Fermions and Loops on Graphs. I. Loop Calculus for Determinant

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(Dated: November 21, 2008)

This paper is the first in the series devoted to evaluation of the partition function in statistical models on graphs with loops in terms of the Berezin/fermion integrals. The paper focuses on a representation of the determinant of a square matrix in terms of a finite series, where each term corresponds to a loop on the graph. The representation is based on a fermion version of the Loop Calculus, previously introduced by the authors for graphical models with finite alphabets. Our construction contains two levels. First, we represent the determinant in terms of an integral over anti-commuting Grassman variables, with some reparametrization/gauge freedom hidden in the formulation. Second, we show that a special choice of the gauge, called BP (Bethe-Peierls or Belief Propagation) gauge, yields the desired loop representation. The set of gauge-fixing BP conditions is equivalent to the Gaussian BP equations, discussed in the past as efficient (linear scaling) heuristics for estimating the covariance of a sparse positive matrix.

PACS numbers: 02.50.Tt, 64.60.Cn, 05.50.+q

The series in general, and this paper in particular, belongs to the new emerging field of statistical inference and graphical models born on the cross-roads of statistical/mathematical physics, computer science, and information theory (see the following recent books as introductory reviews \cite{1,2,3}). A typical problem in the field can be stated as follows. Given a graph (trees, sparse graphs or lattices are the three most popular examples), finite or infinite alphabet with the variables defined on the graph elements (typically vertexes or edges) and a cost function (probability) associated with any given variables configuration, one should find the marginals, correlation functions, or solve the weighted counting problem (calculate the partition function). The main problems that define the field are: (a) to estimate the efficiency of exact evaluation for a typical or worst case problem for a class in terms of its dependence on the problem size; and (b) when an exact evaluation is not feasible, as it requires an unacceptably large number of steps, to suggest an approximation and a corresponding efficient algorithmic implementation.

A powerful approach in the field of statistical inference is to build an efficient scheme based on a simple case or limit, where the evaluation is easy, i.e. complexity is polynomial in the number of variables. One simple case corresponds to trees, i.e. graphs without loops. Following the so-called Bethe-Peierls approach inspired by \cite{4,5}, one can show that the computational effort for the partition function on a tree is linear in its size. Furthermore, one anticipates that the tree-based methods and corresponding algorithms should perform reasonably well on sparse graphs with relatively few loops. This approach was reinvented and successfully explored in coding theory \cite{6} (see e.g. \cite{7} for a modern discussion of the graph-based codes) and artificial intelligence \cite{8}, where the corresponding algorithm was coined Belief Propagation (BP) and this name is now commonly accepted across the disciplines. In a recent development we suggested an approach, called Loop Calculus (LC) \cite{9,10}, which establishes an explicit relation between the BP (previously thought of as just heuristics) and exact results. Formally, LC expresses the partition function of a graphical model in terms of a series over certain subgraphs (referred to as generalized loops), where each individual term (that corresponds to a generalized loop) is expressed explicitly in terms of the BP solution (strictly speaking, a fixed point of the corresponding BP equations). LC, originally formulated for a binary alphabet, has been extended to an arbitrary finite alphabet in \cite{11}, and the corresponding approach has been called the Loop Tower.

There is also a class of problems that are easy in spite of a large number of loops contained in the underlying graphical structure. A so-called Gaussian Graphical Model (GGM) that belongs to this class is closely related to the subject of this paper. Consider a graphical model with continuous variables defined on vertexes of a graph with a Gaussian pair-wise interaction. The corresponding partition function, represented simply by a multi-dimensional Gaussian integral, is, therefore, reduced to evaluating the covariance (inverse) of the interaction matrix (note that this operation is well defined only if the matrix is positive definite). For an arbitrary interaction matrix, expressed in terms of a dense graph, this is a problem of \(N^3\) complexity, where \(N\) is the number of the graph vertexes. However, as shown in \cite{12,13}, one can also use a more efficient, linear in \(N\), Gaussian Belief Propagation (GBP) algorithm for exact calculations of the marginals within the Gaussian continuous-alphabet model. The GBP can also be used for finding the covariance or evaluating the determinant of the interaction matrix. However, it does not give an exact result, but rather provide an approximation, which is conjectured to be a reasonably accurate heuristics at least for sufficiently sparse graphs. An intuitive (but also mathematically rigorous) explanation for the exactness of GBP in the case of marginals and its insufficiency for the covariance and related object has been given in \cite{14} via the so-called Walk-Sum Approach (WSA). WSA relates the exact result for the covariance to the sum over all possible oriented paths on the
graph, while GBP (strictly speaking asymptotic GBP, evaluated on an infinite computational tree) corresponds to the summation over a special sub-family of directed walks, specifically backtracking directed walks. This approach has also been extended to evaluate the partition function of GGM (related to the determinant of the interaction matrix) in [15]. The majority of these and other recent studies of GGM have focused on the analysis of the conditions for the Gaussian BP convergence [13] [14] [15] [17] [18] [19] or practical implementations of the GBP algorithm [17] [20].

However, and in spite of this important progress and practical significance, a systematic analysis of the accurateness of GBP as an approximation and a possibility for systematic improvements of the GBP so far has been largely left unexplored. Even though we are still unable to provide the full answers, this paper reports some progress towards the future resolution of these important questions.

In this paper we introduce a fermion-based version of the Loop Calculus approach of [9] [10] [11] that provides an exact representation for a determinant as a finite loop series, where the first term corresponds to a fixed point of the GBP algorithm for the corresponding GGM (that we will also be calling a solution of the GBP equations, or simply GBP solution). Each subsequent term of the Loop Series is associated with a loop on the underlying graph and is expressed explicitly in terms of the GBP solution. Our approach explores the power of the Berezin representation for a determinant in terms of symbolic integrals over anti-commuting Grassman variables. [21] [22]. Note, that a relation between some binary graphical models on a planar graph and Grassman integrals was briefly discussed in [23]. In this and subsequent papers of the series [24] we take broader perspectives and do not limit our discussions to planar graphs.

The paper is organized as follows. In Section I we start our discussion with an extensive introduction (reminder) to the Berezin integral approach, that will culminate in a Grassman integral representation for the determinant of the underlying correlation matrix. Section II is split in three Subsections and forms the core of the paper. Gauge transformations that keep the partition function of the fermion model (the determinant) invariant are introduced in Section II A. In Section II B the Belief Propagation is interpreted as a gauge fixing condition. Section II C finalizes the construction of the Loop Series. The manuscript also contains two Appendices. Appendix I is auxiliary to Section II A. In Section II D the construction of the Loop Series. The manuscript also contains two Appendices. Appendix I is auxiliary to Section II A.

