Stroboscopic quantum walks

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Discrete time (coined) quantum walks are produced by the repeated application of a constant unitary transformation to a quantum system. By recasting these walks into the setting of periodic perturbations to an otherwise freely evolving system we introduce the concept of a stroboscopic quantum walk. Through numerical simulation, we establish the link between families of stroboscopic walks and quantum resonances. These are observed in the nonlinear systems of quantum chaos theory such as the δ-kicked rotator or the δ-kicked accelerator.

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Quantum walks were first mentioned in [1] and have been a topic of considerable interest in recent years as they exhibit properties that have led to the construction of a new class of quantum algorithms. For computationally difficult problems which can be solved classically by probabilistic algorithms [2] several quantum algorithms have been found which make use of quantum walks as natural counterparts of classical Markov chains. At present these new algorithms fall into two categories: algorithms achieving exponentially faster hitting times on graphs [3, 4, 5, 6, 7] and searching based on quantum walks [8, 9, 10]. The initial discussion of quantum walks and their algorithmic applications was purely theoretical but there have also been proposals for physical realisations of such walks, using ion traps [11], cold atoms in optical lattices [12] and cavity quantum electrodynamics [13].

In real systems the quantum evolution between steps will clearly be important but this has not so far been included in the description of discrete time quantum walks. In this paper we shall outline the theoretical description of the discrete time quantum walk in one dimension and explain our generalization of the dynamics along with its implications.

The discrete time model, introduced in [14], was the first scheme proposed for an abstract quantum walk. In analogy with the classical random walk a particle with internal states |0⟩ and |1⟩ (equivalent to heads and tails of a coin) was supposed to move on an infinite line of positions or dissipation, produces the distinctive features, in particular enhanced diffusion, that have driven interest in these models of quantum evolution.

We now introduce a new model which we call the stroboscopic quantum walk. The idea is that the walker, being a physical system, should also undergo time evolution and thus acquire position- and time-dependent phases, governed by the Schrödinger equation, in between the steps of the original discrete time quantum walk. Introducing this stroboscopic scheme as a natural feature of the dynamics also casts new light on the nature of the physical description of coined quantum walks away from purely algorithmic issues. In essence we are studying the free time evolution of a quantum system in the presence of periodic perturbations. This is, of course, also a central theme in the investigation of quantum chaos [15, 16, 17].

Let us now briefly review the mathematical structure of discrete time quantum walks on the line. The total Hilbert space of the particle $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$ is spanned by $\{|0\rangle, |1\rangle\}$ and $\{|n\rangle \mid n \in \mathbb{Z}\}$ respectively where $|n\rangle$ denote distinct positions on the line. Hence a general state $|\psi(t)\rangle$ of the walker can be expanded in terms of these basis states, thus $|\psi(t)\rangle = \sum_{i,n} c_{in}(t) |i,n\rangle$. The coin toss $\hat{C}$ is an arbitrary unitary transformation in $\mathcal{H}_C$ which is usually taken to be the balanced Hadamard coin $\hat{H} = \frac{1}{\sqrt{2}} (\hat{\sigma}_x + \hat{\sigma}_z)$ with $\hat{\sigma}_i$ denoting the Pauli matrices in $SU(2)$.

The translation operators $\hat{T}$ and $\hat{T}^{-1}$ move the particle by one vertex $\hat{T}|n\rangle = |n+1\rangle$ and $\hat{T}^{-1}|n\rangle = |n-1\rangle$. One step of the walk is thus defined as

$$\hat{U} = (|0\rangle\langle 0| \otimes \hat{T} + |1\rangle\langle 1| \otimes \hat{T}^{-1})(\hat{C} \otimes \hat{I}),$$

(1)

repeatedly applying $\hat{U}$ to the initial state $|\psi_0\rangle$ gives the whole walk $|\psi(t)\rangle = \hat{U}^t|\psi_0\rangle$. The probability of finding the particle at vertex $n$ after $t$ steps is then given by $P(n,t) = \sum_j |\langle i,n|\psi(t)\rangle|^2$. In order to get a symmetric probability distribution on the line the initial state is chosen to be $|\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle) \otimes |0\rangle$, as the Hadamard coin treats $|0\rangle$ and $|1\rangle$ differently. It can be shown [14]...
that the probability distribution of a Hadamard walker 
has a standard deviation (i.e. expected distance from 
the origin) of $\sigma \propto t$ in the asymptotic limit, in other 
words the Hadamard walk spreads quadratically faster 
than its classical counterpart with $\sigma = \sqrt{t}$. The position 
probability distribution of the ordinary Hadamard walk is plotted in Fig. 4 together with numerical results of 
some stroboscopic Hadamard walks.

