Compact Three Dimensional Black Hole:

Topology Change and Closed Timelike Curve

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Abstract

We present a compactified version of the 3-dimensional black hole recently found by considering extra identifications and determine the analytical continuation of the solution beyond its coordinate singularity by extending the identifications to the extended region of the spacetime. In the extended region of the spacetime, we find a topology change and non-trivial closed timelike curves both in the ordinary 3-dimensional black hole and in the compactified one. Especially, in the case of the compactified 3-dimensional black hole, we show an example of topology change from one double torus to eight spheres with three punctures.

I. INTRODUCTION

The topology change of the universe is a fascinating subject in general relativity. Fujiwara, Higuchi, Hosoya, Mishima and the present author [1] have recently found various quantum topology change solutions. Their quantum topology change solutions are tunneling

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manifolds with the Euclidean signature. Such quantum effects occur only on the Planck mass scale.

In this paper we present a new topology change solution with the Lorentzian signature. It means that the topology change will occur in classical gravity. There are some works treating the possibility of topology change in classical gravity [2]. Especially, Geroch showed that in the case of a compact universe there must exist a closed timelike curve in the spacetime representing a topology changing process. In the same case, Tipler also pointed out that spacetime singularity necessarily appears in the spacetime if matter field satisfies the weak energy condition.

To study the topology change we must treat the complicated topology of spacetime. If we go down to the (2+1)-dimensional gravity, the problem becomes much simpler while the topology change is still non-trivial. Further, the (2+1)-dimensional gravity is useful for studying roles of global structure of spacetime in classical and quantum gravity since there are no local dynamical degrees of freedom in the (2+1)-dimensional spacetime. Therefore we adopt the (2+1)-dimensional gravity as a good toy-model to study the Lorentzian topology change.

To formulate the (2+1)-dimensional gravity, we consider the Teichmüller parameters or moduli parameters as dynamical variables. Hosoya and Nakao, and Moncrief [3] investigated a quantum dynamics of the Teichmüller parameters of a torus universe in the ADM-formalism with the York timeslice. The Teichmüller parameters represent the identifications of points in the universal covering space of the torus and determines the shape of the torus. In the case of point particles, Deser, Jackiw and 'tHooft [4] treated the mass of a point particle as a deficit angle of a Minkowski spacetime, which is determined by the change of period for the angular coordinate. We can see that in the (2+1)-dimensional gravity the identification of points of spacetime is crucial in the study of the dynamics of global structure.

Recently Bañados, Teitelboim and Zanelli [5] have found a (2+1)-dimensional black hole solution with a negative cosmological constant. Though the circular symmetric (2+1)-
dimensional vacuum solution has no event horizon, their black hole solution constructed also by making certain identifications in the anti-de Sitter space has an event horizon. Since the Hawking temperature of the 3-dimensional black hole \( T_H \propto \sqrt{M} \) becomes zero in the mass zero limit of the 3-dimensional black hole, it is expected that the Hawking radiation dies away when the mass of black hole becomes zero while it blows up in the 4-dimensional black hole. This suggests that the black hole evaporation can be treated semi-classically in (2+1)-dimensions. A 3-spacetime with a negative cosmological constant has been studied since the variety of topology is rich in that space. Further, the violation of energy condition due to the negative cosmological constant allows topology change \([1]\) \([2]\).

In this paper, we will investigate the 3-dimensional black hole solution and its compactification, where the shape of the space is parametrized by a set of global parameters which becomes extra degrees of freedom. These 3-dimensional black hole solutions will be analytically extended. We analyze topology and the global causal structure of the whole of these spacetimes by studying the identification of points in the 3-anti-de Sitter spacetime. The similarity with the Misner solution suggests existence of closed timelike curves.

In the section 2, the 3-dimensional black hole solution is compactified to give a double torus space. The section 3 shows their analytical extensions. The topology is discussed there. The closed curves are analyzed in section 4. They make the global causal structure clear. The final section is devoted to summary and discussions.

