Spatial dynamics methods for solitary waves on a ferrofluid jet

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Abstract

This paper presents existence theories for several families of axisymmetric solitary waves on the surface of an otherwise cylindrical ferrofluid jet surrounding a stationary metal rod. The ferrofluid, which is governed by a general (nonlinear) magnetisation law, is subject to an azimuthal magnetic field generated by an electric current flowing along the rod.

The ferrohydrodynamic problem for axisymmetric travelling waves is formulated as an infinite-dimensional Hamiltonian system in which the axial direction is the time-like variable. A centre-manifold reduction technique is employed to reduce the system to a locally equivalent Hamiltonian system with a finite number of degrees of freedom, and homoclinic solutions to the reduced system, which correspond to solitary waves, are detected by dynamical-systems methods.

1 Introduction

Figure 1: Waves on the surface of a ferrofluid jet surrounding a current-carrying wire.

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We consider an incompressible, inviscid ferrofluid of unit density in the region

$$S_1 := \{ 0 < r < R + \eta(\theta, z, t) \}$$

bounded by the free interface $$\{ r = R + \eta(\theta, z, t) \}$$ and a current-carrying wire at $$\{ r = 0 \}$$, where $$(r, \theta, z)$$ are cylindrical polar coordinates. The fluid is subject to a static magnetic field and the surrounding region

$$S_2 = \{ r > R + \eta(\theta, z, t) \}$$
is a vacuum (see Figure 1). Travelling waves move in the axial direction with constant speed $c$ and without change of shape, so that $\eta(\theta, z, t) = \eta(\theta, z - ct)$. We are interested in particular in axisymmetric solitary waves for which $\eta$ does not depend upon $\theta$ and $\eta(z - ct) \to 0$ as $z - ct \to \pm \infty$. Waves of this kind for ferrofluids with a linear magnetisation law have been investigated using a weakly nonlinear approximation by Rannacher & Engel [18], experimentally by Bourdin, Bacri & Falcon [4] and numerically by Blyth & Parau [3]. In this paper we present a rigorous existence theory for small-amplitude solitary waves and consider fluids with a general (nonlinear) magnetisation law.

Our starting point is a formulation of the hydrodynamic problem as a reversible Hamiltonian system

$$\eta_z = \frac{\delta H}{\delta \omega}, \quad \omega_z = -\frac{\delta H}{\delta \eta}, \quad \hat{\phi}_z = \frac{\delta H}{\delta \hat{\zeta}}, \quad \hat{\zeta}_z = -\frac{\delta H}{\delta \hat{\phi}} \quad (1.1)$$
in which the axial coordinate $z$ plays the role of time, $\hat{\phi}$ is a variable related to the fluid velocity potential $\phi$ and $\omega, \hat{\zeta}$ are the momenta associated with the coordinates $\eta, \hat{\phi}$. The spatial Hamiltonian system (1.1) is derived from a variational principle for the governing equations in Section 3; it depends upon two dimensionless physical parameters $\alpha$ and $\beta$ (see equation (2.6) for precise definitions) and the (dimensionless) magnitude $m_1(|H_1|)$ of the magnetic intensity corresponding to the magnetic field $H_1$ in the ferrofluid.

Homoclinic solutions of (1.1) (solutions with $$(\eta, \omega, \hat{\phi}, \hat{\zeta}) \to 0$$ as $z \to \pm \infty$$) are of particular interest since they correspond to solitary waves. We detect such solutions using a technique known as the Kirchgässner reduction (Section 4), in which a centre-manifold reduction principle is used to show that all small, globally bounded solutions of a spatial (Hamiltonian) evolutionary system solve a (Hamiltonian) system of ordinary differential equations, whose solution set can in principle be determined. In this fashion we reduce (1.1) to a Hamiltonian system with finitely many degrees of freedom which can be treated by well-developed dynamical-systems methods, in particular normal-form theory. We proceed by perturbing the physical parameters $\beta, \alpha$ around fixed reference values $\beta_0, \alpha_0$ and thus introducing bifurcation parameters $\varepsilon_1, \varepsilon_2$. The Kirchgässner reduction delivers an $\varepsilon$-dependent reduced system which captures the small-amplitude dynamics for small values of these parameters; its dimension is the number of purely imaginary eigenvalues of the corresponding linearised system at $$(\varepsilon_1, \varepsilon_2) = (0, 0)$$. The reduction procedure is therefore especially helpful in detecting bifurcations which are associated with a change in the number of purely imaginary eigenvalues.

Working in the $$(\beta, \gamma)$$ parameter plane, where $\gamma = \alpha - \beta$, one finds that there are three critical curves $C_2, C_3, C_4$ at which the number of purely imaginary eigenvalues changes (see Figure 2(a)), together with a fourth curve $C_1$ at which the number of real eigenvalues changes. (In fact $C_3 = \{ (\beta, 2) : \beta < \frac{1}{4} \}$, $C_4 = \{ (\beta, 2) : \beta > \frac{1}{4} \}$ and explicit formulae for $C_1$ and
$C_2$ are given in Section 4.) A similar diagram arises in the study of gravity-capillary travelling water waves (see Iooss [14], Groves & Wahlén [13] and the references therein), and there the curves corresponding to $C_1$, $C_2$ and $C_4$ are associated with homoclinic bifurcation: homoclinic solutions of the reduced Hamiltonian system (corresponding to solitary water waves) bifurcate from the trivial solution. Figure 2(a) illustrates the parameter regions I, II and III adjacent to $C_1$, $C_2$ and $C_4$ in which the existence of homoclinic solutions is to be expected. In Section 5 we study these regions using the Kirchgässner reduction; the basic types of solitary wave found there are sketched in Figures 2(b)–(d).

In Section 5.1 we examine region I, choosing $(\beta_0, \gamma_0) \in C_4$, so that $\alpha_0 = 2 + \beta_0$, and writing $\alpha = \alpha_0 + \mu$ with $0 < \mu \ll 1$. According to the Kirchgässner reduction small-amplitude solitary waves are given by

$$\eta(z) = \frac{1}{2}\mu(\beta_0 - \frac{1}{4})^{1/2}Q \left(\mu^{1/2}(\beta_0 - \frac{1}{4})^{-1/2}z\right) + O(\mu^{3/2}),$$

where $(Q, P)$ is a homoclinic solution of the reversible Hamiltonian system

$$\dot{Q} = P + O(\mu^{1/2}),$$

$$\dot{P} = Q - \tilde{c}_1 Q^2 + O(\mu^{1/2})$$

with $\tilde{c}_1 := \frac{1}{2}(\alpha_0 m'_1(1) - 6)$. This system admits a homoclinic solution which corresponds to a monotonically decaying, symmetric solitary wave of elevation for $\tilde{c}_1 > 0$ and depression for $\tilde{c}_1 < 0$. For $m'_1(1)$ close to the critical value $6\alpha_0^{-1}$ we write $m'_1(1) = \alpha_0^{-1}(6 + 2\mu^{1/2}\tilde{\kappa})$ with $0 < \tilde{\kappa} \ll 1$ and find that small-amplitude solitary waves are given by

$$\eta(z) = \frac{1}{2}\mu^{1/2}(\beta_0 - \frac{1}{4})^{1/2}Q \left(\mu^{1/2}(\beta_0 - \frac{1}{4})^{-1/2}z\right) + O(\mu^{3/2}),$$

where $(Q, P)$ is a homoclinic solution of the reversible Hamiltonian system

$$\dot{Q} = P + O(\mu^{1/2}),$$

$$\dot{P} = Q - \tilde{\kappa} Q^2 - \tilde{d}_1 Q^2 + O(\mu^{1/2})$$

with $\tilde{d}_1 = \frac{1}{6}(12 - \alpha_0 m''_1(1))$. For $\tilde{d}_1 > 0$ this system admits a pair of homoclinic solutions which correspond to monotonically decaying, symmetric solitary waves; one is a wave of depression, the other a wave of elevation. Note that in the limit $\mu = 0$ or $(\mu, \tilde{\kappa}) = (0, 0)$ the variable $Q$ solves a travelling-wave version of the (generalised) Korteweg-de Vries equation.

In Section 5.2 we apply the Kirchgässner reduction in region II, finding that small-amplitude solitary waves are given by

$$\eta(z) = \frac{1}{2}\mu^4 P_1(\mu z) + O(\mu^5),$$

where $(Q, P)$ is a homoclinic solution of the reversible Hamiltonian system

$$\dot{Q}_1 = -P_1 + \frac{2}{3}(1 + \delta)P_2 + \frac{4}{3}(1 + \delta)^2 P_1 + 3c_1 P_1^2 + O(\mu),$$

$$\dot{Q}_2 = P_2 + \frac{2}{3}(1 + \delta)P_1 + O(\mu),$$

$$\dot{P}_1 = Q_2 + O(\mu),$$

$$\dot{P}_2 = Q_1 + \frac{2}{3}(1 + \delta)Q_2 + O(\mu)$$

$$\dot{P}_3 = Q_3 + O(\mu),$$

$$\dot{P}_4 = Q_4 + O(\mu),$$

$$\dot{Q}_3 = -P_3 + \frac{2}{3}(1 + \delta)P_4 + \frac{4}{3}(1 + \delta)^2 P_3 + 3c_1 P_3^2 + O(\mu),$$

$$\dot{Q}_4 = P_4 + \frac{2}{3}(1 + \delta)P_3 + O(\mu),$$

$$\dot{P}_3 = Q_4 + O(\mu),$$

$$\dot{P}_4 = Q_3 + \frac{2}{3}(1 + \delta)Q_4 + O(\mu).$$

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(a) Bifurcation curves in the \((\beta, \gamma)\)-plane; the shaded regions indicate the parameter regimes in which homoclinic bifurcation is detected.

(b) Solitary waves of elevation (left) and depression (right) in region I.

(c) Primary solitary waves of elevation (left) and depression (right) in region II.

(d) Primary solitary waves of elevation (left) and depression (right) in region III.

Figure 2: Summary of the basic types of solitary wave whose existence is established in the present paper by the Kirchgässner reduction.
with \( c_1 = 48\sqrt{6}(3m'_1(1) - 8) \); the parameters \( 0 < \mu, \delta \ll 1 \) measure the distance from respectively the point \((\beta_0, \gamma_0) = (\frac{1}{4}, 2)\) and the curve \(C_1\). This system admits a homoclinic solution which corresponds to a solitary wave of elevation for \( c_1 > 0 \) and depression for \( c_1 < 0 \); the wave is symmetric with an oscillatory decaying tail. For \( m'_1(1) \) close to the critical value \( \frac{2}{3} \) we write \( m'_1(1) = \frac{1}{3}(8 + \frac{1}{144\sqrt{6}}\kappa^\mu) \) with \( 0 < \kappa \ll 1 \) and find that small-amplitude solitary waves are given by

\[
\eta(z) = \frac{1}{2} \mu^2 P_1(\mu z) + O(\mu^2),
\]

where \((Q, P)\) is a homoclinic solution of the reversible Hamiltonian system

\[
\begin{align*}
\dot{Q}_1 &= -P_1 + \frac{2}{3}(1 + \delta)P_2 + \frac{4}{9}(1 + \delta)^2 P_1 + \kappa P_1^3 + 4d_1 P_1^3 + O(\mu), \\
\dot{Q}_2 &= P_2 + \frac{2}{3}(1 + \delta)P_1 + O(\mu), \\
\dot{P}_1 &= Q_2 + O(\mu), \\
\dot{P}_2 &= Q_1 + \frac{2}{3}(1 + \delta)Q_2 + O(\mu)
\end{align*}
\]

with \( d_1 = 864\left(\frac{1204}{75} - m''_1(1)\right) \). For \( d_1 > 0 \) this system admits a a pair of homoclinic solutions which correspond to symmetric solitary waves with oscillatory decaying tails; one is a wave of depression, the other a wave of elevation. Note that in the limit \( \mu = 0 \) or \((\mu, \kappa) = (0, 0)\) the variable \( P_1 \) solves a travelling-wave version of the (generalised) Kawahara equation.

It is instructive to interpret the above results for two well-studied magnetic intensities.

(i) The linear magnetisation law

\[ m_1(s) = s. \]

In region I we find that \( \check{c}_1 < 0 \) for \( \alpha_0 < 6 \) (solitary waves of depression) and \( \check{c}_1 > 0 \) for \( \alpha_0 > 6 \) (solitary waves of elevation); furthermore \( \check{d}_1 = 2 \), so that both types of waves exist for \( \alpha_0 \) near 6. This region has also been studied by Rannacher & Engel [18] using a weakly nonlinear approximation. In terms of the magnetic Bond number \( B = \alpha_0/\beta \) (corresponding to \( B < 9 \)) they derived a Korteweg-de Vries equation equivalent to \((1.2), (1.3)\) and found solitary waves of depression for \( \frac{3}{2} < B < 9 \) (that is, \( \alpha_0 < 6 \)) and of elevation for \( 1 < B < \frac{3}{2} \) (that is, \( \alpha_0 > 6 \)), in agreement with our results. (Continuing their weakly nonlinear analysis to the next order in this region would lead to a cubic Korteweg-de Vries equation equivalent to \((1.4), (1.5)\) and the prediction of both types of waves for \( B \) near \( \frac{3}{2} \).) In region II we find that \( c_1 = -240\sqrt{6} \) (solitary waves of depression).

(ii) The Langevin magnetisation law

\[ m_1(s) = \frac{\coth(\lambda s) - (\lambda s)^{-1}}{\coth \lambda - \lambda^{-1}}, \]

where \( \lambda > 0 \) is a dimensionless parameter. In Region I we find that \( \check{c}_1 < 0 \) for \( \alpha_0 < 6 \) and \( \alpha_0 > 6 \), \( \lambda \in (\lambda^*(\alpha_0), \infty) \) (solitary waves of depression), while \( \check{c}_1 < 0 \) for \( \alpha_0 > 6 \), \( \lambda \in (0, \lambda^*(\alpha_0)) \) (solitary waves of elevation), where \( \lambda^*(\alpha_0) \) is the unique solution of the equation

\[ \frac{\lambda^{-1} - \lambda \cosech^2 \lambda}{\coth \lambda - \lambda^{-1}} = 6\alpha_0^{-1} \]

(so that \( \lambda^*(6) = 0 \)). Furthermore \( \check{d}_1 > 0 \), so that both types of waves exist for \((\lambda, \alpha_0)\) near \((\lambda^*, \alpha_0(\lambda^*))\) (with \( \alpha_0(0) = 6 \)). In region II we find that \( c_1 < 0 \) (solitary waves of depression).
In Section 5.3 we turn to region III. Introducing a bifurcation parameter \( \mu \) so that positive values of \( \mu \) correspond to points on the ‘complex’ side of \( C_2 \), one obtains the reduced (reversible) Hamiltonian system

\[
\dot{A} = \frac{\partial \tilde{H}^\mu}{\partial B}, \quad \dot{B} = -\frac{\partial \tilde{H}^\mu}{\partial A},
\]

\[
\tilde{H}^\mu = i s (A \dot{B} - \dot{A} B) + |B|^2 + \tilde{H}^{0}_{\text{NF}}(|A|^2, i(A \dot{B} - \dot{A} B), \mu) + O(|(A, B)|^2)(\mu, A, B)^{n_0},
\]

where \( \tilde{H}^{0}_{\text{NF}} \) is a real polynomial which satisfies \( \tilde{H}^{0}_{\text{NF}} = 0 \); it contains the terms of order 3, \ldots, \( n_0+1 \) in the Taylor expansion of \( \tilde{H}^\mu \). The substitution \( A(z) = e^{i sz} a(z), B(z) = e^{i sz} b(z) \) converts the ‘truncated normal form’ obtained by neglecting the remainder term into the system

\[
\dot{a} = b + \partial_a \tilde{H}^{0}_{\text{NF}}(|a|^2, i(ab - \bar{a} \bar{b}), \mu),
\]

\[
\dot{b} = -\partial_b \tilde{H}^{0}_{\text{NF}}(|a|^2, i(ab - \bar{a} \bar{b}), \mu)
\]

(which, as evidenced by the scaling \( z \mapsto \mu^{1/2} z, (a, b) \mapsto (\mu^{1/2} a, \mu b) \), is at leading order equivalent to the nonlinear Schrödinger equation). Supposing that the coefficients of certain terms in \( \tilde{H}^{0}_{\text{NF}} \) have the correct sign, one finds that the latter system admits a circle of homoclinic solutions, two of which are real. The corresponding pair of homoclinic solutions to the original ‘truncated normal form’ are reversible and persist when the remainder terms are reinstated (see Iooss & Pérouème [15]). They generate symmetric solitary waves which take the form of periodic wave trains modulated by exponentially decaying envelopes; one is a wave of depression, the other a wave of elevation.

