ON UNIVERSALITY AND CONVERGENCE OF THE FOURIER SERIES OF FUNCTIONS IN THE DISC ALGEBRA.

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Abstract. We construct functions in the disc algebra with point-wise universal Fourier series on sets which are $G_\delta$ and dense and at the same time with Fourier series whose set of diverge is of Hausdorff dimension zero. We also see that some classes of closed sets of measure zero do not accept uniformly universal Fourier series, although all such sets accept divergent Fourier series.

1. Introduction and notation.

Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} | |z| = 1\} = \mathbb{R}/2\pi$. We denote by $C(\mathbb{T})$ the set of complex continuous functions with the supremum norm $\| \cdot \|$ and, for $f \in C(\mathbb{T})$, by $S_n(f, t)$ the $n$-th partial sum of the Fourier series of $f$ at the point $t \in \mathbb{T}$,

$$S_n(f, t) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikt},$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} dt, \quad k \in \mathbb{Z},$$

is the $k$-th Fourier coefficient of $f$.

Also, let $A(\mathbb{D}) = \{f \in C(\mathbb{T}) | \hat{f}(k) = 0 \text{ for } k < 0\}$ be the disc algebra with the supremum norm.

If $(X, d)$ is a complete metric space, a property is said to be satisfied at quasi all points of $X$ if it is satisfied at a $G_\delta$ and dense set, i.e. at a topologically large set.

First we recall the following propositions regarding the divergence of the partial sums $S_n(f, t)$ that the reader must have in mind.

Key words and phrases. Universality, disc algebra.

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Proposition 1.1. (See [4]) Quasi all \( f \in C(\mathbb{T}) \) have the property that their Fourier series diverge at quasi all points of \( \mathbb{T} \).

Proposition 1.2. (See [3]) (a) Let \( E \subseteq \mathbb{T} \). The following are equivalent.

(i) \( E \) is a set of divergence for \( C(\mathbb{T}) \), i.e. there is a continuous function whose Fourier series diverges at all points of \( E \).

(ii) \( E \) is a set of infinite divergence for \( C(\mathbb{T}) \), i.e. there is a continuous function \( f \) such that \( \lim_{n \to \infty} |S_n(f, t)| = +\infty \) for all \( t \in E \).

(b) Every set of Lebesgue measure zero is a set of divergence for \( C(\mathbb{T}) \).

It is not hard to see that a set \( E \subseteq \mathbb{T} \) may be \( G_\delta \) and dense and simultaneously of Lebesgue measure zero (see [8]). Of course, by Carleson’s theorem a set of divergence for \( C(\mathbb{T}) \) has necessarily measure zero.

Proposition 1.3. (See [1]) The set \( \{ t \mid \lim_{n \to \infty} |S_n(f, t)| = +\infty \} \) has Hausdorff dimension equal to 1 for quasi all \( f \in C(\mathbb{T}) \).

Recently, in [7] and [2], a different notion of divergence has been studied. We present the definitions and the basic results of these papers.

Definition 1.1. Let \( E \subseteq \mathbb{T} \). We say that \( f \in C(\mathbb{T}) \) is pointwise universal on \( E \) if for every \( g : E \to \mathbb{C} \) belonging to the Baire-1 class there exists a strictly increasing sequence of positive integers \( (k_n) \) such that \( S_{k_n}(f, t) \to g(t) \) for all \( t \in E \).

We denote the class of these functions by \( U_p(E) \).

Definition 1.2. Let \( K \subseteq \mathbb{T} \) be a compact set. We say that \( f \in C(\mathbb{T}) \) is uniformly universal on \( K \) if for every continuous \( g : K \to \mathbb{C} \) there exists a strictly increasing sequence of positive integers \( (k_n) \) such that \( \| S_{k_n}(f, \cdot) - g \|_K \to 0 \), where \( \| \cdot \|_K \) is the supremum norm on \( K \).

We denote the class of these functions by \( U(K) \).

Of course the notions of pointwise universality and uniform universality coincide when the set \( E = K \) is finite. In this case we speak about universality on \( E = K \).

