Absorption properties of stochastic equations with Hölder diffusion coefficients

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In this article, we address the absorption properties of a class of stochastic differential equations around singular points where both the drift and diffusion functions vanish. According to the Hölder coefficient $\alpha$ of the diffusion function around the singular point, we identify different regimes. Stability of the absorbing state, large deviations for the absorption time, existence of stationary or quasi-stationary distributions are discussed. In particular, we show that quasi-stationary distributions only exist for $\alpha < 3/4$, and for $\alpha \in (3/4, 1)$, no quasi-stationary distribution is found and numerical simulations tend to show that the process conditioned on not being absorbed initiates an almost sure exponential convergence towards the absorbing state (as is demonstrated to be true for $\alpha = 1$). Applications of these results to stochastic bifurcations and a type of population model are discussed.

Contents

1 Introduction 2

2 General Theory 4
   2.1 Existence and uniqueness of solutions 5
   2.2 Stability of $\delta_0$ 6
   2.3 Absorption time 8
      2.3.1 Case $\alpha < 1$ 8
      2.3.2 For $\alpha \geq 1$ the solution a.s. never reaches 0 12
   2.4 Stationary and quasi-stationary solutions 13
   2.5 Heuristic discussion 17

3 Applications 18
   3.1 Analysis of the stochastic pitchfork bifurcation with Hölder diffusion 18
   3.2 The stochastic saddle-node bifurcation 21

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1 Introduction

A large class of phenomena can be described using non-linear dynamical, in such distinct domains as economics, physics or biology. Investigating the impact of noisy perturbations on such systems is of great interest, and is currently an active field of research. In the last decades, it has been shown in many different areas that applying noise on a non-linear system can lead to many counter-intuitive phenomena, such as noise-induced stabilization [2], oscillations [23] or transport [22] [10] and stochastic resonance [13] [24]. From a mathematical perspective, understanding the interplay between non-linearities and noisy perturbations is a great challenge, with many applications. Several tools have been introduced, ranging from Lyapunov functionals, stationary solutions of Fokker-Planck equations and Feller scale functions (see e.g. [15] [6] for an extensive account on these concepts), the theory of random dynamical systems (RDS) [1] [8], the study of moment equations [27] [28] or multiscale stochastic methods for slow-fast systems [3] [4] [21].

We focus here on the dynamics of stochastic differential equations (SDE) in which both the drift and the diffusion functions vanish at some particular absorbing points, also called singular points. Such SDEs arise in several applications including diffusion approximations of population models in ecology [5] [18], neurosciences [20], and mathematical finance [11] [7]. Beyond the classical multiplicative noise case (geometric Brownian motion), a number of state-dependent diffusion functions arise involving generally power functions. The question we address is how the interplay between the drift and the shape of the diffusion function affects the behavior of the system. In particular, such systems present an absorbing state, whose qualitative properties will be shown to tightly depend on the local behavior of the drift and diffusion functions around this equilibrium. Moreover, the choice of a vanishing diffusion coefficient provides a framework to study subtle competitions between the drift and the noise, leading to a rich and generic phenomenology.

In detail, we analyze a general one dimensional stochastic differential equation

$$dx_t = f(x_t) \, dt + \gamma(x_t) \, dW_t$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the natural filtration $(\mathcal{F}_t)$ of $W$. We assume that there exists a singular point $x^*$ such that $f(x^*) = \gamma(x^*) = 0$. The system has a trivial solution distributed as a Dirac measure localized at the singular point $x^*$, the absorbing state (or absorbing boundary). We shall investigate the stochastic stability of this absorbing state, probabilities of absorption and the behavior of the system prior from absorption (or conditioned on not hitting the absorbing boundary). In contrast to the case of ODEs, there are at least three different notions of stochastic stability:

1. The almost sure exponential stability defined by the property

$$\limsup_{t \to \infty} \frac{1}{t} \log |x_t - x^*| < 0 \quad a.s.$$
for any initial condition $x_0 \in \mathbb{R}$. The *almost sure exponentially instability* defined by the property

$$\liminf_{t \to \infty} \frac{1}{t} \log |x_t - x^*| > 0 \text{ a.s.}$$

for any initial condition $x_0 \in \mathbb{R}$ (see e.g. [19]). This definition implies a strong notion of stability close to the usual exponential stability property of deterministic dynamical systems: almost all trajectories exponentially fast converge to or diverge from $x^*$. In our stochastic context, this notion can appear quite restrictive, and we will hence also discuss a weaker notion of stability:

2. The *stochastic stability* (or stability in probability), defined by the property that for all $\mu \in (0, 1)$ and $r > 0$ there exists $\delta$ depending on $\mu$ and $r$ such that

$$\mathbb{P}(\{|x_t - x^*| < r \ \forall t \geq 0\}) \geq 1 - \mu$$

whenever $|x_0 - x^*| < \delta$ and otherwise it is said to be unstable in probability. The solution $x^*$ is *stochastically asymptotically stable* (or asymptotically stable in probability) if it is stable in probability and for every $\mu \in (0, 1)$ there exists $\eta_0$ depending on $\mu$ such that

$$\mathbb{P}(\lim_{t \to \infty} |x_t - x^*| = 0) \geq 1 - \mu$$

whenever $|x_0 - x^*| < \eta_0$.

3. The *stability* (resp. *instability*) in the first approximation of a stationary solution $\bar{X}$ ([15, Chap. 7.1], notion widely used in the RDS theory [1], consists in that the identically zero solution of the linearized equation around $\bar{X}$ is almost surely exponentially stable (resp. unstable). This criterion will be chiefly used when dealing with non-Dirac stationary solutions in the applications section.

To be more specific, we now consider that $x^* = 0$ and assume that $f(x)$ is a smooth function, which is often the case in applications and a basis of the deterministic bifurcation theory. In detail, we assume that $f(x) = \lambda x + g(x)$ where $g(x)/x \to 0$ when $x \to 0$ (i.e. $f'(0) = \lambda$), and hence consider the following class of SDEs on $\mathbb{R}$:

$$dx_t = (\lambda x_t + g(x_t)) \, dt + \gamma(x_t) \, dW_t$$  \hspace{1cm} (1)

with $\gamma(x) \sim \sigma|x|^{\alpha}$ at $x = 0$. The properties of the absorbing state will be described as a function of the local behavior of $g$ and $\gamma$ around 0.

The behavior of the solutions to such one-dimensional singular SDEs has been intensively studied. In particular, the bifurcations of such systems with smooth diffusion functions was analyzed in the context of RDS theory in [8, 1]. These methods were developed in the context of Stratonovich SDEs (in which case the diffusion function appears in the effective drift in the Itô formulation).
The theory did not address cases where the diffusion is only Hölder continuous at the singular point. In another context, mainly using stochastic calculus theory [26], existence, uniqueness and type of singular points was addressed in [6], and the study includes the case of power drift and diffusion functions.

In the present article, we shall consider both stability and absorption properties for a general class of singular SDEs, and aim at characterizing precisely the absorption time and global behavior of the solutions. New mathematical results proved here include almost sure absorption at the singular points and large deviations estimates of the absorption time, characterizations of possible stationary distributions, existence or non-existence of quasi-stationary distributions, and dependence of the stability of singular points on the local Hölder coefficient of the diffusion function. The methodology is also extended to the case of the generic saddle-node bifurcation for which we completely describe the behavior as a function of drift and diffusion parameters.

The paper is organized as follows. We start by analyzing in detail in section 2 the behavior of the solution of a general diffusion equation (1). We investigate the existence, uniqueness and blow up of the solutions. The stochastic stability of the singular point is characterized and the absorption time at the singular point is shown to be almost surely finite if and only if $\alpha < 1$. In that case, we show that quasi-stationary distributions exist when $\alpha < \frac{3}{4}$. In the case $\alpha \geq 1$, we study the existence of other stationary distributions. Application and extensions to the stochastic pitchfork and saddle-node bifurcations is discussed in section 3, and universality properties of these behaviors is characterized.

2 General Theory

In this section we analyze the behavior of the solutions of a general equation (1) under a few assumptions on the function $g$ and $\gamma$:

(H1). $g(x) \sim \mu x^{1+\kappa}$ at $x = 0$, for some $\mu \in \mathbb{R}$ and $\kappa > 0$

(H2). $g(x) \leq -\nu x^{1+\beta}$ when $|x| \geq A$ for some $A > 0$, for $\nu \in \mathbb{R}$ (generally assumed non-negative) and $\beta > 0$.

(H3). $\gamma(x) \sim \sigma x^\alpha$ at $x = 0$, for some $\sigma \geq 0$ and $\alpha > 0$

(H4). $\gamma(x) \sim dx^\delta$ at $x = \pm \infty$ with $d > 0$ and $\delta \geq 0$.

(H5). $\gamma(x)$ only vanishes at $x = 0$

(H6). Either (i) $\nu \geq 0$ or (ii) $\nu < 0$ and $\delta > 1 + \beta/2$.

Remark. In the above notations, for $x < 0$, $x^\alpha$ shall be understood as $-|x|^\alpha$.

Assumptions (H1) and (H3) describe the behavior of the solutions close from zero and will govern the stability of the singular point 0. The assumptions (H2) (H4) and (H5) are global properties that will be chiefly used for explosion matters, and existence of stationary distributions. Such assumptions
are generally not necessary in the deterministic case: indeed, in the local unfolding of a bifurcation, the presence of a non-zero fixed point implies the local existence of a manifold of fixed points under some non-degeneracy conditions. However, in the stochastic case, as soon as the singular point (where noise vanishes) is unstable, the system will visit very regions very remote from the singular point with positive probability, and these latter conditions will come into play.

The stochastic process distributed as a Dirac at zero \( X_t = 0 \text{ a.s.} \) is solution of the equation whatever the parameters. Studying the stationary solutions and their stability will make essential use of the differential operator \( \mathcal{L} \) associated with Itô’s representation of the diffusion acting on twice differentiable functions \( V \in C^2(\mathbb{R}, \mathbb{R}) \):

\[
\mathcal{L}V(x) = (\lambda x + g(x)) V'(x) + \frac{1}{2} \gamma(x)^2 V''(x).
\]

2.1 Existence and uniqueness of solutions

The first question arising with equation (1) is its well-posedness. Indeed, this equation does not satisfy the standard conditions for existence and uniqueness of solutions: the drift function is neither globally Lipschitz-continuous, nor holds a linear growth condition. Moreover, the diffusion coefficient is also singular: for \( \alpha < 1 \), the diffusion coefficient is not Lipschitz continuous at 0, and for \( \alpha > 1 \) it does not satisfies the linear growth condition generally imposed for the existence and uniqueness. Based on existing results [26], we prove the following property:

**Proposition 1.** For any \( \alpha \in \left[ \frac{1}{2}, \infty \right) \), there exists a unique strong solution to the equation (1). Under the assumptions \((H2), (H4)\) and \((H6)\), the solution never blows up in finite time.

