Skyrmion dynamics in quantum Hall ferromagnets

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(May 8, 2018)

Abstract

Exploring a classical solution of the non-linear sigma model for a quantum Hall ferromagnet, a skyrmion-magnon effective hamiltonian is obtained via the collective coordinates method. Using the Feynman-Vernon functional integral formalism for this model we find the temperature dependent transport coefficients which characterize a single skyrmion dynamics.

I. INTRODUCTION

The concept of skyrmion was introduced by T. Skyrme when he represented the nucleon, a fermion, as a topological excitation (soliton) of the bosonic pion field \([1]\). In the quantum Hall ferromagnet, skyrmions are the solitons in the spin space of the non-linear field theory for electrons, and have spin much larger than \(\frac{1}{2}\).

The theory for the skyrmions, in the 2-D electron gas subject to strong magnetic fields, was developed by Shondi et al \([2]\) based on an effective generalized non-linear sigma model \((O(3))\) which was deduced from the Landau-Ginzburg theory of spinful electrons \([3]\). Shondi’s conjecture was experimentally confirmed by Barrett and co-workers in the optically pumped nuclear magnetic resonance experiments \([4]\). From both works, \([2]\) and \([4]\), it can be seen that this topological excitations are detectable in the quantum Hall ferromagnet when the exchange interaction dominates over the Zeeman coupling. This situation is possible due to the smallness of the effective Landé factor \((g)\) in some particular materials, as in the GaAs case. If this is so, the simple creation or destruction of an electron in the fully occupied spin polarized Landau level, \(\nu = 1\) for instance, do not produce the low energy configuration to be considered the lowest lying charged excitation. On the contrary, a non-uniform cylindrically symmetric spin configuration is produced, in which the spin at the centre is pointing in the wrong direction and smoothly rotates as one moves radially outwards until it reaches the right direction according to the polarized state of the 2-dimensional electron gas. Later on, Hartree-Fock numerical calculations \([5]\) showed that the energy of the quantum Hall skyrmion excitation is always smaller than the excitation energy of the localized spin \(\frac{1}{2}\).

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quasiparticle and quasihole, supporting the idea that the skyrmions are for sure the relevant quasiparticles in the \( g \rightarrow 0 \) limit.

In practice it is convenient to regard the skyrmion as a topological object. From this point of view they can be described as the non-singular mapping of the compactified 2-D real space \( S^2_{\text{phy}} \) onto the 2-D internal space \( S^2_{\text{int}} \) whose elements are the unitarily normalized spin vectors which cover a 2-D spherical surface. All non-singular maps of \( S_2 \) onto \( S_2 \) can be classified into homotopy classes that form a group isomorphic to the group of integers. And, it is using this set of integers, that we define the topological charge or Pontryagin index \( Q \), that classifies the skyrmions. This topological charge can be shown to be a simple measure of the degree of mapping of one surface onto the other and, as a conserved number, helps in finding explicit expressions for the spin configurations (see [8], for instance).

Until now many theoretical studies were devoted to the properties of the quantum Hall skyrmions. Most of them are done following two main approaches; the quantum computational calculations and the non-linear semi-classical field theory. In the first case, the quantum Hartree-Fock calculations are shown to be in excellent agreement with exact diagonalizations for small skyrmion sizes, as is demonstrated in [6]. However, when hydrostatic pressure is applied to the quantum Hall ferromagnet the \( g \) factor almost vanishes. In that case there is strong experimental evidence [9] that skyrmions of large size (about 33 reversed spins) are dominant. In this particular situation computational calculations are not appropriate, making the development of new approaches in the semi-classical field theoretical methods the natural way out to explain the experimental data.

In this paper, starting from a specific Lagragian density [2], we investigate the influence of the ferromagnetic spin wave modes in the temperature dependent dynamical properties of an isolated skyrmion. Our model basically consists of an effective field theory in which the single skyrmion momentum is coupled to the magnon bath total momentum. To begin with, sec II is devoted to obtaining the ferromagnetic spin wave modes by computing the quantum fluctuations of the quantum Hall ferromagnet around the non-linear sigma model classical solutions. Later on, using the standard collective coordinate method a second quantized hamiltonian that simulates the scattering of the spin waves by the skyrmion in 2-D is obtained.

In sec. III, using the well known Feynman-Vernon formalism we obtain an effective equation of motion for the centre of mass of the skyrmion. This equation, that contains all the information about the skyrmion-magnon interaction, is written in terms of a temperature dependent damping matrix function. In our treatment the possible contribution of inelastic process (Cherenkov like radiation) to the soliton dynamics is considered. Finally secs. IV and V are devoted to the explicit calculation of the damping constant matrix elements that define the skyrmion mobility and to present our conclusions.
II. EFFECTIVE SKYRMION-MAGNON HAMILTONIAN

As we said, the quantum Hall skyrmions can be described in terms of a unit vector field associated to the local pseudo-spin orientation as was proposed by Shondi [2]. The effective lagrangian density of the system can be written as

\[ L(n) = \alpha A [n(r)] \cdot \partial_t n(r) - V[n], \]  

(1)

where

\[ A[n(r)] = (\partial_t n \times \partial_r n) \]  

(2)

is the vector potential of the unit monopole, and the effective potential functional is that of a ferromagnet with long range interaction arising from the Coulomb repulsion \( V(r) = \frac{e^2}{r} \) between the underlying electrons. Explicitly,

\[ V[n] = \alpha' [\nabla n(r)]^2 + g\mu_B n(r) \cdot B + \frac{1}{2} \int d^2 r' V(r - r') q(r) q(r') \]  

(3)

where the deviation from the uniform background density \( \bar{\rho} = 1/2\pi l_0^2 \) is the skyrmion density \( q(r) \), which in terms of the \( n \) field has the following form

\[ q(r) = \frac{1}{8\pi} \varepsilon_{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n). \]  

(4)

Here \( \alpha' \) is the spin stiffness and \( l_0 \) stands for the magnetic length.

