Long range scattering for the
Wave-Schrödinger system revisited

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Abstract

We reconsider the theory of scattering for the Wave-Schrödinger system and more precisely the local Cauchy problem with infinite initial time, which is the main step in the construction of the wave operators. Using a method due to Nakanishi, we eliminate a loss of regularity between the Schrödinger asymptotic data and the Schrödinger solution in the treatment of that problem, in the special case of vanishing asymptotic data for the wave field.

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1 Introduction

This paper is devoted to the theory of scattering for the Wave-Schrödinger (WS) system

\[ \begin{cases} 
  i \partial_t u + (1/2) \Delta u = -A \ u \\
  \Box A = |u|^2 
\end{cases} \tag{1.1} \tag{1.2} \]

where \( u \) and \( A \) are respectively a complex valued function and a real valued function defined in space time \( \mathbb{R}^{3+1} \), \( \Delta \) is the Laplacian in \( \mathbb{R}^3 \) and \( \Box = \partial_t^2 - \Delta \) is the d’Alembertian. More precisely, for arbitrarily large asymptotic data, we study the Cauchy problem at infinite initial time, which is the first step in the construction of the wave operators. This problem is complicated by the fact that the WS system is borderline long range, so that the relevant solutions of the Schrödinger equation contain a logarithmically diverging phase factor when \( t \) tends to infinity. We previously studied that problem in [3] and we refer to the introduction of the first paper in [3] for general background. The method used in [3] is an extension of a method previously used in [2] to treat the similar case of the Hartree equation with long range potential \( |x|^{-\gamma} \) with \( \gamma \leq 1 \). A drawback of the method used in [2,3] is a loss of regularity of the solutions as compared with that of the asymptotic data. That defect was remedied by Nakanishi in [4,5] for the Hartree equation in the cases \( \gamma = 1 \) and \( 1/2 < \gamma < 1 \) respectively. The improvement in [4] results basically from the use of a different asymptotic parametrization of the solutions and from a clever use of the local mass conservation law for the Schrödinger equation. It turns out that the method used in [4] can be extended to the WS system, unfortunately (so far) only in the special case where the wave field has zero asymptotic data. The purpose of the present paper is to present that extension, namely to solve the local Cauchy problem at infinite initial time without loss of regularity for the WS system in that special case.

In the same way as in [3], the first step of the method consists in eliminating the wave equation by solving it for \( A \) in terms of \( u \). Restricting our attention to positive time and imposing the condition of vanishing asymptotic data for \( A \), we obtain

\[ A = A(u,t) = -\int_t^\infty dt' \ \omega^{-1} \sin(\omega(t-t'))|u(t')|^2 \tag{1.3} \]

where \( \omega = (-\Delta)^{1/2} \). We henceforth replace (1.2) by (1.3) and restrict our attention to the system (1.1) (1.3). We now introduce the relevant parametrization of \( u \) needed
to study the Cauchy problem at infinite time. The unitary group
\[ U(t) = \exp(i(t/2)\Delta) \]  
which solves the free Schrödinger equation can be written as
\[ U(t) = M(t) \ D(t) \ F \ M(t) \]  
where \( M(t) \) is the operator of multiplication by the function
\[ M(t) = \exp(ix^2/2t) \] , \( F \) is the Fourier transform and \( D(t) \) is the dilation operator
\[ D(t) = (it)^{-3/2} \ D_0(t) \]  
where
\[ (D_0(t)f)(x) = f(x/t) \] .

For any function \( g \) of space time, we define
\[ \tilde{g}(t) = U(t)^* g(t) \] .

We first perform a pseudoconformal inversion on \( u \), namely
\[ u(t) = M(t) \ D(t) \ \overline{u_c}(1/t) \]  
or equivalently
\[ \tilde{u}(t) = F \overline{u_c}(1/t) \] ,

thereby replacing the Cauchy problem at infinite initial time for \( u \) by the Cauchy problem at initial time zero for \( u_c \). Correspondingly we replace \( A(t) \) by \( B(t) \) defined by
\[ A(t) = t^{-1} \ D_0(t) \ B(1/t) \]  
The system (1.1) (1.3) is then replaced by
\[ i \partial_t \ u_c = -(1/2)\Delta u_c - t^{-1} \ B(u_c)u_c \] ,
\[ B(u_c, t) = \int_{1}^{\infty} dv \ \nu^{-3} \omega^{-1} \sin(\omega(\nu - 1)) \ D_0(\nu) |u_c(t/\nu)|^2 \] .
The variables $u_c$ and/or $t$ in $B$ will be partly omitted when no confusion can arise, as for instance in (1.13). We now parametrize $u_c$ in terms of an amplitude $v$ and a phase $\varphi$ according to

$$\tilde{u}_c(t) = \exp(-i\varphi(t))v(t) \quad (1.15)$$

or equivalently

$$u_c(t) = U_\varphi(t) \ v(t) \quad (1.16)$$

where

$$U_\varphi(t) = U(t) \exp(-i\varphi(t)) \ . \quad (1.17)$$

That parametrization is the same as that used in [4] and differs from that used in [3] where the phase factor was introduced in $u_c$ instead of $\tilde{u}_c$. The equation (1.13) then becomes the following equation for $v$:

$$i\partial_t v = \ - \ t^{-1} U_\varphi^* \ B(u_c)U_\varphi + \partial_t \varphi \ v \ . \quad (1.18)$$

The role of the phase $\varphi$ is to cancel the singularity at $t = 0$ of the last term in (1.13), so that (1.18) can be solved with $v$ continuous at $t = 0$ and with initial condition $v(0) = v_0$. It will then turn out that $B(u_c)$ tends to $B(v_0)$ when $t$ tends to zero. The cancellation will be ensured by imposing

$$\partial_t \varphi = -t^{-1}B(v_0) \quad (1.19)$$

which together with the (arbitrary) initial condition $\varphi(1) = 0$, yields

$$\varphi(t) = -(\ln t)B(v_0) \ . \quad (1.20)$$

The equation for $v$ then becomes

$$i\partial_t v = -t^{-1} \left( U_\varphi^* \ B(u_c)U_\varphi - B(v_0) \right) v \equiv t^{-1} \ L(v) v \quad (1.21)$$

with

$$L(v) = - \left( U_\varphi^* \ B(u_c)U_\varphi - B(v_0) \right) \ . \quad (1.22)$$

We shall also need the partially linearized equation for $v'$

$$i\partial_t v' = t^{-1} \ L(v) v' \ . \quad (1.23)$$

The method consists in first solving the Cauchy problem with initial time zero for the linearized equation (1.23). One then shows that the map $v \rightarrow v'$ thereby
defined is a contraction in a suitable space in a sufficiently small time interval. This solves the Cauchy problem with initial time zero for the nonlinear equation (1.21). One then translates the results through the change of variables (1.15) to solve the Cauchy problem with initial time zero for the system (1.13) (1.14) or equivalently with infinite initial time for the system (1.1) (1.3). The final result can be stated as the following proposition, which is a slightly shortened rewriting of Propositions 6.1-6.3. We need the notation

\[ FH^\rho = \left\{ u \in S' : F^{-1}u \in H^\rho \right\}. \]

**Proposition 1.1.** Let \( 1 < \rho < 3/2 \).

(1) Let \( u_0 \in FH^\rho \) and define

\[ \varphi(t) = -\elln t B(Fu_0). \]  

Then there exists \( T_{\infty} > 0 \) and there exists a unique solution \( u \) of the system (1.1) (1.3) such that \( \tilde{u} \in C([T_{\infty}, \infty), FH^\rho) \) and such that \( w \) defined by

\[ w(t) = F^{-1} \exp(-i\varphi(1/t)) F\tilde{u}(t) \]  

satisfy

\[ w(t) \to u_0 \text{ in } FH^\rho \text{ when } t \to \infty. \]  

Furthermore \( w \in C([T_{\infty}, \infty), FH^\rho) \), the map \( u_0 \to w \) is continuous from \( FH^\rho \) to \( L^\infty([T_{\infty}, \infty), FH^\rho) \) and \( u \) satisfies the estimate

\[ \| \tilde{u}(t); FH^\rho \| \leq a_1 (1 + |\elln t|^2) \]

for some \( a_1 \geq 0 \) and for all \( t \geq T_{\infty} \).

(2) Let \( T_{\infty} > 0 \). Let \( u \) be a solution of the system (1.1) (1.3) such that \( \tilde{u} \in C([T_{\infty}, \infty), FH^\rho) \) and that \( u \) satisfy the estimate

\[ \| \tilde{u}(t); FH^\rho \| \leq a_1 (1 + |\elln t|)^\alpha \]

for some \( a_1, \alpha \geq 0 \) and for all \( t \geq T_{\infty} \). Then there exists \( u_0 \in FH^\rho \) such that \( w \) defined by (1.25) (1.24) satisfies (1.26). Furthermore \( w \in C([T_{\infty}, \infty), FH^\rho). \)
(3) Let $T_\infty > 0$. Let $u_i$, $i = 1, 2$ satisfy the assumptions of Part (2) and assume that $u_1(t) - u_2(t) \to 0$ in $L^2$ when $t \to \infty$. Then $u_1 = u_2$.

Part (2) of Proposition 1.1 is the converse of Part (1) in the sense that all sufficiently regular solutions of the system (1.1) (1.3) can be recovered by the construction of Part (1). As an application of that result, one obtains in Part (3) a uniqueness result for that system not making any reference to the parametrization (1.15).

The lower bound $\rho > 1$ in Proposition 1.1 is essential for the success of the method. Actually the value $\rho = 1$ is critical in a natural sense. On the other hand the upper bound $\rho < 3/2$ is imposed for convenience only and could be dispensed with at the expense of complicating the estimates.

We then briefly comment on two questions which we do not consider in this paper. Firstly we do not extend the solutions of the local Cauchy problem to global ones, since the norms in Proposition 1.1 are ill adapted to the wave equation and therefore do not readily allow for globalisation. Secondly we do not consider the converse question of solving the Cauchy problem for (1.1) (1.3) up to infinity in time, starting from a (sufficiently large) finite initial time. The reason is that this problem is no longer a Cauchy problem for the original system (1.1) (1.2) after the latter has been reduced to (1.1) (1.3) by imposing that the wave field vanishes at infinity in time.

This paper is organized as follows. In Section 2, we introduce some notation and we collect a number of estimates which are used throughout this paper. In Section 3, we study the Cauchy problem for the linearized equation (1.23) with initial time $t_0 \geq 0$. In Section 4, we solve the Cauchy problem with initial time zero for the nonlinear equation (1.21). In Section 5, we prove the continuity of the solutions of (1.21) with respect to the initial data. In Section 6 we reformulate the previous results as results on the Cauchy problem for the system (1.13) (1.14) and we derive an additional uniqueness result for that system not making any reference to $v$.

2 Notation and preliminary estimates

In this section we introduce some notation and we collect a number of estimates which will be used throughout this paper. We denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$. For any interval $I$ and any Banach space $X$ we denote by $\mathcal{C}(I, X)$
(resp. \( C_w(I, X), C^k(I, X) \)) the space of strongly (resp. weakly, Lipschitz) continuous functions from \( I \) to \( X \) and by \( L^\infty(I, X) \) the space of measurable essentially bounded functions from \( I \) to \( X \). For real numbers \( a \) and \( b \) we use the notation \( a \vee b = \text{Max}(a, b) \) and \( a \wedge b = \text{Min}(a, b) \).

