Abstract

We study the quantization of Chern-Simons theory with group $G$ coupled to dynamical sources. We first study the dynamics of Chern-Simons sources in the Hamiltonian framework. The gauge group of this system is reduced to the Cartan subgroup of $G$. We show that the Dirac bracket between the basic dynamical variables can be expressed in term of dynamical $r$–matrix of rational type. We then couple minimally these sources to Chern-Simons theory with the use of a regularisation at the location of the sources. In this case, the gauge symmetries of this theory split in two classes, the bulk gauge transformation associated to the group $G$ and world lines gauge transformations associated to the Cartan subgroup of $G$. We give a complete hamiltonian analysis of this system and analyze in detail the Poisson algebras of functions invariant under the action of bulk gauge transformations. This algebra is larger than the algebra of Dirac observables because it contains in particular functions which are not invariant under reparametrization of the world line of the sources. We show that the elements of this Poisson algebra have Poisson brackets expressed in term of dynamical $r$–matrix of trigonometric type. This algebra is a dynamical generalization of Fock-Rosly structure. We analyze the quantization of these structures and describe different star structures on these algebras, with a special care to the case where $G = SL(2, \mathbb{R})$ and $G = SL(2, \mathbb{C})_{\mathbb{R}}$, having in mind to apply these results to the study of the quantization of massive spinning point particles coupled to gravity with a cosmological constant in 2+1 dimensions.

1 Introduction

The aim of the present work is to provide new methods for the study of Chern-Simons theory associated to non-compact group coupled to dynamical sources. The combinatorial quantization program of Chern-Simons theory works perfectly well in the compact case as shown in [5] and for the case where $G = SL(2, \mathbb{C})_{\mathbb{R}}$ as shown in [7]. This method provides a Hamiltonian quantization of Chern-Simons theory on $\Sigma_{g,n} \times \mathbb{R}$ where $\Sigma_{g,n}$ is a topological surface of genus $g$ with $n$ punctures: it provides a complete quantization of the algebra of Dirac Observables, i.e. the algebra of constants of motion of this theory as well as a unitary representation of it.
Having in mind potential application to quantum gravity, this method needs further improvements to address "physical questions" related to the spectrum of important partial observables such as, for example, the radar time between two particles world-lines. The present work is a step in this direction.

In the second section of this paper, we study the dynamics of Chern-Simons sources in the Hamiltonian framework. The gauge group of this system is reduced to the Cartan subgroup of $G$. We show that the Dirac bracket between the basic dynamical variables can be expressed in term of dynamical $r-$matrix of rational type. In the third section, we then couple minimally these sources to Chern-Simons theory with the use of a regularisation at the location of the sources. In this case, the gauge symmetries of this theory split in two classes, the bulk gauge transformation associated to the group $G$ and world lines gauge transformations associated to the Cartan subgroup of $G$. We give a complete hamiltonian analysis of this system and analyze in detail the Poisson algebras of functions invariant under the action of bulk gauge transformations. This algebra is larger than the algebra of Dirac observables because it contains in particular functions which are not invariant under reparametrization of the world-line of the sources. We show that the elements of this Poisson algebra have Poisson brackets expressed in term of dynamical $r-$matrix of trigonometric type. This algebra is a dynamical generalization of Fock-Rosly structure.

In the fourth section, we analyze the quantization of the Poisson algebras involved in the study of free sources, and in the fifth section we analyze the quantization of the Poisson algebras involved in the study of sources coupled to Chern-Simons theory. In the study of these structures, we describe different star structures, with a special care to the case where $G = SL(2, \mathbb{R})$ and $G = SL(2, \mathbb{C})_\mathbb{R}$, having in mind to apply these results to the study of the quantization of massive spinning point particles coupled to gravity with a cosmological constant in $2 + 1$ dimensions.

2 Hamiltonian Dynamic of a Chern-Simons Source.

Let $G$ be a real connected simply-connected simple Lie group and $\mathfrak{g}$ be its real Lie algebra and denote $\langle \cdot, \cdot \rangle$ the Killing form on $\mathfrak{g}$.

Let $\chi \in \mathfrak{g}$, we consider the action for a dynamical variable $g \in G$ defined as:

$$ S[g(t)] = \int_{t_1}^{t_2} dt \langle \chi, g^{-1} \frac{dg}{dt} \rangle. \quad (1) $$

We will now proceed to the Hamiltonian analysis of this system. This system has been analyzed in the case where $G = SL(2, \mathbb{R})$ in [1] where it describes the dynamic of a relativistic massive spinning particle on de Sitter space $dS_3$.

The cotangent bundle $T^*G$ is a trivial bundle and we have $T^*G \simeq \mathfrak{g}^* \times G$. Let $X$ be an element of $\mathfrak{g}$, this defines a linear function on $\mathfrak{g}^*$, which defines via the isomorphism of bundle a function on $T^*G$ denoted $f_X$. If $\pi$ is a representation of $G$ acting on $V_\pi$, we will view its matrix elements denoted $g^\pi$ as functions on $T^*G$. The canonical symplectic
structure on $T^*G$ induces the following Poisson brackets:

$$\{g_1^\pi, g_2^\pi\} = 0, \quad \{f_X, g^\pi\} = \pi(X) g^\pi, \quad \{f_X, f_Y\} = f_{[X,Y]},$$

(2)

for all $\pi, \pi'$ representations of $G$.

In the previous formulas, $g^\pi$ has to be viewed as an element of $\text{End}(V_\pi) \otimes \text{Pol}(G)$, where $\text{Pol}(G)$ is the Hopf algebra of matrix elements of finite dimensional representations of $G$. We denote $\text{Irr}(G)$ the equivalence classes of irreducible finite dimensional representations of $G$.

$\text{Pol}(G)^* = \prod_{\pi \in \text{Irr}(G)} \text{End}(V_\pi)$ is a (multiplier) Hopf algebra containing $U(\mathfrak{g})$ as a Hopf subalgebra, therefore we can view $\text{Pol}(G)^*$ as a completion of $U(\mathfrak{g})$ that we denote $U(\mathfrak{g})^c$. Any irreducible representation of $G$ defines a representation of $U(\mathfrak{g})$ and then extends to a representation of $\text{Pol}(G)^*$. In order to obtain a universal description, i.e independent of the choice of representation, we will collect all the $g^\pi$ in an object $M \in U(\mathfrak{g})^c \otimes \text{Pol}(G)$ such that

$$\pi \otimes \text{id}(M) = g^\pi.$$

(3)

By construction this object satisfies:

$$(\Delta \otimes \text{id})(M) = M_1 M_2,$$

(4)

where $\Delta$ is the canonical coproduct on $\text{Pol}(G)^* = U(\mathfrak{g})^c$.

The relations (2) can be equivalently written as:

$$\{M_1, M_2\} = 0, \quad \{P_1, M_2\} = t_{12} M_2, \quad \{P_1, P_2\} = [P_1, t_{12}],$$

(5)

where $t \in \mathfrak{g} \otimes \mathfrak{g}$ is defined by $t = \sum a, b t^a b X_a \otimes X_b$ where $(X_a)$ is any basis of $\mathfrak{g}$, $(t^{ab})$ is the inverse matrix of $(X_a, X_b)$ and $P : T^*G \rightarrow \mathfrak{g}, P = t^{ab} X_a f_{X_b}$. As a remark, $P$ being in $\mathfrak{g} \otimes F(T^*G)$, it obeys the relation $(\Delta \otimes \text{id})(P) = P_1 + P_2$.

This Hamiltonian system is a constrained system and the primary constraints are written as:

$$\phi = M^{-1} PM + \chi \approx 0.$$ 

(6)

Note that $(\Delta \otimes \text{id})(\phi) = \phi_1 + \phi_2$, as a result $\phi \in \mathfrak{g} \otimes F(T^*G)$. If $\xi$ is an element of $\mathfrak{g}$ we denote $\phi(\xi) = \langle \phi, \xi \rangle$. The canonical hamiltonian is equal to zero, whereas the total Hamiltonian is equal to:

$$H^\text{tot}[M, P, \mu] = \phi(\mu),$$

(7)

where $\mu$ is a Lagrange multiplier function with values in $\mathfrak{g}$.

The Poisson brackets of the constraints are

$$\{\phi_1, \phi_2\} = [\phi_2 - \chi_2, t_{12}],$$

(8)

which can be recast as

$$\{\phi(\xi), \phi(\xi')\} = \phi([\xi, \xi']) - \langle \chi, [\xi, \xi'] \rangle.$$ 

(9)
Therefore we have:
\[ \dot{\phi} = \{ \phi, H^{\text{tot}} \} = [\mu, \phi - \chi]. \tag{10} \]

As a result, conservation of the constraints under time evolution imposes no secondary constraints and fixes \( \mu \) to commute with \( \chi \). We denote \( C_{\chi} \) the centralizer in \( g \) of the element \( \chi \). For all \( \xi \in C_{\chi}, \phi(\xi) \) is a first class constraint, and for every \( \xi \) we have:
\[ \delta_{\xi} M = \{ M, \phi(\xi) \} = -M \xi, \quad \delta_{\xi} P = \{ P, \phi(\xi) \} = 0. \tag{11} \]

The transformation of Lagrange multiplier, allowing for an invariance of the total action
\[ S[M, P, \mu] = -\int_{t_1}^{t_2} dt \langle P, \frac{dM}{dt} M^{-1} \rangle + H^{\text{tot}}[M, P, \mu], \tag{12} \]
up to boundary terms, is given by
\[ \delta_{\xi} \mu = \dot{\xi} + [\xi, \mu]. \tag{13} \]

Up to gauge symmetries which are trivial on-shell, the gauge symmetry corresponding to the reparametrization of the world line of the particle \( t \mapsto t + \zeta(t) \) can be recovered from the gauge transformations generated by \( \phi(\mu)\zeta(t) \). The Noether symmetry of this system is given by
\[ M \mapsto g^{-1} M, \quad P \mapsto g^{-1} P g, \quad \mu \mapsto \mu \tag{14} \]
where \( g \) is a constant element in \( G \).

We will assume that \( \chi \) is a regular semisimple element, therefore \( \mathfrak{h} = C_{\chi} \) is a real Cartan subalgebra. We will now prefer to work with \( \mathfrak{g} = g^C \). We have collected in the Appendix 1 the relevant conventions on complex Lie algebras that we use throughout the rest of our work. We have also gathered, in Appendix 2, relevant notions about real structures. We will still denote by \( M \) (resp. \( P \)) the complexification of \( M \) (resp. \( P \)) and impose on it additional reality conditions imposed by the real form on \( \mathfrak{g} \) selecting \( g \). Once a star \( * \) is defined on \( \mathfrak{g} \), a star (denoted with the same symbol) can then be straightforwardly defined on \( F(T^*G) \) by the following requirement:
\[ (\ast \otimes \ast)(M) = M^{-1}, \quad (\ast \otimes \ast)(P) = -P. \tag{15} \]

As a result, coming back to the canonical analysis, the action (11) is real as soon as \( \chi^* = -\chi \).

We now compute the Dirac bracket of the previous Hamiltonian system. Assume that a constrained Hamiltonian system is given with a set of first class constraints \( \varphi_i \) and second class constraints \( \psi_A \). We will assume
\[ \{ \varphi_i, \varphi_j \} \approx 0, \quad \{ \varphi_i, \psi_A \} \approx 0, \quad \{ \psi_A, \psi_B \} = \mathcal{D}_{AB} \tag{16} \]
where coefficients \( \mathcal{D}_{AB} \) are functions on the phase space and \( \mathcal{D}_{AB} \) is weakly invertible, i.e. there exist functions \( \Delta_{AB}, (\Delta^{-1})_{AB}, \nu^C_{AB}, \rho^i_{AB} \) on the phase space such that
\[ \mathcal{D}_{AB} = \Delta_{AB} + \nu^C_{AB} \psi_C + \rho^i_{AB} \varphi_i, \quad \Delta_{AB}(\Delta^{-1})_{BC} = \delta^C_A. \tag{17} \]
We define the Dirac bracket as follows:

$$\{f, g\}_D = \{f, g\} - \{f, \psi_A\}(\Delta^{-1})^{AB}\{\psi_B, g\}. \quad (18)$$

The Dirac Bracket verifies the following properties, for any functions $f_1, f_2, f_3$ on the phase space:

$$\begin{align*}
\{f_1, f_2\}_D &= -\{f_2, f_1\}_D \quad \{f_1, f_2f_3\}_D = \{f_1, f_2\}_Df_3 + f_2\{f_1, f_3\}_D, \\
\{f_1, \varphi_i\}_D &\approx \{f_1, \varphi_i\}, \quad \{f_1, \{\varphi_i, \varphi_j\}\}_D \approx \{f_1, \{\varphi_i, \varphi_j\}\}, \\
\{f_1, \psi_A\}_D &\approx 0, \\
\{f_1, \{f_2, f_3\}\}_D + \text{cycl. perm.} &\approx 0 \text{ if } \rho_{AB}^i\{\varphi_i, f_a\} = 0, \forall a, A, B. \quad (22)
\end{align*}$$

Due to the property (21), one can strongly impose the constraints $\psi_A = 0$.

Let $\mathfrak{G} = \mathfrak{N}^+ \oplus \mathfrak{N}^-$ where $\mathfrak{N}^\perp = \mathfrak{N}^+ \oplus \mathfrak{N}^-$ and for all $\xi \in \mathfrak{G}$ we denote by $\xi_{\mathfrak{N}}$ (resp. $\xi_{\mathfrak{N}^\perp}$) its projection on $\mathfrak{N}$ (resp. $\mathfrak{N}^\perp$). As a result we obtain:

$$\{\phi(\xi), \phi(\xi')\} \approx \langle \phi_{\mathfrak{N}} - \chi, [\xi, \xi'] \rangle \approx -\langle \text{ad}(\tilde{\chi})(\xi), \xi' \rangle \quad (23)$$

where we have denoted $\tilde{\chi} = \chi - \phi_{\mathfrak{N}}$.

We denote $Q(\tilde{\chi})$ the endomorphism of $\mathfrak{G}$ with kernel $\mathfrak{N}$ and which restriction on $\mathfrak{N}^\perp$ is the inverse of $-\text{ad}(\tilde{\chi})$ i.e $Q(\tilde{\chi})(e_\alpha) = -\alpha(\tilde{\chi})^{-1}e_\alpha$. We denote $r(\tilde{\chi}) \in \mathfrak{G}^{A2}$ by $\langle \xi \otimes \xi', r(\tilde{\chi}) \rangle = -\langle Q(\tilde{\chi})(\xi), \xi' \rangle$, it satisfies:

$$r(\tilde{\chi}) = -\Delta^{-1}^{AB}\xi_A \otimes \xi_B \quad (24)$$

where $(\xi_A)$ is any basis of $\mathfrak{N}^\perp$.

