Chimera states in ring–star network of Chua circuits

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Abstract We investigate the emergence of amplitude and frequency chimera states in ring–star networks consisting of identical Chua circuits connected via nonlocal diffusive, bidirectional coupling. We first identify single-well chimera patterns in a ring network under nonlocal coupling schemes. When a central node is added to the network, forming a ring–star network, the central node acts as the distributor of information, increasing the chances of synchronization. Numerical simulations show that the radial coupling strength $k$ between the central and the peripheral nodes acts as an order parameter leading from a lower- to a higher-frequency domain. The transition between the domains takes place for intermediate coupling values, $0.5 < k < 2$, where the frequency chimera states prevail. The transition region (width and boundaries) depends on the Chua oscillator parameters and the network specifics. Potential applications of star connectivity can be found in the control of Chua networks and in other coupled chaotic dynamical systems. By adding one central node and without further modifications to the individual network parameters, it is possible to entrain the system to lower- or higher-frequency domains as desired by the particular applications.

Keywords Chimera states · Chua circuit · Nonlocal diffusive coupling · Ring–star network

1 Introduction

Chimera states are characterized by the coexistence of synchronous and asynchronous areas when identical dynamical units are coupled equivalently in some network topology. Most commonly studied are the frequency chimeras, which are distinguished by the difference in frequency of the oscillatory elements. Although all oscillators have the same intrinsic frequency, it is the coupling between them which creates a distribution of frequencies in the network, a reason why the chimera states are nontrivial and unexpected. Kuramoto and Battogtokh first observed these patterns in nonlocally coupled phase oscillators described in [1,2]. Following these first reports, chimera states have been studied extensively during the past two decades. Researchers found these states in many different types of dynamical systems, like coupled phase oscillatory flows and discrete maps connected in different network topologies and coupling forms, as described in recent review articles [3–6]. Besides the typical chimeras as reported by Kuramoto, where the synchronous and asynchronous domains remain fixed in space, more complex chimera patterns were recently reported, such
as breathing chimeras, alternating chimeras [7], spiral wave chimeras [8], amplitude chimeras [9], and many more.

Applications of chimera states were first reported in systems of coupled nonlinear neuronal oscillators [5]. These oscillators are known to exhibit highly nonlinear behavior, and chimera states were found in systems that mimic the neuronal activity like the FitzHugh–Nagumo (FHN), the Hindmarsh–Rose (HR), and the Hodgkin–Huxley (HH) systems [10–13]. Apart from the numerical studies, experimental evidence of chimera states related to neuronal activity has also been reported in [14], in which nine FHN oscillators in a ring were considered. Synchronization/coordination aspects related to neurological disorders like epilepsy were studied via this system. The results indicated that epilepsy is not only a dynamical disease but also a topological disease that depends on the type of connection between the neurons. Further indications on the presence of chimera states at the onset of epileptic seizures are reported in references [15–17]. It has also been claimed that chimera states are deeply connected with the causes of creating various kinds of neuronal diseases like Parkinson’s disease, schizophrenia, brain tumors, etc. [18]. While performing experiments, Tognoli et al. reported in reference [19] the presence of synchronous and asynchronous activity during left and right finger movements. Also in relation to brain activity, chimera states have been associated with the unihemispheric sleep pattern in aquatic mammals and birds, where they sleep with one eye open leading one half of the brain in the synchronous state and the other half in the asynchronous [20–22].

Besides experiments related to neuronal activity, experimental evidence of chimera states has been reported in diverse disciplines and notably in the domains of mechanical oscillators [23], coupled map lattices [24], nonlocally coupled FitzHugh–Nagumo and Stuart–Landau oscillator networks [25,26], and chemical oscillators [27–29]. Recent numerical evidence indicates that metamaterials are also a promising domain for chimera applications [30,31]. Apart from the confirmation of the chimera states in experiments and simulations, it still remains an open problem to deal analytically with the mechanism behind the formation of these hybrid states and their control, making the study of chimeras an active research area.

