Scattering of Closed String States
from a Quantized D-particle

S. Hirano† and Y. Kazama ‡
Institute of Physics, University of Tokyo,
Komaba, Meguro-ku, Tokyo 153 Japan

Abstract

By developing an appropriate path-integral formalism, we compute, in bosonic string theory, the disk amplitude for the scattering of closed string states from a D-particle, in which the collective coordinate of the D-particle is fully quantized. As a consequence, the recoil of the D-particle is naturally taken into account. Our result can be readily factorized in the closed string channel to yield the boundary state describing the recoiling D-particle. This turned out to agree with the BRST invariant vertex recently proposed by Ishibashi to the leading order in the derivative expansion, but it will receive corrections in subsequent orders. The advantage of our formalism is that it is extendable to deal with more general processes involving multiple D-particles. A viewpoint regarding our work as describing a dynamical transition of CFT’s is also discussed.

†hirano@hep1.c.u-tokyo.ac.jp; JSPS Research Fellow.
‡kazama@hep3.c.u-tokyo.ac.jp
1 Introduction

D-branes have by now acquired indisputable citizenship in the world of string theory \[1\]. Born as somewhat exotic objects \[2\], their status was dramatically promoted by the Polchinski’s discovery \[3\] that they are the stringy representations of an important class of solitons which carry Ramond-Ramond charges. Besides playing key roles in numerous duality relations among string theories \[4\] and in the understanding of the statistical entropy of a black hole \[5\], they are expected to provide bridges leading us into the enigmatic world of 11 dimensional M-theory \[6, 7\]. Thus their importance may well be even greater in future developments.

One of the many aspects of D-brane physics that need better understanding is that of the dynamics of D-branes. Indeed there have been a large number of investigations made on this subject in the past year \[8\] – \[22\], which may be classified into several categories according to the viewpoints and methods employed.

To one category belong studies of the interaction of a D-brane with another D-brane or with elementary string states by means of open string calculations \[9\] – \[16\]. They revealed the nature of the velocity dependence of the inter-D-brane forces, the internal structure of the D-brane through its form factor and the decay characteristics, and many other interesting properties. In these calculations, the D-branes are treated as infinitely heavy background objects and hence the recoil effects have so far not been taken into account.

Closely related are the investigations from the closed string channel. By the duality property of the string worldsheet, open string calculations can be reinterpreted in the closed string language, yielding a useful notion of boundary states \[23, 24\]. Using the T-duality transformations, the components of the background gauge field strength transverse to the D-brane worldvolume can be interpreted as those of the velocity of the D-brane and for constant velocity one can construct the boundary state for a D-brane moving along a straight line. This yields a complimentary view of the inter-D-brane forces \[21, 22\]. Just as in the open string calculations, however, collective coordinates of the D-brane are not quantized in such a treatment.

The third category is rather different in spirit from the above two. It comprises the studies of low energy interactions of D-branes using the dimensionally reduced super Yang-Mills theory as the effective theory \[17 – 20\]. The D-brane collective coordinates appear
as the Higgs fields on the worldvolume and are thus dynamical. This framework has been successfully employed for uncovering the bound states and resonances of D-branes. It can also be used for the scattering of D-branes provided that D-branes are nonrelativistic. Furthermore, it has been noted that the 11 dimensional Planck scale naturally makes its appearance in the regime treated in this formalism.

Although many intriguing characteristics of D-brane dynamics have been uncovered through these studies, we are still at a primitive stage, lacking in particular a more systematic formalism to deal with the collective dynamical degrees of freedom of the D-branes themselves. In the first two categories sketched above, D-branes are treated as backgrounds and hence not fully dynamical. Attempts have been made to treat the recoil effects within this type of setting, with only a moderate success \cite{25,26,27}. The situation is improved in the effective theory approach of the third category, but its applicability is limited to the low energy domain. Moreover the connection between these two types of approaches is understood only partially.

In this work, as a step toward more systematic treatment of D-brane dynamics, we develop a path integral formalism in which the collective coordinates of a D0-brane, \textit{i.e.} a D-particle, are quantized. Specifically, we compute the disk amplitude for the scattering of closed string states from a quantized D-particle in bosonic string theory.

The calculation is performed in two steps: First, in Sec. 2, we set up a formalism with a D-particle moving along a fixed yet arbitrary trajectory parametrized as $f^\mu(t)$, where $t$ is a worldline coordinate. The effect of the bulk fluctuations of the open string attached to the D-particle is computed to all order in $\alpha'$ in the usual fashion. On the other hand, to incorporate the fluctuation of the ends of the string along the trajectory, we need to employ a generally covariant expansion in the number of derivatives of $f^\mu(t)$, regarded at this stage as a background field. As the only scale in the problem is $\alpha'$, this takes the form of an $\alpha'$ expansion, though distinct from the usual sense. With this setting, we compute the scattering amplitude with closed string tachyon states to the leading order in the above expansion. In the course of the calculation, one finds that the conformal invariance requires the trajectory to satisfy the current conservation condition $k_\mu \dot{f}^\mu = 0$, where $k_\mu$ is the total momentum of the closed string states and $\dot{f}^\mu = df^\mu/dt$. Another point to be mentioned is that the integration over the fluctuation of the ends of the string produces a divergence. It is of such a form that it can be absorbed by the renormalization of $f^\mu(t)$ when there are no vertex insertions. With vertex insertions, a part of the diver-
gence remains in the amplitude but this will turn out to vanish upon quantization of the trajectory. Aside from this piece, the resultant amplitude can be readily factorized in the closed string channel and we obtain the boundary state describing the D-particle moving along $f^\mu(t)$. This turned out to coincide with the particle-particle-string vertex recently proposed by Ishibashi [28], who started with an ansatz which depends only on $\dot{f}^\mu$ and imposed BRST invariance. Our treatment makes it clear that in general the boundary state will receive corrections involving higher derivatives of $f^\mu(t)$ in subsequent orders.

In the second step, described in Sec. 3, we perform the integration over the trajectory $f^\mu(t)$ itself. The calculation requires regularization, upon which the divergent piece mentioned earlier drops out. An important outcome is that, as it should be the case, the current conservation condition $k_\mu \dot{f}^\mu = 0$ automatically comes out when the initial and the final D-particles are on shell. This was anticipated in [28], where, however, the quantization was not explicitly performed. The final result is a relativistic scattering amplitude with the recoil of the D-particle fully taken into account. As an application of our formalism, we compute the amplitudes with two tachyons and with two gravitons. Besides the usual closed string poles in the $t$-channel, these amplitudes exhibit poles in a peculiar channel, which for small momentum transfer is likely to be interpretable as $s$-channel excitations of an open string with a heavy mass attached at the ends. Our result for the two graviton case agrees with the one in [13] in the limit of infinitely heavy D-particle.

As our framework is rather general, the present work can be extended in many directions. Some of these possibilities will be discussed in Sec. 4. Finally, we will close this article by advocating an intriguing viewpoint which regards our work as describing a dynamical transition of CFT’s.

## 2 Scattering Amplitude with Fixed D-particle Trajectory

**The setup**

Let us begin by computing the amplitude for closed string states scattering from a classical D-particle with a specified trajectory as a preliminary step to full quantization.

