Guessing Cost: Bounds and Applications to Data Repair in Distributed Storage
Suayb S. Arslan, Senior Member, IEEE, and Elif Haytaoglu, Member, IEEE

Abstract—The guesswork refers to the distribution of the minimum number of trials needed to guess a realization of a random variable accurately. In this study, a non-trivial generalization of the guesswork called guessing cost (also referred to as cost of guessing) is introduced, and an optimal strategy for finding the \( p \)-th moment of guessing cost is provided for a random variable defined on a finite set whereby each choice is associated with a positive finite cost value (unit cost corresponds to the original guesswork). Moreover, we drive asymptotically tight upper and lower bounds on the logarithm of guessing cost moments. Similar to previous studies on the guesswork, established bounds on the moments of guessing cost quantify the accumulated cost of guesses required for correctly identifying the unknown choice and are expressed in terms of Rényi's entropy. Moreover, new random variables are introduced to establish connections between the guessing cost and the guesswork, leading to induced strategies. Establishing this implicit connection helped us obtain improved bounds for the non-asymptotic region. As a consequence, we establish the guessing cost exponent in terms of Rényi entropy rate on the moments of the guessing cost using the optimal strategy by considering a sequence of independent random variables with different cost distributions. Finally, with slight modifications to the original problem, these results are shown to be applicable for bounding the overall repair bandwidth for distributed data storage systems backed up by base stations and protected by bipartite graph codes.

Index Terms—Guessing, entropy, moments, bounds, cellular networks, sparse graph codes, LDPC, repair bandwidth.

I. INTRODUCTION

The typical guessing framework involves finding the value of a realization of a random variable \( X \) from a finite or countably infinite set \( \mathcal{X} \) by asking a series of questions “Is \( X \) equal to \( x \in \mathcal{X} \)” until the answer becomes “Yes”. What makes guessing framework challenging is that each answer typically affects the following questions and associated answers in which the number of questions is not necessarily fixed a priori, whereas the questions are determined based on a fixed strategy before the decision about the guess is finalized.

Given the distribution of \( X \), denoted by \( P_X(x) \), the ultimate objective of guessing framework is to find the distribution of the number of questions (guesses) before identifying the right answer. In an attempt to optimize the order of these questions, an optimal guessing strategy i.e., a bijective function from \( \mathcal{X} \) to a finite or countably infinite set \( |\mathcal{X}| \equiv \{1, \ldots, |\mathcal{X}|\} \) is adapted to typically minimize the average number of guesses, also known as the average guessing number. In [1], this problem is named as guesswork and lower and upper bounds are investigated on the guessing number in terms of Shannon’s entropy by Massey [2] and later on by McEliece and Yu [3]. A sequence of independent and identically distributed random variables \( X_1, \ldots, X_n \) are considered for practical applications and asymptotically tight bounds are derived on the moments of the expected number of guesses for the guesswork [4]. This study has related the asymptotic exponent of the best achievable guessing moment to the Rényi’s entropy. Later, bounds on the moments of optimal guessing are improved in [5] and subsequently in [6]. Particularly, the relationship between Rényi’s entropy and average guessing number is interesting and useful in different engineering contexts. In fact, Rényi’s entropy was a frequently used information measure in different contexts such as source coding to be able to generalize coding theorems in the past [7]. Such findings on the derived bounds are successfully applied to various recent applications of data compression [8], channel coding [9], networking and data storage security [10] through tweaking the original guesswork problem so that it fits within the requirements of the application at hand.

In many practical scenarios, making a guess about the state of a system (in a physical realm) or the unknown value of a random variable (both in presence or absence of side information [4] or compressed side information [11]) might lead to a certain amount of cost. In our work, using the same set-up, unlike the guesswork, the random variable will be associated with the cost set \( C = \{c_1, \ldots, c_{|C|}\} \) and the additive term in the computation of average cost for \( x \in \mathcal{X} \) would be \( \sum_c c P_X(x) \). Hypothesizing this cost measure to typically represent the potential risks and consequences that may arise from making an incorrect guess, unity costs will correspond to the guesswork as the baseline. With this generalization, the impact of making an error could be high enough to justify taking the time and effort to gather more information or perform additional measurements to reduce the uncertainty.
Consequently, making a choice among multiple possibilities may lead to different types and amounts of costs overall where we would refer it as the cost of guessing or simply guessing cost throughout. In general, these costs may dynamically be changing after making subsequent guesses about a series of random variables $\{X_i\}_{i=1}^n$ not necessarily independent. Independent and identically distributed random variables within the context of guesswork is thoroughly studied in the literature and some extensions to ergodic Markovian dependencies are also considered [12]. More complex dependencies such as the one formed by shift spaces are considered for a sequence of random variables in [13]. To our best of knowledge, the cost of guessing is only mentioned recently in [8] in a limited context whereby the guesser is allowed to stop guessing and declare an error and only then a fixed amount of cost is applied, otherwise the mechanism is identical to the guesswork. In addition, the definition of “cost” is expanded in the context of guessing in [14] to cover each choice to have an individual numerical cost value and a few improved bounds are provided later in [15].

### A. Background and Past Applications

There are numerous applications involving guesswork and its various uses, with many of them being related to cryptography and data error correction. For instance, in the field of security known as the (public) keyword guessing [16] around the cryptographic notion called searchable encryption [17]. On the other hand, guessing is recognized to be a useful analysis tool for data detection and error correction coding as well. For instance, it is shown that the cut-off rate of sequential decoding can easily be characterized if guessing theory is applied to the general idea of decoding of a tree code [4]. The application of guessing to coding theory dates back to Ulam’s problem [18] where one is allowed to lie in their responses [19]. More recently, capacity-achieving maximum likelihood decoding algorithms are developed in a data communication context based on guessing [9]. Later, these studies evolved to develop universal decoders especially for low-latency communication scenarios [20]. Moreover, thanks to the optimal strategy that lists most likely noise sequences and low implementation complexity, guessing framework is demonstrated to be a viable decoding option for the control channel of 5G networks [21]. Therefore, we believe that extending the idea of guessing by associating a cost with each choice would be quite powerful and will find plenty of interesting applications in communications engineering of future generation standards. Distributed systems constitute yet another application area in which cost of data communication depends on the link loads, node availabilities and current traffic at the time of data communication etc. Moreover base stations (BS) could also be used to help with the network data reconstruction processes at the expense of increased costs [22]. BS can help reduce the time needed to reconstruct data, as well as reduce the average cost of the process. Such costs can be expressed in terms of latency, bandwidth used to transfer information or computation complexity depending on the context.

Recently, there have been a multitude of diverse research efforts undertaken to investigate the general concept of guessing across various domains. For instance, in [23], security attacks on distributed storage systems in which an attacker can use one hint on the sensitive data is analyzed. In another study [24], bounds for a specific setting in which simultaneous guesses can be made is investigated. Furthermore, the study presented in [25] delves into the subject of restricted-memory guessing, wherein the individual making the guesses is limited by their ability to recall their previous attempts. In a more recent study, [26] focuses on developing general framework on well-known problems related to guesswork such as, source coding, task partitioning, etc. Among these studies, both [24] and [25] can be extended with the assignment of “costs” for guessing random variables. As for the case in [25], the representation size of each variable can be associated with a form of cost value since guessing a particular value can incur different usage patterns in memory, whereas the number of simultaneous guesses, i.e. the number of attacking computers, can cause additional cost in the system. The study in [24] can be further extended by taking into account the same specific parameter.

### B. Motivating Example and Contribution

In various applications of dynamic cellular networks, data is partitioned and disseminated among multiple nodes of the system, with nodes joining and leaving the cell at unpredictable times. An example scenario is provided in Fig. 1. Because of the frequently changing status of the nodes in the cell, it is difficult to be informed of data location in real time. The data on the departed nodes may be lost indefinitely, necessitating data regeneration, while the system only has minimal knowledge about the cached data whereabouts. Data may be downloaded from either local nodes or the base station, with the cost of each option depending on the physical distance of the nodes, potential obstructions for the line of sight, available bandwidth, or even the popularity of the file being regenerated/cached.

As can be seen in Fig. 1, a guessing protocol is required to run in a smartphone to regenerate the needed data piece. As can be seen, the protocol makes two attempts based on a predefined strategy to locate the data. Upon unavailability, the third attempt has been to download it from the base station. Another intriguing use case could be searchable encryption [17]. One can typically spend more time searching for a cipher keyword in an encrypted document depending on its size or number of defined keywords, leading to varying processing requirements. We finally notice that all such cost considerations can be integrated into our generalized guessing cost framework.

Motivated from such examples, in this study, we introduce the guessing cost, and derive optimal guessing strategies (the ones that minimize the various statistics about the cost) as well as asymptotically tight bounds by using a quantity related to the Rényi’s entropy for the expected values of real powers a.k.a. moments of the guessing cost. Note that numerical calculation of moments of guessing cost might be computationally feasible for small $|\mathcal{X}|$, however when we consider a series of random variables $\{X_i\}_{i=1}^n$ each defined on the same set $\mathcal{X}$,
finding the optimal guessing strategy for minimum average or the moments of the guessing cost would be computationally intractable (exponential in n), motivating us for finding tight bounds. We also have shown that bounds on the guessing cost of a sequence of independent random variables (not necessarily identically distributed) can be expressed in terms of Rényi entropy rate (Theorem 10), which is defined for order-α (α ∈ ℝ⁺) as

\[ \mathcal{R}_\alpha(\{X_i\}) = \begin{cases} \lim_{n \to \infty} \frac{H(\{X_i\})}{\log n}, & \text{if } \alpha = 1, \\ \lim_{n \to \infty} \frac{H_\alpha(\{X_i\})}{\log n}, & \text{Otherwise} \end{cases} \]

(1)
as long as the limits exist, where \( H_\alpha(\{X_i\}) \) is the joint Rényi entropy [28] and \( H(\{X_i\}) \) is the Shannon entropy of the sequence \( \{X_i\}_{i=1}^n \). On the other hand, the computation of the moments of guessing cost for independent sequence of random variables \( \{X_i\}_{i=1}^n \) is observed to be linear in n. Our results are asymptotically tight i.e., as \( n \to \infty \) we characterize the exponential growth rate of the moments of guessing cost. Several improved bounds are conjectured for the non-asymptotic region based on an established connection with the guesswork. Moreover, we realized that our findings for guessing cost can be easily applied to an example distributed data storage scenario, as depicted in Fig. 1, where nodes are repaired using a primary 1-based regeneration protocol and graph-based codes such as low density parity check (LDPC) codes [29] in the event of node failures or unexpected node departures from a base station’s coverage [22].

C. Organization

The rest of the paper is organized as follows. In Section II, the problem is formally stated and necessary and sufficient conditions are laid out for an optimal guessing strategy that minimizes the moments of guessing cost. In addition, distinct guesses for costs are discussed along with an algorithm that describes an optimal guessing strategy for non-configurable costs. In Section III, tight upper and lower bounds are provided for guessing cost moments of a random variable and the logarithm of the guessing cost moments of a series of random variables. While deriving the upper bound, new random variables are introduced and the connection with the guesswork is established. This connection helped us observe that the previous findings may be utilized to find/characterize tighter bounds in the non-asymptotic regime for the guessing cost. In Section IV, an example distributed storage scenario is considered where a long–blocklength LDPC code is utilized along with a guessing protocol which uses a primary node/base station to help with the data regeneration process. It is shown that the data repair problem of an LDPC code can be considered within the context of guessing cost. Several numerical results are provided before we conclude our paper in Section V.

D. Notation

In our work, \( X \) denotes the random variable whose values are selected from the set \( \mathcal{X} \) according to a probability distribution i.e., \( X \sim P_X(x) \) where \( \sim \) denotes “distributed”. We use \( | \cdot | \) to denote the cardinality of a set or absolute value depending on the context and \( |M| \) to mean the index set \{1, 2, ..., M\}. \( \mathbb{E}[\cdot] \) is the expectation operator. We denote the order-α Rényi’s entropy by \( H_\alpha(X) \) and Rényi’s rate by \( R_\alpha(X) \) for a given random variable \( X \). Also, \((z)^a \triangleq \max\{z, 0\}\) for \( z \in \mathbb{R} \). For a given length-n vector \( \mathbf{x} = (x_1, \ldots, x_n) \), the \( a \)-norm is defined to be \( ||\mathbf{x}||_a = (\sum_{i=1}^n |x_i|^a)^{1/a} \) for any positive real \( a \geq 1 \). Also for given scalar integers \( c \) and \( l \leq n \), we define \( \mathbf{x}^{(l)} \triangleq (x_1, x_2, \ldots, x_l) \) and \( \mathbf{x}^{(l)} + c \triangleq (x_1 + c, x_2 + c, \ldots, x_l + c) \).

II. PROBLEM STATEMENT AND GUESSING STRATEGY

Let \( C_\mathcal{G}(x) \) denote the guessing cost required by a particular guessing strategy \( \mathcal{G} : \mathcal{X} \to |M| \) when \( X = x \) (the realization of the random variable \( X \) is \( x \)). If the cost of making the guess \( X = x \) is independent of other guesses, each having unit cost, then this problem would be the same as the characterization of the average number guesses (expected guessing number) and
is identical to Massey’s original guessing problem introduced earlier [2].

Let us assume that the random variable $X$ can take on values from a finite set $\mathcal{X} = \{x_1, \ldots, x_M\}$ according to a distribution $P_X(x)$ with the associated sets of costs $C = \{c_i \in \mathbb{R}^+, c_i \geq 1 \mid 1 \leq i \leq M\}$ and $M \in \mathbb{Z}^+$. Without loss of generality, sets are assumed to have cardinalities $|\mathcal{X}| = |C| = M$ in which using a particular guessing strategy $G$, the probability that a randomly selected element of $\mathcal{X}$ can be found in the $i$-th guess is $p_i = P_X(G^{-1}(i))$ with the associated cost $c_i = c_{G^{-1}(i)}$, independently of already made guesses. Then, the average guessing cost using the strategy $G$ can be expressed as follows

$$E[C_G(X)] = \sum_{i=1}^{M} \sum_{j=1}^{i} c_j p_i = \sum_{i=1}^{M} f_i p_i = \sum_{i=1}^{M} c_i (1 - g_{i-1})$$

where $f_i = \sum_{j=1}^{i} c_j$ and $g_i = \sum_{j=1}^{i} p_j$ are cumulative cost and probability distributions. The minimization of this value is a function of both guessing strategy $G$ and the probability distribution of $X$.

In the context of guessing, one of the fundamental questions arises: For a given probability distribution $P_X(x)$ and $\rho > 0$, what strategy, denoted as $G^\rho$, will minimize $E[C_G(X)^\rho]$, i.e.,

$$G^\rho \triangleq \arg \min_{G} E[C_G(X)^\rho].$$

For $c_i = 1, \forall i \in [M], \rho = 1$, the optimal strategy is studied and well known i.e., guess the possible values of $X$ in the order of non-increasing probabilities [2]. In other words, without loss of generality, we can assume $p^{(M)}$ with $p_1 \geq p_2 \geq \cdots \geq p_M$ being the probabilities of choosing values from $[M]$ and $G(X = x_i) = i$. Then with this choice, the quantity $\sum_{i}^i p_i$ would be minimized. However, the same conclusion could not be easily drawn for an arbitrary vector of costs $c^{(M)} \in \{c_i \mid c_i \geq 1 \}^M$. Let us consider two possible cost selection scenarios based on the timing with respect to when exactly the guesses are made, that will have different implications.

