A NOTE ON SHARP SPECTRAL ESTIMATES FOR PERIODIC JACOBI MATRICES

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Abstract. The spectrum of three-diagonal self-adjoint $p$-periodic Jacobi matrix with positive off-diagonal elements $a_n$ and real diagonal elements $b_n$ consist of intervals separated by $p - 1$ gaps $\gamma_i$, where some of the gaps can be degenerated. The following estimate is true

$$p - 1 \sum_{i=1}^{p-1} |\gamma_i| \geq \max(\max(a_1...a_p)^{\frac{1}{p}}, 2 \max a_n) - 4 \min a_n, \max b_n - \min b_n).$$

We show that for any $p \in \mathbb{N}$ there are Jacobi matrices of minimal period $p$ for which the spectral estimate is sharp. The estimate is sharp for both: strongly and weakly oscillated $a_n$, $b_n$. Moreover, it improves some recent spectral estimates.

1. Introduction

The periodic Jacobi matrices corresponds to the finite-difference approximation of second-order partial differential operators with periodic coefficients, e.g. periodic Schrödinger operators. The $p$-periodic Jacobi matrix $J \equiv J(a, b)$ is the self-adjoint operator acting on $\ell^2(\mathbb{Z})$ of the form

$$(Jy)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \quad y = (y_n)_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}),$$

where $a = (a_n)_{n \in \mathbb{Z}}, b = (b_n)_{n \in \mathbb{Z}}$ are $p$-periodic sequences of real numbers, $a_{n+p} = a_n, b_{n+p} = b_n$ for all $n \in \mathbb{Z}$. We assume also that all $a_n > 0$ and, for simplicity, $p \geq 2$. It is well-known, see, e.g., [vM, Te], that the spectrum of $J$ is absolutely continuous and consists of $p$ intervals $\sigma_i$ separated by $p - 1$ gaps $\gamma_i$. Some of the gaps may be degenerated. Let us denote the distance between maximal and minimal spectral points by $r \equiv r(a, b) = \lambda_{\max} - \lambda_{\min}$. There is an estimate, see [KKr],

$$r \geq 4(a_1...a_p)^{\frac{1}{p}}. \quad (1.1)$$

There is another estimate for the Lebesgue measure of the spectrum, see, e.g., [L, KKr, DS],

$$p \sum_{i=1}^{p} |\sigma_i| \leq 4(a_1...a_p)^{\frac{1}{p}} \quad (1.2)$$

Some similar estimates were obtained also for quasi-periodic cases in [PR]. It is not possible to combine (1.1) and (1.2) to obtain some non-trivial estimates for the lengths of spectral gaps. For a long time, (1.2) was the best estimate, while in [Ku] it was improved

$$p \sum_{i=1}^{p} |\sigma_i| \leq 4 \min a_n. \quad (1.3)$$
Although (1.3) improves significantly (1.2), it was very hard to publish it somewhere. Now, it is possible to combine (1.1) and (1.3) to obtain the estimate for the Lebesgue measure of the spectral gaps
\[
\sum_{i=1}^{p-1} |\gamma_i| = r - \sum_{i=1}^{p} |\sigma_i| \geq 4(a_1...a_p)^{\frac{1}{p}} - 4 \min a_n. \tag{1.4}
\]

The estimate (1.4) is sharp in the following sense.

**Theorem 1.1.** Let \( p \in \mathbb{N} \), \( b = (0)_{n \in \mathbb{Z}} \), \( a = (1 - c \delta_{np})_{n \in \mathbb{Z}} \), where \( c > 0 \) and \( \delta_{np} = 1 \) if \( m = mp \) for \( m \in \mathbb{Z} \) and \( \delta_{np} = 0 \) otherwise. Then, for the gaps \( \gamma_i \) of Jacobi matrix \( J(a,b) \), we have
\[
\sum_{i=1}^{p-1} |\gamma_i| = \frac{4(p-1)}{p} c + o(c), \quad 4(a_1...a_p)^{\frac{1}{p}} - 4 \min a_n = \frac{4(p-1)}{p} c + o(c). \tag{1.5}
\]

Hence, (1.4) is sharp for small \( c \to +0 \). Moreover, the minimal period of \( a \) is \( p \).