II. COMPACTIFIED 3-DIMENSIONAL BLACK HOLE

As shown by Bañados, Teitelboim and Zanelli \([3]\), there exists a black hole solution in the (2+1)-dimensional pure gravity with a negative cosmological constant. The metric of non-rotating black hole is given by

\[
ds^2 = -(r^2/l^2 - M)dt^2 + \frac{dr^2}{(r^2/l^2 - M)} + r^2d\phi^2,
\]  

(1)
where $M$ is the mass of the black hole and $-1/l^2$ is equal to the cosmological constant. This solution is a maximally symmetric spacetime with a constant curvature $-2/l^2$ (3-anti-de Sitter) with certain identifications of points in spacetime. In this sense, this solution is regarded as the topological one. More precisely, we consider the identifications of points by a discrete subgroup of the isometry group $SO(2, 2)$ acting on the 3-pseudo-sphere $x^2 + y^2 - z^2 - w^2 = l^2$, which is a 3-anti-de Sitter spacetime embedded into the flat spacetime with a signature $(- - + +)$. The coordinate parametrization for the 3-dimensional black hole (3D-BH) in the external region ($r \geq l$) is given as

$$
    ds^2 = -dx^2 - dy^2 + dz^2 + dw^2,
    
    x = \sqrt{r^2/M - l^2} \sinh \frac{\sqrt{M}}{l} t,
    
    y = \frac{r}{\sqrt{M}} \cosh \sqrt{M} \phi,
    
    z = \sqrt{r^2/M - l^2} \cosh \frac{\sqrt{M}}{l} t,
    
    w = \frac{r}{\sqrt{M}} \sinh \sqrt{M} \phi.
$$

The internal solution ($r < l$) is given by an analytical continuation of the expression written above as

$$
    x = \sqrt{l^2 - r^2/M} \cosh \frac{\sqrt{M}}{l} t,
    
    y = \frac{r}{\sqrt{M}} \cosh \sqrt{M} \phi,
    
    z = \sqrt{l^2 - r^2/M} \sinh \frac{\sqrt{M}}{l} t,
    
    w = \frac{r}{\sqrt{M}} \sinh \sqrt{M} \phi.
$$

This chart of the 3D-BH solution covers the part of the pseudo-sphere in which $y^2 - w^2 > 0$. In this region the periodic boundary condition of the angular coordinate, $(t, r, \phi) \leftrightarrow (t, r, \phi + 2\pi)$, defines an identification of points and restricts its fundamental region. This identification is written in terms of $(x, y, z, w)$ as
This transformation is an element of the $SO(2, 1)$ group, the subgroup of the isometry group $SO(2, 2)$ of the 3-anti-de Sitter spacetime. We quotient the 3-anti-de Sitter spacetime by the discrete subgroup of $SO(2, 2)$ generated by (2) to obtain the 3D-BH solution. Considering also the coordinate parametrization for the Robertson-Walker chart of the anti-de Sitter spacetime, we find that this element of the group preserves not only the timeslice of the 3D-BH but also the timeslice of the RW-chart. Since each spatial hypersurface of the RW-chart of the 3-anti-de Sitter spacetime is a 2-hyperbolic space, we can construct various handle-body universes by certain tessellations on them so that the tessellation of the handle-body becomes consistent with the 3D-BH solution. For example, we consider a double torus universe with a discrete subgroup of $SO(2, 1) = \text{Isom}(H_2)$ generated by

\begin{align*}
T_1 &= T, \\
T_2 &= R^{-1}TR, \\
T_3 &= R^{-2}TR^2, \\
T_4 &= RTR^{-1},
\end{align*}

(3)

where $T$ is a Lorentz boost and $R$ is a rotation by $\pi/4$. The actions of these transformations on $H_2$ are shown in Fig.1. In terms of $(x, y, z, w)$, these $T$ and $R$ are represented by the matrices,

\begin{equation}
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cosh \alpha & 0 & \sinh \alpha \\
0 & 0 & 1 & 0 \\
0 & \sinh \alpha & 0 & \cosh \alpha
\end{pmatrix},
\end{equation}

