A Conversion Procedure for NNC Polyhedra

Anna Becchi, Enea Zaffanella
Department of Mathematical, Physical and Computer Sciences
University of Parma, Italy
anna.becchi@studenti.unipr.it
enea.zaffanella@unipr.it

Abstract. We present an alternative Double Description representation for the domain of NNC (not necessarily topologically closed) polyhedra, together with the corresponding Chernikova-like conversion procedure. The representation differs from the ones adopted in the currently available implementations of the Double Description method in that it uses no slack variable at all: this new approach provides a solution to a few technical issues caused by the encoding of an NNC polyhedron as a closed polyhedron in a higher dimension space. A preliminary experimental evaluation shows that the new conversion algorithm is able to achieve significant efficiency improvements with respect to state-of-the-art implementations.

1 Introduction

The Double Description (DD) method [28] allows for the representation and manipulation of convex polyhedra by using two different geometric representations: one based on a finite collection of constraints, the other based on a finite collection of generators. Starting from any one of these representations, the other can be derived by application of a conversion procedure [11][12][13], thereby obtaining a DD pair; the procedure allows for the identification and removal of redundant elements from both representations, yielding a DD pair in minimal form; moreover, it is incremental, allowing for capitalizing on the work already done when new constraints and/or generators need to be added to an input DD pair.

The DD method lies at the foundation of several software libraries and tools. The following is an incomplete list of available implementations:

- cdd (www.inf.ethz.ch/personal/fukudak/cdd_home);
- PolyLib (http://icps.u-strasbg.fr/PolyLib);
- NewPolka, part of Apron (http://apron.cri.ensmp.fr/library);
- Parma Polyhedra Library (http://bugseng.com/products/ppl);
- 4ti2 (www.4ti2.de);
- Skeleton (www.uic.unn.ru/~zny/skeleton);
- Addibit (www.informatik.uni-bremen.de/agbs/bgenov/addibit).

Despite the intrinsic exponential complexity of the conversion procedure, these implementations turn out to be surprisingly effective in many contexts. As
a consequence, the range of applicability of the DD method keeps widening, also due to several incremental improvements in the efficiency of the most critical processing phases \cite{20,21,27,35}. Quoting from \cite{20}:

The double description method is a simple and useful algorithm \ldots despite the fact that we can hardly state any interesting theorems on its time and space complexities.

Implementations of the DD method are actively used, either directly or indirectly, in several research fields, with applications as diverse as bioinformatics \cite{32,33}, computational geometry \cite{112}, analysis of analog and hybrid systems \cite{9,19,23,24}, automatic parallelization \cite{7,30}, scheduling \cite{17}, static analysis of software \cite{5,14,16,18,22,25}.

In the classical setting, the DD method is meant to compute geometric representations for topologically closed polyhedra in an $n$-dimensional vector space. However, there are applications requiring the ability to also deal with linear strict inequality constraints, leading to the definition of not necessarily closed (NNC) polyhedra. For example, this is the case for some of the analysis tools developed for the verification of hybrid systems \cite{9,19,23,24}; other examples of the use of NNC polyhedra include static analysis tools such as Pagai \cite{25}, where strict inequality constraints are used to model the semantics of conditional tests acting on program variables of floating point type, as well as the automatic discovery of ranking functions \cite{14} for proving the termination of program fragments.

The few DD method implementations providing support for NNC polyhedra are all based on an indirect representation of the strict inequalities, which are encoded by adding an additional space dimension playing the role of a slack variable. The main advantage of this approach is the possibility of reusing, almost unchanged, all of the well-studied algorithms and optimizations that have been developed for the classical case of closed polyhedra \cite{4,6,23,24}. However, the addition of a slack variable carries with itself an obvious overhead, as well as a few technical issues.

In this paper, we pursue a different approach for the handling of NNC polyhedra in the DD method. Namely, we specify a direct representation, dispensing with the need of the slack variable. The main insight of this new approach is the separation of the (constraints or generators) geometric representation into two components, the skeleton and the non-skeleton of the representation, playing quite different roles: while keeping a geometric encoding for the skeleton component, we will adopt a combinatorial encoding for the non-skeleton one. For this new representation, we propose the corresponding variant of the Chernikova’s conversion procedure, where both components are handled by respective processing phases, so as to take advantage of their peculiarities. In particular, we develop ad hoc functions and procedures for the combinatorial non-skeleton part.

The new representation and conversion procedure, in principle, can be integrated into any of the available implementations of the DD method. Our implementation and experimental evaluation, conducted in the context of the Parma Polyhedra Library, show that the new algorithm, while computing the correct
results for all of the considered tests, achieves impressive efficiency improvements
with respect to the implementation based on the slack variable.

The paper is structured as follows. Section 2, after introducing the required
notation and terminology, briefly describes the Double Description method for
the representation of closed polyhedra, also sketching the Chernikova’s conver-
sion algorithm. Section 3 summarizes the encoding of NNC polyhedra into closed
polyhedra based on the addition of a slack variable, highlighting a few technical
issues. Section 4 proposes the new representation for NNC polyhedra, which
uses no slack variable and distinguishes between a geometric and a combinatorial
component. Section 5 is devoted to the extension of the Chernikova’s conversion
algorithm to the case of NNC polyhedra adopting this new representation. Sec-
tion 6 shows how, by applying duality arguments, all the concepts and results
presented in Sections 4 and 5 for the case of generators can be generalized to also
deal with the case of constraints. Section 7 reports the results obtained by the
experimental evaluation of the new algorithm. We conclude in Section 8. Proof
sketches for the stated results can be found in Appendix A.

This paper is a revision and extension of [8], where the new representation
was introduced and the conversion procedure from constraints to generators was
initially proposed and experimentally evaluated.

2 Preliminaries

We assume some familiarity with the basic notions of lattice theory [10].

For a lattice \( (L, \sqsubseteq, \sqcap, \sqcup, \bot, \top) \), an element \( a \in L \) is an atom if \( \bot \sqsubseteq a \)
and there exists no element \( b \in L \) such that \( \bot \sqsubseteq b \sqsubseteq a \). The lattice \( L \) is said to be
atomistic if every element of \( L \) can be obtained as the join of a set of atoms. For
\( S \subseteq L \), the upward closure of \( S \) is defined as \( \uparrow S \overset{\text{def}}{=} \{ x \in L \mid \exists s \in S . s \sqsubseteq x \} \).
The set \( S \subseteq L \) is upward closed if \( S = \uparrow S \); we denote by \( \wp(\uparrow) \) the set of all the
upward closed subsets of \( L \). For \( x \in L \), \( \uparrow x \) is a shorthand for \( \uparrow \{ x \} \). The notation
for downward closure is similar.

Given two posets \( (L, \sqsubseteq) \) and \( (L^\#, \sqsubseteq^\#) \) and two monotonic functions \( \alpha: L \to L^\# \)
and \( \gamma: L^\# \to L \), the pair \( (\alpha, \gamma) \) is a Galois connection [11] (between \( L \) and \( L^\# \)) if
\[
\forall x \in L, x^\# \in L^\#: \alpha(x) \sqsubseteq^\# x^\# \iff x \sqsubseteq \gamma(x^\#).
\]

We write \( \mathbb{R}^n \) to denote the Euclidean topological space of dimension \( n > 0 \)
and \( \mathbb{R}_+ \), for the set of non-negative reals; for \( S \subseteq \mathbb{R}^n \), \( \text{cl}(S) \) and \( \text{relint}(S) \) denote
the topological closure and the relative interior of \( S \), respectively. The scalar
product of two vectors \( \mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n \) is denoted by \( \mathbf{a}_1^T \mathbf{a}_2 \). For each vector \( \mathbf{a} \in \mathbb{R}^n \),
where \( \mathbf{a} \neq \mathbf{0} \), and scalar \( b \in \mathbb{R} \), the linear non-strict inequality constraint \( \beta = (\mathbf{a}^T x \geq b) \) defines a closed affine half-space of \( \mathbb{R}^n \); similarly, the linear equality
constraint \( \delta = (\mathbf{a}^T x = b) \) defines an affine hyperplane of \( \mathbb{R}^n \).

A topologically closed convex polyhedron (for short, closed polyhedron) is
defined as the set of solutions of a finite system \( C \) of linear non-strict inequality
and linear equality constraints; namely, \( \mathcal{P} = \text{con}(C) \) where
\[
\text{con}(C) \overset{\text{def}}{=} \{ \mathbf{p} \in \mathbb{R}^n \mid \forall \beta = (\mathbf{a}^T x \geq b) \in C, \mathbf{a} \in \{ \geq, = \} : \mathbf{a}^T \mathbf{p} \geq b \}.
\]
A vector \( \mathbf{r} \in \mathbb{R}^n \) such that \( \mathbf{r} \neq \mathbf{0} \) is a ray of a non-empty polyhedron \( \mathcal{P} \subseteq \mathbb{R}^n \) if, for every point \( \mathbf{p} \in \mathcal{P} \) and every non-negative scalar \( \rho \in \mathbb{R}_+ \), it holds \( \mathbf{p} + \rho \mathbf{r} \in \mathcal{P} \). The empty polyhedron has no rays. If both \( \mathbf{r} \) and \( -\mathbf{r} \) are rays of \( \mathcal{P} \), then we say that \( \mathbf{r} \) is a line of \( \mathcal{P} \). By Minkowski and Weyl theorems [31], the set \( \mathcal{P} \subseteq \mathbb{R}^n \) is a closed polyhedron if and only if there exist finite sets \( L, R, P \subseteq \mathbb{R}^n \) of cardinality \( \ell, r, \) and \( m \), respectively, such that \( \mathbf{0} \notin (L \cup R) \) and \( \mathcal{P} = \text{gen}(\langle L, R, P \rangle) \), where

\[
\text{gen}(\langle L, R, P \rangle) \overset{\text{def}}{=} \{ L\lambda + R\rho + P\pi \in \mathbb{R}^n \mid \lambda \in \mathbb{R}^\ell, \rho \in \mathbb{R}^r, \pi \in \mathbb{R}_+^m, \sum_{i=1}^m \pi_i = 1 \}.
\]

When \( \mathcal{P} \neq \emptyset \), we say that \( \mathcal{P} \) is described by the generator system \( \mathcal{G} = \langle L, R, P \rangle \).

In the following, we will abuse notation by adopting the usual set operator and relation symbols to denote the corresponding component-wise extensions on generator systems. For instance, for \( \mathcal{G} = \langle L, R, P \rangle \) and \( \mathcal{G}' = \langle L', R', P' \rangle \), we will write \( \mathcal{G} \subseteq \mathcal{G}' \) to mean \( L \subseteq L', R \subseteq R' \) and \( P \subseteq P' \); similarly, we may write \( \varphi(\mathcal{G}) \) to denote the set of all generator systems \( \mathcal{G}' \) such that \( \mathcal{G}' \subseteq \mathcal{G} \).

The Double Description method due to Motzkin et al. [28], by exploiting the duality principle, allows to combine the constraints and the generators of a polyhedron \( \mathcal{P} \) into a DD pair \( \langle C, \mathcal{G} \rangle \); a conversion procedure is used to obtain each description starting from the other one, also removing the redundant elements. For presentation purposes, we focus on the conversion from constraints to generators; the conversion from generators to constraints works in the same way, using duality to switch the roles of constraints and generators.

