Spherically symmetric, static black holes with scalar hair, and naked singularities in nonminimally coupled k-essence

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(Dated: October 4, 2022)

We apply a recently developed 2+1+1 decomposition of spacetime, based on a nonorthogonal double foliation for the study of spherically symmetric, static black hole solutions of Horndeski scalar-tensor theory. Our discussion proceeds in an effective field theory (EFT) of modified gravity approach, with the action depending on metric and embedding scalars adapted to the nonorthogonal 2+1+1 decomposition. We prove that the most generic class of Horndeski Lagrangians compatible with observations can be expressed in this EFT form. By studying the first order perturbation of the EFT action we derive three equations of motion, which reduce to those derived earlier in an orthogonal 2+1+1 decomposition, and a fourth equation for the metric parameter $N$ related to the nonorthogonality of the foliation. For the Horndeski class of theories with vanishing $G_3$ and $G_5$, but generic functions $G_2(\phi, X)$ (k-essence) and $G_4(\phi)$ (nonminimal coupling to the metric) we prove the unicity theorem that no action beyond Einstein–Hilbert allows for the Schwarzchild solution. Next we integrate the EFT field equations for the case with only one independent metric function and we obtain new solutions characterized by a parameter, which in the simplest cases has the interpretation of mass or tidal charge, the cosmological constant and a third parameter. These solutions represent naked singularities, black holes with scalar hair or have the double horizon structure of the Schwarzchild–de Sitter spacetime. Solutions with homogeneous Kantowski–Sachs type regions also emerge. Finally, one of the solutions obtained for the function $G_4$ linear in the curvature coordinate, in certain parameter range exhibits an intriguing logarithmic singularity lying outside the horizon. The newly derived hairy black hole solutions evade previously known unicity theorems by being asymptotically nonflat, even in the absence of the cosmological constant.

I. INTRODUCTION

Extensions of general relativity are well-motivated by its inability to match astrophysical and cosmological observations unless dark matter and dark energy are introduced, both undetectable otherwise than gravitationally; the need for an inflationary universe with at least one scalar added to the gravitational sector; or the study of the infrared limit of the yet to be developed quantum theory of gravity. The addition of a scalar field $\phi$ to the metric tensor as a gravitational variable is among the simplest possible modifications. In order to avoid Ostrogradski instabilities, such theories should be of second differential order for both the scalar and the tensor. This condition is satisfied by the Horndeski class of theories \cite{1,2}. A more encompassing class of admissible theories restrict to second order only the evolution of the degrees of freedom \cite{3,4}. As the validity of general relativity has been accurately confirmed on the Solar System scale, there is need for a mechanism to switch on or off any of its modifications only at a larger radius. The viability of such a Vainshtein mechanism restricts the Horndeski class \cite{5,7}. The recent confirmation, at least at LIGO frequencies, that gravitational waves propagate with the speed of light \cite{8} adds further restrictions \cite{9,12}, leaving as viable theories only the so-called generalized kinetic gravity braiding \cite{13,14} subset of Horndeski theories. This subset however is still fairly rich, especially concerning the dynamics of the scalar.

Such models were extensively investigated in a cosmological context, including the junction along spacelike hypersurfaces \cite{13,14}. Similar junction conditions hold for timelike hypersurfaces, while techniques developed for null hypersurfaces in general relativity \cite{17,18} were applied \cite{19} for the case of null junction hypersurfaces of kinetic gravity braiding models.

Black hole solutions were also sought for in these type of scalar-tensor theories. The simplest of them is the Brans–Dicke theory \cite{20}, in which the gravitational constant scales with $\phi^{-1}$ (where $\phi$ is a massless scalar field) and a kinetic term $X = \partial^a \phi \partial_a \phi$ couples trough $\omega \phi^{-1}$ (with $\omega$ a parameter) to the curvature part of the action. Penrose has suggested as early as 1968 that gravitational collapse (discussed in the Einstein frame) proceeds similarly \cite{21}; hence, Brans–Dicke black holes are identical to general relativistic ones \cite{22} and his no-hair conjecture has been supported in the large $\omega$ expansion scheme \cite{23}. Hawking had proven in 1972 that stationary black hole solutions of the Brans–Dicke theory exist only for a constant scalar; hence, they are also solutions of the Einstein field equations \cite{24}. Key in the derivation is the existence of a horizon, hence this result does not forbid stationary stellar solutions different from their general relativistic counterparts. In fact, Brans has found in 1962 all spherically symmetric, static vacuum solutions of the Brans–Dicke theory \cite{25,26}. However, none of those with $\phi \neq \text{const.}$ possess event horizons; instead they have naked singularities \cite{22}. Solutions were also found in the stationary, axisymmetric case as generalizations of the Kerr–Newman family of solutions to the Brans–Dicke–Maxwell scenario \cite{27,28}, and also solutions corresponding to the observer’s two modes of description of the static electromagnetic field: as axially symmetric mag-
netic field or axially symmetric radial electric field \[30\]. The Kerr–Newman type solution was further analyzed \[31\]. Interestingly, when switching off both the rotation and the electric charge, it fails to become spherically symmetric, a quadrupolar deformation (disappearing for \(\omega \to \infty\), the general relativistic limit) being induced by the scalar field, which itself exhibits a quadrupolar deformation from spherical symmetry. The analysis has shown that the curvature invariants \(R, R_{abjk}\) and the Kretschmann scalar \(R_{abcd}R^{abcd}\) vanish at the horizon candidate \(\Delta = 0\) when \(\omega \in (-5/2, -3/2)\), while they diverge otherwise, and concluded that the Kerr–Newman type solutions of the Brans–Dicke–Maxwell theory in the above range represent black holes with scalar hair. However the rest of the curvature invariants (there are 17 in total, related algebraically by 8 syzygies; see Ref. \[32\]) were not checked. They also remained unchecked for a spherically symmetric and static black hole candidate in Brans–Dicke theory proposed in Ref. \[33\], based on the regularity (vanishing) of the Kretschmann scalar on the horizon. The collapse of collisionless matter to a black hole in Brans–Dicke theory has been also investigated numerically \[34\]. Contrary to general relativity, in Brans–Dicke theory the Oppenheimer–Snyder collapse leads to a dynamical black hole, in which the condition \(R_{ab}l^a l^b \geq 0\) (for all null vectors \(l^a\)) assumed in the derivation by Hawking can be violated; nevertheless, the end result is the same as in general relativity \[35\].

The no-hair theorem for asymptotically flat, static, spherically symmetric black holes in Brans–Dicke theory has been extended to multicomponent scalar field configurations \[36\]. Numerical evidence ruling out spherical scalar hair of static four-dimensional black holes has been presented for scalar fields satisfying the Positive Energy Theorem, with a potential derivative from a superpotential motivated by supergravity \[37\].

Brans–Dicke theory has been generalized to include a potential (allowing for massive scalar fields) and a scalar field dependent coupling \(\omega(\phi)\) (lifting the constraint \(\omega > 40000\) established for a constant \(\omega\) from the frequency shift of radio photons to and from the Cassini spacecraft as they passed near the Sun \[38\]). Question comes whether in this class of scalar-tensor theories stationary black holes different from general relativistic ones could exist.

This has been investigated by Sotiriou and Faraoni, who extended Hawking’s result to stationary black holes in this class of theories \[39\]. When there is no potential, however \(\omega = \omega(\phi) \neq -3/2\) and does not diverge, Hawking’s original proof still holds. Otherwise they show, that for stationary and isolated black holes (asymptotically flat and with a constant \(\phi_0\) value for the scalar field at spatial infinity, implying the vanishing of the potential at infinity through the tensorial equations of motion), by imposing linear stability for the scalar in the Einstein frame, the scalar field ought to be a constant. Thus the stationary solutions of these generalized Brans–Dicke theories include only the general relativistic black holes. By employing a 1+1+2 decomposition based on kinematical quantities, in the particular case of a Klein–Gordon scalar field \(\omega(\phi) = \phi/2\) with arbitrary potential and coupled nonminimally to the metric, Ref. \[40\] confirmed, that the Schwarzschild solution implies a constant scalar field.

Hawking’s no hair theorem in the particular case of spherically symmetric, static black holes was also generalized for a wide class of Hordenski theories dubbed as Galileon, which are invariant under the shift \(\phi \to \phi + \text{const.}\) of the scalar field \[41\], as the radial component of the Noether current vanishes \(J^r = 0\) implying in general the vanishing of the radial derivative of the scalar field \(\phi'(r) = 0\); hence, a constant scalar. A notable exception to this proof was presented in Ref. \[42\], where a nontrivial Galileon couples to the Gauss–Bonnet invariant. Another exception arises when the Galileon couples to the Einstein tensor \(G_{ab}\) (in the form \(G_{ab} \partial_\phi \partial_\phi \phi\)), with the metric either does (has primary hair) or does not deviate from the general relativistic solutions in the presence of a nontrivial scalar (representing secondary hair) \[43\]. This includes the Schwarzschild black hole unaffected by a time and radial dependent scalar, an example of a stealth black hole. \[2\] Other hairy black holes with derivative coupling (to the Einstein tensor) were found in Refs. \[44\]–\[46\], while for coupling to the Gauss–Bonnet invariant in Refs. \[47\]–\[50\]. Neither of these exceptional cases are however observationally preferred, as the generalized kinetic gravity braiding subset has \(G_4(\phi) = 0\).

Both black holes with primary hair and stealth solutions with nontrivial scalar have been identified in the framework of generalized kinetic gravity braiding subclass with nonvanishing \(G_3\) \[51\]. They evade the no-hair theorem of Ref. \[41\] also due to avoiding \(\phi'(r) = 0\) while \(J^r = 0\) holds.

Early investigations of stellar solutions showed that certain scalar-tensor theories could pass the weak-field tests; however, the predictions in strong field would differ from the general relativistic ones, opening the opportunity of testing them with double pulsar experiments \[52\]–\[54\]. Indeed, the scalar field is sourced by the trace of the energy-momentum tensor of the neutron star and this spontaneous scalarization affects the dynamics of the double pulsar. The existence and modifications of spherically symmetric and static neutron stars in both \(f(R)\) gravity and Brans-Dicke theory with scalar po-

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1. In terms of the Hordneski coefficients \(G_i(\phi, X)\) \((i = 2, ..., 5)\), where \(X = \partial^\phi \partial_\phi \phi\) is the kinetic term, the occurrence of the \(G_i \partial_\phi \partial_\phi X\) coupling requires \(\partial G_i/\partial X \neq 0\), while for the linear coupling to the Gauss–Bonnet invariant, \(G_5\) has to include a contribution proportional to \(\ln|X|\).

