A new kind of symmetric matrix

O Babarinsa$^1$ and H Kamarulhaili$^2$

$^1$Department of Mathematical Sciences, Federal University Lokoja, 1154 Lokoja, Nigeria
$^2$School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Pulau Pinang, Malaysia

E-mail: babs3in1@gmail.com

Abstract. In this paper, we investigate a new category of symmetric matrix, denoted as $B_{n \times n}$, which can be considered as obtained from an improper integral $B_{n \times n} = \lim_{b \to -\infty} \int_{b}^{1} cx^{-2} dx$; where $c = ij - (i + j)$. The elements of the matrix are integers and are in sequence. Thus the matrix is singular (except for $B_{2 \times 2}$) but nonsingular in its $2 \times 2$ connected minors. We give some deductions on its properties that other symmetric matrices do not possess.

1. Introduction

Over a century, symmetric matrix has been the research interest of matrix theorists. Though, several symmetric matrices have been studied with the method of computing their entries. Recently, the properties, applications and different forms of symmetric matrices have spread their tentacles to many fields of studies which are well studied in [1, 2, 3, 4, 5, 6, 7, 8]. An $n \times n$ matrix $A = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$ such that $A = A^T$, where $A^T$ the transpose of $A$ [9]. Symmetric matrix must be a square matrix, then the matrix may be singular or not [10]. A symmetric matrix with $\frac{n(n+1)}{2}$ scalars determines the number of entries above the main diagonal.

The matrix discuss in this paper originates from the notion of Hilbert matrix. Hilbert matrix is obtained from the integral $H_{ij} = \int_{0}^{1} cx^{i+j-2} dx$ [11]. The Hilbert matrix is positive definite and symmetric and which is a typical example of Hankel matrix and particular example of a Cauchy Matrix [12]. Like Hilbert matrix, this matrix also arises from the arbitrary functions by polynomials in the least squares approximation. We, therefore, view to start the first element ($b_{11}$) of the symmetric matrix with 1 such that the method for computing the entries contains only the entry $i$ and $j$. Unlike Hilbert matrix that its entries is $0 \leq h_{i,j} \leq 1$, the matrix has $-\infty < b_{i,j} \leq 1$.

The rest of Section (2) provides details about the derivation of the symmetric matrix. To understand this symmetric matrix better, we briefly discuss it in terms of its properties instead of its coordinates.

2. The new symmetric matrix

The understanding of a particular area of interest is suddenly advanced by the discovery of a single basic equation or idea [13]. Thus, it might be another verification of an old actuality or it might be another way to deal with a few realities in the meantime. In the event that the new confirmation builds up same beforehand unsuspected associations between two thoughts;
it regularly prompts a speculation \cite{14, 15}. The symmetric matrix discussed in this paper is established from unsuspected relationship with Hilbert matrix

Now, let \( B_{n \times n} \) be the new symmetric matrix and \( \{ b_{ij} \} \) its elements. To proceed in establishing this matrix, we let \( c = ij - (i + j) \) where \( c \in \mathbb{Z} \). Unlike Hilbert matrix, the intention of establishing \( c \) in that form is to start the first entry of \( B_{n \times n} \) with 1 (i.e \( b_{11} = 1 \)) such that \( b_{11} > b_{22} > \ldots > b_{nn} \) and to ensure that \( b_{nn} \) is divisible by \( n \). Since the entries in the main diagonal are decreasing and the matrix is an integer matrix, then we can deduce the range of the integer to be \((-\infty, 1] \). Unlike Hilbert matrix that has fraction entries, to obtain non-fraction entries for the new matrix the power of \( x \) in Hilbert matrix must reduce to -2 (i.e \( i + j = 0 \)). Therefore, combining the polynomial form of the matrix with integral we have

\[
B_{n \times n} = \int_{-\infty}^{1} cx^{-2} \, dx
\]

which implies

\[
B_{n \times n} = \lim_{b \to -\infty} \int_{b}^{1} cx^{-2} \, dx
\]

where \( c = ij - (i + j) \).

Solving the improper integral of Equation (1), it is obvious that \( B_{n \times n} = -c \). Therefore, we have

\[
B_{n \times n} = (i + j) - ij.
\]

where \( i, j = 1, 2, \ldots, n \).