**Proof.** The proof uses classical results of stochastic analysis that can be found in [26]. We provide a short description of the proof for the sake of completeness.

For \( \alpha \geq 1 \) is very classical. First, both the drift and diffusion functions are locally Lipschitz continuous, implying strong uniqueness of solutions (see e.g. [26, Chap. IX.2]). Second, the drift and the diffusion functions are locally bounded, ensuring the existence of solutions up to an explosion time.

For \( \alpha < 1 \), the linear growth condition is clearly satisfied, but the diffusion coefficient is not Lipschitz at 0. Existence and uniqueness was proved in [26, theorem IX.3.5] based on the properties of the local time at zero of the diffusion for \( \alpha \geq \frac{1}{2} \) and non-uniqueness of solutions for \( \alpha < \frac{1}{2} \).

Explosion properties are investigate through Feller’s test (see [16, Proposition 5.22.]). The function \( \frac{\lambda x + g(x) + 1}{\gamma(x)^2} \) is locally integrable on both the intervals \((\infty, 0)\) and \((0, \infty)\). We consider the interval \((0, \infty)\) for instance (the same principle applies for the interval \((\infty, 0)\)), and for some \( c > 0 \), we define the scale function:

\[
p(x) := \int_c^x \exp \left(-2 \int_c^y \frac{\lambda \xi + g(\xi)}{\gamma^2(\xi)} d\xi \right) dy.
\]
The process almost surely never diverges before possibly reaching zero as soon as $p(x)$ diverges at $\infty$. At infinity, we have:

$$-2 \int_c^y \frac{\lambda_\xi + g(\xi)}{\gamma^2(\xi)} \geq \frac{2\nu}{d} x^{2+\beta-2\delta}$$

for some $\varepsilon > 0$, arbitrarily small when $x$ is sufficiently large. This quantity is lower bounded at infinity in the case where $\nu > 0$ or in the case $\delta > 1 + \beta/2$, implying in both case that $p(x) \to \infty$ when $x \to \infty$. We hence conclude that the process does not blow up in finite time in these cases, which ends the proof. 

### 2.2 Stability of $\delta_0$

We now address the stability of the equilibrium $\delta_0$, and prove the following:

**Theorem 1.** Let us denote by $\delta_0$ the stationary solution $X_t = 0$ a.s. for all $t$. We have:

- For $\alpha < 1$, $\delta_0$ is asymptotically stable in probability whatever $\lambda \in \mathbb{R}$ and $\sigma \neq 0$.
- For $\alpha = 1$, $\delta_0$ is:
  1. asymptotically stable in probability for $\lambda < \frac{\sigma^2}{2}$. Moreover, if $g(x) = -\mu x^{1+\kappa}$ for $\mu, \kappa > 0$ and $\gamma(x) = \sigma x$, the stationary solution $\delta_0$ is almost surely exponentially stable for $\lambda < \frac{\sigma^2}{2}$.
  2. unstable in probability for $\lambda > \frac{\sigma^2}{2}$
- For $\alpha > 1$, $\delta_0$ is:
  - stable in probability for $\lambda < 0$
  - unstable in probability for $\lambda > 0$

**Proof.** The case $\alpha = 1$ was already understood in the context of stochastic bifurcations, and proofs of the result can be found in [2, 19]. The cases $\alpha \neq 1$ are simple applications of Lyapunov functional theory. The proof proceeds as follows: **Case $\alpha < 1$:** Let us consider $V(x) = x^\alpha$ as a Lyapunov functional for the dynamics. This function is clearly $C_2^0(\mathbb{R})$, i.e. it is twice continuously differentiable except at $x = 0$. Moreover, we have:

$$\mathcal{L}V = \alpha x^{\alpha-1} (\lambda x + g(x)) - \frac{\alpha(1-\alpha)}{2} \gamma(x)^2 x^{\alpha-2}$$

Since we assume that $\alpha < 1$ and $g(x)/x \to 0$ at 0, the leading term close to 0 is of order $-\alpha(1-\alpha)/2 \sigma^2 x^{2\alpha-2}$ which is strictly negative for sufficiently small $x$. More precisely, there exists $r > 0$ such that $V \in C_2^0(U_r)$ where $U_r$ is the open ball of radius $r$ and such that $\mathcal{L}V < 0$ for all $x \in U_r$. Moreover, for any
such that $0 < \varepsilon < r$ we have $V(r) > 0$ and $L V < -c_\varepsilon < 0$ for all $r > x > \varepsilon$. Theorem V.4.1 of [15] therefore applies and concludes the proof of the stability in probability of the solution $\delta_0$ whatever the parameters $\lambda \in \mathbb{R}$, $\sigma > 0$ and $\alpha \in [1/2, 1)$.

**Case $\alpha = 1$:** We start considering the case $\lambda < \sigma^2/2$ and define $\eta = \frac{1}{2} - \frac{\lambda}{\sigma^2} > 0$. The Lyapunov function $V : x \mapsto x^n$ satisfies:

$$L V (x) = \eta x^n (\lambda + \frac{1}{2} (\eta - 1) \gamma^2(x)) + \eta x^{n-1} g(x).$$

Under assumption (H1) we have $x g(x) \sim \mu x^{2+\kappa}$, hence is equivalent, close from zero, to:

$$x^n \left( \lambda + \frac{(\eta - 1) \gamma^2(x)}{x^2} \right) = \frac{1}{2} (\lambda - \frac{\sigma^2}{2}) \eta x^n.$$

Since $(\lambda - \sigma^2/2) < 0$, the same argument as in the case $\alpha < 1$ ensures asymptotic stability of $\delta_0$ in probability for $\lambda < \sigma^2/2$. If we now further assume $g(x) = -\mu x^{1+\kappa}$ and $\gamma(x) = \sigma x$, considering $V(x) = x^2$, we have:

- $L V (x) = (2\lambda + \sigma^2) x^2 - 2\mu x^{2+\kappa} \leq (2\lambda + \sigma^2) V(x)$,
- $|V'(x)\sigma x|^2 = |2\sigma x^2|^2 = 4\sigma^2 V(x)$

Using the stochastic stability theorem [19, Theorem 3.3], we deduce that:

$$\limsup_{t \to \infty} \frac{1}{t} \log |X(t)| \leq -\frac{4\sigma^2 - 2(2\lambda + \sigma^2)}{4} = \lambda - \frac{\sigma^2}{2}$$

almost surely.

By definition of the almost sure exponential stability, we conclude that the solution 0 is almost surely exponentially stable for any $\lambda < \frac{\sigma^2}{2}$.

Let us now deal with the case $\lambda > \sigma^2/2$. We define $V(x) = -\log(x)$. This function tends to infinity as $x \to 0$ and belongs to $C^2_0(\mathbb{R})$. Moreover,

$$L V = (-\lambda + \frac{\gamma(x)^2}{2x}) + \frac{g(x)}{x},$$

which is equivalent at zero to $-\lambda + \frac{\sigma^2}{2} < 0$. Hence there exists $r > 0$ such that $L V \leq 0$ for any $x < r$, and moreover we have for any $\varepsilon > 0$ and $r > x > \varepsilon$, $V(x) \geq 0$ and $L V(x) < -c_\varepsilon < 0$. Theorem V.4.2 of [13] applies and ensures that the solution $\delta_0$ is not stable in probability.

**Case $\alpha > 1$:** For $V(x) = x^2$ around zero, we have:

$$L V = 2\lambda x^2 + 2x g(x) + \gamma(x)^2$$

and under the assumption that $g(x) \sim \mu x^{1+\kappa}$ at zero and $\alpha > 1$, the term $L V$ is equivalent to $2\lambda x^2$ and hence locally has the sign of $\lambda$. For $\lambda < 0$, we can directly apply theorem V.4.1 of [13]. For $\lambda > 0$, we use $V(x) = -\log(x)$ again and conclude that 0 is not stable in probability.
2.3 Absorption time

We know that \( X_t = 0 \) a.s. is a stationary solution of the SDE. It is an absorbing state of the diffusion, i.e. as soon as the solution reaches the singular point, the solution is almost surely identically stuck at the singular point for the whole subsequent evolution. We are now interested in the absorption probability, namely the probability of reaching zero in finite time. We distinguish between the cases \( \alpha < 1 \) and \( \alpha \geq 1 \).

2.3.1 Case \( \alpha < 1 \)

**Proposition 2.** Under assumptions [H1] [H6], the first hitting time of zero of the solution of

\[
    dx_t = (\lambda x + g(x)) \, dt + \gamma(x) \, dW_t
\]

is almost surely finite for any \( \alpha < 1 \) and \( \sigma > 0 \). This is also the case for \( \nu = 0 \) and \( \lambda < 0 \).

**Proof.** In order to establish this property, we use a sufficient condition derived from Feller’s test for expositions and evidence the almost sure finite exit time of the interval \([0, \infty[\). The finiteness of the exit time of \([0, \infty[\) will readily imply the finiteness of the hitting time of 0, since the process is almost surely bounded for all finite time as proved in proposition [1] under the assumptions of the proposition.

