Self-similarity in quantum states and long-range entanglement in fractional dimension

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Quantum matter with fractal geometry has been recently observed and studied in experiments. The fractal self-similarity and the fractional nature of the spatial dimension in such systems suggest exotic quantum order lying beyond paradigms developed in integer dimension. However, the existence of such uncomprehended novelty is still to be established in terms of quantum states with nontrivial many-body entanglement. To lay a foundation for exploring such states and to provide example that demonstrates such novelty, we study many-body entanglement patterns in entanglement-renormalization fixed points of qudit systems with Sierpiński lattice geometry, and introduce corresponding toy models. We show that the interplay between entanglement and self-similarity can be realized in fixed-point states with well-defined self-similar entanglement patterns. These fixed-point states exhibit distinct orders and are all given by single tensors as solutions to a scale invariance equation. In particular, we prove that an example of fixed-point state possesses long-range entanglement, i.e., it is short-range correlated but cannot be completely disentangled through constant-depth local quantum circuit. While the existence of long-range entanglement implies topological order in 2D and is disproved in 1D, our example proves the coexistence between long-range entanglement and fractality, a paradigm in fractional dimension. We show that such long-range entanglement pattern cannot be read as an extension of topological order, but rather exhibits novel quantum ordering inherent in fractional dimension, an evidence of new paradigm describing long-range entanglement.

I. INTRODUCTION

Quantum matter with fractal geometry exhibits the captivating fractal self-similarity in a fractional spatial dimension. While translational symmetry makes integer-dimension system regular, self-similarity iterates the irregularity of the system geometry and complicates the description of locality. This difference conjures up novelty in quantum emergent phenomena. Recently, fractal quantum systems have been realized in experiments of quantum materials and optical systems with fractality-dependent behavior observed [1–5]. These inspiring advancements have motivated exploration of novel quantum phases of matter in fractal geometry, and more generally of fractional dimension [6–17].

Characterizing such novelty in lattice systems is challenging since theoretical tools based on geometric or topological concepts like bulk and boundary appear to be incompatible with fractality. For example, in the famous Sierpiński fractal with fractional dimension 1.58 (see Fig. 1(a)) in between 1D and 2D, bulk and boundary, if defined, get ambiguous as system size increases, since any region claimed as bulk contains region inside which can be claimed as boundary. Here, we take a quantum-information perspective that has been extensively demonstrated and is compatible with fractality: Quantum order that characterizes quantum phase is encoded in the entanglement pattern of quantum state which can be read off the lattice locality [18–24]. A prominent example for this perspective is long-range entangled states of topological order where topological features and anyonic excitations emerge out of global “dancing” patterns of entangled local degrees of freedom in the ground state [19–21, 23, 24]. This perspective emphasizes the importance of studying quantum order directly from quantum states without resorting to a specific Hamiltonian.

In this work, we try to find nontrivial patterns [25] in entanglement-renormalization fixed-point states of qudit (C^d, or equivalently spin) systems on fractal lattice, off which quantum orders are expected to be read. The basic consideration is that fixed points with zero (infinite) correlation length possess the “cleanest” entanglement patterns to represent quantum order (critical points) since the short-range part of entanglement is washed out as much as possible in the renormalization flow [26, 27]. We consider qudits on the Sierpiński lattice (which has attracted most attention in experiments [1–5]). Fixed points are represented in tensor network as solutions to a scale-invariance equation. The coarse-graining scheme which respects the lattice geometry naturally resembles the zoom-out operation of the lattice and hence endows the fixed-point states with well-defined self-similar entanglement patterns. To illustrate such patterns, we study three examples of fixed points with essentially different entanglement. We also introduce toy models of strongly interacting hardcore bosons which can effectively realize self-similar entanglement patterns as in ground states and may shed light on finding realistic models harboring quantum order in fractional dimension.

An important part of our work is the proof of long-range entanglement pattern (LRE) in one fixed-point state. Here, we use LRE to refer to the entanglement in state which has short-range correlation but cannot be completely disentangled through constant-depth local quantum circuit. Note that it is the paradoxical combination of the absent correlation and the nonlocal information shared by local components which reveals LRE as emergent topological order or anyonic excitations in 2D, and distinguishes LRE from long-range ordered states...
like the GHZ state [23, 24, 28]. Indeed, the correlation and
the quantum-circuit arguments only require a lattice or a graph structure to specify the locality properties of vertices about its adjacency so that the existence of LRE can be studied in any dimension [29]. However, the existence of such highly nontrivial entanglement pattern and accordingly emergent quantum order depend on the dimension, e.g., topological order emerges out of LRE in 2D while LRE in 1D is proved not to exist [24]. In this work, we prove the coexistence between LRE and fractal self-similarity which can be viewed as a paradigm for LRE in fractional dimension. Furthermore, as we will show, the LRE in our example cannot be simply read as an extension of topological order, but rather suggests novel quantum order inherent in fractional dimension.

II. ENTANGLEMENT-RENORMALIZATION FIXED POINTS

Our framework for entanglement renormalization is essentially a scheme of wavefunction renormalization without resorting to local Hamiltonians but parameterizing quantum states with tensors. We define the coarse-graining operation which generates renormalization flow as shown in Fig. 1(b), i.e., an operator $(W^+) \otimes (N/3)$ mapping $N$ qudits into $N/3$ qudits in a larger scale with the same geometry. In general, the isometry $W : \mathbb{C}^d \to \mathbb{C}^{d^3}$ is defined in terms of the state in the entanglement renormalization flow so that $W^+ : \mathbb{C}^{d^3} \to \mathbb{C}^d$ (see Fig. 1(b)) extracts exactly the degrees of freedom that is in charge of the entanglement between one qudit block and its neighboring blocks. Following the spirit of the infinite-tensor-network ansatz [26, 30–32], we represent a fixed-point state $|\Psi\rangle$ on the infinite Sierpiński lattice as contracted from infinite copies of a single (or a finite number of) tensor $A = \sum_{\alpha\beta\beta'} A(\alpha\beta\beta') |\alpha\beta\beta'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{D^3}$ as shown in Fig. 1(c). Here $|\alpha\rangle \in \mathbb{C}^d$ is the physical index representing the local degrees of freedom, and $|\beta\rangle, |\beta'\rangle, |\beta''\rangle \in \mathbb{C}^D$ are virtual indices resembling the linking between nearest neighboring vertices. Fixed-point states are simply solutions to the scale-invariance equation of the coarse graining $(W^+)\otimes$. In our setup, the equation is

$$W^+ \text{Tr}_{\{C\}} [A \otimes A \otimes A] = \lambda A,$$

where the contraction on each block of three copies of $A$ followed by $W^+$ results in $A$ itself times a constant $\lambda$, and hence $|\Psi\rangle$ as represented by the tensor is invariant (see Fig. 1(c)). Note that in cases where $|\Psi\rangle$ is determined by multiple tensors, there are correspondingly more than one scale-invariance equations as will be shown in example.