We shall use the Sobolev spaces \( \dot{H}^\sigma_r \) and \( H^\sigma_r \) defined for \(-\infty < \sigma < +\infty, 1 \leq r \leq \infty \) by

\[
\dot{H}^\sigma_r = \{ u : \| u; \dot{H}^\sigma_r \| \equiv \| \omega^\sigma u \|_r < \infty \}
\]

and

\[
H^\sigma_r = \{ u : \| u; H^\sigma_r \| \equiv \| \omega >^\sigma u \|_r < \infty \}
\]

where \( \omega = (-\Delta)^{1/2} \) and \( \langle \cdot, \cdot \rangle = (1 + | \cdot |^2)^{1/2} \). The subscript \( r \) will be omitted if \( r = 2 \) and we shall use the notation

\[
\| \omega^{\sigma \pm \epsilon} u \|_2 = \left( \| \omega^{\sigma \pm \epsilon} u \|_2 \| \omega^{\sigma \mp \epsilon} u \|_2 \right)^{1/2}
\]

for some \( \epsilon > 0 \).

We shall use extensively the following Sobolev inequalities, stated here in \( IR^n \), but to be used only for \( n = 3 \).

**Lemma 2.1.** Let \( 1 < q, r < \infty, 1 < p \leq \infty \) and \( 0 \leq \sigma < \rho \). If \( p = \infty \), assume that \( \rho - \sigma > n/r \). Let \( \theta \) satisfy \( \sigma/\rho \leq \theta \leq 1 \) and

\[
n/p - \sigma = (1 - \theta)n/q + \theta(n/r - \rho).
\]

Then the following inequality holds

\[
\| \omega^\sigma u \|_p \leq C \| u \|_q^{-\theta} \| \omega^\rho u \|_p^\theta.
\]

(2.1)

We shall also use extensively the following Leibnitz and commutator estimates.

**Lemma 2.2.** Let \( 1 < r, r_1, r_3 < \infty \) and

\[
1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4.
\]

Then the following estimates hold for \( \sigma \geq 0 \):

\[
\| \omega^\sigma (uv) \|_r \leq C \| \omega^\sigma u \|_{r_1} \| v \|_{r_2} + \| \omega^\sigma v \|_{r_3} \| u \|_{r_4}.
\]

(2.2)

An easy consequence of Lemmas 2.1 and 2.2 is the following estimate.
Lemma 2.3. Let \( 1 < r < \infty \), \( \sigma_1 \geq 0 \), \( \sigma_2 \geq 0 \), \( \sigma_1 + \sigma_2 = \sigma + n - n/r \), such that \(-n + n/r < \sigma \leq \sigma_1 \land \sigma_2 \leq \sigma_1 \lor \sigma_2 < n/2\). Then
\[
\| \omega^\sigma(uv) \|_r \leq C \| \omega^{\sigma_1} u \|_2 \| \omega^{\sigma_2} v \|_2.
\] (2.3)

For \( 0 \leq \sigma < n/2 \), we define the space \( M^\sigma = \mathcal{L}^\infty \cap \dot{\mathcal{H}}_{n/\sigma}^\sigma \) with norm
\[
\| f; M^\sigma \| = \| f \|_\infty + \| \omega^\sigma f \|_{n/\sigma}.
\] (2.4)

By Lemmas 2.1 and 2.2
\[
\| f; M^\sigma \| \leq \| f \|_\infty + C \| \omega^{n/2} f \|_2 \leq C \left( \| \omega^{n/2+\varepsilon} f \|_2 \| \omega^{n/2-\varepsilon} f \|_2 \right)^{1/2} \equiv C \| \omega^{n/2\pm 0} f \|_2
\] (2.5)

and for \( 0 \leq \sigma' \leq \sigma < n/2 \)
\[
\| f; \dot{H}^\sigma' \| \leq C \| f; M^\sigma \| \| f; \dot{H}^\sigma' \|.
\] (2.6)

We shall also need some commutator estimates.

Lemma 2.4. Let \( \lambda \geq 0 \), let \( \sigma_1, \sigma_2 \) satisfy \( 0 \leq \sigma_1, \sigma_2 < n/2 \) and \( (\lambda - 1) \lor 0 < \sigma_1 + \sigma_2 \leq \lambda + n/2 \).

Then the following estimate holds
\[
| < u, [\omega^\lambda, f]v > | \leq C \| \omega^\beta f \|_2 \| \omega^{\sigma_1} u \|_2 \| \omega^{\sigma_2} v \|_2,
\] (2.7)

where \( \beta = \lambda + n/2 - \sigma_1 - \sigma_2 \).

Lemma 2.4 is a variant of Lemma 3.6 in [4] and is proved by a minor variation of the proof of the latter.

We now give some estimates of \( B \) defined by (1.14). From now on we take \( n = 3 \).

Lemma 2.5. Let \( 0 < \sigma < 3/2 \) and \( u \in \mathcal{C}((0,1], \dot{H}^\sigma) \). Then
\[
\| \omega^{2\sigma-1/2} B(u,t) \|_2 \leq C \int_1^\infty dv \nu^{-2\sigma} \| \omega^{\sigma} u(t/\nu) \|_2^2.
\] (2.8)

Let \( 2 \leq r \leq 7 \). Then
\[
\| B(u,t) \|_r \leq C \int_1^\infty dv \nu^{-1+1/r}(\nu - 1)^{-1+2/r} \| \omega^{\sigma} u(t/\nu) \|_2^2
\] (2.9)
with \( \sigma = 1/2 - 1/2r \).

**Proof.** From (1.14) and from the identity

\[
\| \omega^\alpha D_0(\nu) f \|_r = \nu^{-\alpha+n/r} \| \omega^\alpha f \|_r
\]  

we obtain

\[
\| \omega^{2\sigma-1/2} B(u, t) \|_2 \leq C \int_1^\infty d\nu \, \nu^{-2\sigma} \| \omega^{2\sigma-3/2} |u(t/\nu)|^2 \|_2
\]  

from which (2.8) follows by Lemma 2.3.

From the well known dispersive estimates for the wave equation [6], we obtain

\[
\| B(u, t) \|_r \leq C \int_1^\infty d\nu \, \nu^{-3} (\nu - 1)^{-1+2/r} \| \omega^{1-4/r} D_0(\nu)|u(t/\nu)|^2 \|_r .
\]  

for \( 2 \leq r \leq \infty \), where \( 1/r + 1/\sigma = 1 \), from which (2.9) follows by (2.10) and Lemma 2.3.

\( \square \)

**Remark 2.1.** Lemma 2.5 says nothing on the convergence of the integrals over \( \nu \), which may well be infinite. In the applications, \( u(t) \) will be bounded in time up to logarithmic factors near \( t = 0 \), so that convergence will be ensured for \( \sigma > 1/2 \) in (2.8) and for \( r > 3 \) in (2.9).

In order to obtain further estimates on \( B(u) \), we need additional assumptions on \( u \), namely the fact that \( u \) satisfies some linear Schrödinger equation. The following estimate, which holds for any space dimension \( n \geq 2 \), plays an essential role in [4] and in this paper.

**Lemma 2.6.** Let \( n \geq 2 \), let \( 1/2 < \sigma < n/2 \), let \( I \) be an interval and let \( u \in C(I, H^\sigma) \) be a solution of the equation

\[
i \partial_t u + (1/2) \Delta u = Vu
\]  

in \( I \) for some real \( V \in L^\infty_{loc}(I, L^\infty) \). Then for any \( t_1, t \in I \), \( t_1 \leq t \), the following estimate holds:

\[
\| \omega^{2\sigma-2-n/2} (|u(t)|^2 - |u(t_1)|^2) \|_2 \leq C \int_{t_1}^t dt' \| \omega^\sigma u(t') \|_2^2 .
\]  

**Sketch of proof.** The formal local conservation law

\[
\partial_t |u|^2 = - \text{Im} \pi \Delta u
\]  

(2.15)
implies
\[ \partial_t < |u|^2, \psi > = -i/2 < u, [\Delta, \psi]u > \]
for any test function \( \psi \) of the space variable. Integrating over time and estimating the right hand side by Lemma 2.4 with \( \lambda = 2, \sigma_1 = \sigma_2 = \sigma \) yields
\[ | < |u(t)|^2 - |u(t_1)|^2, \psi > | \leq C \int_{t_1}^t dt' \| \omega^\sigma u(t') \|_2^2 \| \omega^{n/2-2\sigma} \psi \|_2 \]
from which (2.14) follows by duality. The formal proof can be made rigorous under the regularity assumptions made on \( u \) and \( V \).

\[ \square \]

We now exploit Lemma 2.6 to derive another estimate on \( B \).

**Lemma 2.7.** Let \( 1/2 < \sigma < 3/2 \), let \( I = (0, T] \) and let \( u \in C(I, H^\sigma) \) be a solution of the equation (2.13) in \( I \) for some real \( V \in L^{\infty}_{\text{loc}}(I, L^\infty) \). Then for \( 0 < t_1 \leq t \leq T \), the following estimate holds:
\[ \| \omega^{2\sigma-5/2} (B(u, t) - B(u, t_1)) \|_2 \leq C (3 - 2\sigma)^{-1} t \int_1^\infty d\nu \nu^{1-2\sigma} \| \omega^\sigma u(t/\nu) \|_2^2 . \]  

**Proof.** From (1.14) and (2.10) we estimate
\[ \| \omega^\beta (B(u, t) - B(u, t_1)) \|_2 \leq \int_1^\infty d\nu \nu^{-1/2-\beta} \| \omega^{-1} (|u(t/\nu)|^2 - |u(t_1/\nu)|^2) \|_2 \]
which by (2.14) is continued as
\[ \cdots \leq C \int_1^\infty d\nu \nu^{2-2\sigma} \int_{t_1/\nu}^{t/\nu} dt' \| \omega^\sigma u(t') \|_2^2 \]
with \( \beta = 2\sigma - 5/2 \). Changing variables from \((t', \nu)\) to \((s, \nu')\) defined by \( t' = t/\nu' \), \( \nu = \nu' s \) yields
\[ \cdots = C t \int_1^\infty d\nu' \nu'^{1-2\sigma} \| \omega^\sigma u(t/\nu') \|_2^2 \int_1^{(1/\nu') \land (t_1/t)} ds s^{2-2\sigma} \]
\[ = C t \int_1^\infty d\nu \nu^{1-2\sigma} \| \omega^\sigma u(t/\nu) \|_2^2 (3 - 2\sigma)^{-1} \left( 1 - \left( (1/\nu) \land (t_1/t) \right)^{3-2\sigma} \right) \]
which implies (2.16).

\[ \square \]

The great advantage of Lemma 2.7 over Lemma 2.5 is the fact that the RHS of (2.16) tends to zero when \( t_1, t \to 0 \) under suitable assumptions on \( u \). In order to allow for more flexibility, we interpolate between (2.8) and (2.16), as stated in the
Lemma 2.8. Under the assumptions of Lemma 2.7, the following estimate holds for $0 < t_1 \leq t \leq T$ and $0 \leq \theta \leq 1$:

$$
\| \omega^{2\sigma-1/2-2\beta} (B(u,t) - B(u,t_1)) \|_2 \leq C(3 - 2\sigma)^{-\theta} t^\theta 
$$

$$
\int_1^\infty d\nu \nu^{1-2\sigma} \left( \| \omega^\sigma u(t/\nu) \|_2^2 + \| \omega^\sigma u(t_1/\nu) \|_2^2 \right) .
$$

(2.18)

Proof. Interpolating between (2.8) and (2.16) yields

$$
\| \omega^{2\sigma-1/2-2\beta} (B(u,t) - B(u,t_1)) \|_2 \leq C(3 - 2\sigma)^{-\theta} t^\theta 
$$

$$
\left\{ \int_1^\infty d\nu \nu^{1-2\sigma} f(t/\nu) \right\}^{\theta} \left\{ \int_1^\infty d\nu \nu^{-2\sigma} (f(t/\nu) + f(t_1/\nu)) \right\}^{1-\theta}
$$

with $f(t) = \| \omega^\sigma u(t) \|_2^2$, from which (2.18) follows.

