THE SHAPE OF INFINITY

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1. First Introduction: for students

What is there at infinity?

Well, it depends on what there is not at infinity. Consider first a plane; what is there at the infinity of a plane?

**Answer #1**: a point. In *stereographic projection* we place a sphere tangent to the plane (at the origin, say) and draw lines from the north pole, mapping each point on the plane to the point where the corresponding line intersects the sphere. Now the plane has been transformed into the sphere minus the North pole, and the North pole is a ‘point at infinity’.

This is cool, but it’s also weird in some ways. Since there is just one point at infinity, if I go off to infinity in one direction I can then stand at infinity, turn around, and come back from infinity from any other direction I want to. Shouldn’t, maybe, *this* infinity be a little different from *that* infinity?

**Answer #2**: a circle. We can add one point in each direction. A way to visualize this, similar to stereographic projection, is *hemispherical central projection*, drawing lines from the center of the sphere. Now the plane is transformed into the Southern hemisphere, and the equator is a circle at infinity. Here we have one point for each direction; for instance, parallel lines, or asymptotic curves, meet off at infinity.

Actually, however, parallel lines now meet in *two* points: one in each direction. One might argue that one purpose for adding points at infinity is to make parallel lines not so different from intersecting ones (because they should intersect “at infinity”), but surely having them intersect in two points makes them rather different from other lines that intersect only in one point.

Moreover, in this model the points at infinity are weird in another way: they are different from all other points. If I go off in one direction and end up at a point at infinity, I can’t keep going any further in that direction; I have to turn around and come back. Maybe this is the way you think of infinity, as a ‘boundary’ of space, but it’s also nice to have all points of the resulting space be the same.

**Answer #3**: a ‘line’. We do spherical central projection again, but now we consider a point and its antipodal point to be the ‘same’ point of the resulting space. Now the two points at infinity in opposite directions are actually the same point. This is called the *projective plane*, in which parallel lines intersect at a single point at infinity, making them just like all other pairs of lines that intersect in a single finite point. And if I go to infinity in one direction, I can keep on going and come back from the opposite direction.

That means, by the way, that all the ‘lines’ in projective space are actually circles, since they close up at infinity. Thus, the circle at infinity can be called a ‘(projective) line’ at infinity, and usually is.
Now, what about spaces other than the plane? We can do similar things for 3-space or \( n \)-space, of course. But what about, say, an infinite cylinder?

- We could add a single point at infinity, getting a ‘pinched torus’.
- We could add a circle at infinity, getting an ordinary torus.
- We could add a circle at infinity in a different way to get a Klein bottle.
- We could add one point at infinity in each direction, getting—well, actually it’s topologically a sphere.
- We could add a circle at infinity in each direction, getting a finite cylinder with boundary.

What about a sphere? How could we add points at infinity to it? A torus? There isn’t any place to add points at infinity to these spaces, because there is no way to “get off to infinity” from them. We say they are already **compact**, like the projective plane or the closed hemisphere, and our process of adding points at infinity is called **compactification**.

Often in mathematics, when there are many ways to do something, it is useful to look for the **best** possible way. Usually if there is a best possible way, then it contains the most information. We talk about these “best ways” having a **universal property**.

Here, we can see that different compactifications “distinguish” different points at infinity. The stereographic projection sees only one point at infinity. The projective plane sees a whole projective line at infinity, making distinctions between different ways to go off to infinity that the stereographic compactification can’t tell the difference between. The hemispherical compactification makes even more distinctions: it can tell the difference between going off to infinity in exactly opposite directions. We can even see geometrically the process of “losing information” as we pass between these compactifications. If we start from a hemisphere and squash all the equator together into a point, we end up with a sphere.

Thus, it is natural to think that the **best** possible compactification, in a formal sense, would be one which makes the **most possible** distinctions between points at infinity. This would give us the most possible information about how a space behaves as we go off to infinity, and would let us recover any other compactification by squashing some points together. For instance, can we do any better than the hemispherical compactification? Can we distinguish between, say, going off to infinity as \( x \to \infty \) along the lines \( y = 0 \) and \( y = 1 \)?

Our goal in these notes is to construct this “best possible” compactification of any space, which is called the **Stone-Čech compactification**. In order to do this, we’ll need to define “spaces” in a very general sense, and make precise what we mean by “compact”.

2. **Second introduction: for experts**

*If you already know what a topological space is, and perhaps have seen the Stone-Čech compactification before, then you may be interested in the following comments. If not, then this section probably won’t make a whole lot of sense, so I recommend you skip ahead to §3 on page 5.*

The goal of these notes is to develop the basic theory of the Stone-Čech compactification without reference to open sets, closed sets, filters, or nets. In particular, this means we cannot use any of the usual definitions of topological space. This may seem like proposing to run a marathon while hopping on one foot, but I hope to convince you that it is easier than it may appear, and not devoid of interest.
These notes began as a course taught at Canada/USA Mathcamp, a summer program for mathematically talented high school students. Mathcampers are generally very quick and can handle quite abstract concepts, but are (unsurprisingly) lacking in formal mathematical background. Thus, reducing the prerequisite abstraction is important when designing a Mathcamp class.

It is common in introductory topology courses to use metric spaces instead of, or at least prior to, abstract topological spaces. The idea of “measuring the distance between points” is generally regarded as more intuitive than abstract notions such as “open set” or “closed set”. In particular, metric spaces can be used to bridge the gap between $\varepsilon$-$\delta$ notions of continuity that students may have seen in calculus and an abstract definition in terms of open sets.

The main disadvantage of metric spaces, compared to topological ones, is that they are not very general. In particular, Stone-Čech compactifications are not usually metrizable, so metric spaces alone are unsuitable for our purposes here. However, I was pleased to discover a notion which I think is not much harder to understand than a metric space, but which is significantly more general: a set equipped with a family of metrics.

In these notes, I have called such a set (plus an inessential closure condition on its metrics) a gauge space. Some authors call them (pseudo)metric uniformities, due to the amazing theorem (a version of Urysohn’s lemma) that any uniform space can be presented using a family of metrics. In particular, since all completely regular spaces are uniformizable, they are also “gaugeable”. Thus gauge spaces are a much wider class than metric spaces: they include in particular all compact Hausdorff spaces, and thus all Stone-Čech compactifications. Pleasingly, complete regularity is also precisely the necessary condition for a space to embed into its Stone-Čech compactification, so this is a very appropriate level of generality for us.

Gauge spaces are also better than topological spaces, in that they automatically give us more structure than a topology: they give a uniformity. Thus we can discuss “uniform” concepts such as uniform continuity and completeness, at a much more general level than that of metric spaces, but using $\varepsilon$-$\delta$ definitions that are again just the same as the standard ones for real numbers that students may have seen before.

(I also believe there is something to be said for gauge spaces over the more classical “entourage” or “covering” notions of uniformity even aside from pedagogy. In particular, the somewhat obscure and difficult “star-refinement” property of a uniformity is replaced by the simple operation of dividing $\varepsilon$ by two.)

The presence of uniform structure in a gauge space also provides a solution to the next problem which arises: how to define compactness. Compactness is also traditionally a difficult concept for students. In a metric space, compactness is equivalent to sequential compactness, which is easier to understand; but in a gauge space an analogous statement would have to involve nets or filters, which are themselves difficult concepts.

Fortunately, we can take an end run around the whole issue, because compactness of a uniform space is also equivalent to the conjunction of total boundedness and completeness. Interestingly, these are both uniform rather than topological properties, but their conjunction

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1 [http://www.mathcamp.org](http://www.mathcamp.org)

2 In these notes I depart from the venerable tradition according to which all metrics are Hausdorff. I prefer the philosophy of point-set topology, that separation properties should be considered as properties, rather than part of a definition. Thus, my “metrics” are what are traditionally called “pseudometrics”.

3 Uniform spaces, sadly often lacking from undergraduate and even graduate curricula, are to uniformly continuous functions as topological spaces are to continuous functions. References include [4–6, 9].
is equivalent to the topological property of compactness. This description of compactness also provides a pleasing way to construct compactifications (including the Stone-Čech compactification): pass to a topologically equivalent totally bounded gauge, then complete it.

Total boundedness is not so hard to understand as a strong form of being “finite in extent”: no matter how small a mesh we want to draw on our space, we only need finitely many grid points. But the final hurdle is defining completeness and completion. Completeness of metric spaces is usually defined using Cauchy sequences, while for uniform spaces one has to use Cauchy nets or filters instead—something we wanted to avoid!

The solution to this last puzzle comes from Lawvere’s description [7] of metric spaces as enriched categories, and completeness as representability of certain profunctors (or equivalently, existence of certain weighted limits). This involves no Cauchy sequences: instead a “Cauchy point” is specified simply by giving what its distances to all actual points ought to be, plus some very obvious axioms. It has the further advantage that no passage to equivalence classes is necessary in constructing the completion (quotient sets being another traditionally difficult concept for students).

Clementino, Hofmann, and Tholen [1, 3] have shown that uniform spaces, and their completions [2], can be described in a framework which generalizes Lawvere’s. I have not been able to find such a framework which reproduces gauge spaces exactly, although they are a special case of the prometric spaces of [3]. However, a naive generalization of Lawvere’s completeness criterion from metric spaces to gauge spaces seems to work quite well.

This, then, is how we can define the Stone-Čech compactification $\beta X$ without ever mentioning open and closed sets, nets, or filters: work with gauge spaces, define compactness to mean total boundedness plus completeness, construct completions using Lawvere-style Cauchy points, and build $\beta X$ by completing an appropriate totally bounded gauge. The astute reader will notice that filters make a somewhat disguised appearance in §17, but overall I believe I can declare victory in the stated aim of avoiding point-set topological notions.

It turns out that this construction of $\beta X$ is also convenient for proving basic facts about compactness, such as the fact that it is a topological property (which is not obvious from the definition we are using!) I have left this and other interesting facts to the reader as exercises—of which there are many. The starred exercises are generally the more difficult ones (though there is a wide variety of difficulty within the starred exercises). But all of the exercises should be doable using only the concepts and results that have been introduced in the paper up to that point (perhaps including previous exercises).

One last unusual concept that appears in these notes is that of proximity. This is a level of structure that lies in between uniformity and topology, and can be studied abstractly in its own right [8]. In fact, a proximity is essentially the same as a totally bounded uniformity, and every uniformity has an “underlying” proximity which can be thought of as its “totally-boundedification”. Proximity is a natural concept to introduce when discussing compactifications, because totally bounded uniformities compatible with a given topological space (or equivalently proximities compatible with such a space) are equivalent to compactifications thereof. Proximity is also a natural midway-point when trying to motivate the notion of “topology” or “continuity”, starting from metric or gauge notions. However, to avoid too much proliferation of concepts (and because the notion of proximity is somewhat weird to think about), I have mostly relegated proximity notions to the exercises.
3. Gauge spaces

A space has a set of points, but it is more than that. What is it that makes the plane cohesive, rather than just a collection of points? We need to have a way to judge how close together or far apart points are. Here’s a natural such abstract notion.

Definition 3.1. A metric on a set $X$ is a function $d: X \times X \to [0, \infty)$ such that

(i) $d(x, x) = 0$ for each $x \in X$ (reflexivity).
(ii) $d(x, y) = d(y, x)$ for each $x, y \in X$ (symmetry).
(iii) $d(x, y) + d(y, z) \geq d(x, z)$ for each $x, y, z \in X$ (transitivity or the triangle inequality).

A space equipped with a metric is called a metric space.

Reflexivity says that a point is as close as possible to itself. Symmetry says that if $x$ is close to $y$, then $y$ is just as close to $x$. Transitivity says that if $x$ is close to $y$ and $y$ is close to $z$, then $x$ is close to $z$.

Examples 3.2.

- $\mathbb{R}$ with $d(x, y) = |x - y|$. You are probably familiar with the triangle inequality in this case.
- $\mathbb{R}^2$ with $d(x, y) = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}$. I use $x$ and $y$ for points, and natural numbers to label coordinates. This is where the triangle inequality gets its name.
- More generally, $\mathbb{R}^n$.
- Let $d$ be a metric on $X$ and let $A$ be a subset of $X$. Then $d|_{A \times A}$ is a metric on $A$.
- This includes $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$, for example.
- Any set $X$ with $d(x, y) = 0$ for all $x, y \in X$. This is called the indiscrete metric; all the points are squished so close together as to be indistinguishable.
- Any set $X$ with

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Here no two distinct points are close together at all. This is called the discrete metric.

We say a metric is separated (or Hausdorff) if $d(x, y) = 0$ only when $x = y$. Many people include this in the definition of a metric (calling our more general metrics “pseudometrics”).

A good way to get a feel for a particular metric is to look at balls around points. Given a point $x \in X$ and a real number $\varepsilon > 0$, we define the (open) ball around $x$ of radius $\varepsilon$ to be the set

$$B_d(x, \varepsilon) = \{ y \in X \mid d(y, x) < \varepsilon \}.$$

In $\mathbb{R}^2$, a ball is the interior of a circle centered at $x$ with radius $\varepsilon$. In $\mathbb{R}$, it is an open interval centered at $x$ of length $2\varepsilon$. In the indiscrete metric, every open ball is the whole space. In the discrete metric, every open ball is just a single point. Here are some other examples:

- Consider $\mathbb{R}^2$ with $d'(x, y) = |x_0 - y_0| + |x_1 - y_1|$. This is also a perfectly good metric, different from the usual one. It is sometimes called the taxicab or Manhattan metric.
  Now an open ball around $x$ is a diamond-shape.
- Another metric on $\mathbb{R}^2$ is $d''(x, y) = \max(|x_0 - y_0|, |x_1 - y_1|)$. Now an open ball around $x$ is a square centered at $x$ with side length $2\varepsilon$. 
Now a set equipped with a metric is a good notion of ‘space’, but not quite good enough for our purposes. Here is one example of why. We have seen that \( \mathbb{R}^n \) has a natural metric for all finite \( n \), but what about \( \mathbb{R}^\omega \)?