The conversion procedure starts from a DD pair \( \langle C_0, \mathcal{G}_0 \rangle \) representing the whole vector space and adds, one at a time, the elements of the input constraint system \( C = \{ \beta_0, \ldots, \beta_m \} \), producing a sequence of DD pairs \( \{ \langle C_k, \mathcal{G}_k \rangle \}_{0 \leq k \leq m+1} \) representing the polyhedra

\[
\mathbb{R}^n = \mathcal{P}_0 \xrightarrow{\beta_0} \cdots \xrightarrow{\beta_k^{-1}} \mathcal{P}_k \xrightarrow{\beta_k} \mathcal{P}_{k+1} \xrightarrow{\beta_{k+1}} \cdots \xrightarrow{\beta_m} \mathcal{P}_{m+1} = \mathcal{P}.
\]

At each iteration, when adding the constraint \( \beta_k \) to polyhedron \( \mathcal{P}_k = \text{gen}(\mathcal{G}_k) \), the generator system \( \mathcal{G}_k \) is partitioned into the three components \( \mathcal{G}^+_k, \mathcal{G}^0_k, \mathcal{G}^-_k \), according to the sign of the scalar products of the generators with \( \beta_k \) (those in \( \mathcal{G}^0_k \) are the saturators of \( \beta_k \)); the new generator system for polyhedron \( \mathcal{P}_{k+1} \) is computed as \( \mathcal{G}_{k+1} = \text{gen}(\mathcal{G}^+_k \cup \mathcal{G}^0_k \cup \mathcal{G}^-_k) \), where

\[
\mathcal{G}^*_k = \text{combine}\_\text{adj}_{\beta_k}(\mathcal{G}^+_k, \mathcal{G}^-_k)
\]

\[
\overset{\text{def}}{=} \{ \text{combine}_{\text{adj}_{\beta_k}}(g^+, g^-) \mid g^+ \in \mathcal{G}^+_k, g^- \in \mathcal{G}^-_k, \text{adjacent}_{\mathcal{P}_k}(g^+, g^-) \}.
\]

Function ‘\( \text{combine}_{\text{adj}_{\beta_k}} \)’ computes a linear combination of its arguments, yielding a generator that saturates the constraint \( \beta_k \); predicate ‘\( \text{adjacent}_{\mathcal{P}_k} \)’ is used to discard those pairs of generators that are not adjacent in \( \mathcal{P}_k \) (since these would only produce redundant generators).

The conversion procedure is usually followed by a simplification step, where the DD pair is modified, without affecting the represented polyhedron, so as
to achieve some form of minimality. For instance, the implicit linear equality
constraints (encoded by non-strict inequalities) are detected and represented
explicitly; similarly, rays are combined to produce lines. We will not provide a
formalization of these details, assuming anyway that these simplifications are
implicitly taken into proper account when needed.

Similarly, it is worth noting that the one sketched above is a high level de-
scription of the conversion procedure; at the implementation level, each closed
polyhedron $P \subseteq \mathbb{R}^n$ is mapped, by homogenization, into a (topologically closed)
convex polyhedral cone $C \subseteq \mathbb{R}^{n+1}$. This process associates a new space dimen-
sion, usually denoted as $\xi$, to the inhomogeneous term of constraints; the new
space dimension is constrained to only assume non-negative values, i.e., the posi-
tivity constraint $\xi \geq 0$ is added to the constraint representation of the polyhedral
cone. When reinterpreted in the $n$ dimensional vector space, this constraint can
be read as the tautology $1 \geq 0$. The inverse map from a convex polyhedral cone
$C$ to the represented convex polyhedron $P$ is obtained by only considering the
points of the cone having a strictly positive coordinate for the $\xi$ dimension:

$$P = \{ x \in \mathbb{R}^n \mid (x^T, \xi)^T \in C, \xi > 0 \}.$$

By homogenization, all of the vertices of the convex polyhedron are mapped into
rays of the convex polyhedral cone: this also allows for a more uniform handling
of the rays and vertices, a property which is suitably exploited in most imple-
mentations. The rays of the convex polyhedral cone can be easily reinterpreted:
those having a zero (resp., positive) coordinate for the $\xi$ space dimension are the
rays (resp., points) of the represented polyhedron.

The set $\mathbb{CP}_n$ of all closed polyhedra on the vector space $\mathbb{R}^n$, partially ordered
by set inclusion, is a lattice $\langle \mathbb{CP}_n, \subseteq, \emptyset, \mathbb{R}^n, \cap, \cup \rangle$, where the emptyset and $\mathbb{R}^n$
are the bottom and top elements, the binary meet operator is set intersection
and the binary join operator $\cup$ is the convex polyhedral hull.

A linear strict inequality constraint $\beta = (a^T x > 0)$ defines an open affine
half-space of $\mathbb{R}^n$. When the constraint system $C$ is extended to also allow for strict
inequalities, the convex polyhedron $P = \text{con}(C)$ is not necessarily (topologically)
closed. The set $\mathbb{P}_n$ of all NNC polyhedra on the vector space $\mathbb{R}^n$ is a lattice
$\langle \mathbb{P}_n, \subseteq, \emptyset, \mathbb{R}^n, \cap, \cup \rangle$ and $\mathbb{CP}_n$ is a sublattice of $\mathbb{P}_n$.

As shown in [4,6], a description of an NNC polyhedron $P \in \mathbb{P}_n$ in terms of
generators can be obtained by also taking into account its closure points, i.e.,
points that belong to the topological closure of the polyhedron, but are not
necessarily included in the polyhedron itself. Namely, the results by Minkowski
and Weil can be generalized to the case of NNC polyhedra [4 Theorem 4.4]: we
can extend the generator system with a finite set $C$ of closure points, obtaining
$\mathcal{G} = \langle L, R, C, P \rangle$ and $P = \text{gen}(\mathcal{G})$, where

$$\text{gen}(\langle L, R, C, P \rangle) \equiv \left\{ L\lambda + R\rho + C\gamma + P\pi \in \mathbb{R}^n \mid \lambda \in \mathbb{R}^l, \rho \in \mathbb{R}^r,
\gamma \in \mathbb{R}^c, \pi \in \mathbb{R}^p, \pi \neq 0, \sum_{i=1}^c \gamma_i + \sum_{i=1}^p \pi_i = 1 \right\}.$$
When needed for notational convenience, we will split a constraint system into three components $C = \langle C_\leq, C_\geq, C_\succ \rangle$; even in this case, as done for the generators, we will abuse the notation for set operator and relation symbols.

3 NNC Polyhedra as Closed Polyhedra

The DD method provides a solid theoretical base for the representation and manipulation of topologically closed convex polyhedra in $\mathbb{CP}_n$. As mentioned in Section 2 at the implementation level the polyhedra are actually mapped into polyhedral cones in $\mathbb{CP}_{n+1}$ by homogenization, but it is not difficult for software libraries to make this detail completely transparent to the end user: in practice, the library developers have to add some syntactic sugar to the input and output routines for constraints and generators, also hiding the positivity constraint.

Things are less straightforward when considering the case of NNC polyhedra. To start with, many implementations of the DD method do not support NNC polyhedra at all. Also, to the best of our knowledge, the few supported implementations of the domain of NNC polyhedra based on the DD method (that is, the NewPolka domain embedded in the Apron library and the NNC_Polyhedron domain in the Parma Polyhedra Library) adopt an indirect representation: namely, each NNC polyhedron $P \in \mathbb{P}_n$ is mapped into a closed polyhedron $R \in \mathbb{CP}_{n+1}$. The mapping encodes the strict inequality constraints by means of an additional space dimension (playing the role of a slack variable); the new space dimension, usually denoted as $\epsilon$, needs to be non-negative and bounded from above, i.e., the constraints $0 \leq \epsilon \leq 1$ are added to the topologically closed representation $R$ (called $\epsilon$-representation) of the NNC polyhedron $P$.

The inverse map $\lfloor \cdot \rceil_{\epsilon}: \mathbb{CP}_{n+1} \rightarrow \mathbb{P}_n$ from an $\epsilon$-representation $R$ to the represented NNC polyhedron $P$ is obtained by only considering the points of $R$ having a strictly positive coordinate for the $\epsilon$ dimension:

$$ P = \lfloor R \rceil_{\epsilon} \overset{\text{def}}{=} \{ x \in \mathbb{R}^n \mid (x^T, \epsilon)^T \in R, \epsilon > 0 \}.$$

This encoding of NNC polyhedra into closed polyhedra was initially proposed in [23,24] and later reconsidered and studied in more detail in [4,6], where a proper interpretation of the $\epsilon$ dimension for the (extended) generator representation was provided.

Besides showing its strengths, the work in [4,6] highlighted the main weakness of the approach: the DD pair in minimal form computed for an $\epsilon$-representation $R$, when reinterpreted as encoding the NNC polyhedron $P = \lfloor R \rceil_{\epsilon}$, typically includes many redundant constraints and/or generators, leading to a possibly high computational overhead. To avoid this problem, strong minimization procedures were defined in [4,6] that are able to detect and remove those redundancies; in practice, these procedures map the representation $R$ into a different representation $R'$ such that $P = \lfloor R' \rceil_{\epsilon}$, where $R'$ encodes no $\epsilon$-redundancies.

\footnote{An alternative representation can be adopted where the $\epsilon$ dimension is unbounded from below [34].}
Example 1. Figure 1 shows two different $\epsilon$-representations for the NNC polyhedron defined by constraints $C = \{1 \leq x, x < 3\}$. In the $\epsilon$-representations, the constraints having a zero coefficient for the slack variable $\epsilon$ encode a non-strict inequality, such as the one defining facet $[p_0, p_3]$, corresponding to $1 \leq x$; the constraints having a non-zero coefficient for $\epsilon$ encode either the slack variable bounds $0 \leq \epsilon \leq 1$ or the proper strict inequalities, such as the one defining facet $[p_1, p_2]$ (resp., $[p_1, p'_2]$ on the right hand side $\epsilon$-representation), corresponding to $x < 3$. Note that the facet $[p'_2, p_3]$ in the right hand side $\epsilon$-representation is an example of $\epsilon$-redundant constraint, since it is encoding the redundant strict inequality $x < 4$.

When carefully applying strong minimization procedures, most of the overhead of the $\epsilon$-representation is thus avoided, leading to implementations that easily meet the efficiency requirements of many application contexts. For the users of the libraries, the addition of the $\epsilon$ dimension is almost unnoticed, to the point that quite often the domain of NNC polyhedra is adopted even when not really needed (i.e., when a domain of topologically closed polyhedra would be enough). However, the approach described above still suffers from a few issues.

1. At the implementation level, more work is needed to make the $\epsilon$ dimension transparent to the end user and, as a matter of fact, its adoption can sometimes become evident. For instance, a strict constraint such as $x > 30$ may be encoded as $2x - \epsilon \geq 60$, which is then shown to the user as the (unsimplified) strict constraint $2x > 60$. Besides being annoying, the growth in the magnitude of the integer coefficients may cause a computational overhead.

2. The $\epsilon$-representation brings with itself an intrinsic overhead: in any generator system for an $\epsilon$-polyhedron, most of the “proper” points (those having a positive $\epsilon$ coordinate) need to be paired with the corresponding “closure” point (having a zero $\epsilon$ coordinate); this systematically leads to almost doubling the size of the generator system.

3. The strong minimization procedures, even though effective, interfere with the incremental approach of the DD conversion procedures. After applying the strong minimization procedure on the constraint (resp., generator) representation of a DD pair, the dual generator (resp., constraint) representation is lost and, in order to recover it, the non-incremental conversion

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**Fig. 1.** Two $\epsilon$-representations in $\mathbb{CP}_2$ for $P = \text{con}(\mathcal{C}) \in \mathcal{P}_1$, where $\mathcal{C} = \{1 \leq x, x < 3\}$. 
procedure needs to be applied once again. This also implies that the strong 
minimization procedures can not be fully integrated into the DD conversion 
procedures: they are applied after the conversions. As a consequence, during 
the iterations of the conversion procedure, the redundancies caused by the 
$\epsilon$-representation are not removed, causing the computation of bigger inter-
mediate results. For the reasons above, the strong minimization procedures 
are not systematically used in the implementation of the Parma Polyhedra 
Library; rather, they are applied only when strictly needed for correctness. 
Therefore, the end user is left with the responsibility of guessing whether or 
not the strong minimization procedures are going to improve efficiency.