In the first scenario, although the cost values are given i.e., $C$, the assignments are not made a priori i.e., costs can be associated with each choice as it minimizes the average/moments of guessing cost in the beginning of or during the guessing process. To illustrate this scenario, let us assume that we are in the situation of transporting a water tank to a fireplace. The collection of $M$ tanks at our disposal, despite having the same capacity, is made of different materials, incurring different expenses. Furthermore, we have $M$ different vehicles, each with a different chance of successfully reaching the firing zone. In this situation, our primary purpose is to transport a single tank to the given site. Such a work gives us the freedom to attach any tank to any vehicle, with preset prices associated to each choice but costs that may vary based on the precise tank and vehicle coupling chosen. Given the configurable costs, the best strategy for $\rho = 1$ is to guess the possible values of $X$ in the order of non-increasing probabilities and associate the most probable choice with the smallest cost value. In other words, for any assignment (a permutation of $c^{(M)}$) $\tilde{c}^{(M)} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_M)$ with $\tilde{c}_1 \leq \tilde{c}_2 \leq \cdots \leq \tilde{c}_M$ and $\tilde{c}_i \in c^{(M)}$, it is easy to see that for any $\rho > 0$, we have

$$\sum_{i=1}^{M} \left( \sum_{j=1}^{i} c_j \right)^\rho p_i \geq \sum_{i=1}^{M} \left( \sum_{j=1}^{i} \tilde{c}_j \right)^\rho p_i. \quad (4)$$

In case the cumulative costs are given by the moments of the guessing number i.e., $f_i = i^\rho$ for any $\rho \geq 1$, then it is easy to see that $c_i = i^\rho - (i - 1)^\rho$ which implies that $c_1 \leq c_2 \leq \cdots \leq c_M$ is satisfied. Thus, the optimal strategy would again be to guess the possible values of $X$ in the order of non-increasing probabilities as argued in [4].

The second scenario, in which costs associated with each choice are externally determined, involves finite costs and choices predetermined before the guessing process begins. This typical scenario is examined throughout the article. In this case, the best strategy for $\rho = 1$ would not necessarily be guessing the possible values of $X$ in the order of non-increasing probabilities.

**Example:** Suppose that there are three choices $1, 2, 3 (M = 3)$ and $\rho = 1$ with $\{1, p_1 = 0.5, c_1 = 20\}$, $\{2, p_2 = 0.4, c_2 = 2\}$ and $\{3, p_3 = 0.1, c_3 = 1\}$. In that case the guessing order expressed as the index set $\{2, 3, 1\}$ would be preferable over the set $\{1, 2, 3\}$ with the average costs being 12.6 and 21.1, respectively. Note that the latter choice, which is based on the order of non-increasing probabilities, is clearly not optimal.

We next provide the following proposition establishing a necessary condition for the optimal guessing strategy $G^\rho$ for non-configurable costs.

**Proposition 1:** For a given $\rho > 0$ and an optimal guessing strategy, namely $G^\rho$ (or $G^{\rho^2}$) for the $\rho$-th moment of guessing cost, we have the following necessary condition for all $i, j \in [M]$ satisfying $i \leq j$,

$$\left[||c^{(i)}||_1^\rho - ||c^{(j)}||_1^\rho\right] p_i + \left[||c^{(j)}||_1^\rho - ||c^{(i)}||_1^\rho - c_i + c_j\right]^\rho p_j \leq \sum_{l=i+1}^{j+1} \left[||c^{(j)}||_1^\rho - c_i + c_j\right] p_l + \left[||c^{(i)}||_1^\rho - ||c^{(l)}||_1^\rho\right] p_i. \quad (5)$$

**Corollary 1:** Furthermore, if $\rho \geq 1$, then for any $i \leq j$ the condition (5) can be simplified to

$$\left[||c^{(i)}||_1^\rho - (||c^{(i)}||_1^\rho - c_i)^\rho\right] p_j \leq \left[||c^{(j)}||_1^\rho - ||c^{(j-1)}||_1^\rho\right] p_i. \quad (6)$$

**Proof:** The proof of Proposition 1 as well as Corollary 1 can be found in Appendix A.

**Remark 2:** It is clear from Corollary 1 that for a given optimal guessing strategy for the mean guessing cost (i.e., $\rho = 1$), we must have $c_i p_j \leq c_j p_i$ for all $i, j \in [M]$ satisfying $i \leq j$.

**Remark 3:** We also note that there may be more than one optimal guessing strategy that would satisfy the conditions given above. Of these, one or more specific selections will result in the minimum guessing cost. The solution is unique

---

2The subscript indicates the dependency of the optimal strategy on the choice of $\rho$. However, we omit this notation unless it is absolutely necessary to simplify the notation.
if and only if the relation in the necessary condition (5) is a strict inequality.

Remark 4: Note that in our setting due to the freedom of choosing cost values arbitrarily, the choice of ρ may change the stochastic nature of guessing strategies i.e., the strategies for different ρ’s satisfying the condition in Proposition 2.1 are not necessarily the same and hence making a stochastic dominance argument across the optimal guessing strategies for distinct moments would not be possible.

In observation of Proposition 1, let us provide an algorithmic solution to finding optimal strategy for the minimum guessing cost. We notice that if the order based on the arithmetic solution to finding optimal strategy for the minimum moments would not be possible.

Algorithm 1

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{function} \text{OptimalCostGuess}(p, c, ρ) 
\State \State M ← |p| 
\State I ← \{1, 2, 3, \ldots, M\} 
\Comment Selection Order
\State swapped ← \text{false} 
\While{swapped} 
\For{j = 1 : M − 1} 
\State if \(\left|\left|c_{(j+1)}\right|^p - \left|\left|c_j\right|^p\right|\right| P_j > \left|\left|c_{(j+1)}\right|^p - \left|\left|c_{(j)}\right|^p\right|\right| P_{j+1}\) then \Comment If the condition does not hold
\State swap(\(c_j\), \(c_{(j+1)}\))
\State swap(\(p_j\), \(P_{j+1}\))
\State swap(\(I_j\), \(I_{j+1}\))
\State swapped ← \text{true}
\EndFor
\EndWhile
\State return I
\end{algorithmic}
\end{algorithm}

In Fig. 2, An example cost set \(C = \{c_{x_1}, c_{x_2}, \ldots, c_{x_M}\}\) is shown (right plot) and used with a uniform distribution (i.e., \(P_X(x) = 1/6\)). Based on the definitions 3.1 and 3.2, plot on the left demonstrates the calculated distributions \(P_Y(y)\) and \(P_Z(z)\). Also shown are the index thresholds (cumulative sum of (floor/ceiling of) costs) for \(y \in \mathcal{Y}\) (red) and \(z \in \mathcal{Z}\) (blue) across which the assigned probabilities may change. Note that the support for random variables \(Y\) and \(Z\) are different due to floor/ceiling operations.

III. BOUNDS ON MOMENTS OF THE GUESSING COST

Throughout this section and the following sections, we shall assume static costs determined a priori and focus on moments of guessing as the average guessing cost would be a special case. Let us begin by defining two auxiliary random variables \(Y\) and \(Z\) based on previously defined random variable \(X\) and its associated set of costs \(C = \{c_{x_1}, c_{x_2}, \ldots, c_{x_M}\}\).

Definition 5: Let us define the random variable \(Y\) that takes on values from a finite set \(\mathcal{Y} = \{y_1, y_2, \ldots, y_{\sum_{i=1}^{M} [c_{x_i}]}\}\), and the probability distribution of \(Y\) is defined as \(P_Y(y) = \frac{P_X(x_i)/[c_{x_i}]}{}\) for all positive reals \(c_{x_i} \geq 1, i \in [M]\) and \(t \in [[\mathcal{Y}]]\) satisfying

\begin{equation}
\sum_{x \in \mathcal{X}^{i-1}} [c_{x_i}] < t \leq \sum_{x \in \mathcal{X}^i} [c_{x_i}] \tag{7}
\end{equation}

where \(\mathcal{X}^i = \{x_1, \ldots, x_i\}\) with \(i = 0\) corresponding to empty set and the random variable \(Z\) to take on values from a finite set \(\mathcal{Z} = \{z_1, z_2, \ldots, z_{\sum_{i=1}^{M} [c_{x_i}]}\}\) with probabilities \(P_Z(z_i) = P_X(x_i)/[c_{x_i}]\) for all \(c_{x_i} > 1, i \in [M]\) and \(t \in [[\mathcal{Z}]]\) satisfying

\begin{equation}
\sum_{x \in \mathcal{X}^{i-1}} [c_{x_i}] < t \leq \sum_{x \in \mathcal{X}^i} [c_{x_i}] \tag{8}
\end{equation}

In Fig. 2, An example cost set \(\mathcal{C}\) (to the right) along with a uniform \(P_X(x)\) is assumed where the corresponding distributions \(P_Y(y)\) and \(P_Z(z)\) are illustrated (to the left). The same plot also shows the threshold points (red for \(Y\), blue for \(Z\)) where different realizations may have a different probability of occurring. Let us continue with the definition of induced guessing strategy. Note that the induced strategy is defined for integer costs to make it applicable to both random variables \(Y\) and \(Z\) at the same time since they are defined based on either ceiling or floor of real costs.
Any guessing strategy function \( G \) defined on \( \mathcal{X} \) can be transformed to one specific guessing strategy using the induced guessing strategy described below, for \( Y \) and \( Z \), named as \( \mathcal{H}(Y) \) and \( \mathcal{F}(Z) \), respectively.

**Definition 6 (Induced Guessing Strategy):** Let us consider a new random variable \( \mathcal{X} \) to take on values from a finite set \( \mathcal{X} = \{ x_1, x_2, \ldots, x_M \} \) with probabilities \( P_X(x) = P_X(x)/c_x \) for \( x \in \mathcal{X} \) and \( c_x \in \mathbb{Z}^+ \). Let us further assume that the guessing strategy \( G \) is used to guess the values of \( X \). For a given index \( i \in \{ \sum x_x \} \) there exists a positive integer \( k(i) \leq M \) satisfying
\[
\sum_{x \in \mathcal{X}^{(i)}-1} c_x < i \leq \sum_{x \in \mathcal{X}^{(i)}} c_x.
\]

The induced guessing strategy \( \mathcal{G} \) for guessing the values of \( \mathcal{X} \) is defined to be
\[
\mathcal{G}(x = x_i) \triangleq \sum_{x \in \mathcal{G}^{-1}(x_i)} c_x - \sum_{x \in \mathcal{X}^{(i)}-1} c_x + i.
\]

**Proposition 7:** The induced guessing strategy, namely \( \mathcal{G} \), is a valid strategy (bijection).

**Proof:** The proof can be found in Appendix B. \( \square \)

Using the Definition 6, since \( \{ c_x \} \) and \( \{ c_x \} \) are integers, we can define \( \mathcal{F}(Z) \) and \( \mathcal{H}(Y) \) to be the induced guessing strategies for random variables \( Z \) and \( Y \), respectively.

**Example:** Let us consider the example in Fig. 2. Note that the optimal strategy \( \mathcal{G}^*(X) \) is to perform selections in order of non-decreasing costs due to uniform \( P_X(X) = 1/6 \). In other words, \( \mathcal{G}^*(X = x_1) = 3 \), \( \mathcal{G}^*(X = x_2) = 2 \), \( \mathcal{G}^*(X = x_3) = 1 \), \( \mathcal{G}^*(X = x_4) = 4 \), \( \mathcal{G}^*(X = x_5) = 6 \), \( \mathcal{G}^*(X = x_6) = 5 \). For instance, say \( i = 15 \), it is not hard to verify \( k(i) = 5 \) using inequalities (7) since \( \sum x \mathcal{X}^{(i)}-1 c_x = 14 < i \leq 21 = \sum x \mathcal{X}^{(i)} \). Now using (10), we compute
\[
\mathcal{H}^*(Y = y_{15}) = \sum_{x : \mathcal{G}^*(x) < \mathcal{G}^*(x)} [c_x] - \sum_{x \in \mathcal{X}} [c_x] + 15
= \sum_{x : \mathcal{G}^*(x) < 6} [c_x] - \sum_{i=1}^4 [c_{x_i}] + 15 = 18.
\]

which is in line with the optimal guessing strategy \( \mathcal{H}^* \) induced for \( Y \) using the arguments of the guesswork i.e., guessing in the order of non-increasing probabilities (see Fig. 2). Let us now state the main results of this subsection.

### A. Lower and Upper Bounds

Let \( P_X(x) \) to denote the probability distribution of \( X \) and define the moments of the guessing cost using a particular guessing function \( G \) as
\[
\mathbb{E}[C^G(X)^\rho] = \sum_{i=1}^M P_X(G^{-1}(i)) \left[ \sum_{j=1}^{i} c_{G^{-1}(j)} \right]^\rho
\]
where the costs are not necessarily integers. Let us use the previous notation \( c_i = c_{G^{-1}(i)} \) and define \( c^* = \{ c_1, c_2, \ldots, c_M \} \) to be the order of costs obtained by running Algorithm 1 for a given \( \rho > 0 \) to find the optimal guessing strategy \( G^* \). This shall be useful in expressing the lower and upper bounds in the following two theorems.

**Theorem 8:** For any guessing function \( G \), \( \rho \geq 0 \) and costs \( c_j > 1 \), \( \rho \)-th moment of the guessing cost is lower bounded by
\[
\mathbb{E}[C^G(X)^\rho] \geq \mathbb{E}[C^{G^*}(X)^\rho] \geq \left( \frac{M}{1 + \gamma^*} \right)^\rho \exp \{ \rho H^{\frac{1}{\gamma^*}}(X) \}
\]
where \( \gamma^* \) is the harmonic mean of \( \{ \sum c_i - 1 \} \)'s for \( i = \{ 1, 2, \ldots, M \} \) and \( H_\alpha(X) \) is Rényi’s entropy of order \( \alpha \) for a given random variable \( X \) as long as the limit for Rényi’s entropy exists.

**Proof:** The proof can be found in Appendix C. \( \square \)

This lower bound, as will be illustrated in numerical results, is not tight particularly for large \( \rho \). However, this theorem would be useful for asymptotic analysis. For instance, using this result we can demonstrate in the next theorem that the bound given in Theorem 8 is tight within a factor of \( (M/(1 + \gamma^*))^\rho \).

**Theorem 9:** For the optimal guessing function \( G^* \), and \( \rho \geq 0 \), \( \rho \)-th moment of the guessing cost is upper bounded by
\[
\mathbb{E}[C^{G^*}(X)^\rho] \leq \exp \{ \rho H^{\frac{1}{\gamma^*}}(Y) \}
\]
where \( H_\alpha(Y) \) is Rényi’s entropy of order \( \alpha \) for a given random variable \( Y \).