Each of the basic types of solitary waves in regions II and III is the primary member of an infinite family of multipulse solitary waves which resemble multiple copies of the primary. These waves are generated by corresponding multipulse homoclinic solutions which make several large excursions away from the origin in their four-dimensional phase space. A more precise description of the multipulse waves, together with a discussion of the relevant existence theories (which are based on variational and dynamical-systems arguments) is given in Sections 5.2 and 5.3.

Although the techniques used in the present paper are generalisations of those developed for the water-wave problem (see Iooss [14], Groves & Wahlén [13] and the references therein), we employ different methods to compute the reduced Hamiltonian systems. The spatial Hamiltonian system (1.1) is invariant under the transformation \( \phi \mapsto \phi + c, c \in \mathbb{R} \) (‘variation of potential base-level’), and the quantity \( \int_0^1 r \zeta \, dr \) is conserved. In many hydrodynamic problems it is possible to eliminate a symmetry of this kind before applying the Kirchgässner reduction (see e.g. Groves, Lloyd & Stylianou [11] §3.1 for an example in stationary ferrofluids), but here we retain it. It is inherited by the reduced systems: one of the canonical coordinates is cyclic and its conjugate is conserved. According to the classical theory, the next step is to set the conserved variable to zero, solve the resulting decoupled system for the other variables and recover the cyclic variable by quadrature; the lower-order system is typically studied using a canonical change of variables which simplifies its Hamiltonian (a ‘normal-form’ transformation). In the present context it is convenient to use a normal-form transformation before lowering the order of the system since it can be ‘absorbed’ into the changes of variable associated with the Kirchgässner reduction; this procedure greatly simplifies our later calculations. We present a general result for this purpose (Theorem 4.4), whose proof is based upon the method given by Bridges & Mielke [5] Theorem 4.3] and which may also be helpful in other applications.
2 The ferrohydrodynamic problem

We consider an incompressible, inviscid ferrofluid of unit density in the region

\[ S_1 := \{ 0 < r < R + \eta(\theta, z, t) \} \]

bounded by the free interface \( \{ r = R + \eta(\theta, z, t) \} \) and a current-carrying wire at \( \{ r = 0 \} \), where \((r, \theta, z)\) are cylindrical polar coordinates. The fluid is subject to a static magnetic field and the surrounding region

\[ S_2 = \{ r > R + \eta(\theta, z, t) \} \]

is a vacuum (see Figure 1).

We denote the magnetic and induction fields in the fluid and vacuum by respectively \( B_1, H_1 \) and \( B_2, H_2 \), and suppose that the relationships between them are given by the identities

\[ B_1 = \mu_0(H_1 + M_1(H_1)), \quad B_2 = \mu_0 H_2, \]

where \( \mu_0 \) is the magnetic permeability of free space and \( M_1 \) is the (prescribed) magnetic intensity of the ferrofluid. We suppose that \( M_1(H_1) = m_1(\|H_1\|) \frac{H_1}{\|H_1\|} \)

where \( m_1 \) is a (prescribed) nonnegative function, so that in particular \( M_1 \) and \( H_1 \) are collinear. According to Maxwell’s equations the magnetic and induction fields are respectively irrotational and solenoidal, and introducing magnetic potential functions \( \psi_1, \psi_2 \) with \( H_1 = -\nabla \psi_1, \quad H_2 = -\nabla \psi_2 \), we therefore find that

\[ \nabla \cdot (\mu(\|\nabla \psi_1\|) \nabla \psi_1) = 0 \quad \text{in } S_1, \]
\[ \Delta \psi_2 = 0 \quad \text{in } S_2, \]

in which

\[ \mu(s) = 1 + \frac{m_1(s)}{s} \]

is the magnetic permeability of the ferrofluid relative to that of free space. We suppose that the ferrofluid flow is irrotational, so that its velocity field \( \mathbf{v} \) is the gradient of a scalar velocity potential \( \phi \). The Euler equation for the ferrofluid is given by

\[ \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p^* + \mu_0 (M_1 \cdot \nabla) H_1 \]

(Rosensweig [19 §5.1]), where \( p^* \) is its composite pressure, and the calculations

\[ (M_1 \cdot \nabla) H_1 = |M_1| \nabla(\|H_1\|) = \nabla \left( \int_0^{|H_1|} m_1(t) \, dt \right), \quad (\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left( \frac{1}{2} |\mathbf{v}|^2 \right) \]

show that this equation is equivalent to

\[ \phi_t + \frac{1}{2} |\nabla \phi|^2 - \mu_0 \int_0^{|H_1|} m_1(t) \, dt + p^* = c_0, \quad (2.1) \]
where $c_0$ is a constant.

Next we turn to the boundary conditions at $\{r = R + \eta(\theta, z, t)\}$. The magnetic boundary conditions are

$$H_1 \cdot t = H_2 \cdot t, \quad B_1 \cdot n = B_2 \cdot n,$$

where $t$ and $n$ denote tangent and normal vectors to the free surface; it follows that

$$\psi_2 - \psi_1 \bigg|_{r=R+\eta(\theta,z,t)} = 0, \quad \psi_{2n} - \mu (|\nabla \psi_1|) \psi_{1n} \bigg|_{r=R+\eta(\theta,z,t)} = 0.$$

The (hydro-)dynamical boundary condition is given by

$$p^* + \frac{\mu_0}{2} (M_1 \cdot n)^2 = 2\sigma \kappa,$$

(Rosensweig [19 §5.2]), in which $\sigma > 0$ is the coefficient of surface tension and

$$2\kappa = \frac{-2\eta_\theta^2 - (R + \eta)^2(1 + \eta_z^2) + (R + \eta)^3\eta_{zz} + (R + \eta)\eta_\theta^2\eta_{zz} - 2(R + \eta)\eta_\theta\eta_z\eta_{zz} + (R + \eta)(1 + \eta_z^2)\eta_{\theta\theta}}{((R + \eta)(1 + \eta_z^2) + \eta_\theta^2)^{3/2}}$$

is the mean curvature of the interface; using (2.1), we find that

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 - \mu_0 \nu(|\nabla \psi_1|) + 2\sigma \kappa - \frac{\mu_0}{2} (\mu(|\nabla \psi_1|) - 1)^2 \bigg|_{r=R+\eta(\theta,z,t)} = c_0,$$

where

$$\nu(s) = \int_0^s m_1(t) \, dt.$$

Finally, the (hydro-)kinematic boundary condition is

$$(\partial_t + v \cdot \nabla)(r - R - \eta(\theta, z, t)) = 0,$$

that is

$$-\eta_t + \phi_r - \frac{1}{r^2} \phi_\theta \eta_\theta - \phi_z \eta_z = 0 \bigg|_{r=R+\eta(\theta,z,t)}.$$

The relevant conditions at $r = 0$ and in the far field are $v \cdot e_r, B_1 \cdot e_r \to 0$ as $r \to 0$, so that $\phi_r, \psi_1 \to 0$ as $r \to 0$, and $B_2 \cdot e_r \to 0$ as $r \to \infty$, so that $\psi_2 \to 0$ as $r \to \infty$.

The constant $c_0$ is selected so that

$$H_1 = \frac{J}{2\pi r} e_\theta, \quad H_2 = \frac{J}{2\pi r} e_\theta, \quad v = 0, \quad \eta = 0$$

(that is $\psi_1 = \psi_2 = -J\theta/2\pi, \phi = 0, \eta = 0$) is a solution to the above equations (corresponding to a uniform magnetic field and a circular cylindrical jet with radius $R$); we therefore set $c_0 = -\mu_0 \nu(J/2\pi r) + \sigma/R$. Seeking axisymmetric waves for which $\eta$ and $\phi$ are independent of $\theta$, one finds that $\psi_1 = \psi_2 = -J\theta/2\pi$, so that the hydrodynamic problem decouples from the magnetic problem and is given by

$$\phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} = 0, \quad 0 < r < R + \eta(z, t),$$

$$\phi_r = 0, \quad r = 0.$$
and

\[-\eta_t + \phi_r - \phi_z \eta_z = 0, \]
\[\phi_t + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \mu_0 \nu \left( \frac{J}{2 \pi (R + \eta)} \right) + \mu_0 \nu \left( \frac{J}{2 \pi R} \right) + \frac{\sigma}{(R + \eta)(1 + \eta_z^2)^{1/2}} - \frac{\sigma \eta_{zz}}{(1 + \eta_z^2)^{3/2}} - \frac{\sigma}{R} = 0\]

for \(r = R + \eta(z, t)\).

The next step is to seek travelling wave solutions for which \(\eta\) and \(\phi\) depend upon \(z\) and \(t\) only through the combination \(z - ct\), and to introduce dimensionless variables

\[
(\hat{z}, \hat{r}) := \frac{1}{R}(z - ct, r), \quad \hat{\phi} := \frac{1}{cR} \phi, \quad \hat{\eta} := \frac{1}{R} \eta.
\]

and functions

\[
\hat{m}_1(s) := 2\pi R \frac{J}{J \chi} m_1 \left( \frac{J}{2 \pi R} s \right), \quad \hat{\nu}(s) := \frac{4\pi^2 R^2}{J^2 \chi} \nu \left( \frac{J}{2 \pi R} s \right),
\]

where

\[
\chi = \frac{2\pi R}{J} m_1 \left( \frac{J}{2 \pi R} \right)
\]

(note that \(\hat{m}(1) = \hat{\nu}'(1) = 1\)). Dropping the hats for notational simplicity, we find that

\[
\phi_{rr} + \frac{1}{r} \phi_r + \phi_{zz} = 0, \quad 0 < r < 1 + \eta(z, t), \tag{2.2}
\]
\[
\phi_r = 0, \quad r = 0 \tag{2.3}
\]

and

\[
\eta_z + \phi_r - \phi_z \eta_z = 0, \tag{2.4}
\]
\[-\phi_z + \frac{1}{2}(\phi_r^2 + \phi_z^2) - \frac{T'(\eta)}{1 + \eta} + \beta \left( \frac{1}{(1 + \eta)(1 + \eta_z^2)^{1/2}} - \frac{\eta_{zz}}{(1 + \eta_z^2)^{3/2}} - 1 \right) = 0 \tag{2.5}
\]

for \(r = 1 + \eta(z, t)\), where

\[
T(\eta) = \int_0^\eta \left( \nu \left( \frac{1}{1 + s} \right) - \nu(1) \right) (1 + s) \, ds
\]

and

\[
\alpha = \frac{\mu_0 J^2 \chi}{4\pi^2 R^2 c^2}, \quad \beta = \frac{\sigma}{c^2 R} \tag{2.6}
\]

are dimensionless parameters. Solitary waves are nontrivial solutions of (2.2)–(2.5) with \(\eta(z)\), \(\phi(r, z) \to 0\) as \(z \to \pm \infty\). Finally, note that equations (2.2), (2.4) and (2.5) follow from the formal variational principle

\[
\delta \int \left\{ \int_0^{1+\eta} \left( \frac{1}{2} \phi_r^2 + \frac{1}{2} r \phi_z^2 - r \phi_z \right) \, dr - \alpha T(\eta) + \beta (1 + \eta)(1 + \eta_z^2)^{1/2} - \frac{1}{2} \beta (1 + \eta)^2 \right\} \, dz = 0,
\]

where the variations are taken with respect to \(\eta\) and \(\phi\).
3 Spatial dynamics

3.1 Formulation as a spatial Hamiltonian system

The first step is to use the ‘flattening’ transformation

\[ \hat{r} = \frac{r}{1 + \eta} \]

to map the variable domain \( \{0 < r < 1 + \eta\} \) into a fixed strip \((0, 1) \times \mathbb{R}\) and the free interface \( \{r = 1 + \eta(z)\} \) into \( \{\hat{r} = 1\} \). Dropping the hat for notational simplicity, we find that the corresponding ‘flattened’ variable

\[ \hat{\phi}(\hat{r}, z) = \phi(r, z) \]

tsatisfies the equations

\[ \left( \frac{r \phi_r}{(1 + \eta)^2} + r \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right) \right)_z - \frac{r^2 \eta_z}{1 + \eta} \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right)_r = 0, \quad 0 < r < 1 \]  

with boundary conditions

\[ \phi_r |_{r=0} = 0 \]

and

\[ \eta_z + \frac{\phi_r}{1 + \eta} - \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right) \eta_z \bigg|_{r=1} = 0, \]

\[ - \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right) + \frac{1}{2(1 + \eta)^2} \phi_r^2 + \frac{1}{2} \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right)^2 \bigg|_{r=1} - \alpha T'(\eta) + \beta \left( \frac{1}{(1 + \eta)(1 + \eta_z^2)^{1/2}} - \frac{\eta_z}{(1 + \eta_z^2)^{3/2}} \right) = 0. \]

Observe that equations (3.1), (3.3) and (3.4) follow from the new variational principle \( \delta \mathcal{L} = 0 \), where

\[ \mathcal{L}(\eta, \phi) := \int \left\{ \int_0^1 \left( \frac{1}{2} \left( r \phi_r^2 + \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right)^2 (1 + \eta)^2 r \right) - \left( \phi_z - \frac{r \eta_z \phi_r}{1 + \eta} \right) (1 + \eta)^2 r \right) \right\} dr 
- \alpha T(\eta) + \beta (1 + \eta)(1 + \eta_z^2)^{1/2} - \frac{1}{2} \beta (1 + \eta)^2 \right\} dz \]

and the variations are taken in \( \eta \) and \( \phi \) (the functional \( \mathcal{L} \) is obtained from the variational functional for (2.2), (2.4) and (2.5) by ‘flattening’).

