The Auslander bijections:
How morphisms are determined by modules.

Claus Michael Ringel

Abstract. Let Λ be an artin algebra. In his seminal Philadelphia Notes published in 1978, M. Auslander introduced the concept of morphisms being determined by modules. Auslander was very passionate about these investigations (they also form part of the final chapter of the Auslander-Reiten-Smalø book and could and should be seen as its culmination), but the feedback until now seems to be somewhat meager. The theory presented by Auslander has to be considered as an exciting frame for working with the category of Λ-modules, incorporating all what is known about irreducible maps (the usual Auslander-Reiten theory), but the frame is much wider and allows for example to take into account families of modules — an important feature of module categories. What Auslander has achieved is a clear description of the poset structure of the category of Λ-modules as well as a blueprint for interrelating individual modules and families of modules. Auslander has subsumed his considerations under the heading of “morphism being determined by modules”. Unfortunately, the wording in itself seems to be somewhat misleading, and the basic definition may look quite technical and unattractive, at least at first sight. This could be the reason that for over 30 years, Auslander’s powerful results did not gain the attention they deserve. The aim of this survey is to outline the general setting for Auslander’s ideas and to show the wealth of these ideas by exhibiting some examples.

1. Introduction.

There are two basic mathematical structures: groups and lattices, or, more generally, semigroups and posets. A first glance at any category should focus the attention to these two structures: to symmetry groups (for example the automorphism groups of the individual objects), as well as to the posets given by suitable sets of morphisms, for example by looking at inclusion maps (thus dealing with the poset of all subobjects of an object), or at the possible factorizations of morphisms. In this way, one distinguishes between local symmetries and global directedness.

The present survey deals with the category mod Λ of finite length modules over an artin algebra Λ. Its aim is to report on the work of M. Auslander in his seminal Philadelphia Notes published in 1978. Auslander was very passionate about these investigations and they also form part of the final chapter of the Auslander-Reiten-Smalø book: there, they could (and should) be seen as a kind of culmination. It seems to be surprising that the feedback until now is quite meager. After all, the theory presented by Auslander has to be considered as an exciting frame for working with the category mod Λ, incorporating
what is called the Auslander-Reiten theory (to deal with the irreducible maps), but this frame is much wider and allows for example to take into account families of modules — an important feature of a module category. Indeed, many of the concepts which are relevant when considering the categories mod Λ fit into the frame! What Auslander has achieved (but he himself may not have realized it) was a clear description of the poset structure of mod Λ and of the interplay between families of modules.

Auslander’s considerations are subsumed under the heading of morphisms being determined by modules, but the wording in itself seems to be somewhat misleading, and the basic definition looks quite technical and unattractive. As a consequence, for over 30 years, the powerful results of Auslander did not gain the attention they deserve.

Here is a short summary: Auslander asks for a description of the class of maps ending in a fixed module \( Y \). Looking at such maps \( f : X \to Y \), we may (and will) assume that \( f \) is right minimal, thus that there is no non-zero direct summand \( X' \) of \( X \) with \( f(X') = 0 \). Right minimal maps \( f : X \to Y \) and \( f' : X' \to Y \) are said to be right equivalent provided there is an isomorphism \( h : X \to X' \) such that \( f = f'h \). The object studied by Auslander is the set of right equivalence classes of right minimal maps ending in \( Y \) which we denote by

\[
\langle \to Y \rangle
\]

It is a poset (even a lattice) via the relation \( \leq \) which is defined as follows:

\[
(f : X \to Y) \leq (f' : X' \to Y)
\]

provided there is a homomorphism \( h : X \to X' \) with \( f = f'h \), thus provided the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{f'} \\
X' & \xrightarrow{f'} & Y
\end{array}
\]

Of course, to analyze the poset \( \langle \to Y \rangle \) is strongly related to a study of the contravariant Hom-functor \( \text{Hom}(-, Y) \), however the different nature of these two mathematical structures should be stressed: \( \text{Hom}(-, Y) \) is an additive functor whereas \( \langle \to Y \rangle \) is a poset, and it is the collection of these posets \( \langle \to Y \rangle \) which demonstrate the global directedness.

In general, the poset \( \langle \to Y \rangle \) is very large and does not satisfy any chain condition. But it is possible, and this is the main idea of Auslander, to write \( \langle \to Y \rangle \) as the filtered union of subsets \( C \langle \to Y \rangle \), where \( C \langle \to Y \rangle \) is given by those maps \( f \) which are “right \( C \)-determined”. Since the concept of “right determination” looks (at least at first sight) technical and unattractive, let us first describe the set \( C \langle \to Y \rangle \) only in the important case when \( C \) is a generator: in this case, \( C \langle \to Y \rangle \) consists of the (right equivalence classes of the right minimal) maps \( f \) ending in \( Y \) with kernel in \( \text{add} \tau C \) (where \( \tau = D \text{Tr} \) is the Auslander-Reiten translation). Here is the first main assertion:

(1) \[
\langle \to Y \rangle = \bigcup_C C \langle \to Y \rangle,
\]

or, in the formulation of Auslander: any map in mod \( \Lambda \) is right determined by some module.
The second main assertion describes these posets $C[\to Y]$ as follows: There is a lattice isomorphism

$$\eta_{CY}: C[\to Y] \longrightarrow \mathcal{S}\text{Hom}(C, Y),$$

where $\text{Hom}(C, Y)$ is considered as an $\text{End}(C)\text{op}$-module, and where $\mathcal{S}M$ denotes the submodule lattice of a module $M$. Actually, the map $\eta_{CY}$ is easy to describe, namely $\eta_{CY}(f) = \text{Im} \text{Hom}(C, f)$ for $f$ a morphism ending in $Y$. The essential assertion is the surjectivity of $\eta_{CY}$, thus to say that any submodule of $\text{Hom}(C, Y)$ is of the form $\text{Im} \text{Hom}(C, f)$ for some $f$.

What is the relevance? As we have mentioned, usually the poset $[\to Y]$ itself will not satisfy any chain conditions, but all the posets $C[\to Y]$ are of finite height and often can be displayed very nicely: according to (2) we deal with the submodule lattice $\mathcal{S}M$ of some finite length module $M$ over an artin algebra (namely over $\Gamma(C) = \text{End}(C)\text{op}$) and it is easy to see that any submodule lattice arises in this way. Of course, submodule lattices have interesting combinatorial features, but we should stress that we really are in the realm of algebraic geometry: Let us assume that $A$ is a $k$-algebra where $k$ is an algebraically closed field and $M$ a finite-dimensional $A$-module. The submodule lattice $\mathcal{S}M$ comprises the varieties $S_dM$ of all submodules of $M$ with length $d$, and it is well-known that these are projective varieties (and conversely, any projective variety occurs in this way, as Reineke recently has pointed out). It seems curious that Auslander himself did not focus the attention to the geometrical point of view, but restricted his attention just to combinatorial features (for example to possible waists in submodule lattices). Note that the inclusion of $C[\to Y]$ into $[\to Y]$ preserves meets, but usually not joins.

We end this summary by an outline in which way the Auslander bijection (2) incorporates the existence of minimal right almost split maps: We have to look at the special case where $Y$ is indecomposable and $C = Y$ and to deal with the submodule $\text{rad}(Y, Y)$ of $\text{Hom}(Y, Y)$. The bijection (2) yields an element $f: X \to Y$ in $[\to Y]$ such that $\eta_{Y, Y}(f) = \text{rad}(Y, Y)$ and to say that $f$ is right $Y$-determined means that $f$ is right almost split.

Acknowledgment. Some of the material presented here has been exhibited in 2012 in lectures at SJTU, Shanghai, at USTC, Hefei, and at the ICRA conference at Bielefeld. The author is grateful to the audience for questions and remarks. He also wants to thank Lutz Hille and Henning Krause for helpful comments.

I. The setting.

2. The poset $[\to Y]$.

Let $Y$ be a $A$-module. Let $(\to Y)$ be the class of all homomorphisms $f: X \to Y$ (such homomorphisms will be said to be the homomorphisms ending in $Y$). We denote a preorder $\preceq$ on $(\to Y)$ as follows: Given $f: X \to Y$ and $f': X' \to Y$, we write $f \preceq f'$ provided there is a homomorphism $h: X \to X'$ such that $f = f'h$ (clearly, this relation is reflexive and transitive). As usual, this preorder defines an equivalence relation (we call it right...
equivalence) by saying that \( f, f' \) are right equivalent provided both \( f \leq f' \) and \( f' \leq f \), and induces a poset relation \( \leq \) on the set \( \rightarrow \) of right equivalence classes of homomorphisms ending in \( Y \). Given a morphism \( f: X \rightarrow Y \), we denote its right equivalence class by \([f]\) and by definition \([f] \leq [f']\) if and only if \( f \leq f' \).

It should be stressed that \( \rightarrow \) is a set, not only a class: namely, the isomorphism classes of \( \Lambda \)-modules form a set and for every module \( X \), the homomorphisms \( X \rightarrow Y \) form a set; we may choose a representative from each isomorphism class of \( \Lambda \)-modules and given an element \( f: X \rightarrow Y \) in \( \rightarrow \), then there is an isomorphism \( h: X' \rightarrow X \) where \( X' \) is such a representative, and \( f \) is right equivalent to \( fh \).

Recall that a map \( f: X \rightarrow Y \) is said to be right minimal provided any direct summand \( X' \) of \( X \) with \( f(X') = 0 \) is equal to zero. If \( f: X \rightarrow Y \) is a morphism and \( X = X' \oplus X'' \) such that \( f(X''') = 0 \) and \( f|X': X' \rightarrow Y \) is right minimal, then \( f|X' \) is called a right minimalisation of \( f \).

**Proposition 2.1.** Every right equivalence class \([f]\) in \( \rightarrow \) contains a right minimal morphism, namely \([f']\), where \( f' \) is a right minimalisation of \( f \). Given right minimal morphisms \( f: X \rightarrow Y \) and \( f': X' \rightarrow Y \), then \( f, f' \) are right equivalent if and only if there is an isomorphism \( h: X \rightarrow X' \) such that \( f = f'h \).

Proof: Let \( f: X \rightarrow Y \) be an element of \( \rightarrow \). Write \( X = X_1 \oplus X_2 \) such that \( f(X_2) = 0 \) and \( f|X_1: X_1 \rightarrow Y \) is right minimal. Let \( u: X_1 \rightarrow X_1 \oplus X_2 \) be the canonical inclusion, \( p: X_1 \oplus X_2 \rightarrow X_1 \) the canonical projection. Then \( pu = 1_{X_1} \) and \( f = fup \) (since \( f(X_2) = 0 \)). We see that \( fu \leq f \) and \( f = fup \leq fu \), thus \( f \) and \( fu \) are right equivalent and \( fu = f|X_1 \) is right minimal. If the right minimal morphisms \( f: X \rightarrow Y \) and \( f': X' \rightarrow Y \) are right equivalent, then there are morphisms \( h: X \rightarrow X' \) and \( h': X' \rightarrow Y \) such that \( f = f'h \) and \( f' = fh' \). But \( f = fh'h \) implies that \( hh' \) is an automorphism, and \( f' = f'hh' \) implies that \( hh' \) is an automorphism, thus \( h, h' \) have to be isomorphisms (see [ARS] I.2).

**Remark.** Monomorphisms \( X \rightarrow Y \) are always right minimal, and the right equivalence classes of monomorphisms ending in \( Y \) may be identified with the submodules of \( Y \) (here, we identify the right equivalence class of the monomorphism \( f: X \rightarrow Y \) with the image of \( Y \)).

**Proposition 2.2.** The poset \( \rightarrow \) is a lattice with zero and one. Given \( f_1: X_1 \rightarrow Y \) and \( f_2: X_2 \rightarrow Y \), say with pullback \( g_1: X \rightarrow X_1, g_2: X \rightarrow X_2 \), the meet of \([f_1]\) and \([f_2]\) is given by the map \( g_1f_1: X \rightarrow Y \), the join of \([f_1]\) and \([f_2]\) is given by \([f_1, f_2]: X_1 \oplus X_2 \rightarrow Y \).

**Examples 1.** It should be stressed that if \( f_1, f_2 \) are right minimal, neither the pullback map \( f_1g_1 \) nor the direct sum map \([f_1, f_2]\) have to be right minimal. Thus if one wants to work with right minimal maps, one has to right minimalise the maps in question.

Here are corresponding examples: Let \( \Lambda \) be the path algebra of the quiver \( 1 \leftarrow 2 \). When dealing with an algebra with quiver \( Q \), and \( i \) is a vertex of \( Q \), then we denote by \( S(i) \) the simple module corresponding to \( i \), by \( P(i) \) and \( I(i) \) the projective cover or injective envelope of \( S(i) \), respectively. If \( f_1 = f_2: P(2) \rightarrow S(2) \) is the canonical projection, then the pullback \( U \) of \( f_1 \) and \( f_2 \) is a submodule of \( P(2) \oplus P(2) \) with a direct summand of the form \( S(1) \); but any map \( S(1) \rightarrow S(2) \) is zero, thus there is no right minimal map \( U \rightarrow S(2) \). Also, the map \([f_1, f_2]: P(2) \oplus P(2) \rightarrow S(2)\) is not right minimal, since \( \dim \text{Hom}(P(2), S(2)) = 1 \).
Proof of the proposition (trivial verification): Write \( f = f_1 g_1 = f_2 g_2 \). We have \( f = f_1 g_1 \preceq f_1 \) and \( f = f_2 g_2 \preceq f_2 \), thus \([f] \leq [f_1]\) and \([f] \leq [f_2]\). If \( f' : X' \to Y \) is a morphism with \([f'] \leq [f_1]\) and \([f'] \leq [f_2]\), then \( f' \preceq f_1 \) and \( f' \preceq f_2 \), thus there are morphisms \( \phi_i \) with \( f' = f_i \phi_i \), for \( i = 1, 2 \). Since \( f_1 \phi_1 = f_2 \phi_2 \), the pullback property yields a morphism \( \phi : X' \to X \) such that \( \phi_i = g_i \phi \) for \( i = 1, 2 \). Thus \( f' = f_1 \phi_1 = f_2 \phi_2 \phi = f \phi \) shows that \( f' \preceq f \), thus \([f'] \leq [f]\). This shows that \([f]\) is the meet of \([f_1]\) and \([f_2]\).

Second, denote the canonical inclusion maps \( X_i \to X_1 \oplus X_2 \) by \( u_i \), for \( i = 1, 2 \), thus \([f_1, f_2] u_i = f_i \) and therefore \([f_i]\) \leq \([f_1, f_2]\) for \( i = 1, 2 \). Assume that there is given a morphism \( g : X'' \to Y \) with \([f_i] \leq [g]\) for \( i = 1, 2 \). This means that there are morphisms \( \psi_i : X_i \to X'' \) such that \( f_i = g \psi_i \) for \( i = 1, 2 \). Let \( \psi = [\psi_1, \psi_2] : X_1 \oplus X_2 \to X'' \) (with \( \psi u_i = \psi_i \)). Then \([f_1, f_2] = g [\psi_1, \psi_2] = g \psi \) shows that \([f_1, f_2] \preceq g\), thus \([f_1, f_2]\) \leq [g]}. This shows that \([f_1, f_2]\) is the join of \([f_1]\) and \([f_2]\).

It is easy to check that the map \( 0 \to Y \) is the zero element of \([\to Y]\) and that the identity map \( Y \to Y \) is its unit element.

As we have seen, the poset \([\to Y]\) is a lattice. What will be important in the following discussion is the fact that we deal with a meet-semilattice (these are the posets such that any pair has a meet). Note that all the posets which we consider turn out to be lattices, however the poset maps to be considered will preserve meets but usually not joins, thus we really work in the category of meet-semilattices.

**General convention.** The elements of \([\to Y]\) will sometimes be written as short exact sequences \( U \to V \to W \), with \( W \) a submodule of \( Y \), so that the composition \( V \to W \subseteq Y \) is a right minimal map.

**Examples 2. Failure of the chain conditions.** Here are examples which show that \([\to Y]\) *neither satisfies the ascending nor the descending chain condition*. Let \( \Lambda \) be the Kronecker algebra, this is the path algebra of the quiver

\[
\begin{array}{c}
1 \\
\preceq \\
2
\end{array}
\]

with coefficients in a field \( k \). The \( \Lambda \)-modules are also called *Kronecker modules*. Basic facts concerning the Kronecker modules will be recalled in section 12. Let \( Y = S(2) \), the simple injective module.

We denote by \( Q_n \) the indecomposable preinjective module of length \( 2n + 1 \) (thus \( Q_0 = S(2), Q_1 = S(1) \)). There is a chain of epimorphisms

\[
\cdots \to Q_2 \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 = Y,
\]

thus we have the descending chain

\[
\cdots [f_1 f_2 f_3] \prec [f_1 f_2] \prec [f_1]
\]

in \([\to Y]\).

Here, we can assume that all the kernels \( f_n : Q_n \to Q_{n-1} \) are equal to \( R \), where \( R \) is a fixed indecomposable of length 2. Also, there is such a chain of epimorphisms such that all
the kernels are pairwise different and of length 2 (if the ground field \( k \) is infinite). In the first case, the kernels of the maps to \( Y \) are all indecomposable (namely \( R[n] \) for \( n \in \mathbb{N} \)), in the second, they are direct sums of pairwise non-isomorphic modules of length 2.

In order to look at the ascending chain condition, let \( P_n \) be indecomposable preprojective of length \( 2n + 1 \). For \( i \geq 1 \), there are epimorphisms \( f_i: P_i \to Y \) and monomorphisms \( u_i: P_i \to P_{i+1} \) such that \( f_iu_{i-1} = f_{i-1} \). Thus, looking at the sequence of maps

\[
P_1 \xrightarrow{u_1} P_2 \xrightarrow{u_2} \cdots \to Y
\]

we obtain an ascending chain in \( \to Y \).

\[
[f_1] < [f_2] < [f_3] < \cdots
\]

There are also such chains inside the category \( \mathcal{R} \) of the regular Kronecker modules.

3. Morphisms determined by modules: Auslander’s First Theorem.

Here is the decisive definition. Let \( f: X \to Y \) be a morphism and \( C \) a module. Then \( f \) is said to be right \( C \)-determined by \( C \) (or right determined by \( C \)) provided the following condition is satisfied: given any morphism \( f': X' \to Y \) such that \( f'\phi \) factors through \( f \) for any \( \phi: C \to X' \), then \( f' \) itself factors through \( f \).

**Lemma 3.1.** If \( f \) is right \( C \)-determined, then also right \((C \oplus C')\)-determined for any module \( C' \).

Proof. Trivial.

The first main theorem asserts that any morphism is determined by some module. Actually, there is a precise formula which yields for \( f \) the smallest possible module \( C(f) \) which right determines \( f \). We call it the minimal right determiner of \( f \) (and any other right determiner of \( f \) will have \( C(f) \) as a direct summand).

We need two definitions. The intrinsic kernel of \( f \) is the kernel of a right minimalisation of \( f \). An indecomposable projective module \( P \) is said to almost factor through \( f \), provided there is a commutative diagram of the following form

\[
\begin{array}{ccc}
\text{rad} P & \longrightarrow & P \\
\downarrow & & \downarrow \eta \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that the image of \( \eta \) is not contained in the image of \( f \).

**Theorem 3.2 (Determiner formula of Auslander-Reiten-Smalø)** Let \( f \) be a morphism ending in \( Y \). Let \( C(f) \) be the direct sum of the indecomposable modules of the form \( \tau^{-k}K \), where \( K \) is a direct summand of the intrinsic kernel of \( f \) and of the
indecomposable projective modules which almost factor through $f$. Then $f$ is right $C$-determined if and only if $C(f) \in \text{add } C$.

For the proof, see [ARS].

**Corollary 3.3.** Any $f$ is right $C$-determined by some $C$, for example by the module

$$\tau^{-1} \text{Ker}(f) \oplus P(\text{soc Cok}(f)).$$

**Corollary 3.3 (Auslander’s First Theorem).** The module $\tau^{-1} \text{Ker}(f) \oplus \Lambda$ right determines $f$.

**Corollary 3.4.** Let $P$ be a projective module and $f: X \rightarrow Y$ a right minimal morphism. Then $f$ is right $P$-determined if and only if $f$ is a monomorphism and the socle of the cokernel of $f$ is generated by $P$.

This is an immediate consequence of the determiner formula: First, assume that $f$ is right $P$-determined. Then the intrinsic kernel of $f$ has to be zero. Since we assume that $f$ is right minimal, $f$ must be a monomorphism. If $S$ is a simple submodule of the cokernel of $f$, then $P(S)$ almost factors through $f$, thus $P(S)$ is a direct summand of $P$. This shows that the socle of the cokernel of $f$ is generated by $P$. Conversely, assume that $f$ is a monomorphism and the socle of the cokernel of $f$ is generated by $P$. Since $f$ is a monomorphism, $C(f)$ is the direct sum of all indecomposable projective modules which almost factor through $f$. It follows that $C(f)$ is in $\text{add } P$, thus $f$ is right $P$-determined.

**Corollary 3.5.** A right minimal morphism $f: X \rightarrow Y$ is a monomorphism if and only if it is right $\Lambda$-determined.

**4. The posets $C[\rightarrow Y]$.**

We denote by $C(\rightarrow Y)$ the class of all morphisms ending in $Y$ which are right $C$-determined and $C[\rightarrow Y]$ is the set of the right equivalence classes of the morphisms ending in $Y$ which are right $C$-determined.

**Proposition 4.1.** The subset $C[\rightarrow Y]$ of $[\rightarrow Y]$ is closed under meets.

Proof (trivial verification): Let $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$ be right $C$-determined. As we know, the meet of $[f_1]$ and $[f_2]$ is given by forming the pullback of $f_1$ and $f_2$. Thus assume that $X$ is the pullback with maps $g_1: X \rightarrow X_1$ and $g_2: X \rightarrow X_2$ and let $f = f_1g_1 = f_2g_2$. We want to show that $f$ is right $C$-determined. Thus, assume that there is given $f': X' \rightarrow Y$ such that for any $\phi: C \rightarrow X'$, there exists $\phi': C \rightarrow X$ such that $f' \phi = f \phi'$. Then we see that for any $\phi: C \rightarrow X'$, we have $f' \phi = f \phi = f_1(g_1 \phi)$, thus $f' \phi$ factors through $f_1$. Since $f_1$ is right $C$-determined, it follows that $f'$ factors through $f_1$, say $f' = f_1h_1$ for some $h_1: X' \rightarrow X_1$. Similarly, for any $\phi: C \rightarrow X'$, the morphism $f' \phi$ factors through $f_2$ and therefore $f' = f_2h_2$ for some $h_2: X' \rightarrow X_2$. Now $f_1h_1 = f' = f_2h_2$ implies
that there is $h: X' \to X$ such that $g_1 h = h_1$, and $g_2 h = h_2$. Thus $f' = f_1 h_1 = f_1 g_1 h = fh$ shows that $f'$ factors through $f$.

**Example 3.** The subset $C[\to Y]$ of $[\to Y]$ is usually not closed under joins, as the following example shows: Let $\Lambda$ be the path algebra of the quiver of type $A_3$ with one sink labeled 0 and two sources labeled 1 and 2. Let $f_i: P(i) \to I(0)$ be non-zero maps for $i = 1, 2$, these are monomorphisms, thus they are right $\Lambda$-determined. The join of $[f_1]$ and $[f_2]$ in $[\to Y]$ is given by the map $[f_1, f_2]: P(1) \oplus P(2) \to I(0)$. Clearly, this map is right minimal, but it is not injective. Thus $[f_1, f_2]$ is not right $\Lambda$-determined.