Strictly speaking, we can define states $|\Psi\rangle_n$ on the $n$th-generation of the Sierpiński lattice as contracted from finite copies of $A$ together with three constant tensors (with trivial physical degrees of freedom and independent on the system size) at the three corners of the lattice. Then, when identifying $|\Psi\rangle_n$ and $W^+ \otimes |\Psi\rangle_n \propto |\Psi\rangle_{n-1}$ for scale invariance in the entanglement renormalization, only the tensor $A$ matters, since tensor networks giving $|\Psi\rangle_n$ and $|\Psi\rangle_{n-1}$ have the same geometry and are contracted with the same corner tensors respectively. As a result, we can safely speak of $|\Psi\rangle$ as contracted from infinite copies of $A$. This way, we avoid working on infinite lattice where states are not well-defined, but grasp the pattern of entanglement as will be discussed below and shown in examples. In the following, $|\Psi\rangle$ is understood either as the state at an arbitrary finite system size or the state at infinite system size, depending on the context.

For simplicity in all examples, we consider only cases where the constant corner tensor is $\sum_{\beta} (\beta \otimes 1) \in \mathbb{C}^D \otimes \mathbb{C}$ so that $|\Psi_n\rangle$ is a superposition of qudit-product-states $|\{\alpha\}\rangle = |\alpha\alpha'\cdots\rangle$ with coefficient $\sum_{\beta_1, \beta_2, \beta_3} A_n (|\alpha\rangle, |\beta_1\rangle, |\beta_2\rangle, |\beta_3\rangle)$, where the “big” tensor $A_n$ with many physics indices $|\alpha\rangle, |\alpha'\rangle, |\alpha''\rangle\ldots$ results from the contraction of copies of $A$ in the $n$th generation of the lattice.

III. SELF-SIMILAR ENTANGLEMENT PATTERN

Eq. 1 ensures certain structure in the entanglement of $|\Psi\rangle$ that we can call a self-similar entanglement pattern which realizes the interplay between self-similarity and entanglement. Indeed, while the three virtual legs of $A$ represent the entanglement between one qudit and its neighbors, the three virtual legs of $\text{Tr}_{\{C\}} [A \otimes A \otimes A]$ represent the same thing between one block of qudits and its neighboring blocks (see Fig. 1(c), 1(e)). Then, since $W^+$...
only removes entanglement inside each blocks, resembling the zoom-out operation of the lattice as \( W \) resembling the zoom-in operation. Eq. 1 implies that blocks of qudits are entangled in the same way as qudits, i.e. \( |\Psi\rangle \) has certain self-similar entanglement pattern (zoom-in and zoom-out invariant). This argument can be formalized with the transfer matrix \( E = A^+ A \) [33] (see Fig. 2(d)) as discussed in the appendix. In short, characteristic of a fixed-point state including both the tensor \( A \) and the coarse graining operator \( W \) can be uniquely specified solely from a \( E \) with an equivalent scale invariance equation \( \text{Tr}_{\{C^D\}} [E \otimes E \otimes E] = |A|^2 E \) which does not explicitly include \( W \) and perfectly captures the zoom-in the zoom-out invariance (see Fig. 1(e)).

Our framework establishes the identity between the scale invariance of \( |\Psi\rangle \) and the fractal self-similar entanglement pattern in \( |\Psi\rangle \) which distinguishes from integer dimensions. Firstly, our scale-invariance equation distinguishes from that in the tensor-network representation in 2D where further operations on virtual indices following the coarse graining operator, e.g., removal of indices, is required to keep the lattice geometry [24]. Then in 1D, while the lattice geometry is also preserved by the coarse graining operator [26], fractal geometry harbors essentially different physics from those in 1D, e.g., numerical evidence of anyon braiding [12] which does not exist in 1D. These distinctions are all revealed in the entanglement patterns. Therefore, as will be further demonstrated in examples, non-triviality and novelty in quantum order of fractal lattice systems emerge from non-trivial interplay between self-similarity and entanglement properties as given by solutions to Eq. 1.

IV. CHIRALITY AND LONG-RANGE ORDER

We firstly give a simple solution to Eq. 1 to illustrate the above framework. This example gives a 2D-chiral state with self-similar entanglement pattern.

In the tensor \( A \), the physical index \( |\alpha\rangle = |1\rangle, \ldots, |6\rangle \in \mathbb{C}^5 \) and each virtual index \( |\beta\rangle = |a\rangle, |b\rangle, |c\rangle \in \mathbb{C}^3 \). We specify \( A \) by listing the nonzero elements of \( A(\alpha \beta \beta' \beta'') \) which are all equal to 1 in Fig. 2(a). In contracting copies of \( A \), we respect the rotational symmetry and always take the convention that each copy of tensor inside a block is outward (Fig. 2(a)). The contraction gives the fixed-point state \( |\Psi\rangle = \sum_{m=1}^6 |\psi_m\rangle \) where \( |\psi_m\rangle \) is specified by certain constraints which determine the long-range order in \( |\Psi\rangle \) [34] (see Fig. 2(b)). The self-similar pattern or the scale invariance of \( |\Psi\rangle \) is realized with \( W^+ \) (see Fig. 2(c)) which makes \( A \) a solution to Eq. 1 [35].

When applying \( (W^+ \otimes E) \), we always take the convention that the three-qudit-state configuration in a block or in \( \text{Tr}_{\{C^D\}} [A \otimes A \otimes A] \) is read as if the top of the configuration is the outermost in the larger block (see Fig. 2(c)). Fig. 2(d) gives the mirror image \( A' \) and \( |\psi'_m\rangle \), and hence \( |\Psi'\rangle \). Obviously, \( \{|\psi'_m\rangle\} \) form a representation of rotational symmetry orthogonal to that formed from \( \{|\psi_m\rangle\} \).

Therefore, \( |\Psi\rangle \) is 2D-chiral, i.e. \( |\Psi'\rangle \) cannot be superimposed with \( |\Psi\rangle \) through 2D translation or rotation.

V. SHORT-RANGE ENTANGLEMENT PATTERN (SRE)

Now we study fixed-point states in which short-range correlation is completely washed out through the coarse graining, but certain genuine multipartite entanglement still remains. As a prelude for describing long-range entanglement pattern and corresponding model, we start with an example of short-range entanglement pattern (SRE).

