State-dependent Trotter Limits and their approximations

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In this paper, we study how to discretize the Trotter product formula of operators with continuous degrees of freedom – such as the position of particles in molecules, or the amplitude of electromagnetic fields – to make them amenable to digital simulations. Usually, such systems are simulated numerically via the Trotter product formula of their discretized approximations, but this approach can potentially lead to fallacious results. Here, we find sufficient conditions to conclude the validity of this approximate discretized physics. We develop a scheme to verify numerical results involving Trotterization of truncated operators. Essentially, it depends on the state-dependent Trotter error of the latter, for which we establish explicit bounds. These bounds are of independent interest and may also find applications in quantum chemistry.

Introduction.—The Trotter product formula is a central tool in the study of quantum dynamics with a vast amount of applications, such as quantum computing [1–4], quantum field theory [5–8] and its path integral formulation [9], quantum control [10–12], open quantum systems [13], quantum optics [14], Floquet dynamics [15–17] and quantum many body systems [18–21]. It allows to split the time-evolution of a quantum system, that is hard or cost-expensive to implement, into simpler components. Due to this feature, it has become the standard convention for numerically simulating the dynamics of large quantum systems. Nevertheless, two important questions regarding these simulations stay open. On one hand, it is not clear how the error due to the Trotter approximation depends on the input state of the system. For example, one would expect that the approximation becomes better when only low-energy states are considered. On the other hand, digital computers – no matter if classical or quantum – can only simulate a finite number of levels. However, many relevant physical systems are actually represented by continuous degrees of freedom, i.e. infinite dimensions. Therefore, the common practice is to truncate the Hilbert space on some finite level [17, 22–24] and deal with the finite-dimensional truncated operators. These truncations, however, are only approximations to the full system. Hence in total, this simulation procedure involves two simultaneous approximations: First, approximating the time-evolution of the full quantum system by some finite-dimensional truncations; second, approximating the full time-evolution of the quantum system by the Trotter product formula. We summarize the interplay of the two approximations in Fig. 1. A priori, it is not self-evident whether they are compatible with each other. In fact, we will discuss a simple and well-known example, where it seems that they are not. In this context, the main obstacle is the question of convergence, i.e.: Is the infinite-dimensional Trotter product formula a valid approximation? Whilst it always converges in finite dimensions [4, 25], it only converges under certain conditions in the infinite-dimensional case [26, 27]. Therefore, a rigorous numerical treatment of infinite-dimensional truncated operators would need to additionally deliver a reasoning why the Trotterization and truncation procedures are nonconflicting. Such a discussion about the validity of the simulations is miss-

\[ H_d^{(1)} = P_d H^{(1)} P_d \]

\[ H_d^{(2)} = P_d H^{(2)} P_d \]

\[ H = H^{(1)} + H^{(2)} \]

\[ H_d = H_d^{(1)} + H_d^{(2)} \]
In this paper, we address both of the aforementioned questions and show that they are closely related. That is, we first establish state-dependent bounds for the Trotter error. Afterwards, we use them to provide sufficient conditions on the truncations, which ensure the validity of simultaneous Trotterization. Firstly, as a technical condition, the Hamiltonians of consideration have to be jointly approximable by the same truncation scheme. Secondly, as one would expect intuitively, the state-dependent error of the finite-dimensional Trotter approximation has to eventually saturate in the truncation dimension. In this case, our method allows to infer bounds for the Trotter error in infinite dimensions from the state-dependent ones of their finite-dimensional truncations. These results have applications to a wide class of physical models. Not only do even the simplest textbook examples such as the quantum harmonic oscillator and the particle in a box have continuous degrees. But also e.g., in quantum optics, the Rabi-model [28, 29] and the spin-boson model [30], which describe the interaction of matter with light, are infinite-dimensional Hamiltonians. In quantum chemistry, atoms and molecules have continuous positions and momenta. The simulation of their chemical behaviours has been proposed as an impactful application of the emerging field of quantum simulation [1, 2]. In addition, modern quantum information makes heavy use of infinite-dimensional systems, such as GKP qubits [31] or superconducting transmon qubits [32, 33], which were the platform for the “quantum supremacy” experiment [34].

Additionally, our state-dependent Trotter bound is of independent and broader interest in the field of quantum information and computation beyond the scope of this paper. Recently, such state-dependent Trotter error bounds enjoy a considerable interest in particular in the area of Hamiltonian simulation. For instance in [35, 36], the asymptotic scaling of the finite-dimensional Trotter error is derived when the input state is only supported in a (low-energy) subspace. These results rely on strong assumptions on the respective Hamiltonians in order to bound the leakage from this (low-energy) subspace. The authors in [37, 38] study energy-constrained distances between time evolutions generated by Hamiltonians with continuous degrees of freedom. Their results have applications in the realm of Trotterization but only hold for special Hamiltonians generating a particular type of dynamics. In addition, in [39, 40] state-dependent Trotter error bounds are derived for certain (time-dependent) Hamiltonians. These bounds rely on restrictive assumptions on the structure of the commutator applied to the input state. The advantage of our bound, compared to the existing bounds in the literature, is its validity for all finite-dimensional Hamiltonians. Furthermore, it is an explicit bound, which shows a particularly simple dependence on the input state. By taking linear combinations of input vectors, we are immediately able to bound the Trotter convergence speed in arbitrary subspaces.

**Setting.**—Before turning to our main results, let us briefly summarise the setting, in which the Trotter product formula and its truncated versions are studied. A more elaborate introduction is given in the Supplementary Material, Sec. A. We will consider two Hamiltonians \( H^{(1)} \) and \( H^{(2)} \) acting on an infinite-dimensional Hilbert space \( \mathcal{H} \) with countable basis. This is, two self-adjoint operators, which are defined on a dense domain of input states \( \mathcal{D}(H^{(1)}), \mathcal{D}(H^{(2)}) \), respectively. The domains encode important features of the physical system, such as boundary conditions. It often suffices to look at a core, which is a particular subspace of the domain that already carries all the relevant information. See the Supplementary Material, Sec. A for details. We do not make any assumptions on the spectrum of the Hamiltonians in order to accommodate for both purely continuous and (infinitely many) discrete degrees of freedom. The time-evolution under each Hamiltonian is given by a unitary operator \( U^{(1)}(t) = e^{-itH^{(1)}}, U^{(2)}(t) = e^{-itH^{(2)}}, \) respectively. Now, the Trotter product formula, if converging, approximates the dynamics \( U(t) = e^{-itH} \) of a self-adjoint operator \( H \) by the \( n \)-fold product of the individual unitaries \( X(t/n)^n \equiv [U^{(1)}(t/n)U^{(2)}(t/n)]^n \). On the common core \( \mathcal{H} \) restricts to the sum of \( H^{(1)} \) and \( H^{(2)} \), see the Supplementary Material, Sec. C. In order to constrain finite-dimensional approximations of the above operators, we will first fix a basis \( \{ |j\rangle : j \in \mathbb{N}_0 \} \) of the Hilbert space \( \mathcal{H} \). Then, we obtain a truncated, finite-dimensional Hilbert space \( \mathcal{V}_d = \text{span}\{0\}, \ldots, |d-1\rangle \} \) of dimension \( d < \infty \) simply by gathering the first \( d \) basis vectors. We can project any vector in \( \mathcal{H} \) onto \( \mathcal{V}_d \) with \( P_d = \sum_{j=0}^{d-1} \langle j | j \rangle |j\rangle \). Similarly, for our Hamiltonians we receive their truncated versions by \( H^{(1)}_d = P_d H^{(1)} P_d, H^{(2)}_d = P_d H^{(2)} P_d \) and \( H_d = P_d H P_d, \) if existing. All the truncated Hamiltonians only act on the finite-dimensional \( \mathcal{V}_d \) and can, therefore, be represented by Hermitian \( d \times d \) matrices. They generate finite-dimensional unitary dynamics on \( \mathcal{V}_d \) via \( U^{(1)}_d(t) = e^{-itH^{(1)}_d}, U^{(2)}_d(t) = e^{-itH^{(2)}_d} \) and \( U_d(t) = e^{-itH_d}, \) respectively. One goal of this paper is to give conditions on the finite-dimensional truncated Trotter product formula which imply the convergence of the full infinite-dimensional case. We will explain in the Supplementary Material, Sec. A that it is necessary to include the dependence on an input state (also see [41, Sec. 3] and [42, Sec. VIII.8]). That is, by studying \( [U^{(1)}_d(t/n)U^{(2)}_d(t/n)]^n |\psi\rangle \xrightarrow{n \to \infty} U_d(t)|\psi\rangle \), we would like to draw conclusions about \( [U^{(1)}(t/n)U^{(2)}(t/n)]^n |\psi\rangle \xrightarrow{n \to \infty} U(t)|\psi\rangle, \) where \( |\psi_d\rangle = P_d|\psi\rangle \). To this end, we introduce the state-dependent
Trotter error in the finite-dimensional case

\[ b_d^{(n)}(\psi; t) := \left\| \left( [U_d^{(1)}(t/n)U_d^{(2)}(t/n)]^n - U_d(t) \right) \psi_d \right\| , \tag{1} \]

from which we can inherit a dimension-independent Trotter error by taking the limit \( d \to \infty \)

\[ b^{(n)}(\psi; t) := \lim_{d \to \infty} b_d^{(n)}(\psi; t). \tag{2} \]

Notice that \( b^{(n)}(\psi; t) \) might reach the maximum distance of 2 even though \( b_d^{(n)}(\psi; t) \) is considerably smaller for a relatively large \( d \).

