Strongly Disordered Floquet Topological Systems

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based on joint work with Clément Tauber
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Recent progress in mathematics of topological insulators

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Floquet systems

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- Long time dynamics of the system determined by $U(1)$ because $U(n + t) = U(1)^n U(t)$ for $t \in (0, 1), n \in \mathbb{N}$. 
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- Main object however is $U$, not $H$, and all the questions (such as existence of a gap) are asked w.r.t. $U(1)$.
Simple example in zero dimensions

- In zero dimensions, \( \mathcal{H} = \mathbb{C}^N \) (atom with \( N \) internal levels); get a cont. map \( U : [0, 1] \rightarrow \mathcal{U}(N) \).

Cannot use the winding number of \( \text{det} U \) since \( U \) is not a loop!

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\[ - \imath \log \circ U \]
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- **Relative construction**: straight line to next integer value below; get loop on the circle in whose winding may be computed.
In $d > 1$, $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ with $N$ the internal levels; We ask that $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ be piecewise continuous in time and \textit{local} in the sense that $\|\langle \delta_x, H(t)\delta_y \rangle\|$ is exp. decaying in $\|x - y\|$ (uniformly in $t \in \mathbb{S}^1$).
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Higher dimensions

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![Branch cut](image)

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```latex
\begin{center}
\begin{tikzpicture}
\draw[blue, thick, fill=blue!20] (0,0) circle (1.5cm);
\draw[blue, thick] (0,0) circle (1.5cm);
\draw[blue, thick, ->] (0,0) -- (0:1.5cm);
\node at (0,-2) {\text{branch cut}};
\node at (1.5,0) {$\sigma(U(1))$};
\end{tikzpicture}
\end{center}
```

- Topology depends on choice of gap, but not on branch within it!
- In IQHE Chern $\#$ also depends on choice of gap.
- Gap condition is *not* related to insulator property (unlike static case)!
Higher dimensions (cont.)

- In transl. invar. case we get a cont. loop $U^\text{rel} : \mathbb{S}^1 \times \mathbb{T}^d \to U(N)$ based at $\mathbb{1}$, i.e. an element in suspension of C-star algebra $C(\mathbb{T}^d)$. Hence such unitary loops are classified by $K_1(SC(\mathbb{T}^d)) \cong K_0(C(\mathbb{T}^d))$; get same classification as static top. insulators of class A in $d$ dim.
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| Dimension | 1  | 2  | 3  | 4  | …  |
|-----------|----|----|----|----|-----|
| Invariant | 0  | $\mathbb{Z}$ | 0  | $\mathbb{Z}$ | …  |

which has Bott periodicity of two in $d$, like class A row in Kitaev table.
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As in static case, $\exists$ bulk picture (on $\mathcal{H} \equiv \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$) and edge picture on half-space $\mathcal{H}_E := \ell^2(\mathbb{Z}^{d-1} \times \mathbb{N}) \otimes \mathbb{C}^N$ obtained by truncating a given bulk Hamiltonian with some B.C. (truncation always on $H$, not $U!$).
What we studied and previous results

We study the 2D no-symmetries case in the bulk and on the edge. The input is a bulk Hamiltonian $H : \mathbb{S}^1 \to \mathcal{B}(\mathcal{H})$ (piecewise) cont. in time and local in space. It induces a bulk evolution $U : [0, 1] \to U(\mathcal{H})$ via Schrödinger, an edge Hamiltonian $H_E : \mathbb{S}^1 \to \mathcal{H}_E$ (via truncation to half-space with Dirichlet) and an edge evolution $U_E : [0, 1] \to U(\mathcal{H}_E)$ via Schrödinger from $H_E$.  

Previous studies

Physics: Rudner, Lindner, et al (2013)
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K-theoretic classification says this case should have a $\mathbb{Z}$ strong invariant.

Previous studies assume a spectral gap for $U(1)$ which allows one to take a $\log(U(1))$ which is local, then $U_{\text{rel}} : \mathbb{S}^1 \to B(\mathcal{H})$ is $U$ concat. with static $e \cdot \log(U(1))$.

Bulk invariant is 3D winding of the loop $U_{\text{rel}}$.

