We consider the problem of determining the Lévy exponent in a Lévy model for asset prices given the price data of derivatives. The model, formulated under the real-world measure $\mathbb{P}$, consists of a pricing kernel $\{\pi_t\}_{t \geq 0}$ together with one or more non-dividend-paying risky assets driven by the same Lévy process. If $\{S_t\}_{t \geq 0}$ denotes the price process of such an asset then $\{\pi_t S_t\}_{t \geq 0}$ is a $\mathbb{P}$-martingale. The Lévy process $\{\xi_t\}_{t \geq 0}$ is assumed to have exponential moments, implying the existence of a Lévy exponent $\psi(\alpha) = t^{-1} \log \mathbb{E}(e^{\alpha \xi_t})$ for $\alpha$ in an interval $A \subset \mathbb{R}$ containing the origin as a proper subset. We show that if the initial prices of power-payoff derivatives, for which the payoff is $H_T = (\zeta_T)^q$ for some time $T > 0$, are given for a range of values of $q$, where $\{\zeta_t\}_{t \geq 0}$ is the so-called benchmark portfolio defined by $\zeta_t = 1/\pi_t$, then the Lévy exponent is determined up to an irrelevant linear term. In such a setting, derivative prices embody complete information about price jumps: in particular, the spectrum of the price jumps can be worked out from current market prices of derivatives. More generally, if $H_T = (S_T)^q$ for a general non-dividend-paying risky asset driven by a Lévy process, and if we know that the pricing kernel is driven by the same Lévy process, up to a factor of proportionality, then from the current prices of power-payoff derivatives we can infer the structure of the Lévy exponent up to a transformation $\psi(\alpha) \rightarrow \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where $c$ and $\mu$ are constants.

I. INTRODUCTION

We are concerned with determining the extent to which derivative prices, taken at time zero, can be used to infer the nature of the underlying jump process in models where asset prices can jump. To this end we consider geometric Lévy models for the prices of financial assets and ask to what degree the present values of derivatives can be used to determine the Lévy processes driving the models. The paper is structured as follows. In Section II we summarize a few basic facts concerning Lévy processes. Then in Section III we introduce the condition that the Lévy process should admit exponential moments and we explore some of the implications of this assumption. In Section IV we introduce the class of so-called geometric Lévy models. These models generalize the standard geometric Brownian motion model for asset prices, enabling one to see explicitly the form that the excess rate of return takes as a function of the overall level of risk (measured by a volatility parameter $\sigma$) and the overall level of market risk aversion (measured by a risk aversion parameter $\lambda$). After recalling a pair of preliminary lemmas, we proceed in Section V to develop the theory of geometric Lévy models further by showing in Proposition I that a Lévy process admits exponential moments if and only if the probability distribution for the value of the process at any time in the future has thin tails. Thus, from a modern point of view the use of
Lévy processes in finance stems not so much from a desire to model the distributions of the tails of returns but rather to account for the characteristics of the jumps that asset prices can undertake. In Section VI we present a method for ascertaining the underlying jump process given a family of derivative prices at time zero. The result is summarized in Proposition 2. In Section VII we elaborate further on various aspects of the representation of the Lévy exponent in terms of the prices of power-payoff derivatives, and we comment on the possibility of using call option prices for a similar purpose. Finally, in Section VIII we establish a corresponding result for the case of imaginary power-payoff derivatives where we make use of the techniques of Fourier analysis to show that a general European-style derivative can be expressed as a portfolio of imaginary power payoff derivatives, providing that the payoff is sufficiently smooth and has good asymptotic properties.

II. LÉVY PROCESSES

We shall assume the reader is familiar with Lévy processes and their applications to problems in finance (Andersen & Lipton 2013, Appelbaum 2004, Bertoin 2004, Brody, Hughston & Mackie 2012, Chan 1999, Cont & Tankov 2004, Gerber & Shiu 1994, Hubalek & Sgarra 2006, Kyprianou 2006, Norberg 2004, Protter 2005, Sato 1999, Schoutens 2004). We mainly work with one-dimensional Lévy processes. For convenience we recall some definitions and classical results. A random process \( \{ \xi_t \} \) taking values in \( \mathbb{R} \) on a probability space \( (\Omega, \mathcal{F}, P) \) is said to be a Lévy process if (a) \( \xi_s + t - \xi_s \) and \( \{ \xi_u \}_{0 \leq u \leq s} \) are independent for all \( s, t \geq 0 \) (independent increments), (b) \( \xi_s + t - \xi_s \) has the same law as \( \xi_t \) for all \( s, t \geq 0 \) (stationary increments), (c) \( \lim_{t \to 0} P(|\xi_s + t - \xi_s| > \epsilon) = 0 \) (continuity in probability), and (d) there exists an \( \Omega' \in \mathcal{F} \) satisfying \( P(\Omega') = 1 \) such that for \( \omega \in \Omega' \) the path \( \{ \xi_t(\omega) \}_{t \geq 0} \) is right-continuous for \( t \geq 0 \) and has left limits for \( t > 0 \) (càdlàg property). Note that (b) implies that \( \xi_0 = 0 \) almost surely. It follows from this definition that for all \( t \geq 0 \) and all \( \kappa \in \mathbb{R} \) the Fourier transform of \( \xi_t \) can be represented in the form

\[
\frac{1}{t} \log \mathbb{E}[\exp(i\kappa \xi_t)] = ip\kappa - \frac{1}{2} q\kappa^2 + \int_{-\infty}^{\infty} (e^{ix\kappa} - 1 - i\kappa x 1\{|x| < 1\}) \nu(dx). \tag{1}
\]

Here \( p > 0 \) and \( q > 0 \) are constants and \( \nu(dx) \) is a so-called Lévy measure. A Borel measure \( \nu(dx) \) on \( \mathbb{R} \) is called a Lévy measure if \( \nu(\{0\}) = 0 \) and

\[
\int_{-\infty}^{\infty} 1 \wedge x^2 \nu(dx) < \infty, \tag{2}
\]

where \( a \wedge b = \min(a, b) \). The Lévy measure associated with a Lévy process has the property that for any measurable set \( B \subseteq \mathbb{R} \) the expected rate at which jumps occur for which the jump size lies in the range \( B \) is \( \nu(B) \). The sample paths of a Lévy process have bounded variation on every compact interval of time almost surely if and only if \( q = 0 \) and

\[
\int_{-\infty}^{\infty} 1 \wedge |x| \nu(dx) < \infty. \tag{3}
\]

In that case we say that the Lévy process has bounded variation. Let us write \( \xi_t- = \lim_{s \to t} \xi_s \) for the left limit of the process at time \( t \). The discontinuity at time \( t \) is then defined by \( \Delta \xi_t = \xi_t - \xi_t- \), and for the Lévy measure we have

\[
\nu(B) = \frac{1}{t} \mathbb{E} \sum_{0 \leq s \leq t} 1\{\Delta \xi_s \in B\} \tag{4}
\]
for any \( t > 0 \). If \( \nu(\mathbb{R}) < \infty \) we say that \( \{\xi_t\} \) has finite activity, whereas if \( \nu(\mathbb{R}) = \infty \) we say that \( \{\xi_t\} \) has infinite activity. A necessary and sufficient condition for \( (3) \) to hold is that
\[
\sum_{0 \leq s \leq t} |\Delta \xi_s| < \infty
\]  
for every \( t > 0 \) almost surely. If \( \sup_t |\Delta \xi_t| \leq c \) almost surely for some constant \( c > 0 \), then we say that \( \{\xi_t\} \) has bounded jumps.