I. INTRODUCTION: INTEGRAL REPRESENTATION

We start with introducing a convenient integral representation for the determinant of an $N \times N$ matrix $H_{ab}$ that will allow to apply the Loop Calculus approach [9] [10] to represent $\text{det} \ H$ in terms of a finite loop series. Although formally the proposed scheme can be applied to any matrix, it becomes algorithmically practical in the case when the matrix $H$ is sparse.

We associate with our $N \times N$ matrix $H$ a graph $\mathcal{G}(H)$ with $N$ nodes $a \in \mathcal{G}_0$. The nodes $a$ and $b$ are connected by edge $\alpha = \{a,b\} \in G_1$, where $a \neq b$, when $H_{ab} \neq 0$, or $H_{ba} \neq 0$. In other words the nodes $a \in \mathcal{G}_0$ represent the diagonal elements $H_{aa}$, whereas the edges $\{a,b\} \in G_1$, correspond to non-zero off diagonal matrix elements $H_{ab}$ and $H_{ba}$. Hereafter we will also use a convenient notation $a \sim b$ for $\{a,b\} \in G_1$. To avoid confusion we emphasize that $\{a,b\}$ naturally denotes a set that consists of (two) elements $a$ and $b$, i.e., $\{a,b\} \in 2^\mathcal{G}_0$ is a subset of the vertex set $\mathcal{G}_0$. In particular $\{a,b\} = \{b,a\}$, which means that we are dealing with a non-oriented graph, and $\mathcal{G}_1 \subset \mathcal{G}_0^2$ denotes the set of graph (non-oriented) edges. The notation $(a,b)$ with $a \neq b$ stands for ordered pairs, i.e. $(a,b) \neq (b,a)$. Ordered pairs can be utilized to denote oriented edges $(a,b) \in \mathcal{G}_0 \times \mathcal{G}_0$, if we decide to choose an orientation on our non-oriented graph. Note that the sparseness of $H$ means that the valences (degrees of connectivity at nodes), $\text{val}(a) = \text{card}(\{b \in \mathcal{G}_0(H) | b \sim a\})$, are small compared to $N$.

To develop a finite loop decomposition for the determinant we represent $\text{det} \ H$ as a Berezin integral over anti-commuting Grassman variables [21]. Specifically, we introduce a set $\{\theta, \bar{\theta}\} = \{\theta_a, \bar{\theta}_a\}_{a \in \mathcal{G}_0}$ with $a = 1, \ldots, N$ of Grassman variables that anti-commute, i.e.,

$$\theta_a \bar{\theta}_b = -\theta_b \bar{\theta}_a, \quad \bar{\theta}_a \theta_b = -\bar{\theta}_b \theta_a, \quad \bar{\theta}_a \bar{\theta}_b = -\bar{\theta}_b \bar{\theta}_a, \quad \forall a,b \in \mathcal{G}_0. \quad (1)$$

A function $F(\theta, \bar{\theta})$ of the Grassman variables is understood as a Taylor series, which turns out to be finite since, due to the anti-commuting relations Eq. (1), each term of the Taylor series can contain any component $\theta_a$ or $\bar{\theta}_a$ not more than once. The Berezin integral is defined via the Berezin measure

$$\mathcal{D}\theta \mathcal{D}\bar{\theta} = \prod_{a \in \mathcal{G}_0} d\theta_a d\bar{\theta}_a, \quad (2)$$
where the differential variables $\{d\theta_a, d\bar{\theta}_a\}_{a \in \mathcal{G}_0}$ anti-commute with each other and with the original Grassman variables. The Berezin measure is fully defined by the integration rules

$$
\int \theta_a d\theta_a = \int \bar{\theta}_a d\bar{\theta}_a = 1, \quad \int d\theta_a = \int d\bar{\theta}_a = 0. \tag{3}
$$

For those who seek more rigorous definitions: We introduce a Grassman algebra $\text{Gr}(G_0)$ as an algebra over $\mathbb{R}$ (or $\mathbb{C}$) generated by $\{\theta_a, \bar{\theta}_a\}_{a \in \mathcal{G}_0}$ with the relations given by Eq. (1). A function $F$ of Grassman variables should be interpreted as an element $F \in \text{Gr}(G_0)$ of the Grassman algebra. The Berezin integral is a measure $\mu : \text{Gr}(G_0) \to \mathbb{R}$ that associates with any element of the Grassman algebra (or simply a “function of Grassman variables”) the value of its integral, according to the rules given by Eqs. (2) and (3).

A well-known property of the Gaussian Berezin integrals, applied to our case, reads

$$
\det H = \int \mathcal{D}\theta \mathcal{D}\bar{\theta} e^{S_0(\theta, \bar{\theta})}, \quad S_0(\theta, \bar{\theta}) = \sum_{a \in \mathcal{G}_0} H_{aa} \bar{\theta}_a \theta_a + \sum_{a \sim b} H_{ab} \bar{\theta}_a \theta_b. \tag{4}
$$

According to Eq. (4) we interpret the determinant as partition function of a statistical fermion model defined on the graph $\mathcal{G}$, where the fermion (Grassman) variables reside on the graph nodes. To apply the loop decomposition we convert the resulting statistical model into a vertex model, i.e., the one with the variables residing on the graph edges. This task can be easily accomplished with the help of a Hubbard-Stratanovich (HS) type transformation, defined as follows. We introduce a set of Grassman variables $\{\bar{\chi}_{ab}, \chi_{ab}\}_{a,b \in \mathcal{G}_0}$, describing the HS decoupling field representing the off-diagonal terms in the action of Eq. (4). These variables express interaction of the original variables with the decoupling field. This is achieved by making use of the set of identities

$$
e^{H_{ab} \bar{\theta}_a \theta_b + H_{ba} \bar{\theta}_b \theta_a} = -H_{ab} H_{ba} \int d\chi_{ba} d\bar{\chi}_{ba} e^{(H_{ab})^{-1} \chi_{ab} \bar{\chi}_{ba} + (H_{ba})^{-1} \bar{\chi}_{ba} \chi_{ab}} \times e^{\chi_{ba} \bar{\theta}_a + \bar{\chi}_{ba} \theta_b + \bar{\chi}_{ba} \theta_a}, \quad \forall (a,b) \in \mathcal{G}_1, \tag{5}
$$