We exploit this variational principle by regarding \( \mathcal{L} \) as an action functional of the form

\[ \mathcal{L} = \int L(\eta, \phi, \eta_z, \phi_z) \, dz, \]
in which $L$ is the integrand on the right-hand side of equation (3.5), and deriving a canonical Hamiltonian formulation of (3.1)–(3.4) by means of the Legendre transform. To this end, let us introduce new variables $\omega$ and $\xi$ by the formulae

$$
\omega = \frac{\delta L}{\delta \eta_z} = \int_0^1 \left\{ - \left( \phi_z - \frac{r\eta_z \phi_r}{1 + \eta} \right) (1 + \eta) r^2 \phi_r + (1 + \eta) r^2 \phi_r \right\} \, dr + \beta \frac{(1 + \eta) \eta_z}{(1 + \eta^2)^{1/2}},
$$

$$
\xi = \frac{\delta L}{\delta \phi_z} = \left( \phi_z - \frac{r\eta_z \phi_r}{1 + \eta} \right) (1 + \eta)^2 - (1 + \eta)^2
$$

and define the Hamiltonian function by

$$
H(\eta, \omega, \phi, \xi)
= \eta_z \omega + \int_0^1 r\phi_z \xi \, dr - L(\eta, \phi, \eta_z, \phi_z)
= \int_0^1 \left\{ \frac{1}{2} \left( \frac{\xi}{(1 + \eta)^2} + 1 \right)^2 (1 + \eta)^2 r - \frac{1}{2} r\phi_r^2 \right\} \, dr + \alpha T(\eta) - (1 + \eta) \sqrt{\beta^2 - W^2} + \frac{1}{2} \beta (1 + \eta)^2,
$$

(3.6)
in which

$$
W = \frac{1}{1 + \eta} \left( \omega + \frac{1}{1 + \eta} \int_0^1 r^2 \phi_r \xi \, dr \right).
$$

Writing $(\beta, \alpha) = (\beta_0 + \varepsilon_1, \alpha_0 + \varepsilon_2)$, where $(\beta_0, \alpha_0)$ are fixed, and $\xi = \zeta - 1$ (since $(\eta, \omega, \phi, \xi) = (0, 0, -1, 0)$ is the ‘trivial’ solution of Hamilton’s equations), we find that Hamilton’s equations are given explicitly by

$$
\eta_z = \frac{\delta H^e}{\delta \omega} = \frac{W}{\sqrt{\beta_0 + \varepsilon_1)^2 - W^2}},
$$

(3.7)

$$
\omega_z = -\frac{\delta H^e}{\delta \eta} = \int_0^1 \left\{ \left( \frac{\zeta - 1}{1 + \eta} \right)^2 + 1 \right\} (1 + \eta) r + \frac{W r^2 \phi_r (\zeta - 1)}{(1 + \eta)^2 \sqrt{\beta_0 + \varepsilon_1^2 - W^2}} \right\} \, dr
- \left( \alpha_0 + \varepsilon_2 \right) T'(\eta) + \frac{(\beta_0 + \varepsilon_1)^2}{\sqrt{\beta_0 + \varepsilon_1^2 - W^2}} - (\beta_0 + \varepsilon_1)(1 + \eta),
$$

(3.8)

$$
\phi_z = \frac{\delta H^e}{\delta \zeta} = \left( \frac{\zeta - 1}{1 + \eta} \right)^2 + 1 \right) \frac{W}{\sqrt{\beta_0 + \varepsilon_1^2 - W^2}} \left( \frac{r \phi_r}{1 + \eta} \right),
$$

(3.9)

$$
\zeta_z = -\frac{\delta H^e}{\delta \phi} = -\frac{1}{r} (r \phi_r)_r + \frac{W}{\sqrt{\beta_0 + \varepsilon_1^2 - W^2}} \left( \frac{1}{r^2 (\zeta - 1))_r} \right),
$$

(3.10)

where the superscript denotes the dependence upon $\varepsilon$, with boundary condition

$$
r \phi_r - \left. \frac{W}{\sqrt{\beta_0 + \varepsilon_1^2 - W^2}} \frac{r^2 (\zeta - 1)}{1 + \eta} \right|_{r=1} = 0,
$$

(3.11)

the second of which arises from the integration by parts necessary to compute (3.10). Note that our equations are reversible, that is invariant under the transformation $(\eta, \omega, \phi, \zeta)(z) \mapsto S(\eta, \omega, \phi, \zeta)(-z)$, where the reverser is defined by $S(\eta, \omega, \phi, \zeta) = (\eta, -\omega, -\phi, \zeta)$. 

11
To make this construction rigorous we recall the differential-geometric definitions of a Hamiltonian system and Hamilton’s equations for its associated vector field.

**Definition 3.1.** A Hamiltonian system consists of a triple \((M, \Omega, H)\), where \(M\) is a manifold, \(\Omega : TM \times TM \to \mathbb{R}\) is a closed, weakly nondegenerate bilinear form (the symplectic 2-form) and the Hamiltonian \(H : N \to \mathbb{R}\) is a smooth function on a manifold domain \(N\) of \(M\) (that is, a manifold \(N\) which is smoothly embedded in \(M\) and has the property that \(TN|_n\) is densely embedded in \(TM|_n\) for each \(n \in N\)).

Its Hamiltonian vector field \(v_H\) with domain \(\mathcal{D}(v_H) \subseteq N\) is defined as follows. The point \(n \in N\) belongs to \(\mathcal{D}(v_H)\) if and only if

\[
\Omega|_n(w, v) = \text{d}H|_n(v)
\]

for all tangent vectors \(v \in TM|_n\) (by construction \(\text{d}H|_n \in T^*N|_n\) admits a unique extension \(\text{d}H|_n \in T^*M|_n\)). Hamilton’s equations for \((M, \Omega, H)\) are the differential equations

\[
\dot{u} = v_H|_u
\]

which determine the trajectories \(u \in C^1(\mathbb{R}, M) \cap C(\mathbb{R}, N)\) of its Hamiltonian vector field.

**Definition 3.1** applies to the above formulation. Note that the identity mapping is (up to the scaling factor \(\sqrt{2\pi}\)) an isometry \(\tilde{L}^2(B_1(0)) \to L^2_r(0, 1), \tilde{H}^1(B_1(0)) \to H^1_r(0, 1)\) and \(\tilde{H}^2(B_1(0)) \to \{\phi \in H^2_r(0, 1) : \phi_r \in L^2_{r-1}(0, 1)\}\), where \(B_1(0)\) is the unit ball in \(\mathbb{R}^2\) and \(\tilde{H}^s(B_1(0))\) denotes the closed subspace of \(H^s(B_1(0))\) consisting of axisymmetric functions (see Bernardi, Dauge & Maday [2, Theorem II.2.1]). We therefore let \(M\) be a neighbourhood of the origin in

\[
X := \{(\eta, \omega, \phi, \zeta) \in \mathbb{R} \times \mathbb{R} \times H^1_r(0, 1) \times L^2_r(0, 1)\}
\]

and \(N = Y \cap M\) with

\[
Y := \{(\eta, \omega, \phi, \zeta) \in \mathbb{R} \times \mathbb{R} \times H^2_r(0, 1) \times H^1_r(0, 1) : \phi_r \in L^2_{r-1}(0, 1)\},
\]

so that elements \((\eta, \omega, \phi, \zeta) \in Y\) satisfy \(\phi_r|_{r=0} = 0\) (see Bernardi, Dauge & Maday [2, Remark II.1.1]). We consider values of \((\varepsilon_1, \varepsilon_2)\) in a neighbourhood \(\Lambda\) of the origin in \(\mathbb{R}^2\) and choose \(M\) and \(\Lambda\) small enough so that

\[
|\varepsilon_1| < \frac{\beta_0}{4}, \quad \eta > -\frac{1}{2} > -1, \quad |W| < \frac{\beta_0}{2} < \beta_0 + \varepsilon_1.
\]

The formula

\[
\Omega((\eta_1, \omega_1, \phi_1, \zeta_1), (\eta_2, \omega_2, \phi_2, \zeta_2)) = \omega_2\eta_1 - \eta_2\omega_1 + \int_0^1 r(\zeta_2\phi_1 - \phi_2\zeta_1) \, dr
\]

defines a weakly nondegenerate bilinear form \(M \times M \to \mathbb{R}\) and hence a constant symplectic 2-form \(TM \times TM \to \mathbb{R}\) (its closure follows from the fact that it is constant), and the function \(H^\varepsilon\) given by \((3.6)\) belongs to \(C^\infty(N, \mathbb{R})\), so that the triple \((M, \Omega, H)\) is a Hamiltonian system. Applying the criterion in the definition, one finds that

\[
\mathcal{D}(v_{H^\varepsilon}) = \left\{ (\eta, \omega, \phi, \zeta) \in N : r\phi_r - \frac{W}{\sqrt{((\beta_0 + \varepsilon_1)^2 - W^2(\zeta - 1)} \bigg|_{r=1} = 0 \right\}
\]
and that Hamilton’s equations are given explicitly by (3.7)–(3.10).

It remains to confirm the relationship between a solution to Hamilton’s equations for $(M, \Omega, H^\epsilon)$ and a solution to the ‘flattened’ hydrodynamic problem (3.1)–(3.4). Suppose that $(\eta, \omega, \phi, \zeta)$ is a smooth solution of Hamilton’s equations. An explicit calculation shows that the variables $\tilde{\eta}, \tilde{\phi}$ given by $\tilde{\eta}(z) = \eta(z), \tilde{\phi}(r,z) = \phi(z)(r)$ solve (3.1)–(3.4) (see Groves & Toland [12, pp. 212-214] for a discussion of this procedure in the context of water waves).

4 Centre-manifold reduction

Our strategy in finding solutions to Hamilton’s equations (3.7)–(3.10) for $(M, \Omega, H^\epsilon)$ consists in applying a reduction principle which asserts that $(M, \Omega, H^\epsilon)$ is locally equivalent to a finite-dimensional Hamiltonian system. The key result is the following theorem, which is a parametrised, Hamiltonian version of a reduction principle for quasilinear evolutionary equations presented by Mielke [17, Theorem 4.1] (see Buffoni, Groves & Toland [8, Theorem 4.1]).

**Theorem 4.1.** Consider the differential equation

$$\dot{u} = Lu + N(u; \lambda),$$

which represents Hamilton’s equations for the reversible Hamiltonian system $(M, \Omega^\lambda, H^\lambda)$. Here $u$ belongs to a Hilbert space $\mathcal{X}$, $\lambda \in \mathbb{R}^\ell$ is a parameter and $L : D(L) \subset \mathcal{X} \to \mathcal{X}$ is a densely defined, closed linear operator. Regarding $D(L)$ as a Hilbert space equipped with the graph norm, suppose that $0$ is an equilibrium point of (4.1) when $\lambda = 0$ and that

(H1) The part of the spectrum $\sigma(L)$ of $L$ which lies on the imaginary axis consists of a finite number of eigenvalues of finite multiplicity and is separated from the rest of $\sigma(L)$ in the sense of Kato, so that $\mathcal{X}$ admits the decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1 = \mathcal{P}(\mathcal{X})$, $\mathcal{X}_2 = (I - \mathcal{P})(\mathcal{X})$ and $\mathcal{P}$ is the spectral projection corresponding the purely imaginary part of $\sigma(L)$.

(H2) The operator $L_2 = L|_{\mathcal{X}_2}$ satisfies the estimate

$$\|(L_2 - isI)^{-1}\|_{\mathcal{X}_2 \to \mathcal{X}_2} \leq \frac{C}{1 + |s|}, \quad s \in \mathbb{R},$$

for some constant $C$ that is independent of $s$.

(H3) There exists a natural number $k$ and neighbourhoods $\Lambda \subset \mathbb{R}^\ell$ of $0$ and $U \subset D(L)$ of $0$ such that $N$ is $(k + 1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and $N(0, 0) = 0$, $d_1 N[0, 0] = 0$.

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of $0$ and $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ of $0$ and a reduction function $r : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2$ with the following properties. The reduction function $r$ is $k$ times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$, its derivatives are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ and $r(0; 0) = 0$, $d_1 r[0; 0] = 0$. The graph $\tilde{M}^\lambda = \{u_1 + r(u_1; \lambda) \in \mathcal{X}_1 \oplus \mathcal{X}_2 : u_1 \in \tilde{U}_1\}$ is a Hamiltonian centre manifold for (4.1), so that
(i) $\tilde{M}^\lambda$ is a locally invariant manifold of (4.1): through every point in $\tilde{M}^\lambda$ there passes a unique solution of (4.1) that remains on $M^\lambda$ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$.

(ii) Every small bounded solution $u(x), x \in \mathbb{R}$ of (4.1) that satisfies $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$ lies completely in $\tilde{M}^\lambda$.

(iii) Every solution $u_1: (x_1, x_2) \rightarrow \tilde{U}_1$ of the reduced equation

$$\dot{u}_1 = L u_1 + \mathcal{P} \mathcal{N}(u_1 + r(u_1; \lambda); \lambda)$$

(4.2)

generates a solution

$$u(x) = u_1(x) + r(u_1(x); \lambda)$$

(4.3)

of the full equation (4.1).

(iv) $\tilde{M}^\lambda$ is a symplectic submanifold of $M$ and the flow determined by the Hamiltonian system $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, where the tilde denotes restriction to $\tilde{M}^\lambda$, coincides with the flow on $\tilde{M}^\lambda$ determined by $(M, \Omega^\lambda, H^\lambda)$. The reduced equation (4.2) is reversible and represents Hamilton’s equations for $(\tilde{M}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$.

Mielke’s theorem cannot be applied directly to (3.7)–(3.10) because of the nonlinear boundary condition (3.11) in the domain of the Hamiltonian vector field $v_{H^\varepsilon}$ (the right-hand sides of (3.7)–(3.10) define a smooth mapping $g_{H^\varepsilon}: Y \rightarrow X$ with $v_{H^\varepsilon}|_u = g_{H^\varepsilon}(u)$ for any $u \in \mathcal{D}(v_{H^\varepsilon})$). We overcome this difficulty using the change of variable $G^\varepsilon: (\eta, \omega, \phi, \zeta) \mapsto (\eta, \hat{\omega}, \hat{\phi}, \zeta)$, where

$$\hat{\omega} = \int_0^1 r^2 \phi_r \, dr, \quad \hat{\phi} = \phi - \frac{W}{\sqrt{(\beta_0 + \varepsilon_1)^2 - W^2}} \frac{1}{1 + \eta} \int_0^r s(\zeta - 1) \, ds,$$

(4.4)

which transforms the boundary condition in $\mathcal{D}(v_{H^\varepsilon})$ into

$$r \hat{\phi}_r |_{r=1} = 0.$$

Lemma 4.2. For each $\varepsilon \in \Lambda$ the mapping $G^\varepsilon$ is a smooth diffeomorphism from the neighbourhood $M$ of the origin in $X$ onto a neighbourhood $\tilde{M}$ of the origin in $X$, and from $N = M \cap Y$ onto $\tilde{N} = M \cap Y$. The diffeomorphisms and their inverses depend smoothly upon $\varepsilon \in \Lambda$.

Proof. These results follow from the explicit formulae (4.4) and

$$\omega = \frac{(\beta_0 + \varepsilon_1) \Gamma(1 + \eta)}{\sqrt{1 + \Gamma^2}} - \frac{1}{1 + \eta} \int_0^1 r^2 \hat{\phi}_r (\zeta - 1) \, dr - \frac{\Gamma}{(1 + \eta)^2} \int_0^1 r^3 (\zeta - 1)^2 \, dr;$$

$$\phi = \hat{\phi} + \frac{\Gamma}{1 + \eta} \int_0^r s(\zeta - 1) \, ds,$$

where

$$\Gamma = (1 + \eta) \left( \hat{\omega} - \int_0^1 r^2 \hat{\phi}_r \, dr \right) / \int_0^1 r^3 (\zeta - 1) \, dr,$$

for $G^\varepsilon$ and its inverse $(G^\varepsilon)^{-1}: (\eta, \hat{\omega}, \hat{\phi}, \zeta) \mapsto (\eta, \phi, \zeta)$. \qed
A simple calculation shows that the diffeomorphism $G$ transforms

$$u_z = g^\varepsilon(u)$$

into

$$u_z = \hat{g}^\varepsilon(u), \quad (4.5)$$

where $\hat{g}^\varepsilon : Y \to X$ is the smooth vector field defined by

$$\hat{g}^\varepsilon(u) = \text{d}G^\varepsilon\left[ (G^\varepsilon)^{-1}(u) \right] \left( g^\varepsilon((G^\varepsilon)^{-1}(u)) \right).$$

Formula (4.5) represents Hamilton’s equations for the Hamiltonian system $(\hat{M}, \Upsilon^\varepsilon, \hat{H}^\varepsilon)$, where

$$\Upsilon^\varepsilon\mid_m(v_1, v_2) = \Omega(\text{d}G^\varepsilon[(G^\varepsilon)^{-1}(m)]^{-1}(v_1), \text{d}G^\varepsilon[(G^\varepsilon)^{-1}(m)]^{-1}(v_2)), \quad m \in \hat{M}, \ v_1, v_2 \in T\hat{M}\mid_m,$$

and

$$\hat{H}^\varepsilon(n) = H^\varepsilon((G^\varepsilon)^{-1}(n)), \quad n \in \hat{N}.$$

The domain of the Hamiltonian vector field $v_{\hat{H}^\varepsilon}$ is

$$\mathcal{D}(v_{\hat{H}^\varepsilon}) = \{ (\eta, \omega, \phi, \zeta) \in \hat{N} : r\hat{\phi}_r\mid_{r=1} = 0 \}$$

and $v_{\hat{H}^\varepsilon}\mid_n = \hat{g}^\varepsilon(n)$ for any $n \in \mathcal{D}(v_{\hat{H}^\varepsilon})$.

