Standard Character Condition for C-algebras

J. Bagherian and A. Rahnamai Barghi

Institute for Advanced Studies in Basic Sciences
P.O. Box: 45195-1159, Zanjan-Iran

bagherian@iasbs.ac.ir
rahnama@iasbs.ac.ir

March 17, 2008

Abstract

It is well known that the adjacency algebra of an association scheme has the standard character. In this paper we first define the concept of standard character for C-algebras and we say that a C-algebra has the standard character condition if it has the standard character. Then we investigate some properties of C-algebras which have the standard character condition and prove that under some conditions a C-algebra has an adjacency algebra homomorphic image. In particular, we obtain a necessary and sufficient condition for which a commutative table algebra comes from an association scheme.

Key words: cellular algebra; C-algebra; table algebra; standard character.
AMS Classification: 20C99; 16G30; 05E30.

1 Introduction

A table algebra is a C-algebra with nonnegative structure constants was introduced by [2]. As a folklore example, the adjacency algebra of an association scheme (or homogeneous coherent configuration) is an integral table algebra. On the other hand, the adjacency algebra of an association scheme has a special character which is called the standard character, see [10]. We generalize the concept of standard character from adjacency algebras to C-algebras. This generalization enables us to find a necessary and sufficient condition for which a commutative table algebra comes from an association scheme.

In section 2 we recall the concept of C-algebras and some related properties which we will use in this paper.

*Corresponding author: rahnama@iasbs.ac.ir
In section \(3\) we first define the standard feasible trace for C-algebras which is a generalization of the standard character in the theory of association schemes. Thereafter, we show that the standard feasible multiplicities of the characters of a table algebra and its quotient are the same. Furthermore, we prove that the set of standard feasible multiplicities preserve under C-algebras isomorphism.

In section \(4\) we give an example of C-algebra for which the standard feasible trace is a character, such character is called the standard character. By using the standard character we obtain a necessary and sufficient condition for which a commutative table algebra comes from an association scheme.

## 2 Preliminaries

Although in algebraic combinatorics the concept of C-algebra is used for commutative algebras, in this paper we will also consider non-commutative algebras. Hence we deal with non-commutative C-algebras in the sense of [7] as the following:

Let \(A\) be a finite dimensional associative algebra over the complex field \(\mathbb{C}\) with the identity element \(1_A\) and a base \(B\) in the linear space sense. Then the pair \((A, B)\) is called a non-commutative C-algebra if the following conditions (I)-(IV) hold:

(I) \(1_A \in B\) and the structure constants of \(B\) are real numbers, i.e., for \(a, b \in B\):

\[
ab = \sum_{c \in B} \lambda_{abc}c, \quad \lambda_{abc} \in \mathbb{R}.
\]

(II) There is a semilinear involutory anti-automorphism (denoted by \(\ast\)) of \(A\) such that \(B^\ast = B\).

(III) For \(a, b \in B\) the equality \(\lambda_{ab1_A} = \delta_{ab}\ast |a|\) holds where \(|a| > 0\) and \(\delta\) is the Kronecker symbol.

(IV) The mapping \(b \to |b|, b \in B\) is a one dimensional \(\ast\)-linear representation of the algebra \(A\), which is called the degree map.

**Remark 2.1.** In the definition above we should mention that if the algebra \(A\) is commutative, then \((A, B)\) becomes a C-algebra in the sense of [4].

If the structure constants of a given C-algebra (resp. commutative) are nonnegative real numbers, then it is called a table algebra (resp. commutative) in the sense of [2] (resp. [1]).

Throughout this paper a C-algebra (resp. table algebra) will mean a non-commutative C-algebra (resp. non-commutative table algebra).

A C-algebra (table algebra) is called integral if all its structure constants \(\lambda_{abc}\) are integers. The value \(|b|\) is called the degree of the basis element \(b\). From condition (IV) we see that \(|b| = |b^\ast|\) for all \(b \in B\), and from condition (II) for \(a = \sum_{b \in B} x_b^b\) we have \(a^\ast = \sum_{b \in B} x_b^b b^\ast\), where \(x_b^b\) means the complex conjugate to \(x_b\). This implies that the Jacobson radical \(J(A)\) of the algebra \(A\) is equal to \(\{0\}\) which means \(A\) is semisimple.
Let \((A, B)\) and \((A', B')\) be two C-algebras. A C-algebra homomorphism from \((A, B)\) to \((A', B')\) is an \(*\)-algebra homomorphism \(f : A \to A'\) such that \(f(B) = B'\). Such C-algebra homomorphism is called C-algebra epimorphism (resp. monomorphism) if \(f\) is onto (resp. into). A C-algebra epimorphism \(f\) is called a C-algebra isomorphism if \(f\) is monomorphism too. Two C-algebras \((A, B)\) and \((A', B')\) are called isomorphic, if there exists a C-algebra isomorphism between them.

Given a table algebra \((A, B)\), the bilinear form \(\langle \cdot, \cdot \rangle\) on \(A\) is defined in [2] by setting \(\langle x, y \rangle = t(xy^*)\), for \(x, y \in A\), where \(t : A \to \mathbb{C}\) is a linear function defined by \(t(\sum_{b \in B} x_b b) = x_{1A}\). Then one can see that \(\langle \cdot, \cdot \rangle\) is a Hermitian positively definite form on \(A\).