**Main Results.**—Our first main result is an explicit state-dependent Trotter error bound for arbitrary finite-dimensional systems. Usually, when computing the norm between the particular input state quantifying a worst-case scenario. Instead, this state-dependent error by taking the limit \( d \to \infty \), this is a rather unfavourable property and state-dependent error bounds are mandatory. We study these truncated product formulas in our next main result. It manifests a method to check convergence of the Trotter product formula which is particularly well-suited to numerical simulations. Intuitively, if one seeks out that the state-dependent Trotter error \( b_d^{(n)}(\psi; t) \) saturates with increasing \( d \), one would expect that the infinite-dimensional analogue converges for this particular \( \psi \). Indeed, this is what we find.

**Main Result 2** (Truncation of an infinite-dimensional Trotter product). The infinite-dimensional Trotter product formula converges for all \( \psi \in \mathcal{H} \),

\[ (U^{(1)}(t/n)U^{(2)}(t/n))^n \psi \xrightarrow{n \to \infty} U(t) \psi, \tag{4} \]

if the following two conditions are satisfied:

(i) \( H^{(1)} \) and \( H^{(2)} \) can be simultaneously approximated with \( H_d^{(1)} \) and \( H_d^{(2)} \) by the same truncation scheme. That is, \( \mathcal{V} = \bigcup_d V_d \) is a common core of \( H^{(1)} \) and \( H^{(2)} \).

(ii) For all total evolution times \( t \in \mathbb{R} \), the dimension-independent Trotter error goes to zero,

\[ b^{(n)}(\psi; t) \to 0, \text{ as } n \to \infty. \]

In this case, \( U(t) \) gives a unitary dynamics, whose generator \( H \) is self-adjoint and agrees with \( (H^{(1)} + H^{(2)}) \), wherever both are defined. A generalized version to non-unitary dynamics can be found in the Supplementary Material, Sec. C, see Thm. 7.

In order to prove this, we first show that \( U_d(t) \psi \) has a limit as \( d \to \infty \), i.e. the finite-dimensional approximations applied to a state \( \psi \) converge to a well-defined \( \psi(t) \). Since the Trotter limit \( n \to \infty \) is always well-defined in finite dimensions [25], the \( U_d(t) \psi \) can be obtained by Trotterization. Then, we prove that the infinite-dimensional Trotter formula, \( X(t/n)^n \psi \to \psi(t) \) as \( n \to \infty \), converges to the same limit \( \psi(t) \). The last step is to show that this limit is indeed governed by a unitary time-evolution, whose generating Hamiltonian agrees with \( H^{(1)} + H^{(2)} \) on \( \mathcal{V} \). These ramifications are derived using a modification of the Trotter-Kato approximation theorems [44, Thm. 4.8, Thm. 4.9] inspired by [45, Thm. 2.3]. Notice that we only have \( \mathcal{V} \subsetneq \mathcal{H} \), thus in particular \( \mathcal{V} \neq \mathcal{H} \). Furthermore, two Hamiltonians may have a non-overlapping domains, see e.g. [11].
Example 3.8]. This situation is excluded by our common core assumption.

Our truncation main result establishes a way to obtain explicit error bounds for the infinite-dimensional Trotter product formula once the finite-dimensional case is under control: A simple calculation shows that (see the Supplementary Material, Sec. C)

\[ b^{(n)}(|\psi\rangle; t) \leq \left( \sum_{j=1}^{d} |\langle j|\psi\rangle|^2 b^{(n)}(|j\rangle; t)^2 \right)^{1/2}. \]  \( (5) \)

Thus, in order to conclude that the Trotter product formula converges for all $|\psi\rangle \in \mathcal{H}$, it suffices to require $b^{(n)}(|j\rangle; t) \to 0$ as $n \to \infty$ for all basis vectors $|j\rangle$. This is because if $|\psi\rangle \in V$ and hence $|\psi\rangle \in V_d$ for sufficiently large $d$ – then Eq. (5) also goes to zero.

Examples.—Let us numerically study an example of a convergent and a non-convergent Trotter scenario. We denote the position operator with $Q$ and the momentum operator with $P$. The truncation is done in the Fock basis \{$|m\rangle\$}, which is a common core of all the considered operators. That is, we project with $P_d = \sum_{m=0}^{d-1} |m\rangle\langle m|$ onto the finite-dimensional Hilbert space $V_d$. Fig. 2 shows the case of $H^{(3)} = \frac{1}{2} (Q^2 + P^2)$, i.e. the quantum harmonic oscillator, and $H^{(2)} = \frac{1}{2} (QP + PQ)$, i.e. the Hamiltonian which generates the squeezing transformation. It is known that this infinite-dimensional Trotter problem converges for all $|\psi\rangle \in \mathcal{H}$, see for instance [27]. Indeed, we numerically find that the Trotter error on the finite-dimensional truncations $V_d$ is bounded independent of the dimension of truncation for all considered Fock basis states $|m\rangle$. For a non-convergent Trotter scenario, we look at the two operators $H^{(1)} = Q^3$ and $H^{(2)} = P^2$ in Fig. 3. The corresponding sum $H^{(1)} + H^{(2)} = P^2 + Q^3$ describes a particle in a $Q^3$ potential. This Trotter problem does not converge as even in the classical version of this Hamiltonian, the particle would escape to infinity at finite times [46]. Our numerical study also reveals that the Trotter error on the finite-dimensional truncations $V_d$ does not saturate in the truncation dimension $d$. Instead, we observe a phase transition to a chaotic behaviour similar to [15, 17].

Through our state-dependent error bound, we can also perform an analytical examination in certain cases. For this, consider $H^{(1)} = \frac{1}{2} Q^2$ and $H^{(2)} = \frac{1}{2} P^2$ and truncate at level $n$ in the Fock basis \{$|m\rangle\$} as before. We then find the analytic bound

\[ b_d^{(n)}(|m\rangle; t) \leq \frac{t^2}{4n} \sqrt{\frac{3}{8} (m(m + 1) (m^2 + m + 14) + 10)}, \]  \( (6) \)

see the Supplementary Material, Sec. D. This bound does not depend on the cutoff dimension $d$. Hence, by our truncation main result the full continuous Trotter problem converges and its Trotter error can be quantified for Fock states explicitly by Eq. (6). For a numerical comparison, see Fig. 4.

