NOTE ON THE BONDAL-ORLOV FUNCTORS FOR TORIC DM STACKS

YUNFENG JIANG

Abstract. We calculate explicit formulas for the general equivariant Bondal-Orlov functors on the localized K-theory groups for a crepant birational transformation of toric DM stacks. We recall some facts that the Bondal-Orlov functors give equivalences on the bounded derived categories. Applying twice of these functors we get the Seidel-Thomas spherical twists for the derived category.

1. Introduction

In this short note we calculate explicit formulas for the general equivariant Bondal-Orlov functors for a crepant birational transformation of toric Deligne–Mumford (DM) stacks.

Toric DM stacks were introduced by Borisov–Chen–Smith [2] using stacky fans. The notion of extended stacky fan was introduced by Jiang in [12], and it turns out that there is a one-to-one correspondence between the extended stacky fans and GIT data construction of toric DM stacks. Given GIT data determined by a stability parameter $\omega$, we denote the toric DM stack by $X_\omega$, whose construction is reviewed in §2. More details can be found in [8]. Birational transformation of toric DM stacks can be understood as changing the GIT stability parameters in the space of GIT stability conditions.

We study a special case of birational transformation of toric DM stacks: the crepant birational transformations. We consider a special class of crepant birational transformations ($K$-equivalences) of toric DM stacks by a single wall crossing. The construction of such wall crossing can be found in [8, §5.1]. There is a big torus $T$ action on the toric DM stack $X_\omega$, and we work on the $T$-equivariant K-theory and bounded derived category on $X_\omega$. Y. Kawamata in [14] proves that a natural Fourier–Mukai transform induces equivalences of the bounded derived categories of $K$-equivalent toric DM stacks. It was shown in [9] that the $T$-equivariant derived categories are also equivalent. In [8, §6], the authors calculated the equivariant Fourier–Mukai transform for K-theory basis of $X_\omega$ when restricted to torus fixed points.

In this paper we calculate explicit formulas for the general Bondal-Orlov functors in terms of equivariant K-theory basis for a single toric wall crossing. Let

$$\varphi : X_+ := X_{\omega_+} \rightarrow X_- := X_{\omega_-}$$

be a crepant transformation by a single wall crossing corresponding to the stability conditions $\omega_+$ and $\omega_-$. The $T$-equivariant $K$-theory $K^*_T(X_{\pm})$ are generated by equivariant line bundles corresponding to the lattice in the secondary fan. There is a common blow-up $\tilde{X}$ for both $X_+$ and $X_-$ and two contract maps $f_{\pm} : \tilde{X} \rightarrow X_{\pm}$. 
Let $E \subset \tilde{X}$ be the exceptional divisor. The general Bondal-Orlov functors are defined by:

$$BO_k = (f_+)_{\ast}(O_{\tilde{X}}(kE) \otimes (f_-)^{\ast}(-\ldots )) : D^b_T(X-) \to D^b_T(X+)$$

for any integer $k \in \mathbb{Z}$. We prove that $BO_k$ is an equivalence on the equivariant bounded derived categories for any $k$. When $k = 0$, $BO_0$ is the usual Fourier-Mukai transform $FM = (f_+)_{\ast}((f_-)^{\ast}(-\ldots ))$. So $BO_k$ can be taken as generalized Fourier–Mukai transforms. The functors $BO_k$, of course, induce isomorphisms on the equivariant $K$-theory groups. Our computation gives explicit formulas of the Bondal-Orlov functors $BO_k$ on the localized $K$-theory basis. See Theorem 3.2. This generalizes the calculation of Theorem 6.19 in [8] for the Fourier-Mukai transform $BO_0 = FM$, although the proof is basically the same as in [8]. In Theorem 6.23 of [8], the authors prove that the Fourier-Mukai transform $BO_0$ matches the analytic continuation of the $H$-functions for $X_{\pm}$, which implies the invariance of big quantum cohomology of $X_{\pm}$, see [8, §5, 6] for details. It is pretty interesting if the general Bondal-Orlov functors $BO_k$ can match the analytic continuation of some hypergeometric functions for $X_{\pm}$.

We also recall the fact that the Bondal-Orlov functors give an equivalence on the bounded derived categories of a single toric wall crossing. The proof is based on the method of window shifted functor for the derived categories under GIT quotients by [10], [1] and [18]. We completely follow the proof of §5 in [9]. Applying back for the Bondal-Orlov functor we get an autoequivalence of the bounded derived category which is called the spherical twist functor associated with a line bundle on the contraction locus in the sense of Seidel-Thomas in [19]. We also give a proof that for a crepant birational transformation of toric DM stacks via a single wall crossing, the contraction locus are always weighted projective stacks. This result is hidden somewhere in [8], but there is no explicit explanation. The result presented here is related to the monodromy conjecture in [6] for Gromov-Witten theory of symplectic smooth DM stacks, see [13]. The result of the spherical twists can also be applied to find a correspondence for the Chen-Ruan cohomology for quasi-simple orbifold flops, see [7].

This short note is organized as follows. In §2 we review the construction of the crepant transformation of toric DM stacks by a single wall crossing. We calculate the general equivariant Bondal-Orlov functor on the localized $K$-theory basis for the wall crossing of toric DM stacks in §3. In §4 we recall the fact that the general equivariant Bondal-Orlov functors give an equivalence on the bounded derived categories for the wall crossing of toric DM stacks, and relate them to spherical twist associated with line bundles on the contraction locus.

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2. CRePANT TRANSFORMATION OF TORIC DM STACKS

In this section we review some basic facts and establish notations. The main reference is [8].
2.1. Toric Deligne–Mumford stack and GIT quotient. An S-extended stacky fan is a quadruple \( \Sigma = (N, \Sigma, \beta, S) \), where:

- \( N \) is a finitely generated abelian group (torsions allowed);
- \( \Sigma \) is a rational simplicial fan in \( N \otimes \mathbb{R} \);
- \( \beta : \mathbb{Z}^m \to N \) is a homomorphism; we write \( b_i = \beta(e_i) \in N \) for the image of the \( i \)th standard basis vector \( e_i \in \mathbb{Z}^m \), and write \( b_i \) for the image of \( b_i \) in \( N \otimes \mathbb{R} \);
- \( S \subset \{1, \ldots, m\} \) is a subset, such that:
  - each one-dimensional cone of \( \Sigma \) is spanned by \( b_i \) for a unique \( i \in \{1, \ldots, m\} \setminus S \), and each \( b_i \) with \( i \in \{1, \ldots, m\} \setminus S \) spans a one-dimensional cone of \( \Sigma \);
  - for \( i \in S \), \( b_i \) lies in the support \( |\Sigma| \) of the fan.

The vectors \( b_i \) for \( i \in S \) are called extended vectors.

The toric DM stack associated to an extended stacky fan \( (N, \Sigma, \beta, S) \) depends only on the underlying stacky fan and is defined as the quotient stack

\[ X_\Sigma := [U/K], \quad \text{with} \quad U = \mathbb{C}^m \setminus \mathcal{V}(I_\Sigma), \]

where \( I_\Sigma \) is the irrelevant ideal of the fan and \( K := \text{Hom}(L^\vee, \mathbb{C}^\times) \) acts on \( \mathbb{C}^m \) through the data of extended stacky fan.

