Transition fronts for inhomogeneous monostable reaction–diffusion equations via linearization at zero

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Abstract
We prove the existence of transition fronts for a large class of reaction–diffusion equations in one dimension, with inhomogeneous monostable reactions. We construct these as perturbations of corresponding front-like solutions to the linearization of the PDE at \( u = 0 \). While a close relationship between the solutions to the two PDEs is well known and has been exploited for KPP reactions (and our method is an extension of such ideas from Zlatoš A 2012 (J. Math. Pure Appl. 98 89–102)), to the best of our knowledge this is the first time such an approach has been used in the construction and study of fronts for non-KPP monostable reactions.

Keywords: reaction–diffusion equations, transition fronts, monostable reactions
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1. Introduction
We study transition fronts for the one-dimensional reaction–diffusion equation

\[ u_t = u_{xx} + f(x, u), \]

with an inhomogeneous non-negative reaction \( f \geq 0 \) satisfying \( f(x, 0) = f(x, 1) = 0 \), and with \( u \in [0, 1] \). Such PDEs model a host of natural processes such as combustion, chemical reactions, population dynamics and others, with \( u \) representing (normalized) temperature, concentration of a reactant, or population density.
Both $u \equiv 0$ and $u \equiv 1$ are equilibrium solutions of (1.1) and we are interested in studying the propagation of reaction in space, that is, the invasion of the state $u = 0$ by the state $u = 1$. An important class of solutions modeling the propagation of reaction are transition fronts. A (right-moving) transition front is any entire solution $u : \mathbb{R}^2 \to [0, 1]$ of (1.1) which satisfies

$$\lim_{x \to -\infty} u(t, x) = 1 \quad \text{and} \quad \lim_{x \to +\infty} u(t, x) = 0$$

for each $t \in \mathbb{R}$. In addition, we also require that for any $\varepsilon > 0$ there exists $L_\varepsilon < \infty$ such that

$$\sup_{t \in \mathbb{R}} \text{diam}\{x \in \mathbb{R} | \varepsilon \leq u(t, x) \leq 1 - \varepsilon\} \leq L_\varepsilon.$$  

(1.3)

The definition of a left-moving transition front is similar, with the limits in (1.2) exchanged.

We will consider here the case of monostable reactions, for which $u \equiv 1$ is an asymptotically stable solution while $u \equiv 0$ is unstable. We assume that $f$ is Lipschitz,

$$f(x, 0) = f(x, 1) = 0 \quad \text{for} \quad x \in \mathbb{R},$$

(1.4)

$$a(x) := f_a(x, 0) > 0 \quad \text{for} \quad x \in \mathbb{R}$$

(1.5)

(in particular, $f_a(x, 0)$ exists for $x \in \mathbb{R}$), and

$$a(x)g_0(u) \leq f(x, u) \leq a(x)g_1(u) \quad \text{for} \quad (x, u) \in \mathbb{R} \times [0, 1],$$

(1.6)

where $g_0, g_1 \in C^1([0, 1])$ satisfy

$$g_0(0) = g_0(1) = 0, \quad g_0'(0) = 1, \quad g_0(u) > 0 \quad \text{and} \quad g'_0(u) \leq 1 \quad \text{for} \quad u \in (0, 1),$$

(1.7)

$$g_1(1) = 0, \quad g_1'(0) = 1, \quad g_1'(u) \geq 1 \quad \text{for} \quad u \in [0, 1],$$

(1.8)

$$\int_0^1 \frac{g_1(u) - g_0(u)}{u^2} \, du < \infty.$$  

(1.9)

Finally, we let

$$a_- := \inf_{x \in \mathbb{R}} a(x) \leq \sup_{x \in \mathbb{R}} a(x) =: a_+.$$  

(1.10)

