Simulating dam-break over an erodible embankment using SWE-Exner model and semi-implicit staggered scheme

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Abstract. The impact of a dam-break wave on an erodible embankment with a steep slope has been studied recently using both experimental and numerical approaches. In this paper, the semi-implicit staggered scheme for approximating the shallow water-Exner model will be elaborated to describe the erodible sediment on a steep slope. This scheme is known as a robust scheme to approximate shallow water-Exner model. The results are shown in a good agreement with the experimental data. The comparisons of numerical results with data experiment using slopes $\Phi = 59.04$ and $\Phi = 41.42$ by coefficient of Grass formula $Ag = 2 \times 10^{-5}$ and $Ag = 10^{-5}$ respectively are found the closest results to the experiment. This paper can be seen as the additional validation of semi-implicit staggered scheme in the paper of Gunawan, et al (2015).

1. Introduction

Floods from heavy rains can create flood waves that can damage or even destroy river dikes in the form of sediments. The propagation of flood waves that pass through such erosion-like areas are greatly influenced by water flow and sediment. The interaction of flood with the sediments causes damage to the surrounding environment, even can threaten human life. Therefore, a simulation is needed to find out more about the influence of water waves on the sediment such that the possible impacts can be detected and prevented earlier.

Experimental activities to find out more about the impact of water flow on sediments have been discussed and conducted at the LIA Lab (Laboratory of Water Engineering), University of Cassino and Southern Lazio, 2016 [1, 2]. The configurations of this experiment is shown in Figure 1.

In the paper of Di Cristo et al [1], numerical solutions using mixed cell-centered finite-volume (CCFV) and node-centred finite-volume (NCFV) discretization methods have been proposed. The results show that the model allows for a logical simulation of free surface elevation experimental trends independently of the geofailure operator. In this paper, the another model will be used to simulate this experiment. This model is known as the SWE-Exner model which has been applied in many applications such as erodible dambreak, subcritical steady and transcritical flow over a bump, etc.
Figure 1. The lateral view of the experiment with the parameters $H_0 = 0.20$ m, $H_S = 0.10$ m, $L_u = 3.00$ m, $L_d = 3.03$ m and $L_1 = 1.00$ m. The value of $\Phi$ are depend on $X_{TOP}$ and $X_{TOE}$.

Moreover, in order to approximate the solution of SWE-Exner model, a robust semi-implicit staggered grid scheme will be used (this scheme is introduced in [3] in detail). This scheme is a very appropriate method for estimating the Exner-shallow water equation for bedload sediment modeling. Several numerical tests in [3] are shown in good convergence properties to analytical solutions and match well-designed data experiments in dam-break on an erodible embankment.

2. The 1D SWE-Exner and Staggered Scheme

2.1. Water-bedload sediment model

Sediment transport model by fluid flow can be described with a couple of hydrodynamic and morphological sediment change model. Hydrodynamic component can be used to simulate the fluid flow, in this case shallow water equations can be considered. Meanwhile, the morphological sediment change can be given by transport equation.

In hydrodynamic model, the shallow water equations are written as follows

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0,$$  \hspace{1cm} (1)

$$\frac{\partial q^2}{\partial t} + \frac{\partial}{\partial x} \left( q^2 \frac{1}{h} + \frac{1}{2}gh^2 \right) + gh \frac{\partial (z - H)}{\partial x} + S_f = 0,$$  \hspace{1cm} (2)

where $h(x,t)$ is the water depth, $q(x,t) = h(x,t)u(x,t)$ the discharge, $u(x,t)$ the average velocity, $H$ the fix bedrock bottom, $z(x,t)$ the sediment, $S_f = \eta |u|_u R_h^{-\frac{1}{4}}$ the friction term, and $R_h$ is hydraulic ratio that can be approached by $h$ [4]. The numerical methods for approximating the solution of Equations (1) and (2) are available in several references [5, 6, 7, 8].

Here, the morphodynamic model is given as follows:

$$\frac{\partial z_b}{\partial t} + \xi \frac{\partial q_b}{\partial x} = 0,$$  \hspace{1cm} (3)
Figure 2. Sketch of 1D shallow water equations and their primitive variables.

where \( Q \) is solid transport discharge, where \( \xi \) obtained from \( \xi = (1 - \phi)^{-1} \), where \( \phi \) is the size of the empty space between the basic sediments (bed sediment porosity)[3].

There are many empirical formulas from some researchers, one of them is called Grass formula,

\[
q_b = A_g \frac{q}{h} \frac{m_g - 1}{m_g - 1}, \quad 1 \leq m_g \leq 4,
\]

where \( m_g \) value commonly used is \( m_g = 3 \), so that

\[
Q = A_g u^3.
\]

where \( A_g \) range from 0 and 1. The interaction between the sediment and the fluid becomes smaller if the value of \( A_g \) decreases, and vice versa.