A nonempty subset \(C \subseteq B\) is called a closed subset, if \(C^*C \subseteq C\). We denote by \(C(B)\) the set of all closed subsets of \(B\).

Let \((A, B)\) be a table algebra with the basis \(B\) and let \(C \in C(B)\). From [3, Proposition 4.7], it follows that \(\{CbC \mid b \in B\}\) is a partition of \(B\). A subset \(CbC\) is called a C-double coset or double coset with respect to the closed subset \(C\). Let

\[ b/C := \frac{|C^+|^{-1}}{\sum_{x \in CbC} x} \]

where \(C^+ = \sum_{c \in C} c\) and \(|C^+| = \sum_{c \in C} |c|\). Then the following theorem is an immediate consequence of [3, Theorem 4.9]:

**Theorem 2.2.** Let \((A, B)\) be a table algebra and let \(C \in C(B)\). Suppose that \(\{b_1 = 1_A, \ldots, b_k\}\) be a complete set of representatives of C-double cosets. Then the vector space spanned by the elements \(b_i/C\), \(1 \leq i \leq k\), is a table algebra (which is denoted by \(A/C\)) with a distinguished basis \(B/C = \{b_i/C \mid 1 \leq i \leq k\}\). The structure constants of this algebra are given by the following formula:

\[ \gamma_{ijk} = |C^+|^{-1} \sum_{r \in Cb_iC, s \in Cb_jC} \lambda_{rst} \]

where \(t \in Cb_kC\) is an arbitrary element.

The table algebra \((A/C, B/C)\) is called the quotient table algebra of \((A, B)\) modulo \(C\).

We refer the reader to [15] for the background of association schemes.

### 3 The standard feasible trace for C-algebras

In this section we first define the standard feasible trace for C-algebras and then we show that the standard feasible multiplicities of the characters of a table algebra and its quotient are the same. Furthermore, we prove that the set of standard feasible multiplicities preserve under C-algebras isomorphism.

Let \((A, B)\) be a C-algebra and let \(\text{Irr}(A)\) be the set of irreducible characters of \(A\). We define a linear function \(\zeta \in \text{Hom}_C(A, \mathbb{C})\) by \(\zeta(b) = \delta_{1A}|B^+|\), for \(b \in B\), where
It is easily seen that $\zeta(bc) = \zeta(cb)$, for all $b, c \in B$. This shows that $\zeta$ is a feasible trace in the sense of [11]. In addition, since $\text{rad}(\zeta) = \{0\}$, where $\text{rad}(\zeta) = \{x \in A : \zeta(xy) = 0, \forall y \in A\}$, it is a non-degenerate feasible trace on $A$. Therefore, from [11] it follows that $\zeta = \sum_{\chi \in \text{Irr}(A)} \zeta_{\chi} \chi$ where $\zeta_{\chi} \in C$ and all $\zeta_{\chi}$ are non-zero. We call $\zeta$ the standard feasible trace, $\zeta_{\chi}$ the standard feasible multiplicity of $\chi$ and $\{\zeta_{\chi} | \chi \in \text{Irr}(A)\}$, the set of standard feasible multiplicities of the $C$-algebra $(A, B)$.

Since $A$ is a semisimple algebra

$$A = \bigoplus_{\chi \in \text{Irr}(A)} A\varepsilon_{\chi}$$

where $\varepsilon_{\chi}$’s are the central primitive idempotents.

**Lemma 3.1.** (i) Let $\chi \in \text{Irr}(A)$. Then

$$\varepsilon_{\chi} = \frac{1}{|B^+|} \sum_{b \in B} \zeta_{\chi}\chi(b^*) \frac{b}{|b^*|}.$$  \hfill (1)

(ii) (Orthogonality Relation) For every $\phi, \psi \in \text{Irr}(A)$

$$\frac{1}{|B^+|} \sum_{b \in B} \frac{1}{|b^*|} \phi(b^*) \psi(b) = \delta_{\phi\psi} \frac{\phi(1)}{\zeta_{\phi}}.$$  \hfill (2)

(iii) In commutative case, for every $b, c \in B$

$$\sum_{\chi \in \text{Irr}(A)} \zeta_{\chi}\chi(b)\chi(c^*) = \delta_{bc} |b||B^+|.$$  

**Proof.** Part (i) and (iii) follow from [11, 5.7] and [11, 5.5′], respectively, by using the concept of dual basis relative to a non-degenerate feasible trace, indeed the dual basis of $b$ relative to standard feasible trace $\zeta$ is $b^*$ for $b \in B$. Part (ii) follows from equality $\varepsilon_{\phi}\varepsilon_{\psi} = \delta_{\phi\psi}\varepsilon_{\phi}$ by replacing $b^*$ by $1_A$.  

**Remark 3.2.** From [11] one can see that in commutative case, $\zeta_{\chi}$ is the coefficient of $1_A$ in the linear combination of $|B^+|\varepsilon_{\chi}$ in terms of the basis elements of $B$.