The Chua circuit is considered as the simplest nonlinear circuit to exhibit chaotic behavior [32]. It is composed of an inductor, two capacitors, a resistor, and a Chua diode. The circuit’s temporal evolution is described by a three-dimensional continuous dynamical system. The Chua diode, being a nonlinear local active resistor, is mainly responsible for the chaotic behavior of the circuit, which is well known for its double-scroll attractor [33]. Regarding applications, Chua circuits are used in secure communications [34], in improving performance of ultrasonic devices in the presence of cross talk [35] and in the generation of Gaussian white noise which is essentially useful in many engineering systems [36]. Other applications include avant-garde music compositions [37], storage of analog patterns and managing the problem of handwritten recognition [38]. Many present-day applications of the Chua circuit are described in reference [39].

Star networks and ring networks are usually common in social systems, hubs, social networks, and computer networks [40]. In the latter, the information flow needs to be transmitted securely, else there is a compromise in the security leading to cybercrimes. Chua circuits connected in the computer networks, in particular, can help in achieving secure communications as the Chua circuits have proven to be useful in many cryptographic systems [41].

Synchronization of Chua oscillators was studied in star networks, and their properties were established using different kinds of coupling, such as diffusive, conjugate, and mean-field couplings [42]. The regimes of full synchronization of the Chua oscillator networks were mainly studied using these different coupling forms. A variety of chimera structures have been reported in the case of Chua circuits connected in a ring [43]. Chimera states were also observed in the star network consisting of synchronized and desynchronized oscillators in the group. In a two-dimensional lattice of Chua oscillators, spiral waves are obtained in [44]. The present study is an extension of previous work [42], in the sense that we are using a composite connectivity scheme: Starting with a ring of Chua circuits with nonlocal connectivity and common coupling strength $\sigma$ (see [43]), we apply an additional radial connectivity where every Chua circuit is bidirectionally connected to a central node with a variable coupling strength $k$. By varying the values of $k$ and $\sigma$, we can transit from a pure ring connectivity, when $k = 0$ and $\sigma \neq 0$, to a pure star (central) connectivity, when $k \neq 0$ and $\sigma = 0$. We investigate the prevalence of different chimera pat-
In the following, when we refer to the exchange of information between nodes $i$ and $j$ in the system, we mean that at a certain time $t$ the state variables $x_i(t)$ and $y_j(t)$ of node $i$ receive and use the values of the state variables $x_j(t - \Delta t)$ and $y_j(t - \Delta t)$ of node $j$ in the previous time step $t - \Delta t$. This must not be confused with the notion of global information, energy, or entropy exchange between the nodes as discussed in the literature of synchronization between interacting units [45–51] and more recently on synchronization in the form of chimera states [52,53].

The transitions from asynchronous patterns to chimera states and to synchronized states are quantitatively studied with the help of the different synchronization measures [54]. The mean phase velocity [55] is a common measure considered to demonstrate the presence of chimera structures. However, it frequently fails to identify them, mainly in the cases of traveling or diffusing chimeras. In those cases, there is a need for different measures to be considered, which can work as alternatives to the mean phase velocity. Alternative measures analyzing the relative size of the coherent/incoherent domains, the degree of coherence, etc., were considered in [54]. We establish the chimera kingdom in the ring–star network of Chua circuits using these measures, to avoid the problems of pattern displacement in space and for quantitative comparison between the different chimera morphologies.

