Stable Exact Solutions in Cosmological Models with Two Scalar Fields

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Abstract

The stability of isotropic cosmological solutions for two-field models in the Bianchi I metric is considered. We prove that the sufficient conditions for the Lyapunov stability in the Friedmann–Robertson–Walker metric provide the stability with respect to anisotropic perturbations in the Bianchi I metric and with respect to the cold dark matter energy density fluctuations. Sufficient conditions for the Lyapunov stability of the isotropic fixed points of the system of the Einstein equations have been found. We use the superpotential method to construct stable kink-type solutions and obtain sufficient conditions on the superpotential for the Lyapunov stability of the corresponding exact solutions. We analyze the stability of isotropic kink-type solutions for string field theory inspired cosmological models.

1 Introduction

To specify different components of the cosmic fluid one typically uses a phenomenological relation \( p = w\varrho \) between the pressure (Lagrangian density) \( p \) and the energy density \( \varrho \) corresponding to each component of the fluid. The function \( w \) is called the state parameter. Contemporary observations \([1]\) give strong support that in the Universe, the uniformly distributed cosmic fluid with negative pressure, the so-called dark energy, currently dominates with a state parameter value approximately equal to \(-1\):

\[ w_{DE} = -1 \pm 0.2. \]

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Strong restrictions on the anisotropy were found using observations \[2, 3\], and it was also shown that the Universe is spatially flat at large distances. It has been shown in \[4, 5, 6\] (see also reviews \[7, 8\] and references therein), that the recent analysis of the observation data indicates that the time dependent state parameter gives a better fit than \(w_{DE} = -1\), corresponding to the cosmological constant. It, in particular, gives reasons for the interest in models with \(w_{DE} < -1\). The field theory with \(w_{DE} < -1\) is also interesting as a possible solution of the cosmological singularity problem \[9, 10, 11\].

The standard way to obtain an evolving state parameter is to add scalar fields into a cosmological model. Two-field models with the crossing of the cosmological constant barrier \(w_{DE} = -1\) are known as quintom models and include one phantom scalar field and one ordinary scalar field. Quintom models are being actively studied at present time \[7, 8, 12, 13, 14, 15, 16\]. The cosmological models with \(w_{DE} < -1\) violate the null energy condition (NEC), this violation is generally related to the phantom fields appearing. The standard quantization of these models leads to instability, which is physically unacceptable. In \[13\], the theory with \(w_{DE} < -1\) was interpreted as an approximation in the framework of the fundamental theory. Because the fundamental theory must be stable and must admit quantization, this instability can be considered an artefact of the approximation.

We note the problems of quantum instability that arise in effective theories and are related to high order derivatives \[17, 18\]. In \[19, 20\] the instability problem has been reduced to the problem of such choice of the effective theory parameters that the instability turns out to be essential only at times that are not described in the framework of the effective theory approximation. In the mathematical language it means that the terms with higher-order derivatives can be treated as corrections essential only at small energies below the physical cut-off. This approach implies the possibility to construct a UV completion of the theory and to consider these effective theories physically acceptable with the presumption that an effective theory admits immersion into a fundamental theory.

In the Friedmann–Robertson–Walker (FRW) metric the NEC violating models can have classically stable solutions cosmology. In particular, there are classically stable solutions for ghost models with minimal coupling to gravity. Moreover, there exists an attractor behavior in a class of the phantom cosmological models \[21, 22, 14\] (attractor solutions for inhomogeneous cosmological models were considered in \[23\]).

The stability of isotropic solutions in the Bianchi models \[24, 25, 26\] (see also \[27\]) has been considered in inflationary models (see \[28, 29\] and references therein for details of anisotropic slow-roll inflation). Assuming that the energy conditions are satisfied, it has been proved that all initially expanding Bianchi models except type IX approach the de Sitter space-time \[30\] (see also \[31, 32, 33, 34\]). The Wald theorem \[30\] shows that for space-time of Bianchi types I–VIII with a positive cosmological constant and matter satisfying the dominant and strong energy conditions, solutions which exist globally in the future have certain asymptotic properties at \(t \to \infty\).

The standard way to analyse the stability of solutions for Friedmann equations in quintom models uses the change of variables \[12, 14, 15\] (see also \[8\]). In the case of exponential potentials such a transformation is useful, because it transforms the class of nontrivial solutions to fixed points of the new system of equations \[12\]. We show that the stability conditions
for an arbitrary potential that were found in [14] can be obtained without introducing any new variables.

Studying the stability of solutions in the FRW metric, we specify a form of fluctuations. It is interesting to know whether these (isotropic) solutions are stable under the deformation of the FRW metric to an anisotropic one, for example, to the Bianchi I metric. In comparison with general fluctuations we can get an explicit form of fluctuations in the Bianchi I metric, which can probably clarify some nontrivial issues of theories with the NEC violation.

In this paper we consider the stability of isotropic solutions in the Bianchi I metric. Interpreting the solutions of the Friedmann equation as isotropic solutions in the Bianchi I metric, we include anisotropic perturbations in our consideration. The stability analysis is essentially simplified by a suitable choice of variables [25, 27, 35]. In this paper we show that for an arbitrary potential the stability conditions, obtained in [14], are sufficient for stability not only in the FRW metric, but also in the Bianchi I metric. We also analyse the stability with respect to small fluctuations of the initial value of the cold dark matter energy density. We consider quintom cosmological models, as well as models with two scalar (or two phantom scalar) fields.

The stability of a continuous solution tending to a fixed point implies the stability of this fixed point. Using the Lyapunov theorem [36, 37] we find conditions under which the fixed point and the corresponding kink (or lump) solution are stable. For one field models the sufficient conditions for stability of isotropic solutions, which tend to fixed points, have been obtained in [38].

In [13, 39] the superpotential method has been used to construct quintom models with exact solutions. In this paper we get the stability conditions in terms of the superpotential and use the superpotential for construction of two-field models with stable exact solutions. We also verify the stability of solutions, obtained in the string field theory (SFT) inspired quintom models [13, 39].

The paper is organized as follows. In Section 2 we consider two-field model with an arbitrary potential and we find the sufficient stability conditions in the FRW and Bianchi I metrics. In Section 3 we remind the superpotential method and obtain conditions on the superpotential which are sufficient for the stability of exact solutions. In Section 4 we check the stability of kink-type solutions in a few SFT inspired cosmological models. In Section 5 we summarize the results and make a conclusion. In Appendix we present the stability conditions on the superpotential in the case of one-field cosmological models.

## 2 Sufficient stability conditions

### 2.1 The Einstein equations in the Bianchi I metric

We consider a two-field cosmological model with the following action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G_N} - \frac{C_1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{C_2}{2} g^{\mu\nu} \partial_\mu \xi \partial_\nu \xi - V(\phi, \xi) \right], \quad (1) \]
where the potential $V(\phi, \xi)$ is a twice continuously differentiable function, which can include the cosmological constant $\Lambda$, $G_N$ is the Newtonian gravitational constant ($8\pi G_N = 1/M_P^2$, where $M_P$ is the Planck mass). Each of the fields $\phi$ and $\xi$ is either scalar or phantom scalar fields in dependence on signs of the constants $C_1$ and $C_2$.