where, for the sake of simplicity, we assume that, if $a \sim b$, both matrix elements $H_{ab}$ and $H_{ba}$ are non-zero. (The latter condition can be actually relaxed, however this goes beyond the scope of this manuscript.) Using the integral representation (5) for the terms that originate from the off-diagonal terms of the action $S_0$ in Eq. (4) we arrive at the following HS representation for the determinant

$$
\det H = Z \left( \prod_{(a,b) \in \mathcal{G}_1} (-H_{ab} H_{ba}) \right) \int \mathcal{D}\chi \mathcal{D}\bar{\chi} e^{S_{HS}(\theta, \bar{\theta}; \chi, \bar{\chi})}, \quad S_{HS}(\theta, \bar{\theta}; \chi, \bar{\chi}) = \sum_{a \in \mathcal{G}_0} H_{aa} \bar{\theta}_a \theta_a + \sum_{a \sim b} (H_{ab})^{-1} \chi_{ab} \bar{\chi}_{ba} + \sum_{b \sim a} \sum_{a \in \mathcal{G}_0} (\chi_{ba} \bar{\theta}_a + \bar{\chi}_{ba} \theta_a). \tag{6}
$$

As it always happens for the HS transformation, integration over the HS field in Eq. (6) reproduces the original integral representation (4) due to the identities (5). To accomplish the HS trick we integrate over the original variables $\{\theta, \bar{\theta}\}$. This can be readily done, since the integration is local (i.e., can be performed on each node independently). Finally, we arrive at the following desired expression for the determinant in a form of the partition function of a vertex model

$$
\det H = Z = \left( \prod_{(a,b) \in \mathcal{G}_1} (-H_{ab} H_{ba}) \right) \left( \prod_{a \in \mathcal{G}_0} H_{aa} \right) \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \prod_{a \in \mathcal{G}_0} f_a(\chi_a, \bar{\chi}_a) \prod_{a \in \mathcal{G}_1} g_a(\chi_a, \bar{\chi}_a) \tag{7}
$$

where $\chi_a = \{\chi_{ba}\}_{b \in \mathcal{G}_0, b \sim a}$ is a set of edge variables attached to the node $a$, and $\chi_a = \{\chi_{ab}, \chi_{ba}\}$ is the set of variables that reside on edge $\alpha = \{a, b\}$. The edge functions

$$
g_a(\chi_a, \bar{\chi}_a) = g_{ab}(\chi_a, \bar{\chi}_a) = g_{ba}(\bar{\chi}_a, \chi_a) = e^{(H_{ab})^{-1} \bar{\chi}_{ab} \chi_{ba} + (H_{ba})^{-1} \bar{\chi}_{ba} \chi_{ab}}, \quad \alpha = \{a, b\} \tag{8}
$$

define the proper scalar products of the local states that belong to the same edge and different nodes. The vertex factor-functions are obtained via the local integrations described above

$$
f_a(\chi_a, \bar{\chi}_a) = (H_{aa})^{-1} \int d\bar{\theta}_a d\theta_a e^{H_{aa} \bar{\theta}_a \theta_a + \sum_{b \in \mathcal{G}_0} (\chi_{ba} \bar{\theta}_a + \bar{\chi}_{ba} \theta_a)} = e^{(H_{aa})^{-1} \sum_{b \sim a} \bar{\chi}_{ba} \chi_{ab} + \sum_{b' \in \mathcal{G}_0} \chi_{a}^{b' a}}, \tag{9}
$$
II. GAUGE TRANSFORMATION, BELIEF PROPAGATION AND LOOP SERIES

This Section is broken in three Subsections. In Section II.A we introduce a freedom (gauge) allowing to represent an edge $g$-function as a sum of four terms each constituting a product of two vertex terms. Section II.B introduces a way of fixing the gauge freedom in accordance with the Belief Propagation Principle. We further derive the Fermi-BP equations which turn out to be fully equivalent to the (standard) Bose-BP equations discussed in Appendix B. The last Subsection II.C culminates in a derivation of a finite Loop Series representation for the determinant, with each term of the series expressed explicitly via the solution of the BP equations.

A. Gauge Transformation

The $g$-terms in the integrand of Eqs. (7) mix contributions associated with different vertices. Our next step will aim at the decomposing of the $g$-contribution at any edge into a sum of bi-local expressions. Specifically, we will be seeking for a decomposition of the following type:

$$g_{ab} = c_{ab} \left( e^{\gamma_{ab} \chi_{ab} e^{\gamma_{ba} \chi_{ba}}} + \kappa_{ab} e^{\gamma_{ab} \chi_{ab} e^{\gamma_{ba} \chi_{ba}}} \right) + \zeta_{ab} \bar{\chi}_{ab} \chi_{ba} + \zeta_{ba} \bar{\chi}_{ba} \chi_{ab},$$

(10)

with $\kappa_{ab} = \kappa_{ba}$ and $c_{ab} = c_{ba}$. All the newly introduced parameters in Eq. (10) are to be defined by comparison with Eq. (8). Therefore, expanding Eq. (10) and Eq. (8) into a series over the Grassman variables and comparing the results term by term one establishes the following relations for the coefficients entering Eq. (10):

$$c_{ab}(1 + \kappa_{ab}) = 1, \quad \gamma_{ab} + \kappa_{ab} \gamma_{ba} = 0, \quad \gamma_{ba} + \kappa_{ba} \gamma_{ab} = 0, \quad \zeta_{ab} = (H_{ab})^{-1}, \quad \zeta_{ba} = (H_{ba})^{-1},$$

$$c_{ab}(\gamma_{ab} \gamma_{ba} + \kappa_{ab} \gamma_{ab} \gamma_{ba}) = -(H_{ab} H_{ba})^{-1}.$$  

(11)

The relations (11) allow all the coefficients to be expressed in terms of $\gamma_{ab}$:

$${\kappa}_{ab} = -{H}_{ab} {H}_{ba} {\gamma}_{ab} {\gamma}_{ba}, \quad c_{ab} = \frac{1}{1 + \kappa_{ab}}, \quad \gamma_{ab}' = -(\kappa_{ab})^{-1} \gamma_{ab} = \frac{1}{H_{ab} H_{ba} \gamma_{ba}}, \quad \zeta_{ab} = (H_{ab})^{-1}.$$  

(12)

We call the freedom in choosing the $\gamma$ variables the gauge freedom, as any choice of $\gamma$ does not change the value of the partition function defined by Eqs. (7,8,9,10,12).