In this article we exclusively deal with the amplitude due to the disk topology. Under
certain conditions to be discussed later, conformal invariance will be seen to hold \[1\] and this allows us to take the string worldsheet to be a unit disk denoted by $\Sigma$. Let $\theta$ be the polar angle describing its boundary $\partial \Sigma$. A D-particle is characterized by the condition that the ends of the open string terminate on the worldline of the D-particle, which we parametrize by $f^\mu(t)$. As the ends may terminate anywhere on the worldline, the precise Lorentz-covariant condition for a D-particle should be expressed as \[2\]

$$X^\mu(\theta) = f^\mu(t(\theta)) \text{ on } \partial \Sigma,$$

where $X^\mu$ denote the open string coordinates and the function $t(\theta)$ is arbitrary. This means that in the path integral formulation we seek, we must integrate over $X^\mu(z)$ in the bulk and over $t(\theta)$ on the boundary. Thus the relevant amplitude is given by

$$\mathcal{V}(f^\mu, \{k_i\}) = \frac{1}{g_s} \int D^2 X(\zeta) \delta(X^\mu(\theta) - f^\mu(t(\theta))) e^{-S[X]} \prod_i g_s V_i(k_i),$$

where $g_s$ is the string coupling constant,

$$S[X] = \frac{1}{4\pi\alpha'} \int \Sigma d^2 z \partial_\alpha X^\mu \partial_\alpha X_\mu$$

is the open string action \[3\] and $V_i(k_i)$ are the vertex operators for closed string states carrying momenta $k_i$. As will become clear later, closed string states can be factorized so that we will be able to deal with any states on equal footing. With this in mind, we will take tachyon emission vertices $V_i(k_i) = \int d^2 z e^{i k_i \cdot X(z)}$ for illustration purpose.

Consider first the $t(\theta)$ integral. It is convenient to split it into the one over the $\theta$-independent mode, to be denoted by $t$, and the one over the remaining non-constant mode. Further, in order to preserve the general coordinate invariance along the trajectory, we use geodesic normal coordinate expansion \[4\] for $f^\mu(t(\theta))$ around $f^\mu(t)$. For the present case of one dimensional submanifold embedded in a flat space, it is easy to find

$$f^\mu(t(\theta)) = f^\mu(t) + \hat{f}^\mu(t) \zeta(\theta) + \frac{1}{2} K^\mu \zeta(\theta)^2 + \frac{1}{3!} \left( -\hat{f}^\mu \hat{K}^\nu K_\nu - \frac{3}{2} \hat{K}^\mu + P^\mu_{\nu\rho} \partial_\rho f_\nu \right) \zeta(\theta)^3 + O(\zeta^4).$$

\[1\] Precisely speaking, we will be able to maintain BRST invariance. See also the discussion at the end of Sec. 4.

\[2\] We use Euclidean worldsheet and the space-favored Minkowski metric $\eta_{\mu\nu} = \text{diag} (-, +, +, \cdots, +)$ for the target space. Also we write $X^\mu(z)$ for $X^\mu(z, \bar{z})$, etc.
Here, $\zeta(\theta)$ is the normal coordinate, a dot stands for $t$-derivative, and $h(t)$, $K^\mu(t)$ and $P^{\mu\nu}(t)$ are, respectively, the one-dimensional induced metric on the trajectory, the extrinsic curvature and a projection operator normal to the trajectory. Their explicit expressions are

\begin{align}
h & \equiv \dot{f}^\mu \dot{f}_\mu, \\
K^\mu & \equiv \ddot{f}^\mu - \frac{1}{2} \dot{h} \dot{f}^\mu = P^{\mu\nu} \ddot{f}_\nu, \\
P^{\mu\nu} & \equiv \eta^{\mu\nu} - h^{\mu\nu}, \\
h^{\mu\nu} & \equiv \frac{\dot{f}_\mu \dot{f}_\nu}{h},
\end{align}

where we have also introduced the projection operator $h^{\mu\nu}$ along the trajectory. We will take the functional measure $Dt(\theta)$ to mean $dtD\zeta(\theta)$ in the following. For lack of means to perform the $\zeta(\theta)$-integration exactly, we will treat the non-Gaussian higher order corrections pertubatively. Since this can be regarded as an expansion in the number of derivatives in $t$, the basic picture of our treatment is that to the zero-th order the boundary of the disk is attached to a point $t$ (to be integrated) on the trajectory and the effects of the non-local spread is then taken into account by the subsequent integration over $\zeta(\theta)$.

It should be emphasized that although we must assume the smoothness of the trajectory in order to be able to truncate the expansion, the approximation nevertheless is fully covariant.

As for $X^\mu(z)$ we likewise split it into the constant and the non-constant modes:

$$X^\mu(z) = x^\mu + \xi^\mu(z).$$

Then the $\delta$-function at the boundary decomposes into the product

$$\delta(X^\mu(\theta) - f^\mu(t) - \dot{f}^\mu(t)\zeta(\theta) - \cdots) = \delta(x^\mu - f^\mu(t))\delta(\xi^\mu(\theta) - \dot{f}^\mu(t)\zeta(\theta) - \cdots).$$

**Integration over $X^\mu(z)$**

With this setup, we now perform the integration over $X^\mu(z)$. $x^\mu$ integral trivially replaces the zero mode $x^\mu$ in the vertex operators by $f^\mu(t)$, giving

$$e^{ik^\mu f_\mu(t)},$$

where

$$k^\mu \equiv \sum_i k^\mu_i.$$
To perform the one over $\xi^\mu(z)$, it is convenient to set up a complete orthonormal moving frame $\{\hat{e}^\mu_A\}$, $A = (0, a)$, $a = 1, 2, \ldots D - 1$, where $D = 26$ is the dimension of the target spacetime:

\[
\hat{e}_0^\mu = \frac{\dot{f}_\mu}{\sqrt{-h}}, \quad \hat{e}_A^\mu \hat{e}_B^\nu = \eta_{AB},
\]

\[
\hat{e}_A^\mu \hat{e}_B^\nu \eta^{AB} = \hat{e}_A^\nu \hat{e}_A^\mu = \eta^{\mu\nu} = -\hat{e}_0^\mu \hat{e}_0^\nu + \sum_a \hat{e}_a^\mu \hat{e}_a^\nu,
\]

\[
\hat{e}_0^\mu \hat{e}_0^\nu = -h^{\mu\nu}, \quad \sum_a \hat{e}_a^\mu \hat{e}_a^\nu = P^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}.
\]

Then we can decompose $\xi^\mu(z)$ in this frame as

\[
\xi^\mu(z) = \sum_A \hat{e}_A^\mu \rho_A(z),
\]

\[
\rho_A(z) = \hat{e}_A^\mu \xi^\mu(z).
\]

$\rho_0$ ($\rho_a$) describes the fluctuation in the tangential (transverse) direction. At the boundary, the constraint keeps the fluctuations truly along the curved trajectory. In terms of $\rho_A$, this reads

\[
\rho_0(\theta) = -\sqrt{-h}\zeta(\theta) + \frac{1}{3!} \frac{K^2}{\sqrt{-h}} \zeta(\theta)^3 + \mathcal{O}(\zeta^4),
\]

\[
\rho_a(\theta) = \hat{e}_a^\mu \left( \frac{1}{2} \frac{\dot{f}_\mu}{f_\mu} \zeta(\theta)^2 + \frac{1}{9!} \left( \frac{3h}{2h} \frac{\dot{f}_\mu}{f_\mu} + \partial_\mu f_\mu \right) \zeta(\theta)^3 \right) + \mathcal{O}(\zeta^4).
\]

