Phases of Kaluza-Klein Black Holes: 
A Brief Review

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Abstract 
We review the latest progress in understanding the phase structure of static and neutral Kaluza-Klein black holes, i.e. static and neutral solutions of pure gravity with an event horizon that asymptote to a $d$-dimensional Minkowski-space times a circle. We start by reviewing the $(\mu, n)$ phase diagram and the split-up of the phase structure into solutions with an internal $SO(d - 1)$ symmetry and solutions with Kaluza-Klein bubbles. We then discuss the uniform black string, non-uniform black string and localized black hole phases, and how those three phases are connected, involving issues such as classical instability and horizon-topology changing transitions. Finally, we review the bubble-black hole sequences, their place in the phase structure and interesting aspects such as the continuously infinite non-uniqueness of solutions for a given mass and relative tension.
1 Introduction

In this brief review we go through recent progress in the understanding of static and neutral Kaluza-Klein black holes.\(^1\) A \(d + 1\) dimensional static and neutral Kaluza-Klein black hole is defined here as a pure gravity solution that asymptotes to \(d\)-dimensional Minkowski-space times a circle \(\mathcal{M}^d \times S^1\) at infinity, with at least one event horizon.\(^2\) Since we consider only static and neutral solutions they do not have charges or angular momenta.

Static and neutral Kaluza-Klein black holes are interesting to study for a variety of reasons:

- Four-dimensional static and neutral black holes have a very simple phase-structure. For a given mass, there is only one phase available: The Schwarzschild black hole. The phase structure of Kaluza-Klein black holes, on the other hand, is very rich, as we review below. Indeed, for some choice of masses we have a continuous amount of different Kaluza-Klein black holes with that mass.

- The phase structure of Kaluza-Klein black holes contain examples of phase transitions, which are therefore purely gravitational phase transitions between different solutions with event horizons. Specifically, one has examples of phase transitions involving topology change of the event horizon. One of the most interesting examples is the decay of the uniform black string to the localized black hole solution.

- If we have large extra dimensions in Nature the Kaluza-Klein black hole solutions, or generalizations thereof, will become relevant for experiments involving microscopic black holes and observations of macroscopic black holes.

As will be discussed in the Conclusion, Kaluza-Klein black holes are also intimately related to non- and near-extremal branes of String/M-Theory with a circle in the transverse space. Via the gauge/gravity duality this means that the study of Kaluza-Klein black holes is also relevant for the thermal phase structure of the non-gravitational theories that are dual to near-extremal branes with a circle in the transverse space. This development is very briefly reviewed in the proceedings [1].

See Ref. [2] for another recent review of Kaluza-Klein black holes which contains complimentary motivations and discussions.

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\(^1\)This review is an extended and updated version of the proceedings published in Class.Quant.Grav.21:S1509-S1516,2004, for a talk given by T. Harmark at the EC-RTN workshop “The quantum structure of spacetime and the geometric nature of fundamental interactions” held September 2003 in Copenhagen.

\(^2\)In other words, we consider solutions of the vacuum Einstein equations for which we have at least one event horizon present, and such that the asymptotics of the solution is \((d + 1)\)-dimensional Kaluza-Klein space \(\mathcal{M}^d \times S^1\).
2 Physical parameters and definition of phase diagram

In this section we show how to measure the mass $\mu$ and relative tension $n$ of a Kaluza-Klein black hole, so that each phase can be plotted in the $(\mu, n)$ phase diagram. We review the main features of the split-up of this phase diagram into two regions. Finally, we discuss some general results for the thermodynamics of Kaluza-Klein black holes.

Measuring mass and tension and defining phase diagram

For any space-time which is asymptotically $M^d \times S^1$ we can measure the mass $M$ and the tension $T$. These quantities can then conveniently be used to display the various phases of Kaluza-Klein black holes, as we review below.

Define the Cartesian coordinates for $d$-dimensional Minkowski space $M^d$ as $t, x^1, \ldots, x^{d-1}$ and the radius $r = \sqrt{\sum_i (x^i)^2}$. In addition we have the coordinate $z$ for the $S^1$, of period $L$. Thus, the total space-time dimension is $D = d + 1$. Now, for any localized static object we have the asymptotics \cite{[3]}

\[ g_{tt} \simeq -1 + \frac{c_t}{r^{d-3}}, \quad g_{zz} \simeq 1 + \frac{c_z}{r^{d-3}}, \quad (2.1) \]

for $r \to \infty$. The mass $M$ and tension $T$ are then given by \cite{[3] [4]}

\[ M = \frac{\Omega_{d-2}L}{16\pi G_N} [(d-2)c_t - c_z], \quad T = \frac{\Omega_{d-2}}{16\pi G_N} [c_t - (d-2)c_z]. \quad (2.2) \]

The tension has previously been defined in \cite{[5] [6]}. The tension in (2.2) can also be obtained from the general formula for tension in terms of the extrinsic curvature \cite{[7]}, analogous to the Hawking-Horowitz mass formula \cite{[8]}. The mass and tension formulas have been generalized to non-extremal and near-extremal branes in \cite{[9]}.

We now define the relative tension (also called the relative binding energy) as \cite{[3]}

\[ n = \frac{T}{M} = \frac{c_t - (d-2)c_z}{(d-2)c_t - c_z}. \quad (2.3) \]

This measures how large the tension (or binding energy) is relative to the mass. This is a useful quantity since it is dimensionless and since it is bounded as \cite{[3]}

\[ 0 \leq n \leq d - 2. \quad (2.4) \]

Here the upper bound is due to the Strong Energy Condition while the lower bound was found in \cite{[10] [11]}. The upper bound can also be physically understood in a more direct way. If we consider a test particle at infinity it is easy to see that the gravitational force on the particle is attractive for $n < d - 2$ but repulsive for $n > d - 2$.