### III. EXPONENTIAL MOMENTS

In order for \( \{\xi_t\} \) to give rise to a Lévy model for asset prices, we require additionally that for every \( t > 0 \) the random variable \( \xi_t \) should satisfy a moment condition of the form
\[
E(e^{\alpha \xi_t}) < \infty
\]  
for \( \alpha \) in an interval \( A = (\beta, \gamma) \subset \mathbb{R} \) containing the origin. Here we set \( \beta = \inf \alpha : E(e^{\alpha \xi_t}) < \infty \) and \( \gamma = \sup \alpha : E(e^{\alpha \xi_t}) < \infty \). If a Lévy process satisfies this condition, we say it possesses exponential moments. In that case, it follows (Sato 1999, Theorem 25.17) that there exists a so-called Lévy exponent \( \psi : \mathbb{C}_A \to \mathbb{R} \), for \( \mathbb{C}_A = \{ \alpha \in \mathbb{C} \mid \text{Re}(\alpha) \in A \} \), such that
\[
E(e^{\alpha \xi_t}) = e^{\psi(\alpha)t},
\]  
where \( \psi(\alpha) \) admits a Lévy-Khinchin representation of the form
\[
\psi(\alpha) = p\alpha + \frac{1}{2}q\alpha^2 + \int_{-\infty}^{\infty} (e^{\alpha x} - 1 - \alpha x 1_{\{|x| < 1\}}) \nu(dx).
\]  
A necessary and sufficient condition for a Lévy process to satisfy \( (6) \) for \( \alpha \in A \) is that the associated Lévy measure should satisfy
\[
\int_{-\infty}^{\infty} e^{\alpha x} 1_{\{|x| > 1\}} \nu(dx) < \infty
\]  
for \( \alpha \in A \) (Sato 1999, Theorem 25.3). If \( \{\xi_t\} \) admits exponential moments, then one can check that for all \( p > 0 \) we have
\[
E(|\xi_t|^p) < \infty.
\]  
The argument is as follows. Now, for any \( \alpha \in \mathbb{R} \) we have
\[
cosh(\alpha \xi_t) = \sum_{k=0}^{\infty} \frac{(\alpha \xi_t)^{2k}}{(2k)!}.
\]  
Therefore for any \( k \in \mathbb{N} \) we have
\[
cosh(\alpha \xi_t) > \frac{(\alpha \xi_t)^{2k}}{(2k)!}.
\]
If we choose $\alpha$ so that $|\alpha| < \min(|\beta|, \gamma)$, which ensures that $\alpha$ and $-\alpha$ are in $A$, then $\mathbb{E}(\cosh(\alpha \xi_t)) < \infty$. Therefore, $\mathbb{E}(\xi_t^n) < \infty$ for even $n \in \mathbb{N}$, which implies that $\mathbb{E}(|\xi_t|^p) < \infty$ for all $p \in \mathbb{R}^+$, since for each $n$ and any random variable $X$ it holds that $\mathbb{E}(|X|^n) < \infty$ implies $\mathbb{E}(|X|^p) < \infty$ for $0 \leq p \leq n$. More generally, for any $\alpha \in A$ and any $p \in \mathbb{R}^+$ we have

$$\mathbb{E}(e^{\alpha \xi_t} | \xi_t|^p) < \infty. \tag{13}$$

This can be seen as follows. Since $A$ is open, for any $\alpha \in A$ we can choose $\epsilon > 0$ so that $\alpha(1 + \epsilon)$ is still in $A$. Then by Hölder’s inequality we have

$$\mathbb{E}(e^{\alpha \xi_t} | \xi_t|^p) \leq \left(\mathbb{E}(e^{\alpha(1+\epsilon)\xi_t})\right)^{1/(1+\epsilon)} \left(\mathbb{E}(|\xi_t|^p(1+\epsilon)/\epsilon)\right)^{\epsilon/(1+\epsilon)}. \tag{14}$$

But we have already established that the terms on the right are finite, and that gives $\tag{13}$. A similar argument shows that if $\{\xi_t\}$ admits exponential moments then

$$\int_{-\infty}^{\infty} e^{\alpha x} |x|^p \mathbf{1}\{|x| > 1\} \nu(dx) < \infty. \tag{15}$$

for all $\alpha \in A$ and all $p > 0$. Setting $\alpha = 0$ and $p = 1$, we see in particular that

$$\int_{-\infty}^{\infty} |x| \mathbf{1}\{|x| > 1\} \nu(dx) < \infty, \tag{16}$$

which implies that one can extend the truncated term on the right side of $\tag{8}$ to an integral of the form $\int |x| \nu(dx)$, over the whole of $\mathbb{R}$, by dropping the indicator function and redefining the constant $p$ in equation $\tag{8}$. The finiteness of integrals $\tag{13}$ and $\tag{15}$ allows one to compute the Greeks for various derivative payouts in exponential Lévy models.

Examples of Lévy processes admitting exponential moments include: (a) Brownian motion, for which $\psi(\alpha) = 1/2 \alpha^2$, $\alpha \in \mathbb{R}$; (b) the Poisson process with rate $m$, for which $\psi(\alpha) = m(e^\alpha - 1)$ and $\nu(dz) = m\delta_1(dz)$; (c) the compound Poisson process with rate $m$, for which $\psi(\alpha) = m(\theta(\alpha) - 1)$, where $\theta(\alpha)$ is the moment generating function for the distribution $\mu(dx)$ of a typical jump and $\nu(dz) = m\mu(dx)$; (d) the gamma process with rate $m$, for which $\psi(\alpha) = -m \log(1 - \alpha)$, $\alpha < 1$, where $\nu(dz) = \mathbf{1}\{z > 0\} z^{-1} \exp(-z) dz$ (Dickson & Waters 1993, Heston 1993, Brody, Macrina & Hughston 2008, Yor 2017); and (e) the variance gamma (VG) process, for which $\psi(\alpha) = -m \log(1 - \alpha^2/2m^2)$, where we have $-2^{1/2}m < \alpha < 2^{1/2}m$ (Madan & Seneta 1990, Madan & Milne 1991, Madan, Carr & Chang 1998). We also mention (f) the truncated stable family of Lévy processes, which includes the gamma process and the VG process as special cases (Koponen 1995, Carr, Geman, Madan & Yor 2002, Andersen & Lipton 2013, Küchler & Tappe 2014).