Our natural inclination is to write
\[
d(x, y) = \sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2 + \ldots}
\]
but of course this makes no sense; what is
\[
d((0, 0, 0, 0 \ldots), (1, 1, 1, 1 \ldots))?
\]

We could try
\[
d(x, y) = \max(|x_0 - y_0|, |x_1 - y_1|, \ldots)
\]
but then what about
\[
d((0, 0, 0, 0 \ldots), (1, 2, 3, 4 \ldots))?
\]

And yet, it does seem to make sense to talk about two points in \( \mathbb{R}^\omega \) being close together. Two points in \( \mathbb{R}^n \) are close together when all their coordinates are close together, so it makes sense to say the same in \( \mathbb{R}^\omega \).

In fact, \( \mathbb{R}^\omega \) can be given sensible metrics. One possibility is to choose a function \( g : [0, \infty) \rightarrow [0, 1) \) with the property that \( g(a + b) \leq g(a) + g(b) \), and define
\[
(3.3) \quad d_\infty(x, y) = \sup_i g(|x_i - y_i|).
\]

Another somewhat fancier possibility is
\[
(3.4) \quad d_\Sigma(x, y) = \sum_i g(|x_i - y_i|) \cdot 2^{-i}.
\]

One \( g \) which works is \( g(a) = \min(a, 1) \); I’ll generally use this one in the future.

Note that (3.3) still makes sense if we replace \( \omega \) by an uncountable set, but (3.4) does not. On the other hand, we will see in the next section that (3.4) is “topologically correct” but not “uniformly correct”, whereas (3.3) is not even topologically correct. Therefore, to treat the uncountable case correctly, we need to generalize the notion of metric space. This generalization will also be necessary for the Stone-Čech compactification, later on.

If \( d, d' \) are two metrics on a set \( X \), we say that \( d \geq d' \) if for all \( x, y \in X \), \( d(x, y) \geq d'(x, y) \).

**Definition 3.5.** A **gauge** on a set \( X \) is a nonempty set \( \mathcal{G} = \{d_\alpha\} \) of metrics on \( X \), called **gauge metrics** or **\( X \)-metrics**, such that
\[
(\ast) \quad \text{for any two gauge metrics } d_1 \text{ and } d_2 \text{, there exists a gauge metric } d_3 \text{ with } d_3 \geq d_1 \text{ and } d_3 \geq d_2.
\]

A **gauge space** is a set \( X \) equipped with a gauge \( \mathcal{G} \); we write it as \((X, \mathcal{G})\) or just \( X \) if there is no danger of confusion.

In particular, a metric space is a gauge space. However, we also have other examples.

- Consider \( \mathbb{R}^2 \) with three metrics: \( d_0(x, y) = |x_0 - y_0| \), \( d_1(x, y) = |x_1 - y_1| \), and \( d_{01}(x, y) = \max(|x_0 - y_0|, |x_1 - y_1|) \). This generalizes easily to a gauge on \( \mathbb{R}^n \).
• We equip $\mathbb{R}^\omega$ with a countably infinite set of metrics defined by
\[ d_N(x,y) = \max_{1 \leq i \leq N} |x_i - y_i|. \]
The intuition is that two points in a gauge space are close when the distance between them is small in all the metrics. This is why the above gauge on $\mathbb{R}^\omega$ gives what we want. For example, we call a gauge space separated (or Hausdorff) if whenever $d(x,y) = 0$ for all gauge metrics $d$, then $x = y$.

In a gauge space, by an open ball we mean simply an open ball with respect to some metric in the gauge. We write $B_d(x,\varepsilon)$ to specify the metric $d$ being used.

We call condition (⋆) the filteredness property. We can omit it (and some people do), but in that case there is a “whack-a-mole” behavior that comes into play: in some other places the definitions become more complicated. We’ll come back to it later; for now we note that given two open balls $B_{d_1}(x,\varepsilon_1)$ and $B_{d_2}(x,\varepsilon_2)$, the filteredness condition ensures that we can find a metric $d_3$ such that $B_{d_3}(x,\min(\varepsilon_1,\varepsilon_2))$ is contained in both $B_{d_1}(x,\varepsilon_1)$ and $B_{d_2}(x,\varepsilon_2)$.

Note that any set $\mathcal{M}$ of metrics generates a gauge by adding to it the metrics $\max(d_0,\ldots,d_n)$ for any finite set of metrics $d_0,\ldots,d_n \in \mathcal{M}$. But we can’t we allow infinite subsets of $\mathcal{M}$, because then the maximum might not exist—even a supremum might not exist! We already had this problem with $\mathbb{R}^\omega$. We could try to circumvent it as in (3.3), but we’ll see in the next section that this would be the wrong thing to do.

4. Topology

Now, when we did stereographic projection, we identified the plane with the points on the sphere that their lines meet. But that bijection does not preserve distance! What does it preserve? A word for it is topology, but what does that mean? We’d like to say that it preserves “infinite closeness,” but no two points can be infinitely close without being at distance zero (which, in most spaces, is only possible if they are the same point). However, sets can be infinitely close without being identical (or even intersecting).

**Definition 4.1.** For $d$ a metric on $X$ and nonempty subsets $A, B \subseteq X$ we define
\[ d(A, B) = \inf_{a \in A, b \in B} d(a, b) \]
For $x \in X$ we write
\[ d(x, B) = d(\{x\}, B) \]
In a gauge space, we write $A \approx B$ if $d(A, B) = 0$ for all gauge metrics $d$. By convention, we say $\emptyset \not\approx A$ for all sets $A$.

Evidently if $A \cap B \neq \emptyset$, then $A \approx B$. Can you think of an example of two sets with $A \cap B = \emptyset$ and $A \approx B$?

**Examples 4.2.**
- Let $A$ be the open disc in the plane with center $(-1,0)$ and radius 1, and $B$ the similar disc with center $(1,0)$.
- Let $A$ be the curve $y = 1/x$ in the plane, and $B$ the curve $y = -1/x$.
- Let $A$ be the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ and $B$ the set $\{0\}$.

**Lemma 4.3.** $A \approx B$ iff for every gauge metric $d$ and every $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ with $d(a,b) < \varepsilon$.

**Proof.** Easy.□
Does the stereographic bijection preserve \( \approx \)? No; two parallel lines in \( \mathbb{R}^2 \) have \( d(A, B) \) positive (their distance apart) but their stereographic projections are at distance 0. But it \textit{does} preserve \( \approx \) between a point and a set (though this may not be obvious yet). This leads us to separate the following two notions.

- The \textbf{proximity} of a gauge space is the relation \( A \approx B \) on pairs of subsets.
- The \textbf{topology} of a gauge space is the relation \( x \approx A \) on (point, subset) pairs. (We write \( x \approx A \) to mean \{ \( x \) \} \( \approx \) \{ \( A \) \}).

When we say that a concept or definition is \textit{topological}, we mean that it depends only on the relation \( "x \approx A" \) between points and subsets. Similarly, we can talk about a concept or definition being \textit{proximal}.

Two different gauges on a set can define the same topology or proximity. In the case of topologies, there is a nice criterion for this.

\textbf{Lemma 4.4.} Two gauges \( G \) and \( G' \) on a set \( X \) define the same topology (are topologically equivalent) if and only if both

- For every metric \( d \in G \) and open ball \( B_d(x, \varepsilon) \), there exists a metric \( d' \in G' \) and an open ball \( B_{d'}(x, \varepsilon') \subseteq B_d(x, \varepsilon) \), and
- For every metric \( d' \in G' \) and open ball \( B_{d'}(x, \varepsilon') \), there exists a metric \( d \in G \) and an open ball \( B_d(x, \varepsilon) \subseteq B_{d'}(x, \varepsilon') \).

\textit{Proof.} Suppose they define the same topology, and given \( B_d(x, \varepsilon) \), and suppose contrarily that every open ball \( B_{d'}(x, \varepsilon') \) contains a point outside \( B_d(x, \varepsilon) \). Then for every \( d' \in G' \) and \( \varepsilon' > 0 \), there exists a point \( y \) in \( X \setminus B_d(x, \varepsilon) \) with \( d'(x, y) < \varepsilon' \), so \( x \approx (X \setminus B_d(x, \varepsilon)) \) relative to \( G' \). But \( G \) and \( G' \) define the same topology, so \( x \approx (X \setminus B_d(x, \varepsilon)) \) also relative to \( G \). In particular, there exists a point \( y \) outside of \( B(x, \varepsilon) \) such that \( d(x, y) < \varepsilon \), clearly absurd.

Conversely, suppose the ball conditions and that \( x \approx A \) with respect to \( G \), i.e. for every \( d \in G \) and \( \varepsilon > 0 \) there is a \( y \in A \) with \( d(x, y) < \varepsilon \). Suppose given a \( d' \in G' \) and an \( \varepsilon' > 0 \); then there is a \( B_d(x, \varepsilon) \subseteq B_{d'}(x, \varepsilon') \), hence a \( y \in A \cap B_d(x, \varepsilon) \subseteq Y \cap B_{d'}(x, \varepsilon') \), which is what we want. \( \square \)

The relation \( \approx \) is a good way to \textit{think} about topological equivalence, but the conditions in this lemma are usually the best way to \textit{work} with it.

\textit{Example 4.5.} The usual metric, the taxicab metric, and the maximum metric on \( \mathbb{R}^n \) are all topologically equivalent. They are also equivalent to the gauge generated by the metrics \( d_i(x, y) = |x_i - y_i| \).

Similarly, we say that a bijection \( f : X \leftrightarrow Y \) is a \textbf{topological isomorphism} or a \textbf{homeomorphism} if we have \( x \approx A \iff f(x) \approx f(A) \). There is an analogous condition for this. Some examples:

- Stereographic projection is a topological isomorphism from the plane \( \mathbb{R}^2 \) to \( S^2 \setminus \{N\} \).
- The real line \( \mathbb{R} \) is topologically isomorphic to the open interval \( (0, 1) \), via \( f(x) = \frac{1}{2} + \frac{1}{2} \arctan x \) whose inverse is \( f^{-1}(x) = \tan(\pi(x - \frac{1}{2})) \).
- In the discrete metric, replacing 1 with any other positive real number produces a different, but topologically equivalent, metric.

So the first step in the process of compactification involves passing to a new topologically isomorphic space, in which it is somehow “easier to see” where to put points at infinity.
Soon we’ll think about what property of the new space that is, and how to add the missing points.

5. Exercises on gauges

Exercise 5.1. Does the distance between subsets

\[ d(A, B) = \inf_{a \in A, b \in B} d(a, b) \]

define a metric on the power set \( \mathcal{P}X \) of \( X \)?

Exercise 5.2. There is a gauge on any set \( X \) which consists of all possible metrics on \( X \). Prove that this gauge is topologically equivalent to a discrete metric.

Exercise 5.3. Prove that any gauge containing only a finite number of metrics is topologically equivalent to some gauge with only a single metric.

Exercise 5.4. Prove that there is a largest gauge that is topologically equivalent to any given gauge. That is, prove that for any gauge \( \mathcal{G} \) there is a gauge \( \mathcal{H} \), which is equivalent to \( \mathcal{G} \), and such that if \( \mathcal{G}' \) is topologically equivalent to \( \mathcal{G} \), then \( \mathcal{G}' \subseteq \mathcal{H} \).

Exercise 5.5. A metric is bounded if there is an \( N \in [0, \infty) \) such that \( d(x, y) < N \) for all points \( x, y \). Prove that every gauge is topologically equivalent to a gauge containing only bounded metrics.

Exercise 5.6. Consider the following two metrics on \( \mathbb{R}^\omega \):

\[ d_\infty (x, y) = \sup_i \left\{ \min_i (|x_i - y_i|, 1) \right\} \]

\[ d_\Sigma (x, y) = \sum_{i=0}^{\infty} \min_i (|x_i - y_i|, 1) \cdot 2^{-i} \]

Convince yourself that both are, indeed, metrics. Is either of them (that is, the singleton gauges \( \{d_\infty\} \) and \( \{d_\Sigma\} \)) topologically equivalent to the gauge on \( \mathbb{R}^\omega \) defined above, which consisted of metrics

\[ d_N (x, y) = \max_{1 \leq i \leq N} |x_i - y_i| \]

* Exercise 5.7. Find a set \( X \) admitting a gauge which you can prove is not topologically equivalent to any single metric.

* Exercise 5.8. A topology on a set \( X \) is usually defined to be a collection of subsets of \( X \), called open sets, which is closed under finite intersections and arbitrary unions. (This includes the intersection of no sets, which is \( X \), and the union of no sets, which is \( \emptyset \).) Prove that every gauge on \( X \) gives rise to a topology on \( X \), and that this topology is exactly characterized by the relation “\( x \approx A \)” between points and subsets.

* Exercise 5.9. Suppose \( \mathcal{G} \) and \( \mathcal{G}' \) are two gauges on the same set \( X \).

(i) Find an \( \varepsilon \)-style condition which is sufficient to ensure that \( \mathcal{G} \) and \( \mathcal{G}' \) define the same proximity.

(ii) Find an \( \varepsilon \)-style condition which is both sufficient and necessary to ensure that \( \mathcal{G} \) and \( \mathcal{G}' \) define the same proximity.

(iii) Which of the previous topological equivalences are actually proximal equivalences?
Exercise 5.10. A quasi-metric on a set $X$ is a function $d: X \times X \to \mathbb{R}$ satisfying the reflexivity and transitivity properties of a metric, but not necessarily symmetry. A quasi-gauge is a set of quasi-metrics satisfying the same “filteredness” condition as a gauge. See how much of the theory of gauge spaces you can mimic for quasi-gauges (you may want to come back to this exercise as we develop more of this theory).

*Exercise 5.11. Prove that every topology on a set $X$ (see *Exercise 5.8) is induced by some quasi-gauge on $X$ (see Exercise 5.10).

6. Continuity

The most important kind of morphism between gauge spaces is one that preserves the topology (only).

**Definition 6.1.** A function $f: X \to Y$ between gauge spaces is **continuous** if whenever $x \approx A$ in $X$, then $f(x) \approx f(A)$ in $Y$.

As we did for topological equivalence, we can reformulate this in a less intuitive, but more useful, way.