The explicit expression of $r(\tilde{\chi})$ is given by:

$$r(\tilde{\chi}) = -\sum_{\alpha \in \Phi} \frac{1}{\alpha(\tilde{\chi})} e_\alpha \otimes e_{-\alpha}. \quad (25)$$

We denote $\tilde{\chi} = \sum_i \chi_\alpha \lambda^i$ with $\lambda^i$ the basis of $\mathfrak{N}$ dual to $(h_\alpha)$ with respect to $(..,..)$, $r(\tilde{\chi})$ satisfies the classical dynamical Yang Baxter equation, i.e:

$$[r_{12}(\tilde{\chi}), r_{13}(\tilde{\chi}) + r_{23}(\tilde{\chi})] + [r_{13}(\tilde{\chi}), r_{23}(\tilde{\chi})] =$$

$$\begin{align*}
&= \sum_i \left( h_{\alpha_1}^{(1)} \frac{\partial r_{23}(\tilde{\chi})}{\partial \chi_{\alpha_1}} - h_{\alpha_2}^{(2)} \frac{\partial r_{13}(\tilde{\chi})}{\partial \chi_{\alpha_2}} + h_{\alpha_3}^{(3)} \frac{\partial r_{12}(\tilde{\chi})}{\partial \chi_{\alpha_3}} \right). \quad (26)
\end{align*}$$

This solution is the basic rational dynamical $r$-matrix solution [2].

As a result we get:

$$\begin{align*}
\{M_1, M_2\}_D &= M_1 M_2 r_{12}(\tilde{\chi}), \quad \{M, \langle \tilde{\chi}, h \rangle\}_D = M h, \quad h \in \mathfrak{N} \\
\{P_1, M_2\}_D &= t_{12} M_2, \quad \{P_1, P_2\}_D = [P_1, t_{12}].
\end{align*} \quad (27)$$

This is straightforward from [3][4][24].
We can now impose strongly the second class constraints i.e. \((M^{-1}PM + \chi)_{\mathfrak{h}} = 0\), as a result we obtain the strong equality:

\[
P = -M\tilde{\chi}M^{-1}.
\]

(28)

In conclusion, this Hamiltonian system is entirely described by \(M, \tilde{\chi} = \sum_i \tilde{\chi}_i \lambda^i\) with \(\lambda^i\) the basis of \(\mathfrak{h}\) dual to \((h_{\alpha_i})\) with respect to \((\langle ., \cdot \rangle)\), satisfying the relations:

\[
\{M_1, M_2\}_D = M_1 M_2 r_{12}(\tilde{\chi}), \quad \{M, \tilde{\chi}_{\alpha_i}\}_D = M h_{\alpha_i},
\]

(29)

\[
\{\tilde{\chi}_1, \tilde{\chi}_2\}_D = 0, \quad \chi - \tilde{\chi} \approx 0,
\]

(30)

\[
H^{\text{tot}} = \langle \mu, \chi - \tilde{\chi} \rangle \quad (\Delta \otimes \text{id})(M) = M_1 M_2.
\]

(31)

We will denote by \(S(\mathfrak{g})\) the Poisson algebra generated by \(M, \tilde{\chi}\) and will call it the *Poisson algebra of the free source*.

We denote \(S(\mathfrak{g})^\mathbb{R}\) the Poisson subalgebra of \(S(\mathfrak{g})\) Poisson commuting with the components of \(\tilde{\chi}\). This Poisson algebra is generated by the components of \(P\).

We now specialize to the case where \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})\) and denote \(\tilde{\chi} = \tilde{\chi}_0 \lambda\). In the appendix 2, a brief summary of basic properties of star structures is given, in particular star involutions corresponding to the real forms of \(\mathfrak{sl}(2, \mathbb{C})\) are given by equations \([177,178,179]\). As said before, these star structures and automorphisms can be straightforwardly extended to functions on the phase space \(S(\mathfrak{g})\) by equation \((15)\). More precisely, if \(\frac{1}{\pi}\) is the two dimensional unitary representation of \(\mathfrak{su}(2)\), we define

\[
g = \left(\frac{1}{\pi} \otimes \text{id}\right)(M) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(32)

and the following linear automorphisms of \(S(\mathfrak{g})\):

\[
\begin{align*}
\sigma_1(a) &= a & \sigma_1(b) &= -b & \sigma_1(c) &= -c & \sigma_1(d) &= d & \sigma_1(\tilde{\chi}_0) &= \tilde{\chi}_0 \\
\sigma_2(a) &= d & \sigma_2(b) &= -c & \sigma_2(c) &= -b & \sigma_2(d) &= a & \sigma_2(\tilde{\chi}_0) &= -\tilde{\chi}_0.
\end{align*}
\]

(33)

(34)

The corresponding star structures are then given by

\[
su(2) : \ a^* = d \quad b^* = -c \quad \tilde{\chi}^* = -\tilde{\chi}, \quad su(1,1) : \tilde{\mathfrak{f}} = \sigma_1 \circ \ast, \quad sl(2, \mathbb{R}) : \ast = \sigma_2 \circ \ast.
\]

(35)

As a remark, note that \(\sigma_i \circ \ast = \ast \circ \sigma_i^{-1}\) ensuring that these stars are involutions. We will denote \(S(\mathfrak{g})\) the algebra \(S(\mathfrak{g})^\mathbb{R}\) endowed with the corresponding star structure selecting the real form \(\mathfrak{g}\).

In order to apply our study to the case of relativistic particles on a three dimensional deSitter space, it is also fruitful to analyze the action \([\text{II}]\) in the slightly more general situation, i.e. \(G = SL(2, \mathbb{C})^\mathbb{R}\). We denote \(sl(2, \mathbb{C})^\mathbb{R}\) the real Lie algebra of \(sl(2, \mathbb{C})\), although this algebra is not simple, our whole construction can be straightforwardly generalized to this case. A complete analysis of the corresponding system, in this case, as well as the coupling to Chern-Simons theory has been done in \([\text{II}]\). The complexification
is such that $g^C = sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ and its Cartan basis will be denoted $e^{(l,r)}$, $f^{(l,r)}$, $h^{(l,r)}$ where the $l$ (resp. $r$) generates the first (resp.second) component of the direct sum. Any of the star structures (180,181,182) can be used to recover the real Lie algebra $sl_{\mathbb{R}}$. Any real bilinear form on $sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ invariant under the adjoint action is given by: $< x, y > = a < x^{l}, y^{l} > + \bar{a} < x^{r}, y^{r} >$ with $a \in \mathbb{C}$. Following [1], we have chosen $a = \frac{1}{2}$ in order to describe the dynamic of a particle on deSitter space. This action will couple minimally to deSitter Gravity. The real action for the free source $\mathbb{C}$ is entirely defined in terms of a Cartan element $\chi = \chi^{(l)}h^{(l)} + \chi^{(r)}h^{(r)}$ and depends on dynamical variables $M^{(l)}, M^{(r)}, \bar{\chi}^{(l)}, \bar{\chi}^{(r)}$. The star structure on these elements can be easily described if we denote

$$g^{(l)} = \begin{pmatrix} \begin{pmatrix} \frac{1}{2}, 0 \\ \pi \end{pmatrix} \otimes id \end{pmatrix} (M^{(l)}) \otimes id) (M^{(r)}),$$

indeed

$$a^{(l)} \star = d^{(l)} \quad b^{(l)} \star = -c^{(l)} \quad d^{(l)} \star = a^{(l)} \quad c^{(l)} \star = -b^{(l)} \quad \bar{\chi}^{(l)} \star = -\bar{\chi}^{(l)} \quad (36)$$

$$a^{(l)} \bar{\star} = d^{(l)} \quad b^{(l)} \bar{\star} = c^{(l)} \quad d^{(l)} \bar{\star} = a^{(l)} \quad c^{(l)} \bar{\star} = b^{(l)} \quad \bar{\chi}^{(l)} \bar{\star} = -\chi^{(l)} \quad (37)$$

$$a^{(l)} \check{\star} = a^{(l)} \quad b^{(l)} \check{\star} = b^{(l)} \quad d^{(l)} \check{\star} = d^{(l)} \quad c^{(l)} \check{\star} = c^{(l)} \quad \check{\chi}^{(l)} \check{\star} = \check{\chi}^{(l)} \quad (38)$$

The corresponding hamiltonionian system is described by

$$\{M^{(l)} \sigma, M^{(l)} \sigma \}_{D} = M^{(l)} \sigma, M^{(l)} \sigma \}_{D} = -10^\sigma \delta_{\sigma, \epsilon}, \quad \{M^{(l)} \sigma, \bar{\chi}^{(l)} \}_{D} = M^{(l)} \sigma, \bar{\chi}^{(l)} \}_{D} = 0,$$

$$\chi^{(l)} - \bar{\chi}^{(l)} \approx 0 \quad H^{tot} = \langle \rho, (\chi - \bar{\chi}) \rangle, \quad \sigma, \epsilon \in \{l, r\} \quad (39)$$

where the Lagrange multipliers $\rho = \rho^{(l)}h^{(l)} + \rho^{(r)}h^{(r)}$ satisfy reality conditions $\rho^{(l)} \star = -\rho^{(r)}$ (resp.$\rho^{(l)} \star = \rho^{(r)}$) in the case $36, 37$ (resp. $38$).

3 Chern-Simons theory coupled with sources: Hamiltonian analysis

3.1 Hamiltonian reduction and Dirac Brackets

We now proceed to the Hamiltonian analysis of Chern-Simons theory coupled to classical sources. We have generalized the method developed in [1] to higher rank case. Let $\Sigma$ be an orientable topological compact surface of genus $g$ with $p$ punctures $x_1, \ldots, x_p$. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. We denote $M = \Sigma \times [t_1, t_2]$, and to each puncture $x_k$ we assign a regular semisimple element $\chi^{(k)} \in \mathfrak{h}$. The minimal coupling of Chern-Simons theory to pointlike sources located on the punctures would give the following action:

$$S[A, M_{(1)}, \ldots, M_{(p)}] = \theta \int_{\mathcal{M}} (A \wedge dA + \frac{2}{3} A \wedge A) +$$

$$+ \sum_{k=1}^{p} \int_{t_1}^{t_2} dt (\chi^{(k)}, M_{(k)}(t)^{-1}(\frac{d}{dt} + A_c(t, x_k))M_{(k)}(t)), \quad (40)$$

where $\theta$ is a coupling constant parameter.
where $A$ is a $g$–connection on $\mathcal{M}$ and the real parameter $\theta$ is the coupling constant.

The naive field equations obtained from the previous action are:

$$
\epsilon^{ij}(\partial_i A_j - \partial_j A_i + [A_i, A_j]) = -\sum_{k=1}^{p} \frac{1}{4\pi \theta} M(k) \chi(k) M_k^{-1}(x - x_k),
$$

where $i, j$ run along spatial indices.

Because the right-hand side contains $\delta$ distributions, $A$ has to belong to the set of distributions. In the non abelian case, the commutator term $A \wedge A$, involving the product of two distributions is therefore ill-defined. In order to solve this problem one can remove the points $x_k$ and impose, as in [3], the condition $A_\phi = -\frac{1}{4\pi \theta} M(k) \chi(k) M_k^{-1}$ as additional constraint on a small neighborhood of $x_k$ to the obvious bulk constraint $F = 0$, with the gauge group acting continuously at $x_k$. The drawback of this method is that the constraint on $A_\phi$ is imposed by hand and does not come from the canonical analysis of a well defined action principle.

In view of these difficulties we will modify the previous action as explained in [1].

To each puncture $x_k$ we associate a small closed disk $D_k \subset \Sigma$ containing $x_k$ such that these disks do not intersect. Let $S_k$ be the closed curve defined by $S_k = \partial D_k$ and denote by $T_k = \tilde{D}_k \times [t_1, t_2]$ a tubular neighborhood of the vertical line passing through $x_k$, and $T = \cup_k T_k$. We denote $\mathcal{M}^- = \mathcal{M} \setminus T$, its boundary contains $B_k = S_k \times [t_1, t_2]$ and $\Sigma^+ \cup \Sigma_2^-$ where $\Sigma^- = \Sigma \setminus \{t\}$ with $\Sigma^- = \Sigma \setminus (\cup_k \tilde{D}_k)$. The spatial boundary is $\mathcal{B} = \cup_{k=1}^{p} B_k$. $S_k$ being diffeomorphic to a circle, we choose a parametrisation of $S_k$ by an angle $\phi \in [0, 2\pi]$ respecting the orientation and if $f$ is a function on $S_k$ we denote $f^{av}$ its mean value, i.e: $f^{av} = \frac{1}{2\pi} \int f(\phi) d\phi$. For any function $f : \Sigma^- \to \mathbb{C}$ we denote $f_{\Sigma^-} = f|_{S_k}$.
If $A$ is a connection on $\mathcal{M}^-$, $S_k \times [t_1, t_2]$ being embedded in $\mathcal{M}^-$, the pullback of $A$ under this embedding gives a one form on $B_k$ denoted $A^{(k)}_t = A^{(k)}_t dt + A^{(k)}_\varphi d\varphi$. We will also denote $S_t = \partial \Sigma^- = \cup_j S^j_t$.