(4)
\[ R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \pi/4 & -\sin \pi/4 \\ 0 & 0 & \sin \pi/4 & \cos \pi/4 \end{bmatrix} . \] (5)
Compared with (2) this double torus universe is consistent with the 3D-BH solution provided that $\alpha$ is equal to $2\pi\sqrt{M}$. To get a regular double torus (the angle $\theta$ in Fig. 1 is $\pi/4$), however, $\alpha$ should be $2\tanh^{-1}\sqrt{(2\sqrt{2} - 2)/3}$; otherwise the double torus has a conical singularity. From now on, we suppose that

$$2\pi\sqrt{M} = \alpha = 2\tanh^{-1}\sqrt{\frac{2\sqrt{2} - 2}{3}},$$

for the regular double torus. We call this regular double torus spacetime with a negative cosmological constant as a compactified 3D-BH. Because the double torus is constructed on the RW-chart, the foliation by the 2-hyperbolic spaces which are the universal covering spaces of the double tori covers only the region $y^2 - z^2 - w^2 \geq 0$. The compactified 3D-BH is the quotient of this region by the discrete subgroup. The compactified 3D-BH is singular only on a cone $y^2 - z^2 - w^2 = 0$ where the RW-chart is singular. The property of these singularities will be discussed in the next section.
III. SINGULARITY AND MISNER-LIKE EXTENSION

In this section, we consider the causal structure of the 3D-BH solution which is similar to that of the Misner spacetime. We extend the 3D-BH spacetime in the same way as we normally treat the Misner solution [6].

A. Ordinary 3D-BH

In the Misner solution the apparent singularity is merely a coordinate singularity. We can extend the solution to the region beyond the coordinate singularity. It is well known that there are two possible extensions of the solution corresponding to the two directions along the spatial axis. To make all geodesics in these directions complete we should do both of the extensions. The resultant spacetime, however, becomes non-Hausdorff at the boundary between the original region and the extended region as shown in [7].

Now, we extend the 3D-BH by analogy with the Misner solution. By a coordinate transformation \( \rho = r^2 \), the metric of the ordinary 3D-BH changes to the following form which has the Misner-like section:

\[
\begin{align*}
    ds^2 &= \left\{ - \left( M - \rho/l^2 \right)^{-1} \frac{d\rho^2}{4\rho} + \rho d\phi^2 \right\} + \left( M - \rho/l^2 \right) dt^2.
\end{align*}
\]  

When one approaches the singularity (\( \rho = 0 \)), this section behaves like the Misner solution. There are two extensions of spacetime, corresponding to the two directions along the \( \phi \)-axis, which are similar to the extensions of the Misner solution. We consider the following two coordinate transformations corresponding to the two extensions,

\[
    \phi' = \phi \pm \frac{1}{2\sqrt{M}} \log \frac{\sqrt{M + \sqrt{M - \rho/l^2}}}{\sqrt{M - \sqrt{M - \rho/l^2}}}. 
\]

They give the two expressions for the metric,

\[
\begin{align*}
    ds^2 &= \pm \frac{d\phi' d\rho}{\sqrt{M - \rho/l^2}} + \rho d\phi'^2 + \left( M - \rho/l^2 \right) dt^2, 
\end{align*}
\]  

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respectively. Since these metrics are not degenerate on $\rho = 0$, we can extend the spacetime from the region with $\rho > 0$ to the region with $\rho < 0$. By one of the two extensions a certain set of incomplete geodesics become complete, while the other extension makes the other set of incomplete geodesics complete.

The identification ($\phi \leftrightarrow \phi + 2\pi$) should exist also in the extended region. The action of the transformation $T$ (see (2)) in the extended region generates this identification. Fig.2 shows this identification in the original region and also in the extended region for a $x, z = \text{constant}$ section of the pseudo-sphere $x^2 + y^2 - z^2 - w^2 = l^2$. Thus we find that the topology of the space changes from a single $R \times S_1$ (the original region) to two $R \times S_1$’s (the extended region). It is noted that the spacetime is non-Hausdorff on the boundary between the original region and the extended region $y^2 - w^2 = 0$, if we do both of the extensions.

![Diagram](image)

**FIG. 2.** A section of the pseudo-sphere with $x, z = \text{constant}$ is a hyperbola parametrized by $\phi = 1/\sqrt{M} \tanh^{-1}(w/y)$. A point $\phi = \phi_0$ is identified with a point $\phi = \phi_0 + 2\pi$.