**Proof:** The proof can be found in Appendix D. \( \square \)

### B. Relation to Guesswork and Guessing Cost Exponent

In this section, we present tight bounds for the logarithm of guessing cost moments, for a series of \( M \) random variables, which would be useful for our later data storage application. We primarily realize that the introduction of a random variable \( Z \) is useful for establishing a relationship with the guesswork. From the earlier discussions on the random variable \( Z \), we can express a looser lower bound (compared to (13)) for any guessing function \( G(\cdot) \) by observing the following for \( c_j > 0 \),
\[
\mathbb{E}[C^G(X)^\rho] = \sum_{x} P_X(x) C^G(x)^\rho = \sum_{i=1}^{M} P_X(G^{-1}(i)) \left[ \sum_{j=1}^{i} c_{G^{-1}(j)} \right]^\rho
\]
where
\[
\geq \sum_{i=1}^{M} \sum_{j=1}^{\frac{[c_{G^{-1}(i)}]}{[c_{G^{-1}(i)}]}} P_X(G^{-1}(i)) \left[ \sum_{k=1}^{[c_{G^{-1}(i)}]} + [c_{G^{-1}(i)}] \right]^\rho
\]
\[
\geq \sum_{i=1}^{M} \sum_{j=1}^{[c_{G^{-1}(i)}]} P_X(G^{-1}(i)) \left[ \sum_{k=1}^{[c_{G^{-1}(i)}]} + [c_{G^{-1}(i)}] \right]^\rho
\]
\[
\geq \mathbb{E}[C^F(Z)^\rho] \geq \left( 1 + \ln \left( \sum_{x} [c_x] \right) \right)^\rho \exp \{ \rho H^{\frac{1}{\gamma^*}}(Z) \}
\]
where the last inequality is due to guesswork and follows directly from [4] based on the definition of the random variable $Z$. Better lower bounds can be given, however this loose lower bound is enough to prove the following asymptotic result. Here the guessing function $\mathcal{F}(Z)$ for the random variable $Z$ defined earlier is directly induced from $\mathcal{G}(X)$. Next, the guessing cost exponent is given by the following theorem.

**Theorem 10:** Let $\{X_1, \ldots, X_n\}$ be a sequence of independent random variables where each is defined over the set $\mathcal{X}$ with the associated cost distribution $C_i$, and random variables $\{Y_i, Z_i\}$ based on Definition 5. Let $\mathcal{G}^*(X_1, \ldots, X_n)$ be an optimal guessing function. Then, for any $\rho > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}[C_{\mathcal{G}^*}(X_1, X_2, \ldots, X_n)^\rho]^{1/\rho} = R_{\frac{1}{\rho}}(\{Y_i\})$$

(19)

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}[C_{\mathcal{G}^*}(X_1, X_2, \ldots, X_n)^\rho]^{1/\rho} = R_{\frac{1}{\rho}}(\{Z_i\})$$

(20)

where $R_{\frac{1}{\rho}}(.)$ denotes the order-$1/(1 + \rho)$ Rényi rate which is assumed to exist, $Y_i$s and $Z_i$s are random variables induced from random variables $X_i$s as defined before. Moreover, if the costs are integers, then the limits converge and we will have

$$\lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}[C_{\mathcal{G}^*}(X_1, X_2, \ldots, X_n)^\rho]^{1/\rho} = R_{\frac{1}{\rho}}(\{Y_i\})$$

(21)

**Proof:** The proof can be found in Appendix E. □

These results indicate that the complexity of guessing cost of a random variable $X$ with strategy $\mathcal{G}$ can be tied to the complexity of guessing two related random variables $Z$ and $Y$ with the induced strategies $\mathcal{F}$ and $\mathcal{H}$, respectively, which are derived based on the cost distribution $C$ defined earlier.

**C. Improved Bounds: Non-Asymptotic Regime**

One of the observations is that the provided bounds have the potential for improvement particularly in the non-asymptotic regime similar in spirit to works such as [5], [6], and [37]. These improvements can easily be made after we recognize the relationship between guessing cost and the standard guesswork. In the following, we go through these extensions by referring to related past works. We shall also demonstrate how these bounds play out with varying $\rho$.

1) **Extension of Boztas’ Bounds [5]:** Let us extend Boztas’ upper bound by deriving the analog for the guessing cost. Let us first start with the following definition.

**Definition 11 (Balancing Cost):** For a given random variable $X$ and $\rho > 0$, the balancing cost $\tau_X(\rho)$ is defined to satisfy the following equality

$$\sum_{i=1}^{M} \frac{i-1}{i} c_i \rho_i = \sum_{i=1}^{M} \left( \sum_{j=1}^{i} c_j - \tau_X(\rho) \right) \rho_i$$

(22)

and equals a constant if costs are fixed i.e., $c_1 = \cdots = c_M = c$ for some constant $c \in \mathbb{R}$.

**Remark 12:** Note that for the special case $\rho = 1$, we will have $\tau_X(1) = \sum_i c_i \rho_i$, i.e., balancing cost would be equivalent to the average (expected) cost of guessing.

Now considering telescoping sequence argument for $\rho \geq 1$, we observe the following relation

$$\sum_{i=1}^{M} \left[ \left( \sum_{j=1}^{i} c_j \right)^\rho - \left( \sum_{j=1}^{i-1} c_j \right)^\rho \right] \frac{\rho_i}{c_i} \leq \sum_{i=1}^{M} \left[ \left( \sum_{j=1}^{i} c_j \right)^\rho - \left( \sum_{j=1}^{i-1} c_j \right)^\rho \right] \frac{\rho_i}{c_i}$$

(23)

$$= \sum_{i=1}^{M} \sum_{z=1}^{i-1} \left[ \left( \sum_{k=1}^{i-1} c_k + z \right)^\rho - \left( \sum_{k=1}^{i-1} c_k + z - 1 \right)^\rho \right] \frac{\rho_i}{c_i}$$

(24)

Finally using the equality provided in Eqn. (22), we get

$$\mathbb{E}[C_{\mathcal{G}}(X)^\rho] - \mathbb{E}[(C_{\mathcal{G}}(X) - \tau_X(\rho))^\rho]$$

$$= \sum_{i=1}^{M} \left[ \left( \sum_{j=1}^{i} c_j \right)^\rho - \left( \sum_{j=1}^{i} c_j - \tau_X(\rho) \right)^\rho \right] \rho_i$$

(25)

$$= \sum_{i=1}^{M} \left[ \left( \sum_{j=1}^{i} c_j \right)^\rho - \left( \sum_{j=1}^{i} c_j \right)^\rho \right] \rho_i$$

(26)

$$\leq \sum_{i=1}^{M} \sum_{z=1}^{i-1} \left[ \left( \sum_{k=1}^{i-1} c_k + z \right)^\rho - \left( \sum_{k=1}^{i-1} c_k + z - 1 \right)^\rho \right] \rho_i$$

(27)

$$= \sum_{k=1}^{M} (k^\rho - (k-1)^\rho) q_k \leq \left[ \sum_{k=1}^{M} q_k^\frac{1}{\rho} \right]^\rho$$

(28)

where $M' = \sum_{i=1}^{M} |c_i|$ and

$$q_k = \rho_i \text{ for } \frac{i-1}{i} c_i < z \leq \frac{i}{i} c_i \text{ and } i = 1, \ldots, M.$$  

(29)

$$q_k^{1/\rho} \leq \frac{1}{k} (q_1^{1/\rho} + \cdots + q_k^{1/\rho}), \text{ for } k = 1, \ldots, M'. \text{ (30)}$$

Note that the inequality in (28) follows from the Lemma in [5] as long as the “weights” $q_1, \ldots, q_M$ are non-negative reals satisfying the inequality given in Eqn. (30). We can show that the necessary condition for optimal strategy derived earlier will satisfy this inequality. Hence, this is a looser condition making the inequality apply to a broader range of guessing functions other than the optimal. Note here that $\sum k q_k = \sum_i |c_i| p_i \neq 1$ unless $c_i = 1$ for all $i = 1, \ldots, M$ i.e., $q_k$s are not forming a probability distribution for non-unity costs. Next, let us provide our theorem as an extension/generalization of Boztas’ arguments.

**Theorem 13:** For $\bar{c}_X(.)$ as given in Definition 11 and all guessing functions $\mathcal{G}$ for a random variable $X$ inducing $\{q_k\}$s which satisfy the relation in Eqn. (30) for $\rho = m + 1$ where $m \geq 1$ is an integer, the $m$-th moment of the guessing cost can be upper bounded by the recursive relation

$$\mathbb{E}[C_{\mathcal{G}}(X)^m] \leq \frac{1}{\tau_X(m + 1)(m + 1)} \left[ \sum_{k=1}^{M'} q_k^{1/\rho} \right]^{m+1}$$

(31)
where \( m \geq 1 \) is a positive integer and \( M' = \sum_{i=1}^{M} c_i \).

**Proof:** Using equations (22), (28) and the Binomial expansion, we have the following inequalities for integer \( m \),

\[
E[C_F(X)^{m+1}] - E[(C_F(X) - \bar{\tau}X(m+1))^{m+1}] = E[C_F(X)^{m+1}]
\]

\[
- \sum_{l=0}^{m-1} \binom{m}{l} E[C_F(X)^l](-\bar{\tau}X(m+1))^{m+1-l}
\]

\[
\leq \left[ \sum_{k=1}^{M'} \frac{q_k}{k^{1.5}} \right]^{m+1}
\]

which implies that \( \bar{\tau}X(m+1)(m+1)E[C_F(X)^m] \) is less than

\[
\sum_{k=1}^{m-1} \binom{m}{l} E[C_F(X)^l](-\bar{\tau}X(m+1))^{m+1-l}
\]

from which the result follows.

The main difference of our result compared to that of Boztas is the introduction of \( \{q_k\} \) s and the term \( \bar{\tau}X(m+1) \). In case of \( m = 1 \), we have

\[
E[C_F(X)] \leq \frac{1}{2\bar{\tau}X(2)} \left[ \sum_{k=1}^{M'} \frac{q_k}{k^{1.5}} \right]^{2} + \frac{\bar{\tau}X(2)}{2}
\]

subject to \( q_{k+1}^{1/2} \leq \frac{1}{3} (q_1^{1/2} + \cdots + q_k^{1/2}) \), for \( k = 1, \ldots, M' - 1 \).

Similarly for \( m = 2 \), we shall have

\[
E[C_F(X)] \leq \frac{1}{3\bar{\tau}X(3)} \left[ \sum_{k=1}^{M'} \frac{q_k}{k^{1.5}} \right]^{3} + \frac{\bar{\tau}X(3)}{3}
\]

subject to the conditions \( q_{k+1}^{1/2} \leq \frac{1}{3} (q_1^{1/2} + \cdots + q_k^{1/2}) \) and \( q_{k+1}^{1/3} \leq \frac{1}{2} (q_1^{1/3} + \cdots + q_k^{1/3}) \), for \( k = 1, \ldots, M' - 1 \).

We finally note that these expressions/bounds form a generalization of Boztas’ results and requires the calculation of the balancing cost for integer \( \rho \). We provide a gradient descent scheme in Algorithm 2 for efficiently finding the balancing cost for a given integer \( \rho \).

2) **Extension of Sason’s Bounds [6]:** In particular, we have the following improved lower bounds for any guessing strategy \( G \) and \( \rho > 0 \) that show better performance in the non-asymptotic regime of \( \rho \).

\[
E[C_F(X)^\rho] \geq E[C_F(Z)^\rho]
\]

\[
\geq \sup_{\beta \in (-\rho, \infty) - \{0\}} \exp \left\{ \frac{\rho}{\beta} \left[ H_{\hat{\rho}}(Z) - \log u_{\sum_{|c|}(\beta)} \right] \right\}
\]

\[
= \sup_{\beta \in (-\rho, \infty) - \{0\}} [u_{\sum_{|c|}(\beta)}]^{-\frac{\rho}{\beta}} \exp \left( \frac{\rho}{\beta} H_{\hat{\rho}}(Z) \right)
\]

where \( u_{\beta}(\beta) \) is defined similarly as in [6] and given by

\[
\beta = 1 + \frac{1}{1-\beta} \left( \frac{1}{\beta} \left( \frac{1}{|Z|^\beta} \right) - \frac{1}{2} \right)
\]

\[
\beta < 1
\]

\[
\beta \geq 1
\]

\[
\beta \leq -1
\]

where \( |Z| = \sum_{|c|}, \gamma \approx 0.5772 \) is the Euler-Mascheroni constant and \( \zeta(\beta) = \sum_{n=1}^{\infty} \frac{1}{n^\beta} \) is the Riemann zeta function for \( \beta > 1 \). Here the first inequality follows due to equations (15)-(18). Moreover, the second inequality follows due to guesswork arguments given in [6] which are directly applicable to random variable \( Z \) as its cost distribution assumes only unity values. As an extension of the upper bound, we provide the following theorem.

**Theorem 14:** For any guessing function \( G \), \( \rho > 0 \) and costs \( c_j > 1 \) associated with \( \{q_k\} \) s for a random variable \( X \) satisfying both \( q_{k+1}^{1/2} \leq \frac{1}{3} (q_1^{1/2} + \cdots + q_k^{1/2}) \) and \( q_{k+1}^{1/3} \leq \frac{1}{2} (q_1^{1/3} + \cdots + q_k^{1/3}) \), for \( k = 1, \ldots, M' - 1 \), then the \( \rho \)-th
moment of the guessing cost is upper bounded by
\[
E[C_G(X)^\rho] \leq \frac{1}{\tau_{\min X}(\rho)(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} \right]^{1+\rho}
\]
\[
+ \tau_{\min X}(\rho) \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} (\sum_{k=1}^{M'} q_k^{1/\rho}) - \tau_{\min X}(\rho) \right]^{1+\rho} - \frac{\tau_{\min X}(\rho)}{1 + \rho}
\]
(43)
where \( \tau_{\min X}(\rho) = \min \{ \tau_X(\rho), \tau_X(1 + \rho) \} \), \( M' = \sum_{i=1}^{c_j} [c_i] \), \( I_A \) is the indicator function and equals 1 if the condition \( A \) is true otherwise 0, and \( \tau_X(\rho), \tau_X(1 + \rho) \) are as defined before for a given \( \rho \) and can be found using Algorithm 2.

Proof: The proof can be found in Appendix F.

Remark 15: Theorem 14 may be loose for a given parameter set compared to previous upper bounds. However, we note that Theorem 14 is in similar form to Theorem 13 except it is non-recursive and assumes any real \( \rho \geq 0 \) rather than an integer.

It will become evident that by utilizing the claim given above, we will be able to improve the upper bound, particularly for values of \( \rho \) that are relatively small. Additionally, it is worth noting that the subsequent theorem provides an opportunity to refine this bound even further, specifically for \( \rho \in (0, 2] \).

