Multi-Bunch Solutions of Differential-Difference Equation for Traffic Flow

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Newell-Whitham type car-following model with hyperbolic tangent optimal velocity function in a one-lane circuit has a finite set of the exact solutions for steady traveling wave, which expressed by elliptic theta function. Each solution of the set describes a density wave with definite number of car-bunches in the circuit. By the numerical simulation, we observe a transition process from a uniform flow to the one-bunch analytic solution, which seems to be an attractor of the system. In the process, the system shows a series of cascade transitions visiting the configurations closely similar to the higher multi-bunch solutions in the set.

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I. INTRODUCTION

Traffic flow in a no-passing freeway has extensively study with the car-following models. Some of these models can describe the spontaneous generation of density wave along the road, the traffic congestion, as the collective motion of cars. The optimal velocity (OV) model reveals the features of the congested flow. In a certain range of the mean headway, cars gradually compose bunch of cars, which moves backward with a constant velocity. The most of the studies are achieved by the numerical simulations under the cyclic boundary condition, which solve the equation of motion of the model;

\[ \ddot{x}_n(t) = a[V(\Delta x_n(t)) - \dot{x}_n(t)]. \]  

\( x_n \) and \( \Delta x_n \) denote the position of the \( n \)-th car and the headway of the car, \( x_{n-1} - x_n \), respectively. Drivers try to adjust their car’s velocity to the optimal velocity \( V(\Delta x) \) according to the headway. The constant \( a \) parameterizes the sensitivity in the adjustment, which should be less than a critical value to generate the congestion.

In the density wave of the congested flow, high density region, the bunch of cars, alternates with low density one. The cars run almost same headway and velocity in each of the regions, and in the interface of the regions the density changes rapidly exhibiting the characteristic kink-like shape. The shape of the kink on the interface seems to be determined only by the sensitivity \( a \) and the OV function \( V(\Delta x) \), irrespective of the initial condition or the mean headway. In the vicinity of the critical point, the interface can be approximately described by the kink solution of the modified Korteweg-de Vries equation. For some particular choices of the OV function, the exact solution which describes the interface has been obtained. However, in the numerical simulations, the generation process of the congestion displays much complicated aspects. A number of bunches with various length may arise in the circuit, and the fusion of bunches is often observed. One can predict neither the number of the bunches nor the lengths of them.

On the other hand, the traditional car-following model, which is described by the first order differential difference equation,

\[ \dot{x}_n(t + \tau) = V(\Delta x_n(t)), \]  

has been studied in the traffic engineering for a long time. The reaction time \( \tau \) represents the time lag which it takes the car to respond to the change of motion. Since the OV model can be considered as the truncated Taylor expansion, \( \dot{x}_n(t + \tau) \approx \dot{x}_n(t) + \tau \ddot{x}_n(t) \), of Eq. (1), these models rather resemble each other in the qualitative behavior, especially in the generation of the steady traveling wave. It is shown numerically that Eq. (2) with a hyperbolic tangent OV function has the congested flow solutions quite similar to ones for the corresponding OV model.

Early in the 90’s, Whitham showed that the model with \( V(\Delta x) \) given by an exponential function has the pulse-like exact solution for steady traveling wave described by the elliptic functions. Although his choice of the function \( V(\Delta x) \) admits no existence of the car-bunching solution mentioned above, he pointed out a crucial relation between the time lag \( \tau \) and the propagating velocity of the traveling wave. (See Eq. (3).) We may call it Whitham condition. Recently, it is demonstrated analytically that a class of the car-bunching solutions represented by the elliptic theta functions exactly satisfies Eq. (3) with the hyperbolic tangent OV function,

\[ V(\Delta x) = \xi + \eta \tanh \left( \frac{\Delta x - \rho}{2\sigma} \right). \]  

The authors of Ref. report that the Whitham condition holds in the numerical simulation for much wider choice of the OV function.

In this paper, we will investigate the structure of the class of the exact traveling wave solutions of Eq. (3). Under the cyclic boundary condition, each solutions corresponds to a periodic density wave with definite length and number of bunches. Generally, some solutions different in the number of bunches are possible. They will be mentioned as the multi-bunch solutions. We also perform...
the numerical simulation and observe the generation process of the density wave which shows a relaxation to one of the analytic solutions.