**Lemma 6.2.** $f: X \to Y$ is continuous iff for any $x \in X$, any $Y$-metric $d_Y$, and any $\varepsilon > 0$, there exists an $X$-metric $d_X$ and a $\delta > 0$ such that for any $x' \in X$, if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$.

**Proof.** Suppose $f$ is continuous, and given $x$, $d_Y$, and $\varepsilon$, and suppose the contrary: for every $d_X$ and $\delta$ there exists an $x' \in X$ with $d_X(x, x') < \delta$ but $d_Y(f(x), f(x')) \geq \varepsilon$. Define

$$A = \{ x' \in X \mid d_Y(f(x), f(x')) \geq \varepsilon \}$$

Then the assumption implies that $x \approx A$. Hence $f(x) \approx f(A)$, so there should be a point $y \in f(A)$ with $d_Y(f(x), y) < \varepsilon$, which is absurd.

Conversely, suppose the condition and that $x \approx A$. To prove $f(x) \approx f(A)$, say given $d_Y$ and $\varepsilon > 0$; we want to find a $y \in f(A)$ with $d_Y(f(x), y) < \varepsilon$. Choose $d_X$ and $\delta$ as in the condition; then since $x \approx A$ there exists $x' \in A$ with $d_X(x, x') < \delta$. Take $y = f(x')$. □

Examples:

- The identity function $id_X: X \to X$ is always continuous.
- Any constant function $f: X \to Y$, $f(x) = a$ for some fixed $a \in Y$, is continuous.
- Addition $+: \mathbb{R}^2 \to \mathbb{R}$ is continuous; take $\delta = \varepsilon/2$.
- Multiplication $\cdot: \mathbb{R}^2 \to \mathbb{R}$ is continuous. Given $(x_0, x_1) \in \mathbb{R}^2$, take

$$\delta = \min \left( \frac{\varepsilon}{3|x_0|}, \frac{\varepsilon}{3|x_1|}, \sqrt{\frac{\varepsilon}{3}} \right).$$

- The composite of continuous functions is continuous.
- Therefore, any polynomial $p: \mathbb{R} \to \mathbb{R}$ is continuous (using the diagonal as well).
- If $X$ has a discrete gauge or $Y$ has an indiscrete gauge, then any function $X \to Y$ is continuous.

Non-examples:
The step function
\[ \theta(x) = \begin{cases} 
0 & x \leq 0 \\
1 & x > 0 
\end{cases} \]
is not continuous. Take \( x = 0 \) and \( \varepsilon < 1 \). Then there is no ball around 0 such that everything in that ball is mapped to within \( \varepsilon \) of \( \theta(0) = 0 \).

Similarly for the blip function
\[ \delta(x) = \begin{cases} 
0 & x \neq 0 \\
1 & x = 0 
\end{cases} \]

A continuous function preserves all the topological properties of a space. It extends our previous notion of topological isomorphism, as follows.

**Theorem 6.3.** A bijection \( f : X \leftrightarrow Y \) is a topological isomorphism if and only if both it and its inverse are continuous.

**Proof.** Essentially by definition. \( \square \)

We saw lots of topological isomorphisms last time. Here is an important sort of counterexample to keep in mind.

**Example 6.4.** The map \( t \mapsto (\cos t, \sin t) \) from \([0, 2\pi)\) to \( S^1 \) is continuous and bijective, but not a topological isomorphism.

The notion of continuous function gives a reason why we don’t want to use \( d_\infty(x, y) = \max_{i \in \mathbb{N}} |x_i - y_i| \) as a single metric on \( \mathbb{R}^\omega \), and similarly why we don’t want to close up a set of metrics under infinite suprema.

**Proposition 6.5.** Suppose that \( \mathbb{R}^\omega \) has the metric
\[ d_\infty(x, y) = \sup_i \left( \min(|x_i - y_i|, 1) \right) \]
and \( \mathbb{R} \) the usual one. Then the function \( f(t) = (t, \sqrt{|t|}, \sqrt[3]{|t|}, \ldots) \) from \( \mathbb{R} \) to \( \mathbb{R}^\omega \) is not continuous at \( t = 0 \).

**Proof.** Fix \( \varepsilon > 0 \). Then \( |\sqrt[2n]{t}| < \varepsilon \) is equivalent to \( |t| < \varepsilon^{2n} \). Since for any \( t \neq 0 \), there is an \( n \) such that \( |t| > \varepsilon^n \), there is no \( \delta > 0 \) such that \( |t| < \delta \) implies \( |\sqrt[n]{t}| < \varepsilon \) for all \( n \). \( \square \)

However, the actual gauge on \( \mathbb{R}^\omega \) we used does work. More generally, we have the following. Let \( X \) be a gauge space, \( A \) a set, and \( X^A \) the set of all functions \( A \to X \), which we write as \( (x_a)_{a \in A} \). Let \( \pi_a : X^A \to X \) be defined by \( \pi_a(x) = x_a \) for each \( a \in A \). For each \( X \)-metric \( d \) and each finite subset \( B \subseteq A \), define a metric \( d_B \) on \( X^A \) by \( d_B(x, y) = \max_{a \in B} d(x_a, y_a) \). The set of all these metrics, as \( B \) varies over finite subsets of \( A \), is called the pointwise gauge on \( X^A \).

**Proposition 6.6.** For any gauge space \( Z \), a function \( f : Z \to X^A \) is continuous (where \( X^A \) has the pointwise gauge) if and only if for each \( a \in A \), the function \( \pi_a \circ f : Z \to X \) is continuous.

In other words, a function is continuous if and only if all its coordinates are continuous.
Proof. Firstly, the function \( \pi_a : X^A \to X \) defined by \( \pi_a(x) = x_a \) is evidently continuous; the preimage of each \( B_d(x, \varepsilon) \) contains \( B_{d_{\{a\}}}(x, \varepsilon) \). Thus, \( \pi_a \circ f \) is continuous if \( f \) is. (This much is also true for the metric \( d_\infty \).)

Now suppose that each \( \pi_a \circ f \) is continuous. Thus, for each \( a \in A \), \( X \)-metric \( d \), and ball \( B_d(f(z)_a, \varepsilon) \) there is a \( Z \)-metric \( d_a \) and a ball \( B_{d_a}(z, \delta_a) \) contained in the preimage of \( B_d(f(z)_a, \varepsilon) \). Choose any ball around \( f(z) \) in \( X^A \); it is of the form \( B_{d_B}(f(z), \varepsilon) \) for some finite \( B \subseteq A \). By filteredness of the gauge on \( Z \), there is a \( Z \)-metric \( d' \) with \( d' \geq d_a \) for all \( a \in B \). If we let \( \delta = \min_{a \in B} \delta_a \), then it follows that \( B_{d'}(z, \delta) \) is contained in the preimage of \( B_{d_B}(f(z), \varepsilon) \), as desired. \( \square \)

Here is the first place we’ve used filteredness, but this may not be satisfying; we only used it because we decided to close up the metrics on \( X^A \) under finite maxes. What if we didn’t close them up under any maxes at all?

Consider \( \mathbb{R} \times \mathbb{R} \) with the two metrics \( d_0(x, y) = |x_0 - y_0| \) and \( d_1(x, y) = |x_1 - y_1| \). This is not a gauge by our definition, but it would be if we omitted the filteredness condition. But in this “gauge”, and with our definition of \( \approx \), we would have

\[
(0, 0) \approx \{(1, 0), (0, 1)\}
\]

because for each of \( d_0 \) and \( d_1 \), there is an element of the right-hand set which is at distance zero from the left-hand point. Clearly this is not the right topology on \( \mathbb{R} \times \mathbb{R} \). Thus, if we didn’t require the filteredness condition, we would have to define \( A \approx B \) with reference to arbitrary finite sets of metrics, which is more of a pain to think about. (We will, however, have to do something similar later on when we talk about evaluation data in §17.)

Remark 6.7. Lest the reader come away with too negative an impression of \( d_\infty \), let me say that it also has important uses. It is called the \textbf{supremum metric} or \textbf{sup-metric}, and induces the \textbf{topology of uniform convergence} on \( X^A \).

7. Exercises on continuity

Exercise 7.1. Let \( X \) and \( Y \) be metric spaces (that is, gauge spaces whose gauge contains only one metric). A \textbf{contraction} is a function \( f : X \to Y \) such that \( d(f(x), f(x')) \leq d(x, x') \). Prove that any contraction is continuous.

Exercise 7.2. Let \( A \subseteq X \) be nonempty. Prove that for any \( X \)-metric \( d \), the function \( d_A : X \to \mathbb{R} \) defined by \( d_A(y) = d(A, y) \) is continuous.

Exercise 7.3. Prove that two gauges on the same set \( X \) are topologically equivalent if and only if they agree about which functions \( f : X \to \mathbb{R} \) are continuous.

Exercise 7.4. We can consider the relation \( \approx \) between pairs of \textit{points} of a gauge space as well: we have \( x \approx y \) iff \( d(x, y) = 0 \) for all gauge metrics \( d \). Prove that:

(i) \( \approx \) is an equivalence relation on \( X \).

(ii) The quotient \( X/ \approx \) is a separated gauge space.

(iii) The quotient map \( X \to X/ \approx \) is continuous.

Exercise 7.5. Let \((X, \mathcal{G})\) be a gauge space and \( A \subseteq X \) a subset. For each gauge metric \( d \), define a new metric by

\[
d_A(x, y) = \min \left( d(x, y), d(x, A) + d(A, y) \right).
\]
(i) Prove that $d_A$ is a metric on $X$, and that the set of these metrics defines a new gauge $G_A$ on $X$.

(ii) Prove that the identity is a continuous function $(X, G) \to (X, G_A)$.

* Exercise 7.6. If $X$ and $Y$ are sets equipped with topologies as in * Exercise 5.8, a function $f : X \to Y$ is usually defined to be continuous if for every open subset $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is also open. Prove that a function between gauge spaces is continuous in our sense if and only if it is continuous in this sense with respect to the underlying topologies you defined in * Exercise 5.8.

* Exercise 7.7. A function $f : X \to Y$ is called proximally continuous if $A \approx B$ in $X$ implies $f(A) \approx f(B)$ in $Y$. Find an $\varepsilon$-style condition which is sufficient (or, better, necessary and sufficient) for $f$ to be proximally continuous. Which of the continuous functions we have looked at are proximally continuous? (See * Exercise 5.9.)

8. Total boundedness

Our examples of compactification of $\mathbb{R}^2$ all had two steps. First we mapped $\mathbb{R}^2$ to a space that was ‘finite in extent’, making the desired ‘points at infinity’ into ‘holes’ at a finite location. Second, we then added those points to make the space compact. Our goal is to formalize both of those steps to apply them to arbitrary gauge spaces.

Here is a rough roadmap of the concepts that we will introduce.

- A space is called compact if you can’t escape from it, either to infinity or into a hole.
- A space is called totally bounded if it is ‘finite in extent’, or equivalently if you can’t escape to infinity (although you might escape into a hole).
- A space is called complete if it has no holes, so the only way to escape from it is to infinity.

Clearly a space should be compact if and only if it is both totally bounded and complete. Some examples:

- Sphere, toruses, discs, $RP^n$ are all compact.
- A sphere minus a point is totally bounded, but not complete, since it has a hole.
- An ordinary plane is complete, but not totally bounded, since it is ‘infinite in extent’.
- A plane minus a point is neither complete nor totally bounded.

It turns out that compactness is a topological property, i.e. it is invariant under topological equivalence. (This will not be obvious from our definition, but it is true.) But total boundedness is not a topological property, and neither will completeness be. This is a good thing for our process of compactification:

1. Given a gauge space, find a topologically equivalent gauge which is totally bounded.
2. ‘Complete’ the resulting totally bounded gauge space, to produce a space which is both totally bounded and complete, hence compact.

It is possible to compactify ‘in one step’ by directly adding points at infinity. However, it turns out to be easier, and more comprehensible, to first bring those points ‘in from infinity’ and then add them at some finite location.

Now let’s try to define “total boundedness” precisely. What important properties does, say, the sphere-minus-a-point have which distinguish it from the plane for our purposes?
Intuitively, it is “bounded” in some sense. The obvious notion of boundedness for a metric space $X$ is that $d(x, y) < N$ for some $N$. Equivalently, if $X = B(x, N)$ for some $x$ and $N$. However, this is not a very well-behaved concept. Suppose $d$ is any metric; then we can define a new metric by

$$d'(x, y) = \min(d(x, y), 1)$$

This is bounded, but at small distance scales it looks like $d$. For instance, on $\mathbb{R}^2$ we have

$$d'(x, y) = \min\left(\sqrt{(x_0 - y_0)^2 + (x_1 - y_1)^2}, 1\right)$$

Then we have $\mathbb{R}^2 = B_{d'}(0, N)$ for any $N > 1$. But if we shrink $N$ just a little, this becomes wildly false: it takes infinitely many $d'$-balls of radius 1 to cover $\mathbb{R}^2$.

This doesn’t happen on a sphere: for any $\varepsilon > 0$, it only takes finitely many balls of radius $\varepsilon$ to cover the sphere. Thus, the following definition captures this notion of being “bounded at arbitrarily small distance scales.”

**Definition 8.1.** A gauge space $X$ is **totally bounded** if for any $X$-metric $d$ and $\varepsilon > 0$, we can write $X$ as the union of finitely many balls $B_d(x_i, \varepsilon)$:

$$X = B_d(x_0, \varepsilon) \cup \cdots \cup B_d(x_n, \varepsilon).$$

Of course, $X$ can always be covered by some number of $(d, \varepsilon)$-balls. Take, for instance, all the balls $B_d(x, \varepsilon)$ for $x \in X$. The point is whether we can do it with finitely many.

Equivalently, $X$ is totally bounded if for any $d$ and $\varepsilon > 0$, there exists a finite subset $A \subseteq X$ such that $d(x, A) < \varepsilon$ for all $x \in X$.