The next step is to verify that (4.5) satisfies the hypotheses of Theorem 4.1 (with $\mathcal{X} = X$), so that we obtain a finite-dimensional reduced Hamiltonian system $(\tilde{M}^\varepsilon, \tilde{\Gamma}^\varepsilon, \tilde{H}^\varepsilon)$. We write (4.5) as

$$u_z = Lu + N^\varepsilon(u),$$

in which $L = \text{d}v_{\hat{H}^\varepsilon}[0]$ and verify the spectral hypotheses on $L$ by considering the operator $K : \mathcal{D}(K) \subseteq X \to X$, where

$$K \begin{pmatrix} \eta \\ \omega \\ \phi \\ \zeta \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_0} \left( \omega - \int_0^1 r^2\phi_r \, dr \right) \\ -2 \int_0^1 r\zeta \, dr - 2\eta + (\alpha_0 - \beta_0)\eta \\ \zeta + 2\eta \\ -\frac{1}{r}(r\phi_r)_r - \frac{2}{\beta_0} \left( \omega - \int_0^1 r^2\phi_r \, dr \right) \end{pmatrix}, \quad (4.6)$$

and

$$\mathcal{D}(K) = \left\{ (\eta, \omega, \phi, \zeta) \in Y : -r\phi_r - \frac{r^2}{\beta_0} \left( \omega - \int_0^1 r^2\phi_r \, dr \right) \bigg|_{r=1} \right\}$$

(the formal linearisation of $v_{\hat{H}^\varepsilon}$ at the origin); the formula $K = \text{d}G^0[0]^{-1}L\text{d}G^0[0]$ shows that the spectral properties of $K$ and $L$ are identical. It follows from Lemma 4.3 below that $L$ satisfies hypotheses (H1) and (H2); hypothesis (H3) is clearly satisfied for an arbitrary value of $k$. Part (i) of Lemma 4.3 is proved using the elementary theory of ordinary differential equations, while part (ii) is established using arguments similar to those employed for other problems treated using centre-manifold reduction (e.g. see Buffoni, Groves & Toland [8, Proposition 3.2] or Groves & Wahlén [13, Lemma 3.4]).
Lemma 4.3.

(i) The spectrum $\sigma(L)$ of $L$ consists entirely of isolated eigenvalues of finite algebraic multiplicity. A complex number $\lambda$ is an eigenvalue of $L$ if and only if

$$\lambda J_0(\lambda) = (\gamma_0 - \beta_0 \lambda^2) J_1(\lambda),$$

where $\gamma_0 = \alpha_0 - \beta_0$. (In particular, 0 is an eigenvalue of $L$ and $\sigma(L) \cap i\mathbb{R}$ is a finite set.)

(ii) There exist real constants $C$, $s_0 > 0$ such that

$$\|(L - isI)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{C}{|s|}$$

for each real number $s$ with $|s| > s_0$.

According to Lemma 4.3(i), a purely imaginary number $\lambda = is$ is an eigenvalue of $L$ if and only if

$$s I_0(s) = (\gamma_0 + \beta_0 s^2) I_1(s).$$

Straightforward computations show that there are three critical curves

$$C_2 = \left\{ (\beta_0, \gamma_0) = \left( \frac{1}{2} \left( 1 - \frac{I_0(s) I_2(s)}{I_1(s)^2} \right), \frac{1}{2} s^2 \left( -1 + \frac{I_0(s)^2}{I_1(s)^2} \right) \right) : s \in (0, \infty) \right\}$$

and

$$C_3 = \{ (\beta_0, \gamma_0) : \beta_0 < \frac{1}{4}, \gamma_0 = 2 \}, \quad C_4 = \{ (\beta_0, \gamma_0) : \beta_0 > \frac{1}{4}, \gamma_0 = 2 \}$$

in the $(\beta_0, \gamma_0)$ parameter plane at which purely imaginary eigenvalues of $L$ collide, together with a fourth curve

$$C_1 = \left\{ (\beta_0, \gamma_0) = \left( \frac{1}{2} \left( 1 - \frac{J_0(k) J_2(k)}{J_1(k)^2} \right), \frac{k^2 (J_0(k)^2 + J_1(k)^2)}{2 J_1(k)^2} \right) : k \in (0, j_{1,1}) \right\}$$

at which real eigenvalues collide (see Figure 3). Here $J_0$, $J_1$, ... and $I_0$, $I_1$, ... denote respectively the Bessel functions and modified Bessel functions of the first kind, and $j_{1,1} > 0$ is the smallest zero of $J_1$. Furthermore, $L$ has a geometrically simple zero eigenvalue whose algebraic multiplicity is two for $\gamma_0 \neq 2$, four for $\gamma_0 = 2$, $\beta_0 \neq \frac{1}{4}$ and and six for $(\beta_0, \gamma_0) = \left( \frac{1}{4}, 2 \right)$.

The centre manifold $\tilde{M}^\varepsilon$ is equipped with the single coordinate chart $\tilde{U}_1 \subset \mathcal{X}_1$ and coordinate map $\pi : \tilde{M}^\varepsilon \to \tilde{U}_1$ defined by $\pi^{-1}(u_1) = u_1 + r(w_1; \varepsilon)$. It is however more convenient to use an alternative coordinate map for calculations. We define the function $\tilde{r} : \tilde{W}_1 \times \Lambda \to \tilde{U}_1 \times \tilde{U}_2$ with $\tilde{W}_1 = \mathcal{P}(G^\varepsilon)^{-1}(\tilde{U}_1 \times \tilde{U}_2)$ (which in general has components in $\mathcal{X}_1$ and $\mathcal{X}_2$) by the formula

$$w_1 + \tilde{r}(w_1; \varepsilon) = (G^\varepsilon)^{-1}(w_1 + r(w_1; \varepsilon)), \quad (4.7)$$

where $\tilde{r}(0; 0) = 0$, $d_1 \tilde{r}(0; 0) = 0$, and equip $\tilde{M}^\varepsilon$ with the coordinate map $\tilde{\pi} : \tilde{M}^\varepsilon \to \tilde{W}_1$ given by $\tilde{\pi}^{-1}(w_1) = w_1 + \tilde{r}(w_1; \varepsilon)$, so that

$$\tilde{H}^\varepsilon(w_1) = H^\varepsilon(w_1 + \tilde{r}(w_1; \varepsilon)),$$

$$\tilde{\Omega}^\varepsilon|_{w_1}(v_1, v_2) = \Omega(v_1 + d_1 \tilde{r}[w_1; \varepsilon](v_1), v_2 + d_1 \tilde{r}[w_1; \varepsilon](v_2)) = \Omega(v_1, v_2) + O(|(\varepsilon, w_1)|) \quad (4.8)$$

as $(\varepsilon, w_1) \to 0$. Furthermore, using a parameter-dependent version of Darboux’s theorem (e.g. see Buffoni & Groves 17 Theorem 4), we may assume that the remainder term in (4.8) vanishes identically.
We proceed by choosing a symplectic basis \( \{ f_0^1, \ldots, f_n^1, f_0^2, \ldots, f_n^2 \} \) for the centre subspace of \( K \) (so that \( \Omega(f_i^1, f_i^2) = 1 \) for \( i = 0, \ldots, n \) and the symplectic product of any other combination of these vectors is zero); here either \( f_0^1 \) or \( f_0^2 \) is the eigenvector \((0, 0, 1, 0)^T\) corresponding to the zero eigenvalue of \( K \). Using coordinates \( q_0, \ldots, q_n, p_0, \ldots, p_n \), where

\[
 w_1 = q_0 f_0^1 + q_1 f_1^1 + \cdots + q_n f_n^1 + p_0 f_0^2 + p_1 f_1^2 + \cdots + p_n f_n^2,
\]

we find that \( \tilde{\Omega}^\varepsilon \) is the canonical 2-form. Note that equations (3.7)–(3.11) are invariant under the transformation \( \phi \mapsto \phi + c, \ c \in \mathbb{R} \), and the quantity \( \int_0^1 r \zeta \, dr \) is conserved. This symmetry is inherited by the reduced system: one of the variables \( q_0, p_0 \) is cyclic (that is, \( \tilde{\tau} \) and \( \tilde{H}^\varepsilon \) do not depend upon it), so that the other is conserved.

According to the classical theory, the next step is to lower the dimension of the reduced system by two by setting the conserved variable to zero, solving the resulting decoupled system for \( q_1, \ldots, q_n, p_1, \ldots, p_n \), and recovering the cyclic variable by quadrature; the lower-order system is typically studied using a canonical change of variables which simplifies its Hamiltonian \( \tilde{H}^\varepsilon \big|_{q_0=0} \) (a ‘normal-form’ transformation). For our purposes it is convenient to use a normal-form transformation before lowering the order of the system since it can be ‘absorbed’ into \( \tilde{\tau} \) in the same way as the Darboux transformation; this procedure greatly simplifies our later calculations. The following general result (whose proof is based upon the method given by Bridges & Mielke [5, Theorem 4.3]) shows that this procedure is possible; we assume for definiteness that \( p_0 \) is cyclic and use the construction by Elphick [10] as our ‘usual’ normal form. The result is applied to the specific parameter regimes shown in Figure 2(a) in Section 5 below, where we denote the nonlinear part of the reduced Hamiltonian vector field \( v_{\tilde{H}^\varepsilon} \) by \( P^\varepsilon(w_1) \).

---

**Figure 3:** Eigenvalues of \( L \); solid and hollow dots denote respectively algebraically simple and multiple eigenvalues. The curves \( C_j, j = 1, \ldots, 4 \) consist of points in \((\beta_0, \gamma_0)\) parameter space at which the qualitative nature of the eigenvalue picture changes.
Theorem 4.4. Consider the \((n+1)\)-degree-of-freedom Hamiltonian system

\[
\begin{align*}
\dot{q}_i &= \frac{\partial \tilde{H}^\varepsilon}{\partial p_i}, & \dot{p}_i &= -\frac{\partial \tilde{H}^\varepsilon}{\partial q_i}, & i = 1, \ldots, n, \\
\dot{q}_0 &= \frac{\partial \tilde{H}^\varepsilon}{\partial p_0}, & \dot{p}_0 &= -\frac{\partial \tilde{H}^\varepsilon}{\partial q_0},
\end{align*}
\tag{4.9}
\]

where \(\tilde{H}^\varepsilon(q,p,q_0) = O(||(\varepsilon,q_0,q,p)||(q_0,q,p)||)\) and \(p_0\) is cyclic (so that \(q_0\) is conserved).

There exists a near-identity canonical change of variables \((q,p,q_0,p_0) \mapsto (Q,P,Q_0,P_0)\) with the properties that \(P_0\) is cyclic, \(Q_0 = q_0\) and the lower-order Hamiltonian system

\[
\begin{align*}
\dot{Q}_i &= \frac{\partial \tilde{H}^\varepsilon}{\partial P_i}(Q,P,0), & \dot{P}_i &= -\frac{\partial \tilde{H}^\varepsilon}{\partial Q_i}(Q,P,0), & i = 1, \ldots, n,
\end{align*}
\tag{4.10}
\]

adopts its usual normal form. (Here, with a slight abuse of notation, we denote the transformed Hamiltonian by \(\tilde{H}^\varepsilon(Q,P,Q_0)\).)

Proof. Consider the \(n\)-degree of freedom Hamiltonian system

\[
\begin{align*}
\dot{q}_i &= \frac{\partial \tilde{H}^\varepsilon}{\partial p_i}, & \dot{p}_i &= -\frac{\partial \tilde{H}^\varepsilon}{\partial q_i}, & i = 1, \ldots, n,
\end{align*}
\tag{4.11}
\]

in which \(q_0\) and \(\varepsilon\) are parameters. The standard theory asserts the existence of a canonical change of variables

\[
\begin{align*}
Q &= q + h_1^\varepsilon(q,p,q_0), \\
P &= p + h_2^\varepsilon(q,p,q_0)
\end{align*}
\]

with

\[
h_j^\varepsilon(q,p,q_0) = O(||(\varepsilon,q_0,q,p)||(q,p)||), \quad j = 1, 2,
\]

which converts (4.11) into its parameter-dependent normal form; note that

\[
M_1^T J_1 M_1 = J_1,
\]

where

\[
M_1 = \begin{pmatrix} Q_q & Q_p \\ P_q & P_p \end{pmatrix} = \begin{pmatrix} I + \partial_q h_1^\varepsilon & \partial_p h_1^\varepsilon \\ \partial_q h_2^\varepsilon & I + \partial_p h_2^\varepsilon \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

and this condition may also be written as

\[
(I + \partial_q h_1^\varepsilon)(I + \partial_p h_2^\varepsilon) - \partial q h_2^\varepsilon \partial_p h_1^\varepsilon = I. \tag{4.12}
\]

We seek a change of variable for (4.9), (4.10) of the form

\[
\begin{align*}
Q &= q + h_1^\varepsilon(q,p,q_0), \\
P &= p + h_2^\varepsilon(q,p,q_0), \\
Q_0 &= q_0, \\
P_0 &= p_0 + h_4^\varepsilon(q,p,q_0);
\end{align*}
\]
the new function

\[ h_4^\varepsilon(q, p, q_0) = O(||(\varepsilon, q_0, q, p)||) \]

is subject to the requirement that

\[ M_2^T J_2 M_2 = J_2, \]

where

\[
M_2 = \begin{pmatrix}
Q_q & Q_p & Q_{q_0} & Q_{p_0} \\
Q_{q_0} & Q_{p_0} & P_{q_0} & P_{p_0} \\
P_{q_0} & P_{p_0} & Q_0 & Q_p \\
P_{q_0} & P_{p_0} & Q_0 & Q_p
\end{pmatrix} = \begin{pmatrix}
I + \partial_q h_1^\varepsilon & \partial_p h_1^\varepsilon & \partial_{q_0} h_1^\varepsilon & 0 \\
\partial_q h_2^\varepsilon & I + \partial_p h_2^\varepsilon & \partial_{q_0} h_2^\varepsilon & 0 \\
0 & 0 & 1 & 0 \\
\partial_q h_4^\varepsilon & I + \partial_p h_4^\varepsilon & \partial_{q_0} h_4^\varepsilon & 1
\end{pmatrix},
\]

\[
J_2 = \begin{pmatrix}
0 & I & 0 & 0 \\
-I & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\]

and this condition may be written as

\[
(I + \partial_q h_1^\varepsilon)(I + \partial_p h_2^\varepsilon) - \partial_q h_2^\varepsilon \partial_p h_1^\varepsilon = I, \tag{4.13}
\]

\[
(I + \partial_q h_1^\varepsilon) \partial_{q_0} h_2^\varepsilon - \partial_q h_2^\varepsilon \partial_{q_0} h_1^\varepsilon = \partial_q h_4^\varepsilon, \tag{4.14}
\]

\[
\partial_p h_1^\varepsilon \partial_{q_0} h_2^\varepsilon - (I + \partial_p h_1^\varepsilon) \partial_{q_0} h_4^\varepsilon = \partial_p h_4^\varepsilon. \tag{4.15}
\]

It is possible to find \( h_4^\varepsilon \) satisfying these conditions since the compatibility condition for (4.14), (4.15) is the derivative of (4.13) with respect to \( q_0 \), and (4.13) is automatically satisfied because of (4.12).

\[
\Box
\]

5 The reduced Hamiltonian systems

5.1 Homoclinic bifurcation at \( C_4 \)

At each point of the curve \( C_4 \) in Figure 3 two real eigenvalues become purely imaginary by colliding at the origin and increasing the algebraic multiplicity of the zero eigenvalue from two to four. This resonance is associated with the bifurcation of a branch of homoclinic solutions into the region with real eigenvalues (the parameter regime marked I in Figure 2). Let us therefore fix reference values \((\beta_0, \gamma_0) \in C_4\), so that \( \beta_0 > \frac{1}{4}, \alpha_0 = 2 + \beta_0 \), and introduce a bifurcation parameter by choosing \((\varepsilon_1, \varepsilon_2) = (0, \mu)\), where \( 0 < \mu \ll 1 \).

