UNIFORMIZATION OF SEMISTABLE BUNDLES
ON ELLIPTIC CURVES

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Abstract. Let $G$ be a connected reductive complex algebraic group, and $E$ a complex elliptic curve. Let $G_E$ denote the connected component of the trivial bundle in the stack of semistable $G$-bundles on $E$. We introduce a complex analytic uniformization of $G_E$ by adjoint quotients of reductive subgroups of the loop group of $G$. This can be viewed as a nonabelian version of the classical complex analytic uniformization $E \simeq \mathbb{C}^*/\mathbb{Q}^2$. We similarly construct a complex analytic uniformization of $G$ itself via the exponential map, providing a nonabelian version of the standard isomorphism $\mathbb{C}^* \simeq \mathbb{C}/\mathbb{Z}$, and a complex analytic uniformization of $G_E$ generalizing the standard presentation $E \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau)$. Finally, we apply these results to the study of sheaves with nilpotent singular support.

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1. Introduction

1.1. **Background.** Let $G$ be a connected complex reductive algebraic group, and $E$ a complex elliptic curve. The moduli of $G$-bundles on $E$ plays a distinguished role in representation theory, gauge theory and algebraic combinatorics (for example [BS12, Sch12] as the setting of the elliptic Hall algebra), and its geometry has been the subject of a long and fruitful study. Atiyah [Ati57] classified vector bundles on $E$ in terms of line bundles and their extensions. In particular, he showed rank $n$ vector bundles with trivial Jordan-Holder factors are in bijection with unipotent adjoint orbits in $GL(n)$, with the unique irreducible such vector bundle corresponding to the regular unipotent orbit. This initiated the organizing viewpoint that vector bundles on $E$ form an analogue of the adjoint quotient of $GL(n)$, where the “eigenvalues” of a vector bundle are the line bundles appearing as its Jordan-Holder factors. In a beautiful series of papers, Friedman, Morgan and Witten [FMW97, FM98, FM00] extended this to any $G$, definitively describing the Jordan-Holder patterns and the geometry of the coarse moduli of semistable bundles. Our focus here is the moduli stack of semistable bundles, and specifically the construction of an analytic uniformization of it by finite-dimensional subvarieties of the loop group of $G$. We discuss motivations and applications at the end of the introduction.

1.1.1. **Holomorphic loop group with twisted conjugation.** Thanks to complex function theory, the uniformization $E \simeq \mathbb{C}^*/q^\mathbb{Z}$, with $|q| < 1$, has been known since the 19th century. Let $\text{Jac}(E)$ be the Jacobian variety parameterizing degree zero line bundles on $E$. (Thanks to Serre’s GAGA, one can equivalently consider algebraic or holomorphic bundles.) The Abel-Jacobi map $E \to \text{Jac}(E)$, $x \mapsto \mathcal{O}_E(x - x_0)$ is an isomorphism, inducing a similar uniformization $\text{Jac}(E) \simeq \mathbb{C}^*/q^\mathbb{Z}$. This isomorphism also results from the following geometric observations. By the uniformization $E \simeq \mathbb{C}^*/q^\mathbb{Z}$, holomorphic line bundles on $E$ are equivalent to equivariant holomorphic line bundles on $\mathbb{C}^*$. Since every holomorphic line bundle on $\mathbb{C}^*$ is trivializable, equivariant holomorphic line bundles are encoded by their equivariance up to gauge. Such data can be represented by elements of the holomorphic loop group $L_{hol}\mathbb{C}^*$ up to $q$-twisted conjugacy. Within this identification, one finds the uniformization
of \(\text{Jac}(E)\) by the constant loops \(\mathbb{C}^* \subset L_{\text{hol}}\mathbb{C}^*\) up to \(q\)-twisted conjugacy by the coweights \(\mathbb{Z} \simeq \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) \subset L_{\text{hol}}\mathbb{C}^*\).

Now let \(G_E\) denote the connected component of the trivial bundle in the stack of semistable \(G\)-bundles on \(E\). By the uniformization \(E \simeq \mathbb{C}^*/q^\mathbb{Z}\), isomorphism classes of \(G\)-bundles on \(E\) are in bijection with \(q\)-twisted conjugacy classes in the holomorphic loop group \(L_{\text{hol}}G\) (see for example [BG96] where this is attributed to Looijenga). We would like to enhance this to an analytic uniformization of \(G_E\) by finite-dimensional subvarieties of \(L_{\text{hol}}G\). As a first attempt, we could take the constant loops \(G \subset L_{\text{hol}}G\), but unfortunately, in general, the natural map \(G/G \to G_E\) from the adjoint-quotient is neither surjective nor étale. We will correct for both of these shortcomings by considering multiple charts together with their gluing; see the main results as described in Sect. 1.2.

1.1.2. Connections on circle with gauge transformation. Our arguments also apply to an easier situation to give a similar uniformization of \(G/G\) in terms of (open subsets of) adjoint quotients of reductive Lie algebras. In this case, the role of \(L_{\text{hol}}G\) with twisted conjugation is replaced by the affine space of connections on a circle with gauge transformation.

1.2. Main results. Assume for simplicity that \(G\) is almost simple and simply-connected. Let \(T \subset G\) be a maximal torus, with coweight lattice \(X_*(T) = \text{Hom}(\mathbb{C}^*, T)\).

The real affine space \(t_{R} := X_*(T) \otimes \mathbb{R}\) has a natural stratification by simplices coming from the hyperplanes
\[ H_{\alpha,n} := \{ x \in t_{R} \mid \alpha(x) = n \}, \text{ for } \alpha \text{ a root of } G, n \in \mathbb{Z} \]

Let \(C\) be an alcove of \(t_{R}\), i.e. a top dimensional simplex. There is a naturally defined category \(\mathcal{F}_C\) of faces of \(C\), whose objects are faces of \(C\), i.e. simplices in \(\overline{C}\), and whose morphisms are given by the closure relation.

For any \(J \in \mathcal{F}_C\), we associate a finite-dimensional connected reductive subgroup \(G_J \subset L_{\text{hol}}G\), whose Lie algebra \(\mathfrak{g}_J \subset L_{\text{hol}}\mathfrak{g}\) is spanned by \(t\) and those affine root spaces whose affine root vanishes on \(J\).

We introduce an (analytic) twisted adjoint-invariant open subset \(\mathfrak{g}^e_J \subset \mathfrak{g}_J\) (resp. \(G^e_J \subset G_J\)) of elements with “small eigenvalues” with respect to \(J\). Roughly speaking, an eigenvalue in \(t\) (resp. \(T\)) is small with respect to \(J\) if its real part (resp. \(q\)-part) lies in a simplex whose closure contains \(J\) (for detail, see definition before Proposition 4.38 for \(G_J^e\), and Theorem 6.1(6) for \(\mathfrak{g}_J^e\)). Denote by \(G_{J}/G_{J}(\text{resp. } \mathfrak{g}_{J}/G_{J})\) the quotient stack w.r.t the twisted conjugation (Sect 1.1.1) (resp. the gauge action (Sect 1.1.2, which we shall also refer as a “twisted” action)). Then we have:

**Theorem 1.1.** (Theorem 6.1(6), Theorem 4.40) The natural maps \(\mathfrak{g}^e_J/G_J \to G/G, G_J^e/G_J \to G_E\) are open embeddings, and
\[
\begin{align*}
(1) \quad \coprod_{J \in \{ \text{vertices of } C \}} \mathfrak{g}^e_J/G_J & \longrightarrow G/G \\
(2) \quad \coprod_{J \in \{ \text{vertices of } C \}} G^e_J/G_J & \longrightarrow G_E
\end{align*}
\]
are surjective.

To describe the gluing of the above charts, for any face \(J\) of \(C\), we have
\[
\mathfrak{g}^e_J/G_J = \bigcap_{J_0 \in \{ \text{vertices of } J \}} \mathfrak{g}^e_{J_0}/G_{J_0}
\]
\[ G^{\sigma}_{J_0} / G_J = \bigcap_{J_0 \in \{ \text{vertices of } J \}} G^{\sigma}_{J_0} / G_{J_0} \]

Taking descent into account, we have the following extension of Theorem 1.1:

**Theorem 1.2.** (Theorem 6.1(7), Theorem 4.41) There are isomorphisms of complex analytic stacks

\[(1) \colim_{J \in \{ \text{faces of } C \}} g^{\sigma}_{J} / G_J \sim \to G / G\]

\[(2) \colim_{J \in \{ \text{faces of } C \}} G^{\sigma}_{J} / G_J \sim \to G_E\]

One of our motivations for the above is to study dg-categories of complexes of sheaves with nilpotent singular support. To this end, we show for such complexes restriction along the open inclusions induces equivalences:

\[ \Sh_N(g^{\sigma}_{J} / G_J) \sim \to \Sh_N(g^{\sigma}_{J} / G_J) \]

\[ \Sh_N(G^{\sigma}_{J} / G_J) \sim \to \Sh_N(G^{\sigma}_{J} / G_J) \]

From this and an untwisting argument (see Remark 1.4 (2) below), we deduce the following:

**Theorem 1.3.** (Theorem 6.13, Theorem 7.4) There are equivalences

\[(1) \Sh_N(G / G) \sim \to \lim_{J \in \{ \text{faces of } C \}^{\op}} \Sh_N(g^{\sigma}_{J} / G_j) \]

\[(2) \Sh_N(G_E) \sim \to \lim_{J \in \{ \text{faces of } C \}^{\op}} \Sh_N(G^{\sigma}_{J} / G_j) \]

where \( G_j / G_j \) and \( g^{\sigma}_{J} / G_J \) are the quotient stacks by usual conjugations.

**Remark 1.4.**

1. In the limit, the arrow from \( J \) to \( J' \) is identified as parabolic restriction w.r.t the parabolic subalgebra (resp. subgroup) defined by the relative position between \( J \) and \( J' \). Similarly, the (higher) commutativities are given by (higher) transitivity isomorphisms between parabolic restrictions.

2. We remind the reader that the conjugation actions in Theorem 1.2 are twisted. Nevertheless, in the last theorem, the conjugations are the usual (untwisted) ones. To achieve this, one need to untwist all the conjugations compatibly with the diagram. (This essentially comes down to the fact that the simplices \( J \)'s are contractible, and the nilpotent cone in \( g^{\sigma}_{J} / G_J \) (resp. \( G_J / G_J \)) is constant along the direction of \( J \).) Hence the right hand sides of Theorem 1.3 are completely Lie theoretic, and in particular, the right hand side in (2) is irrelevant to the elliptic curve \( E \).

3. For a torus \( T \), let \( \Loc(T / T) \) (resp. \( \Loc(T_E) \)) be the dg-category of local systems on the adjoint quotient (resp. on the degree zero component of bundles on \( E \)). Theorem 1.3 can be thought of as an analogy for a simple, simply-connected group of the statement:

\[ \Loc(T / T) \sim \to \lim_{BX_\ast(T)} \Loc(t / T) \]

\[ \Loc(T_E) \sim \to \lim_{BX_\ast(T)} \Loc(T / T) \]

where \( BX_\ast(T) \) denotes the classifying space of the coweight lattice (viewed as an \( \infty \)-groupoid), the object in \( BX_\ast(T) \) goes to \( \Loc(t / T) \) (resp. \( \Loc(T / T) \)),
and all (higher) morphisms go to the identity. Combining the statements for a almost simple, simply-connected group and a torus, one can obtain a general statement for any reductive group.

**Example 1.5.** \((G = SL_2)\) Theorem 1.3 (1) gives

\[ Sh_{\mathcal{N}}(SL_2/SL_2) = \]

\[ \lim \{ Sh_{\mathcal{N}}([\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{bmatrix}]/\sim) \to Sh_{\mathcal{N}}([\begin{bmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}]/\sim) \to Sh_{\mathcal{N}}([\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{bmatrix}]/\sim) \} \]

where the matrices stand for the corresponding sub-Lie algebras of \(Lsl_2\), and \(/ \sim\) is shorthand for taking the quotient by the corresponding adjoint action. The arrows “\(\to\)” in the diagram are parabolic restrictions with respect to the indicated parabolic subalgebras.

**Example 1.6.** \((G = SL_3)\) Theorem 1.3 (1) gives

\[ Sh_{\mathcal{N}}(SL_3/SL_3) = \lim \]

where the 2-arrows “\(\Rightarrow\)” in the diagram are the transitivity natural isomorphisms between parabolic restrictions.
Remark 1.7. The theorem is compatible with Springer theory in the sense that there is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[W_{\text{aff}}]\text{-mod} & \xrightarrow{\sim} & \lim_{J \in \{\text{faces of } C\}} \mathbb{C}[W_J]\text{-mod} \\
\downarrow & & \downarrow \\
Perv_{\mathcal{X}}(G/G) & \xrightarrow{\sim} & \lim_{J \in \{\text{faces of } C\}} Perv_{\mathcal{X}}(g_J/G_J) \\
\downarrow & & \downarrow \\
\text{Sh}_{\mathcal{X}}(G/G) & \xrightarrow{\sim} & \lim_{J \in \{\text{faces of } C\}} \text{Sh}_{\mathcal{X}}(g_J/G_J)
\end{array}
\]

where $W_J$ be the Weyl group of $G_J$ (which equals the centralizer/stabilizer of $J$ in the affine Weyl group $W_{\text{aff}} := W \ltimes X_*(T)$); Perv$(X)$ denote the category of perverse sheaves on $X$; the first and second limit is taken inside $\text{Cat}_{\text{Ab}}$ the category of abelian categories, and the last limit is taken inside $\text{dg-Cat}$ the category of dg categories. Note the first isomorphism follows from the Coxeter presentation of $W_{\text{aff}}$, hence Theorem 1.3 can be thought of as a Coxeter presentation of character sheaves.

We also define more general descent diagrams for general reductive groups. See Section 1.5 below for more details.

1.3. Applications.

1.3.1. Topological nature of $\text{Sh}_{\mathcal{X}}(G_E)$.

Corollary 1.8. The dg-category $\text{Sh}_{\mathcal{X}}(G_E)$ of complexes of sheaves with nilpotent singular support is locally constant in the parameter $q$.

Remark 1.9. As remarked above, the right hand side in Theorem 1.3 (2) is irrelevant to the elliptic curve $E$. However, the equivalence there depends on a choice of basis in $H^1(E, \mathbb{Z})$ (and a point in $E$). Hence Theorem 1.3 (2) does NOT imply $\text{Sh}_{\mathcal{X}}(G_E)$ is constant over the moduli of elliptic curves $\mathcal{M}_{1,1}$. It is only constant after making the choice of basis (i.e. after a base change to the upper half plane $\mathcal{H}$). And in fact the resulting sheaf of categories on $\mathcal{M}_{1,1}$ has interesting monodromy. For $G = SL_n$, this sheaf contains the monodromy of the $SL(2, \mathbb{Z})$ action on $E[n]:=$ the set of $n$-torsion points of $E$, by considering the cuspidal objects.

With modest further effort, and similar applications of the above results, one can extend the corollary to the dg category $\text{Sh}_{\mathcal{X}}(\text{Bun}_G(E))$ of complexes of sheaves with nilpotent singular support on the entire moduli of all $G$-bundles on $E$. This category contains the Hecke eigensheaves of the geometric Langlands program, and we expect it to offer also a theory of affine character sheaves. Furthermore, under Langlands duality/mirror symmetry, it is expected to correspond to a derived category of coherent sheaves on the commuting stack. (Note that the commuting stack, and hence its coherent sheaves as well, is evidently a topological invariant, only depending on the fundamental group of the elliptic curve.) This is in turn the subject of beautiful recent developments (Schiffmann-Vasserot [SV12, SV11, SV13] on Macdonald polynomials and double affine Hecke algebras; Ginzburg [Gin12] on Cherednik algebras and the Harish Chandra system) and in particular its role as affine character sheaves was established in [BZN13].
1.3.2. **Classification of character sheaves.** The dg-category $\text{Sh}_N(g/G)$ is classified in [RR14], [Rid13]. Based on that, we shall classify $\text{Sh}_N(G/G), \text{Sh}_N(G_E)$ using Theorem 1.3 in the future. As a first step, the cuspidal sheaves in $\text{Sh}_N(G/E)$ is studied in a coming paper [Li].

1.3.3. **Dependence of restriction functor on parabolic subgroups.**

**Corollary 1.10.** Let $P_1, P_2 \subset G$ be two parabolic subgroups of a $G$ with the same Levi $L \subset G$. Then there is a (non-canonical) natural isomorphism between the parabolic restrictions

$$\text{Res}_{P_1} \simeq \text{Res}_{P_2} : \text{Sh}_N(G/G) \longrightarrow \text{Sh}_N(L/L)$$

Such statements has been proved for orbital sheaves on Lie algebras in [Mir04], and for perverse character sheaves on Lie groups in [Gin93]. During the proof of our main theorem, we define a base restriction functor $R_U : \text{Sh}_N(G/G) \rightarrow \text{Sh}_N(L/L)$ depending on a base open subset $U$. The idea is that both parabolic restriction functors are canonically isomorphic to $R_U$. Hence each choice of $U$ gives such a natural isomorphism. This is explained in detail in Section 6.4.

1.4. **Outline of argument in an example.** To illustrate the ideas, we give a first example in its most plain form.

Let $G = SL_2$, $g = sl_2$, $t, t$ the diagonal matrices in $G, g$. Let $U := \{X \in g : |\text{Re}(\lambda(X))| < 1/2\}$, for $\lambda(X)$ an eigenvalue of $X$. Let $V$ be another copy of $U$. We have $U \rightarrow G$ by $X \mapsto \exp(2\pi i X)$ and $V \rightarrow G$ by $Y \mapsto \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right) \exp(2\pi i Y)$. Let $D = \{H \in t : 0 < \lambda_1(H) < 1/2\}$, where $\lambda_1(H)$ is the first eigenvalue of $H$. We have $D \times G/T \rightarrow U$ by $(H, g) \mapsto gHg^{-1}$ and $D \times G/T \rightarrow V$ by $(H, g) \mapsto g(H - \left(\begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array}\right))g^{-1}$. Notice that $[H, \left(\begin{array}{cc} 1/2 & 0 \\ 0 & -1/2 \end{array}\right)] = 0$. We have the commutative diagram

$$\begin{array}{ccc}
D \times G/T & \rightarrow & V \\
\downarrow & & \downarrow \\
U & \rightarrow & G
\end{array}$$

which is a Cartesian. All arrows are open embeddings and $U \bigsqcup V \rightarrow G$ is surjective. The diagram is $G$-equivariant, passing to the quotient, we have

$$\begin{array}{ccc}
D/T & \rightarrow & V/G \\
\downarrow & & \downarrow \\
U/G & \rightarrow & V/G
\end{array}$$

(1.11)
with all actions being adjoint actions. Passing to the sheaves, we have all the pullback preserve nilpotent singular support. Hence we have

\[(1.12) \quad Sh_N(G/G) = \lim_{\to \downarrow} Sh_N(D/T)\]

\[
\begin{array}{c}
Sh_N(U/G) \\
\downarrow
\end{array}
\begin{array}{c}
Sh_N(D/T) \\
\downarrow
\end{array}
\begin{array}{c}
Sh_N(V/G) \\
\end{array}
\]

The above diagram can be identified with

\[(1.13) \quad Sh_N(t/T) \to Sh_N(g/G) \\
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
Sh_N(N) \\
\end{array}
\]

with the pullback identified with parabolic restriction. Hence this gives a description of the category of character sheaves on a Lie group in terms of categories of character sheaves on the Lie algebras. It turns out that the charts \(U/G, V/G\) and \(D/T\) above appears naturally in side the infinity dimension gauge uniformizations.

1.5. Outline of the general strategies. We also define more general descent diagrams for \(G_F\) and \(G/G\) in this paper.

We start with some general construction of (diagrams of) maps into a quotient stack. Let \(X \in \text{Mfld}\) the category of smooth manifolds, we have tautologically \(X = \text{colim}_{Y \in \text{Mfld}} Y\) in the presheaf category (valued in \(\infty\)-groupoids). Now if \(\mathcal{X}\) is a 1-stack over \(\text{Mfld}\), we have similarly \(\mathcal{X} = \text{colim}_{\mathcal{Y} \in \text{1-stack}/\mathcal{X}} \mathcal{Y}\). Notice that 1-stack \(\mathcal{Y}\) is a (2,1)-category.

For each functor \(F : \mathcal{C} \to \text{1-stack}/\mathcal{X}\), we have a natural morphism \(\text{colim} F \to \mathcal{X}\).

We could understand \(\mathcal{X}\) in terms of \(F\). E.g., if \(\mathcal{Y} \to \mathcal{X}\) is (representable) smooth surjective, let \(\mathcal{C} = \Delta^{op}\) the opposite of simplex category and \(F : \mathcal{C} \to \text{1-stack}/\mathcal{X}\) by \(F([n]) = \mathcal{Y}^n\). Then smooth descent implies that \(\text{colim} F \to \mathcal{X}\) is an isomorphism. Note that \(\Delta^{op}\) is essentially an ordinary category, it is also natural to consider other category \(\mathcal{C}\) with 2-morphisms.

Now suppose \(\mathcal{X} = X/G\), we can define a (2,1) category \(\mathcal{X}\) and a functor \(S : \mathcal{X} \to \text{1-stack}/\mathcal{X}\) as follows: The object of \(\mathcal{X}\) are of the form \((X, G)\), for \(X \subset X, G \subset G\) such that \(G\) acts on \(X\). \(\text{Mor}_\mathcal{X}((X_1, G_1), (X_2, G_2)) = \{g \in G : g(X_1) \subset X_2, gG_1g^{-1} \subset G_2\}\).

And a 2-morphism for each triangle (note that we do not require \(f = hg\) in \(G\)).

\[(1.14) \quad f \quad \quad g \quad \quad h \]

Then for \(g : (X_1, G_1) \to (X_2, G_2)\) in \(\mathcal{X}\), then \(g\) induces \(S_g : X_1 \to X_2, x_1 \mapsto g \cdot x_1\) and a group homomorphism \(\phi_g : G_1 \to G_2, g_1 \mapsto g g_1^{-1}\). The two maps \((S_g, \phi_g)\) are compatible in the sense that \(S_g(g_1 x_1) = \phi_g(g_1) S_g(x_1)\). Hence they induces a map of quotient stacks \(S_g : X_1/G_1 \to X_2/G_2\). Similarly, we have \(S_h : (X_2, G_2) \to (X_3, G_3)\) and \(S_f : X_1/G_1 \to X_3/G_3\). We have an natural (invertible) 2-morphism

\[(1.15) \quad S^{h,g}_f : S_h \circ S_g \Rightarrow S_f\]

Now there is a functor \(S : \mathcal{X} \to \text{1-stack}/\mathcal{X}\). By taking (1.14) to
Now for any $J : \mathcal{C} \to \mathcal{X}$, denote by $F := S \circ J$, then we have $\text{colim} F \to \mathcal{X}$.

In practice, $\mathcal{X}$ is assumed to be finite dimensional, but $\mathcal{X}, \mathcal{G}$ could be infinite dimensional. And all objects $(X, G)$ in $\mathcal{X}$ are taken to be finite dimensional.