**B. Compactified 3D-BH**

We extend also the compactified 3D-BH by the above mentioned method. The original region is the region with $x^2 < l^2$; it is easily confirmed that this region coincides
with the region covered by the RW-chart. The coordinate extension is carried out in a similar way as the ordinary 3D-BH. This time there are eight extensions corresponding to the eight directions of the identification. By considering the covering space of the RW-anti-de Sitter spacetime, we find that this extended region is the region $x^2 - l^2 = z^2 + w^2 - y^2 > 0$ of the 3-anti-de Sitter not covered by the RW-chart (see Fig.3).

\[
H_2(x_0^2) = \{ y^2 - z^2 - w^2 = \lambda^2 \equiv l^2 - x_0^2 > 0 \},
\]

with $x_0^2(< l^2)$ being a parameter. The identifications shown in Fig.1 produce a double torus at each surface. As stated in the previous section, these identifications are defined by a discrete subgroup of $SO(2, 2) = \text{Isom}(3\text{-anti-de Sitter})$ generated by $T_i$ in (3).

We also make such identifications in the extended region in the following way. The extended region is foliated by the (1+1)-de Sitter spacetimes:

\[
dS_2(x_0^2) = \{ z^2 + w^2 - y^2 = \lambda^2 \equiv x_0^2 - l^2 > 0 \},
\]

parametrized by $x_0^2(> l^2)$. Since the discrete subgroup preserves each $dS_2(x_0^2)$, the iden-
tifications are induced for each hypersurface. We can find the topology of this region by determining the fundamental region in the hypersurface $dS_2(x_0^2)$. The action of $T_1$ on the surfaces $dS_2(x_0^2)$ is shown in Fig.4.

\[ z^2 = x^2 - 1^2 \]

FIG. 4. The action of $T_1$ on a $dS_2(x_0^2)$ is shown by the arrows.

The other actions of transformations are given by a rotation of $T_i$ by $\pi(1 - i)/4$ with $i = 1 \sim 4$. The fundamental regions are shown in Fig.5 both for $H_2(x_0^2)$ and for $dS_2(x_0^2)$.
FIG. 5. While the fundamental region is an octagon, in the $H_2(x_0^2)$ region of spacetime it changes to eight quadrilaterals in $dS_2(x_0^2)$.

The fundamental region of the $dS_2(x_0^2)$ consists of eight quadrilaterals. Fig.6 shows that such a quadrilateral is topologically equivalent to a sphere with three conical singularities.

To summarize, the analytical continuation of the compactified 3D-BH produces the topology changing geometry from a double torus to eight spheres with three punctures. This topology change is quite different from the quantum topology change investigated in Ref. [1]. Previously the quantum topology change was described by a Euclidean tunneling manifold whereas the present topology changing solution has the Lorentzian signature. The topology change presented in this paper happens in classical gravity. We also note that the spacetime is non-Hausdorff, and it is discussed in detail in [9]. As stated in [7], this kind of non-Hausdorff topology is not so uncomfortable as the case shown in Fig.8, since we have no continuous curve which bifurcates.
FIG. 6. The quadrilateral becomes a sphere with three punctures by the identifications.

IV. CLOSED GEODESIC CURVE

Generally an identification of points creates new closed curves; these closed curves link the two identified points on the boundary of the fundamental region. Since, in the case of the Misner solution, such closed curves become closed timelike curves in its extended region, one can naturally expect that the ordinary 3D-BH and the compactified 3D-BH contain at least a closed timelike curve in their extended regions.

To see this more precisely we consider the homotopy class of smooth curves emanating from one point on the boundary of the fundamental region and ending at the point identified with it, on each hypersurface (see Fig.7). If this homotopy class contains a curve which is everywhere timelike (spacelike), a closed timelike (spacelike) curve exists and passes through the point on the boundary of the fundamental region. As the metric is continuous, when each homotopy class contains its own everywhere timelike (spacelike) curve, the homotopy classes contain no everywhere spacelike (timelike) curve. Because all non-trivial curves are homotopic to the combination of these closed curves crossing the boundary of the fundamental region once, we can see the causal property of the all non-trivial closed curves, finding out the everywhere spacelike or everywhere timelike curves.
FIG. 7. A homotopy class of smooth curves emanating from one point on the boundary of the fundamental region and ending at the point identified with it.

