A variational approach to relativistic superfluid vortex elasticity

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Abstract

It is well known that a superfluid rotates by forming an array of quantized vortices. A relativistic formulation for superfluid vortex dynamics is required for a range of problems in astrophysics and cosmology, from neutron star interiors and radio pulsar glitches to possible dark matter condensates on galactic scales. This paper develops a formalism for such systems, extending the well-established variational approach to relativistic fluids to account for the presence of a collection of quantized vortices. The model is firmly anchored in the geometry of the problem (drawing on aspects from basic string dynamics) and accounts for elastic aspects associated with a vortex array, providing a precise foundation for applications which have so far been based on somewhat ad hoc phenomenology.

Keywords: neutron stars, fluid dynamics, vortices, elasticity

(Some figures may appear in colour only in the online journal)

1. Introduction

Superfluids mimic bulk rotation by forming an array of quantized vortices—a collection of slim 'tornadoes', the distribution of which determines the macroscopic angular momentum of the system. The associated dynamics plays a key role in the description of superfluid systems, both in the laboratory setting and in astrophysics [1, 2]. In particular, an understanding of
vortex dynamics is thought to be essential for an explanation of the enigmatic spin glitches seen in many young pulsars. The presence of a large-scale neutron superfluid in both the core and the crust (where it co-exists with a lattice of neutron-rich nuclei) is a prerequisite for any realistic description of a mature neutron star, and the simple fact that these systems involve extreme densities (reaching several times the nuclear saturation density) means that the relevant modelling has to be done in the context of relativistic gravity. Given this, there have been efforts to extend models for relativistic fluid dynamics to account for superfluid components and the presence of quantized vortices. A recent example—with close connection to the discussion in this paper—provided the first description of the vortex-mediated mutual friction [3], a dissipation channel that is unique to superfluids and which is known to be important for models of macroscopic neutron star dynamics [4, 5].

The fact that vortices are associated with a long-range interaction—which is how they contribute to bulk rotation—implies that the vortex lattice has elastic properties [6–8]. This is an interesting aspect, which may be of observational/experimental relevance. In particular, the vortex lattice supports a set of elastic oscillation modes. These so-called Tkachenko modes, first proposed in a seminal set of papers in the 1960s [9, 10], have been discussed for superfluid helium, superfluid atomic condensates [11–13] and neutron stars [14–17]. The experimental verification of the idea is, however, quite recent [18]. The main difficulty is that other aspects of the physics (vortex pinning to the surface of a laboratory container, or the neutron star crust, and the mixing with inertial modes of a rotating body) tend to overwhelm the subtle effect of the vortex elasticity [19].

Despite the experimental issues, the problem of vortex elasticity is of obvious conceptual interest. Nevertheless, the problem has neither been considered within general relativity, nor within the approach of ‘modern’ elasticity theory (where the elastic properties are viewed as due to the deviation from a relaxed/unstrained equilibrium configuration [20–22]). With this paper, we aim to fill this gap in the discussion. Drawing on the variational approach to relativistic fluid dynamics [23, 24], and basic aspects of string dynamics [25–27], we provide a description of vortex elasticity (notably based on a two-dimensional subspace, orthogonal to the vortex array). As this description is—at least in principle—fully nonlinear, it goes beyond previous (more phenomenological) discussions of the problem [6–8], which tend to be rooted in perturbation theory. Moreover, by considering our final results from the perturbative point of view, we shed light on the origin of expressions that have been used in applications and indicate how these models can be extended to the curved spacetime setting.

The paper is laid out as follows: section 2 introduces the variational approach to relativistic fluids. This material is standard, but is useful as we want to compare and contrast to the Kalb-Ramond variation that follows in section 3. Section 4 provides the connection between the problem for relativistic vortices and the standard action used in (basic) string theory, emphasizing the close links between the two problems. Section 5 then develops the formalism for the vortex problem. Section 6 extends the model to account for the elasticity of a vortex array, while section 7 comments on perturbations and the Newtonian limit of the results, facilitating a comparison with the existing literature. Section 8 provides a few final remarks.

2. Variational fluid model

In order to set the scene, and provide both context and inspiration for the discussion, it makes sense to review the standard variational approach to relativistic fluids (following the approach from [23, 24]). It may seem somewhat odd to go over supposedly familiar material in detail, but it turns out to be relevant to compare and contrast the derivation of the fluid equations with the strategy for the vortex lattice. In essence, the fluid derivation makes use of a three-dimensional
matterspace in order to ensure that the matter flux na is conserved (thus constraining the variation). In the vortex case, we will introduce an analogous—now two-dimensional—subspace, in order to enforce constraints on the vorticity.

Let us first consider a single matter component, represented by a (conserved) flux na (with spacetime indices a, b, c, ... = {0, 1, 2, 3}). For an isotropic system the matter Lagrangian, which we will call Λ, should be a relativistic invariant and hence depend only on n² = −gabnαnβ. In effect, this means that the Lagrangian depends on the flux and the spacetime metric. An arbitrary variation of Λ = Λ(n²) = Λ(na, gab) then gives (ignoring terms that can be written as total derivatives, representing ‘surface terms’ in the action)

\[ \delta \left( \sqrt{-g} \Lambda \right) = \sqrt{-g} \left[ \mu_a \delta n^a + \frac{1}{2} \left( \Lambda g^{ab} + n^a \mu_b \right) \delta g_{ab} \right], \]

where g is the determinant of the spacetime metric and \( \mu_a \) is the canonical momentum:

\[ \mu_a = \frac{\partial \Lambda}{\partial n^a} = -2 \frac{\partial \Lambda}{\partial n^2} g^{ab} n^b. \]

We have also used

\[ \delta \sqrt{-g} = \frac{1}{2} g^{ab} \delta g_{ab}. \]

Equation (1) illustrates why we need to develop a constrained variational principle. As it stands, the variation of Λ suggests that the equations of motion would be \( \mu_a = 0 \), which means that the fluid carries neither energy nor momentum. This problem is resolved by constraining the flux. A natural way to do this involves introducing a three-dimensional ‘matter space’, the coordinates of which, \( X^A \) with \( A, B, C, ... = \{1, 2, 3\} \), serve as labels that distinguish fluid element worldlines, assigned at the initial time of the evolution, say \( t = 0 \). The matter space coordinates can be considered as scalar fields on spacetime, with a unique map (obtained by a pull-back construction) relating them to the spacetime coordinates:

\[ \psi^A_a = \frac{\partial X^A}{\partial n^a}. \]

With this set-up, the conservation of the matter flux is ensured provided that the dual three-form

\[ \epsilon_{abc} n^d = -n^d \epsilon_{dabc}, \quad n^a = \frac{1}{3!} \epsilon_{abcd} n_{bcd}, \]

(where \( \epsilon_{abcd} \) is the volume form associated with spacetime) is closed. It is easy to see that

\[ \nabla_{[a} n_{bcd]} = 0 \implies \nabla_a n^a = 0. \]

Let us consider this argument in more detail. The closure is guaranteed by introducing

\[ n_{abc} = \psi^A_a \psi^B_b \psi^C_c \eta_{ABC}, \]

where the (anti-symmetric) matter-space \( \eta_{ABC} \) depends only on the \( X^A \) coordinates (and the Einstein summation convention applies to repeated matter-space indices). In order for this to make sense, \( n_{abc} \) must be a ‘fixed’ tensor, in the sense that

\[ \text{Note that the four-velocity } u^a \text{ is associated with the fluid flow in this instance. This will be different when we focus on the vorticity later.} \]
\[ u^a n_{abc} = 0, \] (8)

and

\[ \mathcal{L}_u n_{abc} = 0, \] (9)

where \( \mathcal{L}_u \) is the Lie derivative along the flow. The latter is equivalent to requiring

\[ \nabla_{[a} n_{bcd]} = \partial_{[a} n_{bcd]} = 0. \] (10)

The final step involves noting that

\[ \partial_{[a} n_{bcd]} = \psi^A_a \psi^B_b \psi^C_c \psi^D_d \partial_{[A} n_{BCD]} = 0, \] (11)

is automatically satisfied if

\[ \partial_{[A} n_{BCD]} = 0, \] (12)

which, in turn, follows if \( n_{ABC} \) is a function only of the \( X^A \) coordinates. This completes the argument.