We are mainly interested in the situation

(a) $\text{Loc}_G(S^1) = \Omega^1(S^1, g)/C^\infty(S^1, G)$,
(b) $\text{Bun}_0^G(E) = \Omega^0(E, g)/C^\infty(E, G)$, or
(c) $\text{Bun}_G(E) = \text{L}^\text{hol}_G/\text{L}^\text{hol}_G$,

We shall construct some $\mathcal{C}, F$ as above such that $\text{colim} F \to X'$ is an isomorphism, where $X'$ is case (a), and $X' = GE \subset X$ in case (b), (c). They are discussed in the following places of the paper:

1. The category $\mathcal{C} = \int_{\Delta^{op}} S^*/\Gamma$ is defined in Section 2.4, when $G$ is simply connected, and in the case (a)(c), we can also take an easier category as in Section 2.1.
2. The functor $F$ is defined for case (c) in Section 4.3, and similar in case (a)(b).
3. We prove the fact that $\text{colim} F \simeq X'$ in case (c) in Theorem 4.34. This is the main theorem of the first part. The proof relies on the particular geometry of adjoint quotient. And the corresponding statement is stated in Theorem 6.1 for (a) and in Theorem 7.5 for (b).

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2. Category preliminaries

In this section, we will define the categories that organize our descent diagram. The content is presented with increasing generality. Sections 2.2-2.4 are not necessary for readers mainly interested in the concrete Lie theoretic diagram given in the introduction. However, many statements in the paper are of a purely local nature which does not depend on the explicit knowledge of alcove geometry. So they are natural from the perspective of Section 2.4.

2.1. The categories $\mathcal{F}_C$ and $\mathcal{P}_S$.

Definition 2.1. For $C$ a polytope, the category of (non-empty) faces $\mathcal{F}_C$ of $C$ is defined as follows:

$\text{Ob}(\mathcal{F}_C) = \{ J : J \text{ a non-empty face of } C \}$

$\text{Mor}(J, J') = \begin{cases} \{ J \to J' \} \text{ a single element set}, & \text{if } J \subset \mathcal{F}' \\ \emptyset, & \text{otherwise} \end{cases}$

The composition is given by the obvious one: if $J \subset \mathcal{F}$ and $J' \subset \mathcal{F}'$, then $J \subset \mathcal{F}'$ and $(J' \to J') \circ (J \to J') = J \to J''$. □

The following proposition is easy to check.
Proposition 2.2. If $C = C_1 \times C_2$, then $\mathcal{F}_C \simeq \mathcal{F}_{C_1} \times \mathcal{F}_{C_2}$.

Definition 2.3. For a set $S$, the unaugmented power category $\mathcal{P}_S$ of $S$ is the category of non-empty subsets of $S$, with morphisms given by inclusions. The power category $\mathcal{P}_{S,+}$ of $S$ is the category of subsets of $S$, with morphism inclusions.

Let $V_C$ be the set of vertices of $C$. There is a natural functor

$$Ver : \mathcal{F}_C \rightarrow \mathcal{P}_{V_C} \quad J \rightarrow \{\text{set of vertices of } J\}$$

Proposition 2.4. If $C$ is a simplex, then $Ver$ is an isomorphism.

The power category $\mathcal{P}_{S,+}$ can be used to organize descent by open covers: let $\text{Space}$ be the category of manifolds/schemes/stacks. For $S \in \text{Space}$, let $Sh(S)$ be the category of sheaves on $S$ (See Appendix A.2 for our convention on sheaves). We have:

Proposition 2.5. Let $X : \mathcal{P}_{S,+}^{\text{op}} \rightarrow \text{Space}$ be a functor, such that:

1. All arrows in $\mathcal{P}_{S,+}^{\text{op}}$ map to open embeddings under $X$.
2. The map $\coprod_{s \in S} X(s) \rightarrow X(\emptyset)$ is surjective.
3. $X$ preserves Cartesian product. I.e, for any $A, B \subset S$, the following commutative diagram is Cartesian:

$$\begin{array}{ccc}
X(A \cup B) & \xrightarrow{\sim} & X(\emptyset) \\
\downarrow & & \downarrow \\
X(A) & \square & X(B) \\
\downarrow & & \downarrow \\
X(A \cap B) & & 
\end{array}$$

Then (i) the natural map is an isomorphism:

$$\text{colim}_{A \in \mathcal{P}_{S}^{\text{op}}} X(A) \xrightarrow{\sim} X(\emptyset)$$

(ii) the natural functor induced by pull back is an equivalence:

$$\text{lim}_{A \in \mathcal{P}_{S}^{\text{op}}} Sh(X(A)) \xleftarrow{\sim} Sh(X(\emptyset))$$

Proof. This is proved later as a special case of Proposition 2.14. \qed

2.2. The category $\Delta_S$. In this section, we consider a more general category $\Delta_S$ that could organize descent data of etale maps, just as $\mathcal{P}_S$ does for open maps. Let us first recall the simplex category and etale descent.

2.2.1. The simplex category $\Delta$.

Definition 2.6. (1) The simplex category $\Delta$ is the category with objects \{finite totally ordered sets $[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n-1\} : n \in \mathbb{N}\}$, and with morphism set maps preserving the order.

(2) The augmented simplex category $\Delta_+$ is the category with objects \{finite totally ordered sets $[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n-1\} : n \in \mathbb{N} \cup \{-1\}\}$, where $[-1] := \emptyset$, and with morphism set maps preserving the order.
Example 2.7. (The simplicial objects $U_X^*$) Let $U \to X$ be a morphism in $\mathcal{T}$. Then we can define a functor $U_X^* : \Delta^{op} \to \mathcal{T}$. On objects, $U_X^*([n]) := U_X^{n+1}$. And for morphisms $\tau : [m] \to [n]$, $U_X^*(\tau) : U_X^n \to U_X^m$ via $(u_0, u_1, \ldots, u_n) \mapsto (u_{\tau(0)}, u_{\tau(1)}, \ldots, u_{\tau(m)})$. Denote by $U_{X,+}^* : \Delta^{op,+} \to \mathcal{T}$ the augmented functor by sending $\emptyset$ to $X$. The functors $U_{X,+}^*$ clearly preserve Cartesian product.

In fact, it follows from the universal property of $\Delta_+$ that all Cartesian product preserving functors from $\Delta^{op,+}$ are of the above form:

**Proposition 2.8.** Let $F : \Delta^{op,+} \to \mathcal{T}$ be a Cartesian product preserving functor. Then there a canonical isomorphism of functors $F \sim F([0])^* F(\emptyset)$. We have the following smooth decent organized by $\Delta$, we used the term smooth map for submersive map as in algebraic geometry.

**Proposition 2.9.** (1) Let $U \to X$ be a smooth surjective map in $\text{Space}$. Then

(i) the natural map is an isomorphism:

\[
\text{colim}_{\Delta^{op,+}} U_X^* \sim \to X
\]

(ii) the natural functor induced by pull back is an equivalence:

\[
\text{Sh}(X) \sim \to \text{lim}_{\Delta^{op}} \text{Sh}(U_X^*)
\]

(2) Let $X^* : \Delta^{op} \to \text{Space}$ be a Cartesian functor, such that the map $X([0]) \to X(\emptyset)$ is etale and surjective. Then

(i) the natural map is an isomorphism:

\[
\text{colim}_{\Delta^{op}} X^* \sim \to X(\emptyset)
\]

(ii) the natural functor induced by pull back is an equivalence:

\[
\text{Sh}(X(\emptyset)) \sim \to \text{lim}_{\Delta^{op}} \text{Sh}(X^*)
\]

**Proof.** (1)(i) For any $Y$, the Yoneda embedding $\text{Hom}(\cdot, Y)$ defines a sheaves on $X_{sm}$. (ii) This is Proposition A.7. (2) follows from (1) by Proposition 2.8. \qed

2.2.2. The category $\Delta_S$. There is a category $\Delta_S$ that connects the two types of descent given by $\Delta$ and $\mathcal{P}_S$. Roughly speaking, if we write a space $U = \coprod_{s \in S} U_s$ as the disjoint union of connected components, then $\Delta_S$ organizes the connected components of the simplicial space $U_X^*$. The category $\Delta_S$ is also naturally defined via the Grothendieck construction recalled in the next section.

**Definition 2.10.** Let $S$ be a set.

(1) A sequence in $S$ is a pair $(I, s : I \to S)$, where $I$ is a totally order set and $s : I \to S$ a set map. We denote the sequence by $(s_i)_{i \in I}$, where $s_i := s(i)$. A sequence is finite if $|I|$ is finite.

(2) The category $\Delta_S$ (resp. $\Delta_S,+)$ is defined as the category of non-empty finite (resp. finite) sequences in $S$. The morphisms between two sequences $I \to S$ and $J \to S$ are order preserving maps $I \to J$ that respect the map to $S$.

**Remark 2.11.** When $S = \{pt\}$ a one point set, a finite sequence in $S$ is the same as a finite totally ordered set, and $\Delta_{(pt)} \simeq \Delta$. The following example is the main motivation for us to consider $\Delta_S$:
Example 2.12. Let \( \{U_s \to X\}_{s \in S} \) be a set of morphisms in \( \mathcal{T} \). Then we can define a functor \( U^\bullet_{s,s \in S; X : \Delta^\text{op}_S \to \mathcal{T}} \). On objects, \( U^\bullet_{s,s \in S; X}((s_i)_{i \in I}) := \prod_{i \in I} U_{s_i} \).

And for morphism \( \tau : (s_i)_{i \in I} \to (t_j)_{j \in J} \), define \( U^\bullet_X(\tau) : \prod_{j \in J} U_{t_j} \to \prod_{i \in I} U_{s_i} \), via \((u_j)_{j \in J} \mapsto (u_{\tau(j)}(i))_{i \in I}\). Also define the augmented functor \( U^\bullet_{s,s \in S; X, (+)} \) by further sending \( \emptyset \) to \( X \). The functor \( U^\bullet_{s,s \in S; X, (+)} \) preserve Cartesian product.

The category \( \Delta_S \) has properties analogous to those of \( \Delta \):

**Proposition 2.13.** Let \( F : \Delta^\text{op}_{S,+} \to \mathcal{T} \) be a Cartesian product perserving functor. Then there a canonical isomorphism of functors \( F \cong F((s))_{s \in S; F(\emptyset), +} \).

**Proposition 2.14.** Let \( X^\bullet : \Delta^\text{op}_{S,+} \to \text{Space} \) be a Cartesian product perserving functor, such that the map \( \prod_{s \in S} X((s)) \to X(\emptyset) \) is smooth and surjective. Then (i) the natural map is an isomorphism:

\[
\text{colim}_{\Delta^\text{op}_S} X^\bullet \cong X(\emptyset)
\]

(ii) the natural functor induced by pull back is an equivalence:

\[
\text{Sh}(X(\emptyset)) \cong \lim_{\Delta^\text{op}_S} \text{Sh}(X^\bullet)
\]

**Proof of Proposition 2.5.** The idea is same as showing the non-degenerate Cech complex and the degenerate Cech complex having the same cohomology. There is a natural functor given by taking the underlying set of a sequence: \( \pi : \Delta_S \to \mathcal{P}_S \), by \( (s_i)_{i \in I} \mapsto \{s_i\}_{i \in I} \). Let \( M \) consist of all morphisms \( m \) in \( \Delta_S \), such that \( \pi(m) = \text{Id} \). Then \( \pi \) induces an equivalence: \( \Delta_S[M^{-1}] \cong \mathcal{P}_S \). Now suppose given \( X \) as in Proposition 2.5, define the functor \( X(s)_{s \in S; X : \Delta^\text{op}_S \to \text{Space}} \), then for any \( m \in M \), \( X(s)_{s \in S; X}(m) = \text{Id} \). Hence \( X(s)_{s \in S; X} \) naturally factor through \( (s)_{s \in S; X}^{\bullet} : \Delta_{S}[M^{-1}] \to \text{Space} \), and \( \text{colim}_{\Delta^\text{op}_S} X(s)_{s \in S; X} \cong \text{colim}_{\Delta_{S}[M^{-1}]} X(s)_{s \in S; X} \cong X(\emptyset) \). This proves (1), and (2) can be proved similarly.

### 2.3. Grothendieck construction.

The reference for Grothendieck construction (also referred as unstraighten functor) can be found in [Lur12, Sect 3.2], [GR16, I.1.1.4]. The passage from \( \Delta^\text{op} \) to \( \Delta^\text{op}_S \) is a special case of the Grothendieck construction. In this section, we will work in the natural setting of \( \infty \)-categories. By categories, we mean \( \infty \)-categories. For two categories \( A, B \), denote by \([A, B]\) the category of functors between \( A \) and \( B \).

**Definition 2.15.** The Grothendieck construction is the functor

\[
\int_B : [B, \text{Cat}] \to \text{Cat}/B,
\]

where \( \int_B \) is defined so that the fiber over \( b \in B \) is \( F(b) \).

**Example 2.16.** Let \( S^\bullet := S_{\mathbf{2}}^{(p)} : \Delta^\text{op} \to \text{Set} \) be the simplicial set as in Example 2.7, then \( \int_{\Delta^\text{op}} S^\bullet \cong \Delta^\text{op}_S \).
2.3.1. Kan extension.

**Definition 2.17.** Given a functor $\rho : A \to B$ and a category $T$, denote $\rho^* : [B,T] \to [A,T]$ the induced functor.

1. The left Kan extension $\rho_l$ is the left adjoint to $\rho^*$.
2. The right Kan extension $\rho_r$ is the right adjoint to $\rho^*$.

We collect some basic properties of Kan extension:

**Proposition 2.18.** Let $K : A \to T$ a functor.

1. Let $\pi : A \to pt$ (for $pt$ being the one point category), then $\pi A(pt) \simeq \text{colim}_A K$ and $\pi_* A(pt) \simeq \text{lim}_A K$, provided either side of the equation exist.
2. Let $\rho : A \to B, \varphi : B \to C$, and assume that $\rho_{\varphi_*}(K), \varphi_{\varphi_*}(\rho_{\varphi_*}(K))$ exist, then $(\varphi \circ \rho)_{\varphi_*}(K) \simeq \rho_{\varphi_*}(\varphi_{\varphi_*}(\rho_{\varphi_*}(K))$.

**Proof.** (1) follows from the definition of (co)limit. (2) follows from the fact that $\rho^* \circ \varphi^* \simeq (\varphi \circ \rho)^*$ and that adjoints are canonical. □

**Example 2.19.** We use the notation in Example 2.16. Take $K = U^{\bullet}_{s,s\in S;X} : \Delta_{S}^{\text{op}} \to T$, then $\pi K = U^{\bullet}_{X} : \Delta^{\text{op}} \to T$, for $U = \coprod_{s\in S} U_s$.

In the situation of Grothendieck construction, the Kan extension can be calculated as follows:

**Proposition 2.20.** Denote $\pi : \int_B F \to B$, and $\pi^{\text{op}} : (\int_B F)^{\text{op}} \to B^{\text{op}}$ its opposite. Then $\pi, \pi^{\text{op}}$ can be calculated provided that the following colimits or limits exist:

1. $\pi_t(K)(b) = \text{colim}_{F(b)} \pi K$, for $K \in [\int_B F, T]$.
2. $\pi^{\text{op}}(K')(b) = \text{lim}_{F(b)^{\text{op}}} K'$, for $K' \in [\int_B F^{\text{op}}, T]$.

**Proof.** It is known that $\int_B F \to B$ is a coCartesian fibration with fiber $\int_B F \times_B b$ canonically isomorphic to $F(b)$. Then (1) follows from [GR16, I.1.2.2.4]. And (2) follows from similar argument. □

For later use, we spell out (1) in the above Proposition explicitly when $F$ takes values in $\text{Set} \subset \text{Cat}$, and $B = D$ the category of commutative diagram:

$$D = \begin{array}{ccc}
  w & \rightarrow & x \\
  \downarrow k & & \downarrow f \\
  y & \rightarrow & z
\end{array}$$

**Corollary 2.21.** For $b \in F(x), c \in F(y), d \in F(z)$, such that $F(f)b = F(g)c = d$, denote by $S_{b,c} := \{a \in F(w) : F(h)a = b, F(k)a = c\}$. The following are equivalent:

1. $\pi K(D)$ is Cartesian,
2. the following diagram is Cartesian, for all such triple $(b,c,d)$ as above:

$$\begin{array}{ccc}
  \coprod_{a\in S_{b,c}} K(a) & \xrightarrow{K(h)} & K(b) \\
  \downarrow K(k) & & \downarrow K(f) \\
  K(c) & \xrightarrow{K(g)} & K(d)
\end{array}$$

**Proof.** This follows from Proposition 2.20, since $F$ take values in Set, the colimit reduces to a coproduct. □
2.4. The category $\int_{\Delta^+} S^*/\Gamma$.

**Definition 2.22.** Let $S$ be a set, $\Gamma$ a group acting on $S$.

1. The (2-)category $S//\Gamma$ has object $s$ for every $s \in S$: morphism $\gamma: s \to t$ for every $\gamma \in \Gamma, s, t \in S$, such that $\gamma(s) = t$; and 2-morphism $\gamma' \gamma^{-1}: \gamma \Rightarrow \gamma'$, for every $\gamma, \gamma': s \to t$. The identity and composition are given by the obvious ones.

2. The set $S//\Gamma$ is the set of orbits of $\Gamma$ in $S$.

3. Define $S^*/\Gamma: \Delta^{op}_+ \to \text{Cat}$, by $[n] \mapsto S^{n+1}/\Gamma$ (for the diagonal action of $\Gamma$), where $S^0 := pt$ the one point set.

4. Define $\int_{\Delta^+} S^*/\Gamma: \Delta^{op}_+ \to \text{Set}$, by $[n] \mapsto S^{n+1}/\Gamma$.

**Remark 2.23.**

1. The natural functor $S//\Gamma \to S//\Gamma$ is an equivalence of categories. And hence $\int_{\Delta^+} S^*/\Gamma \Longrightarrow \int_{\Delta^+} S^*/\Gamma$.

2. The category $\int_{\Delta^+} S^*/\Gamma$ has a final object, denote by $pt$. And $\int_{\Delta^+} S^*/\Gamma \simeq (\int_{\Delta^+} S^*/\Gamma)_{x*}$, where $C_*$ standards for the category by adding one final object to the category $C$. The same is true for $\int_{\Delta^+} S^*/\Gamma$.

**Theorem 2.24.** Let $X: \int_{\Delta^+} S^*/\Gamma \to \text{Space}$ be a functor, and denote by $\underline{X}: \int_{\Delta^+} S^*/\Gamma \to \text{Space}$ the corresponding functor. Assume that:

1. $\prod_{s \in S} X(s) \xrightarrow{\sim} X(pt)$ is etale and surjective.

2. The assertion (2) in Corollary 2.21 holds for $F = S^*/\Gamma, K = X$ and any Cartesian square $D$ in $\Delta^{op}_+$. Then:

   1. the natural map is an isomorphism:

      \[
      \colim_{\int_{\Delta^+} S^*/\Gamma} X(s) \xrightarrow{\sim} X(pt)
      \]

   2. the natural functor induced by pull back is an equivalence:

      \[
      Sh(X(pt)) \xrightarrow{\sim} \lim_{\int_{\Delta^+} S^*/\Gamma} Sh(X(s))
      \]

**Proof:** We only prove (1), and (2) can be proved similarly. Let $\pi: \int_{\Delta^+} S^*/\Gamma \to \Delta^{op}_+$. Then the functor $\pi_*X: \Delta^{op}_+ \to \text{Space}$ satisfies the assumption of Theorem 2.9 (2) by Corollary 2.21. Hence we have $\colim_{\int_{\Delta^+} S^*/\Gamma} X \simeq \colim_{\Delta^{op}_+} \pi_*X \simeq \pi_*X(pt) \simeq X(pt)$. \(\square\)

The category $\int_{\Delta^+} S^*/\Gamma$ can be used to organize some descent diagrams with non-Galois covers, as in the following:

**Construction 2.25.** Let $X$ be a real analytic/complex manifold, and $\Gamma$ a discrete group acting on $X$ properly discontinuously. For $x \in X^n$, denote by $\Gamma x$ the stabilizer of $\Gamma$ at $x$. And let $X'_x := \{ y \in X : \Gamma_y \subset \Gamma_x \}$. Choose a $\Gamma$ invariant subset $S \subset X$, and open subset $U_s \subset X'_s$, for each $s \in S$, such that the collection of charts $\{U_s : s \in S\}$ are $\Gamma$-invariant, i.e. $\gamma(U_s) = U_{\gamma s}$, for any $\gamma \in \Gamma, s \in S$. Denote by $U_s := \bigcap_{s \in S} U_s$. Then we define a functor

\[
U: \int_{\Delta^+} S^*/\Gamma \longrightarrow \text{Space}
\]
(1) $U(s) := U_s / \Gamma_s$, where $U(pt) := X / \Gamma$;
(2) $U(\gamma) := A_{\gamma} : U_s / \Gamma_s \to U_{\gamma s} / \Gamma_{\gamma s}$ the action by $\gamma$;
(3) for two morphism $\gamma' : s \to t$. Define $U(\gamma') := \eta_{\gamma' - 1} \circ A_{\gamma'} :$ $A_{\gamma} := A_{\gamma' \gamma}$, where $\eta_{\gamma' - 1} : Id_{U_t} \Rightarrow A_{\gamma' \gamma - 1} : U_t / \Gamma_t \to U_t / \Gamma_t$ is the canonical trivialization of the action of $\gamma' \gamma - 1 \in \Gamma_t$ as inner automorphism.
(4) for $\delta : s \to s'$, then $U_s \subset U_{s'}$ and $\Gamma_s \subset \Gamma_{s'}$, this gives $U(\delta) := U_s / \Gamma_s \to U_{s'} / \Gamma_{s'}$.

where "/" stands for the stacky quotient, it can also be replaced by "//" the categorical quotient in real/complex analytic varieties.

**Proposition 2.26.** Assume that $\bigcup_{s \in S} U_s = X$, then the natural map
\[
\text{colim}_{\Delta^{op}} S^*/\Gamma U(s) \xrightarrow{\sim} U(pt) = X / \Gamma
\]
is an isomorphism. Moreover, the stacky quotient // in (1)-(4) above can be replaced by the categorical quotient //, and the statement still holds.

**Remark 2.27.** When $\Gamma$ acts on $X$ freely, take $S = \Gamma x$ for some $x \in X$, and $U_s = X$, for all $s \in S$. Then $\int_{\Delta^{op}} S^*/\Gamma \simeq \int_{\Delta^{op}} \Gamma^{* - 1}$ and we recover the usual Galois decent colim $\Delta^{op}(\Gamma^{* - 1} \times X) \simeq X / \Gamma$.

