On the eigenvalues of some non-Hermitian oscillators

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Abstract

We consider a class of one-dimensional non-Hermitian oscillators and discuss the relationship between the real eigenvalues of PT-symmetric oscillators and the resonances obtained by different authors. We also show the relationship between the strong-coupling expansions for the eigenvalues of those oscillators. A comparison of the results of the complex rotation and the Riccati–Padé methods reveals that the optimal rotation angle converts the oscillator into either a PT-symmetric or a Hermitian one. In addition to the real positive eigenvalues, the PT-symmetric oscillators exhibit real positive resonances under different boundary conditions. These can be calculated by means of the straightforward diagonalization method. The Riccati–Padé method yields not only the resonances of the non-Hermitian oscillators but also the eigenvalues of the PT-symmetric ones.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In a recent paper Jentschura et al [1] discussed the resonances for the anharmonic oscillator $$H = -\frac{1}{2} \frac{d^2}{dt^2} + \frac{1}{2} \alpha t^2 + \sqrt{\gamma t^3}$$ and their weak- and strong-coupling expansions. They showed analytical expressions for the coefficients of the former and numerical estimates for those of the latter. In particular, the leading coefficients of the strong-coupling expansions are the eigenvalues of $$H = -\frac{1}{2} \frac{d^2}{dt^2} + t^3$$.

Some time ago, Bender and Boettcher [2] discussed the eigenvalues of PT-symmetric oscillators of the form $$H = -\frac{d^2}{dx^2} - (ix)^N$$ that exhibit a finite number of real positive eigenvalues for $$1 < N < 2$$ and an infinite number when $$N \geq 2$$.

Alvarez [3] discussed the analytical properties of the solutions of the Hamiltonian operator $$H = \frac{1}{2} p^2 + \frac{1}{2} k x^2 + g x^3$$ and showed that it supports real and complex resonances depending on the complex values of the coupling constant $$g$$. His results suggest that the resonances calculated by Jentschura et al [1] and the real eigenvalues obtained by Bender and Boettcher [2] (see also...
may by related in a simple way by means of the Symanzik scaling [4] already invoked by Alvarez in his investigation [3]. In exactly the same way, the strong-coupling expansion obtained by Jentschura et al [1] may be related to that obtained some time earlier by Fernández et al [5] for the PT-symmetric oscillator $H = p^2 + ix^3 + \lambda x^2$. The purpose of this paper is to explore such relationships as well as other properties of a class of non-Hermitian oscillators.

In section 2, we investigate the relationship among some of the earlier results on one-dimensional non-Hermitian oscillators. In section 3, we discuss the application of the complex-rotation method (CRM) [6, 7] and the Riccati–Padé method (RPM) [8–10] to those oscillators. Finally, in section 4 we summarize the main results and draw conclusions.

2. Real and complex eigenvalues

As outlined above, Jentschura et al [1] discussed several properties of the resonances for the anharmonic oscillator

$$H_c = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} q^2 + \sqrt{g} q^3$$

as well as their weak-coupling

$$E_n(g) = \sum_{k=0}^{\infty} E_{n,k} g^k$$

and strong-coupling expansions

$$E_n(g) = g^{1/5} \sum_{k=0}^{\infty} L_{n,k} g^{-2 k/5}.$$  

The coefficients of the former can be obtained exactly by means of perturbation theory and those of the latter in a numerical way. In particular, the leading coefficients of the strong-coupling expansions $L_{n,0}$ are the eigenvalues of the pure cubic anharmonic oscillator

$$H_l = -\frac{1}{2} \frac{d^2}{dq^2} + q^3.$$  

On the other hand, the closely related PT-symmetric oscillators

$$H_{PT} = -\frac{d^2}{dx^2} - (ix)^N$$

exhibit an infinite number of real positive eigenvalues when $N \geq 2$ [2] and the accurate results for $N = 3$ and $N = 4$ are available for comparison [11].