Formally, we have changed perspective by taking the (scalar fields) \( X^A \) to be the fundamental variables. The construction also provides matter space with a geometric structure. If integrated over a volume in matter space, \( n_{ABC} \) provides a measure of the number of particles in that volume. In essence, we have

\[ n_{ABC} = n_{ABC}. \] (13)

The final step in the derivation of the fluid equations involves introducing the Lagrangian displacement \( \xi^a \), tracking the motion of a given fluid element. From the standard definition of Lagrangian variations, we have

\[ \Delta X^A = \delta X^A + \mathcal{L}_\xi X^A = 0, \] (14)

where \( \delta X^A \) is the Eulerian variation and \( \mathcal{L}_\xi \) is the Lie derivative along the displacement \( \xi^a \). This means that we have

\[ \delta X^A = -\mathcal{L}_\xi X^A = -\xi^a \frac{\partial X^A}{\partial x^a} = -\xi^a \psi^A_a. \] (15)

It also follows that

\[ \Delta \psi^A_a = 0, \] (16)

and

\[ \mathcal{L}_u \psi^A_a = 0. \] (17)

Given these results, it is easy to show that

\[ \Delta n_{abc} = 0, \] (18)

a fact that will be useful later.
Making use of the standard relations
\[ \delta g_{db} = -g_{da}g_{bc} \delta g^{ac}, \tag{19} \]
and
\[ \delta T^{abcd} = \frac{1}{2} \delta^{abcd} g_{ef} \delta g^{ef}, \tag{20} \]
we now have
\[ \delta n^a = \frac{1}{3!} \delta (\epsilon^{abcd} n_{bcd}) = n^b \nabla_b \xi^a - \xi^b \nabla_b n^a - n^a \left( \nabla_b \xi^b - \frac{1}{2} g_{bc} \delta g^{bc} \right). \tag{21} \]
Expressing the variations of the matter Lagrangian in terms of the displacement \( \xi^a \), rather than the perturbed flux, we ensure that the flux conservation is accounted for in the equations of motion. The variation of \( \Lambda \) then leads to
\[ \delta \left( \sqrt{-g} \Lambda \right) = \sqrt{-g} \left\{ f_a \xi^a - \frac{1}{2} \left[ (\Lambda - n^i \mu_i) g_{ab} + n_a \mu_b \right] \delta g^{ab} \right\}, \tag{22} \]
and the fluid equations of motion are given by
\[ f_b \equiv 2 n^a \nabla_{[a} \mu_{b]} = 0, \tag{23} \]
(wheresquarebracketsindicante-antisymmetrization, as usual). Finally, introducing the vorticity two-form
\[ \omega_{ab} = 2 \nabla_{[a} \mu_{b]}, \tag{24} \]
we have the simple relation
\[ n^a \omega_{ab} = 0. \tag{25} \]
We can also read off the stress-energy tensor from (22). We need
\[ T_{ab} = -2 \sqrt{-g} \frac{\delta \left( \sqrt{-g} \Lambda \right)}{\delta g^{ab}} = \Lambda g_{ab} - 2 \frac{\delta \Lambda}{\delta g^{ab}}. \tag{26} \]
Introducing the matter four-velocity, such that \( n^a = n u^a \) and \( \mu_i = \mu u^i \), where \( \mu \) is the chemical potential, we see that the energy is
\[ \varepsilon = u_a u^a T^{ab} = -\Lambda. \tag{27} \]
Moreover, we identify the pressure from the thermodynamic relation:
\[ p = -\varepsilon + n \mu = \Lambda - n^i \mu_i. \tag{28} \]
This means that we have the usual perfect fluid result
$$T^{ab} = p g^{ab} + n^a \mu^b = \varepsilon u^a u^b + \rho h^{ab},$$

where we have used the standard projection

$$h^{ab} = g^{ab} + u^a u^b.$$  

Finally, it is straightforward to confirm that

$$\nabla_a T^{ab} = - f^b + \nabla^b \Lambda - \mu^b \nabla_a n^a = - f^b = 0,$$

since (i) $\Lambda$ is a function only of $n^a$ and $g^{ab}$, and (ii) the definition of the momentum $\mu^a$.

Up to this point we have rehearsed standard arguments, but it turns out that it pays off to keep the detailed steps in mind as we proceed.

3. The Kalb–Ramond approach

In order to make cautious progress, we now set out to derive the fluid results from a different perspective. The ultimate aim is to arrive at an intuitive description of the (suitably averaged) dynamics of a collection of quantized superfluid vortices.

The strategy builds on efforts to relate string dynamics to the forces acting on a superfluid vortex, first considered in [28, 29] and developed further in [30, 31]. We start by noting that the superfluid fluid velocity (technically, the momentum [24]) can be linked to the gradient of a scalar potential $\alpha$ such that

$$\tilde{H}_a = \frac{1}{3!} \epsilon_{abcd} H^{bcd},$$

and introduce the so-called Kalb–Ramond field [29], such that

$$H^{abc} = \nabla^{[a} B^{bc]} = \partial^{[a} B^{bc]}.$$  

It is now easy to see that the scalar wave equation

$$\Box \alpha = 0,$$

is automatically satisfied, as long as

$$\nabla_a (\nabla^a B^{bc} + \nabla^c B^{ab} + \nabla^b B^{ca}) = 0.$$  

In effect, we can shift the focus from $\alpha$ to $B^{ab}$. Alternatively, we could treat $B^{ab}$ as an independent field (and try to solve the more complicated wave equation (35)). The relevant dynamical equations are then automatically solved by expressing this field in terms of a scalar potential—the two descriptions are complementary [30]. The advantage of the Kalb–Ramond representation may not be particularly clear at this point, but we will soon see that it makes the introduction of topological defects (vortices/strings) intuitive.

As a first step in this direction, we return to the fluid problem but shift the attention from the matter flux to the vorticity. Following [32–34], we do this by noting that we can ensure

5 From this point on, we use tildes to indicate Hodge duals.
that the conservation law (6) is automatically satisfied by introducing a two-form \( B_{ab} \) (the Kalb–Ramond field) such that

\[
n_{abc} = 3 \nabla_{[a} B_{bc]}.
\] (36)

That is, we have

\[
n^{a} = \frac{1}{2} \epsilon^{abcd} \nabla_{b} B_{cd},
\] (37)

and the flux conservation (6) follows as an identity\(^6\)—we no longer need to introduce the three-dimensional matter space.

Next, in order to find an action that reproduces the known perfect fluid results, we elevate the vorticity \( \omega_{ab} \) to an additional variable. A Legendre transformation [33] leads to the Lagrangian\(^7\)

\[
\bar{\Lambda} = \Lambda - \frac{1}{4} \epsilon^{abcd} B_{ab} \omega_{cd} = \Lambda - \frac{1}{2} \bar{\omega}_{ab} B_{ab},
\] (38)

where we have used the dual

\[
\bar{\omega}_{ab} = \frac{1}{2} \epsilon^{abcd} \omega_{cd}.
\] (39)

Assuming that \( \Lambda = \Lambda(n) \) we get (ignoring the perturbed metric for the moment)

\[
\delta \bar{\Lambda} = - \frac{1}{3} \mu^{abc} \delta n_{abc} - \frac{1}{2} B_{ab} \delta \bar{\omega}^{ab} - \frac{1}{2} \bar{\omega}_{ab} \delta B_{ab},
\] (40)

where we have introduced

\[
\frac{\partial \Lambda}{\partial n_{abc}} = - \frac{1}{3} \mu^{abc}.
\] (41)

However, we also have

\[
\delta n_{abc} = 3 \nabla_{[a} \delta B_{bc]},
\] (42)

which means that

\[
\delta \bar{\Lambda} = \frac{1}{2} \left( \nabla_{a} \mu^{abc} - \bar{\omega}^{bc} \right) \delta B_{bc} - \frac{1}{2} B_{ab} \delta \bar{\omega}^{ab} - \frac{1}{2} \nabla_{a} \left( \mu^{abc} \delta B_{bc} \right).
\] (43)

Ignoring the surface term (as usual), we see that a variation with respect to \( B_{ab} \) requires

\[
\bar{\omega}^{bc} = \nabla_{a} \mu^{abc},
\] (44)

which leads us back to (24). However, with a free variation we would also have \( B_{ab} = 0 \). With the free lunch as elusive as ever, we need to constrain the variation of \( \bar{\omega}_{ab} \) (or rather \( \omega_{ab} \)).