It is also useful to define a rescaled dimensionless quantity from the mass as

\[ \mu = \frac{16\pi G_N}{L^{d-2}} M. \quad (2.5) \]

We can now formulate the program set forth in \cite{[3] [12]}: To plot all phases of Kaluza-Klein black holes in a $(\mu, n)$ diagram.
The split-up of the phase diagram

According to the present knowledge of phases of static and neutral Kaluza-Klein black holes, the \((\mu, n)\) phase diagram appears to be divided into two separate regions [13]:

- The region \(0 \leq n \leq 1/(d-2)\) contains solutions without Kaluza-Klein bubbles, and the solutions have a local \(SO(d-1)\) symmetry. We review what is known about solutions in this part of the phase diagram in Section 3. The general properties of this class of solutions, which we in the following call black holes and strings on cylinders (since \(\mathcal{M}^d \times S^1\) is the cylinder \(\mathbb{R}^{d-1} \times S^1\) for a fixed time), are the subject of [3, 12]. Due to the \(SO(d-1)\) symmetry there are only two types of event horizon topologies: The event horizon topology is \(S^{d-1}(S^{d-2} \times S^1)\) for the black hole (string) on a cylinder.

- The region \(1/(d-2) < n \leq d - 2\) contains solutions with Kaluza-Klein bubbles. We review this class of solutions in Section 4. This part of the phase diagram is the subject of [13]. It turns out that this part of the phase diagram is much more densely populated with solutions than the lower part.

Thermodynamics, first law and the Smarr formula

For a neutral Kaluza-Klein black hole with a single connected horizon, we can find the temperature \(T\) and entropy \(S\) directly from the metric. Together with the mass \(M\) and relative tension \(n\), these quantities obey the Smarr formula [3, 4] \((d-1)TS = (d-2-n)M\) and the first law of thermodynamics [4, 14] \(dM = TdS + nMdL/L\). It is useful to define the rescaled temperature \(t\) and entropy \(s\) by

\[
    t = LT, \quad s = \frac{16\pi G_N}{L^{d-1}} S .
\]

In terms of these quantities, the Smarr formula for Kaluza-Klein black holes and the first law of thermodynamics take the form

\[
    (d-1)ts = (d-2-n)\mu , \quad \delta \mu = t \delta s .
\]

Combining the Smarr formula and the first law, we get the useful relation

\[
    \frac{\delta \log s}{\delta \log \mu} = \frac{d-1}{d-2-n} ,
\]

so that, given a curve \(n(\mu)\) in the phase diagram, the entire thermodynamics can be obtained.

From [2, 8] it is also possible to derive the Intersection Rule of [3]: For two branches that intersect in the same solution, we have the property that for masses below the intersection point the branch with the lower relative tension has the highest entropy, whereas for masses above the intersection point it is the branch with the higher relative tension that has the highest entropy. We have illustrated the Intersection Rule in Figure 1.
As seen in [13], there are also neutral Kaluza-Klein black hole solution with more than one connected event horizon. The generalization of the Smarr formula and first law were found in [13] for the specific class of solutions considered there.

3  Black holes and strings on cylinders

In this section we review the neutral and static black objects without Kaluza-Klein bubbles. These turn out to have local $SO(d - 1)$ symmetry, which means that the solutions fall into two classes, black holes with event horizon topology $S^{d-1}$ and black strings with event horizon topology $S^{d-2} \times S^1$. We first discuss the ansatz that follows from the local $SO(d - 1)$ symmetry. We then review in turn the uniform black string, non-uniform black string and localized black hole phases. These phases are drawn in the $(\mu, n)$ phase diagram for the five- and six-dimensional cases. Finally we consider various topics, including copies of solutions, the endpoint of the decay of the uniform black string and an observation related to the large $d$ behavior.

The ansatz

As mentioned above, all solutions with $0 \leq n \leq 1/(d-2)$ have, to our present knowledge, a local $SO(d-1)$ symmetry. Using this symmetry it has been shown [15] [12] that the metric of these solutions can be written in the form

$$ds^2 = -f dt^2 + \frac{L^2}{(2\pi)^2} \left[ \frac{A}{f} dR^2 + \frac{A}{K^{d-2}} dv^2 + KR^2 d\Omega_{d-2}^2 \right], \quad f = 1 - \frac{R_0^{d-3}}{R^{d-3}}, \quad (3.1)$$

where $R_0$ is a dimensionless parameter, $R$ and $v$ are dimensionless coordinates and the metric is determined by the two functions $A = A(R, v)$ and $K = K(R, v)$. The form (3.1) was originally proposed in Ref. [16] [17] as an ansatz for the metric of black holes on cylinders.

The properties of the ansatz (3.1) were extensively considered in [16]. It was found that the function $A = A(R, v)$ can be written explicitly in terms of the function $K(R, v)$ thus reducing the number of free unknown functions to one. The functions $A(R, v)$ and $K(R, v)$ are periodic in $v$ with the period $2\pi$. Note that $R = R_0$ is the location of the horizon in (3.1).
As already stated, all phases without Kaluza-Klein bubbles have, to our present knowledge, $0 \leq n \leq 1/(d-2)$ and can be described by the ansatz (3.1) due to their local $SO(d-1)$ symmetry. In the following we review the three known phases and describe their properties.

**Uniform string branch**

The uniform string branch consists of neutral black strings which are translationally invariant along the circle direction. The metric of a uniform string in $d+1$ dimensions is

$$\mathbb{R}^3$$

$$ds^2 = -\left(1 - \frac{r_0^{d-3}}{r^{d-3}}\right)dt^2 + \left(1 - \frac{r_0^{d-3}}{r^{d-3}}\right)^{-1}dr^2 + r^2d\Omega_{d-2}^2 + dz^2.$$  \hspace{1cm} (3.2)

In terms of the relative binding energy $n$ defined above, we note that $n = 1/(d-2)$ for all of the uniform string branch, as can be seen from the metric (3.2) using (2.1) and (2.3). Thus, the uniform string branch is a horizontal line in the $(\mu, n)$ diagram. The rescaled entropy of the uniform black string is given by

$$s_u(\mu) = C_1^{(d-1)}\mu^{\frac{d}{d-2}},$$  \hspace{1cm} (3.3)

where the constant $C_1^{(q)}$ is defined as

$$C_1^{(q)} \equiv 4\pi(\Omega_{q-1})^{-\frac{1}{q-2}}(q-1)^{-\frac{q+1}{q-2}}.$$  \hspace{1cm} (3.4)

The horizon topology of the uniform black string is clearly $S^1 \times S^{d-2}$, where the $S^1$ is along the circle-direction.