**Examples 8.2.**

- The unit square $[0, 1]^2$ is totally bounded; given any $\varepsilon$ we can take a finite $\varepsilon$-spaced grid as the centers of $\varepsilon$-sized balls.
- The plane $\mathbb{R}^2$ is not totally bounded; any finite number of balls has only finite area.
- Any subspace of a totally bounded space is totally bounded. Thus, for instance, $\mathbb{Q} \cap [0, 1]$ is totally bounded, even though it has lots and lots of holes.

9. Uniformity

Note that total boundedness is not a topological property: the plane is not totally bounded, but the sphere-minus-a-point is, and they are topologically isomorphic. In fact, total boundedness is not even a proximity concept!

**Example 9.1.** Let $X$ be an arbitrary set, and consider the following two gauges on $X$. Let $G$ be the discrete gauge, containing only one metric with $d(x, y) = 1$ iff $x \neq y$.

To define $G'$, suppose we have a finite partition of $X$:

$$X = A_1 \sqcup \cdots \sqcup A_n.$$

(Writing $\sqcup$ instead of $\cup$ means that $A_i \cap A_j = \emptyset$ for $i \neq j$.) Then define a metric on $X$ by

$$d_{A_1,\ldots,A_n}(x, y) = \begin{cases} 0 & \text{if } x, y \in A_i \text{ for some } i \\ 1 & \text{if } x \in A_i, y \in A_j \text{ for } i \neq j. \end{cases}$$

The collection of these metrics, for all partitions of $X$, defines a gauge $G'$ on $X$.

I claim that these two gauges are proximally equivalent. First of all, in the discrete gauge, we clearly have $A \approx B$ iff $A \cap B \neq \emptyset$. Since $A \cap B \neq \emptyset$ always implies $A \approx B$, it suffices
to prove that if $A \cap B = \emptyset$, then $A \not\approx B$ in $G'$. But defining $C = X \setminus (A \cup B)$, we have a partition $X = A \cup B \cup C$, and clearly

$$d_{A,B,C}(A, B) = 1$$

so that $A \not\approx B$.

However, $G'$ is totally bounded, but (if $X$ is infinite) $G$ is not. The latter is obvious: there is no finite cover of $X$ by $\varepsilon$-sized balls for any $\varepsilon < 1$. The former is almost as obvious: for any partition and $\varepsilon > 0$, each $A_i$ is contained in some $(d_{A_1,\ldots,A_n}, \varepsilon)$-ball (in fact, if $\varepsilon < 1$ it is equal to some such ball), so that $n$ such balls suffice.

There ought to be some notion of equivalence which does preserve total boundedness, though. It turns out that the following is the right one.

**Definition 9.2.** A function $f : X \to Y$ between gauge spaces is uniformly continuous iff for any $Y$-metric $d_Y$ and $\varepsilon > 0$, there exists an $X$-metric $d_X$ and $\delta > 0$ such that for all $x, x' \in X$, we have

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

The difference between uniform continuity and continuity is that in the uniform case, $\delta$ is only allowed to depend on $\varepsilon$, not on $x$.

**Example 9.3.** $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous.

Thus, uniform continuity is a very strong concept!

**Definition 9.4.** A uniform isomorphism is a bijection $X \leftrightarrow Y$ which is uniformly continuous in both directions.

**Proposition 9.5.** If $f : X \to Y$ is a uniform surjection and $X$ is totally bounded, so is $Y$.

**Proof.** Suppose given a $Y$-metric $d_Y$ and $\varepsilon > 0$, and choose $d_X$ and $\delta > 0$ as in uniform continuity. Since $X$ is totally bounded, it is the union of finitely many $(d_X, \delta)$-balls. By definition, the $f$-image of each of those balls is contained in a $(d_Y, \varepsilon)$-ball, and $f$ is surjective, so finitely many of the latter suffice to cover $Y$. \qed

**Corollary 9.6.** If $X \leftrightarrow Y$ is a uniform isomorphism and $X$ is totally bounded, so is $Y$.

10. **Exercises on total boundedness and uniformity**

**Exercise 10.1.** Which of the three topologically metrics on $\mathbb{R}^2$ that we considered in §3 are uniformly equivalent?

**Exercise 10.2.** Prove that the functions $d_A$ from Exercise 7.2 are, in fact, uniformly continuous.

**Exercise 10.3.** Show that the metric $d_\Sigma$ on $\mathbb{R}^\omega$ is not uniformly equivalent to the many-metric gauge defined in §3.

**Exercise 10.4.** Prove that any uniformly continuous function is proximally continuous. Conclude that any two uniformly equivalent gauges are also proximally equivalent.

**Exercise 10.5.** Which of the other continuous functions (or topological isomorphisms) that we have seen so far are actually uniformly continuous (or uniform isomorphisms)?
A sequence \((x_0, x_1, x_2, \ldots)\) in a gauge space \(X\) is said to **converge** to \(x_\infty \in X\) if for any gauge metric \(d\) and any \(\varepsilon > 0\), there exists an \(N \in \mathbb{N}\) such that \(n > N \implies d(x_n, x_\infty) < \varepsilon\).

**Exercise 10.6**. Prove that if \((x_0, x_1, x_2, \ldots)\) converges to \(x_\infty\) in \(X\), and \(f : X \to Y\) is continuous, then \((f(x_0), f(x_1), f(x_2), \ldots)\) converges to \(f(x_\infty)\) in \(Y\).

A sequence \((x_0, x_1, x_2, \ldots)\) is said to be **Cauchy** if for any gauge metric \(d\) and any \(\varepsilon > 0\), there exists an \(N \in \mathbb{N}\) such that if \(n, m > N\), then \(d(x_n, x_m) < \varepsilon\).

**Exercise 10.7**. Prove that if a sequence in a gauge space converges, then it is Cauchy.

**Exercise 10.8**. Prove that if \((x_0, x_1, x_2, \ldots)\) is Cauchy in \(X\), and \(f : X \to Y\) is **uniformly** continuous, then \((f(x_0), f(x_1), f(x_2), \ldots)\) is Cauchy in \(Y\).

**Exercise 10.9**. Prove that \(X\) is a gauge space such that every sequence in \(X\) has a Cauchy subsequence, then \(X\) is totally bounded.

**Exercise 10.10**. Prove that if \(X\) is a totally bounded **metric** space (a gauge space with exactly one gauge metric), then every sequence in \(X\) has a Cauchy subsequence.

* **Exercise 10.11**. Find an example of a totally bounded gauge space which contains a sequence that has no Cauchy subsequence.

* **Exercise 10.12**. Prove that if \(f : X \to Y\) is proximally continuous and \(Y\) is totally bounded, then \(f\) is uniformly continuous. Conclude that if two totally bounded gauges on a set \(X\) are proximally equivalent, then they are in fact uniformly equivalent.

* **Exercise 10.13**. Prove that if \(f : X \to Y\) is proximally continuous and \(X\) is a metric space, then \(f\) is uniformly continuous.

* **Exercise 10.14**. A gauge space \(X\) is **pseudocompact** if every continuous \(f : X \to \mathbb{R}\) is bounded.

  (i) Prove that if \(X\) is pseudocompact, then every continuous \(f : X \to \mathbb{R}\) achieves a maximum and a minimum.

  (ii) Prove that any pseudocompact gauge space is totally bounded.

### 11. Completeness and completion

Recall from §8 that we compactify in two steps: first we make a space totally bounded, then we complete it. We’ve already defined total boundedness; let’s look now at completeness, or “filling in the holes”. What does it mean for a space to “have a hole”? That is, looking at a gauge space \(X\), how can we characterize a “point which should exist in \(X\), but doesn’t”? One obvious way is to give its distances from all the other points of \(X\).

**Definition 11.1**. A **Cauchy point** of a gauge space \(X\) consists of, for each gauge metric \(d\), a function \(\xi_d : X \to [0, \infty)\) such that

(i) \(d(x, y) + \xi_d(y) \geq \xi_d(x)\) for each \(d\) and each \(x, y \in X\),

(ii) \(\xi_d(x) + \xi_d(y) \geq d(x, y)\) for each \(d\) and each \(x, y \in X\),

(iii) \(\inf_{x \in X} \xi_d(x) = 0\) for each \(d\) (locatedness), and

(iv) For any two gauge metrics \(d_1\) and \(d_2\), there exists a gauge metric \(d_3\) such that \(d_3 \geq \max(d_1, d_2)\) and \(\xi_{d_3} \geq \max(\xi_{d_1}, \xi_{d_2})\).
We think of $\xi d(x)$ as “$d(\xi, x)$”. The first two conditions are then just two versions of the triangle inequality, and can be jointly rephrased as the ‘reverse triangle inequality’

$$|\xi d(x) - d(x, y)| \leq \xi d(y).$$

The locatedness condition can be phrased as “every ball $B_d(\xi, \varepsilon)$ contains a point of $X$”. This ensures that the ‘new point’ $\xi$ is ‘right next to $X$’; we don’t want to think about potential points that are sitting far away from $X$. And the fourth condition ensures that the filteredness condition on gauge metrics carries over to the functions $\xi d$, since they are supposed to be like values of the metrics.

We note that in conjunction with the first three conditions, the fourth implies the following stronger version of itself.

Lemma 11.2. If $\xi$ is a Cauchy point and $d_1$ and $d_2$ are gauge metrics with $d_1 \leq d_2$, then $\xi d_1 \leq \xi d_2$.

Proof. Choose $d_3$ as assumed in the fourth condition. Now for any $\varepsilon > 0$, use locatedness to choose an $x \in X$ with $\xi d_3(x) < \varepsilon$, hence also $\xi d_1(x) < \varepsilon$ and $\xi d_2 < \varepsilon$. Thus for any $y \in X$,

$$\begin{align*}
\xi d_1(y) &\leq d_1(x, y) + \xi d_1(x) \\
&\leq d_2(x, y) + \varepsilon \\
&\leq \xi d_2(y) + \xi d_2(x) + \varepsilon \\
&\leq \xi d_2(y) + 2\varepsilon.
\end{align*}$$

Since this is true for all $\varepsilon > 0$, we have $\xi d_1 \leq \xi d_2$. □

If $z \in X$, then we have a Cauchy point called $\hat{z}$ defined by $\hat{z} d(x) = d(z, x)$. We say a Cauchy point is represented by $z$ if it is equal to $\hat{z}$. We say that a gauge space is complete if every Cauchy point is represented (by some point).

Lemma 11.3. A Cauchy point $\xi$ is represented by $z$ iff $\xi d(z) = 0$ for all $d$.

Proof. Clearly if $\xi$ is represented by $z$, then $\xi d(z) = d(z, z) = 0$. Conversely, if $\xi d(z) = 0$, then for any $x$ we have

$$\xi d(x) \leq d(x, z) + \xi d(z) = d(x, z)$$

and

$$\xi d(x) = \xi d(x) + \xi d(z) \geq d(x, z)$$

and thus $\xi d(x) = d(x, z)$, so $\xi = \hat{z}$. □

It’s easy to come up with silly examples of non-represented Cauchy points.

- If $X = \mathbb{R}^2 \setminus \{0\}$, then $\xi d(0) = d(0, 0)$ is a Cauchy point which is not represented (it should be represented by 0). Thus $\mathbb{R}^2 \setminus \{0\}$ is not complete.
- More generally, if $X$ is any gauge space and $y \in X$, then on $Y = X \setminus \{y\}$ the collection of functions $\xi d(z) = d(y, z)$ (the distance taken in $X$) defines a non-representable Cauchy point of $Y$, so $Y$ is not complete.

In these cases, it is trivial to see how to ‘complete’ the incomplete spaces; add back in the missing point! In general, what we can do is use the Cauchy points themselves to stand for the missing points.
Let $\hat{X}$ be the set of Cauchy points in $X$. Each point of $X$ gives a Cauchy point $\hat{x}$, so we have an inclusion $X \to \hat{X}$. We want to make $\hat{X}$ a gauge space; that is, we need to somehow extend the metrics on $X$ to $\hat{X}$.

By definition, a Cauchy point knows what its distances are to any point of $X$, but not its distances to another Cauchy point. However, each Cauchy point is approximated arbitrarily closely by points of $X$, so we should be able to get pretty close to the distance between two Cauchy points $\xi$ and $\zeta$ by going first from $\xi$ to some point of $X$, then to $\zeta$.

This motivates the following definition. For each metric $d$ on $X$, define a metric $\hat{d}$ on $\hat{X}$ by setting

$$\hat{d}(\xi, \zeta) = \inf_{x \in X} (\xi_d(x) + \zeta_d(x)).$$

The locatedness of Cauchy points implies $\hat{d}(\xi, \xi) = 0$. Symmetry is clear. For transitivity, we have

$$\hat{d}(\xi, \zeta) + \hat{d}(\zeta, \chi) = \inf_x (\xi_d(x) + \zeta_d(x)) + \inf_y (\zeta_d(y) + \chi_d(y))$$
$$= \inf_{x,y} (\xi_d(x) + \zeta_d(x) + \zeta_d(y) + \chi_d(y))$$
$$\geq \inf_{x,y} (\xi_d(x) + d(x,y) + \chi_d(y))$$
$$\geq \inf_x (\xi_d(x) + \chi_d(x))$$
$$= \hat{d}(\xi, \chi).$$

Finally, Lemma 11.2 implies that if $d \geq d'$, then $\hat{d} \geq \hat{d}'$. Therefore, the collection of metrics $\{\hat{d}\}$ forms a gauge on $\hat{X}$, since given $\hat{d}_1$ and $\hat{d}_2$ we can find $d_3$ with $d_3 \geq d_1$ and $d_3 \geq d_1$, hence $\hat{d}_3 \geq \hat{d}_1$ and $\hat{d}_3 \geq \hat{d}_1$.

Lemma 11.4 (Yoneda). For any $z \in X$ and $\xi \in \hat{X}$, and any $X$-metric $d$, we have $\hat{d}(\hat{z}, \xi) = \xi_d(z)$. In particular, for $z, w \in X$ we have $\hat{d}(\hat{z}, \hat{w}) = d(z, w)$.

