Nagata embedding and $\mathcal{A}$-schemes

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Abstract

We define the notion of normal $\mathcal{A}$-schemes, and approximable $\mathcal{A}$-schemes. Approximable $\mathcal{A}$-schemes inherit many good properties of ordinary schemes. As a consequence, we see that the Zariski-Riemann space can be regarded in two ways – either as the limit space of admissible blow ups, or as the universal compactification of a given non-proper scheme. We can prove Nagata embedding using Zariski-Riemann spaces.

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0 Introduction

We introduced the concept of $\mathcal{A}$-schemes in [T]. In this paper, we will investigate further properties of $\mathcal{A}$-schemes, mainly focusing on Zariski-Riemann spaces.
First, we will show that there is a normalization of $\mathcal{A}$-schemes, just as for ordinary schemes. This is important, since we are aiming for an analog of Zariski’s main theorem.

One of the advantage of introducing $\mathcal{A}$-schemes is that we can simplify the proof of Nagata embedding theorem; it can be proven intuitively, as in the original paper of Nagata [N1]. Note that the essential part is already proven in Corollary 4.4.6 of [T]. Compare with the proof of Conrad [C], which only uses ordinary schemes, but is long (approximately 50 pages).

Also, we introduce the notion of approximable $\mathcal{A}$-schemes: an $\mathcal{A}$-scheme is approximable if it is a (filtered) projective limit of ordinary schemes. This notion is convenient, since locally free sheaves on approximable schemes always come from a pull back of a locally free sheaves on ordinary schemes. At the same time, we see that the Zariski-Riemann space defined in [T] is identified with the conventional one, namely the limit space of $U$-admissible blow ups along the exceptional locus $X \setminus U$ where $U$ is an open subscheme of a scheme $X$. This shows that the conventional Zariski-Riemann space has the desirable universal property in the category of $\mathcal{A}$-schemes, not only with schemes.

This paper is organized as follows. In section 1, we quickly review the definitions and properties of $\mathcal{A}$-schemes, which plays the central role in this paper. In section 2, we construct the normalization functor of $\mathcal{A}$-schemes. In section 3, we give a notion of approximable $\mathcal{A}$-schemes. This actually determines the complete hull of the category $(\mathcal{Q}\text{-Sch})$ of ordinary schemes in the category $(\mathcal{A}\text{-Sch})$ of $\mathcal{A}$-schemes, namely the smallest complete full subcategory of $(\mathcal{A}\text{-Sch})$ containing $(\mathcal{Q}\text{-Sch})$. In section 4, we will give a proof of the original version of Nagata embedding, which says that any separated scheme of finite type can be embedded as an open subscheme of a proper scheme.

**Notation and conventions.** In this paper, the algebraic type $\mathcal{A}$ is always that of rings when we talk of $\mathcal{A}$-schemes. When we say ordinary schemes, we treat only coherent schemes and quasi-compact morphisms between them; to emphasize this assumption and to distinguish ordinary coherent schemes from $\mathcal{A}$-schemes, we will say $\mathcal{Q}$-schemes instead of coherent schemes.

For an $\mathcal{A}$-scheme $X$, the description $|X|$ stands for the underlying topological space, which is coherent.

An $\mathcal{A}$-scheme $X$ will be called *integral* if it is irreducible and reduced. This condition is in fact, stronger than assuming any section ring $\mathcal{O}_X(U)$ is
A morphism of \( \mathcal{S} \)-schemes is **proper**, if it is separated and universally closed. We do not include the condition “of finite type”.

An open covering of an \( \mathcal{S} \)-scheme \( X \) is denoted by \( \{ \bigcup U_{ijk} \Rightarrow igcup U_i \} \); here, \( \{ U_i \} \) is a quasi-compact open covering of \( X \), and \( \{ U_{ijk} \} \) is a quasi-compact open covering of \( U_i \cap U_j \) for each \( i, j \). Hence, there are open immersions \( U_{ijk} \rightarrow U_i \) and \( U_{ijk} \rightarrow U_j \) for each \( i, j, k \). Moreover, if \( X \) is a \( \mathcal{D} \)-scheme, we usually take \( U_i \) and \( U_{ijk} \) as open affine subschemes of \( X \).

### 1 A brief review of \( \mathcal{S} \)-schemes

In this section, we will recall some terminologies and definitions in [T]. A good reference for general lattice theories is [S].

A topological space \( X \) is **coherent**, if it is sober, quasi-compact, quasi-separated, and has a quasi-compact open basis.

