1. Introduction

In this paper, we consider the following problem

\[
\begin{align*}
(u_{tt} + \mu \partial^{1+\alpha}_t u + \Delta^2 u + a(x,y,t)u)u &= |u|^{p-1}u, \quad \text{in} \quad \Omega \times (0,T), \\
u(x,y,0) &= u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad \text{in} \quad \Omega
\end{align*}
\]

(1)

with partially hinged boundary condition

\[
\begin{align*}
u(0,y,t) &= u_{x0}(0,y,t) = 0, \quad \text{for} \quad (y,t) \in (-\ell, \ell) \times (0,T), \\
u(L,y,t) &= u_{xL}(L,y,t) = 0, \quad \text{for} \quad (y,t) \in (-\ell, \ell) \times (0,T), \\
u_{yy}(x,\pm\ell,t) + \nu u_{x0}(x,\pm\ell,t) &= 0, \quad \text{for} \quad (x,t) \in (0,L) \times (0,T), \\
\nu_{yyy}(x,\pm\ell,t) + (2-v)u_{xxy}(x,\pm\ell,t) &= 0, \quad \text{for} \quad (x,t) \in (0,L) \times (0,T),
\end{align*}
\]

(2)

where \(\Omega = (0,L) \times (-\ell, \ell) \subset \mathbb{R}^2\) represent a thin rectangular plate as a model of a suspension bridge and \(u = u(x,y,t)\) is the downward displacement of the rectangular plate, see [1,2] for detail description of suspension bridge models. The function \(a = a(x,y,t)\) is bounded, continuous and sign changing. For instance, if \(h : [0,\infty) \rightarrow (-\infty, \infty)\) be any function and \(g : \Omega \rightarrow (-\infty, \infty)\) be a bounded function, then \(\tilde{a}(x,t) = (\text{sign} h)(t)g(x)\) is example of a sign changing function. Furthermore, \(\mu > 0, 0 < \nu < \frac{1}{2}, 1 < p < \infty\) and \(-1 < \alpha < 1\). The notation \(\partial^{1+\alpha}_t\) stand for the Capito’s fractional derivative (see [3,4]) of order \(1 + \alpha\) with respect to \(t\) defined by

\[
\partial^{1+\alpha}_t u(t) = \begin{cases} 
I^{-\alpha} \frac{du(t)}{dt}, & \text{if } -1 < \alpha < 0 \\
I^{1-\alpha} \frac{d^2u(t)}{dt^2}, & \text{if } 0 < \alpha < 1,
\end{cases}
\]

(3)

where \(I^\beta (\beta > 0)\) is the fractional derivative defined by

\[
I^\beta \frac{du(t)}{dt} = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(\tau) d\tau.
\]

(4)
For $-1 < \alpha < 0$, the term $\partial_t^{1+\alpha} u$ is called the fractional damping while for $\alpha = -1$ and $\alpha = 0$, it represent respectively the weak and strong damping. We should mention here that the fractional damping plays a dissipative role that is sandwich between the weak and the strong damping (see [5]). Concerning blow up results for plate equations, we mention among others the result of Messaoudi [6], where he studied the Petrovsky equation

$$u_{tt} + \Delta^2 u + a|u_t|^{m-2}u_t = b|u|^{p-2}u,$$

where $a, b > 0$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial \Omega$. He established local existence and uniqueness of a weak local solution and that for negative initial energy ($E(0) < 0$) the local solution blows up in finite time when $p > m$. In addition, established the existence of global solution when $m \geq p$. The result in [6] was later improved by Chen and Zhou in [7].

Li et al. [8] considered

$$u_{tt} + \Delta^2 u - \Delta u + |u_t|^{m-1}u_t = |u|^{p-1}u$$

and established global existence and blow up of solutions. Piskin and Polat [9] considered (6) and investigated the decay of solutions. Alaimia and Tatar [10] studied

$$
\begin{align*}
&u_{tt} - \Delta u + \partial_t^{1+\alpha} u = |u|^{p-1}u, x \in \Omega, t > 0 \\
&u = 0, \partial \Omega, t > 0, \\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega
\end{align*}
$$

and proved blow up of the solutions for negative initial energy. For related results with fractional damping, we refer the reader to [11–18] and references therein. The article is organized as follows: In Section 2, we recall some fundamental materials and useful assumptions on the relaxation function $g$. In Section 3, we state and prove some technical lemmas. Finally, in section 4, we establish a blow-up result for problem 1.

