Type $\text{III}_1$ factors generated by regular representations of infinite dimensional nilpotent group $B_0^\mathbb{N}$

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Abstract

We study the von Neumann algebra, generated by the unitary representations of infinite-dimensional groups nilpotent group $B_0^{ni}$. The conditions of the irreducibility of the regular and quasiregular representations of infinite-dimensional groups (associated with some quasi-invariant measures) are given by the so-called Ismagilov conjecture (see [1,2,9,10,11]). In this case the corresponding von Neumann algebra is type $I_{\infty}$ factor. When the regular representation is reducible we find the sufficient conditions on the measure for the von Neumann algebra to be factor (see [13,14]).

In the present article we determine the type of corresponding factors. Namely we prove that the von Neumann algebra generated by the regular representations of infinite-dimensional nilpotent group $B_0^{ni}$ is type $\text{III}_1$ hyperfinite factor. The case of the nilpotent group $B_0^{ni}$ of infinite in both directions matrices will be studied in [6].

Key words: von Neumann algebra, type $\text{III}_1$ factor, unitary representation, infinite-dimensional groups, nilpotent groups, regular representations, irreducibility, infinite tensor products, Gaussian measures, Ismagilov conjecture

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1 Regular representations

Let us consider the group $\tilde{G} = B^\mathbb{N}$ of all upper-triangular real matrices of infinite order with unities on the diagonal

$$\tilde{G} = B^\mathbb{N} = \{I + x \mid x = \sum_{1 \leq k < n} x_{kn}E_{kn}\},$$

and its subgroup

$$G = B_0^\mathbb{N} = \{I + x \in B^\mathbb{N} \mid x \text{ is finite}\},$$

where $E_{kn}$ is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k,n}$ is finite means that $x_{kn} = 0$ for all $(k, n)$ except for a finite number of indices $k, n \in \mathbb{N}$.

Obviously, $B_0^\mathbb{N} = \varprojlim_{n} B(n, \mathbb{R})$ is the inductive limit of the group $B(n, \mathbb{R})$ of real upper-triangular matrices with unities on the principal diagonal

$$B(n, \mathbb{R}) = \{I + \sum_{1 \leq k < r \leq n} x_{kr}E_{kr} \mid x_{kr} \in \mathbb{R}\}$$

with respect to the imbedding $B(n, \mathbb{R}) \ni x \mapsto x + E_{n+1n+1} \in B(n + 1, \mathbb{R})$.

We define the Gaussian measure $\mu_b$ on the group $B^\mathbb{N}$ in the following way

$$d\mu_b(x) = \otimes_{1 \leq k < n} (b_{kn}/\pi)^{1/2} \exp(-b_{kn}x_{kn}^2)dx_{kn} = \otimes_{k<n}d\mu_{b_{kn}}(x_{kn}), \quad (1)$$

where $b = (b_{kn})_{k,n}$ is some set of positive numbers.
Let us denote by $R$ and $L$ the right and the left action of the group $B^N$ on itself: $R_s(t) = ts^{-1}$, $L_t(s) = st$, $s, t \in B^N$ and by $\Phi : B^N \mapsto B^N$, $\Phi(I + x) := (I + x)^{-1}$ the inverse mapping. It is known \cite{9,10} that

**Lemma 1** $\mu^R_{b_t} \sim \mu_b \forall t \in B_0^N$ for any set $b = (b_{kn})_{k<n}$.

**Lemma 2** $\mu^L_{b} \sim \mu_b \forall t \in B_0^N$ if and only if $S_{kn}^L(b) < \infty$, $\forall k < n$, where

$$S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}.$$

**Lemma 3** $\mu^L_{b} \perp \mu_b \forall t \in B_0^N \setminus \{e\} \iff S_{kn}^L(b) = \infty \forall k < n$.

**Lemma 4** \cite{12} If $E(b) = \sum_{k<n} S_{kn}^L(b) (b_{kn})^{-1} < \infty$, then $\mu^\Phi_b \sim \mu_b$.

**Lemma 5** \cite{12} The measure $\mu_b$ on $B^N$ is $B^N_0$ ergodic with respect to the right action.

Let $\alpha : G \rightarrow \text{Aut}(X)$ be a measurable action of a group $G$ on the measurable space $X$. We recall that a measure $\mu$ on the space $X$ is $G$-ergodic if $f(\alpha_t(x)) = f(x) \forall t \in G$ implies $f(x) = \text{const} \mu$ a.e. for all functions $f \in L^1(X, \mu)$.

**Remark 6** \cite{13} If $\mu^\Phi_b \sim \mu_b$ then $\mu^L_{b_t} \sim \mu_b \forall t \in B_0^N$.

**Proof.** This follows from the fact that the inversion $\Phi$ replace the right and the left action: $R_t \circ \Phi = \Phi \circ L_t \forall t \in B^N$. Indeed, if we denote $\mu^f(\cdot) = \mu(f^{-1}(\cdot))$ we have $(\mu^f)^\Phi = \mu^{f \circ g}$. Hence

$$\mu_b \sim \mu^R_{b_t} \sim (\mu_b^R)^\Phi = \mu_b^{R_t \circ \Phi} = \mu_b^{\Phi \circ L_t} = (\mu_b^\Phi)^L_t \sim \mu^L_{b_t}.$$

$\square$

If $\mu^R_{b_t} \sim \mu_b$ and $\mu^L_{b_t} \sim \mu_b \forall t \in B_0^N$, one can define in a natural way (see \cite{9,10}), an analogue of the right $T^{R,b}$ and left $T^{L,b}$ representation of the group $B_0^N$ in Hilbert space $H_b = L_2(B^N, d\mu_b)$

$$T^{R,b}, T^{L,b} : B_0^N \rightarrow U(H_b = L_2(B^N, d\mu_b)),$$

$$T^{R,b}_t f(x) = (d\mu_b(xt)/d\mu_b(x))^{1/2} f(xt),$$

$$T^{L,b}_s f(x) = (d\mu_b(s^{-1}x)/d\mu_b(x))^{1/2} f(s^{-1}x).$$
2 Von Neuman algebras generated by the regular representations

Let \( \mathfrak{A}^{R,b} = (T^{R,b}_t \mid t \in B^N_0)'' \) (resp. \( \mathfrak{A}^{L,b} = (T^{L,b}_s \mid s \in B^N_0)'' \)) be the von Neumann algebras generated by the right \( T^{R,b} \) (resp. the left \( T^{L,b} \)) regular representation of the group \( B^N_0 \).

**Theorem 7** [12] If \( E(b) < \infty \) then \( \mu^L_b \sim \mu^b \). In this case the left regular representation is well defined and the commutation theorem holds:

\[ (\mathfrak{A}^{R,b})' = \mathfrak{A}^{L,b}. \] (2)

Moreover, the operator \( J_{\mu^b} \) given by

\[ (J_{\mu^b} f)(x) = (d\mu^b(x^{-1})/d\mu^b(x))^{1/2} \overline{f(x^{-1})} \] (3)

is an intertwining operator:

\[ T^{L,b}_t = J_{\mu^b} T^{R,b}_t J_{\mu^b}, \quad t \in B^N_0 \quad \text{and} \quad J_{\mu^b} \mathfrak{A}^{R,b} J_{\mu^b} = \mathfrak{A}^{L,b}. \]

If \( \mu^L_b \perp \mu_b \) \( \forall t \in B^N_0 \setminus \{e\} \) one can’t define the left regular representation of the group \( B^N_0 \). Moreover the following theorem holds

**Theorem 8** The right regular representation \( T^{R,b} : B^N_0 \mapsto U(H_b) \) is irreducible if and only if \( \mu^L_b \perp \mu_b \) \( \forall s \in B^N_0 \setminus \{0\} \).

**Corollary 9** The von Neumann algebra \( \mathfrak{A}^{R,b} \) is a type \( I_\infty \) factor if

\[ \mu^L_b \perp \mu_b \quad \forall s \in B^N_0 \setminus \{0\}. \]

Let us assume now that \( \mu^L_b \sim \mu_b \) \( \forall t \in B^N_0 \setminus \{e\} \). In this case the right regular representation and the left regular representation of the group \( B^N_0 \) are well defined.