2. Other examples of stealth black holes, with the same metric as in general relativity, but nonconstant scalar field configurations were discussed in Refs. \[44\]–\[46\]. Both solutions with primary hair or secondary hair only (stealth black holes) were identified in beyond Hordenski theories \[51\].
tential were investigated for two particular neutron star equations of state [71]. The stability of spherically symmetric solutions of the cubic covariant Galileon model with a matter source has been studied in the test scalar field approximation [72], while the stability of relativistic stars composed of perfect fluid minimally coupled in the Jordan frame has been analyzed in Ref. [74] for a subclass of Horndeski theories with linear dependence on the kinetic term.

In discussing geometries with spherical symmetry, it is natural to single out a radial direction. Temporal evolution selects another direction; hence, a $2+1+1$ decomposition of spacetime could be useful. Such a formalism was developed in Refs. [74, 75] and explored in the context of braneworlds. It was based on geometrical quantities characterizing the embedding of the 2-surface: extrinsic curvatures, normal fundamental forms and normal fundamental scalars constructed with both singled-out vectors. Some of them were related to temporal and radial derivatives of the metric, playing an important role in the Hamiltonian treatment. The formalism of Refs. [74, 75] was based on an orthogonal double foliation. This unnecessary restriction was lifted in our previous work, Ref. [76], allowing for the nonorthogonal double foliation of spacetime. Hence, a new degree of freedom, a measure of the nonorthogonality, represented by the metric function $N$, emerged, reestablishing generic gauge invariance. The price to pay was that the number of geometric embedding variables is doubled (as there are two orthogonal bases adapted to each hypersurface normal); nevertheless the two sets were expressible in terms of each other.

In this paper we apply this $2+1+1$ decomposition of spacetime, based on a nonorthogonal double foliation for the study of spherically symmetric, static black hole solutions of Horndeski scalar-tensor theories [77]. As a check, the Schwarzschild solution is recovered through these EFT equations from the Einstein–Hilbert Lagrangian, rewritten in terms of the embedding and metric scalars. For comparison, in Appendix C we present the alternative set of equations of motion obtained in the basis adapted to the temporal foliation.

In Sec. VI we give the equations of motion for spherically symmetric, static backgrounds in terms of generic Horndeski functions $G_2(\phi, X)$ and $G_4(\phi)$ (but vanishing $G_3$ and $G_5$) and prove the unicity theorem that no action beyond Einstein–Hilbert allows for the Schwarzschild solution in this class. With this, we generalize the previously announced unicity theorem [39] for the case of spherical symmetry and staticity.

In Sec. VII we discuss the EFT field equations for the case with only one independent metric function and we formally integrate them to obtain the metric function in terms of the nonvanishing Horndeski function $G_4(\phi)$.

In Sec. VIII we obtain new solutions characterized by a parameter, which can be interpreted as mass or tidal charge in the simplest particular cases, the cosmological constant and a third parameter, emerging for various choices of $G_4(\phi)$. These solutions represent naked singularities, hairy black holes or have the double horizon structure of the Schwarzschild–de Sitter spacetime. Solutions with homogeneous Kantowski–Sachs type regions also emerge.

In Sec. IX we rewrite these solutions as the conformally related metrics in the Einstein frame. Finally we give the concluding remarks in Sec. X.

II. NONORTHOGONAL $2+1+1$ SPACETIME DECOMPOSITION: A SUMMARY

This section summarizes the quantities and notations applied in the nonorthogonal $2+1+1$ decomposition of doubly foliable spacetimes worked out in Ref. [76]. The spacetime is foliated by 3-dimensional spacelike $\mathbb{M}_X$ ($\chi =$ const.) and timelike $\mathcal{S}_t$ ($t =$ const.) hypersurfaces, whose 2-dimensional intersection is $\Sigma_{tx}$. The 4-dimensional metric can be decomposed as

$$\tilde{g}_{ab} = -n_a n_b + m_a m_b + g_{ab}$$

$$= -k_a k_b + l_a l_b + g_{ab} .$$

(1)

Here $n_a$ and $l_a$ are normals to $\mathcal{S}_t$ and $\mathbb{M}_X$, respectively. The metric tensor on $\Sigma_{tx}$ is $g_{ab}$, while the 1-form $m_a (k_a)$ is perpendicular to both $\Sigma_{tx}$ and $n_a (l_a)$.

The tangent vectors of the coordinate lines $t$ and $\chi$ in the $(n^a, m^a)$ basis can be given as

$$\left( \frac{\partial}{\partial t} \right)^a = N n^a + N^a + N m^a ,$$

(2)

$$\left( \frac{\partial}{\partial \chi} \right)^a = M m^a + M^a .$$

(3)

3 We do not deal here with the most complicated contribution $L_5$, as it makes the Vainshtein mechanism unfeasible.
Here $N$ is the lapse function while $N^a$ (obeying $N^am_a = 0$) and $\mathcal{N}$ are the components of the 3-dimensional shift vector. In addition $M$ is the lapse function of $\partial/\partial \chi$ in $\mathcal{S}_t$ and $M^a$ is its 2-dimensional shift vector which is tangent to $\Sigma_t \chi$. The decompositions of $\partial/\partial t$ and $\partial/\partial \chi$ in the $(k^a, l^a)$ basis are
\[
\frac{\partial}{\partial t} = N k^a + N^a, \quad \frac{\partial}{\partial \chi} = M (-\mathfrak{s} k^a + \mathfrak{c} l^a) + M^a,
\]
where $\mathfrak{s} = \sinh \psi$, $\mathfrak{c} = \cosh \psi$ are defined by the Lorentz-rotation between the two bases,
\[
\left(\begin{array}{c}
k^a \\
l^a
\end{array}\right) = \left(\begin{array}{cc}
\mathfrak{c} & \mathfrak{s} \\
\mathfrak{s} & \mathfrak{c}
\end{array}\right) \left(\begin{array}{c}
n^a \\
m^a
\end{array}\right),
\]
with rapidity $\psi$ obeying
\[
\mathcal{N} = N \tanh \psi.
\]

The covariant derivatives of the normals $n^a$, $l^a$ to the hypersurfaces are decomposed in their naturally associated bases as
\[
\nabla_a n_b = K_{ab} + 2m_b K_{cb} + m_a m_b \mathcal{L}^* - n_a D_b (\ln N),
\]
\[
\nabla_a l_b = L_{ab} + 2k_b l_{cb} + k_a k_b \mathcal{K}^* - l_a D_b (\ln \mathcal{M}),
\]
Here we have introduced $D$-derivatives representing co-derivatives of any 4-dimensional tensor $T_{b_1 \ldots b_q}$ projected onto $\Sigma_{t \chi}$ as
\[
D_a T_{b_1 \ldots b_q} \equiv g^c_{b_1} \ldots g^{a_r}_{b_q} \nabla_c \ldots \nabla_{a_r} T_{c_1 \ldots c_r}.\tag{10}
\]
The 2-tensors $K_{ab} \equiv D_a n_b$ and $L_{ab} \equiv D_a l_b$ are the extrinsic curvatures of $\Sigma_{t \chi}$, while
\[
K_a \equiv g^c_\alpha m^d \nabla_c n_d \equiv g^c_\alpha m^d \nabla_c n_d,
\]
\[
L_a \equiv -g^c_\alpha k^d \nabla_c l_d = -g^c_\alpha k^d \nabla_c l_d = K_a + D_a \psi
\]
are normal fundamental forms, and
\[
K \equiv m^d m^e \nabla_d n_e, \quad \mathcal{L} \equiv k^d k^e \nabla_d l_e, \tag{12}
\]
the normal fundamental scalars defined by the hypersurface-orthogonal vectors $n^a$ and $l^a$. The corresponding quantities for the basis vectors $k^a$ and $m^a$ are\footnote{The embedding variables of the $\Sigma_{t \chi}$ surface with respect to the normals of the hypersurfaces $n^a$, $l^a$, and those with respect to the complementary orthogonal basis vectors $m^a$, $k^a$ will be distinguished by a star on the latter set.}
\[
K^* \equiv l^d l^e \nabla_d k_e, \quad \mathcal{L}^* \equiv n^d n^e \nabla_d m_e. \tag{13}
\]
The vectorial and tensorial embedding variables generate additional scalars:
\[
\mathcal{R} \equiv K^a K_{ab}, \quad \lambda \equiv L^a L_{ab}, \quad \mathcal{R}^* \equiv K^a K_{a}, \quad \mathcal{T} \equiv \mathcal{L}^a L_{a}, \quad K \equiv K_a, \quad L \equiv L_a. \tag{14}
\]
The covariant derivatives of the complementary basis vectors $m^a$, $k^a$ can also be decomposed in their naturally associated bases,
\[
\tilde{\nabla}_a k_b = K_{ab} + l_a K^* + l_b L + l_a l_b K^* + k_a l_b \mathcal{L} - k_a D_b \left(\ln \mathcal{N}\right),
\]
\[
\tilde{\nabla}_a m_b = L_{ab} + n_a L^* + n_b K + n_a n_b \mathcal{L}^* + n_a k_b \mathcal{K} + n_b D_a (\ln \mathcal{M}),
\]
in terms of the extrinsic curvatures $K^*_{ab} \equiv D_a k_b$, $L^*_{ab} \equiv D_a m_b$ and quantities defined in a similar manner to the normal fundamental forms,
\[
K^*_a \equiv g^d_\alpha l^e \nabla_c k_{d e}, \quad \mathcal{L}^*_a \equiv -g^d_\alpha c^e \nabla_c m_{d e}. \tag{17}
\]
Scalars can be formed once again from the embedding tensors and vectors as
\[
\mathcal{R}^* \equiv K^{ab} K^*_{ab}, \quad \lambda^* \equiv L^{ab} L^*_{ab}, \quad \mathcal{R}^* \equiv K^{ab} K^*_{a}, \quad \mathcal{T}^* \equiv \mathcal{L}^a \mathcal{L}^*_a, \quad K^* \equiv K^*_a, \quad L^* \equiv L^*_a. \tag{18}
\]
The scalars introduced for the two bases are interconnected; the starry ones can be expressed in terms of their unstarred versions together with the metric functions and their derivatives, as shown in Appendix A.