In the next section, we investigate the parameter space of the exact solutions and find the finite set of the multi-bunch solutions. In section III, allowed parameters for the multi-bunch solution are determined. These parameters will be used in the numerical simulation in section IV. The final section is devoted to summary and discussions. Some mathematical formula for the elliptic functions and the exact solutions are summarized in the appendix.

II. MULTI-BUNCH SOLUTIONS IN A CIRCUIT

We begin by summarizing our previous results \([9]\) of the exact solution with a width parameter \(\delta\):

\[
x_n(t) = Ct - nh + A \ln \frac{\vartheta_0(vt - \frac{\pi}{\lambda} + \delta, q)}{\vartheta_0(vt - \frac{\pi}{\lambda} - \delta, q)},
\]

where \(\vartheta_0(v, q)\) is the elliptic theta function and \(q, A, \lambda, \nu, \delta, C\) and \(h\) are ansatz parameters which characterize the solution. \(q\) is modulus parameter of the theta function. \(h\) is the mean headway of \(N\) cars in the circuit with length \(L \equiv Nh\). The traffic flow expressed by \([1]\) displays alternate appearance of high density region and low density region. Thus, the traffic in the circuit is divided into several number of “bunches”, which move backward. The cyclic boundary condition, \(x_{n+N}(t) = x_n(t) - Nh\), and the periodicity of theta function, \(\vartheta_0(v + 1, q) = \vartheta_0(v, q)\), imply that \(N/\lambda\) must be an integer, which coincides with the number of bunches \(n_b\):

\[
\lambda = \frac{N}{n_b}, \quad (n_b : \text{integer}).
\]

Then, the “wavelength” \(\lambda\) is the number of cars within a successive pair of high and low density region, approximately. The width parameter \(\delta\), which ranges in \(0 < 2\delta < 1\), determines the ratio of low density region in a wavelength. \(2\delta\lambda\) and \((1 - 2\delta)\lambda\) are the approximate number of cars in the low and high density regions, respectively.

The exact solution \([1]\) satisfies the equation of motion \([1]\) with the OV function \([3]\). As discussed in \([1]\), the traffic model with given time lag constant \(\tau\) can admit the exact solution \([1]\) only if the Whitham condition

\[
2\nu\lambda\tau = 1,
\]

is met. Under the condition, we find out four relations between the ansatz parameters and the coefficients of the equation of motion, \(\xi, \eta, \rho\) and \(\sigma\), as stated in the appendix \([14]\)–\([17]\). Among the seven ansatz parameters, the mean headway \(h\) is already determined by the system size \(L\) and \(N\). Furthermore, by the Whitham condition, we find that \(\nu\) is not an independent parameter but proportional to the inverse of \(\lambda\). The remaining five parameters \(q, A, \lambda, \delta\) and \(C\) are subject to the above four relations. It may seem that a parameter remains free and the system have a one-parameter family of the exact solutions. However, we notice that \(\lambda\) is discretized as \([3]\) so that the bunch number \(n_b\) is an integer. Moreover, the number of the bunches cannot exceed the total number of the cars \(N\) at least. (We will see that much stronger restriction exists for the maximum number of the bunches later.) Consequently, the ansatz \([3]\) gives us the finite set of the exact solutions which correspond to the \(n_b\)-bunch states of traffic congestion. We call them the “multi-bunch solutions”.

To examine the multi-bunch solutions, let us precisely analyze the four relations \([14]\)–\([17]\):

\[
\xi = C + \frac{A\beta}{2\tau} \frac{d}{d\beta} \ln \frac{\vartheta_1(2\delta + \beta)}{\vartheta_1(2\delta - \beta)}, \quad (7)
\]

\[
\eta = \frac{A\beta}{2\tau} \frac{d}{d\beta} \ln \frac{\vartheta_1^2(\beta)}{\vartheta_1(2\delta + \beta)\vartheta_1(2\delta - \beta)}, \quad (8)
\]

\[
\rho = h - A \ln \frac{\vartheta_1(2\delta - \beta)}{\vartheta_1(2\delta + \beta)}, \quad (9)
\]

\[
\sigma = A. \quad (10)
\]

Here and hereafter, \(\lambda\) is changed to a much convenient variable, \(\beta\), which defined in the appendix as

\[
\beta = \frac{1}{2\lambda}. \quad (11)
\]

Since the variable is also equal to \(n_b/(2N)\), it will be mentioned as the “bunch parameter”. (Note that, in the expressions \([4]\) and \([8]\), \(\beta/\tau\) is substituted for \(\nu\) by using the Whitham condition.) The equation \([9]\) simply says that \(A\) is identical to \(\sigma\). From the equation \([10]\), we realize that the velocity parameter \(C\) can be expressed as a function of \(q, \delta\) and \(\beta\) (or \(\lambda\)). Thus, the two nontrivial relations \([8]\) and \([9]\) should be solved to establish the existence of the multi-bunch solutions and to find out the allowed region of the parameters.