Conclusion.—From a practical perspective, our truncation main result reduces the complexity of determining Trotter convergence and finding infinite-dimensional error bounds to just computing finite-dimensional error bounds, which saturate in the truncation dimension. The latter may be obtained through our state-dependent error bounds from the first main result. This makes the common practice of numerical Trotter simulations rigorous. From a fundamental perspective, it would be great
Figure 4. State-dependent error for the operators $H^{(1)} = \frac{i}{2} Q^2$ and $H^{(2)} = \frac{i}{2} \hat{P}^2$. We consider the first five Fock-basis states $\{ |m\rangle \} = \{ |0\rangle, \ldots, |4\rangle \}$ for different dimensions of truncations $d = 1, \ldots, 50$. The total evolution time is fixed to $t = 1$ and the number of Trotter steps is $n = 1000$. The dots are a numerical simulation, whereas the lines show the explicit error bounds from Eq. (6). Since the Trotter error can be bounded independently of the truncation dimension, this Trotter problem converges.

to have a generic method in order to analytically check the convergence of the infinite-dimensional Trotter product formula in a general setting. This is a hard problem and only very little literature in the mathematical physics community exists on this topic. Our result can extend to such a method in certain cases. For this, one has to be able to apply our finite-dimensional Trotter bounds to the truncation scenario. In particular, one needs that there are eigenstates of the infinite-dimensional target Hamiltonian that stay eigenstates of the truncated operators. This property does not hold in general and therefore sets a practical limitation on our method. For example, one would expect this property to break down if the infinite-dimensional target Hamiltonian does not have eigenstates. In such a case, when our first main result is not applicable to the truncation scenario but the assumptions of our second main result still hold, one might get indication for Trotter convergence by numerically studying the finite-dimensional state-dependent Trotter error. Of course, generalizing our Trotter error bounds to arbitrary input states would make this obsolete: If such a bound existed, our truncation main result would give a generic method to analytically check Trotter convergence. Both of our main results naturally generalize to Trotter products of more than two operators.

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Supplementary Material

A. MATHEMATICAL PRELIMINARIES

In this section, we introduce all the mathematical concepts and the notation needed for the proof of our main results. We also elaborate more on the mathematical setting and motivate why we have to consider state-dependent error bounds instead of norm error bounds.

Consider a separable infinite-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|j\rangle\}$. A corresponding sequence of truncated finite-dimensional Hilbert spaces of increasing dimension $d < \infty$ is then given by $V_d = \text{span}\{0, \ldots, d - 1\} \subset \mathcal{H}$. Denote the orthogonal projection onto $V_d$ by $P_d = \sum_{j=0}^{d-1} |j\rangle\langle j|$. We denote the union of the truncated Hilbert spaces by $V := \bigcup_{d} V_d$. Since $\{|j\rangle\}$ is a basis of $\mathcal{H}$, $V$ lies dense in $\mathcal{H}$, i.e. $V^\perp = \{0\}$.

The scalar product of $\mathcal{H}$ is denoted by $\langle \cdot | \cdot \rangle$ and its induced norm by $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$, $\psi \in \mathcal{H}$.

A strongly continuous contraction semigroup is a set of linear operators $T(t)$ on $\mathcal{H}$, $t \geq 0$, with the following properties

1. semigroup: for all $t, s \in \mathbb{R}_\geq 0$, $T(t)T(s) = T(t+s)$ and $T(0) = \text{id}$,

2. contraction: $\|T(t)\psi\| \leq \|\psi\|$, for all $\psi \in \mathcal{H}$,

3. strong continuity: For all $\psi \in \mathcal{H}$, $\lim_{t\downarrow 0} \|T(t)\psi - \psi\| = 0$.

Every strongly continuous contraction semigroup defines an unbounded operator $K : \mathcal{D}(K) \rightarrow \mathcal{H}$ by setting $\mathcal{D}(K)$ to be the set such that $K\psi := \lim_{t\downarrow 0} \frac{i}{\hbar} (T(t)\psi - \psi)$ exists [44]. As is typical for unbounded operators $\mathcal{D}(K)$ will usually be a proper subset of $\mathcal{H}$. In fact, $\mathcal{D}(K) = \mathcal{H}$ if and only if the semigroup is uniformly continuous. Here, uniform refers to the topology induced by the operator norm $\|A\|_\infty = \sup_{\|\psi\|=1} \|A\psi\|$. In finite dimensions, the uniform and the strong operator topology coincide, but in infinite dimensions this is wrong and the uniform topology is stronger. The operator $K$ is then called the generator of the semigroup and (in principle) encodes all information necessary to compute $T(t)$. Self-adjoint (unbounded) operators $H$ on $\mathcal{H}$ are precisely the generators of unitary groups $U(t) = e^{-itH}$. We will prove a version of Main Result 2 from the main text for strongly continuous contraction semigroups. The unitary case as considered in the main text then follows from that. See Thm. 7 in Sec. C of this Supplementary Material.

Next, we recall the Trotter product formula. This discussion will motivate the use of state-dependent error bounds. For this purpose, let us focus on the finite-dimensional unitary case first: Let $\dim(\mathcal{H}) < \infty$ and let $H^{(1)} : \mathcal{H} \rightarrow \mathcal{H}$, $H^{(2)} : \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint, hence generators of two continuous unitary groups. Then the uniform Trotter error satisfies the bound [4, 25]

$$
\left\| \left( e^{-\frac{i}{\hbar}H^{(1)}} e^{-\frac{i}{\hbar}H^{(2)}} \right)^n - e^{-it(H^{(1)} + H^{(2)})} \right\|_\infty \leq \frac{t^2}{2n} \left\| [H^{(1)}, H^{(2)}] \right\|_\infty.
$$

(7)

Due to the explicit bound with $\mathcal{O}(1/N)$ scaling, we can conclude from Eq. (7) that the finite-dimensional Trotter product formula always converges uniformly, i.e. the uniform Trotter error goes to zero as $n \rightarrow \infty$. But what does Eq. (7) imply in the situation where the Hamiltonians are finite-dimensional truncations of infinite-dimensional operators? Let $H^{(1)} : \mathcal{D}(H^{(1)}) \rightarrow \mathcal{H}$ and $H^{(2)} : \mathcal{D}(H^{(2)}) \rightarrow \mathcal{H}$ be two self-adjoint operators acting on a separable infinite-dimensional Hilbert space $\mathcal{H}$. We do not make any assumptions on the spectra of these two operators, in particular we allow for continuous spectra. Denote their finite-dimensional approximations as $H^{(i)}_d := P_d H^{(i)} P_d : V_d \rightarrow V_d$ with $i = 1, 2$. Obviously, Eq. (7) holds for $H^{(1)}_d$ and $H^{(2)}_d$, so that

$$
\beta_d^n(t) := \left\| \left( e^{-\frac{i}{\hbar}H^{(1)}_d} e^{-\frac{i}{\hbar}H^{(2)}_d} \right)^n - e^{-it(H^{(1)}_d + H^{(2)}_d)} \right\|_\infty \leq \frac{t^2}{2n} \left\| [H^{(1)}_d, H^{(2)}_d] \right\|_\infty.
$$

(8)

In this case, for a fixed total evolution time $t$ and a fixed number of Trotter steps $n$, the commutator error $\left\| [H^{(1)}_d, H^{(2)}_d] \right\|_\infty$ will typically diverge with the dimension $d$ of truncation. This is due to the fact that the operator norm $\| \cdot \|_\infty$ gives a worst-case error by taking the supremum over all normalised $|\psi\rangle \in V_d$. By increasing $d$, new vectors will be added so that the Trotter error in Eq. (8) becomes larger. Consequently, the Trotter approximants might fail to converge uniformly. In fact:

**Proposition 3.** If the uniform Trotter error $\beta_d^n(t)$ is bounded independently of $d$ then the infinite-dimensional Trotter product converges uniformly.
from the main text. Throughout, we will assume that $D$ is dense in $H$ and that $\langle \psi, H\phi \rangle$ is well-defined and symmetric. If $H_1$ is another operator, then we write $H \subset H_1$ and call $H_1$ an extension of $H$ if $D(H) \subset D(H_1)$ and if $H_1|\psi = H|\psi$ for $|\psi \rangle \in D(H)$. $H$ is called densely defined if $D(H)$ is dense in $\mathcal{H}$. In that case, its adjoint $H^*$ is a well-defined closed operator. If $H$ is symmetric, i.e. $\langle \psi, H\varphi \rangle = \langle H\psi, \varphi \rangle$, then $H^{**}$ is the smallest closed extension of $H$, called the closure.

We say that a subspace $W \subset D(H)$ is a core for $H$ if $W$ is dense in $D(H)$ with respect to the graph norm of $H$.