III. NONORTHOGONAL 2+1+1 DECOMPOSITION OF THE HORNDESKI ACTION

We assume that the scalar has nowhere vanishing gradient and it solely depends on $\chi$; hence, it defines a foliation through the $\phi=\text{const}$ level hypersurfaces. Hence the normal to the $\chi=\text{const}$ hypersurfaces can be expressed in terms of the scalar field gradient,
\[
l_a = \nabla_a \phi, \tag{19}
\]
where $X = g^{ab} \nabla_a \phi \nabla_b \psi$ is the kinetic term of the scalar field.

Time evolution along $\partial/\partial t$ proceeds on the $\chi=\text{const}$ hypersurfaces. In a perturbational setup this can be insured both on the background and after perturbation, by absorbing the scalar field variation into a proper gauge choice, dubbed radial unitary gauge \cite{76,77}.
Although for discussing the general relativistic Hamiltonian dynamics the basis \((n^a, m^a)\) turned more advantageous, as fewer embedding variables were related to time derivatives of the metric components \([76]\), Eq. \([11]\) clearly indicates that for the purpose of monitoring spherically symmetric, static configurations in scalar-tensor gravity the \((k^a, l^a)\) basis is better suited.

Hence we proceed in rewriting the second covariant derivative of \(\phi\) in the \((k^a, l^a)\) basis,

\[
\nabla_a \nabla_b \phi = \frac{\tilde{\nabla}^a \phi \nabla_a X}{2X} l_a l_b + \sqrt{X} [L_{ab} + L k_a k_b] + 2K^a l_a k_b + 2k_a \mathcal{L}_b - 2l_a D_b \ln (\mathcal{c}M) \]  

(20)

This generalizes Eq. \((4.2)\) of Ref. \([77]\) for the case of nonorthogonal double foliation.

The Horndeski action is

\[
S^H = \int d^4 x \sqrt{-g} L^H ,
\]

where the Lagrangian,

\[
L^H = \sum_{i=2}^{5} L^H_i ,
\]

is a sum of the contributions

\[
L^H_2 = G_2(\phi, X) ,
\]

providing scenarios with accelerated expansion,

\[
L^H_3 = G_3(\phi, X) \tilde{\phi} ,
\]

enabling the Vainshtein screening-mechanism,

\[
L^H_4 = G_4(\phi, X) \tilde{R} - 2G_4X(\phi, X) \times \left[ (\tilde{\phi}^2 - \tilde{\nabla}^a \tilde{\nabla}_a \phi) \nabla_a \nabla_b \phi \right] ,
\]

the first term of which leads to a time-evolving gravitational constant and the much more involved \(L^H_3\), which has been disused by both the requirement of a working gravitational screening \([9, 7]\) and the observation of the propagation of gravitational waves equaling the speed of light \([8]\). In fact \(L^H_4\) was also simplified by the latter observation, disallowing the \(X\)-dependence of \(G_4\). Nevertheless, we keep this dependence for the time being as it does not add considerable difficulty to our calculations.

Now we need to 2+1+1 decompose the respective parts of the Horndeski Lagrangian and rewrite them in terms of the variables employed in the decomposition. The trace of Eq. \((20)\) gives the d’Alembertian

\[
\Box \phi = \frac{\tilde{\nabla}^a \phi \nabla_a X}{2X} + \sqrt{X} (L - \mathcal{L}) \]

(26)

Now \(L^H_3\) can be written in a more convenient form following Ref. \([1]\) by taking

\[
G_3(\phi, X) = F_3(\phi, X) + 2X F_3X(\phi, X) ,
\]

then integrating \(F_3 \tilde{\phi}\) by parts and using \((26)\) in the term \(2XF_3X \tilde{\phi}\). Then \(L^H_3\) reduces to the sum of

\[
L^H_4 = 2X^{3/2} F_3X (L - \mathcal{L}) - F_3X \phi
\]

(28)

This can be rewritten in terms of the scalars defined in Eqs. \((14), (18)\) as

\[
L^H_4 = G_4 \left( R + K^{ab} K^a b - K^{*2} \right) + 2\sqrt{X} G_4X (L - \mathcal{L}) - \left( G_4 - 2X G_4X \right) [L a b \mathcal{L} - 2\mathcal{L}^a \mathcal{L}_a + 2K^* K^*]
\]

(29)

\[
L^H_4 = 4 \left[ (\tilde{\phi}^2 - \tilde{\nabla}^a \tilde{\nabla}_a \phi) \nabla_a \nabla_b \phi \right] ,
\]

the term \(G_3(\phi, X) \tilde{R} - 2G_4X(\phi, X) \times \left[ (\tilde{\phi}^2 - \tilde{\nabla}^a \tilde{\nabla}_a \phi) \nabla_a \nabla_b \phi \right] \),

the expression of the inverse metric \((\mathcal{B}3)\) of Ref. \([74]\) in the coordinate basis. In the Radial unitary gauge \(\phi = \phi(\chi)\), only \(\tilde{g}^{XX} = \left( N^2 - N'^2 \right) / N^2 M^2 = (\mathcal{c}M)^2 \) matters in calculating

\[
X = \tilde{g}^{XX} (\partial_\chi \phi)^2 = \left( \frac{\partial_\chi \phi}{\mathcal{c}M} \right)^2 .
\]

(32)

Beside the embedding variables the induced curvature scalar, calculated from the induced metric \(g_{ab}\), also appears in the formalism. We can assume that \(g_{ab}\) is diagonal, as all 2-metrics can be diagonalized. Further, such a
metric is locally conformally flat. Globally this might not be the case as there may be singular points of the conformal factor. Hence for spherical symmetry we take any 2-metric conformal to the unit sphere, while for cylindrical symmetry to the plane, in both cases with conformal factor \( \exp(2\zeta) \). The two-metric hence is expressible by a conformal exponent \( \zeta \) alone. On the other hand, the 2-dimensional curvature tensor built from this metric has only one independent component, the curvature scalar \( R \), also expressible in terms of the conformal exponent. In summary, the Horndeski Lagrangian (without \( L^R_0 \)) only depends on the generalized coordinates

\[ \mathcal{N}, N, M, K^*, \xi, K^*, x^*, \mathcal{L}, L, \lambda, \zeta, \phi \] (33)

and their derivatives.

On a technical note, the Horndeski Lagrangian derived above is much simpler than the one derivable in the \((n^a, m^a)\) basis, which is presented in the Appendix [B] confirming the rightness of our basis choice.

IV. THE FIRST ORDER VARIATION OF THE EFT ACTION ON SPHERICALLY SYMMETRIC, STATIC BACKGROUND

Having the Horndeski action rewritten in terms of the variables (33) our aim now is to derive the equations of motion by taking the perturbations of the variables to first order about a background. We summarize the procedure to be followed below. Quantities on the background will be denoted by an overbar.

A. First order variation of the action

The variation of the action \( S = \int d^4x \sqrt{-g} L \) to first order formally reads

\[ \delta S = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\partial S}{\partial G} \right) \delta G \] (34)

where \( \delta G \equiv G - \bar{G} \) for any variable \( G \) (either related to the metric and scalar field) on which the action functional depends and \( \delta S/\delta G \) is the functional derivative of the action with respect to \( G \), evaluated on the background. For a first differential order dependence of the action in \( G \) it can be expressed as the Euler–Lagrange expression

\[ \frac{\sqrt{-g}}{\sqrt{-\tilde{g}}} \frac{\delta S}{\delta G} = \frac{\partial (\sqrt{-g} L)}{\partial G} - \partial_a \left( \sqrt{-g} \frac{\partial L}{\partial (\partial_a G)} \right) \] (35)

As \( \sqrt{-g} \) does not depend on the derivatives of any variable,

\[ \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} L)}{\partial (\partial_a G)} = \frac{1}{\sqrt{-g}} \frac{\partial_0 \left( \sqrt{-g} \frac{\partial L}{\partial (\partial_0 G)} \right)}{\partial_0 (\partial_0 G)} = \nabla_a \frac{\partial L}{\partial (\partial_a G)} \] (36)

and the functional derivative [35] can be further expanded as

\[ \frac{\delta S}{\delta G} = \frac{\partial L}{\partial G} - \nabla_a \frac{\partial L}{\partial (\partial_a G)} + L \frac{\partial \ln \sqrt{-\tilde{g}}}{\partial G} \] (37)

This enables us to write the generic form of the variation of the action,

\[ \delta S = \int d^4x \sqrt{-\tilde{g}} \left( \delta L + L \delta \ln \sqrt{-\tilde{g}} \right) \] (38)

with

\[ \delta L = \left( \frac{\partial L}{\partial G} - \nabla_a \frac{\partial L}{\partial (\partial_a G)} \right) \delta G \] (39)

and

\[ \delta \ln \sqrt{-\tilde{g}} = \frac{\partial \left( \ln \sqrt{-\tilde{g}} \right)}{\partial G} \delta G \] (40)

We will employ the expressions [38–40] in what follows.

On a spherically symmetric, static background we replace \( \chi \) by the radial coordinate \( r \) and we denote the derivatives with respect to \( r \) by a prime.

When rewriting the Horndeski Lagrangian into the 2+1+1 formalism, most of the variables appeared only algebraically. Even if the action depends on the derivatives of the variables, hence the last term of Eq. (39) is non-vanishing, when evaluated on the spherically symmetric and static background, only the \( r \)-dependence survives; hence,

\[ \delta L = \left( \frac{\partial L}{\partial G} - \nabla_r \frac{\partial L}{\partial (\partial_r G)} \right) \delta G \] (41)

In what follows we further explore the particularities of the spherically symmetric, static background.

B. Metric and embedding variables on the spherically symmetric, static background

On the spherically symmetric, static background the metric can be chosen as

\[ ds^2 = -\bar{N}^2 dt^2 + \bar{M}^2 dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \] (42)

implying that the following relations hold

\[ \bar{N}^a = \bar{M}^a = \bar{N} = 0 \] (43)

The latter assures that the two bases coincide on the background and

\[ \bar{n}_a = \bar{k}_a = (-\bar{N}, 0, 0, 0) \] (44)

\[ \bar{m}_a = \bar{l}_a = (0, \bar{M}, 0, 0) \] (44)
hence
\[ \bar{K}^s = \bar{\kappa} = \bar{K}^* = \bar{\chi}^s = 0 . \] (45)

Further,
\[ \bar{\bar{L}} = -\frac{\partial_r \bar{N}}{MN} , \]
\[ \bar{L} = \frac{1}{2M} \bar{g}^{ab} \partial_r \bar{g}_{ab} = \frac{2}{Mr} , \]
\[ \bar{\lambda} = \frac{2}{M^2 r^2} , \] (46)

also
\[ \bar{\bar{L}}_{ab} = \frac{1}{2} \bar{\bar{g}}_{ab} \] (47)

hold. From the 2-dimensional metric expressed in polar coordinates, \( \bar{g}_{ab} = r^2 \text{diag}(1, \sin^2 \theta) \), the conformal exponent (relating it to the unit sphere) \( \bar{\zeta} = \ln r \) is found, also the simple expression,
\[ \bar{R} = \frac{2}{r^2} \] (48)
of the 2-dimensional curvature scalar emerges.