In [13] the condition were studied when the von Neumann algebra \( \mathfrak{A}^{R,b} \) is factor, i.e.

\[ \mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' = \{\lambda I \mid \lambda \in \mathbb{C}\}. \]

Since \( T^{L,b}_t \in (\mathfrak{A}^{R,b})' \) \( \forall t \in B^N_0 \), we have \( \mathfrak{A}^{L,b} \subset (\mathfrak{A}^{R,b})' \), hence

\[ \mathfrak{A}^{R,b} \cap (\mathfrak{A}^{R,b})' \subset (\mathfrak{A}^{L,b})' \cap (\mathfrak{A}^{R,b})' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})'. \]

(4)

The last relation shows that \( \mathfrak{A}^{R,b} \) is factor if the representation

\[ B^N_0 \times B^N_0 \ni (t, s) \rightarrow T^{R,b}_t T^{L,b}_s \in U(H_b) \]

is irreducible.
Let us denote by $\mathfrak{A}^{R,L,b}$ the von Neumann algebras generated by the right $T_{t}^{R,b}$ and the left $T_{s}^{L,b}$ regular representations of the group $B_{0}^{N}$:

$$
\mathfrak{A}^{R,L,b} = (T_{t}^{R,b}, T_{s}^{L,b} \mid t, s \in B_{0}^{N})'' = (\mathfrak{A}^{R,b} \cup \mathfrak{A}^{L,b})''.
$$

Let us denote

$$
S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S_{nm}^{L}(b)}, \ k < n.
$$

**Theorem 10** [13] The representation

$$
B_{0}^{N} \times B_{0}^{N} \ni (t, s) \mapsto T_{t}^{R,b}T_{s}^{L,b} \in U(H_{b})
$$

is irreducible if $S_{kn}^{R,L}(b) = \infty$, $\forall k < n$.

**Corollary 11** The von Neumann algebra $\mathfrak{A}^{R,b}$ is factor if $S_{kn}^{R,L}(b) = \infty$ $\forall k < n$.

3 Type III$_{1}$ factor

Let us denote as before $M = \mathfrak{A}^{L,b} = (T_{s}^{L,b} \mid s \in B_{0}^{N})''$, $\mathfrak{A}^{R,b} = (T_{t}^{R,b} \mid t \in B_{0}^{N})''$.

**Theorem 12** If $S_{kn}^{R,L}(b) = \infty$, $\forall k < n$ then the von Neumann algebra $\mathfrak{A}^{L,b}$ (and hence $\mathfrak{A}^{R,b}$) is III$_{1}$ factor.

**PROOF.** The proof is based on Lemma 13 and 14 we shall prove them later.

Using (3) we conclude that the modular operator $\Delta$ is defined as follows

$$(\Delta f)(x) = (d\mu_{b}(x)/d\mu_{b}(x^{-1}))f(x). \quad (5)$$

**Lemma 13** We have

$$
Sp\Delta = [0, \infty).
$$

We have $Sp\Delta\phi = Sp\Delta = [0, \infty)$, where $\phi(a) = (a1, 1)_{H_{b}}$, $a \in M = \mathfrak{A}^{L,b}$. The centralizer $M_{\phi}$ of $\phi$ is defined by the equality

$$
M_{\phi} = \{a \in M \mid \sigma_{t}^{\phi}(a) \forall t \in \mathbb{R}\}
$$

where $\sigma_{t}^{\phi}(a) = \Delta^{it}a\Delta^{-it}$. For every projection $e \neq 0$, $e \in M_{\phi}$, a faithful semifinite normal weight $\phi_{e}$ on the reduced von Neumann algebra $eMe =$
\{a \in M; \, ea = ae = a\} \text{ is defined by the equality } \\
\phi_e(a) = \phi(a) \, \forall a \in eMe, \, a \geq 0.

One has the formula

\[ S(M) = \bigcap_{e \neq 0} Sp\Delta_{\phi_e}, \tag{6} \]

where \( e \) varies over the nonzero projection of \( M_\phi \) (see [4] p.472).

**Lemma 14** The von Neumann algebra \( M_\phi \) is trivial.

In this case

\[ S(M) = Sp\Delta = [0, \infty), \]

so the von Neumann algebra \( A_{L,b} \) (and hence algebra \( A_{R,b} \)) is type III \( 1 \) factor. \( \square \)

**Proof of Lemma 14** We show that

\[ M_\phi = (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R}, \, s \in B^N_0)', \tag{7} \]

So \( M_\phi \) is trivial means that the set of operators

\[ (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R}, \, s \in B^N_0) \]

is irreducible. To prove (7) we get

\[ M_\phi = (a \in A_{L,b} \mid \Delta^{it}a = a\Delta^{it}, \, \forall t \in \mathbb{R}) = (\Delta^{it} \mid t \in \mathbb{R})' \cap A_{L,b} \]

\[ = (\Delta^{it} \mid t \in \mathbb{R})' \cap (A_{R,b}')' = (\Delta^{it} \mid t \in \mathbb{R})' \cap (T^{R,b}_s \mid s \in B^N_0)' = (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R}, \, s \in B^N_0)' \]

**Definition.** Recall (c.f. e.g. [5]) that a non necessarily bounded self-adjoint operator \( A \) in a Hilbert space \( H \) is said to be affiliated with a von Neumann algebra \( M \) of operators in this Hilbert space \( H \), if \( \exp(itA) \in M \) for all \( t \in \mathbb{R} \). One then writes \( A \eta M \).

To prove the irreducibility of \( (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R}, \, s \in B^N_0) \) it is sufficient to prove (see [10] p.258) that operators \( f(x) \mapsto x_{kn}f(x) \) of multiplication in the space \( H_b \) by the independent variables \( x_{kn} \) are affiliated to the von Neumann algebra

\[ (M_\phi)' = (\Delta^{it}, T^{R,b}_s \mid t \in \mathbb{R}, \, s \in B^N_0)' \gamma. \]

In this case the operator \( A \) commuting with \( \Delta^{it} \) and \( T^{R,b}_s \) is operator of multiplication by some function \( a(x) \). If we use commutation relation \( [A, T^{R,b}_s] = 0, \, s \in B^N_0 \) we obtain \( a(x) = a(xs) \mod \mu \). Using the ergodicity of the measure \( \mu_0 \) with respect of the right action of the group \( B^N_0 \) we conclude that \( a(x) = \text{const} \mod \mu \) i.e. \( A \) is scalar operator.
If we denote
\[ A_{kn}^R = \left( \frac{d}{dt} \right) T_{I+tE_{kn}} \bigg|_{t=0} \]
we have (see for example [9,10,11])
\[ A_{kn}^R = \sum_{r=1}^{k-1} x_{kr} D_{rn} + D_{kn}, \quad 1 \leq k < n. \] (9)

The direct calculation shows that
\[ [A_{13}^R, [A_{23}^R, \ln \Delta]] = 2b_{13} x_{12}, \] (10)
\[ [A_{12}^R, [A_{23}^R, \ln \Delta]] = 2b_{13} x_{13}. \] (11)

Idea: to obtain in a similar way all variables \( x_{kn} \).

Let us denote by \( X^{-1} \) the inverse matrix to the upper triangular matrix \( X = I + x = I + \sum_{k<n} x_{kn} E_{kn} \in B^{\mathbb{N}} \)
\[ X^{-1} = (I + x)^{-1} = I + \sum_{k<n} x_{kn}^{-1} E_{kn} \in B^{\mathbb{N}}. \]

We have by definition \( X^{-1}X = XX^{-1} = I \) hence
\[ \left( XX^{-1} \right)_{kn} = \sum_{r=k}^{n} x_{kr} x_{rn}^{-1} = \delta_{kn} = \sum_{r=k}^{n} x^{-1}_{kr} x_{rn} = \left( X^{-1}X \right)_{kn}, \quad k \leq n, \] (12)

hence
\[ x_{kn}^{-1} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} + x_{kn} = 0 = x_{kn} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} + x_{kn}^{-1}, \quad k < n, \]
and
\[ x_{kn}^{-1} = -x_{kn} - \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1} = -x_{kn} - \sum_{r=k+1}^{n-1} x_{kr}^{-1} x_{rn}. \] (13)

We can write also
\[ x_{kn}^{-1} = -\sum_{r=k+1}^{n} x_{kr} x_{rn}^{-1} = -\sum_{r=k}^{n-1} x_{kr}^{-1} x_{rn}. \] (14)

There is also the explicit formula for \( x_{kn}^{-1} \) (see [8] formula (4.4)) \( x_{kk+1}^{-1} = -x_{kk+1} \)
\[ x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r+1} \sum_{k \leq i_1 < i_2 < \ldots < i_r \leq n} x_{k i_1 i_2 \ldots i_r n}, \quad k < n - 1. \] (15)

Remark 15 Using (15) we see that \( x_{kn}^{-1} \) depends only on \( x_{rs} \) with \( k \leq r < s \leq n. \)
Using (14) we have

\[ x_{kn} + x_{kn}^{-1} = - \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1}, \quad x_{kn} - x_{kn}^{-1} = 2x_{kn} - \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1}. \]  

(16)

Let us denote

\[ w_{kn} := w_{kn} (x) := (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1}). \]  

(17)

Using (1) we get

\[
\Delta (x) = \frac{d\mu_{b}(x)}{d\mu_{b}(x^{-1})} = \exp \left[ - \sum_{k<n} b_{kn} \left( x_{kn}^2 - (x_{kn}^{-1})^2 \right) \right] = \exp \left[ - \sum_{k<n} b_{kn} w_{kn} (x) \right].
\]

\[
- \ln \Delta (x) = \sum_{k<n} b_{kn} \left( x_{kn}^2 - (x_{kn}^{-1})^2 \right) = \sum_{k<n} b_{kn} (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})
\]

\[
= \sum_{k<n} b_{kn} (x_{kn} + x_{kn}^{-1})[2x_{kn} - (x_{kn} + x_{kn}^{-1})] = \sum_{k<n} b_{kn} w_{kn} (x).
\]

To study the action of the operators \( A_{kn}^R = \sum_{r=k+1}^{n-1} x_{kr} D_{rn} + D_{kn} \) on the function \( \ln \Delta (x) \) we need to know the action of \( D_{pq} \) on \( x_{kn}^{-1} \).