### B. PROOF OF THE FIRST MAIN RESULT

This section demonstrates the proof of our Main Result 1 from the main text. Throughout, we will assume that $\mathcal{H}$ is a finite-dimensional Hilbert space. We then consider two Hamiltonians on $\mathcal{H}$, i.e. self-adjoint operators $H^{(1)} : \mathcal{H} \rightarrow \mathcal{H}$ and $H^{(2)} : \mathcal{H} \rightarrow \mathcal{H}$.
and \( H^{(2)} : \mathcal{H} \to \mathcal{H} \), whose Trotter product is examined. Denote
\[
U(t) = e^{-i(H^{(1)} + H^{(2)})t}
\]
(12)
and
\[
W^{(n)}(t) = \left( e^{-iH^{(1)}t} e^{-iH^{(2)}t} \right)^n.
\]
(13)
We now recapitulate [43, Lemma 1], in particular Eq. (2.4) therein. This will be needed in order to prove our Main Result 1.

Lemma 4. Let \( \tilde{H}_1(t), \tilde{H}_2(t) \) be two families of time-dependent, self-adjoint and locally integrable operators. Define the corresponding unitaries they generate
\[
\tilde{U}_j(t) = \mathcal{T} \exp \left( -i \int_0^t \tilde{H}_j(s) ds \right), \quad j = 1, 2,
\]
(14)
where \( \mathcal{T} \) denotes time-ordering. Furthermore, define the integral action \( S_{21}(t) \) as
\[
S_{21}(t) = \int_0^t \left[ H_2(s) - \tilde{H}_1(s) \right] ds.
\]
(15)
Then for all \( t \in \mathbb{R}_{\geq 0} \),
\[
\tilde{U}_2(t) - \tilde{U}_1(t) = -i S_{21}(t) \tilde{U}_2(t) - \int_0^t \tilde{U}_1(t) \tilde{U}_1(s)^\dagger \left[ \tilde{H}_1(s) S_{21}(s) - S_{21}(s) \tilde{H}_2(s) \right] \tilde{U}_2(s) ds.
\]
(16)
Proof. This is proved in [43, Lemma 1].

Let us restate our Main Result 1 in a slightly generalized version.

First Main Result. Let \( \mathcal{H} \) be a finite-dimensional Hilbert space and let \( H^{(1)} : \mathcal{H} \to \mathcal{H} \) and \( H^{(2)} : \mathcal{H} \to \mathcal{H} \) be self-adjoint operators on \( \mathcal{H} \). Define \( U(t) \) and \( W^{(n)}(t) \) as in Eq. (12) and Eq. (13), respectively. If \( |\phi\rangle \) is an eigenvector of the self-adjoint operator \( H^{(1)} + H^{(2)} \) according to the eigenvalue equation \( (H^{(1)} + H^{(2)}) |\phi\rangle = h |\phi\rangle \), then we have for all \( t \in \mathbb{R}_{\geq 0} \) and \( n \in \mathbb{N} \),
\[
\left\| \left( U(t) - W^{(n)}(t) \right) |\phi\rangle \right\| \leq \frac{t^2}{2n} \inf_{\alpha \in \mathbb{R}} \left( \left\| H^{(1)} - \alpha h \right\|^2 + \left\| H^{(2)} - (1 - \alpha)h \right\|^2 \right).
\]
(17)
Proof. We start by applying Lemma 4. For this, take \( \tilde{U}_1(2t) = W^{(n)}(t) \) and \( \tilde{U}_2(2t) = U(t) \). Notice that \( \tilde{U}_1 \) is generated by a piece-wise constant Hamiltonian \( \tilde{H}_1(s) \) over the time interval \([0, 2t]\). This Hamiltonian is defined by
\[
\tilde{H}_1(s) = \begin{cases} H^{(2)}, & t \in \left[ 0, \frac{t}{2} \right] \\ H^{(1)}, & t \in \left[ \frac{t}{2}, 2t \right] \end{cases}
\]
(18)
and then periodically extended by \( \tilde{H}_1(s + \frac{2t}{n}) = \tilde{H}_1(s) \). The unitary \( \tilde{U}_2 \) is simply generated by the average constant Hamiltonian \( \tilde{H}_2(s) = \frac{H^{(1)} + H^{(2)}}{2} \). Since both Hamiltonians are locally integrable, Lemma 4 can be applied. We want to estimate the error on an eigenvector \( (H^{(1)} + H^{(2)}) |\phi\rangle = h |\phi\rangle \) :
\[
\left( U(t) - W^{(n)}(t) \right) |\phi\rangle = -ie^{-ith} S_{21}(2t) |\phi\rangle - \int_0^{2t} e^{-i\frac{3}{2}h} \tilde{U}_1(2t) \tilde{U}_1(s) \left[ \tilde{H}_1(s) - \frac{h}{2} \right] S_{21}(s) |\phi\rangle ds.
\]
(19)
The action reads

\[ S_{21}(s) = \int_0^s \left( \frac{H^{(1)} + H^{(2)}}{2} - \tilde{H}_1(\tau) \right) d\tau \]

\[ = \int_0^{[s/\frac{\pi}{n}]} \left( \frac{H^{(1)} + H^{(2)}}{2} - \tilde{H}_1(\tau) \right) d\tau + \int_{[s/\frac{\pi}{n}]}^{s} \left( \frac{H^{(1)} + H^{(2)}}{2} - \tilde{H}_1(\tau) \right) d\tau \]

\[ = \int_{0}^{\frac{s}{\frac{2\pi}{n}}} \left( \frac{H^{(1)} + H^{(2)}}{2} - \tilde{H}_1(\tau) \right) d\tau \]

\[ = \int_0^{\{s/\frac{\pi}{n}\}} \left( \frac{H^{(1)} + H^{(2)}}{2} - \tilde{H}_1(\tau) \right) d\tau, \quad (20) \]

where we split the integral into full periods and a remainder and used the periodicity of \( \tilde{H}_1(\tau) \). In particular, \( S_{21}(2t) = 0 \) and we can bound

\[ \| (U(t) - W^{(n)}(t)) |\varphi| \| \leq \int_0^{2t} \| \tilde{H}_1(s) - \frac{h}{2} \|S_{21}(s)|\varphi| \| ds. \quad (21) \]

For \( s \in \left[0, \frac{t}{n}\right) \), we have

\[ S_{21}(s) = \int_0^s \left( \frac{H^{(1)} + H^{(2)}}{2} - H^{(2)} \right) d\tau = s \frac{H^{(1)} - H^{(2)}}{2}. \quad (22) \]

Analogously for \( s \in \left[\frac{t}{n}, \frac{2t}{n}\right) \), we obtain

\[ S_{21}(s) = \frac{H^{(1)} - H^{(2)}}{2} - \frac{H^{(1)}}{2} \int_{\frac{t}{n}}^{\frac{s}{\frac{2\pi}{n}}} \left( \frac{H^{(1)} + H^{(2)}}{2} - H^{(1)} \right) d\tau = \left( \frac{2t}{n} - s \right) \frac{H^{(1)} - H^{(2)}}{2}. \quad (23) \]

Therefore,

\[ \| (U(t) - W^{(n)}(t)) |\varphi| \| \leq n \int_0^{\frac{t}{n}} \| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \| ds \]

\[ + \int_{\frac{t}{n}}^{\frac{2t}{n}} \left( \frac{t}{n} - \frac{s}{\frac{2\pi}{n}} \right) \left( \frac{H^{(1)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \| ds \]

\[ = \frac{t^2}{4n} \left( \left\| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \right\| + \left\| \left( \frac{H^{(1)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \right\| \right), \quad (24) \]

where we used the periodicity of \( \tilde{H}_1(\tau) \) and of \( S_{21}(s) \) in the first step. We expand

\[ \left\| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \right\| = \left\| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right) \left( H^{(1)} + H^{(2)} - 2H^{(2)} \right) |\varphi| \right\| \]

\[ = \left\| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right) \left( h - 2H^{(2)} \right) |\varphi| \right\| \]

\[ = 2 \left\| \left( \frac{H^{(2)} - \frac{h}{2}}{2} \right)^2 |\varphi| \right\| \quad (25) \]

