A LOCAL CURVATURE ESTIMATE FOR THE RICCI-HARMONIC FLOW
ON COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper we consider the local $L^p$ estimate of Riemannian curvature for the Ricci-harmonic flow or List’s flow introduced by List [21] on complete noncompact manifolds. As an application, under the assumption that the flow exists on a finite time interval $[0, T)$ and the Ricci curvature is uniformly bounded, we prove that the $L^p$ norm of Riemannian curvature is bounded, and then, applying the De Giorgi-Nash-Moser iteration method, obtain the local boundedness of Riemannian curvature and consequently the flow can be continuously extended past $T$.

1. INTRODUCTION

The Ricci-harmonic flow is defined to be the following system:

\[
\begin{cases}
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(g(t)) + 4 du(t) \otimes du(t), \\
\frac{\partial}{\partial t} u(t) = \Delta_g u(t), \\
g(0) = g_0, \quad u(0) = u_0,
\end{cases}
\]

(1.1)

where $g_0$ is a fixed Riemannian metric, $u_0$ is a fixed smooth function, $t \in [0, T)$, $g(t)$ is a family of metrics, $u = u(t)$ is a family of smooth functions on an $n$-dimensional manifold $M$. It was first introduced in [21] and also called extended Ricci flow in [3, 13, 21, 25]. The flow equations, as the motivation for studying it, were proved to characterize the static Einstein vacuum metrics [7, 21]. Under the assumption that $M$ is compact, List [21,22] prove the short time existence, and also proved that if the Riemann curvature is uniformly bounded for all $t \in [0, T)$, then the solution can be extended beyond $T$. For a more general setting, see [23,24]. In the complete noncompact case, the long time existence of manifolds with bounded scalar curvature was given by the first author [19].

Over the last decade, there are lots of works on both compact and noncompact manifolds about eigenvalues, entropies, functionals, and solitons, see, for example, [1,2,3,5,8,10,11,12,13,16,20,25,27]. In this paper, we mainly focus on the estimate of curvature. List [22] proved that, $M$ being compact, the Ricci-harmonic flow can be extended if the Riemannian curvature is bounded, as an application to see the importance of curvature estimate. Unfortunately, counterexamples show that the Riemannian curvature (see [21]) and Ricci curvature (see [6]) could not be bounded without any restrictions. On the other hand, those curvatures are $L^2$.

Key words and phrases. Ricci-harmonic flow, Parabolic system, Curvature estimate.
bounded in certain cases (e.g. \( n = 4 \)) if scalar curvature is bounded (see [18]). Furthermore, the pseudo-locality theorem corresponding to the Ricci-harmonic flow was be given in [9]. However, as in the Ricci flow case, whether the scalar curvature is bounded in certain cases (e.g. \( n = 4 \)) if scalar curvature is bounded (see [18]).

In the following, we often omit \( t \) variable, for example, \( g = g(t), u = u(t), \Delta = \Delta_g(t), \) etc. The operator \( \Box := \partial_t - \Delta \) will be frequently used later. \( C \) represents positive finite constants that we don’t care about their value.

The first result of this paper is

**Theorem 1.1.** (also see Theorem 2.7) Let \((g(t), u(t))_{t \in [0, T]} \) be a solution to the Ricci-harmonic flow on \( M \times [0, T] \), where \( M \) is a complete \( n \)-dimensional manifold and \( T \in (0, +\infty) \). Suppose there exist constants \( \rho, K, L > 0 \) and a point \( x_0 \in M \) such that the geodesic ball \( B_{g(0)}(x_0, \rho/\sqrt{K}) \) is compactly contained on \( M \) and

\[
|\text{Ric}_g(g(t))|_{g(t)} \leq K, \quad |\nabla_g u(t)|_{g(t)} \leq L.
\]

For any \( p \geq 3 \), there exist constants \( \Gamma_1, \Gamma_2 \) depending only on \( n, p, \rho, K, L \) and \( T \), such that

\[
\int_{B_{g(0)}(x_0, \rho/2\sqrt{K})} |Rm_{g(t)}|_{g(t)}^p dv_{g(t)} \leq \Gamma_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |Rm_{g(0)}|_{g(0)}^p dv_{g(0)} + \Gamma_2 \text{Vol}_{g(0)}(B_{g(0)}(x_0, \rho/\sqrt{K})).
\]

Actually the explicit expressions for \( \Gamma_1 \) and \( \Gamma_2 \) can be found in the proof of Theorem 2.7.

Under the additional condition that \( |\nabla_{g(t)}^2 u|_{g(t)} \) is bounded, Theorem 1.1 was proved in [19]. Theorem 1.1 shows that this additional condition can be removed. According to the following remark, the boundedness of \( |\nabla_{g(t)} u(t)|_{g(t)} \) can also be removed. We include the condition \( |\nabla_{g(t)} u(t)|_{g(t)} \leq L \) in Theorem 1.1 is in order to see how \( K \) and \( L \) involve in the \( L^p \) estimate of \( Rm \).

**Remark 1.2.** (see Theorem B.2 in [19]) Suppose that \((g(t), u(t))_{t \in [0, T]} \) is a solution to (1) on \( M \times [0, T] \), where \( M \) is a complete \( n \)-dimensional manifold. If the estimate

\[
\sup_{M \times [0, T]} |\text{Ric}_g(g(t))|_{g(t)} \leq K
\]

holds for some positive constant \( K \), then we have

\[
\sup_{M \times [0, T]} |\nabla_{g(t)} u(t)|_{g(t)}^2 \leq 2KC(n).
\]
where $C(n)$ is a positive number depends only on $n$.

Theorem 1.3 and Remark 1.2 imply

**Theorem 1.3.** Let $(g(t), u(t))_{t \in [0,T]}$ be a solution to the Ricci-harmonic flow on $M \times [0,T]$, where $M$ is a complete $n$-dimensional manifold and $T \in (0, +\infty)$. Suppose there exist constants $\rho, K$ and a point $x_0 \in M$ such that the geodesic ball $B_{g(0)}(x_0, \rho/\sqrt{K})$ is compactly contained on $M$ and

\[ (1.3) \quad |\text{Ric}(g(t))|_{g(t)} \leq K. \]

For any $p \geq 3$, there exist constants $\Gamma_1, \Gamma_2$ depending only on $n, \rho, K, T$ such that

\[ \int_{B_{g(0)}(x_0, \rho/2\sqrt{K})} |\text{Rm}(g(t))|_{g(t)}^p dV_{g(t)} \leq \Gamma_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}(g(0))|_{g(0)}^p dV_{g(0)} + \Gamma_2 \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right). \]

Finally we state our main theorem.