Define $H_{\text{rel}}$ as the concatenation of $H_E$ and the truncation of $-i \log(U(1))$.

Induces evol. $U_{\text{rel}} : [0, 1] \to U(\mathcal{H}_E)$ (not a loop). Edge invar. is charge pumped along 1 direction after one period of $U_{\text{rel}}$: depends only on endpoint $U_{\text{rel}}(1)$!
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- Define $H_E^{\text{rel}}$ as the concatenation of $H_E$ and the truncation of $-i \log(U(1))$. Induces evol. $U_E^{\text{rel}} : [0, 1] \rightarrow U(\mathcal{H}_E)$ (*not* a loop). Edge invar. is charge pumped along 1 direction after one period of $U_E^{\text{rel}}$: depends only on endpoint $U_E^{\text{rel}}(1)$!
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1st result: mobility gap

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Investigate the physical meaning of the invariants in completely localized case.

4th result: equality
All invariants are equal, including bulk-edge correspondence. Uses continuity argument.
The mobility gap regime

- Via Combes-Thomas, $S^1 \neq \sigma(U(1))$ implies that $\|h(U(1))_{xy}\|$ decays in $\|x - y\|$ for $h$ holomorphic. This off-diagonal decay is apparently all we need for a well-defined topological phase.

Hamza, Joye, Stolz (2009) e.g. prove that certain random unitary ops. have dyn. loc. We assume the a.-s. results of loc. deterministically, i.e. we assume that $\exists \mu > 0$ s.t. for any $\epsilon > 0$ $\exists C_\epsilon < \infty$ with $\sup_{g \in B_1(\Delta)} \|g(U(1))_{xy}\| \leq C_\epsilon e^{-\mu \|x - y\| + \epsilon \|x\|}$ with $B_1(\Delta)$ the set of Borel bdd. maps $|g| \leq 1$ constant outside of $\Delta \subseteq S^1$, which is called the mobility gap. Implies spectral localization in $\Delta$ via RAGE.
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The mobility gap regime (cont.)

\[ \Delta \]

- Spec. gap

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- Mobility gap
  \[ \sigma(U(1)) = \mathbb{S}^1 \]

- No gap
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Theorem
If \( \Delta \) is a mobility gap for \( U(1) \), placing the branch cut of the logarithm in \( \Delta \), the relative construction still goes through, as well as its bulk-edge correspondence proof.

Main point over [GT18]: Use loc. instead of Combes-Thomas to get (weak) locality of \( \log(U(1)) \); then generalize all notions from uniform decay in \( \|x - y\| \) to allow possible explosion in \( \|x\| \) simultaneously, which we call weakly-local operators:

\[ \|A_{xy}\| \leq C\varepsilon e^{-\mu\|x - y\|} + \varepsilon\|x\| \]
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$$\mathcal{W}(U^{\text{rel}}) \equiv -\frac{1}{2} \int_{S^1} \text{tr} \ U^{\text{rel}}(U^{\text{rel}})^* [U^{\text{rel}}_1(U^{\text{rel}})^*, U^{\text{rel}}_2(U^{\text{rel}})^*]$$

where $A_{\lambda i} \equiv i[\Lambda_i, A]$ with $\Lambda_i$ a switch function. We have

$$\mathcal{W}(U^{\text{rel}}) = \mathcal{W}(U) - \mathcal{W}(e^{i \log \lambda(U(1))})$$

so that some winding of $e^{i \log \lambda(U(1))}$ is removed, but what does it mean physically? (non-top. transport contributions?)
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- Edge invariant contains significant information from the bulk, namely, it depends on $U^{\text{rel}}_E$ which is the evolution of $H^{\text{rel}}_E$, which is the concatenation of $H_E$ and the truncation of $-i \log(U(1))$. The latter is a bulk object. Want bulk-edge correspondence where bulk and edge invariants depend on $H$ and $H_E$ alone, without intertwining their evolutions during the proof.
The stretch function construction

- The stretch function (used by Sadel, Schulz-Baldes (2017) only for the edge in spec. gap case) smooth map $F_\Delta : \mathbb{C} \setminus \{0\} \to \mathbb{C}$; restricted to $S^1$: constant 1 outside $\Delta$, has winding number 1.
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![Diagrams showing $\sigma(U(1))$ and $\sigma(F_\Delta(U(1)))$.]
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- Idea: If we can understand the situation for completely localized operators then we could work with $F_{\Delta} \circ U$ and $F_{\Delta} \circ U_E$ for bulk and edge respectively. The application of $F_{\Delta}$ on $U_E$ uses no information from the bulk except the position of the chosen gap!