Let \( J = J(a,b) \) be some periodic Jacobi matrix. We can always shift the spectrum of \( J \) to have \( \lambda^{\text{max}} = -\lambda^{\min} \). Let \( \lambda \in \mathbb{R} \) be such that \( J(a,b + \lambda(1)) = J + \lambda I \) have the property \( \lambda^{\text{max}} = -\lambda^{\min} \), where \( I \) is the identity operator. It is easy to see that \( \lambda^{\text{max}} \geq \max_i a_i \), since \( \lambda^{\text{max}} = \sup_{h \neq 0} \frac{\|Jh\|}{\|h\|} \) for \( h \in \ell^2(\mathbb{Z}) \) and we can always choose a unit vector \( h \) for which \( \|Jh\| \geq \max a_n \). Hence \( r \geq 2 \max_i a_i \) and we can write
\[
\sum_{i=1}^{p-1} |\gamma_i| = r - \sum_{i=1}^{p} |\sigma_i| \geq 2 \max a_n - 4 \min a_n, \tag{1.6}
\]
see (1.3), (1.4), since the shifting \( J + \lambda I \) does not change the off-diagonal elements \( a \) and the spectral lengths \( |\gamma_i|, |\sigma_i|, r \). Estimate (1.6) is sharp for some strongly oscillating large \( a \) (and \( b = 0 \)), where only one \( a_i \) is large while the other are small. This is because \( \lambda^{\text{max}} \sim \max_i a_i \) for such \( a \) (and \( b = 0 \)). But, (1.6) is useless for slightly oscillating \( a \), since RHS in (1.6) is negative for such \( a \).

Let us modify (1.4) as follows:
\[
\sum_{i=1}^{p-1} |\gamma_i| \geq 4(a_1...a_p)^{\frac{1}{p}} - 4 \min a_n \geq 4(\min a_n)^{\frac{p-1}{p}} \left( \max a_n \right)^{\frac{1}{p}} - (\min a_n)^{\frac{1}{p}} \geq \frac{4}{p} \left( \min a_n \right)^{\frac{p-1}{p}} \left( \max a_n - \min a_n \right) \geq \frac{2}{p} \left( \max a_n - \min a_n \right),
\]
for \( p \geq 2 \), where in the last inequality we use
\[
\sum_{i=1}^{p-1} |\gamma_i| \geq \max a_n - \min a_n \quad \text{for} \quad \max a_n \geq 4 \min a_n,
\]
see (1.6). Hence, we obtain
\[
\sum_{i=1}^{p-1} |\gamma_i| \geq \frac{2}{p} \left( \max a_n - \min a_n \right), \tag{1.7}
\]
which improves a little bit one of the estimates from \([G]\), namely
\[
p^{2} \sqrt{p} \max_{i=1} \gamma_i \geq \max a_n - \min a_n.
\]
Note that two-sided spectral estimates from [G] supplement essentially the Borg-type results on the existence of spectral gaps, see also [CGR, KK].

Finally, note the estimate, see (1.4) in [KK]

\[
\sum_{i=1}^{p-1} |\gamma_i| \geq \max b_n - \min b_n
\]  

which is obviously sharp for a weakly and strongly oscillated \( b \) when \( a \) tends to 0. Combining (1.4), (1.6), and (1.8) we obtain the estimate announced in the Abstract

\[
\sum_{i=1}^{p-1} |\gamma_i| \geq \max(4(a_1...a_p)^{1/p}, 2 \max a_n) - 4 \min a_n, \max b_n - \min b_n).
\]  

Following the discussion above and the results of Theorem 1.1 we conclude that (1.9) is sharp for weakly and strongly oscillated \( a \) and \( b \) of arbitrary minimal periods \( p \in \mathbb{N} \).