We will study the following regularized action:

$$S[A, M^{(1)}, \ldots, M^{(p)}] = \theta \int_{\mathcal{M}^-} \langle A \wedge, dA + \frac{2}{3} A \wedge A \rangle +$$

$$+ \sum_{k=1}^p \int_{t_1}^{t_2} dt \langle \chi^{(k)}, M^{(k)}_t \left( \frac{d}{dt} + (A^{(k)}_t)^{av} M^{(k)}_t \right) \rangle + \sum_{k=1}^p \theta \int_{B_k} \langle A^{(k)}_t | A^{(k)}_\varphi \rangle,$$

(42)

where $A$ is a smooth $g$-connection on $\mathcal{M}^-$. We will study the following regularized action:

We first analyze the gauge symmetry of this action. We define

$$G = \{ g \in C^\infty(\mathcal{M}^-, G), \forall t \in [t_1, t_2], \ g(\cdot, t) \text{ is a constant function on each } S_k \}.$$ \hfill (43)

The Lie algebra $\mathfrak{g}$ of this group is:

$$\mathfrak{g} = \{ \Gamma \in C^\infty(\mathcal{M}^-, g), \forall t \in [t_1, t_2], \ \Gamma(\cdot, t) \text{ is a constant function on each } S_k \}.$$ \hfill (44)

and the action of $\xi \in \mathfrak{g}$ on the gauge field $A$ and on dynamical variables $M^{(k)}$, for $k \in \{1, \ldots, p\}$ will be respectively given by:

$$\delta[\Gamma]A_\mu = D^A_\mu \Gamma, \quad \delta[\Gamma]M^{(k)} = \Gamma|_{S_k} M^{(k)}, \quad \forall \Gamma \in \mathfrak{g}.$$ \hfill (45)

As a result the infinitesimal action of the gauge group preserves the total action up to boundary terms on $\Sigma^-_{t_1} \cup \Sigma^-_{t_2}$:

$$\delta[\Gamma]S[A, M^{(1)}, \ldots, M^{(p)}] = \theta \int_{\Sigma^-_{t_2}} d^2 x \varepsilon^{ij} \langle A_i, \partial_j \Gamma \rangle - \theta \int_{\Sigma^-_{-t_1}} d^2 x \varepsilon^{ij} \langle A_i, \partial_j \Gamma \rangle.$$ \hfill (46)

We now study the Hamiltonian analysis of this action. We reexpress the action as:

$$S[A, M^{(1)}, \ldots, M^{(p)}] = \int dt \left( \theta \int_{\Sigma^-} d^2 x \varepsilon^{ij} \langle A_j, \partial_i A_t \rangle + \langle A_t, \varepsilon^{ij} F_{ij}(A) \rangle \right)$$

$$+ \sum_{k=1}^p \langle \chi^{(k)}, M^{(k)}_t \frac{dM^{(k)}_t}{dt} \rangle + 2\theta \sum_{k=1}^p \int_{S^j_t} \langle A^{(k)}_t, A^{(k)}_\varphi - X^{(k)}_t \rangle \right)$$ \hfill (46)

where the mapping from the spatial boundary $B$ to $g$ denoted $X^{(k)}_t$ is defined by

$$X^{(k)}_t = -\frac{1}{4\pi\theta} M^{(k)}_t \chi^{(k)} M^{(k)}_t.$$ \hfill (47)

As in the free case we introduce the momenta $P^{(k)} \in g$ of the sources which satisfy:

$$\{M^{(k)}_t, M^{(l)}_t\} = 0, \quad \{P^{(k)}_t, M^{(l)}_t\} = t_{12} M^{(k)}_t 2 \delta_{kl}, \quad \{P^{(k)}_t, P^{(l)}_t\} = [P^{(k)}_t, t_{12}] \delta_{kl},$$ \hfill (48)
and the primary constraints are:

\[ \phi(k) = M^{-1}_k P_k M_k + \chi(k) \approx 0, \quad k = 1, ..., p. \] (49)

We denote by \( C_\phi \) this set of constraints. \( A_t \) is considered as a Lagrange multiplier and the Poisson structure on the spatial connection satisfies:

\[ \{ A_i(x), A_j(y) \} = \epsilon_{ij} t_{12} \delta^{(2)}(x - y), \quad \forall x, y \in \Sigma^-. \] (50)

Variation with respect to \( A_t \) implements the primary constraint:

\[
\Omega(v) = \frac{1}{2} \int_{\Sigma^-} d^2 x \epsilon^{ij} \langle v, F_{ij}(A) \rangle + \sum_{k=1}^p \int_{S_k} \langle v(k), A_{\phi}^{(k)} - X(k) \rangle \approx 0, \quad \forall v \in C^\infty(\Sigma^-, \mathfrak{g}).
\] (51)

We will denote by \( C_\Omega \) this set of constraints.

It has to be noticed that the previous constraints can be written as bulk constraints and boundary constraints:

\[ F_{ij}(x) = 0 \quad \forall x \in \Sigma^-.
\] (52)

\[ A_{\phi}^{(k)} = X(k). \] (53)

We deduce, from the action, the total Hamiltonian:

\[
H_{tot}[A, M(k), P(k), \mu(k), \rho] = -\theta \left( \int_{\Sigma^-} d^2 x \epsilon^{ij} \langle \rho, F_{ij}(A) \rangle + 2 \sum_{k=1}^p \int_{S_k} \langle \rho(k), A_{\phi}^{(k)} - X(k) \rangle \right)
+ \sum_{k=1}^p \langle \mu(k), \phi(k) \rangle,
\] (54)

where we have introduced Lagrange multipliers \( \mu(k), \rho \) with \( \mu(k) \in \mathfrak{g} \), and \( \rho \) a smooth function on \( \Sigma^- \) with value in \( \mathfrak{g} \).

Conservation of the constraints (49) under time evolution imposes the following conditions:

\[
0 \approx \frac{d\phi(k)}{dt} = \{ \phi(k), H_{tot} \} = [\mu(k) - M^{-1}_k \rho(k) M_k, \chi(k)].
\] (55)

As a result, the equations (55) do not impose any secondary constraint. We define \( v(k) = \mu(k) - M^{-1}_k \rho(k) M_k \), the previous equation imposes \( v(k) \in C_{\chi_k} \).

The requirement that the constraint \( \Omega(v) \) must be preserved in time implies no secondary constraints but imposes conditions on the Lagrange multiplier \( \rho(k) \) as now explained.

Time evolution of \( \Omega(v) \) is given by:

\[
0 \approx \frac{d\Omega(v)}{dt} = \Omega([\rho, v]) + \sum_{k=1}^p \int_{S_k} \langle v(k), \partial_{\phi} \rho(k) + [X(k), \rho(k) - \rho_{\phi}^{(k)}] \rangle.
\] (56)
For $\xi \in \mathfrak{g}$ we define the operator
\[
K^\xi : C^\infty(S^1, \mathfrak{g}) \to C^\infty(S^1, \mathfrak{g}), \ u \mapsto \partial_\varphi u + [\xi, u - u^{av}].
\] (57)

Let $X \in \mathfrak{h}$, we will say that $X$ is special if $\exists \alpha \in \Phi, e^{2\pi \alpha(X)} = 1$.

The condition on the Lagrange multiplier is:
\[
K^{X(k)}(\rho(\cdot)) = 0,
\] (58)
and this property has a very simple meaning: in the case where $\frac{1}{4\pi^2} \chi(k)$ is not special, $\rho(k)$ is constant on $S_k$.

Indeed from the equivariance property $K^g^{-1} \circ Ad_g = Ad_g \circ K^\xi$, it is sufficient to show that if $X \in \mathfrak{h}$ is not special the kernel of $K^X$ is the set of constant functions on $S^1$.

To prove this, we define $G(\varphi) = e^{\varphi X}(u - u^{av})e^{-\varphi X}$. If $u$ lies in the kernel of $K^X$, $G(\varphi)$ is equal to a constant $G$. $2\pi$ periodicity of $u$ imposes that $G$ commutes with $e^{2\pi X}$. The condition $e^{2\pi \alpha(X)} \neq 1$, ensures that $G_{\Phi_\alpha} = 0$. Therefore $G \in \mathfrak{h}$ which implies, as announced, $u - u^{av} = G = 0$.

We will assume in the sequel that every $\frac{1}{4\pi^2} \chi(k)$ is not special.

The Dirac process ends and we are left with the set of constraints $\mathcal{C} = \mathcal{C}_\phi \cup \mathcal{C}_\Omega$.

We replace the set of constraints $\mathcal{C}$ by $\mathcal{C}_\phi \cup \mathcal{C}_\Omega$ where
\[
\tilde{X}(k) = -\frac{1}{4\pi^2} M(k) \tilde{x}(k) M(k)^{-1} \quad \text{with} \quad \tilde{x}(k) = x(k) - \phi(k)|_{\mathfrak{h}},
\] (59)
\[
\tilde{\Omega}(\cdot) = \frac{1}{2} \int_{\Sigma^-} d^2x \ e^{ij} \langle v,F_{ij}(A) \rangle + \sum_{k=1}^{p} \int_{S_k} \langle v(k), A^{(k)} - \tilde{x}(k) \rangle.
\] (60)

In order to compute the Dirac bracket we proceed iteratively.

We treat first the subset of second class constraints in $\mathcal{C}_\phi$ as in the first section and compute the resulting intermediary Poisson bracket denoted $\{\cdot, \cdot\}_d$:
\[
\{M(k)_1, M(l)_2\}_d = M(k)_1 M(k)_2 r_{12} (-\tilde{x}(k)) \delta_{kl}, \quad \{M(k), \tilde{x}(l)\}_d = M(k) h_{\alpha, \delta_{kl}},
\]
\[
\{\tilde{x}(k)_1, \tilde{x}(l)_2\}_d = 0, \quad \chi(k) - \tilde{x}(k) \approx 0, \quad P(\cdot) = -M(k)^{-1} \tilde{x}(k) M(k).
\] (61)

The Poisson brackets involving the connection are left unchanged.

We now compute the Poisson brackets of the remaining constraints:
\[
\{\tilde{\Omega}(u), \tilde{\Omega}(v)\}_d = \frac{1}{2p} \left( \tilde{\Omega}([u,v]) - \sum_{k=1}^{p} \int_{S_k} \langle u, K^X(k) v \rangle \right)
\] (62)
\[
\{\tilde{x}(k) - \chi(k), \tilde{\Omega}(u)\}_d = 0 \quad \forall u, v \in C^\infty(\Sigma^-, \mathfrak{g}).
\] (63)

We now have to distinguish first class from second class constraints.

From the Poisson brackets \[62\], we see that, given $u \in C^\infty(\Sigma^-, \mathfrak{g})$, the constraint $\tilde{\Omega}(u)$ is first class if and only if $u|_{S_k}$ is constant for each $k$.

Moreover the constraints $\tilde{x}(k) - \chi(k)$ are also first class.
There is no canonical way to select the set of second class constraints, however, the Dirac bracket does not depend on this choice. This procedure is achieved by choosing a space of functions $\mathcal{P}$ such that $C^\infty(\Sigma^- G) = \mathcal{P} \oplus \mathcal{P}^\perp$, and $\forall u \in \mathcal{P}^\perp, u^a_{\mid S_k} = 0$ for any $k = 1, \ldots, p$. The detail of this space in the bulk is in fact irrelevant. The key property of this space is the fact that for any couple $u, v \in \mathcal{P}^\perp$, if $\forall k = 1, \ldots, p, u^a_{\mid S_k} = v^a_{\mid S_k}$ then $u = v$. As a result we can identify $\mathcal{P}^\perp$ with the vector space of functions of zero mean value from the boundary circles $\cup_{k=1}^p S_k$ to the Lie algebra. Hence, it can be naturally equipped with a pre-Hilbert space structure using the $L^2$ norm.

In order to compute the final Dirac bracket, it will be useful to introduce the antisymmetric bilinear form $\mathcal{K}^X$ as follows:

$$\mathcal{P} \times \mathcal{P} \rightarrow \mathbb{C}$$

$$(u, v) \mapsto \mathcal{K}^X(u, v) = \sum_{k=1}^p \int_{S_k} \langle u^a_{\mid (k)}, K^X_{\mid (k)} v^a_{\mid (k)} \rangle.$$  \hfill (64)

Given an element $u \in \mathcal{P}^\perp$, $\mathcal{K}^X(u, \cdot) : \mathcal{P}^\perp \rightarrow \mathbb{C}$ is invertible and we denote $(\mathcal{K}^X)^{-1}(\cdot, u)$ its inverse, i.e.

$$\int_{\mathbb{R}} [Dw] \mathcal{K}^X(u, w)(\mathcal{K}^X)^{-1}(w, v) = \delta_{\mathbb{R}}(u - v), \quad \forall u, v \in \mathcal{P}^\perp$$  \hfill (65)

where the previous functional integration is defined via the integration on the Fourier modes of functions on $\cup_{k=1}^p S_k$.

From the expression (62), the Poisson bracket of two second class constraints is strongly equal to the sum of three terms

$$2\theta\{\hat{\Omega}(u), \hat{\Omega}(v)\}_d = \hat{\Omega}([u, v]_{\mathbb{G}}^-) + \hat{\Omega}([u, v]_{\mathbb{G}}) - \mathcal{K}^X(u, v).$$  \hfill (66)

To obtain the expression of the Dirac matrix it is sufficient to compute it up to second class constraints: we can therefore eliminate the first term appearing in the previous expression. Moreover, to compute the expression of the Dirac bracket on functions which are invariant under the first class constraints $\hat{\Omega}(w)$ for $w \in \mathcal{P}$, it is sufficient to invert the Dirac matrix computed only up to these first class constraints. As a result we obtain that:

$$\{f, g\}_D = \{f, g\}_d + 2\theta \int_{\mathbb{R} \times \mathbb{R}} [Du] [Dv] \{f, \hat{\Omega}(u)\}_d (\mathcal{K}^X)^{-1}(u, v) \{\hat{\Omega}(v), g\}_d$$  \hfill (67)

for any $f, g$ which Poisson commute with $\hat{\Omega}(w)$, $w \in \mathcal{P}$.

Let us now consider three different punctures $x_l, x_m, x_n$ and let $\gamma$ (resp. $\gamma'$) be an oriented curve joining $S_l$ to $S_m$ (resp. $S_l$ to $S_n$). We denote by $\varphi$ and $\varphi'$ the angles associated respectively to the departure point of $\gamma$ and $\gamma'$.

Let $U_{\gamma}$ be the holonomy of the connection $A$ along $\gamma$. The functions $V_{\gamma}[A, M] = M_{(m)}^{-1} U_{\gamma} M_{(l)}$ are invariant under gauge transformation belonging to $\mathcal{G}$. We can compute

$$\{f, g\}_D = \{f, g\}_d + 2\theta \int_{\mathbb{R} \times \mathbb{R}} [Du] [Dv] \{f, \hat{\Omega}(u)\}_d (\mathcal{K}^X)^{-1}(u, v) \{\hat{\Omega}(v), g\}_d$$  \hfill (67)

for any $f, g$ which Poisson commute with $\hat{\Omega}(w)$, $w \in \mathcal{P}$.
explicitly the Dirac bracket of these functions, computation done in the Appendix 4, which can be expressed as a dynamical quadratic Poisson bracket as

\[ \{V_{\gamma_1}, V_{\gamma_2}\}_D = V_{\gamma_1}V_{\gamma_2}r_{12}^\theta(\varphi - \varphi'; -\tilde{\chi}(k)) \]  \tag{68}

with \( r_{12}^\theta(\varphi; -\tilde{\chi}) \) given by:

\[ r_{12}^\theta(\varphi; -\tilde{\chi}) = \frac{1}{4\pi\theta} \left( (\pi - \varphi) \sum_j h_{\alpha_j} \otimes \chi^j + \sum_{\alpha \in \Phi} e_{\alpha} \otimes e_{-\alpha} \frac{\pi e^{\tilde{\chi}(\alpha)(\pi-\varphi)/4\pi\theta}}{\sinh(\tilde{\chi}(\alpha)/4\theta)} \right). \]  \tag{69}

Note that there is a discontinuity of this function at \( \varphi = 0 \) and that \( r_{12}^\theta(0+; -\tilde{\chi}) = r_{12}^\theta(+) (-\tilde{\chi}) \) and \( r_{12}^\theta(0-; -\tilde{\chi}) = r_{12}^\theta(-) (-\tilde{\chi}) \) are the basic trigonometric solutions of the classical dynamical Yang-Baxter equation which satisfy

\[ r_{12}^\theta(+) (-\tilde{\chi}) - r_{12}^\theta(-) (-\tilde{\chi}) = \frac{1}{2\theta} t_{12}. \]  \tag{70}

We will denote \( r_{12}^\theta(\pm) = r_{12}^\theta(\pm)(\tilde{\chi} \to -\infty) \) and note that \( \theta r_{12}^\theta(\pm) \) is the standard solution of the classical Yang-Baxter equation.