The paper is organized as follows: Sect. 2 introduces the Chua circuit and the ring–star network topology. In a separate Sect. 2.1, the various synchronization measures are introduced. Section 3 discusses the simulation results obtained in the case of Chua circuits coupled in a ring geometry for parameter values where single-well chimeras emerge. Section 4 is devoted to the ring–star network connectivity. The deformation of the single-well chimeras as a function of the radial coupling $k$ is discussed in this section. In Sect. 5, the transition of the system from the lower-frequency to the higher-frequency domain is discussed where the frequency chimera states prevail. In all cases, alternative synchronization measures are considered for the quantitative study of the mean phase velocity profiles. In the conclusions, the main results of this study are recapitulated and some challenges ahead in analyzing the ring–star network are proposed.

2 Chua ring–star network model

A sketch of the ring–star network is shown in Fig. 1. A number of $N$ Chua oscillators are connected in a ring–star network with nonlocal diffusive coupling. Oscillators are indexed as $i = 1$ for the central node and $i = 2, \ldots, N$ for the peripheral (end) nodes. The central node ($i = 1$) is connected to all the peripherals with the same coupling strength $k$. Each peripheral oscillator is nonlocally connected to $R$ nodes to its left and $R$ nodes to its right with common coupling strength $\sigma$ and is also linked to the central node with coupling strength $k$. To enforce uniformity of the end nodes, ...periodic (ring) boundary... conditions are considered.

The dynamical equations of the ring–star network are given by Eqs. (1) and (2). For $i = 2, \ldots, N$, the dynamical equations of the end nodes are given by:
where

$$x_i = f_x + k(x_1 - x_i) + \frac{\sigma}{2R} \sum_{k=i-R}^{k=i+R} (x_k - x_i),$$

$$y_i = f_y + \frac{\sigma}{2R} \sum_{k=i-R}^{k=i+R} (y_k - y_i),$$

$$z_i = f_z.$$  \hspace{1cm} (1)

For \(i = 1\) (central node), the dynamical equations are:

$$\dot{x}_1 = f_x + \sum_{j=1}^{N} k(x_j - x_1),$$

$$\dot{y}_1 = f_y,$$

$$\dot{z}_1 = f_z,$$  \hspace{1cm} (2)

where

$$f_x = \alpha (y_i - x_i - (Bx$$

$$+ \frac{1}{2} (A - B)(|x_1| - |x - 1|))),$$

$$f_y = x_i - y_i + z_i,$$

$$f_z = -\beta y_i$$

with periodic boundary conditions:

$$x_{i+N}(t) = x_i(t),$$

$$y_{i+N}(t) = y_i(t),$$

$$z_{i+N}(t) = z_i(t),$$

for \(i = 2, 3, \ldots, N\). Following references [42,43], we have used coupling only in the \(x\)- and \(y\)-variables and not in the \(z\)-variable of the Chua coupled elements. Similar coupling only via one variable is used in reference [56] for coupled Rössler oscillators.

From the interaction scheme, it is now clear that oscillators \(i = 2, \ldots, N\) exchange information via their \(x\)- and \(y\)-variable with \(2R\) neighbors symmetrically set around \(i\), while the central unit \(i = 1\) exchanges information with all other units \(j = 2, \ldots, N\) (only via their \(x\)-variable, for simplicity).

Due to the Euler integration scheme used, the variables \(x_i(t), y_i(t), z_i(t)\) are updated using the values \(x_i(t - \Delta t), y_i(t - \Delta t), z_i(t - \Delta t)\) at the previous time step, as also stated in Sect. 1.

As working parameter set, the following values are used throughout this study: The parameters of the identical Chua circuits are set to \(A = -1.143, B = -0.714, \alpha = 9.4\) and \(\beta = 14.28\) in order to keep the circuit in the oscillatory, double-scroll regime. The system size is set to \(N = 300\) and the coupling range to \(R = 100\). The rest of the parameters, the coupling strength \(\sigma\) between the peripheral nodes, and the coupling strength \(k\) between the central node and the peripheral ones are varied to explore their influence in the network synchronization patterns.

