FOUR CLASSIC PROBLEMS

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ABSTRACT. In this work we survey four classic problems: Borsuk’s partition problem, Tarski’s plank problem, the Kneser–Poulsen problem on the monotonicity of the union of balls under a contraction of their centers, and the Hadwiger–Levi problem on covering convex bodies by their smaller positively homothetic copies.

1. The Borsuk Problem

Borsuk [1933] asked the question whether every bounded set in $n$-dimensional space can be partitioned into $n + 1$ subsets of smaller diameter. Although Borsuk did not suggest a positive solution of the problem, for a long while there was general belief that the answer is yes, so, the problem became known as Borsuk’s conjecture.

The truth of the conjecture for $n = 2$ follows easily by the theorem of Pál [1920] stating that every set of unit diameter can be covered by a regular hexagon of side length $1/\sqrt{3}$. Dissecting the hexagon into three pentagons yields the sharp upper bound $\sqrt{3}/2$ for the diameter of the pieces. The same upper bound was obtained by Gale [1953], who obtained it by dissecting a suitable truncation of an equilateral triangle of side-length $\sqrt{3}$.

For sets of constant width some stronger theorems were proved. For a convex disk $K$, let $d(K)$ denote the smallest number with the property that $K$ can be covered by three sets of diameter $d(K)$. Lenz [1956] proved that for a set $K$ of constant width 1 the inequality $\sqrt{3} - 1 \leq d(K) \leq \sqrt{3}/2$ holds, where equality is reached in the upper bound only for the circle, and in the lower bound only for the Reuleaux triangle. Let $l(K)$ and $L(K)$ denote the side-length of the largest equilateral triangle inscribed in $K$ and the side-length of the smallest equilateral triangle circumscribed about $K$, respectively. Melzak [1963] proved that a set $K$ of constant width 1 can be covered by three sets of diameter at most $\min\{l(K), \sqrt{3} - l(K)\}$ and Schopp [1977] proved that it can be covered by three circular disks of diameter $L(K)/2$. Schopp also proved that $\sqrt{12} - 2 \leq L(K) \leq \sqrt{3}$ for every set of constant width 1. In a theorem of Chakerian and Sallee [1969] the roles of disks is changed to the opposite: They proved that every convex disk of unit diameter can be covered by three copies of any set of constant width 0.9101.

Eggleston [1955] settled the three-dimensional case of Borsuk’s conjecture by a rather complicated analytic argument. Simpler proofs were given by Grünbaum...
Both Grünbaum and Heppes started with the result of Gale [1953] that every set of diameter 1 can be imbedded in a regular octahedron the distance between whose opposite faces is 1. Then they observed that suitably truncating the octahedron at three vertices, the resulting polyhedron still can contain every body of diameter 1, and can be partitioned into four sets of diameter less than 1. For the diameter of the pieces Heppes proved the bound 0.9977..., while with a more detailed analysis Grünbaum got the bound 0.9885... The presently best known bound, 0.98, is due to Makeev [1997], who obtained it by proving that every convex body of diameter 1 is contained in a rhombic dodecahedron with parallel faces at distance 1 apart. This last statement was also proved by Hausel, Makai and Szücs [2000] and by G. Kuperberg [1999]. Katzarowa-Karanowa [1967] proved that every set of diameter 2 in three-dimensional space can be covered by four unit balls and she stated that her method can be used to lower the radii of the balls to 0.999983.

For finite sets of points in three dimensions a simple proof was given by Heppes and Révész [1956]. The proof of this case follows by induction from the following conjecture of Vázsonyi (see Erdős [1946]): Among a set of \( n \geq 4 \) points in \( E^3 \) there are at most \( 2n - 2 \) pairs realizing the diameter of the set. Proofs of Vázsonyi’s conjecture were given by Grünbaum [1956], Heppes [1956], Straszewicz [1957], and Dol’nikov [2000]. These proofs use the ball polytope obtained by taking the intersection of the balls centred at the points of the set with radius equal to the diameter of the set. A proof avoiding the use of ball polytopes was given by Swanepoel [2008].