### IV. GEOMETRIC LÉVY MODEL

The geometric Lévy model for asset prices can be viewed as an extension of the well-known geometric Brownian motion model to the Lévy regime. For simplicity, we consider a model driven by a one-dimensional Lévy process $\{\xi_t\}_{t \geq 0}$. The generalization to higher dimensional Lévy processes is straightforward. The process $\{\xi_t\}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is interpreted as the real-world probability measure. We assume that
\{\xi_t\} admits exponential moments and denote the associated Lévy exponent by \(\psi(\alpha)\), for \(\alpha\) in the set \(A = (\beta, \gamma)\), as before, where \(\beta < 0 < \gamma\). The process \(\{m_t\}_{t \geq 0}\) defined by
\[
m_t = e^{\alpha \xi_t - \psi(\alpha)t}
\] (17)
for some choice of \(\alpha \in A\) is called the geometric Lévy martingale associated with \(\{\xi_t\}\), with parameter \(\alpha\). We refer to \(\alpha\) as the volatility of \(\{m_t\}\). By the stationary and independent increments properties we find that \(E_s(m_t) = m_s\). Here we write \(E_t(\cdot) = E(\cdot | F_t)\), where \(\{F_t\}\) denotes the augmented filtration generated by \(\{\xi_t\}\). The associated geometric Lévy model consists of a pricing kernel, a money market account, and one or more so-called investment-grade assets. See Duffie (1992), Hunt & Kennedy (2004), Cochrane (2005) for general aspects of the theory of pricing kernels (equivalently, stochastic discount factors, state price densities) in arbitrage-free asset pricing models. For the construction of the pricing kernel \(\{\pi_t\}_{t \geq 0}\) in the context of a Lévy model we let \(r \in \mathbb{R}\) and \(\lambda > 0\) be constants, and assume that \(-\lambda \in A\). Then we set
\[
\pi_t = e^{-rt - \lambda \xi_t - \psi(-\lambda)t}.
\] (18)
We refer to the related process \(\{\zeta_t\}_{t \geq 0}\) defined by \(\zeta_t = 1/\pi_t\) as the growth-optimal portfolio or natural numeraire asset (Flesaker & Hughston 1997). It serves as a benchmark, relative to which other non-dividend-paying assets are martingales. In some calculations it is convenient to refer to the natural numeraire instead of the pricing kernel. The money market account \(\{B_t\}_{t \geq 0}\) is taken to have the value \(B_t = B_0 e^{rt}\) at time \(t\), where \(B_0\) denotes its initial value in some fixed unit of account. The idea of an investment-grade asset is that it should offer a rate of return that is strictly greater than the interest rate. Ordinary stocks and bonds are in this sense investment-grade, whereas put options and short positions in ordinary stocks and bonds are not. We assume for the moment that the assets pay no dividends over the time horizons considered (dividends will be treated shortly), and we write \(\{S_t\}_{t \geq 0}\) for the value process of a typical non-dividend paying risky asset in the geometric Lévy model. We require that the product of the pricing kernel and the asset price should be a martingale, which we take to be geometric Lévy martingale of the form
\[
\pi_t S_t = S_0 e^{\beta \xi_t - \psi(\beta)t}
\] (19)
for some \(\beta \in A\). From the formulae above we deduce that
\[
S_t = S_0 e^{rt + \sigma \xi_t + \psi(-\lambda)t - \psi(\sigma - \lambda)t},
\] (20)
where \(\sigma = \beta + \lambda\). We assume that \(\sigma > 0\) and that \(\sigma \in A\). It follows that the price can be expressed in the form
\[
S_t = S_0 e^{rt + R(\lambda, \sigma)t + \sigma \xi_t - \psi(\sigma)t},
\] (21)
where
\[
R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda).
\] (22)
Thus we see that \(\sigma\) is the volatility of the asset price relative to the given Lévy process and that \(R(\lambda, \sigma)\) is the excess rate of return above the interest rate. The parameter \(\lambda\) can be interpreted as a measure of the overall level of market risk aversion with respect to the risk
associated with movements in the underlying Lévy process. A calculation shows that the excess rate of return is bilinear in \( \lambda \) and \( \sigma \) if and only if \( \{\xi_t\} \) is a Brownian motion (Brody et al 2012). It follows that the interpretation of \( \lambda \) as a “market price of risk”, which is valid for models based on a Brownian filtration, does not carry through to the general Lévy regime. Nevertheless, the notion of excess rate of return is well defined, and under the assumptions that we have made the strict convexity of the Lévy exponent implies that the excess rate of return is strictly positive and is an increasing function of \( \lambda \) and \( \sigma \). To show that \( R(\lambda, \sigma) > 0 \) we can use the Lévy-Khinchin formula (3) to deduce that

\[
R(\lambda, \sigma) = \int_{-\infty}^{\infty} \frac{e^{\sigma x} - 1}{1 - e^{-\lambda x}} \nu(dx).
\]

It should be evident that the integrand is \( O(x^2) \) as \( x \to 0 \) and hence that the integral is finite. It follows by inspection of (23) that the excess rate of return is an increasing function of the volatility and the level of risk aversion.

In the case of a single asset driven by a single Lévy process one can without loss of generality set \( \sigma = 1 \). This can be achieved by defining a rescaled Lévy process \( \{\bar{\xi}_t\} \) by setting \( \bar{\xi}_t = \sigma \xi_t \). Then we define \( \bar{\psi}(\alpha) = \psi(\sigma \alpha) \) and set \( \bar{\lambda} = \lambda/\sigma \), and we have

\[
\pi_t = e^{-rt - \bar{\lambda}\xi_t - \bar{\psi}(\bar{\lambda})t}, \quad S_t = S_0 e^{rt + \bar{\lambda}(1) t + \bar{\xi}_t - \bar{\psi}(1)t},
\]

where \( \bar{R}(\bar{\lambda}, 1) = \bar{\psi}(1) + \psi(1 - \bar{\lambda}) - \bar{\psi}(1 - \bar{\lambda}) \). If we then drop the bars, we are led back to a version of the model already set up, but with \( \sigma = 1 \). Nevertheless, it can be helpful to leave the parameter \( \sigma \) intact as part of the theory, since there are situations where one would like to compare versions of the model for different values of the parameter, e.g. for sensitivity analysis and the calculation of Greeks. On the other hand, there are also situations where it is desirable to make use of simplifications resulting from setting \( \sigma = 1 \); an example of this can be found in the proof of Proposition 2. The value of the asset given by (23) does not depend on the drift of the Lévy process. This is because if we replace \( \xi_t \) with \( \xi_t + \epsilon t \) for some \( \epsilon \in \mathbb{R} \) then the Lévy exponent \( \psi(\alpha) \) gets replaced with \( \psi(\alpha) + \epsilon \alpha \), and the combination \( \sigma \xi_t - \psi(\sigma) t \) appearing in the formula for the asset price is left unchanged. Thus without loss of generality we can set the drift of the Lévy process to zero. In that case we refer to \( \{\xi_t\} \) as a compensated Lévy process. This implies that \( \mathbb{E}[\xi_t] = 0 \) and that \( \{\xi_t\} \) is a martingale. For instance, if \( \{N_t\} \) is the standard Poisson process, with jump rate \( m \), then the associated compensated process is given by \( \xi_t = N_t - mt \). With these conventions in mind we observe the following (Brody et al 2018), which will be useful as we proceed.

**Lemma 1** Let \( \psi : A \to \mathbb{R} \) be the Lévy exponent of a compensated Lévy process that admits exponential moments. Then \( \psi \) is strictly positive except at the origin, where it vanishes.

**Proof.** Differentiating each side of \( \psi(\alpha) = t^{-1} \log \mathbb{E}(e^{\alpha \xi_t}) \), we obtain \( \mathbb{E}[\xi_t] = \psi'(0)t \). Since \( \psi \) is the Lévy exponent of a compensated Lévy process, we have \( \psi'(0) = 0 \). Hence, the curve \( \psi : A \to \mathbb{R} \) has a horizontal tangent at the origin. Since \( \psi \) is convex, and therefore lies above any tangent except at the point where the tangent touches it, one sees that \( \psi \) is strictly positive except at the origin, where \( \psi(0) = 0 \). \( \square \)

For example, for the Poisson process with rate \( m \), we have \( \psi(\alpha) = m(e^\alpha - 1) \), and thus for the compensated Poisson process the Lévy exponent is \( \psi(\alpha) = m(e^\alpha - 1 - \alpha) \). Then \( \psi(0) = 0 \), \( \psi'(0) = 0 \), and clearly \( \psi(\alpha) > 0 \) for \( |\alpha| > 0 \). For the standard gamma process with
rate \( m \) we have \( \psi(\alpha) = -m \log(1 - \alpha) \), and thus for the compensated gamma process we have \( \psi(\alpha) = -m \log(1 - \alpha) - ma \). In this case, it is obvious that \( \psi'(0) = 0 \) though perhaps less obvious that \( \psi(\alpha) > 0 \) for \( \psi(\alpha) < 1 \) and \( |\alpha| > 0 \). But recall the logarithmic inequality \( \log(1 + x) \leq x \) for \( x > -1 \), which holds as a strict inequality when \( x \neq 0 \).