Here we consider a curve which is an orbit of an element of the discrete subgroup bearing in mind that this curve is smooth even after the identification. In particular, if we vary the boost angle of a Lorentz transformation $T_i$, this one parameter family of transformation generates a smooth curve. For example, the Lorentz boost (2) generates such a curve,

$$C(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cosh t & 0 & \sinh t \\
0 & 0 & 1 & 0 \\
0 & \sinh t & 0 & \cosh t
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix},$$

with $t$ being a boost angle. The expression (2) can be rewritten as

(9)
\[ C(t) = \exp \left[ t \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \]  

On the other hand, the integral curve of the vector field \( X^a \) is generally given by the exponential map,

\[ \gamma(s) = \exp_q(sX^a), \]

so that we obtain the tangent vector to the curve \( C(t) \) given in Eq.\( (10) \) as

\[ X^a = \begin{pmatrix} 0 \\ w \\ 0 \\ y \end{pmatrix}. \]

The square \( X^aX_a \) is \( -w^2 + y^2 \).

In the ordinary 3D-BH closed non-trivial curves are homotopic to the curves \( C(t) \)'s with fixed starting points. In the original region of the ordinary 3D-BH \( (y^2 - w^2 > 0) \), as \( X^a \) is always spacelike there is a closed non-trivial spacelike curve but no closed timelike curve. On the contrary the extended region \( (y^2 - w^2 < 0) \) contains a closed non-trivial timelike curve but no closed spacelike curve as we shall see below. On the boundary surface \( (y^2 - w^2 = 0) \) such closed curves become null.

For the identifications of the compactified 3D-BH, the calculations are similar. We determine the tangent vectors \( X^a[T_1] \)'s as follows,

\[ X^a[T_1] = \begin{pmatrix} 0 \\ w \\ 0 \\ y \end{pmatrix}, \quad X^aX_a = -w^2 + y^2 \]

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In the original region of the compactified 3D-BH $X^a[T_i]$ is everywhere spacelike because $y^2 - w^2 - z^2 = l^2 - x^2$ is positive there. Then each homotopy class of closed curves contains a closed spacelike curve but no closed timelike curve. The case of the extended region of the compactified 3D-BH $(y^2 - w^2 - z^2 < 0)$ is not so straightforward so that we need closer examination of the curves. For instance, we consider the curve generated by $T_1$. $X^a[T_1]$ is spacelike only in the region $z^2 > \lambda^2 \equiv l^2 - x_0^2$. The fundamental region of the extended region, however, is contained in the region where $-z^2 + \lambda^2$ is negative as depicted in Fig.4. Therefore $X^a[T_1]$ is everywhere timelike in the fundamental region of the extended region of the compactified 3D-BH. As this aspect is the same for all the other $T_i$’s, we can conclude that the extended region contains non-trivial closed timelike curves corresponding to each Lorentz boost $T_i$ and therefore there is no non-trivial closed spacelike curve.

V. SUMMARY AND DISCUSSIONS

In this paper, we have investigated the analytical continuations of the ordinary 3D-BH and of the compactified 3D-BH. The compactified 3D-BH is isometric to a regular double
torus universe with a negative cosmological constant if the mass of the black hole is equal to 

\[ m_c = \left(1/\pi \tanh^{-1} \sqrt{2} - 2\right)/3 \]

If the mass of the black hole were different from \( m_c \), however, the double torus would contain a conical singularity. Even if so, all the results of this paper will not be affected.

One might argue that the compactified 3D-BH would not be a proper black hole. Any compactification would remove the spatial infinity and the null infinity from the spacetime. Then, as the argument may go, we could not define any event horizon. In this paper, however, we treat the compactified 3D-BH solution as a black hole because the double torus universe can be regarded as an open 2-hyperbolic space which possesses a certain periodicity. The horizon can be defined in the universal covering spacetime. To summarize we can define a compactified space as a black hole if its universal covering space is a black hole spacetime.