When the reaction $f(x, u) = f(u) \geq 0$ is homogeneous, a special case of transition fronts are travelling fronts. These are of the form $u(t, x) = U(x - ct)$, with front speed $c$ and front profile $U$ such that $\lim_{s \to -\infty} U(s) = 1$ and $\lim_{s \to -\infty} U(s) = 0$, and their study goes back to the seminal works of Kolmogorov et al [5], and Fisher [4]. They considered KPP reactions, a special case of monostable reactions with $g_1(u) = u$, and found that for each $c \geq c_0 := 2\sqrt{f'(0)}$ there is a unique travelling front $u(t, x) = U_c(x - ct)$. A simple phase-plane analysis argument (see Aronson and Weinberger [1]) shows that this turns out to be the case for general homogeneous monostable reactions, although with a different $c_0 \geq 2\sqrt{f'(0)}$. In contrast, ignition reactions, satisfying $f(u) = 0$ for $u \in [0, \theta] \cup [1]$ and $f(u) > 0$ for $u \in (\theta, 1)$ (for some ignition temperature $\theta \in (0, 1)$), give rise to a single speed $c_0 > 0$ and a single travelling front [1].

Despite many developments for homogeneous and space-periodic reactions in the almost eight decades since [4, 5] (see the reviews [2, 15] and references therein), transition fronts in spatially non-periodic media have only been studied relatively recently. The first existence of transition fronts result, for small perturbations of homogeneous bistable reactions (the latter are such that $f(u) < 0$ for $u \in (0, \theta)$ and $f(u) > 0$ for $u \in (\theta, 1)$), was obtained by Vakulenko and Volpert [14]. Existence without a hypothesis of closeness to a homogeneous reaction, for ignition reactions of the form $f(x, u) = a(x)g(u)$ with some ignition $g$, was proved by Mellet et al [8], and by Nolen and Ryzhik [11] (see also [7] for uniqueness and
stability results for these reactions). Existence, uniqueness, and stability of fronts for general inhomogeneous ignition reactions was proved by Zlatoš [17]. He also proved the existence of fronts for some monostable reactions which are in some sense not too far from ignition ones (satisfying, in particular, sup_{x < R} f_0(x, 0) ≤ \frac{1}{2} c_0^2, with c_0 the unique speed for some ignition f_0 with f(x, u) ≥ f_0(u) for all (x, u) ∈ \mathbb{R}^d × [0, 1]). All these results are based on recovering a front as a locally uniform limit, along a subsequence, of solutions u_n of the Cauchy problem with initial data u_n(\tau_n, x) = \chi_{(−∞,−n)}(x), where \tau_n → −∞ are such that |u_n(0, 0)| = \frac{1}{2}. The existence of a limit u on \mathbb{R}^2 is guaranteed by parabolic regularity, and the challenge is to show that u is a transition front. We note that even in the monostable case in [17], when one expects multiple transition fronts, the existence of only a single transition front was obtained.

A very different approach has been used by Nolen et al [10], and by Zlatoš [16] to prove the existence of multiple transition fronts for inhomogeneous KPP reactions. It is well known that when f is KPP, then there is a close relationship between the solutions of (1.1) and those of its linearization

\[ v_t = v_{xx} + a(x)v \]  

(1.11)

at u = 0. The reason for this is that all KPP fronts are pulled, with the front speeds determined by the reaction at u = 0, which is due to the reaction strength \frac{f'(u)}{u} being largest at u = 0 for any fixed x ∈ \mathbb{R}. This is in stark contrast with ignition fronts, which are always pushed because they are ‘driven’ by the reaction at intermediate values of u.

One can therefore consider the simpler front-like solutions of (1.11), which are of the form

\[ v_\lambda(t, x) = e^{\lambda t} \phi_\lambda(x). \]  

(1.12)

Here \phi_\lambda > 0 is a generalized eigenfunction of the operator \mathcal{L} := \partial_{xx} + a(x), satisfying

\[ \phi_\lambda'' + a(x)\phi_\lambda = \lambda \phi_\lambda \]  

(1.13)

on \mathbb{R}, which exponentially grows to \infty as x → −\infty and exponentially decays to 0 as x → \infty. If we let \lambda_0 := \sup \sigma(\mathcal{L}) be the supremum of the spectrum of \mathcal{L}, which satisfies \lambda_0 ∈ [a−, a+] because \sigma(\partial_{xx}) = (−\infty, 0], then it is a well known spectral theory result that such \phi_\lambda exists precisely when \lambda > \lambda_0, and is unique if we also require \phi_\lambda(0) = 1.

