HOMOGENEOUS VECTOR BUNDLES
ON SYMPLECTIC GRASSMANNIANS

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INTRODUCTION

Let $V$ be a vector space of dimension $N$ over an algebraically closed field $k$ of characteristic 0 and denote by $Gr(k, V)$ the grassmannian of $k$-planes in $V$. Let

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and denote by $X = SpGr(k, V) = Gr(k, V)$ the symplectic grassmannian of isotropic $k$-planes with respect to the symplectic form $<, >_J$.

In this paper, we will discuss the K-theory and the category of homogeneous vector bundles on the symplectic grassmannian $SpGr(2, N)$ of isotropic 2-planes. Atiyah and Hirzebruch [AH], Araki [A] and Hodgkin [Ho] have shown that the topological $K_0$ of homogeneous spaces is generated by homogeneous bundles. This implies that the algebraic $K_0$ is also generated by algebraic homogeneous bundles, however this fact relies on the topological methods of Atiyah and Hirzebruch [AH], Araki [A] and Hodgkin [Ho]; at present there is no purely algebraic proof.

Kapranov gives a description of the bounded derived category $D^b_{coh}$ of coherent sheaves of $Gr(k, V)$ in [KI], and in [L-S-W], Marc Levine, V.Srinivas, Jerzy Weyman compute the $K$-groups of Twisted Grassmannians. In both articles this is done by constructing a minimal resolution of the structure sheaf $O_{\Delta_{Gr}}$ of the diagonal $\Delta_{Gr}$ inside of the product $Gr(k, V) \times Gr(k, V)$. The explicit resolution of the diagonal gives both a purely algebraic computation of the K-theory of the grassmannian.
Grassmannians

Let $S_{Gr}$ be the tautological bundle on $Gr(k, V)$ and let $S_{Gr}^\perp$ be the dual of the quotient bundle, that is $((V \otimes O_{Gr})/S_{Gr})^*$. Denote by $p_1$ and $p_2$ the projections on the first respectively second factor from $Gr(k, V) \times Gr(k, V)$ to $Gr(k, V)$, let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $Gr(k, V)$ and denote by $\mathcal{F} \boxtimes \mathcal{G}$ the external tensor product $p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$.

We have the well known Koszul complex $C_i$:

$$0 \to \bigwedge^{k(N-k)} (S \boxtimes S^\perp) \to \cdots \to \bigwedge^2 (S \boxtimes S^\perp) \to S \boxtimes S^\perp \xrightarrow{d} O_{Gr \times Gr}$$

This is a resolution of $O_{\Delta_{Gr}}$, in addition the Cauchy formula gives

$$C_i = \bigwedge^i (S \boxtimes S^\perp) \cong \bigoplus_{N-k \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0} |\alpha| = i \Sigma^\alpha S \boxtimes \Sigma^{\alpha^*} S^\perp,$$

where $\alpha^*$ is the conjugate Young diagram to $\alpha$, and $\Sigma^\alpha S$ and $\Sigma^{\alpha^*} S^\perp$ are the Schur functors. Let $\mathcal{F}$ be a locally free sheaf such that $H^i(Gr, \mathcal{F} \otimes \Sigma^\alpha S^\perp) = 0$ for all $i \geq 1$ and all $N-k \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0$, then the Koszul complex gives rise to a finite resolution of $\mathcal{F}$:

$$0 \to H^0(Gr, \mathcal{F} \otimes \Sigma^{(k,\ldots,k)} S^\perp) \otimes \Sigma^{(N-k,\ldots,N-k)} S \to \cdots \to \bigoplus_{|\alpha|=i} H^0(Gr, \mathcal{F} \otimes \Sigma^{\alpha^*} S^\perp) \otimes \Sigma^\alpha S \to \cdots \to H^0(Gr, \mathcal{F}) \otimes O_X \to \mathcal{F} \to 0$$
This resolution can be applied to give a computation of the algebraic K-theory as well as to describe the structure of the derived category of the grassmannian \( (Gr(k, V)) \). For the K-theory this results in a description of \( K_\ast(Gr(k, V)) \) as a free \( K_\ast(k) \)-module, with generators

\[
\mathcal{X} = \{ [\Sigma^\alpha S_{Gr}] \mid \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \& N - k \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0 \}
\]

and

\[
\mathcal{Y} = \{ [\Sigma^\beta S_{Gr}] \mid \beta = (\beta_1, \beta_2, \ldots, \beta_{N-k}) \& k \geq \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{N-k} \geq 0 \}.
\]

**The symplectic Grassmannian \( X \) of isotropic 2-planes in \( V \)**

In this work we attempt to extend these calculations to the symplectic Grassmannian \( X = SpGr(2, V) \subset Gr(2, V) \). We will use the above Koszul complex \( C \) of \( O_{\Delta_{Gr}} \) and restrict it to \( X \times X \), but now \( C \otimes O_{X \times X} \) is no longer exact. The obstruction to the exactness results from the torsion in degree one. We use the Tate construction to kill the extra torsion and obtain a resolution of \( O_{\Delta} \). Specifically, in degree two,

\[
C_2 = \bigwedge^2 (S \boxtimes S^\perp) \cong \text{Sym}_2 S \boxtimes \wedge^2 S^\perp \oplus \wedge^2 S \boxtimes \text{Sym}_2 S^\perp,
\]

we extend \( \text{Sym}_2 S^\perp \) to the non-trivial extension \( \Psi \) of \( \text{Sym}_2 S^\perp \) and \( O_X \). This kills the extra torsion in degree one. Then we define non-trivial extensions \( \Psi_\beta \) of \( \Sigma^\beta S \) and \( \Psi_{\beta-2} \) to extend this construction to the whole complex.

Let us denote the thus obtained complex by \( D_\ast \).

**Theorem.** \( D_\ast \) is a resolution of \( O_{\Delta} \).

(Chapter 2, Theorem 2.10)

Now this complex does not terminate but instead becomes periodic in large degrees. This is analogous to Kapranov’s construction in the case of quadrics.
In the case of a quadric $Q$, when terminating this resolution, the last term can be understood using the Clifford algebra of $Q$. It turns out that this terminated resolution has the right number of generators.

However, a similar construction applied to the the resolution $D$, does not lead to a minimal resolution. There are summands in $D$, which are not independent in $K_0$. Nevertheless, it turns out that $D$ contains a finite sub-complex, let us call it $B$, which seems to be a good candidate for a minimal resolution. The main result of this work is now to show that this sub-complex is exact.

**Main-Theorem.** The sub-complex $B$ is exact.

(Chapter 3, Theorem 3.2)

We will prove this in chapter 6 by showing that the quotient complex $D/B$ is exact up to degree $2N - 7$, and therefore the sub-complex is exact up to degree $2N - 8$. The proof of exactness of the quotient complex is an application of the geometric techniques of calculating syzygies, see [K] and [PW], which we discuss in chapter 4.

Since the codimension of the diagonal $\Delta \subset X \times X$ is equal to the dimension of $X$, thus $\text{codim}_{X \times X} \Delta = 2N - 5$, it follows that the Kernel $K$ of $B_{2N-6} \to B_{2N-7}$ has to be a vector bundle. This results in a finite resolution of $\mathcal{O}_\Delta$, with i-th entry

$$B_i = \bigoplus_{|\alpha| = i} \Sigma^\alpha S \boxtimes \Psi_{\alpha^*}, \quad N - 2 > \alpha_1 \geq \alpha_2 \geq 0$$

for $i \leq m = 2N - 6$ and $(2N - 5)$-th entry $K$. Although we don’t prove it here, the vector bundles $\Psi_{\alpha^*}, \quad N - 2 > \alpha_1 \geq \alpha_2 \geq 0$ have no relations in $K_0$. The same applies to the bundles $\Sigma^\alpha S, \quad N - 2 > \alpha_1 \geq \alpha_2 \geq 0$.

The hope is that we can filter the kernel $K$ as a direct sum of tensor products of the form $F \boxtimes G$ and that these plus the terms in $B$ will lead to a purely algebraic computation of the K-theory and a description of the derived category $D_{coh}$ of symplectic grassmannians.
As for now we will complete the computation of the K-theory for the symplectic Grassmannian of isotropic 2-planes in 4-space, Chapter 5, $X = \text{SpGr}(2, 4)$, and leave the other cases for future research.
CHAPTER I  

K-THEORY OF ORDINARY GRASSMANNIANS

In this chapter we will discuss the K-theory of the Grassmannians by illustrating the work of Kapranov [KI], On the derived category of coherent sheaves on Grassmann Manifolds, and the work of Marc Levine, V. Srinivas and Jerzy Weyman [LSW], K-theory of some twisted Grassmannians.

Notations

Let $V$ be a vector space of dimension $N$ over an algebraically closed field $k$ of characteristic 0, let $Gr = Gr(k, V)$ be the grassmannian of $k$-planes in $V$ and denote by $\Delta_{Gr}$ the diagonal inside of $Gr(k, V) \times Gr(k, V)$. Let $S$ be the tautological bundle of the grassmannian $Gr(k, V)$ and denote by $S^\perp$ the dual of the quotient of the tautological bundle, that is

$$S^\perp \cong ((V \otimes \mathcal{O}_X)/S)^*.$$  

Thus $S^\perp$ is the vector bundle that fits into the short exact sequences:

$$0 \to S \to V \otimes \mathcal{O}_X \to (S^\perp)^* \to 0 \quad \text{and}$$

$$0 \to S^\perp \to V^* \otimes \mathcal{O}_X \to S^* \to 0.$$  

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_{N-k})$ be ordered partitions, that is $\alpha_1 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \cdots \geq \beta_{N-k}$. Denote by $\alpha^*$ the conjugate partition defined by interchanging rows and columns in the Young diagram $\alpha$, that is if $\alpha$ is represented by a Young diagram with $\alpha_i$ boxes in the $i$-th row, then the Young diagram of the conjugate partition has $\alpha_i$ boxes in the $i$-th column.
Let \( \Sigma^\alpha S \) and \( \Sigma^\beta S^\perp \) be the corresponding Schur functors, see appendix for the definition of Young symmetrizers and the Schur functors.

Denote by \( p_1 \) and \( p_2 \) the projections on the first respectively second factor from \( \text{Gr}(k,V) \times \text{Gr}(k,V) \) to \( \text{Gr}(k,V) \), let \( \mathcal{F} \) and \( \mathcal{G} \) be vector bundles on \( \text{Gr}(k,V) \) and denote by \( \mathcal{F} \boxtimes \mathcal{G} \) the external tensor product \( p_1^* \mathcal{F} \otimes p_2^* \mathcal{G} \).

The main objective for the K-theory of the Grassmannian \( \text{Gr}(k,V) \) is to define a minimal resolution of the structure sheaf \( \mathcal{O}_{\Delta_{\text{Gr}}} \) of the diagonal \( \Delta_{\text{Gr}} \subset \text{Gr}(k,V) \times \text{Gr}(k,V) \). The terms of this resolution decompose as a sum of external tensor products of the form \( \Sigma^\alpha S \boxtimes \Sigma^\alpha S^\perp \). Using a standard trick, we will construct resolutions for certain sheaves \( \mathcal{F} \), which in turn can be used to find a generating system of the K-theory of the Grassmannian \( \text{Gr}(k,V) \). This leads to a description of the K-theory of the grassmannian as a direct sum over the generating system of the K-theory of the field \( k \), see [LSW]. See [KI] for a description of the bounded derived category \( D^b_{\text{coh}} \) of coherent sheaves of \( \text{Gr}(k,V) \).

**Resolution of \( \mathcal{O}_{\Delta_{\text{Gr}}} \)**

We will define a homomorphism from \( S \boxtimes S^\perp \) to \( \mathcal{O}_{\text{Gr} \times \text{Gr}} \) with cokernel \( \mathcal{O}_{\Delta_{\text{Gr}}} \) and then define the associated Koszul complex. We will use some well known facts about Koszul complexes and demonstrate that this defines a resolution of \( \mathcal{O}_{\Delta_{\text{Gr}}} \).

Define a homomorphism of coherent sheaves \( d : S \boxtimes S^\perp \to \mathcal{O}_{\text{Gr} \times \text{Gr}} \) as follows: Choose a basis \( v_1, v_2 \ldots v_N \) of \( V \) and the corresponding dual basis \( x_1, x_2 \ldots x_N \) of \( V^* \). Then on the fibers over the point \( W_1 \times W_2 \in \text{Gr} \times \text{Gr} \) define the map

\[
d : W_1 \boxtimes (V/W_2)^* \to k
\]

\[
d(v \boxtimes f) = \sum_{i=1}^{N} x_i(v) \boxtimes f(v_i) = f(v).
\]

This definition does not depend on the base choice. Note over a fiber this map is just the map given by evaluation.
Remark. An alternative definition is given by Kapranov [KI], page 185:

Consider the cohomology of the dual bundle of $S \boxtimes S^\perp$:

$$H^\ast(Gr \times Gr, (S \boxtimes S^\perp)^*) \cong H^0(Gr, S^*) \otimes H^0(Gr, (S^\perp)^*) \cong V^* \otimes V \cong \text{End}(V)$$

Choose the section $s : Gr \times Gr \to (S \boxtimes S^\perp)^*$ corresponding to the identity element in $\text{End}(V)$, such that the set of zeroes of $s$ is equal to the diagonal $\Delta_{Gr}$. Therefore $s^*$ gives rise to the homomorphism $d : S \boxtimes S^\perp \to \mathcal{O}_{Gr \times Gr}$ with cokernel $\mathcal{O}_{\Delta_{Gr}}$.

Note that these two different definitions in fact define the same map, because the identity element in $V^* \otimes V$ is given by $\sum_{i=1}^N x_i \boxdot v_i$ and $f(v) = \sum_{i=1}^N x_i(v) \boxdot f(v_i)$ for $v \in V$ and $f \in V^*$.

Denote by $C_i$ the Koszul complex of the map $d$, that is the complex

$$0 \to \bigwedge^{k(N-k)}(S \boxtimes S^\perp) \to \cdots \to \bigwedge^2(S \boxtimes S^\perp) \to S \boxtimes S^\perp \xrightarrow{d} \mathcal{O}_{Gr \times Gr}$$

with

$$d((v_1 \boxdot f_1) \wedge \cdots \wedge (v_k \boxdot f_k)) = \sum_{0 \leq i \leq k} (-1)^i d(v_i \boxdot f_i)(v_1 \boxdot f_1) \wedge \cdots \wedge (v_i \boxdot f_i) \wedge \cdots \wedge (v_k \boxdot f_k).$$

This is a well defined complex. For ordinary grassmannians this Koszul complex is exact as well. The differences in the symplectic case arise from the fact that this complex will no longer be exact.

**Theorem 1.1.** The Koszul complex

$$C_i : 0 \to C_{k(N-k)} \to \cdots \to C_i \to \cdots \to \mathcal{O}_{X \times X}$$

is a resolution of $\mathcal{O}_{\Delta_{Gr}}$.

**Proof.** It will be enough to show that the image of $d$ is an ideal sheaf and that $S \boxtimes S^\perp$ has the right rank. Note that the image of $d$ is nothing else but the ideal sheaf of $\Delta_{Gr}$, and the rank of $S \boxtimes S^\perp$ is equal to $k(N-k)$ which is the same as the codimension of $\Delta_{Gr} \subset Gr \times Gr$.

First let us discuss in general terms when a Koszul complex is exact.
Koszul complexes

Let $X$ be a noetherian scheme over $k$, and $F$ a vector bundle on $X$. Given a homomorphism $\Psi : F \to \mathcal{O}_X$, we can form the Koszul complex $K.(F, \Psi)$ of $F$ and $\Psi$.

We define the Koszul complex to be the complex:

$$K.(F, \Psi) : 0 \to \Lambda^k F \to \ldots \to \Lambda^1 F \to \Lambda^0 F \to \mathcal{O}_X$$

where $d(f_1 \wedge f_2 \wedge \ldots \wedge f_k) = \sum_{0 \leq i \leq k} (-1)^i \Psi(f_i) f_1 \wedge \ldots \hat{f}_i \wedge \ldots \wedge f_k$.

This is a well defined complex.

Lemma 1.2. Suppose that the image of $\Psi$ is equal to the ideal sheaf $\mathcal{I}_Y$ of a locally complete intersection $Y \subset X$. Further more if $\text{rank}(F) = \text{codim}_X Y$, then the Koszul complex $K.(F, \Psi)$ is exact.

Proof. This is just a local question, therefore choose an open cover $U = \{U_i\}$ of $X$, such that for all open sets $U \in U$: $Y|_U$ is a complete intersection and $F|_U$ is free. Then locally,

$$U = \text{Spec}(A), F|_U = \widetilde{M}, M \cong Ax_1 \oplus \ldots \oplus Ax_r$$

for a noetherian ring $A$ and denote by $\widetilde{M}$ the sheaf associated to $M$ on $U$. The map $\Psi|_U : F|_U \to \mathcal{O}_U$ is equal to $(\psi : M \to A)^\sim$.

Let $\psi(x_i) = f_i$, then

$$K(U, \Psi|_U) \cong K(A, f_1, \ldots, f_r)^\sim,$$

which is given by:

$$K.(A, f_1, \ldots, f_r) : 0 \to \Lambda^k M \to \ldots \to \Lambda^1 M \to \Lambda^0 M \to \mathcal{O}_X$$

where $d(x_1 \wedge x_2 \wedge \ldots \wedge x_r) = \sum_{0 \leq j \leq r} (-1)^j f_{i_j} x_1 \wedge \ldots \wedge \hat{x}_{i_j} \wedge \ldots \wedge x_r.$
This is exact if and only if \( f_1, \ldots, f_r \) form a regular sequence, see [H II, 7.10A]. Let \( I_Y \) be the ideal generated by \( f_1, \ldots, f_r \), then since \( Y |_U \) is a complete intersection, the \( f_i \)'s form a regular sequence [H] II, 8.21 A(c). Thus the Koszul complex of \( f_1, \ldots, f_r \) is exact. q.e.d.

**Proof of Theorem 1.1.** Since the image of \( d \) is equal to the ideal sheaf of \( \Delta_{Gr} \), and the rank of \( S \otimes S^\perp \) is equal to the codimension of \( \Delta_{Gr} \subset Gr \times Gr \), the exactness follows from Lemma 1.2.

q.e.d.

**The Cauchy Formula.**

The \( i \)-th term of the Koszul complex

\[ C_i : 0 \rightarrow C_{k(N-k)} \rightarrow \cdots \rightarrow C_i \rightarrow \cdots \rightarrow \mathcal{O}_{X \times X} \]

can be expressed as a direct sum of external tensor products of vector bundles over the grassmannian \( Gr(k, V) \). We'll do this by applying the Cauchy formula to this complex, (char \( k = 0 \)):

\[ C_i = \bigwedge^i (S \otimes S^\perp) \cong \bigoplus_{|\alpha|=i} \Sigma^\alpha S \otimes \Sigma^{\alpha^*} S^\perp \]

(see [M] I,4,Ex.6 and [KI], Lemma 0.5), where

\[ \alpha \in I = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \mid N - k \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq 0 \} \]

runs over all Young diagrams with \( i \)-cells, and \( \alpha^* \) is the conjugate Young diagram to \( \alpha \).

Set

\[ C_\alpha = \Sigma^\alpha S \otimes \Sigma^{\alpha^*} S^\perp. \]

This implies:

**Corollary 1.3.** The Koszul complex \( C_i \) is isomorphic to the complex

\[ 0 \rightarrow C_{(k, \ldots, k)} \rightarrow \cdots \rightarrow \bigoplus_{|\alpha|=i} C_\alpha \rightarrow \cdots \rightarrow \mathcal{O}_{X \times X}. \]

We will apply this resolution to the K-theory of the grassmannian \( Gr(k, V) \).
**K-theory of** $Gr(k,V)$

Let $\mathcal{P}$ be the category of vector bundles on Spec $k$. Following Quillen’s proof for the K-theory of projective spaces $[Q]$, define

$$U_\alpha : \mathcal{P} \to \mathcal{C}_{Gr}, \ U_\alpha(W) = q^*W \otimes \Sigma^\alpha S$$

for all Young diagrams $\alpha \in I$. Now $U_\alpha$ is inducing a homomorphism $u_\alpha$ on the K-theory, that is

$$u_\alpha : K_*(k) \to K_*(Gr(k,V))$$

Set $u = \bigoplus_{\alpha \in I} u_\alpha$, then

**Theorem 1.4.**

$$u : \bigoplus_{\alpha \in I} K_*(k) \to K_*(Gr(k,V)) \text{ is an isomorphism.}$$

**Proof.** We will prove this in several steps.

Define

$$I = \{ \alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \mid N - k \geq \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k \geq 0 \}.$$ 

Let $\mathcal{C}_{Gr}$ be the category of locally free coherent sheaves on $Gr(k,V)$ and define $\mathcal{C}$ as the full subcategory of the category $\mathcal{C}_{Gr}$ as follows:

$$\mathcal{C} = \left\{ F \in \mathcal{C}_{Gr} \mid \begin{array}{l} \mathrm{H}^i(Gr, F \otimes \Sigma^\alpha S^\perp) = 0 \text{ for all } i \geq 1 \smallskip \text{ and all } \alpha \in I \end{array} \right\}.$$ 

First we prove that $u$ is surjective by using a standard trick to construct resolutions for all sheaves $F \in \mathcal{C}$ from the above resolution of $\mathcal{O}_{\Delta_{Gr}}$.

Then we will use cohomology computations in order to show that $u$ is injective.
We will apply the standard method to construct resolutions for all sheaves $F \in C$. We will tensor the resolution of $O_{\Delta_{Gr}}$ with the pullback of $F$ to the product of $Gr \times Gr$ on one factor and then push this new complex forward on the other factor.

The pushforward of a complex of coherent sheaves.

Lemma 1.5. Let $q : X \to Y$ be a projective morphism of noetherian schemes. Let $0 \to F_m \xrightarrow{d_m} \cdots \xrightarrow{d_1} F_0$ be an exact sequence of coherent $O_X$-modules. Suppose $R^i q_* F_j = 0$ for all $i > 0$ and all $j$, then

$$0 \to q_* F_m \to \cdots \to q_* F_0$$

is an exact complex of coherent $O_Y$-modules.

Proof. First we split the complex $F_*$ into short exact sequences,

$$0 \to \ker d_{m-k} \to F_{m-k} \to \text{Im } d_{m-k} \to 0.$$ 

In order to show that $q_* F_*$ is exact, we also split this complex into short sequences, $0 \to \ker q_* d_{m-k} \to q_* F_{m-k} \to q_* \text{Im } d_{m-k} \to 0$, and show that these are exact for all $k$. The sequence $0 \to \ker q_* d_{m-k} \to q_* F_{m-k} \to q_* \text{Im } d_{m-k} \to 0$ is exact if $R^1 q_* \ker d_{m-k} = 0$, or since $\ker d_{m-k} \cong \text{Im } d_{m-(k-1)}$, this is exact if $R^1 q_* \text{Im } d_{m-(k-1)} = 0$.

We show by induction over $k$ that $R^i q_* \text{Im } d_{m-(k-1)} = 0$ for all $0 \leq k \leq m$ and all $i > 0$.

$k = 0$: The first case of $k = 0$ is obvious, because $\text{Im } d_{m+1} = 0$ and therefore

$$R^i q_* \text{Im } d_{m+1} = 0.$$ 

$k \Rightarrow k + 1$: Now suppose $R^i q_* \text{Im } d_{m-(k-1)} = 0$ for all $i > 0$, then we need to show that $R^i q_* \text{Im } d_{m-k} = 0$.

Consider the short exact sequence:

$$0 \to \text{Im } d_{m-(k-1)} \to F_{m-k} \to \text{Im } d_{m-k} \to 0.$$
and consider the resulting long exact sequence:

\[ \ldots \to R^i q_\ast f_{m-k} \to R^i q_\ast \text{Im } d_{m-k} \to R^{i+1} q_\ast \text{Im } d_{m-(k-1)} \to \ldots . \]

From the assumptions we know that \( R^i q_\ast f_{m-k} = 0 \) and from the induction hypothesis it follows that \( R^{i+1} q_\ast \text{Im } d_{m-(k-1)} = 0 \), therefore \( R^i q_\ast \text{Im } d_{m-k} = 0 \).

\textbf{q.e.d.}

\textbf{Resolution for } \mathcal{F} \in \mathcal{C}

Let \( \mathcal{F} \in \mathcal{C} \) be a locally free sheaf, that is \( \mathcal{F} \in \mathcal{C}_{Gr} \) such that

\[ H^i(Gr, \mathcal{F} \otimes \Sigma^\alpha S^\perp) = 0 \text{ for all } i \geq 1 \text{ and all } \alpha \in I. \]

Recall that \( p_1 \) and \( p_2 \) are the projections

\[ p_i : Gr(k, V) \times Gr(k, V) \to Gr(k, V) \]

to the \( i \)-th factor. Denote by \( q \) the structure map \( q : Gr(k, V) \to \text{Spec}(k) \). Then

\[ R^i p_{1\ast}(p_2^\ast \mathcal{F} \otimes \mathcal{O}_{\Delta_{Gr}}) \cong \begin{cases} \mathcal{F} & \text{for } i = 0, \\ 0 & \text{otherwise} \end{cases} \]

and

\[ R^i p_{1\ast}(p_2^\ast \mathcal{F} \otimes C_\alpha) \cong R^i p_{1\ast}(p_2^\ast \mathcal{F} \otimes p_2^\ast \Sigma^\alpha S^\perp \otimes p_1^\ast \Sigma^\alpha S) \]
\[ \cong R^i p_{1\ast}(p_2^\ast (\mathcal{F} \otimes \Sigma^\alpha S^\perp) \otimes p_1^\ast \Sigma^\alpha S) \]
\[ \cong R^i p_{1\ast}(p_2^\ast (\mathcal{F} \otimes \Sigma^\alpha S^\perp)) \otimes \Sigma^\alpha S \quad \text{(by projection formula)} \]
\[ \cong q^\ast R^i q_\ast (\mathcal{F} \otimes \Sigma^\alpha S^\perp) \otimes \Sigma^\alpha S \quad \text{(q flat)} \]
\[ \cong H^i(\mathcal{F} \otimes \Sigma^\alpha S^\perp) \otimes \Sigma^\alpha S, \]

which in turn implies:
Proposition 1.6.

For $F \in C$, $R^i p_1^* (p_2^* F \otimes C)$ gives rise to a finite resolution of $F$:

$$0 \to H^0(Gr, F \otimes \Sigma(k_{-k},...,N-k)) S^\perp \otimes \Sigma^{(N-k,...,N-k)} S \to ... \to \bigoplus_{|\alpha|=i} H^0(Gr, F \otimes \Sigma^\alpha S^\perp) \otimes \Sigma^\alpha S \to ... \to H^0(Gr, F) \otimes O_X \to F \to 0$$

Proof. For $F \in C$, $H^i(F \otimes \Sigma^\alpha S^\perp) = 0$, thus all the higher direct images vanish. The statement follows from Lemma 1.4. q.e.d.

We will show next that it suffices to define $K_* (Gr(k,V))$ on the subcategory $C$ rather than defining it on the whole category $C_{Gr}$.

$K_* (C) \cong K_* (Gr)$. 