The most important physical feature of the neutral uniform black string branch is that it can be classically unstable. Gregory and Laflamme showed in [19, 20] that a neutral uniform black string is classically unstable for $\mu < \mu_{GL}$, i.e. for sufficiently small masses. See Table 1 for numerical values of the critical mass $\mu_{GL}$ for $d \leq 14$. For $\mu > \mu_{GL}$ the string is believed to be classically stable. We comment on the endpoint of the classical instability of the uniform black string below.

**Non-uniform string branch**

As realized in [21, 22] the classical instability of the uniform black string for $\mu < \mu_{GL}$ points to the existence of a marginal mode at $\mu = \mu_{GL}$ which again suggests the existence of a new branch of solutions. Since this new branch of solutions should be continuously connected to the uniform black string it should have the same horizon topology, at least when the deformation away from the uniform black string is sufficiently small. Moreover, the solution should be non-uniformly distributed in the circle-direction $z$ since the marginal mode has a dependence on this direction.

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3The metric (3.2) corresponds to $A(R, v) = K(R, v) = 1$ in the ansatz (3.1).
This new branch, here called the non-uniform string branch, has been found numerically [22, 23, 24]. Essential features are:

- The horizon topology is $S^1 \times S^{d-2}$ with the $S^1$ being supported by the circle of the Kaluza-Klein space-time $M^d \times S^1$. Therefore the solutions describe black strings.
- The solutions are non-uniformly distributed along $z$.
- The non-uniform black strings have a local $SO(d-1)$ symmetry and can therefore be written in terms of the ansatz [6, 15, 12].
- The non-uniform string branch meets the uniform string branch at $\mu = \mu_{GL}$, i.e. with $n = 1/(d-2)$.
- The branch has $n < 1/(d-2)$.

If we consider the non-uniform black string branch for $|\mu - \mu_{GL}| \ll 1$, we have

$$n(\mu) = \frac{1}{d-2} - \gamma(\mu - \mu_{GL}) + \mathcal{O}((\mu - \mu_{GL})^2) .$$

Here $\gamma$ is a number representing the slope of the curve corresponding to the non-uniform string branch near $\mu = \mu_{GL}$. In Table 1 we have listed numerical data for $\mu_{GL}$ and $\gamma$ for $4 \leq d \leq 14$. These data were computed in [19, 20, 22, 23, 24].

| $d$ | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-----|----|----|----|----|----|----|----|----|----|----|----|
| $\mu_{GL}$ | 3.52 | 2.31 | 1.74 | 1.19 | 0.79 | 0.55 | 0.37 | 0.26 | 0.18 | 0.12 | 0.08 |
| $\gamma$ | 0.14 | 0.17 | 0.21 | 0.31 | 0.47 | 0.74 | 1.4 | 2.8 | 7.9 | -40 | -9.2 |

Table 1: The critical masses $\mu_{GL}$ for the Gregory-Laflamme instability and the constant $\gamma$ determining the behavior of the non-uniform branch for $|\mu - \mu_{GL}| \ll 1$.

In [21] the larger $d$ behavior of $\mu_{GL}$ is also examined numerically. It is found that the $\mu_{GL}$ versus $d$ curve for $d \leq 50$ obeys $\mu_{GL} \simeq 16.21 \cdot 0.686^d$ to a good approximation. That $\mu_{GL}$ behaves exponentially for large $d$ is confirmed by [25] where the large $d$ behavior of $\mu_{GL}$ is found analytically to be $\log \mu_{GL} \simeq d \log \sqrt{e^2}$.

The qualitative behavior of the non-uniform string branch depends the sign of $\gamma$. If $\gamma$ is positive, we have that the branch starts out from $\mu = \mu_{GL}$ with increasing $\mu$ and decreasing $n$. If $\gamma$ is negative the branch instead starts out from $\mu = \mu_{GL}$ with decreasing $\mu$ and decreasing $n$. Moreover, one can use the Intersection Rule of [3] (see Section 2) to see that if $\gamma$ is positive (negative) then the entropy of the uniform string branch for $|\mu - \mu_{GL}| \ll 1$.

\footnote{\noindent In Table 1, $\gamma$ is found in terms of $\eta_1$ and $\sigma_2$ given in Figure 2 of [24] by the formula

$$\gamma = -\frac{2(d-1)(d-3)^2}{(d-2)^2} \frac{\sigma_2}{\eta_1^2} \frac{1}{\mu_{GL}} .$$

$\eta_1$ and $\sigma_2$ are also found in [22, 23] for $d = 4, 5$.}
given mass is higher (lower) than the entropy of the non-uniform string branch for that mass. This can also be derived directly from (3.5) using (2.8). This gives

\[
\frac{s_{nu}(\mu)}{s_u(\mu)} = 1 - \frac{(d-2)^2}{2(d-1)(d-3)^2} \frac{\gamma}{\mu_{GL}} (\mu - \mu_{GL})^2 + \mathcal{O}((\mu - \mu_{GL})^3),
\]

from which we clearly see the significance of the sign of \( \gamma \). Here \( s_u(\mu) \) (\( s_{nu}(\mu) \)) refers to the rescaled entropy of the uniform (non-uniform) black string branch. We have listed the numerical data for \( \gamma \) for \( 4 \leq d \leq 14 \) in Table 1. From these we see that for \( d \leq 12 \) we have that \( \gamma \) is positive, while for \( d \geq 13 \) we have that \( \gamma \) is negative [24]. Therefore, as discovered in [24], we have qualitatively different behavior of the non-uniform black string branch for small \( d \) and large \( d \), i.e. the system exhibits a critical dimension \( D = 14 \).

In six dimensions, i.e. for \( d = 5 \), Wiseman found in [23] a large body of numerical data for the non-uniform string branch. These data are displayed in the \((\mu, n)\) phase diagram in the right side of Figure 2. This was originally done in [3]. While the branch starts in \( \mu = \mu_{GL} \) the data ends around \( \mu \simeq 2.3 \mu_{GL} \simeq 5.3 \).

In the recent paper [27] numerical evidence has been found that suggest that the non-uniform string branch more or less ends where the data of [23] end, i.e. around \( \mu \simeq 5.3 \) for \( d = 5 \), supporting the considerations of [28]. As we discuss more below, this suggests that the branch has a topology changing transition into the localized black hole branch.