**Theorem 1.4.** (also see Theorem 3.2) Let $(g(t), u(t))_{t \in [0,T]}$ be a smooth solution to the Ricci-harmonic flow on $M \times [0,T]$ with $T \in (0, +\infty)$, where $M$ is a complete $n$-dimensional manifold. If $(M, g(0))$ is complete and:

\[ \text{sup}_M |\text{Rm}(g(0))|_{g(0)} < \infty, \quad \text{sup}_{M \times [0,T]} |\text{Ric}(g(t))|_{g(t)} < \infty \]

then the flow can be extended over $T$.

This paper is organized as follow: In Sect. 2.1, we state our main idea and prove Theorem 1.1 i.e., the $L^p$ norm estimate of Riemannian curvature. We supply the details of the proof in Sect. 2.2. In Sect. 3, We discuss the extension of (1.1) and prove Theorem 1.4.

## 2. $L^p$ ESTIMATE OF RIEMANNIAN CURVATURE

We start with the proof of Theorem 1.1. As in [13-19], we let $\phi$ be a (time independent) Lipschitz function with compact support in a domain $\Omega \subset M$. Throughout this section, we always assume the condition (1.2) holds.

### 2.1. Main idea.

Given a real number $p \geq 1$ that is determined later. We introduce the following integrals:

\[ B_1 := \frac{1}{K} \int_M |\nabla \text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV, \quad B_2 := \int_M |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV, \]

and also

\[ A_1 := \int_M |\text{Rm}|^p \phi^{2p} dV, \quad A_2 := \int_M |\text{Rm}|^{p-1} \phi^{2p} dV, \]
\[ A_3 := \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-1} dV, \quad A_4 := \int_M |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV. \]

In order to control the second derivative of $u$, we need another type of integrals

\[ T_k := \int_M |\text{Rm}|^{k-1} |\nabla^2 u|^2 \phi^{2p} dV, \quad k = 1, 2, \ldots, p. \]

Then we have following inequalities, proved in Sect. 2.2.
Proposition 2.1. We have
\[
\frac{d}{dt} A_1 \leq B_1 + CKB_2 + CKA_4 + C(K + L^2)A_1 + CT_p.
\]

Proposition 2.2.
\[
B_1 \leq CKB_2 + C(K + L^2)A_1 + CKL^2A_2 + CKA_4 + CT_p - \frac{1}{2K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} d\nu \right).
\]

We observe that all $T_k$ can be controlled by $T_p$ and $T_1$.

Lemma 2.3. For any positive constant C and any $k = 1, 2, ..., p$,
\[
T_k \leq \frac{1}{C^{p-k}} T_p + (p-k)C^{k-1}T_1
\]

Proof. We can easily find that, for any positive constant $C$, the following inequality
\[(|\text{Rm}| - C)(|\text{Rm}|^{k-1} - C^{k-1}) \geq 0,
\]
holds, which implies
\[|\text{Rm}|^k - C|\text{Rm}|^{k-1} + C^k \geq C^{k-1}|\text{Rm}| \geq 0.
\]
Integrating on both sides yields
\[
T_k \leq \int_M \left( \frac{1}{C} |\text{Rm}|^k + C^{k-1} \right) |\nabla^2 u|^2 \phi^{2p} d\nu = \frac{1}{C} T_{k+1} + C^{k-1}T_1.
\]

We now use the induction method to prove this lemma. For $k = p$, $T_p \leq T_p$ satisfied. If the lemma is satisfied for some $k \leq p$, then
\[
T_{k-1} \leq \frac{1}{C} T_k + C^{k-2}T_1 \leq \frac{1}{C} \left( \frac{1}{C^{p-k}} T_p + (p-k)C^{k-1}T_1 \right) + C^{k-2}T_1 = \frac{1}{C^{p-(k-1)}} T_p + [p - (k-1)]C^{k-2}T_1.
\]
Therefore the above mentioned estimate hold.

According to Lemma 2.3, we can estimate all $T_k$’s in terms of $T_p$ and $T_1$. However, from the definition, we see that $T_p$ and $T_1$ contain the second derivative of $u$ so that we can not use the condition (1.2) to bound them. More precisely,
\[
T_p = \int_M |\text{Rm}|^{p-1} |\nabla^2 u|^2 \phi^{2p} d\nu, \quad T_1 = \int_M |\nabla^2 u|^2 \phi^{2p} d\nu.
\]

Motivated by these two integrals, by replacing the second derivative of $u$ by its first derivative, we set
\[
S := \int_M |\text{Rm}|^{p-1} |\nabla u|^2 \phi^{2p} d\nu, \quad \tilde{S} = \int_M |\nabla u|^2 \phi^{2p} d\nu.
\]

It is clear from the condition (1.2) that $S \leq L^2 A_2$. 

Proposition 2.4. We have
\[ B_2 \leq -\frac{1}{p-1} \frac{d}{dt} A_2 + CA_1 + CA_4 + CL^2 A_2 + CT_{p-1} \]

Proposition 2.5. For each \( p \geq 2 \), \( T_p \) satisfies the following estimate
\[ T_p \leq \frac{d}{dt} S - \frac{C}{p-1} \frac{d}{dt} A_2 - \frac{C}{K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV \right) + C(K + L^2) A_1 + CKL^2 A_2 + C(K + L^2) A_4 + C^{p-1} T_1 \]

Proposition 2.6. \( T_1 \) satisfies the following estimate
\[ T_1 \leq -\frac{d}{dt} \tilde{S} + CL^2 \text{Vol}_{g(t)}(\Omega). \]

We will give proofs for Proposition 2.4 – Proposition 2.6 in Sect. 2.2. Now we can prove Theorem 1.1.

Theorem 2.7. Let \( (g(t), u(t))_{t \in [0,T]} \) be a solution to the Ricci-harmonic flow on \( M \times [0,T] \), where \( M \) is a complete n-dimensional manifold with \( T \in (0, +\infty) \). Suppose that there exist constants \( \rho, K, L > 0 \) and a point \( x_0 \in M \) such that the geodesic ball \( B_{g(0)}(x_0, \rho/\sqrt{K}) \) is compactly contained on \( M \) and \( (\text{Ric}(g(t)), \nabla u(t)) \) satisfies (1.2). For any \( p \geq 3 \), there exist constants \( \Gamma_1, \Gamma_2 \) depending only on \( n, p, \rho, K, L \) and \( T \), such that
\[ \int_{B_{g(0)}(x_0, \rho/2\sqrt{K})} |\text{Rm}(g(t))|^p dV_{g(t)} \leq \Gamma_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}(g(0))|^p dV_{g(0)} + \Gamma_2 \text{Vol}_{g(0)} \left( B_{g(0)}(x_0, \rho/\sqrt{K}) \right). \]

Actually the explicit expressions for \( \Gamma_1 \) and \( \Gamma_2 \) can be found in the proof.