The action for $\xi^\mu(z)$ now becomes

\[
\frac{1}{4\pi\alpha'} \int_\Sigma d^2 z \partial_\alpha \xi^\mu \partial_\mu \xi^\mu = \frac{1}{4\pi\alpha'} \left( -\int_\Sigma d^2 z \rho_A \partial_\alpha \rho_A + \int_{\partial\Sigma} d\theta \rho_A \partial_\alpha \rho_A \right).
\]

In order to separate the bulk and the boundary interactions, the surface integral should vanish, namely either $\partial_\alpha \rho_A = 0$ or $\rho_A = 0$ on the boundary. Since our expansion scheme is such that to the leading order the boundary $\partial\Sigma$ is mapped to a point on a trajectory, we must adopt the boundary condition so that $\rho_0(\theta)$ can fluctuate only along the trajectory while $\rho_a(\theta)$ cannot fluctuate. This leads to the condition

\[
\partial_\alpha \rho_0(\theta) = 0 \quad \text{Neumann},
\]

\[
\rho_a(\theta) = 0 \quad \text{Dirichlet},
\]

which is precisely the familiar boundary condition for a D-particle.
Now let us integrate over $\rho_A(z)$. To this end we write the $\delta$-function constraint in the form

$$
\delta(\xi^\mu(\theta) - \hat{f}^\mu(\theta) - \cdots) = \int \mathcal{D}\nu^\mu(\theta) \exp \left( i \int d\theta \nu_\mu(\theta)(\hat{e}^\mu \rho_0(\theta) - \hat{f}^\mu(\theta) - \cdots) \right),
$$

where we have used the boundary condition $\rho_a(\theta) = 0$. Again it is convenient to introduce the projected quantities $\nu^A = \nu_\mu \hat{e}^\mu_A$. Then the above expression reduces to

$$
\int \prod\mathcal{D}\nu^A(\theta) \exp \left( i \int d\theta \nu_\mu(\theta)(\hat{e}^\mu \rho_0(\theta) - \hat{f}^\mu(\theta) - \cdots) \right).
$$

(25)

It is important to note that all the functions of $\theta$ do not possess constant modes since such modes have already been separated.

Now it is straightforward to perform the integration over $\rho_A$. Together with the tachyon vertex insertions, the relevant integral is

$$
I = \int \mathcal{D}\rho_A \exp \left( \frac{1}{4\pi \alpha'} \int d^2z \rho_A \partial^2 \rho^A + i \int d^2z J_A \rho^A \right),
$$

$$
J_A(z) = \sum_i k_i A \delta^2(z - z_i) + \delta_{A0} \delta(|z| - 1) \nu_0(\theta),
$$

(26)

where we have introduced the projected momenta $k_i A = k_i \hat{e}_A^\mu$. The result is

$$
I = \exp \left( \frac{\alpha'}{2} \int d^2z d^2z' J_A(z) G_A(z, z') J_A(z') \eta_{AA} \right)
$$

$$
= \exp \left( \frac{\alpha'}{2} \sum_{i,j} k_i A k_j A G_A(z_i, z_j) \eta_{AA} \right)
$$

$$
\cdot \exp \left( -\frac{\alpha'}{2} \int d\theta d\theta' \nu_0(\theta) N(\theta, \theta') \nu_0(\theta') \right)
$$

$$
\cdot \exp \left( -\alpha' \int d\theta \sum_i k_i A \tilde{N}(z_i, \theta) \nu_0(\theta) \right).
$$

(27)

Here $G_A(z, z')$ are the Neumann and the Dirichlet functions on the unit disk given by

$$
\partial^2 G_A(z, z') = 2\pi \delta^2(z - z'),
$$

(28)

$$
G_0(z, z') = N(z, z') = \ln |z - z'| + \ln |1 - \frac{1}{zz'}|,
$$

(29)

$$
G_a(z, z') = D(z, z') = \ln |z - z'| - \ln |1 - zz'|.
$$

(30)

$\tilde{N}(z_i, \theta)$ is the Neumann function with one argument on the boundary with the zero mode part omitted. This follows from the remark made earlier, in particular from the lack of zero mode for $\nu_0(\theta)$. Explicitly, $\tilde{N}(z, \theta) = N(z, \theta) - \ln(1/|z|)$ and it vanishes for $z = 0$.
possesses no zero mode and has a regularized representation \([30, 31]\) which will be needed to deal with the divergence at \(\theta = \theta'\):

\[
N(\theta, \theta') = -2 \sum_{n=1}^{\infty} \frac{e^{-cn}}{n} \cos n(\theta - \theta'). \tag{31}
\]

It is easy to check that its inverse \(N^{-1}(\theta, \theta')\) in the space of non-zero modes on a circle is given by

\[
N^{-1}(\theta, \theta') = -\frac{1}{4\pi^2} \partial_{\theta}^2 N(\theta, \theta'), \tag{32}
\]

\[
\int d\phi N(\theta, \phi) N^{-1}(\phi, \theta') = \tilde{\delta}(\theta - \theta') \equiv \delta(\theta - \theta') - \frac{1}{2\pi}. \tag{33}
\]

At this stage we perform the integration over \(\nu_0(\theta)\). Assembling terms containing \(\nu_0\) from (25) and (27), the relevant integral is

\[
\int \mathcal{D}\nu_0 \exp \left(\frac{-\alpha'}{2} \int d\theta d\theta' \nu_0(\theta) N(\theta, \theta') \nu_0(\theta') + i \int d\theta \tilde{j}_0(\theta) \nu_0(\theta)\right)
= \exp \left(-\frac{1}{2} \int d\theta d\theta' \tilde{j}_0(\theta) N^{-1}(\theta, \theta') \tilde{j}_0(\theta')\right) \tag{34}
\]

where we have absorbed the factor of \(\alpha'\) and defined

\[
\tilde{j}_0(\theta) \equiv \frac{j_0(\theta)}{\sqrt{\alpha'}} = i \sqrt{\alpha'} \tilde{k}(\theta) + \sqrt{-\frac{h}{\alpha'}} \zeta(\theta) - \frac{1}{3!} \sqrt{-h\alpha'} \zeta^3(\theta) + O(\zeta^4), \tag{35}
\]

\[
\tilde{k}(\theta) \equiv \sum_{i} k_{i0} N(z_i, \theta). \tag{36}
\]

**Integration over \(\zeta\)**

We are now ready to perform the \(\zeta\)-integration, which takes the form of an \(\alpha'\)-expansion. As was already emphasized in the introduction, this is simply an organization of the derivative expansion and should not be confused with the usual \(\alpha'\) expansion in the context of non-linear \(\sigma\) model. To facilitate this expansion, we rescale the normal coordinate and introduce \(\bar{\zeta}\) defined by

\[
\zeta = \sqrt{\frac{\alpha'}{-h}} \bar{\zeta}. \tag{37}
\]

