Regularity of solutions in semilinear elliptic theory

E. Indrei, A. Minne, L. Nurbekyan

Abstract
We study the semilinear Poisson equation
\[ \Delta u = f(x, u) \quad \text{in} \quad B_1. \] (1)

Our main results provide conditions on \( f \) which ensure that weak solutions of (1) belong to \( C^{1,1}(B_{1/2}) \). In some configurations, the conditions are sharp.

1 Introduction

The semilinear Poisson equation (1) encodes stationary states of the nonlinear heat, wave, and Schrödinger equation. In the case when \( f \) is the Heaviside function in the \( u \)-variable, (1) reduces to the classical obstacle problem. For an introduction to classical semilinear theory, see [BS11, Caz06].

It is well-known that weak solutions of (1) belong to the usual Sobolev space \( W^{2,p}(B_{1/2}) \) for any \( 1 \leq p < \infty \) provided \( f \in L^\infty \). Recent research activity has thus focused on identifying conditions on \( f \) which ensure \( W^{2,\infty}(B_{1/2}) \) regularity of \( u \).

1.1 The classical theory

There are simple examples which illustrate that continuity of \( f = f(x) \) does not necessarily imply that \( u \) has bounded second derivatives: for \( p \in (0,1) \) and \( x \in \mathbb{R}^2 \) such that \( |x| < 1 \), the function
\[ u(x) = x_1 x_2 (-\log |x|)^p \]
has a continuous Laplacian but is not in \( C^{1,1} \) [Sha15]. However, if \( f \) is Hölder continuous, then it is well-known that \( u \in C^{2,\alpha} \); if \( f \) is Dini continuous, then \( u \in C^2 \) [GT01, Kov99]. The sharp condition which guarantees bounded second derivatives of \( u \) is the \( C^{1,1} \) regularity of \( f \ast N \) where \( N \) is the Newtonian potential and \( \ast \) denotes convolution; this requirement is strictly weaker than Dini continuity of \( f \).

In the general case, the state-of-the-art is a theorem of Shahgholian [Sha03] which states that \( u \in C^{1,1} \) whenever \( f = f(x,u) \) is Lipschitz in \( x \), uniformly
in $u$, and $\partial_u f \geq -C$ weakly for some $C \in \mathbb{R}$. In some configurations this illustrates regularity for continuous functions $f = f(u)$ which are strictly below the classical Dini-threshold in the $u$-variable, e.g. the odd reflection of

$$f(t) = -\frac{1}{\log(t)}$$

about the origin. Shahgholian’s theorem is proved via the celebrated Alt-Caffarelli-Friedman (ACF) monotonicity formula and it seems difficult to weaken the assumptions by this method. On the other hand, Koch and Nadirashvili [KN] recently constructed an example which illustrates that the continuity of $f$ is not sufficient to deduce that weak solutions of $\Delta u = f(u)$ are in $C^{1,1}$.

We say $f = f(x,u)$ satisfies assumption $A$ provided that $f$ is Dini continuous in $u$, uniformly in $x$, and has a $C^{1,1}$ Newtonian potential in $x$, uniformly in $u$ (see §3). One of our main results is the following statement.

**Theorem 1.1.** Suppose $f$ satisfies assumption $A$. Then any solution of (1) is $C^{1,1}$ in $B_{1/2}$.

Our assumption includes functions which fail to satisfy both conditions in Shahgholian’s theorem, e.g.

$$f(x_1, x_2, t) = \frac{x_1}{\log(|x_2|)(-\log|t|)^p},$$

for $p > 1$, $x = (x_1, x_2) \in B_1$ and $t \in (-1, 1)$. The Newtonian potential assumption in the $x$-variable is essentially sharp whereas the condition in the $t$-variable is in general not comparable with Shahgholian’s assumption.

The proof of Theorem 1.1 does not invoke monotonicity formulas and is self-contained. We consider the $L^2$ projection of $D^2u$ on the space of Hessians generated by second order homogeneous harmonic polynomials on balls with radius $r > 0$ and show that the projections stay uniformly bounded as $r \to 0^+$. Although this approach has proven effective in dealing with a variety of free boundary problems [ALS13, FS14, IM15, IM], Theorem 1.1 illustrates that it is also useful in extending and refining the classical elliptic theory.

1.2 Singular case: the free boundary theory

In §4 we study the PDE (1) for functions $f = f(x,u)$ which are discontinuous in the $u$-variable at the origin.

If the discontinuity of $f$ is a jump discontinuity, (1) has the structure

$$f(x,u) = g_1(x,u)\chi_{\{u>0\}} + g_2(x,u)\chi_{\{u<0\}},$$

where $g_1, g_2$ are continuous functions such that

$$g_1(x,0) \neq g_2(x,0), \quad \forall x \in B_1,$$

and $\chi_{\Omega}$ defines the indicator function of the set $\Omega$.  

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Our aim is to find the most general class of coefficients $g_i$ which generate interior $C^{1,1}$ regularity.

The classical obstacle problem is obtained by letting $g_1 = 1, g_2 = 0$, and it is well-known that solutions have second derivatives in $L^\infty$ [PSU12]. Nevertheless, by selecting $g_1 = -1, g_2 = 0$, one obtains the so-called unstable obstacle problem. Elliptic theory and the Sobolev embedding theorem imply that any weak solution belongs to $C^{1,1}$, see also [LSE09, Remark 1.3]. Theorem 1.3 improves and extends this result. Hence, if there is a jump at the origin, $C^{1,1}$ regularity can hold only if the jump is positive and this gives rise to:

**Assumption B.** $g_1(x,0) - g_2(x,0) \geq \sigma_0$, $x \in B_1$ for some $\sigma_0 > 0$.

The free boundary $\Gamma = \partial\{u \neq 0\}$ consists of two parts: $\Gamma^0 = \Gamma \cap \{\nabla u = 0\}$ and $\Gamma^1 = \Gamma \cap \{\nabla u \neq 0\}$. The main difficulty in proving $C^{1,1}$ regularity is the analysis of points where the gradient of the function vanishes. In this direction we establish the following result.

**Theorem 1.2.** Suppose $g_1, g_2$ satisfy A and B. Then if $u$ is a solution of (1), $\|u\|_{C^{1,1}(K)} < \infty$ for any $K \in B_{1/2}(0) \setminus \Gamma^1$.

At points where the gradient does not vanish, the implicit function theorem yields that the free boundary is locally a $C^{1,\alpha}$ graph for any $0 < \alpha < 1$. The solution $u$ changes sign across the free boundary, hence it locally solves the equation $\Delta u = g_1(x,u)$ on the side where it is positive and $\Delta u = g_2(x,u)$ on the side where it is negative. If the coefficients $g_i$ are regular enough to provide $C^{1,1}$ solutions up to the boundary – this is encoded in assumption C, see §4 – then we obtain full $C^{1,1}$ regularity.