Furthermore, it is obvious that the entries in Equation (2) follow a sequential step of arithmetic progression. The general matrix form of the matrix is

\[
B_{n \times n} = \begin{bmatrix}
1 & 1 & 1 & 1 & \ldots & 1 \\
1 & 0 & -1 & -2 & \ldots & 2 - n \\
1 & -1 & -3 & -5 & \ldots & 3 - 2n \\
1 & -2 & -5 & -8 & \ldots & 4 - 3n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 - n & 3 - 2n & 4 - 3n & \ldots & 2n - n^2
\end{bmatrix}
\]

Example 2.1. We give few examples of the symmetric matrix as follows:

\[
B_{2 \times 2} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}, \quad B_{3 \times 3} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & -1 \\
1 & -1 & -3
\end{bmatrix} \quad \text{and} \quad B_{4 \times 4} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & -1 & -2 \\
1 & -1 & -3 & -5 \\
1 & -2 & -5 & -8
\end{bmatrix}
\]

2.1. Properties of the new symmetric matrix

If given an \( n \times n \) matrix \( A \), a minor is a submatrix formed by deleting \( i \) rows and \( j \) columns from \( A \) while a connected minor is one in which all of the rows and columns are adjacent \cite{16}. The determinant of \( B_{n \times n} \) is zero, except for order 2. Since the matrix is singular, hence it is not invertible. However, any of its \( 2 \times 2 \) connected minors have determinant of -1 while its \((n - 1) \times (n - 1) \) connected minors have determinant of 0, for \( n \geq 4 \). Therefore, unlike Hilbert matrix which is totally positive this symmetric matrix is not.

More so, the aesthetic parts of the matrix are that the first row and first column entries of the matrix is 1. The entry \( b_{22} = 0 \), the entry \( b_{nn} \) is always divisible by \( n \) and the total number of negative entries in the matrix is \( n(n - 2) \). The absolute largest element in the entire matrix occurs in the diagonal as \( |B_{ij}| < \max(B_{ii}, B_{jj}) \). Besides, the sum of all entries in the matrix is less than the order of the matrix, that is \( B_{n \times n} \) then

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} \leq n, \text{ whenever } n > 2.
\]
There are many properties of symmetric matrices that are applicable to this matrix such as the sum or product of two symmetric matrices is symmetric [6]. However, the property that states if $A^{-1}$ is symmetric, then $A$ is an invertible symmetric matrix is not generally true for this new symmetric matrix. For instance, let $B$ be the new symmetric matrix then $B^{-1}$ does not exist when $n > 2$. Thus, $B^{-1}B \neq I$ and $B^{-1}B^T \neq I$

Now, we know that the characteristic polynomial of matrix $A$ is $P_A(\lambda) = |\lambda I - A|$. Then, the characteristic polynomial of $B$ is the polynomial $P_B(\lambda)$ that satisfies

$$P_B(\lambda) = |\lambda I - B| = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} tr(\wedge^k B)$$

where $tr(\wedge^k B)$ is the trace of the $k$th exterior power of $B$ with dimension $\binom{n}{k}$. Obviously, simplifying and factorizing $\lambda$ on the right side of Equation (3) turns the equation to a quadratic form. The quadratic form ensures that there are two nonzero solutions while other values of $\lambda$ are zero. Thus, we have $\lambda^{(n-2)}P_B'(\lambda)$ where

$$P_B'(\lambda) = \sum_{k=n-2}^{n} (-1)^k \lambda^2 tr(\wedge^k B).$$

Therefore,

$$P_B(\lambda) = \lambda^{(n-2)} \sum_{k=n-2}^{n} (-1)^k \lambda^2 tr(\wedge^k B).$$

The two nonzero eigenvalues from $\lambda^{(n-2)}P_B'(\lambda)$ denotes that the rank of the matrix is 2 irrespective of the order of the matrix.

**Definition 2.1.** If $A$ is a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then spectral norm of $A$ is given as $\|A\| = \max_i |\lambda_i|$.