The change of the functional measure due to this transformation can easily be computed by expanding \(\zeta(\theta)\) into Fourier modes and employing the \(\zeta\)-function regularization. Remembering that the zero mode has been removed, we get

\[
\mathcal{D}\zeta = \mathcal{D} \left(\sqrt{\frac{\alpha'}{-h}} \bar{\zeta}\right) = \left(\prod_{n=1}^{\infty} \sqrt{\frac{\alpha'}{-h}}\right)^2 \mathcal{D}\bar{\zeta}
= \exp \left(2 \ln \sqrt{\frac{\alpha'}{-h}} \sum_{n=1}^{\infty} 1\right) \mathcal{D}\bar{\zeta} = \sqrt{\frac{-h}{\alpha'}} \mathcal{D}\bar{\zeta}. \tag{38}
\]
With the rescaling (37), \( \tilde{j}_0(\theta) \) becomes

\[
\tilde{j}_0(\theta) = \bar{\zeta}(\theta) + i\sqrt{\alpha'} \tilde{k}(\theta) - \frac{1}{3!}\alpha' K^2 h^2 \bar{\zeta}(\theta)^3 + \mathcal{O}(\alpha'^3/2). \tag{39}
\]

The integrand for the \( \zeta \)-integration consists of (34) plus the boundary contribution due to the transverse fluctuation. Introducing the variables \( \hat{\nu}^\mu \equiv \nu^\mu \hat{e}_a \) and taking into account the rescaling (37), the expression to be added to the exponent of (34) is

\[
\frac{i\alpha'}{2h} \hat{\nu}^\mu \hat{f}_\mu \bar{\zeta}^2 + \mathcal{O}(\alpha'^3/2). \tag{40}
\]

Substituting (39) into (34), adding the above contribution and keeping terms up to \( \mathcal{O}(\alpha') \), the \( \bar{\zeta} \) integration to be performed becomes

\[
\int D\bar{\zeta} \exp \left( -\frac{1}{2} \bar{\zeta}D\bar{\zeta} - i j_\bar{\zeta} \bar{\zeta} + \frac{\alpha'}{3!} K^2 h^2 \bar{\zeta}N^{-1} \bar{\zeta}^3 + \frac{\alpha'}{2} \bar{k}N^{-1} \bar{k} \right), \tag{41}
\]

where we have used condensed notations such as \( \bar{\zeta}D\bar{\zeta} = \int d\theta d\theta' \bar{\zeta}(\theta)D(\theta, \theta')\bar{\zeta}(\theta') \), etc.. \( D \) and \( j_\bar{\zeta} \) appearing in this expression are defined as

\[
D \equiv N^{-1} - i\alpha' \frac{\hat{\nu}^\mu \hat{f}_\mu}{h} = N^{-1} \left( 1 - i\alpha' N \frac{\hat{\nu}^\mu \hat{f}_\mu}{h} \right), \tag{42}
\]

\[
j_\bar{\zeta} \equiv \sqrt{\alpha'} N^{-1} \bar{k}. \tag{43}
\]

To perform this integral, we make a shift \( \bar{\zeta} = \bar{\zeta} - iD^{-1} j_\bar{\zeta} \). This produces a term which cancels the last term in (41). Again keeping terms up to \( \mathcal{O}(\alpha') \), the exponent becomes

\[
-\frac{1}{2} \bar{\zeta}D\bar{\zeta} + \frac{\alpha'}{3!} K^2 h^2 \bar{\zeta}N^{-1} \bar{\zeta}^3. \tag{44}
\]

The second term, quartic in \( \bar{\zeta} \), will be treated perturbatively. Then the integration produces two terms. One is the determinant factor proportional to

\[
\text{Tr} \left( i\alpha' N \frac{\hat{\nu}^\mu}{h} \hat{f}_\mu \right) = \frac{i\alpha'}{h} \int d\theta N(\theta, \theta) \hat{\nu}^\mu(\theta). \tag{45}
\]

With the regularization discussed before, \( N(\theta, \theta) \) is a (divergent) constant. The remaining \( \theta \) integral of \( \hat{\nu}^\mu(\theta) \) vanishes since it does not contain the zero mode. Therefore, to this order, \( \hat{\nu}^\mu \) integral is inert.

The other contribution is due to the quartic term. Contractions of \( \bar{\zeta} \)’s produce a divergent contribution

\[
\frac{\alpha'}{2} \frac{K^2}{h^2} N(\theta', \theta') \int d\theta d\theta' N^{-1}(\theta, \theta') N(\theta, \theta'). \tag{45}
\]
As for \( \int d\theta d\theta' N^{-1}(\theta, \theta') N(\theta, \theta') \), we can employ \( \zeta \)-function regularization and find that it is actually finite:

\[
\int d\theta d\theta' N^{-1}(\theta, \theta') N(\theta, \theta') = \int d\theta \frac{1}{\pi} \sum_{n=1}^{\infty} 1 = -1.
\] (46)

On the other hand \( N(\theta', \theta') \) is truly divergent and using the regularized form given in (31) we get

\[
N(\theta', \theta') = 2 \ln \epsilon + \mathcal{O}(\epsilon).
\] (47)

This divergence signals the breakdown of conformal invariance for a general trajectory.

Assembling all together (and appending a \( 1/g_s \) factor indicating the disk amplitude), we find that to \( \mathcal{O}(\alpha') \) the \( \zeta \)-integration yields

\[
\frac{1}{g_s} \sqrt{-h} \left( 1 - \alpha' K_{\mu}^2 h^2 \ln \epsilon \right).
\] (48)

If we recall the definition \( h = \dot{f}_{\mu} \dot{f}_{\mu} \), the first factor is nothing but the action of a relativistic particle of mass \( 1/(g_s \sqrt{\alpha'}) \), expected of a D-particle, and the second factor is the correction produced by the boundary interaction. We now show that the divergence produced above is precisely of such a form that it can be absorbed by the renormalization (shift) of the collective coordinate \( f_{\mu}(t) \) and yields the renormalized action. Let \( f_{\mu}^R \) denote the renormalized trajectory function and set

\[
f_{\mu}^R = f_{\mu}^R + \delta f_{\mu}^R,
\]

where

\[
\delta f_{\mu}^R = -\alpha' K_{\mu}^R h_R^2 \ln \epsilon.
\] (50)

Then by a simple calculation one can check that \( \sqrt{-h} \) becomes

\[
\sqrt{-h} = \sqrt{-h_R} + \delta \sqrt{-h_R}
\]

\[
\delta \sqrt{-h_R} = -\frac{1}{\sqrt{-h_R}} \delta \sqrt{-h_R} = \alpha' K_{\mu}^R \sqrt{-h_R}.
\] (51)

showing that (18) simply becomes \( \sqrt{-h_R}/\alpha' \). This is in parallel with the renormalization of the Dirac-Born-Infeld action discussed in [31].

Note that the \( \beta \) function read off from (50) is proportional to \( K_{\mu}^R \) and if we set this to zero we obtain a straight line classical trajectory. As we wish to describe a scattering which necessarily requires non-straight path, we cannot set \( K_{\mu}^R \) to zero. This does not
mean, however, that the consistency of the theory is impaired. It will be shown that the on-shell amplitude will be fully BRST invariant. Further clarifying discussion will be given at the end of Sec. 4

**The amplitude and the condition on** \( f^\mu(t) \)

Putting everything together, the amplitude for the scattering of tachyons from a D-particle becomes

\[
\mathcal{V}_T(f_R^\mu, \{k_i\}) = \frac{1}{g_s} \int dt \sqrt{-h_R} e^{ik\cdot f} \cdot \prod_i (g_s d^2 z_i) \exp \left( \frac{\alpha'}{2} \sum_{i,j} k^\mu_i k^\nu_j (h_{\mu\nu} N(z_i, z_j) + P_{\mu\nu} D(z_i, z_j)) \right),
\]

where \( k^\mu = \sum_i k^\mu_i \) is the total momentum of the tachyons, and the prime on the summation means that, as usual, we omit the singular part of the Green’s functions for \( i = j \). On the right hand side, except for the factor \( \sqrt{-h_R} \) just discussed, \( f^\mu(t) \) and its derivative are still bare quantities and hence contain divergence.