The four-dimensional centre subspace of \( K \) is spanned by the generalised eigenvectors

\[
e_1 = \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}, \quad e_2 = \begin{pmatrix}
\frac{1}{2} \\
0 \\
0 \\
0
\end{pmatrix}, \quad e_3 = \begin{pmatrix}
0 \\
\frac{1}{2}(\beta_0 - \frac{1}{4}) \\
-\frac{1}{4}r^2 + A_4 \\
0
\end{pmatrix}, \quad e_4 = \begin{pmatrix}
\frac{1}{4}(\beta_0 - \frac{1}{2}) + \frac{1}{2}A_4 \\
0 \\
0 \\
-\frac{1}{4}r^2 - \frac{1}{2}(\beta_0 - \frac{1}{2})
\end{pmatrix},
\]

where \( A_4 = -\left(\beta_0 - \frac{1}{4}\right)^{-1}\left(\frac{1}{24} + \frac{1}{4}\beta_0(\beta_0 - 1)\right) \) has been chosen so that \( Ke_1 = 0, Ke_j = e_{j-1} \) for \( j = 2, 3, 4 \),

\[
\Omega(e_1, e_4) = -\frac{1}{4}(\beta_0 - \frac{1}{4}), \quad \Omega(e_2, e_3) = \frac{1}{4}(\beta_0 - \frac{1}{4})
\]
and the symplectic product of any other combination of the vectors $e_1, \ldots, e_4$ is zero. Writing
\[ w_1 = q_0 f_4 + p_0 f_1 + q f_2 + p f_3, \quad f_i := \frac{1}{2} (\beta_0 - \frac{1}{3})^{-1/2} e_i, \]
we therefore find that $q_0$, $q$, $p_0$ and $p$ are canonical coordinates for the reduced Hamiltonian system, which has the cyclic variable $p_0$ and reverser $S : (q_0, q, p_0, p) \mapsto (q_0, q, -p_0, -p)$; with a slight abuse of notation we abbreviate $\tilde{H}^\varepsilon|_{(\varepsilon_1, \varepsilon_2) = (0, \mu)}$ to $\tilde{H}^\mu$.

The usual normal-form theory for the two-dimensional system with Hamiltonian $\tilde{H}^\mu(q, p, 0)$ asserts that, after a canonical change of variables,
\[ \tilde{H}^\mu(q, p, 0) = \frac{1}{2} p^2 + \tilde{H}_{NF}^0(q, \mu) + O((q, p)^2|\mu, q, p|^{n_0}), \]
where $\tilde{H}_{NF}^0(q, \mu)$ is a polynomial of order $n_0 + 1$ in $(q, \mu)$ with
\[ \tilde{H}_{NF}^0(q, \mu) = O(|q|^2|\mu(q, p)|). \]
It follows that, after a canonical change of variables,
\[ \tilde{H}^\mu(q, p, q_0) = \frac{1}{2} p^2 - q q_0 + \tilde{H}_{NF}^\mu(q, p, q_0) \]
with
\[ \tilde{H}_{NF}^\mu(q, p, q_0) = \tilde{H}_{NF}(q, q_0, \mu) + \tilde{H}_{r}(q, q_0, \mu) + O((q, p, q_0)^2|\mu, q, p, q_0|^{n_0}); \]
here $\tilde{H}_{NF}(q, q_0, \mu)$ is a polynomial of order $n_0 + 1$ with
\[ \tilde{H}_{NF}(q, q_0, \mu) = O(|q|^2|\mu(q, p)|) \]
and $\tilde{H}_{NF}(q, 0, \mu) = \tilde{H}_{NF}^0(q, \mu)$, and $\tilde{H}_{r}(q, q_0, \mu)$ is an affine function of its first argument which satisfies
\[ \tilde{H}_{r}(q, q_0, \mu) = O((q, q_0)|q_0|(|\mu, q, p, q_0|). \]
Note that
\[ P^\mu(q, p, q_0) = -\partial_q \tilde{H}_{NF}^\mu(q, p, q_0) f_3 - \partial_{q_0} \tilde{H}_{NF}^\mu(q, p, q_0) f_1. \]
Writing
\[ \tilde{H}^0_3(q, p, q_0) = c_1 q^3 + c_2 q^2 q_0 + c_3 q q_0^2 + c_4 q_0^3, \]
\[ \tilde{H}^1_3(q, p, q_0) = c_4 q^2 + c_4 q q_0 + c_4 q_0^2, \]
where $\mu^j \tilde{H}^j_k(q, p, q_0)$ denotes the part of the Taylor expansion of $\tilde{H}^\mu(q, p, q_0)$ which is homogeneous of order $j$ in $\mu$ and $k$ in $(q, p, q_0)$, one finds that
\[ c_1 = \frac{1}{6} (\beta_0 - \frac{1}{3})^{-3/2} (\alpha_0 m'(1) - 6), \quad c_4^1 = -\frac{1}{2} (\beta_0 - \frac{1}{3})^{-1} \]
(see Appendix (i)). Setting $q_0 = 0$ and introducing scaled variables
\[ Z = \mu^{1/2}(\beta_0 - \frac{1}{3})^{-1/2} z, \quad q(z) = \mu(\beta_0 - \frac{1}{3})^{1/2} Q(Z), \quad p(z) = \mu^{3/2} P(Z), \]
yields
\[ \tilde{H}^\mu(q, p, 0) = \mu^3 \left[ \frac{1}{2} P^2 - \frac{1}{2} Q^2 + \frac{1}{3} \tilde{c}_1 Q^3 \right] + O(\mu^{7/2}). \]
where
\[ \tilde{c}_1 = \frac{1}{2}(\alpha_0 m'_1(1) - 6), \]
and the lower-order Hamiltonian system
\[ \dot{Q} = P + O(\mu^{1/2}), \tag{5.1} \]
\[ \dot{P} = Q - \tilde{c}_1 Q^2 + O(\mu^{1/2}), \tag{5.2} \]
which is reversible with reverser \( S : (Q, P) \mapsto (Q, -P) \). Suppose \( \tilde{c}_1 \neq 0 \). In the limit \( \mu = 0 \) equations (5.1), (5.2) are equivalent to the single equation
\[ \partial_Z^2 u - u + u^2 = 0 \]
for the variable \( u = \tilde{c}_1 Q \).

Let us now suppose that \( m'_1(1) \) is close to the critical value \( 6\alpha_0^{-1} \) and introduce a second bifurcation parameter \( \kappa \) by setting
\[ m'_1(1) = \alpha_0^{-1}(6 + \kappa) \]
and observing that
\[ \tilde{r}(q, p, q_0; \mu, \kappa) = O(|(q, p, q_0)||(|\mu, q, p, q_0)|) + O(|\kappa||q, p, q_0|^3), \]
\[ \tilde{H}^{\mu, \kappa}(q, p, q_0) = O(|(q, p, q_0)||(|\mu, q, p, q_0)|) + O(|\kappa||q, p, q_0|^3) \]
(with a slight change of notation). Writing
\[ \tilde{H}^{0, 0, \kappa}(q, p, q_0) = d_1 q^4 + d_2 q^3 q_0 + d_3 q^2 q_0^2 + d_4 q q_0^3 + d_5 q_0^4, \]
where \( \mu^i \kappa^j \tilde{H}^{i, j}(q, p, q_0) \) denotes the part of the Taylor expansion of \( \tilde{H}^{\mu, \kappa}(q, p, q_0) \) which is homogeneous of order \( i \) in \( \mu \), \( j \) in \( \kappa \) and \( k \) in \( (q, p, q_0) \), one finds that
\[ d_1 = \frac{1}{24}(\beta_0 - \frac{1}{4})^{-2}(12 - \alpha_0 m''_1(1)) \]
(see Appendix (ii)). Setting \( q_0 = 0 \), introducing scaled variables
\[ Z = \mu^{1/2}(\beta_0 - \frac{1}{4})^{-1/2}z, \quad q(z) = \mu^{1/2}(\beta_0 - \frac{1}{4})^{1/2}Q(Z), \quad p(z) = \mu P(Z) \]
and writing
\[ \kappa = 2\mu^{1/2}\tilde{\kappa}, \]
thus yields
\[ \tilde{H}^{\mu, \kappa}(q, p, 0) = \mu^2 \left[ \frac{1}{2} P^2 - \frac{1}{2} Q^2 + \frac{1}{3} \kappa Q^3 + \frac{1}{4} d_1 Q^4 \right] + O(\mu^{5/2}), \]
where
\[ \tilde{d}_1 = \frac{1}{6}(12 - \alpha_0 m''_1(1)), \]
and the lower-order Hamiltonian system
\[ \dot{Q} = P + O(\mu^{1/2}), \tag{5.3} \]
\[ \dot{P} = Q - \kappa Q^2 - \tilde{d}_1 Q^3 + O(\mu^{1/2}), \tag{5.4} \]
which is of course reversible with reverser $S: (Q, P) \mapsto (Q, -P)$. Suppose that $d_1 > 0$. In the
limit $(\mu, \kappa) = 0$ equations (5.3), (5.4) are equivalent to the single equation
\[
\partial_z^2 u - u + u^3 = 0
\]
for the variable $u = d_1^{1/2} Q$.

The phase portrait of the equation
\[
\ddot{u} - u + u^m = 0
\]
for a fixed natural number $m$ (which is a travelling-wave version of the generalised Korteweg-de
Vries equation) is readily obtained by elementary calculations and is sketched in Figure 4; the
homoclinic orbits are of particular interest.

**Lemma 5.1.**

(i) Suppose that $m$ is even. Equation (5.5) has precisely one homoclinic solution $h$ (up to
translations). This solution is positive and symmetric, and monotone increasing to the left,
monotone decreasing to the right of its point of symmetry.

(ii) Suppose that $m$ is odd. Equation (5.5) has precisely two homoclinic solutions $\pm h$, where $h$
is symmetric, and monotone increasing to the left, monotone decreasing to the right of its
point of symmetry.

In both cases the homoclinic solutions intersect the symmetric section $\{\dot{u} = 0\}$ in the
two-dimensional phase space $\{(u, \dot{u}) \in \mathbb{R}^2\}$ transversally.

A familiar argument shows that Lemma 5.1(i) also applies to (5.1), (5.2) for small, positive
values of $\mu$, while Lemma 5.1(ii) applies to (5.3), (5.4) for small, positive values of $\mu$ and small,
values of $\kappa$ (that is, small values of $\kappa$); the qualitative statements apply to the variable $\tilde{c}_1 Q$ or
$\tilde{d}_1^{1/2} Q$. The homoclinic orbits at $\mu = 0$ (and $\kappa = 0$) intersect the symmetric section Fix $R = \{P = 0\}$ transversally, and these orbits therefore persist (as small, uniform perturbations of their
limits) for small, positive values of $\mu$ (and small values of $\kappa$).

Altogether we have established the existence of a symmetric, monotonically decaying soli-
ditary wave of depression for $m_1'(1) < 6\alpha_0^{-1}$ and elevation for $m_1'(1) > 6\alpha_0^{-1}$; the corresponding
ferrofluid surface $\{r = 1 + \eta(z)\}$ is obtained from the homoclinic solution of (5.1), (5.2) by the
formula
\[
\eta(z) = \frac{1}{2} \mu (\beta_0 - \frac{1}{6})^{1/2} Q \left( \mu^{1/2} (\beta_0 - \frac{1}{3})^{-1/2} z \right) + O(\mu^{3/2}).
\]

Furthermore, a pair of symmetric, monotonically decaying solitary waves exists for small values
of $m_1'(1) - 6\alpha_0^{-1}$ provided that $m_1'(1) < 12\alpha_0^{-1}$; one is a wave of depression, the other a wave of
elevation. The corresponding ferrofluid surface $\{r = 1 + \eta(z)\}$ is obtained from a homoclinic
solution of (5.3), (5.4) by the formula
\[
\eta(z) = \frac{1}{2} \mu^{1/2} (\beta_0 - \frac{1}{3})^{1/2} Q \left( \mu^{1/2} (\beta_0 - \frac{1}{6})^{-1/2} z \right) + O(\mu^{3/2}).
\]

(A more detailed analysis of a codimension-two bifurcation of this kind was given by Kirrmann
[16, §4] in the context of two-layer fluid flow.) Figure 5 shows a sketch of the ferrofluid surface
corresponding to solitary waves of the present type.
Figure 4: Phase portrait of equation (5.5) for even (left) and odd (right) values of $m$.

Figure 5: A solitary wave of elevation (left) and depression (right) generated by a homoclinic solution (top) in region I.

### 5.2 Homoclinic bifurcation at $C_1$

At each point of the curve $C_1$ in Figure 3, two pairs of real eigenvalues become complex by colliding at non-zero points on the real axis. Of particular interest here is the local part of $C_1$ near the point $(\beta_0, \gamma_0) = (\frac{1}{4}, 2)$ (which is given by $\beta_0 = \frac{1}{4} + \frac{1}{16} \mu^2 + O(\mu^4), \gamma_0 = 2 + \frac{1}{36} \mu^4 + O(\mu^6)$ for $0 < \mu \ll 1$) since we can access this curve using the centre-manifold technique. To this end we choose $\beta_0 = \frac{1}{4}, \alpha_0 = \frac{3}{4}$ and

$$
\varepsilon_1 = \mu_1, \quad \varepsilon_2 = \mu_1 + \mu_2, \quad \mu_1 = \frac{1}{48} (1 + \delta) \mu^2, \quad \mu_2 = \frac{1}{96} \mu^4.
$$

(5.6)

Notice that $\mu$ indicates the distance in $(\beta_0, \gamma_0)$ parameter space from the point $(\frac{1}{4}, 2)$, while $\delta$ plays the role of a bifurcation parameter (varying $\delta$ through zero from above we cross the critical curve $C_1$ from above); the parameter regime marked II in Figure 2 corresponds to small, positive values of $\delta$ and $\mu$. 

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The six-dimensional centre subspace of $K$ is spanned by the generalised eigenvectors

\[ e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{32} - \frac{1}{4}r^2 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} -\frac{1}{64} \\ 0 \\ 0 \\ \frac{1}{8} - \frac{1}{4}r^2 \end{pmatrix}, \]

\[ e_5 = \begin{pmatrix} 0 \\ -\frac{0.192}{128} \\ \frac{0.192}{128} - \frac{1}{4}r^2 \\ 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} -\frac{33}{20480} \\ 0 \\ 0 \\ \frac{3}{256} - \frac{3}{128}r^2 + \frac{1}{64}r^4 \end{pmatrix} \]

where $Ke_1 = 0$, $Ke_j = e_{j-1}$ for $j = 2, \ldots, 6$,

\[ \Omega(e_1, e_6) = \frac{1}{384}, \quad \Omega(e_2, e_5) = -\frac{1}{384}, \quad \Omega(e_3, e_4) = \frac{1}{384} \]

and the symplectic product of any other combination of the vectors $e_1, \ldots, e_6$ is zero. Writing

\[ w_1 = q_0f_1 + p_0f_6 + q_1f_5 + p_1f_2 + q_2f_3 + p_2f_4, \quad f_i := 8\sqrt{6}e_i, \]

we therefore find that $q_0, q_1, q_2, p_0, p_1$ and $p_2$ are canonical coordinates for the reduced Hamiltonian system, which has the cyclic variable $q_0$ and reverser $S : (q_0, q_1, q_2, p_0, p_1, p_2) \mapsto (-q_0, -q_1, -q_2, p_0, p_1, p_2)$; with a slight abuse of notation we abbreviate $\tilde{H}^{\mu_1, \mu_2}(q, p, 0)$ to $\tilde{H}^{\mu_1, \mu_2}$.