A. Toy model of trilayer hardcore bosons

In this case, a qudit can be decomposed into individual degrees of freedom corresponding to the entanglement between the qudit and its different neighbors. We can effectively realize the fixed point as the ground state in a model of trilayer-lattice (with Sierpinski geometry and label \( s = 1, 2, 3 \)) hardcore boson system. Then, a basis of the qudit \( |\alpha\rangle \in \mathbb{C}^8, \alpha = 1, \ldots, 8 \) on a vertex is identi-
fixed with $|x_1 x_2 x_3\rangle$, $x_n = 0, 1$ for occupations on the same vertex but in different layers representing the individual degrees of freedom (see Fig. 3(a)). The Hamiltonian is

$$H = -t \sum_{(k,l)} (a_k^+ a_l^1 + a_k^1 a_l^3) + c.c.$$

$$+ v \sum_{(i,j)} n_i^1 n_j^1 + v \sum_{(i',i'',\ell')} (n_i^2 n_i'^2 + n_i^2 n_i''^2 + n_i^2 n_i''^2),$$

where $t,v > 0$; $(k,l)$ is a pair of nearest neighboring vertices; $(i,j)$ is a pair of neighboring vertices linking two blocks; $(i',i'',\ell')$ is a pair of vertices linking two blocks; bosons on layer-1 have intralayer hopping on nearest neighbors and repulsive intralayer interaction on vertices linking two blocks; bosons on layer-2 and layer-3 have interlayer hopping on nearest neighbors and repulsive interlayer interactions on vertices inside each block (see Fig. 3(a)).

Note that to stabilize renormalization fixed point, $H$ distinguishes from “standard” Bose-Hubbard-type models mainly in the form of interactions. Accordingly, we expect that more realistic models with similar hopping terms, or equivalently with similar scheme for generating quantum fluctuations, stabilize states flowing to the fixed-point state studied here.

We fix the filling factor for each layer as $N^1/N = 1/2$, $(N^2 + N^3)/N = 1/3$ [36] ($N^s = \sum n_i^s$, $s = 1, 2, 3$ and $N$ the number of vertices in one layer) and solve this model in the strong-interaction limit $t \ll v$ by treating the hopping $H^1$ as a perturbation to the interaction $H_0$. Due to the interaction energy gap, the physics of the system is captured by an effective Hamiltonian $H_{eff}$ within the subspace $\mathcal{H}_0 \subset C^8 \otimes C^8$ spanned by qudit-product-states $|\psi_m\rangle$ with the lowest interaction energy. According to the perturbation theory [20], $H_{eff}$ is simply the lowest-order non-vanishing term in the sequence $P(H_0 + H^1 G_0^1 H_1 + H^1 G_0^2 H_1 G_0^1 H_1 + \cdots)P$ where $G_0^1$ is the unperturbed Green function for $H_0$ acting only on excited states (of $H_0$), and $P = \sum_m |\psi_m\rangle\langle\psi_m|$ is the projection onto $\mathcal{H}_0$ [37].

Minimizing the repulsive interaction energy specifies $|\psi_m\rangle$ with the constraints: for a pair of neighboring vertices linking two blocks, the qudit states are $|x_1 x_2 x_3\rangle$ and $|(x_1 + 1) y_2 y_3\rangle$; in a block, the qudit-state configuration is clockwise $|x_1 x_2 (z_2 + 1)\rangle$, $|y_1 y_2 (x_2 + 1)\rangle$, $|z_1 z_2 (y_2 + 1)\rangle$. Here, we use $x, y, z = 0, 1 \in \mathbb{Z}_2$, i.e. $1 + 1 = 0, 0 + 1 = 1$. These constraints are illustrated in Fig. 3(b), where a qudit basis state $|x_1 x_2 x_3\rangle$ in $|\psi_m\rangle$ is labeled by a triangle with $x_1, x_2, x_3$ clockwise represented by red (=1) or yellow (=0) and with $x_1$ outermost in the block. This way, the constraints simply read as: the pair of neighboring colors in two neighboring vertices are opposite. A red-yellow (yellow-red) pair can be viewed as $|\uparrow\rangle\langle\downarrow|$ of a 1/2-spin on a link $n'$. These pairs are independent and cover all local degrees of freedom except for the corners so that a $|\psi_m\rangle$ can be identified with $|\uparrow\downarrow\cdots\uparrow\rangle$ up to the occupation $x_1$ in the three corner qudits of the finite lattice which are not engaged in the interaction (see Fig. 3(b)). In this way, non-vanishing terms in $PH_1 P$ are hopping or equivalently $\sigma_{n'}^n$’s which exchange red-yellow and yellow-red pairs while leave the $x_1$ in the corner qudits unchanged.

As a result, $\mathcal{H}_0$ has 8 sectors (the 8 choices of the occupation $x_1$ of the three corner qudits) and each of the sector is isomorphic to $(C^2)^{\otimes N'}$ ($N'$ the number of links of the finite Sierpiński lattice). $H_{eff}$ is invariant within each sector and has the equivalent form $-\sum_{n'} \sigma_{n'}^n$ describing uncoupled spins located on links of the lattice. Hence, the system is exactly solvable with 8-fold-degenerate gapped ground state $|\Psi\rangle = 1/\sqrt{2^{N'}} \sum |\psi_m\rangle$, i.e. the equal-weight sum of all $|\psi_m\rangle$ in a sector, or equivalently the state $\otimes_{n'} (|\uparrow\rangle_{n'} + |\downarrow\rangle_{n'})$ of 1/2-spins.

**B. Valence-bond-type order**

The self-similar entanglement pattern or the scale invariance of $|\Psi\rangle$ can be shown with $W^+ = 1/\sqrt{8}$

$$\sum |x_1 y_1 z_1 x_2 (z_2 + 1), y_1 y_2 (x_2 + 1), z_1 z_2 (y_2 + 1)\rangle,$$

where the two summation run through $x_1, y_1, z_1 \in \mathbb{Z}_2$ and $x_2, y_2, z_2 \in \mathbb{Z}_2$. As shown in Fig. 3(c), coarse graining on $|\psi_m\rangle$ only keeps the degrees of freedom on occupation $x_1$ at each corner of a block so as to form a new local degrees of freedom in $|\psi'_m\rangle$ which is specified by the same constraints. The tensor-network representation of $|\Psi\rangle$ is given in Fig. 3(d) with copies of two tensors $A_0, A_1$ with $|\alpha\rangle = |x y z\rangle \in C^8$ for $x, y, z = 0, 1$ and $|\beta\rangle = |0\rangle, |1\rangle \in C^2$ satisfying scale-invariance equations with $W^+$ (see Fig. 3(d)). In the contraction, copies of
the two tensors appears in a fixed pattern (see Fig. 3(e)) which is invariant to the coarse graining [38].

As shown in Fig. 3(b) and 3(c), |Ψ⟩ possesses valence-bond type of entanglement where the independent pairs of opposite colors (the valence bonds) guarantees the genuine multipartite entanglement in |Ψ⟩ and also determines its zero correlation length, i.e. two distant qudits not sharing a bond is not correlated. Similar to fixed points with valence-bond type of entanglement in 1D [24, 26], the tensor A₀, A₁ can be viewed as product states in \( \mathbb{C}^{\sqrt{2}xD} \otimes \mathbb{C}^{\sqrt{2}xD} \otimes \mathbb{C}^{\sqrt{2}xD} \), and are obviously injective tensors so that |Ψ⟩ has SRE.