\qed

In order to handle the phase factor occurring in (1.15), we shall need some phase estimates. The basic estimate is best expressed in terms of homogeneous Besov spaces [1]. The following lemma is a variant of Lemma 3.3 in [4].

Lemma 2.9. Let $\varphi$ be a real function. Let $\sigma > 0$ and $1 \leq q, r < \infty$. Then the following estimate holds:

$$
\| (\exp(i\varphi) - 1) \|_2 \leq C \| \varphi \|_2 \| \varphi \|_2 \left( 1 + \| \varphi \|_{\omega^{n/2} \varphi} \right)^{[\sigma]} .
$$

(2.19)

where $[\sigma]$ is the integral part of $\sigma$.

We shall use extensively the following special case of (2.19)

$$
\| \varphi \|_2 \leq C \left( 1 + \| \varphi \|_{\omega^{n/2} \varphi} \right)^{[\sigma]} .
$$

(2.20)

Lemma 2.9 implies the following estimate of the $M^\sigma$ norm of $\exp(i\varphi) - 1$ defined by (2.4) for $0 < \sigma < n/2$:

$$
\| (\exp(i\varphi) - 1); M^\sigma \| \leq C \left( 1 + \| \varphi \|_{\omega^{n/2} \varphi} \right)^{[\sigma]} .
$$

(2.21)
We now exploit the previous abstract lemmas to derive some estimates of $B(u_c, t)$ defined by (1.14) with $u_c$ expressed by (1.16) and (1.20) with $v_0 = v(0)$.

**Lemma 2.10.** Let $1 < \sigma < 3/2$, let $I = (0, T]$ and let $v \in L^\infty(I, H^\sigma) \cap C([0, T], L^2)$. Let $u_c$ be defined by (1.16) (1.20) with $v_0 = v(0)$ and satisfy the equation

$$i\partial_t u_c + (1/2)\Delta u_c = Vu_c$$

for some real $V \in L^\infty_{loc}(I, L^\infty)$. Let $0 < \theta \leq 1$. Then $B(u_c, t)$ defined by (1.14) satisfies the estimate

$$\| \omega^{2\sigma-1/2-2\beta} (B(u_c, t) - B(u_c, t_1)) \|_2 \leq C(\sigma, \theta) t^{\theta}$$

$$\times \left(1 + \| \nabla v_0 \|_2^2 (1 + |\ln t|)\right)^4 \| v; L^\infty((0, t], \dot{H}^\sigma) \|_N$$

for $0 \leq t_1 \leq t \leq T$ and

$$B(u_c, 0) \equiv \lim_{t \to 0} B(u_c, t) = B(v_0).$$

The limit in (2.24) holds in all the norms appearing in (2.23), in the sense that $B(u_c, t) - B(v_0)$ tends to zero when $t \to 0$ in $\dot{H}^\beta$ for $-1/2 < \beta < 2\sigma - 1/2$.

**Remark 2.2.** By Lemma 2.5, $B(v_0) \in \dot{H}^{1/2+0} \cap \dot{H}^{2\sigma-1/2}$. From that fact and from the convergence of $B(u_c)$ to $B(v_0)$, it follows that $B(u_c) \in L^\infty(I, \dot{H}^{1/2+0} \cap \dot{H}^{2\sigma-1/2-0})$. Note however that the limit in (2.24) holds in some norms which are not expected to be finite for $B(v_0)$, typically the $\dot{H}^\beta$ norm for $\beta \leq 1/2$.

**Proof.** We first estimate

$$\| \omega^\sigma u_c(t) \|_2 \leq \| e^{i\varphi}; M^\sigma \| \| \omega^\sigma v(t) \|_2$$

$$\leq \left(1 + a_0^2 |\ln t|\right)^2 \| \omega^\sigma v(t) \|_2$$

with $a_0 = \| \nabla v_0 \|_2$, by (2.21) and Lemma 2.5. Substituting (2.25) into (2.18) yields

$$\| \omega^\beta (B(t) - B(t_1)) \|_2 \leq C(3 - 2\sigma)^{-\theta} t^{\theta} \| v; L^\infty((0, t], \dot{H}^\sigma) \| N$$

where $\beta = 2\sigma - 1/2 - 2\theta$ and

$$N = \int_1^\infty d\nu \nu^{1-2\sigma} \left(1 + a_0^2 (|\ln t| + |\ln t_1| + |\ln \nu|)\right)^4$$

$$\leq C(\sigma) \left(1 + a_0^2 (1 + |\ln t| + |\ln t_1|)\right)^4$$

(2.26)
where

\[ C(\sigma) = C \int_1^\infty d\nu \, \nu^{1-2\sigma} (1 + \ell \ln \nu)^4. \]

We next estimate

\[
\| \omega^\beta (B(t) - B(t_1)) \|_2 \leq \sum_{0 \leq j \leq \ell - 1} \| \omega^\beta (B(t2^{-j}) - B(t2^{-(j+1)})) \|_2 
+ \| \omega^\beta (B(t2^{-\ell}) - B(t_1)) \|_2
\]

for \(2^{-(\ell+1)} \leq t_1/t \leq 2^{-\ell}\). We estimate each term in the sum by (2.26) with \((t,t_1)\) replaced by \((t2^{-j}, t2^{-(j+1)})\) or by \((t2^{-\ell}, t_1)\) thereby obtaining

\[
\| \omega^\beta (B(t) - B(t_1)) \|_2 \leq C(\sigma)(3 - 2\sigma)^{-\theta} t^\theta \| v; L^\infty((0,t], \dot{H}^\sigma) \|
\times \sum_{j \geq 0} 2^{-j\theta} \left(1 + a_0^2(1 + |\ln t| + j\ln 2)\right)^4
\]

from which (2.23) follows for \(0 < t_1 \leq t\) with

\[ C(\sigma, \theta) = C(\sigma)(3 - 2\sigma)^{-\theta} \sum_{j \geq 0} 2^{-j\theta} j^4. \]

Note that \(C(\sigma, \theta)\) blows up when \(\sigma \to 1\) or \(\sigma \to 3/2\) or \(\theta \to 0\).

We next prove that \(B(u_c, t)\) tends to \(B(v_0)\) in \(\dot{H}^1\). In fact

\[
\| \omega (B(t) - B(v_0)) \|_2 \leq \int d\nu \, \nu^{-3/2} \| |u_c(t/\nu)|^2 - |v_0|^2 \|_2 \tag{2.27}
\]

by (2.11) with \(\sigma = 3/4\). Now

\[
|u_c(t/\nu)|^2 - |v_0|^2 = \text{Re} \left( (U^* - 1)e^{i\varphi} \varpi \right) (U + 1)e^{-i\varphi} v + \text{Re}(\varpi - \varpi_0)(v + v_0)
\]

so that

\[
\| |u_c(t/\nu)|^2 - |v_0|^2 \|_2 \leq \| (U^* - 1)e^{i\varphi} \varpi \|_3 \| (U + 1)e^{-i\varphi} v \|_6 
+ \| v - v_0 \|_2^{1/2} \| v - v_0 \|_6^{1/2} \| v + v_0 \|_6
\leq C \left\{ (t/\nu)^{1/4} \| \nabla e^{-i\varphi} v \|_6^2 + \| v - v_0 \|_2^{1/2} (\| \nabla v \|_2 + \| \nabla v_0 \|_2^3/2) \right\}
\leq C \left\{ (t/\nu)^{1/4} (1 + |\ln t|) \| \nabla B_0 \|_3^2 \| \nabla v \|_2^2 + \cdots \right\}
\]

which tends to zero when \(t \to 0\) for \(v \in L^\infty(I, H^1)\) and \(v(t)\) tending to \(v_0\) in \(L^2\). Together with the estimate (2.23) for \(0 < t_1 \leq t \leq T\), this proves that the same estimate also holds for \(t_1 = 0\) with \(B(u_c, 0) = B(v_0)\) by an appropriate abstract argument.
3 The linearized Cauchy problem for $v$

In this section we study the Cauchy problem for the linearized equation (1.23) with $L(v)$ defined by (1.22) (1.14) (1.16) (1.17) (1.20) for a given $v$, with initial time $t_0 \geq 0$. We first give a preliminary result with $t_0 > 0$ where we do not study the behaviour of the solution as $t$ tends to zero.

**Proposition 3.1.** Let $\rho > 1$, let $I = (0, T]$, let $v_0 \in H^\rho$ and let $v \in L^\infty(I, H^\rho)$. Let $0 \leq \rho' < 3/2$, let $0 < t_0 \leq T$ and let $v'_0 \in H^{\rho'}$. Then the equation (1.23) has a unique solution $v' \in C^1(I) H^{\rho'}$ with $v'(t_0) = v'_0$. The solution satisfies

$$\| v'(t) \|_2 = \| v'_0 \|_2$$

for all $t \in I$ and is unique in $C(I, L^2)$.

**Sketch of proof.** The result follows from the fact that the operator $L(v)$ is bounded in $\dot{H}^{\sigma}$ for $0 \leq \sigma < 3/2$ for all $t \in I$ and is self-adjoint in $L^2$. In fact for any $v' \in \dot{H}^{\rho}$

$$\| \omega^* L(v) v' \|_2 \leq \left(\| e^{i\phi}; M^* \|_2 \| B(u_c); M^* \| + \| B(v_0); M^* \| \right) \| \omega^* v' \|_2$$

and the norms in the right hand side are estimated by (2.21) (2.5) and (2.8).

We next study the boundedness and continuity properties near $t = 0$ of the generic solutions of (1.23) obtained in Proposition 3.1. In view of later applications with $\rho' = \rho$, we henceforth restrict our attention to the case $\rho' > 1$.

**Proposition 3.2.** Let $\rho > 1$, let $I = (0, T]$ and let $v \in L^\infty(I, H^\rho) \cap C([0, T], L^2)$ be such that $u_c$ defined by (1.16) (1.20) with $v_0 = v(0)$ satisfy the equation (2.22) for some real $V \in L^\infty_{loc}(I, L^\infty)$. Let $1 < \rho' < 3/2$ and let $v' \in C(I, H^\rho)$ be a solution of the equation (1.23) in $I$. Then

1. $v' \in (C^1 \cap L^\infty)(I, H^\rho) \cap C_w([0, T], H^\rho) \cap C([0, T], H^\rho)$ for $0 \leq \sigma < \rho'$.