The usual normal-form theory for the four-dimensional system with Hamiltonian $\tilde{H}^{\mu_1, \mu_2}(q, p, 0)$, where $q = (q_1, q_2), p = (p_1, p_2)$ asserts that, after a canonical change of variables,

\[ \tilde{H}^{\mu_1, \mu_2}(q, p, 0) = \frac{1}{2}p_2^2 - q_1q_2 + \tilde{H}^{0}_{NF}(q, p, \mu_1, \mu_2) + O((|q, p|^2)(|\mu_1, \mu_2, q, p|)^{n_0}), \]

where $\tilde{H}^{0}_{NF}(q, p, \mu_1, \mu_2)$ is a polynomial of order $n_0 + 1$ which depends upon $q_1, q_2, p_1, p_2$ through the combinations

\[ p_1, \quad q_2^2 - 2p_1p_2, \quad q_3^3 + 3p_1^2q_1 - 3p_1p_2q_1, \quad -8p_1p_2^3 + 3p_2^2q_2^2 - 9p_1^2q_1^2 - 6q_1q_2^3 + 18p_1p_2q_1q_2 \]

and satisfies

\[ \tilde{H}^{0}_{NF}(q, p, \mu_1, \mu_2) = O((|q, p|^2)(|\mu_1, \mu_2, q, p|)). \]

It follows that, after a canonical change of variables,

\[ \tilde{H}^{\mu_1, \mu_2}(q, p, p_0) = \frac{1}{2}p_2^2 - q_1q_2 + p_0p_1 + \tilde{H}^{\mu_1, \mu_2}_{nl}(q, p, p_0) \]

with

\[ \tilde{H}^{\mu_1, \mu_2}_{nl}(q, p, p_0) = \tilde{H}^{\mu_1, \mu_2}_{NF}(q, p, p_0, \mu_1, \mu_2) + \tilde{H}_{r}(q, p, p_0, \mu_1, \mu_2) + O((|q, p, p_0|^2)(|\mu_1, \mu_2, q, p, p_0|)^{n_0}); \]

here $\tilde{H}^{\mu_1, \mu_2}_{NF}(q, p, p_0, \mu_1, \mu_2)$ is a polynomial of order $n_0 + 1$ which depends upon $q_1, q_2, p_1, p_2$ through the above combinations and satisfies

\[ \tilde{H}^{\mu_1, \mu_2}_{NF}(q, p, q_0, \mu_1, \mu_2) = O((|q, p|^2)(|\mu_1, \mu_2, q, p, p_0|)). \]
and $\tilde{H}_{\text{NF}}(q, p, 0, \mu_1, \mu_2) = \tilde{H}_{\text{NF}}^0(q, p, \mu_1, \mu_2)$, and $\tilde{H}(q, p, p_0, \mu_1, \mu_2)$ is an affine function of its first two arguments which satisfies

$$\tilde{H}(q, p, p_0, \mu_1, \mu_2) = O(||(q, p, p_0)||p_0||(\mu_1, \mu_2, q, p, p_0)||).$$

Note that

$$P^{\mu_1, \mu_2}(q, p, q_0) = \partial_{p_1} \tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, p_0)f_5 + \partial_{p_2} \tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, p_0)f_3$$
$$- \partial_{q_1} \tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, p_0)f_2 - \partial_{q_2} \tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, p_0)f_4 + \partial_{p_0} \tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, p_0)f_1.$$ 

Writing

$$\tilde{H}_{\text{nl}}^{0,0}(q, p, p_0) = c_1 p_1^3 + c_2 p_0 p_1 + c_3 p_0^2 p_1 + c_4 p_0^3 + c_5 p_1(q_2^2 - 2p_1p_2) + c_6 p_0(q_2^2 - 2p_1p_2) + c_7 p_0^2 p_2,$$
$$\tilde{H}_{\text{nl}}^{1,0}(q, p, p_0) = c_1^{1,0} p_1 + c_2^{1,0} p_0 p_1 + c_3^{1,0} p_0^2 + c_4^{1,0} p_0^3 + c_5^{1,0} p_0^2 p_2,$$
$$\tilde{H}_{\text{nl}}^{2,0}(q, p, p_0) = c_1^{2,0} p_1 + c_2^{2,0} p_0 p_1 + c_3^{2,0} p_0^2 + c_4^{2,0} p_0^3 + c_5^{2,0} p_0^2 p_2,$$
$$\tilde{H}_{\text{nl}}^{1,1}(q, p, p_0) = c_1^{1,1} p_1 + c_2^{1,1} p_0 p_1 + c_3^{1,1} p_0^2 + c_4^{1,1} p_0^3 + c_5^{1,1} p_0^2 p_2,$$

where $\mu_i \mu_j \tilde{H}_{k}^{i,j}(q, p, p_0)$ denotes the part of the Taylor expansion of $\tilde{H}_{\text{nl}}^{\mu_1, \mu_2}(q, p, q_0)$ which is homogeneous of order $i$ in $\mu_1$, $j$ in $\mu_2$ and $k$ in $(q, p, p_0)$. One finds that

$$c_1 = 48\sqrt{6}(3m_1'(1) - 8), \quad c_1^{1,0} = 0, \quad c_4^{1,0} = -16, \quad c_1^{2,0} = 512, \quad c_1^{0,1} = -48$$

(see Appendix (iii)). Setting $p_0 = 0$, choosing $\mu_1, \mu_2$ according to (5.6) and introducing the scaled variables

$$Z = \mu z, \quad q_1(z) = \mu^7 Q_1(Z), \quad q_2(z) = \mu^5 Q_2(Z), \quad p_1(z) = \mu^4 P_1(Z), \quad p_2(z) = \mu^6 P_2(Z),$$

thus yields

$$\tilde{H}^{\mu_1, \mu_2}(q, p, 0) = \mu^{12} \left[ \frac{1}{2} P_2^2 - \frac{1}{2} P_1^2 - Q_1 Q_2 - \frac{1}{3}(1 + \delta)(Q_2^2 - 2P_1 P_2) + \frac{2}{9}(1 + \delta)^2 P_1^2 + c_1 P_1^3 \right] + O(\mu^{13})$$

and the lower-order Hamiltonian system

$$Q_{1Z} = -P_1 + \frac{2}{3}(1 + \delta) P_2 + \frac{4}{9}(1 + \delta)^2 P_1 + 3c_1 P_1^2 + O(\mu), \quad (5.7)$$
$$Q_{2Z} = P_2 + \frac{2}{3}(1 + \delta) P_1 + O(\mu), \quad (5.8)$$
$$P_{1Z} = Q_2 + O(\mu), \quad (5.9)$$
$$P_{2Z} = Q_1 + \frac{2}{3}(1 + \delta) Q_2 + O(\mu), \quad (5.10)$$

which is reversible with reverser $S : (Q, P) \mapsto (-Q, P)$. Suppose $c_1 \neq 0$. In the limit $\mu = 0$ equations (5.7)–(5.10) are equivalent to the single fourth-order ordinary differential equation

$$\partial^4_Z u - 2(1 + \delta) \partial^2_Z u + u - u^2 = 0$$

for the variable $u = 3c_1 P_1$. 

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Let us now suppose that \( m'(1) \) is close to the critical value \( \frac{\delta}{3} \) and introduce a further bifurcation parameter \( \kappa \) by setting
\[
m'(1) = \frac{1}{3}(8 + \kappa)
\]
and observing that
\[
\tilde{r}(q, p, p_0; \mu_1, \mu_2, \kappa) = O((|q|, |p|, |p_0|)(|\mu_1, \mu_2, q, p, p_0|)) + O(|\kappa|(|q, p, p_0|)^2),
\]
\[
\tilde{H}^{\mu_1, \mu_2, \kappa}(q, p) = O((|q, p|)^2(|\mu_1, \mu_2, q, p, p_0|)) + O(|\kappa||q, p, p_0|)^3)
\]
(with a slight change of notation). Writing
\[
\tilde{H}_4^{0,0,0}(q, p, p_0) = d_3 p_1^4 + d_2 p_1^2 p_0 + d_3 p_1^2 p_0^2 + d_4 p_1 p_0^3 + d_5 p_0^4 + d_6 (q_2^2 - 2p_1 p_2)^2
\]
\[
+ d_7 p_1^2(q_2^2 - 2p_1 p_2) + d_8 p_0^2(q_2^2 - 2p_1 p_2) + d_9 p_0 p_1(q_2^2 - 2p_1 p_2)
\]
\[
+ d_{10} - 8p_1 p_2^3 + 3p_2 q_2^2 - 9q_1 q_2^2 - 6q_1 q_2^2 + 18p_1 p_2 q_1 q_2 + d_{11} p_2 p_0^3
\]
where \( \mu_1^i \mu_2^j \kappa^k \tilde{H}_4^{i,j,k}(q, p, q_0) \) denotes the part of the Taylor expansion of \( \tilde{H}^{\mu_1, \mu_2}(q, p, q_0) \) which is homogeneous of order \( i \) in \( \mu_1, j \) in \( \mu_2, k \) in \( \kappa \) and \( \ell \) in \( (q, p, q_0) \), one finds that
\[
d_1 = 864 \left( \frac{1264}{75} - m''(1) \right)
\]
(see Appendix (iv)). Setting \( p_0 = 0 \), choosing \( \mu_1, \mu_2 \) according to \( (5.6) \), introducing the scaled variables
\[
Z = \mu z, \quad q_1(z) = \mu^3 Q_1(Z), \quad q_2(z) = \mu^3 Q_2(Z), \quad p_1(z) = \mu^2 P_1(Z), \quad p_2(z) = \mu^4 P_2(Z),
\]
and writing
\[
\kappa = \frac{1}{144\sqrt{6}} \tilde{\kappa} \mu^2
\]
thus yields
\[
\tilde{H}^{\mu_1, \mu_2, \kappa}(q, p, 0)
\]
\[
= \mu^8 \left[ \frac{1}{2} P_2^2 - \frac{1}{2} P_1^2 - Q_1 Q_2 - \frac{1}{3}(1 + \delta)(Q_2^2 - 2P_1 P_2) + \frac{2}{9}(1 + \delta)^2 P_1^2 + \frac{1}{3} \kappa P_3^2 + d_1 P_4^4 \right] + O(\mu^9)
\]
and the lower-order Hamiltonian system
\[
Q_{1Z} = -P_1 + \frac{2}{3}(1 + \delta) P_2 + \frac{2}{3}(1 + \delta)^2 P_1 + \kappa P_1^2 + 4d_1 P_3 + O(\mu), \quad (5.11)
\]
\[
Q_{2Z} = P_2 + \frac{2}{3}(1 + \delta) P_1 + O(\mu), \quad (5.12)
\]
\[
P_{1Z} = Q_2 + O(\mu), \quad (5.13)
\]
\[
P_{2Z} = Q_1 + \frac{2}{3}(1 + \delta) Q_2 + O(\mu), \quad (5.14)
\]
which is of course reversible with reverser \( S : (Q, P) \mapsto (-Q, P) \). Suppose \( d_1 > 0 \). In the limit \( (\mu, \kappa) \to 0 \) equations \( (5.11)-(5.14) \) are equivalent to the single fourth-order ordinary differential equation
\[
\partial u^2 - 2(1 + \delta) \partial u^3 + u - u^3 = 0
\]
for the variable \( u = 2d_1^{1/2} P_1 \).
Existence theories for homoclinic solutions to the equation

\[ \ddot{u} - 2(1 + \delta)\dot{u} + u - u^m = 0. \]  

(5.15)

for a fixed natural number \( m \geq 2 \) (which is a travelling-wave version of the generalised Kawahara equation) are given in Theorems 5.2 and 5.3 below. These theorems are generalisations of results given by Buffoni, Champneys & Toland [6] (see also Devaney [9]) for the special case \( m = 2 \); a full discussion of their generalisation to \( m \geq 2 \) is given by Ahmad [1].

**Theorem 5.2.** Suppose that \( \delta \geq 0 \).

(i) Suppose that \( m \) is even. Equation (5.15) has precisely one homoclinic solution \( h \) (up to translations). This solution is positive and symmetric, and monotone increasing to the left, monotone decreasing to the right of its point of symmetry.

(ii) Suppose that \( m \) is odd. Equation (5.15) has precisely two homoclinic solutions \( \pm h \), where \( h \) is symmetric, and monotone increasing to the left, monotone decreasing to the right of its point of symmetry.

In both cases the homoclinic solutions are transverse, that is, the stable and unstable manifolds of the zero equilibrium intersect transversally with respect to the zero level surface of the Hamiltonian at their point of symmetry.

**Theorem 5.3.** The primary homoclinic solutions found in the previous theorem persist (as small, uniform perturbations of their limits at \( \delta = 0 \)) for small, negative values of \( \delta \).

Furthermore, each primary homoclinic solution \( h \) in the region \( \delta < 0 \) generates a family of transverse multipulse homoclinic solutions which resemble multiple copies of \( h \) ‘glued’ together with small oscillations in between. More precisely, for each all natural numbers \( \ell_1, \ldots, \ell_{n-1} \) with \( n = 1, 2, \ldots \) there exists a homoclinic solution \( n(\ell_1, \ldots, \ell_{n-1}) \) associated with \( h \) which

(i) has \( n \) local extrema at \( t_1, \ldots, t_n \),

(ii) oscillates \( \left\lfloor \frac{\ell_k}{2} \right\rfloor \) times and has \( 2 \left\lfloor \frac{\ell_k - 1}{2} \right\rfloor \) extrema in each interval \((t_k, t_{k+1})\),

(iii) oscillates infinitely often in the intervals \((-\infty, t_1)\) and \((t_n, \infty)\).

Theorem 5.2(i) also applies to (5.7)–(5.10) for small, positive values of \( \mu \), while Theorem 5.2(ii) applies to (5.11)–(5.14) for small, positive values of \( \mu \) and small, values of \( \kappa \) (that is, small values of \( \kappa \)); the qualitative statements apply to the variable \( \hat{c}_1 P_1 \) or \( d_1^{1/2} P_1 \). The homoclinic orbits at \( \mu = 0 \) (and \( \kappa = 0 \)) are transverse and therefore persist (as small, uniform perturbations of their limits) for small, positive values of \( \mu \) (and small values of \( \kappa \)). Similarly, Theorem 5.3 applies to any of these persistent primary homoclinic orbits.

Altogether we have established the existence of a primary and accompanying multipulse family of solitary waves of depression for \( m_1(1) < \frac{8}{3} \) and elevation for \( m_1(1) > \frac{8}{3} \); the corresponding ferrofluid surface \( \{r = 1 + \eta(z)\} \) is obtained from the homoclinic solution of (5.1), (5.2) by the formula

\[ \eta(z) = \frac{1}{2} \mu^4 P_1(\mu z) + O(\mu^5). \]
Figure 6: A solitary wave of elevation (left) and depression (right) generated by a ‘primary’ homoclinic solution (top) in region II

Figure 7: A solitary wave of elevation (left) and depression (right) generated by a ‘$2(2)$’ homoclinic solution (top) in region II
Furthermore, two multipulse families of solitary waves exist for small values of \( m''_1(1) - \frac{8}{3} \) provided that \( m''_1(1) \neq \frac{1264}{75} \); one consists of waves of depression, the other of waves of elevation. The corresponding ferrofluid surface \( \{ r = 1 + \eta(z) \} \) is obtained from a homoclinic solution of (5.3), (5.4) by the formula

\[
\eta(z) = \frac{1}{2} \mu^2 P_1(\mu z) + O(\mu^3).
\]

### 5.3 Homoclinic bifurcation at \( C_2 \)

At each point of the curve \( C_2 \) in Figure 3 two pairs of purely imaginary eigenvalues become complex by colliding at non-zero points \( \pm is \) on the imaginary axis and forming two Jordan chains of length 2. This resonance is associated with the bifurcation of a branch of homoclinic solutions into the region with complex eigenvalues (the parameter regime marked III in Figure 2). Let us therefore choose

\[
\beta_0 = \frac{1}{2} \left( 1 - \frac{I_0(s)I_2(s)}{I_1(s)^2} \right), \quad \gamma_0 = \frac{1}{2} s^2 \left( -1 + \frac{I_0(s)^2}{I_1(s)^2} \right)
\]

(so that \( \alpha_0 = \gamma_0 - \beta_0 \)) and introduce a bifurcation parameter \( \mu \) by writing \( (\epsilon_1, \epsilon_2) = (0, \mu) \), where \( 0 < \mu \ll 1 \).