We now turn to the mathematical description of strobo-
soscopic walks. As with the discrete time model we con-
sider a total Hilbert space of $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_P$ but now let 
the position states $\{|n\rangle\}$ in $\mathcal{H}_P$ be eigenstates of some 
time-independent Hamiltonian $\hat{H}_P$ with $\hat{H}_P|n\rangle = E_n|n\rangle$. 
In between the steps, separated by the time interval $\tau$, 
the states in position space evolve according to the 
Schrödinger equation. Thus we define one step of the 
stroboscopic walk as:

$$\hat{U}(\tau) = (|0\rangle\langle 0| \otimes \hat{T} + |1\rangle\langle 1| \otimes \hat{T}^{-1}) (\hat{C} \otimes e^{-\frac{i}{\hbar} \hat{H}_P \tau}).$$

We see that at every vertex on the line the walker effec-
tively acquires a relative phase depending on the time 
interval $\tau$. If the relative phase differences between eigen-
states of adjacent occupied vertices are an integer multi-
ple of $2\pi$ a complete rephasing occurs and no difference to 
the ordinary Hadamard walk will be seen. The smallest 
possible time interval for this to happen is called the 
Talbot time $T$ given by

$$T = \frac{2\pi \hbar \lambda}{E_n - E_{n+2}}, \quad (3)$$

where $\lambda$ is an integer depending on the energy spectrum 
of the particular system. Hence we can restrict our anal-
ysis to $\tau \in [0, T)$.

In order to investigate the properties of the strobo-
soscopic walk a range of numerical simulations were per-
fomed using the simple harmonic oscillator Hamiltonian 
$\hat{H}_P = \hbar \omega (\hat{n} + \frac{1}{2})$ which has the Talbot time $T = \frac{\pi}{\omega}$. We start 
the walks at a high number state $|n\rangle$ so that the ground state $|0\rangle$ is never reached in our simulations.

We first observe that there is linear spreading of the 
walker’s probability distribution on the line not just for 
the Hadamard walk case but also for certain time in-
tervals $\tau$ after the initial oscillations have settled down 
(Fig. 2). Most interestingly some of these slower walks 
still outperform classical walks up to the simulated num-
er of steps. However, for other values of $\tau$ the par-
ticle is tightly localised around the origin of the walk 
(Fig. 1). There are other intriguing features to be seen in 
the time evolution. Firstly the time intervals $\tau$ for which 
significant spreading occurs exhibit an interesting pat-
tern (Fig. 3) and appear to be symmetric about $\tau = \frac{T}{2}$. 
Comparing the whole range of stroboscopic walks (as $\tau$ 
is varied) even allows to distinguish between super- and 
subclassical spreading behaviour (Fig. 4).

In our discussion of numerical simulations we have 
merely focused on the harmonic oscillator Hamiltonian

FIG. 1: Probability distribution of the positions (black 
curves) after $t = 200$ steps for the ordinary Hadamard walk 
($\tau = 0$) as well as for stroboscopic Hadamard walks (har-
monic oscillator model) with various values of $\tau$. Only even 
positions are drawn. For comparison the gray curves show 
the limiting distribution of a classical random walk.
so far to point out the striking features of stroboscopic quantum walks. However, it is important to emphasize that this choice is just a special case of the Hamiltonians $\hat{H}_P$ allowed in Eq. 2 and can therefore only illustrate a fraction of the possible phenomena. In order to demonstrate this we also carried out numerical studies for the Hamiltonian $\hat{H}_P = \hat{p}^2 / 2m$ describing a free non-relativistic particle. Receiving discrete momentum kicks $\Delta p$ the particle walks in the space $\mathcal{H}_P$ spanned by the discrete momentum eigenstates $\{|n\Delta p\rangle | n \in \mathbb{Z}\}$, the Talbot time is given by $T = \frac{\pi m}{\Delta p^2}$. The results for this family of stroboscopic walks can be seen in Fig. 5.

Our results indicate that the concept of quantum walks may actually be embedded in a broader framework. The dynamics of the stroboscopic walks shows all the features of a nonlinear dynamical system in the presence of periodic perturbation. This extends from dynamical localisation—spreading more slowly than in the classical case—to linear diffusion as the period of the perturbation is

FIG. 2: Standard deviation $\sigma$ (black curves) and mean $\langle n \rangle$ (gray curves) of the probability distribution for the ordinary Hadamard walk ($\tau = 0$) as well as for stroboscopic Hadamard walks (harmonic oscillator model) with various values of $\tau$. 

FIG. 3: Standard deviation $\sigma$ of stroboscopic Hadamard walks (harmonic oscillator model) after $t = 100$ steps with a time resolution of $\Delta \tau = \frac{T}{100}$. 

FIG. 4: Normalized standard deviation $\sigma / \sqrt{t}$ of stroboscopic Hadamard walks (harmonic oscillator model) for $t = 20$ steps with a time resolution of $\Delta \tau = \frac{T}{100}$. Sub- and superclassical spreading can clearly be seen.
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FIG. 5: Standard deviation $\sigma$ of stroboscopic Hadamard walks (free non-relativistic particle model) after $t = 100$ steps with a time resolution of $\Delta \tau = \frac{T}{100}$.