We require that the extended stacky fans \( (N, \Sigma, \beta, S) \) satisfy the following conditions:

- (C1) the support \( |\Sigma| \) of the fan is convex and full-dimensional;
- (C2) there is a strictly convex piecewise-linear function \( f : |\Sigma| \to \mathbb{R} \) that is linear on each cone of \( \Sigma \); 
- (C3) the map \( \beta : \mathbb{Z}^m \to N \) is surjective.

The first two conditions are geometric constraints on \( X_\Sigma \): they are equivalent to saying that the corresponding toric stack \( X_\Sigma \) is semi-projective and has a torus fixed point. The third condition can be always achieved by adding enough extended vectors.

We explain the GIT construction of \( X_\Sigma \) from the extended stacky fan \( \Sigma = (N, \Sigma, \beta, S) \) satisfying (C1-C3). First we define a free \( \mathbb{Z} \)-module \( L \) by the exact sequence

\[
0 \longrightarrow L \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} N \longrightarrow 0
\]

and define \( K := L \otimes \mathbb{C}^\times \). The dual of (2.1) is an exact sequence:

\[
0 \longrightarrow N^\vee \longrightarrow (\mathbb{Z}^m)^\vee \longrightarrow L^\vee
\]

and we define the character \( D_i \in L^\vee \) of \( K \) to be the image of the \( i \)th standard basis vector in \((\mathbb{Z}^m)^\vee \) under the third arrow \((\mathbb{Z}^m)^\vee \to L^\vee \). Set

\[
\mathcal{A}_\omega = \left\{ I \subset \{1, 2, \cdots, m\} \mid S \subset I, \sigma_T \text{ is a cone of } \Sigma \right\}.
\]

to be the collection of anticones. The stability condition \( \omega \in L^\vee \otimes \mathbb{R} \) lies in \( \bigcap_{I \in \mathcal{A}_\omega} \mathcal{L}_I \), where

\[
\mathcal{L}_I = \left\{ \sum_{i \in I} a_i D_i \mid a_i \in \mathbb{R}, a_i > 0 \right\} \subset L^\vee \otimes \mathbb{R}.
\]
The condition (C2) ensures that this intersection is non-empty. We understand \( \emptyset = \{0\} \). Let
\[
U_\omega = \bigcup_{i \in A_\omega} (C^\times)^I \times C^T := (C^\times)^I \times C^T = \{(z_1, \ldots, z_m) \in \mathbb{C}^m | z_i \neq 0 \text{ for } i \in I\}.
\]
The GIT data consists of
- \( K \cong (\mathbb{C}^\times)^r \), a connected torus of rank \( r \);
- \( \mathbb{L} = \text{Hom}(C^\times, K) \), the character lattice of \( K \);
- \( D_1, \ldots, D_m \in \mathbb{L}^X = \text{Hom}(K, C^\times) \), characters of \( K \);
- stability condition \( \omega \in \mathbb{L}^\times \otimes \mathbb{R} \);
- \( A_\omega = \{I \subset \{1, 2, \ldots, m\} : \omega \in \angle_I\} \).

The stability condition \( \omega \) satisfies the following assumptions:

**Assumption 2.1.**

(A1) \( \{1, 2, \ldots, m\} \in A_\omega \);

(A2) for each \( I \in A_\omega \), the set \( \{D_i : i \in I\} \) spans \( \mathbb{L}^Y \otimes \mathbb{R} \) over \( \mathbb{R} \).

(A1) ensures that \( X_\omega \) is non-empty; (A2) ensures that \( X_\omega \) is a DM stack. Under these assumptions, \( A_\omega \) is closed under enlargement of sets; i.e., if \( I \in A_\omega \) and \( I \subseteq J \), then \( J \in A_\omega \). The toric DM stack is the quotient stack \( X_\Sigma = X_\omega = [U_\omega/K] \).

Conversely, to obtain an extended stacky fan from GIT data, consider the exact sequence (\ref{eq:exact_sequence}). Let \( b_i = \beta(e_i) \in \mathbb{N} \) and \( b_i \in \mathbb{N} \otimes \mathbb{R} \) be as above and, given a subset \( I \) of \( \{1, \ldots, m\} \), let \( \sigma_I \) denote the cone in \( \mathbb{N} \otimes \mathbb{R} \) generated by \( \{b_i : i \in I\} \). The extended stacky fan \( \Sigma_\omega = (\mathbb{N}, \Sigma_\omega, \beta, S) \) corresponding to our data consists of the group \( \mathbb{N} \) and the map \( \beta \) defined above, together with a fan \( \Sigma_\omega \) in \( \mathbb{N} \otimes \mathbb{R} \) and \( S \) given by
\[
\Sigma_\omega = \{\sigma_I : \overline{I} \in A_\omega\}, \quad S = \{i \in \{1, \ldots, m\} : \{i\} \notin A_\omega\}.
\]

The quotient construction in (\ref{eq:quotient_construction}) coincides with the GIT quotient construction, and therefore \( X_\omega \) is the toric DM stack corresponding to \( \Sigma_\omega \).

### 2.2. Wall crossing and birational transformation

The space \( \mathbb{L}^Y \otimes \mathbb{R} \) of stability conditions is divided into chambers by the closures of the sets \( \angle_I \), \( |I| = r - 1 \), and the DM stack \( X_\omega \) depends on \( \omega \) only via the chamber containing \( \omega \). For any stability condition \( \omega \), the set \( U_\omega \) contains the big torus \( T = (C^\times)^m \). Thus for any two such stability conditions \( \omega_1, \omega_2 \) there is a canonical birational map \( X_{\omega_1} \to X_{\omega_2} \), induced by the identity transformation between \( T/K \subset X_{\omega_1} \) and \( T/K \subset X_{\omega_2} \).

Let \( C_+, C_- \) be chambers in \( \mathbb{L}^Y \otimes \mathbb{R} \) that are separated by a hyperplane wall \( W \), so that \( W \cap \overline{C}_+ \) is a facet of \( \overline{C}_+ \), \( W \cap \overline{C}_- \) is a facet of \( \overline{C}_- \), and \( W \cap \overline{C}_- = W \cap \overline{C}_+ \). Choose stability conditions \( \omega_+ \in C_+, \omega_- \in C_- \) satisfying (A1-A2) and set \( U_+ := U_{\omega_+}, U_- := U_{\omega_-}, X_+ := X_{\omega_+}, X_- := X_{\omega_-}, \) and
\[
A_\pm := A_{\omega_\pm} = \{I \subset \{1, 2, \ldots, m\} : \omega_\pm \in \angle_I\}.
\]

Then \( C_\pm = \bigcap_{I \in A_\pm} \angle_I \). Let \( \varphi : X_+ \to X_- \) be the birational transformation induced by the toric wall-crossing from \( C_+ \) to \( C_- \) and suppose that \( \sum_{i=1}^m D_i \in W \) which implies that \( \varphi \) is crepant. Let \( e \in \mathbb{L} \) denote the primitive lattice vector in \( W^\perp \) such that \( e \) is positive on \( C_+ \) and negative on \( C_- \). We fix the notations
- \( M_+ := \{i \in \{1, \ldots, m\} | D_i \cdot e > 0\} \);
- \( M_- := \{i \in \{1, \ldots, m\} | D_i \cdot e < 0\} \);
- \( M_0 := \{i \in \{1, \ldots, m\} | D_i \cdot e = 0\} \).