The six-dimensional centre subspace of \( K \) is spanned by the generalised eigenvectors

\[
e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} \gamma_0^{-1} \\ 0 \\ 0 \\ 1 - 2\gamma_0^{-1} \end{pmatrix}, \quad e, \quad \bar{e}, \quad f, \quad \bar{f},
\]

where

\[
e = \begin{pmatrix} I_1(s) \\ is\beta_0 I_1(s) - iI_2(s) \\ -iI_0(sr) \\ sI_0(sr) - 2I_1(s) \end{pmatrix}, \quad f = \begin{pmatrix} -iI_0(s) + \frac{1}{s} I_1(s) \\ \beta_0 I_0(s) - \frac{2}{s} I_2(s) - I_3(s) \\ -rI_1(sr) \\ -iI_0(sr) - irsI_1(sr) + 2iI_0(s) - \frac{2}{s} I_1(s) \end{pmatrix} - \frac{i\tau_2}{2\tau_1} e
\]

and

\[
\tau_1 = 2I_0(s)^2 - s\frac{I_0(s)^3}{I_1(s)} + sI_0(s)I_1(s) - I_1(s)^2,
\]

\[
\tau_2 = -\frac{1}{3} \left( \frac{2}{s} (-3 + s^2) I_0(s)^2 - 3s\frac{I_0(s)^4}{I_1(s)^2} + 9\frac{I_0(s)^3}{I_1(s)} - 5I_0(s)I_1(s) + \frac{1}{s} (5 + s^2) I_1(s)^2 \right);
\]

note that \( Ke_1 = 0, Ke_2 = e_1, (K - isI)e = 0, (K - isI)f = e \),

\[
\Omega(e_1, e_2) = \frac{1}{2} - 2\gamma_0^{-1}, \quad \Omega(e, \bar{f}) = \tau_1, \quad \Omega(\bar{e}, f) = \tau_1
\]

and the symplectic product of any other combination of the vectors \( e_1, e, f, \bar{e}, \bar{f} \) is zero. Writing

\[
w_1 = q_0 f_1 + p_0 f_2 + AE + BF + \bar{A}E + \bar{B}F,
\]

29
where
\[ f_1 = \left( \frac{1}{2} - 2\gamma_0^{-1/2} \right)^{-1} e_1, \quad f_2 = \left( \frac{1}{2} - 2\gamma_0^{-1} \right)^{-1/2} e_2, \quad E = \tau_1^{-1/2} e, \quad F = \tau_1^{-1/2} f, \]
we therefore find that \( q_0, p_0, A \) and \( B \) are canonical coordinates for the reduced Hamiltonian system, which has the cyclic variable \( q_0 \) and reverser \( S : (q_0, p_0, A, B) \mapsto (-q_0, p_0, \bar{A}, -\bar{B}) \); with a slight abuse of notation we abbreviate \( \tilde{H}^\varepsilon_{|(e_1, e_2) = (0, 0)} \) to \( \tilde{H}^\mu \).

The usual normal-form theory for the two-dimensional system with Hamiltonian \( \tilde{H}^\mu(A, B, \bar{A}, \bar{B}, 0) \) asserts that, after a canonical change of variables,
\[
\tilde{H}^\mu(A, B, \bar{A}, \bar{B}, 0) = is(A\bar{B} - \bar{A}B) + |B|^2 + \tilde{H}_{\text{NF}}^0(|A|^2, i(A\bar{B} - \bar{A}B), \mu) + O((|A, B|)^2((\mu, A, B)|^0),
\]
where \( \tilde{H}_{\text{NF}}^0 \) is a real polynomial function of its arguments which satisfies
\[
\tilde{H}_{\text{NF}}^0(|A|^2, i(A\bar{B} - \bar{A}B), \mu) = O((|A, B|)^2((\mu, A, B)|).
\]
It follows that, after a canonical change of variables,
\[
\tilde{H}^\mu(A, B, \bar{A}, \bar{B}, p_0) = is(A\bar{B} - \bar{A}B) + |B|^2 + \frac{1}{2}p_0^2 + \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0)
\]
with
\[
\tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) = \tilde{H}_{\text{NF}}(|A|^2, i(A\bar{B} - \bar{A}B), p_0, \mu) + \bar{H}_r(A, B, \bar{A}, \bar{B}, p_0, \mu) + O((|A, B, p_0|)^2((\mu, A, B, p_0)|^0);
\]
here \( \tilde{H}_{\text{NF}} \) is a real polynomial function of its arguments which satisfies
\[
\tilde{H}_{\text{NF}}(|A|^2, i(A\bar{B} - \bar{A}B), p_0, \mu) = O((|A, B|)^2((\mu, A, B, p_0)|)
\]
and \( \tilde{H}_{\text{NF}}(|A|^2, i(A\bar{B} - \bar{A}B), 0, \mu) = \tilde{H}_{\text{NF}}^0(|A|^2, i(A\bar{B} - \bar{A}B), \mu) \), and \( \bar{H}_r(A, B, \bar{A}, \bar{B}, p_0, \mu) \) is an affine function of its first four arguments which satisfies
\[
\bar{H}_r(|A|^2, i(A\bar{B} - \bar{A}B), p_0, \mu) = O((|A, B, p_0|) |p_0||(|\mu, A, B, p_0)|)
\]
Note that
\[
P^\mu(A, B, \bar{A}, \bar{B}, p_0) = \partial_B \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) E + \partial_B \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) \bar{E}
- \partial_A \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) F - \partial_A \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) \bar{F}
+ \partial_{p_0} \tilde{H}_{\text{nl}}^\mu(A, B, \bar{A}, \bar{B}, p_0) f_1.
\]
Writing
\[
\tilde{H}_1^\mu(A, B, p_0) = c_1^1p_0^4 + c_2^1|A|^2 + c_3^1i(A\bar{B} - \bar{A}B) + c_4^1p_0A + c_5^1p_0\bar{A} + c_6^1p_0B + c_7^1p_0\bar{B},
\]
\[
\tilde{H}_2^\mu(A, B, p_0) = c_1^2p_0^4 + c_2^2|A|^2 + c_3^2i(A\bar{B} - \bar{A}B) + c_4^2p_0A + c_5^2p_0\bar{A} + c_6^2p_0B + c_7^2p_0\bar{B}
\]
\[
\tilde{H}_3^\mu(A, B, p_0) = d_1^4p_0^4 + d_2^4|A|^2 + d_3^4p_0^2(A\bar{B} - \bar{A}B) + d_4^4|A|^4 + d_6^4(A\bar{B} - \bar{A}B)|A|^2
- d_6(A\bar{B} - \bar{A}B)^2 + d_7^4p_0A + d_7^4p_0\bar{A} + d_8^4p_0B + d_8^4p_0\bar{B},
\]
30
where $\mu^j \tilde{H}^j_k(A, B, p_0)$ denotes the part of the Taylor expansion of $\tilde{H}^\mu(A, B, p_0)$ which is homogeneous of order $j$ in $\mu$ and $k$ in $(A, B, p_0)$, one finds that

$$
d_4 = \frac{I_1(s)^2}{2\tau_1^2} \left( \frac{(-2s^2 + s^2 \beta_0 - 2sT + 4s^2 ST - \alpha_0 m_1'(1))(-2s^2 - 2s^2 \beta_0 - sS + 4s^2 ST - \alpha_0 m_1'(1))}{2(\gamma_0 + 4s^2 \beta_0 - 2sT)} \right)
$$

$$
- \left( \frac{(s^2 \beta_0 - 4sS + 2 + \alpha_0 m_1'(1))(3sS - \alpha_0 m_1'(1))}{\gamma_0 - 2} \right)
$$

$$
+ 7s^2 - \frac{21}{2}s^2 \beta_0 + \frac{3}{2}s^4 \beta_0 + 6sS - 6s^3 S + 4s^3 S^2 T - 2s^2 ST - 3\alpha_0 m_1'(1) - \frac{1}{2}\alpha_0 m_1''(1),
$$

$$
c_2^1 = -\frac{I_1(s)^2}{\tau_1},
$$

(5.16)

where

$$
S = \frac{I_0(s)}{I_1(s)}, \quad T = \frac{I_0(2s)}{I_1(2s)}
$$

(see Appendix (v)).

The lower-order Hamiltonian system

$$
A_Z = \partial_B \tilde{H}^\mu(A, B, \bar{A}, B, 0),
$$

$$
B_Z = -\partial_A \tilde{H}^\mu(A, B, \bar{A}, B, 0)
$$

(5.17)

(5.18)

has been examined in detail by Iooss & Pérouème [15]. The ‘truncated normal form’ obtained by ignoring the remainder terms in $\tilde{H}^\mu(A, B, \bar{A}, B, 0)$ is conveniently handled using the substitution $A(z) = e^{is} a(z), B(z) = e^{is} b(z)$, which converts it into the system

$$
\dot{a} = b + \partial_b \tilde{H}^\mu_{NF}(|a|^2, i(a\bar{b} - \bar{a}b), \mu),
$$

$$
\dot{b} = -\partial_a \tilde{H}^\mu_{NF}(|a|^2, i(a\bar{b} - \bar{a}b), \mu).
$$

(5.19)

(5.20)

Supposing that the coefficients $c_2^1$ and $d_4$ are respectively negative and positive, one finds that (5.19), (5.20) admits a real, reversible homoclinic solution $(a_h, b_h)$, which evidently generates a circle $\{e^{i\theta}(a_h, b_h) : \theta \in [0, 2\pi)\}$ of further homoclinic solutions, two of which (those with $\theta = 0$ and $\theta = \pi$) are reversible. The corresponding pair of homoclinic solutions to the original ‘truncated normal form’ are reversible and persist when the remainder terms are reinstated. A theory of multipulse homoclinic solutions to (5.17), (5.18) has also been given by Buffoni & Groves [7] (under the same hypotheses on the normal-form coefficients).

**Theorem 5.4.**

(i) (Iooss & Pérouème) For each sufficiently small, positive value of $\mu$ the two-degree-of-freedom Hamiltonian system (5.17), (5.18) has two distinct symmetric homoclinic solutions.

(ii) (Buffoni & Groves) For each sufficiently small, positive value of $\mu$ the two-degree-of-freedom Hamiltonian system (5.17), (5.18) has an infinite number of geometrically distinct homoclinic solutions which generically resemble multiple copies of one of the homoclinic solutions in part (i).
The homoclinic solutions identified above correspond to envelope solitary waves whose amplitude is $O(\mu^{1/2})$ and which decay exponentially as $z \to \pm \infty$; they are sketched in Figure 8.

**Appendix: Calculation of the normal-form coefficients**

The coefficients in the reduced Hamiltonian $\tilde{H}^\varepsilon(w_1)$ are determined using the equations

$$K\tilde{r}(w_1;\varepsilon) - d_1\tilde{r}[w_1;\varepsilon](Kw_1) = P^\varepsilon(w_1) + d_1\tilde{r}[w_1;\varepsilon](P^\varepsilon(w_1)) - g^\varepsilon_n(w_1 + \tilde{r}(w_1;\varepsilon)), \quad (A.1)$$

$$B_l\tilde{r}(w_1;\varepsilon) = -B^\varepsilon_l(w_1 + \tilde{r}(w_1;\varepsilon)), \quad (A.2)$$

to compute the Taylor series of $\tilde{H}^\varepsilon(w_1)$ and $\tilde{r}(w_1;\varepsilon)$ systematically in powers of $(q,p,q_0)$ or $(q, p, p_0)$. Here $K = dg^0[0]$ and $g^\varepsilon_n = g^\varepsilon - K$ are the linear and nonlinear parts of $g^\varepsilon$ (with this slight abuse of notation $K$ is given by the explicit formula (4.6)), and $B_l, B^\varepsilon_l$ are the linear and nonlinear parts of the boundary-value operator $B^\varepsilon : N \to \mathbb{R}$ defined by the left-hand side of (3.11). Throughout these calculations we also make use of the identity

$$\Omega(Ku + g_n^\varepsilon(u), v) + (B_l(u) + B^\varepsilon_n(u))\phi^v|_{r=1} = dH^\varepsilon[u](v),$$

in which $v = (\eta^v, \omega^v, \phi^v, \zeta^v)$. We denote the parts of $H^\varepsilon(w), B^\varepsilon_n(w), g^\varepsilon_n(w)$ which are homogeneous of order $m$ in $\varepsilon$ and $n$ in $w$ by $\varepsilon^mH^m_n(w), \varepsilon^mB^m_{n,n}(w), \varepsilon^m g^m_{n,n}(w)$, and the part of $\tilde{r}(w_1;\varepsilon)$ which is homogeneous of order $m$ in $\varepsilon$ and $n$ in $w_1$ by $\tilde{r}^m_n(w_1;\varepsilon)$; the notation is modified in the natural fashion when $\varepsilon$ is replaced by a more specific parameterisation. Finally, arbitrary constants arising from solving differential equations are denoted by $a_i$. 
Homoclinic bifurcation at $C_4$

(i) Write

$$\tau_m^n(w_1; \mu) = \sum_{h+i+j=m} \mu^n \tau^{n}_{hij0} q^h p^i q^j_0$$

and consider the $q^2$ and $\mu q$ components of (A.1, (A.2), namely

$$q^2 : \begin{cases} K \tilde{r}_{2000}^2 = -3c_1 f_3 - c_2 f_1 - g_{nl,2}^0(f_2, f_2), \\ B_l \tilde{r}_{2000}^1 = -B_{nl,2}^0(f_2, f_2), \end{cases} \quad (A.3)$$

$$\mu q : \begin{cases} K \tilde{r}_{1000}^1 = -2c_1^2 f_3 - c_2^2 f_1 - g_{nl,1}^1(f_2), \\ B_l \tilde{r}_{1000}^1 = 0. \end{cases}$$

Using these equations we find that

$$c_1 = H_3^0(f_2, f_2, f_2) + 2H_2^0(\tilde{r}_{2000}^2, f_2) = H_3^0(f_2, f_2, f_2) + \Omega(K \tilde{r}_{2000}^2, f_2) + B_l \tilde{r}_{2000}^0 f_2 \mid_r = 1 + 3c_1 - \Omega(g_{nl,2}^0(f_2, f_2), f_2) = -2H_3^0(f_2, f_2, f_2) + 3c_1,$$

which implies that

$$c_1 = H_3^0(f_2, f_2, f_2) = \frac{1}{6} (\beta_0 - \frac{1}{4})^{-3/2} (\alpha_0 m_1^4(1) - 6),$$

and

$$c_1 = H_2^1(f_2, f_2) + 2H_2^0(\tilde{r}_{1000}^1, f_2) = H_2^1(f_2, f_2) + 2c_1 - \Omega(g_{nl,1}^1(f_2), f_2) = -H_2^1(f_2, f_2) + 2c_1,$$

which implies that

$$c_1^1 = H_2^1(f_2, f_2) = -\frac{1}{2} (\beta_0 - \frac{1}{4})^{-1}.$$

(ii) Write

$$\tilde{r}_m^{0,0}(w_1; \mu, \kappa) = \sum_{h+i+j=m} \tilde{r}^{0,0}_{hij0} q^h p^i q^j_0$$

and consider the $q^3$ component of (A.1, (A.2), namely

$$q^3 : \begin{cases} K \tilde{r}_{3000}^{0,0} = -4c_1 f_3 - c_2 f_1 - g_{nl,3}^{0,0}(f_2, f_2, f_2) - 2g_{nl,2}^{0,0}(f_2, \tilde{r}_{2000}^0), \\ B_l \tilde{r}_{3000}^{0,0} = -B_{nl,3}^{0,0}(f_2, f_2, f_2) - 2B_{nl,2}^{0,0}(f_2, \tilde{r}_{2000}^0). \end{cases} \quad (A.4)$$