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\(^6\)By construction \( n_{abc} \) is exact, which means that it is automatically closed.

\(^7\)The motivation for the transformation is that we take the vorticity \( \omega_{ab} \) to be the conjugate variable associated with the flux \( n^{a} \). As the vorticity is defined in terms of the momentum \( \mu_{bc} \), which would be the ‘usual’ conjugate, this is not every different from the traditional approach. Moreover, as we will see later, the specific form of (38) is chosen to reproduce the fluid result (see (44)) and leaves the stress-energy tensor unaffected.
Fortunately, the matter space argument from the original fluid derivation provides us with the strategy for doing this.

In order for the vorticity to be a purely spatial object—orthogonal to the flow—we must have
\[ u^a \omega_{ab} = 0. \]  
(45)

In addition, we want it to be ‘fixed’ in the (new) matter space, in the sense that
\[ \mathcal{L}_u \omega_{ab} = 0. \]  
(46)

Since \( \omega_{ab} \) is anti-symmetric, this leads to
\[ u^c \nabla_{[a} \omega_{bc]} = 0. \]  
(47)

Clearly, this condition will be satisfied if
\[ \nabla_{[a} \omega_{bc]} = \partial_{[a} \omega_{bc]} = 0. \]  
(48)

The key difference is that we now make use of a two-dimensional space with coordinates \( \chi^I \) (here, and in the following \( I, J, \ldots \) represent two-dimensional coordinates). We obtain this two-dimensional space either via a map from the original matter space
\[ \hat{\psi}_I^A = \frac{\partial \chi^I}{\partial X^A}, \]  
(49)
or directly from spacetime, using
\[ \bar{\psi}_I^a = \frac{\partial \chi^I}{\partial x^a}. \]  
(50)

The two descriptions are (obviously) consistent since
\[ \bar{\psi}_I^a = \hat{\psi}_I^A \frac{\partial \chi^A}{\partial X^A} = \frac{\partial \chi^I}{\partial x^a}, \]  
(51)

via the chain rule. The coordinates and the corresponding maps are illustrated in figure 1.

Adapting the logic that led to the conserved matter flux, we introduce the matter space tensor \( \omega_{IJ} \), such that
\[ \omega_{ab} = \hat{\psi}_A^a \hat{\psi}_B^b \omega_{AB} = \bar{\psi}_a^I \bar{\psi}_b^J \omega_{IJ}. \]  
(52)

Noting that (48) becomes

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8 At this point, we need to appreciate the difference between fluid elements and topological defects like vortices/strings. The former are naturally associated with worldlines, the tangent vector of which provides the four-velocity \( u^a \). In contrast, a vortex is associated with a two-dimensional world sheet. This world sheet is spanned by two vectors, one timelike and one spacelike. In our discussion, we take the timelike vector be the four velocity. This is an important distinction because it means that \( u^a \) is less directly linked to the motion of the ‘fluid’, which (still) follows from \( u^a \). These notions should become clear as we progress.
it follows that the required condition holds provided $\omega_{IJ}$ only depends on the $\chi^I$ coordinates. The logic is quite familiar.

Next, with $\Delta \chi^I = 0 \implies \delta \chi^I = -\mathcal{L}_\xi \chi^I$, (54)

we have

$\Delta \omega_{ab} = 0$, (55)

which, again ignoring the metric perturbations, leads to

$$\delta \bar{\omega}^{ab} = \frac{1}{2} \epsilon^{abde} \delta \omega_{cd} = -\xi^c \nabla_c \bar{\omega}^{ab} - \epsilon^{abde} \omega_{ed} \nabla_c \xi^e.$$  

(56)

After some algebra, we find that the middle term in (43) leads to (leaving out surface terms)

$$-\frac{1}{2} B_{ab} \delta \bar{\omega}^{ab} = \frac{3}{2} \xi^c \bar{\omega}^{ab} \nabla_c [B_{ab}],$$  

(57)

where have have noted that (44) implies the conservation law

$$\nabla_a \bar{\omega}^{ab} = 0.$$  

(58)

We now see that a variation with respect to $\xi^a$ leads to

$$\frac{3}{2} \bar{\omega}^{ab} \nabla_c [B_{ab}] = \frac{1}{4} \epsilon^{abde} \omega_{de} n_{cab} = n^d \omega_{dc} = 0,$$  

(59)

and we recover the usual fluid equations of motion.

We still do not seem to have made much progress, but the introduction of a two-dimensional ‘vortex space’ is essential if we want to explore the dynamics of a collection of quantized vortices. This will become clear shortly.

For convenience, let us also derive the stress-energy tensor within the new ‘strategy’. Taking (38) as our starting point and noting that
\[ n^2 = -n^a n_a = -\frac{1}{(3!)^2} g_{ab} \epsilon^{bcda} \epsilon^{efgh} n_{bc} n_{ef} n_{gh}. \]  

We have

\[ \delta \Lambda = \frac{\partial \Lambda}{\partial n^2} \delta n^2 = -\frac{2}{3!} \frac{\partial \Lambda}{\partial n^2} (n_a \epsilon^{abc}) \delta n_{abc} + \frac{\partial \Lambda}{\partial n^2} (n_a n_b + n^2 g_{ab}) \delta g^{ab}. \]  

We also need

\[ \frac{1}{4} \delta \epsilon^{abcd} B_{ab} \omega_{cd} = -\frac{1}{4} \tilde{\omega}_{cd} B_{cd} g_{ab} \delta g^{ab}, \]  

to get

\[ \frac{\delta \bar{\Lambda}}{\delta g^{ab}} = \frac{\partial \Lambda}{\partial n^2} (n_a n_b + n^2 g_{ab}) - \frac{1}{4} \tilde{\omega}^{cd} B_{cd} g_{ab}. \]  

Finally,

\[ T_{ab} = \bar{\Lambda} g_{ab} - 2 \frac{\delta \bar{\Lambda}}{\delta g^{ab}} = \Lambda g_{ab} - 2 \frac{\partial \Lambda}{\partial n^2} (n_a n_b + n^2 g_{ab}), \]  

leads us back to (29) once we recall the definition of the momentum (2).

This completes the argument. The introduction of the Kalb–Ramond field shifts the focus onto the vorticity, which is naturally associated with a two-dimensional subspace (replacing the usual three-dimensional matter space). The key point is that we arrive at the fluid equations without explicitly associating the fluid flux \( n^a \) with the four-velocity \( u^a \). Let us now consider this point in more detail.

4. String interlude

In order to form a complete picture—including connections with related problems—and develop some of the tools we need to make progress, it is (perhaps not surprisingly) natural to take a detour in the direction of string theory.

A one-dimensional string moving through spacetime traces out a two-dimensional world sheet\(^9\). This world sheet is naturally spanned by two vectors, one timelike (intuitively taken to be the four velocity of the string \( u^a \)) and one spacelike (naturally, the normalized tangent vector to the string, represented by \( \hat{\kappa}^a \)). These vectors are associated with two-dimensional coordinates\(^10\) such that \( x^a = x^a(\phi^I) \), leading to the tangent surface element

\[ S^{ab} = \epsilon^{IJ} \frac{\partial x^a}{\partial \phi^I} \frac{\partial x^b}{\partial \phi^J}, \]  

with \( \epsilon^{IJ} \) the (normalised) two-dimensional Levi-Civita tensor (density).