Theorem 16: For any guessing function \( G \) and the cost of guessing \( C_G(.) \), \( \rho \in (0, 2] \) and costs \( c_j > 1 \) associated with \( \{ q_k \} \) for a random variable \( X \) satisfying both \( q_{k+1} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \) and \( q_{k+1} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \) for \( k = 1, \ldots, M' - 1 \), then the \( \rho \)-th moment of the guessing cost is upper bounded by
\[
E[C_G(X)^\rho] \leq \frac{1}{\tau_X(1 + \rho)(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} \right]^{1+\rho}
\]
\[
+ \tau_{\min X}(\rho) \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} (\sum_{k=1}^{M'} q_k^{1/\rho}) - \tau_{\min X}(\rho) \right]^{1+\rho} - \frac{\tau_{\min X}(\rho)}{1 + \rho}
\]
\[
+ \frac{\rho^\rho_{\min X}(\rho)}{(\rho - 1)^2} \tau_{\min X}(\rho) - \frac{\rho(\rho - 1)}{2(1 + \rho)}
\]
(44)

Proof: The proof can be found in Appendix H.

Remark 18: Using Theorem 17, we can find an explicit bound for \( \rho \) satisfying \( i + 1 \geq \rho > i \) for all integers \( i \geq 2 \). We can obtain these bounds by applying Equation (44) for \( i - 2 \) times using the result of Theorem 16. In this case however, the set of conditions would be more restrictive i.e., we would require to satisfy \( \frac{q_{k+1}}{\rho} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \) for \( \rho, \rho - 1, \ldots, \rho - i \) and \( k = 1, \ldots, M' - 1 \) all at the same time.

In order to help understand Remark 18 with an example, let us consider for instance \( \rho \in (2, 3] \). In this case we can apply the result of Theorem 16 to get
\[
E[C_G(X)^{\rho - 1}] \leq \frac{1}{\tau_X(1 + \rho)(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} \right]^{1+\rho}
\]
\[
+ \tau_{\min X}(\rho) \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} (\sum_{k=1}^{M'} q_k^{1/\rho}) - \tau_{\min X}(\rho) \right]^{1+\rho} - \frac{\tau_{\min X}(\rho)}{1 + \rho}
\]
\[
+ \frac{\rho^\rho_{\min X}(\rho)}{(\rho - 1)^2} \tau_{\min X}(\rho) - \frac{\rho(\rho - 1)}{2(1 + \rho)}
\]
(45)
which is subject to \( q_{k+1} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \) and \( q_{k+1} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \). Then using Theorem 17 the upper bound for \( \rho \in (2, 3] \) can be expressed in a closed form as
\[
E[C_G(X)^\rho] \leq \frac{1}{\tau_X(1 + \rho)(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} \right]^{1+\rho}
\]
\[
+ \tau_{\min X}(\rho) \left[ \sum_{k=1}^{M'} \frac{q_k^{1/\rho}}{\rho} (\sum_{k=1}^{M'} q_k^{1/\rho}) - \tau_{\min X}(\rho) \right]^{1+\rho} - \frac{\tau_{\min X}(\rho)}{1 + \rho}
\]
\[
+ \frac{\rho^\rho_{\min X}(\rho)}{(\rho - 1)^2} \tau_{\min X}(\rho) - \frac{\rho(\rho - 1)}{2(1 + \rho)}
\]
(46)
with the additional constraint \( \frac{q_{k+1}}{\rho} \leq \frac{1}{\rho} \left( q_1^{1/\rho} + \cdots + q_k^{1/\rho} \right) \) for \( k = 1, \ldots, M' - 1 \).

Remark 19: It is not hard to verify that the bounds given in Theorems 14, 16 and 17 will be reduced to Sason’s bounds given in [6] if we assume constant and unit costs. Hence these bounds are useful extensions and characterize a more general scenario.

D. Extension of Dragomir’s Bounds [37]

Finally, we would like to remark on the Dragomir’s bounds which was originally presented in the context of guesswork. These bounds have been introduced right after Boztas’ bounds are published [37]. Unfortunately these bounds are quite loose particularly in the context of guessing cost. The proposed bounds were based on the following theorem.

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Theorem 20: Let $a_i, b_i \in \mathbb{R}$ for $i \in [n]$ such that

$$a_{\min} \leq a_i \leq a_{\max}, \quad b_{\min} \leq b_i \leq b_{\max} \quad \text{for all} \quad i = 1, \ldots, n$$

with $a_{\max} = \min\{a_i\}$ and $b_{\max} = \min\{b_i\}$. Then, we have the inequality

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} b_i \right) \right| \leq \frac{1}{4} (a_{\max} - a_{\min})(b_{\max} - b_{\min})$$

Proof: The proof can be found in [37].

Let $a_i = f_i^p = \left( \sum_{j=1}^{i} c_j \right)^p$ and $b_i = p_i$ in equation (48). Also, let us define random variable $U$ with exactly the same cost distribution $C$ of $X$ and uniform probability distribution, then for any gussing strategy $\mathcal{G}$ we have

$$|\mathbb{E}[C_{\mathcal{G}}(X)^p] - \mathbb{E}[C_{\mathcal{G}}(U)^p]| \leq \frac{M (p_{\max} - p_{\min})}{4} \left( \sum_{j=1}^{M} c_j - c_{\min} \right)^p$$

where $c_{\min} = \min\{c_i\}$. Note that this relation defines both an upper and a lower bound for $\mathbb{E}[C_{\mathcal{G}}(X)^p]$. The bound can be tightened using the optimal guessing strategy $\mathcal{G}^*$. However, Dragomir’s bounds are generally looser compared to that of Sason’s and hence we omit to present numerical results for this bound.

E. Numerical Results

First, let us provide several numerical results to be able to illustrate how close the provided bounds are for finite values of costs, $\rho$ and $M$. The exact moments for the optimal guessing strategy are calculated using Algorithm 1 and denoted by OPT. The results are provided in Fig. 3. More specifically, inspired from the past research [6], we consider the quantity

$\frac{1}{\rho} \ln \mathbb{E}[C_{\mathcal{G}^*}(X)^\rho]$ in our comparisons where $\rho \in [0.25, 10]$. The probability of each choice is generated using geometric distribution as assumed in [6] with the restricted probability distribution $P_X(x) = (1 - a)x^{a-1}/(1 - a^M)$ using $M = 32$ and the parameter $a = 0.9$. The non-integer cost values are generated based on a truncated normal distribution defined in the range $(1, 100)$ with the same mean and variance i.e., $\mu = \sigma^2 = 16$.

As shown in Fig. 3, the closest values to $\frac{1}{\rho} \ln \mathbb{E}[C_{\mathcal{G}^*}(X)^\rho]$ for $\rho \in (0.25, 10]$, are given by the Eq. (40), which are followed by the bounds provided in Theorem 8 and Eq. (18). On average, the lower bounds of $\frac{1}{\rho} \ln \mathbb{E}[C_{\mathcal{G}^*}(X)^\rho]$ using Eq. (40) is 16.3% and 30.5% higher than that of bounds due to Theorem 8 and Eq. (18), respectively. In fact, it is interesting to show that bounds of Theorem 8 and Eq. (18) are not asymptotically tight. The tightest bound is achieved by the bound given in Theorem 13 among other alternative upper bounds. The bounds given in Theorem 13 are 5.98% and 0.391% less than the bounds given in Theorem 9 and Theorem 14 for $\rho \in \{1, 2, \ldots, 10\}$, respectively. Moreover, for $\rho \in \{4, 5, \ldots, 10\}$ the bounds of Theorem 13 are 0.023% less than that of Theorem 17. Notice also that bounds given in Theorem 13 and Theorem 17 are only valid for integer values of $\rho$ and Theorems 16 and 17 are complementary and should be considered together.

IV. AN APPLICATION: DISTRIBUTED DATA REGENERATION

In this section, we provide an application of the guessing cost within the context of a distributed data storage in which data content regeneration and repair are necessary to maintain the data durability. Such a data repair application scenario involves a slight variation of the guessing cost problem (introduced earlier), which is shown to be quite useful in this section in deriving optimal protocol design for highly dynamic...
In an alternative context, the physical distances between nodes could have been scaled with the link weight for a more realistic communication scenario.

Remark 2 for the standard cost of guessing is that the probabilities are functions of costs as will be explored next. The following proposition establishes a condition for contacting the primary node under optimal guessing context and independent node loss model.

**Theorem 21**: Let each secondary node to be independently unavailable/failed with probability $q > 0$. Assuming a degree-$d_v$ node is lost, let also $c_M$ be the cost of contacting the back-up node and $c_{\text{max}} = \max\{c_1, c_2, \ldots, c_d_v\}$ satisfying

$$c_M \geq c_{\text{max}}((1 - q)^{-c_{\text{max}}} - 1) \geq c_{\text{max}}$$

where $M = d_v + 1$. Then, guessing check relations as well as the back-up primary in the order of non-decreasing costs minimizes the average cost of downloaded symbols in the node repair process.5

**Proof**: Assuming independence, the probability that $j$-th check node will successfully repair the gray-colored node of Fig. 4 can be shown to be of the form

$$p_j = (1 - q)^{c_j M j - 1} \prod_{i=1}^{j-1} (1 - (1 - q)^{c_i})$$

where

$$p_M = 1 - \sum_{j=1}^{d_v} p_j = \prod_{i=1}^{M-1} (1 - (1 - q)^{c_i})$$

from which we realize that the probabilities are dependent on the costs. In a more general version of the problem, the costs of the check nodes may take values independent of the degrees (e.g., the communication cost required for obtaining a variable node may be different). In search of an optimal strategy, we need to think about $p_j$'s and $c_j$'s at the same time. Fortunately from equation (51), we can express $p_j$'s recursively for $j < M - 1$,

$$p_j = p_{j-1} \left( (1 - q)^{c_j - c_{j-1}} - (1 - q)^{c_j} \right)$$

which implies that if $c_{j-1} \leq c_j$, due to $0 < (1 - q)^s \leq (1 - q)^t \leq 1$ for all positive $t \leq s$ and $q \in (0, 1)$, we shall have $p_j \leq p_{j-1}$. Therefore, rearranging costs in non-decreasing order leads to rearrangement of probabilities in non-increasing order. But this result implies that the necessary condition of Remark 2 for $\rho = 1$ i.e., $c_j p_j \leq c_i p_i$ is satisfied for all $i, j \in \{1, 2, \ldots, M - 1\}$ and $i \leq j$. Note that if $c_{j-1} > c_j$ was the case, we would not be able to satisfy the necessary condition. In order to contact the primary (back-up) node when no neighboring nodes are able to help, we then have to satisfy the necessary condition $c_M p_{M-1} \geq c_{M-1} p_M$. Using equation (51), this condition can be reexpressed as

$$c_M (1 - q)^{c_M - 1} \geq c_{M-1} (1 - q)^{c_{M-1}}$$

which is the upper bound in inequality (50) with $c_{\text{max}} = c_{M-1}$. We finally recognize that lower bound in inequality (50) while costs satisfying $c_i > 1$ means $(1 - q)^{c_{\text{max}}} \leq 1/2$.6

Consider it with inequality in (53), this condition reduces to $c_M \geq c_{M-1}$ which completes the proof of the optimality of $c_{\text{max}}$.

---

4Here, due to large block length assumption, it is assumed that subsequent guesses cannot help each other. In addition, the cost of download can also be scaled with the link weight for a more realistic communication scenario. In an alternative context, the physical distances between nodes could have also been part of this cost definition, making the rest of our discussion more general.

5Here we use the guessing term for trying these relations until the lost symbol is repaired or using back-up primary if this symbol could not be repaired using local nodes’ check relations.

6This implies an upper bound on $q$ i.e., $q \geq 1 - 2^{-1/c_{\text{max}}}$.
the non-decreasing cost order. We finally note that the lower bound inequality need not be satisfied for the back-up primary to be the last resort. In fact, the upper bound inequality is a sufficient condition for that. However \( c_M \geq c_{M-1} \) becomes necessary only if the lower bound inequality is satisfied and hence the assertion of the theorem follows.

Let us associate the random variable \( X_u \), with a variable node \( u \) (having degree \( -1 \)) that characterizes the identification of the right check node for a successful repair. Note that a specific node failure pattern determines usable options of repair for that variable node. For instance given the finite set \( X_u = \{ u_1, u_2, \ldots, u_{d_u} \} \) associated with the costs \( C_u = \{ d_{u_1} - 1, d_{u_2} - 1, \ldots, d_{u_{d_u}} - 1 \} \), \( X_u = u_j \) indicates that the check relation \( u_j \) would be the first option for repair (i.e., \( G(X_u = u_j) = 1 \)) if \( d_{u_i} \leq d_{u_j} \) for all \( i \in [d_u], i \neq j \) (due to Theorem 21) for a successful regeneration. Furthermore, let \( G^*(X_1, X_2, \ldots, X_n) \) denote the optimal guessing function for the value of a joint realization of independent random variables \( X_1, X_2, \ldots, X_n \). Then due to Theorem 10, for large enough block length (number of nodes \( n \) tends large), the moments of repair bandwidth (cost in terms of downloaded symbols) using the optimal guessing strategy can be well approximated by the Rényi entropy rate (with equality in the ideal case),

\[
\mathbb{E}[C^*[X_1, X_2, \ldots, X_n]] \approx \prod_{i=1}^{n} \exp\{\rho H_{\frac{1}{1}}(X_i)\}
\approx \exp\{n \rho R_{\frac{1}{1}}(\{X_i\})\}
\]

due to costs are defined to be integers in our application scenario.

B. Data Repair With Multiple Passes: Density and Cost Evolution

In the previous subsection we have considered a static case i.e., a realization of an LDPC code ensemble i.e., a fixed bipartite graph representation. On the other hand, check and variable node degrees of a typical LDPC code ensemble is governed by degree distributions. As can be seen in Figure 4, the variable node of interest as well as its neighboring check nodes of degrees \( d_{u_1}, d_{u_2}, \ldots, d_{u_{d_u}} \), are shown. One can think of these values as realizations of the variable and check node degree distributions of LDPC codes typically expressed in polynomial forms as \( \Lambda(x) = \sum_{d=1}^{D_v} \lambda_d x^d \) and \( \Phi(x) = \sum_{d=1}^{D_c} \phi_d x^d \), respectively. Furthermore, we can define edge-perspective degree distributions for variable and check nodes in terms of node-perspective ones as follows [34],

\[
\begin{align*}
\lambda(x) &= \frac{\Lambda(x)}{\Lambda(1)} = \sum_{d=1}^{D_v} \lambda_d x^{d-1}, \\
\phi(x) &= \frac{\Phi(x)}{\Phi(1)} = \sum_{d=1}^{D_c} \phi_d x^{d-1}.
\end{align*}
\]

where the code rate \( r_{LDPC} \) can be described in terms of edge-perspective degree distributions as follows

\[
r_{LDPC} = \frac{k}{n} = 1 - \frac{\int_{0}^{1} \phi(x) dx}{\int_{0}^{1} \lambda(x) dx} = 1 - \frac{\sum_{d} \phi_d/d}{\sum_{d} \lambda_d/d}.
\]

where \( \phi_d(\lambda_d) \) is the probability that when we select an edge from the underlying bipartite graph randomly, it belongs to the set of the edges of a degree-\( d \) check (variable) node.