Let us consider the Bianchi I metric

$$ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2.$$  

(2)

It is convenient to express $a_i$ in terms of new variables $a$ and $\beta_i$ (we use the notation in [35]):

$$a_i(t) = a(t)e^{\beta_i(t)}.$$  

(3)

Imposing the constraint

$$\beta_1 + \beta_2 + \beta_3 = 0,$$  

(4)

one has the following relations

$$a(t) = (a_1(t)a_2(t)a_3(t))^{1/3},$$  

(5)

$$H_i \equiv \dot{a}_i/a_i = H + \dot{\beta}_i, \quad \text{and} \quad H \equiv \dot{a}/a = \frac{1}{3}(H_1 + H_2 + H_3),$$  

(6)

where the dot denotes the time derivative. To obtain (6) we have used the following consequence of (4):

$$\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3 = 0.$$  

(7)

Note that $\beta_i$ are not components of a vector and, therefore, are not subjected to the Einstein summation rule. In the case of the FRW metric all $\beta_i$ are equal to zero and $H$ is the Hubble parameter. Following [25, 35] (see also [27]) we introduce the shear

$$\sigma^2 \equiv \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2.$$  

(8)

The Einstein equations have the following form:

$$3H^2 - \frac{1}{2}\sigma^2 = 8\pi G_N \varrho,$$  

(9)

$$2\dot{H} + 3H^2 + \frac{1}{2}\sigma^2 = -8\pi G_N p,$$  

(10)

$$\dot{\phi} = \psi, \quad \dot{\psi} = -3H\psi - \frac{1}{C_1} \frac{\partial V}{\partial \phi},$$  

(11)

$$\dot{\xi} = \zeta, \quad \dot{\zeta} = -3H\zeta - \frac{1}{C_2} \frac{\partial V}{\partial \xi},$$  

(12)

where

$$\varrho = \frac{C_1}{2} \dot{\phi}^2 + \frac{C_2}{2} \dot{\xi}^2 + V(\phi, \xi), \quad p = \frac{C_1}{2} \dot{\phi}^2 + \frac{C_2}{2} \dot{\xi}^2 - V(\phi, \xi).$$  

(13)

For $\beta_i$ and $\sigma^2$ we obtain the following equations

$$\ddot{\beta}_i = -3H\dot{\beta}_i,$$  

(14)
\[ \frac{d}{dt} (\sigma^2) = -6H\sigma^2. \quad (15) \]

Functions \( H(t), \sigma^2(t), \phi(t), \xi(t), \) and \( \rho(t) \) can be obtained from equations (9)–(12) and (15). If \( H(t) \) is known then \( \beta_i \) can be trivially obtained from (14). We show in the next subsection that functions \( H(t), \dot{\beta}_i(t) \) and \( \sigma^2(t) \) are very suitable to analyse the stability of isotropic solutions in the Bianchi I metric.

### 2.2 Sufficient conditions for the Lyapunov stability of a fixed point

Summing equations (9) and (10) we obtain the following system

\[ \begin{align*}
\dot{H} &= -3H^2 + 8\pi G_N V(\phi, \xi), \\
\dot{\phi} &= \psi, \\
\dot{\psi} &= -3H\psi - \frac{1}{C_1} \frac{\partial V}{\partial \phi}, \\
\dot{\xi} &= \zeta, \\
\dot{\zeta} &= -3H\zeta - \frac{1}{C_2} \frac{\partial V}{\partial \xi}.
\end{align*} \quad (16) \]

Let the fields \( \phi \) and \( \xi \) tend to finite limits as \( t \to +\infty \). System (16) has a fixed point \( y_f = (H_f, \phi_f, \psi_f, \xi_f, \zeta_f) \) if and only if

\[ \begin{align*}
H_f^2 &= \frac{8\pi G_N}{3} V(\phi_f, \xi_f), \\
\psi_f &= 0, \\
\zeta_f &= 0, \\
V'_{\phi} &\equiv \frac{\partial V}{\partial \phi}(\phi_f, \xi_f) = 0, \\
V'_{\xi} &\equiv \frac{\partial V}{\partial \xi}(\phi_f, \xi_f) = 0.
\end{align*} \quad (17) \]

All fixed points \( y_f \) correspond to \( \psi_f = 0 \) and \( \zeta_f = 0 \). We denote the fixed point \( y_f = (H_f, \phi_f, \psi_f, 0, 0) \) as \( y_f = (H_f, \phi_f, \psi_f) \). To analyse the stability of \( y_f \) we study the stability of this fixed point for the corresponding linearized system of equations and use the Lyapunov theorem [36, 37]. In the neighborhood of \( y_f \) we have

\[ \begin{align*}
H(t) &= H_f + \varepsilon h_1(t) + \mathcal{O}(\varepsilon^2), \\
\phi(t) &= \phi_f + \varepsilon \phi_1(t) + \mathcal{O}(\varepsilon^2), \\
\psi(t) &= \varepsilon \psi_1(t) + \mathcal{O}(\varepsilon^2), \\
\xi(t) &= \xi_f + \varepsilon \xi_1(t) + \mathcal{O}(\varepsilon^2), \\
\zeta(t) &= \varepsilon \zeta_1(t) + \mathcal{O}(\varepsilon^2),
\end{align*} \quad (22) \]

where \( \varepsilon \) is a small parameter.
To first order in $\varepsilon$ we obtain the following system of equations

\[
\begin{align*}
\dot{h}_1(t) &= -6H_fh_1(t), \\
\dot{\phi}_1(t) &= \psi_1(t), \\
\dot{\psi}_1(t) &= -3H_f\psi_1(t) - \frac{1}{C_1} \left( V''_{\phi\phi}\phi_1(t) + V''_{\phi\xi}\xi_1(t) \right), \\
\dot{\xi}_1(t) &= \zeta_1(t), \\
\dot{\zeta}_1(t) &= -3H_f\zeta_1(t) - \frac{1}{C_2} \left( V''_{\xi\phi}\phi_1(t) + V''_{\xi\xi}\xi_1(t) \right),
\end{align*}
\]

where

\[
V''_{\phi\phi} \equiv \frac{\partial^2 V}{\partial \phi^2}(\phi_f, \xi_f), \quad V''_{\phi\xi} \equiv \frac{\partial^2 V}{\partial \xi \partial \phi}(\phi_f, \xi_f), \quad V''_{\xi\xi} \equiv \frac{\partial^2 V}{\partial \phi \partial \xi}(\phi_f, \xi_f).
\]

Equation (27) has the following solution

\[
h(t) = b_0e^{-6H_ft},
\]

where $b_0$ is a constant. For asymptotic stability of the fixed point $y_f$ the function $h(t)$ should tend to zero at $t \to \infty$, therefore, the asymptotic stability requires that the condition $H_f > 0$ be satisfied.