Let $(A, B)$ be a table algebra and $C \in \mathcal{C}(B)$. Set $e = |C^+|^{-1}C^+$. Then $e$ is an idempotent for the table algebra $A$ and the subalgebra $eAe$ denoted by $H$, is equal to the quotient table algebra $(A/C, B/C)$ modulo $C$, see [3]. Let $\zeta$ be the standard feasible trace of the table algebra $(A, B)$. Then $\zeta|_H$ is the standard feasible trace for $(A/C, B/C)$. Indeed, assume that $T \subseteq B$ be a complete set of representatives of $C$-double cosets of $A$. Then $B = \bigcup_{b \in T} CbC$ and $|C^+|^{-1}|B^+| = \sum_{b \in T} |b/C|$. Since

$$\zeta|_H(b/C) = \begin{cases} |C^+|^{-1}|B^+|, & \text{if } b = 1_A \\ 0, & \text{if } b \neq 1_A \end{cases}$$

it follows that $\zeta|_H$ is the standard feasible trace for $(A/C, B/C)$. Thus we proved the following lemma:
Lemma 3.3. Let \((A, B)\) be a table algebra with the standard feasible trace \(\zeta\) and let \(C \in \mathcal{C}(B)\). Then \(\zeta|_H\) is the standard feasible trace of \((A/C, B/C)\).

In the following we will show that the standard feasible multiplicities of the characters of a table algebra and its quotient are the same. For this, we need to observe a relationship between the characters of a table algebra and its quotient. The next three theorems and corollaries are proved for adjacency algebra of an association scheme, see [9]. Now we generalize them for table algebras.

Theorem 3.4. Let \((A, B)\) be a table algebra and let \(P = \{\varepsilon \chi \mid \chi \in \text{Irr}(A)\}\) be the set of central primitive idempotents of \((A, B)\). Then \(P_C = \{e\varepsilon \chi \mid \chi \in \text{Irr}(A)\} \setminus \{0\}\) is the set of central primitive idempotents of the quotient table algebra \((A/C, B/C)\) where \(C \in \mathcal{C}(B)\) and \(e = |C^+|^{-1}C^+\).

Proof. Suppose that \(\varepsilon \chi \in P\) such that \(e\varepsilon \chi \neq 0\). Then the algebra \(\varepsilon \chi A\) is isomorphic to \(\text{End}_A(V)\) where \(V = \varepsilon \chi A\). Let \(T\) be the image of the idempotent \(e\varepsilon \chi\) with respect to this isomorphism. From [12, Theorem 5.4], it follows that the algebra \(e\varepsilon \chi H\) is isomorphic to \(\text{End}_A(TV)\). Since the latter algebra is simple, so \(e\varepsilon \chi\) is a central primitive idempotent of the algebra \(H\). On the other hand, since \(e\) is the unit element of \(H\) and \(e = \sum e\varepsilon \chi\) where \(e\varepsilon \chi\) runs over the set \(\{e\varepsilon \chi \mid \chi \in \text{Irr}(A)\} \setminus \{0\}\), we conclude that \(P_C\) is the set of all central primitive idempotents of the quotient table algebra \((A/C, B/C)\) and we are done.

Corollary 3.5. There is a one to one correspondence between \(\{\chi \in \text{Irr}(A) \mid \chi|_H \neq 0\}\) and \(\text{Irr}(H)\) by the map \(\chi \to \chi|_H\).

Proof. This is an immediate consequence of Theorem 3.4.

Corollary 3.6. Let \((A, B)\) be a table algebra and \(C \in \mathcal{C}(B)\). Then every irreducible \(A\)-module \(V\) is an irreducible \(H\)-module iff \(\dim_C(eV) \neq 0\), where \(e = |C^+|^{-1}C^+\).

Proof. Let \(V\) be an irreducible \(A\)-module and let \(D\) be a matrix representation of \(A\) defined by \(V\). Since \(e\) is an idempotent, rank \(D(e) = \chi(e)\), where \(\chi\) is the irreducible character afforded by \(D\). On the other hand, rank \(D(e) = \dim_C(eV)\). Hence

\[
\chi(e) = \dim_C(eV). \quad (3)
\]

We first suppose that \(V\) is an irreducible \(H\)-module. Then \(\chi|_H \neq 0\), and so \(\chi(e) \neq 0\). Thus equality (3) implies that \(\dim_C(eV) \neq 0\).

Conversely, let \(\dim_C(eV) \neq 0\). From equality (3) and Corollary 3.5, we deduce that \(\chi|_H\) is an irreducible character of \(H\). Thus \(V\) is an irreducible \(H\)-module and we are done.

The theorem below gives the relationship between the standard feasible multiplicity of a character of a table algebra \((A, B)\) and the quotient table algebra \((A/C, B/C)\).

Theorem 3.7. The standard feasible multiplicity of \(\chi|_H\) is equal to that of \(\chi\) if \(\chi|_H \neq 0\).
Proof. Let \( \{ \varepsilon_\chi \mid \chi \in \text{Irr}(A) \} \) be the set of central primitive idempotents of \( A \). Then \( \zeta(e \varepsilon_\chi) = \zeta \chi(e \varepsilon_\chi) \). On the other hand, from Theorem \( \ref{thm:standard-traces} \) we conclude that \( \zeta(e \varepsilon_\chi) = \zeta|_H(e \varepsilon_\chi) \). But from Lemma \( \ref{lem:standard-traces} \) it follows that \( \zeta|_H(e \varepsilon_\chi) = \zeta|_{\chi|_H}(e \varepsilon_\chi) \), where \( \zeta|_{\chi|_H} \) is the standard feasible multiplicity of \( \chi|_H \). This implies that \( \zeta \chi(e \varepsilon_\chi) = \zeta|_{\chi|_H}(e \varepsilon_\chi) \). Thus \( \zeta \chi = \zeta|_{\chi|_H} \), as claimed.