The category \((\text{Coh})\) of coherent spaces and quasi-compact morphisms is isomorphic to the opposite category \((\text{DLat})^{\text{op}}\) of distributive lattices by the functor \( C(\mathcal{S})^{\text{cpt}} \); for a coherent space \( X \), we may regard \( C(X)^{\text{cpt}} \) as the set of quasi-compact open subsets of \( X \), or the set of their complements. A quasi-compact morphism \( f : X \rightarrow Y \) induces a morphism \( f^{-1} : C(Y)^{\text{cpt}} \rightarrow C(X)^{\text{cpt}} \) of lattices.

Therefore, for any algebraic type \( \mathcal{S} \), we can regard an \( \mathcal{S} \)-valued sheaf on a coherent space \( X \) as a continuous covariant functor \( C(X)^{\text{cpt}} \rightarrow \mathcal{S} \).

On a coherent space \( X \), there is a canonical \( (\text{DLat}) \)-valued sheaf \( \tau_X \) on \( X \), which is defined by \( U \mapsto C(U)^{\text{cpt}} \) for quasi-compact open subsets of \( X \), or the entire Zariski site of \( X \).

We have a functor \( \alpha_1 : (\text{Rng}) \rightarrow (\text{Dlat}) \) from the category of commutative rings, which sends a ring \( R \) to the set of finitely generated ideals of \( R \) modulo the relation \( I^2 = I \). Note that this gives the usual spectrum of rings, when combined with the previous isomorphism \( C(\mathcal{S})^{\text{cpt}} \).

Also, we have a natural homomorphism \( R \rightarrow \alpha_1(R) \) of multiplicative monoids, sending \( a \in R \) to the principal ideal generated by \( a \). This homomorphism commutes with localizations.

An \( \mathcal{S} \)-scheme is a triple \( X = (|X|, \mathcal{O}_X, \beta_X) \), where

(i) \( |X| \) is a coherent space (the “underlying space”),

(ii) \( \mathcal{O}_X \) is a ring-valued sheaf on \( |X| \) (the “structure sheaf”), and
(iii) $\beta_X : \alpha_1 \mathcal{O}_X \to \tau_X$ is a morphism of $(\text{DLat})$-valued sheaves (the “support morphism”),

which satisfies the following condition: for an inclusion $V \hookrightarrow U$ of quasi-compact open subsets of $|X|$, the restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ factors through $\mathcal{O}_X(U)_Z$, where $\mathcal{O}_X(U)_Z$ is the localization along the multiplicative system

$$\{ a \in \mathcal{O}_X(U) \mid \beta_X(a) \subset Z \},$$

where $Z = U \setminus V$ is the complement closed subset of $V$ in $U$. By this property, $\mathcal{A}$-schemes are locally ringed spaces. A morphism of $\mathcal{A}$-schemes $f = (f, f^\#) : X \to Y$ is a morphism of ringed spaces, which commutes with the support morphism:

$$\begin{array}{ccc}
\alpha_1 \mathcal{O}_Y & \xrightarrow{\alpha_1 f^\#} & f^* \alpha_1 \mathcal{O}_X \\
\beta_Y \downarrow & & \downarrow \beta_X \\
\tau_Y & \xrightarrow{f_*} & f_* \tau_X
\end{array}$$

The category $(\mathcal{A}-\text{Sch})$ of $\mathcal{A}$-schemes is complete, and co-complete.

We have a fully faithful functor $(\mathcal{B}-\text{Sch}) \to (\mathcal{A}-\text{Sch})$, which preserves pull backs and finite patchings by quasi-compact opens.

Let Spec $K \to S$ be a dominant morphism of $\mathcal{A}$-schemes, where $S$ is integral and $K$ is a field. The Zariski-Riemann space $\text{ZR}^f(K, S)$ is a proper $\mathcal{A}$-scheme over $S$, defined as follows: the points of the underlying space $|\text{ZR}^f(K, S)|$ corresponds to the set of dominant morphisms $\mathcal{O}_{S,s} \to R$ of local rings, where $R$’s are valuation rings of $K$, and $\mathcal{O}_{S,s}$’ are local rings of $S$. The map $|\text{ZR}^f(K, S)| \to |S|$ is defined naturally. The topology of $|\text{ZR}^f(K, S)|$ is generated by the domains $\{ R \in |\text{ZR}^f(K, S)| \mid a \in R \}$ for $a \in K \setminus \{0\}$ and the inverse images of the open subsets of $S$. For a quasi-compact open subset $U$ of $|\text{ZR}^f(K, S)|$, $\mathcal{O}_{\text{ZR}^f(K, S)}(U)$ is the ring which is the intersection of all the valuation rings corresponding to the points in $U$.

