Stochastic Maximum Principle for Mean-field Controls and Non-Zero Sum Mean-field Game Problems for Forward-Backward Systems

Ruimin Xu\textsuperscript{1,2,*}, Liangquan Zhang\textsuperscript{1,3}

1. School of Mathematics, Shandong University
Jinan 250100, People’s Republic of China.
2. School of Mathematics, Shandong Polytechnic University,
Jinan, 250353, People’s Republic of China.
E-mail: ruiminx@126.com.
3. Laboratoire de Mathématiques,
Université de Bretagne Occidentale, 29285 Brest Cédex, France.
E-mail: xiaoquan51011@163.com

27 June 2012

Abstract

The objective of the present paper is to investigate the solution of fully coupled mean-field forward-backward stochastic differential equations (FBSDEs in short) and to study the stochastic control problems of mean-field type as well as the mean-field stochastic game problems both in which state processes are described as FBSDEs. By combining classical FBSDEs methods introduced by Hu and Peng [Y. Hu, S. Peng, Solution of forward-backward stochastic differential equations, Probab. Theory Relat. Fields 103 (1995)] with specific arguments for fully coupled mean-field FBSDEs, we prove the existence and uniqueness of the solution to this kind of fully coupled mean-field FBSDEs under a certain “monotonicity” condition. Next, we are interested in optimal control problems for (fully coupled respectively) FBSDEs of mean-field type with a convex control domain. Note that the control problems are time inconsistent in the sense that the Bellman optimality principle does not hold. The stochastic maximum principle (SMP) in integral form for mean-field controls, which is different from the classical one, is derived, specifying the necessary conditions for optimality. Sufficient conditions for the optimality of a control is also obtained under additional assumptions. Then we are concerned the maximum principle for a new class of non-zero sum

\*Corresponding author.
stochastic differential games. This game system differs from the existing literature in the sense that the game systems here are characterized by (fully coupled respectively) FBSDEs in the mean-field framework. Our paper deduces necessary conditions as well as sufficient conditions in the form of maximum principle for open equilibrium point of this class of games respectively.

Key words: Mean-field; forward-backward stochastic differential equation (FBSDEs); forward-backward stochastic control systems; stochastic maximum principle; non-zero sum stochastic differential game.

1 Introduction

In this paper, we consider the fully coupled forward-backward stochastic differential equations (FBSDEs) of mean-field type

\[ X_t = x + \int_0^t \mathbb{E}'[b(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds + \int_0^t \mathbb{E}'[\sigma(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]dW_s, \]
\[ Y_t = \Phi(X_T) + \int_t^T \mathbb{E}'[f(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds - \int_t^T Z_s dW_s, \]

where \(b, f : \Omega \times \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}\) and \(\sigma : \Omega \times \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) satisfy the “monotone” condition introduced firstly by Hu and Peng \cite{1} and \(W\) is a \(d\)-dimensional Brownian motion. Here, the coefficients \(\mathbb{E}'[\phi(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]\) \((\phi = b, \sigma, f)\), which are different from the classical coefficients of fully coupled FBSDEs, can be interpreted as

\[ \mathbb{E}'[\phi(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)] = \int_\Omega \phi(\omega', \omega, s, X_s(\omega'), Y_s(\omega'), Z_s(\omega'), X_s(\omega), Y_s(\omega), Z_s(\omega))P(d\omega'). \]

(Fully coupled) FBSDEs are encountered in the probabilistic interpretation (Feynman-Kac formula) of a large kind of second order quasi-linear PDEs, mathematical economics, mathematical finance and especially in the stochastic control problems (cf. \cite{2}–\cite{3}). There have been many results on the solvability of fully-coupled FBSDEs. Antonelli \cite{6} first studied these equations, and he proved the existence and uniqueness with the help of the fixed point theorem when the time duration \(T\) is sufficiently small. Among others, to our knowledge, there exist three main methods to investigate the solvability of an FBSDEs on an arbitrarily prescribed time duration. The first one concerns a kind of “four step scheme” by Ma et al. \cite{7} which can be regarded as a sort of combination of methods of PDEs and probability. The second one is the purely probabilistic method by Hu and Peng \cite{1}, Peng and Wu \cite{4}, Yong \cite{8} and Pardoux and Tang \cite{9}. They required the “monotonicity” condition on the coefficients. The third one is motivated by the study of numerical methods for some linear FBSDEs (see Delarue \cite{10} and Zhang \cite{11}). Delarue \cite{10} relied on PDEs arguments, so its coefficients have
to be deterministic while Zhang [11] imposed some assumptions on the derivatives of the coefficients instead of the monotonicity condition.

Buckdahn, Djehiche, Li, and Peng [12] and Buckdahn and Li et al. [13] investigated a new kind of BSDEs-Mean-field BSDEs, inspired by Lasry and Lions [14]. In the present work, we adapt the methods developed by Hu and Peng [1] in order to establish the existence and uniqueness result for the fully coupled mean-field FBSDEs under the "monotone" condition. The two technical lemmas, aiming to prove the existence result of fully coupled mean-field BSDEs, differ from the classical lemma in [1] because of the mean-field type. When the coefficients \( b, \sigma \) and \( f \) do not depend on \( \omega' \), the fully coupled equation (1) reduces to the standard one. So our result is nontrivially more general of [1].

We also consider stochastic optimal control problems and stochastic differential games (SDGs) in which the state variables are described by a system of mean-field FBSDEs. Mean-field control problems were recently studied by many researchers, such as Andersson, Djehiche [15], Buckdahn, Djehiche and Li [16], Meyer-Brandis, Øsendal, and Zhou [17] and Li [18]. Andersson, Djehiche [15] use the methods in Bensoussan [19] to obtain the necessary conditions of the optimality of a control, i.e. they suppose that the control state space is convex so as to make a convex perturbation of the optimal control and obtain a maximum principle of local condition. Buckdahn, Djehiche and Li [16] get a Peng’s type maximum principle for a general action space where the action space is not convex, using a spike variation of the optimal control. In Meyer-Brandis, Øsendal and Zhou [17], a stochastic maximum principle of mean-field type in a similar setting is studied, but by using Malliavin calculus. Li [18], also using the convex perturbation technology with the convex assumption for control domain, has a different controlled system and state equation of mean-field type from [15].

However, the results above are all on the forward control system. As far as we know, Peng [20] originally studied one kind of forward-backward stochastic control system which has the economic background and could be used to study the recursive optimal control problem in the mathematical finance. He obtained the maximum principle for this kind of control system with the control domain being convex. Later, Shi and Wu [21] applied the spike variational technique to derive the maximum principle for fully coupled forward-backward stochastic control system in the global form and indicated that the control domain is not necessarily convex but the control variable can’t enter into the diffusion term. In order to study the forward-backward stochastic control problem under the mean-field framework, we apply the convex perturbation methods introduced in Bensoussan [19] and analytical technique provided by [13] to establish a necessary condition for optimality of the control in the form of the maximum principle for the (fully coupled respectively) mean-field forward-backward stochastic control system in which the state equation is mean-field FBSDE (fully coupled mean-field FBSDE respectively). The adjoint equation, playing an important role in deriving the SMP, is a (fully coupled respectively) mean-field backward SDE and has a unique adapted solution under the given assumptions with the help of the conclusion in [13] (or the conclusion in Theorem 3.1 respectively). Also, we obtain the corresponding sufficient condition, which can check whether the candidate optimal control is optimal or not. Our
results can be reviewed as an extension of Peng [20] and Li [18].

Inspired by Wang and Yu [22], which gave the maximum principle for non-zero sum differential games of BSDE system, we study the non-zero sum stochastic differential games (SDGs in short) of mean-field type. Differential games, originally studied by Isaacs [23], are ones in which the position, being controlled by players, evolves continuously. Fleming and Souganidis [24] were the first to study in a rigorous manner two-player zero sum SDGs. Their work has translated former results on differential games by Isaacs [23], Friedman [25], and, in particular, Evans and Souganidis [26] from the purely deterministic into the stochastic framework and has given an important impulse for the research in the theory of stochastic differential games. Next, the advances in SDGs appear over a large number of fields (cf. [27]-[29]).

We notice that the game literature is mainly restricted to forward (stochastic) systems, i.e., these game systems are described by forward (stochastic) differential equations. Recently, Wang and Yu [22] concerned the theory of backward stochastic differential games and obtained the maximum principle as well as the verification theorem for non-zero sum SDGs of BSDEs in which game systems are described by BSDEs. It is remarkable that this topic about the forward-backward system is quite lacking in literature. To fill the gap, we investigate the theory of forward-backward SDG problems under the mean-field framework. Similar to our stochastic control problems, we study the SDGs with the state equation having two different forms: mean-field FBSDEs and fully coupled mean-field FBSDE. By virtue of an argument of the convex perturbation, we deduce the stochastic maximum principle for the equilibrium point of Problem (FBNZ) (Problem (CFBNZ) respectively), which gives the candidate equilibrium points. By extending classical approaches to the mean-field framework, we prove, under some restrictive assumptions (but comparable with those in the classical case), the sufficiency of the necessary conditions. It is necessary to point that our SDGs conclusion not only extends the result of Wang and Yu [22] but also includes the situation where the state equation of the stochastic game system is classical (i.e. in no mean-field form) FBSDE (fully coupled FBSDE respectively).

Our paper is organized as follows. Section 2 recalls some elements of the theory of FBSDEs and mean-field BSDEs which are needed in what follows. Section 3 investigates the uniqueness and existence of the solution of fully coupled mean-field FBSDEs under the “monotonicity” condition in which two technical lemmas are used to prove the existence result. In Section 4, we study the forward-backward stochastic control system of mean-field type. Specifically, the maximum principle, specifying the necessary condition for optimality, is deduced and we get, under additional assumptions, the corresponding sufficient condition which can check whether the candidate control is optimal or not. Similar results about fully coupled forward-backward stochastic control system of mean-field type are obtained in Section 5. Following the idea introduced in Section 4 and Section 5, we analyze the non-zero sum stochastic differential games of FBSDEs and fully coupled FBSDEs in Section 6 and Section 7, respectively, and derive the necessary condition in the form of the maximum principle as well as the sufficient condition–verification theorem for the equilibrium point.
2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a given complete filtered probability space on which a $d$-dimensional standard Brownian motion $W = (W_t)_{t \geq 0}$ is defined. By $\mathbb{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}$ we denote the natural filtration of $W$ augmented by $P$-null sets of $\mathcal{F}$, i.e.,

$$\mathcal{F}_t = \sigma\{W_s, s \leq t\} \vee N_P, \; t \in [0, T],$$

where $N_P$ is the set of all $P$-null sets and $T > 0$ is a fixed time horizon.

We shall introduce the following two processes which can be used frequently in what follows:

$$\mathcal{S}_P^2(0, T; \mathbb{R}) := \left\{ (\phi_t)_{0 \leq t \leq T} \text{ real-valued } \mathbb{F} - \text{adapted càdlàg process : } \mathbb{E}\left[ \sup_{0 \leq t \leq T} |\phi_t|^2 \right] < +\infty \right\};$$

$$\mathcal{M}_P^2(0, T; \mathbb{R}^n) := \left\{ (\phi_t)_{0 \leq t \leq T} \mathbb{R}^n\text{-valued } \mathbb{F} - \text{adapted process : } \mathbb{E}\left[ \int_0^T |\phi_t|^2 dt \right] < +\infty \right\}.$$

2.1 The classical FBSDEs

We first recall some results on FBSDEs, for its proof the reader is referred to Hu and Peng [1]. The FBSDEs they considered has the form

$$x_t = x_0 + \int_0^t b(s, x_s, y_s, z_s) ds + \int_0^t \sigma(s, x_s, y_s, z_s) dW_s,$$

$$y_t = g(x_T) + \int_0^T f(s, x_s, y_s, z_s) ds - \int_0^T z_s dW_s, \; t \in [0, T].$$

Function $b, f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\sigma : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ with the property that $b(t, x, y, z)_{t \in [0, T]}, \sigma(t, x, y, z)_{t \in [0, T]}$ and $f(t, x, y, z)_{t \in [0, T]}$ are $\mathbb{F}$-progressively measurable for each $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.

Some notations and conditions are needed before giving the existence and uniqueness of the solution of such FBSDEs. Let $\langle , \rangle$ denote the usual inner product in $\mathbb{R}^n$, and for $u = (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, we define

$$F(t, u) := (-f(t, u), b(t, u), \sigma(t, u)).$$

(H1) (i) For each $u = (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, $F(\cdot, u) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)$, and for each $x \in \mathbb{R}, g(x) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$; there exists a constant $c_1 > 0$, such that

$$|F(t, u_1) - F(t, u_2)| \leq c_1|u_1 - u_2|, \; P - a.s., \; a.e. t \in \mathbb{R}^+,$$

$$\forall u_i \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \; (i = 1, 2);$$

$$|g(x_1) - g(x_2)| \leq c_1|x_1 - x_2|, \; P - a.s., \; \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R}.$$

(ii) There exists a constant $c_2 > 0$, such that

$$\langle F(t, u_1) - F(t, u_2), u_1 - u_2 \rangle \leq -c_2|u_1 - u_2|^2, \; \; P - a.s., \; \; a.e. t \in \mathbb{R}^+,$$

$$\forall u_i \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \; (i = 1, 2);$$

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \geq c_2|x_1 - x_2|^2, \; P - a.s., \; \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R}.$$
Lemma 1. Let assumptions (H1) hold, then there exists a unique adapted solution $(x, y, z)$ for the FBSDEs (1)

2.2 Mean-field BSDEs and McKean-Vlasov SDEs

This section is devoted to the recall of some basic results on a new type of BSDEs, the so-called mean-field BSDEs; the reader interested in more details is referred to Buckdahn, Djehiche, Li, and Peng [12] and Buckdahn and Li et al. [13].

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ be the (non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself. We endow this product space with the filtration $\tilde{\mathcal{F}} = \{\mathcal{F}_t = \mathcal{F} \otimes \mathcal{F}_t, 0 \leq t \leq T\}$. Any random variable $\xi \in L^0(\Omega, \mathcal{F}, P)$ originally defined on $\Omega$ is extended canonically to $\tilde{\Omega} : \xi'(\omega', \omega) = \xi(\omega')$, $(\omega', \omega) \in \tilde{\Omega} = \tilde{\Omega} \times \Omega$. For any $\theta \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ the variable $\theta(\cdot, \omega) : \tilde{\Omega} \rightarrow \mathbb{R}$ belongs to $L^1(\tilde{\Omega}, \mathcal{F}, P), P(d\omega) - a.s.$; we denote its expectation by

$$\mathbb{E}'[\theta(\cdot, \omega)] = \int_\tilde{\Omega} \theta(\omega', \omega)P(d\omega').$$

Notice that $\mathbb{E}'[\theta] = \mathbb{E}'(\Omega, \mathcal{F}, P)$, and

$$\tilde{E}[\theta] = \int_\Omega \theta d\tilde{P} = \int_\Omega \mathbb{E}'[\theta(\cdot, \omega)]P(d\omega) = \mathbb{E}[^{\mathcal{E}}[\theta]].$$

The driver of mean-field BSDE is a function $f = f(\omega', \omega, t, \tilde{y}, \tilde{z}, y, z) : \tilde{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ which is $\tilde{\mathcal{F}}$-progressively measurable for all $(\tilde{y}, \tilde{z}, y, z)$, and satisfies the following assumptions.

(H2) (i) There exists a constant $C \geq 0$ such that, $\tilde{P}$-a.s., for all $t \in [0, T], y_1, y_2, \tilde{y}_1, \tilde{y}_2 \in \mathbb{R}$, $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^d$, $|f(t, \tilde{y}_1, \tilde{z}_1, y_1, z_1) - f(t, \tilde{y}_2, \tilde{z}_2, y_2, z_2)| \leq C(|\tilde{y}_1 - \tilde{y}_2| + |\tilde{z}_1 - \tilde{z}_2| + |y_1 - y_2| + |z_1 - z_2|)$.

(ii) $f(\cdot, 0, 0, 0, 0) \in \mathcal{H}_\tilde{P}(0, T; \mathbb{R})$.

