Nontopological Methods for Determining Topological Charge for Bosons and Fermions in Flat Spacetime:

Joseph Saaty, PhD
The Union Institute
School of Interdisciplinary Arts and Sciences

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Abstract

An alternative method to the topological instanton solution for deriving an expression for the topological charge is presented. This alternative method involves the use of relativistic quantum field theory and covariant electrodynamics. In the case of bosons, this method is consistent with the instanton solution in predicting that topological charge is quantized. But furthermore, this method led to the new results that topological charge for fermions cannot be quantized, whereas the instanton solution cannot distinguish between bosons (quantized) and fermions (not quantized). Thus the new technique produced results that were previously unobtainable. Mathematics Subject Classifications (1991): // Key words: Topological charge, Dirac quantization condition, Klein-Gordon equation
To prove that the topological charge \( Q = \frac{1}{8\pi} \int \epsilon_{\mu\nu} \hat{\phi} \cdot (\partial_\mu \hat{\phi} \times \partial_\nu \hat{\phi}) \, d^3x \) is quantized \([12]\) by using covariant electrodynamics and relativistic quantum field theory, we proceed as follows:

**Proof:** \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e|\phi|} \epsilon_{abc} \phi^a (\partial_\mu \phi^b)(\partial_\nu \phi^c) \),

\[
A_\mu = \frac{1}{|\phi|} \phi^a A_\mu^a \quad \text{and} \quad \hat{\phi} = \frac{\phi}{|\phi|}
\]

\( K^\mu = \partial_\nu \tilde{F}^{\mu\nu} \) is the magnetic current

\[
\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}
\]

so that,

\[
K^\mu = -\frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c
\]

\[
K^\sigma = \frac{1}{2} \epsilon^{\nu\sigma\rho} \partial_\nu F_{\sigma\rho}
\]

Since \( \partial_\mu K^\mu = 0 \), the conserved magnetic charge can be written as \( M = \frac{1}{4\pi} \int K^\sigma d^3x \).

\[
M = -\frac{1}{8e\pi} \int \epsilon^{ijk} \epsilon_{abc} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c d^3x
\]

\[
M = \frac{1}{8e\pi} \oint \epsilon^{ijk} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c dS_i
\]

i.e.,

\[
K^\sigma = -\frac{1}{2e} \epsilon^{ijk} \epsilon_{abc} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c
\]  

(1)

but, \( K^\mu = -\frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c \). If \( i = \mu, \quad j = \nu, \quad k = \rho \)

then we will get

\[
K^\mu = -\frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c
\]  

(2)
Maxwell’s equation in covariant form is $\partial_\mu F^{\mu\nu} = j^\nu$.

$$
B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}, \quad \vec{B} \text{ is the magnetic field}
$$

$$
\nabla \cdot \vec{B} = \partial_i B_i \text{ so that } \nabla \cdot \vec{B} = \frac{1}{2} \epsilon_{ijk} \partial_i F^{jk}
$$

$$
K^\mu = \partial_\nu \tilde{F}^{\mu\nu}, \quad K^\nu = \partial_\mu \tilde{F}^{\mu\nu}, \quad B^i = -\frac{1}{2} \epsilon_{ijk} F^{jk}
$$

$$
K^\mu = -\frac{1}{2} e \epsilon^{\mu\rho\sigma} \epsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c
$$

$$
\partial_\nu \tilde{F}^{\mu\nu} = -\frac{1}{2} e \epsilon^{\mu\rho\sigma} \epsilon_{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c
$$

$$
K^\nu = -\frac{1}{2} e \epsilon^{ijk} \epsilon_{abc} \partial_\xi \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c
$$

$$
\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\nu\sigma\rho} \partial_\nu F^{\sigma\rho}
$$

$$
K^\nu = \frac{1}{2} \epsilon^{\nu\rho\sigma} \partial_\nu F^{\sigma\rho} \quad \text{for a fix } \mu = 0
$$

where $\nu, \sigma, \rho$ are variables [14].