**Example 4.** This example indicates that in general, it may not be advisable to ask for closure under joins. Consider again the Kronecker algebra $\Lambda$, let $Y = S(2)$. If $C = \Lambda$, then all the maps in $C[\to Y]$ are given by inclusion maps. In fact, here we deal with the submodule lattice of $Y$

![Lattice diagram]

it is a lattice of height 3. The layer of height 2 consists of the regular modules $R$ of length 2, or better of the corresponding inclusion maps $R \to Y$. The join in $C[\to Y]$ of two different maps $f_1, f_2$ in the height 2 layer is just the identity map $Y \to Y$. Now, for two such maps $f_1, f_2$ the join in $[\to Y]$, is the direct sum map $[f_1, f_2]: R_1 \oplus R_2 \to Y$. More generally, if there are given $n$ pairwise different regular modules $R_1, \ldots, R_n$ of length 2 with inclusion maps $f_i: R_i \to Y$, then the join in $[\to Y]$ is just $[f_1, \ldots, f_n]: R_1 \oplus \cdots \oplus R_n \to Y$. Thus, if the base field $k$ is infinite, the smallest subposet of $[\to Y]$ closed under meets and joins and containing the inclusion maps $R \to Y$ with $R$ regular of length 2 will have infinite height.

5. **The Auslander bijection. Auslander’s Second Theorem.**

Let $C, Y$ be objects. Let $\Gamma(C) = \text{End}(C)^{op}$. We always will consider $\text{Hom}(C, Y)$ as a $\Gamma(C)$-module. For any module $M$, we denote by $SM$ the set of all submodules (it is a lattice with respect to intersection and sum of submodules).

Define

$$\eta_{CY}: (\to Y) \to S(\text{Hom}(C, Y))$$

by $\eta_{CY}(f) = \text{Im}\text{Hom}(C, f) = f \cdot \text{Hom}(C, X)$ for $f: X \to Y$ (note that $f \cdot \text{Hom}(C, X)$ clearly is a $\Gamma(C)$-submodule).

Here is a reformulation of the definition of $\eta_{CY}$.

**Proposition 5.1.** Let $f: X \to Y$. Then $\eta_{CY}(f)$ is the set of all $h \in \text{Hom}(C, Y)$ which factor through $f$. This subset of $\text{Hom}(C, Y)$ is a $\Gamma(C)$-submodule.

Proof. We have mentioned already, that $\eta_{CY}(f)$ is a $\Gamma(C)$-submodule of $\text{Hom}(C, Y)$. Also, if $h \in \eta_{CY}(f) = f \text{Hom}(C, X)$, then $h$ factors through $f$. And conversely, if $h$ factors through $f$, then $h$ belongs to $f \text{Hom}(C, X) = \eta_{CY}(f)$.

8
Lemma 5.2. If $X = X_0 \oplus X_1$ and $f(X_0) = 0$, then $\eta_{CY}(f) = \eta_{CY}(f|X_1)$.

Proof: Let $X = X_0 \oplus X_1$ and write $f = [f_0 \ \ f_1] = [0 \ \ f_1]$ with $f_i : X_i \to Y$. Then, for $\phi_i : C \to X_i$, we have $[0 \ \ f_1] \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = f_1 \phi_1$. Thus

$$\eta_{CY}(f) = f \text{Hom}(C, X_0 \oplus X_1) = f_1 \text{Hom}(C, X_1) = \eta_{CY}(f_1).$$

In particular: If $f_1$ is a right minimal version of $f$, then $\eta_{CY}(f) = \eta_{CY}(f_1)$. Altogether, we see that $\eta_{CY}$ is constant on right equivalence classes, thus we can define $\eta_{CY}([f]) = \eta_{CY}(f)$ and obtain in this way a map

$$\eta_{CY} : [\to Y) \to S(\text{Hom}(C, Y)).$$

Of special interest is the restriction of $\eta_{CY}$ to $C[\to Y)$.

Proposition 5.3. Let $C, Y$ be modules. The map

$$\eta_{CY} : C[\to Y) \to S \text{Hom}(C, Y).$$

is injective and preserves meets. As a consequence, it preserves and reflects the ordering.

Proof (trivial verification): First, let us show that $\eta_{CZ}$ is injective. Consider maps $f : X \to Y$ and $f' : X' \to Y$ such that $f \text{Hom}(C, X) = f' \text{Hom}(C, X')$. Since $f$ is right $C$-determined and $f' \text{Hom}(C, X') \subseteq f \text{Hom}(C, X)$, we see that $f' \in f \text{Hom}(X', X)$. Since $f'$ is right $C$-determined and $f \text{Hom}(C, X) \subseteq f' \text{Hom}(C, X')$, we see that $f \in f \text{Hom}(X, X')$. But this means that $f' \preceq f \preceq f'$, thus $[f] = [f']$.

Next, consider the following pullback diagram

$$\begin{array}{ccc}
X & \xrightarrow{g_1} & X_1 \\
g_2 \downarrow & & \downarrow f_1 \\
X_2 & \xrightarrow{f_2} & Y
\end{array}$$

such that $f_1, f_2$ both are right $C$-determined. Let $f = f_1 g_1 = f_2 g_2$, thus the meet of $[f_1]$ and $[f_2]$ is $[f]$. Now $f = f_1 g_1$ shows that $f \text{Hom}(C, X) \subseteq f_1 \text{Hom}(C, X_1)$. Similarly, $f = f_2 g_2$ shows that $f \text{Hom}(C, X) \subseteq f_2 \text{Hom}(C, X_2)$. Both assertions together yield

$$f \text{Hom}(C, X) \subseteq f_1 \text{Hom}(C, X_1) \cap f_2 \text{Hom}(C, X_2).$$

Conversely, take an element in $f_1 \text{Hom}(C, X_1) \cap f_2 \text{Hom}(C, X_2)$, say $f_1 \phi_1 = f_2 \phi_2$ with $\phi_i : C \to X_i$, for $i = 1, 2$. The pullback property yields a morphism $\phi : C \to X$ such that $g_i \phi = \phi_i$ for $i = 1, 2$. Therefore $f_1 \phi_1 = f_1 g_1 \phi = f \phi$ belongs to $f \text{Hom}(C, X)$. Thus,

$$f_1 \text{Hom}(C, X_1) \cap f_2 \text{Hom}(C, X_2) \subseteq f \text{Hom}(C, X),$$

and therefore

$$f_1 \text{Hom}(C, X_1) \cap f_2 \text{Hom}(C, X_2) = f \text{Hom}(C, X).$$
In general, assume that $L, L'$ are posets with meets and $\eta: L \to L'$ is a set-theoretical map which preserves meets. Then $a \leq b$ in $L$ implies $\eta(a) \leq \eta(b)$ in $L'$. Namely, $a \leq b$ gives $a \land b = a$, thus $\eta(a) \land \eta(b) = \eta(a)$ and therefore $\eta(a) \leq \eta(b)$. Conversely, if $a, b$ are arbitrary elements in $L$ with $\eta(a) \leq \eta(b)$, let $c = a \land b$. Then $\eta(c) = \eta(a) \land \eta(b) = \eta(a)$. Thus, if $\eta$ is injective, then $c = a$, and $a \land b = a$ implies $a \leq b$. Altogether, we see that $\eta_{CY}$ preserves and reflects the ordering.

Auslander’s second theorem (as established in [A1]) asserts:

**Theorem 5.4 (Auslander).** The map $\eta_{CY}: [\to Y] \to S(\text{Hom}(C, Y))$ is surjective.

Altogether we see: The map $\eta_{CY}$ defined by $\eta_{CY}(f) = \text{Im}\text{Hom}(C, f)$ yields a lattice isomorphism

\[ \eta_{CY}: \ C\to Y \longrightarrow S \text{Hom}(C, Y). \]

Note that the set $S\text{Hom}(C, Y)$ of submodules of $\text{Hom}(C, Y)$ has two distinguished elements, namely the zero submodule, as well as $\text{Hom}(C, Y)$ itself. Under (the inverse of) $\eta_{CY}$, the latter element just corresponds to the identity map of $Y$ (namely, $\eta_{CY}(1_Y) = 1_Y \text{Hom}(C, Y) = \text{Hom}(C, Y)$), thus here we obtain nothing spectacular. But already the zero subspace $0$ of $\text{Hom}(C, Y)$ yields a surprise, often this is a quite non-trivial map, a description in general seems to be unknown.

Better: The composition

\[ C \to Y \subseteq [\to Y] \xrightarrow{\eta_{CY}} S_{\Gamma(C)} \text{Hom}(C, Y) \]

of the inclusion map and the map $\eta_{CY}$ defined by $\eta_{CY}(f) = \text{Im}\text{Hom}(C, f)$ is a lattice isomorphism.

**Convention.** In the following, several examples of Auslander bijections will be presented. When looking at the submodule lattice $SM$ of a module $M$, we usually will mark the elements of $SM$ (or some of the elements) by bullets • and connect comparable elements by a solid line. Here, going upwards corresponds to the inclusion relation.

For the corresponding lattices $C[\to Y)$, we often will mark an element $[f: X \to Y)$ (with $f$ a right minimal map) by just writing $X$ and we will connect neighboring pairs $[f: X \to Y) \leq [f': X' \to Y)$ by drawing an (upwards) arrow $X \to Y$. On the other hand, sometimes it seems to be more appropriate to refer to the right minimal map $f: X \to Y$ with kernel $K'$ and image $Y'$ by using the exact sequence notation $K' \to X \to Y'$.

Note that the lattice $S\text{Hom}(C, Y)$ has two distinguished elements, namely $\text{Hom}(C, Y)$ itself as well as its zero submodule. Under the bijection $\eta_{CY}$ the total submodule $\text{Hom}(C, Y)$ correspond to the identity map $1_Y$ of $Y$, this is not at all exciting. But of interest seems to be the map $\eta_{\text{CY}}^{-1}(0)$.
Corollary 5.5. The poset $\mathcal{C}(\rightarrow Y)$ has a zero element, namely $\eta_{CY}^{-1}(0)$.

In general it seems to be quite difficult to describe such a map $f$ such that $[f] \in \mathcal{C}(\rightarrow Y) = \eta_{CY}^{-1}(0)$. But one should be aware that any pair $C, Y$ of $\Lambda$-modules yields uniquely (up to right equivalence) a map $f$ ending in $Y$, namely the right minimal, right $\Lambda$-determined map $f$ with $\eta_{CY}(f) = 0$. Here is a further reformulation:

Given modules $C$ and $Y$, there is a unique submodule $Y'$ of $Y$ which is minimal with respect of having an exact sequence $0 \rightarrow K \rightarrow X \rightarrow Y' \rightarrow 0$ such that the composition $X \rightarrow Y' \subseteq Y$ is right $\Lambda$-determined. This submodule is the largest submodule $Y'$ of $Y$ such that $\mathcal{P}(C, Y') = 0$.

Note: Assume that $\mathcal{P}(C, Y_1) = 0$ and $\mathcal{P}(C, Y_2) = 0$. Let $\psi: C \rightarrow Y_1 + Y_2$ belong to $\mathcal{P}(C, Y_1 + Y_2)$, thus there is a projective module $P$ and maps $\psi': C \rightarrow P$, $\psi'': P \rightarrow Y_1 + Y_2$ with $\psi = \psi'' \psi'$. Now lift $\psi''$ to a map $\beta: P \rightarrow Y_1 \oplus Y_2$ (thus $\psi''$ is the composition of $\beta$ and the projection map $Y_1 \oplus Y_2 \rightarrow Y_1 + Y_2$ and $\psi$ is the composition of $\beta \psi'$ and the projection map $Y_1 \oplus Y_2 \rightarrow Y_1 + Y_2$. But the two components of $\beta \psi'$ are maps in $\mathcal{P}(C, Y_i)$, thus zero. Therefore $\psi = 0$.

The special case $C = \Lambda \Lambda$. It is worthwhile to draw the attention to the special case when $C = \Lambda \Lambda$.

Proposition 5.6. The special case of the Auslander bijection $\eta_{\Lambda Y}$ is the obvious identification of both $\Lambda(\rightarrow Y)$ and $\mathcal{S} \operatorname{Hom}(\Lambda, Y)$ with $\mathcal{S}Y$.

Proof. First, consider $\Lambda(\rightarrow Y)$: The determiner formula asserts: a right minimal morphism is right $\Lambda$-determined if and only if it is a monomorphism. Thus $\Lambda(\rightarrow Y)$ is just the set of right equivalence classes of monomorphisms ending in $Y$, and the map $f \mapsto \operatorname{Im}(f)$ yields an identification between the the set of right equivalence classes of monomorphisms ending in $Y$ and the submodules of $\Lambda$.

Next, we deal with $\mathcal{S} \operatorname{Hom}(\Lambda, Y)$. Note that $\Gamma(\Lambda \Lambda) = \operatorname{End}(\Lambda \Lambda)^{\operatorname{op}} = \Lambda$ and there is a canonical identification $\epsilon: \operatorname{Hom}(\Lambda, Y) \simeq Y$ (given by $\epsilon(h) = h(1)$ for $h \in \operatorname{Hom}(\Lambda, Y)$), thus $\mathcal{S} \epsilon: \mathcal{S} \operatorname{Hom}(\Lambda, Y) \simeq \mathcal{S}Y$ (with $\mathcal{S} \epsilon(U) = \{h(1) \mid h \in U\}$ for $U$ a submodule of $\operatorname{Hom}(\Lambda, Y)$).

The Auslander bijection $\eta_{\Lambda Y}$ attaches to $f: X \rightarrow Y$ the submodule $f \operatorname{Hom}(\Lambda, X)$ and there is the following commutative diagram:

$$
\begin{array}{ccc}
\Lambda(\rightarrow Y) & \xrightarrow{\eta_{\Lambda Y}} & \mathcal{S} \operatorname{Hom}(\Lambda, Y) \\
\downarrow \operatorname{Im} & & \downarrow \epsilon \\
\mathcal{S}Y & & \operatorname{Hom}(\Lambda, Y) \\
\mathcal{S} \epsilon & & \\
\end{array}
$$

Namely, for $f: X \rightarrow Y$ we have

$$(\mathcal{S} \epsilon) \eta_{\Lambda Y}(f) = (\mathcal{S} \epsilon)(f \operatorname{Hom}(\Lambda, X)) = \{fh(1) \mid h \in \operatorname{Hom}(\Lambda, X)\} = \{f(x) \mid x \in X\} = \operatorname{Im}(f).$$

As a consequence, we see that all possible submodule lattices $\mathcal{S}Y$ occur as images under the Auslander bijections.
Remark. Both objects $C \to Y$ and $\mathcal{S}\text{Hom}(C,Y)$ related by the Auslander bijection $\eta_{CY}$ concern morphisms ending in $Y$: Looking at $C \to Y$, we deal with morphisms ending in $Y$ and right $C$-determined. Looking at $\text{Hom}(C,Y)$, we deal with maps ending in $Y$ and starting in $C$. One should be aware that a right minimal map ending in $Y$ and right $C$-determined usually will not start at $C$, thus the relationship between the elements of $C \to Y$ and the submodules of $\text{Hom}(C,Y)$ is really of interest! Note however that in case we deal with a map $f: C \to Y$ which is right $C$-determined (and starts in $C$), then

$$\eta_{CY}(f) = f \text{ Hom}(C,Y)$$

is just the submodule of $\text{Hom}(C,Y)$ generated by $f$.

6. Neighbors in $C \to Y$, composition factors of $\text{Hom}(C,Y)$.

The height of the lattice $C \to Y$ is just the length of a maximal chain of non-invertible maps

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = Y$$

such that all the compositions $X_i \to \cdots \to X_0 = Y$ are right minimal and right $C$-determined. Of course, using $\eta_{CY}$ we see that the lattices $C \to Y$ and $\mathcal{S}\text{Hom}(C,Y)$ have the same height. We want to study the relationship in more detail.

The Auslander bijection asserts that the lattice $C \to Y$ is a modular lattice of finite height, thus there is a Jordan-Hölder Theorem for $C \to Y$; it can be obtained from the corresponding Jordan-Hölder Theorem for the submodule lattice $\mathcal{S}\text{Hom}(C,Y)$. Here we are going to formulate the assertions for $C \to Y$ explicitly.

Let $h_i: X_i \to X_{i-1}$ be maps, where $1 \leq i \leq t$, with composition $f = h_1 \cdots h_t$. The sequence $(h_1, h_2, \ldots, h_t)$ is called a right $C$-factorization of $f$ of length $t$ provided the maps $h_i$ are non-invertible and the compositions $f_i = h_i \cdots h_1$ are right minimal and right $C$-determined, for $1 \leq i \leq t$. It sometimes may be helpful to deal also with right $C$-factorizations of length 0; by definition these are just the identity maps (or, if you prefer, the isomorphisms).

If $(h_1, \ldots, h_t)$ is a right $C$-factorization of a map $f$, then any integer sequence $0 = i(0) < i(1) < \cdots < i(s) = t$ defines a sequence of maps $(h'_1, h'_2, \ldots, h'_s)$ with $h'_j = h_i(j-1)+1 \cdots h_i(j)$ for $1 \leq j \leq s$. The following lemma asserts (by induction) that $(h'_1, h'_2, \ldots, h'_s)$ is again a right $C$-factorization of $f$ and we say that $(h_1, h_2, \ldots, h_t)$ is a refinement of $(h'_1, h'_2, \ldots, h'_s)$. In particular, any right $C$-factorization $(h_1, \ldots, h_t)$ of $f$ is a refinement of $f$.

Lemma 6.1. If $(h_1, \ldots, h_t)$ is a right $C$-factorization of length $t \geq 2$, then

$$(h_1, \ldots, h_{i-1}, h_i h_{i+1}, h_{i+2}, \ldots, h_t)$$

is a right $C$-factorization (of length $t - 1$).

Proof: We only have to check that $h_i h_{i+1}$ cannot be invertible. Assume $h_i h_{i+1}$ is invertible. Then $h_i$ is a split epimorphism. Since $f_i = h_i \cdots h_1$ is right minimal, it follows that $h_i$ is invertible, a contradiction.
We say that a right $C$-factorization $(h_1, h_2, \ldots, h_t)$ is maximal provided it does not have a refinement of length $t + 1$.

**Proposition 6.2.** Any right $C$-factorization $(h_1, \ldots, h_t)$ has a refinement which is a maximal right $C$-factorization and all maximal right $C$-factorizations of $(h_1, \ldots, h_t)$ have the same length.

In particular, any right minimal right $C$-determined map $f$ has a refinement which is a maximal right $C$-factorization, say $(h_1, \ldots, h_t)$ and its length $t$ will be called the $C$-length of $f$, we write $|f|_C$ for the $C$-length of $f$.

There is the following formula:

**Proposition 6.3.** Let $f : X \to Y$ be right minimal and right $C$-determined. Then

$$|f|_C = |\text{Hom}(C, Y)| - |\eta_{CY}(f)|,$$

where $|\text{Hom}(C, Y)|$ denotes the length of the $\Gamma(C)$-module $\text{Hom}(C, Y)$ and $|\eta_{CY}(f)|$ the length of its $\Gamma(C)$-submodule $\eta_{CY}(f)$.

**Corollary 6.4.** Let $f : X \to Y$ be right minimal and right $C$-determined. Then $|f|_C = 1$ if and only if $\eta_{CY}(f)$ is a maximal $\Gamma(C)$-submodule of $\text{Hom}(C, Y)$.

Let $f = h f'$ and $f'$ be right minimal, right $C$-determined maps. We say that the pair $(f, f')$ is a pair of $C$-neighbors provided $|f|_C = |f'|_C + 1$. Note that the pair $(f, f')$ in $C[	o Y]$ is a pair of $C$-neighbors provided $[f] < [f']$ and there is no $f''$ with $[f] < [f''] < [f']$ (of course, it is the condition $[f] < [f']$ which implies that there is a map $h$ with $f = f'h$).

**Examples 5.** Let us assume that $f = f'h$, where $f'$ is right minimal and right $C$-determined. It can happen that $h$ is also right minimal and right $C$-determined, but $f = f'h$ is not right $C$-determined. Also it can happen that (besides $f'$) also $f = f'h$ is right minimal and right $C$-determined, whereas $h$ is not right $C$-determined.

To see examples, consider the linearly directed quiver of type $A_3$ with simple modules $1, 2, 3$ such that $3$ is projective and $1$ is injective. Let $X = I(3), X' = I(2), Y = 1$. There are non-zero maps

$$I(3) \xrightarrow{h} I(2) \xrightarrow{f'} 1,$$

and we let $f = f'h$. All three maps $f', h, f$ are surjective, thus right minimal. The kernel of $f'$ is the simple module $2$, the kernel of $h$ is the simple module $3$ and the kernel of $f$ is $P(2)$.

First, let $C = 1 \oplus 2$, thus $\tau C = 2 \oplus 3$ and both $f'$ and $h$ are right $C$-determined, whereas $f$ is not right $C$-determined.

Second, let $C = I(2) \oplus 1$, thus $\tau C = P(2) \oplus 2$. Then both $f$ and $f'$ are right $C$-determined, whereas $h$ is not right $C$-determined.
Let $Y = \bigoplus Y_i$ with pairwise non-isomorphic indecomposable modules $Y_i$. Then the lattice $C[\to Y]$ has height $\sum |\text{Hom}(C, Y_i)|$.

Proof. This is a direct consequence of the Auslander bijection: the subsets $\text{Hom}(C, Y_i)$ of $\text{Hom}(C, Y)$ are actually $\Gamma(C)$-submodules and the length of $\text{Hom}(C, Y)$ is $\sum |\text{Hom}(C, Y_i)|$ and

We always can assume that $C$ is multiplicity-free and supporting, here supporting means that $\text{Hom}(C_i, Y) \neq 0$ for any indecomposable direct summand $C_i$ of $C$.

If $C$ is a module, let $\Gamma(C) = \text{End}(C)^{\text{op}}$. Note that the indecomposable projective $\Gamma(C)$-modules are of the form $\text{Hom}(C, C_0)$, where $C_0$ is an indecomposable direct summand of $C$, thus the simple $\Gamma(C)$-modules are of the form $\text{top}\text{Hom}(C, C_0)$.

Let $f : X \to Y$ and $f' : X' \to Y$ such that $(f, f')$ is a pair of neighbors. We say that $(f, f')$ is of type $C_0$ where $C_0$ is an indecomposable direct summand of $C$, provided there is a map $\phi : C_0 \to X'$ such that $f'\phi$ does not factor through $f$. Such a summand $C_0$ must exist, since otherwise $f'$ would factor through $f$, due to the fact that $f$ is right $C_0$-determined. The following proposition shows that $C_0$ is uniquely determined.

**Proposition 6.5.** If $(f, f')$ is of type $C_0$, then $\eta_{CY}(f')/\eta_{CY}(f)$ is isomorphic to the simple $\Gamma(C)$-module $\text{top}\text{Hom}(C, C_0)$.