To take advantage of the internal space spherical symmetry, it is convenient to define the \( n \) vector field by means of two scalar fields \( \Theta(r, \theta) \) and \( \Phi(r, \theta) \) such that,

\[ n^x = \sin[\Theta(r, \theta)] \cos[\Phi(r, \theta)] \]
\[ n^y = \sin[\Theta(r, \theta)] \sin[\Phi(r, \theta)] \]
\[ n^z = \cos[\Theta(r, \theta)]. \]  

(5)

In terms of these new fields the effective potential functional (3) normalized to the Zeeman energy reads,

\[ V = \alpha' \left[ (\nabla \Theta)^2 - \sin^2 \Theta (\nabla \Phi)^2 \right] + g\mu_B \bar{\rho} (1 - \cos \Theta) + \frac{e^2}{2} \int \frac{q(r')}{|r - r'|} \left| \frac{\partial(\Theta, \Phi)}{\partial(r, \theta)} \right| \left| \frac{\partial(\Theta, \Phi)}{\partial(r', \theta')} \right| dr'^2, \]  

(6)

where \( \left| \frac{\partial(\Theta, \Phi)}{\partial(r, \theta)} \right| \) stands for the Jacobian of the transformation and the skyrmion density is simply \( q(r) = \sin[\Theta]/4\pi r \).

From (3), the localized static solutions in spin space (skyrmions) can be derived from the variational equations \( \delta V/\delta \Theta = \delta V/\delta \Phi = 0 \). Explicitly,

\[ 0 = \alpha' \left[ \nabla^2 \Theta - \frac{\sin 2\Theta}{2} (\nabla \Phi)^2 \right] - g\mu_B B \bar{\rho} \sin \Theta + e^2 q(r) \int q(r') \left| \frac{\partial(\Phi)}{\partial(r', \theta')} \right| \left| \frac{\partial(\Phi)}{\partial(r', \theta')} \right| dr'^2, \]  

(7)
\[0 = \alpha' \left[ \sin^2 \Theta \nabla^2 \Phi + \sin 2\Theta (\nabla \Phi) \cdot (\nabla \Theta) \right] + e^2 q(r) \int q(r') \left| \frac{\partial(|r - r'|^{-1}, \Theta)}{\partial(r, \theta)} \right| dr'^2. \quad (8)\]

This set of equations with the boundary condition \(\Theta \to 0\) as \(r \to \infty\) and cylindrical symmetry was already solved by numerical methods \(\mathbb{3}\). However, we will use a variational approach, as in \(\mathbb{7}\), in which, starting from the non-linear sigma model (NL\(\sigma\)) solutions \(\mathbb{8}\)

\[\Theta = \arccos \left\{ \frac{r^{2Q} - 4 \lambda^{2Q}}{r^{2Q} + 4 \lambda^{2Q}} \right\} \quad (9)\]

and

\[\Phi = \arctan \left\{ \left( \frac{y}{x} \right)^Q \right\}, \quad (10)\]

an optimum value of \(\lambda\) is determined by balancing the Coulomb and the Zeeman contributions in \(\mathbb{3}\). For the topological charge \(Q = \int q(r) d^2r = 1\), \(\lambda\) has the value

\[\lambda = 0.558 l_0 \left( \tilde{g} |\ln \tilde{g}| \right)^{-\frac{1}{2}}, \quad \tilde{g} = \frac{g \mu_B B}{e l_0} \quad (11)\]

where \(\tilde{g}\) is the reduced Landé factor which varies from 0.02 in the normal GaAs material to 0.002 when an hydrostatic pressure is applied \(\mathbb{3}\).

Therefore, at \(T = 0\) and near \(\nu = 1\), we will assume that the quantum Hall ferromagnet is a non-interacting skyrmion gas given by \(\mathbb{3}\)-(\(\mathbb{14}\)). However, when \(T \neq 0\) the ferromagnetic spin wave modes of the system are excited. This kind of quantum fluctuations are given essentially by the functional expansion of the effective potential \(V[\Theta, \Phi]\) around the NL\(\sigma\) model solution \(\mathbb{3}\) and \(\mathbb{14}\) up to second order. As the optimized size for the skyrmion gives a negligible first functional derivative contribution, the expansion generates the following equation for the fluctuations of the \(\Theta\) field,

\[\{\alpha' \nabla^2 + U[\Theta, \Phi]\} u_{m,k}(r) = E_m(k) u_{m,k}(r), \quad (12)\]

where

\[U[\Theta, \Phi] = \alpha' \cos(2\Theta)(\nabla \Phi)^2 + g \mu_B B \tilde{\rho} \cos \Theta + e^2 q(r) \int q(r') K[\Theta, \Phi] dr'^2 \quad (13)\]

with

\[K = \left| \frac{\partial \left( \Phi(r), \frac{\partial(|r - r'|^{-1}, \Theta(r'))}{\partial(r', \theta')} \right)}{\partial(r, \theta)} \right|. \quad (14)\]

Substituting \(\mathbb{9}\) and \(\mathbb{10}\) in \(\mathbb{13}\) the following Schrödinger-like equation, for \(Q = 1\), is obtained

\[\{-\nabla^2 + U(r)\} u_{m,k}(r) = e_m u_{m,k}(r), \quad (15)\]
where
\[ e_m = E_m - \frac{g\mu_B B}{2\pi}, \]  

(16)

and
\[ U(r) = \frac{\alpha' l_0^2}{r^2} - \frac{2^5 \alpha' \lambda^2 l_0^2}{(r^2 + 4\lambda^2)^2} - \frac{2g\mu_B B\lambda^2}{\pi(r^2 + 4\lambda^2)}. \]  

(17)

Notice that due to the rotational invariance of (11) and the radial symmetry of (10), the Coulomb term does not contribute at all to the fluctuations within this approximation, so, far from the scattering centre, the fluctuations will be mainly dominated by the potential generated by the skyrmion itself and by the presence of the magnetic field.