For each \( i \in M_+ \), let \( \alpha_+^i \) be the bounding hyperplane of \( C_+ \) at \( e \). If \( i \in M_- \), let \( \alpha_-^i \) be the bounding hyperplane of \( C_- \) at \( e \). If \( i \in M_0 \), let \( \alpha_0^i := \alpha_+^i = \alpha_-^i \). Then
\[
\ell_\varphi^i = \sum_{j \in M_+} \alpha_+^j + \sum_{j \in M_-} \alpha_-^j + \sum_{j \in M_0} \alpha_0^j
\]
for \( \varphi \), where \( \ell_\varphi^i : X_+ \to X_- \) is the log-contraction.
Choose $\omega_0$ from the relative interior of $W \cap C_+ = W \cap C^-$. The stability condition $\omega_0$ does not satisfy (A1-A2) on GIT data, but consider
$$A_0 := A_{\omega_0} = \{ I \subset \{ 1, \ldots, m \} : \omega_0 \in \angle I \}$$
and the corresponding toric Artin stack $X_0 := X_{\omega_0} = [U_{\omega_0}/K]$. Here $X_0$ is not a DM stack, as the $C^\times$-subgroup of $K$ corresponding to $e \in \mathbb{L}$ (the defining equation of the wall $W$) has a fixed point in $U_0 := U_{\omega_0}$. The stack $X_0$ contains both $X_+$ and $X_-$ as open substacks and the canonical line bundles of $X_+$ and $X_-$ are the restrictions of the same line bundle $L$ - satisfying assumptions (A1-A2). The $T$-equivariant Artin stack $X_0$ is generated by the torus fixed points of $X_0$ does not satisfy (A1-A2) on GIT data, but consider
$$\delta \in \mathfrak{A}_{\omega_0}$$
and the corresponding toric Artin stack $X_0 := X_{\omega_0} = [U_{\omega_0}/K]$. Here $X_0$ is not a DM stack, as the $C^\times$-subgroup of $K$ corresponding to $e \in \mathbb{L}$ (the defining equation of the wall $W$) has a fixed point in $U_0 := U_{\omega_0}$. The stack $X_0$ contains both $X_+$ and $X_-$ as open substacks and the canonical line bundles of $X_+$ and $X_-$ are the restrictions of the same line bundle $L_0 \rightarrow X_0$ given by the character $\sum_{j=1}^m D_j$ of $K$.

The condition $\sum_{j=1}^m D_j \in W$ ensures that $L_0$ comes from a $Q$-Cartier divisor on the underlying singular toric variety $X_0 = C^m//\omega_0 K$. There are canonical blow-down maps $g_{\pm} : X_+ \rightarrow X_0$, and $K_{X_{\pm}} = s_{\pm} L_0$. We have a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f_+} & \bar{X} \\
\downarrow{f_-} & & \downarrow{\varphi} \\
X_+ & \xrightarrow{s_+} & \bar{X}_0 \\
\end{array}
\]

This shows that $f_+^*(K_{X_+}) = f_-^*(K_{X_-})$ and birational map $\varphi$ is crepant, since they are the pull-backs of the same $Q$-Cartier divisor on $\bar{X}_0$.

To construct $\bar{X}$, consider the action of $K \times C^\times$ on $C^{m+1}$ defined by the characters $\bar{D}_1, \ldots, \bar{D}_{m+1}$ of $K \times C^\times$, where:

\[
\bar{D}_j = \begin{cases}
D_j \oplus 0 & \text{if } j < m + 1 \text{ and } D_j \cdot e \leq 0 \\
D_j \oplus (-D_j \cdot e) & \text{if } j < m + 1 \text{ and } D_j \cdot e > 0 \\
0 \oplus 1 & \text{if } j = m + 1
\end{cases}
\]

Consider the chambers $\bar{C}_+, \bar{C}_-$, and $\bar{C}$ in $(\mathbb{L} \oplus \mathbb{Z})^\vee \otimes \mathbb{R}$ that contain, respectively, the stability conditions

$$\bar{\omega}_+ = (\omega_+, 1) \quad \bar{\omega}_- = (\omega_-, 1) \quad \text{and} \quad \bar{\omega} = (\omega_0, -\epsilon)$$

where $\epsilon$ is a very small positive real number. Let $\bar{X}$ denote the toric DM stack defined by the stability condition $\bar{\omega}$. We have, by [8 Lemma 6.16], that the toric DM stack corresponding to the chamber $\bar{C}_\pm$ is $X_\pm$. Furthermore, there is a commutative diagram as in (2.3), where: $f_\pm : \bar{X} \rightarrow X_\pm$ is a toric blow-up, arising from the wall-crossing from $C$ to $\bar{C}_\pm$.

3. Generalized Bondal-Orlov Transforms

3.1. Equivariant $K$-theory of toric DM stacks. The big torus $T := (C^\times)^m$ acts on the toric DM stack $X_\omega$ corresponding to a stability condition $\omega \in \mathbb{L}^\vee \otimes \mathbb{R}$ satisfying assumptions (A1-A2). The $T$-equivariant $K$-theory group $K^T_0(X_\omega)$ of $X_\omega$ is generated by the $T$-equivariant line bundles $R_i$ corresponding to the ray $\rho_i$ for each $i \in \{ 1, \ldots, m \}$.

Recall that the torus fixed points of $X_\omega$ are in one-to-one correspondence with minimal anticones $\delta \in \mathfrak{A}_\omega$. A minimal anticone $\delta$ determines a torus fixed point stack $x_\delta = \cdots$
$BG_\delta \in X_\omega$, where $G_\delta$ is the isotropy group of the fixed point $x_\delta$. Let $i_{\delta}: x_\delta \to X_\omega$ denote the inclusion. We have

\begin{equation}
(3.1) \quad i_{\delta}^* R_j = 1, \quad \forall j \in \delta.
\end{equation}

We recall the Lefschetz fixed point theorem (c.f. [9] Theorem 3.3) in this formulation.

**Theorem 3.1.** Let $X_\omega = [U_\omega/K]$ be a toric DM stack. The torus $T$ acts on $X_\omega$. Given $\delta \in A_\omega$, write $x_\delta$ for the corresponding T-fixed point of $X_\omega$. Let $N_\delta$ denote the normal bundle to $i_{\delta}$. Let $\mathbb{Z}[T] = \mathbb{K}^T_0(pt)$ denote the ring of regular functions (over $\mathbb{Z}$) on $T$ and let $\text{Frac} \mathbb{Z}[T]$ denote the field of fractions. Then for $\alpha \in \mathbb{K}^T_0(X_\omega)$, we have

$$\alpha = \sum_{\delta \in A_\omega} (i_\delta)^* \left( \frac{i_{\delta}^* \alpha}{\lambda_{-1} N_\delta^\vee} \right) \in \mathbb{K}^T_0(X_\omega) \otimes_{\mathbb{Z}[T]} \text{Frac}(\mathbb{Z}[T])$$

where $\lambda_{-1} N_\delta^\vee := \sum_{i=0}^{\dim X_\omega} (-1)^i N_\delta^i$ is invertible in $\mathbb{K}^T_0(X_\omega) \otimes_{\mathbb{Z}[T]} \text{Frac}(\mathbb{Z}[T])$.

### 3.2. The localized $K$-theory basis

Consider the toric wall crossing diagram (2.3). The torus $T$ acts on $X_\pm$ through the diagonal action of $T$ on $\mathbb{C}^m$. There is an action of $T$ on $\bar{X}$ induced from the inclusion $T = T \times \{1\} \subset T \times \mathbb{C}^\times$ and the $T \times \mathbb{C}^\times$ action on $\mathbb{C}^{m+1}$. So all the maps in (2.3) are $T$-equivariant. The $T$-equivariant $K$-groups $K^T_0(X_\pm), K^T_0(\bar{X})$ are modules over $K^T_0(pt) = \mathbb{Z}[T]$.