First, we treat the equation \([4]\), whose variables in the theta function can be decomposed by using the addition formula as shown in the appendix. Replacing \(\vartheta_1(v)/\vartheta_0(v)\) by the Jacobi’s elliptic function \(\sqrt{k}\, \text{sn} 2K\nu\) in \([19]\), it becomes

\[
\frac{\tau}{\tau_c} = \frac{\beta}{2} \frac{d}{d\beta} \ln \left(\frac{1}{\text{sn}^2 2K\beta} - \frac{1}{\text{sn}^2 4K\delta}\right), \quad (12)
\]

where \(A/\eta = (\sigma/\eta)\) is denoted by \(\tau_c\) because of the following reason. A linear analysis \([11]\) gives the instability condition for a uniform flow described by \(x_n^{(0)}(t) = V(h)t - nh\),

\[
2\tau V'(h) \equiv \frac{\eta}{\sigma} \text{sech}^2 \left[\frac{h - \rho}{2\sigma}\right] > \frac{\pi/N}{\sin \pi/N}. \quad (13)
\]

(Note that the right hand side of \([13]\) is almost equal to one, providing that \(N\) is not too small.) It follows...
that $\eta/\sigma$, the maximum value of $2V'(h)$, determines the minimum value of time lag, $\tau_c$, for which the uniform flow becomes linearly unstable. One expects that when the condition,

$$\tau > \tau_c \equiv \frac{\eta}{\sigma}, \quad (14)$$

is satisfied, there exist the uniform flow which decays to develop a congested flow. The $\tau_c$ may be referred to as the critical time lag. By performing $\beta$-derivative, the equation (12) can be easily solved with respect to $\text{sn}4K\delta$ and gives

$$\text{sn}^24K\delta = \frac{\text{sn}^22K\beta}{1 - \frac{\tau_c}{\tau} \frac{2K\beta \text{cn}2K\beta \text{dn}2K\beta}{\text{sn}2K\beta}}. \quad (15)$$

Since $\text{sn}^24K\delta \leq 1$, the right hand side of the above equation should not exceed 1. Thus, we obtain the solvability condition, which guarantees the existence of $\delta$, as

$$\frac{\tau_c}{\tau} \leq \frac{\text{sn}2K\beta \text{cn}2K\beta}{2\beta \text{dn}2K\beta}. \quad (16)$$

The inequality gives us the restriction for the number of bunches for given $\tau$. The right hand side of it only depends on $\beta$ and $q$, where the complete elliptic integral of the first kind, $K$, and the modulus of Jacobi’s elliptic functions, $k$, are given in (A3) as the function of $q$. In FIG. 1, we show the allowed region of (16) in the $(\beta, q)$-plane for $\tau_c/\tau = 0.85869$. Note here that $\beta(= n_b/(2N))$ ranges in $0 < \beta < 1/2$, since $n_b < N$. It can be easily checked that the allowed region actually disappears for $\tau < \tau_c$, which agrees with the linear analysis $[14]$. The boundary curve in FIG. 1 crosses the $\beta$-axis at $\beta_0$, the maximum value of $\beta$. It is given by a solution of the equation obtained in the $q \to 0$ ($k \to 0$, $K \to \pi/2$) limit of the equality of (14):

$$\frac{\tau_c}{\tau} = \lim_{q \to 0} \frac{\text{sn}2K\beta_0 \text{cn}2K\beta_0}{2K\beta_0 \text{dn}2K\beta_0} = \frac{\sin 2\pi\beta_0}{2\pi\beta_0}. \quad (17)$$

Thus, $\beta_0$ is determined only by the value of $\tau_c/\tau$. The existence of the nontrivial upper bound of $\beta$ implies that the number of the bunches is also restricted much more than $n_b < N$. The maximum value of the number of the bunches $n_b^{\text{max}}$ is given by

$$n_b^{\text{max}} = [2N\beta_0], \quad (18)$$

where $[x]$ is the maximal integer that does not exceed $x$. Consequently, when the time lag $\tau$ and the total car number $N$ are given, the system has $n_b^{\text{max}}$ exact multi-bunch solutions with the bunch parameters

$$\beta = \frac{1}{2N}, \frac{2}{2N}, \ldots, \frac{n_b}{2N}, \ldots, \frac{n_b^{\text{max}}}{2N}. \quad (19)$$