It is not difficult to obtain a connection between the results outlined above by means of the Symanzik scaling

$$U^\dagger p U = \gamma^{-1} p, \quad U^\dagger x U = \gamma x,$$

where $U$ is a well-known unitary operator [4, 7]. This transformation was already used by Alvarez in his investigation of the cubic anharmonic oscillator [3]. For example, if we take into account that $2\gamma^2 U^\dagger H_l U = H_{PT}$ when $\gamma = (i/2)^{1/5}$, then we realize that the complex eigenvalues $L_{n,0}$ of $H_l$ calculated by Jentschura et al [1] and the real positive eigenvalues $E_n^{PT}$ of $H_{PT}$ for $N = 3$ calculated by Bender and Boettcher [2] and Bender [11] are related by

$$L_{n,0} = 2^{-3/5} \gamma^{-2/5} E^{PT}_n.$$  

Some time ago, Fernández et al [5] obtained the perturbation expansion for the PT-symmetric oscillator

$$H_F = p^2 + ix^3 + \lambda x^2$$
Table 1. Coefficients $L_{n,j}$ of the strong-coupling expansion (3).

| $j$ | From [1] | From [5] |
|-----|----------|----------|
| 0   | 0.617 160 050 − 0.448 393 023i | 0.617 160 049 536 − 0.448 393 022 571i |
| 1   | −0.013 228 193 + 0.040 712 191i | −0.013 228 192 867 + 0.040 712 191 4135i |
| 2   | −0.009 259 259 + 0.000 000 000i | 0.009 259 258 68 |
| 3   | −0.000 294 361 − 0.000 905 951i | −0.000 294 361 224 639 − 0.000 905 950 695 052i |

in the form

$$E_n(\lambda) = \sum_{j=0}^{\infty} W_{n,j} \lambda^j.$$ (9)

Arguing as before, we can obtain the coefficients of the strong-coupling expansion (3) from those of the perturbation series (9) as follows:

$$L_{n,j} = 2^{-4(j+3)/5} \left[ i^{-4(j-2)/5} W_{n,j} \right].$$ (10)

The first coefficients are shown in table 1 as an illustrative example.

3. Complex-rotation and Riccati–Padé methods

For concreteness, we consider the family of anharmonic oscillators

$$H_K = \frac{1}{2} p^2 - x^K,$$

$$H_K \psi = E \psi.$$ (11)

The CRM consists of the diagonalization of the rotated Hamiltonian operator

$$U^\dagger H_K U = \gamma^{-2} \left( \frac{1}{2} p^2 - \gamma^{K+2} x^K \right),$$ (12)

where $\gamma = \eta e^{i\theta}$. The parameter $\eta > 0$ produces a dilatation or contraction of the scale and $\theta$ a rotation of the coordinate in the complex $x$-plane. On tuning $\eta$, we improve the rate of convergence of the diagonalization method as the matrix dimension increases and the value of $\theta$ enables us to uncover the resonances [6, 7]. The present adjustable parameter $\eta$ plays the role of the parameter $k$ used by Yaris et al [7]. For the diagonalization method, we choose the basis set of eigenfunctions of the harmonic oscillator $H = p^2 + x^2$. As a result of the complex rotation, the eigenvalues of the truncated matrix depend on $\theta$ and one should determine this optimal rotation angle. Instead of the standard optimization procedures [6, 7], in what follows we resort to an alternative strategy.

For comparison purposes, we also apply the RPM for nonsymmetric potentials [10]. It consists of the expansion of the logarithmic derivative of the eigenfunction $\psi(x)$

$$f(x) = -\frac{\psi'(x)}{\psi(x)}$$ (13)

in a Taylor series about the origin

$$f(x) = \sum_{j=0}^{\infty} f_j x^j,$$ (14)

where the coefficients $f_j$ depend on the two unknowns $E$ and $f_0 = -\psi'(0)/\psi(0)$. From the coefficients of the even and odd powers of the coordinate $f_E, j = f_{2j}$ and $f_o, j = f_{2j+1}$, $j = 1, 2, \ldots$, respectively, we construct the Hankel determinants $H^{(2)}_D (E, f_0) = |f_{E,i+j+d-1}|_{i,j=1}^{D}$, $H^{(2)}_D (E, f_0) = |f_{O,i+j+d-1}|_{i,j=1}^{D}$ and obtain both $E$ and $f_0$ from the roots of the system of
nonlinear equations \( \{H_D^{d}(E, f_0) = 0, \ H_D^{0}(E, f_0) = 0\} \). They are polynomial functions of these variables. For every fixed value of \( d = 0, 1, \ldots \), we look for convergent sequences of roots \( E^{(d, j)}, D = 2, 3, \ldots \). Commonly, we obtain acceptable results for \( d = 0 \), but calculations with other values of \( d \) enable us to test the consistency of the method.