Let us define the interval \( I = (0, \infty) \), and let \( c \in I \) arbitrarily chosen. Under assumption [H5] we have that for all \( x \in I \), the diffusion coefficient is always strictly positive, hence the function \( (1 + |\lambda x + g(x)|)/\gamma(x)^2 \) is locally integrable on \( I \). The finiteness of the exit time of the process from the interval \( I \) can be established through the analysis of Feller’s scale functions. For convenience, we introduce the function:

\[
    G(x) = \int_c^x \frac{(\lambda \xi + g(\xi))}{\gamma^2(\xi)} \, d\xi.
\]

Feller’s scale functions written as a function of \( G \) read:

\[
\begin{cases}
    p(x) & \overset{\text{def}}{=} \int_c^x \exp \left\{ -2G(y) \right\} \, dy \\
    m(dx) & \overset{\text{def}}{=} \frac{2 \exp(2G(x))}{\gamma^2(x)} \, dx \\
    v(x) & \overset{\text{def}}{=} \int_c^x (p(x) - p(y)) \, m(dy)
\end{cases}
\]

We prove that (a) \( \lim_{x \to \infty} p(x) = \infty \) and (b) \( \lim_{x \to 0^+} v(x) < \infty \). The property (a) has been addressed in the proof of proposition [1]. We hence only need to prove the property (b) which is slightly more subtle. The function \( v(x) \) is indeed defined as an integral, which is well behaved except possibly at \( x = 0 \). Close to this point, we have:

\[
    G(y) - G(z) = \int_z^y \frac{\lambda \xi + g(\xi)}{\gamma^2(\xi)} \, d\xi
\]

\[
    \sim_{0^+} \frac{\lambda}{2\sigma^2(1 - \alpha)} (y^{2(1 - \alpha)} - z^{2(1 - \alpha)}) + \frac{\mu}{\sigma^2 \phi} (y^\phi - z^\phi)
\]

8
with \( \phi = 2 + \kappa - 2\alpha \) and this function is smooth and bounded close to zero. Let us denote by \( M \) the supremum of \( |G(y) - G(z)| \) on \([0, c]\), and by \( F(x, y) \) the function defined for \( 0 < y \leq x \leq c \):

\[
F(x, y) = \int_y^x \exp(2(G(y) - G(z))) \, dz \leq e^{2M}(x - y).
\]

It is then clear that:

\[
v(x) = 2 \int_c^x F(x, y) \, dy \leq 2 \int_c^x e^{2M}(x - y) \, dy
\]

which is integrable at zero since the integrand is of order \( x^{1-2\alpha} \) and hence integrable at zero under the condition that \( \alpha < 1 \). By application of Feller’s test for explosion [16, Prop. 5.32 (iii)] we conclude that if \( S \) denotes the exit time of the process from the interval \( I \), we have \( \mathbb{P}(S < \infty) = 1 \). Moreover, we have seen in proposition 1 that under the assumptions of the proposition, the process is almost surely bounded. We hence conclude that necessarily the process reaches 0 in finite time and is absorbed (this is also a direct consequence of Feller’s test for explosion [16, Prop. 5.22] under the condition that \( p(x) \to \infty \) when \( x \to \infty \) if \( p(x) > -\infty \) when \( x \to 0^+ \), which is a simple consequence of the above estimates).

Let us now deal with the case \( \nu = 0 \) and \( \lambda < 0 \), we have \( G(x) = \lambda x^{2(1-\alpha)} + C \) for some constant \( C \). The same reasoning as above ensures that \( p(x) \) tends to infinity as \( x \to \infty \). Demonstrating that \( v(x) \) is bounded at zero did not use the fact that \( \nu \neq 0 \) and hence applies in the present case. We conclude that \( \mathbb{P}(S < \infty) = 1 \), hence the process almost surely reaches 0 in finite time.

We hence proved that for \( \alpha < 1 \), the solutions almost surely hit zero in finite time, where they stay trapped for the subsequent dynamics, as soon as the diffusion coefficient \( \sigma \) is nonzero.

When \( \sigma = 0 \), this is never the case: in the case \( \lambda < 0 \), the solutions of the related ODE converge exponentially fast towards zero but never actually reach it, and in the case \( \lambda > 0 \), the solutions escape from zero exponentially. It is hence of interest to analyze the behavior of the system in the small noise limit. We focus on the case \( \lambda > 0 \) (in which case 0 is unstable in the noiseless system and a.s. reached in finite time in the stochastic case) and use Freidlin and Wentzell large deviations theory [12] to analyze in detail the behavior of the system in the small noise limit. In order to obtain precise quantitative estimates, we will assume that \( g(x) = -\mu x^{1+k} \) with \( k > 0 \) and \( \mu > 0 \) and \( \gamma(x) = \sigma |x|^{\alpha} \), corresponding to a double-well potential. Note that the method developed here does not depend on the choice of \( g(x) \).

Remark. The Freidlin and Wentzell theory [12] was developed for constant diffusion coefficients. Extensive works have been performed to generalize this result, and for our purposes, we may cite [25] who showed the validity of Freidlin and Wentzell estimates for continuous and unbounded coefficients with possibly singular diffusion coefficients,
assuming that the drift $\lambda x + g(x)$ and $\gamma(x)^2$ Lipschitz-continuous, which is the case under our current assumptions.

We analyze the properties of the absorption time of the system with initial condition chosen as the fixed point of the noiseless system:

$$\tau_0 := \inf \left\{ t > 0; \ X_t = 0 \mid X_0 = \left( \frac{\lambda}{\mu} \right)^{1/\kappa} \right\}$$

In the case $\alpha < 1$, one can expect from the large deviation theory that $E[\tau_0]$ becomes exponentially large as $\sigma \to 0$, i.e. behaves in the leading order as $e^{V/\sigma^2}$ where $V$ is a positive constant. More precisely, we have:

**Proposition 3.** For all $\frac{1}{2} < \alpha < 1$ and $\lambda > 0$:

$$\lim_{\sigma \to 0} \sigma^2 \log E[\tau_0] = \lambda \left( \frac{\lambda}{\mu} \right)^{2(1-\alpha)/\kappa} \left[ \frac{1}{2(1-\alpha)} - \frac{1}{2(1-\alpha)+\kappa} \right] > 0$$

*Proof. By application [9, Theorem 5.7.11.], the problem is reduced to compute at the boundary $x = 0$ the quasipotential $U(x)$, solution of the Hamilton-Jacobi equation:

$$(\lambda x - \mu x^{1+\kappa})U'(x) + \frac{x^{2\alpha}}{2}(U'(x))^2 = 0$$

with boundary condition $U((\frac{\lambda}{\mu})^{\frac{\kappa}{2}}) = 0$. That is, one has to compute:

$$U(x) = \int_{x}^{(\frac{\lambda}{\mu})^{1/\kappa}} \frac{\lambda u - \mu u^{1+\kappa}}{u^{2\alpha}} du$$

Simple algebra yields the expression:

$$U(0) = \lambda \left( \frac{\lambda}{\mu} \right)^{2(1-\alpha)/\kappa} \left[ \frac{1}{2(1-\alpha)} - \frac{1}{2(1-\alpha)+\kappa} \right]$$

As a consequence, for $\alpha < 1$ fixed, the absorption time becomes exponentially large, which explains the transition between the regime $\sigma > 0$ (where the system a.s. hits zero) and $\sigma = 0$. A critical boundary of this domain is the point $\alpha = 1$: in that case, the system a.s. never hits zero in finite time as shown in section 2.3.2.

In particular, proposition 3 ensures that the convergence as $\alpha \to 1^-$ with $\lambda > 0$ fixed displays a super-exponential time behavior:

$$\lim_{\alpha \to 1^-} \lim_{\sigma \to 0} \sigma^2 \log E[\tau_0] = \infty$$

Going deeper into the analysis of the above convergence allows to understand more precisely what happens in the critical parameter region where both $\alpha \to 1^-$ and $\lambda \to 0^+$, in the limit of small noise $\sigma \to 0$. Such a “blow up” reveals the following picture around the point $(\alpha, \lambda, \sigma) = (1, 0, 0)$ in parameter space:
Corollary 1. Assume that \( \lambda \to 0^+ \) and \( \alpha \to 1^- \) such that
\[
\frac{\lambda}{(1 - \alpha)} \to c \in [0, \infty]
\] (5)

Then, one has the following three regimes:

- if \( c = 0 \), the absorption time is sub-exponential:
  \[
  \lim_{\alpha \to 1, \lambda \to 0} \lim_{\sigma \to 0} \sigma^2 \log \mathbb{E}[\tau_0] = 0
  \]

- if \( 0 < c < \infty \), the absorption time is exponential:
  \[
  \lim_{\alpha \to 1, \lambda \to 0} \lim_{\sigma \to 0} \sigma^2 \log \mathbb{E}[\tau_0] = \frac{c}{2\mu^2(1-\alpha)/\kappa}
  \]

- if \( c = \infty \), the absorption time is super-exponential:
  \[
  \lim_{\alpha \to 1, \lambda \to 0} \lim_{\sigma \to 0} \sigma^2 \log \mathbb{E}[\tau_0] = \infty
  \]

where \( \lim_{\alpha \to 1, \lambda \to 0} \) is constrained by Eq. (5).

Proof. This proposition is a consequence of the analysis of the different asymptotic regimes of the critical exponent \( c^* \) obtained in proposition 3:
\[
c^* = \lambda \left( \frac{\lambda}{\mu} \right)^{2(1-\alpha)/\kappa} \left( \frac{1}{2(1-\alpha)} - \frac{1}{2(1-\alpha) + \kappa} \right).
\]
The leading term in the considered limit is proportional to \( \lambda^{1+2(1-\alpha)/\kappa}/(1-\alpha) \). A series expansion provides the following expression:
\[
c^* \sim \frac{1}{2\mu^2(1-\alpha)/\kappa} \left( \frac{\lambda}{\alpha - 1} + \frac{2}{\kappa} \lambda \log(\lambda) \right) + O(\lambda \log(\lambda)^2(1-\alpha)).
\]
The different regimes identified in the corollary immediately follow from the analysis of the limits of this exponent as \( \lambda \to 0^+ \) and \( \alpha \to 1^- \) subject to the condition (5). \[\square\]

Remark. Note that the regimes do not depend on \( \mu \) or \( \kappa \) qualitatively and the property is purely local: as \( \lambda \) goes to zero, the integration range becomes increasingly small. The property is hence valid for equation (1) with function \( g(x) \) equivalent at zero to \( -\mu x^{1+\kappa} \). However, the asymptotic exponential divergence rate quantitatively depends on the local behavior of the flow close to zero, as observed in the case \( 0 < c < \infty \).

From a dynamical systems viewpoint, this type of solutions corresponds to a relatively sharp convergence towards the equilibrium. Indeed, the dynamics stops at the almost sure hitting time of the singular point and the whole subsequent dynamics is deterministic and constant. This will not be the case for more regular diffusion coefficients (\( \alpha \geq 1 \)) as we now show.
2.3.2 For $\alpha \geq 1$ the solution a.s. never reaches 0

We now deal with the case $\alpha \geq 1$ and show that in contrast to the case $\alpha < 1$, even when $\delta_0$ is stable in probability or almost surely exponentially stable, the solutions of the SDE almost surely never reaches zero.