When we specify one of the possible bunch numbers or $\beta$’s, the solvability condition (14) gives us an allowed range of modulus parameter $q$,

$$0 \leq q \leq q_{\text{max}}, \quad (20)$$

where $q_{\text{max}}$ is the value such that the equality of the solvability condition is held for the selected $\beta$. The vertical dashed lines in FIG. 1 show the ranges of $q$ for some possible $\beta$ with $N = 20$, where $n_b^{\text{max}} = 5$.

### III. CONSTRUCTION OF THE MULTI-BUNCH SOLUTION

In this section, we determine the width parameter $\delta$ and the modulus parameter $q$ for any possible $n_b$ (or $\beta$) to construct the $n_b$-bunch solution. In the allowed range $[24]$, the equation (15) has two branches of $\delta$ as functions of $q$. One, which stays in $0 < 2\delta_- < 1/2$, is given by

$$2\delta_-(q) = \frac{1}{2K} \text{sn}^{-1} \frac{\text{sn}2K\beta}{\sqrt{1 - \frac{\tau_c}{\tau} \frac{2K\beta \text{cn}2K\beta \text{dn}2K\beta}{\text{sn}2K\beta}}} \quad (21)$$

where we take the branch $0 < \text{sn}^{-1} < K$ for the inverse Jacobi’s function $\text{sn}^{-1}$. The other branch $1/2 < 2\delta_+ < 1$, which corresponds to $K < \text{sn}^{-1} < 2K$, is obtained as $2\delta_+(q) \equiv 1 - 2\delta_-(q)$. These two branches are connected each other at $q = q_{\text{max}}$ with $2\delta = 1/2$ as depicted in FIG. 2. On $2\delta = 1/2$, the low and high density regions in the traffic flow contains same number of cars. Eq. (11) tells
us that such a “symmetric flow” can be yielded when the mean headway \( h \) is equal to \( \rho \), the inflection point of the OV function, since \( \vartheta_1(1/2 + \beta) = \vartheta_1(1/2 - \beta) \) regardless of \( \beta \) or \( q \). As long as \( 2\delta \neq 1/2 \), the traffic flow becomes asymmetric. For the first branch \( 0 < 2\delta_+ < 1/2 \), the low density region contains more cars than high density one and the mean headway \( h \) is less than \( \rho \). Contrary, for the second branch \( 1/2 < 2\delta_+ < 1 \), the congested region has more cars and \( h \) exceeds \( \rho \).

\[
2\delta(q) = \rho + \sigma \ln \frac{\vartheta_1(2\delta(q) - \beta, q)}{\vartheta_1(2\delta(q) + \beta, q)},
\]

where \( h_-(< \rho) \) and \( h_+(> \rho) \) correspond to the two branches of \( \delta \). (Note here that \( h_-(q) + h_+(q) = 2\rho \).) The combined entire function \( h(q) \) is shown in FIG. 3. We can determine the modulus parameter \( q \), within a certain range of mean headway \( L/N \), by solving \( h(q) = L/N \), and obtain a exact multi-bunch solution of (2) by calculating \( \delta \) and other parameters through the equations above.

In some case, we may find several solutions for \( q \), which correspond to same or different bunch number. Although the exact solution can construct for each \( q \), the stability of these solutions is other problem. We will discuss the issue in another work [12]. In FIG. 3, the dashed lines indicate the linearly unstable region of the (common) headway of the uniform flow \( x_n^{(0)}(t) \);

\[
|h - \rho| < 2\sigma \text{Arccosh} \sqrt{\frac{\pi \sin \pi/N}{\tau_c \pi/N}}.
\]

Within the range, we may expect that the unstable uniform flow will grow into one of the analytic solutions. However, the generation process of the multi-bunch solution from a given initial configuration and how one of the possible analytic solutions is selected may be more complicated problem.

\[
h(q)
\]

\[
\tau_c = 85869, \quad \rho = 2, \quad \sigma = 1/2 \text{ with } N = 20. \quad \text{Each curve corresponds to the possible bunch number } n_b \text{ in FIG. 1 and 2. The horizontal dashed lines indicate the critical values of headway for the linear instability, which coincide with the } q \rightarrow 0 \text{ limits of } h(q) \text{ for } n_b = 1.
\]