and similar for the second term

\[ \left\| \left( \frac{H^{(1)} - \frac{h}{2}}{2} \right) \left( H^{(1)} - H^{(2)} \right) |\varphi| \right\| = 2 \left\| \left( \frac{H^{(1)} - \frac{h}{2}}{2} \right)^2 |\varphi| \right\|. \quad (26) \]
Therefore, in total we receive
\[
\left\| \left( U(t) - W^{(n)}(t) \right) |\phi| \right\| \leq \frac{t^2}{2n} \left( \left\| \left[ H(1) - \frac{h}{2} \right] |\phi| \right\| + \left\| \left[ H(2) - \frac{h}{2} \right] |\phi| \right\| \right),
\]
(27)
which is the bound from our Main Result 1 in the main text. In order to show the slightly generalized version of this bound stated in Eq. (17), notice that we can always rescale the Hamiltonians \( H(1) \) and \( H(2) \) by a multiple of the identity \( I \) without changing the Trotter error. That is, by rescaling \( H(1) \rightarrow H(1) + aI \) and \( H(2) \rightarrow H(2) + bI \) we will not change \( \| (U(t) - W^{(n)}(t)) |\phi| \| \) since \( I \) commutes with any Hamiltonian. This procedure leads to a rescaling \( h \rightarrow h + a + b \) and Eq. (27) becomes
\[
\left\| \left( U(t) - W^{(n)}(t) \right) |\phi| \right\| \leq \frac{t^2}{2n} \left( \left\| \left[ H(1) - \frac{h + b - a}{2} \right] |\phi| \right\| + \left\| \left[ H(2) - \frac{h + a - b}{2} \right] |\phi| \right\| \right),
\]
(28)
which is true for any \( a, b \in \mathbb{R} \). The assertion then follows by defining \( \alpha = (h + b - a)/(2h) \) and taking the infimum over \( \alpha \). The latter is possible since the inequality holds for all \( \alpha \in \mathbb{R} \).
\[ \square \]

C. PROOF OF THE SECOND MAIN RESULT

In this section, we present the proof of Main Result 2 from the main text. For this, we will use the notation introduced in Sec. A of this Supplementary Material. We will first prove a more general result, which holds for strongly continuous contraction semigroups. This is given by Thm. 7. Our Main Result 2 will then follow as a special case. In the proofs of the theorems, we will use the following consequence of the Trotter-Kato approximation theorem [44, Sec. 4]:

**Theorem 5.** For each \( n \in \mathbb{N} \), let \( T_d(t) \) be a contraction semigroup on \( V_d \). Let \( H_d \) be the sequence of generators. The following statements are equivalent
(a) The strong limit \( T(t) = \text{s-lim}_{d \to \infty} T_d(t) \) exists for all times \( t > 0 \) and defines a strongly continuous contraction semigroup.
(b) The operator \( H : \mathcal{D}(H) \to \mathcal{H} \), defined by
\[
|\psi\rangle \in \mathcal{D}(H) \iff \text{there exists a sequence } (|\psi_d\rangle)_{d \in \mathbb{N}} \in \mathcal{C}, \text{ such that } |\psi_d\rangle \to |\psi\rangle \in \mathcal{H},
\]
and \( H|\psi\rangle := \lim_{d \to \infty} H_d|\psi_d\rangle \) exists,

is actually well-defined in the sense that \( H|\psi\rangle \) does not depend on the chosen sequence \( (|\psi_d\rangle) \). Additionally, \( H \) generates a contraction semigroup.

Furthermore, if these equivalent statements hold then \( H \) coincides with the generator of \( T(t) \).

For the proof, we extend \( T_d(t) \) to act as the identity on the orthogonal complement \( V_d^\perp \) of \( V_d \). This turns \( T_d(t) \) into a strongly continuous contraction semigroup on \( \mathcal{H} \), which would be wrong for the extension by 0. Notice that this does not affect strong convergence: The difference of the extension by zero and the extension by the identity is the projection \( P_d^\perp \) onto \( V_d^\perp \) which vanishes strongly as \( d \to \infty \). The proof is inspired by [45, Thm. 2.3]:

**Proof.** For the proof, choose \( z \in \mathbb{C} \) with \( \text{Im} \ z > 0 \).

(a) \( \Rightarrow \) (b): Our setting is a special case of the second Trotter-Kato approximation theorem [44, Thm. 4.9]. Therefore, (a) is equivalent to the statement that the resolvents \( R(H_d; z) = (z - H_d)^{-1} \) are strongly convergent with \( \text{s-lim}_{d \to \infty} R(H_d; z) \) having dense range. For a sequence \( (|\psi_d\rangle) \) with \( |\psi_d\rangle \in V_d \), we write \( (|\psi_d\rangle) \in \mathcal{C} \) if it converges in \( \mathcal{H} \), i.e.,
\[
\mathcal{C} = \left\{ (|\psi_d\rangle) \subset \mathcal{H} \mid |\psi_d\rangle \in V_d, \exists |\psi\rangle \in \mathcal{H} : \lim_{d \to \infty} |\psi_d\rangle = |\psi\rangle \right\}.
\]
(29)
Furthermore, we write
\[
\mathcal{D} := \left\{ (|\psi_d\rangle) \in \mathcal{C} \mid (H_d|\psi_d\rangle) \in \mathcal{C} \right\}.
\]
(30)
Consider the resolvent operator $\mathcal{R}(z)$ which maps a sequence $((\psi_d))_d \subset \mathcal{H}$ to the sequence $(R(H_d; z)\psi_d)_d$. On one hand, by definition of $\mathcal{D}$, it holds that $((z - H_d)\psi_d)_d$ is a sequence in $\mathcal{C}$ if $(\psi_d)_d \in \mathcal{D}$, so

$$\mathcal{D} \subset \mathcal{R}(z)\mathcal{C} = \{(R(H_d; z)\psi_d)_d \mid (\psi_d)_d \in \mathcal{C}\}.$$  

(31)

On the other hand, since the $R(H_d; z)$ are uniformly bounded [44, Thm. II.1.10] and strongly convergent in $d$, we have that $(R(H_d; z)\psi_d)_d \in \mathcal{C}$ if $(\psi_d)_d \in \mathcal{C}$. But then $(\psi_d)_d \in \mathcal{D}$ due to $(z - H_d)R(H_d; z) = 1$ or, equivalently, $H_dR(H_d; z) = zR(H_d; z) - 1$. This shows

$$\mathcal{D} = \mathcal{R}(z)\mathcal{C}.$$  

(32)

For now, we denote the generator of the strong limit $T(t)$ by $A$ to avoid confusion with the operator $H$ defined in (b). Of course, we will show later that the two coincide. Note that by the Trotter-Kato Theorem, we have $\lim_d R(H_d; z) = R(A; z)$. For $(\psi_d)_d \in \mathcal{C}$ with $\lim_d |\psi_d| = |\psi|$, we have

$$\lim_{d \to \infty} H_dR(H_d; z)|\psi_d| = \lim_{d \to \infty} \left(z R(H_d; z)\psi_d - |\psi_d|\right) = z R(A; z)|\psi| - |\psi| = AR(A; z)|\psi|.$$  

(33)

Since $\mathcal{D} = \mathcal{R}(z)\mathcal{C}$, this proves that $\lim_d H_d|\phi_d| = A(\lim_d |\phi_d|)$ for all $(|\phi_d|)_d \in \mathcal{D}$. In particular, this shows the well-definedness of $H$ and that $A$ is an extension of $H$. Finally, we prove $H = A$ via

$$\mathcal{D}(H) = \left\{ \lim_{d \to \infty} |\psi_d| \mid (|\psi_d|)_d \in \mathcal{D} \right\} = \left\{ \lim_d R(H_d; z)|\phi_d| \mid (|\phi_d|)_d \in \mathcal{C} \right\} = R(A; z)\mathcal{H} = \mathcal{D}(A),$$  

(34)

where the third equality uses that $\lim_d R(H_d; z)|\phi_d| = R(A; z)(\lim_d |\phi_d|)$ and that any $|\psi| \in \mathcal{H}$ is the limit of a sequence in $\mathcal{C}$.

(b) $\Rightarrow$ (a): Due to uniform boundedness, we only have to show strong convergence on the dense subset $(z - H)\mathcal{D}(H) \subset \mathcal{H}$. Let $|\phi| \in \mathcal{D}(H)$ and put $|\psi| = (z - H)|\phi| \in \mathcal{H}$. We know that $|\psi| = \lim_{d \to \infty}(z - H_d)|\phi|$. This implies that

$$\|R(H_d; z)|\psi| - R(H; z)|\psi|\| \leq \|R(H_d; z)(|\psi| - (z - H_d)|\phi|)\| + \|R(H_d; z)(z - H_d)|\phi| - |\phi|\|$$  

(35)

goes to zero as $d \to \infty$: The first summand on the right-hand side is dominated by $|z|^{-1}\|\psi| - (z - H_d)|\phi\| \to 0$. For the second summand, $R(H_d; z)(z - H_d)|\phi| = |\phi|$ for sufficiently large $d$. This shows that the resolvents $R(H_d; z)$ are strongly convergent. By construction, their limit $R(z) = \lim_{d \to \infty} R(H_d; z)$ has dense range. However, as already pointed out before, this is equivalent to (a) due to the second Trotter-Kato approximation theorem [44, Thm. 4.9].