Proof. By the triangle inequality for $\xi$, for any $x$ we have

$$\hat{z}_d(x) + \xi_d(x) = d(z, x) + \xi_d(x) \geq \xi_d(z)$$

so that $\hat{d}(\hat{z}, \xi) \geq \xi_d(z)$. But on the other hand, we also have

$$\hat{d}(\hat{z}, \xi) \leq \hat{z}_d(z) + \xi_d(z) = \xi_d(z).$$

Thus, the function $(\hat{-}) : X \to \hat{X}$ sending $z$ to $\hat{z}$ is an “embedding” — though it may not be injective. In fact, we have:

Lemma 11.5. $(\hat{-}) : X \to \hat{X}$ is injective if and only if $X$ is separated. Moreover, $\hat{X}$ is always separated.

Proof. It’s easy to see that $\hat{x} = \hat{y}$ if and only if $x \approx y$, which proves the first statement. The second is an exercise. □

The locatedness of Cauchy points also implies that $X$ is dense in $\hat{X}$, i.e. that $\xi \approx X$ for any $\xi \in \hat{X}$.
Finally, we show \( \hat{X} \) is complete. Suppose \( \Xi \) is a Cauchy point of \( \hat{X} \), and define a Cauchy point \( \xi \) of \( X \) by 
\[
\xi_d(x) = \Xi_{\hat{d}}(\hat{x}).
\]
Now we check the axioms for \( \xi \) to be a Cauchy point of \( X \).

(i) \( d(x, y) + \xi_d(y) = \hat{d}(\hat{x}, \hat{y}) + \Xi_{\hat{d}}(\hat{y}) \geq \Xi_{\hat{d}}(\hat{x}) = \xi_d(x) \).

(ii) \( \xi_d(x) + \xi_d(y) = \Xi_{\hat{d}}(\hat{x}) + \Xi_{\hat{d}}(\hat{y}) \geq \hat{d}(\hat{x}, \hat{y}) = d(x, y) \)

(iii) Since \( \inf_{x} \Xi_{\hat{d}}(\xi) = 0 \), for any \( \varepsilon > 0 \) we have an \( \xi \in \hat{X} \) with \( \Xi_{\hat{d}}(\xi) < \frac{\varepsilon}{2} \). And since \( \xi \) is a Cauchy point of \( X \), we have an \( x \in X \) with \( \xi_d(x) < \frac{\varepsilon}{2} \). Thus,
\[
\xi_d(x) = \Xi_{\hat{d}}(\hat{x}) \leq \hat{d}(\hat{x}, \xi) + \Xi_{\hat{d}}(\xi) = \xi_d(x) + \Xi_{\hat{d}}(\xi) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Since this is true for all \( \varepsilon > 0 \), we have \( \inf_{x} \xi_d(x) = 0 \).

(iv) If \( d \geq d' \), then \( \hat{d} \geq \hat{d}' \), so
\[
\xi_d(x) = \Xi_{\hat{d}}(\hat{x}) \geq \Xi_{\hat{d}'}(\hat{x}) = \xi_{d'}(x)
\]

Now I claim that \( \Xi = \hat{\xi} \). We know it’s enough to show \( \Xi_{\hat{d}}(\xi) = 0 \) for all \( d \). To show this, fix an \( \varepsilon > 0 \) and find an \( x \) with \( \xi_d(x) < \varepsilon \). Then we have
\[
\Xi_{\hat{d}}(\xi) \leq \Xi_{\hat{d}}(\hat{x}) + \hat{d}(\hat{x}, \xi) = \xi_d(x) + \xi_d(x) < 2\varepsilon.
\]
Since this is true for all \( \varepsilon > 0 \), we must have \( \Xi_{\hat{d}}(\xi) = 0 \), hence \( \Xi = \hat{\xi} \).

We can now give a definition of compactness.

**Definition 11.6.** A gauge space is **compact** if it is totally bounded and complete.

If you’ve seen compactness before, you may have seen a different definition. Ours is a bit of a cheat, but it is correct.

Finally, remember that we want to construct compactifications by finding an equivalent totally bounded gauge, then completing it. For this to work, we need the following.

**Proposition 11.7.** The completion of a totally bounded space is totally bounded.

*Proof. Exercise. □*

12. **EXERCISES ON COMPLETENESS**

**Exercise 12.1.** Prove that the completion of a totally bounded gauge space is totally bounded (hence compact).

**Exercise 12.2.** Prove that \( \hat{X} \) is always separated.

**Exercise 12.3.** Prove that a Cauchy point \( \xi \) is represented by \( z \) if and only if for every gauge metric \( d \) and every \( \varepsilon > 0 \), there is a point \( x \) with \( \xi_d(x) \leq \varepsilon \) and \( d(x, z) \leq \varepsilon \).

**Exercise 12.4.** Prove that \( \mathbb{R} \) is complete with the usual metric \( d(x, y) = |x - y| \), and that it is uniformly equivalent to the completion of its subspace \( \mathbb{Q} \). (Hint: you’ll need Dedekind-completeness or Cauchy-completeness of \( \mathbb{R} \).)

**Exercise 12.5.** Prove that \( \mathbb{R} \) is not complete with the metric
\[
d^*(x, y) = |\arctan x - \arctan y|
\]
(which is topologically equivalent to the usual metric).
Exercise 12.6. Prove that if \( X \) is complete, then every Cauchy sequence converges to some point.

Exercise 12.7. Suppose \( X \) is a metric space, i.e. a gauge space with only one gauge metric. Prove that if every Cauchy sequence in \( X \) converges to some point, then \( X \) is complete.

* Exercise 12.8. Find an example of a non-complete gauge space in which every Cauchy sequence converges.

* Exercise 12.9. Suppose \( X \) is a totally bounded gauge space and \( \xi, \zeta \) are two Cauchy points of \( X \). Prove that if \( \xi \approx A \iff \zeta \approx A \) for all \( A \subseteq X \), then \( \xi = \zeta \). (Of course, \( \xi \approx A \) means that \( \inf_{a \in A} \xi_d(a) = 0 \) for all \( X \)-metrics \( d \).)

* Exercise 12.10. Prove that if \( f : X \to Y \) is uniformly continuous, then it extends uniquely to a uniformly continuous function \( \hat{f} : \hat{X} \to \hat{Y} \).

* Exercise 12.11. Prove that completeness is a uniform property: if \( X \) and \( Y \) are uniformly isomorphic and \( X \) is complete, then so is \( Y \).

13. Exercises on compactness

Exercise 13.1. Prove that a metric space is compact if and only if every sequence has a convergent subsequence.

Exercise 13.2. Let \( X \) be \([0,1]^\omega\); that is, the set of infinite sequences of real numbers in \([0,1]\). Equip it with the gauge defined by the metrics \( d_N(x,y) = \max_{1 \leq i \leq N} |x_i - y_i| \) for \( N = 0, 1, 2, \ldots \) (this is the restriction of the gauge on \( \mathbb{R}^\omega \) considered in class). Prove that \( X \) is compact.

Exercise 13.3. Let \( K, L \) be compact subsets of \( X \) (with respect to their induced gauges). Prove that \( K \cup L \) is compact. Conclude by induction that the union of any finite number of compact subsets of \( X \) is compact.

Exercise 13.4. Prove that a gauge space \( X \) is compact if and only if its separated quotient from Exercise 7.4 is compact.

* Exercise 13.5 (Tychonoff’s Theorem). Let \( \{X_\alpha\}_{\alpha \in A} \) be a family of gauge spaces and \( X = \prod_{\alpha \in A} X_\alpha \) their cartesian product. By definition, that means the elements of \( X \) are families \( (x_\alpha)_{\alpha \in A} \) where \( x_\alpha \in X_\alpha \) for all \( \alpha \). Each metric \( d \) on some \( X_\alpha \) defines a metric on \( X \) via \( d_\alpha(x,y) = d(x_\alpha, y_\alpha) \), and the collection of all these metrics generates a gauge on \( X \) (by taking finite maxes). For each of completeness, total boundedness, and compactness, prove that if each \( X_\alpha \) has the property in question, so does \( X \).

14. The one-point compactification

Recall our procedure for constructing compactifications: first find a topologically equivalent, totally bounded gauge, then complete it. In the next section we’ll construct the Stone-Cech compactification. But first, as a warm-up, let’s try adding exactly one point at infinity, as in stereographic projection.
It turns out that we need some condition on $X$ for this to work. Define a gauge space to be **locally compact** if for every point $x$ and open ball $B$ containing $x$, there exists a compact closed ball $\overline{B}$ with $x \in \overline{B} \subseteq B$. Here a closed ball is a set of the form
\[ \overline{B} = \overline{B}_d(x, \varepsilon) = \{ y \mid d(x, y) \leq \varepsilon \}. \]

Let $X$ be a locally compact, but noncompact, gauge space. For every compact $K \subseteq X$ and gauge metric $d$, denote by $\neg K$ the complement of $K$ in $X$, and define $d_K$ by
\[ d_K(x, y) = \min\left( d(x, y), d(x, \neg K) + d(\neg K, y) \right). \]
By Exercise 7.5, $d_K$ is a metric. Clearly if $d \geq d'$ and $K \supseteq K'$, then $d_K \geq d_K'$. Thus, using Exercise 13.3, we conclude that the collection of all the $d_K$, as $d$ and $K$ vary, is a new gauge on $X$.

**Theorem 14.1.** When $X$ is locally compact, this new gauge is totally bounded and topologically equivalent to the original gauge.

**Proof.** Given any $d_K$ and $\varepsilon > 0$, choose a finite $(d, \varepsilon)$-sized cover of $K$ (which exists since $K$ is totally bounded). Then together with $\neg K$, which is certainly $(d_K, \varepsilon)$-sized (all points in it are $d_K$-distance 0 from each other), we have a $(d_K, \varepsilon)$-sized cover of $X$ in the new gauge. Thus, the new gauge is totally bounded.

For topological equivalence, since $d_K(x, y) \leq d(x, y)$, every ball in the new gauge trivially contains a ball in the old gauge. Conversely, suppose given $x$ and an open ball $x \in B$ in the old gauge. Choose $x \in \overline{B}_d(x, \varepsilon) \subseteq B$ as in the definition of local compactness, and consider the ball $B_{d_K}(x, \varepsilon)$ in the new gauge, where $K = \overline{B}_d(x, \varepsilon)$. Since $d(x, \neg K) \geq \varepsilon$, we have
\[ B_{d_K}(x, \varepsilon) = B_d(x, \varepsilon) \subseteq B. \]
This proves topological equivalence.

**Theorem 14.2.** There is exactly one non-represented Cauchy point in this new gauge space.

**Proof.** For existence, set $\xi_{d_K}(x) = d(x, \neg K)$. It is easy to check that this defines a non-represented Cauchy point. For uniqueness, suppose that $\xi$ is a Cauchy point. Pick $K \subseteq X$ compact. If $\inf_{k \in K} \xi_{d_L}(k) = 0$ for all $d_L$, then $\xi$ would induce a Cauchy point of $K$, and would thus be represented since $K$ is compact. Therefore, there must be some $X$-metric $d$, compact $L \subseteq X$, and $\delta > 0$ such that $\xi_{d_L}(k) > \delta$ for all $k \in K$.

Now since $\xi$ is a Cauchy point, for any $\varepsilon > 0$, there is some $z \in X$ such that both $\xi_{d_L}(z) < \varepsilon$ and $\xi_{d_K}(z) < \varepsilon$. But by the above, as long as $\varepsilon < \delta$, this $z$ cannot be in $K$, so that $d(z, \neg K) = 0$. Thus, for any $x \in X$ we have
\[ d_K(x, z) \leq d(x, \neg K). \]
We therefore have
\[ \xi_{d_K}(x) \leq d_K(x, z) + \xi_{d_K}(z) \leq d(x, \neg K) + \varepsilon \]
As this is true for all $\varepsilon > 0$, we have $\xi_{d_K} \leq d(\neg, \neg K)$. In particular, we have $\xi_{d_K}(y) \leq d(y, \neg K) = 0$ for any $y \notin K$. Thus, fixing such a $y$, for any $x \in X$ we have
\[ d(x, \neg K) = d_K(x, y) \leq \xi_{d_K}(x) + \xi_{d_K}(y) = \xi_{d_K}(x). \]
and thus $\xi_{d_K} = d(\neg, \neg K)$. \qed
We call the completion of this new gauge the **one-point compactification** of $X$, since it contains only one point in addition to $X$. We write it as $X_\infty$ or $\alpha X$. For example:

- $S^n$ is topologically isomorphic to the one-point compactification of $\mathbb{R}^n$.
- The one-point compactification of the infinite cylinder is the pinched torus.

## 15. The Stone-Čech Compactification

Finally, we’re ready to consider the question: what is the best compactification? Since we compactify by finding a totally bounded gauge and then completing, this question essentially boils down to finding the best totally bounded gauge which is topologically equivalent to the one we started with. We proceed as follows.

Let $n \in \mathbb{N}$, let $\varphi: X \to [0, 1]^n$ be any continuous function, and define a metric on $X$ by

$$d_\varphi(x, y) = d_n(\varphi(x), \varphi(y)),$$

where $d_n(a, b) = \max_{1 \leq i \leq n} |a_i - b_i|$ is one of the canonical metrics on $[0, 1]^n$. The function $d_\varphi$ is clearly a metric on $X$, and the collection of all such metrics defines a gauge on $X$. (For filteredness, given $\varphi_1: X \to [0, 1]^n$ and $\varphi_2: X \to [0, 1]^m$ let $\varphi_3 = (\varphi_1, \varphi_2): X \to [0, 1]^{n+m}$.)

**Theorem 15.1.** This gauge is totally bounded and topologically equivalent to the original one.

**Proof.** Suppose first we are given $x \in X$, $\varphi: X \to [0, 1]^n$ and $\varepsilon > 0$; we want to show that $B_{d_\varphi}(x, \varepsilon)$ contains a ball in the original gauge. But

$$B_{d_\varphi}(x, \varepsilon) = \{y \mid d_n(\varphi(x), \varphi(y)) < \varepsilon\} = \varphi^{-1}(B(\varphi(x), \varepsilon))$$

so this follows since $\varphi$ is continuous.