**C. The effective field theory type action**

The perturbed 2-dimensional metric and the background metric are related to each other through an infinitesimal conformal factor \( \delta \zeta \) as
\[ g_{ab} = e^{2\delta \zeta} \bar{g}_{ab} ; \] (49)
hence, the variation of the 4-dimensional metric and of the 2-dimensional Ricci scalar, under the assumption of spherical symmetric background emerges as
\[ \delta \ln \sqrt{-\tilde{g}} = 2 \delta \zeta + \delta \ln N + \delta \ln M , \] (50)
and
\[ \delta R = R - \bar{R} = -2 \bar{R} \delta \zeta - 2 \bar{g}^{ab} \bar{D}_a \bar{D}_b \delta \zeta . \] (51)

Based on (50) and assuming for \( \mathcal{L}_{\text{EFT}} \) the same functional dependence holds as for the Horndeski Lagrangians, we arrive to the following effective field theory (EFT) type action
\[ S_{\text{EFT}} [N, N, M, K^*, \bar{\kappa}, K^*, \bar{\chi}^s, \bar{\zeta}, L, \lambda, \xi, \phi] \] (52)
which can be viewed as a low-energy approximation of a yet unknown quantum gravity.

Compared to the action (3.1) of Ref. [17], the EFT action [52] does not contain the dependence of the variable \( \mathcal{M} = M^a M_a \), as it did not appear in the Horndeski action; nevertheless, it includes an explicit dependence of the radial shift \( N \), nonexistent in the formalism of Ref. [17]. Further differences are hidden in the definitions of the variables \( (K^*, \bar{\kappa}, K^*, \bar{\chi}^s) \) which reduce to the corresponding ones \( (\bar{K}, \bar{\kappa}, K, \bar{\xi}) \) of Ref. [17] in the orthogonal double foliation limit.

**V. The field equations on spherically symmetric, static background**

From the definitions of \( \bar{\kappa} \) and \( \bar{\chi}^s \) given in Eqs. (14) and (15), respectively, together with their vanishing on the background the variations \( \delta \bar{\kappa} \) and \( \delta \bar{\chi}^s \) are second order, to be dropped from the first order variation leading to the equations of motion. The \( X \)-dependence of the Horndeski Lagrangians do not depend on the derivatives of \( \bar{N} \) and \( \bar{M} \) have only radial dependence, hence \( \bar{D}_a N \) and \( \bar{D}_a M \) are each first order).

As a result, for the purpose of first order variation we can assume the Lagrangian in the form [we do not explore yet Eq. (21)],
\[ \mathcal{L}_{\text{EFT}} (N, N, M, K^*, \bar{\kappa}, K^*, \bar{\chi}^s, L, \lambda, \xi, \phi, \phi^\prime) , \] (53)
with
\[ \delta \mathcal{L}_{\text{EFT}} = L_{N} \delta N + L_{N} \delta N + L_{M} \delta M + L_{K^*} \delta K^* + L_{\bar{\kappa}} \delta \bar{\kappa} + L_{K^*} \delta K^* + L_{\bar{\chi}^s} \delta \bar{\chi}^s + L_{\bar{\kappa}} \delta \bar{\kappa} + L_{M} \delta M , \] (54)
where for all geometric variables \( \mathcal{G}_B \) we denote
\[ L_{\mathcal{G}_B} = \frac{\partial (L_{\text{EFT}})}{\partial \mathcal{G}_B} \] (55)
and for the scalar
\[ L^\phi_{\phi} = \frac{\partial (L^\phi_{\text{EFT}})}{\partial \phi} - \frac{1}{\sqrt{-g}} \partial_r \left( \sqrt{-\tilde{g}} \frac{\partial (L^\phi_{\text{EFT}})}{\partial \phi^\prime} \right) . \] (56)

These are nothing but the expansion coefficients from Eq. (41), taking into account that there is no dependence on the derivatives of the geometric variables \( \mathcal{G}_B \) of the Lagrangian (53).

Now we proceed to compute the variation of the action with the contributions (14)–(53). From Eqs. (14), (15) and (47), to first order,
\[ \delta \lambda = \frac{2}{M \bar{r}} \delta L , \] (57)
thus,
\[ \int d^4x \sqrt{-\tilde{g}} (L^\text{EFT} \delta L + L^\text{EFT} \delta \lambda) = \int d^4x \sqrt{\mathcal{G}_M} \mathcal{F} \delta L , \] (58)
where we have replaced \( \sqrt{-\tilde{g}} \) by its background value, as \( \delta L \) is already first order and the quantity,
\[ \mathcal{F} = L^\text{EFT} + \frac{2}{M \bar{r}} L^\text{EFT} , \] (59)
defined similarly as in Ref. [77], is evaluated on the background.

Next from Eqs. (49) and (50)
\[ L = \nabla_a l^a + \bar{\mathcal{L}} + \delta\mathcal{L} , \]
\[ \delta\mathcal{L} = \left( \nabla_a l^a - \frac{\partial_r N}{M N} - \frac{2}{M r} \right) + \delta\mathcal{L} . \] (60)

From Eqs. (41)–(53) for any scalar \( G \) its directional derivative can be expressed as
\[ l^a \nabla_a G = \left[ \frac{g}{N} \partial_t + \frac{1}{c M} \partial_r \right] G + \left( \frac{1}{c M} M^a + \frac{g}{N} N^a \right) D_a G ; \] (61)
thus,
\[ l^a \nabla_a \mathcal{F} = \frac{1}{c M} \partial_r \mathcal{F} . \] (62)

Also the expansion of \( \epsilon \) on the background gives \( \epsilon = \cosh \psi = 1 + \psi^2/2 + O (\psi^4) \); thus, one can safely replace it by 1 in a first order calculation. Inserting Eq. (60) in Eq. (58), employing integration by parts and Eq. (62) we obtain
\[ \int d^4x \sqrt{-\tilde{g}} \mathcal{F} \delta L = \]
\[ \int d^4x \left[ \sqrt{g} \bar{M} \bar{N} \nabla_a (\mathcal{F} l^a) - \partial_r (\sqrt{g} \bar{N} \mathcal{F}) \right] \]
\[ + \int d^4x \sqrt{g} \bar{M} \bar{N} \left( \frac{\partial_r \mathcal{F}}{M^2} \delta M + \mathcal{F} \delta\mathcal{L} \right) . \] (63)

Further as \( \bar{t} = (0, \bar{M}^{-1}, 0, 0) \) and exploring
\[ \sqrt{-\tilde{g}} = \sqrt{-g} \left( 1 - \ln \sqrt{-\tilde{g}} \right) , \]
\[ \nabla_a \bar{t}^a = \frac{1}{M} \left( \frac{2}{r} + \partial_r \bar{N} \right) , \]
the variational term (63) can be rewritten as a total covariant divergence and a sum of variations \( \delta M \), \( \delta\mathcal{L} \) and \( \delta \ln \sqrt{-\tilde{g}} \):
\[ \int d^4x \sqrt{-\tilde{g}} \mathcal{F} \delta L = \]
\[ \int d^4x \sqrt{-\tilde{g}} \nabla_a (\mathcal{F} l^a) \delta \ln \sqrt{-\tilde{g}} \]
\[ + \int d^4x \sqrt{g} \bar{M} \bar{N} \left( \frac{\partial_r \mathcal{F}}{M^2} \delta M + \mathcal{F} \delta\mathcal{L} \right) \]
\[ - \int d^4x \partial_r (\sqrt{g} \bar{N} \mathcal{F}) \delta \ln \sqrt{-\tilde{g}} . \] (64)

From the relations among embedding variables and coordinate derivatives, given as Eqs. (35) and (36) of Ref. [76], the variation \( \delta\mathcal{L} \) can be expressed as
\[ \delta\mathcal{L} = -\frac{\partial_r \delta N}{N^2} - \frac{\partial_r \delta N}{M N} + \partial_r \bar{N} \left( \frac{\delta N}{N} + \frac{\delta M}{M} \right) , \] (65)
The variation \( \delta K^a \) and \( \delta K^* \) are also related, since
\[ \int d^4x \sqrt{-\tilde{g}} L_{K,\text{EFT}}^\ast \delta K^* = \int d^4x \sqrt{-g} \nabla_a \left( L_{K,\text{EFT}}^\ast k^a \right) \]
\[ - \int d^4x \sqrt{-g} L_{K,\text{EFT}}^\ast \delta K^* . \] (66)

In addition from Eq. (35) of Ref. [76] we find
\[ \delta K^* = \frac{\partial_r \delta M}{M N} . \] (67)

These generate \( \partial_r \left[ \sqrt{g} \left( L_{K,\text{EFT}} - L_{K,\text{EFT}}^\ast \right) \right] \) as the prefactor of \( \delta M \) in the first order variation of the action, which vanishes on a static background.