The main result about mean-field BSDEs of Buckdahn and Li et al. [13] is:

Lemma 2. Under the assumptions (H2), for any random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the mean-field BSDEs

$$Y_t = \xi + \int_t^T \mathbb{E}'[f(s, Y'_s, Z'_s, Y_s, Z_s)]ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

has a unique adapted solution

$$(Y_t, Z_t) \in \mathcal{S}_\tilde{P}^2(0, T; \mathbb{R}) \times \mathcal{M}_\tilde{P}^2(0, T; \mathbb{R}^d).$$
Remark 3. The driving coefficient of (3) has to be interpreted as follows:
\[
\mathbb{E}'[f(s, Y'_s, Z'_s, Y_s, Z_s)](\omega)
\]
\[
= \mathbb{E}'[f(s, Y'_s(\omega'), Z'_s(\omega'), Y_s(\omega), Z_s(\omega))]
\]
\[
= \int_{\Omega} f(\omega', \omega, s, Y_s(\omega'), Z'_s(\omega'), Y_s(\omega), Z_s(\omega)) P(d\omega').
\]

We shall also consider McKean-Vlasov SDEs (see, e.g., Buckdahn and Li et al. [13]). Let
\[ b : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \] and \( \sigma : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) be two measurable functions supposed to satisfy the following conditions:

(H3) (i) \( b(\cdot, \bar{x}, x) \) and \( \sigma(\cdot, \bar{x}, x) \) are \( \bar{\mathbb{F}} \)-progressively measurable continuous processes for all \( \bar{x}, x \in \mathbb{R} \), and there exists some constant \( C > 0 \) such that
\[
|b(t, \bar{x}, x)| + |\sigma(t, \bar{x}, x)| \leq C(1 + |\bar{x}| + |x|), \text{ a.s.,}
\]
for all \( 0 \leq t \leq T \);

(ii) \( b \) and \( \sigma \) are Lipschitz in \( \bar{x}, x \), i.e., there is some constant \( C > 0 \) such that
\[
|b(t, \bar{x}_1, x_1) - b(t, \bar{x}_2, x_2)| + |\sigma(t, \bar{x}_1, x_1) - \sigma(t, \bar{x}_2, x_2)| \leq C( |\bar{x}_1 - \bar{x}_2| + |x_1 - x_2|), \text{ a.s.}
\]
for all \( 0 \leq t \leq T, \bar{x}_1, \bar{x}_2, x_1, x_2 \in \mathbb{R} \).

The McKean-Vlasov SDEs parameterized by the initial condition \( (t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}) \) is given as follows:

\[
\begin{cases}
    dX^{t,\zeta}_s = \mathbb{E}' \left[ b \left( s, (X^{t,\zeta}_s)' , X^{t,\zeta}_s \right) \right] ds + \mathbb{E}' \left[ \sigma \left( s, (X^{t,\zeta}_s)' , X^{t,\zeta}_s \right) \right] dW_s, \\
    X^{t,\zeta}_t = \zeta, \quad s \in [t, T].
\end{cases}
\]

We recall that, due to our notational convention,
\[
\mathbb{E}' \left[ b \left( s, (X^{t,\zeta}_s)' , X^{t,\zeta}_s \right) \right] (\omega) = \int_{\Omega} b (\omega', \omega, s, X^{t,\zeta}_s (\omega'), X^{t,\zeta}_s (\omega)) P(d\omega'), \quad \omega \in \Omega.
\]

Lemma 4. Under Assumption (H3), SDEs (i) has a unique strong solution.

Remark 5. From standard arguments we also get that, for any \( p \geq 2 \), there exists \( C_p \in \mathbb{R} \), which only depends on the Lipschitz and the growth constants of \( b \) and \( \sigma \), such that for all \( t \in [0, T] \) and \( \zeta, \zeta' \in (\Omega, \mathcal{F}_t, P; \mathbb{R}) \),
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{t,\zeta}_s - X^{t,\zeta'}_s|^p | \mathcal{F}_t \right] \leq C_p |\zeta - \zeta'|^p, \quad \text{a.s.,}
\]
\[
\mathbb{E} \left[ \sup_{t \leq s \leq T} |X^{t,\zeta}_s|^p | \mathcal{F}_t \right] \leq C_p (1 + |\zeta|^p), \quad \text{a.s.,}
\]
\[
\mathbb{E} \left[ \sup_{t \leq s \leq t + \delta} |X^{t,\zeta}_s - \zeta|^p | \mathcal{F}_t \right] \leq C_p (1 + |\zeta|^p)^{\frac{p}{2}}, \quad P\text{-a.s., for all } \delta > 0 \text{ with } t + \delta \leq T.
\]

These, in the classical case, well-known standard estimates can be consulted, for instance, in Ikeda and Watanabe [30] (pp. 166-168) and also in Karatzas and Shreve [31] (pp. 289-290).
3 Fully coupled Mean-field FBSDEs

In this section, we shall investigate a new type of FBSDEs called fully coupled mean-field FBSDEs as follows:

$$X_t = X_0 + \int_0^t \mathbb{E}'[b(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds + \int_0^t \mathbb{E}'[\sigma(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]dW_s,$$

$$Y_t = \Phi(X_T) + \int_t^T \mathbb{E}'[f(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)]ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

Here the processes $X, Y, Z$ take values in $\mathbb{R}, \mathbb{R}, \mathbb{R}^d$ respectively; and $b, \sigma, \Phi$ and $f$ take values in $\mathbb{R}, \mathbb{R}^d, \mathbb{R}$ and $\mathbb{R}$ respectively.

**Remark 6.** The driving coefficient here has to the same interpretation as Lemma 2:

$$\mathbb{E}'[\psi(s, X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)](\omega) = \mathbb{E}'[\psi(s, X_s(\omega'), Y_s(\omega'), Z_s(\omega'), X_s(\omega), Y_s(\omega), Z_s(\omega))],$$

$$= \int_{\Omega} \psi(\omega', \omega, s, X_s(\omega'), Y_s(\omega'), Z_s(\omega'), X_s(\omega), Y_s(\omega), Z_s(\omega)) P(\omega').$$

for $\psi = b, \sigma, f$.

For convenience, we will use the following notations in this section: Let $\langle, \rangle$ denote the usual inner product in $\mathbb{R}^n$ and we use the usual Euclidean norm in $\mathbb{R}^n$. For $\Theta = (\bar{x}, \bar{y}, \bar{z}, x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$,

$$F(t, \Theta) = (-f(t, \Theta), b(t, \Theta), \sigma(t, \Theta)).$$

Now we give the standard assumptions on the coefficients of mean-field FBSDE:

**H4** For each $\Theta \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, $F(\cdot, \Theta) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)$, and for each $x \in \mathbb{R}$, $g(x) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$; there exists a constant $C > 0$, such that:

$$|F(t, \Theta_1) - F(t, \Theta_2)| \leq C|\Theta_1 - \Theta_2|, \quad P - a.s., \ a.e. t \in \mathbb{R}^+,$$

$$\Theta_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i, x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, \quad (i = 1, 2),$$

and

$$|\Phi(x_1) - \Phi(x_2)| \leq C|x_1 - x_2|, \quad P - a.s., \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R}.$$

The following monotone conditions are our main assumptions:

**H5** For $\Theta_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i, x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, let $u_i = (x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, then $\Theta_i = (\hat{u}_i, u_i)$ $(i = 1, 2)$. We assume that

$$\mathbb{E} < F(t, \Theta_1) - F(t, \Theta_2), u_1 - u_2 > \leq -C_1 \mathbb{E}(|u_1 - u_2|^2), \quad P - a.s., a.e. t \in \mathbb{R}^+,$$

$$< \Phi(x_1) - \Phi(x_2), x_1 - x_2 > \geq \mu_1|x_1 - x_2|^2, \quad P - a.s., \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R},$$

where $C_1$ and $\mu_1$ are given positive constants.
For the mean-field FBSDE \(3\), we have the the following main result of this section.

**Theorem 7.** Under the assumptions \((H4)\) and \((H5)\), there exists a unique adapted solution \((X,Y,Z)\) for mean-field FBSDEs \(3\).

The proof of this theorem is similar to that of Theorem 3.1 in [1] except the mean-field term. However, to be self-contained, we intend to give the proof. Before giving the proof of this theorem, we need the two technical lemmas below whose proof will be given in the sequel.

**Lemma 8.** Suppose that \((\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}), \xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})\), then the following linear mean-field forward-backward stochastic differential equations

\[
X_t = X_0 + \int_0^t \left(-E[Y'_s] - Y_s + \gamma(s)\right)ds + \int_0^t \left(-E[Z'_s] - Z_s + \phi(s)\right)dW_s, \tag{4}
\]

\[
Y_t = \xi + X_T + \int_t^T \left[E[X'_s] + X_s - \varphi(s)\right]ds - \int_t^T Z_s dW_s, \tag{5}
\]

have a unique adapted solution: \((X, Y, Z) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)\).

Now, we define, for any given \(\alpha \in \mathbb{R},\)

\[
\begin{align*}
  b^\alpha(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) &= \alpha b(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) + (1 - \alpha)(-\bar{y} - y), \\
  \sigma^\alpha(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) &= \alpha \sigma(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) + (1 - \alpha)(-\bar{z} - z), \\
  f^\alpha(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) &= \alpha f(t, \bar{x}, \bar{y}, \bar{z}, x, y, z) + (\alpha - 1)(-\bar{x} - x), \\
  \Phi^\alpha(x) &= \alpha \Phi(x) + (1 - \alpha)(x).
\end{align*}
\]

and consider the following equations:

\[
X_t = X_0 + \int_0^t \left[\tilde{b}^\alpha(s, \Lambda_s) + \gamma(s)\right]ds + \int_0^t \left[\tilde{\sigma}^\alpha(s, \Lambda_s) + \phi(s)\right]dW_s, \tag{6}
\]

\[
Y_t = (\Phi^\alpha(X_T) + \xi) + \int_t^T \left[\tilde{f}^\alpha(s, \Lambda_s) - \varphi(s)\right]ds - \int_t^T Z_s dW_s, \tag{7}
\]

where we use the notation

\[
\Lambda_s = (X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s),
\]

and

\[
\tilde{\psi}(s, \Lambda_s) = E'[\psi(X'_s, Y'_s, Z'_s, X_s, Y_s, Z_s)],
\]

for \(\psi = b, \sigma, f\). Then we can rewrite

\[
\begin{align*}
  \tilde{b}^\alpha(s, \Lambda_s) &= \alpha \tilde{b}(s, \Lambda_s) + (1 - \alpha)(-E'[Y'_s] - Y_s), \\
  \tilde{\sigma}^\alpha(s, \Lambda_s) &= \alpha \tilde{\sigma}(s, \Lambda_s) + (1 - \alpha)(-E'[Z'_s] - Z_s), \\
  \tilde{f}^\alpha(s, \Lambda_s) &= \alpha \tilde{f}(s, \Lambda_s) + (\alpha - 1)(-E'[X'_s] - X_s).
\end{align*}
\]
Lemma 9. For a given $\alpha_0 \in [0,1)$ and for any $(\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, assume that Eqs (6) and (7) have an adapted solution. Then there exists a $\delta_0 \in (0,1)$ which depends only on $c_1, c_2$ and $T$, such that for all $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, and for any $(\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, Eqs (6) and (7) have an adapted solution.

Proof of Theorem 7. 
Uniqueness. If $U = (X, Y, Z)$ and $\bar{U} = (\bar{X}, \bar{Y}, \bar{Z})$ are two adapted solutions of (3), we set

$$(\hat{X}', \hat{Y}', \hat{Z}', \hat{X}, \hat{Y}, \hat{Z}) = (X' - X', Y' - Y', Z' - Z', X - \bar{X}, Y - \bar{Y}, Z - \bar{Z}),$$
$$\hat{b}(t) = b(t, U', U) - b(t, \bar{U}', \bar{U}),$$
$$\hat{\sigma}(t) = \sigma(t, U', U) - \sigma(t, \bar{U}', \bar{U}),$$
$$\hat{f}(t) = f(t, U', U) - f(t, \bar{U}', \bar{U}).$$

From Assumption (A1), it follows that $\{\hat{X}_t\}$ and $\{\hat{Y}_t\}$ are continuous, and

$$\mathbb{E}(\sup_{t \in [0,T]} |\hat{X}_t|^2 + \sup_{t \in [0,T]} |\hat{Y}_t|^2) < +\infty.$$ 

Applying the Itô’s formula to $\hat{X}_t \hat{Y}_t$ on $[0, T]$, we have

$$\mathbb{E}[(\Phi(X_T) - \Phi(X_T))\hat{X}_T]$$
$$= \mathbb{E} \int_0^T \left\{ \mathbb{E}[\hat{b}(t)]\hat{Y}_t - \mathbb{E}[\hat{f}(t)]\hat{X}_t + \mathbb{E}[\hat{\sigma}(t)]\hat{Z}_t \right\} dt.$$ 

$$= \mathbb{E} \int_0^T \left\{ < (-\mathbb{E}[\hat{f}(t)], \mathbb{E}[\hat{b}(t)], \mathbb{E}[\hat{\sigma}(t)]), (\hat{X}_t, \hat{Y}_t, \hat{Z}_t) > \right\} dt.$$ 

By assumptions (A1) and (A2), we get then

$$\mu_2|X_T - \bar{X}_T|^2 \leq \mathbb{E}[(\Phi(X_T) - \Phi(X_T))\hat{X}_T] \leq -C_1 \mathbb{E} \int_0^T |U - \bar{U}|^2 dt.$$ 

So, we get $U = \bar{U}$.

Existence. According to Lemma 8, we see immediately that, when $\alpha = 0$, for any $(\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, Eqs (6) and (7) have an adapted solution. From Lemma 3.2, for any $(\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, we can solve Eqs (6) and (7) successively for the case $\alpha \in [0, \delta_0], [\delta_0, 2\delta_0], \cdots$. When $\alpha = 1$, for any $(\gamma(\cdot), \phi(\cdot), \varphi(\cdot)) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}, \xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, the adapted solution of Eqs. (6) and (7) exists, then we deduce immediately that the adapted solution of Eqs. (3) exists. \(\square\)

Proof of Lemma 8
Proof. We consider the following BSDEs:

\[
\tilde{Y}_t = \xi + \int_t^T [-\mathbb{E}'[\tilde{Y}_s'] - \tilde{Y}_s - \varphi(s) + \gamma(s)]ds - \int_t^T (\mathbb{E}'[\tilde{Z}_s'] + 2\tilde{Z}_s - \phi(s))dW_s.
\]

By Lemma 1, the above equation has a unique adapted solution \((\tilde{Y}, \tilde{Z})\).

Then we solve the following forward equation

\[
X_t = x + \int_0^t \big( -\mathbb{E}'[X_s'] - X_s - \mathbb{E}'[\tilde{Y}_s'] - \tilde{Y}_s + \gamma(s) \big)ds + \int_0^t \big( -\mathbb{E}'[\tilde{Z}_s'] - \tilde{Z}_s + \phi(s) \big)dW_s,
\]

and set \(Y = \tilde{Y} + X, \ Z = \tilde{Z}\), we get

\[
\begin{align*}
X_t &= x + \int_0^t \big( -\mathbb{E}'[X_s'] - X_s + \gamma(s) \big)ds + \int_0^t \big( -\mathbb{E}'[Z_s'] - Z_s + \phi(s) \big)dW_s, \\
Y_t - X_t &= \xi + \int_t^T [\mathbb{E}'[X_s'] - \mathbb{E}'[\tilde{Y}_s'] + X_s - Y_s - \varphi(s) + \gamma(s)]ds \\
&- \int_t^T (\mathbb{E}'[Z_s'] + 2Z_s - \phi(s))dW_s, \\
X_T - X_t &= \int_t^T ( -\mathbb{E}'[\tilde{Y}_s'] - \tilde{Y}_s + \gamma(s) )ds + \int_t^T ( -\mathbb{E}'[\tilde{Z}_s'] - \tilde{Z}_s + \phi(s) )dW_s.
\end{align*}
\]

Then we have

\[
Y_t = \xi + X_T + \int_t^T [\mathbb{E}'[X_s'] + X_s - \varphi(s)]ds - \int_t^T Z_s dW_s.
\]

So \((X, Y, Z)\) is a solution of Eqs. (4) and (5). Thus the existence is proved.