$$
B_i = \frac{1}{2} \epsilon_{ijk} F^{jk} \quad \text{and} \quad B^i = -\frac{1}{2} \epsilon^{ijk} F_{jk},
$$

or $B_\nu = \frac{1}{2} \epsilon_{\nu\sigma\rho} F^{\sigma\rho}$ or $B^\nu = -\frac{1}{2} \epsilon^{\nu\sigma\rho} F_{\sigma\rho}$.

$$
\nabla \cdot \vec{B} = \partial_\nu B_\nu = \partial_i B_i = \partial_\nu B_\nu
$$

$$
\nabla \cdot \vec{B} = \frac{1}{2} \epsilon_{\nu\sigma\rho} \partial_\nu F^{\sigma\rho} \quad \text{Clearly, we get } \nabla \cdot \vec{B} = K^\nu
$$

$$
M = \frac{1}{4\pi} \int \nabla \cdot \vec{B} \ d^3 x \quad \text{so that } \nabla \cdot \vec{B} = 4\pi g \delta^3(x).
$$

Therefore $M = \frac{1}{4\pi} \int 4\pi g \delta^3(x) \ d^3 x = g \int \delta^3(x) \ d^3 x = 1 \text{ Dirac Delta}$

so we can simply conclude that

$$
M = g \quad (3)
$$

$$
Q = \frac{1}{4\pi} \int dS_a \hat{\phi}^a = \frac{1}{8\pi} \int \epsilon^{\mu\nu} \epsilon^{\alpha\beta\gamma} \partial_\mu \hat{\phi}^a \partial_\nu \hat{\phi}^b \partial_\gamma \hat{\phi}^c dS_a
$$

$$
Q = \left( \int d^3 x K_\alpha \right) \times \text{const.} \quad \text{const} = \frac{1}{4\pi e} \quad \text{and } \quad dS_a = \frac{1}{2} \epsilon^{\mu\nu} \epsilon^{\alpha\beta\gamma} \partial_\mu x_b \partial_\nu x_c d^2 \sigma
$$

$$
Q = \frac{1}{8\pi} \int \epsilon_{ijk} \epsilon^{\alpha\beta\gamma} \partial_\xi \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c d^3 x
$$
\[ Q = \frac{1}{8\pi} \int \epsilon_{ijk} \epsilon^{abc} \partial_i (\hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c) d^3x \]

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since \( \hat{\phi} = \frac{\phi}{|\phi|} \) therefore

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e} \epsilon_{abc} \hat{\phi}^a (\partial_\mu \hat{\phi}^b)(\partial_\nu \hat{\phi}^c) \]

\[ B^i = -B_i = -\frac{1}{2} \epsilon^{ijk} F_{jk} \text{ and we get,} \]

\[ B_i = \frac{1}{2} \epsilon^{ijk} F_{jk} \]

\[ \nabla \cdot \vec{B} = \partial_i B_i = \frac{1}{2} \epsilon_{ijk} \partial_i F_{jk} \]

\[ \nabla \cdot \vec{B} = \partial_i B_i = \frac{1}{2} \epsilon_{ijk} \partial_i \left[ (\partial_j A_k - \partial_k A_j) - \frac{1}{e} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c \right] \]

\[ F_{jk} = \partial_j A_k - \partial_k A_j - \frac{1}{e} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c. \]

\[ \partial_i (\partial_j A_k - \partial_k A_j) = 0. \text{ This gives,} \]

\[ \partial_i F_{jk} = -\frac{1}{2e} \epsilon_{abc} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c \text{ to get,} \]

\[ \nabla \cdot \vec{B} = \partial_i B_i = -\frac{1}{2e} \epsilon_{abc} \epsilon^{ijk} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c \]

But \( \nabla \cdot \vec{B} = K^o \) so that

\[ K^o = -\frac{1}{2e} \epsilon_{abc} \epsilon^{ijk} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c \]

The above is the derivation of equation (1).