Finally, the variation the action with respect to the geometrical variables takes the form,
\[ \delta g S_{\text{EFT}} = \delta g S_{\text{EFT}}^{(2)} + \int d^4x \sqrt{-g} \{ L_N^\ast \delta N \]
\[ + \left[ \bar{L}^\ast + \bar{N} L_N^\ast \right] \]
\[ + \frac{1}{M} \left( \frac{2}{r} + \partial_r \bar{N} + \partial_r \right) \bar{L}_L \}
\[ + \bar{L}^{\ast} + \bar{M} L_M^{\ast} \]
\[ + \partial_r \bar{N} \bar{L}_L^{\ast} - \frac{2}{r^2} L_{\text{R}}^{\ast} \delta M \]
\[ + 2 \left[ L_{\text{EFT}} - \frac{2}{r^2} L_{\text{R}}^{\ast} \right] \frac{\delta M}{M} \]
\[ - \frac{1}{M} \left( \frac{2}{r} + \partial_r \bar{N} + \partial_r \right) F \delta \zeta \} , \] (68)
where
\[ \delta g S_{\text{EFT}}^{(2)} = \int d^4x \sqrt{-g} \nabla_a \left( F l^a \delta \ln \sqrt{-g} \right) \]
\[ + L_k^{\ast} k^a + \left( L_N^{\ast} - L_{K,\text{EFT}}^\ast \right) k^a \]
\[ - \left( F + L_N^{\ast} \right) \left( k^a \delta N \frac{N}{N} + r^a \delta N \frac{N}{N} \right) \] (69)
is a boundary term. In the derivation we have used that \( \sqrt{g} = r^2 \sin \theta \) and
\[ \int_0^\pi \int_0^{2\pi} d\varphi \sqrt{g} \mathcal{D}_a \left( \bar{g}^{ab} \bar{D}_b \delta \zeta \right) = 0 . \] (70)
(The integral being a covariant divergence can be transformed to another integral on the boundary of the sphere, which is zero.) The field equations arising from (68) are
\[ L_N^{\ast} = 0 , \] (71)
\[ L_{\text{EFT}} + \bar{N} L_N^{\ast} + \frac{1}{M} \left( \frac{2}{r} + \partial_r \bar{N} + \partial_r \right) L_{\text{EFT}} = 0 , \] (72)
Using Eqs. (46)–(48) and (59), the field equations reduce and (31), whenever an orthogonal double foliation is chosen, containing only \( N \) and \( T \) on the background, \( N \) becomes first order and \( \epsilon \), containing only \( N^2 \), of second order. As the only place \( N \) enters these Lagrangians is through \( \epsilon \), Eq. (71) becomes trivial. Then the dynamical equations are only (72)–(74).

Despite starting with a modified set of variables and employing a nonorthogonal double foliation, Eqs. (72)–(74) are identical with those derived in the orthogonal double foliation [77]. For more generic, beyond Horndeski \( L^{\text{EFT}} \), however, Eq. (71) may carry information.

For comparison, Appendix C also enlists the equations of motion obtained in the \((n,m)\) basis. There, the analog of Eq. (71) is nontrivial. The other equations are quite similar, but expressed in the complementary set of variables.

A. Derivation of the Schwarzschild solution from the EFT form of the Einstein–Hilbert action

As a check of the equations derived we first derive the Schwarzschild solution of general relativity. As in this case there is no scalar field, Eq. (73) is trivially satisfied and in light of the closing remark of the previous section Eq. (71) is also trivial. For Eqs. (72)–(74) one needs the Einstein–Hilbert Lagrangian density \( L^{\text{EH}} \) up to first order perturbations on static and spherically symmetric background. The two-contragradient Gauss equation (22) in the \((k,l)\) basis contains bilinear expressions of \( K^* \), \( K^* \), which due to Eq. (45) are second order. We can also drop a covariant four-divergence term; hence, \( L^{\text{EH}} = R - \lambda + L (L - 2L) \),

\[
L^{\text{EH}} = R - \lambda + L (L - 2L) , \tag{76}
\]

which on the background gives

\[
L^{\text{EH}} = \frac{2}{r^2} + \frac{4N'}{M^2N_r} . \tag{77}
\]

Using Eqs. (50)–(53) and (56) the field equations reduce to

\[
M^2 - 1 + 2r \frac{M'}{M} = 0 , \tag{78}
\]

\[
M^2 - 1 - 2r \frac{N'}{N} = 0 , \tag{79}
\]

\[
\frac{r}{N} (r \tilde{N}') - r^2 \frac{M'}{M} \left( 1 + \frac{N'}{N} \right) = 0 . \tag{80}
\]

The first two equations immediately give \( \tilde{N} \propto M^{-1} \).

The proportionality coefficient can be chosen as 1 by redefining the coordinate \( r \). Then the left-hand side of Eq. (80) can be rewritten as

\[
\frac{r}{N} (r \tilde{N}') + r^2 \frac{N'}{N} \left( 1 + \frac{N'}{N} \right) = \left[ \frac{r^2 (N^2)' - 2N^2}{2N^2} \right] \tag{81}\]

such that Eq. (80) immediately gives

\[
\tilde{N}^2 = K \left( 1 - \frac{C}{r} \right) . \tag{82}\]

The factor \( K \) can be chosen as 1 by redefining the time coordinate, and the weak field limit leads to \( C = 2m \).

VI. EFT EQUATIONS OF MOTION FOR NONMINIMALLY COUPLED K-ESSENCE

In this section we discuss the Horndeski theories with \( G_3 = G_5 = 0 \), generic \( G_2 (\phi, X) \) (k-essence) and \( G_4 (\phi) \) (nonminimal coupling to the metric).

The Lagrangian density at first order in perturbations on static and spherically symmetric background reduces to

\[
L^{\text{EFT}} = G_2 (\phi, X) + G_4 (\phi) \left( R - \lambda + L^2 - 2L \right) + 2\sqrt{X} G_4 (\phi) (L - \mathcal{L}) . \tag{83}\]

On the background

\[
L^{\text{EFT}} = \tilde{G}_2 + \tilde{G}_4 \left( \frac{1}{r^2} + \frac{1}{M^2r^2} + \frac{2N'}{M^2N_r} \right) + \frac{2\phi'}{M^2} \tilde{G}_4 (\phi) \left( \frac{1}{r} + \frac{N'}{N} \right) . \tag{84}\]

The nontrivial field equations (72)–(74) are

\[
M^2 - 1 + 2r \frac{M'}{M} = \frac{r^2}{G_4} \left[ - \frac{M^2}{2} G_2 \right] + 2\frac{\phi'}{M^2} \tilde{G}_4 (\phi) \left( \frac{M'}{M} + \partial_r \right) \tag{85}\]

\[
M^2 - 1 - 2r \frac{N'}{N} = \frac{r^2}{G_4} \left[ - \frac{M^2}{2} G_2 + \frac{N'}{N} \tilde{G}_4 (\phi) \left( \frac{M'}{M} + \partial_r \right) \right] . \tag{86}\]
and

\begin{equation}
\frac{r^2}{N} \left( r \dot{N'} + \frac{1}{r} \left( \frac{N'}{N} + \frac{M'}{M} + \partial_r \right) (\bar{G}_4 \phi') \right) = \frac{r^2}{G_4} \left[ \frac{M^2}{2} \bar{G}_2 - \left( \frac{1}{r} + \frac{N'}{N} - \frac{M'}{M} + \partial_r \right) (\bar{G}_4 \phi') \right],
\end{equation}

while the scalar equation (76) becomes

\begin{equation}
\left( \frac{r^2 \dot{N}}{M} \phi \bar{G}_{2X} \right)' - \frac{r^2 M N}{2} \bar{G}_{2 \phi} = \frac{\dot{N}}{M} \left[ M^2 - 1 - \frac{2r \dot{N}}{N} \right] + \frac{r^2 \dot{M}}{M} \left( \frac{2}{r} + \frac{N'}{N} \right) \bar{G}_{4 \phi}.
\end{equation}

For \( G_4 = \phi \) and \( G_2 = 3X/2X - V(\phi) \) the system reproduces \( f(R) \)-gravity, while the even more generic setup with \( G_4 = \phi \) and \( G_2 = -\omega(\phi) X - V(\phi) \) was considered by Sotiriou and Faraoni. Imposing asymptotic flatness and the vanishing of \( V(\phi) \) at infinity, also forbidding linear instabilities of the scalar in the Einstein frame, they proved that i) the scalar field ought to be constant (then the conformal factor is also constant and asymptotic flatness also holds in the Einstein frame), ii) \( V(\phi) = 0 \) holds in the entire spacetime; therefore \( G_2 = 0 \). With these, the surviving theories are described by the Einstein–Hilbert action.

The consistency of the EFT formalism discussed above can be also verified by imposing the outcome of the Sotiriou–Faraoni unicity theorem, \( G_2 = 0 = \phi' \) leading to the Einstein–Hilbert action. With these conditions the right-hand sides of Eqs. (80)-(83) vanish, hence they are the same as Eqs. (76)-(79), leading immediately to the Schwarzschild solution. Since \( \phi' = 0 \) and \( G_2 = 0 \) the left-hand side of the scalar equation vanishes. On the right-hand side the coefficient of \( G_{4 \phi} \) is a linear combination of the left-hand sides of Eqs. (76), (88) and \( (N'/N + M'/M) \). For the Schwarzschild solution the first two obviously vanish, while the third vanishes due to Eq. (81).

### A. Unicity theorem for the Schwarzschild solution

The equations for the geometric variables (80)-(83) reduce to the corresponding ones for the Schwarzschild solution, Eqs. (76)-(79), whenever the conditions,

\begin{equation}
\frac{M^2}{2} \bar{G}_2 = \left( \frac{2}{r} - \frac{M'}{M} + \partial_r \right) (\bar{G}_4 \phi') , \tag{90}
\end{equation}

\begin{equation}
\frac{M^2}{2} \bar{G}_2 = \frac{1}{r} + \frac{N'}{N} - \frac{M'}{M} + \partial_r (\bar{G}_4 \phi') , \tag{91}
\end{equation}

and

\begin{equation}
\frac{M^2}{2} \bar{G}_2 = \left( \frac{1}{r} + \frac{N'}{N} - \frac{M'}{M} + \partial_r \right) (\bar{G}_4 \phi') , \tag{92}
\end{equation}

hold. Taking the difference of Eqs. (92) and (90) one gets

\begin{equation}
\left( \frac{N'}{N} - \frac{1}{r} \right) \bar{G}_4 \phi' = 0 . \tag{93}
\end{equation}

As for the Schwarzschild solution the first factor does not vanish, \( G_{4 \phi} \phi' = \bar{G}_4 = 0 \); hence \( G_4 \) is a constant. Then from Eq. (92) \( \bar{G}_2 = 0 \) also follows.

With this, we proved the following unicity theorem. In the Horndeski class of theories with generic \( G_2, G_4 \) functions (but \( G_3 \) and \( G_5 \) vanishing) only the Einstein-Hilbert action can allow for the Schwarzschild solution.

### VII. A CLASS OF EXACT SOLUTIONS FOR NONMINIMALLY COUPLED K-ESSENCE

We continue our discussion in the framework of Horndeski theories with \( G_4 = G_5 = 0 \), generic \( G_2, G_4 \) (k-essence) and \( G_4(\phi) \) (nonminimal coupling to the metric) by imposing

\begin{equation}
N = M^{-1} , \tag{94}
\end{equation}

thus, allowing for only one undetermined metric function. With this choice in the weak and stationary field limit there is only one potential appearing in the metric. If that deviates only slightly from the Newtonian potential, both Solar System and gravitational lensing tests of general relativity could be reproduced. One can rewrite the line element in the Eddington–Finkelstein form

\begin{equation}
ds^2 = -\bar{N}^2 du^2 - 2du dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
= -\bar{N}^2 du^2 + 2dr dv + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \tag{95}
\end{equation}

with \( r^* = \int dr/\bar{N}^2 (r) \) the tortoise coordinate, \( u = t - r^* \) and \( v = t + r^* \) as the retarded (outgoing) and advanced (ingoing) time, respectively. For outgoing radial light rays in the ingoing Eddington–Finkelstein coordinates \( dr/dv = \bar{N}^2/2 \); hence at \( \bar{N} = 0 \) there is an apparent and event horizon.