**Theorem 1.3.** Suppose $g_1, g_2$ satisfy A, B and C. Let $u$ be a solution of (1) and $0 \in \Gamma^0$. Then $u \in C^{1,1}(B_{\rho_0}(0))$, for some $\rho_0 > 0$.

Equation (1) with right-hand side of the form (2) is a generalization of the well-studied two-phase membrane problem, where $g_i(x,u) = \lambda_i(x)$, $i = 1, 2$. The $C^{1,1}$ regularity in the case when $\lambda_1 \geq 0$, $\lambda_2 \leq 0$ are two constants satisfying B was obtained by Uraltseva [Ura01] via the ACF monotonicity formula. Moreover, Shahgholian proved this result for Lipschitz coefficients which satisfy B [Sha03, Example 2]. If the coefficients are Hölder continuous, the ACF method does not directly apply and under the stronger assumption that $\inf \lambda_1 > 0$ and $\inf -\lambda_2 > 0$, Edquist, Lindgren, Shahgholian [LS09] obtained the $C^{1,1}$ regularity via an analysis of blow-up limits and a classification of global solutions (see also [LS09, Remark 1.3]). Theorem 1.3 improves and extends this result.

The difficulty in the case when $g_i$ depend also on $u$ is that if $v := u + L$ for some linear function $L$, then $v$ is no longer a solution to the same equation, so one has to get around the lack of linear invariance. Our technique exploits that linear perturbations do not affect certain $L^2$ projections.

The proof of Theorem 1.3 does not rely on classical monotonicity formulas or classification of global solutions. Rather, our method is based on an identity...
which provides monotonicity in $r$ of the square of the $L^2$ norm of the projection of $u$ onto the space of second order homogeneous harmonic polynomials on the sphere of radius $r$.

Theorems 1.2 & 1.3 deal with the case when $f$ has a jump discontinuity. If $f$ has a removable discontinuity, (1) has the structure

$$
\Delta u = g(x, u)\chi_{u\neq 0}.
$$

(3)

In this case, one may merge some observations in the proofs of the previous results with the method in [ALS13] and prove the following theorem.

**Theorem 1.4.** If $g$ satisfies assumption A, then every solution of (3) is in $C^{1,1}(B_{1/2})$.

Theorems 1.1 - 1.4 provide a comprehensive theory for the general semilinear Poisson equation where the free boundary theory is encoded in the regularity assumption of $f$ in the $u$-variable.

## 2 Technical tools

Throughout the text, the right-hand side of (1) is assumed to be bounded. Moreover, $P_2$ denotes the space of second order homogeneous harmonic polynomials. A useful elementary fact is that all norms on $P_2$ are equivalent.

**Lemma 2.1.** The space $P_2$ is a finite dimensional linear space. Consequently, all norms on $P_2$ are equivalent.

For $u \in W^{2,2}(B_1)$, $y \in B_1$ and $r \in (0, \text{dist}(y, \partial B_1))$, $\Pi_y(u, r)$ is defined to be the $L^2$ projection operator on $P_2$ given by

$$
\inf_{h \in P_2} \int_{B_1} \left| D^2 \frac{u(rx + y)}{r^2} - D^2 h \right|^2 dx = \int_{B_1} \left| D^2 \frac{u(rx + y)}{r^2} - D^2 \Pi_y(u, r) \right|^2 dx.
$$

Calderon-Zygmund theory yields the following useful inequality for re-scalings of weak solutions of (1).

**Lemma 2.2.** Let $u$ solve (1), $y \in B_{1/2}$, and $r \leq 1/4$. Then for

$$
\tilde{u}_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}
$$

it follows that for $1 \leq p < \infty$ and $0 < \alpha < 1$,

$$
\| \tilde{u}_r - \Pi_y(u, r) \|_{W^{2,p}(B_1)} \leq C(n, \| f \|_{L^\infty(B_1 \times \mathbb{R})}, \| u \|_{L^\infty(B_1)}, p),
$$

and

$$
\| \tilde{u}_r - \Pi_y(u, r) \|_{C^{1,\alpha}(B_1)} \leq C(n, \| f \|_{L^\infty(B_1 \times \mathbb{R})}, \| u \|_{L^\infty(B_1)}, \alpha).
$$
Proof. By Calderon-Zygmund theory (e.g. [ALS13, Theorem 2.2]),
\[ \| D^2 u \|_{BMO(B_1/2)} \leq C; \]
in particular,
\[ \int_{B_3/2} |D^2 \bar{u}_r - \overline{D^2 \bar{u}_r}|^2 \leq C, \]
where \( \overline{D^2 \bar{u}_r} \) is the average of \( D^2 \bar{u}_r \) on \( B_{3/2} \). Now let
\[ a = a(f, r, y) = \int_{B_{3/2}} f(rx + y, u(rx + y)) \, dx \]
and note that this quantity is uniformly controlled by \( \| f \|_{L^\infty(B_1 \times \mathbb{R})} \); this fact, and the definition of \( \Pi \) yields (note: trace \((D^2 u - a \text{Id}) = 0\)),
\[ \hat{B}_{3/2} \| D^2 (\tilde{u}_r - \Pi_0(\tilde{u}_r, 3/2)) \|_{L^2(B_{3/2})} \leq C. \]
Two applications of Poincaré’s inequality together with the above estimate implies
\[ \| \tilde{u}_r - \Pi_0(\tilde{u}_r, 3/2) \|_{W^{2,p}(B_{3/2})} \leq C, \]
where the averages are taken over \( B_{3/2} \). Elliptic theory (e.g. [GT01, Theorem 9.1]) yields that for any \( 1 \leq p < \infty \),
\[ \| \tilde{u}_r - \Pi_0(\tilde{u}_r, r) \|_{W^{2,p}(B_{3/2})} \leq C. \]
Let \( \phi := \tilde{u}_r - \overline{\nabla \bar{u}_r} \cdot x - \bar{u}_r \). We have that \( \phi(0) = -\bar{u}_r \) and \( \nabla \phi(0) = -\nabla \bar{u}_r \); however, by the Sobolev embedding theorem, \( \phi \) is \( C^{1,\alpha} \) and thus
\[ |\phi(0)| + |\nabla \phi(0)| \leq C \]
completing the proof of the \( W^{2,p} \) estimate. The \( C^{1,\alpha} \) estimate likewise follows from the Sobolev embedding theorem.

Our analysis requires several additional simple technical lemmas involving the projection operator.