### 2.1 Synchronization measures

As discussed in Sect. 1, the mean phase velocity or average frequency \(\omega\) is a valuable measure to quantify the synchronization of the oscillators [55]. For the \(i\)th oscillator, the mean phase velocity is denoted by \(\omega_i\). For a large computational time interval \(T\), \(\omega_i\) expresses the number of times the variable \(x_i\) crosses a certain fixed constant value, say \(c\). If the variable \(x_i\) crosses the constant \(c\), \(M_i\) times with positive slope, then the mean phase velocity of the \(i\)th oscillator is calculated as:

$$\omega_i = \frac{2\pi M_i}{T} = 2\pi f_i.$$  \hspace{1cm} (3)

The positive slope considered in the counting of \(M_i\) in Eq. (3) is needed to avoid double counting the number of periods calculated within the time interval \(T\). The quantity \(f_i\) denotes the average frequency, which differs from the mean phase velocity by a factor \(2\pi\). Due to this simple relation, in the following the terms
mean frequency” and “mean phase velocity” will be used interchangeably.

Chimera states are characterized by the difference in frequency of the identical oscillatory circuit elements. The coupling is responsible for the change in frequency in some oscillators. Different synchronization measures come to play as additional quantitative indices when inconclusive information is conveyed by the mean phase velocity. Such synchronization measures are formed. Having started with initial conditions, \((x(t = 0), y(t = 0), z(t = 0))\), randomly distributed in the interval \([0 < x(t = 0) < 1, 0 < y(t = 0) < 1, 0 < z(t = 0) < 1]\), the system remains always in the positive side of the axes (single-well) and all elements oscillate around the value \(x = 1.5\), but amplitude variations are observed in the different domains formed. For clarity, a single snapshot is presented in Fig. 2b. The mean phase velocity (Fig. 2c) clearly identifies the domains where oscillators act coherently and the incoherent domains.

In Sect. 4, we introduce a central node to the system, creating the ring–star network, and we investigate the system’s response by varying the radial coupling strength \(k\). We study the system response in the case of single-well chimeras, using a coupling constant parameter \(\sigma = 0.75\) and variable \(k\), as discussed above.
Fig. 2 Ring network: single-well chimera structures and measures for $\sigma = 0.75$, $k = 0$ with nonlocal diffusive coupling. a 25 snapshots of the $x_i$-variables at time intervals of 40 units, b typical single snapshot of the $x_i$-variables and c mean phase velocities. Chua circuit parameters are $A = -1.143$, $B = -0.714$, $\alpha = 9.4$, $\beta = 14.28$ and network parameters are $N = 300$ and $R = 100$

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We now extend the Chua ring network model by adding a central node in the center of the ring which is linked equally to all the external nodes. We consider the mixed dynamics of the system and record its transition between chimera states and the full synchronization regimes. Simulations were carried out by considering random initial conditions for $x$, $y$, $z$ state variables in the interval $[0, 1]$, as in the previous section.

To investigate the influence of the central node in the Chua ring network, we gradually vary the coupling strength $k$ between the central node and the peripheral ones. All parameters of the identical Chua circuits are kept to the working parameter set, while the network contains $N = 300$ nodes and each Chua oscillator is connected to $R = 100$ neighbors to the left and $R = 100$ neighbors to the right. The ring coupling strength is fixed to $\sigma = 0.75$. [The pure ring network briefly discussed in the previous section is equivalent to the case of $k = 0$ (no central node), while the ring–star network is realized when $k \neq 0$.]

The central node of the network plays a double role: First, it receives “information” from the peripheral nodes and integrates it forming its own dynamics. Second, it redistributes the obtained information to the peripheral nodes, in such a way that each peripheral node receives information about the average (mean-field) dynamics of the ensemble of all peripheral nodes. Therefore, the central node acts as a modulator of the local dynamics using information over the ensemble dynamics. Based on this view of the system, we ask the question: How does the strength of the central coupling $k$ influences the distribution of information and the overall synchronization properties of the network? To answer this question, we performed numerical simulations using the same parameters as in the case of the single-well chimera and varied the central coupling parameter in the range $0 \leq k \leq 4$.