V. PROBABILITY BOUNDS

A probabilistic interpretation of the condition that a Lévy process admits exponential moments is given in Proposition (\( \Pi \)) below. We require the following (Brody et al 2018):

Lemma 2 Let \( A \subset \mathbb{R} \) be an open interval containing the origin as a proper subset. Let \( \phi : A \to \mathbb{R}^+ \) be a nonnegative, convex function that is differentiable on \( A \) and vanishes at the origin. Then \( a \phi'(\alpha) > \phi(\alpha) \) for all \( \alpha \in A \) except at \( \alpha = 0 \).

Proof. Let \( \alpha \in A \). If \( \alpha > 0 \), then by the mean value theorem there exists a \( \delta \in (0, \alpha) \) such that \( \phi(\alpha) = \alpha \phi'(\delta) \). Since \( \phi \) takes its minimum at the origin and is strictly convex, we have \( \phi'(\delta) < \phi'(\alpha) \). Therefore \( \phi(\alpha) < \alpha \phi'(\alpha) \) for \( \alpha > 0 \). On the other hand, if \( \alpha < 0 \), then by the mean value theorem there exists a \( \delta \in (\alpha, 0) \) such that \( \phi(\alpha) = \alpha \phi'(\delta) \). Since \( \phi \) is strictly convex with a minimum at the origin, it follows that \( \phi'(\alpha) < \phi'(\delta) < 0 \), and hence \( \alpha \phi'(\delta) < \alpha \phi'(\alpha) \), since \( \alpha < 0 \). Therefore we have \( \phi(\alpha) < \alpha \phi'(\alpha) \) for \( \alpha < 0 \). \( \square \)

Then we are led to the following characterization of the tail distribution of a Lévy process:

Proposition 1 A compensated Lévy process \( \{\xi_t\}_{t \geq 0} \) admits exponential moments if and only if for any \( x_0 > 0 \) we can find a \( t_0 \) and functions \( C : [t_0, \infty) \to (0, \infty) \) and \( \gamma : [t_0, \infty) \to (0, \infty) \) with the property that for any \( x \geq x_0 \) and \( t \geq t_0 \) we have \( \mathbb{P}(\xi_t > x) \leq C_t e^{-\gamma x} < 1 \) and for \( x_0 < 0 \) we can find a \( t_0 \) and functions \( D : [t_0, \infty) \to (0, \infty) \) and \( \delta : [t_0, \infty) \to (0, \infty) \) with the property that for \( x \leq x_0 \) and \( t \geq t_0 \) we have \( \mathbb{P}(\xi_t < x) \leq D_t e^{\delta x} < 1 \).

Proof. Under the condition that \( \mathbb{E}(e^{\alpha \xi_t}) < \infty \) for \( \alpha \in A = (\beta, \gamma) \), \( \beta < 0 < \gamma \), we have \( \mathbb{E}(e^{\alpha \xi_t}) = e^{\psi(\alpha) t} \) for all \( t \geq 0 \). Then for every \( \alpha \in (0, \gamma) \) we have a so-called Chernoff bound,

\[
\mathbb{P}(\xi_t > x) \leq e^{-\alpha x + \psi(\alpha) t},
\]

which shows that for fixed \( t \) the probability that \( \xi_t \) exceeds \( x \) falls off exponentially. The Chernoff bound is obtained by noting that for any \( \alpha \in (0, \gamma) \) it holds that \( \mathbb{P}(\xi_t > x) = P(e^{\alpha \xi_t} > e^{\alpha x}) = \mathbb{E}(1_{\{e^{-\alpha x + \alpha \xi_t} > 1\}}) \), and hence \( \mathbb{P}(\xi_t > x) \leq \mathbb{E}(e^{-\alpha x + \alpha \xi_t}) = \mathbb{E}(1_{\{e^{-\alpha x + \alpha \xi_t} > 1\}}) \) \( \leq \mathbb{E}(e^{-\alpha x + \alpha \xi_t}) = e^{-\alpha x + \psi(\alpha) t} \). Since the Lévy process is compensated, it follows by virtue of Lemma 1 that \( \psi(\alpha) \) is strictly positive for \( \alpha \in (0, \gamma) \). Therefore the right-hand side of (25) is strictly less than unity only if \( x \) is sufficiently large. To obtain an inequality that bounds \( \mathbb{P}(\xi_t > x) \) at a level strictly less than unity we optimize with respect to \( \alpha \) by choosing \( \alpha \) so that \( \psi'(\alpha) = x/t \). Since \( \psi' \) is an increasing function on \( (0, \gamma) \) such that \( \psi'(0) = 0 \) and \( \lim_{\alpha, \gamma, \gamma} \psi'(\alpha) = \psi'(\gamma^-) \), we see that \( \psi' \) has an inverse \( J : [0, \psi'(\gamma^-)) \to J(y) \in (0, \gamma) \) such that \( J(\psi'(\alpha)) = \alpha \) and \( \psi'(J(y)) = y \) for \( \alpha \in (0, \gamma) \) and \( y \in [0, \psi'(\gamma^-)) \). In common examples such as Brownian motion, the compound Poisson process, the gamma process, and the variance gamma process, we have \( \psi'(\gamma^-) = \infty \). Let us therefore consider the two cases (a) \( \psi'(\gamma^-) = \infty \) and (b) \( \psi'(\gamma^-) < \infty \) in turn. In case (a), the inverse function \( J(y) \) is
defined for all \( y > 0 \). The optimal bound, for given values of \( x \) and \( t \), is then obtained by setting \( \alpha = J(x/t) \), and we have

\[
P(\xi_t > x) \leq e^{-J(x/t)x + \psi(J(x/t))t}. \tag{26}
\]

We observe that if we choose \( \alpha \) in \((25)\) such that \( \psi'(\alpha) = x/t \), then

\[
P(\xi_t > x) \leq e^{[\alpha \psi'(\alpha) + \psi(\alpha)]t}, \tag{27}
\]

the right-hand side of which is by Lemma 2 strictly less than unity. Thus if for any \( t > 0 \) and for some fixed value of \( x \), say \( x_0 \), we set \( \alpha_0(t) = J(x_0/t) \), then for that choice of \( \alpha \) in \((25)\) the probability is bounded below unity and will decrease if we increase \( x \). In particular, for \( x \geq x_0 \) we have

\[
P(\xi_t > x) \leq e^{-J(x_0/t)x + \psi(J(x_0/t))t} < 1. \tag{28}
\]

That shows that for any choice of \( x_0 \) there exists a function \( C_t = \exp(\psi(J(x_0/t))t) \) and function \( \gamma_t = J(x_0/t) \), defined for \( t > 0 \), such that for \( x \geq x_0 \) we have

\[
P(\xi_t > x) \leq C_t e^{-\gamma tx} < 1. \tag{29}
\]