**Lemma 16** We have

\[
[D_{pq}, x_{kn}^{-1}] = \begin{cases} 
-x_{kp}^{-1} x_{qn}^{-1}, & \text{if } k \leq p < q \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]  

(19)

**PROOF.** We prove (19) by induction in \( p : k \leq p < q \leq n \). For \( p = k \) using (16) we have

\[
[D_{kq}, x_{kn}^{-1}] = -[D_{kq}, x_{kn} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1}] = -[D_{kq}, x_{kq} x_{qn}^{-1}] = -x_{qn}^{-1} = -x_{kk}^{-1} x_{qn}^{-1},
\]

so (19) holds for \( p = k \).

Let us suppose that (19) holds for all \( (p, q) \) with \( k \leq p < s \leq n, \ k \leq p < q \leq n \). We prove that than (19) holds also for \( (s, q) : s < q \leq n \). Indeed we have

\[
[D_{sq}, x_{kn}^{-1}] = -[D_{sq}, x_{kn}^{-1} + \sum_{r=k+1}^{n-1} x_{kr} x_{rn}^{-1}] = - \sum_{r=k+1}^{s} x_{kr} [D_{sq}, x_{rn}^{-1}]
\]

\[
= \sum_{r=k+1}^{s} x_{kr} x_{rs}^{-1} x_{qn}^{-1} = x_{ks}^{-1} x_{qn}^{-1}.
\]

\[\Box\]
Using (19) we get

\[
[D_{pq}, x_{kn} + x_{kn}^{-1}] = \begin{cases} 
-x_{kp}^{-1}x_{qn}^{-1}, & \text{if } k \leq p < q \leq n, \ (p,q) \neq (k,n) \\
0, & \text{otherwise}.
\end{cases}
\]

(20)

Using (20) we have

\[
[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] = \begin{cases} 
2x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1}, & \text{if } k \leq p < q \leq n, \ (p,q) \neq (k,n) \\
2(x_{kn} + x_{kn}^{-1}), & \text{if } (p,q) = (k,n) \\
0, & \text{otherwise}.
\end{cases}
\]

(21)

Indeed, if \( k \leq p < q \leq n, \ (p,q) \neq (k,n) \) we have

\[
[D_{pq}, (x_{kn} + x_{kn}^{-1})(x_{kn} - x_{kn}^{-1})] = [D_{pq}, (x_{kn} + x_{kn}^{-1})(2x_{kn} - (x_{kn} + x_{kn}^{-1}))]
\]

\[
= [D_{pq}, (x_{kn} + x_{kn}^{-1})](2x_{kn} - (x_{kn} + x_{kn}^{-1})) - (x_{kn} + x_{kn}^{-1})[D_{pq}, (x_{kn} + x_{kn}^{-1})] = -2x_{kn}^{-1}[D_{pq}, (x_{kn} + x_{kn}^{-1})] = 2x_{kp}^{-1}x_{qn}^{-1}x_{kn}^{-1}.
\]

Lemma 17 We have

\[
[A_{mm+1}^R, w_{kn}] = \begin{cases} 
0, & \text{if } k < n \leq m \\
2x_{km}x_{km+1}, & \text{if } n = m + 1, \ 1 \leq k \leq m - 1 \\
0, & \text{if } 1 \leq k \leq m - 1, \ m + 1 < n \\
2x_{mn}^{-1}x_{m+1n}, & \text{if } k = m, \ n \geq m + 2 \\
0, & \text{if } m + 1 \leq k < n.
\end{cases}
\]

(22)

hence

\[
- [A_{mm+1}^R, \ln \Delta] = 2 \sum_{r=1}^{m-1} b_{rm+1}x_{rm}x_{rm+1} + 2 \sum_{n=m+2}^{\infty} b_{mn}x_{mn}^{-1}x_{m+1n}^{-1}.
\]

(23)

PROOF. Since

\[
A_{mm+1}^R = \sum_{r=1}^{m-1} x_{rm}D_{rm+1} + D_{mm+1}
\]

and \( w_{kn}, \ k < n \leq m \) do not depend on \( x_{rm+1}, \ 1 \leq r \leq m + 1 \) we conclude that \([A_{mm+1}^R, w_{kn}] = 0\) for \( k < n \leq m \) and \( m + 1 \leq k < n \).
Let $n = m + 1$, since $[D_{rm+1}, w_{km+1}] = 0$ for $1 \leq r < k$ we get

$$[A_{mm+1}^R, w_{km+1}] = \sum_{r=k}^{m-1} x_{rm}[D_{rm+1}, w_{km+1}] + [D_{mm+1}, w_{km+1}] =$$

$$2 \left( x_{km}(x_{km+1} + x_{km-1}^{-1}) + \sum_{r=k+1}^{m-1} x_{rm}x_{kr}^{-1} x_{km+1}^{-1} + x_{km}x_{km+1}^{-1} \right)$$

$$= 2 \left( x_{km}x_{km+1} + \left( x_{km} + \sum_{r=k+1}^{m-1} x_{kr}^{-1} x_{rm} + x_{km}^{-1} \right) x_{km+1}^{-1} \right) \overset{13}{=} 2x_{km}x_{km+1}.$$  

Similarly, for $1 \leq k \leq m - 1$, $m + 1 < n$ we get

$$[A_{mm+1}^R, w_{kn}] = \sum_{r=k}^{m-1} x_{rm}[D_{rm+1}, w_{kn}] + [D_{mm+1}, w_{kn}] =$$

$$2 \left( x_{km}x_{m+1n}^{-1} + \sum_{r=k+1}^{m-1} x_{rm}x_{kr}^{-1} x_{m+1n}^{-1} + x_{km}x_{m+1n}^{-1} \right)$$

$$= 2 \left( x_{km} + \sum_{r=k+1}^{m-1} x_{rm}x_{kr}^{-1} + x_{km}^{-1} \right) x_{m+1n}^{-1} \overset{13}{=} 0.$$  

Finally if $k = m$ and $n \geq m + 2$ we have as before

$$[A_{mm+1}^R, w_{mn}] = [D_{mm+1}, w_{mn}] \overset{21}{=} 2x_{mn}^{-1}x_{m+1n}^{-1}.$$  

We consider the action of $A_{mm+1}^R$ on $\ln \Delta$.

Let $m = 2$. Since

$$[A_{23}^R, w_{13}] = 2b_{13}x_{12}x_{13}, \quad [A_{23}^R, w_{1n}] = 0, \quad n \geq 4, \quad [A_{23}^R, w_{kn}] = 0, \quad 3 \leq k < n,$$

we have

$$-[A_{23}^R, \ln \Delta] = 2b_{13}x_{12}x_{13} + 2 \sum_{n=4}^{\infty} b_{2n}x_{2n}^{-1}x_{3n}^{-1},$$

hence

$$-[A_{12}^R, [A_{23}^R, \ln \Delta]] = 2b_{13}x_{13},$$

$$-[A_{13}^R, [A_{23}^R, \ln \Delta]] = 2b_{13}x_{12}.$$  

The last two equations gives us $x_{12}, x_{13} \eta 2x$.