First consider the factor \( f^\mu k^\mu_{\mu} \) in \( e^{ik\cdot f} \). The divergent piece is proportional to

\[
k^\mu K^\mu_R = k^\mu P^{\mu\nu}_R \dot{f}^\nu_R = \frac{d}{dt} (k^\mu \dot{f}^\mu_R) - (k^\mu \dot{f}^\mu_R) \ddot{f}^\nu_R h_R,
\]

which vanishes if the *conservation of the D-particle current*

\[
k^\mu \dot{f}^\mu_R = 0 \tag{53}
\]

holds.

The same condition is seen to arise from the requirement of the \( SU(1, 1) \simeq SL(2, R) \) invariance of the integrand. The \( SL(2, R) \) transformation of the unit disk can be written as

\[
z \longrightarrow \bar{z} = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 = 1. \tag{55}
\]

The factors containing a particular coordinate, say \( z_1 \), produced by this transformation are

\[
F_1 = |\bar{\beta} z_1 + \bar{\alpha}|^{-4}, \\
F_2 = \left( |\bar{\beta} z_1 + \bar{\alpha}| |\bar{\beta} z_1^{-1} + \bar{\alpha}| \right)^{-\alpha' k_{10} \cdot k_{20}} |\bar{\beta} z_1^{-1} + \bar{\alpha}|^{-\alpha' k_{10}^2}, \\
F_3 = |\bar{\beta} z_1 + \bar{\alpha}|^{\alpha' \sum k_{2a}^2}.
\]

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$F_1$ is from the integration measure, $F_2$ from the Neumann function and $F_3$ from the Dirichlet function. Putting them together, we easily see that they cancel if and only if the following two conditions are met:

$$0 = k \cdot \dot{f},$$

$$0 = \alpha' \left( \sum_a k_{ja}^2 + k_{j0}^2 \right) - 4 = \alpha' k_j^2 - 4 \quad \text{for all } j.$$  

The first is the current conservation, as promised, and the second is the on-shell condition for the tachyons.

At this stage, the current conservation is as yet a condition that can only be imposed by hand for consistency. When the dynamics of the D-particle itself is taken into account, however, we will see that it arises naturally.

Next consider the $\ln \epsilon$'s residing in $h_{\mu\nu}$ and $P_{\mu\nu}$ in the last exponential factor of (52). Even with the current conservation, they remain in the form $c_1(\partial_1^2 f_R(t))^2 \ln \epsilon + c_2 \partial_3^2 f_R(t) \ln \epsilon$, where $c_1, c_2$ are functions of $k_i, z_i$ and $\dot{f}_R$. These unwanted terms, however, will be seen to vanish by quantum averaging over the trajectory, to be performed in the next section.

With the divergent terms put aside, the rest of the calculation is completely standard. As an illustration, consider the case of two tachyons. The $SL(2, R)$ invariance allows us to fix the position of one of the tachyon, say $z_1$, (and if desired the angle of $z_2$) to 0 and the amplitude becomes

$$V_T(f_{R}, \{k_i\}) = g_s \int dt \sqrt{-h_R} e^{ik f_R} \int_{0}^{1} dy y^{(\alpha'/2)k_1} y^{(1-y)\alpha'(k_1 \cdot \dot{f}_R)^2 / j_R^2 - 2},$$

where on the right hand side all the $f^\mu$'s are renormalized. Hereafter we shall omit the subscript R throughout.

**Boundary state representation**

Looking at the expression (52), one immediately notices that the exponential in the second line is nothing but the usual open string amplitude except that for the directions normal to the trajectory the Neummann function is replaced by the Dirichlet function. This means that we can factorize the closed string vertices and isolate the D0-D0-string interaction vertex in the form of a boundary state as seen from the closed string channel.
Let us demonstrate this for the $N$ tachyon amplitude. In order to compare with (52), where $|z_i| \leq 1$, we should use the “bra” boundary state prepared at $z = 1$ given by

$$\langle B; f | = | f \rangle \exp \left(- \sum_{n \geq 1} \frac{1}{n} \alpha_n^\mu D_\mu^\nu \tilde{\alpha}_{n,\nu} \right),$$

(62)

$$D_\mu^\nu = h_\mu^\nu - P_\mu^\nu,$$

(63)

where $\langle f \rangle$ represents the vacuum for the non-zero modes and the position eigenstate for the zero mode.\footnote{Since the zero mode of $X^\mu(z)$ is identified with $f^\mu(t)$, we should take the position eigenstate for \textit{all directions}.} It satisfies

$$\langle B; f | \alpha_{-n}^\mu = \langle B; f | (-D_\nu^\mu \tilde{\alpha}_{n,\nu}) ,$$

(64)

$$\langle B; f | \tilde{\alpha}_{-n}^\mu = \langle B; f | (-D_\nu^\mu \alpha_{n,\nu}) ,$$

(65)

$$\langle B; f | x^\mu = \langle B; f | f^\mu .$$

(66)

It is then straightforward to get

$$\langle B; f | e^{ik_1 \cdot X(z_1, \bar{z}_1)} e^{ik_2 \cdot X(z_2, \bar{z}_2)} \cdots e^{ik_N \cdot X(z_N, \bar{z}_N)} | 0 \rangle = e^{ik \cdot f} e^{\frac{1}{2} \left( \sum_{i,j} k_i^\mu P_{\mu\nu} k_j^\nu D(z_i, z_j) + \sum_{i,j} k_i^\mu h_{\mu\nu} k_j^\nu N(z_i, z_j) + 2 \sum_i k_i^\mu \ln |z_i| h_{\mu\nu} k_i^\nu \right)},$$

(67)

where $k^\mu$ is, as before, the total momentum of the tachyons. Note that this agrees with (52) except for the last term in the second exponent. This term vanishes if and only if the current conservation condition $k_\mu \dot{f}^\mu = 0$ is satisfied. Thus with this condition, the amplitude with the insertion of arbitrary closed string vertices $V_i(k_i)$ can be written as

$$\mathcal{V}(f^\mu, \{k_i\}) = \frac{1}{g_s} \int dt \sqrt{-h} \alpha' \int (g_s d^2 z) \langle B; f | V_1(k_1, z_1) V_2(k_2, z_2) \cdots V_N(k_N, z_N) | 0 \rangle ,$$

(68)

where appropriate symmetrization for $V_i(k_i)$ is understood. If one wishes, one can of course add the ghost part of the boundary state to make it into BRST invariant form.