For KPP reactions one can try to use these solutions to find transition fronts for (1.1) with

\[ \lim_{x → ∞} \frac{u_\lambda(t, x)}{v_\lambda(t, x)} = 1 \]  

(1.14)

for each t ∈ \mathbb{R}, at least for some \lambda > \lambda_0. This has been achieved in [10] for KPP reactions which converge to a homogeneous KPP reaction as |x| → ∞, and for more general KPP reactions in [16]. In both cases one needs \lambda_0 < 2a− (otherwise it is possible that no transition fronts exist [10]) and \lambda ∈ (\lambda_0, 2a−).

In the present paper we show that this linearization approach can be extended to general non-KPP monostable reactions. Our method is an extension of the (relatively simple and robust) approach from [16]. There it was discovered that while v_\lambda is obviously a super-solution of (1.1) when g_1(u) = u (i.e., in the KPP case), one can also use v_\lambda to find a sub-solution of the form \bar{w}_\lambda(t, x) = \bar{h}_\lambda(v_\lambda(t, x)), for \lambda ∈ (\lambda_0, 2a−) and an appropriate g_0-dependent increasing function \bar{h}_\lambda : [0, \infty) → [0, 1) with

\[ \bar{h}_\lambda(0) = 0, \quad \bar{h}_\lambda'(0) = 1, \quad \lim_{v → ∞} \bar{h}_\lambda(v) = 1, \quad \bar{h}_\lambda(v) ≤ v \quad \text{on} \quad [0, \infty). \]  

(1.15)

It follows that \bar{w}_\lambda ≤ v_\lambda, and one can then find a transition front u_\lambda between the two using parabolic regularity (see below).

Since this \bar{w}_\lambda depends on g_0 but not on g_1, it remains a sub-solution even when g_1(u) ≥ u (the latter follows from (1.8)). But, v_\lambda need not be a super-solution anymore.
However, we prove here that sometimes one can also construct a super-solution of the form $w_\lambda(t, x) = h_\lambda(v_\lambda(t, x))$, for an appropriate increasing $h_\lambda : [0, \infty) \to [0, \infty)$ such that

$$h_\lambda(0) = 0, \quad h_\lambda'(0) = 1, \quad h_\lambda'(v) \geq 0 \quad \text{on } h_\lambda^{-1}([0, 1]).$$

(1.16)

Once again, we then find a transition front $u_\lambda$ between $w_\lambda$ and $\min\{w_\lambda, 1\}$. Moreover, a result of Nadin [9] (see also [12]) shows that once some front exists, then a (time-increasing) critical front also exists. The latter is a transition front $u_C$ for (1.1) such that if $u \neq u_C$ is any other transition front and $u(t, x) = u_C(t, x)$ for some $(t, x) \in \mathbb{R}^2$, then

$$[u_C(t, y) - u(t, y)](y - x) < 0$$

for all $y \neq x$. That is, a critical front is the (unique up to time translation) ‘steepest’ transition front for (1.1), and is the inhomogeneous version of the minimal speed front for homogeneous reactions. Indeed, if $f$ is homogeneous, then $u_C$ is precisely the travelling front with the minimal speed $c_0$.

Thus we obtain the following result.

**Theorem 1.** Assume (1.4)–(1.9), let $v := \sup_{x \in (0, 1)} \frac{g'(x)}{a(x)} \geq 1$, and let the supremum of the spectrum of $L := \delta_{xx} + a(x)$ be $\lambda_0 := \sup \sigma(L) \in [a_-, a_+]$. If $\lambda \in (\lambda_0, 2a_-)$ satisfies

$$\lambda \leq 2a_+ - \frac{2\sqrt{v - 1}}{\sqrt{v + \sqrt{v - 1}a_+}},$$

(1.17)

then (1.1) has a transition front $u_\lambda$ with $(u_\lambda)_t > 0$, satisfying (1.14). In particular, if $\lambda_0$ is smaller than the right-hand side of (1.17), then a critical front $u_C$ also exists and $(u_C)_t > 0$.