For arbitrary dominant morphism $X \to S$ of integral $\mathcal{A}$-schemes, $\text{ZR}^f(X, S)$ is defined by the pushout of $X \leftarrow \text{ZR}^f(K, X) \to \text{ZR}^f(K, S)$, where $K = Q(X)$ is the rational function field of $X$. $X$ is of profinite type (resp. strongly of profinite type) over $S$, if the canonical map $\text{ZR}^f(K, X) \to \text{ZR}^f(K, S)$ is a scheme-theoretic immersion (resp. open immersion).

Here, we would like to list up some important properties which is going to be used in the sequel:
A separated, of finite type morphism of $\mathcal{D}$-schemes is always strongly of profinite type.

The map $X \mapsto \mathcal{R}^f(X, S)$ gives the universal proper $\mathcal{A}$-scheme of profinite type over $S$ for each integral $\mathcal{A}$-scheme $X$ of profinite type over $S$.

2 Normalization

In this section, we fix an integral base $\mathcal{A}$-scheme $S$, and any $\mathcal{A}$-scheme is integral, of profinite type over $S$. We denote by $(\text{Int. } \mathcal{A}\text{-Sch})$ the category of integral $\mathcal{A}$-schemes of profinite type over $S$ and dominant morphisms.

Definition 2.1. An $\mathcal{A}$-scheme $X$ is normal, if the ring of every stalk $\mathcal{O}_{X,x}$ is integrally closed.

Remark 2.2. We do not assume Noetherian property on normal rings (or schemes) in this paper.

Theorem 2.3. Let $(\text{N. } \mathcal{A}\text{-Sch})$ be the full subcategory of $(\text{Int. } \mathcal{A}\text{-Sch})$, consisting of normal schemes, and $U : (\text{N. } \mathcal{A}\text{-Sch}) \to (\text{Int. } \mathcal{A}\text{-Sch})$ be the underlying functor. Then, $U$ has a right adjoint ‘nor’. Moreover, the counit $\eta : U \circ \text{nor} \Rightarrow \text{Id}$ is proper dominant.

We will refer to this right adjoint as the normalization functor.

Proof. The proof is somewhat long, so we will divide it into several steps. The construction of the normalization functor is analogous to that of Zariski-Riemann spaces, described in detail in [T]. We will denote by $R^{\text{nor}}$ the integral closure of a given integral domain $R$ in the sequel.

Step 1: First, we will construct the underlying space of the normalization of a given integral $\mathcal{A}$-scheme $X$. Let $\mathcal{N}_0^X$ be the set of finite sets of pairs $(U, \alpha)$, where

(a) $U$ is a quasi-compact open subset of $X$, and

(b) $\alpha \in \mathcal{O}_X(U)^{\text{nor}} \setminus \{0\}$.
Let \( a = \{(U_i, \alpha_i)\}_i, b = \{(V_j, \beta_j)\}_j \) be two elements of \( N^X_0 \). We define two operations \(+, \cdot\) on \( N^X_0 \) by
\[
a + b = a \cup b, \quad a \cdot b = \{(U_i \cap V_j, \alpha_i \beta_j)\}_{ij}
\]
For a pair \((U, \alpha)\), define \( U[\alpha] \) as \( U[\alpha] = \{x \in U \mid x \text{ is in the image of } \text{Spec } O_{X,x}[\alpha^{-1}] \to \text{Spec } O_{X,x}\} \).

For two elements \( a = \{(U_i, \alpha_i)\}_i, b = \{(V_j, \beta_j)\}_j \), the relation \( a \prec b \) holds if
(a) \( U_i[\alpha_i] \subset \cup_j V_j[\beta_j] \) for any \( i \), and
(b) For any \( x \in U_i[\alpha_i] \), set \( J_x = \{j \mid x \in V_j[\beta_j]\} \). Then \( (\beta_j)_{j \in J_x} \) generates the unit ideal in \( O_{X,x}[\alpha_i^{-1}] \).

Let \( \approx \) be the equivalence relation generated by \( \prec \), and set \( \mathcal{N}^X = N^X_0/\approx \). The addition and multiplication of \( N^X_0 \) descends to \( \mathcal{N}^X \), which makes \( \mathcal{N}^X \) into a distributive lattice. Set \( |X^\text{nor}| = \text{Spec } \mathcal{N}^X \).

This is the underlying space of the normalization \( X^\text{nor} \).