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$F_\Delta$ chooses the gap for Floquet just like $\chi(-\infty,E_F)$ chooses the gap for the IQHE, so $F_\Delta$ is like the Floquet’s Fermi projection.
The completely localization case

Let $V : [0, 1] \to \mathcal{U}(\mathcal{H})$ be some bulk evolution s.t. $V(1)$ is completely localized, in the sense that it obeys a det. dyn. loc. estimate on $S^1$ except some finitely many special points; we ask that the Chern $\sharp$ assoc. to each such point vanish.
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- Define the bulk magnetization operator $M(V) := \int_{[0,1]} \text{Im} \ V^* \Lambda_1 i \dot{V} V^* \Lambda_2 V$ and the total (orbital) magnetization $\mathcal{M}(V) := \int_{z \in S^1} \text{tr} \ M(V) \ d P(z)$ with $P$ the proj. valued spectral measure of $V(1)$. Related to magnetization studied by Rudner, Lindner et al (2017). If $\Lambda_i \sim x_i$ then like orbital angular momentum $\frac{1}{2} r(t) \times \dot{r}(t)$. 
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- Let $V : [0, 1] \to \mathcal{U}(\mathcal{H})$ be some bulk evolution s.t. $V(1)$ is completely localized, in the sense that it obeys a det. dyn. loc. estimate on $S^1$ except some finitely many special points; we ask that the Chern # assoc. to each such point vanish.

- Define the **bulk magnetization operator** $M(V) := \int_{[0,1]} \text{Im} \ V^* \Lambda_1 \ i \ V \ V^* \Lambda_2 \ V$ and the total (orbital) magnetization $\mathcal{M}(V) := \int_{z \in S^1} \text{tr} \ M(V) \ d P(z)$ with $P$ the proj. valued spectral measure of $V(1)$. Related to magnetization studied by Rudner, Lindner et al (2017). If $\Lambda_i \sim x_i$ then like orbital angular momentum $\frac{1}{2} r(t) \times \dot{r}(t)$.

- Define the **edge time-avg. charge pumping** assoc. to $V_E(1)$, the evolution of the truncated Hamiltonian assoc. to $V$: $\mathcal{P}_E(V_E(1)) := \lim_{n \to \infty} \lim_{r \to \infty} \frac{1}{n} \text{tr}(V_E(1)^n)[\Lambda_1, V_E(1)^n]\Lambda_{2,r}^\perp$ where $\Lambda_{2,r}^\perp$ restricts to a vertical band from zero to $r$. 
Theorem

If \( U : [0, 1] \rightarrow U(H) \) is s.t. \( U(1) \) is completely loc. as above, then \( M(U) = W(U_{rel}). \)
If $U : [0, 1] \to \mathcal{U}(\mathcal{H})$ is s.t. $U(1)$ is completely loc. as above, then $\mathcal{M}(U) = \mathcal{W}(U^{\text{rel}})$.

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Theorem

If $U : [0, 1] \rightarrow \mathcal{U}(\mathcal{H})$ has a mobility gap at $\Delta$, and $U^{\text{rel}} : \mathbb{S}^1 \rightarrow \mathcal{U}(\mathcal{H})$ is the rel. construction w.r.t. a cut in $\Delta$ then

$$\mathcal{W}(U^{\text{rel}}) = \mathcal{W}((F_\Delta \circ U)^{\text{rel}}) = \mathcal{M}(F_\Delta \circ U) = \mathcal{P}_E(F_\Delta(U_E(1)))$$.
Idea for proof

- We start with

\[
W(U_{\text{rel}}) = W(U) - W(e^{\cdot \log_{\lambda}(U(1))})
\]

\[
(\delta_{\alpha} := -i \, U^* U_{,\alpha})
\]

\[
= \frac{1}{2} \text{tr} \int_{[0,1]} \varepsilon_{\alpha\beta} (\delta_{\alpha}\delta_{\beta} - \delta_{\alpha}^{\lambda}\delta_{\beta}^{\lambda})
\]

\[
(U_{,\alpha} \equiv i[\Lambda_{\alpha}, U] \wedge \delta_{\alpha}(t) = \delta_{\alpha}^{\lambda}(t) \forall t \in \{0, 1\})
\]

\[
= \text{tr} M(U) - M(e^{\cdot \log_{\lambda}(U(1))})
\]

Now use localization to prove (the regularized) trace of \( M(e^{\cdot \log(U(1))}) \) is finite and actually zero.
Idea for proof

- We start with

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\mathcal{W}(U^\text{rel}) = \mathcal{W}(U) - \mathcal{W}(e^{\cdot \log_\lambda(U(1))})
\]
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\]
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(U,_{\alpha} \equiv i[\Lambda_{\alpha}, U] \wedge \delta_\alpha(t) = \delta_\lambda^\lambda(t) \forall t \in \{0, 1\})
\]
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= \text{tr} M(U) - M(e^{\cdot \log_\lambda(U(1))})
\]

Now use localization to prove (the regularized) trace of \( M(e^{\cdot \log(U(1))}) \) is finite and actually zero.

- For \( \mathcal{W}(U^\text{rel}) = \mathcal{W}((F_\Delta \circ U)^\text{rel}) \) we use continuity of \( \mathcal{W} \) under interpolation from the smooth \( F_\Delta \) to the identity map, \textit{in the mobility gap regime}. 