2. Let $0 < \theta < (1/2) \wedge (\rho - 1)$. Then for all $t \in [0, T]$, $t_1 \in I$, the following estimate holds

$$\| \omega^{\rho'} v'(t) \|_2 \leq \| \omega^{\rho'} v'(t_1) \|_2 E(|t - t_1|)$$

with

$$E(t) = E(t, a) = \exp \left( C(\theta)t^\theta a^2 \left( 1 + a^2 (1 + |\ln t|) \right)^8 \right),$$

(3.1)
Proof. We know already that the $L^2$ norm of $v'$ is conserved. The bulk of the proof consists in deriving the estimates (3.1) and (3.4) for $t, t_1 \in I$. We begin with (3.1). From (1.23) we obtain

$$ a = \| v; L^\infty(I, H^\rho) \|. \quad (3.3) $$

(3) For all $t, t_1 \in [0, T]$, the following estimate holds

$$ \| v'(t) - v'(t_1) \|_2 \leq C |t - t_1|^\rho/2 \ a^2 \left( 1 + a^2 (1 + |\ell n| t - t_1) \right)^6 \| \omega^\rho v'(t \vee t_1) \|_2. \quad (3.4) $$

Proof. We know already that the $L^2$ norm of $v'$ is conserved. The bulk of the proof consists in deriving the estimates (3.1) and (3.4) for $t, t_1 \in I$. We begin with (3.1).

From (1.23) we obtain

$$ t \partial_t \| \omega^\rho v' \|_2^2 = 2 \Im < \omega^\rho v', \omega^\rho L(v)v' > = J_0 + J_1 + J_2 \quad (3.5) $$

where

$$ J_0 = -2 \Im < \omega^\rho v', \omega^\rho U_\varphi^* (B(u_c) - B_0) U_\varphi v' >, \quad (3.6) $$

$$ J_1 = -2 \Im < v', \left[ \omega^{2\rho'}, e^{i\varphi} \right] (U^* B_0 U - B_0) e^{-i\varphi} v' >, \quad (3.7) $$

$$ J_2 = i < v', e^{i\varphi} \left[ U^* \left[ \omega^{2\rho'}, B_0 \right] U - \left[ \omega^{2\rho'}, B_0 \right] \right] e^{-i\varphi} v' > \quad (3.8) $$

with $B_0 = B(v_0)$. We estimate $J_0$ by

$$ |J_0| \leq \| e^{i\varphi}; M^\rho \|_2 \| B(u_c) - B_0; M^\rho \| \| \omega^\rho v' \|_2^2 $$

$$ \leq C(\theta) t^\theta \left( 1 + \| \nabla v_0 \|_2^2 \left( 1 + |\ell n t| \right) \right)^8 $$

$$ \times \prod_{\pm} \| v; L^\infty \left( (0, t], \dot{H}^{1+\theta \pm 0} \right) \| \| \omega^\rho v' \|_2^2 \quad (3.9) $$

by (2.5), (2.8), (2.21) and Lemma 2.10 with $2\sigma - 1/2 - 2\theta = 3/2 \pm 0$ or equivalently $\sigma = 1 + \theta \pm 0$ which allows for $\sigma < 3/2$ for $\theta < 1/2$. In order to estimate $J_1$ and $J_2$ we use the identity

$$ U^* AU - A = (U^* - 1)AU + A(U - 1) \quad (3.10) $$

and the estimate

$$ \| (U - 1)f \|_2 \leq t^\mu \| \omega^{2\mu} f \|_2 \quad (3.11) $$

with $0 \leq \mu \leq 1$. We also use Lemma 2.4 in the case

$$ < v_1, [\omega^{2\rho'}, f] v_2 > \leq C \| \omega^{3/2+2\mu} f \|_2 \| \omega^{\rho'-2\mu} v_1 \|_2 \| \omega^{\rho'} v_2 \|_2 \quad (3.12) $$

with $0 < \mu < 1/2 (< \rho'/2)$. We obtain

$$ |J_1| \leq C t^\mu \| \omega^{3/2+2\mu} e^{i\varphi} \|_2 \| B_0; M^\rho \| \| e^{i\varphi}; M^\rho \| \| \omega^\rho v' \|_2^2 $$

$$ \leq C t^\mu |\ell n t| \| \omega^{1+\mu} v_0 \|_2^2 \| \omega^{1\pm 0} v_0 \|_2^2 \left( 1 + \| \nabla v_0 \|_2^2 |\ell n t| \right)^4 \| \omega^{\rho'} v' \|_2^2 \quad (3.13) $$

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by (2.5), (2.8), (2.20) with $\sigma = 3/2 + 2\mu$ and (2.21), for $0 < \mu < 1/2$. Similarly, we estimate
\[
|J_2| \leq C t^\mu \| \omega^{3/2+2\mu} B_0 \|_2 \| e^{i\varphi} ; M^{\rho'} \|_2 \| \omega^{\rho'} v' \|_2^2 \\
\leq C t^\mu \| \omega^{1+\mu} v_0 \|_2^2 \left( 1 + \| \nabla v_0 \|_2^2 |\elln t| \right)^4 \| \omega^{\rho'} v' \|_2^2 .
\]  
(3.14)
Collecting (3.9) (3.13) (3.14) and using the fact that all the norms of $v$ occurring therein are bounded by $a$ for $0 < \mu$, $\theta < 1/2 \land (\rho - 1)$, we obtain
\[
t \left| \partial_t \| \omega^{\rho'} v' \|_2^2 \right| \leq \left\{ C_1(\theta) t^\theta a^2 \left( 1 + a^2(1 + |\elln t|) \right)^5 \right\} \| \omega^{\rho'} v' \|_2^2 .
\]  
(3.15)
Taking $\mu = \theta$ and integrating over time yields (3.1) for $t, t_1 \in I$ by an elementary computation using the fact that \[\int_t^{t_1} dt \ t^{-1+\theta} |\elln t|^p \leq \int_0^{t-t_1} dt \ t^{-1+\theta} |\elln t|^p \]
for $p \geq 1$ and $0 \leq t_1 \leq t \leq 1$.

We next derive the estimates (3.4) for $t, t_1 \in I$. From (1.23) we obtain
\[
\partial_t \| v'(t) - v'_1 \|_2^2 = 2t^{-1} \Im < v'(t) - v'_1, L(v)v'_1 >
\]  
(3.16)
where $v'_1 = v'(t_1)$ so that
\[
|\partial_t \| v'(t) - v'_1 \|_2 | \leq t^{-1} \| L(v)v'_1 \|_2 \leq t^{-1}(K_0 + K_1)
\]  
(3.17)
where
\[
K_0 = \| (B(u_c) - B_0)U_{\varphi} v'_1 \|_2 ,
\]
(3.18)
\[
K_1 = \| (U^*B_0 U - B_0) \ e^{-i\varphi} v'_1 \|_2 .
\]
(3.19)
We estimate
\[
K_0 \leq \| \omega^{3/2-\rho'} (B(u_c) - B_0) \|_2 \| e^{i\varphi}; M^{\rho'} \| \| \omega^{\rho'} v'_1 \|_2 \\
\leq C(\sigma) \ t^\theta \left( 1 + \| \nabla v_0 \|_2^2 (1 + |\elln t|) \right)^6 \| v; L^\infty((0,t], \tilde{H}) \|_2 \| \omega^{\rho'} v'_1 \|_2
\]  
(3.20)
by (2.8) (2.21) and Lemma 2.10 with
\[
1 < \sigma < 3/2 , \quad 0 < \theta = \rho'/2 + \sigma - 1 \leq 1 .
\]
We next estimate
\[ K_1 \leq t^{\rho'/2} \| B_0; M^{\rho'} \| \| e^{i\omega}; M^{\rho'} \| \| \omega^{\rho'} v'_1 \|_2 \]
\[ \leq C \, t^{\rho'/2} \| \omega^{1+0} \|_{L^2} \left(1 + \| \nabla v_0 \|_2 \| \ell n t \| \right)^2 \| \omega^{\rho'} v'_1 \|_2 \]
(3.21)
by (3.10) (3.11) (2.5) (2.8) (2.21). Substituting (3.20) (3.21) into (3.17) yields
\[ |\partial_t \| v'(t) - v'_1 \|_2 | \leq C \, t^{-1+\rho'/2} \, a^2 \left(1 + a^2 (1 + |\ell n t|) \right)^6 \| \omega^{\rho'} v'_1 \|_2 \]
(3.22)
from which (3.4) follows by integration over time for \( t, t_1 \in I \). We next exploit (3.1) and (3.4) in \( I \) to complete the proof of the proposition. From (3.1) it follows that \( v' \in L^\infty(I,H^{\rho'}) \). From (3.4) and (3.1) it then follows that \( v' \) has a limit \( v'(0) \) in \( L^2 \) and that (3.4) holds for \( t, t_1 \in [0,T] \). It then follows by a standard abstract argument that \( v'(0) \in H^{\rho'} \), that \( v' \in C_w([0,T],H^{\rho'}) \cap C([0,T],H^\sigma) \) for \( 0 \leq \sigma < \rho' \) and that (3.1) holds for all \( t \in [0,T], t_1 \in I \).

We have not proved so far that \( v' \in C([0,T],H^{\rho'}) \). This is true but requires a separate argument.

**Proposition 3.3.** Under the assumptions of Proposition 3.2, \( v' \in C([0,T],H^{\rho'}) \) and (3.1) holds for all \( t, t_1 \) in \([0,T]\).

**Proof.** Let \( t_1 \in I, t_1 > 0 \) and let \( A(t) \) be the linear map \( v'_1 \to v'(t) \) defined by Propositions 3.1 and 3.2 for \( 0 \leq t \leq T \), with \( v'_1 = v'(t_1) \). Let \( 0 < \varepsilon < 3/2 - \rho' \). It follows from (3.1) that \( A(t) \) is bounded in \( H^\sigma \) for \( 1 < \sigma < 3/2 \) and in particular satisfies the estimates
\[ \| A(t)v'_1; H^{\rho'} \| \leq \overline{E} \| v'_1; H^{\rho'} \| \]
(3.23)
\[ \| A(t)v'_1; H^{\rho'+\varepsilon} \| \leq \overline{E} \| v'_1; H^{\rho'+\varepsilon} \| \]
(3.24)
for all \( t \in [0,T] \), for some constant \( \overline{E} \) independent of \( t \).

We decompose a function \( f \) into its high and low frequency parts \( f_> \) and \( f_< \) according to
\[ \hat{f}_>(\xi) = \chi \left( |\xi| \geq N \right) \hat{f}(\xi) \]
(3.25)
for some (large) \( N > 0 \).
Let now \( v' \) satisfy the assumptions of Proposition 3.2. We first show that
\[
\| v'(t) \| \leq \| (A(t)v'_1) \|_{H^\rho} \leq \| (A(t)v'_1) \|_{\tilde{H}^\rho} + \| (A(t)v'_1) \|_{H^\rho}
\]
by (3.23) while
\[
\| (A(t)v'_1) \|_{\tilde{H}^\rho} \leq \| (A(t)v'_1) \|_{H^{\rho + \varepsilon}} \leq N^{-\varepsilon} \| A(t)v'_1 \|_{H^{\rho + \varepsilon}} \leq \| v'_1 \|_{H^{\rho + \varepsilon}}
\]
by (3.24), so that uniformly for \( t \in [0, T] \)
\[
\| v'(t) \|_{H^\rho} \leq \| v'_1 \|_{H^\rho} + N^{-\varepsilon} \| v'_1 \|_{H^{\rho + \varepsilon}} \equiv \varepsilon_N. \tag{3.26}
\]
The two terms in the second member of (3.26) tend to zero when \( N \to \infty \) for fixed \( v'_1 \) by definition and by the Lebesgue dominated convergence theorem respectively. We now estimate
\[
\| v'(t) - v'(0) \|_{H^\rho} \leq \| v'(t) - v'(0) \|_{H^\rho} + \| v'(t) - v'(0) \|_{H^{\rho - \varepsilon}} \leq 2\varepsilon_N + (1 + N)^\varepsilon \| v'(t) - v'(0) \|_{H^{\rho - \varepsilon}}
\]
The first term in the right hand side tends to zero when \( N \to \infty \) uniformly in \( t \), while the second term tends to zero for fixed \( N \) when \( t \to 0 \) since \( v' \in C([0, T], H^{\rho - \varepsilon}) \) by Proposition 3.2.