Associated with this world sheet we have a bivector (read: an anti-symmetric tensor of rank 2), let us call it \( \Sigma^{ab} \), mathematically parameterised in terms of the two linearly independent

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\(^9\) The world sheet aspect is common between the string problem and that of vortex dynamics. The key difference is that strings tend to be taken to move through a vacuum, whereas a vortex lives in a medium—typically a superfluid condensate. The geometric aspects of the two problems are close, even though some of the physics aspects are different.

\(^10\) When combined, the two sets of coordinates \( \phi^I \) and \( \chi^I \) provide us with the means to represent spacetime.
vectors that span the surface (as the bivector represents a surface, it is natural to think of it as a contravariant object). Noting that a simple timelike bivector can be written as the alternating product of a timelike and a spacelike vector [26] (such that its dual will be a simple spacelike bivector) and assuming the normalisation

$$\sum_{ab} \Sigma^{ab} = -2,$$

we use

$$\Sigma^{ab} = u^a \hat{\kappa}^b - u^b \hat{\kappa}^a,$$

such that

$$\hat{\kappa}^a = \sum_{ab} u_b.$$

The projection into the two-dimensional space spanned by $u^a$ and $\hat{\kappa}^a$ is then given by

$$\Sigma^{ac} \sum_{cb} = \hat{\kappa}^a \hat{\kappa}^b - u^a u_b.$$

Introducing the dual

$$\tilde{\Sigma}_{ab} = \frac{1}{2} \epsilon_{abcd} \sum_{cd} = \epsilon_{abcd} u^d \hat{\kappa}^c,$$

we also have the orthogonal projection

$$\perp_{ab} = \tilde{\Sigma}^{ac} \sum_{cb} = \delta^a_b + u^a u_b - \hat{\kappa}^a \hat{\kappa}_b,$$

and it follows that

$$\tilde{\Sigma}_{ab} \sum_{bc} = 0.$$

Note, for later convenience, that this result follows immediately from the condition that the bivector is simple:

$$\Sigma^{[ab} \Sigma^{cd]} = 0 \iff \sum_{ab} \sum_{cd} \epsilon_{abce} = 0.$$

Finally, the bivector is surface forming, provided that [26]

$$\tilde{\Sigma}_{ab} \nabla_c \sum_{bc} = \tilde{\Sigma}_{ab} \partial_c \sum_{bc} = 0.$$

With this set-up, we may take the bivector to be proportional to the surface element. Letting

$$\Sigma^{ab} = \alpha^{-1/2} \sum_{ab},$$

we have

$$\Sigma^{IJ} = \alpha^{-1/2} \sum_{IJ} = \alpha^{-1/2} \epsilon^{IJ}.$$
\[
\gamma_{IJ} = g_{ab} \frac{\partial \chi^a}{\partial \phi^I} \frac{\partial \chi^b}{\partial \phi^J},
\]  

(77)

we have

\[
\gamma_{IK} \epsilon_{JL} \epsilon_{KL} = -2\alpha,
\]  

(78)

and we identify

\[
\alpha = -\gamma = -\det \gamma_{IJ}.
\]  

(79)

That is, we have

\[
\Sigma^{ab} = \sqrt{-\gamma} \frac{\partial}{\partial \phi^I} \frac{\partial}{\partial \phi^J} \epsilon_{IJ}.
\]  

(80)

Geometrically, the dual of \(\Sigma^{ab}\) is a two-form that represents (when integrated) the flux carried by vortices (strings) across a surface. The variable \(\gamma\) is a measure of this flux.

It is now natural to assume\(^{11}\) that the Lagrangian of the system depends on \(\gamma\), with

\[
\gamma = \frac{1}{2} \Sigma^{ab} \Sigma_{ab} = \frac{1}{2} \Sigma^{IJ} \Sigma_{IJ} \quad (= -1).
\]  

(81)

Moreover, as we want to compare and contrast with a model based on averaging over a network of vortices—essentially treated as a fluid described by a small number of fields (density, velocity, tension etcetera)—it is natural to consider the example of a coarse-grained 'string fluid’ [35–37]. Hence, we take \(\sqrt{-\gamma} \Lambda(\gamma)\) to be the matter contribution to the action, noting that, if we let \(\Lambda = M \sqrt{-\gamma}\) this leads to the coarse-grained version of the standard Nambu–Goto string action [25, 27], with \(M\) the string tension.

For the stress-energy tensor we need

\[
\delta \Lambda = \frac{d\Lambda}{d\gamma} \left( \frac{\partial \gamma}{\partial \Sigma^{ab}} \delta \Sigma^{ab} + \frac{\partial \gamma}{\partial g_{ab}} \delta g_{ab} \right) + \frac{d\Lambda}{d\gamma} \left( \Sigma_{ab} \delta \Sigma^{ab} + \Sigma^{a}_{\ c} \Sigma^{cb} \delta g_{ab} \right),
\]  

(82)

so

\[
T^{ab} = \Lambda g^{ab} + 2 \frac{\delta \Lambda}{\delta g_{ab}} = \Lambda g^{ab} + 2 \frac{d\Lambda}{d\gamma} \Sigma^{a}_{\ c} \Sigma^{cb}.
\]  

(83)

From this it follows that the equations of motion are

\[
\nabla_a T^{ab} = g^{ad} \nabla_d \Lambda + 2 \Sigma^{a}_{\ c} \Sigma^{cb} \nabla_a \left( \frac{d\Lambda}{d\gamma} \right) + 2 \frac{d\Lambda}{d\gamma} \nabla_a \left( \Sigma^{a}_{\ c} \Sigma^{cb} \right) = 0.
\]  

(84)

However, since \(\gamma = -1\) we have \(\nabla_a \gamma = 0\), which means that we only need

\[
\nabla_a \left( \Sigma^{a}_{\ c} \Sigma^{cb} \right) = \Sigma^{cb} \nabla_a \Sigma^{a}_{\ c} + \frac{1}{2} \Sigma_{ca} \left( \nabla^a \Sigma^{cb} + \nabla^c \Sigma^{ba} + \nabla^b \Sigma^{ca} \right)
\]

\[
= \Sigma^{cb} \nabla_a \Sigma^{a}_{\ c} + 3 \Sigma_{ca} \nabla^{[a} \Sigma^{cb]} = 0,
\]  

(85)

\(^{11}\) The essence of the argument is that we want to find the metric that minimizes the area. In two dimensions all metrics are conformally flat, so there is only one degree of freedom to vary, and it is natural to associate this with the determinant \(\gamma\).
where we have used (66). Following [26], we contract with \( \Sigma_{db} \) to get
\[
\Sigma_{db} \Sigma^{db} \nabla_a \Sigma^c_a + 3 \Sigma_{[da} \Sigma_{b]c} \nabla^a \Sigma^{ac} = 0, \tag{86}
\]
where the second term vanishes since the bivector is simple, cf (73). Noting also that
\[
\Sigma_{db} \Sigma^{db} \nabla_a \Sigma^c_a = 0 \quad \implies \quad \Sigma_{ab} \nabla_a \Sigma^{ac} = 0, \tag{87}
\]
and considering (74), we infer the conservation law [26, 37]
\[
\nabla_a \Sigma^{ab} = 0. \tag{88}
\]
In essence, if the contractions of a vector with both the bivector and its dual vanish then the vector must be zero. Returning to the equations of motion, we are left with
\[
\Sigma^c_e \nabla_a \Sigma^{eb} = 0, \tag{89}
\]
or
\[
\Sigma^c_e \nabla_a \Sigma^{eb} = 0. \tag{90}
\]
This is the simplest version of the model—describing how the surface tension serves to drive the system towards a minimum area—which will be sufficient for our purposes. Still, it is interesting to note extensions like the dissipative case considered in [37] and the discussion of charged cosmic strings in [38].