The gauge transformation, formally described above, utilizes a decomposition of the graphical trace over the vector spaces $V_{ab}$, where each $V_{ab}$ is associated with the link $\{a,b\} \in G_1$ and node $b \in G_0$. $V_{ab}$ represents the Grassman algebra generated by $\bar{\chi}_{ab}$ and $\chi_{ab}$. It has a super-dimension $(2|2)$, i.e., an even dimension 2 (the first number) and odd dimension 2 (the second number), since it has 2 even states, namely $1$ and $\bar{\chi}_{ab} \chi_{ab}$, and 2 odd states, namely $\bar{\chi}_{ab}$ and $\chi_{ab}$.

The skew-orthogonality conditions (10,12) are different from these we have introduced in the context of the finite-alphabet graphical models [9,10,11]. Indeed, since the local state spaces $V_{ab}$ with the super-dimension $(2|2)$ have total dimension $4 = 2 + 2$, a possible approach would be to build a tower hierarchy in the spirit of [11]. Then, a more general set of the skew-orthogonality conditions (compared to those discussed above) should be introduced. However, the additional symmetry, i.e. the superstructure (strictly speaking it should be referred to as a $\mathbb{Z}_2$-graded structure) of the underlying linear algebra problem defines our choice of the more stringent skew-orthogonality constraint. The details of (and actual reasons for) the choice will become clear in the next Subsection when we discuss the additional BP constraints for the gauges.

The gauge transformation results in an explicit representation for the whole partition function in terms of a series where each term corresponds to a choice of one of the four aforementioned states at each edge. The expansion is derived via a direct substitution of Eqs. (10,12) into Eq. (8), followed by expanding the expression into monoms, substituting it back into the integrand of Eq. (7) followed by the evaluation of the resulting integrals term by term. Each of the elementary integrals is vertex-local, thus turning the expression under evaluation into a product of simple vertex-related contributions. For any choice of the edge local parameters $\gamma$ one expects a gauge dependence of the individual contributions to the resulting series for the partition function, while the cumulative result (the entire sum) will be gauge insensitive/invariant by construction.

In the following Subsection we discuss a special choice of the gauge, related to the BP approach, which essentially restricts all the contributions in the aforementioned series over the edge-states to those that correspond to generalized loops (to be defined later) on the graph. Note that another (non-BP) choice of the gauge that leads to an interesting explicit expression for the partition function of the monomer-dimer model as a finite series expansion over determinants (which is also a loop series of a kind, but in another sense) is discussed in the second paper of the series [24].
B. Belief-Propagation Equations

In this Subsection we extend our general approach, coined Loop Calculus \cite{9,10}, to the Grassman integral for the partition function (determinant) of the Gaussian model.

We will impose additional constraints on the $\gamma$-gauges following two complementary approaches. First, we describe the BP gauge as a result of the ground state optimization. Then, we derive the same BP equations as a set of the no-loose-end constraints on the excited states.

1. Variational derivation of BP equations

The ground state contribution to the partition function (determinant) is naturally given by

$$Z_0 = \left( \Pi_{(a,b) \in \mathcal{G}_1} \frac{H_{ab} H_{ba}}{H_{ab} H_{ba} \gamma_{ab} \gamma_{ba} - 1} \right) \left( \Pi_{c \in \mathcal{G}_0} H_{cc} z_c \right), \quad z_c = \int d\chi_c d\bar{\chi}_c f_c(\bar{\chi}_c, \chi_c) \prod_{a' \sim c} e^{\gamma_{a'c} \bar{\chi}_{a'} \chi_c},$$

(13)

where the dependence on the $\gamma$-gauges is spelled out explicitly. Since Eq. (13) interprets $z_c$ as a Gaussian integral over the Grassman variables associated with vertex $c$, a direct evaluation of the integral results in $z_c = \det(M_c)$, where each element of the newly introduced matrix $M_c$ is defined by $M_{c,ab} = (H_{cc})^{-1} + \gamma_{bc} \delta_{ab}$. Evaluating the determinant explicitly one finds

$$z_c = \det(M_c) = \left( \Pi_{a' \sim c} \gamma_{a'c} \right) \left( 1 + (H_{cc})^{-1} \sum_{a' \sim c} (\gamma_{a'c})^{-1} \right),$$

(14)

and the resulting expression for the ground state contribution adopts a form

$$Z_0 = \left( \Pi_{(a,b) \in \mathcal{G}_1} \frac{H_{ab} H_{ba}}{H_{ab} H_{ba} \gamma_{ab} \gamma_{ba} - 1} \right) \left( \Pi_{c \in \mathcal{G}_0} H_{cc} \left( 1 + (H_{cc})^{-1} \sum_{a' \sim c} (\gamma_{a'c})^{-1} \right) \right),$$

(15)

Considering $Z_0(\gamma)$ as a $\gamma$-dependent approximation for the full partition function (by construction the latter does not depend on $\gamma$) one can define the BP-conditions as an adjustment of $\gamma$ that minimizes the dependence of $Z_0$ on it. Formally, one looks for a stationary point of $Z_0(\gamma)$:

$$\forall a \in \mathcal{G}_0 \text{ and } \{a, b\} \in \mathcal{G}_1 : \frac{\partial Z_0}{\partial \gamma_{ab}}|_{\gamma_{ab}^{(bp)}} = 0 \Rightarrow -H_{ab} H_{ba} \gamma_{ab}^{(bp)} = H_{bb} + \sum_{a' \sim b} (\gamma_{a'b})^{-1}.$$  \hspace{1cm} (16)

Then, the actual value of the ground state contribution at the BP stationary point becomes

$$Z_{BP;Fermi} = Z_0(\gamma^{(bp)}) = \left( \Pi_{(a,b) \in \mathcal{G}_1} H_{ab} H_{ba} \gamma_{ab}^{(bp)} \gamma_{ba}^{(bp)} - 1 \right) \left( \Pi_{c \in \mathcal{G}_0} H_{cc} + \sum_{a' \sim c} (\gamma_{a'c})^{-1} \right),$$

(17)

where $\gamma^{(bp)}$ is defined implicitly by Eqs. (16).