**Theorem 2.** Let $\alpha \geq 1$. Then for any $x_0 \in \mathbb{R}$ we have

$$\mathbb{P}[x_t \neq 0 \text{ on } t \geq 0] = 1$$

That is, almost any sample path starting from a non-zero initial condition will never reach zero.

**Proof.** The result is already known for $\alpha = 1$ (see e.g. [19]). We extend the demonstration to our general case. Denote by $\tau$ the first hitting time of zero of the process with initial condition $x_0 \neq 0$ at $t = 0$. Assuming that the conclusion is false, we can find a pair of constants $T > 0$ and $\theta > 1$ sufficiently large such that

$$\mathbb{P}(B) > 0$$

where $B$ is defined as:

$$B = \{ \tau \leq T \text{ and } |x(t)| \leq \theta - 1 \text{ for all } t \leq \tau \}$$

Moreover, under the assumption $\alpha \geq 1$, both the drift and diffusion functions are locally Lipschitz-continuous, hence there exists a positive constant $K_\theta$ such that:

$$|\lambda x + g(x) \vee \gamma(x)| \leq K_\theta |x| \text{ for all } |x| \leq \theta, t \leq T.$$ 

Let now $V(x) := |x|^{-1}$. We have for $0 < |x| \leq \theta$ and $t \leq T$

$$\mathcal{L}V \leq K_\theta (1 + K_\theta)V(x)$$

Let us now define $\tau_\varepsilon = \inf\{t \geq 0; |x_t| \notin (\varepsilon, \theta)\}$. This is a stopping time, so by Itô’s formula and the optimal sampling theorem with the bounded stopping time $\tau_\varepsilon \wedge T$, we have:

$$\mathbb{E}[e^{-K_\theta(1 + K_\theta)\tau_\varepsilon V(x_{\tau_\varepsilon \wedge T})}] = V(x_0) +$$

$$+ \mathbb{E}\left[ \int_0^{\tau_\varepsilon \wedge T} e^{-K_\theta(1 + K_\theta)s} [-K_{\theta}(1 + K_\theta)V(x_s) + (\mathcal{L}V)(x_s)] \, ds \right]$$

$$\leq |x_0|^{-1}$$

We know that for any $\omega \in B$, the time $\tau_\varepsilon$ is smaller than $T$ and moreover that $x_{\tau_\varepsilon} = \varepsilon$ by definition of $B$, hence we have:

$$|x_0|^{-1} \geq \mathbb{E}[e^{-K_\theta(1 + K_\theta)(\tau_\varepsilon \wedge T)V(x_{\tau_\varepsilon \wedge T})}] \geq \mathbb{E}\left[ e^{-K_\theta(1 + K_\theta)T} \varepsilon^{-1} 1_B \right]$$

i.e.

$$\mathbb{P}(B) \leq \varepsilon |x_0|^{-1} e^{K_\theta(1 + K_\theta)T}$$

This inequality is true whatever $\varepsilon$ (we recall that the definition of $B$ did not involve $\varepsilon$), hence valid in particular when $\varepsilon$ is arbitrarily small, and necessarily $\mathbb{P}(B) = 0$ which contradicts the definition of $B$. \qed
2.4 Stationary and quasi-stationary solutions

We are now interested in identifying possible stationary solutions. We already mentioned that the null stochastic process (distributed as a Dirac delta measure at 0) is always a stationary solution of the system. Moreover, for $\alpha < 1$, we have shown that the process is almost surely absorbed at zero in finite time. This property prevents from the existence of any stationary solution, and as far as the permanent dynamics is concerned, it is perfectly characterized by the absorption property. In order to further describe the behavior of the solutions, one may wonder how the solutions that did not yet reach zero are distributed. Stationary solutions of the process conditioned on never hitting zero are called quasi-stationary distributions. In the case $\gamma(x) = \sigma|x|^\alpha$ with $\alpha < 3/4$, we can show that there exists a quasi-stationary distribution that attracts exponentially fast the conditioned process. In detail, we have:

**Proposition 4.** Let us assume that $\nu > 0$, $g$ is differentiable, $g'$ is bounded at zero and smaller than $C_x^2$ for some $C \in \mathbb{R}$ at infinity, that assumptions $(H1)$-$(H6)$ hold and $\alpha \in [1/2, 3/4)$. The process conditioned on never reaching zero and with a bounded initial distribution converges exponentially fast towards a probability measure $p_\pm$ for any positive (resp. negative) initial condition. These are called quasi-stationary solutions, or Yaglom limit.

**Proof.** We apply a result due to Cattiaux and collaborators [5] in the domain of diffusion models in population dynamics. In their article, they prove consider a diffusion of type:

$$dX_t = dB_t - q(X_t) dt$$

under a few conditions on $q$. Our problem can be brought to their framework changing variables and considering: $X_t = x_t^{1-\alpha}/(\sigma(1-\alpha))$. Itô’s formula allows putting our equations in that form, with

$$q(x) = -\frac{1}{\sigma}x^{-\alpha}f(x) + \frac{\alpha}{2(1-\alpha)} \frac{1}{x}$$

and where $f(x) = \lambda x + q(x)$ and $Z = (\sigma(1-\alpha)x)^{1/(1-\alpha)}$. Under the conditions that (i) the first hitting time of zero is almost surely finite and (ii) that the conditions: $\Delta = -\inf_{y \in (0,\infty)} q^2(y) - q'(y) < \infty$ and $\lim_{y \to \infty} q^2(y) - q'(y) = \infty$, they show that the spectrum of the Kolmogorov backward equation (for the related operator $L$) has a purely discrete spectrum with no zero eigenvalue. Moreover, they show that a rescaled version of the related eigenfunction is integrable and exponentially attract all initial conditions under the conditions:

$$\int_0^1 \frac{1}{q(y)^2 - q'(y)} + \Delta + 2 e^{-G(y)} dy < \infty$$

with $G(y) = \int_y^\infty 2q(y) dy$.

$$\int_1^\infty e^{-G(y)} dy < \infty, \int_0^1 xe^{-G(x)/2} dx < \infty \text{ and } \int_1^\infty e^{G(y)} \int_y^\infty e^{G(z)} dz dy < \infty$$
In order to apply their result, we demonstrate that all these conditions are satisfied in our framework. First of all, under the condition $\alpha < 1$ and $\nu > 0$, we showed that the process $x_t$ solution of $\text{(1)}$ almost surely reach zero in finite time (proposition $\text{2}$). This is hence also the case of $X_t$ since $(1 - \alpha) > 0$ in that case. Moreover, the derivative of the function $q(x)$ reads:

$$q'(x) = \alpha Z^{-1} f(Z) - \alpha f'(Z) - \frac{\alpha}{2(1 - \alpha)} \frac{1}{x^2}.$$ 

Around zero, because of our assumptions on $g$, we have:

$$q(x) \sim \frac{\alpha}{2(1 - \alpha)} \frac{1}{x} \quad \text{and} \quad q'(x) \sim -\frac{\alpha}{2(1 - \alpha)} \frac{1}{x^2}$$

and hence

$$q^2(x) - q'(x) \sim_{x \to 0^+} \frac{\alpha(2 - \alpha)}{4(1 - \alpha)^2} \frac{1}{x^2}$$

which tends to infinity as $x \to 0^+$. At $x = \infty$, we have:

$$q^2(x) - q'(x) \geq \frac{\mu^2}{\sigma^2} (1 - \varepsilon) Z^{2(1 - \alpha) + 2}\beta$$

for any $\varepsilon > 0$ and hence tends to infinity when $x \to \infty$. In particular, $\Delta = -\inf_{y \in (0, \infty)} q^2(y) - q'(y) < \infty$.

Let $G(x) = \int_1^x 2q(y) \, dy$. We now show that the condition $\text{(6)}$ is satisfied. At infinity, $q(x)$ is greater or equal to $C'x^{1+\beta/(1-\alpha)}$ with $C' > 0$. It is then easy to show that $\lim_{x \to \infty} \frac{G(x)}{x} = \infty$. Moreover, we have

$$A := \lim_{x \to 0^+} \left( G(x) - \frac{\alpha}{1 - \alpha} \log(x) \right) \in (-\infty, \infty).$$

Therefore, near zero, the integrand involved in the expression of $Q(x)$ is proportional to $y^{2 - \frac{1}{1 - \alpha}}$. Under the condition $\alpha < \frac{3}{4}$, condition $\text{(6)}$ holds. Showing that condition $\text{(7)}$ is valid is straightforward at this point: $G$ grows at least linearly at $\infty$ ensuring that $\int_1^\infty e^{-G(y)} \, dy < \infty$. Moreover, around zero, the function $xe^{-G(x)/2}$ is of order $x^{1 - \frac{\alpha}{2(1 - \alpha)}}$ which is integrable at zero for $\alpha < \frac{3}{4}$, hence in particular for $\alpha < \frac{3}{4}$. The main results of [5] (namely their theorem 5.2, proposition 5.5, corollaries 6.1 and 6.2) apply, and ensure the existence and uniqueness of a quasi-stationary distribution attracting exponentially fast the law of the diffusion conditioned on not hitting zero with initial laws having bounded support.

**Remark.** The above proposition ensures existence of a quasi-stationary solution for $\alpha < 3/4$, using a sufficient condition for existence of such measures, hence does not preclude existence of quasi-stationary distributions for $\alpha \in [3/4, 1)$. Numerical explorations of such system tend to show that the bound $\alpha = 3/4$ is actually optimal. For instance, Figure $\text{1}$ shows that the solutions to the stochastic subcritical pitchfork equation:

$$dx_t = (\lambda x - x^3) \, dt + \sigma x_t \, dW_t$$

(8)
do present invariant distributions of the process conditioned on not hitting zero for
\(\alpha < 3/4\), but for \(\alpha > 3/4\), the distribution does not reach a stationary state but
tends to approach the stable solution \(\delta_0\) (this is also what happens for \(\alpha = 1\) when no
absorption is present but an a.s. exponential convergence occurs).

![Empirical Distribution](image)

**Figure 1:** Distribution (ordinate truncated for legibility) of the trajectories that
did not reach zero in the subcritical pitchfork model (8) with \(\lambda = -0.5, \sigma = 0.5\)
different values of \(\alpha\), computed at 6 times. (a) \(\alpha = 0.6\): stabilization on a quasi-
stationary distribution that do not depend on time, (b) \(\alpha = 0.85\): the repartition
varies as a function of time and all trajectories approach zero, arguing for the
non-existence of quasi-stationary distribution. Simulations were done using a
particle method with 100 000 particles, using the Euler-Maruyama scheme with
dt = 0.01.