2. Preliminaries

Throughout the paper, $C_i$, $i = 1, 2, 3, \ldots$ or $c$ are generic positive constants that may change within lines and $(\cdot, \cdot)_2$ and $\| \cdot \|_2$ denote respectively the inner product and norm in $L^2(\Omega)$. We recall some useful materials. We consider the Hilbert space (see [1])

$$H^2(\Omega) = \left\{ w \in H^2(\Omega) : w = 0 \text{ on } \{0, L\} \times (-\ell, \ell) \right\},$$

together with the inner product

$$(u, v)_{H^2} = \int_{\Omega} (\Delta u \Delta v + (1 - \nu)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))dxdy,$$

and denote by $H(\Omega)$ the dual of $H^2(\Omega)$.

**Lemma 1.** (Embedding, see [19]) Suppose $1 < p < +\infty$. Then for any $u \in H^2(\Omega)$, there exists an embedding constant $S_p = S_p(\Omega, p) > 0$ such that

$$||u||_{L^p(\Omega)} \leq S_p ||u||_{H^2(\Omega)},$$

where $S_p = \left(\frac{L^2}{2\pi} + \frac{\lambda^2}{2}\right)^{\frac{p+2}{p} \left(\frac{1}{1-p}\right)^\frac{1}{2}}$.

The eigenvalue problem

$$
\begin{align*}
\Delta^2 u = \lambda u, & \quad (x, y) \in \Omega \\
 u(0, y) = u(0, y) = u(L, y) = u_{xx}(L, y) = 0, & \text{for } y \in (-\ell, \ell), \\
u_{yy}(x, \pm \ell) + \nu u_{xx}(x, \pm \ell) = 0, & \text{for } x \in (0, L), \\
u_{yy}(x, \pm \ell) + (2 - \nu)u_{xy}(x, \pm \ell) = 0, & \text{for } x \in (0, L)
\end{align*}
$$

"
which has been studied in [1], has a unique eigenvalue \( \lambda_1 \in (1 - \nu, 1) \), \( 0 < \nu < \frac{1}{2} \) and \( \lambda = \lambda_1^2 \) is the least eigenvalue. As a consequent, we have the following lemma

**Lemma 2.** Suppose \(-\lambda_1 < a_1 \leq a \leq a_2\). Then, the following inequality holds

\[
A_1 \|u\|^2_{H^2(\Omega)} \leq \|u\|^2_{H^2(\Omega)} + (au, u) + A_2 \|u\|^2_{H^2(\Omega)},
\]

where \( A_1 = \begin{cases} 1 + \frac{a_1}{\lambda_1}, & a_1 < 0, \\ 1, & a_1 \geq 0 \end{cases} \) and \( A_2 = \begin{cases} 1, & a_2 < 0, \\ 1 + \frac{a_2}{\lambda_2}, & a_2 \geq 0 \end{cases} \) which has been proved in [19].

For completeness, we state without proof a local existence result for problem (1)-(2) (see [19,20] for more on existence).

**Theorem 1.** Let \((u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)\) be given and assume \(-\lambda_1 < a_1 \leq a \leq a_2\). Then, there exists a weak unique local solution to problem (1)-(2) in the class

\[
u \in L^\infty ([0, T), H^2(\Omega)) , u_t \in L^\infty ([0, T), L^2(\Omega)) , u_{tt} \in L^\infty ([0, T), \mathcal{H}(\Omega)),
\]

for some \( T > 0 \).

**Definition 1.** A function \( u \) satisfying (11) is called a weak solution of (1) if

\[
\frac{d}{dt}(u_t(t), w) + \frac{\mu}{\Gamma(-\alpha)} \int_{\Omega} w \int_0^t (t-s)^{-\alpha+1} u_s(s) ds dx dy + (u(t), w)_{H^2(\Omega)} + (au(t), w) = \int_\Omega |u|^{p-1} u dx dy
\]

a.e \( t \in (0, T) \) and \( \forall w \in H^2(\Omega) \).

