Statistics transmutations in two-dimensional systems and the fractional quantum Hall effect

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Abstract

Statistics transmutations to composite fermions in fractional-quantum-Hall-effect systems are considered in the Hartree-Fock approximation and in the RPA. The Hartree-Fock ground-state energy shows that the transmutations are not energetically preferable. Within the RPA it is found that the system exhibits a fractional quantum Hall effect.
1. Introduction

The fractional quantum Hall effect (FQHE) brings many intrinsic questions to physics of two-dimensional systems. Owing to the idea of Laughlin [1] one can explain experimentally observed phenomena, however, an origin of the Laughlin wave function still remains somewhat mysterious.

An alternative approach to understanding the nature of the FQHE was proposed by Jain [2, 3] within the idea of transmutability of statistics in two-dimensional systems. Jain noted that in 2D the antisymmetry of the many-particle wave function is not an unambiguous criterion of quantum statistics. The antisymmetry is held by a class of quantum-statistics particles called composite fermions [2]. An interchange of two composite fermions produces a phase factor of $e^{i(2p+1)\pi}$ (p - an integer number). Note that this is also the case of the Laughlin wave function.

Statistics transmutations to composite fermions are described by a Chern-Simons theory with an even number of flux quanta attached to each electron [4, 5]. In the mean field approximation Chern-Simons interactions produce the average statistical field $B^s = -2p\frac{hc}{e}\rho$ which partially cancels the external magnetic field. Thus, the effective field acting on electrons is $B^{ef} = B^{ex} + B^s$. We predict an analog of the quantum Hall effect when, effectively, $n$ Landau levels a completely filled, i.e. $B^{ef} = \frac{1}{n}\frac{hc}{e}\rho$. Hence, $B^{ex} = \frac{2p+1}{n}\frac{hc}{e}\rho$, which means that from the point of view of the external magnetic field the lowest Landau level is filled in the fraction $\nu = \frac{n}{2p+1}$. We do not consider the case of $|B^s| > B^{ex}$ when $B^{ef} \uparrow\downarrow B^{ex}$ and one expects an opposite Hall effect.

The aim of this paper is to find, within the Hartree-Fock approximation (HFA), the ground state and the ground-state energy of the system. We obtain also the RPA response function, and thus collective modes and a Hall conductance.

2. Hartree-Fock ground state

Let us start from the Hamiltonian of anyons (in the fermion representation) in the
external magnetic field [6]:

\[
H = \frac{1}{2m} \sum_{i=1}^{N} \left( \mathbf{p}_i + \frac{e}{c} \mathbf{A}_i + \frac{e}{c} \mathbf{A}_{i \text{ex}} \right)^2
\]

where Chern-Simons interactions are given by

\[
\mathbf{A}_i = -\frac{2\hbar c}{e} \hat{z} \times \sum_{j \neq i} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2},
\]

\( \hat{z} \) - a perpendicular to the plane unit vector. The average statistical field vector potential is found via the equation:

\[
\bar{\mathbf{A}}_i = -\frac{2\hbar c \rho}{e} \hat{z} \times \int \frac{d^2r}{|\mathbf{r}_i - \mathbf{r}|^2} = \frac{1}{2} B^s \hat{z} \times \mathbf{r}_i.
\]

The Hamiltonian (1) for integer values of \( p \) describes the statistics transmutations to composite fermions (in the system of electrons in the external magnetic field) and for other values of \( p \) - the transmutations to anyons with statistics parameters \( \Theta = (1+2p)\pi \) (an exchange of two anyons produces a phase factor of \( e^{i\Theta} \)). In this paper we assume that \( B_{ef} = \nabla \times \mathbf{A}_{ef} = B_{ex} + B^s = \frac{\hbar e}{n} \rho \) and in the ground state one has \( n \) completely filled Landau levels.