### 3.2 Poisson algebras of dynamical holonomies

Let us stress that the function \( V_\gamma \) is not a Dirac observable because it does not Poisson commute with \( \tilde{x}_m, \tilde{x}_l \) and we have:

\[ \{V_\gamma, \tilde{x}_m\}_{\alpha_i} = V_\gamma h_{\alpha_i}, \quad \{V_\gamma, \tilde{x}_m\}_{\alpha_i} = -h_{\alpha_i} V_\gamma. \]  \tag{71}

Let us fix for each \( S_k \) a point \( z_k \) on it, and fix \( p - 1 \) curves \( \gamma(2), \ldots, \gamma(p) \) where \( \gamma(k) \) goes from \( z_1 \) to \( z_k \). We will assume that \( \gamma(k) \) does not touch any circle except its end points and that two different curves have only \( z_1 \) as intersection. We will study the Poisson algebra of polynomials of matrix elements of the holonomies \( (V_{\gamma(k)})_{k=2, \ldots, p} \) with coefficients in the algebra of functions of \( (\tilde{x}(k))_{k=1, \ldots, p} \). The orientation of the surface \( \Sigma \) at the point \( z_1 \) fixes an order \( < \) on the set of curves \( \gamma(2), \ldots, \gamma(p) \). Up to relabelling of the sources it is always possible to assume that \( \gamma(2) < \cdots < \gamma(p) \).

The Poisson brackets between these elements are then given by:

\[ \{V_{\gamma(k)}, \tilde{x}(1)_{\alpha_i}\} = V_{\gamma(k)} h_{\alpha_i}, \quad \{V_{\gamma(k)}, \tilde{x}(l)_{\alpha_i}\} = -h_{\alpha_i} V_{\gamma(k)} \delta_{kl}, \]  \tag{72}

\[ \{V_{\gamma(k)1}, V_{\gamma(l)2}\} = V_{\gamma(k)1} V_{\gamma(l)2} r_{12}^\theta(-) (-\tilde{x}(1)) \quad \text{with \( k < l \)}, \]  \tag{73}

\[ \{V_{\gamma(k)1}, V_{\gamma(k)\gamma(2)}\} = V_{\gamma(k)1} V_{\gamma(k)2} r_{12}^\theta(-) (-\tilde{x}(1)) + r_{12}^\theta(-) (-\tilde{x}(k)) V_{\gamma(k)1} V_{\gamma(k)2}, \]  \tag{74}

\[ (\Delta \otimes \text{id})(V_{\gamma(k)}) = V_{\gamma(k)1} V_{\gamma(k)2} \]  \tag{75}

We denote \( Hol_p(\mathfrak{g}) \) this Poisson algebra and call it the Poisson algebra of dynamical boundary-boundary holonomies.
Remark: We could have generalized straightforwardly the function $V_\gamma[A,M]$ to embedded spin networks with open legs attached to the circles $S_k$. The Poisson bracket of these objects have two types of contributions: those associated to intersecting points in the bulk which are computed using Goldman bracket and those associated to the boundary which are computed using (73).

In the case where $\Sigma$ is the sphere a similar construction based on spin-networks is equivalent to the previous one. In the case where $\Sigma$ is a genus $n$-surface the complete description of observables requires the introduction of holonomies corresponding to non-trivial cycles. This description can be developed straightforwardly along the guidelines of previous derivations. It is not the aim of the present paper to develop these aspects, we will then stick to the case where $\Sigma$ is the sphere.

At this point, we still have to impose the remaining relations corresponding to the requirement of flatness of the connection on the sphere. This requirement is implemented by the relation

$$\Upsilon - 1 \approx 0,$$

with

$$\Upsilon = e^{-\frac{s_{(1)}}{2\theta}} \prod_{k=p}^{2} \left( V_{\gamma(k)}^{-1} e^{-\frac{s_{(k)}}{2\theta}} V_{\gamma(k)} \right). \quad (77)$$

It is important to emphasize that these relations are not remnants of some large bulk gauge transformations $\tilde{\Omega}$ ($\Upsilon$ is even strictly invariant under $\tilde{\Omega}$!). These relations have to be considered as additional relations to be implemented on $V_{\gamma(k)}, \tilde{\chi}_{(k)}$ in order to recover the true degrees of freedom and not as canonical constraints generating any gauge transformations. We have to note that, using previous Poisson brackets as well as the relation

$$r_{12}^{\theta}(\cdot)(-\tilde{\chi}_{(k)}) - \frac{1}{2\theta} \sum_j h_{\alpha_j} \otimes \lambda^j = e^{-\frac{s_{(k)}}{2\theta}} r_{12}^{\theta}(\cdot)(-\tilde{\chi}_{(k)}) e^{-\frac{s_{(k)}}{2\theta}}, \quad (78)$$

the Poisson brackets between dynamical variables and $\Upsilon$ can be straightforwardly computed and are:

$$\{\Upsilon_1, V_{\gamma(k)} \} = V_{\gamma(k)} \{\Upsilon_1, r_{12}^{\theta}(\cdot)(-\tilde{\chi}_{(1)})\}$$

$$\{\Upsilon, \tilde{\chi}_{(1)} \alpha_i \} = [\Upsilon, h_{\alpha_i}], \quad \{\Upsilon_1, \tilde{\chi}_{(k)} \} = 0, \quad \forall k = 2, ..., p. \quad (80)$$

As a result $(\Upsilon - 1)\text{Hol}_p(S^2, \mathfrak{G})$ is a Poisson ideal, therefore we can define the Poisson algebra $\text{Hol}_p(S^2, \mathfrak{G})$ obtained from $\text{Hol}_p(\mathfrak{G})$ by moding out the relations $\Upsilon = 1$. We call $\text{Hol}_p(S^2, \mathfrak{G})$ the algebra of dynamical boundary-boundary holonomies on $S^2$ of zero curvature.

In order to implement these constraints in a way adapted to quantization, we will give an alternative description of $\text{Hol}_p(S^2, \mathfrak{G})$ by embedding $\text{Hol}_p(\mathfrak{G})$ in a larger Poisson algebra $\text{Hol}_p(\bullet, \mathfrak{G})$, the algebra of bulk-boundary dynamical holonomies defined
as follows: \( \text{Hol}_p(\bullet, \mathfrak{g}) \) is generated by polynomials of matrix elements of the matrices \((U(k))_{k=1,\ldots,p}\) with coefficients in the algebra of functions of \((\tilde{x}(k))_{k=1,\ldots,p}\), with Poisson brackets

\[
\{U(k), \tilde{x}(l)\alpha\} = U(k)h_{\alpha l}, \tag{81}
\]
\[
\{U(k), U(l)\} = \hat{r}^g_{12}U(k)U(l) \quad \text{with } k < l, \tag{82}
\]
\[
\{U(k), U(l)\} = U(k)U(l)\hat{r}_{12}^g(-\tilde{x}(1)) + U(k)U(l)\hat{r}_{12}^g(\tilde{x}(1)), \tag{83}
\]
\[
(\Delta \otimes \text{id})(U(k)) = U(k)U(k), \tag{84}
\]

In order for this Poisson algebra to be well defined the \(r\)-matrix \(\hat{r}^g\) has to be a solution of classical Yang-Baxter equation such that

\[
\hat{r}_{12}^g(+) - \hat{r}_{12}^g(-) = \frac{1}{2\theta_{12}}. \tag{85}
\]

The Poisson-Lie group \(F(G^C)\) which Poisson bracket is defined by

\[
\{g_1, g_2\} = [g_1, g_2, \hat{r}_{12}^g] \tag{86}
\]

coacts on \(\text{Hol}_p(\bullet, \mathfrak{g})\) by the Poisson map

\[
\text{Hol}_p(\bullet, \mathfrak{g}) \rightarrow F(G^C) \otimes \text{Hol}_p(\bullet, \mathfrak{g}) \tag{87}
\]
\[
U(k) \rightarrow g^{-1}(U(k)) \tag{88}
\]

where the Poisson structure on \(F(G^C) \otimes \text{Hol}_p(\bullet, \mathfrak{g})\) is defined as \(\{f \otimes a, f' \otimes a'\} = \{f, f'\} \otimes aa' + ff' \otimes \{a, a'\}\).

As a result the coinvariant elements of \(\text{Hol}_p(\bullet, \mathfrak{g})\) is a Poisson subalgebra \(\text{Hol}_p(\bullet, \mathfrak{g})^\mathfrak{g}\) which is moreover the image of the following injective Poisson map:

\[
\text{Hol}_p(\mathfrak{g}) \rightarrow \text{Hol}_p(\bullet, \mathfrak{g}) \quad (V_{\tilde{x}(k)}, \tilde{x}(1)) \mapsto (U^{-1}_{(k)}U(1), \tilde{x}(k), \tilde{x}(1)), \quad k \geq 2. \tag{89}
\]

It can easily be shown that different choices of \(r\)-matrix \(\hat{r}^g\) fulfilling the required condition \([5, 8]\) give rise to the same Poisson algebra of gauge coinvariant elements. We will then generically choose \(\hat{r}^g = r^g\) (however, in the \(sl(2, \mathbb{C})_\mathbb{R}\) case, a different choice will be more fruitful for our purpose).

Our aim is now to implement the flatness condition \([70]\) in \(\text{Hol}_p(\bullet, \mathfrak{g})\). We define

\[
\Gamma = \prod_{k=p}^{1} \left( U(k)e^{-\frac{\tilde{x}(k)}{2\theta}} U^{-1}(k) \right). \tag{90}
\]

It can easily be shown that the elements Poisson commuting with \(\Gamma\) are exactly the gauge coinvariant elements in \(\text{Hol}_p(\bullet, \mathfrak{g})\). As a result we obtain \(\text{Hol}_p(\bullet, \mathfrak{g})^\mathfrak{g} = \text{Hol}_p(\mathfrak{g})\).
The elements of $H_{ol,p}(\bullet, \mathfrak{g})$ Poisson commuting with $\Gamma$ modded by the relations $\Gamma = 1$ is endowed with a natural Poisson algebra which is isomorphic to the Poisson algebra $H_{ol,p}(\mathbb{S}^2, \mathfrak{g})$.

It will be useful, in order to deal with the invariance under the remaining first class constraints:

$$\pi(k) = \tilde{x}(k) - x(k) \approx 0, \quad k = 1, \ldots, p$$

(91)

to introduce the following objects

$$\mathcal{P}(k) = U(k)e^{-\frac{\chi(k)}{2\pi}}U(k)^{-1}$$

(92)

generating the Poisson subalgebra of elements invariant under the remaining first class constraints $\pi(k)$. The Poisson algebra generated by $(\mathcal{P}(k))$ is the classical multi-loop Poisson algebra $L_{0,p}(\mathfrak{g})$ of Fock-Rosly [3]. If $\pi$ is a finite dimensional representation of $\mathfrak{g}$, we define $C_{(k)}^{\pi} = (tr_\pi \otimes id)(\mathcal{P}(k))$, which are Poisson central elements of $L_{0,p}(\mathfrak{g})$. For $\chi(1), \ldots, \chi(p)$ we denote $I_{\chi(1), \ldots, \chi(p)}$ the ideal generated by $C_{(k)}^{\pi} - tr\pi(e^{-\frac{\chi(k)}{2\pi}}), \forall \pi, \forall k$.

We then define the Poisson algebra $L_{0,p}(\mathfrak{g}; \chi(1), \ldots, \chi(p)) = L_{0,p}(\mathfrak{g})/I_{\chi(1), \ldots, \chi(p)}$.

We can now define the Poisson algebra of elements of $H_{ol,p}(\mathbb{S}^2, \mathfrak{g})$ Poisson commuting with all the remaining first class constraints $(\pi(k))_{k=1, \ldots, p}$ modded out by the relation $\pi(k) \approx 0, \quad k = 1, \ldots, p$. This Poisson algebra is isomorphic to the moduli space of flat connection on $\mathbb{S}^2$ with p-punctures associated to conjugacy classes which representative are $e^{-\frac{\chi(k)}{2\pi}}$. This Poisson algebra is usually denoted $M(\mathbb{S}^2, \mathfrak{g}; \chi(1), \ldots, \chi(p))$ and is obtained from the Poisson algebra $L_{0,p}(\mathfrak{g})$ by modding out the flatness condition, i.e.

$$M(\mathbb{S}^2, \mathfrak{g}; \chi(1), \ldots, \chi(p)) = L_{0,p}(\mathfrak{g}; \chi(1), \ldots, \chi(p))^{\mathfrak{g}}/((\Gamma = 1)).$$

Note that we have worked with the complexification of $\mathfrak{g}$ throughout this section. One can define the analogous Poisson algebras $H_{ol,p}(\mathfrak{g}), H_{ol,p}(\bullet, \mathfrak{g}), H_{ol,p}(\mathbb{S}^2, \mathfrak{g})$, by defining on $H_{ol,p}(\mathfrak{g}), H_{ol,p}(\bullet, \mathfrak{g}), H_{ol,p}(\mathbb{S}^2, \mathfrak{g})$, star structures selecting the real form $\mathfrak{g}$. We are sketchy at this point but we will develop this in the case of the quantization of these Poisson algebras.

The Dirac quantization program has been successfully constructed in [5] for the case of the Poisson algebra $M(\mathbb{S}^2, \mathfrak{g}; \chi(1), \ldots, \chi(p))$ through the use of the multiloop algebra and its representations. In this framework, all the first class gauge constraints are promoted to projectors selecting the space of physical states.

However, we do not want to stick necessarily to the Dirac formalism, in particular it can be interesting to study the intermediate step where particular roles are given to some of the sources and to study partial observables which do not commute with some of the $\pi(k)$. Indeed, in the $\mathfrak{g} = sl(2, \mathbb{C})$ case a Chern-Simons source is a free massive spining point particle evolving in deSitter space. In this case the weight $\chi$ is expressed in term of the mass and the spin of the particle. The formalism of Chern-Simons theory coupled to sources describes the particles coupled to deSitter gravity. Dirac quantization program, which can be fully understood using the combinatorial
quantization techniques, allow us to quantize and analyze only the quantum constants of motion of the system. This leads inevitably to the frozen time problem. It is therefore important to design alternative quantization schemes in order to quantize larger class of observables which are not Dirac observables, for examples partial observables involved in a conditional probability description of quantum gravity. The hierarchy of Poisson algebras that we have introduced in this section has been designed for this purpose.