Rissling [1971] proved that every set in the three-dimensional hyperbolic space can be divided into four parts of smaller diameter and the same is true for the spherical space for sets whose diameter is smaller than \( \pi/3 \).

The theorem that Borsuk’s conjecture holds for smooth convex bodies is generally credited to Hadwiger [1946]. Hadwiger only proved that Borsuk’s conjecture holds for a smooth body of constant width. From this, the validity of Borsuk’s conjecture for general smooth convex bodies follows, if we know that every smooth convex body can be enclosed in a smooth body of constant width of the same diameter. However, this was proved only later by Falconer [1981] and Schulte [1981]. Direct proofs of the Borsuk conjecture for smooth convex bodies were given by Lenz [1956] and Melzak [1967]. The condition of smoothness was weakened by Dekster [1993] who proved that the conjecture holds for every convex body for which there exists a direction in which any line tangent to the body contains at least one point of the body’s boundary at which the tangent hyperplane is unique.

Consider a convex body \( K \) in \( E^n \) for which to any boundary point \( x \) of \( K \) there is a ball of radius \( r \) contained in \( K \) and containing the point \( x \). This means that a ball of radius \( r \) can freely roll in \( K \). Hadwiger [1948] proved the upper bound \( d - 2r \left( 1 - \sqrt{1 - \frac{1}{n}} \right) \) for the diameter of the pieces in an optimal partition of such bodies into \( n + 1 \) parts. Dekster [1989] proved a similar bound, namely \( \sqrt{d^2 - r^2 \frac{1 - \sqrt{1 - \frac{4(n+3)}{1+\sqrt{1+4/(n+3)}}}}{1+\sqrt{1+4/(n+3)}}} \) for odd \( n \), and \( \sqrt{d^2 - r^2 \frac{1}{n+1}} \) for even \( n \).

The Borsuk conjecture is known to be true for some further special cases: Rissling [1971] proved it for centrally symmetric sets, Rogers [1971, 1981] for sets whose symmetry group contains that of the regular simplex, and Kołodziejczyk [1988]
for sets of revolution. The result of Kołodziejczyk was obtained independently by Dekster 1995, who proved the same also for hyperbolic and spherical spaces.

The obvious simplicial decomposition shows that $B^n$ can be decomposed into $n + 1$ subsets of diameter $(\frac{n+1}{n+2})^{1/2}$, if $n$ is even, and $(\frac{1}{2} + \frac{1}{2} (\frac{n-1}{n+3})^{1/2})^{1/2}$ if $n$ is odd. It is conjectured that this is the lower bound for the diameter $d$ of the pieces. Hadwiger 1954 confirmed this for $n \leq 3$ and proved the lower bound $d \geq \left(\frac{1}{2} + \frac{1}{2} (\frac{n-1}{2n})^{1/2}\right)$ for $n \geq 4$. Larman and Tamvakis 1984 improved Hadwiger’s bound to $d \geq 1 - \frac{3}{2n} \log n + O\left(\frac{1}{n}\right)$.

Let $b(n)$ denote the $n$-th Borsuk number, that is, the smallest integer such that every bounded set in $n$-dimensional space can be partitioned into $b(n)$ subsets of smaller diameter. Lassak 1982 established the upper bound $b(n) \leq 2^{n-1} + 1$. Lassak’s result was improved significantly by Schramm 1988 and Bourgain and Lindenstrauss 1991 to $b(n) \leq 1^{\sqrt{n}} + o(1)$, which is the best presently known bound.