Despite of the complex form of (17), the solutions of (15) for the \( m \)-th cylindrical wave component of the ferromagnetic spin waves confined in a cylindrical region of radius \( L \) can be written as the sum of two contributions [10]
\[ u_{mn} \propto \frac{1}{2} \{ H^{(1)}_{|m|}(k_{mn}r)e^{im\phi} + \sum_l e^{-2i\delta_{ml}}H^{(2)}_{|l|}(k_{ln}r)e^{il\phi}\}, \]  

(18)

where \( H^{(1,2)}_m(kr) \) are the Hankel functions of first and second kind respectively and \( \delta_{ml} \) is the phase shift matrix which connects the \( m \) and \( l \) angular momentum channels. The first term of the right hand side of (18) correspond to the incident cylindrically symmetric wave whereas the second term is associated to the shifted \( l \)-th component after the interaction with the potential \( U(r) \) has taken place. As can be seen from (17), the potential involved in the scattering processes is cylindrically symmetric, therefore \( \delta_{ml} \) is a diagonal matrix and (18) becomes
\[ u_{mn} \propto \frac{1}{2} \{ H^{(1)}_{|m|}(k_{mn}r) + e^{-2i\delta_{m,l}}H^{(2)}_{|m|}(k_{mn}r)\} e^{im\phi}. \]  

(19)

To calculate the phase shifts for each angular momentum component we used the Fredholm method [11]. In so doing it is convenient to start from the well-known relation,
\[ \pi A(E) \cot \delta(E) = 1 + \mathcal{P} \int_0^\infty \frac{A(E')}{E - E'} dE', \]  

(20)

where \( \mathcal{P} \) means the Cauchy principal value, and \( A(E) \) can be obtained in first approximation as
\[ A(E) = -\langle E|U|E \rangle. \]  

(21)

In computing \( A(E) \) we will use a cylindrical basis solution of the non perturbed Schrödinger problem with circular boundary of radius \( L \). When \( L \) is large enough, the 2-D cylindrical basis can be written as
\[ \langle r|E \rangle = \left( \frac{2M}{\hbar^2} \right)^{\frac{1}{2}} J_m(kr)e^{im\theta}. \]  

(22)
However looking at (17) it can be seen that, as \( r \to 0 \), \( U(r) \to \infty \) and hence every component of \((22)\) must vanish at \( r = 0 \). This condition is automatically satisfied by the \( m \neq 0 \) \( J_m(kr) \) functions. Therefore substituting \((22)\) into \((21)\) we get

\[
A_m(x) = \frac{\pi e_c}{8\sqrt{2\pi}} \left\{ I_m(x)[K_{m+1}(x) + K_{m-1}(x)] - K_m(x)[I_{m+1}(x) + I_{m-1}(x)] - (4mx)^{-1} \right\} + 2ezI_m(x)K_m(x),
\]

with

\[
e_c = \frac{e^2/\ell_0}{\hbar^2/ML_0^2}, \quad e_z = \frac{g\mu_B B}{\hbar^2/M\lambda^2},
\]

where we have used the value of the spin stiffness \( e^2/32\sqrt{2\pi}\ell_0 \) of the quantum Hall ferromagnet. On the other hand \( I_m \) and \( K_m \) are the modified Bessel functions and \( x = 2k\lambda \). Then, the phase shift for each \( m \) angular momentum component is given by

\[
\delta_m = \arctan \left( \frac{\pi A_m(2\lambda k)}{1 + \Delta_m(2\lambda k)} \right),
\]

where we have used \( \Delta(2\lambda k) \) to denote the expression

\[
\Delta_m(k) = \mathcal{P} \int_0^\infty \frac{A_m(x)}{(2\lambda k)^2 - x^2} x dx.
\]

So, until now, our model for the quantum Hall ferromagnet is composed by a set of static cylindrically symmetric skyrmion excitations given by \((9)-(11)\) and non interacting spin waves modes described by \((19)\) and \((23)-(26)\) with frequencies

\[
\omega_{mn} = \frac{\hbar k_{mn}^2}{2M} + \frac{g\mu_B B}{\hbar},
\]

where the momenta \( k_{mn} \) can be determined from the boundary condition \( u_{mn}(kL) = 0 \). Therefore using the asymptotic expansion of \((19)\) we get

\[
k_{mn} = \frac{2n + 1}{2L} + \frac{m\pi}{2L} - \frac{\delta_m}{L}.
\]

On the classical level, due to the translational invariance of the system, the skyrmions can move freely, and the position of its centre of mass will be enough to describe its motion. However, if we look at the problem from the quantum field theoretical point of view, the skyrmion will be a particle and its centre of mass a dynamical variable. In this picture, due to the excitation of the spin wave modes at finite \( T \), not all of the degrees of freedom of the system will contribute to the skyrmion formation, this can be interpreted as the origin of the residual skyrmion-magnon interaction that leads to a non trivial dynamics of these objects. To show that, we explicitly evaluate the equation of motion for the \( \Theta \) field (our skyrmion field now) from the equations \((1),(5)\) and \((6)\). Considering cylindrical symmetry for non static skyrmions and neglecting higher order spatial derivatives, the effective equation of motion for \( \Theta(\mathbf{r},t) \) can be written as,
\[
\frac{1}{c^2} \ddot{\Theta} + 2\alpha' \left[ \nabla^2 \Theta - \frac{\sin 2\Theta}{2} \right] - g\mu_B B \sin \Theta = 0,
\]

where \( c = \alpha' / \alpha \). Comparing (7) and (29) it can be seen that to describe the skyrmion dynamics interacting with a magnon thermal bath in a simple way, we can start from the one field Lagrangian density associated with the \( z \) pseudo-spin component which reads

\[
\mathcal{L} = \frac{1}{2c^2} \left( \dot{\Theta} \right)^2 - V[\Theta],
\]

where

\[
V[\Theta] = \alpha' \left( (\nabla \Theta)^2 - \frac{\sin^2 \Theta}{r^2} \right) + g\mu_B B \mathbf{\rho}(1 - \cos \Theta).
\]

Therefore, our assumptions have simplified the initial problem given by (1)-(4) into an effective two dimensional one scalar field theory that describes the dynamical skyrmion and the possible scattering states given by ferromagnetic spin waves.