We provide below some examples of the modifications which take place when the central coupling strength $k$ becomes nonzero. When a small deviation is applied leading from the ring network, $k = 0$, to the ring–star network, $k = 0.25$, as in Fig. 3, we observe that the single-well chimera persists: The $x$ state variable keeps oscillating in the positive part of the axis and does not transverse below $x = 0$. For this low $k$ value, all oscillators have very similar frequencies, as shown in Fig. 3c. The green arrow in panel (c) points to the mean phase velocity $\omega_1$ of the central node, which is slightly lower than the rest of the elements. Increasing the radial strength to $k = 1$ in Fig. 4 and $k = 1.3$ in Fig. 5, the single-well chimeras change to double-well ones. In both cases, two domains of oscillators are formed: one domain where the $x$-variables oscillate in the positive axis and one in the negative axis. Related
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**Fig. 3** Ring–star network: single-well chimera structures and measures for $\sigma = 0.75$, $k = 0.25$ with nonlocal diffusive coupling. **a** 25 snapshots of the $x_i$-variables at time intervals of 40 units, **b** typical single snapshot of the $x_i$-variables, and **c** mean phase velocities. The arrow in panel **c** indicates the mean phase velocity of the central node. All other parameters as in Fig. 2.

**Fig. 4** Ring–star network: double-well chimera structures and measures for $\sigma = 0.75$, $k = 1$ with nonlocal diffusive coupling. **a** 25 snapshots of the $x_i$-variable at time intervals of 40 units, **b** typical single snapshot of the $x_i$-variable and **c** mean phase velocities. The arrow in panel **c** indicates the mean phase velocity of the central node. All other parameters as in Fig. 2.

To the mean phase velocity values, in the case of $k = 1$ (Fig. 4c), all peripheral nodes have the same mean phase velocity, while the central node has increased its mean phase velocity, $\omega_1 \sim 1.6$, indicating a tendency of the system to transit to higher frequencies. (See position of the green arrow in Fig. 4c.) Increasing further the $k$ values (see the case $k = 1.3$ in Fig. 5c), the nodes occupying the transition regions, between the negative and the positive $x$ value domains, follow the central node (see position of the green arrow) and attain in their turn higher mean phase velocities. With a further increase in the radial coupling strength $k$, for example $k = 2.5$, we observe that the $x$ state variables still traverse both the positive and negative part of the axis as in Fig. 6. Furthermore, in panel (c) we observe that all mean-phase velocities have increased as compared to lower values of $k$.

In Sect. 5, we study how the transition from the single- to double-well chimera takes place as we gradually increase the radial coupling strength $k$. 
Fig. 5  Ring–star network: double-well chimera structures and measures for $\sigma = 0.75$, $k = 1.3$ with nonlocal diffusive coupling. a 25 snapshots of the $x_i$-variables at time intervals of 40 units, b typical single snapshot of the $x_i$-variables, and c mean phase velocities. The arrow in panel c indicates the mean phase velocity of the central node. All other parameters as in Fig. 2.

Fig. 6  Ring–star network: double-well chimera structures and measures for $\sigma = 0.75$, $k = 2.5$ with nonlocal diffusive coupling. a 25 snapshots of the $x_i$-variables at time intervals of 40 units, b typical single snapshot of the $x_i$-variables, and c mean phase velocities. The arrow in panel c indicates the mean phase velocity of the central node. All other parameters as in Fig. 2.