If we compare the expression for \( Q \) and \( M \) we find that

\[ Q = \frac{M}{e} \]

\[ M = \frac{1}{4\pi} \int K^o d^3x = \frac{1}{4\pi} \int \nabla \cdot \vec{B} d^3x \]
\[ \nabla \cdot \vec{B} = 4\pi g \delta^3(x) \]

therefore \[ M = \frac{1}{4\pi} \int 4\pi g \delta^3(x) d^3x = g \int \delta^3(x) d^3x. \]

Thus \( M = g \) since \( \int \delta^3(x) d^3x = 1 \) (Dirac Delta), thus,

\[ Q = \frac{g}{e} \tag{4} \]

where \( g \) is the magnetic strength, and, according to (Dirac Quantization Condition)

\[ g = \frac{hc}{2e} n, \quad n = 0, \pm 1, \ldots \]

therefore \( Q = \frac{hc}{2e^2} n, \quad n = 0, \pm 1, \ldots \) Thus, the topological charge \( Q \) is quantized.

Let us next prove the Dirac Quantization Condition is derived from the Klein-Gordon equation.

**Proof:** The solution for Klein-Gordon (K-G) equation is

\[ \Psi = |\Psi| \exp \left( \frac{i}{\hbar} (\vec{P} \cdot \vec{r} - Et) \right), \]

but for the time independent equation, \( \Psi = |\Psi| \exp \left( \frac{i}{\hbar} (\vec{P} \cdot \vec{r}) \right). \)

In presence of an electromagnetic field \( \vec{P} \rightarrow \vec{P} - e \vec{A} \), and, due to the influence of magnetic monopole we will get \[ [12] [13] \]

\[ \Psi \rightarrow |\Psi| \exp \left( \frac{i}{\hbar} \vec{P} \cdot \vec{r} \right) \exp \frac{-ie}{\hbar c} \vec{A} \cdot \vec{r} \]

\[ \Psi \rightarrow |\Psi| \exp \left( \frac{i}{\hbar} \vec{P} \cdot \vec{r} \right) \exp \frac{-ie}{\hbar c} \oint \vec{A} \cdot d\vec{l} \]

where \( \vec{A} \cdot \vec{r} \rightarrow \oint \vec{A} \cdot d\vec{l} \) close cycle for a periodic function. Therefore

\[ \Psi \rightarrow \Psi \exp \left( \frac{-ie}{\hbar c} \oint \vec{A} \cdot d\vec{l} \right) \text{ so that } \exp \left( \frac{-ie}{\hbar c} \oint \vec{A} \cdot d\vec{l} \right) = 1, \]

\[ e^{-\frac{ie}{\hbar c} \oint \vec{A} \cdot d\vec{l}} = 1 \text{ then } \frac{-ie}{\hbar c} \oint \vec{A} \cdot d\vec{l} = -2\pi n i \]

changing from line integral to surface integral we have

\[ \oint \vec{A} \cdot d\vec{l} = \int \nabla \times \vec{A} \cdot d\vec{S}, \quad \vec{B} = \nabla \times \vec{A} \text{ gives } \]
\[ \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot dS \]

changing from surface integral to volume integral

\[ \int \mathbf{B} \cdot dS = \int \nabla \cdot \mathbf{B} \, d^3x \]

to get

\[ \oint \mathbf{A} \cdot d\mathbf{l} = \int \nabla \cdot \mathbf{B} \, d^3x \]

or \( \frac{e}{\hbar c} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{e}{\hbar c} \int \nabla \cdot \mathbf{B} \, d^3x, \)

therefore \( 2\pi n = \frac{e}{\hbar c} \int \nabla \cdot \mathbf{B} \, d^3x. \)