The most important of the issues listed above were known since [6]. As a mat-
ter of fact, both [4] and [5] put forward the possibility of devising an alternative 
approach regarding the representation and manipulation of NNC polyhedra in 
the DD framework. Quoting from [5]:

It would be interesting, from both a theoretical and practical point of 
view, to provide a more direct encoding of NNC polyhedra, i.e., one that 
is not based on the use of slack variables […]

The main obstacle on the road towards such a goal is the definition of a con-
version procedure that is not only correct, but also competitive with respect to 
the highly tuned implementations available in software libraries such as Apron 
and the Parma Polyhedra Library. It is worth stressing that several experimental 
evaluations, including recent ones [2], confirm that the Parma Polyhedra Library 
is a state-of-the-art implementation of the DD method for a wide spectrum of 
application contexts.

4 Direct Representations for NNC Polyhedra

As briefly recalled in Section 2, an NNC polyhedron can be described by us-
ing an extended constraint system $C = \langle C_e, C_\geq, C_> \rangle$, possibly containing strict 
inequalities, and/or an extended generator system $G = \langle L, R, C, P \rangle$, possibly 
containing closure points. These representations are said to be geometric, mean-
ing that they provide a precise description of the position of all the elements in 
the constraint/generator system.

For a closed polyhedron $P \in \mathbb{RP}_n$, the use of completely geometric repre-
sentations is an adequate choice: it is possible to provide a DD pair $(C, G)$ that is 
“canonical”[4] In the case of an NNC polyhedron $P \in \mathbb{P}_n$, the adoption of a com-
pletely geometric representation can be seen as an overkill, since the knowledge 
of the precise geometric position of some of the elements is not really needed.

4 Strictly speaking, the canonical form for constraints (resp., generators) still depends 
on the specific representation chosen for the non-redundant set of equality constraints 
(resp., generating lines). Even those can be made canonical and each software library 
typically provides its own canonical form.
Example 2. Consider the NNC polyhedron $\mathcal{P} \in \mathbb{P}_2$ in Figure 2, where the (strict) inequality constraints are denoted by (dashed) lines and the (closure) points are denoted by (unfilled) circles. The polyhedron can be seen to be described by generator system $\mathcal{G} = \langle L, R, C, P \rangle$, where $L = R = \emptyset$, $C = \{ c_0, c_1, c_2 \}$ and $P = \{ p_0, p_1 \}$. However, there is no need to know the precise position of point $p_1$, since it can be replaced by any other point on the open segment $(c_0, c_1)$. Similarly, when considering the constraint representation, there is no need to know the exact slope of the strict inequality constraint $\beta$, as it can be replaced by any other strict inequality that is satisfied by all the points in $\mathcal{P}$ and saturated by closure point $c_0$.

In other words, some of the elements in the geometric representations of NNC polyhedra are better described by combinatorial information, rather than geometric. The following section introduces the terminology and notation needed to reason on this combinatorial information.

4.1 The combinatorial structure of convex polyhedra

A linear inequality or equality constraint $\beta = (a^T x \preceq b)$ is said to be valid for the polyhedron $\mathcal{P} \in \mathbb{P}_n$ if all the points in $\mathcal{P}$ satisfy $\beta$; for each such $\beta$, the subset $F = \{ p \in \mathcal{P} \mid a^T p = b \}$ is a face of $\mathcal{P}$. We write $c\text{Faces}_\mathcal{P}$, omitting the subscript when clear from context, to denote the finite set of faces of $\mathcal{P} \in \mathbb{P}_n$; the set $c\text{Faces}_\mathcal{P}$ is a sublattice of $\mathbb{P}_n$, having the empty face as bottom element and the whole polyhedron $\mathcal{P}$ as top element. Note that we have

$$\mathcal{P} = \bigcup \{ \text{relint}(F) \mid F \in c\text{Faces}_\mathcal{P} \}.$$

The face lattice is also known as the combinatorial structure of the polyhedron. If the polyhedron is bounded (i.e., it is a polytope, having no rays and lines), then the lattice is atomistic, meaning that each face can be obtained as the convex polyhedral hull of the vertices contained in the face.

Even in the case of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ it is possible to define the finite set $nnc\text{Faces}_\mathcal{P}$ of its faces, which is a sublattice of $\mathbb{P}_n$; hence, each face is an NNC polyhedron and, as before, we have

$$\mathcal{P} = \bigcup \{ \text{relint}(F) \mid F \in nnc\text{Faces}_\mathcal{P} \}.$$
In this case, however, the lattice may be non-atomistic even when the polyhedron is bounded. Letting $Q = \text{cl}(P)$, the closure operator $\text{cl}: \text{nncFaces}_P \rightarrow \text{cFaces}_Q$ maps each NNC face of $P$ into a distinct, corresponding (closed) face of $Q$. The image $\text{cl}(\text{nncFaces}_P)$ is a join sublattice of $\text{cFaces}_Q$; meets are generally not preserved, since there may exist $F_1, F_2 \in \text{nncFaces}_P$ such that

$$F_1 \cap F_2 = \emptyset \neq \text{cl}(F_1) \cap \text{cl}(F_2).$$

The image of the set of non-empty faces $\text{cl}(\text{nncFaces}_P \setminus \{\emptyset\})$ is an upward closed subset of $\text{cFaces}_Q$; hence, it can be efficiently described by recording just the set of its minimal elements. For each NNC face $F \subseteq P$ corresponding to one of these minimal elements (that is, for each atom of the $\text{nncFaces}_P$ lattice), we have $F = \text{relint}(F)$. As a consequence, the combinatorial structure of $P \in \mathbb{P}_n$ can be described by integrating the combinatorial structure of its topological closure $Q \in \mathbb{C}\mathbb{P}_n$ with the information identifying the atoms of $\text{nncFaces}_P$.

**Example 3.** Consider the polyhedron $P$ in Figure 2. The lattice $\text{nncFaces}_P$ has two atoms: the 0-dimension face $\{p_0\}$ and the 1-dimension open segment $(c_0, c_1)$; note that both atoms are relatively open sets. Also note that, even if $P$ is an NNC polytope, the lattice is not atomistic: for instance, the half-open segment $(c_0, p_0]$ is a 1-dimension face that can not be obtained by joining the atoms.

### 4.2 Skeleton and non-skeleton of an NNC polyhedron

Let $P \in \mathbb{P}_n$ be an NNC polyhedron and $Q = \text{cl}(P) \in \mathbb{C}\mathbb{P}_n$ be its topological closure. As explained above, a description of $P$ can be obtained by combining a geometric representation of $Q$, which will be called the skeleton component, with some combinatorial information related to $\text{nncFaces}_P$ (the non-skeleton component). We now provide formal definitions that allow for splitting a fully geometric representation for $P$ into these two components. For exposition purposes, here we will consider the generator system representation only; the definitions for the constraint system representation are similar and will be briefly described in a later section.

**Definition 1 (Skeleton of a generator system).** Let $G = \langle L, R, C, P \rangle$ be a generator system in minimal form, $P = \text{gen}(G)$ and $Q = \text{cl}(P)$. The skeleton of $G$ is the generator system

$$\text{SK}_Q = \text{skel}(G) \overset{\text{def}}{=} \langle L, R, C \cup \text{SP}, \emptyset \rangle,$$

where $\text{SP} \subseteq P$ is the set of points that can not be obtained as a combination of the other generators in $G$.

Note that the skeleton has no points at all, so that $\text{gen}(\text{SK}_Q) = \emptyset$. However, we can define a variant function ‘full.gen’, that reinterprets the closure points to be points,

$$\text{full.gen}(\langle L, R, C, P \rangle) \overset{\text{def}}{=} \text{gen}(\langle L, R, \emptyset, C \cup P \rangle),$$

so as to obtain the following result.

---

5 This term is unrelated to the concept of $p$-skeleton used in algebraic topology.
Proposition 1. Let \( P = \text{gen}(G) \) and \( Q = \text{cl}(P) \). Then
\[
\text{full} \cdot \text{gen}(G) = \text{full} \cdot \text{gen}(SK_Q) = Q.
\]
Also, there does not exist \( G' \subset SK_Q \) such that \( \text{full} \cdot \text{gen}(G') = Q \).

In other words, the skeleton of an NNC polyhedron can be seen to provide a non-redundant representation of its topological closure. The elements of \( SP \subseteq P \) are called skeleton points; the non-skeleton points in \( P \setminus SP \) are redundant when representing the topological closure, since they can be obtained by combining the lines in \( L \), the rays in \( R \) and the closure points in \( C \); these non-skeleton points are the elements in \( G \) that need not be represented geometrically.

Example 4. For the polyhedron in Figure 2, \( SK_Q = \langle \emptyset, \emptyset, \{ c_0, c_1, c_2, p_0 \}, \emptyset \rangle \), so that \( p_0 \) is a skeleton point and \( p_1 \) is a non-skeleton point (it can be generated by combining \( c_0 \) and \( c_1 \)).

Having modeled the skeleton component for \( P = \text{gen}(L, R, C, P) \), we now turn our attention to the non-skeleton component. As discussed in Section 4.1, our goal is to provide a combinatorial representation for the set of points \( P \). Reasoning slightly more generally, consider a point \( p \in Q = \text{cl}(P) \) (not necessarily in \( P \)). There exists a single face \( F \in \text{cFaces}_Q \) such that \( p \in \text{relint}(F) \).

By definition of function ‘gen’, point \( p \) behaves as a filler for \( \text{relint}(F) \), meaning that, when combined with the skeleton, it generates \( \text{relint}(F) \). Note that \( p \) also behaves as a filler for the relative interiors of all the faces in the set \( \uparrow F \). The choice of \( p \in \text{relint}(F) \) is actually arbitrary: any other point of \( \text{relint}(F) \) would be equivalent as a filler.

Proposition 2. Consider a polyhedron \( P = \text{gen}(L, R, C, P) \). For \( p \in P \), let \( F \) be the face of \( Q = \text{cl}(P) \) such that \( p \in \text{relint}(F) \); let \( p' \in \text{relint}(F) \), and \( P' = P \setminus \{ p \} \cup \{ p' \} \). Then \( P = \text{gen}(L, R, C, P') \).

A less arbitrary representation for \( \text{relint}(F) \) is thus provided by its own skeleton \( SK_F \subseteq SK_Q \); namely, each (geometric) filler \( p \in P \) can be mapped into a more abstract (combinatorial) representation, the subset of \( SK_Q \) identifying the corresponding face. For each face \( F \in \text{cFaces}_Q \), we say that the skeleton subset \( SK_F \subseteq SK_Q \) is the support for the points in \( \text{relint}(F) \) and that any point \( p' \in \text{relint}(\text{full} \cdot \text{gen}(SK_F)) = \text{relint}(F) \) is a materialization of \( SK_F \).

Definition 2 (Support sets for a skeleton). Let \( SK \) be the skeleton of an NNC polyhedron and let \( Q = \text{full} \cdot \text{gen}(SK) \in \mathbb{CP}_n \). Then the set \( NS_{SK} \) of all supports for \( SK \) is defined as
\[
\text{NS}_{SK} \overset{\text{def}}{=} \{ SK_F \subseteq SK \mid F \in \text{cFaces}_Q \}.
\]

By definition, the set \( NS_{SK} \) is a lattice isomorphic to \( \text{cFaces}_Q \); we will drop the subscripts \( SK \) and \( Q \) when clear from context.

We now define a pair of abstraction and concretization functions mapping a subset of the (geometric) points of an NNC polyhedron into the set of supports that are filled by these points, and vice versa.
\textbf{Definition 3 (Filled supports).} Let $SK$ be the skeleton of the polyhedron $P \in \mathbb{P}_n$, $Q = cl(P)$ and $NS$ be the corresponding set of supports. The abstraction function $\alpha_{SK}: \wp(Q) \to \wp(NS)$ is defined, for each $S \subseteq Q$, as

$$\alpha_{SK}(S) \overset{\text{def}}{=} \bigcup \{ \uparrow_{SK} F \mid \exists \theta \in S, F \in cFaces. \theta \in \text{relint}(F) \}.$$ 

The concretization function $\gamma_{SK}: \wp_{\uparrow}(NS) \to \wp(Q)$, for each $NS \in \wp_{\uparrow}(NS)$, is defined as

$$\gamma_{SK}(NS) \overset{\text{def}}{=} \bigcup \{ \text{relint(full.gen}(ns) \} \mid ns \in NS \}.$$ 

\textbf{Proposition 3.} The pair of functions $(\alpha_{SK}, \gamma_{SK})$ is a Galois connection.