Suppose that \( (A, B) \) is a C-algebra and \( \rho \in \text{Hom}_\mathbb{C}(A, \mathbb{C}) \) such that \( \rho(b) = |b| \). Then \( \rho \) is an irreducible character of \( A \), which is called the principle character of \( (A, B) \). From \( \ref{eq:principle-character} \) by replacing \( \phi \) and \( \psi \) by \( \rho \) we conclude that \( \zeta_\rho = 1 \). Moreover, if \( (A, B) \) is a commutative C-algebra, then \( \ref{cor:commutative-traces} \) shows that \( \zeta_\chi > 0 \). In the following lemma we give a lower bound for the standard feasible multiplicities of the characters of a table algebra.

**Lemma 3.8.** Let \( (A, B) \) be a table algebra. Then \( |\zeta_\chi| \geq \chi(1_A)^{-1} \), for every \( \chi \in \text{Irr}(A) \). In particular, if \( (A, B) \) is commutative table algebra then \( |\zeta_\chi| \geq 1 \).

**Proof.** We first claim that \( |\chi(a)| \leq |a|\chi(1) \), where \( a \in B \) and \( \chi \) is a character of \( A \). To do this, let \( D \) be a representation of \( A \) which affords character \( \chi \) and let \( a \in A \). Suppose that \( m_a(x) \) is the minimal polynomial of \( a \) and \( \text{Spec}(a) \) is the set of all roots of \( m_a(x) \). Let \( \lambda \in \text{Spec}(a) \). Then \( a - \lambda 1_A \) can not be invertible, see \( \ref{cor:invertibility} \) Corollary 2.25]. So there exists a non zero element \( x \in A \) such that \( (a - \lambda 1_A)x = 0 \) or equivalently \( ax = \lambda x \). But by \( \ref{prop:traces} \) Proposition 2.3] we have \( |\langle ax, x \rangle| \leq |a|\langle x, x \rangle \) and so the latter equality implies that \( |\lambda(x,x)| \leq |a|\langle x, x \rangle \). Therefore \( |\lambda| \leq |a| \). This fact along with the obvious inclusion \( \text{Spec}(D(a)) \subseteq \text{Spec}(a) \) prove the claim. Now the result follows by applying the degree map \( | \cdot | \) on the both sides of the equation \( \ref{eq:principle-character} \).

The second statement is an immediate consequence of the first one, since \( \chi(1_A) = 1 \) for commutative case.

**Lemma 3.9.** The set of standard feasible multiplicities of two isomorphic C-algebras are the same.

**Proof.** Let \( (A, B) \) and \( (A', B') \) be two C-algebras and \( f : (A, B) \to (A', B') \) be an isomorphism. Let \( \zeta \) and \( \zeta' \) be the standard feasible traces of \( (A, B) \) and \( (A', B') \), respectively. Let \( P = \{ \varepsilon_\chi \mid \chi \in \text{Irr}(A) \} \) be the set of central primitive idempotents of \( A \). Then it is easily seen that the set \( P' = \{ \varepsilon_{\chi'} \mid \chi' \in \text{Irr}(A) \} \) is the set of central primitive idempotents of \( A' \), where \( \chi'(a') = \chi(f^{-1}(a')) \) and \( a' \in A' \). It follows that for any \( \chi \in \text{Irr}(A) \) there exists \( \psi \in \text{Irr}(A) \) such that \( (\varepsilon_\psi)^f = \varepsilon_{\chi'} \), and so \( \psi(1) = \chi(1) \). Therefore, by comparing the coefficient of \( 1_{A'} \) in the both sides of the former equality we get

\[
\frac{\psi(1)}{|B^+|} \zeta_\psi = \frac{\chi(1)}{|B'^+|} \zeta'_{\chi'}
\]

where \( \zeta_\psi \) and \( \zeta'_{\chi'} \) are the standard feasible multiplicities of \( \psi \) and \( \chi' \) with respect to standard feasible traces \( \zeta \) and \( \zeta' \), respectively. This implies that \( \zeta_\psi = \zeta'_{\chi'} \). Therefore the set of standard feasible multiplicities of the C-algebras \( (A, B) \) and \( (A', B') \) are the same, as desired.
4 The standard character

Let $X$ be a set with $n$ elements. According to [14] a linear subspace $W$ of the algebra $\text{Mat}_n(C)$, the set of all $n \times n$ matrices with entries in $C$, is called a cellular algebra on $X$ if $I_n, J_n \in W$; $W$ is closed under the matrix and the Hadamard (componentwise) multiplications and $W$ is closed under transpose, where $I_n$ is the identity matrix and $J_n$ is the matrix all of whose entries are ones. For example, any complex adjacency algebra of an association scheme is a cellular algebra. Conversely, in the sense of [10] for a given cellular algebra $W$ on a finite set $X$, there is a coherent configuration on $X$ whose adjacency algebra is $W$. So cellular algebras and adjacency algebras are equivalent objects, see [14]. On the other hand, any cellular algebra is a table algebra but the converse is not true, see Example 4.2. In this section we are interested in finding a necessary and sufficient condition for which a commutative table algebra becomes a cellular algebra.

