Transitions from the Quantum Hall State to the Anderson Insulator: Fate of Delocalized States

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Abstract

Transitions between the quantum Hall state and the Anderson insulator are studied in a two dimensional tight binding model with a uniform magnetic field and a random potential. By the string (anyon) gauge, the weak magnetic field regime is explored numerically. The regime is closely related to the continuum model. The change of the Hall conductance and the trajectory of the delocalized states are investigated by the topological arguments and the Thouless number study.

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I. INTRODUCTION

In the integer quantum Hall effect, delocalized states near the center of each Landau band play a crucial role \[1\]. On the other hand, when the randomness strength becomes as large as the energy cutoff, it is expected that all the states become localized \[2\]. An appealing scenario for the transitions, which is called the 'floating' or 'levitation' of the delocalized states, was proposed in refs. \[3\] \[5\]. It claims that the delocalized states do not disappear discontinuously but float upward in energy with the increase of the randomness strength. Finally, when all the delocalized states rise above the Fermi energy, the system becomes the usual Anderson insulator. Then a crossover occurs from the unitary class to the orthogonal class. Furthermore, based on the scenario, the global phase diagram was proposed for transitions between different quantum Hall states \[6\]. Recently some controversies arise about the trajectory of the delocalized states \[7\] \[11\]. There are also experimental discussions on the validity of the global phase diagram \[12\] \[15\].

In this paper, we study transitions between the quantum Hall state and the Anderson insulator in a tight binding model. The weak magnetic field regime is especially focused on. It is closely related to the continuum model. The topological invariant for each energy band (Chern number) and the Thouless number are used as probes for the transitions. The summation of the Chern number below the Fermi energy gives the Hall conductance. The Thouless number tells us the location of the delocalized states. This paper is an extended version of our previous letter \[10\] including new data and more detailed discussions.

The present article is organized as follows. In the next section, the tight binding Hamiltonian is defined and the string (anyon) gauge is introduced. The string gauge is essential for the numerical realization of the weak magnetic field regime. In section III, key observables in our study are defined: the topological invariant for each energy band (Chern number) and the Thouless number. Section IV is devoted to the statement of a sum rule in the change of the Hall conductance. In Section V, the weak magnetic field regime is studied in connection with the continuum model. The breakdown of the quantized Hall conductance due to randomness is demonstrated and the trajectory of the delocalized states is investigated.

II. MODEL

A. Hamiltonian

The tight binding Hamiltonian is

\[ \mathcal{H} = \sum_{\langle i, j \rangle} c_i^\dagger t_{ij} c_j + h.c. + \sum_i w_i c_i^\dagger c_i, \]

where \( t_{ij} = \exp(ia_{ij}) \), \( a_{ij} \) denotes a gauge potential and \( \langle i, j \rangle \) refers to a nearest neighbor. A magnetic field per plaquette is given by \( \phi = \sum_{\text{a}} a_{ij}/2\pi = p/q \) (\( p \) and \( q \) are coprime integers) where the summation runs over four links around a plaquette. The operator \( c_i^\dagger \) (\( c_i \)) creates (annihilates) a spinless fermion at a site \( i \). A random potential at a site \( i \) is expressed by \( w_i = W f_i \) where \( f_i \)'s are uniform random numbers and chosen from \([-1/2, 1/2]\). See, for example, refs. \[7\] \[8\] \[16\] \[17\] for previous works on this model.
The integer quantum Hall effect is believed to be described by the continuum model. As will be discussed later, the weak magnetic field regime in the tight binding model is closely related to it. Therefore, in this paper, we focus on the weak magnetic field regime (e.g. $\phi = p/q = 1/64$). In order to realize the regime numerically, the choice of a gauge potential plays an important role. Here we employ the string (anyon) gauge [10]. An example of the string gauge is shown in Fig.1. The extension to other geometries is straightforward.

Choosing a plaquette $S$ as a starting one, we draw outgoing arrows (strings) from the plaquette $S$. $a_{ij}$ on a link $ij$ is given by $\phi n_{ij}$ where $n_{ij}$ is the number of strings cutting the link $ij$ (the orientation is taken into account). In the string gauge, the compatible magnetic field $\phi$ with the periodicity of the system $L_x \times L_y$ is $\phi = n/L_x L_y$, $n = 1, 2, \cdots, L_x L_y$ (on the other hand, $\phi = n/L_{x(y)}$, $n = 1, 2, \cdots, L_{x(y)}$ in the Landau gauge). Therefore, for a square $L \times L$ geometry, one is able to use $L$ times smaller magnetic field in the string gauge than in the Landau gauge.