Before we move on, let us establish two useful results. First of all, we have
\[
\Sigma^a_e \nabla_a \Sigma^{eb} = 0, \tag{91}
\]
by virtue of (88). Similarly
\[
\Sigma^a_e \nabla_a \Sigma^{eb} = 0. \tag{92}
\]

5. Vortex dynamics

A natural extension to the model developed in section 3 allows \( \Lambda \) to depend on both \( n_{abc} \) and \( \omega_{ab} \) from the outset. Intuitively, such a model represents a superfluid condensate with (averaged) vorticity represented by a collection of vortices. At the quantum level, the dynamics would be represented by a single wave function, but at the fluid level we can always describe the problem in terms of an irrotational condensate and a contribution from vortices.

Starting from \( \Lambda = \Lambda(n_{abc}, \omega_{ab}, g^{ab}) \) we immediately have (using the convention from [33])
\[
\delta \Lambda = \frac{1}{3!} \mu^{abc} \delta n_{abc} - \frac{1}{2} \chi^{ab} \delta \omega_{ab} + \frac{\delta \Lambda}{\delta g^{ab}} \delta g^{ab}, \tag{93}
\]
where
\[
\chi^{ab} = -2 \frac{\partial \Lambda}{\partial \omega_{ab}}. \tag{94}
\]
The first term in (93) may be interpreted as the energy cost associated with introducing additional particles in the system, while the second term is associated with rotational energy. From (38) it then follows that (ignoring the metric variation and a surface term, as before)
\[ \delta \Lambda = \frac{1}{2} \left( \nabla_i \mu^{abc} - \tilde{\omega}^{abc} \right) \delta B_{db} - \frac{1}{2} \left( \lambda^{cd} + \frac{1}{2} \epsilon^{abcd} B_{db} \right) \delta \omega_{cd}, \]  
\tag{95} \]

which leads us back to (43) and (44). However, we now have an additional term involving \( \delta \omega_{ab} \).

Making use of (48), this new term can be written
\[ -\frac{1}{2} \lambda^{cd} \delta \omega_{cd} = \frac{1}{2} \lambda^{cd} \left( \xi^a \nabla_a \omega_{cd} + 2 \omega_{ca} \nabla_c \xi^a \right) = -\xi^a \omega_{ca} \nabla_c \lambda^{cd}. \]  
\tag{96} \]

Combining this with the result from the previous section, we see that a variation with respect to the displacement leads to (see [32–34])
\[ n^a \omega_{ab} = \omega_{ab} \nabla^c \Lambda^{ca} = -2 \omega_{ab} \nabla_c \left( \frac{\partial \Lambda}{\partial \omega_{cd}} \right). \]  
\tag{97} \]

Basically, the explicit dependence on the vorticity has led to amended equations of motion. However, if we want to interpret the term on the right-hand side of (97) we need to do a little bit more work.

First of all, it is worth noting that we may write (97) as
\[ \left[ n^a + 2 \nabla_c \left( \frac{\partial \Lambda}{\partial \omega_{cd}} \right) \right] \omega_{ab} \equiv \tilde{n}^a \omega_{ab} = 0, \]  
\tag{98} \]
with
\[ \tilde{n}^a = n^a + 2 \nabla_c \left( \frac{\partial \Lambda}{\partial \omega_{cd}} \right). \]  
\tag{99} \]

This means that \( \tilde{n}^a \) must be proportional to \( u^a \), which makes the result appear more ‘familiar’ (see section 2) but it does not really help us understand the ingredients in (97).

Instead, let us consider the implications of the two-dimensional matter space. Intuitively, the idea makes sense for a collection of (locally) aligned quantized vortices as one can always introduce a two-dimensional surface orthogonal to the vortex array (that is, orthogonal to the world sheet we used in the discussion of strings). Points in this surface are described by the \( \chi^I \) coordinates. Not surprisingly, we can adapt the logic from the usual matter-space construction to this new setting—although in doing so we would focus on the map from the original three-dimensional space to the two-dimensional one. As is evident from (53), we also need the map from spacetime to either of the two lower-dimensional spaces. In essence, the original fluid derivation involved
\[ \psi^A B^a = \delta^a_b + u^a \psi_b. \]  
\tag{100} \]

Meanwhile, the corresponding map to the two-dimensional stage takes the form
\[ \tilde{\psi}^A \tilde{\psi}^B = \delta^A_B - \hat{k}^A \hat{k}_B, \]  
\tag{101} \]
with a suitable spatial unit vector \( \hat{k}^a \), automatically orthogonal to the four velocity \( u^a \) since
\[ u^a \hat{k}_a = (u^a \psi_A^a) \hat{k}^A = 0. \]  
\tag{102} \]

We take the new vector \( \hat{k}^a \) to be normal to the area spanned by the \( \chi^I \) coordinates (and identify it with the spacelike coordinate we used in the discussion of the string world sheet). That is, we have
\[ \hat{\kappa}^A \hat{\psi}^I_A = 0. \]  

(103)

In essence, \( \hat{\kappa}^A \) is assumed to be aligned with the quantized vortices. It also follows that

\[ \bar{\psi}^I_a \bar{\psi}^b_I = (\psi^I_a \hat{\psi}^I_A) (\psi^b_B \hat{\psi}^I_B) = \psi^I_a \psi^b_B (\delta^I_A - \hat{\kappa}^A \hat{\kappa}^B) = \delta^b_a + u^a du^b - \hat{\kappa}^A \hat{\kappa}^B \equiv \frac{1}{u^I}. \]  

(104)

This will be relevant later.

In order to stress the close resemblance to the various relations for \( n_{ABC} \) from section 2, we first of all introduce a vector

\[ W^A = \frac{1}{2} e^{ABC} \omega_{BC} \implies \omega_{AB} = \epsilon_{ABC} W^C. \]  

(105)

In spacetime, we then have

\[ W^a = \frac{1}{2} \psi^a_A e^{ABC} \omega_{BC} = \frac{1}{2} \psi^b_A \psi^c_B e^{ABC} \omega_{bc} = \frac{1}{2} u_{d ef} e^{dabc} \omega_{bc}. \]  

(106)

We recognize this as the vorticity vector [3] and note that it is simply related to the dual of the vorticity:

\[ W^a = u_d \omega^d a. \]  

(107)

We may also work in the two-dimensional space, where we must have

\[ \omega_{IJ} = N \kappa_{IJ} \implies \omega_{AB} = N \kappa_{AB}. \]  

(108)

with (for future reference)

\[ \epsilon_{IJ} \epsilon^{JK} = \delta^K_I, \]  

(109)

\[ \epsilon_{IJ} \epsilon^{IJ} = 2, \]  

(110)

and

\[ \epsilon_{AB} = \kappa^C \epsilon_{CAB}. \]  

(111)

Letting \( \kappa^A = \kappa \kappa^A \), we now have

\[ \omega_{AB} = N \kappa^C \epsilon_{CAB}. \]  

(112)

so

\[ \kappa^A \omega_{AB} = 0. \]  

(113)

In fact, we have

\[ W^A = N \kappa^A. \]  

(114)

The interpretation of this is intuitive—we have a collection of vortices, each associated with a quantum \( \kappa \) of circulation—with number density (per unit area) \( N \).