We have shown that the analytical continuations of the 3D-BH’s produce a topology changing spacetime geometry. Especially the topology change of the ordinary 3D-BH is easily attributed to the singularity where the spacetime is non-Hausdorff. A well-known simple example of non-Hausdorff topological space is shown in Fig.8, and is the simplest example of a topology change, from one point to two points, if one imagines that the \( x \) and \( y \) axes are the time axes. The non-Hausdorff topology causes the branching of the space which is one type of the topology change. In the ordinary 3D-BH, the topology of a hypersurface changes from a single \( R \times S_1 \) to two \( R \times S_1 \)'s. This topology change belongs to the branch-type topology change corresponding to the change of the number of connected components. However, the topology change of the compactified 3D-BH (a double torus universe) is not so trivial, since the Euler number changes from \(-2\) (a double torus) to \( 8 \times (-1) \) (eight spheres with three punctures). This can be seen by considering the geodesics running from the original region to the extended region. Anyway these topology change is closely related to non-Hausdorff topologies. We may speculate that all Lorentzian topology changes are related to the non-Hausdorff topology.
FIG. 8. An example of a non-Hausdorff manifold. The two lines above are identical for $x = y < 0$. However, the two points $a \ (x = 0)$ and $a' \ (y=0)$ are not identified.

As for the problem of the topology change in a Lorentzian spacetime, some authors have discussed its possibility or the condition for it to occur [2] [8]. These works, however, considered spacelike hypersurfaces as an initial and a final states. As shown in this paper there is a new type of topology change if we consider timelike hypersurfaces.

The investigation of this paper about a double torus is also applicable to a 2-surface with higher genus. When we consider the case of genus $n > 2$, the mass of the black hole should be $(1/\pi \tanh^{-1} \sqrt{2/(1 + \sec(\pi/2n))})^2$ to form a regular spatial hypersurface with a genus $n$. In this case the topology changes from a single $n$-torus to $4n$ spheres with three conical singularities.

The existence of the closed timelike curve means the break down of causality. Hawking conjectured that such a closed timelike curve cannot be formed since the energy momentum tensor for quantum matter fields diverges near the cauchy horizon which should appear if the closed timelike curve is created [11]. This problem is still an open question [11] [12]. The spacetime shown in this paper may be an appropriate background spacetime to discuss this problem.

For the open chart of 3-de Sitter spacetime or the 3-Milne universe, the same compactification and extension can be done. In these cases we will also find a topology change and
a closed timelike curve. It is our future plan to study the cosmological implication in the Friedmann spacetimes with compactified hyperbolic spatial hypersurface. Furthermore we can study the Teichmüller deformation and its quantum theory of a further simplified model, for example a torus with a conical singularity in the 3D-BH \cite{10}.

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REFERENCES

[1] Y. Fujiwara, S. Higuchi, A. Hosoya, T. Mishima and M. Siino, Phys. Rev. D44 (1991) 1756, Phys. Rev. D44 (1991) 1763, Class. Quantum Grav. 7 (1992) 163.

[2] R. P. Geroch, J. Math. Phys. 8 (1967) 782, P. Yodzis, Gen. Rel. Grav. 4 (1973) 299, D. Gannon, J. Math. Phys. 16 (1975) 2364, F. J. Tipler, Ann. Phys. 108 (1977) 1, R. D. Sorkin, Phys. Rev. D33 (1986) 978.

[3] A. Hosoya and K. Nakao, Class. Quantum Grav. 7 (1990) 163, Prog. Theor. Phys. 84 (1990) 739, V. Moncrief, J. Math. Phys. 30 (1989) 2907.

[4] S. Deser, R. Jackiw and G. ’tHooft, Ann. of Phys. 152 (1984) 220.

[5] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.

[6] C. W. Misner Relativity Theory and Astrophysics I: Relativity and Cosmology, ed. J. Ehlers, Lectures in Applied Mathematics, Volume 8 (American Mathematical Society, 1967) 160-9.

[7] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time (Cambridge University Press, Cambridge, 1973).

[8] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. 148 (1992) 345.

[9] M. Siino, Ph.D. Thesis, T.I.Tech. (1994).

[10] M. Siino, in preparation.

[11] S. W. Hawking Phys. Rev. D46 (1992) 603.

[12] M. S. Morris, K. S. Thorne and U. Yurtsever, Phys. Rev. Lett. 61 (1988) 1446.