2. BP equations as no-loose-end constraints

We reiterate that the gauge fixing boils down to a particular choice for the set of parameters $\{\gamma_{ab}\}_{a,b \in \mathcal{G}_0; (a,b) \in \mathcal{G}_1}$. This can be done by imposing an additional set of constraints that forbid the excited state structures with loose ends (more precisely nodes of valence one). Note that loose ends that correspond to odd local excited states are automatically forbidden due to the $Z_2$-grading. Actually, this rationalizes our choice of the skew-orthogonality constraints in Eqs. (10,12). Therefore, the BP conditions enforce a cancellation of a large set of contributions to the partition function. The forbidden contributions are those that contain loose end with even excited states at any vertex of the graph (while all other edges adjusted to the vertex being in the ground state). Formalization of the BP constraints results in

$$\forall a \in \mathcal{G}_0 \text{ and } \{a, c\} \in \mathcal{G}_1 : \int d\chi_a d\bar{\chi}_a f_a(\bar{\chi}_a, \chi_a) e^{\bar{\chi}_a \chi_a / (H_{cc} H_{ca} \gamma_{ac})} \prod_{b \sim a} e^{\gamma_{ba} \bar{\chi}_b \chi_b} \bigg|_{\gamma^{(bp)}} = 0.$$  \hspace{1cm} (18)
The condition in Eq. \[18\] can also be restated as \(\det(M_c') = 0\), where the matrix \(M_c'\) is obtained from \(M_c\) by replacing \(\gamma_{ac}\) with \(\gamma_{ac}' = H_{ac}H_{ca}/\gamma_{ac}\). Utilizing Eq. \[14\] we observe that Eqs. \[18\] turn explicitly into Eqs. \[16\].

3. Brief discussion of BP equations

In the two preceding Subsections the BP equations were derived in two different ways, via the variational and loose-end approaches respectively. Appendix \[23\] also contain a relevant information. It is shown there that the problem of finding the Fermi-BP-gauge (here we emphasize that BP conditions follow from the Grassman/Fermi formulation) is completely equivalent to finding a stationary point of the so-called Gaussian BP equations stated within the standard Gaussian integrals. We call the standard BP approach Bose-BP to contrast it with the Fermi BP discussed above.

Note, that BP equations can also be stated as defining extrema of the so-called Bethe Free Energy functional, introduced for a general finite alphabet graphical model in \[25\], and also discussed in the context of the Gaussian (continuous alphabet) graphical model in \[19\].

C. Loop Series for the Determinant

The loop series is obtained in a standard way by considering local excited states that correspond to the choice of the second, third, or fourth term in Eq. \[10\]. Therefore, each contribution to the correction for \(Z_0\) determines a subgraph \(C \subset G\) that consists of the edges on which excited states have been chosen. Due to the BP equations a subgraph that provides a non-zero contribution does not have loose ends. Such subgraphs are referred to as generalized loops. To describe a contribution fully we partition the edges of \(C \subset G\) into the neutral ones that correspond to the choice of the even local excited states and the oriented edges that correspond to the odd local excited states. The choice of the second, third, or fourth term in Eq. \[10\] will be denoted by an arrow on the edge \{\(a, b\)\} that goes from \(a\) to \(b\), the choice of the fourth term is denoted by an opposite arrow that goes from \(b\) to \(a\). The \(Z_2\) grading implies that in order to provide a non-zero contribution the number of incoming arrows coincides with the number of the outgoing counterparts for all nodes of \(C\).

We will further demonstrate that to provide a non-zero contribution each node of \(C\) can actually have no more than one incoming/outgoing arrow. This follows from the following form of the matrices \(M_c^{-1}\) inverse to \(M_c\)

\[
(M_c^{-1})_{ab} = -\frac{1}{\gamma_{ac}\gamma_{bc}(H_{cc} + \sum_{a'\sim c}(\gamma_{a'c})^{-1})} \quad \text{for } a \neq b
\]

and to their modified counterparts \(M_c'\) obtained by replacing a certain number of \(\gamma_{ac}\)-terms with \(\gamma_{ac}'\)-terms. We can further make use of the properties of the Gaussian integrals and Eq. \[19\] to derive

\[
\int d\chi_c d\bar{\chi}_c \chi_{ac} \bar{\chi}_{bc} e^{\sum_{a', a'' \sim c} M_{c,a'} a'' \bar{x}_{a'} x_{a''} e^{\sum_{a', a'' \sim c} M_{c,a'} a'' \bar{x}_{a'} x_{a''} e^{-H_{cc}}} = (M_c^{-1})_{ab} \det(M_c) = -(H_{cc})^{-1} \prod_{a' \sim c} \gamma_{a'c} \quad \text{for } a \neq b.
\]

The property described above follows from the fact

\[
\int d\chi_c d\bar{\chi}_c \chi_{a_1 c} \ldots \chi_{a_k c} \bar{\chi}_{b_1 c} \ldots \bar{\chi}_{b_k c} e^{\sum_{a', a'' \sim c} M_{c,a'} a'' \bar{x}_{a'} x_{a''} e^{-H_{cc}}} = 0 \quad \text{for } \{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\} = \emptyset, \quad k \geq 2.
\]

To prove Eq. \[21\] we apply the Wick’s theorem that represents the Grassman integral in Eq. \[21\] as a sum of \(k!\) contributions that correspond to \(k!\) possible pairings between the \(\bar{\chi}\) and \(\chi\) variables in the pre-exponents. Each contribution consists of a product of \(k\) pair correlation functions given explicitly by Eq. \[21\]. Due to the form of the pair correlation functions, their product does not depend on a particular choice of the pairing. On the other hand, the signs in front of the contributions are alternating and the number of negative signs among the \(k!\) contributions is the same as the number of positive signs, provided \(k \geq 2\). This leads to Eq. \[21\]. For \(k = 1\) the (non-zero) result is given by Eq. \[20\]. These results apply as well to the modified matrices \(M'_c\).

We are now in a position to summarize the results of Eqs. \[19,20,21\] in the finite loop series expression

\[
\det(H) = Z = Z_{BP;Fermi} \left(1 + \sum_{C \in \text{GL}(G)} \sum_{C' \in \text{DOC}(C)} r(C, C')\right)
\]

where \(\text{GL}(G)\) is the set of generalized loops of the graph \(G\) (subgraphs with all nodes of valence 2 or higher), and \(\text{DOC}(C)\) is the set of Disjoint Oriented Cycles (DOC) of the graph \(C\), i.e., subgraphs of \(C\) whose all nodes have
FIG. 1: Example of the set of generalized loops for a sample graph, consisting of 6 vertexes and 7 edges. Generalized loops are shown in solid lines, while part of the original graph which does not belong to a generalized loop is shown in dashed gray. Oriented disjoint circles, which are also subgraphs of respective generalized loops, are marked with arrows and shown in red.