**Proof.** We need to check the assumption of Theorem 2.24 is satisfied. (1) is satisfied by assumption $\bigcup_{s \in S} U_s = X$. For (2), by Corollary 2.21, we need to show that for any Cartesian square
\[
\begin{array}{ccc}
[n] & \rightarrow & [m] \\
\downarrow & & \downarrow \\
[k] & \rightarrow & [l]
\end{array}
\]
in $\Delta^{op}$, and any $b \in S^m / \Gamma, c \in S^k / \Gamma$ and $d \in S^l / \Gamma$, the following diagram is Cartesian:
\[
\begin{array}{ccc}
\coprod_{a \in S_{b,c}} U(a) & \rightarrow & U(b) \\
& & \downarrow \\
U(c) & \rightarrow & U(d)
\end{array}
\]

We shall check the case when $l = 0, m = k = 1$ and $n = 2$, the general case are similar. We have $R := \Gamma \backslash \Gamma \times \Gamma / \Gamma \times \Gamma \twoheadrightarrow S_{b,c}$, by $(\gamma_1, \gamma_2) \mapsto (\gamma_1 b, \gamma_2 c)$, hence we reduced to show the following Lemma.

**Lemma 2.28.** The diagram
\[
\begin{array}{ccc}
\coprod_{(\gamma_1, \gamma_2) \in R} U(\gamma_1 b, \gamma_2 c) / \Gamma_{(\gamma_1 b, \gamma_2 c)} & \xrightarrow{A_{\gamma_1 - 1}} & U_b / \Gamma_b \\
& \downarrow & \downarrow \\
U_c / \Gamma_c & \rightarrow & U / \Gamma
\end{array}
\]
is Cartesian.
Proof. For stacky quotient, we have $U_b/\Gamma_b \times_{U/T} U_c/\Gamma_c \simeq (U_b \times_{U/T} U_c)/(\Gamma_b \times \Gamma_c) = \coprod_{\gamma \in \Gamma} U_{(\gamma b,c)}/(\Gamma_b \times \Gamma_c)$. If $\beta = \gamma_2 \alpha \gamma_1^{-1}$, then the action of $(\gamma_1, \gamma_2)$ identifies $U_{(\alpha b,c)}$ and $U_{(\beta b,c)}$. And $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ stabilize $U_{(b,c)}$ if and only if $\gamma_1 = \gamma_2$, all such elements can be identified with $\Gamma_1 \Gamma_2^{-1} \cap \Gamma_c$. Hence $(\coprod_{\gamma \in \Gamma} U_{(\gamma b,c)}/(\Gamma_b \times \Gamma_2) \simeq \coprod_{\gamma \in \Gamma} \Gamma_1 \Gamma_2^{-1} \cap \Gamma_c) \simeq \coprod_{(\gamma_1, \gamma_2)} U_{(\gamma_1 b, \gamma_2 c)}/(\Gamma_b \times \Gamma_c)$. For categorical quotient, the induced map $F : \coprod_{(\gamma_1, \gamma_2) \in R} U_{(\gamma_1 b, \gamma_2 c)}/(\Gamma_b \times \Gamma_c) \longrightarrow U_b/\Gamma_b \times_{U/T} U_c/\Gamma_c$ is a local isomorphism, and by consider generic locus in $X$ and use the result on stacky quotient, we see $F$ is generically an isomorphism. This implies $F$ is an open embedding. It remains to show that $F$ is isomorphism on the set of points. To this end, we have $U_b/(\Gamma_b \times_{U/T} U_c)/\Gamma_c \simeq [U_b/\Gamma_b] \times_{[U/T]} [U_c/\Gamma_c] \simeq [U_b/\Gamma_b \times_{U/T} U_c/\Gamma_c] \simeq \coprod_{(\gamma_1, \gamma_2) \in R} U_{(\gamma_b, \gamma_2 c)}/(\Gamma_b \times \Gamma_2^{-1}) \simeq \coprod_{(\gamma_1, \gamma_2) \in R} U_{(\gamma_1 b, \gamma_2 c)}/(\Gamma_1 \Gamma_2^{-1} \cap \Gamma_c)$, where the first and last $\simeq$ hold because $\Gamma$ is discrete, and the action is properly discontinuous, the second $\simeq$ holds because $U_b/\Gamma_b \rightarrow U/\Gamma(pt)$ is by construction a full embedding of groupoids. □

Remark 2.29. The upshot of this construction is that the charts $U_x/\Gamma_x \rightarrow X/\Gamma$ is usually non-Galois. The category $\int_{\Delta \rightarrow S} S^*/T$ gives a way to organized these non-Galois charts. Then main example we have in mind is when $X = t$ and $\Gamma = W_{aff}$, then $t/W_{aff} \simeq T/W$. We want to cover $G/G$ which is a stacky version of $T/W$. And the stacky version of the Galois cover $T \rightarrow T/W$ is usually thought as the Grothendieck-Springer resolution $G\!/G \rightarrow G/G$. However, this map is not etale and hence may lose information. Instead, we use various exponential maps such as $G/G \rightarrow G$, this could be thought as stacky version of the non-Galois cover $t/W \rightarrow t/W_{aff}$.

3. Lie theoretic preliminaries

3.1. Groups generated by reflections. A reference for this section is [Bou02, V].

We will denote by $A$ a real affine space of finite dimension, and by $L$ the vector space of translations of $A$. Assume that $L$ is provided with an inner product. Let $\mathcal{H}$ be a set of hyperplanes of $A$, and $W = W_{\mathcal{H}}$ be the subgroup of automorphism of $A$ generated by orthogonal reflections $r_H$ with respect to the hyperplanes $H \in \mathcal{H}$. We assume the following conditions are satisfied:

(1) For any $w \in W$ and $H \in \mathcal{H}$, the hyperplane $w(H)$ belongs to $\mathcal{H}$;
(2) The group $W$, provided with the discrete topology, acts properly on $A$.

Given two points $x$ and $y$ of $\mathcal{H}$, denote by $R(x, y)$ the equivalence relation: For any hyperplane $H \in \mathcal{H}$, either $x \in H$ and $y \in H$ or $x$ and $y$ are strictly on the same side of $H$.

Definition 3.1. (1) A facet of $A$ is an equivalence class of the equivalence relation defined above.
(2) A chamber of $A$ is a facet that is not contained in any hyperplane $H \in \mathcal{H}$.
(3) A vertex of $A$ is a facet that consists of a single point.
(4) For $S \subset A$ subset, the star of $S$ is $St_S := \bigcup_{J \subset S} \mathcal{H}_J \setminus \emptyset$; and $W_S := \{w \in W : w|_S = id\}$ denotes the group of elements fixing $S$.

We collect some facts:
Theorem 3.2.  
(1) For $J \subset A$ a facet, the group $W_J$ is generated by $\{r_H : J \subset H\}$

(2) For any chamber $C$, the closure $\overline{C}$ of $C$ is a fundamental domain for the action of $W$ on $A$, i.e., every orbit of $W$ in $A$ meets $\overline{C}$ in exactly one point.

Fix a chamber $C$, for faces $J, J'$ of $C$ (which are automatically facets of $E$), such that $J \subset \overline{J'}$, we have $St_{J'} \subset St_J$ and $W_{J'} \subset W_J$. The maps $St_{J'}/W_{J'} \to St_J/W_J$ give a functor $\mathcal{F}_C \to \text{Space}$. And similar to Construction 2.25, the stacky quotient $//\cdot$ can be replaced by categorical quotient $/\cdot$.

Proposition 3.3. The natural map $p_J : St_J/W_J \to A/W$ is an open embedding, and the induced map

$$\text{colim}_{\mathcal{F}_C} St_J/W_J \xrightarrow{\sim} A/W$$

is an isomorphism. Moreover, the stacky quotient $//\cdot$ can be replaced by the categorical quotient $/\cdot$.

Proof. By proposition 2.2, we can assume that $(A, \mathfrak{g})$ is reduced, i.e., it is not a product of nontrivial factors $(A_1, \mathfrak{g}_1) \times (A_2, \mathfrak{g}_2)$. Furthermore, we could assume $A$ is of affine type, since for finite reflection group $W$, the statement trivially holds ($A/W = St_0/W_0$ is final for the diagram). Now any chamber $C$ is a (bounded) simplex. Theorem 3.2 (1) implies $p_J$ is an open embedding. To show $p_J$ is an open embedding, take $J_1, J_2$ two simplex in $St_J$, $w \in W$, such that $w(J_1) = J_2$, we need to show that $w \in W_J$. Let $C_1$ be a chamber whose closure contains $J_1$, then $J_2 \subset \overline{C_2}$, for $C_2 := w(C_1)$, hence $C_2 \subset St_J$. Let $C'_i$ be the corresponding chamber of $A$ equipped with the hyperplanes $\{r_H : J \subset H\}$ containing $C_i$, $i = 1, 2$, and fix $x \in J$. then there exist $\epsilon > 0$, such that $B(x, \epsilon) \cap C'_i \subset C_i$. By Theorem 3.2, there $w' \in W_J$, such that $w'(C'_i) = C'_2$. We also have $w'(B(x, \epsilon)) = B(x, \epsilon)$ because $w'(x) = x$ and $w'$ is an isometry. Hence $w'(y) \in C_2$ for any $y \in B(x, \epsilon) \cap C'_i \subset C_i$ hence $w = w' \in W_J$. (2) implies $\bigcap_{i \in \mathcal{F}_C} St_i/W_i \to A/W$ is surjective. We are left to show $St_J/W_J = \bigcap_{\alpha \in \mathcal{F}_C} St_\alpha/W_\alpha$ (inside $A/W$). This follows from the definition of star. \hfill \Box

3.2. Lie theoretic reminder.

Notation 3.4. Let $G$ be a reductive algebraic group, $T \subset G$ a maximal torus. Denote by $\Phi = \Phi(G, T)$ the set of roots, and by $X_*(T) := \text{Hom}(\mathbb{C}^*, T)$ the coweight lattice. Let $\mathfrak{t}_G := X_*(T) \otimes \mathbb{R}$, the Weyl group $W := N_G(T)/T$, it acts naturally on $T, X_*(T)$ and $\mathfrak{t}_G$. Let $W_{\text{aff}} := W \ltimes X_*(T)$ be the affine Weyl group, and $\Phi_{\text{aff}} := \{\alpha - n : \alpha_0 \in \Phi, n \in \mathbb{Z}\} \subset \text{Map}(\mathfrak{t}_G, \mathbb{R})$ be the set of affine roots. Denote by $\mathfrak{g}$ the Lie algebra of $G$, by $L\mathfrak{g}$ and $LG$ the polynomial loop algebra and loop group. For any $\alpha_0 \in \Phi$, denote by $\mathfrak{g}_{\alpha_0} \subset \mathfrak{g}$ the root space of $\alpha$, and for $\alpha = \alpha_0 - n \in \Phi_{\text{aff}}$, denote by $\mathfrak{g}_\alpha := \mathfrak{g}_{\alpha_0} z^n \subset L\mathfrak{g}$. Fix a lift of set $W \to N_G(T) \subset G$. It gives a lift $W_{\text{aff}} \to LG$. For $w \in W_{\text{aff}}$, denote its lift by $\bar{w}$.

Assume further that $G$ is simply-connected and semisimple. Then $\mathfrak{t}_G$ carries an inner product induced by the Killing form. Denote by $\mathfrak{h} := \{\{\alpha(x) = 0 : x \in \mathfrak{t}_G\}_{\alpha \in \Phi}\}$ and $\mathfrak{h}_{\text{aff}} := \{\{\alpha(x) = 0 : x \in \mathfrak{t}_G\}_{\alpha \in \Phi_{\text{aff}}}\}$ two collections of hyperplanes in $\mathfrak{t}_G$, let $W_{\mathfrak{h}}, W_{\mathfrak{h}_{\text{aff}}}$ be the corresponding groups generated by reflections. The inclusion $\mathbb{Z} \subset \mathbb{R}$ induces $X_*(T) \subset \mathfrak{t}_G$.

Theorem 3.5. Viewing $X_*(T)$ as translations of $\mathfrak{t}_G$, we have the following equality as subgroup of affine linear transformation of $\mathfrak{t}_G$:
(1) $W_R = W$
(2) $W_{G_{\text{aff}}} = W_{\text{aff}}$.

3.2.1. Levi and Parabolic subgroups associated to facet geometry.

**Definition 3.6.** Let $J$ be a facet of $\mathfrak{t}_R$ equipped with $\mathfrak{f}_J$ (resp. $\mathfrak{f}_{\text{aff}}$).
1. $\Phi_J := \{ \alpha \in \Phi \text{ (resp. } \Phi_{\text{aff}}) : \alpha(J) = 0 \}$.
2. Denote by $\mathfrak{g}_J \subset \mathfrak{g}$ (resp. $\mathfrak{g}_J \subset \mathfrak{L}$) the subalgebra:
   $$
   \mathfrak{g}_J := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha,
   $$
   where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ (resp. $\mathfrak{L}$) is the root space of $\alpha$. Denote by $G_J$ the corresponding Levi subgroup of $G$ (resp. $L_G$).
3. Let $J, J'$ two facet with $J \subset J'$, denote by $\mathfrak{p}_{J' \cap J} \subset \mathfrak{g}_J$ the subalgebra:
   $$
   \mathfrak{p}_{J' \cap J} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_J, \alpha(J') > 0} \mathfrak{g}_\alpha.
   $$
   Denote by $P_{J' \cap J}$ the corresponding subgroup of $G_J$.

Theorem 3.2 (1) implies:

**Proposition 3.7.** $W_J \subset G_J$, i.e., $G_J = < T, \exp \mathfrak{g}_\alpha, \dot{w} \mid \alpha \in \Phi_J, w \in W_J >$. In particular, the latter group is connected.

3.2.2. Transitivity of parabolic subgroups. Let $L \subset K \subset G$ be a sequence of Levi subgroups for parabolic subgroups $P \subset K, R \subset G$. Denote by $U_R$ the unipotent radical of $R$, and by $Q := R \circ P := < P, U_R >$. The notation of composition is explained in the following:

**Proposition 3.8.** There is a commutative diagram of stacks with the middle squares being Cartesian:

Diagram:

\[ \begin{array}{ccc}
BQ & \xrightarrow{\tilde{q}_2} & BR \\
\downarrow & & \downarrow \\
BP & \xrightarrow{p_1} & BR \\
\downarrow & & \downarrow \\
BL & & BK \\
\downarrow & & \downarrow \\
L & & G
\end{array} \]

Proof. The fibers of $q_2$ and $\tilde{q}_2$ are naturally isomorphic (and are non-canonically isomorphic to $BU_R$).

Denote by $\mathfrak{l}, \mathfrak{p}, \mathfrak{q}, \mathfrak{t}, \mathfrak{r}, \mathfrak{g}$ the corresponding Lie algebras.

**Corollary 3.9.** There are commutative diagrams of stacks with the middle squares being Cartesian, where all actions are adjoint actions:

Diagram:

\[ \begin{array}{ccc}
\mathfrak{q}/Q & \xrightarrow{\tilde{q}_2} & \mathfrak{t}/R \\
\downarrow & & \downarrow \\
\mathfrak{p}/P & \xrightarrow{\dot{p}_1} & \mathfrak{t}/R \\
\downarrow & & \downarrow \\
\mathfrak{t}/L & & \mathfrak{g}/G
\end{array} \]
Proof. By Proposition 3.8, for any stack $X$, we have

\[ \text{Map}(X, BQ) \xrightarrow{\tilde{q}_2} \text{Map}(X, BP) \xrightarrow{\tilde{p}_1} \text{Map}(X, BR) \xrightarrow{\tilde{p}_2} \text{Map}(X, BG) \]

Then we obtain the first diagram by taking $X = \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$ (up to a shift), and second diagram by taking $X = S^1$. \qed

The following proposition is easy to check from root datum:

**Proposition 3.10.**

1. $g_J' \subset p_J' \subset g_J$, and $p_J'$ is a parabolic subalgebra of $g_J$ with Levi factor $g_J'$.
2. $p_J' \circ p_J' = p_J'$.

4. Twisted conjugacy classes in loop group

Fix $q \in \mathbb{C}^*$ with $|q| < 1$, and let $E = \mathbb{C}^*/q^2$ be the corresponding elliptic curve.

In this section, we focus on the connected component $G_E$ of the trivial bundle in the moduli stack $\text{Bun}_G(E)$ of semistable $G$-bundles on $E$. We describe the geometry of $G_E$ in terms of the Lie theory of $q$-twisted conjugacy classes in the holomorphic loop group. We work in the context of complex analytic stacks (see Appendix A for the facts used).

4.1. Automorphism groups. The aim of this subsection is to calculate the automorphism groups of semisimple semistable bundles. The main result is Corollary 4.10, stating that the automorphism group can be calculated in terms of affine root systems. This was previously obtained in [BEG03, Theorem 5.6], though the approaches to the component groups differ somewhat. Our approach makes the role of the affine Weyl group transparent.

We adopt Notation 3.4. Let $\text{Bun}_G(E)$ be the moduli stack of $G$-bundles on $E$.

Let $L_{hol}G$ be the holomorphic loop group of holomorphic maps $g(z) : \mathbb{C}^* \to G$. It acts on itself by $q$-twisted conjugation

\[ Ad^q_{k(z)}g(z) := k(qz)g(z)k(z)^{-1} \]

Unless otherwise stated, we will always use $q$-twisted conjugation instead of usual conjugation throughout this section.
For any $g(z) \in L_{hol}G$, we can define a $G$-bundle

$$\mathcal{P}_{g(z)} := \mathbb{C}^\times \times_{q^z} G \longrightarrow E = \mathbb{C}^\times / q^z$$

where the $q^z$-action of is given by

$$q \cdot (z, x) = (qz, g(z)x)$$

Note if $g(z), h(z)$ are $q$-twisted conjugate, then their associated bundles $\mathcal{P}_{g(z)}, \mathcal{P}_{h(z)}$ are isomorphic.

The automorphism group of $\mathcal{P}_{g(z)}$ admits the description

$$Aut(\mathcal{P}_{g(z)}) \simeq \{k(z)|k(qz) = g(z)k(z)g(z)^{-1}\} = C_{L_{hol}G}(g(z))$$

as a $q$-twisted centralizer, since the automorphisms of $\mathcal{P}_{g(z)}$ are isomorphic to the automorphisms of the corresponding $q^z$-equivariant $G$-bundle over $\mathbb{C}^\times$.

Since any $G$-bundle on $\mathbb{C}^\times$ is trivializable, we have an isomorphism of groupoids

$$L_{hol}G / L_{hol}G \simeq \text{Bun}_G(E)(\mathbb{C})$$

For $s \in T$, set $G_s := C_{L_{hol}G}(s)$. Note that $G_s$ is preserved by $q$-twisted conjugation on itself: for $f(z), g(z) \in G_s, f(qz)g(z)f(z)^{-1} \in G_s$ by direct calculation.

For the moment, $G_s$ is simply an abstract group. In the rest of this subsection, we will calculate $G_s$ explicitly and equip it with the structure of algebraic group, which is compatible with the one coming from the automorphism group of a $G$-bundle.

4.1.1. Calculation of $G_0^0$. We start with the calculation of the neutral component $G_0^0$, the result is given in Corollary 4.6.

Recall $LG \subset L_{hol}G$ denote the subgroup of polynomial loops $g(z) : \mathbb{C}^\times \rightarrow G$. We will regard it as an ind-scheme, more specifically, as the increasing union of its closed subschemes of prescribed zeros and poles. Let $L_0$ denote its Lie algebra.

We will begin with $G = GL_N$. Let $T_N \subset GL_N$ be the invertible diagonal matrices.

**Lemma 4.1.** Let $f : \mathbb{C}^\times \rightarrow \mathbb{C}$ be a holomorphic function. Assume that $f(qz) = af(z)$, for some $a \in \mathbb{C}$. If $a \in q^\mathbb{Z}$, then $f(z) = cz^n$, for some $c \in \mathbb{C}$ and $n = \log_q(a)$; otherwise, $f(z) \equiv 0$.

**Proof.** Follows from an elementary comparison of the coefficients of the Laurent expansion of $f$. \qed

**Proposition 4.2.** Let $s = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \in T_N$.

Let $I_s$ consist of those $(i, j)$ such that $\lambda_i/\lambda_j \in q^{\mathbb{Z}}$, where $n_{ij} = \log_q(\lambda_i/\lambda_j)$.

Set $\mathfrak{gl}_{N,s} = \bigoplus_{(i,j) \in I_s} \mathbb{C} z^{n_{ij}} E_{ij} \subset L_{poly}\mathfrak{gl}_N$, where $E_{ij} \in \mathfrak{gl}_N$ is the elementary matrix with non-zero $(i, j)$-entry.

Then under the standard embedding $GL_N \rightarrow \mathfrak{gl}_N$ as invertible matrices, $GL_{N,s}$ consists of the invertible matrices in $\mathfrak{gl}_{N,s}$.

**Proof.** For $g(z) = \bigoplus_{(i,j)} g_{ij}(z)E_{ij} \in GL_{N,s}$, observe that $g(qz) = s \cdot g(z) \cdot s^{-1}$ is equivalent to $g_{ij}(qz) = (\lambda_i/\lambda_j)g_{ij}(z)$ for all $(i, j)$. By the previous lemma, $g_{ij}(z) = cz^{n_{ij}}$ if $\lambda_i/\lambda_j = q^{n_{ij}}$ and $g_{ij}(z) = 0$ otherwise. In particular, $g_{ii}(z)$ is constant. \qed
Corollary 4.3. $GL_{N,s} \subset L_{\text{hol}}GL_N$ lies in $LGL_N \subset L_{\text{hol}}GL_N$ and is Zariski-closed therein. With its reduced subscheme structure, $GL_{N,s}$ is a reductive algebraic group, its $q$-twisted conjugation is an algebraic action, and the evaluation map

$$ev_1 : GL_{N,s} \longrightarrow GL_N \quad g(z) \longmapsto g(1)$$

is an injective homomorphism of algebraic groups. Furthermore, the Lie algebra of $GL_{N,s}$ is precisely $\mathfrak{gl}_{N,s}$.

For a general reductive algebraic group $G$ with maximal torus $T \subset G$, choose an embedding of pairs $i : (G,T) \rightarrow (GL_N,T_N)$. This induces embeddings $LGL \subset L_{\text{poly}}GL_N$, $G_s \subset GL_{N,s}$, with $G_s = LGL \cap GL_{N,s}$. Hence $G_s$ is Zariski-closed in both $GL_{N,s}$ and $LG$. Thus we have the following generalization of the previous corollary.

Proposition 4.4. $G_s \subset L_{\text{hol}}G$ lies in $LG \subset L_{\text{hol}}G$ and is Zariski-closed therein. With its reduced subscheme structure, $G_s$ is a reductive algebraic group, its $q$-twisted conjugation is an algebraic action, and the evaluation map

$$ev_1 : G_s \longrightarrow G \quad g(z) \longmapsto g(1)$$

is an injective homomorphism of algebraic groups. Moreover, the natural map $G_s \rightarrow Aut(\mathcal{P}_s)$ is an isomorphism of algebraic groups.

Proof. Only the last statement needs proof. We have a commutative diagram of abstract groups:

$$\begin{array}{ccc}
G_s & \cong & Aut(\mathcal{P}) \\
\downarrow \quad ev_1 & & \\
G = Aut(\mathcal{P}_0) & & \\
\end{array}$$

where $\mathcal{P}_0$ is the fiber of $\mathcal{P}$ over $0 \in E$, and the “$\cong$” means canonical isomorphism. The two vertical maps are injective morphisms of algebraic groups, so the top arrow is also a morphism of algebraic groups.