Despite the fact that some doubt in the truth of the conjecture was announced by Erdős 1981, Larman 1984, and Rogers 1971, it came as a surprise when Kahn and Kalai 1993 proved that the conjecture fails in all dimensions $n \geq 1015$. Moreover, they proved that $b(n) \geq (1.2)^{\sqrt{n}}$ for sufficiently large $n$. The best lower bound for $b(n)$ presently known is $b(n) \geq (\sqrt{\frac{2}{\sqrt{3}}} + o(1))^{\sqrt{n}} = (1.2255 \ldots + o(1))^{\sqrt{n}}$, due to Raigorodskii 1999. Kahn and Kalai 1993 claimed without giving any details that the Borsuk conjecture fails in dimension $n = 1325$. Weissbach 2000 pointed out that this statement does not follow from the argument of Kahn and Kalai (see also Jenrich 2018).

The lower bound for the dimension $n$ in which the Borsuk conjecture fails was lowered by Nill 1994 to 946, by Grey and Weissbach 1997 to 903, by Raigorodskii, 1997 to 561, by Weissbach 2000 to 560, by Hinrichs 2002 to 323, by Pikhurko 2002 to 321, by Hinrichs and Richter 2003 to 298, and by Bondarenko 2014 to 65. The last step thus far was done by Jenrich and Brouwer 2014, who found a 64-dimensional subset of 352 points of the set constructed by Bondarenko that cannot be divided into fewer than 71 parts of smaller diameter.

The notion of the $k$-fold Borsuk number of a set was introduced by Hujter and Lángi 2014 as follows. Let $S$ be a set of diameter $d > 0$. The smallest positive integer $m$ such that there is a $k$-fold covering of $S$ with $m$ sets of diameters strictly smaller than $d$, is called the $k$-fold Borsuk number of $S$. Besides presenting a few other results concerning this notion, Hujter and Lángi determined the $k$-fold Borsuk number for every bounded planar set. Lángi and Naszódi 2017 investigated multiple Borsuk numbers in normed spaces.

For comprehensive surveys of the topic see Grünbaum 1963, Raigorodskii 2004, 2007, 2008, Kalai 2015, and the corresponding chapters of the books.
2. Tarski’s plank problem

In [1932], Tarski raised the following problem: Is it true that if a convex body $C$ of width $w$ is covered by parallel slabs with the widths $w_1, \ldots, w_l$, then $w_1 + \ldots + w_l \geq w$? If $C$ is a circle the solution was given by Moese [1932]. Straszewicz [1948] solved the problem in the plane for two strips. An affirmative answer to the question for general convex bodies was given by Bang [1950, 1951]. Variations of Bang’s proof were given by Fenchel [1951] and Bognár [1961].

The width of a slab relative to a convex body $C$ is the width of the slab divided by the width of $C$ in the direction perpendicular to the slab. Bang [1950] asked whether the following generalization of his theorem is true. If some $s$ labs cover a convex body $C$ then the sum of the widths of the slabs relative to $C$ is at least 1. For centrally symmetric bodies Bang’s question was answered in the affirmative by Ball [1991]. As a corollary Ball proved the following theorem. Given a centrally symmetric convex body $C$ and $n$ hyperplanes in $n$-dimensional Euclidean space, then there is a translate of $\frac{1}{n+1}C$ inside $C$ whose interior does not meet any of the hyperplanes. The result is obviously sharp for every $n$ and $C$ and is a generalization of a result by Davenport [1962] who considered the special case when $C$ is a cube. For non-symmetric sets $C$ Bang’s problem was solved only for coverings of $C$ by two slabs (see Bang [1954], Moser [1958], Alexander [1968], and Hunter [1964]).

Related to the above corollary is Conway’s fried potato problem, phrased by Croft, Falconer and Guy [1994, Problem C1, p. 80] as follows. “In order to fry it as expeditiously as possible Conway wishes to slice a given convex potato into $n$ pieces by $n-1$ successive plane cuts (just one piece being divided by each cut) so as to minimize the greatest inradius of the pieces.” This problem was solved by A. Bezdek and K. Bezdek [1995]. In A. Bezdek and K. Bezdek [1996] the problem is generalized and solved for the case in which the role of the inradius is played by the maximum positive coefficient of homothety of a given convex body contained in the slices.