Let $(X, G)$ be an association scheme and let $CG = \bigoplus_{g \in G} C\sigma_g$ be the complex adjacency algebra of $G$. Then the representation of $CG$ which sends $\sigma_g$ to itself for every $g \in G$ affords a character $\gamma_G$ which is called standard character of $CG$, see [15]. Moreover, $\gamma_G(\sigma_{1_X}) = |X|$ and $\gamma_G(\sigma_g) = 0$ for $1_X \neq g \in G$ and

$$\gamma_G = \sum_{\chi \in \text{Irr}(G)} m_\chi \chi. \tag{4}$$

In this case by setting $A = CG$ and $B = \{\sigma_g : g \in G\}$, the pair $(A, B)$ is a C-algebra with the standard feasible trace $\zeta = \gamma_G$ given in (4). Therefore, the standard feasible multiplicities $\zeta_\chi = m_\chi$ for $\chi \in \text{Irr}(G)$ are nonnegative integers.

In general, we do not know if $\zeta_\chi$'s are nonnegative integers, or equivalently wether or not $\zeta$ is a character. It is interesting to find some examples of C-algebras apart from association schemes, for which $\zeta$ is a character. In example below we give a commutative table algebra which does not come from association schemes and for it $\zeta$ is a character. In the case that the standard feasible trace $\zeta$ of a C-algebra $(A, B)$ is a character, by pattern of the theory of association schemes we call $\zeta$ the standard character of $(A, B)$.

**Definition 4.1.** We say that a C-algebra has standard character condition, if it possesses the standard character. We denote by $S$ the class of all such C-algebras.

Clearly association schemes belong to the class $S$ and Example 4.2 below shows that the class $S$ is larger than the class of association schemes. But this class is not equal to the class of integral table algebras, in fact in Example 4.3 below we give an integral table algebra does not belong to $S$.

For a given strongly regular graph $(X, E)$ with parameters $(n, k, \lambda, \mu)$ one can find an association scheme $\mathcal{C} = (X, G)$ where $G = \{1_X, g, h\}$ with structure constants $\lambda_{gg1_X} = k, \lambda_{ggh} = \lambda, \lambda_{ggh} = \mu$. In [5] some of the necessary conditions for the existence of a strongly regular graph with parameters $(n, k, \lambda, \mu)$ are given. One of them is integrality condition. If we consider adjacency algebra of association scheme $\mathcal{C}$, which is a C-algebra $(A, B)$ of dimension 3, then one can see that the standard character
condition for \((A, B)\) is equivalent to integrality condition for the existence strongly regular graphs with parameters \((n, k, \lambda, \mu)\), see [5].

In Example 4.2 we will use the definition of a finite affine plane in the sense of [6]. We recall that for every finite affine plane \(P = (P, L, I)\), i.e., \(P \cup L\) is finite, there exists an integer \(q \geq 2\), called the order of affine plane such that \(|P| = q^2\) and \(|L| = q^2 + q\), and each line is incident to exactly \(q + 1\) points. Besides, there are exactly \(q + 1\) classes \(B_1, \ldots, B_{q+1}\) of pairwise parallel (nonintersecting) lines, each of which is of cardinality \(q\).

**Example 4.2.** (cf. [13]) Let \(P\) be a finite affine plane with point set \(P\) and line set \(L\). Let \(\text{Mat}_P(\mathbb{C})\) be the algebra of all \(|P| \times |P|\) complex matrices whose rows and columns are indexed by the elements of \(P\). Define a \((0, 1)\)-matrix \(r_i \in \text{Mat}_P(\mathbb{C})\) by

\[
(r_i)_{u,v} = \begin{cases} 
1, & \text{if } u \neq v \text{ and } l(u, v) \in B_i \\
0, & \text{otherwise}
\end{cases}
\]

where \(l(u, v)\) is the line incident with both \(u\) and \(v\). Then for all \(i, j = 1, \ldots, q + 1\) we have

\[
r_i r_j = \begin{cases} 
(q - 1)r_0 + (q - 2)r_i, & \text{if } i = j \\
\sum_{k \neq 0, i, j} r_k, & \text{if } i \neq j
\end{cases}
\]

where \(r_0\) is the identity matrix. So the set \(B = \{r_0, \ldots, r_{q+1}\}\) is a linear base of the subalgebra \(A\) of the algebra \(\text{Mat}_P(\mathbb{C})\) generated by \(B\). Then it is easily seen that \((A, B)\) is a table algebra (with \(*\) being the Hermitian conjugation in \(\text{Mat}_P(\mathbb{C})\)). An easy computation shows that the character table of the table algebra \((A, B)\) is as follows:

| \(r_0\) | \(r_1\) | \(r_2\) | \ldots | \(r_{q+1}\) | \(\zeta_{\chi_1}\) |
|--------|--------|--------|--------|-----------------|------------------|
| \(\chi_1\) | 1 | \(q - 1\) | \(q - 1\) | \ldots | \(q - 1\) | 1 |
| \(\chi_2\) | 1 | \(q - 1\) | -1 | \ldots | -1 | \(q - 1\) |
| \(\chi_3\) | 1 | -1 | \(q - 1\) | \ldots | -1 | \(q - 1\) |
| \(\ldots\) | \(\ldots\) | \(\ldots\) | \(\ldots\) | \ldots | \(\ldots\) | \(\ldots\) |
| \(\chi_{q+2}\) | 1 | -1 | -1 | \ldots | \(q - 1\) | \(q - 1\) |