By Proposition 3, the composition $(\gamma_{SK} \circ \alpha_{SK})$ is an upper closure operator mapping each non-empty set of points $S \subseteq Q$ into the smallest NNC polyhedron containing $S$ and having $SK$ as the skeleton component. In particular, the following result holds.

\textbf{Proposition 4.} Let $P = \text{gen}(\{L, R, C, P\}) \in \mathbb{P}_n$ and let $SK$ be the corresponding skeleton component. Then $P = (\gamma_{SK} \circ \alpha_{SK})(P)$.

The non-skeleton component of a geometrical generator system, can be abstracted by ‘$\alpha_{SK}$’ and described as a combination of skeleton generators.

\textbf{Definition 4 (Non-skeleton of a generator system).} Let $P \in \mathbb{P}_n$ be defined by generator system $G = \langle L, R, C, P \rangle$ and let $SK$ be the corresponding skeleton component. The non-skeleton component of $G$ is defined as $NS_G \overset{\text{def}}{=} \alpha_{SK}(P)$.

Even in this case, we will drop the subscript when clear from context. Note that, by definition of the abstraction function ‘$\alpha_{SK}$’, the non-skeleton component $NS$ contains an upward closed set of supports, therefore representing all the faces of the NNC polyhedron.

\textbf{Example 5.} We now show the non-skeleton component for the polyhedron in Figure 2. Since in this polyhedron we have no rays and no lines, we will adopt a simplified notation, identifying each support with the set of its closure points.

By Definition 3, we have:

$$\alpha_{SK}(\{p_0\}) = \{ \{p_0\}, \{c_0, p_0\}, \{c_2, p_0\}, \{c_0, c_1, c_2, p_0\} \},$$

$$\alpha_{SK}(\{p_1\}) = \{ \{c_0, c_1\}, \{c_0, c_1, c_2, p_0\} \};$$

hence, the non-skeleton component is computed as

$$NS_G = \alpha_{SK}(\{p_0, p_1\}) = \{ \{p_0\}, \{c_0, p_0\}, \{c_2, p_0\}, \{c_0, c_1, c_2, p_0\}, \{c_0, c_1\} \}.$$ 

The minimal elements in $NS_G$ are the supports $\{p_0\}$ and $\{c_0, c_1\}$, which can be seen to describe the atoms of the face lattice $\text{nncFaces}_P$.

By combining Definition 4 with Proposition 4 we obtain the following result, stating that the new representation is semantically equivalent to the fully geometric one.

\textbf{Corollary 1.} For a polyhedron $P = \text{gen}(G) \in \mathbb{P}_n$, let $(SK, NS)$ be the skeleton and non-skeleton components for $G$. Then $P = \gamma_{SK}(NS)$. 

12
5 The New Conversion Algorithm

When working with direct representations of NNC polyhedra, the Chernikova’s conversion algorithm needs to be extended to properly handle closure points and strict inequalities.

A first attempt in this direction was developed in [29]. In that case, the $\epsilon$-less encoding for constraints and generators was not distinguishing the skeleton and non-skeleton components, thereby adopting geometric-only representations. The main difference with respect to the classical conversion algorithm for closed polyhedra was in the combination phase, where the sets of generators $G^+$, $G^-$ are processed to produce the new set of generators $G^\star$: this phase was extended in [29] to perform a systematic case analysis on the generator kinds and to also consider the set $G^0$. Even though the resulting algorithm is correctly specified, it suffers from a high computational overhead because, as highlighted in [34], the new combination phase needs to also consider pairs of generators that are not adjacent; this prevents the adoption of the key optimizations that were developed for the closed polyhedra case, making the overall approach infeasible from a practical point of view.

The new representation described in Section 4, by distinguishing the skeleton and non-skeleton components, allows for a corresponding separation in the conversion procedure: while the skeleton component can be handled following the classical combination procedure for closed polyhedra, the non-skeleton will be managed using a few brand new procedures that can correctly deal with closure points and strict inequalities without incurring into a significant overhead.

As already pointed out in Section 4, we will focus on the conversion from constraints to generators. The conversion working the other way round will be obtained, as usual, by applying duality arguments.

The conversion function is shown as Pseudocode 1. In the following, we will describe its main steps, first introducing some implementation details and then explaining the auxiliary functions and procedures.

5.1 Encoding the new representation

In Section 4 it was shown how the geometric generator system $G$ can be equivalently represented by the pair $\langle SK, NS \rangle$, where $SK = \langle L, R, C \cup SP, \emptyset \rangle$ is the skeleton component and $NS \subseteq \wp_\uparrow(NS)$ is the non-skeleton component. We now discuss a few minor adaptations to this representation that are meant to result in efficiency improvements at the implementation level.

First, observe that every support $ns \in NS$ always includes all of the lines in the $L$ skeleton component; hence, these lines can be left implicit in the representation of the supports in $NS$. Note that, even after removing the lines, each $ns \in NS$ is still a non-empty set, since it includes at least one closure point.

When lines are implicit, those supports $ns \in NS$ that happen to be singletons can be seen to play a special role: they correspond to the combinatorial

---

6 Since the support $ns$ is a subset of the skeleton $SK$, by ‘singleton’ here we mean a system $ns = \langle \emptyset, \emptyset, \{p\}, \emptyset \rangle$. 

encoding of the skeleton points in $SP$ (see Definition 1). These points are not going to benefit from the combinatorial representation, since their geometric position is uniquely identified (modulo the lines component). Therefore, we will remove them from the non-skeleton $NS$ and directly include them in the point component of the skeleton $SK$: namely, the skeleton $SK = \langle L, R, C \cup SP, \emptyset \rangle$ will be actually represented as $SK = \langle L, R, C, SP \rangle$. We stress that this is only done as an optimization: the formalization presented in Section 4 is still valid, with just a minor adaptation to the definition of the function $\gamma$, which is replaced by the following:

$$\gamma_{SK}'(NS) \overset{\text{def}}{=} \text{gen}(SK) \cup \gamma_{SK}(NS).$$

We also remark that, at the implementation level, each support $ns \in NS$ can be encoded by using a set of indices on the data structure representing the skeleton component $SK$. Since $NS$ is a finite upward closed set, the representation only needs to record its minimal elements. In this low level representation, the non-minimal elements can be efficiently identified (and removed) by performing appropriate inclusion tests on these sets of indices. When also considering the optimization for skeleton points mentioned before, we can adopt the following definition of redundancy.

**Definition 5 (Redundant support).** A support $ns \in NS$ is said to be redundant in $\langle SK, NS \rangle$ if there exists $ns' \in NS$ such that $ns' \subset ns$ or if $ns \cap SP \neq \emptyset$, where $SK = \langle L, R, C, SP \rangle$.

In the following, we will write $NS_1 \oplus NS_2$ to denote the non-redundant union of the support sets $NS_1, NS_2 \subseteq NS_{SK}$.

### 5.2 Processing the skeleton component

From a high level point of view, the conversion function in Pseudocode follows the same structure as the classical conversion procedure for closed polyhedra: it incrementally processes each of the input constraints $\beta \in C_{in}$ keeping the generator system $\langle SK, NS \rangle$ up-to-date. In this section, we focus on the handling of the skeleton component $SK = \langle L, R, C, SP \rangle$.

The first processing step (line 3) of the main loop is the partitioning of the skeleton $SK$ according to the signs of the scalar products with constraint $\beta$. Since the skeleton component is entirely geometric, it can be split into $SK^+, SK^0$ and $SK^-$ exactly as done in the Chernikova’s algorithm. In the pseudocode, this partition info is kept implicit inside the data structure encoding $SK$: we will freely use the superscripts to refer to each component when needed.

Note that lines 8 to 10 of the conversion function are meant to take care of a line violating $\beta$, whereas lines 11 to 22 are meant to efficiently handle those special cases when $SK^+$ or $SK^-$ happens to be empty; these will be briefly discussed later on. Hence, the second main processing step for the skeleton component occurs in lines 24 to 25 where the generators in $SK^+$ and $SK^-$ are combined to produce $SK^*$, which is then merged into $SK^0$. This step too is quite similar to
Pseudocode 1  Incremental conversion from constraints to generators.

function conversion($C_{in}$, $⟨SK, NS⟩$)

2: for all $β ∈ C_{in}$ do
3:     skel_partition($β$, $SK$);
4:     nonskel_partition($⟨SK, NS⟩$);
5:     if line $l ∈ SK^+ ∪ SK^−$ then $β$ violates line $l$
6:         violating-line($β$, $l$, $⟨SK, NS⟩$);
7:     else if $SK^− = ∅$ then
8:         if is_equality($β$) then
9:             if $SK^0 = ∅$ then $⊿ P$ is empty
10:                return $⟨∅, ∅⟩$;
11:         else if is_strict_ineq($β$) then
12:             if $SK^+ = ∅$ then $⊿ P$ is empty
13:                return $⟨∅, ∅⟩$;
14:         else if $SK^0 ≠ ∅$ then
15:             strict_on_eq_points($β$, $⟨SK, NS⟩$);
16:     else
17:         $⟨SK, NS⟩ ← ⟨SK^0, NS^0⟩$;
18:     end of loop on $C_{in}$
19:     return $⟨SK, NS⟩$;

20: end of loop on $C_{in}$

21: $SK^* ←$ combine_adj($SK^+, SK^-$);
22: $SK^0 ← SK^0 ∪ SK^*$;
23: $NS^* ←$ move-ns($β$, $⟨SK, NS⟩$);
24: $NS^0 ← NS^0 ∪$ create-ns($β$, $⟨SK, NS⟩$);
25: if is_equality($β$) then
26:     $⟨SK, NS⟩ ← ⟨SK^0, NS^0 ∪ NS^*⟩$;
27: else if is_nonstrict_ineq($β$) then
28:     $⟨SK, NS⟩ ← ⟨SK^0 ∪ SK^0, (NS^0 ∪ NS^0) ∪ NS^*⟩$;
29: else
30:     $⟨SK, NS⟩ ←$ points_become_closure_points($SK^0$);
31:     $⟨SK, NS⟩ ← ⟨SK^+ ∪ SK^0, (NS^0 ∪ NS^0) ∪ NS^*⟩$;
32:     promote-singletons($⟨SK, NS⟩$);
33: else
34:     promote-singletons($⟨SK, NS⟩$);
35:     $⊿ P$ is empty
36: return $⟨SK, NS⟩$;
the one for closed polyhedra described in Section 2, except that we now have to consider how the different generator kinds combine with each other, according to the kind of constraint $\beta$: the systematic case analysis is presented in Table 1.

The table shows that, for instance, when processing a non-strict inequality $\beta \geq$, if we combine a closure point in $\mathcal{SK}^+$ with a ray in $\mathcal{SK}^-$ we shall obtain a closure point in $\mathcal{SK}^*$ (row 3, column 6).

| $\beta = \text{or} \beta \geq$ | $\mathcal{SK}^+$ | $\mathcal{SK}^-$ | $\mathcal{SK}^*$ |
|-------------------------------|------------------|------------------|------------------|
| $\beta >$                     | R C C C C C C SP SP SP | R C SP R C SP R C SP | R C C C C C C C C |

Table 1. Case analysis for function ‘combine’ when adding an equality ($\beta =$), a non-strict ($\beta \geq$) or a strict ($\beta >$) inequality constraint to a pair of generators from $\mathcal{SK}^+$ and $\mathcal{SK}^-$ ($R =$ ray, $C =$ closure point, $SP =$ skeleton point).