Since $G_s = LGL \cap GL_{N,s}$, its Lie algebra satisfies $\mathfrak{g}_s = L\mathfrak{g} \cap \mathfrak{gl}_{N,s}$. More explicitly, it admits the following description.

Proposition 4.5. Denote by $\Phi_s := \{ \alpha \in \Phi_{\text{aff}} \mid \alpha(s) = 1 \}$, where the affine roots $\Phi_{\text{aff}}$ is regarded as a subset of $Map(T, \mathbb{C}^*)$ via $\alpha = \alpha_0 + n \rightarrow \{ s \mapsto \alpha(s)q^n \}$. Then the Lie algebra of $G_s$ is precisely $\mathfrak{g}_s = t \oplus \bigoplus_{\alpha \in \Phi_s} \mathfrak{g}_0 \subset L\mathfrak{g}$.

Proof. $\mathfrak{g}_{N,s}$ is a finite dimensional subalgebra of $L\mathfrak{g}_{N}$ satisfying $\{ X(z) \in L\mathfrak{g} \mid X(qz) = Ad(s)X(z) \}$. So $\mathfrak{g}_s = L_{\text{poly}}\mathfrak{g} \cap \mathfrak{gl}_{N,s} = \{ X(z) \in L\mathfrak{g} \mid X(qz) = Ad(s)X(z) \}$. Write $X(z) = h(z) + \sum_{\alpha_0 \in \Phi} f_{\alpha_0}(z)$, with respect to the root decomposition of $t$, i.e $h(z) : \mathbb{C}^* \rightarrow t, f_{\alpha_0}(z) : \mathbb{C}^* \rightarrow \mathfrak{g}_{\alpha_0}$. Now the condition $X(qz) = Ad(s)X(z)$ is equivalent to $h(qz) = h(z)$, and $f_{\alpha_0}(qz) = \alpha_0(s)f_{\alpha_0}(z)$. Let $f_{\alpha_0}(z) = \sum_{n=-K}^{K} a_n z^n X_{\alpha_0}$, where $X_{\alpha_0}$ is the root vector of $\alpha_0$. The above condition become $\sum_{n=-K}^{K} a_n q^n z^n = \sum_{n=-K}^{K} a_n \alpha_0(s)z^n$ compare coefficients, get the only nonvanishing $f_{\alpha_0}$ are those
with \( \alpha_0(s) = q^{n_0} \) for some \( n_0 \in \mathbb{Z} \) and in this case \( f_{\alpha_0} = z^{n_0}X_{\alpha_0} \in \mathfrak{g}_\alpha \). h(z) corresponds to the case \( \alpha_0(s) = 1 \) so \( h(z) \) is constant function. So \( \mathfrak{g}_\alpha = t \oplus \bigoplus_{\alpha_0 \in \Phi_\alpha, \alpha_0(s) \in \mathbb{Z}} \mathfrak{g}_{\alpha_0} \cdot z^{n_0} \). By compare this expression with the definition of \( \mathfrak{g}_\alpha \) for affine root \( \alpha \), the proposition follows. \( \square \)

**Corollary 4.6.** \( G_s^0 \subset LG \) is generated by \( T \cup \{ \exp \mathfrak{g}_\alpha \mid \alpha \in \Phi_s \} \).

**Example 4.7.** \( G = SL_2 \), and \( T \) diagonal matrices, with roots \( \{ \alpha, -\alpha \} \) take \( s = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q}^{-1} \end{pmatrix} \in T \), hence \( \alpha(s) = \sqrt{q}/(\sqrt{q}^{-1}) = q \), so \( n_\alpha = 1 \), similarly \( n_{-\alpha} = -1 \), we have \( X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), then by the proposition, \( G_s^0 \) is generated by \( T, \exp \begin{pmatrix} 0 & bz \\ 0 & 0 \end{pmatrix}, \exp \begin{pmatrix} 0 & 0 \\ cz^{-1} & 0 \end{pmatrix} \) which equal the subgroup \( \begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix} \) of \( LG \). In fact, in this case we have \( G_s^0 = G_s = C_{LG}(s) \). \( \square \)

**4.1.2. Calculation of \( G_s \).** We proceed to calculate \( G_s \), the result is given in Corollary 4.10, which stating that the component group can be controlled by affine Weyl group.

Let \( M_1, M_2 \) be smooth/complex manifolds, and write \( Map(M_1, M_2) \) for the set of infinitely differentiable/holomorphic maps \( M_1 \rightarrow M_2 \).

**Lemma 4.8.** Suppose \( M \) is connected.

Regard \( Map(M, G) \) as a group, and \( T \subset N_T(G) \subset G \subset Map(M, G) \) as subgroups of constant maps. Then \( N_{Map(M, G)}(T) = Map(M, T) \cdot N_G(T) \) as subgroups of \( Map(M, G) \).

**Proof.** Let \( f(x) \in N_{Map(M, G)}(T) \). Then \( f(x)Tf(x)^{-1} = T \), for any \( x \in M \), hence \( f(x) \in Map(M, N_G(T)) \). Hence \( N_{Map(M, G)}(T) \subset Map(M, N_G(T)) \). Since \( M \) is connected, \( Map(M, N_G(T)) = Map(M, T) \cdot N_G(T) \). Now \( Map(M, T) \subset N_{Map(M, G)}(T) \), and \( N_G(T) \subset N_{Map(M, G)}(T) \). Hence \( Map(M, T) \cdot N_G(T) \subset N_{Map(M, G)}(T) \).

**Lemma 4.9.** \( (L_{hol}T \cdot N_G(T)) \cap G_s = T \cdot C_{W_{aff}}(s) \) as subgroups of \( L_{hol}G \).

**Proof.** The right hand side does not depend on the lifting of \( W \) and equals \( (X_s(T) \cdot N_G(T)) \cap G_s \) which naturally sits inside the left hand side. We need to show that \( (L_{hol}T \cdot N_G(T)) \cap G_s \subset X_s(T) \cdot N_G(T) \). Suppose \( f \in (L_{hol}T \cdot N_G(T)) \cap G_s \), for \( f \in L_{hol}T, w \in N_G(T) \). Then we have \( s = f \cdot (w(s)) \). However, \( w(s), s \in T \), and this implies \( f \in X_s(T) \) by Lemma 4.1. \( \square \)

Let \( W_s := C_{W_{aff}}(s) \) be the stabilizer, by Lemma 4.8, 4.9, we have \( N_{G_s}(T) = N_{L_{hol}G}(T) \cap G_s = (L_{hol}T \cdot N_G(T)) \cap G_s = T \cdot W_s \) and the weyl group \( W(G_s, T) := N_{G_s}(T)/T \simeq W_s \), now \( T \subset G_s \) is a maximal torus, hence we have

**Corollary 4.10.** \( G_s = G_s^0 \cdot N_{G_s}(T) = \langle G_s^0, \tilde{w} \mid w \in W_s \rangle = \langle T, \exp \mathfrak{g}_\alpha, \tilde{w} \mid \alpha \in \Phi_s, w \in W_s \rangle \).

**Example 4.11.** \( G = PGL_2 \), and \( T \) diagonal matrices, with roots \( \{ \alpha, -\alpha \} \) take \( s = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & 1 \end{pmatrix} \). We have \( G_s^0 = T, G_s = \langle T, \tilde{w} \rangle \), where \( \tilde{w} = \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix} \).

**Remark 4.12.** The same method can be use to calculate the automorphism group of any semisimple (not only semistable) bundles. And theta functions naturally
show up in the calculation for non-semistable bundles, so $C_{L_{hol}G}(g(z))$ is not contained in $LG$ in general.

4.1.3. Untwist twisted conjugation. The twisted conjugation of $L_{hol}G$ is very different from the usual conjugation. However, when restricted to the action of $G_s$, the twisted conjugation is isomorphic to usual conjugation:

**Proposition 4.13.** The left multiplication by $s^{-1}: G_s \to G_s$ is a $G_s$-equivariant isomorphism of algebraic varieties, where the first action is $q$-twisted conjugation, and second action is usual conjugation. Hence we have an isomorphism of stacks $s^{-1}: G_s/G_s \to G_s/\text{ad}G_s$. □

Proof. For any $k(z) \in G_s$, we have $k(qz)sk(z)^{-1} = s$, hence $s^{-1}\text{Ad}_{k(z)}^{q}g(z) = s^{-1}k(qz)g(z)k(z)^{-1} = k(z)s^{-1}g(z)k(z)^{-1} = \text{Ad}_{k(z)}(s^{-1}g(z))$. □

$G_s^0$ is stable under the twisted conjugation of $G_s$. Hence $G_s^0/G_s$ has the usual properties of adjoint quotient (on its neutral component.)

**Corollary 4.14.** $\mathbb{C}[G_s^0]^G_s = \mathbb{C}[T]^W_s$. So there is a map $\chi_s: G_s^0 \to T^W_s$. Let $U \subset T$ be an $W_s$ invariant open subset, let $V := \chi_s^{-1}(U^W_s)$

**Definition 4.15.** Let $S$ be a topological space, and $A \subset S$ a subset. We say $A$ is abundant (in $S$) if the only open subset of $S$ containing $A$ is $S$

Examples of abundant subsets we will use are given in the next corollary. Note that $A \subset S$ is abundant if and only if for any $s \in S$, $\{s\} \cap A$ is nonempty.

**Corollary 4.16.** The image of $U$ in $|V/G_s|$ is abundant. □

Let $T^{s\text{-reg}}$ be the locus where the action of $W_s$ is free.

**Corollary 4.17.** Assume further that $U \subset T^{s\text{-reg}}$, then $V/G_s \xrightarrow{\sim} (U/N_{G_s}(T)) \xrightarrow{\sim} (U \times BT)/W_s$. □

In particular, let $T^{q\text{-reg}}$ be the locus where the action of $W_{aff}$ is free. Let $G_s^{0,q\text{-reg}} := \chi_s^{-1}(T^{q\text{-reg}}/W_s)$. We have:

**Corollary 4.18.** $G_s^{0,q\text{-reg}}/G_s \xrightarrow{\sim} T^{q\text{-reg}}/N_{G_s}(T) \xrightarrow{\sim} (T^{q\text{-reg}} \times BT)/W_s$. □

4.2. Étale charts. In this section, we will define some étale charts of $G_E$. The main result in this section is Theorem 4.29. Facts about semistable bundles on elliptic curves are collected in Appendix B.

4.2.1. **Definition and representibility.** There are three mutually commuting action of $q^Z, G_s, G$ on $\mathbb{C}^* \times G_s^0 \times G$:

$$q \cdot (z, h, g) := (qz, h, h(z)g), \quad q \in q^Z$$

$$k \cdot (z, h, g) := (z, \text{Ad}_k(h), k(z)g), \quad k \in G_s$$

$$g' \cdot (z, h, g) := (z, h, gg'^{-1}), \quad g' \in G$$

Let $\mathcal{P}_s := (\mathbb{C}^* \times G_s^0 \times G)/q^Z$. Then $\mathcal{P}_s$ maps naturally to $E \times G_s^0$ which is a $G_s^0$ equivariant (with respect to the twisted conjugation on $G_s^0$) principal $G$-bundle. The following definition extends the map on $\mathbb{C}$-points $G_s^0/G_s \to L_{hol}G/L_{hol}G \to \text{Bun}_G(E)(\mathbb{C})$ to arbitrary $S$-points.

Definition 4.19. There is a map $p_s : G^0_s/G_s \to \text{Bun}_G(E)$ defined as follows: given a $S$ point of $G^0_s/G_s$:

\[
\begin{array}{ccc}
P & \longrightarrow & G^0_s \\
\downarrow & & \\
S & \longleftarrow & 
\end{array}
\]

where $P$ is an $G_s$-bundle on $S$, and $P \to G^0_s$ is a $G_s$ equivariant map. We have $Y := (E \times P) \times_{E \times G^0_s} \mathcal{P}_s \longrightarrow \mathcal{P}_s$

\[
\begin{array}{ccc}
E \times P & \longrightarrow & E \times G^0_s \\
\downarrow & & \\
E \times S & \longrightarrow & 
\end{array}
\]

$Y$ has an induced transitive $G_s$ action, such that $Y \to E \times P$ is $G_s$ equivariant, and the induced map $Y/G_s \to E \times P/G_s = E \times S$ is a principal $G$ bundle. This gives an $S$ point of $\text{Bun}_G(E)$.

Denote by $G_E := \text{Bun}_G(E)^{0,ss}$ the stack of degree 0 semistable $G$-bundles.

Proposition 4.20. The image of $p_s$ lies in $G_E$.

Proof. Since $G^0_s/G_s$ is connected, the image lies in $\text{Bun}_G(E)^0$. $T \subset G^0_s$ maps to the degree 0 semisimple semistable bundles. By Lemma 4.16, any other point in $G^0_s$ has closure containing points in $T$. So by Proposition B.2, $G_s$ maps to $G_E \subset \text{Bun}_G(E)^0$.

Proposition 4.21. $p_s$ is representable.

Proof. Let $G_{E,0}$ be the stack classifying $\{(\mathcal{P}, \beta)\}$, where $\mathcal{P}$ is a semistable $G$ bundle of degree 0, $\beta$ is a trivialization of $\mathcal{P}$ at $0 \in E$. $G_{E,0}$ is representable by Proposition B.7. There is a natural map $p'_s : G^0_s \to G_{E,0}$ defined by $\mathcal{P}_s$ with the natural trivialization by identifying the fiber over $0 \in E$ with the fiber over $1 \in \mathbb{C}^*$. Then $p'_s$ is $G_s$ equivariant, where the $G_s$ acts on $G_{E,0}$ via $\text{ev}_1 : G_s \to G$ and $G$ acts via change of trivialization. So $p'_s$ induces $\overline{p}'_s : G^0_s/G_s \to G_{E,0}/G$. Hence $\overline{p}'_s$ is representable. We have the following commutative diagram of stacks:

\[
\begin{array}{ccc}
G^0_s/G_s & \longrightarrow & G_{E,0}/G \\
\downarrow & \searrow & \\
G_E & \longrightarrow & 
\end{array}
\]

By Proposition 4.4, $\text{ev}_1 : G_s \to G$ is injective, so the top arrows are representable and hence $p_s$ is representable.

4.2.2. 1-shifted symplectic stacks. In this section, we show that the morphism $p_s$ is a symplectomorphism.

Definition 4.22. Let $X$ be a smooth stack, $TX$ its tangent complex
A weak 1-shifted symplectic structure is an 1-shifted non-degenerate 2-form
\(\omega_X\), i.e. a non-degenerate \(\mathcal{O}_X\)-bilinear antisymmetric pairing
\[\omega_X : TX[-1] \times TX[-1] \to \mathcal{O}_X[-1].\]

A symplectomorphism \(f : (X, \omega_X) \to (Y, \omega_Y)\) between smooth stacks with
weak 1-shifted symplectic structure is a morphism of stacks \(f : X \to Y\)
with an isomorphism \(f^*\omega_Y \simeq \omega_X\).

**Remark 4.23.** To define the actual shifted symplectic structure, the notion of
closed forms is needed and requires more careful definition, see [PTVV13].
The weak version above is sufficient for our purpose. For smooth stacks with positive
dimensional automorphism group, \(n = 1\) is the only possible value for \(n\)-shifted
symplectic structure to exist.

Our main motivation to use the shifted symplectic structures is that it relates
the stacky and infinitesimal behaviours:

**Proposition 4.24.** Let \(f : (X, \omega_X) \to (Y, \omega_Y)\) be a symplectomorphism, and \(x \in X\).
Assume that \(f_x : \text{Aut}(x)^0 \to \text{Aut}(f(x))^0\) is an isomorphism, then \(f\) is étale at \(x\).

**Proof.** We need to show that \(df_x : T_x X \to T_{f(x)} Y\) is a quasi-isomorphism. The
tangent complex is concentrated in degree \(-1, 0\) since the stacks are smooth. For
degree \(-1\), we have \(H^{-1}(df_x) = d(f_x)\), so it is an isomorphism. For degree 0, the
map \(H^0(df_x)\) is also an isomorphism since the (weak) 1-shifted symplectic structure
pairs \(H^{-1}\) and \(H^0\).

**Example 4.25.** Fix \(\kappa\) an invariant bilinear form on \(\mathfrak{g}\). For \(P \in \text{Bun}_G(E)\), we
have a natural identification \(T_P \text{Bun}_G(E)[-1] \simeq R\Gamma(E, \mathfrak{g}_P)\), and \(\text{Bun}_G(E)\) (hence
\(G_E\)) has a natural weak 1-shifted symplectic structure given by the Serre duality
pairing:
\[
R\Gamma(E, \mathfrak{g}_{P_E(z)}) \times R\Gamma(E, \mathfrak{g}_{P_E(z)}) \xrightarrow{\kappa} \tau^\geq 1 R\Gamma(E, \mathcal{O}_E)
\]

Similarly, \(G/G \simeq \text{Loc}_G(S^1)\) has a natural weak 1-shifted symplectic structure given
by Poincaré duality. In general, [PTVV13] shows that \(\text{Bun}_G(X)\) has a \(2 - n\) shifted
symplectic structure for \(X\) a \(n\)-dimensional Calabi-Yau manifold and \(\text{Loc}_G(M)\)
has a \(2 - n\) shifted symplectic structure for \(M\) a \(n\)-dimensional oriented smooth
manifold.

The uniformization \(p : L_{\text{hol}}G/L_{\text{hol}}G \to \text{Bun}_G(E)\) can be thought as a non-
linear Cech resolution associated to the cover \(\mathbb{C}^* \to E\), in the sense that, after
linearization:
\[
dp_{P_E(z)} : T_{P_E(z)} L_{\text{hol}}G/L_{\text{hol}}G[-1] \xrightarrow{\sim} T_{P_E(z)} \text{Bun}_G(E)[-1]
\]
\[
\{ L_{\text{hol}}G \xrightarrow{\phi(z)} L_{\text{hol}}G \} \xrightarrow{\sim} R\Gamma(E, \mathfrak{g}_{P_E(z)})
\]

the tangent map in the first row can be identified with the Cech resolution in the
second row above, where \(\phi(z)(X(z)) = \text{Ad}_{g(z)}^{-1}X(qz) - X(z)\), and also complexes
are (cohomologically) concentrated in degree 0, 1.
There is a natural pairing:

\[ \kappa : \{ L_{\text{hol}} g \to L_{\text{hol}} g \} \times \{ L_{\text{hol}} g \to L_{\text{hol}} g \} \to \tau_{\geq 1} \{ L_{\text{hol}} C \to L_{\text{hol}} C \} \]

\[(X^*(z), Y^*(z)) \quad \kappa (X^*(z), Y^*(z)) \]

It follows from definition that this pairing resolve the Serre duality pairing, i.e:

**Proposition 4.26.** The diagram naturally commute:

\[
\begin{array}{ccc}
\{ L_{\text{hol}} g \to L_{\text{hol}} g \} \times \{ L_{\text{hol}} g \to L_{\text{hol}} g \} & \xrightarrow{\sim} & \{ L_{\text{hol}} C \to L_{\text{hol}} C \} \\
\downarrow \kappa & & \downarrow \kappa \\
R\Gamma(E, g_{P(t)}) \times R\Gamma(E, g_{P(t)}) & \xrightarrow{\sim} & R\Gamma(E, O_E) \\
\end{array}
\]

If we view the tangent complex \( T_{g(z)} G_s^0/G_s \) as a subcomplex of \( T_{g(z)} L_{\text{hol}} G/L_{\text{hol}} G \), then \( G^0_s/G_s \) has an induced 1-shifted 2-form \( \omega \).

**Proposition 4.27.** The 1-shifted 2-form \( \omega \) on \( G^0_s/G_s \) is non-degenerate. And the map \( p_s : G^0_s/G_s \to G_E \) is a 1-shifted symplectomorphism.

**Proof.** The second statement follows from Proposition 4.26. For the first statement, we first prove that the pairing \( \kappa (\phi, \psi) \) is non-degenerate. This is because \( g_s = t \oplus \bigoplus_{a \in \Phi} g_{s_0} \), and the pairing pairs \( t \) with \( t \), pairs \( g_{s_0} \) with \( g_{s_0} \). Now the non-degeneracy of \( \omega \) follows from the following tautological Lemma:

**Lemma 4.28.** Let \( \langle - , - \rangle \) : \( V^0 \times V^1 \to C \) a non-degenerate pairing between finite dimensional vector spaces, and let \( \phi : V^0 \to V^1 \), such that \( \langle \ker(\phi), \phi(\ker(\phi)) \rangle = 0 \), then the induced pairing \( \ker(\phi) \times (V^1/\ker(\phi)) \to C \) is also non-degenerate.

To complete the proof of Proposition, take \( V^0 = g_s, V^1 = g_s \), and \( \phi = \phi_{g(z)} \). \( \square \)

**4.2.3. Étale charts.** Let \( T^e_s := \{ t \in T : G_t \subset G_s \} \), then \( T^e_s \) is an \( W \) invariant open subset of \( T \), also note that \( T^e_s \) can be computed in terms of root datum and the elliptic parameter \( q \) thanks to Corollary 4.10. Denote \( G^0_s^{\text{et}} := \chi_e^{-1}(T^0_s/W_s) \).

**Theorem 4.29.** \( p_s : G^0_s^{\text{et}}/G_s \to G_E \) is étale. And \( p : \bigsqcup_{s \in T} G^0_s^{\text{et}}/G_s \to G_E \) is surjective.

**Proof.** We first prove that \( p_s \) is étale for \( t \in T^e_s \). By Proposition 4.24 and 4.27, we need to show that \( (p_s)_t : Aut(t) \to Aut(P_t) \) is an isomorphism (on the neutral component). This is true because \( (p_s)_t \) is identified as \( Aut(t) = C_{G_s}(t) = G_s \cap C_{L_{\text{hol}} G(t)} = G_s \cap G_t = G_t \). Now the first assertion follows since \( T^e_s \) is abundant in \( |G^0_s^{\text{et}}/G_s| \) and the and étale locus is open. For the second assertion, note that \( p_s \) is étale at \( s \), so the image of \( p \) contains all semisimple bundles, by Proposition B.2 the set of semisimple bundles is abundant in \( |G_E| \), and \( p \) has open image since it is étale, so we conclude that \( p \) is surjective. \( \square \)

Let \( X \) be a algebraic variety, \( K \) an algebraic group, then the group automorphism \( Aut(K) \) acts naturally on \( BK \) and hence on \( Bun_K(X) \). The action induced by \( Inn(G) \) is canonical isomorphic to identity morphism on \( Bun_K(X) \). Hence \( Aut(G) \) acts on \( Bun_G(X) \). Let \( H < K, Bun_H(X) \to Bun_K(X) \) is \( N_K(H) \) equivariant, and \( N_K(H) \) action on \( Bun_K(X) \) is canonical isomorphic to identity since \( N_K(H) \) acts via inner automorphism. Take \( X = E, K = G, H = T, \) then \( W = N_G(T) \) acts
on $T_E$. Let $T^\text{reg}_E$ be the locus where this action is free (on the set of points). So we have $T^\text{reg}_E/W \to G_E$, which factor through $T^\text{reg}_E/W \to G^\text{reg}_E \subset G_E$, where $G^\text{reg}_E$ is the (Zariski) open substack of $G_E$ consisting of regular semisimple bundles with connected automorphism group. And $T^\text{reg}_E/W \to G^\text{reg}_E$ is an isomorphism.