In Thm. 5, we started with a sequence of generators $H_d$ and defined $H$ as a certain limit. However, for the proof of Thm. 7, we also need to look at the reversed setting, i.e. start from $H$ and define the $H_d$ as its finite-dimensional approximations. This scenario is elaborated in the following lemma.

**Lemma 6.** If $\mathcal{V} \subset \mathcal{D}(H)$ is a core of $H$, then it holds that $P_dHP_d =: H_d \to H$ strongly on $\mathcal{V}$. Furthermore, we have $T_d(t) \to T(t)$ strongly on all of $\mathcal{H}$.

**Proof.** Let $|\psi| \in \mathcal{V}$. Then there exists an $d \in \mathbb{N}$ such that $|\psi| \in V_d$. Hence, for $d \geq d$ it holds that $|\psi_d| = P_d|\psi| = |\psi|$ and, thus, $H_d|\psi_d| = P_dH|\psi| \to H|\psi|$. By the first Trotter-Kato approximation theorem [44, Thm. 4.8], the assertion follows.

Before we turn towards strong convergence of the Trotter product, we provide the proof of Prop. 3. It also introduces some of the notation and techniques that we will employ later on. For the proof, recall Eq. (8).

**Proof of Prop. 3.** Let $H^{(i)} : \mathcal{D}(H^{(i)}) \to \mathcal{H}$, $i = 1, 2$, be Hamiltonians such that $\mathcal{V} = \bigcup_d V_d \subset \mathcal{D}(H^{(1)}) \cap \mathcal{D}(H^{(2)})$. As before, $H_d^{(i)} = P_dH^{(i)}P_d$. Denote $X_d(t) = e^{-itH_d^{(i)}}e^{-itH_d^{(i)}}$ and $U_d(t) = e^{-it(H_d^{(i)} + H_d^{(2)})}$ and by $X(t)$ and $U(t)$ the analogous operators on the infinite dimensional space. Assume that there exists $c_n = c_n(t) > 0$ such that $\beta_d^{(n)}(t) < c_n$.
with \( c_n \to 0 \) if \( n \to \infty \). Let \( \psi \in \mathcal{V} \), w.l.o.g. \( \psi \in V_d \), and let \( \tilde{d} > d \). Then \( P_d \psi = \psi \), so

\[
\| (X(t/n)^n - U(t))\psi \| \leq \| X_d(t/n)^n - U_d(t)\psi \| + \| (1 - P_d)(X(t/n)^n - U(t))\psi \|
\]

\[
\leq c_n \| \psi \| + \| (1 - P_d)(X(t/n)^n - U(t))\psi \|.
\]  

(36)

Taking \( \tilde{d} \to \infty \), we thus obtain \( \| (X(t/n)^n - U(t))\psi \| \leq c_n \| \psi \| \). This shows uniform convergence on \( \mathcal{V} \). Approximating \( \psi \in \mathcal{H} \) by elements in \( \mathcal{V} \) and using the triangle inequality and unitarity of \( X(t) \) and \( U(t) \) then shows \( \| X(t/n)^n - U(t)\|_\infty < c_n \to 0 \).

Recall the definitions in (9) and (10). We are now ready to present and prove:

**Theorem 7.** Let \( H^{(1)} : \mathcal{D}(H^{(1)}) \to \mathcal{H} \), \( i = 1, 2 \), be generators of two strongly continuous contraction semigroups \( T^{(i)}(t) \), \( i = 1, 2 \). Assume that \( \mathcal{V} = \bigcup_{d} V_d \subset \mathcal{D}(H^{(1)}) \cap \mathcal{D}(H^{(2)}) \) is a common core of \( H^{(1)} \) and \( H^{(2)} \). Denote by \( T_d(t) \) the strongly continuous contraction semigroup on \( V_d \) generated by \( H^{(1)}_d + H^{(2)}_d \). Then

1. If for some \( \psi \in \mathcal{V} \), \( t > 0 \) one has \( \psi(t); t \to 0 \) as \( n \to \infty \), then both limits \( \lim_{d \to \infty} T_d(t)\psi \) and \( \lim_{n \to \infty} (T^{(1)}(t/n)T^{(2)}(t/n))^n\psi \) exist in \( \mathcal{H} \) and coincide.

2. If for all \( t \geq 0 \) and all \( \psi \in \mathcal{V} \) one has \( \psi(t); t \to 0 \), then there is a strongly continuous contraction semigroup \( T(t) \) on \( \mathcal{H} \) with \( T(t)\psi = \lim_{d \to \infty} T_d(t)\psi \), \( \psi \in \mathcal{V} \). The generator \( H \) of \( T(t) \) is an extension of \( H^{(1)}|_{\mathcal{V}} + H^{(2)}|_{\mathcal{V}} \). The Trotter product of \( T^{(1)} \) and \( T^{(2)} \) converges strongly to \( T(t) \).

**Proof.** We use the notations \( X_d(t) = T_d^{(1)}(t)T_d^{(2)}(t) \) and \( X(t) = T^{(1)}(t)T^{(2)}(t) \).

1. We have to prove two statements. First, we prove that for the series of finite-dimensional approximations, \( T_d(t)\psi \) converges to a well-defined vector \( |\phi\rangle \) as the level of truncation goes to infinity, \( d \to \infty \). Second, we show that for the Trotter product, \( X(t/n)^n\psi \) converges to exactly this vector \( |\phi\rangle \) in the Trotter limit \( n \to \infty \). Let us start with the first assertion assuming \( \psi \in \mathcal{V} \). By the triangle inequality, it holds that

\[
\| (T_d(t) - T_d^{(1)}(t))\psi \| \leq b^{(n)}(|\psi\rangle; t) + \| (X_d(t/n)^n - X_d^{(1)}(t/n)^n)|\psi\| + b^{(n)}(|\psi\rangle; t),
\]  

(37)

Due to Lemma 6, we have \( \lim_{d \to \infty} \lim_{n \to \infty} \| (X_d(t/n)^n - X_d^{(1)}(t/n)^n)|\psi\| = 0 \), and therefore

\[
\lim_{d \to \infty} \sup_{d \to \infty} \sup_{d \to \infty} \| (T_d(t) - T_d^{(1)}(t))\psi \| \leq 2b^{(n)}(|\psi\rangle; t) \overset{n \to \infty}{\longrightarrow} 0.
\]  

(38)

Hence, \( T_d(t)|\psi\rangle \) forms a Cauchy sequence and thus converges to some \( |\phi\rangle \in \mathcal{H} \). For the second assertion, let \( |\psi\rangle \in \mathcal{V} \) and use the triangle inequality again in order to receive

\[
\| X(t/n)^n|\psi\rangle - |\phi\rangle\| \leq \| (X(t/n)^n - X_d(t/n)^n)|\psi\|
\]

\[
+ b^{(n)}(|\psi\rangle; t)
\]

\[
+ \| T_d(t)|\psi\rangle - |\phi\rangle\|.
\]  

(39)

By convergence of \( T_d(t)|\psi\rangle \) and, therefore, \( X_d(t)|\psi\rangle \) and by taking the \( \limsup_{d \to \infty} \), we obtain

\[
\| X(t/n)^n|\psi\rangle - |\phi\rangle\| \leq b^{(n)}(|\psi\rangle; t) \overset{n \to \infty}{\longrightarrow} 0,
\]  

(40)

which proves the claim.