Denoting by \( \mathcal{K}(\mathbb{T}) \) the complete metric space of all compact nonempty subsets of \( \mathbb{T} \) with the Hausdorff metric, we have the following result.

Proposition 1.4. Quasi all \( f \in C(\mathbb{T}) \) (or \( f \in A(\mathbb{D}) \)) are uniformly universal on quasi all sets \( K \in \mathcal{K}(\mathbb{T}) \). (For \( C(\mathbb{T}) \) see [7] and for \( A(\mathbb{D}) \) see [2].)

Since quasi all sets in \( \mathcal{K}(\mathbb{T}) \) are perfect sets (see [6]), we have the following corollary.
Corollary 1.1. There are perfect $K \subseteq \mathbb{T}$ such that $A(\mathbb{D}) \cap U(K) \neq \emptyset$.

Regarding pointwise universality we have the following.

Proposition 1.5. For each countable $E \subseteq \mathbb{T}$ quasi all $f \in C(\mathbb{T})$ (or $f \in A(\mathbb{D})$) are pointwise universal on $E$. (For $C(\mathbb{T})$ see [7] and for $A(\mathbb{D})$ see [2].)

The proofs of the above propositions are not constructive. They use Baire’s category theorem. Hence a first question which arises is to construct a uniformly or pointwise universal function. A second question is what can we say about the convergence of the Fourier series outside $E$ of a function in $U_p(E)$. If for example $f \in A(\mathbb{D}) \cap U_p(E)$, where $E$ is a countable dense set in $\mathbb{T}$, then the Fourier series of $f$ cannot converge at all points outside $E$, since $E \subseteq G \subseteq D$, where $G = \cap_{N=1}^{+\infty} \cup_{n=1}^{+\infty} \{ t \mid S_n(f, t) > N \} \text{ and } D = \{ t \mid S_n(f, t) \text{ diverges} \}$, and $G$ is $G_δ$ and dense and hence uncountable. Also, taking into account that the pointwise universal functions are highly divergent, we may ask whether it is possible that the above set $D$ has Hausdorff dimension less than 1. We deal with these questions in sections 2 and 3. More precisely, we give a method to construct pointwise universal functions in $A(\mathbb{D})$ on finite and countably infinite sets. We also give a criterion for convergence of the Fourier series outside the set of pointwise universality and we see that the above set $D$ can even be of Hausdorff dimension equal to 0. Of course, the above method can also be applied for functions in $C(\mathbb{T})$.

In section 4 we turn to the study of uniform universality (see definition 1.2). By Carleson’s theorem the perfect sets which accept uniform universality must have Lebesgue measure zero. Of course the first class of such perfect sets that comes to mind consists of the familiar Cantor type sets. We prove that these sets do not accept uniform universality. Moreover, we prove that the same is true for a class of compact countable sets. This is in contrast with the well known fact that all sets of measure zero are sets of divergence for $C(\mathbb{T})$.

Finally, we close this paper with some open problems.

In the following the symbol $C$ will denote an absolute constant which may change from one relation to the next.

2. UNIVERSALITY ON FINITE $K \subseteq \mathbb{T}$ AND CONVERGENCE ON $\mathbb{T} \setminus K$.

Let $N > n$. We consider the Fejer polynomials

$$Q_{N,n}(t) = 2 \sin Nt \sum_{k=1}^{n} \frac{\sin kt}{k} = \sum_{m=-(N+n)}^{N+n} \hat{Q}_{N,n}(m) e^{imt},$$

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where
\[ \hat{Q}_{N,n}(\pm m) = \begin{cases} \frac{1}{2k}, & m = N - k, k = 1, \ldots, n \\ -\frac{1}{2k}, & m = N + k, k = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases} \]

Obviously,
\[ Q_{N,n}(0) = 0. \]

It is well known that the Fejer polynomials are uniformly bounded, i.e.
\[ \|Q_{N,n}\| \leq C. \]

Also,
\[ S_N(Q_{N,n}, 0) = \frac{1}{n} + \cdots + 1 \sim \log n \]

and the sequence of the Fourier coefficients of \( Q_{N,n} \) are of uniform bounded variation, i.e.
\[ \sum_{m=-(N+n)}^{N+n} |\hat{Q}_{N,n}(m-1) - \hat{Q}_{N,n}(m)| \leq C. \]

Finally, we have the estimate
\[ |S_k(Q_{N,n}, t)| \leq \frac{C}{|t|} \text{ for all } k \text{ and } t \neq 0. \]

We now get the following result.