View $X_s(T)$ as a subgroup of $LT$, and it acts freely on the constant loops $T \subset LT$ via twisted conjugation, we have $T_E \simeq T/X_s(T) \times BT$. The group $W_\text{aff} = X_s(T) \rtimes W$ acts on $T$, and let $T^{q\text{-reg}}$ be the open dense locus where the action of $W_\text{aff}$ is free. Let $G_s^{0,q\text{-reg}} := \chi_s^{-1}(T^{q\text{-reg}}/W_s)$. Using the identification $T^{q\text{-reg}}_E/W \simeq (T^{q\text{-reg}}/X_s(T) \times BT)/W \simeq (T^{q\text{-reg}} \times BT)/W_\text{aff}$, we have a commutative diagram:

\[
\begin{array}{ccc}
(T^{q\text{-reg}} \times BT)/W_s & \xrightarrow{\sim} & G_s^{0,q\text{-reg}}/G_s \\
\downarrow & & \downarrow p_s \\
(T^{q\text{-reg}} \times BT)/W_\text{aff} & \xrightarrow{\sim} & G^\text{reg}_E
\end{array}
\]

Recall the semi-simplification map $\chi_E : G_E \to \mathfrak{e}_E$ as in (B.3).

**Proposition 4.31.** The following commutative diagram is Cartesian:

\[
G_s^{0,\text{et}}/G_s \xrightarrow{\chi_s} T^\text{et}/W_s \\
\downarrow p_s \quad \Box \quad \downarrow \\
G_E \xrightarrow{\chi_E} \mathfrak{e}_E \simeq T/\text{Waff}
\]

**Proof.** Suffices to show for each small open $U \subset T^\text{et}/W_s$, the diagram obtained by restricting to $U$ is Cartesian:

\[
\chi_s^{-1}(U) \xrightarrow{\chi_s} U \\
\downarrow p_s \quad \Box \quad \downarrow p'
\]

Assume $U$ is small so that the $p'$ is an open embedding. Now by (4.30), the top arrow $p$ is generically open embedding, and it is also étale, so by Lemma A.2, the map $p$ is an open embedding. Now we need to check the image of $p$ equal $\chi_E^{-1}(p'(U))$. This is because the image contains all the semi-simple bundles in $\chi_E^{-1}(p'(U))$ by construction and hence consist all $\chi_E^{-1}(p'(U))$ by Proposition B.2. \hfill \Box

**4.3. Gluing.** In this section, we will glue the charts defined in Section 4.2, i.e we will calculate the fiber products of the charts. The combinatorics of higher descent data is naturally organized in diagrams introduced in Section 2. The main result of this section is Theorem 4.34.

For $s = (s_1, s_2, \ldots, s_k) \in T^k$, let $G_s := \bigcap_{i=1}^k G_{s_i}$, $W_s := N_{G_s}(T)/T$, define $\chi_s, T^\text{et}_s, G^\text{et}_s$ analogously. We have all the above statement for $G_s$ still holds for $G_s$, and $W_s = \bigcap_{i=1}^k W_{s_i}, T^\text{et}_s = \bigcap_{i=1}^k T^\text{et}_{s_i}, T^{q\text{-reg}} \subset T^\text{et}_s$ for all $s$ by the connectedness of $G$.

**Proposition 4.32.** The twisted conjugation $\text{Ad}^\text{et}_s : G^0_s \to G^0_w(s)$ intertwines the action $\text{Ad}_w : G_s \to G_{w(s)}$. Hence we have an isomorphism of stack $\text{Ad}^\text{et}_w : G^0_s/G_s \xrightarrow{\sim} G^0_{w(s)}/G_{w(s)}$. 
Proof. For $s \in T$, $w = u\lambda \in W_{\text{aff}} = W \times X_s(T)$. Since $G_{\tilde{w}(s)} = C_{LG}(\tilde{w}(s)) = Ad_u C_{LG}(s)$, so we have a isomorphism of algebraic groups: $Ad_{\tilde{w}} : G_s \to G_{\tilde{w}(s)}$ where $Ad_{\tilde{w}}$ is the usual conjugation of $\tilde{w}$ in $LG$.

$Ad_{\tilde{w}}^0(G_s) = \tilde{w}(qz)G_s\lambda(z)^{-1}\tilde{w}^{-1} = \tilde{w}\lambda(z)\lambda(q)G_s\lambda(z)^{-1}\tilde{w}^{-1} = Ad_{\tilde{w}}(G_s) = G_{\tilde{w}(s)}$ since $\lambda(q) \in T \subset G_s$. Since $Ad_{\tilde{w}}^0$ stabilize $T$, we have $Ad_{\tilde{w}}^0 : G_s^0 \to G_{\tilde{w}(s)}^0$ isomorphism of algebraic varieties. The pair of isomorphism of algebraic varieties and algebraic groups $(Ad_{\tilde{w}}^0, Ad_{\tilde{w}}) : (G_s^0, G_s) \to (G_{\tilde{w}(s)}^0, G_{\tilde{w}(s)})$ intertwine the twisted conjugation action on both sides. Hence we have an induced isomorphism of quotient stacks, still denoted by $Ad_{\tilde{w}}^0 : G_s^0/G_s \to G_{\tilde{w}(s)}^0/G_{\tilde{w}(s)}$. It’s also easy to see that the above map takes étale locus to étale locus, so we have $Ad_{\tilde{w}}^0 : G_s^0,et/G_s \to G_{\tilde{w}(s)}^0,et/G_{\tilde{w}(s)}$. □

Let $S \subset T$ be a $W_{\text{aff}}$ invariant subset, and let $\{U_s, s \in S\}$ be a collection of open subset of $T$ and $U_{w(s)} = w(U_s)$, for all $w \in W_{\text{aff}}$. Let $U_s := \bigcap_{s \in S} U_s$, $V_s := \lambda_s^{-1}(U_s/W_s)$. Then we have a a functor

$$V : \int_{\Delta_s^0} S^*///W_{\text{aff}} \longrightarrow \text{Stack}$$

defined similarly to Construction 2.25 by:

1. $V(s) := V_s/G_s$;
2. $V(w) := Ad_{\tilde{w}}^0 : V_s/G_s \xrightarrow{\sim} V_{w(s)}/G_{w(s)}$;
3. $V(w'w^{-1}) := \eta_{w'w^{-1}} \circ Ad_{\tilde{w}}^0 : Ad_{\tilde{w}}^0 \longrightarrow Ad_{\tilde{w}'}^0;$
4. $V(\delta : s \to s') := i : V_s/G_s \xrightarrow{\sim} V_s'/G_s'$.

The augmentation morphism $p_s$ and 2-morphism $\varphi_{\tilde{w}}$ defined below extends the functor $V$ to $V_+ : \int_{\Delta_s^0} S^*///W_{\text{aff}} \longrightarrow \text{Stack}$ by sending the final object to $G_E$.

**Definition 4.33.** The lifting $\tilde{w} = \tilde{w}(z)$ of $w \in W_{\text{aff}}$ induces $Ad_{\tilde{w}}^0 : G_s^0 \to G_{\tilde{w}(s)}^0$. There is commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C} \times G_s^0 \times G & \xrightarrow{\varphi_{\tilde{w}}} & \mathbb{C} \times G_{\tilde{w}(s)}^0 \times G \\
\downarrow & & \downarrow \\
\mathbb{C} \times G_s^0 & \xrightarrow{Id \times Ad_{\tilde{w}}^0} & \mathbb{C} \times G_{\tilde{w}(s)}^0
\end{array}
\]

where $\varphi_{\tilde{w}}(z, h, g) := (z, Ad_{\tilde{w}}^0(h), \tilde{w}(z)g)$. The diagram is $q^X$-equivariant and hence induces:

\[
\begin{array}{ccc}
\mathcal{P}_s & \xrightarrow{\varphi_{\tilde{w}}} & \mathcal{P}_{\tilde{w}(s)} \\
\downarrow & & \downarrow \\
E \times G_s^0 & \xrightarrow{Id \times Ad_{\tilde{w}}^0} & E \times G_{\tilde{w}(s)}^0
\end{array}
\]
And induces an isomorphism between the diagrams below, intertwining the $G_s$ action and $G_{w(s)}$ action, factorial in $S$:

\[
\begin{array}{ccc}
(E \times P) \times E \times G_0^0 \mathcal{P}_s & \cong & (E \times P_w) \times E \times G_{w(s)}^0 \mathcal{P}_{w(s)} \\
E \times P & \to & E \times G_0^0 \\
E \times S & \to & E \times S
\end{array}
\]

Hence $\varphi_w : p_s \Rightarrow p_{w(s)} \circ \text{Ad}_{\dot{w}} : G_0^0/G_s \to G_E$ an isomorphism between the morphisms of stacks.

We have the main theorem of this section:

**Theorem 4.34.** Assume $U_s \subset T^\text{et}$ and $\bigcup_{s \in S} U_s = T$, then the natural map is an isomorphism:

\[
\colim_{\Delta^\text{op}} S^* / W_{\text{aff}} V \xrightarrow{\sim} G_E
\]

**Proof.** By Construction 2.25, we have

\[
U_+ : \int_{\Delta^\text{op}} S^* / W_{\text{aff}} \longrightarrow \text{Var} \subset \text{Stack}
\]

which satisfies the assumption of Theorem 2.24, by (the proof of) Proposition 2.26. The character polynomial map $\chi_s$ and $\chi_E$ gives a natural transformation $\chi : V_+ \Rightarrow U_+$. which is Cartesian by argument similar to Proposition 4.31. Hence the functor $V_+$ also satisfies the assumption of Theorem 2.24. \qed

### 4.3.1. A Lie theoretic choice of charts

In this section, we will simplify the previous general discussions to concrete Lie theoretic data involving alcove geometry. For simplicity, we will assume $G$ is almost simple and simply-connected throughout this section. The statements also works for semisimple and simply-connected groups.

Choose $\tau \in \mathcal{H}$, such that $q = \exp(2\pi i \tau)$. The identification $\mathbb{Z} \simeq \mathbb{Z}\tau$ gives $t_R = X_*(T) \otimes \mathbb{R} \simeq X_*(T) \otimes \mathbb{R}\tau$. And hence gives a natural wall stratification on $X_*(T) \otimes \mathbb{R}\tau$. Under the identification $\mathbb{C} = \mathbb{R} \times \mathbb{R}\tau$,

\[
(X_*(T) \otimes \mathbb{R}/\mathbb{Z}) \times (X_*(T) \otimes \mathbb{R}\tau) = X_*(T) \otimes \mathbb{C}/\mathbb{Z} \xrightarrow{\text{Exp} = \exp(2\pi i \tau^-)} X_*(T) \otimes \mathbb{C}^* = T
\]

Note that the restriction of exponential map: $X_*(T) \otimes \mathbb{R}\tau \to T$ is an embedding. We show that the groups defined in Section 3.2 and in Section 4.1 coincide under this embedding:

**Proposition 4.35.** (1) For $a \in t_R$, we have $G_a = G_{\text{Exp}(0,a\tau)}$.

(2) For $a \in t_\mathbb{R}$, and $\theta \in X_*(T) \otimes \mathbb{R}/\mathbb{Z}$, we have $G_{\text{Exp}(\theta,a\tau)} \subset G_{\text{Exp}(0,a\tau)}$. 

Proof:

\[
\begin{array}{c}
\left( X_*(T) \otimes \mathbb{R}/\mathbb{Z} \right) \times \left( X_*(T) \otimes \mathbb{R}\tau \right) \xrightarrow{\alpha} \left( X_*(T) \otimes \mathbb{R}\tau \right) \xrightarrow{\text{Exp}} \left( X_*(T) \otimes \mathbb{R}\tau \right) \xrightarrow{\text{Exp}} \mathbb{C}^*
\end{array}
\]

Denote \( \Phi_\theta := \{ \alpha = \alpha_0 - n \in \Phi_{\text{aff}} \mid \alpha_0(\theta) = 0 \} \) and \( W_\theta := C^{\text{aff}}(\theta) \). For \( s := \text{Exp}(\theta, a\tau) \), then \( \Phi_s = \Phi_{\theta} \cap \Phi_a \) as subset of \( \Phi_{\text{aff}} \) and \( W_s = W_\theta \cap W_a \) as group of \( \Phi_{\text{aff}} \). Hence (1), (2) follow since \( \Phi_{\theta=0} = \Phi_{\text{aff}} \) and \( W_{\theta=0} = \Phi_{\text{aff}} \), c.f Proposition 3.7.

Now we assume that \( G \) is simply-connected.

Corollary 4.36. For \( s = \text{Exp}(0, a\tau) \), the group \( G_s \) is connected.

Remark 4.37. For general \( s \), the group \( G_s \) may not be connected, a counterexample is given in [BEG03, page 18].

Denote \( t_{r,j}^s := \text{St}_J \) (Definition 3.1 (4)), and \( T_j^s := (X_*(T) \otimes \mathbb{R}/\mathbb{Z}) \times t_{r,j}^s \tau \subset T \) and \( G_j^s := \chi_j^{-1}(T_j^s/W_J) \subset G_J \) be the set elements with “small eigenvalues”.

Proposition 4.38. \( T_j^s \) is contained in \( T_j^s \).

Proof. Need to prove that for any \( s \in T_j^s \), the group \( G_s \) is contained in \( G_J \). By Proposition 4.35, we can assume \( s = (0, a\tau) \), i.e. we need to prove \( G_a \subset G_J \) for \( a \in t_{r,j}^s = \text{St}_J \), and this can be easily checked.

Fix \( C \) an alcove, then similar to Proposition 3.3, we have

Proposition 4.39. \( \text{colim}_{J \in \mathcal{C}_J} T_j^s / W_J \xrightarrow{\sim} T / W_{\text{aff}} \)

Proof. For \( s_2 = (\theta_2, a_2\tau) \), \( s_2 = (\theta_2, a_2\tau) \in T_j^s \), \( w \in W_{\text{aff}} \), such that \( w(s_2) = s_2 \), we have \( w(a_1) = a_2 \), and \( a_1 \in \text{St}_J \). Hence \( w \in W_J \). So \( T_j^s / W_J \rightarrow T / W_{\text{aff}} \) is an open embedding. The surjectivity and requirement on intersection can be checked similarly as before.

By Proposition 4.38 and 4.39, we have the following more explicit version of Theorem 4.34:

Theorem 4.40. The natural map \( G_j^s / G_J \rightarrow G_E \) is an open embedding, and

\[
\prod_{J \in \{	ext{vertices of } C_J\}} G_j^s / G_J \rightarrow G_E
\]

is surjective.

Theorem 4.41. There is an isomorphism of stacks:

\[
\text{colim}_{J \in \mathcal{C}_J} G_j^s / G_J \rightarrow G_E
\]

Remark 4.42. The locus of small eigenvalues \( G_j^s \) depends on the choice of \( \tau \). Nevertheless, as we will see in Corollary 6.15, the category of nilpotent sheaves \( \text{Sh}_{\mathcal{N}}(G_j^s / G_J) \) does not depend on \( \tau \) and it is equivalence to \( \text{Sh}_{\mathcal{N}}(G_J / G_J) \).
5. Character sheaves on Lie algebras

The theory of character sheaves on a reductive algebraic group $G$ was introduced in a series of papers by Lusztig [Lus85a, Lus85b, Lus85c, Lus86a, Lus86b]. In characteristic $p$, it play important roles in the representation theory of the finite group $G(F_p)$. In particular, Lusztig shows that under Grothendieck sheaf-function correspondence, the (irreducible) character sheaves gives an orthonormal basis for the space of class functions on $G(F_p)$. In characteristic 0, [MV88, Gin89] show that character sheaves agree with adjoint-equivariant sheaves with singular support in nilpotent cone. Parallel theory of character sheaves on reductive Lie algebras was also developed in [Lus87, Mir04].

In this section, we shall focus on the microlocal description of nilpotent singular support, to study the category of character sheaves of a reductive Lie algebra together with various restriction functors between them. The viewpoint in [Gin93] is also helpful for us. Results in this section serve as local geometry for Sect 6 and 7.

5.1. A rescaling lemma. We start with some preliminaries on singular support of sheaves. For any $F \in Sh(X)$, we can associate to it the singular support $SS(F)$ of $F$, where $SS(F)$ is a conical closed coisotropic subset of $H^0(T^*X)$. Many properties of the sheaves can be seen from the microlocal measurement. For example, $F$ is constructible if and only if the microlocal stalk $F_x$ vanish outside degree 0, for $x$ in some generic locus of $SS(F)$.

Let $\mathcal{L} \subset H^0(T^*X)$ a closed conical isotropic subset. Let $Sh_{\mathcal{L}}(X) := \{ F \in Sh(X) : SS(F) \subset \mathcal{L} \}$, $D^c_{\mathcal{L}}(X) := \{ F \in Sh_{\mathcal{L}}(X) : F_x \text{ is perfect, for any } x \in X \}$ the category of constructible sheaves, and $Perv_{\mathcal{L}}(X) := \{ F \in D^c_{\mathcal{L}}(X) : F \text{ is perverse} \}$ the category of perverse sheaves. Our results in the remaining of the paper will be stated for $Sh_{\mathcal{L}}(X)$, but they work equally well for $D^c_{\mathcal{L}}(X)$ and $Perv_{\mathcal{L}}(X)$.

Let $A$ be a Lie group acting on a smooth manifold $X$, $\mathcal{L} \subset T^*X$ be a closed $A$-invariant conical isotropic subset. Let $\pi : X \to X/A$ and $\mu : T^*X \to g^*$ the moment map, then $\pi^*(T^*(X/A)) = \mu^{-1}(0) = \prod_{x \in X} T^*_{Ax}X$, and $\mu^{-1}(0)$.

Proposition 5.1. $\mathcal{L}$ is contained in $\mu^{-1}(0)$.

Proof. Suffices to check on the smooth locus of $\mathcal{L}$, where it follows from definition of isotropic submanifold.

Proposition 5.2. Let $U \subset X$ open subset, $F \in Sh_{\mathcal{L}}(U) := Sh_{\mathcal{L}|U}(U)$, then $F|_{Ax \cap U}$ is locally constant for all $x \in U$.

Proof. $SS(F) \subset \mathcal{L} \subset \pi^*_U(T^*(X/A))$, hence by [KS90, Prop. 6.6.2], in a neighborhood of $x \in U$, $F = \pi^*|_U(F')$, for some $F' \in Sh(X/A)$.

Lemma 5.3. Let $X$ be a smooth manifold with $\mathbb{R}^+$ action, $\mathcal{L}$ be a closed biconical isotropic subset of $T^*X$. Let $j : U \to X$ open embedding, such that $U \cap \mathbb{R}^+x$ is contractible, for any $x \in X$, then the restriction functor:

$$j^* : Sh_{\mathcal{L}}(X) \to Sh_{\mathcal{L}}(U)$$

is an equivalence of categories.

Proof. By Proposition 5.2, any $F \in Sh_{\mathcal{L}}(X)$ is conic, i.e. satisfies $F|_{\mathbb{R}^+x}$ is locally constant for all $x \in X$. Hence for $F_1, F_2 \in Sh_{\mathcal{L}}(X)$ by [KS90, Prop. 3.7.4(iii), Cor. 3.7.3], $\text{Hom}_{Sh_{\mathcal{L}}(X)}(F_1, F_2) \cong \text{Hom}_{Sh_{\mathcal{L}}(U)}(j^*F_1, j^*F_2)$ is an isomorphism, hence $j^*$
is fully faithful. Let $F' \in Sh_L(U)$, the by Proposition 5.2, for any $x \in U$, $F'|_{\mathbb{R}^+ x \cap U}$ is locally constant. We have natural maps

$$
\begin{array}{cc}
U \times \mathbb{R}^+ & U \\
\downarrow \alpha' & \downarrow j \\
X \times \mathbb{R}^+ & X
\end{array}
$$

Where $j = j \times Id$, $\alpha$ is the action map, $p$ is the projection, $i (resp. i')$ is inclusion to $X (resp. U) \times \{1\}$. $\alpha' := a \circ j : U \times \mathbb{R}^+ \rightarrow X$, then $\alpha'$ is $\mathbb{R}^+$-equivariant, has contractible fibers and $F' \boxtimes k_{\mathbb{R}^+}$ is constructible along the fibers of $\alpha'$. $F := a'_*(F' \boxtimes k_{\mathbb{R}^+}) \in Sh(X)$, then $F$ is conic by [KS90, Prop. 3.7.4(ii)]. There is a chain of isomorphisms

\begin{equation}
F' \simeq i'^*(F' \boxtimes k_{\mathbb{R}^+}) \simeq i'^* a'^*(F) \simeq i'^* \jmath^* a^*(F) \simeq i'^* \jmath^* p^*(F) \simeq j^*(F)
\end{equation}

where the second isomorphism is by [KS90, Prop. 2.7.8], the fourth isomorphism is by [KS90, Prop. 3.7.2]. Now $a'$ is smooth and surjective, and $SS(a'^*F) = SS(F' \boxtimes k_{\mathbb{R}^+}) \subset L \times T_{\mathbb{R}^+}^* \mathbb{R}^+ = a'^*(L)$ since $L$ is biconical. Hence $SS(F) \subset L$ by descent [KS90, Prop. 5.4.5]. Combining with (5.4), $j^*$ is essentially surjective. \qed

5.2. Restriction functors. The stack $g/G$ has a (weak) 1-shifted symplectic structure given by a non-degenerate bilinear pairing $\kappa$, hence $T_X^*(g/G) \simeq T_X(g/G)[-1] \simeq \{g \xrightarrow{[\cdot, \cdot]} g\}$ in degree 0,1.

Let $\mathcal{N} = \mathcal{N}_g \subset H^0(T^*(g/G))$ be the nilpotent cone, i.e., under the above identification $\mathcal{N}_{g,X} := \{Y \in g : [X,Y] = 0, Y \text{ is nilpotent}\}$. The definition does not depend on the choice of $\kappa$. We will use nilpotent cones lying in different cotangent bundles in the paper. When the context is understood, we shall drop the indices and write all nilpotent cones as $\mathcal{N}$.

Let $L \subset G$ be a Levi subgroup, denote by $\mathfrak{l}$ the Lie algebra of $L$. There are various restriction functors $R : Sh_{\mathcal{N}}(g/G) \rightarrow Sh_{\mathcal{N}_L}(l)$, however, all of them depends on some extra choices. In this section, we will study two kind of restriction functors: base restriction which depends on the choice of a base open set, and parabolic/hyperbolic restriction which depends on the choice of a parabolic subgroup. Finally, we prove that these two kinds of functors are isomorphic.

Remark 5.5. $Sh_{\mathcal{N}}(\mathfrak{l}/K)$ is by definition the category of character sheaves on $\mathfrak{l}$. Fourier transform gives an equivalence $Fr : Sh_{\mathcal{N}}(\mathfrak{l}/K) \xrightarrow{\sim} Sh(\mathcal{N}/K)$. The latter category is studied in the generalized Springer theory initiated in [Lus84]. For characteristic 0 coefficient, $Sh(\mathcal{N}/K)$ is explicitly calculated in [Rid13, RR14]. For characteristic $p$ coefficient, the abelian category $\text{Perv}(\mathcal{N}/K)$ is the subject of recent developed modular generalized Springer theory [AHJR13].

