Geometrical Aspects of Cosmic Magnetic Fields

Christos G. Tsagas
Relativity and Cosmology Group, University of Portsmouth, Portsmouth PO1 2EG, England

Abstract

We discuss how the vector nature of magnetic fields, and the geometrical interpretation of gravity introduced by general relativity, lead to a special coupling between magnetism and spacetime curvature. This magneto-geometrical interaction effectively transfers the tension properties of the field into the spacetime fabric, triggering a variety of effects with potentially far-reaching implications.

Keywords: Magnetic Fields, Early Universe, Large-Scale Structure

1 Introduction

It has long been thought that magnetic fields might have played a role during the formation and the evolution of the observed large scale structure [1]. Recently, this idea has received renewed interest manifested by the increasing number of related papers that have appeared in the literature [2, 3]. Nevertheless, there are still only a few fully relativistic approaches available. Most treatments are either Newtonian or semi-relativistic. As such, they are bound to exclude certain features of the magnetic nature. Two key features are the vector nature of the field and the tension properties of magnetic force lines. In general relativity vectors have quite a different status than ordinary scalar sources, such as the energy density and pressure of matter. The geometrical nature of Einstein’s theory means that vector fields are directly coupled to the spacetime curvature, as manifested by the Ricci identity

\[ 2\nabla_{[a} \nabla_{b]} B_c = R_{abcd} B^d, \]  

applied here to the magnetic vector \( B_a \), where \( R_{abcd} \) is the spacetime Riemann tensor. The Ricci identity plays a fundamental role in the mathematical formulation of general relativity. Essentially, it is the definition of spacetime curvature itself. We call the special interaction reflected in the right hand side of Eq. (1) the magneto-curvature coupling. This coupling goes beyond the standard interplay between matter and geometry as introduced by the Einstein field equations. In fact, it makes the magnetic field an inseparable part of the spacetime fabric by effectively transferring its properties to the spacetime itself. The key property appears to be the tension of the magnetic lines of force.

Magnetic fields transmit stresses between regions of material particles and fluids. The field exerts an isotropic pressure in all directions and carries a tension along the magnetic lines of force. Each small flux-tube behaves like an infinitely elastic rubber band, while neighbouring

*Present address: Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa.
tubes expand against each other under their own pressure. Equilibrium exists only when a balance between pressure and tension is possible. To unravel these tension properties, consider the energy momentum tensor of a pure magnetic field

\[ T_{ab} = \frac{1}{2}B^2u_au_b + \frac{1}{6}B^2h_{ab} + \Pi_{ab}, \]  

where \( B^2 = B_aB^a \) and \( \Pi_{ab} = (B^2/3)h_{ab} - B_aB_b \). Thus, the field behaves as an imperfect fluid with energy density \( \rho_m = B^2/3 \), isotropic pressure \( p_m = B^2/6 \) and anisotropic pressure \( \Pi_{ab} \). The tension properties of the field are incorporated in the eigenvalues of the symmetric trace-free tensor \( \Pi_{ab} \). Orthogonal to \( B_a \) one finds two positive eigenvalues equal \( 1/3 \) each. Thus, the magnetic pressure perpendicular to the field lines is positive, reflecting their tendency to push each other apart. In the \( B_a \) direction, however, the associated eigenvalue is \( -2/3 \) and the magnetic pressure is negative. The minus sign reflects the tension properties of the field lines, namely their tendency to remain as ‘straight’ as possible.

These elastic magnetic properties are effectively injected into the spacetime fabric itself, through the aforementioned coupling between magnetism and geometry. The result is a magneto-curvature stress which always reacts to curvature distortions and tries to restore spatial flatness \[ 4, 5 \]. This relativistic magneto-curvature stress closely resembles the classical one exerted by distorted magnetic field lines (see e.g. \[ 6 \]). The difference is that, in the relativistic case, the distortion of the field pattern is triggered by the spacetime geometry itself. The implications are widespread and far from trivial.

2 The magnetic impact on universal expansion

Consider a general spacetime filled with a magnetised, highly conductive, perfect fluid. Its volume expansion is governed by the non-linear Raychaudhuri equation \[ 3 \]

\[ \frac{1}{3} \Theta^2 q = \frac{1}{2} (\rho + 3p + B^2) + 2 (\sigma^2 - \omega^2) - \nabla^a A_a - \Lambda, \]  