2.2. Semi-implicit staggered grid scheme

In this paper, the semi-implicit staggered grid scheme will be used to approximate the solution of water-sediment model (1-3). This numerical scheme is introduced in the paper of Gunawan P. H. et al [3] and Doyen et al [8]. Following are the discrete properties of staggered grid scheme for SWE-Exner model:

- Time domain: the time interval \((0, T)\) is divided by time step \(\Delta t\) such that the discrete time \(t^n\) is obtained for all \(n \in 0, ..., N_t, t^n := n\Delta t\).
- Space domain: the domain \(\Omega := (0, L)\) is divided into \(N_x\) cells by uniform discrete space step \(\Delta x = L/N_x\).
- Full grid: variables \(h_i\) and \(z_i\) are defined in the full grids \(i \in \mathcal{M} := \{1, ..., N_x\}\).
- Half grid: the variables \(u_i\) and \(q_i\) are defined in the half grids \(i \in \mathcal{E}_{int} := \{1, ..., N_x - 1\}\).
- The rest of sets is defined as \(\mathcal{E}_b := \{0, N_x\}\), and \(\mathcal{E} := \mathcal{E}_{int} \cup \mathcal{E}_b\).

The space discretization can be shown in Figure 3.

Giving the average of the initial conditions by,

\[
h_i^0 = \frac{\int_{x_{i-1/2}}^{x_{i+1/2}} h_0(x) dx}{\Delta x}, \quad \forall i \in \mathcal{M},
\]
Figure 3. 1D staggered grids. The orange area represents volume control for mass conservation, association with unknown $h_i$ and $z_i$. Volume control for momentum conservation ($u_{i+\frac{1}{2}}$) shown in red area.

$$z_i^0 = \frac{\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} z_0(x)dx}{\Delta x}, \quad \forall i \in \mathcal{M},$$

and

$$u_{i+\frac{1}{2}}^0 = \frac{\int_{x_i}^{x_{i+1}} u_0(x)dx}{\Delta x}, \quad \forall i \in \varepsilon_{int}.$$  

The discretization of mass and momentum conservation is given as

$$h_{i+1}^{n+1} - h_i^n + \frac{\Delta t}{\Delta x}(q_{i+\frac{1}{2}}^n - q_{i-\frac{1}{2}}^n) = 0, \quad \forall i \in \mathcal{M},$$

and

$$h_{i+\frac{1}{2}}^{n+1}u_{i+\frac{1}{2}}^{n+1} - h_{i+\frac{1}{2}}^nu_{i+\frac{1}{2}}^n + \frac{\Delta t}{\Delta x}[q_{i+1}^n\hat{u}_{i+1}^n - q_i^n\hat{u}_i^n + \frac{1}{2}g((h_{i+1}^{n+1})^2 - (h_i^{n+1})^2) + gh_{i+\frac{1}{2}}^n(z_{i+1}^n - z_i^n) - gh_{i+\frac{1}{2}}^n(H_{i+1} - H_i)] + \Delta tS_{fi+1/2}^n = 0, \quad \forall i \in \varepsilon_{int},$$

respectively. The momentum balance equation is discretized with explicit upwind fluxes for the convection term and implicit centered fluxes for the pressure and topography term. The fluxes $q_{i+\frac{1}{2}}^n$ and $q_i^n$ can be found in the paper [3] in detail.

The friction term $S_{fi+1/2}^n$ form is given in the case of Manning’s friction (see [9]) by

$$S_{fi+1/2}^n = k^2|u_{i+\frac{1}{2}}^n|u_{i+\frac{1}{2}}^{n+1}\left(h_{i+\frac{1}{2}}^{n+1}\right)^{-\frac{3}{2}},$$

and at the case of Darcy-Weisbach’s friction by

$$S_{fi+1/2}^n = f|u_{i+\frac{1}{2}}^n|u_{i+\frac{1}{2}}^{n+1}\left(8gh_{i+\frac{1}{2}}^{n+1}\right)^{-1}. $$

At the sediment equation, the discretization of Exner equation is given as

$$z_{i+1}^{n+1} - z_i^n + \frac{\Delta t}{\Delta x}\xi(Q_{i+\frac{1}{2}}^{n+1} - Q_{i-\frac{1}{2}}^{n+1}) = 0, \quad \forall i \in \mathcal{M},$$

where $Q_{i+\frac{1}{2}}^{n+1} = Q_s(h_{i+\frac{1}{2}}^{n+1}, u_{i+\frac{1}{2}}^{n+1}).$

Therefore, since sediment transport has no effect on fluid flow, sufficient stability conditions are obtained by determining conditions Courant-Friedrichs-Lewy (CFL) only on fluid part. For the Courant number $\nu \geq 1$, this condition is expressed as
\[
\max_{i \in M} \left( \sqrt{gh_i^n} + \frac{|q_{i-\frac{1}{2}}^n + q_{i+\frac{1}{2}}^n|}{2h_i^n} \right) = \frac{\nu \Delta x}{\Delta t}. \tag{14}
\]

3. Flowchart algorithm

This is the flowchart algorithm of the simulation. There are some steps that must be done so the simulation works well. It can be seen in Figure 4.