As for uniqueness, it only has to use the method of the proof of uniqueness in Theorem 3.1 and we omit it. \(\square\)

Proof of Lemma 9

Proof. For simplicity, we set

\[
\begin{align*}
U^i &= (X^i, Y^i, Z^i), \\
\Lambda^0 &= ((X^0)', (Y^0)', (Z^0)', X^0, Y^0, Z^0) = 0, \\
\Lambda^i &= ((X^i)', (Y^i)', (Z^i)', X^i, Y^i, Z^i), \\
\hat{\Lambda}^{i+1} &= \Lambda^{i+1} - \Lambda^i = ((\hat{X}^{i+1})', (\hat{Y}^{i+1})', (\hat{Z}^{i+1})', \hat{X}^{i+1}, \hat{Y}^{i+1}, \hat{Z}^{i+1}), \\
\hat{U}^{i+1} &= U^{i+1} - U^i = (\hat{X}^{i+1}, \hat{Y}^{i+1}, \hat{Z}^{i+1}),
\end{align*}
\]

for all \(i \in N^+\).
For any given \( \alpha_0 \in [0, 1] \) and any \( \delta > 0 \), we solve iteratively the following equations:

\[
X^{i+1}_t = a + \int_0^t \left( \bar{b}^{\alpha_0}(s, \Lambda^{i+1}_s) + \delta [Y^i_s + \mathbb{E}'((Y^i_s)')] + \bar{b}(s, \Lambda^i_s) \right) ds + \int_0^t \left( \bar{\sigma}^{\alpha_0}(s, \Lambda^{i+1}_s) + \delta [Z^i_s + \mathbb{E}'((Z^i_s)')] + \bar{\sigma}(s, \Lambda^i_s) \right) \phi(s) dW_s,
\]

\[
Y^{i+1}_t = (\Phi^{\alpha_0}(X^{i+1}_T) + \delta (\Phi(X^i_T) - X^i_T) + \xi) + \int_t^T \left( \bar{f}^{\alpha_0}(s, \Lambda^{i+1}_s) + \delta [\bar{f}(s, \Lambda^i_s) - \mathbb{E}'((X^i_s)')] - \varphi(s) \right) ds - \int_t^T Z^{i+1}_s dW_s.
\]

Applying the Itô formula to \( \hat{X}^{i+1}_t \hat{Y}^{i+1}_t \), on and noticing that \( \mathbb{E}'[Y'] = \mathbb{E}[Y] \), we have

\[
\mathbb{E} \left( (\Phi^{\alpha_0}(X^{i+1}_T) - \Phi^{\alpha_0}(X^i_T)) \hat{X}^{i+1}_T \right) = -\delta \mathbb{E} \left[ (\Phi(X^i_T) - \Phi(X^i_T)) \hat{X}^{i+1}_T \right] + \mathbb{E} \int_0^T \left\{ \hat{Y}^{i+1}_s [\bar{b}^{\alpha_0}(s, \Lambda^{i+1}_s) - \bar{b}(s, \Lambda^i_s)] - \hat{X}^{i+1}_s [\bar{f}^{\alpha_0}(s, \Lambda^{i+1}_s) - \bar{f}(s, \Lambda^i_s)] + \hat{Z}^{i+1}_s [\bar{\sigma}^{\alpha_0}(s, \Lambda^{i+1}_s) - \bar{\sigma}(s, \Lambda^i_s)] \right\} ds
\]

\[
- \hat{X}^{i+1}_s [\bar{f}(s, \Lambda^i_s) - \bar{f}(s, \Lambda^{i-1}_s) - \hat{X}^{i}_s - \mathbb{E}[\hat{X}^{i}_s]] + \hat{Z}^{i+1}_s [\bar{\sigma}(s, \Lambda^i_s) - \bar{\sigma}(s, \Lambda^{i-1}_s)] \right\} ds
\]

\[
= -\delta \mathbb{E} \left[ (\Phi(X^i_T) - \Phi(X^{i-1}_T)) \hat{X}^{i+1}_T \right] + \mathbb{E} \int_0^T \left\{ \bar{F}^{\alpha_0}(s, \Lambda^{i+1}_s) - \bar{F}^{\alpha_0}(s, \Lambda^i_s), \hat{U}^{i+1}_s \right\] ds
\]

\[
+ \delta \mathbb{E} \int_0^T \left\{ \bar{U}^{i}_s + \mathbb{E}[\hat{U}^{i}_s] + \bar{F}(s, \Lambda^i_s) - \bar{F}(s, \Lambda^{i-1}_s), \hat{U}^{i+1}_s \right\} ds.
\]

Using the definition of \( \bar{F}(t, \Lambda) \), we have the following inequality:

\[
\bar{F}(s, \Lambda^i_s) - \bar{F}(s, \Lambda^{i-1}_s) = \mathbb{E}' \left[ F(s, \Lambda^i_s) - F(s, \Lambda^{i-1}_s) \right] \leq C \mathbb{E}' \lambda^i_s - \hat{\lambda}^{i-1}_s \leq C(\mathbb{E}[\|U^i\|] + |\hat{U}^i|).
\]

From assumptions (A1), (A2), Eq (10) and the notation

\[
\bar{F}^{\alpha_0}(s, \Lambda^{i+1}_s) = \alpha_0 \bar{F}(s, \Lambda^{i+1}_s) - (1 - \alpha_0) \left( \mathbb{E}[\hat{U}^{i+1}_s] + \hat{U}^{i+1}_s \right).
\]
for any $\alpha_0 \in [0, 1]$, we deduce easily that
\[
(\mu_1 \alpha_0 + 1 - \alpha_0)E[\hat{X}_T^{i+1}]^2 + (C_1 \alpha_0 + 2 - 2\alpha_0)E \int_0^T |\hat{U}_t^{i+1}|^2 dt \\
\leq \delta E[\hat{X}_T^{i+1}]^2 - \delta E[(\Phi(X_t^{i}) - \Phi(X_t^{i-1}))\hat{X}_T^{i+1}] + \delta E \int_0^T <\hat{U}_t^i, \hat{U}_t^{i+1}> dt \\
+ \delta E \int_0^T <E[\hat{U}_t^i], \hat{U}_t^{i+1}> dt + \delta E \int_0^T <\bar{F}(t, \Lambda_t^i) - \bar{F}(t, \Lambda_t^{i-1}), \hat{U}_t^{i+1}> dt \\
\leq 2\delta(1 + C) \left( E[\hat{X}_T^{i}]^2 + \int_0^T |\hat{U}_t^i||\hat{U}_t^{i+1}| dt \right).
\]

In virtue of
\[
\min (\mu_1 \alpha_0 + 1 - \alpha_0, C_1 \alpha_0 + 2 - 2\alpha_0) \geq \mu_2 = \min (1, \mu_1, C_1) > 0,
\]
the above inequality yields
\[
E[\hat{X}_T^{i+1}]^2 + E \int_0^T |\hat{U}_t^{i+1}|^2 dt \leq \frac{2\delta(1 + C)}{\mu_2} E \left( |\hat{X}_T^{i}]^2 + \int_0^T |\hat{U}_t^i||\hat{U}_t^{i+1}| dt \right).
\]

For $\varepsilon = \frac{\mu_2}{4\delta(1+C)} > 0$, with the help of $ab \leq \frac{a^2}{4\varepsilon} + \varepsilon b^2$, we can derive
\[
E[\hat{X}_T^{i+1}]^2 + E \int_0^T |\hat{U}_t^{i+1}|^2 dt \leq \left( \frac{2\delta(1 + C)}{\mu_2} \right)^2 E \left( |\hat{X}_T^{i}]^2 + \int_0^T |\hat{U}_t^i|^2 dt \right).
\]

Note that there exists a constant $C' > 0$ which depends only on $C$ and $T$, such that
\[
E[\hat{X}_T^{i}]^2 \leq C' E \int_0^T \left( |\hat{U}_s^{i}]^2 + |\hat{U}_s^{i-1}]^2 \right) ds, \quad \forall i \geq 1.
\]

Indeed, for $i \geq 1$,
\[
E[\hat{X}_T^{i}]^2 = 2E \int_0^T |\hat{X}_s^{i}| \left\{ \hat{b}(s, \Lambda_s^i) - \hat{b}(s, \Lambda_s^{i-1}) + \delta \left( \hat{Y}_s^{i-1} + E'([\hat{Y}_s^{i-1}]') + \hat{b}(s, \Lambda_s^{i-1}) - \hat{b}(s, \Lambda_s^{i-2}) \right) \right\} ds \\
+ E \int_0^T \left\{ \hat{\sigma}(s, \Lambda_s^i) - \hat{\sigma}(s, \Lambda_s^{i-1}) + \delta \left( \hat{Z}_s^{i-1} + E'([\hat{Z}_s^{i-1}]') + \hat{\sigma}(s, \Lambda_s^{i-1}) - \hat{\sigma}(s, \Lambda_s^{i-2}) \right) \right\}^2 ds \\
\leq E \int_0^T |\hat{X}_s^{i}]^2 ds + 2E \int_0^T \left\{ \hat{b}(s, \Lambda_s^i) - \hat{b}(s, \Lambda_s^{i-1}) \right\}^2 + \left\{ \hat{\sigma}(s, \Lambda_s^i) - \hat{\sigma}(s, \Lambda_s^{i-1}) \right\}^2 \right\} ds \\
+ 2\delta^2 E \int_0^T \left\{ \hat{Y}_s^{i-1} + E'([\hat{Y}_s^{i-1}]') + \hat{b}(s, \Lambda_s^{i-1}) - \hat{b}(s, \Lambda_s^{i-2}) \right\}^2 ds \\
+ 2\delta^2 E \int_0^T \left\{ \hat{Z}_s^{i-1} + E'([\hat{Z}_s^{i-1}]') + \hat{\sigma}(s, \Lambda_s^{i-1}) - \hat{\sigma}(s, \Lambda_s^{i-2}) \right\}^2 ds \\
\leq C E \int_0^T \left\{ |\hat{U}_s^{i}]^2 + |\hat{U}_s^{i-1}]^2 \right\} ds.
\]
Similar to (10), in the last inequality, we use the fact that
\[ |b(s, \Lambda_i^{-1}) - b(s, \Lambda_i^{-2})|^2 \leq E' \left( [b(s, \Lambda_i^{-1}) - b(s, \Lambda_i^{-2})]^2 \right) \]
\[ \leq c \left( \mathbb{E}(|\hat{U}_i^{-1}|^2) + |\hat{U}_i^{-1}|^2 \right), \]
as well as
\[ \bar{b}^{\alpha_0}(s, \Lambda_i^i) - \bar{b}^{\alpha_0}(s, \Lambda_i^{i-1}) \leq \alpha_0 C|\Lambda_i^i - \Lambda_i^{i-1}| + (1 - \alpha_0)(-\mathbb{E}[\hat{Y}^i] - \hat{Y}^i), \]
where \( c \) is a constant which depends on \( C \). One can show that \( \bar{\sigma}(s, \Lambda_i^{i-1}) - \bar{\sigma}(s, \Lambda_i^{i-2}) \) and \( \bar{\sigma}^{\alpha_0}(s, \Lambda_i^i) - \bar{\sigma}^{\alpha_0}(s, \Lambda_i^{i-1}) \) have the similar results. By a standard method of estimation, the desired result (12) can be derived easily.

From (11) and (12), we know that there exists a constant \( K > 0 \) which depends only on \( C, C_1, \mu_1 \) and \( T \), such that
\[ \mathbb{E} \int_0^T |\hat{U}_i^{i+1}| ds \leq K\delta^2 \left( \mathbb{E} \int_0^T \left\{ |\hat{U}_i^i|^2 + |\hat{U}_i^{i-1}|^2 \right\} ds \right). \]
Hence there exists a \( \delta_0 \in (0, 1) \), which depends only on \( C, C_1, \mu_1 \) and \( T \), such that when \( 0 < \delta \leq \delta_0 \),
\[ \mathbb{E} \int_0^T |\hat{U}_i^{i+1}|^2 ds \leq \frac{1}{4} \mathbb{E} \int_0^T |\hat{U}_i^i|^2 ds + \frac{1}{8} \mathbb{E} \int_0^T |\hat{U}_i^{i-1}|^2 ds. \]
That is
\[ \mathbb{E} \int_0^T \left( |\hat{U}_i^{i+1}|^2 + \frac{1}{4} |\hat{U}_i^i|^2 \right) ds \leq \frac{1}{2} \mathbb{E} \int_0^T \left( |\hat{U}_i^i|^2 + \frac{1}{4} |\hat{U}_i^{i-1}|^2 \right) ds. \]
Repeat the above inequality as many times as you desire, there holds
\[ \mathbb{E} \int_0^T \left( |\hat{U}_i^{i+1}|^2 + \frac{1}{4} |\hat{U}_i^i|^2 \right) ds \leq \left( \frac{1}{2} \right)^{i-1} \mathbb{E} \int_0^T \left( |\hat{U}_i^2|^2 + \frac{1}{4} |\hat{U}_i^1|^2 \right) ds, \quad i \geq 1. \]
It turns out that \( U^i \) is a Cauchy sequence in \( \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d) \) and its limit is denoted by \( U = (X, Y, Z) \). Passing to the limit in Eqs.(8) and (9), we see that, when \( 0 < \delta \leq \delta_0 \), \( U = (X, Y, Z) \) solves Eqs.(6) and (7) for \( \alpha = \alpha_0 + \delta \). The proof is completed. \( \square \)

The condition (A2) can be replaced by the following condition.

(H6) For \( \Theta_i = (\tilde{x}_i, \tilde{y}_i, \tilde{z}_i, x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \), let \( u_i = (x_i, y_i, z_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \), then \( \Theta_i = (\tilde{u}_i, u_i) \) (\( i = 1, 2 \)). We assume that
\[ \mathbb{E} < F(t, \Theta_1) - F(t, \Theta_2), u_1 - u_2 > \geq C_1 \mathbb{E}(|u_1 - u_2|^2), \quad P - a.s., a.e. \ t \in \mathbb{R}^+, \]
\[ < \Phi(x_1) - \Phi(x_2), x_1 - x_2 > \leq -\mu_1 |x_1 - x_2|^2, \quad P - a.s., \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \]
where \( C_1 \) and \( \mu_1 \) are given positive constants.
We have another parallel existence and uniqueness theorem for mean-field FBSDEs.

**Theorem 10.** Let (H4) and (H6) hold. Then there exists a unique adapted solution \((X,Y,Z)\) of mean-field FBSDEs (13).

The method to prove the existence is similar to Theorem 7. We now consider the following (13) for each \(\alpha \in [0,1] :\)

\[
\begin{align*}
    dX^\alpha_s &= [\alpha \hat{b}(s, \Lambda_s) + \gamma(s)]ds + [\alpha \hat{\sigma}(s, \Lambda_s) + \phi(s)]dW_s, \\
    -dY_s &= \left[ - (1 - \alpha)c_2X_s + \alpha \hat{f}(s, \Lambda_s) + \varphi(s) \right]ds - Z_s dW_s, \\
    X^\alpha_0 &= a, \quad Y^\alpha_T = \alpha \Phi(X_T) - (1 - \alpha)X_T + \xi,
\end{align*}
\]

where \(\gamma, \phi\) and \(\varphi\) are given processes in \(\mathcal{M}^2(0,T)\) with values in \(\mathbb{R}, \mathbb{R}^d\), and \(\mathbb{R}\), resp., \(\xi \in L^2(\Omega, \mathcal{F}_T, P)\). Clearly the existence of (13) for \(\alpha = 1\) implies the existence of FBSDEs (3). From the existence and uniqueness of SDEs and BSDEs, when \(\alpha = 0\), the equation (13) has a unique solution.

In order to obtain this conclusion, we also need the following lemma. This lemma gives a priori estimate for the existence interval of (13) with respect to \(\alpha \in [0,1]\).