From the wall crossing construction in (2.3) there are two types of minimal anticones for $\bar{X}$. The first type, called flopping type, is given by $\tilde{\delta} = (j_1, \cdots, j_{r-1}, j_+, j_-)$, where $j_1, \cdots, j_{r-1} \in M_0$, and $j_+ \in M_+, j_- \in M_-$. This type of minimal anticones induce the maps from the fixed point stack of $\bar{X}$ to the fixed point stacks of $X_+$ and $X_-$ by

$$f_{+\tilde{\delta}}: x_{\tilde{\delta}} \to x_{\delta_+}, \quad f_{-\tilde{\delta}}: x_{\tilde{\delta}} \to x_{\delta_-},$$

where $\delta_+ = (j_1, \cdots, j_{r-1}, j_+, m+1)$ and $\delta_- = (j_1, \cdots, j_{r-1}, j_-, m+1)$. We use the following notations: $\delta_\pm$ means that the fixed point $x_{\tilde{\delta}}$ maps to the fixed point $x_{\delta_\pm}$ corresponding to flopping minimal anticone $\delta_\pm$ for $X_\pm$.

The second type of minimal anticone, called nonflopping type, is given by $\tilde{\delta}$ containing the last, $m+1$-st, ray corresponding to the common blow-up. The nonflopping minimal anticones map isomorphically to minimal anticones of $X_+$ and $X_-$. Such minimal anticones ($\delta$ and $\delta_\pm$) are of the form $(j_1, \cdots, j_{r-2}, j_+, j_-, m+1)$.

The $T$-invariant divisor $\{z_i = 0\}$ on $X_\omega$ determines a $T$-equivariant line bundle $\mathcal{O}(\{z_i = 0\})$ on $X_\omega$, and we denote the class of this line bundle in the $T$-equivariant $K$-theory by $R_i$. For $K^T_0(X_\pm), K^T_0(\bar{X})$ we write these classes as:

$$\{ R_i^- : 1 \leq i \leq m \} : \quad \text{for } K^T_0(X_-);$$
$$\{ R_i^+ : 1 \leq i \leq m \} : \quad \text{for } K^T_0(X_+);$$
$$\{ \bar{R}_i : 1 \leq i \leq m+1 \} : \quad \text{for } K^T_0(\bar{X}).$$

From §6.3.2 in [8], each character $p \in \text{Hom}(K, \mathbb{C}^\times) = \mathbb{L}^\vee$ define a line bundle $L_-(p)$ over $X_-$. This line bundle $L_-(p)$ is equipped with a $T$-linearized action, thus make it a $T$-equivariant line bundle. The line bundles $R_i^- = L_-(D_i) \otimes e^{\lambda_i}$, where $e^{\lambda_i}$ is the standard $i$-th irreducible $T$-representation $T \to \mathbb{C}^\times$. Similar construction works for the $K$-theory ring $K^T_0(X_+)$. 

\[ \]
For a character \((p,n) \in \mathbb{H}om(K \times C^\times, C^\times) = L^Y \oplus \mathbb{Z}\) we define a \(T\)-equivariant line bundle \(L(p,n) \to \mathbb{X}\) ans we have:
\[
R_i = L(D_i) \otimes e^{\delta_i}, (1 \leq i \leq m); \quad R_{m+1} = L(D_{m+1}) = L(0,1).
\]
The classes \(L_{\pm}(X_{\pm})\) (the classes \((L(p,n))\)) generate the equivariant \(K\)-group \(K_0^T(X_{\pm})\) (\(K_0^T(\mathbb{X})\)) over \(\mathbb{Z}[T]\).

We describe the localized \(T\)-equivariant \(K\)-theory basis for \(K_0^T(X_-)\). Let \(\delta_- \in \mathcal{A}_-\) be a minimal cone and \(x_{\delta_-}\) be the corresponding \(T\)-fixed point. Let
\[
i_{\delta_-} : x_{\delta_-} \to X_-\]
be the inclusion of the fixed point, and \(G_{\delta_-}\) the isotropy group of \(x_{\delta_-}\). We have \(x_{\delta_-} = BG_{\delta_-}\). A basis for \(K_0^T(X_-)\), after inverting nonzero elements of \(\mathbb{Z}[T]\), is given by
\[
(3.2) \quad \{ (i_{\delta_-},q) : q \text{ an irreducible representation of } G_{\delta_-}, \delta_- \in \mathcal{A}_- \}
\]
Choose a lift \(\hat{\delta} \in \mathbb{H}om(K, C^\times) = L^Y\) of each \(G_{\delta_-}\)-representation \(q : G_{\delta_-} \to C^\times\), an element in \((3.2)\) can be written in the form:
\[
e_{\delta_-} := L_-(\hat{\delta}) \prod_{i \in \delta_-} (1 - S^-_i).
\]
Then \(\{ e_{\delta_-}\} \) is a basis for the localized \(T\)-equivariant \(K\)-theory of \(X_-\). There is a similar basis \(\{ e_{\delta_+}\} \) for the localized \(T\)-equivariant \(K\)-theory of \(X_+\).

3.3. The Bondal-Orlov functors. The general Bondal-Orlov functor on the bounded derived categories \(D^b_T(X_{\pm})\):
\[
\text{BO}_k : D^b_T(X_-) \to D^b_T(X_+)
\]
is defined by:
\[
\text{BO}_k(a) = (f_+)_*(\text{O}_{\mathbb{X}}(KE) \otimes (f_-)^*(a)).
\]
We consider the induced functor on the \(K\)-theory of \(X_-\):
\[
\text{BO}_k : K_0^T(X_-) \to K_0^T(X_+).
\]
We explicitly calculate \(\text{BO}_k\) in terms of the localized \(T\)-equivariant \(K\)-theory basis for \(X_-\). Let
\[
S^+_i := (R^+_i)^{-1}, \quad S^-_i := (R^-_i)^{-1}, \quad S_i := (\bar{R}_i)^{-1},
\]
and let
\[
k_i := \max(D_i \cdot e, 0), \quad l_i := \max(-D_i \cdot e, 0).
\]

**Theorem 3.2.** Let \(\delta_- \in \mathcal{A}_-\) be a minimal anticone such that \(\delta_- \in \mathcal{A}_+\), then \(\text{BO}_k(e_{\delta_-}) = e_{\delta_-}\), where on the right side \(\delta_-\) is taken as a minimal anticone in \(\mathcal{A}_+\); If \(\delta_- \in \mathcal{A}_-\) is a minimal anticone such that \(\delta_- \notin \mathcal{A}_+\), then
\[
\text{BO}_k(e_{\delta_-}) =
1 \sum_{k \in T} \left( \frac{\hat{t}^k(1 - S^+_i)}{1 - t^{-1}} \cdot L_+(\hat{\delta}) \cdot \prod_{i \notin \delta_+} \prod_{D_i \cdot e < 0} (1 - S^+_i) \cdot \prod_{i \in \delta_-} \prod_{D_i \cdot e \geq 0} (1 - t^{-D_i \cdot e} S^-_i) \right)
\]
where \( j_+ \in \delta_+ \) is the unique element such that \( D_{j_+} \cdot e < 0, I = -D_{j_+} \cdot e \) and
\[
T := \{ \zeta \cdot (R^+_j)^{\frac{1}{l}} : \zeta \in \mu_L \}.
\]

**Proof.** The proof is similar to the proof of Theorem 6.19 in [8], except that we take into account of the role of the line bundle \( \mathcal{O}_X(kE) \). The line bundle \( \mathcal{O}_X(E) \) corresponds to the line bundle \( \mathcal{O}_{\tilde{X}}(kE) \) over \( \tilde{X} \). So
\[
\mathcal{O}_X(kE) \cong \mathcal{O}_{\tilde{X}}(kE).
\]

We calculate \( BO_k \) for any \( k \in \mathbb{Z} \). For \( \delta_- \in \mathcal{A}_+ \), \( \varphi \) is an isomorphism in an neighbourhood of the fixed points of \( x_{\delta_-} \in X_{\pm} \). So \( BO_k(e_{\delta_- \varphi}) = e_{\delta_- \varphi} \).