Let $m = 3$. Since

$$[A_{34}^R, w_{13}] = 0, \quad [A_{34}^R, w_{14}] = 2x_{13}x_{14}, \quad [A_{34}^R, w_{24}] = 2x_{23}x_{24},$$

$$[A_{34}^R, w_{15}] = 2x_{13}x_{14}x_{15}, \quad [A_{34}^R, w_{25}] = 2x_{23}x_{24}x_{25},$$
\[ A_{34}^{R}, w_{1n} = [A_{34}^{R}, w_{1n}] = 0, \quad [A_{34}^{R}, w_{3n}] = b_{3n} x_{3n}^{-1} x_{4n}^{-1}, \quad n \geq 5, \]
\[ [A_{34}^{R}, w_{kn}] = 0, \quad 4 \leq k < n, \]
we have
\[-[A_{34}^{R}, \ln \Delta] = 2b_{14} x_{13} x_{14} + 2b_{24} x_{23} x_{24} + 2 \sum_{n=5}^{\infty} b_{3n} x_{3n}^{-1} x_{4n}^{-1}, \]
hence
\[-[A_{23}^{R}, [A_{34}^{R}, \ln \Delta]] = 2b_{14} x_{12} x_{14} + 2b_{24} x_{24} \]
\[-[A_{12}^{R}, [A_{34}^{R}, \ln \Delta]]] = 2b_{14} x_{14}, \]
\[-[A_{24}^{R}, [A_{34}^{R}, \ln \Delta]] = 2[b_{12} D_{14} + D_{24}, b_{14} x_{13} x_{14} + b_{24} x_{23} x_{24}] = 2b_{14} x_{12} x_{13} + 2b_{24} x_{23}. \]
Since \( x_{12}, x_{13} \eta A \) from the latter equation we conclude that \( x_{23} \eta A \). The previous equation gives us \( x_{14} \eta A \) and the equation before gives \( x_{24} \eta A \). Finally we conclude that \( x_{14}, x_{24}, x_{23} \eta A \).

Let us suppose that we have obtained the variables \( x_{rm}, 1 \leq r \leq m - 2 \) and \( x_{m-2,m-1} \). We prove that we can obtain the following variables \( x_{rm+1}, 1 \leq r \leq m - 1 \) and \( x_{m-1,m} \).

Indeed we calculate the action of the following sequence of operators on the result: \( A_{m-1,m}^{R}, A_{m-2,m-1}^{R} \) etc. till \( A_{12}^{R} \). We obtain
\[-[A_{m-1,m}^{R}, [A_{mm+1}^{R}, \ln \Delta]] = 2 \left( \sum_{r=1}^{m-2} b_{r,m+1} x_{r-1,m} x_{r,m+1} + b_{m-1,m+1} x_{m-1,m+1} \right), \]
\[-[A_{m-2,m-1}^{R}, [A_{m-1,m}^{R}, [A_{mm+1}^{R}, \ln \Delta]]] \]
\[= 2 \left( \sum_{r=1}^{m-3} b_{r,m+1} x_{r-2,m} x_{r,m+1} + b_{m-2,m+1} x_{m-2,m+1} \right), \]
\[-[A_{m-s,m-s+1}^{R}, [A_{m-s+1,m-s+2}^{R}, \ldots, [A_{m-1,m}^{R}[A_{mm+1}^{R}, \ln \Delta]] \ldots]] \]
\[= 2 \left( \sum_{r=1}^{m-s-1} b_{r,m+1} x_{r,m-s} x_{r,m+1} + b_{m-s,m+1} x_{m-s,m+1} \right), \quad 1 \leq s \leq m, \]
\[-[A_{34}^{R}, \ldots, [A_{mm+1}^{R}, \ln \Delta]] \ldots] = 2(b_{1,m+1} x_{13} x_{1,m+1} + b_{2,m+1} x_{23} x_{2,m+1} + b_{3,m+1} x_{3,m+1}), \]
\[-[A_{24}^{R}, \ldots, [A_{mm+1}^{R}, \ln \Delta]] \ldots] = 2(b_{1,m+1} x_{12} x_{1,m+1} + b_{2,m+1} x_{22} x_{2,m+1}), \]
\[-[A_{12}^{R}, [A_{23}^{R}, \ldots, [A_{mm+1}^{R}, \ln \Delta]] \ldots]] = 2b_{1,m+1} x_{1,m+1}. \]
From the latter equation we conclude that \( x_{1,m+1} \eta A \). The last but one equation gives us \( x_{2,m+1} \eta A \) (since \( x_{12}, x_{1,m+1} \eta A \)) etc. i.e. \( x_{rm+1} \eta A, 1 \leq r \leq m - 1 \).

\[-[A_{m-1,m+1}^{R}, [A_{mm+1}^{R}, \ln \Delta]] = \left[ \sum_{r=1}^{m-2} x_{rm-1} D_{rm+1} + D_{m-1,m+1}, 2 \sum_{r=1}^{m-1} b_{r,m+1} x_{rm} x_{rm+1} \right] = 2 \sum_{r=1}^{m-2} b_{r,m+1} x_{rm-1} x_{rm} + b_{m-1,m+1} x_{m-1,m}. \]
since $x_{rm-1}, x_{rm} \in A$ for $1 \leq r \leq m-2$ hence $x_{m-1,m} \eta A$. □

To be sure that all this argument works we should prove that all involved operators are affiliated to the von Neumann algebra $M'_\phi$ defined by (7). For example if $A_{23}^R$ and $\Delta$ (and hence $\ln \Delta$) are affiliated to the von Neumann algebra $M'_\phi$, why the operator $[A_{23}^R, \ln \Delta]$ is also affiliated. In general, why the operators $[A_{12}^R, [A_{23}^R, [A_{34}^R, ... [A_{m_{m+1}}^R, \ln \Delta]...]]$ are affiliated?

**Remark 18** In general we do not know whether the commutator $[A, B]$ of two operators $A$ and $B$ affiliated to the von Neumann algebra is also affiliated.

This is the reason, why we use another approach to prove that the algebra $M_\phi$ is trivial.

## 4 The von Neumann algebra $M_\phi$ is trivial

Since $M_\phi = (\Delta^t, T^R_s | t \in \mathbb{R}, s \in B_0^N)'$ (see (7)) it is sufficient to prove that the set of operators

$$(\Delta^s, T^R_t | s \in \mathbb{R}, t \in B_0^N) \subset M'_\phi$$

is irreducible.

**Idea of the proof.** We show that the von Neumann subalgebra in the algebra $M'_\phi$, generated by the following operators

$$\{\{T^R_{t_n}, T^R_{t_{n-1}}, ..., T^R_{t_1}, \Delta^s\}...\} \mid s \in \mathbb{R}, t_1, ..., t_n \in B_0^N,$$

where $\{a, b\} := aba^{-1}b^{-1}$ is the maximal abelian subalgebra. More precisely we prove that this subalgebra contains all functions $\exp(isx_{kn})$, $k < n$, $s \in \mathbb{R}$.

To prove the irreducibility of the algebra $M'_\phi$ (see proof of the Lemma 14) we observe that if an bounded operator commute with all $\exp(isx_{kn})$, $k < n$, $s \in \mathbb{R}$ then this operator itself is an operator of multiplication by some essentially bounded function $A = a(x)$. Commutation relation $[T^R_t, A] = 0$ for all $t \in B_0^N$ gives us $a(xt) = a(x) \text{mod } \mu_b$ for all $t$. Since the measure $\mu_b$ is $B_0^N$-right ergodic we conclude that $A$ is trivial i.e. $A = a(x) = CI$.

We note that expressions in (24) are the "right" analog of the left hand side of the expressions (10) and (11)

$$[A^R_{13}, [A^R_{23}, \ln \Delta]] = 2b_{13}x_{12},$$
\[ [A_{12}^R, [A_{23}^R, \ln \Delta]] = 2b_{13}x_{13}, \]

involving generators \( A_{kn}^R \). In general, if we have two subgroups of unitary operators \( U(t) \) and \( V(s) \) with the generators \( A \) and \( B \), to obtain the commutator \( [iA, iB] \) it is sufficient to differentiate the following expression \( U(t)V(s)U(-t) \):

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} U(t)V(s)U(-t) \bigg|_{t=s=0} = [iA, iB].
\]

Indeed we have

\[
\frac{\partial}{\partial s} U(t)V(s)U(-t) = U(t)iBV(s)U(-t), \quad \frac{\partial}{\partial t} U(t)iBV(s)U(-t) \bigg|_{t=s=0} = (iAU(t)iBV(s)U(-t) - U(t)iBV(s)iAU(-t)) \bigg|_{t=s=0} = [iA, iB].
\]

We show that more convenient analog of the commutator \( [iA, iB] \) is commutator (in the group sense) of two one-parameter groups

\[
\{U(t), V(s)\} := U(t)V(s)U(t)^{-1}V(s)^{-1} = U(t)V(s)U(-t)V(-s).
\]

**Lemma 19** For the operator \( g \) of multiplication on the function \( g : f(x) \mapsto g(x)f(x) \) in the space \( H_b = L_2(B^N, d\mu_b) \) we have

\[
T_t^Rg(x)T_{t^{-1}}^R = g(tx), \quad t \in B^N_0.
\]

**PROOF.** We have

\[
f(x) \xrightarrow{g(x)T_{t^{-1}}^R g(x) \left( \frac{d\mu(xt^{-1})}{d\mu(x)} \right)^{1/2} f(xt^{-1}) T_{t^{-1}}^R f(x) = g(x)f(x).
\]

Using the lemma we have

\[
T_t^R \Delta^{is}(x) T_{t^{-1}}^R = \Delta^{is}(tx).
\]

Using (18) we have

\[
\Delta^{is}(x) = \exp \left( -is \sum_{k+1<n} b_{kn}(x_{kn} + x_{kn}^{-1})[2x_{kn} - (x_{kn} + x_{kn}^{-1})] \right) = \\
\exp \left( -is \sum_{k+1<n} b_{kn}w_{kn}(x) \right), \quad (25)
\]
where \( w_{kn}(x) = (x_{kn} + x_{kn}^{-1})[2x_{kn} - (x_{kn} + x_{kn}^{-1})] \) (see [17]).