We consider the energy functional \( E(t) \) defined by

\[
E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|u(t)\|^2_{H^2(\Omega)} + \frac{1}{2} (au(t), u(t))_2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx dy.
\]

Multiplying (1) by \( u_t \) and integrating over \( \Omega \), using integration by part, definition of fractional derivative (4) and recalling that \( a_1 \leq a \leq a_2 \), we obtain

\[
E'(t) = -\frac{\mu}{\Gamma(-\alpha)} \int_{\Omega} u_t \int_0^t (t-s)^{-\alpha+1} u_s(s) ds dx dy
\]

for almost all \( t \in [0, T) \). The result in (14) is for any regular solutions. However, this result remains valid for weak solutions by simple density argument. We define a modify energy functional:

\[
E_\epsilon(t) = E(t) - \epsilon (u, u_t)_2,
\]

for some \( \epsilon \) to be specified later. Differentiating (15) and making use of (1)_1 and (14), we arrive at

\[
E'_\epsilon(t) = -\frac{\epsilon \mu}{\Gamma(-\alpha)} \int_{\Omega} u_t \int_0^t (t-s)^{-\alpha+1} u_s(s) ds dx dy - \epsilon \|u_t(t)\|^2 + \epsilon \|u(t)\|^2_{H^2(\Omega)}
\]

\[
+ \frac{\epsilon \mu}{\Gamma(-\alpha)} \int_{\Omega} u \int_0^t (t-s)^{-\alpha+1} u_s(s) ds dx dy + \epsilon (au, u)_2 - \epsilon \int_{\Omega} |u|^{p+1} dx dy.
\]

Also, we define the functional

\[
H(t) = - (e^{-\gamma t} E_\epsilon(t) + \theta F(t) + \lambda),
\]

where

\[
F(t) = \int_0^t \int_{\Omega} M(t-s)e^{-\gamma s} u_s^2(x,y,s) ds dx dy
\]
with
\[ M(t) = e^{\theta t} \int_t^{+\infty} e^{-\beta s} s^{-(\alpha + 1)} \, ds, \]
where \( \gamma = \frac{p+1}{2} \) and \( \theta, \lambda, \beta \) are positive constants to be specified later. The differentiation of (18) gives the relation
\[ F'(t) = \beta \Gamma(-\alpha) e^{-\gamma t} \|u_t(t)\|^2 + \beta F(t) - \int _{\Omega} \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma s} u_t^2(s) \, ds \, dx \, dy. \]  
In the next section, we state and prove some useful Lemmas.

3. Technical lemma

Lemma 3. Suppose \( E_k(0) < 0 \) and \( p \) is sufficiently large, then \( H(t) \) and \( H'(t) \) are strictly positive.