The valence exactly 2, equipped with orientation. For a DOC $C'$ let $\deg(C') = \text{card}(\pi_0(C'))$ be the number of its connected components and for an edge $\alpha \in C'_1$ let $\partial_0 \alpha \in C'_0$ and $\partial_1 \alpha \in C'_0$ be its left and right ends, respectively (orientation arrows go from the left to the right). A relative contribution $r(C, C')$ can be represented as a product of the node factors $r_a(C, C')$, edge factors $r_\alpha(C, C')$, and the parity factor:

$$r(C, C') = (-1)^{\deg(C')} \prod_{a \in C_0} r_a(C, C') \prod_{\alpha \in C_1} r_\alpha(C, C')$$

(23)

where the factors are calculated in a straightforward way (see Appendix A for the details).

$$a \in C_0 \setminus C'_0 : \quad r_a(C, C') = \frac{H_{aa} + \sum_{a' \sim a} H_{a'a} \gamma_{a'(bp)} + \sum_{a' \sim a} \gamma_{a'(bp)}^{-1}}{H_{aa} + \sum_{a' \sim a} \gamma_{a'(bp)}^{-1}}$$

$$a \in C'_0 : \quad r_a(C, C') = -\frac{1}{H_{aa} + \sum_{a' \sim a} \gamma_{a'(bp)}^{-1}}$$

$$\alpha \in C_1 \setminus C'_1 : \quad r_\alpha(C, C') = -\frac{1}{H_{cd} \gamma_{cd}^{-1}}$$

$$\alpha \in C'_1 : \quad r_\alpha(C, C') = \frac{1 - H_{cd} \gamma_{cd}^{-1}}{H_{cd} \gamma_{cd}^{-1}}$$

(24)

where in the last two formulas $c = \partial_0 \alpha$ and $d = \partial_1 \alpha$. 
III. SUMMARY AND CONCLUSIONS

The manuscript describes an explicitly constructed Loop Series for the Fermion Graphical Model. The LS expresses determinant of an arbitrary square matrix in terms of a finite series. Four important features of the series are:

- The first term in the series corresponds to an approximation associated with solution of the GBP equations, identical to those that emerged in the standard GGM discussed before [12, 13, 14, 15, 16, 17, 18, 19, 20]. Note, however, that while the standard GGM approach would requires the matrix to be (at least) semi-definite (so that the normal variables Gaussian integral would converge), our approach does not have this restriction as the Berezin integrals are defined for any (even zero determinant!) square matrices.

- Each term of the Loop Series is associated with a generalized loop and an oriented disjoint cycle defined on the generalized loop.

- Each term is expressed explicitly in terms of the chosen solution of the GBP equations. Computation of any of the contributions (once the GBP solution is known) is the task of linear complexity in the size of the underlying generalized loop.

- The Loop Series can be constructed around any solution of the GBP equations, e.g., around these solutions that are unstable with respect to the standard iterative GBP.

Let us now briefly discuss how the fermion based Loop Series could potentially be used in the future. First of all, one hopes that the LS formula can clarify the accuracy of the GBP approximation for different classes of matrices (sparse, walk-summable, planar, etc.). Second, aiming to improve GBP one might be interested to identify problems (matrices) where accounting for a relatively small, $O(N^\gamma)$ with $\gamma < 3$, number of loops, will significantly improve the GBP. In the context of these two general problems it will also be important to extend further analysis of the Bethe Free energy for the Gaussian models initiated in [19, 25]. The Bethe Free energy is a functional whose stationary points coincide with the GBP solutions.

Finally, let us briefly overview the main idea of the second paper in the series [24], and its relation to the results discussed above. [24] describes a construction generalizing LS for determinant discussed in this paper. The construction starts with a $\mathbb{Z}_2$-gauge theory, stated in terms of binary/Ising spins (that represent a gauge field with the discrete gauge group $\mathbb{Z}_2$) and fermions on an arbitrary graph. It is shown that, on the one hand, the graphical gauge model is reduced to a monomer-dimer model on the graph, while on the other hand it turns into a series over disjoint oriented cycles on the graph, where the corresponding coefficient is given by determinant of a matrix related to to the full graph with excluded disjoint. We find that this relation (between the monomer-dimer model and the Cycle Series) also follows (via a certain type of inversion) from the Loop Series for the determinant discussed in this paper.

IV. ACKNOWLEDGMENTS

We are grateful to J. Johnson for useful comments. This material is based upon work supported by the National Science Foundation under CHE-0808910. The work at LANL was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396.

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APPENDIX A: BEREZIN GAUSSIAN INTEGRALS

In this technical Appendix we present some details of the calculations of the node $r_a(C,C')$ and edge $r_{ab}(C,C')$ factors for the relative loop contributions $r(C,C')$ given by Eq. (24). We start with defining the Grassman variables correlation function
\[ \langle \tilde{x}_{a_1c} \cdots \tilde{x}_{a_kc} \tilde{\chi} \rangle_x = Q_c^{-1} \int d\tilde{x} \tilde{x} \chi \tilde{\chi} \cdots \tilde{x} \tilde{\chi} \tilde{x} \cdots \tilde{x} \tilde{\chi} \tilde{x} = \det(M_c). \] (A1)

The main feature of the Gaussian integral is expressed in the following Wick’s theorem formula
\[ \langle \tilde{x}_{a_1c} \cdots \tilde{x}_{a_kc} \tilde{\chi} \rangle_x = \sum_P (-1)^{\deg(P)} \prod_j \langle \tilde{x}_{a_jc} \chi P(b_j)c \chi \rangle_x, \] (A2)

with the pair correlation function $\langle \tilde{x}_{ac} \chi bc \rangle_x = (M^{-1})_{ab}$. Note, that Eq. (A2) follows naturally from a direct expansion of the following expression for the generating function
\[ F(\xi_c, \xi_b) = \langle e^{\sum_{a,b} (\xi_{ac} \chi_{ab} + \xi_{ac} \chi_{ab})} \rangle_x = e^{\sum_{a,b} (M^{-1})_{ab} \xi_{ac} \xi_{bc}}, \] (A3)
in powers of its argument.