Step 2: There is a natural homomorphism \( C(X)_{\text{cpt}} \to \mathcal{N}^X \) of distributive lattices, defined by \( Z \mapsto \{(Z, 1)\} \). This defines a quasi-compact morphism \( \pi : |X^\text{nor}| \to |X| \) of coherent spaces.

Step 3: Let \( p \) be a point of \( |X^\text{nor}| \), and set \( x = \pi(p) \). Then,
\[
p = \{a \in O_{X,x}^\text{nor} \mid (X, a) \leq p\}
\]
becomes a prime ideal of \( O_{X,x}^\text{nor} \). Let \( R_p \) be the localization of \( O_{X,x}^\text{nor} \) by \( p \). Then, \( R_p \) dominates \( O_{X,x} \).

Step 4: The structure sheaf \( O_{X^\text{nor}} \) is defined by
\[
U \mapsto \{a \in K \mid a \in R_p \quad (p \in U)\},
\]
where \( K \) is the function field of \( X \). The support morphism \( \beta_{X^\text{nor}} : \alpha_1 O_{X^\text{nor}} \to \tau_{X^\text{nor}} \) is defined by
\[
\alpha_1 O_{X^\text{nor}}(U) \ni (a_1, \ldots, a_n) \mapsto \{(U, a_i)\}_i.
\]
This defines an \( \mathcal{A} \)-scheme \( X^\text{nor} = (|X^\text{nor}|, O_{X^\text{nor}}, \beta_{X^\text{nor}}) \).
Step 5: We have a canonical morphism of sheaves \( \mathcal{O}_X \to \pi_* \mathcal{O}_{X_{\text{nor}}} \), defined by the identity \( a \mapsto a \). This yields a morphism \( \pi : X_{\text{nor}} \to X \) of \( \mathcal{A} \)-schemes. It is of profinite type, by the criterion 4.3.3 in [T].

Step 6: Let us show that \( \pi \) is proper.

We can see from the construction that we have a natural morphism \( ZR^f(K, X) \to X_{\text{nor}} \): the morphism \( |ZR^f(K, X)| \to |X_{\text{nor}}| \) of underlying spaces is defined by

\[
\mathcal{N}^X \to \mathcal{M}^X \quad \{(U_i, \alpha_i)\}_i \mapsto \{(X \setminus U_i, \{\alpha_i^{-1}\})\}_i,
\]

where \( \mathcal{M}^X = C(ZR^f(K, X))_{\text{cpt}} \), and the morphism between the structure sheaves is canonical. Note that \( ZR^f(K, X) \) is already normal. This shows that \( X_{\text{nor}} \) is proper over \( X \) by the valuative criterion.

Step 7: We will show that the normalization is a functor. Let \( f : X \to Y \) be a dominant morphism of \( \mathcal{A} \)-schemes. \( |f_{\text{nor}}| : X_{\text{nor}} \to Y_{\text{nor}} \) is defined by

\[
\mathcal{N}^Y \to \mathcal{N}^X \quad \{(U_i, \alpha_i)\}_i \mapsto \{(f^{-1}U_i, f^\# \alpha_i)\}_i.
\]

The morphism \( f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X \) extends canonically to \( f_{\text{nor}}^\# : \mathcal{O}_{Y_{\text{nor}}} \to f_{\text{nor}}^* \mathcal{O}_{X_{\text{nor}}} \). This gives a functor \( \text{nor} : (\text{Int. } \mathcal{A}\text{-Sch}) \to (\text{N. } \mathcal{A}\text{-Sch}) \).

Step 8: It remains to show that the normalization functor is indeed the right adjoint of the underlying functor. The unit \( \epsilon : \text{Id} \Rightarrow \text{nor} \circ U \) is the identity, since the normalization of a normal \( \mathcal{A} \)-scheme is trivial. The counit \( \eta : U \circ \text{nor} \Rightarrow \text{Id} \) is given by \( \pi \) defined above.

\[\square\]

Remark 2.4. We know that the integral closure \( \tilde{R} \) of a Noetherian domain \( R \) is not Noetherian [N2]. Therefore, we must drop the ‘of finite type’ condition from the definition of properness, if we wish to say that “the normalization \( \text{Spec } \tilde{R} \to \text{Spec } R \) is proper”.

Lemma 2.5. Let \( X \) be a normal \( \mathcal{A} \)-scheme. Then, \( \mathcal{O}_X(U) \) is normal for any open \( U \).