Suppose now that \( \delta_- \in \mathcal{A}_- \), but \( \delta_- \notin \mathcal{A}_+ \). Let \( \delta_- = \{ j_1, \ldots, j_{r-1}, j_- \} \). Then \( D_{j_1} \cdot e = D_{j_2} \cdot e = \cdots = D_{j_{r-1}} \cdot e = 0 \) and \( D_{j_-} \cdot e < 0 \). We have from [8] Proposition 6.21,
\[
(f_-)^*(e_{\delta_- \varphi}) = L(\hat{\delta}, 0) \prod_{i \in \delta_-} (1 - \tilde{S}_{m+1}^{-j_i} \tilde{S}_i).
\]

Then
\[
\mathcal{O}_X(kE) \otimes (f_-)^*(e_{\delta_- \varphi}) = \mathcal{O}_{\tilde{X}}(kE) \cong \mathcal{O}_{\tilde{X}}(kE).
\]

We use the localized Theorem 3.1 in the \( T \)-equivariant K-theory restricting above to all torus fixed points \( x_{\overline{\delta}} \in f_-(x_{\delta_-}) \), where \( \overline{\delta} = \delta_- \cup \{ j_+ \} \) for \( D_{j_+} \cdot e > 0 \). So (3.3)
\[
\mathcal{O}_X(kE) \otimes (f_-)^*(e_{\delta_- \varphi}) = \sum_{\delta \in \mathcal{A}} (i_{\delta_+})_*(i_{\overline{\delta}})_* \left[ \frac{\tilde{R}_{m+1} \cdot L(\hat{\delta}, 0) \prod_{i \notin \delta_-} (1 - \tilde{R}_{m+1}^{j_i} \tilde{S}_i)}{(1 - \tilde{S}_m) \prod_{i \notin \delta_-} (1 - \tilde{S}_i)} \right]
\]

For \( j_+ \in \delta_+ \), \( \tilde{R}_{j_+} \) is trivial when restricted to \( x_{\overline{\delta}} \). So: \( (1 - \tilde{R}_{m+1}^{j_+} \tilde{S}_i) = (1 - \tilde{S}_m) \) and (3.3) is actually a polynomial on \( \mathcal{R}_{m+1} \) on the numerator. Then applying the pushforward
\[
(f_+)_*(\mathcal{O}_X(kE) \otimes (f_-)^*(e_{\delta_- \varphi})) = \sum_{\delta \in \mathcal{A}} (i_{\delta_+})_*(i_{\overline{\delta}})_* \left[ \frac{\tilde{R}_{m+1} \cdot L(\hat{\delta}, 0) \prod_{i \notin \delta_-} (1 - \tilde{R}_{m+1}^{j_i} \tilde{S}_i)}{(1 - \tilde{S}_m) \prod_{i \notin \delta_-} (1 - \tilde{S}_i)} \right]
\]

here we use the formula (3) in Proposition 6.22 of [8]. Hence we get:
\[
(f_+)_*(\mathcal{O}_X(kE) \otimes (f_-)^*(e_{\delta_- \varphi})) = \sum_{\delta \in \mathcal{A}} (i_{\delta_+})_*(i_{\overline{\delta}})_* \left[ \frac{\tilde{R}_{m+1} \cdot L(\hat{\delta}, 0) \prod_{i \notin \delta_-} (1 - \tilde{R}_{m+1}^{j_i} \tilde{S}_i)}{(1 - \tilde{S}_m) \prod_{i \notin \delta_-} (1 - \tilde{S}_i)} \right]
\]

By localization again we get the result in the Theorem. The only thing we need to check is that \( t^k \cdot (1 - S_j^+) \cdot L(\hat{\delta}, 0) \cdot t^k \prod_{i \notin \delta_-} (1 - t^{-k_i} S_i^+) \) vanishes on \( x_{\overline{\delta}} \) for \( \delta \in \mathcal{A}_+ \cap \mathcal{A}_- \). But this is a similar check as in the proof of Theorem 6.19 of [8]. \( \square \)
Remark 3.3. In Theorem 6.23 of [8], we prove that \( \mathcal{BO}_0 \) actually matches the analytic continuation of I-functions of \( X_\pm \). Since the I-functions of \( X_\pm \) determine the bid quantum cohomology for \( X_\pm \), the result of Theorem 6.23 in [8] tells us that the Fourier–Mukai transform preserves the big quantum cohomology of a single toric wall crossing. It is of course interesting to see if the general Bondal-Orlov transforms \( \mathcal{BO}_k \) preserves some analytic continuation of hypergeometric function of \( X_\pm \).

4. Derived equivalence and spherical twists

In this section we recall some facts that the general Bondal-Orlov functors give equivalences on the bounded derived categories.

4.1. Derived equivalence. Let \( Q := T/K \) be the quotient torus since \( K \subset T \) is a subtorus. Both \( X_+ \) and \( X_- \) carry effective actions of \( Q \). In this section we prove the following:

**Theorem 4.1.** Let (2.3) be a toric crepant transformation. Then

\[
\mathcal{BO}_k : D^b_Q(X_-) \rightarrow D^b_Q(X_+)
\]

gives an equivalence on the equivariant bounded derived categories.

**Remark 4.2.** We use the same proof as in [9, §5], which uses the idea of Halpern-Leistner [17] and Halpern-Leistner-Shipman [18].

**Proof.** We mainly follow the construction and notations in §5 of [9]. First we recall the variation of the GIT quotients of \( X_\pm \) and \( \tilde{X} \). They correspond to chambers \( \tilde{C}_\pm, \tilde{C}, W_\pm |_{\pm}, W_\pm |_{\sim} \) respectively. Let

\[
W_0 = W_+ |_{-} \cap W_+ |_{\sim} \cap W_- |_{\sim}.
\]