5 Dynamics with an increase in the radial coupling strength

We analyze here the dynamics of the ring–star Chua network as the coupling strength $k$ (the ray coupling strength) increases between $0 \leq k \leq 4$. The ring coupling strength is fixed as $\sigma = 0.75$. We study the behavior of the mean phase velocity of the central node, $\omega_{\text{central}}$, of the coherent oscillators, $\omega_{\text{coh}}$, and of the “leader” incoherent oscillator, $\omega_{\text{leader}}$. From Fig. 5 in the previous section, we record in the incoherent regions a continuous distribution of frequencies and not a single one. In these regions, we call “leader” the oscillator which demonstrates the maximum mean phase velocity, which for this reason is called $\omega_{\text{leader}}$. Note that there can be more than one leaders in the system, one for each incoherent region, as Fig. 5 indicates. In a way, the leader oscillators can be considered as the ones which lead the deviations from coherence, while
the difference $\Delta \omega = \omega_{\text{leader}} - \omega_{\text{coh}}$ is indicative of the total incoherence in the system.

In Fig. 7, we plot the mean phase velocities of the central node $\omega_{\text{central}}$ (blue color), of the coherent nodes $\omega_{\text{coh}}$ (green color), and of the leader incoherent nodes $\omega_{\text{leader}}$ (red color). Initial conditions were chosen randomly in all simulations within the positive interval, $[0, 1]$, for the $x$, $y$, $z$ state variables. This figure indicates the presence of a phase transition taking place in the parameter region $0.5 \leq k_{\text{trans}} \leq 1$. In particular, for small values of the radial coupling, $k \leq 0.5$, all oscillators present similar $\omega$ values, around $\omega \sim 1$. In this region ($k \leq 0.5$), the central oscillator, $i = 1$, has the smallest frequency, slightly below the coherent ones, while the leaders have frequencies slightly above the coherent. As $k$ increases above 0.5, the abrupt transition occurs. First, ... central node doubles (almost) its mean phase velocity which becomes close to 1.6, while the rest of the oscillators remain close to the values $\omega \sim 1$. This behavior holds in the intermediate coupling region, $0.5 \leq k \leq 1.0$ (for the parameter values $\sigma = 0.75$, $N = 300$ oscillators and $R = 100$ neighbors). Above this transition region, and for $k > 1$, the coherent and incoherent nodes also increase gradually their mean phase velocities, which also attain values around $\omega \sim 1.6$. In particular, for values $1.25 < k < 1.5$, the central node frequency is located between the coherent and the leader ones. When $k$ reaches strengths $> 2.0$, the oscillator regions stabilize and the system attains constant frequencies, $\omega_{\text{central}} \sim 1.55$, $\omega_{\text{coh}} \sim 1.6$ and $\omega_{\text{leader}} \sim 1.72$ independent of $k$. The above discussion tells us that the frequency “chimera kingdom” characterized by considerable differences in the frequencies between coherent and incoherent domains is established for $k$-parameter values in the transition region $1.0 < k < 2.0$.

Figure 8 shows the ring–star network in action. We represent the central (blue) node, coherent (green) and incoherent (red) nodes for $k = 1$, $\sigma = 0.75$ in the ring–star network with different colors. As we see from the figure, the coherent nodes are not isolated but form clusters in the ring. At the same instant, the incoherent nodes are most abundant.

The transition shown in Fig. 7 is corroborated by the plots depicting $\omega_{\text{central}} - \omega_{\text{coh}}$ and $\omega_{\text{leader}} - \omega_{\text{coh}}$ in Figs. 9 and 10, respectively.

Namely, in Fig. 9 we note that the values $\omega_{\text{central}} - \omega_{\text{coh}}$ remain slightly below 0, for $k < 0.5$, indicating that the central node has constantly lower frequency...
The difference $\omega_{\text{central}} - \omega_{\text{coh}}$ as a function of the radial coupling strength $k$. Ring coupling strength $\sigma = 0.75$ and other parameters as in Fig. 4.

The difference $\omega_{\text{leader}} - \omega_{\text{coh}}$ as a function of the radial coupling strength $k$. Ring coupling strength $\sigma = 0.75$ and other parameters as in Fig. 4.