Ohmann [1953] proved the following generalization of the planar case of Bang’s theorem: If a convex disk is covered by a finite family of convex disks, then the sum of the inradii of the covering disks is at least as large as the inradius of the covered domain. Theorem 1 in A. Bezdek [2007] directly implies the same result. Kadets [2005] extended this result to $n$ dimensions. Instead of the inradius Akopyan and Karasev [2012] measured a convex body $B$ by the size $r_K(B) = \sup\{h \geq 0 : h K + t \subset B\}$ of the greatest positively homothetic copy of a given convex body $K$ contained in it. They proved that if in the plane $C_1, \ldots, C_k$ form a convex partition of the convex disk $K$, then $\sum_{i=1}^k r_K(C_i) \geq 1$. In higher dimensions they proved the analogous statement for special partitions only. It should be mentioned that not all coverings can be reduced to a partition. The question whether an analogous result holds for coverings remains open.

A. Bezdek [2003] made the following conjecture: For every convex disk there exists an $\varepsilon > 0$ such that the minimum total width of planks needed to cover the annulus obtained by removing from the disk its $\varepsilon$-homothetic copy contained in its interior is the same as for the whole disk. He verified the conjecture for the case of a square with $\varepsilon = 1 - \sqrt{2}/2$. He further supported the conjecture by proving
that it is true for every polygon whose incircle is tangent to two of its parallel sides. It turned out that the conjecture stated in such generality is false, as White and Wisewell [2007] noticed. In fact, they characterized all convex polygons for which Bezdek’s conjecture holds as the polygons with no minimum-width chord that meets a vertex and divides the angle at that vertex into two acute angles.

Zhang and Ding [2008] showed, with a very short proof, that the equilateral triangle with an arbitrarily small hole placed anywhere in its interior is a counterexample to Bezdek’s conjecture. Also, they gave a positive result for parallelograms. Smurov, Bogataya and Bogatyĭ [2010] proved that Bezdek’s conjecture holds for the cube of dimension \( n \geq 2 \), even with infinitely many cubical holes, each homothetic to the covered cube, if the hole’s total edge length is sufficiently small.

L. Fejes Tóth [1973c] considered the following problem: Place \( k \) great circles on a sphere so that the maximum inradius of the regions into which the circles partition the sphere is as small as possible. He conjectured that in the optimal arrangement the great circles dissect the sphere into the regular tiling \( \{2,2k\} \) with congruent digonal faces. Rosta [1972] confirmed the conjecture for \( k = 3 \) and Linhart [1974a] proved it for \( k = 4 \).

A great circle of the unit sphere and a number \( r > 0 \) define a zone of width \( 2r \) on the sphere, consisting of points that are at a distance at most \( r \) from the great circle. A different way to state the same problem is: Find the smallest number \( r_k \) such that the sphere can be covered by \( k \) zones of width \( 2r_k \). A zone of width \( 2r_k \) has area \( \sin r_k \), thus \( w_k \leq \arcsin 1/k \). Fodor, Vígh and Zarnócz [2016a] gave an improvement of this trivial bound. This reformulation of the problem gives rise to the following more general conjecture: If the sphere is covered by a finite number of zones, then the total width of the zones is at least \( \pi \). Jiang and Polyanskii [2017] gave a short, elegant proof of this conjecture valid in all dimensions. Ortega-Moreno [2019] gave an alternative proof for the special case of zones of the same width. As a corollary of their theorem Jiang and Polyanskii also proved that if a centrally symmetric convex body on the sphere is covered by zones of total width \( w \), then it can be covered by one zone of width \( w \).

Another spherical version of Tarski’s plank problem, on covering the \( n \)-dimensional spherical ball with a family of spherical convex bodies was considered by K. Bezdek and Schneider [2010]. In their theorem, the inradius of a set used for the covering plays the role of the width of a plank: If on the \( n \)-dimensional sphere the spherically convex bodies \( C_1, \ldots, C_m \) cover a spherical ball \( B \) of radius \( r(B) \geq \pi/2 \), then the sum of their inradii is greater than or equal to \( r(B) \).

