Glassy Dynamics from Quark Confinement: 
Atomic Quantum Simulation of Gauge-Higgs Model on Lattice

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In the previous works, we proposed atomic quantum simulations of the U(1) gauge-Higgs model by ultra-cold Bose gases. By studying extended Bose-Hubbard models (EBHMs) including long-range repulsions, we clarified the locations of the confinement, Coulomb and Higgs phases. In this paper, we study the EBHM with nearest-neighbor repulsions in one and two dimensions at large fillings by the Gutzwiller variational method. We obtain phase diagrams and investigate dynamical behavior of electric flux from the gauge-theoretical point of view. We also study if the system exhibits glassy quantum dynamics in the absence and presence of quenched disorder. We explain that the obtained results have a natural interpretation in the gauge theory framework. Our results suggest important perspective on many-body localization in strongly-correlated systems. They are also closely related to anomalously slow dynamics observed by recent experiments performed on Rydberg atom chain, and our study indicates existence of similar phenomenon in two-dimensional space.

I. INTRODUCTION

Ultra-cold atomic gas systems are one of the most actively studied subjects in physics these days.\cite{1} By their high-controllability and versatility, the ultra-cold atoms provide an important playground for study on interesting problems in quantum physics. In particular, dynamical properties of the many-body quantum systems can be investigated by controlling physical parameters of the systems. Most of these investigations are beyond the reach of the conventional research methods such as various numerical methods including the Monte-Carlo simulations, density-matrix renormalization group, etc. From this point of view, the ultra-cold atom systems are sometimes called ideal quantum simulators\cite{2,3}.

Among them, numerous interesting studies on quantum simulations of the lattice gauge theory (LGT) have been reported\cite{4-18}. Various setups using internal degrees of freedoms of atoms have been proposed. In these studies, one of the most important point is how to realize the local gauge symmetry in charge-neutral atomic systems. In the previous works\cite{19-22}, we considered single-component Bose gas systems described by an extended Bose-Hubbard model (EBHM)\cite{23}, and show that the U(1) gauge-Higgs model with the exact local gauge symmetry can be quantum simulated by the EBHM. The gauge-Higgs model (GHM) is one of the most fundamental gauge theories\cite{24,25} in not only high-energy physics but also condensed matter physics. The GHM has (at least) two distinct phases, one is the confinement phase and the other is the Higgs phase. In our works, we clarified phase diagrams by using the Monte-Carlo (MC) simulations. Dynamical variables such as the electric field exhibit very different behaviors in the above two phases, and we studied their dynamics by using the Gross-Pitaevskii equations.

In this paper, we continue the above study and investigate the EBHM and GHM by the Gutzwiller (GW) variational method. In particular, we are interested in case of relatively large fillings with the average particle number per site $\rho_0 = 7 \sim 30$, as large filling legitimates the use of the GW variational method and the EBHM-GHM correspondence.

This paper is organized as follows. In Sec. II, we introduce the EBHM and explain how it quantum simulates the GHM on the lattice. We also briefly summarize the previous works. In Sec. III, we show the numerical results for the model in one and two dimensions. We first clarify the phase diagrams of the EBHM, and identify the parameter regions corresponding to the confinement and Higgs phases. Then, we investigate the dynamical behavior of the electric flux put in the central region of the lattice. In the confinement phase, the electric flux is stable although it exhibits string-breaking-like fluctuations. On the other hand in the Higgs phase, it spreads in the empty space and breaks into bits. This result is in good agreement with the previous result obtained by the Gross-Pitaevskii equations. In Sec. IV, we study the robustness of confinement state in the GHM and the effect of the random chemical potential on it. In particular, we observe a kind of glassy dynamics of configurations with a finite synthetic electric field in the confinement phase. This behavior is closely related to anomalously slow dynamics observed by recent experiments performed on Rydberg atom chain\cite{26}, as indicated by Ref.\cite{18}. Then, it is interesting to study the effect of quenched disorder induced by the random chemical potential on the glassy state. We calculate life time of high-energy states with density-wave (DW)-type configurations for various the strength of the disorder. We obtain somewhat ‘unexpected’ results, that it, a weak disorder hinders the glassy state first, whereas further increase of disorder enhances the glassy nature. This means that there exists a critical strength of the disorder at which the glassy nature is hindered maximally. Section V is devoted for discussion and conclusion. We discuss the observed glassy behavior of the confinement phase from the gauge-theoretical point of view, and clarify the origin of the above ‘un-
expected results. We also suggest certain experiments for examining our observation and searching many-body localization (MBL) in ultra-cold gases with a dipole moment.