where \( q \) is the deceleration parameter, \( \sigma^2 \) and \( \omega^2 \) are the shear and vorticity magnitudes respectively and \( \Lambda \) is the cosmological constant. The state of the expansion is determined by the sign of the right-hand side of Eq. \( 3 \). Positive terms decelerate the universe while negative ones lead to acceleration. Clearly, conventional matter and shear effects slow the expansion down. On the other hand, vorticity and a positive cosmological constant accelerate the universe. Hence, every term on the right-hand side of Eq. \( 3 \) has a clear kinematical role with the exception of \( \nabla^a A_a \). The latter can be either positive or negative, depending on the specific form of the 4-acceleration. In our case \( A_a \) obeys the non-linear Euler formula given by \[ 3 \]

\[ (\rho + p + \frac{2}{3}B^2) A_a = -c_s^2 D_a \rho - \varepsilon_{abc} B^b \text{curl} B^c - A^b \Pi_{ba}, \]  

where \( A_a \) is the fluid 4-acceleration, \( c_s^2 = \dot{\rho}/\rho \) is the sound speed squared and \( \varepsilon_{abc} \) the spatial alternating tensor. In a weakly magnetised, slightly inhomogeneous and anisotropic, almost-FRW universe Eqs. \[ 3 \] and \[ 3 \] linearise to give \[ 3 \]

\[ \frac{1}{3} \Theta^2 q = \frac{1}{2} \rho(1 + 3w) + \frac{c_s^2 \Delta}{(1 + w)a^2} + \frac{c_s^2 B}{2(1 + w)a^2} - \frac{2k\rho^2}{(1 + w)a^2} - \Lambda. \]  

In the above \( \Delta \) and \( B \) describe scalar perturbations in the fluid and the magnetic energy density respectively, \( c_s^2 = B^2/\rho \) is the square of the Alfvén speed, \( k = 0, \pm 1 \) is the background curvature
index and $a$ is the scale factor. Given that in the linear regime the mean values of $\Delta$ and $B$ are zero, one expects that on average Eq. (5) looks like

$$\frac{1}{3} \Theta^2 q = \frac{1}{2} \rho (1 + 3w) - \frac{2k \rho a}{(1 + w)a^2},$$

(6)

where $\Lambda = 0$ from now on. Note the magneto-curvature term in the right-hand side which results from the coupling between magnetism and geometry as manifested in Eq. (1). This term affects the expansion in two completely different ways depending on the sign of the background curvature. In particular, the magneto-geometrical effects slow the expansion down when $k = -1$ but tend to accelerate the expansion if $k = +1$. Such a behaviour seems odd, especially since positive curvature is always associated with gravitational collapse. The explanation lies in the elastic properties of the field lines. As curvature distorts the magnetic force lines their tension backreacts giving rise to a restoring magneto-curvature stress [4]. The magnetic backreaction has kinematical, dynamical as well as geometrical implications. In Eq. (6), for example, the tension of the field adjusts the expansion rate of the universe to minimize the kinematical effects of curvature. As a result the expansion rate is brought closer to that of a flat FRW model. Overall, it looks as though the elastic properties of the field have been transferred into space.

According to Eq. (6), the magneto-curvature effects also depend on the material component of the universe. When dealing with conventional matter (i.e. for $0 \leq w \leq 1$) the most intriguing cases occur in positively curved spaces. In particular, when $w = 1$ (i.e. for stiff matter) the Alfvén speed grows as $c_a^2 \propto a^2$ and the magneto-curvature term in Eq. (6) becomes time-independent. In this case the field acts as an effective positive cosmological constant. For radiation and dust, on the other hand, $c_a^2 = \text{const}$ and $c_a^2 \propto a^{-1}$ respectively. In these cases the magneto-curvature term is no longer time independent but drops with time mimicking a time-decaying quintessence [5].

The magneto-curvature effects discussed so far, subtle though they may be, remain secondary unless the field is relatively strong. However, the coupling between magnetism and geometry means that even weak fields can have a strong overall impact when the curvature is strong. To demonstrate how this might happen, we consider a weakly magnetised spatially open cosmology. For $k = -1$ Eq. (6) becomes

$$\frac{1}{3} \Theta^2 q = \frac{1}{2} \rho (1 + 3w) + \frac{2c_a^2}{(1 + w)a^2},$$

(7)

where now the magneto-curvature term tends to decelerate the expansion. Let us assume a spacetime filled with non-conventional matter, namely that $-1 \leq w < 0$. Scalar fields, for example, can have an effective equation of state that satisfies this requirement. Such models allow for an early curvature dominated regime with $\Omega \ll 1$. Given that $\rho \propto a^{-3(1+w)}$ and $c_a^2 \propto a^{-1+3w}$, the magneto-curvature term in Eq. (6) can dominate the early expansion, even when the field is weak, if $-1 \leq w \leq -1/3$. In this case the accelerated inflationary phase, which otherwise would have been inevitable, is suppressed. Instead of inflating the magnetised universe remains in a state of decelerated expansion. For $w = -1$, in particular, the mere presence of the field can inhibit the de Sitter inflationary regime if $\Omega < 0.5$ [5]. Clearly, this result challenges the widespread perception that magnetic fields are relatively unimportant for cosmology. Even weak fields can play a decisive role when the curvature is strong. Moreover, it casts doubt on the efficiency of standard inflation in the presence of primordial magnetism.
3 Magnetic fields and gravity waves