The system of four first order equations (28)-(31) can be written as the following system of two second order equations

\[
\begin{align*}
\ddot{\phi}_1(t) + 3H_f\dot{\phi}_1(t) + \frac{1}{C_1} \left( V''_{\phi\phi}\phi_1(t) + V''_{\phi\xi}\xi_1(t) \right) &= 0, \\
\ddot{\xi}_1(t) + 3H_f\dot{\xi}_1(t) + \frac{1}{C_2} \left( V''_{\xi\phi}\phi_1(t) + V''_{\xi\xi}\xi_1(t) \right) &= 0.
\end{align*}
\]

In the case when $V''_{\phi\xi} = 0$ the system of equations (33)-(34) becomes a system of two independent second order equations. The general solution of this system is as follows:

- $\phi_1(t) = \bar{D}_1e^{-\frac{3}{2}H_ft - \frac{1}{2}\sqrt{9H_f^2 - 4V''_{\phi\phi}/C_1}t} + \bar{D}_2e^{-\frac{3}{2}H_f + \frac{1}{2}\sqrt{9H_f^2 - 4V''_{\phi\phi}/C_1}t}$ if $9H_f^2 \neq 4V''_{\phi\phi}/C_1$,

- $\phi_1(t) = (\bar{D}_1 + \bar{D}_2t)e^{-\frac{3}{2}H_ft}$ if $9H_f^2 = 4V''_{\phi\phi}/C_1$,

- $\xi_1(t) = \bar{D}_3e^{-\frac{3}{2}H_f - \frac{1}{2}\sqrt{9H_f^2 - 4V''_{\phi\xi}/C_2}t} + \bar{D}_4e^{-\frac{3}{2}H_f + \frac{1}{2}\sqrt{9H_f^2 - 4V''_{\phi\xi}/C_2}t}$ if $9H_f^2 \neq 4V''_{\phi\xi}/C_2$,

- $\xi_1(t) = (\bar{D}_3 + \bar{D}_4t)e^{-\frac{3}{2}H_ft}$ if $9H_f^2 = 4V''_{\phi\xi}/C_2$,

where $\bar{D}_1, \bar{D}_2, \bar{D}_3$ and $\bar{D}_4$ are arbitrary constants.
For the asymptotical stability of the considered fixed point the functions $\phi_1(t)$ and $\xi_1(t)$ must converge to 0 at $t \to \infty$. We obtain that at $V_{\phi_1}' = 0$ the sufficient conditions for the asymptotical stability are

$$H_f > 0, \quad \frac{V_{\xi_1}''}{C_2} > 0, \quad \frac{V_{\phi_1}''}{C_1} > 0. \quad (35)$$

Let us consider the case $V_{\phi_1}'' \neq 0$.

1. For

$$V_{\phi_1}'' \neq \frac{C_1 C_2}{16} \left(9 H_f^2 - \frac{4 V_{\xi_1}''}{C_2}\right) \left(9 H_f^2 - \frac{4 V_{\phi_1}''}{C_1}\right) \quad \text{and} \quad V_{\phi_1}'' \neq -\frac{C_1 C_2}{4} \left(\frac{V_{\xi_1}''}{C_2} - \frac{V_{\phi_1}''}{C_1}\right)^2,$$

the general solution of (33)–(34) is

$$\phi_1(t) = \bar{D}_1 e^{-(\frac{1}{2} H_f + \frac{1}{2} \sqrt{\Delta_1 - 2 \sqrt{\Delta_2}}) t} + \bar{D}_2 e^{-(\frac{1}{2} H_f - \frac{1}{2} \sqrt{\Delta_1 - 2 \sqrt{\Delta_2}}) t} + \bar{D}_3 e^{-(\frac{1}{2} H_f + \frac{1}{2} \sqrt{\Delta_1 + 2 \sqrt{\Delta_2}}) t} + \bar{D}_4 e^{-(\frac{1}{2} H_f - \frac{1}{2} \sqrt{\Delta_1 + 2 \sqrt{\Delta_2}}) t}, \quad (36)$$

$$\xi_1(t) = \frac{C_1 V_{\xi_1}'' - C_2 V_{\phi_1}'' + \sqrt{(C_1 V_{\xi_1}'' - C_2 V_{\phi_1}'')^2 + 4 C_1 C_2 V_{\phi_1}''^2}}{2 C_2 V_{\phi_1}''} \times \left(\bar{D}_1 e^{-(\frac{1}{2} H_f + \frac{1}{2} \sqrt{\Delta_1 - 2 \sqrt{\Delta_2}}) t} + \bar{D}_2 e^{-(\frac{1}{2} H_f - \frac{1}{2} \sqrt{\Delta_1 - 2 \sqrt{\Delta_2}}) t}\right) + \left(\bar{D}_3 e^{-(\frac{1}{2} H_f + \frac{1}{2} \sqrt{\Delta_1 + 2 \sqrt{\Delta_2}}) t} + \bar{D}_4 e^{-(\frac{1}{2} H_f - \frac{1}{2} \sqrt{\Delta_1 + 2 \sqrt{\Delta_2}}) t}\right), \quad (37)$$

here and further $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_3$, and $\bar{D}_4$ are arbitrary constants and

$$\Delta_1 = 9 H_f^2 - 2 \frac{V_{\xi_1}''}{C_2} - 2 \frac{V_{\phi_1}''}{C_1}, \quad \Delta_2 = \left(\frac{V_{\xi_1}''}{C_2} - \frac{V_{\phi_1}''}{C_1}\right)^2 + 4 \frac{V_{\phi_1}''^2}{C_1 C_2}. \quad (38)$$

For asymptotical stability of the considered fixed point these functions must converge to 0. As we can see both functions are just linear combinations of exponents to some degrees. To satisfy this condition all these degrees must be negative. It is easy to obtain that the sufficient conditions for asymptotic stability are

$$H_f > 0, \quad \frac{V_{\xi_1}''}{C_2} + \frac{V_{\phi_1}''}{C_1} > 0, \quad \frac{V_{\xi_1}'' V_{\phi_1}''}{C_1 C_2} > \frac{V_{\phi_1}''^2}{C_1 C_2}. \quad (39)$$

2. In the case

$$V_{\phi_1}'' = \frac{C_1 C_2}{16} \left(9 H_f^2 - \frac{4 V_{\xi_1}''}{C_2}\right) \left(9 H_f^2 - \frac{4 V_{\phi_1}''}{C_1}\right) \quad \text{and} \quad V_{\phi_1}'' \neq -\frac{C_1 C_2}{4} \left(\frac{V_{\xi_1}''}{C_2} - \frac{V_{\phi_1}''}{C_1}\right)^2,$$
the inequality $\Delta_1 \neq 0$ is valid and the general solution of (33–34) is

$$
\phi_1(t) = \tilde{D}_1 e^{-\frac{4}{3}H_f t} + \tilde{D}_2 e^{-\frac{4}{3}H_f t} + \tilde{D}_3 e^{(-\frac{4}{3}H_f + \frac{4}{3}\sqrt{\Delta}i)t} + \tilde{D}_4 e^{(-\frac{4}{3}H_f - \frac{4}{3}\sqrt{\Delta}i)t},
$$

(40)

$$
\xi_1(t) = \frac{C_1}{4V''_{\phi\xi}} \left( 9H^2 - \frac{4V''_{\phi\phi}}{C_1} \right) \left( \tilde{D}_1 e^{-\frac{4}{3}H_f t} + \tilde{D}_2 e^{-\frac{4}{3}H_f t} - \left( 9H^2 - \frac{4V''_{\phi\xi}}{C_2} \right) \left( \tilde{D}_3 e^{(-\frac{4}{3}H_f + \frac{4}{3}\sqrt{\Delta}i)t} + \tilde{D}_4 e^{(-\frac{4}{3}H_f - \frac{4}{3}\sqrt{\Delta}i)t} \right) \right). 
$$

(41)

It is easy to show that in this case the sufficient conditions for asymptotic stability of the considered fixed point coincide with (35).