II. EXTENDED BOSE-HUBBARD MODEL AND GAUGE-HIGGS MODEL

In the previous works [13,22], we showed that the GHM appears as a low-energy effective theory from the EBHM. For the simplicity of the presentation, here we consider the one-dimensional (1D) EBHM, and explain its relation to the GHM. Extension to higher-dimensional cases are rather straightforward although long-range repulsions are necessary. Hamiltonian of the EBHM in 1D is given as follows,

\[
H_{\text{EBH}} = -J \sum_i (\hat{b}_i^\dagger \hat{b}_{i+1} + \hat{b}_{i+1}^\dagger \hat{b}_i) + U \sum_i \hat{n}_i (\hat{n}_i - 1) + V \sum_i \hat{n}_i \hat{n}_{i+1} - \mu \sum_i \hat{n}_i,
\]

where \(\hat{b}_i (\hat{b}_i^\dagger)\) is the boson annihilation (creation) operator at site \(i\), \(\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i\), and \(\mu\) is the chemical potential. The \(U\)-term and \(V\)-term in Eq. (1) are one-site and nearest-neighbor repulsions, respectively. We introduce the density \((\rho_i)\) and phase \((\theta_i)\) operators as follows, \(\hat{\rho}_i = e^{i\theta_i} \sqrt{\rho_i}\). By controlling the chemical potential, we consider the case of relatively large fillings such as \(\rho_0 = \frac{1}{L} \sum_i (\hat{\rho}_i) = (7 \sim 30)\) in this paper, where \(L\) is the linear system size. To relate the boson operator to the gauge field, we introduce a dual lattice with time slice \(\tau\), which corresponds to link \((i,i+1)\) of the original lattice. Artificial electric field, \(E_\tau\), and vector potential, \(A_\tau\), are given by \(E_r = -(\sim)^r (\hat{\rho}_r - \rho_0) \equiv -(\sim)^r \eta_r\), and \(A_{r,1} = -(\sim)^r \hat{\theta}_r\). It is verified that \(E_r\) and \(A_{r,1}\) satisfy the ordinary canonical commutation relations such as \([E_r, A_{r',1}] = -i \hbar d_{r,r'}\). Then, the following Hamiltonian is derived from \(H_{\text{EBH}}\) [Eq. (1)] by ignoring the third or higher-order terms of \(\eta_r\),

\[
H_{\text{GH}} = \sum_r \left[ \frac{V}{2} (E_{r+1} - E_r)^2 + \frac{g^2}{2} E_r^2 - 2J \rho_0 \cos (A_{r+1,1} + A_{r,1}) \right],
\]

\begin{equation}
E_r = \frac{1}{2} \int [dA_\tau] [dE] \exp \left[ \sum_r (\Delta r H_{\text{GH}}) \right],
\end{equation}

where \(g^2 = U - 2V\).

From Eq. (2), the partition function of the system, \(Z\), is given by

\[
Z = \int [dA_\tau] [dE] \times \exp \left[ \sum_r (\Delta r H_{\text{GH}}) \right],
\]

where we have introduced the imaginary time \(\tau\) and the corresponding lattice with time slice \(\Delta \tau\). It is obvious that the Hamiltonian \(H_{\text{GH}}\) in Eq. (2) and the partition function \(Z\) in Eq. (3) are not invariant under a local gauge transformation such as \(A_{r,1} \rightarrow A_{r,1} - \nabla_1 \alpha_i\), where \(\nabla_1 \alpha_i = \alpha_{i+1} - \alpha_i\) \([r = (i+1,i)]\) and \(\{\alpha_i\}\) are arbitrary real parameters at original sites. In Ref. [13], we showed that the system given by Eqs. (2) and (3) can be regarded as the U(1) GHM with the exact local gauge symmetry. In order to express the partition function \(Z\) in a gauge-invariant form, we introduce two-component compact gauge potential on the link \((x,x+\nu)\), \(U_{x,\nu} = e^{iA_{x,\nu}}\) \([x = (x_0,x_1) = (\tau,r)]\) and Higgs field \(\phi_x = e^{i\phi_x}\). Then, we can prove the following equation,

\[
Z = \int [dA_0] [dA_1] [d\phi] \exp[A_{\text{GH}}],
\]

\[
A_{\text{GH}} = A_f + A_P + A_H,
\]

\[
A_f = \frac{1}{2V \Delta \tau} \sum_x \phi_{x+0} U_{x,0} \phi_x + c.c.,
\]

\[
A_P = \frac{1}{2g^2 \Delta \tau} \sum_x \bar{U}_{x,0} \bar{U}_{x+1,0} U_{x+1,0} U_{x,1} + c.c.,
\]

\[
A_H = J \rho_0 \Delta \tau \sum_x \phi_{x+2} U_{x+1,1} U_{x,1} \phi_x + c.c.,
\]

where \(A_f\) is the hopping term of the Higgs field in the \(\tau\)-direction (the kinetic term), \(A_P\) is the plaquette term of the gauge field (the electro-magnetic term), and \(A_H\) is the spatial hopping term of the Higgs field. The time-component of the gauge field \(A_{x,0}\) has been introduced as an auxiliary field in order to perform the integration over the electric field \(E_r\). It is easy to show that the system described by Eq. (2) is gauge-invariant. By fixing the gauge freedom with the gauge condition such as \(\phi_x = 1\), which is so-called unitary gauge, the system Eq. (4) reduces to the one derived from the original system Eq. (4) by integrating out \(E_r\) with the auxiliary field \(A_{x,0}\). From the action in Eq. (4), it is shown that for large \(J\rho_0\), the Higgs phase is realized, whereas the (homogeneous) confinement phase forms for large \(U, V\) and \(g^2 > 0\).