With the choice (94), Eqs. (80)-(83) reduce to

\begin{equation}
1 - \bar{N}^2 - r (\bar{N})' = \frac{r^2}{G_4} \left[ \frac{1}{2} \bar{G}_2 + \left( \frac{2\bar{N}^2}{r} + \frac{(\bar{N})'}{2} + \bar{N}^2 \partial_r \right) (\bar{G}_4 \phi') \right] , \tag{96}
\end{equation}

\begin{equation}
1 - \bar{N}^2 - r (\bar{N})' = \frac{r^2}{G_4} \left[ \frac{1}{2} \bar{G}_2 + \left( \frac{2\bar{N}^2}{r} + \frac{(\bar{N})'}{2} \right) \bar{G}_4 \phi' + \bar{N}^2 \phi'^2 \bar{G}_{2X} \right] , \tag{97}
\end{equation}

preliminary to the weak equivalence principle and the accuracy of gravitational experiments.
\[
\frac{[r^2 (\bar{N}'^2)']}{2} = \frac{r^2}{G_4} \left[ \frac{1}{2} \bar{G}_2 - \left( \frac{1}{r} \bar{N}'^2 + (\bar{N}')^2 + \bar{N}^2 \bar{G}_{\theta\theta} \right) (\bar{G}_{\phi\phi}') \right],
\]
(98)

and

\[
(r^2 \bar{N}' \bar{G}_{2X})' - \frac{r^2}{2} \bar{G}_{2\phi} = \left( 1 - \bar{N}'^2 - r (\bar{N}'^2)' - \frac{[r^2 (\bar{N}')^2]}{2} \right) \bar{G}_{4\phi}.
\]
(99)

Both for the left-hand side of Eq. (98) and the right-hand side of Eq. (99) we have explored Eq. (82). By taking the difference of Eqs. (98) and (97) the following simple side of Eq. (99) we have explored Eq. (82). By taking

\[
\pi \bar{g} \frac{d\sigma}{G_4 (\phi (\sigma))} \int_{\sigma} d\rho \bar{G}_4 (\phi (\rho)) = \text{constant}.
\]
(100)

with the solution

\[
\bar{N}^2 = -2r^2 \int_{\sigma} d\sigma \frac{1}{G_4 (\phi (\sigma))} \int_{\sigma} d\rho \bar{G}_4 (\phi (\rho)).
\]
(103)

With this, the metric is fully determined in terms of \(\bar{G}_4\) and two integration constants. Then Eqs. (100) and (99) give \(\bar{G}_{2X}\) and \(\bar{G}_{2\phi}\), respectively, while \(\bar{G}_2\) itself is given by Eq. (97).

\section{VIII. BLACK HOLE, NAKED SINGULARITY AND HOMOGENEOUS SOLUTIONS}

Let us explore a few particular cases of the general solution derived in the previous section.

\subsection{A. Constant \(\bar{G}_4\)}

Assuming \(\bar{G}_4 = (16\pi G)^{-1}\) a constant, a minimal coupling of constant scalar and metric is realized, hence the Einstein and Jordan frames coincide. In this case the metric function becomes

\[
\bar{N}^2 = 1 - \frac{2m}{r} - \Lambda r^2.
\]
(104)

Here \(m\) and \(\Lambda\) are integration constants and we obtained the Schwarzschild-de Sitter metric for \(\Lambda > 0\) and Schwarzschild-anti de Sitter metric for \(\Lambda < 0\). Then Eq. (100) gives \(\phi' \bar{G}_{2X} = 0\). With \(\phi' = 0\) also \(X = 0\) holds, hence \(\bar{G}_{2X} = 0\). Next Eq. (99) yields \(\bar{G}_{2\phi} = 0\), yielding \(\bar{G}_2\) to a constant, to be found from Eq. (97) as \(\bar{G}_2 = -6\Lambda/(16\pi G)\). Thus \(\bar{G}_2\) contributes a cosmological constant term to the action. This is why the Schwarzschild-(anti) de Sitter metric emerged.\(^5\)

\subsection{B. The case \(\bar{G}_4 = \phi = r\)}

When both \(\bar{G}_4\) and its inverse are regular, \(\bar{G}_4\) can be identified with the scalar \(\bar{G}_4\). If further, the scalar is a monotonic function of the radial coordinate with a nowhere vanishing derivative, it can be chosen as the radial coordinate itself. Here we explore the case when the scalar is the curvature coordinate. The metric function in this case becomes

\[
\bar{N}^2 = \frac{1}{2} + \frac{Q}{\bar{r}^2} - \Lambda r^2.
\]
(105)

Here \(Q\) and \(\Lambda\) are integration constants, interpreted as tidal charge and cosmological constant. The metric is a curvature singularity in the origin and evades asymptotic flatness due to the term \(1/2\) even in the absence of the cosmological constant, as

\[
\lim_{\lambda \to 0} R_{\theta \phi \theta} = \frac{1}{2}, \quad \lim_{\lambda \to 0} R_{\theta \phi \phi} = -\sin^2 \theta - \frac{2}{\varphi}.
\]
(106)

Equation (100) gives \(\bar{G}_{2X} = 0\). Next Eq. (99) yields \(\bar{G}_{2\phi} = -12\Lambda - 1/\bar{r}^2\), which easily integrates to

\[
\bar{G}_2 (\phi) = -12\Lambda \phi + \frac{1}{\phi}.
\]
(107)

Equation (97) fixed the integration constant to zero.

\subsubsection{1. Horizons}

The metric function \(\bar{N}^2\) vanishes for

\[
r_{1,2}^2 = \frac{1 \pm \sqrt{1 + 16\Lambda Q}}{4\Lambda},
\]

provided \(16\Lambda Q \geq -1\). In this range for \(\Lambda > 0\) and \(Q < 0\) there are two positive roots, hence horizons. The metric function \(\bar{N}^2\) is positive only between them. In this case the metric represents a Kantowski-Sachs type homogeneous solution inside the smaller horizon, a spherically

\(^5\) In another context, in the particular case of a constant \(\bar{G}_4\) and \(\bar{G}_2\) (X) Ref. also arrived to a constant \(\bar{G}_2\), leading to the Schwarzschild-(anti) de Sitter metric.
symmetric, static black hole in between, while outside the larger, cosmological horizon it is homogeneous again, asymptotically approaching anti de Sitter.

For \( \Lambda < 0 \) and \( Q > 0 \) there is no horizon hiding the central singularity.

For \( \Lambda < 0 \) and \( Q < 0 \) there is a positive root (hence horizon) at

\[ r_1 = \frac{1}{2} \sqrt{\frac{1 - \sqrt{1 + 16Q\Lambda}}{\Lambda}}. \tag{108} \]

Above this horizon \( \bar{N}^2 > 0 \) and below \( \bar{N}^2 < 0 \); hence the solution represents a spherically symmetric, static black hole with homogeneous interior.

Finally, for \( \Lambda > 0 \) and \( Q > 0 \) a horizon appears at

\[ r_2 = \frac{1}{2} \sqrt{\frac{1 + \sqrt{1 + 16Q\Lambda}}{\Lambda}}. \tag{109} \]

Then \( \bar{N}^2 > 0 \) holds below the anti de Sitter type cosmological horizon.

C. The case \( G_4 = \phi = r^\alpha \)

The metric function becomes

\[ \bar{N}^2 = \frac{1}{1 + \alpha} + \frac{C}{r_1^{1+\alpha}} - \Lambda r^2, \tag{110} \]

with \( C \) and \( \Lambda \) integration constants. This metric includes the previous two cases for \( \alpha = 0, 1 \) (with \( C = 2m, Q \) in these cases, respectively). Here we allow all \( \alpha \geq 0 \); hence the metric is singular in the origin. The metric has a curvature singularity in the origin and evades asymptotic flatness when \( \alpha \neq 0 \), even in the absence of the cosmological constant, as

\[ \lim_{\Lambda \to 0} R_{\theta\varphi\theta}^\varphi = \frac{\alpha}{1 + \alpha}, \quad \lim_{\Lambda \to 0} R_{\theta\varphi\varphi}^\theta = -\frac{\alpha \sin^2 \theta}{1 + \alpha}. \tag{111} \]

Then Eq. (100) gives

\[ \bar{G}_{2X} = \frac{\alpha - 1}{\alpha r^{\alpha}}. \tag{112} \]

Next Eq. (97) gives

\[ \bar{G}_2 = \frac{2\alpha^2 r^{\alpha-2}}{1 + \alpha} + \frac{\alpha (\alpha - 1) C}{r^3} - 2 \left( 3 + 2\alpha + \alpha^2 \right) \Lambda r^\alpha. \tag{113} \]

In the particular case \( \alpha = 1 \) Eq. (104) is recovered, while for \( \alpha = 0 \) the desired cosmological constant type contribution to the action emerges.

From Eq. (99) we find

\[ \bar{G}_{2\phi} = -\frac{2}{(1 + \alpha) r^2} - \frac{\alpha (\alpha - 1) C}{r^{\alpha+3}} - 6 (\alpha + 1) \Lambda. \tag{114} \]

With

\[ X = N^2 \phi^2 = \alpha^2 \left( \frac{r^{2\alpha-2}}{1 + \alpha} + C r^{\alpha-3} - \Lambda r^{2\alpha} \right), \tag{115} \]

Eq. (113) can be rewritten in terms of the scalar field and kinetic term as

\[ \bar{G}_2 = \frac{\alpha - 1}{\alpha} \frac{X}{\phi} + \alpha \frac{\phi''}{\phi} - \left( 6 + 5\alpha + \alpha^2 \right) \Lambda \phi, \tag{116} \]

which correctly reproduces both \( \bar{G}_{2X} \) and \( \bar{G}_{2\phi} \). For \( \alpha = 1 \) we recover Eq. (107) while for \( \alpha = 0 \) (taking into account that \( X \propto \alpha^2 \)) and for constant \( G_4 \) the scalar is also a constant) the cosmological constant type contribution to the action reemerges.