Let us now deal with the general case with diffusion coefficient \(\gamma(x)\) and
\(\alpha \geq 1\). In that case, the process is almost surely never absorbed at zero. Possible
stationary solutions with smooth probability density function \(p\) with respect
to Lebesgue’s measure satisfy the forward Kolmogorov (or Fokker-Planck) equation:

\[
\frac{d}{dx} \left( (\lambda x + g(x))p(x) - \frac{1}{2} \frac{d}{dx} (\gamma(x)^2 p(x)) \right) = 0.
\]  

i.e.

\[
(\lambda x + g(x) - \gamma'(x)\gamma(x))p(x) - \frac{1}{2} \gamma(x)^2 p'(x) = K
\]

for some \(K \in \mathbb{R}\). If we can find solutions to these equations subject to the
conditions \(p(x) \geq 0\) for all \(x \in \mathbb{R}\) and \(\int_{\mathbb{R}} p(x) \, dx = 1\) (acceptable solutions), then
these are probability density functions of stationary solutions of the diffusion
equation. For simplicity, we consider that assumption [H5] is valid, i.e. \(\gamma(x)\)
only vanishes at \(x = 0\). In that case, since \(\delta_0\) is a solution to the SDE (1),
stationary solutions will either have support on \(\mathbb{R}^+\) or \(\mathbb{R}^-\). In the case where
the dynamics has additional singular points, the Kolmogorov equation will define
solutions in each interval delimited by the singular points (see section 3.2). The
Kolmogorov equation is a linear ordinary differential equation with constant
source term. The homogeneous solutions (with \(K = 0\)) is given by:

\[
p_0(x) = C\gamma(x)^{-2} \exp \left( \int_x^\infty \frac{2\lambda y}{\gamma(y)^2} + 2 \frac{g(y)}{\gamma(y)^2} \, dy \right)
\]
defined for \( x \neq 0 \). Particular solutions are given by:

\[
p_K(x) = \left( -2K \int_1^x \exp \left( -2 \int_1^y \frac{2\lambda z}{\gamma(z)^2} + 2 \frac{g(z)}{\gamma(z)^2} \, dz \right) \right) \gamma(x)^{-2} \exp \left( \int_1^x \frac{2\lambda y}{\gamma(y)^2} + 2 \frac{g(y)}{\gamma(y)^2} \, dy \right)
\]

(10)

We have the following:

**Proposition 5.** In addition to the stationary distribution \( \delta_0 \), we have:

- For \( \alpha = 1 \) and:
  - \( \lambda < \sigma^2/2 \) there exists no integrable solution of the Kolmogorov equation.
  - \( \lambda \geq \sigma^2/2 \), under assumption \([H6]\) there exist two integrable solutions of the Kolmogorov equation given by \( Z^+ p_0(x) \chi_{x>0} \) and \( Z^- p_0(x) \chi_{x<0} \) where \( Z^+ \) and \( Z^- \) generically denote normalization constants.
  - \( \lambda \geq \sigma^2/2, \nu > 0 \) and \( \delta < 1 + \beta/2 \) (case where assumption \([H6]\) is not satisfied) there does not exist any stationary solution (we saw that the solutions a.s. blow up in finite time).

- For \( \alpha > 1 \) and:
  - \( \lambda < 0 \) the Kolmogorov equation has no integrable solution.
  - \( \lambda > 0 \) and under assumption \([H6]\), the Kolmogorov equation has two integrable solution given by \( Z^+ p_0(x) \chi_{x>0} \) and \( Z^- p_0(x) \chi_{x<0} \).
  - \( \lambda > 0, \nu < 0 \) and \( \delta \leq 1 + \frac{\beta}{2} \) (i.e. when \([H6]\) is not satisfied), the Kolmogorov equation has no integrable solution.

**Remark.** In the case \( \alpha < 1 \), we already mentioned that no stationary solution can exist. We can show using the expressions of the solutions to the Kolmogorov equation that no solution is integrable.

**Proof.** For \( \alpha > 1 \), let us analyze the integrability of \( p_0(x) \). Around 0, the function \( x^{-2\alpha} \) is not integrable. The leading term of the exponential around zero is given by \( \frac{\lambda}{\sigma^2(1-\alpha)} x^{2(1-\alpha)} \), hence diverges at zero. If \( \lambda < 0 \), this term tends to \( +\infty \) and is not integrable, and if \( \lambda > 0 \), the exponential term tends to zero and the function is hence integrable at 0 and \( p(0) = 0 \). For \( x \to \infty \), we distinguish two cases:

- for \( \delta < 1 + \beta/2 \) and \( \nu > 0 \), the leading term of the exponential in the expression of \( p_0(x) \) is given by \( -\nu z^{1+\beta-2\delta}/(1 + \beta - 2\delta) \) which tends to \( -\infty \) for \( \nu > 0 \) fast enough ensuring integrability of \( p_0(x) \) at \( \infty \). For \( \nu < 0 \), this term tends to \( \infty \) and \( p_0(x) \) is not integrable at infinity.

- when \( \nu < 0 \), the leading term of the exponential at infinity is of order \( -\nu z^{1+\beta-2\delta}/(1 + \beta - 2\delta) \) which tends to \( +\infty \) and hence the solution to the Kolmogorov equation is not integrable at infinity.
For $\delta \geq 1 + \beta/2$, the terms inside the exponential tend to zero, and the integrability relies on the integrability property of $x^{-2\delta}$, always integrable at infinity.

We hence proved that acceptable solutions related to $K = 0$ exist for $\lambda > 0$ and no solution existed when $\lambda < 0$. Moreover, there is no possible choice of $K \neq 0$ that can overcome the exponential divergence at zero. Indeed, the only possible choice of $K$ would be the only constant preventing divergence at zero, i.e.

$$p(1) = 2 \frac{K}{\sigma^2} \int_1^0 \exp \left( -2 \int_1^y \frac{2\lambda z}{\gamma(z)^2} + 2 \frac{g(z)}{\gamma(z)^2} \, dz \right) \, dy.$$  

However, the integral term of the righthand side diverges, hence the above relationship cannot be satisfied and there is no acceptable solution for $\lambda < 0$. Eventually let us remark that for $\lambda > 0$, any choice of $K$ prevents integrability at infinity.

For $\alpha = 1$, we again analyze the integrability properties of $p_0(x)$. Around $x = 0$, $p_0$ behaves like $\frac{x^{2\lambda}}{\sigma^2} \exp \left( 2 \int_1^x \frac{g(y)}{\gamma(y)^2} \, dy \right)$. The term in the exponential behaves like $\frac{1}{\sigma^2} y^\kappa$. The integrability at zero hence depends on the exponent $-2(1 - \frac{1}{\sigma^2})$: $p_0$ is integrable if the exponent is strictly larger than $-1$, i.e. $2\lambda/\sigma^2 > 1$. In the case $2\lambda/\sigma^2 < 1$, the non-integrability is due to a polynomial divergence, and hence might be compensated by a suitable choice of $K$. The only possible choice of $K$ compensating the divergence at zero would correspond to:

$$p(1) = 2K \int_1^0 y^{-2\lambda/\sigma^2} \exp \left( -2 \int_1^y \frac{g(z)}{\gamma(z)^2} \, dz \right) \, dy,$$

and here again the integral diverges at zero, and hence there is no such solution.

At infinity, the exact same analysis as done in the case $\alpha > 1$ applies, and hence we obtain that $p_0$ is integrable at $\infty$ under assumption [H6] \[ \Box \]

### 2.5 Heuristic discussion

We hence proved that as a function of the value of the coefficient $\alpha$, the system can be in one of two substantially different regimes: in the case $\alpha < 1$ the singular point is always stable, and reached in finite time, and for $\alpha > 1$ the stability of the singular point is not affected by the presence of noise and only depends on the stability of the fixed point in the noiseless system. The case $\alpha = 1$ constitutes the transition between these two regimes, and in that case the stability of the singular point depends both on the eigenvalue of the Jacobian matrix of the drift at the singular point and on the level of noise.

In order to understand heuristically these stabilization and destabilization phenomena, let us focus on the behavior of the system in the neighborhood of the singular point. In the case $\alpha > 1$, the diffusion coefficient vanishes faster than the drift coefficient at the singular point, and the system locally behaves as if there were no noise in the system. In particular, the stability of the singular
point remains unchanged from that of the noiseless system. On the contrary, when \(\alpha < 1\), the diffusion coefficient vanishes slower than the drift. When the system approaches the singular point, it is hence mainly driven by random fluctuations related to the diffusion coefficient. When the system is brought away from the singular point, noise increases, and when it is brought towards the singular point noise decreases. This is at the origin of a ratchet-like phenomenon: consider to points \(A\) and \(B\) close of the singular point, \(A\) being closer than \(B\). The time it takes to go from \(A\) to \(B\) with power diffusion coefficient is way larger than the time needed to go from \(B\) to \(A\). This behavior of the noise close of zero explains the stability in probability, and acts as a weak drift by ‘stabilizing’ the solutions when they approach the singular point. The intermediate case \(\alpha = 1\) is precisely the transition where these two effects compensate. When the noise coefficient is large enough compared to the eigenvalue of the Jacobian matrix at the singular point, the noise effects described for \(\alpha < 1\) dominate and stabilize the fixed point, and on the contrary the deterministic phenomena govern the dynamics similarly to the case \(\alpha > 1\).

3 Applications

The detailed characterization of the solutions of equation (1) has several implications in applied mathematics. We focus here on generic descriptions of the dynamics of one-dimensional SDEs in the flavor of bifurcations theory. We will specifically discuss the behavior of SDEs locally reducible (in a sense that we make precise) to the pitchfork or saddle-node bifurcation with power diffusion coefficients.

3.1 Analysis of the stochastic pitchfork bifurcation with Hölder diffusion

In this section we first study the dynamics of an SDE with drift given by the normal form of the pitchfork bifurcation with Hölder diffusion coefficients, before addressing the universality of these behaviors. This problem was widely addressed in the theory of random dynamical system by several authors [8, 1] in the case of multiplicative noise (\(\alpha = 1\) in our notations). We extend these results to the cases \(\alpha \neq 1\), and show that actually the multiplicative noise case is a singular transition case when varying \(\alpha\).