Steinerberger [2018] gave lower bounds for the sum of the \( s \)’th power of the areas of pairwise intersections of \( n \) congruent zones on \( S^2 \). His bound is asymptotically sharp for \( 0 < s < 1 \). Bezdek, Fodor, Vígh and Zarnócz [2019] investigated the multiplicity of points covered by zones. They showed that it is possible to arrange \( n \) congruent zones of suitable width on \( S^{d-1} \) such that no point belongs to more than a constant number of zones, where the constant depends only on the dimension and the width of the zones. They also proved that it is possible to cover \( S^{d-1} \) by \( n \) equal zones such that each point of the sphere belongs to at most \( cd\ln n \) zones.

Concerning the Tarski plank problem and its generalizations, we refer the reader to the book and survey by K. Bezdek [2010, 2014].
3. The Kneser-Poulsen problem

The following attractive problem was stated independently by Poulsen [1954] and Kneser [1955]: If in $n$-dimensional Euclidean space finitely many balls are rearranged so that no distance between their centers increases, then the volume of their union does not increase. The problem turned out to be more difficult than it appears, even in the plane. The first result supporting the conjecture was obtained by Habicht (see Kneser [1955, p. 388]) and Bollorás [1968], who proved it for congruent circular disks under the assumption that the rearrangement is the result of a continuous motion during which all distances between the disks’ centers change monotonically. Csikós [1997] and, independently, Bern and Sahai [1998] extended this result to arbitrary circular disks. Soon after he published this result, Csikós [1998] generalized it to balls in every dimensions.

Under the assumption of a continuous motion of the ball’s centers the monotonicity of the volume of the intersection also holds in spherical and hyperbolic space (see Csikós [2001]). However, Csikós and Moussong [2006] showed that in the $n$-dimensional elliptical (real projective) space the conjecture is false. Yet, in spite of the counterexample, a configuration of $n+1$ balls reaches maximum volume of their union if the distances between their centers become equal to $\pi/2$, the diameter of the space.

In his proof, Csikós represents the moving configuration of $N$ balls of radii $r_1, \ldots, r_N$, centered at $P_1(t), \ldots, P_N(t)$ in $E^n$ ($0 \leq t \leq 1$), by a single moving point $P(t) = (P_1(t), \ldots, P_N(t))$ in the configuration space $R^{nN}$ and he assumes that, while the distances between $P_i(t)$ and $P_j(t)$ do not increase for $i, j = 1, \ldots, N$, the function $P(t)$ is analytic. Then he derives a formula expressing the derivative of the volume of the union of the balls as a linear combination of the derivatives of the distances between their centers with nonnegative coefficients. This yields directly that the volume of the union does not increase. K. Bezdek and Connelly [2002], suitably modifying Lemma 1 of Alexander [1985, p. 664] and using the volume formula derived by Csikós [1998], succeeded in confirming the planar case of the Kneser-Poulsen conjecture.

The conjecture that under a contraction of the centers the volume of the intersection of a set of balls does not decrease was stated by Gromov [1987] and Klee and Wagon [1991] Problem 3.1]. The special case of the conjecture for congruent circular disks and continuous motion was established by Capoyleas [1996] before K. Bezdek and Connelly [2002] proved it in full generality for the plane.