**IV. NUMERICAL SIMULATIONS**

To investigate the generation process of the multi-bunch solution from a uniform flow, we perform numerical simulations by solving the differential-difference equation (2). We adopt an OV function

\[
V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2,
\]

where the coefficients are chosen as \( \xi = \tanh 2, \quad \eta = 1, \quad \rho = 2 \) and \( \sigma = 0.5 \) (\( \tau_c = 0.5 \)). The time lag \( \tau = 0.58828 \) gives \( \tau_c/\tau = 0.85869 < 1 \). We prepare 20 cars and arrange them so that they form a uniform flow with a common headway \( h = 1.88571 \) and an initial velocity \( V(h) = 0.850233 \) in a circuit whose length is \( L = 37.7142 \). The uniform flow is sustained for a duration of \( \tau \) to prepare the initial function of the differential-difference equation. The solvability condition (14) tells us that maximally five-bunch mode is allowed. The modulus parameters \( q \) for the possible bunch numbers \( n_b = 1, 2, 3, 4, 5 \) are 0.70792140328755, 0.50113376, 0.3536167, 0.2418044, 0.140292, respectively. Since the uniform flow is linearly unstable, it starts developing the density wave. In this case, the one-bunch mode is generated after enough relaxation time, \( t \simeq 6 \times 10^4 \), ultimately. The result, which is displayed in FIG. 4 by dots, agrees quite well with the analytic one-bunch solution with \( q = 0.70792140328755 \) as shown by the thin line in the figure.
FIG. 4. The result of a numerical simulation with $N = 20$, $\tau_c/\tau = 0.85869$, $V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2$ after Dots show the headway of a car. The line shows the corresponding one-bunch solution with $q = 0.70792140328755$.

We observe much interesting phenomena in the generation process of the analytic exact solution. At first, until $t \approx 300$, the initial uniform flow grows into a three-bunch configuration which closely resembles the exact three-bunch solution. Although the configuration keeps its shape for a while, it is gradually distorting and one of the bunches get closer to the other. At $t \approx 4680$, the fusion of bunches occurs and it is transformed into a two-bunch configuration. The two-bunch configuration lives about ten times longer than the three-bunch one. Subsequently, a bunch starts shrinking. The bunch is absorbed by the other at $t \approx 51560$ as shown in Fig. 5, and then the exact one-bunch solution is finally accomplished. The simulation has been continued until $t = 2 \times 10^5$, at which the one-bunch configuration has still survived. These phenomena suggest that the one-bunch solution is an attractor of the system [13].

FIG. 5. The fusion process at $t \approx 51560$. A bunch starting to shrink at $t \approx 48000$ is absorbed by the other

V. SUMMARY AND DISCUSSIONS

We find a finite set of exact multi-bunch solutions for a car-following model with driver’s reaction time lag in a circuit, which is described by a differential-difference equation under the cyclic boundary condition. When the circuit length and the total number of cars are given, we can calculate the possible number of car-bunches in the circuit and the profile of the density wave which describes the car-bunching. The numerical simulation shows that the linearly unstable uniform flow develops into the one-bunch solution, which seems to be an attractor of the system. In the relaxation process from the uniform flow to the one-bunch solution, the system stays for a while on the configurations which are similar to the higher multi-bunch solutions. For smaller number of bunches, the durations of these quasi stable configurations become longer exponentially (See Fig. 1 in [13]).

The series of cascade transitions among the multi-bunch configurations suggests that the multi-bunch solutions correspond to heteroclinic points of the system. However, any flow out of the one-bunch solution does not observed up to the present. As the other possibility, each of the multi-bunch solution may be a kind of the Milnor attractor [14], which is unstable for any small perturbations, but globally attracts the orbits. To see the problem, we are going to have to investigate the stability and the attracting domain of the multi-bunch solutions more precisely.

The density wave formation from a uniform flow on the OV model is very similar to the present model. The car-bunching and the fusion of bunches are observed also in the OV model. It may be possible to expect that the existence of the multi-bunch solutions and the qualitative feature of the cascade transitions are common to both models, although the analytic solutions for the OV model have not been obtained.