The coefficient $d_1$ can be expressed as

$$d_1 = H_4^{0,0}(f_2, f_2, f_2, f_2) + 3H_3^{0,0}(f_2, f_2, \tilde{r}_{2000}^0) + 2H_2^{0,0}(f_2, \tilde{r}_{2000}^0) + H_2^{0,0}(f_2, \tilde{r}_{3000}^0), \quad (A.5)$$
and it follows from (A.5) that
\[ 2H_2^{0,0}(r_{3000}, f_2) = \Omega(K r_{3000}, f_2) + B_1 r_{3000} \phi f_2 |_{r=1} \]
\[ = 4d_1 - \Omega(g_{1,3}^{0,0}(f_2, f_2, f_2, f_2) - 2\Omega(g_{0,2}^{0,0}(f_2, f_2, f_2)) + B_1 r_{3000} \phi f_2 |_{r=1} \]
\[ = 4d_1 - 4H_4^{0,0}(f_2, f_2, f_2, f_2) - 6H_3^{0,0}(f_2, f_2, r_{3000}) \]
\[ + (B_1 r_{3000} + 2B_{n,l}^{1,0}(f_2, r_{2000}) + B_{n,l}^{1,0}(f_2, f_2)) \phi f_2 |_{r=1}, \]
and we find from the boundary condition in (A.4) that the sum inside the parentheses vanishes. From (A.3) we find that
\[ \tau_{2000}^{0,0} = (6(\beta_0 - \frac{1}{4})^{-1/2} - c_2) f_2 + a_1 f_1, \]
and it follows from (A.5) that
\[ d_1 = H_4^{0,0}(f_2, f_2, f_2, f_2) + H_3^{0,0}(f_2, f_2, \tau_{2000}^{0,0}) - \frac{1}{3}H_2^{0,0}(\tau_{2000}^{0,0}, \tau_{2000}^{0,0}) = \frac{1}{24}(\beta_0 - \frac{1}{4})^{-2}(12 - \alpha_0 m_1''(1)). \]

**Homoclinic bifurcation at C_1**

(iii) Write
\[ \tau_{m}^{n_1,n_2}(w; \mu_1, \mu_2) = \sum_{h+i+j+k+l=m} \mu_1^{h_1} \mu_2^{i_1} \tau_{h,i,j,k,l}^{n_1,n_2} \]
and consider the \( \mu_1 p_1 \) and \( \mu_2 p_1 \) components of (A.1), (A.2), namely
\[ p_1^2 : \begin{cases} K r_{020000} = 3c_1 f_5 - 2c_5 f_3 + c_2 f_1 - g_{0,2}^{0,0}(f_2, f_2), \\ B r_{020000} = -B_{n,l}^{0,0}(f_2, f_2), \end{cases} \]  
(A.6)
\[ \mu_2 p_1 : \begin{cases} K r_{010000} = 2c_1^{0,1} f_5 - 2c_4^{1,1} f_3 + c_2^{0,1} f_1 - g_{0,1}^{0,1}(f_2), \\ B r_{010000} = 0, \end{cases} \]  
\[ \mu_1 p_1 : \begin{cases} K r_{010000} = 2c_1^{1,0} f_5 - 2c_4^{1,0} f_3 + c_2^{1,0} f_1 - g_{0,1}^{1,0}(f_2), \\ B r_{010000} = -B_{n,l}^{1,0}(f_2). \end{cases} \]  
(A.7)

Using the method described in part (i) above, we find from these equations that
\[ c_1 = H_3^{0,0}(f_2, f_2, f_2) = 48\sqrt{6}(3m_1'(1) - 8), \]
\[ c_1^{0,1} = H_2^{0,1}(f_2, f_2) = -48, \]
\[ c_1^{1,0} = H_2^{1,0}(f_2, f_2) = 0. \]

Combining
\[ \mu_1 q_2 : \begin{cases} K r_{010000} - r_{010000}^{1,0} = -2c_4^{1,0} f_4 - g_{0,1}^{1,0}(f_3), \\ B r_{010000}^{1,0} = -B_{n,l}^{1,0}(f_3), \end{cases} \]  
(A.8)
with
\[ r_{010000}^{1,0} = -2c_4^{1,0} f_4 + c_2^{1,0} f_2 + a_2 f_1, \]
which is obtained from \((A.7)\), one finds by the usual argument that
\[
c_4^{1,0} = \frac{1}{3} H_2^{1,0}(f_3, f_3) = -16.
\]
Similarly, combining
\[
\mu_1^2 p_1 : \begin{cases} \ K\tilde{r}_{010000}^{2,0} = 2c_4^{1,0} f_5 - 2c_4^{2,0} f_3 + c_2^{2,0} f_1 - 2c_4^{1,0}\tilde{r}_{001000}^{1,0}, \\ B_1\tilde{r}_{010000}^{2,0} = 0 \end{cases}
\]
with
\[
\tilde{r}_{001000}^{1,0} = -4c_4^{1,0} f_5 + c_2^{1,0} f_3 + a_2 f_2 + a_3 f_1 + \begin{pmatrix} 0 \\ 4\sqrt{6} \\ 0 \\ 0 \end{pmatrix},
\]
which is obtained from \((A.8)\), yields
\[
c_1^{2,0} = H_2^{2,0}(f_2, f_2) + 2H_2^{0,0}(\tilde{r}_{010000}^{2,0}, f_2) + 2H_2^{1,0}(\tilde{r}_{010000}^{1,0}, f_2) + H_2^{0,0}(\tilde{r}_{010000}^{1,0}, \tilde{r}_{010000}^{1,0}) = 2c_1^{2,0} = 512,
\]
so that \(c_1^{2,0} = 512\).

(iv) Write
\[
r^{0,0,0}_m(w_1; \mu_1, \mu_2, \kappa) = \sum_{h+i+j+k+l=m} \tilde{r}_{hijk0}^{0,0,0} q_i^1 q_j^2 q_k^3 q_l^4,
\]
and note that
\[
d_1 = H_4^{0,0,0}(f_2, f_2, f_2, f_2) + 3H_3^{0,0,0}(f_2, f_2, \tilde{r}_{020000}^{0,0,0}) + 2H_2^{0,0,0}(f_2, \tilde{r}_{030000}^{0,0,0}) + H_2^{0,0,0}(\tilde{r}_{020000}^{0,0,0}, \tilde{r}_{020000}^{0,0,0}).
\]
Since
\[
2H_2^{0,0,0}(f_2, \tilde{r}_{030000}^{0,0,0}) = 4d_1 - 4H_4^{0,0,0}(f_2, f_2, f_2, f_2) - 6H_3^{0,0,0}(f_2, f_2, \tilde{r}_{020000}^{0,0,0}) - 2c_5 \Omega(\tilde{r}_{010000}^{0,0,0}, f_2),
\]
where we have used
\[
p_1^2 : \begin{cases} K\tilde{r}_{030000}^{0,0,0} = 4d_1 f_5 + d_2 f_1 - 2d_7 f_3 - g_{nl}^{0,0,0}(f_2, f_2) - 2g_{nl}^{0,0,0}(f_2, \tilde{r}_{020000}^{0,0,0}) - 2c_5 \tilde{r}_{010000}^{0,0,0}, \\ B_1\tilde{r}_{030000}^{0,0,0} = -B_{nl}^{0,0,0}(f_2, f_2, f_2) - 2B_{nl}^{0,0,0}(f_2, \tilde{r}_{020000}^{0,0,0}), \end{cases}
\]
it follows that
\[
3d_1 = 3H_4^{0,0,0}(f_2, f_2, f_2, f_2) + 3H_3^{0,0,0}(f_2, f_2, \tilde{r}_{020000}^{0,0,0}) - H_2^{0,0,0}(\tilde{r}_{020000}^{0,0,0}, \tilde{r}_{020000}^{0,0,0}) + 2c_5 \Omega(\tilde{r}_{010000}^{0,0,0}, f_2).
\]
In order to compute \(d_1\) it is therefore necessary to compute \(\tilde{r}_{020000}^{0,0,0}, \tilde{r}_{011000}^{0,0,0}\) and \(c_5\).

From \((A.6)\) one finds that
\[
\tilde{r}_{020000}^{0,0,0} = -2c_5 f_4 + (c_2 + 6\sqrt{6}) f_2 + a_4 f_1,
\]
and
\[
p_1 q_2 : \begin{cases} K\tilde{r}_{011000}^{0,0,0} = -2\tilde{r}_{020000}^{0,0,0} = -2c_5 f_4 - 2g_{nl}^{0,0,0}(f_2, f_3), \\ B_1\tilde{r}_{011000}^{0,0,0} = -2B_{nl}^{0,0,0}(f_2, f_3) \end{cases}
\]
yields
\[ \tilde{r}_{011000}^{0,0,0} = -6c_5f_5 + (2c_2 + 12\sqrt{6})f_3 + a_4f_2 + a_5f_1 + \begin{pmatrix} 0 \\ 72 \\ 48r^2 \\ 0 \end{pmatrix}. \]

Furthermore,
\[ q_2^2 : \begin{cases} K\tilde{r}_{002000}^{0,0,0} - \tilde{r}_{011000}^{0,0,0} = c_5f_5 + c_6f_1 - g_{n_{l2}}^{0,0,0}(f_3, f_3), \\ B_i\tilde{r}_{002000}^{0,0,0} = -B_{n_{l2}}^{0,0,0}(f_3, f_3), \end{cases} \]

and
\[ p_1p_2 : \begin{cases} K\tilde{r}_{010100}^{0,0,0} - \tilde{r}_{011000}^{0,0,0} = -4c_5f_5 - 2c_6 - 2g_{n_{l2}}^{0,0,0}(f_2, f_4), \\ B_i\tilde{r}_{010100}^{0,0,0} = -2B_{n_{l2}}^{0,0,0}(f_2, f_4) \end{cases} \]
yield
\[ \begin{align*}
\tilde{r}_{002000}^{000} &= -5c_5f_6 + (2c_2 + 12\sqrt{6})f_4 + c_6f_2 + a_4f_3 + a_5f_2 + a_6f_1 + \begin{pmatrix} 54 \\ 0 \\ 0 \\ -108 + 144r^2 \end{pmatrix}, \\
\tilde{r}_{010100}^{0,0,0} &= -10c_5f_6 + (2c_2 + 12\sqrt{6})f_4 - 2c_6f_2 + a_4f_3 + a_5f_2 + a_7f_1 + \begin{pmatrix} 39 \\ 0 \\ 0 \\ -48(1 + r^2) \end{pmatrix},
\end{align*} \]

and using these results we find that the solvability condition for
\[ q_2p_2 : \begin{cases} K\tilde{r}_{010100}^{0,0,0} - 2\tilde{r}_{002000}^{0,0,0} - \tilde{r}_{011000}^{0,0,0} = -2g_{n_{l2}}^{0,0,0}(f_3, f_4), \\ B_i\tilde{r}_{010100}^{0,0,0} = -2B_{n_{l2}}^{0,0,0}(f_3, f_4) \end{cases} \]
is \( c_5 = -\frac{144\sqrt{6}}{5} \). Inserting these expressions for \( \tilde{r}_{020000}^{0,0,0}, \tilde{r}_{011000}^{0,0,0} \) and \( c_5 \) into (A.10), we obtain
\[ d_1 = 864 \left( \frac{1264}{75} - m'(1) \right). \]

**Homoclinic bifurcation at C_2**

(v) Here we write
\[ \tilde{r}_m^n(w_1; \mu) = \sum_{h+i+j+k+l=m} \tilde{r}_{hijkl}^n \mu^h A^i B^j \bar{A}^i \bar{B}^j p_l. \]

The coefficient \( c_2^i \) is found from
\[ \mu A : \begin{cases} (K - i\sigma L)\tilde{r}_{100000}^{1} &= c_3^iE - c_2^iF + c_4^i f_1 - g_{n_{l1}}^{1}(E), \\ B_i\tilde{r}_{100000}^1 &= 0. \end{cases} \]  \( (A.11) \)
Noting that
\[ \Omega(K\tilde{r}_{100000}^1, \bar{E}) = 2H_2^0(\tilde{r}_{100000}^0, \bar{E}) = \Omega(K\bar{E}, \tilde{r}_{100000}^0), \]
we find from (A.11) that
\[ c_2^1 = -\Omega(\tilde{r}_{100000}^1, (K + isI)\bar{E}) + \Omega(g_{n,l,1}^1(E), \bar{E}) = 2H_2^1(E, \bar{E}) = -\frac{I_1(s)^2}{\tau_1}. \]

Finally, to compute \( d_4 \) we consider
\[
A|A|^2 : \begin{cases} 
(K - isI)\tilde{r}_{201000}^0 = id_5E - 2d_4F - 3g_{n,l,3}^0(E, E, \bar{E}) - 2g_{n,l,2}^0(E, \bar{r}_{200000}^0) - 2g_{n,l,2}^0(E, \tilde{r}_{101000}^0), \\
B_1\tilde{r}_{201000} = -3B_{n,l,3}^0(E, E, \bar{E}) - 2B_{n,l,2}^0(E, \tilde{r}_{200000}^0) - 2B_{n,l,2}^0(E, \tilde{r}_{101000}^0).
\end{cases}
\]

Taking the symplectic product with \( \bar{E} \) and simplifying in the usual fashion, we find that
\[ d_4 = 6H_4^0(E, E, \bar{E}, \bar{E}) + 3H_3^0(\tilde{r}_{200000}^0, \bar{E}, \bar{E}) + 3H_3^0(\tilde{r}_{101000}^0, E, \bar{E}), \quad (A.12) \]
where \( \tilde{r}_{200000}^0 \) and \( \tilde{r}_{101000}^0 \) are obtained from
\[
A^2 : \begin{cases} 
(K - 2isI)\tilde{r}_{200000}^0 = -g_{n,l,2}^0(E, E), \\
B_1\tilde{r}_{200000}^0 = -B_{n,l,2}^0(E, E)
\end{cases}
\]
and
\[
|A|^2 : \begin{cases} 
K(\tilde{r}_{101000}^0 - c_2f_2) = -2g_{n,l,2}^0(E, \bar{E}), \\
B_1(\tilde{r}_{101000}^0 - c_2f_2) = 0,
\end{cases}
\]
where
\[ c_2 = 6H_3^0(E, \bar{E}, f_2) \]
because of
\[
p_0A : \begin{cases} 
(K - isI)\tilde{r}_{100001}^0 = ic_3E - c_2F + 2c_4f_1 - 2g_{n,l,2}^0(E, f_2), \\
B_1\tilde{r}_{100001}^0 = -2B_{n,l,2}^0(E, f_2)
\end{cases}
\]
(note that \( \tilde{r}_{101000}^0 \) is determined up to addition of \( a_8f_1 \)). Altogether (A.12) shows that
\[ d_4 = 6H_4^0(E, E, \bar{E}, \bar{E}) + 3H_3^0(\tilde{r}_{200000}^0, \bar{E}, \bar{E}) + 3H_3^0(\tilde{r}_{101000}^0 - c_2f_2, E, \bar{E}) + 18H_3^0(E, \bar{E}, f_2)^2, \]
and the result of this calculation is given in equation (5.16).

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