But \( \nabla \cdot \mathbf{B} = 4\pi g \delta^3(x) \) gives \( 2\pi n = \frac{e}{\hbar c} \int 4\pi g \delta^3(x) \, d^3x. \)

thus \( n = \frac{2e}{\hbar c} g \int \delta^3(x) \, d^3x. \)

But \( \int \delta^3(x) \, d^3x = 1, \) Dirac Delta, whence

\[ n = \frac{2eg}{\hbar c} \quad \text{therefore} \quad g = \frac{\hbar cn}{2e}, \quad \text{Dirac Quantization Condition} \]

The result thus far applies to bosons, because the Dirac quantization condition is derived from the Klein-Gordon equation, which describes only spin-zero particles (bosons). We will now proceed to address the fermion case.
Discussion on the Quantization of
Topological Charge for Fermions

To prove that each of the four components $\Psi_i$ of Dirac’s equation satisfies the Klein-Gordon equation.

**Proof:** The covariant Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0 \quad \text{or} \quad i\gamma^\mu \partial_\mu \Psi = m\Psi.$$  

Operate on the covariant Dirac equation by $\gamma^\nu \partial_\nu$, to get:

$$\gamma^\nu \partial_\nu (i\gamma^\mu \partial_\mu - m)\Psi = 0, \text{or}$$

$$(i\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu)\Psi = 0$$

but

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \sum_{\mu\nu} \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \sum_{\nu,\mu} \left(\frac{1}{2} \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu\right)\partial_\nu \partial_\mu$$

so that

$$i\frac{1}{2} (\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)\partial_\nu \partial_\mu \Psi - m\gamma^\nu \partial_\nu \Psi = 0$$

By the anticommutation relation: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

the preceding equation becomes

$$ig^{\mu\nu} \partial_\nu \partial_\mu \Psi - m\gamma^\nu \partial_\nu \Psi = 0$$

Note that the covariant Dirac equation will result in $i\gamma^\nu \partial_\nu \Psi = m\Psi$ or $\gamma^\nu \partial_\nu \Psi = \frac{m\Psi}{i}$.

so that,

$$ig^{\mu\nu} \partial_\nu \partial_\mu \Psi - m\left(\frac{m\Psi}{i}\right) = 0$$

$g^{\mu\nu} \partial_\nu \partial_\mu = \partial^\mu \partial_\mu \quad \text{where} \quad g^{\mu\nu}$ is the metric tensor, therefore $i\partial^\mu \partial_\mu \Psi - m\left(\frac{m\Psi}{i}\right) = 0, \quad i = -\frac{1}{i}$

$$i\partial^\mu \partial_\mu \Psi + m^2 \Psi = 0 \quad \text{or} \quad i(\partial^\mu \partial_\mu + m^2)\Psi = 0.$$
But since $\partial^\mu \partial_\mu = \Box^2$ then $(\Box^2 + m^2)\Psi = 0$, or $(\Box^2 + m^2)\Psi_i = 0$, therefore each of the four components $\Psi_i$ satisfies the Klein-Gordon equation. However in the presence of electromagnetic field that may not be the case.

In the presence of electromagnetic field, Dirac’s equation in flat spacetime is [17]

$$\left[ -\left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 + \left( \frac{1}{i} \nabla + e\vec{A} \right)^2 + m^2 + \vec{\sigma} \cdot \vec{B} - ie\rho_1(\vec{\sigma} \cdot \vec{E}) \right] \Psi(\vec{x}, t) = 0$$

$\vec{\sigma} \cdot \vec{B} - ie\rho_1(\vec{\sigma} \cdot \vec{E})$ = function of the electromagnetic field $(\vec{A}(x), i\phi(t))$ so that

$$\vec{\sigma} \cdot \vec{B} - ie\rho_1(\vec{\sigma} \cdot \vec{E}) = f(\vec{x}, t) = f(x, t)$$