In the previous work [22], we investigated the phase diagrams of the EBHM [Eq. (1)] and the GHM [Eq. (4)] by means of the MC simulations separately, and verified that the phase diagrams of two models are consistent with each other. There exist three phases in the phase diagram, i.e., the superfluid (SF), Mott insulator (MI) and DW. It was shown that the SF corresponds to Higgs phase of the gauge theory, whereas the MI in the vicinity of the DW corresponds to the confinement phase of the gauge theory. We also studied the 2D and 3D EBHM from the view point of a quantum simulation for the lattice GHM, and obtained interesting results [19,21]. In this paper, we shall study the EBHM in 1D and 2D at relatively large fillings by means of the GW variational method. At large fillings, the GW variational method is reliable even for the 1D system, as it is expected that a quasi-Bose-Einstein condensation forms at each site of the optical lattice at large fillings and the GW variational method can describe dynamics of both the MI and SF.
III. NUMERICAL RESULTS: SYSTEMS WITHOUT DISORDER

In this section, we show the numerical results for the EBHM in 1D and 2D obtained by the GW variational method. The Hamiltonian of the EBHM in Eq. (1) is factorized into single-site local Hamiltonian with the maximum particle number at each site, \( n_c \). In this work, we set \( n_c = 30 \) for 1D and \( n_c = 50 \) for 2D systems. While so far the 1D EBHM has been extensively studied under unit filling condition \( \frac{\mu}{J} \in [28, 30] \), our focus is large filling regime, thus it is worth characterizing the large filling ground state. We also employ the periodic boundary condition for the practical calculation.

We first study the phase diagrams and identify the parameter regions of the confinement and Higgs phases. Then, we investigate dynamical properties of the gauge field in these phases.

A. Phase diagram of 1D EBHM

In this subsection, we study equilibrium properties of the system, in particular, the ground-state phase diagrams of the 1D EBHM. To this end, we obtain the lowest-energy states for \( H_{\text{EBHM}} \) by the GW variational method. Order parameters, which are used for identification of phases, are followings;

\[
\Phi = \frac{1}{N_s} \sum_i \Phi_i = \frac{1}{N_s} \sum_i \langle \hat{b}_i \rangle, \quad \Delta n = \bar{n}_e - \bar{n}_o, \quad (5)
\]

where \( \bar{n}_e(o) \) is the average density of atom at even (odd) sites, and \( N_s \) is the total number of sites and \( N_s = L = 200 \) in the present calculation. Finite value of \( \Phi \) indicates the existence of the SF, and \( \Delta n \) measures the DW. As we fix the chemical potential \( (\mu) \) in the calculation, the total average density of atom, \( \rho_0 \), varies under a change of the parameters in the Hamiltonian \( H_{\text{EBHM}} \).

In Fig. 1 we show the calculation of \( \Delta n \) in the \((V/J - U/J)-\text{plain}\). \( J = 0.01 \) and chemical potential is fixed as \( \mu/J = 950 \) to obtain relatively large fillings. In Fig. 2 we show the calculation of \( \Phi \). From the results in Figs. 1 and 2, we obtain the phase diagram of the 1D EBHM as in Fig. 3. SF forms in the regions of relatively small \( U/J \) and \( V/J \). MIs for large \( U/J \) have large integer filling factors such as \( \rho_0 = 7 \) for \( U/J = 71 \) and \( V/J = 35 \). In Fig. 3 the three parameter regions indicated by the arrows refer to the confinement [(a)], Higgs close to confinement [(b)], and genuine Higgs phases [(c)], respectively. Here, we should comment that in Fig. 2 there are many lines where the finite SF density appears. These lines exist between the MIs with different fillings or the DW phase. Supersolid (SS) also exists in some parameter regions including narrow line regions between the MIs and DW. Similar tendency was reported in Ref. [20]. In the subsequent section, we shall study physical properties of the above phases from the viewpoint of the gauge theory.

B. Behavior of electric flux in quantum simulation of gauge-Higgs model: 1D case

In this subsection, we shall study the time evolution of “electric flux” put on a straight line. To this end, we employ the time-dependent GW methods [31–38]. Behavior of the electric flux is a very important quantity in the gauge theory, which discriminates the confinement, Coulomb and Higgs phases [19]. In the EBHM, an artificial electric flux at \( r \) is produced by the configuration such as \( \Delta = -(-)^n \langle \hat{\rho}_1 \rangle \), located at the edges of the electric flux string. In the GHM, this configuration is explicitly given by \( \prod_{r_1 < r < r_2} (U_0 r) \Delta |0\rangle \), where a pair of static charge \( \pm \Delta \) are located at \( r_1 \) and \( r_2 \) and \( |0\rangle \) is the ‘vacuum’ without electric fluxes. In the practical
FIG. 3. Phase diagram of the 1D EBHM. For large $U/J, V/J$, the MI and DW occupy the phase diagram, and the SF is located between the MI and DW phases. The arrows indicate the locations in which the behavior of electric flux is measured.

calculation, we add very small but finite fluctuations in local density of boson (i.e., local electric field) for initial states in order to perform smooth calculations by the time-dependent GW method. We set unit of time with $\hbar/J$.