**Lemma 11.** We assume (H4) and (H6). Then there exists a positive constant \(\delta_0\) such that if, a priori, for a \(\alpha_0 \in [0,1]\) there exists a triple of solution \((X^{\alpha_0}, Y^{\alpha_0}, Z^{\alpha_0})\) of (13), then for each \(\delta \in [0,\delta_0]\) there exists a solution \((X^{\alpha_0+\delta}, Y^{\alpha_0+\delta}, Z^{\alpha_0+\delta})\) of FBSDEs (13) for \(\alpha = \alpha_0 + \delta\).

**Proof.** We use the notations

\[
\begin{align*}
    u &= (x,y,z), & \quad \hat{u} &= (\hat{x},\hat{y},\hat{z}), \\
    U &= (X,Y,Z), & \quad \hat{U} &= (\hat{X},\hat{Y},\hat{Z}), \\
    \Theta &= (x',y',z',x,y,z), & \quad \hat{\Theta} &= (\hat{x}',\hat{y}',\hat{z}',\hat{x},\hat{y},\hat{z}), \\
    \Lambda &= (X',Y',Z',X,Y,Z), & \quad \hat{\Lambda} &= (\hat{X}',\hat{Y}',\hat{Z}',\hat{X},\hat{Y},\hat{Z}), \\
    \hat{\Theta} &= \Theta - \hat{\Theta}, & \quad \hat{\Lambda} &= \Lambda - \hat{\Lambda}, \\
    \hat{u} &= (\hat{x},\hat{y},\hat{z}) = (x - \hat{x},y - \hat{y},z - \hat{z}), \\
    \hat{U} &= (\hat{X},\hat{Y},\hat{Z}) = (X - \hat{X},Y - \hat{Y},Z - \hat{Z}).
\end{align*}
\]

Since for \((\gamma,\phi,\varphi) \in \mathcal{M}^2(0,T;\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}), \xi \in L^2(\Omega, \mathcal{F}_T, P), \alpha_0 \in [0,1]\) there exists a unique solution of (13), thus, for each \(x_T \in L^2(\Omega, \mathcal{F}_T, P)\) and a triple \(u_s = (x_s,y_s,z_s) \in \mathcal{M}^2(0,T;\mathbb{R} \times \mathbb{R}^d \times \mathbb{R})\) there exists a unique triple \(U_s = (X_s,Y_s,Z_s) \in \mathcal{M}^2(0,T;\mathbb{R} \times \mathbb{R}^d \times \mathbb{R})\) satisfying the following FBSDE:

\[
\begin{align*}
    dX_s &= [\alpha_0 \hat{b}(s, \Lambda_s) + \delta \hat{b}(s, \Theta_s) + \gamma(s)]ds + [\alpha_0 \hat{\sigma}(s, \Lambda_s) + \delta \hat{\sigma}(s, \Theta_s) + \phi(s)]dW_s, \\
    -dY_s &= \left[ - (1 - \alpha_0)c_1X_s + \alpha_0 \hat{f}(s, \Lambda_s) + \delta (C_1 x_s + \hat{f}(s, \Theta_s)) + \varphi(s) \right]ds - Z_s dW_s, \\
    X_0 &= a, \quad Y_T = \alpha_0 \Phi(X_T) - (1 - \alpha_0)X_T + \delta(\Phi(x_T) + x_T) + \xi.
\end{align*}
\]
We now proceed to prove that, if $\delta$ is sufficiently small, the mapping defined by

$$I_{\alpha_0+\delta}(u \times x_T) = U \times X_T : \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}) \to \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}) \times L^2(\Omega, \mathcal{F}_T, P)$$

is a contraction.

Let $\bar{u} = (\bar{x}, \bar{g}, \bar{z}) \in \mathcal{M}^2(0, T; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$, and let $\bar{U} \times \bar{X}_T = I_{\alpha_0+\delta}(\bar{u} \times \bar{x}_T)$.

Using Itô’s formula to $\dot{X}_sZ_s$ yields

$$\alpha_0 E < \Phi(X_T) - \Phi(\bar{X}_T), \dot{X}_T > + (\alpha_0 - 1) E|\dot{X}_T|^2 + \delta E < \Phi(x_T) - \Phi(\bar{x}_T) + \dot{x}_T, \bar{X}_T >$$

$$= - E \int_0^T \dot{X}_s \left[ \alpha_0 [\bar{f}(s, \Lambda_s) - \bar{f}(s, \bar{\Lambda}_s)] + \delta [\bar{f}(s, \Theta_s) - \bar{f}(s, \bar{\Theta}_s)] - (1 - \alpha_0) C_1 \bar{X}_s + \delta C_1 \dot{\bar{X}}_s \right] ds$$

$$+ E \int_0^T \dot{Y}_s \left[ \alpha_0 [\bar{b}(s, \Lambda_s) - \bar{b}(s, \bar{\Lambda}_s)] + \delta [\bar{b}(s, \Theta_s) - \bar{b}(s, \bar{\Theta}_s)] \right] ds$$

$$+ E \int_0^T \dot{Z}_s \left[ \alpha_0 [\bar{\sigma}(s, \Lambda_s) - \bar{\sigma}(s, \bar{\Lambda}_s)] + \delta [\bar{\sigma}(s, \Theta_s) - \bar{\sigma}(s, \bar{\Theta}_s)] \right] ds$$

$$= \alpha_0 E \int_0^T < \bar{F}(s, \Lambda_s) - \bar{F}(s, \bar{\Lambda}_s), \hat{U}_s > + ds + (1 - \alpha_0) C_1 E \int_0^T |\dot{\bar{X}}_s|^2 ds - \delta C_1 E \int_0^T < \hat{\bar{X}}_s, \hat{x}_s > ds$$

$$+ \delta E \int_0^T < \hat{F}(s, \Theta_s) - \hat{F}(s, \bar{\Theta}_s), \hat{U}_s > ds$$.

From (H4) and (H6), we can get

$$(\mu_1 \alpha_0 + (1 - \alpha_0)) E[|\dot{X}_T|^2] + C_1 E \int_0^T |\dot{X}_s|^2 ds + C_1 \alpha_0 E \int_0^T (|\dot{Y}_s|^2 + |\dot{Z}_s|^2) ds$$

$$\leq \delta E \Phi(x_T) - \Phi(\bar{x}_T) + \dot{x}_T, \bar{X}_T > + \delta C_1 E \int_0^T < \hat{\bar{X}}_s, \hat{x}_s > ds$$

$$- \delta E \int_0^T < \hat{F}(s, \Theta_s) - \hat{F}(s, \bar{\Theta}_s), \hat{U}_s > ds$$

$$\leq \delta K_1 E(|\dot{x}_T|^2 + |\dot{X}_T|^2) + \delta K_1 E \int_0^T \left( |\dot{u}_s|^2 + |\dot{U}_s|^2 \right) ds$$.

This means

$$\mu E[|\dot{X}_T|^2] + C_1 E \int_0^T |\dot{X}_s|^2 ds \leq \delta K_1 E(|\dot{x}_T|^2 + |\dot{X}_T|^2) + \delta K_1 E \int_0^T \left( |\dot{u}_s|^2 + |\dot{U}_s|^2 \right) ds,$$

where $\mu_1 \alpha_0 + (1 - \alpha_0) \geq \mu = \min(1, \mu_1) > 0$.

On the other hand, for the difference of the solutions $(\hat{Y}, \hat{Z}) = (Y - \bar{Y}, Z - \bar{Z})$, we apply the usual technique to the BSDE part:

$$E \int_0^T \left( |\hat{Y}_s|^2 + |\hat{Z}_s|^2 \right) ds \leq K_2 \delta E \int_0^T |\hat{u}_s|^2 ds + K_2 \delta E |\hat{x}_T|^2 + K_2 E \int_0^T |\hat{X}_s|^2 ds + C_1 E |\hat{X}_T|^2.$$
Here the constant $K_2$ depends on the Lipschitz constants $C, C_1,$ and $T$.

Combining the above two estimates, it is clear that, we have

$$
\mathbb{E}[|\hat{X}_T|^2] + \mathbb{E}\int_0^T |\hat{U}_s|^2 ds \leq \delta K \mathbb{E}\left( \int_0^T |\hat{u}_s|^2 ds + |\hat{x}_T|^2 \right).
$$

Here the constant $K$ depends only on $C, C_1, \mu_1, K_1,$ and $T$. We now choose $\delta_0 = \frac{1}{2K}$. It is clear that, for each fixed $\delta \in [0, \delta_0]$, the mapping $I_{\alpha_0+\delta}$ is a contraction in the sense that

$$
\mathbb{E}[|\hat{X}_T|^2] + \mathbb{E}\int_0^T |\hat{U}_s|^2 ds \leq \frac{1}{2} \left( \mathbb{E}\int_0^T |\hat{u}_s|^2 ds + \mathbb{E}|\hat{x}_T|^2 \right).
$$

It follows that this mapping has a unique fixed point $U^{\alpha_0+\delta} = (X^{\alpha_0+\delta}, Y^{\alpha_0+\delta}, Z^{\alpha_0+\delta})$ which is the solution of (13) for $\alpha = \alpha_0 + \delta$. The proof is complete. \qed

We now give the proof of Theorem 10.

Proof of Theorem 10. The uniqueness is obvious from Theorem 7. When $\alpha = 0$, the equation (13) has a unique solution. It then follows from Lemma 3.3 that there exists a positive constant $\delta_0$ depending on Lipschitz constants $C, C_1, \mu_1, K_1$ and $T$ such that, for each $\delta \in [0, \delta_0]$, equation (13) for $\alpha = \alpha_0 + \delta$ has a unique solution. We can repeat this process for $N$-times with $1 \leq N\delta_0 < 1 + \delta_0$. It then follows that, in particular, FBSDEs (13) for $\alpha = 1$ with $\xi = 0$ has a unique solution. The proof is complete. \qed

Theorem 10 can ensure the existence and uniqueness of solution to the adjoint forward-backward systems in Section 5 and Section 7.

4 Stochastic maximum principle in mean-field controls of FBSDEs

In this section, we study the stochastic maximum principle for mean-field control problem of FBSDEs. The action space $U$ is a non-empty, closed and convex subset of $\mathbb{R}^k$ ($k \in \mathbb{N}^+$), and we define the admissible control set as

$$
U = \{v_t \in L_2^2(0, T; U)|v_t(\omega', \omega) : [0, T] \times \Omega \times \Omega \to U, t \in [0, T]\}.
$$

For any $v(\cdot) \in U$, we consider the following forward-backward stochastic control system of Mean-field type:

$$
\begin{align*}
\begin{cases}
    dX_t = \mathbb{E}'[b(t, X'_t, X_t, v_t)]dt + \mathbb{E}'[\sigma(t, X'_t, X_t, v_t)]dW_t, \\
    X(0) = x_0, \\
    -dY_t = \mathbb{E}'[f(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_t)]dt - Z_t dW_t, \\
    Y_T = \Phi(X_T),
\end{cases}
\end{align*}
$$

17
where
\[ b : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \]
\[ \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}^d, \]
\[ f : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \]
\[ \Phi : \mathbb{R} \to \mathbb{R}. \]

The optimal control problem is to minimize the following expected cost functional over \( U \):
\[
J(v(\cdot)) = \mathbb{E} \left( \int_0^T \mathbb{E}' [h(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, v(t))] dt \right) \\
+ \mathbb{E} \left( g(X_T) + \gamma(Y(0)) \right),
\]
where
\[ g : \mathbb{R} \to \mathbb{R}, \]
\[ \gamma : \mathbb{R} \to \mathbb{R}, \]
\[ h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}. \]

An admissible control \( u \in U \) is said to be optimal if
\[
J(u) = \min_{v \in U} J(v).
\]

Now we give the following conditions in this section.

\textbf{(A1)} The given functions \( b(t, \tilde{x}, x, v), \sigma(t, \tilde{x}, x, v), f(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v), h(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v), g(x) \) and \( \gamma(y) \) are continuously differentiable with respect to all of their components respectively.

\textbf{(A2)} All the derivatives in (A1) are Lipschitz continuous and bounded.

For any admissible controls \( v(\cdot) \in U \), due to Lemma 2, the mean-field FBSDEs (4) admits a unique solution under assumptions (A1) and (A2), which is denoted by \( (X_t, Y_t, Z_t) \).

\section*{4.1 Variational equations and variational inequality}

Let \( u(\cdot) \) be an optimal control and \( (X^n(\cdot), Y^n(\cdot), Z^n(\cdot)) \) be the corresponding state trajectory of stochastic control system. For any \( 0 \leq \theta \leq 1 \), we denote by \( (X^\theta_t, Y^\theta_t, Z^\theta_t) \) the state trajectory corresponding the following perturbation \( u^\theta_t \) of \( u_t \).
\[
u^\theta_t = u_t + \theta(v_t - u_t), \quad v_t \in U.
\]
Since $\mathcal{U}$ is convex, then $u^\theta(\cdot)$ is also in $\mathcal{U}$. Let $(k(\cdot), m(\cdot), n(\cdot))$ be a solution of the variational equation

\[
\begin{align*}
\begin{aligned}
dk_t &= \mathbb{E}' \left[ b_z(t, (X^u_t)^{\theta}_t, u_t)(k_t) + b_x(t, (X^u_t)^{\theta}_t, u_t) dk_t ight] + \mathbb{E}' \left[ \sigma_z(t, (X^u_t)^{\theta}_t, u_t) (k_t) \right] \, dt + \mathbb{E}' \left[ \sigma_x(t, (X^u_t)^{\theta}_t, X, u_t) (v_t - u_t) \right] \, dW_t, \\
k_0 &= 0,
\end{aligned}
\end{align*}
\]  

(17)

\[
\begin{align*}
\begin{aligned}
dm_t &= -\mathbb{E}' \left[ \tilde{f}_z(t, u) (k_t) + \tilde{f}_x(t, u) k_t + \tilde{f}_\nu(t, u) (m_t) + \tilde{f}_\gamma(t, u) m_t + \tilde{f}_z(t, u) (n_t) \right] + \tilde{f}_\nu(t, u) (v_t - u_t) \, dt + n_t dW_t, \\
m_T &= k_T \Phi(x(T)).
\end{aligned}
\end{align*}
\]  

(18)

where we use the notation $\tilde{f}(t, u) = f(t, (X^u_t)^{\theta}_t, \theta, Y^u_t, Z^u_t, u_t)$.

Set

\[
\begin{align*}
\tilde{X}_t^\theta &= \theta^{-1}(X_t^\theta - X_t^u) - k_t, \\
\tilde{Y}_t^\theta &= \theta^{-1}(Y_t^\theta - Y_t^u) - m_t, \\
\tilde{Z}_t^\theta &= \theta^{-1}(Z_t^\theta - Z_t^u) - n_t.
\end{align*}
\]  

(19)

Then, we have the following convergence result:

**Lemma 12.** We suppose (A3) and (A4) hold. Then

\[
\begin{align*}
\lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|\tilde{X}_t^\theta|^2 &= 0, \\
\lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|\tilde{Y}_t^\theta|^2 &= 0, \\
\lim_{\theta \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|\tilde{Z}_t^\theta|^2 &= 0.
\end{align*}
\]  

(20)

**Proof.** Since the coefficients in linear mean-field FBSDE (17) and (18) are bounded, it follows from Proposition 1.2 in [13] that there exists a unique solution $(k(t), m(t), n(t))$ for equations (17) and (18).