SRE is defined in terms of local quantum circuit. A local quantum circuit is a unitary operator \( U = U_L \cdots U_2 U_1 \) where each \( U_i = \otimes_p U_i(p) \) is a product of unitary operators acting on nonoverlapping local patches \( \{p\} \) of qudits with sizes smaller than some number \( P \), and \( L \) is the depth of the circuit (see Fig. 4(a)). In the above example, |Ψ⟩ can be completely disentangled through a quantum circuit \( U \) with \( L = 2 \) and \( P = 3 \) as shown in Fig. 4(b). In \( U_i \), each \( U_i(p) \) acts on a pair of neighboring qudits linking two blocks, disentangling the color pairs \( \{0x2x3, 1y2y3\} \mapsto 0x2x3 \otimes 1y2y3, \{1x2x3, 0y2y3\} \mapsto 1x2x3 \otimes 0y2y3 \) so that blocks of qudits are disentangled in \( U_i |Ψ⟩ \). Then \( U_2 \) simply disentangles the three qudits within each block in a similar manner and hence completely disentangles |Ψ⟩. In the quantum circuit \( U \), the independence of \( L \) and \( P \) on the system size defines the SRE of |Ψ⟩.

FIG. 4. (a) Local quantum circuit as layers of non-overlapping local unitary operators. The vertical lines stand for qudits and each block is a local unitary operator. The purple unitary operators form a causal structure. (b) Local quantum circuit \( \tilde{U}_2 \tilde{U}_1 \) for completely disentangling the |Ψ⟩ with SRE.

VI. LONG-RANGE ENTANGLEMENT PATTERN (LRE)

LRE of a state |Ψ⟩ is then defined by its short-range correlation and the dependence of the lower bounds of \( L \) and \( P \) on the system size for a local quantum circuit to completely disentangle |Ψ⟩. In other words, a short-range correlated |Ψ⟩ possesses LRE if and only if for arbitrarily given \( L \) and \( P \), any local quantum circuit characterized by \( L \) and \( P \) cannot completely disentangle |Ψ⟩ when the system is over certain size. Note that the characterization of LRE only requires a lattice or a graph structure to specify the locality properties of vertices about their adjacency and is hence reliable exploring highly nontrivial entangled states in fractional dimension.

FIG. 5. (a) Tensors \( A_0, A_1, A_2 \) and \( A_3 \) which sum into \( A \). (b) Example for nonvanishing (tick) and vanishing (cross) contraction of copies from \( A_0, A_1, A_2 \) and \( A_3 \). (c) Illustration of |ψ_m⟩. The qudit basis states are labeled by triangles with qudits linking two blocks, disentangling the color pairs \( \{0x2x3, 1y2y3\} \mapsto 0x2x3 \otimes 1y2y3, \{1x2x3, 0y2y3\} \mapsto 1x2x3 \otimes 0y2y3 \) so that blocks of qudits are disentangled in \( U_i |ψ_m⟩ \). Then \( U_2 \) simply disentangles the three qudits within each block in a similar manner and hence completely disentangles |Ψ⟩. In the quantum circuit \( U \), the independence of \( L \) and \( P \) on the system size defines the SRE of |Ψ⟩.

Now we study LRE in a state |Ψ⟩ as given by a tensor \( A \), a solution to Eq. 1. We firstly prove that |Ψ⟩ possesses LRE and then introduce a toy model which can effectively stabilize |Ψ⟩ as the unique ground state. Here \( A \) still has the minimal nontrivial virtual index \( \beta = 0\} \), \( |1⟩ \in \mathbb{C}^2 \), but has the physical index \( |α⟩ = |0⟩, |1⟩, |2⟩, |3⟩ \in \mathbb{C}^4 \) so that the physical degrees of freedom cannot be decomposed into parts corresponding to virtual legs. We define the tensor

\[
A = A_0 + A_1 + A_2 + A_3, \\
A_0 = |0⟩ \otimes |000 + 111⟩, \quad A_1 = |1⟩ \otimes |100 + 011⟩, \quad (3) \quad A_2 = |2⟩ \otimes |010 + 101⟩, \quad A_3 = |3⟩ \otimes |001 + 110⟩,
\]

as illustrated in Fig. 5(a) where for convenience we represent \( 000 + 111 \) by \( 0001 \cdot 100 + 011 \) by \( 0101 \cdot 001 + 010 \), etc. According to this convenience, contraction of any three copies from \( A_0, A_1, A_2 \) or \( A_3 \) (following the convention introduced previously) does not vanish if and only if there are even number of flips between \( 0 \) and \( 1 \) along the circle formed by contracting indices of the tensors (see Fig. 5(b)), or equivalently, there are even number of copies from \( A_2 \) or \( A_3 \).

The above condition for tensor contraction determines constraints that specify qudit-product-states \( |ψ_m⟩ \)'s in the expansion |Ψ⟩ = \( 1/√M \sum_m |ψ_m⟩ \). Here \( M \) is the total number of \( |ψ_m⟩ \)'s satisfying the constraints and |Ψ⟩ is
contrasted from copies of $A$. Note that the equal-weight summation results from our choice of the constant corner tensors as described previously. The constraints can be elucidated pictorially in the illustration of $|\psi_m\rangle$ (see Fig. 5(c)) by mapping the basis states $\{|\alpha\rangle\}$ of the qudit $C^4$ to images of triangles with odd number of pink sides and even number of red sides. Then as shown in Fig. 5(c) and 5(d), the constraints are associated to each circle of vertices on which the colored sides form a chain; and the constraint on a circle is simply that the number of red sides in the chain is even. Note that when representing $|\alpha\rangle$ on a vertex by a triangle in Fig. 5(d), we choose the orientation such that the triangle is outward in the block containing the vertex.

Fig. 5(f) illustrates the self-similar entanglement pattern, i.e. the scale invariance of $|\Psi\rangle$. In the illustration, the coarse graining simply removes the inner sides of a block and only keeps the six outer sides which are eventually reduced into three sides according to the rule: two red (pink) sides on one side are reduced into one pink side; one red and one pink side are reduced into one red side. Formally, $W^+$ is defined as $1/\sqrt{8}$ times the sum

$$
|0\rangle\langle 0| 000 + 111 + 023 + 132 + 213 + 230 + 302 + 321|
+ |1\rangle\langle 1| 011 + 032 + 100 + 123 + 202 + 211 + 313 + 330|
+ |2\rangle\langle 2| 010 + 033 + 101 + 122 + 203 + 220 + 312 + 331|
+ |3\rangle\langle 3| 001 + 110 + 022 + 135 + 213 + 230 + 302 + 320|
$$

Here, each term in $W^+$ corresponds to a nonvanishing contraction of three copies from $A_0, A_1, A_2, A_3$. For example, $|1\rangle\langle 1| 011$ corresponds to the contraction of $A_1, A_2, A_3$ (see Fig. 5(e)) which results in a tensor $|123\rangle\otimes|100\rangle$ that is eventually mapped to $A_1 = |1\rangle\otimes|010\rangle$ (up to a constant) through $W^+$. Obviously, the tensor $A$ together with $W^+$ satisfies Eq. 1 ($\lambda = 1/\sqrt{8}$).