Now suppose we are given $x \in X$, $d$, and $\varepsilon > 0$, and we want to show that $B_d(x, \varepsilon)$ contains a ball in the new gauge. But by **Exercise 7.2**, the function $\psi(y) = \max(d(x, y), 1)$ is a continuous map $X \to [0, 1]$, and it is easy to check that if $d_\psi(x, y) < \varepsilon$, then $d(x, y) < \varepsilon$.

Finally, to show the new gauge is totally bounded, let $\varepsilon > 0$ and $\varphi: X \to [0, 1]^n$ be given. Cover $[0, 1]^n$ by finitely many $\varepsilon$-sized sets $A_j$; then the sets $\varphi^{-1}(A_j)$ are a finite $(d_\varphi, \varepsilon)$-sized cover of $X$. \(\square\)

**Corollary 15.2.** Every gauge space is topologically equivalent to a totally bounded one.

Call this new gauge $\mathcal{G}_\beta$. The completion of $(X, \mathcal{G}_\beta)$ is the **Stone-Čech compactification** of $X$; we write it as $\beta X$. Since $\mathcal{G}_\beta$ is totally bounded, $\beta X$ is compact. Note that the inclusion $X \to \beta X$ is uniformly continuous when $X$ has the new gauge $\mathcal{G}_\beta$, but only continuous when $X$ has the original gauge.

The points of $\beta X$ are, of course, the Cauchy points of $(X, \mathcal{G}_\beta)$. Such a Cauchy point consists of, for each continuous $\varphi: X \to [0, 1]^n$, a function $\xi_\varphi: X \to [0, \infty)$, such that

- $\xi_\varphi(x) + d_n(\varphi(x), \varphi(y)) \geq \xi_\varphi(y)$,
- $\xi_\varphi(x) + \xi_\varphi(y) \geq d_n(\varphi(x), \varphi(y))$,
- $\inf_x \xi_\varphi(x) = 0$, and
- If $d_\varphi \leq d_\psi$, then $\xi_\varphi \leq \xi_\psi$.  

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We can reformulate this in a convenient way as follows. For each continuous \( \varphi : X \to [0, 1] \) and \( \varepsilon > 0 \), define

\[
A_{\varphi, \varepsilon} = \inf_{\xi \in \varphi(x) \leq \varepsilon} \varphi(x)
\]

\[
B_{\varphi, \varepsilon} = \sup_{\xi \in \varphi(x) \leq \varepsilon} \varphi(x).
\]

Clearly \( A_{\varphi, \varepsilon} \leq B_{\varphi, \varepsilon} \), and moreover by the triangle inequality, \( B_{\varphi, \varepsilon} - A_{\varphi, \varepsilon} \leq 2\varepsilon \). Therefore, by a standard property of real numbers, the intersection \( \bigcap_{\varepsilon} [A_{\varphi, \varepsilon}, B_{\varphi, \varepsilon}] \) consists of a single point. Call that point \( \varphi(\xi) \). More generally, if \( \varphi = (\varphi_1, \ldots, \varphi_n) : X \to [0, 1]^n \), then we write \( \varphi(\xi) = (\varphi_1(\xi), \ldots, \varphi_n(\xi)) \).

Of course, a point \( x \in X \) induces a Cauchy point of \((X, G_\beta)\), which we write as \( \bar{x} \) to avoid confusion with \( \hat{x} \), the induced Cauchy point of \( X \) in its original gauge. It is easy to see that we have \( \varphi(\bar{x}) = \varphi(x) \) for any such \( x \).

We have shown that any Cauchy point of \((X, G_\beta)\) gives a way to ‘evaluate’ functions \( \varphi \). We now want to show that the Cauchy point \( \xi \) is determined by the operation \( \varphi \mapsto \varphi(\xi) \). Specifically, we want to show that \( \xi_{\varphi}(y) = d_{\varepsilon}(\varphi(y), \varphi(\xi)) \) for any \( y \). To show this, let \( \varepsilon > 0 \) and choose some \( x \) with \( \xi_{\varphi}(x) \leq \varepsilon \) (possible by locatedness of \( \xi \)). Then the reverse triangle inequality gives

\[
\left| \xi_{\varphi}(y) - d_{\varepsilon}(\varphi(x), \varphi(y)) \right| \leq \varepsilon.
\]

Now, by definition of \( x \) and the filteredness of a Cauchy point, we have \( \xi_{\varphi}(x) \leq \varepsilon \) for all coordinates \( 1 \leq i \leq n \), and therefore \( \varphi_i(x) \in [A_{\varphi, \varepsilon}, B_{\varphi, \varepsilon}] \). Hence, by definition of \( \varphi_i(\xi) \), we have \( |\varphi_i(x) - \varphi_i(\xi)| \leq 2\varepsilon \), and thus \( d_{\varepsilon}(\varphi(x), \varphi(\xi)) \leq 2\varepsilon \). It follows that

\[
\left| \xi_{\varphi}(y) - d_{\varepsilon}(\varphi(\xi), \varphi(y)) \right| \leq 3\varepsilon.
\]

But since this holds for all \( \varepsilon > 0 \), and the left-hand side is independent of \( \varepsilon \), it must be zero; thus \( \xi_{\varphi}(y) = d_n(\varphi(\xi), \varphi(y)) \) as desired.

Now suppose we have an “evaluation” operation taking any continuous function \( \varphi : X \to [0, 1] \) to a number \( \varphi(\xi) \in [0, 1] \). As before, if \( \varphi = (\varphi_1, \ldots, \varphi_n) : X \to [0, 1]^n \), then we write \( \varphi(\xi) = (\varphi_1(\xi), \ldots, \varphi_n(\xi)) \). Then if we define \( \xi_{\varphi}(y) = d_n(\varphi(y), \varphi(\xi)) \), we get a collection of functions satisfying the first two conditions to be a Cauchy point, and the definition of \( \varphi(\xi) \) when \( n > 1 \) immediately implies the fourth condition. Thus, to obtain a Cauchy point in this way we only need to ensure locatedness. We summarize this as follows:

**Lemma 15.3.** To give a Cauchy point of \((X, G_\beta)\) is equivalent to giving, for every continuous \( \varphi : X \to [0, 1] \), a number \( \varphi(\xi) \in [0, 1] \), such that

For any finite family \((\varphi_i)_{1 \leq i \leq n}\) and any \( \varepsilon > 0 \), there exists a point \( x \in X \) such that \( |\varphi_i(x) - \varphi_i(\xi)| < \varepsilon \) for all \( i \).

In other words, the points of \( \beta X \) are the operations \( \varphi \mapsto \varphi(\xi) \) which are arbitrarily closely approximable by (evaluation at) points of \( X \). This gives another way of viewing them as “virtual points” of the space \( X \).

We note in passing that Lemma 15.3 can be reformulated in the following concise way. Let \( C(X, [0, 1]) \) denote the set of continuous functions \( X \to [0, 1] \), and consider the space \([0, 1]^{C(X, [0, 1])}\) with the pointwise gauge as in Proposition 6.6. There is a continuous function
\( e : X \to [0, 1]^{C(X,[0,1])} \) defined by \( e(x)_{\varphi} = \varphi(x) \), and the points of \( \beta X \) can be identified with the points \( \xi \in [0, 1]^{C(X,[0,1])} \) such that \( e(X) \approx \xi \).

The following lemma will also be useful.

**Lemma 15.4.** Let \( \xi \) be a Cauchy point of \((X, \mathcal{G}_\beta)\).

(i) If \( \varphi, \psi : X \to [0, 1] \) are continuous with \( \varphi \leq \psi \), then \( \varphi(\xi) \leq \psi(\xi) \).

(ii) If \( \psi : [0, 1]^n \to [0, 1] \) and \( \varphi : X \to [0, 1]^n \) are continuous, then we have \( \psi(\varphi(\xi)) = (\psi\varphi)(\xi) \).

**Proof.** For the first statement, observe that for any \( \varepsilon > 0 \) there is an \( x \) with \( \xi(\varphi_\psi(x)) < \varepsilon \), and hence both \( \xi(\varphi(x)) < \varepsilon \) and \( \xi(\psi(x)) < \varepsilon \). Therefore \( \varphi(x) \in [A_{\varphi,\varepsilon}, B_{\varphi,\varepsilon}] \) and \( \psi(x) \in [A_{\psi,\varepsilon}, B_{\psi,\varepsilon}] \). Since \( \varphi(x) \leq \psi(x) \), we have \( A_{\varphi,\varepsilon} \leq B_{\psi,\varepsilon} \), from which \( \varphi(\xi) \leq \psi(\xi) \) follows (taking \( \varepsilon \to 0 \)).

Now consider the second statement. Fix an \( \varepsilon > 0 \) until further notice. By definition of \( (\psi \varphi)(\xi) \), there is a \( \delta > 0 \) such that if \( \xi(\varphi(x)) < \delta \), then \( |\psi(\varphi(x)) - \varphi(\xi)| < \varepsilon \). And since \( \psi \) is continuous, there is a \( \delta' > 0 \) such that if \( d(\varphi(x), \varphi(\xi)) < \delta' \), then \( |\psi(\varphi(x)) - \psi(\varphi(\xi))| < \varepsilon \). But by definition of \( \varphi(\xi) \), there is then a \( \delta'' > 0 \) such that if \( \varphi(x) < \delta'' \), then \( d(\varphi(x), \varphi(\xi)) < \delta' \), and hence also \( |\psi(\varphi(x)) - \varphi(\xi)| < \varepsilon \).

Now, since \( \xi \) is a Cauchy point, there is an \( x \) such that \( \xi(\varphi_\psi(x)) < \min(\delta, \delta'') \). It follows that we have both \( |\psi(\varphi(x)) - \varphi(\xi)| < \varepsilon \) and \( |\psi(\varphi(x)) - \psi(\varphi(\xi))| < \varepsilon \), and hence, by the triangle inequality, \( |\psi(\varphi(\xi)) - \psi(\varphi(\xi))| < 2\varepsilon \). Since this is true for all \( \varepsilon > 0 \), we must have \( (\psi \varphi)(\xi) = \psi(\varphi(\xi)) \), as desired. \( \square \)

Finally, the following theorem is one way to express the idea that the Stone-Čech compactification is the best compactification.

**Theorem 15.5** (Extension Theorem). Let \( g : X \to Y \) be any continuous map, where \( Y \) is compact. Then there exists a continuous map \( \overline{g} : \beta X \to Y \) such that for all \( x \in X \), we have \( \overline{g}(x) \approx g(x) \). Moreover, if \( h : \beta X \to Y \) is any other continuous map with this property, then for all \( \xi \in \beta X \) we have \( \overline{g}(\xi) \approx h(\xi) \).

Let us consider how to prove this theorem. Given \( \xi \in \beta X \), represented as above by the operation \( \varphi \mapsto \varphi(\xi) \), we would like to define a Cauchy point \( \overline{g}(\xi) \) of \( Y \) by

\[
(\overline{g}(\xi))_d(y) = \psi_{d,g,y}(\xi) \quad \text{where} \quad \psi_{d,g,y}(x) = d(g(x), y).
\]

Note that \( \psi_{d,g,y} \) is continuous since \( g \) and \( d \) are. Unfortunately, it will not generally take values in \([0,1]\). We can remedy this by using instead

\[
(\overline{g}(\xi))'_d(y) = \psi'_{d,g,y}(\xi) \quad \text{where} \quad \psi'_{d,g,y}(x) = \min(d(g(x), y), 1).
\]

but now we cannot expect \( \overline{g}(\xi)' \) to be a Cauchy point; at large distances it will fail the triangle inequality. The solution to this conundrum is the following lemma.

**Lemma 15.6.** Suppose \( X \) is a gauge space and we have a family of subsets \( A_d \subseteq X \), one for each \( X \)-metric \( d \), and also functions \( \xi_d : A_d \to [0, \infty) \) such that

(i) \( d(x, y) + \xi_d(y) \geq \xi_d(x) \) for each \( d \) and each \( x, y \in A_d \).

(ii) \( \xi_d(x) + \xi_d(y) \geq d(x, y) \) for each \( d \) and each \( x, y \in A_d \).

(iii) \( \inf_{\xi \in A_{d_1}} \xi_d(x) = 0 \) for each \( d \).

(iv) For any two gauge metrics \( d_1 \) and \( d_2 \), there exists a gauge metric \( d_3 \) such that \( A_{d_3} \subseteq A_{d_1} \cap A_{d_2} \) and \( d_3 \geq \max(d_1, d_2) \) and \( \xi_{d_3} \geq \max(\xi_{d_1}, \xi_{d_2}) \).
Then there exists a unique Cauchy point \( \zeta \) of \( X \) such that \( \zeta_d(x) = \xi_d(x) \) for all gauge metrics \( d \) and all \( x \in A_d \).

**Proof.** I claim that if \( \zeta \) is as asserted, then we must have

\[
(15.7) \quad \zeta_d(x) = \inf_{a \in A_d} (d(x, a) + \xi_d(a))
\]

for all \( d \). It is easy to check that this definition gives a Cauchy point, so it suffices to show that any such \( \zeta \) must be equal to this one. The triangle inequality immediately gives

\[
\zeta_d(x) \leq d(x, a) + \zeta_d(a) = d(x, a) + \xi_d(a)
\]

so it remains to prove the direction \( \geq \) of (15.7). For this, let \( \varepsilon > 0 \) and let \( a_0 \in A_d \) be such that \( \xi_d(a_0) < \varepsilon \). Then

\[
\inf_{a \in A_d} (d(x, a) + \xi_d(a)) \leq d(x, a_0) + \xi_d(a_0)
\]

\[
\leq \zeta_d(x) + \xi_d(a_0) + \xi_d(a_0)
\]

\[
= \zeta_d(x) + \xi_d(a_0) + \xi_d(a_0)
\]

\[
\leq \zeta_d(x) + 2\varepsilon.
\]

Taking \( \varepsilon \to 0 \) we obtain the desired inequality. \( \square \)

**Proof of Theorem 15.5.** Given \( \xi \in \beta X \), for any \( Y \)-metric \( d \) define \( A_d = \{ y \in Y \mid \psi_{d,g,y}(\xi) < 1 \} \). where \( \psi_{d,g,y}(x) = d(g(x), y) \). Now as suggested above, for \( y \in A_d \) define

\[
(\overline{\gamma}(\xi))_d(y) = \psi'_{d,g,y}(\xi) \quad \text{where} \quad \psi'_{d,g,y}(x) = \min(d(g(x), y_1), 1).
\]

We will show that \( \overline{\gamma}(\xi) \) with these sets satisfies the hypotheses of Lemma 15.6.