**Remarks.**

1. If $a(x) = f_0(x, 0)$ is constant on $\mathbb{R}$, then $\lambda_0 = a_- = a_+$, so the right-hand side of (1.17) is always greater than $\lambda_0$. Thus a transition front exists for any $g_0, g_1$ in this case.

2. The front $u_\lambda$ does not have a constant speed in general, but when $f$ is stationary ergodic in $x$, then it almost surely has an asymptotic speed $c_\lambda > 0$ in the sense that if $X(t)$ is the rightmost point such that $u(t, X(t)) = \frac{1}{2}$, then

$$\lim_{|t| \to \infty} \frac{X(t)}{t} = c_\lambda.$$

This is because the same claim holds for $v_\lambda$ [16] and $\hat{h}_\lambda(v_\lambda) \leq u_\lambda \leq \hat{h}_\lambda(v_\lambda)$.

3. The result also holds with $v_\lambda$ replaced by more general solutions of (1.11) of the form $v_\mu(t, x) = \int v_\lambda(t, x) \, d\mu(\lambda)$, with $\mu$ a finite non-negative non-zero Borel measure supported on a compact subset of $(\lambda_0, 2a_+ - 2\sqrt{v - 1}(\sqrt{v + \sqrt{v - 1})^{-1}a_+})$ (or of $(\lambda_0, 2a_-)$ if $v = 1$).

4. The result also applies to the more general equation

$$u_t = (A(x)u_x)_x + q(x)u_x + f(x, u)$$

with

$$0 < A_- \leq A(x) \leq A_+ < \infty \quad \text{and} \quad |q(x)| \leq q_* < \infty$$

for all $x \in \mathbb{R}$, provided that $q_* \leq 2\sqrt{(aA)_-}$ with $(aA)_- := \inf_{x \in \mathbb{R}} [a(x)A(x)]$, where

$$\lambda_0 := \sup_{\phi \in H^1(\mathbb{R})} \frac{\int_{\mathbb{R}} [-A(x)\psi'(x)^2 + q(x)\psi'(x)\psi(x) + a(x)\psi(x)^2] \, dx}{\int_{\mathbb{R}} \psi(x)^2 \, dx} \quad (\geq a_-)$$

and $2a_-$ is replaced in (1.17) by

$$\lambda_1 := \inf_{x \in \mathbb{R}} \left[ a(x) + \sqrt{(aA)_-} \left\{ \sqrt{(aA)_- - |q(x)|} \right\} A(x)^{-1} \right] \quad (\leq 2a_-).$$
We indicate the proofs of remarks 2–4 after the proof of the theorem.

Our construction of the super-solution $u_0$ is of independent interest and extends to more general equations in several dimensions, possibly with time-dependent coefficients. Hence we state it here as a separate result.

**Lemma 1.2.** Let the function $f(t,x,u) \geq 0$, positive definite matrix $A(t,x)$, and vector field $q(t,x)$ be all Lipschitz, with $(t,x,u) \in (t_0,t_1) \times \mathbb{R}^d \times [0,1]$ and some $-\infty < t_0 < t_1 < \infty$. Assume that $a(t,x) \equiv f_u(t,x,0) > 0$ exists, (1.4)–(1.9) hold with $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$ in place of $x \in \mathbb{R}^d$, and define $v := \sup_{u \in (0,1)} \frac{f(u)}{a} \geq 1$. Let $v > 0$ be a solution of

$$u_t = \nabla \cdot (A(t,x) \nabla v) + q(t,x) \cdot \nabla v + a(t,x)v$$

on $(t_0,t_1) \times \mathbb{R}^d$.