Since the rate of convergence of the RPM is considerably greater than that for the CRM, we choose the results of the former as exact or reference eigenvalues. Figure 1 shows \( \log |(E_n^{\text{RPM}} - E_n^{\text{CRM}})/E_n^{\text{RPM}}| \) as a function of \( \theta \) for the first resonances of the cubic oscillator \((K = 3)\). These results suggest that the minimum of the logarithmic deviation appears at \( \theta = \pi/10 \), which we consider to be the optimal rotation angle (in all our calculations we have chosen \( \eta = e^{-1} \), which provides a reasonable rate of convergence). In order to understand this empirical result we resort to the scaling transformation (12) for \( K = 3, 5, \ldots \). It is clear that \( U^\dagger H_K U = \gamma_j^{-2}\left[\frac{1}{4}p^2 - \gamma_j^{K+2}x^K\right] \) is proportional to the PT-symmetric oscillator \( \frac{1}{4}p^2 - (1)i_x^K \) when \( \gamma_j = e^{(2j+1)i\pi/[2(K+2)]} \) and \( j = 0, 1, \ldots, K + 1 \). For \( K = 3 \) and \( j = 0 \), we obtain \( \theta = \pi/10 \) as suggested by figure 1. The obvious conclusion is that the optimal rotation angle converts each of the anharmonic oscillators of this particular class into a PT-symmetric one.

It also follows from equation (12) that \( \gamma_j^{-2}U^\dagger H_K U = H_K \) when \( \gamma_j = e^{2\pi ij/(K+2)} \). Therefore, instead of just one eigenvalue \( E_n \) we expect \( K + 1 \) replicas located at

\[
E_{n,j} = e^{2\pi ij/(K+2)}E_n, \ j = 0, 1, \ldots, K + 1.
\]  
(15)

The RPM yields all these eigenvalues simultaneously as limits of different sequences of roots of the same sequence of pairs of Hankel determinants. On the other hand, the CRM uncovers them at different values of \( \theta \). Figure 2 shows \( \log |(E_{0,j}^{\text{RPM}} - E_{0,j}^{\text{CRM}})/E_{0,j}^{\text{RPM}}| \) as a function of \( \theta \) for the lowest resonance of the cubic oscillator. We appreciate that the closest agreement between both methods takes place exactly at the rotation angles \( \theta_j = (2j + 1)\pi/10 \) derived above. Table 2 shows these results more precisely and table 3 shows a similar calculation for the quintic oscillator.

The case \( j = 0 \) for the cubic oscillator agrees with the resonance calculated by Jentschura et al [1]. These authors claimed to have chosen the rotation angle \( \theta = \pi/5 \) for all their calculations on the cubic oscillator (in particular for the strong-coupling expansion). However, we could not obtain acceptable results for this rotation angle. In fact, our calculations for the cubic oscillator suggest that the multiples of \( \theta = \pi/5 \) are the worst choices. Figure 3 shows the real and imaginary parts of the first resonance as functions of \( \theta \). We appreciate that the regions

\[\text{Figure 1. Logarithmic error, log} |(E_n^{\text{RPM}} - E_n^{\text{CRM}})/E_n^{\text{RPM}}|, \text{for the first resonances of the cubic oscillator.}\]
Figure 2. Logarithmic error, $\log\left(\frac{|E_{\text{RPM}}^0 - E_{\text{CRM}}^0|}{E_{\text{RPM}}^0}\right)$, for the first set of eigenvalues of the cubic oscillator ($K = 3$) as functions of $\theta$.