We also have

\[ W^2 = (N \kappa)^2 = \frac{1}{2} \omega_{IJ} \omega^{IJ} = \frac{1}{2} \omega_{AB} \omega^{AB} = \frac{1}{2} \omega_{ab} \omega^{ab} = \frac{1}{2} \delta^{abc} \omega_{ab} \omega_{cd}. \]  

(115)
Finally, the spacetime vorticity takes the (expected) form
\[ \omega_{ab} = N u^c \kappa^d \epsilon_{cdab}, \quad (116) \]
(explicitly connecting to the dual \( \tilde{\Sigma}_{ab} \) used to describe string dynamics in section 4). We also have
\[ L_u \kappa_a = L_u (\psi^A \kappa_A) = \psi^A L_u \kappa_A = \psi^A (u^c \psi^b) \frac{\partial \kappa_A}{\partial X^b} = 0, \quad (117) \]
\[ u^b \nabla_b N = (u^b \tilde{\psi}^I) \frac{\partial N}{\partial \chi^I} = 0, \quad (118) \]
and
\[ \kappa^a \nabla_a N = \kappa^a \tilde{\psi}^I \frac{\partial N}{\partial \chi^I} = \psi^A \kappa^a \tilde{\psi}^I \frac{\partial N}{\partial \chi^I} = \kappa^A \frac{\partial \tilde{\psi}^I}{\partial \chi^I} = 0. \quad (119) \]
These results are quite intuitive, and (for later convenience) it is worth noting that
\[ (u^a u_b - \hat{\kappa}^a \hat{\kappa}_b) \nabla_a N = 0 , \quad (120) \]
and we will also need to recall (91) and (92). Let us now return to the equations of motion (97). If we consider an explicit model where \( \Lambda = \Lambda(n^2, \dot{N}^2) \), we have
\[ \frac{\partial \Lambda}{\partial \omega_{ab}} = \frac{\partial \Lambda}{\partial N^2} \frac{\partial N^2}{\partial \omega_{ab}} = \frac{\partial \Lambda}{\partial N^2} \omega_{ab} = -\frac{1}{2} \lambda^{ab}, \quad (121) \]
and we arrive at
\[ n^a \omega_{ab} = -\frac{2}{\kappa^2} \omega_{ab} \nabla_c \left( \frac{\partial \Lambda}{\partial N^2} \omega^{ca} \right) = -\frac{1}{\kappa} \omega_{ab} \nabla_c \left( \frac{\partial \Lambda}{\partial N} \frac{1}{\dot{N}} \omega^{ca} \right). \quad (122) \]
Making use of (116) we have
\[ \frac{1}{\kappa} \omega_{ab} \nabla_c \left( \frac{\partial \Lambda}{\partial N} \frac{1}{\dot{N}} \omega^{ca} \right) = -\dot{N} \frac{1}{\kappa} \left[ \nabla_a \left( \frac{\partial \Lambda}{\partial N} \frac{1}{\dot{N}} \omega^{ca} \right) - \frac{\partial \Lambda}{\partial N} (\dot{\kappa}^c \epsilon_{ca} - u^c \nabla_c u_a) \right]. \quad (123) \]
Here it is worth noting that \(-\partial \Lambda/\partial \dot{N}\) is naturally interpreted as the energy per vortex (assuming that all vortices carry the same circulation and that the averaged energy is simply proportional to the vortex density. It is straightforward to make a connection with the ‘thin vortex’ limit considered in [34] but we will not do so here. Suppose that we also introduce a (distinct) four-velocity associated with the matter flux (the condensate), i.e. let
\[ n^a = n a^\alpha \]  

such that

\[ u^a_n = \gamma (u^a + v^a), \quad u^a v_a = 0, \quad \gamma = (1 - v^2)^{-1/2}. \]

We then have

\[ n^a \omega_{ab} = n \gamma N \hat{v}^a \kappa_c \delta^c_{ab} = n \gamma N \epsilon_{bac} \kappa^a v^c. \]  

This represents the Magnus force that acts on a set of vortices moving relative to a superfluid condensate (represented by \( n^a \)) (see, for example, [39]). Also recognizing the tension associated with the bending of the vortices, we have the final equations of motion

\[ n \gamma \epsilon_{bac} \kappa^a v^c = \nabla_a \left( \frac{\partial \Lambda}{\partial N} - \frac{\partial \Lambda}{\partial N} \kappa^c \nabla_c \hat{v}_a + \frac{\partial \Lambda}{\partial N} u^d \nabla_d u_a \right). \]

For completeness, and immediate benefit for the discussion of elasticity, we should also work out stress-energy tensor for this model. This is fairly straightforward, as the required calculation repeats (61), apart from that we now need to account for the \( N^2 \) dependence on \( \Lambda \). With \( \Lambda = \Lambda(n^2, N^2) = \Lambda(n_{abc}, \omega_{ab}, g^{ab}) \) we need

\[ \frac{\partial \Lambda}{\partial N} \kappa^c \delta^c_{ab} = \frac{1}{2} \frac{\partial \Lambda}{\partial N} \frac{\partial N}{\partial \Lambda} \nabla^a \nabla^b. \]  

Combining this with the previous (fluid) result, we have

\[ T_{ab} = (\Lambda - n^a \mu_c) g_{ab} + n_a n_b - N^2 \frac{\partial \Lambda}{\partial N} \nabla^a \nabla^b. \]

A direct calculation verifies that the divergence of this expression leads us back to (127). It is a straightforward exercise, which involves (91), (92), (118) and (119).

6. Two-dimensional elasticity

The role of the two-dimensional vortex space should be clear from the derivation of (127). The developments relied heavily on results that are easy to obtain once we introduce the \( X^I \) coordinates. The power of this approach becomes even more apparent when we consider elastic aspects of the vortex lattice. This should be expected from the corresponding problem for the neutron star crust—where the geometry of the configuration space associated with the \( X^I \) coordinates plays a central role [22, 40]. Adapting the argument to the two-dimensional case, we can account for stresses and strains of the vortex lattice.

We focus our attention on two new matter-space tensors. First of all, we introduce another object that remains fixed along the flow;

\[ k_{ab} = \tilde{v}_a^I \tilde{v}_b^J k_{IJ}, \]  

such that
\[ u^b k_{ab} = 0, \quad (132) \]

and

\[ \mathcal{L}_u k_{ab} = 0, \quad (133) \]

In essence, it follows (as in the case of \( \omega_{ab} \)) that

\[ \Delta k_{ab} = 0. \quad (134) \]

Next, we introduce

\[ \eta_{ab} = \bar{\psi}_a \bar{\psi}_b \eta_{IJ}, \quad (135) \]

to represent the relaxed lattice configuration. This simply means that, in absence of elastic stresses we have \([22, 40]\)

\[ g_{IJ} \eta_{JK} = \delta^I_J, \quad (136) \]

where

\[ g^{IJ} = \bar{\psi}_a \bar{\psi}_b \eta_{ab} \]

As the spacetime evolves with the system, we can represent the elastic strain associated with any deformation by

\[ s_{AB} = \frac{1}{2} (g_{IJ} - \eta_{IJ}) \quad \Rightarrow \quad s_{ab} = \frac{1}{2} (\perp_{ab} - \eta_{ab}). \quad (138) \]

Finally, we focus on conformal deformations, for which (adapting the argument from the appendix in \([40]\) to the present two-dimensional setting) we have

\[ k_{ab} = W \eta_{ab} = N \kappa \eta_{ab}, \quad (139) \]

and—as we are mainly interested in modest effects—we also make the Hookean approximation

\[ \Lambda = \tilde{\Lambda} (n^2, N^2) - \tilde{\mu} (N) s^2, \quad (140) \]

where the sign is motivated by the fact that the energy measured by an observer moving along with the vortex array (with four velocity \( u^a \)) is \( \varepsilon = -\Lambda \) and \( \tilde{\mu} \) represents the shear modulus. As in \([40]\) we use checks to indicate that quantities are evaluated for the unstrained configuration. In effect, the first term in \((140)\) remains as in the previous section, so we may focus on the second contribution. Clearly, this leads to

\[ \delta \Lambda = \delta \tilde{\Lambda} - \frac{\partial \tilde{\mu}}{\partial N} \delta N - \tilde{\mu} \delta s^2. \quad (141) \]

The middle term is readily evaluated using results we already have at hand. The final term is different, as we have to provide a form for the strain scalar \( s^2 \) in order to make progress.
In general, we will have $s^2 = s^2(N, k_{ab}, g^{ab})$, which means that

$$
\delta \Lambda = \frac{\partial \delta \Lambda}{\partial n^2} n^2 + \left[ \frac{\partial \Lambda}{\partial N} \delta N + \frac{\partial \mu}{\partial N} s^2 + \frac{\partial \xi^2}{\partial N} \right] \delta N - \frac{\partial \mu}{\partial k_{ab}} \delta k_{ab} + \frac{\partial \xi^2}{\partial g^{ab}} \delta g^{ab} \right].
$$

(142)