A crucial observation regarding this combination phase is that, since it is restricted to work on the skeleton component only, it can safely apply the adjacency tests to quickly get rid of all those combinations that would introduce redundant elements (for the skeleton component). Also note how the direct inclusion of the skeleton points $SP$ in $\mathcal{SK}$ (as discussed in Section 5.1), besides simplifying the non-skeleton representation, allows for processing them using the adjacency tests. Nonetheless, since the points in $SP$ should behave as fillers, they will have to be properly reconsidered when processing the non-skeleton component $NS$.

The final processing steps for the skeleton component, occurring in lines 28 to 34, are those meant to update the generator system for the next iteration. The new skeleton is computed according to the constraint kind, similarly to what done in the closed polyhedra case. However, an additional processing step (line 33) is needed for the case of a strict inequality constraint: the helper function

$$\text{points\_become\_closure\_points}(\langle L, R, C, SP \rangle) \overset{\text{def}}{=} \langle L, R, C \cup SP, \emptyset \rangle$$

applied to $\mathcal{SK}^0$, makes sure that all of the skeleton points saturating $\beta$ are transformed into closure points having the same position.

### 5.3 Processing the non-skeleton component

We now consider the handling of the non-skeleton component $NS$, which is clearly where the new algorithm significantly differs from the corresponding algorithm for closed polyhedra.

**Partitioning.** The first processing step (line 4) is the partitioning of the supports in $NS$, so as to detect their position with respect to the constraint $\beta$. To
this end, we can exploit the partition info already computed for the skeleton $S\mathcal{K}$ to obtain the corresponding partition info for $NS$, without computing any additional scalar product. Namely, each support $ns \in NS$ is classified as follows:

$$
ns \in NS^+ \iff ns \subseteq (S\mathcal{K}^+ \cup S\mathcal{K}^0) \land ns \cap SK^+ \neq \emptyset;
$$

$$
ns \in NS^0 \iff ns \subseteq SK^0;
$$

$$
ns \in NS^- \iff ns \subseteq (S\mathcal{K}^- \cup S\mathcal{K}^0) \land ns \cap SK^- \neq \emptyset;
$$

$$
ns \in NS^{\pm} \iff ns \cap SK^+ \neq \emptyset \land ns \cap SK^- \neq \emptyset.
$$

Note that the partitioning above is fully consistent with respect to the one computed for skeleton elements. For instance, if $ns \in NS^+$, then for every possible materialization $p \in \text{relint}(\text{full.gen}(ns))$ the scalar product of $p$ and $\beta$ is strictly positive. Things are similar when $ns \in NS^0$ and $ns \in NS^-$. The supports in $NS^{\pm}$ are those whose materializations can indifferently satisfy, saturate or violate the constraint $\beta$ (i.e., the corresponding face crosses the constraint hyperplane). As did for the skeleton, even in this case the partition info is kept implicit inside the data structure encoding $NS$.

**Pseudocode 2** Helper procedure for promoting singleton supports.

```plaintext
procedure PROMOTE-SINGLETONS((SK, NS))
let SK = ⟨L, R, C, SP⟩;
for all ns ∈ NS such that ns = ⟨∅, ∅, {c}, ∅⟩ do
    NS ← NS \ {ns};
    C ← C \ {c};
    SP ← SP ∪ {c};
```

As said before, we delay for the moment the discussion of lines 5 to 22 of the conversion function, proceeding directly to explain lines 26 and 27, where we find the calls of the two main functions processing the non-skeleton component. A set $NS^*$ of brand new supports is built as the union of the contributes provided by functions $\text{MOVE-NS}$ and $\text{CREATE-NS}$. This set, which contains the supports generated in a given iteration of the main loop, will be later merged into the appropriate portions of the non-skeleton component, chosen according to the constraint kind (see lines 28 to 34). The final processing step of the main loop (line 35) calls helper procedure $\text{PROMOTE-SINGLETONS}$ (shown in Pseudocode 2), making sure that all singleton supports get promoted to skeleton points.

**Moving supports.** The $\text{MOVE-NS}$ function, shown in Pseudocode 3, processes the supports in $NS^{\pm}$. As hinted by its name, the goal of this function is to “move” the fillers of the faces that are crossed by the new constraint, making sure they lie on the correct side.

Let $ns \in NS^{\pm}$ and consider the face $F = \text{relint}(\text{full.gen}(ns))$. Note that $F$ is a face of the polyhedron before the addition of the new constraint $\beta$; at this
Pseudocode 3 Helper function for moving supports.

```
function move-ns(β, ⟨SK, NS⟩)
2: NS* ← ∅;
for all ns ∈ NS± do
4: NS* ← NS* ∪ {proj_βSK(supp.clSK(ns))};
return NS*;
```

Point, the elements in SK* have been added to SK0, but this change still has to be propagated to the non-skeleton component NS. Therefore, we compute the support closure ‘supp.cl_SK(ns)’ of the support ns according to the updated skeleton SK. Intuitively, supp.cl_SK(ns) ⊆ SK is the subset of all the skeleton elements that are included in face F.

At the implementation level, the support closure operator can be efficiently computed by exploiting the same saturation information that is needed to quickly perform the adjacency tests. Namely, given the constraints C and the generators G, we can define the functions

sat.inter_C(G) = {β' ∈ C | ∀g ∈ G : g saturates β'},
sat.inter_C(C) = {g ∈ G | ∃β' ∈ C : g saturates β'}.

Then, if C and SK = ⟨L,R,C,SP⟩ are the constraint system and the skeleton generator system defining the polyhedron, for each ns ∈ NS we can compute the support closure as follows [26]:

$$supp.cl_{SK}(ns) \equiv sat.inter_{SK}(sat.inter_{C}(ns)) \setminus L.$$ 

Face F is intuitively split by constraint β into the three subsets $F^+$, $F^0$, and $F^-$. When β is a strict inequality, only $F^+$ shall be kept in the polyhedron; when the new constraint is a non-strict inequality, both $F^+$ and $F^0$ shall be kept. When working with the updated support, a non-skeleton representation for these subsets can be obtained by projecting the support on the corresponding portions of the skeleton. Namely, we can define the function

$$proj_{β}^{SK}(ns) \equiv \begin{cases} ns \setminus SK^-, & \text{if } β \text{ is a strict inequality;} \\ ns \cap SK^0, & \text{otherwise.} \end{cases}$$

Since the projection operator is applied after having computed the support closure, when β is a non-strict inequality we have $ns \cap SK^0 \neq \emptyset$; hence, the support of $F^0$ is a subset of the support of $F^+$ and $proj_{β}^{SK}(ns)$ will be a filler for $F^+$ too.

To summarize, by composing support closure and projection in line 4 of move-ns, each support in $NS^\pm$ is moved to the correct side of β.

Example 6. Consider the polyhedron $P \in P_2$ in the left hand side of Figure 3 described by the skeleton and non-skeleton components $⟨SK, NS⟩$. The skeleton $SK = ⟨∅, ∅, C, ∅⟩$ is composed by the four closure points in $C = \{c_0, c_1, c_2, c_3\}$;
the non-skeleton \( NS = \{ ns \} \) contains a single support \( ns = \{ c_0, c_3 \} \), which makes sure that the open segment \((c_0, c_3)\) is included in \( P \); in the figure, we show just one of the many possible materializations for \( ns \).

When processing the strict inequality constraint \( \beta = (y < 1) \), we obtain the polyhedron in the right hand side of the figure. In the skeleton phase of the \textsc{conversion} function the adjacent skeleton generators are combined: \( c_4 \) (combining \( c_0 \in SK^+ \) and \( c_3 \in SK^- \)) and \( c_5 \) (combining \( c_1 \in SK^+ \) and \( c_2 \in SK^- \)) are added to \( SK^0 \). Since the non-skeleton support \( ns \) belongs to \( NS^\pm \), it is processed in the \textsc{move-ns} function:

\[
ns^* = \text{proj}^\beta_{SK} \left( \text{supp} \cdot \text{cl}_{SK}(ns) \right) \\
= \text{supp} \cdot \text{cl}_{SK}(\{c_0, c_3\}) \setminus SK^- \\
= \{c_0, c_3, c_4\} \setminus \{c_2, c_3\} \\
= \{c_0, c_4\}.
\]

Intuitively, we have moved \( ns \) to \( ns^* \): again, for the new support we show only one of its many possible materializations, but it is clear that now they all satisfy constraint \( \beta \).

Example 7. In the left hand side of Figure 4 we reconsider the same polyhedron of Example 6 but we now add the non-strict inequality \( \beta' = (y \leq 1) \). The skeleton phase of the \textsc{conversion} procedure behaves exactly as shown before, producing closure points \( c_4 \) and \( c_5 \). We then process \( ns \in NS^\pm \) in the \textsc{move-ns}
function:

\[ \begin{align*}
ns^* &= \text{proj}_{\mathcal{SK}}(\text{supp.cl}_{\mathcal{SK}}(ns)) \\
&= \text{supp.cl}_{\mathcal{SK}}(\{c_0, c_3\}) \cap \mathcal{SK}^0 \\
&= \{c_0, c_3, c_4\} \cap \{c_4, c_5\} \\
&= \{c_4\}.
\end{align*} \]

Since \( ns^* \) is a singleton, it will be upgraded to become a skeleton point by procedure `promote-singletons`, thereby obtaining the new skeleton component \( \mathcal{SK} = (\emptyset, \emptyset, C, SP) \), where \( C = \{c_0, c_1, c_5\} \) and \( SP = \{c_4\} \), and the new non-skeleton component \( NS = \emptyset \). Hence, we obtain the polyhedron in the right hand side of the figure; note that the skeleton point \( c_4 \) is responsible for the inclusion of the facets \( (c_0, c_4] \) and \( [c_4, c_5) \) in the polyhedron.

**Creating new supports.** On the one hand, the choice of representing only the minimal elements of the upward closed set \( NS \) enables many efficiency improvements; on the other hand, it also means that some care has to be taken before removing these minimal elements.

As an example, consider the case of a support \( ns \in NS^- \) when dealing with a non-strict inequality constraint \( \beta \): this support is going to be removed from \( NS \) in line 31 of the conversion function. However, by doing so, we are also implicitly removing other supports from the set \( \uparrow ns \), here included some supports that do not belong to \( NS^- \) and hence should be kept in \( NS \). Therefore, at each iteration, we have to explore the set of filled faces and detect the ones that are going to lose their filler: the corresponding minimal supports will be added to \( NS^+ \). Moreover, when processing a non-strict inequality constraint, we also need to consider the new faces introduced by the constraint: the corresponding supports can be found by projecting on the constraint hyperplane those faces that are possibly filled by an element in \( SP^+ \) or \( NS^+ \).

This is the task of the `create-ns` function, shown in Pseudocode 4. This function uses `enumerate-faces` as a helper; the latter provides an enumeration of all the (higher dimensional) faces that contain the initial support \( ns \). The new faces are obtained by adding to \( ns \) a new generator \( g \) and then composing the projection and support closure functions, as done in function `move-ns`.

For efficiency purposes, in function `create-ns` a case analysis is performed so as to suitably restrict the search area of the enumeration phase. Since the faces we are going to compute have to be projected, it is enough to consider those that can cross the constraint: hence, when adding a new generator \( g \) to a non-skeleton support \( ns \), we consider only those coming from the opposite side of the constraint (for instance, when processing \( ns \in NS^- \) we consider \( g \in \mathcal{SK}^+ \), disregarding the generators in \( \mathcal{SK}^- \) and \( \mathcal{SK}^0 \)). We also avoid adding a point to \( ns \), since this would definitely yield a redundant support.