There are 7 stability conditions on \( W_0, \tilde{C}_\pm, \tilde{C}, W_+ |_{-}, W_+ |_{\sim}, W_- |_{\sim} \) respectively. If we let \( V_0 \subset C^{m+1} \) be the semi-stable locus of \( W_0 \), then

\[
V_0 = U_0 \times C = C^{m+1} \setminus \left( \bigcup_{l \in A_0} C^l \times C \right)
\]

where \( U_0 \) is in [2.2] As in [9], the other 6 stability conditions are as follows:

| Location of stability condition | Semi-stable locus |
|--------------------------------|-------------------|
| \( \tilde{C}_+ \) | \( V_+ = V_0 \setminus (C^{M_{\geq 0}} \times C) \setminus C^m \) |
| \( \tilde{C}_- \) | \( V_- = V_0 \setminus (C^{M_{\leq 0}} \times C) \setminus C^m \) |
| \( \tilde{C} \) | \( V_+ |_{-} = V_0 \setminus (C^{M_{\leq 0}} \times C) \setminus (C^{M_{\geq 0}} \times C) \) |
| \( W_+ |_{-} \) | \( V_+ |_{-} = V_0 \setminus (C^{M_{\geq 0}} \times C) \) |
| \( W_+ |_{\sim} \) | \( V_+ |_{\sim} = V_0 \setminus (C^{M_{\leq 0}} \times C) \) |
| \( W_- |_{\sim} \) | \( V_- |_{\sim} = V_0 \setminus (C^{M_{\geq 0}} \times C) \) |

We have the GIT quotients

\[
X_+ = [V_+ / K], \quad X_- = [V_- / K], \quad \tilde{X} = [V_\sim / K].
\]
Now we recall the KN stratum introduced in [9, §5.1]. A KN stratum \((\lambda, Z, S)\) contains a one-parameter subgroup \(\lambda \subset K \times \mathbb{C}^\times\), a connected component \(Z\) of the fixed locus, and the associated blade \(S\) defined as:
\[
S = \{ y \in \mathbb{C}^{m+1} : \lim_{t \to \infty} \lambda(t)(y) \in Z \}.
\]
To a KN stratum, there is a numerical invariant
\[
\eta = \text{Weight}_\lambda (\det(N_S/\mathbb{C}^{m+1})).
\]
In our cases let
\[
d := \sum_{i \in M_+} D_i \cdot e = - \sum_{i \in M_-} D_i \cdot e
\]
and consider the KN-strata:
\[
((e, 1), C^{M_{\geq 0}} \cap V_{+\sim}, C^m \cap V_{+\sim}), \quad \eta = 1
\]
and
\[
((-e, -1), C^{M_{\geq 0}} \cap V_{+\sim}, C^m \cap V_{+\sim}), \quad \eta = d
\]
Then \(V_+\) and \(V_-\) are open subsets of \(V_{+\sim}\), which are the complements of the above KN strata. Then from [9, §5] and [17], let
\[
F \subset \tilde{F} \subset D^b_{T \times \mathbb{C}^\times}(V_{+\sim})
\]
be the subcategories by imposing the grade-restriction rule on the subvariety \(C^{M_{\geq 0}} \cap V_{+\sim}\), where for \(F\) we require that the \((e, 1)\)-weights lie in \([0, 1)\), and for \(\tilde{F}\) we require that the \((e, 1)\)-weights lie in \([0, d)\). The we have the following diagram:
\[
\begin{tikzcd}
\tilde{F} \arrow[swap]{d}{\cong} \arrow{rr}{(f_+)^*} & & D^b_Q(X_+) \arrow{dl}{D^b_Q(X_)}
\end{tikzcd}
\]
and the diagonal map is the restriction of functors. Similarly, take \(V_-\) as an open subset of \(V_{-\sim}\) and taking into account of the KN-stratum:
\[
((0, 1), C^{M_{\geq 0}} \cap V_{-\sim}, C^m \cap V_{-\sim})
\]
which has numerical invariant \(\eta = 1\). There is a subcategory
\[
H \subset D^b_{T \times \mathbb{C}^\times}(V_{-\sim})
\]
such that the \((0, 1)\)-weights lie in \([0, 1)\). We have the commuting triangle:
\[
\begin{tikzcd}
H \arrow[swap]{d}{\cong} \arrow{rr}{(f_-)^*} & & D^b_Q(\tilde{X}) \arrow{dl}{D^b_Q(\tilde{X})}
\end{tikzcd}
\]
and the diagonal map is the restriction of functors.
Let us recall the definition of the functor \(\text{GR}_k\) for each integer \(k \in \mathbb{Z}\) in [9, §5.1]. Note that [9] only discusses the case \(\text{GR}_0\), but general \(\text{GR}_k\) are similar. For the KN stratum \((e, Z, S_-)\) with numerical invariant \(\eta_+ = \sum_{i \in M_+} D_i \cdot e, Z = U_0 \cap C^{M_0}\) and
$S_\times = U_0 \cap C^{M \geq 0}$, where the toric DM stack $X_\times = [(U_0 \setminus S_\times) / K]$, there exists a subcategory

$$G_k \subset D^b_T(U_0)$$

using the grade restriction rule and requiring the $e$-weights lying in $[k, k + \eta_\times]$. We have $G_k \cong D^b(X_\times)$.

On the other hand, for the KN stratum $(-e, Z, S_\times)$ with numerical invariant

$$\eta_- = - \sum_{i \in M} D_i \cdot e, \quad Z = U_0 \cap C^{M_0}$$

and $S_\times = U_0 \cap C^{M_\geq 0}$, where the toric DM stack $X_- = [(U_0 \setminus S_\times) / K]$, there exists a subcategory

$$G_k \subset D^b_T(U_0)$$

using the grade restriction rule and requiring the $(-e)$-weights lying in $[-\eta_- + k + 1, k + 1)$. Then we have $G_k \cong D^b(X_-)$. Thus the functor $GR_k : D^b_Q(X_-) \rightarrow D^b_Q(X_\times)$ are defined by the diagram:

$$G_k \cong \begin{array}{c} \cong \\ \cong \\ \end{array} \begin{array}{c} D^b_Q(X_-) \\ \downarrow \\ \downarrow \\ \end{array} \rightarrow \begin{array}{c} \cong \\ \cong \\ \end{array} \begin{array}{c} D^b_Q(X_\times) \\ \downarrow \\ \downarrow \\ \end{array}$$

by inverting the right isomorphism.

Consider the subcategory $(\pi_-)^*G_0 \subset D^b_{T \times C^\times}(V_0)$, where

$$\pi_- : \left[ V_0 / (T \times C^\times) \right] \rightarrow [U_0 / T]$$

is the natural morphism. Under the restriction functor from $V_0 \rightarrow V_\times|\sim$, the subcategory $(\pi_-)^*G_0$ maps to $\tilde{F}$. Under the restriction functor from $V_0 \rightarrow V_\times|\sim$, the subcategory $(\pi_-)^*G_0$ maps to $H$, which is an isomorphism.

The line bundle $O_{\tilde{X}}(kE) \rightarrow \tilde{X}$ corresponds to an $T \times C^\times$-equivariant line bundle $L_k$ on $C^{m+1}$. Let

$$\otimes L_k : D^b_Q(\tilde{X}) \rightarrow D^b_Q(\tilde{X})$$

be the tensor product morphism. Then since the line bundle has $e$-weight $k$, the tensor product sends $(\pi_-)^*G_0$ to $(\pi_-)^*G_k$. We have the following modified diagram as for the last diagram in [9]:

$$\begin{array}{c} (\pi_-)^*G_0 \otimes L_k \rightarrow (\pi_-)^*G_k \\ \downarrow \\ \downarrow \\ \end{array} \begin{array}{c} H \\ \cong \\ \cong \\ \end{array} \begin{array}{c} D^b_Q(X_-) \\ (f_-)^* \\ \otimes L_k \\ (f_\times)^* \\ \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow \\ \end{array} \begin{array}{c} \tilde{F} \\ \cong \\ \cong \\ \end{array} \begin{array}{c} D^b_Q(\tilde{X}) \\ \otimes L_k \\ D^b_Q(\tilde{X}) \\ \end{array} \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \end{array} \begin{array}{c} D^b_Q(X_\times) \\ (f_\times)^* \\ \end{array}$$

The result is easily seen from the above diagram since the bottom represents the Bondal-Orlov functor $BO_k$. \qed
4.2. The spherical twist. Let us fix a single toric wall crossing \([2,3]\). We first classify the exceptional locus of the contractions \(g_\pm\).