Proof. Applying Lemma 2.3 with \( C = 1 \) and \( k = p - 1 \) to Proposition 2.4 yields
\[ B_2 \leq -\frac{1}{p-1} \frac{d}{dt} A_2 + CA_1 + CA_4 + CL^2 A_2 + CT_p + CT_1 \]

Plugging Proposition 2.2 the above inequality into Proposition 2.1 successively to replace \( B_1 \) and \( B_2 \): \[
\frac{d}{dt} \left[ A_1 + \frac{CK}{p-1} A_2 + \frac{1}{2K} \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV \right]
\leq C(K + L^2) A_1 + CKL^2 A_2 + CKA_4 + CKT_p + CKT_1
\]

Then apply proposition 2.5 and Proposition 2.6 to replace \( T_p \) and \( T_1 \), we obtain
\[ \frac{d}{dt} \left[ A_1 + \frac{CK}{p-1} A_2 + CK\tilde{S} + CKS + C \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV \right]
\leq CK(K + L^2) A_1 + CK^2 L^2 A_2 + CK(K + L^2) A_4 + CKC^p L^2 \text{Vol}_{g(t)}(\Omega). \]

Choose \( \Omega := B_{g(0)}(x_0, \rho/\sqrt{K}) \) and
\[ \phi := \left( \frac{\rho/\sqrt{K} - d_{g(0)}(x_0, \cdot)}{\rho/\sqrt{K}} \right)_+ \].
Define
\[
U := \int_M |\mathcal{Rm}|^p \phi^2 p^p dV_t + \frac{CK}{p-1} \int_M |\mathcal{Rm}|^{p-1} \phi^2 p^p dV_t + C \int_M |\text{Ric}|^2 |\mathcal{Rm}|^{p-1} \phi^2 p^p dV_t \\
+ CK \int_M |\mathcal{Rm}|^{p-1} |\nabla u|^2 \phi^2 p^p dV_t + KCp \int_M |\nabla u|^2 \phi^2 p^p dV_t
\]
then \(U\) satisfies the following estimate
\[
U' \leq \left[CK^2 + CKL^2 + C(p-1)KL^2\right] U + CK(K + L^2)A_4 + CKCpL^2 \text{Vol}_{g(t)}(\Omega).
\]
using
\[
e^{-2Kt} g(0) \leq g(t) \leq e^{2Kt} g(0)
\]
and
\[
|\nabla_{g(t)} \phi|_{g(t)} \leq e^{KT} |\nabla_{g(0)} \phi|_{g(0)} \leq \sqrt{\text{Vol}_K} e^{KT}/p.
\]
we can estimate \(A_4\) as follows:
\[
A_4 = \int_M |\mathcal{Rm}|^{p-1} |\nabla \phi|^2 \phi^2 p^{2p-2} dV_t \leq \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\mathcal{Rm}|^{p-1} \phi^2 p^{2p-2} dV_t \leq A_1 + Kp^2 \phi g_{i\bar{j}} \rho^{-2p} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right)
\]
Hence
\[
U' \leq \Lambda_1 U + \left[CK(K + L^2)K^p e^{2pKt} \rho^{-2p} + CKCpL^2\right] \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right),
\]
where \(\Lambda_1 := C(p-1)KL^2 + CK(K + L^2)\) is a constant. The Bishop-Gromov volume comparison theorem shows that the inequality
\[
\text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right) \leq e^{T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right)
\]
holds for all \(0 \leq t \leq \tau \leq T\). consequently, we arrive at
\[
U' \leq \Lambda_1 U + \Lambda_2 e^{T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right),
\]
with \(\Lambda_2 := CK(K + L^2)K^p e^{2pKt} \rho^{-2p} + CKCpL^2\). This implies that
\[
\frac{d}{dt} \left(e^{-\Lambda_1 t} U(t)\right) \leq \Lambda_2 e^{T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right).
\]
Upon integration over \([0, \tau]\), it yields
\[
U(\tau) \leq e^{\Lambda_1 T} \left(U(0) + \Lambda_2 \text{Vol}_{g(\tau)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}}\right)\right)\right).
\]
Now we consider
\[
U(0) = \left(A_1 + \frac{CK}{p-1} A_2 + KCpS + CKS + C \int_M |\text{Ric}|^2 |\mathcal{Rm}|^{p-1} \phi^2 p^p dV_t\right)_{t=0}
\]
We have proved that
\[ A_4 \leq A_1 + \Lambda_2^2 \rho p K T \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right). \]

According to the definition, it is clear that
\[
S = \int_M |\text{Rm}|^{p-1} |\nabla u|^2 \phi^{2p} dV \leq \int M A_2,
\]
\[
\tilde{S} = \int_M |\nabla u|^2 \phi^{2p} dV \leq C L^2 \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right).
\]

Applying Young’s inequality to \( A_2 \), we get
\[
A_2 = \int_M |\text{Rm}|^{p-1} \phi^{2p} dV = \int_M \left( |\text{Rm}|^{p-1} \phi^{2p-2} \right) \phi^2 dV \\
\leq \frac{p-1}{p} \int_M |\text{Rm}|^p \phi^{2p} dV + \frac{1}{p} \int_M \phi^2 dV \\
\leq A_1 + C \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right).
\]

The obvious estimate
\[
\int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV \leq K^2 A_2
\]
tells us that
\[
U(0) \leq \left( \frac{CK}{p-1} + CK^2 + CKL^2 \right) \int_M |\text{Rm}(g(0))|^p \phi^{2p} dV_{g(0)} \\
+ \left( \frac{CK}{p-1} + C + CKL^2 \right) \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right)
= \Gamma_1 e^{-\Lambda_1 T} \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}(g(0))|^p \phi^{2p} dV_{g(0)} \\
+ \left( \Gamma_2 e^{-\Lambda_1 T} - \Lambda_2 \right) \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right),
\]
where
\[
\Gamma_1 := e^{\Lambda_1 T} \left( \frac{CK}{p-1} + CK^2 + CKL^2 \right), \quad \Gamma_2 := e^{\Lambda_1 T} \left( \frac{CK}{p-1} + C + CKL^2 + \Lambda_2 \right).
\]

Plug it into the differential inequality and we obtain for \( p \geq 2 \)
\[
\int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}(g(t))|^p dV_{g(t)} \\
\leq \Gamma_1 \int_M |\text{Rm}(g(0))|^p \phi^{2p} dV_{g(0)} + \Gamma_2 \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right) \\
\leq \Gamma_1 \int_{B_{g(0)}(x_0, \rho/2\sqrt{K})} |\text{Rm}(g(0))|^p dV_{g(0)} + \Gamma_2 \text{Vol}_{g(0)} \left( B_{g(0)} \left( x_0, \frac{\rho}{\sqrt{K}} \right) \right).
\]
We finished the proof. \( \square \)