Dynamics of the canonical stochastic pitchfork equation We consider the supercritical stochastic pitchfork bifurcation with power diffusion.

\[
\frac{dx_t}{dt} = (\lambda x_t - x_t^3) dt + \sigma |x_t|^\alpha dW_t
\]

for \(x_t \in \mathbb{R}\), with initial condition \(X_0\) at \(t = 0\). In this equation, \(\lambda\) is a real parameter, \(\sigma\) is a non-negative parameter and \(\alpha \geq \frac{1}{2}\). In the case \(\sigma = 0\), it is well known that the solution \(x = 0\) is a stable fixed point for any \(\lambda < 0\), and
unstable for $\lambda > 0$, and two additional stable equilibria $\pm \sqrt{\lambda}$ exist in the region $\lambda > 0$ (see e.g. [14]).

For $\sigma \neq 0$, the model clearly satisfies assumptions [H1](H6) with $\nu = -\mu = 1 > 0$, $\kappa = \beta = 2$, $d = \sigma$ and $\delta = \alpha$. Moreover, it satisfies the additional assumptions used in proposition 4 since $g(x) = -x^3$ is differentiable with bounded derivative at zero and divergence at infinity upperbounded by $x^4$. The analysis of the model is therefore a direct application of section 2.

The problem was widely analyzed in the literature in the case $\alpha = 1$ (see e.g. [1, 8]). In this multiplicative noise case, it was proved that $\delta_0$

- $\lambda < \sigma^2/2$, $\delta_0$ is almost surely exponentially stable and no additional stationary solution exist.
- $\lambda \geq \sigma^2/2$, $\delta_0$ is unstable in probability, and two stationary solutions stable in the first approximation exist.

Therefore, in the multiplicative noise case, stability of zero depends on the level of noise.

Our analysis shows that this is a singular phenomenon that disappears upon variation of the Hölder exponent $\alpha$:

- For $\alpha < 1$, the fixed point $\delta_0$ is stable in probability. Moreover, any solution almost surely reaches zero in finite time. Quasi-stationary distributions exists for $\alpha \in (1/2, 3/4)$.
- For $\alpha > 1$ and:
  - $\lambda < 0$, $\delta_0$ is stable in probability and no other stationary solution exist.
  - $\lambda > 0$, $\delta_0$ unstable in probability and stationary solutions exist.

Let us now further describe the dynamics of the non-trivial solutions as a function of $\alpha$.

- **Case $\alpha = 1$**: It was observed in the literature (see e.g. [1, Chap. 9]) for a comprehensive account) that for any $\lambda > 0$, the unstable deterministic fixed point becomes asymptotically exponentially stable when the noise parameter $\sigma$ is large enough. For $\lambda > \frac{\sigma^2}{2}$, two symmetrical stationary distributions appear. For $\frac{\sigma^2}{2} < \lambda \leq \sigma^2$ the stationary distribution concentrates at zero and has a non increasing density diverging at zero. For $\lambda \geq \sigma^2$, the probability density function of the stationary distribution vanishes at zero and has a unique maximum reached for $x = \pm \sqrt{\lambda - \sigma^2}$ (see Figure 2). There is hence a qualitative transition at $\lambda = \sigma^2$, or $P$-bifurcation. In comparison with the deterministic bifurcation, the loss of stability is delayed and noise tends to stabilize the saddle point.

- **Case $\alpha > 1$**: the deterministic picture is qualitatively and quantitatively recovered: for $\lambda < 0$, $\delta_0$ is stable in probability and is the unique stationary solution, and for $\lambda > 0$, $\delta_0$ is unstable in probability, two additional
stationary solutions appear, presumably stable as shown for \( \alpha = 2 \). The stationary probability density functions for \( \lambda > 0 \) vanish at zero and reach a maximum at \( X_m \) the solution of

\[
\left( \sigma^2 \alpha - \lambda X^{2-2} \alpha + X^{-2} \alpha^4 \right) = 0.
\]

The behavior of \( x_m \) when \( \lambda \) is close to zero follows one of the two regimes:

- for \( \alpha < 2 \), \( X_m \sim \left( \frac{\lambda}{\alpha \sigma^2} \right)^{1/(2\alpha-2)} \). For instance for \( \alpha = \frac{3}{2} \) we get
  \[ X_m = -\frac{1}{2} \sigma^2 + \frac{1}{4} \sqrt{9 \sigma^4 + 16 \lambda} \]
- for \( \alpha > 2 \), \( X_m \sim \sqrt{\lambda} \) and is insensitive to noise parameters \( \sigma \) and \( \alpha \).

For \( \alpha = 3 \) we obtain the explicit form \( X_m = \frac{1}{\sqrt{\sigma^2}} \sqrt{\frac{\lambda}{1+2\sigma^2}} \)

Surprisingly, in the limit \( \lambda \to 0 \) there is a discontinuity in the behavior of \( X_m \) as a function of \( \alpha \) at \( \alpha = 2 \). Nevertheless, when \( \lambda > 0 \) this discontinuity is smoothed out. We conclude that in the case \( \alpha > 2 \), the behavior of the system is close from the one of the deterministic pitchfork bifurcation and the stationary distribution escapes zero as a square root of the parameter. But when \( 1 < \alpha < 2 \), this is no more the case: the scaling of distribution peak behaves as \( \lambda^{1/(2\alpha-2)} \), hence much slower than \( \sqrt{\lambda} \). As \( \sigma \) is increased, the peak of the stationary distribution gets closer to zero.

- **Case \( \alpha < 1 \):** the solution \( \delta_0 \) is always stable in probability. This result can appear relatively surprising at first sight. Indeed, in the deterministic case, \( \lambda \) is the exponential rate of divergence from the solution 0. However, adding a (possibly small) diffusion term proportional to \( |x|^\alpha \) with \( \alpha < 1 \) stabilizes \( \delta_0 \) in probability whatever the value of the noise intensity \( \sigma \neq 0 \). This observation, added to the fact that the solution is not exponentially asymptotically stable (though stable in probability) raises the question of how this convergence occurs. Figure 2(b1) presents a typical sample path of the process. Starting from a positive initial condition, we observe that the solution is evolving in the half-plane \( x > 0 \) and does not show any absorption clue. However, it suddenly reaches zero where it is absorbed after that random transient phase. This perfectly illustrates a typical behavior of the solutions of the pitchfork equation for \( \alpha < 1 \): the absorption time was shown to be almost surely finite. Moreover, trajectories that did not hit zero are distributed according to a quasi-stationary distributions as long as \( \alpha < 3/4 \).

**Universality properties** Similarly to the analysis of nonlinear dynamical systems, the analysis of the pitchfork equation reveals universal features of a
wide class of systems. We formally show the universality property of the pitchfork bifurcation for general SDE on \( \mathbb{R} \):

\[
dx_t = f(x_t, \lambda) \, dt + \gamma(x_t) \, dW_t
\]

with a smooth \( f \) (at least three times continuously differentiable). We consider the dynamics of the system for parameter values \( \lambda \) in a neighborhood of \( \lambda_0 \in \mathbb{R} \).

Let us assume that:

- there exists \( \nu > 0 \) and \( \kappa > 0 \) such that for all \( \lambda \) in a neighborhood of \( \lambda_0 \),
  
  \( f(x, \lambda) \leq -\nu x^{1+\kappa} \),

- for any \( \lambda \), \( f \) is an odd function: \( f(-x, \lambda) = -f(x, \lambda) \)

- \( \frac{\partial^3 f}{\partial x^3}(0, \lambda_0) \neq 0 \)

Defining \( \xi_t = \beta(\lambda) x_t \) with \( \beta(\lambda) = \sqrt{\frac{1}{6} \left| \frac{\partial^3 f}{\partial x^3}(0, \lambda) \right|} \), we have:

\[
d\xi_t = \left( \frac{\partial f}{\partial x}(0, \lambda) \xi_t + \varepsilon \xi_t^3 + \Psi(\xi_t, \lambda) \right) \, dt + \beta(\lambda) \gamma(\xi_t) \, dW_t.
\]

where \( \Psi(x, \lambda) = O(x^5) \) and \( \varepsilon = \text{sign}(\frac{\partial^3 f}{\partial x^3}(0, \lambda)) \) which is well defined around \( \lambda = \lambda_0 \).

Let us now assume that \( \gamma(x) \sim \sigma |x|^\alpha \) around \( x = 0 \) and \( \gamma(x) > 0 \) for \( x \neq 0 \).

We can hence apply the results developed in the previous section to characterize the dynamics of this general system. Pitchfork-like transitions occur at \( \lambda = \lambda_0 \) when:

- \( \alpha > 1 \), \( \frac{\partial f}{\partial x}(0, \lambda_0) = 0 \) and \( \frac{\partial^2 f}{\partial x \partial \lambda}(0, \lambda_0) \neq 0 \) or

- \( \alpha = 1 \), \( \frac{\partial f}{\partial x}(0, \lambda_0) = \frac{\sigma^2}{2} \) and \( \frac{\partial^2 f}{\partial x \partial \lambda}(0, \lambda_0) \neq 0 \).

### 3.2 The stochastic saddle-node bifurcation

Another prominent universal codimension one bifurcation of equilibria in deterministic ODEs is the saddle-node (or fold) bifurcation. It corresponds to the normal form \( \dot{x} = -x^2 + a \) (sometimes considered \( x^2 - a \) which is equivalent to the normal form considered here through a change of time). This dynamical system has no equilibrium when \( a < 0 \) (solutions blows up in finite time) and two equilibria for \( a > 0 \): \( \sqrt{a} \) which is stable and \( -\sqrt{a} \) which is unstable. The system has multiple singular points, and these depends on the bifurcation parameter. Choices of diffusion functions vanishing only at \( \sqrt{a} \) or at \( -\sqrt{a} \) enter the application domain of the general theory developed in section 2. We will consider a slightly different case in this section.
Dynamics of a canonical stochastic saddle-node equation  Let us consider now a case, contrasting with the previous cases treated, where the diffusion function vanishes at both singular points:

\[ dX_t = (-X_t^2 + a) \, dt + \sigma |X_t^2 - a|^\alpha \, dW_t \]  

(12)

For \( a < 0 \), there are two stationary distributions corresponding to the equilibria \( \sqrt{a} \) and \( -\sqrt{a} \). Denoting \( y_t^+ = x_t - \sqrt{a} \) (vanishing at \( x = \sqrt{a} \)) and \( y_t^- = x_t + \sqrt{a} \), it is easy to show that these variable satisfy the equations:

\[
\begin{align*}
\left\{ 
\begin{aligned}
dy_t^+ &= (-2\sqrt{a}y_t^+ - (y_t^+)^3) \, dt + \sigma - 2\sqrt{a}y_t^+ - (y_t^+)^2|\alpha \, dW_t \\
dy_t^- &= (2\sqrt{a}y_t^- - (y_t^-)^3) \, dt + \sigma |2\sqrt{a}y_t^- - (y_t^-)^2|^\alpha \, dW_t 
\end{aligned}
\right.
\]

In both case, the drift and diffusion coefficients satisfy the assumptions \([H1], [H2], [H3], [H4]\) and \([H6]\), and obviously assumption \([H5]\) is not valid. Assumptions \([H1], [H2]\) and \([H6]\) are trivial since the function corresponding to \( g(x) \) in the general cases is equal to \(-x^2 \), hence \( \mu = -\nu = -1 < 0 \) and \( \beta = \kappa = 1 \).