In case (b), for a given value of \( x \) we need to choose \( t \) to be large enough to ensure that \( x/t \in [0, \psi'(\gamma -)) \). Suppose then that \( x_0 \) and \( t_0 \) are such that \( x_0/t_0 \in [0, \psi'(\gamma -)) \). Then clearly \( x_0/t \in [0, \psi'(\gamma -)) \) for any \( t \geq t_0 \). Therefore if we set \( \alpha_0(t) = J(x_0/t) \) for \( t \geq t_0 \), then for that choice of \( \alpha \) in \((25)\) the probability is bounded below unity and will decrease if we increase \( x \). In particular, for \( x \geq x_0 \) and \( t \geq t_0 \) we are led back to \((28)\). That shows that the right tail of the distribution of \( \xi_t \) is exponentially bounded. A similar argument allows one to show that the left tail is exponentially bounded. Conversely, suppose that \( \{\xi_t\} \) is a Lévy process and that \((29)\) holds for \( x \geq x_0 \) and \( t \geq t_0 \) for strictly positive \( \{C_t\}_{t \geq t_0} \) and \( \{\gamma_t\}_{t \geq t_0} \). Let \( t > t_0 \) be fixed, and let \( \alpha \) satisfy \( 0 < \alpha < \gamma_t \). Recall that for any random variable \( X : \Omega \to \mathbb{R}^+ \cup \{\infty\} \) it holds that

\[
E(X) = \int_0^\infty P(X > y) \, dy. \tag{30}
\]

Then for \( y_0 > 0 \) we have

\[
E(e^{\alpha \xi_t}) = \int_0^\infty P(e^{\alpha \xi_t} > y) \, dy \leq y_0 + \int_{y_0}^\infty P\left(\xi_t > \frac{1}{\alpha} \log y\right) \, dy. \tag{31}
\]

Now setting \( y_0 = \exp(\alpha x_0) \), we see that for \( y \geq y_0 \) we have

\[
\frac{1}{\alpha} \log y \geq x_0, \tag{32}
\]

from which it follows that \((29)\) holds with \( x = \alpha^{-1} \log y \), and hence

\[
E(e^{\alpha \xi_t}) \leq y_0 + \int_{y_0}^\infty C_t \exp\left(-\gamma_t \frac{1}{\alpha} \log y\right) \, dy. \tag{33}
\]
Carrying out the integration, we obtain

\[ E(e^{\alpha \xi_t}) \leq e^{\alpha x_0} \left( 1 + \frac{\alpha C_t}{\gamma_t - \alpha} e^{-\gamma_t x_0} \right) < \infty, \]

(34)

which shows that \( E(e^{\alpha \xi_t}) < \infty \) for \( 0 \leq \alpha < \gamma_t \) for the given value of \( t \). But since \( \{\xi_t\}_{t \geq 0} \)

is by assumption a Lévy process, this implies that \( E(e^{\alpha \xi_t}) < \infty \) for \( 0 \leq \alpha < \sup_{u > 0} \gamma_u \).

A similar argument shows that \( E(e^{\alpha \xi_t}) < \infty \) for \( \inf_{u > 0} \gamma_u < \alpha < 0 \), and we conclude that \( \{\xi_t\}_{t \geq 0} \) admits exponential moments. \( \square \)

It is worth recalling that one of the motivations indicated by Mandelbrot (1963) for the introduction of Lévy models in finance is the possibility of offering an explanation for the apparent existence of “fat tails” in the distributions of returns. But it seems that what he had in mind was not the construction of specific dynamical models for price processes, but rather the introduction of infinitely-divisible distributions with infinite moments to model the returns on such assets, an assumption that makes the construction of dynamical models difficult. From an empirical point of view, however, the requirement a Lévy process should have “thin tails” is a relatively mild one: a sufficient condition for a Lévy process to admit exponential moments (and hence to have thin tails) is that the jumps should be bounded (Protter 2005, Theorem 34), even if the bounds are set at arbitrarily high values.

VI. DERIVATION OF LÉVY EXPONENT FROM DERIVATIVE PRICES

The price \( H_0 \) at time 0 of a European style derivative that delivers a single random payment \( H_T \) at time \( T \) is given by

\[ H_0 = E(\pi_T H_T). \]

(35)

The expectation is, of course, with respect to the real-world probability measure. The pricing kernel takes care of both the discounting and the probability weighting needed to give the answer. The use of such a formula for derivative pricing is well known, but it may be useful to recall the argument. In the general theory of asset pricing one fixes, as we have done, a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), where \( \mathbb{P} \) is interpreted as the real-world measure, together with a filtration \( \{\mathcal{F}_t\} \), and one assumes the existence of an adapted process \( \{\pi_t\}_{t \geq 0} \) satisfying \( \pi_t > 0 \) for all \( t \geq 0 \) such that for any asset with non-negative price process \( \{S_t\}_{t \geq 0} \) and non-decreasing cumulative dividend process \( \{K_t\}_{t \geq 0} \), the associated discounted, risk-adjusted total value process \( \{\bar{S}_t\}_{t \geq 0} \) defined by

\[ \bar{S}_t = \pi_t S_t + \int_0^t \pi_s \, dK_s \]

(36)

is a martingale. The process \( \{\pi_t\} \) is called the pricing kernel, and we call \( \{\bar{S}_t\} \) the deflated total value process. In defining the pricing kernel we have considered limited liability assets, for which the prices are non-negative and the cumulative dividend process is increasing. By a general asset, not necessarily of limited liability, we mean an asset with the property that its price can be expressed as the difference between the prices of two limited liability assets, and for which the associated cumulative dividend process can be expressed as the difference between two increasing cumulative dividend processes. It then follows that the deflated total value process of a general asset is a martingale.
The formula given above allows for the possibility of both continuously paid and discretely paid dividends. If the dividends are entirely discrete (and paid at possibly random times), then the deflated total value process can be expressed in the form

\[ \bar{S}_t = \pi_t S_t + \sum_{0 \leq s \leq t} \pi_s \Delta(K_s). \]  

(37)

At each time at which a dividend is paid, the cumulative dividend process jumps, and the value of the jump at that time is equal to the dividend. In particular, in the case of a European style derivative with a single payoff \( H_T \) made at time \( T \), we think of the single payoff as a dividend, and hence the cumulative dividend process is zero up to time \( T \), then jumps to \( H_T \) at time \( T \). The sum in (37) reduces to a single term, given by the jump \( \Delta(K_T) = H_T \), and we have \( \bar{S}_T = \pi_T H_T \) at time \( T \). Since the value of the derivative itself drops to zero the instant that the dividend is paid, it follows by the martingale condition that \( \bar{S}_0 = \mathbb{E}(\pi_T H_T) \), which gives us (35). In the literature, by virtue of a conventional slight abuse of notation, one often writes the price process of the derivative in the form \( \{H_s\}_{0 \leq s \leq T} \), as if somehow the terminal value of the derivative is what is paid; but if that were actually the case we would have a situation where one has ones cake and eats it too; it will always therefore be understood therefore that the value of the derivative itself at maturity is 0, whereas the payoff (or dividend) at time \( T \) is \( H_T \). This is consistent with the notion that the price process of the derivative should be right continuous.

We are also in a position to check that if a risky asset pays a proportional dividend at the constant rate \( \delta \) then its price in the geometric Lévy model will be given by

\[ S_t = S_0 e^{(r-\delta)t + R(\lambda, \sigma)t + \sigma \xi_t - \psi(\sigma)t}. \]  

(38)

The expression is of course intuitively plausible, perhaps even obvious by analogy with the corresponding result in the geometric Brownian motion model. Nevertheless, we need to check that the process \( \{\bar{S}_t\} \) defined by (36) is a martingale. A calculation making use of (18), (36) and (38) gives

\[ \mathbb{E}_s \bar{S}_t = S_0 e^{-\delta t + (\sigma - \lambda)\xi_t - \psi(\sigma - \lambda)t} + \delta S_0 \int_0^t e^{-\delta u + (\sigma - \lambda)\xi_u - \psi(\sigma - \lambda)u} \, du. \]  

(39)

Splitting the integral at time \( s < t \) and taking a conditional expectation making use of Fubini’s theorem, we get

\[ \mathbb{E}_s \bar{S}_t = S_0 e^{-\delta t + (\sigma - \lambda)\xi_s - \psi(\sigma - \lambda)s} + \delta S_0 \int_0^s e^{-\delta u + (\sigma - \lambda)\xi_u - \psi(\sigma - \lambda)u} \, du \]

\[ + \delta S_0 \int_s^t e^{-\delta u + (\sigma - \lambda)\xi_u - \psi(\sigma - \lambda)u} \, du, \]  

(40)

from which it follows immediately that \( \mathbb{E}_s \bar{S}_t = \bar{S}_s \). Thus we have shown that the price defined by (38) together with the proportional dividend rate \( \delta \) is such that the resulting deflated total value process is a martingale, which demonstrates that (38) is indeed the correct expression in the geometric Lévy model for the price of a risky asset that pays a proportional dividend at a constant rate.