Let $g : Y \to Z$ be a quasi-projective map of schemes, and let $X$ be a set of locally free sheaves on $Y$. Denote by $C_Y$ the category of locally free sheaves $F$ on $Y$ and let $C_X$ be the subcategory of $C_Y$ consisting of all sheaves $F$ with

$$R^i g_* (F \otimes G) = 0$$

for all $i > 0$ and for all $G \in X$.

Lemma 1.7. The inclusion $C_X \subset C_Y$ induces an isomorphism on the $K$-theory, that is

$$K_* (C_X) \cong K_* (C_Y) .$$

Proof. This follows from [LSW] Lemma 4.3, which implies that $q$ induces a homotopy equivalence on the classifying spaces $BQC_X \to BQC_Y$. q.e.d.

$u$ surjective.

Since $K_* (Gr) \cong K_* (C)$ by Lemma 1.7, it is enough to show that $u$ is surjective on $K_* (C)$. Let $F \in C$. Consider the finite resolution of $F$. Denote by $[F]$ the class of $F$ in $K_* (C)$. Then

$$[F] = \sum_{0 \neq \alpha \in I} (-1)^{|\alpha|} [H^0(F \otimes \Sigma^\alpha S^\perp) \otimes \Sigma^\alpha S],$$

thus $u$ is surjective.

q.e.d.
U INJECTIVE

To show that $u$ is injective we define a homomorphism

$$v : K_\ast(Gr(k, V)) \to \bigoplus_{\alpha \in I} K_\ast(k),$$

and then show that $v \circ u$ is injective, hence that $u$ is an isomorphism.

Let

$$C^\ast = \left\{ \mathcal{F} \in C_{Gr} \mid \begin{array}{l} H^i(Gr, \mathcal{F} \otimes \Sigma^\alpha S^*) = 0 \text{ for all } i \geq 1 \\ \text{and all } \alpha \in I \end{array} \right\}.$$ 

By Lemma 1.7, it is enough to define the K-theory on the subcategory $C^\ast$.

Set

$$V_\alpha : C^\ast \to P, \quad V_\alpha(\mathcal{F}) = q_*(\mathcal{F} \otimes \Sigma^\alpha S^*),$$

for all $\alpha \in I$. Note that $V_\alpha$ on the category $C_{Gr}$ is not exact, because $R^i q_*(\mathcal{F} \otimes \Sigma^\alpha S^*)$ is in general not zero for $i > 0$, but $V_\alpha$ is exact on the subcategory $C^\ast$. The $V_\alpha$’s induce homomorphisms $v_\alpha$ on the K-theory.

$$v_\alpha : K_\ast(Gr(k, V)) \to K_\ast(k).$$

Let $v = (\ldots, v_\alpha, \ldots)$. Suppose that the higher cohomology of $\Sigma^\alpha S \otimes (\Sigma^\beta S)^*$ vanishes and consider $v \circ u :$

$$V_\beta \circ U_\alpha(W) = V_\beta(q^*W \otimes \Sigma^\alpha S) = [H^0(W \otimes \Sigma^\alpha S \otimes (\Sigma^\beta S)^*)].$$

We will show that the matrix of $v \circ u$ with respect to the lexicographical ordering of the weights $\alpha$, is upper triangular with ones down the diagonal, thereby showing that $u$ is an isomorphism.

First we will recall some cohomology computations. Since we will need the same techniques for the symplectic Grassmannians we will do this more detailed then needed at this point.
COHOMOLOGY COMPUTATIONS

Let \( G \) be a Lie group, \( B \) a Borel subgroup containing the torus \( T \) of \( G \). We denote by \( R \subset X \) the set of all roots of \( G \) with respect to \( T \) and call negative the roots of \( B \). Let \( \rho \) be the half sum over all positive roots \( \alpha \), i.e. \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \Sigma \lambda_i \), where the last sum is over the fundamental weights \( \lambda_i \). Let \( \lambda \) be a character of \( T \), then the one dimensional representation \( L_\lambda \) of \( T \), extends trivially to \( B \), and we can form the holomorphic line bundle

\[
L_\lambda = G \times_B L_\lambda = \{(g,v) \in G \times L_\lambda | (g,v) \sim (gb,b^{-1}v), b \in B \}.
\]

We have Bott’s theorem for the vanishing of the cohomology of this line bundle:

**Borel-Weil-Bott Theorem 1.8.**

1. If \( \lambda \) is a dominant weight, then \( L_\lambda \) has no higher cohomology and the space of sections \( H^0(G/B, L_\lambda) \) is the dual of the irreducible representation with highest weight \( \lambda \).

2. If \( \lambda \) is not dominant add \( \rho \) to \( \lambda \), then find \( \sigma \) in the Weyl group such that \( \sigma(\lambda + \rho) \) is dominant, then subtract again \( \rho \).

   If \( \sigma(\lambda + \rho) - \rho \) is dominant, then there is only cohomology in degree \( l(\sigma) = \text{length}(\sigma) \) and \( H^l(\sigma)(G/B, L_\lambda) \cong H^0(G/B, L_{\sigma(\lambda+\rho)-\rho}) \) which by (1) is given by the dual of the irreducible representation with highest weight \( \sigma(\lambda + \rho) - \rho \).

   If \( \sigma(\lambda + \rho) - \rho \) is not dominant, then \( H^i(G/B, L_\lambda) = 0 \) for all \( i \geq 0 \).

**Proof.** See [D].

Here we will use \( G = Gl_N(V) \) for our cohomology computations for \( Gr(k,N) \). Let \( F = G/B \) be the total flag manifold, i.e.

\[
F = \{0 \subset W_1 \subset W_2 \subset ... \subset W_N = V \text{ with dimension } W_i = i\}.
\]
Then $Gr(k,N)$ is isomorphic to $G/P$, where $P \supset B$ is the parabolic subgroup preserving the space $V_k$ in the standard representation. $P$ can be described as the parabolic subgroup, corresponding to omitting one node of the Dynkin diagram: (see also [F-H] § 23.3 Homogeneous spaces)

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \bullet & \circ & \circ \\
\end{array}
\]

From this description it is easy to get the Levi decomposition of $P$ into Levi factors

\[ P \cong (L_1 \times L_2) \ltimes U_P \]

where $U_P$ is the unipotent radical of $P$, $L_1 \cong GL(V_k)$, and $L_2 \cong GL(V/V_k)$ are the Levi factors with $V_k$ the vector space of dimension $k$ of the standard representation.

We denote by $\pi$ the projection $\pi : G/B \to G/P$. Now the vector bundles $\Sigma^{\alpha}S$ and $\Sigma^{\beta}S^\perp$ on $Gr(k,V)$ come from line bundles on $F = G/B$. Set

\[
\lambda(\alpha) = (-\alpha_k, -\alpha_{k-1}, \ldots, -\alpha_1, 0, \ldots, 0) \\
\lambda(\beta) = (0, \ldots, 0, \beta_1, \ldots, \beta_{N-k-1}, \beta_{N-k}).
\]

**Lemma 1.9.**

1. $\pi_* L_{\lambda(\alpha)} \cong \Sigma^{\alpha}S$
2. $\pi_* L_{\lambda(\beta)} \cong \Sigma^{\beta}S^\perp$
3. $\pi_* L_{\lambda(\alpha)+\lambda(\beta)} \cong \Sigma^{\alpha}S \otimes \Sigma^{\beta}S^\perp$

**Proof.** See [KI] 2.5.

We order the partitions $\alpha = (\alpha_1, \ldots, \alpha_m)$ lexicographically, that is $\alpha \leq \beta$ if $\alpha_1 < \beta_1$, or $\alpha_1 = \beta_1$ but $\alpha_2 < \beta_2$, etc.

**Lemma 1.10.** For $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ with $N-k \geq \alpha_1 \geq \cdots \geq \alpha_k \geq 0$ and $N-k \geq \beta_1 \geq \cdots \geq \beta_k \geq 0$:

1. $H^i(Gr, (\Sigma^{\beta}S)^* \otimes \Sigma^{\alpha}S) = 0$ for all $i > 0$
2. $H^0(Gr, (\Sigma^{\beta}S)^* \otimes \Sigma^{\alpha}S) = 0$ for $\alpha > \beta$
3. $H^0(Gr, (\Sigma^{\alpha}S)^* \otimes \Sigma^{\alpha}S) = \mathbb{k}$
Proof. See [KI] 2.2.

Since we will be using these techniques again, we will outline Kapranov’s proof. Note that \((\Sigma^\beta S)^* \cong \Sigma^{(-\beta_k, \ldots, -\beta_1)} S\). Consider the tensor product:

\[
\Sigma^\alpha S \otimes (\Sigma^\beta S)^* \cong \Sigma^\alpha S \otimes \Sigma^{(-\beta_k, \ldots, -\beta_1)} S \cong \bigoplus \Sigma^\gamma S
\]

Since all the \(\alpha_i\)’s and \(\beta_i\)’s are between 0 and \(N - k\), it follows that every summand \(\gamma\) occurring in this product satisfies \(-(N - k) \leq \gamma_i \leq N - k\), see appendix, Remark (3).

Consider the cohomology of

\[
\Sigma^\gamma S \cong \pi_*(\mathcal{O}_F(-\gamma_k, \ldots, -\gamma_1, 0 \ldots 0))
\]

Set \(\hat{\gamma} = (-\gamma_k, \ldots, -\gamma_1, 0 \ldots 0)\).

(a) If \(\gamma_1 \leq 0\), thus \(-\gamma_k \geq \cdots \geq -\gamma_1 \geq 0 = \cdots = 0\), then \(\hat{\gamma}\) is a dominant weight and by the Borel-Weil-Bott Theorem has only \(H^0 \cong \Sigma^{(\gamma_1, \ldots, \gamma_k)} V^*\).

(b) If \(\gamma_1 > 0\), then \(\hat{\gamma}\) is not dominant. Consider

\[
\hat{\gamma} + \rho = (N - \gamma_k, \ldots, N - k + 1 - \gamma_1, N - k, \ldots, 1).
\]

Since \(\gamma_1 > 0\), it follows that \(N - k + 1 - \gamma_1\) is between 1 and \(N - k\), thus \(\hat{\gamma} + \rho\) has a repetition and therefore \(\mathcal{O}_F(\hat{\gamma})\) has no cohomology.

(1) From (a) and (b) it follows that for all summands \(\gamma\), \(\Sigma^\gamma S\) can have at most \(H^0\).

(2) If \(\alpha > \beta\), then \(\gamma_1 > 0\), thus \(\Sigma^\gamma S\) has no cohomology.

(3) If \(\alpha = \beta\), consider the tensor product

\[
\Sigma^\alpha S \otimes (\Sigma^\alpha S)^* \cong \bigoplus \Sigma^\gamma S,
\]

then \(\gamma\) is either \((0, \ldots, 0)\) or at least \(\gamma_1 > 0\). Thus

\[
H^0(Gr, \Sigma^\alpha S \otimes (\Sigma^\alpha S)^*) \cong H^0(Gr, \mathcal{O}_{Gr}) \cong k.
\]

q.e.d.
Lemma 1.11. For $\alpha = (\alpha_1, \ldots, \alpha_{N-k})$ and $\beta = (\beta_1, \ldots, \beta_{N-k})$ with $k \geq \alpha_1 \geq \cdots \geq \alpha_{N-k} \geq 0$ and $k \geq \beta_1 \geq \cdots \geq \beta_{N-k} \geq 0$ we have

\[ (1) \ H^i(Gr, (\Sigma^\beta S^\perp)^* \otimes \Sigma^\alpha S^\perp) = 0 \quad \text{for all } i > 0 \]
\[ (2) \ H^0(Gr, (\Sigma^\beta S^\perp)^* \otimes \Sigma^\alpha S^\perp) = 0 \quad \text{for } \alpha > \beta \]
\[ (3) \ H^0(Gr, (\Sigma^\alpha S^\perp)^* \otimes \Sigma^\alpha S^\perp) = k \]

Proof. See [LSW] Lemma 4.2.

\textbf{u injective.}

Now we finish the proof of Theorem 1.3. Consider the matrix of $v \circ u$ with respect to the lexicographical ordering of the weights $\alpha$. Then Lemma 1.9 implies:

\[ H^i(W \otimes \Sigma^\alpha S \otimes (\Sigma^\beta S)^*) = 0 \quad \text{for } i > 0 \quad \text{and} \]
\[ V_\beta \circ U_\alpha(W) = [H^0(W \otimes \Sigma^\alpha S \otimes (\Sigma^\beta S)^*)] = \begin{cases} 0 & \text{for } \alpha > \beta \\ k & \text{for } \alpha = \beta \end{cases} \]

Thus the matrix of $v \circ u$ is upper triangular with ones down the diagonal, therefore $u$ is injective, hence an isomorphism.
CHAPTER II

THE TATE CONSTRUCTION

Let $V$ be an $N = 2n$ dimensional vector space over an algebraically closed field $k$ of characteristic 0. In this section $X$ shall denote the symplectic grassmannian $SpGr(2,V)$ of two dimensional symplectic planes and $Gr(k,V)$ shall denote the grassmannian of $k$-planes in $V$. In the previous chapter we discussed the definition of the Koszul complex $C$ of the structure sheaf $\mathcal{O}_{\Delta_{Gr}}$ of the diagonal $\Delta_{Gr}$ inside the product $Gr(k,V) \times Gr(k,V)$. Using the inclusion $X \subset Gr(2,V)$ and the complex $C$ restricted to $X \times X$, we will construct a resolution of the structure sheaf $\mathcal{O}_{\Delta}$ of the diagonal $\Delta$ inside $X \times X$.

Consider the Koszul complex $C$ restricted to $X \times X$. In degree $i$ the terms are:

$$C_i \otimes \mathcal{O}_{X \times X} = \bigwedge^i (S \boxtimes S^\perp) \cong \bigoplus_{|\alpha| = i} \Sigma^\alpha S \boxtimes \Sigma^{\alpha^*} S^\perp$$

This Koszul complex is no longer exact. In this section we will use Tate’s techniques of adjoining an element in degree two to kill the extra torsion in degree one.

Accordingly we will define a non-trivial extension $\Psi$ of $Sym_2 S^\perp$ and $\mathcal{O}_X$ and then extend this construction to the whole complex, that is we will extend $\bigwedge^i (S \boxtimes S^\perp)$ to vector bundles $\Phi_i$. This extended complex defines a resolution of $\mathcal{O}_{\Delta}$.

Analogous to the case of the ordinary grassmannians, the bundles $\Phi_i$ break up into direct sums over external tensor products of homogeneous bundles, that is $\Phi_i \cong \bigoplus \Sigma^\alpha S \boxtimes \Psi_{\alpha^*}$ for some homogeneous bundles $\Psi_{\alpha^*}$ on $X$.

Notations and Preliminaries

Let $G = Sp_N(V)$ be the symplectic group of $V$ consisting of all $x \in Gl_N(V)$
satisfying
\[ \tau xJx = J, \]
where \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \).

Let \( < , >_J \) be the symplectic form defined by \( < v, w >_J = \tau vJw \). We call a subspace \( W \) of \( V \) isotropic if it is isotropic with respect to the form \( < , >_J \).

Let \( X = \text{SpGr}(2, V) \) be the symplectic grassmannian of isotropic two planes in \( V \) inside the ordinary grassmannian \( \text{Gr}(2, V) \) of two planes in \( V \). Let us fix a symplectic basis \( v_1, v_2, \ldots, v_{2n} \) of \( V \), i.e.

\[ < v_i, v_j >_J = 0 \text{ if } j \neq i \pm n \text{ and } < v_i, v_i + n >_J = 1 \text{ for } 1 \leq i \leq n. \]

Let \( x_1, x_2, \ldots, x_{2n} \) be the corresponding dual basis of \( V^* \). Set

\[ z = x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n}. \]

**Lemma 2.1.** The symplectic grassmannian \( X \) is given by the hyperplane section \( z \) inside the grassmannian \( \text{Gr}(2, V) \).

_Proof._ Consider the Plücker embedding that embeds \( \text{Gr}(2, V) \) in the projective space \( \mathbb{P}(\wedge^2 V) \). The condition for an element \( v \wedge w \) in \( \text{Gr}(2, V) \) to be isotropic is given by \( < v, w >_J = 0 \). This is equivalent to \( z(v \wedge w) = 0 \). Thus \( z \) is the defining equation of \( X \).

Let us show that \( z \) is independent of the base choice:

Let \( A = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (s_{ij})_{1 \leq i, j \leq 2n} \) be a symplectic matrix and \( y' = A \cdot y \), then

\[
y_1 \wedge y_{n+1} + y_2 \wedge y_{n+2} + \cdots + y_n \wedge y_{2n} = \sum_{1 \leq j, k \leq 2n} \sum_{1 \leq i \leq n} s_{ij}y'_j \wedge s_{n+i}k y'_k
\]

\[
= \sum_{1 \leq j < k \leq 2n} \sum_{1 \leq i \leq n} (s_{ij}s_{n+i}k - s_{i}k s_{n+i} j)y'_j \wedge y'_k
\]
If $1 \leq j < k \leq n$, then the coefficient of $y_j \wedge y_k$ is given by $(\tau AC - \tau CA)_{j \ k} = 0$, if $n + 1 \leq j < k \leq 2n$, then the coefficient equals $(\tau BD - \tau DB)_{j-n \ k-n} = 0$ and finally if $1 \leq j \leq n, n + 1 \leq k \leq 2n$, then the coefficient is $(\tau AD - \tau CB)_{j \ k} = \delta_{j \ k-n}$, hence

$$y_1 \wedge y_{n+1} + y_2 \wedge y_{n+2} + \ldots + y_n \wedge y_{2n} = y_1' \wedge y'_{n+1} + y_2' \wedge y'_{n+2} + \ldots + y_n' \wedge y'_{2n},$$

thus $z$ is well defined.

q.e.d.

Since $X$ is a hyperplane in $Gr(2, V)$, we obtain the short exact sequence:

$$0 \rightarrow \mathcal{O}_{Gr}(-1) \xrightarrow{\cdot z} \mathcal{O}_{Gr} \rightarrow \mathcal{O}_X \rightarrow 0,$$

which we will use for cohomology computations.

For our purposes it will be convenient to give a description of $X$ as homogeneous space. Let $V_i = \text{Span}\{v_1, \ldots, v_i\}$ and denote by $0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset V$ the standard symplectic flag. Let $B$ be the Borel preserving the standard symplectic flag and let $P$ be the parabolic subgroup preserving the point $V_2 \in X$. Then

$$X \cong G/P.$$ 

Let $\mathcal{V}$ be a homogeneous vector bundle on $X$. Denote by $\mathcal{V}_{eP} \cong \mathcal{V}_{V_2}$ the zero fiber of $\mathcal{V}$ over the point $V_2 \in X$. Recall that homogeneous vector bundles are completely determined by its representation of $P$ on the zero fiber $\mathcal{V}_{eP} \cong \mathcal{V}_{V_2}$, that is $\mathcal{V} \cong G \times_P \mathcal{V}_{eP}$. Conversely each representation $W$ of $P$ defines a homogeneous vector bundle

$$G \times_P W = G \times W/(gp, w) \sim (g, pw), p \in P.$$ 

Let $S$ be the tautological bundle on $X$ and $S^\perp$ the dual of the quotient bundle on $X$, i.e. the bundle that fits into the short exact sequence:

$$0 \rightarrow S^\perp \rightarrow V^* \otimes \mathcal{O}_X \rightarrow S^* \rightarrow 0.$$
Analogously denote by $S_{Gr}$ and $S_{Gr}^\perp$ the corresponding bundles on the grassmannian $Gr(2, V)$.

Following the notations of the previous chapter, let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \ldots, \beta_{N-2})$ be ordered partitions, and let $\Sigma^\alpha S$, $\Sigma^\beta S^\perp$, $\Sigma^\alpha S_{Gr}$ and $\Sigma^\beta S_{Gr}^\perp$ be the associated Schur functors. Denote by $\alpha^*$ the conjugate or dual of $\alpha$, with $\alpha_j$ = number of entries in the $j$-th column of the Young diagram given by $\alpha$.

Note that all these bundles are homogeneous:

$$S \cong G \times_P V_2 \quad S^\perp \cong G \times_P (V/V_2)^*$$

$$\Sigma^\alpha S \cong G \times_P \Sigma^\alpha V_2 \quad \Sigma^\beta S^\perp \cong G \times_P \Sigma^\beta (V/V_2)^* .$$

Let $C_\alpha = \Sigma^\alpha S \boxtimes \Sigma^{\alpha^*} S^\perp$ for $N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0$,

$$C_i = \Lambda(S \boxtimes S^\perp) \cong \bigoplus_{|\alpha| = i} C_\alpha \quad \text{and} \quad C = \bigoplus_{i \geq 0} C_i .$$

**The Tate construction $\Phi$ in degree 2**

Let us first discuss the Tate construction in degree 2. We will adjoin an element in degree 2 in order to kill the extra torsion in degree 1, that is we will define an extension $\Psi \subset V^* \otimes S^\perp$ of $Sym_2 S^\perp$ and $O_X$. Then

$$\Phi = \Lambda^2 S \boxtimes \Psi \oplus Sym_2 S \boxtimes \Lambda^2 S^\perp$$

is an extension of

$$C_2 = \Lambda(S \boxtimes S^\perp) \cong \Lambda^2 S \boxtimes Sym_2 S \boxtimes Sym_2 S \boxtimes \Lambda^2 S^\perp$$

in the Koszul complex $C$. At the end of this chapter we will complete the discussion of $\Psi$ and show that $\Psi$ is a non-trivial extension.

Let

$$\eta = (x_1 \otimes x_{n+1} + x_2 \otimes x_{n+2} + \cdots + x_n \otimes x_{2n}) \in V^* \otimes (V/V_2)^* ,$$
and define $W$ to be the subspace of $V^* \otimes (V/V_2)^*$,

$$W = \text{Sym}_2(V/V_2)^* \oplus k\eta .$$

Recall that the natural action of $P$ on $V^*$ is given by:

$$p \cdot f(v) = f(p^{-1}v) .$$

The corresponding action on $V^* \otimes V^*$ leaves $\text{Sym}_2(V/V_2)^*$ invariant, which is defining the vector bundle $\text{Sym}_2S^\perp \cong G \times_p \text{Sym}_2(V/V_2)^*$. Suppose that this action is leaving $W$ invariant as well, then the $P-$ representation $W$ defines the homogeneous vector bundles:

**Definition 2.2.**

$$\Psi = G \times_P W \subset V^* \otimes S^\perp \text{ and } \Phi = \mathcal{L}S \boxtimes \Psi \oplus \text{Sym}_2S \boxtimes \mathcal{L}S^\perp .$$

These are well defined as long as $W$ is a representation of $P$, which we will show next.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$ and let $\mathfrak{p}$ be the parabolic algebra corresponding to the parabolic group $P \subset G$. The action of $P$ on $V^*$ translates to the natural action of $\mathfrak{p}$ on $V^*$, that is the action of $\mathfrak{p}$ on $V^*$ is given by $p \cdot f(v) = f(-\tau pv)$.

**Lemma 2.3.**

1. The Levi factors of the parabolic $P$ are $L_1 \cong \text{Gl}_2$ and $L_2 \cong \text{Sp}_{2n-4}$.
2. For $p \in \mathfrak{p}, p \cdot \eta \in \text{Sym}_2(V/V_2)^*$ and
3. $P \rightarrow \text{Gl}(W)$ is a subrepresentation of $P$ of $V^* \otimes (V/V_2)^*$.

**Proof.** It is enough to do these computations on the algebra level.

1. Let us first study the Levi factors of $P$ and $\mathfrak{p}$. Consider the Levi decomposition of $P$ into Levi factors:

$$P \cong (L_1 \times L_2) \ltimes U_P ,$$
where $L_1$ and $L_2$ are the Levi factors and $U_P$ is the unipotent radical. This composition corresponds to the Levi decomposition of $p$ as

$$p \cong l_1 \oplus l_2 \oplus u_p.$$  

Observe that the condition for an element $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathfrak{gl}_{2n}$ to be symplectic is that

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} -\tau m & -\tau p \\ -\tau n & -\tau q \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

i.e. that $\tau n = n, \tau p = p,$ and $\tau m = -q.$

Thus for $p \in p$, this Levi decomposition corresponds to:

$$p = \begin{pmatrix} A & F & B & G \\ 0 & C & \tau G & D \\ 0 & 0 & -\tau A & 0 \\ 0 & E & -\tau F & -\tau C \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C & 0 & D \\ 0 & 0 & 0 & 0 \\ 0 & E & 0 & -\tau C \end{pmatrix} + \begin{pmatrix} 0 & F & B & G \\ 0 & 0 & \tau G & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau F & 0 \end{pmatrix},$$

where $A$ and $B$ are $2 \times 2$ matrices, $F$ and $G$ are $2 \times (n-2)$ matrices and $C, D$ and $E$ are $(n-2) \times (n-2)$ matrices. Since $p$ is a symplectic matrix, $B, D$ and $E$ are symmetric, i.e. $B = \tau B, D = \tau D$ and $E = \tau E.$ Therefore the Levi factors of the parabolic algebra $p$ are

$$l_1 \cong \mathfrak{gl}_2$$ and $$l_2 \cong \mathfrak{sp}_{2n-4}$$

and the Levi factors of the parabolic group $P$ are

$$L_1 \cong Gl_2$$ and $$L_2 \cong Sp_{2n-4}.$$  

(2) The action of $p$ on $V^* \otimes V^*$ is given by:

$$p \cdot (f \otimes g) = (p \cdot f) \otimes g + f \otimes (p \cdot g),$$
and the action of \( p \) on \( f = \sum a_i x_i = \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix} \):

\[
(p \cdot f) = -\tau p \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix}.
\]

We will use the Levi decomposition of \( p \) into the two levi factors \( l_1 \) and \( l_2 \) and the unipotent ideal \( u_p \). Let \( p \in \mathfrak{p} \) be as above with \( p = p_1 + p_2 + u \). Then

\[
p \cdot \eta = p_1 \cdot \eta + p_2 \cdot \eta + u \cdot \eta.
\]

All the following matrix multiplications are the obvious ones, for example:

\[
\tau A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = (a_{1,1}x_1 + a_{2,1}x_2) \otimes x_{n+1} + (a_{1,2}x_1 + a_{2,2}x_2) \otimes x_{n+2}
\]

Using the symmetry of \( B, D \) and \( E \), we get:

\[
p_1 \cdot \eta = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\tau A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot (x_1 \otimes x_{n+1} + x_2 \otimes x_{n+2} + \cdots + x_n \otimes x_{2n})
\]

\[
= -\tau A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes A \begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix}
\]

\[
= 0
\]
This implies that \( p \cdot \eta \) is in \( \text{Sym}_2(V/V_2)^* \), and thus that \( P \rightarrow \text{Gl}(W) \) is an indecomposable representation of \( P \).
(3) Since \( p \) leaves the subspace \((V/V_2)^*\) and thus the space \( \text{Sym}_2(V/V_2)^* \) invariant, we only need to check the action of \( p \) on \( \mathbb{K}\eta \). Since \( p \cdot \eta \) is in \( \text{Sym}_2(V/V_2)^* \subset W \) for \( p \in p \), it follows that \( W \) is a representation of \( P \).

q.e.d.

**Remark.**

(1) \( \Psi \) is a subbundle of the homogeneous bundle \( V^* \otimes S^1 \), this follows from Lemma 2.3(3).