We will therefore study in the next sections the quantization of the Poisson algebras $\text{Hol}_p(\mathfrak{g})$, $\text{Hol}_p(\mathfrak{g},\mathfrak{g})$, $\text{Hol}_p(S^2,\mathfrak{g})$. We will study as well their star structures, which implement the choice of real structure on $\mathfrak{g}$ when $\mathfrak{g}$ is of rank 1 or when $\mathfrak{g}$ is $\text{sl}(2,\mathbb{C})$. This will be important for the construction and analysis of unitary representations of these algebras which will be the subject of a second article [8].

The canonical analysis can be pursued for the case of sources coupled to Chern-Simons gauge theory. In the case where $G = \text{SL}(2,\mathbb{C})$, the connection is a $\text{sl}(2,\mathbb{C}) \oplus \text{sl}(2,\mathbb{C})$ connection with a left and right component denoted $A^{(l)}$, $A^{(r)}$. The Chern-Simons action coupled to sources is written as:

$$S[A, M_{(1)}, \ldots, M_{(p)}] = \frac{1}{\theta} (S[A^{(l)}, M_{(1)}, \ldots, M_{(p)}] - S[A^{(r)}, M_{(1)}, \ldots, M_{(p)}])$$

with coupling constant $\theta \in \mathbb{R}$. The hamiltonian analysis has already been done and the final result is that the dynamical $r$-matrix in this case is simply:

$$r(\varphi; -\bar{\chi}) = r^{(ll)}_{12} \half(\varphi; -\bar{\chi}^{(l)}) + r^{(rr)}_{12} \frac{\varphi}{2} (\varphi; -\bar{\chi}^{(r)})$$

with $r^{(rr)}_{12} = r^{(ll)}_{12}$. In the following we will use the shortened notation $r^{(ll)}_{12}(0\pm) = r^{(ll)}_{12}(0; -\bar{\chi}^{(l)})$ and $r^{(rr)}_{12}(0\pm) = r^{(rr)}_{12}(0; -\bar{\chi}^{(r)})$.

The corresponding Poisson algebra of dynamical boundary-boundary holonomies is generated by elements $V^{(l)}_{\gamma(k)}$, $V^{(r)}_{\gamma(k)}$, $\bar{X}^{(l)}_{\gamma(k)}$, $\bar{X}^{(r)}_{\gamma(k)}$ such that

$$\{V^{(s)}_{\gamma(k)}, \bar{X}^{(l)}_{\gamma(m)}\} = V^{(s)}_{\gamma(m)} h^{(s)}_{\delta \sigma \epsilon} , \{V^{(s)}_{\gamma(k)}, \bar{X}^{(m)}_{\gamma(m)}\} = -\delta_{k,m} h^{(s)}_{\sigma \delta \epsilon} V^{(s)}_{\gamma(k)}$$

$$\{V^{(l)}_{\gamma(k)}, V^{(r)}_{\gamma(m)}\} = 0$$

$$\{V^{(l)}_{\gamma(k)}, V^{(l)}_{\gamma(m)}\} = V^{(l)}_{\gamma(k)} V^{(l)}_{\gamma(m)} r^{(ll)}_{12}(-\bar{\chi}^{(l)}) \quad \text{with } k < m,$$

$$\{V^{(r)}_{\gamma(k)}, V^{(r)}_{\gamma(m)}\} = V^{(r)}_{\gamma(k)} V^{(r)}_{\gamma(m)} r^{(rr)}_{12}(-\bar{\chi}^{(r)}) \quad \text{with } k < m,$$

$$\{V^{(s)}_{\gamma(k)}, V^{(e)}_{\gamma(k)}\} = \delta_{\sigma,\epsilon} V^{(s)}_{\gamma(k)} V^{(s)}_{\gamma(k)} r^{(ss)}_{12}(-\bar{\chi}^{(s)}) +$$

$$+ r^{(ss)}_{12}(\bar{\chi}^{(e)}_{\gamma(k)} V^{(s)}_{\gamma(k)} V^{(s)}_{\gamma(k)}).$$

Note that in this case we can define a star structure on $\text{Hol}_p(\text{sl}(2,\mathbb{C}) \oplus \text{sl}(2,\mathbb{C}))$ as

$$(\star \otimes \star)(V^{(l)}_{\gamma(k)}) = V^{(r)}_{\gamma(k)}^{-1}, (\star \otimes \star)(\bar{X}^{(l)}_{\gamma(k)}) = -\bar{X}^{(r)}_{\gamma(k)}.$$
4 Quantization of Chern-Simons sources

We now proceed to the quantization of a free Chern-Simons source i.e to the quantization of the Poisson algebra $S(G)$.

The quantization of dynamical quadratic Poisson brackets is a well studied subject and the quantization of the Poisson brackets of the free source gives:

\[ M_1 M_2 = M_2 M_1 R_{12} (\hat{\mu}) \]
\[ [\hat{\mu}_\alpha, M] = M h_{\alpha} \]

with the following semiclassical behaviour $\hat{\mu} \sim -\frac{1}{\hbar}$ and $R_{12}(\hat{\mu}) = 1 + i \hbar r_{12} (-\hat{\chi}) + o(\hbar)$.

We will denote $\mu = -\frac{1}{\hbar} \chi$.

It is natural, due to the associativity of the operator algebra, to require that the matrix $R(\mu)$ satisfies the dynamical Yang-Baxter equation with $R(\mu) \in U(G) \otimes U(G)$. We define an algebra denoted $\hat{S}(G)$, generated by the components of $M \in U(G) \otimes U(G)$ and components of $\hat{\mu} \in \hat{S}(G)$, with relations:

\[ M_1 M_2 = (\Delta \otimes id) (M) F_{12} (\hat{\mu}) \]
\[ [\hat{\mu}_\alpha, M] = M h_{\alpha} \]

Note that the relation implies the exchange relation.

$\hat{\mu}_\alpha$ are elements which generate gauge transformations. The algebra of gauge invariant elements, denoted $\hat{S}(G)^G$, is the subalgebra of $\hat{S}(G)$ commuting with components of $\hat{\mu}$. It is generated by the components of

\[ P = \frac{i \hbar}{2} M (b(\hat{\mu}) + c) M^{-1} \]

where $b(\hat{\mu}) = \sum_j (2\hat{\mu}_j + h_{\alpha_j}) \lambda^j$ and $c$ is the quadratic Casimir of $U(G)$ normalized by $\Delta(c) = c_1 + c_2 + 2t_{12}$.

$P$ satisfy

\[ \Delta(P) = P_1 + P_2 \]
\[ [P_1, M_2] = i \hbar t_{12} M_2 \]
\[ [P_1, P_2] = i \hbar [P_1, t_{12}] \]
\[ [P_1, \hat{\mu}_2] = 0 \]

As a result, we obtain that $P = t^{ab} X_a \otimes \hat{P}_b$, with $[\hat{P}_a, \hat{P}_b] = i \hbar f_{ab}^b \hat{P}_c$, where $[X_a, X_b] = f_{ab}^c X_c$. The algebra generated by the components of $P$, i.e $(\hat{P}_a)$ is therefore isomorphic to $U(G)$. The proof of the results is contained in Appendix 4.

In a Dirac quantization process the study of unitary representations of this algebra is sufficient. However, in order to follow other quantization schemes, it is necessary to study representation of the algebra $\hat{S}(G)$ itself.

In the case where $G = sl(2, \mathbb{C})$, we have the formula:

\[ F(\mu) = 1 - \frac{1}{\mu_0} e \otimes f \] in the fundamental representation,
where we have denoted by $\mu_0$ the quantity $\mu_\alpha$ where $\alpha$ is the positive root of $sl(2)$.

As a result $R(\mu) = 1 - \frac{1}{\mu_\alpha} e \wedge f - \frac{\mu_0}{\mu_\alpha} fe \otimes ef$ in the fundamental representation.

The equations (101,102) defining $\hat{S}(G)$ can be written:

$$M_1 M_2 = M_2 M_1 R_{12}(\hat{\mu}_0) \quad \hat{\mu}_0 M = M(\hat{\mu}_0 + h).$$  \hfill (109)

Denoting $g = \left( \hat{\tau} \otimes id \right)(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the commutation relations (109) can be rewritten as:

$$\hat{\mu}_0 b = b(\hat{\mu}_0 - 1), \quad \hat{\mu}_0 d = d(\hat{\mu}_0 - 1), \quad \hat{\mu}_0 a = a(\hat{\mu}_0 + 1), \quad \hat{\mu}_0 c = c(\hat{\mu}_0 + 1)$$

$$ac = ca, \quad bd = db, \quad ab(1 - \frac{1}{\hat{\mu}_0}) = ba, \quad cd(1 - \frac{1}{\hat{\mu}_0}) = dc$$

$$ad - da = cb \frac{1}{\hat{\mu}_0}, \quad cb - bc = ad \frac{1}{\hat{\mu}_0}.$$  \hfill (110)

The quantum determinant $ad - cb$ is a central element which is fixed to

$$ad - cb = 1$$  \hfill (111)

from the fusion equation (103).

In the Dirac scheme, the first class constraints $\Pi = \hat{\mu}_0 - \mu_0$ are promoted to constraint operators. The algebra of quantum observables $\hat{S}(G)^H$ is the subalgebra of operators commuting with $\Pi$. As in the classical case, the algebra of quantum observables $\hat{S}(G)^H$ is then generated by momentum variables $P$ and is isomorphic to the algebra $U(sl(2, \mathbb{C}))$ using

$$e = -c\hat{\mu}_0 d, \quad f = a\hat{\mu}_0 b, \quad h = -(a\hat{\mu}_0 d + c\hat{\mu}_0 b).$$  \hfill (112)

The center of $\hat{S}(G)^H$ is generated by the quadratic Casimir element which has a very simple expression in term of $\hat{\mu}_0 : C = h^2 + 2(ef + fe) = \hat{\mu}_0^2 - 1$.

Along the lines of the classical treatment, we introduce the following linear automorphisms of $\hat{S}(G)$:

$$\sigma_1(a) = a, \quad \sigma_1(b) = -b, \quad \sigma_1(c) = -c, \quad \sigma_1(d) = d, \quad \sigma_1(\hat{\mu}_0) = \hat{\mu}_0$$

$$\sigma_2(a) = \frac{\hat{\mu}_0}{\mu_0 + 1} d, \quad \sigma_2(b) = -c, \quad \sigma_2(c) = -\frac{\hat{\mu}_0}{\mu_0 + 1} b, \quad \sigma_2(d) = a, \quad \sigma_2(\hat{\mu}_0) = -\hat{\mu}_0.$$  \hfill (113)

In order to identify hermitian operators associated to real functions on the classical phase space, we have to define a star structure (involutive antilinear antimorphism) on the complex algebra defined by the relations (103,104) quantizing the function algebra.
according to its classical counterpart defined in previous section. In the case of first rank algebras, there are only three cases of interest:

\[
\hat{S}(su(2)) : a^* = d, \quad b^* = -c, \quad \hat{\mu}_0^* = \hat{\mu}_0, \\
\hat{S}(su(1, 1)) : \tau = \sigma_1 \circ *, \\
\hat{S}(sl(2, \mathbb{R})) : \varpi = \sigma_2 \circ *.
\]

(115) (116) (117)

As a remark we have the properties \(\sigma_i \circ * = * \circ \sigma_i^{-1}\) necessary for these stars to be involutive. Using definitions [112] we obtain

\[
su(2) : e^* = f, \quad f^* = e, \quad h^* = h, \\
su(1, 1) : c^* = -f, \quad f^* = -e, \quad h^* = h, \\
sl(2, \mathbb{R}) : c^* = -e, \quad f^* = -f, \quad h^* = -h.
\]

(118) (119) (120)

In the \(sl(2, \mathbb{C})_\mathbb{R}\) case, the algebra of interest is \(\hat{S}(sl(2, \mathbb{C})_\mathbb{R}) = \hat{S}(sl(2, \mathbb{C})) \otimes 2\) equipped with one of the following star structures:

\[
a^{(l)} \star = d^{(r)}, \quad b^{(l)} \star = -c^{(r)}, \quad c^{(l)} \star = -b^{(r)}, \quad d^{(l)} \star = a^{(r)}, \quad \hat{\mu}^{(l)} \star = \hat{\mu}^{(r)}, \\
\varpi = (id \otimes \sigma_1) \circ * \quad * = (id \otimes \sigma_2) \circ *.
\]

(121)

In each of these cases, it can be shown that the star Lie algebra generated by the components of \(P\) are all isomorphic to \(sl(2, \mathbb{C})_\mathbb{R}\).

5 Quantization of Chern-Simons theory coupled to Sources

In this section we define and study the quantization of the Poisson algebras defined in section [14].

The quantization of the Poisson algebra \(Hol_p(G)\) is the algebra denoted \(\hat{Hol}_p(G)\) and defined as:

\[
[\hat{\mu}(1)_{\alpha_i}, V_{\gamma(k)}] = V_{\gamma(k)} h_{\alpha_i}, \quad [\hat{\mu}(1)_{\alpha_i}, V_{\gamma(k)}] = -h_{\alpha_i} V_{\gamma(k)} \delta_{kl}, \\
V_{\gamma(k)} V_{\gamma(l)} = V_{\gamma(l)} V_{\gamma(k)} R^{\theta(-)}(\hat{\mu}(1)) \text{ with } k < l,
\]

(122)

where \(R^\theta(\hat{\mu})\) is a solution of dynamical Yang-Baxter equation, with \(R^{\theta(+)}(\hat{\mu}) = R^\theta(\hat{\mu})\), \(R^{\theta(-)}(\hat{\mu}) = R^{\theta}_{21}(\hat{\mu})^{-1}\), and the following semiclassical behaviour \(\hat{\mu}(k) \sim -\frac{\chi(k)}{ih}\) and \(R_{12}(\hat{\mu}(k)) = 1 + ihr^{\theta}_{12}(-\hat{\chi}(k)) + o(h)\) and define \(q = exp(\frac{ih}{\hbar})\).

We will denote \(\mu(k) = \frac{-\chi(k)}{ih}\).

The quantization of (141) leads to

\[
R^{\theta(e)}_{21}(\hat{\mu}(k)) V_{\gamma(k)} V_{\gamma(k)} = V_{\gamma(k)} V_{\gamma(k)} R^{\theta(e)}_{12}(\hat{\mu}(1)),
\]

(123)

which is implied by

\[
V_{\gamma(k)} V_{\gamma(k)} = R^{\theta}_{21}(\hat{\mu}(k)) R^{\theta}_{12}(\Delta \otimes id)(V_{\gamma(k)}) F^{\theta}_{12}(\hat{\mu}(1)).
\]

(124)

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These identities are straightforwardly derived from basic definitions, and we give, as an example, the proof of \(\ref{eq:122}, \ref{eq:124}\) in the appendix 4.