III. OBSERVABLES

In this section, we define key observables to study transitions between the quantum Hall state and the Anderson insulator. One is the topological invariant for each energy band (the Chern number). It is closely related to the Hall conductance. The other is the Thouless number. It tells us how the wavefunctions are extended spatially.

A. Topological invariant (Chern number)

The topological invariant for each energy band, Chern number, plays an important role in the following arguments. Here the periodicity of $L_x \times L_y$ is imposed on $a_{ij}$ and $w_i$ and the momentum $k$ is well-defined (the infinite size limit corresponds to $L_x, L_y \to \infty$). The Chern number for the $n$-th band, $C_n$, is defined by

$$C_n = \frac{1}{2\pi i} \int d\mathbf{k} \hat{z} \cdot (\nabla \times \mathbf{A}_n), \quad \mathbf{A}_n = \langle u_n(k)|\nabla|u_n(k)\rangle,$$

where $\nabla = \partial/\partial \mathbf{k}$, $\langle u_n(k)\rangle$ is a Bloch wavefunction of the $n$-th band with $L_x L_y$ components and the $\gamma$-th component is $u_n^\gamma(k)$. The integration $\int d\mathbf{k}$ runs over the Brillouin zone. When the Fermi energy lies in the lowest $j$-th energy gap, the Hall conductance $\sigma_{xy}$ is given by

$$\sigma_{xy} = \sum_{n=1}^{j} C_n \ [18,19].$$

When all states in a band are localized, the Chern number vanishes. On the other hand, states with a non-zero Chern number contribute to the $\sigma_{xy}$ and we call them 'extended states'.

Arbitrarily choosing the $\alpha$ and $\beta$-th components of the wavefunction, another expression of the Chern number is written as

$$C_n = \sum_\ell N_{n\ell}, \quad N_{n\ell} = \frac{1}{2\pi i} \oint_{\partial R_\ell} d\mathbf{k} \cdot \nabla \text{Im} \ln \left( \frac{u_n^\alpha(k)}{u_n^\beta(k)} \right).$$
Here $k_\ell$ is a zero point (vortex) of $w_n^\alpha(k)$ in the Brillouin zone. $N_n^\ell$ is a winding number (charge) of the vortex, $R_\ell$ is a region around $k_\ell$ which does not include either zero of the $\alpha$-th or $\beta$-th component, and $\partial R_\ell$ is the boundary. The arbitrariness in the choice of indices $\alpha$ and $\beta$ is a kind of gauge degree of freedom. The configuration of the vortices depends on the gauge choice but the physical observables, e.g. the Hall conductance and the position of each extended state, are gauge invariant.

B. Thouless number

The Thouless number $g(E)$ is widely used in the study of the Anderson localization [20] (see also ref. [16]). It tells us how the wavefunctions at energy $E$ are extended spatially. Here a finite $L\times L$ cluster is used with a periodic or an antiperiodic boundary condition. It is defined by

$$g(E) = v(E)/w(E)$$

where the $v(E)$ is an energy shift by replacing the boundary condition and $w(E)$ is a local level spacing. By fitting the $g(E)$’s to the form $g(E) = g_0 \exp(-L/\xi(E))$, one can determine the $\xi(E)$ which is the localization length when $L$ is sufficiently large. However, when the condition $L > \xi(E)$ is not satisfied, the $\xi(E)$ can not be identified with the localization length. For example, in the intermediate region discussed below, the localization length is extremely long and beyond our available system size.

IV. CHANGE OF THE HALL CONDUCTANCE AND SUM RULE

Before going to the weak magnetic field regime, let us explain the change of $\sigma_{xy}$ due to randomness and a sum rule in the transitions [10,21–23]. By fixing the ‘gauge’ in the definition of the Chern number, the configuration of vortices are determined in each energy band. Results for $\phi = p/q = 1/3$ and 2/5 are shown as examples (Figs.2). As the randomness strength $W$ is increased, vortices in each band move in energy. The motion of each vortex is continuous as a function of $W$ and it forms a vortex line. Therefore the only way for vortices to appear or vanish is the pair-creation or pair-annihilation of them with opposite charges. Although vortices move with the change of $W$, the Chern number does not change generally due to its topological stability. When the two bands touch at a critical value of $W = W_0$, the spectrum generally becomes that of the Dirac fermion [24–26]. Define $(k_x^0, k_y^0)$ by the gap-closing point in the Brillouin zone and focus on the neighborhood i.e. $p = \frac{q}{2} (k_x - k_x^0, k_y - k_y^0, W - W_0) \sim 0$. Then the leading part of the Hamiltonian is, in generic, given by