Proof. Let \( b \in K \) be an element which is integral over \( \mathcal{O}_X(U) \), where \( K \) is the function field of \( X \). Since \( b_x \) is integral over the stalk \( \mathcal{O}_{X,x} \) for any \( x \in U \) and \( \mathcal{O}_{X,x} \) is integrally closed, we have \( b_x \in \mathcal{O}_{X,x} \). Hence, \( b \in \mathcal{O}_X(U) \). \[\square\]
Proposition 2.6. The normalization functor coincides with the usual normalization, when restricted to \(\mathcal{D}\)-schemes.

Proof. First, we will show for affine schemes \(X = \text{Spec} \, A\). The universality of the normalization functor gives a canonical morphism \(f : \text{Spec} (A^\text{nor}) \to X^\text{nor}\). Since \(\Gamma(X^\text{nor}, \mathcal{O}_{X,x})\) is normal, we have a canonical homomorphism \(A^\text{nor} \to \Gamma(X^\text{nor}, \mathcal{O}_{X,x})\). This yields a morphism \(g : X^\text{nor} \to \text{Spec} (A^\text{nor})\). It is easy to check that these two morphisms \(f, g\) are inverse to each other.

It is obvious from the construction that normalization commutes with localizations. This shows that the normalization of any \(\mathcal{D}\)-scheme coincides with the usual definition.

3 Approximations by ordinary schemes

We fix an integral base \(\mathcal{D}\)-scheme \(S\) in the sequel. The next proposition is pure category-theoretical and easy, so we will omit the proof.

Proposition 3.1. Let \(\mathcal{B}, \mathcal{C}\) be two categories, with \(\mathcal{B}\) finite complete and \(\mathcal{C}\) small complete. Let \(F : \mathcal{B} \to \mathcal{C}\) be a finite continuous functor, namely \(F\) preserves fiber products. For any object \(a\) of \(\mathcal{C}\), the followings are equivalent:

(i) \(a\) is isomorphic to a limit of the objects in \(\text{Im} F\).

(ii) \(a\) is isomorphic to a filtered limit of the objects in \(\text{Im} F\).

Definition 3.2. Let \(X\) be an \(\mathcal{A}\)-scheme, and \(\mathcal{P}\) be a class of \(\mathcal{D}\)-schemes.

(1) \(X\) is approximable by \(\mathcal{P}\), if \(X\) is isomorphic to a filtered limit of some objects of \(\mathcal{P}\).

(2) \(X\) is approximable, if \(X\) is isomorphic to a filtered limit of some \(\mathcal{D}\)-schemes.

Proposition 3.3. Any approximable \(\mathcal{A}\)-scheme is approximable by \(\mathcal{D}\)-schemes of finite type.

Proof. It suffices to show that any \(\mathcal{D}\)-scheme is approximable by \(\mathcal{D}\)-schemes of finite type.

Let \(X\) be any \(\mathcal{D}\)-scheme, and \(\bigcup U_{ijk} \to \bigcup U_i\) be a finite affine covering of \(X\). Since \(U_{ijk} \to U_i\) is quasi-compact, \(U_{ijk}\) is of finite type over \(U_i\). Thus, we
have approximations $U_i = \lim_{\lambda} U^\lambda_i$ and $U_{ijk} = \lim_{\lambda} U^\lambda_{ijk}$ so that $U^\lambda_i$ and $U^\lambda_{ijk}$ are of finite type and $U^\lambda_{ijk} \to U^\lambda_i$ are open immersions. We may also assume that the above limits are filtered. Since filtered limits and finite colimits commute, we have

$$X = \lim_{-\to} U_i = \lim_{i} \lim_{\lambda} U^\lambda_i = \lim_{\lambda} \lim_{i} U^\lambda_i$$

and $\lim_{-\to} U^\lambda_i$ is a $\mathcal{D}$-scheme of finite type.

**Definition 3.4.** Let $X, Y$ be two integral $\mathcal{A}$-schemes. A morphism $f : X \to Y$ is *birational*, if $f$ induces an isomorphism $\mathbb{Q}(X) \simeq \mathbb{Q}(Y)$ between the rational function fields.

**Remark 3.5.** Note that, the morphism being birational does not imply the existence of an open dense subset $U$ of $X$ such that $U \simeq f(U)$.

**Proposition 3.6.** Let $X$ be an approximable $\mathcal{A}$-scheme, say $X = \lim_{\lambda} X^\lambda$ where $X^\lambda$'s are $\mathcal{D}$-schemes.

1. If $X$ is reduced, then $X$ is approximable by reduced $\mathcal{D}$-schemes.
2. If $X$ is integral, then $X$ is approximable by integral $\mathcal{D}$-schemes.
3. Further, if the rational function field $\mathbb{Q}(X)$ is finitely generated over an integral base $\mathcal{D}$-scheme, then $X$ is approximable by integral $\mathcal{D}$-schemes birational to $X$.
4. If $X$ is normal, then $X$ is approximable by normal $\mathcal{D}$-schemes.
5. If $X$ is proper and approximable by separated $\mathcal{D}$-schemes, then $X$ is approximable by proper (and of finite type) $\mathcal{D}$-schemes.