Lemma 2.3. For any \( u \in W^{2,2}(B_1) \) and \( s \in [1/2, 1] \),
\[ \| \Pi_0(u, s) - \Pi_0(u, 1) \|_{L^2(B_1)} \leq C \| \Delta u \|_{L^2(B_1)}, \]
and
\[ \| \Pi_0(u, s) - \Pi_0(u, 1) \|_{L^\infty(B_1)} \leq C \| \Delta u \|_{L^2(B_1)}, \]
for some constant \( C = C(n) \).
Proof. Let \( f = \Delta u \) and \( v \) be the Newtonian potential of \( f \), i.e.

\[
v(x) = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)\chi_{B_1}(y)}{|x-y|^{n-2}} \, dx,
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Since \( u - v \) is harmonic,

\[
\Pi_0(u - v, s) = \Pi_0(u - v, 1);
\]

therefore

\[
\Pi_0(u, s) - \Pi_0(u, 1) = \Pi_0(v, s) - \Pi_0(v, 1).
\]

Invoking bounds on the projection (e.g. \cite[Lemma 3.2]{ALS13}) and Calderon-Zygmund theory (e.g. \cite[Theorem 2.2]{ALS13}), it follows that

\[
\|\Pi_0(u, s) - \Pi_0(u, 1)\|_{L^2(B_1)} = \|\Pi_0(v, s) - \Pi_0(v, 1)\|_{L^2(B_1)} \leq C\|\Delta v\|_{L^2(B_1)} = C\|\Delta u\|_{L^2(B_1)}.
\]

The \( L^\infty \) bound follows from the equivalence of the norms in the space \( P_2 \).

Lemma 2.4. Let \( u \) solve (1). Then for all \( 0 < r \leq 1/4 \), \( s \in [1/2, 1] \) and \( y \in B_{1/2} \),

\[
\sup_{B_1} |\Pi_y(u, rs) - \Pi_y(u, r)| \leq C,
\]

and

\[
\sup_{B_1} |\Pi_y(u, r)| \leq C \log(1/r),
\]

for some constant \( C = C(n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}) \).

Proof. Note that

\[
\Pi_y(u, rs) - \Pi_y(u, r) = \Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1),
\]

where

\[
\tilde{u}_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}
\]

as before. From Lemma 2.3 we have that

\[
\|\Pi_0(\tilde{u}_r, s) - \Pi_0(\tilde{u}_r, 1)\|_{L^\infty(B_1 \times \mathbb{R})} \leq C\|\Delta \tilde{u}_r\|_{L^2(B_1)} \leq C\|f\|_{L^\infty(B_1)}.
\]

As for the second inequality in the statement of the lemma let \( r_0 = 1/4 \) and \( s \in [1/2, 1] \). Then we have that

\[
\sup_{B_1} |\Pi_y(u, sr_0/2^j)| \leq \sup_{B_1} |\Pi_y(u, sr_0/2^j) - \Pi_y(u, r_0/2^j)|
\]

\[
+ \sum_{k=0}^{j-1} \sup_{B_1} |\Pi_y(u, r/2^{k+1}) - \Pi_y(u, r/2^k)|
\]

\[
+ \sup_{B_1} |\Pi_y(u, r_0)| \leq Cj \leq C \log \left( \frac{2^j}{sr_0} \right),
\]

for all \( j \geq 1 \).
The previous tools imply a growth estimate on weak solutions solution of (1).

**Lemma 2.5.** Let $u$ solve (1). Then for $y \in B_{1/2}$ and $r > 0$ small enough, $$\sup_{B_r(y)} |u(x) - u(y) - (x - y)\nabla u(y)| \leq Cr^2 \log(1/r).$$

**Proof.** Let $$\tilde{u}_r = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2}.$$ The assertion of the Lemma is equivalent to the estimate $$\|\tilde{u}_r\|_{L^\infty(B_1)} \leq C \log(1/r),$$ for $r$ small enough. Lemma 2.4 and the $C^{1,\alpha}$ estimates of Lemma 2.2 imply $$\|\tilde{u}_r\|_{L^\infty(B_1)} \leq \|\tilde{u}_r - \Pi_y(u, r)\|_{L^\infty(B_1)} + \|\Pi_y(u, r)\|_{L^\infty(B_1)} \leq C + C \log(1/r) \leq C \log(1/r),$$ provided $r$ is small enough. \qed

Next lemma relates the boundedness of the projection operator and the boundedness of second derivatives of weak solutions of (1).

**Lemma 2.6.** Let $u$ be a solution to (1). If for each $y \in B_{1/2}$ there is a sequence $r_j(y) \to 0^+$ as $j \to \infty$ such that $$M := \sup_{y \in B_{1/2}} \sup_{j \in \mathbb{N}} \|D^2 \Pi_y(u, r_j(y))\|_{L^\infty(B_{1/2})} < \infty,$$ then $$|D^2 u| \leq C \quad \text{a.e. in } B_{1/2},$$ for some constant $C = C(M, n, \|f\|_{L^\infty(B_1 \times \mathbb{R})}, \|u\|_{L^\infty(B_1)}) > 0$.

**Proof.** Let $y \in B_{1/2}$ be a Lebesgue point for $D^2 u$ and $r_j = r_j(y) \to 0^+$ as $j \to \infty$. Then by utilizing Lemma 2.2

$$|D^2 u(y)| = \lim_{j \to \infty} \int_{B_{r_j}(y)} |D^2 u(z)|dz \leq \limsup_{j \to \infty} \int_{B_{r_j}(y)} |D^2 u(z) - D^2 \Pi_y(u, r_j)|dz + M \leq C.$$ Since a.e. $z \in B_{1/2}$ is a Lebesgue point for $D^2 u$, the proof is complete. \qed

Next, we introduce another projection that we need for our analysis. Define $Q_y(u, r)$ to be the minimizer of $$\inf_{q \in P_2} \int_{\partial B_1} \left| \frac{u(rx + y)}{r^2} - q(x) \right|^2 dH^{n-1}.$$ The following lemma records the basic properties enjoyed by this projection, cf. [ALS13, Lemma 3.2].


Lemma 2.7.  

i. $Q_y(\cdot, r)$ is linear;

ii. if $u$ is harmonic $Q_y(u, s) = Q_y(u, r)$ for all $s < r$;

iii. if $u$ is a linear function then $Q_y(u, r) = 0$;

iv. if $u$ is a second order homogeneous polynomial then $Q_y(u, r) = u$;

v. $\|Q_0(u, s) - Q_0(u, 1)\|_{L^2(\partial B_1)} \leq C_s\|\Delta u\|_{L^2(B_1)}$, for $0 < s < 1$;

vi. $\|Q_0(u, 1)\|_{L^2(\partial B_1)} \leq \|u\|_{L^2(\partial B_1)}$.