For a compact set $M$ in a space of constant curvature, consider those balls $B \subset M$ (of possibly zero radius) that are not a proper subset of any ball $B' \subset M$. The set of centers of these balls is called the center set of $M$. Gorbovichkis [2018] invented a method dealing with Kneser–Poulsen-type problems based on the investigation of the properties of central sets. He proved that if on a plane of constant curvature the union of a finite set of (not necessarily congruent) closed circular disks has a simply connected interior, then the area of the union of these disks cannot increase after any contractive rearrangement. We emphasise the following corollary of the spherical case of this theorem: (i) If a finite set of circular disks on the sphere with radii not smaller than $\pi/2$ is rearranged so that the distance between each pair of centers does not increase, then the area of the union of the disks does not increase. (ii) If a finite set of disks on the sphere with radii not greater than $\pi/2$ is rearranged so that the distance between each pair of centers does not increase, then the area of
the intersection of the disks does not decrease. For two dimensions, this generalizes the result of K. Bezdek and Connelly [2004] in which they proved the analogous statement for \( n \) dimensions but only for hemispheres. The limiting case of the application of claim (ii) when the radii approach zero yields an alternative proof of the corresponding theorem in the Euclidean plane. With a suitable adaptation of Gorbovickis' method Csikós and Horváth [2018] also proved the monotonicity of the area of the intersection of circular disks on the hyperbolic plane.

Alexander [1985] conjectured that, under an arbitrary contraction of the centers of finitely many congruent circles, the perimeter of the intersection of the circles does not decrease. The proof of this conjecture does not seem to lie within reach. K. Bezdek, Connelly and Csikós [2008] settled some special cases of the conjecture, among other cases, they proved it for four circles. The weaker result concerning the perimeter of the convex hull of the circles was proved by Sudakov [1971], rediscovered by Alexander [1985] and extended to the hyperbolic plane and to the hemisphere by Csikós and Horváth [2018]. For the Euclidean case, a simpler proof was given by Capoyleas and Pach [1991], who also established a similar result in the case where the Euclidean norm is replaced by the maximum norm.

In dimensions higher than 2, for non-continuous contractions, there are only a few partial results concerning the Kneser–Poulsen conjecture. In \( n \) dimensions, the conjecture was verified for \( n + 1 \) balls by Gromov [1987]. K. Bezdek and Connelly [2002] extended Grovov’s result to at most \( n + 3 \) balls. The conjecture is proved for special arrangements of balls, e.g. for a small number of intersections or large radii by Gorbovickis [2013, 2014]. K. Bezdek and Naszódi [2018a] proved that none of the intrinsic volumes of the intersection of \( k \geq (1 + \sqrt{2})^d \) congruent balls in \( E^n \) decreases under uniform contractions of the centers, that is under contractions where all the pairwise distances in the first set of points are larger than all the pairwise distances in the second set of points. K. Bezdek [2019a] gave an alternative proof of a slightly stronger theorem. Moreover, in [2020] he proved that in a Minkowski space, under uniform contraction of the centers, the volume of both the intersection and the union of the balls changes monotonously.

The Kneser-Poulsen conjecture makes sense in any connected Riemannian manifold. However, Csikós and Kunszenti-Kovács [2010] proved that if the conjecture is true in a Riemannian manifold, then the manifold must be of constant curvature. Moreover, Csikós and Horváth [2014] showed that the same consequence holds even for balls of the same radii.

Further results and generalizations on this topic are found in Csikós [2001] and K. Bezdek and Connelly [2008]. The articles by K. Bezdek [2008] and Csikós [2018] contain surveys on the subject of the Kneser-Poulsen conjecture.

4. Covering a convex body by smaller homothetic copies

Hadwiger [1957], Levi [1955] and Gohberg and Markus [1960], independently of each other, asked for the smallest integer \( h(K) \) such that a given \( n \)-dimensional convex body \( K \) can be covered by \( h(K) \) smaller positively homothetic copies of \( K \). A boundary point \( x \) of the convex body \( K \) is illuminated from the direction of a unit vector \( u \) if the ray issuing from \( x \) in the direction of \( u \) intersects the interior of \( K \). Let \( i(K) \) denote the minimum number of directions from which the boundary of \( K \) can be illuminated. The problem of finding the maximum value of
$i(K)$ was raised by Boltjanski˘ı [1960] and in a slightly different, but for compact sets equivalent form, by Hadwiger [1960]. Boltjanski˘ı observed that for convex bodies $h(K) = i(K)$. Both Boltjanski˘ı and Hadwiger conjectured that

$$i(K) \leq 2^n$$

for every $n$-dimensional convex body $K$ and that equality holds only for paralleloptopes.