Table 2. Lowest resonance for the cubic oscillator ($K = 3$). The first column shows the value of $j$ which determines the optimal rotation angle $\theta_j = \frac{2j+1}{14} \pi$ for the CRM. The first and second entries in the second and third columns correspond to the CRM and RPM eigenvalues, respectively.

| $j$ | $\Re(E)$ | $\Im(E)$ |
|-----|-----------|-----------|
| 0   | 0.617 160 049 5373 | -0.448 393 022 575 |
|     | 0.617 160 049 538 936 737 54 | -0.448 393 022 575 932 856 33 |
| 1   | -0.235 734 162 4247 | -0.725 515 150 844 |
|     | -0.235 734 162 425 304 962 69 | -0.725 515 150 846 158 289 94 |
| 2   | -0.762 851 774 225 | 0.000 000 000 000 00 |
|     | -0.762 851 774 227 263 549 70 | 0.000 000 000 000 00 000 00 |
| 3   | -0.235 734 162 4247 | 0.725 515 150 844 |
|     | -0.235 734 162 425 304 962 69 | 0.725 515 150 846 158 289 94 |
| 4   | 0.617 160 049 537 | 0.448 393 022 575 |
|     | 0.617 160 049 538 936 737 54 | 0.448 393 022 575 932 856 33 |

Table 3. The same as table 2 for the quintic oscillator ($K = 5$); in this case the optimal rotation angle is $\theta_j = \frac{2j+1}{24} \pi$.

| $j$ | $\Re(E)$ | $\Im(E)$ |
|-----|-----------|-----------|
| 0   | 0.639 629 797 817 | -0.308 029 476 062 |
|     | 0.639 629 797 817 251 829 20 | -0.308 029 476 061 779 666 96 |
| 1   | 0.157 975 513 9908 | -0.692 135 950 055 |
|     | 0.157 975 513 990 788 434 52 | -0.692 135 950 054 596 682 45 |
| 2   | -0.442 637 553 984 | -0.555 049 936 656 |
|     | -0.442 637 553 983 955 293 72 | -0.555 049 936 655 913 878 38 |
| 3   | -0.709 935 515 648 | 0.000 000 000 000 00 |
|     | -0.709 935 515 648 169 940 02 | 0.000 000 000 000 00 000 00 |
| 4   | -0.442 637 553 984 | 0.555 049 936 656 |
|     | -0.442 637 553 983 955 293 72 | 0.555 049 936 655 913 878 38 |
| 5   | 0.157 975 513 9909 | 0.692 135 950 055 |
|     | 0.157 975 513 990 788 434 52 | 0.692 135 950 054 596 682 45 |
| 6   | 0.639 629 797 817 | 0.308 029 476 062 |
|     | 0.639 629 797 817 251 830 08 | 0.308 029 476 061 779 667 39 |
Figure 3. Real and imaginary parts of the first resonance $E(\theta)$ for the cubic oscillator ($K = 3$). The vertical lines mark multiples of $\pi/5$.

Table 4. First resonances for the cubic and quintic oscillators calculated by means of the RPM.

| n | $\Re E_n$ | $\Im E_n$ |
|---|---|---|
| $K = 3$ |
| 0 | 0.617 160 049 538 936 737 543 | -0.448 393 022 575 932 856 3 |
| 1 | 2.193 309 731 021 120 867 6 | -1.593 532 796 674 843 259 7 |
| 2 | 4.036 380 019 834 828 325 2 | -2.932 601 743 601 124 886 6 |
| 3 | 6.039 097 108 464 794 53 | -4.387 660 880 053 876 93 |
| 4 | 8.161 899 874 821 120 867 6 | -5.929 967 368 379 177 80 |
| 5 | 10.382 295 727 979 694 2 | -7.543 179 384 705 625 61 |
| $K = 5$ |
| 0 | 0.639 629 797 817 251 829 2 | -0.308 029 476 061 779 666 9 |
| 1 | 2.396 357 680 279 750 382 | -1.154 025 036 407 214 392 |
| 2 | 4.917 700 190 046 959 8 | -2.368 239 594 431 575 8 |
| 3 | 7.917 462 140 328 48 | -3.812 848 812 151 728 |
| 4 | 11.317 988 505 404 04 | -5.450 456 000 157 942 |
| 5 | 15.062 218 927 774 | -7.253 582 338 538 |