2. Again, two claims are to be proved. For the first claim, we have to show that \( T(t) \) is a strongly continuous contraction semigroup on \( \mathcal{H} \). For the second claim, we have to show that the generator \( H \) of \( T(t) \) is an extension of \( H^{(1)}|_{\mathcal{V}} + H^{(2)}|_{\mathcal{V}} \), i.e. the sum of the individual generators of \( T^{(1)}(t) \) and \( T^{(2)}(t) \). For the first claim, notice that by (1), \( T(t) \) is the strong limit of the semigroups \( T_d(t) \). Hence, it is a semigroup itself. For strong continuity, let \( |\psi\rangle \in \mathcal{V} \). Then for \( d \) large enough we have that \( |\psi\rangle \in V_d \) and it holds that [44, Lemma 1.3]

\[
\| T_d(t)|\psi\rangle - |\psi\rangle\| = \left\| \int_0^t T_d(s)(H^{(1)}_d + H^{(2)}_d)|\psi\rangle \ ds \right\|
\]

\[
\leq \int_0^t \| (H^{(1)}_d + H^{(2)}_d)|\psi\rangle \| \ ds
\]

\[
= t\| P_d(H^{(1)} + H^{(2)})|\psi\rangle \|.
\]  

(41)
By continuity of \( \| \cdot \| \) we have
\[
\| T(t)\psi - \psi \| = \lim_{d \to \infty} \| T_d(t)\psi - \psi \| \leq t \| (H^{(1)} + H^{(2)})\psi \| \to 0.
\] (42)

By density of \( \mathcal{V} \) in \( \mathcal{H} \) and, again, uniform continuity, strong continuity of \( T(t) \) follows on all of \( \mathcal{H} \). Now, let \( \psi \in \mathcal{V} \subset \mathcal{D}(H^{(1)}) \cap \mathcal{D}(H^{(2)}) \) and \( b_d(\psi) = U_d(\psi) \). Then as \( d \to \infty \), \( b_d(\psi) \to \psi \) and \( H^{(1)}b_d(\psi) \to H^{(i)}\psi \), since for \( d \) large enough \( b_d(\psi) = \psi \). Hence, \( (H^{(1)} + H^{(2)})b_d(\psi) \to (H^{(1)} + H^{(2)})\psi \), so \( \psi \in \mathcal{D}(H) \) and \( H\psi = (H^{(1)} + H^{(2)})\psi \) by Thm. 5 (b).

Thus \( H^{(1)}|_{\mathcal{V}} + H^{(2)}|_{\mathcal{V}} \subset H \). \( \square \)

Now, our Main Result 2 follows as a special case of Thm. 7:

**Second Main Result.** Let \( H^{(1)} : \mathcal{D}(A^{(1)}) \to \mathcal{H} \) and \( H^{(2)} : \mathcal{D}(A^{(2)}) \to \mathcal{H} \) be self-adjoint, hence generators of two strongly continuous unitary groups \( T^{(i)}(t), \ i = 1, 2 \). Assume that \( \mathcal{V} = \bigcup_d \mathcal{V}_d \subset \mathcal{D}(H^{(1)}) \cap \mathcal{D}(H^{(2)}) \) is a common core of \( H^{(1)} \) and \( H^{(2)} \). If for all \( t \in \mathbb{R} \), \( b_d(\psi); t \to 0 \) as \( n \to \infty \), then
\[
(T^{(1)}(t/n)T^{(2)}(t/n))^{n/2}|\psi\rangle \xrightarrow{n \to \infty} T(t)|\psi\rangle,
\] (43)
where \( T(t) \) is a strongly continuous unitary group, whose generator \( H \) is a self-adjoint and agrees with \( (H_1 + H_2) \) wherever both are defined.

**Proof.** Since \( H^{(1)} \) and \( H^{(2)} \) are self-adjoint, so are \( H_d^{(i)}, i = 1, 2 \) and \( H_d \). All these operators generate unitary groups that we will denote by \( U^{(i)}_d \), \( U^{(i)}_d \) and \( U_d \), respectively. Since \( b^{(n)}(\psi); t \to 0 \) as \( n \to \infty \) for \( t \geq 0 \), we obtain a contraction semigroup \( U(t), t \geq 0 \), generated by \( H \). Now, \( U(t) \) is isometric: \( \| U(t)|\psi\rangle \| = \lim_{d \to \infty} \| U_d(t)|\psi\rangle \| = \| |\psi\rangle \| \). Hence, \( H \) is symmetric as the generator of a strongly continuous isometry semigroup. To see this, let \( |\psi\rangle, |\phi\rangle \in \mathcal{D}(H) \) and differentiate \( |\psi\rangle|\phi\rangle = \langle U(t)|\psi\rangle|U(t)|\phi\rangle \) at \( t = 0 \). The same argument can be applied for \( t \leq 0 \), in which case we obtain a semigroup generated by \( -H^* \). Hence, \( H^* \) is symmetric as well. Notice that an operator \( A \) is symmetric if and only if (i) \( A \subset A^* \), and (ii) \( A \subset B \) implies \( B^* \subset A^* \) [42]. Applying this to \( H \) and \( H^* \) gives \( H \subset H^* \subset H^{**} = H \), which shows that \( H \) is self-adjoint. In order to see that \( H \) is equal to \( (H^{(1)} + H^{(2)}) \) wherever both are defined, we consider the difference: Let \( Y \) be the closure of the symmetric operator \( (H - H^{(1)} + H^{(2)}) \) defined on the intersection of the domains. We know from Thm. 7 (2) that \( Y \mathcal{V} = 0 \). But a closed operator which vanishes on a dense subset must vanish everywhere. To see this, note that since \( Y \) is closed, we have that \( Y \) is an extension of \( \overline{Y|\mathcal{V}} \). But \( Y|\mathcal{V} = 0 \) and since the zero operator is bounded this means that \( \overline{Y|\mathcal{V}} = 0 \). As an extension of the zero operator, \( Y \) is itself zero, i.e., \( H = H^{(1)} + H^{(2)} \) wherever all are defined. \( \square \)

It remains to prove Eq. (5) from the main text.

**Proof of Eq. (5).** By the triangle inequality, \( b^{(n)}(\psi); t)^2 \leq \lim sup_{d \to \infty} \sum_{j=1}^d |\langle j|\psi\rangle|^2 b_d^{(n)}(|j\rangle; t)^2 \). If \( |\psi\rangle \in \mathcal{V} \), then there exists an \( \tilde{d} \in \mathbb{N} \), such that \( |\psi\rangle \in \mathcal{V}_{\tilde{d}} \) and therefore \( |\psi\rangle = \sum_{j=1}^{\tilde{d}} |\langle j|\psi\rangle|j\rangle \). As a consequence for \( d \geq \tilde{d} \), \( b_d^{(n)}(|\psi\rangle; t)^2 \leq \sum_{j=1}^{\tilde{d}} |\langle j|\psi\rangle|^2 b_d^{(n)}(|j\rangle; t)^2 \), and therefore,
\[
b^{(n)}(|\psi\rangle; t) \leq \left( \sum_{j=1}^{\tilde{d}} |\langle j|\psi\rangle|^2 b_d^{(n)}(|j\rangle; t)^2 \right)^{1/2},
\] (44)
which proves the statement. \( \square \)

**D. EXAMPLE: THE QUANTUM HARMONIC OSCILLATOR**

The bounds from our Main Result 1 hold for arbitrary Hamiltonians acting on a finite-dimensional Hilbert space. In particular, we can use them for bounding strong Trotter errors for the finite-dimensional truncated operators, in which we are interested. For instance, consider the case where \( H^{(1)} = \frac{1}{2}Q^2 \) and \( H^{(2)} = \frac{1}{2}P^2 \), where where \( Q : \mathcal{D}(Q) \to L^2(\mathbb{R}) \) is the position operator and \( P : \mathcal{D}(P) \to L^2(\mathbb{R}) \) is the momentum operator. We can rewrite \( Q \) and \( P \) in terms of the
creation operator $a^\dagger$ and annihilation operator $a$ as follows

$$Q = \frac{1}{\sqrt{2}} (a + a^\dagger),$$

$$P = \frac{i}{\sqrt{2}} (a^\dagger - a).$$

The operators $a$ and $a^\dagger$ satisfy the commutation relation $[a, a^\dagger] = 1$. In this particular Trotter scenario, we would like to implement the target Hamiltonian $H^{\text{osc}} = \frac{1}{2} (Q^2 + P^2)$ by switching between the time evolutions generated by $H^{(1)} = \frac{1}{4}Q^2$ and $H^{(2)} = \frac{1}{2}P^2$, respectively. $H^{\text{osc}}$ is the quantum harmonic oscillator and is related to the number operator $N = a^\dagger a$ by $H^{\text{osc}} = N + \frac{1}{2}$. The eigenstates $|m\rangle$ of $N$ according to the eigenvalue equation $N|m\rangle = m|m\rangle$ are the Fock states. In turn, the $|m\rangle$ are eigenstates of $H^{\text{osc}}$ as well and satisfy the eigenvalue equation $H^{\text{osc}} |m\rangle = (m + \frac{1}{2}) |m\rangle$. Let us use our Main Result to compute a strong error bound for this Trotter problem for the corresponding finite-dimensional approximations. Obviously, we have

$$Q^2 = \frac{1}{2} (2N + aa + a^\dagger a^\dagger + 1),$$

$$P^2 = \frac{1}{2} (2N - aa - a^\dagger a^\dagger + 1)$$

and the Fock states are a common core of $H^{(1)} = \frac{1}{4}Q^2$ and $H^{(2)} = \frac{1}{2}P^2$. Define $R^\pm = \frac{1}{2} (2N \pm aa \pm a^\dagger a^\dagger + 1)$, so that $R^+ = Q^2$ and $R^- = P^2$. Then the action of $R^\pm$ onto a Fock state $|m\rangle$ with $m \geq 2$ computes to

$$R^\pm |m\rangle = \left( m + \frac{1}{2} \right) |m\rangle \pm \frac{1}{2} \sqrt{m (m - 1)} |m - 2\rangle \pm \frac{1}{2} \sqrt{(m + 1) (m + 2)} |m + 2\rangle.$$  