(2) Since \( X \) is a homogeneous space isomorphic to \( G/P \), where \( P \supset B \) is the parabolic subgroup preserving the point \( V_2 \in X \), \( P \) can also be described as the parabolic subgroup, corresponding to omitting one node of the Dynkin diagram:

\[
\begin{array}{cccccccc}
\circ & \bullet & \circ & \circ & \circ & \circ & \circ & \circ \quad \leftrightarrow \quad \circ \\
\end{array}
\]

Thus the Levi factors of \( P \) correspond to the two Dynkin diagrams:

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \quad \leftrightarrow \quad \circ \\
\end{array}
\]

Thus the Levi factors of \( P \) correspond to the two Dynkin diagrams:

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \quad \leftrightarrow \quad \circ \\
\end{array}
\]

Thus the Levi factors of \( P \) correspond to the two Dynkin diagrams:

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \quad \leftrightarrow \quad \circ \\
\end{array}
\]

Therefore \( L_1 \cong GL_2 \) and \( L_2 \cong Sp_{2n-4} \).

**The Tate construction in higher degrees**

We will extend the construction of \( \Psi \) and \( \Phi \) to the whole complex. We will do this in several steps, let us outline the procedure.

Let \( \beta \) be an ordered partition with \( 2 \geq \beta_1 \geq \cdots \geq \beta_{N-2} \geq 0 \).

In the first part we define the homogeneous bundles \( \Psi_\beta = G \times_P W_\beta \) for some representations \( W_\beta \) of \( P \).

In the second part we discuss the homogeneous \( \mathcal{O}_{X \times X} \)-bundles

\[
\Phi_k = (G \times G) \times_P M_k.
\]

These will be extensions of \( \Lambda(S \boxtimes S^1) \) in the Koszul complex and are needed to extend the Koszul complex to a resolution of \( \mathcal{O}_\Delta \).
In the third part we will relate these two definitions. We will show that
\[ \Phi_k \cong \bigoplus_{|\beta|=k} \Sigma^\beta S \boxtimes \Psi_\beta. \]

Finally in the last part, we extend the map \( d \) to the Tate construction
\[ D_\ast = \bigoplus_{k \leq 0} \Phi_k \]
and show that this complex defines a resolution of the structure sheaf of the diagonal.

**The extensions \( \Psi_\beta \)**

We will now define the extensions \( \Psi_\beta \) of \( \Sigma_\beta S^\perp \) for \( 2 \geq \beta_1 \geq \cdots \geq \beta_{N-2} \geq 0 \).

Denote by \( \beta - 2 \) the ordered partition
\[ \beta - 2 = \begin{cases} (\beta_2, \beta_3, \ldots, \beta_{N-2}, 0) & \text{if } \beta_1 = 2 \\ 0 & \text{otherwise} \end{cases}. \]

Set \( \Sigma_\beta = \Sigma^\beta (V/V_2)^* \) and let \( Sym_2 = Sym_2 (V/V_2)^* \).

**Definition 2.4.**

Define \( W_\beta \) as
\[ W_\beta = \bigoplus_{i \geq 0} (\Sigma_{\beta-2i} \otimes \eta^i \subset \bigoplus_{i \geq 0} \Sigma_{\beta-2i} \otimes Sym_i (V^* \otimes (V/V_2)^*)). \]

Note that this definition is extending the definition of \( W \) in the previous section, since
\[ W_2 = \Sigma_{(2,0,0,0)} \oplus \Sigma_{(0,0,0)} \otimes \eta \cong Sym_2 (V/V_2)^* \oplus k \eta = W. \]

We will define an action on \( W_\beta \). We cannot use the natural action of \( P \) on \( W_\beta \) as subspace of \( \bigoplus \Sigma_{\beta-2i} \otimes Sym_i (V^* \otimes (V/V_2)^*) \), since this action does not leave \( W_\beta \) invariant, for example:
Let $\beta = (2, 1, 0, \ldots, 0)$, then $W_\beta = \Sigma^{(2, 1, 0, \ldots, 0)}(V/V_2)^* \oplus (V/V_2)^* \otimes \eta$. Choose $p \in p$ and $f \in (V/V_2)^*$, then

$$p \cdot (f \otimes \eta) = (p \cdot f) \otimes \eta + f \otimes (p \cdot \eta).$$

Since $p \cdot \eta \in Sym_2(V/V_2)^*$, see Lemma 2.3, it follows that

$$f \otimes (p \cdot \eta) \in (V/V_2)^* \otimes Sym_2(V/V_2)^*$$

which is isomorphic to $\Sigma^{(2, 1, 0, \ldots, 0)}(V/V_2)^* \oplus Sym_3(V/V_2)^*$, see appendix on the tensor product of Young diagrams. Here the tensor product is associated to the tensor product

$$\boxtimes \boxplus \boxplus \cong \boxplus \boxplus \oplus \boxplus \boxplus .$$

Note that

$$f \otimes (p \cdot \eta) \in \Sigma^{(2, 1, 0, \ldots, 0)}(V/V_2)^* \oplus Sym_3(V/V_2)^*$$

is not contained in $W_{(2, 1, 0, \ldots, 0)}$, but the projection of $f \otimes (p \cdot \eta)$ in

$$\Sigma^{(2, 1, 0, \ldots, 0)}(V/V_2)^* \oplus Sym_3(V/V_2)^* \rightarrow \Sigma^{(2, 1, 0, \ldots, 0)}(V/V_2)^*$$

is. In order to define an action of $P$ on $W_\beta$, we need to define a projection

$$\pi_\beta : \Sigma_{\beta-2} \otimes Sym_2 \rightarrow \Sigma_\beta$$

in general.

**The $*$-multiplication.**

Let us first discuss the tensor product of $\Sigma_{\beta-2}$ and $Sym_2$.

**Lemma 2.5.** Let $\beta$ be an ordered partition $N - 2 \geq \beta_1 \geq \cdots \geq \beta_{N-2} \geq 0$ such that $\Sigma_\beta$ and $\Sigma_{\beta-2}$ are non-zero.

Then $\Sigma_\beta$ is a summand of $\Sigma_{\beta-2} \otimes Sym_2$ with multiplicity one.
Proof.
Consider the corresponding tensor product of the Young diagrams:

\[ \beta - 2 \otimes \square \]

This is given by adding the two boxes of \( \square \) to \( \beta - 2 \) according to the Littlewood-Richardson rule, (see appendix), that is at most one box to each column. This adds either no, one or two boxes to the first two columns of the Young diagram \( \beta - 2 \). If both boxes are added to the first two columns, then there is only one way of doing so and the resulting weight is equal to \( \beta \). Thus \( \beta \) appears with multiplicity one.

\[ \text{q.e.d.} \]

Since \( \Sigma_\beta \) is a direct summand of \( \Sigma_{\beta - 2} \otimes \text{Sym}_2 \) with multiplicity one, we can define the projections

\[ \pi_\beta : \Sigma_{\beta - 2} \otimes \text{Sym}_2 \rightarrow \Sigma_\beta . \]

These are well defined and determined up to a constant. Note that these projections project \( \Sigma_{\beta - 2} \otimes \text{Sym}_2 \) onto the factors \( \Sigma_\beta \) for all partitions \( \beta \) that do appear in the Koszul complex.

**Definition 2.6.** Let \( N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0 \).

1. (a) If \( \alpha^* - 2 \) is not zero, then let \( \pi_\alpha \) be the projection

\[ \pi_\alpha : \Sigma_{\alpha^* - 2} \otimes \text{Sym}_2 \rightarrow \Sigma_{\alpha^*} . \]

(b) if \( \alpha^* - 2 \) is zero, then set \( \pi_\alpha = 0 \).

2. Set \( \pi = \Sigma \pi_\alpha : \bigoplus \Sigma_{\alpha^* - 2} \otimes \text{Sym}_2 \rightarrow \bigoplus \Sigma_{\alpha^*} , \) where the sum goes over all \( N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0 \) and

3. for \( F \in \bigoplus \Sigma_{\alpha^* - 2} \), define the \(*\)-multiplication:

\[ F \ast (f \cdot g) = \pi (F \otimes (f \cdot g)) . \]

\( \pi \) is well defined.
\(\pi\) is well defined since \(\Sigma_\alpha\) appears as the summand of the tensor product \(\Sigma_\alpha^{-2} \otimes \text{Sym}_2\) with multiplicity one, Lemma 2.5, which determines the projections \(\pi_\alpha\) up to multiplication by a constant.

We will see later that the choice of the scalars does not effect our computations. We will use the \(*\)-multiplication to define the Lie algebra action on \(W_\beta\).

**The action of \(P\) on \(W_\beta\).**

**Definition 2.7.**

For \(p \in \mathfrak{p}, f_\beta \cdot \eta^i \in W_\beta\) let

\[
\begin{align*}
(1) & \quad p \cdot \eta^i = i(p \cdot \eta) \cdot \eta^{i-1}, \\
(2) & \quad p \cdot (f_\beta \cdot \eta^i) = (p \cdot f_\beta) \cdot \eta^i + i(f_\beta * (p \cdot \eta)) \cdot \eta^{i-1}, i \geq 1 \\
& \quad p \cdot f_\beta = p \cdot f_\beta, f_\beta \in W_\beta.
\end{align*}
\]

The action of \(P\) is well defined.

This is a well defined action leaving \(W_\beta\) invariant. However we need to show that this is a Lie algebra action.

Claim:

(a) \(\pi_\beta\) commutes with the action of \(P\) and

(b) \((f_\beta * (p \cdot \eta)) * (g \cdot \eta) = (f_\beta * (g \cdot \eta)) * (p \cdot \eta)\)

(a) \(\Sigma_{\beta-2} \otimes \text{Sym}_2 \cong \Sigma_\beta \oplus \text{other representations of } (V/V_2)^*\). This is an isomorphism of representations of \(Gl((V/V_2)^*)\). Since the action of \(P\) leaves \(V_2\) invariant, this is also an isomorphism of representations of \(P\).

(b) Consider the tensor product of \(\Sigma_{\gamma-4}\) by \(\Sigma_{(2,2,0,\ldots,0)}\), for \(2 \geq \gamma_1 \geq \cdots \geq \gamma_{N-2} \geq 0\). This is given by adding the four boxes of the Young diagram

\[\begin{array}{c}
\end{array}\]

to the diagram of \(\gamma-4\), according to the Littlewood-Richardson rule.
Similar to the proof of the previous Lemma, there is precisely one way of adding two boxes to each of the two columns of \( \gamma - 4 \), thus \( \Sigma_\gamma \) appears with multiplicity one in the tensor product of \( \Sigma_{\gamma-4} \) by \( \Sigma_{(2,2,0,...,0)} \).

Denote by \( \wedge \pi \) the projection

\[
\wedge \pi : \Sigma_{\gamma-4} \otimes \Sigma_{(2,2,0,...,0)} \to \Sigma_\gamma.
\]

This is determined uniquely up to a non-zero constant. Note that \( \Sigma_\gamma \) also appears with multiplicity one in \( \Sigma_{\gamma-2} \otimes \text{Sym}_2 \). Therefore the two projections

\[
\pi_\gamma \circ (\pi_{\gamma-2} \otimes \text{id}_{\text{Sym}_2}) , \quad \wedge \pi \circ (\text{id}_{\Sigma_{\gamma-4}} \otimes \pi_{(2,2,0,...,0)}) : \Sigma_{\gamma-4} \otimes \text{Sym}_2 \otimes \text{Sym}_2 \to \Sigma_\gamma
\]

only differ by a constant \( c \). Let \( f \in \Sigma_{\gamma-4} \) and \( x, y \in \text{Sym}_2 \), then it is enough to prove that

\[
x * y = y * x
\]

since \( (f * x) * y = \pi(f \otimes (x * y)) = c \cdot \wedge \pi(f \otimes (x * y)) \)

and \( (f * y) * x = \pi(f \otimes (y * x)) = c \cdot \wedge \pi(f \otimes (y * x)) \).

Consider the projection

\[
\pi_{(2,0...0)} : \text{Sym}_2 \otimes \text{Sym}_2 \to \Sigma_{(2,2,0,...,0)}.
\]

The Young symmetrizer of \( (2,2,0, \ldots, 0) \) is equal to

\[
c_{(2,2,0,...,0)} = [(1 + \epsilon_{(1\,2)})(1 + \epsilon_{(3\,4)})][(1 - \epsilon_{(1\,3)})(1 - \epsilon_{(2\,4)})]
\]

\[
= [(1 + \epsilon_{(1\,2)})(1 + \epsilon_{(3\,4)})][1 + \epsilon_{(1\,3)} - \epsilon_{(2\,4)} + \epsilon_{(1\,3)(2\,4)}],
\]

thus we get an explicit description of the projection \( \pi_{(2,0...0)} \) up to a multiplication by a non-zero constant \( k \) as follows:

\[
\pi_{(2,0...0)}((f_1 \cdot f_2) \otimes (g_1 \cdot g_2)) = k[(f_1 \cdot f_2) \otimes (g_1 \cdot g_2) - (g_1 \cdot f_2) \otimes (f_1 \cdot g_2) - (f_1 \cdot g_2) \otimes (g_1 \cdot f_2) + (g_1 \cdot g_2) \otimes (f_1 \cdot f_2)]
\]

\[
= \pi_{(2,0...0)}((g_1 \cdot g_2) \otimes (f_1 \cdot f_2)).
\]
This proves the claim. q.e.d.

For \(g, p \in \mathfrak{P}\), denote by \([p, g] = pg - gp\) the Lie algebra bracket. Consider

\[
p \cdot (g \cdot (f_\beta \otimes \eta^i)) - g \cdot (p \cdot (f_\beta \otimes \eta^i)) = p \cdot ((g \cdot f_\beta) \otimes \eta^i + (f_\beta \ast (g \cdot \eta)) \otimes i\eta^{i-1}) - g \cdot ((p \cdot f_\beta) \otimes \eta^i + (f_\beta \ast (p \cdot \eta)) \otimes i\eta^{i-1}) = (pg \cdot f_\beta) \otimes \eta^i + (g \cdot f_\beta) \ast (p \cdot \eta) \otimes i\eta^{i-1} + p \cdot (f_\beta \ast (g \cdot \eta)) \otimes i\eta^{i-1} + (f_\beta \ast (g \cdot \eta) \ast (p \cdot \eta)) \otimes i(i-1)\eta^{i-2} - (gp \cdot f_\beta) \otimes \eta^i - (p \cdot f_\beta) \ast (g \cdot \eta) \otimes i\eta^{i-1} - g \cdot (f_\beta \ast (p \cdot \eta)) \otimes i\eta^{i-1} - (f_\beta \ast (p \cdot \eta) \ast (g \cdot \eta)) \otimes i(i-1)\eta^{i-2}
\]

using (a) and (b) above

\[
= ([p, g] \cdot f_\beta) \otimes \eta^i + "f_\beta \otimes [p, g] \cdot \eta^i = [p, g] \cdot (f_\beta \otimes \eta^i)"
\]

Thus the \(\ast\)-multiplication does define a Lie algebra action.

q.e.d.

**Remark.** Note that the choice of the scalars for the projections \(\pi_\beta\) define isomorphisms of representations of \(P\). We can realize this in two ways.

Let \(W_\beta\) and \(\hat{W}_\beta\) be representations of \(P\) that differ by the choices of the projections \(\pi_\beta\). Suppose \(\hat{\pi}_\beta = \lambda_\beta \cdot \pi_\beta\).

1. The representation \(P \to Gl(W_\beta)\) and \(P \to Gl(\hat{W}_\beta)\) are defined by

   \[
   \rho, \hat{\rho} : P \to \bigoplus_{i \geq 0} Hom(\Sigma_{\beta-2i}, \Sigma_{\beta-2i}) \oplus Hom(\Sigma_{\beta-2(i-1)}, \Sigma_{\beta-2(i-1)})
   \]

   The difference between \(\rho\) and \(\hat{\rho}\) is given by the relation:

   \[
   \hat{\rho} = \bigoplus_{i \geq 0} (id|_{Hom(\Sigma_{\beta-2i}, \Sigma_{\beta-2i})} \oplus \lambda_{\beta-2i} \cdot id|_{Hom(\Sigma_{\beta-2(i-1)}, \Sigma_{\beta-2(i-1)})}) \circ \rho.
   \]

2. We can define an isomorphism between \(W_\beta\) and \(\hat{W}_\beta\) directly. Set

   \[
   \mu_\gamma = \prod_{j \geq 0} \lambda_{\gamma-2j}.
   \]
Define $i : W_\beta \to \widehat{W}_\beta$ by

$$i(f_{\beta-2i} \eta^i) = \mu_{\beta-2i} \cdot f_{\beta-2i} \eta^i.$$  

Then

$$p \cdot i(f_{\beta-2i} \eta^i)$$

$$= \mu_{\beta-2i} p \cdot f_{\beta-2i} \eta^i$$

$$= \mu_{\beta-2i}(p \cdot f_{\beta-2i}) \eta^i + \mu_{\beta-2i}(f_{\beta-2i} \cdot p \cdot \eta) \eta^{i-1}$$

$$= \mu_{\beta-2i}(p \cdot f_{\beta-2i}) \eta^i + \mu_{\beta-2i+2}(f_{\beta-2i} \cdot p \cdot \eta) \eta^{i-1}$$

$$= \mu_{\beta-2i}(p \cdot f_{\beta-2i}) \eta^i + \mu_{\beta-2i+2}(f_{\beta-2i} \cdot p \cdot \eta) \eta^{i-1}$$

$$= i(p \cdot (f_{\beta-2i} \eta^i)).$$

Thus $i$ defines an isomorphism of representations of $P$ between $W_\beta$ and $\widehat{W}_\beta$.

Let us define the homogeneous bundles associated to the representations $W_\beta$ of $P$, that is

**Definition 2.8.**

$$\Psi_\beta = G \times_\rho W_\beta$$

Next we define the homogeneous bundles $\Phi_k$ and then show how this definition relates to the definition of the bundles $\Psi_\beta$.

**The homogeneous bundles $\Phi_k$**

We will define $P \times P$ representations $M_k$ and and use these to define the homogeneous bundles $\Phi_k$.  

Notations. Recall that $\eta = x_1 \otimes x_{n+1} + \ldots x_n \otimes x_{2n} \in V^* \otimes (V/V_2)^*$. Let $\eta^i \in Sym_i(V^* \otimes (V/V_2)^*)$ and set

$$\Lambda_\alpha = \Sigma^\alpha V_2 \otimes \Sigma^\alpha (V/V_2)^*,$$

$$\Lambda_i = \Lambda(V_2 \otimes (V/V_2)^*) \cong \bigoplus_{|\alpha|=i} \Lambda_\alpha \text{ and }$$

$$\Lambda = \bigoplus_{i \geq 0} \Lambda_i,$$

$$\theta^i = (\Lambda V_2)^{\otimes i} \boxtimes \eta^i \subset (\Lambda V_2)^{\otimes i} \boxtimes Sym_i(V^* \otimes (V/V_2)^*) \text{ and }$$

$$M = \bigoplus_{i \geq 0} \Lambda[-2i] \otimes \theta^i.$$

Remark.

(1) There is a natural action of $P \times P$ on $M$ coming from the inclusion of $M$ in the representation of $P \times P$:

$$N = \bigoplus_{i \geq 0} \Lambda[-2i] \otimes ((\Lambda V_2)^{\otimes i} \boxtimes Sym_i(V^* \otimes (V/V_2)^*)) .$$

(2) $N$ has a natural multiplication induced by the wedge product on $\Lambda$ and the symmetric product on $Sym(V^* \otimes (V/V_2)^*)$, that is for

$$\lambda_1, \lambda_2 \in \Lambda, \mu_1 \in Sym_i(V^* \otimes (V/V_2)^*), \mu_2 \in Sym_j(V^* \otimes (V/V_2)^*),$$

$$(\lambda_1 \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \mu_1)) \cdot (\lambda_2 \otimes ((v_1 \wedge v_2)^{\otimes j} \boxtimes \mu_2))$$

$$= (\lambda_1 \cdot \lambda_2) \otimes ((v_1 \wedge v_2)^{\otimes i+j} \boxtimes (\mu_1 \cdot \mu_2)) .$$

This restricts to give a product on $M$.

The multiplicative structure of $\Lambda$ is essential for the Tate construction.
The Tate construction only applies to skew-commutative, graded $R$-algebras over a commutative noetherian ring $R$ satisfying certain conditions.
Lemma 2.9. The $P \times P$ action leaves $M$ and $M_k$, the $k$-th graded piece of $M$, that is

$$M_k = \bigoplus_{i \geq 0} \Lambda_{k-2i} \otimes \theta^i$$

invariant.

Proof. It is sufficient to do these computations on the algebra level. Let

$$\lambda_k \in \Lambda_k, v_1 \wedge v_2 \in \Lambda V_2, g, h \in p$$

and $i \geq 1$, then

$$w = \lambda_k \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i) \in \Lambda_k \otimes \theta^i$$

and

$$(g, h) \cdot w$$

$$= ((g, h) \cdot \lambda_k) \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i) + \lambda_k \cdot (g, h) \cdot ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i)$$

$$= ((g, h) \cdot \lambda_k) \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i) + \lambda_k \cdot ((g \cdot (v_1 \wedge v_2)^{\otimes i}) \boxtimes \eta^i)$$

$$+ \lambda_k \cdot ((v_1 \wedge v_2)^{\otimes i} \boxtimes h \cdot \eta^i)$$

$$= ((g, h) \cdot \lambda_k) \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i) + \lambda_k \cdot ((g \cdot (v_1 \wedge v_2)^{\otimes i}) \boxtimes \eta^i)$$

$$+ i \lambda_k \cdot ((v_1 \wedge v_2) \boxtimes h \cdot \eta) \otimes ((v_1 \wedge v_2)^{\otimes i-1} \boxtimes \eta^{i-1})$$

Note that $(v_1 \wedge v_2) \boxtimes h \cdot \eta \in 2 \Lambda V_2 \boxtimes Sym_2(V/V_2)^* \subset \Lambda_2$, see Lemma 2.3. Therefore

$$i \lambda_k \cdot ((v_1 \wedge v_2) \boxtimes h \cdot \eta) \in \Lambda_k \cdot \Lambda_2 \subset \Lambda_{k+2}$$

and thus for $i \geq 1$

$$(g, h) \cdot w \in (\Lambda_k \otimes \theta^i) \oplus (\Lambda_{k+2} \otimes \theta^{i-1})$$

and for $w \in \Lambda_k, (g, h) \cdot w \in \Lambda_k$.

This proves that $M$ and $M_k, k \geq 0$ are invariant under the action of $P \times P$.

q.e.d.

Since the $M_k$’s are representations of $P \times P$, we can define the corresponding homogeneous vector bundles.
Definition 2.10. Set

\[ \Phi_k = (G \times G)_{P \times P} M_k \]

and set

\[ D = \bigoplus_{k \geq 0} \Phi_k . \]

The bundles \( \Phi_k \) are the extensions needed to extend the Koszul complex \( C \) to an exact complex. Before extending the map \( d \) to \( D \), we show that these bundles are isomorphic to a direct sum of homogeneous bundles, that is

\[ \Psi_k \cong \bigoplus_{|\alpha| = k} \sum^\alpha \otimes \Psi_{\alpha^*} . \]

Remark 2.11.

(1) Note that

\[ \Phi_2 \cong \Phi , \]

since the representations

\[ M_2 = \Lambda_2 \oplus \theta = \frac{2}{2} \Lambda(V_2 \otimes (V/V_2)^*) \oplus \frac{2}{2} \Lambda V_2 \otimes \eta = Sym_2 V_2 \otimes \Lambda(V/V_2)^* \oplus \frac{2}{2} \Lambda V_2 \otimes Sym_2 (V/V_2)^* \oplus \frac{2}{2} \Lambda V_2 \otimes \eta = Sym_2 V_2 \otimes \frac{2}{2} \Lambda(V/V_2)^* \oplus \frac{2}{2} \Lambda V_2 \otimes W \]

of \( P \times P \) are isomorphic as subrepresentations of \( V^* \otimes V^* \), thus \( \Phi_2 \cong \Phi \).

(2) The inclusion of \( \Lambda_k \subset M_k \) defines a short exact sequence of representations of \( P \times P \):

\[ 0 \to \Lambda_k \to M_k \to \frac{2}{2} \Lambda V_2 \otimes M_{k-2} \to 0 , \]

which in turn defines the short exact sequence of homogeneous vector bundles.

\[ 0 \to C_k \to \Phi_k \to \mathcal{O}_X(-1) \otimes \Phi_{k-2} . \]
This follows from
(a) $M_k/\Lambda_k \cong \Lambda_{k-2} \otimes \theta \oplus \Lambda_{k-4} \otimes \theta^2 \oplus \ldots$
and
(b) the action of $P \times P$ on $M_k/\Lambda_k$ is compatible to the action of $P \times P$
on $\tilde{\Lambda} V_2 \otimes M_{k-2}$.

Similarly to the definition of $\Phi_k$, we will define homogeneous bundles $\Phi_\alpha$ andthen discuss the relation to the homogeneous bundles $\Psi_\alpha^*$.

**The extensions $\Phi_\alpha$ and $\Psi_\alpha^*$**

**Notations.** Set $M_\alpha = \bigoplus_{i \geq 0} \Lambda_{\alpha - 2i} \otimes \theta^i$ for $N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0$ and recall that

$$\beta - 2 = \begin{cases} (\beta_2, \beta_3, \ldots, \beta_{N-2}, 0) & \text{if } \beta_1 = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\Sigma_\beta = \Sigma^\beta(V/V_2)^*$ and set $Sym_2 = Sym_2(V/V_2)^*$.

**Definition of $\Phi_\alpha$**

**Lemma 2.12.**

$M_\alpha$ is invariant under the action of $P \times P$.

**Proof.**

It is sufficient to do these computations on the algebra level. Let

$\lambda \in \Lambda_\gamma, v_1 \land v_2 \in \tilde{\Lambda} V_2, g, h \in p$ and $i \geq 1$, then

$$w = \lambda \otimes ((v_1 \land v_2)^{\otimes i} \otimes \eta^i) \in \Lambda_\gamma \otimes \theta^i$$

and

$$(g, h) \cdot w$$

$$= ((g, h) \cdot \lambda) \otimes ((v_1 \land v_2)^{\otimes i} \otimes \eta^i) + \lambda \cdot ((g \cdot (v_1 \land v_2)^{\otimes i}) \otimes \eta^i)$$

$$+ i\lambda \cdot ((v_1 \land v_2) \otimes h \cdot \eta) \otimes ((v_1 \land v_2)^{\otimes i-1} \otimes \eta^{i-1})$$
Note that
\[ i\lambda \cdot ((v_1 \wedge v_2) \boxtimes h \cdot \eta) \in \Lambda_\gamma \cdot \Lambda_{(1,1)} \]
and \( \Lambda_{(1,1)} = (\Lambda^2 \boxtimes \text{Sym}_2) \). Before we finish the proof, we will discuss the multiplication of \( \Lambda_\gamma \) by \( \Lambda_{(1,1)} \).

Let \( |\alpha| = k, N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0, \gamma^* = \alpha^* - 2 \), that is \( \gamma = (\alpha_1 - 1, \alpha_2 - 1) \).

Assume that \( \gamma \geq (0,0) \).