As it will be needed in the following discussion, We also restate the Theorem \[1.1\] to emphasize the power of \( p \), which can be easily obtained from \( \Gamma_1, \Gamma_2, \Lambda_1, \Lambda_2 \):
From (2.5), (2.6) and (2.7) in [14], we have:

Proposition 2.9. We have

Combine them and we prove the proposition. □

2.2. Proof of Propositions 2.1–2.5 In this subsection we give proofs of Proposition 2.1–2.5.

Proposition 2.8. We have

\[
\frac{d}{dt} A_1 \leq B_1 + CKB_2 + CKA_4 + C(K + L^2)A_1 + CT_p
\]

Proof. Compute

\[
\frac{d}{dt} \left( \int_M |\text{Rm}|^p \phi^2 dV_i \right) = \int_M (\partial_t |\text{Rm}|^p) \phi^2 dV_i + \int_M |\text{Rm}|^p \phi^2 (-R + 2|\nabla u|^2) dV_i
\]

\[
= \frac{p}{2} \int_M |\text{Rm}|^{p-2} |\nabla^2 \text{Ric} \ast \text{Rm} + \text{Ric} \ast \text{Rm} \ast \text{Rm} + \text{Rm} \ast \nabla^2 u \ast \nabla^2 u + \text{Rm} \ast \nabla u \ast \nabla u \ast \phi^2| dV_i
\]

\[
\leq \int_M |\text{Rm}|^{p-2} (\nabla^2 \text{Ric} \ast \text{Rm}) \phi^2 dV_i + CKA_1 + CT_p + CL^2 A_1
\]

From (2.5), (2.6) and (2.7) in [14], we have:

\[
C \int_M |\text{Rm}|^{p-2} (\nabla^2 \text{Ric} \ast \text{Rm}) \phi^2 dV_i \leq B_1 + CKB_2 + CKA_4.
\]

Combine them and we prove the proposition. □

Proposition 2.9. We have

\[
B_1 \leq CKB_2 + C(K + L^2)A_1 + CKL^2 A_2 + CKA_4
\]

\[
+ C_0 T_p - \frac{1}{2K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^2 dV_i \right).
\]
Proof. From the evolution equation of $|\text{Ric}|^2$ (see \[21\]), we can deduce that

$$
|\nabla \text{Ric}|^2 = -\frac{1}{2}|\nabla|\text{Ric}|^2 + 2R_{pjqk} R^{pq} R^{k} - 4R_{pjqk} R^{ij} \nabla^p u \nabla^q u \\
+ 4\Delta u R^{ij} \nabla_i \nabla_j u - 4R^{ij} \nabla_i \nabla_k \nabla^k \nabla_j u - 4R_{ij} R_k^i \nabla^i u \nabla^j u \\
\leq -\frac{1}{2}|\text{Ric}|^2 + CK(L^2 + K)|\text{Rm}| + CK|\nabla^2 u|^2 + CK^2 L^2,
$$

in which we used the fact that $|\Delta u| \leq \sqrt{\pi}|\nabla^2 u|$. Hence we have

$$
B_1 \leq \int_M \left[ \frac{1}{2K}(\Delta - \partial_t)|\text{Ric}|^2 + C(L^2 + K)|\text{Rm}| \\
+ CKL^2 + C|\nabla^2 u|^2 \right] |\text{Rm}|^{-1} \phi^2 u \, dV_t
$$

$$
= \frac{1}{2K} \int_M \left[ (\Delta - \partial_t)|\text{Ric}|^2 \right] |\text{Rm}|^{-1} \phi^2 u \, dV_t \\
+ C(L^2 + K)A_1 + CKL^2 A_2 + CT_p
$$

$$
= \frac{1}{2K} \int_M (\Delta|\text{Ric}|^2) |\text{Rm}|^{-1} \phi^2 u \, dV_t + C(L^2 + K)A_1 + CKL^2 A_2 + CT_p
$$

$$
- \frac{1}{2K} \int_M \left[ \partial_1(|\text{Ric}|^2|\text{Rm}|^{-1} \phi^2 u \, dV_t) \\
- |\text{Ric}|^2 (\partial_1|\text{Rm}|^{-1} \phi^2 u \, dV_t) - |\text{Ric}|^2 |\text{Rm}|^{-1} \phi^2 (-R + 2|\nabla u|^2) \, dV_t \right]
$$

$$
= -\frac{1}{2K} \left( \int_M \langle \nabla|\text{Ric}|^2, \nabla|\text{Rm}|^{-1} \phi^2 u \, dV_t + \int_M \langle \nabla|\text{Ric}|^2, \nabla \phi^2 u \rangle |\text{Rm}|^{-1} \phi^2 \, dV_t \right)
$$

$$
- \frac{1}{2K} \left( \frac{d}{dt} \int_M |\text{Ric}|^2 |\text{Rm}|^{-1} \phi^2 u \, dV_t \right) + C(L^2 + K)A_1 + CKL^2 A_2
$$

$$
+ CT_p + \frac{1}{2K} \int_M |\text{Ric}|^2 (\partial_1|\text{Rm}|^{-1} \phi^2 u \, dV_t
$$

From the proof of (2.13)-(2.15) in \[14\], we can deduce:

$$
\frac{C}{K} \int_M |\text{Ric}|^2 |\text{Rm}|^{-3} \phi^2 u (\nabla^2 \text{Ric} \ast \text{Rm}) \, dV_t \leq \frac{1}{5} B_1 + CKB_2 + CKA_4
$$

Then we can write:

$$
\frac{1}{2K} \int_M (\partial_1|\text{Rm}|^{-1} \phi^2 u \, dV_t = \frac{p - 1}{4K} \int_M |\text{Ric}|^2 (|\text{Rm}|^{-3} \partial_1|\text{Rm}|^2) \phi^2 u \, dV_t
$$

$$
= \frac{C}{K} \int_M |\text{Ric}|^2 |\text{Rm}|^{-3} \phi^2 u (\nabla^2 \text{Ric} \ast \text{Rm} + \text{Ric} \ast \text{Rm} \ast \text{Rm} +
$$

$$
\text{Rm} \ast \nabla^2 u \nabla^2 u \ast \text{Rm} \ast \text{Rm} \ast \nabla u \nabla u \, dV_t
$$

$$
\leq \frac{1}{5} B_1 + CKB_2 + CKA_1 + CKL^2 A_2 + CKA_4 + CT_p
$$
From (2.10) and (2.11) in \[14\], we have:

\[
-\frac{1}{2K} \int_M (\nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1}) \phi^{2p} dV_t \leq \frac{1}{10} B_1 + CK B_2
\]

\[
-\frac{1}{2K} \int_M (\nabla |\text{Ric}|^2, \nabla \phi^{2p}) |\text{Rm}|^{p-1} dV_t \leq \frac{1}{10} B_1 + CA_4
\]

Plugging them all together and we arrive at Proposition 2.2.