Proof.  

i. This is evident.

ii. It suffices to prove $Q_y(u, r) = Q_y(u, 1)$ for $r < 1$. Let

$$\sigma_2 = \frac{Q_y(u, 1)}{\|Q_y(u, 1)\|_{L^2(\partial B_1)}}$$

and for $i \neq 2$, let $\sigma_i$ be an $i$th degree harmonic polynomial. Then there exist coefficients $a_i$ such that

$$u(x + y) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in \partial B_1;$$

in particular, $a_2 = \|Q_y(u, 1)\|$. Let

$$v(x) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in B_1.$$  

Then $v$ is a harmonic and $u(x + y) = v(x)$ for $x \in \partial B_1$. Hence, we have that $u(x + y) = v(x)$ for $x \in B_1$ and in particular

$$u(x + y) = \sum_{i=0}^{\infty} a_i \sigma_i(x), \quad x \in B_1.$$  

Therefore

$$\frac{u(rx + y)}{r^2} = \sum_{i=0}^{\infty} a_i \frac{\sigma_i(rx)}{r^2} = \sum_{i=0}^{\infty} a_i r^{-i} \sigma_i(x), \quad x \in B_1,$$

so $Q_y(u, r) = a_2 \sigma_2(x) = Q_y(u, 1)$.

iii. & iv. These are evident.

v. Similar to Lemma 2.3.

vi. This follows from the fact that $Q_0(u, 1)$ is the $L^2$ projection of $u$.  

\[ \square \]
Next we prove some technical results for $Q_y(u, r)$ and establish a precise connection between $\Pi_y(u, r)$ and $Q_y(u, r)$ by showing that the difference is uniformly bounded in $r$.

**Lemma 2.8.** For $u \in W^{2,p}(B_1(y))$ with $p$ large enough and $r \in (0, 1]$,
\[
\frac{d}{dr} Q_y(u, r) = \frac{1}{r} Q_0(x \cdot \nabla u(x + y) - 2u(x + y), r).
\]

*Proof.* Firstly,
\[
Q_y(u, r) = Q_0\left(\frac{u(rx + y)}{r^2}, 1\right).
\]
Since $u$ is $C^{1,\alpha}$ if $p$ large enough and $Q$ is linear bounded operator, it follows that
\[
\frac{d}{dr} Q_y(u, r) = Q_0\left(\frac{d}{dr} \frac{u(rx + y)}{r^2}, 1\right) = Q_0\left(\frac{rx \cdot \nabla u(rx + y) - 2u(rx + y)}{r^3}, 1\right) = \frac{1}{r} Q_0(x \cdot \nabla u(x + y) - 2u(x + y), r).
\]

**Lemma 2.9.** Let $u \in W^{2,p}(B_1(y))$ with $p$ large enough and $q \in \mathcal{P}_2$. Then
\[
\int_{B_1} q(x) \Delta u(x + y) \, dx = \int_{\partial B_1} q(x)(x \cdot \nabla u(x + y) - 2u(x + y)) d\mathcal{H}^{n-1}. \tag{4}
\]

*Proof.* Integration by parts implies
\[
\int_{B_1} q(x) \Delta u(x + y) \, dx = \int_{B_1} \Delta q(x) u(x + y) \, dx + \int_{\partial B_1} q(x) \frac{\partial u(x + y)}{\partial n} - u(x + y) \frac{\partial q(x)}{\partial n} d\mathcal{H}^{n-1}.
\]
By taking into account that $q$ is a second order homogeneous polynomial it follows that
\[
\frac{\partial q(x)}{\partial n} = 2q(x), \quad x \in \partial B_1.
\]
Moreover,
\[
\frac{\partial u(x + y)}{\partial n} = x \cdot \nabla u(x + y), \quad x \in \partial B_1.
\]
Combining these equations yields (4).

**Lemma 2.10.** Let $u \in W^{2,p}(B_1(y))$ with $p$ large enough and $0 < r \leq 1$. Then for every $q \in \mathcal{P}_2$,
\[
\int_{\partial B_1} q(x) \frac{d}{dr} Q_y(u, r)(x) d\mathcal{H}^{n-1} = \frac{1}{r} \int_{B_1} q(x) \Delta u(rx + y) \, dx.
\]
Proof. Let \( \tilde{u}_r(x) = u(rx + y)/r^2 \). From Lemmas 2.8 and 2.9 we obtain

\[
\int_{\partial B_1} q(x) \frac{d}{dr} Q_y(u, r)(x) d\mathcal{H}^{n-1} = \frac{1}{r} \int_{\partial B_1} q(x) Q_0 \left( \frac{rx \cdot \nabla u(rx + y) - 2u(rx + y)}{r^2}, 1 \right) d\mathcal{H}^{n-1}
\]

\[
= \frac{1}{r} \int_{\partial B_1} q(x) Q_0 \left( x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x), 1 \right) d\mathcal{H}^{n-1}
\]

\[
= \frac{1}{r} \int_{\partial B_1} q(x) (x \cdot \nabla \tilde{u}_r(x) - 2\tilde{u}_r(x)) d\mathcal{H}^{n-1}
\]

\[
= \frac{1}{r} \int_{B_1} q(x) \Delta \tilde{u}_r(x) dx = \frac{1}{r} \int_{B_1} q(x) \Delta u(rx + y) dx.
\]

\[
\square
\]

Lemma 2.11. For \( u \in W^{2,p}(B_1(y)) \) with \( p \) large enough and \( 0 < r \leq 1 \),

\[
\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} = \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.
\]

Proof. By Lemmas 2.8 2.10 we get

\[
\frac{d}{dr} \int_{\partial B_1} Q_y^2(u, r) d\mathcal{H}^{n-1} = 2 \int_{\partial B_1} Q_y(u, r) \frac{d}{dr} Q_y(u, r) d\mathcal{H}^{n-1}
\]

\[
= \frac{2}{r} \int_{B_1} Q_y(u, r) \Delta u(rx + y) dx.
\]

\[
\square
\]

Lemma 2.12. Let \( u \) be a solution of (1) and \( y \in B_{1/2} \). For \( 0 < r < 1/2 \) consider

\[
u_r(x) := \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - Q_y(u, r), \]

\[
v_r(x) := \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - Q_y(u, r).
\]

Then

i. \( u_r - v_r \) is bounded in \( C^\infty \), uniformly in \( r \);

ii. the family \( \{v_r\} \) is bounded in \( C^{1,\alpha}(B_1) \cap W^{2,p}(B_1) \), for every \( 0 < \alpha < 1 \) and \( p > 1 \).
Proof. i. For each $r$, the difference $u_r - v_r = Q_y(u, r) - \Pi_y(u, r)$ is a second order harmonic polynomial. Therefore, it suffices to show that $L^\infty$ norm of that difference admits a bound independent of $r$. Note that

$$u_r - v_r = Q_y(u, r) - \Pi_y(u, r) = Q_0 \left( \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r), 1 \right) = Q_0(u_r, 1).$$

Hence,

$$\sup_r \sup_{B_1} |Q_0(u_r, 1)| \leq C \sup_r |u_r| < \infty.$$

ii. Lemma 2.2 implies that $\{u_r\}_{r>0}$ is bounded in $C^{1,\alpha}(B_1) \cap W^{2,p}(B_1)$ for every $\alpha < 1$ and $p > 1$. Hence, the result follows from i. \qed

3 $C^{1,1}$ regularity: general case

In this section we utilize the previous technical tools and prove $C^{1,1}$ regularity provided that $f = f(x, t)$ satisfies assumption $A$:

**Assumption A.**

(i) $|f(x, t_2) - f(x, t_1)| \leq h(x)\omega(|t_2 - t_1|)$, where $h \in L^\infty(B_1)$ and

$$\int_0^\epsilon \frac{\omega(t)}{t} \, dt < \infty,$$

for some $\epsilon > 0$;

(ii) The Newtonian potential of $x \mapsto f(x, t)$ is $C^{1,1}$ locally uniformly in $t$: for $v_t := f(\cdot, t) \ast N$ where $N$ is the Newtonian potential,

$$\sup_{a \leq t \leq b} \|D^2 v_t\|_{L^\infty(B_1)} < \infty, \quad \text{for all } a, b \in \mathbb{R}.$$

**Proof of Theorem 1.1.** Let $y \in B_{1/2}$ and $v = v_{u(y)} = f(x, u(y)) \ast N$. Note that if

$$u_r(x) = \frac{u(rx + y) - rx \cdot \nabla u(y) - u(y)}{r^2} - \Pi_y(u, r),$$

then

$$\Pi_y(u_r/2) - \Pi_y(u, r) = \Pi_y(u_r, 1/2) - \Pi_y(u_r, 1) = \Pi_y(u_r, 1/2).$$
We conclude via Lemma 2.6 and Lemma 2.4.

To generate examples, consider \( f(x, t) = \phi(x)\psi(t) \). If \( \phi \in L^\infty \) and \( \psi \) is Dini, then \( f \) satisfies condition (i). If \( \phi \ast N \) is \( C^{1, 1} \) and \( \psi \) is locally bounded, then \( f \) satisfies (ii). Thus if \( \phi \ast N \) is \( C^{1, 1} \) and \( \psi \) is Dini, then \( f \) satisfies both conditions. In particular, \( f \) may be strictly weaker than Dini in the \( x \)-variable.

**Remark 2.** The projection \( Q_y \) has similar properties to \( \Pi_y \). Consequently, if \( f \) satisfies assumption A, (5) holds for \( \Pi_y \) replaced by \( Q_y \).

### 4 \( C^{1, 1} \) regularity: discontinuous case

The goal of this section is to investigate the optimal regularity for solutions of (1) with \( f \) having a jump discontinuity in the \( t \)-variable. This case may be viewed as a free boundary problem. The idea is to employ again an \( L^2 \) projection operator.
4.1 Two-phase obstacle problem

Suppose $f = f(x, u)$ has the form

$$f(x, u) = g_1(x, u)\chi_{\{u > 0\}} + g_2(x, u)\chi_{\{u < 0\}},$$

where $g_1, g_2$ are continuous. We recall from the introduction that if $f$ has a jump in $u$ at the origin, then we assume it to be a positive jump:

**Assumption B.** $g_1(x, 0) - g_2(x, 0) \geq \sigma_0$, $x \in B_1$ for some $\sigma_0 > 0$.

**Remark 3.** In the unstable obstacle problem, i.e. $g_1 = -1$, $g_2 = 0$, there exists a solution which is $C^{1,\alpha}$ for any $\alpha \in (0, 1)$ but not $C^{1,1}$.

Let $\Gamma^0 := \Gamma \cap \{|\nabla u| = u = 0\}$ and $\Gamma^1 := \Gamma \cap \{|\nabla u| \neq 0\}$. Our main result provides optimal growth away from points with sufficiently small gradients.

**Theorem 4.1.** Suppose $g_1, g_2 \in C^0$ satisfy B. Then for all constants $\theta, M > 0$ there exist $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $C_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for any solution of (11) with $\|u\|_{L^\infty(B_1)} \leq M$

$$\|Q_g(u, r)\|_{L^2(\partial B_1(0))} \leq C_0,$$

for all $r \leq r_0$ and $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$. Consequently, for the same choice of $r$ and $y$ we have that

$$\sup_{x \in B_r} |u(x + y) - x \cdot \nabla u(y)| \leq C_1 r^2,$$

for some constant $C_1(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$.

The proof of the theorem is carried out in several steps. A crucial ingredient is the following monotonicity result.

**Lemma 4.2.** Suppose $g_1, g_2 \in C^0$ satisfy B. Then for all constants $\theta, M > 0$ there exist $\kappa_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $r_0(\theta, M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for any solution $u$ of (11) with $\|u\|_{L^\infty(B_1)} \leq M$ if

$$\|Q_g(u, r)\|_{L^2(\partial B_1)} \geq \kappa_0,$$

for some $0 < r < r_0$ and $y \in B_{1/2} \cap \Gamma \cap \{|\nabla u(y)| < \theta r\}$, then

$$\frac{d}{dr} \int_{\partial B_1} Q_g^2(u, r) d\mathcal{H}^{n-1} > 0.$$

**Proof.** If the conclusion is not true, then there exist radii $r_k \to 0$, solutions $u_k$ and points $y_k \in B_{1/2} \cap \Gamma_k \cap \{|\nabla u_k(y_k)| < \theta r_k\}$ such that $\|u_k\|_{L^\infty(B_1)} \leq M$, and $\|Q_g(u_k, r_k)\|_{L^2(\partial B_1)} \to \infty$, and

$$\frac{d}{dr} \int_{\partial B_1} Q_g^2(u_k, r) d\mathcal{H}^{n-1} \bigg|_{r=r_k} \leq 0.$$
Let

\[ T_k := \|Q_{y_k}(u_k, r_k)\|_{L^2(\partial B_1)}, \]

and consider the sequence

\[ v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k). \]

Without loss of generality we can assume that \( y_k \to y_0 \) for some \( y_0 \in B_{1/2} \).

Lemma 2.11 implies the existence of a function \( v \) such that up to a subsequence

\[ v_k(x) = \frac{u_k(r_k x + y_k) - r_k x \cdot \nabla u_k(y_k)}{r_k^2} - Q_{y_k}(u_k, r_k) \to v, \text{ in } C^{1,\alpha}_\text{loc}(\mathbb{R}^n) \cap W^{2,p}_\text{loc}(\mathbb{R}^n). \]

Evidently, \( v(y_0) = |\nabla v(y_0)| = 0 \). Moreover, for \( q_k(x) := Q_{y_k}(u_k, r_k)/T_k \), we can assume that up to a further subsequence, \( q_k \to q \) in \( C^\infty \) for some \( q \in \mathcal{P}_2 \). Note that

\[ \Delta v_k(x) = g_1(r_k x + y_k, u_k(r_k x + y_k))\chi_{\{u_k(r_k x + y_k) > 0\}} + g_2(r_k x + y_k, u_k(r_k x + y_k))\chi_{\{u_k(r_k x + y_k) < 0\}} \]

hence

\[ \Delta v_k \to \Delta v = g_1(y_0, 0)\chi_{\{q(x) > 0\}} + g_2(y_0, 0)\chi_{\{q(x) < 0\}}. \]