**Table (1)**

From Table (1) one can see that the standard feasible multiplicities of the characters of the table algebra \((A, B)\) are positive integers. Thus \(\zeta\) is a character. Now we claim that by a suitable integer \(q\) the table algebra \((A, B)\) is not the complex adjacency algebra of any association scheme. To do so, suppose on the contrary that the table algebra \((A, B)\) is the complex adjacency algebra of an association scheme \((X, G)\), where \(G = \{g_0, \ldots, g_{q+1}\}\). Then for each \(i, 1 \leq i \leq q + 1\), the subset \(\{g_0, g_i\}\) of \(G\) is a closed subset and so \(E_i = g_0 \cup g_i\) is an equivalence relation on \(X\). Now let \(L\) be the set of all equivalence classes of the equivalence relations \(E_i, 1 \leq i \leq q + 1\). Then it is easily seen that the sets \(X\) and \(L\) form an affine plane of order \(q\) consisting \(X\) as the set of points.
and \( L \) as the set of lines. But we can choose a suitable integer \( q \) in such a way that there is no affine plane of degree \( q \) (see [6]), we get a contradiction. Thus \((A,B)\) can not come from an association scheme.

**Example 4.3.** Let \( A \) be a \( \mathbb{C} \)-linear space with the basis \( B = \{1_A, b, c\} \) such that

\[
\begin{align*}
b^2 &= 2 \ 1_A + b \\
c^2 &= 25 \ 1_A + 25b + 22c \\
bc &= cb = 2c
\end{align*}
\]

Then one can see that the pair \((A,B)\) is an integral table algebra. By using the orthogonality relation given in Lemma 3.1 part (ii) the character table of \((A,B)\) is as the following:

|   | \(1_A\) | \(b\) | \(c\) | \(\zeta_{\chi_{\lambda}}\) |
|---|---|---|---|---|
| \(\chi_1\) | 1 | 2 | 25 | 1 |
| \(\chi_2\) | 1 | 2 | -3 | \(
\frac{25}{3}\) |
| \(\chi_3\) | 1 | -1 | 0 | \(
\frac{25}{3}\) |

Table (2)

Thus from Table (2) the standard feasible multiplicities of the characters of \((A,B)\) are not integers. This shows that \((A,B) \notin \mathcal{S}\).

**Lemma 4.4.** Let \((A,B) \in \mathcal{S}\) be a commutative \( \mathbb{C}\)-algebra. Then any matrix representation \(D\) of \(A\) which affords \(\zeta\) is faithful.

**Proof.** Let \(D\) be a matrix representation of \(A\) which affords \(\zeta\). Suppose that \(a = \sum_{b_i \in B} x_i b_i \in A\) is in the kernel of \(D\), so \(D(a) = 0\). Since \(A\) is commutative, there is a non-singular matrix \(P\) such that for all \(b_i \in B\) the following equality holds:

\[
PD(b_i)P^{-1} = \text{diag}(\chi_1(b_i), \chi_2(b_i), \ldots, \chi_2(b_i), \ldots, \chi_n(b_i), \ldots, \chi_n(b_i))
\]

\[
= \zeta_{\chi_\lambda} \times \zeta_{\chi_n} - \text{times} \zeta_{\chi_\lambda} \times \zeta_{\chi_n} - \text{times}
\]

where \(\text{Irr}(A) = \{\chi_1, \ldots, \chi_n\}\). It follows that \(\sum_{b_i \in B} x_i PD(b_i)P^{-1} = 0\). Therefore, \(MX = 0\), where \(M\) is an \(n \times n\) matrix whose \((i,j)\) entry is \(\zeta_{\chi_i}(b_j)\) and \(X\) is a column matrix whose \(i\)-th entry is \(x_i\). Now since \(M\) is a non-singular matrix, it follows that \(X = 0\) which implies that \(a = 0\). This completes the proof of the lemma.

**Remark 4.5.** Let \((A,B) \in \mathcal{S}\) be a table algebra and let \(D\) be a matrix representation of \(A\) which affords the standard character \(\zeta\). Then one can check that \((D(A), D(B))\) is a table algebra by defining \(D(b)^* = D(b^*)\) and \(|D(b)| = |b|\).

We say that a table algebra \((A,B)\) has an adjacency algebra homomorphic image, if there are an association scheme \((X,G)\) and a \(\mathbb{C}\)-algebra epimorphism \(T : (A,B) \rightarrow (\mathbb{C}G, C)\), where \(C = \{\sigma_g : g \in G\}\) is the basis of the adjacency algebra \(\mathbb{C}G\).
**Theorem 4.6.** Let \((A, B)\) be a table algebra. Then \((A, B)\) has an adjacency algebra homomorphic image iff \((A, B) \in \mathcal{S}\) and a matrix representation \(D\) which affords \(\zeta\) satisfies the following conditions for any \(b \in B\):

1. \(D(b^*) = D(b)^t\).
2. \(D(b)\) is a \((0, 1)\)-matrix.