---

7 This enumeration phase is inspired by the algorithm in [26].
Pseudocode 4 Helper functions for creating new supports.

function \textsc{create-ns}(\beta, (SK, NS))
\begin{align*}
2: & \quad NS^* \leftarrow \emptyset; \\
& \quad \text{let } SK = (L, R, C, SP); \\
4: & \quad \text{for all } p \in SP^- \text{ do} \\
& \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, \{p\}, SK); \\
6: & \quad \text{for all } ns \in NS^- \text{ do} \\
& \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, ns, SK^+, SK); \\
8: & \quad \text{if is\_strict\_ineq}(\beta) \text{ then} \\
& \quad \quad \text{for all } p \in SP^0 \text{ do} \\
& \quad \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, \{p\}, SK^+, SK); \\
& \quad \quad \text{for all } ns \in NS^0 \text{ do} \\
& \quad \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, ns, SK^+, SK); \\
& \quad \text{else if is\_nonstrict\_ineq}(\beta) \text{ then} \\
& \quad \quad \text{for all } p \in SP^+ \text{ do} \\
& \quad \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, \{p\}, SK^-, SK); \\
& \quad \quad \text{for all } ns \in NS^+ \text{ do} \\
& \quad \quad \quad NS^* \leftarrow NS^* \cup \text{enumerate-faces}(\beta, ns, SK^-, SK); \\
18: & \quad \text{return } NS^*; \\
\end{align*}

function \textsc{enumerate-faces}(\beta, ns, SK', SK)
\begin{align*}
2: & \quad NS^* \leftarrow \emptyset; \\
& \quad \text{let } SK' = (L', R', C', SP'); \\
4: & \quad \text{for all } g \in (R' \cup C') \text{ do} \\
& \quad \quad ns' \leftarrow ns \cup \{g\}; \\
6: & \quad NS^* \leftarrow NS^* \cup \{\text{proj}_SK(\text{supp.cl}_{SK}(ns'))\}; \\
& \quad \text{return } NS^*; \\
\end{align*}

---

Fig. 5. Application of \textsc{create-ns} when adding a strict inequality.
Example 8. Consider the polyhedron \( P \in \mathbb{P}_2 \) on the left hand side of Figure 5. The skeleton \( SK = \langle \emptyset, C, \emptyset \rangle \) is composed by the four closure points \( C = \{c_0, c_1, c_2, c_3\} \); the non-skeleton \( NS = \{ns\} \) contains a single support \( ns = \{c_2, c_3\} \), which makes sure that the open segment \((c_2, c_3)\) is included in \( P \). By upward closure, this non-skeleton point is also the filler for the whole polyhedron; in particular, it fills \( \text{relint}(P) \).

The strict inequality makes \( ns \in NS^- \), since all the generators in the support are in \( SK^- \); hence, support \( ns \) is processed by line 7 of function \textsc{create-ns}. The call to function \textsc{enumerate-faces} will produce new supports by adding to \( ns \) a generator from \( SK^+ \) and then computing the corresponding support closure and projection. Namely, it will compute

\[
\text{proj}_{\beta}^SK(\text{supp.cl}_{SK}(ns \cup \{c_0\})) = \text{supp.cl}_{SK}(ns \cup \{c_0\}) \setminus SK^-
\]
\[
= \{c_0, c_1, c_2, c_3, c_4, c_5\} \setminus \{c_2, c_3\}
\]
\[
= \{c_0, c_1, c_4, c_5\},
\]
\[
\text{proj}_{\beta}^SK(\text{supp.cl}_{SK}(ns \cup \{c_1\})) = \text{supp.cl}_{SK}(ns \cup \{c_1\}) \setminus SK^-
\]
\[
= \{c_0, c_1, c_2, c_3, c_4, c_5\} \setminus \{c_2, c_3\}
\]
\[
= \{c_0, c_1, c_4, c_5\}.
\]

Hence, the new (minimal) support \( ns^* = \{c_0, c_1, c_4, c_5\} \) will be added to \( NS^* \).

The resulting polyhedron, shown in the right hand side of the figure, is described by the skeleton \( SK = \langle \emptyset, \emptyset, \{c_0, c_4, c_5\}, \emptyset \rangle \) and the non-skeleton \( NS = \{ns^*\} \).

Fig. 6. Application of \textsc{create-ns} when adding a non-strict inequality.

Example 9. Consider polyhedron \( P \in \mathbb{P}_2 \) in the left hand side of Figure 6 described by skeleton \( SK = \langle \emptyset, \emptyset, \{c_0, c_1, c_2\}, \{p\} \rangle \) and non-skeleton \( NS = \emptyset \). The partition for \( SK \) induced by the non-strict inequality is as follows:

\[
SK^+ = \langle \emptyset, \emptyset, \{p\} \rangle,
\]
\[
SK^0 = \langle \emptyset, \emptyset, \{c_0, c_2\}, \emptyset \rangle,
\]
\[
SK^- = \langle \emptyset, \emptyset, \{c_1\}, \emptyset \rangle.
\]

There are no adjacent generators in \( SK^+ \) and \( SK^- \), so that the call to function ‘\text{combine_adj}(SK^+, SK^-)’ on line 24 of \textsc{conversion} leaves \( SK^* \) empty.
When processing the non-skeleton component, the skeleton point in $SK^+$ will be considered in line 15 of function `create-ns`. The corresponding call to function `enumerate-faces` produces new supports by first adding to $\{p\}$ each generator in $SK^-$ and then computing the corresponding support closure and projection. Namely, it will compute

$$ns^* = \text{proj}_{SK}^\beta (\text{supp.cl}_{SK}(\{p\} \cup \{c_1\}))$$

$$= \text{supp.cl}_{SK}(\{c_1, p\}) \cap SK^0$$

$$= \{c_0, c_1, c_2, p\} \cap \{c_0, c_2\}$$

$$= \{c_0, c_2\},$$

thereby producing the filler for the open segment $(c_0, c_2)$. The resulting polyhedron, shown in the right hand side of the figure, is thus described by the skeleton $SK = \langle \emptyset, \emptyset, \{c_0, c_2\}, \{p\} \rangle$ and the non-skeleton $NS = \{ns^*\}$.

It is worth noting that, when handling Example 9 adopting an entirely geometrical representation (as done in [29]), closure point $c_1$ needs to be geometrically combined with point $p$ even if these two generators are not adjacent. In general, this leads to a significant efficiency penalty. Similarly, an implementation based on the $\epsilon$-representation will have to geometrically combine closure point $c_1$ with point $p$ (and/or with some other $\epsilon$-redundant points), because the addition of the slack variable makes them adjacent.

In contrast, an implementation based on the new approach is going to obtain a twofold benefit: first, the distinction between the skeleton and non-skeleton components allows for restricting the handling of non-adjacent combinations to the non-skeleton phase, thereby recovering the corresponding optimizations on the skeleton part; second, by exploiting the combinatorial representation, the non-skeleton component can be processed by using set index operations only, i.e., computing no linear combination at all. As a consequence, the implementation is able to correctly deal with closure points and strict inequalities without a significant increase in the number of computationally heavy operations.

**Handling special cases.** In the previous paragraphs we have provided an explanation of the core of the `conversion` function. We conclude by briefly discussing those portions of Pseudocode 1 that are meant to efficiently handle some special cases. Note that, being just optimizations, these portions could be removed without compromising correctness.

In lines 5 to 6 of `conversion` we consider the case when constraint $\beta$ is violated by a line. This special case is handled in procedure `violating-line` in Pseudocode 4. The pseudocode is similar to the corresponding special case for topologically closed polyhedra except that, when processing a strict inequality constraint, the helper procedure `strict-on-eq-points` gets called; this can be seen as a tailored version of the `create-ns` function, also including the final updating of $SK$ and $NS$.

In lines 7 to 22 of `conversion` we consider instead the cases when $SK^+$ or $SK^-$ (or both) are empty. Here we perform a few additional checks to see if
Pseudocode 5 Processing a line violating constraint $\beta$.

```
procedure violating-line(\beta, l, \langle SK, NS \rangle)
2: split $l$ into rays $r^+$ satisfying $\beta$ and $r^-$ violating $\beta$;
   $l \leftarrow r^+$;
4: for all $g \in SK$ do
   $g \leftarrow $ combine$\beta$(g, l);
4: for all $g \in SK$ do
   $g \leftarrow $ combine$\beta$(g, l);
6: if is_equality($\beta$) then
   $SK \leftarrow SK^0$;
8: else if is_strict_ineq($\beta$) then
   $SK \leftarrow $ strict-on-eq-points($\beta$, \langle SK, NS \rangle);
```

Pseudocode 6 Processing points saturating a strict inequality.

```
procedure strict-on-eq-points($\beta$, \langle SK, NS \rangle)
2: $NS^* \leftarrow \emptyset$;
4: let $SK^0 = \langle L^0, R^0, C^0, SP^0 \rangle$;
4: for all $p \in SP^0$ do
   $NS^* \leftarrow NS^* \cup $ enumerate-faces($\beta$, \{p\}, $SK^+$, $SK$);
6: for all $ns \in NS^0$ do
   $NS^* \leftarrow NS^* \cup $ enumerate-faces($\beta$, $ns$, $SK^+$, $SK$);
8: $SK^0 \leftarrow $ points-become-closure-points($SK^0$);
   $\langle SK, NS \rangle \leftarrow \langle SK^+ \cup SK^0, NS^+ \oplus NS^* \rangle$;
```

an inconsistency has been detected, making the polyhedron empty and thereby allowing for an early exit from the main loop. If this is not the case, we efficiently update the $SK$ and $NS$ components, possibly calling helper procedure strict-on-eq-points.

6 Duality

The definitions and observations given in Section 4 for a geometric generator system have their dual versions working on a geometric constraint system. In the following we provide a brief overview of these correspondences, which are also summarized in Table 2.

For a non-empty $\mathcal{P} = \text{con}(\mathcal{C}) \in \mathbb{P}_n$, the skeleton component of the geometric constraint system $\mathcal{C} = (C_\leq, C_\geq, C_\succ)$ includes the non-redundant constraints defining the topological closure $\mathcal{Q} = \text{cl}(\mathcal{P})$. Denoting by $SC_\succ$ the set of skeleton strict inequalities (i.e., those in $C_\succ$ whose corresponding non-strict inequality is not redundant for $\mathcal{Q}$), we can define $SK_{\mathcal{Q}} \overset{\text{def}}{=} \langle C_\leq, C_\geq \cup SC_\succ, \emptyset \rangle$, so that $\mathcal{Q} = \text{con}(SK_{\mathcal{Q}})$.

The ghost faces of $\mathcal{P}$ are the faces of the topological closure $\mathcal{Q}$ that do not intersect $\mathcal{P}$:

$$g\text{Faces}_\mathcal{P} \overset{\text{def}}{=} \{ F \in c\text{Faces}_\mathcal{Q} \mid F \cap \mathcal{P} = \emptyset \};$$
as a consequence, we obtain
\[
P = \text{con}(SK_Q) \setminus \bigcup gFaces_P.
\]

With the only exception of the empty face, the elements in \(gFaces\) are exactly those not occurring in \(\text{cl}(nnncFaces)\). The set \(gFaces' \overset{\text{def}}{=} gFaces \cup \{Q\}\) is a meet sublattice of \(cFaces\); moreover, \(gFaces\) is downward closed and thus can be efficiently represented by its maximal elements (with respect to the set inclusion relation on faces), which are the dual-atoms of \(gFaces'\).

The skeleton support \(SK_F\) of a face \(F \in cFaces_Q\) is defined as the set of all the skeleton constraints that are saturated by all the points in \(F\). Each face \(F \in gFaces\) saturates a strict inequality \(\beta_\succ \in C_\succ\): we can represent such a face using its skeleton support \(SK_F\) of which \(\beta_\succ\) is a possible materialization. A constraint system non-skeleton component \(NS \subseteq NS\) is thus a combinatorial representation of the \textit{strict inequalities} of the polyhedron.

Hence, the non-skeleton components for generators and constraints have a complementary role: in the case of generators they are face fillers, marking the minimal faces that are included in \(nnncFaces\); in the case of constraints they are face cutters, marking the maximal faces that are excluded from \(nnncFaces\). Note however that, when representing a cutter in \(gFaces\) using its skeleton support the non-redundant cutters are again those having a minimal skeleton support, as is the case for the fillers.