of stability appear at $j\pi/5 < \theta < (j + 1)\pi/5$, $j = 0, 1, 2, 3, 4$ (the boundaries are marked by vertical dashed lines). The optimal rotation angles discussed above (those that convert the anharmonic oscillator into a PT-symmetric one) bisect each of these regions and the rotation angle chosen by Jentschura et al corresponds to one of the boundaries. The present results agree with those of Alvarez [3] who proposed integrating the differential equation along the rays $\arg(\pm x) = \pi/10 - \arg(g)/5$ in the case of a harmonic oscillator perturbed by the cubic term $gx^3$. More precisely, he also showed that the left and right boundary conditions for the resonances hold in the common sector $0 < \frac{1}{2} \arg(g) + \frac{1}{2} \arg(x) < \frac{\pi}{2}$, so that $0 < \arg(x) < \frac{\pi}{2}$ for $g = 1$, in agreement with the first region of stability shown in figure 3. The appearance of the optimal rotation angle $\theta = \pi/5$ in the paper by Jentschura et al [1] is merely due to a misprint [13]. In general, the optimal integration rays for the resonances of the oscillators (11) are given by $\arg(x) = \pi/(2K + 4)$, which contain the particular case $K = 3$ just discussed.

Table 4 shows the first resonances for the cubic and quintic oscillators calculated by means of the RPM. They may be useful as a benchmark for testing other approximate methods. For example, the first three of them for $K = 3$ agree with those of Jentschura et al [1].
Table 5. The Rayleigh–Ritz method for $H = p^2 + i\lambda x$ with basis sets of $M$ harmonic-oscillator eigenfunction.

| $M$ | $N = 5$ | $N = 7$ |
|-----|---------|---------|
| 10  | 1.137 702 766 619 76 | 1.297 856 565 125 58 |
| 20  | 1.165 710 289 071 53 | 1.225 994 998 518 04 |
| 30  | 1.164 772 393 472 23 | 1.224 709 898 074 91 |
| 40  | 1.164 770 426 778 32 | 1.224 711 627 414 09 |
| 50  | 1.164 770 408 157 80 | 1.224 711 686 447 15 |
| 60  | 1.164 770 407 943 14 | 1.224 711 689 048 64 |
| 70  | 1.164 770 407 943 43 | 1.224 711 689 659 77 |
| 80  | 1.164 770 407 943 42 | 1.224 711 689 368 49 |
| RPM | 1.164 770 407 943 414 994 19 | 1.224 711 689 331 1451 |

Table 6. First eigenvalues of the PT-symmetric oscillators $H = p^2 + i\lambda x$ calculated by means of the RPM, CRM and WKB method.

| RPM  | CRM ($\theta = 0, \eta = 0.4$) | WKB |
|------|--------------------------------|-----|
| $N = 5$ |                                |     |
| 1.164 770 407 943 414 994 19 | 1.164 770 407 943 415 0203 | 1.771 244 715 |
| 4.363 784 367 712 109 1602 | 4.363 784 367 712 107 3149 | 8.509 035 978 |
| 8.955 166 998 240 671 6852 | 8.955 166 998 240 678 966 | 17.652 537 59 |
| $N = 7$ |                                |     |
| 1.224 711 689 331 1451 | 1.224 711 689 659 769 4535 | 2.855 548 625 |
| 4.721 462 535 392 46 | 4.721 447 691 270 68 | 15.771 688 04 |
| 10.075 449 563 0818 | 10.075 762 341 7291 | 34.912 120 93 |

From the results just discussed, one may be tempted to conclude that the CMR with $\theta = 0$ should yield the eigenvalues of the PT-symmetric oscillators. This conjecture is supported by the convergence of this method toward the accurate RPM eigenvalues shown in table 5. However, such a conclusion would be wrong. Although the eigenvalues produced by two quite different methods such as the RPM and CRM agree accurately for all $N = 3, 5, 7, \ldots$, only in the case $N = 3$ are they those of the PT-symmetric oscillators. For $N = 5, 7, \ldots$, both methods yield the resonances discussed above rotated in the complex plane. In an earlier version of their published paper [2], Bender and Boettcher [12] discussed the diagonalization method in somewhat more detail. There, they clearly stated that the diagonalization method is useful only for $1 < N < 4$ because in the other cases the wedges in which the eigenfunction vanishes as $|x| \to \infty$ do not contain the real $x$ axis. Consequently, the dilatation transformation outlined above is insufficient to take into account both the left and right PT boundary conditions [2].