**Corollary 6.6.** The type of a pair of neighbors in $C[\to Y]$ is well-defined.

Proof of the proposition. Let $\phi : C_0 \to X'$ be a map such that $f'\phi$ does not factor through $f$. We obtain a homomorphism of $\Gamma(C)$-modules

$$\text{Hom}(C, f'\phi) : \text{Hom}(C, C_0) \to \text{Hom}(C, Y)$$

which maps into $f'\text{Hom}(C, X')$ (since the image consists of the maps $f'\phi\psi$ with $\psi : C \to C_0$). We claim that $\text{Hom}(C, f'\phi)$ does not map into $f\text{Hom}(C, X)$. Assume, for the contrary, that $\text{Hom}(C, f'\phi)$ maps into $f\text{Hom}(C, X)$. Choose $m : C_0 \to C$ and $e : C \to C_0$ with $em = 1$. By assumption, the element $\text{Hom}(C, f'\phi)(e) = f'\phi e$ belongs to $f\text{Hom}(C, X)$, thus there is $\phi' : C \to X$ with $f'\phi e = f\phi' \phi$ and therefore

$$f'\phi = f'\phi em = f\phi' m$$

shows that $f'\phi$ factor through $f$, a contradiction.

Thus, the image of $\text{Hom}(C, f'\phi)$ is a $\Gamma(C)$-submodule of $\text{Im}\text{Hom}(C, f')$ which is not contained in $\text{Im}(C, f')$ and which is an epimorphic image of the projective module $\text{Hom}(C, C_0)$. Since we know that $\text{Im}\text{Hom}(C, f)$ is a maximal submodule of $\text{Im}\text{Hom}(C, f')$, it follows that

$$\text{Im}\text{Hom}(C, f')/\text{Im}\text{Hom}(C, f') \simeq \text{top}\text{Im}\text{Hom}(C, f'\phi) \simeq \text{top}\text{Hom}(C, C_0).$$

This is what we wanted to show.

**Examples 6.** If $(f, f')$ is a pair of neighbors and $f = f'h$, then $h$ may be neither injective nor surjective. A typical example of such a pair with $f' = 1_Y$ is the following:
Let \( Q \) be the linearly ordered quiver \( 1 \leftarrow 2 \leftarrow 3 \) of type \( A_3 \) and \( \Lambda \) the path algebra of \( Q \) modulo the zero relation. Let \( Y = \mathcal{P}(3) \) and \( C = \mathcal{S}(2) \), then

\[
\mathcal{S}(2)[\rightarrow \mathcal{P}(3)] \leftrightarrow \mathcal{S}\text{Hom}(\mathcal{S}(2),\mathcal{P}(3)),
\]

are lattices with precisely two elements: in \( \mathcal{S}(2)[\rightarrow \mathcal{P}(3)] \), there is the right equivalence class of the identity map \( 1_{\mathcal{P}(3)} \) as well as \([f]\), where \( f: \mathcal{P}(2) \rightarrow \mathcal{P}(3) \) is a non-zero map. Note that \( f \) is right minimal and right \( \mathcal{S}(2) \)-determined, and it is neither mono nor epi.

For an arbitrary projective module \( C \), there is the following description of the \( C \)-length of a right minimal, right \( C \)-determined morphism \( f: X \rightarrow Y \). Here, we denote by \([M: S]\) the Jordan-Hölder multiplicity of the simple module \( S \) in the module \( M \), this is the number of factors in a composition series of \( M \) which are isomorphic to \( S \).

**Proposition 6.7.** Let \( C \) be projective. The right minimal, right \( C \)-determined maps \( f: X \rightarrow Y \) are up to right equivalence just the inclusion maps of submodules \( X \) of \( Y \) such that the socle of \( Y/X \) is generated by \( C \).

If \( f: X \rightarrow Y \) is right minimal and right \( C \)-determined, then \( f \) is injective and

\[
|f|_C = \sum_{P(S)|C} [\text{Cok}(f): S].
\]

The minimal element \( \eta_{CY}(0) \) of \( C[\rightarrow Y] \) is the inclusion map \( X \rightarrow Y \), where \( X \) is the intersection of the kernels of all maps \( Y \rightarrow I(S) \), where \( S \) is a simple module with \( P(S) \) a direct summand of \( C \).

**Proof.** Let \( \mathcal{I} \) be the set of modules \( I(S) \), where \( S \) is a simple module with \( P(S) \) a direct summand of \( C \). Let \( X \) be the intersection of the kernels of all maps \( Y \rightarrow I \) with \( I \in \mathcal{I} \). Since \( Y \) is of finite length, there are finitely many maps \( g_i: Y \rightarrow Q_i \) with \( Q_i \in \mathcal{I} \), say \( 1 \leq i \leq m \), such that \( X = \bigcap_{i=1}^{m} \text{Ker}(g_i) \). Then \( Y/X \) embeds into \( \bigoplus_{i=1}^{m} Q_i \), thus its socle if generated by \( C \). It follows that the inclusion map \( X \rightarrow Y \) is right \( C \)-determined. On the other hand, if \( X' \rightarrow X \) is right minimal and right \( C \)-determined, then it is a monomorphism, thus we can assume that it is an inclusion map. In addition, we know that the socle of \( Y/X' \) is generated by \( C \), thus \( Y/X' \) embeds into a finite direct sum of modules in \( \mathcal{I} \). It follows that \( X' \) is the intersection of some maps \( Y \rightarrow I \), where \( I \in \mathcal{I} \), thus \( X \subseteq X' \).

Of course, it follows that for \( C = \Lambda \Lambda \), and any inclusion map \( X \rightarrow Y \) (such a map is right minimal and right \( \Lambda \)-determined), the \( \Lambda \Lambda \)-length of \( f \) is precisely the length of \( Y/X \).

### 7. Epimorphisms in \([\rightarrow Y]\).

If \( f \) is a monomorphism, then \( f \) is right minimal and right \( \Lambda \)-determined (see 3.5), thus right \( C \)-determined for some projective module \( C \). Conversely, if \( C \) is projective, then any right minimal, right \( C \)-determined morphism is a monomorphism (see 3.4). Namely,
if $K$ is an indecomposable direct summand of the kernel of $f$, where $f$ is right minimal, then $K$ is not injective and $\tau^{-1}K$ is a direct summand of any module $C$ such that $f$ is right $C$-determined. Of course, since $K$ is not injective, $\tau^{-1}K$ is an indecomposable non-projective module. One should be aware that a morphism may be right $C$-determined for some module $C$ without any indecomposable projective direct summand, without being surjective.

**Example 7.** Let $\Lambda$ be given by the linearly directed quiver of type $A_3$, say $1 \to 2 \to 3$ with radical square zero. Then the non-zero map $P(2) \to P(3)$ is not surjective, but right $S(2)$-determined, and, of course, $S(2)$ is not projective. (As we will see below, it is essential for this feature that the kernel of $f$ has injective dimension at least 2; for a general discussion of maps which are not surjective, but right $C$-determined by a module $C$ without any indecomposable projective direct summands, we refer to [R8]).

The submodule $\mathcal{P}(C,Y)$ of $\text{Hom}(C,Y)$. Recall that $\mathcal{P}(C,Y)$ denotes the set of morphisms $C \to Y$ which factor through a projective module. Note that $\mathcal{P}(C,Y)$ is a $\Gamma(C)$-submodule of $\text{Hom}(C,Y)$.

**Proposition 7.1.** Assume that $f: X \to Y$ is right $C$-determined. Then $f$ is surjective if and only if $\eta_{CY}(f) \supseteq \mathcal{P}(C,Y)$.

Proof. One direction is a trivial verification: First assume that $f$ is surjective. Let $h$ belong to $\mathcal{P}(C,Y)$, thus $h = h_2 h_1$ where $h_1: C \to P$ and $h_2: P \to Y$ with $P$ projective. Since $f$ is surjective and $P$ is projective, there is $h_2': P \to X$ such that $h_2 = fh_2'$. Thus shows that $h = h_2 h_1 = fh_2' h_1$ belongs to $f \text{Hom}(C,X) = \eta_{CY}(f)$.

The converse is more interesting, here we have to use that $f$ is right $C$-determined. We assume that $\eta_{CY}(f) \supseteq \mathcal{P}(C,Y)$. Let $p: P(Y) \to Y$ be a projective cover of $Y$. Consider an arbitrary morphism $\phi: C \to P(Y)$. The composition $p \phi$ belongs to $\mathcal{P}(C,Y)$, thus to $\eta_{CY}(f) = f \text{Hom}(C,X)$. Since $f$ is right $C$-determined, it follows that $p$ itself factors through $f$, say $p = fp'$ for some $p': P(Y) \to X$. Now the composition $fp' = p$ is surjective, there $f$ has to be surjective.

Let us denote by $C[\to Y]_{\text{epi}}$ the subset of $C[\to Y]$ given by all elements $[f]$ with $f$ an epimorphism.

**Proposition 7.2.** The Auslander bijection $\eta_{CY}$ yields a poset isomorphism

$$\eta_{CY}: C[\to Y]_{\text{epi}} \longrightarrow S(\text{Hom}(C,Y)/\mathcal{P}(C,Y)) = S\text{Hom}(C,Y).$$

Proof: It is well-known that given a module $M$ and a submodule $M'$, then the lattice of submodules of the factor module $M/M'$ is canonically isomorphic to the lattice of the submodules $U$ of $M$ satisfying $M' \subseteq U$.

More generally, dealing with a morphism $f$ which is right $C$-determined, we can recover the image of $f$ as follows:
Proposition 7.3. Let $f: X \to Y$ be right $C$-determined. Then one recovers the image of $f$ as the largest submodule $Y'$ of $Y$ (with inclusion map $u: Y' \to Y$) such that $uP(C,Y') \subseteq f\text{Hom}(C,X)$.

Proof. Let $Y'$ be the image of $f$ with inclusion map $u$ and $uf' = f$ (with $f'$ surjective). First of all, we show that $uP(C,Y') \subseteq f\text{Hom}(C,X)$. Let $\phi': C \to \Lambda$ and $\phi'': \Lambda \to Y'$ (the maps $\phi''\phi'$ obtained in this way generate $P(C,Y')$ additively). We want to show that $u\phi''\phi$ factors through $f$. Since $f': X \to Y'$ is surjective, there is $\psi: \Lambda \to X$ such that $\phi'' = f'\psi$ (since $\Lambda$ is projective). Thus $u\phi''\phi' = uf'\psi\phi' = f\psi\phi'$. Thus $u\phi''\phi'$ factors through $f$.

On the other hand, let $u'': Y'' \to Y$ be a submodule of $Y$ such that $u''P(C,Y'') \subseteq f\text{Hom}(C,X)$. Let $p: P(Y'') \to Y''$ be a projective cover. Consider the map $f' = u''p: P(Y'') \to Y$. It has the property that for all maps $\phi: C \to P(Y'')$ the composition $f'\phi$ factors through $f$ (namely $f'\phi = u''p\phi$ belongs to $u''P(C,Y'') \subseteq f\text{Hom}(C,X)$). But $f$ is right $C$-determined, thus we conclude that $f'$ factors through $\alpha_{\phi}$, say $f' = f\phi'$ for some $\phi': C \to P(Y'')$. Thus the image $Y''$ of $f'$ is contained in the image $Y'$ of $f$. This is what we wanted to prove.

We recover in this way Proposition 7.1. Namely, if $f$ is surjective, then $Y$ is the image of $f$, thus $Y$ is one of the submodule $Y'$ with $uP(C,Y') \subseteq f\text{Hom}(C,X)$, thus $P(C,Y) \subseteq f\text{Hom}(C,Y)$. Conversely, if $P(C,Y) \subseteq f\text{Hom}(C,Y)$, then $Y$ is one of the submodules $Y'$ with $uP(C,Y') \subseteq f\text{Hom}(C,X)$ and therefore the image of $f$ contains $Y$, thus is equal to $Y$. This shows: If $f$ is right $C$-determined, then $f$ is surjective if and only if $P(C,Y) \subseteq f\text{Hom}(C,X)$.

Corollary 7.4.
(a) All maps in $C\to Y$ are epimorphisms if and only of $P(C,Y) = 0$.
(b) $P(C,Y) = \text{Hom}(C,Y)$ if and only if the only element $[f]$ in $C\to Y$ with $f$ surjective is the right equivalence class of the identity map $Y \to Y$.

Kernels with injective dimension at most 1.

Proposition 7.5. Let $K$ be a module and $C = \tau^{-1}K$. The following conditions are equivalent:
(i) The injective dimension of $K$ is at most 1.
(ii) If $Y$ is any module, then all maps in $C\to Y$ are epimorphisms.

Proof. Recall from [R3, 2.4] that $K$ has injective dimension at most 1 if and only if $\text{Hom}(C,\Lambda) = 0$. Thus, if $K$ has injective dimension at most 1 and $Y$ is an arbitrary module, then $P(C,Y) = 0$, thus any map in $C\to Y$ is surjective. On the other hand, assume that the injective dimension of $K$ is not bounded by 1, thus $\text{Hom}(C,\Lambda) \neq 0$. The Auslander bijection shows that $C\to \Lambda\Lambda$ contains non-trivial elements $[f]$. Since any surjective map $f: X \to \Lambda$ is surjective.

Corollary 7.6. Let $\Lambda$ be hereditary and $C$ a module without any indecomposable projective direct summand. Then any right $C$-determined morphism $f: X \to Y$ is an epimorphism.

Proof. Let $K = \tau C$. Since $C$ has no indecomposable projective direct summand, it follows that $C = \tau^{-1} K$. Since $\Lambda$ is hereditary, the injective dimension of any module is at
most 1. Since the injective dimension of $K$ is at most 1, it follows from the proposition that all right $C$-determined maps are epimorphisms.

Of course, we also can show directly that $\mathcal{P}(C,Y) = 0$. Namely, let $g: C \to Y$ be in $\mathcal{P}(C,Y)$. Then $g = g_2g_1$ with $g_1: C \to P$, where $P$ is a projective module. The image $P'\prime$ of $g_1$ is a submodule of $P$, thus, since $\Lambda$ is hereditary, the module $P'$ is also projective. Thus, we have a surjective map $C \to P'$ with $P'$ projective. Such a map splits. This shows that $P'$ is isomorphic to a direct summand of $C$. It follows that $P' = 0$ and therefore $g = 0$.

**Modules $K$ with semisimple endomorphism ring.** Here is a well-known characterization of such modules.

**Lemma 7.7.** Let $K$ be a module. The following conditions are equivalent:

(i) The endomorphism ring of $K$ is semisimple.

(ii) There are pairwise orthogonal bricks $K_1, \ldots, K_n$ such that $\text{add} \, K = \text{add} \{K_1, \ldots, L_n\}$.

**Lemma 7.8.** Let $K$ be a module with semisimple endomorphism ring. Let $f = f'h$, where $f, f'$ are right minimal epimorphisms with kernels in $\text{add} \, K$. Then $h$ is surjective and its kernel belongs to $\text{add} \, K$.

**Proof.** By assumption, there is a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K & \overset{u}{\longrightarrow} & X & \overset{f}{\longrightarrow} & Y & \longrightarrow & 0 \\
\downarrow{h'} & & \downarrow{h} & & \downarrow & & \\
0 & \longrightarrow & K' & \overset{u'}{\longrightarrow} & X & \overset{f'}{\longrightarrow} & Y & \longrightarrow & 0
\end{array}
$$

with inclusion maps $u, u'$ such that both $K, K'$ belong to $\text{add} \, K$. Since $K, K'$ belong to $\text{add} \, K$ and the endomorphism ring of $K$ is semisimple, there is a submodule $K''$ of $K'$ such that $K' = \text{Im}(h') \oplus K''$. Let us denote by $u'': K'' \to K'$ the inclusion map and by $q': K' \to K'/\text{Im}(h') = K''$ the canonical projection, thus $q'u'' = 1_{K''}$. Now $q': K' \to K''$ is the cokernel of $h'$, and we can identify $K''$ and we can complete the diagram above by inserting the cokernels of $h$ and $h'$ as follows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & K & \overset{u}{\longrightarrow} & X & \overset{f}{\longrightarrow} & Y & \longrightarrow & 0 \\
\downarrow{h'} & & \downarrow{h} & & \downarrow & & \\
0 & \longrightarrow & K' & \overset{u'}{\longrightarrow} & X & \overset{f'}{\longrightarrow} & Y & \longrightarrow & 0 \\
\downarrow{q'} & & \downarrow{q} & & \downarrow & & \\
K'' & \longrightarrow & K'' \\
\downarrow & & \downarrow & & \\
0 & & 0
\end{array}
$$
Since $qu' = q'$, we see that $qu'u'' = q'u'' = 1_{K''}$, thus $K''$ is a direct summand of $X$ which lies inside the kernel of $f'$. Since we assume that $f'$ is right minimal, it follows that $K'' = 0$, thus $h'$ is surjective. Therefore also $h$ is surjective.

On the other hand, the kernel of $h$ can be identified with the kernel of $h'$, and again using that $K, K'$ belong to add $K$ and that the endomorphism ring of $K$ is semisimple, we see that the kernel of $h$ belongs to add $K$.

**Lemma 7.9.** Assume that $f: X \to Y$ and $f': X' \to Y$ are epimorphisms with $f = f'h$. Then we have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & K' & \rightarrow & X' & \xrightarrow{f'} Y & \rightarrow & 0 \\
\uparrow h' & & \uparrow h & & \uparrow & & \\
0 & \rightarrow & K & \rightarrow & X & \xrightarrow{f} Y & \rightarrow & 0.
\end{array}
$$

If $f$ is right minimal and $h'$ is a split epimorphism, then also $f'$ is right minimal.

**Remark.** Observe that it is not enough to assume that $h'$ is an epimorphism. As an example, take the indecomposable injective Kronecker module $X = Q_1$ of length 3, let $K$ be a submodule of length 2, and $K' = K/\text{soc}$. Then $Q_1 \rightarrow Q_1/K$ is right minimal. But the induced sequence is just the short exact sequence $K/\text{soc} \rightarrow Q_1/\text{soc} \rightarrow Y$ which splits.

**Proof.** Denote the kernel of $h'$ by $K''$, thus we can assume that $K = K' \oplus K''$ such that $h'$ is the canonical projection $K \rightarrow K'$ with kernel $K''$. Assume that $X' = U \oplus V$, where $U$ is contained in the kernel of $f'$, thus $U \subseteq K'$. Since $X' = X/K''$, there are submodule $U', V'$ of $X'$ both containing $K''$ such that $U' + V' = X'$ and $U' \cap V' = K''$, with $U = U'/K''$ and $V = V'/K''$.

Consider $U'' = U' \cap K'$, this is a submodule of the kernel $K$ of $f$. Also, $U' = K'' + U''$ (using the modular law). Thus we have $U'' \cap V' = U' \cap K' \cap V' \subseteq K' \cap K'' = 0$ and $U'' + V' = U'' + V' + K'' = U' + V' = X$. This shows that $U''$ is a direct summand of $X$ which is contained in the kernel of $f$. Since $f$ is right minimal, we see that $U'' = 0$. Since $U' = K'' + U'' = K''$, it follows that $U = 0$.

**Proposition 7.10.** Let $K$ be a module with injective dimension at most 1 and assume that the endomorphism ring of $K$ is semisimple. Let $C = \tau^{-1}K$ and assume that $\mathcal{P}(C,Y) = 0$. If $f: X \rightarrow Y$ is right minimal and right $C$-determined, then $f$ is surjective and

$$|f|_C = \mu(\text{Ker}(f))$$

where $\mu(M)$ denotes the number of direct summands when $M$ is written as a direct sum of indecomposable modules.

Also, $\eta_{CY}^1(0)$ is given by the universal extension from below

$$0 \rightarrow K' \rightarrow X \rightarrow Y \rightarrow 0$$

19
with $K' \in \text{add } K$.

Proof. Since we assume that $\mathcal{P}(C, Y) = 0$, all right minimal right $C$-determined maps $f : X \to Y$ are surjective, and the kernel of such a map is in $\text{add } K$. Let $|f|_C = n$ and consider a (maximal) chain

$$X_n \xrightarrow{h_n} X_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} X_0 = Y$$

of non-invertible maps such that the compositions $f_t = h_1 \cdots h_t$ for $1 \leq t \leq n$ are right minimal and right $C$-determined. Now all the maps $f_t$ are right minimal epimorphisms with kernels in $\text{add } K$, thus, according to Lemma 7.8, also the maps $h_t$ are epimorphisms with kernels $K_t$ in $\text{add } K$. Since we assume that the endomorphism ring of $K$ is semisimple, we see that the kernel of $f_t$ is just $\bigoplus_{i=1}^t K_i$. Now let $K_t = K' \oplus K''$ be a direct decomposition with $K'$ indecomposable, and let $h : X_t \to X_t/K'$ be the canonical projection. Since $K'$ is contained in the kernel of $h_t$, we can factor $h_t$ through $h$ and obtain a map $h'_t : X_t/K' \to X_{t-1}$ with kernel $X''$ such that $h_t = h'_t h$. Altogether we have obtained a refinement

$$X_n \xrightarrow{h_n} X_{n-1} \xrightarrow{h_{n-1}} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} X_0 = Y$$

with $h_t = h'_t h$. Let $f' = f_{t-1} h'_t = h_1 \cdots h_{t-1} \cdots h'_t$. We apply Lemma 7.9 to $f_t$ and $f'$. Note that $f_t = f_{t-1} h_t = f_{t-1} h'_t h = f' h$ and the corresponding map from the kernel of $f_t$ to the kernel of $f'$ is just the split epimorphism $K_t \to K_t/K'$. Thus Lemma 7.9 asserts that $f'$ is right minimal (and of course also right $C$-determined). The maximality of the chain $(h_1, \ldots, h_t)$ implies that $h'_t$ has to be invertible, thus $K'' = 0$. This shows that $K_t$ is indecomposable. Altogether, we see that $\mu(\text{Ker}(f_t)) = t$, thus $\mu(\text{Ker}(f)) = n = |f|_C$.

The last assertion is obvious: If

$$0 \to K' \to X \xrightarrow{f} Y \to 0$$

universal extension from below with $K' \in \text{add } K$, then $[f]$ belongs to $C[\to Y]$ and any other extension of $Y$ from below with kernel in $\text{add } K$ is induced from it. Thus $[f]$ has to be the zero element of the lattice $C[\to Y]$.

**Corollary 7.11.** Let $K$ be a module with injective dimension at most 1 and assume that the endomorphism ring of $K$ is semisimple. Let $C = \tau^{-1} K$. If $f : X \to Y$ is right minimal and right $C$-determined, then $f$ is surjective and $|f|_C = \mu(\text{Ker}(f))$. Also, $\eta_{CY}(0)$ is given by the universal extension from below using modules in $\text{add } K$.

Proof. Combine Proposition 7.5 and Proposition 7.10.

**Example 8.** Consider the 3-subspace quiver with sink 0 and sources 1, 2, 3. Let
Note that here we have $\mathcal{K} = \tau \mathcal{C} = \mathcal{P}(1) \oplus \mathcal{P}(2) \oplus \mathcal{P}(3)$.

**Riedtmann-Zwara degenerations.** Recall that $M'$ is a Riedtmann-Zwara degeneration of $M$ if and only if there is an exact sequence of the form

$$0 \to K \to K \oplus M \to M' \to 0,$$

or, equivalently, if and only if there is an exact sequence of the form

$$0 \to M' \to M \oplus L \to L \to 0;$$

(in both sequences we can assume that the maps $K \to K$ and $L \to L$, respectively, are in the radical).

In terms of the Auslander bijection, we may deal with these data in several different ways: namely, we may look at the right equivalence classes of both $[M \to M']$ and $[K \oplus M \to M']$ in $[\to M']$ as well as at the right equivalence class $[M' \to M]$ in $[\to M]$. In case we deal with $[K \oplus M \to M']$, one should be aware that this map $[K \oplus M \to M']$ is the union of the two maps $[K \to M']$ and $[M \to M']$ in $[\to M']$ (thus already in $\mathcal{C}[\to M']$ provided both maps $[K \to M']$ and $[M \to M']$ are right $C$-determined.