Dividing the Hamiltonian (1) into one-, two- and three-body parts one finds the Hartree-Fock equations of the ground-state energy [8, 9, 10]:

\[
\langle H_a \rangle = \frac{1}{2m} \int d^1\phi_i^+(1) (\mathbf{p}_1 + \frac{e}{c} \mathbf{A}^{ef})^2 \phi_i(1),
\]

\[
\langle H_b \rangle = -\frac{e}{mc} \sum_{l,m} \int d1d2 \phi_i^+(1) \phi_m^+(2) \mathbf{A}_{12} (\mathbf{p}_1 + \frac{e}{c} \mathbf{A}^{ef}) \phi_l(2) \phi_m(1),
\]

\[
\langle H_c \rangle = \frac{e^2}{2c^2 m} \sum_{l,m} \int d1d2 \phi_i^+(1) \phi_m^+(2) |\mathbf{A}_{12}|^2 \phi_l(2) \phi_m(1),
\]

\[
\langle H_d \rangle = -\frac{e^2}{2c^2 m} \sum_{l,m} \int d1d2 \phi_i^+(1) \phi_m^+(2) |\mathbf{A}_{12}|^2 \phi_l(2) \phi_m(1),
\]

\[
\langle H_e \rangle = \frac{e^2}{c^2 m} \sum_{l,m,k} \int d1d2d3 \phi_i^+(1) \phi_m^+(2) \phi_k^+(3) \mathbf{A}_{12} \mathbf{A}_{13} \phi_l(3) \phi_m(1) \phi_k(2),
\]

\[
\langle H_f \rangle = -\frac{e^2}{2c^2 m} \sum_{l,m,k} \int d1d2d3 \phi_i(1) |\mathbf{A}_{12}|^2 \phi_m^+(2) \phi_k^+(3) \mathbf{A}_{12} \mathbf{A}_{13} \phi_m(3) \phi_k(2).
\]
where \( d_1 = d^2 \mathbf{r}_1 \), sums extend over occupied Landau levels. We assume that
\[
\sum_l |\phi_{kl}><\phi_{kl}| = \Pi_k
\]
is the k-th Landau level projector. Hartree-Fock self energies are found from the first variation of the ground-state energy [9, 11]. Using methods developed for anyons one obtains (in units of \( E_F = \frac{2\pi\hbar^2}{m} \) - the Fermi energy of the 2D electron gas) [9, 10]:

\[
H_{HF} = \frac{1}{n} \sum_{k=0}^{n-1} \left( k + \frac{1}{2} + 2p + 2p^2n - 2p^2nE_R + p^2nS_{n-1} - 2p^2nS_{n-1-k} \right) \Pi_k
\]
\[
+ \frac{1}{n} \sum_{k=n}^{\infty} \left( k + \frac{1}{2} - p^2 + 2p^2nE_R + p^2n^2\left( \frac{1}{k} + \frac{1}{k+1} \right) - p^2nS_{n-1} + 2p^2nS_{k-n} \right) \Pi_k
\]

(10)

where \( E_R = \ln R \sqrt{\frac{\mu eB}{\hbar c}} + \frac{1}{2}(\gamma - \ln 2) \) (\( \gamma \approx 0.577... \) - the Euler constant), \( R \) - a sample radius, \( S_m = \sum_{k=1}^{m} \frac{1}{k} \) and \( S_0 = 0 \).

Above results confirm the choice of the ground state. We find the large energy gap \((\approx 4p^2E_R)\) between the last occupied Landau level and the next ones. Taking \( n = 1 \) we obtain the same results as Cabo and Martinez [11]. The ground-state energy is given by:

\[
< H_a > = \frac{N}{2}, \quad < H_b > = \frac{pN}{n} \quad (11)
\]
\[
< H_c > = < H_d > = 2p^2NE_R + p^2nS_{n-1} - \frac{1}{2n}p^2N(n-1), \quad (12)
\]
\[
< H_e > = p^2N, \quad < H_f > = -2p^2NE_R - \frac{p^2}{n}(nS_{n-1} - n + 1)n. \quad (13)
\]

Hence, the total result is finite and equals:

\[
< H > = \frac{1}{2}NE_F(1 + 3p^2 + \frac{2p}{n} - \frac{p^2}{n}). \quad (14)
\]

In the limit \( n \to \infty \) one finds the system in the zero effective field and the ground state is the Fermi sea [12]. The direct calculations of the Hartree-Fock energy of this system [13] give the same result as obtained from Eq.(14), i.e. \( < H > = \frac{1}{2}NE_F(1 + 3p^2) \) as \( n \to \infty \).