Let us now turn our attention to geometry and examine the implications of the magnetic presence for propagating gravitational radiation. The production of gravity waves by stochastic magnetic fields has been investigated in [9]. Here we will take a more geometrical approach and look at the implications of the tension properties of the field for gravity waves passing through a magnetised region. Our starting point is a spatially flat FRW background universe filled with a highly conductive perfect fluid. We will then perturb this background by allowing for weak gravitational waves and a weak magnetic field. Covariantly, gravity waves are described via the electric ($E_{ab}$) and the magnetic ($H_{ab}$) parts of the Weyl tensor [8]. Their magnitudes, $E^2 = E_{ab}E^{ab}/2$ and $H^2 = H_{ab}H^{ab}/2$, provide a measure of the wave’s energy density and amplitude. Given that $H_{ab} = \text{curl} \sigma_{ab}$, we can simplify the problem by replacing the magnetic Weyl tensor with the shear. Note that the field couples to gravitational radiation directly via the anisotropic magnetic stresses, which affect the propagation of both $E_{ab}$ and $\sigma_{ab}$ [9]. Having set the constraints that isolate tensor perturbations in a magnetised universe (see [9]), we arrive at the system

\begin{align*}
(E^2)' &= -2\Theta E^2 - \frac{1}{2} \rho (1 + w) \chi - \frac{1}{2} \Theta B^2 \mathcal{E}, \\
(\sigma^2)' &= -\frac{4}{3} \Theta \sigma^2 - \chi - \frac{1}{2} B^2 \Sigma, \\
\dot{\chi} &= -\frac{5}{3} \Theta \chi - 2 E^2 - \rho (1 + w) \sigma^2 - \frac{1}{2} B^2 \mathcal{E} - \frac{1}{2} \Theta B^2 \Sigma, \\
\dot{\mathcal{E}} &= -\Theta \mathcal{E} - \frac{1}{2} \rho (1 + w) \Sigma - \frac{1}{3} \Theta B^2, \\
\dot{\Sigma} &= -\frac{2}{3} \Theta \Sigma - \mathcal{E} - \frac{1}{3} B^2,
\end{align*}

with $B^2 \propto a^{-4}$, $\chi = E_{ab} \sigma^{ab}$, $\mathcal{E} = E_{ab} \eta^a \eta^b$ and $\Sigma = \sigma_{ab} \eta^a \eta^b$ ($\eta_a = B_a/\sqrt{B^2}$). The last two scalars are related via the Gauss-Codacci equation by

$$\mathcal{E} = \frac{1}{3} \Theta \Sigma + \frac{1}{3} B^2 + R,$$

where $R = [R_{(ab)} - (R/3) h_{ab}] \eta^a \eta^b$ describes spatial curvature distortions in the direction of the magnetic field lines. For radiation, the late-time solutions for $E^2$ and $\sigma^2$ are [9]

\begin{align*}
E^2 &= \frac{4}{9} \left[ E_0^2 + \frac{\sigma_0^2}{4 t_0^2} - \frac{\chi_0}{2 t_0} \right] \left( \frac{t_0}{t} \right)^2 - \frac{2}{9} \left( \frac{1}{6} B_0^2 + R_0 \right) B_0^2 \left( \frac{t_0}{t} \right)^2, \\
\sigma^2 &= \frac{1}{9} \left[ \sigma_0^2 + 4 E_0^2 t_0^2 - 2 \chi_0 t_0 \right] - \frac{2}{9} \left( \frac{1}{6} B_0^2 + R_0 \right) B_0^2 t_0^2,
\end{align*}