**Example 9.** Let $\Lambda$ be given by the quiver with one loop $\alpha$ at the vertex 1 and an arrow $2 \to 1$, with the relation $\alpha^2 = 0$.

Let $K = \mathcal{P}(1)$ and $Y = \mathcal{S}(2)$. Then $\dim \text{Ext}^1(Y, K) = 2$. Then $C = \tau^{-1} K$ is of length 4 with socle 1 and top $2 \oplus 2$, thus $\dim \text{Hom}(C, Y) = 2$. Note that $\Gamma(C) = \text{End}(C)^{\text{op}} = k[t]/t^2$ and that $\text{Hom}(C, Y)$ as a $\Gamma(C)$-module is cyclic.
The universal cover of the Auslander-Reiten quiver of $\Lambda$ looks as follows:

The region in-between the dashed lines is a fundamental domain of the Auslander-Reiten quiver of $\Lambda$; the Auslander-Reiten quiver of $\Lambda$ is obtained by identifying these lines in order to form a Möbius strip.

The encircled vertices yield $C[\to Y]$. Here is $C[\to Y]$ as well as $S\Hom(C, Y)$:

In addition, we also may concentrate on the possible maps $K \to K$ and $L \to L$ (sometimes called steering maps).

We deal with an epimorphism $K \oplus X \to Y$, and this is a map which belongs to $C[\to Y]$ for $C = \tau^{-1}K$. Note that for some $C'$, the map $K \oplus X \to Y$ is the union of $K \to Y$ and $X \to Y$ (the $C'$ has to be a determiner of both maps $K \to Y$ and $X \to Y$).

When dealing with epimorphisms in $C[\to Y]$, Riedtmann-Zwara degenerations play a decisive role, as the following lemma shows:

**Proposition 7.12.** Let $f : X \to Y$ and $f' : X' \to Y$ be epimorphisms with isomorphic kernels. If $[f] \leq [f']$, then $X'$ is a Riedtmann-Zwara degeneration of $X$. 

22
Proof. Let \( h : X \to X' \) with \( f = f' h \). Let \( u : K \to X \) and \( u' : K \to X' \) be the kernel maps. Since \( f = f' h \), there is \( h' : K \to K' \) such that \( u' h' = h u \), thus we deal with the following commutative diagram with exact sequences:

\[
\begin{array}{c}
0 \longrightarrow K \overset{u}{\longrightarrow} X \overset{f}{\longrightarrow} Y \longrightarrow 0 \\
\downarrow h' \quad \downarrow h \\
0 \longrightarrow K \overset{u'}{\longrightarrow} X \overset{f'}{\longrightarrow} Y \longrightarrow 0.
\end{array}
\]

The diagram shows that the lower exact sequence is induced from the upper one by \( h' \). But this means that the following sequence is exact:

\[
0 \to K \left[ \frac{u}{-h'} \right] \longrightarrow K \oplus X \left[ \frac{h u'}{} \right] X' \to 0.
\]

This is a Riedtmann-Zwara sequence, thus \( X' \) is a Riedtmann-Zwara degeneration of \( X \).

For example, let us consider for \( \Lambda \) hereditary and \( M \) an indecomposable non-injective module the relationship between

\[ \text{rad}(M, M) \text{ and } \text{Ext}^1(M, M), \]

thus we look at the set

\[ \tau^{-M}[-\to M] \]

of right equivalence classes of right minimal maps \( f \) ending in \( M \) and right \((\tau^{-1}M)\)-determined. Since \( \Lambda \) is hereditary and \( \tau^{-1}M \) has no indecomposable projective direct summand, \( f \) has to be surjective with kernel in \( \text{add } M \). Of special interest will be the surjective maps \( f : X \to M \) with kernel in \( M \); such a map is right minimal if and only if it is not a split epimorphism. The lemma shows that the order relation \([f : X \to M] \leq [f' : X' \to M]\) on this subset implies that \( X' \) is a Riedtmann-Zwara degeneration of \( X \).

8. Comparison with Auslander-Reiten theory.

Let \( C = Y \) be indecomposable and consider the Auslander bijection for \( C = Y \):

\[
Y[-\to Y] \leftrightarrow S \text{Hom}(Y, Y)
\]

the subspace \( \text{rad}(Y, Y) \) of \( \text{Hom}(Y, Y) \) on the right corresponds to the right almost split map ending in \( Y \). Two possible cases have to be distinguished:

If \( Y = P \) is projective, then we get a morphism which is right determined by a projective module, thus we must get a monomorphism. Of course, what we obtain is just the embedding of the radical \( \text{rad } P \) into \( P \).

If \( Y \) is not projective, then we get an epimorphism with kernel in \( \text{add } \tau Y \). Actually, we get the epimorphism of the Auslander-Reiten sequence ending in \( Y \), thus the kernel is precisely \( \tau Y \).
What we see is that the minimal right almost split map ending in \( Y \) is a waist in \( Y \to \) and it just corresponds to the waist \( \text{rad}(Y,Y) \subset \text{End}(Y) \).

More generally, let \( Y \) be indecomposable and \( Y \) a direct summand of \( C \), and consider
\[
C[\to Y] \leftrightarrow \mathcal{S}\text{Hom}(C,Y).
\]
Then \( \mathcal{S}\text{Hom}(C,Y) \) is a local module (with maximal submodule \( \text{Im}\text{Hom}(C,g) \), where \( g \) is minimal right almost split ending in \( Y \)).

Proof: Let \( \tau Y \to \mu Y \xrightarrow{g} Y \) be the Auslander-Reiten sequence ending in \( Y \). Then the right-equivalence class \([g]\) of \( g \) belongs to \( C[\to Y] \) and every map \( X \to Y \) which is not a split epimorphism, factors through \( g \). This means that every element of \( C \to Y \) different from the identity map \( Y \to Y \) is less or equal to \([g]\). This means that \( C[\to Y] \) has a unique maximal submodule, namely \( \text{Hom}(C,g) \).

The **Auslander-Reiten formula** \( \text{Ext}^1(Y,K) \cong D\text{Hom}(\tau^{-K},Y) \). The Auslander bijection provides a bijection between the submodules of \( \text{Hom}(C,Y) \) with the right equivalence classes of surjective maps \( M \to Y \) with kernel in \( \text{add}(K) \) (namely, the submodules of \( \text{Hom}(\tau^{-K},Y) \) just correspond bijectively to the submodules of \( \text{Hom}(\tau^{-K},Y) \) which contain \( \mathcal{P}(\tau^{-K},Y) \)). Consider the following triangle:

\[
\begin{array}{ccc}
(f,g) & \{ \xrightarrow{f} K \xrightarrow{g} M \xrightarrow{g} Y \mid \text{short exact} \} \\
\downarrow & & \downarrow \\
[f,g] & \text{Ext}^1(Y,K) & \\
\downarrow & & \downarrow \\
[g] & \tau^{-K}[\to Y] \xrightarrow{\text{epi}} & \mathcal{S}\text{Hom}(\tau^{-K},Y)
\end{array}
\]

Here, the two left hand columns concern the canonical way of attaching to a short exact sequence \( \epsilon = (f,g) \) the corresponding element \([\epsilon] = [f,g] \) in \( \text{Ext}^1 \); if \((f,g)\) and \((f',g')\) are short exact sequences with \([f,g] = [f',g'] \) in \( \text{Ext}^1 \), then the maps \( f, f' \) are right equivalent, thus \([f,g] \mapsto [g] \) is a well-defined map \( \text{Ext}^1(Y,K) \to \tau^{-K}[\to Y] \xrightarrow{\text{epi}} \mathcal{S}\text{Hom}(\tau^{-K},Y) \) and we obtain a commutative triangle as shown.

Having another look at the left columns of the triangle, the reader should observe that the split short exact sequence which yields the zero element of \( \text{Ext}^1 \) gives the unit element of the lattice \( \tau^{-K}[\to Y] \xrightarrow{\text{epi}} \), namely the identity map \( 1: Y \to Y \).

**Proposition 8.1.** Let \( g: M \to Y \) be surjective with kernel \( K \) and \( C = \tau^{-K} \). Then an element \( h \in \text{Hom}(C,Y) \) belongs to the set \( \eta_{C,Y}(g) \) if and only if the induced sequence \( h_*(\langle f,g \rangle) \) splits. Also, the set \( \eta_{C,Y}(g) \) is a \( \Gamma(C) \)-submodule of \( \text{Hom}(C,Y) \), where \( \Gamma(C) = \text{End}(C)^{\text{op}} \).

Proof. Given an element \( h \in \text{Hom}(C,Y) \), the induced sequence \( h_*(\langle f,g \rangle) \) splits if and only if \( h \) factors through \( g \). But we have already noted that \( \eta_{C,Y}(g) \) is the set of all elements
$h \in \text{Hom}(C,Y)$ which factor through $g$. And we know that $\eta_{C,Y}(g)$ is a $\Gamma(C)$-submodule of $\text{Hom}(C,Y)$.

Now let us invoke the Auslander-Reiten formula: $\text{Ext}^1(Y,K) \simeq D\text{Hom}(\tau^{-K},Y)$. Here we use that we deal with an artin algebra $\Lambda$, thus the center $k$ of $\Lambda$ is a (commutative) artinian ring and $D$ is the duality functor $\text{mod} \rightarrow \text{mod} k$ given by a minimal cogenerator of $\text{mod} k$.

There are the following two horizontal bijections as well as the vertical map on the left:

\[
\begin{array}{ccc}
[f,g] & \text{Ext}^1(Y,K) & \simeq D\text{Hom}(\tau^{-K},Y) \\
\downarrow & \downarrow & \downarrow \\
[g] & \tau^{-K}[\rightarrow Y]_{\text{epi}} & S\text{Hom}(\tau^{-K},Y)
\end{array}
\]

For the composition of the three maps, we have inserted a dashed arrow on the right. It seems to be of interest to specify in detail this map! One may conjecture that here one attaches to a linear map $\alpha \in D\text{Hom}(\tau^{-K},Y)$ the largest $\Gamma$-submodule of $\text{Hom}(\tau^{-K},Y)$ lying in the kernel of $\alpha$.

Let us now assume that $K$ is indecomposable so that $\text{add}K$ consists of direct sums of copies of $K$. Given a short exact sequence $K \xrightarrow{f} M \xrightarrow{g} Y$, the map $g$ is right minimal if and only if the sequence does not split. On the other hand, given $[g] \in \tau^{-K}[\rightarrow Y]_{\text{epi}}$ with $g$ right minimal, the kernel of $g$ is the direct sum of say $t$ copies of $K$ and $t \leq 1$ just means that $[g]$ is obtained from a short exact sequence $K \xrightarrow{f} M \xrightarrow{g} Y$. It follows that under the assumption that $K$ is indecomposable, it is easy to identify the elements of $S\text{Hom}(\tau^{-K},Y)$ which are images of the elements of $\text{Ext}^1(Y,K)$ under $\eta_{\tau^{-K},Y}$.

**Comparison.** In which way are the Auslander bijections better than the Auslander-Reiten formula? What is the advantage of the Auslander bijection compared to the Auslander-Reiten formula?

1) We do not only deal with the set $\text{Hom}(C,Y)$ but with all of $\text{Hom}(C,Y)$. To extend such a bijection to a larger setting should always be of interest. But also note that the set $\text{Hom}(C,Y)$ depends on the module category which we consider, not just on the modules themselves.

2) The duality is replaced by a covariant bijection.

3) The usual Auslander-Reiten picture concerns indecomposable modules, and almost split sequences, thus indecomposable modules and irreducible maps. In the language of the Auslander bijection, we only deal with $C = Y$ indecomposable and only with the submodule $\text{rad}(C,C) \subset \text{Hom}(C,C)$, whereas

- we should not restrict to indecomposable modules,
- and not to the condition $C = Y$,
- and we want to deal with all submodules of $\text{Hom}(C,Y)$, not just the radical subspace.

Concerning the classical AR-theory, there is an essential difference whether $C$ is projective or not. If $C$ is projective, we obtain an inclusion map, if $C$ is not projective, then
an extension. — This feature dominates also the Auslander bijections: the extreme cases are: $C$ is projective, then we consider submodules (and we consider arbitrarily ones, not just the radicals of the indecomposable projective modules. If $C$ has no indecomposable projective direct summand, then we deal with extensions ending in $Y$; and we deal with all possible extension of $Y$ from below using modules in add $\tau C$.

4) The Auslander-Reiten theory only deals with the factor category of $\text{mod } \Lambda$ modulo the infinite radical. The Auslander bijection takes care of any morphism.

5) Families of modules do not play any role in the Auslander-Reiten theory. As we will see soon, families of modules are an essential features in the frame of the Auslander bijections.

Part II. Families of modules.

9. The modules present in $C[\rightarrow Y]$ are of bounded length.

We say that a module $M$ is present in $C[\rightarrow Y]$ provided there exists a right minimal map $f: M \rightarrow Y$ which is right $C$-determined.

Proposition 9.1. There is a constant $\lambda = \lambda(\Lambda)$ such that for any short exact sequence $K' \rightarrow M \rightarrow Y'$ in $C[\rightarrow Y]$, we have $|K'| \leq \lambda |C|^2 |Y|$.

Proof: Write $C = \bigoplus C_i$ with indecomposable direct summands $C_i$ and let $K_i = \tau C_i$. Note that $|K_i| \leq d^2 |C_i|$, where $d = |\Lambda \Lambda|$ (see for example [R1]). Of course, we see in this way that there is also the weaker bound $|K_i| \leq d^2 |C|$.

Let $\Lambda$ be an artin $k$-algebra, where $k$ is a commutative artinian ring. Since there are only finitely many simple modules and $\text{Ext}^1(Y, X)$ is a $k$-module of finite length, for any $\Lambda$-modules $X, Y$ of finite length, we may denote by $e$ the maximum of the length of the $k$-modules $\text{Ext}^1(S, S')$, where $S, S'$ are simple $\Lambda$-modules. The long exact Hom-sequences imply that the length of the $k$-module $\text{Ext}^1(Y, X)$ is bounded by $e |X||Y|$, for any $\Lambda$-modules $X, Y$ of finite length.

Consider a short exact sequence $K' \rightarrow M \rightarrow Y'$ in $C[\rightarrow Y]$. Thus, $Y'$ is a submodule of $Y$, the module $K'$ belongs to add $\tau C$ and the map $M \rightarrow Y'$ is right minimal. Write $K' = \bigoplus_{i=1}^s K_i$. The Ext-Lemma of [R4] asserts that $t_i$ is bounded by the $k$-length of $\text{Ext}^1(Y', K_i)$, since the map $M \rightarrow Y'$ is right minimal. Thus

$$t_i \leq e |Y'||K_i| \leq d^2 e |Y'||C_i| \leq d^2 e |Y||C_i|,$$

and therefore $\sum t_i \leq d^2 e |Y||C|$. Thus

$$|K'| = \sum t_i |K_i| \leq d^2 |C| \sum t_i \leq \lambda |C|^2 |Y|,$$

with $\lambda = d^4 e$.

There is the following converse:
Proposition 9.2. Let $I = (DA)^b$. Then any module of length at most $b$ is present in $\Lambda[\to I]$.

Proof. Let $M$ be a module of length at most $b$, thus the socle $\text{soc} M$ of $M$ is a semisimple module of length at most $b$ and therefore a submodule of $I = (DA)^b$. It follows that $M$ itself can be embedded into $I$. Such an embedding $f: M \to I$ is right minimal and right $\Lambda$-determined, thus $M$ is present in $\Lambda[\to I]$.

Corollary 9.3. Let $\mathcal{M}$ be a set of modules. Then there exist modules $C,Y$ such that any module $M$ in $\mathcal{M}$ is present in $C[\to Y]$ if and only if the modules in $\mathcal{M}$ are of bounded length.

Proof. If all the modules in $\mathcal{M}$ are present in $C[\to Y]$, then they have to be of length at most $\lambda(\Lambda)|C||Y|$, thus of bounded length, see Proposition 9.1. Conversely, if the modules in $\mathcal{M}$ are of length at most $b$, then they are present in $\Lambda[\to I]$ where $I = (DA)^b$, according to Proposition 9.2.

Remark: Modules versus Morphisms. As we have seen, given any infinite set $\mathcal{M}$ of modules of bounded length, there are modules $C,Y$ such that all the modules in $\mathcal{M}$ are present in $C[\to Y]$. On the other hand, given modules $X,Y$, one cannot expect that the right equivalence classes $[f]$ of all (or at least infinitely many) non-zero morphisms $f: X \to Y$ belong to some $C[\to Y]$, since the kernels of these maps $f$ may belong to infinitely many isomorphism classes.

10. Minimal infinite families.

Recall that a Krull-Remak-Schmidt category $\mathcal{C}$ is said to be finite provided the number of isomorphism classes of indecomposable objects in $\mathcal{C}$ is finite, otherwise $\mathcal{C}$ is said to be infinite.

Let $\mathcal{M}$ be a family of modules. We say that $\mathcal{M}$ is minimal infinite provided $\text{add} \mathcal{M}$ is infinite whereas $\text{add} \mathcal{M}'$ is finite, where $\mathcal{M}'$ is the set of modules $M'$ which are proper submodules or proper factor modules of modules in $\mathcal{M}$.

Lemma 10.1. If $\mathcal{M}$ is a minimal infinite family of modules (not necessarily of the same length), then there is a subset $\mathcal{N} \subseteq \mathcal{M}$ which is infinite and consists of pairwise non-isomorphic indecomposable modules.

Of course, $\mathcal{N}$ is again minimal infinite.

Proof. Since $\text{add} \mathcal{M}$ is infinite, there is a sequence of modules $M_i \in \mathcal{M}$ with $i \in \mathbb{N}$ such that $M_i$ does not belong to $\text{add}\{M_1, \ldots, M_{i-1}\}$ for all $i$. Write $M_i = N_i \oplus N'_i$ such that $N_i$ is indecomposable and does not belong to $\text{add}\{M_1, \ldots, M_{i-1}\}$. It follows that the modules $N_1, N_2, \ldots$ are indecomposable and pairwise non-isomorphic. Let $I$ be the set of natural numbers such that $N'_i \neq 0$. If $i \in I$, then $N_i$ is a proper submodule of $M_i$, thus belongs to $\text{add} \mathcal{M}'$. Since $\text{add} \mathcal{M}'$ is finite and the modules $N_i$ with $i \in I$ are indecomposable and pairwise non-isomorphic, it follows that $I$ is finite. Let $\mathcal{N}$ be the set of modules $N_i$ with $i \notin I$, then $\mathcal{N}$ is an infinite subset of $\mathcal{M}$ and consists of pairwise non-isomorphic indecomposable modules.
We will need some basic facts concerning the Gabriel-Roiter measure of finite length modules, see [R4]. We write $\gamma(N)$ for the Gabriel-Roiter measure of a module $N$. We recall that any indecomposable module $M$ which is not simple has a Gabriel-Roiter submodule $M'$, this is an indecomposable submodule of $M$ such that also $M/M'$ is indecomposable. In addition, the embedding $M' \to M$ is mono-irreducible: this means that for any proper submodule $M''$ of $M$ with $M' \subseteq M''$, the inclusion $M' \subseteq M''$ splits. As a consequence, given any nilpotent endomorphism $f$ of $M/M'$, the sequence induced from $0 \to M' \to M \to M/M' \to 0$ using $f$ splits.

Of course, we may use duality and consider Gabriel-Roiter submodules $U$ of $DM$, the corresponding factor module $DU$ of $D^2M = M$ will be said to be a Gabriel-Roiter factor module of $M$.

**Proposition 10.2.** Let $M$ be a minimal infinite family of modules of fixed length $b$. Then there are indecomposable non-projective modules $C,Y$ and infinitely many indecomposable modules $M_i \in M$ with short exact sequences

$$0 \to \tau C \to M_i \to Y \to 0$$

such that $Y$ is a Gabriel-Roiter factor module of $M_i$.

**Addendum.** The sequences

$$0 \to \tau C \to M_i \to Y \to 0$$

belong to the socle of $\text{Ext}^1(Y,K)$ as an $(\text{End} K)^{op}$-module, where $K = \tau C$.

Proof. Let us now start with the proof of the proposition. According to the Lemma, we can assume that all the modules in $M$ are indecomposable. Let $M'$ be the set of modules $M$ which are proper submodules or proper factor modules of modules in $M$. By assumption, there are only finitely many isomorphism classes of modules in $M'$. Since there are only finitely many simple modules, we must have $b \geq 2$, thus any $M \in M$ has a Gabriel-Roiter factor-module $Q_M = M/K_M$; here, $K_M$ is a submodule of $M$. Note that $K_M$ is a proper non-zero submodule, that $Y_M$ is a proper factor module of $M$, and that both modules $K_M$ and $Y_M = M/K_M$ are indecomposable. Of course, both $K_M$ and $Y_M$ belong to $M'$. Since there are (up to isomorphism) only finitely many pairs $(K,Y)$ in $(M')^2$, it follows that there is a pair $(K,M)$ such that there are infinitely many pairwise non-isomorphic indecomposable modules $M$ with $K = K_M$ and $Y = Y_M$. The exact sequences $0 \to K \to M \to Y \to 0$ with $M$ indecomposable show that $K$ cannot be injective, thus $C = \tau^{-1}K$ is again indecomposable. This completes the proof of the proposition.

Proof of the addendum. Given any nilpotent endomorphism $f$ of $K$, the sequence induced from $0 \to K \to M \to Y \to 0$ using $f$ splits, thus this sequence belongs to the socle of $\text{Ext}^1(Y,K)$ as an $(\text{End} K)^{op}$-module.

**11. Forks and coforks.**

We call a family of maps $f_i : X \to M_i$ with $i \in I$ a fork provided for any finite subset $I' \subseteq I$, the map $(f_i)_{i \in I'} : X \to \bigoplus_{i \in I'} M_i$ is left minimal. The dual notion will
be that of a cofork, this is a set of maps \( g_i : M_i \to Y \) with \( i \in I \) such that the maps \((g_i)_{i \in I'} : \bigoplus_{i \in I'} M_i \to Y \) with \( I' \) any finite subset of \( I \) is right minimal.

**Lemma 11.1.** Let \( f_i : X \to M_i \) with \( i \in I \) be a family of non-zero maps with indecomposable modules \( M_i \). Then \((f_i : X \to M_i)_i\) is a fork if and only if \( f_i \notin \sum_{j \neq i} \text{Hom}(M_j, M_i)f_j \) for all \( i \in I \).

Proof: First, assume that there is some \( i \) with \( f_i \in \sum_{j \neq i} \text{Hom}(M_j, M_i)f_j \), say there is the subset \( \{1, 2, \ldots, t\} \subseteq I \) such that \( f_1 \in \sum_{j=2}^t \text{Hom}(M_j, M_1)f_j \). Thus, for \( 2 \leq j \leq t \) there are maps \( p_j : M_j \to M_1 \) such that \( f_1 = \sum_{j=2}^t p_j f_j \). We want to show that the map \( f' = (f_i)_{i=1}^t : X \to M = \bigoplus_{i=1}^t M_i \) is not left minimal. Let \( N \) be the kernel of the map

\[
(-1, p_2, \ldots, p_t) : \bigoplus_{i=1}^t M_i \to M_1.
\]

It is easy to check that \( M \) is the direct sum of \( M_1 \) and \( N \). But the equality \(-f_1 + \sum_{j=2}^t p_j f_j\) shows that the image of \( f' \) is contained in \( N \). Since the image of \( f' \) is contained in a proper direct summand of \( M \), we see that \( f' \) is not left minimal.