**Proof.** The proofs are all similar, so let us just see (1).

Since $X$ is reduced, $X \to X^\lambda$ factors through the reduced $\mathcal{D}$-scheme $(X^\lambda)_{\text{red}}$. This shows that $X \simeq \lim_{\lambda}(X^\lambda)_{\text{red}}$. \qed

**Proposition 3.7.** Let $f : X \to Y$ be a morphism of $\mathcal{A}$-schemes over $S$, with $X$ approximable and $Y$ a $\mathcal{D}$-scheme, of finite type over $S$.

1. Suppose $X$ is a filtered projective limit $\lim_{\lambda} X^\lambda$ of $\mathcal{D}$-schemes. Then, $f$ factors through $X \to X^\lambda$ for some $\lambda$. 

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(2) Furthermore, if $X$ is proper over $S$ and approximable by separated $\mathcal{Q}$-schemes, and $Y$ is separated over $S$, then the above $X^\lambda$ can be chosen to be a proper scheme over $Y$.

**Proof.** (1) We may assume that $Y$ is affine. Since $Y$ is of finite type and $\Gamma(X, \mathcal{O}_X)$ is a filtered colimit of $\Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$, $f$ factors through $X^\lambda$ for some $\lambda$.

(2) By the above proposition, we may assume that $X^\lambda$'s are proper over the base scheme $S$. Since $Y$ is separated, the morphism $X^\lambda \to Y$ is proper.

**Theorem 3.8.** Let $f : X \to Y$ be a proper birational morphism, where $X$ is an integral $\mathcal{A}$-scheme approximable by separated $\mathcal{Q}$-schemes, and $Y$ a normal $\mathcal{Q}$-scheme separated and of finite type over $S$. Then, $f_* \mathcal{O}_X = \mathcal{O}_Y$.

**Proof.** The previous proposition shows that $f$ factors through proper morphisms $f_\lambda : X^\lambda \to Y$, where $X = \lim_{\leftarrow \lambda} X^\lambda$ and $\{X^\lambda\}$ is a filtered system of integral $\mathcal{Q}$-schemes, proper birational and of finite type over $Y$. Since $Y$ is normal, the usual Zariski’s main theorem tells that $\mathcal{O}_Y \to (f_\lambda)_* \mathcal{O}_{X^\lambda}$ is an isomorphism (Corollary III. 11.4 of [H]), and $f_* \mathcal{O}_X$ coincides with the right hand side, since it is a colimit of $(f_\lambda)_* \mathcal{O}_{X^\lambda}$’s.

Since “of profinite type” morphisms are stable under taking limits, approximable $\mathcal{A}$-schemes are necessarily of profinite type over $S$.

**Theorem 3.9.** Let $X$ be a normal $\mathcal{A}$-scheme, proper and of profinite type over the integral base $\mathcal{Q}$-scheme $S$. Assume that the rational function field $Q(X)$ is finitely generated over $Q(S)$. The followings are equivalent:

(i) $X$ is approximable by separated $\mathcal{Q}$-schemes.

(ii) Let $\mathcal{U} = \{\Pi U_{ijk} \supseteq \Pi U_i\}$ be any quasi-compact open covering of $X$. Then, there exists a refinement $\Pi V_{ijk} \supseteq \Pi V_i$ of $\mathcal{U}$ such that Spec $\mathcal{O}_X(V_{ijk}) \to$ Spec $\mathcal{O}_X(V_i)$ are open immersions.

**Proof.** (i)$\Rightarrow$(ii): $X$ can be written as a filtered limit $X = \lim_{\leftarrow \lambda} X^\lambda$, where $X^\lambda$’s are normal $\mathcal{Q}$-schemes, proper and of finite type over $S$. Since the number of $U_i$’s and $U_{ijk}$’s are finite, $U_i = \pi^{-1}\bar{U}_i$, $U_{ijk} = \pi^{-1}\bar{U}_{ijk}$ for some $\pi : X \to X^\lambda$, where $\bar{U}_i$’s and $\bar{U}_{ijk}$’s are quasi-compact open subsets of $X^\lambda$.  