3. In the case $V_{\phi\xi}^2 = -\frac{C_1C_2}{4} \left( \frac{V_{\phi\xi}}{C_2} - \frac{V_{\phi\phi}}{C_1} \right)^2$, in other words $\Delta_2 = 0$, and $V_{\phi\xi}^2 \neq \frac{C_1C_2}{16} \left( 9H^2 - \frac{4V''_{\phi\xi}}{C_2} \right) \left( 9H^2 - \frac{4V''_{\phi\phi}}{C_1} \right)$ the inequality $\Delta_1 \neq 0$ holds and therefore

$$
\phi_1(t) = \left( \tilde{D}_1 + \tilde{D}_3 t \right) e^{-(3H_f - \sqrt{\Delta}i)t/2} + \left( \tilde{D}_2 + \tilde{D}_4 t \right) e^{-(H_f + \sqrt{\Delta}i)t/2},
$$

$$
\xi_1(t) = \frac{C_1}{\sqrt{-C_1C_2}} \left\{ \left( \tilde{D}_1 + \left( 1 - \frac{C_1\sqrt{\Delta}}{V''_{\phi\xi}} \right) \tilde{D}_3 \right) e^{-(3H_f - \sqrt{\Delta}i)t/2} + \left( \tilde{D}_2 + \left( 1 + \frac{C_1\sqrt{\Delta}}{V''_{\phi\xi}} \right) \tilde{D}_4 \right) e^{-(H_f + \sqrt{\Delta}i)t/2} \right\}.
$$

The sufficient conditions for the asymptotic stability of the considered fixed point are

$$
H_f > 0, \quad \frac{V_{\phi\xi}}{C_2} + \frac{V_{\phi\phi}}{C_1} > 0.
$$

(42)

4. In the case

$$
V''_{\phi\xi}^2 = \frac{C_1C_2}{16} \left( 9H^2 - \frac{4V''_{\phi\xi}}{C_2} \right) \left( 9H^2 - \frac{4V''_{\phi\phi}}{C_1} \right) = -\frac{C_1C_2}{4} \left( \frac{V''_{\phi\xi}}{C_2} - \frac{V''_{\phi\phi}}{C_1} \right)^2
$$

it is easy to check that the equality $\Delta_1 = 0$ is valid automatically and, therefore, the solution of (33–34) is

$$
\phi_1(t) = \left( \tilde{D}_1 + \tilde{D}_2 t + \tilde{D}_3 t^2 + \tilde{D}_4 t^3 \right) e^{-\frac{3}{4}H_f t},
$$

(43)

$$
\xi_1(t) = \frac{C_1}{2V''_{\phi\xi}} \left\{ \tilde{D}_1 \left( \frac{V''_{\phi\xi}}{C_2} - \frac{V''_{\phi\phi}}{C_1} \right) + \tilde{D}_2 \left( \frac{V''_{\phi\xi}}{C_2} - \frac{V''_{\phi\phi}}{C_1} \right) t + \tilde{D}_3 \left[ \left( \frac{V''_{\phi\xi}}{C_2} - \frac{V''_{\phi\phi}}{C_1} \right) t^2 - 4 \right] + \tilde{D}_4 \left[ \left( \frac{V''_{\phi\xi}}{C_2} - \frac{V''_{\phi\phi}}{C_1} \right) t^3 - 12t \right] \right\} e^{-\frac{3}{4}H_f t}.
$$

(44)

For the asymptotic stability of the considered fixed point it is sufficient that the inequality $H_f > 0$ hold.
So, we obtained that in the case \( V_{\phi\xi} \neq 0 \) the general sufficient conditions of asymptotical stability of the fixed point \( y_f = (H_f, \phi_f, \psi_f) \) of the system of equations (16) are as follows:

\[
H_f > 0, \quad \frac{V''_{\xi\xi}}{C_2} + \frac{V''_{\phi\phi}}{C_1} > 0, \quad \frac{V''_{\xi\phi} V''_{\phi\phi}}{C_1 C_2} > \frac{V''_{\phi\xi}^2}{C_1 C_2}.
\]  \hfill (45)

It is easy to see that the conditions (35) obtained in the case \( V''_{\phi\xi} = 0 \) are equivalent to the conditions (45), if the equality \( V''_{\phi\xi} = 0 \) is substituted in them.

So, in the general case including all the above particular cases with definite relations between the parameters, the sufficient conditions for asymptotic stability of the fixed point \( y_f = (H_f, \phi_f, \psi_f) \) of the system of equations (16) are (45).

Let us add the cold dark matter (CDM) to our model. One-field models with the CDM in the Bianchi I metric have been considered in [38]. The generalization for two-field models is straightforward, so we point only the most important steps. If one adds into consideration the CDM energy density \( \rho_m \), then system (16) should be modified as follows:

\[
\dot{H} = -3H^2 + 8\pi G_N (V(\phi, \xi) + \rho_m),
\]

\[
\dot{\phi} = \psi,
\]

\[
\dot{\psi} = -3H\psi - \frac{1}{C_1} \frac{\partial V}{\partial \phi},
\]

\[
\dot{\xi} = \zeta,
\]

\[
\dot{\zeta} = -3H\zeta - \frac{1}{C_2} \frac{\partial V}{\partial \xi},
\]

\[
\dot{\rho}_m = -3H\rho_m.
\]  \hfill (46)

Let us consider the possible fixed points of system (46). From the last equation of this system, it follows that at the fixed point we have either \( H_f = 0 \) or \( \rho_{mf} = 0 \). Substituting (22)–(26) and

\[
\rho_m(t) = \rho_{mf} + \varepsilon \tilde{\rho}_m(t) + \mathcal{O}(\varepsilon^2),
\]  \hfill (47)

into the system (46), we obtain the following system to first order in \( \varepsilon \):

\[
\dot{h}(t) = -6H_f h(t) + 8\pi G_N \tilde{\rho}_m(t),
\]

\[
\dot{\tilde{\rho}}_m(t) = -3H_f \tilde{\rho}_m(t) - 3\rho_{mf} h(t),
\]

\[
\dot{\phi}_1(t) = \psi_1(t),
\]

\[
\dot{\psi}_1(t) = -3H_f \psi_1(t) - \frac{1}{C_1} (V''_{\phi\phi} \phi_1(t) + V''_{\phi\xi} \xi_1(t)),
\]

\[
\dot{\xi}_1(t) = \zeta_1(t),
\]

\[
\dot{\zeta}_1(t) = -3H_f \xi_1(t) - \frac{1}{C_2} (V''_{\xi\phi} \phi_1(t) + V''_{\xi\xi} \xi_1(t)),
\]  \hfill (53)

It is easy to see that the last four equations of this system coincide with the equations (28)–(31). Therefore, the case \( H_f = 0 \) can not be analysed with the Lyapunov theorem. Let us prove that conditions (45) are sufficient for the stability of fixed points for models with
the CDM. First of all, it follows from $H_f \neq 0$ that $\rho_{mf} = 0$. Solving equations (48)–(50), we obtain

$$
\tilde{\rho}_m(t) = b_1 e^{-3H_f t}, \quad h(t) = b_0 e^{-6H_f t} + \frac{b_1}{3H_f} e^{-3H_f t},
$$

where $b_0$ and $b_1$ are arbitrary constants.