**Localized black hole branch**

On physical grounds we clearly expect that there exists a branch of neutral black holes in the space-time \( M^d \times S^1 \). One defining feature of these solutions is that their event horizons should have topology \( S^{d-1} \). We call this branch the *localized black hole branch* in the following since the \( S^{d-1} \) horizon is localized on the \( S^1 \) of the Kaluza-Klein space.

Neutral black hole solutions in the space-time \( M^3 \times S^1 \) were found and studied in [29, 30, 31, 32]. However, the study of black holes in the space-time \( M^d \times S^1 \) for \( d \geq 4 \) has only recently begun. A reason for this is that it is a rather difficult problem to solve since such black holes are not algebraically special [33] and, in particular, the solution cannot be found using a Weyl ansatz since the number of Killing vectors is too small.

**Analytical results:**

Progress towards finding an analytical solution for the localized black hole was made in [16] where, as reviewed above, the ansatz [31] was proposed. As stated above, it was subsequently proven in [15, 12] that the localized black hole can be put in this ansatz.

5The first occurrence of a critical dimension in this system was given in [26], where evidence was given that the merger point between the black hole and the string depends on a critical dimension \( D = 10 \), such that for \( D < 10 \) there are local tachyonic modes around the tip of the cone (the conjectured local geometry close to the thin “waist” of the string) which are absent for \( D > 10 \).
In [14] the metric for small black holes, i.e. for $\mu \ll 1$, was found using the ansatz (3.1) of [16] to first order in $\mu$. The first order metric of the localized black hole branch was also found analytically in Ref. [34], using a different method. An important feature of the localized black hole solution is that $n \to 0$ for $\mu \to 0$. This means that the black hole solution becomes more and more like a $(d + 1)$-dimensional Schwarzschild black hole as the mass goes to zero.

One of the main results of [14] is the first correction to the relative tension $n$ as function of $\mu$ for the localized black hole branch, which was found to be:

$$n = \frac{(d - 2)\zeta(d - 2)}{2(d - 1)\Omega_{d-1}} \mu + O(\mu^2). \quad (3.7)$$

Using this in (2.8) one can find the leading correction to the thermodynamics as

$$s_{bh}(\mu) = C_1^{(d)} \frac{d-1}{2} \left( 1 + \frac{\zeta(d - 2)}{2(d - 2)\Omega_{d-1}} \mu + O(\mu^2) \right), \quad (3.8)$$

where $C_1^{(d)}$ is defined in (3.4). This constant of integration is fixed by the physical requirement that in the limit of vanishing mass we should recover the thermodynamics of a Schwarzschild black hole in $(d + 1)$-dimensional Minkowski space. For $d = 4$, the second order correction to the metric has recently been studied [35].

**Numerical results:**

The black hole branch has been studied numerically, for $d = 4$ in [36, 27] and for $d = 5$ in [37, 27]. For small $\mu$, the impressively accurate data of [27] is consistent with the analytical results of [14, 34, 35]. We have displayed the results for $d = 4, 5$ of [27] in a $(\mu, n)$ phase diagram in Figure 2.

Amazingly, the work of [27] seems to give an answer to the question: “Where does the localized black hole branch end?”. Several scenarios have been suggested, see [12] for a list. The scenario that [27] points to, is the scenario suggested by Kol [26] in which the localized black hole branch meets with the non-uniform black string branch in a topology changing transition point. This seems highly evident from the $(\mu, n)$ phase diagram for $d = 5$ in Figure 2 and is moreover supported by examining the geometry of the two branches near the transition point, and also by examining the thermodynamics [27].

Therefore, it seems reasonable to expect that the localized black hole branch is connected with the non-uniform string branch in any dimension. This means we can go from the uniform black string branch to the localized black hole branch through a connected series of static classical geometries. The critical behavior near the point in which the localized black hole and the non-uniform string branch merge has recently been considered in [38, 39].

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6Here $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ is the Riemann zeta function.
Phase diagrams for $d = 4$ and $d = 5$

We display here in Figure 2 the $(\mu, n)$ diagram for $d = 4$ and $d = 5$, since these are the cases where we have the most knowledge of the various branches of black holes and strings on cylinders. For $d = 4$ we only know the leading linear behavior of the non-uniform branch, which emanates at $\mu_{GL} = 3.52$ from the uniform branch which is given by the horizontal line $n = 1/2$. For $d = 5$ on the other hand, we have shown the complete non-uniform branch, as numerically obtained by Wiseman [23], which emanates at $\mu_{GL} = 2.31$ from the uniform branch which has $n = 1/3$. These data were first put in a $(\mu, n)$ diagram in Ref. [3]. For the black hole branch we have plotted in the figure the recently obtained numerical data of Kudoh and Wiseman [27], both for $d = 4$ and 5. As seen from the figure this branch has an approximate linear behavior for a fairly large range of $\mu$ close to the origin and the numerically obtained slope agrees very well with the analytic result (3.7).

![Figure 2: Black hole and string phases for $d = 4$ and $d = 5$, drawn in the $(\mu, n)$ phase diagram. The horizontal (red) line at $n = 1/2$ and 1/3 respectively is the uniform string branch. The (blue) branch emanating from this at the Gregory-Laflamme mass is the non-uniform string branch. For $d = 4$ only the linear behavior close to the Gregory-Laflamme mass is known, while for $d = 5$ the entire behavior has been obtained numerically by Wiseman [23]. The (purple) branch starting in the point $(\mu, n) = (0, 0)$ is the black hole branch which was numerically obtained by Kudoh and Wiseman [27]. In particular for $d = 5$ we observe the remarkable result that the black hole and non-uniform black string branch meet.](image)

Copies of solutions

In [12] it is discussed that one can generate new solutions by copying solutions on the circle several times, following an idea of Horowitz [40]. This works for solutions which vary along the circle direction (i.e. the $z$ direction), so it works both for the black hole branch and the non-uniform string branch. Let $k$ be a positive integer, then if we copy a
solution \( k \) times along the circle we get a new solution with the following parameters:

\[
\tilde{\mu} = \frac{\mu}{k^{d-3}} , \quad \tilde{n} = n , \quad \tilde{t} = k t , \quad \tilde{s} = \frac{s}{k^{d-2}} .
\] (3.9)

See Ref. [12] for the corresponding expression of the metric of the copies, as given in the ansatz (3.1). Using the transformation (3.9), one easily sees that the non-uniform and the localized black hole branches depicted in Figure 2 are copied infinitely many times in the \((\mu, n)\) phase diagrams.