Conversely, assume that \( f_i \notin \sum_{j \neq i} \text{Hom}(M_j, M_i)f_j \) for all \( i \in I \). We want to show that for any finite subset \( I' \subseteq I \), the map \((f_i)_{i \in I'} : X \to \bigoplus_{i \in I'} M_i \) is left minimal. If \( I' \) is empty, then we deal with the zero map \( X \to 0 \) which of course is left minimal. If \( I' \) consists of the single element \( i \), then we deal with \( f_i : X \to M_i \). Since \( f_i \neq 0 \) and \( M_i \) is indecomposable, the map \( f_i \) is left minimal. Thus we can assume that \( I' \) contains \( t \geq 2 \) elements, say the elements \( \{1, 2, \ldots, t\} \). Assume that the map \( f' = (f_i)_{i=1}^t : X \to M \) with \( M = \bigoplus_{i=1}^t M_i \) is not left minimal. Then there is a proper direct decomposition \( M = N \oplus N' \) such that the image of \( f \) is contained in \( N \). We can assume that \( N' \) is indecomposable, thus isomorphic to some \( M_i \), say to \( M_1 \). Let \( p : M \to N' = M_1 \) be the projection with kernel \( N \), write \( p = (p_i)_i \) with \( p_i : M_i \to M_1 \). Note that \( p_1 \) has to be an isomorphism. Replacing any \( p_i \) by \( (p_1)^{-1}p_i \), we can assume that \( p_1 = 1 \). Since the image of \( f' \) is contained in the kernel \( N \) of \( p \), we have \( \sum_{i=1}^t p_i f_i = 0 \), thus

\[
f_1 = -\sum_{i=2}^t p_i f_i.
\]

This completes the proof.

**Lemma 11.2 (Gabriel-Roiter forks).** Let \( M_i \) with \( i \in I \) be pairwise non-isomorphic indecomposable modules of fixed length with isomorphic Gabriel-Roiter submodule \( U \), say with embeddings \( u_i : U \to M_i \). Then the set of maps \( u_i : U \to M_i \) is a fork.

Proof. The maps \( u_i : U \to M_i \) are non-zero maps and the modules \( M_i \) are indecomposable. Thus, if the family \((u_i : U \to M_i)_i\) is not a fork, then the Lemma asserts that there is some \( i \) with \( u_i \notin \sum_{j \neq i} \text{Hom}(M_j, M_1)u_j \), thus we can assume that there is the subset \( \{1, 2, \ldots, t\} \subseteq I \) and maps \( p_j : M_j \to M_1 \) for \( 2 \leq j \leq t \) such that \( u_1 = \sum_{j=2}^t p_j u_j \).

Let \( M'' = \bigoplus_{j=2}^t M_j \). For \( 2 \leq j \leq t \), consider the submodule \( p_j u_j(U) \) of \( M_1 \). It is a proper submodule of \( M_1 \), thus \( \gamma(p_j u_j(U)) \leq \gamma(U) \), since \( U \) is a Gabriel-Roiter submodule of \( M_1 \). On the other hand, the image of \((p_j u_j)_{j=2}^t : U \to M''\) is contained in
\( \bigoplus_{j=2}^{t} p_j u_j(U) \). Since \( u_1 = \sum_{j=2}^{t} p_j u_j \), this map \( (p_j u_j)_{j=2}^{t} \) is injective, thus \( U \) is a submodule of \( \bigoplus_{j=2}^{t} p_j u_j(U) \). It follows that \( \gamma(U) \leq \max \gamma(p_j u_j(U)) \). Thus there is some \( s \) with \( 2 \leq s \leq t \) such that \( \gamma(U) = \gamma(p_s u_s(U)) \). Since \( U \) is indecomposable, \( U \) has to be a direct summand of \( p_s u_s(U) \). But \( p_s u_s(U) \) is a factor module of \( U \), thus \( p_s u_s : U \to p_s u_s(U) \) is an isomorphism. As a consequence, \( u_s \) is a split monomorphism. But this is impossible, since \( u_s \) is the inclusion of a Gabriel-Roiter submodule.

A similar proof shows that starting with any infinite family \( \mathcal{M} \) of indecomposable modules with fixed length, there are infinite forks consisting of maps \( S \to M_i \), where \( S \) is a simple module and \( M_i \in \mathcal{M} \).

**Lemma 11.3.** Let \( \mathcal{M} \) be an infinite set of pairwise non-isomorphic indecomposable modules of fixed length. Then there exists a simple module \( S \) and infinitely many maps \( u_i : S \to M_i \) with \( M_i \in \mathcal{M} \) such that the family of these maps \( u_i : S \to M_i \) is a fork.

Proof. Let us assume that the modules in \( \mathcal{M} \) have length \( b \). Of course, \( b > 1 \). Since there are only finitely many possible Gabriel-Roiter measures, we can assume that all the modules in \( \mathcal{M} \) have the same Gabriel-Roiter measure. Consider a module \( M \in \mathcal{M} \). It is not cogenerated by the remaining modules, thus the intersection of the kernels of all maps \( \phi : M \to M' \) is non-zero, where \( M' \in \mathcal{M} \) is not isomorphic to \( M \). Let \( S(M) \) be a simple submodule of \( M \) which is contained in this intersection (thus \( \phi(S(M)) = 0 \) for all maps \( \phi : M \to M' \) with \( M' \in \mathcal{M} \) and \( M \) not isomorphic to \( M \)). Since there are only finitely many simple module \( S \), we may assume that \( S(M) = S \) for all \( M \in \mathcal{M} \). Denote by \( u_M : S = S(M) \to M \) the inclusion map. We claim that this set \( u_M : S \to M \) with \( M \in \mathcal{M} \) is a fork.

We have to show the following: Assume that there are given inclusion maps \( u_i : S \to M_i \) with \( 1 \leq i \leq t \) such that \( \text{Hom}(M_i, M_j) u_i = 0 \) for \( i \neq j \). Then the map \( (u_i)_i : S \to M \) with \( M = \bigoplus_{i=1}^{t} \) is left minimal. Again, this is clear for \( t = 1 \), since the maps \( u_i \) are non-zero and the modules \( M_i \) are indecomposable. Thus we can assume that \( t \geq 2 \). If the map \( (u_i)_i \) is not left minimal, then, up to permutation of the indices, there are maps \( p_i : M_i \to M_1 \) with \( 2 \leq i \leq t \) such that

\[
  u_1 = \sum_{i=2}^{t} p_i u_i.
\]

However, by construction, \( \text{Hom}(M_i, M_1) u_i = 0 \) for \( i \neq 1 \), thus all the summands \( p_i u_i \) on the right are zero. Since \( u_1 \neq 0 \), we obtain a contradiction.

The setting developed here allows to provide a proof of the following result which first was established in [R4]. Note that this result strengthens the assertion of the first Brauer-Thrall conjecture. We start with the following proposition:

**Proposition 11.4.** Let \( g_i : M_i \to Y \) with \( i \in \mathbb{N} \) be a fork, and assume that all the modules \( M_i \) are indecomposable with Gabriel-Roiter measure \( \gamma_0 \). Let \( K_t \) be the kernel of \( (g_i)_i : \bigoplus_{i=1}^{t} M_i \to Y \). Let \( K \) be the additive subcategory generated by the modules \( K_t \) with \( t \in \mathbb{N} \). Then \( K \) is infinite and all the indecomposable modules \( K \) in \( K \) satisfy \( \gamma(K) < \gamma_0 \).
Proof. First, assume that $K = \text{add } K$ for some module $K$, let $C = \tau^{-1} K$. Then all the modules $\bigoplus_{i=1}^t M_i$ are present in $\mathcal{C}[- \rightarrow Y]$. But the modules which are present in $\mathcal{C}[- \rightarrow Y]$ are of bounded length. This shows that $K$ is infinite.

Let $K$ be an indecomposable module in $\mathcal{K}$, say a direct summand of $K_t$ for some $t$. Assume that $\gamma(K) \geq \gamma_0$. This implies that the inclusion map $K \subseteq K_t \subset \bigoplus_{i=1}^t M_i$ splits. But this contradicts the fact that the map $(g_i)_i: \bigoplus_{i=1}^t M_i \rightarrow Y$ is right minimal.

**Corollary 11.5 (First Brauer-Thrall conjecture).** Let $\mathcal{M}$ be an infinite set of indecomposable modules of a fixed length. Then there are indecomposable modules of arbitrarily large length which are cogenerated by modules in $\mathcal{M}$.

Proof. We can assume that all the modules in $\mathcal{M} = \mathcal{M}_0$ have the same Gabriel-Roiter measure $\gamma_0$. Take an infinite cofork consisting of maps $g_i: M_i \rightarrow Y$ and let $K_t$ be the kernel of $(g_i)_i: \bigoplus_{i=1}^t M_i \rightarrow Y$. Let $\mathcal{K}$ be the additive subcategory generated by the modules $K_t$ with $t \in \mathbb{N}$. Of course, all the modules in $\mathcal{K}$ are cogenerated by $\mathcal{M}_0$. According to the proposition, $\mathcal{K}$ is infinite and all the indecomposable modules in $\mathcal{K}$ have Gabriel-Roiter measure greater than $\gamma_0$. Now either the indecomposable modules in $\mathcal{K}$ of unbounded length, then we are done. Or else they are of bounded length: then we find in $\mathcal{K}$ an infinite set $\mathcal{M}_1$ of indecomposable modules having the same Gabriel-Roiter measure, say $\gamma_1$ and $\gamma_1 > \gamma_0$. Inductively, we construct a sequence of sets of indecomposable modules

$$\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_i$$

such that the modules in $\mathcal{M}_i$ are cogenerated by $\mathcal{M}_{i-1}$ and have fixed Gabriel-Roiter measure $\gamma_i > \gamma_{i-1}$, for $i \geq 1$. The procedure stops in case there are indecomposable modules of unbounded length which are cogenerated by $\mathcal{M}_i$, and then these modules are cogenerated by $\mathcal{M}$. Otherwise the procedure can be continued indefinitely. But then the modules in $\mathcal{M}_i$ have Gabriel-Roiter measure $\gamma_i$ and are cogenerated by $\mathcal{M}$. Since the measures $\gamma_i$ are pairwise different, the modules in $\bigoplus_i \mathcal{M}_i$ cannot be of bounded length.

12. The Kronecker algebra.

Let us consider in detail the Kronecker algebra $\Lambda$ which was mentioned already in section 2. It is a very important artin algebra and a clear understanding of its module category $\text{mod } \Lambda$ seems to be of interest.

For all pairs $C, Y$ of indecomposable $\Lambda$-modules, we are going to describe the lattices $\mathcal{C}[- \rightarrow Y]$ as well as all the modules present in $\mathcal{C}[- \rightarrow Y]^1$, this is the subset of $\mathcal{C}[- \rightarrow Y]$ of elements of $C$-length 1.

Let us recall the structure of the category $\text{mod } \Lambda$ (see for example [R3], or [ARS, section VIII.7]). There are the preprojective and the preinjective modules, Modules without an indecomposable direct summand which is preprojective or preinjective are said to be regular. For any $\Lambda$-module $M$, its defect is defined by $\delta(M) = \dim \text{Hom}(M, Q_0) - \dim \text{Hom}(P_0, M)$. Any indecomposable preprojective $\Lambda$-module has defect -1, the indecomposable preinjective modules have defect 1, all the regular modules have defect 0. There are countably many indecomposable preprojective modules, they are labeled $P_i$, and also countably many indecomposable preinjective modules, they are labeled $Q_i$; both $P_i$ and $Q_i$ have length $2i + 1$. 


The indecomposable regular modules are those modules which belong to stable Auslander-Reiten components, and all these components are stable tubes of rank 1. The full subcategory $\mathcal{R}$ of all regular modules is abelian. By definition, the simple regular modules are the regular modules which are simple objects in this subcategory. Given any indecomposable regular module $R$, its endomorphism ring $\text{End}(R)$ is a commutative ring (namely a ring of the form $k[T]/(f)$, where $k[T]$ is the polynomial ring in one variable $T$ with coefficients in $k$ and $f$ is a power of an irreducible polynomial) and $\dim R = 2 \dim \text{End}(R)$.

As we have mentioned, we are interested in pairs $C,Y$ of indecomposable $\Lambda$-modules such that a family of modules is present $C[\rightarrow Y]$. It turns out that only the case of $C$ being preprojective, $Y$ being preinjective is relevant, as the following proposition shows.

**Proposition 12.1.** Let $\Lambda$ be the Kronecker algebra and $C,Y$ indecomposable $\Lambda$-modules. If $C$ is preprojective or preinjective, then $C[\rightarrow Y]$ is a projective geometry. If $C$ is regular, then $C[\rightarrow Y]$ is a chain.

By definition, a projective geometry $\mathbb{G}(d)$ over the field $k$ is the lattice of subspaces of the $k$-space of dimension $d$. The chain $\mathbb{I}(d)$ is the set of integers $i$ with $0 \leq i \leq d$ with the usual ordering. The labels have been chosen in such a way that the height of $\mathbb{G}(d)$ as well as of $\mathbb{I}(d)$ is just $d$.

The following table provides the precise data: Here, an indecomposable regular module of regular length $t$ and with regular socle $R$ is denoted by $R[t]$.

| row | $C$ | $Y$ | $C[\rightarrow Y]$ |
|-----|-----|-----|---------------------|
| 1)  | $P_i$ | $P_j$ | $(i \leq j)$ |
| 2)  | $P_i$ | $R[t]$ | $\mathbb{G}(\frac{1}{2} \dim R[t])$ |
| 3)  | $P_i$ | $Q_j$ | $\mathbb{G}(i + j)$ |
| 4)  | $R[s]$ | $R[t]$ | $\mathbb{I}(\text{min}(s, t))$ |
| 5)  | $R[s]$ | $Q_j$ | $\mathbb{I}(\frac{1}{2} \dim R[s])$ |
| 6)  | $Q_i$ | $Q_j$ | $(i \geq j)$ |

Let us stress that in row 4), the regular modules $C,Y$ are supposed to belong to the same tube, namely to the tube containing a fixed simple regular module $R$. For all pairs $C,Y$ of indecomposable Kronecker modules which are not contained in the table, one has $\text{Hom}(C,Y) = 0$, thus $C[\rightarrow Y]$ consists of a single element.

**Proof.** First, let us calculate $\dim \text{Hom}(C,Y)$. The reflection functors of [BGP] yield

\[
\begin{align*}
\dim \text{Hom}(P_i, P_j) &= \dim \text{Hom}(P_0, P_{j-i}) = j - i + 1 \\
\dim \text{Hom}(P_i, R[t]) &= \dim \text{Hom}(P_0, R[t]) = \frac{1}{2} \dim R[t] \\
\dim \text{Hom}(P_i, Q_j) &= \dim \text{Hom}(P_0, Q_{j+i}) = i + j \\
\dim \text{Hom}(R[s], Q_j) &= \dim \text{Hom}(R[s], Q_0) = \frac{1}{2} \dim R[s] \\
\dim \text{Hom}(Q_i, Q_j) &= \dim \text{Hom}(Q_{i-j}, Q_0) = i - j + 1
\end{align*}
\]
As we have mentioned, we have \( \dim R = 2 \dim \End(R) \). It follows that
\[
\dim \Hom(R[s], Q_j) = \frac{1}{2} \dim R[s] = s \dim \End(R).
\]

Finally,
\[
\dim \Hom(R[s], R[t]) = \min(s, t) \dim \End(R).
\]

In case \( C = P_i \) or \( Q_j \) one has \( \End(C) = k \), thus Auslander’s second theorem asserts that \( C[-\rightarrow Y] \) is of the form \( \mathbb{G}(\dim \Hom(C, Y)) \). This yields the rows 1), 2), 3) and 6) of the table.

It remains to look at the rows 4) and 5), thus we assume now that \( C = R[s] \) and \( Y = R[t] \) or \( Y = Q_j \). We show that \( \Hom(C, Y) \) is a cyclic \( \Gamma(C) \)-module, thus we have to find an element \( g \in \Hom(C, Y) \) such that \( g \End(C) = \Hom(C, Y) \), or, equivalently, such that \( \rad \End(C)^{r-1} \neq 0 \), where \( r \) is the length of \( \Hom(C, Y) \) (as a \( \Gamma(C) \)-module). Note that \( \Gamma(C) \) is a local ring, thus there is a unique simple \( \Gamma(C) \)-module \( S \). Since \( S \) has \( k \)-dimension \( \dim \End(R) \), the calculations above show that \( r = s \) in case \( Y = Q_j \) and \( r = \min(s, t) \) in case \( Y = R[t] \). On the other hand, \( \rad \End(C)^{r-1} \) is generated by any endomorphism of \( R[s] \) with image \( R[s-r+1] \). Thus let \( p: R[s] \rightarrow R[s-r+1] \) be the canonical projection, \( u: R[s-r+1] \rightarrow R[s] \) the canonical inclusion, then \( gu \) generates \( \rad \End(C)^{r-1} \). Our aim is to exhibit \( g: R[s] \rightarrow Y \) such that \( gup \neq 0 \).

First, let \( Y = R[t] \) and \( s \leq t \). Then \( r = \min(s, t) = s \) and \( R[s-r+1] = R[1] = R \). Let \( g: R[s] \rightarrow R[t] \) be the canonical inclusion, thus \( gu: R \rightarrow R[t] \) is an inclusion, in particular non-zero, and therefore also \( gup \neq 0 \).

Second, let \( Y = R[t] \) and \( s > t \). Then \( r = \min(s, t) = t \) and \( R[s-r+1] = R[s-t+1] \). Let \( g: R[s] \rightarrow R[t] \) be the canonical projection. Then \( gu: R[s-t+1] \rightarrow R[t] \) has image \( R \) (since the kernel of \( g \) is \( R[s-t] \)). This shows that \( gu \) is non-zero, and therefore also \( gup \neq 0 \).

Finally, we have to deal with the case \( Y = Q_j \). Take a non-zero map \( g': R \rightarrow Q_j \). Since \( \Ext^1(R[s]/R, Q_j) = 0 \), there exists \( g: R[s] \rightarrow Q_j \) such that \( gu = g' \). Since \( gu \neq 0 \), it follows that also \( gup \neq 0 \).

Thus, always we have found \( g: R[s] \rightarrow Y \) such that \( gup \neq 0 \). As a consequence, \( \Hom(C, Y) \) is a cyclic \( \Gamma(C) \)-module. Since \( \Gamma(C) \) is a uniserial ring, it follows that \( S \Hom(C, Y) \) is of the form \( \mathbb{I}(r) \), where \( r \) is the length of \( \Hom(C, Y) \). According to our calculations, \( r = \min(s, t) \) in case \( Y = R[t] \) and \( r = \frac{1}{2} \dim R[s] \) in case \( Y = Q_j \).

In this way, we have verified the assertions presented in the table. If the pair \( C, Y \) does not occur in the table, then it is well-known that \( \Hom(C, Y) = 0 \), thus \( S \Hom(C, Y) \) consists of a single element and therefore is of the form \( \mathbb{I}(0) = \mathbb{G}(0) \). This completes the proof.

The following assertion which has been shown in the proof will be of further interest:

**Lemma 12.2.** Any non-zero map \( g': R \rightarrow Q_j \) can be extended to a map \( g: R[t] \rightarrow Q_j \), and any such \( g: R[t] \rightarrow Q_j \) generates the \( \Gamma(C) \)-module \( \Hom(R[t], Q_j) \).

**Remark.** We have mentioned in section 4 that both sides of the Auslander bijection concern maps with target \( Y \), but that they invoke these maps in quite different ways. A nice illustration seems to be Lemma 12.1. The map \( g: R[t] \rightarrow Q_j \) constructed there
is a generator of the maximal submodule of \( \text{Hom}(R[t], Q_j) \) and is used in the proof of proposition 12.1 in order to show that \( \mathcal{S} \text{Hom}(R[t], Q_j) \) is of the form \( \mathbb{I}(d) \) for some \( d \). On the other hand, in proposition 12.9, we will consider the right equivalence class \([g]\) as an element of \([\to Q_j]\).

**Proposition 12.3.** Let \( \Lambda \) be the Kronecker algebra, let \( C, Y \) be indecomposable \( \Lambda \)-modules. Then either \( C \) is preprojective and \( Y \) is preinjective, or else there is a tube \( T \) such that for any module \( M \) present in \( C[\to Y] \), all the indecomposable direct summands of \( M \) are preprojective, or preinjective or belong to \( T \).

Proof. First, assume that \( Y \) is preinjective. If \( f : X \to Y \) is right minimal, then \( X \) has to be preprojective. Thus, for any module \( C \), all the modules present in \( C[\to Y] \) are preprojective.

Second, assume that \( Y \) is regular, say belonging to the tube \( T \). Again, assume that \( f : X \to Y \) is right minimal. Then \( X \) is the direct sum of a preprojective module and a module in \( T \). Thus, for any module \( C \), all the modules present in \( C[\to Y] \) are direct sums of preprojective modules and modules in \( T \).

Finally, assume that \( Y \) is preinjective and \( C \) is regular or preinjective. Let \( f : X \to Y \) be right minimal and right \( C \)-determined. Since \( C \) has no indecomposable projective direct summand and \( \Lambda \) is hereditary, we see that \( f \) is surjective and its kernel is isomorphic to \( \tau C \). Now, \( C \) is preinjective, then also \( \tau C \) is preinjective and \( X \) is an extension of a preinjective module by a preinjective module is preinjective again. On the other hand, if \( C \) is regular, say belonging to the tube \( T \), then also \( \tau C \) belongs to \( T \). Since \( X \) is an extension of a module in \( T \) by a preinjective module, it is the direct sum of a module in \( T \) and a preinjective module. This completes the proof.

**Remark.** Let us stress that the cases \( C = P_i, Y = R[t] \) and \( C = R[s], Y = Q_j \) are of completely different nature. Of course, we have already seen that \( C[\to Y] \) is in the first case a projective geometry, in the second case a chain. But also if we look at the different layers, we encounter clear differences. In the chain case, all the elements of \( C[\to Y] \) are given by short exact sequences of the form \((R[s] \to \ast \to Q_j)\), thus by elements of \( \text{Ext}_{\Lambda}^1(Q_j, R[s])\). In this case, we may interpret \( C[\to Y] \) as a display of the various orbits in \( \text{Ext}_{\Lambda}^1(Q_j, R[s])\) with respect to the action of the automorphism group of \( R[s] \). In contrast, in the projective geometry case, only the elements of \( C[\to Y] \) of \( C \)-length at most 1 are given by short exact sequences of the form \((P_i \to \ast \to R[t])\), whereas for the elements of \( C \)-length at least 2, we need short exact sequences of the form \((P_i^a \to \ast \to R[t])\) with \( a \geq 2 \).

In the proof of proposition 12.3, we have seen that for \( C = P_i, Y = R[t] \), the modules \( M \) present in \( C[\to Y] \) are direct sums of preprojective modules and modules in the tube which contains \( T \), and that for \( C = R[s], Y = Q_j \) the modules present in \( C[\to Y] \) are direct sums of preinjective modules and modules in \( T \). However, in the first case the number of indecomposable preprojective direct summands of \( M \) may be large, whereas in the second case there is just one direct summand of \( M \) which is indecomposable preinjective.