**Proposition 2.1.** For every \( c \in \mathbb{C}, \epsilon > 0 \) and \( N_0 \) there exist \( N > n \geq N_0 \) and a polynomial \( P_{N,n} \) of degree \( N + n \)
\[ P_{N,n}(t) = \sum_{m=-(N+n)}^{N+n} \hat{P}_{N,n}(m)e^{int} \]

such that
\[ P_{N,n}(0) = 0, \]
\[ S_N(P_{N,n}, 0) = c, \]
\[ \|P_{N,n}\| < \epsilon, \]
\[ \sum_{m=-(N+n)}^{N+n} |\hat{P}_{N,n}(m-1) - \hat{P}_{N,n}(m)| \leq \epsilon, \]
\[ |S_k(P_{N,n}, t)| \leq \frac{\epsilon}{|t|} \text{ for all } k \text{ and } t \neq 0. \]
Proof. We consider
\[ P_{N,n} = \frac{c}{S_N(Q_{N,n}, 0)} Q_{N,n}, \]
taking \( N, n \) large enough with \( N > n \). \( \square \)

Let \( \{c_j \mid j \in \mathbb{N} \} \) be a countable dense set in \( \mathbb{C} \) and let \( \epsilon, \epsilon_j > 0 \) with
\[ \sum_{j=1}^{+\infty} \epsilon_j < \epsilon. \]
Then by Proposition 2.1, for each \( j \) we can choose arbitrarily large \( N_j > n_j \) and polynomials \( P_{N_j,n_j} \) such that
\begin{align*}
(1) & \quad P_{N_j,n_j}(0) = 0, \\
(2) & \quad S_N(P_{N_j,n_j}, 0) = c_j, \\
(3) & \quad \|P_{N_j,n_j}\| < \epsilon_j, \\
(4) & \quad \sum_{m=-(N_j+n_j)}^{N_j+n_j+1} |\hat{P}_{N_j,n_j}(m-1) - \hat{P}_{N_j,n_j}(m)| \leq \epsilon_j. \\
(5) & \quad |S_k(P_{N_j,n_j}, t)| \leq \frac{\epsilon_j}{|t|} \quad \text{for all } k \text{ and } t \neq 0.
\end{align*}

We also choose the \( N_j > n_j \) to satisfy the inequalities
\[ 2N_j + (N_j + n_j) < 2N_{j+1} - (N_{j+1} + n_{j+1}) \]
for all \( j \) and we set
\[ P_j(t) = e^{2iN_j t}P_{N_j,n_j}(t) \]
\[ \mathcal{B}_j = \{N_j - n_j, \ldots, 3N_j + n_j\} \supset \text{spectral}(P_j). \]

From (6) we have
\[ \mathcal{B}_j \prec \mathcal{B}_{j+1}, \]
where \( \prec \) means that the block \( \mathcal{B}_j \) lies to the left of \( \mathcal{B}_{j+1} \), that is \( \max \mathcal{B}_j < \min \mathcal{B}_{j+1} \).

Hence (3) implies that the series
\[ \sum_{j=1}^{+\infty} P_j = f \]
converges uniformly to a function \( f \in C(\mathbb{T}) \) such that
\[ \|f\| < \epsilon. \]
Also, from (7) and (8) we get

\[
\hat{f}(n) = \begin{cases} 
\hat{P}_j(n) = \hat{P}_{N_j, n_j}(n - 2N_j), & \text{if } n \in B_j, \; j \in \mathbb{N} \\
0, & \text{otherwise}
\end{cases}
\]  

(9)

In particular \( \hat{f}(n) = 0 \) for \( n < 0 \) and thus \( f \) belongs to \( A(\mathbb{D}) \).