Consider the multiplication \( m_\gamma \):

\[
\Lambda_\gamma \otimes \Lambda_{(1,1)} \to \Lambda_k \\
\cap \quad \cap \quad \| \\
\Lambda_\gamma \otimes \Lambda_2 \to \Lambda_k \\
(\lambda) \otimes (\lambda_1 \wedge \lambda_2) \mapsto \lambda \wedge \lambda_1 \wedge \lambda_2
\]

**Lemma 2.13.** Let \( |\alpha| = k, N - 2 \geq \alpha_1 \geq \alpha_2 \geq 1, \gamma^* = \alpha^* - 2 \), then the image of \( m_\gamma : \Lambda_\gamma \otimes \Lambda_{(1,1)} \to \Lambda_k \)

(1) is contained in \( \Lambda_\alpha \) and

(2) is not zero.

**Proof.**

(1) Consider the tensor product

\[
\Lambda_\gamma \otimes \Lambda_{(1,1)} \\
\cong (\Lambda^2 \boxtimes \text{Sym}_2) \otimes (\Sigma^\gamma V_2 \boxtimes \Sigma_{\gamma^*}) \\
\cong (\Sigma(\gamma_1+1,\gamma_2+1)V_2) \boxtimes (\Sigma(\gamma_1+1,\gamma_2+1)^* \oplus \Sigma(\gamma_1,\gamma_2+1)^* \oplus \Sigma(\gamma_1,\gamma_2+1) \oplus \Sigma(\gamma_1,\gamma_2,1)^*) \\
= (\Sigma(\alpha_1,\alpha_2)V_2) \boxtimes (\Sigma(\alpha_1,\alpha_2)^* \oplus \Sigma(\alpha_1,\alpha_2-1,1)^* \oplus \Sigma(\alpha_1-1,\alpha_2,1)^* \oplus \Sigma(\alpha_1-1,\alpha_2-1,1,1)^*)
\]

See also appendix, Lemma 2(b), for this tensor product.
The image is contained in $\Lambda_\gamma \otimes \Lambda_{(1,1)}$ modulo the relations of the $k$-th exterior power $\Lambda^k$. Since $\Lambda^k \cong \bigoplus_{|\beta|=k} \Sigma^\beta V_2 \boxtimes \Sigma_{\beta^*}$, it follows that the image is inside of $\Sigma^\alpha V_2 \boxtimes \Sigma_{\alpha^*}$.

(2) Let $v_1 \ldots v_N$ be the chosen and fixed symplectic basis with its dual basis $x_1 \ldots x_N$. Recall that $V_j = \text{Span}\{v_1 \ldots v_j\}$.

Let $f = x_3$, then $f \in (V/V_2)^*$. Consider

$$(v \wedge w) \boxtimes f^2 \in \hat{\Lambda} V_2 \boxtimes 

\begin{array}{ccc}
\| & \cap \\
2(v \boxtimes f) \wedge (w \boxtimes f) \in \hat{\Lambda}(V_2 \boxtimes (V/V_2)^*)
\end{array}$$

Recall that $\gamma = (\alpha_1 - 1, \alpha_2 - 1)$, therefore $\gamma^*$ is of the form $(\gamma_1^*, \ldots, \gamma_{N-3}^*, 0)$. Set

$$\overset{\wedge}{\gamma} = (\gamma_1^*, \ldots, \gamma_{N-3}^*)$$

Then $V_2 \subset V_3$ and $\gamma \geq (0, 0)$ imply

$$0 \neq \Sigma^\gamma S \boxtimes \Sigma^{\overset{\wedge}{\gamma}^*} (V/V_3)^* \subset \Sigma^\gamma S \boxtimes \Sigma^{\overset{\wedge}{\gamma}^*} (V/V_2)^*$$

Finally consider the multiplication of $\Sigma^\gamma S \boxtimes \Sigma^{\overset{\wedge}{\gamma}^*} (V/V_3)^*$ by $(v \wedge w) \boxtimes f^2$:

$$(v \wedge w) \boxtimes f^2 \cdot w_{\overset{\wedge}{\gamma}}$$

$$= 2(v \boxtimes f) \wedge (w \boxtimes f) \wedge w_{\overset{\wedge}{\gamma}}$$

The image is not zero, because $w_{\overset{\wedge}{\gamma}}$ is a sum over elements of the form $(v_1 \boxtimes f_1) \wedge \cdots \wedge (v_{i-2} \boxtimes f_{i-2}) \neq 0$ and all of these elements $v_j \boxtimes f_j$ and $v \boxtimes f$ and $w \boxtimes f$ are linearly independent.

Thus multiplication of $\Sigma^\gamma S \boxtimes \Sigma^{\overset{\wedge}{\gamma}^*} (V/V_3)^*$ by $(v \wedge w) \boxtimes f^2$ is non-zero and therefore the multiplication $\Lambda_{(1,1)} \otimes \Lambda_\gamma \rightarrow \Lambda_\alpha$ is non-zero.

q.e.d.
Proof of Lemma 2.12.

Since \( \Lambda^\gamma \cdot \Lambda_{(1,1)} \subset \Lambda_{(\gamma_1+1,\gamma_2+1)} \), it follows that for \( i \geq 1, w \in \Lambda^\gamma \otimes \theta^i, \gamma = (\alpha_1 - i, \alpha_2 - i) \),

\[(g, h) \cdot w \in (\Lambda^\gamma \otimes \theta^i) \oplus (\Lambda_{(\gamma_1+1,\gamma_2+1)} \otimes \theta^{i-1})\]

and for \( w \in \Lambda^\alpha, (g, h) \cdot w \in \Lambda^\alpha \). This proves that \( M^\alpha \) is invariant under the action of \( P \times P \).

q.e.d

Definition 2.14.

Define \( \Phi^\alpha = (G \times G)_{P \times P} M^\alpha \).

Relation between \( M^\alpha \) and \( W_{\beta^*} \).

Proposition 2.15. Let \( N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0, \gamma = (\alpha_1 - 1, \alpha_2 - 1) \geq (0,0) \) and let \( \alpha^* - 2 = \gamma^* \). The multiplication \( m^\gamma : \Lambda^\gamma \otimes \Lambda_{(1,1)} \to \Lambda^\alpha \) and the \(*\)-multiplication are compatible, that is:

Denote by \( r \) the isomorphism

\[ r : \bigoplus \Sigma^\alpha V_2 \otimes \Sigma^\alpha^* (V/V_2)^* \cong \Lambda, \]

then there is an isomorphism

\[ i = \oplus i_\alpha : \bigoplus \Lambda^\alpha \to \Lambda^\alpha \text{ such that } m^\gamma (((v \wedge w) \otimes f \cdot g) \otimes (v_\gamma \otimes f_\gamma)) = i(r((v \wedge w \otimes v_\gamma) \otimes (f_\gamma \ast (f \cdot g)))) . \]

Proof. The multiplication \( m^\gamma \) is given by the map

\[ m^\gamma : \Lambda^\gamma \otimes \Lambda_{(1,1)} \to \Lambda^\gamma \otimes \Lambda_{(1,1)} \text{ modulo the relations of the } i\text{-th exterior power } \Lambda_i . \]
Consider once more the tensor product (see also Lemma 2.13)

\[ \Lambda \otimes \Lambda_{(1,1)} \]

\[ \cong (\Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes \Sigma^{(\alpha_1,\alpha_2)*} ) + \Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes (\Sigma^{(\alpha_1,\alpha_2-1,1)*} + \Sigma^{(\alpha_1-1,\alpha_2,1)*} + \Sigma^{(\alpha_1-1,\alpha_2-1,1)*} ) \]

Since \( \Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes (\Sigma^{(\alpha_1+1,\alpha_2-1)*} + \Sigma^{(\alpha_1-1,\alpha_2+1)*}) \) modulo the relations of the i-th exterior product is zero, it follows that \( m_\gamma \circ r \) is given by the projection

\[ (\Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes \Sigma^{(\alpha_1,\alpha_2)*} ) + \Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes (\Sigma^{(\alpha_1,\alpha_2-1,1)*} + \Sigma^{(\alpha_1-1,\alpha_2,1)*} + \Sigma^{(\alpha_1-1,\alpha_2-1,1)*} ) \to (\Sigma^{(\alpha_1,\alpha_2)} V_2 \boxtimes \Sigma^{(\alpha_1,\alpha_2)*} ) . \]

This projection is determined up to a constant \( k_\alpha \), thus

\[ m_\gamma ( (v \wedge w) \boxtimes f \cdot g) \otimes (v_\gamma \boxtimes f_\gamma) ) = k_\alpha r ( (v \wedge w \otimes v_\gamma) \boxtimes (f \cdot g) ) . \]

Set \( i_\alpha = k_\alpha \cdot \text{id}|_{\Lambda_\alpha} \).

\text{q.e.d.}

\textbf{Remark.} We will now identify

\[ \Lambda = \Lambda(V_2 \boxtimes (V/V_2)^*) \quad \text{with} \quad \bigoplus_{|\alpha|=k} \Sigma^\alpha V_2 \boxtimes \Sigma^\alpha (V/V_2)^* . \]

\textbf{Theorem 2.16.}

For \( N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0 \),

\[ M_\alpha \cong \Sigma^\alpha V_2 \boxtimes W_\alpha^* \]

\[ \Phi_\alpha \cong \Sigma^\alpha S \boxtimes \Psi_\alpha^* . \]
Proof.

Let \( \alpha^* = \beta - 2i, v_\alpha \boxtimes f_\alpha \in \Sigma^a V_2 \boxtimes \Sigma_{\alpha^*} \) and \( (g, h) \in p \times p \). Consider

\[
\begin{align*}
(g, h) \cdot ((v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i}) \boxtimes (f_\alpha \otimes \eta^i)) & = (g \cdot v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i} + v_\alpha \otimes g \cdot (v_1 \wedge v_2)^{\otimes i}) \boxtimes (f_\alpha \otimes \eta^i) \\
& + ((v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i}) \boxtimes (h \cdot f_\alpha \otimes \eta^i)) \\
& + ((v_1 \wedge v_2)^{\otimes i-1} \boxtimes (v_\alpha \boxtimes f_\alpha) \cdot (v_1 \wedge v_2 \boxtimes h \cdot \eta) \otimes \eta^{i-1}) \\
& = (g \cdot v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i} + v_\alpha \otimes g \cdot (v_1 \wedge v_2)^{\otimes i}) \boxtimes (f_\alpha \otimes \eta^i) \\
& + ((v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i}) \boxtimes (h \cdot f_\alpha \otimes \eta^i + (f_\alpha \ast h \cdot \eta) \cdot \eta^{i-1}) \\
& = (g \cdot (v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i}) \boxtimes (f_\alpha \otimes \eta^i)) \\
& + ((v_\alpha \otimes (v_1 \wedge v_2)^{\otimes i}) \boxtimes h \cdot (f_\alpha \otimes \eta^i))
\end{align*}
\]

The second equality follows from Proposition 2.15. These computations imply that \( W_\beta \) is a representation of \( p \) coming from the representation \( M_{\beta^*} \) of \( p \times p \) and thus

\[
M_{\beta^*} \cong \Sigma^a V_2 \boxtimes W_\beta.
\]

The isomorphism of the bundles follows immediately from here, since

\[
\Phi_{\alpha} \cong (G \times G)_{P \times P} M_{\alpha} \cong (G \times P \Sigma^a V_2) \boxtimes (G \times P W_{\alpha^*}) \cong \Sigma^a S \boxtimes \Psi_{\alpha^*}.
\]

q.e.d.

We complete the Tate construction by extending the connecting morphism \( d \) from the Koszul complex \( C_\ast \otimes \mathcal{O}_{X \times X} \) to the new construction \( D_\ast \), thus extending the Koszul complex \( C_\ast \) to the complex \( D_\ast \).

**The map \( D \)**

Recall that \( \eta = x_1 \otimes x_{n+1} + x_2 \otimes x_{n+2} + \cdots + x_n \otimes x_{2n} \) for the standard symplectic basis \( v_1, \ldots, v_{2n} \) and its dual basis \( x_1, \ldots, x_{2n} \).
Definition 2.17.

(1) For \( g, h \in G, i > 0 \), set

\[
T_{(g,h)} = (g, v_1 \wedge v_2) \boxtimes (h, \eta) \in \Phi_{2, (gP, hP)} \quad \text{and}
\]

\[
T^i_{(g,h)} = (g, (v_1 \wedge v_2)^{\otimes i}) \boxtimes (h, \eta^i) \in \Phi_{2i, (gP, hP)} .
\]

(2) Let \( \hat{d} : D. \rightarrow D. \) be the extension of the map \( d \) of the Koszul complex \( C. \otimes \mathcal{O}_{X \times X} \) satisfying over a point \( (gP, hP) \in G/P \times G/P \cong X \times X \):

(a) \( \hat{d}(T_{(g,h)}) = (g, h \cdot x_1 (g \cdot v_1) v_2 - h \cdot x_1 (g \cdot v_2) v_1) \boxtimes (h, x_{n+1}) \)

\[
+ (g, h \cdot x_2 (g \cdot v_1) v_2 - h \cdot x_2 (g \cdot v_2) v_1) \boxtimes (h, x_{n+2})
\]

\[
+ \cdots +
\]

\[
+ (g, h \cdot x_n (g \cdot v_1) v_2 - h \cdot x_n (g \cdot v_2) v_1) \boxtimes (h, x_{2n})
\]

\( =: t_{(g,h)} \in S_{gP} \boxtimes S^\perp_{hP} , \)

(b) \( \hat{d}(T^i_{(g,h)}) = i \ t_{(g,h)} \ T^{i-1}_{(g,h)} \) and

(c) \( \hat{d}(w T^i_{(g,h)}) = (dw) T^i_{(g,h)} + (-1)^{|w|} i w \cdot t_{(g,h)} T^{i-1}_{(g,h)} ) \) for \( w \in (C_j \otimes \mathcal{O}_{X \times X})_{(gP, hP)} . \)

Lemma 2.18. \( \hat{d} \) is well defined and uniquely determined and \( \hat{d} \) restricted to the Koszul complex \( C. \otimes \mathcal{O}_{X \times X} \) is equal to \( d \).

Proof.

(a) Consider \( \hat{d}(T_{(g,h)}) \). Note that \( T_{(g,h)} \) is an element in \( ((G \times G)_{P \times P} M_2)_{(gP, hP)} \) and note that the \( P \times P \)-representation \( M_2 = \Lambda V_2 \boxtimes W \) is contained in \( \Lambda V_2 \boxtimes V^* \otimes (V/V_2)^* \). Define

\[
\tilde{d} : (G \times_p \Lambda^2 V_2) \otimes (G \times_p (V^* \otimes (V/V_2)^*)) \rightarrow (G \times_p V_2) \otimes (G \times_p (V/V_2)^*)
\]

\[
(g, v \wedge w) \boxtimes (h, F \otimes G) \rightarrow (g, h \cdot F(g \cdot v) - h \cdot F(g \cdot w)) \boxtimes (h, G) .
\]
Clearly $\tilde{d} = \hat{d}$ on $\Phi = (G \times G)_{P \times P} M_2$. $\tilde{d}$ is the evaluation map on the first factor and it is well defined, since:

$$\tilde{d}((g_1, v \wedge w) \boxtimes (h_1, F \otimes G))$$

$$= (g_1, h_1 F(g_1 v) w - h_1 F(g_1 w) v) \times (h_1, G)$$

and

$$\tilde{d}((g, p_1(v \wedge w)) \boxtimes (h, p_2(F \otimes G)))$$

$$= (g, p_1(h_1 F(g_1 v) w - h_1 F(g_1 w) v) \times (h, p_2 G)$$

are the same for all $p_1, p_2 \in P$.
Thus $\hat{d} = \tilde{d}\vert_\Phi$ is well defined on $\Phi$.

(b) The condition in (2) (b) assures that $\hat{d}$ respects the relation $T^i T^j = T^{i+j}$, that is

$$\hat{d}(T^i_{(g,h)} T^j_{(g,h)}) = \hat{d}(T^i_{(g,h)}) T^j_{(g,h)} + T^i_{(g,h)} \hat{d}(T^j_{(g,h)}) = (i + j) t_{(g,h)} T^{i+j-1}_{(g,h)}$$

(c) $\Phi_k = (G \times G)_{P \times P} M_k$ and elements in $M_k$ are sums over elements of the form

$$\lambda_{k-2i} \otimes ((v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i).$$

Thus over the point $(gP, hP) \in X \times X$, the elements of $\Phi_k\vert_{(gP, hP)}$ are sums over elements of the form

$$w_{k-2i} \otimes ((g, h), (v_1 \wedge v_2)^{\otimes i} \boxtimes \eta^i) = w_{k-2i} \otimes T^i_{(g,h)}$$

This determines $\hat{d}$ uniquely and it is well defined since $\hat{d}\vert_\Phi$ is well defined .

q.e.d.

Set $d = \hat{d}$.

**Remark 2.18.** Note that

1. $d^2 = 0$ and
(2) \(d\) maps \(\Phi_\alpha\) to \(\Phi_{\alpha-1}\), where \(\Phi_{\alpha-1}\) denotes the sum of \(\Phi_\gamma\) over all Young diagrams \(\gamma\) that are contained in \(\alpha\), i.e. \(\gamma_i \leq \alpha_i\), and of length \(|\alpha|-1\).

(3) There is a short exact sequence of homogeneous vector bundles

\[0 \to \sum^\beta S^\perp \to \Psi_\beta \to \Psi_{\beta-2} \to 0,\]

for \(2 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{N-2} \geq 0\).

(4) The Tate complex \(D_k\) in degree \(k\) is given by:

\[D_k \cong \bigoplus_{|\alpha|=k} \sum^\alpha S \otimes \Psi_\alpha.\]

**Proof.**

(1) This follows from \(t \cdot t = t \wedge t = 0\) and

\[dd(wT^i) = d((dw)T^i + (-1)^{|w|}iw \cdot tT^{i-1}) = ddwT^i + (\frac{-1}{2} \cdot \frac{i}{w} \cdot iw \cdot tT^{i-1}) \]

\[= 0 + 0 + iwd(tT^{i-1}) = iwdtT^{i-1} - (i-1)t \cdot t T^{i-2} = 0.\]

(2) Consider the map \(d : M_\alpha \to M_{\alpha-1}\) of \(p\) representations. Let \(|\alpha| = kw \in \Lambda_\gamma, \gamma = (\alpha_1 - i, \alpha_2 - i)\), then

\[d(w\eta^i) = (dw)\eta^i + (-1)^{|w|}w \cdot d\eta T^{i-1} \in \Lambda_\gamma \eta^i \oplus \Lambda_\gamma \eta^{i-1},\]

because \(w \cdot t \in \Lambda_\gamma \cdot \Lambda_{(i,0)} \subset \Lambda_\gamma \oplus \Lambda_{(i+1,0)} \oplus \Lambda_{(i+1,0)}.\)

Note that \(d\) is not a homogeneous map.
(3) The inclusion $\Sigma^\beta (V/V_2)^* \subset W_\beta$ defines a short exact sequence of $P$ representations:

$$0 \to \Sigma^\beta (V/V_2)^* \to W_\beta \to W_{\beta-2} \to 0,$$

which implies the exactness of the corresponding sequence of homogeneous vector bundles.

(4) This is obvious, since

$$D_i = \bigoplus_{k \geq 0} \Phi_k \cong \bigoplus_{|\alpha| = k} \Phi_{\alpha^*},$$

q.e.d.

We will show next that the complex $D_i$, that we have defined, is exact and thus does resolve $O_\Delta$.

The resolution

**Theorem 2.19.**

$$D_i : \cdots \to \Phi_k \xrightarrow{d} \Phi_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} \Phi \xrightarrow{S \otimes S^\perp} \delta \xrightarrow{S \otimes S^\perp} \delta \xrightarrow{S \otimes S^\perp} \delta \xrightarrow{S \otimes S^\perp} \delta \to O_{X \times X}$$

is a left resolution of the structure sheaf $O_\Delta$ of the diagonal $\Delta$ inside of $X \times X$.

**Proof.** This is a local question. Let $U_1, U_2$ be open sets in $X$. We may assume that all of the vector bundles $D_i|_{U_1 \times U_2}$ are trivial. Thus

$$\Psi|_{U_2} \cong \text{Sym}_2 S|_{U_2} \oplus T \quad \text{and} \quad \Phi_{k|_{U_1 \times U_2}} \cong \bigoplus_{i \geq 0} S|_{U_1} \otimes S^\perp|_{U_2} \delta^i$$

Therefore $D_i|_{U_1 \times U_2}$ is isomorphic to the polynomial ring in one variable $T$ of degree 2 over $(C. \otimes O_{X \times X})|_{U_1 \times U_2}$. Thus

$$D_i|_{U_1 \times U_2} \cong (C. \otimes O_{X \times X})|_{U_1 \times U_2}[T].$$
We will show that this is the same construction as in Tate [T]. Since \( D|_{U_1 \times U_2} \) is gotten by adjoining the variable \( T \) in degree 2 to \((C \otimes \mathcal{O}_{X \times X})|_{U_1 \times U_2}\), this complex will kill the class of the cycle \( dT \), see [T], Theorem 2 or Lemma 2.20 below.

We will prove this theorem in two parts:

First we discuss the Tate construction of adjoining a variable in even degree and then we will show that the homology of the complex \( C \) restricted to \( X \times X \) is given by the cycle \( dT \).

**Adjoining a variable in even degree**

Let \( A \) be an \( R \)-algebra, that is a graded algebra over a commutative noetherian ring \( R \) with unit element and an \( R \)-linear mapping \( d : A_i \to A_{i-1} \) satisfying

1. \( A_i = 0 \) for \( i < 0 \), \( A_0 = R \cdot 1 \) and \( A_i \) is a finitely generated \( R \)-module for all \( i > 0 \).
2. \( A \) is skew-commutative and
3. \( d \) is a skew derivation, that is

\[
d(xy) = (dx)y + (-1)^{deg(x)}x(dy).
\]

Let \( t \) be a cycle of odd degree \( \rho - 1 \). We will adjoin a variable in even degree \( \rho \) in order to kill the cycle \( t \).

Let \( Y = A[T] \) be the ring of polynomials in one commuting variable \( T \) of degree \( \rho \) over \( A \). Extend the derivation \( d \) in \( A \) to \( Y \) in the unique way such that \( dT = t \).

Then \( Y \) is an \( R \)-algebra.

**Lemma 2.20.** Assume that the homology class \( [t] \in H(A) \) is a skew non-zero divisor, that is

\[
[t] \xi = 0 \Rightarrow \xi \in [t]H(A) \quad \text{for all } \xi \in H(A).
\]

Then \( H(Y) \cong H(A)/ [t]H(A) \).

**Proof.** See [T], Theorem 2, page 17.
We continue with the proof of the theorem.

Set \( \Lambda = (C \otimes \mathcal{O}_{X \times X})|_{U_1 \times U_2}, Y = D.|_{U_1 \times U_2} \) and \( t = dT \). Then \( Y = \Lambda[T] \).

Suppose that the homology of \( \Lambda \) is generated by \([t]\). Then the condition of the previous Lemma is satisfied, which implies that \( Y \) has no homology, thus that \( D. \) is exact.

Thus left to show that the homology of the complex \((C \otimes \mathcal{O}_{X \times X})|_{U_1 \times U_2}\) is generated by \([t]\).

The torsion

Lemma 2.21.

\[
\text{Tor}_i^{O_{Gr \times Gr}}(\mathcal{O}_{\Delta_{Gr}}, \mathcal{O}_{X \times X}) \cong \begin{cases} 
\mathcal{O}_\Delta & i = 0 \\
\mathcal{O}_\Delta(-1) & i = 1 \\
0 & i > 1 
\end{cases}
\]

Proof.

Recall that \( X \) in \( Gr(2, V) \) is given by the hyperplane section \( z = 0 \) with

\[
z = x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \ldots + x_n \wedge x_{2n}
\]

where \( x_1 \ldots x_{2n} \) is the symplectic basis of \( V^* \). Let \( Gr(2, V) = \text{Proj}(A) \), then \( X = \text{Proj}(A/z) \). Consider the short exact sequence:

\[
0 \to \mathcal{O}_{Gr}(-1) \xrightarrow{\pi} \mathcal{O}_{Gr} \to \mathcal{O}_X \to 0.
\]

Define \( z_1 = z \boxtimes 1, z_2 = 1 \boxtimes z \). Now \( z_1 \) and \( z_2 \) form a regular sequence in \( A \boxtimes A \), hence the Koszul complex of \( z_1 \) and \( z_2 \) is exact. Since \( X \times X = \text{Proj}(A \boxtimes A/(z_1, z_2)) \) this gives a resolution of \( \mathcal{O}_{X \times X} \):

\[
0 \to \mathcal{O}_{Gr \times Gr}(-1, -1) \to \mathcal{O}_{Gr \times Gr}(-1, 0) \oplus \mathcal{O}_{Gr \times Gr}(0, -1) \to \mathcal{O}_{Gr \times Gr} \to \mathcal{O}_{X \times X} \to 0.
\]
To compute the Tor’s we tensor this complex by $\mathcal{O}_{\Delta_G}$ and compute cohomology of the thus obtained complex. Since $\Delta_G \cong \Sigma_\Delta$ and $\Sigma_\Delta \cong X$ we get

$$0 \to \mathcal{O}_{\Sigma_\Delta}(-2) \xrightarrow{\beta} \mathcal{O}_{\Sigma_\Delta}(-1) \oplus \mathcal{O}_{\Sigma_\Delta}(-1) \xrightarrow{\alpha} \mathcal{O}_{\Sigma_\Delta} \to \mathcal{O}_X \to 0$$

This complex has only cohomology in degree 0 and 1. In degree 0, $H^0 \cong \mathcal{O}_X$ and in degree 1, we have

$$\alpha(\sum_i (f_i \oplus g_i)) = z \cdot (\sum_i (f_i + g_i)) = 0 \iff \sum_i (f_i + g_i) = 0$$

Thus $\ker\alpha \cong (\mathcal{A}_{[-1]}) \cong \mathcal{O}_{\Sigma_\Delta}(-1)$, and since $\beta(f) = z \cdot f \oplus (-z \cdot f)$, we get $\text{Im}\beta \cong z \cdot (\mathcal{A}_{[-2]})$.

Therefore $H^1 \cong (\mathcal{A}_{[-1]}/z\mathcal{A}_{[-2]}) = \mathcal{O}_X(-1) \cong \mathcal{O}_\Delta(-1)$. q.e.d.

**Corollary 2.22.**

Let $y_1, \ldots, y_N$ be a symplectic basis of $V^*$, such that over a point $W_1 \times W_2 \in X \times X, (V/W_2)^*$ is generated by $y_3, \ldots, y_N$. Let $W_1 = v \wedge w$. Then the torsion over the point $W_1 \times W_2$ is generated by the cycle $\gamma_{W_1 \times W_2}$:

$$\gamma_{W_1 \times W_2} = (y_1(v)w - y_1(w)v) \boxtimes y_{n+1} + \ldots + (y_n(v)w - y_n(w)v) \boxtimes y_{2n}.$$ 

**Proof.** $W_1$ is an isotropic subspace of $V$. Thus for $v, w \in W_1$:

$$d\gamma_{W_1 \times W_2} = y_1(v) y_{n+1}(w) - y_1(w) y_{n+1}(v) + \ldots + y_n(v) y_{2n}(w) - y_n(w) y_{2n}(v) = <v, w> = 0.$$

q.e.d.