A central result is the following factorization theorem:

\[ \widetilde{Hol}_p(\bullet, G) \text{ is isomorphic to } \widetilde{Hol}_1(\bullet, G)^\otimes p. \]

We will now prove this proposition. \(\widetilde{Hol}_1(\bullet, G)\) is the algebra defined by

\[
\begin{align*}
[\hat{\mu}_{\alpha_i}, W] &= W h_{\alpha_i}, \\
W_1 W_2 &= J_{21}^{-1} R_{12}^{\theta}(\Delta \otimes \text{id})(W) F_{12}^\theta(\hat{\mu}), 
\end{align*}
\]
where we have denoted $W = U_1, \tilde{\mu} = \tilde{\mu}_1$.

This algebra contains $U_q(\mathfrak{g})$ as a subalgebra. Indeed if we define $L = Wv^{-1}B(\tilde{\mu})W^{-1}$ we verify

$$L_2 \tilde{R}_1^{\theta} L_1 \tilde{R}_1^\theta = J_1^{12}(\Delta \otimes id)(L)J_{12}. \quad (137)$$

As usual we introduce the Gauss factorization $L = L^{(+)} - L^{(-)}$ with

$$L_1^{(\pm)} L_2^{(\pm)} = J_1^{12}(\Delta(L^{(\pm)})J_{12} \quad (138)$$

$$\tilde{R}^{(\pm)}_{12} L_2^{(+) - L_2^{(-)} = L_2^{-1}(\tilde{R}^{(+)_{12}} \quad (139)$$

Moreover $U_q(\mathfrak{g})$ acts on $\hat{Hol}_1(\bullet, \mathfrak{g})$ by adjoint action as follows

$$L_2^{(\pm)} W_1 L_2^{(\pm)} = \tilde{R}_2^{(\pm)} W_1 \quad (140)$$

Let us now consider the algebra $\hat{Hol}_1(\bullet, \mathfrak{g})^{gp}$ as being generated by $W_k, \tilde{\mu}_k$ associated to $p$ commuting copies of $\hat{Hol}_1(\bullet, \mathfrak{g})$ (we will also denote $L_1^{(\pm)}_{(k)}$ the corresponding elements). We introduce one more element by

$$\Gamma_k^{(\pm)} = \prod_{j=k}^PL_k^{(\pm)} \quad (141)$$

It is easy to show that following map

$$\kappa : \hat{Hol}_p(\bullet, \mathfrak{g}) \rightarrow \hat{Hol}_1(\bullet, \mathfrak{g})^{gp}$$

$$(U(k), \tilde{\mu}(k)) \rightarrow (\Gamma^{(-)}_{(k+1)}W_k, \tilde{\mu}(k)) \quad (142)$$

is an isomorphism of algebra.

In order to implement the conditions of the flatness of the connection it is useful to remark that

$$\kappa(\mathcal{P}) = \Gamma^{(-)}_{(k+1)}L_k \Gamma^{(-)}_{(k+1)}, \quad (143)$$

$$\kappa(\prod_{j=p}^{1} \mathcal{P}(j)) = \Gamma^{(+)_{(1)}(-}_{(1)} \quad (144)$$

The subalgebra of elements of $\hat{Hol}_u(\bullet, \mathfrak{g})$ commuting with $\prod_{j=p}^{1} \mathcal{P}(j)$ will be denoted $\hat{Hol}_p(\bullet, \mathfrak{g})^U_q(\mathfrak{g}) \simeq \hat{Hol}_p(\mathfrak{g})$.

Note that if $\mathcal{H}$ is a $\hat{Hol}_1(\bullet, \mathfrak{g})$ module, using the explicit isomorphism between $\hat{Hol}_1(\bullet, \mathfrak{g})$ and $\hat{Hol}_1(\bullet, \mathfrak{g})^{gp}$, we obtain that a representation of $\hat{Hol}_p(\bullet, \mathfrak{g})$ is provided by the module $\mathcal{H}^{gp}$. Because $U_q(\mathfrak{g})$ acts on $\mathcal{H}^{gp}$ via $\Delta^p$, this action corresponds to the action of $\Gamma^{(\pm)}_{(1)}$. Therefore a representation of $\hat{Hol}_p(\bullet, \mathfrak{g})^U_q(\mathfrak{g})$ implementing the condition $\prod_{j=p}^{1} \mathcal{P}(j) = 1$ consists in the module $(\mathcal{H}^{gp})^U_q(\mathfrak{g})$ i.e the vector space of invariant vectors under the action of $U_q(\mathfrak{g})$. 
We will construct an explicit unitary representation of \( \widehat{Hol}_p(\mathfrak{g}) \) when \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \) or \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \) in [8].

We will now provide, in the case where \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \), an explicit analysis of the structure of \( \widehat{Hol}_1(\bullet, \mathfrak{g}) \). Let us denote \( \hat{x} = q^{\mu_0} \), and \( \frac{1}{\pi} \otimes \text{id})(W) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). In the fundamental representation we have:

\[
\mathcal{F}_{12}^0(x) = 1 - (e \otimes f) \frac{q - q^{-1}}{1 - x^{-2}}.
\]

(145)

As a result the exchange relations implied by (136) can be explicitly recast as:

\[
\begin{align*}
\hat{x}a &= qa\hat{x}, & \hat{x}c &= q\hat{x}c, & \hat{x}b &= q^{-1}b\hat{x}, & \hat{x}d &= q^{-1}d\hat{x}, \\
ac &= qca, & bd &= qdb, & ab(q^{-1} - \frac{q - q^{-1}}{x^2 - 1}) &= ba, & cd(q^{-1} - \frac{q - q^{-1}}{x^2 - 1}) &= dc, \\
ad - da &= cb\frac{q - q^{-1}}{1 - \hat{x}^{-2}}, & cb - bc &= ad\frac{q - q^{-1}}{1 - \hat{x}^{-2}}.
\end{align*}
\]

(146)

\(ad - qcb\) is a central element which can be computed using the fusion relation (136)

\[
ad - qcb = q^{-1/2}.
\]

(147)

The subalgebra \( U_q(\mathfrak{sl}(2)) \) is expressed as:

\[
\frac{1}{\pi} \otimes \text{id})(L) = \begin{pmatrix} (q - q^{-1})^2 fe + q^{-h} & -(q - q^{-1})q^2f \phi \\ -q(q - q^{-1})e & q^h \end{pmatrix} = \frac{1}{\pi} \otimes \text{id})(Wv^{-1}B(\mu)W^{-1}),
\]

(148)

which can be written

\[
c = -cd\frac{q^{\hat{x}}(q^{-1} \hat{x} - q\hat{x}^{-1})}{q - q^{-1}}, & q^h = ab\frac{q^{-\frac{1}{2}}(q^{-1} \hat{x} - \hat{x}^{-1}q)}{q - q^{-1}},
\]

(149)

The Casimir element \( C = (q - q^{-1})^2 fe + q^h + q^{-1}q^{-h} \) is given by \( C = \hat{x} + \hat{x}^{-1} \).

We will now study when \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \), the star structures that can be defined on \( \widehat{Hol}_1(\bullet, \mathfrak{g}) \). In the case where \( \mathfrak{g} \) is a simple Lie algebra, our quantization procedure leads to the expression \( q = exp(\frac{\theta}{\pi}) \), therefore \( q \) has to be unimodular.

We will not dwell on the case where \( \mathfrak{g} = \mathfrak{su}(2) \). In that case numerous technical problem arise, in this formalism as well as others (need of weak quasi-Hopf algebras at the start to ensure truncation of the spectrum for \( q \) root of unit, etc...) and we do not want to adress them in this work.

We will only give here the \( \star \) structure associated to \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \). Let us now consider the star algebra \( (\widehat{Hol}_1(\bullet, \mathfrak{sl}(2, \mathbb{C})), \star) \) associated to the \( \mathfrak{sl}(2, \mathbb{R}) \) case. The star structure, in this case, is defined by:

\[
\begin{align*}
|q| &= 1 & a^\star &= q^2a & b^\star &= q\frac{x^2 - 1}{q^2x^2 - 1}b & c^\star &= qc & d^\star &= \frac{x^2 - 1}{q^2x^2 - 1}d & \hat{x}^\star &= x.
\end{align*}
\]

(150)
As a consequence, the star structure on the subalgebra \( U_q(sl(2, \mathbb{R})) \) is given by

\[
e^\ast = -q^{-1}e, \quad (q^h)^\ast = q^h, \quad f^\ast = -qf.
\] (151)

In the case where \( g = sl(2, \mathbb{C})_\mathbb{R} \) our choice of classical invariant bilinear form imposes \( q \) to be fixed to the real value \( q = \exp(\frac{\pi i}{2}) \). A detailed study of \( U_q(sl(2, \mathbb{C})_\mathbb{R}) \) has been done in [9]. In this work we have chosen \( U_q(sl(2, \mathbb{C})_\mathbb{R}) \) being the quantum double of \( U_q(su(2)) \). This choice implements naturally a quantum analog of Iwasawa decomposition allowing to develop an harmonic analysis on \( U_q(sl(2, \mathbb{C})_\mathbb{R}) \). We have \( U_q(sl(2, \mathbb{C})_\mathbb{R}) = U_q(sl(2))^{(l)} \otimes U_q(sl(2))^{(r)} \) as an algebra. The isomorphism of coalgebra is true up to a twist \( J = (R_{12}^\theta)^{(rl)} \). As a result the \( R \) matrix of the quantum double is twist equivalent to \( (R_{12}^\theta)^{(l)}(R_{12}^{-\theta})^{(rr)} \) with the twist \( J = (R_{12}^\theta)^{(rl)} \).

As a result we define \( \widetilde{Hol}_1(\bullet, (sl(2, \mathbb{C})_\mathbb{R})^\mathbb{C}) \) to be the algebra:

\[
\begin{align*}
&\lbrack \hat{\mu}^{(\sigma)}, W^{(r)} \rbrack = W^{(\sigma)} h^{(\sigma)} \delta_{\sigma, \epsilon}, \\
&W_1^{(l)} W_2^{(l)} = (R_{12}^\theta)^{(l)}(\Delta \otimes id)(W^{(l)}) F_{12}^\theta(\hat{\mu}^{(l)}) \\
&W_1^{(r)} W_2^{(r)} = (R_{12}^\theta)^{(-\sigma)}(\Delta \otimes id)(W^{(r)}) F_{12}^\theta(\hat{\mu}^{(r)}) \\
&(R_{21}^\theta)^{(l)} W_1^{(l)} W_2^{(r)} = W_2^{(r)} W_1^{(l)}.
\end{align*}
\] (152)

As a result this algebra is generated by the elements \( \hat{x}^{(l)} = q^{d^{(l)}}, \hat{x}^{(r)} = q^{b^{(r)}} \) as well as the matrix elements of

\[
(\begin{pmatrix} \frac{1}{2} \\ \pi \end{pmatrix} \otimes id)(W^{(l)}) = \begin{pmatrix} a^{(l)} \\ b^{(l)} \\ c^{(l)} \\ d^{(l)} \end{pmatrix}, \quad (\begin{pmatrix} 0 & \frac{1}{2} \\ \pi \end{pmatrix} \otimes id)(W^{(r)}) = \begin{pmatrix} a^{(r)} \\ b^{(r)} \\ c^{(r)} \\ d^{(r)} \end{pmatrix}.
\] (156)

The precise commutation relations are

1. the left variables satisfy the relations [146] with \( a^{(l)} d^{(l)} - q c^{(l)} b^{(l)} = q^{-1/2} \)
2. the right variables satisfy the relations [146] with \( a^{(r)} d^{(r)} - q c^{(r)} b^{(r)} = q^{5/2} \)
3. the left and right variables satisfy the exchange relation [153].

The subalgebra \( U_q(sl(2, \mathbb{C})) \otimes U_q(sl(2, \mathbb{C})) \) is defined as:

\[
\begin{align*}
&e^{(l)} = -q^2 \frac{q^2 q^{-1} c^{(l)} d^{(l)} (q^{-1} d^{(l)} - q^{2} c^{(l)} - 1)}{q - q^{-1}}, \quad (q^h)^{(l)} = a^{(l)} b^{(l)} q^{-\frac{1}{2}} (q^{-1} d^{(l)} - q^{2} c^{(l)} - 1), \\
&q^{(l)} = q^2 (a^{(l)} c^{(l)} - 1) d^{(l)} - q^{2} c^{(l)} b^{(l)}), \quad (q^h)^{(r)} = q^{-3} q^2 (a^{(r)} c^{(r)} - 1) d^{(r)} - q^{2} c^{(r)} b^{(r)}), \\
&e^{(r)} = -q^{-3} c^{(r)} d^{(r)} q^{-2} (q^{-1} d^{(r)} - q^{2} c^{(r)} - 1), \quad (q^h)^{(r)} = q^{-3} a^{(r)} b^{(r)} q^{-\frac{1}{2}} (q^{-1} d^{(r)} - q^{2} c^{(r)} - 1).
\end{align*}
\] (157)

The star structure on \( \widetilde{Hol}_1(\bullet, (sl(2, \mathbb{C})_\mathbb{R})^\mathbb{C}) \) is defined using

\[
\begin{align*}
a^{(l)} \ast &= q^{-3/2} a^{(r)}, \quad b^{(l)} \ast = -q^{-1/2} c^{(r)}, \quad c^{(l)} \ast = -q^{-5/2} b^{(r)}, \quad d^{(l)} \ast = q^{-3/2} a^{(r)}, \quad x^{(l)} \ast = x^{(r)}.
\end{align*}
\] (158)

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Proving that this star is an antilinear antihomorphism of $\widehat{Hol}_1(\bullet, sl(2, \mathbb{C})_{\mathbb{C}})$ is done by a direct verification of the commutation relations. With this star we have, in particular,

$$e^{(t)} \star = q q^{h(r)} f^{(r)} \quad f^{(t)} \star = q^{-1} e^{(r)} q^{-h(r)} \quad (q^{h(t)}) \star = q^{h(r)}. \quad (159)$$

Remark. We end this section of the definition and study of quantization of these dynamical algebras by looking at the subalgebra of $\hat{\text{Hol}}_p(\mathcal{G})$. We call this algebra the \textit{quantum algebra of dynamical monodromies originating from the first source}, and we denote it $\hat{\text{Mon}}_p(\mathcal{G})$. The commutation relations satisfied by these elements are:

$$[\hat{\mu}_{(1)} \alpha_i, \mathcal{M}_{(k)}] = [\mathcal{M}_{(k)}, h_{\alpha_i}], \quad (160)$$

$$R_{21}(\hat{\mu}_{(1)}) \mathcal{M}_{(k)} 1 R_{21}^{-1}(\hat{\mu}_{(1)}) \mathcal{M}_{(l)} 2 = \mathcal{M}_{(l)} 2 R_{21}(\hat{\mu}_{(1)}) \mathcal{M}_{(k)} 1 R_{21}^{-1}(\hat{\mu}_{(1)}), \quad k < l. \quad (161)$$

$$\mathcal{M}_{(k)} 2 R_{12}^{-1}(\hat{\mu}_{(1)}) \mathcal{M}_{(k)} 1 R_{12}(\hat{\mu}_{(1)}) = F_{12}^{-1}(\hat{\mu}_{(1)}) (\Delta \otimes id)(\mathcal{M}_{(k)}) F_{12}(\hat{\mu}_{(1)}). \quad (162)$$

As a result the algebra generated by $\hat{\mu}_{(1)}$ and $\mathcal{M}_{(k)}$ for $k$ fixed is a dynamical reflection algebra which has been recently studied in [13].