We consider the 1D case. In the phase diagram shown in Fig. 3, we exhibit three typical parameter regions corresponding to (a) MI in the vicinity of the DW ($U/J = 71, V/J = 35$), (b) SF close to MI ($U/J = 70, V/J = 14$), and (c) SF ($U/J = 27, V/J = 10$). For all cases, $J = 0.01$ and $\mu/J = 950$. System size $N_s = 200$, and the electric flux is put from $r = 70$ to $r = 130$. The confinement phase corresponds to the case (a), and the Higgs phase to the case (c).

For the case (a), we performed numerical simulations for two cases, i.e., the first one for the background particle density $\rho_0 = 10$ and the magnitude of the electric flux $\Delta = 3$, and the second one for $\rho_0 = 7$ and $\Delta = 1$. The equilibrium filling of the MI at this parameter is $\rho_0 = 7$. As we explained above, this parameter region corresponds to the confinement phase, and therefore we expect that the electric flux is rather stable and remains in the original position without breaking up small pieces for rather long period.

In Fig. 4, we show the results of the simulation for $\rho_0 = 10$ and $\Delta = 3$. The electric flux is quite stable as we expected. Close look at the inside of the electric flux reveals that small but finite fluctuations of the electric field take place there. We studied the fluctuations of the electric field in the central region,

$$E_{in} = \frac{1}{N_i} \sum_{70 \leq r \leq 130} E_r,$$

where $N_i$ is the length of the initial electric string, and the result is shown in the middle and the lower panels in Fig. 4. Averaged electric field first decreases slightly, and then keeps constant with small fluctuations. In the gauge theoretical point of view, the stability means that the system is in confinement phase as we expected.

Recently, closely related experiments were done on Rydberg atom chains [26]. By the strong NN repulsion between Rydberg states, the system is nearly unit-filling, and the DW type configurations exhibit anomalous slow dynamics. In Ref. [18], this phenomenon is interpreted as reminiscence of string-breaking of electric flux in the confined gauge theory [22, 39]. We will discuss this gauge-theoretical interpretations somewhat in detail in Sec. V.

This stability of the electric field implies that the original EBHM exhibits a glassy behavior in the parameter region corresponding to the confinement phase of the corresponding GHM. This observation will be examined in the subsequent section.

We also performed numerical calculations for $\rho_0 = 7$ and $\Delta = 1$. The obtained results are quite similar to those for $\rho_0 = 10$ and $\Delta = 3$ in Fig. 4. The electric flux is quite stable even for $\Delta = 1$, as the background particle density, $\rho_0 = 7$, is equal to that of the equilibrium value.

Let us turn to case (b). We show the numerical results
FIG. 5. Time evolution of electric flux located in the center of the 1D system. \( \rho_0 = 10 \) and \( \Delta = 3 \). The system exists in the SF close to the MI.

FIG. 6. Time evolution of electric flux located in the center of the 1D system in the Higgs (SF) phase. \( \rho_0 = 20 \) and \( \Delta = 3 \).

in Fig. 5. It is obvious that the electric flux string keeps the original configuration for a while, but it breaks into small pieces and these pieces spread the whole system. This indicates the instability of the electric flux. In the Higgs phase of the gauge theory, electric charge is not conserved, and the electric fluxes are destroyed and also generated in various places.

Finally, we show the evolution of the electric flux in the case (c) in Fig. 6. It is obvious that the electric flux decays quite easily, and the whole system is full of large fluctuations of electric field. This means that the system is in deep Higgs phase.

FIG. 7. Time evolution of electric flux in the outside region. Cases (a), (b) and (c).

In order to verify the above behavior of the electric flux, we measured average of electric field in the outside of the original location of the electric flux, i.e.,

\[
E_{\text{out}} = \frac{1}{N_0} \sum_{0 < r < 70, 130 < r < 200} E_r^2,
\]

where \( N_0 \) is the number of sites in which the electric fluxes do not exist in the initial configuration. We show the results in Fig. 7. It is obvious that \( E_{\text{out}} \) is getting larger for smaller \( V/J \) as we expected.

C. Phase diagram of 2D EBHM and behavior of electric flux

FIG. 8. DW order (upper panel) and SF (lower panel) in the \((V/J - U/J)\) plain. \( J = 0.01 \) and \( \mu/J = 2000 \). (a) ((b)) corresponds to confinement (Higgs) phase.