Turning to the problem of the determination of the Lévy exponent from price data, we proceed to consider a one-parameter family of so-called power-payoff derivatives, for which

\[ H_T = (S_T)\theta, \]  

(41)
where \( q \in \mathbb{R} \). The value \( H_0 \in \mathbb{R}^+ \cup \infty \) of such a derivative at time zero is given by
\[
H_0 = \mathbb{E}(\pi_T S_T^q).
\] (42)

Here we allow the possibility that the value of the derivative may not be finite for some values of \( q \). In a model driven by Brownian motion, the asset price takes the form
\[
S_T = S_0 e^{(r + \lambda \sigma)T + \sigma W_T - \frac{1}{2}\sigma^2 T},
\] (43)
and for the pricing kernel we have
\[
\pi_T = e^{-rT - \lambda W_T - \frac{1}{2}\lambda^2 T},
\] (44)
where for convenience we set \( \pi_0 = 1 \). It follows that
\[
\pi_T S_T^q = S_0^q e^{(q-1)r T + q\lambda \sigma T + (q\sigma - \lambda) W_T - \frac{1}{2}(q\sigma^2 + \lambda^2) T}.
\] (45)

A short calculation then allows one to deduce that the value of the power payoff derivative, regarded as a function \( q \), takes the form
\[
H_0(q) = S_0^q e^{(q-1)r T + \frac{1}{2}q(q-1)\sigma^2 T}
\] (46)
in the Brownian case. One notes that the terms involving \( \lambda \) cancel when the expectation is taken, so the value of the derivative only depends on \( S_0, r, \sigma, \) and \( q \), and that \( H_0(q) \) is finite for all values of \( q \) in this example.

In the case of a geometric Lévy model, the asset price is given by (21) and the pricing kernel is given by (18). Thus we have
\[
\pi_T S_T^q = S_0^q e^{(q-1)r T + q\lambda \sigma T + (q\sigma - \lambda) W_T - \frac{1}{2}(q\sigma^2 + \lambda^2) T}.
\] (47)

The value of the power-payoff derivative regarded as a function of \( q \) then takes the form
\[
H_0(q) = S_0^q e^{(q-1)r T + \psi(q\sigma - \lambda) T + (q-1)\psi(-\lambda) T - q\psi(q\sigma - \lambda) T}.
\] (48)

It is perhaps remarkable that an explicit expression is obtained, but this allows one to study in detail the relation between the type of Lévy model under consideration and the resulting values of derivatives. We note that \( H_0(0) = e^{-rT} \) and that \( H_0(1) = S_0 \), as one would expect. It should be evident that in general \( H_0(q) \) is finite only for a certain range of values of the parameter \( q \). In particular, suppose that \( \sigma > \lambda > 0 \) and that \( \sigma \in A \) and \( -\lambda \in A \). Then for \( A = (\beta, \gamma) \) clearly \( q\sigma - \lambda \in A \) if and only if
\[
\frac{1}{\sigma}(\beta + \lambda) < q < \frac{1}{\sigma}(\gamma + \lambda).
\] (49)

Since \( \beta < -\lambda \) and \( \gamma > 0 \), these inequalities ensure that the interior of the set of values of \( q \) for which \( H_0(q) < \infty \) is an open set \( B \) that includes the origin.

Now we are in a position to ask to what extent specification of the family of derivative prices \( \{H_0(q)\}_{q \in B} \) allows one to infer the nature of the Lévy process driving the model. To this end we note that from observations of \( H_0(0) \) and \( H_0(1) \) one can infer the value of \( r \) and \( S_0 \). Thus without loss of generality it suffices to regard the function
\[
D_0(q) = \frac{1}{T} \log \frac{H_0(q)}{S_0^q e^{(q-1)r T}} = \psi(q\sigma - \lambda) + (q - 1)\psi(-\lambda) - q\psi(q\sigma - \lambda),
\] (50)
which is finite for \( q \in B \), as representing the data supplied by the family of derivative prices under consideration.
Proposition 2 Let the prices of power payoff derivatives with payoffs $H_T = (S_T)^q$ for $q \in \mathbb{R}$ be given for a non-dividend-paying risky asset with a price process $\{S_t\}_{t \geq 0}$ that is known to be a geometric Lévy process, and suppose it is known that the pricing kernel is a geometric Lévy process driven by the same Lévy process, up to a constant factor of proportionality. Then the Lévy exponent can be inferred up to a transformation $\psi(\alpha) \to \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where $c$ and $\mu$ are constants.

Proof. Without loss of generality one can set $\sigma = 1$. Then we have

$$D_0(q) = \psi(q - \lambda) + (q - 1)\psi(-\lambda) - q\psi(1 - \lambda).$$

In the setting of the problem we take $D_0(q)$ to be given for all $q \in \mathbb{R}$ and finite in some open set $B$, and we consider $\lambda$ to be unknown. The goal is to determine the Lévy exponent. Writing $\hat{\psi}(\alpha) = \psi(\alpha + \lambda) - \hat{\psi}(\lambda)$, we have $D_0(q) = \hat{\psi}(q) - q\hat{\psi}(1)$. This implies that $\hat{\psi}(q) = D_0(q) + qb$ for some $b \in \mathbb{R}$. Now, it is easy to see that $\psi(\alpha) = \hat{\psi}(\alpha + \lambda) - \hat{\psi}(\lambda)$. We conclude that for some choice of $\lambda$ and $b$ the Lévy exponent takes the form

$$\psi(\alpha) = D_0(\alpha + \lambda) - D_0(\lambda) + b\alpha.$$

Substituting (52) into the right-hand side of (51), one can check that the solution is valid. Finally, we note that under a transformation of the form $\psi(\alpha) \to \hat{\psi}(\alpha)$, with $\hat{\psi}(\alpha) = \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where $c, \mu \in \mathbb{R}$, we find that $\hat{\psi}(\alpha) = D_0(\alpha + \lambda + \mu) - D_0(\lambda + \mu) + b\alpha$. The effect of the transformation is $\lambda \to \hat{\lambda} = \lambda + \mu$. Since $\lambda$ is unknown, this shows that the Lévy exponent is determined only up to a transformation of the type indicated. \hfill \Box