Table 6 shows the first eigenvalues for the PT-symmetric oscillators (5) with $N = 5$ and $N = 7$ calculated by means of the RPM, CRM ($\theta = 0, \eta = 0.4$) and WKB method. The first two approaches agree between themselves but not with the WKB method that provides estimates to the actual eigenvalues of the PT-symmetric oscillators [2]. Note that the discrepancy increases with the quantum number which makes the WKB increasingly accurate. On the other hand, it is well known that the eigenvalues of the Hamiltonian matrix agree with the WKB ones for the $N = 3$ case [14]. For additional confirmation that the eigenvalues of the CRM are not those of the PT-symmetric oscillators, compare the results of table 6 with the accurate upper and lower bounds derived by Yan and Handy [15]. Although the functional form of the operators is the same, the boundary conditions are different [2]. For simplicity, from now on we will refer to resonance [3] and PT-symmetric boundary conditions [2]. Although the CRM takes into account only the former, it is interesting that it yields real positive eigenvalues for the
PT-symmetric oscillators (5) with $N = 5, 7, \ldots$. In what follows, we will discuss this point somewhat further.

In order to understand the results just outlined, we inspect the form of the eigenfunctions provided by the CRM for the PT-symmetric oscillators. We calculated the eigenfunctions $\psi_n(x)$, $n = 0, 1, 2$, and their absolute squares are shown in figure 4 for $N = 3, 5, 7$. We appreciate that they all look similar and satisfy $|\psi_n(-x)|^2 = |\psi_n(x)|^2$ as expected from the fact that $\psi_n(-x)^* = \lambda \psi_n(x)$, where $|\lambda| = 1$ [11]. More precisely, our numerical calculations suggest that in these particular cases $\psi_n(-x)^* = (-1)^n \psi_n(x)$. Even though the resonance boundary conditions are different from the PT-symmetric ones for $N = 5, 7, \ldots$, there appears to be an unbroken symmetry that produces real eigenvalues. Besides, all those eigenfunctions are strongly localized about $x = 0$ as expected for a resonance. It is interesting that both the RPM and the CRM yield real and positive eigenvalues with localized eigenfunctions for the PT-symmetric oscillators, although they are not the true eigenvalues and eigenfunctions of the PT-symmetric oscillators for $N > 3$.

In addition to the resonances just discussed, the RPM also yields the true eigenvalues of the PT-symmetric oscillators for all $N = 3, 5, \ldots$. For example, for $N = 5$ we estimated $E_0 = 1.908 2646$ from determinants of dimension $D = 10, \ldots, 20$. Note that this eigenvalue is considerably greater than that in table 6 obtained from the resonance boundary condition. We will discuss this issue again below.

The situation is remarkably different for $K = 4, 6, \ldots$. Because of the location of the wedges in the complex $x$-plane [2], one should apply the RPM for nonsymmetric potentials outlined above. However, in the case of resonances the boundary conditions are symmetric (for example, outgoing waves to the right and left)

$$
\psi \sim \exp \left[ \frac{i2^{3/2}}{K + 2} x^{(K+2)/2} \right], \quad x \rightarrow \infty
$$

$$
\psi \sim \exp \left[ -i \frac{2^{3/2}}{K + 2} x^{(K+2)/2} \right], \quad x \rightarrow -\infty
$$

and the oscillator exhibits true even parity. In this case, the appropriate logarithmic derivative of the wavefunction is of the form

$$
f(x) = \frac{s}{x} - \frac{\psi'(|x|)}{\psi(|x|)},
$$

where $s = 0$ or $s = 1$ for even or odd eigenfunctions, respectively. From the coefficients of the Taylor expansion

$$
f(x) = \sum_{j=0}^{\infty} f_j x^{2j+1}
$$

we construct the Hankel determinants $H^D_{n}(E) = |f_{j+1} + i D f_j|_{j=1}^{D}$ that depend on the only unknown $E$ and obtain the eigenvalues from sequences of roots of $H^D_{n}(E) = 0$ [8, 9].