The proof for the convergence of $\tilde{X}_t^\theta$ can be found in Lemma 3.2 of [18]. We need only to deal with $\tilde{Y}_t^\theta$ and $\tilde{Z}_t^\theta$. From the definition of $\tilde{Y}_t^\theta$, it fulfills the following BSDE,

\[
\begin{align*}
-d\tilde{Y}_t^\theta &= \frac{1}{\theta} \left( dY_t^\theta - dY_t^u \right) + dm_t \\
&= \frac{1}{\theta} \mathbb{E}' \left[ f(t, (X_t^\theta)^{\prime}, (Y_t^\theta)^{\prime}, (Z_t^\theta)^{\prime}, X_t^\theta, Y_t^\theta, Z_t^\theta, u_t^\theta) - \tilde{f}(t, u) \right] \, dt \\
&\quad - \mathbb{E}' \left[ \tilde{f}_z(t, u) (k_t) + \tilde{f}_x(t, u) k_t + \tilde{f}_\nu(t, u) (m_t) + \tilde{f}_\gamma(t, u) m_t + \tilde{f}_z(t, u) (n_t) \right] + \tilde{f}_\nu(t, u) (v_t - u_t) \, dt - \tilde{Z}_t^\theta dW_t.
\end{align*}
\]
Denote by \( X_t^{\lambda, \theta} = X_t^u + \lambda \theta (X_t^\theta + k_t), Y_t^{\lambda, \theta} = Y_t^u + \lambda \theta (Y_t^\theta + m_t), Z_t^{\lambda, \theta} = Z_t^u + \lambda \theta (Z_t^\theta + n_t) \) and \( u^{\lambda, \theta}(t) = u_t + \lambda \theta (v_t - u_t) \). For convenience, we introduce the notation

\[
f(\lambda) = f(t, (X_t^{\lambda, \theta})', (Y_t^{\lambda, \theta})', (Z_t^{\lambda, \theta})', X_t^\theta, Y_t^\theta, Z_t^\theta, u^{\lambda, \theta}(t)).
\]

Then, we have

\[
\frac{1}{\theta} \left( f(t, (X_t^\theta)'), (Y_t^\theta)'), (Z_t^\theta)', X_t^\theta, Y_t^\theta, Z_t^\theta, u_t) - \bar{f}(t, u) \right) dt
\]

\[
= \int_0^1 f_x(\lambda)((\bar{X}_t^\theta)' + (k_t)')d\lambda + \int_0^1 f_x(\lambda)(\bar{X}_t^\theta) + k_t)d\lambda + \int_0^1 f_y(\lambda)((\bar{Y}_t^\theta)' + (m_t)')d\lambda
\]

\[
+ \int_0^1 f_y(\lambda)(\bar{Y}_t^\theta + m_t)d\lambda + \int_0^1 f_z(\lambda)((\bar{Z}_t^\theta)' + (n_t)')d\lambda + \int_0^1 f_z(\lambda)(\bar{Z}_t^\theta + n_t)d\lambda
\]

\[
+ \int_0^1 f_v(\lambda)(v_t - u_t)d\lambda,
\]

and \( \bar{Y}_t^\theta \) satisfies

\[
- d\bar{Y}_t^\theta = \left\{ \int_0^1 \mathbb{E}' \left\{ (f_x(\lambda)(\bar{X}_t^\theta)' + f_x(\lambda)(\bar{X}_t^\theta) + f_y(\lambda)(\bar{Y}_t^\theta)' + f_y(\lambda)(\bar{Y}_t^\theta) + f_z(\lambda)(\bar{Z}_t^\theta)' + f_z(\lambda)(\bar{Z}_t^\theta)) \right\} d\lambda
\]

\[
+ A_t + B_t + C_t + G_t \right\} dt - \bar{Z}_t^\theta dW_t,
\]

where we denote

\[
A_t = \int_0^1 \mathbb{E}' \left\{ (f_x(\lambda) - \bar{f}_x(t,u))(k_t)' + (f_x(\lambda) - \bar{f}_x(t,u))k_t \right\} d\lambda,
\]

\[
B_t = \int_0^1 \mathbb{E}' \left\{ (f_y(\lambda) - \bar{f}_y(t,u))(m_t)' + (f_y(\lambda) - \bar{f}_y(t,u))m_t \right\} d\lambda,
\]

\[
C_t = \int_0^1 \mathbb{E}' \left\{ (f_z(\lambda) - \bar{f}_z(t,u))(n_t)' + (f_z(\lambda) - \bar{f}_z(t,u))n_t \right\} d\lambda,
\]

\[
G_t = \int_0^1 \mathbb{E}' \left\{ (f_v(\lambda) - \bar{f}_v(t,u))(v_t - u_t) \right\} d\lambda.
\]

\( A_t \) tends to 0 in \( L^2(\Omega \times [0, T]) \) as \( \theta \to 0 \). Indeed, since the Lipschitz continuity of \( f_x(\bar{x}, \bar{y}, \bar{z}, x, y, z) \), \( f_x(\bar{x}, \bar{y}, \bar{z}, x, y, z) \) with respect to \((\bar{x}, \bar{y}, \bar{z}, x, y, z)\), there exists a positive constant \( C \), which may differ from line to line if not specified, such that:

\[
|f_x(\lambda) - \bar{f}_x(t,u)| \leq C \lambda \theta \beta_t, \quad |f_x(\lambda) - \bar{f}_x(t,u)| \leq C \lambda \theta \beta_t,
\]

with

\[
\beta_t = |(\bar{X}_t^\theta + k_t)'| + |(\bar{Y}_t^\theta + m_t)'| + |(\bar{Z}_t^\theta + n_t)'| + |\bar{X}_t^\theta + k_t| + |\bar{Y}_t^\theta + m_t| + |\bar{Z}_t^\theta + n_t| + |v_t - u_t|.
\]
Then

\[ |A_t|^2 = \int_0^1 E' \left\{ \left( f_x(\lambda) - \bar{f}_x(t,u) \right)(k_t)' + \left( f_x(\lambda) - \bar{f}_x(t,u) \right) k_t \right\} d\lambda \]

\[ \leq C\theta^2 \left( E'[(|k_t|') + |k_t|]\beta_t] \right)^2 \]

\[ \leq C\theta^2 \left( E'[(|k_t|')^2]E'[\beta_t^2] + |k_t|^2 E'[\beta_t^2] \right) \]

\[ = C\theta^2 \left( E[|k_t|^2] + |k_t|^2 \right)E'[\beta_t^2]. \]

So

\[ E \int_0^T |A_t|^2 dt \leq C\theta^2 E \left( \int_0^T E[|k_t|^2]E'[\beta_t^2]dt \right) \]

\[ \leq C\theta^2 E \left( \int_0^T |k_t|^4 dt \right)^{\frac{1}{2}} \left( \int_0^T E'[\beta_t^2]dt \right)^{\frac{1}{2}}, \]

which converges to 0 as \( \theta \to 0 \) since the expected values are finite. Similar estimations for \( B_t, C_t \) and \( G_t \) in (21) show that these terms also converge to 0 in \( L^2(\Omega \times [0,T]) \). For simplicity, we let \( I_t = A_t + B_t + C_t + G_t \). Using Ito’s formula to \( |\bar{Y}_t^\theta|^2 \) and noting that assumption (A2), we have

\[ E|\bar{Y}_t^\theta|^2 + E \int_t^T |\tilde{Z}_s^\theta|^2 ds \leq CE \int_t^T |\bar{Y}_s^\theta| E(\bar{X}_s^\theta + \bar{Y}_s^\theta + \bar{Z}_s^\theta) + E(\tilde{Z}_s^\theta) + I_s ds \]

\[ \leq CE \int_t^T |\bar{Y}_s^\theta|^2 ds + \frac{1}{2} E \int_t^T |\tilde{Z}_s^\theta|^2 ds + J_\theta, \]

with

\[ J_\theta = E \int_t^T |\bar{X}_s|^2 ds + E \int_t^T |I_s|^2 ds, \]

where \( C > 0 \) is a constant and \( J_\theta \to 0 \) as \( \theta \to 0 \). Applying Gronwall’s lemma gives the last two results of (20). \( \square \)

**Lemma 13.** Under the assumptions (A1) and (A2), for any \( v(\cdot) \in U \), the following variational inequality holds:

\[ E \int_0^T E' \left( h_x(t)(k_t)' + h_x(t)k_t + h_y(t)(m_t)' + h_y(t)m_t + h_z(t)(n_t)' + h_z(t)n_t + h_v(t)(v(t) - u(t)) \right) dt \]

\[ + E \left( g_x(X_T^u)k_T \right) \geq 0. \]

(22)

where we denote \( h(t, (X_t^u)^n_t, Y_t^u, Z_t^u, u_t) \) by \( \tilde{h}(t) \).
Proof. Since \( u(\cdot) \) is an optimal control of the problem, then

\[
\theta^{-1} \left[ J(u(\cdot) + \theta(v(\cdot) - u(\cdot))) - J(u(\cdot)) \right] \geq 0. \tag{23}
\]

From the estimate of (20), when \( \theta \to 0 \), it follows that

\[
\frac{1}{\theta} \mathbb{E}[g(X^\theta_T) - g(X^u_T)] \to \mathbb{E}[g_x(X^\theta_T)k_T],
\]

\[
\frac{1}{\theta} \mathbb{E}[\gamma(Y^\theta(0)) - \gamma(Y^u(0))] \to \mathbb{E}[\gamma_y(Y^u(0))m_0],
\]

\[
\frac{1}{\theta} \left\{ \mathbb{E} \int_0^T \mathbb{E}'[h(t, (X^\theta_t)^t, (Z^\theta_t)^t, X^\theta_t X^\theta_t, Z^\theta_t, u(t) + \theta(v(t) - u(t))) - \bar{h}(t)] dt \right\} \to
\]

\[
\mathbb{E} \int_0^T \mathbb{E}'(\bar{h}_x(t,k_t) + \bar{h}_y(t) + \bar{h}_z(t)(n_t) + \bar{h}_{x}(t)n_t + \bar{h}_y(t)v(t) - u(t)) dt.
\]

Combining the limits above with (23) and the definition of the cost functional, we derive (22) easily. \qed

4.2 Adjoint equation and Maximum principle

For deriving the maximum principle, we introduce the following adjoint equation corresponding to mean-field FBSDEs (4), which is a mean-field FBSDEs and whose solution is denoted by \( (p(\cdot), q(\cdot), Q(\cdot)) \),

\[
\begin{cases}
-dp_t = \mathbb{E}'(b_{x}(t, (X^u_t)^t, u_t)(p_t)' + b_x(t, (X^u_t)^t, u_t)p_t \\
+\sigma_x(t, (X^u_t)^t, u_t)(q_t)' + \sigma_x(t, (X^u_t)^t, u_t)q_t) dt \\
+\mathbb{E}'(\bar{h}_x(t) + \bar{h}_z(t) - \bar{f}_x(t, u)(Q_t)' - \bar{f}_x(t, u)Q_t) dt - q_{t}dW_t, \\
\end{cases}
\]

\[
dQ_t = \mathbb{E}'(\bar{f}_y(t, u)(Q_t)' + \bar{f}_y(t, u)Q_t - \bar{h}_y(t) - \bar{h}_y(t)) dt \\
+\mathbb{E}'(\bar{f}_z(t, u)(Q_t)' + \bar{f}_z(t, u)Q_t - \bar{h}_z(t) - \bar{h}_z(t))dW_t, \\
p_T = g_x(X^\theta_T) - \Phi_x(X^\theta_T)Q_T, \quad Q_0 = -\gamma_y(Y^u(0)).
\tag{24}
\]

This equation reduces to the standard one, when the coefficients do not depend explicitly on \( \omega' \) of the underlying diffusion. Under the assumptions (A1) and (A2), this is a linear mean-field FBSDEs with bounded coefficients. Moreover, due to Lemma 2.2 and Theorem 4.1 in [13], it admits a unique \( \mathbb{F} \)-adapted solution \( (Q, p, q) \) such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Q(t)|^2 + \sup_{0 \leq t \leq T} |p(t)|^2 + \int_0^T |q(t)|^2 dt \right] < +\infty.
\]

Next, we define the Hamiltonian function as follows:

\[
H(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, p, q, Q, v) = b(t, \bar{x}, x, v)p + \sigma(t, \bar{x}, x, v)q - f(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v)Q + h(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v).
\]

22
The following theorem constitutes the main result of this section.

**Theorem 14.** (SMP in Integral Form). Suppose (A1)-(A2) hold. Let \( u(\cdot) \) be an optimal control of the problem, and \((X^u(\cdot), Y^u(\cdot), Z^u(\cdot))\) denote the corresponding trajectory. Then, for all \( v \in \mathcal{U} \), there holds

\[
\mathbb{E} \int_0^T \mathbb{E}' [H_v(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, p_t, q_t, Q_t, u(t))(v(t) - u(t))] dt \geq 0, \tag{25}
\]

a.e., a.s., where \((p(\cdot), q(\cdot), Q(\cdot))\) is the the solution of adjoint equation \(\text{[24]}\).

**Proof.** Applying Itô’s formula to \(k_t p_t + m_t Q_t\) yields

\[
\mathbb{E} \left[ k_T p_T + m_T Q_T - k_0 p_0 - m_0 Q_0 \right] = \mathbb{E} \left[ g(X^u_T)k_T + m_0 \gamma_y(Y^u(0)) \right]
\]

\[
= \mathbb{E} \int_0^T \mathbb{E}' \left[ (p_t b_v(t, (X'^u_t)' , X'^u_t, u_t) + q_t \sigma_v(t, (X'^u_t)' , X'^u_t, u_t) - Q_t \bar{f}_v(t, u)) (v(t) - u(t)) \right] dt
\]

\[
- \mathbb{E} \int_0^T \mathbb{E}' \left( k_t \bar{h}_z(t) + k_t \bar{h}_x(t) + m_t \bar{h}_y(t) + n_t \bar{h}_z(t) + n_t \bar{h}_x(t) \right) dt.
\]

Together with Lemma 13, we derive

\[
\mathbb{E} \int_0^T \mathbb{E}' \left[ (p_t b_v(t, (X'^u_t)' , X'^u_t, u_t) + q_t \sigma_v(t, (X'^u_t)' , X'^u_t, u_t) - Q_t \bar{f}_v(t, u) + \bar{h}_v(t)) (v(t) - u(t)) \right] dt
\]

\[
= \mathbb{E} \int_0^T \mathbb{E}' [H_v(t, (X'^u_t)' , (Y'^u_t)' , (Z'^u_t)' , X'^u_t, Y'^u_t, Z'^u_t, p_t, q_t, Q_t, u_t)(v(t) - u(t))] dt \geq 0.
\]

Thus, we come to the conclusion of this theorem. \( \square \)

**Remark 15.** From \(\text{[24]}\), we can get that

\[
\mathbb{E}' [H_v(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, p_t, q_t, Q_t, u(t))(v(t) - u(t))] \geq 0, \tag{26}
\]

dtdP-a.e., for any \( v \in \mathcal{U} \).

### 4.3 Sufficient conditions for maximum principle

This section is devoted to establish the sufficient maximum principle (also called verification theorem) of the mean-field stochastic control problem.

We need the following additional assumptions.

\(\text{(A3)}\) The function \( \Phi \) is convex in \( x \). \( g \) is convex in \( x \) and \( \gamma \) is convex in \( y \).
Theorem 16. (Sufficient Conditions for the Optimality of the control) Assume that the conditions (A1)-(A3) are satisfied. Let \( u(\cdot) \in U \) with state trajectory \((X^u_t, Y^u_t, Z^u_t)\) and \((p(\cdot), q(\cdot), Q(\cdot))\) be the solution of Mean-field FBSDE (24). Suppose
\[
\mathbb{E}[H(t, (X^u_t), (Y^u_t), (Z^u_t), X^u_t, Y^u_t, Z^u_t, p_t, q_t, Q_t, u(t))]
= \min_{u \in U} \mathbb{E}^{(u)}[\gamma(\cdot)], Y^u_t, Z^u_t, p_t, q_t, Q_t, v(t)]
\]
hold for all \( t \in [0, T] \). Moreover, suppose function \( H(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, p, q, Q, v) \) is convex with respect to \((t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v)\). Then \( u \) is an optimal control of problem (4)-(16).