Proof. Differentiating (17) with respect to \( t \) and using (15) yields
\[ H'(t) = \gamma e^{-\gamma t} E_k(t) - e^{-\gamma t} E_k'(t) - \theta F'(t) \]
\[ = \gamma e^{-\gamma t} E(t) - \gamma e^{2} e^{-\gamma t} (u, u)_2 - e^{-\gamma t} E_k'(t) - \theta F'(t). \]  
Substituting (13), (16) and (20) into (21), we arrive at
\[ H'(t) = \left[ \frac{\gamma e}{2} - e - \beta \Gamma(-\alpha) \right] e^{-\gamma t} \|u_t(t)\|^2 + \left[ \frac{\gamma e}{2} - e \right] e^{-\gamma t} \|u(t)\|^2_{H^2(\Omega)} + \gamma e^{-\gamma t} (au, u)_2 + \left( e - \frac{\gamma e}{p+1} \right) e^{-\gamma t} \int_\Omega |u|^{p+1} \, dx \, dy \]
\[ - \gamma e^{2} e^{-\gamma t} (u, u)_2 + \frac{\mu \gamma e^{-\gamma t}}{\Gamma(-\alpha)} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} u_0(s) ds \, dx \, dy \]
\[ + \theta \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma s} u_t^2(s) ds \, dx \, dy - \beta \theta F(t). \]
Using Young’s inequality and Lemma 1, we obtain
\[ (u, u)_2 \leq \delta_1 S_2^2 \|u(t)\|^2_{H^2(\Omega)} + \frac{1}{4 \delta_1} \|u_t(t)\|^2_2, \quad \delta_1 > 0. \]  
Again, Young’s and Cauchy-Schwarz inequalities, we get
\[ e^{-\gamma t} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u(s) ds \, dx \, dy \]
\[ \leq \delta_2 e^{-\gamma t} \|u_t(t)\|^2_2 + \frac{\gamma e^{-\gamma t}}{4 \delta_2} \int_\Omega \left( \int_0^t (t-s)^{-(\alpha+1)} \frac{u(s) ds}{2} \right)^2 \, dx \, dy \]
\[ \leq \delta_2 e^{-\gamma t} \|u_t(t)\|^2_2 + \frac{\gamma e^{-\gamma t} \Gamma(-\alpha)}{4 \delta_2} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma s} u_t^2(s) ds \, dx \, dy, \quad \delta_2 > 0. \]  
In a similar way, with the help of Lemma 1, we find
\[ e^{-\gamma t} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u(s) ds \, dx \, dy \]
\[ \leq \delta_3 S_2^2 e^{-\gamma t} \|u(t)\|^2_{H^2(\Omega)} + \frac{\gamma e^{-\gamma t} \Gamma(-\alpha)}{4 \delta_3} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma s} u_t^2(s) ds \, dx \, dy, \quad \delta_3 > 0. \]  
Substitution of (23)-(25) into (22) and using lemma 2, we obtain
\[ H'(t) \geq \left[ \frac{\gamma e}{2} + e - \beta^a \theta \Gamma(-a) - \frac{\gamma e^2}{4\delta_1} - \frac{\delta_2}{\Gamma(-a)} \right] e^{-\gamma \epsilon t} \|u(t)\|^2_H \\
+ \left[ A_1 \frac{\gamma e}{2} - A_1 e - \delta_1 S_2^2 \gamma^2 - \frac{\delta_3 S_2^2 e \mu}{\Gamma(-a)} \right] e^{-\gamma \epsilon t} \|u(t)\|^2_{H^2(\Omega)} \\
+ \left[ e - \gamma e \frac{\epsilon}{p+1} \right] e^{-\gamma \epsilon t} \int_{\Omega} |u|^{p+1} dx dy - \beta \theta F(t) \\
+ \left[ \theta - \mu (\gamma e)^a - \frac{\mu e (\gamma e)^a}{4\delta_3} \right] \int_{\Omega} \int_0^t \frac{1}{(t-s)^{-(\alpha+1)}} e^{-\gamma \epsilon s} u^2(s) ds dx dy. \]

Adding \( C_1 H(t) - C_1 H(t) \) to the right hand of (26), for some \( C_1 \) to be precise, we arrive

\[ H'(t) \geq C_1 H(t) + \left[ C_1 \frac{\gamma e}{2} + C_1 e - \beta^a \theta \Gamma(-a) - \frac{\gamma e^2}{4\delta_1} - \frac{\delta_2}{\Gamma(-a)} \right] e^{-\gamma \epsilon t} \|u(t)\|^2_H \\
+ \left[ A_1 C_1 \frac{\gamma e}{2} + A_1 C_1 e - A_1 e - \delta_1 S_2^2 \gamma^2 - C_1 \delta_1 S_2^2 e - \frac{\delta_3 S_2^2 e \mu}{\Gamma(-a)} \right] e^{-\gamma \epsilon t} \|u(t)\|^2_{H^2(\Omega)} \\
- C_1 e e^{-\gamma \epsilon t} (u, u) + \left[ e - \gamma e \frac{\epsilon}{p+1} - C_1 \frac{\epsilon}{p+1} \right] e^{-\gamma \epsilon t} \int_{\Omega} |u|^{p+1} dx dy \\
+ \left[ \theta - \mu (\gamma e)^a - \frac{\mu e (\gamma e)^a}{4\delta_3} \right] \int_{\Omega} \int_0^t \frac{1}{(t-s)^{-(\alpha+1)}} e^{-\gamma \epsilon s} u^2(s) ds dx dy \\
+ (C_1 - \beta) \theta F(t) + C_1 \lambda. \]