We can now state the main result on the Cauchy problem for the linearized equation (1.23).

**Proposition 3.4.** Let \( \rho > 1 \), let \( I = (0, T] \) and let \( v \in L^\infty(I, H^\rho) \cap C([0, T], L^2) \) be such that \( u' \) defined by (1.16) (1.20) with \( v_0 = v(0) \) satisfy the equation (2.22) for some real \( V \in L^\infty(I, L^\infty) \). Let \( 1 < \rho' < 3/2 \) and let \( v'_0 \in H^{\rho'} \). Let \( t_0 \in [0, T] \). Then there exists a unique solution \( v' \in C([0, T], H^{\rho'}) \) of the equation (1.23) with \( v'(t_0) = v'_0 \). Furthermore \( v' \) satisfies the estimates (3.1) and (3.4) for all \( t, t_1 \in [0, T] \).
The solution is actually unique in $C([0,T], L^2)$.

**Proof.** For $t_0 > 0$, the result follows from Propositions 3.1, 3.2 and 3.3. For $t_0 = 0$, it will be proved by a limiting procedure on $t_0$. For any $t_1 \in I$, let $v'_t$ be the solution of (1.23) with $v'_t(t_1) = v'_0$ given by Propositions 3.1 and 3.2. Let now $0 < t_1 < t_2 \leq T$. It follows from (3.1) that

$$
\| \omega' v'_0(t) \|_2 \leq E(|t - t_1|) \| \omega' v'_0 \|_2
$$

(3.27)

for $i = 1, 2$ and for all $t \in [0,T]$. Furthermore, from (3.4) and (3.27) and from $L^2$-norm conservation, it follows that

$$
\| v'_2(t) - v'_1(t) \|_2 = \| v'_2(t_1) - v'_0 \|_2 = \| v'_2(t_1) - v'_2(t_2) \|_2
\leq C|t_2 - t_1|^{\rho'/2} a^2 \left(1 + a^2(1 + |\ell n(t_2 - t_1)|)\right)^6 E(t_2) \| \omega' v'_0 \|_2 .
$$

(3.28)

From (3.28) it follows that $v'_t$ converges in $L^\infty(I, L^2)$ norm to some $v' \in C([0,T], L^2)$ when $t_1 \to 0$. From the uniform estimate (3.27) it follows by abstract arguments that $v' \in (C_w \cap L^\infty)\big([0,T], H^{\rho'}\big) \cap C\big([0,T], H^{\sigma}\big)$ for $0 \leq \sigma < \rho'$, that $v'$ satisfies the estimates of Proposition 3.2 and that $v'(0) = v'_0$. Furthermore $v'$ is easily seen to satisfy (1.23) in $I$, so that $v' \in C^L(I, H^{\rho'})$. It remains to be proved that actually $v'$ is strongly continuous in $H^{\rho'}$ at $t = 0$. This follows from Proposition 3.3, which has not been used so far. Alternatively it follows from the estimate (3.27) with $t_1 = 0$ that

$$
\lim_{t \to 0} \sup \| \omega' v'(t) \|_2 \leq \| \omega' v'_0 \|_2 E(0) = \| \omega' v'(0) \|_2
$$

which together with weak continuity implies strong continuity at $t = 0$.

\[\Box\]

**Remark 3.1.** Note that in the case where $t_0 = 0$, Proposition 3.3 is not needed for the proof of Proposition 3.4.

## 4 The nonlinear Cauchy problem at time zero for $v$

In this section we prove that the non linear equation (1.21) for $v$ with initial data at time $t_0$ has a unique solution in a small time interval by showing that the map $\Gamma : v \to v'$ defined by Proposition 3.4 with $t_0 = 0$ is a contraction. For that purpose,
we need to estimate the difference of two solutions of the linearized equation (1.23).

**Lemma 4.1.** Let $\rho > 1$, let $0 < \theta < (1/2) \wedge (\rho - 1)$, let $I = (0, T]$ and let $v_i$, $i = 1, 2$ satisfy the assumptions of Proposition 3.4 with $v_i(0) = v_0 \in H^\rho$. Let $1 < \rho' < 3/2$ and let $v'_i$, $i = 1, 2$ be the solutions of the equation (1.23) with $v'_i(0) = v'_0 \in H^\rho'$ obtained in Proposition 3.4. Let $v_+ = v_2 - v_1$ and $v'_+ = v'_2 - v'_1$. Then the following estimate holds for all $t$, $0 < t \leq T$:

$$
\|v'_+; L^\infty((0,t), H^\rho')\| \leq C t^\theta E(t,a) a \phi \left(1 + a^2(1 + |\ell n t|)\right) \|v_+; L^\infty((0,t), H^\rho)\|
$$

where $E(t,a)$ is defined by (3.1) and

$$
a = \text{Max} \|v_i; L^\infty((0,T], H^\rho)\|, \quad a' = \text{Max} \|v'_i; L^\infty((0,T], H^\rho')\| .
$$

**Proof.** From (1.23) we obtain

$$
i \partial_t v'_+ = t^{-1} (L_2 v'_2 - L_1 v'_1) = t^{-1} (L_2 v'_+ + L_- v'_1)
$$

where $L_i = L(v_i)$ and

$$
L_- = L_2 - L_1 = -U_0^\ast B_- U_0, \\
B_- = B(u_{c2}) - B(u_{c1}) .
$$

We estimate for $0 \leq \sigma' < 3/2$

$$
t \partial_t \|\omega^\sigma' v'_-(t)\|_2^2 = 2 \text{Im} \left(\langle \omega^\sigma' v'_-, \omega^\sigma' L_2 v'_- \rangle + \langle \omega^\sigma' v'_-, \omega^\sigma' L_- v'_1 \rangle\right). \quad (4.2)
$$

By the estimates in the proof of Proposition 3.2 (see in particular (3.15)) we obtain

$$
\|\omega^\sigma' v'_-(t)\|_2 \leq E(t,a) \int_0^t dt' t'^{-1} \|\omega^\sigma' L_- v'_1(t')\|_2. \quad (4.3)
$$

We next estimate

$$
\|\omega^\sigma' L_- v'_1\|_2 \leq \|e^{i\varphi}; M^\sigma'\|_2 \|B_-; M^\sigma'\| \|\omega^\sigma' v'_1\|_2 \\
\leq C(1 + a^2|\ell n t|)^4 \|\omega^{3/2 \pm 0} B_-\|_2 \|\omega^\sigma v'_1\|_2 . \quad (4.4)
$$

by (2.5) (2.21). From (1.14) we obtain

$$
B_-(t) = \int_1^\infty d\nu \nu^{-3} \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu)(|u_{c2}|^2 - |u_{c1}|^2)(t/\nu). \quad (4.5)
$$
By the same method as in the proof of (2.8), we estimate
\[ \| \omega^{2\sigma-1/2} B_-(t) \|_2 \leq C \int_1^{\infty} d\nu \nu^{-2\sigma} \| \omega^{\sigma} u_{c+}(t/\nu) \|_2 \| \omega^{\sigma} u_{c-}(t/\nu) \|_2 \] (4.6)
for 1/2 < \sigma < 3/2, with \( u_{c\pm} = u_{c2} \pm u_{c1} \). On the other hand, in the same way as in Lemma 2.6, the local conservation law (2.15) for \( u \) implies
\[ \partial_t < |u_{c2}|^2 - |u_{c1}|^2, \psi > = -(i/4) \left( < u_{c+}, [\Delta, \psi] u_{c-} > + < u_{c-}, [\Delta, \psi] u_{c+} > \right) \] (4.7)
for any test function \( \psi \), so that
\[ \| \omega^{2\sigma-7/2} \left( |u_{c2}(t)|^2 - |u_{c1}(t)|^2 - \left( |u_{c2}(t_1)|^2 - |u_{c1}(t_1)|^2 \right) \right) \|_2 \leq C \int_1^{t} dt' \| \omega^{\sigma} u_{c+}(t') \|_2 \| \omega^{\sigma} u_{c-}(t') \|_2 \] (4.8)
for 1/2 < \sigma < 3/2. Substituting (4.8) into the definition of \( B_- \) yields
\[ \| \omega^{2\sigma-5/2} (B_-(t) - B_-(t_1)) \|_2 \leq \int_1^{\infty} d\nu \nu^{-2\sigma} \int_1^{t/\nu} dt' \| \omega^{\sigma} u_{c+}(t') \|_2 \| \omega^{\sigma} u_{c-}(t') \|_2 \leq C t \int_1^{\infty} d\nu \nu^{1-2\sigma} \| \omega^{\sigma} u_{c+}(t/\nu) \|_2 \| \omega^{\sigma} u_{c-}(t/\nu) \|_2 \] (4.9)
for 1 < \sigma < 3/2, in the same way as in the proof of Lemma 2.7. We know from Lemma 2.10 that \( B_-(t_1) \) tends to \( B_-(0) \) when \( t_1 \to 0 \) in the norms appearing in (4.9) and that \( B_-(0) = 0 \) since \( v_1(0) = v_2(0) \). Therefore
\[ \| \omega^{2\sigma-5/2} B_-(t) \|_2 \leq C t \int_1^{\infty} d\nu \nu^{1-2\sigma} \| \omega^{\sigma} u_{c+}(t/\nu) \|_2 \| \omega^{\sigma} u_{c-}(t/\nu) \|_2 \] (4.10)
Interpolating between (4.6) and (4.10) we obtain
\[ \| \omega^{2\sigma-1/2-2\theta} B_-(t) \|_2 \leq C t^\theta \int_1^{\infty} d\nu \nu^{1-2\sigma} \| \omega^{\sigma} u_{c+}(t/\nu) \|_2 \| \omega^{\sigma} u_{c-}(t/\nu) \|_2 \] (4.11)
for 1 < \sigma < 3/2 and 0 \leq \theta \leq 1,
\[ \cdots \leq C t^\theta (1 + a^2(1 + |\ell n t|)^4) \| v_-; L^\infty((0, t], \hat{H}^\sigma) \| \] (4.11)
(see (2.25) (2.26)). Substituting (4.11) into (4.4) with \( \sigma = 1 + \theta \pm 0 \) and substituting the result into (4.3) yields (1.1).

\[ \square \]

We can now state the main result on the Cauchy problem at time zero for the equation (1.21).
Proposition 4.1. Let $1 < \rho < 3/2$ and let $v_0 \in H^\rho$. Then there exists $T > 0$ and there exists a unique solution $v \in \mathcal{C}([0,T], H^\rho)$ of the equation (1.21) with $v(0) = v_0$. One can ensure that

$$\| v; L^\infty([0,T], H^\rho) \| \leq R = 2 \| v_0; H^\rho \|$$

(4.12)

$$C T^\theta R^2 \left(1 + R^2 (1 + |\ln T|)\right)^8 = 1/2$$

(4.13)

for some $\theta$, $0 < \theta < \rho - 1$, and some $C$ independent of $v_0$.