At this point, our model to treat the transport properties of the skyrmions is given by (30) and (31). The correct quantization of (30) can be done using the canonical formalism via the method of collective coordinates [12]. Doing this we appropriately treat the zero frequency mode associated to the translational invariance, and the second quantized version of \( \hat{H} \) (obtained from (30) and (31)) which describes the momentum-momentum coupling between skyrmions and magnons (see [13] for details) can be written as

\[
\hat{H} = \frac{1}{2M} \left( \hat{\mathbf{P}}_s - \hat{\mathbf{P}}_{mg} \right)^2 + \sum_{mn} \hbar \omega_{mn} b_{mn}^\dagger b_{mn},
\]

In (32), \( \hat{\mathbf{P}}_s \) and \( \hat{\mathbf{P}}_{mg} \) denotes the skyrmion and magnon momentum operators and \( \omega_{mn} \) is given by (27). Explicitly,

\[
\hat{\mathbf{P}}_{mg} = \sum_{mn,kl} \mathbf{D}_{mn,kl} b_{mn}^\dagger b_{kl} + \sum_{mn,kl} \mathbf{C}_{mn,kl} \left( b_{mn} b_{kl} - b_{mn}^\dagger b_{kl}^\dagger \right),
\]

with

\[
\mathbf{D}_{mn,kl} = \frac{1}{2} \left[ \sqrt{\frac{\omega_{kl}}{\omega_{mn}}} + \sqrt{\frac{\omega_{mn}}{\omega_{kl}}} \right] \mathbf{G}_{mn,kl},
\]

\[
\mathbf{C}_{mn,kl} = \frac{1}{4} \left[ \sqrt{\frac{\omega_{kl}}{\omega_{mn}}} - \sqrt{\frac{\omega_{mn}}{\omega_{kl}}} \right] \mathbf{G}_{mn,kl},
\]

where \( \mathbf{G}_{mn,kl} \) is a 2-D vector given by

\[
\mathbf{G}_{mn,kl} = \int u_{kl}(\mathbf{r}) \nabla u_{mn}(\mathbf{r}) d^2r
\]

and
\[ M = \int (\nabla u_0(r))^2 d^2 r, \quad (37) \]

where \( u_0 \) denote the localized solution (3) with \( Q = 1 \) whose gradient is associated to the zero frequency mode.

The relations (32)-(37) are a generalization for the 2-D case of the results obtained in (13). Notice from (33), that the magnon momentum operator consists of two contributions. A block diagonal term, which only couples excitations with different \( k \), as can be seen from the antisymmetric behaviour of \( D_{mn,kl} \) under inversion of index, and that is responsible for the low energy scattering of magnons by the skyrmions. This term commutes with the total number of magnons, so any kind of transition involving processes which do not conserve the number of magnons is not allowed. Actually, this contribution remind us of the problem of scattering by a hard (but not fixed) object because there are no internal degrees of freedom in our model. On the other hand, the block off-diagonal terms in (33) describes the emission or absorption of magnons by the skyrmion. In some specific situations, during the Hall measurements or in the high temperature limit, for instance, the effective motion of the skyrmion can be affected by the off-diagonal contribution. Although we do not attempt to study the contribution of the inelastic process to the skyrmion dynamics in this paper, we will show that in the long time approximation this term do not contribute at all to the transport properties.

III. THE FEYNMAN-VERNON FORMALISM

Based on the functional integral method of the Feynman-Vernon formalism [14], Castro Neto and Caldeira developed a new model for dissipation in quantum mechanics [15] to study the polaron dynamics [13] without appealling to the kinetic theory. In this work we will follow the same kind of approach, but now in the 2-D physical space, in our study of the effective skyrmion dynamics in the interacting system (skyrmion plus magnons).

The key point of the method is always to compute the reduced density operator (\( \hat{\rho}_s \)) for the particle of interest (the skyrmion) by tracing the mesons (magnons) coordinates out of the problem. The density operator of the interacting system evolves in time according to

\[ \hat{\rho}(t) = e^{-\frac{\pi}{\hbar^2} \hat{H}t} \hat{\rho}(0) e^{\frac{\pi}{\hbar^2} \hat{H}t}, \quad (38) \]

where \( \hat{H} \) is given by (32) and \( \hat{\rho}(0) \) is assumed to be decoupled at \( t = 0 \) as

\[ \hat{\rho}(0) = \hat{\rho}_s(0) \hat{\rho}_{mg}(0), \quad (39) \]

where \( s \) and \( mg \) denotes skyrmion and magnon respectively and

\[ \hat{\rho}_{mg}(0) = \frac{e^{-\beta \hat{H}_{mg}}}{Tr(e^{-\beta \hat{H}_{mg}})}. \quad (40) \]

As it was shown in (13), the reduced density operator in the coordinate representation for the particle of interest (skyrmion) can be written as
\[ \tilde{\rho}(x, y, t) = \int \int d'x' dy' \mathcal{J}(x, y, t; x', y', 0) \tilde{\rho}_s(x', y', 0), \]  

(41)

where \( \mathcal{J} \) is the skyrmion superpropagator which can be explicitly written in terms of the influence functional \( \mathcal{F}[x, y] \) as

\[
\mathcal{J} = \int \mathcal{D}x \int \mathcal{D}y e^{\hat{\mathcal{H}}(S_0[x] - S_0[y])} \mathcal{F}[x, y],
\]

(42)

where the action associated to the free skyrmion motion is

\[
S_0[x] = \int_0^t \frac{M(x)^2}{2} dt,
\]

(43)

with \( M \) given by (37), and the influence functional in the coherent state representation is written as

\[
\mathcal{F} = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta}{\pi} \int \frac{d^2 \gamma}{\pi} \{ e^{-|\alpha|^2} e^{-\frac{|\beta|^2}{2}} e^{-\frac{1}{2} \rho_{mg} \int \mathcal{D} \alpha \int \mathcal{D} \gamma e^{\hat{\mathcal{H}}(S_I[x, \alpha] - S_I[y, \gamma])} \times e^{(\alpha^*(0) + \alpha^*(t) + \gamma(0) + \gamma^*(t))} \},
\]