As it happens with lines, all the equalities in \(C_=\) are included in all the supports \(ns \in NS\) so that, for efficiency, they are not represented explicitly. After removing the equalities, a singleton \(ns = \{\beta\} \in NS\) stands for a skeleton strict inequality constraint, which is better represented in the skeleton component, thereby obtaining \(SK = \langle C_=, C_\succ, SC_\succ\rangle\). Hence, a support \(ns \in NS\) is redundant if there exists \(ns' \in NS\) such that \(ns' \subset ns\) or if \(ns \cap SC_\succ \neq \emptyset\).

The handling of the empty face deserves a technical observation (which can be skipped when adopting a higher level point of view). The empty face is al-
ways cut away from the polyhedron, hence it belongs to \textit{gFaces} even when \( P \) is topologically closed. The skeleton support for the empty face can be given by a set of skeleton constraints whose hyperplanes have an empty intersection or by a constraint that is saturated by no points or closure points: the latter happens to be the case for the positivity constraint \( 1 \geq 0 \) (see Section 2). It follows that, when the positivity constraint is not redundant, the empty face should be represented by the non-skeleton support \( ns = \{ 1 \geq 0 \} \); being a singleton, this will be promoted into the skeleton component, thereby encoding the positivity constraint as a \textit{strict} inequality \( 1 > 0 \). Otherwise, when the positivity constraint is redundant, the empty face will be cut by the maximal support \( ns = C_{\geq} \).

When the concepts underlying the skeleton and non-skeleton representation are reinterpreted as discussed above, it is possible to define a conversion procedure mapping a generator representation into a constraint representation which is very similar to the one from constraints to generators shown in Section 6. One of the few differences, only occurring when performing a non-incremental conversion, can be seen in the \textit{initialization} phase. While in Section 5 we are starting from a representation of the universe polyhedron, having preprocessed the positivity constraint only, when converting from generators to constraints we look for a point in \( G_n \): if such a point does not exist, the polyhedron is empty; otherwise, we preprocess it to obtain a skeleton constraint system being made of \( n \) linear equality constraints (plus the strict positivity constraint). Another difference is in the handling of the special cases in lines 7 to 22 of \textsc{conversion}. When converting from generators to constraints, since we incrementally add new generators to a non-empty polyhedron, there is no way we can obtain an inconsistency: hence, the checks corresponding to the comments \( \text{‘} P \text{ is empty} \text{’} \) can be omitted. The rest of the code is almost unchanged: as a matter of fact, for each of the functions and procedures in Pseudocodes 2, 3, 4, 5 and 6, the corresponding C++ implementation is based on either a single function or a function template (which is then instantiated for both cases).

7 Experimental Evaluation

The new representation and conversion algorithms for NNC polyhedra presented in the previous sections have been implemented and tested, for both correctness and efficiency, in the context of the Parma Polyhedra Library (PPL)\footnote{All experiments have been performed on a laptop with an Intel Core i7-3632QM CPU, 16 GB of RAM and running GNU/Linux 4.13.0-16.}.

Due to the adoption of the direct encoding for constraints and generators, a full integration of the new algorithm in the domain of NNC polyhedra provided by the PPL is not possible, since the latter assumes the presence of the slack variable \( \epsilon \). Rather, the approach adopted is to intercept every call to the PPL’s conversion procedures (working on the \( \epsilon \)-representations in \( \mathbb{CP}_{n+1} \)) and pair it with a corresponding call to the newly defined conversion algorithms (working on the new representations in \( \mathbb{P}_n \)).
The diagram in Figure 7, where we consider the case of a non-incremental conversion, provides a more detailed description of the experimental setting. On the left hand side of the diagram we see the application of the standard PPL conversion procedure: the input constraint system $C_{in}$ (resp., generator system $G_{in}$) for the $\epsilon$-representation of the NNC polyhedron is processed by the three computational phases (‘init DD’, ‘conversion’ and ‘simplify’) so as to produce the output $\epsilon$-representation DD pair ($C_{out}$, $G_{out}$). A copy of the input system is processed by the ‘$\epsilon$-less encoding’ phase so as to remove the slack variable and produce a corresponding $\epsilon$-less version $C'_{in}$ (resp., $G'_{in}$); this is processed by the three computational phases of the new algorithm (‘new init DD’, ‘new conversion’ and ‘new simplify’) to produce the output DD pair, which is based on the new skeleton/non-skeleton representation $(\langle C_{SK}, C_{NS} \rangle, \langle G_{SK}, G_{NS} \rangle)$. After both the old and new conversions are completed, the two outputs are passed to a checking phase, where the new output is tested for both semantic equivalence and non-redundancy.

A similar diagram could be shown for an incremental conversion: in this case, the input is a DD pair for an $\epsilon$-representation together with some new constraints/generators to be processed by the standard conversion phases (skipping the ‘init DD’ phase). The ‘$\epsilon$-less encoding’ phase translates all the inputs into the corresponding $\epsilon$-less representations, including an input skeleton/non-skeleton DD pair, to be processed by the new algorithms (again, skipping the ‘new init DD’ phase).
As far as correctness is concerned, the final checking phase was successful on all the experiments performed, which includes all of the tests present in the PPL library itself, as well as several new tests explicitly written to stress specific portions of the new algorithms.

In order to assess the efficiency of the new algorithm, additional code was added so as to measure the time spent inside the standard and new computational phases, disregarding the input encoding and output checking phases.

The first experiment on efficiency is meant to evaluate the overhead incurred by the new representation and algorithm for NNC polyhedra when processing topologically closed polyhedra, so as to compare it with the corresponding overhead incurred by the $\epsilon$-representation. To this end, we considered the `ppl_lcedd` demo application of the Parma Polyhedra Library, which solves the vertex/facet enumeration problem. In Table 3 we report the results obtained on a selection of the test benchmarks, whose name is reported in the first column of the table. Note that, for each benchmark, the application performs a single conversion of representation, taking as input a system of constraints (for those tests having ‘.ine’ as file extension) or a system of generators (for those tests having ‘.ext’ as extension). For each of these tests we show the efficiency measures obtained in the following cases: when using the standard conversion algorithm for closed polyhedra (columns 2–4); when using the standard conversion algorithm for the $\epsilon$-representation of NNC polyhedra (columns 5–7); and when using the new conversion algorithm for the new representation of NNC polyhedra (columns 8–10). The three values measured are:

|   | time | vec ops | sat ops |
|---|------|---------|---------|
|   | milliseconds | thousands | millions |

Also note that in each case we report, in different rows, two sets of values: the first row shows the results for the ‘conversion’ phase, while the second row shows the results for the ‘simplify’ phase; the latter is shown just to stress that it is usually negligible, since most of the computation time is spent in the ‘conversion’ phase proper.

The inspection of the results in Table 3 leads to a few observations. As mentioned in Section 3, the use of the $\epsilon$-representation for topologically closed polyhedra incurs a significant overhead, which on the considered tests ranges from 53% (cross12.ine) to 317% (trunc10.ine). In contrast, the new representation and algorithm go beyond all expectations: in almost all of the tests there is no overhead at all (that is, any overhead incurred is so small to be masked by the improvements obtained in other parts of the algorithm); the efficiency gain ranges from 25% (reg600-5.m.ext) to 70% (kdd38_6.ine); the only slowdown, measuring 25%, is obtained on a test (cross12.ine) where the time spent

---

9 We only show those tests where the absolute difference between the PPL closed polyhedron time and the new algorithm time is bigger than 10 milliseconds.
| test               | closed poly | ε-repr NNC | (SK, NS) NNC |
|-------------------|-------------|------------|--------------|
|                   | time | vec ops | sat ops | time | vec ops | sat ops | time | vec ops | sat ops |
| cp6.ext           | 24   | 6.4     | 1.1     | 52   | 14.1    | 5.3     | 12   | 6.4     | 1.1     |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| cross12.in        | 48   | 112.8   | 0.3     | 124  | 172.2   | 1.3     | 56   | 112.7   | 0.5     |
|                   | 104  | —       | 16.8    | 108  | —       | 167.9   | 132  | —       | 16.8    |
| in7.in            | 56   | 8.7     | 1.7     | 136  | 13.9    | 4.7     | 24   | 8.7     | 0.9     |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| kkd38_6.in        | 656  | 64.7    | 28.3    | 2700 | 129.2   | 113.2   | 200  | 64.6    | 14.2    |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| kq20_11_m.in      | 56   | 8.7     | 1.7     | 132  | 13.9    | 4.7     | 24   | 8.7     | 0.9     |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| metric80_16.in    | 44   | 20.9    | 2.3     | 84   | 32.1    | 5.4     | 24   | 20.4    | 2.0     |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| mit31-20.in       | 1308 | 69.4    | 88.7    | 5100 | 102.1   | 358.0   | 724  | 69.3    | 60.2    |
|                   | 4    | —       | 0.0     | 4    | —       | 0.0     | 12   | —       | 0.0     |
| mp6.in            | 100  | 35.1    | 6.4     | 260  | 60.3    | 17.6    | 68   | 38.4    | 8.0     |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |
| reg600-5_m.ext    | 956  | 725.3   | 24.3    | 3508 | 1460.9  | 117.7   | 688  | 725.3   | 12.9    |
|                   | 16   | —       | 0.4     | 40   | —       | 1.4     | 44   | —       | 0.4     |
| sampleh8.in       | 7184 | 543.8   | 307.4   | 28940| 1086.7  | 1228.7  | 2904 | 543.8   | 153.8   |
|                   | 8    | —       | 0.0     | 16   | —       | 0.0     | 32   | —       | 0.0     |
| trunc10.in        | 1660 | 213.3   | 91.7    | 6928 | 423.8   | 396.6   | 784  | 212.8   | 89.9    |
|                   | 0    | —       | 0.0     | 0    | —       | 0.0     | 0    | —       | 0.0     |

**Table 3.** Measuring the overhead of the conversion procedure for NNC polyhedra; the topologically closed polyhedra used as tests are part of the **ppl-lcdd** test suite. Units: time (ms), vec ops (K), sat ops (M).
in the ‘simplify’ phase dominates the ‘conversion’ phase. It is worth stressing that we are comparing the time obtained for the new algorithm for NNC polyhedra against the time obtained by the standard algorithm for closed polyhedra. A direct comparison against the $\epsilon$-representation NNC polyhedra results in much bigger efficiency gains (and no slowdown at all).

Table 4. Comparison between the $\epsilon$-representation based (standard and enhanced) computations for NNC polyhedra and the one based on the new representations and conversion procedures.

| Algorithm                     | Iter count | Iter repr sizes | Full Conv | Incr Conv | Time |
|-------------------------------|------------|-----------------|-----------|-----------|------|
| $\epsilon$-repr standard     | 1142       | 3381            | 3706      | 7259      | 4    |
| $\epsilon$-repr enhanced     | 525        | 169             | 109       | 1661      | 7    |
| $\langle \mathcal{SK}, \mathcal{NS} \rangle$ standard | 314        | 56              | 62        | 156       | 4    |

The second experiment is meant to evaluate the efficiency gains obtained by the application of the new representation and algorithm in a more appropriate context, i.e., when processing NNC polyhedra that are not topologically closed. To this end, we reconsider the same benchmark that was discussed at length in [4, Table 2]: in this test, four NNC dual-hypercubes are combined by a few convex polyhedral hull and intersection operations. This test was meant to highlight the efficiency improvement resulting from the adoption of an enhanced evaluation strategy (where a knowledgeable user of the library explicitly invokes, when appropriate, the strong minimization procedures for $\epsilon$-representations) with respect to the standard evaluation strategy (where the user simply performs the required computation, leaving the burden of optimization to the library developers). In Table 4, we report the results obtained for the most expensive test among those described in [4, Table 2], comparing the standard and enhanced evaluation strategies for the $\epsilon$-representation (rows 1 and 2) with the new algorithm (row 3). For each algorithm, whose name is reported in column 1, we show in column 2 the total number of iterations of the conversion procedures and, in the next three columns, the average, median and maximum sizes of the representations computed at each iteration (i.e., the size of the intermediate results); in columns from 6 to 9 we show the numbers of incremental and non-incremental calls to the conversion procedures, together with the corresponding time spent (in milliseconds); in the final column, we show the overall time ratio, computed with respect to the time spent by the new algorithm.