Applying (23) to (27), we arrive at

\[ H'(t) \geq \left[ \frac{C_1 \gamma e}{2} + \frac{C_1 e}{2} - \beta^a \theta \Gamma(-a) - \frac{C_1 \gamma e^2}{4\delta_1} - \frac{C_1 \delta_2}{\Gamma(-a)} - C_1 \frac{\epsilon}{p+1} \right] e^{-\gamma \epsilon t} \|u(t)\|^2_H \\
+ \left[ A_1 C_1 \gamma e + A_1 C_1 e - A_1 e - \delta_1 S_2^2 \gamma^2 - C_1 \delta_1 S_2^2 e - C_1 \delta_3 S_2^2 e \mu \right] e^{-\gamma \epsilon t} \|u(t)\|^2_{H^2(\Omega)} \\
+ \left[ e - \gamma e \frac{\epsilon}{p+1} - C_1 \frac{\epsilon}{p+1} \right] e^{-\gamma \epsilon t} \int_{\Omega} |u|^{p+1} dx dy + C_1 H(t) + (C_1 - \beta) \theta F(t) + C_1 \lambda \\
+ \left[ \theta - \mu (\gamma e)^a - \frac{\mu e (\gamma e)^a}{4\delta_3} \right] \int_{\Omega} \int_0^t \frac{1}{(t-s)^{-(\alpha+1)}} e^{-\gamma \epsilon s} u^2(s) ds dx dy. \]

Recalling that \( \gamma = \frac{p+1}{2} \) and choosing \( \delta_1 = \frac{1}{2}, \delta_2 = \delta_3 = \frac{\Gamma(-a) \epsilon}{2} \) and \( C_1 = \frac{(p+1) \epsilon}{2} \), we get

\[ H'(t) \geq \left[ \frac{(p+1) \epsilon}{2} H(t) + \left[ \frac{p+1}{2} e \frac{e}{p+1} (1-e) - \beta^a \theta \Gamma(-a) \right] e^{-\gamma \epsilon t} \|u(t)\|^2_H \\
+ \left[ A_1 (p-1) - A_1 e \frac{e}{p+1} + \mu \right] e^{-\gamma \epsilon t} \|u(t)\|^2_{H^2(\Omega)} \\
+ \left[ p+1 e \frac{e}{p+1} - \beta \theta \right] F(t) + \left[ \frac{p+1}{2} e \frac{e}{p+1} \lambda \right] \int_{\Omega} \int_0^t \frac{1}{(t-s)^{-(\alpha+1)}} e^{-\gamma \epsilon s} u^2(s) ds dx dy. \]

Now, choosing \( \epsilon < \epsilon_1 := \min \left\{ 1, \frac{A_1 (p-1)}{2 S_2^2 ((p+1) + \mu)} \right\} \), we get

\[ \epsilon \left[ A_1 (p-1) - A_1 e \frac{e}{p+1} + \mu \right] > \frac{A_1 (p-1) \epsilon}{4}. \]
Next, we select $\beta = 1$, we see that for sufficiently large values of $p$

\[
\frac{(p+1)e}{2} - \beta > 0.
\]

Finally, we choose $\theta$ such that the coefficient of the second term is non-negative and the coefficient of the last term is greater than $\frac{\mu(p+1)^{a}}{2^{a+1}e^{1-a}\Gamma(-a)}$. Thus, we arrive at

\[
H'(t) \geq \frac{(p+1)e}{2}H(t) + \frac{A_1(p-1)e}{4}e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)} + \frac{\mu(p+1)^{a}}{2^{a+1}e^{1-a}\Gamma(-a)} \int_0^t \int_0^1 (t-s)^{-(\alpha+1)}e^{-\gamma es}u^2_2(s)dsdxdy.
\] (31)

If we choose $\lambda < -E_{\ast}(0)$, then $H(0) > 0$. Consequently, it follows from (31) that $H(t) > 0$ and $H'(t) > 0$. This completes the proof. \(\Box\)

4. Main results

In this section, we show that the solutions of 1-2 blows up in finite time for negative initial energy.

**Theorem 2.** Assume that $-\lambda_1 < a < a_2$, $-1 < \alpha < 0$, $E(0) < 0$ and $(u_0,u_1)_2 \geq 0$. Then the solutions of 1-2 blows up in finite time for sufficiently large values of $p$.