Focussing on terms associated with the vortex elasticity, we need

$$
\delta N = \frac{1}{2N \kappa^2} \left( g^{ac} g^{bd} \omega_{ad} \delta \omega_{ab} + g^{cd} \omega_{ad} \omega_{bc} \delta g^{ab} \right),
$$

(143)

where $\Delta \omega_{ab} = 0$ allows us to shift the focus onto a variation with respect to the displacement $\xi^a$ (as before):

$$
\delta \omega_{ab} \to 0 \implies \delta \omega_{ab} = -\xi \nabla \omega_{ab} - \omega_{ab} \nabla \xi - \omega_{ac} \nabla b \xi.
$$

(144)

Similarly, we have

$$
\Delta k_{ab} = 0 \implies \delta k_{ab} = -\xi \nabla \kappa - k_{ab} \nabla \xi - k_{ac} \nabla \xi + k_{bc} \nabla \xi,
$$

(145)

and it follows that the elastic contributions to the stress-energy tensor are [with the ... representing terms that remain as in (130)]

$$
\frac{\delta \Lambda}{\delta g^{ab}} = \ldots + \frac{N}{2} \left[ \frac{\partial \mu}{\partial N} s^2 + \frac{\partial \xi^2}{\partial N} \right] \nabla \mu - \frac{\partial \mu}{\partial \kappa^{ab}}
$$

$$
= \ldots + \frac{N}{2} \frac{\partial \mu}{\partial N} s^2 - \frac{\partial \xi^2}{\partial N} \nabla \mu + \frac{\partial \xi^2}{\partial g^{ab}}
$$

(146)

This may be as far as we can get without specifying the strain scalar $s^2$. An intuitive approach to that part of the problem $\{22, 40\}$ is to build $s^2$ out of ‘invariants’ of $\eta_{ab}$. In two dimensions it makes sense to use

$$
I_1 = \eta_1 = \frac{1}{N \kappa} s^2 k_{IJ},
$$

(147)

and

$$
I_2 = \eta_1 \eta_1 = \frac{1}{(N \kappa)^2} s^2 g^{KL} k_{IK} k_{JI}.
$$

(148)

We then have

$$
\frac{\partial s^2}{\partial N} = \frac{\partial s^2}{\partial I_1} \frac{\partial I_1}{\partial N} + \frac{\partial s^2}{\partial I_2} \frac{\partial I_2}{\partial N} = -\frac{\partial s^2}{\partial I_1} \frac{\partial I_1}{\partial N} - 2 \frac{\partial s^2}{\partial I_2} \frac{\partial I_2}{\partial N},
$$

(149)

and

$$
\frac{\partial s^2}{\partial g^{ab}} = \frac{\partial s^2}{\partial I_1} \frac{\partial I_1}{\partial g^{ab}} + \frac{\partial s^2}{\partial I_2} \frac{\partial I_2}{\partial g^{ab}} = \eta_1 \eta_1 \frac{\partial s^2}{\partial I_1} \frac{\partial I_1}{\partial g^{ab}} + \frac{\partial s^2}{\partial I_2} \frac{\partial I_2}{\partial g^{ab}}
$$

$$
= \eta_1 \eta_1 \frac{\partial s^2}{\partial I_1} \eta_{IJ} + 2 \frac{\partial s^2}{\partial I_2} \eta_{IJ} \eta_{IJ},
$$

(150)

which leads to

$$
\frac{N}{2} \frac{\partial s^2}{\partial N} \nabla \mu + \frac{\partial s^2}{\partial g^{ab}} = \frac{\partial s^2}{\partial g^{IJ}} \eta_{IJ} + 2 \frac{\partial s^2}{\partial g^{IJ}} \eta_{IJ},
$$

(151)
where the $\langle \ldots \rangle$ indicate that the trace is removed. The fact that this object is trace-free indicates that it represents anisotropic stresses. The result is (naturally) similar to that from the three-dimensional problem [see, e.g. equation (80) in [40]].

Putting the pieces together, the complete stress-energy tensor takes the form

$$
T_{ab} = (\ddot{\Lambda} - n^a \dot{\mu}_c - \ddot{\mu} \delta_a^c) g_{ab} + n_a \mu_b - N^c \left( \frac{\partial \ddot{\Lambda}}{\partial \eta} - \frac{\partial \dot{\mu}}{\partial \eta} \delta^c_{ab} \right) + 2 \ddot{\mu} \left( \frac{\partial \delta^2}{\partial \eta_{(ab)}} + 2 \frac{\partial \delta^2}{\partial \eta_{(c)} \eta_{(b)}} \right).
$$

(152)

In order to work out the new terms in the equations of motions, we first write the elastic contributions

$$
T_{ab} = \ldots - \ddot{\mu} \delta^c_{ab} g_{ab} + N^c \frac{\partial \dot{\mu}}{\partial \eta} \delta^c_{ab} + 2 \ddot{\mu} \left( \frac{\partial \delta^2}{\partial \eta} + 2 \frac{\partial \delta^2}{\partial \eta_{(c)} \eta_{(b)}} \right) + \ldots \ldots - \ddot{\mu} \delta^c_{ab} g_{ab} + N^c \frac{\partial \dot{\mu}}{\partial \eta} \delta^c_{ab} + 2 \ddot{\mu} \left( \frac{\partial \delta^2}{\partial \eta} + 2 \frac{\partial \delta^2}{\partial \eta_{(c)} \eta_{(b)}} \right) + \pi_{ab},
$$

(153)

That is, we need

$$
\nabla_a \left[ - \ddot{\mu} \delta^c_{ab} + N^c \frac{\partial \dot{\mu}}{\partial \eta} \delta^c_{ab} + \frac{\partial \delta^2}{\partial \eta} \right] = - s^2 \frac{\partial \ddot{\mu}}{\partial \eta} \nabla_b N - \ddot{\mu} \nabla_b s^2 + \frac{\partial \ddot{\mu}}{\partial \eta} \left[ s^2 \nabla_b N + N s^2 \nabla_a \nabla b + N \nabla b s^2 \right],
$$

(154)

where we have used (120) and an analogous argument for $s^2$, which is also a matter space object. It follows that the elastic contribution is

$$
\nabla_a T^b_{ab} = \ldots + \nabla_a \left[ \left( N^c \frac{\partial \dot{\mu}}{\partial \eta} - \ddot{\mu} \right) \nabla_b s^2 + \frac{\partial \ddot{\mu}}{\partial \eta} \right] + \nabla_a \pi^b_{ab},
$$

(155)

and the complete equations of motion take the form

$$
n^a \omega_{ab} = \ldots - \nabla_a \left[ \left( N^c \frac{\partial \dot{\mu}}{\partial \eta} - \ddot{\mu} \right) \nabla_b s^2 + \frac{\partial \ddot{\mu}}{\partial \eta} \right] + \nabla_a \pi^b_{ab},
$$

(156)

This is the final result, describing the dynamics of an elastic vortex array in full general relativity. It can be meaningfully compared to the corresponding relation for an elastic nuclear lattice, e.g. equation (85) in [40]. Notably, the model is nonlinear (although the Hookean assumption (140) obviously implies a Taylor expansion for weak strains). This is in contrast to previous (Newtonian) models, which have exclusively been perturbative.

7. Perturbations and the Newtonian limit

If we want to compare the final equations of motion (156) to the corresponding expression in the Newtonian context [6–8] we need to do a bit more work. The usual expression for vortex elasticity tends to be given in terms of displacement vectors, i.e. at the perturbative level. In order to facilitate a comparison we need to reframe the result in terms of explicit (Lagrangian)
perturbations with respect to an unstrained background configuration. The strategy for doing this is the same as in the case of an elastic nuclear lattice, e.g. the problem considered in [40]. The fact that the elastic contribution is two-dimensional makes little conceptual difference. However, we need to pay careful attention to the unperturbed configuration.