If $v > 1$ and for some $\alpha \leq (\sqrt{v} - \sqrt{v-1})^2$ (or for some $\alpha < 1$ if $v = 1$),

$$\nabla v(t,x) \cdot A(t,x) \nabla v(t,x) \leq \alpha a(t,x)v(t,x)^2$$

holds for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$, then there exist increasing functions $\tilde{h}$ satisfying (1.15) and $h$ satisfying (1.16) such that $\tilde{h} := \tilde{h}(v)$ is a sub-solution of

$$u_t = \nabla \cdot (A(t,x) \nabla u) + q(t,x) \cdot \nabla u + f(t,x,u)$$

on $(t_0,t_1) \times \mathbb{R}^d$ and $w := h(v)$ is a super-solution on $((t_0,t_1) \times \mathbb{R}^d) \cap \{v(t,x) \leq 1\}$. Therefore, if $u$ solves (1.19) with

$$\tilde{u}(t_0,x) \leq u(t_0,x) \leq \min\{w(t_0,x),1\}$$

for all $x \in \mathbb{R}^d$, then for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$ we have

$$\tilde{u}(t,x) \leq u(t,x) \leq \min\{w(t,x),1\}. \tag{1.21}$$



**2. Proof of theorem 1.1 (using lemma 1.2)**

Let $\lambda \in (\lambda_0,2a_-)$ and $v = v_0$ be from (1.12), with $\phi = \phi_0 \geq 0$ from (1.13) with $\lim_{x \to \infty} \phi(x) = 0$ and $\phi(0) = 1$. It is proved in [16, proof of theorem 1.1] that there exists such a $\phi$ and it satisfies

$$\phi'(x)^2 \leq a_+ \phi(x)^2 \tag{2.1}$$

for $\alpha := 1 - (2a_- - \lambda)a_+^{-1} < 1$ and all $x \in \mathbb{R}$, as well as

$$\phi(x) \geq 2\phi(y) \tag{2.2}$$

for some $L < \infty$ and any $y - x \geq L$.

Since $\alpha \leq (\sqrt{v} - \sqrt{v-1})^2$ is, by the definition of $\alpha$, equivalent to

$$\lambda \leq 2a_- - \left[1 - \left(\sqrt{v} - \sqrt{v-1}\right)^2\right]a_+ = 2a_- - \frac{2\sqrt{v} - 1}{\sqrt{v} + \sqrt{v-1}}a_+$$

(which is (1.17)), lemma 1.2 applies to $v$ and (1.1). Thus we have (1.21) and a standard limiting argument now recovers an entire solution to (1.1) between $\tilde{u}$ and $\min\{v,1\}$. We let $u_n$ be the solution of (1.1) on $(-n,\infty) \times \mathbb{R}$ with $u_n(-n,x) := \tilde{w}(-n,x)$. Since $\tilde{u}(t,x) \leq \min\{w(t,x),1\}$ because $h(v) \geq v$ for $v \in h^{-1}(0,1)$, (1.20) is satisfied with $t_0 := -n$ and we have (1.21) on $(-n,\infty) \times \mathbb{R}$. By parabolic regularity, there is a subsequence of $\{u_n\}$ which converges, locally uniformly on $\mathbb{R}^2$, to an entire solution $u$ of (1.1). We obviously have

$$\tilde{w} \leq u \leq \min\{w,1\}. \tag{2.3}$$
and (1.14) for $u_\lambda := u$ follows from $\tilde{h}'(0) = h'(0) = 1$. We also have $u_t \geq 0$, because $(u_n)_t \geq 0$ due to $\dot{w}_t = h'(v)u_t \geq 0$ and the maximum principle for $(u_n)_t$ (which satisfies a linear equation and is non-negative at $t = -\infty$). The strong maximum principle then gives $u_t > 0$ because obviously $u_t \not\equiv 0$. Finally, $u$ is a transition front because the second limit in (1.2) follows from $\lim_{t \to -\infty} \phi(x) = 0$ and (1.16), and (1.3) holds with $L_\lambda := L \left[ \log_2 \left( \tilde{h}^{-1}(1 - \varepsilon) - h^{-1}(\varepsilon) \right) \right]$ due to (2.3) and (2.2) (with $\tilde{h}$, $h$ from the lemma). The first limit in (1.2) is then obvious from $u \leq 1$, and the proof is finished by using the result from [9] for critical fronts.

The claim in remark 2 is proved as an analogous statement in [16, theorem 1.2].