Also, (1) and (2) imply

\[
S_{3N_j}(f, 0) = c_j.
\]

(10)

Consequently, \( f \) is universal on \( K = \{0\} \).

Moreover, (4), (8) and (9) imply

\[
\sum_{n=0}^{+\infty} |\hat{f}(n - 1) - \hat{f}(n)| < \epsilon
\]

i.e. the sequence \( (\hat{f}(n)) \) is of bounded variation. Hence the Fourier series \( \sum_{n=0}^{+\infty} \hat{f}(n)e^{int} \) converges for \( t \neq 0 \) and uniformly in each closed interval of \( \mathbb{T} \) which does not contain 0. Moreover, (5) implies

\[
|S_k(f, t)| \leq \frac{\epsilon}{|t|} \quad \text{for all } k \text{ and } t \neq 0.
\]

From (7) and the uniform convergence of the series \( \sum_{j=1}^{+\infty} P_j \) we get

\[
S_n(f, t) \rightarrow f(t) \quad \text{as } n \rightarrow +\infty \text{ and } n \notin \bigcup_{j=1}^{+\infty} B_j.
\]

Summing up all the above, we have the following.

**Theorem 2.1.** For every \( \epsilon > 0 \) there are blocks \( B_j \) in \( \mathbb{N} \) such that

\( B_1 \prec B_2 \prec \ldots \) and corresponding polynomials \( P_j \) with spectrum \( (P_j) \subseteq B_j \) so that the function \( f = \sum_{j=1}^{+\infty} P_j \) has spectrum \( (f) \subseteq \bigcup_{j=1}^{+\infty} B_j \) and the following properties:

(i) \( f \in A(\mathbb{D}) \cap U(\{0\}) \),

(ii) \( \|f\| \leq \sum_{j=1}^{+\infty} ||P_j|| < \epsilon \),

(iii) \( S_n(f, t) \rightarrow f(t) \) for every \( t \neq 0 \) and uniformly on every closed interval in \( \mathbb{T} \) which does not contain 0.

(iv) \( |S_k(f, t)| \leq \frac{\epsilon}{|t|} \) for all \( k \) and \( t \neq 0 \).

(v) \( S_n(f, t) \rightarrow f(t) \) for every \( t \) as \( n \rightarrow +\infty \) and \( n \notin \bigcup_{j=1}^{+\infty} B_j \).

**Corollary 2.1.** If \( f \) is the function of Theorem 2.1 and we define \( f_0(t) = f(t - t_0) \), then spectrum \( (f_0) \subseteq \bigcup_{j=1}^{+\infty} B_j \) and

(i) \( f_0 \in A(\mathbb{D}) \cap U(\{t_0\}) \)

(ii) \( \|f_0\| < \epsilon \),

(iii) \( S_n(f_0, t) \rightarrow f_0(t) \) for every \( t \neq t_0 \) and uniformly on every closed
interval in $\mathbb{T}$ which does not contain $t_0$.

(iv) $|S_k(f_0, t)| \leq \frac{\epsilon}{|t - t_0|}$ for all $k$ and $t \neq t_0$.

(v) $S_n(f_0, t) \to f_0(t)$ for every $t$ as $n \to +\infty$ and $n \notin \bigcup_{j=1}^{+\infty} B_j$.

The previous constructions can be extended for any finite number of points $t_1, \ldots, t_n \in \mathbb{T}$. For simplicity we present the construction for two points.

Let $t_1 \neq t_2$ and $\{(a_j, b_j) \mid j \in \mathbb{N}\}$ be a countable dense set in $\mathbb{C}^2$. We consider the functions $f_1(t) = f(t-t_1)$ and $f_2(t) = f(t-t_2)$ of Corollary 2.1 with $(a_j), (b_j)$ in place of $(c_j)$ in relation (10), which now becomes

$$S_{3N_j}(f_1, t_1) = a_j, \quad S_{3N_j}(f_2, t_2) = b_j.$$ Then

$$f_1 \in A(\mathbb{D}) \cap U(\{t_1\}), \quad f_2 \in A(\mathbb{D}) \cap U(\{t_2\}).$$

Also $S_n(f_1, t) \to f_1(t)$ and $S_n(f_2, t) \to f_2(t)$ for $t \neq t_1, t_2$ and uniformly on every closed interval in $\mathbb{T}$ which does not contain $t_1, t_2$ and $S_n(f_1, t) \to f_1(t)$ and $S_n(f_2, t) \to f_2(t)$ for every $t$ as $n \to +\infty$ and $n \notin \bigcup_{j=1}^{+\infty} B_j$.