As already stated in notations that all \(C\) are irrelevant constants, while \(C_0\) in Proposition 2.2 is a special constant used latter.

**Proposition 2.10.** We have

\[
B_2 \leq -\frac{1}{p-1} \frac{d}{dt} \left( \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + CA_1 + CA_4 + CL^2 A_2 + CT_{p-1}
\]

**Proof.** Using the evolution inequality of \(|\text{Rm}|\) (see \[21\]), we can obtain:

\[
B_2 \leq \int_M \left[ \frac{1}{2} (\Delta - \partial_t) |\text{Rm}|^2 + C |\text{Rm}|^3 + CL^2 |\text{Rm}|^2 + C |\nabla^2 u|^2 |\text{Rm}| \right] |\text{Rm}|^{p-3} \phi^{2p} dV_t
\]

\[
= \frac{1}{2} \int_M (\Delta |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t + CA_1 + CL^2 A_2
\]

\[
+ T_{p-1} - \frac{1}{2} \int_M (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t
\]

\[
\leq C \int_M |\nabla \text{Rm}| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t + CA_1 + CL^2 A_2
\]

\[
+ T_{p-1} - \frac{1}{2} \int_M (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t
\]

Following the proof of (2.18)-(2.19) in \[14\],

\[
-\frac{1}{2} \int_M (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t
\]

\[
= -\frac{1}{2} \int_M (\partial_t (|\text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t)
\]

\[
= -|\text{Rm}|^2 (\partial_t |\text{Rm}|^{p-3} \phi^{2p} dV_t - |\text{Rm}|^{p-1} \phi^{2p} \partial_t dV_t)
\]

\[
= -\frac{1}{2} \partial_t A_2 + \frac{p-3}{4} \int_M (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t
\]

\[
- \frac{1}{2} \int_M R |\text{Rm}|^{p-1} \phi^{2p} dV_t + \int_M |\text{Rm}|^{p-1} |\nabla u|^2 \phi^{2p} dV_t.
\]

Therefore, we can find:

\[
-\frac{1}{2} \int_M (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t \leq -\frac{1}{p-1} \partial_t A_2 + CA_1 + CA_4 + CL^2 A_2 + CT_{p-1}
\]

In summary we can find

\[
B_2 \leq -\frac{1}{p-1} \frac{d}{dt} \left( \int_M |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + CA_1 + CA_4 + CL^2 A_2 + CT_{p-1}
\]

and finish the proof. \(\square\)
Proposition 2.11. For any \( p \geq 2 \), \( T_p \) satisfy the following estimate

\[
T_p \leq -\frac{d}{dt} S - \frac{C}{p-1} \frac{d}{dt} A_2 - \frac{C d}{K} \frac{d}{dt} \left( \int_M |Ric|^2 |Rm|^{p-1} \phi^{2p} dV_t \right) + C(K + L^2) A_1 + CKL^2 A_2 + C(K + L^2) A_4 + C^{p-1} T_1
\]

**Proof.** We consider the quantity:

\[
\frac{d}{dt} \left( \int_M |Rm|^{p-1} |\nabla u|^2 \phi^{2p} dV_t \right)
\]

\[
= \int_M (\partial_t |Rm|^{p-1}) |\nabla u|^2 \phi^{2p} dV_t
\]

\[
- \int_M |Rm|^{p-1} |\nabla u|^2 (R - 2 |\nabla u|^2) \phi^{2p} dV_t
\]

\[
+ \int_M |Rm|^{p-1} (\Delta |\nabla u|^2 - 2 |\nabla^2 u|^2 - 4 |\nabla u|^4) \phi^{2p} dV_t,
\]

which infer:

\[
T_p = \int_M |Rm|^{p-1} |\nabla^2 u|^2 \phi^{2p} dV_t
\]

\[
= -\frac{1}{2} \int_M |Rm|^{p-1} |\nabla u|^2 \phi^{2p} dV_t
\]

\[
+ \frac{1}{2} \int_M (\partial_t |Rm|^{p-1}) |\nabla u|^2 \phi^{2p} dV_t
\]

\[
- \frac{1}{2} \int_M |Rm|^{p-1} |\nabla u|^2 (R - 2 |\nabla u|^2) \phi^{2p} dV_t
\]

\[
+ \int_M |Rm|^{p-1} \left( \frac{1}{2} \Delta |\nabla u|^2 - 2 |\nabla u|^4 \right) \phi^{2p} dV_t.
\]

Using

\[
(2.1) \quad \Box |\nabla u|^2 = -2 |\nabla^2 u|^2 - 4 |\nabla u|^4,
\]

from [21] we yields that \( R - 2 |\nabla u|^2 \geq -C \) and then

\[
-\frac{1}{2} \int_M |Rm|^{p-1} |\nabla u|^2 (R - 2 |\nabla u|^2) \phi^{2p} dV_t \leq CS.
\]

Therefore, we arrive at

\[
T_p \leq -\frac{1}{2} \frac{d}{dt} \int_M |Rm|^{p-1} |\nabla u|^2 \phi^{2p} dV_t
\]

\[
+ \frac{1}{2} \int_M |\nabla u|^2 (\partial_t |Rm|^{p-1}) \phi^{2p} dV_t
\]

\[
+ CS + \frac{1}{2} \int_M |Rm|^{p-1} \Delta |\nabla u|^2 \phi^{2p} dV_t.
\]
Notice that by the evolution equation of \(|Rm|^2\) (see [21])

\[
\frac{1}{2} \int_M |\nabla u|^2 (\partial_t |Rm|^p - 1) \phi^2 p dV_t
\]

\[
= \frac{1}{2} \int_M |\nabla u|^2 (\nabla^2 \text{Ric} \ast \text{Rm} + \text{Ric} \ast \text{Rm} \ast \text{Rm} + \nabla^2 u \ast \nabla^2 u

+ \text{Rm} \ast \text{Rm} \ast \nabla u \ast \nabla u) |\text{Rm}|^{p-3} \phi^2 p dV_t
\]

\[
\leq C \int_M |\nabla u|^2 \ast \nabla^2 \text{Ric} \ast |\text{Rm}|^{p-2} \phi^2 p dV_t

+ CL^2 A_1 + CL^2 T_{p-1} + CL^2 S
\]

\[
= -C \int_M \langle \nabla |\nabla u|^2, \nabla \text{Ric} \rangle |\text{Rm}|^{p-2} \phi^2 p dV_t

- CL^2 \int_M \langle \nabla |\text{Rm}|^2, \nabla \text{Ric} \rangle |\text{Rm}|^{p-4} \phi^2 p dV_t

- CL^2 \int_M \langle \nabla \phi, \nabla \text{Ric} \rangle |\text{Rm}|^{p-2} \phi^2 p dV_t