**Preprojective \( C \), preinjective \( Y \).** Let us focus now the attention to \( C[\to Y] \), where \( C \) is indecomposable preprojective and \( Y \) is indecomposable preinjective. Here is
the general behavior as seen in Proposition 12.1:

\[ \begin{array}{cc|cccc}
K & C & Y & Q_0 & Q_1 & Q_2 & Q_3 & Q_4 \\
\hline
P_0 & 0 & 1 & 2 & 3 & 4 \\
P_1 & 1 & 2 & 3 & 4 \\
P_0 & 2 & 3 & 4 \\
\hline
P_1 & 3 & 4 \\
\hline
P_2 & 4 \\
\end{array} \]

We want to know which modules are present in \( \mathcal{C}[\rightarrow Y] \). As we will show, these are certain regular modules.

**Lemma 12.4.** Let \( \Lambda \) be the Kronecker algebra and \( M \) a regular module. The following conditions are equivalent.

(i) If \( M = M' \oplus M'' \), then \( \text{Ext}^1(M', M'') = 0 \).

(ii) The regular socle of \( M \) is multiplicity-free.

(ii*) The regular top of \( M \) is multiplicity-free.

(iii) \( M = \bigoplus_{i=1}^n M_i \), with indecomposable modules \( M_i \) belonging to pairwise different tubes.

(iv) \( \text{End}(M) \) is commutative.

Proof. The equivalence of (ii) and (iii), and dually of (ii*) and (iii) is straightforward. The implication (iii) \( \implies \) (i) follows from the fact that \( \text{Ext}^1(M_i, M_j) = 0 \) in case \( M_i, M_j \) are regular and belong to different Auslander-Reiten components. The converse implication (i) \( \implies \) (iii) follows from the fact that \( \text{Ext}^1(M_i, M_j) \neq 0 \) in case \( M_i, M_j \) are nonzero regular and belong to the same Auslander-Reiten component. In order to see the implication (iii) \( \implies \) (iv), one should be aware that for \( M = \bigoplus_{i=1}^n M_i \), with indecomposable modules \( M_i \) belonging to pairwise different tubes, one has \( \text{End}(M) = \prod_i \text{End}(M_i) \) and that \( \text{End}(R) \) is commutative for any indecomposable regular module \( R \). Conversely, in order to show the implication (iv) \( \implies \) (iii), assume that \( M \) is regular and assume that \( M = M' \oplus M'' \oplus M''' \) with \( M', M'' \) non-zero modules belonging to some tube. Then \( \text{Hom}(M'', M''') \neq 0 \),
thus there is a non-zero homomorphism $\phi: M' \to M''$ and we may consider this as an endomorphism of $M$ by setting $\phi$ to be zero on $M'' \oplus M'''$. Let $e': M \to M'$ be the projection of $M$ to $M'$ with kernel $M'' \oplus M'''$. Then $\phi e' = \phi \neq 0$, whereas $e' \phi = 0$. This shows that $\text{End}(M)$ is not commutative.

A regular Kronecker-module $M$ will be said to be strongly regular provided the equivalent conditions of the Lemma are satisfied. Let $R(i)$ be the set of isomorphism classes of strongly regular modules of length $i$. Note that $R(i)$ is empty in case $i$ is odd or negative, and $R(0)$ has just one element, namely the isomorphism class of the zero module.

As we have mentioned, we want to see in which way families of modules may be present in $C[\to Y]^{-1}$, with $C$ and $Y$ indecomposable. Here is the description of these sets.

**Proposition 12.5.** Let $\Lambda$ be the Kronecker algebra, $C$ indecomposable preprojective, $Y$ indecomposable preinjective.

If $C = P(S)$ for some simple module $S$, then $C[\to Y]^{-1}$ may be identified with the set of inclusion maps $X \to Y$ such that the socle of $Y/X$ is equal to $S$.

If $C = \tau^{-}K$ for some indecomposable module $K$, then $C[\to Y]^{-1}$ consists of the right equivalence classes of surjective maps $X \to Y$ with kernel $K$.

Proof: See 6.7 and 7.10.

Given a morphism $f: X \to Y$, let $\sigma(f) = [X]$, the isomorphism class of the source $X$ of $f$. We study the function $\sigma$ defined on $C[\to Y]^{-1}$ with values in the set of isomorphism classes of modules.

**Proposition 12.6.** Let $\Lambda$ be the Kronecker algebra, $C$ indecomposable preprojective, $Y$ indecomposable preinjective. Then $\sigma$ is a bijection

$$\sigma: C[\to Y]^{-1} \to R(i) \quad \text{with} \quad i = |C| + |Y| - 4.$$  

For the proof, we need the following Lemma.

**Lemma 12.7.** Let $\Lambda$ be the Kronecker algebra, let $0 \to U' \to X \to Y \to 0$ be a non-split exact sequence with $Y$ indecomposable preinjective, $U$ preprojective. Then no indecomposable direct summand of $X$ is preinjective.

Proof. Let $X = X' \oplus X''$ with $X'$ indecomposable. Assume that $X'$ is preinjective. Denote the map $X \to Y$ by $f$ and let $f'$ be its restriction to $X'$. Then $f' \neq 0$, since otherwise $X'$ is a direct summand of the kernel of $f$, thus equal to $K$, so that the sequence splits. Non-zero maps between indecomposable preinjective modules are surjective, thus $f'$ is surjective. Of course, $f'$ is not an isomorphism, since otherwise the sequence would split. Let $U'$ be the kernel of $f'$. As we see, $U' \neq 0$. We have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & U' & \longrightarrow & U & \longrightarrow & X'' & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & U' & \longrightarrow & X' & \overset{f'}{\longrightarrow} & Y & \longrightarrow & 0.
\end{array}
$$
Now $U'$ is a submodule of $U$, thus it is preprojective. Since $U' \neq 0$, we must have $\delta(U') \leq -1$, where $\delta$ denotes the defect. It follows that $\delta(X') = \delta(U') + \delta(Y) \leq 0$, since $\delta(Y) = 1$. This contradicts our assumption that $X'$ is preinjective.

**Lemma 12.8.** Let $X$ be regular, $Y$ indecomposable preinjective. Then the following conditions are equivalent:

(i) $X$ is strongly regular,

(ii) There exists $f : X \to Y$ such that the kernel of $f$ does not contain a simple regular submodule.

Any map $f : X \to Y$ with no simple regular submodule in its kernel is a monomorphism or an epimorphism. If $f : X \to Y$ and $f' : X' \to Y$ are maps with no simple regular submodule in the kernel, and $X, X'$ are isomorphic, then $f, f'$ are right equivalent.

Proof: First, we show that (ii) implies (i). Assume that there is $f : X \to Y$ such that the kernel of $f$ does not contain a simple regular submodule of $X$. In order to show that $X$ is strongly regular, we show that its regular socle is multiplicity-free. Assume, for the contrary, that $X$ has a submodule $Y$ such that $U = R \oplus R$ with $R$ simple regular. Let $f_1, f_2$ be the restrictions of $f$ to $R \oplus 0$ or $0 \oplus R$, respectively. Since no simple regular submodule of $X$ is contained in the kernel of $f$, we see that both $f_1, f_2$ are non-zero maps. According to Lemma 12.2, there is a map $h : R \to R$ such that $f_1 = f_2 h$. But then $R' = \{(-h(x), x) \mid x \in R\}$ belongs to the kernel of $f$. Of course, $R'$ is isomorphic to $R$, thus $R'$ is a simple regular submodule of $X$ which belongs to the kernel of $f$, a contradiction.

Conversely, assume that $X = \bigoplus_i R_i[t_i]$ is a strongly regular module, with pairwise different simple regular modules $R_i$. According to Lemma 12.2, there is a map $f_i : R[t_i] \to Y$ such that the restriction of $f_i$ to $R_i$ is non-zero. Since the simple regular submodules $R_i$ are pairwise non-isomorphic, these are the only simple regular submodules of $X$, thus no simple regular submodule of $X$ lies in the kernel of $f = (f_i)_i : X \to Y$.

Now assume that $f : X \to Y$ is a map such that the kernel of $f$ does not contain a simple regular submodule. Assume that $f$ is not an epimorphism. The image of $f$ is a factor module of $X$, thus the direct sum of a regular and a preinjective module. But $Y$ has no non-zero proper submodule which is preinjective. Thus, the image of $f$ is regular. The kernel of a map between regular modules is regular, thus either a monomorphism, or it contains a simple regular submodule. Since the latter is not possible, we see that $f$ has to be a monomorphism.

Finally, assume that $f : X \to Y$ and $f' : X' \to Y$ are maps with no simple regular submodule in the kernel. Let $g : X \to X'$ be an isomorphism. Write $X = \bigoplus R_i[t_i]$ with pairwise non-isomorphic simple regular modules $R_i$. Let $f_i$ and $f'_i$ be the restriction of $f$ and $f'g$, respectively, to $R_i[t_i]$. Since $R_i$ is not in the kernel of $f$, the restriction of $f_i$ to $R_i$ is non-zero. According to Lemma 12.2, there is a map $h_i : R_i \to R_i$ such that $f_i = f'_ih_i$. But also the restriction of $f'_i$ to $R_i$ is non-zero, thus $R_i$ is not in the kernel of $h_i$ and therefore $h_i$ is an automorphism. Let $h = (h_i)_i : X \to X$. This is an automorphism of $X$ with $f = f'gh$. Since $f = f'gh$ and $gh$ is an isomorphism, we see that $f, f'$ are right equivalent.

**Proposition 12.9.** Let $Y$ be indecomposable projective. The maximal submodules of $Y$ are pairwise non-isomorphic and these are, up to isomorphism, all the strongly regular
modules of length \(|Y| - 1\).

The kernels of the non-zero maps \(Y \to Q_1\) are pairwise non-isomorphic and these are, up to isomorphism, all the strongly regular modules of length \(|Y| - 3\).

Proof. Let \(X\) be a maximal submodule of \(Y\). Then \(Y/X\) is simple injective, thus \(\delta(X) = \delta(Y) - \delta(Y/X) = 0\). Since \(Y\) has no proper non-zero preinjective submodule, we see that \(X\) has to be regular. According to Lemma 12.8, the inclusion map \(X \to Y\) shows that \(X\) is even strongly regular, and of course of length \(|Y| - 1\). Conversely, if \(X\) is strongly regular and of length \(|Y| - 1\), Lemma 12.8 yields a monomorphism \(X \to Y\). Now assume that two maximal submodules \(X, X'\) are isomorphic, let \(f: X \to Y\) and \(f': X' \to Y\) be the inclusion maps. Then, according to Lemma 12.8, \(f, f'\) are right equivalent, thus \(X = X'\).

In the same way, we consider the kernels \(X\) of the non-zero maps \(Y \to Q_1\). Clearly, all non-zero maps \(Y \to Q_1\) are surjective, thus \(\delta(X) = 0\) and again, \(X\) has to be regular, and according to Lemma 12.8 even strongly regular, of course of length \(|Y| - 3\). Conversely, if \(X\) is strongly regular and of length \(|Y| - 1\), Lemma 12.8 yields a monomorphism \(X \to Y\). The factor module \(Y/X\) is of length 3 and has the same composition factors as \(Q_1\). But the only factor module of \(Y\) of length 3 with the same composition factors as \(Q_1\) itself, thus \(X\) is the kernel of a non-zero map \(Y \to Q_1\). Finally, we use again Lemma 12.8 in order to see that isomorphic kernels of non-zero maps \(Y \to Q_1\) are actually identical.

Remark. It should be stressed that here the inverse \(\eta_{CY}^{-1}\) of the Auslander bijection can be seen very well.

If \(M\) is a Kronecker module, we may write \(M\) in the form

\[ M = (M_1, M_2; \alpha: M_2 \to M_1, \beta: M_2 \to M_1) \]

and we can make the identification \(M_1 = \text{Hom}(P_0, M)\) and \(M_2 = \text{Hom}(P_1, M)\), since \(P_0 = \mathbb{P}(1), P_1 = \mathbb{P}(2)\).

Consider the case \(C = P_1\). Given a maximal submodule \(U\) of \(\text{Hom}(C, Y)\), we may interpret it as a maximal submodule of \(Y_2\), thus we may consider the submodule \(\Lambda U\) of \(Y\) generated by \(U\), let \(u: \Lambda U \to Y\) be the inclusion map. Then \([u: \Lambda U \to Y]\) belongs to \(C[\to Y]_1\) and \(\eta_{CV}(u) = U\). This shows that \(u = \eta_{CV}^{-1}(U)\).

Similarly, for \(C = P_0\), starting with a maximal submodule \(V_1\) of \(\text{Hom}(C, Y) = Y_1\), we may construct \(V_2 = M_{\alpha}^{-1}(V) \cap M_{\beta}^{-1}(V)\), then the pair \((V_1, V_2)\) may be considered as a submodule of \(Y\), say with inclusion map \(v: (V_1, V_2) \to Y\). Then \([v: (V_1, V_2) \to Y]\) belongs to \(C[\to Y]_1\) and \(\eta_{CV}(u) = V_1\). This shows that \(v = \eta_{CV}^{-1}(V_1)\).

Proposition 12.10. Let \(\Lambda\) be the Kronecker algebra and \(Y\) indecomposable preinjective, \(C\) indecomposable preprojective. Let \(d = |C| + |Y| - 4\). Let \(X\) be any module.

There is \(f: X \to Y\) right minimal, right \(C\)-determined with \(|f|_C = 1\), if any only if the isomorphism class of \(X\) belongs to \(R(d)\).

If \(f: X \to Y\) and \(f': X' \to Y\) are right minimal, right \(C\)-determined maps with \(|f|_C = |f'|_C = 1\). Then \(f, f'\) are right equivalent if and only if \(X, X'\) are isomorphic.

Proof: First, consider the case when \(C\) is not projective, thus \(|C| \geq 5\), therefore \(d > |Y|\).

Since \(\Lambda\) is hereditary and \(C\) has no indecomposable projective direct summand, any right \(C\)-determined map \(f: X \to Y\) is surjective and its kernel belongs to add \(K\), where
$K = \tau C$. If we assume that $|f|_C = 1$, then the kernel of $f$ has to be equal to $K$. It follows that $\delta(X) = \delta(Y) + \delta(K) = 0$. Since all indecomposable submodules of $X$ are preprojective or regular, we see that $X$ has to be regular. Of course, its length is just $d$. Note that no simple regular submodule $R$ of $X$ can be contained in the kernel of $f$, since by assumption the kernel of $f$ is $K$, thus preprojective. The Lemma 12.8 shows that $X$ is strongly regular, thus the isomorphism class of $X$ belongs to $\mathcal{R}(d)$.

Conversely, assume that $X$ is a strongly regular module of length $|X| = d$. According to the Lemma 12.8 there exists a morphism $f: X \to Y$ such that its kernel contains no simple regular module. Since $|X| = d > |Y|$, the map $f$ cannot be a monomorphism, thus it is an epimorphism. Since the kernel of $f$ does not contain a simple regular module, it is the direct sum of indecomposable preprojective modules. Since the defect of the kernel is $\delta(X) - \delta(Y) = -1$, we see that the kernel of $f$ is indecomposable preprojective. The length of the kernel is $|X| - |Y| = |C| - 4 = |\tau C|$, thus the kernel of $f$ is isomorphic to $\tau C$. Altogether, we have shown: if the isomorphism class of $X$ is strongly regular, then clearly $X, X'$ are isomorphic. Conversely, assume that $X, X'$ are isomorphic. In order to show that $f, f'$ are right equivalent, we may assume that $X' = X$. Lemma 12.8 asserts that $f, f'$ are right equivalent.

Now assume that $C$ is projective, thus $|C| \leq 3$, therefore $d < |Y|$. In case $C = P_1$, we have to consider the submodules $X$ of $Y$ with $Y/X = I_0$, in case $C = P_0$, we have to consider the submodules $X$ of $Y$ with $Y/X = I_1$. This has been done in the previous proposition.

**Corollary 12.11.** Let $\Lambda$ be the Kronecker algebra, $C$ indecomposable preprojective, $Y$ indecomposable preinjective. Then $\eta_{CY}^{-1}$ yields a bijection

$$\eta_{CY}^{-1}: \mathcal{S}_m \text{Hom}(C, Y) \longrightarrow \mathcal{R}(|C| + |Y| - 4).$$

Note that $\Gamma(C) = k$, thus Hom$(C, Y)$, considered as a $\Gamma(C)$-module is just a vector space, thus $\mathcal{S}_m \text{Hom}(C, Y)$ is the set of the maximal subspaces of a vector space, and therefore a projective space. We see that we obtain a parameterization of the set $\mathcal{R}(2i)$ of all strongly regular Kronecker modules of length $2i$ by the projective space $\mathbb{P}_i$.

**Remark.** This parameterization of $\mathcal{R}(d)$ extends the well-known description of the geometric quotient of the “open sheet”, when dealing with conjugacy classes of $(n \times n)$-matrices with coefficients in a field, see for example Kraft [KX].

Let us mention some details: Let $R = k[T]$ be the polynomial ring in one variable $T$ with coefficients in the field $k$, and let us assume that $k$ is algebraically closed. We consider $R$ as the path algebra of the quiver with one vertex and one loop; in this way, the $R$-modules of dimension $n$ are just pairs $(V, \phi)$, where $V$ is a $k$-space of dimension $n$ and $\phi: V \to V$ an endomorphism of $V$, or, after choosing a basis of $V$, we just deal
with \((n \times n)\)-matrices with coefficients in \(k\). Isomorphism of \(R\)-modules translates to equivalence (or conjugacy) of matrices. The assertions of Lemma 12.4 can be reformulated in this context, the properties mentioned there characterize just the cyclic \(R\)-modules of finite length. Note that the category of \(k[T]\)-modules of finite length is equivalent to the full subcategory \(R'\) of all regular Kronecker modules without eigenvalue \(\infty\) (to be precise: let us denote the two arrows of the Kronecker quiver by \(\alpha, \beta\) and let \(T_\infty\) be the Auslander-Reiten component which contains the indecomposable regular representation \(M\) with \(M_\alpha = 0\). The subcategory \(R'\) consists of all regular representations with no indecomposable direct summand in \(T_\infty\), or equivalently, it is the full subcategory of all representations \(M\) such that \(M_\alpha: N \to N\) is the identity map, and \(M_\beta: N \to N\) the multiplication by \(T\). A representation \(M\) in \(R'\) is strongly regular if and only if \(M\) corresponds under this equivalent to a cyclic \(k[T]\)-module. Thus, one may be tempted to call the strongly regular Kronecker modules “cyclic” modules, but this would be in conflict with standard terminology.

**Remark.** “Modules determined by morphisms”. A bijection of two sets can always be read in two different directions. This survey is concerned with the Auslander bijection

\[
\eta_{CY}: C[\to Y] \longrightarrow S\text{Hom}(C, Y),
\]

In the previous sections, the focus was going from left to right: Any right minimal morphism \(f \in C[\to Y]\) yields under \(\eta_{CY}\) a submodule of \(\text{Hom}(C, Y)\) and is uniquely determined by this submodule, this is the philosophy of saying that morphisms as elements of \([\to Y]\)) are determined by modules.

The considerations in the present section point into the reverse direction: we use sets of morphisms as convenient indices for parameterizing isomorphism classes of modules. We let us take the restriction

\[
\eta_{CY}: C[\to Y]^1 \longrightarrow S_m\text{Hom}(C, Y),
\]

where \(C[\to Y]^1\) are the right equivalence classes of the maps ending in \(Y\) which have \(C\)-length 1, and where \(S_m\text{Hom}(C, Y)\) is the set of maximal submodules of \(\text{Hom}(C, Y)\). According to 5.4, the map \(\eta_{CY}\) furnishes a bijection between \(C[\to Y]^1\) and \(S_m\text{Hom}(C, Y)\), thus we can use the right hand set in order to parametrize the left hand set.

Our interest lies in the special case where maps \(f: M \to Y\) and \(f': M' \to Y\) of \(C\)-length 1 are right equivalent only in case \(M\) and \(M'\) are isomorphic. In this case, the maximal submodule \(\eta_{CY}(f: M \to Y)\) (thus a set of morphisms) uniquely determines the module \(M\). In addition, we will assume that \(C\) is a brick, or at least that \(C\) is indecomposable and \(\text{rad End}(C)\) annihilates \(\text{Hom}(C, Y)\). In this case, \(S_m\text{Hom}(C, Y)\) is a projective space (namely the projective \(d\)-space over the division ring \(\text{End}(C)/\text{rad End}(C)\), provided \(\text{Hom}(C, Y)\) is a module of length \(d + 1\)). Here is the relevant definition:

Let us determine \(\eta_{CY}^{-1}(0)\) for \(C = P_i, Y = Q_j\) with \(i, j \in \mathbb{N}_0\). Note that

\[
\dim \text{Hom}(P_{i-1}, Q_j) = i + j - 1,
\]

40
where $P_{-1} = 0$. Thus, the universal map from $\text{add} P_{i-1}$ to $Q_j$ is of the form $P^{i+j-1}_{i-1} \to Q_j$ (for $j = 0$, it is a map of the form $P^{i-1}_{i-1} \to Q_0$).

**Proposition 12.12.** Let $i, j \in \mathbb{N}_0$. Let $f: P^{i+j-1}_{i-1} \to Q_j$ be the universal map from $\text{add} P_{i-1}$ to $Q_j$. Then

$$\eta^{-1}_{CY}(0) = [f: P^{i+j-1}_{i-1} \to Q_j].$$

Proof: For $i = 0$, the intersection of the kernels of the maps $Q_j \to Q_1$ is zero. For $i = 1$, the module $P_0 = P_{i-1}$ is simple projective, thus, for $j \geq 1$, the universal map $f: P^j_0 \to Q_j$ is just the embedding of the socle of $Q_j$ into $Q_j$ and the socle is the intersection of the kernels of the maps $Q_j \to Q_0$. For $i = 1$ and $j = 0$, the intersection of the kernels of the maps $Q_j \to Q_0$ is zero, but also $P^{i+j-1}_{i-1}$ is zero.

Assume now that $i \geq 2$. We have

$$\dim \text{Ext}^1(Q_j, P_{i-2}) = \dim \text{Hom}(P_i, Q_j) = \dim \text{Hom}(P_{i+j}, Q_0) = i + j,$
$$
thus the universal extension of $Q_j$ from below using copies of $P_{i-2}$ looks as follows

$$0 \to P^{i+j}_{i-2} \to X \xrightarrow{f} Q_j \to 0$$

with a module $X$ such that $\text{Ext}^1(X, X) = 0$. The dimension vector of $X$ is

$$\dim X = (i + j) \dim P_{i-2} + \dim Q_j = (i + j - 1) \dim P_{i-1}.$$

Since also $\text{Ext}^1(P_{i-1}, P_{i-1}) = 0$, it follows that $X = P^{i+j-1}_{i-1}$. Since $f$ is right minimal and $\dim \text{Hom}(P_{i-1}, Q_j) = i + j - 1$, we see that $f$ has to be the universal map from $\text{add} P_{i-1}$ to $Q_j$.

**Example 10:** $C = P_4$, $Y = Q_0$. Note that $\dim Y = (0, 1)$ and $\dim P_4 = (5, 4)$, thus $\tau C = P_2$ has dimension vector $(3, 2)$ and $\text{Hom}(\tau C, Y) = 2$. Here is a sketch of $C[\to Y]$. In any layer $C[\to Y]_t$ with $1 \leq t \leq 3$, we indicate the form of the elements $[f: X \to Y]$ in the form $P^t_2 \to \dim X \to Y$. The map in the layer $t = 0$ has been described in Proposition 12.12.