2. Proof of Theorem 1.1

Let \( J = J(a, b) \) be \( p \)-periodic Jacobi matrix. Let us define the matrix

\[
J(a, b, e^{ik}) = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & e^{ik} a_p \\ a_1 & b_2 & a_2 & \ldots & 0 \\ 0 & a_2 & b_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-ik} a_p & 0 & 0 & \ldots & b_p \end{pmatrix}
\]

Denote its eigenvalues as \( \lambda_i(k), i = 1, \ldots, p \). Let us recall some well-known facts about the spectral properties of scalar periodic Jacobi matrices, see, e.g., [12]. It is well known that the spectral components of \( J \) consist of \( \lambda_i(k) \), i.e. \( \sigma_i = \cup_{k \in [0, \pi]} \{ \lambda_i(k) \} \). It is possible to take the union up to \( k = \pi \) instead of \( k = 2\pi \), since \( \lambda_i(k) \) is symmetric relatively to the center \( k = \pi \) of the Brillouin zone \([0, 2\pi]\). The values \( \lambda_i(0) \) and \( \lambda_i(\pi) \) are the edges of \( \sigma_i \). Inside \( k \in (0, \pi) \) the functions \( \lambda_i(k) \) are strictly monotonic for the case \( a_n > 0, n \in \mathbb{N} \).

Suppose that \( p \) is even, the odd \( p \) can be treated similarly. To study the spectrum of the Jacobi matrix \( J(a, b) \) with \( b = (0)_{n \in \mathbb{Z}}, a = (1 - \cos np)_{n \in \mathbb{Z}} \), it is useful to apply the regular perturbation theory, since \( c \to 0 \). Let us start with the unperturbed Jacobi matrix \( J^0 = J(a^0, b) \), where \( a^0 = (1)_{n \in \mathbb{Z}} \). The are no gaps in the spectrum of \( J^0 \), since in fact it is 1-periodic Jacobi matrix. Nevertheless, the perturbed Jacobi matrix \( J(a, b) \) is \( p \)-periodic. Thus, we should consider \( J(a^0, b) \) as \( p \)-periodic also. Following the general spectral theory for Jacobi matrices, the edges of spectral components \( \sigma_i^0 \) are eigenvalues of the following two matrices

\[
J_-^0 = J(a^0, b, 1) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad J_+^0 = J(a^0, b, -1) = \begin{pmatrix} 0 & 1 & 0 & \ldots & -1 \\ 1 & 0 & 1 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \ldots & 0 \end{pmatrix}.
\]

Let us consider the first matrix. The first matrix \( J_-^0 \) has eigenvalues \( \lambda_n^0 = 2 \cos \frac{2\pi n}{p}, n = 0, \ldots, p/2 \). The eigenvalues \( \lambda_0^0 = 2 \) and \( \lambda_{p/2}^0 = -2 \) are simple eigenvalues with the orthonormal
eigenvectors
\[ v_0 = \frac{1}{\sqrt{p}}(1_j^p)_{j=1}, \quad v_{p/2} = \frac{1}{\sqrt{p}}((-1)^j)_j^p. \]

Other eigenvalues \( \lambda_n^0 = 2\cos \frac{2\pi n}{p}, \ n = 1, ..., p/2 - 1 \) are double eigenvalues with the corresponding orthonormal eigenvectors
\[ v_{n1} = \frac{1}{\sqrt{p}}(e^{\frac{2\pi inj}{p}})_j^p, \quad v_{n2} = \frac{1}{\sqrt{p}}(e^{-\frac{2\pi inj}{p}})_j^p. \]

The edges of spectral components \( \sigma_n \) of \( J(a, b) \) are eigenvalues of the following matrices
\[ J_\pm \equiv J(a, b, 1) = J_0 - cJ_1, \quad J_\mp \equiv J(a, b, -1) = J_0^* + cJ_1, \]
where
\[ J_1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}. \]