+ CL^2 A_1 + CL^2 T_{p-1} + CL^2 S
\]

\[
\leq C \int_M |\nabla^2 u||\nabla u||\nabla \text{Ric}||\text{Rm}|^{p-2} \phi^2 p

+ CL^2 \int_M |\nabla \text{Rm}||\nabla \text{Ric}||\text{Rm}|^{p-3} \phi^2 p dV_t

+ CL^2 \int_M |\nabla \phi||\nabla \text{Rm}||\text{Rm}|^{p-2} \phi^2 p dV_t

+ CL^2 A_1 + CL^2 T_{p-1} + CL^2 S
\]

\[
\leq CT_{p-2} + \frac{1}{8C_0} B_1 + CL^2 B_2 + CA_4 + CL^2 A_1 + CL^2 T_{p-1} + CL^2 S
\]

Applying integrating by parts, the last term becomes

\[
\int_M |\text{Rm}|^{p-1} \Delta |\nabla u|^2 \phi^2 p dV_t
\]

\[
= -\int_M \langle \nabla |\nabla u|^2, \nabla |\text{Rm}|^{p-1} \phi + 2p |\text{Rm}|^{p-1} \nabla \phi \rangle \phi^{2p-1} dV_t
\]

\[
\leq 2C \int_M |\nabla^2 u||\nabla u||\nabla \text{Rm}||\text{Rm}|^{p-2} \phi^2 p dV_t

+ 2C \int_M |\nabla^2 u||\nabla u||\nabla \phi||\text{Rm}|^{p-1} \phi^{2p-1} dV_t
\]

\[
\leq \frac{1}{8} T_p + 8CL^2 B_2 + \frac{1}{8} T_p + 8CL^2 A_4
\]

Plugging them into the inequality of \(T_p\), we obtain

\[
T_p \leq -\frac{1}{2} \partial_t S + CT_{p-2} + \frac{1}{8C_0} B_1 + CL^2 A_1

+ CL^2 T_{p-1} + CL^2 S \frac{1}{8} T_p + CL^2 B_2 + CL^2 A_4
\]
Replacing $B_1$ by using Proposition 2.9 yields

\[
T_p \leq -\frac{1}{2} \partial_t S - \frac{p}{p-1} \partial_t A_2 - \frac{1}{16C_0K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dv \right) + \frac{3}{8} T_p + 2(8C)^2 T_1 + \frac{\partial}{\partial t} \left[ C(K + L^2) A_1 \right] + C K L^2 A_2
\]

Using the relationship between $T_k$ (see Lemma 2.3), we can write inequalities:

\[
CT_{p-2} \leq C \left[ \frac{1}{8C} T_p + 2 \frac{\partial}{\partial t} A_1 \right] \leq \frac{1}{8} T_p + 2(8C)^2 T_1
\]

to replace $CT_{p-2}$ and we will get:

\[
T_p \leq -\frac{1}{2} \partial_t S + \left( \frac{3}{8} - \frac{1}{16C_0K} \right) \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dv \right) + 2(8C)^2 T_1 + C(K + L^2) A_1 + C K L^2 A_2
\]

Replacing $B_2$ by using Proposition 2.4 we obtain

\[
T_p \leq -\frac{1}{2} \partial_t S - \frac{p}{p-1} \partial_t A_2 - \frac{1}{16C_0K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dv \right) + \frac{3}{8} T_p + 2(8C)^2 T_1 + C(K + L^2) A_1 + C K L^2 A_2
\]

Again we can write

\[
CL^2 T_{p-1} \leq CL^2 \left[ \frac{1}{8CL^2} T_p + (8CL^2)^{p-2} T_1 \right] = \frac{1}{8} T_p + (8CL^2)^p T_1
\]

Plugging it into the inequality and we finally have

\[
T_p \leq -\frac{1}{2} \partial_t S - \frac{p}{p-1} \partial_t A_2 - \frac{1}{16C_0K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dv \right) + \frac{1}{2} T_p + C(K + L^2) A_1 + C K L^2 A_2 + C^{p-1} T_1 + CL^2 S + C(K + L^2) A_4
\]

which infer:

\[
T_p \leq -\partial_t S + CL^2 \left[ \frac{C}{K} \frac{d}{dt} \left( \int_M |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dv \right) \right] + C(K + L^2) A_1 + C K L^2 A_2 + C^{p-1} T_1 + CL^2 S + C(K + L^2) A_4
\]

Then we finish the proof. \(\square\)

**Proposition 2.12.** $T_1$ satisfy the following estimate

\[
T_1 \leq -\partial_t S + CL^2 \text{Vol}_{\partial(t)}(\Omega)
\]
Proof. Consider the quantity:

\[ \partial_t S = \partial_t \int_M |\nabla u|^2 \phi^2 \, dV \]

\[ = \int_M (\Delta |\nabla u|^2 - 2 |\nabla^2 u|^2 - 4 |\nabla u|^4) \phi^2 \, dV + \int_M |\nabla u|^2 \phi^2 (-R + 2 |\nabla u|^2) \, dV \]

\[ \leq -2T_1 + \int_M |\nabla u|^2 \phi^2 \, dV + C L^2 \int_M \phi^2 \, dV \]

\[ \leq -2T_1 + 2C \int_M |\nabla u|^2 |\nabla u||\nabla \phi| \phi^{2p} \, dV + CL^2 \text{Vol}_{g(t)}(\Omega) \]

\[ \leq -T_1 + C \int_M |\nabla u|^2 |\nabla \phi| \phi^{2p-2} \, dV + CL^2 \text{Vol}_{g(t)}(\Omega) \]

\[ \leq -T_1 + CL^2 \text{Vol}_{g(t)}(\Omega) \]

\[ \square \]

3. The Extension of the Ricci-Harmonic Flow

As [22] has proved, the flow can be extended over \( T \) if the Riemannian curvature is bounded at each point. First we prove

**Lemma 3.1.** There exist constants \( C \) such that the following estimate

\[ \Box |Rm| \leq C |Rm|^2 + C |\nabla^2 u|^2 + C \]

holds.