**Proof.** We first prove the necessity of conditions (1) and (2). Let \(CG\) be an adjacency algebra homomorphism \(T\) from \(A\) onto \(CG\). Then \(T(A) = CG\) and \(T(b^*) = T(b)^t\). It follows that \(|b| = |T(b)|\), for \(b \in B\). So \(T\) induces a matrix representation \(D\) of degree \(|B^+|\) and conditions (1) and (2) are valid for \(D\). Then the character \(\chi\) which is afforded by \(D\) has values \(|B^+|\) at \(1_A\) and 0 at any \(b \in B \setminus \{1_A\}\). This implies that \(\chi\) is the standard character \(\zeta\) of \((A, B)\) and so \((A, B) \in \mathcal{S}\), as desired.

Conversely, suppose that \((A, B) \in \mathcal{S}\) and conditions (1) and (2) hold for a matrix representation \(D\) of \(A\) which affords the standard character \(\zeta\). Since

\[
D(b)D(b)^t = D(b)D(b^*) = |b|D(1_A) + \sum_{1_A \neq b \in B} \lambda_{bb^*}D(d)
\]

we conclude that the matrix \(D(b)\) contains \(|b|\) ones in each rows and columns. On the other hand, let \(b \in B\) such that \(D(b)_{ij} = 1\). If \(D(c)_{ij} = 1\) for some \(c \in B\), then the \((i, i)\) entry of matrix \(D(b)D(c)^t = 1\). It follows that \(b = c\). Thus the matrices \(D(b)\), \(b \in B\) are disjoint and the sum of them is the matrix \(J_n\) where \(n = |B^+|\). This implies that \((D(A), D(B))\) is a cellular algebra ( or an adjacency algebra), as desired. \(\blacksquare\)

**Corollary 4.7.** Let \((A, B)\) be a commutative table algebra. Then \((A, B)\) comes from an association scheme iff \((A, B) \in \mathcal{S}\) and a matrix representation \(D\) which affords \(\zeta\) satisfies the following conditions for any \(b \in B\):

1. \(D(b^*) = D(b)^t\).
2. \(D(b)\) is a \((0, 1)\)-matrix.

**Proof.** This is an immediate consequence from Theorem 4.6 and Lemma 4.4. \(\blacksquare\)

Let \((A, B)\) be a C-algebra. The coordinate-wise multiplication \(\circ\) with respect to the basis \(B\) by \(b \circ c = \delta_{bc}b\), for \(b, c \in B\) is defined in the sense of [7]. We say that a matrix representation \(D\) of \(A\) preserves Hadamard products if \(D(b \circ c) = D(b) \circ D(c)\), for \(b, c \in B\).

For a matrix \(A\), \(\tau(A)\) denotes the sum of all entries \(A\). One can see that for any two square matrices \(A\) and \(B\) of the same size:

\[
\tau(A \circ B) = \text{tr}(AB^t) = \text{tr}(A^tB).
\]

**Corollary 4.8.** Let \((A, B) \in \mathcal{S}\) be a table algebra and let \(D\) be a matrix representation of \(A\) which affords \(\zeta\). Then table algebra the \((D(A), D(B))\) is a cellular algebra iff \(D\) perseveres Hadamard products.
Proof. The necessity is obvious. For the sufficiency, since $D(b), b \in B$ persevere Hadamard products, each $D(b), b \in B$ is $(0,1)$-matrix. On the other hand,

$$\tau(D(b^*) \circ D(c)^t) = \text{tr}(D(b^*)D(c)) \quad b,c \in B.$$  

But $\text{tr}(D(b^*)D(c)) = 0$ iff $b \neq c$. Thus $D(b^*) = D(b)^t$. Now the result follows from Theorem 4.6 and we are done.

In the rest of this section, we suppose that $(A, B)$ is a commutative $C$-algebra of dimension $d$ with the set of the primitive idempotents $\{\varepsilon_\chi | \chi \in \text{Irr}(A)\}$. Then from [4 Section 2.5] there are two matrices $P = (p_\chi(\chi))$ and $Q = (q_\chi(b))$ in $\text{Mat}_d(\mathbb{C})$, where $b \in B$ and $\chi \in \text{Irr}(A)$ such that $PQ = QP = |B^+|I$ and

$$b = \sum_{\chi \in \text{Irr}(A)} p_\chi(\chi)\varepsilon_\chi \quad \text{and} \quad \varepsilon_\chi = \frac{1}{|B^+|} \sum_{b \in B} q_\chi(b)b.$$  

Then from Remark (3.2) and (4) we get

$$q_\chi(1_A) = \zeta_\chi \quad \text{and} \quad \chi(b) = p_\chi(\chi),$$

where $b \in B$ and $\chi \in \text{Irr}(A)$. The dual of $(A, B)$ in the sense of [4] is as follows: with each linear representation $\Delta_\chi : b \mapsto p_\chi(\chi)$, we associate the linear mapping $\Delta_\chi^* : b \mapsto q_\chi(b)$. Since $Q = (q_\chi(b))$ is non-singular, the set $\hat{B} = \{\Delta_\chi^* : \chi \in \text{Irr}(A)\}$ is a linearly independent and so form a base of the set of all linear mapping $\hat{A}$ of $A$ into $\mathbb{C}$. From [4, Theorem 5.9] the pair $(\hat{A}, \hat{B})$ is a $C$-algebra with the identity $1_{\hat{A}} = \Delta_\chi^*$ and involutory automorphism which maps $\Delta_\chi^*$ to $\Delta_\chi^*$, where $\chi$ is complex conjugate to $\chi$. The $C$-algebra $(\hat{A}, \hat{B})$ is called the dual $C$-algebra of $(A, B)$. Moreover, the structure constants of $(\hat{A}, \hat{B})$ which are given in [4 (5.26)] can be written as the following