As a remark, we note that the equation (160) and the linear equation (159) can be equivalently written as:

$$(vB(\mu_{(1)}))^{-1} R_{12}^{-1}(\hat{\mu}_{(1)}) \mathcal{M}_{(k)} 1 R_{12}(\hat{\mu}_{(1)}) = R_{21}(\hat{\mu}_{(1)}) \mathcal{M}_{(k)} 1 R_{21}^{-1}(\hat{\mu}_{(1)})(vB(\mu_{(1)}))^{-1} \quad (163)$$

$$(vB(\mu_{(1)}))^{-1} R_{12}^{-1}(\hat{\mu}_{(1)})(vB(\mu_{(1)}))^{-1} R_{12}(\hat{\mu}_{(1)}) = F_{12}^{-1}(\hat{\mu}_{(1)}) (\Delta \otimes id)((vB(\mu_{(1)}))^{-1}) F_{12}(\hat{\mu}_{(1)}).$$

6 Conclusion

In this work, we have defined and studied algebras associated to the quantization of Chern-Simons theory with sources. We have emphasized the example $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ and $G = SL(2, \mathbb{C})_{\mathbb{R}}$ having in mind potential applications to Lorentzian quantum gravity in $2 + 1$ dimensions with $\Lambda < 0$ or $\Lambda > 0$. These algebras are bigger than the algebra of constants of motion and include partial observables depending on the parametrization of the world-line of the sources. These algebras are nice and new examples of dynamical exchange algebras.

Our work can be extended in different directions that we are now exploring.

1. A direct continuation of our work is the representation side of the present work $\mathcal{S}$. As explained in section $3.1$ the structure of $\widehat{Hol}_p(\bullet, G)$ is reduced to the study of $\widehat{Hol}_1(\bullet, G)$. We can construct therefore unitary representation of this algebra allowing to obtain unitary irreducible representation of $\widehat{Hol}_p(\bullet, G)$ and of its different quotients introduced in our work.
2. From the mathematical perspective, we have pointed numerous links between these algebras. Our work can give new insights in the theory of dynamical quantum groups. In particular the study of the quantum algebra of dynamical monodromies will give new results in the explicit construction of the dynamical coboundary in the $sl(n)$ case.

3. The application of the present work to the study of physical questions that can be addressed in $2 + 1$ quantum gravity has to be developed and is currently in progress.

References

[1] E. Buffenoir, K. Noui, “Unfashionable observations about three dimensional gravity” [arXiv: gr-qc/0305079].

[2] P. Etingof, O. Schiffmann, “Lectures on the dynamical Yang-Baxter equations”, [arXiv: math.QA/9908064].

[3] A. Alekseev, A. Malkin, “Symplectic structure of the moduli space of flat connections on a Riemann surface”, Commun.Math.Phys. 169 (1995) 99 [arXiv: hep-th/9312004].

[4] A. Alekseev, “Integrability in the Hamiltonian Chern-Simons Theory,” st.Petersburg.Math.J, Vol.6 (1995),No.2, [arXiv: hep-th/9311074].

[5] A. Alekseev, V. Schomerus, “Representation Theory of Chern-Simons Observables,” Duke Math. J. 85 (1996), no. 2, 447-510. [arXiv: q-alg/9503016].

[6] V.V. Fock, A.A. Rosly, “Poisson structure on moduli of flat connections on Riemann surfaces and $r$-matrix”, [arXiv: math.QA/9802054].

[7] E. Buffenoir, K. Noui, Ph. Roche, ” Hamitonian Quantization of $SL(2,C)$ Chern-Simons Theory.” Class.Quant.Grav. 19 (2002) 4953. [arXiv:hep-th/0202121].

[8] E. Buffenoir, Ph. Roche, “Chern-Simons Theory with Sources and Dynamical Quantum Groups II: Unitary representations.” In preparation.

[9] E. Buffenoir, Ph. Roche, “Harmonic analysis on the quantum Lorentz group”, Comm.Math.Phys, 207, 499-555, (1999). [arXiv: q-alg/9710022].

[10] D. Arnaudon, E. Buffenoir, E. Ragoucy, Ph. Roche, “Universal Solutions of Quantum Dynamical Yang-Baxter Equations”, Lett.Math.Phys. 44 (1998) 201-214, [arXiv: q-alg/9712037].

[11] M. Jimbo, H. Konno, S. Odake, J. Shiraishi, “Quasi-Hopf twistors for elliptic quantum groups”, [arXiv: q-alg/9712029].
Appendix 1: Conventions on $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and denote $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote $\Phi \subset \mathfrak{h}^*$ the set of roots of $\mathfrak{g}$. We select a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and denote $\Phi^+ = \mathfrak{h}^* \cap \Phi$ the corresponding decomposition of $\mathfrak{g}$ and let $\mathfrak{g}_\alpha$ be the root subspace of $\mathfrak{g}$. We denote by $\langle \cdot, \cdot \rangle$ the Killing form on $\mathfrak{g}$.

If $\alpha \in \mathfrak{h}^*$ we will denote by $t_\alpha \in \mathfrak{h}$ the element defined by $\langle t_\alpha, h \rangle = \alpha(h), \forall h \in \mathfrak{h}$, and still by $\langle \cdot, \cdot \rangle$ the scalar product on $\mathfrak{h}^*$ defined by duality. To each root $\alpha$ we will associate the element $h_\alpha = \frac{2}{\alpha(\alpha)} t_\alpha$.

A presentation of $U_q(\mathfrak{g})$ by generators and relations is given by:

\begin{align*}
[t_\alpha, t_\beta] &= 0 \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{q^{t_\alpha_i} - q^{-t_\alpha_i}}{q - q^{-1}} \\
\quad [t_\alpha, e_{\alpha_j}] &= a_{ij}^{\text{sym}} e_{\alpha_j} \\
\quad [t_\alpha, e_{-\alpha_j}] &= -a_{ij}^{\text{sym}} e_{-\alpha_j} \\
(\text{ad}_{q'} e_{\alpha_i})^{n_{ij}} (e_{\alpha_j}) &= 0 \quad \text{if } i \neq j \text{ and } n_{ij} = 1 - 2 a_{ij}^{\text{sym}} q' = q \text{ or } q^{-1}
\end{align*}

where we have introduced:

\begin{equation}
(\text{ad}_{q^\pm 1} x)(y) = \sum_{(x)} x(1) y S^\pm 1(x(2)).
\end{equation}

The Hopf algebra structure is defined by:

\begin{align*}
\Delta(t_\alpha) &= t_\alpha \otimes 1 + 1 \otimes t_\alpha \\
\Delta(e_{\alpha_i}) &= e_{\alpha_i} \otimes q^{t_\alpha_i} + 1 \otimes e_{\alpha_i} \\
\Delta(e_{-\alpha_i}) &= e_{-\alpha_i} \otimes 1 + q^{-t_\alpha_i} \otimes e_{-\alpha_i}
\end{align*}

As usual, we denote by $U_q(\mathfrak{b}_+)$ (resp. $U_q(\mathfrak{b}_-)$) the algebra generated by $h_\alpha, e_\alpha, \alpha \in \Pi$ (resp. $h_\alpha, e_\alpha, \alpha \in \Pi$).

We denote $\pi^+$ (resp. $\pi^-$) the projections of $U_q(\mathfrak{b}_+)$ (resp. $U_q(\mathfrak{b}_-)$) onto $U_q(\mathfrak{h})$.

We define $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the kernel of $\pi^+$ (resp. $\pi^-$).

$U_q(\mathfrak{g})$ is quasi-triangular, i.e. there exists an $R-$matrix which obeys the standard quasitriangularity equations

\begin{equation}
(\Delta \otimes \text{id})(R) = R_{13} R_{23} \quad (\text{id} \otimes \Delta)(R) = R_{13} R_{12}
\end{equation}

\begin{equation}
R \Delta = \Delta' R.
\end{equation}
As usual we denote \( R^{(+)} = R, R^{(-)} = R_{21}^{-1} \).

As a remark, the expression of \( R \) for \( U_q(sl(2)) \) is simply
\[
R = q^{\frac{h \otimes h}{2}} \exp_q((q - q^{-1}) e \otimes f)
\]
with the \( q \)-exponential being defined as:
\[
\exp_q(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n)_q!} \tag{171}
\]
where \((n)_q! = (n)_q \cdots (1)_q\), with \((z)_q = q^z - 1 + [z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}\).

\( U_q(\mathfrak{g}) \) is a ribbon Hopf algebra, which means that it exists an invertible element \( v \in U_q(\mathfrak{g}) \) such that:
\[
\begin{align*}
& v \text{ is a central element,} \\
& v^2 = uS(u), \epsilon(v) = 1, S(v) = v, \\
& \Delta(v) = (R_{21} R_{12})^{-1}(v \otimes v), \tag{172}
\end{align*}
\]
where we have denoted \( u = \sum_i S(b_i)a_i \) where \( S \) is the antipode and \( R = \sum_i a_i \otimes b_i \). A fundamental property of \( u \) is
\[
S^2(x) = uxu^{-1}, \quad \forall x \in U_q(\mathfrak{g}). \tag{173}
\]

The explicit values of these elements as well as the conventions concerning Clebsch-Gordan coefficients for finite dimensional representations of \( U_q(sl(2)) \) are summarized in [9].

**Appendix 2: Real forms and Star structures**

A real vector space \( V \) can be equivalently described in terms of its complexification \( V^C = V \otimes \mathbb{C} \) equipped with a star structure, i.e. an antilinear involution, denoted \( \ast \) and chosen such that \( V \) is the real vector space of elements \( x \in V^C \) such that \( x^* = -x \). A complex bilinear form \( \langle \cdot, \cdot \rangle \) on \( V^C \) is the complexification of a real bilinear form on \( V \) if and only if
\[
\langle x, y \rangle = \langle x^*, y^* \rangle, \quad \forall x, y \in V^C.
\]

If \( V = \mathfrak{g} \) is a Lie algebra we have moreover to require the star to be an antimorphism of \( \mathbb{C} \)-Lie algebra. A real form of a complex Hopf algebra is defined by choosing a star structure which is an antilinear involutive antimorphism of algebra and which property with respect to the coalgebra is usually chosen as being a morphism or antimorphism of coalgebra (see the review in [9].)

Once a star \( \ast \) is defined on \( \mathfrak{g} \), a star (denoted with the same symbol) can then be straightforwardly defined on \( F(G^C) \) by the following requirement added to the definitions [8,9]:
\[
(\ast \otimes \ast)(M) = M^{-1} \tag{174}
\]

We recall here the different real form of \( sl(2, \mathbb{C}) \) namely \( su(2), su(1,1), sl(2, \mathbb{R}) \). Note that although classically \( su(1,1) \) is isomorphic as a real Lie algebra to \( sl(2, \mathbb{R}) \), in the quantum case one obtains two different real forms. These structures are simpler to
describe using the following linear involutive automorphism \( \sigma_1, \sigma_2 : sl(2, \mathbb{C}) \to sl(2, \mathbb{C}) \) defined by

\[
\begin{align*}
\sigma_1(h) &= h & \sigma_1(e) &= -e & \sigma_1(f) &= -f \\
\sigma_2(h) &= -h & \sigma_2(e) &= -f & \sigma_2(f) &= -e.
\end{align*}
\] (175)

We can now define the following star structures on the Lie algebra:

\[
\begin{align*}
su(2) : e^* &= f & f^* &= e & h^* &= h, \\
su(1, 1) : \overline{r} = \sigma_1 \circ * \\
sl(2, \mathbb{R}) : \overline{r} &= \sigma_2 \circ *.
\end{align*}
\] (177)

\( \mathfrak{g} = sl(2, \mathbb{C})_\mathbb{R} \) is the real Lie algebra of \( sl(2, \mathbb{C}) \). Its complexification is such that \( \mathfrak{g}^\mathbb{C} = sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \). Let us define \( e^{(l,r)}, f^{(l,r)}, h^{(l,r)} \) to be a Cartan basis of \( \mathfrak{g}^\mathbb{C} \) where the \( l \) (resp. \( r \)) generate the first (resp.second) component of the direct sum. Any of the following star structure

\[
\begin{align*}
(e^{(l)})^* &= f^{(r)} & (f^{(l)})^* &= e^{(r)} & (h^{(l)})^* &= h^{(r)} \\
(e^{(l)})\overline{r} &= -f^{(r)} & (f^{(l)})\overline{r} &= -e^{(r)} & (h^{(l)})\overline{r} &= h^{(r)} \\
(e^{(l)})\overline{*} &= -e^{(r)} & (f^{(l)})\overline{*} &= -f^{(r)} & (h^{(l)})\overline{*} &= -h^{(r)}
\end{align*}
\] (180)

is a real form on \( sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \) selecting the same real Lie algebra \( sl(2, \mathbb{C})_\mathbb{R} \).

### Appendix 3: Dynamical Quantum Groups

The purpose of dynamical quantum group theory (for a review see [2]) is to give an algebraic framework to the study of solutions of the \textit{Dynamical Quantum Yang Baxter Equation} i.e:

\[
R_{12}(\mu)R_{13}(\mu + h_2)R_{23}(\mu) = R_{23}(\mu + h_1)R_{13}(\mu)R_{12}(\mu + h_3),
\] (183)

where \( R : \mathfrak{g}^* \to U_q(\mathfrak{g})^{\otimes 2} \) and \([R_{12}(\mu), h \otimes 1 + 1 \otimes h] = 0, \forall h \in \mathfrak{g} \).

This last equation is a quantization of the classical dynamical Yang Baxter equation, i.e \( R \) satisfies \( R(\mu) = 1 + i\hbar r(-\tilde{\chi}) + o(\hbar) \) with \( \mu \sim -\frac{\tilde{\chi}}{\hbar} \) and \( r \) satisfies the Classical Dynamical Yang Baxter quation, i.e:

\[
[r_{12}(\tilde{\chi}), r_{13}(\tilde{\chi}) + r_{23}(\tilde{\chi})] + [r_{13}(\tilde{\chi}), r_{23}(\tilde{\chi})] = \\
\sum_i (h^{(1)}_{\alpha_i} \frac{\partial r_{23}(\tilde{\chi})}{\partial \chi_{\alpha_i}} - h^{(2)}_{\alpha_i} \frac{\partial r_{13}(\tilde{\chi})}{\partial \chi_{\alpha_i}} + h^{(3)}_{\alpha_i} \frac{\partial r_{12}(\tilde{\chi})}{\partial \chi_{\alpha_i}}).
\] (184)

When \( r_{12}(\tilde{\chi}) \) is the rational solution [25], it has been shown that the quantization \( R(\mu) \) lies in \( U(\mathfrak{g})^{\otimes 2} \) and can be expressed as:

\[
R(\mu) = F_{21}(\mu)^{-1} F_{12}(\mu),
\]
where \( F(\mu) \in U(\mathfrak{g})^{\otimes 2} \) is the solution of the dynamical 2-cocycle equation
\[
(id \otimes \Delta)(F(\mu))F_{23}(\mu) = (\Delta \otimes id)(F(\mu))F_{12}(\mu + h_3),
\]
with \([F_{12}(\mu), h \otimes 1 + 1 \otimes h] = 0, \forall h \in \mathfrak{g} \).