In Theorem 21, we have 1) conditioned on the node degrees of variable and check nodes and 2) considered only a single pass of the iterative repair strategy. Also, depending on the node failure patterns, it is likely that none of the check relations would be able to help with the repair process in the initial pass which would require us to decide on the successful completion of the repair process. One option is to download the missing content from the backup primary and cease the repair process. The alternative option is to execute one more iteration to reduce the secondary node unavailability/failure probability. Note that in this scenario, the node repairs are decentralized and take place in the absence of node unavailability/failure information.

Let \( \{ c_{u_1}, \ldots, c_{u_{d_u}} \} \) be the list of random variables characterizing the costs of contacting the check nodes \( u_1, \ldots, u_{d_u} \), and \( \{ c_{u_1} \leq \cdots \leq c_{u_{d_u}} \} \) denote these random variables rearranged in non-decreasing order of magnitude with \( c_{\text{max}} = c_{u_{d_u}} \) representing the maximum of the cost values. Based on Theorem 21, an optimal guessing strategy shall order the check nodes on the basis of their degrees (i.e., costs) assuming independent node failures. Accordingly, let us define

\[
\phi^{(z)}(x) = \sum_{d=1}^{D_c} \phi_d^{(z)} x^{d-1}
\]

where \( \phi_d^{(z)}(\lambda_d) \) is the probability that when we select an edge from the underlying bipartite graph randomly, it belongs to the set of the edges of a degree-\( d \) check node.

For this option to be reliable, we have to have the simultaneous repair successes of the other secondary nodes which were to be repaired in the previous pass and the assumption that no further node losses occur during the consecutive iterations.

Here, we note that the repairing variable node does not download the corresponding symbols unless the check relations ensure that the repair can complete successfully.
the repair process if none of the check relations are able to help, which happens with probability at the $l$-th iteration

$$1 - d_v \sum_{j=1}^{d_v} p_j^{(l)}(c^{(j)}) = \prod_{i=1}^{d_v} \left(1 - \sum_{d \in c^{(j)} + 1} \phi_{d|d_v}^{(i)} (1 - \epsilon_i)^{d-1}\right).$$  (60)

To remove conditioning, we sum over all possibilities of both sides and obtain

$$1 - \sum_{d_v=1}^{D_v} \lambda_{d_v} d_v \sum_{j=1}^{d_v} p_j^{(l)}(c^{(j)})$$

$$= \sum_{d_v=1}^{D_v} \lambda_{d_v} d_v \prod_{i=1}^{d_v} \left(1 - \sum_{d \in c^{(j)} + 1} \phi_{d|d_v}^{(i)} (1 - \epsilon_i)^{d-1}\right)$$

$$= \sum_{d_v=1}^{D_v} \lambda_{d_v} d_v \prod_{i=1}^{d_v} \left(1 - \sum_{d \in c^{(j)} + 1} \phi_{d}^{(i)} (1 - \epsilon_i)^{d-1}\right)$$

$$= \sum_{d_v=1}^{D_v} \lambda_{d_v} d_v (1 - \phi(1 - \epsilon_i))^{d_v} = \lambda (1 - \phi(1 - \epsilon_i))$$  (65)

which clearly does not depend on the guessing strategy since the local repair process already fails. From these equalities we observe that the availability of neighbors of the next check node and so on. The lost node is repaired (actual download happens) using neighbors of a check node whose all neighboring variable nodes are intact and reachable. However, if this attempt is not successful at the current iteration, we have to decide whether to reach the backup for the completion of the repair process or take another round of iteration, unless a predetermined maximum number of iterations is reached. An example is shown in Fig. 5 for $d_v = 3$. As can be seen, at the end of each iteration, a decision is made whether to complete the repair process with the help of a backup or continue with another round of iteration. Since we may have downloaded the contents of the repaired node directly from the backup, we first check if the backup node is to be contacted at the end of contacting all local nodes for a given iteration round. If so, we allow another round of iteration and if not, we cease iterations and complete the repair with the help of the backup node. Therefore, with this deferred primary-node communication protocol, to be able to ease the analysis, the backup node is allowed to be contacted only at the end of each iteration round and the advantage of conducting data regeneration using local nodes is maximized.

Accordingly, to contact the backup at the beginning of the $l$-th iteration for $l \geq 1$, we need to make sure that the backup node would not be contacted last within the same iteration i.e., the cost of contacting back-up is not too costly compared to local downloads. Recall from Theorem 21 that the back-up is not considered as a last resort only when the condition $C_{\pi} < c_{\pi} (1 - q)^{c_{\pi}} - 1$ is met. In other words, when the repair process comprises several iterations, to contact the back-up at the beginning of the $l$-th iteration for $l \geq 0$, we need to make sure that it would not be contacted at the end of the current iteration. For the LDPC code ensembles, the above condition happens with a non-zero probability since $c_{\pi}$ is a random variable. For a given variable node degree $d$, we contact the backup node at the beginning of the $l$-th

$$C. A. D e f e r r e d ~ P r i m a r y - N o d e ~ C o m m u n i c a t i o n ~ P r o t o c o l$$

Let us assume there is a backup primary (a base station) with the repair process within the cell. In this case, we assume that we can directly download the contents from the backup at the expense of a fixed cost $C_\pi (> 1)$ per symbol download. In this simple protocol, we aim at maximizing (minimizing) the use of local nodes (primary nodes) in the repair process. More specifically, we order the check relations based on the degrees i.e., bandwidth cost of repair, and utilize multiple iterations to ensure the regeneration. Since the node availability information is missing at the time of the repair, we confirm whether all connections of the neighbors of the first check node can successfully be established. If at least one of the variable nodes can not be reached, we next check the availability of neighbors of the next check node and so on. The lost node is repaired (actual download happens) using neighbors of a check node whose all neighboring variable nodes are intact and reachable. However, if this attempt is not successful at the current iteration, we have to decide whether to reach the backup for the completion of the repair process or take another round of iteration, unless a predetermined maximum number of iterations is reached. An example is shown in Fig. 5 for $d_v = 3$. As can be seen, at the end of each iteration, a decision is made whether to complete the repair process with the help of a backup or continue with another round of iteration. Since we may have downloaded the contents of the repaired node directly from the backup, we first check if the backup node is to be contacted at the end of contacting all local nodes for a given iteration round. If so, we allow another round of iteration and if not, we cease iterations and complete the repair with the help of the backup node. Therefore, with this deferred primary-node communication protocol, to be able to ease the analysis, the backup node is allowed to be contacted only at the end of each iteration round and the advantage of conducting data regeneration using local nodes is maximized.

Accordingly, to contact the backup at the beginning of the $l$-th iteration for $l \geq 1$, we need to make sure that the backup node would not be contacted last within the same iteration i.e., the cost of contacting back-up is not too costly compared to local downloads. Recall from Theorem 21 that the back-up is not considered as a last resort only when the condition $C_{\pi} < c_{\pi} (1 - q)^{c_{\pi}} - 1$ is met. In other words, when the repair process comprises several iterations, to contact the back-up at the beginning of the $l$-th iteration for $l \geq 0$, we need to make sure that it would not be contacted at the end of the current iteration. For the LDPC code ensembles, the above condition happens with a non-zero probability since $c_{\pi}$ is a random variable. For a given variable node degree $d$, we contact the backup node at the beginning of the $l$-th
iteration with conditional probability $\tau_{l|d}$ given by

$$
\tau_{l|d} = Pr \left( \left(C_{\tau} < c_{\max} \left((1 - \epsilon_{l})^{-c_{\max}} - 1\right) \land \left(C_{\pi} \geq c_{\max} \left((1 - \epsilon_{l-1})^{-c_{\max}} - 1\right) \forall t \in [l]\right) \right) \right)
$$

(68)

which can be simplified due to the monotonicity of $c_{\max} \left((1 - \epsilon_{l})^{-c_{\max}} - 1\right)$ as (see also Remark 22)

$$
\tau_{l|d} = Pr \left( \left(C_{\tau} < c_{\max} \left((1 - \epsilon_{l})^{-c_{\max}} - 1\right) \land \left(C_{\pi} \geq c_{\max} \left((1 - \epsilon_{l-1})^{-c_{\max}} - 1\right)\right) \right) \right)
$$

(69)

Note that since successful repair is guaranteed when the back-up node is involved, based on the above formulation, the evolution formula in (67) can be rewritten as (again with the And-Or tree assumption [31] in the decoding path)

$$
\epsilon_{l+1} = \epsilon_{0} \sum_{d=1}^{D_{v}} \lambda_{d} (1 - \tau_{l|d}) (1 - \phi(1 - \epsilon_{l}))^{d-1}. \tag{70}
$$

On the other hand, we notice that the conditional probability that we contact the back-up at the end of the $l$-th iteration i.e., $1 - \rho_{l|d}$ can be approximated for small $\epsilon_{0} c_{\max}$ (i.e., $\epsilon_{0} D_{v}$) as

$$
1 - \tau_{l|d} = Pr \left( \left(c_{\max} \left((1 - \epsilon_{l+1})^{-c_{\max}} - 1\right) \leq C_{\pi}\right) \right)
$$

$$
= Pr \left( \left(c_{\max}^{-1} c_{\max} \left((1 - \epsilon_{l+1})^{-c_{\max}} - 1\right) \leq C_{\pi}\right) \right)
$$

$$
\approx Pr \left( \left(c_{\max} \leq \sqrt{C_{\pi}} c_{\max}^{-1}\right) \right)
$$

$$
= \left( \frac{1}{\sqrt{C_{\pi}}} \right)^{d} \tag{71}
$$

(71)

$$
= \left( \sum_{i=1}^{\min\{D_{v}, \sqrt{C_{\pi}} c_{\max}^{-1}\}} \Phi_{i} \right)^{d} \tag{72}
$$

(72)

where (72) follows due to independence of $c_{\tau}$.

Remark 22: Note that in this setting, as long as $\epsilon_{l} \to 0$ as $l$ tends large, we have $\epsilon_{l+1} \leq \epsilon_{l}$ which leads to the relationship $\tau_{l|d} \geq \tau_{l+1|d}$ i.e., as the iterations get deeper, it becomes less likely to contact the back-up node for the repair due to reduced loss probabilities of the neighboring nodes.

D. Decoding Threshold With Back-up

For ease of analysis, let us assume small $\epsilon_{0} D_{v}$ and not put any limit on the number of iterations with a predefined threshold. In this case, we notice that if $\sqrt{C_{\pi}} \epsilon_{0} \geq D_{v}$, then this would result in standard density evolution and the decoding threshold, in that case, would be defined to be

$$
\epsilon_{0} = \sup \{ \epsilon_{0} (1 - \phi(1 - x)) < x, \forall x, \epsilon_{0} \}
$$

(73)

i.e., the maximal erasure probability below which error-free repair is possible through solely using iterations/local nodes. On the other hand, if $\sqrt{C_{\pi}} \epsilon_{0} < D_{v}$, suppose in one of the iterations of the BP (say $l^{*}$-th iteration), we have $\epsilon_{l} \geq \epsilon_{l-1} \geq \epsilon_{l}$ such that $\left[C_{\pi}/\epsilon_{l-1} \leq D_{v} \leq \left[C_{\pi}/\epsilon_{l}\right]\right]$, then for all $l \leq l^{*} - 2$ we would have $\epsilon_{l+1}$ to be the solution to the following equation

$$
\epsilon_{l+1} = \epsilon_{0} \sum_{d=1}^{D_{v}} \lambda_{d} \left( \sum_{i=1}^{\min\{D_{v}, \sqrt{C_{\pi}} \epsilon_{l+1}\}} \Phi_{i} \right)^{d} \left(1 - \phi(1 - \epsilon_{l})\right)^{d-1}
$$

(74)

and finally for $l \geq l^{*} - 1$, $\epsilon_{l+1}$ is given by the standard density evolution formula. Accordingly, the decoding threshold in that case is given by $\forall x, \epsilon(0, 1)$,

$$
\epsilon_{0}^* (C_{\pi}) = \sup \left\{ \epsilon_{0} \sum_{d=1}^{D_{v}} \lambda_{d} \left( \min\{D_{v}, \sqrt{C_{\pi}} \epsilon_{l+1}\}\right)^{d} \right. \tag{75}
$$

$$
\times \left(1 - \phi(1 - x)\right)^{d-1} < x, \forall x, \epsilon(0, 1) \right\}
$$

$$
= \inf \left\{ \frac{x(1 - \phi(1 - x))^{1-d}}{\sum_{d=1}^{D_{v}} \lambda_{d}} \left(1 - \sum_{i=1}^{\min\{D_{v}, \sqrt{C_{\pi}} \epsilon_{l+1}\}} \Phi_{i}\right)^{d} \right\} \tag{76}
$$

(76)

Here we immediately realize the relationship $\epsilon_{0}^* (C_{\pi}) \leq 1$ i.e., the decoding threshold can be improved with
the help of a primary back-up node in the context of data reconstruction.

E. Numerical Demonstration

We consider an irregular LDPC code that performs provably close to the optimal (achieving minimum gap to the channel capacity) under BEC [33]. The edge-perspective degree distributions of this code ensemble are given by

\[
\phi(x) = 0.60829x^5 + 0.39171x^6 \tag{77}
\]

\[
\lambda(x) = 0.20503x^3 + 0.45572x^2 + 0.19325x^{13} + 0.146x^{14} \tag{78}
\]

from which the rate of the code can be calculated to be 0.4339 with \(D_e = 7\) and \(D_v = 15\). The results of our cost evolution process are presented in which \(\tau_{\ell|i}\) is estimated numerically based on

\[
\tau_{\ell|i} \approx Pr \left( C \pi < c_{max} \left( (1 - \epsilon_i)^{-c_{max} - 1} \right) \right) \times Pr \left( C \pi \geq c_{max} \left( (1 - \epsilon_{l-1})^{-c_{max} - 1} \right) \right) \tag{79}
\]

rather than the approximation given by the Eqn. (71) since \(\max \{\epsilon_0 D_e\} = 3.037\) is not small enough.

The cost of a symbol repair using the strategy with back-up primary node having \(C \pi \in \{50, 700, 1000\}\), respectively, is provided in Fig. 6. When \(C \pi = 50\), the use of the base station has commenced in early stages of the node repair process (iterations). In other words, in case of \(C \pi = 50\), the primary node is used as the last resort without further iterations are performed for \(\epsilon \in [0.53, 0.56]\). When \(C \pi = 700\) and \(\epsilon_0 > 0.544\), the use of backup primary is preferred, which in turn leads to higher node repair cost. Moreover, when \(C \pi = 1000\) and \(\epsilon_0\) attains the value of 0.551, the value of \(c_{max}\) is increased substantially. For the value of \(C \pi = 1000\), the use of BS before the last iteration is preferred when \(\epsilon_0 \geq 0.575\).

In Table I, the results of actual optimal LDPC repair cost as well as upper and lower bounds obtained through numerical evaluations of Thm. 13, and Eq. (41) are presented for all combinations of \(\rho \in \{1, 2, 3\}\) and \(\epsilon_0 \in \{0.01, 0.05, 0.1\}\). For this data repair scenario, the base station cost is assumed to be \(C \pi = 1000\). Based on our numerical results for \(\rho = 1\), \(\rho = 2\), \(\rho = 3\), the evaluation of Thm. 13 gives 64%, 22%, 17% higher results than the actual results on average, whereas Eq. (41) provides 24%, 12%, 11% lower values than the actual on average, respectively.