(44)

Here we have introduced \( S_I \) as the interacting skyrmion-magnon action, and \( \rho_{mg} \) stands for the magnon density operator in the coherent state representation,

\[
S_I = \int_0^t \left\{ \frac{i\hbar}{2} \sum_{mn} \alpha_{mn}^* \dot{\alpha}_{mn} - \alpha_{mn} \dot{\alpha}_{mn}^* - \sum_{mn} \hbar \omega_{mn} \alpha_{mn} \alpha_{mn}^* \\
+ \hbar M \cdot \sum_{mn,kl} D_{mn,kl} \alpha_{mn}^* \alpha_{kl} \\
+ \hbar M \cdot \sum_{mn,kl} C_{mn,kl} (\alpha_{mn} \alpha_{kl} + \alpha_{mn}^* \alpha_{kl}^*) \right\} dt',
\]

(45)

and

\[
\rho_{mg}(\beta^*, \gamma) = \prod_{mn} e^{\beta_{mn}^* \gamma_{mn} e^{-\beta \hbar \omega_{mn} - \frac{|\beta_{mn}|^2}{2} - \frac{|\beta_{mn}|^2}{2}}} \frac{1}{(1 - e^{-\beta \hbar \omega_{mn}})^{-1}}.
\]

(46)

Notice from (43) that changing to the Lagrangian formalism through the coherent state representation we have simplified the problem (32), obtaining an interacting skyrmion-magnon action quadratic in \( \alpha \). Therefore the equation of motion obtained from the variational problems \( \delta S_I/\delta \alpha = \delta S_I/\delta \alpha^* = 0 \) can be formally solved.

The linear Euler-Lagrange equations obtained from (45) can be written as

\[
\dot{\alpha}_{mn} + i \omega_{mn} \alpha_{mn} - i M \cdot \sum_{kl} D_{mn,kl} \alpha_{kl} + 2 C_{mn,kl} \alpha_{kl}^* = 0,
\]

(47)
\[ \alpha_{mn}^* - i\omega_{mn} \alpha_{mn} + iM \dot{x} \cdot \sum_{kl} D_{mn,kl} \alpha_{kl}^* + 2C_{mn,kl} \alpha_{kl} = 0, \]  

(48)

whose solutions, in terms of the functional operator \( W_{nm} \) are of the form,

\[ \alpha_{mn}(\tau) = e^{-i\omega_{mn}\tau} \left[ \alpha_{nm} + \sum_{m \neq n} W_{nm}(\tau) \alpha_m \right], \]  

(49)

\[ \alpha_{n}(\tau) = e^{i\omega_{n}\tau} \left[ \alpha_{n} + \sum_{mn \neq kl} (W_{mn,kl}(\tau) \alpha_{kl})^* e^{-i\omega_{kl} t} \right]. \]  

(50)

The functional operator introduced above obeys the integral equation

\[ W_{mn}[x(\tau)] = \int_0^{\tau} W_{mn,kl}(\tau') d\tau' \]  

\[ + \int_0^{\tau} \sum_{pq \neq mn} W_{mn,pq}(\tau') W_{pq,kl}(\tau') d\tau' \]  

(51)

with

\[ W_{mn,kl} = iM \dot{x} \cdot \left\{ D_{mn,kl} e^{-i(\omega_{mn} - \omega_{kl})t} + 2C_{mn,kl}^* e^{i(\omega_{mn} + \omega_{kl})t} \right\}. \]  

(52)

Now, using (49) and (50) we are able to solve the functional integrals in (44) in the stationary phase approximation. After evaluating some integrals, the influence functional can be further expressed as

\[ F[x, y] = \frac{1}{\det \left( 1 - \mathbf{N}_{mn} \Gamma_{mn,kl} \right)}, \]  

(53)

where \( \Gamma_{mn,kl} \) is given by

\[ \Gamma_{mn,kl} = W_{mn,kl}[y] + W_{mn,kl}^*[x] + \sum_{pq \neq mn} W_{mn,pq}[y] W_{pq,kl}^*[x]. \]  

(54)

and the ocupation number for magnons is,

\[ \mathbf{N}_{mn} = \frac{1}{\exp(\beta \hbar \omega_{mn}) - 1}. \]  

(55)

To go on in computing the influence functional we need to explicitly solve (51). This can be done up to any order by iteration, however as this relation is nothing but the amplitude of scattering from one state \( mn \) to another \( kl \) through virtual transitions involving the state \( pq \), we hope that in describing the low velocity skyrmion dynamics only small energy processes are involved and therefore a second order iteration will be sufficient to describe the main features of the transport properties of the system. Doing this, and after some algebra, the influence functional can be expressed as
\[ F[x, y] = e^{\frac{1}{2}(\Phi^1 + \Phi^1)} e^{-\frac{1}{2}(\Phi^R + \Phi^R)}. \] (56)

Where the following definitions were used,

\[ \Phi^R = \int_0^t dt' \int_0^{t'} dt'' \sum_{i,j=1}^2 \left\{ [\ddot{x}_i(t') + \dot{y}_i(t')] \Gamma_{i,j}^R(t' - t'') \left[ \ddot{x}_j(t'') - \dot{y}_j(t'') \right] \right\}, \] (57)

\[ \Phi^I = \int_0^t dt' \int_0^{t'} dt'' \sum_{i,j=1}^2 \left\{ [\ddot{x}_i(t') + \dot{y}_i(t')] \Gamma_{i,j}^I(t' - t'') \left[ \ddot{x}_j(t'') - \dot{y}_j(t'') \right] \right\}, \] (58)

\[ \Psi^R = \int_0^t dt' \int_0^{t'} dt'' \sum_{i,j=1}^2 \left[ \dot{x}_i(t'') \Delta_{i,j}^R(t' - t'') \dot{y}_j(t') - \dot{x}_i(t') \Delta_{i,j}^R(t' - t'') \dot{y}_j(t'') \right] \] (59)