$$q_{\chi \psi}^\phi = \frac{\zeta_\phi \zeta_\psi}{|B^+|} \sum_{b \in B} \frac{1}{|b|^2} p_\phi(\phi)p_\psi(\psi)p_\chi(\chi)$$  

which are real numbers, where $\overline{p_\phi(\phi)}$ is the complex conjugate to $p_\phi(\phi)$. From (7) and (2) one can see that $q_{\chi \chi}^\phi = \zeta_\chi$. Then $|\hat{B}^+| = \sum_{\chi \in \text{Irr}(A)} \zeta_\chi$. The primitive idempotents $f_b, b \in B$ of $\hat{A}$ are given by [4, 5.23] as the following

$$f_b = \frac{1}{|\hat{B}^+|} \sum_{\chi \in \text{Irr}(A)} p_\chi(\chi)\Delta_\chi^*.$$  

Lemma 4.9. Keeping the notation above, there is a bijection correspondence between the standard feasible multiplicities of the characters of $(\hat{A}, \hat{B})$ and the degrees of basis elements $B$.

Proof. From (3), one can see that the coefficient of the unit element $1_{\hat{A}}$ of $\hat{A}$ in the linear decomposition of $|\hat{B}^+|f_b$ in terms of the basis elements $\hat{B}$ is equal to $p_\chi(\chi)$. On the other hand, from the equation of the right hand side of (6) we get $p_\chi(\chi) = \rho(b) = |b|$. But from Remark 3.2 any standard feasible multiplicity of the characters of $(\hat{A}, \hat{B})$ corresponds to the number $p_\chi(\chi)$ for some $b \in B$, as desired.
A C-algebra is called integral degree if its all degrees $|b|, b \in B$, are integer.

**Corollary 4.10.** Let $(A, B)$ be a C-algebra. Then $(A, B)$ is integral degree and belongs to $S$ iff so is $(\hat{A}, \hat{B})$.

**Proof.** Let $(A, B)$ be a C-algebra and $(\hat{A}, \hat{B})$ be its dual with the standard feasible traces $\zeta$ and $\hat{\zeta}$, respectively. To prove the necessity, since $(A, B)$ is in $S$ the equality $q^\zeta = \zeta$ implies that $(\hat{A}, \hat{B})$ is integral degree. Since $(A, B)$ is integral degree, from Lemma 4.9 we conclude that $(\hat{A}, \hat{B})$ is in $S$.

To prove the sufficiency, by the necessity we see that $(\hat{A}, \hat{B}) \in S$ is integral degree. Now the proof follows from Lemma 4.9 and the Duality Theorem [4, Theorem 5.10], i.e., $(A, B) \simeq (\hat{A}, \hat{B})$.

**References**

[1] Z. Arad, H. Blau, On Table Algebras and their Applications to Finite Group Theory, J. Algebra, 138, 137-185, 1991.

[2] Z. Arad, E. Fisman, M. Muzychuk, On the Product of Two Elements in Noncommutative C-Algebras, Algebra Colloquium, 5:1, 85-97, 1998.

[3] Z. Arad, E. Fisman, M. Muzychuk, Generalized Table Algebras, Israel J. Math. 114, 29-60, 1999.

[4] E. Bannai, T. Ito, Algebraic Combinatorics I. Association Schemes. Benjamin-Cumming Lecture Notes Ser., v. 58, The Benjamin/Cumming Publishing Company Inc, London, 1984.

[5] A. E. Brouwer and J. H. van Lint, Strongly Regular Graphs and Partial Geometries, In Enumeration and Design, Waterloo, Ont., 1982.

[6] T. Beth, D. Jungnickel and H. Lenz, Design Theory , Cambridge University Press, Cambridge, Vol. 1, 1999.

[7] S. Evdokimov, I. Ponomarenko and A. Vershik, Algebras in Plancherel Duality and Algebraic Combinatorics, Functional Analysis and its Applications, 31, no. 4, 34-46, 1997.

[8] D. R. Farenick, Algebras of Linear Transformations, Universitex, Springer-Verlag New York, Inc 2001.

[9] A. Hanaki and M. Hirasaka, Theory of Hecke Algebras to Association Schemes, SUT Journal of Mathematics Vol. 38, No. 1, 61-66, 2002.

[10] D. G. Higman, Coherent Configurations, Part(I): Ordinary Representation Theory, Geometriae Dedicata, 4, 1-32, 1975.

[11] D. G. Higman, Coherent Algebras, J. Linear Algebra and its Applications, 93, 209-239, 1987.
[12] H. Nagao, Y. Tsushima, Representations of Finite Groups, New York, Academic Press 1989.

[13] I. Ponomarenko, A. Rahnamai Barghi, On Amorphic C-Algebras, Journal of Mathematical Sciences, Vol. 145, No. 3, 2007.

[14] B. Ju. Weisfeiler, On Construction and Identification of Graphs, Springer Lecture Notes, 558, 1976.

[15] P-H. Zieschang, An Algebraic Approach to Association Schemes, in: Lecture Notes in Math., vol. 1628, Springer-Verlag, Berlin, 1996.