We would like to obtain the functions \( \exp(isx_{kn}) \) using the expressions (21). To simplify the situation we consider firstly the projections of all considered object: the measure \( \mu^{(k)}_{M} \), the generators \( A_{kn}^{R} \), operator \( \Delta_{(k)} \) algebra \( M_{(k)} := (M_{o})^{(k)} \) etc. on the following subspace \( X^{(k)}, \ k \geq 2 \) of the space \( B^{N} \):

\[
X^{(2)} = \begin{pmatrix} 1 & x_{12} & x_{13} & \ldots & x_{1n} & \ldots \\ 0 & 1 & x_{23} & \ldots & x_{2n} & \ldots \\ \end{pmatrix}, \quad X^{(3)} = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & \ldots & x_{1n} & \ldots \\ 0 & 1 & x_{23} & x_{24} & \ldots & x_{2n} & \ldots \\ \end{pmatrix}, \quad \text{etc.}
\]

Note that

\[
\begin{pmatrix} 1 & x_{12} & x_{13} & \ldots & x_{1n} & \ldots \\ 0 & 1 & x_{23} & \ldots & x_{2n} & \ldots \\ \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x_{12} & -x_{13} & -x_{14} & \ldots & -x_{1n} & \ldots \\ 0 & 1 & x_{23} & \ldots & -x_{2n} & \ldots \\ \end{pmatrix}.
\]

We have for the corresponding projections on \( X^{(2)} \):

\[
A_{1n}^{R} = D_{1n}, \quad A_{2n}^{R} = x_{12}D_{1n} + D_{2n}, \quad A_{kn}^{R(2)} = x_{1k}D_{1n} + x_{2k}D_{2n}, \quad 2 < k < n,
\]

\[
w_{1n}(x) = (x_{1n} + x_{1n}^{-1})(x_{1n} - x_{1n}^{-1}) = x_{12}x_{2n}(2x_{1n} - x_{12}x_{2n}), \quad w_{2n}(x) = 0,
\]

hence

\[
\Delta^{is}_{(2)}(x) := \exp\left(-is \sum_{k=3}^{\infty} b_{ln}w_{1n}(x)\right) = \exp\left(-is \sum_{k=3}^{\infty} b_{ln}x_{12}x_{2n}(2x_{1n} - x_{12}x_{2n})\right).
\]

Let us denote by

\[
E_{kn}(t) := I + tE_{kn}, \quad T_{kn}(t) = T_{E_{kn}(t)}^{R}, \quad k < n, \quad t \in \mathbb{R}
\]
the corresponding one-parameter subgroups. We have

\[
\begin{pmatrix} x_{12} & x_{1m} \\ 1 & x_{2m} \end{pmatrix} E_{2m}(t) \mapsto \begin{pmatrix} x_{12} & x_{1m} + tx_{12} \\ 1 & x_{2m} + t \end{pmatrix}, \quad w_{1n}(xE_{2m}(t)) = \begin{cases} w_{1n}(x) & \text{if } n \neq m \\ w_{1m}(xE_{2m}(t)) & \text{if } n = m \end{cases}
\]

so using Lemma [19] we get

\[
\{T_{2m}(t), \Delta^{is}_{(2)}(x)\} = T_{2m}(t)\Delta^{is}_{(2)}(x)T_{2m}(-t)\Delta^{-is}_{(2)}(x) = \Delta^{is}_{(2)}(xE_{2m}(t))\Delta^{-is}_{(2)}(x) = \exp\left(-is \sum_{k=3,k \neq m}^{\infty} b_{ln}w_{1n}(x) + b_{lm}w_{1m}(xE_{2m}(t))\right)\exp\left(is \sum_{k=3}^{\infty} b_{ln}w_{1n}(x)\right) = \exp\left(-isb_{1m}[w_{1m}(xE_{2m}(t)) - w_{1m}(x)]\right) = \exp\left(isb_{1m}(2tx_{12}x_{1m} + t^2x_{12}^2)\right),
\]

since

\[
w_{1m}(xE_{2m}(t)) - w_{1m}(x) = x_{12}(x_{2m} + t)[2(x_{1m} + tx_{12}) - x_{12}(x_{2m} + t)] - x_{12}x_{2m}(2x_{1m} - x_{12}x_{2m}) = x_{12}t[x_{12}x_{2m} + t(2x_{1m} - x_{12}x_{2m}) + t^2x_{12}] = 2tx_{12}x_{1m} + t^2x_{12}^2.
\]

Let us denote

\[
\phi_{t,s}(x) := \{T_{2m}(t), \Delta^{is}_{(2)}(x)\} = \exp\left(isb_{1m}(2tx_{12}x_{1m} + t^2x_{12}^2)\right).
\]

(28)
Using Lemma 19 we get
\[ \{T_{1m}(t_1), \{T_{2n}(t), \Delta_{[2]}^i(x)\} = \{T_{1m}(t_1), \phi_{t,s}(x)\} = \]
\[ T_{1m}(t_1)\phi_{t,s}(x)T_{1m}(-t_1)(\phi_{t,s}(x))^{-1} = \phi_{t,s}(xE_{1m}(t_1))(\phi_{t,s}(x))^{-1} = \]
\[ \exp\left[i\sigma b_{1m}(2tx_{12}(x_{1m} + t) + t^2x_{12}^2) - i\sigma b_{1m}(2tx_{12}x_{1m} + t^2x_{12}^2)\right] = \exp(i\sigma b_{1m}x_{12}2tt_1). \]

Finally we get for \( X^{(2)} \)
\[ \exp(isx_{12}) \in M^{(2)} := (M'_{\phi})^{(2)}. \]

Using (28) we conclude that
\[ \exp(isx_{12}x_{1m}) \in M^{(2)}. \]

Applying again \( T_{12}(t) \) and \( T_{1m}(t) \) we get
\[ \{T_{12}(t), \exp(isx_{12}x_{1m})\} = T_{12}(t)\exp(isx_{12}x_{1m})T_{12}(-t)\exp(-isx_{12}x_{1m}) = \]
\[ \exp(is(x_{12} + t)x_{1m} - isx_{12}x_{1m}) = \exp(isx_{12}), \]
\[ \{T_{1m}(t), \exp(isx_{12}x_{1m})\} = T_{1m}(t)\exp(isx_{12}x_{1m})T_{1m}(-t)\exp(-isx_{12}x_{1m}) = \]
\[ \exp(isx_{12}(x_{1m} + t) - isx_{12}x_{1m}) = \exp(isx_{12}x_{1m}). \]

At last we conclude that for \( X^{(2)} \) we have \( \exp(isx_{12}), \exp(isx_{1m}) \in M^{(2)} \) in particular
\[ \exp(isx_{12}), \exp(isx_{13}) \in M^{(2)} \] (29)

For \( X^{(3)} \) and the corresponding projections we have
\[ \left( \begin{array}{cccc} 1 & x_{12} & x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & x_{23} & \cdots & x_{2n} & \cdots \\ 0 & 0 & 1 & x_{34} & \cdots & x_{3n} & \cdots \\ \end{array} \right)^{-1} = \]
\[ \left( \begin{array}{cccc} 1 & -x_{12} & -x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & -x_{23} & \cdots & -x_{2n} & \cdots \\ 0 & 0 & 1 & -x_{34} & \cdots & -x_{3n} & \cdots \\ \end{array} \right) = \left( \begin{array}{cccc} 1 & -x_{12} & -x_{13} & \cdots & x_{1n} & \cdots \\ 0 & 1 & -x_{23} & \cdots & -x_{2n} & \cdots \\ 0 & 0 & 1 & -x_{34} & \cdots & -x_{3n} & \cdots \\ \end{array} \right), \] (30)

\[ A_{1n}^R = D_{1n}, \quad A_{2n}^R = x_{12}D_{1n} + D_{2n}, \quad A_{3n}^R = x_{13}D_{1n} + x_{23}D_{2n} + D_{3n}, \quad 3 < n. \]