In this subsection, we shall study the 2D EBHM and 2D GHM. We first show the phase diagram of the 2D EBHM at large fillings obtained by the GW methods. Used order parameters are the superfluidity, \( \Phi \), and DW, \( \Delta n \) as in the study of the 1D system. The obtained numerical calculations and phase diagram are shown in Fig. 8 and we also indicate the parameter regions in which stability of the electric flux will be examined. As in the 1D case, the MI and DW occupy most of the phase diagram for large \( U/J \) and \( V/J \), and the SF forms in narrow regions between the MI and DW. Most of the calculations were performed for the system size \( N_x = 20 \times 20 \).

In the 2D case, electric flux is initially put on the central region. In the practical calculation, the initial con-
FIG. 9. Time evolution of electric flux put in the center of the system. Upper panel: Confinement region with $\rho_0 = 7$ and $\Delta = 1$. Electric flux is stable for long period. Lower panel: Higgs region with $\rho_0 = 30$ and $\Delta = 3$. Electric flux decays rapidly.

This configuration is prepared as follows,

$$\rho_{x,y} - \rho_0 = (-)^x \Delta, \quad \text{for } 6 \leq x \leq 15 \text{ and } y = 10,$$
$$\rho_{x,y} = \rho_0, \quad \text{otherwise.} \quad (8)$$

FIG. 10. $E_{\text{out}}$ for the Higgs (upper panel) and confinement (lower panel) phases, respectively.

In order to verify the universality of the above result, we investigated various parameter regions of the EBHM. In particular, the stability of the electric flux is a very important phenomenon. In Fig. 11, we show the calculations of the case of relatively large hopping $J = 0.05$ and $U/J = 175, V/J = 30, \mu/J = 2000$, which corresponds to point (a) in Fig. 8. The parameter point is in the confinement phase. The electric flux is again quite stable even for larger value of $J$.

FIG. 11. Time evolution of electric flux put in the center of the system. $J = 0.05, \rho_0 = 7$ and $\Delta = 2$. Electric flux is stable for long period.

In Fig. 12, we show the behavior of the electric flux (a) in the confinement region for $J = 0.01, \mu/J = 2000$ and $U/J = 175, V/J = 30$ (MI close to DW), and also (b) in the Higgs region $U/J = 45, V/J = 5$ (SF close to MI). For the confinement region, we put $\rho_0 = 7$, which is the equilibrium value for the above parameters. The source electric charge at $x = 6$ and $15$ are $\pm \Delta = \pm 1$, respectively. Even for the smallest unit charge, the electric flux is stable up to $t = 300$, and it gradually decays after that. For larger source charges such as $\Delta = 3$, the electric flux is quite stable. On the other hand for the Higgs region, we put $\rho_0 = 30$, which is again the equilibrium value for the above parameters. We show the calculations for $\Delta = 3$. Even for this relatively large value of $\Delta$, the electric flux breaks after a very small period, and it spreads whole region and fluctuates strongly. In Fig. 10 we also show the the square of the electric flux outside of the region $\vec{r} \neq (6 \leq x \leq 15, y = 10)$, $E_{\text{out}}$, which is defined similarly to the 1D case in Eq. 7. The results in Fig. 10 obviously support the results in Fig. 9.

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In Fig. 12, 2D profiles of the electric flux in the whole $20 \times 20$ region are shown. For the confinement phase for $J = 0.01$ (Fig. 9), the electric flux starts to get shorter at $t \approx 300$. For the Higgs phase in Fig. 9, the electric flux decays quite rapidly. The initial electric flux ‘melts’ and fluctuation of electric field (i.e., particle density) starts to develop immediately. Time evolution of the electric field located out of the original place, $E_{\text{out}}$, in Fig. 10 again supports the above behavior.
FIG. 12. Upper panels: Time evolution of DW-type density modulation corresponding to electric flux put in the confinement phase (MI close to DW). $J = 0.01, \rho_0 = 7$ and $\Delta = 1$. Electric flux shortens but is stable for long period. Lower panels: Time evolution of electric flux put in the Higgs phase (SF close to MI). Electric flux ‘melts’ and its fluctuations start immediately.

IV. GLASSY DYNAMICS AND EFFECT OF QUENCHED DISORDER

In the previous section, we observed that the electric-flux string is quite stable in the confinement phase. Then, it is interesting to study how higher-energy states of the DW type evolve in that parameter region. To examine effects of a random chemical potential on this phenomenon is also an important problem. In real experiments in an optical lattice, a similar random chemical potential can be implemented by using a laser speckle [40, 41]. Closely related phenomenon to the above was recently investigated by experiments on ultra-cold atomic gases, and it was observed that life time of states with higher energies is lengthened by the quenched disorder induced by the random chemical potential [42]. To reveal the origin of this glassy phenomenon and its relation to the MBL is an interesting problem. For the $(1 + 1)$D quantum electrodynamics (QED), in which electron is always confined, an extremely slow evolution of entropy was observed for configurations with background charges [43]. From the gauge-theoretical point of view, this configuration is nothing but the state filled with a bunch of electric flux tubes pointing to opposite directions alternatively. From the above observation indicating the stability of the uniform electric field in the confinement phase, how the genuine DW state evolves is an interesting problem, and if this configuration is stable, we can conclude that the confinement phase has the genuine glassy dynamics.