**Proof.** For any \( v(\cdot) \in U \), we consider
\[
J(v(\cdot)) - J(u(\cdot)) = I + II + III
\]  
with
\[
I = \mathbb{E} \int_0^T \mathbb{E}^{(v)} \left[ h(t, (X^v_t'), (Y^v_t'), (Z^v_t'), X^v_t, Y^v_t, Z^v_t, v(t)) \right. \\
- \left. h(t, (X^u_t'), (Y^u_t'), (Z^u_t'), X^u_t, Y^u_t, Z^u_t, u(t)) \right] dt,
\]
\[
II = \mathbb{E} \left[ g(X^v_T) - g(X^u_T) \right],
\]
\[
III = \mathbb{E} \left[ \gamma(Y^v(0)) - \gamma(Y^u(0)) \right].
\]

Since \( g \) is convex, it holds that
\[
II = \mathbb{E}(g(X^v_T) - g(X^u_T)) \geq \mathbb{E}[g_x(X^v_T)(X^v_T - X^u_T)].
\]  
(28)

Due to \( \gamma \) is convex on \( y \), we have
\[
III \geq \gamma_y(Y^v(0))(Y^v(0) - Y^u(0)) = -Q_0(Y^v(0) - Y^u(0)).
\]  
(29)

From (28) and (29), we get
\[
II + III \geq \mathbb{E}[g_x(X^v_T)(X^v_T - X^u_T) + \gamma_y(Y^v(0))(Y^v(0) - Y^u(0))].
\]  
(30)

Since \( \Phi \) is convex,
\[
Y^v_T - Y^u_T = \Phi(X^v_T) - \Phi(X^u_T) \geq \Phi_x(X^u_T)(X^v_T - X^u_T).
\]  
(31)

By applying Itô’s formula to \( Q_t(Y^v_t - Y^u_t) + p_t(X^v_t - X^u_t) \), we get
\[
\mathbb{E} \left[ Q_T(Y^v_T - Y^u_T) - Q_0(Y^v(0) - Y^u(0)) + p_T(X^v_T - X^u_T) - p(0)(X^v(0) - X^u(0)) \right]
= \mathbb{E} \left[ Q_T(Y^v_T - Y^u_T) + \gamma_y(Y^v(0))(Y^v(0) - Y^u(0)) + (g_x(X^v_T) - \Phi_x(X^u_T)Q_T)(X^v_T - X^u_T) \right]
= \mathbb{E} \int_0^T \mathbb{E} \left\{ p_t(b(t, (X^v_t)'), X^v_t, v(t)) - b(t, (X^u_t)', X^u_t, u(t))) - Q_t(\tilde{f}(t, v) - \tilde{f}(t, u)) \\
+ q_t(\sigma(t, (X^v_t)'), X^v_t, v(t)) - \sigma(t, (X^u_t)', X^u_t, u(t))) \right\} dt + IV
\]  
(32)
with

\[ IV = E \int_0^T E' \left\{ \left( \bar{f}_\delta(t, u)(Q_t)' + \bar{f}_\delta(t, u)Q_t - \bar{h}_\delta(t) - \bar{h}_y(t) \right)(Y_t^v - Y_t^u) \\
+ \left( \bar{f}_\z(t, u)(Q_t)' + \bar{f}_\z(t, u)Q_t - \bar{h}_\z(t) - \bar{h}_\z(t) \right)(Z_t^v - Z_t^u) \\
- \left( b_\z(t, (X_t^u)', X_t^u, u_t)(p_t)' + b_x(t, (X_t^u)', X_t^u, u_t)p_t \\
+ \sigma_\z(t, (X_t^u)', X_t^u, u_t)(q_t)' + \sigma_x(t, (X_t^u)', X_t^u, u_t)q_t \right)(X_t^v - X_t^u) \\
- \left( \bar{h}_\z(t) + \bar{h}_x(t) - \bar{f}_\z(t, u)(Q_t)' - \bar{f}_\z(t, u)Q_t \right)(X_t^v - X_t^u) \right\} dt \]

\[ = -E \int_0^T E' \left\{ H_\z(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^u, Y_t^u, Z_t^u, p_t, q_t, Q_t, u(t))(X_t^v - X_t^u)' \\
+ H_\z(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^u, Y_t^u, Z_t^u, p_t, q_t, Q_t, u(t))(X_t^v - X_t^u)' \\
+ H_\z(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^u, Y_t^u, Z_t^u, p_t, q_t, Q_t, u(t))(X_t^v - X_t^u)' \\
+ H_\z(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^u, Y_t^u, Z_t^u, p_t, q_t, Q_t, u(t))(X_t^v - X_t^u)' \\
+ H_\z(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^u, Y_t^u, Z_t^u, p_t, q_t, Q_t, u(t))(X_t^v - X_t^u)' \right\} dt. \]

Together with (27), (31), (31) and (32), we get

\[ J(v(\cdot)) - J(u(\cdot)) = I + II + III \]

\[ \geq I + E \int_0^T E' \left\{ p_t(b(t, (X_t^u)'), X_t^v, v(t)) - b(t, (X_t^u)'), X_t^u, u(t)) \right\} dt + IV \]

\[ = E \int_0^T E' \left\{ H(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^v, Y_t^v, Z_t^v, p_t, q_t, Q_t, v(t)) \\
- H(t, (X_t^u)', (Y_t^u)', (Z_t^u)', X_t^v, Y_t^v, Z_t^v, p_t, q_t, Q_t, u(t)) \right\} dt + IV. \]

Noticing \( H \) is convex with respect to \((\tilde{x}, \tilde{y}, \tilde{z}, x, y, z)\), the use of the Clark generalized gradient
of $H$, evaluated at $((X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, v(t))$, yields
\[
H(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, v(t)) - H(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t)) \geq H_{x}(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t))(X^u_t - X^u_t') + H_{y}(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t))(Y^u_t - Y^u_t') + H_{z}(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t))(Z^u_t - Z^u_t') + H_{v}(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t))(v(t) - u(t)).
\]

Combined the above inequality with (33), we have
\[
J(v(\cdot)) - J(u(\cdot)) \geq \mathbb{E} \int_0^T \mathbb{E}' \left[ H_{v}(t, (X^u)'_t, (Y^u)'_t, (Z^u)'_t, X^u_t, Y^u_t, Z^u_t, pt, q_t, Q_t, u(t))(v(t) - u(t)) \right] dt \geq 0.
\]

Hence, we draw the desired conclusion.
\[
\square
\]

5 Stochastic maximum principle for fully coupled forward-backward stochastic control systems of mean-field type

In this section, we extend control problems to the fully coupled mean-field FBSDEs. For any $v(\cdot) \in U$, the state equation consists of the following forward-backward control system of mean-field type:
\[
\begin{aligned}
&dX_t = \mathbb{E}[b(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_t)]dt + \mathbb{E}'[\sigma(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_t)]dW_t, \\
&X(0) = x_0, \\
&-dY_t = \mathbb{E}[f(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_t)]dt - Z_t dW_t, \\
&Y_T = \Phi(X_T),
\end{aligned}
\]

where
\[
\begin{aligned}
b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}, \\
\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}^d, \\
f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}, \\
\Phi : \mathbb{R} \to \mathbb{R}.
\end{aligned}
\]

The expected cost function is given by:
\[
J(v(\cdot)) = \mathbb{E} \left( \int_0^T \mathbb{E}'[h(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v(t))]dt + g(X_T) + \gamma(Y(0)) \right),
\]

(35)
where
\[ g : \mathbb{R} \to \mathbb{R}, \]
\[ \gamma : \mathbb{R} \to \mathbb{R}, \]
\[ h : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}. \]

The optimal control problem is to minimize the functional \( J(\cdot) \) over \( U \). A control that solves this problem is called optimal.

We assume:

\( \text{(A4)} \)

\[
\begin{align*}
(i) \quad & b, \sigma, f, \Phi, h, g \text{ and } \gamma \text{ are continuously differentiable;} \\
(ii) \quad & \text{The derivatives of } b, \sigma, f \text{ and } \Phi \text{ are bounded;} \\
(iii) \quad & \text{The derivatives of } h \text{ are bounded by } C(1 + |\bar{x}| + |\bar{y}| + |\bar{z}| + |x| + |y| + |z|); \\
(iv) \quad & \text{The derivatives of } g \text{ and } \gamma \text{ with respect to } x \text{ and } y \text{ are bounded by } C(1 + |x|) \text{ and } C(1 + |y|) \text{ respectively;} \\
(v) \quad & \text{For any given admissible control } v(\cdot), \text{ the equation (33) satisfies (H4) and (H5).}
\end{align*}
\]

According to Theorem 3.1, for any given admissible control \( v(\cdot) \in U \), there exists a unique adapted solution \((X^v_t, Y^v_t, Z^v_t)\) satisfying the fully coupled mean-field FBSDEs (34).

Let \( u(\cdot) \) be an optimal control and \((X^{u(\cdot)}, Y^{u(\cdot)}, Z^{u(\cdot)})\) be the corresponding state trajectory of stochastic control system. In this case, the corresponding adjoint equation becomes

\[
\begin{align*}
-dp(t) &= \mathbb{E}'\left( b_x(t)(p_t) + \bar{b}_x(t)p_t + \sigma_x(t)(q_t) + \bar{\sigma}_x(t)q_t \right) dt \\
&\quad + \mathbb{E}'\left( \bar{h}_x(t) + \bar{h}_x(t) - \bar{f}_x(t)Q_t \right) dt - q_t dW_t, \\
\] \quad + \mathbb{E}'\left( \bar{f}_y(t)(Q_t) + \bar{f}_y(t)Q_t - \bar{b}_y(t)p_t + \bar{\sigma}_y(t)(q_t) - \bar{\sigma}_y(t)q_t - \bar{h}_y(t) - \bar{h}_y(t) \right) dt \\
\] \quad + \mathbb{E}'\left( \bar{f}_z(t)(Q_t) + \bar{f}_z(t)Q_t - \bar{b}_z(t)p_t + \bar{\sigma}_z(t)(q_t) - \bar{\sigma}_z(t)q_t - \bar{h}_z(t) - \bar{h}_z(t) \right) dW_t, \\
pt = g_x(X^u_t) - \Phi_x(X^u_t)Q_T, \quad q_0 = -\gamma_y(Y^u(0)),
\end{eqnarray}
\]

in which we use the notation \( \tilde{\psi}(t) = \psi(t, (X^u_t)^\omega, Y^u_t, Z^u_t) \) for \( \psi = b, \sigma, f, h \). When the coefficients \( b, \sigma \) and \( f \) do not depend explicitly on \( \omega' \), the adjoint equation (36) reduces to the standard adjoint equation (see Shi and Wu [21]) corresponding to fully coupled FBSDE.

On the other hand, from the assumption (A4) and the fact that (34) satisfies (H4) and (H5), we can easily verify that this adjoint equation (36) satisfies (H4) and (H6). Then, from Theorem 3.2, we know that (36) has a unique \( \mathbb{F} \)-adapted solution \((Q, p, q)\) such that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |Q(t)|^2 + \sup_{0 \leq t \leq T} |p(t)|^2 + \int_0^T |q(t)|^2 dt \right] < +\infty.
\]

Define the Hamiltonian function as

\[
H(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, p, q, Q, v) = b(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v)p + \sigma(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v)q \\
- f(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v)Q + h(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v). \quad (37)
\]
Theorem 17. (SMP in Integral Form). Under assumptions (A4), if \( u(\cdot) \) is an optimal control with state trajectory \( (X^u(\cdot), Y^u(\cdot), Z^u(\cdot)) \), then there exists a pair \((p(\cdot), q(\cdot), Q(\cdot))\) of adapted processes which satisfies (36), such that

\[
\mathbb{E} \int_0^T \mathbb{E}'[H_u(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, p_t, q_t, Q_t, u(t))(v(t) - u(t))]dt \geq 0,
\]

\( \mathbb{P}\text{-a.s., for all } t \in [0, T] \).

Theorem 18. (Sufficient Conditions for the Optimality of the Control) Assume the condition (A4) is satisfied and let \( u(\cdot) \in \mathcal{U} \) with state trajectory \( (X^u_t, Y^u_t, Z^u_t) \) be given such that there exist solutions \((p(\cdot), q(\cdot), Q(\cdot))\) to the adjoint equation (36). Moreover, suppose the functions \( g, \gamma, \Phi \) are convex and \( H(t, x, \tilde{y}, \tilde{z}, x, y, z, p, Q, v) \) is convex in \((\tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v)\). Then, if

\[
\mathbb{E}'[H(t, X^u_t), (Y^u_t), (Z^u_t), X^u_t, Y^u_t, Z^u_t, p_t, q_t, Q_t, u(t)]
\]

\[
= \min_{v \in \mathcal{U}} \mathbb{E}'[\nu_t(u, v)]_t, X^u_t, Y^u_t, Z^u_t, p_t, q_t, Q_t, v],
\]

for all \( t \in [0, T], \mathbb{P}\text{-a.s.}, u \) is an optimal control of problem (34)-(35).

6 Maximum principle for mean-field stochastic games of FBSDEs

In this section, we consider a class of non-zero sum differential games where state variables are described by the system of FBSDEs of mean-field type. Our objective is to derive necessary conditions for optimality in the form of a stochastic maximum principle and the corresponding verification theorem.

We always use the subscript 1 (respectively, subscript 2) to characterize the variables corresponding to Player 1 (respectively, Player 2).

Let action space \( U_i \) be a non-empty, closed and convex subset of \( \mathbb{R}^k \) \((i = 1, 2, k \in \mathbb{N}^+)\). The admissible control set is defined as

\[
\mathcal{U}_i = \{v_i \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^k)|v_i \in U_i, t \in [0, T]\} \quad (i = 1, 2).
\]

For any \( v_i(\cdot) \in \mathcal{U}_i \) \((i = 1, 2)\), we consider the following mean-field FBSDE:

\[
\begin{aligned}
&\begin{cases}
    dX_t = \mathbb{E}'[b(t, X'_t, X_t, v_1(t), v_2(t))]dt + \mathbb{E}'[\sigma(t, X'_t, X_t, v_1(t), v_2(t))]dW_t, \\
    X(0) = x_0, \\
    -dY_t = \mathbb{E}'[f(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))]dt - Z_tdW_t, \\
    Y_T = \Phi(X_T),
  \end{cases}
\end{aligned}
\]

(39)
The given functions
\[ \Phi(\cdot) \]
where
\[ \minimizing \text{ the following expected cost functionals:} \]
\[ J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E} \int_0^T \mathbb{E}'[h_i(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))] dt \]
\[ + \mathbb{E}\left(g_i(X_T) + \gamma_i(Y(0))\right), \] (40)
where
\[ g_i : \mathbb{R} \rightarrow \mathbb{R} \quad (i = 1, 2), \]
\[ \gamma_i : \mathbb{R} \rightarrow \mathbb{R} \quad (i = 1, 2), \]
\[ h_i : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R} \quad (i = 1, 2). \]

Ensuring to achieve the goal \( \Phi(x_T) \), Player \( i \) \( (i = 1, 2) \), who has his own benefits, aims at minimizing the following expected cost functionals:

\[ J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E} \int_0^T \mathbb{E}'[h_i(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))] dt \]
\[ + \mathbb{E}\left(g_i(X_T) + \gamma_i(Y(0))\right), \] (40)

Suppose each player hopes to minimize her/his cost functional \( J_i(v_1(\cdot), v_2(\cdot)) \) by selecting an appropriate admissible control \( v_i(\cdot) \) \( (i = 1, 2) \). The problem is then to find a pair of admissible controls \((u_1(\cdot), u_2(\cdot)) \in U_1 \times U_2\), called a Nash equilibrium point for the non-zero sum game, such that

\[ \begin{align*}
J_1(u_1(\cdot), u_2(\cdot)) &= \min_{v_1(\cdot) \in \mathcal{L}_u} J_1(v_1(\cdot), u_2(\cdot)) \\
J_2(u_1(\cdot), u_2(\cdot)) &= \min_{v_2(\cdot) \in \mathcal{L}_u} J_2(u_1(\cdot), v_2(\cdot)) \end{align*} \] (41)

We call the problem above a forward-backward non-zero sum stochastic differential game of mean-field type, where the word “forward-backward” means that the game system is described by a FBSDE and the reason for calling “mean-field” is the coefficients of the state equation and cost functionals depend on the law of the state process. For simplicity, we denote it by Problem (FBNM).

We assume that the following hypothesis holds.

(A5) (i) The given functions \( b(t, \bar{x}, x, v_1, v_2), \sigma(t, \bar{x}, x, v_1, v_2), f(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v_1, v_2), \Phi(x), \]
\[ h_i(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v_1, v_2), g_i(x) \] and \( \gamma_i(y) \) \( (i = 1, 2) \) are continuously differentiable with respect to all of the components in these functions.

(ii) All the derivatives in (i) are Lipschitz continuous and bounded.