**Proof.** We begin by defining the functional $G$ by

\[
G(t) = H^{1-\sigma}(t) + \eta e^{-\gamma et}(u,u_1)_2,
\] (32)

where $\sigma = \frac{p-1}{2(p+1)}$ and $\eta > 0$ to be specified later. Then differentiating $G(t)$ and using (1) yields

\[
G'(t) = (1-\sigma)H^{1-\sigma}(t)H'(t) - \eta\gamma ee^{-\gamma et}(u,u_1)_2 + \eta e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)} + \eta e^{-\gamma et}(u,u_2)_2
\]

\[
= (1-\sigma)H^{1-\sigma}(t)H'(t) - \eta\gamma ee^{-\gamma et}(u,u_1)_2 + \eta e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)} + \eta e^{-\gamma et}\int_0^t |u|^{p+1}dxdy
\]

\[
- \eta e^{-\gamma et}(au,u)_2 - \eta e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)} - \frac{\mu \eta e^{-\gamma et}}{\Gamma(-\alpha)} \int_0^t \int_0^1 (t-s)^{-(\alpha+1)}u_2(s)dsdxdy.
\] (33)

Similarly as in the inequalities (23) and (25), we have that

\[
(u,u_1)_2 \leq \delta_4 S_2^2 \|u(t)\|^2_{H^2(\Omega)} + \frac{1}{4\delta_4} \|u_t(t)\|^2_{2}, \quad \delta_4 > 0.
\] (34)

and

\[
e^{-\gamma et}\int_0^1 \int_0^t (t-s)^{-(\alpha+1)}u_2(s)dsdxdy
\]

\[
\leq \delta_5 e^{-\gamma et}\|u(t)\|^2_{2} + \frac{(\gamma e)\Gamma(-\alpha)}{4\delta_5} \int_0^t \int_0^1 (t-s)^{-(\alpha+1)}e^{-\gamma es}u^2_2(s)dsdxdy, \quad \delta_5 > 0.
\] (35)

From Lemma 2, we get

\[
A_1 e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)} \leq e^{-\gamma et}\left(\|u(t)\|^2_{H^2(\Omega)} + (au,u)_2\right).
\] (36)

Substituting (34)-(36) into (33), we obtain

\[
G'(t) \geq (1-\sigma)H^{1-\sigma}(t)H'(t) + \eta \left(1 - \frac{\gamma e}{4\delta_4}\right) e^{-\gamma et}\|u(t)\|^2_{2} - \eta \left(A_1 + \delta_4 \gamma e S_2^2\right) e^{-\gamma et}\|u(t)\|^2_{H^2(\Omega)}
\]

\[
+ \eta e^{-\gamma et}\int_0^t |u|^{p+1}dxdy - \frac{\eta \mu \delta_5}{\Gamma(-\alpha)} e^{-\gamma et}\|u(t)\|^2_{2} - \frac{\mu \eta \gamma e}{4\delta_5} \int_0^t \int_0^1 (t-s)^{-(\alpha+1)}e^{-\gamma es}u^2_2(s)dsdxdy.
\] (37)

Using (31), we obtain
\[ G'(t) \geq (1 - \sigma) H^{-\sigma}(t) H'(t) + \eta \left( 1 - \frac{\gamma e}{4\delta_4} \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} - \eta \left( A_1 + \delta_4 \gamma e S^2_2 \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{\eta \mu \delta_5}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{2^{\alpha^{-1}} \chi \gamma e \Gamma(-\alpha)}{\delta_5 (p+1)^a} \left( -H'(t) + \frac{p+1}{2} \epsilon H(t) + \frac{A_1 (p-1) \epsilon e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)}}{4} \right). \]

From the last inequality, we get
\[ G'(t) \geq \left[ (1 - \sigma) H^{-\sigma}(t) - \frac{\eta 2^{\alpha^{-1}} \chi \gamma e \Gamma(-\alpha)}{\delta_5 (p+1)^a} \right] H'(t) + \frac{\eta 2^{\alpha^{-2}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} H^1(t) - \eta \left( A_1 + \delta_4 \gamma e S^2_2 \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{\eta \mu B}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} H'(t) \| u(t) \|^2_{H^2(\Omega)}. \]

Now, we choose \( \delta_5 = BH^\sigma(t) \) for some \( B \) positive to be precise later. Then, (39) becomes
\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{\alpha^{-1}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} \right] H^{-\sigma}(t) H'(t) + \frac{\eta 2^{\alpha^{-2}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} H^{1-\sigma}(t) - \eta \left( A_1 + \delta_4 \gamma e S^2_2 \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{\eta \mu B}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} H'(t) \| u(t) \|^2_{H^2(\Omega)}. \]

Adding and subtracting \( H(t) \) on the right hand side of (40) and making use of lemma 2 leads to
\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{\alpha^{-1}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} \right] H^{-\sigma}(t) H'(t) + \left[ 1 + \frac{\eta 2^{\alpha^{-2}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} \right] H^{1-\sigma}(t) - \eta \left( A_1 + \delta_4 \gamma e S^2_2 \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{\eta \mu B}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} H'(t) \| u(t) \|^2_{H^2(\Omega)} \]

- \epsilon e^{-\gamma \epsilon t} (u, u_2) + \theta \Gamma(t) + \lambda - \eta \frac{\mu B}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} H'(t) \| u(t) \|^2_{H^2(\Omega)}.