We come to the conclusion that if conditions (45) hold, then a solution of system of equations (46) is stable. In other words if conditions (45) are satisfied, then the solution, which is stable in the model without the CDM, is stable with respect to the CDM energy density fluctuations as well.

Let us compare the obtained results with the known results. In [14] the stability of fixed points in quintom models ($C_1 = -1$, $C_2 = 1$) with an arbitrary potential has been considered in the FRW metric. The authors have presented the Einstein equations in the form of a dynamical system having introduced the Hubble-normalized variables [40]

$$
\begin{align*}
  x_\phi &= \frac{\dot{\phi}}{\sqrt{6H}}, \quad x_\xi = \frac{\dot{\xi}}{\sqrt{6H}}, \\
  y &= \frac{\sqrt{V}}{\sqrt{3H}},
\end{align*}
$$

and considered a system of ordinary differential equations with differentiation with respect to the new variable $\tau = \log a^3$ instead of $t$. In our paper we have demonstrated that such change of variables is not necessary, because the stability conditions can be easily found in the initial variables.

Proceeding from some physical considerations, the authors of [14] assumed that $H_f > 0$, we proved that the condition $H_f < 0$ is sufficient for instability of the fixed point. Note that the case $H_f = 0$, or equivalently $V(\phi_f, \xi_f) = 0$, has not been analysed both in [14] and in our paper. In the case $H_f > 0$ we and the authors of [14] obtained the same conditions on the potential, which guarantee the stability of the fixed point. In our paper we have proved that the obtained conditions are sufficient for the stability not only in the FRW metric, but also in the Bianchi I metric.

3 Construction of stable solutions via the superpotential method

3.1 The superpotential for two-field models

Let us consider the superpotential method [41] (see also [42, 43]) for two-field models. We consider the FRW metric and assume that the Hubble parameter $H(t)$ is a function (superpotential) of $\phi(t)$ and $\xi(t)$:

$$
H(t) = W(\phi(t), \xi(t)),
$$

and that functions $\phi(t)$ and $\xi(t)$ are solutions of the following system of two ordinary differential equations

$$
\begin{align*}
  \dot{\phi} &= F(\phi, \xi), \\
  \dot{\xi} &= G(\phi, \xi).
\end{align*}
$$
Therefore,
\[
\dot{\phi} = \frac{\partial F}{\partial \phi} F + \frac{\partial F}{\partial \xi} G, \quad \dot{\xi} = \frac{\partial G}{\partial \phi} F + \frac{\partial G}{\partial \xi} G, \tag{58}
\]
and we can rewrite the Friedmann equations as follows:
\[
3W^2 = 8\pi G_N \left( \frac{C_1}{2} F^2 + \frac{C_2}{2} G^2 + V \right), \tag{59}
\]
\[
\frac{\partial W}{\partial \phi} F + \frac{\partial W}{\partial \xi} G = -4\pi G_N \left( C_1 F^2 + C_2 G^2 \right), \tag{60}
\]
\[
\frac{\partial F}{\partial \phi} F + \frac{\partial F}{\partial \xi} G + 3WF + \frac{1}{C_1} \frac{\partial V}{\partial \phi} = 0, \tag{61}
\]
\[
\frac{\partial G}{\partial \phi} F + \frac{\partial G}{\partial \xi} G + 3WG + \frac{1}{C_2} \frac{\partial V}{\partial \xi} = 0. \tag{62}
\]
From (59) we obtain
\[
6W \frac{\partial W}{\partial \phi} = 8\pi G_N \left( C_1 F \frac{\partial F}{\partial \phi} + C_2 G \frac{\partial G}{\partial \phi} + \frac{\partial V}{\partial \phi} \right), \tag{63}
\]
therefore,
\[
G \left( \frac{C_2}{C_1} \frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \xi} \right) = 3W \left( \frac{1}{4\pi G_N C_1} \frac{\partial W}{\partial \phi} + F \right). \tag{64}
\]
Also, we have
\[
F \left( \frac{C_1}{C_2} \frac{\partial F}{\partial \xi} - \frac{\partial G}{\partial \phi} \right) = 3W \left( \frac{1}{4\pi G_N C_2} \frac{\partial W}{\partial \xi} + G \right). \tag{65}
\]
If the functions $F$ and $G$ satisfy the following condition:
\[
\frac{\partial F}{\partial \xi} = \frac{C_2}{C_1} \frac{\partial G}{\partial \phi}, \tag{66}
\]
then
\[
\frac{\partial W}{\partial \phi} = -4\pi G_N C_1 F, \quad \frac{\partial W}{\partial \xi} = -4\pi G_N C_2 G. \tag{67}
\]
Note that using condition (66) and formulas (67), one can verify that
\[
\frac{\partial^2 W}{\partial \phi \partial \xi} = \frac{\partial^2 W}{\partial \xi \partial \phi}. \tag{68}
\]
Therefore, to obtain particular solutions of system (9)–(12) it suffices to require that the relations
\[
\frac{\partial W}{\partial \phi} = -4\pi G_N C_1 F, \quad \frac{\partial W}{\partial \xi} = -4\pi G_N C_2 G, \tag{69}
\]
\[
V = \frac{3}{8\pi G_N} W^2 - \frac{1}{32\pi^2 G_N^2} \left( \frac{1}{C_1} \left( \frac{\partial W}{\partial \phi} \right)^2 + \frac{1}{C_2} \left( \frac{\partial W}{\partial \xi} \right)^2 \right) \tag{70}
\]
be satisfied.
3.2 Stability conditions in the superpotential method

Let us obtain conditions on the superpotential $W$ that are equivalent to conditions (45) for the corresponding potential $V$. At the fixed point $y_f = (H_f, \phi_f, \psi_f)$ we get

$$W_f \equiv W(\phi_f, \xi_f) = H_f, \quad W'_\phi \equiv \frac{\partial W}{\partial \phi}(\phi_f, \xi_f) = 0, \quad W'_\xi \equiv \frac{\partial W}{\partial \xi}(\phi_f, \xi_f) = 0. \quad (71)$$

It is easy to see that from conditions (71) it follows that

$$V'_\phi = 0, \quad V'_\xi = 0$$

and

$$V''_{\phi\phi} = \frac{1}{16C_1C_2\pi^2G_N}(12C_1C_2\pi^2G_NW_fW''_{\phi\phi} - C_2W''_{\phi\phi}^2 - C_1W''_{\phi\xi}^2), \quad (73)$$

$$V''_{\xi\xi} = \frac{1}{16C_1C_2\pi^2G_N}(12C_1C_2\pi^2G_NW_fW''_{\xi\xi} - C_1W''_{\xi\xi}^2 - C_2W''_{\phi\xi}^2), \quad (74)$$

$$V''_{\phi\xi} = \frac{1}{16C_1C_2\pi^2G_N}W''_{\phi\xi}(12C_1C_2\pi^2G_NW_f - C_2W''_{\phi\phi} - C_1W''_{\xi\xi}). \quad (75)$$

The condition $H_f > 0$ can be rewritten as $W_f > 0$.