**Proof of Theorem 2.19**

In order to finish the proof we have to show that $dT$ generates the homology of the complex $C \otimes \mathcal{O}_{X \times X}$. This follows immediately from the last corollary, since $dT_{(gP, hP)} = dT_{(g, h)} = t_{(g, h)} = \gamma_{(gP, hP)} = \gamma_{W_1 \times W_2}$, for the parabolic $gP$ and $hP$ that are preserving $W_1$ respectively $W_2$.

q.e.d.
The periodic resolution

We started with the Koszul resolution of $\mathcal{O}_{\Delta_{Gr}}$ and we used the Tate construction to define the resolution $D$ of $\mathcal{O}_{\Delta}$. The first thing we note is that this complex is not finite anymore, but instead it becomes periodic in large degrees of period two. To be more precise:

Recall from remark 2.18 that

$$0 \to \Lambda(S \otimes S^\perp) \to \Phi_k \to \Phi_{k-2} \otimes (\mathcal{O}_X(-1) \boxtimes \mathcal{O}_X) \to 0$$

is exact. In particular if $k > \text{rank } (S \boxtimes S^\perp) = 2 \cdot (N - 2)$, then

$$\Phi_k \cong \Phi_{k-2} \otimes (\mathcal{O}_X(-1) \boxtimes \mathcal{O}_X) = \mathcal{O}_X(-1) \otimes \Phi_{k-2}$$

Set

$$\Phi_+ = \Phi_{2N-4}, \quad \Phi_- = \Phi_{2N-5}$$

and denote by $\mathcal{F}(-k) = \mathcal{F} \otimes (\mathcal{O}_X(-k) \boxtimes \mathcal{O}_X)$

Then the complex $D$ becomes:

**Corollary 2.23.**

The complex $D$ in large degrees is periodic of degree 2, that is $D$ is equal to:

$$\ldots \to \Phi_+(-i) \to \Phi_-(i) \to \Phi_+(-i+1) \to \Phi_-(i+1) \to \ldots$$

$$\to \ldots$$

$$\Phi_+ \to \Phi_- \to \Phi_{2N-6} \to \ldots \to S \boxtimes S^\perp \to \mathcal{O}_X \times X.$$  

This completes the Tate construction of the resolution $D$.

**Remark.** Although we will not use it in this work, we will discuss one more issue. The extensions $\Psi_\beta$ are extensions of $\Sigma^\beta S^\perp$ and of $\Psi_\beta^{-2}$, see Remark 2.18. It turns out that these are non-trivial extensions.
This is a very essential comment for the $K$-theory of $X$. The goal is to show algebraically that

$$K_\ast(X) \cong \bigoplus_{g \in X} K_\ast(k),$$

for a generating system $X$. Particularly the elements of a generating system should not have any cohomology. We note without proof that, for example $\text{Sym}_2 S^\perp$ has higher cohomology and that is why we chose to extend $\text{Sym}_2 S^\perp$ to $\Psi$. If $\Psi$ were a trivial extension, it would still have higher cohomology.

Non-triviality of the extensions $\Psi_\beta$

**Theorem 2.24.** $\Psi$ is a non-trivial extension of $\text{Sym}_2 S^\perp$ and $\mathcal{O}_X$, i.e.

$$0 \to \text{Sym}_2 S^\perp \to \Psi \to \mathcal{O}_X \to 0 \text{ does not split}.$$ 

**Proof.** Consider the short exact sequence

$$0 \to \mathcal{O}_{Gr}(-1) \to \mathcal{O}_{Gr} \to \mathcal{O}_X \to 0.$$

Tensor this sequence by $V^* \otimes S^\perp_{Gr}$:

$$0 \to V^* \otimes S^\perp_{Gr}(-1) \to V^* \otimes S^\perp_{Gr} \to V^* \otimes S^\perp \to 0.$$

Consider the long exact cohomology sequence:

$$0 \to H^0(V^* \otimes S^\perp_{Gr}(-1)) \to H^0(V^* \otimes S^\perp_{Gr}) \to H^0(V^* \otimes S^\perp)$$

$$\to H^1(V^* \otimes S^\perp_{Gr}(-1)) \to H^1(V^* \otimes S^\perp_{Gr}) \to H^1(V^* \otimes S^\perp) \to \ldots$$

Note that

$$S^\perp_{Gr}(-1) \cong S^\perp_{Gr} \otimes^2 S^\perp_{Gr} \cong \Sigma(2,1) S^\perp_{Gr} \oplus 3 S^\perp_{Gr}$$

(see appendix for the tensor product of two $\text{Gl}_M$ representations).
Since $\Sigma^{(2,1)}S_{Gr}^\perp$ and $\Lambda^3 S_{Gr}^\perp$ have no cohomology, see Lemma 1.9, chapter 1, it follows that $V^* \otimes S_{Gr}^\perp (-1)$ has no cohomology. Also $V^* \otimes S_{Gr}^\perp$ has no cohomology. From the long exact cohomology sequence it follows that $V^* \otimes S_{Gr}^\perp$ has no cohomology, particularly $V^* \otimes S_{Gr}$ has no non-trivial sections.

Suppose the sequence $0 \to Sym_2 S^\perp \to \Psi \to \mathcal{O}_X \to 0$ splits, then $\Psi$ has a non-trivial section. Since $\Psi$ is inside of $V^* \otimes S_{Gr}^\perp$, it follows that $V^* \otimes S_{Gr}^\perp$ has a non-trivial section, which is a contradiction to the cohomology computations above. q.e.d.

**Proposition 2.25.** The short exact sequence

$$0 \to \Sigma^\beta S_{Gr}^\perp \to \Psi_\beta \to \Psi_{\beta-2} \to 0$$

does not split, i.e. $\Psi_\beta$ is a non-trivial extension of $\Sigma^\beta S_{Gr}^\perp$ and $\Psi_{\beta-2}$.

**Proof.** We have to show that $\Psi_\beta \in Ext^1(\Psi_{\beta-2}, \Sigma^\beta S_{Gr}^\perp)$ is not zero. Tensor the short exact sequence above by $\Sigma^{\beta-2}(S_{Gr}^\perp)^*$, then it is enough to show that

$$\Psi_\beta \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^* \in Ext^1(\Psi_{\beta-2} \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^*, \Sigma^\beta S_{Gr}^\perp \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^*)$$

is not zero.

Consider

(a) the identity inclusion

$$i : \mathcal{O}_X \hookrightarrow \Sigma^{\beta-2} S_{Gr}^\perp \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^* \subset \Psi_\beta \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^*$$

and

(b) the evaluation map:

$$ev : \Sigma^\beta S_{Gr}^\perp \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^* \to Sym_2 S_{Gr}^\perp$$

defined by

$$\Sigma^\beta S_{Gr}^\perp \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^* \subset Sym_2 S_{Gr}^\perp \otimes \Sigma^{\beta-2} S_{Gr}^\perp \otimes \Sigma^{\beta-2}(S_{Gr}^\perp)^* \to Sym_2 S_{Gr}^\perp \to f \cdot g \cdot f \beta-2 \otimes v \beta-2 \mapsto f \beta-2(v \beta-2) \cdot f \cdot g .$$
These maps induce maps on the extensions:

\[
\begin{array}{ccc}
\Psi_\beta \otimes \Sigma^{\beta-2}(S^\perp)^* & \in & Ext^1(\Psi_{\beta-2} \otimes \Sigma^{\beta-2}(S^\perp)^*, \Sigma^\beta S^\perp \otimes \Sigma^{\beta-2}(S^\perp)^*) \\
\downarrow & & \downarrow \\
F & \in & Ext^1(\Psi_{\beta-2} \otimes \Sigma^{\beta-2}(S^\perp)^*, \mathcal{O}_X) \\
\downarrow & & \downarrow \\
G & \in & Ext^1(Sym^2 S^\perp, \mathcal{O}_X)
\end{array}
\]

We will show that \(G\) is a non-trivial extension, thus \(\Psi_\beta \otimes \Sigma^{\beta-2}(S^\perp)^*\) and \(\Psi_\beta\) are non-trivial extensions.

Let us first discuss \(F\), set

\[
W_F = \Sigma^\beta(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2) \otimes \eta \cdot 1 \subset W_\beta \otimes \Sigma^{\beta-2}(V/V_2),
\]

where \(1 \in \Sigma^{\beta-2}(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2)\) is the element corresponding to the identity element in \(End(\Sigma^{\beta-2}(V/V_2)^*) \cong \Sigma^{\beta-2}(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2)\).

We claim that the action of \(P\) leaves \(W_F\) invariant. Again we do these computations on the algebra level:

For \(p \in \mathfrak{p}: p \cdot (\eta \cdot 1) = \pi((p \cdot \eta) \cdot 1) + \eta \cdot (p \cdot 1) = \pi((p \cdot \eta) \cdot 1) + 0\)

\[
\in \pi(Sym^2(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2))
\subset \Sigma^\beta(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2)
\subset W_F .
\]

Claim: \(F \cong G \times_P W_F\). This follows from the commutative diagram of representations of \(P\):

\[
\begin{array}{ccc}
0 & \rightarrow & \Sigma^\beta(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2) \rightarrow W_\beta \otimes \Sigma^{\beta-2}(V/V_2) \rightarrow W_{\beta-2} \otimes \Sigma^{\beta-2}(V/V_2) \rightarrow 0 \\
\| & & \cup \uparrow \\
0 & \rightarrow & \Sigma^\beta(V/V_2)^* \otimes \Sigma^{\beta-2}(V/V_2) \rightarrow W_F \rightarrow \mathcal{O}_X \rightarrow k \rightarrow 0
\end{array}
\]
Next consider $\mathcal{G}$:

We will see that $\mathcal{G} \cong \Psi = G \times_P W$, which by Theorem 2.4 is a non-trivial extension of $\text{Sym}_2 S^\perp$ and $\mathcal{O}_X$.

Let $W_\mathcal{F} \to W$ be the map of representations of $P$:

$$
\begin{align*}
\Sigma^\beta(V/V_2)^* \otimes \Sigma^\beta-2(V/V_2) \oplus \eta \cdot 1 & \to \text{Sym}_2(V/V_2)^* \oplus k\eta \\
 f_\beta \otimes v_{\beta-2} + c\eta \cdot 1 & \mapsto ev(f_\beta \otimes v_{\beta-2}) + c\eta.
\end{align*}
$$

This map induces the commutative diagram of representations of $P$:

$$
\begin{array}{cccccc}
0 & \to & \Sigma^\beta(V/V_2)^* \otimes \Sigma^\beta-2(V/V_2) & \to & W_\mathcal{F} & \to & k & \to 0 \\
\downarrow ev & & \downarrow & & \parallel & & \\
0 & \to & \text{Sym}_2(V/V_2)^* & \to & W & \to & k & \to 0.
\end{array}
$$

which in turn induces the commutative diagram of homogeneous bundles:

$$
\begin{array}{cccccc}
0 & \to & \Sigma^\beta S^\perp \otimes \Sigma^\beta-2(S^\perp)^* & \to & \mathcal{F} & \to & O_X & \to 0 \\
\downarrow ev & & \downarrow & & \parallel & & \\
0 & \to & \text{Sym}_2(V/V_2)^* & \to & \Psi & \to & O_X & \to 0.
\end{array}
$$

Thus $\mathcal{G} \cong \Psi$ and therefore $\mathcal{G}$ is a non-zero element in $\text{Ext}^1(\mathcal{O}_X, \text{Sym}_2 S^\perp)$.

q.e.d.
CHAPTER III

THE MAIN THEOREM

Let $X$ be the symplectic Grassmannian $X = SpGr(2, V)$. In the previous chapter we used Tate’s techniques to define the complex $D_\gamma$, which turned out to be a resolution of the structure sheaf of the diagonal $\Delta \subset X \times X$.

In the case of ordinary Grassmannians, chapter 1, we were able to define a finite resolution $C_\gamma$ of $O_{\Delta_{Gr}}$ of the diagonal $\Delta_{Gr} \subset Gr \times Gr$. In the symplectic case of the symplectic grassmannian $X$ we constructed the infinite resolution $D_\gamma$ of $O_{\Delta}$. This resolution does not terminate, instead it becomes periodic, see Corollary 2.23.

However it does contain a finite sub-complex $B_\gamma$.

**Definition 3.1.** For $i \leq 2N - 6$, let

$$B_i = \bigoplus \Sigma^\alpha S \boxtimes \Psi_\alpha^* ,$$

where the sum goes over all $\alpha$ with $l(\alpha) = i$ and $N - 2 \geq \alpha_1 \geq \alpha_2 \geq 0$.

Note that this sum excludes all $\alpha$ with $\alpha_1 = N - 2$.

Let us check that $B_\gamma$ is a sub-complex. Recall that $\alpha - 1 = \{ \gamma \subset \alpha$ and $l(\gamma) = l(\alpha) - 1 \}$. Let $\gamma \in \alpha - 1$, then $\alpha_1 \neq N - 2$ implies $\gamma_1 \neq N - 2$. Since $d$ maps $\Sigma^\alpha S \boxtimes \Psi_\alpha^*$ to $\bigoplus_{\gamma \in \alpha - 1} \Sigma^\gamma S \boxtimes \Psi_\gamma^*$, Remark 2.18 (2), $d$ maps $B_\gamma$ to itself. (We will devote the next three chapters to the proof of the Main Theorem:

**Theorem 3.2.** The sub-complex $B_\gamma$:

$$B_{2N-6} \to ... \to B_2 \to S \otimes S^\perp \to O_{X \times X}$$

is exact.
First part of the proof:

Let \( Q \) be the quotient complex of the total complex and the sub-complex. In the following section on ”exactness of sub- and quotient complexes”, we prove that it is enough to show that the quotient complex is exact up to degree \( 2N - 6 \), and in the section of ”fiberwise exactness”, we prove that it is enough to check exactness of the quotient complex on fibers over the diagonal \( \Delta \subset X \times X \).

**Exactness of sub-complexes and quotient complexes**

Let \( D \) be an arbitrary exact complex, and suppose \( B \subset D \) is a sub-complex. Since this sub-complex is not necessarily exact, we would like to find a way to decide whether or not \( B \) is exact. One possibility is by looking at the quotient-complex: \( Q = D/B \).

**Lemma 3.3.** \( B \) is exact up to degree \( m \) if and only if \( Q \) is exact up to degree \( m + 1 \).

**Proof.** This can be easily proven by a diagram chase. Consider the following diagram:
Suppose $B_{i}$ is exact up to degree $m$. If $q \in \ker Q_{i+1} \rightarrow Q_{i}$, for $i \leq m$, then we will show that $q = d\tilde{q}$, for some $\tilde{q} \in Q_{i+2}$. Denote by $\overline{c}$ the image of $c \in D_{j}$ in $Q_{j}$. Then $q = \overline{c}$, for some $c \in D_{i+1}$. Since $d\overline{c} = 0$, it follows that $dc = b \in B_{i}$ with $db = 0$. Using the fact that $B_{i}$ is exact in degree $i$, there exist an element $b' \in B_{i+1}$ with $b = db'$. Consider $c - b'$, then $d(c - b') = 0$ and since $C_{i}$ is exact, $c - b' = d\tilde{c}$ for some $\tilde{c} \in C_{i+2}$. Set $\tilde{q} = \tilde{c}$.

Now suppose $Q_{i}$ is exact up to degree $m + 1$. If $b \in \ker B_{i} \rightarrow B_{i-1}$, for $i \leq m$, then we need to show that $b = d\tilde{b}$, for some $\tilde{b} \in B_{i+1}$. Since $d\overline{c} = 0$, it follows that $dc = b \in B_{i}$ with $db = 0$. Using the fact that $B_{i}$ is exact in degree $i$, there exist an element $b' \in B_{i+1}$ with $b = db'$. Consider $c - b'$, then $d(c - b') = 0$ and since $C_{i}$ is exact, $c - b' = d\tilde{c}$ for some $\tilde{c} \in C_{i+2}$. Consider $c - d\tilde{c}$, then $c - d\tilde{c} = 0$, therefore $c - d\tilde{c} \in B_{i+1}$. Set $\tilde{b} = c - d\tilde{c}$, then $d\tilde{b} = d(c - d\tilde{c}) = b - 0 = b$, as desired. q.e.d.

**Fiberwise exactness**

In this section we will show that it is enough to check the exactness of the quotient complex on the diagonal $\Delta \subset X \times X$. Since $Q_{i}$ is a complex that resolves a vector bundle, it is enough to check exactness fiberwise. Suppose for $p \times q$, we can find an element $g$ in the isotropy group of $p$, $Iso_p \subset Sp(V)$, such that $g(q) = p$. Then the cohomology of the complex $Q_{p \times q}$, and the cohomology of $Q_{g(p) \times g(q)} = Q_{p \times p}$, for $g \in Iso_p$, are just the same, because $Q_{i}$ is a complex of homogeneous vector bundles over $X \times X$ and $X \cong Sp(V)/P_X$ where $P_X$ is the parabolic $P_X = Iso_e$ and $e = e_1 \wedge e_2$ for some fixed symplectic basis $\{e_1, e_2, ..., e_{2n}\}$ of $X$.

Suppose we can find, if not a $g$, at least a family $g_{\lambda}$ of elements in $Iso_p$, such that the limit of $g_{\lambda}(q)$ over $\lambda$ is equal to $p$. Then from the Upper-Semi-Continuity Theorem for Cohomology,[H] III, Theorem 12.8, it follows, that the cohomology of $Q_{p \times q}$, and $Q_{p \times p}$, are the same, that is $Q_{i}$ is exact up to degree $m$ if it is exact on the diagonal up to degree $m$.

**Lemma 3.4.** If $p \times q \in X \times X$, then we can use the isotropy group of $p$ to move $q$ to $p$, i.e. there is a family $g_{\lambda} \in Iso_p$, such that $\lim_{\lambda} g_{\lambda}(q) = p$. 
Proof. Let $p \times q$ be in $X \times X$. Then we may assume that $p = e_1 \wedge e_2$, for some symplectic basis $e_1, e_2, \ldots, e_{2n}$. Without loss of generality, we may even assume that $p$ and $q$ fit into one of the following five cases:

If $p \cap q = \{0\}$, then
a) $q = e_{n+1} \wedge e_{n+2}$, or
b) $q = e_{n+1} \wedge e_{n+3}$, or
c) $q = e_{n+3} \wedge e_{n+4}$.

If $p \cap q = k v$, for some vector $v \in V$, then we may assume that $v = e_1$, and
d) $q = e_1 \wedge e_{n+2}$, or
e) $q = e_1 \wedge e_{n+3}$

Let $\gamma = (1 - \lambda)\lambda^{-1}$

a) Define

$$g_1 = \begin{pmatrix}
\lambda^{-1} & 1 - \lambda & 1 - \lambda \\
\lambda^{-1} & 1 & 1 - \lambda \\
& & & & \ddots \\
& & & & & & & & \lambda \\
& & & & & & & & 1 \\
& & & & & & & & 1 \\
& & & & & & & & 1 \\
& & & & & & & & 1 \\
\end{pmatrix}.$$ 

Then $g_1(q) = e_{n+1} \wedge e_{n+2} = q$ and $g_\lambda(q) = g_\lambda(e_{n+1} \wedge e_{n+2}) = ((1 - \lambda)e_1 + \lambda e_{n+1}) \wedge ((1 - \lambda)e_2 + \lambda e_{n+2})$. Therefore the limit $\lim_{\lambda \to 0} g_\lambda(q) = e_1 \wedge e_2$. 
b) Define $g_\lambda = \begin{pmatrix}
\lambda^{-1} & 1 & 1-\lambda & 1-\lambda \\
1 & \lambda^{-1} & \gamma & \\
 & 1 & \ddots & 1 \\
 & & 1 & \lambda \\
 & & & \ddots & 1 \\
 & & & & 1
\end{pmatrix}.

Then $g_1(q) = e_{n+1} \wedge e_{n+3} = q$ and $g_\lambda(q) = g_\lambda(e_{n+1} \wedge e_{n+3}) = ((1-\lambda)e_1 + \lambda e_{n+1}) \wedge ((1-\lambda)e_2 + \lambda e_{n+3})$. Therefore the limit $\lim_{\lambda \to 0} g_\lambda(q) = e_1 \wedge e_2$.

c) Define $g_\lambda = \begin{pmatrix}
1 & 1-\lambda & 1-\lambda \\
1 & \lambda^{-1} & \gamma & \\
 & 1 & \ddots & 1 \\
 & & 1 & \lambda \\
 & & & \ddots & 1 \\
 & & & & 1
\end{pmatrix}.

Then $g_1(q) = e_{n+3} \wedge e_{n+4} = q$ and $g_\lambda(q) = g_\lambda(e_{n+3} \wedge e_{n+4}) = ((1-\lambda)e_1 + \lambda e_{n+3}) \wedge ((1-\lambda)e_2 + \lambda e_{n+4})$. Therefore the limit $\lim_{\lambda \to 0} g_\lambda(q) = e_1 \wedge e_2$. 
d) Define

\[
 g_\lambda = \begin{pmatrix}
 1 & 1 - \lambda \\
 \lambda^{-1} & 1 \\
 & 1 \\
 & \ddots \\
 & & 1 \\
 & & & 1 \\
 & & & & \lambda \\
 & & & & & 1 \\
 & & & & & & \ddots \\
 & & & & & & & & 1 
\end{pmatrix}.
\]

Then \( g_1(q) = e_1 \wedge e_{n+2} = q \) and \( g_\lambda(q) = g_\lambda(e_1 \wedge e_{n+2}) = e_1 \wedge ((1 - \lambda)e_2 + \lambda e_{n+2}) \).

Therefore the limit \( \lim_{\lambda \to 0} g_\lambda(q) = e_1 \wedge e_2 \).

e) Define \( g_\lambda = \)

\[
 g_\lambda = \begin{pmatrix}
 1 & \lambda^{-1} & 1 - \lambda \\
 1 & 1 & \gamma \\
 & 1 & 1 \\
 & \ddots & \ddots \\
 & & 1 \\
 & & & 1 \\
 & & & \lambda \\
 & & & & 1 \\
 & & & & & \ddots \\
 & & & & & & 1 
\end{pmatrix}.
\]

Then \( g_1(q) = e_{n+1} \wedge e_{n+3} = q \) and \( g_\lambda(q) = g_\lambda(e_{n+1} \wedge e_{n+3}) = ((1 - \lambda)e_1 + \lambda e_{n+1}) \wedge ((1 - \lambda)e_2 + \lambda e_{n+3}) \). Therefore the limit \( \lim_{\lambda \to 0} g_\lambda(q) = e_1 \wedge e_2 \).

Left to check that all the above matrices are indeed symplectic. The symplectic form that we are using, is given by:

\[
 <v, w> = \tau v \begin{pmatrix}
 0 & I \\
 -I & 0 
\end{pmatrix} w.
\]

It is easy to verify that the columns of all the above defined matrices are in fact a symplectic basis. q.e.d.
Before we finish the proof of the Main Theorem, we will discuss some exact sequences, chapter 4, and apply these to the K-theory of the symplectic grassmannian of isotropic two planes in four space, chapter 5, and finally prove the Main Theorem, chapter 6.
CHAPTER IV

SOME EXACT SEQUENCES

In this chapter we will discuss some special exact sequences which play an important role in proving that the sub-complex $B_i$ is exact. These sequences are a special case of Lascoux Resolutions. The techniques that we are applying in this chapter are the geometric techniques of calculating syzygies, which go back to G. Kempf [K] and which were developed by P. Pragacz and J. Weyman [PW]. Similar methods can also be found in J. Weyman’s book [W], under preparation, On the cohomology of vector bundles and syzygies, chapter 5 and 6.

The idea of using these complexes to prove the main theorem and the construction of these complexes is entirely due to J. Weyman. Nevertheless the proof here is different from J. Weyman’s proof, since we are interested only in a very special case of these sequences and for this special case we can proof the results more directly.

Notations

Let $T, T'$ be vector spaces of dimension $n$ and $m$ and set

\[ X = \text{Hom}(T, T') \cong T^* \otimes T' \]
\[ X_r \subset X, X_r = \{ \phi \in X | \text{rank}(\phi) \leq r \} \quad \text{and} \]
\[ \hat{X} = X - X_r. \]

Note, that $X$ can also be described as the set of all $m \times n$ matrices, thus $X_r$ is the subset of matrices with all $(r + 1) \times (r + 1)$ minors vanishing. This gives a description of $X_r$ as a determinantal variety. A more general case of determinantal varieties can be found in Chapter 6 [W].
Set $V = \text{Gr}(n-r, T)$, let $\mathcal{R}$ be the tautological bundle and consider the product $V \times X$. Denote the projections of $V \times X$ to $V$ and $X$ by $p$ respectively $q$, i.e. $p : V \times X \rightarrow V$ and $q : V \times X \rightarrow X$. Let 

$$Z_r = \{ (l, \phi) \in V \times X \mid l \subset \ker \phi \}. $$

Note, that $Z_r$ is just the pullback of $X_r$ under $q$. Set $\hat{Z} = V \times X - Z_r$. Denote the projections of $\hat{Z}$ to $V$ and $\hat{X}$ by $\hat{p}$ respectively $\hat{q}$, that is:

$$\begin{array}{ccc}
\hat{Z} & \xleftarrow{\hat{p}} & \hat{X} \\
\downarrow & & \downarrow \\
V & & \hat{X}
\end{array}$$

We will now apply the geometric techniques of calculating syzygies to define an exact Koszul complex of $\mathcal{O}_Z$-modules, which after twisting we will then push forward under the map $\hat{q}$ to get an exact complex of $\mathcal{O}_\hat{X}$-modules.

**A special Koszul complex**

View $T^{\ast\ast}$ as trivial bundle over $V$ and consider the bundle $\mathcal{R} \otimes T^{\ast\ast}$ over $V$. Define a map of vector bundles:

$$\Psi : \hat{p}^*(\mathcal{R} \otimes T^{\ast\ast}) \rightarrow \mathcal{O}_\hat{Z},$$

over a fiber $l \times \psi : \Psi(r \otimes T) = T(\psi(r))$.

$\Psi$ is surjective, because for a point $(l, \psi) \in \hat{Z}, l$ is not contained in the kernel of $\psi$, thus the image of $\Psi_{l \times \psi} = \mathbb{C}$.

Let $K = K(\hat{p}^*(\mathcal{R} \otimes T^{\ast\ast}), \Psi)$ be the Koszul complex of $\hat{p}^*(\mathcal{R} \otimes T^{\ast\ast})$ and $\Psi : \hat{p}^*(\mathcal{R} \otimes T^{\ast\ast}) \rightarrow \mathcal{O}_\hat{Z}$, see also chapter 1, Koszul complexes.
Proposition 4.1.

\[ K : 0 \rightarrow \Lambda^p (R \otimes T^{**}) \rightarrow \ldots \rightarrow \Lambda^2 (R \otimes T^{**}) \rightarrow \hat{p}^* (R \otimes T^{**}) \rightarrow O \rightarrow 0 \]

is exact.