Fig. 11 Ratio of coherent elements as a function of the radial coupling range $k$. All other parameters are as in Fig. 2

The ratios of coherent and incoherent elements were calculated using Eqs. (4) and (6), and the results are depicted in Figs. 11 and 12, respectively. We have a similar picture of the transition from small to large frequency values with increasing strength $k$ of radial coupling. For small ($k < 0.5$) and large ($k > 2$) radial coupling strengths, the ratio of coherent elements stays low, $r_{\text{coh}} \sim 0.1$, and the ratio of incoherent stays large, $r_{\text{incoh}} \sim 0.9 = 1 - r_{\text{coh}}$. In the intermediate region, $0.5 < k < 2$ a transitive behavior is recorded, where the coherent ratio increases, while the incoherent one decreases. This reorganization of the system taking place in the intermediate $k$ regions where the frequency chimera states prevail marks the passage from the lower- to the higher-frequency domain.

As a general conclusion, in the transition between low and high frequencies, first the central node makes the jump to the higher frequencies at $k \sim 0.5$ entrain-
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Fig. 12 Ratio of incoherent elements as a function of the central coupling range \( k \). All other parameters are as in Fig. 2.

ing the rest of the nodes. Following the central node, the incoherent nodes are entrained. Their leaders make the transition at \( k \sim 1 \), while the coherent nodes attain the transition at \( k \sim 1.5 \). Note that the present values are indicative and hold for the working parameter set. For different parameters \((\sigma, N, R)\), the transition values as well as the transition regions are expected to vary depending on these parameters.

6 Conclusions

We have studied a ring network of Chua circuits, equipped with a central node which serves to redistribute to the peripheral nodes information about the mean field state of all nodes. For small values of the radial coupling strengths, single-well amplitude chimeras are observed. At intermediate radial couplings, a transition region is observed where the frequency chimeras prevail with a large difference in the frequencies between coherent and incoherent nodes. This region mediates the transition between the lower- and higher-frequency domains. For large radial coupling strengths, the system attains the higher-frequency domain and keeps constant mean phase velocities and ratios of coherent to incoherent nodes, independent of the radial coupling range. The frequency chimera kingdom is established for the intermediate radial couplings \( k \) values, as evidenced by the plots of all different synchronization measures.

The above results have potential applications in the control of Chua networks as well as other coupled chaotic dynamical systems. By just adding one central node, identical to all peripheral ones, and without further modifications to the individual oscillators or to the network parameters, it is possible to entrain the system to lower- or higher-frequency domains as desired by the particular applications by only adjusting the radial coupling. We must stress here that the transition described above is an example of transitions taking place in nonequilibrium systems (nonequilibrium transitions); the Chua system (1) is a characteristic example of such systems since it presents chaotic, nonconservative dynamics [32–34,39,44]. This transition cannot be directly related to the known phase transitions in equilibrium systems at criticality, such as the Ising model phase transitions (see reference [58]).

For future studies, it would be interesting to understand how the dynamics of the Chua network changes with different coupling forms such as conjugate coupling or mean-field coupling, or by strengthening the role of the central node and endowing it with interactions to the peripheral nodes using all three \( x \)-, \( y \)-, and \( z \)-variables. Transitions in different network types may also be considered as, for example, in the case of a 2D lattice of Chua oscillators equipped with a central element, or extensions to Chua circuits in multilayer arrangements.

A different study concerns the connection between star and ring–star networks. In order to achieve a complete star network (central node connected to peripheral nodes and no connection in between the peripheral nodes), mathematically we need to fix a finite value of \( k \) while letting \( \sigma \to 0 \). It would be interesting to account for chimera states and potential transitions in this limiting \( \sigma \) case and to investigate the link to the phenomenon of remote synchronization (RS) [56,59], a nontrivial phenomenon in star networks, where the peripheral oscillators synchronize (without being directly linked), while the central, relay node remains asynchronous.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.
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