For any admissible controls \( v_1(\cdot) \) and \( v_2(\cdot) \), we suppose that (A5) hold. Then we know mean-field FBSDE (39) admits a unique solution \((x^{v_1,v_2}(\cdot), y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot)) \) by Lemma 2.2 and Lemma 2.3, which is called the corresponding trajectory.
6.1 A Pontryagin’s stochastic maximum principle

Let \((u_1(\cdot), u_2(\cdot))\) be a Nash equilibrium point of Problem (FBNM) and \((X(\cdot), Y(\cdot), Z(\cdot))\) be the corresponding state trajectory of game system. For any given \(v_i(\cdot) \in \mathcal{U}_i \ (i = 1, 2)\), since \(\mathcal{U}_i\) is convex, then \(u^*_i(\cdot) = u_i(\cdot) + \theta(v_i(\cdot) - u_i(\cdot)) \in \mathcal{U}_i \ (i = 1, 2), \ \forall \theta \in [0, 1]\).

We introduce the short-hand notation which will be in force in this section

\[
\begin{align*}
\bar{b}(t) &= b(t, X'_t, X_t, u_1(t), u_2(t)), \\
\bar{\sigma}(t) &= \sigma(t, X'_t, X_t, u_1(t), u_2(t)), \\
\bar{f}(t) &= f(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, u_1(t), u_2(t)), \\
\bar{h}_i(t) &= h_i(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, u_1(t), u_2(t)).
\end{align*}
\]

Let \((k(\cdot), m(\cdot), n(\cdot))\) be the solution of the following variational equation which is a linear mean-field FBSDE:

\[
\begin{bmatrix}
dk_t = \mathbb{E}'\left[\bar{b}_x(t)(k_t) + \bar{b}_x(t)k_t + \bar{b}_v(t)(v_1(t) - u_1(t)) + \bar{b}_v(t)(v_2(t) - u_2(t))\right]dt \\
+ \mathbb{E}'\left[\bar{\sigma}_x(t)(k_t) + \bar{\sigma}_x(t)k_t + \bar{\sigma}_v(t)(v_1(t) - u_1(t)) + \bar{\sigma}_v(t)(v_2(t) - u_2(t))\right]dW_t,

dm_t = -\mathbb{E}'\left[\bar{f}_x(t)(k_t) + \bar{f}_x(t)k_t + \bar{f}_v(t)(m_t) + \bar{f}_v(t)(n_t) + \bar{f}_z(t)(n_t)\right]dt + n_tdW_t,

dm_t = k_T\Phi_x(X_T),
\end{bmatrix}
\]

The adjoint equation corresponding to state trajectory \((X^{v_1,v_2}(\cdot), Y^{v_1,v_2}(\cdot), Z^{v_1,v_2}(\cdot)),\) which is a mean-field FBSDE and whose solution is denoted by \((p^{v_1,v_2}_i(\cdot), q^{v_1,v_2}_i(\cdot), Q^{v_1,v_2}_i(\cdot)),\) satisfies

\[
\begin{bmatrix}
-dp^{v_1,v_2}_i(t) = \mathbb{E}'\left(\bar{b}_x(t)(p^{v_1,v_2}_i(t)) + \bar{b}_v(t)q^{v_1,v_2}_i(t) + \bar{\sigma}_x(t)(q^{v_1,v_2}_i(t))' + \bar{\sigma}_v(t)q^{v_1,v_2}_i(t)\right)dt \\
+ \mathbb{E}'\left(\bar{h}_{iz}(t) + \bar{h}_{iz}(t) - \bar{h}_z(t)(Q^{v_1,v_2}_i(t))' - \bar{h}_z(t)Q^{v_1,v_2}_i(t)\right)dt - q^{v_1,v_2}_i(t)dt,

dQ^{v_1,v_2}_i(t) = \mathbb{E}'\left(\bar{f}_y(t)(Q^{v_1,v_2}_i(t)) + \bar{f}_z(t)(Q^{v_1,v_2}_i(t)) - \bar{h}_{iz}(t) + \bar{h}_z(t)\right)dt + \mathbb{E}'\left(\bar{f}_z(t)(Q^{v_1,v_2}_i(t)) + \bar{f}_z(t)(Q^{v_1,v_2}_i(t)) - \bar{h}_{iz}(t) + \bar{h}_z(t)\right)dt,dW_t,
\end{bmatrix}
\]

\(p^{v_1,v_2}_i(T) = g_x(X_T) - \Phi_x(X_T)Q^{v_1,v_2}_i(T), \ Q_0 = -\gamma_y(Y(0)),\)

where \(h_{iz}\) denotes the partial derivatives of \(h_i\) with respect to \(x\).

By (A5) and Lemma 2.2, we can easily verify that the linear FBSDE of mean-field type [2] admits a unique solution \((p^{v_1,v_2}_i(\cdot), q^{v_1,v_2}_i(\cdot), Q^{v_1,v_2}_i(\cdot)),\)

The Hamiltonian function associated with random variables is defined as follows:

\[
H_i(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, p_i, q_i, Q_i, v_1, v_2) = p_i(t)b(t, \bar{x}, x, v_1, v_2) + q_i(t)\sigma(t, \bar{x}, x, v_1, v_2) \\
- f(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v_1, v_2)Q_i(t) \\
+ h_i(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, v_1, v_2).
\]
Fix \( u_2(\cdot) \) (respectively, \( u_1(\cdot) \)), to minimize the cost functional \( J_1(v_1(\cdot), u_2(\cdot)) \) (respectively, \( J_2(u_1(\cdot), v_2(\cdot)) \)) subject to \((39)\) over \( \mathcal{U}_1 \) (respectively, \( \mathcal{U}_2 \)) is an optimal control problem of mean-field FBSDEs. Following the idea developed in Section 4, it is not difficult to analyze the game problem. Thus, we omit the detailed deduction and only state the main result for simplicity.

**Theorem 19.** (Stochastic Maximum Principle for SDGs) Suppose (A5) hold. Let \((u_1(\cdot), u_2(\cdot))\) be a Nash equilibrium point for our stochastic game problem (FBNM), \((X(\cdot), Y(\cdot), Z(\cdot))\) be the corresponding trajectory and \((p_i^{u_1,u_2}(\cdot), q_i^{u_1,u_2}(\cdot), Q_i^{u_1,u_2}(\cdot))\) be the solution of adjoint equation \((43)\). Then we have

\[
\mathbb{E}\int_0^T \mathbb{E}'[H_{v_1}(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_1(t), q_1(t), Q_1(t), u_1(t), u_2(t))(v_1(t) - u_1(t))] dt \geq 0, \\
\mathbb{E}\int_0^T \mathbb{E}'[H_{v_2}(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_2(t), q_2(t), Q_2(t), u_1(t), u_2(t))(v_2(t) - u_2(t))] dt \geq 0, \\
\forall (v_1, v_2) \in U_1 \times U_2, \quad a.e. \in [0,T], \quad a.s.,
\]

where the Hamiltonian function \( H_i \) is defined by \((43)\).

6.2 **Sufficient conditions for maximum principle**

We will establish the sufficient maximum principle (also called verification theorem) of Problem (FBNM).

**Theorem 20.** (Sufficient Conditions for the equilibrium point of Problem (FBNM)) Let (A5) hold and suppose that \((u_1(\cdot), u_2(\cdot))\) \(\in\) \( U_1 \times U_2 \) with state trajectory \((X_t, Y_t, Z_t)\) satisfies:

\[
\mathbb{E}'[H_1(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_1^{u_1,u_2}(t), q_1^{u_1,u_2}(t), Q_1^{u_1,u_2}(t), u_1(t), u_2(t))] = \min_{v_1 \in U_1} \mathbb{E}'[H_1(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_1^{u_1,u_2}(t), q_1^{u_1,u_2}(t), Q_1^{u_1,u_2}(t), v_1, u_2(t))],
\]

\[
\mathbb{E}'[H_2(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_2^{u_1,u_2}(t), q_2^{u_1,u_2}(t), Q_2^{u_1,u_2}(t), u_1(t), u_2(t))] = \min_{v_2 \in U_2} \mathbb{E}'[H_2(t, X_t', Y_t', Z_t', X_t, Y_t, Z_t, p_2^{u_1,u_2}(t), q_2^{u_1,u_2}(t), Q_2^{u_1,u_2}(t), v_2, u_1(t))],
\]

for all \( t \in [0,T] \), where \( p_i^{u_1,u_2}(t), q_i^{u_1,u_2}(t), Q_i^{u_1,u_2}(t) \) is the solution of adjoint equation \((42)\).

We further assume that the functions \( \Phi(x), g_i(x), \gamma_i(y) \) and Hamiltonian function \( H_i \ (i = 1, 2) \) are convex in \( (\tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v_1, v_2) \). Then, \((u_1(\cdot), u_2(\cdot))\) is an equilibrium point of problem \((39)-(47)\).

**Proof.** From the proof of Theorem 16, we affirm that

\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq 0
\]

holds for any \( v_1(\cdot) \in \mathcal{U}_1 \), and

\[
J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in \mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot))
\]

holds for any \( v_2(\cdot) \in \mathcal{U}_2 \). Hence, we draw the desired conclusion. \(\square\)
Remark 21. Note that if Eq. (39) does not include the “forward” part and without the influence of $\omega'$, then the stochastic game problem and the corresponding conclusion reduce to the case introduced by Wang and Yu \[22\].

7 Maximum principle for Mean-field stochastic games of fully coupled FBSDEs

In this section, we study the mean-field games of fully coupled FBSDEs. That is, the state equation is characterized by following fully coupled FBSDEs:

$$
\begin{align*}
\frac{dX_t}{dt} &= \mathbb{E}'[b(t, X'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))]dt + \mathbb{E}'[\sigma(t, X'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))]dW_t, \\
X(0) &= x_0, \\
-dY'_t &= \mathbb{E}'[f(t, X'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))]dt - Z_t dW_t, \\
Y_T &= \Phi(X_T),
\end{align*}
$$

(44)

where

$$
\begin{align*}
b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}, \\
\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}^d, \\
f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}, \\
\Phi : \mathbb{R} \to \mathbb{R}.
\end{align*}
$$

For any admissible $v_i(\cdot) \in U_i$ ($i = 1, 2$), if conditions (H4) and (H5) hold, the fully coupled mean-field FBSDE (44) has a unique $\mathbb{F}$-adapted solution $(X^{v_1,v_2}(\cdot), Y^{v_1,v_2}(\cdot), Z^{v_1,v_2}(\cdot))$ according to Theorem 3.1.

For Player $i$ ($i = 1, 2$), the expected cost functionals is defined as follows:

$$
J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E} \int_0^T \mathbb{E}'[h_i(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, v_1(t), v_2(t))]dt + \mathbb{E}(g_i(X_T) + \gamma_i(Y(0)))
$$

(45)

where

$$
\begin{align*}
g_i : \mathbb{R} \to \mathbb{R} \ (i = 1, 2), \\
\gamma_i : \mathbb{R} \to \mathbb{R} \ (i = 1, 2), \\
h_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times U_1 \times U_2 \to \mathbb{R}, \ (i = 1, 2).
\end{align*}
$$

Each player, having the same goal $\Phi(X_T)$, aims at minimizing her/his cost functional $J_i(v_1(\cdot), v_2(\cdot))$ by selecting an appropriate admissible control $v_i(\cdot) \in U_i$ ($i = 1, 2$). The problem is to find a Nash equilibrium point $(u_1(\cdot), u_2(\cdot)) \in U_1 \times U_2$ for the non-zero sum game, such that

$$
\begin{align*}
J_1(u_1(\cdot), u_2(\cdot)) &= \min_{v_1(\cdot) \in U_1} J_1(v_1(\cdot), u_2(\cdot)), \\
J_2(u_1(\cdot), u_2(\cdot)) &= \min_{v_2(\cdot) \in U_2} J_2(u_1(\cdot), v_2(\cdot)).
\end{align*}
$$

(46)
For simplicity, we denote the problem above by Problem (CFBNM).

In order to give the maximum principle, we assume that the following hypothesis holds.

\begin{equation}
\begin{aligned}
& \text{(A6)} \\
& \text{(i) } b, \sigma, f, \Phi, h_i, g_i \text{ and } \gamma_i \text{ are continuously differentiable;} \\
& \text{(ii) The derivatives of } b, \sigma, f \text{ and } \Phi \text{ are bounded;} \\
& \text{(iii) The derivatives of } h_i \text{ are bounded by } C(1 + |x| + |y| + |z| + |x| + |y| + |z|); \\
& \text{(iv) The derivatives of } g_i \text{ and } \gamma_i \text{ with respect to } x \text{ and } y \text{ are bounded by } C(1 + |x|) \\
& \quad \text{and } C(1 + |y|) \text{ respectively;} \\
& \text{(v) For any given pair of control } (v_1(\cdot), v_2(\cdot)), \text{ equation } (44) \text{ satisfies (H4) and (H5).}
\end{aligned}
\end{equation}

Let \((u_1(\cdot), u_2(\cdot))\) be a Nash equilibrium point of Problem (CFBNM) and let \((X(\cdot), Y(\cdot), Z(\cdot))\) be the corresponding trajectory of game system. In this fully coupled case, the adjoint equation, different from the case in Section 6, has the form: for \(i = 1, 2,\)

\begin{equation}
\begin{aligned}
- dp_i(t) &= \mathbb{E}'\left(\tilde{b}_x(t)(p_i(t))' + \tilde{b}_x(t)p_i(t) + \tilde{\sigma}_x(t)(q_i(t))' + \tilde{\sigma}_x(t)q_i(t)\right)dt \\
& \quad + \mathbb{E}'\left(\tilde{h}_{ix}(t) + \tilde{h}_{ix}(t) - \tilde{f}_x(t)(Q_i(t))' - \tilde{f}_x(t)Q_i(t)\right)dt - q_i(t)dW_t, \\
\text{d}Q_i(t) &= \mathbb{E}'\left(\tilde{f}_y(t)(Q_i(t))' + \tilde{f}_y(t)Q_i(t) - \tilde{\sigma}_y(t)(p_i(t))' - \tilde{\sigma}_y(t)p_i(t) - \tilde{\sigma}_y(t)(q_i(t))' \right. \\
& \quad - \tilde{\sigma}_y(t)q_i(t) - \tilde{h}_{iy}(t) - \tilde{h}_{iy}(t)\right)dt + \mathbb{E}'\left(\tilde{f}_x(t)(Q_i(t))' + \tilde{f}_x(t)Q_i(t) - \tilde{\sigma}_x(t)(q_i(t))' \right. \\
& \quad - \tilde{\sigma}_x(t)q_i(t) - \tilde{h}_{ix}(t) - \tilde{h}_{ix}(t)\right)dW_t, \\
& \quad p_i(T) = g_{ix}(X_T) - \Phi_x(X_T)Q_i(T), \ Q_i(0) = -\gamma_{iy}(Y(0)),
\end{aligned}
\end{equation}

with \(\tilde{\psi} = \psi(t, X_i', Y_i', Z_i', X_t, Y_t, Z_t, u_1(t), u_2(t)), \) for \(\psi = b, \sigma, f, h_1, h_2.\)

This is a linear fully coupled mean-field FBSDE with bounded coefficients under assumption (A6). It is easy to know that the adjoint equation (47) satisfies (H4) and (H6) since condition (A6) and equation (44) satisfying (H4) and (H5). From Theorem 3.2, this equation has a unique \(\mathbb{F}\)-adapted solution \((p_i(\cdot), q_i(\cdot), Q_i(\cdot))\) such that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |Q_i(t)|^2 + \sup_{0 \leq t \leq T} |p_i(t)|^2 + \int_0^T |q_i(t)|^2 dt \right] < +\infty, \quad (i = 1, 2).
\]

We define the Hamiltonian function \(H_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\)

\begin{equation}
\begin{aligned}
H_i(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, p_i, q_i, Q_i, v_1, v_2) &= p_i(t)b(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v_1, v_2) + q_i(t)\sigma(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v_1, v_2) \\
& \quad - f(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v_1, v_2)Q_i(t) + h_i(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, v_1, v_2), \quad (i = 1, 2).
\end{aligned}
\end{equation}

The proof of the maximum principle and verification theorem in this case is practically similar to Section 5. Thus we present these theorems without proof.