Moreover,

$$|S_k(f_1, t)| \leq \frac{\epsilon}{|t - t_1|}, \quad |S_k(f_2, t)| \leq \frac{\epsilon}{|t - t_2|} \quad \text{for all } k \text{ and } t \neq t_1, t_2.$$ Taking into account that the set $\{(a_j + f_2(t_1), b_j + f_1(t_2))\}$ is dense in $\mathbb{C}^2$ we get the following for the function $f = f_1 + f_2$ or, more generally, for the function $f = f_1 + \cdots + f_m$ when $K = \{t_1, \ldots, t_m\}$.

**Theorem 2.2.** Let $K = \{t_1, \ldots, t_m\}$ be a finite set in $\mathbb{T}$. For every $\epsilon > 0$ there are blocks $B_j$ in $\mathbb{N}$ such that $B_1 \prec B_2 \prec \ldots$ and corresponding polynomials $P_j$ with spectrum($P_j$) $\subseteq B_j$ so that the function $f = \sum_{j=1}^{+\infty} P_j$ has spectrum($f$) $\subseteq \bigcup_{j=1}^{+\infty} B_j$ and the following properties:

(i) $f \in A(\mathbb{D}) \cap U(K)$,

(ii) $\|f\| \leq \sum_{j=1}^{+\infty} \|P_j\| < \epsilon$,

(iii) $S_n(f, t) \to f(t)$ for every $t \notin K$ and uniformly on every closed interval in $\mathbb{T}$ which does not intersect $K$.

(iv) $|S_k(f, t)| \leq \epsilon \sum_{t=1}^{m} \frac{1}{|t - t_0|}$ for all $k$ and $t \neq t_1, \ldots, t_m$.

(v) $S_n(f, t) \to f(t)$ for every $t$ as $n \to +\infty$ and $n \notin \bigcup_{j=1}^{+\infty} B_j$.

We note that the blocks $B_j$ can be taken arbitrarily far to the right in $\mathbb{N}$.
3. Pointwise Universality on Countably Infinite \( E \subseteq \mathbb{T} \) and Convergence on \( \mathbb{T} \setminus E \).

Let \( E = \{ t_l | l \in \mathbb{N} \} \subseteq \mathbb{T} \) be a countably infinite set. We begin with the construction of a function \( f \in A(\mathbb{D}) \cap U_p(E) \).

Let \( E_m = \{ t_1, \ldots, t_m \} \). By theorem 2.2 we have that for each \( m \) and for each \( \epsilon_m > 0 \) there are blocks \( B_{m,j} \in \mathbb{N} \) such that \( B_{m,1} \prec B_{m,2} \prec \ldots \) and corresponding polynomials \( P_{m,j} \) with \( \text{spec}(P_{m,j}) \subseteq B_{m,j} \) so that the function

\[
  f_m = \sum_{j=1}^{+\infty} P_{m,j}
\]

belongs to \( A(\mathbb{D}) \cap U(E_m) \) and satisfies

\[
  \| f_m \| \leq \sum_{j=1}^{+\infty} \| P_{m,j} \| < \epsilon_m
\]

and

\[
  |S_k(f_m, t)| \leq \epsilon_m \sum_{l=1}^{m} \frac{1}{|t - t_l|} \quad \text{for all } k \text{ and } t \neq t_1, \ldots, t_m.
\]

Since \( B_{m,j} \) can be chosen to be arbitrarily far to the right, we may take them in the following diagonal order:

\[
  B_{1,1} \prec B_{2,1} \prec B_{1,2} \prec B_{3,1} \prec B_{2,2} \prec B_{1,3} \prec \ldots.
\]

We may also assume that

\[
  \sum_{m=1}^{+\infty} \epsilon_m < +\infty.
\]

We set

\[
  f = \sum_{m=1}^{+\infty} f_m = \sum_{m=1}^{+\infty} \sum_{j=1}^{+\infty} P_{m,j}.
\]

By (11), (13) it follows that \( f \in A(\mathbb{D}) \). We now prove that \( f \) is pointwise universal on \( E \).