The energy of electrons in the external magnetic field \( B_{ex} = \frac{2p+1}{n} \frac{\hbar c}{e} \rho \) is \( \frac{1}{2}NE_F(2p + \frac{1}{n}) \). Comparing with the result (14) one finds the difference:

\[
\Delta E = \frac{1}{2}NE_F[2p^2 + (p - 1)^2(1 - \frac{1}{n})] \quad (15)
\]
which is always greater than zero. For example when \( p = 1 \) the cost of the generation of fluxes is \( \Delta E = N E_F \), i.e. twice the energy of the 2D electron gas (for \( n = 1 \), \( \Delta E = N E_F p^2 \)). Hence, the HFA suggests that transmutations to composite fermions in fractional-quantum-Hall-effect systems are not energetically favourable at filling fractions \( \nu = \frac{n}{2pm+1} \).

3. RPA response function

The Hamiltonian \( H \) can be separated into two parts: \( H = H_0 + H_1 \) where
\[
H_0 = \frac{1}{2m} \sum_i (p_i + A_i^e)^2
\]
is treated as the unperturbed term and \( H_1 \) is the interaction Hamiltonian [6, 7]. Let us define current densities:
\[
J(r) = \frac{1}{2m} \sum_j \left\{ p_j + \frac{e}{c} A_j^e, \delta(r - r_j) \right\}
\]
and
\[
j(r) = \frac{1}{2m} \sum_j \left\{ p_j + \frac{e}{c} A_j^e, \delta(r - r_j) \right\}
\]
where braces denote an anticommutator. \( J, j \) are the vector parts of \( J^\mu, j^\mu \), respectively, with \( \mu = 0, x, y \). We define \( j^0, J^0 \) as density fluctuations:
\[
j^0 = J^0 = \sum_j \delta(r - r_j) - \rho.
\] The problem of interest is the linear response of the system to a weak external electromagnetic field, \( J^\mu = K^{\mu\nu} A_\nu \), which is characterized by:
\[
K^{\mu\nu}(q, \omega) = \frac{4\pi e^2}{mc^2} \delta^{\mu\nu}(1 - \delta^{00}) + \frac{4\pi e^2}{\hbar c^2} \Delta^{\mu\nu}_R(q, \omega).
\]
where (in the real space)
\[
\Delta^{\mu\nu}_R(r, r') = -i \theta(t - t') < [J^\mu(\mathbf{r}), J^\nu(\mathbf{r'})] >.
\]
Choosing \( q = q \mathbf{\hat{x}} \) and the Coulomb gauge one has \( J^x = \frac{\omega}{q} J^0 \) and we can restrict our problem to \( 2 \times 2 \) \( K^{\mu\nu} \) matrix [12]. It is convenient first to consider the retarded correlation function of two effective field currents:
\[
D^{\mu\nu}_R(r, r') = -i \theta(t - t') < [j^\mu(\mathbf{r}), j^\nu(\mathbf{r'})] >.
\]
The random phase approximation consists in approaching $D_R$ by a sum of bubble graphs which can be written as:

$$D_R^{RPA}(q, \omega) = [I - D_R^0(q, \omega)V(q)]^{-1}D_R^0(q, \omega)$$

(22)

where $V$ is the interaction matrix obtained from the Hamiltonian $H_1$. We have

$$(\hbar, c = 1, \omega_c = \frac{eBef}{m})$$

$$V(q) = \frac{4p\pi}{q^2} \begin{pmatrix} 2pm\omega_c & -iq \\ iq & 0 \end{pmatrix}.$$  

(23)

As in the case of anyons [14] one finds

$$D_R^0(q, \omega) = \frac{n}{2\pi\omega_c} \begin{pmatrix} q^2\Sigma_0 - i\omega_c\Sigma_1 \\ i\omega_c\Sigma_1 \omega_c\Sigma_2 \end{pmatrix}$$