We have
\[ \Delta_{[3]}^i(x) = \exp\left(-is\left[\sum_{n=3}^{\infty} b_{1n}w_{1n}(x) + \sum_{n=4}^{\infty} b_{2n}w_{2n}(x)\right]\right) = \]
\[ \exp\left(-is\left[\sum_{n=3}^{\infty} b_{1n}(x_{1n} + x_{1n}^{-1})[2x_{1n} - (x_{1n} + x_{1n}^{-1})]\right]\right) \times \]
\[ \exp\left(-is\left[\sum_{n=1}^{\infty} b_{2n}(x_{2n} + x_{2n}^{-1})[2x_{2n} - (x_{2n} + x_{2n}^{-1})]\right]\right). \]
By the same procedure as in the case of the space $X^{(2)}$ we can obtain that
\[ \exp(isx_{12}), \exp(isx_{13}) \in M^{(3)}. \]  
(31)

We show that
\[ \{T_{34}(t), \Delta_{(3)}^{is}(x)\} = \exp \left( is \left[ b_{14}(2tx_{13}x_{14} + t^2x_{13}^2) + b_{24}(2tx_{23}x_{24} + t^2x_{23}^2) \right] \right). \]
(32)
(compare with (28)). Indeed we have
\[ \{T_{34}(t), \Delta_{(3)}^{is}(x)\} = T_{34}(t)\Delta_{(3)}^{is}(x)T_{34}(-t)\Delta_{(3)}^{-is}(x) = \]
\[ \exp \left( -is \left( b_{14}[w_{14}(x E_{34}(t)) - w_{14}(x)] + b_{24}[w_{24}(x E_{34}(t)) - w_{24}(x)] \right) \right), \]
which implies (32), since
\[ w_{14}(x) = (x_{14} + x_{14}^{-1})[2x_{14} - (x_{14} + x_{14}^{-1})] = -(x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34})[2x_{14} + x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34}], \]
and
\[ w_{14}(x E_{34}(t)) - w_{14}(x) = -[x_{12}^{-1}(x_{24} + tx_{23}) + x_{13}^{-1}(x_{34} + t)][2(x_{14} + tx_{13}) + x_{12}^{-1}(x_{24} + tx_{23}) + x_{13}^{-1}(x_{34} + t)] + (x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34})[2x_{14} + x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34}] = \\
- t \left[ (x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34})(2x_{13} + x_{12}^{-1}x_{23} + x_{13}^{-1}) + (x_{12}^{-1}x_{23} + x_{13}^{-1})(2x_{14} + x_{12}^{-1}x_{24} + x_{13}^{-1}x_{34}) \right] \\
- t^2(x_{12}^{-1}x_{23} + x_{13}^{-1})(2x_{13} + x_{12}^{-1}x_{23} + x_{13}^{-1}) = -t[-(x_{14} + x_{14}^{-1})x_{13} - x_{13}(x_{14} + x_{14}^{-1})] + t^2x_{13}x_{13} = \\
2tx_{13}x_{14} + t^2x_{13}^2. \]

Using (31) and (32) we get
\[ \phi_{t,s}^{(3)}(x) := \exp \left( is \left[ b_{14}2tx_{13}x_{14} + b_{24}(2tx_{23}x_{24} + t^2x_{23}^2) \right] \right) \in M^{(3)}, \]
hence
\[ \{T_{13}(t_1), \phi_{t,s}^{(3)}(x)\} = T_{13}(t_1)\phi_{t,s}^{(3)}(x)T_{13}(-t_1)(\phi_{t,s}^{(3)}(x))^{-1} = \exp (istt_1b_{14}2tx_{14}), \]
so $\exp(isx_{14}) \in M^{(3)}$ and $\exp[isb_{24}(2tx_{23}x_{24} + t^2x_{23}^2)] \in M^{(3)}$. Similarly we get
\[ \{T_{24}(t_1), \exp[isb_{24}(2tx_{23}x_{24} + t^2x_{23}^2)]\} = \exp(isb_{24}tt_1x_{23}), \]
so $\exp(isx_{23}), \exp(isx_{23}x_{24}) \in M^{(3)}$. At last we get
\[ \{T_{24}(t_1), \exp(isx_{23}x_{24})\} = \exp(isx_{24}). \]

Finally we can obtain $\exp(isx_{kn})$ in the following order on the first step:
\[ \exp(isx_{12}), \exp(isx_{13}); \]
on the second step:

\[ \exp(isx_{14}), \exp(isx_{23}), \exp(isx_{24}) \in M^{(3)}, \]

or symbolically in the following order:

\[
\begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} \\
0 & 1 & x_{23} & x_{24}
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 2 & 2 \\
0 & 0 & 3 & 2
\end{pmatrix}.
\]

In general we get the order

\[
\begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\
0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} & x_{27}
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 2 & 1 & 4 & 1 & 5 \\
0 & 0 & 2 & 3 & 2 & 4 & 2
\end{pmatrix}.
\]

This order is right in the general case (without any projections on \(X^{(k)}\)). To obtain \(\exp(isx_{12})\) and \(\exp(isx_{13})\) on the first step we get by Lemma 19

\[
\{T_{23}(t), \Delta^{is}(x)\} = T_{23}(t)\Delta^{is}(x)T_{23}(-t)\Delta^{-is}(x) = \Delta^{is}(xE_{23}(t))\Delta^{-is}(x) = \exp\left\{ -is\left( \sum_{n=3}^{\infty} b_{1n}[w_{1n}(xE_{23}(t)) - w_{1n}(x)] + \sum_{n=4}^{\infty} b_{2n}[w_{2n}(xE_{23}(t)) - w_{2n}(x)] \right) \right\}.
\]

Now we shall calculate \(w_{1n}(xE_{23}(t)) - w_{1n}(x)\) and \(w_{2n}(xE_{23}(t)) - w_{2n}(x)\). We have by (16)

\[
x_{1n} + x_{1n}^{-1} = -\sum_{r=2}^{n-1} x_{1r}x_{rn}^{-1}, \quad x_{2n} + x_{2n}^{-1} = -\sum_{r=3}^{n-1} x_{2r}x_{rn}^{-1}
\]

so we conclude that for \(n > 3\) holds

\[
(x_{1n} + x_{1n}^{-1})E_{23}(t) = -\left( \sum_{r=2}^{n-1} x_{1r}x_{rn}^{-1} \right)E_{23}(t) = -\left( x_{12}x_{2n}^{-1} + x_{13}x_{3n}^{-1} + \sum_{r=4}^{n-1} x_{1r}x_{rn}^{-1} \right)E_{23}(t) =
\]

\[
- \left( x_{12}(-x_{2n} - [x_{23} + t]x_{3n}^{-1} - \sum_{r=4}^{n-1} x_{2r}x_{rn}^{-1}) + [x_{13} + tx_{12}]x_{3n}^{-1} + \sum_{r=4}^{n-1} x_{1r}x_{rn}^{-1} \right) =
\]

\[
- \left( \sum_{r=2}^{n-1} x_{1r}x_{rn}^{-1} - tx_{12}x_{3n}^{-1} + tx_{12}x_{rn}^{-1} \right) = x_{1n} + x_{1n}^{-1}.
\]

For \(n = 3\) we get \(x_{13} + x_{13}^{-1} = -x_{12}x_{23}^{-1} = x_{12}x_{23}\) hence

\[
(x_{13} + x_{13}^{-1})E_{23}(t) = (x_{12}x_{23})E_{23}(t) =
\]

\[
x_{12}[x_{23} + t] = x_{12}x_{23} + tx_{12} = x_{13} + x_{13}^{-1} - tx_{12}^{-1}.
\]
Finally we conclude that

\[(x_{1n} + x_{1n}^{-1})^{E_{23}(t)} = \begin{cases} 
  x_{1n} + x_{1n}^{-1}, & \text{if } 3 < n, \\
  x_{13} + x_{13}^{-1} + tx_{12}, & \text{if } n = 3 
\end{cases} \quad (35)\]

and

\[(x_{1n} \pm x_{1n}^{-1})^{E_{23}(t)} = \begin{cases} 
  x_{1n} \pm x_{1n}^{-1}, & \text{if } 3 < n, \\
  x_{13} \pm x_{13}^{-1} + tx_{12}, & \text{if } n = 3 
\end{cases} \quad (36)\]

since

\[(x_{13} - x_{13}^{-1})^{E_{23}(t)} = (2x_{13} - (x_{13} + x_{13}^{-1}))^{E_{23}(t)} = 2[x_{13} + tx_{12}] - (x_{13} + x_{13}^{-1} + tx_{12}) \]

\[= x_{13} - x_{13}^{-1} + tx_{12}. \]