The diffusion coefficients denoted \( \gamma(x) \) in the general case satisfy for both the equation on \( y^+ \) and on \( y^- \):

\[
\gamma(x) \sim x \to 0 + \sigma(2\sqrt{a})^\alpha x^\alpha, \text{ ensuring that assumption } [H3] \text{ is satisfied, and } \gamma(x) \sim x \to \infty \sigma|y|^{2\alpha} \text{ ensuring that assumption } [H4] \text{ is satisfied with } d = \sigma \text{ and } \delta = 2\alpha.
\]

In the deterministic case, for \( a > 0 \) and an initial condition smaller than \(-\sqrt{a} \) or for any initial condition when \( a < 0 \), the solutions blow up in finite time to \(-\infty \), and in usual applications additional, higher order confining terms prevent the blow up. Interestingly, in the present case, the presence of noise can prevent blow up as we now show:

**Proposition 6.** The solutions of the stochastic saddle-node equation (12) are defined for all times or blow up in finite time under the conditions:

- For \( \frac{1}{2} \leq \alpha < 1 \) we have:
  - if \( a < 0 \) the solutions almost surely blow up in finite time to \(-\infty \)
  - if \( a > 0 \), the solutions with initial condition smaller than \(-\sqrt{a} \) almost surely exit the interval \((-\infty, -\sqrt{a}) \) in finite time for any initial condition \( |x| < \sqrt{a} \), the solution almost surely reaches one of the boundaries of the interval in finite time. The probability of reaching \(-\sqrt{a} \) prior to reaching \( \sqrt{a} \) is given by

\[
\frac{p(\sqrt{a}^-) - p(x)}{p(\sqrt{a}^-) - p(-\sqrt{a}^+)}
\]

and is plotted in Fig. 3(c).

- for \( \alpha \geq 1 \), the solutions almost surely never blow up, and for \( a > 0 \), the exit time of \((\sqrt{a}, \sqrt{a}) \) is almost surely infinite.
Proof. We demonstrate this proposition using Feller’s test for explosion.

Let us start by considering $a < 0$. The diffusion is then defined on $\mathbb{R}$ with no singular point, and Feller’s scale function reads:

$$p(x) = \int_c^x \exp \left( -2 \int_c^y \frac{2(2\alpha - 1) - y^2}{2(2\alpha - 1) - 1} dy \right) dy.$$ 

At infinity, the integrand inside the exponential is equivalent at infinity to $\frac{2}{2(2\alpha - 1) - 1} y^{-2(2\alpha - 1) - 1}$. This quantity has a finite limit when $\alpha \geq 1$, implying that $p(x) \to \pm\infty$ when $x \to \pm\infty$. Feller’s test implies that the first exit time of $(-\infty, \infty)$ is infinite with probability one.

For $\alpha < 1$, $p(x)$ tends to $\infty$ at $\infty$, and has a finite limit at $-\infty$. This property ensures that the probability of the process diverge towards $-\infty$ is equal to one. It remains to prove that the explosion occurs in finite time. To this purpose, we need to show that the limit of Feller’s scale function $v(x)$ is finite at $-\infty$. This is the case under the assumption $\alpha < 1$. Indeed, $v(x)$ is defined as:

$$v(x) = \int_c^x \int_c^y \exp \left( 2 \int_c^z (u^2 - a) 1^{-2\alpha} \right)$$

The integral in the argument of the exponential function is equivalent at $-\infty$ to $\frac{2}{2(2\alpha - 1)} (y - z)^{2\alpha + 1}$ which tends to $-\infty$ as a power function, hence the exponential has moments of any order, and in particular is twice integrable at $-\infty$.

For $a > 0$, we consider the intervals $(-\infty, -\sqrt{a})$ and $(\sqrt{a}, \infty)$. The above analysis proved that for $\alpha \geq 1$, $p(x)$ tends to $\infty$ when $x \to \infty$ and towards a finite value at $x \to -\infty$. At $\pm\sqrt{a}$, the argument of the exponential term is equivalent to $\int x^{-2\alpha + 1}$ which diverges for $\alpha \geq 1$, implying that the exit time of $(-\infty, -\sqrt{a})$, $(-\sqrt{a}, \sqrt{a})$ and $(\sqrt{a}, \infty)$ are all almost surely infinite and that there is no blow up in finite time. For $\alpha < 1$, both $p(-\sqrt{a}^-)$ and $p(\sqrt{a}^+)$ are finite. For $c \in (-\sqrt{a}, \sqrt{a})$, we have:

$$\begin{align*}
G(x) &= \int_c^x - (\xi + \sqrt{a})^{-2\alpha} (-\xi + \sqrt{a})^{-2\alpha} d\xi \\
v(x) &= 2 \int_c^x \int_y^{\sqrt{a}^+} \exp \left( -2 \left( G(y) - G(z) \right) \right) dz dy
\end{align*}$$

and using a similar argument as used in the proof of proposition 2, we show that $\lim_{x \to \sqrt{a}^-} v(x) = \infty$, and by symmetry we hence have $\lim_{x \to -\sqrt{a}^+} v(x) = \infty$. Feller’s test for explosion hence ensures that the first exit time of the interval $(-\sqrt{a}, \sqrt{a})$ is almost surely finite. We hence converge in finite time towards $\sqrt{a}$ or $-\sqrt{a}$. Moreover, the probability for a trajectory starting with an initial condition $|x| < \sqrt{a}$ to reach $\sqrt{a}$ (resp. $-\sqrt{a}$) prior to reaching $-\sqrt{a}$ (resp. $\sqrt{a}$), denoted $p_+(x)$ (resp. $p_-(x)$) have the expression:

$$p_-(x) = 1 - p_+(x) = \frac{p(\sqrt{a}^-) - p(x)}{p(\sqrt{a}^+) - p(-\sqrt{a}^+)}$$

This quantity is plotted in Figure 3(c).
Application of the general results of section 2 yield the following:

**Proposition 7.** For \( a > 0 \), the saddle-node equation (12) has two Dirac delta distributed functions \( \delta_{\sqrt{a}} \) and \( \delta_{-\sqrt{a}} \), which enjoy the stability properties:

- for \( \alpha < 1 \), both \( \delta_{\sqrt{a}} \) and \( \delta_{-\sqrt{a}} \) are stable in probability, and the first hitting time \( \tau \) of one of the equilibria is almost surely finite.

- for \( \alpha = 1 \), the stationary distribution \( \delta_{\sqrt{a}} \) is always stable in probability. The stationary distribution \( \delta_{-\sqrt{a}} \) is stable in probability for \( a > \sigma^{-4} \) and unstable in probability otherwise.

- for \( \alpha > 1 \), the stationary distribution \( \delta_{\sqrt{a}} \) is stable in probability and \( \delta_{-\sqrt{a}} \) is unstable in probability.

**Proof.** This is a direct application of the results of section 2 applied to \( y^+ \) to characterize the properties of \( \sqrt{a} \) and to \( y^- \) for \( -\sqrt{a} \). For instance for the fixed point \( -\sqrt{a} \) and \( \alpha = 1 \), the fixed point is stable if and only if \( 2\sqrt{a} < 2\sigma^2 \alpha \) i.e. \( \sigma^2 \sqrt{a} > 1 \). The finite absorption time property is a consequence of proposition 6.

For \( \alpha < 1 \), we know that the trajectories either reach one of the equilibria or blow up in finite time, which describe perfectly the permanent regime of the saddle-node bifurcation. We are now interested in stationary distributions of the saddle-node equation distinct from \( \delta_{\sqrt{a}} \) and \( \delta_{-\sqrt{a}} \). Since the system presents two singular points, the results of proposition 5 based on integrability properties on \( R^+ \) or \( R^- \) of the solutions of the Kolmogorov equation, do not directly apply. However, a similar approach allows to demonstrate the following:

**Proposition 8.** Stationary distributions of the saddle-node equation (12) distinct from \( \delta_{\sqrt{a}} \) and \( \delta_{-\sqrt{a}} \) (when \( a > 0 \)) enjoy the following classification:

- For \( \alpha = 1 \) and \( 0 < a < \sigma^{-4} \), the system has a stationary distribution, stable in the first approximation, charging only the interval \((-\infty, -\sqrt{a})\) that undergoes a P-bifurcation at \( \sigma^2 \sqrt{a} = \frac{1}{2} \). On \((-\sqrt{a}, \infty)\), any solution converges almost surely exponentially towards \( \sqrt{a} \) (see Figure 3). For \( \alpha = 1 \) and \( a < 0 \), there exists a unique distribution charging the whole real line.

- For \( \alpha > 1 \), the system has no additional stationary solution.

**Proof.** In the case \( \alpha = 1 \), the Kolmogorov equation has the solution:

\[
p_0(x) \propto |x + \sqrt{a}|^{-2+ \frac{1}{\sigma^2 \sqrt{a}}} |x - \sqrt{a}|^{-2- \frac{1}{\sigma^2 \sqrt{a}}}
\]

For \( a > 0 \), this function is integrable at \( \sqrt{a} \) if and only if \( -2 - \frac{1}{\sigma^2 \sqrt{a}} > -1 \) which is not possible. At \( -\sqrt{a} \), this function is integrable when \( -2 + \frac{1}{\sigma^2 \sqrt{a}} > 1 \) i.e. \( \sigma^2 \sqrt{a} < 1 \) (i.e. when \( -\sqrt{a} \) is unstable). Eventually, this function is integrable at
infinity whatever the parameters, since it behaves like $x^{-4}$. Hence there exists a unique stationary distribution charging $(-\infty, -\sqrt{a})$. In both the interval $(-\sqrt{a}, \sqrt{a})$ and $(\sqrt{a}, \infty)$ the only stationary distribution is $\delta_{\sqrt{a}}$ which is stable.