Therefore

$$\left[ -\left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 + \left( \frac{1}{i} \nabla + e\vec{A} \right)^2 + m^2 + f(x, t) \right] \Psi(\vec{x}, t) = 0. \quad (5)$$

Klein-Gordon’s equation in flat spacetime in the presence of electromagnetic field is [17]:

$$\left[ -\left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 - \left( \frac{1}{i} \nabla + e\vec{A} \right)^2 \right] \Psi(\vec{x}, t) = m^2 \Psi(\vec{x}, t)$$

$$(E + e\phi)^2 - (\vec{P} + e\vec{A})^2 = m^2$$

The above two equations lead us to

$$\left[ -\left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 - \left( \frac{1}{i} \nabla + e\vec{A} \right)^2 \right] \Psi(\vec{x}, t) = \left[ (E + e\phi)^2 - (\vec{P} + e\vec{A})^2 \right] \Psi(\vec{x}, t)$$

Thus the time independent equation is:

$$\left( \frac{1}{i} \nabla + e\vec{A} \right)^2 \Psi(\vec{x}) = (\vec{P} + e\vec{A})^2 \Psi(\vec{x})$$

and the time dependent equation is:

$$\left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 \Psi(t) = (E + e\phi)^2 \Psi(t)$$

Equation (5) can be written as

$$\left[ \left( \frac{1}{i} \nabla + e\vec{A} \right)^2 - \left( \frac{1}{i} \frac{\partial}{\partial t} + e\phi \right)^2 + m^2 \right] \Psi(\vec{x}, t) = -f(x, t)\Psi(\vec{x}, t).$$
This is regarded as nonhomogeneous equation whose solution is

$$\Psi(\vec{x}, t) = \phi(\vec{x}, t) + \int G_p(\vec{x}, \vec{x}')[-f(x', t)]\Psi(\vec{x}', t)d^3x'$$

where $\phi(\vec{x}, t)$ is the solution of the homogeneous equation.

$$\left[\left(-\frac{1}{i}\nabla + e\vec{A}\right)^2 - \left(-\frac{1}{i}\frac{\partial}{\partial t} + e\phi\right)^2\right] \phi(\vec{x}, t) = 0$$

is the homogeneous equation which is Klein-Gordon’s equation representing a particle in an electromagnetic field. The time independent equation corresponding to the above equation is

$$\left(-\frac{1}{i}\nabla + e\vec{A}\right)^2\Psi(x) = (\vec{P} + e\vec{A})^2\Psi(x)$$

there will be a restriction on the above equation due to the presence of magnetic monopole.

One restriction on $\vec{A}$ is that it must be singular along the negative $z$-axis, there must be an unphysical singularity (Dirac String). Such a descriptive corresponds only to a wave function in a presence of magnetic monopole which can be obtained by making the standard substitution $\vec{P} \rightarrow \vec{P} + e\vec{A}$, and also demanding the wave function to be single valued when going around the loop, i.e.,

$$\phi(\vec{P} \cdot \vec{x} + 2\pi n) = \phi(\vec{P} \cdot \vec{x}),$$

although $\vec{P} \cdot \vec{x} \neq \vec{P} \cdot \vec{x} + 2\pi n$ so that, the line integral around the Dirac String must be $2\pi n$, $n =$ integer. This condition will give rise to $\exp +ie\vec{A} \cdot \vec{x} = 1$ so that, $2\pi n = e \oint \vec{A} \cdot d\vec{l}$. Under these conditions we examine the equation

$$\left(-\frac{1}{i}\nabla + e\vec{A}\right)^2\phi(\vec{x}) = (\vec{P} + e\vec{A})^2\phi(\vec{x})$$

to find that the only way these conditions can be satisfied if

$$\left(-\frac{1}{i}\nabla + e\vec{A}\right)^2 \rightarrow \left(-\frac{1}{i}\nabla'\right)^2 \quad \text{and} \quad \phi(\vec{x}) \rightarrow \phi'(\vec{x'})$$