**Theorem 3.1.** The function \( f \) constructed above is in \( A(\mathbb{D}) \cap U_p(E) \).

**Proof.** Let \( h : E \to \mathbb{C} \) be an arbitrary function.

We first choose \( m_1 \) such that

\[
  \sum_{m=m_1+1}^{+\infty} \epsilon_m < \frac{\delta_1}{3},
\]

where \( \delta_1 = 1 \).
From the fact that the blocks $B_{m,j}$ are mutually disjoint and from theorem 2.2, we get

$$S_n(f_m, t_l) \to f_m(t_l) \quad \text{as } n \to +\infty, n \in \bigcup_{j=1}^{+\infty} B_{m_j},$$

$$1 \leq l \leq m_1, 1 \leq m \leq m_1 - 1.$$  \hspace{1cm} (14)

From (11) it follows that

$$\sum_{m=m_1+1}^{+\infty} |S_n(f_m, t_l)| \leq \sum_{m=m_1+1}^{+\infty} \sum_{j=1}^{+\infty} |P_{m,j}(t_l)| < \frac{\delta_1}{3},$$

$$1 \leq l \leq m_1, n \in \bigcup_{j=1}^{+\infty} B_{m_j}. \hspace{1cm} (15)$$

Also, from the universality of $f_{m_1}$ on $E_{m_1} = \{t_1, \ldots, t_{m_1}\}$ and from (14) we get that there exists $n_1 \in \bigcup_{j=1}^{+\infty} B_{m_j}$ so that

$$|S_{n_1}(f_{m_1}, t_l) - (h(t_l) - \sum_{m=1}^{m_1-1} f_m(t_l))| < \frac{\delta_1}{3}, \quad 1 \leq l \leq m_1 \hspace{1cm} (16)$$

$$|S_{n_1}(f_m, t_l) - f_m(t_l)| < \frac{\delta_1}{3(m_1 - 1)},$$

$$1 \leq l \leq m_1, 1 \leq m \leq m_1 - 1. \hspace{1cm} (17)$$

Now we observe that the $n_1$-th Fourier sum of $f$ is a finite sum of $n_1$-th Fourier sums of the functions $f_1, f_2, \ldots, f_{m_1}$ for some $m_1' \geq m_1$. Hence

$$S_{n_1}(f, t_l) - h(t_l)$$

$$= (S_{n_1}(f_1, t_l) - f_1(t_l)) + \cdots + (S_{n_1}(f_{m_1-1}, t_l) - f_{m_1-1}(t_l))$$

$$+ \left(S_{n_1}(f_m, t_l) - (h(t_l) - \sum_{m=1}^{m_1-1} f_m(t_l))\right)$$

$$+ S_{n_1}(f_{m_1+1}, t_l) + \cdots + S_{n_1}(f_{m_1'}, t_l).$$

Finally, from (15), (16), (17) we get

$$|S_{n_1}(f, t_l) - h(t_l)| < \delta_1, \quad 1 \leq l \leq m_1.$$

Similarly, for $\delta_2 = \frac{1}{2}$ there exists $m_2 > m_1$ such that $\sum_{m=m_2+1}^{+\infty} \epsilon_m < \frac{\delta_2}{3}$ and there exists $n_2 > n_1, n_2 \in \bigcup_{j=1}^{+\infty} B_{m_2,j}$ such that

$$|S_{n_2}(f, t_l) - h(t_l)| < \delta_2, \quad 1 \leq l \leq m_2.$$
Continuing in this manner, we construct strictly increasing sequences of positive integers \((m_N), (n_N)\) such that
\[
|S_{n_N}(f, t_l) - h(t_l)| < \delta_N = \frac{1}{N}, \quad 1 \leq l \leq m_N.
\]
This implies that
\[
S_{n_N}(f, t) \to h(t), \quad t \in E
\]
and the proof of pointwise universality is complete. \(\square\)

Now we turn to the study of the convergence of \(S_n(f, t)\) when \(t \notin E\).