### A. Proof for the zero correlation length

To prove that $|\Psi\rangle$ possesses LRE, we firstly prove that $|\Psi\rangle$ has zero correlation length, i.e. $\langle \Psi|O_iO_j|\Psi\rangle - \langle \Psi|O_j|\Psi\rangle \langle \Psi|O_i|\Psi\rangle = 0$ for any local operators $O_i$ and $O_j$ acting on nonadjacent qudits located at vertices $i$ and $j$. According to the expansion $O_i = \sum_{\alpha'\alpha''} c_{\alpha''}\alpha' |\alpha'\rangle\langle\alpha|$, it suffices to show that $\langle \Psi|(|\alpha''\rangle\langle\alpha'|\alpha\rangle)\langle\alpha|\rangle = 0$ for any qudit basis states $|\alpha\rangle, |\alpha'\rangle, |\alpha''\rangle$.

The first case to be considered is that $|\alpha\rangle \neq |\alpha'\rangle$ (the case of $|\alpha\rangle = |\alpha'\rangle$ can be treated in a similar way). We observe that in the illustration of an arbitrary $|\psi_m\rangle$ (see Fig. 5(c), 5(d) and 6(a)), each triangle (a qudit basis state) is engaged in constraints on three circles respectively as each triangle has three sides. It follows that the change from $|\alpha\rangle$ to $|\alpha'\rangle$ (as the color change on two sides of the triangle) on a vertex $i$ will break the constraints on two circles (see Fig. 6(a)), resulting in a qudit-product-state $|\phi\rangle$ orthogonal to all the qudit-product-states expanding $|\Psi\rangle$. Then, if $|\bar{\alpha}\rangle = |\alpha'\rangle$, we have $(|\bar{\alpha}\rangle\langle\bar{\alpha}|)\langle\psi_m\rangle$ (either unchanged or equal to 0) is orthogonal to $|\phi\rangle = (|\alpha''\rangle\langle\alpha'|\psi_m\rangle$ and hence both $\langle \Psi|(|\bar{\alpha}\rangle\langle\bar{\alpha}|)\langle\alpha''\rangle\langle\alpha'|\rangle = 0$ and $\langle \Psi|(|\bar{\alpha}\rangle\langle\bar{\alpha}|)\langle\alpha''\rangle\langle\alpha'|\rangle\langle\alpha'\rangle\langle\alpha''\rangle = 0$. On the other hand, if $|\bar{\alpha}\rangle \neq |\alpha'\rangle$, we also observe that the vertex $j$ which is nonadjacent to $i$ share at most one circle with $i$ and the qudits on the two vertices respectively are at most commonly engaged in one constraint. Consequently, the change from $|\alpha''\rangle\alpha'\rangle$ to $|\bar{\alpha}\rangle$ on $j$ maps $|\psi_m\rangle$ to another qudit-product-state $|\phi\rangle$ which is orthogonal to both $|\psi_m\rangle$ and $|\phi\rangle$ (as they differ by at least the constraint on one circle). Hence, both $\langle \Psi|(|\bar{\alpha}\rangle\langle\bar{\alpha}|)\langle\alpha''\rangle\langle\alpha'|\rangle = 0$ and $\langle \Psi|(|\bar{\alpha}\rangle\langle\bar{\alpha}|)\langle\alpha''\rangle\langle\alpha'|\rangle = 0$, and we have the desired condition.

![Fig. 6.](image)

FIG. 6. (a) Operator $|3\rangle\langle 0|$ for the qudit on vertex $i$ maps $|\psi_m\rangle$ to $|\phi\rangle$ which breaks the constraints on two circles. “+$” for constraints satisfied on the corresponding circles and “$-$” for not satisfied. (b) Operator $T_{ij'}$ maps $|\psi_m\rangle$ to $|\psi_m'\rangle$, preserving the constraints on all circles. (c) The content of $T_{ij'}$ acting on a pair of neighboring qudits. (d) $T_{ii'}, T_{i'j}, T_{ij'}$ acting on different neighboring pairs of qudits map the single-qudit-state $|1\rangle$ on vertex $i$ to $|0\rangle$, $|3\rangle$ and $|2\rangle$ respectively.
the other three basis states respectively (see Fig. 6(d)). Now, as shown in Fig. 6(b), if in $|\psi_m\rangle$ the qudit on vertex $i$ is in $|\alpha\rangle$, certain $T_{ij}$ maps $|\psi_m\rangle$ exactly to a $|\psi'_{m'}\rangle$ with the single-qudit-state changed from $|\alpha\rangle$ to $|\alpha'\rangle$. Since $T_{ij}$ is unitary, it maps all $|\psi_m\rangle$'s with nonzero $\langle\psi_m|(|\alpha\rangle|\alpha\rangle|\psi_m\rangle$ one-to-one to all $|\psi'_{m'}\rangle$'s with nonzero $\langle\psi'_{m'}|(|\alpha\rangle|\alpha'\rangle|\psi'_{m'}\rangle$. Therefore, since $\sum_{n=0}^{3} \langle\psi|(|\alpha\rangle|\alpha\rangle|\psi\rangle = \sum_{n=0}^{3} \langle\psi'|(|\alpha\rangle|\alpha\rangle|\psi'\rangle = 1$, we have $\langle\psi|(|\alpha\rangle|\alpha\rangle|\psi\rangle = 1/4$ and similarly $\langle\psi'|(|\alpha\rangle|\alpha\rangle|\psi\rangle = 1/4$. Next, we show $\langle\Psi|(|\alpha\rangle|\alpha\rangle|\Psi\rangle = 1/16$. Indeed, following the spirit of the above arguments, we only need to notice that vertices $i$ and $j$ are nonadjacent so that $T_{ij}$ leave the qudit on $j$ unchanged (see Fig. 6(b)), and hence we have $\langle\Psi|(|\alpha\rangle|\alpha\rangle|\alpha\rangle|\Psi\rangle = \langle\Psi|(|\alpha\rangle|\alpha\rangle|\alpha\rangle|\Psi\rangle = 1/4 \times 1/4 = 1/16$. Obviously, it means that $\langle\Psi|(|\alpha\rangle|\alpha\rangle|\alpha\rangle|\Psi\rangle - \langle\Psi|(|\alpha\rangle|\alpha\rangle|\alpha\rangle|\Psi\rangle = 1/16 - 1/16 = 0$. We have hence proved that $|\Psi\rangle$ has zero correlation length.