Proof. Lemma 1.2 gives a criteria for a Koszul complex to be exact. Since \( \Psi : \hat{p}^* (R \otimes T^{**}) \rightarrow O \wedge Z \) is surjective, we have to check that \( \hat{p}^* (R \otimes T^{**}) \) has the right rank, that is:

\[ \text{rank}(\hat{p}^* (R \otimes T^{**})) = \dim Z = \dim Z - \dim Z_r. \]

Fix \( l \in V \) and consider the subspace \( X_{l,r} \) of \( X \), consisting of all \( \varphi \in X = Hom(T, T') \) with \( l \subset \ker \varphi \). Recall that \( \dim l = n - r \), \( \dim T = n \) and \( \dim T' = m \). Thus \( X_{l,r} \) is a subspace of \( X \) of codimension \( rm \), and therefore \( \dim_{V \times X} Z_r = (\dim V) rm = \dim Z \).

The rank of \( \hat{p}^* (R \otimes T^{**}) \) is equal to \( (\dim V) \text{rank}(R \otimes T^{**}) = (\dim V) rm \) as desired. q.e.d.

We will twist \( K \) and push this twisted Koszul complex forward under the projection \( \hat{q} : \hat{Z} \rightarrow \hat{X} \). Since there might be higher direct images we can not always do this. For our purposes it will be enough to look at special cases where the higher direct images vanish automatically. For a broader approach, see appendix or [W].

In Chapter 1, Lemma 1.5, we got a criteria to when the pushforward of a complex is exact. In the above situation the hypothesis of Lemma 1.5 that all the higher direct images under the projection \( \hat{q} : \hat{Z} \rightarrow \hat{X} \) of all the twisted \( K_j(V)'s \), \( V \) a vector bundle on \( V \), are zero, are not satisfied. Here only certain direct images vanish, but as we will see next, this weaker condition is enough to imply the exactness of the complex \( \hat{q}_* K(V) \).

\( R^i q_* \) of a complex of coherent sheaves

Lemma 4.2. Let \( q : X \rightarrow Y \) be a projective morphism of noetherian schemes. Let

\[ 0 \rightarrow F_{top} \xrightarrow{d_{top}} \ldots \rightarrow F_m \xrightarrow{d_m} \ldots \xrightarrow{d_1} F_0 \rightarrow 0 \]

be an exact sequence of coherent
$O_X$ - modules. Suppose $R^i q_* F_{l+i} = 0$ for all $i > 0$ and all $0 \leq l \leq m$, then

$$q_* F_m \to \cdots \to q_* F_0 \to 0$$

is an exact complex of coherent $O_Y$ - modules.

**Proof.** We will follow the same lines as in the proof of Lemma 1.3. First we split the complex $F_.$ into short exact sequences,

$$0 \to \ker d_{m-k} \to F_{m-k} \to \text{Im } d_{m-k} \to 0.$$  

Next, in order to show that $q_* F_.$ is exact, we also split this complex into sequences,

$$0 \to \ker q_* d_{m-k} \to q_* F_{m-k} \to q_* \text{Im } d_{m-k} \to 0,$$

and show that these are exact for all $k$.

Again it suffices to show that $R^i q_* \ker d_{(m-k)} = 0$ for all $i > 0, 0 \leq k \leq m$.

As before we will use induction over $k$, but now the case of $k = 0$ is not obvious anymore.

$k = 0$: Set $A = \ker d_m$. We will show $R^i q_* \ker d_{(m+l)} \cong R^{i+1} q_* \ker d_{(m+l+1)}$ for all $l \geq 0$, which in turn implies

$$R^i q_* A \cong R^{i+1} q_* \ker d_{m+1} \cong R^{i+2} q_* \ker d_{m+2} \cong R^{i+3} q_* \ker d_{m+3} \cong \cdots \cong R^{\text{top}} q_* \ker d_{\text{top}} = 0.$$  

$l = 0$: $A \cong \text{Im } d_{m+1}$ and $A$ fits into the short exact sequence: $0 \to \ker d_{m+1} \to F_{m+1} \to A \to 0$. Consider the corresponding long exact sequence of $R q_*$:

$$\cdots \to R^i q_* F_{m+1} \to R^i q_* A \to R^{i+1} q_* \ker d_{m+1} \to R^{i+1} q_* F_{m+1} \to \cdots$$

Since the higher direct images of $F_{m+1}$ are all zero, it follows that $R^i q_* A \cong R^{i+1} q_* \ker d_{m+1}$.

$l > 0$: Consider the short exact sequence $0 \to \ker d_{m+l+1} \to F_{m+l+1} \to \ker d_{m+l} \to 0$, and consider the resulting long exact sequence of $R q_*$:

$$\cdots \to R^i q_* F_{m+l+1} \to R^i q_* \ker d_{m+l} \to R^{i+1} q_* \ker d_{m+l+1} \to R^{i+1} q_* F_{m+l+1} \to \cdots$$
Here the additional assumption of $R^i q_* \mathcal{F}_{m+1} = 0, 0 \leq l \leq m$ is needed. These imply $R^i q_* \mathcal{F}_{m+l+1} = 0$ for $i \geq l$, hence

$$R^i q_* \ker d_{m+l} \cong R^{i+1} q_* \ker d_{m+l+1}$$

for $i \geq l + 1$. q.e.d.

$k \sim k + 1$: This is just the same proof as in Lemma 1.5, since the only facts used here are $R^i q_* A = 0$ and $R^i q_* \mathcal{F}_j = 0$ for $0 \leq j \leq m$. q.e.d.

**Some exact sequences**

Recall the notations. $\hat{X} = X - X_r, X = Hom(T, T'), V = Gr(n - r, T)$ and $\hat{q}$ is the projection $\hat{q}: \hat{Z} \rightarrow \hat{X}$.

Let $\mathcal{V}$ be a vector bundle on $V$. Denote by $K_*(\mathcal{V})$ the tensor product of the Koszul complex $K$ by the vector bundle $\mathcal{V}$.

We will use Lemma 4.2 to show that the pushforward $\hat{q}_* K_*(\mathcal{V})$ under the projection $\hat{q}$ is exact up to a certain degree, depending on the cohomology of the vector bundle $\mathcal{V}$:

**Corollary 4.3.** Let $\mathcal{V}$ be a vector bundle on $V$ and suppose that

$$H^i(V, \Lambda^{i+l}(\mathcal{R} \otimes T'^*) \otimes \mathcal{V}) = 0$$

for $i > 0$ and all $0 \leq l \leq m$.

then $\hat{q}_* K_*(\mathcal{V})$ is exact up to degree $m - 1$, i.e.

$$\hat{q}_* (\Lambda^m (\mathcal{R} \otimes T'^*) \otimes \mathcal{V}) \rightarrow ... \rightarrow \hat{q}_* (\Lambda^1 (\mathcal{R} \otimes T'^*) \otimes \mathcal{V}) \rightarrow \Gamma(V, \mathcal{V}) \otimes O_{\hat{X}} \rightarrow 0$$

is exact.

**Proof.** This is a special case of Lemma 4.2, thus we only need to check that $R^i \hat{q}_* K_{l+1}(\mathcal{V}) = 0$ for all $i > 0$ and all $0 \leq l \leq m$. Consider the commutative
For all $0 \leq l \leq m$, the higher direct images of $K_\ast(V)$ are:

$$R^i\hat{q}_\ast(\hat{p}_\ast(\Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V})) \cong \hat{q}_\ast R^i\hat{p}_\ast(\Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \cong$$

$$H^i(V, \Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X} = \begin{cases} H^0(V, \Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}.$$  

Thus the higher direct images that are supposed to vanish, are zero, and by Lemma 4.2 $\hat{q}_\ast K_\ast(V)$ is exact up to degree $m - 1$. q.e.d.

Under the same assumption of the Corollary, note that $\hat{q}_\ast(\Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \cong H^i(V, \Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X}$. Thus $\hat{q}_\ast K_\ast(V)$ is given by:

$$H^0(V, \Lambda^i (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X} \to \ldots \to H^0(V, \Lambda^2 (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X} \to H^0(V, \Lambda^1 (\mathcal{R} \otimes T'') \otimes \mathcal{V}) \otimes O_\hat{X} \to H^0(V, \mathcal{V}) \otimes O_\hat{X} \to 0,$$

which is exact by the previous Corollary.

Next we consider the case, when $V$ is a projective space, to show that the subcomplex is exact. In particular, in this case $\mathcal{V} = O_{\mathbb{P}^{n-1}(j)}$ for some $j$. 

\[
\begin{align*}
\hat{Z} & \quad \hat{p} \quad \hat{q} \\
V & \quad \hat{X} \\
\hat{p}' & \quad \hat{q}' \\
\text{Spec } k
\end{align*}
\]
Two special sequences

Corollary 4.4. Suppose $V \cong \mathbb{P}(T) \cong \mathbb{P}^{n-1}$ and let $V = O_{\mathbb{P}^{n-1}}(j)$ for $j \geq 0$. Then

(a) $j \Lambda T^* \otimes O_{\hat{X}} \rightarrow T^* \otimes j^{-1} \Lambda T^* \otimes O_{\hat{X}} \rightarrow \text{Sym}_2(T^*) \otimes j^{-2} \Lambda T^* \otimes O_{\hat{X}} \rightarrow ...$

$\rightarrow \text{Sym}_{j-1}(T^*) \otimes T^* \otimes O_{\hat{X}} \rightarrow \text{Sym}_j(T^*) \otimes O_{\hat{X}} \rightarrow 0$

is exact.

(b) Over a point $\varphi = s \otimes f \in \hat{X} \subset X = \text{Hom}(T, T') \cong T^* \otimes T'$, the maps of this complex are given by:

$d_\varphi : \text{Sym}_k(T^*) \otimes j^{-k} \Lambda T^* \otimes k(\varphi) \rightarrow \text{Sym}_{k+1}(T^*) \otimes j^{-k+1} \Lambda T^* \otimes k(\varphi)$

$d_\varphi((s_1...s_k) \otimes (t_1 \wedge ... \wedge t_{j-k})) = s_1...s_k s \otimes \sum_{i \geq 0} (-1)^i f(t_i) t_1 \wedge ... \wedge t_i \wedge ... \wedge t_{j-k}$

Proof.

(a) This is a very special case of the previous Corollary. We need to check that the hypothesis are satisfied. The tautological bundle $\mathcal{R}$ in this case is equal to $O_{\mathbb{P}^{n-1}}(-1)$. The higher cohomology,

$H^i(i+l)(\mathcal{R} \otimes T^* \otimes V) = H^i(i+l)(O_{\mathbb{P}^{n-1}}(-1) \otimes T^*) \otimes O_{\mathbb{P}^{n-1}}(j))$

$\cong H^i(O_{\mathbb{P}^{n-1}}(j - (i + l)) \otimes i+l \Lambda T^* = 0 \text{ for } i > 0 \text{ and all } 0 \leq l \leq m.$

Thus the hypothesis of Corollary 4.2 are satisfied and for $i = 0$, the sections are given by:

$H^0(i)(\mathcal{R} \otimes T^* \otimes O_{\mathbb{P}^{n-1}}(j)) \cong H^0(O_{\mathbb{P}^{n-1}}(j - l) \otimes i \Lambda T^*)$

$\cong \text{Sym}_{j-l}(T^*) \otimes i \Lambda T^*$

(b) We need to trace the map through the isomorphism of

$H^0(j^{-k} \Lambda (O_{\mathbb{P}^{n-1}}(-1) \otimes T^*) \otimes O_{\mathbb{P}^{n-1}}(j))$

with $\text{Sym}_k(T^*) \otimes j^{-k} \Lambda T^*$
Set \( m = j - k \). Let us start with the map \( d \) of the Koszul complex \( \mathcal{K}_\mathcal{V} \):

\[
d : O_{\mathbb{P}^{n-1}}(-1) \otimes T'^* \to \Lambda^{m-1} (O_{\mathbb{P}^{n-1}}(-1) \otimes T'^*)
\]

\[
d((r_1 \otimes T_1) \wedge \ldots \wedge (r_m \otimes T_m))
\]

\[
= \sum_{i \geq 0} (-1)^i s(r_i) f(T_i)(r_1 \otimes T_1) \wedge \ldots \wedge (r_i \otimes T_i) \wedge \ldots \wedge (r_m \otimes T_m).
\]

Use the isomorphism of \( \Lambda^l (O_{\mathbb{P}^{n-1}}(-1) \otimes T'^*) \) with \( O_{\mathbb{P}^{n-1}}(-l) \otimes \Lambda^l (T'^*) \). Then the map \( d \) is given by:

\[
d : O_{\mathbb{P}^{n-1}}(-m) \otimes \Lambda^m (T'^*) \otimes k(\varphi) \to O_{\mathbb{P}^{n-1}}(-m - 1) \otimes \Lambda^{m-1} (T'^*) \otimes k(\varphi)
\]

\[
d((s_1 \ldots s_m) \otimes (T_1 \wedge \ldots \wedge T_m))
\]

\[
= \sum_{i,j \geq 0} (-1)^{i+j} s(r_j) f(T_i)(r_1 \ldots r_j \ldots r_m) \otimes (T_1 \wedge \ldots \wedge T_i \wedge \ldots \wedge T_m)
\]

\[
= r_1 \ldots r_m s \otimes \sum_{i \geq 0} (-1)^i f(T_i)(T_1 \wedge \ldots \wedge T_i \wedge \ldots \wedge T_m).
\]

The last equality follows from the isomorphism: \( O_{\mathbb{P}^{n-1}} \cong O_{\mathbb{P}^{n-1}}(-1) \otimes O_{\mathbb{P}^{n-1}}(1) \), which sends \( s \) to \( \Sigma s(t_i) \otimes t^i \) for a basis \( t_1 \ldots t_n \) of \( T \) and its dual basis \( t^1 \ldots t^n \).

This implies that the map of \( \mathcal{H}_s \mathcal{K}_\mathcal{V} \) is given by:

\[
d : O_{\mathbb{P}^{n-1}}(k) \otimes \Lambda^m (T'^*) \otimes k(\varphi) \to O_{\mathbb{P}^{n-1}}(k + 1) \otimes \Lambda^{m-1} (T'^*) \otimes k(\varphi)
\]

\[
d((s_1 \ldots s_k) \otimes (T_1 \wedge \ldots \wedge T_m))
\]

\[
= s_1 \ldots s_k s \otimes \sum_{i \geq 0} (-1)^i f(T_i)(T_1 \wedge \ldots \wedge T_i \wedge \ldots \wedge T_m).
\]

q.e.d.

Tensor this complex of the Corollary 4.4 by \( \Lambda^\text{top} (T') \), then this exact sequence is the sequence that we will use in the proof of the Main Theorem 3.2.
Corollary 4.5. Let $V \cong \mathbb{P}(T) \cong \mathbb{P}^{n-1}$ and set $\mathcal{V} = \mathcal{O}_{\mathbb{P}^{n-1}}(j)$ for $j \geq 0$.

(a) $\Lambda^{m-j} T' \otimes \mathcal{O}_X^* \rightarrow T^* \otimes \Lambda^{m-j+1} T' \otimes \mathcal{O}_X^* \rightarrow \text{Sym}_2(T^*) \otimes \Lambda^{m-j+2} T' \otimes \mathcal{O}_X^* \rightarrow \ldots$

\[ \rightarrow \text{Sym}_{j-1}(T^*) \otimes \Lambda^{m-j+1} T' \otimes \mathcal{O}_X^* \rightarrow \text{Sym}_j(T^*) \otimes \Lambda T' \otimes \mathcal{O}_X^* \rightarrow 0 \]

is exact.

(b) Over a point $\varphi = s \otimes f \in X \subset X = \text{Hom}(T,T') \cong T^* \otimes T'$, the maps of this complex are given by:

\[ d\varphi : \text{Sym}_k(T^*) \otimes \Lambda^{m-j+k} T' \otimes \mathbb{k}(\varphi) \rightarrow \text{Sym}_{k+1}(T^*) \otimes \Lambda^{m-j+k+1} T' \otimes \mathbb{k}(\varphi) \]

\[ d\varphi((s_1 \ldots s_k) \otimes (f_1 \wedge \ldots \wedge f_{m-j+k})) = s_1 \ldots s_k s \otimes f \wedge f_1 \wedge \ldots \wedge f_{m-j+k} \]

Proof. This follows from:

\[ \Lambda^{l} T'^* \otimes \Lambda^{top} T' \cong \Lambda^{top-l} T' \]

q.e.d.

We can now prove the Main Theorem, but first we discuss the example of the symplectic grassmannian of 2-planes in four space,

\[ X = \text{SpGr}(2,4) \).

CHAPTER V

THE SYMPLECTIC GRASSMANNIAN X OF TWO DIMENSIONAL PLANES IN FOUR DIMENSIONAL SPACE

Denote by $X$ the symplectic grassmannian of two-planes in the four dimensional vector space $V$ over an algebraically closed field $k$ of characteristic 0. We keep the notations of chapter 2 and 3.

In this chapter we will study the K-theory of $X$. Since $X$ is isomorphic to a quadric in $\mathbb{P}^4$ this case has been proven by Kapranov in [KII] and by Swan in [S] using the Clifford algebra of $2\Lambda V$.

Our approach here is different, since we discuss $X$ as a symplectic grassmannian inside a grassmannian, and not as a quadric inside a projective space. Therefore we also get a different generating system for the K-theory of $X$, but overall all of these approaches give similar algebraic descriptions of the K-theory.

Similar to the case of ordinary grassmannians, chapter 1, we will construct a finite resolution of $\mathcal{O}_\Delta$ and use this to show that

$$X = \{\mathcal{O}_X(-2), \mathcal{O}_X(-1), S, \mathcal{O}_X\}$$

is a generating system for the K-theory of $X$, that is:

**Theorem 5.1.**

$$K_*(SpGr(2,4)) \cong \bigoplus_{\mathcal{G} \in X} K_*(k)$$

**Proof.** The idea of the proof is similar to the proof in the case of ordinary grassmannians. We proof this in two steps. First we construct the resolution, then we use some cohomology computations in order to show that $X$ is a generating system.
The resolution

Consider the Tate construction in this case:

\[ \Sigma^{(2,2)} S \boxtimes \Phi_+ \to \Sigma^{(2,1)} S \boxtimes \Phi_- \]

\[ Sym_2 S \boxtimes \Lambda S^\perp \]

\[ S \boxtimes S^\perp \to O_{X \times X} \]

\[ \Lambda S \boxtimes \Psi \]

and consider the sub-complex:

\[ 2 \Lambda S \boxtimes \Psi \to S \boxtimes S^\perp \to O_{X \times X} \]

In this section we show that this sub-complex is exact and furthermore that the kernel is given by \( (\Lambda S)^{\otimes 2} \boxtimes \Lambda S^\perp \simeq O_{X \times X}(-2, -1) \), that is:

**Theorem 5.2.**

\[ 0 \to (\Lambda S)^{\otimes 2} \boxtimes \Lambda S^\perp \to 2 \Lambda S \boxtimes \Psi \to S \boxtimes S^\perp \to O_{X \times X} \]

is a finite resolution of \( O_\Delta \).

**Proof.** We split this proof into two parts. We show first that the sub-complex is exact, then we show that the kernel is equal to \( (\Lambda S)^{\otimes 2} \boxtimes \Lambda S^\perp \).

The sub-complex

Let us represent the representations of \( S \) and \( S^\perp \) by Young diagrams, i.e. let us represent \( \Sigma^\alpha S \boxtimes \Sigma^\beta S^\perp \) by the external tensor product of the Young diagrams corresponding to the weights \( \alpha \) and \( \beta \). For example:

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \boxtimes \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = Sym_2 S \boxtimes \Lambda S^\perp
\]

Over an open set \( U \), where all the \( \Psi_\alpha \)'s are trivial extensions, that is \( \Psi_\alpha \) is a direct sum over all \( \Sigma^{\alpha-2k} S^\perp \mid_U, k \geq 0 \), consider the resolution \( D \) of \( O_\Delta \) from chapter 2:
Recall that in order to show that the sub-complex is exact up to degree 1, it is enough to show that the quotient complex is exact up to degree 2, Lemma 3.3. In addition it is enough to show that the quotient complex is exact over points on the diagonal $\Delta \subset X \times X$, Lemma 3.4.

Consider the quotient complex over the open set $U$:

Over the open set $U$, the map $d$ of the quotient complex, splits into two kinds of maps, $ev$ and $\varphi$. Let us discuss these maps.

The maps $ev$ and $\varphi$. 

Let \((W_1, W_2)\) be a point in \(X \times X\) and let \(gP\) respectively \(hP\) be the parabolic leaving \(W_1\) respectively \(W_2\) invariant.

Recall the definitions of \(T_{(g,h)}^i\) and \(t_{(g,h)}\).

For \(g, h \in G, i > 0\), let

\[ T_{(g,h)} = (g, v_1 \wedge v_2) \boxtimes (h, \eta), \quad T_{(g,h)}^i = (g, (v_1 \wedge v_2)^{\otimes i}) \boxtimes (h, \eta^i) \]

and \(t_{(g,h)} = dT_{(g,h)}\).

Recall that an element in \(\Sigma^\alpha S \boxtimes \Psi_{\alpha^*}\) over the point \((gP, hP)\) can be written uniquely as a sum of elements of the form

\[ wT_{(g,h)}^i, \quad w \in (\Sigma^{(\alpha_1-i, \alpha_2-i)}S)_{gP} \boxtimes (\Sigma^{(\alpha^*-2i)}S^\perp)_{hP}. \]

The map \(d\) is defined as:

\[ d(wT_{(g,h)}^i) = (dw)T_{(g,h)}^i + (-1)^{|w|}i w \cdot t_{(g,h)} T_{(g,h)}^{i-1}. \]

This translates to the two maps \(ev\) and \(\varphi\) as follows.

Let \(\beta = (\alpha_1 - i, \alpha_2 - i)^*\).

\[ \Sigma^{\alpha-1}W_1 \boxtimes \Sigma^{\beta-1}(V/W_2)^* \xrightarrow{ev} \]

\[ d : \Sigma^\alpha W_1 \boxtimes \Sigma^\beta (V/W_2)^* \oplus \]

\[ \xleftarrow{\varphi} \]

\[ \Sigma^{\alpha-1}W_1 \boxtimes \Sigma^{\beta+1}(V/W_2)^* \]

The map \(ev\) is induced by the evaluation map or

\[ ev((w_1 \wedge w_2)^{\otimes i} \otimes w) = (w_1 \wedge w_2)^{\otimes i} \otimes dw \]
and
\[ \varphi((w_1 \wedge w_2)^{\otimes i} \otimes w) = (-1)^{|w|i}(w_1 \wedge w_2)^{\otimes i-1}(w \cdot t_{(g,h)}) . \]

Over a point $W \times W$ on the diagonal $\Delta$, all the evaluation maps $ev$ are zero, since for $v \in W$ and $f \in (V/W)^*$, $f(v) = 0$. Therefore the quotient complex splits into several subsequences.

Hence to show that the quotient complex is exact in degree 2, it is enough to show that over a point $W \times W \in \Delta$:

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
\boxtimes \\
\end{array}
\end{array}
\end{array} \]

is surjective.

This on the other hand is a special case of our sequences in chapter 4, Corollary 4.5. Recall the definition of $Y$:

Some exact sequences .

Let $W \times W$ be a point on the diagonal inside of $X \times X$. Let $Y = Hom(T, T')$, $Y_r = \{ x \in Y \mid \text{rank}(x) \leq r \}$ and $\hat{Y} = Y - Y_r$. Set $V = Gr(n-r, T)$, $T = W^*$ and $T' = (V/W)^*$. 
Then $Y = \text{Hom}(W^*, (V/W)^*)$. Consider the case when $r = 1$, that is when $V$ is equal to the projective space $\mathbb{P}(W^*) \cong \mathbb{P}^1$.

**Lemma 5.3.** Let $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(2)$, then
\[
\mathcal{O}_{\hat{Y}} \xrightarrow{d} W \otimes (V/W)^* \otimes \mathcal{O}_{\hat{Y}} \xrightarrow{d} \text{Sym}_2(W) \otimes \Lambda(V/W)^* \otimes \mathcal{O}_{\hat{Y}} \rightarrow 0
\]
is exact and over a point $x \in \hat{Y}$, $d_x$ is given by
\[
d_x(w_1 w_2 \ldots w_m \otimes f_1 \wedge \ldots \wedge f_l) = w_1 w_2 \ldots w_m x \wedge f_1 \wedge \ldots \wedge f_l
\]

Proof.

Recall the statement of Corollary 4.5, using $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(j)$:
\[
\begin{align*}
\Lambda^{m-j} T' \otimes \mathcal{O}_{\hat{Y}} & \rightarrow T^* \otimes \Lambda^{m-j+1} T' \otimes \mathcal{O}_{\hat{Y}} \rightarrow \text{Sym}_2(T^*) \otimes \Lambda^{m-j+2} T' \otimes \mathcal{O}_{\hat{Y}} \rightarrow \\
& \cdots \rightarrow \text{Sym}_{j-1}(T^*) \otimes \Lambda^{m-1} T' \otimes \mathcal{O}_{\hat{Y}} \rightarrow \text{Sym}_j(T^*) \otimes \Lambda T' \otimes \mathcal{O}_{\hat{Y}} \rightarrow 0
\end{align*}
\]
is exact.

Over a point $x = s \otimes f \in \hat{Y} \subset Y = \text{Hom}(T, T') \cong T^* \otimes T'$, the maps of this complex are given by:
\[
d_x : \text{Sym}_k(T^*) \otimes \Lambda^{m-j+k} T^* \otimes \mathbb{k}(\varphi) \rightarrow \text{Sym}_{k+1}(T^*) \otimes \Lambda^{m-j+k+1} T^* \otimes \mathbb{k}(\varphi)
\]
\[
d_x((s_1 \ldots s_k) \otimes (f_1 \wedge \ldots \wedge f_{m-j+k})) = s_1 \ldots s_k \cdot s \otimes f \wedge f_1 \wedge \ldots \wedge f_{m-j+k}
\]

In particular for $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(2)$, we get the exact complex
\[
\Lambda^2(V/W) \otimes \mathcal{O}_{\hat{Y}} \rightarrow W \otimes V/W \otimes \mathcal{O}_{\hat{Y}} \rightarrow \text{Sym}_2(W) \otimes \mathcal{O}_{\hat{Y}} \rightarrow 0
\]

Twist this last complex by $\Lambda^2(V/W)^*$, then
\[
\mathcal{O}_{\hat{Y}} \xrightarrow{d} W \otimes (V/W)^* \otimes \mathcal{O}_{\hat{Y}} \xrightarrow{d} \text{Sym}_2(W) \otimes \Lambda^2(V/W)^* \otimes \mathcal{O}_{\hat{Y}} \rightarrow 0
\]
is exact.

q.e.d.
Exactness of the sub-complex. 

Choose \( g \in G \) in such a way that the parabolic \( gP \) preserves the space \( W \). We will set \( x = t \) with \( t = t_{(g, g)} \) in the above sequence.

Let \( e_1, ..., e_4 \) be a symplectic basis of \( V \), and \( y_1, ..., y_4 \) the corresponding dual basis of \( V^* \), such that \( W = e_1 \wedge e_2 \) and \( g \cdot \eta = y_1 \otimes y_3 + y_2 \otimes y_4 \).

Recall that \( \eta \) is defined as \( \eta = x_1 \otimes x_3 + x_2 \otimes x_4 \).