V. CONCLUSION AND FUTURE WORK

In this work, the general notion of guessing cost is introduced and an optimal strategy is provided for guessing a random variable defined on a finite set with each choice may be associated with a positive finite cost. Upper and lower bounds on the logarithm of guessing cost moments are derived and expressed in terms of the Rényi’s entropy and entropy rate. We have established connections with the guesswork through introducing novel random variables. Thanks to this connection, previous works on the improvements of upper/lower bounds for the guesswork become readily applicable. Accordingly, we provided improved bounds on the moments of guessing cost without lengthy proofs. Finally, we established the guessing cost exponent on the moments of the optimal guessing by considering a sequence of random variables. These bounds are shown to serve quite useful for bounding the overall repair latency cost (data repair complexity) for distributed data storage systems in which sparse graph codes may be utilized. We have assumed a simple protocol to derive initial results and demonstrated the usefulness of the previously derived bounds. It’s important to highlight that in the design of a distributed storage protocol, there may be value in giving up the prediction of the next value based on conditions like the total accumulated cost. Characterization of the guessing cost, in that case, would have to be expressed in terms of smooth Rényi’s entropy. Recent studies such as [8] considered similar constraints for the guesswork within the context of source coding. Such scenarios would be considered in our future work to be able to improve protocol design towards better system maintenance in presence/assistance of an external back-up/base station primary.

APPENDIX A

PROOF OF PROPOSITION 1

Let us start with \(\rho = 1\) i.e., mean guessing cost given in Remark 2. Consider swapping the \(i\)-th and \(i+1\)-th guessed values. Let \(G_{i,i+1}\) be the original guessing strategy and \(G_{i+1,i}\)
be the swapped version. Then it is straightforward to show that the difference is

\[
\mathbb{E}[C_{G_i, i+1}(x)] - \mathbb{E}[C_{G_{i+1}, i}(x)] = c_i(1 - g_{i-1}) + c_{i+1}(1 - g_{i-1} - p_i)
\]

\[
+ c_{i+1}(1 - g_{i-1}) - c_i(1 - g_{i-1} - p_{i+1})
\]

\[
= c_{i+1}(1 - g_{i-1}) - c_i(1 - g_{i-1} - p_{i+1})
\]

which implies that if \( c_i p_{i+1} > c_{i+1} p_i \), then we swap \( i \)-th and \((i + 1)\)-th guessed values in order to reduce the average guessing cost, otherwise no swapping is performed. Since each swapping leads to lower cost, for any \( i, j \in \{1, \ldots, M\} \) with \( i \leq j \), the optimal guessing strategy \( G^* \) would satisfy the following series of inequalities

\[
c_{i+1} p_{i+1} \leq c_i p_{i+1}
\]

\[
c_{i+1} p_{i+2} \leq c_{i+2} p_{i+1}
\]

\[
\vdots
\]

\[
c_{j-1} p_j \leq c_j p_{j-1}
\]

where continuing deriving the inequality and multiplying left-hand and right-hand consecutive terms individually would give us the desired result since all \( p_i \) and \( c_i \) are non-negative.

Now, let us consider the general case i.e., for any real \( \rho > 0 \), we have

\[
\mathbb{E}[C_{G_{i, j}}(X)^\rho] = \sum_{i=1}^{M} \sum_{j=1}^{i} c_j \rho^i p_i = \sum_{i=1}^{M} (|c(i)|^\rho |p_i|
\]

\[
\leq \sum_{i=1}^{M} \sum_{j=1}^{i} (|c(i)|^\rho |p_i|)
\]

\[
= \sum_{i=1}^{M} \sum_{j=1}^{i} (|c(i)|^\rho |p_i|)
\]

\[
= \sum_{i=1}^{M} \sum_{j=1}^{i} (|c(i)|^\rho |p_i|)
\]

\[
= \sum_{i=1}^{M} \sum_{j=1}^{i} (|c(i)|^\rho |p_i|)
\]

Note that due to non-negativity of costs, for any \( l \in \{2, \ldots, j - i\} \) and \( \rho \geq 1 \), we always have

\[
||c(i+l)||^\rho_1 - (||c(i+l)||^\rho_1 - c_{i+l-1})^\rho \geq ||c(i+l-1)||^\rho_1 - ||c(i+l-2)||^\rho_1
\]

which can be rewritten as

\[
(||c(i+l-1)||^\rho_1 + c_{i+l})^\rho - (||c(i+l-2)||^\rho_1 + c_{i+l})^\rho \geq ||c(i+l-1)||^\rho_1 - ||c(i+l-2)||^\rho_1
\]

Then in order to satisfy all the inequalities above, we must have

\[
(||c(i+l-1)||^\rho_1 - (||c(i+l)||^\rho_1 - c_{i+l})^\rho \leq ||c(j)||^\rho_1 - ||c(j-1)||^\rho_1
\]

which is a simplified condition as compared to the condition in (87) for \( \rho \in (1, +\infty) \).

**APPENDIX B**

**Proof of Proposition 7**

Let us assume \( c_x \in \mathbb{Z}^+ \). Our objective is to show that \( \overline{G} : \mathcal{X} \rightarrow \{\sum x \} \) is a bijection. Let us begin with one-to-one property and with a trivial case. Suppose that \( j = i + 1 \neq i \) for \( i < \sum x \), and assume that the following relation is true.

\[
\overline{G}(\mathcal{X} = \mathcal{X}_i) = \overline{G}(\mathcal{X} = \mathcal{X}_j) = \overline{G}(\mathcal{X} = \mathcal{X}_{i+j})
\]

which obviously violates the 1-to-1 property. For a given index \( i \in \{\sum x \} \), due its construction in the manuscript, there exists a positive integer \( k(i) \leq M \) satisfying

\[
\sum_{x \in \mathcal{X}} c_x \leq \sum_{x \in \mathcal{X}^{k(i)-1}} c_x
\]

where \( \mathcal{X}^i = \{x_1, \ldots, x_i\} \) with \( \mathcal{X}^0 = \emptyset \). If \( k(i) = k(i+1) \), then it is easy to see that the equality (94) would be impossible to hold since \( i \neq i + 1 \) in the definition of \( \overline{G} \). If \( k(i) \neq k(i+1) \) we notice that we can use the expression given for \( \overline{G} \) and rewrite both sides of the equation (94) as

\[
\overline{G}(\mathcal{X} = \mathcal{X}_i) = \sum_{x \in \mathcal{G}(\mathcal{X}_i)} c_x - \sum_{x \in \mathcal{X}^{k(i)-1}} c_x + i
\]

\[
= \sum_{x \in \mathcal{G}(\mathcal{X}_i)} c_x - \sum_{x \in \mathcal{X}^{k(i)-1}} c_x + i + c_{x_{k(i)}}
\]

\[
\overline{G}(\mathcal{X} = \mathcal{X}_{i+j}) = \sum_{x \in \mathcal{G}(\mathcal{X}_{i+j})} c_x - \sum_{x \in \mathcal{X}^{k(i+1)-1}} c_x + i + 1
\]

\[
= \sum_{x \in \mathcal{G}(\mathcal{X}_{i+j})} c_x - \sum_{x \in \mathcal{X}^{k(i)+1}} c_x + i + 1
\]

where we used the fact that \( \mathcal{X}^{k(i+1)+1} = \mathcal{X}^{k(i)+1} \) due to the way \( \mathcal{X}^i \) is defined. Note that since \( k(i) \neq k(i+1) \) is assumed, the first terms in the expressions of (96) and (97) cannot be equal to satisfy the equation (94). Therefore, we have two possible cases.
• Case $\sum x : \mathcal{X} = \mathcal{X}_{i+1} \leq \sum x : \mathcal{X} < \mathcal{X}_{i+1}$: In this case however, it must be clearly seen that we must have $\mathcal{X} = \mathcal{X}_{i} > \mathcal{X} = \mathcal{X}_{i+1}$ since $c_{x_{i+1}} \geq 1$.

• Case $\sum x : \mathcal{X} = \mathcal{X}_{i+1} \leq \sum x : \mathcal{X} < \mathcal{X}_{i+1}$: In this case we must have

$$\sum x : \mathcal{X} = \mathcal{X}_{i+1} \leq \sum x : \mathcal{X} < \mathcal{X}_{i+1} \geq c_{x_{i+1}} \geq c_{x_{i}} \geq c_{x_{i}} \quad (98)$$

due to the ordering of costs. This result implies that $\mathcal{X} = \mathcal{X}_{i} = \mathcal{X} = \mathcal{X}_{i+1}$, which is necessarily a strict inequality due to the assumption $j = i + 1 \neq i$.

As a result, our initial assumption that $\mathcal{X} = \mathcal{X}_{i} = \mathcal{X} = \mathcal{X}_{i+1}$ cannot be true. Using this observation, we can extend our arguments to any $(i, j)$ pair with $i \neq j$. WLOG, assume that $j > i$, by considering the pairs $(i, i+1), (i+1, i+2), \ldots, (j-1, j)$ in this particular order, it is not hard to show $\mathcal{X} = \mathcal{X}_{i} \neq \mathcal{X} = \mathcal{X}_{j}$ i.e., $\mathcal{X}$ is one-to-one. On the other hand, we also notice that

$$\mathcal{X} = \mathcal{X}_{i} = \sum x : \mathcal{X} = \mathcal{X}_{i+1} \geq 0 \quad (99)$$

due to non-negativity of costs. Hence, the minimum integer the strategy could map to is $1$. In addition,

$$\mathcal{X} = \mathcal{X}_{i} = \sum x : \mathcal{X} = \mathcal{X}_{i+1} \geq 0 \quad (99)$$

which, together with (99), implies $1 \leq \mathcal{X} = \mathcal{X}_{i} \leq \sum x : \mathcal{X} = \mathcal{X}_{i}$. As a result, we can induct that $\mathcal{X}$ must be a bijection and hence a valid strategy/mapping.

**APPENDIX C\nPROOF OF THEOREM 8**

Before giving the formal proof let us state a well known lemma.

Lemma 23 (Hölder’s Inequality): Let $a_{i}$ and $b_{i}$ for $(i = 1, \ldots, n)$ be positive real sequences. If $q > 1$ and $1/q + 1/r = 1$, then

$$\left(\sum_{i=1}^{n} a_{i}^{q}\right)^{1/q} \left(\sum_{i=1}^{n} b_{i}^{q}\right)^{1/r} \geq \sum_{i=1}^{n} a_{i} b_{i} \quad (101)$$

Let $a_{i}$ be a positive real number for all $i$, $M$ be a natural number, and $\gamma$ be the harmonic mean of $\{a_{1}, \ldots, a_{n}\}$, then we have

$$\sum_{i=1}^{M} a_{i} \leq M \frac{1 + \gamma}{1 + \gamma} \quad (102)$$

which can easily be proved using Radon’s inequality [35]. Now, let us express the lower bound of the moments of the guessing cost as follows,

$$\mathbb{E}[C_{G}^{\alpha}(X)] \geq \mathbb{E}[C_{G}(X)^{\alpha}] \quad (103)$$

where easily follows from a direct application of Hölder’s inequality, where $\{c_{i}^{\alpha}\}$ are the optimal ordering of cost values. To see this, let us set $r = 1 + \rho, q = (1 + \rho)/\rho$ in Hölder’s inequality so that $1/q + 1/r = 1$ is satisfied for $\rho > 0$. We also let

$$a_{i} = \left[\sum_{j=1}^{i} c_{G}^{-1}(j)\right]^{-\rho/(1+\rho)} \quad \text{and} \quad b_{i} = \left[\sum_{j=1}^{i} c_{G}^{-1}(j)\right]^{\rho/(1+\rho)}$$

Now, using Hölder’s inequality, it would be easy to obtain

$$\left[\sum_{i=1}^{M} \frac{1}{\sum_{i=1}^{M} a_{i}^{-\rho}} \right]^{1/(1+\rho)} \geq \sum_{i=1}^{M} P_{X}(G^{-1}(i))^{1/(1+\rho)} \quad (104)$$

from which inequality (103) follows for the optimal strategy $G^{*}$. Now, considering the ordering of costs that minimizes the right hand side, we shall have,

$$\mathbb{E}[C_{G}(X)^{\alpha}] \geq \left[\sum_{i=1}^{M} \frac{1}{\sum_{i=1}^{M} a_{i}^{-\rho}} \right]^{1/(1+\gamma)} \left[\sum_{i=1}^{M} P_{X}(G^{-1}(i))^{1/\gamma} \right]^{1/(1+\gamma)} \quad (105)$$

where $\gamma^{*}$ is the harmonic mean of $\{a_{i}^{-1} - 1\}$’s for $i = 1, 2, \ldots, M$ and $H_{\alpha}(X)$ is Rényi’s entropy of order $\alpha$ ($\alpha > 0, \alpha \neq 1$) for random variable $X$ defined as,

$$H_{\alpha}(X) = \frac{\alpha}{1-\alpha} \ln \left[\sum_{x} P_{X}(x)^{\alpha}\right]^{1/\alpha} \quad (107)$$

Note that inequality (105) followed from the inequality (28).

**APPENDIX D\nPROOF OF THEOREM 9**

Let us first observe that with the optimal guessing strategy $G^{*}$ that minimizes the expected guessing cost $x$,

$$C_{G^{*}}(x) = \sum_{x : C_{G^{*}}(x) \leq C_{G}(x)} \frac{c_{x}}{c_{G^{*}}(x)} \quad (108)$$

$$\leq \sum_{x : C_{G^{*}}(x) \leq C_{G}(x)} \frac{c_{x} P_{X}(x')}{c_{X^{*}} P_{X}(x)} \quad (109)$$

$$= \sum_{x : C_{G^{*}}(x) \leq C_{G}(x)} \frac{c_{x} P_{X}(x')}{P_{X}(x)} \quad (110)$$
where the inequality (109) follows from the necessary condition of Proposition 1 with \( \rho = 1 \) i.e., \( c_{x'} P_X(x) \leq c_x P_X(x') \) for all \( x' : C_{G^*}(x') \leq C_{G^*}(x) \) that needs to hold for the optimal guessing strategy \( G^* \). Also, although the exponent \( 1/(1 + \rho) \) decreases the value, it is still greater than 1 due to \( c_{x'} P_X(x') / c_x P_X(x) \geq 1 \). Using the inequality given in (111) in equation (12), we get

\[
E[C_{G^*}(X)^\rho] = \sum_x P_X(x) C_{G^*}(x)^\rho 
\leq \sum_x P_X(x) \left( \sum_{x'} c_{x'} P_X(x') \right)^{\frac{\rho}{1+\rho}} 
= \sum_x \left( \sum_{x'} c_{x'} P_X(x') \right)^{\frac{\rho}{1+\rho}} 
= \sum_x \left( \sum_{x'} c_x (P_X(x) / c_x) \right)^{\frac{\rho}{1+\rho}} 
\leq \sum_x \left( \sum_{x'} c_x (P_X(x) / c_x) \right)^{\frac{\rho}{1+\rho}} p_i 
\leq \sum_{l=1}^{j-1} \left( ||c(l)||_1 - c_i + c_j \right)^{\frac{\rho}{1+\rho}} p_i \tag{114}
\]

is satisfied, then \( c_{i,j} \leq c_{j,i} \) may or may not hold. However, in the worst case the strategy satisfying \( c_{i,j} \leq c_{j,i} \) may not be optimal for a given \( \rho \). Hence, our argument in Eqn. (109) is still valid since we are generating an upper bound for the optimal guessing strategy. However, with the general condition the upper bound can be tightened at the expense of ending up with more complex expressions. For the asymptotic result of the paper, this simpler upper bound would be just sufficient.