The endpoint of the decay of the uniform black string

As mentioned above, the uniform black string is classically unstable for \( \mu < \mu_{GL} \). A natural question to ask is: “What is the endpoint of the classical instability?”.

The entropy for a small localized black hole branch is much larger than that of the than the entropy of a black string of same mass, i.e. \( s_{bh}(\mu) \gg s_{u}(\mu) \) for \( \mu \ll 1 \), as can be easily seen by comparing (3.3) and (3.8). This suggests that a light uniform string will decay to a localized black hole. However, one can imagine other possibilities, for example the uniform black string could decay to another static geometry, or it could even keep decaying without reaching an endpoint.

For \( d = 5 \) we can be more precise about this issue. In Figure 3 we have displayed the entropy \( s \) versus the mass \( \mu \) diagram for the localized black hole, uniform string and non-uniform string branches. We see from this that in six dimensions the localized black hole always has greater entropy than the uniform strings, in the mass range where the uniform string is classically unstable. Combining this with the fact that the non-uniform string branch does not exist in this mass range, it suggest that the unstable uniform black string decays classically to the localized black hole branch in six dimensions.

![Figure 3: Entropy \( s/s_u \) versus the mass \( \mu/\mu_{GL} \) diagram for the uniform string (red), non-uniform string (blue) and localized black hole (purple) branches.](image)

The viewpoint that an unstable uniform string decays to a localized black hole has been challenged in [41]. Here it is shown that in a classical evolution an event horizon
cannot change topology, i.e. cannot pinch off, in finite affine parameter (on the event horizon).

However, this does not exclude the possibility that a classically unstable horizon pinches off in infinite affine parameter. Indeed, in [42] the numerical study [43] of the classical decay of a uniform black string in five dimensions was reexamined, suggesting that the horizon of the string pinches off in infinite affine parameter.

Interestingly, the classical decay of the uniform string is quite different for \(d \geq 13\). As we reviewed above, the results of [24] show that for \(d \geq 13\) the non-uniform string has decreasing mass \(\mu\) for decreasing \(n\), as it emanates from the uniform string in the Gregory-Laflamme point at \(\mu = \mu_{GL}\). This in addition means that the entropy of the a non-uniform black string is higher than the entropy of a uniform string with same mass. Therefore, for \(d \geq 13\) we have a certain range of masses in which the uniform black string is unstable, and for which we have a non-uniform black string with higher entropy. This obviously suggests that a uniform black string in that mass range will decay classically to a non-uniform black string. Such a decay can then be possible in a finite affine parameter, according to [41], since the horizon topology is fixed in the decay.

Note that the range of masses for which we have a non-uniform string branch with higher entropy is extended by the fact that we have copies of the non-uniform string branch. The copies, which have the quantities given by the transformation rule (3.9), can easily be seen from the Intersection Rule of [3] (see Section 2) to have higher entropy than that of a uniform string of same mass, since they also have decreasing \(\mu\) for decreasing \(n\). Thus, for \(d \geq 13\), it is even possible that there exists a non-uniform black string for any given \(\mu < \mu_{GL}\), with higher entropy than that of the uniform black string with mass \(\mu\). This will occur if the non-uniform string branch extends to masses lower than \(2^{3-d}\mu_{GL}\). Otherwise, the question of the endpoint of the decay of the uniform black string will involve a quite complicated pattern of ranges.

More on large \(d\) behavior

It is interesting to consider the slope of the localized black hole branch (3.7) for large \(d\), measured in units normalized with respect to the Gregory-Laflamme point \((\mu_{GL}, n_{GL} = 1/(d-2))\), with \(\mu_{GL}\) given by the large \(d\) formula of [25] reviewed above. We get
\[
\frac{n}{n_{GL}} \simeq d^{d/2} \frac{\mu}{\mu_{GL}},
\]
(3.10)
for \(d \gg 1\). We see from this that the slope become infinitely steep as \(d \to \infty\). This suggests that the curve describing the localized black hole and non-uniform black string branches should behave as sketched in Figure 4. Thus, the large \(d\) behavior of the phase structure is expected to be such that the non-uniform string gets closer and closer to having \(n = n_{GL} = 1/(d-2)\).
4 Phases with Kaluza-Klein bubbles

Until now all the solution we have been discussing lie in the range $0 \leq n \leq 1/(d-2)$. But, there are no direct physical restrictions on $n$ preventing it from being in the range $1/(d-2) < n \leq d-2$. The question is therefore whether there exists solutions in this range or not. This was answered in [13] where it was shown that pure gravity solutions with Kaluza-Klein bubbles can realize all values of $n$ in the range $1/(d-2) < n \leq d-2$.

In this section we first review the static Kaluza-Klein bubble and its place in the $(\mu, n)$ phase diagram. We then discuss the main properties of bubble-black hole sequences, which are phases of Kaluza-Klein black holes that involve Kaluza-Klein bubbles. In particular, we comment on the thermodynamics of these solutions. Subsequently, we present the five- and six-dimensional phase diagrams as obtained by including the simplest bubble-black hole sequences. Finally, we comment on non-uniqueness in the phase diagram.

Static Kaluza-Klein bubble

Kaluza-Klein bubbles were discovered in [44] by Witten. In [44] it was explained that the Kaluza-Klein vacuum $\mathcal{M}^4 \times S^1$ is unstable semi-classically, at least in absence of fundamental fermions. The semi-classical instability of $\mathcal{M}^4 \times S^1$ is in terms of a spontaneous creation of expanding Kaluza-Klein bubbles, which are Wick rotated 5D Schwarzschild black hole solutions. The Kaluza-Klein bubble is essentially a minimal $S^2$ somewhere in the space-time, i.e. a “bubble of nothing”. In the expanding Kaluza-Klein bubble solution the $S^2$ bubble then expands until all of the space-time is gone. However, apart from these time-dependent bubble solutions there are also solutions with static bubbles, as we now first review.