The Loop Series is obtained upon the substitution of the skew-orthogonality condition [10] into the integral representation for the determinant that results in the following expression
\[ \det(H) = Z_{BP};Fermi + \sum_{C \in GL(G)} \sum_{C' \in SL(C)} \prod_{\alpha \in \Phi_1} H_{\partial \alpha, \partial_1 \alpha} H_{\partial_1 \alpha, \partial \alpha} z_{\partial \alpha, \partial_1 \alpha} (C, C') \prod_{a \in \Phi_0} (-H_{aa} z_a (C, C')) \] (A4)

The quantities $z_{ab}(C,C')$ with $a = \partial \alpha$, $b = \partial_1 \alpha$ are determined by the coefficients in the skew-orthogonal representation [10] and are given by
\[ z_{\alpha}(C,C') = c_{ab} \text{ for } \alpha \in \Phi_1 \setminus C_1, \quad z_{\alpha}(C,C') = c_{ab} \ell_{ab} \text{ for } \alpha \in C_1 \setminus C_1', \quad z_{\alpha}(C,C') = \zeta_{ab} \text{ for } \alpha \in C_1' \] (A5)
The explicit expressions are obtained by combining Eq. (A3) with Eq. (12). The quantities $z_{\epsilon}(C,C')$ are obtained by attaching the functions of the local Grassman variables in [10] to the corresponding vertices followed by the
integration over the local Grassman variables. This results in

\[ c \in G_0 \setminus C_0 : \quad z_c(C, C') = \int d\chi_c d\bar{\chi}_c \exp \sum_{a,b,c,M_c,ab} \chi_c \bar{\chi}_c = \det(M_c) = \prod_{a \sim c} \gamma_{a,c} (H_{cc})^{-1} \left( H_{cc} + \sum_{b \sim c} \gamma_{bc} \right)^{-1}, \]

\[ c \in C_0 \setminus C'_0 : \quad z_c(C, C') = \int d\chi_c d\bar{\chi}_c \exp \sum_{a,b,c,M_c,ab} \chi_c \bar{\chi}_c = \det(M'_c) \]

\[ = \prod_{a \sim c} \gamma_{a,c} \prod_{a \sim c} \gamma'_{a,c} (H_{cc})^{-1} \left( H_{cc} + \sum_{b \sim c} \gamma_{bc} \right)^{-1} + \sum_{b \sim c} \gamma'_{bc} \right)^{-1}, \]

\[ c, s', s'' \in C'_0 : \quad z_c(C, C') = \int d\chi_c d\bar{\chi}_c \exp \sum_{a,b,c,M_c,ab} \chi_c \bar{\chi}_c = \det(M'_c)(M'_c)^{-1} \]

\[ = -(H_{cc})^{-1} \prod_{a \sim c} \gamma_{ac} \prod_{a \sim c} \gamma'_{ac}. \]

The loop series (22) is obtained from the representation of Eq. (A4) with the ingredients given by Eqs. (A5) and (A6) by regrouping the factors into the node and edge ones in a very obvious way.

Finally, we demonstrate that the vertices that have more than one incoming/outgoing arrow do not contribute to the determinant. Such contribution is proportional to \( \langle \chi_{a_1 c} \cdots \chi_{a_k c} \chi_{b_1 c} \cdots \chi_{b_k c} \rangle \), with \( \{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\} = \emptyset \) and vanishes, since applying the Wick’s theorem Eq. (A2) we derive

\[ \langle \chi_{a_1 c} \cdots \chi_{a_k c} \chi_{b_1 c} \cdots \chi_{b_k c} \rangle = \sum_{P} (-1)^{\deg(P)} \prod_{j=1}^{k} \langle \chi_{a_j c} \chi_{P(b_j)c} \rangle \chi_{a_j c} = \sum_{P} (-1)^{\deg(P)} \prod_{j=1}^{k} (M'_{a_j c})^{-1} \]

\[ = -\sum_{P} (-1)^{\deg(P)} \left( H_{cc} + \sum_{a' \sim c} \gamma_{a'c}^{-1} + \sum_{a' \sim c} \gamma'_{a'c}^{-1} \right) \prod_{j=1}^{k} \gamma_{a_j c} \gamma_{b_j c} = 0, \]

and the alternating sum over \( k! \) permutations \( P \) in the numerator vanishes for \( k \geq 2 \).

**APPENDIX B: BELIEF-PROPAGATION GAUGE IN BOSE-REPRESENTATION**

In the normal (Bose) integral representation the determinant can be represented as follows

\[ (\det H)^{-1} = \prod_{a} \left( \frac{d\bar{\psi}_a d\psi_a}{2\pi} \right) \exp \left( -\frac{1}{2} \sum_{a \in G_0} H_{aa} \bar{\psi}_a \psi_a - \frac{1}{2} \sum_{(a,b) \in G_1} H_{ab} \bar{\psi}_a \psi_b \right), \]

where the integration goes over the pairs of the complex conjugated variables (normal c-numbers). Here, we restrict ourselves to the case when the integral is well-defined (convergent). To decouple the terms associated with different vertices of \( G_0 \) we introduce the following Bose-version of the HS transformation

\[ \exp \left( -\frac{H_{ab}}{2} \bar{\psi}_a \psi_b - \frac{H_{ba}}{2} \bar{\psi}_b \psi_a \right) = \frac{1}{(2\pi)^2 H_{ab} H_{ba}} \int d\tilde{\varphi}_{ab} d\varphi_{ab} d\tilde{\varphi}_{ba} d\varphi_{ba} \exp \left( -\frac{H_{ab}}{2} \bar{\psi}_ab \varphi_{ab} - \frac{H_{ba}}{2} \bar{\psi}_ba \varphi_{ba} \right) \]

\[ \times \exp \left( i \bar{\varphi}_{ba} \bar{\psi}_a / 2 + i \bar{\varphi}_{ab} \bar{\psi}_b / 2 + i \varphi_{ab} \psi_{ba} / 2 + i \varphi_{ba} \psi_{ab} / 2 \right). \]