**Proof.** Using the evolution equation of \( |Rm|^2 \) (see Chapter 2.7 in [21]), we obtain:

\[ \Box |Rm|^2 = 2 |Rm| (\partial_t |Rm|) - 2 |Rm| (\Delta |Rm|) - 2 |\nabla |Rm| |^2 \]

\[ = 2 |Rm| (\Box |Rm|) - 2 |\nabla |Rm| |^2 \]

\[ \leq -2 |\nabla Rm|^2 + C |Rm|^3 + C |Rm||\nabla^2 u|^2 + C |\nabla u|^2 |Rm|^2 \]

From \( |\nabla Rm| \geq |\nabla |Rm|| \) and assumption (2), we can get

\[ \Box |Rm| \leq C |Rm|^2 + C |\nabla^2 u|^2 + CL^2 |Rm| \]

\[ \leq C |Rm|^2 + C |\nabla^2 u|^2 + CL^2 (|Rm|^2 + 1) \]

\[ = C |Rm|^2 + C |\nabla^2 u|^2 + C \]

which gives the desired estimate. \( \square \)

Now we prove Theorem [1.4]

**Theorem 3.2.** Let \((g(t), u(t))\) be a smooth solution to the Ricci-harmonic flow on \( M \times [0, T) \) with \( T < \infty \), where \( M \) is a complete \( n \)-dimensional manifold. If \((M, g(0))\) is complete and:

\[ \sup_M |Rm(g(0))|_{g(0)} < \infty, \quad \sup_{M \times [0, T)} |Ric(g(t))|_{g(t)} < \infty, \]

then \( |Rm| \) is locally bounded and \( g(t) \) extends smoothly to a complete solution on \([0, T + \epsilon)\) for some constants \( \epsilon > 0 \).
Proof. According to Remark 1.2 we can denote
\[ K := \sup_{M \times [0, T)} |\text{Ric}|(x, t) < \infty, \quad L := \sup_{M \times [0, T)} |\nabla u|(x, t) < \infty. \]
According to Lemma 3.1 we can pick a constant \( C_m \geq 2 \) that is sufficiently large so that
\[ \square |\text{Rm}| \leq C_m(|\text{Rm}|^2 + 2|\nabla^2 u|^2 + 1) \]
Plugging it with evolution equation (2.1) we can find
\[
(\partial_t - \Delta)(|\text{Rm}| + C_m|\nabla u|^2 + 1) = (\partial_t - \Delta)(|\text{Rm}| + C_m|\nabla^2 u|^2)
\]
\[
= C_m(|\text{Rm}|^2 - 4|\nabla u|^4 + 1)
\]
\[
\leq C_m(|\text{Rm}|^2 + C_m^2|\nabla u|^4 + 1)
\]
\[
\leq C_m(|\text{Rm}| + C_m|\nabla u|^2 + 1)^2
\]
On the other hand,
\[
\int_{\Omega} (|\text{Rm}| + C_m|\nabla u|^2 + 1)^p dV_{g(t)} \leq 3^{p-1} \int_{\Omega} (|\text{Rm}|^p + C_m^p|\nabla u|^{2p} + 1) dV_{g(t)}
\]
\[
\leq 3^{p-1} \int_{\Omega} |\text{Rm}|^p dV_{g(t)} + 3^{p-1}(C_m^pL^{2p} + 1) \text{Vol}_{g(t)}(\Omega)
\]
Define
\[ \Phi := |\text{Rm}| + C_m|\nabla u|^2 + 1 \]
and then the above propositions gives
\[
\left( \int_{\Omega} \Phi^p dV_{g(t)} \right)^{\frac{1}{p}} \leq \left( 3^{p-1} \int_{\Omega} |\text{Rm}|^p dV_{g(t)} + 3^{p-1}(C_m^pL^{2p} + 1) \right)^{\frac{1}{p}}
\]
\[
\leq 3 \left( \int_{\Omega} |\text{Rm}|^p dV_{g(t)} \right)^{\frac{1}{p}} + 3C_mL^2 + 3
\]
\[
\leq 3 \left[ C e^{\Lambda \rho^{-1} (\Lambda + K \rho^{-2p})} \right]^{\frac{1}{p}} + 3C_mL^2 + 3
\]
\[
\leq C(1 + \Lambda) + 3K\rho^{-2} + 3C_mL^2 + 3
\]
\[
:= C_n,
\]
which is a constant independent of \( p \). We also have
\[
(\partial_t - \Delta)\Phi \leq C_m\Phi^2.
\]
The progress to give uniform bound from \( L^p \) estimate is an essentially routine applying De Giorgi-Nash-Moser iteration presented in Lemma 19.1 of [15]. We write \( f = u = \Phi \) and the above inequality shows that
\[
\partial_t u \leq \Delta u + Cf u
\]
weakly on \( M \times [0, T] \). It is equivalent to say that for fixed \( a \geq 1 \)
\[
(3.1) \quad -\int_M q^2 u^{2a-1} \Delta u dV_{g(t)} + \frac{1}{2a} \int_M q^2 \partial_t (u^{2a}) dV_{g(t)} \leq C \int_M q^2 u^{2a} f dV_{g(t)}
\]
for any $t \in [0, T]$ and non-negative Lipschitz function $\varphi$ whose support is compactly contained in $B_{g(0)}(x_0, \rho/2\sqrt{K})$. Integrate by part and notice that $a \geq 1$, we obtain

$$\int_M \varphi^2 u^{2a-1} \Delta u dV_{g(t)}$$

$$= 2 \int_M \varphi u^{2a-1} (\nabla u, \nabla \varphi) dV_{g(t)} + (2a - 1) \int_M \varphi^2 u^{2a-2} |\nabla u|^2 dV_{g(t)}$$

$$\geq \frac{1}{a} \int_M 2a \varphi u^{2a-1} (\nabla u, \nabla \varphi) dV_{g(t)} + \frac{1}{a} \int_M a^2 \varphi^2 u^{2a-2} |\nabla u|^2 dV_{g(t)}$$

$$= \frac{1}{a} \int_M |\nabla (\varphi u^a)|^2 dV_{g(t)} - \frac{1}{a} \int_M |\nabla \varphi|^2 u^{2a} dV_{g(t)}$$

For Ricci-Harmonic flow, we have $\partial_t dV_{g(t)} = (-R + 2|\nabla u|^2) dV_{g(t)}$, and furthermore

$$|R - 2|\nabla u|^2| \leq |R| + 2|\nabla u|^2 \leq C \left(|Rm| + C_m |\nabla u|^2 + 1\right) = C\Phi = Cf,$$

we then arrive at

$$\int_M \varphi^2 \partial_t (u^{2a}) dV_{g(t)} = \frac{d}{dt} \left( \int_M \varphi^2 u^{2a} dV_{g(t)} \right) - \int_M \varphi^2 u^{2a} (R - 2|\nabla u|^2) dV_{g(t)}$$

$$\geq \frac{d}{dt} \left( \int_M \varphi^2 u^{2a} dV_{g(t)} \right) - C \int_M \varphi^2 u^{2a} f dV_{g(t)}.$$

Plugging the above two inequalities into (3.1) implies

$$\int_M |\nabla (\varphi u^a)|^2 dV_{g(t)} + \frac{1}{2} \frac{d}{dt} \left( \int_M \varphi^2 u^{2a} dV_{g(t)} \right)$$

$$\leq Ca \int_M \varphi^2 u^{2a} f dV_{g(t)} + \int_M |\nabla \varphi|^2 u^{2a} dV_{g(t)}.$$