On the other hand, we notice that

\[
\frac{P_X(x)}{c_x} = \frac{c_x P_X(x)}{c_x c_x} \geq \frac{P_X(x)}{c_x} \left( \frac{c_x}{c_x} \right)^{1+\rho} \tag{115}
\]

from which the following inequality follows for \( \rho \geq 0 \),

\[
\left[ c_x \right] (P_X(x) / c_x) \geq c_x (P_X(x) / c_x)^{1+\rho}. \tag{116}
\]

Thus, using the inequality (116) and the pre-defined auxiliary random variable \( Y \) earlier, we finally express the upper bound in a more compact form

\[
E[C_{G^*}(X)^\rho] \leq \left( \sum_x c_x (P_X(x) / c_x) \right)^{1+\rho} \tag{117}
\]

\[
\leq \left( \sum_x \left[ c_x \right] (P_X(x) / c_x) \right)^{1+\rho} \tag{118}
\]

Notice that this upper bound will reduce to Arikan’s upper bound i.e., \( \exp(\rho H_{1+\rho}(Y)) \) with all costs set to unity.

**Appendix E**

**Proof of Theorem 10**

Let us consider the general case and first define the induced random variables \( Z_1 \sim Z \) and \( Y_i \sim Y \) for the corresponding random variables \( X_i \sim X \) for \( i = 1, \ldots, n \) each with cost distributions \( C_i \) based on Definition 5. Also let \( F^* \) and \( H^* \) be the induced optimal guessing strategies from \( G^* \) for random variables \( \{Z_1, \ldots, Z_n\} \) and \( \{Y_1, \ldots, Y_n\} \), respectively.

Now, consider the upper bound for i.i.d. random variables and observe

\[
C_{G^*}(x_1, \ldots, x_n) \leq \sum_{x_1, \ldots, x_n} \prod_{i=1}^{n} P_{X_i}(x_i) \tag{119}
\]

\[
\prod_{i=1}^{n} \sum_{x_i} \sum_{x_i} \left( \prod_{i=1}^{n} P_{X_i}(x_i) \right)^{1+\rho} \tag{120}
\]

\[
= \prod_{i=1}^{n} \sum_{x_i} \left( c_{x_i} P_{X_i}(x_i) \right)^{1+\rho} \tag{121}
\]

due to independence and series of inequalities \( c_{x_i} P_{X_i}(x_i) \leq c_{x_i} P_{X_i}(x_i') \leq c_{x_i} P_{X_i}(x_i') \leq \cdots \leq c_{x_n} P_{X_n}(x_n') \) for all \( x_i' : C_{G^*}(x_i') \leq C_{G^*}(x_i) \) where \( i = 1, \ldots, n \) that needs to hold for the optimal strategy \( G^* \) required by the necessary condition. Finally, we can upper bound the expected guessing cost for a sequence of i.i.d. random variables as

\[
E[C_{G^*}(X_1, \ldots, X_n)^\rho] \leq \sum_x P_X(x_1, \ldots, x_n) C_{G^*}(x_1, \ldots, x_n)^\rho \tag{122}
\]

\[
\leq \prod_{i=1}^{n} \sum_{x_i} \left( c_{x_i} P_{X_i}(x_i) \right)^{1+\rho} \tag{123}
\]

where the last inequality follows due to inequalities similar to (116) for each random variable \( X_i \). If the cost and probability distributions of \( X_i \)’s are arranged such that the induced \( Y_i \)’s are identically distributed (for instance \( X_i \)’s are i.i.d. with identical
cost distributions i.e., $C_1 \equiv C_2 \equiv \cdots \equiv C_n \triangleq C$) then we can further simplify (123) as

$$\mathbb{E}[C_g^*(X_1, \ldots, X_n)] \leq \exp\{\rho n H_{\frac{1}{1+\rho}}(Y_1)\}. \quad (124)$$

Let us define $s_n = \mathbb{E}[C_g^*(X_1, \ldots, X_n)]$ and $\beta_k = \inf\{s_n : n \geq k\}$ for $k \geq 1$. Note that $\beta_k$ is an increasing sequence ($\beta_{k+1} \geq \beta_k$) bounded above by (123). Then we have

$$\liminf_{n \to \infty} \frac{1}{n}\ln(s_n) = \lim_{k \to \infty} \left\{ \frac{1}{n}\ln(\beta_k) \right\} = \lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} H_{\frac{1}{1+\rho}}(Y_i) = \mathcal{R}_{\frac{1}{1+\rho}}(\{Y_i\})$$

which is defined to be the order-$1/(1+\rho)$ Rényi entropy rate [30] as long as the limit exists. In addition to the upper bound, we can extend the lower bound given in (18) for a sequence of random variables as

$$\mathbb{E}[C_g^*(X_1, \ldots, X_n)] \geq \mathbb{E}[C_{\mathcal{F}^*}(Z_1, \ldots, Z_n)] \geq \left(1 + \ln\left(\prod_{i=1}^{n} [c_{x_i}]\right)\right)^{-\rho} \exp\left\{\rho \sum_{i=1}^{n} H_{\frac{1}{1+\rho}}(Z_i)\right\}.$$ \quad (126)

where the first inequality can be shown to be true through induction and the second inequality follows from [4] through a bit of generalization. Note that $\mathcal{F}^*$ is the optimal induced strategy from $\mathcal{F}^\ast$. If the cost and probability distributions of $X_i$'s are arranged such that the induced $Z_i$'s are identically distributed (for instance $X_i$'s are i.i.d. with identical cost distributions i.e., $C_1 \equiv C_2 \equiv \cdots \equiv C_n \triangleq C$) then we can further simplify (126) as

$$\mathbb{E}[C_g^*(X_1, \ldots, X_n)] \leq \mathbb{E}[C_{\mathcal{F}^*}(Z_1, \ldots, Z_n)] \geq \left(1 + \ln\left(\prod_{i=1}^{n} [c_{x_i}]\right)\right)^{-\rho} \exp\left\{\rho \sum_{i=1}^{n} H_{\frac{1}{1+\rho}}(Z_i)\right\}. \quad (127)$$

Similarly, we further define $\alpha_k = \sup\{s_n : n \geq k\}$ for $k \geq 1$ which makes it a decreasing sequence lower bounded by (127). As a consequence, we have

$$\limsup_{n \to \infty} \frac{1}{n}\ln(s_n) = \lim_{k \to \infty} \left\{ \frac{1}{n}\ln(\alpha_k) \right\} = \lim_{n \to \infty} \ln \left(1 + \ln\left(\prod_{i=1}^{n} [c_{x_i}]\right)\right) - \frac{1}{n} \sum_{i=1}^{n} H_{\frac{1}{1+\rho}}(Z_i) \leq \lim_{n \to \infty} \frac{1}{n}\sum_{i=1}^{n} H_{\frac{1}{1+\rho}}(Z_i) = \mathcal{R}_{\frac{1}{1+\rho}}(\{Z_i\}). \quad (128)$$

If \{X_i\} are identically distributed with the same cost distribution $C$, then order-$1/(1+\rho)$ Rényi entropy rates would be equal to $H_{\frac{1}{1+\rho}}(Y)$ and $H_{\frac{1}{1+\rho}}(Z)$, respectively. Note that in general, these rates are not necessarily equal. However, if the costs are integers, it would not be hard to verify $\mathcal{R}_{\frac{1}{1+\rho}}(\{Y_i\}) = \mathcal{R}_{\frac{1}{1+\rho}}(\{Z_i\})$. Thus, combining equations (128) with (125), we shall have

$$\limsup_{n \to \infty} \frac{1}{n}\ln(s_n) = \liminf_{n \to \infty} \frac{1}{n}\ln(s_n) = \mathcal{R}_{\frac{1}{1+\rho}}(\{Y_i\}) = \mathcal{R}_{\frac{1}{1+\rho}}(\{Z_i\}) = \mathcal{R}_{\frac{1}{1+\rho}}(\{X_i\})$$

which completes the proof.

**APPENDIX F**

**PROOF OF THEOREM 14**

Let us first consider $\rho \geq 1$, and for a given real constant $c \geq 1$ we let $r(u; c)$ be the parametric function given for $u \geq c$ by

$$r(u; c) = \frac{1}{c(1+\rho)}(u^{1+\rho} - (u - c)^{1+\rho} - (u - c)^\rho). \quad (130)$$

One of the things we realize about this function is that its derivative is non-negative, i.e.,

$$\frac{\partial}{\partial u} r(u; c) = \frac{1}{c}(u^\rho - (u - c)^\rho) - \rho(u - c)^{\rho - 1} \geq 0 \quad (131)$$

which is not hard to see by invoking mean value theorem from standard calculus. Moreover, we have

$$\frac{\partial}{\partial c} \frac{\partial}{\partial u} r(u; c) = \frac{u^\rho}{c^2}(u^\rho - (u - c)^\rho) + \rho(u - c)^{\rho - 2} \geq 0, \quad (132)$$

i.e., it is always non-negative for $u \geq c \geq 1$ and $\rho \geq 1$. Thus $\frac{\partial}{\partial c} r(u; c)$ is an increasing function of $c$ which therefore leads to the conclusion that for $c \geq 1$, it is non-negative. Since $r(c, c) = 0$, it follows that $r(u; c)$ is non-negative for $u \geq c \geq 1$.

Remember that for a given random variable $X$ associated with costs $C = c_1, \ldots, c_M$, we have from equation (28)

$$\mathbb{E}[C_g^*(X)] - \mathbb{E}[C_g^*(X) - \tau_X(\rho)] \leq \left[ \sum_{k=1}^{M} \frac{1}{q_k} \right]^\rho, \quad (133)$$

where $M' = \sum_{i=1}^{M} [c_i]_1$, $q_k = p_i$ for $\sum_{i=1}^{k-1} [c_i]_1 < k \leq \sum_{i=1}^{k} [c_i]_1$ and $i = 1, \ldots, M$ and balancing cost $\tau_X(\rho)$ is as defined in Definition 11 for as long as $q_k$'s satisfy the relation given in equation (30). Note that since $\mathbb{E}[r(C_g^*(X), \tau_X(\rho))] \geq 0$, it implies for $\rho \geq 1$ that if $\tau_X(\rho) \leq \tau_X(1 + \rho)$,

$$\mathbb{E}[(C_g^*(X) - \tau_X(\rho))] \leq \frac{1}{\tau_X(\rho)(1 + \rho)} (\mathbb{E}[C_g^*(X)]^\rho - \mathbb{E}[(C_g^*(X) - \tau_X(\rho)]^\rho) - \frac{\tau_X(\rho)}{1 + \rho} \quad (134)$$

$$\leq \frac{1}{\tau_X(\rho)(1 + \rho)} (\mathbb{E}[C_g^*(X)]^\rho$$
Therefore combining (137) and (142), we get the final expression

\[
\frac{\rho}{1 + \rho} \geq \frac{\min\{\xi_X(\rho), \xi_X(1 + \rho)\}^{1+\rho}}{1 + \rho} - \frac{\rho}{1 + \rho}. 
\]  

Hence using equation (133) and (139), we finally obtain

\[
\min\{\xi_X(\rho), \xi_X(1 + \rho)\}^{1+\rho} - \frac{\rho}{1 + \rho} 
\]

Thus combining (137) and (142), we get the final expression

\[
E[G(X)] \leq \frac{1}{\xi_X(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{1}{\xi_X(1 + \rho)} \right]^{1+\rho} 
\]

Let us now consider the case \( \rho \in (0, 1) \). Now, we introduce the following function for \( u \geq c \geq 1 \),

\[
r(u, c) = \frac{1}{c(1 + \rho)}(u^{1+\rho} - (u - c)^{1+\rho} + \rho c^{1+\rho}) - u^\rho. 
\]  

When we take the derivative with respect to \( u \), we get

\[
\frac{\partial}{\partial u} r(u, c) = \frac{1}{c^2} (u^\rho - (u - c)^\rho) - \rho u^{\rho-1} + \rho c^{\rho-1}, \quad x \in (u - c, u) 
\]

where the equation (146) holds due to mean value theorem for all \( \rho \in (0, 1) \). Since \( r(c, c) = 0 \), we always have \( r(u, c) \geq 0 \) for all \( u \geq c \geq 1 \). Applying this function, we will have

\[
E[r(C(X)), \xi_X(1 + \rho)] \geq 0 
\]

where the inequality (149) follows since \( 1 + \rho \geq 1 \). Similarly for \( E[r(C(X)), \xi_X(\rho)] \geq 0 \), we get

\[
E[G(X)] \leq \frac{1}{\xi_X(1 + \rho)} \left( E[G(X)]^{1+\rho} 
\]

if \( \xi_X(\rho) \leq \xi_X(1 + \rho) \). Thus,

\[
E[G(X)] \leq \frac{1}{\xi_X(1 + \rho)} \left[ \sum_{k=1}^{M'} \frac{1}{\xi_X(1 + \rho)} \right]^{1+\rho} 
\]

Thus, using an indicator function \( \mathbb{1}_{\rho \geq 1} \) to be able to combine (143) and (152), the result follows.

**APPENDIX G**

**PROOF OF THEOREM 16**

Let us consider the case \( \rho \in (0, 1) \) case first and state the following Lemma.