(49)

Analogously, for $(R^\pm)^2$ we obtain

$$(R^\pm)^2 = \frac{1}{4} \left( 4N \pm aa \pm a^\dagger a^\dagger + 1 \right)^2$$

$$= \frac{1}{4} \left( 4N^2 \pm 2aaN \pm 4a^\dagger a^\dagger + 4N \pm 2aaN + aaaa + aaaa + 1 \right)$$

$$+ \frac{1}{4} \left( 4N^2 \pm 2aaN \pm 4a^\dagger a^\dagger + 4N \pm 2aaN + aaaa + 1 \right)$$

$$= \frac{1}{4} \left( 4N^2 \pm 4aaN \pm 4a^\dagger a^\dagger N + 2aa^\dagger a^\dagger + 2aa^\dagger a^\dagger + 2aa^\dagger a^\dagger + 2aa + 6a^\dagger a^\dagger - 1 \right),$$

(50)

where we have used $[N, aa] = -2aa$, $[N, a^\dagger a] = 2a^\dagger a$ and $[aa, a^\dagger a] = 4N + 2$. Hence, by acting on a Fock state $|m\rangle$ with $m \geq 4$,

$$(R^\pm)^2 |m\rangle = \left( N^2 \pm aaN \pm a^\dagger a^\dagger + \frac{1}{4} aaaa + \frac{1}{4} aaaa + \frac{1}{4} aaaa + \frac{1}{4} aaaa + \frac{1}{4} aaaa + \frac{1}{4} aaaa \right) |m\rangle$$

$$= m^2 |m\rangle \pm m \sqrt{m (m - 1)} |m - 2\rangle \pm m \sqrt{(m + 1) (m + 2)} |m + 2\rangle$$

$$+ \frac{1}{2} (m + 1) (m + 2) |m\rangle \pm \frac{1}{2} \sqrt{m (m - 1) (m - 2)} |m - 4\rangle$$

$$+ \frac{1}{2} \sqrt{(m + 1) (m + 2) (m + 3) (m + 4) |m + 4\rangle \pm \frac{1}{2} \sqrt{m (m - 1) |m - 2\rangle$$

$$= \frac{3}{4} \left( 2m (m + 1) + 1 \right) |m\rangle \pm \frac{1}{2} \left( 2m - 1 \right) \sqrt{m (m - 1)} |m - 2\rangle$$

$$\pm \frac{1}{2} \left( 2m + 3 \right) \sqrt{(m + 1) (m + 2) |m + 4\rangle \pm \frac{1}{2} \sqrt{m (m - 1) (m - 2) (m - 3) |m - 4\rangle$$

$$+ \frac{1}{4} \sqrt{(m + 1) (m + 2) (m + 3) (m + 4) |m + 4\rangle.$$  

(51)
In order to apply our Main Result 1, we have to compute the norm of
\[
\left( (R^\pm)^2 - 2 \left( m + \frac{1}{2} \right) R^\pm + \left( m + \frac{1}{2} \right)^2 \right) |m\rangle = \frac{3}{4} (2m(m+1)+1)|m\rangle \pm \frac{1}{2} (2m-1) \sqrt{m(m-1)} |m-2\rangle \\
\pm \frac{1}{2} (2m+3) \sqrt{(m+1)(m+2)} |m+2\rangle \\
+ \frac{1}{4} \sqrt{m(m-1)(m-2)(m-3)} |m-4\rangle \\
+ \frac{1}{4} \sqrt{(m+1)(m+2)(m+3)(m+4)} |m+4\rangle \\
- \left( m + \frac{1}{2} \right)^2 |m\rangle \mp \left( m + \frac{1}{2} \right) \sqrt{m(m-1)} |m-2\rangle \\
\mp \left( m + \frac{1}{2} \right) \sqrt{(m+1)(m+2)} |m+2\rangle \\
\mp \left( m + \frac{1}{2} \right) \sqrt{(m+1)(m+2)} |m+2\rangle \\
= \frac{1}{2} (m^2 + m + 1) |m\rangle \mp \sqrt{m(m-1)} |m-2\rangle \\
\pm \sqrt{(m+1)(m+2)} |m+2\rangle + \frac{1}{4} \sqrt{m(m-1)(m-2)} |m-4\rangle \\
+ \frac{1}{4} \sqrt{(m+1)(m+2)(m+3)(m+4)} |m+4\rangle. \tag{52}
\]

Since the \{ |m\rangle \} form an orthonormal basis,
\[
\left\| \left( (R^\pm)^2 - 2 \left( m + \frac{1}{2} \right) R^\pm + \left( m + \frac{1}{2} \right)^2 \right) |m\rangle \right\|^2 = \frac{3}{8} (m(m+1) (m^2 + m + 14) + 10). \tag{53}
\]

We now truncate the operators at dimension \( d \) in the Fock basis. That is, we use the orthogonal projector \( P_d = \sum_{m=0}^{d-1} |m\rangle \langle m| \) as a truncation scheme. Denote the truncated vectors and operators with a subscript \( d \) and assume that \( d \geq m + 4 \). Then
\[
\left\| \left( U_d(t) - W_d^{(m)}(t) \right) |m_d\rangle \right\| \leq \frac{t^2}{4n} \sqrt{\frac{3}{8} (m(m+1) (m^2 + m + 14) + 10)}, \tag{54}
\]
which is Eq. (6) of the main text. The assumption \( d \geq m + 4 \) is made in order to ensure that \((R^\pm)^2 \) (and \( R^\pm \)) acts correctly on the truncated Fock states \( |m_d\rangle \): Since \((R^\pm)^2 \) involves terms up to fourth order in \( a_d^\dagger \), we have to make sure that all vectors in the linear combination of \((R^\pm)^2 |m\rangle \) in Eq. (52) are actually again vectors in the truncated Hilbert space \( V_d \). If we drop this assumption and consider \( m \leq d < m + 4 \) instead, all terms in \((R^\pm)^2 |m\rangle \), which give Fock states \( |m+j\rangle \) with \( j \geq d - m + 1 \), will not contribute to \((R^\pm)^2 |m_d\rangle \). In this case, Eq. (53) will become smaller and Eq. (54) will still give an upper bound. Similarly, for the validity of Eq. (49)–(53), we made the assumption \( m \geq 4 \). However, our bound in Eq. (54) stays valid even for Fock states \( |m\rangle \) with \( m < 4 \). In this case, all terms for which \( m - 4 < 0 \) vanish in the derivation of the bound. Nevertheless, adding these additional terms only increases the norm so that Eq. (54) is still an upper bound. This bound is independent of the truncation dimension \( d \), which shows that the full infinite-dimensional Trotter problem \( H^{(1)} = \frac{1}{2} Q^2 \), \( H^{(2)} = \frac{1}{2} P^2 \) converges strongly to \( H^{osc} = \frac{1}{2} (X^2 + P^2) \). Furthermore, since \( Q^2 \) and \( P^2 \) are self-adjoint operators when acting on the Fock space \( F = \text{span} \{|m\rangle\} \), \( H^{osc} = \frac{1}{2} (Q^2 + P^2) \) is a self-adjoint operator on the Fock space.