To construct the static Kaluza-Klein bubble in $d+1$ dimensions we take the Euclidean section of the $d$-dimensional Schwarzschild black hole and add a trivial time-direction. This gives the metric

$$ds^2 = -dt^2 + \left(1 - \frac{R^{d-3}}{r^{d-3}}\right)dz^2 + \frac{1}{1 - \frac{R^{d-3}}{r^{d-3}}}dr^2 + r^2 \Omega^2_{d-2}.$$ (4.1)
We see that there is a minimal \((d-2)\)-sphere of radius \(R\) located at \(r = R\). To avoid a conical singularity we need that \(z\) is a periodic coordinate with period
\[
L = \frac{4\pi R}{d-3}.
\]
Clearly, the solution asymptotes to \(\mathcal{M}^d \times S^1\) for \(r \to \infty\). We see that the only free parameter in the solution is the circumference of the \(S^1\).

Now, since the static Kaluza-Klein bubble is a static solution of pure gravity that asymptotes to \(\mathcal{M}^d \times S^1\) it belongs to the class of solutions we are interested in. Thus, it is part of our phase diagram, and using (2.1), (2.2), (2.3), (2.5) and (4.2) we can read off \(\mu\) and \(n\) as
\[
\mu = \mu_b = \Omega_{d-2} \left( \frac{d-3}{4\pi} \right)^{d-3}, \quad n = d-2.
\]
We see from this that the static Kaluza-Klein bubble is a specific point in the \((\mu, n)\) phase diagram, which also follows from the fact that it does not have any free dimensionless parameters. Notice that \(n = d-2\) precisely saturates the upper bound on \(n\) in (2.4).

In fact, a test particle at infinity will not experience any force from the bubble (in the Newtonian limit).

The static Kaluza-Klein bubble is known to be classically unstable. This can be seen from the fact that the static bubble is the Euclidean section of the Schwarzschild black hole times a trivial time direction.\(^7\) The Euclidean flat space \(\mathbb{R}^3 \times S^1\) (hot flat space) is semi-classically unstable to nucleation of Schwarzschild black holes. This was shown by Gross, Perry, and Yaffe \[46\], who found that the Euclidean Lichnerowicz operator \(\Delta_E\) for the Euclidean section of the four dimensional Schwarzschild solution with mass \(M\) has a negative eigenvalue: \(\Delta_E u_{ab} = \lambda u_{ab}\) with \(\lambda = -0.19(GM)^{-2}\). The Lichnerowicz equation for the perturbations of the Lorentzian static bubble space-time is \(\Delta_L h_{ab} = 0\) (in the transverse traceless gauge), so taking the ansatz \(h_{ab} = u_{ab} e^{i\Omega t}\), the Lichnerowicz equation requires \(\Omega^2 = \lambda\), i.e. \(\Omega = \pm \sqrt{-\lambda}\).

The classical instability of the static Kaluza-Klein bubble causes the bubble to either expand or collapse exponentially fast. For five-dimensional Kaluza-Klein space-times, there exists initial data \[47\] for massive bubbles that are initially expanding or collapsing \[48\], and numerical studies \[49\] shows that there exist massive expanding bubbles and furthermore indicates that contracting massive bubbles collapse to a black hole with an event horizon.

If the bubble is collapsing the endstate is presumably an object with an event horizon. In this connection, it is noteworthy that the value of \(\mu\) in (4.3) is smaller than \(\mu_{GL}\), i.e.

\(^7\)Another way to understand the classical instability of the static Kaluza-Klein bubble is to relate it to the marginal mode of the uniform black string. See e.g. Ref. \[15\] where the linear stability of static bubble solutions of Einstein-Maxwell theory was examined. Here a unique unstable mode was found and shown to be related, by double analytic continuation, to marginally stable stationary modes of perturbed charged black strings.
the Gregory-Laflamme mass, for $4 \leq d \leq 9$ (as can be seen by comparing $\mu_b$ in (4.3) to $\mu_{GL}$ in Table 1). This means that the static Kaluza-Klein bubble does not decay to the uniform black string. It is therefore likely that the Kaluza-Klein bubble in that case decays to whatever is the endstate of the uniform black string (see end of Section 3).

Bubble-black hole sequences

We have now a solution at $n = d - 2$, where it should be emphasized that this solution does not have any event horizon, and hence no entropy or temperature. But so far in this review we have not mentioned any solutions lying in the range $1/(d - 2) < n < d - 2$. However, as shown in [13], the solutions in that range have the property that they contain both Kaluza-Klein bubbles and black hole event horizons with various topologies.

For $d = 4, 5$ Emparan and Reall constructed in [50] exact solutions describing a black hole attached to a Kaluza-Klein bubble using a generalized Weyl ansatz, describing axisymmetric static space-times in higher-dimensional gravity.\footnote{See Ref. [51] for the generalization of this class to stationary solutions.} For $d = 4$ this was generalized in [52] to two black holes plus one bubble or two bubbles plus one black string. There, it was also argued that the bubble balances the gravitational attraction between the two black holes, thus keeping the configuration in static equilibrium.

In [13] these solutions were generalized to a family of exact metrics for configurations with $p$ bubbles and $q = p, p \pm 1$ black holes in $D = 5, 6$ dimensions. These are regular and static solutions of the vacuum Einstein equations, describing sequences of Kaluza-Klein bubbles and black holes placed alternately, e.g. for $(p, q) = (2, 3)$ we have the sequence:

\begin{center}
black hole - bubble - black hole - bubble - black hole.
\end{center}

This class of solutions is called bubble-black hole sequences and we refer to particular elements of this class as $(p, q)$ solutions. This large class of solutions, which was anticipated in Ref. [50], thus includes as particular cases the $(1, 1), (1, 2)$ and $(2, 1)$ solutions obtained and analyzed in [50] [52].

We refer the reader to [13] for the explicit construction of these bubble-black hole sequences and a comprehensive analysis of their properties. Here we list a number of essential features:

- All values of $n$ in the range $1/(d - 2) < n < d - 2$ are realized.
- The mass $\mu$ can become arbitrarily large, and for $\mu \to \infty$ we have $n \to 1/(d - 2)$.
- The solutions contain bubbles and black holes of various topologies. In the five-dimensional case we find black holes with spherical $S^3$ and ring $S^2 \times S^1$ topology, depending on whether the black hole is at the end of the sequence or not. Similarly, in the six-dimensional case we find black holes with ring topology $S^3 \times S^1$ and tuboid topology $S^2 \times S^1 \times S^1$, depending on whether the black hole is at the end of the
sequence or not. The bubbles support the $S^1$’s of the horizons against gravitational collapse.