Let \( y_1, y_2 \in A_d \), and let \( \varepsilon > 0 \) be smaller than \( 1 - \psi_{d,g,y_i}(\xi) \) for \( i = 1, 2 \). Now choose an \( x \) with \( |\psi_{d,g,y_i}(x) - \psi_{d,g,y_i}(\xi)| < \frac{\varepsilon}{2} \) for \( i = 1, 2 \); then we also have \( \psi_{d,g,y_i}(x) < 1 \) and hence \( \psi'_{d,g,y_i}(x) = \psi_{d,g,y_i}(x) = d(g(x), y_i) \). Thus we can compute

\[
(\overline{\gamma}(\xi))_d(y_1) + d(y_1, y_2) = \psi'_{d,g,y_1}(\xi) + d(y_1, y_2)
\]

\[
= \psi_{d,g,y_1}(\xi) + d(y_1, y_2) \geq \psi_{d,g,y_1}(x) - \varepsilon + d(y_1, y_2) = d(g(x), y_1) + d(y_1, y_2) - \varepsilon
\]

\[
= d(g(x), y_2) - \varepsilon = \psi_{d,g,y_2}(x) - \varepsilon \geq \psi_{d,g,y_2}(\xi) - \varepsilon
\]

\[
= \psi'_{d,g,y_2}(\xi) - \varepsilon = (\overline{\gamma}(\xi))_d(y_2) - \varepsilon.
\]

Since this is true for all \( \varepsilon > 0 \), we obtain

\[
(\overline{\gamma}(\xi))_d(y_1) + d(y_1, y_2) \geq (\overline{\gamma}(\xi))_d(y_2).
\]

Similarly, with the same \( \varepsilon \) and \( x \), we have

\[
(\overline{\gamma}(\xi))_d(y_1) + (\overline{\gamma}(\xi))_d(y_2) = \psi'_{d,g,y_1}(\xi) + \psi'_{d,g,y_2}(\xi) = \psi_{d,g,y_1}(\xi) + \psi_{d,g,y_2}(\xi)
\]

\[
\geq \psi_{d,g,y_1}(x) + \psi_{d,g,y_2}(x) - \varepsilon = d(g(x), y_1) + d(g(x), y_2) - \varepsilon \geq d(y_1, y_2) - \varepsilon.
\]

Again, taking \( \varepsilon \to 0 \) we obtain

\[
(\overline{\gamma}(\xi))_d(y_1) + (\overline{\gamma}(\xi))_d(y_2) \geq d(y_1, y_2).
\]
Thus \( \overline{g}(\xi) \) satisfies the first two conditions of Lemma 15.6. For the third, we want to show that given any \( Y \)-metric \( d \) and any \( \varepsilon > 0 \), there exists a \( y \in A_d \) with \( (\overline{g}(\xi'))_d(y) < \varepsilon \). We may assume \( \varepsilon < 1 \). Now since \( Y \) is totally bounded, we can cover it by finitely many balls \( B_d(z_j, \frac{\varepsilon}{2}) \). Since \( \xi \) is a Cauchy point of \((X, G)\), we can choose an \( x \in X \) such that for all \( j \) we have \( |\psi'_{d,g,z_j}(x) - \psi'_{d,g,z_j}(\xi)| < \frac{\varepsilon}{2} \). Now since our balls covered \( Y \), we must have \( g(x) \in B_d(z_j, \frac{\varepsilon}{2}) \) for some \( j \); let \( y \) be that \( z_j \). Then since \( \psi_{d,g,y}(x) = d(g(x), y) < \frac{\varepsilon}{2} \), we have \( \psi'_{d,g,y}(x) = \psi_{d,g,y}(x) = d(g(x), y) \). Thus:

\[
(\overline{g}(\xi'))_d(y) = \psi'_{d,g,y}(\xi) \leq \psi_{d,g,y}(\xi) + \frac{\varepsilon}{2} = d(g(x), y) + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

as desired.

Finally, suppose that \( d \leq d' \) are \( Y \)-metrics. Then clearly \( \psi'_{d,g,y} \leq \psi'_{d',g,y} \), so by Lemma 15.4 we have \( A_d \subseteq A_d \) and \( \overline{g}(\xi')_d \leq \overline{g}(\xi')_{d'} \).

Note that \( \overline{g}(\xi') \) is a \((Y, G)\)-metric on \( Y \). Let \( \hat{x} \) be a \((Y, G)\)-Cauchy point of \( Y \), which we may denote \( \overline{g}(\xi) \), with the property that \( \overline{g}(\xi)_d(y) = \psi_{d,g,y}(\xi) \) whenever the latter is \( < 1 \). Since \( Y \) is complete, this \( \overline{g}(\xi) \) is a \((Y, G)\)-Cauchy point which must be represented by some actual point; choose such a point and denote it also by \( \overline{g}(\xi) \). Thus for any \( y \in Y \) and \( Y \)-metric \( d \) we have

\[
d(\overline{g}(\xi), y) = \psi_{d,g,y}(\xi)
\]

if the latter is \( < 1 \). In particular, for any \( x \in X \), since \( \psi_{d,g,g(x)}(\hat{x}) = d(g(x), g(x)) = 0 < 1 \), we have

\[
d(\overline{g}(\hat{x}), g(x)) = \psi_{d,g,y}(\hat{x}) = 0
\]

for any \( d \), and therefore \( \overline{g}(\hat{x}) \approx g(x) \) as desired.

We now show that \( \overline{g} \) is continuous. Let \( \xi \in \hat{X} \), let \( d \) be a \( Y \)-metric, and let \( \varepsilon > 0 \). Choose a \( y \in Y \) which satisfies \( y \in A_d \) (i.e. \( \psi_{d,g,y}(\xi) < 1 \)) and also \( (\overline{g}(\xi'))_d(y) < \varepsilon \). Such a \( y \) exists by the third condition of Lemma 15.6, which we proved above for \( \overline{g}(\xi') \). Then since \( y \in A_d \), we have \( d(\overline{g}(\xi), y) = \psi_{d,g,y}(\xi) < \varepsilon \).

In particular, for any other \( \zeta \in \hat{X} \), by Lemma 15.4(i) and the triangle inequality for \( d \),

\[
\psi_{d,g,\overline{g}(\xi)}(\zeta) \leq d(\overline{g}(\xi), y) + \psi_{d,g,y}(\xi)
\]

\[
< d(\overline{g}(\xi), y) - y(\overline{g}(\xi)) + 2\varepsilon
\]

\[
\leq d(\overline{g}(\xi), \overline{g}(\xi)) + 2\varepsilon.
\]

Since this holds for any \( \varepsilon > 0 \), we have \( \psi_{d,g,\overline{g}(\xi)}(\zeta) \leq d(\overline{g}(\xi), \overline{g}(\xi)) \). Since we similarly have

\[
\psi_{d,g,\overline{g}(\xi)}(\zeta) \geq d(\overline{g}(\xi), y) - \psi_{d,g,y}(\xi)
\]

\[
> d(\overline{g}(\xi), y) + y(\overline{g}(\xi)) - 2\varepsilon
\]

\[
\geq d(\overline{g}(\xi), \overline{g}(\xi)) - 2\varepsilon.
\]

for all \( \varepsilon > 0 \), we must in fact have \( \psi_{d,g,\overline{g}(\xi)}(\zeta) = d(\overline{g}(\xi), \overline{g}(\xi)) \).

Let \( d_X \) be the metric on \( X \) in \( G \) induced by the function \( \psi_{d,g,\overline{g}(\xi)} \), and let \( \delta = \varepsilon \). Then if \( d_X(\xi, \zeta) < \delta \), we have

\[
|\psi_{d,g,\overline{g}(\xi)}(\zeta) - \psi_{d,g,\overline{g}(\xi)}(\xi)| = |d(\overline{g}(\xi), \overline{g}(\xi)) - d(\overline{g}(\xi), \overline{g}(\xi))| = d(\overline{g}(\xi), \overline{g}(\xi)) < \varepsilon,
\]

which is exactly what we need for continuity of \( \overline{g} \).

Finally, if \( h : \hat{X} \to Y \) is continuous and satisfies \( h(\hat{x}) \approx g(x) \) for all \( x \), then we have \( h(\hat{x}) \approx \overline{g}(\hat{x}) \) for all \( x \). Fix \( \xi \in \hat{X} \), a \( Y \)-metric \( d_Y \), and \( \varepsilon > 0 \); we want to show \( d_Y(h(\xi), \overline{g}(\xi)) < \varepsilon \).
Since $h$ is continuous, there exists an $\hat{X}$-metric $d_1$ and $\delta_1 > 0$ such that (in particular) if $d_1(\xi, \hat{x}) < \delta_1$, then $d_Y(h(\xi), h(\hat{x})) < \frac{\delta}{2}$. Similarly, since $\overline{g}$ is continuous, there exists an $\hat{X}$-metric $d_2$ and $\delta_2 > 0$ such that if $d_2(\xi, \hat{x}) < \delta_2$, then $d_Y(\overline{g}(\xi), \overline{g}(\hat{x})) < \frac{\delta}{2}$.

Pick $d_3 \geq \max(d_1, d_2)$ and $\delta_3 = \min(\delta_1, \delta_2)$. By locatedness of $\xi$, there exists $x \in X$ such that $d_3(\xi, \hat{x}) < \delta_3$, hence $d_Y(h(\xi), h(\hat{x})) < \frac{\delta}{2}$ and $d_Y(\overline{g}(\xi), \overline{g}(\hat{x})) < \frac{\delta}{2}$. Since $d_Y(\overline{g}(\hat{x}), h(\hat{x})) = 0$, the triangle inequality implies $d_Y(h(\xi), \overline{g}(\xi)) < \varepsilon$ as desired. \hfill $\square$

If $Y$ is itself a (separated) compactification of $X$, this says that we have a unique map from $\beta X$ to $Y$ fixing the image of $X$. That is, every point at infinity in $\beta X$ becomes a point in $Y$, but one point in $Y$ may come from multiple points in $\beta X$. This is the sense in which $\beta X$ is the best compactification of $X$. (The technical term is initial. One also says that $\beta$ is the left adjoint to the inclusion of compact separated gauge spaces in all gauge spaces.)

16. Exercises on the Stone-Čech compactification

Exercise 16.1. Prove that if $X$ is already compact and separated, then $\beta X$ is topologically isomorphic to $X$.

Exercise 16.2. For any map $g : X \to Y$, the composite $X \xrightarrow{\hat{g}} Y \to \beta Y$ induces a map $\beta g : \beta X \to \beta Y$. Prove that the composites $X \to \beta X \to \beta Y$ and $X \to Y \to \beta Y$ are equal. (This is called naturality.)

* Exercise 16.3. Use *Exercise 12.10 to give an alternative proof of Theorem 15.5.

Exercise 16.4. Let $X$ be a gauge space, and define a new gauge $G_e$ similarly to $G_\beta$, but using continuous functions $X \to \mathbb{R}^n$ instead of $X \to [0,1]^n$. Prove that:

(i) $G_e$ is topologically equivalent to the original gauge.

(ii) If $G_e$ is totally bounded, then $X$ is pseudocompact (see *Exercise 10.14).

Exercise 16.5. Suppose that $X$ has a discrete metric. Prove that the points of $\beta X$ can be identified with ultrafilters on $X$; that is, subsets $F \subseteq \mathcal{P}X$ such that

- $X \in F$ and $\emptyset \notin F$,
- If $A, B \in F$ then $A \cap B \in F$,
- If $A \in F$ and $A \subseteq B$, then $B \in F$, and
- For all $A \subseteq X$, either $A \in F$ or $\neg A \in F$.

* Exercise 16.6. Prove that no point of $\beta \mathbb{N} \setminus \mathbb{N}$ (where $\mathbb{N}$ has the discrete metric) is the limit of a sequence of points of $\mathbb{N}$. Conclude that $\beta \mathbb{N}$ is not metrizable (that is, not topologically equivalent to a space with a single metric).

* Exercise 16.7. Let $X$ be a gauge space, $\xi$ a point of $\beta X \setminus X$, and let $Y = \beta X \setminus \{\xi\}$. Prove that $\beta Y$ is topologically isomorphic to $\beta X$.

* Exercise 16.8. Let $\Omega$ be an uncountable well-ordered set (that is, it has a total order $<$ and any subset of it has a least element) such that for any $a \in \Omega$, the set $\{b \in \Omega \mid b < a\}$ is countable. For any $a \in \Omega$ define

$$d_a(x, y) = \begin{cases} 0 & \text{if } x = y \text{ or } (x > a \text{ and } y > a) \\ 1 & \text{otherwise.} \end{cases}$$

(i) Verify that the set of all these metrics, $\{d_a \mid a \in \Omega\}$, is a gauge on $\Omega$. 

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(ii) Prove that any continuous function \( f : \Omega \rightarrow [0, 1] \) is eventually constant, i.e. there is an \( a \in \Omega \) such that \( f(x) = f(a) \) whenever \( x \geq a \).

(iii) Prove that \( \beta \Omega \) contains exactly one point not in \( \Omega \).

*Exercise 16.9.* Suppose \( X \) is equipped with a topology as in *Exercise 5.8.* Classically, \( X \) is said to be **completely regular** if for any point \( x \in X \) and open subset \( U \subseteq X \) such that \( x \in U \), there exists a continuous function \( \varphi : X \rightarrow [0, 1] \) (in the sense of *Exercise 7.6*) such that \( \varphi(x) = 1 \) and \( \varphi(X \setminus U) = \{0\} \). Prove that any such topology is induced by some gauge on \( X \).