We have \(w_{1n}(xE_{23}(t)) - w_{1n}(x) = 0\) for \(n > 3\). For \(n = 3\) holds

\[w_{13}(xE_{23}(t)) - w_{13}(x) = (x_{13} + x_{13}^{-1} + tx_{12})(x_{13} - x_{13}^{-1} + tx_{12}) - (x_{13} + x_{13}^{-1}) (x_{13} - x_{13}^{-1}) \]

\[= tx_{12}(x_{13} + x_{13}^{-1} + x_{13} - x_{13}^{-1}) + t^2 x_{12}^2 = 2tx_{12}x_{13} + t^2 x_{12}^2. \]

Finally

\[w_{1n}(xE_{23}(t)) - w_{1n}(x) = \begin{cases} 
  0, & \text{if } 3 < n \\
  2tx_{12}x_{13} + t^2 x_{12}^2, & \text{if } n = 3. 
\end{cases} \quad (37)\]

For \((x_{2n} + x_{2n}^{-1})^{E_{23}(t)}\) we have

\[(x_{2n} + x_{2n}^{-1})^{E_{23}(t)} = - \left( \sum_{r=3}^{n-1} x_{2r} x_{r^{-1}} \right)^{E_{23}(t)} = - \left( x_{23} x_{3n}^{-1} + \sum_{r=4}^{n-1} x_{2r} x_{r^{-1}} \right)^{E_{23}(t)} = \]

\[- \left( x_{23} + t \right) x_{3n}^{-1} + \sum_{r=4}^{n-1} x_{2r} x_{r^{-1}} \right) = x_{2n} + x_{2n}^{-1} - tx_{3n}^{-1}. \]

Since

\[(x_{2n} - x_{2n}^{-1})^{E_{23}(t)} = [2x_{2n} - (x_{2n} + x_{2n}^{-1})]^{E_{23}(t)} = [2x_{2n} - (x_{2n} + x_{2n}^{-1} - tx_{3n}^{-1})] \]

\[= x_{2n} - x_{2n}^{-1} + tx_{3n}^{-1} \]

we conclude that

\[(x_{2n} \pm x_{2n}^{-1})^{E_{23}(t)} = x_{2n} \pm x_{2n}^{-1} \mp tx_{3n}^{-1}. \quad (38)\]

Finally we have

\[w_{2n}(xE_{23}(t)) - w_{2n}(x) = (x_{2n} + x_{2n}^{-1} - tx_{3n}^{-1})(x_{2n} - x_{2n}^{-1} + tx_{3n}^{-1}) - (x_{2n} + x_{2n}^{-1})(x_{2n} - x_{2n}^{-1}) = \]

\[tx_{3n}^{-1}(x_{2n} + x_{2n}^{-1} + x_{2n} - x_{2n}^{-1}) - t^2 (x_{3n}^{-1})^2 = 2tx_{2n}^{-1}x_{3n}^{-1} - t^2 (x_{3n}^{-1})^2, \]
Using (37) and (39) we get

\[ w_{kn}(x E_{23}(t)) - w_{kn}(x) = \begin{cases} 2tx_{12}x_{13} + t^2x_{12}^2, & \text{if } n = 3, \ k = 1 \\ 2tx_{2n}x_{3n} - t^2(x_{3n}^{-1})^2, & \text{if } k = 2, \ n \geq 4 \\ 0, & \text{otherwise.} \end{cases} \]  

(40)

At last using (34) and (40) we have

\[ \{T_{23}(t), \Delta^{is}(x)\} = \exp \left( -is \left[ b_{13}(2tx_{12}x_{13} + t^2x_{12}^2) + \sum_{n=4}^{\infty} b_{2n}(2tx_{2n}x_{3n} - t^2(x_{3n}^{-1})^2) \right] \right). \]

Further we get

\[ \{T_{13}(t_2)\{T_{23}(t_1), \Delta^{is}(x)\}\} = \exp (-isb_{13}2t_1t_2x_{12}). \]  

(41)

Indeed

\[ \{T_{13}(t_2)\{T_{23}(t_1), \Delta^{is}(x)\}\} = \exp (-isb_{13} \left[ (2tx_{12}x_{13} + t^2x_{12}^2) - (2tx_{12}x_{13} - t^2x_{12}^2) \right]) = \exp (-isb_{13}2t_1t_2x_{12}), \]

compare with (10): \(-[A_{13}^R, [A_{23}^R, \ln \Delta]] = 2b_{13}x_{12}\) ! We have \(\exp(itx_{12}) \in M_\phi'\) and hence \(\exp(itx_{12}^2) \in M_\phi'\). Using expression for \(\{T_{23}(t_1), \Delta^{is}(x)\}\) we conclude that

\[ M_\phi' \ni \{T_{23}(t_1), \Delta^{is}(x)\} \exp(isb_{13}t^2x_{12}^2) = \exp \left( -is \left[ b_{13}(2tx_{12}x_{13}) + \sum_{n=4}^{\infty} b_{2n}(2tx_{2n}x_{3n} - t^2(x_{3n}^{-1})^2) \right] \right), \]

so

\[ M_\phi' \ni \{T_{23}(t_2), \{T_{23}(t_1), \Delta^{is}(x)\} \exp(isb_{13}t^2x_{12}^2)\} = \exp (-isb_{13}2t_1t_2x_{13}). \]

Compare with the expression \(-[A_{12}^R, [A_{23}^R, \ln \Delta]] = 2b_{13}x_{13}\). Finally we conclude that

\[ \exp(itx_{12}), \ \exp(itx_{13}) \in M_\phi' \]  

(42)

In general (without any projections) the following lemma holds

**Lemma 20** We have

\[ w_{kn}(xE_{mn+1}(t)) - w_{kn}(x) = \begin{cases} 2tx_{rm}x_{rm+1} + t^2x_{rm+1}^2, & \text{if } n = m + 1, \ 1 \leq k \leq m - 1 \\ 2tx_{mn}x_{m+1n} - t^2(x_{m+1n}^{-1})^2, & \text{if } k = m, \ n \geq m + 2 \\ 0, & \text{otherwise.} \end{cases} \]  

(43)
hence

\[ \{T_{mm+1}(t), \Delta^{is}(x)\} = \exp \left( -is \sum_{r=1}^{m-1} b_{rm+1} (2tx_{rm+1} + t^2x_{rm+1}^2) + \sum_{n=m+2}^{\infty} b_{mn} \left( 2tx_{mn} x_{m+1n} - t^2(x_{m+1n})^2 \right) \right). \]

(44)

**PROOF.** The proof is similar to the proof of the Lemma 17. □

To obtain another functions \( \exp(itx_{kn}) \) in the general case we should make all the steps as it was indicated before. For example to obtain \( \exp(isx_{14}) \), \( \exp(isx_{23}) \), \( \exp(isx_{24}) \) we should do the **second step** i.e. consider the operators

\[ \{T_{34}(t), \Delta^{is}(x)\} \]

and all necessary combinations.

To obtain \( \exp(isx_{15}) \), \( \exp(isx_{25}) \), \( \exp(isx_{34}) \), \( \exp(isx_{34}) \) we should consider the following operators

\[ \{T_{45}(t), \Delta^{is}(x)\}, \]

and so on. Finally we shall obtain all functions \( \exp(isx_{kn}), k < n \).