If $W''_{\phi\xi} = 0$, then $V''_{\phi\phi} = 0$ and conditions (35) are

$$(12\pi G_N C_1 W_f - W''_{\phi\phi}) W''_{\phi\phi} > 0, \quad (12\pi G_N C_2 W_f - W''_{\xi\xi}) W''_{\xi\xi} > 0. \quad (76)$$

In the general case conditions (45) are written in the following form

$$12C_1C_2\pi^2G_N(C_2W''_{\phi\phi} + C_1W''_{\xi\xi})W_f > C_2^2(W''_{\phi\phi})^2 + 2C_1C_2(W''_{\phi\xi})^2 + C_1^2(W''_{\xi\xi})^2, \quad (77)$$

$$144C_1C_2\pi^2G_N^2W_f^2 - 12C_1C_2\pi^2G_N(C_2W''_{\phi\phi} + C_1W''_{\xi\xi})W_f + W''_{\xi\xi}W''_{\phi\phi} - (W''_{\phi\xi})^2) \times (W''_{\xi\xi}W''_{\phi\phi} - (W''_{\phi\xi})^2) > 0. \quad (78)$$

In the case of one-field models the stability conditions on the superpotential are considered in Appendix.

4 String field theory inspired cosmological models

4.1 Quintom models with the sixth degree potential

The interest in cosmological models related to the open string field theory [44] is caused by the possibility to get solutions describing transitions from a perturbed vacuum to the true vacuum (so-called rolling solutions [45]). After all massive fields (or some of the lower massive fields) are integrated out by means of equations of motion, the open string tachyon acquires a potential with a nontrivial vacuum, corresponding to a minimum of the energy. The dark energy model [44] (see also [43, 46, 13]) assumes that our Universe is a slowly decaying D3-brane and its dynamics is described by the open string tachyon mode. For
the Neveu–Schwarz–Ramond (NSR) open fermionic string with the GSO(−) sector \cite{47} in a reasonable approximation, one gets the Mexican hat potential for the tachyon field (see \cite{48} for a review). Rolling of the tachyon from the unstable perturbative extremum towards this minimum describes, according to the Sen conjecture \cite{48}, the transition of an unstable D-brane to a true vacuum. In fact one gets a nonlocal potential with a string scale as a parameter of nonlocality. After a suitable field redefinition the potential becomes local, meanwhile, the kinetic term becomes non-local. This nonstandard kinetic term has a so-called phantomlike behavior and can be approximated by a phantom kinetic term. It has been found that the open string tachyon behavior is effectively modelled by a scalar field with a negative kinetic term \cite{52}. The back reaction of the brane is determined by dynamics of the closed string tachyon. This dynamic can be effectively described by a local scalar field ξ with an ordinary kinetic term \cite{53} and possibly a nonpolynomial self-action \cite{54}. An exact form of the open-closed tachyon interaction is not known. So, following \cite{13}, we consider the simplest polynomial interaction.

In the papers \cite{13, 39} quintom models \((C_1 = -C_2 < 0)\) with effective potentials \(V(\phi, \xi)\) have been considered. The form of these potentials are assumed to be given from the SFT within the level truncation scheme. We postulate that the potential is a polynomial. Specifically, we assume that the potential \(V(\phi, \xi)\) should be an even sixth degree polynomial

\[
V(\phi, \xi) = \sum_{k=0}^{6} \sum_{j=0}^{6-k} c_{kj} \phi^k \xi^j, \quad V(\phi, \xi) = V(-\phi, -\xi),
\]

therefore, if the sum \(k + j\) is odd, then \(c_{kj} = 0\).

From the SFT we can also assume asymptotic conditions for solutions \cite{13, 39}. We assume that the phantom field \(\phi(t)\) smoothly rolls from the unstable perturbative vacuum \((\phi = 0)\) to a nonperturbative one, for example \(\phi = A\), and stops there. The field \(\xi(t)\) corresponds to the closed string and is expected to go asymptotically to zero in the infinite future. Namely, we seek such a function \(\phi(t)\) that \(\phi(0) = 0\) and it has a non-zero asymptotic at \(t \to +\infty\): \(\phi(+\infty) = A\). The function \(\xi(t)\) should have zero asymptotic at \(t \to +\infty\). In other words, we analyse the stability of solutions, tending to a fixed point with \(\phi_f = A\) and \(\xi_f = 0\).

### 4.2 Construction of stable solutions

Let us assume that there exists a polynomial superpotential \(W(\phi, \xi)\), which determines potential \((79)\) with formula \((70)\). To construct an even sixth degree polynomial potential \(V(\phi, \xi)\) we should choose \(W(\phi, \xi)\) as an odd third degree polynomial. Obviously, the suitable form of the superpotential is as follows:

\[
W_3(\phi, \xi) = 4\pi G_N \left( a_{1,0} \phi + a_{3,0} \phi^3 + a_{0,1} \xi + a_{0,3} \xi^3 + a_{2,1} \phi^2 \xi + a_{1,2} \phi \xi^2 \right),
\]

\(\footnote{Note that quintom models naturally arise from nonlocal cosmological models with quadratic potential \cite{49, 50, 51}.}

Note that quintom models naturally arise from nonlocal cosmological models with quadratic potential \cite{49, 50, 51}.
where \(a_{i,j}\) are constants. For the superpotential \(W_3\) system (69) is as follows:

\[
\dot{\phi} = \frac{1}{C_2} \left( a_{1,0} + 3a_{3,0}\phi^2 + 2a_{2,1}\phi\xi + a_{1,2}\xi^2 \right), \\
\dot{\xi} = -\frac{1}{C_2} \left( a_{0,1} + 3a_{0,3}\xi^2 + a_{2,1}\phi^2 + 2a_{1,2}\phi\xi \right). 
\]  

(81)

Using asymptotic conditions: \(\phi(+\infty) = A, \xi(+\infty) = 0, \dot{\phi}(+\infty) = \dot{\xi}(+\infty) = 0\) we obtain

\[
a_{1,0} = -3a_{3,0}A^2, \quad a_{0,1} = -a_{2,1}A^2.
\]