B. Proof for the system-size dependence of the circuit depth

Now, we consider arbitrary $L$ and $P$ and an arbitrary local quantum circuit $U$ characterized by $L$ and $P$. We prove that once the system is over certain size, $U$ cannot completely disentangle $|\Psi\rangle$. To that end, we introduce another state $|\Phi\rangle$ which is orthogonal to $|\Psi\rangle$. Here $|\Phi\rangle = 1/\sqrt{M} \sum_{m} |\phi_m\rangle$ is the sum over all qudit-product-states satisfying constraints on all circles except for the largest four circles (see Fig. 7(a) and 7(b)). In the proof, we take advantage of the two-dimensional space spanned by $|\Psi\rangle$ and $|\Phi\rangle$ as a quantum code for detecting errors of the form $U^{+}O_jU$ where $O_j$ is a local operator located at some vertex $j$. Note that the space spanned by the orthonormal $|\Psi\rangle$ and $|\Phi\rangle$ detects an error, i.e., an operator $O$ if and only if $\langle\Psi|O|\Psi\rangle = \langle\Phi|O|\Psi\rangle = 0$ and $\langle\Psi|O|\Phi\rangle = \langle\Phi|O|\Phi\rangle$.

The proof is based on the following consideration. We assume that $U$ completely disentangles $|\Psi\rangle$, i.e., $U|\Psi\rangle = \otimes_i |\chi_i\rangle$ with $|\chi_i\rangle$ being a normalized single-qudit-state on vertex $i$. Then, since $\langle U|\Phi\rangle = \langle U^{+}|O_j|U\rangle = \langle O_j|U\rangle = 0$, it is easy to show that there exists a local operator $O_j$ located at some vertex $j$ such that $U|\Psi\rangle|O_j|U\rangle \neq U|\Phi\rangle|O_j|U\rangle$. Indeed, in any qudit-product-state basis including $|\chi_i\rangle$ and expanding $|U|\Phi\rangle$, qudit-product-states with nonzero coefficients are orthogonal to $\otimes_i |\chi_i\rangle$, and hence there must be some $\otimes_i |\chi_i\rangle$ with $\langle\chi_j'|\chi_j\rangle = 0$ at some vertex $j$. In that case, we can take $O_j$ as $I \otimes \cdots \otimes |\chi_j\rangle\langle\chi_j| \otimes \cdots \otimes I$ so that we have $\langle U|O_j|U\rangle < 1$ and hence $\langle U|O_j|U\rangle \neq \langle U|O_j|U\rangle$. The inequality rewritten as $\langle U^{+}|O_j|U\rangle \neq \langle U^{+}|O_j|U\rangle$ implies that the space spanned by $|\Psi\rangle$ and $|\Phi\rangle$ does not detect the error $U^{+}O_jU$. According to this observation, we can prove that $|\Psi\rangle$ and $|\Phi\rangle$ detect all errors of the form $U^{+}O_jU$ once the system is over certain size, which is more than enough to contradict the assumption, and hence to prove the desired result.

Following the above consideration, we now prove that once the linear size $D$ of the lattice is over $\frac{16}{3}P - \frac{5}{3}P + \frac{5}{3}$ the error-detecting property is satisfied. In the graph-theoretical language, we define the linear size of the lattice (a connected subset) as the diameter of the lattice (subset) in the graph (induced subgraph) metric [39], consistent with the study of LRE in 2D [29]. Then, $D$ of a finite-generation Sierpiński lattice is exactly the number of links on one side of the lattice as a triangle. There is another important size in our proof which is determined by the causal structure in the local quantum circuit (see the dark color in Fig. 7(c)): the operator $O_j = U^{+}O_jU$ is supported on a connected subset $J$ of vertices with size $D_J \leq (2L - 1)P$ [40]. Now assume the inequality $D \geq \frac{16}{3}P - \frac{5}{3}P + \frac{5}{3}$ or equivalently $D_J \leq \frac{3}{8}D - \frac{3}{4}$.

We firstly prove $\langle\Psi|O_j|\Phi\rangle = \langle\Phi|O_j|\Psi\rangle = 0$. Indeed, it is easy to check that the distance between a closest pair of vertices apart in any two separated circles among the four with broken constraints in $|\phi_m\rangle$ is $(2L - 1)/2$, greater than $D_J$ so that $J$ can only cover vertices in one such circle. Consequently, $\langle\psi_m'|O_j|\phi_m\rangle = 0$ since $O_j$ keeps $O_j|\phi_m\rangle$ with at least one constraint broken and different from that in $|\psi_m'\rangle$. Hence we have proved $\langle\Psi|O_j|\Phi\rangle = 0$, and $\langle\Phi|O_j|\Psi\rangle = 0$ as well using the same argument.

To prove $\langle\Psi|O_j|\Psi\rangle = \langle\Phi|O_j|\Phi\rangle$, we define unitary operators

\[
S_1 = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|, \\
S_2 = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|, \\
S_3 = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|,
\]

acting on a single qudit located at the vertex $i$. As illustrated in Fig. 7(a) and 7(d), the effect of these operators is to exchange the red and pink on two sides of the triangle or create two red (pink) sides from two pink (red) sides. Then, since vertex $i$ is simultaneously included in three circles, applying $S_1^I, S_2^I$ or $S_3^I$ to $|\psi_m\rangle$ simply breaks the constraints on two out of the three circles according to the superscript 1,2,3. With these unitary operators, we can define a unitary operator $V_1 = S_1^I S_2^I S_3^I$ where each of the three vertices $i_1, i_2, i_3$ is included in two out of the four largest circles. As shown in Fig. 7(a) and 7(b), $V_1$ simply break the constraints on the four circles and maps $|\psi_m\rangle$ to $|\phi_m\rangle$ and hence maps $|\Psi\rangle$ to $|\Phi\rangle$.