This was in fact the guiding principle used by Ishibashi \footnote{Since the zero mode of $X^\mu(z)$ is identified with $f^\mu(t)$, we should take the position eigenstate for \textit{all directions}.}. Our formalism has given a firm ground for his proposal and at the same time revealed that in general the boundary state receives corrections involving higher derivatives of $f^\mu(t)$, just like in the case of Dirac-Born-Infeld action \footnote{Since the zero mode of $X^\mu(z)$ is identified with $f^\mu(t)$, we should take the position eigenstate for \textit{all directions}.}.
Quantization of D-particle and Amplitude with Recoil

Preliminary

Having obtained the amplitude describing the interaction of closed string states with a D-paraticle along a fixed trajectory, we now quantize the trajectory itself. This means that we integrate over $f^\mu(t)$ with the weight $\exp(-im_0 \int_0^1 dt \sqrt{-h})$, where $m_0 \sim 1/(g_s\sqrt{\alpha'})$ is the bare D-particle mass inversely proportional to the string coupling $g_s$. The action in the exponent is nothing but the previous amplitude without the vertex operator insertions, as it should be for the first quantized formalism \[33\]. It is actually more convenient to turn it into the Polyakov form. How this comes about is well-known \[34\] but for the sake of completeness let us briefly recall the procedure.

First, one introduces a Lagrange multiplier, call it $\alpha(t)$, so that the induced metric $h$ can be treated as an independent variable. Next, define the proper time $\tau(t) \equiv \int_0^t dt' \sqrt{-h(t')}$ and the total proper time $T = \tau(1)$. Then the integration over $h$ reduces to the one over $T$, the modulus of the trajectory. As was argued in \[34\], the fact that the fluctuation of $\alpha(t)$ is short-ranged allows one to replace it by its mean value $<\alpha> \sim 1/\epsilon$, where $\epsilon$ is a short-time cutoff. This divergence and the one arising in the $T$-integration measure are then absorbed by the mass renormalization, producing the renormalized mass term of the form $-\frac{1}{2}m^2T$. This means that effectively the original $\tau$ is replaced by $m\tau$ and the new $\tau$ should be treated as a variable of dimension mass$^{-2}$.

The net result of all this is that, instead of $-m_0 \int_0^1 dt \sqrt{-h}$, we may use the effective action of the Polyakov type, namely

$$S = \int_0^T d\tau \frac{1}{2}(\dot{\tau}^2 - m^2)$$

(69)

and perform an overall $T$-integration $\int_0^\infty dT$. We will take this as our starting point. Hereafter a dot denotes the derivative with respect to $\tau$.

Integration over $f^\mu(\tau)$

With this preliminary, we now begin the calculation of the amplitude with quantized D-particle. Rather than dealing with an amplitude with some definite closed string vertex insertions, it is more advantageous to compute the amputated vertex

$$\langle \mathcal{B}; p^\mu, p'^\nu, k^\mu | = (p^2 + m^2)(p'^2 + m^2) \int d^D f d^D f' e^{i(p\cdot f - p'\cdot f')} \int_0^\infty dT$$
\[
\int_{f^\mu(0)=f^\mu}^{f^\mu(T)=f'^\mu} \mathcal{D}f^\mu(\tau) e^i \int_0^T d\tau \frac{1}{2}(f'^2-m^2) \int_0^T d\tilde{\tau} e^{ik \cdot f(\tilde{\tau})} \langle \tilde{B}; \partial^\mu f(\tilde{\tau}) \rangle , \tag{70}
\]
from which one can compute any scattering amplitude just as in \([68]\). It describes the transition of a D-particle with the initial momentum \(p^\mu\) into one with the final momentum \(p'^\mu\). Here \(f^\mu\) and \(f'^\mu\) are, respectively, the initial and the final position of the D-particle, \(k^\mu = \sum_i k_i^\mu\) is the total momentum of the closed string states. We have also separated the zero-mode part so that \(\langle \tilde{B}; \partial^\mu f(\tilde{\tau}) \rangle\), \((n = 1, 2, 3)\), represents the non-zero mode part of the boundary state for a fixed trajectory, which, in our approximation, depends primarily on \(\dot{f}(\tilde{\tau})\), except for the divergent pieces involving \(\partial^\mu f(\tilde{\tau}), (n = 2, 3)\), mentioned in the previous section.

First, in order to isolate the effect of \(f^\mu(\tau)\) integration on the boundary state, we replace \(\partial^\mu f(\tilde{\tau})\) in the boundary state by new variables \(v_n^\mu\) via the insertion

\[
1 = \prod_n \int dv_n \int d\omega_n \exp \left( i \omega_n \cdot (v_n - \partial^\mu f(\tilde{\tau})) \right) . \tag{71}
\]

Then the \(f^\mu(\tau)\) integration to be performed is

\[
\int \mathcal{D}f^\mu \exp \left( i \int_0^T d\tau \frac{1}{2} f'^2 - \sum_n i \omega_n \cdot \partial^\mu f(\tilde{\tau}) + i k \cdot f(\tilde{\tau}) \right) . \tag{72}
\]

To extract the dependence on the endpoint values \(f^\mu\) and \(f'^\mu\), we expand around the classical solution satisfying the endpoint condition:

\[
f^\mu(\tau) = f^\mu_{cl}(\tau) + \tilde{f}^\mu(\tau) , \tag{73}
\]
\[
f^\mu_{cl}(\tau) = \frac{y^\mu}{T} \tau + f^\mu , \tag{74}
\]
where \(y^\mu = f'^\mu - f^\mu\) and the fluctuation satisfies \(\tilde{f}^\mu(0) = \tilde{f}^\mu(T) = 0\). The contribution from this classical part is

\[
\exp \left( i T \left( \frac{1}{2} y^2 + k \cdot y - y \omega \right) + i k \cdot f \right) . \tag{75}
\]

Combined with the Fourier transform factor \(e^{i p \cdot f - i p' \cdot f'} = e^{-i y \cdot (p' - p)}\), the integration over \(f^\mu\) gives the momentum conserving delta function \(\delta(p + k - p')\), while that over \(y^\mu\) yields

\[
T^{D/2} \exp \left( -\frac{i}{2T} (\omega + p'T - k\tilde{\tau})^2 \right) . \tag{77}
\]
The remaining $\tilde{f}$ integral to be performed is
\[ \int_{\tilde{f}(0) = \tilde{f}(T) = 0} \mathcal{D}\tilde{f} \exp \left( \frac{i}{2} \int_0^T d\tilde{\tau} \tilde{f}^2 - \sum_n i\omega_n \cdot \partial_n^\tau \tilde{f}(\tilde{\tau}) + ik \cdot \tilde{f}(\tilde{\tau}) \right). \] (78)

This is easily done by writing $\tilde{f}(\bar{\tau}) = \int_0^T d\tau \delta(\tau - \bar{\tau}) \tilde{f}(\tau)$. Then the exponent of (78) becomes
\[ E_f = -\frac{i}{2} \int_0^T d\tau \tilde{f} \partial_\tau \tilde{f} + i \int_0^T d\tau \tilde{f}(\tau) j(\tau), \] (79)

where $j(\tau) = k\delta(\tau - \bar{\tau}) - \sum_n (-1)^n \omega_n \partial_n^\tau \delta(\tau - \bar{\tau})$. (80)

Upon integrating over $\tilde{f}(\tau)$, we get
\[ T^{D/2} e^{E[j]}, \] (81)
\[ E[j] = \frac{i}{2} \int d\tau d\tau' j(\tau) G(\tau, \tau') j(\tau'). \] (82)