The claim in remark 3 holds because $L$ can be chosen uniformly for all $\lambda$ in the support of $\mu$ [16] and so (2.2) holds with $\phi(\cdot)$ replaced by $v_\mu(t, \cdot)$. Also, $v_\mu$ satisfies (2.1) with $\alpha$ corresponding to $\lambda := \sup \text{supp } \mu$.

The claim in remark 4 holds because (2.1) and (2.2) continue to hold in that case, albeit with $2a_-$ replaced by $\lambda_1$ in the definition of $\alpha$ [16].

3. Proof of Lemma 1.2

Lemma 2.1 in [16] shows that there is an increasing $\tilde{h} = \tilde{h}_2$ as in (1.15) such that $\tilde{w}(t, x) := \tilde{h}(v)$ is a sub-solution of (1.1) with the reaction $\min \{ f(x, u), a(x)u \}$ (which is a KPP reaction). Then $\tilde{w}$ is also a sub-solution of (1.1), which yields the first inequality in (1.21).

We will next find an increasing $h = h_2$ as in (1.16) such that $w(t, x) := h(v(t, x))$ will be a super-solution on the space-time domain where $w(t, x) \leq 1$, which will yield the second inequality in (1.21) because $u \leq 1$ by the hypotheses. Our proof will be a super-solution counterpart to the sub-solution argument in [16, lemma 2.1]; it was a little surprising to us that such a counterpart argument can be found for non-KPP reactions.

If $h$ is as in (1.16), then (1.18) shows that

$$w_t = \nabla \cdot (A \nabla w) - q \cdot \nabla w = h'(v) \left[ v_t - \nabla \cdot (A \nabla v) - q \cdot \nabla v \right] - h''(v) \nabla v \cdot A \nabla v$$

$$= h'(v)u \nabla h''(v) \nabla v \cdot A \nabla v$$

$$\geq a(vh'(v) - \alpha^2 h''(v))$$

when $w(t, x) \leq 1$. We can then conclude that $w$ is a super-solution of (1.1) where $w(t, x) \leq 1$ once we show that on $h^{-1}([0, 1])$ we also have

$$vh'(v) - \alpha^2 h''(v) \geq g_1(h(v)).$$

(3.1)

It therefore remains to find $h$ satisfying (1.16) and (3.1). We let $c := \alpha^{1/2} + \alpha^{-1/2}$ and notice that since $\gamma + \gamma^{-1} \geq 2\sqrt{\nu}$ for all positive $\gamma \leq \sqrt{\nu} - \sqrt{\nu - 1}$, the hypothesis $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ yields $c \geq 2\sqrt{\nu}$. Next let $U$ be the unique solution to the ODE

$$U'' + cU' + g_1(U) = 0$$

(3.2)
on $[s_0, \infty)$, with

$$U(s_0) = 1 \quad \text{and} \quad U'(s_0) = -\sqrt{\alpha}g_1(1),$$

(3.3)

where $s_0 \in \mathbb{R}$ will be chosen later. (This is the ODE that would be satisfied by the travelling front profile with speed $c$ for the homogeneous reaction $g_1(u)$ if we had $g_1(1) = 0$; this profile would then also satisfy $\lim_{s \to -\infty} U(s) = 1$ and $\lim_{s \to -\infty} U'(s) = 0$ instead of (3.3).)

Notice that $U'(s_0) \geq -\frac{\alpha}{\sqrt{\alpha}}$ because $g_1(1) \leq v$ and

$$\frac{\sqrt{\alpha}}{v} \leq (\sqrt{\nu} + \sqrt{\nu - 1})^{-1} \leq v^{-1/2}.$$