Proof. Let $v_0 \in H^\rho$, let $B_0 = B(v_0)$ and $\phi = -\elln t B_0$. Let $T > 0$. For any $v \in \mathcal{C}([0,T], H^\rho)$, we define $u_v = U_\phi v$. Let $F(T, v_0)$ be the set of $v \in \mathcal{C}([0,T], H^\rho)$ such that $v(0) = v_0$ and that $u_v$ satisfy the equation (2.22) in $(0,T]$ for some real $V \in L^\infty_{loc}((0,T], L^\infty)$. It follows from Proposition 3.4 that $F(T, v_0)$ is stable under the map $\Gamma : v \rightarrow v'$ defined by that proposition with $t_0 = 0$ and $v'_0 = v_0$. Let $B(R)$ be the ball of radius $R$ in $\mathcal{C}([0,T], H^\rho)$. From Proposition 3.2 it follows that $B(R) \cap F(T, v_0)$ is stable under $\Gamma$ if

$$E(T, R) \leq 2$$

(4.14)

with $R = 2 \| v_0; H^\rho \|$. Furthermore by Lemma 4.1, $\Gamma$ is a contraction in the $L^\infty([0,T]), H^\rho$ norm on that set if

$$C T^\theta E(T, R) R^2 \left(1 + R^2 (1 + |\ln T|)\right)^8 \leq 1/2 .$$

(4.15)

Therefore for $T$ sufficiently small to satisfy (4.14) (4.15), namely under a condition of the type (4.13), the map $\Gamma$ has a unique fixed point in $B(R)$ provided $F(T, v_0)$ is non empty. The set $F(T, v_0)$ is non empty because it contains the solution of the linear equation

$$i\partial_t v^{(0)} = t^{-1} \left(U_\phi^* B_0 U_\phi - B_0\right) v^{(0)}$$

with $v^{(0)}(0) = v_0$ obtained by a simplified version of Proposition 3.2. Clearly the fixed point $v$ satisfies the equation (1.21) and therefore belongs to $F(T, v_0)$.

\[\square\]

5 Continuity with respect to initial data

In this section we prove that the map $v_0 \rightarrow v$ defined in Proposition 4.1 is continuous in the natural norms. For that purpose, we need to estimate the difference
of two solutions of the linearized equation (1.23) corresponding to two functions \( v_1 \) and \( v_2 \) not necessarily satisfying the condition \( v_1(0) = v_2(0) \). The following lemma is an extension of Lemma 4.1 where we drop that assumption.

**Lemma 5.1.** Let \( \rho > 1 \), let \( I = (0, T) \) and let \( v_i, i = 1, 2 \) satisfy the assumptions of Proposition 3.4 with \( v_i(0) = v_0i \in H^\rho \). Let \( 1 < \rho' < 3/2 \) and let \( v_i', i = 1, 2 \), be the solutions of the equation (1.23) with \( v_i'(0) = v_0i' \in H^{\rho'} \) obtained in Proposition 3.4. Let \( v_- = v_2 - v_1 \), \( v_{0-} = v_{02} - v_{01} \) and similarly \( v'_- = v'_2 - v'_1 \), \( v'_{0-} = v'_{02} - v'_{01} \). Let \( 0 < 2\mu \leq \rho' \) and \( \theta = \mu + (\rho - 1)/2 \wedge (1/8) \). Then the following estimate holds for all \( t, 0 < t \leq T \):

\[
\begin{align*}
\| \omega^{\rho'-2\mu} v_-'(t) \|_2 & \leq E(t, a) \left( \| \omega^{\rho'-2\mu} v_{0-} \|_2 \\
& + C(\rho) a\rho' \left( \mu^{-7} t^{10} \| v_{0-} \|_{H^\rho} + t^6 L^8 \| v_- \|_{L^\infty((0, t]; H^{\rho'})} \right) \right) \tag{5.1}
\end{align*}
\]

where \( E(t, a) \) is defined by (3.1),

\[
a = \text{Max} \| v_i; L^\infty((0, T], H^\rho) \|, \quad a' = \text{Max} \| v_i'; L^\infty((0, T], H^{\rho'}) \|,
\]

\[
L = 1 + a^2(1 + |\ell n| t).
\]

The constant \( C(\rho) \) depends on \( \rho \) but is independent of \( \mu \).

**Proof.** In the same way as in the proof of Lemma 4.1 we obtain from (1.23)

\[
i \partial_t v_-' = t^{-1} (L_2 v_2' - L_1 v_1')
\]

where \( L_i = L(v_i) \) and \( L_- = L_2 - L_1 \), from which we obtain (4.2) and (4.3). Now however \( L_- \) is more complicated because we do not assume that \( v_i(0) = v_2(0) \). Let \( B_{0i} = B(v_i(0)) \), \( B_{0-} = B_{02} - B_{01} \) and correspondingly \( \phi_i = -(\ell n) B_{0i}, \phi_- = \phi_2 - \phi_1 \). Let \( B_i = B(u_{ci}) \) and \( B_- = B_2 - B_1 \). Then

\[
-L_- = U_{\phi_2}^* B_2 U_{\phi_2} - B_{02} - U_{\phi_1}^* B_1 U_{\phi_1} + B_{01}
\]

\[
= U_{\phi_2}^* (B_2 - B_{02}) U_{\phi_2} - U_{\phi_1}^* (B_1 - B_{01}) U_{\phi_1}
\]

\[
+ e^{i\phi_2} (U^* B_{02} U - B_{02}) e^{-i\phi_2} - e^{i\phi_1} (U^* B_{01} U - B_{01}) e^{-i\phi_1}
\]

\[
= J_0 + J_1 + J_2 + J_3 \tag{5.2}
\]

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where
\begin{align*}
J_0 &= \left( e^{i\varphi_2} - e^{i\varphi_1} \right) U^* (B_2 - B_{02}) U \varphi_2 + U^* (B_2 - B_{02}) \left( e^{-i\varphi_2} - e^{-i\varphi_1} \right), \\
J_1 &= U^* (B_- - B_{0-}) U \varphi_1, \\
J_2 &= \left( e^{i\varphi_2} - e^{i\varphi_1} \right) (U^* B_{02} U - B_{02}) e^{-i\varphi_2} \\
&\quad + e^{i\varphi_1} (U^* B_{02} U - B_{02}) \left( e^{-i\varphi_2} - e^{-i\varphi_1} \right), \\
J_3 &= e^{i\varphi_1} (U^* B_{0-} U - B_{0-}) e^{-i\varphi_1}.
\end{align*}

We now take $0 \leq \sigma' = \rho' - 2\mu < \rho'(<3/2)$ and we estimate the previous quantities as follows
\begin{align}
\| \omega \sigma' J_0 v_1' \|_2 &\leq C \left( \prod_i \| e^{i\varphi_i} ; M^{\rho'} \| \right) \left( \left( e^{i\varphi} - 1 \right) ; M^{\rho'} \right) \\
&\quad \times \prod_\pm \| \omega^{3/2-2\mu \pm} (B_2 - B_{02}) \|^{1/2}_2 \| \omega^{\rho'} v_1' \|_2, \\
\| \omega \sigma' J_1 v_1' \|_2 &\leq C \| e^{i\varphi_1} ; M^{\rho'} \| \prod_\pm \| \omega^{3/2-2\mu \pm} (B_- - B_{0-}) \|^{1/2}_2 \| \omega^{\rho'} v_1' \|_2,
\end{align}
with
\begin{align*}
\begin{cases}
\mu_+ = \mu & \text{for } \mu \geq \varepsilon, \\
\mu_- = \mu \pm \varepsilon & \text{for } \mu < \varepsilon
\end{cases}
\end{align*}
for some fixed $\varepsilon$ and with constants $C$ which depend on $\varepsilon$ but can be taken independent of $\mu$. Here we have used estimates of the type
\begin{align}
\| \omega^{\rho'} B v \|_2 &\leq C \prod_\pm \| \omega^{3/2-2\mu \pm} B \|^{1/2}_2 \| \omega^{\rho'} v \|_2,
\end{align}
which follow from a minor variation of Lemma 2.3, with constants $C$ satisfying the property quoted above.

We next estimate
\begin{align}
\| \omega \sigma' J_2 v_1' \|_2 &\leq C \left( \prod_i \| e^{i\varphi_i} ; M^{\rho'} \| \right) \left( \left( e^{i\varphi} - 1 \right) ; M^{\rho'} \right) \| B_{02} ; M^{\rho'} \| \| \omega^{\rho'} v_1' \|_2, \\
\| \omega \sigma' J_3 v_1' \|_2 &\leq C \left( \prod_i \| e^{i\varphi_i} ; M^{\rho'} \| \right) \left( \left( e^{i\varphi} - 1 \right) ; M^{\rho'} \right) \| B_{0-} ; M^{\rho'} \| \| \omega^{\rho'} v_1' \|_2
\end{align}
where we have used (3.11) and where the constants are again independent of $\mu$. Collecting (5.7) (5.8) (5.10) (5.11), we obtain
\begin{align}
\| \omega^{\rho'} L_- v_1' \|_2 &\leq C \left( \max \| e^{i\varphi_i} ; M^{\rho'} \|^2 \right) K \| \omega^{\rho'} v_1' \|_2,
\end{align}
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where

\[ K = \| (e^{i\varphi} - 1); M^{\rho'} \| \left( \prod_{\pm} \| \omega^{3/2 - 2\mu\pm}(B_2 - B_{02}) \|^{1/2} + t^\mu \| B_{02}; M^{\rho'} \| \right) \]

\[ + \prod_{\pm} \| \omega^{3/2 - 2\mu\pm}(B_- - B_{0-}) \|^{1/2} + t^\mu \| B_{0-}; M^{\rho'} \| \]  

(5.13)

We have already estimated

\[ \| e^{i\varphi}; M^{\rho'} \| \leq C \left( 1 + \| \nabla v_{0i} \|_{2} |\ell n t| \right)^{2} \leq C L^{2} \]  

(5.14)

\[ \| B_{02}; M^{\rho'} \| \leq C a^{2}. \]  

(5.15)

Furthermore, by (2.5) (2.8) (2.21),

\[ \| B_{0-}; M^{\rho'} \| \leq C \| \omega^{1+0}v_{0+} \|_{2} \leq C a \| v_{0-}; H^{\rho} \|, \]  

(5.16)

\[ \| (e^{i\varphi} - 1); M^{\rho'} \| \leq C \left( \| \varphi_{-} \|_{\infty} + \| \omega^{3/2}\varphi_{-} \|_{2} \left( 1 + \| \omega^{3/2}\varphi_{-} \|_{2} \right) \right) \]

\[ \leq C|\ell n t|\left\{ \| B_{0-} \|_{\infty} + \| \omega^{3/2}B_{0-} \|_{2} \left( 1 + |\ell n t| \| \omega^{3/2}B_{0-} \|_{2} \right) \right\} \]

\[ \leq C|\ell n t| \| \omega^{1+0}v_{0+} \|_{2} \| \omega^{1+0}v_{0-} \|_{2} \left( 1 + |\ell n t| \| \nabla v_{0+} \|_{2} \| \nabla v_{0-} \|_{2} \right). \]  

(5.17)

We next estimate

\[ \| \omega^{3/2 - 2\mu\pm}(B_2 - B_{02}) \|_{2} \leq C t^{\theta\pm a^{2}} L^{4} \]  

(5.18)

by Lemma 2.10, especially (2.23), with

\[ 0 < \theta_{\pm} = \mu_{\pm} + \sigma - 1 \leq 1 \]

for some \( \sigma \) satisfying \( 1 < \sigma < 3/2, \sigma \leq \rho \). We choose

\[ 2\varepsilon = \sigma - 1 = (\rho - 1) \land 1/4 \]

so that

\[ \mu + (\sigma - 1)/2 \equiv \theta = \theta_{-} < \theta_{+} \leq 1 \]

and

\[ \| \omega^{3/2 - 2\mu\pm}(B_2 - B_{02}) \|_{2} \leq C t^{\theta a^{2}} L^{4} \]  

(5.19)

for \( 0 \leq 2\mu \leq \rho \), with a constant \( C \) depending only on \( \rho \).