**Theorem 22.** (Stochastic Maximum Principle for SDGs of coupled FBSDEs) Let (A6) hold. If \((u_1(\cdot), u_2(\cdot))\) is a Nash equilibrium point of Problem (CFBNM) and \((X(\cdot), Y(\cdot), Z(\cdot))\)
denotes the corresponding trajectory, then for any \((v_1, v_2) \in U_1 \times U_2\), the following maximum principle
\[
\mathbb{E} \int_0^T \mathbb{E}' \left[ H_{1v_1}(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_1(t), \rho_2(t), u_1(t), u_2(t)) (v_1(t) - u_1(t)) \right] dt \geq 0,
\]
\[
\mathbb{E} \int_0^T \mathbb{E}' \left[ H_{2v_2}(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_2(t), u_1(t), u_2(t)) (v_2(t) - u_2(t)) \right] dt \geq 0,
\]
hold a.s. a.e., where \((\rho_i(\cdot), \rho_1(\cdot), \rho_2(\cdot)) (i = 1, 2)\) is the solution of the adjoint equation \((47)\) and the Hamiltonian function \(H_i (i = 1, 2)\) is defined by \((48)\).

**Theorem 23.** (Sufficient Conditions for the Problem (CFBNM)) Assume that the condition \((A6)\) is satisfied. Let \((u_1(\cdot), u_2(\cdot)) \in U_1 \times U_2\) and \((X_t, Y_t, Z_t)\) be the corresponding state trajectory. Suppose \((\rho_i(\cdot), \rho_1(\cdot), \rho_2(\cdot)) (i = 1, 2)\) is the solution of linear mean-field FBSDE \((47)\). Moreover, we assume functions \(F, g_i (i = 1, 2)\) are convex in \(x\), \(\gamma_i (i = 1, 2)\) is convex in \(y\) and function \(H_i(t, \hat{x}, \hat{y}, \hat{z}, x, y, z, \rho_i, \rho_1, \rho_2, v_1, v_2) (i = 1, 2)\) is convex with respect to \((\hat{x}, \hat{y}, \hat{z}, x, y, z, v_1, v_2)\). Then, if
\[
\mathbb{E}'[H_1(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_1(t), \rho_2(t), u_1(t), u_2(t))]
= \min_{v_1 \in U_1} \mathbb{E}'[H_1(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_1(t), \rho_2(t), v_1(t), u_2(t))]
\]
\[
\mathbb{E}'[H_2(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_2(t), u_1(t), u_2(t))]
= \min_{v_2 \in U_2} \mathbb{E}'[H_2(t, X'_t, Y'_t, Z'_t, X_t, Y_t, Z_t, \rho(t), \rho_t(t), \rho_2(t), v_2(t), Q_2(t), u_1(t), v_2(t))]
\]
hold for all \(t \in [0, T]\), \((u_1(\cdot), u_2(\cdot))\) is an equilibrium point of Problem (CFBNM).

### 8 Applications: Linear-Quadratic Case

In this section, we give two LQ examples to illustrate our theoretical results.

**Example 24.** For notational simplicity, we consider the following one-dimensional stochastic control problem. Our aim is to search for the admissible control \(u(\cdot)\) minimizing
\[
J(v(\cdot)) = \frac{1}{2} \mathbb{E} \left[ \int_0^T v^2(t) dt + X_T^2 + Y_T^2 \right], \tag{49}
\]
subject to the following FBSDE:
\[
\begin{align*}
dX_t &= \left[ \hat{A}(t) \mathbb{E}[X_t] + A(t) X_t + B(t) v(t) \right] dt + \left[ \hat{C}(t) \mathbb{E}[X_t] + C(t) X_t + D(t) v(t) \right] dW_t, \\
-dY_t &= \left[ \hat{a}(t) \mathbb{E}[X_t] + a(t) X_t + \hat{b}(t) \mathbb{E}[Y_t] + b(t) Y_t + \hat{\beta}(t) \mathbb{E}[Z_t] + \beta(t) Z_t + E(t) v(t) \right] dt - Z_t dW_t, \\
X_0 &= a, \quad Y_T = X_T, \quad t \in [0, T],
\end{align*}
\tag{50}
\]
where $\tilde{A}(\cdot)$, $A(\cdot)$, $B(\cdot)$, $\tilde{C}(\cdot)$, $C(\cdot)$, $D(\cdot)$, $\tilde{a}(\cdot)$, $a(\cdot)$, $\tilde{b}(\cdot)$, $b(\cdot)$, $\tilde{\beta}(\cdot)$, $\beta(\cdot)$ and $E(\cdot)$ are bounded and deterministic, and $v(t)$, $0 \leq t \leq T$ takes value in $\mathbb{R}$.

In this process, the Hamiltonian function is in the form of

$$H(t, \tilde{x}, \tilde{y}, \tilde{z}, x, y, z, p, q, Q, v) = p \left[ \tilde{A}(t) \tilde{x} + A(t) x + B(t) v \right] + q \left[ \tilde{C}(t) \tilde{x} + C(t) x + D(t) v \right]$$

$$- Q \left[ \tilde{a}(t) \tilde{x} + a(t) x + \tilde{b}(t) \tilde{y} + b(t) y + \tilde{\beta}(t) \tilde{z} + \beta(t) z + E(t) v \right] + \frac{1}{2} v^2, \quad (51)$$

where $(p(\cdot), q(\cdot), Q(\cdot))$ satisfies

$$\begin{cases}
  dQ_t = \left( \tilde{b}(t) E[Q_t] + b(t) Q_t \right) dt + \left( \tilde{\beta}(t) E[Q_t] + \beta(t) Q_t \right) dW_t \\
  -dp_t = \left( \tilde{A}(t) E[p_t] + A(t) p_t + \tilde{C}(t) E[q_t] + C(t) q_t - \tilde{a}(t) E[Q_t] - a(t) Q_t \right) dt - q_t dW_t,
\end{cases}$$

$$Q_0 = -Y_0, \quad P_T = X_T - Q_T.$$  

If $u(\cdot)$ is optimal, then it follows from Theorem 5.1 and (51) that

$$u(t) = Q_tE(t) - p_t B(t) - q_t D(t), \quad t \in [0, T]. \quad (52)$$

Moreover, it is easy to check that candidate optimal control $(52)$ is really the optimal control since the coefficients of Eq (50) and cost functional (49) satisfy the assumptions of Theorem 4.2.

**Example 25.** Let us consider the following forward-backward stochastic control system:

$$dX_t = \left[ \tilde{b}(t) E[X_t] + b(t) X_t + \tilde{A}(t) E[Y_t] + A(t) Y_t + \tilde{B}(t) E[Z_t] + B(t) Z_t + D(t) v(t) \right] dt$$

$$+ \left[ \tilde{\beta}(t) E[X_t] + \beta(t) X_t - \tilde{B}(t) E[Y_t] - B(t) Y_t + \tilde{C}(t) E[Z_t] + C(t) Z_t + E(t) v(t) \right] dW_t,$$

$$-dY_t = \left[ \tilde{a}(t) E[X_t] + a(t) X_t + \tilde{b}(t) E[Y_t] + b(t) Y_t + \tilde{\beta}(t) E[Z_t] + \beta(t) Z_t + G(t) v(t) \right] dt - Z_t dW_t,$$

$$X_0 = a, \quad Y_T = RX_T, \quad t \in [0, T], \quad (53)$$

where $R > 0$ is a constant and $v \in L^2_2(0, T; U)$. For simplicity we also suppose that $U = \mathbb{R}$.

Functions $\tilde{a}(\cdot) > 0$, $a(\cdot) > 0$, $\tilde{A}(\cdot) < 0$, $\tilde{C}(\cdot) < 0$, $A(\cdot) < 0$, $C(\cdot) < 0$, $\tilde{B}(\cdot)$, $b(\cdot)$, $\tilde{\beta}(\cdot)$, $\beta(\cdot)$, $B(\cdot)$, $D(\cdot)$, $E(\cdot)$, $G(\cdot)$, $b(\cdot)$ and $\beta(\cdot)$ are bounded and deterministic. For any given $v(\cdot)$, it is easy to show that condition (H4) and monotonic condition (H5) hold. Then from Theorem 7, the fully coupled Mean-field FBSDEs (53) has a unique solution $(X(\cdot), Y(\cdot), Z(\cdot))$.

The cost functional is

$$J(v(\cdot)) = \frac{1}{2} E \int_0^T \left[ L(t)v^2(t) \right] dt + E[MX_T^2 + NY_0^2], \quad (54)$$
where constants $M > 0$, $N > 0$. Function $L(\cdot)$ is deterministic and bounded, and $L^{-1}$ is also bounded. By (37), the Hamiltonian function is given by

$$H(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, p, q, Q, v) = p \left[ \bar{b}(t) \bar{x} + b(t)x + \bar{A}(t)\bar{y} + A(t)y + \bar{B}(t)\bar{z} + B(t)z + D(t)v \right] + q \left[ \bar{\beta}(t) \bar{x} + \beta(t)x - \bar{B}(t)\bar{y} - B(t)y + \bar{C}(t)\bar{y} + C(t)z + E(t)v \right] - Q \left[ \bar{a}(t) \bar{x} + a(t)x + \bar{b}(t)\bar{y} + b(t)y + \bar{\beta}(t)\bar{z} + \beta(t)z + G(t)v \right] + \frac{1}{2} L(t)v^2.$$

According to Theorem 17, if $u(\cdot)$ is optimal, then

$$u(t) = -L^{-1}(t)(p_t D(t) + q_t E(t) - Q_t G(t)), \quad 0 \leq t \leq T,$$

where $(p(\cdot), q(\cdot), Q(\cdot))$ is the solution of the following fully coupled Mean-field FBSDEs

$$\begin{cases}
    dQ_t = \left( \bar{b}(t)\mathbb{E}[Q_t] + b(t)Q_t - \bar{A}(t)\mathbb{E}[p_t] - A(t)p_t + \bar{B}(t)\mathbb{E}[q_t] + B(t)q_t \right) dt \\
    \quad \quad \quad \quad + \left( \bar{\beta}(t)\mathbb{E}[Q_t] + \beta(t)Q_t - \bar{B}(t)\mathbb{E}[p_t] - B(t)p_t - \bar{C}(t)\mathbb{E}[q_t] - C(t)q_t \right) dW_t \\
    -dp_t = \left( \bar{b}(t)\mathbb{E}[p_t] + b(t)p_t + \bar{\beta}(t)\mathbb{E}[q_t] + \beta(t)q_t - \bar{a}(t)\mathbb{E}[Q_t] - a(t)Q_t \right) dt - q_t dW_t, \\
    Q_0 = -2NY_0, \quad p_T = 2M_TX_T - RQ_T, \quad t \in [0, T].
\end{cases}$$

Similarly, it is easy to verify that the monotonic condition ($H6$) holds, then from Theorem 10, FBSDEs (37) admits a unique solution $(Q(\cdot), p(\cdot), q(\cdot))$.

Moreover, since $g(x) = M_T x^2$, $\gamma(y) = Ny^2$, $\Phi(x) = Rx$ are convex and $H(t, \bar{x}, \bar{y}, \bar{z}, x, y, z, p, q, Q, v)$ is convex in $(\bar{x}, \bar{y}, \bar{z}, x, y, z, v)$, we can know that the admissible control (55) which satisfying the necessary condition of optimality is really an optimal control.

References

[1] Y. Hu, S. Peng: Solution of forward-backward stochastic differential equations, Probab. Theory Relat. Fields 103 (1995) 273-283.

[2] S. Peng: Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stochastics, 37 (1991), 61-74.

[3] D. Duffie, L. Epstein: Asset pricing with stochastic differential utilities, Rev. Financial Stud, 5 (1992), 411-436.

[4] S. Peng, Z. Wu: Fully coupled forward-backward stochastic differential equations and applications to the optimal control, SIAM J. Control Optim. 37(3), (1999) 825-843
[5] J. Shi, Z. Wu: Maximum principle for partially-observed Optimal control of fully-coupled forward-backward stochastic systems, J Optim Theory Appl (2010) 145: 543-578

[6] F. Antonelli: Backward-forward stochastic differential equations, Ann. Appl. Probab. 3, (1993) 777-793.

[7] J. Ma, P. Protter, J. Yong: Solving forward-backward stochastic differential equations explicitly-a four step scheme. Probab. Theory Relat. Fields 98, (1994) 339-359

[8] J. Yong.: Finding adapted solutions of forward-backward stochastic differential equations: method of continuation. Probab. Theory Relat. Fields 107, (1997) 537-572

[9] E. Pardoux, S. Tang: Forward-backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Relat. Fields 114, (1999) 123-150

[10] F. Delarue: On the Existence and Uniqueness of Solutions to FBSDEs in a Non-Degenerate Case. Stoch. Process. Appl. 99, (2002) 209-286

[11] J. Zhang: The wellposedness of FBSDEs. Discrete Contin. Dyn. Syst., Ser. B 6, (2006) 927-940

[12] R. Buckdahn, B. Djehiche, J. Li, S. Peng: Mean-field backward stochastic differential equations. A limit approach, Ann. Probab. 37 (4) (2009) 1524-1565.

[13] R. Buckdahn, J. Li, S. Peng: Mean-field backward stochastic differential equations and related partial differential equations, Stoch. Process. Appl. 119 (10) (2009) 3133-3154.

[14] J.M. Lasry, P.L. Lions: Mean field games, Japan. J. Math. 2 (2007) 229-260.

[15] D. Andersson, B. Djehiche: A Maximum Principle for SDEs of Mean-Field Type, Appl Math Optim. 63 (2011) 341-356.

[16] R. Buckdahn, B. Djehiche, J. Li: A General Stochastic Maximum Principle for SDEs of Mean-Field Type, Appl Math Optim. (2011).

[17] T. Meyer-Brandis, B. Øsendal, X.Y. Zhou: A mean-field stochastic maximum principle via Malliavin calculus. (A Special issue for Mark Davis, Festschrift) (2010).

[18] J. Li: Stochastic maximum principle in the mean-field controls, Automatica 48(2012) 366-373.

[19] A. Bensoussan: Lectures on stochastic control. In: Mitter, S.K., Moro, A. (eds.) Nonlinear Filtering and Stochastic Control. Springer Lecture Notes in Mathematics, vol. 972. Springer, Berlin (1982)
[20] S. Peng: Backward stochastic differential equations and application to optimal control. Applied Mathematics and Optimization, 27(4) (1993) 125-144

[21] J. Shi, Z. Wu: The maximum principle for fully coupled forward-backward stochastic control system. Acta Autom Sin, 32: (2006) 161-169

[22] G. Wang, Z. Yu: A Pontryagin’s Maximum Principle for Non-Zero Sum Differential Games of BSDEs with Applications. IEEE Transactions on Automatic control, 55 (7), (2010) 1742-1747.

[23] R. Isaacs: Differential Games, Wiley, New York, 1965.

[24] W. H. Fleming and P. E. Souganidis: On the existence of value functions of two-player, zero-sum stochastic differential games, Indiana Univ. Math. J., 38 (1989), 293-314.

[25] A. Friedman: Differential Games, Wiley, New York, 1971.

[26] L. C. Evans and P. E. Souganidis: Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations, Indiana Univ. Math. J., 33 (1984), 773-797.

[27] S. Hamadène: Nonzero-sum linear-quadratic stochastic differential games and backward-forward equations, Stochastic Anal. Appl., vol. 17, (1999) 117-130.

[28] A. E. B. Lim and X. Zhou: Risk-sensitive control with HARA utility, IEEE Trans. Autom. Control, vol. 46, no. 4, (2001) 563-578.

[29] E. Altman: Applications of dynamic games in queues, in Advances in Dynamic Games. Boston, MA: Birkhauser, vol. 7, (2005) 309-342.

[30] N. Ikeda, S. Watanabe: Stochastic differential equations and diffusion processes. Amsterdam-Tokyo: North Holland-Kodansha. (1989)

[31] I. Karatzas, S. E. Shreve: Brownian motion and stochastic calculus. Springer, (1987)