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APPENDIX A: DEFINITIONS AND FORMULA OF ELLIPTIC FUNCTIONS

The definitions of elliptic functions and some mathematical formula used in this paper are listed. The elliptic theta functions are defined as the infinite products;
\[ \vartheta_0(v, q) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi v + q^{4n-2}), \]

\[ \vartheta_1(v, q) = 2q^{1/4} q_0 \sin \pi v \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi v + q^{4n}), \]

where \( q_0 = \prod_{n=1}^{\infty} (1 - q^{2n}). \) \( (A1) \)

The addition formula of the theta functions are:

\[ \vartheta_0(v + w) \vartheta_0(v - w) \vartheta_0^2(0) = \vartheta_0^2(v) \vartheta_0^2(w) - \vartheta_1^2(v) \vartheta_1^2(w), \]
\[ \vartheta_1(v + w) \vartheta_1(v - w) \vartheta_0^2(0) = \vartheta_1^2(v) \vartheta_0^2(w) - \vartheta_0^2(v) \vartheta_1^2(w). \]

\( (A2) \)

Jacobi's elliptic functions and their modulus \( k \) and complementary modulus \( k' \) can be expressed by the theta functions:

\[ k = \frac{\vartheta_2^2(0, q)}{\vartheta_3^2(0, q)}, \quad k' = \frac{\vartheta_3^2(0, q)}{\vartheta_2^2(0, q)}, \quad K = \frac{\pi}{2} \vartheta_3^2(0, q), \]

\[ \text{sn}2Kv = \frac{1}{\sqrt{k}} \frac{\vartheta_1(v)}{\vartheta_0(v)}, \quad \text{cn}2Kv = \sqrt{1 - \text{sn}2Kv^2}, \]

\[ \text{dn}2Kv = \sqrt{1 - k^2 \text{sn}2Kv^2}, \]

where \( K \) is the complete elliptic integral of the first kind.

**APPENDIX B: THETA FUNCTION FORMALISM**

Let us solve the equation of motion,

\[ \dot{x}_n(t + \tau) = \xi + \eta \tanh \left[ \frac{\Delta x_n(t) - \rho}{2\sigma} \right], \]

by the functional ansatz,

\[ x_n(t) = Ct - nh + A \ln \frac{\vartheta_0(v + \beta - \delta)}{\vartheta_0(v - \beta - \delta)}, \]

using the variable notations,

\[ v = \nu t - \frac{n}{\chi}, \quad \beta = \frac{1}{2\lambda}. \]

By virtue of the Whitham condition \( \nu \tau = \beta \), the velocity with time lag \( \tau \), \( \dot{x}_n(t + \tau) \), can be written as

\[ \dot{x}_n(t + \tau) = C + A\nu \frac{d}{dv} \ln \frac{\vartheta_0(v + \delta)}{\vartheta_0(v - \delta)}, \]

where we replace the time derivative \( d/dt \) with \( \nu d/dv \). Converting the \( v \)-derivative into \( d/d\delta \) again, we can apply the addition formula \( (A2) \) to the expression;

\[ \frac{d}{dv} \ln \frac{\vartheta_0(v + \delta)}{\vartheta_0(v - \delta)} = \frac{d}{d\delta} \ln \left[ \vartheta_0(v + \delta) \vartheta_0(v - \delta) \right] = \frac{d}{d\delta} \ln \left[ \vartheta_2^2(v) \vartheta_0^2(\delta) - \vartheta_1^2(v) \vartheta_1^2(\delta) \right]. \]

\( (B5) \)

It allows the velocity to be a rational expression of \( \vartheta_0^2(v) \) and \( \vartheta_1^2(v) \):

\[ \dot{x}_n(t + \tau) = C + A\nu \frac{\vartheta_0^2(v) \vartheta_0^2(\delta) - \vartheta_0^2(v) \vartheta_1^2(\delta)}{\vartheta_0^2(v) \vartheta_0^2(\delta) - \vartheta_1^2(v) \vartheta_1^2(\delta)} \left[ \vartheta_0^2(v) \vartheta_0^2(\delta) - \vartheta_0^2(v) \vartheta_1^2(\delta) \right] . \]

\( (B6) \)