For \(g_+: X_+ \to \overline{\mathcal{X}}_0\), let
\[
\mathbb{L}^\vee_\text{ex} := \mathbb{L}^\vee / \langle D_i : i \in M_0 \rangle
\]
and let \(p : \mathbb{L}^\vee \to \mathbb{L}^\vee_\text{ex}\) be the projection. Then \(p : \mathbb{L}^\vee \otimes \mathbb{R} \to \mathbb{L}^\vee_\text{ex} \otimes \mathbb{R}\) is the projection to the vector spaces. Let \(\omega_\text{ex} = p(\omega_+)_\text{stability condition}\) be the image of the stability condition \(\omega_+\). The lattice \(\mathbb{L}^\vee_\text{ex}\), which is rank one, may have torsion in general. In this section we assume that \(\langle D_i : i \in M_0 \rangle\) generate the lattice wall \(W \cap \mathbb{L}^\vee\). Then \(\mathbb{L}^\vee_\text{ex} \cong \mathbb{Z}\). The elements \(D_i \in \mathbb{L}^\vee\) have images \(p(D_i) = D_i : e \in \mathbb{L}^\vee_\text{ex}\). So only \(D_i\) for \(i \in M_\pm\) survive. Hence we get the GIT data on \(\mathbb{L}^\vee_\text{ex}\):

- \(K \cong \mathbb{C}^\times\), a connected torus of rank 1;
- \(\mathbb{L}_\text{ex} = \text{Hom}(\mathbb{C}^\times, K)\);
- \(D_1 \cdot e, \ldots, D_m \cdot e \in \mathbb{L}_\text{ex} = \text{Hom}(K, \mathbb{C}^\times)\), characters of \(K\);
- stability condition \(\omega_\text{ex}^+ \in \mathbb{L}_{\text{ex}}^\vee \otimes \mathbb{R}\);
- \(\mathcal{A}_\omega^\text{ex} = \{ I \subset \{1, 2, \ldots, m\} : D_i \cdot e > 0, i \in I \} \).

Let \(a_i := D_i \cdot e\) for \(D_i \cdot e > 0\) and \(a = (D_i \cdot e : D_i \cdot e > 0)\).

**Proposition 4.3.** The corresponding toric DM stack \(X_{\omega_\text{ex}^+}\) associated with the above GIT data is the weighted projective stack \(\mathbb{P}(a)\). Moreover, the map \(g_+: X_+ \to \overline{\mathcal{X}}_0\) always contracts the weighted projective stack \(X_{\omega_\text{ex}^+} = \mathbb{P}(a)\).

**Proof.** The first statement is easily seen from the GIT data. For the second statement, look at the map
\[
g_+: X_+ = [U_+ / K] \to \overline{\mathcal{X}}_0 = [U_0 / K],
\]
where \(U_+ = U_0 \setminus (\mathbb{C}^{M_{<0}} \cap U_0)\). The torus \(\mathbb{C}^\times\)-fixed points on \(\overline{\mathcal{X}}_0 = [U_0 / K]\) corresponds to nonsimplicial cones, which are spanned by rays containing \(D_i\)'s for \(i \in M_\pm\). Then from the above map \(g_+\), it must contract the weighted projective stack \(\mathbb{P}(a)\) to this fixed point. \(\square\)

**Remark 4.4.** Similar result holds for the contract map \(g_- : X_- = [U_- / K] \to \overline{\mathcal{X}}_0 = [U_0 / K]\).

Let \(b_i := D_i \cdot e\) for \(D_i \cdot e < 0\) and \(b = (D_i \cdot e : D_i \cdot e < 0)\). Then \(g_-\) contracts the weighted projective stacks \(\mathbb{P}(b)\).

Let \(N := \sum_{i : D_i \cdot e > 0} D_i \cdot e = -\sum_{i : D_i \cdot e < 0} D_i \cdot e\), which is the sum of the weights.

**Proposition 4.5.** We have for any \(k \in \mathbb{Z}\),
\[
\text{GR}_k \cong \mathbb{B}O_{(N-1)+k}.
\]

**Proof.** We generalize the proof of Proposition 3.1 in \([1]\). We show that \(\text{GR}_{k-1} \circ \mathbb{B}O_{(N-1)+k}\) takes \(\mathcal{O}_{\mathcal{X}_-}(l)\) to \(\mathcal{O}_{\mathcal{X}_-}(l)\) and acts as identity on
\[
\mathcal{E}xt^i(\mathcal{O}_{\mathcal{X}_-}(l), \mathcal{O}_{\mathcal{X}_-}(l'))
\]
for \(k \leq l, l' \leq k + (N - 1)\), since these objects split-generate the derived category \(\mathcal{D}^b(\mathcal{X}_-).\)
First \(\text{GR}_k\) takes \(\mathcal{O}_{\mathcal{X}_-}(l)\) to \(\mathcal{O}_{\mathcal{X}_-}(-l)\) for \(k \leq l \leq k + (N-1)\), since the subcategory \(\mathcal{G}_k \subset \mathcal{D}^b(\mathcal{U}_0) = \mathcal{D}^b(\mathcal{X}_0)\) is the full-subcategory split-generated by
\[
\mathcal{O}_{\mathcal{X}_0}(k), \ldots, \mathcal{O}_{\mathcal{X}_0}(k + (N-1)).
Also
\[ \mathbb{B}O_{(N-1)+k}(\mathcal{O}_{X_+}(l)) = (f_+)_*(\mathcal{O}_{\tilde{X}}(((N-1) + k)E) \otimes (f_-)^*(\mathcal{O}_{X_-}(l))) \]
\[ = (f_+)_*(\mathcal{O}_{\tilde{X}}((l - (N-1) - k_2, -(N-1) - k)) \]
\[ = \mathcal{O}_{X_+}(-l) \otimes (f_+)_*(\mathcal{O}_{\tilde{X}}(((N-1) + k - l)E)) \]
and
\[ (f_+)_*(\mathcal{O}_{\tilde{X}}(((N-1) + k - l)E)) \cong \mathcal{O}_{X_+} \]
for \(0 \leq (N-1) + k \leq (N-1)\). So \(GR^{-1}_k \circ \mathbb{B}O_{(N-1)+k}\) takes \(\mathcal{O}_{X_-}(l)\) to \(\mathcal{O}_{X_-}(l)\)
for \(k \leq l \leq k + (N-1)\). The proof that \(GR^{-1}_k \circ \mathbb{B}O_{(N-1)+k}\) acts as identity on
\(\mathcal{E}xt^1(\mathcal{O}_{X_-}(l), \mathcal{O}_{X_-}(l'))\) is the same as [1, Proposition 3.1]. \(\square\)

Let
\[ j_+: \mathbb{P}(a) \hookrightarrow X_+; \quad j_-: \mathbb{P}(b) \hookrightarrow X_- \]
be the closed immersions for the weighted projective stacks \(\mathbb{P}(a)\) and \(\mathbb{P}(b)\). To abuse notations, we understand \(\mathcal{O}_{\mathbb{P}(b)}(k)\) as line bundle over \(\mathbb{P}(b)\), and at the
same time taken as the coherent sheaf \(j_- \mathcal{O}_{\mathbb{P}(b)}(k)\) on \(X_-\). The same situation
holds for \(j_+: \mathbb{P}(a) \hookrightarrow X_+\).