let $e$ in the above equation be replaced by $-e$ to get

$$\left(-\frac{1}{i}\nabla - e\vec{A}\right)^2 \rightarrow \left(-\frac{1}{i}\nabla'\right)^2 \quad \text{and} \quad \phi(\vec{x}) \rightarrow \phi'(\vec{x'})$$
so that if $\phi'(\vec{x}') \equiv \phi(\vec{P} \cdot \vec{x} + 2\pi n)$ then $\phi'(\vec{x}') = \phi(\vec{P} \cdot \vec{x})$ or $\phi'(\vec{x}') = \phi(\vec{x}) \equiv \phi(\vec{P} \cdot \vec{x})$, where $\phi(\vec{P} \cdot \vec{x}) = \phi(\vec{P} \cdot \vec{x} + 2\pi n)$.

If $\phi(\vec{P} \cdot \vec{x}) = \phi[(\vec{P} - e\vec{A}) \cdot \vec{x}]$ which implies that $e\vec{A} \cdot \vec{x} = 2\pi n$, we can clearly see that the only way these conditions are met if $(\frac{1}{i} \nabla - e\vec{A})^2 \rightarrow (\frac{1}{i} \nabla')^2$ and $\phi(\vec{x}) \rightarrow \phi'(\vec{x}')$ so that $\left(\frac{1}{i} \nabla'\right)^2 \phi'(\vec{x}') = (\vec{P} - e\vec{A})\phi'(\vec{x}')$ which gives $\phi'(\vec{x}') = A_1 e^{(\vec{P} - e\vec{A}) \cdot \vec{x}')}$ that satisfies the above equations and conditions, therefore $\phi(\vec{x}, t) = \phi(t)e^{(\vec{P} - e\vec{A}) \cdot \vec{x}}$.

This leads us to $\Psi(\vec{x}, t) = \phi(t)e^{i(\vec{P} - e\vec{A}) \cdot \vec{x}} + \int G_p(\vec{x}, \vec{x}', t) \Psi(\vec{x}', t)d^3x'$ or $\Psi(\vec{x}, t) = \phi(t)e^{i(\vec{P} - e\vec{A}) \cdot \vec{x}} - \int G_p(\vec{x}, \vec{x}', t) f(x', t) \Psi(\vec{x}', t)d^3x'$.

In the absence of electromagnetic field and of magnetic monopole we have seen above that each of the four components of Dirac’s Matrix satisfies the Klein-Gordon equation, therefore

$$\Psi(\vec{x}, t) = A_1 e^{i(\vec{P} \cdot \vec{x} - Et)} = A_1 e^{i\vec{P} \cdot \vec{x} - Et}.$$ 

For $\Psi(\vec{x}, t)$ to be single valued when $\vec{A}$ is singular we should have

$$A_1 e^{i\vec{P} \cdot \vec{x} - Et} = \phi(t)e^{i\vec{P} \cdot \vec{x}} e^{-ie\vec{A} \cdot \vec{x}} - \int G_p(\vec{x}, \vec{x}', t) f(x', t) \Psi(\vec{x}', t) d^3x'$$

$$e^{-ie\vec{A} \cdot \vec{x}} = A_1 \phi^{-1}(t)e^{-iE t} + \phi^{-1}(t)e^{-i\vec{P} \cdot \vec{x}} \int G_p(\vec{x}, \vec{x}', t) f(x', t) \Psi(\vec{x}', t) d^3x'$$

in order for the quantization to take place.

We have seen already that $e^{-ie\vec{A} \cdot \vec{x}}$ (where $x$ here is equivalent to $r$ as used earlier) should have been equal to a constant, when the wave function is single valued. However, we can see from the above equation that $e^{-ie\vec{A} \cdot \vec{x}} = $ function of $x \neq$ constant. Thus, we conclude that, in a case of fermions the topological charge is not quantized.

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