(82)

So, we obtain the following system of equations:

\[
\dot{\phi} = \frac{1}{C_2} \left( 3a_{3,0}(\phi^2 - A^2) + 2a_{2,1}\phi\xi + a_{1,2}\xi^2 \right), \\
\dot{\xi} = -\frac{1}{C_2} \left( a_{2,1}(\phi^2 - A^2) + 3a_{0,3}\xi^2 + 2a_{1,2}\phi\xi \right)
\]

(83)

and the corresponding superpotential:

\[
W_3(\phi, \xi) = 4\pi G_N \left( -3a_{3,0}A^2\phi + a_{3,0}\phi^3 - a_{2,1}A^2\xi + a_{0,3}\xi^3 + a_{2,1}\phi^2\xi + a_{1,2}\phi\xi^2 \right).
\]

(84)

At the fixed point \(\phi_f = A, \xi_f = 0\)

\[
W_f = -8\pi G_N a_{3,0}A^3.
\]

(85)

So, the condition \(W_f > 0\) is equivalent to \(a_{3,0}A < 0\). It is easy to calculate, that

\[
W_{\phi\phi}'' = 24\pi G_N a_{3,0}A, \quad W_{\xi\xi}'' = 8\pi G_N a_{1,2}A, \quad W_{\phi\xi}'' = 8\pi G_N a_{2,1}A.
\]

(86)

Using (73)–(75), we obtain:

\[
V_{\phi\phi}'' = \frac{4A^2}{C_2} \left( 9(1 - 4\pi G_N C_2 A^2) a_{3,0}^2 - a_{2,1}^2 \right), \\
V_{\xi\xi}'' = \frac{4A^2}{C_2} \left( a_{2,1}^2 - a_{1,2}^2 - 12\pi G_N C_2 A^2 a_{3,0} a_{1,2} \right), \\
V_{\phi\xi}'' = \frac{4A^2 a_{2,1}}{C_2} \left( 3(1 - 4\pi G_N C_2 A^2) a_{3,0} - a_{1,2} \right).
\]

(87) (88) (89)

If \(a_{2,1} = 0\), then \(V_{\phi\xi}'' = 0\) and the sufficient conditions for the stability are

\[
a_{3,0}A < 0, \quad 4\pi G_N C_2 A^2 > 1, \quad a_{1,2}(a_{1,2} + 12\pi G_N C_2 a_{3,0} A^2) < 0.
\]

(90)

If \(a_{2,1} \neq 0\), then conditions (145) are equivalent to

\[
a_{3,0}A < 0, \\
2a_{2,1}^2 - a_{1,2}^2 - 12\pi G_N C_2 A^2 a_{3,0} a_{1,2} + 9(4\pi G_N C_2 A^2 - 1)a_{3,0}^2 > 0, \\
[3a_{3,0} a_{1,2} - a_{2,1}^2] [3(4\pi G_N A^2 C_2 - 1)a_{3,0} a_{1,2} - 36\pi G_N A^2 C_2 (4\pi G_N A^2 C_2 - 1)a_{3,0}^2 + a_{2,1}^2] < 0.
\]

(91)
4.3 Examples of stable solutions

The case of superpotential \( W(\phi, \xi) \) with \( a_{2,1} = 0 \) and \( a_{0,3} = 0 \) has been considered in [13]. In this case the system (83) has the following form

\[
\phi = -\frac{C_2}{2a_{1,2}} \dot{\xi},
\]

(92)

\[
\ddot{\xi} = \frac{(2a_{1,2} - 3a_{3,0})}{2a_{1,2}} \xi^2 + \frac{2a_{1,2}}{C_2} \xi \left(3a_{3,0}A^2 - a_{1,2} \xi^2\right).
\]

(93)

Equation (93) can be integrated in quadratures:

\[
\int \frac{\sqrt{\xi^{3B-2}(3B + 2)C_2}}{\sqrt{(12BA^2 + 8A^2 - 4\xi^2)}\xi^{3B}a_{1,2}^2 + (3B + 2)C_2 D_1}} d\xi = \pm(t - t_0),
\]

(94)

where \( D_1 \) and \( t_0 \) are arbitrary constants and \( B = a_{3,0}/a_{1,2} \). At \( B = -1/3, a_{2,1} = 0, \) and \( a_{0,3} = 0, \) the general solution of system (83) can be obtained in explicit form [39]:

\[
\phi_s(t) = \frac{A (C_2^2 e^{2a_{1,2}At/C_2} - 64a_{1,2}^2 C_2^2 A^2 D_1^2 - 4a_{1,2}^2 A^2 D_2^2)}{(C_2 e^{2a_{1,2}At/C_2} - 2D_2 a_{1,2} A)^2 + 64D_1^2 a_{1,2} C_2^2 A^2},
\]

(95)

\[
\xi_s(t) = \frac{16D_1 C_2^2 a_{1,2}^2 A^2 e^{2a_{1,2}At/C_2}}{(C_2 e^{2a_{1,2}At/C_2} - 2D_2 a_{1,2} A)^2 + 64D_1^2 a_{1,2} C_2^2 A^2}.
\]

(96)

Let us analyse the stability of the exact solution. One can see that \( \phi_s(t) \) and \( \xi_s(t) \) are continuous functions, which tend to a fixed point at \( t \to \infty \). Therefore, the obtained exact solution is attractive if and only if the fixed point is asymptotically stable. At \( a_{3,0} = -a_{1,2}/3 \) we obtain that three stability conditions [39] transform into two independent conditions

\[
a_{1,2} A > 0, \quad C_2 > \frac{1}{4\pi G_N A^2},
\]

(97)

From these conditions it follows that \( a_{1,2} A/C_2 > 0 \).

Let us check the stability of solutions, obtained in [39]. In [39] the author has considered the quintom model with the following energy density:

\[
\rho = \frac{8\pi G_N}{m_p^2} \left(-\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \xi^2 + V_1\right),
\]

(98)

where

\[
m_p^2 = \frac{g_o}{8\pi G_N M_s^2},
\]

(99)

\( g_o \) is the open string coupling constant, \( M_s \) is the string mass,

\[
V_1 = \frac{\omega^2}{8A^2} \left( (A^2 - \phi^2 + \xi^2)^2 - 4\phi^2 \xi^2 + \frac{1}{6m_p^2} \phi^2 (3A^2 - \phi^2 + 3\xi^2)^2 \right),
\]

(100)
where $\omega$ is a nonzero constant.

Therefore, we obtain $C_2 = M_s^2/g_o$, the potential

$$V_s = \frac{1}{8\pi G_N m_p^2} V_1 = \frac{M_s^2}{g_o} V_1$$

(101)
corresponds to the superpotential

$$W_s = 4\pi G_N \omega \phi \left( A \left( \frac{1}{2} - \frac{\phi^2}{6A^2} \right) + \frac{\xi^2}{2A} \right).$$

(102)

This choice corresponds to

$$a_{1,2} = \frac{\omega}{2A},$$

(103)
so the stability conditions are

$$\omega > 0, \quad 4\pi G_N A^2 M_s^2 > g_o.$$}

(104)

So, we come to the conclusion that the exact solutions, obtained in [39] are stable for sufficiently large $A$.