$T^{D/2}$ is the determinant factor and it cancels the similar one produced by the zero mode integration in (77). The Green’s function $G(\tau, \tau')$ is defined to satisfy
\[ \partial_\tau G(\tau, \tau') = \delta(\tau - \tau') - \theta(\tau' - \tau - T), \] (83)

where $\theta(x)$ is the usual step function with $\theta(0) \equiv \frac{1}{2}$. Upon expanding $E[j]$ in (82), one is required to evaluate the expressions of the form $\partial_\tau G(\tau, \tau')|_{\tau = \tau'}$, which in general contain $\delta(x)$ and its derivatives and need regularization. If one takes the representation $\delta(x) = (2/T)(\frac{1}{2} + \sum_{n \geq 1} \cos(2\pi nx/T))$ and employ the standard $\zeta$-function regularization, one finds that $\delta(x)$ and all its derivatives vanish at $x = 0$. In this way one easily finds that the only non-vanishing expressions of the above type are
\[ G(\tau, \tau')|_{\tau' = \tau} = \frac{\tau}{T}(\tau - T), \] (84)
\[ \partial_\tau G(\tau, \tau')|_{\tau' = \tau} = \frac{\tau}{T} - \frac{1}{2}, \] (85)
\[ \partial_\tau \partial_\tau G(\tau, \tau')|_{\tau' = \tau} = \frac{1}{T}. \] (86)

This means that effectively contributions of $\partial_\tau^2 \tilde{f}(\bar{\tau})$ for $n \geq 2$ wash out to zero upon quantization of the trajectory\footnote{This situation is expected to change if we take the higher order corrections into account so that the particle action itself is modified.}. With these formulae, the exponent $E[j]$ is easily evaluated to be
\[ E[j] = \frac{i}{2T} \omega^2 - \frac{k \cdot \omega}{T} \left( \bar{\tau} - \frac{1}{2} T \right) + \frac{i}{2T} k^2 \bar{\tau} \left( \bar{\tau} - T \right). \] (87)
Adding this to the contribution from the zero-mode integration (77), we find
\[-\frac{i}{2T}(\omega + p'T - k\bar{\tau})^2\]
\[+i\frac{\omega^2}{2T} - i\frac{k \cdot \omega}{T}\left(\bar{\tau} - \frac{1}{2}T\right) + \frac{i}{2T}k^2\bar{\tau}(\bar{\tau} - T)\]
\[= -i\omega \cdot \left(p + \frac{k}{2}\right) + i\frac{1}{2}(p'^2 - p^2)\bar{\tau} - \frac{i}{2}p'^2T,\] (88)
where we used the momentum conservation. Note that the parts quadratic in \(\omega\) have canceled out. Putting the factor \(e^{i\omega \cdot v}\) (see (71)) back in, we see that the \(\omega\)-integration produces a delta function
\[\delta(v - (p + \frac{k}{2})).\] (89)
Thus, remarkably \(v\) is completely determined to be equal to \(p + \frac{k}{2} = \frac{1}{2}(p + p')\), the mean of the initial and the final D-particle momenta. This result was anticipated in [28] but now we have a proof. As we can write
\[k \cdot v = k \cdot \left(p + \frac{k}{2}\right) = \frac{1}{2}(p'^2 + m^2) - \frac{1}{2}(p^2 + m^2),\] (90)
the crucial consistency condition, \(k \cdot v = 0\), is automatically satisfied when the initial and the final states of the D-particle are on shell.

The remaining integrals over \(\bar{\tau}\) and \(T\) are trivial. They give
\[\int_0^\infty dT \int_0^T d\bar{\tau} e^{\frac{i}{2}(p'^2 - p^2)\bar{\tau} - \frac{i}{2}(p'^2 + m^2)T}\]
\[\propto \frac{1}{p^2 + m^2} \frac{1}{p'^2 + m^2}\] (91)
i.e. the D-particle propagator legs, to be removed for the proper scattering amplitude.

Thus our final result in the form of the vertex is
\[\langle B; p^\mu, p'^\nu, k^\mu| = \langle 0| \exp\left(\sum_{n \geq 1} \frac{1}{\alpha_n} \cdot \tilde{\alpha}_n - 2 \sum_{n \geq 1} \frac{1}{\alpha_{\mu,n}} \tilde{\alpha}_{\nu,n} \frac{(p + p')^\mu(p + p')^\nu}{(p + p')^2}\right)\]
\[\times \delta(p + k - p'),\] (92)
When appended with the ghost contribution it agrees with the one proposed in [28], for which BRST invariance was enforced by construction.

Let us make a brief remark on the \(m \to \infty\) limit. In this limit, in \(p\) and \(p'\) the energy components dominate and hence the last factor in the exponent of (94) tends to
\[ n^\mu n^\nu / n^2 \text{ where } n^\mu = (1, 0, \ldots, 0). \] This gives precisely the boundary state for a stationary D-particle. Alternatively, we can examine this limit in the path integral itself. If we Fourier-transform the amplitude back to the position representation, we have a factor \[ \exp(i(f' - f)^2/2T - im^2T/2). \] Thus as \( m \to \infty \) the \( T \)-integral is dominated by small \( T \) and this in turn forces \( f'_\mu \sim f_\mu \), showing that the D-particle does not move.

**Simple Applications**

As simple applications of our formalism, let us compute the amplitude with two tachyons and the one with two gravitons.

The two-tachyon on-shell amplitude is immediately obtained from (61) and is proportional to

\[
A(p, p', k) = \int_0^1 dy y^{(\alpha'/2)k_1 \cdot k_2} (1 - y)^{\alpha'(k_1 \cdot v)^2/v^2 - 2} \propto \frac{\Gamma\left(\frac{\alpha'}{2}k_1 \cdot k_2 + 1\right) \Gamma\left(\frac{\alpha'(k_1 \cdot v)^2}{v^2} - 1\right)}{\Gamma\left(\frac{\alpha'}{2}k_1 \cdot k_2 + \frac{\alpha'(k_1 \cdot v)^2}{v^2} - 1\right)},
\]

where \( v = p + \frac{k}{2}, \quad k_1^2 = k_2^2 = \frac{4}{\alpha'}. \) (93)

The first \( \Gamma \)-function in the numerator gives the usual \( t \)-channel closed string poles at

\[ t_\equiv -k^2 = -\frac{4}{\alpha'}(1 - n), \quad n = 0, 1, \ldots. \] (95)

On the other hand, the second \( \Gamma \)-function has poles in a peculiar channel, namely,

\[ -\frac{(k_1 \cdot v)^2}{v^2} = \frac{1}{\alpha'}n, \quad n = 0, 1, \ldots. \] (96)

Rewriting this in terms of \( s \equiv -(k_1 + p)^2 \) and \( t \), we find

\[
s = m^2 + \frac{2}{\sqrt{\alpha'}} \sqrt{n} \sqrt{m^2 - \frac{t}{4} - \frac{4}{\alpha'} - \frac{t}{2}},
\]

\[ m^2 \gg t \quad m^2 + \frac{2}{\sqrt{\alpha'}} \sqrt{n} \sqrt{m^2 - \frac{4}{\alpha'} - \frac{t}{2} + O\left(\frac{t}{\sqrt{\alpha'} m}\right)}. \] (98)

Because of the presence of \( t \) on the right hand side, they do not represent genuine poles in the \( s \)-channel. However, for \( t << m^2 \), we see the excitation spectrum for which the energy scale is given by the geometrical mean of \( m \) and the string scale \( 1/\sqrt{\alpha'} \) and the spacing is of square root type. Intuitively, they should represent the excitations of an open string.
with a heavy mass attached at the ends. Although more detailed study is required, they appear to be new objects which can only be seen in this type of relativistic treatment. 