\[ \Psi^I = \int_0^t dt' \int_0^{t'} dt'' \sum_{i,j=1}^2 \left[ \dot{x}_i(t'') \Delta_{i,j}^I(t' - t'') \dot{y}_j(t') + \dot{x}_i(t') \Delta_{i,j}^I(t' - t'') \dot{y}_j(t'') \right] \] (60)

where \( \Gamma_{i,j} \) and \( \Delta_{i,j} \) are 2 \times 2 matrices with elements involving the components of the 2-D vectors \( D_{mn,kl} \) and \( C_{mn,kl} \) given by (34)-(36). Explicitly

\[ \Gamma_{i,j}^R(t) = \sum_{mn,kl} \frac{(N_{mn} + N_{kl} + 2N_{mm}N_{kl})(\omega_{mn} + \omega_{kl})^2}{2\omega_{mn}\omega_{kl}} M_{i,j}(mn, kl) \cos(\omega_{kl} - \omega_{mn})t, \] (61)

\[ \Gamma_{i,j}^I(t) = \sum_{mn,kl} \frac{(N_{mn} - N_{kl})(\omega_{mn} + \omega_{kl})^2}{2\omega_{mn}\omega_{kl}} M_{i,j}(mn, kl) \sin(\omega_{kl} - \omega_{mn})t, \] (62)

\[ \Delta_{i,j}^R(t) = \sum_{mn,kl} \frac{(N_{mn} + N_{kl} + 2N_{mm}N_{kl})(\omega_{mn} - \omega_{kl})^2}{2\omega_{mn}\omega_{kl}} M_{i,j}(mn, kl) \cos(\omega_{kl} + \omega_{mn})t, \] (63)

\[ \Delta_{i,j}^I(t) = \sum_{mn,kl} \frac{(N_{mn} + N_{kl} + 2N_{mm}N_{kl})(\omega_{mn} - \omega_{kl})^2}{2\omega_{mn}\omega_{kl}} M_{i,j}(mn, kl) \cos(\omega_{kl} + \omega_{mn})t, \] (64)

\[ M_{i,j} = \begin{pmatrix} |G_{mn,kl}^{(1)}|^2 & 0 \\ 0 & |G_{mn,kl}^{(2)}|^2 \end{pmatrix} \] (65)

where \( G_{mn,kl}^{(1,2)} \) are the components of the \( G \) vector given by (36). Finally, substituting (56) in (42) the skyrmion superpropagator becomes
\[ \mathcal{J} = \int_{\mathcal{D}x}^{x} \int_{\mathcal{D}y}^{y} e^{iS[x,y]} \ dx \ dy \cdot \ dx \ dy + \int_{\mathcal{D}x}^{x} \int_{\mathcal{D}y}^{y} e^{i(\Phi^R - \Psi^R)} \ dx \ dy. \] (66)

Now, after having traced the magnons coordinates out, the reduced skyrmion action contains all the information of the interaction with the thermal bath and can be written in a simple way as

\[ S[x,y] = S_0[x] - S_0[y] + i\hbar(\Phi^I + \Psi^I). \] (67)

Before analyzing effective skyrmion motion it is convenient to rewrite (67) in terms of a new set of variables, \( R = (x + y)/2 \) and \( r = (x - y) \). The equations of motion associated to the action \( S[R,r] \) can be written as

\[ \ddot{R}_i(t) + \int_0^t \gamma_{i,j}(t - t') \dot{R}_j(t') \ dt' = 0, \] (68)

\[ \ddot{r}_i(t) - \int_0^t \gamma_{i,j}(t - t') \dot{r}_j(t') \ dt' = 0, \] (69)

where we have introduced the matrix damping function, \( \gamma_{i,j}(t - t') \), that characterize in a general form the mobility of the 2-D solitons. In terms of (62), (54) and (65) the matrix \( \gamma_{i,j} \) is given by

\[ \gamma_{i,j}(t - t') = -\frac{2\hbar}{M} \frac{d}{dt} \left\{ \Gamma_{i,j}^I(t - t') + \Delta_{i,j}^I(t - t') \right\}. \] (70)

The equations of motion (68) and (69) are generalizations of those obtained in the case of the soliton dynamics [15] describing a quantum Brownian motion with memory effect. The main differences are that the present treatment allows for the study of the 2-D topological excitations and that the Cherenkov term accounts for the possibility of mesons (magnons) emission or absorption by the soliton (skyrmion). This contribution can be important, even for small velocities, if we are studying the time evolution of the soliton.

On the other hand, as in [15], the real part of the exponent in (67) is related to the matrix diffusion function

\[ D_{i,j}(t) = \hbar \frac{d^2}{dt^2} (\Gamma_{i,j}^R + \Delta_{i,j}^R), \] (71)

that can be computed in close analogy to the matrix damping functions as shown in the next section.

To simplify the analysis of (70) the explicit expression for the damping function will be written as a sum of two terms,

\[ \gamma_{i,j}(t - t') = \gamma_{i,j}^E(t - t') + \gamma_{i,j}^I(t - t'), \] (72)

where
\[ \gamma_{i,j}^E = \frac{\hbar}{M} \sum_{mn,kl} \frac{(N_{mn} - \overline{N}_{kl})(\omega_{mn} + \omega_{kl})^2(\omega_{kl} - \omega_{mn})}{2\omega_{mn}\omega_{kl}} \mathcal{M}_{i,j}(mn, kl) \cos(\omega_{kl} - \omega_{mn})(t - t'), \]  

(73)

and

\[ \gamma_{i,j}^I = \frac{2\hbar}{M} \sum_{mn,kl} \frac{(N_{mn} + \overline{N}_{kl} + 2N_{mn}\overline{N}_{kl})}{\omega_{mn}\omega_{kl}} \mathcal{M}_{i,j}(mn, kl) \times \]

\[ (\omega_{mn} - \omega_{kl})^2(\omega_{kl} + \omega_{mn}) \cos(\omega_{kl} + \omega_{mn})(t - t'). \]  

(74)