### 5 Example of the measure

We show that the set \( b = (b_{kn})_{k<n} \) for which

\[ S_{kn}^L(b) < \infty, \quad E(b) < \infty, \quad \text{and} \quad S_{kn}^{RL}(b) = \infty, \quad 1 \leq k < n, \]

where

\[ S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{b_{nm}}, \quad E(b) = \sum_{k<n} \frac{S_{kn}^L(b)}{b_{kn}}, \quad S_{kn}^{RL}(b) = \sum_{m=n+1}^{\infty} \frac{b_{km}}{S_{kn}^L(b)}. \]

is not empty. Indeed let us take \( b_{kn} = (a_k)^n \). We have

\[ S_{kn}^L(b) = \sum_{m=n+1}^{\infty} \left( \frac{a_k}{a_n} \right)^m = \left( \frac{a_k}{a_n} \right)^{n+1} \sum_{m=0}^{\infty} \left( \frac{a_k}{a_n} \right)^m = \left( \frac{a_k}{a_n} \right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} < \infty \]

iff \( a_k < a_{k+1}, \ k \in \mathbb{N} \), for example \( a_k = s^k \) with \( s > 1 \). Further we get

\[ E(b) = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{S_{kn}^L(b)}{b_{kn}} = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \left( \frac{a_k}{a_n} \right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} \frac{1}{a_k^n} = \]
\[ \sum_{k=1}^{\infty} a_k \sum_{n=k+1}^{\infty} \left( \frac{1}{a_n} \right)^{n+1} \frac{1}{1 - \frac{a_k}{a_n}} < \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left( \frac{1}{a_n} \right)^{n+1} \]

\[ \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \sum_{n=k+1}^{\infty} \left( \frac{1}{a_{k+1}} \right)^{n+1} = \sum_{k=1}^{\infty} \frac{a_k}{1 - \frac{a_k}{a_{k+1}}} \frac{1}{a_{k+1} - 1} < \sum_{k=1}^{\infty} \frac{1}{1 - a_{k+1}} \frac{1}{a_2} \frac{1}{a_2 - 1}. \]

If for example \( a_k = s^k \) with \( s > 1 \) we have
\[ E(b) < \frac{1}{1 - s} \sum_{k=1}^{\infty} \frac{1}{s^{k(k+1)}} \frac{1}{s^{k+1} - 1} < \infty. \]

At last
\[ S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} b_{km} = \sum_{m=n+1}^{\infty} \frac{a_k}{m} \frac{1}{a_m} + \frac{1}{m+1} \frac{1}{a_m} \frac{1}{a_m} = \sum_{m=n+1}^{\infty} \frac{a_k}{a_m} \frac{1}{a_m} + \frac{1}{m+1} \frac{1}{a_m} + \frac{1}{a_m} + \frac{1}{m+1} = \infty, \]
\[ \text{if } \lim_{m} a_m = \infty. \]
For \( a_k = s^k \) with \( s > 1 \) we have
\[ S_{kn}^{R,L}(b) = \sum_{m=n+1}^{\infty} s^{(m+k-n)m} \frac{1}{(s^{m-n} - 1)} = \infty. \]

6 Modular operator

We recall how to find the modular operator and the operator of canonical conjugation for the von Neumann algebra \( \mathcal{A}_G \), generated by the right regular representation \( \rho \) of a locally compact Lie group \( G \). Let \( h \) be a right invariant Haar measure on \( G \) and
\[ \rho, \lambda : G \mapsto U(L^2(G, h)) \]
be the right and the left regular representations of the group \( G \) defined by
\[ (\rho_t f)(x) = f(xt), \quad (\lambda_t f)(x) = (dh(t^{-1}x)/dh(x))^{-1/2} f(t^{-1}x). \]

To define the right Hilbert algebra on \( G \) we can proceed as follows. Let \( M(G) \) be algebra of all probability measures on \( G \) with convolution
\[ (\mu * \nu)(s) = \int_G \mu(x) \nu(x) dx. \]

We define the homomorphism
\[ M(G) \ni \mu \mapsto \rho^\mu = \int_G \rho_t d\mu(t) \in B(L^2(G, h)). \]
We have $\rho^\mu \rho^\nu = \rho^{\mu\nu}$, indeed

$$
\rho^\mu \rho^\nu = \int_G \rho_1 \, d\mu(t) \int_G \rho_s \, d\nu(s) = \int_G \int_G \rho_\ast \, d\mu(t) \, d\nu(s) = \int_G \rho_t \, d(\mu \ast \nu)(t) = \rho^{\mu\nu}.
$$

Let us consider a subalgebra $M_\mu(G) := (\nu \in M(G) \mid \nu \sim h)$ of the algebra $M_\mu(G)$. In the case when $\mu \in M_\mu(G)$ we can associate with the measure $\mu$ its Randon-Nikodim derivative $d\nu(t)/dh(t) = f(t)$. When $f \in C_0^\infty(G)$ or $f \in L^1(G)$ we can write

$$
\rho^f = \int_G f(t) \rho_t \, dh(t),
$$

hence we can replace the algebra $M_\mu(G)$ by its subalgebra identified with algebra of functions $C_0^\infty(G)$ or $L^1(G, h)$ with convolutions. If we replace the Haar measure $h$ with some measure $\mu \in M_\mu(G)$ we obtain the isomorphic image $T^{R,\mu}$ of the right regular representation $\rho$ in the space $L^2(G, \mu): T^{R,\mu}_t = U \rho_t U^{-1}$ where $U : L^2(G, h) \mapsto L^2(G, \mu)$ defined by $(U f)(x) = (\frac{dh(x)}{d\mu(x)})^{1/2} f(x)$. we have

$$
(T^{R,\mu}_t f)(x) = \left(\frac{d\mu(x t)}{d\mu(x)}\right)^{1/2} f(x),
$$

and

$$
T^f = \int_G f(t) T^{R,\mu}_t \, d\mu(t).
$$

We have (see §, p.462) (we shall write $T_t$ instead of $T^{R,\mu}_t$)

$$
S(T^f) := (T^f)^* = \int_G \overline{f(t)} T_{t^{-1}} \, d\mu(t) = \int_G \overline{f(t)} T_{t^{-1}} \, \frac{d\mu(t)}{d\mu(t^{-1})} \, d\mu(t^{-1})
$$

$$
\int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t)} T_{t^{-1}} \, d\mu(t).
$$

Hence

$$
(S f)(t) = \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})}.
$$

To calculate $S^*$ we use the fact that $S$ is antilinear so $(S f, g) = (S^* g, f)$. We have

$$
(S f, g) = \int_G \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} g(t) \, d\mu(t) = \int_G \frac{d\mu(t^{-1})}{d\mu(t^{-1})} g(t) \, d\mu(t^{-1}) =
$$

$$
\int_G g(t^{-1}) \overline{f(t)} \, d\mu(t) = (S^* g, f),
$$

hence $(S^* g)(t) = \overline{g(t^{-1})}$. Finally the modular operator $\Delta$ defined by $\Delta = S^* S$ has the following form $(\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t)$. Indeed we have

$$
f(t) \mapsto \frac{d\mu(t^{-1})}{d\mu(t)} \overline{f(t^{-1})} \mapsto \frac{d\mu(t)}{d\mu(t^{-1})} f(t).
$$
Finally, since $J = S\Delta^{-1/2}$ (see [4] p.462) we get

$$f(t) \overset{\Delta^{-1/2}}{\mapsto} \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} f(t)^J \overset{J}{\mapsto} \left(\frac{d\mu(t)}{d\mu(t^{-1})}\right)^{1/2} \frac{d\mu(t^{-1})}{d\mu(t)} f(t^{-1})$$

$$= \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} f(t^{-1}).$$

Hence

$$(Jf)(t) = \left(\frac{d\mu(t^{-1})}{d\mu(t)}\right)^{1/2} f(t^{-1}), \quad (\Delta f)(t) = \frac{d\mu(t)}{d\mu(t^{-1})} f(t).$$

To prove that $JT_t^{R,\mu} J = T_t^{L,\mu}$ we get

$$f(t) \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \frac{f(x^{-1})}{f(x)} T_t^{R,\mu} \overset{J}{\mapsto} \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} \frac{d\mu((xt)^{-1})}{d\mu(xt)} \frac{1}{f((xt)^{-1})} =$$

$$\left(\frac{d\mu(t^{-1}x^{-1})}{d\mu(x)}\right)^{1/2} \frac{f(t^{-1}x^{-1})}{f(t^{-1}x)} \overset{J}{\mapsto} \left(\frac{d\mu(x^{-1})}{d\mu(x)}\right)^{1/2} \frac{d\mu(t^{-1}x)}{d\mu(x)} \frac{1}{f(t^{-1}x)} =$$

$$\left(\frac{d\mu(t^{-1}x)}{d\mu(x)}\right)^{1/2} f(t^{-1}x) = \left(T_t^{L,\mu} f\right)(x).$$

**Remark 21** The representation $T_{R,\mu}^s$ is the inductive limit of the representations $T_{R,\mu}^s$ of the group $B(m, \mathbb{R})$ where the measure $\mu_m^s$ is the projection of the measure $\mu_b$ onto subgroup $B(m, \mathbb{R})$. Obviously $\mu_m^s$ is equivalent with the Haar measure $h_m$ on $B(m, \mathbb{R})$.

**7 The uniqueness of the constructed factor**

Let $G$ be a solvable separable locally compact group or a connected locally compact group. Then any representation $\pi$ of $G$ in a Hilbert space generates an approximately finite-dimensional von Neumann algebra (see [3]).

Theorem 15 from V.9 p. 504 [4] (Haagerup) There exists up to isomorphism only one amenable factor of type $III_1$, the factor $R_\infty$ of Araki and Woods (see [7]).

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