*Exercise 16.10.* Prove that every gauge space is proximally equivalent to a totally bounded one. (This is a sort of converse to *Exercise 10.12.* )

*Exercise 16.11.* A **proximity** on a set \( X \) is a relation \( \approx \) between subsets of \( X \) such that

- \( A \not\approx \emptyset \) for all \( A \) (nontriviality).
- \( A \cap B \not\approx \emptyset \) implies \( A \approx B \) (reflexivity).
- \( A \approx B \iff B \approx A \) (symmetry).
- If \( A \not\approx B \) then there is a \( C \) with \( B \not\approx C \) and \( A \not\approx (X \setminus C) \) (transitivity).
- \( A \approx (B \cup C) \iff A \approx B \) or \( A \approx C \) (filteredness).

Prove that:

(i) The relation \( \approx \) induced by a gauge is a proximity.

(ii) Every proximity is induced by a gauge.

*Exercise 16.12.* (This exercise continues *Exercise 12.9.*) Suppose \( X \) is a totally bounded gauge space and \( \mathcal{F} \) is a nonempty collection of subsets of \( X \) such that

- \( \emptyset \not\in \mathcal{F} \) (nontriviality).
- \( (A \cup B) \in \mathcal{F} \iff A \in \mathcal{F} \) or \( B \in \mathcal{F} \) (filteredness).
- If \( A \not\in \mathcal{F} \), then there is a \( B \in \mathcal{F} \) such that \( A \not\approx B \) (transitivity).
- If \( A \in \mathcal{F} \) and \( B \in \mathcal{F} \), then \( A \approx B \) (locatedness).

Prove that there is a unique Cauchy point \( \xi \) of \( X \) such that \( \mathcal{F} = \{ A \subseteq X \mid \xi \approx A \} \). Thus, completion of totally bounded gauge spaces (hence, compactifications) can equivalently be expressed using proximities.

*Exercise 16.13.* A **uniformity** on a set \( X \) is a nonempty collection of subsets of \( X \times X \), called **entourages**, such that

- If \( U \) is an entourage and \( U \subseteq V \), then \( V \) is an entourage (saturation).
- If \( U \) and \( V \) are entourages, then so is \( U \cap V \) (filteredness).
- If \( U \) is an entourage, then \( (x, x) \in U \) for all \( x \in X \) (reflexivity).
- If \( U \) is an entourage, then so is \( U^{-1} = \{ (y, x) \mid (x, y) \in U \} \) (symmetry).
- If \( U \) is an entourage, then there exists an entourage \( V \) such that \( V \circ V = \{ (x, z) \mid (x, y) \in V \text{ and } (y, z) \in V \} \) is a subset of \( U \) (transitivity).

A function \( f : X \rightarrow Y \) between sets equipped with uniformities (called **uniform spaces**) is said to be **uniformly continuous** if whenever \( U \) is an entourage of \( Y \), then \((f \times f)^{-1}(U)\) is an entourage of \( X \). Prove that:

(i) Every gauge induces a uniformity.
(ii) A function \( f : X \to Y \) between gauge spaces is uniformly continuous in our sense if and only if it is uniformly continuous in the entourage sense.

(iii) Every uniformity is induced by a gauge.

(iv) Two gauges on a set induce the same uniformity if and only if they are uniformly equivalent.

17. Constructing points in \( \beta X \)

One thing which is still missing is a proof that every point at infinity in some other compactification \( Y \) must come from at least one point at infinity in \( \beta X \). In other words, we want to be able to get any other compactification by squashing together some points in \( \beta X \).

The reason this is hard to prove is that even with Lemma 15.3, the points of \( \beta X \) are still pretty amorphous beasts. However, there is a fairly easy way to produce them (even if the result is not all that explicit). The idea is to start with an approximation and gradually refine it.

Definition 17.1. An evaluation datum on a gauge space \( X \) is an operation \( J \) assigning to each continuous \( \varphi : X \to [0, 1] \) a nonempty closed interval \( J(\varphi) \subseteq [0, 1] \), such that

- For any finite family \( (\varphi_i)_{1 \leq i \leq n} \) and every \( \varepsilon > 0 \), there exists a point \( x \in X \) such that \( d(\varphi_i(x), J(\varphi_i)) < \varepsilon \) for all \( i \).

By Lemma 15.3, a Cauchy point of \((X, \mathcal{G}_\beta)\) is precisely an evaluation datum such that each interval \( J(\varphi) \) has length zero (i.e. is of the form \([\varphi(\xi), \varphi(\xi)]\)). The next lemma says that we can “improve” any evaluation datum to become more like a Cauchy point.

Lemma 17.2. Let \( J \) be an evaluation datum, \( \varphi : X \to [0, 1] \) continuous, and suppose \( J(\varphi) = I_1 \cup I_2 \) is the union of two closed subintervals. Define \( J_1 \) to be \( J \) except that \( J_1(\varphi) = I_1 \), and similarly \( J_2 \) to be \( J \) except that \( J_2(\varphi) = I_2 \). Then either \( J_1 \) or \( J_2 \) is an evaluation datum.

Proof. Suppose that neither is. Then we have an \( \varepsilon_1 > 0 \) and a finite family \( (\psi_i)_{1 \leq i \leq n} \) forming a counterexample for \( J_1 \), and similarly \( \varepsilon_2 > 0 \) and \( (\chi_j)_{1 \leq j \leq m} \) forming a counterexample for \( J_2 \). Since \( J \) is an evaluation datum, we must have \( \psi_i = \varphi \) for some \( i \) and \( \chi_j = \varphi \) for some \( j \). But now \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \) and the union of the \( \psi \)'s and \( \chi \)'s is a counterexample for \( J \), since a point that is within \( \varepsilon \) of \( J(\varphi) \) must either be within \( \varepsilon \) of \( I_1 \) or \( I_2 \). \( \square \)

Using this, we can produce a Cauchy point refining any evaluation datum.

Theorem 17.3. For any evaluation datum \( J \), there exists a point \( \xi \in \beta X \) such that \( \varphi(\xi) \in J(\varphi) \) for all \( \varphi \).

Proof. Consider the partially ordered set of evaluation data, where \( J_1 \leq J_2 \) is defined to mean \( J_2(\varphi) \subseteq J_1(\varphi) \) for all \( \varphi \). Lemma 17.2 implies that if \( J \) is an evaluation datum that is not a \( \beta \)-Cauchy point, then there is an evaluation datum \( J' \) with \( J < J' \). Thus, a maximal element of this poset must be a \( \beta \)-Cauchy point.

We will apply Zorn’s Lemma to show that a maximal element exists, and indeed a maximal element exists above any element; this will prove the theorem. Thus, we must show that chains of evaluation data have upper bounds. Let \( (J_a)_{a \in A} \) be a chain, i.e. a set of evaluation data such that \( \leq \) restricted to it is a total order, and define

\[
J(\varphi) = \bigcap_{a \in A} J_a(\varphi).
\]
Evidently the lower bound of \( J(\varphi) \) is the supremum of the lower bounds of the \( J_a(\varphi) \) and likewise its upper bound is the infimum of their upper bounds.

Clearly \( J \geq J_a \) for all \( a \), so it remains to show that \( J \) is an evaluation datum. Let \( (\varphi_i)_{1 \leq i \leq n} \) and \( \varepsilon > 0 \) be given. Since \( (J_a)_a \) is a chain (or more precisely, filtered), for each \( i \) there exists some \( a_i \) such that the length of \( J_{a_i}(\varphi) \) is no more than \( \frac{\varepsilon}{2} \) greater than the length of \( J_a(\varphi) \). Again, since \( (J_a)_a \) is a chain (or filtered) and there are only finitely many \( i \)'s, there exists some \( b \in A \) which is above all of these \( a_i \)'s. But since \( J_b \) is an evaluation datum, we have an \( x \in X \) such that \( d(\varphi_i(x) , J_b(\varphi)) < \frac{\varepsilon}{2} \) for all \( i \), and hence \( d(\varphi_i(x) , J(\varphi)) < \varepsilon \). Thus \( J \) is an evaluation datum, completing the proof.

\( \square \)

Now we can prove the missing theorem.

**Theorem 17.4.** Suppose \( Y \) is compact and separated, and \( g : X \to Y \) is a continuous map such that for all \( y \in Y \) we have \( g(X) \approx y \). Then the unique extension \( \overline{\varphi} : \beta X \to Y \) is surjective.

**Proof.** Given \( y \in Y \), define an evaluation datum \( J \) on \( X \) as follows. For each \( Y \)-metric \( d \) and \( \varepsilon > 0 \), let \( C_{d,\varepsilon} = \{ x \in X \mid d(g(x), y) < \varepsilon \} \). Since \( g(X) \approx y \) by assumption, \( C_{d,\varepsilon} \) is always nonempty. For \( \varphi : X \to [0,1] \) continuous, let

\[
A(\varphi) = \sup_{d,\varepsilon} \inf_{x \in C_{d,\varepsilon}} \varphi(x) \quad \text{and} \quad B(\varphi) = \inf_{d,\varepsilon} \sup_{x \in C_{d,\varepsilon}} \varphi(x).
\]

and set \( J(\varphi) = [A(\varphi), B(\varphi)] \). To show that \( J \) is an evaluation datum, let \( (\varphi_i)_{1 \leq i \leq n} \) and \( \varepsilon' > 0 \) be given. Since \( Y \)-metrics are filtered and positive real numbers admit minima, we can then choose a single \( Y \)-metric \( d \) and an \( \varepsilon > 0 \) such that for all \( i \), we have

\[
\left| A(\varphi_i) - \inf_{x \in C_{d,\varepsilon}} \varphi_i(x) \right| < \varepsilon' \quad \text{and} \quad \left| B(\varphi_i) - \sup_{x \in C_{d,\varepsilon}} \varphi_i(x) \right| < \varepsilon'
\]

Thus, for any \( x \in C_{d,\varepsilon} \), since \( \varphi_i(x) \) lies between \( \inf_{x \in C_{d,\varepsilon}} \varphi_i(x) \) and \( \sup_{x \in C_{d,\varepsilon}} \varphi_i(x) \) for all \( i \), it must also lie within \( \varepsilon' \) of \( J(\varphi_i) \).

As in the proof of Theorem 15.5, for any \( Y \)-metric \( d \) we define \( \psi_{d,g,y}(x) = d(g(x), y) \). Note that then for any \( \varepsilon > 0 \), we have

\[
\sup_{x \in C_{d,\varepsilon}} \psi_{d,g,y}(x) \leq \varepsilon.
\]

Therefore \( B(\psi_{d,g,y}) = 0 \), and so \( J(\psi_{d,g,y}) = [0,0] \).

Now we apply Theorem 17.3 to obtain a point \( \xi \in \beta X \) such that \( \varphi(\xi) \in J(\varphi) \) for all \( \varphi \). In particular, for any \( Y \)-metric \( d \) we have \( \psi_{d,g,y}(\xi) = 0 \). Since \( 0 < 1 \), by the proof of Theorem 15.5 we also have

\[
d(\overline{\varphi}(\xi) , y) = \psi_{d,g,y}(\xi) = 0.
\]

Since this is true for any \( d \), and \( Y \) is separated, \( \overline{\varphi}(\xi) = y \).

\( \square \)

Thus, for instance, \( \beta(\mathbb{R}^2) \) contains points at infinity which map onto all the points at infinity in the projective plane. In fact, each point at infinity in the projective plane is the image of *many* points at infinity in \( \beta(\mathbb{R}^2) \).
Example 17.5. For any fixed $y_0 \in \mathbb{R}$, define an evaluation datum $J_y$ on $\mathbb{R}^2$ by setting
\[
A_{y_0}(\varphi) = \sup_{z \in \mathbb{R}} \inf_{x > z} \varphi(x, y_0)
\]
\[
B_{y_0}(\varphi) = \inf_{z \in \mathbb{R}} \sup_{x > z} \varphi(x, y_0)
\]
\[
J_{y_0}(\varphi) = [A_{y_0}(\varphi), B_{y_0}(\varphi)]
\]
Since finite sets of real numbers have maxima, the same argument as in the proof of Theorem 17.4 shows that $J_{y_0}$ is an evaluation datum. Thus, it is refined by some point $\xi_{y_0} \in \beta(\mathbb{R}^2)$. It is easy to see that $\xi_{y_0} \neq \xi_{y_1}$ if $y_0 \neq y_1$ (consider a continuous function which is constant at zero on the line $y = y_0$, but constant at 1 on the line $y = y_1$).

Thus, although in the projective plane all pairs of parallel lines meet at exactly one point, in $\beta(\mathbb{R}^2)$, every line contains points at infinity that are not shared by any line parallel to it. Also, we can do the same thing with $x < z$ rather than $x > z$ and obtain different points, so there are different points at infinity in opposite directions on the same line (just as in the lower-hemisphere compactification).

The numbers $A_{y_0}(\varphi)$ and $B_{y_0}(\varphi)$ defined in the previous example are sometimes called the limit inferior and the limit superior, respectively, of $\varphi(x, y_0)$ as $x \to \infty$, and written $\lim_{x \to \infty} \varphi(x, y_0)$ and $\lim_{x \to \infty} \varphi(x, y_0)$. The similar definitions in Theorem 17.4 could also be called limits inferior and superior of a sort.

18. Final exercises

Exercise 18.1. Prove that if the canonical map $X \to \beta X$ is surjective, then $X$ is complete.

* Exercise 18.2. Prove that if $X \to \beta X$ is surjective, then $X$ is totally bounded.

Exercise 18.3. Prove that $X$ is compact if and only if $X \to \beta X$ is surjective. Conclude that compactness is a topological property.

Exercise 18.4. Prove that $X$ is compact if and only if the gauge $G_{\beta}$ is complete.

* Exercise 18.5. Prove that in a compact gauge space, we have $A \approx B$ if and only if there exists a point $x$ with $x \approx A$ and $x \approx B$.

Exercise 18.6. Prove that if $g : X \to Y$ is a continuous surjection with $X$ compact, then $Y$ is also compact.

Exercise 18.7. Prove that any compact gauge space is pseudocompact.

Exercise 18.8. Prove that any continuous map with compact domain is in fact uniformly continuous. Conclude that if two compact gauge spaces are topologically isomorphic, then they are uniformly isomorphic.

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