The term \( (u, u_2) \) is estimated similarly as in (23) as
\[ (u, u_2) \leq \delta_6 S^2_2 \| u(t) \|^2_{H^2(\Omega)} + \frac{1}{4 \delta_6} \| u(t) \|^2_{H^2(\Omega)}, \quad \delta_6 > 0. \]

For the term \( H^\sigma(t) \| u(t) \|^2_{H^2(\Omega)} \), we use the definition of \( H(t) \) in (17) and the choice of \( \epsilon \) in (30) to get (see [10] page 141 for detail computations)
\[ H^\sigma(t) \| u(t) \|^2_{H^2(\Omega)} \leq \frac{C_2}{(p+1)^\sigma} \left( 1 + \int_\Omega |u|^{p+1} \, dx \, dy \right) \]

for some constant \( C_2 > 0 \). Substituting (42) and (43) into (41) yields
\[ G'(t) \geq \left[ (1 - \sigma) - \frac{\eta 2^{\alpha^{-1}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} \right] H^{-\sigma}(t) H'(t) + \left[ 1 + \frac{\eta 2^{\alpha^{-2}} \chi \gamma e \Gamma(-\alpha)}{B(p+1)^a} \right] H^{1-\sigma}(t) - \eta \left( A_1 + \delta_4 \gamma e S^2_2 \right) e^{-\gamma \epsilon t} \| u(t) \|^2_{H^2(\Omega)} + \frac{\eta \mu B}{\Gamma(-\alpha)} e^{-\gamma \epsilon t} H'(t) \| u(t) \|^2_{H^2(\Omega)} \]
we arrive at
\[ \|u\|_{L^p(\Omega)}^2 \]
therefore we select
\[ \epsilon = \frac{1}{2^p} \frac{B(p+1)\alpha}{(p+1)^2S_2^2} \] (45)
we see that the coefficient of the first term is positive. By choosing \( \eta = \frac{p+3}{4(p+1)}, \delta_4 = \frac{1}{2}, \) and \( \epsilon \) small enough so that
\[ \epsilon \leq \epsilon_3 := \frac{4(p-1)}{(p+1)^2S_2^2} \] (46)
we find that the coefficient of \( \|u(t)\|_{L^2(\Omega)}^2 \) is positive. Next, we pick \( \epsilon \) small enough such that
\[ \epsilon \leq \epsilon_4 := \frac{2(3p+5)}{(p+1)(p+11)} \] (47)
to get the coefficient of \( \|u(t)\|_{L^2(\Omega)}^2 \) greater or equal to \( \frac{1}{2} \). We select \( B \) such that
\[ B < \frac{(p+1)^2\alpha}{(p+1)} \min \left\{ \frac{p-1}{2}, 4\lambda(p+1) \right\} \] (48)
to see that the coefficient of \( \int_\Omega |u|^{p+1}dxdy \) is greater than \( \frac{p-1}{4(p+1)} \) and the term
\[ \left( \lambda - \frac{\eta\mu B C_2}{(p+1)^2\alpha} \right) > 0. \]
Thus, for any \( \epsilon \) positive small enough such that
\[ \epsilon < \min \{ \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \} \] (49)
we arrive at
\[ G'(t) \geq H(t) + \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \frac{p-1}{4(p+1)} \int_\Omega |u|^{p+1}dxdy \forall t \geq 0. \] (50)
Using Cauchy-Schwarz and Young's inequalities, we have
\[ |(u, u_2)|^{\frac{1}{r_1}} \leq \|u(t)\|_{L^2(\Omega)}^{\frac{1}{r_1}} \|u(t)\|_{L^2(\Omega)}^{\frac{1}{r_1}} \]
\[ \leq C_2 \|u(t)\|_{L^p(\Omega)}^{\frac{1}{p+1}} \|u(t)\|_{L^2(\Omega)}^{\frac{1}{r_2}} \]
\[ \leq C_3 \left( \|u(t)\|_{L^{p+1}(\Omega)}^{\frac{1}{p+1}} + \|u(t)\|_{L^2(\Omega)}^{\frac{1}{r_2}} \right) \] (51)
where \( C_2 = C_2(|\Omega|, p) > 0, C_3 = C_3(|\Omega|, p, \sigma) > 0 \) are constants and \( \frac{1}{r_1} + \frac{1}{r_2} = 1. \) We recall that \( \sigma = \frac{p-1}{2(p+1)}, \)
therefore we select \( r_1 = \frac{2(1-\sigma)}{1-2\sigma}, r_2 = 2(1-\sigma), \) and arrive at
\[ |(u, u_2)|^{\frac{1}{r_1}} \leq C_3 \left( \|u(t)\|_{L^{p+1}(\Omega)}^{\frac{2}{p+1}} + \|u(t)\|_{L^2(\Omega)}^{\frac{2}{r_2}} \right) \] (52)
We observe that \( \frac{2}{(p+1)(1-2\sigma)} = 1, \) so
\[ \|u(t)\|_{p+1}^{2} = \int_{\Omega} |u|^{p+1} dxdy. \]