**Theorem 3.2.** (a) Let \(f\) be the function of theorem 3.1. For each \(t \in T \setminus E\) satisfying the condition
\[
\sum_{m=1}^{+\infty} \epsilon_m \sum_{l=1}^{m} \frac{1}{|t - t_l|} < +\infty
\]
we have
\[
S_n(f, t) \to f(t).
\]
In particular, if \(\sum_{m=1}^{+\infty} m \epsilon_m < +\infty, d(t, E) > 0\), then \(S_n(f, t) \to f(t)\).

(b) Let \((\delta_m)\) be a decreasing sequence of positive numbers such that \(\sum_{m=1}^{+\infty} \delta_m^a < +\infty\) for all \(a > 0\). Now, if \(\sum_{m=1}^{+\infty} \frac{m}{\delta_m} \epsilon_m < +\infty\), then we have \(S_n(f, t) \to f(t)\) outside a set of Hausdorff dimension zero.

**Proof.** (a) By our diagonal ordering of the blocks \(B_{m,j}\) and the definition of \(f\) we have
\[
S_n(f, t) = P_{1,1}(t) + P_{2,1}(t) + P_{1,2}(t) + \cdots + P_{k,1}(t) + P_{k-1,2}(t) + \cdots + P_{k-j,1,j}(t) + S_n(P_{k-j,j+1}, t)
\]
if \(n \in B_{k-j,j+1}\). Hence, if \(k \leq p\) and \(m \in B_{p-q,q+1}, m \geq n\) and since \(B_{k-j,j+1} \leq B_{p-q,q+1}\), we have that
\[
|S_n(f, t) - S_m(f, t)| \leq |P_{k-j,j+1}(t) - S_n(P_{k-j,j+1}, t)|
\]
\[
+ \left( |P_{k-j-1,j+2}(t)| + \cdots + |P_{p-q+1,q}(t)| \right)
\]
\[
+ |S_m(P_{p-q,q+1}, t)|.
\]
From (11) and (13) it follows that the sum inside the parentheses can be made arbitrarily small by taking \(k\) large enough.

Also, (11) and (12) imply
\[
|S_m(P_{u,v}, t)| \leq |S_m(f_u, t)| + \sum_{j=1}^{v-1} |P_{u,j}(t)| \leq \epsilon_u \sum_{l=1}^{a} \frac{1}{|t - t_l|} + \epsilon_u
\]
and, similarly, \( |P_{u,v}(t) - S_n(P_{u,v}, t)| \leq \epsilon_u \sum_{l=1}^n \frac{1}{|t-t_l|} + 2\epsilon_u \).

(b) We consider

\[ I_m = (t_m - \delta_m, t_m + \delta_m), \quad D = \bigcap_{m=1}^{+\infty} \bigcup_{l=m}^{+\infty} I_l. \]

Since \( \sum_{m=1}^{+\infty} \delta_m < +\infty \) for all \( a > 0 \), the set \( D \) is of Hausdorff dimension zero.

Now, if \( t \) is not in the countable set \( E \) neither in \( D \), then it satisfies (18). Indeed, let \( t \notin E \) and \( t \notin \bigcup_{l=m_0}^{+\infty} I_l \) for some \( m_0 \). Then

\[
\sum_{m=1}^{+\infty} \epsilon_m \sum_{l=1}^{m} \frac{1}{|t-t_l|} = \sum_{m=1}^{m_0-1} \epsilon_m \sum_{l=1}^{m} \frac{1}{|t-t_l|} + \sum_{m=m_0}^{+\infty} \epsilon_m \sum_{l=1}^{m_0-1} \frac{1}{|t-t_l|} + \sum_{m=m_0}^{+\infty} \epsilon_m \sum_{l=m_0}^{m} \frac{1}{|t-t_l|}.
\]

The first term of the right side is finite and the second clearly converges.