Following (3.6)-(3.11) of [14] for the rest of the steps with $B = B_{g(0)}(x_0, \rho/2\sqrt{K})$, we obtain the following inequality

$$\sup_{B_{\hat{g}(0)}(0, \frac{\rho}{2\sqrt{K}}) \times [\frac{T}{4}, T]} u \leq Ce^{C(T + \frac{1}{\sqrt{K}})} \left( A^a + \left( \frac{\rho}{\sqrt{K}} \right)^{-2} + T^{-1} \right)^{-\frac{2\mu - 1}{p(\mu - 1) - p}} A,$$

where $\alpha = \frac{\mu(\mu - 1)}{p(\mu - 1) - p}$ and $\mu = \mu(n) \leq \frac{n}{n-2}$ is given by the Sobolev inequality (see [14]). $A$ is the average $L^p$ estimate of $f$, i.e.

$$A := \sup_{t \in [0, T]} \left( \int_B f^p(t) dV_0 \right)^{\frac{1}{p}}.$$

Apply the following result back to $\Phi$ and we get the local uniform bound for $\Phi$ near $T$:

$$\sup_{B_{g(0)}(0, \frac{\rho}{2\sqrt{K}}) \times [\frac{T}{4}, T]} \Phi \leq Ce^{C(T + \frac{1}{\sqrt{K}})} \left( 1 + C_n^\mu \left( \frac{K}{\rho^\alpha} + T^{-1} \right)^{\beta'} \right),$$
where constants $\alpha', \beta'$ only depend on $n$ and other constants may depend on $n, K, L, \rho, \Lambda, C_m$ but not $p$. Finally, since:

$$\lim_{t \to T} |R_m| \leq \lim_{t \to T} \Phi < \infty$$

satisfied and by the Theorem 6.22 of [21], we immediately yield that the the Ricci-Harmonic flow can be smoothly extended past $T$. □

REFERENCES

[1] Abolarinwa, Abimbola; Adebimpe, Olukayode; Bakare Emmanuel A., Monotonicity formulas for the first eigenvalue of the weighted $p$-Laplacian under the Ricci-harmonic flow, Journals of inequalities and applications, 10(2019), 1-16.

[2] Abolarinwa, Abimbola; Oladejo, Nathaniel K.; Salawu, Sulyman O., On the Entropy Formulas and Solutions for the Ricci-Harmonic Flow, Bulletin of the Iranian Mathematical Society, 45(2019), 1177-1192.

[3] Azami, Shahroud, Some results of evolution of the first eigenvalue of weighted $p$-laplacian along the extended Ricci flow, Commun. Korean Math. Soc., 35(2020), no. 3, 953-966.

[4] Chow, B., Lu, Peng; Ni, Lei, Hamilton’s Ricci flow, Gradient Studies in Mathematics, 77, American Mathematical Society, New York, 2006.

[5] Cao, Xiaodong; Guo Hongxin; Tran Hung, Harnack estimates for conjugate heat kernel on evolving manifolds, Math. Z., 281(2015), 201-214.

[6] Chen, Liang; Zhu, Anqiang, On the extension of the harmonic Ricci flow, Geom. Dedicata, 164(2013), 179-185.

[7] Ehlers, J.; Kundt, W., Exact solutions of the gravitational field equations, Gravitation: An introduction to current research, pages 49–101. John Wiley & Sons, Inc., New York, London, 1962.

[8] Fang, Shouwen; Zheng, Tao, An upper bound of the heat kernel along the harmonic-Ricci flow, Manu. Math., 151(2016), 1-18.

[9] Guo, Bin; Huang, Zhijie; Phong, Duong H., Pseudo-locality for a coupled Ricci flow, Comm. Anal. Geom., 26(2018), no. 3, 585-626.

[10] Guo, Hongxin; Philipowski, Robert; Anton Thalmaier, Entropy and lowest eigenvalue on evolving manifolds, Pacific J. Math., 264(2013), no. 1, 61-81.

[11] Guo, Hongxin; Philipowski, Robert; Anton Thalmaier, A stochastic approach to the harmonic map heat flow on manifolds with time-dependent Riemannian metric, Stochastic Process, 124(2014), no. 11, 3535-3552.

[12] Guo, Hongxin; Philipowski, Robert; Anton Thalmaier, An entropy formula for the heat equation on manifolds with time-dependent metric, application to ancient solutions, Potential Anal., 42(2015), no. 2, 483-497.

[13] Huang, Guangyue; Li Zhi, Monotonicity Formulas of Eigenvalues and Energy Functionals Along the Rescaled List’s Extended Ricci Flow, Mediterr. J. Math. 15(2018), Article number 63.

[14] Kotschwar, Brett; Munteanu, Ovidiu; Wang, Jiaping, A local curvature estimate for the Ricci flow, J. Funct. Anal., 271(2016), no. 9, 2604-2630.

[15] Li, Peter, Geometric Analysis, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Press, Cambridge, 2012.

[16] Li, Yi, Eigenvalues and entropies under the harmonic-Ricci flow, Pacific J. Math., 267(2014), no. 1, 141-184.

[17] Li, Yi, Long time existence of Ricci-harmonic flow, Front. Math. China, 11(2016), no. 5, 1313-1334.

[18] Li, Yi, Long time existence and bounded scalar curvature in the Ricci-harmonic flow, J. Differ. Equ., 265(2018), no. 1, 69-97.

[19] Li, Yi, Local curvature estimates for the Ricci-harmonic flow, arXiv:1810.09760

[20] Liu, Xiangao; Wang, Kui, A Gaussian upper bound of the conjugate heat equation along Ricci-harmonic flow, Pacific J. Math., 287(2017), no. 2, 465-484.

[21] List, B., Evolution of an extended Ricci flow system, PhD thesis, AEI Potsdam, 2005.

[22] List, B., Evolution of an extended Ricci flow system, Commun. Anal. Geom., 16(2008), no. 5, 1007-1048.

[23] Müller, R., The Ricci flow coupled with harmonic map flow, PhD thesis, ETH Zürich, 2012.

[24] Müller, R., Ricci flow coupled with harmonic map flow, Ann. Sci. Ec. Norm. Super., 45(2012), no. 4.

[25] Wu, Guoqiang; Zheng, Yu, Sharp logarithmic sobolev inequalities along an extended Ricci flow and applications, Pacific J. Math., 298(2019), no. 2, 464-509.
[26] Wu, Guoqiang; Zheng, Yu, *On the extension of Ricci harmonic flow*, Results Math, 75(2020), Article number 55.

[27] Yang, Fei; Shen, JingFang, *Volume growth for gradient shrinking solitons of Ricci-harmonic flow*, Science China Mathematics, 55(2012), no. 6, 1221-1228.

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