**Lemma 24.** For \( \rho \in (0, 1) \) and any \( c \in \mathbb{R}, c \geq 1 \) and \( u \geq c \),

\[
u^\rho \leq \frac{u^{1+\rho} - (u - c)^{1+\rho} + \rho c^{1+\rho}}{c(1 + \rho)} \mathbb{1}_{u \geq c+1} + \frac{(c + 1)^{1+\rho} - 1}{c(1 + \rho)} \mathbb{1}_{u \geq c+1}. 
\]
Proof: For \( \rho \in (0, 1) \) and a given real constant \( c \geq 1 \) we define to parametric functions given by
\[
\begin{align*}
    r_1(u, c) &= \frac{u^{1 + \rho} - (u - c)^{1 + \rho}}{c(1 + \rho)} + \frac{\rho u^{\rho^c} - u^\rho}{1 + \rho} - u^\rho, \\
    r_2(u, c) &= \frac{u^{1 + \rho} - (u - c)^{1 + \rho}}{c(1 + \rho)} + (c + 1)^\rho - \frac{(c + 1)^1 - 1}{c(1 + \rho)} - u^\rho.
\end{align*}
\]
(154)
(155)
For \( u \in (c, \infty) \), we have \( \frac{\partial r_1(u, c)}{\partial u} \neq \frac{\partial r_2(u, c)}{\partial u} = \frac{1}{2} (u^\rho - (u - c)^\rho) - \rho u^{\rho - 1} > 0 \) again similarly due to mean value theorem. Moreover, \( r_1(c, c) = r_2(c + 1, c) = 0 \). As a result, \( r_1(u, c) \geq 0 \) for \( u \geq c \) and \( r_2(u, c) \geq 0 \) for \( u \geq c + 1 \). Next we observe that for \( \rho \in (0, 1) \) and \( c \geq 0 \), we have \((c + 1)^\rho - (c + 1)^0 < \rho c^{1 + \rho} - 1\) which implies that
\[
(c + 1)^\rho - \frac{(c + 1)^1 - 1}{c(1 + \rho)} < \frac{\rho c^{1 + \rho} - 1}{1 + \rho}
\]
(156)
which completes the proof by recognizing \( \min\{r_1(u, c), r_2(u, c)\} = r_2(u, c) \).
\( \square \)

Let \( \tau_{\min, X}(\rho) = \min(\tau_X(\rho), \tau_X(1 + \rho)) \). Now using the result of Lemma 24 for \( \rho \in (0, 1) \) and replacing \( u \) with \( C_G(\rho) \), and considering both cases \( \tau_X(1 + \rho) \leq \tau_X(\rho) \leq \tau_X(1 - \rho) \) separately, similar to Appendix E, we obtain
\[
\mathbb{E}[C_G(X)^\rho] \leq \frac{1}{\tau_{\min, X}(\rho)(1 + \rho)} \left( \mathbb{E}[C_G(X)^{1 + \rho}] - \mathbb{E}[(C_G(X) - \tau_X(1 + \rho))^{1 + \rho}] \right)
+ \frac{\rho \tau_{\min, X}(\rho)}{1 + \rho} \mathbb{P}(\tau_{\min, X}(\rho) \leq C_G(X) < \tau_{\min, X}(\rho) + 1)
+ \frac{\rho \tau_{\min, X}(\rho)}{(\tau_{\min, X}(\rho) + 1)^\rho - \frac{\tau_{\min, X}(\rho) + 1}{\tau_{\min, X}(\rho)(1 + \rho)}} \mathbb{P}(C_G(X) \geq \tau_{\min, X}(\rho) + 1)
\]
(157)
where \( (158) \) follows from (133). Next, we state the following Lemma.

Lemma 25: For \( \rho \in [1, 2] \) and any \( c \in \mathbb{R}, c \geq 1 \) and \( u \geq c \),
\[
u^\rho \leq \frac{u^{1 + \rho} - (u - c)^{1 + \rho}}{1 + \rho} + \frac{u^\rho - (u - c)^\rho}{\rho} + \frac{c^\rho(c^2 - c^1 - 1)}{\rho(1 + \rho)}
\]
(159)
Proof: For \( \rho \in [1, 2] \), let \( r(u, c) \) be a parametric function given by
\[
\mathbb{E}[C_G(X)^\rho] \leq \frac{1}{\tau_{\min, X}(\rho)(1 + \rho)} \left( \mathbb{E}[C_G(X)^{1 + \rho}] - \mathbb{E}[(C_G(X) - \tau_X(1 + \rho))^{1 + \rho}] \right)
+ \frac{\rho \tau_{\min, X}(\rho)}{1 + \rho} \mathbb{P}(\tau_{\min, X}(\rho) \leq C_G(X) < \tau_{\min, X}(\rho) + 1)
+ \frac{\rho \tau_{\min, X}(\rho)}{(\tau_{\min, X}(\rho) + 1)^\rho - \frac{\tau_{\min, X}(\rho) + 1}{\tau_{\min, X}(\rho)(1 + \rho)}} \mathbb{P}(C_G(X) \geq \tau_{\min, X}(\rho) + 1)
\]
(157)
One of the things we realize about this function is that its derivative is non-negative for \( \rho \geq 2 \), i.e.,
\[
\frac{\partial}{\partial u} r(u, c) = \frac{1}{c} \frac{u^{\rho} - (u - c)^{\rho}}{\rho} - \rho u^{\rho - 1} + \frac{1}{2} \rho(\rho - 1) c u^{\rho - 2}
\]
(161)
(162)
To be able to shorten the notation let \( v(x) = x^\rho \). Now consider the Taylor series expansion of \( v(x) \) around \( u \) i.e.,
\[
v(x) = v(u) + v'(u)(x - u) + \frac{1}{2} v''(u)(x - u)^2 + \frac{1}{6} v'''(u)(x - u)^3 + \ldots
\]
(163)

If we take the partial derivative of this function with respect to \( u \), we get
\[
\frac{\partial}{\partial u} r(u, c) = u^\rho - (u - c)^\rho + u^{\rho - 1} - (u - c)^{\rho - 1} - \rho u^{\rho - 1}
\]
(164)
(165)
(166)
(167)
(168)
Now, evaluate \( v(x) \) at \( x = u - c \) and observe to truncate the expansion,

\[
(u - c)^\rho = v(u - c) = v(u) - cv'(u) + \frac{c^2}{2} v''(u) - \frac{c^3}{6} v'''(u)'
\]

(169)

for some \( u' \in (u - c, u) \) for \( u \geq c \). By plugging (169) into (167), we obtain

\[
\frac{\partial}{\partial u} r(u, c) = v'(u) - \frac{c}{2} v''(u)
\]

\[
+ \frac{c^2}{6} v'''(u') - \frac{1}{2} v'(u) + \ldots + \frac{c^3}{6} v'''(u')
\]

(170)

since \( u' > 0 \) due to \( u \geq c \) and \( \rho \geq 2 \). It is also not hard to verify that \( r(c, c) = 0 \), which eventually implies that \( r(u, u) \geq 0 \) for \( u \geq c \). If we substitute \( u = C_G(X) \) and \( c = \tau_X(1 + \rho) \) and take the expectation i.e. \( E[r(u, c)] \geq 0 \), we finally obtain

\[
E[C_G(X)^\rho] \leq \frac{1}{\tau_X(1 + \rho)^{1 + \rho}} E[C_G(X)^{1 + \rho}]
\]

\[
- E[(C_G(X) - \tau_X(1 + \rho))^{1 + \rho}]
\]

\[
- \frac{\tau_X(1 + \rho)}{2} E[C_G(X)^{\rho - 1}] - \frac{\rho(\rho - 1)}{2(1 + \rho)}
\]

(172)

from which the result follows using the relationship given in (28).

REFERENCES

[1] J. O. Pliam. (1999). The Disparity Between Work and Entropy in Cryptology. [Online]. Available: http://philby.ucsd.edu/cryptolib/1998/98-24.html

[2] J. L. Massey. “Guessing and entropy,” in Proc. IEEE Int. Symp. Inf. Theory, Trondheim, Norway, Jun. 1994, p. 204.

[3] P. J. McEliece and Z. Yu. “An inequality on entropy,” in Proc. IEEE Int. Symp. Inf. Theory, Whistler, BC, Canada, Sep. 1995, p. 329.

[4] E. Arikan. “An inequality on guessing and its application to sequential decoding,” IEEE Trans. Inf. Theory, vol. 42, no. 1, pp. 99–105, Jan. 1996.

[5] S. Boztas, “Comments on ‘an inequality on guessing and its application to sequential decoding,'” IEEE Trans. Inf. Theory, vol. 43, no. 6, pp. 2062–2063, Nov. 1997.

[6] I. Sason and S. Verdú, “Improved bounds on guessing moments via Rényi measures,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2018, pp. 566–570.

[7] L. L. Campbell, “A coding theorem and Rényi’s entropy,” Inf. Control, vol. 8, no. 4, pp. 423–429, Aug. 1965.

[8] S. Kuzuoka, “On the conditional smooth Rényi entropy and its application in guessing,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Paris, France, Jul. 2019, pp. 647–651.

[9] K. R. Duffy, J. Li, and M. Méard, “Capacity-achieving guessing random additive noise decoding,” IEEE Trans. Inf. Theory, vol. 65, no. 7, pp. 4023–4040, Jul. 2019.

[10] A. Bracher, E. Hof, and A. Lapidoth, “Guessing attacks on distributed-storage systems,” IEEE Trans. Inf. Theory, vol. 65, no. 11, pp. 6975–6998, Nov. 2019.

[11] R. Graczyk, A. Lapidoth, N. Merhav, and C. Pfister, “Guessing based on compressed side information,” IEEE Trans. Inf. Theory, vol. 68, no. 7, pp. 4244–4256, Jul. 2022.

[12] D. Malone and W. G. Sullivan, “Guesswork and entropy,” IEEE Trans. Inf. Theory, vol. 50, no. 3, pp. 525–526, Mar. 2004.

[13] C.-E. Pfister and W. G. Sullivan, “Rényi entropy, guesswork moments, and large deviations,” IEEE Trans. Inf. Theory, vol. 50, no. 11, pp. 2794–2800, Nov. 2004.

[14] S. S. Arslan and E. Haytaoglu. “Cost of guessing: Applications to data repair,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2020, pp. 2194–2198.

[15] S. S. Arslan and E. Haytaoglu. “Improved bounds on the moments of guessing cost,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Espoo, Finland, Jun. 2022, pp. 3351–3356.

[16] M. Noroozi and Z. Eslamian. “Public-key encryption with keyword search: A generic construction secure against online and offline keyword guessing attacks,” J. Ambient Intell. Humanized Comput., vol. 11, no. 2, pp. 879–890, Feb. 2020.

[17] D. X. Song, D. Wagner, and A. Perrig, “Practical techniques for searches on encrypted data,” in Proc. IEEE Symp. Secur. Privacy, May 2000, pp. 44–55.

[18] I. Niven, “Coding theory applied to a problem of Ulam,” Math. Mag., vol. 61, pp. 275–281, Dec. 1988.

[19] E. Arikan and S. Boztas. “Guessing with lies,” in Proc. IEEE Int. Symp. Inf. Theory, Lausanne, Switzerland, Jun. 2002, p. 208.

[20] K. R. Duffy, M. Méard, and W. An, “Guessing random additive noise decoding with symbol reliability information (SRGRAND),” IEEE Trans. Commun., vol. 70, no. 1, pp. 3–18, Jan. 2022.

[21] A. Solomon, K. R. Duffy, and M. Méard, “Soft maximum likelihood decoding using GRAND,” in Proc. IEEE Int. Conf. Commun. (ICC), Jun. 2020, p. 1–6.

[22] E. Haytaoglu, E. Kaya, and S. S. Arslan, “Data repair-efficient fault tolerance for cellular networks using LDPC codes,” IEEE Trans. Commun., vol. 70, no. 1, pp. 19–31, Jan. 2022.

[23] A. Bracher, E. Hof, and A. Lapidoth, “Guessing attacks on distributed-storage systems,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Hong Kong, Jun. 2015, pp. 1585–1589.

[24] M. Christiansen, K. R. Duffy, F. D. P. Calmon, and M. Méard, “Multi-user guesswork and brute force security,” IEEE Trans. Inf. Theory, vol. 61, no. 12, pp. 6876–6886, Dec. 2015.

[25] W. Huleihel, S. Salamatian, and M. Méard, “Guessing with limited memory,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Aachen, Germany, Jun. 2017, pp. 2253–2257.

[26] M. A. Kumar, A. Sunny, A. Thakre, A. Kumar, and G. D. Manohar, “A unified framework for problems on guessing, source coding, and tasks partitioning,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Espoo, Finland, Jun. 2022, pp. 3339–3344.

[27] J. Williams. “Algorithm 232: Heapsort,” Commun. ACM, vol. 7, no. 6, pp. 347–348, 1964.

[28] A. Rényi, “On measures of information and entropy,” in Proc. 4th Berkeley Symp. Math. Statist. Probab., vol. 1, 1961, pp. 547–561.

[29] R. G. Gallagher, “Low-density parity-check codes,” IRE Trans. Inf. Theory, vol. 8, no. 1, pp. 21–28, Jan. 1962.

[30] C. Bunte and A. Lapidoth, “Maximum Rényi entropy rate,” IEEE Trans. Inf. Theory, vol. 62, no. 3, pp. 1193–1205, Mar. 2016.

[31] M. Luby, M. Mitzenmacher, and A. Shokrollahi, “Analysis of random processes via and-or tree evaluation,” in Proc. 33th Annu. ACM-SIAM Symp. Discrete Algorithms, 1998, pp. 364–373.

[32] M. Le, Z. Song, Y.-W. Kwon, and E. Tilevich, “Reliable and efficient mobile edge computing in highly dynamic and volatile environments,” in Proc. 2nd Int. Conf. Fog Mobile Edge Comput. (FMEC), May 2017, pp. 113–120.

[33] A. Amraoui, A. Montanari, and R. Urbanke, “How to find good finite-length codes: From art towards science,” Eur. Trans. Telecommun., vol. 18, no. 5, pp. 491–508, Aug. 2007.

[34] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, “Design of capacity-approaching irregular low-density parity-check codes,” IEEE Trans. Inf. Theory, vol. 47, no. 2, pp. 619–637, Feb. 2001.

[35] W.-K. Lai and E. Kim, “Some inequalities involving geometric and harmonic means,” Int. Math. Forum, vol. 11, no. 4, pp. 163–169, 2016.

[36] H. Park, D. Lee, and J. Moon, “LDPC code design for distributed storage: Balancing repair bandwidth, reliability, and storage overhead,” IEEE Trans. Commun., vol. 66, no. 2, pp. 507–520, Feb. 2018.
[37] S. S. Dragomir and J. V. D. Hoek, “New inequalities for the moments of guessing mapping,” *East Asian Math. J.*, vol. 14, no. 1, pp. 1–14, 1998.

[38] D. Knuth, “Section 5.2.4: Sorting by merging,” in *The Art of Computer Programming*, vol. 3, 2nd ed. Reading, MA, USA: Addison-Wesley, 1998, pp. 158–168.

Suayb S. Arslan (Senior Member, IEEE) received the B.Sc. degree in electrical and electronics engineering from Boğaziçi University, Istanbul, Turkey, in 2006, and the M.Sc. and Ph.D. degrees in electrical engineering from the University of California at San Diego, San Diego, CA, USA, in 2009 and 2012, respectively. In 2009, he was with Mitsubishi Electric Research Laboratories, Boston, MA, USA, where he was involved in research and development of image processing and machine learning algorithms for biomedical applications. In 2011, he joined Quantum Corporation, Irvine, CA, USA, where he conducted research with IBM and HP on advanced data detection and coding algorithms as well as reliability modeling for increased capacity cold storage and cloud systems. He is currently with the Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, MA, USA, and Boğaziçi University. His research interests include digital communications and storage systems, information and reliability theory, image/video processing, neuroscience, artificial intelligence, and the Internet of Things. He has been serving as an Associate Editor for *Internet of Things* journal (Elsevier) since 2018.

Elif Haytaoglu (Member, IEEE) received the B.Sc. degree in computer engineering from Pamukkale University, Denizli, Turkey, and the Ph.D. degree in information technologies from Ege University, Izmir, Turkey. She is currently an Assistant Professor with the Department of Computer Engineering, Pamukkale University. She has lots of studies and projects about distributed storage and distributed graph algorithms. Her research interests include distributed caching, distributed graph algorithms, distributed storage systems, wireless ad hoc networks, and the similarities between cell operations and distributed algorithms.