It remains to estimate the last but one term in (5.13). Taking the limit \( t_{1} \to 0 \) in (4.9) and using again Lemma 2.10 with \( \theta = 1 \), we obtain

\[ \| \omega^{2\sigma - 5/2}(B_{-}(t) - B_{0-}) \|_{2} \leq C t \int_{1}^{\infty} d\nu \nu^{1-2\sigma} \| \omega^{\sigma}u_{c+}(t/\nu) \|_{2} \| \omega^{\sigma}u_{c-}(t/\nu) \|_{2} \]  

(5.20)
which differs from \((4.10)\) by the fact that now \(B_0 \neq 0\). Interpolating between \((5.20)\) and \((4.6)\) together with the simpler estimate
\[
\| \omega^{2 \sigma - 1/2} B_{0-} \|_2 \leq C \| \omega^\sigma v_{0-} \|_2 \tag{5.21}
\]
we obtain
\[
\| \omega^{2 \sigma - 1/2 - 2\theta} (B_-(t) - B_{0-}) \|_2 \leq C \| t^\theta \int_1^\infty dv \nu^{1-2\sigma} \times (\| \omega^\sigma u_c+(t/\nu) \|_2 \| \omega^\sigma u_c-(t/\nu) \|_2 + a \| \omega^\sigma v_{0-} \|_2) \tag{5.22}
\]
for \(1 < \sigma < 3/2\), \(\sigma \leq \rho\) and \(0 < \theta \leq 1\). We estimate \(\| \omega^\sigma u_c+ \|_2\) by \((2.25)\). On the other hand, from
\[
u_c = e^{-i\varphi_1} \left( (e^{-i\varphi} - 1) v_2 + v_- \right)
\]
we obtain
\[
\| \omega^\sigma u_{c-} \|_2 \leq C \| e^{-i\varphi_1}; M^\sigma \| \left( \| (e^{-i\varphi} - 1); M^\sigma \| \| \omega^\sigma v_2 \|_2 + \| \omega^\sigma v_- \|_2 \right) . \tag{5.23}
\]
Substituting \((5.14)\) \((5.17)\) into \((5.23)\) yields
\[
\| \omega^\sigma u_{c-} \|_2 \leq C L^2 \left( a^2 |\ln t| L \| v_{0-}; H^\rho \| + \| v_-; L^\infty((0,t], H^\rho) \| \right) . \tag{5.24}
\]
Substituting \((5.24)\) into \((5.22)\) yields (see \((2.25)\))
\[
\| \omega^{3/2 - 2\mu_\pm} (B_- - B_{0-}) \|_2 \leq C \| t^\theta a \left( L^6 \| v_{0-}; H^\rho \| + L^4 \| v_-; L^\infty((0,t], H^\rho) \| \right) \tag{5.25}
\]
with the same \(\theta\) and with \(C\) depending only on \(\rho\) as in \((5.19)\).

Substituting \((5.15)-(5.19)\) and \((5.25)\) into \((5.13)\) yields
\[
K \leq C a \left( t^\mu L^2 \| v_{0-}; H^\rho \| + t^\theta \left( L^6 \| v_{0-}; H^\rho \| + L^4 \| v_-; L^\infty((0,t], H^\rho) \| \right) \right) . \tag{5.26}
\]
Substituting \((5.26)\) into \((5.12)\), substituting the result into \((4.3)\), integrating over time and using the fact that \(\theta \geq \mu \vee ((\rho - 1)/2 \wedge (1/8))\) and that
\[
\int_0^t dt' t'^{-1} |\ln t'|^p \leq C \lambda^{-(p+1)} t^\lambda (1 + |\ln t|)^p
\]
yields \((5.1)\).

\[\square\]

We can now prove the continuity of the map \(v_0 \to v\) defined in Proposition 4.1.
Proposition 5.1. Let \( 1 < \rho < 3/2 \), let \( R > 0 \) and let \( T \) be defined by (4.13) with \( \theta = (\rho - 1)/2 \wedge (1/8) \). Let \( B(R) \) be the ball of radius \( R \) in \( H^\rho \).

(1) Let \( 1 < \lambda < \rho \). Then the map \( v_0 \to v \) defined by Proposition 4.1 is continuous from \( H^\lambda \) to \( L^\infty((0,T], H^\lambda) \) uniformly for \( v_0 \in B(R/2) \). Furthermore, there exists \( T_\lambda \), \( 0 < T_\lambda \leq T \), such that for two solutions \( v_i \), \( i = 1,2 \) of the equation (1.21) with \( v_i(0) = v_{0i} \in B(R/2) \) as obtained in Proposition 4.1, the following estimate holds:

\[
|| v_-; L^\infty((0,T_\lambda], H^\lambda) || \leq 2E(T_\lambda, R) \left( 1 + C(\lambda)R^2(\rho - \lambda)^{-7}T_\lambda^{(\rho-\lambda)/2} L(T_\lambda)^{10} \right) \times || v_{0-}; H^\lambda ||
\]

(5.27)

where \( v_- = v_2 - v_1 \) and \( v_{0-} = v_{02} - v_{01} \). One can define \( T_\lambda \) by

\[
C(\lambda) E(T_\lambda, R) R^2 T_\lambda^\theta L(T_\lambda)^8 = 1/2
\]

(5.28)

where \( E(T, \cdot) \) is defined by (3.2) and

\[
L(t) = 1 + R^2(1 + |\ln t|)
\]

(2) The map \( v_0 \to v \) defined by Proposition 4.1 is (pointwise) continuous from \( H^\rho \) to \( L^\infty((0,T], H^\rho) \) for \( v_0 \in B(R/2) \).

Proof

Part (1). From (5.1) we obtain

\[
|| v_-; L^\infty((0,t]; H^{\rho-2\mu}) || = \sup_{0 < t' \leq t} \sup_{\mu \leq \mu' \leq \rho/2} || \omega^{\rho-2\mu'}v'_-(t') ||_2
\]

\[
\leq E(t, a) \left( || v_{0-}; H^{\rho-2\mu} || + C(\rho)aa'\left( \mu^{-7}t^\mu L^{10} || v_{0-}; H^\rho || + t^\theta L^\infty || v_-; L^\infty((0,t], H^\rho) || \right) \right).
\]

(5.29)

Changing \( (\rho', \rho) \) to \( (\rho, \lambda) \) satisfying \( 1 < \lambda = \rho - 2\mu < \rho \) and using the fact that \( v'_i(t) = v_i(t) \in B(R) \) for all \( t \in (0,T] \), we obtain

\[
|| v_-; L^\infty((0,t]; H^\lambda) || \leq E(t, R) \left( \left(1 + C(\lambda)R^2\mu^{-7}t^\mu L^{10}\right) || v_{0-}; H^\lambda || + C(\lambda)R^2 t^\theta L^\infty || v_-; L^\infty((0,t], H^\lambda) || \right).
\]

(5.30)

The value of \( \theta \) coming from Lemma 5.1 is

\[
((\lambda - 1)/2 \wedge (1/8)) + \mu \geq (\rho - 1)/2 \wedge (1/8)
\]
and we have therefore replaced it in (5.30) by the latter quantity, which is independent of \( \lambda \). From (5.30) with \( T_\lambda \) satisfying (5.28), we obtain (5.27) which proves the stated continuity in the interval \((0, T_\lambda)\). The extension of the continuity to the whole interval follows by an iteration argument using a simplified version of Lemma 5.1.

**Part (2).** The proof is similar to that of Proposition 3.3. Let \( A(v) \) be the linear map \( v_0' \to v' \) from \( H^\rho \) to \( C([0, T], H^\rho) \) defined by Proposition 3.4. Let \( 0 < \varepsilon < 3/2 - \rho \) (for instance \( \varepsilon = 3/4 - \rho/2 \)). It follows from (3.1) that

\[
\begin{align*}
\| A(v)v_0' \|_{L^\infty((0, T], H^\rho)} &\leq \mathcal{E} \| v_0' \|_{H^\rho} \quad (5.31) \\
\| A(v)v_0' \|_{L^\infty((0, T], H^{\rho+\varepsilon})} &\leq \mathcal{E} \| v_0' \|_{H^{\rho+\varepsilon}} 
\end{align*}
\]

for some constant \( \mathcal{E} = \text{Max } E(T, R) \) where the maximum is taken over \( \rho' = \rho, \rho + \varepsilon \). Let now \( v_i, i = 1, 2 \) be two solutions of the equation (1.21) with \( v_i(0) = v_{0i} \in B(R/2) \) as obtained in Proposition 4.1. Then \( v_i = A_i v_{0i} \) where \( A_i = A(v_i) \). We want to show that \( v_2 \) tends to \( v_1 \) when \( v_{02} \) tends to \( v_{01} \) for fixed \( v_{01} \). We separate again high and low frequency according to (3.25). We estimate

\[
\begin{align*}
\| v_2 - v_1 \|_{L^\infty((0, T], H^\rho)} &\leq \| (A_2 v_{02} > - A_1 v_{01} >) > ; L^\infty(H^\rho) \| \\
&\quad + \| (A_2 v_{02} < - A_1 v_{01} <) > ; L^\infty(H^\rho) \| + \| (v_2 - v_1) < ; L^\infty(H^\rho) \| . 
\end{align*}
\]

Now

\[
\begin{align*}
\| (A_2 v_{02} > - A_1 v_{01} >) > ; L^\infty(H^\rho) \| &\leq \mathcal{E} \sum_i \| v_{0i} > ; H^\rho \| \\
&\leq \mathcal{E} (2 \| v_{01} > ; H^\rho \| + \| v_{02} - v_{01} ; H^\rho \|) . 
\end{align*}
\]

On the other hand

\[
\begin{align*}
\| (A_2 v_{02} < - A_1 v_{01} <) > ; L^\infty(H^\rho) \| &\leq N^{-\varepsilon} \| (A_2 v_{02} < - A_1 v_{01} <) > ; L^\infty(H^{\rho+\varepsilon}) \| \\
&\leq N^{-\varepsilon} \sum_i \| A_i v_{0i} < ; L^\infty(H^{\rho+\varepsilon}) \| \leq \mathcal{E} N^{-\varepsilon} \sum_i \| v_{0i} < ; H^{\rho+\varepsilon} \| \\
&\leq \mathcal{E} \left( 2N^{-\varepsilon} \| v_{01} < ; H^{\rho+\varepsilon} \| + \| v_{02} - v_{01} ; H^\rho \| \right). 
\end{align*}
\]