The additional stationary distribution on $(-\infty, -\sqrt{a})$ reads:

$$p_0(x) = Z \left(-x - \sqrt{a}\right)^{-2+\frac{1}{2}+\frac{1}{2\sqrt{a}}} \left(-x + \sqrt{a}\right)^{-2-\frac{1}{2}+\frac{1}{2\sqrt{a}}} =: Z q_0(x)$$

Denoting $\Phi = \frac{1}{\sigma^2 \sqrt{a}}$, we give the following closed form primitives related:

\[
\begin{align*}
\int q_0(x) \, dx &= -(-\sqrt{a} - x)^{-1+\Phi} (\sqrt{a} - x)^{-1-\Phi} a(2\Phi^2 - 1) - 2\sqrt{a} \Phi x + x^2 \\
\int x q_0(x) \, dx &= (-\sqrt{a} - x)^{-1+\Phi} (\sqrt{a} - x)^{-1-\Phi} - 2\sqrt{a} \Phi x + a + x^2 \\
\int x^2 q_0(x) \, dx &= (-\sqrt{a} - x)^{-1+\Phi} (\sqrt{a} - x)^{-1-\Phi} - 2\sqrt{a} \Phi x + a + (2\Phi^2 - 1)x^2
\end{align*}
\]

Using these formulae, it is straightforward to obtain the normalization constant of $p_0$ on $(-\infty, -\sqrt{a})$:

$$Z = 4a^{3/2} \Phi (\Phi^2 - 1)$$

The expectation $m$ is equal to:

$$m = Z \int_{-\infty}^{-\sqrt{a}} x q_0(x) \, dx = -\Phi \sqrt{a} = -\frac{1}{\sigma^2},$$

and the second moment:

$$s = Z \int_{-\infty}^{-\sqrt{a}} x^2 q_0(x) \, dx = (2\Phi^2 - 1)a$$

Eventually, let us analyze the shape of the obtained stationary distribution. The differential of the distribution reads:

$$p_0'(x) = p_0(x) \left(\frac{-2 - \Phi}{x - \sqrt{a}} + \frac{-2 + \Phi}{x + \sqrt{a}} \right).$$

This quantity is always negative for $\Phi < 2$, the density is decreasing and moreover in that case it is clear that the distribution diverges at $-\sqrt{a}$. For $\Phi > 2$, the distribution vanishes at zero, takes its maximum at $x^* = -\frac{\Phi \sqrt{a}}{2} = -\frac{1}{2\sigma^2}$ and goes back to zero.

Let us now address the stability in the first approximation for this distribution. The linearized equation reads:

$$dv_t = -2X_tv_t \, dt + 2\sigma X_tv_t \, dW_t$$

Integrating the linearized equation as previously done, we obtain that the Lyapunov exponent of the linearized equation reads:

$$l = -2\mathbb{E}[X_t] - 2\sigma^2 \mathbb{E}[X_t^2] = \frac{2(-1 + \sigma^2 a)}{\sigma^2}$$
and since we consider the case $\sigma^2 \sqrt{a} < 1$, the Lyapunov exponent is strictly negative, implying almost sure exponential stability of zero for the linearized equation, hence stochastic linear stability of $p_0(x)$.

For $a < 0$, the solutions of the Kolmogorov equation are given by:

$$p_0(x) = \frac{K}{(x^2 - a)^2} \exp \left(-2 \frac{\arctan(x/\sqrt{-a})}{\sigma^2 \sqrt{-a}}\right)$$

and the definition interval of possible stationary distributions is $\mathbb{R}$. In order for $p_0$ to define a probability distribution, we need this function to be integrable at $\pm \infty$, which is always the case. The obtained stationary distribution reaches its maximum again for $x = -\frac{1}{2\sigma^2}$.

For $\alpha > 1$, the solution of the Kolmogorov equation reads:

$$p(x) = \frac{1}{(a - x^2)^{2\alpha}} \exp \left(\frac{2}{\sigma^2} \int_x^\infty (-y^2 + a)^{1-2\alpha} dy\right)$$

At $-\sqrt{a}$, the exponential term diverges since it behaves as $(x + \sqrt{a})^{1-2\alpha}$, hence the solution is never integrable at zero. It is easy at this point to show that there is no integrable solution to the Kolmogorov equation.

**Remark.** For $\alpha < 1$ one could be interested in the existence of quasi-stationary solutions. An additional difficulty arises from the fact that the solutions can be attracted in finite time towards different solutions: $\pm \sqrt{a}$ whatever $\alpha < 1$, and $-\infty$ for $\alpha < \frac{3}{4}$. Moreover, in the saddle-node bifurcation case as studied here, the diffusion coefficient is exactly equal to a power function. Therefore, for $\alpha < 1$, blow up in finite time as well as absorption at $\pm \sqrt{a}$ can occur are not a consequence of the results of [5], are not in the scope of the present manuscript and require new mathematical developments that will be addressed elsewhere.

The above result characterize the dynamics relatively exhaustively for $\alpha = 1$, except for $a > \sigma^{-4}$ and $x_0 \in (-\sqrt{a}, \sqrt{a})$. In that case, both singular points are stable, and can be reached by the solution. Therefore for such initial conditions, there is a competition between the possible attractors, and the relative stability of one singular point compared to the other will determine the probability, given an initial condition $x_0$, to converge towards this equilibrium. This quantity is plotted in Figure 3(c). We observe, as one can expect, that the attractivity of $\sqrt{a}$ is stronger than that of $-\sqrt{a}$: the probability of converging towards $\sqrt{a}$ is never less than 0.4 even for initial conditions very close from $-\sqrt{a}$, and goes relatively fast towards 1 as the initial condition is chosen closer from $\sqrt{a}$.

**Universality properties**  Let us now discuss the universality of this bifurcation. We consider a general one-dimensional diffusion equation:

$$dx_t = f(x_t, \lambda) \, dt + g(x_t) \, dW_t$$

with $f$ a smooth function such that $f(0, 0) = \frac{\partial f}{\partial x}(0, 0) = 0$ and $\frac{\partial f}{\partial \lambda}(0, 0) \neq 0$. Following the reduction to a normal form of Kuznetsov [17], we use a Taylor
expansion of $f$:

$$f(x, \lambda) = f_0(\lambda) + f_1(\lambda) x + f_2(\lambda) x^2 + h(x)$$

where $h(x)/x^3$ is bounded at $x = 0$. We perform a shift of coordinates by defining a new variable $\xi = x + \delta$ and obtain:

$$d\xi_t = \left( [f_0(\lambda) - f_1(\lambda)\delta] + [f_1(\lambda) - 2f_2(\lambda)\delta] + O(\delta^2) \right) dt + g(\xi - \delta) dW_t$$

Using the inverse function theorem under the assumption that $\frac{\partial f}{\partial \lambda}(0, 0) \neq 0$, we deduce that there exists a smooth function $\delta(\lambda) = \frac{\lambda}{2f_2(0)} \lambda + O(\lambda^2)$ such that the term $[f_1(\lambda) - 2f_2(\lambda)\delta] + O(\delta^2)$ vanishes for all $\lambda$ in a neighborhood of 0. This $\delta(\lambda)$ identified, we can apply the inverse function theorem to find locally an inverse to the constant term of the Taylor development seen as a function of $\lambda$. This inverse function, $\lambda(\mu)$, yields the system to the SDE:

$$d\xi_t = (\mu + b(\mu)\xi_t^2 + O(\xi^3)) dt + g(\xi - \delta(\lambda(\mu))) dW_t$$

with $b(0) = f_2(0) \neq 0$. The final reduction involves a scaling of the variable: $z_t = |b(\mu)|\xi_t$, yielding the SDE:

$$dz_t = \left( \beta + \epsilon z_t^2 + O(z_t^3) \right) dt + g\left( \frac{z_t}{|b(\mu)|} - \delta(\lambda(\mu)) \right) dW_t$$

where $\epsilon$ has the sign of $S^{(3)}(0)$. The stability and bifurcations of the whole system is hence reduced to the local analysis of the behavior of $g$ close of the equilibria.

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27
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Figure 2: (a) Codimension 2 stochastic bifurcation diagram of the pitchfork bifurcation: orange: zero is stable, blue: zero is unstable, yellow: zero is stable and is reached in finite time. (b) Sample path trajectories in all 6 regimes for the stochastic pitchfork bifurcation. Parameters \((\lambda, \sigma, \alpha)\): (b1) \((10, 0.6, 1)\) (quasi-stationary distribution plotted in Fig. 1(a)) (b2): \((2, 1, 1.2)\), (b3): \((1, 1.5, 1.2)\), (b4): \((0.5, 1, 1.2)\), (b5): \((-1, 1.5, 1.2)\). In (b2) and (b3) the peak of the stationary distribution is depicted in dashed black. (c) Representation of stationary densities for the stochastic pitchfork system with linear noise, as a function of the parameter \(\lambda\). Red square: \(\delta_0\) is stable. Blue and colored surfaces display the probability distribution \(p_\lambda(x)\) of the stable stationary distributions, and the black line represents the maximal value of the distribution.
Figure 3: Behaviors of the stochastic saddle-node system with $\alpha = 1$. The bifurcation diagram as a function of $a$ is at the center of the figure, and displays the singular points in blue lines, their stability marked by the type of curve: plain: stable, dashed: unstable. Stationary distributions are encoded in color in the interval $a \leq \sigma^{-4}$. Two typical stationary distributions are plotted in Figs. (e) and (f): one for $a \leq 1/4\sigma^4$ where the pdf has a peak at $x = -\sigma^{-2}/2$, and for $a > \sigma^{-4}/4$ where the distribution diverges at zero. The P-bifurcation between these behaviors is noted P. (a) corresponds to a typical trajectory for $a = 0.4 < \sigma^{-4}/4$ with $\sigma = 0.8$. (b),(c) and (d) correspond to $a > \sigma^{-4}$ and different initial conditions: (c): frequency of the trajectories converging towards $+\sqrt{a}$ as a function of the initial condition in $[-\sqrt{a}, \sqrt{a}]$, (b) and (d) are typical trajectories for initial condition in $(\sqrt{a}, \infty)$ or $(-\infty, -\sqrt{a})$ respectively ($a = 4$ and $\sigma = 0.8$).