$$H_0(p) = 1v_0p + (\sigma_x, \sigma_y, \sigma_z)v_p,$$

where $v_0$ is a $1\times 3$ vector, $\sigma_{x(y,z)}$ is a $2\times 2$ Pauli matrix and $v$ is a $3\times 3$ matrix. By performing the unitary transformation and the redefinition of $p$, the $H_0$ reduces to [22]

$$H_1(p) = 1v_0p + \sigma_x p_x + \sigma_y p_y + \sigma_z p_z \text{sgn}(\det(v)).$$
By diagonalizing the $H_1$, it can be seen that a vortex moves from one band to the other at the gap-closing point and the Chern number for each band changes. Then the continuity requires that the total Chern numbers of the two bands do not change. This is the sum rule \[10,21,22\]. Moreover, the change of $\sigma_{xy}$ with a fixed electron density generally obeys the selection rule $\Delta \sigma_{xy} = \text{sgn}(\det v) = \pm 1$ \[10,10,22\]. However, the change of the observed Hall conductance $\sigma_{xy}^{\text{phys}}$ can break the selection rule (anomalous plateau transition). Here the definition of $\sigma_{xy}^{\text{phys}}$ is

$$\sigma_{xy}^{\text{phys}} = \lim_{T \to 0} \lim_{L \to \infty} \sum_n f(E_n) C_n$$

where $E_n = E_n(k = 0)$ and $f(E)$ is the Fermi distribution function. Then the $\sigma_{xy}^{\text{phys}}(L)$ is obtained as the averaged $\sigma_{xy}(L)$ within an energy window $|E - E_f| \sim O(1/L)$. Therefore we assume

$$\sigma_{xy}^{\text{phys}}(L) \approx \sigma_{xy}(L)$$

where $L$ is sufficiently large and $\sigma_{xy}(L)$ is the averaged $\sigma_{xy}(L)$ over different realizations of randomness \[27\]. The $\sigma_{xy}(L)$ is quantized in each realization of randomness even when the Fermi energy lies in a small gap $\sim O(1/L)$. However, since the small gap is sensitive to the randomness realization, the quantization breaks down after the ensemble average. Therefore the anomalous plateau transition $\sigma_{xy} = n \to 0$ occurs (see also ref. \[10\]). It is in contrast to the plateau transition due to the change of electron density where $\Delta \sigma_{xy} = \pm 1$ holds even after the ensemble average.

V. WEAK MAGNETIC FIELD REGIME

Now we shall explore the weak magnetic field regime, which is closely related to the continuum model. In the following, $\phi$ is fixed as $\phi = p/q = 1/64$. It is realized numerically by the string gauge discussed above. Transitions between the quantum Hall state and the Anderson insulator are investigated by the topological invariant for each energy band (Chern number) and the Thouless number. The summation of the Chern number below the Fermi energy gives the Hall conductance. The Thouless number tells us the location of the delocalized states.

A. Breakdown of quantum Hall states

In Figs.3, the density of states (DOS) and the Thouless number $g(E)$ are shown for different randomness strengths $W$’s. In Fig.4, the vortices are shown as a function of $W$. In Figs.5, the averaged local Chern number $\overline{C(E)}$ (see \[28\] for a precise definition) and its variance $(\overline{\delta C(E)^2})^{1/2}$ are shown with different $W$’s. Here $\overline{O}$ denotes an averaged observable $O$ over different realizations of randomness. Moreover, in Figs.6, the Chern number of the $i$-th band $C_i$ and $\overline{C_i}$ are shown as a function of $W$.

Let us first investigate a region with sufficiently weak randomness. As seen in Fig.3 (a), there are well-defined Landau bands. The Thouless number $g(E)$ shows a sharp peak near the center of each Landau band. It is consistent with the result that all the states are
localized except at a single energy in each Landau band \cite{16}. The number of states extended over a \( L \times L \) sample is \( \sim L^{2-1/\nu} \) and is not macroscopic. The Chern number in each Landau band is +1 except in the center band. The sum of all Chern numbers is always zero and the center band has negative Chern number \(-62\). When the Fermi energy lies in the Landau gap or near the band edge, the \( \sigma_{xy} \) does not depend on the randomness realization generally. Then the \( \sigma_{xy} \) is quantized to an integer even after the ensemble average and the system belongs to the quantum Hall states.