• The $(p, q)$ solutions are subject to constraints enforcing regularity, but this leaves $q$ independent dimensionless parameters allowing for instance the relative sizes of the black holes to vary. The existence of $q$ independent parameters in each $(p, q)$ solution is the reason for the large degree of non-uniqueness in the $(\mu, n)$ phase diagram, when considering bubble-black hole sequences.

An interesting feature of the bubble-black hole sequences is that there exists a map between five- and six-dimensional solutions \[13\]. As a consequence there is a corresponding map between the physical parameters which reads

\[
\mu^{(6D)} = \frac{2\pi}{3L^{(6D)}} \mu^{(5D)} \left(5 - n^{(5D)}\right), \quad n^{(6D)} = \frac{5n^{(6D)} - 1}{5 - n^{(5D)}}, \quad (4.4)
\]

\[
t^{(6D)}_k = t^{(5D)}_k, \quad s^{(6D)}_k = \frac{4\pi}{L^{(5D)}} s^{(5D)}_k, \quad (4.5)
\]

where the superscripts $5D$ and $6D$ label the five- and six-dimensional quantities respectively. This form of the map assumes a certain normalization of the parameters of the solution, or equivalently, a choice of units, as further explained in Ref. \[13\].

**Thermodynamics**

For static space-times with more than one black hole horizon we can associate a temperature to each black hole by analytically continuing the solution to Euclidean space and performing the proper identifications needed to make the Euclidean solution regular where the horizon was located in the Lorentzian solution. The temperatures of the black holes need not be equal, and one can derive a generalized Smarr formula that involves the temperature of each black hole. The Euclidean solution is regular everywhere only when all the temperatures are equal. It is always possible to choose the $q$ free parameters of the $(p, q)$ solution to give a one-parameter family of regular equal temperature solutions, which we denote $(p, q)_t$.

The equal temperature $(p, q)_t$ solutions are of special interest for two reasons: First, the two solutions, $(p, q)_t$ and $(q, p)_t$, are directly related by a double Wick rotation which effectively interchanges the time coordinate and the coordinate parameterizing the Kaluza-Klein circle. This provides a duality map under which bubbles and black holes are interchanged, giving rise to the following explicit map between the physical quantities of the solutions

\[
\mu' = nt^{d-3}\mu, \quad n' = \frac{1}{n}, \quad t' = \frac{1}{t}, \quad s' = \frac{(d-2)n-1}{d-2-n} t^{d-1}s. \quad (4.6)
\]

Secondly, for a given family of $(p, q)$ solutions, the equal temperature solution extremizes the entropy for fixed mass $\mu$ and fixed size of the Kaluza-Klein circle at infinity. For all
explicit cases considered we find that the entropy is minimized for equal temperatures.\(^9\)

Furthermore, the entropy of the \((1,1)\) solution is always lower than the entropy of the uniform black string of the same mass \(\mu\). We expect that all other bubble-black hole sequences \((p, q)\) have entropy lower than the \((1,1)\) solution, and this is confirmed for all explicitly studied examples. The physical reason to expect that all bubble-black hole sequences have lower entropy than a uniform string of same mass, is that some of the mass has gone into the bubble rather than the black holes, giving a smaller horizon area for the same mass.

**Phase diagrams for \(d = 4\) and \(d = 5\)**

The general \((\mu, n)\) phase diagrams for \(d = 4, 5\) can in principle be drawn with all possible values \((p, q)\), though the explicit solution of the constraints becomes increasingly complicated for high \(p, q\). However, the richness of the phase structure and the non-uniqueness in the \((\mu, n)\) phase diagram, is already illustrated by considering some particular examples of the general class of solutions, as was done in \[13\]. As an illustration, we give here the exact form of the curve for the \((1,1)\) solution, corresponding to a bubble on a black hole,

\[
d = 4: \quad n_{(1,1)}(\mu) = \frac{1}{4} \left[ -1 + 3 \sqrt{1 + \frac{8}{\mu^2}} \right]; \quad d = 5: \quad n_{(1,1)}(\mu) = \frac{1}{3} + \frac{4}{3\mu}. \tag{4.7}
\]

in five and six dimensions respectively. These two solutions are related by the map in \[4.4\] and one may also check that these curves are correctly self-dual under the duality map \[4.6\].

In Figure 5 we have drawn for \(d = 4\) and \(5\) the curves in the \((\mu, n)\) phase diagram for the \((p, q) = (1,1), (1,2), (2,1)\) solutions. These correspond to the configurations

\[
(1,1): \quad \text{black hole} - \text{bubble} \\
D = 5 \quad S^3 \quad D \\
D = 6 \quad S^3 \times S^1 \quad D \times S^1 (4.8)
\]

\[
(1,2): \quad \text{black hole} - \text{bubble} - \text{black hole} \\
D = 5 \quad S^3 \quad S^1 \times I \quad S^3 \\
D = 6 \quad S^3 \times S^1 \quad T^2 \times I \quad S^3 \times S^1 (4.9)
\]

\[
(2,1): \quad \text{bubble} - \text{black ring} - \text{bubble} \\
D = 5 \quad D \quad S^2 \times S^1 \quad D \\
D = 6 \quad D \times S^1 \quad S^2 \times T^2 \quad D \times S^1 (4.10)
\]

where the first/second line in each configuration corresponds to the topology in five/six dimensions. Here \(D\) denotes the disc and \(I\) the line interval.

\(^9\)This is a feature that is particular to black holes, independently of the presence of bubbles. As an analog, consider two Schwarzschild black holes very far apart. It is straightforward to see that for fixed total mass, the entropy of such a configuration is minimized when the black holes have the same radius (hence same temperature), while the maximal entropy configuration is the one where all the mass is located in a single black hole.
Figure 5: $(\mu, n)$ phase diagrams for five (left figure) and six (right figure) dimensions. We have drawn the $(p, q) = (1, 1)$, $(1, 2)_t$ and $(2, 1)$ solutions. These curves lie in the region $1/2 < n \leq 2$ for the five-dimensional case and $1/3 < n \leq 3$ for the six-dimensional case. The lowest (red) curve corresponds to the $(1, 1)$ solution. The (blue) curve that has highest $n$ for high values of $\mu$ is the equal temperature $(1, 2)_t$ solution. The (green) curve that has highest $n$ for small values of $\mu$ is the $(2, 1)$ solution. The entire phase space of the $(1, 2)$ configuration is the wedge bounded by the equal temperature $(1, 2)_t$ curve and the $(1, 1)$ curve. For completeness we have also included the uniform (orange) and non-uniform (cyan) black string branch, and the small black hole branch (magenta) displayed in Figure 2.