Substituting Eq. (B2) into Eq. (B1) and performing the integration over the vertex variables \( \psi \) we arrive at the following Gaussian edge factor function formulation

\[ (\det H)^{-1} = (2\pi)^{-2|G_1|} \prod_{a \in G_0} H_{aa}^{-1} \prod_{(a,b) \in G_1} (-H_{ab} H_{ba})^{-1} \int D\bar{\varphi} D\varphi \prod_{a \in G_0} F_a (\varphi_a, \bar{\varphi}_a) \prod_{a \in G_1} G_a (\varphi_a, \bar{\varphi}_a) \]

\[ F_a (\varphi_a, \bar{\varphi}_a) = \exp \left( -\frac{H_{aa}}{2} \bar{\varphi}_a \varphi_a - \frac{H_{ab}}{2} \bar{\varphi}_b \varphi_b \right), \]

\[ G_a (\varphi_a, \bar{\varphi}_a) = \exp \left( -\frac{H_{ab}}{2} \bar{\varphi}_a \varphi_b - \frac{H_{ba}}{2} \bar{\varphi}_b \varphi_a \right). \]
We further introduce the following gauge representation for the skew scalar product
\[ G_{\alpha} = \phi_{ab} \exp \left( -\frac{\lambda_{ab}}{2} \tilde{\varphi}_{ab} \tilde{\varphi}_{ab} - \frac{\lambda_{ba}}{2} \tilde{\varphi}_{ba} \tilde{\varphi}_{ba} \right) + \sum_{n=1}^{\infty} Q_{ab}^{(n)}(\tilde{\varphi}_{ab}, \tilde{\varphi}_{ab})Q_{ba}^{(n)}(\tilde{\varphi}_{ba}, \tilde{\varphi}_{ba}), \] (B6)
where \( \phi_{ab} = \phi_{ba} \); the first term on the rhs of Eq. (B6) corresponds to the local ground state while the sum (remainder) represents some discrete spectrum of the excited states. The remaining freedom in Eq. (B6) is fixed via the following BP-(zero loose end) conditions
\[ \forall \{a,b\} \in \mathcal{G}_1 : \int \mathcal{D} \tilde{\varphi}_a \mathcal{D} \tilde{\varphi}_a F_a(\tilde{\varphi}_a, \varphi_a) \left( \prod_{c \sim a} \exp \left( -\frac{\lambda_{ca}}{2} \tilde{\varphi}_{ca} \tilde{\varphi}_{ca} \right) \right) Q_{ba}^{(n)}(\tilde{\varphi}_{ba}, \tilde{\varphi}_{ba}) = 0. \] (B7)

Multiplying Eqs. (B7) with \( Q_{ba}^{(n)}(\tilde{\varphi}_{ba}, \tilde{\varphi}_{ba}) \), making summation over all excited states \( (n > 0) \), and expressing the \( Q \)-terms via the ground state contribution (with the help of Eqs. (B6)), one arrives at the following BP relations stated solely in terms of the local ground states (local gauges)
\[ \forall a, b \in \mathcal{G}_0, a \sim b : \int \mathcal{D} \tilde{\varphi}_a \mathcal{D} \tilde{\varphi}_a F_a(\tilde{\varphi}_a, \varphi_a) \left( \prod_{c \sim a} \exp \left( -\frac{\lambda_{ca}}{2} \tilde{\varphi}_{ca} \tilde{\varphi}_{ca} \right) \right) \exp \left( -\frac{H_{ab}}{2} \tilde{\varphi}_{ab} \varphi_{ab} - \frac{H_{ba}}{2} \tilde{\varphi}_{ba} \varphi_{ba} \right) = \phi_{ab} \exp \left( \frac{\lambda_{ab}}{2} \tilde{\varphi}_{ab} \tilde{\varphi}_{ab} \right) \int \mathcal{D} \tilde{\varphi}_a \mathcal{D} \tilde{\varphi}_a F_a(\tilde{\varphi}_a, \varphi_a) \left( \prod_{c \sim a} \exp \left( -\frac{\lambda_{ca}}{2} \tilde{\varphi}_{ca} \tilde{\varphi}_{ca} \right) \right) := \phi_{ab} \exp \left( \frac{\lambda_{ab}}{2} \tilde{\varphi}_{ab} \tilde{\varphi}_{ab} \right). \] (B8)

Evaluation of the Gaussian integrals transforms Eqs. (B8) into
\[ \forall a, b \in \mathcal{G}_0, a \sim b : -\lambda_{ab} H_{ab} H_{ba} = H_{aa} + \sum_{c \sim a}(\lambda_{ca})^{-1}, \] (B9)
\[ \phi_{ab} = \lambda_{ba} \left( H_{aa} + \sum_{c \sim a}(\lambda_{ca})^{-1} \right) = 1 - \lambda_{ab} \lambda_{ba} H_{ba} H_{ab}, \] (B10)
where the transformation from the lhs of Eq. (B10) to its rhs also involves Eqs. (B9). The resulting BP expression for the entire partition function (BP expression for the inverse determinant) becomes
\[ Z_{BP; \text{Bose}} = (2\pi)^{-2|\mathcal{G}|} \left( \prod_{a \in \mathcal{G}_0} H_{aa}^{-1} \int \mathcal{D} \tilde{\varphi}_a \mathcal{D} \tilde{\varphi}_a F_a(\tilde{\varphi}_a, \varphi_a) \exp \left( -\sum_{b \sim a} \frac{\lambda_{ba}}{2} \tilde{\varphi}_{ba} \varphi_{ba} \right) \right) \left( \prod_{\{a,b\} \in \mathcal{G}_1} \phi_{ab}/(-H_{ab} H_{ba}) \right) \] (B11)
Comparing Eq. (B11) with Eq. (17) we conclude that BP expressions in the Fermi- and Bose- cases are fully equivalent, i.e.
\[ Z_{BP; \text{Bose}} \cdot Z_{BP; \text{Fermi}} = 1. \] (B12)

To summarize, we have shown in this Appendix that the Bose-BP equations of the standard Gaussian Graphical model, e.g., discussed in [12 13 14 15 16 17 18 19 20], are equivalent to the ones derived within the Fermi-BP approach discussed in the main part of the text. Moreover, the BP partition functions of the Fermi and Bose models are inversely proportional to each other.

One final remark of this Appendix (and of the manuscript) concerns the reconstruction of the full expression for the inverse determinant from BP. The “ground state” part of the BP-gauge is fully defined by Eqs. (B9) [B10], however, a freedom in calculating the “excited state” gauges \( Q^{(n)} \) still remains. The \( Q^{(n)} \) gauges can be fixed in the spirit of [11], resulting in an infinite loop tower hierarchy for the inverse determinant. Note, that this infiniteness is in contrast with the finiteness of the Loop Series, built in the main part of the manuscript for the determinant of the same matrix. Here we do not discuss the Loop Tower reconstruction for the inverse determinant, leaving the question of possible relation between the aforementioned infinite and finite series open for future investigations.