![Diagram of Example 10](image_url)
Let us exhibit some of the short exact sequences $P^t_1 \to X \to Y$.

For (3,3), the module $X$ must be strongly regular module, thus, there are three different kinds: direct sums of three pairwise non-isomorphic indecomposable modules of length 2, direct sums of an indecomposable module $M$ of length 4 and an indecomposable module $M'$ of length 2 such that $\text{Hom}(M,M') = 0$, and finally indecomposable modules of length 6.

For (6,5) and (9,7), we deal with direct sums of preprojective and regular modules. We consider the case $\text{dim} X = (6,5)$, thus the case $t = 2$, in detail.

We start with a basis of $U = P_2^2$ and add an element $x$ in order to obtain a basis of $X$, so that $X_1 = U_1$, and $X_2 = U_2 \oplus \langle x \rangle$. In this way, $U$ is a submodule of $X$ with $X/U = Q_0$. What we have to describe are the elements $\alpha(x)$ and $\beta(x)$ in $U_1$, and we have to provide a decomposition of $X$ into indecomposables. In this way, we also will see that the map $X \to Q_0$ with kernel $U$ is right minimal.

We exhibit $U$ as follows: $U_1$ has the basis $v_0, \ldots, v_5$ and $U_2$ has the basis $u_1, u_2, u_4, u_5$ and we assume that $\alpha(u_i) = v_{-1}$ and $\beta(u_i) = v_i$. In order to define $X$, we have to describe $\alpha(x)$ and $\beta(x)$.

In the first three cases, let $\alpha(x) = v_2$. (1) If we define $\beta(x) = v_3$, then clearly $X = P_3$. (2) If we define $\beta(x) = v_4$, then we get a decomposition of $X$ as follows: The elements $u_1, u_2, x, u_5, v_0, v_1, v_2, v_4, v_5$ yield a submodule of the form $P_4$, the elements $x - u_4, v_2 - v_3$ also yield a submodule (since $\beta(x - u_4) = 0$). These two submodules provide a direct decomposition. (3) If we define $\beta(x) = v_5$, then we see that the elements $u_1, u_2, x; v_0, v_1, v_2, v_5$ yield a submodule of the form $P_3$. The elements $u_2 - u_4, x - u_5, v_1 - v_3, v_2 - v_5$ yield a 4-dimensional indecomposable submodule, and we obtain in this way a direct decomposition.

(4) Finally, let $\alpha(x) = v_1, \beta(x) = v_4$. Then we get a submodule of $X$ with basis $u_1, x, u_5, v_0, v_1, v_4, v_5$ which is of the form $P_3$ as well as two indecomposable submodules of length 2, namely with basis $x - u_1, v_4 - v_2$ and with basis $x - u_4, u_1 - u_3$.

In this way, we obtain short exact sequences

$$0 \to P_2^2 \to X \to Q_0 \to 0$$

such that $X$ is of the form $P_3$, of the form $P_4 \oplus R$, of the form $P_3 \oplus R'$, or finally of the form $P_3 \oplus R \oplus R''$, where $R, R''$ are both regular of length 2, and $R'$ is regular of length 4.

13. Lattices of height at most 2.

The height of $\mathcal{S}\text{Hom}(C,Y)$ is the length of the $\Gamma(C)$-module $\text{Hom}(C,Y)$. If $\Lambda$ is a $k$-algebra, $k$ is algebraically closed, and $C$ is multiplicity-free (what we can assume), then the height of $\mathcal{S}\text{Hom}(C,Y)$ is the $k$-dimension of $\text{Hom}(C,Y)$.

Our main interest will concern the height 2 lattices which are not distributive, since this is the first time that one may encounter infinite families. Here is a discussion of the lattices of height at most 2, in general.

Height 0. The lattice $\mathcal{S}M$ has height 0 if and only if $M = 0$. Thus, to say that $\mathcal{S}\text{Hom}(C,Y)$ has height zero means that $\text{Hom}(C,Y) = 0$. 42
**Height 1.** The lattice $SM$ has height 1 if and only if $M$ is a simple module. Thus, in our case $M = \text{Hom}(C,Y)$, we deal with a simple $\Gamma(C)$-module. Note that $\text{Hom}(C,Y)$ is a simple $\Gamma(C)$-module if any only if there is a right minimal, right $C$-determined map $f: X \rightarrow Y$ which is not an isomorphism, such that for any right minimal, right $C$-determined map $f': X' \rightarrow Y$ which is not an isomorphism, there is an isomorphism $h: X \rightarrow X'$ such that $f = f'h$.

Three cases should be noted:

1. $f$ may be an epimorphism. For example, take the path algebra of the quiver of type $A_2$ with an arrow $2 \rightarrow 1$. Let $C = Y = S(2)$ (thus $\tau C = S(1)$). Then the epimorphism $f: P(2) \rightarrow S(1) = Y$ is, up to isomorphism, the only right minimal, right $C$-determined map ending in $Y$ which is not an isomorphism.

2. $f$ may be a monomorphism. To obtain an example, take again take the path algebra of the quiver of type $A_2$ with an arrow $2 \rightarrow 1$. Let $C = S(1)$ and $Y = P(2)$, thus now $C$ is projective. The monomorphism $f: S(1) \rightarrow P(2) = Y$ is, up to isomorphism, the only right minimal, right $C$-determined map ending in $Y$ which is not an isomorphism.

3. $f$ is neither epi nor mono. As an example, take the radical square zero algebra with the linearly oriented quiver of type $A_3$, say with arrows $3 \rightarrow 2 \rightarrow 1$. Let $Y = P(3)$ and $C = S(2)$ (thus $\tau C = S(1)$). The non-zero map $f: P(2) \rightarrow P(3)$ is, up to isomorphism, the only right minimal, right $C$-determined map ending in $Y$ which is not an isomorphism.

**Height 2.** A lattice of height 2 may be either the chain $I(2)$ or not. If the lattice is not a chain, the lattice still may be distributive (case III) or not (case IV). The submodule lattices $SM$ of type III and IV occur for a semisimple module $M$ of length 2; in case $M$ is the direct sum of two non-isomorphic simple modules, we deal with case III, otherwise $M$ is the direct sum of two isomorphic simple modules and then we deal with case IV. Since the lattices we are interested are submodule lattices (here of a module of length 2), we distinguish also in the case of a serial module $M$ of length 2, whether the two composition factors are isomorphic (case I) or not (case II).

Altogether, we see that for a lattice $S\text{Hom}(C,Y)$ of height 2, there are the following four cases:

```
  C1
 /|
C2 | C1
 /|
  C1
```

Type I: Example 11. Take the linearly oriented quiver of type $A_3$, say $1 \rightarrow 2 \rightarrow 3$. Let $Y = S(1)$ and $C = I(2) \oplus S(1)$. Then $\Gamma(C)$ is the path algebra of the quiver of type $A_2$ and $\text{Hom}(C,Y)$ is the indecomposable $\Gamma(C)$ module of length 2. The lattices $C[\rightarrow Y]$
and $\mathcal{S}\text{Hom}(C, Y)$ look as follows:

$$
\begin{array}{c}
Y = S(1) \\
I(2) \uparrow \\
I(3) \uparrow \\
\text{Hom}(C, Y) \\
f' \downarrow \\
0
\end{array}
$$

**Type II.** We have seen such examples for the Kronecker algebra, namely: if $C, Y$ are modules with $C$ indecomposable, such that $C(\to Y)$ is of the form $I(2)$, then we must be in type II.

Another example has been presented already in section 7 when dealing with Riedtmann-Zwara degenerations, namely the example 9. There, we have chosen (non-projective) indecomposable modules $C$ and $Y$ such that $\text{Hom}(C, Y)$ was a cyclic module of length 2. There, an additional module $Y$ was considered with an epimorphism $M' \to Y$ such that the composition $M \to M' \to Y$ is non-zero. Note that this procedure fits into the consideration of families $M = \{M_i \mid i \in I\}$ of modules which is based on dealing with fixed morphisms $f_i : M_i \to Y$ for some module $Y$.

**Type III. Example 12.** Take the quiver of type $A_3$ with source 2 and sinks 1 and 3. Here, on the left, is the quiver, on the right the Auslander-Reiten quiver:

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & \rightarrow & S(1) \\
& & P(2) \\
& & S(2) \\
& & I(1) \\
& & S(3) \\
& & I(3) \\
\end{array}
\]

Let $Y = S(2)$ and $C = I(1) \oplus I(3)$. As usual, let us show both $C(\to Y)$ as well as $\mathcal{S}\text{Hom}(C, Y)$:

\[
\begin{array}{c}
Y = S(2) \\
I(1) \uparrow \\
P(2) \downarrow \\
I(3) \downarrow \\
\text{Hom}(C, Y) \\
0
\end{array}
\]

**Example 13.** Here is a second example of type III. In contrast to the previous example, here the two incomparable right minimal maps $f : X \to Y$ and $f' : X' \to Y$ have the property that the modules $X$ and $X'$ are isomorphic. We consider the Kronecker algebra. Let $Q_0, Q_1, Q_2, \ldots$ be the indecomposable preinjective modules, with $\text{dim} Q_i = (i, i+1)$. Let $f : Q_1 \to Q_0$ have kernel $R$, and let $f : Q_1 \to Q_0$ have kernel $R'$, with $R, R'$
non-isomorphic regular modules. We consider the case that \( Y = Q_0 \), the simple injective module and \( C = R \oplus R' \).

\[ \begin{array}{c}
\begin{array}{c}
Q_0 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
Q_1 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
Q_2 \\

\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f \\

\end{array}
\begin{array}{c}
\begin{array}{c}
f' \\

\end{array}
\end{array}
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
\text{Hom}(C, Y) \\

\end{array}
\begin{array}{c}
\begin{array}{c}
0 \\

\end{array}
\end{array}
\end{array}
\]

**Type IV.** Let \( C, Y \) be a pair of modules such that \( C[\rightarrow Y] \) is of type IV. As we will see, in this is our main concern, the behaviour of the modules present in \( C[\rightarrow Y]1 \) may be quite different.

**Examples 14. The modules in** \( C[\rightarrow Y]1 \) **are all isomorphic.** We deal with the Kronecker algebra as considered in section 12. As we have seen in the previous section, here we find many examples. As before, let \( P_0, P_1, P_2, \ldots \) be the indecomposable preprojective modules, \( Q_0, Q_1, Q_2, \ldots \) the indecomposable preinjective modules, with both \( P_i \) and \( Q_i \) being of length \( 2i + 1 \).

First, let \( C = P_i \) and \( Y = P_{i+1} \), for some \( i \geq 0 \), thus \( \dim \text{Hom}(C, Y) = 2 \) and therefore \( S \text{Hom}(C, Y) \) is of the form IV.

If \( i = 0 \), then we deal with the lattice of submodules \( U \) of \( P_1 \) such that the socle of \( P_1/U \) is generated by \( P_0 \). Such a submodule is either 0 or simple (thus of the form \( P_0 \)) or equal to \( P_1 \):

\[ \begin{array}{c}
\begin{array}{c}
P_1 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
P_0 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
P_0 \\

\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
P_0 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
P_0 \\

\end{array}
\end{array}
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
0 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Hom}(C, Y) \\

\end{array}
\end{array}
\end{array}
\]

If \( i = 1 \), then we deal with the lattice of all submodules \( U \) of \( P_2 \) which contain the socle of \( P_2 \) (these are just the submodules \( U \) of \( P_2 \) such that the socle of \( P_2/U \) is generated by \( P_1 \)). Note that such a submodule is either the socle itself, thus isomorphic to \( P_0^3 \), or of the form \( N = P_0 \oplus P_1 \), or \( P_2 \) itself.

\[ \begin{array}{c}
\begin{array}{c}
P_2 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
N \\

\end{array}
\begin{array}{c}
\begin{array}{c}
N \\

\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
N \\

\end{array}
\begin{array}{c}
\begin{array}{c}
N \\

\end{array}
\end{array}
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
P_0^3 \\

\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Hom}(C, Y) \\

\end{array}
\end{array}
\end{array}
\]

with \( N = P_0 \oplus P_1 \)
Whereas for the cases \( i = 0, 1 \) all the maps shown in the lattice \( C[\rightarrow Y] \) are inclusion maps, the maps exhibited in \( C[\rightarrow Y] \) for \( i \geq 2 \) are all epimorphisms. So let us assume that \( i \geq 2 \) (see Proposition 7.6 and Lemma 7.8). A map \( f: X \rightarrow Y \) is right \( P_i \)-determined if and only if its kernel is a direct sum of copies of \( P_{i-2} \). Since \( \text{Ext}^1(P_{i+1}, P_{i-2}) = \text{DHom}(P_i, P_{i+1}) \) is a two-dimensional vector space, we see that we deal with short exact sequences of the form

\[
0 \rightarrow P_{i-2} \rightarrow N \rightarrow P_{i+1} \rightarrow 0 \\
0 \rightarrow P_{i-2}^2 \rightarrow M \rightarrow P_{i+1} \rightarrow 0
\]

such that the maps \( N \rightarrow P_{i+1} \) and \( M \rightarrow P_{i+1} \) are right minimal. It follows easily that all these modules \( N \) have to be of the form \( P_{i-1} \oplus P_i \), and \( M \) has to be of the form \( P_{i-1}^3 \). The lattice \( C[\rightarrow Y] \) looks as follows:

Let us describe the right minimal maps \( N = P_1 \oplus P_2 \rightarrow P_3 \) in detail. We have \( \dim \text{Hom}(P_1 \oplus P_2, P_3) = 3 \), but actually only the homomorphisms \( P_2 \rightarrow P_3 \) matter and \( \text{Hom}(P_2, P_3) = 2 \). Why only the homomorphisms \( P_2 \rightarrow P_3 \) matter? We need an epimorphism \( f: P_1 \oplus P_2 \rightarrow P_3 \), write it as \( f = [f_1, f_2] \) with \( f_1: P_1 \rightarrow P_3 \) and \( f_2: P_2 \rightarrow P_3 \). In order that \( f \) is an epimorphism, the following two conditions have to be satisfied:

1. The restriction \( f_2 \) of \( f \) to \( P_2 \) has to be non-zero.
2. The image \( f_1(P_1) \) is not contained in \( f_2(P_2) \), or, equivalently (since \( P_1 \) is projective) \( f_1 \) does not factor through \( f_2 \).

If two such maps \( [f_1, f_2] \) and \( [f_1', f_2'] \) are given with both \( f_2 \) and \( f_2' \) non-zero, and \( f_1(P_1) \nsubseteq f_2(P_2) \) as well as \( f_1'(P_1) \nsubseteq f_2'(P_2) \), then \( [f_1, f_2] \) is right equivalent to \( [f_1', f_2'] \) if and only if \( f_1 \) is right equivalent to \( f_2 \), if and only if there is a scalar \( c \in k^* \) such that \( f_2' = cf_1 \).

It follows that the existence of the one-parameter family of right minimal maps \( N \rightarrow P_3 \) comes from the fact that \( \dim \text{Hom}(P_2, P_3) = 2 \).

Observe that the lattices \( C[\rightarrow Y] \) for \( C = P_i, Y = P_{i+1} \) and all \( i \geq 0 \) have the same form, provided we set \( P_{-1} = 0 \).

**Examples 15.** The modules in \( C[\rightarrow Y] \) are pairwise non-isomorphic. Again, we deal with the Kronecker algebra. Let \( Y = Q_0 \), the simple injective module, and \( C = P_2 \) (thus \( \dim \text{Hom}(C, Y) = 2 \)). Then \( K = \tau C = P_0 \) is the simple projective module. The right minimal right \( C \)-determined morphisms ending in \( Y \) are epimorphisms with kernel
in add $K$. Here are the lattices in question:

\[ \begin{array}{c}
Q_0 \\
R \quad R' \quad R'' \quad \cdots \\
P_1
\end{array} \]

\[ \text{Hom}(C,Y) \]

\[ \mathcal{P}(C,Y) = 0 \]

with pairwise non-isomorphic indecomposable representations $R, R', R'', \ldots$ of length 2.

**Examples 16. The modules in $C[\to Y]^1$ belong to a finite number of isomorphism classes with a fixed dimension vector.** Take the 3-subspace quiver with sink 0 and sources 1, 2, 3.

\[ \begin{array}{c}
1 \\
2 \\
3 \\
\downarrow \\
0
\end{array} \]

Let $Y = I(0)$, and $C$ the maximal indecomposable module, thus $C$ has the dimension vector $1^1 1^1$. Note that $K = \tau C$ is the simple projective module $K = P(0) = S(0)$.

Then we deal with the non-distributive lattice of length 2; in the $\mathbb{P}_1$-family, there are three special elements:

In the frame $C[\to Y]$, this concerns the decomposable modules $M(i) = P(i) \oplus \tau - P(i)$ for $i = 1, 2, 3$; note that the remaining elements of the $\mathbb{P}_1$-family are the surjective maps $C \to Y$. Note that all the modules $M(1), M(2), M(3)$ as well as $C$ have dimension vector $1^1 1^1$.

In the submodule lattice $\mathcal{S}\text{Hom}(C,Y)$, it concerns the three subspaces generated by compositions of two irreducible maps $C \to \tau - P(i) \to Y$.

In the frame $C[\to Y]$, the zero element is the projective cover $P(Y) \to Y$. In the submodule lattice $\mathcal{S}\text{Hom}(C,Y)$, the zero element is $\mathcal{P}(C,Y) = 0$.

**Examples 17. The modules in $C[\to Y]^1$ form a family of modules with varying dimension vectors.** Again, we take the 3-subspace quiver with sink 0 and sources 1, 2, 3, but now let $C = P(0)$ and $Y$ the maximal indecomposable module, as we mentioned, its dimension vector is $1^1 1^1$. Since $C$ is the projective module $P(0)$, the right minimal right $C$-determined maps $f: X \to Y$ are the inclusions maps of submodules $X$ of $Y$ such that the socle of $Y/X$ is a direct sum of copies of $S(0)$. The indecomposable
submodules of $Y$ are $P(1), P(2), P(3)$ as well as many submodules isomorphic to $S(0)$. In order that the socle of $Y/X$ is a direct sum of copies of $S(0)$, either $X = Y$ (and $Y/X = 0$), or $X = 0$ (and $Y/X = Y$), or $X$ has to be indecomposable. If $X$ is an indecomposable proper submodule of $Y$, then the socle of $Y/X$ is either isomorphic to $S(0)$ or else $X$ is contained in one of the modules $P(1), P(2), P(3)$. It follows that the 1-parameter family in the middle of $\mathcal{C}[\rightarrow Y]$ consists of the indecomposable proper submodules $X$ of $Y$ such that the socle of $Y/X$ is isomorphic to $S(0)$, thus either $X$ is one of $P(1), P(2), P(3)$, or else $X$ is a simple submodule of $Y$ not contained in $P(1), P(2), P(3)$.

**Examples 18.** Another example where the modules in $\mathcal{C}[\rightarrow Y]^1$ form a family of modules with varying dimension vectors. Take the one-point extension of the Kronecker algebra using a regular module of length 2, say with vertices $1, 2, 3$ with two arrows $2 \to 1$ and one arrow $3 \to 2$ (and one relation). Let $R = R(\infty) = \text{rad} \ P(3)$.

Let $K = S(1)$ and $C = \tau^{-}K = P_2$ (the indecomposable module with dimension vector $(3, 2, 1)$). Let $Y = P(3)/S(1)$, thus its dimension vector is $(0, 1, 1)$. We have $\dim \hom(C,Y) = 2$. Since $\text{End}(C) = k$, the submodule lattice $\mathcal{S} \hom(C,Y)$ is just the projective line. Under $\eta_{SY}$ we obtain a $\mathbb{P}_1$-family of right minimal maps ending in $Y$, namely those with the following short exact sequences:

$$K \to P(3) \xrightarrow{p} Y$$
$$K \to R(\lambda) \xrightarrow{f_\lambda} S(2) \quad \text{for} \quad \lambda \in k$$

Here is, on the left, $\mathcal{C}[\rightarrow Y]$, as well as, on the right, $\mathcal{S} \hom(C,Y)$

The 0-subspace of $\hom(C,Y)$ corresponds to the projective presentation $g: P(2) \to P(3)$ of $S(3)$, thus to the short exact sequence

$$K^2 \to P(2) \xrightarrow{g'} S(2).$$
It should be noted that the short exact sequence

\[ K \to R(\infty) \xrightarrow{f_\infty} S(2) \]

yields the map \( R(\infty) \xrightarrow{f_\infty} S(2) \subset Y \) which we denote by \( f_\infty \) and which is not \( C \)-determined. Namely, there is the following commutative diagram

\[
\begin{array}{ccc}
\text{rad } P(3) & \longrightarrow & P(3) \\
\downarrow & \downarrow p & \\
R(\infty) & \xrightarrow{f_\infty} & Y
\end{array}
\]

We see that \( p \) almost factorizes through \( f_\infty \), thus the theory asserts that \( P(3) \) has to belong to any determiner of \( f_\infty \) and therefore \( f_\infty \) cannot belong to \( C[\to Y] \).

As we have noted, \( P(3) \) has to belong to any determiner of \( f_\infty \) and therefore \( f_\infty \) cannot belong to \( C[\to Y] \).

Let us add \( P(3) \) to \( C \) and consider the Auslander bijection for \( C \oplus P(3) \) and \( Y \). We have \( \dim \text{Hom}(C \oplus P(3), Y) = 3 \). Note that the endomorphism ring of \( C \oplus P(3) \) is hereditary of type \( A_2 \) and the submodule structure of \( \text{Hom}(C \oplus P(3), Y) \) looks as follows:

![Diagram](image)

For the proof, we only have to verify that a non-trivial map \( C \to P(3) \) does not annihilate the module \( \text{Hom}(C \oplus P(3), Y) \). The encircled vertex is the submodule \( \mathcal{P}(C \oplus P(3), Y) \).

The corresponding diagram in \( C \oplus P(3)[\to Y] \) looks as follows; here, we write its elements as short exact sequences ending in a submodule of \( Y \):

![Diagram](image)
We may label the lines of the Hasse diagram of $S\text{Hom}(C \oplus P(3))$ by the corresponding indecomposable direct summand, thus either by $C$ or by $P(3)$.

\begin{center}
\begin{tikzpicture}
  \node (C) at (0,0) {$C$};
  \node (P3) at (1,1.5) {$P(3)$};
  \node (C2) at (2,0) {$C$};
  \node (C3) at (3,0) {$\ldots$};

  \draw[->] (C) -- (P3);
  \draw[->] (P3) -- (C2);
  \draw[->] (C2) -- (C3);
\end{tikzpicture}
\end{center}

We should add that we obtain $S\text{Hom}(C,Y)$ from $S\text{Hom}(C \oplus P(3), Y)$ by deleting the shaded part:

\begin{center}
\begin{tikzpicture}
  \node (C) at (0,0) {$C$};
  \node (P3) at (1,1.5) {$P(3)$};
  \node (C2) at (2,0) {$C$};
  \node (C3) at (3,0) {$\ldots$};

  \draw[->,dashed] (C) -- (P3);
  \draw[->,dashed] (P3) -- (C2);
  \draw[->,dashed] (C2) -- (C3);
\end{tikzpicture}
\end{center}

\section*{Part III. Special cases.}

\textbf{14. The module $C$ being a generator.}

The special case $C = \Lambda$ has been discussed already at the end of section 5. In this case, $S\text{Hom}(C,Y)$ is just the Grassmannian of all submodules of $Y$.