We find that the headway,

\[ \Delta x_n(t) = h + A \ln \frac{\vartheta_0(v + \beta + \delta)}{\vartheta_0(v + \beta - \delta)}, \]

can also be expressed as a rational form of \( \vartheta_0^2(v) \) and \( \vartheta_1^2(v) \):

\[ e^{2X_n} = \frac{\vartheta_0^2(v) \vartheta_0^2(\delta + \beta) - \vartheta_0^2(v) \vartheta_1^2(\delta + \beta)}{\vartheta_0^2(v) \vartheta_0^2(\delta - \beta) - \vartheta_1^2(v) \vartheta_1^2(\delta - \beta)} . \]

\( (B8) \)

where

\[ X_n = \frac{\Delta x_n(t) - h}{2A} . \]

\( (B9) \)

By eliminating \( \vartheta_0^2(v) \) and \( \vartheta_1^2(v) \) from \( (B6) \) and \( (B8) \), it will be shown that the velocity \( \dot{x}_n(t + \tau) \) is equal to a first order rational expression of \( e^{2X_n} \), which can be rewritten as an hyperbolic tangent function of the headway \( \Delta x_n(t) \).

Performing the elimination, we get a differential-difference equation which the ansatz \( (B2) \) satisfies;

\[ \dot{x}_n(t + \tau) = C + A\nu \frac{N_+ + N_-e^{2X_n}}{D_+ + D_-e^{2X_n}} , \]

\( (B10) \)

where

\[ D_\pm = \vartheta_0^2(0) \vartheta_1(\beta) \vartheta_1(2\delta \pm \beta), \]

\[ N_\pm = \pm \left[ \vartheta_1^2(\delta \pm \beta)(\vartheta_0^2(\delta) - \vartheta_0^2(\delta \pm \beta)(\vartheta_1^2(\delta)) \right] . \]

\( (B11) \)

We can transform \( N_\pm \) into the expressions in terms of \( \beta \)-derivatives, instead of \( \delta \) ones;

\[ N_\pm = \left( \pm \frac{d}{d\delta} - \frac{d}{d\beta} \right) \left[ \vartheta_1^2(\delta \pm \beta)(\vartheta_0^2(\delta) - \vartheta_0^2(\delta \pm \beta)(\vartheta_1^2(\delta)) \right] . \]

\( (B12) \)

As mentioned above, the differential-difference equation can be written by using the hyperbolic tangent function as

\[ \dot{x}_n(t + \tau) = C + A\nu \left( \frac{N_-}{D_-} - \frac{N_+}{D_+} \right) \tanh \left( X_n + \frac{1}{2} \ln \frac{D_-}{D_+} \right) . \]

\( (B13) \)

Thus, we find the equations to which the ansatz parameters are subject, by comparing the above equation with the equation of motion \( (B1) \):

\[ \dot{x}_n(t + \tau) = C + A\nu \frac{N_- + N_+}{D_- + D_+} . \]
\[ \xi = C + \frac{A \nu}{2} \frac{d}{d \beta} \ln \frac{\vartheta_1(2\delta + \beta)}{\vartheta_1(2\delta - \beta)}, \quad (B14) \]

\[ \eta = \frac{A \nu}{2} \frac{d}{d \beta} \ln \frac{\vartheta_1^2(\beta)}{\vartheta_1(2\delta + \beta) \vartheta_1(2\delta - \beta)}, \quad (B15) \]

\[ \rho = h - A \ln \frac{\vartheta_1(2\delta + \beta)}{\vartheta_1(2\delta - \beta)}, \quad (B16) \]

\[ \sigma = A. \quad (B17) \]

In the first two equations, we can decompose the variables \( \delta, \beta \) in the theta functions by the addition formula;

\[ \xi = C + \frac{A \nu}{2} \frac{d}{d \delta} \ln \vartheta_0(2\delta) \]
\[ + \frac{A \nu}{4} \frac{d}{d \delta} \ln \left( \frac{\vartheta_1^2(2\delta)}{\vartheta_0^2(2\delta)} - \frac{\vartheta_1^2(\beta)}{\vartheta_0^2(\beta)} \right), \quad (B18) \]

\[ \eta = -\frac{A \nu}{2} \frac{d}{d \beta} \ln \left( \frac{\vartheta_1^2(\beta)}{\vartheta_1^2(2\delta)} - \frac{\vartheta_0^2(2\delta)}{\vartheta_0^2(\beta)} \right). \quad (B19) \]

For the first one, \( d/d\beta \) is converted into \( d/d\delta \), to apply the addition formula, before the decomposition. All of the equations \((B14)-(B17)\) can be shown that they are equivalent to the ones which we found in the previous paper \([9]\), although they are quite different in the appearances.

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