By Lemma 2.11

\[ 0 \geq \frac{d}{dr} \int_{\partial B_1} Q_{y_k}^2(u_k, r) d\mathcal{H}^{n-1} \bigg|_{r=r_k} = \frac{2}{r_k} \int_{B_1} Q_{y_k}(u_k, r_k) \Delta u_k(r_k x + y_k) dx \]

\[ = \frac{2T_k}{r_k} \int_{B_1} q_k(x) \Delta v_k(x) dx. \]

Therefore

\[ \int_{B_1} q_k(x) \Delta v_k(x) dx \leq 0. \]

On the other hand

\[ \lim_{k \to \infty} \int_{B_1} q_k(x) \Delta v_k(x) dx = \int_{B_1} q(x) \left( g_1(0, y_0)\chi_{\{q(x) > 0\}} + g_2(0, y_0)\chi_{\{q(x) < 0\}} \right) dx \]

\[ = (g_1(0, y_0) - g_2(0, y_0)) \int_{q(x) > 0} q(x) dx > 0, \]

a contradiction. \( \square \)

**Proof of Theorem 4.1** Let \( \kappa_0 \) and \( r_0 \) be the constants from Lemma 4.2. Without loss of generality we can assume that \( r_0 \leq 1/4 \). From Lemmas 2.4 and 2.12 we have that

\[ \|Q_y(u, r_0)\|_{L^2(\partial B_1)} \leq C \log \frac{1}{r_0}. \]
for all \( y \in B_{1/2} \), where \( C = C(M, \|g_1\|_{\infty}, \|g_2\|_{\infty}, n) \) is a constant. Take

\[
C_0 = \max \left( k_0, 2C \log \frac{1}{r_0} \right).
\]

We claim that

\[
\|Q_y(u, r)\|_{L^2(\partial B_1)} \leq C_0,
\]

for \( r \leq r_0 \) and \( y \in B_{1/2} \cap \Gamma \cap \{ |\nabla u(y)| < \theta r \} \). Let us fix \( y \) such that \( |\nabla u(y)| \leq \theta r_0 \) and consider

\[
T_y(r) := \|Q_y(u, r)\|_{L^2(\partial B_1)}
\]

as a function of \( r \) on the interval \( \frac{|\nabla u(y)|}{\theta} \leq r \leq r_0 \). Let

\[
e := \inf \{ r \text{ s.t. } T_y(r) \leq C_0 \}.
\]

We have that \( T_y(r_0) < C_0 \), so \( \frac{|\nabla u(y)|}{\theta} < e < r_0 \). If \( e = \frac{|\nabla u(y)|}{\theta} \) then \( T_y(e) = C_0 \) and by Lemma 4.2 we have that \( T_y(r) < C_0 \) for \( e - \varepsilon < r < e \) which contradicts (8).

Therefore, \( e = \frac{|\nabla u(y)|}{\theta} \) and \( T_y(r) \leq C_0 \) for all \( |\nabla u(y)|/\theta \leq r \leq r_0 \) which proves (6).

Inequality (7) follows from Lemmas 2.2 and 2.12.

Remark 4. Note that \( A \) is the condition given in Theorem 1.1. If \( g_i \) only depend on \( x \), then this reduces to the assumption that the Newtonian potential of \( g_i \) is \( C^{1,1} \), which is sharp.

Proof of Theorem 1.2. Suppose \( A \) and \( B \) hold. We show that for every \( \delta > 0 \) there exists \( C_{\delta} > 0 \) such that for all \( y \in B_{1/2}(0) \) such that \( \text{dist}(y, \Gamma^1) \geq \delta \), there exists \( r_y > 0 \) such that

\[
\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C_{\delta},
\]

for \( r \leq r_y \).

Consequently,

\[
|u(x) - u(y) - \nabla u(y)(x - y)| \leq \tilde{C}_{\delta}|x - y|^2
\]

for \( |x - y| \leq r_y \), \( y \in B_{1/2}(0) \) and \( \text{dist}(y, \Gamma^1) \geq \delta \); this readily yields the desired result.

Note that (10) follows from (9) via Lemmas 2.2 and 2.12.

Without loss of generality assume that \( \delta \leq r_0 \), where \( r_0 > 0 \) is the constant from Theorem 1.1. For every \( y \in B_{1/2}(0) \) consider the ball \( B_{\delta/2}(y) \). Then there are two possibilities.
i. \( B_{\delta/2}(y) \cap \Gamma^0 = \emptyset \).

In this case \( B_{\delta/2} \cap \Gamma = \emptyset \), hence \( u \) satisfies the equation

\[
\Delta u = g_i(x, u)
\]

in \( B_{\delta/2}(y) \) for \( i = 1 \) or \( i = 2 \). Inequality (5) in the Theorem 1.1 assumption \( A \) yields

\[
\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C \log \frac{4}{\delta} + C(\|D^2v^i_{u(y)}\|_{\infty} + 1),
\]

for \( r \leq \delta/4 \).

ii. \( B_{\delta/2}(y) \cap \Gamma^0 \neq \emptyset \).

Let \( w \in \Gamma^0 \) be such that \( d := |y - w| = \text{dist}(y, \Gamma_0) \). We have that \( d \leq \delta/2 \). As before, assumption \( A \) yields

\[
\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2v^i_{u(y)}\|_{\infty} + 1),
\]

for \( r \leq d/2 \). From Theorem 4.1 we have that

\[
\left| u \left( y + \frac{d}{2} z \right) \right| \leq C \left| y + \frac{d}{2} z - w \right|^2 \leq Cd^2,
\]

for all \( |z| \leq 1 \) because \( d \leq \delta/2 \leq r_0 \). On the other hand

\[
Q_y(u, d/2) = \text{Proj}_{P_2} \left( u \left( y + \frac{d}{2} z \right) - \frac{d}{2} z \cdot \nabla u(y) - u(y) \right) - \frac{d}{2} / 4,
\]

where \( \text{Proj}_{P_2} \) is the \( L^2(\partial B_1(0)) \) projection on the space \( P_2 \). We have used the fact that the projection of a linear function is 0. Hence

\[
\|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} \leq \left\| u \left( y + \frac{d}{2} z \right) \right\|_{L^2(\partial B_1(0))} \leq C,
\]

which yields

\[
\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C + C(\|D^2v^i_{u(y)}\|_{\infty} + 1),
\]

for \( r \leq d/2 \).

The proof is now complete.
Lastly we point out that if the coefficients $g_i$ are regular enough to provide $C^{1,1}$ solutions at points where the gradient does not vanish, then we obtain full interior $C^{1,1}$ regularity.