Let now \( 1 < \lambda < \rho \) (for instance \( \lambda = (\rho + 1)/2 \)). It follows from Part (1) that

\[
\begin{align*}
\| (v_2 - v_1) < ; L^\infty((0, T_\lambda], H^\rho) \| &\leq (N + 1)^{\rho - \lambda} \| v_2 - v_1 ; L^\infty((0, T_\lambda], H^\lambda) \| \\
&\leq (N + 1)^{\rho - \lambda} C_-(\lambda) \| v_{02} - v_{01} ; H^\lambda \| 
\end{align*}
\]

\[(5.36)\]
for some \( T_\lambda \) possibly smaller than \( T \) and for some \( C_-(\lambda) \) which can be read from (5.27). Substituting (5.34)-(5.36) into (5.33) yields

\[
\| v_2 - v_1; L^\infty((0, T_\lambda], H^\rho) \| \leq 2E \left\{ \| v_{01 >}; H^\rho \| + N^{-\varepsilon} \| v_{01 <}; H^{\rho + \varepsilon} \| \right\} \\
+ \left( 2E + (N + 1)^{\rho - \lambda} C_-(\lambda) \right) \| v_{02} - v_{01}; H^\rho \|.
\]

The two terms in the bracket tend to zero when \( N \) tends to infinity by definition and by the Lebesgue dominated convergence theorem respectively, while the last term in (5.37) tends to zero for fixed \( N \) when \( v_{02} \) tends to \( v_{01} \) in \( H^\rho \). This completes the proof of continuity from \( H^\rho \) to \( L^\infty((0, T_\lambda], H^\rho) \). The extension to the original \( T \) proceeds by simpler arguments.

\[\square\]

**Remark 5.1.** Using a minor generalization of Lemma 5.1, one could solve the Cauchy problem for the equation (1.21) down to \( t = 0 \), starting from a (sufficiently small) positive initial time.

### 6 The Cauchy problem at time zero for \( u_c \)

In this section, we first translate the results of Sections 4 and 5 on the Cauchy problem for the equation (1.21) for \( v \) into results on the Cauchy problem for the equation (1.13) for \( u_c \). We then show that conversely, any sufficiently regular solution \( u_c \) of (1.13) can be recovered from a suitable \( v \) solution of (1.21). As an application of the latter result, we derive a uniqueness result for the equation (1.13) not making any reference to \( v \).

**Proposition 6.1.** Let \( 1 < \rho < 3/2 \), let \( v_0 \in H^\rho \), define \( \varphi \) by (1.20) (1.14). Then there exists \( T > 0 \) and a unique solution \( u_c \in C((0,T], H^\rho) \) of (1.13) such that \( v(t) \) defined by (1.15) satisfies the equation (1.21) and that \( v \in C([0,T], H^\rho) \) with \( v(0) = v_0 \). The map \( v_0 \rightarrow v \) is continuous from \( H^\rho \) to \( L^\infty((0,T], H^\rho) \) and \( u_c \) satisfies the estimate

\[
\| u_c(t); H^\rho \| \leq a_1 (1 + |\ln t|^2)
\]

for some \( a_1 \geq 0 \) and for all \( t \in (0,T] \).

**Proof.** The results follow from Proposition 4.1 and 5.1 through the change of variables (1.15). The estimate (6.1) follows from (2.25).

\[\square\]
We now derive a converse of Proposition 6.1.

**Proposition 6.2.** Let $1 < \rho < 3/2$, let $I = (0, T]$ and let $u_c \in C(I, H^\rho)$ be a solution of (1.13) in $I$ satisfying
\[
\| u_c(t); H^\rho \| \leq a_1 (1 + |\ell n t|)^\alpha \tag{6.2}
\]
for some $a_1, \alpha \geq 0$ and for all $t \in I$. Then there exists $v \in C([0, T], H^\rho)$ satisfying the equation (1.21) with $v_0 = v(0)$ such that $u_c$ is recovered from $v$ through (1.15) (1.20). Furthermore
\[
B(v_0) = \lim_{t \to 0} B(u_c, t) \tag{6.3}
\]
in $\dot{H}^\beta$ for $1/2 < \beta < 2\rho - 1/2$.

**Proof.** We first prove the existence of the limit of $B(u_c, t)$ when $t$ tends to zero. From (2.8) (6.2), it follows that
\[
\| \omega^{2\sigma-1/2} B(u_c, t) \|_2 \leq C a_1^2 (1 + |\ell n t|)^{2\alpha} \tag{6.4}
\]
for $1/2 < \sigma \leq \rho$. From Lemma 2.7, especially (2.16), with $V = t^{-1} B(u_c)$, it follows that
\[
\| \omega^{2\sigma-5/2} (B(u_c, t) - B(u_c, t_1)) \|_2 \leq C (3 - 2\sigma)^{-1} a_1^2 t (1 + |\ell n t|)^{2\alpha} \tag{6.5}
\]
for $0 < t_1 \leq t$ and $1 < \sigma \leq \rho$. Interpolating between (6.4) and (6.5) yields
\[
\| \omega^{2\sigma-1/2-2\theta} (B(u_c, t) - B(u_c, t_1)) \|_2 \leq C (3 - 2\sigma)^{-\theta} a_1^2 t^\theta (1 + |\ell n t| + |\ell n t_1|)^{2\alpha} \tag{6.6}
\]
for $1 < \sigma \leq \rho$ and $0 \leq \theta \leq 1$. By the same dyadic argument as in the proof of Lemma 2.10, this implies
\[
\| \omega^{2\sigma-1/2-2\theta} (B(u_c, t) - B(u_c, t_1)) \|_2 \leq C (\sigma, \theta) a_1^2 t^\theta (1 + |\ell n t|)^{2\alpha} \tag{6.7}
\]
for $0 < t_1 \leq t$, $1 < \sigma \leq \rho$ and $0 < \theta \leq 1$. From (6.4) (6.7) it follows that $B(u_c, t)$ has a limit $B_0$ in $\dot{H}^\beta$ for $1/2 < \beta < 2\rho - 1/2$ (see however Remark 2.2) and that
\[
\| \omega^{2\sigma-1/2-2\theta} (B(u_c, t) - B_0) \|_2 \leq C (\sigma, \theta) a_1^2 t^\theta (1 + |\ell n t|)^{2\alpha} . \tag{6.8}
\]
From (6.4) and (6.8) taken for some fixed $t$, it follows that
\[
\| B_0; H^\beta \| \leq C a_1^2 \tag{6.9}
\]
for $1/2 < \beta < 2\rho - 1/2$.

We now define $v$ by (1.15) with $\varphi = - (\ln t)B_0$ so that $v$ satisfies the equation

$$i\partial_t v = - t^{-1} \left( U^* B(u_c) U_{\varphi} - B_0 \right) v.$$  \hfill (6.10)

We next prove by a variant of Propositions 3.2 and 3.3 that $v \in C([0, T], H^\rho)$ and that $B(v(0)) = B_0$. We estimate $v$ by (3.5)-(3.8) with $(v', \rho')$ replaced by $(v, \rho)$. In the same way as in the proof of Proposition 3.2, using now (6.8) we obtain

$$|J_0| + |J_1| + |J_2| \leq C a_1^2 (1 + a_1^2 |\ln t|) t^{\theta} \left( (1 + |\ln t|)^{2\alpha} + 1 + a_1^2 |\ln t| \right) \| \omega^\rho v \|_2$$  \hfill (6.11)

for $0 < \theta < \rho - 1$, from which boundedness of $v$ in $L^\infty((0, T], H^\rho)$ follows. Similarly we estimate $\| v(t) - v(t_1) \|_2$ by (3.17)-(3.19) where now

$$K_0 + K_1 \leq C t^{\theta/2} a_1^2 (1 + a_1^2 |\ln t|)^2 (1 + |\ln t|)^{2\alpha} \| \omega^\rho v(t_1) \|_2$$  \hfill (6.12)

which implies the continuity of $v$ in $L^2$ at $t = 0$. The regularity of $v$ then follows by the same arguments as in Propositions 3.2 and 3.3. The fact that $B_0 = B(v_0)$ is proved by the same argument as in the end of the proof of Lemma 2.10.

\[\square\]

We finally state the uniqueness result.

**Proposition 6.3.** Let $1 < \rho < 3/2$, let $I = (0, T]$ and let $u_{ci} \in C(I, H^\rho)$, $i = 1, 2$, be two solutions of the equation (1.13) in $I$ satisfying (6.2) for some $a_1, \alpha \geq 0$ and such that $u_{c1} - u_{c2}(t)$ tends to zero in $L^2$ when $t \to 0$. Then $u_{c1} = u_{c2}$.

**Proof.** The main step consists in proving that $u_{c1}$ and $u_{c2}$ yield the same $B_0$. For that purpose we estimate

$$\| \omega^{2\sigma-1/2} (B(u_{c2}, t) - B(u_{c1}, t)) \|_2 \leq C \int_1^\infty d\nu \nu^{-2\sigma} \| \omega^{2\sigma-3/2} (|u_{c2}(t/\nu)|^2 - |u_{c1}(t/\nu)|^2) \|_2$$

$$\leq \int d\nu \nu^{-2\sigma} \| u_{c2}(t/\nu) - |u_{c1}(t/\nu)| \|_2 \left( \| u_{c1}(t/\nu) \|_r + \| u_{c2}(t/\nu) \|_r \right)$$  \hfill (6.13)

for $1/2 < \sigma \leq 3/4$, with $3/2 - 3/r = 2\sigma$. We next estimate

$$|u_c| \leq |(U - 1) e^{-i\varphi} v| + |v|$$

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for $u_c = u_{ci}$, $i = 1, 2$, so that
\[
\| u_c \|_r \leq C \left( \mu \| \omega^{2(\sigma+\mu)} e^{-i\varphi v} \|_2 + \| \omega^{2\sigma} v \|_2 \right)
\leq C \left( \mu \| e^{-i\varphi v} M^{2(\sigma+\mu)} \|_2 + \| \omega^{2\sigma} v \|_2 \right)
\leq C \left( 1 + t^{\mu}(1 + a^2|\ln t|^2) \| v; L^\infty(I, H^\sigma) \| \right)
\] (6.14)
for $\sigma + \mu \leq \rho/2$, which can be achieved with $\mu > 0$ for $\sigma < \rho/2$. It follows from
\[(6.13) \quad (6.14)\] through the Lebesgue dominated convergence theorem that $B(u_{c2}, t) - B(u_{c1}, t)$ tends to zero in $\dot{H}^\beta$ for $1/2 < \beta < \rho - 1/2$ since $|u_{c2}| - |u_{c1}|$ tends to zero in $L^2$ when $t$ tends to zero. This implies that $B_{02} = B_{01}$ so that $\varphi_2 = \varphi_1$ and $v_i$, $i = 1, 2$, satisfy the same equation (1.21).

On the other hand $\| v_2(t) - v_1(t) \|_2 = \| u_{c2}(t) - u_{c1}(t) \|_2$ tends to zero by assumption, so that $v_{02} = v_{01}$. The uniqueness result of Proposition 4.1 then implies that $v_2 = v_1$ and therefore $u_{c2} = u_{c1}$.

\[\Box\]

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