with an analogous result for dust [9]. Note that the terms in square brackets determine the magnetic-free case. According to [9], the field leaves the evolution rates of both $E^2$ and $\sigma^2$ unchanged but modifies their magnitudes. The magnetic impact depends on the initial conditions and it is twofold. There is what one might call a pure magnetic effect, independent of the spatial curvature, which suppresses the energy of the wave. It becomes apparent when we set $R_0 = 0$ in Eqs. (11). Hence, when one starts from a FRW background, as we did here, the field presence will have a smoothing effect on the waves. Given the inherent weakness of gravitational radiation, these magnetic effects are potentially detectable even when relatively weak fields are involved. Solution (11) also reveals a magneto-curvature effect on gravitational radiation. This is encoded in the $R_0$-term and depends entirely on the spatial curvature. For $R_0 > 0$, namely when the curvature distortion along the field lines is positive, the effect is to decrease the energy...
of the wave. On the other hand, the magneto-curvature term will increase the wave’s energy when $R_0 < 0$. These magneto-geometrical effects result from the tension properties of the field and get stronger with increasing curvature distortion. Let us take a closer look at them. According to Eq. (9), the scalar $R$ describes distortions in the local spatial curvature generated by the propagating magnetised gravity wave. The magneto-geometrical terms in (10) modify the energy density of the wave as though they are trying to minimize the curvature distortion. In other words, the pure magneto-curvature effect shows a tendency to preserve the spatial flatness of the background universe. Earlier, an analogous magneto-curvature effect was also observed on the expansion rate of spatially curved FRW universes. This pattern of behaviour raises the question as to whether it reflects a generic feature of the magnetic nature. More specifically, one wonders if the tension properties of the magnetic force lines and the coupling between magnetism and spacetime curvature, imply an inherent ‘preference’ of the field for flat geometry. Next, we will take a more direct look at this possibility.

4 Magnetic effects on curvature distortions

Consider an almost-FRW magnetised universe and assume that the background spatial geometry is Euclidean. If $R$ is the 3-Ricci scalar of the perturbed spatial sections we obtain

$$\dot{R} = -\frac{2}{3} \left[ 1 + \frac{2c_s^2}{3(1 + w)} \right] \Theta R + \frac{4c_s^2\Theta}{3(1 + w)a^2} \Delta + \frac{2c_s^2\Theta}{3(1 + w)a^2} B,$$

(11)

to linear order. As expected, the expansion dilutes curvature distortions. The latter are caused by fluctuations in the fluid and the magnetic energy densities, represented by the scalars $\Delta$ and $B$ respectively. Interestingly, however, the field, through its coupling with the geometry, also enhances the smoothing effect of the expansion on $R$. This is the direct result of the tension properties of the magnetic force lines, which tend to suppress curvature distortions. Clearly, given the weakness of the field (recall that $c_a^2 \ll 1$), this magnetically induced smoothing is negligible compared to that caused directly by the expansion. Nevertheless, the tendency of the field to maintain the original flatness of the spatial sections is quite intriguing. It seems to support the idea that, given their tension properties and their direct coupling to curvature, magnetic fields might indeed have a natural preference for flat spaces.

5 Discussion

The magneto-curvature effects presented here reveal a side of the magnetic nature which as yet remains unexplored. They derive from the vector nature of the field and from the geometrical approach to gravity adopted by general relativity. The latter allows a direct coupling between magnetism and curvature which effectively transfers the magnetic properties into space itself. The tension of the field lines appears to be the key property. Kinematically speaking, the magneto-curvature effects tend to accelerate spatially closed regions, while they decelerate those with open spatial curvature. Crucially, even weak fields can have a strong overall impact when the curvature input is strong. This challenges the widespread belief that, due to their perceived weakness, magnetic fields are relatively unimportant for cosmology. Inflationary scenarios, for example, allow for strong-curvature regimes during their very early stages. Such initial curvature dominated epochs have never been considered a serious problem for inflation given the vast smoothing power of the accelerated expansion. It is during these early stages, however, that a
weak magnetic presence is found capable of suppressing the accelerated phase in spatially open ‘inflationary’ models. Such a possibility casts doubt on the efficiency, and potentially on the viability, of standard inflation in the presence of primeval magnetism. In fact, every cosmological model that allows for a strong-curvature regime and a weak magnetic field could also be vulnerable to these magneto-curvature effects. The coupling between magnetism and spacetime curvature has also intriguing geometrical implications. It modifies the expansion of spatially closed, and open, FRW universes bringing the rate closer to that of a flat Friedmannian model. The magnetic presence is also found to suppress gravity wave distortions induced in a FRW background universe. Moreover, the tension properties of the field lines tends to smooth out perturbations in the spatial curvature of a flat FRW universe, and modulate the energy of gravity waves as if to preserve the background flatness. In short the magnetised space seems to react to curvature distortions showing, what one might interpret as, a preference for flat geometry. Given the ubiquity of magnetic fields in the universe this unconventional behaviour deserves further investigation, as it could drastically change our views on the role of cosmic magnetism not only in cosmology but also in astrophysics. It is the aim of this article to bring these issues to light and draw attention to their potential implications.

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