As for the third term we have

\[
\sum_{m=m_0}^{+\infty} \epsilon_m \sum_{l=m_0}^{m} \frac{1}{|t-t_l|} \leq \sum_{m=m_0}^{+\infty} \epsilon_m \sum_{l=m_0}^{m} \frac{1}{\delta_l} \leq \sum_{m=m_0}^{+\infty} \frac{m}{\delta_m} \epsilon_m < +\infty.
\]

Note that, if \( E \) is dense in \( \mathbb{T} \), the set of divergence of \( S_n(f; t) \) is necessarily \( G_\delta \) and dense and hence uncountable, although by the proper choice of \( f \) it can be of Hausdorff dimension zero.

4. SUBSETS OF \( \mathbb{T} \) NOT ACCEPTING UNIFORM UNIVERSALITY.

Let \( K \) be a compact subset of \( \mathbb{T} \), \( K \neq \mathbb{T} \). In [5] it was shown that \( H(\mathbb{D}) \cap U(K) \) is a dense-\( G_\delta \) subset of \( H(\mathbb{D}) \) when the latter has the topology of uniform convergence on compacta.

If \( f(z) = \sum_{n=0}^{+\infty} a_n z^n \in H(\mathbb{D}) \cap U(K) \) and \( t_0 \in K \), then (C-1) summability of \( S_n(f; t_0) \) depends on the structure of \( K \) around \( t_0 \). More precisely, suppose for simplicity that \( t_0 = 0 \) and that \( K \) has the following property:

for all infinite \( M \subseteq \mathbb{N} \), there exist arbitrarily small \( a_N, b_N > 0 \)

so that \( \frac{1}{m}(a_N, b_N) \cap K \neq \emptyset \) for infinitely many \( m \in M \).

(19)

Then \( S_n(f; 0) \) is not (C-1) summable. For the proof of this see [5].

On the other hand we have the following.

**Proposition 4.1.** The one third Cantor set \( C \) has the property (19).
Proof. We set \( a_N = \frac{1}{3^N}, b_N = \frac{1}{3^N} \) and \( B_n = \{3^{n-1} + l \mid 1 \leq l \leq 2 \cdot 3^{n-1}\} \).

When \( m = 3^{n-1} + l \in B_n \) we have

\[
\frac{1}{3^N m} < \frac{1}{3^N 3^{n-1}} < \frac{1}{3^{N-1} m}.
\]

Hence,

\[
\frac{1}{3^N 3^{n-1}} \in \frac{1}{m_1}(a_N, b_N) \cap C, \quad m \in B_n, n \in \mathbb{N}.
\]

Hence \( \frac{1}{m}(a_N, b_N) \cap C \neq \emptyset \) for all \( m \in B_n \) and all \( n \in \mathbb{N} \). Since \( \bigcup_{n=1}^{+\infty} B_n = \mathbb{N} \), we get that condition (19) is satisfied by \( C \). \( \square \)

Now, since all \( f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A(\mathbb{D}) \) are (C-1) summable with (C-1) sum \( f(t) \) for all \( t \in \mathbb{T} \), we get the following.

**Corollary 4.1.** If \( C \) is the one third Cantor set, then \( A(\mathbb{D}) \cap U(C) = \emptyset \).

Also, it is not hard to construct countably infinite sets with the property (19). Hence,

**Corollary 4.2.** There are \( E \subseteq \mathbb{T} \) which are countably infinite so that \( A(\mathbb{D}) \cap U(E) = \emptyset \).

We close with the following open problems.

(I) **Construct a perfect set \( K \) and a function \( f \in A(\mathbb{D}) \cap U(K) \).**

(II) **Study pointwise universality on uncountable sets \( E \) with Lebesgue measure zero for functions in \( C(\mathbb{T}) \) or \( A(\mathbb{D}) \).**

Of course the functions \( g \) in Definition 1.1 of pointwise universality must belong to the first class of Baire.

(III) **Is it true that, if \( E \) has positive Hausdorff dimension, then \( E \) does not accept uniform universality?**

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