On the other hand, we can apply the CRM exactly in the same way discussed above. In this case, the optimal rotation angles are given by $\gamma_j = e^{(2j+1)i\pi/(K+2)}$, $j = 0, 1, \ldots, K + 1$ that make $U^* H U = \gamma_j^{-2} (\frac{d}{d} + x^K)$ proportional to the Hermitian operator $\frac{d^2}{d^2} + x^K$. Table 7 shows the comparison between the CRM and RPM results for the first resonance of the quartic oscillator ($K = 4$). There are $K/2 + 1$ replicas of every resonance given by

$$
E_{n,j} = e^{-\pi i j/(K/2+1)} E_n, \quad j = 0, 1, \ldots, \frac{K}{2},
$$

In this case, the RPM for nonsymmetric potentials yields the eigenvalues of the corresponding PT-symmetric oscillator. Table 8 shows the first three eigenvalues of the PT-symmetric
Figure 4. $|\psi_n|^2$, $n = 0$ (solid line, red), $n = 1$ (dashed line, green), $n = 2$ (dotted line, blue) for the PT-symmetric oscillators (5) with $N = 3, 5, 7$. 
oscillators (5) with $N = 4$ and $N = 5$. The results in the first column agree with those obtained earlier by means of numerical integration [2, 11] and those in the second column lie within the upper and lower bounds derived by Yan and Handy [15]. The rate of convergence of the RPM is considerably greater for $N = 4$; in addition, we experienced considerable numerical difficulties in obtaining the roots of the pair of Hankel determinants for $N = 5$ by means of the Newton–Raphson method.

4. Conclusions

In section 2, we have shown that a simple scaling argument enables one to connect the results obtained earlier by several authors for a class of non-Hermitian oscillators. Although this relationship is contained in Alvarez’s work [3], the actual connection formulas have not been made explicit as far as we know. For example, the resonances of the cubic oscillator are straightforwardly related to the eigenvalues of the corresponding PT-symmetric oscillator. Such connection is not possible for other oscillators because the boundary conditions that give rise to the resonances and PT-symmetric eigenvalues are different.

The comparison of the RPM and CRM results enabled us to obtain the optimal rotation angle for the latter approach. We have shown that the effect of the optimal coordinate rotation is to convert the non-Hermitian oscillators (11) into either a PT-symmetric or Hermitian one, for $K$ odd or even, respectively. Such results are consistent with Alvarez’s analysis of the harmonic oscillator with a cubic perturbation [3].

We have also shown that both the RPM and CRM yield real positive eigenvalues for the PT-symmetric oscillators (5) with $N = 3, 5, 7, \ldots$, but these results are not the actual eigenvalues of the PT-symmetric oscillators when $N > 3$ because the resonance and PT-symmetric boundary conditions are different. The CRM eigenfunctions are strongly localized and their absolute squares exhibit the relationship coming from unbroken symmetry. It is also interesting that the eigenfunctions for the case $N = 3$ (where the CRM yields the actual eigenvalues of the PT-symmetric oscillator) are similar to those for the cases $N = 5, 7$ (where the boundary conditions are those for the resonances).

On the other hand, the RPM yields both the real positive resonances mentioned above for odd $N$ and the actual PT-symmetric eigenvalues obtained by Bender and Boettcher [2].
and Bender [11]. In the case of \( N = 3, 5, 7, \ldots \), all the eigenvalues are limits of sequences of roots of the same Hankel determinants given by the RPM for nonsymmetric potentials. In the case of even-parity potentials \( N = 4, 6, \ldots \), the RPM for even-parity potentials yields the resonances and the approach for nonsymmetric potentials provides the eigenvalues of the corresponding PT-symmetric oscillators. The main disadvantage of this approach as a practical tool is that it provides results for so many different problems that it is sometimes difficult to pick up the correct sequence of roots of the system of two Hankel determinants necessary for the treatment of nonsymmetric problems. On the other hand, from a mathematical point of view, this property of the RPM is most intriguing and interesting. For example, it is worth noting that the RPM yields both the positive and negative spectra of the harmonic oscillator [9] discussed by Bender and Turbiner [16].

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