**Figure 4.** Flowchart algorithm of Dam-break over an erodible embankment Simulation.

Following is the explanations of algorithm which can be found in Figure 4.

- The first step is to set the value of variables for the initial conditions. There are 3 values of \( A_g \) and 2 slopes \( \Phi \) which will be compared. Other variables are set according to Figure 1.
- After all variables initialized, compute the value of \( \Delta t \) at the beginning of all iterations by using (14). The value of this \( \Delta t \) may be different for each iterations depend on the stability criteria.
- The next step is to compute the value of \( h_{i+\frac{1}{2}}^{n+1} \) that can be computed by using the \( \Delta t \) and other variables as in water conservation equation (9).
- After that, compute the value of \( h_{i+\frac{1}{2}}^{n+1} \) and update the value of \( u_{i+\frac{1}{2}}^{n+1} \). This is the part of semi-implicit scheme because to compute the value of \( u_{i+\frac{1}{2}}^{n+1} \), the value updated water height \( h_{i+\frac{1}{2}}^{n+1} \) is used instead of old water height \( h_{i+\frac{1}{2}}^n \). This momentum balance equation can be seen in (10).
- After all iteration done, the last step is to plot those variables and compare it to the experimental result.
4. Numerical result vs data experiment
Here, the numerical results of SWE-Exner model using staggered scheme will be compared with the data experiment obtained in [1].

**Figure 5.** The comparison of numerical result and data experiment dambreak over an erodible embankment [1]. Figures (a), (b) and (c) are the comparison results in slope $\Phi = 59.04^\circ$ using $A_g = 9 \times 10^{-6}$, $A_g = 10^{-5}$ and $A_g = 2 \times 10^{-5}$ respectively. Figures (d), (e) and (f) are the comparison results in slope $\Phi = 41.42^\circ$ using $A_g = 9 \times 10^{-6}$, $A_g = 10^{-5}$ and $A_g = 2 \times 10^{-5}$ respectively.

In this simulation, the final time $t = 60$ s is used. Figures 5(a), (b), and (c) are obtained using various $A_g$ for a slope of initial sediment $\Phi = 59.04^\circ$. From the results, the numerical
results of sediment are shown in a good agreement with the data experiment. Moreover, the
discrete norm error for each simulations is given in Table 1. The error is calculated by

$$||Error|| = \sum_{k=1}^{N_x} |Z_k - Z_{experiment}(k)|,$$

where $Z_{experiment}(k)$ is the sediment profile of experiment.

Figure 5(c) is shown the best result compared with the data experiment within slope
$\Phi = 59.04^\circ$. The discrete norm error of Figure 5(c) is obtained 0.093389 (see Table 1).

In another comparison, Figures 5(d), (e), and (f) show the comparison of numerical
simulations and data experiment for slope $\Phi = 41.42^\circ$. The best result is observed using
$A_g = 1 \times 10^{-5}$ which is the closest profile to the experimental results. Here, the discrete
norm error using $A_g = 1 \times 10^{-5}$ in Table 1 is given 0.073335.

| No | Figure 5 | $||Error||$ |
|----|----------|------------|
| 1  | (a)      | 0.128325   |
| 2  | (b)      | 0.095005   |
| 3  | (c)      | 0.093389   |
| 4  | (d)      | 0.084301   |
| 5  | (e)      | 0.073335   |
| 6  | (f)      | 0.094042   |

5. Conclusion
The numerical simulation of erodible dambreak embankment in two different slopes are given.
The results are obtained using SWE-Exner equation which is discretized by semi-implicit
staggered scheme. The results show that the sediment profile of numerical result are match
very well with the data experiment. Here, the value of coefficient Grass formula $A_g$ becomes the
main influence of the results. In the simulations with both slopes $\Phi = 59.04^\circ$ and $\Phi = 41.42^\circ$
the closest results are obtained using $A_g = 2 \times 10^{-5}$ and $A_g = 1 \times 10^{-5}$ respectively. The discrete
norm error for using slope $\Phi = 59.04^\circ$ and $A_g = 2 \times 10^{-5}$ is obtained 0.093389. Meanwhile,
using slope $\Phi = 41.42^\circ$ and $A_g = 1 \times 10^{-5}$ the discrete norm error is observed 0.073335.

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