First, we study time evolution of the initial state in which a half of the system is a DW-type configuration, and the other half is a homogeneous state with $\rho_1 \simeq \rho_0 = 7$. More precisely, the DW-type region of the initial configuration is the state filled with electric field pointing to the $y$-direction. Detailed lattice structure of the gauge system, electric field defined on links of the gauge lattice, and the initial configuration are shown in Fig. 13. Time evolution of this kind of configurations is a good measure for the stability of a bunch of electric flux tubes.

In Fig. 14 we show the time evolution of the 2D system with $W = 0$ and $W = 20$. Data are obtained for a single initial-configuration sample. Other samples give almost the same results. It is obvious that in the case of $W = 0$, the electric field is quite stable. This is an expected result from the stability of electric flux in the confinement phase. On the other hand, in the case of $W = 20$, it tends to spread out the empty space. We examined the case with $W = 10$ and $W = 30$, and obtained similar results. That is, the EBHM and GHM exhibit glassy dynamics in the case without disorder, and disorder hinders glassy nature. This is somehow an ‘unexpected’ result. Disorder often enhances the localization even if there are interactions between particles. However as we explain later, the gauge-theoretical viewpoint gives a clear interpretation to above phenomenon. Before going into discussion on this point, let us consider another example of glassy dynamics of the present system.

Next, we consider the evolution of the initial configurations of the genuine DW type such as $\rho_i = \rho_0 + (-)^i \delta$ in the whole system [$(-)^i = 1(-1)$ for even sites (odd sites)]. From the gauge-theoretical point of view, this configuration is nothing but the state filled with a bunch of electric flux tubes pointing to opposite directions alternatively. From the above observation indicating the stability of the uniform electric field in the confinement phase, how the genuine DW state evolves is an interesting problem, and if this configuration is stable, we can conclude that the confinement phase has the genuine glassy dynamics.

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For $\delta = 2$, calculations of the local DW order parameter $\delta_i \equiv (-)^i (\langle \rho_i \rangle - \rho_0)$ and the difference between the average particle numbers at even-odd sites, $\Delta n$ in Eq. (1),

...
FIG. 14. (a) and (b): Time evolution of electric field in the confinement phase without disorder. Direction of arrows indicate the electric field direction, and length of arrows and color show its magnitude. Electric field is quite stable and keeps its original configuration. (c) and (d): Confinement phase with disorder $W = 20$. Electric field spreads from the original region. Data are for a single sample of initial configurations.

FIG. 15. Upper panel: Time evolution of the local DW order $\{\delta_i\}$ at $y = 10$ in the confinement phase with $W = 0$. DW order is stable for long period. Second panel: The density difference keeps $\Delta n \approx 4$. Third panel: Time evolution of $\{\delta_i\}$ at $y = 10$ for $W = 20$. $\{\delta_i\}$ starts to fluctuate after certain period. Bottom panel: $\Delta n$ as a function of time for various $W$s. The result indicates the existence of critical $W$ at which the DW fades away maximally. The dotted arrow indicates an increase of $W$ as a guide for eyes. Data are for a single sample of initial configurations.

The behavior of the disorder system is reminiscent of the MBL dynamics, and it is expected that there exists a critical value of $W$, $W_c \sim 10$, at which the DW fades away maximally. This interesting observation will be discussed in Sec. V from the gauge-theoretical point of view.
V. DISCUSSION AND CONCLUSION

In this paper, we studied the 1D and 2D EBHM by the GW methods from the view point of the quantum simulation for the gauge theory. We first clarified the phase diagrams at relatively large fillings $\rho_0 = (7 \sim 30)$. The phase diagrams themselves exhibit a rather interesting structure composed of the SF, MI, DW and SS phases. We identified the parameter regions in the phase diagrams corresponding to the confinement and Higgs phases. Then, we studied the time evolution of configurations with electric flux tube, and verified that electric flux tube is stable in the confinement phase but breaks immediately in the Higgs phase. The stability in the confinement phase increases as the magnitude of charge at the edges of the electric flux, $\Delta$, increases.

After the above observations, we studied the effect of disorder caused by the random chemical potential in the 2D system, which generates density inhomogeneity in the system. We first verified the stability of the electric field in the confinement phase by studying time evolution of electric field filling half of the system. For the case without disorder, the electric field remains stable for long periods. Then, we introduced disorder and found that disorder induced by the random chemical potential renders the electric field unstable. This is somehow an ‘unexpected result’, but is plausible from the gauge-theoretical point of view. In the gauge theory, it is established that the quark confinement takes place as a straight electric flux tube forms between a quark-anti-quark pair and it exhibits almost no fluctuations. This confinement picture explains the stability of the electric field filling the half of the system in the case of without disorder. On the other hand, the random chemical potential induces spatial electric field fluctuations as it generates inhomogeneous background charges. The above picture obviously is based on the Gauss law, $\nabla \cdot \vec{E} = \rho$, where $\rho$ is charge density. In the quantum simulations of the GHM, the NN repulsion, as well as the exact gauge symmetry, plays an essential role for the Gauss-law constraint to be satisfied. Recently, some related observation with the above was given in Ref. [18], in which the experiments on Rydberg atom chain in Ref. [20] was interpreted in the gauge-theory framework. There, the strong NN repulsion of the Rydberg state hinders its occupation on NN sites, and this constraint of the Hilbert space can be regarded as the Gauss law. The emergent gauge invariance explains the very slow dynamics observed in Ref. [20].