Next let us consider the strong-randomness effects. With the increase of \( W \), the Landau gaps collapse from the center (\( E = 0 \)) to the bottom (see Figs.3). As seen in Figs.4-6, it is associated with many pair-creations and -annihilations. Extended states with a negative Chern number spread from the center to the bottom through the pair-creations and -annihilations and an energy region where \( \overline{C(E)} \sim 0 \) but \((\delta \overline{C(E)}^2)^{1/2} \neq 0 \) extends over the whole spectrum (see Figs.5 (c) and (d)). We call it the intermediate region. In this region, the mean free path \( \ell \) is comparable with the magnetic length \( \ell_B(\sim 1/\sqrt{\phi}) \) and the internal structure within the magnetic unit cell is irrelevant. Then, since the magnetic unit cell contains multiple of the flux quanta, it is effectively gauge equivalent to the zero magnetic field and a crossover occurs from the unitary class to the orthogonal class. In the region, the Chern number in each band is generally non-zero and quantized to an integer for a given realization of randomness. However, the Chern number depends strongly on the randomness realization. Then the averaged Chern number is not an integer and takes a small value due to the cancellation of positive and negative Chern numbers in different realizations. Therefore, after the ensemble average, the \( \sigma_{xy} \) is no longer quantized to an integer. In the region, the energy position of each extended state depends strongly on the randomness realization. The number of the extended states is not macroscopic and we speculate that the probability measure for the delocalized states to exist is zero in the infinite size limit (similar region was found in the random magnetic field problem \cite{29,30}). When the randomness is stronger, the extended states no longer exist and \( \overline{C(E)} = 0 \) in each realization of randomness. Then the system belongs to the usual Anderson insulator and the \( \sigma_{xy} \) becomes zero.

To summarize: for sufficiently weak randomness, there are delocalized states in each Landau band and the system is in the quantum Hall state. As the randomness strength is increased, the Landau gaps collapse from the center (\( E = 0 \)) to the bottom and the intermediate region extends over the whole spectrum. When the Fermi energy lies in the region, the \( \sigma_{xy} \) is no longer quantized to an integer. With sufficiently strong randomness, the system change into the usual Anderson insulator and the \( \sigma_{xy} = 0 \).

**B. Scaling in the weak magnetic-field limit**

In the above subsection, the magnetic-field strength is kept as \( \phi = p/q = 1/64 \) and the randomness strength \( W \) is varied. In this subsection, we comment on the scaling in the weak magnetic-field limit \((\phi = 1/q \to 0)\).

In the limit \((\phi \to 0)\) without randomness \( W = 0 \), the rescaled energy spectrum \( E/E_g \) becomes the Landau levels and the system reduces to the continuum model \((E_g \sim \Gamma_c/q \approx 1)\) is the energy cut-off in the lattice model\). In the limit, the magnetic length \( l_B \) is much larger than the lattice spacing i.e. \( l_B = 1/\sqrt{\phi} = \sqrt{q} \to \infty \).
When the $W$ is increased to $\sim W_c$, the width of the lowest Landau band becomes comparable with the Landau gap. In the weak field limit, $W_c/\Gamma_c \to 0$ but $W_c/E_g \to \infty$ \cite{9}. It implies that the lowest Landau band is robust in the continuum limit. Therefore, before the lowest Landau gap closes, higher Landau bands merge and the reconstruction of eigenstates occurs through the pair-creation and -annihilation. Then the 'floating' is not the only way for the delocalized states to disappear. For stronger randomness ($\Gamma_c \gtrsim W \gtrsim W_c$), all the states are in the intermediate region. In this region, the localization length is extremely large. Finally, when $W \gtrsim \Gamma_c$, the system belongs to the usual Anderson insulator.