VII. INTERPRETIVE REMARKS

Following on from this result, a few comments may be in order. We recall that a Lévy process is completely characterized by the random variable $\xi_t$ at a single instant of time $t$. This reflects the remarkable fact that there is a one-to-one correspondence between Lévy processes and infinitely divisible distributions, and a Lévy process has the property that the distribution of its value at any particular time is infinitely divisible. Taking $t = 1$ for convenience, we have $\psi(\alpha) = \log \mathbb{E}[e^{\alpha \xi_1}]$ in the case of a Lévy process that admits exponential moments, and we note that the distribution of $\xi_1$ is completely determined by the Lévy exponent $\{\psi(\alpha)\}_{\alpha \in A}$. Each such distribution belongs in a natural way to a certain one-parameter family of distributions, which we call an Esscher family of distributions. The distribution of $\xi_1$ is the function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \mathbb{E}[\mathbb{1}\{\xi_1 \leq x\}]$. For the associated Lévy exponent we then have

$$\psi(\alpha) = \log \int_{-\infty}^{+\infty} e^{\alpha x} dF(x).$$

The corresponding Esscher family $F_\delta : \mathbb{R} \to [0, 1], \delta \in A$, is defined by the measure change

$$F_\delta(x) = \mathbb{E}[e^{\delta \xi_1 - \psi(\delta)} \mathbb{1}\{\xi_1 \leq x\}].$$

We may accordingly ask for the structure of the Lévy exponent associated with $F_\delta$. This is

$$\psi_\delta(\alpha) = \log \int_{-\infty}^{+\infty} e^{\delta x} dF_\delta(x) = \log \int_{-\infty}^{+\infty} e^{\delta x - \psi(\delta)} e^{\alpha x} dF(x) = \psi(\alpha + \delta) - \psi(\delta).$$
So we see that by Proposition 2, the specification of the prices of power payoff derivatives allows one to determine the Esscher family of the Lévy exponent of the underlying Lévy process, modulo an irrelevant linear term. Lévy processes that are equivalent in this sense can be said to belong to the same “noise type” (Brody, Hughston & Yang 2013).

On the other hand, if more information is known a priori about the nature of the underlying asset, then a more precise determination of the Lévy exponent is possible. Consider the case, for instance, where it is known that the asset on which the power payoff derivative is based is the natural numeraire. In that situation, we know that \( \sigma = \lambda \), and for the asset price we have

\[
\zeta_t = \zeta_0 e^{rt + R(\lambda,\lambda)t + \lambda\xi_t - \psi(\lambda)t},
\]

where the excess rate of return is given by \( R(\lambda,\lambda) = \psi(\lambda) + \psi(-\lambda) \). In this case we can without loss of generality set \( \lambda = 1 \). It follows then from (51) and (52) that \( D_0(1) = 0 \) and hence \( \psi(\alpha) = D_0(\alpha + 1) + \lambda \theta \). The Lévy exponent is completely determined up to an irrelevant constant.

In a geometric Lévy model, the pricing kernel can be written in the form

\[
\pi_t = e^{-rt}\Lambda_t
\]

where the martingale \( \{\Lambda_t\}_{t\geq0} \) defined by

\[
\Lambda_t = e^{-\lambda\xi_t - \psi(-\lambda)t},
\]

determines a change of measure. Thus for any \( \mathcal{F}_t \)-measurable random variable \( Z_t \) we have

\[
\tilde{P}(Z_t < z) = \tilde{E} [\mathbbm{1}(Z_t < z)] = E [\Lambda_t \mathbbm{1}(Z_t < z)].
\]

We refer to \( \tilde{P} \) as the risk-neutral measure and write \( \tilde{E} \) for expectation under \( \tilde{P} \). The terminology “risk-neutral” comes from the fact that \( \tilde{E}(S_t) = S_0e^{rt} \) in the geometric Lévy model. Then \( \{\tilde{\psi}(a)\} \) has the interpretation of being the Lévy exponent associated with \( \xi_t \) under the risk-neutral measure. That is to say,

\[
\tilde{\psi}(a) = \frac{1}{t} \log \tilde{E}[e^{a\xi_t}].
\]

The essence of Proposition 2 is that the given family of derivative prices can be used to calculate \( \{\tilde{\psi}(a)\} \), which then fixes the associated exponent \( \{\psi(a)\} \) under \( P \), modulo the freedom indicated.

Let us write \( C_0T(x) \) for the price at time 0 of a call option with maturity \( T \) and strike \( x \). Can one use the data \( \{C_0T(x)\}_{x\geq0} \) for fixed \( T \) to ascertain the Lévy exponent in a geometric Lévy model? The answer is yes, though the method is in some respects less straightforward than the use of power-payoff derivatives, as we shall see. Now, it is known that if the random variable \( S_T \) corresponding to the terminal value of the asset at time \( T \) admits a risk-neutral density function, then we can use the idea of Breeden & Litzenberger (1978) to work out this density in terms of call option data. In particular, if we write

\[
\tilde{\theta}(x) = \frac{d}{dx} \tilde{P}(S_T \leq x)
\]

for the density of \( S_T \) under the risk-neutral measure \( \tilde{P} \), we have

\[
\tilde{\theta}(x) = e^{-rT} \frac{d^2C_0T(x)}{dx^2}.
\]
This follows from the fact that
\[ C_{0T}(x) = e^{-rT} \int_{0}^{\infty} (y - x)^+ \tilde{\theta}(y) \, dy. \] (62)

Then \( \{\tilde{\theta}(x)\}_{x \geq 0} \) can be used to calculate the values of power-payoff derivatives via the relation
\[ \tilde{E}(S_T^q) = \int_{0}^{\infty} x^q \tilde{\theta}(x) \, dx, \] (63)
and from there we can work out \( \tilde{\psi}(\alpha) \), as indicated in the previous section. The difficulty with this approach is that in a geometric Lévy model the distribution of \( S_T \) does not in general admit a density function, and the system of call option prices \( \{C_{0T}(x)\}_{x \geq 0} \) is not differentiable for all \( x \in \mathbb{R}^+ \). The situation can be remedied to some extent if instead we make use of the risk-neutral distribution function \( \{\tilde{F}(x)\}_{x \geq 0} \) and express the option price in the form of a Lebesgue-Stieltjes integral, writing
\[ C_{0T}(x) = e^{-rT} \int_{0}^{\infty} (y - x)^+ \, d\tilde{F}(y), \] (64)
with the understanding that the distribution function is right-continuous. Then the derivative of the option price with respect to the strike is defined for all \( x \in \mathbb{R}^+ \) apart from points of discontinuity of the distribution function, and this is sufficient to enable us to recover the distribution function in its entirety. Once the distribution function has been reconstructed, one can proceed to determine the Lévy exponent by calculating the system of power-payoff prices, given for all \( q \in \mathbb{R} \) by
\[ H_0(q) = e^{-rT} \int_{0}^{\infty} x^q \, d\tilde{F}(x). \] (65)

VIII. IMAGINARY POWER PAYOFFS

As another example of a one-parameter family of derivatives from which information can be extracted concerning the Lévy exponent when the underlying is a geometric Lévy asset we consider a family of imaginary power payoffs, for which the terminal cash flow is given by
\[ F_T(q) = (S_T)^i q, \] (66)
where \( q \in \mathbb{R} \). The value of such a contract at time zero takes the form
\[ F_0(q) = \mathbb{E}(\pi_T S_T^{i q}) = \mathbb{E}(\pi_T e^{i q \log S_T}). \] (67)

Since the payoff is a complex function, we are in effect valuing two different derivatives, each with a bounded payoff. That is to say,
\[ F_0(q) = \mathbb{E}[\pi_T \cos(q \log S_T)] + i \mathbb{E}[\pi_T \sin(q \log S_T)]. \] (68)
In other words, the prices \( \{F_0(q)\} \) for \( q \in \mathbb{R} \) can be thought of as a pair of families of prices \( \{F_0^c(q)\} \) and \( \{F_0^s(q)\} \), for which the corresponding payoff functions are given respectively
by $F_T^p(q) = \cos(q \log S_T)$ and $F_T^s(q) = \sin(q \log S_T)$. Note that the payoffs, and hence the prices, are bounded for all values of $q$. A calculation then shows that

$$F_0(q) = S_T^q e^{(iq-1)T} e^{(-iq \psi(\sigma-\lambda)+(iq-1)\psi(-\lambda)+\psi(iq\sigma-\lambda))T}. \quad (69)$$