5.2.1. Base restriction.

Definition-Proposition 5.6. Let $L \subset K$ be reductive groups of the same rank (where rank is the dimension of a maximal torus). Let $\kappa$ be a non-degenerate invariant bilinear form on $\mathfrak{l}$.

1. The map induced by inclusion $f : l/L \rightarrow \mathfrak{l}/K$ is 1-shifted symplecto-

prrism with resepct to the shifted symplectic structures given by $\kappa$.

2. Denote $\mathfrak{l}' = \mathfrak{l}' := \{X \in l : C_K(X) = C_L(X)\}$, then $\mathfrak{l}'/L \rightarrow \mathfrak{l}/K$ is étale.
(3) $df|_{\mathfrak{f}/L}$ respects nilpotent singular support. I.e., for $Y \in \mathfrak{f}$, $\xi \in H^0(T_Y(\mathfrak{k}/K))$, we have $H^0(df^*_Y)(\xi) \in \mathcal{N}_I$ if and only if $\xi \in \mathcal{N}_I$.

(4) An base open subset of $\mathfrak{l}$ (with respect to $\mathfrak{k}$) is an $L$-invariant (analytic) open subset of $\mathfrak{l}$, such that $U$ is star-shaped centered at $C$ for some $C \in \mathfrak{c}_{l} := \text{center of } \mathfrak{l}$.

(5) For $U$ an base open subset, the pull back is an equivalence:

$$j^*_U : Sh^N_N(\mathfrak{k}/K) \xrightarrow{\sim} Sh^N_N(U/L)$$

(6) The base restriction with respect to $U$ is functor $R_U : Sh^N_N(\mathfrak{g}/G) \to Sh^N_N(\mathfrak{l}/L)$ defined by the following commutative diagram:

$$\begin{array}{ccc}
Sh^N_N(\mathfrak{k}/K) & \xrightarrow{R_U} & Sh^N_N(\mathfrak{l}/L) \\
\downarrow & & \downarrow \sim \\
Sh^N_N(\mathfrak{g}/G) & \xrightarrow{(f \circ j_U)^*} & Sh^N_N(U/L)
\end{array}$$

Proof. (1) follows from similar (and easier) argument as in Proposition 4.27.
(2) follows from (1) and Proposition 4.24.
(3) For any $Y \in \mathfrak{f}$, and under the identification by shift symplectic form, we have $H^0(df^*_Y)|_{\mathfrak{k}} = Id$.
(5) follows from Lemma 5.3 since $\mathcal{N}$ is biconical with respect to the $\mathbb{R}^+$ action on $\mathfrak{k}$ centered at $C$ and also the fact that this $\mathbb{R}^+$ action commute with conjugation by $K$. Note also that the set of all such $C$ is contractible. □

Remark 5.8. For non-empty base open subset to exist, we must have $\mathfrak{l} \cap \mathfrak{c}_l \neq \emptyset$, this implies $L = C_K(X)$, for any $X \in \mathfrak{l} \cap \mathfrak{c}_l$, and hence $L$ is actually a Levi factor of some parabolic subgroup.

Now we discuss the transitivity between base restrictions. Let $L \subset K \subset G$ sequence of reductive subgroups of same rank and $\mathfrak{l} \subset \mathfrak{f} \subset \mathfrak{g}$ the corresponding Lie algebra.

Proposition 5.9. (Transitivity of base restriction) Let $U_1$ be a base open subset of $\mathfrak{f}$ w.r.t $\mathfrak{g}$, and $U_2$ be a base open subset of $\mathfrak{l}$ w.r.t $\mathfrak{f}$, such that $U := U_1 \cap U_2$ is a base open subset of $\mathfrak{l}$ w.r.t. $\mathfrak{g}$. Then we have a natural isomorphism

$$\beta^Y_{U_1, U_2} : R_{U_2} \circ R_{U_1} \xrightarrow{\sim} R_U$$

Given by the following diagram:

$$\begin{array}{ccc}
Sh^N_N(\mathfrak{g}/G) & \xrightarrow{R_{U_2}} & Sh^N_N(U_2/L) \\
\downarrow & & \downarrow \sim \\
Sh^N_N(\mathfrak{f}/K) & \xrightarrow{(f \circ j_U)^*} & Sh^N_N(U_2/L) \\
\downarrow & & \downarrow \sim \\
Sh^N_N(\mathfrak{l}/L) & \xrightarrow{R_{U_1}} & Sh^N_N(U_1/K) \\
\downarrow & & \downarrow \circ \\
Sh^N_N(\mathfrak{k}/K) & \xrightarrow{R_U} & Sh^N_N(U/L)
\end{array}$$

There is a class of examples of base open subset given in the following:
Proposition 5.11. Let $T$ be a maximal torus of $L$, a completely invariant subset $U$ of $l$ is by definition a $L$-invariant open subset such that $L \cdot (U \cap t)$ is abundant in $U$.

(1) There is one to one correspondence between:
\[ \{ \text{completely open invariant subsets of } l \} \longleftrightarrow \{ W_L := W(L, T) \text{ invariant open subsets of } t \} \]

where $L \cap V := \chi^{-1}(V/W_L) \subseteq \text{minimal open subset containing } L \cdot V$.

(2) The correspondence in (1) preserve the property of being star-shape at $C \in c_l$. Hence for any $W_L$-invariant convex subset $V$ of $t \cap l$ with $V \cap c_l \neq \emptyset$, the set $lV$ is a base open subset of $l$.

In the rest of the paper, we shall only consider base open subsets of the form $L \cdot V$, and we shall write $R_V$ for $R_L \cdot V$.

5.2.2. Parabolic restriction. Let $P$ be a parabolic subgroup of $K$ with Levi factor $L \subseteq K$, we have

$$ l/L \leftarrow \frac{q}{P} \nrightarrow \frac{p}{\mathfrak{t}/K} $$

Definition 5.12. (1) The parabolic restriction with respect to $p$ is the functor

$$ \text{Res}_p := q \circ p^* : \text{Sh}(l/K) \longrightarrow \text{Sh}(l/L) $$

(2) In the setting of Corollary 3.8, denote the natural isomorphism by

$$ \alpha_{p*} : \text{Res}_p \circ \text{Res}_\mathfrak{t} \circ \sim \longrightarrow \text{Res}_q $$

which is induced by the base change isomorphism $\tilde{q}_2 ! \circ \tilde{p}_1 ! \sim p_1^* \circ q_2 !$.

The parabolic restriction preserves nilpotent singular support:

Proposition 5.13. Identify $\mathfrak{t}^* \simeq \mathfrak{t}$, $l^* \simeq l$ via $\kappa$.

(1) $\text{Res}_p$ naturally commute with Fourier transformation $Fr$:

$$ \text{Sh}(l/K) \xrightarrow{\text{Res}_p} \text{Sh}(l/L) $$

$$ Fr \sim \longrightarrow \quad Fr \sim \longrightarrow $$

$$ \text{Sh}(l/K) \xrightarrow{\text{Res}_p} \text{Sh}(l/L) $$

(2) $\text{Res}_p$ takes $\text{Sh}_{N^*}(l/K)$ to $\text{Sh}_{N^*}(l/L)$.

Proof. (1) is proved in [Mir04, Lemma 4.1].

(2) The first statement follows from the fact that $q(N_l \cap p) = N_l$. And the second statement follows from that the fact $Fr$ induces isomorphism $\text{Sh}(N/K) \sim \rightarrow \text{Sh}_{N^*}(l/K)$ by [KS90, Theorem 5.5.5].  \[\square\]
5.2.3. Base restriction and parabolic restriction are naturally isomorphic. We continue to use the notation as in last section.

**Proposition 5.14.** Let $U \subset V$ a invariant open subset, denote $^p U := q^{-1}(U) \subset p$.

1. The following diagram commutes, with left square Cartesian and $q_U$ isomorphism:

$$
\begin{array}{ccc}
U/L & \xrightarrow{i_U} & ^p U/P \\
\downarrow j_U & & \downarrow p_U \\
\downarrow \cong & & \downarrow \cong \\
\square & & \square \\
\end{array}
$$

Hence we have

$$
\text{Res}_p \simeq R_U : Sh_N(\mathfrak{g}/G) \rightarrow Sh_N(\mathfrak{g}/L) .
$$

**Proof.** Only need to show that $q_U$ (and hence $i_U$) is an isomorphism. \hfill \square

The isomorphism is also compatible with transitivity:

**Proposition 5.16.** Let $U_1$ be a base open subset of $\mathfrak{l}$ (w.r.t $\mathfrak{k}$), and $U_2$ be a base open subset of $\mathfrak{g}$ (w.r.t $\mathfrak{g}$) and assume that $U := U_1 \cap U_2$ is a base open subset of $\mathfrak{l}$ (w.r.t $\mathfrak{g}$). Then there is a natural isomorphism between the two triangles:

6. Complex gauge theory on $S^1$

Gauge theory on $S^1$ provides the simplest nontrivial example of gauge theories. In this section, we study the stack $G/G \simeq \text{Loc}_G(S^1)$ using the gauge uniformization $A(G)/\text{Aut}(G)$, for $G$ the trivial $G$-bundles on $S^1$. We use complex structure groups which in turn gives nilpotent codirections in $T^*\text{Loc}_G(S^1)$.

The charts in $A(G)/\text{Aut}(G)$ mapping to $G/G$ are a kind of exponential map. The exponential map plays important role in Lie theory, for example in Harish Chandra’s theory of harmonic analysis on (non-compact) semisimple Lie groups. We develop similar method of exponential maps for character sheaves. And one of our main theorems, Theorem 6.13, describing of the category character sheaves on $G/G$ as glued from character sheaves on various Lie algebras, can be viewed as Coxeter presentation of character sheaves on Lie groups.
6.1. **Gauge uniformization on circle.** We will establish the results in section 4 in the present situation. It can be viewed as a nonabelian analog of the uniformization \( \mathbb{C} \to \mathbb{C}^*/\mathbb{Z} \).

Denote by \( \mathcal{G} \) the trivial \( G \)-bundle on \( S^1 \), by \( \mathcal{A}(\mathcal{G}) \) the space of connections on \( \mathcal{G} \), and by \( \text{Conn}_G(S^1) \) the moduli space of smooth \( G \)-bundles on \( S^1 \) with connection. Since every \( G \)-bundle on \( S^1 \) is trivial, we have an isomorphism of groupoids \( \text{Conn}_G(S^1)(pt) = \mathcal{A}(\mathcal{G})/\text{Aut}(\mathcal{G}) \). We have an identification \( \text{Aut}(\mathcal{G}) \simeq C^\infty(S^1, G) =: L_{sm}G \). The trivial connection on \( \mathcal{G} \) gives \( \mathcal{A}(\mathcal{G}) \simeq \Omega^1(S^1, g) \). Fix \( z \in C^\infty(S^1, \mathbb{C}^*) \) a degree 1 map, such that \( dz \) is nowhere vanishing. (For example, take \( S^1 \) to be the unit circle with angle coordinate \( \theta \) and \( z = e^{i\theta} \).) Then we have a identification \( - \wedge d\log(z) : L_{sm}g := C^\infty(S^1, g) \simeq \Omega^1(S^1, g) \).

We have

\[
\text{Conn}_G(S^1)(pt) = L_{sm}G/L_{sm}G
\]

And the action above of \( L_{sm}G \) on \( L_{sm}g \) is identified with the gauge transformation:

\[
G g_A := g A g^{-1} - \frac{dg_A}{d\log(z)} g^{-1} \text{ for } g \in L_{sm}G, A \in L_{sm} \mathfrak{g}.
\]

We have \( X_\ast(T) \to L_{sm}T \) via \( \lambda \mapsto \lambda \circ z \), then \( t \subset L_{sm}t \) is stable under the gauge action of \( X_\ast(T) \) and the action is identified as translation under \( X_\ast(T) \to X_\ast(T) \otimes \mathbb{C} \simeq t \), where the last isomorphism is given by \( (\lambda, c) \mapsto d\lambda(c) \). Fix a lift of \( \text{set } W \to N_G(T) \) gives a lift \( W_{\text{aff}} \to L_{sm}T \). Again denote by \( w \) the lift of \( w \in W_{\text{aff}} \).

**Theorem 6.1.** For \( A \in t \subset L_{sm}g \), let \( G_A := C_{L_{sm}G}(A) \) and \( \mathfrak{g}_A := \text{Lie}(G_A) \).

For \( A = (A_i) \in t^n \), let \( G_A := \bigcap_{i=1}^n G_{A_i}, \mathfrak{g}_A := \bigcap_{i=1}^n \mathfrak{g}_{A_i}, \Phi_A := \bigcap_{i=1}^n \Phi_{A_i} \), and \( W_A := \bigcap_{i=1}^n W_{A_i} \).

1. \( G_A^\alpha = < T, \text{exp} \mathfrak{g}_A | \alpha \in \Phi_A > \).
2. \( G_A = < G_A^\alpha, w | w \in W_A > \), where \( W_A = C_{W_{\text{aff}}}(A) \).
3. \( \mathfrak{g}_A \) is stable under the gauge transformation of \( G_A \). The translation by \( -A \) gives an isomorphism of stacks \(-A : \mathfrak{g}_A/G_A \simeq \mathfrak{g}_A/adG_A, \) where the later action is adjoint action.
4. Let \( \chi_A : \mathfrak{g}_A \to \mathfrak{t}_A/W_A \) the characteristic polynomial map with respect to the gauge action. Let \( \mathfrak{t}_A^\text{ct} := \{ X \in \mathfrak{t} | W_X \subset W_A, \Phi_X \subset \Phi_A \} \), and \( \mathfrak{g}_A^\text{ct} := \chi_A^{-1}(\mathfrak{t}_A^\text{ct}/W_A) \). Then the natural map \( p_A : \mathfrak{g}_A^\text{ct}/G_A \to \text{Loc}_G(S^1) \) is (representable) étale.
5. Let \( S \subset t \) be a \( W_{\text{aff}} \)-invariant subset, for each \( A \in S \), let \( V_A \subset t_A \) be \( W_{\text{aff}} \)-invariant open subset, satisfying \( V_w(A) = w(V_A) \) for all \( w \in W_{\text{aff}} \). Let \( V_A := \bigcap_{A \in S} V_A, \) and \( U_A := \chi_A^{-1}(V_A/W_A) \), then we have a functor by sending \( A \) to \( U_A/G_A \) and \( pt \) to \( \text{Loc}_G(S^1) \):

\[
\begin{array}{ccc}
U : \int_{\Delta^*} S^* \times /W_{\text{aff}} & \longrightarrow & \text{Stack} \\
\end{array}
\]

Assume further more that \( V_A \subset t_A^\text{ct} \), and \( \bigcup_{A \in S} V_A = t \), then the induced map is an isomorphism:

\[
\text{colim}_{\Delta^*} S^*/W_{\text{aff}} U \longrightarrow \text{Loc}_G(S^1) \simeq G/G
\]

6. Assume that \( G \) is simply-connected, let \( t_{ij}^\text{ct} := t_{R,ij}^\text{ct} \times i t_{R} \subset t \), and \( \mathfrak{g}_{ij}^\text{ct} := \chi^{-1}(t_{ij}^\text{ct}/W_{ij}) \). Then \( \mathfrak{g}_{ij}^\text{ct}/G_J \to \text{Loc}_G(S^1) \) is open embedding, and the map

\[
\prod_{J \in \{ \text{vertices of } C \}} \mathfrak{g}_{ij}/G_J \longrightarrow \text{Loc}_G(S^1) \simeq G/G
\]

is surjective.
(7) Assume that $G$ is simply-connected, there is an isomorphism:

\[
\colim_{J \in \mathcal{F}_C} g^*_J / G_J \iso \text{Loc}_G(S^1) \simeq G / G
\]

**Proof.** The proof is similar to that of Section 4, we shall only highlight some difference in the present situation.

(1) We have \( g_A = \{X \in C^\infty(S^1, g) : dX + [A, X] \wedge d\log(z) = 0\} \). Let \( X = H + \sum_{\alpha \in \Phi} f_\alpha X_\alpha \), where \( H : S^1 \to t, f_\alpha : S^1 \to \mathbb{C} \). Then the equation \( dX + [A, X] \wedge d\log(z) = 0 \) is equivalent to \( dH = 0 \) and \( df_\alpha = \alpha(A) f_\alpha \wedge d\log(z) \). The first equation has solution constant functions. The second equation has nontrivial solution only when \( \alpha(A) \in \mathbb{Z} \), and in this case, the solutions are \( f_\alpha = cz^{\alpha(A)}, c \in \mathbb{C} \).

(2) Follow similarly from Lemma 4.9, with \( L_{hol} \) replaced by \( L_{sm} \).

(3) Follow from directly computation similar to 4.13. As a remark, the map \( -A : g_A / G_A \to g_{A/ad} G_A \) can be thought of as untwisting the gauge transformation. Since the gauge transformation is an affine linear action, and the action of \( G_A \) fix \( A \), so re-center the affine space \( g_A \) at \( A \) will make the action of \( G_A \) a linear action (in fact adjoint action).

(4) Similar to Section 4.2. The 1-shifted symplectic structure on \( \text{Loc}_G(S^1) \) is used.

(5) Follows from Proposition 2.26.

(6) (7) use Proposition 3.3, and arguments in Theorem 4.34. \( \square \)

### 6.2. Character sheaves on Lie groups.

Let \( N \subset T^* \text{Loc}_G(S^1) \) be the nilpotent cone.

**Proposition 6.2.** For the map \( p_A : g^*_A / G_A \to \text{Loc}_G(S^1) \), we have \( p^*_A(N) = N \).

**Proof.** Let \( x \in g_A \), then under \( p^*_A \), we have the identification \( N_{p^*_A(x)} = \{ \xi \in H^0(T^*(g_A / G_A)) : \xi(\theta) \text{ is nilpotent, for all } \theta \in S^1 \} \), then the proposition follows from the Lemma below. \( \square \)

**Lemma 6.3.** Let \( X \in g_A \), the following are equivalent:

1. \( X \) is nilpotent in \( g_A \).
2. \( X(\theta) \) is nilpotent in \( g \), for some \( \theta \in S^1 \).
3. \( X(\theta) \) is nilpotent in \( g \), for all \( \theta \in S^1 \).

**Proof.** This follows from the evaluation map \( ev_\theta : g_A \to g \) at \( \theta \) is injective for all \( \theta \in S^1 \), and the fact that the notation of nilpotent element is preserved under embedding of reductive Lie algebras of the same rank. \( \square \)

From Theorem 6.1(5)(7), we have:

**Proposition 6.4.** There is an equivalence:

\[
(1) \lim_{J \in \mathcal{F}_C} \pi_{S^1 / W_A} \text{Sh}_N(U_A / G_A) \iso \text{Sh}_N(G / G) \]

And for \( G \) simply-connected:

\[
(2) \lim_{J \in \mathcal{F}_C} \text{Sh}_N(g^*_J / G_J) \iso \text{Sh}_N(G / G)
\]

**Remark 6.5.** In the proof of above Proposition, we do NOT claim that for arbitrary diagram \( \mathcal{C} \), the statement \( \colim_{X \in \mathcal{C}} X \simeq X \) implies \( \lim_{X \in \mathcal{C}} \text{Sh}(X) \simeq \text{Sh}(X) \), even when all the morphisms envolved are \( \text{étale} \). Instead, we use Theorem 2.24 or Proposition 2.5.
We define the notion of base open subset, base restriction, parabolic restriction in Section 5.2 analogously in the twisted setting, and all statements there has analog to the untwisted case.

**Notation 6.6.** To give a functor $F : \mathcal{C} \to \mathcal{J}$, by definition we need to specify:

1. $F(c)$, for any $c$ object in $\mathcal{C}$;
2. $F(c \to c') : F(c) \to F(c')$, for any $c \to c'$ 1-arrow in $\mathcal{C}$;
3. $F((c' \to c'') \circ (c \to c')) : F(c' \to c'') \circ F(c \to c') \Rightarrow F(c \to c'')$, for any $(c' \to c'') \circ (c \to c') \Rightarrow (c \to c'')$ 2-arrow in $\mathcal{C}$;
4. and so on for higher morphisms.

When the higher morphism are understood, we shall only specify the 0 and 1 morphisms, and denote $F$ by $\{F(c), F(c \to c')\}_{c \in \mathcal{C}}$, and denote $\lim F$ by $\lim_{c \in \mathcal{C}} \{F(c), F(c \to c')\}$.

**Proposition 6.7.** (1) For any $J \to J'$, $\mathfrak{g}_{J'}^{se} \subset \mathfrak{g}_J$ is a base open subset w.r.t $\mathfrak{g}_J$, denote $R^J_1 := R^J_2 : \text{Sh}_N(\mathfrak{g}_J/G_J) \to \text{Sh}_N(\mathfrak{g}_J/G_{J'})$ the base restriction of $\mathfrak{g}_J^{se}$. Then there is an equivalence of categories:

$$\lim_{J \in \mathcal{F}_G} \{\text{Sh}_N(\mathfrak{g}_J/G_J), R^J_1\} \cong \text{Sh}_N(\text{Loc}_G(S^1))$$

See Notation 6.6 for the limit on left hand side. And the higher morphisms in $\mathcal{F}_G$ go to (higher) transivities between base restrictions as in Proposition 5.9.

(2) Let $\text{Res}^J_1 := \text{Res}_{J'}$, there is an equivalence of categories:

$$\lim_{J \in \mathcal{F}_G} \{\text{Sh}_N(\mathfrak{g}_J/G_J), \text{Res}^J_1\} \cong \text{Sh}_N(\text{Loc}_G(S^1))$$

where the higher morphisms in $\mathcal{F}_G$ go to (higher) transivities between parabolic restrictions as in Definition 5.12 (2).

**Proof.** (1) Let $c_{J'} = \{c \in \mathfrak{g}_{J'} : G_g(c) = c, \forall g \in G_J\}$ be the twisted center of $\mathfrak{g}_{J'}$. To show $\mathfrak{g}_{J'}^{se}$ is base open, by Proposition 5.11, suffices to show that: (i) $\mathfrak{g}_{J'}^{se} \subset \mathfrak{g}_{J'} := \{X \in \mathfrak{g}_{J'} : C^N_{\mathfrak{g}_J}(X) = C^N_{\mathfrak{g}_{J'}}(X)\}$, where $C^N$ is the stabilizer with respect to the Gauge transformation, (ii) $t^N_{\mathfrak{g}_{J'}} \cap c_{J'}^{se} \neq \emptyset$, and (iii) $t^N_{\mathfrak{g}_{J'}}$ is convex. Indeed (i) follows because $G_X \subset G_{J'}$ for any $X \in \mathfrak{g}_{J'}^{se}$. (ii) follows since $\emptyset \neq J' \subset t^N_{\mathfrak{g}_{J}} \cap c_{J'}^{se}$. (iii) $St_{J'}$ is convex. Now the equivalence follows from Proposition 6.4 (2).

(2) This follows from twisted version of Proposition 5.16.