Now, there are two cases to be considered regarding the support $J$ of $O_j$: (1) $J$ does not cover any of $i_1, i_2, i_3$; (2) $J$ covers only one of the three vertices. Indeed, as we have shown, $J$ cannot cover more than one of the three vertices since they are farther apart than the size $D_J$. In case (1), we have $[O_j, V_1] = [O_j, V_1^+] = 0$. It follows that $\langle\Psi|O_j|\Phi\rangle = \langle\Psi|V_1 O_j|\Psi\rangle = \langle\Psi|V_1^+ O_j V_1|\Psi\rangle = \langle\Psi|V_1^+ V_1 O_j V_1|\Psi\rangle = \langle\Phi|O_j|\Psi\rangle$ as desired. In case (2), without loss of generality, we suppose the covered vertex...
is large enough. We have also proved that for given
P cannot completely disentangle |P⟩ and that any
U where the two subsets of vertices
\{D_i\}, \{K_i\} are characterized by given
property, and that any
U_i maps |ψ_m⟩ to |φ_m⟩. (c) The causal structure in the local quantum circuit which extends the support of O_j as a single vertex to the support J of O_j as a connected subset of vertices. (d) The size of support J which does not cover the
path linking two neighboring largest circles.

is i_1. Then we can define another two unitary operators

\begin{align*}
V_2 &= S^1_i S^2_i S^2_{k_1(D+1)/4} \cdots S^2_{k_1/2} S^1_{k_1}, \\
V_3 &= S^1_i S^2_i S^2_{k_1(D+1)/4} \cdots S^2_{k_1/2} S^1_{k_1/2}
\end{align*}

where the two subsets of vertices \(K = \{k_1, k_2, \ldots, k_{(D+1)/4}\}\) and \(K' = \{k_1', k_2', \ldots, k_{(D+1)/4}\}\), as specified in Fig. 7(d) and with distance \((D + 1)/4, (D + 1)/4 - 1\) to \(i_1\) respectively, form the two paths connecting two adjacent largest circles. Since \(i_1\) is covered in \(J\), \(J\) cannot have overlap with both \(K\) and \(K'\), otherwise the distance between \(k_1\) and \(k_1'\) within \(J\) exceeds the size \(D_J\) which implies contradiction. Note that due to the inequality \(D_J \leq \frac{3}{2}D - \frac{3}{2}\), \(J\) cannot cover the whole of any of the four circles, since any connected region including such a circle has size at least equal to \(\frac{3}{2}D + \frac{3}{2} > D_J\). Hence, the shortest path connecting \(k_1\) and \(k_1'\) (which defines their distance with \(J\)) must pass through \(i_1\), giving rise to the distance \(\frac{3}{2}D - \frac{3}{2} > D_J\). Consequently, we have either \([O_J, V_2] = [O_J, V_2^{T}] = 0\) or \([O_J, V_3] = [O_J, V_3^{T}] = 0\), which leads to \(⟨Ψ|O_J|Ψ⟩ = ⟨Φ|O_J|Φ⟩\) by the same arguments as in case (1).

Above arguments have proved the error-detecting property, and that any \(U\) characterized by given \(P\) and \(L\) cannot completely disentangle |Ψ⟩ once the system size is large enough. We have also proved that for given \(P\), the depth \(L\) for any local quantum circuit characterized by \(P\) and \(L\) to completely disentangle |Ψ⟩ has a lower bound \(L > \frac{3}{16}D + \frac{1}{2} - \frac{5}{16}P\) which is linear to the lattice size \(D\).

C. Novel quantum order

Now, compared with the fixed-point state of SRE, the pictorial illustrations of the two different entanglement patterns in Fig. 3(b) and 5(c) respectively portray two distinct types of order describing how local degrees of freedom are constrained together in forming the entanglement. While in the SRE case the zero correlation length implies only locally encoded information, the entanglement pattern in |Ψ⟩ appears to be highly nontrivial due to the paradoxical combination of the absent correlation and the nonlocal information shared by local components that cannot be locally removed. The nontriviality of such combination in |Ψ⟩ is partially due to that the combination is proved as a contradiction in 1D [24], i.e., there is no LRE in 1D. The peculiarity of LRE is due to that LRE in integer dimension is believed to provide the microscopic picture of topological order. On the other hand, topological order is accepted as the only known macroscopic characterization of LRE. However, as we will show in the following, the paradigm of topological order cannot
be extended to the fractal lattice systems in our study so that topological order cannot capture the quantum order encoded in $|\Psi\rangle$.

Firstly, as an apparent reason, the meaning of topology in our case is lost or at least intrinsically different from that in the 2D topological order. Indeed, topological order is macroscopically described by an effective topological quantum field theory (TQFT) where the topological spaces in the theory refer to the smooth manifolds of spacetime whose change leads to the change of ground state degeneracy, i.e. the topological degeneracy. While the smooth-manifold topology might have burdensome modifications to fit the fractal geometry and the fractional spatial dimension, it loses the conciseness in capturing the emergent order therein.

Even with complex topological structure compatible with the fractal geometry, an extended version of topologically ordered system cannot be defined on the fractal lattice with dimension between 1 and 2. Note that since the fractional spatial dimension in our study is in between 1 and 2, the complex topological structure is considered as embedded in 2D and only 2D topological order need to be considered. In a recent work [17] which tries to extend topological order to fractional dimension, it is proved that $\mathbb{Z}_N$ topological order cannot be realized in fractal lattice systems of dimension between $1D$ and $2D$, which is claimed to be generalized to all 2D topological order. As applied to our case, a complex topological structure which fits the Sierpinski lattice geometry but remains within the framework of TQFT treats the lattice as generated from a regular lattice by punching holes, i.e. removing the vertices as if circulated by the circles defined in Fig. 5(c) and 5(b). Then, the essence of topological order gets contradicted with the fractality which requires the holes to be punched in all scales to realize the self-similarity. That is, string-like operators acting on the logical “nonlocal” degrees of freedom encoded in ground states can only be defined on vertices connecting two neighboring circles, which are compressed into sizes of constant independent on the system size by the fractal patterns of circles.

To summarize, if the paradigm of topological order applies to the Sierpinski lattice qudit systems, then the information shared by local components in the ground state is encoded in between neighboring circles of different sizes and is hence local so that LRE does not exist within the paradigm. Consequently, the entanglement pattern in $|\Psi\rangle$ with LRE rigorously proved cannot exhibit (extended) topological order in fractional dimension. Indeed, the nonlocal information shared by qudits in $|\Psi\rangle$ is encoded in an opposite way, i.e. encoded along each circle, and the self-similar pattern of circles in various sizes eventually entangles all qudits as each qudit is engaged in three circles with different sizes. Therefore, in the perspective that quantum orders are patterns of entanglement, it is sufficient to conclude that $|\Psi\rangle$ exhibits novel quantum order; and due to the coexistence of LRE and fractal self-similarity which has no counterparts in regular geometry with translational symmetry, the novel quantum order is inherent in fractional dimension.

A comprehensive characterization of the novel quantum order requires systematic study on the excitation properties of possibly different Hamiltonians which stabilize $|\Psi\rangle$ as a ground state, i.e., how the LRE in $|\Psi\rangle$ reveals itself in excitation, and is included in a separated work of the author beyond the scope of the present work which focuses on entanglement patterns. The following section will end our discussion by introducing one such Hamiltonian, more like a proposal to effectively simulate qudits in the unique gapped ground state $|\Psi\rangle$ and to shed light on finding experimentally realizable models.