Denote by \( d \) the map from the Tate construction. Then

\[
t = dT_{(g, g)}(g, g) = d((g, v_1 \wedge v_2) \boxtimes (g, \eta)) = d((e_1 \wedge e_2) \boxtimes (y_1 \otimes y_3 + y_2 \otimes y_4)) = y_1(e_1) e_2 \boxtimes y_3 - y_1(e_2) e_1 \boxtimes y_3 + y_2(e_1) e_2 \boxtimes y_4 - y_2(e_2) e_1 \boxtimes y_4
\]

Set \( x = t = e_2 \boxtimes y_3 - e_1 \boxtimes y_4 \). Then \( x \in Y \) and the rank of \( x \) is equal to 2, because \( x(y_1) = -y_4, x(y_2) = y_3 \) and \( y_3, y_4 \) are linearly independent. Thus \( x \in \wedge Y \), because \( x \notin Y_1 \).

Now evaluate the complex of Lemma 4.3 at \( x \in \wedge Y \):

\[
O_{\wedge Y, x} \xrightarrow{dx} W \otimes (V/W)^* \otimes O_{\wedge Y, x} \xrightarrow{dx} \text{Sym}_2(W) \otimes \overset{2}{\Lambda}(V/W)^* \otimes O_{\wedge Y, x} \to 0
\]

In terms of Young diagrams:

\[
\vdots \xrightarrow{dx} \begin{ytableau} 1 
\end{ytableau} \xrightarrow{dx} \begin{ytableau} 2 
1
\end{ytableau} \xrightarrow{dx} \begin{ytableau} 3 
1
2
\end{ytableau} \xrightarrow{dx} 0
\]

This is almost the sequence we want, except for a twist. Recall the definition of \( \varphi \):

\[
\varphi((e_1 \wedge e_2) \otimes v \boxtimes f) = -(v \boxtimes f) \cdot t = dx(v \boxtimes f)
\]
Hence the difference between \( \varphi \) and \( d_x \) is just a twist by \( \mathcal{O}_{\hat{Y}}(-1) \), which is trivial over the affine variety \( \hat{Y} \).

Therefore

\[
\begin{array}{ccccccccc}
\bigotimes & \xrightarrow{\varphi} & \bigotimes & \xrightarrow{\varphi} & \bigotimes & \rightarrow & 0
\end{array}
\]

is exact, which is just what we needed in order to show that the sub-complex is exact.

q.e.d.

**The Kernel:**

First we define a map \( \vartheta \) from \( K = (\Lambda^2 S) \bigotimes \Lambda^2 S^\perp \cong \mathcal{O}_{X \times X}(-2, -1) \) to the sub-complex and show that this map is well defined and nonzero. Then we discuss the quotient complex and show that the kernel of the sub-complex is indeed isomorphic to \( K \), which then implies the exactness of

\[
0 \rightarrow (\Lambda^2 S)^\bigotimes \bigotimes \Lambda^2 S^\perp \xrightarrow{\vartheta} \Lambda^2 S \bigotimes \Psi \xrightarrow{d} S \bigotimes S^\perp \xrightarrow{d} \mathcal{O}_{X \times X}.
\]

**Definition of the map \( \vartheta \).**

Define

\[
\vartheta : (\Lambda^2 S)^\bigotimes \bigotimes \Lambda^2 S^\perp \rightarrow \Lambda^2 S \bigotimes (V^* \otimes S^\perp)
\]

\[
\vartheta(s_1 \wedge s_2 \otimes v \wedge w \bigotimes f \wedge g) = s_1 \wedge s_2 \bigotimes \left( \begin{array}{c}
[g(w) < v, > -g(v) < w, >] \otimes f \\
-f(w) < v, > -f(v) < w, >] \otimes g
\end{array} \right).
\]

This is certainly well defined and since \( v, w, s_1 \) and \( s_2 \) are all in \( S_{W_1} = W_1 \), it follows that \( d(\vartheta(s_1 \wedge s_2 \otimes v \wedge w \bigotimes f \wedge g)) = 0 \) in \( S \bigotimes S^\perp \). Set

\[
\varrho = \left( \begin{array}{c}
[g(w) < v, > -g(v) < w, >] \otimes f \\
-f(w) < v, > -f(v) < w, >] \otimes g
\end{array} \right).
\]

We need to check that the Image of \( \vartheta \) is contained in \( \Lambda S \bigotimes \Psi \).
Remark. Let us discuss an alternative definition of the extension $\Psi$. Recall that $\Psi \subset V^* \otimes S^1$, Lemma 2.3, Remark (1), is an extension of $\text{Sym}_2 S^1$ and $\mathcal{O}_X$. Consider the exact sequence:

$$0 \to \text{Sym}_2 S^1 \to V^* \otimes S^1 \xrightarrow{\psi} \Lambda^2 V^*$$

Set $\delta = (x_1 \wedge x_3 + x_2 \wedge x_4)$. Then $\Psi$ is the pullback of $\delta \cdot \mathcal{O}_X$ under the map $\psi$.

Thus we have to show that $\psi(\rho) \in \delta \mathcal{O}_X$.

We can do this calculation fiberwise. Over a fiber $W_1 \times W_2$, we choose a symplectic basis of $V : \{e_1, ..., e_4\}$, and its dual basis $\{y_1, ..., y_4\}$, in such a way that $W_2 = e_1 \wedge e_2$. Without loss of generality, we may assume $f = y_3$ and $g = y_4$. Let $v = \Sigma a_i e_i$ and $w = \Sigma b_i e_i$. Then

$$f(v) = a_3, \quad g(v) = a_4, \quad <v, > = a_1 y_3 + a_2 y_4 - a_3 y_1 - a_4 y_2,$$

$$f(w) = b_3, \quad g(w) = b_4, \quad <w, > = b_1 y_3 + b_2 y_4 - b_3 y_1 - b_4 y_2.$$ 

And $<v, w> = 0$ is equivalent to $a_1 b_3 + a_2 b_4 - a_3 b_1 - a_4 b_2 = 0$, thus $a_1 b_3 - a_3 b_1 = -(a_2 b_4 - a_4 b_2)$. Finally:

$$\psi(\rho) = [g(w) < v, > - g(v) < w, >] \wedge f - [f(w) < v, > - f(v) < w, >] \wedge g$$

$$= [b_4 < v, > - a_4 < w, >] \wedge y_3 - [b_3 < v, > - a_3 < w, >] \wedge y_4$$

$$= [b_4 [a_1 y_3 + a_2 y_4 - a_3 y_1 - a_4 y_2] - a_4 [b_1 y_3 + b_2 y_4 - b_3 y_1 - b_4 y_2]] \wedge y_3$$

$$- [b_3 [a_1 y_3 + a_2 y_4 - a_3 y_1 - a_4 y_2] - a_3 [b_1 y_3 + b_2 y_4 - b_3 y_1 - b_4 y_2]] \wedge y_4$$

$$= (b_4 a_2 - a_4 b_2) y_4 \wedge y_3 - (b_3 a_1 - a_3 b_1) y_3 \wedge y_4$$

$$+ (-b_4 a_4 + a_4 b_4) y_2 \wedge y_3 - (-b_3 a_3 + a_3 b_3) y_1 \wedge y_4$$

$$+ (-b_4 a_3 + a_4 b_3) y_1 \wedge y_3 - (-b_3 a_4 + a_3 b_4) y_2 \wedge y_4$$

$$= (a_2 b_3 - b_4 a_3) (y_1 \wedge y_3 + y_2 \wedge y_4)$$

$$= (a_2 b_3 - b_4 a_3) \delta$$

Thus the map from $K$ to the sub-complex is well defined and nonzero. Next we show that $K$ is indeed isomorphic to the kernel.
\[ K \cong \text{kernel}. \]

First we show that we can compute the kernel from the quotient complex:

**Lemma 5.4.** Let \( D \) be an exact complex, \( B \subset D \) a sub-complex, and \( Q = D / B \) the quotient complex. Suppose \( Q_j = D_j \) for \( j \geq m + 3 \), and \( Q \) exact up to degree \( m + 1 \). Then

\[
\text{kernel} (B_{m+1} \to B_m) = H_{m+2}(Q).
\]

**Proof.** This can be proven by a diagram chase. Consider the following diagram:

\[
\begin{array}{cccccccc}
0 & \to & K & \to & B_{m+1} & \to & B_m & \to & \ldots \\
& & \cap & \cap & & & & \\
\ldots & \to & D_{m+3} & \to & D_{m+2} & \to & D_{m+1} & \to & D_m & \to & \ldots \\
& & \| & \| & \downarrow & \downarrow & & & & \\
\ldots & \to & Q_{m+3} & \to & Q_{m+2} & \to & Q_{m+1} & \to & Q_m & \to & \ldots \\
& & \uparrow & \uparrow & \uparrow & & & & & & \\
Q. \text{exact} & Q. \text{not exact} & Q. \text{exact}
\end{array}
\]

Let \( K = \text{kernel} (B_{m+1} \to B_m) \) and \( H = H_{m+2}(Q) \). Then

\[
H = \text{ker}(Q_{m+2} \to Q_{m+1})/\text{im}(Q_{m+3} \to Q_{m+2})
\]

\[
\cong \text{ker}(D_{m+2} \to Q_{m+1})/\text{ker}(D_{m+2} \to D_{m+1}).
\]

For \( b \in K \), there exist a \( c \in D_{m+2} \), unique up to \( \text{ker}(D_{m+2} \to D_{m+1}) \). This defines a map \( f : K \to H \). On the other hand if \( c \in \text{ker}(D_{m+2} \to Q_{m+1}) \), then \( dc \in K \), and if \( c \in \text{ker}(D_{m+2} \to D_{m+1}) \), then \( dc = 0 \). This defines a map \( g : H \to K \). And since \( g \circ f = \text{id}_K \), it follows that \( H \cong K \).

q.e.d.
This implies for our computation:

\[ \ker(\Lambda^2 S \otimes \Psi \xrightarrow{d} S \otimes S^\perp) \cong H_3(Q.) = \ker(Q_3 \to Q_2)/\text{im}(Q_4 \to Q_3). \]

Thus all we need to show is that the rank of \( H_3(Q.) \) is equal to the rank of \( K \) which is equal to 1. Since the quotient complex resolves the vector bundle \( Sym^2 S \otimes \Lambda^2 S^\perp \), the rank of \( H_3(Q.) \) is a constant. Therefore it is enough to compute the rank of the kernel over a point \( W \times W \) on the diagonal inside of \( X \times X \):

Consider once more the quotient complex over a point \( W \times W \):

\[
\begin{array}{ccc}
\text{deg 4} & \text{deg 3} & \text{deg 2} \\
\vdots & \vdots & \vdots \\
& \oplus & \oplus \\
& \oplus & \oplus \\
& \vdots & \vdots \\
\end{array}
\]

Since \( \begin{array}{c}
\begin{array}{c}
\text{deg 4} \ \rightarrow \\
\text{deg 3} \\
\text{deg 2} \\
\end{array}
\end{array} \rightarrow 0 \) is exact, see Lemma 5.3, the rank of the kernel is equal to the rank of

\[ \ker(\begin{array}{c}
\begin{array}{c}
\text{deg 4} \ \rightarrow \\
\text{deg 3} \\
\end{array}
\end{array} \rightarrow 0)/\text{im}(\begin{array}{c}
\begin{array}{c}
\text{deg 4} \ \rightarrow \\
\text{deg 3} \\
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\text{deg 2} \\
\end{array}
\end{array}) \]
We again use Corollary 4.5: Set $\mathcal{V} = \mathcal{O}_\mathcal{X}(1), x = t_{(g,g)}$ as in the section on the sub-complex, then

$$d_x : \begin{array}{c|c} & \\ \hline & \end{array} \cong \begin{array}{c|c} & \\ \hline & \end{array}$$ is an isomorphism.

Tensor this on the right by $\begin{array}{c} & \\ \hline & \end{array}$, then

$$d_x(1) : \begin{array}{c|c} & \\ \hline & \end{array} \oplus \begin{array}{c} & \\ \hline & \end{array} \cong \begin{array}{c|c} & \\ \hline & \end{array} \begin{array}{c|c} & \\ \hline & \end{array}$$ is an isomorphism.

Recall the definition of $\varphi$:

$$\begin{array}{c|c} & \\ \hline & \end{array} \begin{array}{c} \varphi \end{array} \begin{array}{c|c} & \\ \hline & \end{array} \cong \begin{array}{c|c} & \\ \hline & \end{array} \begin{array}{c|c} & \\ \hline & \end{array}$$

$$\varphi((s_1 \wedge s_2) \otimes (t_1 \wedge t_2) \boxtimes f \cdot g) = (s_1 \wedge s_2) \otimes (t_1 \wedge t_2) \cdot t \otimes f \cdot g$$

$$= d_x((t_1 \wedge t_2) \boxtimes f \cdot g)$$

Hence the difference between $\varphi$ and $d_x$ is just a twist by $\mathcal{O}_{\hat{\mathcal{Y}}}(-1)$, which is trivial over the affine variety $\hat{\mathcal{Y}}$.

Therefore
and thus

$$H_3(Q|_{W \times W})$$

\[ \cong \begin{array}{c} \oplus \end{array} / \text{im}(\begin{array}{c} \oplus \\ \rightarrow \end{array}) \]

\[ \cong (\begin{array}{c} \oplus \\ \oplus \end{array}) / \begin{array}{c} \oplus \\ \oplus \end{array} \]

\[ \cong \begin{array}{c} \oplus \end{array} \]

Note that this has rank one and it is clearly isomorphic to $K|_{W \times W}$. This completes the proof of Theorem 5.2.

q.e.d.

Next we discuss the K-theory of $X$.

**K-theory of $X$**

Let $\mathcal{P}$ be the category of vector bundles over $k$, and $\mathcal{C}_X$ be the category of locally free coherent sheaves on $X$. Define the subcategories $\mathcal{C}(X^*)$, $\mathcal{C}(Y)$ as follows:

$$\mathcal{C}(X^*) = \{ \mathcal{F} \in \mathcal{C}_X | H^i(X, \mathcal{F} \otimes \mathcal{G}^*) = 0 \text{ for all } \mathcal{G} \in \mathcal{X} \}$$

$$\mathcal{C}(Y) = \{ \mathcal{F} \in \mathcal{C}_X | H^i(X, \mathcal{F} \otimes \mathcal{G}) = 0 \text{ for all } \mathcal{G} \in \mathcal{Y} \}$$

We will use the same standard trick, that we used in the case of ordinary grassmannians, to construct resolutions for all sheaves $\mathcal{F} \in \mathcal{C}(Y)$. Denote by $p_1$ and $p_2$ the projections to the first and second factor:

$$p_i : X \times X \to X,$$

and denote by $q : X \to \text{Spec}(k)$ the structure map. Then

$$R^i p_2_*(p_1^* \mathcal{F} \otimes \mathcal{O}_\Delta) \cong \begin{cases} \mathcal{F} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$
Lemma 5.5.

For $\mathcal{F} \in \mathcal{C}(\mathcal{Y})$, $R^i p_1^*(p_2^* \mathcal{F} \otimes \mathcal{B})$ gives rise to a finite resolution of $\mathcal{F}$:

$$0 \to H^0(X, \mathcal{F} \otimes (\mathcal{L} \otimes \mathcal{S}) \otimes \mathcal{S} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \Psi) \to H^0(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \Psi) \to H^0(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \Psi) \to 0$$

Proof.

Recall that $\mathcal{B}$ is the sub-complex resolving $\mathcal{O}_\Delta$. For $\mathcal{F} \in \mathcal{C}(\mathcal{Y})$, the higher direct images of $(p_2^* \mathcal{F} \otimes \mathcal{B})$ all vanish, since

$$R^i p_2^*(p_1^* \mathcal{F} \otimes (\mathcal{S} \otimes \mathcal{S} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \Psi)) \cong H^i(\mathcal{F} \otimes \mathcal{S} \otimes \mathcal{S} \otimes \mathcal{L} \otimes \mathcal{S} \otimes \Psi).$$

Since all the higher direct images vanish, it follows that the push-forward

$$p_2^*(p_1^* \mathcal{F} \otimes \mathcal{B})$$

is exact, see also Lemma 1.3.

$q.e.d.$

Similar to the case of ordinary grassmannians, define

$$U_{\mathcal{G}} : \mathcal{P} \to \mathcal{C}_X, \quad U_{\mathcal{G}}(W) = q^* W \otimes \mathcal{G}$$

for all $\mathcal{G} \in \mathcal{X}$. Then the $U_{\mathcal{G}}$’s are inducing homomorphisms $u_{\mathcal{G}}$ on the K-theory, i.e.

$$u_{\mathcal{G}} : K_*(\mathbb{k}) \to K_*(X)$$

Set $u = \bigoplus_{\mathcal{G} \in \mathcal{X}} u_{\mathcal{G}}$, then

Theorem 5.6.

$$u : \bigoplus_{\mathcal{G} \in \mathcal{X}} K_*(\mathbb{k}) \to K_*(X)$$

is an isomorphism.
Proof.

First we show that \( u \) is surjective:

Recall that it is enough to define the K-theory on the subcategory \( \mathcal{C}(\mathcal{Y}) \), or \( \mathcal{C}(\mathcal{X}^*) \), Lemma 1.6. Consider the finite resolution of \( F \in \mathcal{C}(\mathcal{Y}) \), then

\[
[F] = [H^0(F) \otimes \mathcal{O}_X] - [H^0(F \otimes S^\perp) \otimes S]
+ [H^0(F \otimes \Psi) \otimes \Lambda^2 S] - [H^0(F \otimes \Lambda S^\perp) \otimes (\Lambda S)^{\otimes 2}]
\]

which implies that \( u \) is surjective.

To show that \( u \) is injective we define a homomorphism

\[
v : K_*(X) \to \bigoplus_{G \in \mathcal{X}} K_*(k).
\]

and then show that \( v \circ u \) is injective, consequently that \( u \) is an isomorphism.

It suffices to define \( v \) on the subcategory \( \mathcal{C}(\mathcal{X}^*) \) rather than defining it on the whole category \( \mathcal{C}_X \), Lemma 1.7.

Define

\[
V_G : \mathcal{C}(\mathcal{X}^*) \to \mathcal{P}, \quad V_G(F) = q_*(F \otimes G^*) \text{ for all } G \in \mathcal{X}.
\]

Note that \( V_G \) on the category \( \mathcal{C}_X \) is not exact, because \( R^i q_*(F \otimes G^*) \) are in general not zero for \( i > 0 \), but the \( V_G \)'s are exact on the subcategory \( \mathcal{C}(\mathcal{X}^*) \). The \( V_G \)'s induce homomorphisms \( v_G \) on the K-theory.

\[
v_G : K_*(X) \to K_*(k).
\]

Let us give an ordering to \( \mathcal{X} \):

\[
\mathcal{X} : \mathcal{O}_X(-2) \leq \mathcal{O}_X(-1) \leq S \leq \mathcal{O}_X.
\]

Let \( v = (\ldots, v_G, \ldots) \), then consider \( v \circ u \):
\[ V_H \circ U_G(W) = V_H(q^*W \otimes G) = [H^0(W \otimes G \otimes (H)^*)] \]

We will show that the matrix of \( v \circ u \) with respect to the ordering of \( \mathcal{X} \) is upper triangular with ones down the diagonal, thereby showing that \( u \) is an isomorphism. Thus we need to show that \( \mathcal{X} \) is orthogonal, that is:

**Lemma 5.7.**

For all \( G, H \in \mathcal{X} \):

\[
\text{Ext}^i(H, G) = \begin{cases} 
  k & \text{for } G = H \text{ and } i = 0 \\
  0 & \text{for } G < H \text{ and } i = 0 \\
  0 & \text{for all } G, H \in \mathcal{X} \text{ and } i > 0
\end{cases}
\]

**Proof.** We need some cohomology computations.

### Cohomology Computations

Here we use the fact that the symplectic grassmannian \( X = \text{SpGr}(2, 4) \) sits inside the grassmannian \( Gr(2, 4) \). Therefore we can apply the cohomology computations for the grassmannian \( Gr(2, 4) \) to \( X \):

Let \( F = \{0 \subset W_1 \subset W_2 \subset ... \subset W_N = V \text{ with dimension } W_i = i\} \) be the full flag manifold and \( \pi \) the projection \( \pi : F \to Gr(2, V) \). Consider the short exact sequence:

\[
0 \to O_{Gr}(-X) \to O_{Gr} \to O_X \to 0.
\]

Recall that \( O_{Gr}(-X) \cong O_{\mathbb{P}(\Lambda V^*)}(-1) \cong \Lambda S = \pi_* O_F(-1, -1, 0, ..., 0) \).

Tensor the above short exact sequence by a vector bundle \( V \) on the grassmannian \( Gr(2, 4) \) and consider the resulting long cohomology sequence:

\[
0 \to H^0(Gr, V \otimes O_{Gr}(-X)) \to H^0(Gr, V) \to H^0(X, V \otimes O_X) \\
\to H^1(Gr, V \otimes O_{Gr}(-X)) \to H^1(Gr, V) \to H^1(X, V \otimes O_X) \to ...
\]

(*)
In particular we see from here, that if $\mathcal{V}$ has no cohomology, then

$$H^i(X, \mathcal{V} \otimes \mathcal{O}_X) \cong H^{i+1}(Gr, \mathcal{V}(-X)).$$

Recall that $\Sigma^a S_{Gr} \cong \pi_* \mathcal{O}_F(-a_2, -a_1, 0, 0)$ and also recall if $\gamma$ is not a dominant weight and if $\gamma + \rho$ with $\rho = (4, 3, 2, 1)$, has a repetition, then $\mathcal{O}_F(\alpha)$ has no cohomology.

Although we have done most of the cohomology computations, that are needed here, in chapter 1, we will do these computations again in detail:

**Lemma 5.8.**

$$H^i(X, \mathcal{O}_X(k)) = \begin{cases} k & k = 0, \quad i = 0 \\ 0 & k = -1, -2, \quad i = 0 \\ 0 & k \geq -2, \quad i > 0 \end{cases}$$

Proof. Recall that $\mathcal{O}_{Gr}(k) \cong \pi_* (\mathcal{O}_F(k, k, 0, 0))$. The cohomology of these twisted sheaves is given by:

$$H^i(Gr, \mathcal{O}_{Gr}(k)) = \begin{cases} k & k = 0, \quad i = 0 \\ 0 & k = -1, -2, -3, \quad i = 0 \\ 0 & k \geq -3, \quad i > 0 \end{cases}$$

This follows from:

$$(-j, -j, 0, 0) + \rho = (4 - j, 3 - j, 2, 0)$$

has a repetition for $j = 1, 2, 3$, thus $\mathcal{O}_{Gr}(-j)$ has no cohomology for $j = 1, 2, 3$. and for $k > 0 : (k, k, 0, 0)$ is dominant, thus $\mathcal{O}_{Gr}(k)$ has only $H^0$.

The statement follows now from the long exact cohomology sequence (*).

q.e.d.

Let us discuss the terms in Lemma 5.7 involving $S$:
Lemma 5.9.

(a) \( H^i(X, S(k)) = 0 \) for \( k = 0, 1, 2 \) and \( i > 0 \),
(b) \( H^i(X, S^*(-k)) = 0 \) for \( k = 0, 1, 2 \) and \( i > 0 \),
(c) \( H^i(X, S \otimes S^*) = 0 \) for \( i > 0 \),
(d) \( H^0(X, S \otimes S^*) = k \)
(e) \( H^0(X, S) = 0 \)

Proof.

(a),(e) \( S_{Gr}(-1) \cong \pi_* \mathcal{O}_F(-1, -2, 0, 0) \)
\( S_{Gr} \cong \pi_* \mathcal{O}_F(0, -1, 0, 0) \)
\( S_{Gr}(1) \cong \pi_* \mathcal{O}_F(1, 0, 0, 0) \)
\( S_{Gr}(2) \cong \pi_* \mathcal{O}_F(2, 1, 0, 0) \)

After adding \( \rho = (4, 3, 2, 1) \) to the first two weights, these have repetitions and thus \( S_{Gr}(-1) \) and \( S_{Gr} \) have no cohomology. The last two weights are dominant and thus \( S_{Gr}(1) \) and \( S_{Gr}(2) \) have only \( H^0 \). Using the long exact cohomology sequence (*), we get that \( S \) has no cohomology and \( S(1) \) and \( S(2) \) have no higher cohomology.

(b) \( S^*_r(-3) \cong \pi_* \mathcal{O}_F(-2, -3, 0, 0) \)
\( S^*_r(-2) \cong \pi_* \mathcal{O}_F(-1, -2, 0, 0) \)
\( S^*_r(-1) \cong \pi_* \mathcal{O}_F(0, -1, 0, 0) \)
\( S^*_r \cong \pi_* \mathcal{O}_F(1, 0, 0, 0) \)

The first three weights have repetitions after adding \( \rho \) to them, thus they have no cohomology. The last weight is dominant and thus has only \( H^0 \). The claim again follows from the long exact cohomology sequence (*).
(c),(d) \[ S_{Gr} \otimes S_{Gr}^{*}(-1) \cong S_{Gr} \otimes S_{Gr} \cong 2S_{Gr} \oplus Sym_{2}S_{Gr} \]
\[ \cong \pi_{*}\mathcal{O}_{F}(-1, -1, 0, 0) \oplus \pi_{*}\mathcal{O}_{F}(0, -2, 0, 0). \]

Since \((0, -2, 0, 0) + \rho = (4, 1, 2, 1)\) has a repetition, \(S_{Gr} \otimes S_{Gr}^{*}(-1)\) has no cohomology.

\[ S_{Gr} \otimes S_{Gr}^{*} \cong \mathcal{O}_{X} \oplus Sym_{2}S_{Gr}(1) \cong \pi_{*}\mathcal{O}_{F} \oplus \pi_{*}\mathcal{O}_{F}(1, -1, 0, 0). \]

Since \((1, -1, 0, 0) + \rho = (5, 2, 2, 1)\) has a repetition, \(S_{Gr} \otimes S_{Gr}^{*}\) has only \(H^{0} \cong k \cong H^{0}(\pi_{*}\mathcal{O}_{F}).\)

Using the long exact cohomology sequence (*), it follows that \(S \otimes S^{*}\) over \(X\) has only \(H^{0} \cong k.\)

Hence \(\mathcal{X}\) is orthogonal.

q.e.d.

**Proof of Lemma 5.7.** Recall that \(Ext^{i}(\mathcal{F}, \mathcal{G}) \cong H^{i}(\mathcal{X}, \mathcal{F}^{*} \otimes \mathcal{G}).\) Lemma 5.7 then follows from Lemma 5.8 and 5.9. q.e.d.

**Proof of Theorem 5.6.** Consider the matrix of \(v \circ u\) with respect to the ordering of \(\mathcal{X}:\)

\[ V_{\mathcal{H}} \circ U_{\mathcal{G}}(W) = V_{\mathcal{H}}(q^{*}W \otimes \mathcal{G}) = [H^{0}(W \otimes \mathcal{G} \otimes (\mathcal{H}^{*})] \]

Lemma 5.7 implies that
(a) \(V_{\mathcal{H}} \circ U_{\mathcal{G}}\) is well defined for \(\mathcal{G}, \mathcal{H} \in \mathcal{X},\) since \(\mathcal{X} \subset C(\mathcal{X}^{*})\) and
(b) it implies that this matrix with respect to the ordering of \(\mathcal{X}\) is upper triangular with ones down the diagonal.

This proofs that \(u\) is injective, thus that \(u\) is an isomorphism.

q.e.d.
CHAPTER VI

PROOF OF THE MAIN THEOREM

In this chapter we will finish the proof of the Main Theorem 3.2. Let us first recall some notations and the statement of the main theorem.