Even though adopting the standard computation strategy (requiring no clever guess by the end user), the new algorithm is able to outperform not only the standard, but also the enhanced computation strategy for the $\epsilon$-representation. As discussed in Section 3, the reasons for this efficiency improvement is that the $\epsilon$-representations polynomials manage to be smaller in size.

The test dualhypercubes.cc is distributed with the source code of the PPL.
enhanced computation strategy is interfering with incrementality: the figures in Table 4 confirm that the new algorithm performs three of the seven required conversions in an incremental way, while in the enhanced case they are all non-incremental. Moreover, a comparison of the iteration count and the size of the intermediate results provides further evidence that the new algorithm is able to maintain a non-redundant description even during the iterations of a conversion, which justifies the impressive time improvements.

After having discussed the outcome of the experimental evaluation, it is possible to highlight how the adoption of the new representation and conversion procedure provides a solution for all of the issues affecting the $\epsilon$-representation approach, which were listed at the end of Section 3.

1. At the implementation level, no tricks are needed to hide the $\epsilon$ dimension, as in the new representation there is no slack variable at all.
2. The overhead of the $\epsilon$-representation for generators has simply disappeared: the skeleton points need not be matched by corresponding closure points. This claim is backed up by the efficiency results shown in Table 3.
3. The new conversion procedure is fully incremental: it is able to remove the redundant elements from the representation at each iteration of the main loop, thereby keeping the intermediate results smaller. This claim is supported by the efficiency results shown in Table 4.

8 Conclusion

We have presented a new approach for the representation of NNC polyhedra in the Double Description framework. The main difference of the new approach with respect to previous proposals is that it adopts a direct representation, where the strict inequality constraints and the closure points of NNC polyhedra are encoded using no slack variable at all. The new representation also distinguishes between the skeleton component, which is encoded geometrically, and the non-skeleton component, which is provided with a combinatorial encoding.

Based on this new representation, we have proposed and implemented a variant of the Chernikova-like conversion procedure which is able to achieve significant efficiency improvements with respect to state-of-the-art implementations of the domain of NNC polyhedra.

As future work, we plan to provide a full implementation of the domain of NNC polyhedra which is based on this new representation and conversion algorithm. To this end, we will have to reconsider each semantic operator already implemented by the existing libraries (which are based on the addition of a slack variable), so as to propose, implement and experimentally evaluate a corresponding correct specification based on the new approach.

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A Appendix

We provide here proof sketches for the results stated in Section 4.

**Proof of Proposition 1.** Let $\mathcal{G} = \langle L, R, C, P \rangle$ and consider a generator system $\mathcal{G}_m = \langle L_m, R_m, C_m, P_m \rangle$ in minimal form such that $\text{gen}(\mathcal{G}_m) = \text{gen}(\mathcal{G}) = \mathcal{P}$. By Definition 1 we obtain $\mathcal{SK}_Q = \langle L_m, R_m, C_m \cup SP_m, \emptyset \rangle$, where $SP_m \subseteq P_m$ is the set of skeleton points in $P_m$. Since each point $p \in P_m \setminus SP_m$ can be obtained by a combination of the generators in $L_m$, $R_m$ and $C_m \cup SP_m$, we have the following chain of equivalences:

\[
\text{full}\cdot\text{gen}(\mathcal{G}) = \text{full}\cdot\text{gen}(\mathcal{G}_m) = \text{gen}(\langle L_m, R_m, \emptyset, C_m \cup P_m \rangle) = \text{gen}(\langle L_m, R_m, \emptyset, C_m \cup SP_m \rangle) = \text{full}\cdot\text{gen}(\langle L_m, R_m, C_m \cup SP_m, \emptyset \rangle) = \text{full}\cdot\text{gen}(\mathcal{SK}_Q).
\]

Since function ‘full\cdot gen’ interprets closure points as points, it computes a topologically closed polyhedron, so that $\text{full}\cdot\text{gen}(\mathcal{SK}_Q) = Q = \text{cl}(\mathcal{P})$. Moreover, since $\mathcal{SK}_Q$ has been built from the generator system $\mathcal{G}_m$ in minimal form, by construction it only keeps in $C_m \cup SP_m$ the non-redundant points of $Q = \text{cl}(\mathcal{P})$; hence, it is the minimal system such that $\text{full}\cdot\text{gen}(\mathcal{SK}_Q) = Q$. □

**Proof of Proposition 2.** Let $\mathcal{SK}_F = \langle L_F, R_F, C_F, \emptyset \rangle \subseteq \mathcal{SK}_Q$ be the skeleton of the face $F \subseteq Q$, so that $\text{full}\cdot\text{gen}(\mathcal{SK}_F) = F$. By definition of ‘full\cdot gen’, the points $p, p' \in \text{relint}(F)$ can be obtained by combining the generators in $\mathcal{SK}_F$:

\[
p = L_F \lambda + R_F \rho + C_F \gamma,
p' = L_F \lambda' + R_F \rho' + C_F \gamma',
\]

where $\lambda, \lambda' \in \mathbb{R}^e$, $\rho, \rho' \in \mathbb{R}^r$, $\gamma, \gamma' \in \mathbb{R}^c_+$, $\sum_{i=1}^c \gamma_i = \sum_{i=1}^c \gamma_i' = 1$ and, for each $i \in \{1, \ldots, e\}$, both $\gamma_i > 0$ and $\gamma_i' > 0$. Therefore,

\[
\text{relint}(F) = \text{gen}(\langle L_F, R_F, C_F, \{p\} \rangle) = \text{gen}(\langle L_F, R_F, C_F, \{p'\} \rangle).
\]

Hence we have shown that, in order to generate $\text{relint}(F) \subseteq \mathcal{P}$, point $p \in P$ can be replaced by any other point $p' \in \text{relint}(F)$.

By definition of ‘gen’, the contribution of $p \in P$ is to generate the sets $\text{relint}(F') \subseteq \mathcal{P}$, where $F' \in \text{nnFaces}$ is such that $\text{relint}(F) \subseteq F'$ (i.e., all the faces of $\mathcal{P}$ containing $\text{relint}(F)$). It follows that $p \in P$ can be substituted by any other point $p' \in \text{relint}(F)$, obtaining the same polyhedron. □

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Proof of Proposition 3. Let $SK$ be the skeleton of the polyhedron $P \in \mathbb{R}^n$, $Q = \text{cl}(P)$ and $NS$ be the corresponding set of supports. In order to prove that $(\alpha_{SK}, \gamma_{SK})$ is a Galois connection between $\mathcal{P}(Q)$ and $\mathcal{P}(NS)$, we will show that $'\alpha_{SK}'$ and $'\gamma_{SK}'$ are monotonic, $'\alpha_{SK} \circ \gamma_{SK}'$ is reductive and $'\gamma_{SK} \circ \alpha_{SK}'$ is extensive; the result will thus follow from [15, Theorem 5.3.0.4].

The monotonicity of both $'\alpha_{SK}'$ and $'\gamma_{SK}'$ follows trivially from Definition 3. Consider $NS \in \mathcal{P}(NS)$. Note that, for each $ns \in NS$, there exists a face $F \in cFaces$ such that $ns = SK_F$, so that $\text{full.gen}(ns) = F$. Therefore,

$$\alpha_{SK} (\gamma_{SK}(NS))$$

$[\text{by definition of } \gamma_{SK}]$

$$= \alpha_{SK} \left( \bigcup \{ \text{relint}(\text{full.gen}(ns)) \mid ns \in NS \} \right)$$

$[\text{by definition of } \alpha_{SK}]$

$$= \bigcup \{ \text{relint}(F) \mid F = \text{full.gen}(ns) \in cFaces, ns \in NS \}$$

$[\text{by definition of } \alpha_{SK}]$

$$= \bigcup \{ \text{relint}(F) \mid \exists p \in \text{relint}(F), F = \text{full.gen}(ns) \in cFaces, ns \in NS \}$$

$[\text{since } NS \text{ is an upward closed set}]$

$$= NS.$$  

Hence, $'\alpha_{SK} \circ \gamma_{SK}'$ is the identity function, which implies that it is reductive.

In order to show that $'\gamma_{SK} \circ \alpha_{SK}'$ is extensive, let $S \subseteq Q$. Note that for each point $p \in S$, there exists a face $F \in cFaces$ such that $p \in \text{relint}(F)$. Hence:

$$\gamma_{SK} (\alpha_{SK}(S))$$

$[\text{by definition of } \alpha_{SK}]$

$$= \gamma_{SK} \left( \bigcup \{ \text{relint}(\text{full.gen}(ns)) \mid \exists p \in S, F \in cFaces . p \in \text{relint}(F) \} \right)$$

$[\text{by definition of } \gamma_{SK}]$

$$= \bigcup \{ \text{relint}(\text{full.gen}(ns)) \mid \exists p \in S, F \in cFaces . p \in \text{relint}(F), ns \in \text{relint}(F) \}$$

$\supseteq \bigcup \{ \text{relint}(F) \mid \exists p \in S, F \in cFaces . p \in \text{relint}(F) \}$

$\supseteq \bigcup \{ p \in S \mid \exists F \in cFaces . p \in \text{relint}(F) \}$

$$= S.$$  

$\square$

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Proof of Proposition 4. Applying ‘$\gamma_{SK} \circ \alpha_{SK}$’ to the set of points $P$ we obtain:

$$
\gamma_{SK}(\alpha_{SK}(P)) = \bigcup \left\{ \text{relint}(\text{full.gen}(SK_F)) \mid \exists p' \in P, F' \in cFaces . \right. \\
\left. p' \in \text{relint}(F'), SK_F \in \uparrow SK_{F'} \right\}. \tag{1}
$$

By definition of function ‘gen’, a face is included in the polyhedron $P$ if and only if it contains a point in $P$. In particular, letting $nncFaces' = nncFaces \setminus \{\emptyset\}$, this holds for the minimal faces in $nncFaces'$; these are the atoms of the lattice $\text{cl}(nncFaces)$, which is a sublattice of $cFaces$. For these atoms $A \in \text{cl}(nncFaces)$, we have $A = \text{relint}(A)$; hence

$$
\forall A \text{ atom of } \text{cl}(nncFaces) : \exists p \in P . p \in \text{relint}(A). \tag{2}
$$

Moreover, since $\text{cl}(nncFaces')$ is an upward closed set, we have:

$$
\forall F' \in \text{cl}(nncFaces') : \exists A \text{ atom of } \text{cl}(nncFaces) . SK_A \subseteq SK_{F'}. \tag{3}
$$

Therefore, we have the following chain of equations:

$$
P = \bigcup \{ \text{relint}(F) \mid F \in nncFaces' \} \\
= \bigcup \{ \text{relint}(\text{full.gen}(SK_F)) \mid F \in \text{cl}(nncFaces') \} \tag{by property (3)} \\
= \bigcup \{ \text{relint}(\text{full.gen}(SK_F)) \mid \exists A \text{ atom of } \text{cl}(nncFaces) . SK_A \subseteq SK_F \} \tag{by property (2)} \\
= \bigcup \left\{ \text{relint}(\text{full.gen}(SK_F)) \mid \exists p \in P, A \text{ atom of } \text{cl}(nncFaces) . \right. \\
\left. p \in \text{relint}(A), SK_F \in \uparrow SK_A \right\}. \tag{4}
$$

We now show that (4) is equivalent to (1). The inclusion (4) $\subseteq$ (1) follows by simply taking $F' = A$; the other inclusion (4) $\supseteq$ (1) follows by applying property (3) while also observing that, since $p' \in \text{relint}(F')$, then $F' \in \text{cl}(nncFaces)$. \qed