5 Conclusion

We have analysed the stability of isotropic solutions for two-field models in the Bianchi I metric. Using the Lyapunov theorem we have found sufficient conditions of stability of kink-type and lump-type isotropic solutions for two-field models in the Bianchi I metric. The obtained results allow us to prove that the exact solutions, found in string inspired phantom models [13, 39], are stable.

Our study of the stability of isotropic solutions for quintom models in the Bianchi I metric shows that the NEC is not a necessary condition for classical stability of isotropic solutions. In this paper we have shown that the models [13, 39] have stable isotropic solutions and that large anisotropy does not appear in these models. It means that considered models are acceptable, because they do not violate limits on anisotropic models, obtained from the observations [2, 3].

We also have presented the algorithm for construction of kink-type and lump-type isotropic exact stable solutions via the superpotential method. In particular we have formulated the stability conditions in terms of superpotential.

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Appendix. Stability conditions on superpotential in a one-field cosmological model

The main goal of our paper is to consider stable solutions in two-field models. At the same time it is convenient to remind the superpotential method for a cosmological model with one scalar field $\tilde{\phi}$, which is described with the action

$$ S = \int d^4 x \sqrt{-g} \left( \frac{R}{16 \pi G_N} - \frac{\tilde{C}}{2} g^{\mu \nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right), \quad (105) $$

where the potential $\tilde{V}(\tilde{\phi})$ is a twice continuously differentiable function and $\tilde{C}$ is a nonzero real constant.

It has been shown in [38] that to find sufficient conditions for the stability of the isotropic fixed point in the Bianchi I metric one can consider the spatially flat Friedmann–Robertson–Worker Universe with

$$ ds^2 = - dt^2 + \tilde{a}^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right), $$

where $\tilde{a}(t)$ is the scale factor. In the FRW metric the field $\tilde{\phi}$ depends only on time.

The Friedmann equations can be written in the following form:

$$ \dot{\tilde{H}} = - 4 \pi G_N \tilde{C} \left( \dot{\tilde{\phi}} \right)^2, \quad 3\dot{H}^2 = 8 \pi G_N \left( \frac{\tilde{C}}{2} \left( \dot{\tilde{\phi}} \right)^2 + \tilde{V}(\tilde{\phi}) \right). \quad (106) $$

In [38] it has been proven that the fixed point $\tilde{y}_f = (\tilde{H}_f, \tilde{\phi}_f)$ is asymptotically stable and, therefore, the exact solution $(\tilde{\phi}(t), \tilde{H}(t))$, which tends to this fixed point, is attractive if:

$$ \frac{\tilde{V}''(\tilde{\phi}_f)}{\tilde{C}} > 0 \quad \text{and} \quad \tilde{H}_f > 0, \quad (107) $$

in this subsection a prime denotes a derivative with respect to $\tilde{\phi}$.

System of equations (106) with a polynomial potential $\tilde{V}(\tilde{\phi})$ is not integrable. At the same time it is possible to construct the potential $\tilde{V}(\tilde{\phi})$ and to find $\tilde{H}(t)$ if $\tilde{\phi}(t)$ is given explicitly.

Following [41], we assume, that $\tilde{H}(t)$ is a function of $\tilde{\phi}(t)$, called superpotential (for details of the Hamilton–Jacobi formulation of the Friedmann equations and the superpotential method see also [42, 43, 13, 39]), that is $\tilde{H}(t) = \tilde{W}(\tilde{\phi}(t))$. Using equality $\dot{\tilde{H}} = \tilde{W}' \dot{\tilde{\phi}}$, where $\tilde{W}' \equiv \frac{\partial \tilde{W}}{\partial \tilde{\phi}}$, we obtain from system (106):

$$ \dot{\tilde{\phi}} = - \frac{1}{4 \pi G_N \tilde{C}} \tilde{W}', \quad (108) $$

$$ \tilde{V} = \frac{3}{8 \pi G_N} \tilde{W}^2 - \frac{1}{32 \pi^2 G_N^2 \tilde{C}} \left( \tilde{W}' \right)^2. \quad (109) $$
The superpotential method is to choose $\tilde{W}(\tilde{\phi})$ in such form that both $\tilde{\phi}(t)$ and $\tilde{V}(\tilde{\phi})$ have required properties. Equation (108) is always solvable at least in quadratures. Formula (109) allows one to find the potential $\tilde{V}$, provided the superpotential $\tilde{W}$ is given.

Let $\tilde{\phi}(t)$ tend to a finite limit $\tilde{\phi}_f$ at $t \to +\infty$. We assumed that $\tilde{V}(\tilde{\phi})$ is a twice continuously differentiable function, therefore, $\tilde{V}(\tilde{\phi}_f)$ is finite and $\tilde{H}_f = \tilde{W}(\tilde{\phi}_f)$ is finite as well. So, system (106) has the fixed point $\tilde{y}_f = (\tilde{H}_f, \tilde{\phi}_f)$. It is easy to see that

$$\tilde{V}'(\tilde{\phi}_f) = 0, \quad \tilde{H}_f^2 = \frac{8}{3} \pi G_N \left( \tilde{V}(\tilde{\phi}_f) \right).$$

(110)

From (109) we get the condition

$$\tilde{V}'(\tilde{\phi}_f) = \frac{\tilde{W}'(\tilde{\phi}_f)}{16 \pi^2 G_N^2 C} \left( 12 \pi G_N \tilde{C} \tilde{W}(\tilde{\phi}_f) - \tilde{W}''(\tilde{\phi}_f) \right) = 0.$$  

(111)

If $\tilde{W}'(\tilde{\phi}_f) \neq 0$, then from (108) it follows that $\tilde{\phi}_f$ is not a fixed point, so, we analyse only the case $\tilde{W}'(\tilde{\phi}_f) = 0$ and obtain that

$$\tilde{V}''(\tilde{\phi}_f) = \frac{\tilde{W}''(\tilde{\phi}_f)}{16 \pi^2 G_N^2 C} \left( 12 \pi G_N \tilde{C} \tilde{W}(\tilde{\phi}_f) - \tilde{W}''(\tilde{\phi}_f) \right).$$

(112)

Thus, we come to the conclusion that to construct a stable kink-type solution one should find such $\tilde{W}(\tilde{\phi})$ that $\tilde{\phi}(t)$ tends to a fixed point $\phi_f$ and the following conditions are satisfied

$$\tilde{W}'''(\tilde{\phi}_f) \left( 12 \pi G_N \tilde{C} \tilde{W}(\tilde{\phi}_f) - \tilde{W}''(\tilde{\phi}_f) \right) > 0 \quad \text{and} \quad \tilde{H}_f = \tilde{W}(\tilde{\phi}_f) > 0.$$  

(113)

Note that the obtained conditions are sufficient for the stability of the obtained isotropic solution in the Bianchi I metric as well as with respect to small fluctuations of the CDM energy density [38].

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