D. Toy model of multi-layer hardcore bosons

In the framework of realizing fixed-point states in models of hardcore bosons as introduced in Eq. 2, we consider a multi-layer system where the occupations of hardcore bosons on the same vertex but in different layers are coupled into the qudit degrees of freedom $\mathbb{C}^4$.

![FIG. 8.](image)

FIG. 8. (a) The multi-layer structure of hardcore boson system. The intralayer hopping, pair creation and the interlayer long-range interaction are indicated by arrows. (b) and (c) illustrates how coupled degrees of freedom in layers of hardcore bosons are mapped to the qudit degrees of freedom in the state $|\psi_m\rangle$.

The difficulty in realizing $|\Psi\rangle$ is that the constraints on circles can be nonlocal as the size of circles increases. To avoid including long-range interaction or multibody interaction on the Sierpinski lattice, we consider each layer as a sublattice of the Sierpinski lattice, i.e., isolated circles in different sizes (see Fig. 8(a) and 8(b)) so that the constraints on circles can be simulated by the fixed parity of particle numbers therein. We take the convention that the size of circles increases from the top layer to the bottom layer (see Fig. 8(a)). The fixed parity is conserved in the quantum fluctuation driven by the intralayer tunneling $a_i^+a_j + a_i^+a_j^+ + c.c.$ with amplitude $-\bar{t}$ on nearest neighbors which only allows particle exchange and creation (annihilation) in pairs (see Fig. 8(a), the blue arrows). The only interaction term
(n_t^s - 1/2)(n_t^s - 1/2)(n_t^s - 1/2) with amplitude 8\psi and summing over all vertices of the Sierpiński lattice in the model is the three-body long-range interaction in the vertical direction on the same vertex but in three layers in order to guarantees that each effective qudit degrees of freedom is \( \mathbb{C}^4 \).

Applying the same perturbation theory as detailed in solving Eq. 2 to the multi-layer model in the strong-interaction limit \( t < v \), we obtain an effective Hamiltonian \( H_{\text{eff}} \) within the subspace \( \mathcal{H}_0 \subset \mathbb{C}^4^\otimes N \) spanned exactly by all \( |\psi_m\rangle \). Here, \( H_{\text{eff}} \) consists of the second order perturbation terms, i.e. terms of the form \( (a_i^{s+}a_j^{s}a_j^{s+} + \text{c.c.})(a_i^{s+}a_j^{s} + a_i^{s}a_j^{s+} + \text{c.c.}) \) can exactly maps to \( -\sum g_i T_{ij} \) in the qudit system where the unitary \( T_{ij} \) terms with eigenvalue 1 and \(-1\) commute with each other and are ergodic in \( \mathcal{H}_0 \). Consequently, \( H_{\text{eff}} \) is gapped and has the unique ground state \( |\Psi\rangle \).

VII. CONCLUSION

We have developed a framework for exploring quantum states with nontrivial entanglement intrinsic to the fractal lattice geometry based on tensor-network representation of entanglement-renormalization fixed points. In our examples of the fixed-point states, the interplay between entanglement and self-similarity can be realized in distinct orders. In an important example, we rigorously proved the long-range entanglement in such a fixed-point state, i.e., the state is short-range correlated but cannot be completely disentangled through constant-depth local quantum circuit. This long-range entanglement pattern is proved to be coexistent with fractal self-similarity and can be viewed as a paradigm in fractional dimension in addition to that as the microscopic picture of topological order in 2D. More importantly, we have shown that such long-range entanglement lies beyond the picture of topological order, even an extended picture in fractional dimension; it rather exhibits novel ordering inherent in fractional dimension and indicates the existence of a new paradigm for describing long-range entanglement. Our results demonstrate that, fractal geometry, or more generally fractional dimension, provide a much larger but much less visited playground for the study of exotic quantum phases of matter.

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Here we view $A$ as a linear map $A : \mathbb{C}^{3D} \to \mathbb{C}^{d}$ which is equivalent to $A$ as a tensor in $\mathbb{C}^{d} \otimes \mathbb{C}^{3D}$.

The constraints determine one single-qudit state from another arbitrarily distant qudit and hence give rise to infinite correlation length.

Then $(W^+)^{\otimes n}$ maps $|\psi\rangle_{\text{odd}} (|\psi\rangle_{\text{even}})$ to $|\psi\rangle_{\text{even}} (|\psi\rangle_{\text{odd}})$, hence maps $|\Psi\rangle = |\psi\rangle_{\text{even}} + |\psi\rangle_{\text{odd}}$ to itself. Note that $|\psi\rangle_{\text{odd}}$ or $|\psi\rangle_{\text{even}}$ does not possess the self-similarity individually.

At finite system size we set $N^2/(N - 3) = 1/2$ as if we exclude the three vertices at the corners since they are not covered by the interaction in $H$. This way, when system size increase to infinity as in thermodynamic limit, we exactly have $N^2/N = 1/2$.

For the model at finite system size, the interaction does not cover the three vertices at the corners of the lattice, and thus the species-$a$ occupation therein are unspecified in $\mathcal{H}_0$ and lead to certain degeneracy in ground states. However, when the system approaches its thermodynamic limit, all vertices participate the interaction, and degenerate ground states are essentially identical. For this reason, we treat those internal degrees of freedom at the corner as fixed at finite system size as implied in $|\psi_\text{m}\rangle$ such that the map $\iota$ indeed maps into a sector corresponding to such fixing.

$A_0$ and $A_1$ link two neighboring blocks; $A_0$ and $A_1$ form the inner side of a block in a larger block. The choice of $A_0$ or $A_1$ at the three corners specifies $|\Psi\rangle$ among the degenerate ground states.

In a connected graph, the distance between two vertices is the length of, i.e. the number of edges (links) in the shortest path connecting the two vertices. The diameter of a connected subset in a graph is the diameter of the subgraph induced by the subset, i.e., the maximal length of the shortest path within the subset which connects a pair of vertices in the subset.

To be consistent with the case in 2D, a local patch of unitary operator of size $P$ in the quantum circuit is defined as a unitary operator supported on a connected subset with diameter $P$, i.e. it acts as the identity operator on qudits outside the support. The $P$ here confines the distance between any pair of vertices within the subset so that the unitary operator acts locally. Then, starting from a single vertex $j$ as the support of $O_j$, the first layer of non-overlapping local unitary operators extends the size of the support at most by $P$ since only one local patch contains $j$. And each of the following layers extends the support by $2P$ since the distance of the farthest apart vertex pair is extended at most by $2P$. Note that the layer structure in the quantum circuit guarantees the connectivity of $J$. 

Here we view $A$ as a linear map $A : \mathbb{C}^{3D} \to \mathbb{C}^{d}$ which is equivalent to $A$ as a tensor in $\mathbb{C}^{d} \otimes \mathbb{C}^{3D}$.