\textbf{Proposition 14.1.} Let $C$ be a generator. Then $f: X \to Y$ is right $C$-determined if and only if the intrinsic kernel of $f$ is in $\text{add} \tau C$.

\textbf{Proof.} [...] Everyone admits that the concept of being determined is not very intuitive, however in the special case when $C$ is a generator (and this is often the only important case), one knows: The maps in $C[\to Y]$ can be described by the exact sequences

$$0 \to K \to X \to Y' \to 0$$

where $K$ is in $\text{add} \tau C$ and $m: Y' \to Y$ is an inclusion map (thus $Y'$ is just a submodule of $Y$): here, the map in $C[\to Y]$ in question is the composition $me$.

This means: for $C$ a generator, the set $C[\to Y]$ can be visualized very well. Unfortunately, in general not, but the notion of determination just allows to have the bijection $C[\to Y] \leftrightarrow S\text{Hom}(C,Y)$.

To repeat: If $C$ is a generator, then $C[\to Y]$ is easy to describe: \textit{These are just the right-minimal maps to $Y$ (not necessarily surjective) with kernel in $\text{add} \tau C$}. 

50
More about using generators. Let $C$ be a generator and $Z' \subseteq Z$. Then

\[
C \xrightarrow{\to} Z \leftrightarrow S \text{Hom}(C, Z)
\]

where the vertical maps are inclusion maps (and on the right, $S \text{Hom}(C, Z')$ is a lattice ideal in $S \text{Hom}(C, Z)$).

In order to have inclusion maps, we need that $C$ is a generator: [...] 

For $C$ a generator, the submodule 0 of $\text{Hom}(C, Z)$ just corresponds to the map $0 \to Z$ 

\[
\eta_{CZ}(0 \to Z) = 0.
\]

On the other hand, if $P(\text{soc } Z)$ is not in $\text{add } C$, then (and only then) the submodule 0 of $\text{Hom}(C, Z)$ yields a non-trivial element in $\to Z$, namely an exact sequence $\ast \to \ast \to Z'$ with $Z'$ the largest submodule of $Z$ which belongs to the Serre subcategory generated by all simple modules $S$ with $P(S)$ not in $\text{add } C$. (and the sequence itself is a universal extension from below...)

15. The case of $\to Y$ being of finite height.

Assume that there are only finitely many indecomposable modules $X_i$ with $\text{Hom}(X_i, Y) \neq 0$, let $A$ be the direct sum of these modules and consider the Auslander bijection

\[
A \xrightarrow{\to} Y \leftrightarrow S \text{Hom}(A, Y),
\]

it maps $\bigoplus_{i,j} X_i^n \xrightarrow{(f_{ij})} Y$ to the $\Gamma(A)$-submodule of $\text{Hom}(A, Y)$ generated by the maps $f_{ij}$ (considered as maps $A \to X_i \to Y$ where the map $A \to X_i$ is the canonical projection onto the direct summand).

Proposition 15.1. Let $Y$ be module such that $\to Y$ is of height $h$.

(a) The number of isomorphism classes of indecomposable modules $X$ with $\text{Hom}(X, Y) \neq 0$ is at most $h$.

(b) If $X_1, \ldots, X_t$ are all the indecomposable modules $X_i$ with $\text{Hom}(X_i, Y) \neq 0$, one from each isomorphism class, and $X = \bigoplus_i X_i$, then $\to Y = X \to Y$.

(c) If $\to Y = C \to Y$ for some module $C$, then $X \in \text{add } C$.

Proof. Let $X_1, \ldots, X_t$ be pairwise non-isomorphic indecomposable modules with $\text{Hom}(X_i, Y) \neq 0$, and let $X = \bigoplus_i X_i$. Let $\Gamma(X) = \text{End}(X)^{\text{op}}$ and $M = \text{Hom}(X, Y)$. Then the indecomposable projective $\Gamma(X)$-modules $P(i) = \text{Hom}(X, X_i)$ are pairwise non-isomorphic, and $\text{Hom}_{\Gamma(X)}(P(i), M) \neq 0$, thus the $\Gamma(X)$-module $M$ has length at least $t$. Now, according to the Auslander bijection, the poset $SM$ is isomorphic to the subposet

51
$X \to Y$ of $\to Y$. The length of $SM$ is at least $t$, the length of $\to Y$ is $h$. This shows that $t \leq h$. This shows (a).

Next, we show that there is some module $C$ such that $\to Y = \overline{C[\to Y]}$. Take a module $C$ such that $\overline{C[\to Y]}$ is maximal. Assume there is $f : X \to Y$ such that $[f]$ does not belong to $C[\to Y]$. Now $f$ is right determined by some module $C'$, thus $[f]$ belongs to $C' \oplus C[\to Y]$. Since also $C[\to Y] \subset C' \oplus C[\to Y]$, we get a contradiction to the maximality of $C[\to Y]$.

Now assume that $\to Y = \overline{C[\to Y]}$ for some module $C$. If $C = C' \oplus C''$ with $\operatorname{Hom}(C'', Y) = 0$, then we have $\to Y = \overline{C'[\to Y]}$. This shows that we can assume that any indecomposable direct summand $C_i$ of $C$ satisfies $\operatorname{Hom}(C_i, Y) \neq 0$. Thus we can assume that $C$ is a direct summand of $X$. But if $C$ is a proper direct summand of $X$, say $C = \bigoplus_{i=1}^{t'} X_i$, then

$$|[\to Y]| = |\overline{C[\to Y]}| = |\operatorname{Hom}(C, Y)|$$
$$= \sum_{i=1}^{t'} |\operatorname{Hom}(X_i, X)| < \sum_{i=1}^{t} |\operatorname{Hom}(X_i, X)|$$
$$= |\operatorname{Hom}(X, Y)| = |X[\to Y]| \leq ||\to Y||$$

a contradiction. This shows that $C = X$.

Remark. As we see, the indecomposable modules $X_i$ which occur as direct summands of a minimal module $C$ with $\to Y = \overline{C[\to Y]}$ are modules with $\operatorname{Hom}(X_i, Y) \neq 0$. But this is not surprising, since the minimal determiner $C(f)$ of any morphism $f$ has as indecomposable direct summands only modules $C_i$ with $\operatorname{Hom}(C_i, Y) \neq 0$ (namely modules which almost factor through $f$). May be the converse should be stressed, namely the assertion (c): any indecomposable module $X_i$ with $\operatorname{Hom}(X_i, Y) \neq 0$ is needed as a direct summand of $C$.

**Corollary 15.2.** Let $Y$ be a module. The following conditions are equivalent:

(i) There are only finitely many isomorphism classes of indecomposable modules $X$ with $\operatorname{Hom}(X, Y) \neq 0$.

(ii) The lattice $\to Y$ is of finite length.

(iii) There is a module $C$ with $\to Y = \overline{C[\to Y]}$.

If $S$ is simple and $\to I(S)$ is of finite height, then this lattice $\to I(S)$ is called a hammock.

**Example 19.** The lattices $\to Y$ and $S \operatorname{Hom}(\to Y)$ for $Y = I(0)$, where we consider
the 3-subspace quiver with 0 the sink.

\[
\begin{array}{c}
\vdots \\
M \\
N(3) \\
M \\
I(0) \\
M \\
N(2) \\
M \\
N(1) \\
M \\
M \\
P(3) \\
P(2) \\
P(1) \\
P(0) \\
0
\end{array}
\]

The module Hom(A, Y) is of length 1 with 9 different composition factors, they are labeled by the indecomposables

\[P(0); P(1), P(2), P(3); M; N(1), N(2), N(3); I(0)\]

in the hammock for the simple module S(0) (here, \(N(i) = \tau^{-1}P(i)\)). The composition factor labeled M occurs with multiplicity 2.

Only join irreducible elements (as well as the zero element) have been labeled. Note that in the middle layer of the non-distributive interval of length 2, all but three elements are join irreducible, and have the label M; here we deal with the various epimorphisms M → I(0).

Altogether there are 18 non-zero elements which are not join-irreducible.

Note that the picture is obtained from the free modular lattice in 3 generators as presented by Dedekind in 1900 by adding in the non-distributive interval of length 2 further diagonals (one may call it the free \(k\)-modular lattice in 3 generators).

Here is the hammock (with arrows pointing upwards), on the right there is the ham-
mock function:

\[
\begin{array}{ccc}
I(0) & & 1 \\
N(1) & N(2) & N(3) \\
M & & 2 \\
P(1) & P(2) & P(3) \\
P(0) & & 1
\end{array}
\]

We may use the Auslander bijection in order to focus the attention to the non-distributive part: the non-distributive interval of length 2. Here it is:

\[
\begin{array}{ccc}
M[\to M\oplus M] & & M\oplus M \\
M\oplus P(3) & \cdots & \cdots & M & M\oplus P(1) \\
& & & & & P(1)\oplus P(2)\oplus P(3)
\end{array}
\]

16. Some serial modules $\text{Hom}(C, Y)$.

The Auslander bijections are defined for any pair of $\Lambda$-modules $C, Y$ and one of the posets involved is $\mathcal{S}\text{Hom}(C, Y)$. Assume that $I$ is an ideal of $\Lambda$ which annihilates both modules $C, Y$ so that we may consider $C$ and $Y$ as $\Lambda'$-modules, with $\Lambda' = \Lambda/I$. On the one hand, we have $\text{Hom}_\Lambda(C, Y) = \text{Hom}_{\Lambda'}(C, Y)$. On the other hand, we have to distinguish the set $C[\to Y]_{\Lambda'}$ of right equivalence classes of right $C$-determined $\Lambda'$-modules ending in $Y$ from $C[\to Y]_{\Lambda} = C[\to Y]$. Using the Auslander bijections for $\Lambda$ as well as for $\Lambda'$, it is clear that the posets $C[\to Y]_{\Lambda'}$ and $C[\to Y]_{\Lambda}$ are isomorphic, however the modules present in $C[\to Y]_{\Lambda'}$ usually will be completely different from those present in $C[\to Y]_{\Lambda}$. The following examples will show such deviations.

Let $\Lambda$ be a uniserial ring. The indecomposable module of length $n$ will be denoted just be $n$.  

54
Example 20. Let \( C = Y = 4 \). First, let \( \Lambda \) be of length at least 8, so that \( \mathcal{P}(4, 4) = 0 \).

Second, let \( \Lambda \) be of length 6:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & 4 & = & 4 & \quad \text{Hom}(C, C) \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 5 \oplus 3 & \longrightarrow & 4 & \quad \text{rad}(C, C) \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 6 \oplus 2 & \longrightarrow & 4 & \quad \text{rad}(C, C)^2 \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 7 \oplus 1 & \longrightarrow & 4 & \quad \text{rad}(C, C)^3 \\
\downarrow & & \downarrow [0] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{m} & 8 & \longrightarrow & e & \quad 0 = \mathcal{P}(C, C) \\
\end{array}
\]

here, we have denoted by \( m \) the canonical inclusion maps, by \( e \) the canonical projections and \( \phi \) is a radical generator \( \text{End}(C) \). The Auslander-Reiten sequence is marked as (AR).

Second, let \( \Lambda \) be of length 4:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & 4 & = & 4 & \quad \text{Hom}(C, C) \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 5 \oplus 3 & \longrightarrow & 4 & \quad \text{rad}(C, C) \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 6 \oplus 2 & \longrightarrow & 4 & \quad \text{rad}(C, C)^2 = \mathcal{P}(C, C) \\
\downarrow & & \downarrow [e \ m] & & \downarrow [e \ m] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{\phi} & 6 \oplus 1 & \longrightarrow & 3 & \quad \text{rad}(C, C)^3 \\
\downarrow & & \downarrow [1] & & \downarrow \text{(AR)} \\
4 & \xrightarrow{m} & 6 & \longrightarrow & e & \quad 0 \\
\end{array}
\]

again, \( m \) stands for a canonical inclusion map, \( e \) for a canonical projection map and \( \phi \) for a radical generator of the endomorphism ring of a uniserial module.

Finally, let \( \Lambda \) be of length 4, thus \( C \) is projective and all the right minimal, right
Example 21. Let $C = 1 \oplus 2$ and $Y = 3$. The ring $\Gamma(C) = \text{End}(C)^{\text{op}}$ is the Nakayama algebra with Kupisch series 2, 3. The $\Gamma(C)$-module $\text{Hom}(C, Y)$ is the indecomposable projective module of length 3.

If we work over a uniserial ring $\Lambda$ of length at least 5, so that $\mathcal{P}(C, Y) = 0$, then the situation is as follows:
Next, assume that $\Lambda$ is of length 4.

Finally, we consider the case where $\Lambda$ is of length 3.

17. Final remarks.

17.1. Duality. For all the results presented here, there is a dual version which has the same importance. Note that we deal with an artin algebra $\Lambda$ and consider only finitely generated modules, thus there is a duality functor $\text{Hom}(\cdot, I)$. Namely, by assumption, $\Lambda$ is a module-finite $k$-algebra, where $k$ is a commutative artinian ring, thus let $I$ be a minimal injective cogenerator for $\text{mod} \, k$.

We did not attempt to formulate the dual definitions and statements, but the reader is advised to do so. To start with, let us denote by $\langle Y \leftarrow \rangle$ the left equivalence classes of maps starting in $Y$ (or of left minimal maps), and by $\langle Y \rightarrow \rangle^C$ the left equivalence classes of those maps starting in $Y$ which are left $C$-determined.

17.2. Proofs of Auslander’s two Main Theorems. Of course, both results are presented in detail in the Philadelphia notes [A1], see also [A2]. There is a different treatment in the book [ARS] of Auslander-Reiten-Smalø: Auslander’s Second Theorem is presented in Theorem XI.3.9., for the First Theorem, see Theorem XI.2.10 and Proposition
XI.2.4 (this actually improves the assertion given in the Philadelphia Notes. A concise proof of the First Theorem can also be found in [R7]. As one of the main ingredients for the proof of the Second Theorem, one may use Auslander’s defect formula. For a direct approach to the defect formula, we recommend the paper [K] by Krause.

17.3. The universal character of the Auslander bijections. The Auslander bijections are Auslander’s approach to describe say the module category for an artin algebra completely: not just to provide some invariants. The importance of using invariants usually relies on the fact that they will allow to distinguish certain objects, but they may not say much about other ones — the most effective invariants are often those which attach to objects just one of the values 1 or 0 (thus “yes” or “no”), of course, the use of such an invariant is then restricted to some specific problem. So, if Auslander’s approach wants to describe a module category completely, we may ask whether it does not have to be tautological: if we don’t forget some of the structure, we will not see the remaining structure in more detail. Now the Auslander approach indeed is also the focus on parts of the category, namely the sets $C \rightarrow Y$, but we can change the focus by enlarging $C$ (adding direct summands). The universality of this approach is due to the fact that the category is covered completely by such subsets.

17.4. The irritation of the wording “morphisms determined by modules”. Auslander asserts that every morphism in $\text{mod } \Lambda$ is right determined by a $\Lambda$-module, thus one expects a classification of the (right minimal) morphisms in $\text{mod } \Lambda$ using as invariants just $\Lambda$-modules. One even may strengthen Auslander’s assertion by saying every morphism in $\text{mod } \Lambda$ is right determined by the isomorphism class of a multiplicity-free $\Lambda$-module. Clearly, such a formulation is irritating, since the set of isomorphism classes of multiplicity-free modules may outnumber the set of right equivalence classes of right minimal morphisms by far: Consider just the special case of a representation finite artin algebra of uncountable cardinality, then there are only finitely many isomorphism classes of multiplicity-free modules, but usually uncountably many right equivalence classes of right minimal morphisms. So how should it be possible that finitely many modules $C$ determine uncountably many morphisms $\alpha$? The solution is rather simple: It is not just the module $C$ which is needed to recover a morphism $\alpha$: $X \rightarrow Y$ but one actually needs a submodule of $\text{Hom}(C,Y)$, with $\text{Hom}(C,Y)$ being considered as an $\text{End}(C)^{\text{op}}$-module. In the setting where $\Lambda$ is representation-finite and uncountable, one should be aware that usually the modules $\text{Hom}(C,Y)$ will have uncountably many submodules, thus we are no longer in trouble. So if we assert that the morphism $\alpha$: $X \rightarrow Y$ is determined by the module $C$, then one should keep in mind that $C$ is only part of the data which are required to recover $\alpha$; in addition to $C$ one will need a submodule of $\text{Hom}(C,Y)$.

17.5. Modules versus morphisms, again. The following feature seems to be of interest: The concept of the determination of morphisms by modules concerns the category of maps with fixed target $Y$, namely one wants to decide whether two elements in $C \rightarrow Y$ are comparable. The theory asserts that there is a test set of modules, namely the indecomposable direct summands of $C(f)$. For the testing procedure, they are just modules, but any such object $L$ comes equipped with a non-zero (thus right minimal) map $L \rightarrow Y$.

17.6. Logic and category theory. Let us stress that the setting of the Auslander
The power of a prime number, thus if $G$ is a field with $\Lambda \rightarrow [\Lambda \rightarrow]$. In this way, all the results concerning pp-formulae and pp-definable subgroups concern the Auslander setting.

In the terminology of abstract category theory, we deal with a comma category, namely the category of objects over $Y$: its objects are the maps $X \rightarrow Y$, a map from $f: X \rightarrow Y$ to $f': X' \rightarrow Y$ being given by a map $h: X \rightarrow X'$ such that $f = f'h$.

Of course, the use of the representable functors $\text{Hom}(X, -)$ and $\text{Hom}(-, Y)$ is standard in representation theory. To provide here all relevant references would overload our presentation due to the abundance of such papers. So we restrict to mention only few names: of course Auslander himself, but also Gabriel [G] as well as Gelfand-Ponomarev [GP].

17.7. The relevance of submodule lattices in representation theory. [..]

17.8. The operation of the automorphism group of $Y$ on $\mathcal{S}\text{Hom}(C, Y)$. [..]

18. Appendix: Modular lattices.

Let $L$ be a modular lattice of height $h$. The set of elements in $L$ of height $i$ will be denoted by $L_i$ or also $L^{h+1-i}$.

Here are some typical modular lattices:

The chain $\mathbb{I}(h)$ of height $h$, this is the poset of the integers $0, 1 \ldots, h$ with the usual order relation. Note that $\mathbb{I}(h)_i$ consists of a single element, for $0 \leq i \leq h$. Here are the lattices $\mathbb{I}(i)$ with $0 \leq i \leq 4$.

\[
\begin{array}{c}
\mathbb{I}(0) \\
\mathbb{I}(1) \\
\mathbb{I}(2) \\
\mathbb{I}(3) \\
\mathbb{I}(4)
\end{array}
\]

The projective geometry $\mathbb{G}(d) = \mathbb{G}_F(d)$ over the field $F$, this is the lattice of all subspaces of a $(d+1)$-dimensional vector space over $F$, its height is $d$. Here are lattices $\mathbb{G}(d)$ for $0 \leq d \leq 4$:

\[
\begin{array}{c}
\mathbb{G}(0) \\
\mathbb{G}(1) \\
\mathbb{G}(2) \\
\mathbb{G}(3) \\
\mathbb{G}(4)
\end{array}
\]

Of course, for $d \geq 2$, the number of elements of $\mathbb{G}(d)$ depends on the cardinality of $F$. If $F$ is a field with $q$ elements, the cardinality of $\mathbb{G}(2)_1 = \mathbb{G}(2)^1$ is $q + 1$ and $q$ is an arbitrary power of a prime number, thus if $\mathbb{G}(2)$ is finite, then $|\mathbb{G}(2)_1| = 3, 4, 5, 6, 8, 9, 10, 12, 14, \ldots$. 

59
Note that the subsets $\mathbb{G}(d)_i$ are the Grassmannians $\mathbb{G}(i, d - i)$ (this is the set of subspaces of dimension $i$ of a $k$-space of dimension $d$); in particular, the subset $\mathbb{G}(d)_1$ and $\mathbb{G}(d)^1$ are projective spaces $\mathbb{P}^{d-1}$ of dimension $d - 1$:

\[
\mathbb{G}(d)_{d-1} = \mathbb{G}(d)^1 = \mathbb{P}^{d-1}
\]
\[
\mathbb{G}(d)_i = \mathbb{G}(d)^{d-i} = \mathbb{G}(i, d - i)
\]
\[
\mathbb{G}(d)_1 = \mathbb{G}(d)^{d-1} = \mathbb{P}^{d-1}
\]

19. References.

[A1] Auslander, M.: Functors and morphisms determined by objects. In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker, New York (1978), 1-244. Also in: Selected Works of Maurice Auslander, Amer. Math. Soc. (1999).

[A2] Auslander, M.: Applications of morphisms determined by objects. In: Representation Theory of Algebras. Lecture Notes in Pure Appl. Math. 37. Marcel Dekker, New York (1978), 245-327. Also in: Selected Works of Maurice Auslander, Amer. Math. Soc. (1999).

[ARS] Auslander, M., Reiten, I., Smalø, S.: Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics 36. Cambridge University Press. 1997.

[BGP] Bernstein, I.N., Gelfand, I.M, Ponomarev, V.A.: Coxeter functors and Gabriel’s theorem. Uspekhi Mat. Nauk 28(1973), Russian Math. Surveys 28 (1973), 17-32.

[G] Gabriel, P.: Representations indecomposables. Sem. Bourbaki (1973-74). LNM 431, Springer 1975, 143-160.

[GP] Gelfand, I.M., Ponomarev, V.A.: Indecomposable representations of the Lorentz group, Russian Math. Surveys 23 (1968), 1.58.

[Kt] Kraft, H.: Geometric methods in representation theory, In: Representations of Algebras, Lecture Notes in Math. 944, Springer-Verlag, New York (1980), 180.257.

[K1] Krause, H.: A short proof for Auslander’s defect formula. Linear Algebra and its Applications. 365 (2003), 267 – 270.

[K2] Krause, H.: Morphisms determined by objects in triangulated categories. arXiv:1110.5625.

[M] MacLane, S.: Categories for the Working Mathematician. Springer, 1998

[P] Prest, M.: Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.

[Re] Reineke, M.: Every projective variety is a quiver Grassmannian arXiv:1204.5730.

[R1] Ringel, C. M.: Report on the Brauer-Thrall conjectures. Proceedings ICRA 2. Springer LNM 831 (1980), 104-136.

[R2] Ringel, C. M.: The rational invariants of tame quivers. Invent. Math. 58 (1980), 217-239.

[R3] C.M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer-Verlag, New York (1984), xiii+376pp.

[R4] Ringel, C. M.: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R5] Ringel, C. M.: The ladder construction of Prüfer modules. Revista de la Union Matematica Argentina. (2007) Vol 48-2, p.47-65. volume 48

60
[R6] Ringel, C. M.: The first Brauer-Thrall conjecture. In: Models, Modules and Abelian Groups. In Memory of A. L. S. Corner. Walter de Gruyter, Berlin (edt: B. Goldsmith, R. Gobel) (2008), 369-374.

[R7] Ringel, C. M.: Morphisms determined by objects: The case of modules over artin algebras. Illinois Journal (to appear).

[R8] Ringel, C. M.: Kernel-determined morphisms. (to appear)

[RV] Ringel, C. M., Vossieck, D.: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.

[Ro] Roiter, A. V.: Unboundedness of the dimensions of the indecomposable representations of an algebra which has infinitely many indecomposable representations, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 1275.1282.

[S] Smalø, S.: The inductive step of the second Brauer-Thrall conjecture, Canad. J. Math. 32 (1980), no. 2, 342.349.