As it can be seen from (36) and (65) the matrix elements \( \mathcal{M}_{i,j} \) essentially involve the single particle momentum operator between two different ferromagnetic spin wave states given by (19). Therefore in close analogy to the spectral functions introduced in [16] we will define the \( 2 \times 2 \) matrix scattering function, \( S_{i,j} \), as

\[ S_{i,j}(\omega, \omega') = \sum_{mn,kl} \mathcal{M}_{i,j}(mn, kl) \delta(\omega - \omega_{mn})\delta(\omega' - \omega_{kl}), \]  

(75)

that allow us to rewrite the expressions for \( \gamma_{i,j}^I \) and \( \gamma_{i,j}^E \) in the following form

\[ \gamma_{i,j}^E(t) = \frac{\hbar}{M} \Theta(t) \int d\omega \int d\omega' S_{i,j}(\omega, \omega')(\overline{N}(\omega) - \overline{N}(\omega')) \]

\[ \frac{(\omega + \omega')^2(\omega' - \omega)}{\omega\omega'} \cos(\omega' - \omega)t, \]  

(76)

which corresponds to the elastic scattering process with purely instantaneous memory when we assume long time approximation, as in [13]. On the other hand

\[ \gamma_{i,j}^I(t) = \frac{2\hbar}{M} \Theta(t) \int d\omega \int d\omega' \frac{S_{i,j}(\omega, \omega')(\omega - \omega')^2(\omega' + \omega)}{\omega\omega'} \]

\[ \frac{[\overline{N}(\omega) + \overline{N}(\omega') + 2\overline{N}(\omega)\overline{N}(\omega')]}{\cos(\omega + \omega')t}, \]  

(77)

is related to the Cherenkov radiation and only contributes to the soliton dynamics when transitions between states with different energies are taken into account. So, we will refer to this contribution as inelastic. In (76) and (77) \( \Theta(t) \) ensures the causality principle.

IV. THE DAMPING MATRIX

In the first place let us analyze the \( \gamma_{i,j}^I(t) \) contribution to the damped motion of the skyrmion. Notice from (77) that for long times the \( \cos(\omega' + \omega)t \) term oscillates rapidly giving no net contribution to the damping function matrix elements. This is true unless \( \omega' + \omega \) is close to zero, but at the same time the factor \( (\omega - \omega')^2 \) prevents this situation. So, up to first order in the solution of (51), the inelastic contribution \( \gamma_{i,j}^I(t) \) to the damping function can be neglected. However, if we consider processes with characteristic times of the same order as \( (\omega' + \omega)^{-1} \), when the long time approximation is no longer valid, this term will play an important role in the soliton dynamics.
Now, the analysis of (76) becomes simpler if we change the frequency variables to 
\( \phi = (\omega' + \omega)/2 \) and \( \eta = (\omega' - \omega) \). In this way

\[
\gamma_{E}^{i,j}(t) = \frac{2\hbar}{M} \Theta(t) \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\eta S_{i,j}(\phi, \eta) \eta^2 f(\phi, \eta) \frac{N(\phi + \eta/2) - N(\phi - \eta/2)}{\eta} \cos(\eta t). \tag{78}\]

where

\[
f(\phi, \eta) = 2 + \frac{\phi + \eta/2}{\varphi - \eta/2} + \frac{\varphi - \eta/2}{\varphi + \eta/2}.	ag{79}\]

As in (77), the expression (78) vanishes for long times due to the rapid oscillations of
\( \cos(\eta t) \) term unless \( \eta \) is close to zero. If this is so \( f(\phi, \eta) \) can be taken as constant and
we can assume that

\[
\frac{N(\phi + \eta/2) - N(\phi - \eta/2)}{\eta} \approx \frac{\partial N(\phi)}{\partial \phi}, \tag{80}\]

therefore (78) can be rewritten as

\[
\gamma_{E}^{i,j}(t) = \frac{8\hbar}{M} \Theta(t) \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\eta A_{i,j}(\phi) \frac{\partial N(\phi)}{\partial \phi} \cos(\eta t), \tag{81}\]

where we have introduced \( A_{i,j}(\phi) \) given by

\[
A_{i,j}(\phi) = \lim_{\eta \to 0} S_{i,j}(\phi, \eta) \eta^2 \tag{82}\]

After integrating (81) in \( d\eta \) and using (55) we get

\[
\gamma_{E}^{i,j}(t) = \pi_{i,j}(T) \delta(t), \tag{83}\]

where \( \pi_{i,j}(T) \) is the temperature dependent matrix damping parameter that characterizes
the soliton mobility in the 2-D case and \( \delta(t) \) is the Dirac delta function. As usual the
we will refer to the form (83) as the Markovian approximation, because this results have
only instantaneous memory not depending on the previous motion of the particle. Explicitly
\( \pi_{i,j}(T) \) is given by

\[
\pi_{i,j}(T) = \frac{8\hbar}{M} \int_{0}^{\infty} e^{\beta(g\mu BB + h\varphi)} (1 - e^{\beta(g\mu BB + h\varphi)})^2 A_{i,j}(\phi) d\phi. \tag{84}\]

In order to compute the temperature dependence of \( \pi_{i,j}(T) \) we need to explicitly evaluate
the scattering function \( S_{i,j}(\phi, \eta) \). The first step in doing so is to get an analytic expression for
\( S_{i,j}(\omega, \omega') \), and therefore the matrix elements of (65) must be evaluated. As it can be seen
from (19) the \( x \) and \( y \) components of the \( G \) vector defined by (65) are equal, in agreement
with the isotropic character of our model. Therefore the function matrix \( M_{mn,kl} \) can be
written as \( M = G_{nn,kl} \) where \( 1 \) is the unitary matrix. Then we need to compute only one
of its components, for instance,
\[ G_{mn,kl} = \int_0^L rdr \int_0^{2\pi} d\theta u_{kl}^*(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta})u_{mn} \]  