As the second example, consider the amplitude with graviton insertions. The graviton vertices are given by

\[ V_G(k_i) = \int d^2z_i \zeta^i_{\mu\nu} \partial X^\mu(z_i) \overline{\partial X^\nu(z_i)} e^{i k_i \cdot X(z_i)} \quad \text{with} \quad k^2_i = 0, \quad (99) \]

where \( \zeta^i_{\mu\nu} \) is a polarization tensor for a graviton and satisfies the conditions \( k^\mu_i \zeta^i_{\mu\nu} = k^\nu_i \zeta^i_{\mu\nu} = 0 \) and \( \sum_\mu \zeta^i_{\mu\nu} = 0 \). The calculation is facilitated by introducing the source term \( i J_{\mu}(z) \partial X^\mu(z) + i J_{\nu}(z) \overline{\partial X^\nu(z)} \) in the string action. Then the relevant amplitude can be expressed as

\[ V_G = \prod_i \int d^2z_i \zeta^i_{\mu\nu} \frac{\delta}{\delta J_{\mu}(z_i)} \frac{\delta}{\delta J_{\nu}(z_i)} F[J] \bigg|_{J=0} V_T, \quad (100) \]

where \( V_T \) is the amplitude with tachyons and \( F[J] \) is the factor containing the sources:

\[ F[J] = \exp \left\{ \alpha' \sum_{i,j} \eta_{AA} \left( \frac{1}{2} J_{\alpha A}(z_i) \partial^\alpha \partial^\beta G_A(z_i, z_j) J_{\beta A}(z_j) - k_i A J_{\alpha A}(z_j) \partial^\alpha G_A(z_i, z_j) \right) \right\}. \quad (101) \]

Here \( \partial^\beta \) denotes the derivative with respect to the second argument of the Green function.

Restricting to the two graviton case and performing the rest of the calculation in the gauge \( \zeta^i_{0\alpha} = 0 \), we find the amputated amplitude to be proportional to

\[ \frac{\alpha'^2}{8} \int_0^1 dy \left\{ \zeta^i_{AB} \zeta^2_{AB} \left( 1 + \frac{1}{y^2} \right) + \alpha' (\zeta^2_{AB} k_{2B}^2) (\zeta^2_{AC} k_{1C}) \left( 1 - \frac{1}{y^2} \right) \right\} \]

\[ + \left( \frac{\alpha'}{2} \right)^2 (k_{2A} \zeta^i_{AB} k_{2B})(k_{1C} \zeta^2_{CD} k_{1D}) \left( 1 - \frac{2}{y} + \frac{1}{y^2} \right) \right\} (1 - y)^{-\alpha' k_{10}^2 y^2} y_{k_{10} k_{20}} \quad (102) \]

Upon \( y \)-integration, this becomes

\[ \left( \frac{\alpha'}{2} \right)^3 \frac{\Gamma(\frac{\alpha'}{2} k_{10} - k_{20} - 1)}{\Gamma(\frac{\alpha'}{2} k_{10} - k_{20} + 2)} \left\{ \zeta^i_{AB} \zeta^2_{AB} \left( \frac{\alpha'}{2} k_{10}^2 - k_{10}^2 + \frac{\alpha'}{2} \left( \sum_a k_{1a} k_{2a} \right)^2 \right) \right\} \]

\[ - \alpha' (\zeta^i_{AB} k_{2B})(\zeta^2_{AC} k_{1C}) \sum_a k_{1a} k_{2a} (1 - \alpha' k_{10}^2) \]

\[ + \left( \frac{\alpha'}{2} \right)^2 (k_{2A} \zeta^i_{AB} k_{2B})(k_{1C} \zeta^2_{CD} k_{1D}) (1 - \frac{\alpha'}{2} k_{10}^2)(1 - \alpha' k_{10}^2) \right\} . \quad (103) \]

In the limit of infinitely heavy D-particle, this reduces to the result obtained in [3]. Just as in the previous example, poles occur in the exotic channel.

\[ ^6 \text{These peculiar excitations have also been noted by N. Ishibashi and M. Li} \]
4 Discussions

In this work, we have initiated a path integral formalism for the quantization of D-particles. Although this article is devoted to explaining the basic formalism and its applications in the simplest setting, our work should serve as a starting point for investigations of a variety of important problems, both conceptual and technical. Below we shall discuss some of these perspectives.

Let us begin with some immediate extensions. One obvious and necessary task is to extend the formalism to the superstring case. This extension will be reported elsewhere.

Another urgent application is the calculation of the scattering amplitude for two quantum D-particles. This is important in many respects. In particular, this would allow us to compare our approach with that using the low energy effective gauge theory and clarify both the foundation and the limitation of the latter. The key would be to understand the mechanism of the appearance of the enhanced gauge symmetry and its spontaneous breakdown. These matters are currently under investigation and we hope to report our progress in the near future.

Once the two-particle case is understood, the next task will be to extend it to the case involving multiple D-particles. Here, we expect to be able to make contact with the extremely interesting proposal recently made by Banks et al \cite{Banks}, namely the formulation of the M-theory in terms of D-particles in the infinite momentum frame. One of the crucial questions concerning this proposal is whether one can find the action which is covariant with respect to the 11 dimensional Lorentz group. As our path integral formalism (when extended to the superstring) respects covariance at least in the 10 dimensional sense, it may provide a clue to this important question.

Aside from applications to D-particle systems, our formalism can in principle be adapted to the processes involving D-strings, provided they are compactified to carry finite masses. This might shed some light on the nature of 12 dimensional F-theory \cite{F-theory}.

Finally, we wish to discuss a possible conceptual implication that our work may have upon the outstanding problem of the vacuum selection in string theory. From the standpoint of non-linear sigma model, one consistent background corresponds to one conformal field theory (CFT) and the remarkable discovery of Polchinski \cite{Polchinski} is that D-branes provide exact CFT’s of that kind. To make our discussions concrete, let us focus upon the process considered in this work. Form a wavepacket for the D-particle which is initially moving...
along a straightline. As this is a solution of the condition for the vanishing $\beta$-function, it gives a CFT. Similarly, the final packet moving along a different straightline gives another CFT. Therefore the scattering process describes a transition between two different CFT’s. Moreover, the quantum fluctuation necessarily involves non-CFT stages in the middle. Thus, strictly speaking, the conformal invariance is not respected in the usual sense. Indeed, to describe the scattering we could not set the $\beta$-function to zero. This, however, does not imply a disaster: Consistency of the theory is maintained in the form of BRST invariance. This is somewhat reminiscent of the situation that occurs in non-linear sigma model, where string loop corrections effectively modify the $\beta$-function, which in turn can be derived from the requirement of BRST invariance [30, 37, 38]. As in many situations in string theory, the imperative requirement is the unitarity and BRST invariance is the most powerful way to implement this crucial consistency condition.

Thus, to sum up, our work may be interpreted as an instructive example in which the dynamical transition between CFT’s is consistently described: Configuration space of D-particles can be regarded as the moduli space of a class of CFT’s and, upon quantization, points of this moduli space are dynamically connected. It should be extremely interesting to explore the implication of this viewpoint for the problem of vacuum selection in string theory and beyond.

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