In other words, the glassy dynamics in the confinement phase observed in the present work results from strong interactions between atoms in the confinement regime. It was shown in Ref. [42] that MBL and glassy dynamics appear due to frustrating dynamical constraints by interactions.

As the second case with disorder, we investigated the stability of the genuine DW configuration with high energies in the whole system. Interesting enough, we found that the DW configuration is stable for a long period in the case without disorder, whereas the inhomogeneous particle density caused by moderate Ws reduces the robustness of the DW-type configuration. This result means that the disorder-induced spatial density modulations hinder the glassy dynamics. Obviously, this behavior is reminiscent of the quark confinement mechanism explained in the above.

Interestingly enough, we observed that further increase of disorder makes the DW-type configurations tend to dynamically survive again. Recently, certain related experiment was performed on the ultra-cold atoms and similar result was obtained, i.e., the random chemical potential enhances life time of the high-energy configurations [13, 14]. This may be an expected result, i.e., disorder enhances the localization. For stronger disorders with $W > W_c$ in the present system, background charge is modulated strongly, and as a result, the gauge-theoretical picture does not work anymore. We think that the EBHM in the present parameter regime exhibits interesting multiple ‘phase transitions’ or ‘crossovers’ from the interaction induced glassy dynamics to the ordinary MBL by increasing the strength of disorder and in between regime an ergodic phase exists.

From the above observations, it is interesting to study atomic gas systems with NN repulsions by varying strength of disorder. We expect that similar experiment on them to those in Ref. [43] sheds light on our picture of the glassy dynamics of the confined gauge theory obtained in this work. It is also important to examine effects of disorders in the experiments on the Rydberg atom chain [20] by introducing disorder in detuning and/or Rabi frequency. We expect that similar ‘phase transitions’ are to be observed there.

Also, a recent numerical study [18] suggested that the glassy dynamics (slow down to the relaxation) is related to MBL. In our work, as shown in Fig. 15 (d), the tendency of the glassy dynamics becomes stronger as weaker disorder in the confinement phase. That is, we expect that the confinement phase has also the MBL properties. In confinement phase, such a conjecture has been verified for other lattice gauge models [47, 48].

Finally, we would like to comment on some related works in which localization of magnetic flux lines was studied for the type-II superconductors [49, 50]. Although type-II superconductors correspond to the Higgs phase of the gauge theory, there exists duality between confinement and superconductivity. Confinement of the gauge theory takes place as a result of condensation of magnetic charges such as a magnetic monopole. Squeeze of electric flux in the confinement phase is sometimes called dual Meissner effect [51]. Therefore, the localization of magnetic flux lines in superconductors suggests the similar localization of electric flux string in the confinement phase, which is observed in this work.

In the present work, we employed the GW variational method, and we could not calculate the entanglement entropy. We expect that the glassy dynamics of the confined gauge theory is closely related to MBL. We are now
studying the EBHM in 1D by the exact diagonalization from the above point of view, and we hope that the results will be reported in the near future.