**Notations**

Let $X = \text{SpGr}(2,V)$ be the symplectic grassmannian of 2-planes in $V$, and let $V$ be a vector space of dimension $N = 2n$ over an algebraically closed field $k$ of characteristic 0. We keep the notations of chapter 2 and 3.

Let $D$ be the complex of the Tate construction that was defined in chapter 2, thus

$$D_i = \bigoplus_{N-2\geq\alpha_1 \geq \alpha_2 \geq 0} \Sigma^\alpha S \boxtimes \Psi_{\alpha^*}.$$  

and denote by $B_i$ the sub-complex:

$$B_i : B_{2N-6} \to ... \to B_2 \to S \boxtimes S^\perp \to \mathcal{O}_{X \times X},$$

where $B_i \subset D_i, i \leq 2N - 6$ is defined as the direct sum of $\Sigma^\alpha S \boxtimes \Psi_{\alpha^*}$ over all $\alpha$ of length $i$ with $\alpha_1 \neq N - 2$.

**Theorem 6.1.**

The sub-complex $B_i$ is exact.

**Remark.**

We have proven this theorem for $X = \text{SpGr}(2,4)$. In this case the sub-complex is given by

$$2 \Lambda S \boxtimes \Psi \xrightarrow{d} S \boxtimes S^\perp \to \mathcal{O}_{X \times X}.$$
Moreover for $X = \text{SpGr}(2, 4)$ we not only proved the exactness of this complex, but we also found the kernel of the map $d : \Lambda^2 S \boxtimes \Psi \rightarrow S \boxtimes S^\perp$, which is equal to $O_X(-2) \boxtimes O_X(-1)$. This enabled us to find the generating system

$$\mathcal{X} = \{O_X(-2), O_X(-1), S, O_X\}$$

for the $K$-Theory of $X$. In the general case we will only show the exactness of the sub-complex.

**Proof of the Main Theorem:**

Recall that at the end of chapter 3, we reduced the proof to showing that the quotient complex $Q. = D./B.$ is exact up to degree $2N - 6$ over points on the diagonal $\Delta \subset X \times X$. Over a point $W \times W$ on the diagonal, the quotient complex splits into many different sequences, all of which come in one way or another from the sequences of chapter 4.

Let us use the same notations that were used in chapter 2, 3 and 5. Thus the external tensor product of two Young diagrams of weight $\alpha$ respectively $\beta$ represents the external tensor product of the corresponding representations of $W$ and $(V/W)^*$, that is

$$\alpha \boxtimes \beta = \Sigma^\alpha W \boxtimes \Sigma^\beta (V/W)^*.$$

We will study the quotient complex over a point $W \times W$ on the diagonal $\Delta$.

**The maps $ev$ and $\varphi$**

Over a point $W \times W$, $\Psi_\alpha$ splits into a direct sum

$$\Psi_\alpha|_{W \times W} \cong \bigoplus_{i \geq 0} \Sigma^\alpha W \boxtimes \Sigma^{\alpha^* - 2i}(V/W)^*.$$

The map $d$ of the quotient complex, analogously to the case of $X = \text{SpGr}(2, 4)$, splits into two kinds of maps: $ev$ and $\varphi$. 
Let $gP$ be the parabolic leaving $W$ invariant. Recall the definition of $T^i = T^i_{(g,g)}$ and $t = t_{(g,g)}$. For $g \in G, i > 0$,

$$T = (g, v_1 \land v_2) \boxtimes (g, \eta), T^i = (g, (v_1 \land v_2)^{\otimes i}) \boxtimes (g, \eta^i)$$

and $t = dT$.

Let $w \in \Sigma^{(\alpha_1-i, \alpha_2-i)}W \boxtimes \Sigma^{(\alpha^*-2i)}(V/W)^*$, then

$$d(wT^i) = (dw)T^i + (-1)^{|w|}i w \cdot T^i - 1.$$

Then the maps $ev$ and $\varphi$ are given by:

$$\begin{array}{c}
\Sigma^{\alpha-1}W \boxtimes \Sigma^{\beta-1}(V/W)^* \\
\uparrow ev \\
d : \Sigma^{\alpha}W \boxtimes \Sigma^{\beta}(V/W)^* \\
\downarrow \varphi \\
\Sigma^{\alpha-1}W \boxtimes \Sigma^{\beta+1}(V/W)^*
\end{array}$$

$$ev((w_1 \land w_2)^{\otimes i} \otimes w) = (w_1 \land w_2)^{\otimes i} \otimes dw$$

and

$$\varphi((w_1 \land w_2)^{\otimes i} \otimes w) = (-1)^{|w|}i(w_1 \land w_2)^{\otimes i} \otimes (w \cdot t).$$

Over a point $W \times W$ on the diagonal $\Delta$, all the evaluation maps $ev$ are zero, because for $v \in W$ and $f \in (V/W)^*$, $f(v) = 0$, thus $dw = 0$. Therefore the quotient complex splits into several subsequences.
Since all the evaluation maps are zero over points on the diagonal, the quotient complex splits into $N - 2$ sequences

$$A^0, A^1, \ldots A^{N-2}$$

as follows:

$$Q|_{\Delta} \cong A^0 \oplus \ldots \oplus A^{N-2}.$$
$A^{N-5}: \quad\begin{array}{c}
\text{deg } 2N - 5 \\
\text{deg } 2N - 6 \\
\text{deg } 2N - 7
\end{array}$

\[
\begin{array}{cccc}
\{ & N-2 & \oplus & \{ & N-4 & \rightarrow & \{ & N-3 & \rightarrow & \{ & N-2 \\
\} & N-1 & \oplus & \} & \oplus & \} & \oplus & \} & \oplus & \}
\end{array}
\]

\[
\{ & N-1 & \oplus & \} & \oplus & \}
\]

\[
\{ & N \\
\}
\]

\[
\ldots
\]

\[
\ldots
\]

$A^1: \quad\begin{array}{c}
\text{deg } 2N - 5 \\
\text{deg } 2N - 6 \\
\text{deg } N - 1
\end{array}$

\[
\begin{array}{cccc}
\{ & N-2 & \oplus & \{ & N-4 & \rightarrow & \{ & N-3 & \rightarrow & \{ & N-2 \\
\} & N-2 & \oplus & \} & \oplus & \} & \oplus & \}
\end{array}
\]

\[
\{ & N-2 & \oplus & \} & \oplus & \}
\]

\[
\{ & N-2 \rightarrow \ldots \rightarrow N-2 & \oplus & \} & \oplus & \}
\]

\[
\{ & N-2 \rightarrow \ldots \rightarrow N-2 & \oplus & \} & \oplus & \}
\]

\[
\{ & N-2 \rightarrow \ldots \rightarrow N-2 & \oplus & \} & \oplus & \}
\]

\[
\ldots \rightarrow 0
\]

\[
\ldots \rightarrow 0
\]

\[
\ldots \rightarrow 0
\]
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\[
\deg 2N - 5 \quad \deg 2N - 6
\]

\[
A^0 : \begin{array}{c}
N-2 \\
\end{array} \rightarrow \begin{array}{c}
N-2 \\
\end{array}
\]

\[
\deg N - 1 \quad \deg N - 2
\]

\[
\rightarrow \cdots \rightarrow \begin{array}{c}
\end{array} \rightarrow \begin{array}{c}
\end{array} \rightarrow \begin{array}{c}
\end{array} \rightarrow \begin{array}{c}
\end{array} \rightarrow 0
\]

All of these sequences \(A^0, \ldots, A^{N-3}\) are exact up to degree \(2N-6\). This is obvious for \(A^{N-3}\), because \(A^{N-3}\) is zero up to degree \(2N-6\). For the other sequences the proof is similar to the case of \(X = SpGr(2, 4)\). We will use the sequences of Chapter 4, Corollary 4.5, to show that \(A^0, \ldots, A^{N-3}\) are exact.

First recall the construction of these sequences:

**Some exact sequences**

Similar to the previous chapter, let \(Y = Hom(T, T')\), \(Y_r = \{x \in Y \mid \text{rank}(x) \leq r\}\) and \(\hat{Y} = Y - Y_r\). Set \(V = Gr(n-r, T)\), \(T' = W^*\) and \(T = (V/W)^*\).

Then \(Y = Hom(W^*, (V/W)^*)\). Again consider the case when \(V\) is equal to the projective space \(\mathbb{P}(W^*) \cong \mathbb{P}^1\).

Recall the statement of Corollary 4.5 for \(V = \mathcal{O}_{\mathbb{P}}(j)\):

\[
m-j \quad \Lambda T' \rightarrow T^* \otimes T^{' \otimes \mathcal{O}_Y} \rightarrow Sym_2(T^*) \otimes \Lambda T' \otimes \mathcal{O}_Y \rightarrow \ldots
\]

\[
\rightarrow Sym_{j-1}(T^*) \otimes \Lambda T' \otimes \mathcal{O}_Y \rightarrow Sym_j(T^*) \otimes \Lambda T' \otimes \mathcal{O}_Y
\]

is exact.
Over a point $x = s \otimes f \in Y = \text{Hom}(T, T') \cong T^* \otimes T'$, the maps are given by:

$$d_x : \text{Sym}_k(T^*) \otimes \Lambda^{m-j+k} T^* \otimes k(\varphi) \to \text{Sym}_{k+1}(T^*) \otimes \Lambda^{m-j+k+1} T^* \otimes k(\varphi)$$

$$d_x((s_1...s_k) \otimes (f_1 \wedge ... \wedge f_{m-j+k})) = s_1...s_k s \otimes f \wedge f_1 \wedge ... \wedge f_{m-j+k}$$

We will use $x = t$. Let us check that $t \in \wedge Y$:

Let $e_1, ..., e_4$ be a symplectic basis of $V$, and $y_1, ..., y_4$ the corresponding dual basis of $V^*$, such that $W = e_1 \wedge e_2$.

Recall that $\eta$ is defined as $\eta = x_1 \otimes x_3 + x_2 \otimes x_4$ and denote by $d$ the map from the Tate construction. Then

$$t = dT$$

$$= d((g, v_1 \wedge v_2) \boxtimes (g, \eta))$$

$$= d((e_1 \wedge e_2) \boxtimes (y_1 \otimes y_{n+1} + \cdots + y_{n+2} \otimes y_{2n})$$

$$= y_1(e_1)e_2 \boxtimes y_{n+1} - y_1(e_2)e_1 \boxtimes y_{n+1}$$

$$+ y_2(e_1)e_2 \boxtimes y_{n+2} - y_2(e_2)e_1 \boxtimes y_{n+2}$$

$$+ \cdots +$$

$$y_n(e_1)e_2 \boxtimes y_{2n} - y_n(e_2)e_1 \boxtimes y_{2n}$$

$$= e_2 \boxtimes y_{n+1} - e_1 \boxtimes y_{2n}.$$ 

Thus $t$ has rank 2 and therefore $t \in \wedge Y$.

**Remark.** Note that

$$\varphi((e_1 \wedge e_2) \otimes w) = d_x(w),$$

thus $\varphi$ and $d_x$ just differ by a twist of $\mathcal{O}_{\wedge Y}(-1)$. Since $\wedge Y$ is affine, this twist is trivial.

Let us now finish the proof of the theorem:
The sequences $A^0, A^1 \ldots A^{N-3}$ are exact

We will proof this in three steps:

First we show that $A^0$ is exact, then secondly we discuss the exactness of $A^1$, and finally we consider $A^l$ for any $l$ and proof the exactness by induction.

The second step is not really necessary, but it is helpful to recognize a pattern for the proof.

**Step 1:**

**Lemma 6.2.** $A^0$ is exact.

**Proof.**

Set $\mathcal{V} = \mathcal{O}_x(N-2) \otimes \Lambda^2 (V/W)^*$ in 4.5. Then

\[
\begin{array}{c}
\times \xrightarrow{d_x} \square \xrightarrow{d_x} \square \xrightarrow{d_x} \square \xrightarrow{d_x} \ldots \\
\xrightarrow{d_x} \square \xrightarrow{d_x} \square \xrightarrow{d_x} \square \xrightarrow{\phi} \square \\
\end{array}
\]

is exact.

Recall that the difference between $d_x$ and $\phi$ is that $\phi$ adds a shift to the map $d_x$ (see above note), thus

\[
\begin{array}{c}
\times \xrightarrow{\phi} \square \xrightarrow{\phi} \square \xrightarrow{\phi} \square \\
\xrightarrow{\phi} \square \xrightarrow{\phi} \square \xrightarrow{\phi} \square \\
\end{array}
\]

is exact.

q.e.d.
Lemma 6.3.

$A^1$ is exact.

Proof.

Consider $A^1$:

The lower row of $A^1$ is equal to the complex $A^0$ shifted by 2, i.e. $A^0[-2]$, that is in degree $2N - 5$ the element of the lower row is equal to $A^0_{2N-7}$ and so on.

Recall Lemma 3.3.

Let $\hat{D}$ be an exact complex, let $\hat{B}$ be a sub-complex and denote by $\hat{Q}$ the quotient complex of $\hat{D}$ by $\hat{B}$.

Then $\hat{B}$ is exact up to degree $m$ if and only if $\hat{Q}$ is exact up to degree $m + 1$.

Since $A^0[-2]$ is exact up to degree $2N - 3$, this reduces the proof to showing that the upper row of $A^1$ is exact.

Use $\mathcal{V} = \mathcal{O}_\mathcal{F}((N - 2) - 1) \otimes \Lambda^{N-2} (V/W)^*$. Then by 4.5 the following complex is exact:
Tensor this by \((V/W)^*\) on the right:

Then the top row (*) is what we would like to be exact.

The bottom row is the sequence of \(V = \mathcal{O}_F((N-2) - 2) \otimes N^{-2} (V/W)^*\), which is exact up to degree \(2N - 5\) by Corollary 4.5.

Using Lemma 3.3 again this implies the exactness of (*), up to degree \(2N - 6\), as desired. q.e.d.

For the completion of the proof of the Main Theorem, we will proof the exactness in general for these sequences \(A^i\). The theorem then follows.
Lemma 6.4. The sequence $A^l$ is exact for all $0 \leq l \leq N - 3$.

Proof.
Consider $A^l$:

$A^l$:

Then the lower rows are the sequence $A^{l-1}[-2]$, thus using Lemma 3.3 we reduce the proof to showing that the top row is exact.

Let us denote the top row by $T^l$.
Use $\mathcal{V} = \mathcal{O}_p((N - 2) - l) \otimes \Lambda^{N-2}(V/W)^*$, then by 4.5 the following is exact:

$$
\begin{array}{c}
\begin{array}{c}
N-2 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
N-2 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
(N-2-l)+l \\
\end{array}
\end{array}
\rightarrow
0.
\end{array}
$$

Tensor this by $\Lambda^l(V/W)^*$ on the right and call the corresponding sequence (**), which again is exact.

Note that the top row is equal to $T^l$. Apply again Lemma 3.3: To show that $T^l$ and thus $A^l$ is exact up to degree $2N - 6$, it is enough to show that the quotient of (**) by $T^l$ (the bottom rows of (**)) is exact up to degree $2N - 5$.

Consider the sequence for $\mathcal{V} = \mathcal{O}_p((N - 2) - (l + 1)) \otimes \Lambda^{N-2}(V/W)^*$:
Twist this by $l^{-1} \Lambda (V/W)^*$ on the right:

These are just the bottom rows of (**), thus the quotient of (**) by $T^l$. This completes the proof of the Lemma and thereby the proof of the Main Theorem.

q.e.d.

$X=\text{SpGr}(2,6)$:

Last let us look at one example, $X = \text{SpGr}(2,6)$, the symplectic grassmannian of 2-planes in 6-space.

Consider the Tate construction in this case:
The sub-complex starts in degree 6, with the term

\[ \boxtimes \Psi_{2,2,2} \cong \mathcal{O}_X(-3) \boxtimes \Psi_{2,2,2} . \]

Thus Theorem 6.1 implies the exactness of the sub-complex \( B \).

\[ B : B_6 \to B_5 \to ... \to B_1 \to \mathcal{O}_{X \times X} , \]

that is:
Corollary 6.5.

\[ \mathcal{O}_X(-3) \boxtimes \Psi_{2,2,2} \rightarrow S(-2) \boxtimes \Psi_{2,2,1} \]

\[ \rightarrow \text{Sym}_2 S \boxtimes \Lambda S \]

\[ \rightarrow \text{Sym}_3 S \boxtimes \Lambda S \]

\[ \rightarrow O_X \boxtimes O_X \]

\[ \rightarrow \Lambda S \boxtimes \Psi_2 \]

is exact.
APPENDIX: TENSOR PRODUCT OF TWO YOUNG DIAGRAMS

The Young symmetrizer

Let $E$ be a vector space of dimension $M$ over an algebraically closed field $k$ of characteristic 0 and denote by $\mathfrak{S}_d$ the symmetric group.

The symmetric group $\mathfrak{S}_d$ acts on $E^\otimes d$ by permuting the factors, that is for $\sigma \in \mathfrak{S}_d$,

$$(e_1 \otimes e_2 \otimes \cdots \otimes e_d) \cdot \sigma = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(d)}.$$

This action commutes with the left action of $Gl(E)$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$ be an ordered partition of length $d$ and $\lambda_M \geq 0$, that is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_M = d$. To a partition $\lambda$ associate the Young diagram with $\lambda_i$ boxes in the $i$-th row, the rows of boxes are lined up on the left. Denote by $\lambda^*$ the partition defined by interchanging rows and columns in the Young diagram $\lambda$.

Define two subgroups of the symmetric group:

$$P = P_\lambda = \{ p \in \mathfrak{S}_d | p \text{ preserves each row} \}$$

$$Q = Q_\lambda = \{ q \in \mathfrak{S}_d | q \text{ preserves each column} \}$$

Set

$$a_\lambda = \sum_{p \in P} e_p \text{ and } b_\lambda = \sum_{q \in Q} \text{sgn}(q) \cdot e_q.$$

Definition.

(1) Define the Young symmetrizer of $\lambda, c_\lambda$ as

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d \text{ and}$$
(2) denote by $\Sigma^\lambda E$ the image of

$$c_\lambda : E^{\otimes d} \to E^{\otimes d}.$$ 

$E \mapsto \Sigma^\lambda E$ is called the Schur functor of $\lambda$.

Examples.

(1) For $\lambda = (2, 0, \ldots, 0)$ and its associated Young diagram $\square$, the Young symmetrizer is equal to $c_\lambda = 1 + e_{(1 2)}$. Then the image of $c_\lambda$ is the subspace of $E \otimes E$ spanned by all vectors:

$$v_1 \otimes v_2 + v_2 \otimes v_1.$$ 

Thus $\Sigma^{(2, 0, \ldots, 0)} \cong \text{Sym}_2 E$.

(2) For $\lambda = (1, 1, 0 \ldots 0)$, the associated Young diagram is equal to $\square$ and $c_\lambda = 1 - e_{(1 2)}$. The image of $c_\lambda$ is spanned by all vectors

$$v_1 \otimes v_2 - v_2 \otimes v_1.$$ 

Thus $\Sigma^{(1, 1, 0 \ldots 0)} \cong \Lambda^2 E$.

(3) For $\lambda = (2, 1, 0 \ldots 0)$, the associated Young diagram is equal to $\square$ and $c_\lambda = 1 + e_{(1 2)} - e_{(1 3)} - e_{(1 3 2)}$.

The image of $c_\lambda$ is the subspace of $E^{\otimes 3}$ spanned by all vectors

$$v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$ 

Lemma.

(1) $\Sigma^{(m, 0, \ldots, 0)} E \cong \text{Sym}_m E$,

(2) $\Sigma^{(1, \ldots, 1, 0 \ldots 0)} E \cong \Lambda^m E$. 
Proof. This follows immediately from the definition of $c_\lambda$, since in (1) $c_\lambda = a_\lambda$, and in (2) $c_\lambda = b_\lambda$.

q.e.d.

Note that the definition of the Young-symmetrizer $c_\lambda$ and the Schur functor $\Sigma^\lambda E$ also make sense for an ordered partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M, \lambda_{M+1}, \ldots)$, but if $\lambda_{M+1} \neq 0$, then $\Sigma^\lambda E = 0$.

**Littlewood-Richardson Rule**

Let $\lambda$ and $\mu$ be ordered partitions. Then

$$\Sigma^\lambda E \otimes \Sigma^\mu E \cong \bigoplus N_{\lambda \mu \nu} \Sigma^\nu E,$$

where $N_{\lambda \mu \nu}$ can be determined by the Littlewood-Richardson rule.

**Littlewood-Richardson Rule.**

$N_{\lambda \mu \nu} =$ number of ways the Young diagram can be extended to the Young diagram $\nu$ by a strict $\mu$-expansion.

A $\mu$-expansion of a Young diagram is obtained by first adding $\mu_1$ boxes to the rows of the Young diagram $\lambda$, but with no two boxes in the same column and in such a way that one obtains a Young diagram for each box added. Then put the integer 1 in each of these $\mu_1$ boxes. Then adding similarly $\mu_2$ boxes, continuing until finally $\mu_M$ boxes are added with the integer $M$.

The expansion is called strict if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first $t$ entries in this list (for any $t$ between 1 and $\mu_1 + \ldots + \mu_k$), each integer $p$ between 1 and $k - 1$ occurs as least as many times as the next integer $p + 1$.

Proof. See [FH], page 456, and [M], §1.9.
\textbf{Cauchy-Formula.} Let $E$ and $F$ be two vector spaces over $k$. Then

$$\Lambda^p(E \otimes F) \cong \bigoplus_{|\lambda|=p} \Sigma^\lambda E \otimes \Sigma^{\lambda^*} F$$

as $\text{Gl}(E) \times \text{Gl}(F)$ representations.

\textit{Proof.} The proof is essentially an application of the Littlewood-Richardson rule, see [KI], Lemma 0.5.

\textbf{Lemma 1.}

Let $E$ be a vector space of dimension $M$ over $k$, then

$$\Sigma^\lambda E \otimes \Sigma^{(1,\ldots,1)} E \cong \Sigma^{(\lambda_1+1,\ldots,\lambda_M+1)} .$$

Particularly if $\dim E = 2$, then

$$\Sigma^\lambda E \otimes \Sigma^{(1,1)} E \cong \Sigma^{(\lambda_1+1,\lambda_2+1)} .$$

\textit{Proof.} The Young diagram of $(1,\ldots,1)$ is of the form

\[ \begin{array}{c} \\
\end{array} \]

Thus the tensor product of $\lambda$ by $(1,\ldots,1)$ is given by adding $M$ boxes to each $\lambda$ following the combinatorial rule of the Little-Richardson formula. This can only be done in one way, which is to add one box to each row, thus

$$\lambda \otimes (1,\ldots,1) \cong (\lambda_1+1,\ldots,\lambda_M+1) .$$

q.e.d.

Set $\Sigma^\mu = 0$, if $\mu$ is not ordered, that is $\mu_i < \mu_{i+1}$ for some $i$. 
Lemma 2. Let \( 2 \geq \lambda_1 \geq \cdots \geq 0 \) and set \( \alpha = \lambda^* \), then

\[
\Sigma^\alpha \otimes E \cong \Sigma^{(\alpha_1, \alpha_2, 1, 0 \ldots 0)^*} E \oplus \Sigma^{(\alpha_1, \alpha_2 + 1, 1, 0 \ldots 0)^*} E \oplus \Sigma^{(\alpha_1, \alpha_2, 1, 1, 0 \ldots 0)^*} E
\]

and

\[
\Sigma^\alpha \otimes \text{Sym}_2 E \cong \Sigma^{(\alpha_1 + 1, \alpha_2 + 1, 0 \ldots 0)^*} E \oplus \Sigma^{(\alpha_1, \alpha_2 + 1, 1, 0 \ldots 0)^*} E \oplus \Sigma^{(\alpha_1, \alpha_2, 1, 1, 0 \ldots 0)^*} E
\]

with each summand occurring with multiplicity one.

Proof.

(1) Consider the tensor product of \( \lambda \) by \( \Box \).

This is given by adding one box to \( \lambda \). The box can be added either to the first, second or third column of \( \lambda \). Since \( \alpha_i = \text{number of boxes in the } i\text{-th column of } \lambda \), it follows that adding a box in the \( j \)-th column just adds one to \( \alpha_j \). Therefore the summands that occur in the tensor product are

\[
(\alpha_1 + 1, \alpha_2, 0 \ldots 0)^* \oplus (\alpha_1, \alpha_2 + 1, 0 \ldots 0)^* \oplus (\alpha_1, \alpha_2, 1, 0 \ldots 0)^*,
\]

all occurring precisely once.

(2) Consider the tensor product of \( \lambda \) by \( \Box \times \Box \).

This is given by adding the two boxes of \( \Box \times \Box \) to \( \lambda \) according to the combinatorial rules, that is at most one box to each column. This adds either (a) no, (b) one or (c) two boxes to the first two columns of the Young diagram \( \lambda \).

(a) If we add no box to the first two columns, this can only be done by adding 2 boxes to the first row of \( \lambda \). The thus obtained weight is equal to \( (\alpha_1, \alpha_2, 1, 1, 0 \ldots 0)^* \).

(b) Adding one box to the third column and adding a box to the first column or second column of \( \lambda \) results in the weights \( (\alpha_1 + 1, \alpha_2, 1, 0 \ldots 0)^* \) and \( (\alpha_1, \alpha_2 + 1, 1, 1, 0 \ldots 0)^* \), which both just occur once.
(c) If both boxes are added to the first two columns, then there is only one way of doing so and the resulting weight is equal to \((\alpha_1 + 1, \alpha_2 + 1, 0 \ldots 0)^*\).

q.e.d.

**Remark.**

1. Denote by \(\det(E)\) the vector bundle \(\Lambda^M E \cong \Sigma^{(1, \ldots, 1)} E\). Let \(\lambda\) be any ordered partition \((\lambda_1, \ldots, \lambda_M)\). Suppose that the \(\lambda_i\)’s are not all positive. Set

\[
\Sigma^\lambda E = \Sigma^{(\lambda_1 - \lambda_M, \ldots, \lambda_M - \lambda_M)} \otimes \det(E)^{\otimes \lambda_M},
\]

then this is well defined by Lemma 1 and because \(\lambda_1 - \lambda_M \geq \cdots \geq \lambda_M - \lambda_M \geq 0\).

2. The dual of the representation \(\Sigma^\lambda E\) is given by:

\[
(\Sigma^\lambda E)^* \cong \Sigma^\lambda E^* \cong \Sigma^{(-\lambda_M, -\lambda_M - 1, \ldots, -\lambda_1)},
\]

see [KI], (0.1).

3. The maximum number of rows or of columns in each Young diagram \(\nu\) appearing in the tensor product \(\lambda \otimes \mu\) does not exceed the sum of these numbers for the separate factors, that is:

If \(\Sigma^\nu \subset \Sigma^\lambda \otimes \Sigma^\mu\), then \(\nu_i \leq \lambda_1 + \mu_1\) for all \(i\) and \((\nu^*)_i \leq (\lambda^*)_1 + (\mu^*)_1\).

This follows from the Littlewood-Richardson rule, since we add at most \(\mu_1\) boxes to \(\lambda_1\) and since we add at most \((\mu^*)_1\) boxes to the first column of \(\lambda\).
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