It has been shown in [10, 11] that the corresponding solution of the previous equation
is also the unique solution of the following linear equation on \( F \)
\[
[F_{12}(\mu), b(\mu)] = -2\left( \sum_{\alpha \in \Phi^+} e_\alpha \otimes e_{-\alpha} \right) F_{12}(\mu)
\]
(186)
where \( b(\mu) = \sum_j (2\mu_j + h_{\alpha_j}) \lambda^j \) with the condition that \( F(\mu) - 1 \in U^+(\mathfrak{g}) \otimes U^-(\mathfrak{g}) \).

When \( r^\theta_{12}(\hat{x}) \) is the trigonometric solution [195], it has been shown that the quantization \( R^\theta(\mu) \in U_q(\mathfrak{g})^{\otimes 2} \) can be expressed as:
\[
R^\theta(\mu) = F_{21}^\theta(\mu)^{-1} R^\theta F_{12}^\theta(\mu)
\]
(187)
where \( F^\theta(\mu) \in U_q(\mathfrak{g})^{\otimes 2} \) is the solution of the dynamical 2-cocycle equation
\[
(id \otimes \Delta)(F^\theta(\mu))F^\theta_{23}(\mu) = (\Delta \otimes id)(F^\theta(\mu))F^\theta_{12}(\mu + h_3),
\]
(188)
\( q = \exp(\frac{ih}{\hbar}) \) and \( R^\theta \) is the universal \( R \) matrix of \( U_q(\mathfrak{g}) \). It has been shown in [10, 11]
that the corresponding solution of the dynamical 2-cocycle equation is also the unique solution of the dynamical equation on \( F^\theta \)
\[
F^\theta(\mu) B_2(\mu) = R^{-1}_{12} q^{\sum_j h_{\alpha_j} \otimes \lambda^j} B_2(\mu) F^\theta(\mu),
\]
(189)
with \( B(\mu) = q^{\sum_j (2\mu_j + h_{\alpha_j}) \lambda^j} = q^{b(\mu)} \), with the condition that \( F^\theta(\mu) - 1 \in U^+(\mathfrak{g}) \otimes U^-(\mathfrak{g}) \)
and \([F_{12}(\mu), h \otimes 1 + 1 \otimes h] = 0, \forall h \in \mathfrak{g} \).

This implies the following equation on the dynamical \( R \) matrix:
\[
R_{12}(\mu) B_2(\mu) R_{21}(\mu) = B_2(\mu + h_1).
\]
(190)

**Appendix 4: Miscellaneous Results**

We compute the Dirac bracket between \( V_\gamma \) and \( V_{\gamma'} \) and obtain the results [38, 59].

\[
\{V_\gamma, V_{\gamma'}\}_D = \{V_\gamma, V_{\gamma'}\}_d + 2\theta \int_{\mathfrak{g}^+ \times \mathfrak{g}^+} [Du][Dv] \{V_\gamma, \tilde{\Omega}(u)\}_d (K^X)^{-1}(u, v) \{\tilde{\Omega}(v), V_{\gamma'}\}_d
\]

\[
= V_\gamma V_{\gamma'} \left( r_{12}(-\tilde{x}(k)) - \frac{1}{2\theta} \int_{\mathfrak{g}^+ \times \mathfrak{g}^+} [Du][Dv] u(\varphi)_1 (K^{-\frac{1}{2 \pi} \tilde{x}(k)})^{-1}(u, v) v(\varphi')_2 \right)
\]

The last identity is obtained using basic properties already established, as well as the following identities
\[
\{V_\gamma, V_{\gamma'}\}_d = V_\gamma V_{\gamma'} r_{12}(-\tilde{x}(k)), \quad \{\tilde{\Omega}(u), M_{(k)}\}_d = 0, \quad \{\tilde{\Omega}(u), A_i\}_d = \frac{1}{2\theta} D_{A_i} u,
\]
\[
\int_{\mathfrak{g}^+ \times \mathfrak{g}^+} [Du][Dv] M_{(k)}^{-1}_1 u(\varphi)_1 M_{(k)}(K^X)^{-1}(u, v) M_{(k)}^{-1}_2 v(\varphi')_2 M_{(k)} = \int_{\mathfrak{g}^+ \times \mathfrak{g}^+} [Du][Dv] u(\varphi)_1 (K^{-\frac{1}{2 \pi} \tilde{x}(k)})^{-1}(u, v) v(\varphi')_2.
\]
This Dirac bracket can also be written as a dynamical quadratic Poisson bracket as

\[ \{V_{\gamma 1}, V_{\gamma 2}\}_D = V_{\gamma 1}V_{\gamma 2}r_{12}^\theta (\varphi - \varphi' ; - \tilde{\chi}) \]  

(191)

with \( r_{12}^\theta (\varphi; - \tilde{\chi}) \) given by:

\[
r_{12}^\theta (\varphi; - \tilde{\chi}) = \frac{1}{4\pi \theta} \left( (\pi - \varphi) \sum_j h_{\alpha j} \otimes \lambda^j + \sum_{\alpha \in \Phi} e_\alpha \otimes e_{-\alpha} \frac{\pi e^{\tilde{\chi}(\alpha)(\pi - \varphi)/4\pi \theta}}{\sinh(\tilde{\chi}(\alpha)/4 \theta)} \right).
\]

(192)

This is obtained by computing explicitly \((\mathcal{K}^Y)^{-1}\). Let \(u, v \in \mathfrak{h}^\perp\) we define \(u(\varphi) = \sum_{n, \alpha \in \Phi} u_n^{\alpha} e^{in\varphi} + \sum_{n, j = 1, \ldots, r} u_n^{h_{\alpha j}} e^{im\varphi}\), with \(u_n^{\alpha} \in \mathbb{C} e_{\alpha}\) and \(u_n^{h_{\alpha j}} \in \mathbb{C} h_{\alpha j}\), and similarly \(v(\varphi) = \sum_{n, \alpha \in \Phi} v_n^{\alpha} e^{in\varphi} + \sum_{n, j = 1, \ldots, r} v_n^{h_{\alpha j}} e^{im\varphi}\), with \(v_n^{\alpha} \in \mathbb{C} e_{\alpha}\) and \(v_n^{h_{\alpha j}} \in \mathbb{C} h_{\alpha j}\).

Let \(Y \in \mathfrak{h}\) we have

\[
\mathcal{K}^Y (u, v) = \langle u, \partial \varphi v + [Y, v] \rangle
= 2\pi \left( \sum_{n \neq 0, \alpha \in \Phi} \langle u_n^{-\alpha}, v_n^{\alpha} \rangle (\alpha(Y) + in) + \sum_{n \neq 0, j} \langle h_{\alpha j}^{h_{\alpha j}}, v_n^{\lambda_j} \rangle in \right).
\]

(193)

As a result we get:

\[
(\mathcal{K}^Y)^{-1} (u, v) = \frac{1}{2\pi} \left( \sum_{n \neq 0, \alpha \in \Phi} \frac{\langle u_n^{\alpha}, v_n^{-\alpha} \rangle}{\alpha(Y) + in} + \sum_{n \neq 0, j} \frac{\langle h_{\alpha j}^{h_{\alpha j}}, v_n^{\lambda_j} \rangle}{in} \right).
\]

(194)

Therefore we obtain:

\[
4\pi \theta r_{12}^\theta (\varphi; - \tilde{\chi}) = \sum_{m \neq 0, \alpha \in \Phi} \frac{e_\alpha \otimes e_{-\alpha}}{4\pi \tilde{\chi}(\alpha) + im} e^{im\varphi} + \sum_{j, m \neq 0} h_{\alpha j} \otimes \lambda^j e^{im\varphi}
= (\pi - \varphi) \sum_j h_{\alpha j} \otimes \lambda^j + \sum_{\alpha \in \Phi} e_\alpha \otimes e_{-\alpha} \frac{\pi e^{\tilde{\chi}(\alpha)(\pi - \varphi)/4\pi \theta}}{\sinh(\tilde{\chi}(\alpha)/4 \theta)},
\]

(195)

where in the second equality we have extended the function by 2\(\pi\)-periodicity with the convention that \(\varphi \in [0, 2\pi]\).

We now show the result (196 197).

We denote \(k_{12} = \sum_j h_{\alpha j} \otimes \lambda^j, \) and \(a_{12} = \sum_{\alpha \in \Phi^+} e_\alpha \otimes e_{-\alpha}. \)

\[
\frac{2}{i\hbar} P_1 M_2 = M_1 b_1 (\hat{\mu}) M_1^{-1} M_2
= M_1 b_1 (\hat{\mu}) M_2 R_{12}^{-1} (\hat{\mu}) M_1^{-1}
= M_1 M_2 (b_1 (\hat{\mu}) + 2k_{12}) R_{12}^{-1} (\hat{\mu}) M_1^{-1}
= M_2 M_1 R_{12} (\hat{\mu}) (b_1 (\hat{\mu}) + 2k_{12}) R_{12}^{-1} (\hat{\mu}) M_1^{-1}.
\]

31
Using the identity $R_{12}(\hat{\mu})(b_1(\hat{\mu}) + 2k_{12})R_{12}^{-1}(\hat{\mu}) = 2t_{12} + b_1(\hat{\mu})$, which is a consequence of the linear equation 186, we obtain:

$$\frac{2}{i\hbar}P_1M_2 = 2t_{12}M_2 + \frac{2}{i\hbar}M_2P_1. \quad (196)$$

We now prove that $\Delta(P) = P_1 + P_2$. Let $\mathcal{P} = \frac{1}{2}Mb(\hat{\mu})M^{-1}$,

$$2\Delta(\mathcal{P}) = M_1M_2F_{12}^{-1}\Delta(b(\hat{\mu}))F_{12}(\hat{\mu})M_2^{-1}M_1^{-1}$$
$$= M_1M_2\Delta(b(\hat{\mu}))M_2^{-1}M_1^{-1}$$
$$= M_1M_2(b_1(\hat{\mu}) + b_2(\hat{\mu}) + 2k_{12})M_2^{-1}M_1^{-1}$$
$$= M_1b_1(\hat{\mu})M_1^{-1} + M_1M_2b_2(\hat{\mu})M_2^{-1}M_1^{-1}$$
$$= M_1b_1(\hat{\mu})M_1^{-1} + M_2M_1R_{12}(\hat{\mu})b_2(\hat{\mu})R_{12}^{-1}(\hat{\mu})M_1^{-1}M_2^{-1}$$
$$= M_1b_1(\hat{\mu})M_1^{-1} + M_2M_1(b_2(\hat{\mu}) + 2k_{12} - 2t_{12})M_1^{-1}M_2^{-1}$$
$$= M_1b_1(\hat{\mu})M_1^{-1} + M_2b_2(\hat{\mu})M_2^{-1} - 2M_2M_1t_{12}M_1^{-1}M_2^{-1}.$$ 

This closes the proof of $\Delta(P) = P_1 + P_2$.

We now prove the $q$–analog of the two previous relations:

$$\mathcal{P}_2\hat{R}_{12}^{-1} U_1 = (U_2v_2^{-1}B_2(\hat{\mu})U_2^{-1})\hat{R}_{12}^{-1} U_1 =$$
$$= U_2v_2^{-1}B_2(\hat{\mu})U_1R_{21}(\hat{\mu})U_2^{-1} =$$
$$= U_2U_1v_2^{-1}B_2(\hat{\mu} + h_1)R_{21}^{-1}(\hat{\mu})U_2^{-1} =$$
$$= U_2U_1v_2^{-1}R_{12}(\hat{\mu})B_2(\hat{\mu})U_2^{-1} =$$
$$= \hat{R}_{21}U_1U_2v_2^{-1}B_2(\hat{\mu})U_2^{-1} =$$
$$= \hat{R}_{21}U_1\mathcal{P}_2$$

and

$$J_{12}^{-1}\Delta(P)J_{12} = J_{12}^{-1}\Delta(U)v_1^{-1}v_2^{-1}R_{21}R_{12}B_2(\hat{\mu})B_1(\hat{\mu} + h_2)\Delta(U)^{-1}J_{12} =$$
$$= J_{12}^{-1}R_{12}^{-1}J_{12}U_1U_2F_{12}^{-1}(\hat{\mu})v_1^{-1}v_2^{-1}R_{21}R_{12}B_2(\hat{\mu})B_1(\hat{\mu} + h_2)F_{12}(\hat{\mu})U_2^{-1}U_1^{-1}J_{21}^{-1}R_{12}J_{12} =$$
$$= U_2U_1R_{12}^{-1}(\hat{\mu})F_{12}^{-1}(\hat{\mu})v_1^{-1}v_2^{-1}R_{21}R_{12}B_2(\hat{\mu})B_1(\hat{\mu} + h_2)F_{12}(\hat{\mu})U_2^{-1}U_1^{-1}\hat{R}_{12} =$$
$$= U_2U_1F_{21}^{-1}(\hat{\mu})v_1^{-1}v_2^{-1}R_{21}R_{12}B_2(\hat{\mu})B_1(\hat{\mu} + h_2)F_{12}(\hat{\mu})U_2^{-1}U_1^{-1}\hat{R}_{12} =$$
$$= U_2U_1v_1^{-1}v_2^{-1}R_{12}(\hat{\mu})B_2(\hat{\mu})B_1(\hat{\mu} + h_2)U_2^{-1}U_1^{-1}\hat{R}_{12} =$$
$$= U_2U_1v_1^{-1}v_2^{-1}B_2(\hat{\mu} + h_1)R_{21}^{-1}(\hat{\mu})B_1(\hat{\mu} + h_2)U_2^{-1}U_1^{-1}\hat{R}_{12} =$$
$$= U_2v_2^{-1}B_2(\hat{\mu})U_1R_{21}^{-1}(\hat{\mu})U_2^{-1}v_1^{-1}B_1(\hat{\mu})U_1^{-1}\hat{R}_{12} =$$
$$= (U_2v_2^{-1}B_2(\hat{\mu})U_2^{-1})\hat{R}_{12}^{-1}(U_1v_1^{-1}B_1(\hat{\mu})U_1^{-1})\hat{R}_{12} =$$
$$= \mathcal{P}_2\hat{R}_{12}^{-1}\mathcal{P}_1\hat{R}_{12}. $$