Both Levi [1955] and Gohberg and Markus [1960] verified the conjecture for the plane. Lassak [1980] proved the sharp result that every convex disk can be covered by four homothetic copies with ratio $\sqrt{2}/2$. An extreme example is the circle. The conjecture remains open for $n \geq 3$. In three dimensions it was proved for centrally symmetric convex bodies by Lassak [1984], for convex bodies symmetric in a plane by Dekster [2000], and for convex polyhedra with an arbitrary affine symmetry by K. Bezdek [1991].

There is a great variety of results confirming the conjecture for special classes of bodies in $E^n$ by establishing upper bounds for $h(K)$ or $i(K)$ smaller than $2^n$. Schramm [1988] proved that if $K$ is a set of constant width then $i(K) < 5n\sqrt{n}(4 + \ln n)\left(\frac{4}{n}\right)^{n/2}$. K. Bezdek [2011, 2012b] extended Schramm’s inequality to a wider class of convex bodies, namely for those convex bodies $K$ that are the intersection of congruent balls with centers in $K$. Martini [1987] proved the bound $h(K) \leq 3 \cdot 2^{n-2}$ for every zonotope other than a paralleloptope. The bound $h(K) \leq 3 \cdot 2^{n-2}$, $K$ not a paralleloptope, was verified also for zonoids by Boltjanski˘ı and P. S. Soltan [1992], and Boltjanski˘ı and Martini [2001] characterized those belt bodies $K$ for which $h(K) = 3 \cdot 2^{n-2}$. K. Bezdek and Bisztriczky [1997] proved the conjecture for dual cyclic polytopes. For the dual $P$ of an $n$-dimensional cyclic polytope Talata [1999b] proved the inequality $(n + 1)(n + 3)/4 \leq h(P) \leq (n + 1)^2/2$ and, for the case when $n$ is even, he gave the sharp bound $h(P) \leq (n + 2)/2$. Tikhomirov [2014] considered convex bodies $K$ in $E^n$ with the property that for any point $(x_1, \ldots, x_n) \in K$, any choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and any permutation $\sigma$ on $n$ elements $\varepsilon_1x_{\sigma_1}, \ldots, \varepsilon_1x_{\sigma_n} \in K$. He proved that for sufficiently large $n$, we have $i(K) < 2^n$ for every such convex body $K$ different from a cube. Smooth convex bodies in $E^n$ can be illuminated by $n + 1$ directions, see e.g. Boltjanski˘ı and Gohberg [1985, Theorem 9, p. 61]. Dekster [1994] extended this result by replacing the assumption of smoothness of the body by a certain smoothness of just a single belt of the body.

Livshyts and Tikhomirov [2020a] proved that the cube represents a strict local maximum for these problems: If a convex body, that is not a paralleloptope, is close to the cube in the Banach–Mazur metric, then it can be covered by $2^n - 1$ smaller homothetic copies of itself, and $2^n - 1$ light sources suffice to illuminate its boundary.

A simple consequence of the upper bound of Rogers for the translational covering density combined with the Rogers–Shephard inequality for the volume of the difference body is that $h(K) \leq \binom{2n}{n}(n \log n + \log n + 5n)$ for every convex body and $h(K) \leq 2^n(n \log n + \log n + 5n)$ for centrally symmetric convex bodies in $E^n$ (see Rogers and Zong [1997]). An improvement by a sub-exponential factor was recently obtained by Huang, Slomka, Tkocz and Vritsiou [2018]. For general convex bodies their bound is of the order of $(\frac{2n}{n})e^{-c\sqrt{n}}$ for some universal constant.
For low dimensions better bounds were given by Lassak [1988] and Prymak and Shepelska [2020]. In three dimensions Lassak [1998] proved \( h(K) \leq 20 \) which was improved by Papadoperakis [1999] to \( h(K) \leq 16 \) and subsequently by Prymak [2021] to 14.