(24)

where

$$\Sigma_j = e^{-x} \sum_{m=n}^{\infty} \sum_{l=0}^{n-1} \frac{(\omega\omega_c)^2}{m-l-i\eta} \frac{l!}{m!} x^{m-l-1}[L_l^{m-l}(x)]^{2-j}$$

$$\times [(m-l-x)L_l^{m-l}(x) + 2x\frac{dL_l^{m-l}(x)}{dx}]^j$$

(25)

and $x = \frac{q^2}{2eBef}$. Then we have:

$$D_R(q, \omega) = \frac{n}{2\pi\omega_c det} \begin{pmatrix} q^2\Sigma_0 - i\omega_c\Sigma_s \\ i\omega_c\Sigma_s \omega_c\Sigma_p \end{pmatrix}$$

(26)

where $det = det(I-D^0V) = (1-2pn\Sigma_1)^2 - (2pn)^2\Sigma_0(1+\Sigma_2)$, $\Sigma_s = \Sigma_1 - 2pn\Sigma_1^2 + 2pn\Sigma_0\Sigma_2$, $\Sigma_p = (2pn)^2\Sigma_1 + \Sigma_2 - (2pn)^2\Sigma_0\Sigma_2$.

Collective modes are determined by the zeros of the determinant $det$. Because of the denominator in $\Sigma_j$ one finds an infinite set of modes with gaps $\omega_m = m\omega_c$ as $q \to 0$, $m = 2, 3, ...$ [13], which is in agreement with results of Lopez and Fradkin [5]. In contrast to the case of anyons we have no gapless excitations.

The full current correlation function is given by [3]:

$$\Delta_R \simeq \Delta_R^{RPA} = (I + U^+)D_R^{RPA}(I + U)$$

(27)
where
\[ U = \frac{\omega_c 2pn}{q} \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}. \] (28)

Hence, the linear response kernel equals:
\[ K_{RPA} = \frac{e^2n}{2\pi \text{det}} \begin{pmatrix} \frac{\Sigma_0}{\omega_c} & -iq(\Sigma_s + 2pn\Sigma_0) \\ iq(\Sigma_s + 2pn\Sigma_0) & \omega_c(1 + \Sigma_2) \end{pmatrix}. \] (29)

As it is expected there is no Meissner effect in the system, i.e. \( K_{yy}(\omega = 0, q \to 0) = 0 \).

Furthermore, the Hall conductance of the system is given by:
\[ \sigma_{xy}(q, 0) = \frac{e^2n}{2\pi \text{det}} \left(-2pn\Sigma_0 - \Sigma_s\right) \simeq |_{q \to 0} \frac{e^2}{2\pi} \frac{n}{1 + 2pn} = \frac{e^2}{2\pi} \nu \] (30)
which is a fractional multiple of \( \frac{e^2}{h} \). Thus, the system of composite fermions exhibits a fractional quantum Hall effect with the correct value of the Hall conductance. This result was first obtained by Lopez and Fradkin [4, 5] within a field-theoretical approach.

4. Conclusions

Within the formalism developed for anyons we consider the Chern-Simons theory of the FQHE. In the HFA we confirm the choice of the ground state as proposed by Jain [2, 3] within the mean field approximation. We obtain the large energy gap (\( \simeq \ln R \)) between the last occupied Landau level and the next ones. The ground-state energy is found and in the limit of \( B^{\text{eff}} \to 0 \) it agrees with the direct result for the system in the zero effective field [13]. Comparing with the energy of electrons in the external magnetic field \( B^{ex} = \frac{2m+1}{n} \frac{hc}{e} \rho \) we find transmutations to composite fermions in fractional-quantum-Hall-effect systems not to be energetically preferable at filling fractions \( \nu = \frac{n}{2pn+1} \).

The RPA response function is found and gives the expected Hall conductance in agreement with results of Lopez and Fradkin [4]. We find also the set of collective modes with gaps \( m\omega_c \), however, a dependence on \( q \) is not given.

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