(85)

which, after some integration and assuming large values of \( L \) becomes

\[
G_{mk,kl} = \frac{\pi^2 \delta_{l,m+1}}{L^2 \sqrt{k k'}} \left[ \Lambda_{lm}^{(1)} P_{k} + i \pi k \Lambda_{lm}^{(2)} \delta(k - k') - i \pi k \Lambda_{lm}^{(3)} \delta(k + k') \right] - \Lambda_{lm}^{(4)} P_{k - k'} \\
+ \frac{\pi^2 \delta_{l,m}}{L^2 \sqrt{k k'}} \left[ \frac{\pi}{2} \Lambda_{lm}^{(2)} + (C_i(k - k') \Lambda_{lm}^{(4)} - C_i(k + k') \Lambda_{lm}^{(6)}) \right]
\]

(86)

where \( C_i(k) \) stands for

\[
C_i(k) = \int_{\lambda}^L \frac{\cos kt}{t} dt
\]

(87)

and the set of expressions \( \Lambda^{(n)} \) are defined by

\[
\Lambda^{(1)} = e^{i \pi (l+m)} e^{2i\delta_m} + e^{-i \pi (l+m)} e^{2i\delta_l} e^{-i \pi (l+m)}
\]

(88)

\[
\Lambda^{(2)} = e^{i \pi (l-m)} - e^{-i \pi (l-m)} e^{2i\delta_l-\delta_m}
\]

(89)

\[
\Lambda^{(3)} = e^{i \pi (l+m)} e^{2i\delta_m} + e^{2i\delta_l} e^{-i \pi (l+m)}
\]

(90)

\[
\Lambda^{(4)} = e^{i \pi (l-m)} + e^{-i \pi (l-m)} e^{2i\delta_l-\delta_m}
\]

(91)

\[
\Lambda^{(5)} = \Lambda^{(2)} + e^{2i\delta_l} e^{-i \pi (l+m)} - e^{-2i\delta_m} e^{i \pi (l+m)}
\]

(92)

\[
\Lambda^{(6)} = e^{i \pi (l+m)} e^{-2i\delta_m} + e^{2i\delta_l} e^{-i \pi (l+m)}
\]

(93)

Expression (86) allows us to write a continuum version of (75) for the matrix damping function diagonal elements \( S_{i,i} \). Explicitly

\[
S_{i,i}(\omega, \omega') = \frac{L^4}{4\pi^2} \sum_{ml} \int kdk \int k'dk'|G_{mk,kl}|^2 \delta(\omega - \Omega_k) \delta(\omega' - \Omega_{k'})
\]

(94)

where

\[
\omega = \frac{\hbar k^2}{2M}.
\]

(95)

Now, using (86)-(90) we can evaluate (94) and obtain an explicit expression of \( S_{i,i}(\omega, \omega') \) which, by the time, allow us to get the analytic form of \( A_{i,i} \) in terms of the variables \( \varphi \) and \( \eta \) introduced before. Then, the limit given by (82) yields
\[ A_{i,i}(\varphi) = \left(\frac{\pi M}{\hbar}\right)^2 \mathcal{G}(\varphi) \varphi^2 \] (96)

with

\[ \mathcal{G}(\varphi) = 2 \sum_{m=1}^{\infty} \sin^2(\delta_{m+1} - \delta_m). \] (97)

where the phase shifts \( \delta_m \) are also functions of \( \varphi \).

The matrix damping coefficient elements (84) can be written now as

\[ \overline{\tau}_{ii} = \left(\frac{8\pi^2 M}{\hbar}\right) e^{\beta g_\mu B} \int_0^\infty \frac{\varphi^2 e^{\beta \varphi} \mathcal{G}(\varphi)}{\left(e^{\beta(g_\mu B+h\varphi)} - 1\right)^2} d\varphi. \] (98)

In computing this final expression (98) we need a numerical evaluation of the phase shift functions defined by (23)-(26). This was done taking the magnetic field \( B = 9T \) for three different values of the Landé factor \( g \) or skyrmion size. However, the analysis of the high temperature regime of the skyrmion mobility can be done from (98) without explicit evaluation of \( \mathcal{G} \), giving

\[ \overline{\tau}_{ii} = \frac{M}{\hbar^3} \left(\int_0^\infty \mathcal{G}(\varphi) d\varphi\right) \frac{1}{\beta^4}. \] (99)

that increases linearly with temperature independently of the explicit form of \( \mathcal{G}(\varphi) \).

The behaviour of \( \overline{\tau}_{ii}(T) \) in units of \( \gamma_0 = \pi^2 \hbar^2/32M^2\lambda^4 \) for any temperature, is shown in Fig.1. for three different values of the Landé factor. Due to the smallness of the Zeeman energy (0.13 meV, 0.013 meV and 0.0013 meV) in all cases the scattering states are excited leading to a damped motion of the skyrmion. Although we do not have an analytic expression for \( \overline{\tau}_{ii}(T) \) the numerical result shows that near \( T = 0 \) the mobility presents a power law behaviour \( T^\alpha \), as can be seen from the log-log plot in Fig.2, where \( \alpha \) is a strong dependent Landé factor coefficient.

V. CONCLUSIONS

Here we have succeded in applying the collective coordinate method of quantum field theory to the description of the mobility of the skyrmion excitation in a quantum Hall ferromagnet as a function of the temperature. We have been able to treat both elastic and inelastic effects of the skyrmion-magnon interaction and show that only the former plays a major role in the long time approximation for this problem. We have also shown that in this limit the mobility of the skyrmion is linear for high temperatures and behaves as \( T^\alpha \) for low temperatures.

Although we have directly addressed the problem of the quantum Hall ferromagnet, our description can be straightforwardly extended to any system which can be mapped into a NL\( \sigma \) model or any extension thereof. An investigation of the experimental evidences of our findings is now in process.

Finally AVF wishes to thank Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP) for financial support, whereas AOC kindly acknowledges partial support from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).
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FIG. 1. Damping coefficient as a function of temperature for different effective Landé factors. The solid line correspond to the case in which $g = 0.25$, the dotted line to $g = 0.025$, and the dashed line to $g = 0.0025$.

FIG. 2. The circles correspond to the case in which $g = 0.25$, the squares to $g = 0.025$, and the triangles to $g = 0.0025$ respectively.