### 6.2.1. Untwist the gauge transformation.

As in Theorem 6.1 (3), for single stack, the gauge action is isomorphic to the usual adjoint action. In this section, we show that the entire diagram in Proposition 6.7 (2) can be untwisted simultaneously, i.e it is isomorphism to the corresponding diagram of adjoint quotients. Note that this can only be done for nilpotent sheaves, and there is no such simultaneous untwist at the level of algebraic stacks.

We are going to use a special case of the following definition:

**Definition 6.8.** A pair $(X, \Lambda_X)$ consists of a smooth stack $X$ with $\Lambda_X \subset T^*X$ a closed conical substack.

1. A map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a map $f : X \to Y$, such that $f$ is non-characteristic w.r.t. $\Lambda_Y$ and $f^*(\Lambda_Y) \subset \Lambda_X$. 

---

**}}
(2) A \textit{U-family of maps} $F$ is a map of pair

$$F : U \times (X, \Lambda_X) := (U \times X, T_U^0 U \times \Lambda_X) \longrightarrow (Y, \Lambda_Y).$$

**Proposition 6.9.** Assume $U$ is a smooth manifold, and $F$ as above, then there is induced map of categories

$$F^* : U \longrightarrow [\text{Sh}_{\Lambda_Y}(Y), \text{Sh}_{\Lambda_X}(X)]$$

where $U$ is regarded as a topological space (or $\infty$-groupoid).

The following is a special case of the proposition above we are going to use:

**Proposition 6.10.** Denote similarly $c'_{gA} := \{c \in g_A : GgG(c) = c, \forall g \in G_A\}$ to be the twisted center of $g$. Then

$$-A : c'_{gA} \times (g_A/G_A, N) \longrightarrow (g_A/adG_A, N)$$

$$(c, a) \longrightarrow a - c$$

is a family of maps, hence there is induced map:

$$-A^* : c'_{gA} \longrightarrow [\text{Sh}_N(g_A/G_A), \text{Sh}_N(g_A/adG_A)]$$

Since the space $c'_{gA}$ is contractible, $-A^*$ can be regarded as a canonically defined functor, still by the same notation $-A^* : \text{Sh}_N(g_A/G_A) \rightarrow \text{Sh}_N(g_A/adG_A)$. Also denote by $-J^* : \text{Sh}_N(g_J/G_J) \rightarrow \text{Sh}_N(g_J/adG_J)$.

**Proposition 6.11.** $-J^*_{\{J \in F_C\}}$ defines an natural isomorphism of functors:

$$-J^*_{\{J \in F_C\}} : \{\text{Sh}_N(g_J/G_J), \text{Res}_J^J\}_{J \in F_C} \longrightarrow \{\text{Sh}_N(g_J/adG_J), \text{Res}_J^J\}_{J \in F_C}.$$\[Proof.\] For $J \rightarrow J'$, we have $c'_{gJ} \subset c'_{pJ} \subset c'_{gJ'}$. For any $c \in c_{gJ}$, $-c^*$ naturally commute with $\text{Res}_J^J$, because we have naturally commutative diagram:

$$\begin{array}{ccc}
\mathfrak{g}_J/G_J & \xleftarrow{q} & \mathfrak{p}_J/G_J \\
\cong & \downarrow{e} & \cong \\
\mathfrak{g}_J/adG_J & \xrightarrow{q} & \mathfrak{p}_J/adG_J
\end{array}$$

Now $-J^*_{\{J \in F_C\}}$ is well defined because $c'_{gJ}$ is contractible, different choice of $c$ are canonical isomorphic. The Proposition follows since each $-J^*$ is an equivalence. \qed

**Remark 6.12.** The untwisting can be thought of as the analogy of the following statement in the abelian situation $G = \mathbb{C}^*$: the gauge action of $\mathbb{Z} \cong X_*(\mathbb{C}) \subset L\mathbb{C}^*$ on $\mathbb{C} \subset L\mathbb{C}$ is identified with the translation of $\mathbb{Z}$ on $\mathbb{C}$ (up to $2\pi i$), whereas the adjoint action of $\mathbb{Z}$ is identified with the trivial action. The untwisting gives

$$(\text{Loc}(\mathbb{C}^*) \cong) \text{Loc}(\mathbb{C}/\mathbb{Z}) \cong \text{Loc}(\mathbb{C}/ad\mathbb{Z}) (\cong \text{Loc}(\mathbb{C} \times BZ) \cong \text{Loc}(BZ)).$$

Note that as an analytic stack, $\mathbb{C}^* \simeq \mathbb{C}/\mathbb{Z} \neq \mathbb{C}/ad\mathbb{Z} \cong \mathbb{C} \times BZ$.

Combining Proposition 6.7 (2) and 6.11, we get:

**Theorem 6.13.** There is an equivalence of category:

$$\lim_{J \in F_C} \{\text{Sh}_N(g_J/adG_J), \text{Res}\} \longrightarrow \text{Sh}_N(G/G).$$
Example 6.14. For $G = SL_2$, identifying $X_c(T) \subset t_\mathbb{R}$ as $\mathbb{Z} \subset \mathbb{R}$, and take the alcove $C = (0, 1/2) \subset t_\mathbb{R}$ we have

$$Sh_N(G/G) = \lim_{\to v_0, v_1} \text{Res}_{v_0, v_1} Sh_N(t/_{ad}T) \text{ Res}_{v_0, v_1} Sh_N(g_0/_{ad}G_0) \to \text{Res}_{v_0, v_1} Sh_N(g_1/_{ad}G_1/2)$$

If the coefficient $k = \mathbb{C}$, the above diagram for perverse sheaves can be explicitly calculated as:

$$\text{Perv}_N(G/G, \mathbb{C}) = \lim_{\to v_0, v_1} \text{Res}_{v_0, v_1} Sh_N(t/_{ad}T) \text{ Res}_{v_0, v_1} Sh_N(g_0/_{ad}G_0) \to \text{Res}_{v_0, v_1} Sh_N(g_1/_{ad}G_1/2)$$

Restriction functors. We have similar statements as in Section 5.2 for the Lie group case. We shall omit some repeated proofs.

The following Proposition is a Lie group version of Proposition 5.6 (5).

Proposition 6.15. Let $G$ be a reductive group, and $c \in Z(G)$, and $V$ be a $W$-invariant open subset of $T$ containing the maximal compact torus, and such that $\tilde{V} := \text{Exp}^{-1}(V) \subset X_c(T) \oplus \mathbb{C}$ is convex. Then the restriction

$$Sh_N(G/G) \xrightarrow{\sim} Sh_N(GcV/G)$$

is an equivalence, where $cV$ is the translation of $V$ by $c$.

Proof. Since $\mathcal{N} \subset T^*(G/G)$ is invariant under translation by central elements, it is suffice to assume $c = 1$. In Proposition 6.4 (1), we could choose $S$ as a subset of $t_\mathbb{R}$ and $U_s$ small so that $U_s/G_s \to G/G$ is an open embedding. Then restriction map $Sh_N(G/G) \to Sh_N(GcV/G)$ is induced by taking the limit over $\int_{\Delta_s} S^*/W_{aff}$ of the restriction $Sh_N(U_s/G_s) \to Sh_N((U_s \cap Gs\tilde{V})/G_s)$, which is an isomorphism, since $U_s \cap Gs\tilde{V}$ is again star shape centered at $s$.

Definition-Proposition 6.16. Let $L \subset K$ be reductive groups of the same rank (where rank is the dimension of a maximal torus). Let $\kappa$ be a non-degenerate invariant bilinear form on $\mathfrak{t}$.

1. The map induced by inclusion $f : L/L \to K/K$ is 1-shifted symplectomorphism with respect to the shifted symplect structures given by $\kappa$.
2. Denote $L' = L_K := \{X \in L : C_K(X) = C_L(X)\}$, then $L'/L \to K/K$ is étale.
3. $df^*_L|_{L'/L}$ respects nilpotent singular support. I.e, for $Y \in L'$, $\xi \in H^0(T_Y(K/K))$, we have $H^0(df^*_L)(\xi) \in N_\mathfrak{t}_K$ if and only if $\xi \in N_K$.
4. An base open subset of $L$ (with respect to $\mathfrak{t}$) is an $L$-invariant (analytic) open subset of $L'$ of the form $LcV$ as in Proposition 6.15 (for $G = L$).
(5) Let $U$ be a base open subset of $L$, the \emph{base restriction} with respect to $U$ is functor $R_U : Sh_{\mathcal{A}}(K/K) \to Sh_{\mathcal{A}}(L/L)$ defined by the following commutative diagram:

\[
\begin{array}{c}
Sh_{\mathcal{A}}(K/K) \\
\downarrow^{R_U} \\
Sh_{\mathcal{A}}(L/L) \xrightarrow{j^*_U} Sh_{\mathcal{A}}(U/L)
\end{array}
\]

\[\text{Proof.} \quad \text{This is similar to Proposition-Definition 5.6.}\]

**Proposition 6.17.** \emph{(Transitivity of base restriction)} Let $U_1$ be a base open subset of $\mathfrak{k}$ w.r.t $\mathfrak{g}$, and $U_2$ be a base open subset of $\mathfrak{l}$ w.r.t $\mathfrak{g}$, such that $U := U_1 \cap U_2$ is a base open subset of $\mathfrak{l}$ w.r.t. $\mathfrak{g}$. Then we have a natural isomorphism

\[\beta^{U_1, U_2} : R_{U_2} \circ R_{U_1} \cong R_U \]

\[\text{Given by the following diagram:}
\]

\[
\begin{array}{c}
Sh_{\mathcal{A}}(G/G) \\
\downarrow \\
Sh_{\mathcal{A}}(K/K) \xrightarrow{\sim} Sh_{\mathcal{A}}(U_1/K) \\
\downarrow \\
Sh_{\mathcal{A}}(L/L) \xrightarrow{\sim} Sh_{\mathcal{A}}(U_2/L) \xrightarrow{\sim} Sh_{\mathcal{A}}(U/L)
\end{array}
\]

Let $P$ be a parabolic subgroup of $K$ with Levi factor $L \subset K$, we have

\[L/L \xleftarrow{q} P/P \xrightarrow{p} K/K\]

**Definition 6.19.**

(1) The \emph{parabolic restriction} with respect to $P$ is the functor

\[\text{Res}_P := q_! p^* : Sh_{\mathcal{A}}(K/K) \longrightarrow Sh_{\mathcal{A}}(L/L)\]

(2) In the setting of Corollary 3.8, denote the natural isomorphism by

\[\alpha^{P, R} : \text{Res}_P \circ \text{Res}_R \cong \text{Res}_Q\]

which is induced by the base change isomorphism $\tilde{q}_2 ! \circ \tilde{p}_1^* \cong \tilde{p}_1^* \circ q_2 !$.

**Proposition 6.20.**

(1) \emph{There is natural isomorphism}:

\[\text{Res}_P \simeq R_U : Sh_{\mathcal{A}}(K/K) \longrightarrow Sh_{\mathcal{A}}(L/L)\]

(2) \emph{Let $U_1$ be a base open subset of $L$ (w.r.t $K$), and $U_2$ be a base open subset of $K$ (w.r.t $G$ ) and assume that $U := U_1 \cap U_2$ is a base open subset of $L$}
Dependence of restriction functors on parabolic subgroups. Results in this section will not be used later. The main results here is Proposition 6.22, which follows immediately from Proposition 6.20. However, we want to proceed to explain the problem in a more natural point of view that compatible with other previous approaches.

In the theory of finite groups, let \( f : A \hookrightarrow B \) be an inclusion of finite groups. A useful tool is the induction/restriction of characters:

\[
\begin{array}{c}
\mathbb{C}[A/A] \xrightarrow{f_*} \mathbb{C}[B/B] \\
\end{array}
\]

Now let \( G \) be a reductive Lie group, \( L \subset G \) a Levi. It turns out that direct induction/restriction between \( G \) and \( L \) as in finite group case does not behave well. To correct it, the idea is to use an intermediate parabolic subgroup \( P \). And define the parabolic induction/restriction in various context using the diagram

\[
\begin{array}{c}
L \xleftarrow{q} P \xrightarrow{p} G \\
\end{array}
\]

It’s natural to ask what is the dependence of the resulting restriction/induction on the choice of parabolic subgroups. One heuristic reason the restriction (pull back) along \( f : L/L \rightarrow G/G \) does not behave well is that the map \( f \) is not étale (nor smooth). Nevertheless, the map \( f|_{L'} : L'/L \rightarrow G/G \) is étale, so when restricted to \( L' \), the “correct” restriction functor should agree with \( f|_{L'} \). And we are done if we could recover the restriction functor from its information on \( L' \). In the setting of perverse character sheaves, this is what happens, essentially as explained in [Gin93]:

Proposition 6.21. The bottom horizontal arrow is fully faithful and the triangle is naturally commutative.

\[
\begin{array}{c}
\text{Perv}_\mathcal{N}(G/G) \xrightarrow{\text{Res}_P} \text{Perv}_\mathcal{N}(L'/L) \\
\end{array}
\]

In particular, \( \text{Res}_P \) is the unique (up to canonical isomorphism) functor making the diagram commutative.
connected). So two parabolic restrictions are canonical isomorphic since there is a canonical choice of base open subset, namely the largest one $L'$. However, at the level of $\infty$-categories, $L'$ is not a base open subset since the map $H^*(L) \to H^*(L')$ is not an isomorphism. (This is more obvious for Lie algebras, where $I$ is contractible while $I'$ is not.) Nevertheless, we could still use other base open subset to get:

**Proposition 6.22.** Let $P_1, P_2$ be two parabolic subgroup of a reductive group $G$ with the same Levi factor $L$, then there is an isomorphism of functors (depending on a choice of base open subset of $L$ w.r.t $G$):

$$\text{Res}_{P_1} \simeq \text{Res}_{P_2} : Sh_{\mathcal{N}}(G/G) \longrightarrow Sh_{\mathcal{N}}(L/L)$$

**Proof.** This is immediate from Proposition 6.20 (1) since we have $\text{Res}_{P_1} \simeq R_U \simeq \text{Res}_{P_2}$. □

**Remark 6.23.** We leave it to the reader to show that for Levi $L \subset G$ there exists a base open subset. Hence at least $\text{Res}_{P_1} \simeq \text{Res}_{P_2}$ as functors. Note that base open does not always exist for general $H \subset G$ of maximal rank. A counterexample is that for $H = SL_2 \times SL_2 \subset Sp_4 = G$, there is no base open subset of $H$ w.r.t $G$.

Since the choice of base open subset is not canonical, it is natural to understand the space of choices. This is more clear in the situation of Lie algebras. Let $p_1, p_2$ be two parabolic subalgebra of $\mathfrak{g}$ with Levi $I$. The space of choice of a base open subset of $I$ w.r.t $\mathfrak{g}$ is $c^I$. Indeed, for any $x \in c^I$, we could choose a small base open $U_x$ near $x$, and for $x, y$ close enough and $U_x, U_y$ small enough, we could choose a $U_{x,y}$ base open containing $U_x, U_y$. Hence we have constructed

**Proposition 6.24.** Regard $c^I$ as an $\infty$-groupoid. There is an morphism of categories

$$c^I \longrightarrow [\text{Res}_{P_1}, \text{Res}_{P_2}]$$

**Remark 6.25.**

1. Pick any $x \in c^I$, it gives Lie algebra version of Proposition 6.22.

2. Under Fourier transform, for orbital sheaves, such morphism is constructed in [Mir04] using nearby cycle functor of the family given by characteristic polynomial map. Note that the same choice $c^I$ is implicit in the proof.

The same approach does not apply to the elliptic situation, we don’t know yet if the parabolic restriction for nilpotent sheaves on $G_E$ are isomorphic for different choice of parabolics. The problem is that $E$ is compact, restricting to any (proper) open subset will miss the top cell, hence there is no base open subset in this case. This is contrary to the case for $\mathbb{C}^*$ or $\mathbb{C}$ where one could restrict to a smaller open subset where the relevant maps behave well while still retains the topology. We will understand this question for $E$ in a future paper via a different method.

### 7. Holomorphic gauge theory on elliptic curves

7.1. **Twisted conjugation of loop group revisited.** We continue to use the notation Section 4. Recall that $t_2 \simeq t_2 \circ \text{Exp} \longrightarrow T$. We shall identify $t_2$ with its image in $T$. Most of section 6 has direct analog with minor changes, we shall not repeat the proof but only state the facts.
7.1. Base and parabolic restriction.

**Definition-Proposition 7.1.** (1) For any $J \to J'$ in $\mathcal{F}_C$, the nilpotent cones correspond under the map $f : G_J^T/G_J \to G_J/G_J$ and the map $G_J^T/G_J \to G_E$.

(2) $G_J^T$ is base open subset of $G_J$ with respect to $G_J$, denote its base restriction by $R_J$.

(3) Denote by $\text{Res}_{J'}^J := \text{Res}_{J'}$.

From Theorem 4.41, and similarly to Proposition 6.7, we have:

**Proposition 7.2.** There are equivalences of categories:

1. $\lim_{J \in \mathcal{F}_C} \{\text{Sh}_N(G_J/G_J), R_J\} \xlongleftarrow{\sim} \text{Sh}_N(G_E)$

2. $\lim_{J \in \mathcal{F}_C} \{\text{Sh}_N(G_J/G_J), \text{Res}_J^J\} \xlongleftarrow{\sim} \text{Sh}_N(G_E)$

7.1.2. Untwisting the twisted conjugation. $J \subset Z(G_J)$ and hence induces $-J^* : \text{Sh}_N(G_J/G_J) \to \text{Sh}_N(G_J/adG_J)$.

**Proposition 7.3.** $-J^* \{J \in \mathcal{F}_C\}$ induces a natural isomorphism between the functors $\{\text{Sh}_N(G_J/G_J), \text{Res}\}$ and $\{\text{Sh}_N(G_J/adG_J), \text{Res}\}$

The following is the main theorem of this paper:

**Theorem 7.4.** There is an equivalence of categories:

$\lim_{J \in \mathcal{F}_C} \{\text{Sh}_N(G_J/adG_J), \text{Res}_J^J\} \xlongleftarrow{\sim} \text{Sh}_N(G_E)$.

7.2. Holomorphic gauge uniformization. Let $\omega = (\omega_1, \omega_2)$ be a pair of complex numbers not contained in the same real line. $E = E_\omega := \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ an elliptic curve. As we shall established below, similar to Section 6.1, the holomorphic gauge uniformization gives an nonabelian analog of the uniformization $\mathbb{C} \to E$. The proofs are similar, and we will only state facts.

Denote by $G$ the trivial smooth $G$-bundle on $E$, by $\mathcal{A}^0(E)$ the space of $(0,1)$-connections on $G$. Any such connection $\nabla$ defines a holomorphic structure on $G$ by defining the holomorphic sections are those section $s$ satisfying $\nabla(s) = 0$.

Since every degree 0 holomorphic $G$-bundle on $E$ is trivial as smooth bundle, we have an isomorphism of groupoids $\text{Bun}^0_G(E)(pt) = \mathcal{A}^0(G)/\text{Aut}(G)$. We have an identification $\text{Aut}(G) \simeq C^\infty(E, G)$. The $\tilde{\partial}$ operator and the $(0,1)$-form $\bar{\partial}z$ give identifications $\mathcal{A}^0(E) \simeq C^0(E, g)$. Hence we have $\text{Bun}^0_G(E)(pt) = C^\infty(E, g)/C^\infty(E, G)$.

And the action above is identified with the Gauge transformation: $Gq^{-1}_g(B) := gBg^{-1} - \tilde{\partial}g \cdot g^{-1}$ for $g \in C^\infty(E, G)$, and $B \in C^\infty(E, g)$.

Let $S_1, S_2$ be two copies of the unit circle. We have isomorphism of Lie groups $S_1 \times S_2 \cong E$, by $(\theta_1, \theta_2) \mapsto \frac{\theta_1 + \omega \theta_2}{2\pi}$. This induces $X_s(T) \times X_s(T) \simeq \text{Hom}_{\text{Lie}}(E, T)$. An easy calculation shows that under the identification $t_r \times t_r \sim t$, $(A_1, A_2) \mapsto -\frac{2\pi i}{\omega_1 \omega_2 - \omega_1 \omega_2}(\omega_2 A_1 - \omega_1 A_2)$

The translation of $X_s(T) \times X_s(T)$ on $t_r \times t_r$ is identified with the gauge transformation of $\text{Hom}_{\text{Lie}}(E, T) \subset C^\infty(E, T)$ on $t \subset C^\infty(E, t)$ (as constant maps). Let
Proof. By Theorem 7.7.

Remark 7.6. Let \( P \) be a point in \( \mathfrak{e}_E \), assume that \( \text{Aut}(P) \) (resp. \( \text{Aut}(P^{ss}) \), where \( P^{ss} \) is the semisimplification of \( P \)) is connected. Then \( \mathfrak{e}_E \) is smooth at \( P \).

Proof. By Theorem 7.5, near \( P \), the stack \( G_E \) is locally isomorphic to the quotient stack \( \text{Lie}(\text{Aut}(P))/\text{Aut}(P) \) near 0. When \( \text{Aut}(P) \) is connected reductive, we know the coarse moduli of the later stack (i.e. the affine quotient) is smooth (in fact, an affine space). \( \square \)

Remark 7.7. (1) By a theorem of Looijenga, \( \mathfrak{e}_E \) is isomorphic to a weighted projective space (with explicit weights depending on the root datum), and it is not always smooth.

(2) It is possible to deduce Corollary 7.6 from some general slicing theorem such as in [AHR15].
7.2.1. Relation with gauge uniformization on circle.

**Notation 7.8.** We denote by $G_i^A, \Phi_i^A, W_i^A$ the corresponding notation associated to $S_i, i = 1, 2$.

The inclusion \( \{0\} \hookrightarrow S_1 \hookrightarrow S_2, \) induces \( C^\infty(E_\omega, G) \rightarrow C^\infty(S_1, G) \rightarrow C^\infty(S_2, G) \).

**Proposition 7.9.** Under the above map, let $B = (A_1, A_2)$ we have

\[
\begin{array}{ccc}
G_B^\omega & \hookrightarrow & G_1^A \\
\downarrow & & \downarrow \\
G_2^A & \hookrightarrow & G \\
\end{array}
\]

and all the arrows are injective.

**Proof.** It is easy to check that

\[
\begin{array}{ccc}
W_B & \hookrightarrow & \Phi_B \\
\downarrow & & \downarrow \\
W_1^A & \hookrightarrow & \Phi_1^A, \\
\downarrow & & \downarrow \\
W_2^A & \hookrightarrow & \Phi_2^A, \\
\downarrow & & \downarrow \\
W & \hookrightarrow & \Phi \\
\end{array}
\]

The proposition follows since the groups involved are determined by the above data. \(\square\)

**Remark 7.10.** Let $G_c$ be a maximal compact subgroup of $G$. Under Yang-Mills equation, this Proposition can be thought of as an analogue of the fact that for a $G_c$-local system $L$ on $E$, we have

\[
\begin{array}{ccc}
\text{Aut}(L) & \hookrightarrow & \text{Aut}(L|_{S_1}) \\
\downarrow & & \downarrow \\
\text{Aut}(L|_{S_1}) & \hookrightarrow & \text{Aut}(L|_{S_2}) \\
\downarrow & & \downarrow \\
\text{Aut}(L|_0) = G_c \\
\end{array}
\]

Note that both $\text{Aut}(L|_{S_i})$ and $G_i^A$ are of the form $C_G(s)$ for some $s \in T_c$ (or $T$). In particular, they are connected if $G$ is simply-connected.
8. Remarks

8.1. Stratification of compact group. Lusztig stratification is commonly used in the study of geometry of $G/G$. We explain its relation with our charts when restricted to a maximal compact subgroup.