With these ideas at hand we can use the methods of Fourier analysis to investigate more general payoffs. We begin by recalling briefly a few well known facts. Let the map $f: \mathbb{R} \to \mathbb{R}$ be such that $f \in L^1$. The Fourier transform of $f$ is the function $g: \mathbb{R} \to \mathbb{C}$ defined by

$$g(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} f(x) \, dx. \quad (70)$$

Under various further conditions the relation between $f$ and $g$ can then be inverted. For example, if $f \in L^1$ and is continuous, and if $g \in L^1$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} g(q) \, dq. \quad (71)$$

A sufficient condition for these requirements to be satisfied is that $f$ should be a “good” function in the sense of Lighthill (1958), that is to say, that it should be everywhere differentiable any number of times and such that it and all its derivatives are $O(|x|^{-n})$ as $|x| \to \infty$ for all $n \in \mathbb{N}$. We recall that $f(x)$ is said to be $O(h(x))$ as $x \to \infty$ if

$$\limsup_{x \to \infty} \left| \frac{f(x)}{h(x)} \right| < \infty. \quad (72)$$

If $f$ is a good function then its Fourier transform $g$ is also good.

Now consider the situation where the payoff of a European-style derivative with value $H_0$ at time zero takes the form $H_T = f(\log S_T)$ for some $f \in L^1$. If $f$ is continuous and $g \in L^1$, then by use of (71) we can write the payoff in the form

$$H_T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iq\log S_T} g(q) \, dq. \quad (73)$$

This expresses $H_T$ as a portfolio of imaginary power payoffs parameterized by $q$, where $g(q)$ determines the relative portfolio weighting for the given $q$. Multiplying each side of equation (73) with the pricing kernel $\pi_T$ and taking the expectation we obtain

$$\mathbb{E}(\pi_T H_T) = \frac{1}{\sqrt{2\pi}} \mathbb{E} \left( \int_{-\infty}^{\infty} \pi_T e^{i q \log S_T} g(q) \, dq \right), \quad (74)$$

from which it follows that

$$\mathbb{E}(\pi_T H_T) = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} \mathbb{E} \left[ \pi_T e^{i q \log S_T} \right] g(q) \, dq \right). \quad (75)$$

Inserting (67) into (75), we then have

$$H_0 = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} F_0(q) \, g(q) \, dq \right). \quad (76)$$
Thus the price of the derivative can be expressed as the value of a portfolio of imaginary power payoff derivatives. To check that the interchange of the expectation and the integration in equation (74) is valid (Kingman & Taylor 1966, Theorem 6.5), we note that

$$\mathbb{E}\left(\int_{-\infty}^{\infty} \pi_T e^{i q \log S_T} g(q) \, dq\right) = \int_{\omega \in \Omega} \int_{-\infty}^{\infty} \pi_T e^{i q \log S_T} g(q) \, dq \, \mathbb{P}(d\omega), \quad (77)$$

and that

$$\int_{\omega \in \Omega} \int_{-\infty}^{\infty} |\pi_T e^{i q \log S_T} g(q)| \, dq \, \mathbb{P}(d\omega) = \mathbb{E}[\pi_T] \int_{-\infty}^{\infty} |g(q)| \, dq < \infty. \quad (78)$$

As an example we consider a European-style derivative payoff $H_T = f(\log S_T)$ at time $T$ for which $f$ takes the form of a normal density function

$$f(x) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{1}{2} \frac{(x-a)^2}{u}}, \quad (79)$$

with mean $a$ and variance $u$. In the case of a geometric Brownian motion model, the random variable corresponding to the terminal value of the asset is normally distributed with a risk-neutral density of the form

$$\bar{\theta}(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2} \frac{(x-b)^2}{v}}, \quad (80)$$

where $b = (r - \frac{1}{2}\sigma^2) T$ and $v = \sigma^2 T$. The price of the derivative at time zero is given by

$$H_0 = e^{-rT} \mathbb{E}[H_T] = e^{-rT} \int_{-\infty}^{\infty} \bar{\theta}(x) f(x) \, dx. \quad (81)$$

With some calculation, we find that

$$H_0 = e^{-rT} \frac{1}{\sqrt{2\pi (u + v)}} e^{-\frac{1}{2} \frac{(a-b)^2}{u+v}}. \quad (82)$$

For instance, if we set $a = 0, u = 1$ and insert the aforementioned values of $b$ and $v$, we obtain

$$H_0 = e^{-rT} \frac{1}{\sqrt{2\pi (1 + \sigma^2 T)}} e^{-\frac{1}{2} \frac{(r-\frac{1}{2}\sigma^2)^2 T^2}{1+\sigma^2 T}}. \quad (83)$$

Alternatively, we can replicate the payoff of the derivative as a portfolio of imaginary power payoffs using the Fourier technique. Since $f \in L^1$, we can set

$$g(q) = \frac{1}{2\pi \sqrt{u}} \int_{-\infty}^{\infty} e^{-iqx} e^{-\frac{1}{2} \frac{(x-a)^2}{u}} \, dx, \quad (84)$$

and a calculation gives

$$g(q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} q^2 u} e^{-iqa}. \quad (85)$$
By (73) and (85), and using the fact that $f$ is a good function, one sees that the payoff of the derivative can be expressed in the form

$$H_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_T^q e^{-\frac{1}{2}q^2 u} e^{-iq\alpha} dq. \quad (86)$$

Then by (76) we obtain the derivative price as a portfolio of imaginary power-payoff prices:

$$H_0 = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} F_0(q) e^{-\frac{1}{2}q^2 u} e^{-iq\alpha} dq \right). \quad (87)$$

We observe, finally, that if the prices of imaginary power payoff derivatives delivering the cash flows defined by (66) are given for all $q \in \mathbb{R}$, then by adapting the framework of Proposition 2 we can work out the implied Lévy exponent modulo some specified freedom. We recall that the price of a power-payoff derivative at time zero is given for power $q$ by equation (69). Therefore if we consider the function $\{D_0(q)\}_{q\in \mathbb{R}}$ defined by

$$D_0(q) = \frac{1}{T} \log \frac{F_0(q)}{S_0^q e^{r(iq-1)T}}, \quad (88)$$

we find that

$$D_0(q) = -iq (\psi(\sigma - \lambda) - \psi(-\lambda)) + \psi(iq\sigma - \lambda) - \psi(-\lambda) = \tilde{\psi}(iq\sigma) - iq \tilde{\psi}(\sigma), \quad (89)$$

where $\tilde{\psi}(\alpha) = \psi(\alpha - \lambda) - \psi(-\lambda)$. Without loss of generality we can then set $\sigma = 1$ to get

$$D_0(q) = \tilde{\psi}(iq) - iq \tilde{\psi}(1). \quad (90)$$

This implies that $\tilde{\psi}(iq) = D_0(q) + iq b$ for some $b \in \mathbb{R}$. Now, $\psi(\alpha) = \tilde{\psi}(\alpha + \lambda) - \tilde{\psi}(\lambda)$. It follows that for some $\lambda$ and $b$ the Lévy exponent takes the form

$$\psi(iq) = D_0(iq + \lambda) - D_0(\lambda) + ibq. \quad (91)$$

Thus, we see that once we have been given a range of price data for imaginary power-payoff options, we can work out the Lévy exponent modulo a transformation of the form

$$\psi(iq) \rightarrow \psi(iq + \mu) - \psi(\mu) + icq, \quad (92)$$

where $c$ and $\mu$ are constants.

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