A cap body of a ball is the convex hull of a closed ball \( B \) and a countable set \( \{ p_i \} \) of points outside the ball such that for any pair of distinct points \( p_i, p_j \) the line segment \( p_i p_j \) intersects \( B \). The difficulty of the illumination problem is exposed by the following example of Naszódi [2016b]. Clearly, we have \( i(B^d) = d + 1 \). On the other hand, for any \( \varepsilon > 0 \) there is a centrally symmetric cap body \( K \) of \( B^d \) and a positive constant \( c = c(\varepsilon) \) such that \( K \) is \( \varepsilon \)-close to \( B^d \) and \( i(K) \geq c^d \). Ivanov and Strachan [2021] studied the illumination number of cap bodies in 3 and 4 dimensions and proved that \( i(K) \leq 6 \) for centrally symmetric cap bodies of a ball in \( E^3 \), and \( i(K) \leq 8 \) for unconditionally symmetric cap bodies of a ball in \( E^4 \).

Weighted illumination was introduced by Naszódi [2009] and weighted covering was introduced by Artstein-Avidan and Raz [2011] and Artstein-Avidan and Slomka [2015]. A collection of weighted light sources illuminates a convex body \( K \) if for every boundary point \( x \) of \( K \) the total weight of light sources illuminating \( x \) is at least 1. Similarly, a collection of weighted bodies covers \( K \) if for every point \( x \) of \( K \) the total weight of bodies containing \( x \) is at least 1. The weighted or fractional illumination number \( i^*(K) \) of \( K \) is the infimum of the total weight of light sources illuminating \( K \). The weighted covering number \( h^*(K) \) of \( K \) is the infimum of the total weight of smaller weighted homothetic copies of \( K \) covering \( K \). Naszódi [2009] conjectured that \( i^*(K) \leq 2^n \) for every \( n \)-dimensional convex body \( K \). He proved the conjecture for centrally symmetric bodies and established the inequality \( i^*(K) \leq \binom{2n}{n} \) for general convex bodies. Artstein-Avidan and Slomka [2015] proved that \( h^*(K) \leq 2^n \) for centrally symmetric convex bodies \( K \subset E^n \) with equality only for a parallelepiped and \( h^*(K) \leq \binom{2n}{n} \) for general convex bodies. They pointed out that the proof of the equivalence between the illumination problem and the Levi-Hadwiger covering problem carries over to the weighted setting. This way they gave an alternative proof of Naszódi’s result.

Bezdek and Lángi [2020a] considered the illumination problem on the sphere. A boundary point \( q \) of a convex body \( K \) in \( S^d \) is illuminated from a point \( p \in S^n \setminus K \) if it is not antipodal to \( p \), the spherical segment with endpoints \( p \) and \( q \) does not intersect the interior of \( K \), but the great-circle through \( p \) and \( q \) does. The illumination number of \( K \) is the smallest cardinality of a set that illuminates each boundary point of \( K \) and lies on an \((n - 1)\)-dimensional great-sphere of \( S^n \) which is disjoint from \( K \). Bezdek and Lángi proved that the illumination number of every convex polytope in \( S^n \) is \( n + 1 \) and raised the question whether there is a convex body in \( S^n \) whose illumination number is greater than \( n + 1 \).

The paper by Naszódi [2016a] surveys different problems about covering, among others the Hadwiger-Levi problem. For literature and further results concerning the illumination problem we refer to the surveys by K. Bezdek [1993, 2005], K. Bezdek and Khan [2003], Boltjanskii and Gohberg [1995], Szabó [1997], Martini and Soltan [1999] and to the book by Boltjanskii, Martini and P. S. Soltan [1997].
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