The first matrix \( J_- \) has eigenvalues \( \lambda_n \) that can be computed by using the regular perturbation theory for small parameter \( c \), see, e.g. [K]. Namely, applying the regular perturbation theory for single and double eigenvalues \( \lambda_n^0 \), we obtain
\[ \begin{align*} 
\lambda_0 &= \lambda_0^0 + cv_0J_1v_0 + o(c) = 2 + \frac{2c}{p} + o(c), \\
\lambda_{p/2} &= \lambda_{p/2}^0 + v_{p/2}^*J_1v_{p/2} + o(c) = -2 - \frac{2c}{p} + o(c),
\end{align*} \]
(* denotes the conjugation) and
\[ \begin{align*} 
\lambda_{n1} &= \lambda_n^0 + c\lambda_{n1}^0 + o(c), \\
\lambda_{n2} &= \lambda_n^0 + c\lambda_{n2}^0 + o(c),
\end{align*} \]
where \( \lambda_{n1}^0 \) and \( \lambda_{n2}^0 \) are eigenvalues of the matrix \( H_n \) defined by
\[ H_n = \begin{pmatrix} (v_{n1})^*J_1v_{n1} & (v_{n1})^*J_1v_{n2} \\ (v_{n2})^*J_1v_{n1} & (v_{n2})^*J_1v_{n2} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} \cos \frac{2\pi n(p-1)}{p} & 2c \cos \frac{2\pi n(p+1)}{p} \\ 2c \cos \frac{2\pi n(p+1)}{p} & \cos \frac{2\pi n(p-1)}{p} \end{pmatrix}. \]

Hence, the Lebesgue measure of the gap, which appears around the edge \( \lambda_n^0 \) is
\[ |\gamma_n| = |\lambda_{n1} - \lambda_{n2}| = c|\lambda_{n1}^0 - \lambda_{n2}^0| + o(c) = \frac{4c}{p} + o(c). \]

Thus, all the gaps related to the spectral edges corresponding to the eigenvalues of \( J_- \) have the same size equivalent to \( 4c/p \) for small \( c > 0 \). The similar calculations allow us to conclude that the gaps related to the spectral edges corresponding to the eigenvalues of \( J_+ \) have also the same size equivalent to \( 4c/p \) for small \( c > 0 \). This leads to
\[ \sum_{n=1}^{p-1} |\gamma_n| = \frac{4c(p-1)}{p} + o(c), \]
which proves first identity in (1.5). Second identity in (1.5) is trivial, since all \( a_n = 1 \) except one \( a_n = 1 - c \).
REFERENCES

[CGR] Clark, S.; Gesztesy, F.; Renger, W. Trace formulas and Borg-type theorems for matrix-valued Jacobi and Dirac finite difference operators. J. Diff. Eq. 219 (2005), 144–182.

[DS] P. Deift, B. Simon. Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension. Commun. Math. Phys., 90, 389–411 (1983).

[K] Kato T. Perturbation Theory for Linear Operators. Springer (February 15, 1995).

[KKr] Korotyaev, E.; Krasovsky, I. Spectral estimates for periodic Jacobi matrices, Commun. Math. Phys. 234(2003), 517–532.

[KKu] Korotyaev, E., Kutsenko, A. Borg type uniqueness Theorems for periodic Jacobi operators with matrix valued coefficients. Proc. of the AMS, Volume 137, Number 6, June 2009, Pages 1989–1996.

[L] Y. Last. On the measure of gaps and spectra for discrete 1D Schrödinger operators. Commun. Math. Phys., 149, 347–360 (1992).

[PR] A. Poltoratski, C. Remling. Reflectionless Herglotz Functions and Jacobi Matrices, Commun. Math. Phys. Volume 288 Number 3(2009), 1007–1021.

[Ku] Kutsenko, A. A. Sharp spectral estimates for periodic matrix-valued Jacobi operators. Mathematical technology of networks, 133–136, Springer Proc. Math. Stat., 128, Springer, Cham, 2015.

[Te] G. Teschl. Jacobi Operators and Completely Integrable Nonlinear Lattices, Mathematical Surveys and Monographs, vol. 72, AMS, Rhode Island, 2000.

[vM] P. van Moerbeke. The spectrum of Jacobi matrices. Invent. Math. 37 (1976), no. 1, 45–81.

[G] Golinskii, L. On stability in the Borg–Hochstadt theorem for periodic Jacobi matrices, https://arxiv.org/abs/1704.03679, accepted in Journal of Spectral Theory (2017).

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