[1] I. M. Georgescu, S. Ashhab, and F. Nori, Rev. Mod. Phys. 86, 153 (2014).
[2] M. Lewenstein, A. Sanpera, and V. Alufinger, Ultracold Atoms in Optical Lattices: Simulating Quantum Many-body Systems (Oxford University Press, 2012).
[3] I. Bloch, J. Dalibard, and S. Nascimbène, Nat. Phys. 8, 267 (2012).
[4] E. Zohar and B. Reznik, Phys. Rev. Lett. 107, 275301 (2011).
[5] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. 109, 125302 (2012).
[6] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, Ann. Phys. 330, 160 (2013).
[7] D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. 109, 175302 (2012).
[8] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. 110, 055302 (2013).
[9] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. 110, 125304 (2013).
[10] D. Banerjee, M. Bögli, M. Dalmonte, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. 110, 125303 (2013).
[11] L. Tagliacozzo, A. Celi, P. Orland, M. W. Mitchell, and M. Lewenstein, Nat. Commun. 4, 2615 (2013).
[12] E. Zohar, J. I. Cirac, and B. Reznik Phys. Rev. A 88, 023617 (2013).
[13] U.-J. Wiese, Annalen der Physik 525, 777 (2013).
[14] E. Zohar, J. I. Cirac, and B. Reznik, Rep. Prog. Phys. 79, 014401 (2016).
[15] A. Bazavov, Y. Meurice, S.-W. Tsai, J. Ummuth-Yockey, and J. Zhang, Phys. Rev. D 92, 076003 (2015).
[16] D. González-Cuadra, E. Zohar, and J. I. Cirac, New J. Phys. 19, 063038 (2017).
[17] C. Kokail, C. Maier, R. van Bijnen, T. Brydges, M. K. Joshi, P. Jurcevic, C. A. Muschik, P. Silvi, R. Blatt, C. F. Roos, P. Zoller, arXiv: 1810.03421.
[18] F. M. Surace, P. P. Mazza, G. Giudici, A. Lerose, A. Gambassi, M. Dalmonte, arXiv: 1902.09551.
[19] K. Kasamatsu, I. Ichinose, and T. Matsui, Phys. Rev. Lett. 111, 115303 (2013).
[20] Y. Kuno, K. Kasamatsu, Y. Takahashi, I. Ichinose, and T. Matsui, New J. Phys. 17, 063005 (2015).
[21] Y. Kuno, M. Sakane, K. Kasamatsu, I. Ichinose, and T. Matsui, Phys. Rev. A 94, 063641 (2016).
[22] Y. Kuno, S. Sakane, K. Kasamatsu, I. Ichinose, and T. Matsui, Phys. Rev. D 95, 094507 (2017).
[23] O. Dutta, M. Gajda, P. Hauke, M. Lewenstein, D.-S. Luhmann, B. A. Malomed, T. Sowinski, and J. Zakrzewski, Rep. Prog. Phys. 78, 066001 (2015). There, many extended versions of the Bose-Hubbard model are discussed.
[24] E. Fradkin and S. H. Shenker, Phys. Rev. D 19 3682 (1979).
[25] J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979).
[26] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, Nature 551, 579 (2017).
[27] More precisely, the next-NN (NNN) repulsions have to be added to $H_{EBH}$ in Eq. (1) for the quantum simulation of the genuine GHM in 2D. However, low-energy system derived from $H_{EBH}$ in Eq. (1) is gauge-invariant, and the universality class does not change by the existence of the NNN repulsions.
[28] D. Rossini and R. Fazio, New. J Phys. 14, 065012 (2012).
[29] G. G. Batrouni, V. G. Rousseau, R. T. Scalettar, and B. Gremaud, Phys. Rev. B 90, 205123 (2014).
[30] K. Kawaki, Y. Kuno, and I. Ichinose, Phys. Rev. B 95, 195101 (2017).
[31] D. Jaksch, V. Venturi, J. I. Cirac, C. J. Williams, and P. Zoller, Phys. Rev. Lett. 89, 040402 (2002).
[32] J. Zakrzewski, Phys. Rev. A 71, 043601 (2005).
[33] M. Jreissaty, J. Carrasquilla, F. A. Wolf, and M. Rigol, Phys. Rev. A 84, 043610 (2011).
[34] M. Buchhold, U. Bisshorst, S. Will, and W. Hofstetter, Phys. Rev. A 84, 023631 (2011).
[35] S. S. Natu, K. R. A. Hazzard, and E. J. Mueller, Phys. Rev. Lett. 106, 125301 (2011).
[36] H. Fehrmann, M. A. Baranov, B. Damski, M. Lewenstein, and L. Santos, Opt. Commun. 243, 23 (2004).
[37] N. Horiguchi, T. Oka, and H. Aoki, Journal of Physics: Conference Series 150, 032007 (2009).
[38] K. Shimizu, Y. Kuno, T. Hirano, and I. Ichinose, Phys. Rev. A 97, 033626 (2018).
[39] D. Spitz and J. Berges, Phys. Rev. D 99, 036020 (2019).
[40] T. Schulte, S. Drenkelforth, J. Kruse, W. Ertmer, J. Arlt, K. Sacha, J. Zakrzewski, and M. Lewenstein, Phys. Rev. Lett. 95, 170411 (2005).
[41] D. Clement, A. F. Varon, J. A. Retter, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, New J. Phys. 8, 165 (2006).
[42] G. Carleo, F. Becca, M. Schiro, and M. Fabrizio, Sci. Rep. 2, 43 (2012).
[43] J.-y. Choi, S. Hild, J. Zeiher, P. Schaus, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science 352, 1547 (2016); For numerical study of this experiments, see Ref. [22].
[44] M. Yan, H.-Y. Hui, M. Rigol, and V. W. Scarola, Phys. Rev. Lett. 119, 073002 (2017).
[45] M. Brenes, M. Dalmonte, M. Heyl, and A. Scardicchio, Phys. Rev. Lett. 120, 030601 (2018).
[46] A. Smith, J. Knolle, D. L. Kovrizhin, and R. Moessner, Phys. Rev. Lett. 118, 266601 (2017).
[47] A. Smith, J. Knolle, R. Moessner, and D. L. Kovrizhin, Phys. Rev. Lett. 119, 176601 (2017).
[48] P. Sierant and J. Zakrzewski, New J. Phys. 20, 043032 (2018).
[49] R. M. Nandkishore and S. L. Sondhi, Phys. Rev. X 7, 041021 (2017).
[50] M. Pretko and R. M. Nandkishore, Phys. Rev. B 98, 134301 (2018).
[51] See for example, S. Mandelstam, Phys. Rep. 23, 245 (1976).