**Proposition 2.1.** Let $B$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The spectral norm of $B$ is

$$\|B\| = \frac{|tr(B)|}{2}.$$ 

**Proof.** Since $\lambda_i$ produces two similar eigenvalues with real root and $n - 2$ eigenvalues with zeros then $tr(B) = 2\lambda_i$ for $i = 1, 2$. Now taking the absolute value we have

$$\frac{|tr(B)|}{2} = |\lambda_i|.$$ 

**Definition 2.2 (See [17]).** The trace of matrix $A$, $tr(A)$, is the sum of elements in the main diagonal of $A$.

**Proposition 2.2.** Let $tr(B)$ be the trace of symmetric matrix $B$. Then

$$tr(B) = \sum_{k=1}^{n} 2k - k^2$$

**Proof.** Since $b_{11} = 1$ and $b_{ii} = (i + i) - ii = 2i - i^2$ for $i \geq 2$. Obviously, $b_{ii}$ follows arithmetic progression of the form $1, 2 - i, 3 - 2i, 4 - 3i, \ldots, 2i - i^2$ for $1 \leq i \leq n$. Then by summing the sequence we have
\[ tr(B) = \sum b_{kk} = \sum_{k=1}^{n} 2k - k^2. \]

**Theorem 2.3 (Spectral Theorem).** Suppose matrix \( A \) is symmetric. Then \( A \) has \( n \) orthogonal eigenvectors with real eigenvalues.

Although all eigenvalues of symmetric matrix \( B \) are real yet the matrix is neither semi-definite nor positive definite. Any symmetric matrix is diagonalizable if the sum of the dimensions of the eigenspaces is \( n \). However, the dimension of the eigenspace of \( B \) for each \( \lambda_k \) does not equal to the multiplicity of \( \lambda_k \), for \( 1 \leq k \leq n \). Therefore, the eigenvectors of \( B \) have multiple basis of eigenvalues with no linearly independent eigenvectors of \( B \). Hence, matrix \( B \) is non-diagonalizable.

**References**

[1] Golub G H and Van Loan C F 1996 *Matrix computations. Johns Hopkins studies in the mathematical sciences* (Johns Hopkins University Press, Baltimore, MD)
[2] Kronenburg M 2013 *arXiv preprint arXiv:1306.6291*
[3] Ashtiani A A, Raja P and Nikravesh S K Y 2014 *The Journal of Nonlinear Science and Applications (JNSA),* 7(1) 63–69
[4] Fiedler M 1975 *Czechoslovak Mathematical Journal* 25 619–633 ISSN 0011-4642
[5] McCulloch C E 1982 *Journal of the American Statistical Association* 77 679–682 ISSN 0162-1459
[6] Schott J R 2016 *Matrix analysis for statistics* (John Wiley and Sons) ISBN 1119092477
[7] Kirisci M and Polat H 2016 *arXiv preprint arXiv:1609.01543*
[8] Batalshchikov A, Grudsky S and Stukopin V 2015 *Linear Algebra and its Applications* 469 464–486 ISSN 0024-3795
[9] Kim I and Waters C 2010 *Czechoslovak mathematical journal* 60 101–104 ISSN 0011-4642
[10] Debnath L 2013 *International Journal of Mathematical Education in Science and Technology* 45 360–377 ISSN 0020-739X 1464-5211
[11] Brevig O F, Perfekt K M, Seip K, Siskakis A G and Vukotic? D *Advances in Mathematics* 302 410–432 ISSN 0002-9070
[12] Choi M D 1983 *The American Mathematical Monthly* 90 301–312 ISSN 0002-9890
[13] Pickover C A 2011 *A passion for mathematics: numbers, puzzles, madness, religion, and the quest for reality* (John Wiley and Sons) ISBN 1118046072
[14] Halmos P R 1980 *The American Mathematical Monthly* 87 519–524 ISSN 00029890, 19300972
[15] Olayiwola B 2014 *International Journal of Mathematics and Statistics Invention* 2 47–50
[16] Main M, Donor M and Harwood R C 2016 *arXiv preprint arXiv:1607.05352*
[17] Jarvis F 2014 *Fields, Discriminants and Integral Bases* (Springer) pp 39–63