**Assumption C.** For any $M > 0$ there exist $\theta_0(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ and $C_3(M, \|g_1\|_\infty, \|g_2\|_\infty, n) > 0$ such that for all $z \in B_{1/2}$ any solution of

\[
\begin{cases}
\Delta v = g_1(x, v) \chi_{v>0} + g_2(x, v) \chi_{v<0}, & x \in B_{1/2}(z); \\
|v(x)| \leq M, & x \in B_{1/2}(z); \\
v(z) = 0, & 0 < |\nabla v(z)| \leq \theta_0/4; \\
v|_{\partial B_{1/2}(z)} \text{ continuous,}
\end{cases}
\]

admits a bound

\[\|D^2v\|_{L^\infty(B_{1/2}(z))/\theta_0(z)} \leq C_3.\]

**Remark 5.** A sufficient condition which ensures C is that $g_i$ are Hölder continuous, see [LSE09, Proposition 2.6] and [ADN64, Theorem 9.3]. The idea being that at such points, the set $\{u = 0\}$ is locally $C^{1,\alpha}$ (via the implicit function theorem) and one may thereby reduce the problem to a classical PDE for which up to the boundary estimates are known.

Theorem 4.1 and C imply Theorem 1.3.

**Proof of Theorem 1.3.** By Lemmas 2.12 and 2.6 the assertion follows if we show that there exist $\rho_0, C > 0$ such that for every $y \in B_{\rho_0}(0)$ there exists $r_y > 0$ such that

\[\|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C \quad (11)\]

for $0 < r \leq r_y$.

Let $\rho_0$ be such that $|\nabla u(y)| \leq \theta_0$ for $y \in B_{\rho_0}(0)$, where $\theta_0$ is the constant from assumption C (we can do this because $u$ is $C^{1,\alpha}$ and $0 \in \Gamma^0$). For $y \in B_{\rho_0}(0)$ let $d := \text{dist}(y, \Gamma)$ and let $w \in \Gamma$ be such that $d = |y - w|$.

From Corollary 1.2 we can assume that $2d < r_0$. One of the following cases is possible.

i. $d = 0, y \in \Gamma^0$.

In this case we have that (11) holds for $r \leq r_0$ by Theorem 1.4.

ii. $d = 0, y \in \Gamma^1$.

Here, (11) follows from the assumption C.

iii. $d > 0, w \in \Gamma^0$.

$u$ solves $\Delta u = g_i(x, u)$ in $B_{d/2}(y)$ for $i = 1$ or $i = 2$. Then, by the analysis similar to the one in Corollary 1.2 we get that (11) holds for $r \leq d/2$. 

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iv. $d > 0, w \in \Gamma^1$.

From Theorem 4.1 we have that
\[ |u(z + w) - z \cdot \nabla u(w)| \leq C_1|z|^2 \] (12)
for $|\nabla u(w)|/\theta_0 \leq |z| \leq r_0$. On the other hand, by assumption C we obtain that (12) holds for $|z| \leq |\nabla u(w)|/\theta_0$. Hence, (12) holds for all $z$ such that $|z| \leq r_0$.

By assumption A we have that
\[ \|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} + C(\|D^2v_i^u(y)\|_{\infty} + 1), \]
for $r \leq d/2$.

Furthermore,
\[ Q_y(u, d/2) = \text{Proj}_{P^2_2} \left( u \left( y + \frac{d}{2} z \right) - \frac{d}{2} z \cdot \nabla u(y) - u(y) \right) \]
\[ = \text{Proj}_{P^2_2} \left( u \left( y + \frac{d}{2} z \right) - (y + \frac{d}{2} z - w) \cdot \nabla u(w) \right). \]

Hence from (12) we get
\[ \|Q_y(u, d/2)\|_{L^2(\partial B_1(0))} \leq \left\| u \left( y + \frac{d}{2} z \right) - (y + \frac{d}{2} z - w) \cdot \nabla u(w) \right\|_{L^2(\partial B_1(0))} \]
\[ \leq C, \]
which yields
\[ \|Q_y(u, r)\|_{L^2(\partial B_1(0))} \leq C + C(\|D^2v_i^u(y)\|_{\infty} + 1), \]
for $r \leq d/2$.

The previous analysis applies to the following example.

**Example.** Let $g_i(x, u) = \lambda_i(x)$ for $i = 1, 2$, where $\lambda_i$ are such that
i. $\lambda_1(x) - \lambda_2(x) \geq \sigma_0 > 0$ for all $x \in B_1$;
ii. $\lambda_1(x), \lambda_2(x)$ are Hölder continuous.

We recall from the introduction that under the stronger assumption $\inf_{B_1} \lambda_1 > 0, \inf_{B_1} -\lambda_2 > 0$, this problem is studied in [LSE09] and the optimal interior $C^{1,1}$ regularity is established. The authors use a different approach based on monotonicity formulas and an analysis of global solutions via a blow-up procedure.
4.2 No-sign obstacle problem

Here we observe that assumption $A$ implies that the solutions of (3) are in $C^{1,1}(B_{1/2})$. This theorem was proven in [ALS13] (Theorem 1.2) for the case when $g(x,t)$ depends only on $x$. Under assumption $A$, appropriate modifications of the proof in [ALS13] work also for the general case; since the arguments are similar, we provide only a sketch of the proof and highlight the differences.

**Sketch of the proof of Theorem 1.4.**

Let $\tilde{\Gamma} := \{ y \text{ s.t. } u(y) = |\nabla u(y)| = 0 \}$.

For $r > 0$ let $\Lambda_r := \{ x \in B_1 \text{ s.t. } u(rx) = 0 \}$ and $\lambda_r := |\Lambda_r|$.

The proof of Theorem 1.2 in [ALS13] consists of the following ingredients.

- Interior $C^{1,1}$ estimate
- Quadratic growth away from the free boundary
- [ALS13, Proposition 5.1]

Let us recall that the interior $C^{1,1}$ estimate is the inequality

$$\|u\|_{C^{1,1}(B_{d/2})} \leq C \left( \|g\|_{L^\infty(B_d)} + \frac{\|u\|_{L^\infty(B_d)}}{d^2} \right), \quad (13)$$

where $\Delta u(x) = g(x)$ for $x \in B_d$ and the Newtonian potential of $g$ is $C^{1,1}$. This estimate is purely a consequence of $g$ having a $C^{1,1}$ Newtonian potential.

Quadratic growth away from the free boundary is a bound

$$|u(x)| \leq C \text{dist}(x, \tilde{\Gamma})^2. \quad (14)$$

The first observation in [ALS13] is that if $g(x,t) = g(x)$ has a $C^{1,1}$ Newtonian potential, then (14) and (13) yield $C^{1,1}$ regularity for the solution. Indeed, “far” from the free boundary, the solution $u$ solves the equation $\Delta u = g(x)$ and is locally $C^{1,1}$ by assumption. For points close to the free boundary, $u$ solves the same equation but now on a small ball centered at the point of interest and touching the free boundary. At this point one invokes (14) and by (13) obtains that the $C^{1,1}$ bound does not blow up close to the free boundary (see Lemma 4.1 in [ALS13]).