C. Fate of delocalized states

At last, let us discuss the fate of delocalized states. In Fig.7, the trajectory of the delocalized states is shown for the lowest Landau band ($n = 0$) and the second one ($n = 1$). In order to search the trajectory, the Thouless number $g(E)$ is obtained for $L_x \times L_y = 24 \times 24$, $32 \times 32$ and $40 \times 40$ (the ensemble average is performed over 100 different realizations of randomness). The energy position of the delocalized states is obtained by searching the peak position of $g(E)$ in each Landau band. For the full circles, we also confirmed that the localization length $\xi(E)$ is the largest at the point in each Landau band (see the inset of Fig.7 as an example). It can be observed that their positions are going down in energy with the increase of randomness strength $W$. At the same time, as seen in Figs.3, the density of states also shifts downwards. Then, in the lowest Landau band, the delocalized states 'floats' or 'levitates' slightly \cite{31}. However there is no sign of the floating across the Landau gap. The way how the delocalized states disappear is also shown in Fig.7. After the collapse of the Landau gap, the intermediate region extends over the whole spectrum and the delocalized states mix in the region. Even after the collapse, the peak positions of $g(E)$ can be defined for each Landau band and they are also shown by the open circles. In the region, the positions have a large fluctuation and they finally merge when $W$ is sufficiently large. In fact, for a given realization of randomness, we can further trace the trajectory of states which carry non-zero Chern number, and it finally disappear through the pair-annihilation \cite{10}. However, since their positions in the region depend strongly on the randomness realization, we speculate that the probability density for the states to be delocalized is zero.

VI. SUMMARY

In summary, we studied transitions between the quantum Hall state and the Anderson insulator based on the tight-binding model. Aspects of the transitions are revealed by the topological arguments and the Thouless number study. In the weak randomness region, there are delocalized states in each Landau band and the system is in the quantum Hall state. As the randomness becomes stronger, the Landau gaps collapse from the center ($E = 0$) to the bottom. Accompanied by the collapse, the intermediate region with a large localization length extends over the whole spectrum. When the Fermi energy lies in the region, the Hall conductance is not generally quantized to an integer. The delocalized states in each Landau band mix in the region successively and, finally, the system belongs to the Anderson insulator.
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FIGURES

FIG. 1. The definition of the string (anyon) gauge for a $3 \times 3$ square system with periodic boundary condition.

FIG. 2. Zero points (vortices) of the Bloch wavefunctions with their winding numbers (charges). The shaded regions show energy bands. Each circle denotes the energy position of a vortex as a function of randomness strength $W$. The full circle means the charge $+1$ and the open circle $-1$. When the Fermi energy lies in the energy gap, the $\sigma_{xy}$ is quantized to an integer. The integer is also shown in each energy gap. (a) $\phi = 1/3$ and $L_x \times L_y = 3 \times 3$ and (b) $\phi = 2/5$ and $L_x \times L_y = 5 \times 1$.

FIG. 3. The density of states (DOS) and the Thouless number $g(E)$ for $\phi = 1/64$ and $L_x \times L_y = 40 \times 40$, where the ensemble average is performed over 100 different realizations of randomness. (a) $W = 0.346$, (b) $W = 1.386$ and (c) $W = 5.196$.

FIG. 4. Zero points (vortices) of the Bloch wave functions with their winding numbers (charges) for $\phi = 1/64$ and $L_x \times L_y = 24 \times 24$ (see also Figs.2).

FIG. 5. The averaged local Chern number $\langle C(E) \rangle$ (the solid line) and its variance $\langle \delta C(E)^2 \rangle^{1/2}$ (the broken line) for $\phi = 1/64$ and $L_x \times L_y = 8 \times 8$, where the ensemble average is performed over 100 different realizations of randomness. (a) $W = 0.476$, (b) $W = 1.169$, (c) $W = 2.641$ and (d) $W = 6.495$.

FIG. 6. The Chern number of the $i$-th band $C_i$ for $\phi = 1/64$ and $L_x \times L_y = 8 \times 8$ as a function of randomness strength $W$. It is to be noted that $C_i = +1$ except at the center band for sufficiently weak randomness. (a) the realization of randomness is fixed. (b) the ensemble average is performed over 100 different realizations.

FIG. 7. The trajectory of delocalized states for $\phi = 1/64$ is denoted by the full circles. For the open circles, although the Thouless number $g(E)$ has a peak at the point, the corresponding states belong to the intermediate region. The inset shows the density of states (DOS) and the inverse localization length $1/\xi(E)$ for $W = 1.039$. The $\xi(E)$ is obtained by fitting the $g(E)$’s to the form $g_0 \exp(-L/\xi(E))$. 
$C(E) \sqrt{\langle \delta C(E)^2 \rangle^{1/2}}$
$C(E), (\delta C(E)^2)^{1/2}$
$W = 2.641$

$C(E) \left( \delta C(E)^2 \right)^{1/2}$
\[ W = 6.495 \]