1. Discussion of the horizons

The location of the horizons is determined by

\[ -\Lambda r^{3+\alpha} + \frac{1}{(1 + \alpha)^3} r^{1+\alpha} + C = 0. \tag{117} \]

For \( \Lambda = 0 \) and for \( C < 0 \) there is one real solution, hence horizon at

\[ r = \frac{1^{1+\alpha}}{\sqrt{(1 + \alpha) C}}. \tag{118} \]

Similarly, for \( C = 0 \) and \( \Lambda > 0 \) a horizon can be found at

\[ r = \sqrt{\frac{1}{(1 + \alpha) \Lambda}}. \tag{119} \]

For \( \Lambda \neq 0 \) and integer \( \alpha \) the number of positive real roots is given by the Descartes’ rule of sign. According to it for \( \Lambda \) negative and \( C \) positive there is no horizon. When \( \Lambda \) and \( C \) have identical signs, a horizon exists. When \( \Lambda \) is positive and \( C \) negative there is either no horizon or there are two horizons, according to this rule.

D. The linear case \( G_4 = \phi = A (1 + Br) \)

With two integrations the metric function (103) becomes

\[ \bar{N}^2 = 1 + 3Bm - \frac{2m}{r} - B (1 + 6Bm) r - \Lambda r^2 \]

\[ -B^2 (1 + 6Bm) r^2 \ln \left| \frac{Br}{1 + Br} \right|, \tag{120} \]

where \( m \) and \( \Lambda \) are integration constants. The case \( B = 0 \) reproduces the earlier found Schwarzschild–(anti) de Sitter metric. Asymptotically the last term of Eq. (120) vanishes and the solution approaches (anti) de Sitter with the cosmological constant \( \Lambda \). The independent, nonvanishing components of the Riemann tensor at spatially infinity for \( \Lambda = 0 \) are

\[ \lim_{\Lambda \to 0} R_{\theta\varphi\theta}^\varphi = -3Bm, \quad \lim_{\Lambda \to 0} R_{\theta\varphi\varphi}^\theta = 3Bm \sin^2 \theta, \tag{121} \]

hence a nonvanishing parameter \( B \) obstructs asymptotic flatness.
Equation (100) gives \( \bar{G}_{2X} = 0 \) (unless \( B = 0 \), a case already discussed). Then Eq. (97) yields

\[
\frac{\bar{G}_2}{2A} = -\frac{3mB^2}{r(1+Br)} - 3(1+2Br)\Lambda \\
-\frac{B^2[11+72mB+12(1+6mB)Br]}{2(1+Br)} \\
-3B^2(1+6mB)(1+2Br)\ln\left|\frac{Br}{1+Br}\right| \tag{122}
\]

or in terms of the scalar,

\[
\bar{G}_2 = -\frac{6mA^2B^3}{\phi - A} - 6(2\phi - A)\Lambda \\
+AB^2\left(\frac{A}{\phi} - 12\right)(1+6mB) \\
-6B^2(1+6mB)(2\phi - A)\ln\left|\phi - A\right| \tag{123}
\]

and Eq. (99) is also verified.

1. Horizons and singularities

Although the leading terms of (120) at low \( r/m \) values may be negative, for negative \( \Lambda \) and positive \( B \) the cosmological and logarithmic terms are positive, dominating the behavior of the metric at larger distances. To illustrate this we plot the metric function with \( Bm = 1 \) on Fig. 1. In the parameter range with negative \( \Lambda \) this represents a new spacetime with one horizon. For positive values of \( \Lambda \) there is no horizon, \( N^2 < 0 \) and a Kantowski-Sachs type geometry emerges.

A similar plot for \( Bm = -1 \) is represented on Fig. 2. There a horizon emerges for all values of \( \Lambda \), nevertheless an intriguing feature shows up. Outside the horizon there is a singularity generated by the blowing up of the logarithmic term in Eq. (120). Outside the logarithmic singularity the spacetime is spherically symmetric for negative \( \Lambda m^2 \) values, while in the positive regime another horizon appears, rendering the spacetime homogeneous outside it.

The metric coefficient for \( \Lambda m^2 = -1 \) is represented as function of the radial distance and the parameter \( Bm \) on Fig. 3. For positive \( Bm \) values the black hole structure emerges again. For negative values of \( Bm \) the logarithmic singularity (positive for \( Bm < -1/6 \), negative otherwise) appears outside the horizon, at \( Br = -1 \). This feature has already been encountered on Fig. 2.

We illustrate this singularity for \( Bm = -1 \) and \( \Lambda m^2 = -1 \) on Fig. 4. We checked that both the Ricci curvature scalar and the Kretschmann scalar diverge at the logarithmic singularity, confirming the singularity is not a coordinate artifact.

![FIG. 1: The metric function \( \bar{N}^2 \) represented as function of the radial distance from the central singularity (in units of mass) and of the parameter \( \Lambda m^2 \), for \( Bm = 1 \). For negative values of \( \Lambda \) the singularity is hidden by a horizon (represented by the intersection of \( \bar{N}^2 \) with the zero plane, depicted in sky blue). For positive values of \( \Lambda \) the metric function stays negative, representing a homogeneous spacetime with central naked singularity.](image)

**IX. THE EINSTEIN FRAME DESCRIPTION OF THE SOLUTIONS**

In the previous section we have identified black hole, naked singularity and homogeneous solutions for the simple Horndeski-type action

\[
S^{ETF} = \int d^4x\sqrt{-\bar{g}} \left[ G_2(\phi, X) + G_4(\phi) \hat{R} \right] , \tag{124}
\]

with the ansatz (121) of only one independent metric function. For nonconstant \( G_4 \) this action is in Jordan frame, the natural frame of the Horndeski-type theories. The advantage of the Jordan frame is that only the metric couples to matter (minimally), however the coupling of the metric and scalar is intricate.

By a conformal transformation \( \hat{g}_{ab} = \Omega^2\bar{g}_{ab} \) the expression \( \sqrt{-\bar{g}}\hat{R} \) generates \( \Omega^{-2}\sqrt{-\bar{g}}\hat{R} \) as the only curvature term [73]. In order to ensure a minimal coupling of the scalar to the metric tensor, hence achieve the Einstein frame, the conformal factor \( \Omega^2 = G_4(\phi) > 0 \) should be chosen. The line element conformal to (122) in the Einstein frame becomes

\[
\hat{ds}^2 = -\bar{N}^2dt^2 + \bar{M}^2d\hat{r}^2 + \hat{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{125}
\]

with the new curvature coordinate \( \hat{r} = \bar{G}_4^{1/2}r \) and new metric functions

\[
\bar{N}^2 = \bar{G}_4\bar{N}^2 , \quad \bar{M}^2 = \frac{M^2}{\left[ 1 + \frac{\bar{r}^2}{2}(\ln \bar{G}_4)^2 \right]^2} . \tag{126}
\]
FIG. 2: The metric function $\bar{N}^2$ represented as function of the radial distance from the central singularity (in units of mass) and of the parameter $\Lambda m^2$, for $Bm = -1$. For all values of $\Lambda$ the central singularity is hidden by a horizon (represented by the intersection of $\bar{N}^2$ with the zero plane, depicted in sky blue). Another, logarithmic singularity is generated outside the horizon, rendering the spacetime to a naked singularity. Outside the logarithmic singularity the spacetime is spherically symmetric for negative $\Lambda m^2$ values, while after stepping into the positive regime, another horizon appears, the spacetime becoming homogeneous outside it.

In deriving these we have employed the identity

$$1 + \frac{1}{2} \left( \ln G_4 \right)^2 = 1 - \frac{1}{r^2} \frac{d \ln G_4}{d r}.$$ (127)

For the special cases discussed in Sec. VIII the metric becomes more complicated in the Einstein frame, as shown in Table I.

| $\Omega^2 = \tilde{G}_4 = \phi$ | Eq. for $\bar{N}^2$ | $\bar{N}^2$ | $\tilde{M}^2$ |
|---------------------------------|------------------|-------------|-------------|
| $(16\pi G)^{-1} r^\alpha$      | $10^4$           | $10^{-4} N^2$ | $N^{-2}$    |
| $A (1 + Br)$                   | $10^2$           | $A (1 + Br) N^2$ | $\frac{N^2}{(1 + Br)^2} \bar{N}^{-2}$ |

TABLE I: The metric in Einstein frame in the particular cases investigated, the Einstein frame and Jordan frame curvature coordinates being related as $\tilde{\bar{r}} = r G_4^{1/2}$.

X. CONCLUDING REMARKS

In this paper we have explored the recently developed 2+1+1 decomposition of spacetime [76], based on a nonorthogonal double foliation, for the study of spherically symmetric, static solutions of a particular subclass of the Horndeski scalar-tensor theory. This subclass contains k-essence models nonminimally coupled to the metric (however the coupling depends only on the scalar field, not its derivatives).

We started the analysis by employing the approach of the effective field theory (EFT) of modified gravity, in which the action is conceived as a functional of a sufficiently large set of scalars, constructed from metric and embedding variables, all adapted to the nonorthogonal 2+1+1 decomposition. The choice of variables roughly followed and generalized the earlier analysis [77], performed in the orthogonal double foliation particular case, with the notable inclusion of a nonorthogonality param-
First, we proved that the class of Horndeski Lagrangians with $G_5 = 0$ (the inclusion of the latter would be incompatible with observations) can be expressed in this EFT form (in terms of the scalars constructed from the metric and embedding variables adapted to the nonorthogonal double foliation).

Next, by studying the first order perturbation of the EFT action, we derived three equations of motion for the metric and embedding variables, which reduce to those derived earlier in an orthogonal 2+1+1 decomposition, and a fourth equation for the metric parameter $N$ related to the nonorthogonality of the foliation. An additional equation emerged for the scalar field. As a check, we recovered the Schwarzschild solution for the Einstein–Hilbert action rewritten in this framework.

Then, for the Horndeski class of theories with vanishing $G_2$ and $G_5$, but generic functions $G_2 (\phi, X)$ (k-essence) and $G_4 (\phi)$ (scalar field dependent nonminimal coupling to the metric) we proved the unicity theorem, according to which no action beyond Einstein–Hilbert allows for the Schwarzschild solution. With this, for a spherical symmetric and static setup, we extended the unicity theorem previously announced in the literature, which applies for a function $G_2$ linear in $X$ (asymptotically homogeneous in $X$) and asymptotical flatness.