To prove (14), the authors prove in Proposition 5.1 [ALS13] that if the projection $\Pi_y(u, r)$ (for some $y \in \tilde{\Gamma}$) is large enough then the density $\lambda_r$ of the coincidence set diminishes at an exponential rate. On the other hand, if $\lambda_r$ diminishes in an exponential rate, $\Pi_y(u, r)$ has to be bounded. Consequently, by invoking Lemma 2.2 one obtains (14).

Now let $g$ satisfy $A$.

- Interior $C^{1,1}$ estimate

In the general case, (13) is replaced by

$$\|Q_y(u,s)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u,r)\|_{L^2(\partial B_1(0))} + C(\|D^2v_{u(y)}\|_{\infty} + 1), \quad (15)$$

where $0 < s < r < d$, $\Delta v_{u(y)} = g(x, u(y))$ and $\Delta u = f(x, u)$ in $B_d(y)$. Estimate (15) is purely a consequence of assumption $A$ (see [5] in the proof of Theorem 1.1).
In this proposition, it is shown that there exists $C$ such that if $\Pi_y(u,r) \geq C$ then
\[
\lambda_{1/2}^1 \leq \frac{\hat{C}}{\|\Pi_y(u,r)\|_{L^\infty(B_1)}} \lambda_{1/2}^1
\] (16)
for some $\hat{C} > 0$. The inequality is obtained by the decomposition
\[
u(rx + y) = \Pi_y(u,r) + h_r + w_r,
\]
where $h_r, w_r$ are such that
\[
\begin{aligned}
\{ & \Delta h_r = -g(rx + y)\chi_{\Lambda_r} \quad \text{in } B_1, \\
& h_r = 0 \quad \text{on } \partial B_1,
\end{aligned}
\]
and
\[
\begin{aligned}
\{ & \Delta w_r = g(rx + y) \quad \text{in } B_1, \\
& w_r = \frac{u(rx+y)}{r^2} - \Pi_y(u,r) \quad \text{on } \partial B_1.
\end{aligned}
\]
The authors show that
\[
\|D^2 h_r\|_{L^2(B_1/2)} \leq C\|g\|_{L^\infty} \|\chi_{\Lambda_r}\|_{L^2(B_1)},
\]
\[
\|D^2 w_r\|_{L^2(B_1/2)} \leq C \left(\|g\|_{L^\infty} + \|u\|_{L^\infty(B_1)}\right).
\] (17)
In the general case one may consider the decomposition
\[
u(rx + y) = Q_y(u,r) + h_r + w_r + z_r,
\]
where $h_r, w_r, z_r$ are such that
\[
\begin{aligned}
\{ & \Delta h_r = -g(rx + y,0)\chi_{\Lambda_r} \quad \text{in } B_1, \\
& h_r = 0 \quad \text{on } \partial B_1,
\end{aligned}
\]
and
\[
\begin{aligned}
\{ & \Delta w_r = g(rx + y,0) \quad \text{in } B_1, \\
& w_r = \frac{u(rx+y)}{r^2} - Q_y(u,r) \quad \text{on } \partial B_1,
\end{aligned}
\]
and
\[
\begin{aligned}
\{ & \Delta z_r = (g(rx + y, u(rx+y)) - g(rx + y,0))\chi_{B_1 \setminus \Lambda_r} \quad \text{in } B_1, \\
& z_r = 0 \quad \text{on } \partial B_1.
\end{aligned}
\]
Evidently, estimates (17) are still valid. Additionally, we have
\[
\|D^2 z_r\|_{L^2(B_1/2)} \leq C\|\Delta z_r\|_{L^2(B_1)} \leq C\omega(r^2\log \frac{1}{r}),
\] (18)
since $g(x,t)$ is uniformly Dini in $t$.

Combining (17) and (18) and arguing as in [ALS13] one obtains the existence of $C > 0$ such that
\[
\lambda_{r/2}^{1/2} \leq \tilde{C} \frac{\lambda_{r/2}^{1/2}}{\|Q_y(u,r)\|_{L^2(\partial B_1)}} + \omega \left( r^2 \log \frac{1}{r} \right), \tag{19}
\]
whenever $\|Q_y(u,r)\|_{L^2(\partial B_1)} \geq C$.

- **Quadratic growth away from the free boundary**

In [ALS13], the norms of $\Pi_y(u,r/2^j), k \geq 1$ are estimated in terms of the sum $\sum_{j=0}^{\infty} \lambda_{r/2^j}$. If the norms of projections are unbounded, one obtain estimate (16) which implies convergence of the previous sum and hence boundedness of the projections. This is a contradiction.

Similarly, in the general case the norms of $Q_y(u,r/2^j), k \geq 1$ can be estimated by
\[
\sum_{j=0}^{\infty} \lambda_{r/2^j} + \sum_{j=0}^{\infty} \omega \left( \left( \frac{r}{2^j} \right)^2 \log \frac{r}{2^j} \right).
\]

Inequality (19) and Dini continuity imply
\[
\sum_{j=0}^{\infty} \omega \left( \left( \frac{r}{2^j} \right)^2 \log \frac{r}{2^j} \right), \sum_{j=0}^{\infty} \lambda_{r/2^j} < \infty,
\]
if the norms of projections are unbounded. Furthermore, one completes the proof of the quadratic growth as in [ALS13].

To verify that the above ingredients imply $C^{1,1}$ regularity, we split the analysis into two cases. If we are “far” from the free boundary, $u$ locally solves $\Delta u = g(x,u)$ so by Theorem 3.1 $u$ is $C^{1,1}$. If we are close to the free boundary then $u$ solves $\Delta u = g(x,u)$ in a small ball $B_d(y)$ that touches the free boundary. We invoke (15) for $0 < s < r = d/2$ and the quadratic growth to obtain
\[
\|Q_y(u,s)\|_{L^2(\partial B_1(0))} \leq \|Q_y(u,d/2)\|_{L^2(\partial B_1)} + C(\|D^2v_{u(y)}\|_\infty + 1)
\leq C \left| \frac{u(y+d/2x)}{d^2/4} \right|_{L^2(\partial B_1)} + C(\|D^2v_{u(y)}\|_\infty + 1)
\leq C + C(\|D^2v_{u(y)}\|_\infty + 1).
\]
for $s \leq d/2$.

So there exists a constant $C$ such that for all $y \in B_{1/2}$ there exist radii $r_j(y) \to 0$ such that
\[
Q_y(u,r_j(y)) \leq C.
\]

We conclude via Lemma 2.6.
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EMANUEL INDREI
CENTER FOR NONLINEAR ANALYSIS
Carnegie Mellon University
Pittsburgh, PA 15213, USA
email: egi@cmu.edu

ANDREAS MINNE
DEPARTMENT OF MATHEMATICS
KTH Royal Institute of Technology
100 44 Stockholm, Sweden
email: minne@kth.se

LEVON NURBEKYAN
CEMSE Division,
King Abdullah University of Science and Technology (KAUST)
Thuwal 23955-6900, Saudi Arabia
email: levon.nurbekyan@kaust.edu.sa