A suitable background configuration involves two spacetime symmetries. First of all, the assumption that the problem is stationary implies the existence of a timelike Killing vector. Taking the four velocity of the configuration to the aligned with this Killing vector, it follows immediately that

$$u^a \nabla_a u_b = 0. \tag{157}$$

This is natural and intuitive—there is no acceleration associated with the equilibrium configuration. We also need to consider the vortex array, which in equilibrium ought to be associated with axisymmetry. Letting the vortex vector $\hat{\kappa}^a$ be aligned with a second (spatial) Killing vector, we see that we should also have

$$\hat{\kappa}^a \nabla_a \hat{\kappa}_b = 0. \tag{158}$$

The implications are that there is no contribution from the tension—intuitively, the vortices are ‘straight’—and the vortex array moves without expansion/contraction. Again, this makes sense. Finally, the absence of elastic strain implies that background is such that both $s^2$ and $\pi^{ab}$ vanish. In practice, we have

$$\eta_{ab} = \perp_{ab}, \tag{159}$$

for a relaxed configuration.

Turning to the perturbed case and the anisotropic stresses, it is helpful to make the model (even more) specific. In order to construct a suitable combination to represent the strain scalar, we first of all note that $I_1 = I_2 = 2$ in the relaxed configuration. Secondly, we know that $k_{IJ}$ is (by construction) independent of $W$ (or equivalently $\mathcal{N}$) so we can scale out the dependence on this quantity if we work with $I_1^2$ and $I_2$. A simple possibility\footnote{This choice does not in any way represent a restriction of the model. Other combinations of the invariants would be equally acceptable, as long as they lead to the strain vanishing ($s^2 = 0$) in the relaxed configuration. Different choices simply represent different parameterizations of the strain, and these will be normalized once with consider, for example, the breaking strain of the configuration.} would then be

$$s^2 = I_2 - \frac{1}{2} I_1^2, \tag{160}$$

which leads to

$$\frac{\partial s^2}{\partial I_1} = -I_1 \quad \text{and} \quad \frac{\partial s^2}{\partial I_2} = 1. \tag{161}$$

This means that

$$\pi^{ab} = -2 \tilde{\mu} g^{ad} \left( I_1 \eta_{db} - 2 \eta_{[d} \eta_{b]} \right) = -2 \tilde{\mu} g^{ad} \left[ I_1 \left( \eta_{db} - \frac{1}{2} I_1 g_{db} \right) - 2 \left( \eta_{d} \eta_{b}^{[d} - \frac{1}{2} I_1 g_{d} g_{[b]} \right) \right]. \tag{162}$$

As the combined terms in the bracket vanish for an unstrained background, which we perturb with respect to, we have
\[ \Delta \pi^a_b = -2 \hat{\mu} g^{ad} \Delta \left[ I_1 \left( \eta_{db} - \frac{1}{2} I_1 g_{db} \right) - 2 \left( \eta_{cd} \eta^e_b - \frac{1}{2} I_2 g_{db} \right) \right]. \] 

(163)

We now need

\[ \Delta N = \frac{N}{2} \tilde{\psi}_i \tilde{\psi}_j g_{ij} \Delta g^{ac} = \frac{1}{2} N \perp_{ac} \Delta g^{ac}. \]

(164)

Combining this with \( \Delta k_{ab} \), we have

\[ \Delta \eta_{ab} = \Delta \left( \frac{1}{N} k_{ab} \right) = - \frac{1}{N} k_{ab} \Delta N = - \frac{1}{2} \eta_{ab} \perp_{cd} \Delta g_{cd}. \]

(165)

It also follows that

\[ \Delta I_1 = \Delta \left( g^{ab} \eta_{ab} \right) = \perp_{ab} \left( \Delta g^{ab} - \frac{1}{2} \perp_{cd} \Delta g^{cd} \right) = 0, \]

(166)

since \( g^{ab} \perp_{ab} = \delta^a_a - 1 - 1 = 2 \), and

\[ \Delta I_2 = \Delta \left( g^{ac} g^{bd} \eta_{ab} \eta_{cd} \right) = 2 \perp_{ac} \Delta g^{ac} + 2 \perp^{cd} \left( - \frac{1}{2} \perp_{cd} \perp_{ab} \Delta g^{ab} \right) = 0. \]

(167)

Putting things together, we are left with

\[ \Delta \pi^a_b = 4 \hat{\mu} g^{ad} \left[ \perp_{fd} \perp_{be} - \frac{1}{2} \perp_{db} \perp_{ef} \right] \Delta g^{ef}. \]

(168)

This result is notably similar to the corresponding expression of an elastic nuclear lattice, e.g. equation (91) in [40]. The main difference is that the projection is no orthogonal to both \( \gamma^a \) and \( \hat{\kappa}^a \) and the two-dimensional nature of the lattice leads to a factor of \( 1/2 \) rather than the usual \( 1/3 \) (this is simply the inverse of the number of dimensions of the elastic matter). In essence, the result makes intuitive sense.

The final step involves expressing the contribution to the perturbed equations of motion in terms of the displacement vector. In order to do this, we need

\[ \Delta \nabla_a \pi^a_b = \nabla_a \Delta \pi^a_b, \]

(169)

which is true as long as the background is unstrained. This then means that

\[ \Delta \pi^a_b = 4 \hat{\mu} \left[ \perp^a_f \perp_{be} - \frac{1}{2} \perp^a_b \perp_{ef} \right] \delta g^{ef} + 4 \hat{\mu} \left( \perp^a_b \perp^c_e \perp_{ac} \perp_{be} - \perp^a_b \perp^c_e \right) \nabla_c \xi^e. \]

(170)

At this point it is natural to introduce the totally projected derivative (with a projection of each free index)

\[ D_c \xi^f = \perp^a_c \perp^f_b \nabla_d \xi^b. \]

(171)

This is helpful as it facilitates an immediate comparison with Newtonian expressions. We then have
\[ 4\dot{\mu} \left( \dot{x}_a \dot{x}_b - \dot{x}_c \dot{x}_b - \dot{x}_c \dot{x}_a \right) \nabla_c \xi^e = -4\dot{\mu} \dot{x}_a \dot{x}_b \left[ D_i \xi_j + D_j \xi_i - g_{ij} D_k \xi^k \right] = -\dot{x}_a \dot{x}_b \Pi_{ij}, \]

(172)

where we have defined

\[ \Pi_{ij} = 4\dot{\mu} \left[ D_i \xi_j + D_j \xi_i - \dot{\xi}_i \dot{\xi}_j \right], \]

(173)

which is manifestly orthogonal to both \( u^a \) and \( \kappa^a \). This leads to

\[ \nabla_a \left( \dot{x}_a \dot{x}_b \Pi_{ij} \right) = D^\ell \Pi_{ib} + \Pi_{ib} \left( u^a \nabla_a u^b - \kappa^a \nabla_a \kappa^b \right) + \Pi_{ij} \left( u_b \nabla_a u^b - \kappa_b \nabla_a \kappa^b \right). \]

(174)

We have already argued that the second term on the right-hand side should vanish for a suitable equilibrium configuration, and it is easy to show (since \( \Pi_{ab} \) is symmetric) that the Killing vector argument removes the third term, as well. We are left with

\[ \nabla_a \left( \dot{x}_a \dot{x}_b \Pi_{ij} \right) = D^\ell \Pi_{ib} = 4D^\ell \left[ \dot{\mu} \left( D_i \xi_b + D_b \xi_i - \dot{\xi}_i \dot{\xi}_b \right) \right]. \]

(175)

This result resembles the result from the Newtonian setting, e.g. equation (24) in [7], although one has to keep in mind that the derivatives do not commute in a curved spacetime.

8. Final remarks

Starting from the well-established variational approach to relativistic fluids, and bringing in notions from a basic description of string dynamics, we have developed a model for superfluid vortex dynamics. This provides a valuable—if somewhat technical—extension to previous efforts to account for elastic properties of a vortex array. The approach to the problem is of conceptual interest as it highlights the role of the two-dimensional subspace orthogonal to a given vortex array (analogous to the world sheet coordinates used to describe a moving string in spacetime). The discussion also provides a detailed description of concepts that have previously been described in a somewhat phenomenological manner, and which may be applied to interesting problems in astrophysics and cosmology.

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