j > 0$. Every buyer $i$ comes to the market with a certain amount of money, denoted by $w_i > 0$. The goal of a buyer is to obtain a bundle of goods, denoted by $a_i \in \mathbb{R}^n_+$, that maximizes her utility function $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$.

Fisher showed that if all the utility functions $u_i$ satisfy some mild conditions then there always exists an equilibrium price vector $p \in \mathbb{R}^n_+$. At this price, one can find a bundle of goods $a_i$ for each buyer $i$ such that $a_i$ maximizes her utility under the budget constraint that $a_i \cdot p \leq w_i$, and at the same time, the market demands equal to the market supply: $\sum_{i \in [m]} a_{i,j} \leq c_j$ for all $j \in [n]$.

Fisher’s model is a special case of the more general model of exchange economies considered by Arrow and Debreu \cite{arrow-debreu}: In an exchange economy there are

\textsuperscript{1} Recently, Vazirani and Yannakakis independently proved that the problem of computing an approximate Fisher equilibrium in a market with additively separable and PLC utility functions is PPAD-hard \cite{vazirani-yannakakis}. They also showed that the problem of finding an exact Arrow-Debreu equilibrium in such markets is in PPAD and thus, both problems are PPAD-complete.
m traders and n divisible goods. Trader i has an initial endowment of \( w_{i,j} \geq 0 \) of good j, and a utility function \( u_i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \). The individual goal of a trader is to obtain a new bundle of goods that maximizes her utility.

In a sense, Fisher’s model focuses more on spending than trading as in Arrow-Debreu’s model. In his model, money can be viewed as a special kind of good. All but one “special” trader only have money as their endowments, and money has no value to their utilities; the special trader, sometime called the “market”, has all the goods and her interest is to collect all the money.

Over the last two decades we have more algorithmic success in computing a market equilibrium in Fisher’s model than computing a market equilibrium in Arrow-Debreu’s model.

- For the latter, polynomial-time algorithms are only known for markets with utility functions that are linear \([4,5,6,7,8,9,10,11,12]\) or satisfy weak gross substitutability \([13]\). These algorithms critically used the fact that the set of equilibria of these markets is convex. Progress on markets with non-convex set of equilibria has been relatively slow. There are only a few algorithms in this case. Devanur and Kannan \([14]\) gave a polynomial-time algorithm for markets with piecewise-linear and concave (PLC) utilities and a constant number of goods. Codenotti, McCune, Penumatcha, and Varadarajan \([15]\) gave a polynomial-time algorithm for CES markets when the elasticity of substitution \( s \geq 1/2 \).

For Leontief markets, in which each utility function is of the form \( \min_j a_j x_j \), the problem of finding an approximate Arrow-Debreu equilibrium is known to be PPAD-hard \([16,17,18]\). In \([19]\), Chen et al. proved that finding an approximate Arrow-Debreu equilibrium, even if all the utilities are additively separable \([3]\) and PLC, is PPAD-hard. Recently, Vazirani and Yannakakis \([1]\) showed that the problem of computing an exact Arrow-Debreu equilibrium in such markets is a member of PPAD and thus, is complete in PPAD.

- For Fisher’s model, polynomial-time algorithms are given not only for linear markets but also for Leontief and many other markets, e.g., the hybrid linear Leontief markets \([20]\). We know that an (approximate) market equilibrium in any Fisher’s economy with CES utilities can be found in polynomial time \([4,15,12,21,17,22]\). In fact, Ye \([21]\) proved that if every utility function is the minimum of a collection of homogeneous linear functions, then one can find a Fisher equilibrium in polynomial time.

### 1.2 Our Results

It remains open whether there is a family of concave utility functions for which it is PPAD-hard to compute a Fisher equilibrium. The family of utility functions that has drawn most attention is the additively separable, piecewise-linear, and concave (PLC) functions. In \([23]\), Vazirani remarked that obtaining a polynomial-time algorithm for markets with additively separable and concave utilities is

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2 A function \( u(x_1, \ldots, x_m) \) from \( \mathbb{R}_+^m \) to \( \mathbb{R}_+ \) is *additively separable* if there exist m real-valued functions \( f_1, \ldots, f_m \) such that \( u(x_1, \ldots, x_m) = \sum_{j=1}^m f_j(x_j) \).