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Take any refinement \( \{ \Pi V_{ijk} \rightarrow \Pi V_i \} \) of \( \Pi \tilde{V}_{ijk} \rightarrow \Pi \tilde{V}_i \), by affine opens \( \tilde{V}_{ijk} \) and \( \tilde{V}_i \). Set \( V_{ijk} = \pi^{-1} \tilde{V}_{ijk} \) and \( V_i = \pi^{-1} \tilde{V}_i \). Since \( V_i \rightarrow \tilde{V}_i \) is proper and \( \tilde{V}_i \) is normal, of finite type, Theorem 3.8 implies that \( \mathcal{O}_X(V_i) = \mathcal{O}_{X_1}(V_i) \). This shows that Spec \( \mathcal{O}_X(V_{ijk}) \rightarrow \text{Spec} \mathcal{O}_X(V_i) \) are open immersions.

(ii)\( \Rightarrow \) (i): For any covering \( U = \{ \Pi U_{ijk} \Rightarrow \Pi U_i \} \) of \( X \), the refinement \( \Pi V_{ijk} \Rightarrow \Pi V_i \) gives open immersions Spec \( \mathcal{O}_X(V_{ijk}) \rightarrow \text{Spec} \mathcal{O}_X(V_i) \) which patches up to give a \( \mathfrak{A} \)-scheme \( X(U) \) and the canonical morphism \( \pi_U : X \rightarrow X(U) \). The covering \( U \) is a pull back of a covering of \( X(U) \), and ditto for the elements of \( \mathcal{O}_X(U_i) \)’s. From this observation, we see that the induced morphism \( X \rightarrow \lim_{U \leftarrow} X(U) \) is an isomorphism. It is clear from the construction that \( X(U) \) is proper. \( \square \)

4 Another proof of Nagata embedding

In the sequel, any \( \mathfrak{A} \)-schemes are integral.

**Definition 4.1.** Let \( S \) be a \( \mathfrak{A} \)-scheme, and \( X \) be a \( \mathfrak{A} \)-scheme over \( S \). We say that \( X \) is **compactifiable** over \( S \), if there is an open immersion \( X \rightarrow Y \) where \( Y \) is a \( \mathfrak{A} \)-scheme, proper, of finite type over \( S \).

**Proposition 4.2.** Let \( S \) be a \( \mathfrak{A} \)-scheme, and \( X \) be a \( \mathfrak{A} \)-scheme over \( S \). The followings are equivalent:

(i) \( X \) is compactifiable over \( S \).

(ii) \( \text{ZR}^f(X, S) \) is approximable by separated \( \mathfrak{A} \)-schemes, and the natural map \( X \rightarrow \text{ZR}^f(X, S) \) is an open immersion.

**Proof.** (i)\( \Rightarrow \) (ii): There exists an open immersion \( X \rightarrow Y \) into a \( \mathfrak{A} \)-scheme \( Y \), proper of finite type over \( S \). This morphism factors through \( \text{ZR}^f(X, S) \) by the universal property. We will show that for any quasi-compact open subset \( U \) of \( \text{ZR}^f(X, S) \), there exists a proper birational morphism \( Y' \rightarrow Y \), such that \( g^{-1}(V) = U \) for some quasi-compact open subset \( V \) of \( Y' \), where \( g : \text{ZR}^f(X, S) \rightarrow Y' \) is the canonical extension of \( f : \text{ZR}^f(X, S) \rightarrow Y \):

\[
\begin{array}{ccc}
U & \longrightarrow & \text{ZR}^f(X, S) \\
\downarrow & & \downarrow \quad \rotatebox{90}{$\cong$} \\
V & \longrightarrow & Y' \longrightarrow Y
\end{array}
\]
By the construction of $\text{ZR}^I(X, S)$, we may assume $U$ is of the form $U(W, \alpha)$, where $W$ is a quasi-compact open subset of $S$ and $\alpha$ is a finite subset of $K \setminus \{0\}$, and

$$U(W, \alpha) = \pi^{-1}(W) \cap \{ p \in \text{ZR}^I(X, S) \mid \alpha \subset O_{\text{ZR}^I(X, S)} \}.$$  

Note that $f(U \cap X)$ is open in $Y$, since $X \to \text{ZR}^I(X, S)$ is an open immersion. Suppose $\alpha = \{ a_i/b_i \}_i$, where $a_i, b_i \in O_Y$ locally. Let $Y' \to Y$ be the blow up along $(Y \setminus X) \cap \text{Supp}(a_i, b_i)$. Then, either $a_i/b_i$ or $b_i/a_i$ is in $O_Y$, locally, which shows that the domain of $a_i/b_i$ is open in $Y'$. This shows that $U$ is the pull back of some $V$ by the morphism $g: \text{ZR}^I(X, S) \to Y'$. Hence, $\text{ZR}^I(X, S) \to \varprojlim \ Y^\lambda$ becomes a homeomorphism on the underlying space, where $Y^\infty = \varprojlim \ Y^\lambda$ is the filtered projective limit of $X$-admissible blow-ups of $Y$. A similar argument shows that $O_{Y^\infty} \to O_{\text{ZR}^I(X, S)}$ also becomes an isomorphism. Note that $Y^\lambda$’s are separated over $S$, since we only used blow-ups.