**Proposition 4.6.** We have the following result for the autoequivalence:
\[ GR^{-1}_k \circ GR_{k+1} = T_{\mathcal{O}_{\mathbb{P}(b)}(k)} \]
associated with the spherical functor
\[ T_{\mathcal{O}_{\mathbb{P}(b)}(k)}: D^b_Q(X_-) \to D^b_Q(X_-) \]
defined by:
\[ T_{\mathcal{O}_{\mathbb{P}(b)}(k)}(\mathcal{E}) = \text{Cone}(\mathcal{O}_{\mathbb{P}(b)}(k) \otimes R\text{Hom}(\mathcal{O}_{\mathbb{P}(b)}(k), \mathcal{E}) \otimes_{\text{ev}} \mathcal{E}). \]

**Proof.** We generalize the proof in Proposition 3.2 of [1]. We prove the \(k = 0\)
case, since other cases are similar. It suffices to check that both functors act on
\(\mathcal{O}_{X_-}(1), \cdots, \mathcal{O}_{X_-}(N)\), since these objects split-generate the derived category \(D^b_Q(X_-)\). Clearly \(GR^{-1}_0 \circ GR_1\) and \(T_{\mathcal{O}_{\mathbb{P}(b)}}\) act on \(\mathcal{O}_{X_-}(1), \cdots, \mathcal{O}_{X_-}(N - 1)\) as identities. This
is due to the facts that the full subcategories \(G_0 \subset D^b_Q(X_0)\); and \(G_1 \subset D^b_Q(X_0)\)
are split-generated by the objects \(\mathcal{O}_{X_0}, \cdots, \mathcal{O}_{X_0}(N - 1)\); and \(\mathcal{O}_{X_0}(1), \cdots, \mathcal{O}_{X_0}(N)\),
respectively.

We check the case \(\mathcal{O}_{X_-}(N)\). Consider the Koszul resolution of the substack
\([\mathcal{C}^{M<0} \cap U_0/K] \subset X_0\), which is cut out by a transverse section of \(\mathcal{O}_{X_0}(-1) \otimes S_+:\n\mathcal{O}_{X_0}(N) \otimes \det(S_+^*) \to \cdots \to \mathcal{O}_{X_0}(2) \otimes \wedge^2(S_+^*) \to \mathcal{O}_{X_0}(1) \otimes S_+^* \to \mathcal{O}_{X_0} \to \mathcal{O}_{[\mathcal{C}^{M<0} \cap U_0/K]}.
Restrict to \(X_-\) we get:
\[ (\mathcal{O}_{X_-}(N) \otimes \det(S_+^*) \to \cdots \to \mathcal{O}_{X_-}(2) \otimes \wedge^2(S_+^*) \to \mathcal{O}_{X_-}(1) \otimes S_+^* \to \mathcal{O}_{X_-} \to 0_{j_- \mathcal{O}_{\mathbb{P}(b)}}). \]

Then we restrict to \(X_+\), we get: (Note that there is no last term.)
\[ (\mathcal{O}_{X_+}(-N) \otimes \det(S_+^*) \to \cdots \to \mathcal{O}_{X_+}(-2) \otimes \wedge^2(S_+^*) \to \mathcal{O}_{X_+}(-1) \otimes S_+^* \to \mathcal{O}_{X_+}). \]
Now we have
\[ GR_1(\mathcal{O}_{X_-}(N)) = \mathcal{O}_{X_+}(-N). \]
Use (4.2) we get:
\[
\deg_0 \mathcal{O}_{X_+}(-(N-1)) \otimes S_+ \to \mathcal{O}_{X_+}(-(N-2)) \otimes \wedge^2(S_+) \to \cdots \to \mathcal{O}_{X_+} \otimes \det(S_+)
\]

Then applying the functor $\mathcal{G} \mathcal{R}_0$,
\[
\deg_0 \mathcal{O}_{X_+}(N-1) \otimes S_+ \to \mathcal{O}_{X_+}(N-2) \otimes \wedge^2(S_+) \to \cdots \to \mathcal{O}_{X_+} \otimes \det(S_+)
\]
which is the middle $N$-terms of (4.1) tensored with $\det(S_+)$, and this extension is
\[
\text{Cone}(\mathcal{J}_- \mathcal{O}_{P(b)} \otimes \det(S_+)[−N] \to \mathcal{O}_{X_-}(N)).
\]

On the other hand, the spherical twist
\[
\mathcal{T}_{\mathcal{O}_{P(b)}(\mathcal{O}_{X_-}(N))} = \text{Cone}(\mathcal{J}_- \mathcal{O}_{P(b)} \otimes \mathcal{R} \text{Hom}(\mathcal{J}_- \mathcal{O}_{P(b)}, \mathcal{O}_{X_-}(N)) \to \mathcal{O}_{X_-}(N)),
\]
has the same description. These two extensions are the same since the functors $\mathcal{G} \mathcal{R}_0^{-1} \circ \mathcal{G} \mathcal{R}_1$ and $\mathcal{T}_{\mathcal{O}_{P(b)}}$ acts in the same way on the Exts. \qed

Let $\mathcal{B} \mathcal{O}'_k : D^b_Q(X_+) \to D^b_Q(X_-)$ be the general Bondal-Orlov functors other way around by:
\[
\mathcal{B} \mathcal{O}'_k := (f_-)_*(\mathcal{O}_{\mathcal{X}}(kE)) \otimes (f_+)^*(-(-)).
\]
The degree zero $\mathcal{B} \mathcal{O}'_0$ is the Fourier-Mukai transform $\mathcal{F} \mathcal{M}'$.

**Corollary 4.7.** We have:
\[
\mathcal{F} \mathcal{M}' \circ \mathcal{F} \mathcal{M} = \mathcal{T}^{-1}_{\mathcal{O}_{P(b)}(-1)} \circ \cdots \circ \mathcal{T}^{-1}_{\mathcal{O}_{P(b)}(-(N-1))}.
\]

**Proof.** The results in Proposition 4.5 and Proposition 4.6 imply that
\[
\mathcal{B} \mathcal{O}'_{-k} \circ \mathcal{B} \mathcal{O}_{N-1+k+1} = \mathcal{T}_{\mathcal{O}_{P(b)}(k)}^{-1}.
\]
Hence we have:
\[
\mathcal{B} \mathcal{O}'_{-k-1} \circ \mathcal{B} \mathcal{O}_{N-1+k} = \mathcal{T}_{\mathcal{O}_{P(b)}(k)}^{-1}.
\]
By Grothendieck duality, we have that $\mathcal{B} \mathcal{O}_k^{-1} = \mathcal{B} \mathcal{O}'_{(N-1)-k}$. Then the result is a direct calculation. \qed

**Remark 4.8.** By a similar argument we have:
\[
\mathcal{F} \mathcal{M} \circ \mathcal{F} \mathcal{M}' = \mathcal{T}^{-1}_{\mathcal{O}_{P(a)}(-1)} \circ \cdots \circ \mathcal{T}^{-1}_{\mathcal{O}_{P(a)}(-(N-1))}.
\]

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