Non-uniqueness in the phase diagram

We finally remark on non-uniqueness in the $(\mu, n)$ phase diagram for Kaluza-Klein black holes.\(^{10}\) Clearly for a given mass there is a high degree of non-uniqueness. The non-uniqueness is not lifted once we also take into account the relative tension $n$, as there are explicitly known cases of physically distinct solutions with the same mass and tension. For example the $(1, 2)_t$ solution and the $(2, 1)$ solution intersect each other in the phase diagram. This means that we have two physically different solutions in the same point of the $(\mu, n)$ phase diagram.

Moreover, we have in fact a continuously infinite non-uniqueness\(^{11}\) for certain points in the $(\mu, n)$ phase diagram. This is due to the fact that the $(p, q)$ solution has $q$ free parameters\(^{13}\). Hence, for given $p \geq 2$ and $q \geq 3$, we have $q - 2$ free continuous

\(^{10}\)Non-uniqueness in higher dimensional pure gravity has also been found for stationary black hole solutions in asymptotically flat space-time: Here, there exists for a certain range of parameters both a rotating black hole with $S^3$ horizon\(^ {53}\) and rotating black rings with $S^2 \times S^1$ horizons\(^ {53}\).

\(^{11}\)Infinite non-uniqueness has also been found in\(^ {53}\) for black rings with dipole charges in asymptotically flat space.
parameters labelling physically different \((p, q)\) solutions, for certain points in the \((\mu, n)\) phase diagram.

5 Discussion

In this review we have summarized the current state of knowledge on the phases of Kaluza-Klein black holes. It is an open question whether there exist other phases beyond the ones discussed above, but for five and six dimensions it could be that the picture presented above is complete. There are, however, a number of important issues and questions regarding the present phase structure, which we briefly list here:

- At present the complete non-uniform branch is numerically known only for \(d = 5\). It would be interesting to compute it in other dimensions. Similarly, for the black hole branch, the present status is that only for \(d = 4\) and 5 do we numerically know the entire branch, and having the numerical data for other dimensions would be interesting as well.

- If possible, it would obviously be very interesting to find an analytic form for the non-uniform and black hole branch, for example using the ansatz (3.1). A first step in this direction would be to extend the first order analytical results for these branches to second order.\(^\text{12}\)

- The classical stability of all branches (except the uniform one, where we know the Gregory-Laflamme instability) should be examined.

- The bubble-black hole sequences are only known in five and six dimensions. It is interesting to ask if solutions for bubble-black hole sequences exist for \(D \geq 7\), and if so, whether they can be related to the solutions of \([\text{13}]\) by a map similar to the one mapping five- to six-dimensional solutions. For spaces with more than one Kaluza-Klein circle, i.e. asymptotics \(\mathcal{M}^d \times T^k \geq 2\), it is easy construct solutions describing regular bubble-black hole sequences \([\text{13}]\). On the other hand, finding bubble-black hole sequences in \(\mathcal{M}^{d \geq 6} \times S^1\) may be difficult since one cannot use the generalized Weyl ansatz for such solutions, but we do not see any physical obstructions to the existence of them. It would also be interesting to see what type of topologies occur in these higher dimensional bubble-black hole sequences, should they exist.

- In connection with the above points, the general question of the \(d\)-dependence of the phase diagram is also an important one. For example, for the non-uniform branch it is known that there is a critical dimension beyond which the slope changes sign. Likewise one could imagine new phases appearing (or other phases disappearing) as the number of dimensions increases.\(^\text{12}\)

\(^{12}\)For the \(d = 4\) black hole branch this was done in Ref. \([\text{35}]\).
The instability of the uniform black string implies that a naked singularity may be formed when the horizon of the black string pinches. As originally pointed out by Gregory and Laflamme [19, 20], this would entail a violation of the Cosmic Censorship Hypothesis. The results of [23, 27] suggest that, for certain dimensions, the localized black hole is the only solution with higher entropy than that of the uniform black string (for masses where the black string is unstable). Thus, the endpoint of the instability seems to be the localized black hole which means that the horizon should pinch off in the classical evolution. On the other hand, in [41] it was argued that the horizon cannot pinch off in finite affine parameter, thus suggesting that it is not possible for the black string horizon to pinch off. A way to reconcile these two results is if the horizon pinches off in infinite affine parameter. Recently, the numerical analysis of [43, 42] indicates that this indeed is the case. If this is correct it would be interesting to examine the implications for the Cosmic Censorship Hypothesis.

We conclude with a brief discussion of the relevance of the study of Kaluza-Klein black holes in the context of String/M-theory as well as supersymmetric Yang-Mills (SYM) theories and Little String Theory (LST) (see also the short review [1]). Recently a map\(^\text{13}\) was found [9] (see also Refs. [56, 57] for related work) from static and neutral Kaluza-Klein black holes to non- and near-extremal (singly charged) branes of String/M-theory on a transverse circle. This gives a precise connection between phases of static and neutral Kaluza-Klein black holes and the thermodynamic behavior of the non-gravitational theories dual to near-extremal branes on a circle. In this way any phase of Kaluza-Klein black holes has immediate consequences for the phase structure of these dual non-gravitational theories. In particular, for the thermodynamics of strongly-coupled SYM theories on a circle this has led to the prediction of a new non-uniform phase as well as new information about the localized phase [9]. For finite temperature two-dimensional SYM theory on a circle it was shown in [57] that a non-uniform phase at weak coupling indeed appears, with qualitatively similar features as the one predicted at strong coupling. As another application, Ref. [9] also presents evidence for the existence of a new stable phase of (2, 0) LST in the canonical ensemble for temperatures above its Hagedorn temperature. The application of this map to the bubble-black hole sequences, by which non and near-extremal bubble-black hole sequences are generated, is in progress [58].

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\(^{13}\)For the class of solutions described by the ansatz [51], this map was already discovered in Ref. [16].
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