From the definition of \(G(t)\), we have
\[
G(t) = \left( H^{1-\sigma}(t) + \eta e^{-\gamma t} (u,u_1) \right)^{\frac{1}{1-\sigma}} \leq 2^{\frac{1}{1-\sigma}} \left( H(t) + \eta \frac{1}{\sigma} \right)^{\frac{1}{1-\sigma}} (u,u_1)^{\frac{1}{1-\sigma}} \\
\leq 2^{\frac{1}{1-\sigma}} \left( H(t) + C_3 \eta \frac{1}{\sigma} \left( \|u(t)\|_{p+1} + \|u_1(t)\|_{2} \right)^{\frac{1}{1-\sigma}} \right) \\
= 2^{\frac{1}{1-\sigma}} \left( H(t) + C_3 \eta \frac{1}{\sigma} \left( \int_{\Omega} |u|^{p+1} dxdy + \|u_1(t)\|_{2}^{2} \right) \right) \\
\leq C \left( H(t) + \frac{1}{2} \|u(t)\|_{2}^{2} + \frac{p-1}{4(p+1)} \int_{\Omega} |u|^{p+1} dxdy \right),
\]
for some positive constant \(C\) such that
\[ C \geq 2^{\frac{1}{1-\sigma}} \max \left\{ 1, 2C_3 \eta \frac{1}{\sigma}, C_3 \eta \frac{1}{\sigma} \frac{4(p+1)}{p-1} \right\}. \]

A combination of (50) and (53) leads to
\[ (G(t))^{\frac{1}{1-\sigma}} \leq CG'(t), \forall t \geq 0. \] (54)

From (50), we see clearly that \(G'(t) \geq 0\). It follows from the definition of \(G(t)\) and the assumption on \(u_0\) and \(u_1\) that
\[ G(t) \geq G(0) > \eta (u_0,u_1)_2 \geq 0. \] (55)

Hence, \(G(t) > 0\). Integrating (54) over \((0,t)\) yields
\[ (G(t))^{\frac{1}{1-\sigma}} \leq (G(0))^{\frac{1}{1-\sigma}} - \frac{\sigma}{C(1-\sigma)} t \]
which gives
\[ (G(t))^{\frac{1}{1-\sigma}} \geq \frac{1}{(G(0))^{\frac{1}{1-\sigma}}} - \frac{\sigma}{C(1-\sigma)} t. \] (56)

From (56), we obtain that \(G(t)\) blows up in time
\[ T^* \leq \frac{C(1-\sigma)}{\sigma (G(0))^{\frac{1}{1-\sigma}}}. \] (57)

This completes the proof. \(\Box\)

5. Conclusion

In this paper, we have studied a plate equation supplemented with partially hinged boundary conditions as model for suspension bridge in the presence of fractional damping and non-linear source terms. We showed that the solution blows up in finite time. We saw that, even in the present of a weaker damping, the bridge will collapse in infinite time when the power \(p\) of the non-linear source term is sufficiently large. This is a very important factor for engineers to consider when constructing such types of bridges.

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