After that, we integrated the EFT field equations for the case with only one independent metric function. We assumed $G_4 (\phi) = 0$, while $G_2 (\phi, X)$ was fixed by the equations of motion. We discussed in particular scalar fields with polynomial and linear radial dependences, obtaining new solutions characterized by mass, tidal charge, a parameter generalizing both, the cosmological constant and an additional parameter. We have also written them up as conformally related metrics in the Einstein frame.

These solutions represent naked singularities, black holes or have the double horizon structure of the Schwarzschild–de Sitter spacetime. Solutions with homogeneous Kantowski–Sachs type regions also emerged. Finally, one of the solutions obtained for the function $G_4$ linear in the curvature coordinate in certain parameter range exhibits an intriguing logarithmic singularity lying outside the horizon. All these solutions were asymptotically nonflat (even when the cosmological constant was switched off). Hence, the black hole solutions evade previously known unicity theorems, and exhibit scalar hair.

**Acknowledgements**

This work was supported by the Hungarian National Research Development and Innovation Office (NKFIH) in the form of the Grant No. 123996 and has been carried out in the framework of COST actions CA16104 (GWverse) and CA18108 (QG-MM), supported by COST (European Cooperation in Science and Technology). C.N. was supported by the UNKP-19-3 and UNKP-20-3 New National Excellence Programs of the Ministry for Innovation and Technology. Z.K. was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences; also by the UNKP-19-4 and UNKP-20-5 New National Excellence Programs of the Ministry for Innovation and Technology.

**Appendix A: The relation among the scalar variables in the two bases**

From Table III. and Eqs. (34) of Ref. [76], the tensorial and vectorial embedding variables turn out to be related as

\[ K_{ab}^* = \frac{1}{c^2} (K_{ab} + sL_{ab}) , \]
\[ L_{ab}^* = \frac{1}{c^2} (L_{ab} - sK_{ab}) , \]
\[ K_a^* = K_a + \frac{5}{c} D_a \ln \frac{N}{cM} , \]
\[ L_a^* = L_a + \frac{5}{c} D_a \ln \frac{N}{cM} . \]  

(A1)

from which the relations connecting the sets of scalars \([12]\) and \([18]\) emerge:

\[ \kappa^* = \frac{1}{c^2} (\kappa + 2sK_{ab}L^{ab} + s^2 \lambda) , \]
\[ \lambda^* = \frac{1}{c^2} (\lambda - 2sK_{ab}L^{ab} + s^2 \kappa) , \]
\[ \mathcal{R}^* = \mathcal{R} + 2\frac{s}{c} K^a D_a \ln \frac{N}{cM} + \left( \frac{5}{c} \right)^2 \left( D_a \ln \frac{N}{cM} \right)^2 , \]
\[ \mathcal{E}^* = \mathcal{E} + 2\frac{s}{c} L^a D_a \ln \frac{N}{cM} + \left( \frac{5}{c} \right)^2 \left( D_a \ln \frac{N}{cM} \right)^2 , \]
\[ K^* = \frac{1}{c} (K + sL) , \]
\[ L^* = \frac{1}{c} (L - sK) . \]  

(A2)

where $(D_a F)^2 \equiv (D_a F) (D^a F)$ for any function $F$. Similarly, the scalars \([12]\) and \([13]\) are related through

\[ K^* = \frac{1}{c} (K - sL) + \frac{c}{M} (\partial_t - M^a D_a) \left( \frac{N}{N} \right) , \]
\[ L^* = \frac{1}{c} (sK + L) + \frac{c^2}{N} (\partial_t - N^a D_a) \left( \frac{N}{N} \right) . \]  

(A3)

These emerge from Table III. of Ref. [76] and the inverse relations

\[ n^a = \frac{1}{N} \left( \frac{\partial}{\partial t} \right)^a - \frac{s}{c} \frac{1}{M} \left( \frac{\partial}{\partial X} \right)^a - \frac{1}{N} N^a + \frac{5}{c} \frac{1}{M} M^a , \]
\[ m^a = \frac{1}{M} \left( \frac{\partial}{\partial X} \right)^a - \frac{1}{M} M^a . \]  

(A4)
and
\[ l^a = \frac{s}{N} \left( \frac{\partial}{\partial t} \right)^a + \frac{1}{cM} \left( \frac{\partial}{\partial X} \right)^a - \frac{s}{N} N^a - \frac{1}{cM} M^a, \]
\[ k^a = \frac{\epsilon}{N} \left( \frac{\partial}{\partial t} \right)^a - \frac{\epsilon}{N} N^a \]  
(A5)
of Eqs. (2)–(5), giving
\[ l^a - s n^a = \frac{c}{M} \left[ \left( \frac{\partial}{\partial X} \right)^a - M^a \right], \]
\[ s l^a + n^a = \frac{c^2}{N} \left[ \left( \frac{\partial}{\partial X} \right)^a - N^a \right]. \]  
(A6)
Finally, from the last equality of Eq. (11) one obtains
\[ \mathcal{R} = \mathfrak{t} - 2c^2 L^a D_a \left( \frac{N}{N} \right) + c^4 \left[ D_a \left( \frac{N}{N} \right) \right]^2. \]  
(A7)
Remarkably, in the particular subcase of orthogonal foliations (s = 0 = N and c = 1) all starry scalars coincide with their unstarred versions in Eqs. (A2) and (A3), while Eq. (A7) implies a further simplification \( \mathcal{R} = \mathfrak{t} \), further reducing the number of independent scalars.
For nonorthogonal foliations the number of independent embedding scalars is reduced to 7 by Eqs. (A2)-(A7), which also involve the three metric components N, M, and N.

Appendix B: The decomposed Horndeski Lagrangian in the (n, m) basis

We give in this appendix the expressions of \( \nabla \phi \) and \( L^1 \) in terms of the kinematical quantities arising in the (n, m) basis. For this first we calculate
\[
\nabla_a \nabla_b \phi = s \sqrt{X} (K_{ab} + 2m_a \kappa_{b} + m_{a} m_b \kappa - n_a n_b + n_a \kappa_{b} + n_b \kappa_{a} + n_a \kappa_{a} + m_a \kappa_b + m_b \kappa_{a} + \sqrt{N} n_b \nabla_a s + \sqrt{X} m_b \nabla_a \epsilon + \frac{1}{2X} (sn_b + cm_b) (sn_a + cm_a) \nabla_c X + \sqrt{X} (sn_b + cm_b) (sn_c + cm_c) \times \left( n_a \nabla_c s + m_a \nabla_c \epsilon \right) \times \sqrt{X} (cn_a + sm_a) \times (sn_b + cm_b) (\kappa - s L^*) + \sqrt{X} (sn_b + cm_b) \times [s \kappa (\kappa_a - L^*_a) + s^2 D_a (\ln N) - \epsilon^2 D_a (\ln M)].
\]
(B1)
Hence,
\[ \nabla^a \phi \nabla_a X - \nabla^a \phi \nabla_a X = 2X [s (K + \kappa)] + c (L^* - L^*) + c^2 (cn^a + sm^a) \nabla_a \phi \]  
(B2)

and
\[ L^1 = G_4 \left[ R + K_{ab} K^{ab} - L^*_{ab} L^{*ab} - K^2 + (L^*)^2 \right] + 2\sqrt{X} G_\phi [s (K + \kappa) + c (L^* - L^*')] + c^2 (cn^a + sm^a) \nabla_a \phi \]  
(B3)
These equations, when compared with Eqs. (2b) and (31), clearly show that the Horndeski action takes simpler form in terms of the kinematical quantities defined in the \((k, l)\) basis. This is because the scalar function \( \phi \) depends only on \( \chi \); therefore it is adapted to \( l^a \).

Appendix C: The field equations in the (n, m) basis

By choosing the \((n, m)\) basis as primary, the action is naturally rewritten as depending on the variables
\[ S_{EFT}^{(2)} \left[ N, N, M, K, \mathfrak{R}, K, \mathfrak{r}, L^*, L^*, \lambda, R, \phi \right]. \]  
(C1)
This action depends on the scalar \( \phi \), the metric variables \((N, M, M)\), the induced curvature scalar \( R \) and the embedding scalars \((K, \mathfrak{R}, K, \mathfrak{r}, L^*, \lambda^*)\) related to the \((n, m)\) basis. Beside switching the starry / nonstarry variables when going from the \((k, l)\) basis to the \((n, m)\) basis the only step needing additional explanation is the change of the variable \( \mathfrak{t} \) into \( \mathcal{R} \), as the complementary variables \( \mathfrak{t}^* \) into \( \mathcal{R}^* \) are not formed from normal fundamental forms. Note that cf. Appendix A the set \((K, \mathfrak{R}, K, \mathfrak{r}, L^*, \lambda^*)\) cannot be transformed to the alternative set \((K^*, \mathfrak{t}, K^*, \mathfrak{r}^*, L, \lambda)\) employed in the main part of the paper without the use of the new variables \( K_{ab} L^{ab} \) and \( L^a D_a \left( \frac{N}{N} \right) \) appearing in the transformations (A2) and (A7), respectively.
A similar variation procedure of the EFT action (C1)
as described in Sec. V yields the following field equations:

\[
\begin{align*}
L^EFT(N) + \frac{\partial_r}{r \N M} \left( r^2 L^EFT(N) \right) - \frac{2 \mathcal{F}^EFT}{r^2} &= 0, \\
L^EFT(2) + \frac{\partial_r}{r \N M} \left( r^2 L^EFT(2) \right) - \frac{2 \mathcal{F}^EFT}{r^2} &= 0, \\
2 \mathcal{F}^{EFT(2)} &= \frac{\partial_r}{r} \frac{\partial_r \N}{\N} L^EFT(2), \\
L^EFT(2) - \frac{\partial_r \mathcal{F}^{EFT(2)}}{r \N M} &= 2 r^2 \frac{L^EFT(N)}{r^2} = 0, \quad (C2)
\end{align*}
\]

where

\[
\begin{align*}
D_r &= \frac{N}{r} + \frac{\partial_r \N}{\N} + \frac{\partial_r}{r}, \\
\mathcal{F}^{EFT(2)} &= L^EFT(N) + 2 \frac{\N}{M^2} L^EFT(N).
\end{align*}
\]

It is not simple to show the equivalence of these to the set (71)-(74), although the last three equations are formally quite similar in the two approaches and both reduce to the same set in the orthogonal double foliation limit. The comparison is nontrivial as \( L^EFT \) and \( L^EFT(N) \) depend on different sets of variables (and there is need for additional variables to relate them, as emphasized above). Beside also they differ by total covariant divergences (obvious from the procedure they were derived). Taking into account these should lead to the same set of solutions.

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