(ii)$\Rightarrow$(i): The Zariski-Riemann space $\text{ZR}^I(X, S)$ can be written as a form $\varprojlim \ Y^\lambda$, where $Y^\lambda$’s are proper, of finite type $\mathcal{O}$-schemes. Since $X \to \text{ZR}^I(X, S)$ is an open immersion and $X$ is quasi-compact, $X \to \text{ZR}^I(X, S) \to Y^\lambda$ becomes an open immersion for some $\lambda$. 

Now, we are on the stage to give the proof of the Nagata embedding.

**Theorem 4.3** (Nagata). Let $S$ be a $\mathcal{O}$-scheme, and $X$ be a $\mathcal{O}$-scheme, separated and of finite type over $S$. Then, $X$ is compactifiable over $S$.

In this section, we will prove this theorem for the essential case, namely when $S$ and $X$ are integral. This restriction is due to the fact that we simply haven’t established the theorem of Zariski-Riemann spaces for non-integral schemes.

Since $X$ is quasi-compact, and affine schemes of finite type over $S$ is obviously compactifiable, it suffices to prove the following proposition:

**Proposition 4.4.** Let $V_1$ and $V_2$ be compactifiable open sub-$\mathcal{O}$-schemes of a $\mathcal{O}$-scheme $X$ separated over $S$, with $X = V_1 \cup V_2$. Then $X$ is also compactifiable.

**Proof.** Consider $\text{ZR}^I(X, S)$. Since $X$ is separated, of finite type over $S$, the morphism $X \to \text{ZR}^I(X, S)$ is an open immersion by Corollary 4.4.6 of [T].
Let $W_1$ (resp. $W_2$) be the complement of the closure of $V_2 \setminus V_1$ (resp. $V_1 \setminus V_2$) in $\text{ZR}^f(X, S)$.

We can see that $W_1 \cap W_2 = V_1 \cap V_2$, since the interior of the complement of $V_1 \cup V_2$ is empty. Next, we see that $W_1 \cup W_2 = \text{ZR}^f(X, S)$. For this, it suffices to show that $V_2 \setminus V_1 \cap V_1 \setminus V_2 = \emptyset$. Suppose there is a point $p$ in $V_2 \setminus V_1 \cap V_1 \setminus V_2$. Since $V_2 \setminus V_1$ and $V_1 \setminus V_2$ are coherent subsets of $\text{ZR}^f(X, S)$, $p$ must be a specialization of some $x_1 \in V_2 \setminus V_1$ and $x_2 \in V_1 \setminus V_2$ by Corollary 1.2.8 of [T]. Since $\text{ZR}^f(K, S) \to \text{ZR}^f(X, S)$ is surjective, there are inverse images $y_i \in \text{ZR}^f(K, S)$ of $x_i$ such that $y_i$ specializes to $p$. The points in $\text{ZR}^f(K, S)$ are valuation rings, hence $y_2$ must be the specialization of $y_1$, or the converse. In either cases, this contradicts to the fact that $x_1$ and $x_2$ has no specialization-generalization relations. This also shows that $W_1$ and $W_2$ are quasi-compact. The morphism $p_1 : \text{ZR}^f(V_1, S) \to \text{ZR}^f(X, S)$ induces an isomorphism on $W_1$, hence $W_1$ is approximable by $\mathcal{O}$-morphisms of finite type over $S$, ditto for $W_2$.

Take any $\mathcal{O}$-model $Y_i$ of $W_i$ (namely, a morphism $\pi_i : W_i \to Y_i$ where $Y_i$ is a $\mathcal{O}$-scheme) such that the morphism $\pi_i$ induces an isomorphism on $V_i$. Then, $Y_1$ and $Y_2$ can be patched along $\pi_1(W_1 \cap W_2) \simeq \pi_2(W_1 \cap W_2)$ to obtain a $\mathcal{O}$-scheme $Y$ of finite type, and a surjective morphism $\text{ZR}^f(X, S) = W_1 \cup W_2 \to Y$. This shows that $Y$ is proper.

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