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Let \( V(s) := U'(s) \), and consider the curve \( \theta := ((U(s), V(s)))_{s \geq s_0} \). It is easy to see that \( \theta \) cannot leave the closed triangle \( T \) in the \((U, V)\) plane with sides \( V = 0, U = 1, \) and \( V = -\frac{1}{2} U \). This is because \((U(s_0), V(s_0)) \in T \) and on \( \partial T \), the vector field \((V, -cV - g_1(U))\) either points inside \( T \) or is parallel to \( \partial T \). Here we use \( c \geq 2\sqrt{\frac{\epsilon}{\alpha}} \) to obtain on the third side
\[
\left( \frac{c}{2}, 1 \right) \cdot \left( -\frac{c}{2} U, -c \left( -\frac{c}{2} U \right) - g_1(U) \right) = \frac{c^2}{4} U - g_1(U) \geq vU - g_1(U) \geq 0. \tag{3.4}
\]
It follows that \( U'(s) < 0 \) on \([s_0, \infty)\), and since \( g_1(U(s)) > 0, U(s) \) cannot have local minima on \([s_0, \infty)\). Hence \( \lim_{s \to \infty} U(s) \) exists and \( \lim_{s \to \infty} U'(s) = 0 \). Finally, \( g_1 > 0 \) on \((0, 1]\) yields \( \lim_{s \to \infty} U(s) = 0 \).

We now define \( h(0) := 0 \) and
\[
h(v) := U(-\alpha^{-1/2} \ln v) \tag{3.5}\]
for \( v \in (0, e^{-\sqrt{\alpha} s_0}] \), so \( h \) is increasing and continuous at 0, with \( h(e^{-\sqrt{\alpha} s_0}) = 1 \) (we then extend \( h \) onto \([0, \infty)\) arbitrarily, only requiring that it be increasing). Since \( c > 2\sqrt{\frac{\epsilon}{\alpha}} g_1(0) = 2 \) and
\[
\int_0^1 \frac{g_1(u) - u}{u^2} \, du < \infty
\]
by (1.7) and (1.9), a result of Uchiyama [13, lemma 2.1] shows \( \lim_{s \to \infty} U(s)e^{\sqrt{\alpha} s} \in (0, \infty) \).

(This result assumes \( g_1(1) = 0 \) but we can extend \( g_1, U \) to \([0,2]\) so that \( g_1(2) = 0 \) and \( U \) satisfies (3.2), and then apply [13] to \( \tilde{g}(u) := \frac{1}{2} g_1(2u) \) and the function \( \tilde{U}(s) := \frac{1}{2} U(s) \).)

If we now pick the unique \( s_0 \) in (3.3) such that \( \lim_{s \to \infty} U(s)e^{\sqrt{\alpha} s} = 1 \) (notice that (3.2) is an autonomous ODE), we obtain \( h'(0) = 1 \). We also have (3.1) on \([0, e^{-\sqrt{\alpha} s_0}] \) because on that interval, (3.2) immediately yields
\[
\alpha v^2 h'(v) - vh'(v) + g_1(h(v)) = 0. \tag{3.6}
\]

It therefore remains to show that \( h''(v) \geq 0 \) on \([0, e^{-\sqrt{\alpha} s_0}] \). Due to (3.6) and (3.5), this is equivalent to
\[
-U'(s) \geq \sqrt{\alpha} g_1(U(s)) \tag{3.7}
\]
for \( s \geq s_0 \). Thus we need to show that \( \theta \) stays at or below \( \psi := \{(U(s), -\sqrt{\alpha} g_1(U(s)))\}_{s \geq s_0} \).

This is true at \( s = s_0 \) by the definition of \( U \), so it is sufficient to show that on \( \psi \), the vector field \((V, -cV - g_1(U))\) points either below or is parallel to \( \psi \). This holds because the normal vector to \( \psi \) pointing down is \( (\sqrt{\alpha} g'_1(U)V, V) \), so on \( \psi \) we have
\[
(V, -cV - g_1(U)) \cdot (\sqrt{\alpha} g'_1(U)V, V) = \alpha^{1/2} g'_1(U)V^2 - (\alpha^{1/2} + \alpha^{-1/2})V^2 - \alpha^{1/2} g_1(U)\alpha^{-1/2}V = \alpha^{1/2}(g'_1(U) - 1)V^2,
\]
which is non-negative due to (1.8) and \( U(s) \leq 1 \) for \( s \geq s_0 \). It follows that \( h''(v) \geq 0 \) on \([0, e^{-\sqrt{\alpha} s_0}] \) and so \( h \) satisfies (1.16) and (3.1). The proof is finished.

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