Let $G_c$ be a almost simple and simply-connected compact Lie group, and $T_c \subset G_c$ a maximal compact torus. Choose an alcove $c$ in $X_*(T_c) \otimes \mathbb{R}$, we define open cover $\mathcal{C}$ of $G_c$ similarly by $\mathcal{C} = \{G_{c,J} : J \in \text{vertices of } C\}$. Identifying the cover with its image, and the cover does not depend on the choice of $T_c$ and $C$, hence the cover is intrinsic associated to $G_c$. Denote by $\mathcal{I}$ the finest stratification of $G_c$ generated by $\mathcal{C}$ via taking complement and intersection. Then $\mathcal{C}$ can also be recovered from $\mathcal{I}$: a chart in $\mathcal{C}$ is the union of all strata whose closure containing a fixed closed stratum in $\mathcal{I}$. It is clear from the definition that $\mathcal{C}$ and $\mathcal{I}$ are conjugation invariant. The stratification $\mathcal{I}$ can also be described more explicitly:

**Proposition 8.1.** $\mathcal{I} = \{G_c(\text{Exp}(J)) : J \in \text{faces of } C\}$.

Now let $G$ be the complexification of $G_c$, then $G$ has a Lusztig stratification $\mathcal{L}$ by conjugation invariant subvarieties. Let $\mathcal{L}_c$ denote the induced stratification on $G_c$. Note that even each stratum in $\mathcal{I}$ is connected, its intersection with $G_c$ may not be connected.

**Proposition 8.2.** Strata in $\mathcal{I}$ are precisely the connected components of strata in $\mathcal{L}_c$.

**Example 8.3.** For $G_c = SU(3), G = SL(3, \mathbb{C}), \mathcal{L} = \{(\text{connected components of}) \ L_\lambda : \lambda \text{ a partition of } 3\},$ where $L_\lambda = \{g \in G : g^{ss} \text{ has eigenvalue of type } \lambda\}$. The stratum $L_{(2,1)}$ is connected. However $L_{(2,1)} \cap G_c = \coprod_{k=0,1,2} S_k$ has three connected component, where $S_k = \{g \in G_c : g \text{ has eigenvalues } \{a, a, a^{-2}\}, a = e^{2\pi i/3}, \text{ and } \theta \in (k, k + 1)\}$. And $\mathcal{I} = \{\{I\}, \{e^{2\pi i/3}I\}, \{e^{4\pi i/3}I\}, \{e^{4\pi i/3}I\}, \{S_0, S_1, S_2, G_{c,J}\}\}$ consists of 7 strata.

The closed strata in $\mathcal{I}$ (or $\mathcal{L}_c$) are precisely the isolated conjugacy classes in $G_c$, they are in bijection with the vertices of $C$. For a vertex $v$, the corresponding conjugacy class is isomorphic to $G/C_G(\text{Exp}(v))$. For type $\Lambda$, the isolated conjugacy classes are central elements, hence discrete. For other type, the isolated class corresponds to a non-special vertex has positive dimension.

8.2. Nonabelian Weierstrass $\wp$-function. We have understand the nonabelian analog of $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ and $E = \mathbb{C}^*/q^\mathbb{Z}$, we also describe the nonabelian analog of view $E$ as of a cubic equation $y^2 = 4x^3 - g_2x - g_3$ birationally.

Let $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} + \mathbb{Z} \tau$, recall the $\wp$-function is defined as a $\Lambda$-invariant meromorphic function on $C$:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right)$$

**Definition 8.4.** The nonabelian $\wp$-function and its derivative is defined as the following meromorphic functions $\mathfrak{gl}_n \to \mathfrak{gl}_n$:

$$\wp(Z) = \frac{1}{Z^2} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{(Z + \omega I)^2} - \frac{1}{(\omega I)^2} \right)$$
\[ \wp'(Z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(Z + \omega I)^3} \]

We shall consider \( G = GL_n \), recall that \( GL_{n,E,0} \) is representable by a smooth algebraic variety, and there is a map \( p : GL_n \to GL_{n,E,0} \). For \( n = 1 \), \( GL_{1,E,0} = \text{Pic}^0(E) \cong E \), and the map \( p \) is identified as \( C^* \to C^*/e^{2\pi i}Z = E \).

**Theorem 8.5.**

1. Under the maps \( \mathfrak{gl}_n \xrightarrow{\text{Exp}} GL_n \xrightarrow{p} GL_{n,E,0} \), the function \( \wp(Z) \) and \( \wp'(Z) \) descend to a rational function on \( GL_{n,E,0} \).
2. The map \( (\wp, \wp') : GL_{n,E,0} \to \mathfrak{gl}_n \times \mathfrak{gl}_n \), defines a birational isomorphism between \( GL_{n,E,0} \) and the subvariety:

   \[ \{(X,Y) \in \mathfrak{gl}_n \times \mathfrak{gl}_n : [X,Y] = 0, \text{ and } Y^2 = 4X^3 - g_2X - g_3 \} \]

   where \( g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4} \) and \( g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6} \).

**Remark 8.6.** It is more complicated to (partially) compactify the image of the above rational map to give an actual isomorphism. So far we have only use a single chart, and the various other charts may be useful for this purpose.

**Appendices**

**A. Analytic stacks**

Let \( \text{ComSp} \) be the site of complex spaces, where the coverings are collection of étale (:= locally biholomorphic) maps \( \{U_i \to X\}_{i \in I} \), such that \( \coprod_{i \in I} U_i \to X \) is surjective. A presheaf is by definition a functor \( \mathcal{X} : \text{ComSp} \to \text{Grpd} \), and a stack is a presheaf which is a sheaf. An morphism \( f : \mathcal{X} \to \mathcal{Y} \) between stacks is representable if for any morphism from a complex space \( g : Y \to \mathcal{Y} \), \( Y \times_{\mathcal{Y}} \mathcal{X} \) is representable by complex space.

A morphism between two complex spaces is smooth if it is locally isomorphic to \( U \times D^n \to U \), where \( D \) is the unit disk. A representable morphism between two stacks is called smooth if it is so after base change from any complex space. Let \( P \) be a property of morphism that is stable in smooth topology on \( \text{ComSp} \), we say a representable morphism has property \( P \) if it has property \( P \) after base change by smooth morphism from any complex space. Such properties include being surjective, étale, smooth, closed embedding, open embedding, open dense embedding, isomorphism. It’s not hard to see that the two definition of being smooth of a representable morphism agrees.

An analytic stack is by definition a stack \( \mathcal{X} \) such that:

1. \( \Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X} \) is representable, and
2. There is smooth morphism surjective \( \pi : X \to \mathcal{X} \), where \( X \) is a complex space.

An analytic stack is smooth if in the above definition, \( X \) is a complex manifold. To study smooth analytic stack \( \mathcal{X}, \mathcal{Y} \) and smooth morphism between them, it suffices to study functor \( \mathcal{X}, \mathcal{Y} \) after restriction to \( \text{Cplx} \) the site of complex manifolds.

**Definition A.1.** \( f : \mathcal{X} \to \mathcal{Y} \) is generically open if there is \( \mathcal{U} \subset \mathcal{X} \) open dense embedding, such that \( f|_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y} \) is open embedding.
Lemma A.2. An étale and generically open morphism of analytic stacks is open embedding.

Notation A.3. For a groupoid $\mathcal{C}$, let $|\mathcal{C}|$ be the set of isomorphism classes in $\mathcal{C}$. For a stack $\mathcal{X}$, Denote $|\mathcal{X}| := |\mathcal{X}|(\mathbb{C})$. It has a natural topology coming from (representable) étale morphisms.

Let $x \in \text{Ob}(\mathcal{X}(\mathbb{C}))$, we say $f$ is étale at $x$ if for any base change $Y \to \mathcal{Y}$, $f' : X := \mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is étale at any point $x' \in X$ over $x$. By definition, étaleness only depends on isomorphism class of $x$, so we can also speak about $f$ being étale at $x \in |\mathcal{X}|$. We have the locus in $|\mathcal{X}|$ where $f$ is étale is open. It’s easy to see that $f$ is étale if and only if it is étale at every $|\mathcal{X}|$.

A.1. Tangent groupoid and tangent complex. $T_x\mathcal{X}$ the tangent groupoid at $x$ is defined to be the fiber category of $\sigma : \mathcal{X}[\mathbb{C}[[t]]] \to \mathcal{X}(\mathbb{C})$ over $x$. i.e. $\text{Ob}(T_x\mathcal{X}) := \{(v, \phi) : v \in \mathcal{X}(\mathbb{C}[[t]]), \phi : \sigma(v) \sim x\}$, and morphism are those induces identity on $x$. There is an action of $\text{Aut}(x)$ on $T_x\mathcal{X}$ via $(v, \phi) \to (v, g \circ \phi)$ We have natural map of groupoid $df_x : T_x\mathcal{X} \to T_{f(x)}\mathcal{Y}$ by postcomposition with $f$. The map interwine the action of $\text{Aut}(x)$ and $\text{Aut}(f(x))$. By base change, We have

Proposition A.4. Assume $\mathcal{X}, \mathcal{Y}$ smooth analytic stacks, then $f$ is étale at $x$ if and only if $df_x$ is an equivalence.

For smooth analytic stack $\mathcal{X}$, the tangent groupoid $T_x\mathcal{X}$ has a natural structure of category in vector spaces such that the commutativity constraint is trivial. Such datum is equivalent to complex of vector spaces in degree -1,0. The assignment is by associate such a category $\mathcal{C}$ to $H^{-1} \to H^0$ where the differential is trivial and $H^0(\mathcal{C}) := \text{isomorphism classes of objects in } \mathcal{C}$, and $H^{-1}(\mathcal{C}) := \text{the automorphisms of identity object}$. Note that both of them have vector space structures. Under this assignment, $\text{Aut}(x)$ acts linearly on $H^1(T_x\mathcal{X})$, and $df_x$ induces an linear map between $H^1$’s and it is an isomorphism if and only if the original functor between groupoids is an equivalence. We have $H^{-1}(T_x\mathcal{X}) = \text{Lie}(\text{Aut}(x))$, and the action of $\text{Aut}(x)$ is conjugation.

Example A.5. Let $\mathcal{X} = X/G$ be the quotient stack, $T\mathcal{X}$ is represented by the complex $g \otimes_{\mathbb{C}} \mathcal{O}_X \to TX$, for $x \in X$, let $\bar{x}$ be the image of $x$ in $\mathcal{X}$, then $T_{\bar{x}}\mathcal{X}$ is quasi-isomorphic to the complex $g \to T_xX$, where the arrow is $H \mapsto \delta|_{t=0}\exp(tH)x$. We have $\text{Aut}(\bar{x}) = C_G(x)$ the stabilizer at $x$ acts on $T_{\bar{x}}\mathcal{X} = \{g \to T_xX\}$, by conjugation on $g$, and for $g \in \text{Aut}(\bar{x})$, $v = \gamma'(0) \in T_xX$, for $\gamma(t)$ a curve through $x$. $g \cdot v := \delta|_{t=0}g\gamma(t)$. Then $\delta$ is an $\text{Aut}(\bar{x})$ module map. Indeed, $\delta(ad_g(H)) = \delta|_{t=0}\exp(tad_g(H))x = \delta|_{t=0}g\exp(tH)g^{-1}x = g \cdot \delta(H)$, where $g^{-1}x = x$ because $g \in C_G(x)$.

A.2. Sheaves on analytic stacks. We shall work in the framework of [Lur09, Lur12] or [Toë07] for the higher categorical aspect of the theory.

Let $\text{Stk}_{sm}$ be the category of analytic stack with smooth morphisms, an let $X \in \text{Stk}_{sm}$. Denote by $X_{sm}$ the over category $\text{Stk}_{sm}/X$.

Definition A.6. Let $\mathcal{V}$ be a (higher) category, a sheaf on $X$ (in smooth topology) valued in $\mathcal{V}$ is a functor $F : X_{sm}^p \to \mathcal{V}$, such that:

1. $F(\coprod_{i \in I} Y_i) \simeq \prod_{i \in I} F(Y_i)$, for $I$ an index set, and $Y_i \in X_{sm}$
For any surjective arrow \( Y \to Z \) in \( X_{sm} \), the following induced map is an isomorphism in \( \mathcal{V} \):

\[
F(Z) \xrightarrow{\sim} \lim \Delta F(Y^\bullet_Z)
\]

Denote by \( \text{Sh}(X) := \text{Sh}(X, \mathcal{V}) \) be the category of sheaves on \( X \) valued in \( \mathcal{V} \).

Then by definition of sheaf, we have

\textbf{Proposition A.7.}  
(1) \( \text{Sh}(\coprod_{i \in I} X_i) \simeq \prod_{i \in I} \text{Sh}(X_i) \)

(2) For \( Y \to X \) smooth surjective, the induced map is an isomorphism

\[
\text{Sh}(X) \xrightarrow{\sim} \lim \Delta \text{Sh}(Y^\bullet_X)
\]

where the limit is taken inside \( \text{Cat} \) the category of (higher) categories.

Suppose in addition that \( \mathcal{V} \) is a closed symmetric monoidal category, then \( \text{Sh}(X) \) is naturally \( \mathcal{V} \)-enriched, and the above limit can be taken in \( \mathcal{V} \)-\text{Cat}, the category of \( \mathcal{V} \)-enriched categories. Some examples we have in mind are:

(1) \( \mathcal{V} = \text{dg-Vect}_k \), the chain complexes over field \( k \) of characteristic 0 (with the tensor product), then \( \mathcal{V} \)-\text{Cat} = \text{dg-Cat}_k \) is the category of dg categories over \( k \).

(2) \( \mathcal{V} = \infty \text{-Groupoid} \), then \( \mathcal{V} \)-\text{Cat} = \infty \text{-Cat} the \( \infty \)\text{-category of \( \infty \)-categories}.

\textbf{B. Semisimple and semistable bundles}

In this section, we collected some basic facts about \( G \)-bundles on elliptic curve, summarized in Proposition \textbf{B.2}.

Following [BG96, FM98, FM00], we make the following

\textbf{Definition B.1.} Let \( \mathcal{P} \) be a \( G \) bundle on a compact Riemann surface \( C \). \( \mathcal{P} \) is of degree 0 if it lies in the neutral component \( \text{Bun}_G(C)^0 \) of \( \text{Bun}_G(C) \).

\( \mathcal{P} \) is semisimple if \( \mathcal{P} \) has reduction to a maximal torus \( T \).

Let \( C = E \) be an elliptic curve.

\( \mathcal{P} \) is semistable if the associated adjoint bundle \( g_\mathcal{P} \) is semistable.

It’s easy to see that semisimple semistable \( G \) bundles of degree 0 are exactly those in the image of the map \( \text{Bun}_F(C)^0 \to \text{Bun}_G(C)^0 \). Let \( G_E := G_{\text{Bun}_G(E)^0,ss} \) be the stack of degree 0 semistable \( G \) bundles on \( E \). It is an open substack of \( \text{Bun}_G(E)^0 \). We have the following characterization of degree 0 semistable bundle on \( E \).

\textbf{Proposition B.2.} Let \( \mathcal{P} \) be a \( G \) bundle of degree 0 on \( E \), then:

(1) \( \mathcal{P} \) is semisimple semistable if and only if \( \mathcal{P} \) is a closed point in \( |\text{Bun}_G(E)^0,ss| \).

(2a) \( \mathcal{P} \) is semistable if and only if the closure of \( \mathcal{P} \) in \( |\text{Bun}_G(E)^0| \) contains a semisimple semistable bundle. (2b) Moreover, in this case, such semisimple semistable bundle is unique, it is defined to be the semi-simplification of \( \mathcal{P} \).

The proposition is known in the case when \( G \) is semisimple and simply connected [BEG03, FM00]. For the general case, we shall give the proof later.

The set of degree 0 semisimple semistable bundles \( \mathcal{E}_E := (\text{Pic}^0(E) \otimes X_*(T))/W \) has a natural structure of an algebraic variety, and Proposition \textbf{B.2} gives a (non-representable) maps between algebraic stacks:

\[ \chi_E : G_E \to \mathcal{E}_E \]
by taking a bundle to its semi-simplification. In fact, \( \epsilon_E \) is the coarse moduli of \( G_E \), in the sense that for any algebraic space \( X \), the natural map \( \chi_E : \text{Map}(\epsilon_E, X) \to \text{Map}(G_E, X) \) is an isomorphism.

To prove Proposition B.2, we need the following Lemmas.

**Lemma B.4.** Let \( H' \to H \) finite cover of algebraic groups, with kernel \( K \) central in \( H' \). \( H \) a compact Riemann surface. \( \mathcal{P} \) an \( H' \) bundle \( C \). Then

1. \( H^1(C, K) \) acts on \( |\text{Bun}_{H'}(C)^0| \), with quotient \( |\text{Bun}_H(C)^0| \). In particular, \( \pi : |\text{Bun}_{H'}(C)^0| \to |\text{Bun}_H(C)^0| \) is open finite surjective.

2. Assume \( H \) (hence \( H' \)) is reductive, then \( \mathcal{P} \) is semisimple if and only if \( \mathcal{P}_H \) is semistable.

3. Assume \( C = E \), \( H \) is reductive, then \( \mathcal{P} \) is semistable if and only if \( \mathcal{P}_H \) is semistable.

**Proof.** (3) follows directly from definition.

1. There is a short exact sequence \( 1 \to \mathcal{H} \to \mathcal{H}' \to \mathcal{H} \to 1 \), where \( \mathcal{H} \) (resp. \( \mathcal{H}', \mathcal{H} \)) is the sheaf of sections of the constant group scheme \( K \) (resp. \( H', H \)) over \( C \). Notice that \( H^0(C, \mathcal{H}') \to H^0(C, \mathcal{H}) \) is surjective. We have

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & H^1(C, K) & \longrightarrow & |\text{Bun}_{H'}(C)| & \longrightarrow & |\text{Bun}_H(C)| & \longrightarrow & H^2(C, K) \\
 & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & 1 & \longrightarrow & \pi_1(\text{Bun}_{H'}(C)) & \longrightarrow & \pi_1(\text{Bun}_H(C)) & \longrightarrow & K & \longrightarrow & 1 \\
\end{array}
\]

This gives \( 1 \to H^1(C, K) \to |\text{Bun}_{H'}(C)^0| \to |\text{Bun}_H(C)^0| \to 1 \).

2. Let \( T' \) be a maximal torus of \( H' \), then \( K \subset T' \) and \( T := T'/K \) is a maximal torus in \( H \). We have the commutative diagram of short exact sequences of groups.

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & 1 \\
\end{array}
\]

Take the corresponding sheaves and long exact sequences on cohomology, (2) is obtained by diagram chasing. \( \square \)

**Lemma B.5.** Let \( H' \to H \) be a finite central cover of reductive algebraic groups with kernel \( K \). If Proposition B.2 holds for \( H' \), then it holds \( H \).

**Proof.** Suppose Proposition B.2 holds for \( H' \). Let \( \mathcal{P} \in |\text{Bun}_H(E)^0| \). For (1), assume \( \mathcal{P} \) is semisimple semistable. Then by Lemma B.4, \( \pi^{-1}(\mathcal{P}) \) is a finite union of semisimple semistable \( H' \) bundles, hence is closed in \( |\text{Bun}_{H'}(E)^0| \). Hence \( \{\mathcal{P}\} = \pi^{-1}((\pi^{-1}(\mathcal{P}))^c)^c \) is closed (\( \pi \) is open, \( c \) stands for complement). Conversely if \( \mathcal{P} \) is a closed point in \( |\text{Bun}_H(E)^0,ss| \), \( \pi^{-1}(\mathcal{P}) \) is a closed set in \( |\text{Bun}_{H'}(E)^0,ss| \), hence it contains closure of any of its point, hence contains a semisimple semistable \( H' \) bundle. Again by Lemma B.4, \( \mathcal{P} \) is semisimple semistable. To prove (2a), assume \( \mathcal{P} \) is semistable, then there is a semistable \( H' \) bundle \( \mathcal{P}' \), such that \( \pi(\mathcal{P}') = \mathcal{P} \), there is a semisimple semistable \( H' \) bundle \( \mathcal{R}' \in \{\mathcal{P}'\} \), then we have \( \mathcal{R} := \pi(\mathcal{R}') \) is semisimple semistable and \( \mathcal{R} \in \{\mathcal{P}\} \). Conversely, suppose \( \{\mathcal{P}\} \) contains some semisimple semistable bundle. Let \( \pi^{-1}(\mathcal{P}) = \{\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_n\} \), if any of \( \mathcal{P}'_i \) has a semisimple semistable \( H' \) bundle in its closure, then we conclude \( \mathcal{P}'_i \) is semistable, hence so is \( \mathcal{P} \). Suppose none of \( \mathcal{P}'_i \) contains a semisimple semistable bundle in its
closure, we want to reach a contradiction. Indeed, we’ll have \((\pi((\bigcup_{i=1}^{n} \{P_i\}))^c)\) is closed, containing \(\mathcal{P}\) and does not contain any semisimple semistable \(H\) bundle, which contradicts to the assumption. For (2b), let \(\mathcal{P}\) be semistable, take \(\mathcal{P}' \in \pi^{-1}(\mathcal{P})\), then \(\pi(H^1(E,K) \cdot \{\mathcal{P}'\})\) is closed containing \(\mathcal{P}\) and only contains one closed point which the image of the closed point in \(\{\mathcal{P}'\}\) under \(\pi\).

Proof of Proposition B.2. Proposition holds for all semisimple simply-connected groups, Lemma B.5 implies it holds for all semisimple groups. It’s easy to see if proposition holds for \(H\), then it holds for \(T \times H\) for \(T\) a torus. Then for any connected reductive group \(G, G = (Z(G)^0 \times [G,G])/F\) for a finite central \(F\). So by Lemma B.5 again, we conclude proposition holds for \(G\).

Corollary B.6. Let \(K \subset G\) two reductive algebraic groups. Under the natural induction map \(\text{Bun}_K(E) \rightarrow \text{Bun}_G(E)\), the image of \(K_E\) lies in \(G_E\).

Let \(G_{E,0}\) be the moduli stack of degree 0 semistable \(G\)-bundles with trivialization at 0. We have:

**Proposition B.7.** \(G_{E,0}\) is representable.

**Proof.** Suffices to prove that for any degree 0 semistable \(G\)-bundles \(\mathcal{P}\), the natural map \(\text{Aut}(\mathcal{P}) \rightarrow \text{Aut}(\mathcal{P}_0)\) is injective. By Corollary B.6, we can reduce to the case when \(G = GL_n\). Then the above claim follows from Atiyah’s classification.

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